IRREDUCIBLE SUBQUOTIENTS OF GENERIC GELFAND-TSETLIN MODULES OVER $U_q(\mathfrak{gl}_n)$

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Abstract. We provide a classification and explicit bases of tableaux of all irreducible subquotients of generic Gelfand-Tsetlin modules over $U_q(\mathfrak{gl}_n)$ where $q \neq \pm 1$.

1. Introduction

Recently there has been a breakthrough in the theory of Gelfand-Tsetlin modules in the papers [8], [9], [10], [11]. In these papers new classes of simple $\mathfrak{gl}_n$-modules were constructed generalizing the classical Gelfand-Tsetlin bases [14], [23] for finite-dimensional representations. These new representations also have a basis consisting of Gelfand-Tsetlin tableaux but such tableaux are not necessarily eigenvectors of the Gelfand-Tsetlin subalgebra [6]. This fact requires a modified action of the generators of the Lie algebra on this basis. Gelfand-Tsetlin representations are related to the theory of integrable systems [1], [2], [3], [4], [20], [21], general hypergeometric functions on the complex Lie group $GL(n)$, [15], [16]; solutions of the Euler equation, [7], [26] among others.

The purpose of current paper is to study the Gelfand-Tsetlin basis for quantum $\mathfrak{gl}_n$ aiming to generalize the constructions above in the quantum case. Previously, partial results were obtained for example in [17], [12], [22], [24] , [25]. Even though quantization of the Gelfand-Tsetlin basis for generic module in the non-root of unity case may seem straightforward it does require a very careful treatment which is done in this paper. We also include a root of unity case.

Our main result is Theorem 6.2 which provides explicit construction of all irreducible generic Gelfand-Tsetlin modules with tableaux realization. In Section 7 we consider q a root of unity and apply our construction in this case. It yields new explicit constructions of some finite dimensional irreducible modules.

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2. Notation and conventions

Throughout the paper we fix an integer $n \geq 2$. The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers m such that $m \geq a$. Similarly, we define $\mathbb{Z}_{<a}$, etc. We fix the standard Cartan subalgebra $\mathfrak{h}$, the standard triangular decomposition and the corresponding basis of simple roots of $U_q(\mathfrak{gl}_n)$. The weights

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of $U_q(\mathfrak{gl}_n)$ will be written as n-tuples $(\lambda_1, ..., \lambda_n)$. For a commutative ring R, by $\operatorname{Specm} R$ we denote the set of maximal ideals of R. We will write the vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$ in the following form:

$$v = (v_{ij}) = (v_{n1}, ..., v_{nn} \mid v_{n-1,1}, ..., v_{n-1,n-1} \mid \cdots | v_{21}, v_{22} | v_{11}).$$

For $1 \leq j \leq i \leq n$, $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero. For $i > 0$ by S_i we denote the ith symmetric group. Let $1(q)$ be the set of all complex x such that $q^x = 1$, where $q = e^{\mathfrak{h}}$, that is $1(q) = \{\frac{2k\pi i}{\mathfrak{h}} | k \in \mathbb{Z}\}$. Finally, for any complex number x, we set

$$(x)_q = \frac{q^x - 1}{q - 1}, \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

3. Gelfand-Tsetlin modules

Let P be the free $\mathbb{Z}$-lattice of rank n with the canonical basis $\{\epsilon_1, \ldots, \epsilon_n\}$, i.e. $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$, endowed with symmetric bilinear form $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, 2, \ldots\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n-1\}$. Then Φ realizes the root system of type A_{n-1} with Φ a basis of simple roots.

We define $U_q(\mathfrak{gl}_n)$ as the unital associative algebra generated by $e_i, f_i (1 \leq i \leq n-1)$ and $q^h (h \in P)$ with the following relations:

$$q^0 = 1, \ q^h q^{h'} = q^{h+h'} \quad (h, h' \in P), \tag{1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \tag{2}$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \tag{3}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}, \tag{4}$$

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \tag{5}$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \tag{6}$$

$$e_i e_j = e_j e_i, \ f_i f_j = f_j f_i \quad (|i - j| > 1). \tag{7}$$

The quantum special linear algebra $U_q(sl_n)$ is the subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $e_i, \ f_i, \ q^{\pm\alpha_i} (i = 1, 2, \ldots, n-1)$. $U_q(\mathfrak{gl}_n)$ has the following alternative presentation [13, 5]. It is isomorphic to the associative algebra with unit generated by l_{ij}^+, l_{ji}^-, $1 \leq i \leq j \leq n$. Let $L^{\pm} = (l_{ij}^{\pm})$ with $l_{ij}^+ = l_{ji}^- = 0$ for $1 \leq j < i \leq n$. The defining relations are given in matrix form as follows:

$$RL_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R, \tag{8}$$

$$RL_1^+ L_2^- = L_2^- L_1^+ R, \tag{9}$$

$$l_{ii}^+ l_{ii}^- = l_{ii}^- l_{ii}^+ = 1, \tag{10}$$

where $R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$, $e_{ij} \in End(\mathbb{C}^n)$, $L_1^{\pm} = L^{\pm} \otimes I$ and $L_2^{\pm} = I \otimes L^{\pm}$. The isomorphism between this two representations is given by

$$l_{ii}^{\pm} = q^{\pm\epsilon_i}, \ l_{i,i+1}^+ = (q - q^{-1}) q^{\epsilon_i} f_i, \ l_{i+1,i}^- = -(q - q^{-1}) e_i q^{-\epsilon_i}.$$

We have the following chain

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \ldots \subset U_q(\mathfrak{gl}_n).$$

Let Z_m denote the center of $U_q(\mathfrak{gl}_m)$. The subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $\{Z_m \mid m = 1, \ldots, n\}$ will be called the *Gelfand-Tsetlin subalgebra* of U_q and will be denoted by Γ_q. When q is not root of unity, the center of $U_q(\mathfrak{gl}_m)$ is generated by the following $m+1$ elements

$$c_{mk} = \sum_{\sigma,\sigma' \in S_m} (-q)^{-l(\sigma)-l(\sigma')} l^+_{\sigma(1),\sigma'(1)} \cdots l^+_{\sigma(k),\sigma'(k)} l^-_{\sigma(k+1),\sigma'(k+1)} \cdots l^-_{\sigma(m),\sigma'(m)},$$

where $0 \leq k \leq m$.

Remark 3.1. *When q is a root of unity c_{mk} belong to the center $U_q(\mathfrak{gl}_m)$, but they do not generate the center.*

Definition 3.2. *A finitely generated U-module M is called a* Gelfand-Tsetlin module (with respect to Γ_q) *if*

$$M = \bigoplus_{\mathsf{m} \in \operatorname{Specm} \Gamma_q} M(\mathsf{m}), \tag{11}$$

where $M(\mathsf{m}) = \{v \in M \mid \mathsf{m}^k v = 0$ for some $k \geq 0\}$.

4. Finite dimensional modules of U_q

In this section we recall the quantum version of a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite dimensional U_q-module. We assume that q is not root of unity in this section.

Definition 4.1. *For a vector $v = (v_{ij})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(v)$ we will denote the following array with entries $\{v_{ij} \mid 1 \leq j \leq i \leq n\}$*

$$\begin{array}{ccccccc}
\boxed{v_{n1}} & \boxed{v_{n2}} & & \cdots & & \boxed{v_{n,n-1}} & \boxed{v_{nn}} \\
& \boxed{v_{n-1,1}} & & \cdots & & \boxed{v_{n-1,n-1}} & \\
& & \cdots & \cdots & \cdots & & \\
& & \boxed{v_{21}} & & \boxed{v_{22}} & & \\
& & & \boxed{v_{11}} & & &
\end{array}$$

such an array will be called a Gelfand-Tsetlin tableau *of height n. A Gelfand-Tsetlin tableau of height n is called* standard *if $v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n$.*

The following theorem describes the Gelfand-Tsetlin approach for simple finite dimensional U_q modules with a given highest weight.

Theorem 4.2 ([17, 24, 25])**.** *Let $L(\lambda)$ be the finite dimensional irreducible module over U_q of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exist a basis of $L(\lambda)$ consisting of all standard tableaux $T(v)$ with fixed top row $v_{nj} = \lambda_j - j$. Moreover, the* *action of the generators of* U_q *on* $L(\lambda)$ *is given by the* Gelfand-Tsetlin formulae*:*

$$
\begin{aligned}
&q^{\epsilon_k}(T(v)) = q^{a_k}T(v), \quad a_k = \sum_{i=1}^{k} v_{k,i} - \sum_{i=1}^{k-1} v_{k-1,i} + k, \ \ k = 1, \dots, n, \\
&e_k(T(v)) = -\sum_{j=1}^{k} \frac{\prod_i [v_{k+1,i} - v_{k,j}]_q}{\prod_{i\neq j}[v_{k,i} - v_{k,j}]_q} T(v + \delta^{kj}), \qquad (12)\\
&f_k(T(v)) = \sum_{j=1}^{k} \frac{\prod_i [v_{k-1,i} - v_{k,j}]_q}{\prod_{i\neq j}[v_{k,i} - v_{k,j}]_q} T(v - \delta^{kj}).
\end{aligned}
$$

The next theorem gives the explicit action of the generators of Γ_q.

Theorem 4.3. *The generator* c_{mk} *of* Γ_q *acts on* $T(v)$ *as multiplication by*

$$
\gamma_{mk}(v) = (k)_{q^{-2}}!(m-k)_{q^{-2}}!q^{k(k+1)-\frac{m(m+1)}{2}} \sum_{\tau} q^{\sum_{i=1}^{k} v_{m\tau(i)} - \sum_{i=k+1}^{m} v_{m\tau(i)}} \qquad (13)
$$

where $\tau \in S_m$ *is a* $(k, n-k)$*-shuffle such that* $\tau(1) < \cdots < \tau(k), \tau(k+1) < \cdots < \tau(m)$.

Proof. For any tableau $T(z)$ in $L(\lambda)$, Let W be the subspace of $L(\lambda)$ spanned by all the tableaux $T(w)$ such that $w_{ki} = z_{ki}$ for $m \le k \le n$, $1 \le i \le k$. It follows from Theorem 4.2 that W is a finite dimensional irreducible module of $U_q(\mathfrak{gl}_m)$ with highest weight μ, where $\mu_i = z_{mi} + i$.

Let $T(v)$ be the tableau in W such that $v_{ij} = v_{m,j}$ for any $1 \le j \le i \le m$. Note that the generators l_{ij}^+, l_{ji}^- belong to the upper and lower Borel subalgebra generated by f_i, e_i respectively. The element $l^-_{\sigma(k+1),\sigma'(k+1)} \cdots l^-_{\sigma(m),\sigma'(m)}$ kills $T(v)$ unless $\sigma_{k+1} = \sigma'_{k+1}, \dots, \sigma_n = \sigma'_n$. But $\sigma_1 \le \sigma'_1, \dots, \sigma_k \le \sigma'_k$, so one must have $\sigma_i = \sigma'_i$ for all $1 \le i \le n$ in the action of c_{nk} on $T(v)$. We thus have

$$
\begin{aligned}
c_{mk}T(v) &= \sum_{\sigma\in S_m} q^{-2l(\sigma)} q^{v_{\sigma(1)}+\sigma(1)\cdots+v_{\sigma(k)}+\sigma(k)-v_{\sigma(k+1)}-\sigma(k+1)-\cdots-v_{\sigma(n)}-\sigma(n)}T(v)\\
&= \sum_{\sigma\in S_m} q^{-2l(\sigma)} q^{\mu_{\sigma(1)}+\cdots+\mu_{\sigma(k)}-\mu_{\sigma(k+1)}-\cdots-\mu_{\sigma(n)}}T(v).
\end{aligned}
$$

Let us denote the set of $(k, n-k)$-shuffles by Sh. Since $\Sigma_{\sigma\in S_n} q^{\ell(\sigma)} = (n)_q$ and $S_n = \sqcup_{\tau\in \mathrm{S}h}(S_k \times S_{n-k})\tau$, the following equality holds

$$
\begin{aligned}
&\sum_{\sigma\in S_m} q^{-2l(\sigma)} q^{\mu_{\sigma(1)}+\cdots+\mu_{\sigma(k)}-\mu_{\sigma(k+1)}-\cdots-\mu_{\sigma(n)}}\\
&= \sum_{\sigma\in S_k} q^{-2l(\sigma)} \sum_{\sigma'\in S_{m-k}} q^{-2l(\sigma')} \sum_{\tau\in Sh} q^{-2l(\tau)} q^{\mu_{\tau(1)}+\cdots+\mu_{\tau(k)}-\mu_{\tau(k+1)}-\cdots-\mu_{\tau(n)}}\\
&= (k)_{q^{-2}}!(n-k)_{q^{-2}}! \sum_{\tau\in Sh} q^{-2l(\tau)} q^{\mu_{\tau(1)}+\cdots+\mu_{\tau(k)}-\mu_{\tau(k+1)}-\cdots-\mu_{\tau(n)}}\\
&= (k)_{q^{-2}}!(n-k)_{q^{-2}}! \sum_{\tau\in Sh} q^{-2\sum_{i=1}^{k}(\tau(i)-i)+\sum_{i=1}^{k}(v_{m\tau(i)}+\tau(i))-\sum_{i=k+1}^{m}(v_{n\tau(i)}+\tau(i))}\\
&= (k)_{q^{-2}}!(n-k)_{q^{-2}}!q^{k(k+1)-\frac{m(m+1)}{2}} \sum_{\tau\in Sh} q^{\sum_{i=1}^{k} v_{m\tau(i)} - \sum_{i=k+1}^{n} v_{m\tau(i)}}.
\end{aligned}
$$

Since W is finite dimensional irreducible module of $U_q(\mathfrak{gl}_m)$, the action of c_{mk} on any vector in W is given as above. Thus the action of c_{mk} on $T(v)$ only depends on the mth row of $T(v)$. The theorem is proved. □

5. Generic Gelfand-Tsetlin modules of U_q

In this section we assume that q is not a root of unity. Recall that $1(q)$ stands for the set of all complex x such that $q^x = 1$.

Definition 5.1. *A Gelfand-Tsetlin tableau $T(v)$ is called* generic *if it satisfies the following defining conditions:*

$$v_{ki} - v_{kj} \notin \frac{1(q)}{2} + \mathbb{Z} \text{ for all } 1 \leq k \leq n-1 \text{ and } i \neq j.$$

By $\mathcal{B}(T(v))$ we will denote the set of all Gelfand-Tsetlin tableaux $T(R)$ of height n satisfying $r_{nj} = v_{nj}$ and $r_{ij} - v_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n-1$.

Theorem 5.2 ([22] Theorem 2)**.** *Let $T(v)$ be a generic tableau, the vector space $V(T(v)) = \operatorname{span} \mathcal{B}(T(v))$ has a structure of a U_q-module of finite length with action of the generators of U_q given by the Gelfand-Tsetlin formulae (12).*

Proposition 5.3. *The Gelfand-Tsetlin subalgebra Γ_q separates the tableaux in $V(T(v))$. That is, for any two different tableaux in $V(T(v))$, there exists an element in Γ_q with different eigenvalues corresponding to the tableaux.*

Proof. Let $T(R)$ and $T(S)$ be two tableaux with different m-th row. Assume $T(R)$ and $T(S)$ have the same eigenvalue for any element in Γ_q . It is easy to see from (13) that $(q^{2s_{m1}}, \ldots, q^{2s_{mm}})$ is a permutation of $(q^{2r_{m1}}, \ldots, q^{2r_{mm}})$. Therefore, for any r_{mi}, there exists j such that $q^{2r_{mi}} = q^{2s_{mj}}$, which implies that $r_{mi} - s_{mj} \in \frac{1(q)}{2}$. This lead to $i = j$ and $r_{mi} = s_{mj}$ which is a contradiction. □

5.1. **Classification of irreducible generic Gelfand-Tsetlin U_q-modules.**

We recall the following result of Mazorchuk and Turowska.

Theorem 5.4 ([22] Proposition 2)**.** *If $\mathsf{n} \in \operatorname{Specm} \Gamma$ is generic, then there exists a unique irreducible Gelfand-Tsetlin module N such that $N(\mathsf{n}) \neq 0$.*

Definition 5.5. *If $T(R)$ is a generic tableau and $\mathsf{r} \in \operatorname{Specm} \Gamma_q$ corresponds to R, then the unique irreducible module N such that $N(\mathsf{r}) \neq 0$ is called the* irreducible Gelfand-Tsetlin module containing $T(R)$*, or simply, the* irreducible module containing $T(R)$.

This section is devoted to an explicit construction of the irreducible Gelfand-Tsetlin module containing $T(R)$ for every generic tableau $T(R)$. For convenience we introduce and recall some notation.

Notation 5.6. *Let $T(v)$ be a fixed tableau of height n.*

(i) $\mathcal{B}(T(v)) := \{T(v+z) \mid z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ *with* $z_{ni} = 0$ *for* $1 \leq i \leq n\}$.
(ii) $V(T(v)) = \operatorname{span} \mathcal{B}(T(v))$.
(iii) *For any $T(R) \in \mathcal{B}(T(v))$ and for any $1 < p \leq n$, $1 \leq s \leq p$ and $1 \leq u \leq p-1$ we define:*
  (a) $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$.
  (b) $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}\}$.
  (c) $\Omega^+(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}\}$.

(d) $\mathcal{N}(T(R)) := \{T(S) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(S))\}$.
(e) $W(T(R)) := \operatorname{span} \mathcal{N}(T(R))$.
(f) $U_q \cdot T(R)$: *the* U_q*-submodule of* $V(T(v))$ *generated by* $T(R)$.

5.2. **Submodule generated by a single tableau.** In order to find an explicit basis of every irreducible generic module, we first find a basis of $U_q \cdot T(R)$ for any tableau $T(R)$ in $\mathcal{B}(T(v))$.

Definition 5.7. *Given* $T(Q)$ *and* $T(R)$ *in* $\mathcal{B}(T(v))$, *we write* $T(R) \preceq_{(1)} T(Q)$ *if there exists* $g \in U_q$ *such that* $T(Q)$ *appears with nonzero coefficient in the decomposition of* $g \cdot T(R)$ *into a linear combination of tableaux. For any* $p \geq 1$ *we write* $T(R) \preceq_{(p)} T(Q)$ *if there exist tableaux* $T(L^{(1)})$,..., $T(L^{(p)})$, *such that*

$$T(R) = T(v^{(0)}) \preceq_{(1)} T(v^{(1)}) \preceq_{(1)} \cdots \preceq_{(1)} T(v^{(p)}) = T(Q).$$

As an immediate consequence of the definition of $\preceq_{(p)}$ we have the following.

Lemma 5.8. *If* $T(Q)$, $T(Q^{(0)})$, $T(Q^{(1)})$ *and* $T(Q^{(2)})$ *are tableaux in* $\mathcal{B}(T(L))$ *then:*

(i) $T(Q^{(0)}) \preceq_{(r)} T(Q^{(1)})$ *and* $T(Q^{(1)}) \preceq_{(s)} T(Q^{(2)})$ *imply* $T(Q^{(0)}) \preceq_{(r+s)} T(Q^{(2)})$.
(ii) $T(Q) \preceq_{(1)} T(Q)$.

The next theorem describes the submodule of $V(T(v))$ generated by a fixed tableau $T(R)$.

Theorem 5.9. *Let* $T(v)$ *be generic tableau,* $T(R)$ *and* $T(S)$ *be in* $\mathcal{B}(T(v))$.

(i) *The Gelfand-Tsetlin formulas endow* $W(T(R))$ *with a* U_q*-module structure.*
(ii) $U_q \cdot T(R) = W(T(R))$. *In particular,* $\mathcal{N}(T(R))$ *forms a basis of* $U_q \cdot T(R)$, *and the action of* U_q *on* $U_q \cdot T(R)$ *is given by the Gelfand-Tsetlin formulas.*
(iii) $U_q \cdot T(R) = U_q \cdot T(S)$ *if and only if* $\Omega^+(T(S)) = \Omega^+(T(R))$.
(iv) $U_q \cdot T(R) = V(T(L))$ *whenever* $\Omega^+(T(R)) = \emptyset$.
(v) *Every submodule of* $V(T(v))$ *is finitely generated.*

Proof. (i) In order to prove that $W(T(R))$ is a submodule, it is enough to prove $U \cdot T(S) \subseteq W(T(R))$ for any $T(S) \in \mathcal{N}(T(R))$. We will show $g \cdot T(S)$ is in $W(T(R))$ for every generator of U_q.

Suppose $g = e_k$ for some $1 \leq k \leq n-1$. By the Gelfand-Tsetlin formulas, we have

$$e_k(T(S)) = -\sum_{j=1}^{k} \frac{\prod_i [s_{k+1,i} - s_{k,j}]_q}{\prod_{i \neq j} [s_{k,i} - s_{k,j}]_q} T(S + \delta^{kj}).$$

If there exist k and j such that $T(S) \in \mathcal{N}(T(R))$ but $T(S + \delta^{kj}) \notin \mathcal{N}(T(R))$, that implies

$$\Omega^+(T(R)) \subseteq \Omega^+(T(S)), \text{ and } \Omega^+(T(R)) \nsubseteq \Omega^+(T(S + \delta^{kj})),$$

Hence, there exists $(p, s, u) \in \Omega^+(T(R))$ such that $\omega_{p,s,u}(T(S)) \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$ and $\omega_{p,s,u}(T(S + \delta^{kj})) \notin \frac{1(q)}{2} + \mathbb{Z}_{\geq 0}$. The latter holds only in two cases:

$$(p, s, u) \in \{(k, j, u), (k+1, s, j) \mid 1 \leq u \leq k-1; 1 \leq s \leq k+1\}.$$

Note that if neither of these two cases hold, we have $\omega_{p,s,u}(T(R+\delta^{kj})) = \omega_{p,s,u}(T(S))$. We consider now each of the two cases separately.

(a) Suppose $(p,s,u)=(k,j,u)$. Then $\omega_{k,j,u}(T(S))=s_{kj}-s_{k-1,u}\in\frac{1(q)}{2}+\mathbb{Z}_{\geq 0}$ and $\omega_{k,j,u}(T(S+\delta^{kj}))=(s_{kj}+1)-s_{k-1,u}\notin\frac{1(q)}{2}+\mathbb{Z}_{\geq 0}$, which is impossible.

(b) Suppose $(p,s,u)=(k+1,s,j)$. Then $\omega_{k+1,s,j}(T(S))=s_{k+1,s}-s_{ki}\in\frac{1(q)}{2}+\mathbb{Z}_{\geq 0}$ and $\omega_{k+1,s,i}(T(S+\delta^{ki}))=s_{k+1,s}-(s_{ki}+1)\notin\frac{1(q)}{2}+\mathbb{Z}_{\geq 0}$. Hence $s_{k+1,s}-s_{k,i}=0$ and then the coefficient of $T(S+\delta^{ki})$ in the decomposition of $e_k(T(S))$ is

$$-\frac{\prod_i[s_{k+1,i}-s_{k,j}]_q}{\prod_{i\neq j}[s_{k,i}-s_{k,j}]_q}=0.$$

Therefore, the tableaux that appear with nonzero coefficients in the decomposition of $e_k(T(S))$ are elements of $\mathcal{N}(T(R))$. Hence, $e_k(T(S))\in W(T(R))$.

The proof that $f_k(T(S))\in W(T(R))$ is analogous to the one of $e_k(T(S))\in W(T(R))$. The case q^{ϵ_k} is trivial because q^{ϵ_k} acts as a multiplication by a scalar on $T(S)$ and $T(S)\in\mathcal{N}(T(R))\subseteq W(T(R))$.

(ii) As the Gelfand-Tsetlin subalgebra separate tableaux in $\mathcal{B}(T(v))$ (Proposition 5.3), it is sufficient to prove that for any $T(S)\in W(T(R))$, $T(S)\preceq_{(p)}T(R)$ for some $p\in\mathbb{Z}_{>0}$. Let $T(S)=T(R+z)$, we prove the statement by induction on $\sum_{1\leq j\leq i<n}|z_{ij}|$. When $\sum_{1\leq j\leq i<n}|z_{ij}|=1$, there exist i,j such that $z_{ij}=\pm1$ and all other entries are zero. We consider each case separately.

(a) Suppose $z_{ij}=1$. Then the coefficient of $T(S)$ in $e_iT(R)$ is

$$-\frac{\prod_i[r_{i+1,k}-r_{i,j}]_q}{\prod_{i\neq j}[r_{j,k}-r_{i,j}]_q}.$$

If there exist $[r_{i+1,k}-r_{i,j}]_q=0$, one has $s_{i+1,k}-s_{i,j}=\frac{1(q)}{2}-1$, then $T(S)\notin W(T(R))$. Thus $r_{i+1,k}-r_{i,j}\neq 0$ for any k which implies $T(S)\preceq_{(1)}T(R)$

(b) Suppose $z_{ij}=-1$. Similarly the coefficient of $T(S)$ in $f_iT(R)$ is not zero.

When $\sum_{1\leq j\leq i<n}|z_{ij}|>1$, there exist i,j such that $z_{ij}>0$ and $\Omega^+(T(R))\subseteq\Omega^+(T(R+\delta^{ij}))\subseteq\Omega^+(T(S))$ or $z_{ij}<0$ and $\Omega^+(T(R))\subseteq\Omega^+(T(R-\delta^{ij}))\subseteq\Omega^+(T(S))$. For the fist case $T(R+\delta^{ij})\preceq_{(1)}T(R)$ by induction there exists $p\in\mathbb{Z}$ such that $T(S)\preceq_{(p)}T(R+\delta^{ij})$. Thus $T(S)\preceq_{(p+1)}T(R+\delta^{ij})$. For the second case it can be proved similarly.

(iii) (iv) and (v) are easy to see from (i) and (ii). □

6. Main results

Now we are ready to give the main theorem in the paper. In this section we assume that q is not a root of unity.

Definition 6.1. *For any generic tableau $T(v)$, the* block associated with $T(v)$ *is the set of all Gelfand-Tsetlin U_q-modules with Gelfand-Tsetlin support contained in $Supp_{GT}(V(T(v)))$. Also, for any $T(R)\in\mathcal{B}(T(v))$, $1<p\leq n$ and $1\leq u\leq p-1$, define $d_{pu}(T(R))$ to be the number of distinct elements in $\{v_{p,s,u}(T(R))\mid(p,s,u)\in\Omega(T(R))\}$.*

Theorem 6.2. *Let $T(v)$ be a generic tableau, $T(R)\in\mathcal{B}(T(v))$.*

(i) *The irreducible module containing $T(R)$ has a basis of tableaux*

$$\mathcal{I}(T(R))=\{T(S)\in\mathcal{B}(T(R)):\Omega^+(T(S))=\Omega^+(T(R))\}.$$

The action of U_q on this irreducible module is given by the Gelfand-Tsetlin formulas (12).

(ii) *The number of irreducible modules in the block associated with $T(v)$ is:*

$$\prod_{1\leq u\leq p-1<n}(d_{pu}(T(v))+1).$$

In particular, $V(T(v))$ is irreducible if and only if $d_{pu}(T(v))=0$ for any p and u, or equivalently, if and only if $\Omega(T(v))=\emptyset$.

Proof. (i) For each tableau $T(R)$, we have an explicit construction of the module containing $T(R)$ (recall Definition 5.5):

$$M(T(R)) := U_q\cdot T(R)/\left(\sum U_q\cdot T(S)\right)$$

where the sum is taken over tableaux $T(Q)$ such that $\Omega^+(T(R))\subsetneq\Omega^+(T(S))$ and $U_q\cdot T(S)$ is a proper submodule of $U_q\cdot T(R)$. The module $M(T(R))$ is simple. Indeed, this follows from the fact that for any nonzero tableau $T(S)$ in $M(T(R))$ we have $U_q\cdot T(S)=U_q\cdot T(R)$ and, hence, $T(S)$ generates $M(T(R))$ (note that one should see tableaux $T(S)$ with $\Omega^+(T(R))\subsetneq\Omega^+(T(S))$ as zero). By Theorem 5.9, $\mathcal{I}(T(R))$ is a basis for the subquotient $M(T(R))$.

(ii) The irreducible modules are in one-to-one correspondence with the subsets of $\Omega(T(v))$ of the form $\Omega^+(T(v+z))$. For any $T(R)\in\mathcal{B}(T(v))$, we can decompose $\Omega(T(R))$ into a disjoint union $\Omega(T(R))=\bigsqcup_{p,u}\Omega_{pu}(T(R))$, where

$$\Omega_{p,u}(T(R))=\{(p,1,u),(p,2,u),\ldots,(p,p,u)\}\cap\Omega(T(R)).$$

Now, if $\Omega^+_{p,u}(T(R)):=\Omega_{p,u}(T(R))\cap\Omega^+(T(R))$, one can write $\Omega^+(T(R))=\bigsqcup_{p,u}\Omega^+_{pu}(T(R))$. For p,u fixed, let us denote by $s_{p,u}$ the number of different subsets of the form $\Omega^+_{p,u}(T(R))$. So, the number of different subsets of the form $\Omega^+(T(R))$ is $\prod_{p,u}s_{p,u}$. It is easy to see that $s_{pu}=d_{pu}(T(v))+1$. □

7. Root of unity case

In this section we describe irreducible modules of the quantum enveloping algebra U_q when the complex parameter q is a root of unity. Let d be the order of q. Since $q\neq\pm1$, we have $d>2$.

Theorem 7.1. [19] *When q is a root of unity, any irreducible module of U_q is finite dimensional.*

Denote

$$e=\begin{cases} d, & \text{if } d \text{ is odd}\\ d/2, & \text{d is even.}\end{cases}$$

It is easy to verify that

$$[x]_q=0\Longleftrightarrow x \text{ congruent to 0 modulo } e.$$

Remark 7.2. *In the Gelfand-Tsetlin formulae (12), none of the $[v_{ki}-v_{kj}]_q$ is zero if $v_{n1}-v_{nn}\leq e$. Let $g=0$ be any defining relation in $U_q(\mathfrak{gl}_n)$. Then $g\cdot T(v)=0$ for any tableau and any non root of unity q. The coefficients of tableaux in $g.T(v)$ can be regarded as functions of q, they are zero for any non root of unity q. If q is a root of unity but the denominators of the coefficients in $g\cdot T(v)$ are not zero then* *all these coefficients must be zero. Hence Theorem 4.2 holds if $\lambda_1 - \lambda_n \leq e + 1 - n$ when q is a root of unity. For a generic tableau $T(v)$ all $[v_{ki} - v_{kj}]_q$ are not zero and thus Theorem 5.2 also holds when q is a root of unity by the same argument as in the generic case ([19], Chapter 7).*

Quantum Gelfand-Tsetlin subalgebra Γ_q separates the tableaux in the following sense.

Theorem 7.3. *Let q be a root of unity, $T(v)$ a generic tableau. If $T(R), T(S) \in V(T(v))$ and $r_{ij} - s_{ij} \neq 0$ mod $e, 1 \leq j \leq i < n$, then Γ_q separates $T(R)$ and $T(S)$.*

Proof. Let $T(R)$ and $T(S)$ be tableaux with different m-th row. Assume $T(R)$ and $T(S)$ have the same eigenvalue for any element in Γ_q . It is easy to see from (13) that $(q^{2s_{m1}}, \ldots, q^{2s_{mm}})$ is a permutation of $(q^{2r_{m1}}, \ldots, q^{2r_{mm}})$. For any r_{mi}, there exists j such that $q^{2r_{mi}} = q^{2s_{mj}}$. We have that $r_{mi} - s_{mj} \in \frac{1(q)}{2}$. As $T(L)$ is generic, one has that $i = j$. Then $r_{ij} - s_{ij}$ congruent to 0 modulo e which is a contradiction. □

As a consequence we have

Proposition 7.4. *Let q be a root of unity, $T(v)$ a generic tableau, $T(R)$ a tableau in $V(T(v))$ and N the submodule of $V(T(v))$ generated by $T(R)$. If $g \cdot T(R) = \sum_i c_i T(R_i)$ for some distinct tableaux $T(R_i)$ in $\mathcal{B}(T(v))$ and nonzero $c_i \in \mathbb{C}$ then $T(R_i) \in N$ for all i.*

Proof. Suppose $g = e_k$ for some $1 \leq k \leq n-1$. By the Gelfand-Tsetlin formulas, we have

$$e_k(T(R)) = -\sum_{j=1}^{k} \frac{\prod_i [r_{k+1,i} - r_{k,j}]_q}{\prod_{i \neq j} [r_{k,i} - r_{k,j}]_q} T(R + \delta^{kj}).$$

Let $T(R_1)$ and $T(R_2)$ be any two tableaux in the summation with nonzero coefficients. Then $(r_1)_{ij} - (r_2)_{ij} = 0$ or ± 1 for any $1 \leq j \leq i < n$. It follows from Theorem 7.3 that Γ_q separate these two tableaux. Thus $T(R_i) \in N$ for all i.

The proof that $f_k(T(R)) \in N$ is analogous. Since q^{ϵ_k} acts as a multiplication by scalar on $T(R)$, the proof follows. □

7.1. **Submodule generated by a single tableau.**

Notation 7.5. *Let $T(R)$ be a fixed tableau of height n and remember that $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$.*

(a) *If $\omega_{p,s,u}(T(R)) \in \frac{1(q)}{2} + \mathbb{Z}$, we denote $\omega_{p,s,u}(T(R)) = \alpha_{p,s,u}(T(R)) + \beta_{p,s,u}(T(R))$, where $\alpha_{p,s,u}(T(R)) \in \frac{1(q)}{2}$ and $0 \leq \beta_{p,s,u}(T(R)) < e$.*
(b) $\mathcal{N}(T(R)) := \{T(S) \in \mathcal{B}(T(L)) \mid \omega_{p,s,u}(T(S)) - \alpha_{p,s,u}(T(R)) \in \mathbb{Z}_{\geq 0}$ *for all* $(p, s, u) \in \Omega(T(R))\}$.
(c) $W(T(R)) := \operatorname{span} \mathcal{N}(T(R))$.

Theorem 7.6. *Let $T(v)$ be a generic Gelfand-Tsetlin tableau, $T(R), T(S) \in \mathcal{B}(T(v))$.*

(i) *The Gelfand-Tsetlin formulas endow $W(T(R))$ with a U_q-module structure.*
(ii) *$U_q \cdot T(R) = W(T(R))$. In particular, $\mathcal{N}(T(R))$ forms a basis of $U_q \cdot T(R)$, and the action of $U_q(\mathfrak{gl}(n))$ on $U_q \cdot T(R)$ is given by the Gelfand-Tsetlin formulas (12).*
(iii) *$U_q \cdot T(R) = U_q \cdot T(S)$ if and only if $\alpha_{p,s,u}(T(S)) = \alpha_{p,s,u}(T(R))$ for all $(p, s, u) \in \Omega(T(v))$.*

Proof. (i) In order to prove that $W(T(R))$ is a submodule, it is sufficient to show that $g \cdot T(S) \in W(T(R))$ for every generator g of U_q. The proof is similar to the proof of Theorem 5.9 (i).

(ii) It is sufficient to show that for any $T(S) \in W(T(R))$, $T(S) \preceq_{(p)} T(R)$ for some $p \in \mathbb{Z}_{>0}$. The argument is similar to the proof of Theorem 5.9 (ii).

(iii) Follows from (i) and (ii). □

7.2. **New constructions of irreducible Gelfand-Tsetlin modules.** In this section we use Gelfand-Tsetlin basis to give a new realization of some irreducible Gelfand-Tsetlin modules in the root of unity case. We assume d to be odd.

Let $p = (p_{ij}), 1 \leq j \leq i < n$ with nonzero complex entries, $W_{ij}(R)$ the submodule of $V(T(v))$ generated by $T(R + d\delta^{ij}) - p_{ij}T(R)$. By Theorem 7.6, the set $\{T(S + d\delta^{ij}) - p_{ij}T(S) \mid T(S) \in W(T(R))\}$ is a basis for $W_{ij}(R)$.

Let $N = \sum\limits_{\substack{T(R)\in B(T(v))\\ 1\leq j\leq i<n}} W_{ij}(R)$, and $M = V(T(v))/N$.

Theorem 7.7. *M is an irreducible module of dimension $d^{\frac{n(n-1)}{2}}$. Moreover, M has a basis consisting of tableaux $T(v + m_{ij}\delta^{ij})$, $0 \leq m_{ij} < d$, $1 \leq j \leq i < n$.*

Proof. The submodule N has a basis $\{T(R+\delta^{ij})-p_{ij}T(R) : R \in B(T(L)), 1 \leq j \leq i < n\}$. So the subquotient M has basis $T(v+m_{ij}\delta^{ij})$, $0 \leq m_{ij} < d$, $1 \leq j \leq i < n$. We denote this basis of M by I. Suppose M_1 is a nonzero submodule of M. By Theorem 7.3, M_1 has a basis which is a subset of I. From Theorem 7.6 and relations in the quotient module M, one has that $I \subseteq U_qT(R)$ for any tableau $T(R)$ in I. Thus $M_1 = M$ and M is irreducible. □

Remark 7.8. *Module M coincides with the one constructed in §7.5.5 of* [19].

From now on we will denote by Λ the following set

$$\{(i, j) \mid (i + 1, s, j) \in \Omega(T(R)) \text{ for some } 1 \leq s \leq i + 1\}.$$

Definition 7.9. *For any $T(R) \in \mathcal{B}(T(v))$, $1 < p \leq n$ and $1 \leq u \leq p - 1$, for $(i, j) \in \Lambda$ define $a_{ij}(T(R))$ and $b_{ij}(T(R))$ as follows*

$$a_{ij}(T(R)) = \min\{\beta_{i+1,s,j} \mid (i + 1, s, j) \in \Omega(T(R))\},$$
$$b_{ij}(T(R)) = \min\{d - \beta_{i+1,s,j} \mid (i + 1, s, j) \in \Omega(T(R))\}.$$

Define

$$t_{ij}(T(R)) = \begin{cases} a_{ij}(T(R)) + b_{ij}(T(R)), & \text{for } (i,j) \in \Lambda \\ d, & \text{for } (i,j) \notin \Lambda. \end{cases}$$

Definition 7.10. *Let Λ_1 be a subset of Λ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. By $\widetilde{M}(T(R))$ we will denote the quotient of $U_q \cdot T(R)$*

$$\left(\sum_{(i,j)\notin\Lambda} W_{ij}(R) + \sum_{T(S_1)} U_q(T(S_1)) + \sum_{T(S_2)} U_q(T(S_2') - p_{ij}T(S_2)) \right),$$

where $T(S_t), t = 1, 2$ run through the set of tableaux in $\mathcal{N}(T(R))$ such that $(i, j) \in \Lambda_t$, $\omega_{i-1,s,j}T(S_2') - \omega_{i-1,s,j}(T(S_2)) = d$ for some $(i - 1, s, j) \in \Omega(T(R))$ and $\omega_{p,s,u}T(S_2) - \omega_{p,s,u}(T(S_2')) = 0$ for any $(p, s, u) \neq (i - 1, s, j)$.

Theorem 7.11. *$\widetilde{M}(T(R))$ is an irreducible module of dimension $\prod\limits_{1\leq j\leq i<n} t_{ij}(T(R))$.*

Proof. The subquotient $U_qT(R)/\sum_{T(S)}U_q(T(S)))$ has a basis

$$I = \{T(S) \mid \alpha_{p,s,u}(T(S)) = \alpha_{p,s,u}(T(R)) \text{ for all } (p,s,u) \in \Omega(T(L)\}.$$

The module $\widetilde{M}(T(R))$ can be regarded as the subquotient of $U_qT(R)/\sum_{T(S)}U_q(T(S))$. Then it has a basis: $\{T(S) \in I \mid s_{ij} = r_{ij} + m_{ij}, 0 \leq m_{ij} < d, (i,j) \notin \Lambda\}$. Similar to Theorem 7.7, $\widetilde{M}(T(R))$ is irreducible. For any $(i,j) \in \Lambda$, if we fix the $i+1$-th row of the tableau, the number of distinct s_{ij} in I is $t_{ij}(T(R))$. For $(i,j) \notin \Lambda$, there are d different s_{ij}. Thus the dimension of $\widetilde{M}(T(R))$ is $\prod\limits_{1\leq j\leq i<n} t_{ij}(T(R))$.

□

7.3. **Example.** Recall the following two families of d-dimensional modules of $U_q(sl_2)$ [18]. The first family depends on three complex numbers λ, a and b. We assume $\lambda \neq 0$. Consider the d-dimensional vector space with a basis $\{v_0, v_1, \ldots, v_{d-1}\}$. for $0 \leq p \leq d-1$. Set

$$Kv_p = \lambda q^{-2p} v_p, \tag{14}$$

$$Ev_{p+1} = (\frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}}[p+1]_q + ab)v_p, \tag{15}$$

$$F_{v_p} = v_{p+1}, \tag{16}$$

and $Ev_0 = av_{d-1}, Fv_{d-1} = bv_0$, and $Kv_{e-1} = \lambda q^{-2(d-1)}v_p$. These formula endow the vector space with a U_q-module structure, denoted by $V(\lambda, a, b)$.

The second family depends on two scalars $\mu \neq 0$ and c. Let E, F, K act on the vector space with basis $\{v_0, v_1, \ldots, v_{d-1}\}$ by

$$Kv_p = \mu q^{2p} v_p, \tag{17}$$

$$Fv_{p+1} = \frac{q^{-p}\mu^{-1} - q^p\mu}{q-q^{-1}}[p+1]_q v_p, \tag{18}$$

$$E_{v_p} = v_{p+1}, \tag{19}$$

and $Fv_0 = 0, Ev_{d-1} = cv_0$, and $Kv_{e-1} = \mu q^{-2} v_{e-1}$. These formulae endow the vector space with a U_q-module structure, denoted by $\tilde{V}(\mu, c)$.

Theorem 7.12. [18] *Any irreducible $U_q(\mathfrak{sl}_2)$-module of dimension d is isomorphic to one of the following:*

(i) $V(\lambda, a, b)$ *with* $b \neq 0$,
(ii) $V(\lambda, a, 0)$ *where* λ *is not of the form* $\pm q^{j-1}$ *for any* $1 \leq j \leq d-1$,
(iii) $\tilde{V}(\pm q^{1-j}, c)$ *with* $c \neq 0$ *and* $1 \leq j \leq d-1$.

In the following we will compare above modules with modules in Theorems 7.7 and 7.11. Let x, y, z be three complex number, $v_p = (x, y|z-p), 0 \leq p \leq d-1$. Consider the vector space with a basis of tableaux $\{T(v_p) : 0 \leq p \leq d-1\}$. Theorem 7.7 endows the vector space with a U_q-module structure. The actions of E, F, K are given by

$$KT(v_p) = q^{2z-(x+y+1)} q^{-2p} T(v_p), \tag{20}$$

$$ET(v_{p+1}) = -[x+p+1-z]_q[y+p+1-z]_q T(v_p), \tag{21}$$

$$FT(v_p) = T(v_{p+1}), \tag{22}$$

and $ET(v_0) = -s[x-z]_q[y-z]_qT(v_{d-1}), FT(v_{d-1}) = \frac{1}{s}T(v_0)$. Let $\lambda = q^{2z-(x+y+1)}$, $b = \frac{1}{s}$, $a = -s[x-z]_q[y-z]_qv_{d-1}$, this module is isomorphic to $V(\lambda, a, b)$ with $b \neq 0$.

Let x, y, z be three complex number with $x-z$ or $y-z \in \frac{1(q)}{2}$. Consider the vector space with a basis of tableaux $\{T(v_p) : 0 \leq p \leq d-1\}$, where $v_p = (x, y|z-p), 0 \leq p \leq d-1$. Theorem 7.11 endows this vector space with a U_q-module structure. The actions of E, F, K are given by

$$KT(v_p) = q^{2z-(x+y+1)}q^{-2p}T(v_p), \tag{23}$$

$$ET(v_{p+1}) = -[x+p+1-z]_q[y+p+1-z]_qT(v_p), \tag{24}$$

$$FT(v_p) = T(v_{p+1}), \tag{25}$$

and $Ev_0 = 0, Fv_{d-1} = sv_0$. This module is isomorphic to $V(\lambda, 0, s), \lambda = q^{2z-(x+y+1)}$.

There exists an algebra endomorphism of $U_q(sl_2)$ such that $E \longmapsto F, F \longmapsto E, K \longmapsto K^{-1}$. $V(\lambda, a, 0)$ and $\tilde{V}(\mu, c)$ can be obtained from $V(\lambda, 0, b)$ via this algebra endomorphism.

References

[1] M. Colarusso, The Gelfand-Zeitlin Integrable System and Its Action on Generic Elements of $gl(n)$ and $so(n)$. In New Developments in Lie Theory and Geometry (Cruz Chica, Cordoba, Argentina, 2007), v. 491 of Contemp. Math., pages 255-281. Amer. Math. Soc., Providence, RI, 2009.

[2] M. Colarusso, The Orbit Structure of the Gelfand-Zeitlin Group on $n \times n$ Matrices, Pacific J. Math. 250 (2011), no. 1, 109-138.

[3] M. Colarusso, S. Evens, The Gelfand-Zeitlin Integrable System and K-orbits on the Flag Variety in ?Symmetry: representation theory and its applications?, 85-119, Progr. Math., 257, Birkauser/Springer, New York, 2014.

[4] M. Colarusso, S. Evens, On Algebraic Integrability of Gelfand-Zeitlin Fields, Transform. Groups 15 (2010), no.1, 46-71.

[5] J .Ding, I. B. Frenkel, Isomorphism of two realizations of quantum affine algebra, Communications in mathematical physics, 1993, 156(2): 277-300.

[6] Y. Drozd, S. Ovsienko, V. Futorny, Harish-Chandra subalgebras and Gelfand-Zetlin modules, Math. and Phys. Sci. 424 (1994), 72–89.

[7] A. Fomenko, A. Mischenko, Euler equation on finite-dimensional Lie groups, Izv. Akad. Nauk SSSR, Ser. Mat. 42 (1978), 396–415.

[8] V. Futorny, D. Grantcharov, L. E. Ramirez, On the classification of irreducible Gelfand-Tsetlin modules of $\mathfrak{sl}(3)$, Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, 623, (2014), 63–79.

[9] V. Futorny, D. Grantcharov, L. E. Ramirez, Classification of irreducible Gelfand-Tsetlin modules for $\mathfrak{sl}(3)$. In progress.

[10] V. Futorny, D. Grantcharov, L. E. Ramirez, Irreducible Generic Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$. Symmetry, Integrability and Geometry: Methods and Applications, v. 018, (2015).

[11] V. Futorny, D. Grantcharov, L. E. Ramirez, Singular Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$. Advances in Math (2016). 453-482.

[12] V. Futorny, J. Hatrwig, E. Wilson, Irreducible completely pointed modules of quantum groups of type A. Journal of Algebra (Print), (2015), 432, p. 252-279.

[13] L. Faddeev, N. Reshetikhin and L. Takhtadzhyan, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990), 193–225.

[14] I. Gelfand, M. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Doklady Akad. Nauk SSSR (N.s.) 71 (1950), 825–828.

[15] M. Graev, Infinite-dimensional representations of the Lie algebra $gl(n, \mathbb{C})$ related to complex analogs of the Gelfand-Tsetlin patterns and general hupergeometric functions on the Lie group $GL(n, \mathbb{C})$, Acta Appl. Mathematicae 81 (2004), 93–120.

[16] M. Graev, A continuous analogue of Gelfand-Tsetlin schemes and a realization of the principal series of irreducible unitary representations of the group GL(n,C) in the space of functions on the manifold of these schemes, Dokl. Akad. Nauk. 412 no.2 (2007), 154–158.

[17] M. Jimbo *Quantum R matrix related to the generalized Toda system: An algebraic approach*, Field theory, quantum gravity and strings. Springer, Berlin, Heidelberg, 1988: 335-361.
[18] C. Kassel, Quantum groups. Vol. 155. Springer Science & Business Media, 2012.
[19] A. Klimyc, K. Schmüdgen, Quantum groups and their representations, Springer-Verlag, Berlin Heidelberg, 1997.
[20] B. Kostant, N. Wallach, Gelfand-Zeitlin theory from the perspective of classical mechanics I, In Studies in Lie Theory Dedicated to A. Joseph on his Sixtieth Birthday, Progress in Mathematics, 243 (2006), 319–364.
[21] B. Kostant, N. Wallach, Gelfand-Zeitlin theory from the perspective of classical mechanics II. In The Unity of Mathematics In Honor of the Ninetieth Birthday of I. M. Gelfand, Progress in Mathematics, 244 (2006), 387–420.
[22] V. Mazorchuk, L. Turowska, On Gelfand-Tsetlin modules over $U_q(\mathfrak{gl}(n))$, Czechoslovak Journal of Physics, (2000), 139–141
[23] A. Molev, Gelfand-Tsetlin bases for classical Lie algebras, Handbook of Algebra, Vol. 4, (M. Hazewinkel, Ed.), Elsevier, (2006), 109–170.
[24] K. Ueno, Y. Shibukawa, T. Takebayashi, Gelfand-Zetlin basis for $U_q(gl_{n+1})$ modules. Letters in Mathematical Physics 18.3 (1989): 215-221.
[25] K. Ueno, Y. Shibukawa, T. Takebayashi, Construction of Gelfand-Tsetlin Basis for $Uq(\mathfrak{gl}(N+1))$-modules. Publ. RIMS, Kyoto Univ. 26 (1990), 667-679.
[26] E. Vinberg, On certain commutative subalgebras of a universal enveloping algebra, Math. USSR Izvestiya 36 (1991), 1–22.
[27] D. Zhelobenko, Compact Lie groups and their representations, Transl. Math. Monographs, AMS, 40 (1974)

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