# Comparing and characterizing some constructions of canonical bases from Coxeter systems

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**Abstract**

The Iwahori-Hecke algebra $\mathcal{H}$ of a Coxeter system $(W, S)$ has a "standard basis" indexed by the elements of $W$ and a "bar involution" given by a certain antilinear map. Together, these form an example of what Webster calls a pre-canonical structure, relative to which the well-known Kazhdan-Lusztig basis of $\mathcal{H}$ is a canonical basis. Lusztig and Vogan have defined a representation of a modified Iwahori-Hecke algebra on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions in $W$, and shown that this module has a unique pre-canonical structure satisfying a certain compatibility condition, which admits its own canonical basis which can be viewed as a generalization of the Kazhdan-Lusztig basis. Surprisingly, one can modify the parameters defining Lusztig and Vogan's module to obtain other pre-canonical structures, each of which admits a unique canonical basis indexed by twisted involutions. In this work we classify all of the pre-canonical structures which arise in this fashion, and explain the relationships between their resulting canonical bases. We show that up to a certain natural notion of isomorphism, the pre-canonical structures we consider are each represented by one of a finite list of related constructions.

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# 1 Introduction

Let $(W, S)$ be a Coxeter system and write $\mathcal{H}$ for its associated Iwahori-Hecke algebra. This algebra has a "standard basis" indexed by the elements of $W$, whose structure constants have a simple inductive formula. The Kazhdan-Lusztig basis of $\mathcal{H}$ is the unique basis which is invariant under a certain antilinear map $\mathcal{H} \to \mathcal{H}$, referred to as the "bar involution," and whose elements are each unitriangular linear combinations of standard basis elements with respect to the Bruhat order. The standard basis and bar involution of $\mathcal{H}$ are an example of what Webster [17] calls a *pre-canonical structure*, relative to which the Kazhdan-Lusztig basis is a *canonical basis*. This terminology, which is similar to Du's notion of an *IC basis* [2], is useful for organizing several similar constructions attached to Coxeter systems. We review the precise definitions in Section 2.1.

In [11, 12, 13], Lusztig and Vogan study a representation of a modified Iwahori-Hecke algebra $\mathcal{H}_2$ on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions $\mathbf{I} = \mathbf{I}(W, S)$ in a Coxeter group. (See Section 2.3 for the definition of this set; though we mean something more general, in this introduction one can simply assume $\mathbf{I} = \{w \in W : w^2 = 1\}$.) They show that this module has a unique pre-canonical structure which is compatible with the action of $\mathcal{H}_2$ in a certain sense, and that this structure admits a canonical basis, of which the Kazhdan-Lusztig basis can be viewed as a special case. The definition of Lusztig and Vogan's $\mathcal{H}_2$-representation has a particular simple form: independent of $(W, S)$, it is defined by a formula involving eight parameters in $\mathbb{Z}[v, v^{-1}]$. It turns out that by modifying these parameters one can obtain other $\mathcal{H}_2$-module structures on the free $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\mathbf{I}$; some (but not all) of these modules likewise possess a unique pre-canonical structure compatible with the action of $\mathcal{H}_2$; in each such case there is a unique associated canonical basis.

We review Lusztig and Vogan's results in Section 3.2, and derive from them a family of analogous theorems (along the lines just described) in Section 3.3. In Section 3.4 we present another variation of the same results, in which the role of the modified Iwahori-Hecke algebra $\mathcal{H}_2$ is replaced by the usual algebra $\mathcal{H}$. These constructions give three canonical bases indexed by the twisted involutions in a Coxeter group; these bases all can be seen as generalizations of the Kazhdan-Lusztig basis of $\mathcal{H}$, but, somewhat unexpectedly, they do not appear to be related in any simple way.

In Section 4 we describe a precise sense in which these three bases account for all canonical bases indexed by twisted involutions which arise from analogous constructions. Specifically, we define in Section 4.3 a category whose objects are pre-canonical structures on free $\mathbb{Z}[v, v^{-1}]$-modules. Our definition of morphisms in this category has the following appealing properties:

(i) The structure constants of canonical bases arising from pre-canonical structures which are isomorphic in our sense can be transformed to each other by either negating their signs or applying the variable substitution $v \mapsto -v$; see Corollary 4.8.

(ii) Assume the free $\mathbb{Z}[v, v^{-1}]$-module generated by $W$ has a pre-canonical structure in which the natural basis $W$ is standard. If this structure is compatible with a representation of $\mathcal{H}$ of a certain natural form (such as the regular representation), then it is isomorphic to the pre-canonical structure on $\mathcal{H}$ itself, and so it has a unique canonical basis which can be identified in the sense of (i) with the Kazhdan-Lusztig basis; see Theorem 4.11.

With respect to these definitions, our main results are as follows. Suppose we are given a pre-canonical structure on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions in $W$, in which the natural basis $\mathbf{I}$ is the standard one. We prove that

(1) If the structure is compatible with a representation of $\mathcal{H}$ of a certain natural form, then it is isomorphic to the pre-canonical structure we define in Section 3.4; see Theorem 4.20.

(2) If the structure is compatible with a representation of the modified Iwahori-Hecke algebra $\mathcal{H}_2$ of a certain natural form, then it is isomorphic to one of four pre-canonical structures: the one Lusztig and Vogan define in [11, 12], the one we define in Section 3.3, or one of two non-isomorphic structures derived from the one given in Section 3.4; see Theorem 4.24.

These results provide some formal justification for considering the pre-canonical structures described in Sections 3.2, 3.3, and 3.4 to be particularly natural objects. Lusztig and Vogan have given two interpretations of the first structure, in terms of the geometry of an associated algebraic group when $W$ is a Weyl group [12] and in terms of the theory of Soergel bimodules for general $W$ [13]. It remains an open problem to give similar interpretations of the two other pre-canonical structures.


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## 2 Preliminaries

### 2.1 Canonical bases

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials with integer coefficients in a single indeterminant. We write $f \mapsto \overline{f}$ for the ring involution of $\mathcal{A}$ with $v \mapsto v^{-1}$ and say that a map $\varphi : U \to V$ between $\mathcal{A}$-modules is $\mathcal{A}$-*antilinear* if $\varphi(fu) = \overline{f} \cdot \varphi(u)$ for $f \in \mathcal{A}$ and $u \in U$. Let $V$ be a free $\mathcal{A}$-module.

**Definition 2.1.** A *pre-canonical structure* on $V$ consists of

- a "bar involution" $\psi$ given by an $\mathcal{A}$-antilinear map $V \to V$ with $\psi^2 = 1$.
- a "standard basis" $\{a_c\}$ with partially ordered index set $(C, \leq)$ such that

$$\psi(a_c) \in a_c + \sum_{c' < c} \mathcal{A} \cdot a_{c'}.$$

Assume $V$ has a pre-canonical structure $(\psi, \{a_c\})$; we then have this accompanying notion.

**Definition 2.2.** A set of vectors $\{b_c\}$ in $V$ also indexed by $(C, \leq)$ is a *canonical basis* if

(C1) each vector $b_c$ in the basis is invariant under $\psi$.

(C2) each vector $b_c$ in the basis is in the set $b_c = a_c + \sum_{c'<c} v^{-1}\mathbb{Z}[v^{-1}] \cdot a_{c'}$.

We have taken these definitions from Webster's paper [17], where they appear in a slightly more general form. The differences are as follows. In [17], a pre-canonical structure also includes a choice of an $\mathcal{A}$-sesquilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}((v^{-1}))$ with $\langle \psi(x), \psi(y) \rangle = \langle y, x \rangle$ for $x, y \in V$, and a canonical basis $\{b_c\}$ is required to be "almost orthonormal" with respect to this form in the sense of $\langle b_c, b_{c'} \rangle \in \delta_{c,c'} + v^{-1}\mathbb{Z}[[v^{-1}]]$. Moreover, in Webster's definition, the coefficients of $a_{c'}$ in condition (C2) may be arbitrary elements of $\mathcal{A}$. To view our definitions as special cases of those in [17], define $\langle \cdot, \cdot \rangle$ as the $\mathcal{A}$-sesquilinear form with $\langle a_c, \psi(a_{c'}) \rangle = \delta_{c,c'}$ for $c, c' \in C$. By condition (C2), a canonical basis according to Definition 2.2 is almost orthonormal with respect to this form.

**Example 2.3.** We view the ring $\mathcal{A}$ itself as possessing the pre-canonical structure in which the bar involution is the map $f \mapsto \overline{f}$ and the standard basis is the singleton set $\{1\}$. This structure admits a canonical basis, which is again just $\{1\}$.

Canonical bases as we have defined them are also special cases of *IC bases* as formalized by Du in [2]. The latter type of basis is given as follows. (Du's paper [2] does not appear to be easily accessible, so we mention that the following material is also conveniently reviewed in [4, Section 2].) Let $V$ be a free $\mathcal{A}$-module with an $\mathcal{A}$-antilinear involution $\psi : V \to V$ and a basis $\{a_c\}$ with index set $C$. The set $C$ is no longer required to be partially ordered and there is no condition on the action of $\psi$ on $a_c$. A basis $\{b_c\}$ of $V$ is an *IC basis* relative to $(\psi, \{a_c\})$ if it is the *unique* basis such that $\psi(b_c) = b_c \in a_c + \sum_{c' \in C \setminus \{c\}} v^{-1}\mathbb{Z}[v^{-1}] \cdot a_{c'}$ for each $c \in C$. Even though there is no uniqueness requirement in Definition 2.2, a canonical basis is nevertheless an IC basis, as a result of the following proposition from the introduction of [17]. The elementary proof of this result is an instructive exercise.

**Proposition 2.4** (Webster [17])**.** A pre-canonical structure admits at most one canonical basis.

Even though a canonical basis is thus automatically unique, it is usually quite difficult to describe its elements explicitly. Under mild hypotheses on the partial order of the index set, however, one can at least guarantee that a canonical basis exists. Continue to assume $V$ is a free $\mathcal{A}$-module with a pre-canonical structure $(\psi, \{a_c\})$ as in Definition 2.1.

**Theorem 2.5** (Du [2])**.** If all lower intervals $(-\infty, x] = \{c \in C : c \leq x\}$ in the partially ordered index set $(C, \leq)$ are finite then the pre-canonical structure on $V$ admits a canonical basis.

*Proof.* The result follows from [2, Theorem 1.2 and Remark 1.2.1(1)]. One can also adapt the argument Lusztig gives in [11, Section 4.9], which proves the existence of a canonical basis in one particular pre-canonical structure but makes sense in greater generality. These proofs are constructive, and lead to an inductive (but often not very illuminating) formula for computing elements of the canonical basis. $\square$

Webster lists several examples of pre-canonical structures from representation theory in the introduction of [17]. Pre-canonical structures, such as in these examples, arise naturally from categorifications, by which we broadly mean isomorphisms

$$V \xrightarrow{\sim} [\mathscr{C}] \tag{2.1}$$

where $\mathscr{C}$ is an additive category and $[\mathscr{C}]$ denotes its split Grothendieck group: this is the abelian group generated by the symbols $[C]$ for objects $C \in \mathscr{C}$, subject to the relations $[A] + [B] = [C]$ whenever $A \oplus B \cong C$. The isomorphism (2.1) is most meaningful when viewed as between $\mathcal{A}$-modules, which is possible when $\mathscr{C}$ has a notion of $\mathbb{Z}$-grading for its objects. Then $[\mathscr{C}]$ becomes an $\mathcal{A}$-module by defining $v^n[C] = [C(n)]$ where $C(n)$ is the object $C \in \mathscr{C}$ with its grading shifted down by $n$. The bar involution of a pre-canonical structure on $V$ should then correspond via (2.1) to a duality functor on $\mathscr{C}$, and elements of the standard basis should arise as some set of easily located objects in $\mathscr{C}$, each of which contains a unique indecomposable summand not found in smaller objects. A canonical basis in turn should correspond to a representative set of indecomposable objects which are self-dual with respect to some choice of grading shift.

**Example 2.6.** The pre-canonical structure on $V = \mathcal{A}$ comes from the categorification taking $\mathscr{C}$ to be the category of finitely generated $\mathbb{Z}$-graded free $R$-modules (with $R$ any commutative ring), with morphisms given by grading preserving $R$-linear maps. For this category, there is a unique ring isomorphism $\mathcal{A} \xrightarrow{\sim} [\mathscr{C}]$ identifying $1 \in \mathcal{A}$ with $[\mathbb{1}] \in [\mathscr{C}]$, where $\mathbb{1}$ denotes the graded $R$-module whose $n$th component is $R$ when $n = 0$ and is 0 otherwise. The bar involution $f \mapsto \overline{f}$ on $\mathcal{A}$ is the decategorification of the duality functor $M \mapsto \mathrm{Hom}(M, \mathbb{1})$ where $\mathrm{Hom}(M, \mathbb{1})$ denotes the graded $R$-module whose $n$th component is the set of grading preserving $R$-linear maps $M \to \mathbb{1}(n)$.

In general, confronted with some natural pre-canonical structure, it is an interesting problem (which in the present work we do not address) to identify a categorification which can explain the existence and special properties of an associated canonical basis.

## 2.2 Pre-canonical module structures

Suppose $\mathcal{B}$ is an $\mathcal{A}$-algebra with a pre-canonical structure; write $\overline{b}$ for the image of $b \in \mathcal{B}$ under the corresponding bar involution. (For us, all algebras are unital and associative.) Let $V$ be a $\mathcal{B}$-module which is free as an $\mathcal{A}$-module.

**Definition 2.7.** A *pre-canonical $\mathcal{B}$-module structure* on $V$ is a pre-canonical structure whose bar involution $\psi : V \to V$ commutes with the bar involution of $\mathcal{B}$ in the sense that

$$\psi(bx) = \overline{b} \cdot \psi(x) \qquad \text{for all } b \in \mathcal{B} \text{ and } x \in V.$$

Observe that a pre-canonical structure is thus the same thing as a pre-canonical $\mathcal{A}$-module structure. The additional compatibility condition satisfied by a pre-canonical $\mathcal{B}$-module structure can be useful for proving uniqueness statements. In particular, we have the following lemma.

**Lemma 2.8.** Suppose $V$ has a basis $\{a_c\}$ with partially ordered index set $(C, \leq)$. If $V$ is generated as a $\mathcal{B}$-module by the minimal elements of the basis $\{a_c\}$, then there exists at most one pre-canonical $\mathcal{B}$-module structure on $V$ in which $\{a_c\}$ serves as the "standard basis."

*Proof.* Suppose $\psi$ and $\psi'$ are two $\mathcal{A}$-antilinear maps $V \to V$ which, together with $\{a_c\}$, give $V$ a pre-canonical $\mathcal{B}$-module structure. Let $U \subset V$ be the set of elements on which $\psi$ and $\psi'$ agree. Then $U$ is a $\mathcal{B}$-submodule which contains the minimal elements of the basis $\{a_c\}$. Since these elements generate $V$, we have $U = V$ so $\psi = \psi'$. ∎

Combining the preceding lemma with Theorem 2.5 yields this proposition.

**Proposition 2.9.** Suppose $V$ has a pre-canonical $\mathcal{B}$-module structure whose standard basis $\{a_c\}$ is indexed by the partially ordered set $(C, \leq)$. Assume the following conditions hold:

(i) The minimal elements of the basis $\{a_c\}$ generate $V$ as a $\mathcal{B}$-module.

(ii) The lower intervals $(-\infty, x] = \{c \in C : c \leq x\}$ in $(C, \leq)$ are always finite.

Then the pre-canonical $\mathcal{B}$-module structure on $V$ admits a canonical basis.

### 2.3 Twisted involutions

We review here the definition of the set of twisted involutions attached to a Coxeter system. This set has many interesting combinatorial properties; see [5, 6, 7, 8, 9].

Let $(W, S)$ be any Coxeter system. Write $\ell : W \to \mathbb{N}$ for the associated length function and $\leq$ for the Bruhat order. We denote by $\mathrm{Aut}(W, S)$ the group of automorphisms $\theta : W \to W$ such that $\theta(S) = S$, and define

$$W^+ = \{(x, \theta) : x \in W \text{ and } \theta \in \mathrm{Aut}(W, S)\}.$$

We extend the length function and Bruhat order to $W^+$ by setting $\ell(x, \theta) = \ell(x)$ and by setting $(x, \theta) \leq (x', \theta')$ if and only if $\theta = \theta'$ and $x \leq x'$. The set $W^+$ has the structure of a group, in which multiplication of elements is given by

$$(x, \alpha)(y, \beta) = (x \cdot \alpha(y), \alpha\beta).$$

We view $W \subset W^+$ as a subgroup by identifying $x \in W$ with the pair $(x, 1)$. Likewise, we view $\mathrm{Aut}(W, S) \subset W^+$ as a subgroup by identifying $\theta \in \mathrm{Aut}(W, S)$ with the pair $(1, \theta)$. With respect to these inclusions, $W^+$ is a semidirect product $W \rtimes \mathrm{Aut}(W, S)$. Finally, we let

$$\mathbf{I} = \mathbf{I}(W, S) = \{w \in W^+ : w = w^{-1}\}.$$

These elements are the *twisted involutions* or *generalized involutions* of $(W, S)$. A pair $(x, \theta) \in W^+$ belongs to $\mathbf{I}$ if and only if $\theta = \theta^{-1}$ and $\theta(x) = x^{-1}$. In this situation, often in the literature (e.g., in [5, 6, 7, 11, 14, 15]) the element $x \in W$ is referred to as a twisted involution, relative to the automorphism $\theta$. We have defined twisted involutions slightly more generally as ordinary involutions of the extended group $W^+$, since all of the results we will state are true relative to any choice of automorphism $\theta$.

If $s \in S$ and $w = (x, \theta) \in \mathbf{I}$ then $sws = (s \cdot x \cdot \theta(s), \theta)$ is also a twisted involution. The latter may be equal to $w$; in particular, $sws = w$ if and only if $sw = ws$, in which case $sw \in \mathbf{I}$.

**Notation.** Let $s \ltimes w$ denote whichever of $sws$ or $sw$ is in $\mathbf{I} \setminus \{w\}$; i.e., define

$$s \ltimes w = \begin{cases} sws & \text{if } sw \neq ws \\ sw & \text{if } sw = ws \end{cases} \qquad \text{for } s \in S \text{ and } w \in \mathbf{I}. \tag{2.2}$$

While $s \ltimes (s \ltimes w) = w$, this notation does not extend to an action of $W$ of $\mathbf{I}$.

The restriction of the Bruhat order on $W$ to $\mathbf{I}$ forms a poset with many special properties. Concerning this, we will just need the following result, which rephrases [5, Theorem 4.8].

**Theorem 2.10** (Hultman [5])**.** The poset $(\mathbf{I}, \leq)$ is graded, and its rank function $\rho : \mathbf{I} \to \mathbb{N}$ satisfies

$$\rho(s \ltimes w) = \rho(w) - 1 \qquad \Leftrightarrow \qquad \ell(s \ltimes w) < \ell(w) \qquad \Leftrightarrow \qquad \ell(sw) = \ell(w) - 1$$

for all $s \in S$ and $w \in \mathbf{I}$.

We reserve the notation $\rho$ in all later sections to denote the rank function of $(\mathbf{I}, \leq)$. Note that $\rho(1) = 0$, and so one can compute $\rho(w)$ inductively using the equivalent identities in the theorem. As with $\ell$, there are explicit formulas for $\rho$ when $W$ is a classical Weyl group; see [8, 9].

# 3 Existence statements

## 3.1 Kazhdan-Lusztig basis

We recall briefly the definition of the Kazhdan-Lusztig basis of the Iwahori-Hecke algebra of a Coxeter system and the pre-canonical structure from which it arises. As references for this material, we mention [1, 10, 16].

We continue to let $(W, S)$ be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and Bruhat order $\leq$. Let $\mathcal{H} = \mathcal{H}(W, S)$ be the free $\mathcal{A}$-module with a basis given the symbols $H_w$ for $w \in W$. There is a unique $\mathcal{A}$-algebra structure on $\mathcal{H}$ such that

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w \\ H_{sw} + (v - v^{-1}) H_w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

The *Iwahori-Hecke algebra* of $(W, S)$ is $\mathcal{H}$ equipped with this structure. The unit of $\mathcal{H}$ is the basis element $H_1$, which often we write as 1 or simply omit. Observe that $H_s^{-1} = H_s + (v^{-1} - v)$ and that $H_w = H_{s_1} \cdots H_{s_k}$ whenever $w = s_1 \cdots s_k$ is a reduced expression. Hence every basis element $H_w$ for $w \in W$ is invertible. We denote by $H \mapsto \overline{H}$ the $\mathcal{A}$-antilinear map $\mathcal{H} \to \mathcal{H}$ with $\overline{H_w} = (H_{w^{-1}})^{-1}$ for $w \in W$. One checks that this map is a ring involution. We have the following theorem from Kazhdan and Lusztig's seminal work [10].

**Theorem 3.1** (Kazhdan and Lusztig [10])**.** Define

- the "bar involution" of $\mathcal{H}$ to be the map $H \mapsto \overline{H}$.
- the "standard basis" of $\mathcal{H}$ to be $\{H_w\}$ with the partially ordered index set $(W, \leq)$.

This is a pre-canonical structure on $\mathcal{H}$ and it admits a canonical basis $\{\underline{H}_w\}$.

**Notation.** Following the convention of [3, 16], we usually denote elements of the standard basis in a pre-canonical structure by some capital letter with a subscript, say $X_c$, and we underline this letter (i.e., write $\underline{X}_c$) to denote elements of the associated canonical basis.

The canonical basis $\{\underline{H}_w\}$ is the *Kazhdan-Lusztig basis* of $\mathcal{H}$. It is a simple exercise to show for $s \in S$ that $\underline{H}_s = H_s + v^{-1}$. Define $h_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in W$ such that $\underline{H}_w = \sum_{y \in W} h_{y,w} H_y$. We note the following well-known property of these polynomials.

**Proposition 3.2** (Kazhdan and Lusztig [10])**.** If $y \leq w$ then $v^{\ell(w)-\ell(y)} h_{y,w} \in 1 + v^2\mathbb{Z}[v^2]$.

**Remark.** Define $q = v^2$ and $P_{y,w} = v^{\ell(w)-\ell(y)} h_{y,w}$ for $y, w \in W$. The polynomials $P_{y,w} \in \mathbb{Z}[q]$ are usually called the *Kazhdan-Lusztig polynomials* of the Coxeter system $(W, S)$.

The Kazhdan-Lusztig basis displays several remarkable positivity properties; for example, it is now known from work of Elias and Williamson [3] that for all $x, y \in W$ one has $h_{x,y} \in \mathbb{N}[v^{-1}]$ and $\underline{H}_x \underline{H}_y \in \mathbb{N}[v, v^{-1}]$-span$\{\underline{H}_z : z \in W\}$. Available proofs of such phenomena make extensive use of the interpretation of the Iwahori-Hecke algebra $\mathcal{H}$ as the split Grothendieck of an appropriate category (in [3], the category of Soergel bimodules). This is an important motivation for the problem of constructing categorifications which give rise to pre-canonical structures of interest.

### 3.2 A canonical basis for twisted involutions

Lusztig and Vogan [12, 11, 13] describe a pre-canonical structure on the free $\mathcal{A}$-module generated by the set of twisted involutions $\mathbf{I} = \mathbf{I}(W, S)$ attached to an arbitrary Coxeter system $(W, S)$ and prove that this structure admits a canonical basis. Let us review their construction.

Let $\mathcal{H}_2$ be the free $\mathcal{A}$-module with a basis given the symbols $K_w$ for $w \in W$, with the unique $\mathcal{A}$-algebra structure such that

$$K_s K_w = \begin{cases} K_{sw} & \text{if } sw > w \\ K_{sw} + (v^2 - v^{-2}) K_w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

We call this the *Iwahori-Hecke algebra of* $(W, S)$ *with parameter* $v^2$. We again denote by $K \mapsto \overline{K}$ the $\mathcal{A}$-antilinear map $\mathcal{H}_2 \to \mathcal{H}_2$ with $\overline{K_w} = K_{w^{-1}}^{-1}$ for $w \in W$. This "bar involution" together with the "standard basis" $\{K_w\}$ indexed by $(W, \leq)$ forms a pre-canonical structure on $\mathcal{H}_2$, which admits a canonical basis $\{\underline{K}_w\}$. The $\mathbb{Z}$-linear map

$$\Phi : \mathcal{H} \to \mathcal{H}_2 \tag{3.1}$$

with $\Phi(v^n H_w) = v^{2n} K_w$ is an injective ring homomorphism and $\underline{K}_w = \Phi(\underline{H}_w)$ for all $w \in W$.

In this section we let $\mathcal{L} = \mathcal{L}(W, S)$ denote the free $\mathcal{A}$-module with a basis given by the symbols $L_w$ for $w \in \mathbf{I}$. Lusztig and Vogan [12] first proved the following result in the case that $W$ is a Weyl group or affine Weyl group; Lusztig's paper [11] then extended the theorem to arbitrary Coxeter systems by elementary methods. Lusztig and Vogan's preprint [13] provides another proof of this result, using the machinery of Soergel bimodules developed by Elias and Williamson in [3].

**Theorem 3.3** (Lusztig and Vogan [12]; Lusztig [11])**.** There is a unique $\mathcal{H}_2$-module structure on $\mathcal{L}$ such that

$$K_s L_w = \begin{cases} L_{sws} & \text{if } sw \neq ws > w \\ L_{sws} + (v^2 - v^{-2}) L_w & \text{if } sw \neq ws < w \\ (v + v^{-1}) L_{sw} + L_w & \text{if } sw = ws > w \\ (v - v^{-1}) L_{sw} + (v^2 - 1 - v^{-2}) L_w & \text{if } sw = ws < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

*Proof.* This is [11, Theorem 0.1], where $v^2 = u$ and $K_s = u^{-1} T_s$ and $L_w = a'_w = v^{-\ell(w)} a_w$. $\square$

From now on we view $\mathcal{L}$ as an $\mathcal{H}_2$-module according to the preceding result. For $x \in W$ we write $\mathrm{sgn}(x) = (-1)^{\ell(x)}$. We denote by $L \mapsto \overline{L}$ the $\mathcal{A}$-antilinear map $\mathcal{L} \to \mathcal{L}$ with

$$\overline{L_{(x,\theta)}} = \mathrm{sgn}(x) \cdot \overline{K_x} \cdot L_{(x^{-1},\theta)} \qquad \text{for } (x,\theta) \in \mathbf{I}.$$

The following theorem appears in full generality in [11], based on an earlier result in [12]. It combines [11, Theorem 0.2, Theorem 0.4, and Proposition 4.4].

**Theorem 3.4** (Lusztig and Vogan [12]; Lusztig [11])**.** Define

- the "bar involution" of $\mathcal{L}$ to be the map $L \mapsto \overline{L}$.
- the "standard basis" of $\mathcal{L}$ to be $\{L_w\}$ with the partially ordered index set $(\mathbf{I}, \leq)$.

This is a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}$, and it admits a canonical basis $\{\underline{L}_w\}$.

Observe, by Lemma 2.8, that the pre-canonical $\mathcal{H}_2$-module structure thus defined on $\mathcal{L}$ is the unique one in which $\{L_w\}$ serves as the "standard basis." Following the convention in [11], we define $\pi_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ such that $\underline{L}_w = \sum_{y\in\mathbf{I}} \pi_{y,w} L_y$. Note that $\pi_{y,w} = \delta_{y,w}$ if $y \not< w$. We note the following degree bound from [11, Section 4.9(c)].

**Proposition 3.5** (Lusztig [11])**.** If $y, w \in \mathbf{I}$ such that $y \leq w$ then $v^{\ell(w)-\ell(y)}\pi_{y,w} \in 1 + v^2\mathbb{Z}[v^2]$.

**Remark.** The polynomials $v^{\ell(w)-\ell(y)}\pi_{y,w}$ are denoted $P^{\sigma}_{y,w}$ in [11, 12, 14, 15]. Lusztig proves an inductive formula [11, Theorem 6.3] for the action of $\underline{K}_s = K_s + v^{-2} \in \mathcal{H}_2$ on $\underline{L}_w$ which can be used to compute these polynomials; see also [15, Section 2.1].

The polynomials $\pi_{y,w}$ may have negative coefficients; however, they appear to possess another positivity property. Recall that $h_{y,w} \in \mathbb{N}[v^{-1}]$ are the polynomials such that $\underline{H}_w = \sum_{y\in W} h_{y,w} H_y$. Given $y, w \in W$ and $\theta, \theta' \in \mathrm{Aut}(W, S)$, define $h_{(y,\theta),(w,\theta')}$ to be $h_{y,w}$ if $\theta = \theta'$ and zero otherwise. Lusztig [11, Theorem 9.10] has shown that

$$\tfrac{1}{2}\left(h_{y,w} \pm \pi_{y,w}\right) \in \mathbb{Z}[v^{-1}] \qquad \text{for all } y, w \in \mathbf{I}$$

and has conjectured that these polynomials actually belong to $\mathbb{N}[v^{-1}]$. Lusztig and Vogan provide a geometric proof of this conjecture when $W$ is a Weyl group (see [12, Section 3.2]) and outline a proof for arbitrary Coxeter systems in [13]. The canonical basis $\{\underline{L}_w\}$ conjecturally displays some other positivity properties, which are considered in detail in [14, 15].

### 3.3 Another pre-canonical $\mathcal{H}_2$-module structure

In this section we prove a variant of Theorem 3.3 which gives a different $\mathcal{H}_2$-module structure on the free $\mathcal{A}$-module generated by $\mathbf{I}$. For each choice of $(W, S)$, this module has a unique pre-canonical $\mathcal{H}_2$-module structure in which $\mathbf{I}$ is the standard the basis. This pre-canonical structure admits a canonical basis which is not related in any obvious way to the basis $\{\underline{L}_w\}$ in the previous section, although it has similar properties. It is an open problem to find an interpretation of this new canonical basis along the lines of [12, 13].

Let $\mathcal{L}' = \mathcal{L}'(W, S)$ denote the free $\mathcal{A}$-module with a basis given by the symbols $L'_w$ for $w \in \mathbf{I}$. We view $\mathcal{L}'$ as an $\mathcal{H}_2$-module distinct from $\mathcal{L}$ via the following theorem.

**Theorem 3.6.** There is a unique $\mathcal{H}_2$-module structure on $\mathcal{L}'$ such that

$$K_s L'_w = \begin{cases} L'_{sws} & \text{if } sw \neq ws > w \\ L'_{sws} + (v^2 - v^{-2})L'_w & \text{if } sw \neq ws < w \\ (v^{-1}+v)L'_{sw} - L'_w & \text{if } sw = ws > w \\ (v^{-1}-v)L'_{sw} + (v^2+1-v^{-2})L'_w & \text{if } sw = ws < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

*Proof.* Define $f^z_{x,y} \in \mathcal{A}$ for $x \in W$ and $y, z \in W$ such that $(-1)^{\rho(y)} K_x L_y = \sum_{z\in\mathbf{I}} (-1)^{\rho(z)} f^z_{x,y} L_z$. It is a straightforward exercise to check, using the well-known relations defining $\mathcal{H}_2$ (see, e.g., [11, Section 2.1]), that there is a unique $\mathcal{H}_2$-module structure on $\mathcal{L}'$ in which $K_x L'_y = \mathrm{sgn}(x) \sum_{z\in\mathbf{I}} \overline{f^z_{x,y}} L'_z$ for $x \in W$ and $y \in \mathbf{I}$. In this $\mathcal{H}_2$-module structure, the generators $K_s$ for $s \in S$ act on the basis elements $L'_w$ according to the given formula. $\square$

Denote by $L' \mapsto \overline{L'}$ the $\mathcal{A}$-antilinear map $\mathcal{L}' \to \mathcal{L}'$ with

$$\overline{L'_{(x,\theta)}} = \overline{K_x} \cdot L'_{(x^{-1},\theta)} \qquad \text{for } (x,\theta) \in \mathbf{I}.$$

We have this analogue of Theorem 3.4.

**Theorem 3.7.** Define

- the "bar involution" of $\mathcal{L}'$ to be the map $L' \mapsto \overline{L'}$.
- the "standard basis" of $\mathcal{L}'$ to be $\{L'_w\}$ with the partially ordered index set $(\mathbf{I}, \leq)$.

This is a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}'$, and it admits a canonical basis $\{\underline{L}'_w\}$.

By Lemma 2.8, this is the unique pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}'$ in which $\{L'_w\}$ is the "standard basis."

*Proof.* Define $r_{y,w} \in \mathcal{A}$ for $y, w \in I$ such that $\overline{L_w} = \sum_{y\in\mathbf{I}} r_{y,w} L_y$ and let $f^z_{x,y}$ be as in the proof of Theorem 3.6. Let $L \mapsto \widetilde{L}$ be the $\mathcal{A}$-antilinear map with $\widetilde{L'_w} = \sum_{y\in\mathbf{I}} (-1)^{\rho(w)-\rho(y)} \cdot \overline{r_{y,w}} \cdot L'_y$ for $w \in \mathbf{I}$. We claim that $\widetilde{L} = \overline{L}$ for all $L \in \mathcal{L}'$. To prove this, we note that if $w = (x,\theta) \in \mathbf{I}$ then

$$K_{x^{-1}} \widetilde{L'_w} = \mathrm{sgn}(x) \sum_{y\in\mathbf{I}} \sum_{z\in\mathbf{I}} (-1)^{\rho(z)-\rho(y)} \cdot \overline{r_{y,w} \cdot f^z_{x^{-1},y}} \cdot L'_z$$

while

$$L_w = \mathrm{sgn}(x) K_{x^{-1}} \overline{L_w} = \mathrm{sgn}(x) \sum_{y\in\mathbf{I}} \sum_{z\in\mathbf{I}} (-1)^{\rho(z)-\rho(y)} \cdot r_{y,w} \cdot f^z_{x^{-1},y} \cdot L_z.$$

We deduce that $K_{x^{-1}} \widetilde{L'_w} = L'_w = K_{x^{-1}} \overline{L'_w}$ since the right side of the first equation is the image of the right side of the second under the $\mathcal{A}$-antilinear map $\mathcal{L} \to \mathcal{L}'$ with $L_z \mapsto L'_z$ for $z \in \mathbf{I}$. Since $K_{x^{-1}}$ is invertible this shows that $\widetilde{L'_w} = \overline{L'_w}$ for $w \in \mathbf{I}$ which suffices to prove our claim.

Given the claim, it follows from Theorem 3.4 that the bar involution and standard basis of $\mathcal{L}'$ form a pre-canonical structure, and it is easy to show that the identity $\overline{K_s L_w} = \overline{K_s} \cdot \overline{L_w}$ implies $\overline{K_s L'_w} = \overline{K_s} \cdot \overline{L'_w}$ for $s \in S$ and $w \in \mathbf{I}$. Hence the bar involution and standard basis of $\mathcal{L}'$ form a pre-canonical $\mathcal{H}_2$-module structure, which admits a canonical basis $\{\underline{L}'_w\}$ by Proposition 2.9. $\square$

Define $\pi'_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ as the polynomials such that $\underline{L}'_w = \sum_{y\in\mathbf{I}} \pi'_{y,w} L'_y$. We introduce some notation to state a recurrence for computing these polynomials. First, for $y, w \in \mathbf{I}$ let

$$\mu'(y,w) = (\text{the coefficient of } v^{-1} \text{ in } \pi'_{y,w}),$$

$$\mu''(y,w) = (\text{the coefficient of } v^{-2} \text{ in } \pi'_{y,w}) + (v+v^{-1})\mu'(y,w).$$

Next, for $s \in S$ and $y, w \in \mathbf{I}$ define

$$\mu'(s,y,w) = \delta_{sy<y} \cdot \mu''(y,w) + \delta_{sy,ys} \cdot (\ell(y) - \ell(sy)) \cdot \mu'(sy,w) - \sum_{\substack{y<z<w \\ sz<z}} \mu'(y,z)\mu'(z,w).$$

Here $\delta_{sy<y}$ is 1 if $sy < y$ and 0 otherwise. In what follows, recall that $\underline{K}_s = K_s + v^{-2}$ for $s \in S$.

**Proposition 3.8.** Let $w \in \mathbf{I}$ and $s \in S$ such that $w < sw$.

(a) If $sw \neq ws$ then $\underline{K}_s \underline{L}'_w = \underline{L}'_{sws} + \sum_{y<sws} \mu'(s,y,w)\underline{L}'_y$.

(b) If $sw = ws$ then $\underline{K}_s \underline{L}'_w = (v+v^{-1})\underline{L}'_{sw} - \underline{L}'_w + \sum_{y<sw} (\mu'(s,y,w) - \mu'(y,sw))\underline{L}'_y$.

**Remark.** Lusztig [11, Theorem 6.3(c)] shows that the canonical basis $\{\underline{L}_w\} \subset \mathcal{L}$ in the previous section is such that $\underline{K}_s \underline{L}_w = (v^2+v^{-2})\underline{L}_w$ if $s \in S$ and $w \in \mathbf{I}$ and $sw < w$. This property has no analogue for the canonical basis $\{\underline{L}'_w\} \subset \mathcal{L}'$.

*Proof.* Each part of the proposition follows by showing that the difference between the two sides of the desired equality both (i) is an element of the set $\sum_{y<s\ltimes w} v^{-1}\mathbb{Z}[v^{-1}] \cdot L'_y$ and (ii) is invariant under the bar operator of $\mathcal{L}'$. Since the only such element with these two properties is 0, the given identities must hold. The observation (ii) is immediate in either case from Theorem 3.7, while showing that property (i) holds is a straightforward exercise from Theorem 3.6. $\square$

Write $f \equiv g \pmod 2$ if $f, g \in \mathcal{A}$ are such that $f - g \in 2\mathcal{A}$, and define $\pi_{y,w}$ and $h_{y,w}$ for $y, w \in \mathbf{I}$ as in the previous section. We note the following relationship between $\pi'_{y,w}$, $\pi_{y,w}$, and $h_{y,w}$.

**Proposition 3.9.** For all $y, w \in \mathbf{I}$ it holds that $\pi'_{y,w} \equiv \pi_{y,w} \equiv h_{y,w} \pmod 2$.

*Proof.* The second congruence is [11, Theorem 9.10]. For $F \in \mathcal{L}$ and $G \in \mathcal{L}'$, we write $F \equiv G \pmod 2$ if $F = \sum_{y\in\mathbf{I}} f_y L_y$ and $G = \sum_{y\in\mathbf{I}} g_y L'_y$ for some polynomials $f_y, g_y \in \mathcal{A}$ with $f_y \equiv g_y \pmod 2$ for all $y \in \mathbf{I}$. To prove the first congruence we must show that $\underline{L}_w \equiv \underline{L}'_w \pmod 2$ for all $w \in \mathbf{I}$. This automatically holds if $\rho(w) = 0$. Let $w \in \mathbf{I}$ and $s \in S$ such that $w < sw$ and assume $\underline{L}_y \equiv \underline{L}'_y \pmod 2$ if $y < s \ltimes w$. It suffices to show under this hypothesis that

$$\underline{L}_{s\ltimes w} \equiv \underline{L}'_{s\ltimes w} \pmod 2. \tag{3.2}$$

Towards this end, define $\mu(y,w) \in \mathbb{Z}$ for $y, w \in \mathbf{I}$ as the coefficient of $v^{-1}$ in $\pi_{y,w}$, and let

$$X_{s,w} = \begin{cases} \underline{L}_{sws} & \text{if } sw \neq ws \\ (v+v^{-1})\underline{L}_{sw} - \sum_{y<sw} \mu(y,sw)\underline{L}_y & \text{if } sw = ws \end{cases}$$

and

$$X'_{s,w} = \begin{cases} \underline{L}'_{sws} & \text{if } sw \neq ws \\ (v+v^{-1})\underline{L}'_{sw} - \sum_{y<sw} \mu'(y,sw)\underline{L}'_y & \text{if } sw = ws. \end{cases}$$

We claim that to prove the congruence (3.2) it is enough show that $X_{s,w} \equiv X'_{s,w} \pmod 2$. This is obvious if $sw \neq ws$ so assume $sw = ws$ and $X_{s,w} \equiv X'_{s,w} \pmod 2$. We must check that $\pi'_{y,sw} \equiv \pi_{y,sw} \pmod 2$ for all $y \leq sw$; to this end we argue by induction on $\rho(sw) - \rho(y)$. By definition $\pi'_{sw,sw} = \pi_{sw,sw} = 1$. Fix $y < sw$ and suppose $\pi'_{z,sw} \equiv \pi_{z,sw} \pmod 2$ for $y < z \leq sw$. The congruence $X_{s,w} \equiv X'_{s,w} \pmod 2$ implies

$$(v + v^{-1})\pi_{y,sw} - \sum_{y \leq z < sw} \mu(z, sw)\pi_{y,z} \equiv (v + v^{-1})\pi'_{y,sw} - \sum_{y \leq z < sw} \mu'(z, sw)\pi'_{y,z} \pmod 2.$$

By hypothesis, the terms indexed by $z > y$ in the sums on either side of this congruence cancel, and we obtain

$$(v + v^{-1})\pi_{y,sw} - \mu(y, sw) \equiv (v + v^{-1})\pi'_{y,sw} - \mu'(y, sw) \pmod 2.$$

It is an elementary exercise, noting that $\pi_{y,sw}$ and $\pi'_{y,sw}$ both belong to $v^{-1}\mathbb{Z}[v^{-1}]$, to show that this congruence implies $\pi_{y,sw} \equiv \pi'_{y,sw} \pmod 2$, and so we conclude by induction that (3.2) holds. This proves our claim.

We now argue that $X_{s,w} \equiv X'_{s,w} \pmod 2$. For this we observe that there are unique polynomials $a_{s,y,w}, a'_{s,y,w} \in \mathcal{A}$ such that

$$X_{s,w} = \underline{K}_s \underline{L}_w - \sum_{y < s \ltimes w} a_{s,y,w} \underline{L}_y \qquad \text{and} \qquad X'_{s,w} = \underline{K}_s \underline{L}'_w - \sum_{y < s \ltimes w} a'_{s,y,w} \underline{L}'_y.$$

Indeed, the polynomials $a'_{s,y,w}$ are given by Proposition 3.8, and an entirely analogous statement decomposing the product $\underline{K}_s \underline{L}_w$ gives the polynomials $a_{s,y,w}$. It is not difficult to show, by deriving a formula for $a_{s,y,w}$ similar to the one for $\mu'(s, y, w)$, that the hypothesis $\underline{L}_y \equiv \underline{L}'_y \pmod 2$ for $y < s \ltimes w$ implies $a_{s,y,w} \equiv a'_{s,y,w} \pmod 2$. Hence to prove $X_{s,w} \equiv X'_{s,w} \pmod 2$ we need only check that $\underline{K}_s \underline{L}_w \equiv \underline{K}_s \underline{L}'_w \pmod 2$. As we assume $\underline{L}_w \equiv \underline{L}'_w \pmod 2$, this follows by comparing Theorems 3.3 and 3.6, which shows more generally that $\underline{K}_s F \equiv \underline{K}_s G \pmod 2$ whenever $F \in \mathcal{L}$ and $G \in \mathcal{L}'$ such that $F \equiv G \pmod 2$. $\square$

The polynomials $\pi'_{y,w}$ also satisfy the same degree bound as $\pi_{y,w}$ and $h_{y,w}$.

**Proposition 3.10.** If $y, w \in \mathbf{I}$ such that $y \leq w$ then $v^{\ell(w)-\ell(y)}\pi'_{y,w} \in 1 + v^2\mathbb{Z}[v^2]$.

*Proof.* The proposition holds if $\rho(w) = 0$ since then $\pi'_{y,w} = \delta_{y,w}$. Let $w \in \mathbf{I}$ and $s \in S$ such that $w < sw$ and assume $v^{\ell(z)-\ell(y)}\pi'_{y,z} \in 1 + v^2\mathbb{Z}[v^2]$ for all $y \leq z < s \ltimes w$. It suffices to show under this hypothesis that

$$v^{\ell(s \ltimes w)-\ell(y)}\pi'_{y,s\ltimes w} \in 1 + v^2\mathbb{Z}[v^2] \qquad \text{for all } y \in \mathbf{I} \text{ with } y \leq s \ltimes w. \tag{3.3}$$

To this end, define $X'_{s,w}$ as in the proof of Proposition 3.9 and let $p_y \in \mathcal{A}$ for $y \in \mathbf{I}$ be such that $X'_{s,w} = \sum_{y \leq s \ltimes w} p_y \underline{L}'_y$. We claim that to prove (3.3) it is enough to show that

$$v^{\ell(w)-\ell(y)+2}p_y \in 1 + v^2\mathbb{Z}[v^2] \qquad \text{for all } y \in \mathbf{I} \text{ with } y \leq s \ltimes w. \tag{3.4}$$

This follows when $sw \neq ws$ as then $\ell(s \ltimes w) = \ell(w) + 2$ and $p_y = \pi'_{y,s\ltimes w}$. Alternatively, suppose that $sw = ws$ and (3.4) holds. We then have

$$p_y = (v + v^{-1})\pi'_{y,sw} - \mu'(y, sw) - \sum_{y < z < sw} \mu'(z, sw)\pi'_{y,z}. \tag{3.5}$$

To deduce (3.3), we argue by induction on $\ell(sw) - \ell(y)$. If $y = sw$ then the desired containment holds automatically. Let $y < sw$ and suppose $v^{\ell(sw)-\ell(z)}\pi'_{z,sw} \in 1 + v^2\mathbb{Z}[v^2]$ for $y < z \leq sw$. Then $\mu'(z, sw)$ is nonzero for $z > y$ only if $\ell(w) - \ell(z)$ is even, so if we multiply both sides of (3.5) by $v^{\ell(w)-\ell(y)+2}$, then it follows from (3.4) via our inductive hypothesis that

$$(v^2 + 1)v^{\ell(sw)-\ell(y)}\pi'_{y,sw} - v^{\ell(sw)-\ell(y)+1}\mu'(y, sw) \in 1 + v^2\mathbb{Z}[v^2].$$

Since we always have $\pi'_{y,sw} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $\mu'(y, sw) \in \mathbb{Z}$, this containment can only hold if $\mu'(y, sw) = 0$ whenever $\ell(sw) - \ell(y)$ is even. We deduce from this that in fact

$$(v^2 + 1)v^{\ell(sw)-\ell(y)}\pi'_{y,sw} \in 1 + v^2\mathbb{Z}[v^2]$$

and it is easy to see that this implies $v^{\ell(sw)-\ell(y)}\pi'_{y,sw} \in 1 + v^2\mathbb{Z}[v^2]$, which is what we needed to show. We conclude by induction that (3.4) implies (3.3).

We now argue that (3.4) holds. Fix $y \leq s \ltimes w$. Proposition 3.8 then implies

$$p_y = (a + \delta_{sw,ws}) \cdot \pi'_{y,w} + b \cdot \pi'_{s\ltimes y,w} - \Sigma$$

where

$$(a, b) = \begin{cases} (v^{-2},\ 1) & \text{if } sy \neq ys > y \\ (v^2,\ 1) & \text{if } sy \neq ys < y \\ (v^{-2} - 1,\ v^{-1} - v) & \text{if } sy = ys > y \\ (v^2 + 1,\ v^{-1} + v) & \text{if } sy = ys < y \end{cases} \qquad \text{and} \qquad \Sigma = \sum_{z < s\ltimes w} \mu'(s, z, w)\pi'_{y,z}.$$

Since we assume that $v^{\ell(z')-\ell(z)}\pi'_{z,z'} \in 1 + v^2\mathbb{Z}[v^2]$ for $z \leq z' \leq w$, inspecting our definition shows that $\mu'(s, z, w)$ is an integer when $\ell(w) - \ell(z)$ is even and an integer multiple of $v + v^{-1}$ when $\ell(w) - \ell(z)$ is odd. Consequently, it follows that

$$v^{\ell(w)-\ell(y)+2}\Sigma \in v^2\mathbb{Z}[v^2].$$

In turn, since $y \leq s \ltimes w$, [7, Lemma 2.7] implies that $s \ltimes y \leq w$ if $sy < y$ and that $y \leq w$ if $sy > y$. Using this fact and the hypothesis stated in the second sentence of this proof, one checks that

$$v^{\ell(w)-\ell(y)+2}\left((a + \delta_{sw,ws}) \cdot \pi'_{y,w} + b \cdot \pi'_{s\ltimes y,w}\right) \in 1 + v^2\mathbb{Z}[v^2].$$

Combining these observations, we conclude that (3.4) holds. □

Despite these results, there does not appear to be any simple relationship between the polynomials $\pi_{y,w}$ and $\pi'_{y,w}$, and it is unclear what positivity properties the latter polynomials possess, if any. In general, $\pi'_{y,w}$ may have both positive and negative coefficients. The combination of Propositions 3.2, 3.5, 3.9, and 3.10 shows that

$$\tfrac{1}{2}\left(h_{y,w} \pm \pi'_{y,w}\right) \qquad \text{and} \qquad \tfrac{1}{2}\left(\pi_{y,w} \pm \pi'_{y,w}\right) \tag{3.6}$$

are polynomials in $v^{-1}$ with integer coefficients, which become polynomials in $v^2$ when multiplied by $v^{\ell(w)-\ell(y)}$. Unlike the analogous polynomials $\frac{1}{2}(h_{y,w} \pm \pi_{y,w})$ discussed at the end of the previous section (which conjecturally belong to $\mathbb{N}[v^{-1}]$), the four polynomials in (3.6) can each have both positive and negative coefficients.

## 3.4 A third canonical basis for twisted involutions

It is interesting to consider possible analogues of the results in the previous sections in which the modified Iwahori-Hecke algebra $\mathcal{H}_2$ can be replaced by $\mathcal{H}$, which from a formal standpoint seems more natural. In this section, we show that there are such analogues. In particular, the free $\mathcal{A}$-module generated by $\mathbf{I}$ has a natural pre-canonical $\mathcal{H}$-module (rather than $\mathcal{H}_2$-module) structure, which admits a canonical basis unrelated to our other bases $\{\underline{L}_w\}$ and $\{\underline{L}'_w\}$. It is also an open problem to find an interpretation of this third basis.

Let $\mathcal{I} = \mathcal{I}(W,S)$ be the free $\mathcal{A}$-module with a basis given by the symbols $I_w$ for $w \in \mathbf{I}$. We view this as an $\mathcal{H}$-module according to the following result.

**Theorem 3.11.** There is a unique $\mathcal{H}$-module structure on $\mathcal{I}$ such that

$$H_s I_w = \begin{cases} I_{sws} & \text{if } s \ltimes w = sws > w \\ I_{sws} + (v - v^{-1}) I_w & \text{if } s \ltimes w = sws < w \\ I_{sw} + I_w & \text{if } s \ltimes w = sw > w \\ (v - v^{-1}) I_{sw} + (v - 1 - v^{-1}) I_w & \text{if } s \ltimes w = sw < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

*Proof.* Define $J_w = (v + v^{-1})^{2\rho(w) - \ell(w)} L_w \in \mathcal{L}$ and let $\mathcal{J} = \mathbb{Z}[v^2, v^{-2}]\text{-span}\{J_w : w \in \mathbf{I}\}$. Define $\phi : \mathcal{I} \to \mathcal{J}$ as the $\mathbb{Z}$-linear bijection with $v^n I_w \mapsto v^{2n} J_w$ for $w \in \mathbf{I}$. With $\Phi : \mathcal{H} \to \mathcal{H}_2$ the ring homomorphism (3.1), the multiplication formula $HI = \phi^{-1}(\Phi(H)\phi(I))$ for $H \in \mathcal{H}$ and $I \in \mathcal{I}$ makes $\mathcal{I}$ into an $\mathcal{H}$-module, and one checks that relative to this structure the action of $H_s$ on $I_w$ is described by precisely the given formula. This $\mathcal{H}$-module structure is unique since the elements $H_s$ for $s \in S$ generate $\mathcal{H}$ as an $\mathcal{A}$-algebra. $\square$

We denote by $I \mapsto \overline{I}$ the $\mathcal{A}$-antilinear map $\mathcal{I} \to \mathcal{I}$ with the now familiar formula

$$\overline{I_{(x,\theta)}} = \mathrm{sgn}(x) \cdot \overline{H_x} \cdot I_{(x^{-1},\theta)} \qquad \text{for } (x,\theta) \in \mathbf{I}. \tag{3.7}$$

We have this analogue of Theorems 3.4 and 3.7.

**Theorem 3.12.** Define

- the "bar involution" of $\mathcal{I}$ to be the map $I \mapsto \overline{I}$.
- the "standard basis" of $\mathcal{I}$ to be $\{I_w\}$ with the partially ordered index set $(\mathbf{I}, \leq)$.

This is a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ and it admits a canonical basis $\{\underline{I}_w\}$.

Again by Lemma 2.8, this is the unique pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ in which $\{I_w\}$ serves as the "standard basis."

*Proof.* Define $\mathcal{J}$ and $\Phi : \mathcal{H} \to \mathcal{H}_2$ and $\phi : \mathcal{I} \to \mathcal{J}$ as in the proof of Theorem 3.11. Observe that the bar involution defined in Theorem 3.4 for $\mathcal{L}$ restricts to an $\mathcal{A}$-antilinear map $\mathcal{J} \to \mathcal{J}$. Denote this restricted map by $\psi'$, and write $\psi : I \mapsto \overline{I}$ for the bar involution of $\mathcal{I}$. Since $\Phi(\overline{H_x}) = \overline{K_x}$ for all $x \in W$, it follows that $\psi = \phi^{-1} \circ \psi' \circ \phi$, and from this identity the claim that $(\psi, \{I_w\})$ is a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ follows as a consequence of Theorem 3.4. Given this, we conclude that a canonical basis $\{\underline{I}_w\}$ exists by Proposition 2.9. $\square$

**Remark.** Suppose $(W', S')$ is a Coxeter system such that $W = W' \times W'$ and $S = S' \sqcup S'$. Let $\theta \in \mathrm{Aut}(W,S)$ be the automorphism with $\theta(x,y) = (y,w)$. There is then an injective $\mathcal{A}$-module homomorphism $\mathcal{H}(W',S') \to \mathcal{I}(W,S)$ with

$$H_w \mapsto I_{((w,w^{-1}),\theta)} \qquad \text{and} \qquad \underline{H}_w \mapsto \underline{I}_{((w,w^{-1}),\theta)} \qquad \text{for } w \in W'.$$

Via this map, one may view the canonical basis of $\mathcal{I}$ as a generalization of the Kazhdan-Lusztig basis of $\mathcal{H}$. The canonical bases of $\mathcal{L}$ and $\mathcal{L}'$ in the previous sections generalize the canonical basis of $\mathcal{H}_2$ in an entirely analogous fashion.

Define $\iota_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ such that $\underline{I}_w = \sum_{y\in\mathbf{I}} \iota_{y,w} I_y$ and let

$$\nu(s,y,w) = \begin{cases} \text{the coefficient of } v^{-1} \text{ in } \iota_{y,w} & \text{if } sy < y \\ \text{the coefficient of } v^{-1} \text{ in } \iota_{sy,w} & \text{if } sy = ys > y \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } s \in S \text{ and } y, w \in \mathbf{I}.$$

Recall that $\underline{H}_s = H_s + v^{-1}$ for $s \in S$.

**Proposition 3.13.** If $s \in S$ and $w \in \mathbf{I}$ such that $w < sw$ then

$$\underline{H}_s \underline{I}_w = \underline{I}_{s \ltimes w} + \delta_{sw,ws} \underline{I}_w + \sum_{y<w} \nu(s,y,w) \underline{I}_y.$$

**Remark.** Unlike the canonical basis $\{\underline{L}_w\}$ (see the remark after Proposition 3.8), there is no simple formula for $\underline{H}_s \underline{I}_w$ when $s \in S$ such that $sw < w$.

*Proof.* The difference between the two sides of the desired identity is invariant under the bar involution of $\mathcal{I}$ and is also an element of the set $\sum_{y<s\ltimes w} v^{-1}\mathbb{Z}[v^{-1}] \cdot I_y$, as is straightforward to check from the definition of $\nu(s,y,w)$ and Theorem 3.11. The only such element in $\mathcal{I}$ is 0. ☐

We note one other proposition. Recall the definition of $\rho : \mathbf{I} \to \mathbb{N}$ from Theorem 2.10.

**Proposition 3.14.** If $y, w \in \mathbf{I}$ such that $y \leq w$ then $v^{\rho(w)-\rho(y)} \iota_{y,w} \in 1 + v\mathbb{Z}[v]$.

*Proof.* The proposition holds if $\rho(w) = 0$ since then $\iota_{y,w} = \delta_{y,w}$. Let $w \in \mathbf{I}$ and $s \in S$ such that $sw > w$ and assume $v^{\rho(z)-\rho(y)} \iota_{y,z} \in 1 + v\mathbb{Z}[v]$ for all $y \leq z < s \ltimes w$. It suffices to show under this hypothesis that $v^{\rho(s\ltimes w)-\rho(y)} \iota_{y,s\ltimes w} \in 1 + v\mathbb{Z}[v]$ for all $y \in \mathbf{I}$ with $y \leq s \ltimes w$. To this end, let $y \leq s \ltimes w$ and observe that Proposition 3.8 implies

$$\iota_{y,s\ltimes w} = (a - \delta_{sw,ws}) \cdot \iota_{y,w} + b \cdot \iota_{s\ltimes y,w} - \Sigma$$

where

$$(a,b) = \begin{cases} (v^{-1},\ 1) & \text{if } sy \neq ys > y \\ (v,\ 1) & \text{if } sy \neq ys < y \\ (v^{-1}+1,\ v - v^{-1}) & \text{if } sy = ys > y \\ (v-1,\ 1) & \text{if } sy = ys < y \end{cases} \qquad \text{and} \qquad \Sigma = \sum_{z<w} \nu(s,z,w) \iota_{y,z}.$$

By hypothesis $v^{\rho(s\ltimes w)-\rho(y)} \Sigma \in v\mathbb{Z}[v]$, and it is straightforward to check that

$$v^{\rho(s\ltimes w)-\rho(y)} ((a - \delta_{sw,ws}) \cdot \iota_{y,w} + b \cdot \iota_{s\ltimes y,w}) \in 1 + v\mathbb{Z}[v]$$

using the fact (see [7, Lemma 2.7]) that $s \ltimes y \leq w$ if $sy < y$ and $y \leq w$ if $sy > y$. Combining these observations shows that $v^{\rho(s\ltimes w)-\rho(y)} \iota_{y,s\ltimes w} \in 1 + v\mathbb{Z}[v]$ as desired. ☐

As one might expect from the new type of degree bound given in Proposition 3.14, there is no obvious relationship between the polynomials $\iota_{y,w}$ and the other polynomials $h_{y,w}, \pi_{y,w}, \pi'_{y,w} \in \mathbb{Z}[v^{-1}]$ we have seen so far. This becomes clear from computations. For example, suppose $|S| = 2$ so that $(W, S)$ is a dihedral Coxeter system. Then the values of $v^{\ell(w)-\ell(y)} h_{y,w}$ (for $y, w \in W$) and $v^{\ell(w)-\ell(y)} \pi_{y,w}$ (for $y, w \in \mathbf{I}$) are all 0 or 1; see [15, Theorem 4.3]. However, the polynomials $v^{\rho(w)-\rho(y)} \iota_{y,w}$ for $y, w \in \mathbf{I}$ can achieve any of the values 0, 1, $1+v$, $1-v$, or $1-v^2$. The polynomials $\iota_{y,w}$ may thus have negative coefficients, and do not in general satisfy any parity condition analogous to Proposition 3.9.

# 4 Uniqueness statements

## 4.1 Morphisms between pre-canonical structures

Theorems 3.3, 3.6, and 3.11 each give a certain formula, depending on eight parameters in $\mathcal{A}$, which defines an $\mathcal{H}$- or $\mathcal{H}_2$-module structure on $\mathcal{A}\mathbf{I}$ for all Coxeter systems $(W, S)$. The results above show that there are unique pre-canonical structures on $\mathcal{A}\mathbf{I}$ which are compatible with these module structures, in which $\mathbf{I}$ ordered by $\leq$ as the "standard basis." The goal of this section is to show that any pre-canonical structures on $\mathcal{A}\mathbf{I}$ determined in an analogous fashion (from an $\mathcal{H}$- or $\mathcal{H}_2$-module structure given by a different choice of parameters in Theorem 3.3) is in a certain sense "the same" as those defined on $\mathcal{L}$, $\mathcal{L}'$, or $\mathcal{I}$.

Our first step towards this end is to say what it means for two pre-canonical structures to be "the same." This amounts to defining what should comprise a morphism between pre-canonical structures on free $\mathcal{A}$-modules. In this pursuit we are guided by the principle that if a morphism exists from one pre-canonical structure to another, and if the first structure admits a canonical basis, then the second structure should admit a canonical basis which can be described explicitly in terms of the first basis.

The following is a natural but rigid notion of (iso)morphism compatible with this philosophy. Suppose $V$ and $V'$ are free $\mathcal{A}$-modules with respective pre-canonical structures $(\psi, \{a_c\})$ and $(\psi', \{a'_c\})$. We say that an $\mathcal{A}$-linear map $\varphi : V \to V'$ is a *strong isomorphism* of pre-canonical structures if $\varphi$ restricts to an order-preserving bijection $\{a_c\} \to \{a'_c\}$ between standard bases and $\varphi$ commutes with bar involutions in the sense that $\varphi \circ \psi = \psi' \circ \varphi$. Under these conditions, $\varphi$ is necessarily invertible as an $\mathcal{A}$-linear map. The inverse and composition of strong isomorphisms of pre-canonical structures are again strong isomorphisms of pre-canonical structures. Moreover, if $\varphi : V \to V'$ is a strong isomorphism of pre-canonical structures and $V$ admits a canonical basis $\{b_c\}$, then $\{\varphi(b_c)\}$ is a canonical basis of $V'$.

There are other situations in which we would like to consider two pre-canonical structures to be "the same" besides when they are strongly isomorphic. We illustrate this as follows. Continue to let $V$ be a free $\mathcal{A}$-module with a pre-canonical structure $(\psi, \{a_c\})$ whose standard basis is indexed by $(C, \leq)$. Suppose for each index $c \in C$ we have an element $d_c \in \mathcal{A}$. Let $u_c = d_c a_c$ and consider the set of rescaled basis elements $\{u_c\}$, likewise indexed by $(C, \leq)$. These elements are linearly independent if and only if each $d_c \neq 0$, so assume this condition holds and define $U = \mathcal{A}\text{-span}\{u_c : c \in C\}$. One naturally asks when $(\psi, \{u_c\})$ is a pre-canonical structure on the submodule $U \subset V$. Since we have

$$\psi(u_c) \in \tfrac{\overline{d_c}}{d_c} \cdot u_c + \sum_{c' < c} \mathcal{A} \cdot \tfrac{\overline{d_c}}{d_{c'}} \cdot u_{c'}$$

it follows that $(\psi, \{u_c\})$ is a pre-canonical structure on $U$ at least when (i) each $d_c = \overline{d_c}$ and (ii) $d_c = q_{c',c} d_{c'}$ for some $q_{c',c} \in \mathcal{A}$ whenever $c' < c$ in $C$. Moreover, the first of these sufficient conditions is also necessary. Note that if (i) and (ii) hold then $q_{c',c} = \overline{q_{c',c}}$ and so $q_{c',c} \in \mathbb{Z}[v+v^{-1}]$ since $\mathbb{Z}[v+v^{-1}]$ is the set of bar invariant elements of $\mathcal{A}$.

Assume conditions (i) and (ii) hold and further that $V$ admits a canonical basis $\{b_c\}$ with respect to the pre-canonical structure $(\psi, \{a_c\})$. In general, this does not immediately imply that $U$ has a canonical basis, but provided one exists, one asks whether it is related to the basis $\{a_c\}$; in particular, when does some rescaling of $\{b_c\}$ give a canonical basis for $U$? By condition (C2) in Definition 2.2, it follows that the only possible such basis would be given by $\{d_c b_c\}$. Since

$$d_c b_c \in u_c + \sum_{c'<c} v^{-1}\mathbb{Z}[v^{-1}] \cdot q_{c',c} \cdot u_{c'}$$

it follows that $\{d_c b_c\}$ is a canonical basis for $U$ at least when $q_{c',c} \in \mathbb{Z}[v^{-1}]$. Since $\mathbb{Z} = \mathbb{Z}[v^{-1}] \cap \mathbb{Z}[v+v^{-1}]$, we may summarize this discussion with the following lemma.

**Lemma 4.1.** For each index $c \in C$ let $d_c \in \mathcal{A}$ and define

$$u_c = d_c a_c \qquad \text{and} \qquad U = \mathcal{A}\text{-span}\{u_c : c \in C\}.$$

Suppose the following conditions hold:

(i) $d_c \in \mathbb{Z}[v+v^{-1}]$ and $d_c \neq 0$ for all $c \in C$.

(ii) $d_c / d_{c'} \in \mathbb{Z}$ whenever $c' < c$.

Then $(\psi, \{u_c\})$ is a pre-canonical structure on $U$. If $\{b_c\}$ is a canonical basis of $V$ then $\{d_c b_c\}$ is a canonical basis of $U$.

Morphisms between pre-canonical structures should at least include strong isomorphisms and also the $\mathcal{A}$-linear maps $D : V \to V'$ given by $D(a_c) = d_c a_c$ when the conditions hold in the preceding proposition. There is a third kind of map which should form a morphism; in particular, it is natural to consider the map $\Phi$ given by (3.1) to be a morphism between the pre-canonical structures on $\mathcal{H}$ and $\mathcal{H}_2$, as we will see in the following lemma.

Let $\epsilon$ be a ring endomorphism of $\mathcal{A}$. Such a map is $\mathbb{Z}$-linear and completely determined by its value at $v \in \mathcal{A}$, which must be a unit, since $\epsilon(v)\epsilon(v^{-1}) = \epsilon(vv^{-1}) = \epsilon(1) = 1$. It follows that $\epsilon(v) = \pm v^n$ for some $n \in \mathbb{Z}$. Call $n$ the *degree* of the endomorphism $\epsilon$. We say that a map $\varphi : M \to N$ between $\mathcal{A}$-modules is *$\epsilon$-linear* if $\varphi(fm) = \epsilon(f)\varphi(m)$ for $f \in \mathcal{A}$ and $m \in M$.

**Lemma 4.2.** Let $\epsilon$ be a ring endomorphism of $\mathcal{A}$ and write $\tau : V \to V$ and $\phi : V \to V$ for the respective $\epsilon$-linear and $\mathcal{A}$-antilinear maps with

$$\tau(a_c) = a_c \qquad \text{and} \qquad \phi(a_c) = \tau \circ \psi(a_c) \qquad \text{for } c \in C.$$

Then $(\phi, \{a_c\})$ is another pre-canonical structure on $V$. If $\{b_c\}$ is a canonical basis of $V$ relative to $\{\psi, \{a_c\}\}$ and $\epsilon$ has positive degree, then $\{\tau(b_c)\}$ is a canonical basis of $V$ relative to $(\phi, \{a_c\})$.

*Proof.* That $(\phi, \{a_c\})$ is a pre-canonical structure is clear from the definitions. To prove the rest of the lemma, define $r_{x,y} \in \mathcal{A}$ and $p_{x,y} \in \mathbb{Z}[v]$ such that

$$\psi(a_y) = \sum_{x \le y} r_{x,y} a_x \qquad \text{and} \qquad b_y = \sum_{x \le y} \overline{p_{x,y}} a_x \qquad \text{for } y \in C.$$

Since $\epsilon$ has positive degree, we have $\epsilon(\overline{p_{x,y}}) = \overline{\epsilon(p_{x,y})} \in \mathbb{Z}[v^{-1}]$ and it follows that $\tau(b_y) \in a_y + \sum_{x<y} v^{-1}\mathbb{Z}[v^{-1}]a_x$. In turn, we have also

$$\phi(\tau(b_y)) = \sum_{x\leq y} \overline{\epsilon(\overline{p_{x,y}})}\tau \circ \psi(a_y) = \sum_{x\leq y}\sum_{w\leq x} \epsilon(p_{x,y}r_{x,y})a_w = \tau \circ \psi(b_y) = \varphi(b_y).$$

We conclude that $\{\tau(b_c)\}$ is a canonical basis of $V$ relative to $(\phi, \{a_c\})$. $\square$

Motivated by the preceding lemmas, we adopt the following definition. Let $V$ and $V'$ be free $\mathcal{A}$-modules with pre-canonical structures $(\psi, \{a_c\})$ and $(\psi', \{a'_c\})$. Assume the standard bases $\{a_c\}$ and $\{a'_c\}$ have the same partially ordered index set $(C, \leq)$.

**Definition 4.3.** A map $\varphi : V \to V'$ is a *morphism* of pre-canonical structures if

(i) The map $\varphi$ is $\epsilon$-linear for a positive degree ring endomorphism $\epsilon : \mathcal{A} \to \mathcal{A}$.

(ii) There are nonzero polynomials $d_c \in \mathcal{A}$ for $c \in C$ with $d_c/d_{c'} \in \mathbb{Z}$ whenever $c' < c$, such that if $D : V \to V$ is the $\mathcal{A}$-linear map with $D(a_c) = d_c a_c$ for $c \in C$ then $\psi' \circ \varphi = \varphi \circ \psi^D$, where we define $\psi^D = D^{-1} \circ \psi \circ D$.

**Remark.** The polynomials $d_c$ in condition (ii) automatically belong to $\mathbb{Z}[v + v^{-1}]$ since the coefficients of $a_c$ in $\varphi^{-1} \circ \psi' \circ \varphi(a_c)$ and in $\psi^D(a_c)$, which must be equal, are 1 and $\overline{d_c}/d_c$ respectively. This observation and the fact that $d_c/d_{c'} \in \mathbb{Z}$ whenever $c' < c$ in $C$ ensure that $\psi^D$ is a well-defined map $V \to V$, even though $D^{-1}$ may not be.

If $\varphi : V \to V'$ is a morphism of pre-canonical structures then we call a map $D : V \to V$ of the form in condition (ii) of Definition 4.3 a *scaling factor* of $\varphi$. If $V' \subset V$ and $\varphi$ is equal to one of its scaling factors then we call $\varphi$ a *scaling morphism*. We define the *degree* of any morphism $\varphi$ to be the degree of the ring endomorphism $\epsilon$ in condition (i). If $V = V'$ and $\{a_c\} = \{a'_c\}$ and the identity is a scaling factor of $\varphi$, then we call $\varphi$ a *parametric morphism*.

In the rest of this section we describe some properties of morphisms in this sense. We fix some notation. Let $V$ and $V'$ and $V''$ be free $\mathcal{A}$-modules with pre-canonical structures $(\{a_c\}, \psi)$ and $(\{a'_c\}, \psi')$ and $(\{a''_c\}, \psi'')$. Assume the standard bases of these structures all have the same partially ordered index set $(C, \leq)$, and suppose $\varphi : V \to V'$ and $\varphi' : V' \to V''$ are morphisms of pre-canonical structures.

**Proposition 4.4.** The composition $V \xrightarrow{\varphi} V' \xrightarrow{\varphi'} V''$ is a morphism of pre-canonical structures. The collection of pre-canonical structures on free $\mathcal{A}$-modules forms a category.

*Proof.* Condition (i) in Definition 4.3 clearly holds for the composition. Let $D$ and $D'$ be scaling factors for $\varphi$ and $\varphi'$. Define $D'' : V \to V$ as the $\mathcal{A}$-linear map with $D''(a_c) = d_c d'_c a_c$ for $c \in C$ where $d_c, d'_c \in \mathbb{Z}[v+v^{-1}]$ are such that $D(a_c) = d_c a_c$ and $D'(a'_c) = d'_c a'_c$. Since $d'_c/d'_{c'} \in \mathbb{Z}$ (and so is fixed by all ring endomorphisms of $\mathcal{A}$) whenever $c' < c$ in $C$, we have $\varphi' \circ \varphi \circ \psi^{D''} = \varphi' \circ (\psi')^{D'} \circ \varphi = \psi'' \circ \varphi' \circ \varphi$. We conclude that $\varphi' \circ \varphi$ is a morphism for which $D''$ is a scaling factor. $\square$

**Proposition 4.5.** Every morphism of pre-canonical structures can be written as a composition $\iota \circ \sigma \circ \tau$ where $\iota$ is a strong isomorphism, $\sigma$ is a scaling morphism, and $\tau$ is a parametric morphism.

*Proof.* Let $\epsilon$ be the $\mathcal{A}$-endomorphism of positive degree such that $\varphi$ is $\epsilon$-linear. Define $\tau : V \to V$ and $\phi : V \to V$, relative to $(\psi, \{a_c\})$ and $\epsilon$, as in Lemma 4.2. Then $(\psi, \{a_c\})$ and $(\phi, \{a_c\})$ are both pre-canonical structures on $V$ and $\tau : V \to V$ is a parametric morphism from the first to the second.

Next, let $D$ be a scaling factor of $\varphi$ so that $D(a_c) = d_c a_c$ for some $d_c \in \mathbb{Z}[v+v^{-1}]$ for each $c \in C$. Let $d'_c = \epsilon(d_c)$ and write $\sigma : V \to V$ for the $\mathcal{A}$-linear map with $\sigma(a_c) = d'_c a_c$. Define $u_c = d'_c a_c$ and $U = \mathcal{A}\text{-span}\{u_c : c \in C\}$ as in Lemma 4.1. Then $(\phi, \{u_c\})$ is a pre-canonical structure on $U$ and the map $\sigma : V \to U$ is a scaling morphism from $(\phi, \{a_c\})$ to $(\phi, \{u_c\})$.

Finally, define $\iota : U \to V'$ as the $\mathcal{A}$-linear map with $\iota(u_c) = a'_c$ for $c \in C$. This is a strong isomorphism since for any $c \in C$ we have

$$\iota \circ \phi(u_c) = d'_c \cdot \iota \circ \tau \circ \psi(a_c) = \varphi \circ \psi^D(a_c) = \psi' \circ \varphi(a_c) = \psi'(a'_c) = \psi' \circ \iota(u_c).$$

As both $\iota \circ \psi$ and $\psi' \circ \iota$ are $\mathcal{A}$-antilinear, this identity shows that the two maps are equal. The composition $\iota \circ \sigma \circ \tau$ agrees with $\varphi$ at each basis element $a_c$, and both maps are $\epsilon$-linear, so they are equal. ☐

**Proposition 4.6.** Suppose the pre-canonical structure on $V$ admits a canonical basis $\{b_c\}$. Then the pre-canonical structure on $V'$ also admits a canonical basis $\{b'_c\}$. If $D$ is a scaling factor of $\varphi$ and $\beta : V \to V$ is the $\mathcal{A}$-linear map with $\beta(a_c) = b_c$ for each $c \in C$, then the composition

$$\varphi \circ D^{-1} \circ \beta \circ D \circ \beta^{-1}$$

is a well-defined map $V \to V'$ which restricts to an order-preserving bijection $\{b_c\} \to \{b'_c\}$.

*Proof.* Let $b'_c = \varphi \circ D^{-1} \circ \beta \circ D \circ \beta^{-1}(b_c) = \varphi \circ D^{-1} \circ \beta \circ D(a_c)$. It suffices to check that this element satisfies the defining conditions of a canonical basis. To see that $b'_c$ is invariant under $\psi'$, note that $\psi' \circ \varphi = \varphi \circ D^{-1} \circ \psi \circ D$ and $\psi \circ \beta \circ D(a_c) = \beta \circ D(a_c)$ for $c \in C$, and so

$$\psi'(b'_c) = \psi' \circ \varphi \circ D^{-1} \circ \beta \circ D(a_c) = \varphi \circ D^{-1} \circ \psi \circ \beta \circ D(b_c) = b'_c.$$

For condition (C2), note that if $b_c = a_c + \sum_{c'<c} f_{c',c} a_{c'}$ for some $f_{c',c} \in v^{-1}\mathbb{Z}[v^{-1}]$ then

$$b'_c = a'_c + \sum_{c'<c} \epsilon(f_{c',c} \cdot d_c/d_{c'}) a'_{c'}$$

where $\epsilon$ is the ring endomorphism of $\mathcal{A}$ such that $\varphi$ is $\epsilon$-linear. The coefficients $\epsilon(f_{c',c} \cdot d_c/d_{c'})$ all belong to $v^{-1}\mathbb{Z}[v^{-1}]$ since $\epsilon$ is a ring endomorphism of positive degree and each $d_c/d_{c'} \in \mathbb{Z}$. Thus $b'_c$ has the required triangular form. ☐

**Proposition 4.7.** A morphism of pre-canonical structures is an isomorphism (that is, there exists a morphism of pre-canonical structures which is its left and right inverse) if and only if it has degree 1 and it has a scaling factor whose eigenvalues are each $\pm 1$.

*Proof.* If $\varphi$ has degree 1 and a scaling factor $D$ whose eigenvalues are each $\pm 1$, then $D = D^{-1}$ and $\varphi$ is an $\epsilon$-linear bijection (where $\epsilon = \epsilon^{-1}$ is a ring involution of $\mathcal{A}$) and it follows that the inverse map $\varphi^{-1}$ is well-defined and a morphism of pre-canonical structures with scaling factor $\varphi \circ D \circ \varphi^{-1}$. Hence in this case $\varphi$ is an isomorphism of pre-canonical structures. Suppose conversely that $D$ is a scaling factor for $\varphi$ and that $\varphi^{-1}$ exists and is a morphism with scaling factor $D'$. Then $\varphi$ must have degree 1 since otherwise $\varphi$ is not invertible. To show that $\varphi$ has some scaling factor all of whose eigenvalues are $\pm 1$, let $D'' = \varphi \circ D \circ \varphi^{-1}$. Then

$$\psi' = \varphi \circ (\varphi^{-1} \circ \psi' \circ \varphi) \circ \varphi^{-1} = D''^{-1} \circ (\varphi \circ \psi \circ \varphi^{-1}) \circ D'' = (D'D'')^{-1} \circ \psi' \circ (D'D'').$$

For each $c \in C$ let $d_c$ and $d'_c$ be the elements of $\mathbb{Z}[v+v^{-1}]$ such that $D(a_c) = d_c a_c$ and $D'(a'_c) = d'_c a'_c$. Now, write $\sim$ for the minimal equivalence relation on $C$ such that $c \sim c'$ whenever $c, c' \in C$ such that the coefficient $f_{c',c}$ of $a'_{c'}$ in $\psi'(a'_c)$ is nonzero. The equation above implies

$$f_{c',c} = d_c/d_{c'} \cdot d'_c/d'_{c'} \cdot f_{c',c}$$

so since $d_c/d_{c'}$ and $d'_c/d'_{c'}$ are both integers, these quotients must each be $\pm 1$. Hence if $K$ is an equivalence class under $\sim$ then $d_c/d_{c'} \in \{\pm 1\}$ for any $c, c' \in K$. For each such equivalence class $K$, choose an arbitrary $c \in K$ and let $d_K = d_c$. Now let $E : V \to V$ be the $\mathcal{A}$-linear map with $E(a_c) = d_K a_c$ where $K$ is the equivalence class of $c \in C$. We claim that

$$\psi = E^{-1} \circ \psi \circ E.$$

This follows since if the coefficient of $a_{c'}$ in $\psi(a_c)$ is some polynomial $f \in \mathcal{A}$, then the coefficient of $a_{c'}$ in $E^{-1} \circ \psi \circ E(a_c)$ is $d_K/d_{K'} \cdot f$ where $K$ and $K'$ are the equivalence classes of $c$ and $c'$. If $f = 0$ then these coefficients are both zero, and if $f \neq 0$ then the coefficient of $a'_{c'}$ in $\psi'(a'_c)$ is also nonzero, so $K = K'$ and our coefficients are again equal. From this claim, we conclude that $E^{-1}D$ is another scaling factor of $\varphi$. The eigenvalues of this scaling factor are each $\pm 1$ since if $K$ is the equivalence class of $c \in C$ then $d_c/d_K \in \{\pm 1\}$. ∎

The following corollary shows that the structure constants of canonical bases arising from isomorphic pre-canonical structures differ only by a factor of $\pm 1$ or the substitution $v \mapsto -v$.

**Corollary 4.8.** Suppose the pre-canonical structures on $V$ and $V'$ are isomorphic and admit canonical bases $\{b_c\}$ and $\{b'_c\}$. Define $f_{x,y}(t), g_{x,y}(t) \in \mathbb{Z}[t]$ such that

$$b_y = \sum_{x \leq y} f_{x,y}(v^{-1}) a_x \qquad \text{and} \qquad b'_y = \sum_{x \leq y} g_{x,y}(v^{-1}) a'_x.$$

Then for each $x, y \in C$ there are $\varepsilon_i \in \{\pm 1\}$ such that $f_{x,y}(t) = \varepsilon_1 \cdot g_{x,y}(\varepsilon_2 t)$.

*Proof.* Let $\varphi : V \to V'$ be an isomorphism of pre-canonical structures. By the previous proposition, $\varphi$ has a scaling factor $D$ whose eigenvalues are all $\pm 1$, and $\varphi$ is $\epsilon$-linear where $\epsilon \in \mathrm{End}(\mathcal{A})$ is either the identity or the ring homomorphism with $v \mapsto -v$. Given these considerations, the corollary follows from Proposition 4.6. ∎

## 4.2 Generic representations on group elements

The $\mathcal{H}$-module structure in Theorem 3.11 has the following simple form: for each $s \in S$ and $w \in \mathbf{I}$ the generator $H_s \in \mathcal{H}$ maps $I_w \mapsto aI_{s \ltimes w} + bI_w$ where $a, b \in \mathcal{A}$ depend only on the length difference between $w$ and $s \ltimes w$, which can take four possible values. Thus, the $\mathcal{H}$-module structure on $\mathcal{I}$ is completely determined by eight parameters in $\mathcal{A}$. Besides the one in Theorem 3.11, there are many other choices of these parameters which give well-defined $\mathcal{H}$-module structures on $\mathcal{A}\mathbf{I}$ for all Coxeter systems $(W, S)$. It is a natural problem to classify such choices, and to identify for which of the resulting $\mathcal{H}$-modules an analogue of Theorem 3.12 holds. This problem is the main topic of the next section; in the present section, we briefly consider its analogue for $\mathcal{H}$-representations on the free $\mathcal{A}$-module generated by $W$.

Our results here are useful for comparison with the theorems in the next sections. The proofs in this section are only sketched, since they are just simpler versions of the arguments in Section 4.3, which we carry out in detail.

**Notation.** If $X$ is a set then we write $\mathcal{A}X$ for the free $\mathcal{A}$-module generated by $X$, and let $\mathrm{End}(\mathcal{A}X)$ denote the $\mathcal{A}$-module of $\mathcal{A}$-linear maps $\mathcal{A}X \to \mathcal{A}X$. A *representation* of $\mathcal{H}$ in some $\mathcal{A}$-module $\mathcal{M}$ is an $\mathcal{A}$-algebra homomorphism $\mathcal{H} \to \mathrm{End}(\mathcal{M})$.

Consider a $2 \times 2$ matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. Given a Coxeter system $(W, S)$, we let $\rho_\gamma : \{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}W)$ denote the map with

$$\rho_\gamma(H_s)(w) = \begin{cases} \gamma_{11} \cdot sw + \gamma_{12} \cdot w & \text{if } sw > w \\ \gamma_{21} \cdot sw + \gamma_{22} \cdot w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

**Definition 4.9.** The matrix $\gamma$ is an $(\mathcal{H}, W)$*-structure* if for every Coxeter system $(W, S)$, the map $\rho_\gamma$ extends to a representation of $\mathcal{H} = \mathcal{H}(W, S)$ in $\mathcal{A}W$.

An $(\mathcal{H}, W)$-structure $\gamma = (\gamma_{ij})$ is *trivial* if $\gamma_{11} = \gamma_{21} = 0$ and $\gamma_{12} = \gamma_{22} \in \{v, -v^{-1}\}$. Such a structure defines an $\mathcal{H}$-representation which decomposes as a direct sum of free $\mathcal{A}$-modules of rank one. Observe that the units in the ring $\mathcal{A}$ are the monomials of the form $\pm v^n$ for $n \in \mathbb{Z}$.

**Theorem 4.10.** Every nontrivial $(\mathcal{H}, W)$-structure is equal to

$$\begin{bmatrix} \alpha & 0 \\ \alpha^{-1} & v - v^{-1} \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \alpha & v - v^{-1} \\ \alpha^{-1} & 0 \end{bmatrix}$$

for some unit $\alpha$ in $\mathcal{A}$. All nontrivial $(\mathcal{H}, W)$-structures define isomorphic $\mathcal{H}$-representations.

*Proof sketch.* The given matrices are $(\mathcal{H}, W)$-structures, since those on the left (respectively, right) describe the action of $H_s$ for $s \in S$ on the basis $\{\alpha^{-\ell(w)} H_w : w \in W\}$ (respectively, $\{\alpha^{-\ell(w)} \overline{H_w} : w \in W\}$) of $\mathcal{H}$. In particular, these $(\mathcal{H}, W)$-structures define $\mathcal{H}$-representations isomorphic to the regular representation of $\mathcal{H}$ on itself. That there are no other nontrivial $(\mathcal{H}, W)$-structures follows by a simpler version of the argument used in the proof of Theorem 4.16 in the next section. ☐

An $(\mathcal{H}, W)$-structure $\gamma$ defines an $\mathcal{H}$-module structure on $\mathcal{A}W$ for every Coxeter system $(W, S)$. We say that $\gamma$ is *pre-canonical* if each of these $\mathcal{H}$-modules has a pre-canonical $\mathcal{H}$-module structure in which $W$ partially ordered by the Bruhat order is the "standard basis." It follows from the preceding theorem and Lemma 2.8 that if $\gamma$ is nontrivial and pre-canonical, then there is a unique bar involution $\psi : \mathcal{A}W \to \mathcal{A}W$ such that $(\psi, W)$ is a pre-canonical $\mathcal{H}$-module structure.

These definitions lead to the following characterization of the Kazhdan-Lusztig basis of $\mathcal{H}$.

**Theorem 4.11.** Exactly 4 nontrivial $(\mathcal{H}, W)$-structures are pre-canonical. The 4 associated pre-canonical structures on $\mathcal{A}W$ are all isomorphic (in the sense of Definition 4.3) to the pre-canonical structure on $\mathcal{H}$ defined in Theorem 3.1, and hence they each admit a canonical basis.

*Proof sketch.* The proof is similar to that of Theorem 4.20 in the next section. Let $\gamma$ be a nontrivial, pre-canonical $(\mathcal{H}, W)$-structure. Then $\gamma$ must be one of the two matrices in Theorem 4.10 for some unit $\alpha \in \mathcal{A}$. One first argues that $\alpha = \overline{\alpha}$ and hence that $\alpha \in \{\pm 1\}$. Next, one observes that $\gamma$ remains pre-canonical if $\alpha$ is replaced with $-\alpha$, and that the pre-canonical structures associated to these two $(\mathcal{H}, W)$-structures are always isomorphic. One may therefore assume $\alpha = 1$. It remains to prove that if $\gamma$ is the right-hand matrix in Theorem 4.10 then its associated pre-canonical structure is isomorphic to the pre-canonical structure on $\mathcal{H}$ defined in Theorem 3.1. This can be deduced from [10, Lemma 2.1(i)], after noting that the $\mathcal{A}$-linear map with $w \mapsto \overline{H_w}$ defines an isomorphism between $\mathcal{A}W$ viewed as an $\mathcal{H}$-module via $\gamma$ and $\mathcal{H}$ viewed as a left module over itself. $\square$

### 4.3 Generic representations on twisted involutions

We turn to the classification problems described at the beginning of the previous section. Consider a $4 \times 2$ matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. Given a Coxeter system $(W, S)$, writing $\mathbf{I} = \mathbf{I}(W, S)$, we let $\rho_\gamma : \{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}\mathbf{I})$ denote the map with

$$\rho_\gamma(H_s)(w) = \begin{cases} \gamma_{11} \cdot sws + \gamma_{12} \cdot w & \text{if } s \ltimes w = sws > w \\ \gamma_{21} \cdot sws + \gamma_{22} \cdot w & \text{if } s \ltimes w = sws < w \\ \gamma_{31} \cdot sw + \gamma_{32} \cdot w & \text{if } s \ltimes w = sw > w \\ \gamma_{41} \cdot sw + \gamma_{42} \cdot w & \text{if } s \ltimes w = sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in \mathbf{I}.$$

**Definition 4.12.** The matrix $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-*structure* if for every Coxeter system $(W, S)$, the map $\rho_\gamma$ extends to a representation of $\mathcal{H} = \mathcal{H}(W, S)$ in $\mathcal{A}\mathbf{I} = \mathcal{A}\mathbf{I}(W, S)$.

**Remark.** It would make sense to view $\rho_\gamma$ as a map $\{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}W)$ by the same formula. However, combining some computations with the analysis in this section, one can show that $\rho_\gamma$ only extends to a representation of $\mathcal{H}$ in $\mathcal{A}W$ for every Coxeter system $(WS)$ when $\gamma$ is *trivial*, where we say that $\gamma = (\gamma_{ij})$ is trivial if $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 0$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} \in \{v, -v^{-1}\}$.

We will classify all nontrivial $(\mathcal{H}, \mathbf{I})$-structures after proving three lemmas.

**Lemma 4.13.** Let $A, B, C, D, E, F, G, H \in \mathcal{A}$ and suppose $\alpha, \beta \in \mathbb{Q}(v) - \{0\}$ such that $A\alpha^{-1}$ and $C\alpha$ and $E\beta^{-1}$ and $G\beta$ all belong to $\mathcal{A}$. Let

$$\gamma = \begin{bmatrix} A & B \\ C & D \\ E & F \\ G & H \end{bmatrix} \qquad \text{and} \qquad \gamma[\alpha, \beta] = \begin{bmatrix} A\alpha^{-1} & B \\ C\alpha & D \\ E\beta^{-1} & F \\ G\beta & H \end{bmatrix}.$$

If $\gamma$ is a $(\mathcal{H}, \mathbf{I})$-structure then so is $\gamma[\alpha, \beta]$. In this case, we say that $\gamma$ and $\gamma[\alpha, \beta]$ are *diagonally equivalent*. If $\alpha, \beta \in \mathcal{A}$ then $\gamma$ and $\gamma[\alpha, \beta]$ define isomorphic representations of $\mathcal{H}$.

Recall the definition of $\rho : \mathbf{I} \to \mathbb{N}$ from Theorem 2.10.

*Proof.* Assume $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-structure. The $\mathcal{H}$-representation $\rho_\gamma$ extends to a representation in the larger $\mathcal{A}$-module $\mathbb{Q}(v)\mathbf{I}$ by linearity. Define $T : \mathbb{Q}(v)\mathbf{I} \to \mathbb{Q}(v)\mathbf{I}$ as the $\mathbb{Q}(v)$-linear map with

$$T(w) = \alpha^{\ell(w) - \rho(w)} \cdot \beta^{2\rho(w) - \ell(w)} \cdot w \qquad \text{for } w \in \mathbf{I}.$$

Then $\gamma[\alpha, \beta]$ is an $(\mathcal{H}, \mathbf{I})$-structure since $\rho_{\gamma[\alpha,\beta]}(H) = T^{-1} \circ \rho_\gamma(H) \circ T$ for all $H \in \mathcal{H}$. $\square$

We denote by $\Theta$ the $\mathcal{A}$-algebra automorphism of $\mathcal{H}$ with $\Theta(H_s) = -H_s + v - v^{-1}$ for $s \in S$. Observe that more generally $\Theta(H_w) = \mathrm{sgn}(w) \cdot \overline{H_w}$ for $w \in W$.

**Lemma 4.14.** The involution of the set of $4 \times 2$ matrices with entries in $\mathcal{A}$ given by the map

$$\Theta : \left[\begin{array}{cc} A & B \\ C & D \\ E & F \\ G & H \end{array}\right] \mapsto \left[\begin{array}{cc} -A & v+v^{-1}-B \\ -C & v+v^{-1}-D \\ -E & v+v^{-1}-F \\ -G & v+v^{-1}-H \end{array}\right]$$

restricts to an involution of the set of $(\mathcal{H}, \mathbf{I})$-structures.

*Proof.* Observe that if $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-structure then $\rho_{\Theta(\gamma)}$ is the $\mathcal{H}$-representation $\rho_\gamma \circ \Theta$. $\square$

Our third lemma is more technical. Fix a choice of parameters $A, B, C, D, E, F, G, H \in \mathcal{A}$ and define $\gamma$ as in Lemma 4.13.

**Lemma 4.15.** If $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-structure then the following properties hold:

(a) $(B - v)(B + v^{-1}) = (D - v)(D + v^{-1}) = -AC$.

(b) $(F - v)(F + v^{-1}) = (H - v)(H + v^{-1}) = -EG$.

(c) If $A$ or $C$ is nonzero, then $B + D = v - v^{-1}$ and $D - H \in \{\pm 1\}$.

(d) If $E$ or $G$ is nonzero, then $F + H = v - v^{-1}$ and $B - F \in \{\pm 1\}$.

(e) If $A, C, E, G$ are all nonzero, then $B \in \{0, v - v^{-1}\}$.

*Proof.* In this proof we abbreviate by letting $\rho = \rho_\gamma$. Suppose $s, t \in S$ are such that $st$ has order 3. Since $\rho$ defines a representation of $\mathcal{H}$, we have $(\rho(H_s) - v)(\rho(H_s) + v^{-1})w = 0$ for all $w \in \mathbf{I}$. Expanding the left side of this identity for the elements $w \in \{1, s, t, sts\} \subset W \cap \mathbf{I}$ yields the equations in parts (a) and (b), and also the identities

$$X(B + D + v^{-1} - v) = 0 \qquad \text{and} \qquad Y(F + H + v^{-1} - v) = 0$$

for $X \in \{A, C\}$ and $Y \in \{E, G\}$. It follows that if $A$ or $C$ is nonzero then $B + D = v - v^{-1}$ and that if $E$ or $G$ is nonzero then $F + H = v - v^{-1}$.

We also have $\rho(H_s)\rho(H_t)\rho(H_s)w = \rho(H_t)\rho(H_s)\rho(H_t)w$ for all $w \in \mathbf{I}$. Expanding both sides of this identity for $w \in \{1, s, t, sts\} \subset W \cap \mathbf{I}$ and then comparing coefficients yields the identities

$$X(D^2 + (B - D)H - EG) = 0 \qquad \text{and} \qquad Y(F^2 + B(H - F) - AC) = 0 \tag{4.1}$$

again for $X \in \{A, C\}$ and $Y \in \{E, G\}$. Assume $A$ or $C$ is nonzero, so that we can take $X$ to be nonzero. Then $B - D = v - v^{-1} - 2D$ and $-EG = (H - v)(H + v^{-1})$. Substituting these identities into the first equation in (4.1) and dividing both sides by $X$ produces the equation

$$D^2 + (v - v^{-1} - 2D)H + (H - v)(H + v^{-1}) = 0.$$

The left hand sides simplifies to the expression $(D - H)^2 - 1$, and thus $D - H \in \{\pm 1\}$. This establishes part (c). In a similar way one finds that if $E$ or $G$ is nonzero then $B - F \in \{\pm 1\}$, which establishes part (d).

To prove part (e), suppose now that $s, t \in S$ are such that $st$ has order 4. Then $(\rho(H_s)\rho(H_t))^2 w = (\rho(H_t)\rho(H_s))^2 w$ for all $w \in \mathbf{I}$. Expanding both sides of this equation for $w = 1$ and comparing the coefficients of $sts$ yields the identity $AE(DF + BH - EG) = 0$. Assume $A$, $C$, $E$, $G$ are all nonzero. Then, after dividing both sides by $AE$ and applying the substitutions $D = v - v^{-1} - B$ and $H = v - v^{-1} - F$ and $-EG = (F - v)(F + v^{-1})$, our previous identity becomes

$$(v - v^{-1} - B)B + (B - F)^2 - 1 = 0.$$

Since $(B - F)^2 - 1 = 0$ by part (d), either $B = 0$ or $B = v - v^{-1}$, as claimed. $\square$

Let $u = v - v^{-1}$ and define four $4 \times 2$ matrices as follows:

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & u \\ 1 & 1 \\ u & u-1 \end{bmatrix} \qquad \Gamma' = \begin{bmatrix} 1 & u \\ 1 & 0 \\ 1 & u-1 \\ u & 1 \end{bmatrix} \qquad \Gamma'' = \begin{bmatrix} 1 & 0 \\ 1 & u \\ 1 & -1 \\ -u & u+1 \end{bmatrix} \qquad \Gamma''' = \begin{bmatrix} 1 & u \\ 1 & 0 \\ 1 & u+1 \\ -u & -1 \end{bmatrix}.$$

**Theorem 4.16.** Each of $\Gamma$, $\Gamma'$, $\Gamma''$, $\Gamma'''$ is an $(\mathcal{H}, \mathbf{I})$-structure and every nontrivial $(\mathcal{H}, \mathbf{I})$-structure is diagonally equivalent to one of these.

*Proof.* We first show that $\Gamma$, $\Gamma'$, $\Gamma''$, $\Gamma'''$ are all $(\mathcal{H}, \mathbf{I})$-structures. The matrices $\Gamma$ and $\Gamma'''$ are $(\mathcal{H}, \mathbf{I})$-structures since the corresponding representations just describe the action of $\mathcal{H}$ on the respective bases $\{I_w\}$ and $\{\overline{I_w}\}$ of $\mathcal{I}$. The matrices $\Gamma'$ and $\Gamma''$ are $(\mathcal{H}, \mathbf{I})$-structures by Lemmas 4.13 and 4.14, since $\Gamma' = \Theta(\Gamma)[-1, -1]$ and $\Gamma'' = \Theta(\Gamma''')[-1, -1]$.

Fix a choice of parameters $A, B, C, D, E, F, G, H \in \mathcal{A}$ and define the $4 \times 2$ matrix $\gamma$ as in Lemma 4.13. Assume $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-structure. We show that $\gamma$ is diagonally equivalent to $\Gamma$, $\Gamma'$, $\Gamma''$, or $\Gamma'''$. There are four cases to consider:

- Suppose $AC = EG = 0$. Then $B, D, F, H \in \{-v^{-1}, v\}$ by Lemma 4.15, and by Lemma 4.13 we may assume that $A, C, E, G \in \{0, 1\}$. There are 144 choices of parameters satisfying these conditions. With the aid of the computer algebra system MAGMA, we have checked that the only matrices $\gamma$ of this form which are $(\mathcal{H}, \mathbf{I})$-structures are the two trivial ones. (For this calculation, it suffices just to consider finite Coxeter systems of rank three.)

- Suppose $AC \neq 0$ and $EG = 0$. By Lemma 4.13 we may then assume that $E, G \in \{0, 1\}$. By the second and third parts of Lemma 4.15, it follows that $F, H \in \{-v^{-1}, v\}$ and $D \in \{H \pm 1\}$ and $B = v - v^{-1} - D$. By Lemma 4.13 and the first part of Lemma 4.15, finally, we may assume that $A = 1$ and $C = -(D - v)(D + v^{-1}) \neq 0$. This leaves 8 possible choices of parameters, and we have checked (again with the help of a computer) that for each of the resulting matrices $\gamma$, there are finite Coxeter systems $(W, S)$ for which $\rho_\gamma$ fails to define an $\mathcal{H}(W, S)$-representation. Hence it cannot occur that $AC \neq 0$ and $EG = 0$.

- It follows by similar consideration that it cannot happen that $AC = 0$ and $EG \neq 0$.

- Finally suppose $AC \neq 0$ and $EG \neq 0$ so that $A, C, E, G$ are all nonzero. By Lemma 4.15 we then have $B \in \{0, v - v^{-1}\}$ and $D = v - v^{-1} - B$ and $F \in \{B \pm 1\}$ and $H = v - v^{-1} - F$ and $AC = 1$ and $EG \in \{\pm(v - v^{-1}\}$; more specifically, Lemma 4.15 implies that $EG = v - v^{-1}$ when $B = 0 = F - 1$ or $B = v - v^{-1} = F - 1$ while in all other cases $EG = v^{-1} - v$. There are thus four choices for the quadruple $(B, D, F, H)$ and it is easy to see by Lemma 4.13 that in each case $\gamma$ is diagonally equivalent to one of $\Gamma$, $\Gamma'$, $\Gamma''$, or $\Gamma'''$.

This completes the proof of the theorem. $\square$

An $(\mathcal{H},\mathbf{I})$-structure $\gamma$ defines an $\mathcal{H}$-module structure on $\mathcal{A}\mathbf{I}$ for every Coxeter system $(W,S)$. Analogous to our definition for $(\mathcal{H},W)$-structures, we say that $\gamma$ is *pre-canonical* if each of these $\mathcal{H}$-modules has a pre-canonical $\mathcal{H}$-module structure in which $\mathbf{I}$ partially ordered by the Bruhat order is the "standard basis." We have the same remark as concerned pre-canonical $(\mathcal{H},W)$-structures: by the preceding theorem and Lemma 2.8, if $\gamma$ is a nontrivial, pre-canonical $(\mathcal{H},\mathbf{I})$-structure, then for each choice of Coxeter system $(W,S)$ there is a unique bar involution $\psi : \mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ such that $(\psi,\mathbf{I})$ is a pre-canonical $\mathcal{H}$-module structure.

The property of an $(\mathcal{H},\mathbf{I})$-structure being pre-canonical is preserved under the operations in Lemmas 4.13 and 4.14, in the following precise sense.

**Lemma 4.17.** If $\gamma$ is a nontrivial, pre-canonical $(\mathcal{H},\mathbf{I})$-structure, then so is $\gamma[-1,-1]$, and the (unique) associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are isomorphic via the identity map, which has as a scaling factor the $\mathcal{A}$-linear map $\mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ with $w \mapsto (-1)^{\rho(w)}w$ for $w \in \mathbf{I}$.

*Proof.* Let $\gamma$ be a nontrivial, pre-canonical $(\mathcal{H},\mathbf{I})$-structure, and define $\gamma' = \gamma[-1,-1]$. Let $(\psi,\mathbf{I})$ be the unique pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that $\psi\left(\rho_\gamma\left(\overline{H}\right)I\right) = \rho_\gamma\left(\overline{H}\right)\psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$. Let $\psi' = D^{-1}\circ\psi\circ D$ where $D : \mathcal{A}I \to \mathcal{A}I$ is the $\mathcal{A}$-linear map with $D(w) = (-1)^{\rho(w)}w$ for $w \in \mathbf{I}$. Since $\rho_{\gamma'}(H) = D^{-1}\circ\rho_\gamma(H)\circ D$ for $H \in \mathcal{H}$, it follows that $(\psi',\mathbf{I})$ is a pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that

$$\psi'\left(\rho_{\gamma'}\left(\overline{H}\right)I\right) = \rho_{\gamma'}\left(\overline{H}\right)\psi'(I) \qquad \text{for } H \in \mathcal{H} \text{ and } I \in \mathcal{A}\mathbf{I}.$$

Thus $\gamma'$ is pre-canonical. Moreover, the identity map $\mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ is evidently an isomorphism between the pre-canonical structures $(\psi,\mathbf{I})$ and $(\psi',\mathbf{I})$, with $D$ as a scaling factor. $\square$

**Lemma 4.18.** If $\gamma$ is a nontrivial, pre-canonical $(\mathcal{H},\mathbf{I})$-structure, then so is $\Theta(\gamma)$, and the (unique) associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are strongly isomorphic via the identity map.

*Proof.* Let $\gamma$ be a nontrivial, pre-canonical $(\mathcal{H},\mathbf{I})$-structure, and define $\gamma' = \Theta(\gamma)$. Let $(\psi,\mathbf{I})$ be the unique pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that $\psi\left(\rho_\gamma\left(\overline{H}\right)I\right) = \rho_\gamma\left(\overline{H}\right)\psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$. Then it also holds that $\psi\left(\rho_{\gamma'}\left(\overline{H}\right)I\right) = \rho_{\gamma'}\left(\overline{H}\right)\psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$ since $\rho_{\gamma'}(H) = \rho(\Theta(H))$ and $\overline{\Theta(H)} = \Theta(\overline{H})$. Thus $\gamma'$ is also pre-canonical and its associated pre-canonical structure is strongly isomorphic to the one associated to $\gamma$. $\square$

Before we can give the $(\mathcal{H},\mathbf{I})$-structure analogue of Theorem 4.11, we require an additional lemma. To state this, let

$$\mathcal{I}, \qquad \mathcal{I}', \qquad \mathcal{I}'', \qquad \text{and} \qquad \mathcal{I}'''$$

be the free $\mathcal{A}$-module with a basis given by the symbols $I_w$, $I'_w$, $I''_w$, and $I'''_w$ respectively for $w \in \mathbf{I}$. View these as $\mathcal{H}$-modules relative to the $(\mathcal{H},\mathbf{I})$-structure $\Gamma$, $\Gamma'$, $\Gamma''$, and $\Gamma'''$ respectively. Of course, $\mathcal{I}$ defined in this way is the same thing as $\mathcal{I}$ defined by Theorem 3.11. In addition, let $\epsilon$ denote the ring endomorphism of $\mathcal{A}$ with $\epsilon(v) = -v$.

**Lemma 4.19.** There are unique pre-canonical $\mathcal{H}$-module structures on $\mathcal{I}$, $\mathcal{I}'$, $\mathcal{I}''$, $\mathcal{I}'''$, respectively, in which $\{I_w\}$, $\{I'_w\}$, $\{I''_w\}$, $\{I'''_w\}$ indexed by $(\mathbf{I},\leq)$ are the "standard bases." Moreover, these pre-canonical structures are all isomorphic; the following maps are isomorphisms:

(a) The $\mathcal{A}$-linear map $\mathcal{I} \to \mathcal{I}'$ with $I_w \mapsto I'_w$ for $w \in \mathbf{I}$.

(b) The $\mathcal{A}$-linear map $\mathcal{I}'' \to \mathcal{I}'''$ with $I''_w \mapsto I'''_w$ for $w \in \mathbf{I}$.

(c) The $\epsilon$-linear map $\mathcal{I} \to \mathcal{I}'''$ with $I_w \mapsto I'''_w$ for $w \in \mathbf{I}$.

Finally, the morphisms in (a), (b), (c) have as respective scaling factors the $\mathcal{A}$-linear maps with

$$I_w \mapsto (-1)^{\rho(w)} I_w \qquad \text{and} \qquad I''_w \mapsto (-1)^{\rho(w)} I''_w \qquad \text{and} \qquad I_w \mapsto I_w \qquad \text{for } w \in \mathbf{I}.$$

**Remark.** The "bar involution" of $\mathcal{I}$ in the pre-canonical structure mentioned in this result is the one defined before Theorem 3.11. One can show, though we omit the details here, that the "bar involutions" of $\mathcal{I}'$, $\mathcal{I}''$, and $\mathcal{I}'''$ are the respective $\mathcal{A}$-antilinear maps with

$$I'_{(x,\theta)} \mapsto H_x \cdot I'_{(x^{-1},\theta)} \qquad \text{and} \qquad I''_{(x,\theta)} \mapsto \overline{H_x} \cdot I''_{(x^{-1},\theta)} \qquad \text{and} \qquad I'''_{(x,\theta)} \mapsto \mathrm{sgn}(x) \cdot H_x \cdot I'''_{(x^{-1},\theta)}$$

for twisted involutions $(x,\theta) \in \mathbf{I}$.

*Proof.* The uniqueness of the pre-canonical $\mathcal{H}$-module structures is clear from Lemma 2.8. From Theorem 3.12 we already have a bar involution $I \mapsto \overline{I}$ on $\mathcal{I}$ which forms a pre-canonical $\mathcal{H}$-module structure with $\{I_w\}$ as the standard basis. Define $r_{y,w} \in \mathcal{A}$ for $y, w \in \mathbf{I}$ such that $\overline{I_w} = \sum_{y \in \mathbf{I}} r_{y,w} I_y$. In addition, for $x \in W$ and $y, z \in \mathbf{I}$ let $f^x_{y,z} \in \mathcal{A}$ be such that $H_x I_y = \sum_{z \in \mathbf{I}} f^x_{y,z} I_z$.

Let $\mathcal{J}$ be the free $\mathcal{A}$-module with a basis given by the symbols $J_w$ for $w \in \mathbf{I}$. View this as an $\mathcal{H}$-module relative to the $(\mathcal{H}, \mathbf{I})$-structure $\gamma = \Gamma''[-1,-1] = \Theta(\Gamma''')$, and define $J \mapsto \overline{J}$ as the $\mathcal{A}$-antilinear map $\mathcal{J} \to \mathcal{J}$ with $\overline{J_w} = \sum_{y \in \mathbf{I}} \epsilon(r_{y,w}) J_y$ for $w \in \mathbf{I}$. It is immediate that this bar involution forms a pre-canoncal structure on $\mathcal{J}$ with $\{J_w\}$ as the standard basis. Since $H_s J_y = -\sum_{z \in \mathbf{I}} \epsilon(f^s_{y,z}) J_z$ for all $s \in S$ and $y \in \mathbf{I}$, it follows moreover that $\overline{H_s J_y} = \overline{H_s} \cdot \overline{J_y}$, which suffices to show that $\overline{H} \cdot \overline{J} = \overline{HJ}$ for all $H \in \mathcal{H}$ and $J \in \mathcal{J}$. We thus have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{J}$. It is clear that the $\epsilon$-linear map $\mathcal{I} \to \mathcal{J}$ with $I_w \mapsto J_w$ is an isomorphism of the pre-canonical structures on $\mathcal{I}$ and $\mathcal{J}$, which has the identity map as a scaling factor.

One deduces the remaining assertions in the lemma from the existence of these isomorphic pre-canonical structures on $\mathcal{I}$ and $\mathcal{J}$, using Lemmas 4.17 and 4.18 and the fact that

$$\Gamma' = \Theta(\Gamma)[-1,-1] \qquad \text{and} \qquad \Gamma'' = \gamma[-1,-1] \qquad \text{and} \qquad \Gamma''' = \Theta(\gamma).$$

$\square$

The following theorem shows that while the choice of parameters in Theorem 3.11 is not the unique one for which an analogue of Theorem 3.12 holds, this choice has no effect on the isomorphism class of the resulting pre-canonical $\mathcal{H}$-module structure.

**Theorem 4.20.** Exactly 16 nontrivial $(\mathcal{H}, \mathbf{I})$-structures are pre-canonical. The 16 associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are all isomorphic (in the sense of Definition 4.3) to the pre-canonical structure on $\mathcal{I}$ defined in Theorem 3.12, and hence they each admit a canonical basis.

*Proof.* Let $\gamma$ be a nontrivial $(\mathcal{H}, \mathbf{I})$-structure which is pre-canonical, and write $\psi : \mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ for the associated bar involution. We claim that $\gamma_{11}$ and $\gamma_{31}$ must then belong to $\mathbb{Z}[v + v^{-1}]$. To see this let $\theta \in \mathrm{Aut}(W,S)$ be an involution and let $s \in S$. If $s \neq \theta(s)$ then $w = (s \cdot \theta(s), \theta) \in \mathbf{I}$ and we have

$$\overline{\gamma_{11}} \cdot \psi(w) + \overline{\gamma_{12}} \cdot \theta = \psi(\gamma(H_s)\theta) = \gamma(H_s + v^{-1} - v)\theta = \gamma_{11} \cdot w + (\gamma_{12} + v^{-1} - v) \cdot \theta.$$

On the other hand if $s = \theta(s)$ then $w = (s, \theta) \in \mathbf{I}$ and we have

$$\overline{\gamma_{31}} \cdot \psi(w) + \overline{\gamma_{32}} \cdot \theta = \psi(\gamma(H_s)\theta) = \gamma(H_s + v^{-1} - v)\theta = \gamma_{31} \cdot w + (\gamma_{32} + v^{-1} - v) \cdot \theta.$$

These equations, compared with the unitriangular property of the bar involution, imply $\overline{\gamma_{11}} = \gamma_{11}$ and $\overline{\gamma_{31}} = \gamma_{31}$; hence these two parameters must belong to $\mathbb{Z}[v + v^{-1}]$ as claimed. Since Theorem 4.16 implies that

$$\gamma_{11} \cdot \gamma_{21} = 1 \qquad \text{and} \qquad \gamma_{31} \cdot \gamma_{41} \in \{\pm(v - v^{-1})\}$$

it necessarily follows that $\gamma_{11}, \gamma_{31} \in \{\pm 1\}$. From Theorem 4.16 we conclude that for some $\varepsilon_i \in \{\pm 1\}$ we have $\gamma[\varepsilon_1, \varepsilon_2] \in \{\Gamma, \Gamma', \Gamma'', \Gamma'''\}$. Thus $\gamma$ must be one of 16 different $(\mathcal{H}, \mathbf{I})$-structures. It is a simple exercise to show that $\gamma$ is pre-canonical if and only if $\gamma[\varepsilon_1, \varepsilon_2]$ is pre-canonical; moreover, the associated pre-canonical structures are isomorphic. Hence, by Lemma 4.19 we conclude that all 16 possibilities for $\gamma$ are pre-canonical, and that the associated pre-canonical structures are all isomorphic to the one in Theorem 3.12. $\square$

### 4.4 Generic representations of the modified Iwahori-Hecke algebra

In light of the formal similarity between the results in Sections 3.2 and 3.3, it is perhaps even more natural to consider the questions answered in the previous section for $\mathcal{H}_2$-module structures on $\mathcal{A}\mathbf{I}$. In particular, we can ask in what ways the parameters in Theorems 3.3 and 3.6 can be modified to still produce an $\mathcal{H}_2$-module structure on $\mathcal{A}\mathbf{I}$ for all Coxeter systems, and for which of these $\mathcal{H}_2$-modules does there exist a compatible pre-canonical structure. We address these questions here.

Consider a $4 \times 2$ matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. We define $\rho_{\gamma,2} : \{K_s : s \in S\} \to \mathrm{End}(\mathcal{A}\mathbf{I})$ again by the formula (4.3) except with $H_s$ replaced by $K_s$; that is, let $\rho_{\gamma,2}$ be the composition of $\rho_\gamma$ with the obvious bijection $\{K_s : s \in S\} \to \{H_s : s \in S\}$.

**Definition 4.21.** The matrix $\gamma$ is an $(\mathcal{H}_2, \mathbf{I})$-*structure* if for every Coxeter system $(W, S)$, the map $\rho_{\gamma,2}$ extends to a representation of $\mathcal{H}_2 = \mathcal{H}_2(W, S)$ in $\mathcal{A}\mathbf{I} = \mathcal{A}\mathbf{I}(W, S)$.

Given a matrix $\gamma$ over $\mathcal{A}$, define $\gamma_2$ by applying the ring endomorphism of $\mathcal{A}$ with $v \mapsto v^2$ to the entries of $\gamma$. The following observation motivates this notation.

**Observation 4.22.** If $\gamma$ is an $(\mathcal{H}, \mathbf{I})$-structure then $\gamma_2$ is an $(\mathcal{H}_2, \mathbf{I})$-structure.

As usual, we say that an $(\mathcal{H}_2, \mathbf{I})$ structure $\gamma$ is *trivial* if $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 0$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} \in \{v^2, -v^{-2}\}$. Lemma 4.13 holds *mutatis mutandis* with "$(\mathcal{H}, \mathbf{I})$-structure" replaced by "$(\mathcal{H}_2, \mathbf{I})$-structure" and "$\mathcal{H}$" replaced by "$\mathcal{H}_2$." Define two $(\mathcal{H}_2, \mathbf{I})$-structures to be *diagonally equivalent* as in that result. The classification of $(\mathcal{H}_2, \mathbf{I})$-structures up to diagonal equivalence is no different than for $(\mathcal{H}, \mathbf{I})$-structures:

**Theorem 4.23.** Let $\Gamma$, $\Gamma'$, $\Gamma''$, and $\Gamma'''$ be the $(\mathcal{H}, \mathbf{I})$-structures defined before Theorem 4.16. Then every nontrivial $(\mathcal{H}_2, \mathbf{I})$-structure is diagonally equivalent to $\Gamma_2$, $\Gamma'_2$, $\Gamma''_2$, or $\Gamma'''_2$.

*Proof sketch.* The result follows by nearly the same argument as in the proof Theorem 4.16, using three lemmas analogous to Lemmas 4.13, 4.14, and 4.15, *mutatis mutandis*. We omit the details. $\square$

Define an $(\mathcal{H}_2, \mathbf{I})$-structure $\gamma$ to be *pre-canonical* exactly as for $(\mathcal{H}, \mathbf{I})$-structures: namely, say that $\gamma$ is pre-canonical if, for every Coxeter system $(W, S)$, there exists a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{A}\mathbf{I}$ (relative to the $\mathcal{H}_2$-module structure defined by $\gamma$) in which $\mathbf{I}$ partially ordered by the Bruhat order is the "standard basis." Just like for $(\mathcal{H}, W)$-structures and $(\mathcal{H}, \mathbf{I})$-structures, if an $(\mathcal{H}_2, \mathbf{I})$-structure is nontrivial and pre-canonical, then by Lemma 2.8 it associates a unique pre-canonical $\mathcal{H}_2$-structures to $\mathcal{A}\mathbf{I}$ for each Coxeter system $(W, S)$.

We devote the rest of this section to classifying which nontrivial $(\mathcal{H}_2, \mathbf{I})$-structures are pre-canonical. For this purpose, define $\Delta$ and $\Delta'$ as the matrices

$$\Delta = \begin{bmatrix} 1 & 0 \\ 1 & v^2 - v^{-2} \\ v + v^{-1} & 1 \\ v - v^{-1} & v^2 - 1 - v^{-2} \end{bmatrix} \qquad \text{and} \qquad \Delta' = \begin{bmatrix} 1 & 0 \\ 1 & v^2 - v^{-2} \\ v^{-1} + v & -1 \\ v^{-1} - v & v^2 + 1 - v^{-2} \end{bmatrix}.$$

These are $(\mathcal{H}_2, \mathbf{I})$-structures by Theorems 3.3 and 3.6, and by Theorems 3.4 and 3.7 they are pre-canonical. In addition, let $\Delta'' = \Gamma_2$ and $\Delta''' = \Gamma''_2$. These are then also pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structures, since by the proof of Theorem 4.20 both $\Gamma$ and $\Gamma''$ are pre-canonical $(\mathcal{H}, \mathbf{I})$-structures, and it is easy to see that whenever $\gamma$ is a pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, $\gamma_2$ is a pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structure.

**Remark.** Observe that $\Delta$ and $\Delta''$ (respectively, $\Delta'$ and $\Delta'''$) are diagonally equivalent; however, the $\mathcal{H}_2$-module structures they define on $\mathcal{A}\mathbf{I}$ are technically not isomorphic (although they would be if all of our algebras and modules were defined over the field $\mathbb{Q}(v)$ instead of the ring $\mathcal{A}$).

It follows from Corollary 4.8 and the discussion in Sections 3.4 and 3.3 (or more concretely, from small computations) that the pre-canonical structures which $\Delta$, $\Delta'$, $\Delta''$, and $\Delta'''$ associate to $\mathcal{A}\mathbf{I}$ are in general not isomorphic. We will see conversely that these four structures represent all which can arise from a nontrivial pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structure.

**Remark.** The pre-canonical structures on $\mathcal{A}\mathbf{I}$ defined by the $(\mathcal{H}, \mathbf{I})$-structures $\Gamma$ and $\Gamma''$ are isomorphic by Theorem 4.20, so one might expect the same to be true of the pre-canonical structures defined by $\Delta'' = \Gamma_2$ and $\Delta''' = \Gamma''_2$. The reason this is not so is that the latter structures admit canonical bases $\{b_w\}$ and $\{b'_w\}$ with the special form $b_w = \sum_{y \leq w} f_{y,w}(v^{-2})y$ and $b'_w = \sum_{y \leq w} g_{y,w}(v^{-2})y$ where $f_{y,w}(t), g_{y,w}(t) \in \mathbb{Z}[t]$ for $y, w \in \mathbf{I}$ are polynomials related by the identity $f_{y,w}(t) = g_{y,w}(-t)$. Corollary 4.8 shows that such canonical bases cannot arise from isomorphic pre-canonical structures, provided $f_{y,w}(t)$ and $g_{y,w}(t)$ are sufficiently complicated polynomials.

The following theorem shows that the isomorphism classes of pre-canonical structures on $\mathcal{A}\mathbf{I}$ arising from nontrivial pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structures are given as follows: the structures from Theorems 3.4 and 3.7 each represent a distinct class, while the isomorphism class of the structure in Theorem 3.12 splits to contribute two additional classes.

**Theorem 4.24.** Exactly 32 nontrivial $(\mathcal{H}_2, \mathbf{I})$-structures are pre-canonical. Their associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are each isomorphic (in the sense of Definition 4.3) to one of the four structures arising from $\Delta$, $\Delta'$, $\Delta''$, or $\Delta'''$.

*Proof sketch.* One deduces that at most 32 nontrivial $(\mathcal{H}_2, \mathbf{I})$-structures are pre-canonical exactly as in the proof of Theorem 4.20: first argue that any such structure $\gamma$ has $\overline{\gamma_{11}} = \gamma_{11}$ and $\overline{\gamma_{31}} = \gamma_{31}$, and then appeal to Theorem 4.23. The claim that these $(\mathcal{H}_2, \mathbf{I})$-structures are in fact all pre-canonical, along with the second sentence in the theorem, follows from Lemmas 4.17 and 4.18, which hold *mutatis mutandis* with "$(\mathcal{H}, \mathbf{I})$-structure" replaced by $(\mathcal{H}_2, \mathbf{I})$-structure" and $\Theta$ replaced by a slightly different involution on $4 \times 2$ matrices. ☐

## 4.5 Application to inversion formulas

Let $V$ be a free $\mathcal{A}$-module of finite rank, with a pre-canonical structure $(\psi, \{a_c\})$, the standard basis indexed by $(C, \leq)$. Define $V^*$ as the set of $\mathcal{A}$-linear maps $V \to \mathcal{A}$. This is naturally a free $\mathcal{A}$-module: a basis is given by the $\mathcal{A}$-linear maps $a_c^* : V \to V$ for $c \in C$ defined by

$$a_c^*(a_{c'}) = \delta_{c,c'} \qquad \text{for } c' \in C.$$

Define $\psi^* : V^* \to V^*$ as the $\mathcal{A}$-antilinear map such that

$$\psi^*(f)(v) = \overline{f \circ \psi(v)} \qquad \text{for } f \in V^* \text{ and } v \in V.$$

Also let $\leq^{\mathrm{op}}$ denote the partial order on $C$ with $c \leq^{\mathrm{op}} c'$ if and only if $c' \leq c$. The following appears in a slightly more general form as [17, Proposition 7.1].

**Proposition 4.25** (Webster [17])**.** The "bar involution" $\psi^*$ and "standard basis" $\{a_c^*\}$, indexed by the partially ordered set $(C, \leq^{\mathrm{op}})$, form a pre-canonical structure on $V^*$. If $V$ has a canonical basis $\{b_c\}$, then the dual basis $\{b_c^*\}$ of $V^*$ is canonical relative to $(\psi^*, \{a_c^*\})$.

Let $\mathcal{B}$ denote a free $\mathcal{A}$-algebra with a pre-canonical structure; write $\overline{b}$ for the image of $b \in \mathcal{B}$ under the corresponding bar involution. Suppose $V$ is a $\mathcal{B}$-module and $(\psi, \{a_c\})$ is a pre-canonical $\mathcal{B}$-module structure. Assume $\mathcal{B}$ has a distinguished $\mathcal{A}$-algebra antiautomorphism $b \mapsto b^\dagger$. We may then view $V^*$ as a $\mathcal{B}$-module by defining $bf$ for $b \in \mathcal{B}$ and $f \in V^*$ to be the map with the formula

$$(bf)(v) = f(b^\dagger v) \qquad \text{for } v \in V. \tag{4.2}$$

**Proposition 4.26.** Suppose the maps $b \mapsto b^\dagger$ and $b \mapsto \overline{b}$ commute. Then the pre-canonical structure $(\psi^*, \{a_c^*\})$ on $V^*$ is a pre-canonical $\mathcal{B}$-module structure.

*Proof.* If $b \in \mathcal{B}$ and $f \in V^*$ then $\psi^*(bf) = \overline{b} \cdot \psi^*(f)$, since for $v \in V$ one computes

$$\psi^*(bf)(v) = \overline{(bf)(\psi(v))} = \overline{f(b^\dagger \psi(v))} = \overline{f \circ \psi\left(\overline{b^\dagger} v\right)} = \psi^*(f)\left(\overline{b}^\dagger v\right) = (\overline{b} \cdot \psi^*(f))(v).$$

☐

Assume $(W, S)$ is a finite Coxeter system, so that $W$ has a longest element $w_0$. Recall since the longest element is unique, we have $w_0 = w_0^{-1} = \theta(w_0)$ for all $\theta \in \mathrm{Aut}(W, S)$. Write $\theta_0$ for the inner automorphism of $W$ given by $w \mapsto w_0 w w_0$. This map is an automorphism of the poset $(W, \leq)$ and in particular is length-preserving [1, Proposition 2.3.4(ii)]; thus it belongs to $\mathrm{Aut}(W, S)$. In fact, $\theta_0$ lies in the center of $\mathrm{Aut}(W, S)$. Let $w_0^+ = (w_0, \theta_0) \in W^+$. Observe that $w_0^+$ is a central involution in $W^+$, and so if $w = (x, \theta) \in \mathbf{I}$ then $ww_0^+ = (xw_0, \theta\theta_0) \in \mathbf{I}$.

We may use the results in the previous sections to prove an inversion formula for the structure constants of the canonical bases of $\mathcal{L}$, $\mathcal{L}'$, and $\mathcal{I}$ given in Theorems 3.4, 3.7, and 3.12. Part (a) of the following theorem is due to Lusztig [11, Theorem 7.7].

**Theorem 4.27.** Let $x, y \in \mathbf{I}$. The following formulas then hold:

(a) $\sum_{w\in\mathbf{I}}(-1)^{\rho(x)+\rho(w)} \cdot \pi_{x,w} \cdot \pi_{yw_0^+,ww_0^+} = \delta_{x,y}$.

(b) $\sum_{w\in\mathbf{I}}(-1)^{\rho(x)+\rho(w)} \cdot \pi'_{x,w} \cdot \pi'_{yw_0^+,ww_0^+} = \delta_{x,y}$.

(c) $\sum_{w\in\mathbf{I}}(-1)^{\rho(x)+\rho(w)} \cdot \iota_{x,w} \cdot \iota_{yw_0^+,ww_0^+} = \delta_{x,y}$.

*Proof.* We only prove part (c) since the proof of part (b) is similar, and since part (a) appears in [11]. There is a unique antiautomorphism $H \mapsto H^\dagger$ of $\mathcal{H}$ with $H_w \mapsto H_{w^{-1}}$ for $w \in W$. We make $\mathcal{I}^*$ into an $\mathcal{H}$-module relative to this antiautomorphism via the formula (4.2). Let $s \in S$ and $w \in \mathbf{I}$. Since $w_0^+$ is central, we have $sw = ws$ if and only if $sww_0^+ = ww_0^+s$. Since $x \leq y$ if and only if $yw_0 \leq xw_0$ for any $x, y \in W$ (see [1, Proposition 2.3.4(i)]), it follows that $sw < w$ if and only if $sww_0^+ > sww_0^+$, and also that $\rho(xw_0^+) - \rho(yw_0^+) = \rho(y) - \rho(x)$ for $x, y \in \mathbf{I}$. Given these facts it is straightforward to check that if $\mathcal{I}'$ is the $\mathcal{H}$-module defined before Lemma 4.19, then the $\mathcal{A}$-linear map $\varphi : \mathcal{I}' \to \mathcal{I}^*$ with $\varphi(I'_w) = I^*_{ww_0^+}$ for $w \in \mathbf{I}$ is an isomorphism of $\mathcal{H}$-modules.

We have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}'$ from Lemma 4.19. Likewise, since the maps $H \mapsto H^\dagger$ and $H \mapsto \overline{H}$ commute, we have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}^*$ from Proposition 4.26. Write $\psi^*$ for the bar involution of $\mathcal{I}^*$ in this structure. Then $(\varphi^{-1} \circ \psi^* \circ \varphi, \{I'_w\})$ is another pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}'$, so the uniqueness assertion in Lemma 4.19 implies that $\varphi^{-1} \circ \psi^* \circ \varphi$ is equal to the bar involution $I \mapsto \overline{I}$ on $\mathcal{I}'$, and thus $\varphi$ is a strong isomorphism between the pre-canonical structures on $\mathcal{I}'$ and $\mathcal{I}^*$. Composing $\varphi$ with the map in Lemma 4.19(a), it follows that the $\mathcal{A}$-linear map $\mathcal{I} \to \mathcal{I}^*$ with $I_w \mapsto I^*_{ww_0^+}$ is an isomorphism of pre-canonical structures (though not of $\mathcal{H}$-modules), having as a scaling factor the $\mathcal{A}$-linear map $D : \mathcal{I} \to \mathcal{I}$ with $D(I_w) = (-1)^{\rho(w)} I_w$ for $w \in \mathbf{I}$.

From Proposition 4.6, we deduce that elements of the canonical basis $\{\underline{I}^*_w\}$ of $\mathcal{I}^*$ have the form $\underline{I}^*_y = I^*_y + \sum_{w>y}(-1)^{\rho(y)-\rho(w)}\iota_{yw_0^+,ww_0^+} \cdot I^*_w$. Since $\underline{I}^*_y(\underline{I}_x) = \delta_{x,y}$ for $x, y \in \mathbf{I}$ by Proposition 4.25, we deduce that $MN = 1$ where $M$ and $N$ are the $\mathbf{I} \times \mathbf{I}$-indexed matrices with $M_{y,w} = (-1)^{\rho(y)-\rho(w)}\iota_{ww_0^+,yw_0^+}$ and $N_{w,x} = \iota_{w,x}$. Since $M$ and $N$ are finite square matrices, $MN = 1$ implies $NM = 1$; the desired inversion formula is equivalent to the second equality. ☐

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