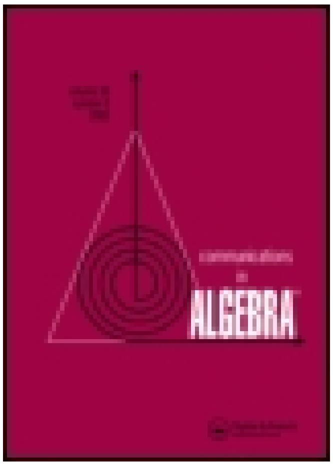

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Twisted hopf comodule algebras

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TWISTED HOPF COMODULE ALGEBRAS

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Abstract

For k a commutative ring, H a k-bialgebra and A a right H-comodule k-algebra, we define a new multiplication on the H-comodule A to obtain a "twisted algebra" A^τ, $\tau \in \mathrm{Hom}(H, \mathrm{End}(A))$. If τ is convolution invertible, the categories of relative right Hopf modules over A and A^τ are isomorphic. Similarly a convolution invertible left twisting gives an isomorphism of the categories of relative left Hopf modules. We show that crossed products are invertible twistings of the tensor product, and obtain, as a corollary, a duality theorem for crossed products.

§0. Introduction

Let k be a commutative ring, H a k-bialgebra and A a right H-comodule k-algebra with comodule structure map $\rho : A \longrightarrow A \otimes H$. For τ an element of the convolution algebra $\mathrm{Hom}(H, \mathrm{End}(A))$, we "twist" the multiplication on A by τ to obtain A^τ, and similarly we define M^τ for $M \in \mathcal{M}_A^H$, the category of relative right Hopf modules. We give necessary and sufficient conditions on τ for A^τ to be an H-comodule algebra and for F_τ, defined

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by $F_\tau(M) = M^\tau$, $F_\tau(f) = f$, to be a functor from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$. In this case we call τ a twisting from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ and A^τ a twisted algebra of A or a twisting of A. If τ is convolution invertible, then $F_\tau : \mathcal{M}_A^H \longrightarrow \mathcal{M}_{A^\tau}^H$ is a category isomorphism.

In the second section, we consider categories of left Hopf modules ${}_A\mathcal{M}^H$, define a left hand version of twistings, and prove that if τ is a convolution invertible twisting of $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ and H is a Hopf algebra with a bijective antipode then ${}_A\mathcal{M}^H$ and ${}_{A^\tau}\mathcal{M}^H$ are also isomorphic.

In the last section we prove that crossed products $B\#_\sigma H$ (with invertible cocycle σ) are precisely the (invertible) twistings of $B \otimes H$, thus obtaining a new description of cleft extensions. Furthermore, we show that if A^τ is an invertible twisting of A, then, for k a field, if $H^{*rat} \neq 0$, $A\#H^{*rat} \cong A^\tau\#H^{*rat}$ as algebras. Combining these results, we obtain the duality statement: If k is a field, H a Hopf algebra with $H^{*rat} \neq 0$, then for any crossed product $B\#_\sigma H$ with invertible cocycle, $(B\#_\sigma H)\#H^{*rat} \cong B \otimes (H\#H^{*rat})$.

§1. Twistings of the category of right (H, A) Hopf modules.

Unless indicated otherwise, all maps are k-linear, $\otimes$ means $\otimes_k$, etc. Throughout, H will be a bialgebra with comultiplication Δ and counit ϵ. The definitions of H-comodule algebra, H-module algebra, and any other basic notation and definitions can be found in [M] or [S]. We use Sweedler's summation notation throughout. If H is a Hopf algebra, S will denote the antipode, and if S is bijective, $\overline{S}$ will denote its composition inverse. Throughout, ρ is reserved for comodule structure maps and these are written ρ_M, ρ_A, etc., or just ρ if the comodule is clear from the context. If M is a right (left) H-comodule we write $\sum m_0 \otimes m_1$ ($\sum m_{-1} \otimes m_0$) for $\rho(m)$. For a right H-comodule M, $M^{coH} = \{m \in M : \rho(m) = m \otimes 1\}$ denotes the coinvariants of M.

For H a bialgebra, H^{op} will denote the bialgebra H with opposite algebra structure. If H is a Hopf algebra and S is bijective, then H^{op} is also a Hopf algebra but with the antipode $\overline{S}$. If H is finite (i.e. finitely generated projective over k) or if H is commutative or cocommutative, then S is bijective.

If (A, ρ_A) is a right H-comodule algebra, then A^{op} will denote the usual opposite algebra; A^{op} is a right H^{op}-comodule algebra. We denote by $\mathcal{M}_A^H$ the category of relative right Hopf modules, i.e. the category of right A-modules, right H-comodules M such that for $m \in M$, $a \in A$,

$$\rho_M(ma) = \sum (ma)_0 \otimes (ma)_1 = \sum m_0 a_0 \otimes m_1 a_1 = \rho_M(m)\rho_A(a),$$

with morphisms being A-module H-comodule maps. Similarly ${}_A\mathcal{M}^H$ denotes the category of left A-modules, right H-comodules M such that $\rho_M(am) = \rho_A(a)\rho_M(m)$ for all $a \in A$, $m \in M$.

For (A, ρ_A) a right H-comodule algebra, let τ be an element of the convolution algebra $\mathrm{Hom}(H, \mathrm{End}(A))$. We will assume throughout that τ satisfies the normality conditions:

$$\tau(1) = id_A \quad \text{and} \quad \tau(h)(1) = \epsilon(h) \tag{N}$$

for all $h \in H$. Note that if τ satisfies (N) and τ has convolution inverse λ, then λ satisfies (N) also.

For all $h \in H$ and $a \in A$, we denote $\tau(h)(a) \in A$ by $h \cdot_\tau a$. We define a new (possibly nonassociative) multiplication $*_\tau$ on A by

$$a *_\tau b = \sum a_0(a_1 \cdot_\tau b)$$

where the multiplication on the right hand side is the usual multiplication on A. We denote this algebra $(A, *_\tau)$ by A^τ. The condition (N) ensures that A^τ has the same multiplicative identity as A. Given $M \in \mathcal{M}_A^H$, we define a map $M \otimes A^\tau \longrightarrow M$ by

$$m \otimes a \longmapsto m *_\tau a = \sum m_0(m_1 \cdot_\tau a)$$

where the right hand side is the original A-module action on M. Note that by (N), $m *_\tau 1 = m$ for all $m \in M$. Let M^τ denote the H-comodule M together with this map. We will often omit τ and write $*$ and $\cdot$ instead of $*_\tau$ and $\cdot_\tau$ respectively.

THEOREM 1.1. *(i) Let τ, A and M^τ be as above. Then for all $M \in \mathcal{M}_A^H$, M^τ lies in $\mathcal{M}_{A^\tau}^H$ if and only if for all $a, b \in A$, $h \in H$,*

$$\sum(1 \otimes h_1)\rho(h_2 \cdot a) = \sum(h_2 \cdot a)_0 \otimes h_1(h_2 \cdot a)_1 = \sum h_1 \cdot a_0 \otimes h_2 a_1, \text{ and} \tag{1.1.1}$$

$$h \cdot_\tau (a *_\tau b) = h \cdot \{\sum a_0(a_1 \cdot b)\} = \sum(h_1 \cdot a_0)((h_2 a_1) \cdot b). \tag{1.1.2}$$

In particular, if (1.1.1) and (1.1.2) hold, then A^τ is a right H-comodule algebra.

(ii) If (1.1.1) and (1.1.2) hold, then we can define a functor F_τ from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ by $F_\tau(M) = M^\tau$ for all $M \in \mathcal{M}_A^H$ and $F_\tau(f) = f$ for all morphisms f in $\mathcal{M}_A^H$.

(iii) If τ as above has convolution inverse λ, then λ satisfies (1.1.1) and (1.1.2) for A^τ and $(A^\tau)^\lambda = A$. In this case the functor $F_\tau : \mathcal{M}_A^H \longrightarrow \mathcal{M}_{A^\tau}^H$ is a category isomorphism with inverse $F_\lambda : \mathcal{M}_{A^\tau}^H \longrightarrow \mathcal{M}_A^H$.

PROOF. First suppose τ satisfies (1.1.1) and (1.1.2). Let $M \in \mathcal{M}_A^H$. For $m \in M$, $a, b \in A$,

$$(m * a) * b = \sum[m_0(m_1 \cdot a)] * b \quad \text{by the definition of } *$$

$$
\begin{aligned}
&= \sum m_0(m_2 \cdot a)_0\{[m_1(m_2 \cdot a)_1] \cdot b\} \quad \text{since } M \in \mathcal{M}_A^H \\
&= \sum m_0(m_1 \cdot a_0)[(m_2 a_1) \cdot b] \quad \text{by } (1.1.1) \\
&= \sum m_0\{m_1 \cdot (a * b)\} \quad \text{by } (1.1.2) \\
&= m * (a * b),
\end{aligned}
$$

and thus M^τ is a right A^τ-module. To see that $M^\tau \in \mathcal{M}_{A^\tau}^H$, note that for $m \in M$, $a \in A$,

$$
\begin{aligned}
\sum (m * a)_0 \otimes (m * a)_1 &= \sum (m_0(m_1 \cdot a))_0 \otimes (m_0(m_1 \cdot a))_1 \\
&= \sum m_0(m_2 \cdot a)_0 \otimes m_1(m_2 \cdot a)_1 \quad \text{since } M \in \mathcal{M}_A^H \\
&= \sum m_0(m_1 \cdot a_0) \otimes m_2 a_1 \quad \text{by } (1.1.1) \\
&= \sum m_0 * a_0 \otimes m_1 a_1.
\end{aligned}
$$

Conversely suppose that for all $M \in \mathcal{M}_A^H$, $M^\tau \in \mathcal{M}_{A^\tau}^H$. We show that then (1.1.1) and (1.1.2) hold. Let $M = A \otimes H$ with $\rho_M = id_A \otimes \Delta$ and right A-module structure given by

$$(b \otimes h)a = \sum b a_0 \otimes h a_1.$$

Then M^τ is a right A^τ-module by

$$(b \otimes h) * a = \sum (b \otimes h_1)(\tau(h_2)(a)) = \sum b(h_2 \cdot a)_0 \otimes h_1(h_2 \cdot a)_1.$$

Note that $(id_A \otimes \epsilon)[(b \otimes h) * a] = b(h \cdot a)$. If $b = 1$, we obtain (1.1.1) by observing that

$$
\begin{aligned}
\sum (h_2 \cdot a)_0 \otimes h_1(h_2 \cdot a)_1 &= (id_A \otimes \epsilon \otimes id_H)(id_A \otimes \Delta)((1 \otimes h) * a) \\
&= (id_A \otimes \epsilon \otimes id_H)(\sum (1 \otimes h_1) * a_0 \otimes h_2 a_1) \\
&= \sum h_1 \cdot a_0 \otimes h_2 a_1.
\end{aligned}
$$

Similarly, to verify (1.1.2), we note that

$$
\begin{aligned}
h \cdot (a * b) &= (id_A \otimes \epsilon)[(1 \otimes h) * (a * b)] \\
&= (id_A \otimes \epsilon)[((1 \otimes h) * a) * b] \\
&= (id_A \otimes \epsilon)[\sum (h_1 \cdot a_0 \otimes h_2 a_1) * b] \\
&= (id_A \otimes \epsilon)[\sum (h_1 \cdot a_0 \otimes h_2 a_1)(h_3 a_2 \cdot b)] \\
&= \sum (h_1 \cdot a_0)(h_2 a_1 \cdot b),
\end{aligned}
$$

and the proof of (i) is complete.

Next let $f : M \longrightarrow N$ be a morphism in $\mathcal{M}_A^H$. To see that F_τ is a functor from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ it suffices to check that f is also a morphism in $\mathcal{M}_{A^\tau}^H$. For $m \in M^\tau$ and $a \in A^\tau$, using the fact that f is an H-comodule A-module map, we have

$$f(m * a) = f(\sum m_0(m_1 \cdot a)) = \sum f(m_0)(m_1 \cdot a) = \sum f(m)_0(f(m)_1 \cdot a) = f(m) * a.$$

Hence f is an A^τ-module homomorphism and (ii) is proved.

Now suppose τ has convolution inverse λ. We already noted that λ satisfies (N). First we show that if (1.1.1) holds for τ, A, it holds for λ, A^τ. For $h \in H$, $a \in A$,

$$\begin{aligned}\sum h_1 \cdot_\lambda a_0 \otimes h_2 a_1 &= \sum h_1 \cdot_\lambda (\tau(h_3)\lambda(h_4)a)_0 \otimes h_2(\tau(h_3)\lambda(h_4)a)_1 \\ &= \sum (\lambda(h_1)\tau(h_2)(\lambda(h_4)a)_0 \otimes h_3(\lambda(h_4)a)_1 \quad \text{by (1.1.1) for } \tau \\ &= \sum (\lambda(h_2)a)_0 \otimes h_1(\lambda(h_2)a)_1 \\ &= \sum (1 \otimes h_1)\rho(h_2 \cdot_\lambda a),\end{aligned}$$

so (1.1.1) holds for λ. The right-hand side of (1.1.2) for λ and A^τ is

$$\begin{aligned}&\sum (h_1 \cdot_\lambda a_0) *_\tau ((h_2 a_1) \cdot_\lambda b) \\ &= \sum \lambda(h_1)\tau(h_2)\{\lambda(h_3)(a_0) *_\tau \lambda(h_4 a_1)(b)\} \\ &= \sum \lambda(h_1)\{\tau(h_2)[\lambda(h_4)(a_0)]_0 \tau(h_3(\lambda(h_4)(a_0))_1)\lambda(h_5 a_1)(b)\} \quad \text{by (1.1.2) for } \tau, A \\ &= \sum \lambda(h_1)\{(\tau(h_2)\lambda(h_3)(a_0))(\tau(h_4 a_1)\lambda(h_5 a_2)(b)\} \quad \text{by (1.1.1) for } \lambda \\ &= \lambda(h)(ab).\end{aligned}$$

But the left-hand side of (1.1.2) for λ is $h \cdot_\lambda (\sum a_0 *_\tau (a_1 \cdot_\lambda b)) = h \cdot_\lambda (ab)$ and thus (1.1.2) holds. It is straightforward to show that for $M \in \mathcal{M}_A^H$, $F_\lambda(M^\tau) = M$ and for $N \in \mathcal{M}_{A^\tau}^H$, $F_\tau(N^\lambda) = N$. Therefore F_λ is the inverse of F_τ and $(A^\tau)^\lambda = A$. ∎

DEFINITION 1.2. *For τ, A satisfying (1.1.1) and (1.1.2) as above, we call τ a twisting of $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ and A^τ a twisted algebra of A or a twisting of A. If τ is convolution invertible, then we call the twisting invertible.*

REMARK 1.3. If we do not assume the normality conditions (N), the algebra A^τ may not have a multiplicative identity, and even if A^τ does have an identity, it may not equal the identity of A. For example, let A be a right H-comodule algebra and let $u \in A^{coH}$, i.e. $\rho(u) = u \otimes 1$. Then we define $\tau \in \mathrm{Hom}(H, \mathrm{End}(A))$ by $\tau(h)(a) = \epsilon(h)ua$ for all $h \in H$, $a \in A$. It is easy to check that τ is a twisting of $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ and that the multiplication of

A^τ is defined by $a * b = aub$. If u is not invertible in A, A^τ has no multiplicative identity. If u is invertible, then u^{-1} is the identity of A^τ, and the convolution inverse λ of τ is defined by $\lambda(h) : a \longmapsto \epsilon(h)u^{-1}a$. The map $a \longmapsto au^{-1}$ is an algebra isomorphism from A to A^τ.

If τ is an invertible twisting of $\mathcal{M}_A^H$ not satisfying (N) but A^τ has a multiplicative identity, then it can be shown that there is a $\sigma \in \operatorname{Hom}(H, \operatorname{End}(A))$ satisfying (N) such that σ is an invertible twisting of $\mathcal{M}_A^H$, and $A^\tau \cong A^\sigma$. ∎

REMARK 1.4. (i) If H is a Hopf algebra, then (1.1.1) becomes

$$\rho(h \cdot a) = \sum h_2 \cdot a_0 \otimes S(h_1)h_3 a_1. \tag{1.4.1}$$

Also if $\tau : H \longrightarrow \operatorname{End}(A)$ is a ring homomorphism, i.e. τ induces an H-module structure on A, then τ has convolution inverse $\tau \cdot S$.

(ii) For H a bialgebra, if A is a left H-module algebra via τ, then (1.1.2) clearly holds. Conversely if A is a left H-module via τ, τ is convolution invertible, and (1.1.2) holds, A is an H-module algebra. For if τ has convolution inverse λ, then

$$\begin{aligned}
h \cdot (ab) &= h \cdot [\sum a_0(\tau(a_1)\lambda(a_2)(b))] \\
&= h \cdot [\sum a_0 *_\tau \lambda(a_1)(b)] \\
&= \sum (h_1 \cdot a_0)(\tau(h_2 a_1)\lambda(a_2)(b)) \quad \text{by (1.1.2)} \\
&= \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{since } \tau \text{ is a ring homomorphism.}
\end{aligned}$$

∎

The motivation for the definition of twistings above was the work of the third author [Z] for H a group or semigroup algebra, in particular for $H = k\mathbf{Z}$. Of course, then H is cocommutative, so that computations are less technical. Note that Definition 1.2 gives a more general definition of twisting than that in [Z].

EXAMPLE 1.5. Note that if H is a cocommutative Hopf algebra, then (1.4.1) is the requirement that τ be a comodule map. However if H is only a bialgebra, then Definition 1.2 is more general than the definition of twistings in [Z] where the maps τ are required to be comodule maps. If G is a noncancellative semigroup and $H = kG$, the two definitions may differ.

For example, let S be the two element semigroup $\{1, f\}$ where 1 is the identity and $f^2 = f$. Let $R = k[x]$, the polynomial ring, and I the ideal generated by $x - x^2$. Let $A = R/I$, and let $\bar{x}$ be the image of x in A. Then A is S-graded by $A_1 = k$ and $A_f = k\bar{x}$.

Now define $\tau \in \mathrm{Hom}(kS, \mathrm{End}(A))$ by $\tau(1) = id_A$, $\tau(f)(k + l\overline{x}) = l + k$. Clearly $\tau(f)$ is not a kS-comodule map. However (1.1.1) and (1.1.2) hold. The verification for $h = 1$ is clear. If $h = f$, then for all $a \in A$, (1.1.1) holds since

$$(\tau(f)(a))_1 \otimes f + (\tau(f)(a))_f \otimes f = \tau(f)(a) \otimes f = \tau(f)(a_1) \otimes f + \tau(f)(a_f) \otimes f.$$

Also, note that A is a left kS-module algebra and thus (1.1.2) holds also. ■

REMARK 1.6. Condition (1.4.1) certainly reminds us of Yetter-Drinfel'd modules [RT, §2], and if H is commutative and τ induces a module structure, then τ is a twisting if and only if (A, τ, ρ) is a left-right Yetter Drinfel'd structure.

If we do not assume H commutative, then "Yetter-Drinfel'd-like" conditions give functors from $\mathcal{M}_A^H$ to a category of relative Hopf modules over H^{op}. Let A, τ be as above. Then for all M in $\mathcal{M}_A^H$, M^τ lies in $\mathcal{M}_{A^\tau}^{H^{op}}$ if and only if for all $h \in H$, $a, b \in A$,

$$\sum (1 \otimes h_1)\rho(h_2 \cdot a) = \sum h_1 \cdot a_0 \otimes a_1 h_2, \tag{1.6.1}$$

$$h \cdot \{\sum a_0(a_1 \cdot b)\} = \sum (h_1 \cdot a_0)((a_1 h_2) \cdot b). \tag{1.6.2}$$

If H is a Hopf algebra, (1.6.1) is equivalent to

$$\rho(h \cdot a) = \sum h_2 \cdot a_0 \otimes S(h_1) a_1 h_3.$$

The proof follows as in Theorem 1.1 (i) and the analogous statements to 1.1 (ii), (iii) follow too. Thus if τ induces a right H-module algebra structure on A then (1.6.1) holds if and only if (A, τ, ρ_A) is a right-right Yetter-Drinfel'd structure. ■

EXAMPLE 1.7. (i) A trivial twisting is given by $\tau = \epsilon$, so that $h \cdot a = \epsilon(h)a$ for all $a \in A$. Then $A^\tau = A^\epsilon = A$ and $F_\epsilon : \mathcal{M}_A^H \longrightarrow \mathcal{M}_A^H$ is the identity.

(ii) If A is an H-dimodule algebra as in Long [L], H commutative and cocommutative, then the H-action on A is an invertible twisting. Then the H-opposite algebra $\overline{A}$, defined by $\bar{a}\bar{b} = \overline{(a_1 \cdot b)(a_0)}$ is just $(A^{op})^\tau$. The generalization of H-dimodule algebra for H not necessarily commutative or cocommutative are the QYB (quantum Yang-Baxter) H-module algebras of [CVOZ].

(iii) We give a simple example of a twisting τ which is a module action but where A is not an H-dimodule algebra. Let G be a finite (not necessarily abelian) group, $H = (kG)^* = \mathrm{Hom}(kG, k)$ with basis $p_g, p_g(h) = \delta_{g,h}$, and A a right H-comodule algebra. Then A is a left kG-module algebra and for $a \in A$, $\rho(a) = \sum_{t \in G} t(a) \otimes p_t$. Suppose A is also a G-graded

k-algebra with $g(A_h) \subseteq A_{ghg^{-1}}$. Define $\tau : (kG)^* \to \mathrm{End}(A)$ by $\tau(p_t)(a) = a_t$. Then A^τ has multiplication $a * b = \sum_{t \in G} t(a)b_t$. The map τ has convolution inverse λ, $\lambda(p_t)(a) = a_{t^{-1}}$.

(iv) Let k be a field of characteristic 0; the Weyl algebra $\mathcal{A}_1$ is a twisted algebra of the commutative polynomial algebra $k[X, Y]$. Let $H = k[X]$ with the usual Hopf algebra structure, let $A = k[X, Y]$ with H-comodule algebra structure defined by $\rho(f(X, Y)) = \sum_{i \geq 0} \frac{1}{i!}\delta^i(f(X, Y)) \otimes X^i, \delta = \partial/\partial X$. Define $\tau : H \to \mathrm{End}(A)$ by $\tau(X) = \partial/\partial Y$. Then

$$X * Y = XY + \tau(X)(Y) = XY + 1 \text{ and } Y * X = YX = XY$$

so $k[X, Y]^\tau \cong k[X, Y]/\langle XY - YX - 1\rangle$, the first Weyl algebra. In fact, we will see in the third section that since $\mathcal{A}_n \cong \mathcal{U}(L)\#\mathcal{U}(L')$ for L, L' n-dimensional abelian Lie algebras, $\mathcal{A}_n = k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]^\tau$, and the twisting τ is convolution invertible. ■

§2 Left hand twistings

Next, we examine a left hand version of twistings. For A a right H-comodule algebra, let ν be an element of the convolution algebra $\mathrm{Hom}(H, \mathrm{End}(A))$ satisfying (N). For all $h \in H$ and $a \in A$, we denote $\nu(h)(a) \in A$ by $a \cdot_\nu h$ or $a \cdot h$ if ν is clear. For any element $\nu \in \mathrm{Hom}(H, \mathrm{End}(A))$, we define a new (possibly nonassociative) multiplication $*_\nu$, (or $*$ if ν is clear) on A by

$$a *_\nu b = \sum (a \cdot_\nu b_1)b_0$$

where multiplication on the right hand side is the usual multiplication on A. We denote this algebra $(A, *)$ by ${}^\nu A$. Given an object M of ${}_A\mathcal{M}^H$, we define a map $A \otimes M \longrightarrow M$ by

$$a \otimes m \longmapsto a * m = \sum (a \cdot m_1)m_0$$

where the multiplication on the right hand side is the usual A-action on M. Let ${}^\nu M$ denote the H-comodule M together with this map. Since ν satisfies (N), ${}^\nu A$ has the same identity as A and $1 * m = m$ for all $m \in {}^\nu M$.

Recall that ${}_A\mathcal{M}^H \cong \mathcal{M}_{A^{op}}^{H^{op}}$, where every left A-module M is a right A^{op}-module in the usual way by $m \cdot a^{op} = am$ for $a \in A$, $m \in M$. It is straightforward to check that

$$(A^\nu)^{op} = {}^\nu(A^{op}), \quad \text{and,} \quad (A^{op})^\nu = ({}^\nu A)^{op}$$

as rings and as right H^{op}-comodules.

Necessary and sufficient conditions for ${}^\nu M$ to lie in ${}_{{}^\nu A}\mathcal{M}^H$ for all M in ${}_A\mathcal{M}^H$ now follow directly from Theorem 1.1.

PROPOSITION 2.1. *Let A, ν be as above. Then for all $M \in {}_A\mathcal{M}^H$, ${}^\nu M$ lies in ${}_{{}^\nu A}\mathcal{M}^H$ if and only if for all $a, b \in A$, $h \in H$,*

$$\sum \rho(a \cdot h_2)(1 \otimes h_1) = \sum (a \cdot h_2)_0 \otimes (a \cdot h_2)_1 h_1 = \sum a_0 \cdot h_1 \otimes a_1 h_2, \quad and \tag{2.1.1}$$

$$(a *_\nu b) \cdot_\nu h = \{\sum (a \cdot b_1) b_0\} \cdot h = \sum (a \cdot (b_1 h_2))(b_0 \cdot h_1). \tag{2.1.2}$$

If (2.1.1) and (2.1.2) hold, then ${}^\tau A$ is an H-comodule algebra, and we can define a functor F_ν from ${}_A\mathcal{M}^H$ to ${}_{{}^\nu A}\mathcal{M}^H$ by $F_\nu(M) = {}^\nu M, F_\nu(f) = f$. If, moreover, ν has convolution inverse μ, then μ satisfies (2.1.1) and (2.1.2) for ${}^\nu A$ and F_ν is a category isomorphism.

PROOF. Note that under the standard isomorphism between ${}_{{}^\nu A}\mathcal{M}^H$ and $\mathcal{M}^{H^{op}}_{(A^{op})^\nu}$, ${}^\nu M$ corresponds to M^ν. Thus for all $M \in {}_A\mathcal{M}^H$, ${}^\nu M \in {}_{{}^\nu A}\mathcal{M}^H$ if and only if ν is a twisting of $\mathcal{M}^{H^{op}}_{A^{op}}$ to $\mathcal{M}^{H^{op}}_{(A^{op})^\nu}$. But it is straightforward to check that $\nu \in \mathrm{Hom}(H, \mathrm{End}(A))$ satisfies (2.1.1) and (2.1.2) if and only if $\nu \in \mathrm{Hom}(H^{op}, \mathrm{End}(A^{op}))$ satisfies (1.1.1) and (1.1.2). The rest of the proposition is immediate. ■

EXAMPLE 2.2. For H a cocommutative Hopf algebra, B a right H-comodule algebra, L a right H-module algebra, let $B\#L$ be the right smash product [B]. Recall that $B\#L = B \otimes L$ as k-modules but multiplication in $B\#L$ is given by

$$(b\#l)(c\#m) = \sum bc_0 \#(l \cdot c_1)m.$$

The right H-coaction of $B \otimes L$ is $\rho(b \otimes l) = \sum (b_0 \otimes l) \otimes b_1$. Let ν be the right action of H on L, i.e. $\nu(h)(b \otimes l) = b \otimes l \cdot h$. Conditions (2.1.1) and (2.1.2) are easily checked and ν has convolution inverse $\nu^{-1}(h)(b \otimes l) = b \otimes l \cdot S(h)$. ■

Now we show that if H has bijective antipode, then $\mathcal{M}^H_A \cong \mathcal{M}^H_{A^\tau}$ via an invertible twisting if and only if there is also an invertible twisting from ${}_A\mathcal{M}^H$ to ${}_{A^\tau}\mathcal{M}^H$.

THEOREM 2.3. *Suppose H is a Hopf algebra with bijective antipode. Then there is an invertible twisting τ from $\mathcal{M}^H_A$ to $\mathcal{M}^H_{A^\tau}$ if and only if there is an invertible left hand twisting ν from ${}_A\mathcal{M}^H$ to ${}_{{}^\nu A}\mathcal{M}^H$ with ${}^\nu A \cong A^\tau$ as H-comodule algebras.*

PROOF. Suppose τ is an invertible twisting from $\mathcal{M}^H_A$ to $\mathcal{M}^H_{A^\tau}$ with convolution inverse λ. Define $\nu : H \longrightarrow \mathrm{End}(A)$ by

$$\nu(h)(a) = \sum \tau(\overline{S}(a_2 h))\lambda(\overline{S}(a_1))(a_0).$$

Since the normality conditions (N) hold for τ, they hold for ν also. We need to verify (2.1.1). Note that, by (1.4.1),

$$\rho(\sum \lambda(\overline{S}(a_1))(a_0)) = \sum \lambda(\overline{S}(a_1))(a_0) \otimes a_2. \tag{2.3.1}$$

Then

$$\begin{aligned}
\rho(\nu(h)(a)) &= \rho(\sum \tau(\overline{S}(a_2h))\lambda(\overline{S}(a_1))(a_0)) \\
&= \sum \tau(\overline{S}(a_4h_2))\lambda(\overline{S}(a_1))(a_0) \otimes a_5h_3\overline{S}(a_3h_1)a_2 \quad \text{by (2.3.1) and (1.4.1) for } \tau \\
&= \sum \tau(\overline{S}(a_2h_2))\lambda(\overline{S}(a_1))(a_0) \otimes a_3h_3\overline{S}(h_1) \\
&= \sum \nu(h_2)(a_0) \otimes a_1h_3\overline{S}(h_1).
\end{aligned}$$

Next, we define $f : {}^{\nu}A \longrightarrow A^{\tau}$ by $f(a) = \sum \lambda(\overline{S}(a_1))(a_0)$. By (2.3.1), f is an H-comodule map. It is easy to see that the inverse of f is $f^{-1} : A^{\tau} \longrightarrow {}^{\nu}A$ defined by $f^{-1}(a) = \sum \tau(\overline{S}(a_1))(a_0)$. Let us show that f is an algebra map.

$$\begin{aligned}
f(a *_\tau b) &= \sum \lambda(\overline{S}(a_1b_1))(a_0 *_\nu b_0) \\
&= \sum \lambda(\overline{S}(a_1b_2))\nu(b_1)(a_0)b_0 \\
&= \sum \lambda(\overline{S}(a_3b_5))(\nu(b_2)(a_0)) *_\tau \lambda(\overline{S}(a_2b_4)a_1b_3\overline{S}(b_1))b_0 \quad \text{by (1.1.2) for } \lambda \\
&= \sum \lambda(\overline{S}(a_1b_3))(\nu(b_2)(a_0)) *_\tau \lambda(\overline{S}(b_1))b_0 \\
&= \sum \lambda(\overline{S}(a_1))(a_0) *_\tau \lambda(\overline{S}(b_1))b_0 \quad \text{by the definition of } \nu \\
&= f(a) *_\tau f(b).
\end{aligned}$$

To prove τ is a twisting, it remains to check (2.1.2). By the above computation for f,

$$\begin{aligned}
\nu(h)(a *_\nu b) &= \sum \tau(\overline{S}(a_2b_2h))[\lambda(\overline{S}(a_1))(a_0) *_\tau \lambda(\overline{S}(b_1))b_0] \\
&= \sum [\tau(\overline{S}(a_4b_3h_2))\lambda(\overline{S}(a_1)(a_0)][\tau(\overline{S}(a_3b_2h_1)a_2)\lambda(\overline{S}(b_1)(b_0)] \quad \text{by (1.1.2) for } \tau \\
&= \sum [\tau(\overline{S}(a_2b_3h_2))\lambda(\overline{S}(a_1)(a_0)][\tau(\overline{S}(b_2h_1))\lambda(\overline{S}(b_1)(b_0)] \\
&= \sum \nu(b_1h_2)(a)\nu(h_1)(b_0).
\end{aligned}$$

Finally we show that ν is convolution invertible. Define $\mu : H \longrightarrow \mathrm{End}(A)$ by

$$\mu(h)(b) = \sum \tau(\overline{S}(b_2h_3\overline{S}(h_1)))\lambda(\overline{S}(b_1h_2))(b_0),$$

for all $h \in H$, $b \in A$. Then, using (2.1.1) for ν, we have

$$\begin{aligned}
\sum \mu(h_1)\nu(h_2)(a) &= \sum \tau(\overline{S}(a_2h_8\overline{S}(h_4)h_3\overline{S}(h_1)))\lambda(\overline{S}(a_1h_7\overline{S}(h_5)h_2))(\nu(h_6)(a_0)) \\
&= \sum \tau(\overline{S}(a_4h_4\overline{S}(h_1)))\lambda(\overline{S}(a_3h_3))\tau(\overline{S}(a_2h_2))\lambda(\overline{S}(a_1))(a_0) \\
&= \epsilon(h)a.
\end{aligned}$$

Similarly, direct computation shows that $\sum \nu(h_1)\mu(h_2)(a) = \epsilon(h)a$.

Thus ${}^{\nu}A \cong A^{\tau}$ and ${}_A\mathcal{M}^H \cong {}_{{}^{\nu}A}\mathcal{M}^H$. Conversely, suppose ν is an invertible twisting from ${}_A\mathcal{M}^H$ to ${}_{{}^{\nu}A}\mathcal{M}^H$, with convolution inverse μ.

Define

$$\tau(h)(a) = \sum \nu(S(ha_2))\mu(S(a_1))(a_0).$$

Similar computations show that τ is a twisting from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$ and that $f : A^\tau \to {}^\nu A$ defined by $f(a) = \sum \mu(S(a_1))(a_0)$ is an H-comodule algebra isomorphism. ∎

QUESTION: For A, B right H-comodule algebras, H a Hopf algebra with bijective antipode, if $\mathcal{M}_A^H$ and $\mathcal{M}_B^H$ are isomorphic (or equivalent), then are ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}^H$?

§3. Crossed products and duality theorems

In this section we prove that crossed products are twistings of the tensor product and obtain some duality theorems. First, we show that if A^τ is an invertible twisting of A, and if L is a right H-module subalgebra of H^* and a rational left H^*-module, then $A\#L \cong A^\tau\#L$ as k-algebras.

THEOREM 3.1. *Let H be a Hopf algebra with bijective antipode. Let L be a right H-module subalgebra of H^*, possibly without 1, which is also a rational left H^*-module under the action $h^* \to l = h^*l$. Also suppose the evaluation map ev from H to $\mathrm{Hom}(H^*, k)$, $ev(h)(h^*) = \langle h^*, h\rangle$ is injective. Let ν be a twisting from ${}_A\mathcal{M}^H$ to ${}_{^\nu A}\mathcal{M}^H$. Then there is a k-algebra map ϕ from ${}^\nu A\#L$ to $A\#L$, and if ν is convolution invertible, ϕ is an isomorphism.*

PROOF. Since L is a rational left H^*-module, L is a right H-comodule and for $l \in L, m \in H^*$,

$$ml = \sum l_0\langle m, l_1\rangle. \tag{3.1.1}$$

Since $\epsilon_H l = l, \sum l_0\epsilon(l_1) = l$. As in Example 2.2, recall that the (right) smash product $A\#L$ has multiplication

$$(a\#l)(b\#m) = \sum ab_0\#(l \leftharpoonup b_1)m,$$

where $\leftharpoonup$ denotes the right H-module algebra structure of $L \subset H^*$, $(l \leftharpoonup h)(g) = \langle l, hg\rangle$.

Define $\phi : {}^\nu A\#L \to A\#L$ by

$$\phi(a\#l) = \sum \nu(l_1)(a)\#l_0.$$

Note that by (3.1.1) and the injectivity of ev, for $l \in L$, $h \in H$,

$$\sum\langle l_0, h\rangle l_1 = \sum h_1\langle l, h_2\rangle. \tag{3.1.2}$$

For $a, b \in A$, $l, m \in L$,

$$\begin{aligned}\phi(a\#l)\phi(b\#m) &= \sum(\nu(l_1)(a)\#l_0)(\nu(m_1)(b)\#m_0)\\ &= \sum \nu(l_1)(a)\nu(m_2)(b_0)\#(l_0 \leftharpoonup b_1m_3\overline{S}(m_1))m_0 \quad \text{by (2.1.1)}\end{aligned}$$

$$
\begin{aligned}
&= \sum \nu(l_1)(a)\nu(m_3)(b_0)\#\langle l_0, b_1 m_4 \overline{S}(m_2)m_1\rangle m_0 \quad \text{by } (3.1.1) \\
&= \sum \nu(l_1\langle l_0, b_1 m_2\rangle)(a)\nu(m_1)(b_0)\# m_0 \\
&= \sum \nu(b_1 m_2\langle l, b_2 m_3\rangle)(a)\nu(m_1)(b_0)\# m_0 \quad \text{by } (3.1.2) \\
&= \sum \nu(m_1)(a * b_0)\#\langle l, b_1 m_2\rangle m_0 \quad \text{by } (2.1.2) \\
&= \phi[\sum a * b_0\#\langle l, b_1 m_1\rangle m_0] \\
&= \phi[\sum a * b_0\#(l \leftharpoonup b_1)m] \text{ by } (3.1.1) \\
&= \phi[(a\# l)(b\# m)].
\end{aligned}
$$

Finally, if ν is convolution invertible, then ϕ has inverse $\phi^{-1}(a\#l) = \sum \nu^{-1}(l_1)(a)\#l_0$. ■

Recall [S,2.1.3(d)] that since H^* is a left H^*-module via the (convolution) multiplication in H^*, H^* contains a unique maximal rational submodule denoted H^{*rat}.

Corollary 3.2. *Let k be a field and suppose $H^{*rat} \neq 0$. Then, if τ is an invertible twisting from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$, $A\#H^{*rat} \cong A^\tau\#H^{*rat}$.*

Proof. By [CC, §3], S is bijective and $L = H^{*rat}$ satisfies the assumptions in Theorem 3.1. By Theorem 2.3, there exists an invertible left hand twisting ν from ${}_A\mathcal{M}^H$ to ${}_{{}^\nu A}\mathcal{M}^H$ with ${}^\nu A \cong A^\tau$ as H-comodule algebras. The statement then follows directly from Theorem 3.1. ■

Corollary 3.3. *Suppose H is finitely generated projective over the commutative ring k. Then if τ is an invertible twisting from $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$, $A\#H^* \cong A^\tau\#H^*$.*

Proof. This follows from Theorems 2.3 and 3.1 with $L = H^*$. ■

Note that for L as in Corollaries 3.2 or 3.3, i.e. $L = H^*$, H finite, or $L = H^{*rat}$, then the functor F_ν from ${}_A\mathcal{M}^H$ to ${}_{{}^\nu A}\mathcal{M}^H$ is just the composition of isomorphisms

$$
{}_A\mathcal{M}^H \cong {}_{A\#L}\mathcal{M} \cong {}_{{}^\nu A\#L}\mathcal{M} \cong {}_{{}^\nu A}\mathcal{M}^H
$$

where the middle isomorphism F_ϕ is defined by $F_\phi(M) = M$ with left ${}^\nu A\#H^*$ structure given by $x \cdot m = \phi(x)m$ for $x \in {}^\nu A\#H^*$, $m \in M$, and the isomorphisms ${}_B\mathcal{M}^H \cong {}_{B\#L}\mathcal{M}$, $B = A$ or ${}^\nu A$, are the usual maps for $L = H^*$, finite, and can be found in [CC,2.3] for $L = H^{*rat}$. In the latter case, ${}_{B\#L}\mathcal{M}$ denotes the category of rational or unitary modules.

Next we discuss an important example of a twisted algebra where the twisting τ does not induce an H-module algebra structure on A. Let H be a Hopf algebra acting weakly on an algebra A on the left, and σ in $\mathrm{Hom}(H \otimes H, A)$, a cocycle with respect to the weak action.

Then we can form the crossed product $A\#_\sigma H$ with $\rho_{A\#_\sigma H} = 1\otimes\Delta$. (See [M, 7.1] for details of the definition.) We show that $A\#_\sigma H$ is a twisting of the tensor product $A \otimes H$, and is invertible if σ is convolution invertible.

Recall that there is a correspondence between H-cleft extensions of A and crossed products $A\#_\sigma H$ with σ invertible. (See [BM], [DT], or [M, §7].) Given a cleft extension $A \subset B$ with $\phi : H \to B$ a convolution invertible H-comodule map, define a weak H-action on A and an invertible cocycle $\sigma : H \otimes H \to A$ by

$$h \cdot a = \sum \phi(h_1)a\phi^{-1}(h_2), \qquad \text{and}$$

$$\sigma(h,g) = \sum \phi(h_1)\phi(g_1)\phi^{-1}(h_2g_2);$$

then $B \cong A\#_\sigma H$ as H-comodule algebras. Conversely, if σ is convolution invertible, the crossed product $A\#_\sigma H$ is cleft via

$$\phi : H \to A\#_\sigma H,\ \phi(h) = 1\#h,\ \phi^{-1}(h) = \sum \sigma^{-1}(Sh_2, h_3)\#Sh_1.$$

THEOREM 3.4. *Let $A\#_\sigma H$ be a crossed product as in [M,§7] but with σ not necessarily invertible. Then there is a twisting τ, $\tau \in Hom(H, \mathrm{End}(A \otimes H))$, with $A\#_\sigma H = (A \otimes H)^\tau$, and, if σ is convolution invertible, so is τ. Conversely, if $(A \otimes H)^\tau$ is a twisting of $A \otimes H$, then $(A \otimes H)^\tau = A\#_\sigma H$ for some weak action of H on A, some σ, and if τ is an invertible twisting, σ is convolution invertible, or, equivalently, $(A \otimes H)^\tau$ is cleft.*

PROOF. Let $\tau : H \longrightarrow \mathrm{End}(A \otimes H)$ be defined by

$$\tau(h)(b \otimes g) = \sum (h_2 \cdot b)\sigma(h_3, g_1) \otimes S(h_1)h_4g_2.$$

Since $\sigma(h,1) = \sigma(1,h) = \epsilon(h)$ for all $h \in H$, (N) holds.

Now $(A \otimes H)^\tau = A\#_\sigma H$ since

$$\begin{aligned}(a \otimes h) * (b \otimes g) &= \sum (a \otimes h_1)[\tau(h_2)(b \otimes g)] \\ &= \sum (a \otimes h_1)((h_3 \cdot b)\sigma(h_4, g_1) \otimes S(h_2)h_5g_2) \\ &= \sum a(h_1 \cdot b)\sigma(h_2, g_1) \otimes h_3g_2 \\ &= (a\#_\sigma h)(b\#_\sigma g).\end{aligned}$$

It is straightforward to verify that (1.4.1) and (1.1.2) hold, the first by direct computation and the second by the associativity of $(A \otimes H)^\tau$ using the fact that

$$(1 \otimes h) * [(b \otimes g) * (c \otimes f)] = [(1 \otimes h) * (b \otimes g)] * (c \otimes f).$$

Now, suppose σ is convolution invertible, so that $A\#_\sigma H = (A \otimes H)^\tau$ is a cleft extension. Then, by direct computation, or by the paragraph preceding the theorem, for $\phi : H \to A\#_\sigma H$ given by $\phi(h) = 1\#h$, $\phi^{-1}(h) = \sum \sigma^{-1}(Sh_2, h_3)\#Sh_1$, note that

$$\tau(h)(b \otimes g) = \sum \phi(h_2)b\phi(g_1)\phi^{-1}(h_3 g_2) \otimes S(h_1)h_4 g_3.$$

Then a tedious computation shows that τ has convolution inverse λ,

$$\lambda(f)(c \otimes m) = \sum \phi^{-1}(f_3)c\phi(f_4 m_1)\phi^{-1}(S(f_2)f_5 m_2) \otimes S(f_1)f_6 m_3.$$

Now let $\tau \in \mathrm{Hom}(H, \mathrm{End}(A \otimes H))$ be a twisting from $\mathcal{M}^H_{A\otimes H}$ to $\mathcal{M}^H_{(A\otimes H)^\tau}$; we show that $(A \otimes H)^\tau$ is a crossed product. Define a map from $H \otimes A$ to A by $h \otimes a \to h \cdot a = (I \otimes \epsilon)\tau(h)(a \otimes 1)$; by (N) and (1.1.2), this map defines a weak action of H on A. Now define $\sigma : H \otimes H \to A$ by $\sigma(h, g) = (I \otimes \epsilon)\tau(h)(1 \otimes g)$. Then

$$\begin{aligned}
\tau(h)(a \otimes g) &= (I \otimes \epsilon \otimes I)\rho(\tau(h)(a \otimes g)) \\
&= \sum (I \otimes \epsilon)[\tau(h_2)((a \otimes 1) *_\tau (1 \otimes g_1))] \otimes S(h_1)h_3 g_2 \quad \text{by (1.1.1)} \\
&= \sum (I \otimes \epsilon)[\tau(h_2)(a \otimes 1)\tau(h_3)(1 \otimes g_1)] \otimes S(h_1)h_4 g_2 \quad \text{by (1.1.2)} \\
&= \sum h_2 \cdot a\sigma(h_3, g_1) \otimes S(h_1)h_4 g_2.
\end{aligned}$$

Therefore $(A \otimes H)^\tau = A\#_\sigma H$ and by the associativity of $(A \otimes H)^\tau$, the cocycle and twisted module conditions [M, 7.13, 7.14] hold and $A\#_\sigma H$ is a crossed product. If, furthermore, τ is invertible with convolution inverse λ, then define σ^{-1} by

$$\sigma^{-1}(h, g) = (I \otimes \epsilon) \sum \tau(h_1 g_1)\lambda(g_2)(1 \otimes Sg_3).$$

Then

$$\begin{aligned}
\sum \sigma(h_1, g_1)\sigma^{-1}(h_2, g_2) &= (I \otimes \epsilon) \sum (\tau(h_1)(1 \otimes g_1))(\tau(h_2 g_2)\lambda(g_3)(1 \otimes Sg_4)) \\
&= (I \otimes \epsilon)[\tau(h)(1 \otimes g_1) *_\tau \lambda(g_2)(1 \otimes Sg_3))] \quad \text{by (1.1.2)} \\
&= (I \otimes \epsilon)\tau(h)(1 \otimes \epsilon(g)) \\
&= \epsilon(h)\epsilon(g).
\end{aligned}$$

Checking the other convolution product requires a bit more work.

$$\begin{aligned}
\sum \sigma^{-1}(h_1, g_1)\sigma(h_2, g_2) &= (I \otimes \epsilon) \sum (\tau(h_1 g_1)\lambda(g_2)(1 \otimes Sg_3))(\tau(h_2)(1 \otimes g_4)) \\
&= (I \otimes \epsilon)[\tau(h_1 g_1)(\lambda(g_2)(1 \otimes Sg_3) *_\tau (1 \otimes g_4)) \quad \text{by (1.1.2)} \\
&= (I \otimes \epsilon) \sum \tau(h_1 g_1)(\lambda(g_2)(1 \otimes Sg_5) *_\tau \lambda(g_3 Sg_4)(1 \otimes g_6)) \\
&= (I \otimes \epsilon) \sum \tau(h_1 g_1)(\lambda(g_2)((1 \otimes Sg_3)(1 \otimes g_4)) \quad \text{by (1.1.2) for } \lambda \\
&= \epsilon(h)\epsilon(g).
\end{aligned}$$

Thus σ is invertible. ■

From Hopf Galois theory [M,§8], if B is a cleft extension of its coinvariants A and is faithfully flat as a left or right A-module, then the category $\mathcal{M}_B^H$ is equivalent to the category of modules $\mathcal{M}_A$, so that $\mathcal{M}_B^H$ and $\mathcal{M}_{A\otimes H}^H$ are equivalent since each is equivalent to $\mathcal{M}_A$. The preceding theorem tells us that $\mathcal{M}_B^H \cong \mathcal{M}_{A\otimes H}^H$ and no flatness assumption is needed for this isomorphism of categories of relative Hopf modules.

The equivalences $G_{A\otimes H}$ and $G_{A\#_\sigma H}$ from $\mathcal{M}_A$ to $\mathcal{M}_{A\otimes H}^H$ and $\mathcal{M}_{A\#_\sigma H}^H$ are given by $N \longrightarrow N\otimes_A (A\otimes H)$, and $N \longrightarrow N \otimes_A (A\#_\sigma H)$ respectively; it is easy to verify that the diagram below commutes:

$$\begin{tikzcd} \mathcal{M}_A \arrow[r, "id_{\mathcal{M}_A}"] \arrow[d, "G_{A\otimes H}"'] & \mathcal{M}_A \arrow[d, "G_{A\#_\sigma H}"] \\ \mathcal{M}_{A\otimes H}^H \arrow[r, "F_\tau"] & \mathcal{M}_{A\#_\sigma H}^H \end{tikzcd}$$

Similarly we note that if A is H-Galois and faithfully flat over A^{coH}, then if τ is an invertible twisting of $\mathcal{M}_A^H$ to $\mathcal{M}_{A^\tau}^H$, A^τ is also H-Galois by a diagram similar to that above and [M, 8.5.6].

Corollary 3.5. For B any cleft extension of A, $\mathcal{M}_B^H \cong \mathcal{M}_{A\otimes H}^H$, and, if H has bijective antipode, ${}_B\mathcal{M}^H \cong {}_{A\otimes H}\mathcal{M}^H$ also.

Proof. This follows directly from Theorems 3.4 and 2.3. ■

The twisting in Theorem 3.4 combines with Theorem 3.1 to yield the following duality results.

Corollary 3.6. Let H be a Hopf algebra over a field k such that $H^{*rat} \neq 0$. For $A\#_\sigma H$ a crossed product with σ invertible,

$$(A\#_\sigma H)\#H^{*rat} \cong (A\otimes H)\#H^{*rat} \cong A\otimes (H\#H^{*rat}).$$

Proof. The first isomorphism follows from Theorem 3.4 and Corollary 3.2, and the second is just $(a\otimes h)\#h^* \rightarrow a\otimes (h\#h^*)$. ■

Note that if $H^{*rat} \neq 0$, $H\#H^{*rat}$ is a central simple k-algebra [CC,3.8] and is a dense ring of linear transformations on H [CC,3.9].

Corollary 3.7. [BM, 2.2]. Suppose H is finitely generated projective over the commutative ring k and $A\#_\sigma H$ is a crossed product with σ invertible. Then

$$(A\#_\sigma H)\#H^* \cong (A\otimes H)\#H^* \cong A\otimes (H\#H^*) \cong A\otimes \mathrm{End}_k(H).$$

PROOF. The statement follows directly from Theorem 3.4 and Corollary 3.3. ■

In fact, crossed products $A\#_\sigma B$ are defined for any A with weak left H-action and B any left H-comodule algebra. If H is cocommutative, then $A\#_\sigma B$ is a right H-comodule algebra with $\rho_{A\#_\sigma B} = 1 \otimes \rho_B$.

PROPOSITION 3.8. *Suppose H is cocommutative, and let $A\#_\sigma B$ be a crossed product. Then there is a twisting τ from $\mathcal{M}^H_{A\otimes B}$ to $\mathcal{M}^H_{A\#_\sigma B}$, and, if σ is convolution invertible, τ is also, and then $\mathcal{M}^H_{A\otimes B} \cong \mathcal{M}^H_{A\#_\sigma B}$.*

PROOF. As in Theorem 3.4, we define τ by

$$\tau(h)(a\otimes b) = \sum (h_1 \cdot a)\sigma(h_2, b_1) \otimes b_0.$$

Since τ is an H-comodule map, (1.1.1) is clear and the verification of (1.1.2) is a straightforward application of [M, (7.1.3) and (7.1.4)]. As in 3.4, $(A\otimes B)^\tau = A\#_\sigma H$.

If σ is convolution invertible, then τ has convolution inverse λ defined by

$$\lambda(h)(a\otimes b) = \sum \sigma^{-1}(Sh_1, h_2)[Sh_3 \cdot (a\sigma^{-1}(h_4, b_1))]\sigma(Sh_5, h_6) \otimes b_0.$$

Thus the categories $\mathcal{M}^H_{A\#_\sigma B}$ and $\mathcal{M}^H_{A\otimes B}$ are isomorphic. ■

COROLLARY 3.9. For $A\#_\sigma B$ a crossed product with H cocommutative and σ invertible, then

$$(A\#_\sigma B)\#L \cong (A\otimes B)\#L \cong A \otimes (B\#L)$$

as k-algebras, for H and L satisfying the conditions of 3.1, i.e. H maps injectively to $\mathrm{Hom}(H^*, k)$, L is a right H-module subalgebra of H^* and is also a rational left H^*-module.

PROOF. This follows from Propositions 3.8 and Theorem 3.1. ■

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