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Hopf Extensions of CM-finite Artin Algebras

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Abstract Let H be a finite-dimensional Hopf algebra over a field k, and A a left H-module k-algebra. We show that $A\#H$ is a CM-finite algebra if and only if A is a CM-finite algebra preserving global dimension of their relative Auslander algebras when A/A^H is an H^*-Galois extension and $A\#H/A$ is separable. As application, we describe all the finitely-generated Gorenstein-projective modules over a triangular matrix artin algebra $\Lambda = \begin{pmatrix} A^H & A \\ 0 & A\#H \end{pmatrix}$, and obtain a criteria for Λ being Gorenstein. We also show that Hopf extensions can induce recollements between categories $A\#H$-Mod and A^H-Mod.

Keywords Hopf Galois extensions · Smash products · Separable functors · Gorenstein-projective modules · CM-finite algebras

Mathematics Subject Classification (2010) 18G25

1 Introduction

Gorenstein-projective modules and CM-finite type of algebras receive a lot of attention in the representation theory of algebras, Gorenstein Homological algebras, the Tate cohomology of algebras, and in the theory of singularity and stable categories, etc. (See e.g. [1, 3–8, 12–17, 20]). Let H be a finite-dimensional Hopf algebra over a field k, A a left H-module k-algebra and A^H the subalgebra of invariants. Then

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A/A^H is an H^*-extension. Perhaps the main interest of Hopf Galois theory of H^*-extensions is that it covers the traditional Galois and representation theories but contains several new interesting cases as well. (See e.g. [10, 18]).

In this paper, we aim to study Gorenstein-projective modules and CM-finiteness for Hopf extensions of CM-finite algebras. By this way we obtain an inductive construction for CM-finite algebras and Gorenstein-projective modules. Also we show the existence of recollements induced by Hopf extensions.

2 Preliminaries

In this section we will fix notation and recall very briefly the definitions needed in the proofs of our main results.

Throughout this paper k will denote a field. Write $\otimes$ for $\otimes_k$. We use Sweedler's notation for a comultiplication: $\Delta(C) = \Sigma C_{(1)} \otimes C_{(2)}$. By definition, a Hopf algebra is a k-vector space H with an associative algebra structure $(H, m, 1)$ and a coassociative coalgebra structure (H, Δ, ε), such that Δ and ε are algebra homomorphisms and there is a linear map $S : H \to H$, called an antipode, such that $m(\mathrm{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathrm{id})(h \otimes 1), m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)(1 \otimes h)$, for all $f, g, h \in H$. For more details see [21].

Recall that a k-algebra A is a left H-module algebra if A is a left H-module such that $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1 = \varepsilon(h)1_A$, for all $a, b \in A$ and $h \in H$. Set $A^H = \{a \in A : h \cdot a = \varepsilon(h)a, \forall h \in H\}$. Note that A^H is a subalgebra of A. Dually, an algebra A is a right H-comodule algebra if A is a right H-comodule via $\rho : a \longmapsto a_{(0)} \otimes a_{(1)}$ such that $\rho(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$ for all $a, b \in A$. Set $A^{coH} := \{a \in A |\ \rho(a) = a \otimes 1\}$. For a finite-dimensional Hopf algebra H the H-comodule category is equivalent to the H^*-module category, where $H^* = \mathrm{Hom}_k(H, k)$, is the dual of H.

Following [21], we say that an algebra inclusion $B \subset A$ is an H-extension, denoted by A/B, whenever A is a right H-comodule algebra and $B = A^{coH}$. Note that A/B is an H-extension if and only if A is an H^*-module algebra with $A^{H^*} = A^{coH} = B$ when H is a finite-dimensional Hopf algebra. Usually, when A is an H-module algebra we say that A/A^H is an H^*-extension. An H-extension A/B is said to be H-Galois if the map $\beta : A \otimes_B A \to A \otimes H$ given by $\beta(a \otimes b) = \Sigma ab_{(0)} \otimes b_{(1)}$ is bijective.

Let A be a left H-module algebra. The smash product algebra $A\#H$ is defined on a k-vector space $A \otimes H$, with multiplication given by $(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g,\ \forall\, a,\ b \in A,\ h,\ g \in H$. Recall from [21] that A is an $A\#H$-A^H-bimodule, also an A^H-$A\#H$-bimodule.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ a covariant functor. Note that we have a natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-, -) \to \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))$. Recall from [9, 22] that F is called separable if $\mathcal{F}$ splits, that is, there exists a natural transformation $\mathcal{G} : \mathrm{Hom}_{\mathcal{D}}(F(-), F(-)) \to \mathrm{Hom}_{\mathcal{C}}(-, -)$ such that $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$.

Recall that an algebra $B \subset A$ is said to be separable when the multiplication map $A \otimes_B A \to A$ is a split A-bimodule epimorphism. For a semisimple Hopf algebra H and A an H-module algebra, $A\#H/A$ is separable. However, for general finite Hopf algebras, $A\#H$ need not be separable over A.

From now on, H will stand for a finite-dimensional Hopf algebra over a field k, and A a left H-module k-algebra. Let M be in $A\#H$-Mod. Denote by ${}_AM$ the image of M by the restriction of the scalar functor ${}_A(-) : A\#H\text{-Mod} \to A\text{-Mod}$. We need the following fact:

Lemma 2.1 [23, Corollary 5.2 and 5.4] *Let A/A^H be an H^*-Galois extension. The following are equivalent:*

(a) *The scalar extension functor $A \otimes_{A^H} - : A^H\text{-Mod} \to A\#H\text{-Mod}$ is separable;*
(b) *The functor ${}_A(-)$ is separable;*
(c) *A has a trace 1 element $c \in C_A(A^H)$;*
(d) *$A\#H/A$ is separable;*
(e) *The functor $A\#H \otimes_A -$ is separable.*

For an artin algebra A, denote by A-Mod and A-mod the categories of left A-modules and of finitely generated left A-modules, respectively. Denote by $A\text{-}\mathcal{P}$ the full subcategory of all projective A-modules. Following [13], an A-module M is said to be Gorenstein-projective in A-Mod (resp. in A-mod), if there is an exact sequence $P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots$ of projective modules in A-Mod (resp. in A-mod) with $\mathrm{Hom_A(P^\bullet, Q)}$ exact for any projective module Q in A-Mod (resp. in A-mod), such that $M \cong \ker d^0$. Denote by $A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{G}proj$) the full subcategory of Gorenstein-projective modules in A-Mod (resp. in A-mod). An artin algebra A is Gorenstein if A has finite injective dimension, both as left and right A-module. If A is a Gorenstein algebra, then $A\text{-}\mathcal{G}proj =^\perp A$, where ${}^\perp A$ is the full subcategory of A-mod given by $\{X \in A\text{-mod} | \mathrm{Ext}^i_A(X, A) = 0, \forall i \geq 1\}$ [13, Corollary 11.5.3].

Recall from [4, 6] that an artin algebra A is called Cohen-Macaulay finite (simply, CM-finite) if there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein-projective A-modules.

3 CM-finiteness of Smash Product Algebras

This section is devoted to the proofs of our main results. We first describe explicitly Gorenstein-projective modules of smash product algebras. We need the following fact

Lemma 3.1 [10] *Let (F, G) be an adjoint pair between categories $\mathcal{C}$ and $\mathcal{D}$. Then F is separable if and only the unit $\eta_Z : Z \to GF(Z)$ of the adjoint pair (F, G) is a split monomorphism for all $Z \in \mathcal{C}$.*

Lemma 3.2 (Compare [19, Lemma 3.1, Proposition 3.2]) *Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then we have $A\#H\text{-}\mathcal{GP}$=add$(A\#H \otimes_A (A\text{-}\mathcal{GP}))$.*

Proof Since H is a finite-dimensional Hopf algebra, it follows that $A\#H \otimes_A -$ is isomorphic to $\mathrm{Hom}_A(A\#H, -)$. So we have adjoint pairs $(A\#H \otimes_A -, {}_A(-))$ and $({}_A(-), A\#H \otimes_A -)$. We claim that functors ${}_A(-) : A\#H\text{-Mod} \to A\text{-Mod}$

and $A\#H \otimes_A - : A\text{-Mod} \to A\#H\text{-Mod}$ induce adjoint pairs between categories $A\#H\text{-}\mathcal{GP}$ and $A\text{-}\mathcal{GP}$ as follows:

$$_A(-) : A\#H\text{-}\mathcal{GP} \to A\text{-}\mathcal{GP}$$

$$A\#H \otimes_A - : A\text{-}\mathcal{GP} \to A\#H\text{-}\mathcal{GP}.$$

Since $A\#H$ is a projective right A-module [23, Proposition 2.1 and 2.3], we get that $A\#H \otimes_A -$ is exact. It follows that $_A(-)$ is exact.

Let $G \in A\text{-}\mathcal{GP}$. We firstly verify $A\#H \otimes_A G \in A\#H\text{-}\mathcal{GP}$. Let $P^\bullet := \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ be exact in $A\text{-}\mathcal{P}$ with $G = \mathrm{Ker}(P^0 \to P^1)$ such that it remains exact whenever $\mathrm{Hom}_A(-, P)$ is applied for every $P \in A\text{-}\mathcal{P}$. Since $A\#H \otimes_A -$ is exact, we get that $A\#H \otimes_A P^\bullet$ is exact and $A\#H \otimes_A G = \mathrm{Ker}(A\#H \otimes_A P^0 \to A\#H \otimes_A P^1)$. Since $_A(-)$ is exact and $A\#H \otimes_A -$ is a left adjoint, it follows that $A\#H \otimes_A P^i$ is projective for each i. Assume that $E \in A\#H\text{-}\mathcal{P}$. Then $_AE$ is projective because $_A(-)$ is a left adjoint of the exact functor $A\#H \otimes_A -$. Since G is a Gorenstein-projective A-module, and $\mathrm{Hom}_{A\#H}(A\#H \otimes_A P^\bullet, E) \cong \mathrm{Hom}_A(P^\bullet, {}_AE)$, we can get that $\mathrm{Hom}_{A\#H}(A\#H \otimes_A P^\bullet, E)$ is exact, and so $A\#H \otimes_A G \in A\#H\text{-}\mathcal{GP}$.

Let $G' \in A\#H\text{-}\mathcal{GP}$. We secondly verify $_AG' \in A\text{-}\mathcal{GP}$. Let $P'^\bullet := \cdots \to P'^{-1} \to P'^0 \to P'^1 \to \cdots$ be exact in $A\#H\text{-}\mathcal{P}$ with $G' = \mathrm{Ker}(P'^0 \to P'^1)$ such that it remains exact whenever $\mathrm{Hom}_A(-, P')$ is applied for every $P' \in A\#H\text{-}\mathcal{P}$. Since $_A(-)$ is exact, it follows that $_AP'^\bullet$ is exact and $_AG' = \mathrm{Ker}({}_AP'^0 \to {}_AP'^1)$. Also we get from the above that $_AP'^i$ is projective for every i. Assume that $E' \in A\text{-}\mathcal{P}$. Then $A\#H \otimes_A E'$ is projective by the above. Since $_{A\#H}G'$ is Gorenstein-projective and $\mathrm{Hom}_{A\#H}(P'^\bullet, A\#H \otimes_A E') \cong \mathrm{Hom}_A({}_AP'^\bullet, E')$, we can get that $\mathrm{Hom}_A({}_AP'^\bullet, E')$ is exact, and so $_AG' \in A\text{-}\mathcal{GP}$.

Finally we verify $G' \in \mathrm{add}(A\#H \otimes_A (A\text{-}\mathcal{GP}))$. By assumption A/A^H is an H^*-Galois extension and $A\#H/A$ is separable, it follows from Lemma 2.1 that $_A(-)$ is separable. By Lemma 3.1 we get that G' is a direct summand of $A\#H \otimes_A G'$ as a left $A\#H$-module, that is, $G' \in \mathrm{add}(A\#H \otimes_A (A\text{-}\mathcal{GP}))$. □

We are now in the position to give our first result.

Theorem 3.3 *Let A be a finite-dimensional k-algebra. Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then A is a CM-finite algebra if and only if $A\#H$ is a CM-finite algebra. In particular, if we suppose $G_1, \cdots, G_m$ are all the pairwise non-isomorphic indecomposable finitely generated Gorenstein-projective A-modules, and $G = \bigoplus_{1 \le i \le m} G_i$. Then $A\#H\text{-}\mathcal{G}proj = \mathrm{add}(A\#H \otimes_A G)$.*

Proof Let $E \in A\#H\text{-}\mathcal{G}proj$. Since A/A^H is an H^*-Galois extension and $A\#H/A$ is separable, it follows from Lemma 3.2 that $E \in \mathrm{add}(A\#H \otimes_A E) \subset \mathrm{add}(A\#H \otimes_A G)$. Clearly, $\mathrm{add}(A\#H \otimes_A G) \subset A\#H\text{-}\mathcal{G}proj$, this follows that $A\#H\text{-}\mathcal{G}proj = \mathrm{add}(A\#H \otimes_A G)$. Obviously, if $A\#H$ is CM-finite, then A is CM-finite. Conversely, let A be CM-finite. We claim that if G and G' are two non-isomorphic Gorenstein-projective $A\#H$-modules, then $_AG$ and $_AG'$ are non-isomorphic Gorenstein-projective A-modules. In fact, if $0 \neq f : G \to G'$ is a non-isomorphic $A\#H$-map, then $_A\mathrm{Ker} f \neq 0$. Otherwise, by $\mathrm{Ker} f$ is a direct summand of $A\#H \otimes_A \mathrm{Ker} f$, we know that $\mathrm{Ker} f = 0$. This is a contradiction. Thus ${}_A f : {}_A G \to {}_A G'$ is a non-isomorphism A-map. It follows that $A\#H$ is CM-finite. □

Example 3.4 Let R be a CM-finite Gorenstein ring, G a finite group of automorphisms of R, and RG the skew group ring. If R has a trace 1 element $c \in C_R(R^G)$, then RG is a CM-finite Gorenstein ring.

Proof By [11, Proposition 1.1] we get that RG is a Gorenstein ring. By the proof of Theorem 3.3 we get that RG is CM-finite ring. This completes the proof. □

Lemma 3.5 [23, Corollary 5.1] *Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then* A^H-mod *and* $A\#H$-mod *are Morita equivalent under the functor* $\mathrm{Hom}_{A^H}(A, -)$, *whose inverse is the functor $A \otimes_{A\#H} -$.*

Corollary 3.6 *Let A be a finite-dimensional k-algebra. Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then A is a CM-finite algebra if and only if A^H is a CM-finite algebra.*

Proof By assumption we know from Theorem 3.3 that A is a CM-finite algebra if and only if $A\#H$ is a CM-finite algebra. Again by assumption we know from Lemma 3.4 that $A\#H$ is a CM-finite algebra if and only if A^H is a CM-finite algebra. Thus A is a CM-finite algebra if and only if A^H is a CM-finite algebra. □

Now let A be a CM-finite finite-dimensional k-algebra. Denote by $G_1, \cdots, G_n$ are all the pairwise non-isomorphic indecomposable finitely generated Gorenstein-projective A-modules, and $G = \bigoplus_{1 \le i \le n} G_i$. Then by Theorem 3.3 we know that $A\#H$ is CM-finite and $A\#H\text{-}\mathcal{G}proj = \mathrm{add}(A\#H \otimes_A G)$. Put $\mathcal{G}p(A) := \mathrm{End}_A(G)$ and $\mathcal{G}p(A\#H) := \mathrm{End}_{A\#H}(A\#H \otimes_A G)$. We respectively call $\mathcal{G}p(A)$ and $\mathcal{G}p(A\#H)$ the relative Auslander algebras of A and $A\#H$.

Next we will show that if A is a CM-finite finite-dimensional algebra, then the relative Auslander algebras of A and $A\#H$ are of same global dimension under some Hopf Galois extension and algebra separable extension.

Lemma 3.7 [25, Theorem 3.6] *Suppose $\mathcal{C}$ and $\mathcal{D}$ are additive k-categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ an adjoint pair of additive functors with F is a left adjoint and G is a right adjoint. We denote by $\widehat{\mathcal{C}}$ the category of coherent functors over $\mathcal{C}$. If there is an endofunctor $E : \mathcal{C} \to \mathcal{C}$ such that GF is naturally equivalent to $\mathrm{Id}_{\mathcal{C}} \oplus E$, then* $\mathrm{gl.dim}(\widehat{\mathcal{C}}) \le \mathrm{gl.dim}(\widehat{\mathcal{D}})$.

Lemma 3.8 [25, Lemma 2.2] *Let A be an artin algebra, and $M \in$* A-mod. *Then the category* $\widehat{\mathrm{add}(M)}$ *and* $\mathrm{End}_A(M)$-mod *are equivalent. In particular,* $\mathrm{gl.dim}(\mathrm{End}_A(M)) = \mathrm{gl.dim}(\widehat{\mathrm{add}(M)})$.

Theorem 3.9 *Use above notation. Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then*

$$\mathrm{gl.dim}\mathcal{G}p(A\#H) = \mathrm{gl.dim}\mathcal{G}p(A).$$

Proof Since A is finite-dimensional, it follows from the proof of Lemma 3.2 that $(A\#H \otimes_A -, {}_A(-))$ and $({}_A(-), A\#H \otimes_A -)$ are adjoint pairs between categories A-$\mathcal{G}proj$ and $A\#H$-$\mathcal{G}proj$. Since A/A^H is an H^*-Galois extension and $A\#H/A$ is separable, it follows from Lemma 2.1 that functors $A\#H \otimes_A -$ and ${}_A(-)$ are separable, and so for each $E \in A$-$\mathcal{G}proj$ the unit $E \to {}_A(A\#H \otimes_A E)$ is a split monomorphism and for each $G \in A\#H$-$\mathcal{G}proj$ the unit $G \to A\#H \otimes_A {}_AG$ is a split monomorphism. It follows from Lemma 3.6 that $\text{gl.dim}(\widehat{A\text{-}\mathcal{G}proj}) \leq \text{gl.dim}(A\#\widehat{H\text{-}\mathcal{G}proj})$ and $\text{gl.dim}(\widehat{A\text{-}\mathcal{G}proj}) \leq \text{gl.dim}(A\#\widehat{H\text{-}\mathcal{G}proj})$. Thus we have $\text{gl.dim}(\widehat{A\text{-}\mathcal{G}proj}) = \text{gl.dim}(A\#\widehat{H\text{-}\mathcal{G}proj})$.

Since A is CM-finite, it follows from Theorem 3.3 that $A\#H$-$\mathcal{G}proj = \text{add}(A\#H \otimes_A G)$ and $A\#H$ is a CM-finite artin algebra. By Lemma 3.7 we get that $\text{gl.dim}(A\#\widehat{H\text{-}\mathcal{G}proj}) = \text{gl.dim}\mathcal{G}p(A\#H)$ and $\text{gl.dim}(\widehat{A\text{-}\mathcal{G}proj}) = \text{gl.dim}\mathcal{G}p(A)$. This completes the proof. □

4 Application to $\begin{pmatrix} A^H & A \\ 0 & A\#H \end{pmatrix}$

Let A be a finite-dimensional k-algebra. Assume that A/A^H is an H^*-Galois extension and $A\#H/A$ is separable. In this section we will describe all the finitely-generated Gorenstein-projective modules over a triangular matrix artin algebra $\Lambda = \begin{pmatrix} A^H & A \\ 0 & A\#H \end{pmatrix}$, and give a criteria for Λ being Gorenstein. For basics on triangular matrix artin algebras we refer to [2, 24]. We need the following facts:

Lemma 4.1 [26, Theorem 2.2] *Let A and B be artin algebras over the same commutative artin ring, and $M = {}_AM_B$ an A-B-bimodule. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. If* $\text{proj.dim}_A M < \infty$ *and* $\text{proj.dim} M_B < \infty$*, then Λ is Gorenstein if and only if A and B are Gorenstein algebras.*

Lemma 4.2 (Compare [23, Theorem 5.2]) *Let A be a finite-dimensional k-algebra. Assume that A/A^H is an H^*-Galois extension and $A\#H/A$ is separable. Then A is a Gorenstein algebra if and only if $A\#H$ is a Gorenstein k-algebra. Furthermore, A is a Gorenstein algebra if and only if A^H is a Gorenstein k-algebra.*

Proof By assumption it is not hard to see from the proof of Theorem 5.2 in [23] that

$$\text{inj.dim}_A A = \text{inj.dim}_{A\#H} A\#H$$

$$\text{inj.dim} A_A = \text{inj.dim} A\#H_{A\#H}.$$

Since A is a finite-dimensional k-algebra, it follows that A is Gorenstein if and only if $A\#H$ is Gorenstein. Again by assumption we get from Lemma 3.4 that $A\#H$ is Gorenstein if and only if A^H is Gorenstein. This completes the proof. □

Now we will give a criterion for $\Lambda = \begin{pmatrix} A^H & A \\ 0 & A\#H \end{pmatrix}$ being Gorenstein.

Theorem 4.3 *Let A be a finite-dimensional k-algebra. Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then $\Lambda = \begin{pmatrix} A^H & A \\ 0 & A\#H \end{pmatrix}$ is a Gorenstein artin algebra if and only if A is a Gorenstein algebra.*

Proof By assumption we get from [23, Proposition 2.3 and 2.1] that A is a projective right $A\#H$-module. It follows that A is a projective left A^H-module by Lemma 3.4. By Lemmas 4.1 and 4.2 we get that Λ is a Gorenstein artin algebra if and only if A is a Gorenstein algebra. □

Lemma 4.4 [26, Corollary 1.7] *Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ ba an artin algebra, where A and B are Gorenstein algebras, and $M =_A M_B$ an A-B-bimodule. If Λ is Gorenstein, then $\left(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \phi\right)$ is a Gorenstein-projective Λ-module if and only if $\phi : M \otimes_B Y \to X$ is an A-monomorphism, ${}_A\mathrm{Coker}\phi$ is a Gorenstein-projective A-module, and ${}_BY$ is a Gorenstein-projective B-module.*

Next we will describe finitely-generated Gorenstein-projective modules over $\left(\begin{smallmatrix} A^H & A \\ 0 & A\#H \end{smallmatrix}\right)$.

Theorem 4.5 *Let A be a finite-dimensional Gorenstein k-algebra, and $\Lambda = \left(\begin{smallmatrix} A^H & A \\ 0 & A\#H \end{smallmatrix}\right)$. Let A/A^H be an H^*-Galois extension and $A\#H/A$ be separable. Then $\left(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \phi\right)$ is a Gorenstein-projective Λ-module if and only if $\phi : A \otimes_{A\#H} Y \to X$ is an A^H-monomorphism, ${}_A(A \otimes_{A^H} \mathrm{Coker}\phi)$ and ${}_AY$ are Gorenstein-projective A-modules. In particular, if Λ is CM-finite, then A is CM-finite.*

Proof By assumption we get from Lemma 4.2 that $A\#H$ and A^H are Gorenstein k-algebras. Again by assumption we get from Theorem 4.3 that Λ is a Gorenstein artin algebra. By Lemma 4.4 we know that $\left(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \phi\right)$ is a Gorenstein-projective Λ-module if and only if $\phi : A \otimes_{A\#H} Y \to X$ is an A^H-monomorphism, $\mathrm{Coker}\phi$ is a Gorenstein-projective A^H-module and Y is a Gorenstein-projective $A\#H$-module. By Lemma 3.4 we know that $\mathrm{Coker}\phi$ is a Gorenstein-projective A^H-module if and only if $A \otimes_{A^H} \mathrm{Coker}\phi$ is a Gorenstein-projective $A\#H$-module. By the proof of Lemma 3.2 we know that $A \otimes_{A^H} \mathrm{Coker}\phi$ and Y are Gorenstein-projective $A\#H$-modules if and only if ${}_A(A \otimes_{A^H} \mathrm{Coker}\phi)$ and ${}_AY$ are Gorenstein-projective A-modules. Thus $\left(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \phi\right)$ is a Gorenstein-projective Λ-module if and only if $\phi : A \otimes_{A\#H} Y \to X$ is an A^H-monomorphism, ${}_A(A \otimes_{A^H} \mathrm{Coker}\phi)$ and ${}_AY$ are Gorenstein-projective A-modules. □

Next we will show that Hopf extensions can induce recollements between categories $A\#H$-Mod and A^H-Mod.

Theorem 4.6 *Let A/A^H be an H^*-extension having a trace 1 element. Let J be the localizing subcategory of $A\#H$-*Mod *consisting of $A\#H$-modules M such that $A \otimes_{A\#H} M = 0$. Then we have a recollement*

$$\begin{tikzcd}[column sep=huge]
J \arrow[r] & A\#H\text{-Mod} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "A\otimes_{(A\#H)}-"] & A^H\text{-Mod} \arrow[l, bend right, swap, "{}_{(A\#H)}A\otimes_{A^H}-"] \arrow[l, bend left, "\mathrm{Hom}_{A^H}(A_{A\#H},-)"]
\end{tikzcd}$$

Proof Since A/A^H is an H^*-extension having a trace 1 element, it follows from Theorem 2.1 and Corollary 2.2 in [23] that $({}_{(A\#H)}A \otimes_{A^H} -,\ {}_{A^H}A \otimes_{(A\#H)} -)$ and $({}_{A^H}A \otimes_{(A\#H)} -,\ \mathrm{Hom}_{A^H}(A_{A\#H}, -))$ are adjoint pairs, also functors ${}_{(A\#H)}A \otimes_{A^H} -$ and $\mathrm{Hom}_{A^H}(A_{A\#H}, -)$ are fully faithful. By the assumption on J we know that $A\#H$-Mod$/J$ is equivalent to A^H-Mod. Thus we have the following recollement

$$\begin{tikzcd}
J \arrow[r] & A\#H\text{-Mod} \arrow[l, shift right=4] \arrow[l, shift left=4] \arrow[r, "A\otimes_{(A\#H)}-"] & A^H\text{-Mod} \arrow[l, shift right=4, "{}_{(A\#H)}A\otimes_{A^H}-"'] \arrow[l, shift left=4, "\mathrm{Hom}_{A^H}(A_{A\#H},-)"]
\end{tikzcd}$$

□

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References

1. Auslander, M., Bridger, M.: Stable Module Theory. Mem. Amer. Math. Soc., vol. 94., Amer. Math. Soc. Providence, RI (1969)
2. Auslander, M., Reiten, I., Smalø, S.O.: Representation Theory of Artin Algebras. Cambridge Studies in Adv. Math., vol. 36. Cambridge Univ. Press (1995)
3. Avramov, L.L., Martsinkovsky, A.: Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. Proc. Lond. Math. Soc. **85**(3), 393–440 (2002)
4. Beligiannis, A.: Cohen–Macaulay modules, (co)tosion pairs and virtually Gorenstein algebras. J. Algebra **288**(1), 137–211 (2005)
5. Beligiannis, A.: On algebras of finite Cohen–Macaulay type. Adv. Math. **226**, 1973–2019 (2011)
6. Bruns, W., Herzog, J.: Cohen–Macaulay rings, revised edition. Cambridge Studies in Advanced Mathematica, vol. 39. Cambridge University Press, Cambridge (1998)
7. Chen, X.W.: An Auslander-type result for Gorenstein-projective modules. Adv. Math. **218**, 2043–2050 (2008)
8. Christensen, L.W., Frankild, A., Holm, H.: On Gorenstein projective, injective and flat dimensions-a functorial description with applications. J. Algebra **302**(1), 231–279 (2006)
9. Rafael, M.D.: Separable functors revisited. Commun. Algebra **18**, 1445–1459 (1990)
10. Doi, Y.: Algebras with total integrals. Commun. Algebra **13**, 2137–2159 (1985)
11. Enochs, E.E., Garcia, J.J., del Rio, A.: When does R Gorenstein imply R^G Gorenstein. J. Algebra **182**, 561–576 (1996)
12. Enochs, E.E., Jenda, O.M.G., Gorenstein injective and projective modules. Math. Z. **220**(4), 611–633 (1995)
13. Enochs, E.E., Jenda, O.M.G.: Relative homological algebra. De Gruyter Exp. Math., vol. 30. Walter De Gruyter Co (2000)
14. Gao, N., Zhang, P.: Gorenstein derived categories. J. Algebra **323**(7), 2041–2057 (2010)
15. Happel, D.: On Gorenstein algebras. In: Representation Theory of Finite Groups and Finite-dimensional Algebras. Prog. Math., vol. 95, pp. 389–404. Birkhüser, Basel (1991)
16. Holm, H.: Gorenstein homological dimensions. J. Pure Appl. Algebra **189**(1–3), 167–193 (2004)
17. Körrer, H.: Cohen–Macaulay modules on hpersurface singularities. Invent. Math **88**(1), 153–164 (1987)
18. Kreimer, H., Takeuchi, M.: Hopf algebras and Galois extensions. Indiana Univ. Math. J. **30**, 675–692 (1981)
19. López-Ramos, J.A.: Gorenstein injective and projective modules and actions of finite-dimensional Hopf algebras. Ark. Math. **46**, 349–361 (2008)

20. Li, Z.W., Zhang, P.: Gorenstein algebras of finite Cohen–Macaulay type. Adv. Math. **223**(2), 728–734 (2010)
21. Montgomery, S.: Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, vol. 82. Amer. Math. Soc., Providence, RI (1993)
22. Năstăsescu, C., Van den Bergh, M., Van Oyaestaeyen, F.: Separable functors applied to graded rings. J. Algebra **123**, 397–413 (1989)
23. Van Oyaestaeyen, F., Xu, Y.H., Zhang, Y.H.: Inductions and coinductions for Hopf extensions. Sci. China. (Ser. A) **39**(3), 246–263 (1996)
24. Ringel, C.M.: Tame algebras and integral quadratic forms. Lecture Notes in Math. vol. 1099. Springer-Verlag (1984)
25. Xi, C.C.: Adjoint functors and representation dimensions. Acta Math. Sinica **22**(2), 625–640 (2006)
26. Xiong, B.L., Zhang, P.: Gorenstein–projective modules over triangular matrix Artin algebras. Preprint (2009)