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Characters of Representations and Paths in $\mathfrak{H}_{\mathbb{R}}^*$

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Introduction

The aim of this note is to give an introduction to a new combinatorial tool in representation theory, the path model. The model is an extension of the usual weight theory of representations of a connected complex semisimple Lie group G, it can also be viewed as a generalization of the classical Young tableaux theory for the group $SL_n(\mathbb{C})$ to arbitrary connected complex semisimple Lie groups.

To construct objects like the tableaux in such a general setting, we consider piecewise linear paths $\pi : [0,1] \to \Lambda_{\mathbb{R}}$ in the real span of the weight lattice Λ of G. The idea is to associate to an irreducible representation V a set of paths $\mathbb{B}$ starting in the origin and ending in an integral weight, such that the character $\operatorname{Char} V$ of V reads as the sum $\sum e^{\eta(1)}$ over all paths in $\mathbb{B}$. The advantage of this approach, in comparison with the usual weight theory, is that we can speak of the "individual" contribution of a path to the character. This makes it possible to avoid the alternating sums in classical formulas like Steinberg's tensor product decomposition formula. In fact, as a consequence of this theory, we get a very simple decomposition formula for tensor products of representations, which can be seen as a generalization of the classical Littlewood-Richardson formula.

The main motivation for the construction of the path model came from the observation [13] of a connection between the work of Lakshmibai and Seshadri on standard monomial theory (see for example [9] for an overview of this work related to the geometry of Schubert varieties), and the work of Kashiwara on crystal bases of representations of quantum groups (see for example [4,5,6,7], or a book on quantum groups, for example [1,3,16]).

The model itself is a purely elementary construction; only some basic knowledge in weight theory and in the combinatoric of Weyl groups is required. In this note we consider only complex semisimple Lie algebras, though (with the appropriate reformulation) the statements hold more generally for arbitrary symmetrizable Kac-Moody algebras (see [12] for an overview).

The restriction to the case of complex semisimple Lie algebras enables us to give complete proofs of the most important statements in this note. The proofs given here are different from those in [10,11] and, I hope, much simpler.

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1. The Paths

The paths considered in this note live in the real span of the weight lattice of a semisimple complex Lie algebra. We assume for simplicity always that the paths are piecewise linear, but it is easy to see (by approximation by piecewise linear paths) that the theorems stated in the following hold also for piecewise smooth or, more generally, rectifiable paths.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, fix a compact form $\mathfrak{g}_o \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{H} \subset \mathfrak{g}$ such that $\mathfrak{H} = \mathfrak{H}_0 \oplus i\mathfrak{H}_0$, where $\mathfrak{H}_0 := \mathfrak{H} \cap \mathfrak{g}_o$. Recall that the restriction of the Killing form $(\cdot\,,\cdot)$ to $\mathfrak{H}_\mathbb{R} := i\mathfrak{H}_0$ is a positive definite form. Denote by $\Delta \subset \mathfrak{H}_\mathbb{R}^*$ the set of roots and by $\Lambda \subset \mathfrak{H}_\mathbb{R}^*$ the lattice of integral weights. Corresponding to the choice of a set of positive roots Δ^+, let Λ^+ be the set of dominant weights.

Definition 1.1. A **piecewise linear path** in $\mathfrak{H}_\mathbb{R}^*$ is a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{H}_\mathbb{R}^*$. We consider two paths as identical if there exists a piecewise linear, nondecreasing, continuous, surjective map $\phi : [0,1] \to [0,1]$ such that $\pi = \eta \circ \phi$. Denote by Π the set of all piecewise linear paths such that $\pi(0) = 0$ and $\pi(1) \in \Lambda$.

Example 1.2. For $\lambda \in \mathfrak{H}_\mathbb{R}^*$ set $\pi_\lambda(t) := t\lambda$. We often write just λ for the path π_λ. Then $\pi_\lambda \in \Pi$ if and only if $\lambda \in \Lambda$.

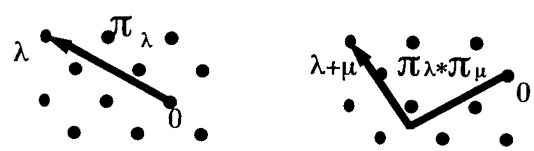


FIGURE 1. The straight line and the concatenation of paths

Example 1.3. Let π_1, π_2 be two piecewise linear paths starting in 0. By the **concatenation** $\pi := \pi_1 * \pi_2$ of the paths π_1 and π_2, we mean the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t) & \text{if } 0 \le t \le 1/2, \\ \pi_1(1) + \pi_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Example 1.4. The piecewise linear paths can be identified with certain finite sequences of elements in $\mathfrak{H}_\mathbb{R}^*$: Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$ be such a finite sequence and set $\pi_{\underline{\lambda}} := \lambda_1 * \dots * \lambda_s$. This is the path that joins successively the weights $0, \lambda_1, \lambda_1 + \lambda_2$, etc. Of course, $\pi_{\underline{\lambda}} \in \Pi$ if and only if $\lambda_1 + \dots + \lambda_s \in \Lambda$. Note that, up to reparametrization, all paths in Π are of this form.

Example 1.5. For $\mathfrak{g} = \mathfrak{sl}_n$ let $\mathfrak{H}$ be the subalgebra of diagonal matrices of trace zero. A classical combinatorial tool in the representation theory of $\mathfrak{sl}_n$ is the semi-standard Young tableaux. These tableaux can be identified with certain paths as follows.

Fix a partition $p = (a_1, \dots, a_n)$; i.e., $a_1 \ge \dots \ge a_n \ge 0$ is a nonincreasing sequence of nonnegative integers. Recall that the **Young diagram** of shape p is a left justified sequence of rows of boxes with a_1 boxes in the first row, a_2 in the second, etc. A **semi-standard Young tableau T** of shape p is a filling of the boxes with numbers $1, \dots, n$ such that the entries are not decreasing in the rows and strictly increasing in the columns.

Let $\epsilon_i : \mathfrak{H} \to \mathbb{C}$ be the projection of a diagonal matrix onto its i-th entry. For a given tableau $\mathbf{T}$ let $(i_1, \dots, i_N)$ be the entries of the boxes, where we read the entries columnwise (from the top to the bottom of each column), starting with the right most column. We associate to $\mathbf{T}$ the path $\pi_{\mathbf{T}} := \epsilon_{i_1} * \dots * \epsilon_{i_N}$.

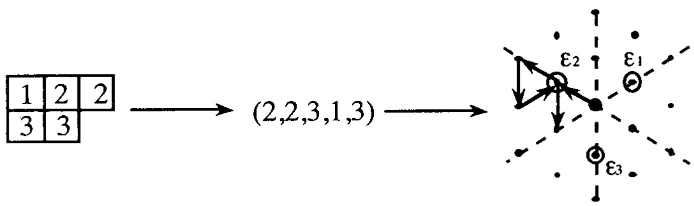


FIGURE 2. A tableau and its path

Example 1.6. We present two different procedures to create new paths from given ones: Fix $\eta \in \Pi$. By the **dual** path $\eta^* \in \Pi$ we mean the path defined by $\eta^*(t) := \eta(1-t) - \eta(1)$. By the **stretching** of paths we mean the multiplication of paths: For $n \in \mathbb{N}$ let $n\eta \in \Pi$ be the path defined by $(n\eta)(t) := n\eta(t)$.

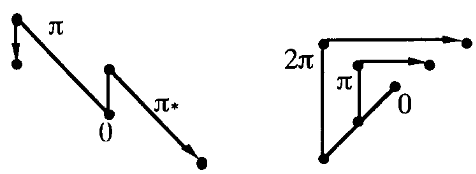


FIGURE 3. The dual path and the stretching of paths

Example 1.7. Here is an approach to the paths using the language of loop groups: Let G be a simply connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. Fix a maximal compact subgroup K such that $\operatorname{Lie} K = \mathfrak{g}_o$, and let $H \subset K$ be a maximal torus such that $\operatorname{Lie} H = \mathfrak{H}_0$. The compact torus $\check{H} := \mathfrak{H}_{\mathbb{R}}/\Lambda$ is a maximal torus of the so-called dual group $\check{K}$ of K. Fix $\eta \in \Pi$, the map $t \mapsto \overline{\eta(t)} \in \check{H}$ induces a loop $\exp \eta : S^1 \to \check{H}$. This correspondence provides a bijection between the set of piecewise linear paths starting in 0 and ending in an integral weight, and the group of "piecewise linear" loops in $\check{H}$ at the identity.

2. The Root Operators

To obtain combinatorial character formulas and multiplicity formulas, we define **lowering** and **raising** operators f_α, e_α for each simple root. Let $(\cdot, \cdot)$ be the Killing-form. The definition of the operators is elementary; it is a cutting and gluing procedure. For convenience we introduce a special element θ, which is not a path but which has the abstract properties $\theta * \pi = \pi * \theta := \theta$ for all $\pi \in \Pi$. Fix $\pi \in \Pi$, let $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ be the co-root of α, and denote by h_α the function:

$$h_\alpha : [0,1] \to \mathbb{R}, \qquad t \mapsto (\pi(t), \alpha^\vee).$$

Let m_α be the minimal value attained by this function. We define nondecreasing functions $l, r : [0,1] \to [0,1]$:

$l(t) := \min\{1, h_\alpha(s) - m_\alpha \mid t \leq s \leq 1\}$, $r(t) := 1 - \min\{1, h_\alpha(s) - m_\alpha \mid 0 \leq s \leq t\}$.

Note that $l(t) = 0$ for $0 \leq t \leq s$, where s is maximal such that $h(s) = m_\alpha$, and $r(t) = 1$ for $s' \leq t \leq 1$, where s' is minimal such that $h(s) = m_\alpha$.

Definition 2.1.

$$e_\alpha \pi := \begin{cases} t \mapsto \pi(t) + r(t)\alpha & \text{if } r(0) = 0; \\ \theta & \text{otherwise.} \end{cases}$$

Definition 2.2.

$$f_\alpha \pi := \begin{cases} t \mapsto \pi(t) - l(t)\alpha & \text{if } l(1) = 1; \\ \theta & \text{otherwise.} \end{cases}$$

We set $e_\alpha \theta = f_\alpha \theta := \theta$. For a path π let $s_\alpha(\pi)$ be defined by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$. If we think of a path as a concatenation of "smaller" paths $\pi = \pi_1 * \ldots * \pi_r$, then we can view e_α and f_α as operators that replace some of the π_j by $s_\alpha(\pi_j)$:

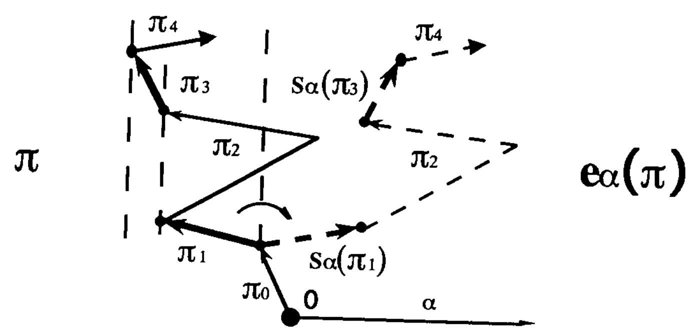


FIGURE 4. The part of $e_\alpha(\pi)$ different from π is drawn as a dashed line

Example 2.3. Suppose $\mathfrak{g} = \mathfrak{sl}_3$ and β is the highest root. The paths obtained from $\pi_\beta : t \mapsto t\beta$ by applying the operators f_α, e_α are the paths $\pi_\gamma(t) := t\gamma$, where γ is an arbitrary root, and for the two simple roots the paths $\eta_\alpha := \pi_{-\alpha/2} * \pi_{\alpha/2}$. The arrow $\xrightarrow{i}$ indicates in the following picture that the operator f_{α_i} transforms the given path into the next one.

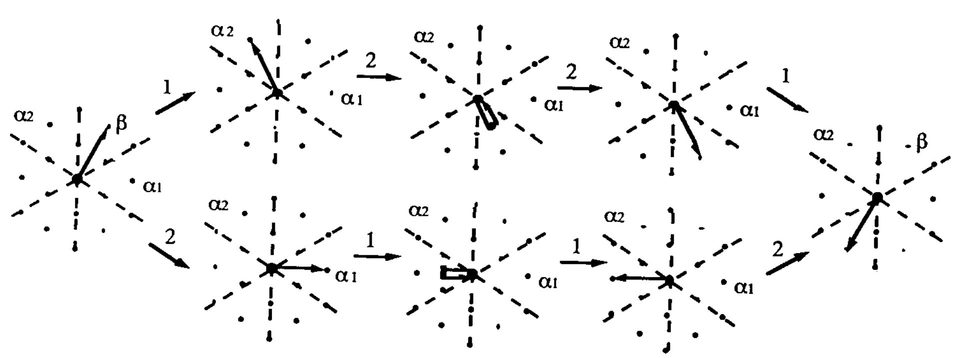


FIGURE 5. The paths generated by $\eta : t \to t\beta$

3. Some Simple Properties.

We list below some simple properties of the operators e_α, f_α that are easy to verify. These properties have also been the "guideline" for the definition of the operators; i.e., the operators are completely determined by these properties. Recall that we identify the weight lattice Λ with the paths of the form $t \mapsto t\lambda$.

i) MOVING, STRETCHING AND DUALIZING. The operators preserve the length of a path and move the endpoint by $\pm\alpha$. Whenever $e_\alpha\eta \neq \theta$ for some $\eta \in \Pi$, then $f_\alpha(e_\alpha\eta) = \eta$ and $(e_\alpha\eta)^* = f_\alpha(\eta^*)$. Similarly, if $f_\alpha\eta \neq \theta$ for some $\eta \in \Pi$, then $e_\alpha(f_\alpha\eta) = \eta$ and $(f_\alpha\eta)^* = e_\alpha(\eta^*)$. Further, the operators are compatible with stretching; i.e., for all $\eta \in \Pi$, we have

$$k(f_\alpha\eta) = f_\alpha^k(k\eta) \quad \text{and} \quad k(e_\alpha\eta) = e_\alpha^k(k\eta).$$

ii) α-STRINGS. For $\eta \in \Pi$ let n be maximal such that $f_\alpha^n\eta \neq \theta$, let m be maximal such that $e_\alpha^m\eta \neq \theta$, and let m_α be the minimum of the function h_α. Then

$$n - m = (\eta(1), \alpha^\vee), \quad m \leq |m_\alpha| < m + 1, \quad n \leq |(\eta(1), \alpha^\vee) - m_\alpha| < n + 1.$$

iii) REFLECTIONS. We define an action of the simple reflection s_α on Π. For $\eta \in \Pi$ set $k := (\eta(1), \alpha^\vee)$. Then:

$$s_\alpha(\eta) := f_\alpha^k(\eta) \text{ if } k > 0 \quad \text{and} \quad s_\alpha(\eta) := e_\alpha^{-k}(\eta) \text{ otherwise.}$$

Note that $s_\alpha^2 = id$, and the restriction of the action to $\Lambda \subset \Pi$ yields the usual action of the simple reflection on the weight lattice.

Remark 3.1. Let $\mathbb{B} \subset \Pi$ be a finite subset, such that $\mathbb{B} \cup \{\theta\}$ is stable under the root operators. Then (iii) implies that its character $\operatorname{Char}\mathbb{B} := \sum_{\eta\in\mathbb{B}} e^{\eta(1)}$ is W-stable.

iv) CONCATENATION. Let $\pi := \lambda_1 * \ldots * \lambda_r$ be such that $\lambda_1, \ldots, \lambda_r$ are integral weights. Set $a_i := (\lambda_1 + \ldots + \lambda_{i-1}, \alpha^\vee)$ and $a_0 := 0$. For the minimum m_α of the a_i fix p minimal with $m_\alpha = a_p$ and q maximal with $m_\alpha = a_q$. If $p = 0$, then $e_\alpha\pi = \theta$, and if $q = r$, then $f_\alpha\pi = \theta$. Otherwise we get for $x \leq \min\{a_i - m_\alpha \mid 0 \leq i \leq p-1\}$ and $y \leq \min\{a_i - m_\alpha \mid q+1 \leq i \leq r\}$:

$$e_\alpha^x\pi = \lambda_1 * \ldots * (e_\alpha^x\lambda_p) * \ldots * \lambda_r, \qquad f_\alpha^y\pi = \lambda_1 * \ldots * (f_\alpha^y\lambda_{q+1}) * \ldots * \lambda_r.$$

Let $\widetilde{\Pi}$ be the set of all piecewise linear paths η as before, but with a fixed parametrization. We define a distance on $\widetilde{\Pi}$:

$$d(\eta, \pi) := \max_{t\in[0,1]}\{\|\eta(t) - \pi(t)\|\}.$$

It is easy to see that if two paths are "close" with respect to $d(\cdot\,,\cdot)$, then the functions h_α and the functions l and r are close. More precisely:

v) CONTINUITY. The operators are "continuous": $\exists c \in \mathbb{R}$ (depending only on $\mathfrak{g}$) such that if $d(\eta, \pi) < y$ and $e_\alpha(\eta), e_\alpha(\pi) \neq \theta$ then $d(e_\alpha(\eta), e_\alpha(\pi)) < cy$, and if $f_\alpha(\eta), f_\alpha(\pi) \neq \theta$, then $d(f_\alpha(\eta), f_\alpha(\pi)) < cy$.

Proposition 3.2. *If $\{f'_\alpha, e'_\alpha \mid \alpha$ a simple root$\}$ is a set of maps $\Pi \to \Pi \cup \{\theta\}$ satisfying the properties* (i) *to* (v)*, then $f_\alpha = f'_\alpha$ and $e_\alpha = e'_\alpha$ for all simple roots.*

PROOF. By a rational path $\eta \in \Pi$ we mean a path such that all turning points are rational weights, in other words: $n\eta = \lambda_1 * \ldots * \lambda_r$ for some $\lambda_1, \ldots, \lambda_r \in \Lambda$ and some $n \in \mathbb{N}$. The properties (ii) and (v) imply that the action on Π is a "continuous" extension of the action on the rational paths: Let m_α be as in (ii). For any given $\epsilon > 0$ we can approximate $\eta \in \Pi$ by a rational path π such that $d(\eta, \pi) < \epsilon$. Further, if η is such that $m_\alpha \geq k$ or $m_\alpha < k$ for some $k \in \mathbb{Z}$, then we can choose a rational approximation π with the same property. It is now easy to see that the definition of the operators is the "continuous" extension of the action on the rational paths.

So it is sufficient to consider rational paths and to prove $f_\alpha = f'_\alpha$ (property (i)). Fix $\lambda \in \Lambda$ such that $a := (\lambda, \alpha^\vee) > 0$. Now (iii) implies ${f'}^a_\alpha(\lambda) = s_\alpha(\lambda)$, and if $k \leq a$, then (iv) implies ${f'}^{ka}_\alpha(a\lambda) = (k{f'}^a_\alpha(\lambda)) * ((a-k)\lambda)$. It follows that

$$ {f'}^k_\alpha(\lambda) = \big(\frac{k}{a} s_\alpha(\lambda)\big) * \big(\frac{a-k}{a}\lambda\big), \text{ and hence: } {f'}^k_\alpha(\lambda) = f^k_\alpha(\lambda). $$

If $\eta = \lambda_1 * \ldots * \lambda_r$ for some $\lambda_1, \ldots, \lambda_r \in \Lambda$, then the turning points are integral weights and the local minima of the function h_α are integers. Now (iv) and the definition of f_α imply that the local minima of the function $t \mapsto (f_\alpha\eta(t), \alpha^\vee)$ are integers. Since ${f'}^k_\alpha(\lambda) = f^k_\alpha(\lambda)$, it follows by (iv) that ${f'}^k_\alpha(\eta) = f^k_\alpha(\eta)$ and the local minima of $t \mapsto (f'_\alpha\eta(t), \alpha^\vee)$ are integers. Using stretching and (iv), it is now easy to see by induction that

$$ {f'}^k_\alpha\eta = ({f'}^{k_1}_\alpha\lambda_1) * \ldots * ({f'}^{k_r}_\alpha\lambda_r) = (f^{k_1}_\alpha\lambda_1) * \ldots * (f^{k_r}_\alpha\lambda_r) = f^k_\alpha\eta. $$

Let $\eta \in \Pi$ be a rational path and fix n such that $n\eta = \lambda_1 * \ldots * \lambda_r$ for some $\lambda_1, \ldots, \lambda_r \in \Lambda$. Since $f_\alpha\eta = \frac{1}{n}f^n_\alpha(\lambda_1 * \ldots * \lambda_r) = \frac{1}{n}{f'}^n_\alpha(\lambda_1 * \ldots * \lambda_r) = {f'}_\alpha(\eta)$, the operators f_α, e_α and f'_α, e'_α coincide on rational paths, and hence on all paths.

4. A First Character Formula

Denote by $\Pi^+ \subset \Pi$ the set of paths η such that $\operatorname{Im}\eta$ is contained in the dominant Weyl chamber C, and let Π^+_0 be the set of paths such that $\operatorname{Im}\eta$ is in the interior of C (for $t > 0$). Let Λ^+ be the set of dominant weights and denote by $\rho \in \Lambda^+$ half the sum of the positive roots. If $\mathbb{B} \subset \Pi$ is a finite subset such that $\mathbb{B} \cup \theta$ is stable under the root operators e_α, f_α, then we have already seen that its character $\operatorname{Char}\mathbb{B} := \sum_{\eta\in\mathbb{B}} e^{\eta(1)}$ is stable under the action of W. In fact, $\operatorname{Char}\mathbb{B}$ can be computed by the following path version of Weyl's character formula:

Proposition 4.1.

$$ \Big(\sum_{w\in W} \operatorname{sgn}(w)e^{w(\rho)}\Big)\operatorname{Char}\mathbb{B} = \sum_{\substack{\eta\in\mathbb{B} \\ \rho*\eta\in\Pi^+_0}} \Big(\sum_{w\in W}\operatorname{sgn}(w)e^{w(\rho+\eta(1))}\Big) $$

Corollary 1. *For $\mu \in \Lambda^+$ let V_μ be the corresponding irreducible representation of $\mathfrak{g}$. Then*

$$ \operatorname{Char}\mathbb{B} = \sum_{\substack{\eta\in\mathbb{B} \\ \rho*\eta\in\Pi^+_0}} \operatorname{Char} V_{\eta(1)} $$

PROOF OF THE PROPOSITION. Both sides are stable under the Weyl group; so it is sufficient to compare the coefficients of the terms corresponding to dominant weights; i.e., we have to prove for $\Omega := \{(w,\pi) \mid w \in W, \pi \in \mathbb{B}, w(\rho)+\pi(1) \in \Lambda^+\}$:

$$\sum_{(w,\pi)\in\Omega} \operatorname{sgn}(w) e^{w(\rho)+\pi(1)} = \sum_{\eta\in\mathbb{B},\, \rho*\eta\in\Pi_0^+} e^{\rho+\eta(1)}.$$

Let Ω_0 be the set of pairs $(w,\pi) \in \Omega$ such that w is the identity and $\rho * \pi \in \Pi_0^+$. Set $\Omega' := \Omega - \Omega_0$. To prove the proposition we have to show:

$$\sum_{(w,\pi)\in\Omega'} \operatorname{sgn}(w) e^{w(\rho)+\pi(1)} = 0. \tag{$*$}$$

We will define an involution $\varphi : \Omega' \to \Omega'$ such that $\varphi(w,\pi) = (w',\pi')$ has the property: $\operatorname{sgn}(w) = -\operatorname{sgn}(w')$ and $w(\rho)+\pi(1) = w'(\rho)+\pi'(1)$. This implies obviously $(*)$ and hence the proposition.

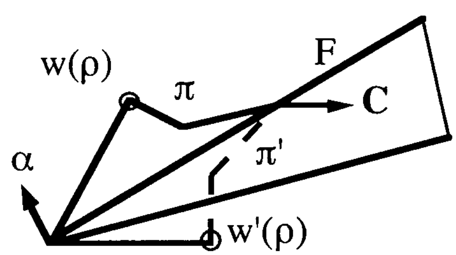


FIGURE 6. The involution φ

The construction of the involution: Suppose first $(w,\pi) \in \Omega'$ is such that w is not the identity. Since $w(\rho)+\pi(1) \in \Lambda^+$, the path $w(\rho)*\pi$ has to meet at least once a proper face of the dominant Weyl chamber C. If w is the identity, then $\rho * \pi$ also has to meet a proper face F of C. (The pair would otherwise be an element of Ω_0.)

For a proper face F of C denote by $\Omega'(F)$ the set of pairs $(w,\pi) \in \Omega'$ that meet F as the last face. More precisely: $w(\rho)*\pi$ meets F, and if $t_0 \in [0,1]$ is maximal with the property such that $w(\rho)+\pi(t_0) \in F$, then $w(\rho)+\pi(t_0)$ is in the interior of F, and $w(\rho)+\pi(t)$ is in the interior of C for all $t > t_0$.

The set Ω' is obviously the disjoint union of the $\Omega'(F)$; so it is sufficient to define the involution for such an $\Omega'(F)$. Let α be a simple root orthogonal to F. For $(w,\pi) \in \Omega'(F)$ set $n := (w(\rho),\alpha^\vee)$. Note that $n \neq 0$.

If $n > 0$, then the minimum m_α of the function $t \mapsto (\pi(t),\alpha^\vee)$ is at least $-n$ (since $w(\rho)*\pi$ meets F for some value of $t > 0$). It follows that $e_\alpha^n(\eta) \neq \theta$ and $w(\rho)+\pi(1) = s_\alpha w(\rho) + e_\alpha^n \pi(1)$. Further, if $t_0 \in [0,1]$ is maximal with the property that $w(\rho)*\pi(t_0) \in F$, then $w(\rho)+\pi(t) = s_\alpha w(\rho) + e_\alpha^n \pi(t)$ for all $t \geq t_0$, and hence

$$\varphi(w,\rho) := (s_\alpha w, e_\alpha^n(\pi)) \in \Omega'(F).$$

Similarly, if $n < 0$, then $f_\alpha^{|n|}(\eta) \neq \theta$, $w(\rho)+\pi(1) = s_\alpha w(\rho) + f_\alpha^{|n|}\pi(1)$, and

$$\varphi(w,\rho) := (s_\alpha w, f_\alpha^{|n|}(\pi)) \in \Omega'(F).$$

Property (i) in section 3 implies that φ is an involution, which finishes the proof.

5. Locally Integral Concatenations

The next aim is to describe the possible sets of paths $\mathbb{B}$ such that $\mathbb{B} \cup \{\theta\}$ is stable under the root operators. Of course, one is particulary interested in those sets such that Corollary 1 in section 4 provides a character formula for an irreducible representation V_λ. A good candidate for such a set is the following: Start with the path λ (recall, we identify the weight λ with the path $\pi_\lambda : t \mapsto t\lambda$), and let $\mathbb{B}_\lambda$ be the set of paths obtained from this line by applying the root operators. Since $\lambda \in \Pi^+$, it is evident that $\rho * \lambda \in \Pi_0^+$. So the character of V_λ will show up on the right side in the character formula for $\mathbb{B}_\lambda$. But, a priori, it is not at all evident that λ is the only path in $\mathbb{B}_\lambda$ with this property, and, even more important, so far it is not even clear that $\mathbb{B}_\lambda$ is a finite set.

To prove that $\mathbb{B}_\lambda$ has in fact these two properties, it turns out that it is much more natural to consider from the very beginning the following special class of paths: $\underline{\lambda} := \lambda_1 * \ldots * \lambda_r$, where the λ_i are rational dominant weights with the following properties: $\lambda := \lambda_1 + \ldots + \lambda_r \in \Lambda^+$, and for $1 \leq i \leq r-1$ there exist $p_i, q_i > 0$ with

$$\lambda_1 + \ldots + \lambda_{i-1} + p_i\lambda_i \in \Lambda^+, \quad q_i\lambda_{i+1} + \lambda_{i+2} + \ldots + \lambda_r \in \Lambda^+.$$

Examples of such paths are those of the form $\underline{\lambda} := \lambda_1 * \ldots * \lambda_r$, where $\lambda_1, \ldots, \lambda_r$ are dominant weights, or paths of the form $\underline{\lambda} := \lambda_1 * \lambda_2$, where $\lambda_1, \lambda_2 \in \Lambda_{\mathbb{Q}}^+$ are such that $\lambda := \lambda_1 + \lambda_2 \in \Lambda^+$. Let $\mathbb{P} \subset \Pi$ be the smallest set that contains all these paths and that has the property that $\mathbb{P} \cup \{\theta\}$ is stable under the operators e_α, f_α.

To give a more intrinsic description of $\mathbb{P}$, one associates to every turning point of a path a root system that "measures" the change of direction. We will show that the set $\mathbb{P}$ consists essentially of those paths for which all turning points are "integral points" for the associated root system.

Let "$\geq$" be the Bruhat order on W. If $\nu_1, \ldots, \nu_r \in \Lambda_{\mathbb{Q}}$ are rational weights, then let $\lambda_1, \ldots, \lambda_r \in \Lambda_{\mathbb{Q}}^+$ be the rational dominant weights such that $\nu_i \in W.\lambda_i$. We write:

$$\nu_1 \succeq \ldots \succeq \nu_r \iff \exists\, w_i \in W \text{ such that } \nu_i = w_i(\lambda_i) \text{ and } w_1 \geq \ldots \geq w_r.$$

Remark 5.1. Note that $\nu_1 \succeq \nu_2$ and $\nu_2 \succeq \nu_3$ does not necessarily imply $\nu_1 \succeq \nu_3$. Let ω_1, ω_2 be fundamental weights, and let α_1, α_2 be the corresponding simple roots. Set $\nu_1 := \omega_1$, $\nu_2 := \omega_2$, and $\nu_3 := s_{\alpha_1}(\omega_1)$. Then $\nu_1 \succeq \nu_2$ (choose $w_1 = w_2 = id$) and $\nu_2 \succeq \nu_3$ (choose $w_1 = w_2 = s_{\alpha_1}$), but, of course, $\nu_1 \not\succeq \nu_3$.

We use the length function $l(\cdot)$ on W also for the Weyl group orbits and cosets in W/W_μ: If $\nu \in W\mu$ for some $\mu \in \Lambda_{\mathbb{Q}}^+$, then let $\tau \in W/W_\mu$ be the unique element such that $\tau(\mu) = \nu$ and let $\tilde{\tau} \in W$ be the unique element of minimal length such that $\tilde{\tau}(\mu) = \nu$. We write then $l(\nu)$ and $l(\tau)$ for $l(\tilde{\tau})$. If β is positive root, then let $\beta^\vee = 2\beta/(\beta,\beta) \in \Phi^\vee$ be the dual root.

Suppose $\nu \succeq \mu$ are rational weights. Let $w \in W$ be the unique element of minimal length such that $w(\mu^+) = \mu$ for some $\mu^+ \in \Lambda_{\mathbb{Q}}^+$, and let $v \in W$ be the unique element of minimal length such that $v \geq w$ and $v(\nu^+) = \nu$ for some $\nu^+ \in \Lambda_{\mathbb{Q}}^+$ [2,9]. There exist positive roots $\beta_1, \ldots, \beta_r$ such that $w := s_{\beta_r} \ldots s_{\beta_1} v$ and

$$v_0 := v > v_1 := s_{\beta_1} v > \ldots > v_r := w, \quad l(v_0) = l(v_1) + 1 = \ldots = l(v_r) + r.$$

Definition 5.2. The root system $\Phi^\vee_{\nu,\mu} \subset \Phi^\vee$ spanned by the roots $\beta_1^\vee, \dots, \beta_r^\vee$ is called the **root system of the pair** (ν, μ).

Note that this root system is independent of the choice of the β_i: This is evident if $l(v) - l(w) \leq 1$. Otherwise one proceeds by induction on $l(v) - l(w)$ and $l(v)$: Let α be a simple root such that $s_\alpha v < v$. If $s_\alpha w < w$, then it is easy to see that from any given sequence of roots for the pair (v, w) one can construct a sequence for the pair $(s_\alpha v, s_\alpha v)$ such that:

$$\Phi^\vee_{s_\alpha(\nu), s_\alpha(\mu)} = s_\alpha(\Phi^\vee_{\nu,\mu}).$$

Since the first is independent of the choice of the positive roots (by induction), so is the latter. Suppose now $s_\alpha w > w$, so that $v > s_\alpha v \geq w$. It is easy to see that, from any given sequence of roots for the pair (v, w), one can construct a sequence for the pair $(s_\alpha v, w)$ such that $\Phi^\vee_{\nu,\mu}$ is spanned by $\alpha^\vee$ and $\Phi^\vee_{s_\alpha v, w}$. Again, by induction, the latter is independent of the choice of the positive roots, and so is the first.

Definition 5.3. Suppose $\nu_1, \dots, \nu_r$ are rational weights such that $\nu_1 + \dots + \nu_r$ is in Λ and $\nu_1 \succeq \dots \succeq \nu_r$. The path $\underline{\nu} := \nu_1 * \dots * \nu_r \in \Pi$ is called a **locally integral concatenation** if it satisfies the following conditions for all $i = 2, \dots, r$:

a) $(\nu_1 + \dots + \nu_{i-1}, \beta^\vee) \in \mathbb{Z}$ for all $\beta^\vee \in \Phi^\vee_{\nu_{i-1}, \nu_i}$,

b) If there exists no $t_i > 0$ such that $w(t_i \nu_i) = \nu_{i+1}$ for some $w \in W$, then there exist $p_i, q_i > 0$ such that $\nu_1 + \dots + \nu_{i-1} + p_i \nu_i$ and $q_i \nu_{i+1} + \nu_{i+2} + \dots + \nu_r$ are in Λ.

Example 5.4. If $\lambda \in \Lambda^+$ or $\lambda_1, \lambda_2 \in \Lambda^+_{\mathbb{Q}}$ are such that $\lambda_1 + \lambda_2 \in \Lambda^+$, then the paths λ and $\lambda_1 * \lambda_2$ are locally integral concatenations.

Remark 5.5. It is easy to see that the property of being a locally integral concatenation is independent of the chosen parametrization.

Lemma 5.6. *Let $\underline{\nu} = \nu_1 * \dots * \nu_r$ be a locally integral concatenation, and let α be a simple root. The local minima of the function $h_\alpha : t \mapsto (\underline{\nu}(t), \alpha^\vee)$ are integers.*

Proof. Let $s \in [0, 1]$ be such that h_α attains in s a local minimum. We may assume that $\underline{\nu}(s) = \nu_1 + \dots + \nu_i$ for some $0 \leq i \leq r$. If $i = r$ or $i = 0$, then $\underline{\nu}(s)$ is an integral weight and hence $h_\alpha(s) \in \mathbb{Z}$. Suppose now $1 \leq i \leq r - 1$. Since h_α attains a local minimum, we may further assume that $(\nu_i, \alpha^\vee) < 0$ and $(\nu_{i+1}, \alpha^\vee) \geq 0$. Let $v, w \in W$ for (ν_i, ν_{i+1}) be as in the definition of the associated root system. The first inequality implies $s_\alpha v < v$, and the latter implies $s_\alpha w > w$ by the minimality of w. So $v > s_\alpha v \geq w$ and hence $\alpha^\vee \in \Phi^\vee_{\nu_i, \nu_{i+1}}$, which implies $h_\alpha(s) \in \mathbb{Z}$.

Lemma 5.7. *If $\underline{\nu} = \nu_1 * \dots * \nu_r$ is a locally integral concatenation such that $\rho * \underline{\nu}$ is in the interior of the dominant Weyl chamber (for $t > 0$), then $\nu_1, \dots, \nu_r \in \Lambda^+_{\mathbb{Q}}$.*

Proof. Let α be a simple root. If $\rho * \underline{\nu}$ is in the interior of the dominant Weyl chamber (for $t > 0$), then the minimum of the function $h_\alpha : t \mapsto (\underline{\nu}(t), \alpha^\vee)$ is > -1. But the minimum is an integer and hence is equal to 0; i.e., the image of $\underline{\nu}$ is in the dominant Weyl chamber. Suppose one of the $\nu_i \notin \Lambda^+_{\mathbb{Q}}$. We may assume that i is minimal with this property.

Let $w_1 \geq \dots \geq w_r$ be such that $\nu_j = w_j(\nu_j^+)$, where ν_j^+ is a rational dominant weight. Choose a simple root α such that $(\nu_i, \alpha^\vee) < 0$. Note that this implies $s_\alpha w_i < w_i$ and hence $w_i \geq s_\alpha$. Since $w_1 \geq \dots \geq w_i$, we know that $w_j \geq s_\alpha$ for

$1 \leq j \leq i$. But the ν_j are dominant. So $w_j \in W_{\nu_j}$ for $1 \leq j < i$ and hence $(\nu_j, \alpha^\vee) = 0$ for $1 \leq j < i$. But this would imply $(\nu_1 + \ldots + \nu_i, \alpha^\vee) = (\nu_i, \alpha^\vee) < 0$, in contradiction to the fact that the path is in the dominant Weyl chamber.

Proposition 5.8. *If $\underline{\nu} = \nu_1 * \ldots * \nu_r$ is a locally integral concatenation and $e_\alpha \underline{\nu} \neq \theta$, then (after reparametrization) $\exists\, i, k$ such that $(\nu_j, \alpha^\vee) < 0$ for $i \leq j \leq k$ and*

$$e_\alpha \underline{\nu} = \nu_1 * \ldots * \nu_{i-1} * s_\alpha(\nu_i) * \ldots * s_\alpha(\nu_k) * \nu_{k+1} * \ldots * \nu_r.$$

If $f_\alpha \underline{\nu} \neq 0$, then $\exists\, i, k$ such that $(\nu_j, \alpha^\vee) > 0$ for $i \leq j \leq k$ and

$$f_\alpha \underline{\nu} = \nu_1 * \ldots * \nu_{i-1} * s_\alpha(\nu_i) * \ldots * s_\alpha(\nu_k) * \nu_{k+1} * \ldots * \nu_r.$$

Proof. We consider only f_α. The proof for e_α is similar. Let s be maximal such that h_α attains in s its minimum m_α, and let i be such that $\underline{\nu}(s) = \nu_1 + \ldots + \nu_{i-1}$. Then $(\nu_{i-1}, \alpha^\vee) \leq 0$ and $(\nu_i, \alpha^\vee) > 0$. Let $p > s$ be minimal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \geq p$. We may assume that $\underline{\nu}(p) = \nu_1 + \ldots + \nu_k$. Since the local minima of h_α are integers, the conditions on the choice of s and p imply that h_α is a nondecreasing function on the interval $[s, p]$. Thus $(\nu_j, \alpha^\vee) \geq 0$ for $i \leq j \leq k$.

Suppose now $(\nu_j, \alpha^\vee) = 0$. By the choice of i, k we know that $i < j < k$. So we may choose j such that $(\nu_{j+1}, \alpha^\vee) > 0$. If $\nu_j \neq w(x\nu_{j+1})$ for some $w \in W$ and $x > 0$, then there exists a $q > 0$ such that $\nu_1 + \ldots + \nu_{j-1} + q\nu_j$ is an integral weight. But this would imply that:

$$(\nu_1 + \ldots + \nu_{j-1} + \nu_j, \alpha^\vee) = (\nu_1 + \ldots + \nu_{j-1} + q\nu_j, \alpha^\vee) \in \mathbb{Z},$$

which is not possible. So $\nu_j = \tau(x\nu_{j+1})$ for some $\tau \in W$. Let $v, w \in W$ be chosen as in the definition of the associated root system. Now $\nu_j = \tau(x\nu_{j+1})$ implies that v, w are just the minimal elements in W such that $v(\nu_j^+) = \nu_j$ and $w(\nu_{j+1}^+) = \nu_{j+1}$ because $\nu_j^+ = x\nu_{j+1}^+$. But $(\nu_j, \alpha^\vee) = 0$ and $(\nu_{j+1}, \alpha^\vee) > 0$ implies $v \geq s_\alpha w > w$. Therefore $\alpha \in \Phi^\vee_{j,j+1}$ and $(\nu_1 + \ldots + \nu_j, \alpha^\vee) \in \mathbb{Z}$, which is not possible.

The proposition follows now by the definition of the operator f_α.

Proposition 5.9. *The set of locally integral concatenations is stable under the root operators.*

Corollary 1. *Let $\lambda_1, \ldots, \lambda_r \in \Lambda^+_{\mathbb{Q}}$ be such that $\underline{\lambda} := \lambda_1 * \ldots * \lambda_r \in \Pi^+$ is a locally integral concatenation. Then $\mathbb{B}_{\underline{\lambda}}$ is a finite set, and if $\underline{\nu} \in \mathbb{B}_{\underline{\lambda}}$ is such that $\rho * \underline{\nu}$ is (for $t > 0$) in the interior of the dominant Weyl chamber, then $\underline{\nu} = \underline{\lambda}$.*

As a consequence we get by Corollary 1, section 4, the following character formula. Note that this a special case of Theorem 6.1 in the next section.

Corollary 2. $\operatorname{Char} \mathbb{B}_{\underline{\lambda}} = \operatorname{Char} V_\lambda$.

Corollary 1 and the proposition above prove also the following characterization of the set of paths $\mathbb{P}$ introduced at the beginning of this section:

Corollary 3. *$\mathbb{P}$ coincides with the set of locally integral concatenations.*

Proof of Corollary 1. Suppose $\underline{\nu} = \nu_1 * \ldots * \nu_r \in \mathbb{B}_{\underline{\lambda}}$. We may assume (after a reparametrization of $\underline{\lambda}$) that the r is the same as for $\underline{\lambda}$ and the ν_i are Weyl group conjugates of the λ_i. If $\rho * \underline{\nu}$ is in the interior of the dominant Weyl chamber (for $t > 0$), then Lemma 5.7 implies that $\nu_1, \ldots, \nu_r$ are rational dominant weights.

So $\lambda_i = \nu_i$ and hence $\underline{\nu} = \underline{\lambda}$. As a consequence we know that all paths in $\mathbb{B}_{\underline{\lambda}}$ are of the form: $\pi = f_\alpha f_{\alpha'} \ldots (\underline{\lambda})$, and therefore the possible endpoints are all of the form $\lambda - \sum a_\alpha \alpha$, where the a_α are nonnegative integers. Property (iii), section 3, then implies that all possible endpoints are in the convex hull of the Weyl group orbit of λ. So the number of possible endpoints is finite. Now for a given weight μ there are only a finite number of monomials in the f_α such that the endpoint of $\pi = f_\alpha f_{\alpha'} \ldots (\underline{\lambda})$ is μ. So $\mathbb{B}_{\underline{\lambda}}$ is a finite set.

Proof of Proposition 5.9. Let $\underline{\nu} := \nu_1 * \ldots * \nu_r$ be a locally integral concatenation. We will show that $f_\alpha \underline{\nu}$ is again a locally integral concatenation. The proof for the operator e_α is similar. By Proposition 5.8 we may assume that the parametrization of $\underline{\nu}$ is such that $(\nu_{i-1}, \alpha^\vee) \leq 0$ and $(\nu_j, \alpha^\vee) > 0$ for some $i \leq j \leq k$, and

$$\underline{\mu} := f_\alpha \underline{\nu} = \nu_1 * \ldots * \nu_{i-1} * s_\alpha(\nu_i) * \ldots * s_\alpha(\nu_k) * \nu_{k+1} * \ldots * \nu_r = \mu_1 * \ldots * \mu_r.$$

To check that $\mu_1 \succeq \ldots \succeq \mu_r$ is a simple exercise in Weyl group combinatorics and is left to the reader. Denote by $P_l = \nu_1 + \ldots + \nu_l$ (resp. $Q_l = \mu_1 + \ldots + \mu_l$) the l-th turning point of $\underline{\nu}$ (resp. $\underline{\mu}$).

Suppose first $l < i-1$ or $k < l < r$. Then $\nu_l = \mu_l$, $\nu_{l+1} = \mu_{l+1}$, $\Phi^\vee_{\mu_l, \mu_{l+1}} = \Phi^\vee_{\nu_l, \nu_{l+1}}$, and $P_l = Q_l$ or $P_l = Q_l + \alpha$, so that the conditions for a locally integral concatenation are obviously satisfied at these points.

If $l = k$, then $\nu_{l+1} = \mu_{l+1}$, $P_l = Q_l + \alpha$, and $\Phi^\vee_{\mu_l, \mu_{l+1}}$ is the root system spanned by $\Phi^\vee_{\nu_l, \nu_{l+1}}$ and $\alpha^\vee$. Since $(P_l, \alpha^\vee) \in \mathbb{Z}$ (Lemma 5.6), we know that $(Q_l, \alpha^\vee) \in \mathbb{Z}$. Thus part (a) of the condition for a locally integral concatenation is satisfied. If ν_k and ν_{k+1} are conjugate (up to multiplication by positive rational number), then so are μ_k and μ_{k+1} Suppose now $p > 0$ is such that $\nu := \nu_1 + \ldots + \nu_{k-1} + p\nu_k \in \Lambda$. Then

$$\mu_1 + \ldots + \mu_{k-1} + p\mu_k = \nu - (\nu - \mu_1 + \ldots + \mu_{i-1}, \alpha^\vee)\alpha.$$

Since $(\nu - \mu_1 + \ldots + \mu_{i-1}, \alpha^\vee)$ is an integer, it follows that $\mu_1 + \ldots + \mu_{k-1} + p\mu_k \in \Lambda$.

If $l = i-1$, then $P_l = Q_l$, $\nu_l = \mu_l$, and either $\Phi^\vee_{\mu_l, \mu_{l+1}} = \Phi^\vee_{\nu_l, \nu_{l+1}}$ or $\Phi^\vee_{\nu_l, \nu_{l+1}}$ is the root system spanned by $\Phi^\vee_{\mu_l, \mu_{l+1}}$ and $\alpha^\vee$. Further, the same calculation as above shows that if $q\nu_{l+1} + \nu_{l+2} + \ldots + \nu_k \in \Lambda$, then $q\mu_{l+1} + \mu_{l+2} + \ldots + \mu_k \in \Lambda$. So the conditions for a locally integral concatenation are also satisfied in this point.

If $i \leq l < k$, then $\Phi_{\mu_l, \mu_{l+1}} = s_\alpha(\Phi_{\nu_l, \nu_{l+1}})$, and $Q_l = s_\alpha(P_l) \pm a\alpha$ for some $a \in \mathbb{N}$. So part (a) of the definition of a locally integral concatenation holds for $f_\alpha \underline{\nu}$. If ν_l, ν_{l+1} are conjugate under the Weyl group (up to multiplication by a positive rational number), then so are μ_l, μ_{l+1}. Suppose now x, y are positive rational numbers such that $\nu_1 + \ldots + \nu_{l-1} + x\nu_l$ and $y\nu_{l+1} + \nu_{l+2} + \ldots + \nu_r$ are integral weights. Since $(\nu_1 + \ldots + \nu_{i-1}, \alpha^\vee)$ is an integer, it follows that $(\nu_i + \ldots + \nu_{l-1} + x\nu_l, \alpha^\vee)$ is an integer, and hence:

$$\mu_1 + \ldots + \mu_{l-1} + x\mu_l = \nu_1 + \ldots + \nu_{i-1} + s_\alpha(\nu_i + \ldots + \nu_{l-1} + x\nu_l) \in \Lambda.$$

Similarly, since $(\nu_{k+1} + \ldots + \nu_r, \alpha^\vee)$ is an integer, it follows that $(y\nu_{l+1} + \ldots + \nu_k, \alpha^\vee)$ is an integer, and hence:

$$y\mu_{l+1} + \mu_{l+2} + \ldots + \mu_r = s_\alpha(y\nu_{l+1} + \nu_{l+2} + \ldots + \nu_k) + \nu_{k+1} + \ldots + \nu_r \in \Lambda,$$

which finishes the proof.

6. The General Case

For $\pi \in \Pi^+$ denote by $\mathbb{B}_\pi$ the set of all paths obtained from π by applying the root operators. In other words, $\mathbb{B}_\pi \cup \{\theta\}$ is the smallest set of paths that is stable under the root operators and contains π. Let $\lambda = \pi(1) \in \Lambda^+$ be the endpoint of π. In the following we present the most important properties of the set $\mathbb{B}_\pi$. Proofs will be given in the following sections.

Theorem 6.1. *$\mathbb{B}_\pi$ is a finite set, and if $\eta \in \mathbb{B}_\pi$ is such that $\rho * \eta \in \Pi_0^+$, then $\eta = \pi$.*

As an immediate consequence we get by Corollary 3.1:

Corollary 1. $\operatorname{Char} \mathbb{B}_\pi = \operatorname{Char} V_\lambda$

Example 6.2: Tableaux and paths. Let $p = (a_1, \dots, a_n)$ be a partition, and denote by $\mathbf{T}_0$ the semi-standard Young tableau of shape p having only 1's as entry in the first row, 2's in the second row etc.

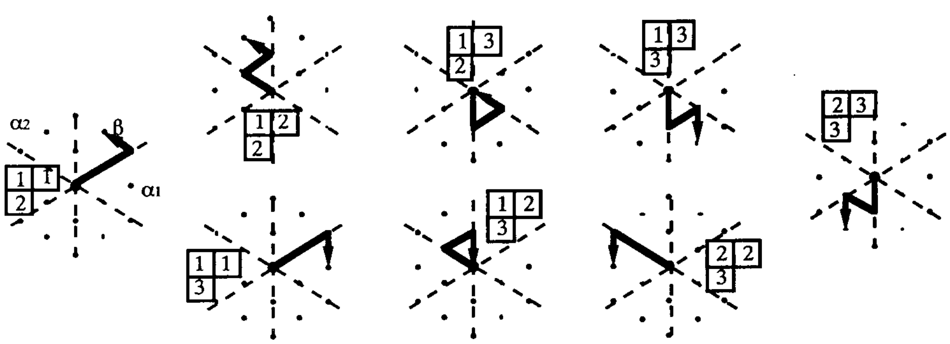


FIGURE 7. The tableaux and the associated paths for the adjoint representation of $\mathfrak{sl}_3$

If π_0 is the associated path (Example 1.5), then $\pi_0(1) = a_1\epsilon_1 + \dots + a_n\epsilon_n$. The condition $a_1 \geq \dots \geq a_n$ implies that the image of the path is contained in the dominant Weyl chamber. So $\pi_0 \in \Pi^+$. It is a nice exercise to check that

$$\mathbb{B}_{\pi_0} = \{\pi_{\mathbf{T}} \mid \mathbf{T} \text{ semi-standard Young tableau of shape } p\}.$$

The classical formula using semi-standard tableaux to calculate dimensions and characters of $\mathfrak{sl}_n$-modules can hence be considered as a special case of Corollary 1.

Theorem 6.1 characterizes the path $\pi \in \Pi^+$ as the unique path in $\mathbb{B}_\pi$ such that the image of $\rho * \pi$ is completely contained in the interior of the dominant Weyl chamber for $t > 0$. Since $\rho * \eta$ meets at least one of the walls for any other path $\eta \in \mathbb{B}_\pi$, this means that for any other path there exist at least one simple root α such that minimum of the function $t \mapsto \langle \eta(t), \alpha^\vee \rangle$ is smaller or equal to -1. By property (iv), section 3, this implies that there exists at least one simple root α such that $e_\alpha(\eta) \neq \theta$. So we get a characterization of π and $\mathbb{B}_\pi$ resembling that of a highest weight vector and a highest weight module (without using the equality of characters above):

Corollary 2.

i) $\mathbb{B}_\pi = \{\eta \in \Pi \mid \exists i_1, \dots, i_s : \eta = f_{\alpha_{i_1}} \cdots f_{\alpha_{i_s}} \pi\}$, *and if* $\eta \in \mathbb{B}_\pi$ *is such that* $e_\alpha(\eta) = \theta$ *for all simple roots, then* $\eta = \pi$.

ii) *Set* $\lambda = \pi(1)$. *For every* $w \in W/W_\lambda$ *there exists a unique path* $\eta \in \mathbb{B}_\pi$ *such that* $\eta(1) = w(\lambda)$.

Since the character is independent of the choice of π, this means that for any choice of a path $\pi \in \Pi^+$ ending in λ we get a different combinatorial model for V_λ. So the next question is: *What do these models have in common?*

Definition 6.3. For $\pi \in \Pi^+$ let $\mathcal{G}_\pi$ be the colored, directed graph having as vertices the elements of $\mathbb{B}_\pi$. We put an arrow $\eta \xrightarrow{\alpha} \eta'$ with color a simple root α between $\eta, \eta' \in \mathbb{B}_\pi$ if and only if $f_\alpha(\eta) = \eta'$ (or, equivalently, $e_\alpha(\eta') = \eta$).

Remark 6.4. Corollary 2 implies that $\mathcal{G}_\pi$ is connected and has a special vertex: π is the unique vertex with no "incoming" arrow.

We call two such graphs $\mathcal{G}_\pi, \mathcal{G}_{\pi'}$ isomorphic if there exists a bijection $\phi : \mathcal{G}_\pi \to \mathcal{G}_{\pi'}$ of the vertices such that we have an arrow $\eta \xrightarrow{\alpha} \eta'$ with color α between $\eta, \eta' \in \mathbb{B}_\pi$ if and only if we have an arrow $\phi(\eta) \xrightarrow{\alpha} \phi(\eta')$ with color α between $\phi(\eta), \phi(\eta') \in \mathbb{B}_{\pi'}$.

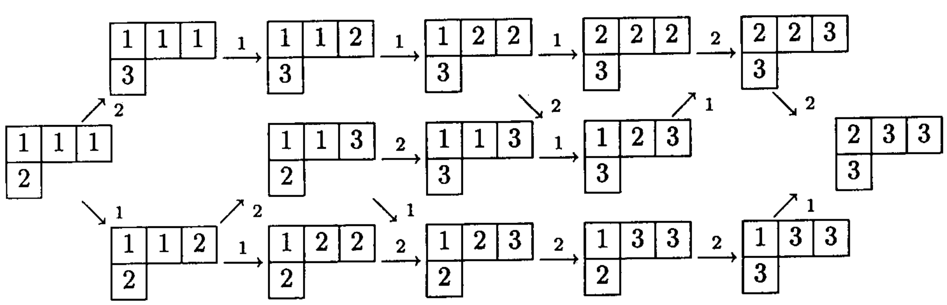
Such an isomorphism maps necessarily the special vertices π and π' onto each other. Further, ϕ maps obviously the α-strings in the graphs onto each other. So property (ii), section 3, implies $(\pi(1), \alpha^\vee) = (\pi'(1), \alpha^\vee)$ for all simple roots, and hence $\pi(1) = \pi'(1)$. In fact, this condition is also sufficient:

Theorem 6.5. *The graphs* $\mathcal{G}_\pi$ *and* $\mathcal{G}_{\pi'}$ *are isomorphic if and only if* $\pi(1) = \pi'(1)$.

Remark 6.7. Since the graph depends only on the endpoint, it makes sense to write just $\mathcal{G}_\lambda$ for the graph $\mathcal{G}_\pi$, where $\lambda = \pi(1)$.

Example 6.8. Figure 5 in Example 2.3 is the graph associated to the adjoint representation of $\mathfrak{sl}_3$.

Example 6.9. We consider again the Lie algebra $\mathfrak{sl}_3$, and we take as highest weight $\lambda = 2\omega_1 + \omega_2$. Using the identification: Young tableaux $\leftrightarrow$ paths (see Example 1.5), we get the following graph $\mathcal{G}_{2\omega_1+\omega_2}$:



The following property is very important for the concept of the path model:

Definition 6.10. A path $\eta \in \Pi$ is called **integral** if the minimum of the function $h_\alpha : t \mapsto (\pi(t), \alpha^\vee)$ is an integer for all simple roots. We call a subset $\mathbb{B} \subset \Pi$ **integral** if all elements of $\mathbb{B}$ are integral.

Suppose $\mathbb{B} \subset \Pi$ is finite and integral, and fix $\eta \in \mathbb{B}$. Then $e_\alpha(\eta) = 0$ for all simple roots implies $h_\alpha(t) \geq 0$ and hence $\eta \in \Pi^+$. Theorem 6.1 and Corollary 2 hence imply:

Lemma 6.11. *If $\mathbb{B} \subset \Pi$ is integral and $\mathbb{B} \cup \{\theta\}$ is stable under the root operators, then $\mathbb{B}$ is the disjoint union $\mathbb{B} = \bigcup \mathbb{B}_\pi$, where the union is taken over all $\pi \in \mathbb{B} \cap \Pi^+$.*

Let $\mathbb{B}, \mathbb{B}' \subset \Pi$ be two integral subsets, and denote by $\mathbb{B} * \mathbb{B}'$ the set of all concatenations $\pi * \pi'$, where $\pi \in \mathbb{B}$, $\pi' \in \mathbb{B}'$. The set $\mathbb{B} * \mathbb{B}'$ is obviously again integral. The following lemma is a simple consequence of section 3, property (ii).

Lemma 6.12. *If $\mathbb{B}, \mathbb{B}' \subset \Pi$ are integral and stable under the root operators, then $\mathbb{B} * \mathbb{B}'$ is stable under the root operators too. More precisely: $e_\alpha(\pi * \eta) = \pi * (e_\alpha \eta)$ if $\exists\, n \geq 1$ such that $e_\alpha^n \eta \neq 0$ but $f_\alpha^n \pi = 0$, and $e_\alpha(\pi * \eta) = (e_\alpha \pi) * \eta$ otherwise, and*

$$f_\alpha(\pi * \eta) = \begin{cases} (f_\alpha \pi) * \eta, & \text{if } \exists n \geq 1 \text{ such that } f_\alpha^n \pi \neq 0 \text{ but } e_\alpha^n \eta = 0; \\ \pi * (f_\alpha \eta), & \text{otherwise.} \end{cases}$$

Theorem 6.13. *If $\pi \in \Pi^+$, then B_π is integral.*

Since $\operatorname{Char} \mathbb{B}_{\pi_1} * \mathbb{B}_{\pi_2} = \operatorname{Char} \mathbb{B}_{\pi_1} \operatorname{Char} \mathbb{B}_{\pi_2} = \operatorname{Char} V_{\pi_1(1)} \otimes V_{\pi_2(1)}$ for $\pi_1, \pi_2 \in \Pi^+$, we get as an immediate consequence:

Generalized Littlewood-Richardson rule. *For $\lambda, \mu \in \Lambda^+$ let $\pi_1, \pi_2 \in \Pi^+$ be such that $\pi_1(1) = \lambda$ and $\pi_2(1) = \mu$. Then the tensor product $V_\lambda \otimes V_\mu$ is isomorphic to the direct sum*

$$V_\lambda \otimes V_\mu \simeq \bigoplus V_{\lambda + \eta(1)},$$

where the sum runs over all paths $\eta \in \mathbb{B}_{\pi_2}$ such that $\pi_1 * \eta \in \Pi^+$.

Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi subalgebra associated to a subset of the set of simple roots. We denote by $C_\mathfrak{l} \supset C$ and $\Lambda_\mathfrak{l}^+$ the dominant Weyl chamber and the set of dominant weights for $\mathfrak{l}$. Let $\Pi_\mathfrak{l}^+$ be the set of all paths in Π such that the image is completely contained in the dominant Weyl chamber $C_\mathfrak{l}$ of $\mathfrak{l}$. For $\nu \in \Lambda_\mathfrak{l}^+$ let U_ν be the associated simple $\mathfrak{l}$-module. The same arguments as above prove:

Restriction formula. *For $\lambda \in \Lambda^+$ let $\pi \in \Pi^+$ be such that $\pi(1) = \lambda$. The simple $\mathfrak{g}$-module V_λ decomposes as $\mathfrak{l}$ module into the direct sum*

$$V_\lambda \simeq \bigoplus U_{\eta(1)},$$

where the sum runs over all paths $\eta \in \mathbb{B}_\pi$ such that $\eta \in \Pi_\mathfrak{l}^+$.

7. The Weyl Group Action

In section 3 we defined an action of the simple reflections on Π.

Proposition 7.1. *The action of the s_α extends to an action of W on Π such that $(w(\eta))(1) = w(\eta(1))$ for $\pi \in \Pi$, $w \in W$.*

PROOF. We have to prove that the braid relations are satisfied. Without loss of generality, we may hence assume that we are in the rank two case. Using the continuity property and approximation by rational paths, it is sufficient to prove that the relations hold for rational paths. By using the stretching property, we can even assume that the path we start with is of the form $\eta = \nu_1 * \nu_2 * \ldots * \nu_r$ for some $\nu_1, \ldots, \nu_r \in \Lambda$.

If $\lambda_1, \ldots, \lambda_r \in \Lambda^+$ are dominant weights such that ν_i is in the Weyl group orbit of λ_i, then $\eta \in \mathbb{B}_{\lambda_1} * \ldots * \mathbb{B}_{\lambda_r}$. But this implies that η and the paths obtained from η by applying the root operators are integral. So to prove the claim, we are reduced to prove the following: Let $\mathbb{B} \subset \Pi$ be integral and stable under the root operators. Then the action of the simple reflections on $\mathbb{B}$ extends to an action of the Weyl group.

Let α, γ be the simple roots. Now s_α, s_γ commute if and only if the roots are orthogonal to each other. The root operators commute in this case too; so there is nothing to be proved. We may hence assume that α is not orthogonal to γ.

For $\eta \in \mathbb{B}$ denote by η^n the path obtained by concatenating the path $\eta * \eta * \ldots * \eta$ with itself n-times. Suppose $(\eta(1), \alpha^\vee) > 0$ and $\pi \in \mathbb{B}$ is arbitrary. It is easy to see that for $n, k \in \mathbb{N}$ big enough we can find k_1, k_2 such that (Lemma 6.12):

$$f_\alpha^k(\pi * \eta^n) = f_\alpha^{k_1}(\pi) * s_\alpha(\eta) * \ldots * s_\alpha(\eta) * f_\alpha^{k_2}(\eta).$$

So if we choose n big enough for given π_1, π_2, then there exist $\pi_1', \pi_2' \in \Pi$ such that

$$s_\alpha(\pi_1 * \eta^n * \pi_2) = \pi_1' * s_\alpha(\eta)^k * \pi_2'$$

for some $k \in \mathbb{N}$, where the k depends linearly on n for $n \gg 0$.

To prove that the braid relations hold, it is sufficient to prove that $s_\alpha s_\gamma \cdots (\eta) = s_\gamma s_\alpha \cdots (\eta)$ for $\eta \in \mathbb{B}$ with $\eta(1) \in \Lambda^+$. Since $s_\alpha(\eta) = \eta$ (resp. $s_\gamma(\eta) = \eta$) if $(\eta(1), \alpha^\vee) = 0$ (resp. $(\eta(1), \gamma^\vee) = 0$), the relation holds trivially if $\eta(1)$ is a multiple of a fundamental weight. So we may assume that $\eta(1)$ is regular.

Let $\lambda \in \Lambda^+$ be a regular dominant weight. Then, for a given $n \in \mathbb{N}$, we can find a $k \in \mathbb{N}$ such that $k\lambda * \eta^n \in \Pi^+$. Now Corollary 2 in section 6 implies:

$$s_\alpha s_\gamma \cdots (k\lambda * \eta^n) = s_\gamma s_\alpha \cdots (k\lambda * \eta^n).$$

But the arguments above show that we can choose n, π_1, π_2 such that the left and right side are of the form:

$$\pi_1 * s_\alpha s_\gamma \cdots (\eta)^k * \pi_2 \quad \text{and} \quad \pi_1 * s_\gamma s_\alpha \cdots (\eta)^k * \pi_2$$

for some $k > 0$. It follows that $s_\alpha s_\gamma \cdots (\eta) = s_\gamma s_\alpha \cdots (\eta)$.

8. The Proofs

It remains to give the proofs for Theorem 6.1, 6.5, and 6.13. A first step is the proof of the following weaker version of Theorem 6.5:

Proposition 8.1. *If λ, μ are dominant weights, then the graphs $\mathcal{G}_{\lambda+\mu}$ and $\mathcal{G}_{\lambda*\mu}$ are isomorphic.*

PROOF. Consider the family of paths $\pi_s := ((1-s)\lambda) * (\mu + s\lambda)$. Note that $\pi_0 = \lambda * \mu$, $\pi_1 = \lambda + \mu$, and $\pi_s \in \Pi^+$ for all $s \in [0,1]$. The results in section 5 imply that for all rational $t \in [0,1]$:

$$\mathbb{B}_t := \mathbb{B}_{\pi_t} \text{ is integral and } \pi_t \text{ is the only path in } \mathbb{B}_t \text{ such that } \rho * \pi_t \in \Pi_0^+. \quad (**)$$

We use now (**) to prove that the graphs are isomorphic: If $s \in [0,1]$, then, for any $\epsilon > 0$, we can find a rational number $s' \in [0,1]$ such that (after choosing appropriate parametrizations) $d(\pi_s, \pi_{s'}) < \epsilon$. So by continuity, it follows that $\mathbb{B}_s$ is integral and π_s is the only path in $\mathbb{B}_s$ such that $\rho * \pi_s \in \Pi_0^+$.

Corollary 1, section 4, implies that $\operatorname{Char} \mathbb{B}_s = \operatorname{Char} V_{\lambda+\mu}$ for all $s \in [0,1]$. Fix $\eta_1 \in \mathbb{B}_1$, and let $\alpha_1, \ldots, \alpha_r$ be such that $\eta_1 = f_{\alpha_1} \ldots f_{\alpha_r} \pi_1$. Since the $\mathbb{B}_s$ are integral, it follows by continuity (and property (ii), section 3) that $\eta_s = f_{\alpha_1} \ldots f_{\alpha_r} \pi_s \neq \theta$. Of course, the $\alpha_1, \ldots, \alpha_r$ are not necessarily uniquely determined. Suppose $\gamma_1, \ldots, \gamma_r$ are simple roots such that $\eta_1 = f_{\gamma_1} \ldots f_{\gamma_r} \pi_1$. Property (i), section 3, implies then $\pi_1 = e_{\gamma_r} \ldots e_{\gamma_1} \eta_1$. Again, since the $\mathbb{B}_s$ are integral, continuity and property (ii), section 3, implies that $e_{\gamma_r} \ldots e_{\gamma_1} \eta_t \neq 0$. The endpoint of this path is $\lambda + \mu$. So the character of $\mathbb{B}_s$ implies that this is the path π_s, and hence $\eta_s = f_{\gamma_1} \ldots f_{\gamma_r} \pi_s$.

This proves that the map $\mathbb{B}_1 \to \mathbb{B}_s$, $\eta_1 \mapsto \eta_s$, is well-defined for all $s \in [0,1]$. The same arguments prove $f_{\alpha_1} \ldots f_{\alpha_r} \pi_1 = f_{\gamma_1} \ldots f_{\gamma_r} \pi_1$ iff $f_{\alpha_1} \ldots f_{\alpha_r} \pi_s = f_{\gamma_1} \ldots f_{\gamma_r} \pi_s$. Therefore the map is bijective and induces an isomorphism of the graphs.

We prepare now the proofs of the theorems. Let λ be a dominant weight. The path $\lambda * (-\lambda)$ has the property that $f_\alpha(\lambda * (-\lambda)) = e_\alpha(\lambda * (-\lambda)) = \theta$. Further, if $\pi \in \Pi^+$, then the map $\mathbb{B}_\pi \to \mathbb{B}_{\pi * \lambda * (-\lambda)}$, $\eta \mapsto \eta * \lambda * (-\lambda)$ induces by Lemma 6.12 obviously an isomorphism of graphs $\mathcal{G}_\pi \to \mathcal{G}_{\pi * \lambda * (-\lambda)}$.

Suppose λ, μ, ν are dominant weights such that $\lambda + \mu = \nu$. By Proposition 8.1, we have an isomorphism $\mathcal{G}_{\lambda * \mu} \to \mathcal{G}_\nu$. Since $-\mu$ is an element of $\mathbb{B}_{-w_0(\mu)}$, the isomorphism above induces an inclusion and a bijection:

$$\mathbb{B}_{\nu * -\mu} \hookrightarrow \mathbb{B}_\nu * \mathbb{B}_{-w_0(\mu)} \xrightarrow{\sim} \mathbb{B}_{\lambda * \mu} * \mathbb{B}_{-w_0(\mu)},$$

such that the image of $\nu * -\mu$ is the path $\lambda * \mu * -\mu$. Since the locally integral concatenations are integral, Lemma 6.12 implies that this map induces an isomorphism of graphs $\mathcal{G}_{\nu * (-\mu)} \to \mathcal{G}_\lambda = \mathcal{G}_{\nu - \mu}$. An easy induction process shows:

Lemma 8.2. *If $\pi = \lambda_1 * \ldots * \lambda_r \in \Pi^+$ is such that the λ_i are in either Λ^+ or $-\Lambda^+$, then $\mathcal{G}_\pi$ is isomorphic to $\mathcal{G}_\lambda$, where $\lambda = \lambda_1 + \ldots + \lambda_r$.*

The proof of the general case will be reduced to paths of the form above. The next lemma is an important step in the reduction procedure:

Lemma 8.3. *If $\pi \in \Pi^+$ and $\lambda \in \Lambda^+$ are such that $\mathcal{G}_{n\pi}$ is isomorphic to $\mathcal{G}_{n\lambda}$ for $n \gg 0$, then $\mathbb{B}_\pi$ is integral and $\mathcal{G}_\pi$ is isomorphic to $\mathcal{G}_\lambda$.*

PROOF OF THE LEMMA. The stretching property ((i), section 3) implies that $\mathcal{G}_\lambda \subset \mathcal{G}_{n\lambda}$ and $\mathcal{G}_\pi \subset \mathcal{G}_{n\pi}$ can be recovered as the subgraphs associated to the operators f_α^n, e_α^n. So the isomorphism $\mathcal{G}_{n\pi} \to \mathcal{G}_{n\lambda}$ implies an isomorphism $\phi : \mathcal{G}_\pi \to \mathcal{G}_\lambda$.

Suppose $\eta \in \mathbb{B}_\pi$ and the minimum m_α of the function $h_\alpha : t \to (\eta(t), \alpha^\vee)$ is not an integer. By Lemma 5.6, we know that the minimum n_α of the function $t \mapsto (\phi(\eta)(t), \alpha^\vee)$ is an integer.

By property (ii), section 3, $|n_\alpha|$ is maximal such that $e_\alpha^{|n_\alpha|}\phi(\eta) \neq \theta$. By the isomorphism of graphs, this implies also that $|n_a|$ is maximal with the property that $e_\alpha^{|n_a|}\eta \neq \theta$ and hence $|n_\alpha| < |m_a|$. Choose $k \in \mathbb{N}$ such that $k|m_a| - k|n_\alpha| > 1$.

The minima of the functions $t \mapsto (k\phi(\eta)(t), \alpha^\vee)$ and $t \mapsto (k\eta(t), \alpha^\vee)$ are kn_α and km_α. This implies $e_\alpha^{|kn_\alpha|+1}\phi(\eta) = \theta$. Since $k|m_a| - k|n_\alpha| > 1$, we get $e_\alpha^{|kn_\alpha|+1}(\eta) \neq \theta$. But the graphs $\mathcal{G}_{k\pi}$ and $\mathcal{G}_{k\lambda}$ are isomorphic, and the stretching property implies that $k\eta$ is mapped onto $k\phi(\eta)$. So $e_\alpha^{|kn_\alpha|+1}\phi(\eta) = \theta$ implies $e_\alpha^{|kn_\alpha|+1}\eta = \theta$. It follows that $n_\alpha = m_\alpha$.

Proof of Theorem 6.5 and 6.13. Let $\pi \in \Pi^+$ be a rational path. By Lemma 8.3 it is sufficient to prove the theorems for $n\pi$, $n \gg 0$. Thus we may in fact assume that $\pi = \lambda_1 * \ldots * \lambda_r$ for some $\lambda_1, \ldots, \lambda_r \in \Lambda$. Set $\lambda = \pi(1)$.

If $\lambda_i \notin \Lambda^+$ and $-\lambda_i \notin \Lambda^+$, then fix $\mu_i, \nu_i \in \Lambda^+$ such that $\lambda_i = \mu_i - \nu_i$. Further, we may assume that for any simple root α we have either $(\nu_i, \alpha^\vee) = 0$ or $(\mu_i, \alpha^\vee) = 0$.

For $\eta \in \Pi$ denote by $\eta^k = \eta * \ldots * \eta$ the path obtained by concatenating η k-times. Note that for any given ϵ we can chose $k \gg 0$ such that (after choosing an appropriate parametrization of the paths) $d(\lambda_i, (\frac{\mu_i}{k} * \frac{-\nu_i}{k})^k) < \epsilon$.

So we can approximate π by paths of the form $\lambda_1 * \ldots * \lambda_r$ such that λ_i or $-\lambda_i$ is a rational dominant weight. By Lemma 8.2 and Lemma 8.3 we know that the corresponding paths are integral, and the graph is isomorphic to $\mathcal{G}_\lambda$. Then it follows by continuity that $\mathbb{B}_\pi$ is integral and $\mathcal{G}_\pi$ isomorphic to $\mathcal{G}_\lambda$.

An arbitrary path $\pi \in \Pi^+$ can be approximated by rational paths in Π^+. Since the structure of the graph is independent of the choice of the approximation, the continuity property implies that $\mathcal{G}_\pi$ is isomorphic to $\mathcal{G}_\lambda$ and $\mathbb{B}_\pi$ is integral.

Proof of Theorem 6.1. Suppose $\lambda = \pi(1)$. The isomorphism of graphs implies that the cardinalities of the sets $\mathbb{B}_\pi$ and $\mathbb{B}_\lambda$ are the same. Since the latter is equal to $\dim V_\lambda$ by Corollary 1, section 5, it follows that $\mathbb{B}_\pi$ is a finite set. Recall that λ is the only path in $\mathbb{B}_\lambda$ such that $\rho * \lambda$ is contained in the interior of the dominant Weyl chamber for $t > 0$ (Corollary 1, section 5). So for any $\eta' \in \mathbb{B}_\lambda$ there exists a simple root α such that $e_\alpha(\eta') \neq \theta$. The isomorphism of graphs hence implies that for any $\eta \in \mathbb{B}_\pi$, $\eta \neq \pi$, there exists a simple root α such that $e_\alpha(\eta) \neq 0$. It follows that there exists a simple root α such that the minimum of the function $t \mapsto (\eta(t), \alpha)$ is ≤ -1. Thus $\rho * \eta$ meets at least one of the walls of the dominant Weyl chamber.

9. A Demazure-Type Character Formula

For a simple root α denote by Λ_α the Demazure operator on the group ring $\mathbb{Z}[\Lambda]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}.$$

In other words:

$$\Lambda_\alpha(e^\mu) = \begin{cases} e^\mu + e^{\mu-\alpha} + \ldots + e^{s_\alpha(\mu)} & \text{if } (\mu, \alpha^\vee) \geq 0; \\ 0 & \text{if } (\mu, \alpha^\vee) = -1; \\ -e^{\mu+\alpha} - \ldots - e^{s_\alpha(\mu)-\alpha} & \text{if } (\mu, \alpha^\vee) < -1. \end{cases}$$

Note that $\Lambda_\alpha \circ \Lambda_\alpha = \Lambda_\alpha$. So Λ_α applied to a root string $e^\mu + e^{\mu-\alpha} + \ldots + e^{s_\alpha(\mu)}$ just reproduces the string.

Fix $\lambda \in \Lambda^+$ and let $\underline{\nu} := \nu_1 * \ldots * \nu_r$ be a path in $\mathbb{B}_\lambda$. The "first direction" ν_1 is (up to multiplication by a positive rational number) of the form $\sigma(\lambda)$ for some $\sigma \in W/W_\lambda$. We define a map $i : \mathbb{B}_\lambda \to W/W_\lambda$ by $i(\underline{\nu}) := \sigma$. For $w \in W/W_\lambda$ denote by $\mathbb{B}_\lambda(w)$ the subset

$$\mathbb{B}_\lambda(w) := \{\underline{\nu} \in \mathbb{B}_\lambda \mid i(\underline{\nu}) \leq w\}$$

Note that $\mathbb{B}_\lambda = \mathbb{B}_\lambda(w_0)$ for the longest word w_0 in the Weyl group.

Theorem 9.1. $\operatorname{Char} \mathbb{B}_\lambda(w) := \sum_{\underline{\nu} \in \mathbb{B}_\lambda(w)} e^{\underline{\nu}(1)} = \Lambda_{\alpha_1} \circ \cdots \circ \Lambda_{\alpha_r}(e^\lambda)$ *for any reduced decomposition* $w = s_{\alpha_1} \cdots s_{\alpha_r}$.

Comments about the proof. The details can be found in [10]. The main idea is to prove the following two properties.

First property: If $s_\alpha w > w$ in W/W_λ and $i(\underline{\nu}) = s_\alpha w$, then there exists an $k > 0$ such that $e_\alpha^{k+1}(\underline{\nu}) = \theta$, $e_\alpha^k(\underline{\nu}) \neq \theta$, $i(e_\alpha^k(\underline{\nu})) = w$, $i(e_\alpha^j(\underline{\nu})) = s_\alpha w$ for all $j < k$ and $i(f_\alpha^j(\underline{\nu})) = s_\alpha w$ for all j such that $f_\alpha^j(\underline{\nu}) \neq \theta$.

Second property: If $s_\alpha w > w$ in W/W_λ and $i(\underline{\nu}) = w$, then either $e_\alpha \underline{\nu} = \theta$ and $i(f_\alpha^j(\underline{\nu})) = s_\alpha w$ for all j such that $f_\alpha^j(\underline{\nu}) \neq \theta$, or $i(f_\alpha^j(\underline{\nu})) = i(e_\alpha^k(\underline{\nu})) = w$ for all j such that $f_\alpha^j(\underline{\nu}) \neq \theta$ and all k such that $e_\alpha^k(\underline{\nu}) \neq \theta$.

The theorem is obviously true for the class of the identity. We proceed now by induction on the length of w. Let α be a simple root such that $s_\alpha w > w$. The second property implies that we can decompose $\mathbb{B}_\lambda(w)$ into $\mathbb{B}_\lambda^+(w) \cup \mathbb{B}_\lambda^0(w)$, where $\mathbb{B}_\lambda^0(w)$ is the set of all paths in $\mathbb{B}_\lambda(w)$ such that $i(f_\alpha^j(\underline{\nu})) = i(e_\alpha^k(\underline{\nu})) = w$ for all j having $f_\alpha^j(\underline{\nu}) \neq \theta$ and all k having $e_\alpha^k(\underline{\nu}) \neq \theta$, and $\mathbb{B}_\lambda^+(w)$ is the set of all paths such that $e_\alpha \underline{\nu} = \theta$ and $i(f_\alpha^j(\underline{\nu})) = s_\alpha w$ for all j having $f_\alpha^j(\underline{\nu}) \neq \theta$.

Obviously $\operatorname{Char} \mathbb{B}_\lambda(w) = \operatorname{Char} \mathbb{B}_\lambda^+(w) + \operatorname{Char} \mathbb{B}_\lambda^0(w)$. Now $\operatorname{Char} \mathbb{B}_\lambda^0(w)$ is just a sum of α-strings. So $\Lambda_\alpha(\operatorname{Char} \mathbb{B}_\lambda^0(w)) = \operatorname{Char} \mathbb{B}_\lambda^0(w)$. The first property implies that

$$\mathbb{B}_\lambda(s_\alpha w) = \mathbb{B}_\lambda^0(w) \cup \{f_\alpha^k \underline{\nu} \mid \underline{\nu} \in \mathbb{B}_\lambda^+(w), 0 \leq k \leq (\underline{\nu}(1), \alpha^\vee)\}.$$

(Note: $f_\alpha^{(\underline{\nu}(1),\alpha^\vee)+1} \underline{\nu} = 0$ since $e_\alpha \underline{\nu} = 0$.) Since $\operatorname{Char}\{\underline{\nu}, f_\alpha \underline{\nu}, \ldots, \tilde{s}_\alpha(\underline{\nu})\} = \Lambda_\alpha(e^{\underline{\nu}(1)})$, we get: $\operatorname{Char} \mathbb{B}_\lambda(s_\alpha w) = \Lambda_\alpha(\operatorname{Char} \mathbb{B}_\lambda(w))$.

10. The P–R–V Conjecture

Consider the tensor product $V_\lambda \otimes V_\mu$ of two simple $\mathfrak{g}$-modules of highest weight λ and μ. The Parthasarathy–Ranga-Rao–Varadarajan conjecture (which has been proved independently in [8] and [17]) states:

Theorem 10.1. *If* $\tau_1, \tau_2 \in W$ *are such that* $\nu := \tau_1(\lambda) + \tau_2(\mu)$ *is a dominant weight, then the module* V_ν *occurs in* $V_\lambda \otimes V_\mu$.

Proof. Using the generalized Littlewood-Richardson rule, one can give a purely combinatorial proof. To say that $\nu := \tau_1(\lambda) + \tau_2(\mu)$ is a dominant weight is the same as to say that ν is the unique dominant weight in the Weyl group orbit of $\lambda + \sigma(\mu)$, where $\sigma \in W/W_\mu$ is such that $\sigma(\mu) = \tau_1^{-1}\tau_2(\mu)$.

We construct now a path $\eta \in \mathbb{B}_\mu$ such that $\lambda + \eta(1) = \nu$ and $\lambda * \eta \in \Pi^+$. The generalized Littlewood-Richardson rule implies then that V_ν occurs in $V_\lambda \otimes V_\mu$ with multiplicity at least one.

Let us start with the path $\eta := \sigma(\mu)$. If $\lambda * \eta \in \Pi^+$, then we are done. So suppose $\lambda * \eta \notin \Pi^+$. Then we can find a $t \in [0,1]$ such that $\lambda + \eta(s)$ is in the dominant Weyl chamber for $s \leq t$ and such that $\lambda + \eta(s)$ is outside the dominant Weyl chamber for $s > t$. We are going to fold the last part of the path back into the dominant Weyl chamber. Let Δ be the set of simple roots orthogonal to $\lambda + \eta(t)$.

Note that $(\lambda + t\sigma(\mu), \alpha^\vee) = 0$ implies $(1-t)(\sigma(\mu), \alpha^\vee) \in \mathbb{Z}$ for $\alpha \in \Delta$. So $(1-t)\sigma(\mu)$ is an integral weight for the sub-root system spanned by the simple roots in Δ. Further, since $(\lambda, \alpha^\vee) \geq 0$, it follows that $(1-t)(\sigma(\mu), \alpha^\vee) \leq 0$. Let w_0' be the longest element in the Weyl group generated by the reflections s_α, $\alpha \in \Delta$, and let $\sigma_1 \in W/W_\mu$ be such that $\sigma_1(\mu) = w_0'(\sigma(\mu))$.

Since $(1-t)\sigma(\mu)$ is an antidominant integral weight for the simple roots in Δ, it is easy to see that $\eta' := t\sigma(\mu) * (1-t)\sigma_1(\mu)$ is an element of $\mathbb{B}_\mu$.

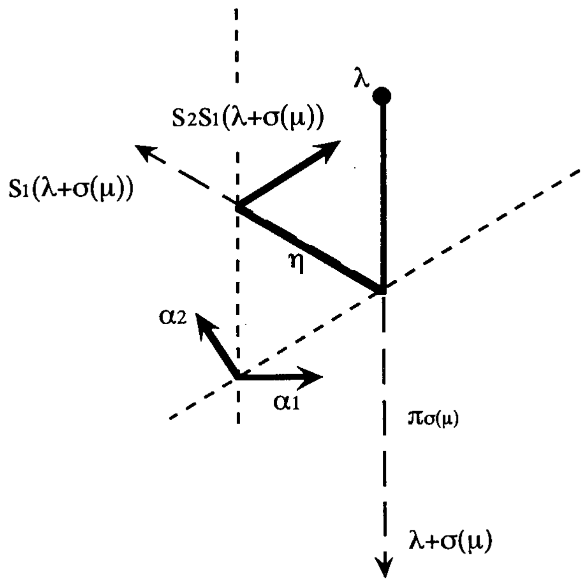


FIGURE 8. A proof of the P–R–V conjecture

Note that the new path is "better" then the old one: First of all, $\lambda + \eta'(1) = w_0'(\lambda + \eta(1))$, so that the endpoint of $\lambda * \eta'$ is in the Weyl group orbit of $\lambda + \sigma(\mu)$. Also the path $\lambda * \eta'$ stays longer in the dominant Weyl chamber than the path $\lambda * \eta$: For $s \leq t_1$ we have of course $\lambda * \eta'(s) = \lambda * \eta(s)$; so these points are in the dominant Weyl chamber. If $\alpha \in \Delta$ and $s > t$, then

$$(\lambda + \eta'(s), \alpha^\vee) = (\lambda + \eta(t), \alpha^\vee) + (s - t)(\sigma_1(\mu), \alpha^\vee) \geq 0.$$

And if $\alpha \notin \Delta$, then $(\lambda + \eta'(t), \alpha^\vee) > 0$; so we can choose $0 < r \ll 1$ such that $(\lambda + \eta'(s), \alpha^\vee) \geq 0$ for all $t \leq s \leq t + r$ and all $\alpha \notin \Delta$. It follows that $\lambda + \eta'(s)$ is in the dominant Weyl chamber for all $0 \leq s \leq t + r$.

If $\lambda * \eta' \in \Pi^+$, we are done; otherwise we proceed as in Figure 8; i.e., we fix $t' > t$ such that $\lambda + \eta'(s)$ is in the dominant Weyl chamber for all $s \leq t'$ and $\lambda + \eta'(s)$ is outside the dominant Weyl chamber for $s > t'$. Using the same procedure as above, we then fold another part of the path back into the dominant Weyl chamber.

Note that this is a finite procedure: The endpoint of the new path is always of the form the endpoint of the old path plus a sum of positive roots. Since the weights that can occur are all of the form $\lambda + \mu$ minus a sum of positive roots, this procedure has to end after a finite number of steps.

11. Paths, the Crystal Graph, and the Plactic Algebra

We would like to conclude this note with two remarks, one concerning the relation of the paths to the crystal graph and one remark on the so-called plactic algebra.

Let $U_q(\mathfrak{g})$ be the q-analogue of the enveloping algebra of $\mathfrak{g}$. For a detailed introduction we mention the books [1,3,16]. A finite-dimensional irreducible representation V_λ of $\mathfrak{g}$ admits a quantum deformation V_λ^q [15]. Kashiwara introduced in [7] the notion of a crystal graph of an $U_q(\mathfrak{g})$-representation (and, using the quantum deformation, one can of course associate such a graph to a $\mathfrak{g}$-representation). This graph can be considered as a refined version of the character of the representation.

The following connection between the paths and the crystal graph was found by Kashiwara [6] and Joseph [3]:

Theorem 11.1. *Suppose $\pi \in \Pi^+$, and set $\lambda := \pi(1)$. Then the crystal graph C_λ of the representation V_q^λ is isomorphic to the graph $\mathcal{G}_\lambda$.*

Using the tensor product of quantum representations, one can make the union of all crystal bases into an algebra. In terms of paths this would be the $\mathbb{C}$-vector space with basis $\bigcup_{\pi\in\Pi^+} \mathbb{B}_\pi$, with product the concatenation of paths, but where we factor out the relations obtained via the isomorphisms $\mathcal{G}_\pi \simeq \mathcal{G}_{\pi'}$ for $\pi(1) = \pi'(1)$.

This algebra contains a great deal of information about the tensor products of the representations. For the groups $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$, such an algebra has been defined before by Lascoux and Schützenberger. Their idea was to define a product structure on the set of all semi-standard Young tableaux such that this product mimics the tensor product of GL_n-representations. It turns out that this "plactic algebra" (as they call it) is precisely the crystal or path algebra defined above.

A description of this algebra in terms of generators and relations (i.e., a description more in the style of [18]) can be found in [14].

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