A BASIS FOR REPRESENTATIONS OF SYMPLECTIC LIE ALGEBRAS

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Mathematics Research Report No. MRR 012-98

Abstract

A basis for each finite-dimensional irreducible representation of the symplectic Lie algebra $\mathfrak{sp}(2n)$ is constructed. The basis vectors are expressed in terms of the Mickelsson lowering operators. Explicit formulas for the matrix elements of generators of $\mathfrak{sp}(2n)$ in this basis are given. The basis is natural from the viewpoint of the representation theory of the Yangians. The key role in the construction is played by the fact that the subspace of $\mathfrak{sp}(2n-2)$ -highest vectors in any finite-dimensional irreducible representation of $\mathfrak{sp}(2n)$ admits a natural structure of a representation of the Yangian $Y(\mathfrak{gl}(2))$.

Mathematics Subject Classifications (1991). 17B10, 81R10

0. Introduction

One of the central problems of the representation theory is to construct a basis in the representation space and to find the representation matrices in the basis. A solution of this problem for the general linear Lie algebra $\mathfrak{gl}(N)$ and the orthogonal Lie algebra $\mathfrak{o}(N)$ was given by Gelfand and Tsetlin [GT1], [GT2]. They proposed a parameterization of basis vectors and gave formulas for the matrix elements of the generators of the Lie algebras in this basis. An explicit construction of the Gelfand–Tsetlin basis vectors in terms of lowering operators was given by Zhelobenko [Z1], [Z2], Nagel–Moshinsky [NM] ($\mathfrak{gl}(N)$ -case); Pang–Hecht [PH], Wong [Wo] ($\mathfrak{o}(N)$ -case). Different formulas for the lowering operators are also obtained by Asherova–Smirnov–Tolstoy [AST2], Gould [G1], [G2], Nazarov–Tarasov [NT1], Molev [Mo2].

A quite different approach to construct modules over the classical Lie algebras is developed in the papers by King–El-Sharkaway [KS], Berele [B], King–Welsh [KW], Koike–Terada [KT], Proctor [P2]. It is based on the Weyl realization of the representations of the classical groups in tensor spaces; see [W]. In particular, bases in the representations of the orthogonal and symplectic Lie algebras parameterized by $\mathfrak{o}(N)$ -standard or $\mathfrak{sp}(2n)$ -standard Young tableaux are constructed. Although the subset of the standard Young tableaux is not preserved by the action of the Lie algebra, explicit trace relations and Garnir relations between the Young tableaux allow one to get an algorithm for calculation the matrix elements of the generators of the Lie algebras.

Bases with special properties in the universal enveloping algebra for a simple Lie algebra $\mathfrak g$ and in some $\mathfrak g$ -modules were constructed by Lakshmibai–Musili–Seshadri [LMS], Littelmann [L] (monomial bases); De Concini–Kazhdan [CK] (combinatorial bases for GL(n)); Gelfand–Zelevinsky [GZ2], Retakh–Zelevinsky [RZ], Mathieu [M1] ('good' bases); Lusztig [Lu], Kashiwara [Ka] (canonical or crystal bases); see also Mathieu [M2] for a review and more references.

The problem of constructing an analog of the Gelfand–Tsetlin basis for the symplectic Lie algebra $\mathfrak{sp}(2n)$ has been addressed by many authors. The branching rule for the reduction $\mathfrak{sp}(2n) \downarrow \mathfrak{sp}(2n-2)$ is obtained by Zhelobenko [Z1]. Contrary to the case of the Lie algebras $\mathfrak{gl}(N)$ and $\mathfrak{o}(N)$ this reduction turns out to be not multiplicity free which makes the problem of constructing a basis for representations of the symplectic Lie algebra $\mathfrak{sp}(2n)$ more complicated.

Raising and lowering operators acting on the subspace $V(\lambda)^+$ of $\mathfrak{sp}(2n-2)$ -highest vectors in a representation $V(\lambda)$ of $\mathfrak{sp}(2n)$ are constructed by Mickelsson [Mi1] (see also Bincer [Bi1], [Bi2]). They are explicitly expressed as elements of the universal enveloping algebra $U(\mathfrak{sp}(2n))$. Applying the lowering operators consequently to the $\mathfrak{sp}(2n)$ -highest vector one obtains a basis in $V(\lambda)^+$ and then by induction one constructs a basis in $V(\lambda)$. However, the monomials in the lowering operators can be chosen arbitrarily (since the operators do not commute) and none of the bases is distinguished. The problem of calculating the matrix elements of generators of $\mathfrak{sp}(2n)$ in such a basis appears to be very difficult.

The algebra generated by the raising and lowering operators, and more general algebras $Z(\mathfrak{g},\mathfrak{g}')$ associated with a Lie algebra \mathfrak{g} and a reductive subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ were studied by Mickelsson [Mi2], Van den Hombergh [Ho]. The theory of these algebras was further developed by Zhelobenko [Z3]–[Z6] with the use of the extremal projection method originated from [AST1]–[AST3].

A basis for the representations of the symplectic Lie algebras was constructed by Gould and Kalnins [GK], [G3] with the use of the restriction $\mathfrak{gl}(2n) \downarrow \mathfrak{sp}(2n)$. The basis vectors are parameterized by a subset of the Gelfand–Tsetlin $\mathfrak{gl}(2n)$ -patterns. Some matrix element formulas are also derived by using the $\mathfrak{gl}(2n)$ -action.

A similar observation is made independently by Kirillov [K] and Proctor [P1]. A description of the Gelfand–Tsetlin patterns for $\mathfrak{sp}(2n)$ and $\mathfrak{o}(N)$ can be obtained by regarding them as fixed points of involutions of the Gelfand–Tsetlin patterns for the corresponding Lie algebra $\mathfrak{gl}(N)$.

The problem of separation of multiplicities in the reduction $\mathfrak{sp}(2n) \downarrow \mathfrak{sp}(2n-2)$ can be approached by investigating the restriction of $\mathfrak{sp}(2n)$ -modules to an intermediate (non-reductive) subalgebra $\mathfrak{sp}(2n-1) \subset \mathfrak{sp}(2n)$. Such subalgebras and their representations are studied by Gelfand–Zelevinsky [GZ1], Proctor [P1], Shtepin [S]. The separation of multiplicities can be achieved by constructing a filtration of $\mathfrak{sp}(2n-1)$ -modules [S].

Matrix elements of generators of $\mathfrak{sp}(2n)$ are obtained by Wong-Yeh [WY] for certain degenerate irreducible representations.

In this paper we give a construction of a weight basis in $V(\lambda)$ and obtain explicit formulas for the matrix elements of generators of $\mathfrak{g}_n = \mathfrak{sp}(2n)$ in this basis; see Theorem 1.1. Our approach is based on the theory of Mickelsson algebras and the representation theory of the Yangians.

It is well-known [D, Section 9.1] that the subspace $V(\lambda)^+_{\mu}$ of \mathfrak{g}_{n-1} -highest vectors of a given weight μ is an irreducible representation of the centralizer algebra $C_n = U(\mathfrak{g}_n)^{\mathfrak{g}_{n-1}}$. However, the algebraic structure of C_n is very complicated which makes the problem of studying their representations very difficult. An approach to solve this problem is developed by Olshanski [O3]; see also [MO]. He constructed a chain of natural homomorphisms

$$C_1 \leftarrow C_2 \leftarrow \cdots \leftarrow C_n \leftarrow C_{n+1} \leftarrow \cdots$$

analogous to the Harish-Chandra homomorphism [D, Section 7.4]. The projective limit of this chain is an algebra isomorphic to the tensor product of an algebra of polynomials and a quantized enveloping algebra $Y^-(2)$ which was called the twisted Yangian. (This centralizer construction can be applied to any pairs of Lie algebras $\mathfrak{a}(N-M)\subset\mathfrak{a}(N)$ of type A-D, where $N\to\infty$ with M fixed. In the result one obtains either the Yangian $Y(M):=Y(\mathfrak{gl}(M))$ for the Lie algebra $\mathfrak{gl}(M)$ (see Olshanski [O1], [O2]), or the orthogonal $Y^+(M)$ or symplectic twisted Yangian $Y^-(M)$; see [O3], [MO]). In particular, one has an algebra homomorphism $Y^-(2)\to C_n$ so that the subspace $V(\lambda)^+_\mu$ admits a structure of a representation of $Y^-(2)$ which can be shown to be irreducible. The algebra $Y^-(2)$ can be either

defined as a subalgebra in the Yangian Y(2) or can be presented by generators and defining relations. The algebraic structure of the twisted Yangians is studied in [O3] and [MNO], and their finite-dimensional irreducible representations are described in [Mo4] in terms of the highest weights. In particular, it is proved that any finite-dimensional irreducible representation of Y⁻(2) can be extended to the Yangian Y(2) thus providing the subspace $V(\lambda)^+_{\mu}$ with a structure of an irreducible Y(2)-module (see Theorem 5.2 below).

Lowering operators for the Yangian reduction $Y(M) \downarrow Y(M-1)$ and Gelfand-Tsetlin-type bases for representations of Y(M) were constructed in [Mo2] and [NT2] (see also [C], [NT1]). We use a special case of these constructions to get a Gelfand-Tsetlin-type basis in the Y(2)-module $V(\lambda)^+_{\mu}$; cf. [T], [Dr], [CP]. The basis corresponds to an inclusion $Y(1) \subset Y(2)$ which can be naturally chosen by at least in two different ways. However, to compute the action of generators of \mathfrak{g}_n in this basis we need to express the basis vectors in terms of the elements of the twisted Yangian $Y^-(2)$. In other words, the two inclusions

$$Y(1) \subset Y(2), \qquad Y^{-}(2) \subset Y(2)$$

must be compatible with each other in some sense (see Remark 4.3) which makes the choice of the first inclusion unique and brings the necessary rigidity into the construction of the basis in $V(\lambda)^+_{\mu}$.

To calculate the matrix elements of generators of \mathfrak{g}_n in this basis we explicitly express the elements of the twisted Yangian Y⁻(2) in terms of the Mickelsson raising and lowering operators. Our main instrument is Theorem 5.1 which provides explicit formulas for the images of generators of Y⁻(2) under the natural homomorphism to the Mickelsson algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$:

$$Y^{-}(2) \to C_n \to Z(\mathfrak{g}_n, \mathfrak{g}_{n-1}).$$

The use of the quadratic relations in both the algebras $Y^{-}(2)$ and $Z(\mathfrak{g}_{n},\mathfrak{g}_{n-1})$ allows us to avoid long calculations.

The sections are organized as follows. The main result (Theorem 1.1) is formulated in Section 1. Sections 2–4 contain preliminary results which are used in the proof of Theorem 1.1. In Section 2 following [Z3]–[Z6] we introduce the Mickelsson raising and lowering operators and describe the algebraic structure of the algebra $Z(\mathfrak{g}_n,\mathfrak{g}_{n-1})$. In Section 3 we formulate some known results on the algebraic structure of the Yangian Y(2n) and the twisted Yangian $Y^-(2n)$; see [O3], [MNO]. In Section 4 we describe a particular case of the construction of Gelfand–Tsetlin-type basis for a certain class of representations of Y(2) and $Y^-(2)$; see [Mo2], [NT2]. Our main arguments are given in Sections 5 and 6. We construct the highest vector and find the highest weight for the representation $V(\lambda)^+_{\mu}$ of $Y^-(2)$. As a corollary we obtain a proof of the Zhelobenko branching rule for representations of the symplectic Lie algebras [Z1] (see also Hegerfeldt [H], King [Ki], Proctor [P2], Okounkov [Ok]). In Section 6 we construct a basis in $V(\lambda)$ and derive the formulas for the

matrix elements of generators of \mathfrak{g}_n in this basis. They have a multiplicative form which exhibits some similarity with the Gelfand–Tsetlin formulas in the case of $\mathfrak{gl}(N)$ and $\mathfrak{o}(N)$.

This project was initiated in collaboration with G. Olshanski to whom I would like to express my deep gratitude. I would like to thank M. Nazarov, V. Tolstoy and D. P. Zhelobenko for useful remarks and discussions.

1. Main Theorem

We shall enumerate the rows and columns of $2n \times 2n$ -matrices over \mathbb{C} by the indices $-n, -n+1, \ldots, -1, 1, \ldots, n$. We shall also assume throughout the paper that the index 0 is skipped in a summation or a product. The canonical basis $\{e_i\}$ in the space \mathbb{C}^{2n} will be enumerated by the same set of indices. We let the E_{ij} , $i, j = -n, \ldots, -1, 1, \ldots, n$ denote the standard basis of the Lie algebra $\mathfrak{gl}(2n)$. Introduce the elements

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i}, \qquad \theta_{ij} = \operatorname{sgn} i \cdot \operatorname{sgn} j. \tag{1.1}$$

The symplectic Lie algebra $\mathfrak{g}_n = \mathfrak{sp}(2n)$ can be identified with the subalgebra in $\mathfrak{gl}(2n)$ spanned by the elements F_{ij} , $i, j = -n, \ldots, n$. They satisfy the following symmetry property

$$F_{-i,-i} = -\theta_{ij}F_{ij}. (1.2)$$

The elements $F_{k,-k}$, $F_{-k,k}$ with $k=1,\ldots,n$ and $F_{k-1,-k}$ with $k=2,\ldots,n$ generate \mathfrak{g}_n as a Lie algebra.

The subalgebra \mathfrak{g}_{n-1} is spanned by the elements (1.1) with the indices i, j running over the set $\{-n+1, \ldots, n-1\}$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in \mathfrak{g}_n . The elements F_{11}, \ldots, F_{nn} form a basis of \mathfrak{h} .

The finite-dimensional irreducible representations of \mathfrak{g}_n are in a one-to-one correspondence with *n*-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying the inequalities

$$0 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$$
.

We denote the corresponding representation by $V(\lambda)$. It contains a unique, up to a multiple, nonzero vector ξ (the highest vector) such that

$$F_{ii} \xi = \lambda_i \xi,$$
 $i = 1, \dots, n,$
 $F_{ij} \xi = 0,$ $-n \le i < j \le n.$

We shall sometimes use the numbers $\lambda_{-i} := -\lambda_i$. They are eigenvalues of ξ with respect to the operators $F_{-i,-i}$.

¹The non-negative labels $\lambda_{-n} \ge \cdots \ge \lambda_{-1} \ge 0$ are usually used to parameterize the irreducible finite-dimensional representations of \mathfrak{g}_n . We have chosen to work with positive subindices. Both parameterizations can be easily obtained from each other.

The restriction of $V(\lambda)$ to the subalgebra \mathfrak{g}_{n-1} is isomorphic to a direct sum of irreducible finite-dimensional representations $V'(\mu)$, $\mu = (\mu_1, \dots, \mu_{n-1})$ of \mathfrak{g}_{n-1} with certain multiplicities:

$$V(\lambda) = \bigoplus_{\mu} c(\mu)V'(\mu). \tag{1.3}$$

The multiplicity $c(\mu)$ is equal to the number of *n*-tuples of integers (ν_1, \ldots, ν_n) satisfying the inequalities [Z1] (see also Corollary 5.3 below):

$$0 \ge \nu_1 \ge \lambda_1 \ge \nu_2 \ge \lambda_2 \ge \dots \ge \nu_{n-1} \ge \lambda_{n-1} \ge \nu_n \ge \lambda_n,$$

$$0 \ge \nu_1 \ge \mu_1 \ge \nu_2 \ge \mu_2 \ge \dots \ge \nu_{n-1} \ge \mu_{n-1} \ge \nu_n.$$
(1.4)

Denote by $V(\lambda)^+$ the subspace of \mathfrak{g}_{n-1} -highest vectors in $V(\lambda)$:

$$V(\lambda)^+ = \{ \eta \in V(\lambda) \mid F_{ij} \eta = 0, \quad -n < i < j < n \}.$$

Given $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $V(\lambda)^+_{\mu}$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)_{\mu}^{+} = \{ \eta \in V(\lambda)^{+} \mid F_{ii} \eta = \mu_{i} \eta, \quad i = 1, \dots, n-1 \}.$$

We obviously have dim $V(\lambda)^+_{\mu} = c(\mu)$. Any nonzero vector $\eta \in V(\lambda)^+_{\mu}$ generates a \mathfrak{g}_{n-1} -submodule in $V(\lambda)$ isomorphic to $V'(\mu)$.

A parameterization of basis vectors in $V(\lambda)$ is obtained by using its further restrictions to the subalgebras of the chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n.$$

Define the pattern Λ associated with λ as an array of integer row vectors of the form

$$\lambda_{n1} \quad \lambda_{n2} \quad \cdots \quad \lambda_{nn}$$

$$\lambda'_{n1} \quad \lambda'_{n2} \quad \cdots \quad \lambda'_{nn}$$

$$\lambda_{n-1,1} \quad \cdots \quad \lambda_{n-1,n-1}$$

$$\lambda'_{n-1,1} \quad \cdots \quad \lambda'_{n-1,n-1}$$

$$\cdots \quad \cdots$$

$$\lambda_{11}$$

$$\lambda'_{11}$$

such that the upper row coincides with λ and the following inequalities hold

$$0 \ge \lambda'_{k1} \ge \lambda_{k1} \ge \lambda'_{k2} \ge \lambda_{k2} \ge \dots \ge \lambda'_{k,k-1} \ge \lambda'_{k,k-1} \ge \lambda'_{kk} \ge \lambda_{kk}$$

for $k = 1, \ldots, n$; and

$$0 \ge \lambda'_{k1} \ge \lambda_{k-1,1} \ge \lambda'_{k2} \ge \lambda_{k-1,2} \ge \cdots \ge \lambda'_{k,k-1} \ge \lambda_{k-1,k-1} \ge \lambda'_{kk}$$

for $k = 2, \ldots, n$. Let us set

$$l_{ki} = \lambda_{ki} - i, \qquad l'_{ki} = \lambda'_{ki} - i, \qquad 1 \le i \le k \le n.$$
 (1.5)

Theorem 1.1. There exists a basis $\{\zeta_{\Lambda}\}$ in $V(\lambda)$ parameterized by all patterns Λ associated with λ such that the action of generators of \mathfrak{g}_n is given by the formulas

$$F_{kk} \, \zeta_{\Lambda} = \left(2 \sum_{i=1}^{k} \lambda'_{ki} - \sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) \zeta_{\Lambda},$$

$$F_{k,-k} \, \zeta_{\Lambda} = \sum_{i=1}^{k} A_{ki}(\Lambda) \, \zeta_{\Lambda + \delta'_{ki}},$$

$$F_{-k,k} \, \zeta_{\Lambda} = \sum_{i=1}^{k} B_{ki}(\Lambda) \, \zeta_{\Lambda - \delta'_{ki}},$$

$$F_{k-1,-k} \, \zeta_{\Lambda} = \sum_{i=1}^{k-1} C_{ki}(\Lambda) \, \zeta_{\Lambda - \delta_{k-1,i}} + \sum_{i=1}^{k} \sum_{j,m=1}^{k-1} D_{kijm}(\Lambda) \, \zeta_{\Lambda + \delta'_{ki} + \delta_{k-1,j} + \delta'_{k-1,m}}.$$

Here

$$A_{ki}(\Lambda) = \prod_{a=1, a \neq i}^{k} \frac{1}{l'_{ka} - l'_{ki}},$$

$$B_{ki}(\Lambda) = 4 A_{ki}(\Lambda) l'_{ki} \prod_{a=1}^{k} (l_{ka} - l'_{ki}) \prod_{a=1}^{k-1} (l_{k-1,a} - l'_{ki}),$$

$$C_{ki}(\Lambda) = \frac{1}{2 l_{k-1,i}} \prod_{a=1, a \neq i}^{k-1} \frac{1}{l'_{k-1,i} - l'_{k-1,a}},$$

and

$$D_{kijm}(\Lambda) = A_{ki}(\Lambda)A_{k-1,m}(\Lambda)C_{kj}(\Lambda)$$

$$\prod_{a=1, a\neq i}^{k} (l_{k-1,j} - l'_{ka})(l_{k-1,j} + l'_{ka} + 1) \prod_{a=1, a\neq m}^{k-1} (l_{k-1,j} - l'_{k-1,a})(l_{k-1,j} + l'_{k-1,a} + 1).$$

The arrays $\Lambda \pm \delta_{ki}$ and $\Lambda \pm \delta'_{ki}$ are obtained from Λ by replacing λ_{ki} and λ'_{ki} by $\lambda_{ki} \pm 1$ and $\lambda'_{ki} \pm 1$ respectively. It is supposed that $\zeta_{\Lambda} = 0$ if the array Λ is not a pattern.

Theorem 1.1 will be proved in Sections 5 and 6.

2. Mickelsson algebra $Z(\mathfrak{g}_n,\mathfrak{g}_{n-1})$

This section contains preliminary results on the algebraic structure of the Mickelsson algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$; see [Z3]–[Z6] for further details.

Consider the extension of the universal enveloping algebra $U(\mathfrak{g}_n)$

$$U'(\mathfrak{g}_n) = U(\mathfrak{g}_n) \otimes_{U(\mathfrak{h})} R(\mathfrak{h}),$$

where $R(\mathfrak{h})$ is the field of fractions of the commutative algebra $U(\mathfrak{h})$. Let J denote the left ideal in $U'(\mathfrak{g}_n)$ generated by the elements F_{ij} with -n < i < j < n. The *Mickelsson algebra* $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ is the quotient algebra of the normalizer

Norm
$$J = \{x \in U'(\mathfrak{g}_n) \mid Jx \subseteq J\}$$

modulo the two-sided ideal J. It is an algebra over \mathbb{C} and an $R(\mathfrak{h})$ -bimodule. The algebraic structure of $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ can be described by using the *extremal projection* $p = p_{n-1}$ for the Lie algebra \mathfrak{g}_{n-1} [AST1]–[AST3]. The projection p is, up to a factor from $R(\mathfrak{h}_{n-1})$, a unique element of an extension of $U'(\mathfrak{g}_{n-1})$ to an algebra of formal series, satisfying the condition

$$F_{ij} p = p F_{ji} = 0$$
 for $-n < i < j < n$. (2.1)

Explicit formulas for p are given in [AST1], [Z3]. The element p is of zero weight (with respect to the adjoint action of \mathfrak{h}_{n-1}) and it can be normalized to satisfy the condition $p^2 = p$. The Mickelsson algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ can also be defined as the image of the quotient $U'(\mathfrak{g}_n)/J$ with respect to the projection p:

$$Z(\mathfrak{g}_n,\mathfrak{g}_{n-1}) = p(U'(\mathfrak{g}_n)/J).$$

An analog of the Poincaré-Birkhoff-Witt theorem holds for the algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ [Z4] so that ordered monomials in the elements

$$F_{n,-n}, F_{-n,n}, pF_{in}, pF_{ni}, i = -n+1, \dots, n-1$$

form a basis of $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ as a left or right $R(\mathfrak{h})$ -module. These elements can also be given by the following explicit formulas:

$$pF_{in} = \sum_{i>i_1>\dots>i_s>-n} F_{ii_1}F_{i_1i_2}\cdots F_{i_{s-1}i_s}F_{i_sn}\frac{1}{(f_i - f_{i_1})\cdots(f_i - f_{i_s})},$$

$$pF_{ni} = \sum_{i
(2.2)$$

where $s = 0, 1, \ldots$ and $f_a := F_{aa} - a$. By (1.2) we have the equalities

$$pF_{i,-n} = \operatorname{sgn} i \cdot pF_{n,-i}, \qquad pF_{-n,i} = \operatorname{sgn} i \cdot pF_{-i,n}.$$

It will be convenient to use the following normalized generators which can be identified with elements of the universal enveloping algebra $U(\mathfrak{g}_n)$: for $i = 1, \ldots, n-1$

$$z_{in} = pF_{in} \prod_{\substack{-n < a < i}} (f_i - f_a), \qquad z_{i,-n} = pF_{i,-n} \prod_{\substack{-n < a < i}} (f_i - f_a),$$

$$z_{ni} = pF_{ni} \prod_{\substack{i < a < n}} (f_i - f_a), \qquad z_{-n,i} = pF_{-n,i} \prod_{\substack{i < a < n}} (f_i - f_a).$$
(2.3)

We also set for all i

$$z_{-i,n} = \operatorname{sgn} i \cdot z_{-n,i}, \qquad z_{n,-i} = \operatorname{sgn} i \cdot z_{i,-n}.$$

The elements z_{ni} can be written in the following equivalent form: for i = 1, ..., n-1

$$z_{ni} = \sum_{i < i_1 < \dots < i_s < n} (f_i - f_{j_1}) \cdots (f_i - f_{j_k}) F_{ni_s} F_{i_s i_{s-1}} \cdots F_{i_2 i_1} F_{i_1 i},$$

$$z_{n,-i} = \sum_{i > i_1 > \dots > i_s > -n} (f_i - f_{j_1}) \cdots (f_i - f_{j_k}) F_{i_s,-n} F_{i_{s-1} i_s} \cdots F_{i_1 i_2} F_{i i_1},$$

$$(2.4)$$

where $s = 0, 1, \ldots$ and $\{j_1, \ldots, j_k\}$ is the complementary subset to $\{i_1, \ldots, i_s\}$ respectively in the set $\{i+1, \ldots, n-1\}$ or $\{-n+1, \ldots, i-1\}$.

We shall use the following quadratic relations satisfied by the elements of the algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ [Z4]. For all i

$$[F_{n,-n}, z_{ni}] = 0, [F_{n,-n}, z_{in}] = -2 z_{i,-n}, [F_{-n,n}, z_{in}] = 0, [F_{-n,n}, z_{ni}] = 2 z_{-n,i}.$$
 (2.5)

Furthermore, for $i + j \neq 0$

$$[z_{ni}, z_{nj}] = 0 (2.6)$$

and

$$z_{in} z_{j,-n} = z_{j,-n} z_{in} \frac{f_i - f_j - 1}{f_i - f_j} + z_{i,-n} z_{jn} \frac{1}{f_i - f_j}.$$
 (2.7)

Introduce the following element of $U(\mathfrak{g}_n)$

$$z_{n,-n} = \sum_{n>i_1>\dots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s,-n} (f_n - f_{j_1}) \cdots (f_n - f_{j_k}), \quad (2.8)$$

where $s = 0, 1, \ldots$ and $\{j_1, \ldots, j_k\}$ is the complementary subset to $\{i_1, \ldots, i_s\}$ in the set $\{-n+1, \ldots, n-1\}$. One easily checks that $z_{n,-n}$ belongs to the normalizer Norm J and so it can be regarded as an element of $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$. The following equivalent formula holds for $z_{n,-n}$ where the notation of (2.8) is used:

$$z_{n,-n} = \sum_{n>i_1>\dots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s,-n} (f_{-n} - f_{j_1} - 1) \cdots (f_{-n} - f_{j_k} - 1).$$
(2.9)

To see this, we can use the standard argument of the extremal projection method; see [Z3]–[Z6]. By the property (2.1) of the extremal projection p for a sequence of indices $n > i_1 > \cdots > i_s > -n$ we have

$$pF_{ni_1}F_{i_1i_2}\cdots F_{i_{s-1}i_s}F_{i_s,-n} = 2pF_{ni_s}F_{i_s,-n}, \qquad (2.10)$$

if $i_m + i_{m+1} = 0$ for a certain m; otherwise this equals $pF_{ni_s}F_{i_s,-n}$. This allows one to write the right hand side (2.8) in the form

$$\sum_{i=1}^{n-1} p F_{ni} F_{i,-n} a_i + F_{n,-n} b, \qquad a_i, b \in \mathbf{R}(\mathfrak{h}).$$
 (2.11)

It remains to check that the coefficients a_i and b remain unchanged if we replace f_n by $f_{-n} - 1 = -f_n - 1$. This can be done by a straightforward calculation which implies that the right hand side of (2.9) coincides with (2.11).

Proposition 2.1. We have the relations in $U'(\mathfrak{g}_n) \mod J$:

$$F_{n,-n} = \sum_{i=-n+1}^{n} z_{ni} z_{n,-i} \prod_{\substack{a=-n+1\\a\neq i}}^{n} \frac{1}{f_i - f_a},$$
 (2.12)

$$F_{n-1,-n} = \sum_{i=-n+1}^{n-1} z_{n-1,i} z_{i,-n} \prod_{\substack{a=-n+1\\a\neq i}}^{n-1} \frac{1}{f_i - f_a},$$
 (2.13)

where $z_{nn} = z_{n-1,n-1} := 1$.

Proof. Both relations are proved in the same way, so we only give a proof of (2.12). The following equality in $U'(\mathfrak{g}_n) \mod J$ is implied by the explicit formulas for the $pF_{i,-n}$ (see (2.2)):

$$F_{n,-n} = z_{n,-n} \frac{1}{(f_n - f_{-n+1}) \cdots (f_n - f_{n-1})} + \sum_{\substack{n \ge i_1 \ge \dots \ge i_r \ge -n}} F_{ni_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} \cdot pF_{i_s,-n} \frac{1}{(f_{i_s} - f_n)(f_{i_s} - f_{i_1}) \cdots (f_{i_s} - f_{i_{s-1}})}$$

where $s = 1, 2, \ldots$ Now (2.12) follows from (2.4). \square

3. Yangian Y(2n) and twisted Yangian $Y^{-}(2n)$

Proofs of the results formulated in this section can be found in [MNO].

The Yangian $Y(2n) = Y(\mathfrak{gl}(2n))$ is the complex associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $i, j = -n, \ldots, -1, 1, \ldots, n$, and the defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u - v} (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)), \tag{3.1}$$

where

$$t_{ij}(u) := \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \dots \in Y(2n)[[u^{-1}]].$$

One can rewrite (3.1) as a ternary relation for the matrix

$$T(u) := \sum_{i,j} t_{ij}(u) \otimes E_{ij} \in Y(2n)[[u^{-1}]] \otimes \operatorname{End} \mathbb{C}^{2n}.$$

To do this introduce the following notation. For an operator $X \in \operatorname{End} \mathbb{C}^{2n}$ and a number $m = 1, 2, \ldots$ we set

$$X_k := 1^{\otimes (k-1)} \otimes X \otimes 1^{\otimes (m-k)} \in \left(\operatorname{End} \mathbb{C}^{2n} \right)^{\otimes m}, \quad 1 \le k \le m.$$
 (3.2)

If $X \in (\operatorname{End} \mathbb{C}^{2n})^{\otimes 2}$ then for any k, l such that $1 \leq k, l \leq m$ and $k \neq l$, we denote by X_{kl} the operator in $(\mathbb{C}^{2n})^{\otimes m}$ which acts as X in the product of kth and lth copies and as 1 in all other copies. That is,

$$X = \sum_{r,s,t,u} a_{rstu} E_{rs} \otimes E_{tu} \quad \Rightarrow \quad X_{kl} = \sum_{r,s,t,u} a_{rstu} (E_{rs})_k (E_{tu})_l, \tag{3.3}$$

where $a_{rstu} \in \mathbb{C}$. The ternary relation has the form

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v), \tag{3.4}$$

where $R(u) = R_{12}(u) = 1 - u^{-1}P$ and P is the permutation operator in $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. The Yangian Y(2n) is a Hopf algebra with the coproduct

$$\Delta(t_{ij}(u)) = \sum_{a=-n}^{n} t_{ia}(u) \otimes t_{aj}(u). \tag{3.5}$$

The twisted Yangian Y⁻(2n) corresponding to the symplectic Lie algebra $\mathfrak{g}_n = \mathfrak{sp}(2n)$ is defined as follows. By $X \mapsto X^t$ we will denote the matrix transposition such that $(E_{ij})^t = \theta_{ij} E_{-j,-i}$. Introduce the matrix $S(u) = (s_{ij}(u))$ by setting $S(u) := T(u)T^t(-u)$, or, in terms of matrix elements,

$$s_{ij}(u) = \sum_{a=-n}^{n} \theta_{aj} t_{ia}(u) t_{-j,-a}(-u).$$
(3.6)

Write $s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots$. The twisted Yangian Y⁻(2n) is the subalgebra of Y(2n) generated by the elements $s_{ij}^{(1)}, s_{ij}^{(2)}, \ldots$, where $-n \leq i, j \leq n$.

The matrix S(u) satisfies the following quaternary relation and symmetry relation which follow from (3.4):

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v),$$

$$S^t(-u) = \frac{2u-1}{2u}S(u) + \frac{1}{2u}S(-u).$$
(3.7)

Here we use the notation (3.2), where $R^t(u)$ is obtained from R(u) by applying the transposition t in either of the two copies of End \mathbb{C}^{2n} :

$$R^{t}(u) = 1 - u^{-1} \sum_{i,j} \theta_{ij} E_{-j,-i} \otimes E_{ji}.$$

Relations (3.7) are defining relations for the algebra $Y^{-}(2n)$ and they can be rewritten in terms of the generating series $s_{ij}(u)$ as follows:

$$[s_{ij}(u), s_{kl}(v)] = \frac{1}{u - v} (s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u))$$

$$-\frac{1}{u + v} (\theta_{k,-j}s_{i,-k}(u)s_{-j,l}(v) - \theta_{i,-l}s_{k,-i}(v)s_{-l,j}(u))$$

$$+\frac{1}{u^2 - v^2} (\theta_{i,-j}s_{k,-i}(u)s_{-j,l}(v) - \theta_{i,-j}s_{k,-i}(v)s_{-j,l}(u))$$
(3.8)

and

$$\theta_{ij}s_{-j,-i}(-u) = \frac{2u-1}{2u}s_{ij}(u) + \frac{1}{2u}s_{ij}(-u). \tag{3.9}$$

This allows one to regard $Y^{-}(2n)$ as an abstract algebra with generators $s_{ij}^{(r)}$ and the relations (3.8), (3.9).

The mapping $F_{ij} \mapsto s_{ij}^{(1)}$ defines an inclusion $U(\mathfrak{g}_n) \hookrightarrow Y^-(2n)$ while the mapping

$$s_{ij}(u) \mapsto \delta_{ij} + \frac{F_{ij}}{u - 1/2} \tag{3.10}$$

defines an algebra homomorphism $Y^-(2n) \to U(\mathfrak{g}_n)$. Any even formal series $c(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ defines an automorphism of $Y^-(2n)$ given by

$$s_{ij}(u) \mapsto c(u) \, s_{ij}(u). \tag{3.11}$$

The Sklyanin determinant sdet S(u) is a formal series in u^{-1} with coefficients from the center of the algebra $Y^{-}(2n)$. It can be defined by the formula (see (3.2), (3.3)):

$$A_{2n}S_1(u)R_{12}^t \cdots R_{1,2n}^t S_2(u-1)R_{23}^t \cdots R_{2,2n}^t S_3(u-2)$$
$$\cdots S_{2n-1}(u-2n+2)R_{2n-1,2n}^t S_{2n}(u-2n+1) = \operatorname{sdet} S(u)A_{2n},$$
(3.12)

where $R_{ij}^t := R_{ij}^t(-2u+i+j-2)$, and A_{2n} is the normalized antisymmetrizer in the tensor space $(\mathbb{C}^{2n})^{\otimes 2n}$ so that $A_{2n}^2 = A_{2n}$. Explicit formulas for sdet S(u) are given in [O3], [MNO], [Mo3].

The Sklyanin comatrix $\widehat{S}(u) = (\widehat{s}_{ij}(u))$ is defined by

$$sdet S(u) = \widehat{S}(u)S(u - 2n + 1). \tag{3.13}$$

The mapping

$$S(u) \mapsto \frac{2u+1}{2u-2n+1} \widehat{S}(-u+n-1)$$
 (3.14)

defines an automorphism of the algebra $Y^{-}(2n)$; see [Mo4, Proposition 1.1].

Let us denote by $S^{(n-1)}(u)$ and $\widetilde{S}(u)$ the submatrices of S(u) whose rows and columns are enumerated by the sets of indices $\{-n+1,\ldots,-1,1,\ldots,n-1\}$ and $\{-n+1,\ldots,-1,1,\ldots,n\}$ respectively. Introduce the *nn*th quasi-determinant of the matrix $\widetilde{S}(u)$ by

$$|\widetilde{S}(u)|_{nn} = \left(\left(\widetilde{S}(u)^{-1}\right)_{nn}\right)^{-1};$$

see [GKLLRT]. We shall need the following expression for the matrix element $\widehat{s}_{nn}(u)$ of the Sklyanin comatrix $\widehat{S}(u)$.

Proposition 3.1. We have the formula

$$\widehat{s}_{nn}(u) = \frac{2u+1}{2u-1} |\widetilde{S}(-u)|_{nn} \operatorname{sdet} S^{(n-1)}(u-1).$$
(3.15)

Proof. Multiplying both sides of (3.12) by $S_{2n}^{-1}(u-2n+1)$ from the right and using (3.13) we obtain the relation

$$A_{2n}S_1(u)R_{12}^t \cdots R_{1,2n}^t S_2(u-1)R_{23}^t \cdots R_{2,2n}^t S_3(u-2)$$
$$\cdots S_{2n-1}(u-2n+2)R_{2n-1,2n}^t = A_{2n}\widehat{S}_{2n}(u).$$
(3.16)

It can be easily verified by using the symmetry relation (3.9) (see also [Mo3]) that

$$A_{2n}S_1(u)R_{12}^t \cdots R_{1,2n}^t = \frac{2u+1}{2u-1}A_{2n}S_1^t(-u).$$

Denote by $A_{2n}^{(2)}$ the normalized antisymmetrizer corresponding to the subgroup $\mathfrak{S}_{\{2,\dots,2n\}}$ of the symmetric group \mathfrak{S}_{2n} . Clearly, $A_{2n} = A_{2n}A_{2n}^{(2)}$. Note that $A_{2n}^{(2)}$ is permutable with $S_1^t(-u)$, while R_{ij}^t is permutable with R_{kl}^t and $S_k(u)$ provided that the indices i, j, k, l are distinct. So, we can rewrite formula (3.16) in the form:

$$\frac{2u+1}{2u-1}A_{2n}S_1^t(-u)A_{2n}^{(2)}S_2(u-1)R_{23}^t\cdots R_{2,2n-1}^tS_3(u-2)
\cdots S_{2n-1}(u-2n+2)R_{2,2n}^t\cdots R_{2n-1,2n}^t = A_{2n}\widehat{S}_{2n}(u).$$
(3.17)

Let us apply the operators in both sides of this formula to the vector $v_i = e_{-i} \otimes e_{-n+1} \otimes e_{-n+2} \otimes \cdots \otimes e_{n-1} \otimes e_n$, where $i \in \{-n+1,\ldots,n\}$. For the right hand side we clearly obtain

$$A_{2n}\widehat{S}_{2n}(u)v_i = \delta_{in}\,\widehat{s}_{nn}(u)\,\zeta,\tag{3.18}$$

where $\zeta := A_{2n}(e_{-n} \otimes e_{-n+1} \otimes \cdots \otimes e_n)$. To calculate the left hand side we note first that

$$R_{2,2n}^t \cdots R_{2n-1,2n}^t v_i = v_i.$$

Further, let us introduce the formal series

$$\Phi_{a_2,...,a_{2n-1}}(u-1) \in Y^-(2n)[[u^{-1}]], -n \le a_i \le n,$$

as follows:

$$A_{2n}^{(2)}S_2(u-1)R_{23}^t \cdots R_{2,2n-1}^t S_3(u-2) \cdots S_{2n-1}(u-2n+2)(e_{-n+1} \otimes \cdots \otimes e_{n-1})$$

$$= \sum_{a_2,\dots,a_{2n-1}} \Phi_{a_2,\dots,a_{2n-1}}(u-1)(e_{a_2} \otimes \cdots \otimes e_{a_{2n-1}}).$$

In particular,

$$(2n-2)! \ \Phi_{-n+1,\dots,n-1}(u-1) = \operatorname{sdet} S^{(n-1)}(u-1), \tag{3.19}$$

and the series $\Phi_{a_2,...,a_{2n-1}}(u-1)$ is skew symmetric with respect to permutations of the indices $a_2,...,a_{2n-1}$; see [MNO, Section 4]. This allows us to write the left hand side of (3.17) applied to v_i in the form:

$$\frac{2u+1}{2u-1} (2n-2)! \sum_{k=1}^{2n-1} (-1)^{k-1} s_{b_k,-i}^t(-u) \Phi_{b_1,\dots,\widehat{b}_k,\dots,b_{2n-1}}(u-1) \zeta$$

$$= \frac{2u+1}{2u-1} (2n-2)! \theta_{in} \sum_{k=1}^{2n-1} s_{i,-b_k}(-u) (-1)^{k-1} \theta_{-b_k,n} \Phi_{b_1,\dots,\widehat{b}_k,\dots,b_{2n-1}}(u-1) \zeta,$$

where $(b_1, \ldots, b_{2n-1}) = (-n, -n+1, \ldots, n-1)$ and the hat indicates the index to be omitted. Put

$$\Phi_{-b_k}(u-1) := (2n-2)! (-1)^{k-1} \theta_{-b_k,n} \Phi_{b_1,\dots,\widehat{b}_k,\dots,b_{2n-1}}(u-1).$$

Then, taking into account (3.18), we get the following matrix relation:

$$\frac{2u+1}{2u-1}\widetilde{S}(-u)\begin{pmatrix} \Phi_{-n+1}(u-1) \\ \vdots \\ \Phi_{n}(u-1) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \widehat{S}_{nn}(u) \end{pmatrix}$$

Multiplying its both sides by the matrix $\widetilde{S}(-u)^{-1}$ from the left and comparing the nth coordinates of the vectors, we obtain using (3.19) that

$$\frac{2u+1}{2u-1} \operatorname{sdet} S^{(n-1)}(u-1) = (\widetilde{S}(-u)^{-1})_{nn} \, \widehat{s}_{nn}(u),$$

which implies (3.15). \square

4. Representations of the algebras Y(2) and $Y^{-}(2)$

Here we formulate some necessary results on representations of the algebras Y(2) and $Y^{-}(2)$; see [T], [Dr], [CP], [NT2], [Mo2], [Mo4]. Having in mind their applications in Sections 5 and 6 we shall enumerate the generators of Y(2) and $Y^{-}(2)$, as well as rows and columns of 2×2 -matrices, by the symbols -n, n instead of the usual -1, 1.

A representation of the Yangian Y(2) is called *highest weight* if it is generated by a nonzero vector η such that

$$t_{i,i}(u) \eta = \lambda_i(u) \eta, \qquad i = -n, n,$$

$$t_{-n,n}(u) \eta = 0,$$

for certain formal series $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The pair $(\lambda_{-n}(u), \lambda_n(u))$ is called the *highest weight* of the representation. Given arbitrary series $\lambda_{-n}(u), \lambda_n(u)$ there exists a unique, up to an isomorphism, irreducible highest weight representation of Y(2) with the highest weight $(\lambda_{-n}(u), \lambda_n(u))$ which will be denoted by $L(\lambda_{-n}(u), \lambda_n(u))$.

Similarly, a representation of the Yangian $Y^{-}(2)$ is highest weight if it is generated by a nonzero vector η such that

$$s_{n,n}(u) \eta = \mu(u) \eta,$$

$$s_{-n,n}(u) \eta = 0,$$

for a certain formal series $\mu(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ called the *highest weight* of the representation. Given an arbitrary series $\mu(u)$ there exists a unique, up to an isomorphism, irreducible highest weight representation of $Y^{-}(2)$ with the highest weight $\mu(u)$ which will be denoted by $V(\mu(u))$. Every irreducible finite-dimensional representation of the algebra $Y^{-}(2)$ is isomorphic to a unique $V(\mu(u))$.

Given a pair of complex numbers (α, β) such that $\alpha - \beta \in \mathbb{Z}_+$ we denote by $L(\alpha, \beta)$ the irreducible representation of the Lie algebra $\mathfrak{gl}(2)$ with the highest weight (α, β) with respect to the upper triangular Borel subalgebra. We have $\dim L(\alpha, \beta) = \alpha - \beta + 1$. We may regard $L(\alpha, \beta)$ as a Y(2)-module by using the algebra homomorphism Y(2) $\to U(\mathfrak{gl}(2))$ given by

$$t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1}, \qquad i, j \in \{-n, n\}.$$
 (4.1)

The coproduct (3.5) allows one to construct representations of Y(2) of the form

$$L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_n, \beta_n). \tag{4.2}$$

One easily obtains from (3.5) and (4.1) that the tensor product

$$\eta = \omega_1 \otimes \cdots \otimes \omega_n \tag{4.3}$$

of the highest weight vectors ω_i of the $L(\alpha_i, \beta_i)$ generates a highest weight submodule in L with the highest weight $(\alpha(u), \beta(u))$, where

$$\alpha(u) = (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_n u^{-1}),$$

$$\beta(u) = (1 + \beta_1 u^{-1}) \cdots (1 + \beta_n u^{-1}).$$
(4.4)

Hence, if the representation L is irreducible then it is isomorphic to $L(\alpha(u), \beta(u))$. A criterion of irreducibility of representation (4.2) is given by Chari and Pressley [CP] and can be also deduced from results of Tarasov [T] (see [Mo4]). To formulate the result, with each $L(\alpha, \beta)$ associate the *string*

$$S(\alpha, \beta) = \{\beta, \beta + 1, \dots, \alpha - 1\} \subset \mathbb{C}.$$

We say that two strings S_1 and S_2 are in general position if either

- (i) $S_1 \cup S_2$ is not a string, or
- (ii) $S_1 \subseteq S_2$, or $S_2 \subseteq S_1$.

The representation (4.2) of Y(2) is irreducible if and only if the strings $S(\alpha_i, \beta_i)$ and $S(\alpha_i, \beta_i)$ are in general position for all i < j [CP].

The tensor product (4.2) can also be regarded as a representation of the subalgebra $Y^{-}(2) \subset Y(2)$. The following criterion of its irreducibility is given in [Mo4] and will be used in Section 5.

Proposition 4.1. The representation (4.2) of Y⁻(2) is irreducible if and only if each pair of strings $(S(\alpha_i, \beta_i), S(\alpha_j, \beta_j))$ and $(S(\alpha_i, \beta_i), S(-\beta_j, -\alpha_j))$ is in general position for all i < j. \square

If the representation L of $Y^{-}(2)$ defined in (4.2) is irreducible then by (3.6) it is isomorphic to $V(\mu(u))$ with

$$\mu(u) = \alpha(-u)\beta(u), \tag{4.5}$$

where $\alpha(u)$ and $\beta(u)$ are given by (4.4).

It follows from (3.5) and (4.1) that the elements $t_{ij}^{(r)} \in Y(2)$ with $r \geq n+1$ act as zero operators in (4.2). Therefore, the operators

$$T_{ij}(u) = u^n t_{ij}(u) (4.6)$$

are polynomials in u. By (3.6) the same is true for the operators $S_{ij}(u) = u^{2n} s_{ij}(u)$. Note that the defining relations (3.1) allow us to rewrite the formula (3.6) for $s_{n,-n}(u)$ in the form

$$s_{n,-n}(u) = \frac{u+1/2}{u} \left(t_{n,-n}(u) t_{nn}(-u) - t_{n,-n}(-u) t_{nn}(u) \right).$$

Therefore we may introduce another polynomial operator in L by

$$S_{n,-n}^{\natural}(u) = \frac{1}{u+1/2} S_{n,-n}(u) = \frac{(-1)^n}{u} \left(T_{n,-n}(u) T_{nn}(-u) - T_{n,-n}(-u) T_{nn}(u) \right). \tag{4.7}$$

Note that by (3.8) we have $[S_{n,-n}^{\sharp}(u), S_{n,-n}^{\sharp}(v)] = 0.$

Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be an *n*-tuple of complex numbers such that

$$\alpha_i - \gamma_i \in \mathbb{Z}_+, \qquad \gamma_i - \beta_i \in \mathbb{Z}_+, \qquad i = 1, \dots, n.$$
 (4.8)

Introduce the following vector in L

$$\eta_{\gamma} = \prod_{i=1}^{n} S_{n,-n}^{\sharp}(-\gamma_{i}+1) \cdots S_{n,-n}^{\sharp}(-\beta_{i}-1) S_{n,-n}^{\sharp}(-\beta_{i}) \eta, \tag{4.9}$$

where η is defined in (4.3).

Proposition 4.2. Suppose that the representation L of $Y^{-}(2)$ given by (4.2) is irreducible and the strings $S(\alpha_i, \beta_i)$ satisfy the condition

$$S(\alpha_i, \beta_i) \cap S(\alpha_j, \beta_j) = \emptyset$$
 for $i \neq j$. (4.10)

Then the vectors η_{γ} with γ satisfying (4.8) form a basis in L. Moreover, one has the formulas

$$T_{nn}(u) \eta_{\gamma} = (u + \gamma_1) \cdots (u + \gamma_n) \eta_{\gamma}, \tag{4.11}$$

$$T_{n,-n}(-\gamma_i)\,\eta_{\gamma} = \frac{1}{2} \prod_{a=1,\, a \neq i}^{n} \frac{1}{-\gamma_i - \gamma_a} \,\eta_{\gamma+\delta_i},\tag{4.12}$$

$$T_{-n,n}(-\gamma_i) \, \eta_{\gamma} = -2 \, \prod_{k=1}^n (\alpha_k - \gamma_i + 1)(\beta_k - \gamma_i) \cdot \prod_{a=1, \, a \neq i}^n (-\gamma_i - \gamma_a + 1) \, \eta_{\gamma - \delta_i}, \tag{4.13}$$

where δ_i is the n-tuple which has 1 on the ith position and zeroes as remaining entries; it is assumed that $\eta_{\gamma} = 0$ if γ does not satisfy (4.8).

Proof. Since L is irreducible as a Y(2)-module it is isomorphic to the highest weight representation $L(\alpha(u), \beta(u))$. For each γ satisfying (4.8) introduce the vector

$$\widetilde{\eta}_{\gamma} = \prod_{i=1}^{n} T_{n,-n}(-\gamma_i + 1) \cdots T_{n,-n}(-\beta_i - 1) T_{n,-n}(-\beta_i) \eta.$$

The vectors $\{\widetilde{\eta}_{\gamma}\}\$ form a basis in L and the following relations hold:

$$T_{nn}(u)\,\widetilde{\eta}_{\gamma} = (u + \gamma_1)\cdots(u + \gamma_n)\,\widetilde{\eta}_{\gamma},\tag{4.14}$$

$$T_{n,-n}(-\gamma_i)\,\widetilde{\eta}_{\gamma} = \widetilde{\eta}_{\gamma+\delta_i},\tag{4.15}$$

$$T_{-n,n}(-\gamma_i)\,\widetilde{\eta}_{\gamma} = -\prod_{k=1}^n (\alpha_k - \gamma_i + 1)(\beta_k - \gamma_i)\,\widetilde{\eta}_{\gamma - \delta_i}.$$
 (4.16)

This is a special case of the construction of Gelfand–Tsetlin-type bases for representations of the Yangian Y(m) [Mo2], [NT2] (see also [T], [NT1]).

The formulas (4.7), (4.14) and (4.15) imply that

$$S_{n,-n}^{\sharp}(-\gamma_i)\,\widetilde{\eta}_{\gamma} = 2\,\prod_{a=1,\,a\neq i}^n (-\gamma_i - \gamma_a)\,\widetilde{\eta}_{\gamma+\delta_i}.\tag{4.17}$$

Hence, for each γ the vectors η_{γ} and $\widetilde{\eta}_{\gamma}$ coincide up to a nonzero multiple. This proves (4.11). Now (4.12) and (4.13) follow from (4.15) and (4.16). \square

Remark 4.3. The above proof of (4.17) relies on the fact that the $\widetilde{\eta}_{\gamma}$ are eigenvectors for the operators $T_{nn}(u)$; see (4.7). That is, the Gelfand–Tsetlin-type basis $\{\widetilde{\eta}_{\gamma}\}$ in L corresponds to the inclusion $Y(1) \subset Y(2)$ with Y(1) generated by the coefficients of $t_{nn}(u)$.

5. Yangian action on $V(\lambda)^+_{\mu}$

Let us introduce the $2n \times 2n$ -matrix $F = (F_{ij})$ whose ijth entry is the element $F_{ij} \in \mathfrak{g}_n$ (see (1.1)) and set

$$\mathcal{F}(u) = 1 + \frac{F}{u - 1/2}.$$

Denote by $\widehat{\mathcal{F}}(u)$ the image of the Sklyanin comatrix $\widehat{S}(u)$ under the homomorphism $S(u) \mapsto \mathcal{F}(u)$; see (3.10). By (3.11) and (3.14) the mapping

$$\pi: S(u) \mapsto c(u) \frac{2u+1}{2u-2n+1} \widehat{\mathcal{F}}(-u+n-1),$$

where

$$c(u) = \prod_{k=1}^{n} (1 - (k - 1/2)^2 u^{-2}),$$

defines a homomorphism $Y^{-}(2n) \to U(\mathfrak{g}_n)$; cf. [O3], [MO]. The series

$$(1+nu^{-1})^{-1}$$
 sdet $S(u+n-1/2)$

is even in u (see [MNO, Section 4.11]) and so by (3.14) the image of the generator $s_{ij}^{(1)}$ with respect to π coincides with F_{ij} . By (3.8) we then have

$$[F_{ij}, s_{kl}^{\pi}(u)] = \delta_{kj} s_{il}^{\pi}(u) - \delta_{il} s_{kj}^{\pi}(u) - \theta_{k,-j} \delta_{i,-k} s_{-j,l}^{\pi}(u) + \theta_{i,-l} \delta_{-l,j} s_{k,-i}^{\pi}(u), \quad (5.1)$$

where $s_{ij}^{\pi}(u) := \pi(s_{ij}(u))$. This implies that the image of the restriction of π to the subalgebra Y⁻(2) generated by the elements $s_{ij}(u)$ with $i, j \in \{-n, n\}$ is contained in the centralizer $C_n = U(\mathfrak{g}_n)^{\mathfrak{g}_{n-1}}$ and thus defines an algebra homomorphism

$$\pi: \mathbf{Y}^{-}(2) \to \mathbf{C}_n. \tag{5.2}$$

However, the subspace $V(\lambda)^+_{\mu}$ is an irreducible representation of the centralizer C_n ; see [D, Section 9.1]. It follows from [MO, Proposition 4.9] that the algebra C_n is generated by the image of π and the center of $U(\mathfrak{g}_n)$. Since the elements of the center of $U(\mathfrak{g}_n)$ act as scalar operators in $V(\lambda)$, the $Y^-(2)$ -module $V(\lambda)^+_{\mu}$ defined via the homomorphism (5.2) is irreducible.

Note that C_n is a subalgebra in the normalizer Norm J (see Section 2):

$$C_n \hookrightarrow \text{Norm J}$$
.

Thus, using (5.2) and the definition of the Mickelsson algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ we obtain an algebra homomorphism which we still denote by π :

$$\pi: \mathbf{Y}^{-}(2) \to \mathbf{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1}). \tag{5.3}$$

In other words, the elements of Y⁻(2), as operators in the space $V(\lambda)^+_{\mu}$, can be expressed as elements of the Mickelsson algebra $Z(\mathfrak{g}_n,\mathfrak{g}_{n-1})$. An explicit form of the images of the generators of Y⁻(2) under the homomorphism (5.3) is given in the following theorem.

Introduce the polynomials $Z_{ij}(u)$, $i, j \in \{-n, n\}$ with coefficients in the Mickelsson algebra $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$:

$$Z_{n,-n}(u) = \sum_{i=1}^{n-1} z_{ni} z_{n,-i} \prod_{a=1, a \neq i}^{n-1} \frac{u^2 - g_a^2}{g_i^2 - g_a^2} + F_{n,-n} \prod_{a=1}^{n-1} (u^2 - g_a^2),$$
 (5.4)

$$Z_{-n,n}(u) = \sum_{i=1}^{n-1} z_{-i,n} z_{in} \prod_{a=1, a \neq i}^{n-1} \frac{u^2 - g_a^2}{g_i^2 - g_a^2} + F_{-n,n} \prod_{a=1}^{n-1} (u^2 - g_a^2),$$
 (5.5)

$$Z_{n,n}(u) = \sum_{i=-n+1}^{n-1} z_{ni} \, z_{-n,-i} \prod_{\substack{a=-n+1\\a\neq i}}^{n-1} \frac{u+g_a}{g_i - g_a} + (u+g_n) \prod_{\substack{a=-n+1}}^{n-1} (u+g_a), \tag{5.6}$$

$$Z_{-n,-n}(u) = -\sum_{i=-n+1}^{n-1} z_{-i,n} z_{i,-n} \prod_{\substack{a=-n+1\\a\neq i}}^{n-1} \frac{u+g_a}{g_i-g_a} + (u+g_n') \prod_{\substack{a=-n+1\\a\neq i}}^{n-1} (u+g_a),$$
(5.7)

where $g_i := f_i + 1/2 = F_{ii} - i + 1/2$ and $g'_n = -g_n - 2n + 1$.

Theorem 5.1. The images of the generators $s_{ij}(u)$, $i, j \in \{-n, n\}$ of $Y^{-}(2)$ under the homomorphism (5.3) are given by the formulas

$$s_{-n,-n}(u) \mapsto \frac{u+1/2}{u^{2n}} Z_{-n,-n}(u), \qquad s_{-n,n}(u) \mapsto \frac{u+1/2}{u^{2n}} Z_{-n,n}(u),$$

$$s_{n,-n}(u) \mapsto \frac{u+1/2}{u^{2n}} Z_{n,-n}(u), \qquad s_{n,n}(u) \mapsto \frac{u+1/2}{u^{2n}} Z_{n,n}(u).$$
(5.8)

Proof. Consider first the generator $s_{n,n}(u)$. Using Proposition 3.1 we obtain the following formula for the *nn*th matrix element of the matrix $\widehat{\mathcal{F}}(u-1/2)$:

$$\widehat{\mathcal{F}}(u-1/2)_{nn} = \frac{u}{u-1} |1 - \widetilde{F} u^{-1}|_{nn} \operatorname{sdet} \mathcal{F}^{(n-1)}(u-3/2), \tag{5.9}$$

where \widetilde{F} is the submatrix of F obtained by removing the row and column enumerated by -n, and sdet $\mathcal{F}^{(n-1)}(u)$ is the image of the Sklyanin determinant sdet $S^{(n-1)}(u)$ under the homomorphism (3.10). Using the combinatorial interpretation of the quasi-determinant $|1 - \widetilde{F} u^{-1}|_{nn}$ [GKLLRT, Proposition 7.20] we obtain the formula:

$$|1 - \widetilde{F} u^{-1}|_{nn} = 1 - \sum_{k=1}^{\infty} F_{nn}^{(k)} u^{-k}, \tag{5.10}$$

where

$$F_{ab}^{(k)} = \sum F_{ai_1} F_{i_1 i_2} \cdots F_{i_{k-1} b},$$

summed over all values of the indices $i_m \in \{-n+1, \ldots, -1, 1, \ldots, n-1\}$. Let us show that for $k \geq 2$ we have the equality in $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$:

$$F_{nn}^{(k)} = \sum_{i=-n+1}^{n-1} p F_{ni}^{(k-1)} \cdot \prod_{i < a < n} (f_i - f_a) \cdot p F_{in} \prod_{i < a < n} \frac{1}{f_i - f_a},$$
 (5.11)

where p is the extremal projection for \mathfrak{g}_{n-1} ; see Section 2. First, we note that

$$[F_{ni}^{(k-1)}, F_{ab}] = \delta_{ai} F_{nb}^{(k-1)} - \theta_{ab} \, \delta_{i,-b} F_{n,-a}^{(k-1)}, \tag{5.12}$$

where $i, a, b \in \{-n+1, \ldots, n-1\}$. Indeed, the coefficients of sdet S(u) are central in $Y^-(2)$ and so by (3.13) and the definition of π the relations (5.1) hold for the $s_{ij}^{\pi}(u)$ replaced with $(1 - Fu^{-1})_{ij}^{-1}$. Taking the coefficient at u^{-m} we get the well-known formula for the commutator $[F_{ij}, (F^m)_{kl}]$ which implies (5.12). Now we transform the right hand side of (5.11) by using the explicit formula for the generators pF_{in} given in (2.2). By the property (2.1) of p we obtain from (5.12) that if $i > i_1 > \cdots > i_s > -n$ then

$$pF_{ni}^{(k-1)} \cdot F_{ii_1}F_{i_1i_2} \cdots F_{i_{s-1}i_s}F_{i_sn} = 2 pF_{ni_s}^{(k-1)}F_{i_sn},$$

if $i+i_1=0$ or $i_m+i_{m+1}=0$ for a certain m; otherwise this equals $pF_{ni_s}^{(k-1)}F_{i_sn}$ (cf. (2.10)). Using this, we verify by a straightforward calculation that the coefficient at each product $pF_{ni}^{(k-1)}F_{in}$ on the right hand side of (5.11) equals 1, and so it is given by

$$\sum_{i=-n+1}^{n-1} pF_{ni}^{(k-1)} F_{in} = pF_{nn}^{(k)} = F_{nn}^{(k)},$$

which proves (5.11). Further, we have in $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$

$$pF_{ni}^{(k-1)} = pF_{ni}(f_i + n - 1)^{k-2}, \qquad i = -n + 1, \dots, n - 1.$$
 (5.13)

Indeed.

$$pF_{ni}^{(k-1)} = \sum_{a=-n+1}^{n-1} pF_{na}^{(k-2)} F_{ai} = \sum_{a=i}^{n-1} pF_{na}^{(k-2)} F_{ai} = pF_{ni}^{(k-2)} (F_{ii} + n - i - 1),$$

where we have used (2.1) and (5.12). Now (5.13) follows by induction. Thus, rewriting (5.11) and (5.13) in terms of the generators z_{ni} and z_{in} (see (2.3)) we obtain from (5.10) that

$$|1 - \widetilde{F} u^{-1}|_{nn} = 1 - F_{nn} u^{-1} + \sum_{i=-n+1}^{n-1} z_{ni} z_{-n,-i} \frac{1}{u(u - f_i - n)} \prod_{\substack{a=-n+1\\a \neq i}}^{n-1} \frac{1}{f_i - f_a}.$$
(5.14)

The coefficients of the series sdet $\mathcal{F}^{(n-1)}(u)$ belong to the center of $U(\mathfrak{g}_{n-1})$ and so its image under π coincides with its image with respect to the Harish-Chandra homomorphism. The latter was found in different ways in [Mo3, Section 5] and [MN, Section 6]. The result can be written as follows: in $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ we have

$$\operatorname{sdet} \mathcal{F}^{(n-1)}(u) = \prod_{a=1}^{n-1} ((u-n+3/2)^2 - f_a^2) \cdot \prod_{a=1}^{2n-2} \frac{1}{u-a+1/2}.$$
 (5.15)

Finally, using (5.9), (5.14) and (5.15) we find that in $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$

$$s_{n,n}^{\pi}(u) = c(u) \frac{2u+1}{2u-2n+1} \widehat{\mathcal{F}}(-u+n-1)_{nn} = \frac{u+1/2}{u^{2n}} Z_{n,n}(u).$$

To find the image of $s_{n,-n}(u)$ under π we note that by (5.1)

$$[F_{n,-n}, s_{n,n}^{\pi}(u)] = -2 s_{n,-n}^{\pi}(u).$$

This implies that the series

$$\frac{u^{2n}}{u+1/2} s_{n,-n}^{\pi}(u) = -\frac{1}{2} \left[F_{n,-n}, Z_{n,n}(u) \right]$$
 (5.16)

is a polynomial in u, and by the symmetry relation (3.9) this polynomial is even. By (5.6) it is of degree n-1 in u^2 with the highest coefficient $F_{n,-n}$. Moreover, we see from (5.6) that²

$$Z_{n,n}(-g_i) = -z_{ni} z_{-n,-i}, \qquad i = -n+1, \dots, n-1.$$

Hence, using (2.5) we obtain

$$[F_{n,-n}, Z_{n,n}(-g_i)] = -2 z_{ni} z_{n,-i}.$$
(5.17)

Applying the Lagrange interpolation formula to the polynomial (5.16) by using its values at the n-1 points $-g_i$, $i=1,\ldots,n-1$ we prove that

$$-\frac{1}{2}\left[F_{n,-n}, Z_{n,n}(u)\right] = Z_{n,-n}(u). \tag{5.18}$$

Similarly, replacing $F_{n,-n}$ by $F_{-n,n}$ in the above argument we get the formula for the image of $s_{-n,n}(u)$ under π .

Finally, to find the image of $s_{-n,-n}(u)$ we use the following formula implied by (5.1):

$$[F_{n,-n}, [F_{-n,n}, s_{n,n}^{\pi}(u)]] = 2 s_{n,n}^{\pi}(u) - 2 s_{-n,-n}^{\pi}(u). \quad \Box$$

Theorem 5.2. We have an isomorphism of $Y^{-}(2)$ -modules:

$$V(\lambda)^+_{\mu} \simeq L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_n, \beta_n),$$
 (5.19)

where

$$\alpha_i = \min\{\lambda_{i-1}, \mu_{i-1}\} - i + 1/2, \quad i = 2, \dots, n, \quad \alpha_1 = -1/2;
\beta_i = \max\{\lambda_i, \mu_i\} - i + 1/2, \quad i = 1, \dots, n-1, \quad \beta_n = \lambda_n - n + 1/2.$$
(5.20)

In particular, $V(\lambda)^+_{\mu}$ is a Y(2)-module.

*Proof.*³ Let us consider the following vector η_{μ} in $V(\lambda)_{\mu}^{+}$:

$$\eta_{\mu} = z_{n1'}^{\lambda_{1'} - \mu_{1'}} \cdots z_{n(n-1)'}^{\lambda_{(n-1)'} - \mu_{(n-1)'}} \xi, \tag{5.21}$$

where

$$a' = \begin{cases} a, & \text{if } \lambda_a - \mu_a \ge 0, \\ -a, & \text{if } \lambda_a - \mu_a < 0, \end{cases}$$

²If the polynomials $Z_{ij}(u)$ are evaluated in $R(\mathfrak{h})$, we assume, to avoid an ambiguity, that they are written in such a way that the elements of $Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ are to the left from coefficients belonging to $R(\mathfrak{h})(u)$, as appears in their definition.

³Theorem 5.2 was announced in [Mo1].

(recall that $\lambda_{-a} = -\lambda_a$, $\mu_{-a} = -\mu_a$). Let us verify that the vector η_{μ} is nonzero and satisfies the relations

$$Z_{-n,n}(u)\,\eta_{\mu} = 0\tag{5.22}$$

and

$$Z_{nn}(u)\,\eta_{\mu} = (u - \alpha_2)\cdots(u - \alpha_n)(u + \beta_1)\cdots(u + \beta_n)\,\eta_{\mu}. \tag{5.23}$$

Using (2.6) and (2.7) we easily obtain by induction on the degree of the monomial (5.21) that

$$z_{-i',n} \eta_{\mu} = 0, \qquad i = 1, \dots, n-1.$$
 (5.24)

Let us verify that for i = 1, ..., n-1

$$z_{in} \eta_{\mu} = -(m_i + \alpha_2) \cdots (m_i + \alpha_n)(m_i - \beta_1) \cdots (m_i - \beta_n) \eta_{\mu + \delta_i}, \tag{5.25}$$

if $\lambda_i \geq \mu_i$; and

$$z_{-i,n} \eta_{\mu} = -(m_i - \alpha_2 - 1) \cdots (m_i - \alpha_n - 1)(m_i + \beta_1 - 1) \cdots (m_i + \beta_n - 1) \eta_{\mu - \delta_i}, (5.26)$$

if $\lambda_i \leq \mu_i$; here we have set $m_i = \mu_i - i + 1/2$. We shall prove (5.23), (5.25) and (5.26) simultaneously by induction on the degree of the monomial (5.21). If the degree is zero all the relations are obvious. If $\lambda_i = \mu_i$ then both (5.25) and (5.26) hold by (5.24) because in this case $\beta_i = \alpha_{i+1} + 1 = m_i$. Suppose now that $\lambda_i > \mu_i$. By (2.6) we can write

$$z_{in} \eta_{\mu} = z_{in} z_{ni} \eta_{\mu+\delta_i}$$
.

Formula (5.7) gives $z_{in} z_{ni} = -Z_{-n,-n}(-g_{-i})$. However,

$$(-g_{-i})\,\eta_{\mu+\delta_i}=m_i\,\eta_{\mu+\delta_i}.$$

Applying Theorem 5.1 and the symmetry relation (3.9) we obtain

$$Z_{-n,-n}(m_i) = \left(\frac{1}{2m_i} - 1\right) Z_{n,n}(-m_i) - \frac{1}{2m_i} Z_{n,n}(m_i).$$

Using the induction hypotheses we can find $Z_{n,n}(\pm m_i) \eta_{\mu+\delta_i}$ by (5.23). Since $m_i = \alpha_{i+1}$ we have $Z_{n,n}(m_i) \eta_{\mu+\delta_i} = 0$ and (5.25) follows.

The same argument can be applied to prove (5.26). Here we use the relation $z_{-i,n} z_{n,-i} = Z_{-n,-n}(-g_i)$ implied by (5.7).

To prove (5.23) it suffices to check this relation for 2n-1 different values of u because $Z_{n,n}(u)$ is a monic polynomial in u of degree 2n-1. For these take the eigenvalues of $-g_i$ with $i=-n+1,\ldots,n$ on the vector η_{μ} and then use (5.6), (5.25) and (5.26).

Relation (5.22) follows from (2.5) and (5.24)–(5.26). The fact that $\eta_{\mu} \neq 0$ is implied by (5.25) and (5.26). Indeed, applying appropriate operators z_{in} or $z_{-i,n}$

to the vector η_{μ} repeatedly we can obtain the highest weight vector ξ of $V(\lambda)$ with a nonzero coefficient.

Finally, using formulas (5.8) we deduce from (5.22) and (5.23) that η_{μ} is the highest vector of the representation $V(\lambda)^{+}_{\mu}$ of the algebra Y⁻(2), and the highest weight is given by

$$\mu(u) = (1 - \alpha_1 u^{-1}) \cdots (1 - \alpha_n u^{-1})(1 + \beta_1 u^{-1}) \cdots (1 + \beta_n u^{-1}).$$

On the other hand, Proposition 4.1 implies that the representation (5.19) of $Y^{-}(2)$ is irreducible because the strings

$$S(\alpha_1, \beta_1), \ldots, S(\alpha_n, \beta_n), S(-\beta_1, -\alpha_1), \ldots, S(-\beta_n, -\alpha_n)$$

are pairwise in general position. Therefore, it is isomorphic to the highest weight representation $V(\mu(u)) \simeq V(\lambda)_{\mu}^+$; see (4.5). \square

The branching rule for representations of the symplectic Lie algebras [Z1] follows immediately from Theorem 5.2.

Corollary 5.3. The restriction of $V(\lambda)$ to the subalgebra \mathfrak{g}_{n-1} is isomorphic to the direct sum (1.3) of irreducible finite-dimensional representations $V'(\mu)$ of \mathfrak{g}_{n-1} where the multiplicity $c(\mu)$ equals the number of n-tuples of integers ν satisfying the inequalities (1.4).

Proof. Examining the weight diagram of $V(\lambda)$ (see, e.g., [D, Section 7.2]) we find that the representation $V'(\mu)$ can occur in the decomposition (1.3) only if the \mathfrak{g}_{n-1} -highest weight μ satisfies the condition

$$\lambda_{i+1} \le \mu_i \le \lambda_{i-1}, \qquad i = 1, \dots, n-1,$$

 $(\lambda_0 := 0)$. In this case the multiplicity $c(\mu)$ coincides with the dimension of the space $V(\lambda)^+_{\mu}$. By Theorem 5.2,

dim
$$V(\lambda)_{\mu}^{+} = \prod_{i=1}^{n} (\alpha_{i} - \beta_{i} + 1),$$

which is equal to the number of solutions of the inequalities (1.4). \square

6. Construction of the basis in $V(\lambda)$

In this section we complete the proof of Theorem 1.1.

We first construct a basis in the space $V(\lambda)^+_{\mu}$. A basis in $V(\lambda)$ will then be obtained by induction with the use of the branching rule (1.3).

Note that by (2.12), (5.6) and (5.18) we have $Z_{n,-n}(g_n) = z_{n,-n}$ and so the polynomial $Z_{n,-n}(u)$ can be written as follows:

$$Z_{n,-n}(u) = \sum_{i=1}^{n} z_{ni} z_{n,-i} \prod_{a=1, a \neq i}^{n} \frac{u^2 - g_a^2}{g_i^2 - g_a^2},$$

where, as before, $z_{nn} = 1$ and $g_i = F_{ii} - i + 1/2$.

We shall use the notation

$$l_i = \lambda_i - i + 1/2, \qquad \gamma_i = \nu_i - i + 1/2, \qquad m_i = \mu_i - i + 1/2$$
 (6.1)

with i ranging over $\{1, \ldots, n\}$ or $\{1, \ldots, n-1\}$ respectively. Given ν satisfying the inequalities (1.4) consider the vector

$$\xi_{\nu} = \prod_{i=1}^{n-1} z_{ni}^{\nu_i - \mu_i} z_{n,-i}^{\nu_i - \lambda_i} \cdot \prod_{k=l_n}^{\gamma_n - 1} Z_{n,-n}(k) \, \xi, \tag{6.2}$$

where the operators z_{nj} and $Z_{n,-n}(u)$ are defined in (2.3) and (5.4). Here the action of elements of $R(\mathfrak{h})$ is obtained by the extension from that of $U(\mathfrak{h})$. The action is well-defined for those elements whose denominators are not zero operators. One easily checks that the vector ξ_{ν} is well-defined. By definition of the algebra $Z(\mathfrak{g}_n,\mathfrak{g}_{n-1})$ we have $\xi_{\nu} \in V(\lambda)^+$. Moreover, ξ_{ν} is clearly of \mathfrak{g}_{n-1} -weight μ .

Proposition 6.1. The vectors ξ_{ν} with ν satisfying the inequalities (1.4) form a basis in the space $V(\lambda)^{+}_{\mu}$.

Proof. We use Theorem 5.2. The strings $S(\alpha_i, \beta_i)$ obviously satisfy the condition (4.10). By Proposition 4.2 the vectors η_{γ} defined by (4.9) with the *n*-tuple $\gamma = (\gamma_i)$ given in (6.1) form a basis in $V(\lambda)^+_{\mu}$. However, the operator $S^{\natural}_{n,-n}(u)$ coincides with $Z_{n,-n}(u)$ by Theorem 5.1. That is, the vectors

$$\eta_{\gamma} = \prod_{i=1}^{n} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i) \, \eta_{\mu}, \tag{6.3}$$

with η_{μ} defined in (5.21) form a basis in $V(\lambda)_{\mu}^{+}$. Let us show that for each ν satisfying (1.4) we have the equality of corresponding vectors:

$$\eta_{\gamma} = \xi_{\nu}.\tag{6.4}$$

Note first that for any $i = -n + 1, \dots, n - 1$ and any value $u \in \mathbb{C}$ one has

$$[Z_{n,-n}(u), z_{ni}] = 0. (6.5)$$

Indeed, by (5.1) and (5.8) we have $[Z_{n,-n}(u), F_{ni}] = 0$. It remains to apply the extremal projection $p = p_{n-1}$ and use the fact that $Z_{n,-n}(u)$ commutes with \mathfrak{g}_{n-1} . Let b_1, \ldots, b_k be all the indices a among $1, \ldots, n-1$ for which the difference $\lambda_a - \mu_a$ is positive and let c_1, \ldots, c_l be the remaining indices; k + l = n - 1. Using (2.6) and (6.5) rewrite the vector (6.3) as follows:

$$\eta_{\gamma} = \prod_{i=1}^{k} z_{nb_{i}}^{\lambda_{b_{i}} - \mu_{b_{i}}} \cdot \prod_{i=1}^{n} Z_{n,-n}(\gamma_{i} - 1) \cdots Z_{n,-n}(\beta_{i}) \cdot \prod_{i=1}^{l} z_{n,-c_{i}}^{\mu_{c_{i}} - \lambda_{c_{i}}} \xi.$$
 (6.6)

Further, by (5.4) we have

$$Z_{n,-n}(g_i) = z_{ni} z_{n,-i}, \qquad i = 1, \dots, n-1.$$

However,

$$g_i \prod_{i=1}^{l} z_{n,-c_i}^{\mu_{c_i} - \lambda_{c_i}} \xi = \beta_i \prod_{i=1}^{l} z_{n,-c_i}^{\mu_{c_i} - \lambda_{c_i}} \xi.$$

Therefore, given $i \in \{1, \ldots, n-1\}$ the operator $Z_{n,-n}(\beta_i)$ in (6.6) can be replaced with $z_{ni} z_{n,-i}$. Moving the z_{ni} to the left permuting it with the operators of the form $Z_{n,-n}(u)$ we represent the vector again in the form (6.6). Proceeding by induction we shall get the expression for the vector η_{γ} which coincides with (6.2). \square

Given a pattern Λ associated with λ (see Section 1) define the vector ξ_{Λ} by the formula

$$\xi_{\Lambda} = \prod_{k=1,\dots,n}^{\rightarrow} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki} - \lambda_{k-1,i}} z_{k,-i}^{\lambda'_{ki} - \lambda_{ki}} \cdot \prod_{a=l_{kk}}^{l'_{kk} - 1} Z_{k,-k}(a+1/2) \right) \xi.$$

Here the z_{kj} and $Z_{k,-k}(u)$ are elements of the Mickelsson algebra $Z(\mathfrak{g}_k,\mathfrak{g}_{k-1})$, and the numbers l_{kk}, l'_{kk} are defined in (1.5). The branching rule (1.3) and Proposition 6.1 immediately imply the following.

Proposition 6.2. The vectors ξ_{Λ} where Λ runs over all patterns associated with λ form a basis in the representation $V(\lambda)$ of \mathfrak{g}_n . \square

Our next task is to calculate the matrix elements of the generators F_{kk} , $F_{k,-k}$, $F_{-k,k}$, $F_{k-1,-k}$ of the Lie algebra \mathfrak{g}_n in the basis $\{\xi_{\Lambda}\}$. Note that the elements F_{kk} , $F_{k,-k}$, $F_{-k,k}$ belong to the centralizer of the subalgebra \mathfrak{g}_{k-1} in $U(\mathfrak{g}_k)$. Therefore, these operators preserve the subspace of \mathfrak{g}_{k-1} -highest vectors in $V(\lambda)$. So, it suffices to compute the action of these operators with k=n in the basis $\{\xi_{\nu}\}$ of the space $V(\lambda)^+_{\mu}$; see Proposition 6.1.

For F_{nn} we immediately get

$$F_{nn}\,\xi_{\nu} = \left(2\sum_{i=1}^{n}\nu_{i} - \sum_{i=1}^{n}\lambda_{i} - \sum_{i=1}^{n-1}\mu_{i}\right)\xi_{\nu}.\tag{6.7}$$

Further, by (6.3) and (6.4)

$$Z_{n,-n}(\gamma_i) \xi_{\nu} = \xi_{\nu+\delta_i}, \qquad i = 1, \dots, n.$$

However, $Z_{n,-n}(u)$ is a polynomial in u^2 of degree n-1 with the highest coefficient $F_{n,-n}$; see (5.4). Applying the Lagrange interpolation formula with the interpolation points γ_i , $i=1,\ldots,n$ we obtain

$$Z_{n,-n}(u)\,\xi_{\nu} = \sum_{i=1}^{n} \prod_{a=1,\,a\neq i}^{n} \frac{u^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2} \,\xi_{\nu+\delta_i}.$$

Taking here the coefficient at u^{2n-2} we get

$$F_{n,-n}\,\xi_{\nu} = \sum_{i=1}^{n} \prod_{a=1}^{n} \frac{1}{\gamma_i^2 - \gamma_a^2} \,\xi_{\nu+\delta_i}.\tag{6.8}$$

Similarly, we see from (5.5) that $Z_{-n,n}(u)$ is a polynomial in u^2 of degree n-1 with the highest coefficient $F_{-n,n}$. Using the defining relations (3.1) we can write the following formula for the operator $S_{-n,n}(u)$ in $V(\lambda)^+_{\mu}$:

$$S_{-n,n}(u) = \frac{(-1)^n (u+1/2)}{u} \left(T_{-n,n}(u) T_{-n,-n}(-u) - T_{-n,n}(-u) T_{-n,-n}(u) \right).$$

Hence, by Theorem 5.1 we obtain the equality of operators in $V(\lambda)_{\mu}^{+}$:

$$Z_{-n,n}(u) = \frac{(-1)^n}{u} \left(T_{-n,n}(u) T_{-n,-n}(-u) - T_{-n,n}(-u) T_{-n,-n}(u) \right).$$

This implies that $F_{-n,n}$, as an operator in $V(\lambda)^+_{\mu}$, coincides with $2t^{(1)}_{-n,n}$, where $t^{(1)}_{-n,n}$ is the highest coefficient of the polynomial $T_{-n,n}(u)$ which has degree n-1; see (4.6). Using (4.13) we find that

$$T_{-n,n}(-\gamma_i)\,\xi_{\nu} = -2\,\prod_{k=1}^n (\alpha_k - \gamma_i + 1)(\beta_k - \gamma_i) \cdot \prod_{a=1,\, a\neq i}^n (-\gamma_i - \gamma_a + 1)\,\xi_{\nu-\delta_i}.$$

Note that by (5.20) we have

$$\prod_{k=1}^{n} (\alpha_k - \gamma_i + 1)(\beta_k - \gamma_i) = (1/2 - \gamma_i) \prod_{k=1}^{n} (l_k - \gamma_i) \prod_{k=1}^{n-1} (m_k - \gamma_i);$$

see (6.1). Applying the Lagrange interpolation formula to the polynomial $T_{-n,n}(u)$ with the interpolation points $-\gamma_i$, i = 1, ..., n and taking the coefficient at u^{n-1} we finally obtain that

$$F_{-n,n} \xi_{\nu} = 2 \sum_{i=1}^{n} \frac{\prod_{a=1}^{n} (l_{a} - \gamma_{i})(\gamma_{a} + \gamma_{i} - 1) \prod_{a=1}^{n-1} (m_{a} - \gamma_{i})}{\prod_{a=1, a \neq i}^{n} (\gamma_{i} - \gamma_{a})} \xi_{\nu - \delta_{i}}.$$
 (6.9)

To compute the action of the elements $F_{k-1,-k}$ we may only consider the case k=n. The operator $F_{n-1,-n}$ preserves the subspace of \mathfrak{g}_{n-2} -highest vectors in $V(\lambda)$. Therefore it suffices to calculate its action on the basis vectors of the form

$$\xi_{\nu\mu\nu'} = \prod_{i=1}^{n-2} z_{n-1,i}^{\nu'_i - \mu'_i} z_{n-1,-i}^{\nu'_i - \mu_i} \cdot \prod_{a=m_{n-1}}^{\gamma'_{n-1}-1} Z_{n-1,-n+1}(a) \, \xi_{\nu\mu},$$

where $\xi_{\nu\mu} = \xi_{\nu}$ is defined in (6.2), μ' is a fixed \mathfrak{g}_{n-2} -highest weight, ν' is an (n-1)-tuple of integers such that the inequalities (1.4) are satisfied with λ , ν , μ respectively replaced by μ , ν' , μ' , and we set $\gamma'_i = \nu'_i - i + 1/2$. The operator $F_{n-1,-n}$ is permutable with the elements $z_{n-1,i}$ and $Z_{n-1,-n+1}(u)$ which follows from the explicit formulas (2.3) and (5.4). Hence, we can write

$$F_{n-1,-n} \, \xi_{\nu\mu\nu'} = X_{\mu\nu'} \, F_{n-1,-n} \, \xi_{\nu\mu},$$

where $X_{\mu\nu'}$ denotes the operator

$$X_{\mu\nu'} = \prod_{i=1}^{n-2} z_{n-1,i}^{\nu'_i - \mu'_i} z_{n-1,-i}^{\nu'_i - \mu_i} \cdot \prod_{a=m_{n-1}}^{\gamma'_{n-1} - 1} Z_{n-1,-n+1}(a).$$

Let us now apply formula (2.13). We have

$$f_a \xi_{\nu\mu} = (m_a - 1/2) \xi_{\nu\mu}, \qquad a = 1, \dots, n-1;$$
 (6.10)

see (6.2). Further, for $i = 1, \ldots, n-1$

$$X_{\mu\nu'} z_{n-1,-i} z_{ni} \xi_{\nu\mu} = \xi_{\nu,\mu-\delta_i,\nu'}. \tag{6.11}$$

Indeed, by (6.2) $z_{ni} \xi_{\nu\mu} = \xi_{\nu,\mu-\delta_i}$, and $X_{\mu\nu'} z_{n-1,-i} = X_{\mu-\delta_i,\nu'}$ for i < n-1, where we have used (2.6) and (6.5). Let us verify that the latter formula holds for i = n-1 as well. By (2.12) and (5.4) we can write $z_{n-1,-n+1} = Z_{n-1,-n+1}(g_{n-1})$. However,

$$g_{n-1} \xi_{\nu,\mu-\delta_{n-1}} = (m_{n-1} - 1) \xi_{\nu,\mu-\delta_{n-1}},$$

and so

$$X_{\mu\nu'} z_{n-1,-n+1} = X_{\mu\nu'} Z_{n-1,-n+1} (m_{n-1} - 1) = X_{\mu-\delta_{n-1},\nu'},$$

as desired.

Finally, for $j = 1, \ldots, n-1$ consider the expression

$$X_{\mu\nu'} z_{n-1,j} z_{n,-j} \xi_{\nu\mu}. \tag{6.12}$$

First transform the vector $z_{n,-j} \xi_{\nu}$. The calculation is trivial if $\nu_j = \mu_j$, so we shall assume that $\nu_j - \mu_j \ge 1$. We have

$$z_{n,-j} \, \xi_{\nu\mu} = z_{n,-j} \, z_{nj} \, \xi_{\nu,\mu+\delta_j}.$$

By (5.17) and (5.18) we have $z_{n,-j} z_{nj} = Z_{n,-n}(g_j - 1)$. However,

$$(g_j - 1) \,\xi_{\nu,\mu+\delta_j} = m_j \,\xi_{\nu,\mu+\delta_j}.$$

To calculate $Z_{n,-n}(m_j)\xi_{\nu,\mu+\delta_j}$ we apply again the Lagrange interpolation formula (cf. the proof of (6.8)) for the polynomial $Z_{n,-n}(u)$ at the interpolation points γ_i , $i=1,\ldots,n$ and then put $u=m_i$. The result is

$$Z_{n,-n}(m_j)\xi_{\nu,\mu+\delta_j} = \sum_{i=1}^n \prod_{a=1, a\neq i}^n \frac{m_j^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2} \,\xi_{\nu+\delta_i,\mu+\delta_j}.$$
 (6.13)

Let us now transform the operator $X_{\mu\nu'} z_{n-1,j}$, j < n-1. Here the calculation is very similar to the previous one. We shall assume that $\nu'_j - \mu_j \ge 1$. We have

$$X_{\mu\nu'} z_{n-1,j} = X_{\mu+\delta_j,\nu'} z_{n-1,-j} z_{n-1,j}.$$

Using (5.17) and (5.18) again, we can write $z_{n-1,-j} z_{n-1,j} = Z_{n-1,-n+1}(g_j - 1)$. We have

$$(g_j - 1) \xi_{\nu + \delta_i, \mu + \delta_j} = m_j \xi_{\nu + \delta_i, \mu + \delta_j}.$$

Exactly as above, we use the Lagrange interpolation formula for the polynomial $Z_{n-1,-n+1}(u)$ with the interpolation points γ'_r , $r=1,\ldots,n-1$ and then put $u=m_j$. This gives

$$X_{\mu+\delta_j,\nu'}Z_{n-1,-n+1}(m_j) = \sum_{r=1}^{n-1} \prod_{a=1}^{n-1} \frac{m_j^2 - {\gamma'_a}^2}{{\gamma'_r}^2 - {\gamma'_a}^2} X_{\mu+\delta_j,\nu'+\delta_r}.$$
 (6.14)

In the case j=n-1 we write $X_{\mu\nu'}=X_{\mu+\delta_{n-1},\nu'}Z_{n-1,-n+1}(m_{n-1})$ and (6.14) holds for this case as well.

Combining (6.10)–(6.14) we obtain from (2.13)

$$F_{n-1,-n} \, \xi_{\nu\mu\nu'} = \sum_{i=1}^{n-1} \frac{1}{2m_i - 1} \prod_{a=1, \, a \neq i}^{n-1} \frac{1}{(m_i - m_a)(m_i + m_a - 1)} \, \xi_{\nu,\mu-\delta_i,\nu'}$$

$$+ \sum_{i=1}^{n} \sum_{j,r=1}^{n-1} \frac{1}{2m_j - 1} \prod_{a=1, \, a \neq j}^{n-1} \frac{1}{(m_j - m_a)(m_j + m_a - 1)}$$

$$\prod_{a=1, \, a \neq i}^{n} \frac{m_j^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2} \prod_{a=1, \, a \neq r}^{n-1} \frac{m_j^2 - {\gamma_a'}^2}{{\gamma_r'}^2 - {\gamma_a'}^2} \, \xi_{\nu+\delta_i,\mu+\delta_j,\nu'+\delta_r}.$$

$$(6.15)$$

To complete the proof of Theorem 1.1 we rewrite the formulas (6.7)–(6.9) and (6.15) in the notation related to the patterns Λ (see Section 1) to get the matrix elements of the generators F_{kk} , $F_{k,-k}$, $F_{-k,k}$, $F_{k-1,-k}$ in the basis $\{\xi_{\Lambda}\}$ of $V(\lambda)$. The formulas of Theorem 1.1 are given in the normalized basis $\{\zeta_{\Lambda}\}$ where

$$\zeta_{\Lambda} = N_{\Lambda} \, \xi_{\Lambda},$$

and

$$N_{\Lambda} = \prod_{k=2}^{n} \prod_{1 \le i < j \le k} (-l'_{ki} - l'_{kj} - 1)!$$

Theorem 1.1 is proved.

Remark 6.3. As it follows from our arguments, the problem of constructing an orthogonal basis in the \mathfrak{g}_n -module $V(\lambda)$ is in fact reduced to the problem of constructing an orthogonal basis in the $Y^-(2)$ -module $V(\lambda)^+_{\mu}$. A natural way to do this is to find the eigenvectors of a commutative subalgebra in $Y^-(2)$. Such subalgebra is generated by the coefficients of the series $s_{-1,-1}(u) + s_{1,1}(u) = \operatorname{tr} S(u)$. The problem can also be reformulated for the Yangian Y(2) since the commutative subalgebra can be identified with that of Y(2).

References

- [AST1] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N.: Projection operators for simple Lie groups. Theor. Math. Phys. 8, 813–825 (1971)
- [AST2] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N.: Projection operators for simple Lie groups. II. General scheme for constructing lowering operators. The groups SU(n). Theor. Math. Phys. **15**, 392–401 (1973)
- [AST3] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N.: Description of a certain class of projection operators for complex semisimple Lie algebras. Math. Notes **26**, no. 1-2, 499 504 (1979)

- [B] Berele, A.: Construction of Sp-modules by tableaux. Linear and Multilinear Algebra 19, 299–307 (1986)
- [Bi1] Bincer, A.: Missing label operators in the reduction $Sp(2n) \downarrow Sp(2n-2)$. J. Math. Phys. **21**, 671–674 (1980)
- [Bi2] Bincer, A.: Mickelsson lowering operators for the symplectic group. J. Math. Phys. 23, 347–349 (1982)
- [CP] Chari, V., Pressley, A.: Yangians and R-matrices. L'Enseign. Math. 36, 267–302 (1990)
 - [C] Cherednik, I. V.: A new interpretation of Gelfand-Tzetlin bases. Duke Math. J. 54, 563–577 (1987)
- [CK] De Concini, C., Kazhdan, D.: Special bases for S_N and GL(n). Israel J. Math. **40**, no. 3-4, 275–290 (1981)
 - [D] Dixmier, J.: Algèbres Enveloppantes. Paris: Gauthier-Villars 1974
- [Dr] Drinfeld, V. G.: A new realization of Yangians and quantized affine algebras. Soviet Math. Dokl. **36**, 212–216 (1988)
- [GKLLRT] Gelfand, I. M., Krob, D., Lascoux, A., Leclerc, B., Retakh, V. S., Thibon, J.-Y.: Noncommutative symmetric functions. Adv. Math. 112, 218–348 (1995)
 - [GT1] Gelfand, I. M., Tsetlin, M. L.: Finite-dimensional representations of the group of unimodular matrices. Dokl. Akad. Nauk SSSR 71, 825–828 (1950) (Russian). English transl. in: Gelfand, I. M. Collected papers. Vol II, Berlin: Springer-Verlag 1988
 - [GT2] Gelfand, I. M., Tsetlin, M. L.: Finite-dimensional representations of groups of orthogonal matrices. Dokl. Akad. Nauk SSSR 71, 1017–1020 (1950) (Russian). English transl. in: Gelfand, I. M. Collected papers. Vol II, Berlin: Springer-Verlag 1988
 - [GZ1] Gelfand, I. M., Zelevinsky, A.: Models of representations of classical groups and their hidden symmetries. Funct. Anal. Appl. 18, 183–198 (1984)
 - [GZ2] Gelfand, I. M., Zelevinsky, A.: Multiplicities and proper bases for gl_n . In: Group theoretical methods in physics. Vol. II, Yurmala 1985, pp. 147–159. Utrecht: VNU Sci. Press 1986
 - [G1] Gould, M. D.: On the matrix elements of the U(n) generators. J. Math. Phys. **22**, 15–22 (1981)
 - [G2] Gould, M. D.: Wigner coefficients for a semisimple Lie group and the matrix elements of the O(n) generators. J. Math. Phys. **22**, 2376–2388 (1981)
 - [G3] Gould, M. D.: Representation theory of the symplectic groups. I. J. Math. Phys. **30**, 1205–1218 (1989)
 - [GK] Gould, M. D., Kalnins, E. G.: A projection-based solution to the Sp(2n) state labeling problem. J. Math. Phys. **26**, 1446–1457 (1985)
 - [H] Hegerfeldt, G. C.: Branching theorem for the symplectic groups. J. Math. Phys. 8, 1195–1196 (1967)
 - [Ho] Van den Hombergh, A.: A note on Mickelsson's step algebra. Indag. Math. 37, no.1, 42–47 (1975)

- [Ka] Kashiwara, M.: Crystalizing the q-analogue of universal enveloping algebras. Comm. Math. Phys. **133**, 249–260 (1990)
- [Ki] King, R. C.: Weight multiplicities for the classical groups. In: Group theoretical methods in physics. Fourth Internat. Colloq., Nijmegen 1975. Lecture Notes in Phys., Vol. 50, pp. 490–499. Berlin: Springer 1976
- [KS] King, R. C., El-Sharkaway, N. G. I.: Standard Young tableaux and weight multiplicities of the classical Lie groups. J. Phys. A 16, 3153–3177 (1983)
- [KW] King, R. C., Welsh, T. A.: Construction of orthogonal group modules using tableaux. Linear and Multilinear Algebra 33, 251–283 (1993)
 - [K] Kirillov, A. A.: A remark on the Gelfand-Tsetlin patterns for symplectic groups. J. Geom. Phys. 5, 473–482 (1988)
- [KT] Koike, K., Terada, I.: Young-diagrammatic methods for the representation theory of the classical groups of type B_n , C_n , D_n . J. Algebra **107**, 466–511 (1987)
- [LMS] Lakshmibai, V., Musili, C., Seshadri, C. S.: Geometry of G/P. IV. Standard monomial theory for classical types. Proc. Indian Acad. Sci. Sect. A Math. Sci. 88, no. 4, 279–362 (1979)
 - [L] Littelmann, P.: An algorithm to compute bases and representation matrices for SL_{n+1} -representations. J. Pure Appl. Algebra 117/118, 447–468 (1997)
 - [Lu] Lusztig, G.: Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. 3, 447–498 (1990)
 - [M1] Mathieu, O.: Good bases for G-modules. Geom. Dedicata 36, 51–66 (1990)
 - [M2] Mathieu, O.: Bases des représentations des groupes simples complexes (d'après Kashiwara, Lusztig, Ringel et al.). Sémin. Bourbaki, Vol. 1990/91. Astérisque no. 201–203. Exp. no. 743, 421–442 (1992)
- [Mi1] Mickelsson, J.: Lowering operators and the symplectic group. Rep. Math. Phys. 3, 193–199 (1972)
- [Mi2] Mickelsson, J.: Step algebras of semi-simple subalgebras of Lie algebras. Rep. Math. Phys. 4, 307–318 (1973)
- [Mo1] Molev, A.: Representations of twisted Yangians. Lett. Math. Phys. 26, 211–218 (1992)
- [Mo2] Moley, A.: Gelfand–Tsetlin basis for representations of Yangians. Lett. Math. Phys. **30**, 53–60 (1994)
- [Mo3] Molev, A.: Sklyanin determinant, Laplace operators, and characteristic identities for classical Lie algebras. J. Math. Phys. 36, 923–943 (1995)
- [Mo4] Molev, A.: Finite-dimensional irreducible representations of twisted Yangians. Preprint CMA 047-97, Austral. Nat. University, Canberra; q-alg/9711022.
- [MN] Molev, A., Nazarov, M.: Capelli identities for classical Lie algebras. Preprint CMA 003-97, Austral. Nat. University, Canberra; q-alg/9712021.
- [MNO] Molev, A., Nazarov, M., Olshanski, G.: Yangians and classical Lie algebras. Russian Math. Surveys 51:2, 205–282 (1996)
 - [MO] Molev, A., Olshanski, G.: Centralizer construction for twisted Yangians. Preprint CMA 065-97, Austral. Nat. University, Canberra; q-alg/9712050.

- [NM] Nagel, J. G., Moshinsky, M.: Operators that lower or raise the irreducible vector spaces of U_{n-1} contained in an irreducible vector space of U_n . J. Math. Phys. **6**, 682–694 (1965)
- [NT1] Nazarov, M., Tarasov, V.: Yangians and Gelfand–Zetlin bases. Publ. RIMS, Kyoto Univ. 30, 459–478 (1994)
- [NT2] Nazarov, M., Tarasov, V.: Representations of Yangians with Gelfand–Zetlin bases. J. Reine Angew. Math. 496, 181–212 (1998)
- [Ok] Okounkov, A.: Multiplicities and Newton polytopes. In: Olshanski, G. (ed.) Kirillov's Seminar on Representation Theory. Amer. Math. Soc. Transl. 181, pp. 231–244. AMS, Providence RI 1998
- [O1] Olshanski, G. I.: Extension of the algebra U(g) for infinite-dimensional classical Lie algebras g, and the Yangians Y(gl(m)). Soviet Math. Dokl. **36**, 569–573 (1988)
- [O2] Olshanski, G. I.: Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians. In: Kirillov, A. A. (ed.) Topics in Representation Theory. Advances in Soviet Math. 2, pp. 1–66. AMS, Providence RI 1991
- [O3] Olshanski, G. I.: Twisted Yangians and infinite-dimensional classical Lie algebras. In: Kulish, P. P. (ed.) Quantum Groups, Lecture Notes in Math. 1510, pp. 103–120. Berlin-Heidelberg: Springer 1992
- [PH] Pang, S. C., Hecht, K. T.: Lowering and raising operators for the orthogonal group in the chain $O(n) \supset O(n-1) \supset \cdots$, and their graphs. J. Math. Phys. 8, 1233–1251 (1967)
- [P1] Proctor, R.: Odd symplectic groups. Invent. Math. 92, 307–332 (1988)
- [P2] Proctor, R.: Young tableaux, Gelfand patterns, and branching rules for classical groups. J. Algebra **164**, 299–360 (1994)
- [RZ] Retakh, V., Zelevinsky, A.: Base affine space and canonical basis in irreducible representations of Sp(4). Dokl. Acad. Nauk USSR **300**, 31–35 (1988)
 - [S] Shtepin, V. V.: Intermediate Lie algebras and their finite-dimensional representations. Russian Akad. Sci. Izv. Math. 43, 559–579 (1994)
 - [T] Tarasov, V. O.: Irreducible monodromy matrices for the R-matrix of the XXZ-model and lattice local quantum Hamiltonians. Theor. Math. Phys. 63, 440–454 (1985)
- [W] Weyl, H.: Classical Groups, their Invariants and Representations. Princeton NJ: Princeton Univ. Press 1946
- [Wo] Wong, M. K. F.: Representations of the orthogonal group. I. Lowering and raising operators of the orthogonal group and matrix elements of the generators. J. Math. Phys. 8, 1899–1911 (1967)
- [WY] Wong, M. K. F., Yeh, H.-Y.: The most degenerate irreducible representations of the symplectic group. J. Math. Phys. **21**, 630–635 (1980)
- [Z1] Zhelobenko, D. P.: The classical groups. Spectral analysis of their finite-dimensional representations. Russ. Math. Surv. 17, 1–94 (1962)

- [Z2] Zhelobenko, D. P.: Compact Lie groups and their representations. Transl. of Math. Monographs 40 AMS, Providence RI 1973
- [Z3] Zhelobenko, D. P.: S-algebras and Verma modules over reductive Lie algebras. Soviet. Math. Dokl. 28, 696–700 (1983)
- [Z4] Zhelobenko, D. P.: Z-algebras over reductive Lie algebras. Soviet. Math. Dokl. 28, 777–781 (1983)
- [Z5] Zhelobenko, D. P.: On Gelfand–Zetlin bases for classical Lie algebras. In: Kirillov, A. A. (ed.) Representations of Lie groups and Lie algebras, pp. 79–106. Budapest: Akademiai Kiado 1985
- [Z6] Zhelobenko, D. P.: Extremal projectors and generalized Mickelsson algebras on reductive Lie algebras. Math. USSR-Izv. **33**, 85–100 (1989)