The AS-Cohen-Macaulay property for quantum flag manifolds of minuscule weight

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Abstract

It is shown that quantum homogeneous coordinate rings of generalised flag manifolds corresponding to minuscule weights, their Schubert varieties, big cells, and determinantal varieties are AS-Cohen-Macaulay. The main ingredient in the proof is the notion of a quantum graded algebra with a straightening law, introduced by T.H. Lenagan and L. Rigal [J. Algebra **301** (2006), 670-702]. Using Stanley's Theorem it is moreover shown that quantum generalised flag manifolds of minuscule weight and their big cells are AS-Gorenstein.

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1 Introduction

The theory of quantum groups provides noncommutative analogues of coordinate rings of affine algebraic groups and their homogeneous spaces. It is a recurring theme that these algebras resemble their undeformed counterparts with respect to properties that allow a Lie theoretic formulation. As pointed out in [GJ07] this similarity is also a guiding principle for homological properties of quantised coordinate rings. Notably Cohen-Macaulay, (Auslander-) Gorenstein, and (Auslander-) regular conditions should be reflected in the quantum setting. Indeed, for the generic quantised coordinate ring $\mathbb{C}_q[G]$ this is verified in [GJ07, Theorem 0.1], [BZ, Proposition 6.5].

The homogeneous coordinate rings of generalised flag manifolds allow straightforward quantum analogues as $\mathbb{N}_0$-graded algebras [LR92], [Soi92]. A graded analogue of the Cohen-Macaulay property is the AS-Cohen Macaulay property introduced in [vdB97, p. 674], [Jør99]. For quantum generalised flag manifolds it is crucial to establish the AS-Cohen-Macaulay property for the following reason. The commutation relations of quantum algebras are governed by the universal R-matrix. The construction of the R-matrix implies that quantum generalised flag manifolds have enough normal elements in the sense of [Zha97]. Hence the methods developed in that paper may be applied. In particular, given AS-Cohen-Macaulayness, one may use Stanley's Theorem [JZ00, Theorem 6.2] to verify AS-Gorensteinness, which in turn implies the Auslander-Gorenstein and the Cohen-Macaulay property [Zha97, Theorem 0.2]. Thus establishing AS-Cohen-Macaulayness is an essential step towards other desirable homological properties.

In the present paper the AS-Cohen-Macaulay property is verified for quantum homogeneous coordinate rings of generalised flag manifolds corresponding to minuscule weights, their Schubert varieties, big cells, and determinantal varieties. Using Stanley's Theorem it is moreover shown that quantum generalised flag manifolds of minuscule weight and their big cells are AS-Gorenstein.

The main ingredient in the proof is the notion of a quantum graded algebra with a straightening law (ASL) introduced in [LR06], in generalisation of a notion from commutative algebra developed by C. De Concini, D. Eisenbud, and C. Procesi (e.g. [DEP82], [BV88]). In their paper T.H. Lenagan and L. Rigal prove that any quantum graded ASL on a wonderful poset is AS-Cohen-Macaulay. They apply this result to quantum Grassmannians, their Schubert varieties, quantum matrices and the corresponding quantum determinantal varieties. Moreover, in [LR06] AS-Gorensteinness is established via Stanley's Theorem.

The original motivation for the present work was to understand the arguments of [LR06] in a more representation theoretic setting. In particular a consequent use of the Bruhat order, standard monomial theory, and the properties of the universal R-matrix shorten the arguments and allow generalisation to all flag manifolds corresponding to minuscule weights. One slight remark of caution, however, is in place. Following [Jan96] we work over an arbitrary field $\mathbb{K}$ and with a deformation parameter $q \in \mathbb{K} \setminus \{0\}$ which is not a root of unity. No attempt is made to include the case where q is a root of unity, although the results of [LR06] also hold in this setting.

The paper is organised as follows. In Section 2 we fix the usual notations for semisimple Lie algebras and recall the notions of wonderful posets, minuscule weights, and Seshadri's classical result on the standard monomial theory for flag manifolds of minuscule weight. In Section 3 we recall notions and results from the theory of quantum groups and the definition and properties of quantum homogeneous coordinate rings of flag manifolds, their

Schubert varieties, big cells, and determinantal varieties. We also consider standard monomial theory in the minuscule case in Section 3.5. Generically these results are well known. Following [Jan96] however, we take great care that all our statements hold over an arbitrary field $\mathbb{K}$ and for any $q \in \mathbb{K} \setminus \{0\}$ which is not a root of unity. Moreover, for $\mathfrak{g} = \mathfrak{sl}_n$ we identify the representation theoretic definition of quantum homogeneous coordinate rings of Grassmann manifolds with the definition in terms of quantum matrices and their quantum minors.

With these preliminary considerations and [LR06, Theorem 2.2.5] at hand we give a short proof of the AS-Cohen-Macaulay property in Section 4. Moreover, the AS-Gorenstein property for quantum homogeneous coordinate rings of flag manifolds of minuscule weight and their big cells is established in 4.4. This argument does not involve any quantum considerations.

Finally, in 4.5 the AS-Cohen Macaulay and AS-Gorenstein properties are established for the simplest family of quantum flag manifolds corresponding to non-minuscule fundamental weights.

2 Preliminaries

Let $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the complex numbers, the rational numbers, the integers, the positive integers, and the nonnegative integers, respectively.

2.1 Semisimple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra of rank r and $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra. Let $\Delta \subseteq \mathfrak{h}^*$ denote the root system associated with $(\mathfrak{g}, \mathfrak{h})$. Choose an ordered basis $\pi = \{\alpha_1, \dots, \alpha_r\}$ of simple roots for Δ and let Δ^+ (resp. Δ^-) be the set of positive (resp. negative) roots with respect to π. Identify $\mathfrak{h}$ with its dual via the Killing form. The induced nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ is denoted by $(\cdot, \cdot)$. For $\alpha \in \Delta$ let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ denote the corresponding coroot. Let $\omega_i \in \mathfrak{h}^*$, $i = 1, \dots, r$, be the fundamental weights with respect to π and let $P^+(\pi)$ denote the set of dominant integral weights. Moreover, let $\leq$ denote the standard partial ordering on $\mathfrak{h}^*$. In particular, $\mu \leq \gamma$ if and only if $\gamma - \mu \in \mathbb{N}_0\pi$. For $\mu \in P^+(\pi)$ let $V(\mu)$ denote the finite dimensional irreducible $\mathfrak{g}$-module of highest weight μ. We will write $\Omega(V)$ to denote the set of weights of a finite dimensional $\mathfrak{g}$-module V.

Let G denote the connected, simply connected complex Lie group with Lie algebra $\mathfrak{g}$. For any subset $S \subset \pi$ let $P_S \subset G$ be the corresponding standard parabolic subgroup with Lie algebra

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+ \cup (\mathbb{Z}S \cap \Delta^-)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space corresponding to $\alpha \in \Delta$.

Let W be the Weyl group associated to the root system Δ. For any $\alpha \in \Delta^+$ let $s_\alpha \in W$ denote the reflection on the hyperplane orthogonal to α with respect to $(\cdot, \cdot)$. Moreover, for any subset $S \subset \pi$ we write $W_S \subset W$ to denote the subgroup generated by the reflections corresponding to simple roots in S, and we write W^S for the set of minimal length coset representatives of W/W_S in W.

For our purposes it is essential that the Weyl group is a poset with the Bruhat order. For $w, w' \in W$ write $w \to w'$ if there exists $\alpha \in \Delta^+$ such that $w = s_\alpha w'$ and $l(w') = l(w) + 1$ where l denotes the length function on W. The Bruhat order $\leq$ on W is then given by the relation

$$\begin{aligned} w \leq w' \Leftrightarrow &\text{ there exists } n \geq 1,\ w_2, \dots, w_{n-1} \in W, \\ &\text{such that } w = w_1 \to w_2 \to \ldots \to w_n = w'. \end{aligned}$$

Note that we are using the same symbol $\leq$ for the standard partial ordering on the weight lattice and for the Bruhat order. We hope that this does not lead to confusion.

2.2 Wonderful posets

Let $(P, \leq)$ be a poset. Inspired by the notation for the Bruhat order we write $x \to y$ for two elements $x, y \in P$ if $x < y$ and there does not exist an element $z \in P$ such that $x < z < y$. In this case y is called an upper neighbour of x. For our purposes the following slightly simplified notion of a definition from [BV88, Chapter 5.D] is sufficient.

Definition 2.1. *A poset $(P, \leq)$ with a smallest and a greatest element is called wonderful if for any $x, y, z, u \in P$ such that $z \to x \leq u$ and $z \to y \leq u$ there exists an element $w \in P$ such that $x \to w$, $y \to w$, and $w \leq u$.*

In the examples we are interested in P is always a subset of the Weyl group and $\leq$ is the Bruhat order. These posets will have a smallest and a greatest element. Note also that the Weyl group itself with the Bruhat order is in general not wonderful.

Example 2.2. Consider the symmetric group S_4 with standard generators $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$. Then the elements $x = s_1 s_2 s_3$, $y = s_3 s_2 s_1$,

$z = s_1 s_3$ satisfy the relations $z \to x$ and $z \to y$. However, one easily verifies that there exists no element w such that $x \to w$ and $y \to w$. Hence S_4 is not wonderful with the Bruhat order.

2.3 Minuscule weights

Let $\lambda \in P^+(\pi) \setminus \{0\}$ be a minuscule weight in the sense of [Hum72, page 72], i.e. $(\lambda, \alpha^\vee) \in \{0, 1\}$ for all positive roots α. Humphreys calls such weights minimal. Recall that this condition implies that the set of weights $\Omega(V(\lambda))$ consists of one single orbit under the action of the Weyl group. In particular all weight spaces of $V(\lambda)$ are one-dimensional. Moreover, $\lambda = \omega_s$ for some simple root $\alpha_s \in \pi$. A complete list of the possible ω_s can be found also in [Hum72, page 72].

Proposition 2.3. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. Then the following hold:*

1. *Assume that $w, w' \in W^S$ satisfy $w \to w'$. Then $w' = s_i w$ for some $\alpha_i \in \pi$.*

2. *Assume that $w, s_i w, s_j w \in W^S$ and $l(s_i w) = l(s_j w) = l(w)+1$ for some $\alpha_i, \alpha_j \in \pi$. Then $s_i s_j w = s_j s_i w \in W^S$ holds. If moreover $\alpha_i \neq \alpha_j$ then $l(s_i s_j w) = l(w) + 2$.*

3. *Assume that $w, s_i w, s_j w \in W^S$ and $l(s_i w) = l(s_j w) = l(w)-1$ for some $\alpha_i, \alpha_j \in \pi$. Then $s_i s_j w = s_j s_i w \in W^S$ holds. If moreover $\alpha_i \neq \alpha_j$ then $l(s_i s_j w) = l(w) - 2$.*

4. *Two elements $w, w' \in W^S$ satisfy $w \le w'$ if and only if $w\lambda \ge w'\lambda$.*

Proof: We leave it to the reader to verify these elementary facts. One way around the proof is to construct the representation $V(\omega_s)$ explicitly, as it is done for instance in [Jan96, 5A.1] in the quantum case. ■

For any $S \subseteq \pi$ define $W^S_{\le w} = \{w' \in W^S \,|\, w' \le w\}$. The first, the second, and the last statement of the above proposition immediately imply the following result.

Corollary 2.4. *Let $\lambda = \omega_s$ be a minuscule weight, $S = \pi \setminus \{\alpha_s\}$, and $w \in W^S$. Then the poset $(W^S_{\le w}, \le)$ is wonderful. In particular the poset $(W^S, \le)$ is wonderful.*

Remark 2.5. In our later conventions it will be advantageous to consider $W^S_{\le w}$ with the inverse Bruhat order $\ge$. The first, the third, and the last statement of the above proposition imply that for minuscule ω_s the poset $(W^S_{\le w}, \ge)$ is also wonderful. □

2.4 A result from standard monomial theory

Among the many useful applications of standard monomial theory are character formulae for irreducible representations $V(\lambda)$, $\lambda \in P^+(\pi)$, in terms of the Weyl group and its subsets W^S where $S \subset \pi$. We introduce the following notation to describe characters for representations of highest weight $n\omega_s$ where $n \in \mathbb{N}_0$ and ω_s is minuscule. For any subset S of π and $n \in \mathbb{N}_0$ define

$$W^S_{n,\geq} := \{(w_1, \dots, w_n) \in (W^S)^n \mid w_j \geq w_{j+1} \text{ for } j = 1, \dots, n-1\}.$$

For any weight μ let e^μ denote the corresponding element in the group algebra of the weight lattice. Moreover, we write $\mathrm{char}(V)$ to denote the character of a finite dimensional $\mathfrak{g}$-module V. The next result follows from the standard monomial theory developed in [Ses78].

Proposition 2.6. *Let $\lambda = \omega_s$ be a minuscule weight, $S = \pi \setminus \{\alpha_s\}$, and $n \in \mathbb{N}_0$. Then*

$$\mathrm{char}(V(n\lambda)) = \sum_{(w_1,\cdots,w_n)\in W^S_{n,\geq}} e^{w_1\lambda} e^{w_2\lambda} \cdots e^{w_n\lambda}.$$

3 Quantum groups and quantum flag manifolds

In this section we recall well known objects from the theory of quantum groups, in particular quantised algebras of functions on flag manifolds and Schubert varieties. All properties used to prove the main result hold over an arbitrary field $\mathbb{K}$ and for any deformation parameter $q \in \mathbb{K} \setminus \{0\}$ which is not a root of unity.

3.1 Quantum groups

We recall some well know definitions and facts along the lines of [Jan96]. Let $\mathbb{K}$ be an arbitrary field and let $q \in \mathbb{K} \setminus \{0\}$ be not a root of unity. Choose $N \in \mathbb{N}$ minimal such that $(\omega_i, \omega_j) \in \mathbb{Z}/N$ and assume that $\mathbb{K}$ also contains $q^{1/N}$. The quantum universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra generated by elements E_i, F_i, K_i, K_i^{-1}, $i = 1, \dots, r$, and relations given for instance in [Jan96, 4.3]. Let $U_q(\mathfrak{b}^+)$ denote the subalgebra generated by E_i, K_i, K_i^{-1} for $i = 1, \dots, r$. For $\lambda \in P^+(\pi)$ we write $V(\lambda)$ to denote the irreducible $U_q(\mathfrak{g})$-module of highest weight λ. In particular $V(\lambda)$ contains a highest weight vector v_λ which satisfies

$$E_i v_\lambda = 0, \qquad K_i v_\lambda = q^{(\lambda, \alpha_i^\vee)} v_\lambda.$$

We hope it will be clear from the context whether $V(\lambda)$ denotes a $\mathfrak{g}$-module or an $U_q(\mathfrak{g})$-module. The character of the $\mathfrak{g}$-module $V(\lambda)$ coincides with the character of the corresponding $U_q(\mathfrak{g})$-module. Hence we may apply Proposition 2.6 and the Weyl character formula also in the quantum case.

Note that the dual $\mathbb{K}$-vector space $V(\lambda)^*$ is a right $U_q(\mathfrak{g})$-module with action given by $fu(v) = f(uv)$ for all $f \in V(\lambda)^*, v \in V(\lambda), u \in U_q(\mathfrak{g})$. For any $v \in V(\lambda)$ and $f \in V(\lambda)^*$ we write $c^\lambda_{f,v}$ to denote the linear functional on $U_q(\mathfrak{g})$ defined by $c^\lambda_{f,v}(u) = f(uv)$ for all $u \in U_q(\mathfrak{g})$. The quantum coordinate ring $\mathbb{K}_q[G]$ is defined to be the linear span of the all matrix coefficients $c^\lambda_{f,v}$ for all $\lambda \in P^+(\pi)$. It is a $U_q(\mathfrak{g})$-bimodule algebra with right and left action given by $uc^\lambda_{f,v}u' = c^\lambda_{fu',uv}$ for all $u, u' \in U_q(\mathfrak{g})$.

It is an important observation due to V. Drinfeld that the category of type one representations of $U_q(\mathfrak{g})$, i.e. of finite direct sums of representations $V(\lambda)$, is a braided monoidal category [Jan96, 7.3–7.8]. Moreover, for any two type one representations V, W and any weight vectors $v \in V$, $w \in W$ the braiding $\hat{R}_{V,W} : V \otimes W \to W \otimes V$ satisfies

$$\hat{R}_{V,W}(v \otimes w) = q^{(\mathrm{wt}(v),\mathrm{wt}(w))} w \otimes v + \sum_i w_i \otimes v_i \tag{1}$$

where w_i, v_i are weight vectors satisfying $\mathrm{wt}(w_i)<\mathrm{wt}(w)$ and $\mathrm{wt}(v_i)>\mathrm{wt}(v)$. This property follows from the construction of $\hat{R}_{V,W}$. Formula (1) implies that for any $\lambda \in P^+(\pi)$ the vector $v_\lambda \otimes v_\lambda \in V(\lambda) \otimes V(\lambda)$ is an eigenvector of $\hat{R}_{V(\lambda),V(\lambda)}$ with eigenvalue $q^{(\lambda,\lambda)}$.

Remark 3.1. It is possible to translate [LR92, Lemma 2.16] into the language of [Jan96, Chapter 7]. This implies, also for general $\mathbb{K}$ and q not a root of unity, that $V(2\lambda) \subset V(\lambda) \otimes V(\lambda)$ is the full eigenspace of $\hat{R}_{V(\lambda),V(\lambda)}$ corresponding to the eigenvalue $q^{(\lambda,\lambda)}$. $\square$

Remark 3.2. Note that [Jan96, Chapter 7] is formulated for $\mathrm{char}(\mathbb{K}) = 0$ and any q which is transcendental over $\mathbb{Q}$. As pointed out in the introduction to that chapter, however, everything can be extended to the case of general characteristic and q not a root of unity. To this end it is essential to note that the pairing [Jan96, 6.12] remains nondegenerate. This follows from [Jan96, 6.23, 8.30]. $\square$

3.2 Quantum flag manifolds

For any $\lambda \in P^+(\pi)$ define $S \subset \pi$ by $S = S(\lambda) = \{\alpha_i \in \pi \mid (\lambda, \alpha_i) \neq 0\}$. The generalised flag manifold G/P_S is a projective algebraic variety. It can be embedded into projective space $\mathbb{P}(V(\lambda))$ by $G \ni g \mapsto [gv_\lambda]$ where $v_\lambda \in V(\lambda)$

is the highest weight vector and $[v]$ denotes the line represented by $v \in V(\lambda)$. The homogeneous coordinate ring $S[G/P_S]$ with respect to this embedding is isomorphic to $\bigoplus_{n=0}^{\infty} V(n\lambda)^*$ endowed with the Cartan multiplication

$$V(n_1\lambda)^* \otimes V(n_2\lambda)^* \to V((n_1+n_2)\lambda)^*. \tag{2}$$

These structures can immediately be translated into the quantum setting. Indeed, following [LR92], [Soi92], [CP94] we define the quantised homogeneous coordinate ring $S_q[G/P_S]$ of the generalised flag manifold G/P_S to be the subalgebra of $\mathbb{K}_q[G]$ generated by the matrix coefficients $\{c^{\lambda}_{f,v_\lambda} \mid f \in V(\lambda)^*\}$, where v_λ is a highest weight vector of the $U_q(\mathfrak{g})$-module $V(\lambda)$. We will freely identify $f \in V(\lambda)^*$ with the generator $c^{\lambda}_{f,v_\lambda}$. It is immediate from the definition of the multiplication on $\mathbb{K}_q[G]$ that the right $U_q(\mathfrak{g})$-module algebra $S_q[G/P_S]$ is again isomorphic to $\bigoplus_{n=0}^{\infty} V(n\lambda)^*$ endowed with the Cartan multiplication (2). Note that $S_q[G/P_S]$ depends on λ and not only on S even if this is not made explicit in our notation.

Let $\hat{R}^*_{V(\lambda),V(\lambda)} : V(\lambda)^* \otimes V(\lambda)^* \to V(\lambda)^* \otimes V(\lambda)^*$ denote the map dual to $\hat{R}_{V(\lambda),V(\lambda)}$. Recall from the previous subsection that the subspace $V(2\lambda) \subset V(\lambda) \otimes V(\lambda)$ is the eigenspace of $\hat{R}_{V(\lambda),V(\lambda)}$ with corresponding eigenvalue $q^{(\lambda,\lambda)}$. It follows that the generators $f, g \in V(\lambda)^*$ of $S_q[G/P_S]$ satisfy the relations

$$m(\hat{R}^*_{V(\lambda),V(\lambda)}(f \otimes g)) = q^{(\lambda,\lambda)} fg \tag{3}$$

where m denotes the multiplication map.

Remark 3.3. It is moreover known [Bra94] that $S_q[G/P_S]$ is a quadratic algebra. Check carefully that in view of Remark 3.1 Braverman's result also holds in general characteristic and for q not a root of unity. Hence the algebra $S_q[G/P_S]$ is given in terms of generators and relations by (3). This is an instance of a quantum effect. In the commutative case the R-matrix becomes the twist of tensor factors, and there is no similar canonical way to encode the defining Plücker relations of $S[G/P_S]$. $\square$

Remark 3.4. To facilitate later reference we explicitly translate property (1) into properties of $\hat{R}^*_{V(\lambda),V(\lambda)}$. For $f \in V(\lambda)^* \setminus \{0\}$ write $\mathrm{wt}(f) = \mu$ if $f(v) = 0$ for all weight vectors $v \in V(\lambda)$ with $\mathrm{wt}(v) \neq \mu$. For $f, g \in V(\lambda)^*$ formula (1) implies

$$\hat{R}^*_{V(\lambda),V(\lambda)}(f \otimes g) = q^{(\mathrm{wt}(f),\mathrm{wt}(g))} g \otimes f + \sum_i g_i \otimes f_i \tag{4}$$

where $g_i, f_i \in V(\lambda)^*$ are weight vectors such that $\mathrm{wt}(g_i) < \mathrm{wt}(g)$ and $\mathrm{wt}(f_i) > \mathrm{wt}(f)$. $\square$

Remark 3.5. Note that for $\mathfrak{g} = \mathfrak{sl}_n$ and $\lambda = \omega_m$ the ring $S_q[G/P_S]$ coincides with the algebra $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ considered in [LR06]. Indeed, recall that by definition $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is the subalgebra generated by quantum $(m \times m)$-minors in the algebra $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ of quantum $(m \times n)$-matrices. Here we consider $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ as a left $U_q(\mathfrak{sl}_m)$-module algebra and a right $U_q(\mathfrak{sl}_n)$-module algebra generated by $V(\omega_1) \otimes V(\omega_1)^*$, where $V(\omega_1)$ and $V(\omega_1)^*$ are considered as left $U_q(\mathfrak{sl}_m)$-module and right $U_q(\mathfrak{sl}_n)$-module, respectively. With respect to the right $U_q(\mathfrak{sl}_n)$-module algebra structure the space spanned by the quantum $(m \times m)$-minors is isomorphic to $V(\omega_m)^*$. The commutation relations of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ are given in terms of $\hat{R}$-matrices and induce the relations (3) on quantum $(m \times m)$-minors. This follows from the naturality of the braiding and the hexagon equation. As $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is a domain it follows from Remark 3.3 that $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is isomorphic to $S_q[G/P_S]$ as a $U_q(\mathfrak{sl}_n)$-module algebra. $\square$

Lemma 3.6. *For any $\lambda \in P^+(\pi)$ and $S = S(\lambda)$ the algebra $S_q[G/P_S]$ is noetherian.*

Proof: Let $x_1, \ldots, x_{\dim V(\lambda)}$ be a weight basis of $V(\lambda)^*$ such that $\mathrm{wt}(x_i) < \mathrm{wt}(x_j)$ implies $i < j$. It follows from the commutation relation (3) and (4) that for all $1 \le i \le j \le \dim V(\lambda)$ there exist scalars $q_{ij} \in \mathbb{K} \setminus \{0\}$ and $\alpha_{ij}^{st} \in \mathbb{K}$ such that

$$x_i x_j = q_{ij} x_j x_i + \sum_{s=1}^{j-1} \sum_{t=1}^{\dim V(\lambda)} \alpha_{ij}^{st} x_s x_t.$$

Now the lemma follows from [BG02, Proposition I.8.17]. $\blacksquare$

3.3 Quantum Schubert Varieties

For any $\mu \in P^+(\pi)$ and $w \in W$ let $v_{w\mu} \in V(\mu)$ be a weight vector of weight $w\mu$ and let $V_w(\mu) = U_q(\mathfrak{b}^+) v_{w\mu}$ denote the corresponding Demazure module. We write $V_w(\mu)^\perp$ to denote the orthogonal complement of $V_w(\mu)$ in $V(\mu)^*$.

Quantum analogues of Schubert varieties were introduced and studied in [LR92] (cp. also [Soi92], [CP94]). Let S and λ be as in Subsection 3.2, and $w \in W$. Let $\mathfrak{I}_q(w, \lambda)$ be the two sided ideal in $S_q[G/P_S]$ generated by all matrix elements c^λ_{f,v_λ} such that $f \in V_w(\lambda)^\perp$. We call the quotient algebra $S_q^w[G/P_S] = S_q[G/P_S]/\mathfrak{I}_q(w, \lambda)$ the quantised algebra of functions on the Schubert variety corresponding to w. Note that $\mathfrak{I}_q(w, \lambda)$ is right $U_q(\mathfrak{b}^+)$-invariant and hence $S_q^w[G/P_S]$ is a right $U_q(\mathfrak{b}^+)$-module algebra.

Following [Jos95a, 10.1.8] define

$$Q_w = \sum_{n\in\mathbb{N}_0} V_w(n\lambda)^\perp.$$

We write $\pi_w : S_q[G/P_S] \to S_q[G/P_S]/Q_w$ to denote the canonical projection. By [Jos95a, 10.1.8] the subspace $Q_w \subset S_q[G/P_S]$ is a completely prime ideal invariant under the right action of $U_q(\mathfrak{b}^+)$. Note that the proof of this result given in [Jos95a] is also valid for an arbitrary field $\mathbb{K}$ an any $q \in \mathbb{K} \setminus \{0\}$ which is not a root of unity. It is moreover proved in [Jos95b] that the ideals $\mathfrak{I}_q(w,\lambda)$ and Q_w coincide for transcendental q. In Subsection 3.5 we give an elementary proof of this fact for minuscule λ and all q which are not a root of unity.

3.4 Quantum big cells and determinantal varieties

Let $f_e \in V(\lambda)^*$ denote the up to a scalar factor uniquely determined element of weight λ. The commutation relations (3) and (4) imply that the element f_e commutes up to a power of q with all generators of $S_q[G/P_S]$. Hence we can form the localisations $S_q[G/P_S](f_e^{-1})$ and $S_q^w[G/P_S](f_e^{-1})$ of the quantum coordinate rings of flag manifolds and Schubert varieties with respect to the multiplicative set $\{f_e^n \mid n \in \mathbb{N}\}$. As f_e is homogeneous of degree one these localisations are $\mathbb{Z}$-graded and we let $S_q[G/P_S](f_e^{-1})_0$ and $S_q^w[G/P_S](f_e^{-1})_0$ denote the homogeneous components of degree zero. We call $S_q[G/P_S](f_e^{-1})_0$ the quantum big cell of G/P_S. Moreover, we call $S_q^w[G/P_S](f_e^{-1})_0$ for $w \in W$ quantum determinantal varieties.

The algebra $S_q[G/P_S](f_e^{-1})_0$ can be realised as the graded dual of a suitable coalgebra. This observation is not necessary for the proof of Corollary 4.9 but we include it here as we believe it to be noteworthy in itself. Moreover, the construction explains how we want to consider $S_q[G/P_S](f_e^{-1})_0$ and $S_q^w[G/P_S](f_e^{-1})_0$ as graded algebras. It is also used to establish the AS-Gorenstein property for $S_q[G/P_S](f_e^{-1})_0$ in Proposition 4.10.

We introduce some notations to formulate the desired duality. First define a relative height function on the root lattice by

$$\mathrm{ht}_S : \mathbb{Z}\pi \to \mathbb{Z}, \qquad \mathrm{ht}_S\Big(\sum_{\alpha_i\in\pi} n_i\alpha_i\Big) = \sum_{\alpha_i\notin S} n_i.$$

Note that the algebra $S_q[G/P_S](f_e^{-1})_0$ is $\mathbb{N}_0$-graded if one defines $\deg(f_e^{-1}f_\mu) = \mathrm{ht}_S(\lambda-\mu)$ for any generator $f_\mu \in V(\lambda)^*$ of weight μ. Moreover, define $U_q(\mathfrak{p}_S)$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $\{K_i^{\pm1}, E_i, F_j, \mid i =$

$1, \ldots, r$ and $\alpha_j \in S\}$ and let $U_q(\mathfrak{p}_S)^+ = U_q(\mathfrak{p}_S) \cap \ker(\varepsilon)$ denote the augmentation ideal. Then

$$\overline{U}_- := U_q(\mathfrak{g})/U_q(\mathfrak{g})U_q(\mathfrak{p}_S)^+$$

is a $\mathbb{N}_0$-graded coalgebra, where the grading is given by $\deg(u) = \mathrm{ht}_S(\mu)$ if $u \in (\overline{U}_-)_{-\mu}$.

Note that both $S_q[G/P_S](f_e^{-1})_0$ and $\overline{U}_-$ have finite dimensional homogeneous components. Let $\delta(\overline{U}_-)$ denote the graded dual algebra of $\overline{U}_-$. The left $U_q(\mathfrak{g})$-module structure on $\overline{U}_-$ induces the structure of a right $U_q(\mathfrak{g})$-module algebra on $\delta(\overline{U}_-)$.

Proposition 3.7. *There is an isomorphism of right $U_q(\mathfrak{g})$-module algebras $S_q[G/P_S](f_e^{-1})_0 \cong \delta(\overline{U}_-)$.*

Proof: Inside $S_q[G/P_S](f_e^{-1})$ one has an inclusion

$$f_e^{-n}V(n\lambda)^* \subseteq f_e^{-(n+1)}V((n+1)\lambda)^*$$

and hence $S_q[G/P_S](f_e^{-1})_0$ can be written as a direct limit

$$S_q[G/P_S](f_e^{-1})_0 \cong \lim_{n\to\infty} f_e^{-n}V(n\lambda)^*. \tag{5}$$

Moreover, the canonical pairing $\mathbb{K}_q[G] \otimes U_q(\mathfrak{g}) \to \mathbb{K}$ induces a pairing

$$S_q[G/P_S](f_e^{-1})_0 \otimes U_q(\mathfrak{g}) \to \mathbb{K}. \tag{6}$$

It follows from (5) that (6) induces a nondegenerate pairing

$$S_q[G/P_S](f_e^{-1})_0 \otimes \overline{U}_- \to \mathbb{K}. \tag{7}$$

between the algebra $S_q[G/P_S](f_e^{-1})_0$ and the coalgebra $\overline{U}_-$ which respects the $\mathbb{N}_0$-grading of both $S_q[G/P_S](f_e^{-1})_0$ and $\overline{U}_-$. Therefore the algebra $S_q[G/P_S](f_e^{-1})_0$ coincides with the graded dual $\delta(\overline{U}_-)$. ■

Remark 3.8. For $\mathfrak{g} = \mathfrak{sl}_n$ and $\lambda = \omega_s$ it is shown in [HK04, Corollaries 1 and 2] that the graded algebra $\delta(\overline{U}_-)$ coincides with the graded algebra $\mathcal{O}_q(M_{s,n-s}(\mathbb{K}))$ of quantum $(s \times (n-s))$-matrices. In view of the above proposition there is also an isomorphism of graded algebras between $S_q[G/P_S](f_e^{-1})_0$ and $\mathcal{O}_q(M_{s,n-s}(\mathbb{K}))$. Analogous results hold for all cominuscule weights. Note that a direct proof of the isomorphism $S_q[G/P_S](f_e^{-1})_0 \cong \mathcal{O}_q(M_{s,n-s}(\mathbb{K}))$ is also given in [KLR04].

For $\mathfrak{g} = \mathfrak{sl}_n$ and $\lambda = \omega_s$ one can moreover verify, that for suitable $w_t \in W^S$ the algebra $S^{w_t}[G/P_S](f_e^{-1})_0$ coincides with the algebra $\mathcal{O}_q(M_{s,n-s}(\mathbb{K}))/\mathcal{I}_t$ defined for instance in [LR06, 3.5]. This explains the name quantum determinantal variety.

3.5 Standard monomials for minuscule weights

Recall the notation $W^S_{n,\geq}$ introduced in Section 2.4. Moreover, for $n \in \mathbb{N}_0$ and $w \in W^S$ define

$$W^S_{n,\geq,w} = \{(w_1, w_2, \dots, w_n) \in W^S_{n,\geq} \mid w_1 \leq w\}.$$

For $\underline{w} = (w_1, \dots, w_n) \in W^S_{n,\geq}$ we define standard monomials $f_{\underline{w}} \in V(n\lambda)^* \subset S_q[G/P_S]$ by $f_{\underline{w}} = f_{w_1} \dots f_{w_n}$. Here $f_w \in V(\lambda)^*$ denotes the up to a scalar factor uniquely determined element of weight $w\lambda$.

Proposition 3.9. *Let $\lambda = \omega_s$ be minuscule, $S = \pi \setminus \{\alpha_s\}$, and $w \in W^S$.*

1. *The equality $\mathfrak{I}_q(w, \lambda) = Q_w$ holds, i.e. $\pi_w(S_q[G/P_S]) = S_q^w[G/P_S]$.*
2. *The standard monomials $\{f_{\underline{w}} \mid \underline{w} \in W^S_{n,\geq}\}$ form a basis of $V(n\lambda)^* \subset S_q[G/P_S]$.*
3. *The standard monomials $\{f_{\underline{w}} \mid \underline{w} \in W^S_{n,\geq,w}\}$ form a basis of $\pi_w(V(n\lambda)^*) \subset \pi_w(S_q[G/P_S])$.*

Proof: Statement 1. is equivalent to $\mathfrak{I}_q(w, \lambda) \cap V(n\lambda)^* = V_w(n\lambda)^\perp$ for all $n \in \mathbb{N}_0$. Note that all statements of the above proposition hold for $n = 0, 1$. We proceed by induction on n.

We first prove linear independence of the standard monomials $f_{\underline{w}} \in S_q[G/P_S]$ for $\underline{w} \in W^S_{n,\geq}$. Assume there exists a nontrivial linear combination

$$\sum_{\underline{w} \in W^S_{n,\geq}} c_{\underline{w}} f_{\underline{w}} = 0.$$

Choose $w \in W^S$ minimal such that $c_{\underline{w}} \neq 0$ for some $\underline{w} = (w, w_2 \dots, w_n)$. Then in $S_q[G/P_S]/Q_w$ one obtains a relation

$$f_w \left(\sum_{\underline{w}' \in W^S_{n-1,\geq,w}} c_{\underline{w}'} f_{\underline{w}'} \right) = 0 \tag{8}$$

where not all coefficients $c_{\underline{w}'}$ vanish. As $S_q[G/P_S]/Q_w$ is an integral domain one obtains a contradiction to 3. for $n-1$. Hence the standard monomials of length n are linearly independent. By Proposition 2.6 they generate a space of dimension $\dim(V(n\lambda))$ and hence form a basis. This proves 2.

The third claim is proved in a very similar fashion, but now one also performs induction over $l(w)$. Assume there exists a nontrivial linear combination

$$f = \sum_{\underline{w} \in W^S_{n,\geq,w}} c_{\underline{w}} f_{\underline{w}}$$

such that $\pi_w(f) = 0$. Choose $w_1 \in W^S$ minimal such that $c_{\underline{w}} \neq 0$ for some $\underline{w} = (w_1, \dots, w_n)$. The relation $\pi_w(f) = 0$ implies $\pi_{w_1}(f) = 0$. By induction hypothesis we may assume $w_1 = w$. Hence in $S_q[G/P_S]/Q_w$ one again obtains the relation (8), where not all coefficients $c_{\underline{w}'}$ vanish. As $S_q[G/P_S]/Q_w$ is an integral domain one obtains a contradiction to 1. and 3. for $n-1$. Hence the standard monomials $\{f_{\underline{w}} \,|\, \underline{w} \in W^S_{n,\geq,w}\}$ are linearly independent in $\pi_w(V(n\lambda)^*)$. As Q_w is an ideal all $f_{\underline{w}}$ such that $\underline{w} \in W^S_{n,\geq} \setminus W^S_{n,\geq,w}$ belong to Q_w. Hence by 2. the set $\{f_{\underline{w}} \,|\, \underline{w} \in W^S_{n,\geq,w}\}$ is a basis of $\pi_w(V(n\lambda)^*)$.

To verify 1. note that $\mathfrak{I}_q(w,\lambda) \subset Q_w$ because Q_w is an ideal. On the other hand $\dim(Q_w \cap V(n\lambda)^*) \leq \dim(\mathfrak{I}_q(w,\lambda) \cap V(n\lambda)^*)$ by 3. ■

Remark 3.10. Statement 1. of the above proposition reproduces [Jos95b, Théorème 3] for minuscule $\lambda \in P^+(\pi)$. Note that Joseph's proof relies on a specialisation argument and hence yields the above result only for transcendental q. However, assuming here λ to be minuscule we only consider a very special case. In particular, it doesn't generally hold that the standard monomials contain a basis of $\ker(\pi_w)$.

The second part of the above proposition reproduces [LR92, 3.8] for minuscule λ, again avoiding specialisation arguments. Finally, the third part of the above proposition is claimed in [LR92, 4.7] again for arbitrary λ but only for transcendental q. □

4 The AS-Cohen-Macaulay property

The AS-Cohen-Macaulay property appeared in noncommutative algebraic geometry as a graded analogue of the Cohen-Macaulay property for commutative local rings [vdB97], [Jør99]. As we are mainly interested in quantum flag manifolds we will formulate this notion not in full generality but only for a slightly restricted class of graded algebras. We refer the reader to [JZ00] for the general definition and more details.

For any algebra B we write B° to denote the opposite algebra, i.e. the vector space B with multiplication defined by $m(a,b) = ba$ for all $a, b \in B^\circ$.

4.1 Gelfand-Kirillov dimension and depth

Throughout this section we assume that $A = \bigoplus_{k=0}^{\infty} A_k$ is a noetherian, $\mathbb{N}_0$-graded, connected algebra over a field $\mathbb{K}$ and that A is generated by the finite dimensional subspace A_1. Define $V_n = \bigoplus_{k=0}^{n} A_n$ and let

$$\mathrm{GKdim}\, A = \limsup \frac{\log \dim_{\mathbb{K}}(V_n)}{\log n}$$

denote the Gelfand-Kirillov dimension of A.

Example 4.1. For quantised flag manifolds one has $\mathrm{GKdim}\,(S_q[G/P_S]) = l(w^S)+1$ where w^S is the longest element in W^S. This follows from the corresponding result for $q = 1$ because for any $\mu \in P^+(\pi)$ the $U_q(\mathfrak{g})$-module $V(\mu)$ has the same dimension as the corresponding $U(\mathfrak{g})$-module. For the classical result consult e.g. [Har77, Exercise 2.6] and recall that for finitely generated commutative $\mathbb{K}$-algebras the GK-dimension coincides with the Krull dimension [KL00, Theorem 4.5 (a)]. $\square$

For any finitely generated left A-module M the depth of M is defined by

$$\mathrm{depth}_A M = \inf\{i \in \mathbb{N} \,|\, \mathrm{Ext}^i_A(\mathbb{K}, M) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Throughout this section we assume that there exists a sequence $x_1, \ldots, x_r$ of elements in A_1 such that the image of x_i in $A/\langle x_1, \ldots, x_{i-1}\rangle$ is normal for all $i = 2, \ldots, r$ and such that $A/\langle x_1, \ldots, x_r\rangle$ is a finite dimensional $\mathbb{K}$ vector space. As noted in [Zha97, p. 392] the existence of such a normalising sequence implies that A has enough normal elements in the sense of [Zha97, p. 392], [LR06, Definition 2.13]. Under the previous assumptions we may use the following definitions.

Definition 4.2. *Let A be a noetherian, $\mathbb{N}_0$-graded, connected $\mathbb{K}$-algebra which has enough normal elements.*

1. *The algebra A is called AS-Cohen-Macaulay if*
$$\mathrm{depth}_A A = \mathrm{GKdim}\, A = \mathrm{depth}_{A^\circ} A^\circ. \tag{9}$$
2. *The algebra A is called AS-Gorenstein if it is of finite left and right injective dimension.*

The equivalence of the first definition with the standard definition of the AS-Cohen-Macaulay property in terms of local cohomology follows for instance from [LR06, Remark 2.2.1]. Similarly, the equivalence of the above notion of AS-Gorensteinness with [JZ00, Definition 0.2] holds by [Zha97, Proposition 2.3(2)].

Remark 4.3. Let $x_1, \dots, x_{\dim V(\lambda)}$ be a weight basis of $V(\lambda)^*$ such that $\mathrm{wt}(x_i) < \mathrm{wt}(x_j)$ implies $i < j$. It follows from the commutation relation (3) and (1) that the image of this sequence in $S_q^w[G/P_S]$ yields a normalising sequence such that $S_q^w[G/P_S]/\langle x_1, \dots, x_{\dim V(\lambda)}\rangle \cong \mathbb{K}$. Moreover, by Lemma 3.6 the algebra $S_q^w[G/P_S]$ is noetherian. Hence $S_q^w[G/P_S]$ satisfies all the assumptions in Definition 4.2 and it remains to verify (9) in order to obtain the AS-Cohen-Macaulay property. $\square$

4.2 Quantum algebras with a straightening law

We recall the following definitions from [LR06, 1.1]. The second definition is a quantum version of a well known notion from commutative algebra [BV88, 4.A], introduced and developed by C. De Concini, D. Eisenbud, and C.Procesi. Consult [DEP82], [BV88, 4.E, 5.F] for references to the original literatur.

Definition 4.4. *Let A be an $\mathbb{N}_0$-graded algebra over a field $\mathbb{K}$ and Π a finite subset of A equipped with a partial order $\leq$. A standard monomial on Π is an element of A which is either 1 or of the form $f_1 \dots f_s$, for some $s \geq 1$, where $f_1, \dots, f_s \in \Pi$ and $f_1 \leq \dots \leq f_s$.*

Definition 4.5. *Let A be an $\mathbb{N}_0$-graded algebra over a field $\mathbb{K}$ and Π a finite subset of A equipped with a partial order $\leq$. The algebra A is called a quantum graded algebra with a straightening law on the poset $(\Pi, \leq)$ if the following conditions are satisfied.*

1. *The elements of Π are homogeneous with positive degree.*
2. *The elements of Π generate A as a $\mathbb{K}$-algebra.*
3. *The set of standard monomials on Π is linearly independent.*
4. *If $f, g \in \Pi$ are not comparable for $\leq$, then fg is a linear combination of terms F or FG, where $F, G \in \Pi$, $F \leq G$, and $F < f, g$.*
5. *For all $f, g \in \Pi$, there exists $c_{fg} \in \mathbb{K} \setminus \{0\}$ such that $fg - c_{fg}gf$ is a linear combination of terms F or FG, where $F, G \in \Pi$, $F \leq G$, and $F < f, g$.*

To shorten notation we also call A a quantum graded ASL on Π.

The relevance of the above notions for our purposes stems from the following quantum version of [BV88, 5.14].

Theorem 4.6. [LR06, Theorem 2.2.5] *If A is a quantum graded ASL on a wonderful poset, then A is AS-Cohen-Macaulay.*

4.3 The main result

We are now in a position to prove the desired generalisation of [LR06, Theorem 3.4.4]. Recall that $W^S_{\le w} = \{w' \in W^S \,|\, w' \le w\}$.

Proposition 4.7. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. For any $w \in W^S$ the algebra $S_q^w[G/P_S]$ is a quantum graded ASL on the poset $W^S_{\le w}$ with the inverse Bruhat order. In particular $S_q[G/P_S]$ is a quantum graded ASL on W^S.*

Proof: The generators of the algebra $S_q^w[G/P_S]$ can be identified with the poset $W^S_{\le w}$. They are homogeneous of degree one and by Proposition 3.9 the standard monomials are a linearly independent set. Property 4 of Definition 4.5 is obtained analogously to the proof of [LR06, Theorem 3.3.8]. More explicitly, let $w', w'' \in W^S_{\le w}$ be not comparable in the Bruhat order. By Proposition 3.9 one can write in $S_q^w[G/P_S]$

$$f_{w'} f_{w''} = \sum_{j=1}^{n} a_j f_{w_j} f_{w'_j} \tag{10}$$

for some $w_j, w'_j \in W^S_{\le w}$, such that $w \ge w_j \ge w'_j$ for all j, and $a_j \in \mathbb{K} \setminus \{0\}$. For any $j = 1, \ldots, n$ apply the projection $S_q^w[G/P_S] \to S_q^{w_j}[G/P_S]$ to both sides of (10). By Proposition 3.9 the image of the right hand side under this projection is nonzero. Hence so is the image of the left hand side. This implies $w' \le w_j$ and $w'' \le w_j$ for all $j = 1, \ldots, n$. This proves property 4 with respect to the inverse Bruhat order.

It remains to prove property 5. Let $w', w'' \in W^S_{\le w}$. If w' and w'' are not comparable then the previous step shows that property 5 holds for $f_{w'}$ and $f_{w''}$. If on the other hand $w' > w''$ then $w'\omega_s < w''\omega_s$. Hence by relations (3) and (4) as well as Proposition 2.3.4 there exist $w'_j, w''_j \in W^S$, $j = 1, \ldots, n$, and $a_j \in \mathbb{K}$ such that $w'_j > w'$, $w''_j < w''$ and

$$q^{(\omega_s, \omega_s)} f_{w''} f_{w'} = q^{(w'\omega_s, w''\omega_s)} f_{w'} f_{w''} + \sum_{j=1}^{n} a_j f_{w'_j} f_{w''_j}.$$

This concludes the proof of property 5 and hence the proof of the proposition. ■

Proposition 4.7 implies an analogous result for quantum big cells and determinantal varieties. To this end define $W^{S,e} = \{w' \in W^S \,|\, e < w'\}$ and $W^{S,e}_{\le w} = \{w' \in W^S \,|\, e < w' \le w\}$. As the highest weight vector $f_e \in V(\lambda)^*$ q-commutes with all generators of $S_q[G/P_S]$ one immediately obtains the following result.

Corollary 4.8. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. For any $w \in W^S$ the algebra $S_q^w[G/P_S](f_e^{-1})_0$ is a quantum graded ASL on the poset $W^{S,e}_{\leq w}$ with the inverse Bruhat order. In particular $S_q[G/P_S](f_e^{-1})_0$ is a quantum graded ASL on $W^{S,e}$.*

Using Theorem 4.6 and Remark 2.5 one now obtains the AS-Cohen-Macaulay property for quantum flag manifolds of minuscule weight, for their big cell, their Schubert varieties, and for their determinantal varieties.

Corollary 4.9. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. For any $w \in W^S$ the graded algebras $S_q^w[G/P_S]$ and $S_q^w[G/P_S](f_e^{-1})_0$ are AS-Cohen-Macaulay. In particular $S_q[G/P_S]$ and $S_q[G/P_S](f_e^{-1})_0$ are AS-Cohen-Macaulay.*

4.4 The AS-Gorenstein property

Let A be a noetherian, $\mathbb{N}_0$-graded, connected, AS-Cohen-Macaulay algebra with enough normal elements. In this case, by Stanley's Theorem [JZ00, Theorem 6.2], the algebra A is AS-Gorenstein if and only if there exists $m \in \mathbb{N}_0$ such that the Hilbert series $H_A(t) = \sum_n \dim(A_n)t^n$ of A satisfies the functional equation

$$H_A(t) = \pm t^{-m} H_A(t^{-1}) \tag{11}$$

as a rational function over $\mathbb{Q}$. For the quantum algebras considered in this paper the Hilbert series coincide with the Hilbert series of the corresponding commutative algebras. Hence these algebras are AS-Gorenstein if and only if their commutative analogues are Gorenstein (cf. [LR06, Remark 2.1.10(ii)]). We use Stanley's Theorem to verify the AS-Gorenstein property for $S_q[G/P_S]$ and for its big cell $S_q[G/P_S](f_e^{-1})_0$ in the minuscule case.

Proposition 4.10. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. Then the graded algebra $S_q[G/P_S](f_e^{-1})_0$ is AS-Gorenstein.*

Proof: It follows from Proposition 3.7, the PBW-theorem for $U_q(\mathfrak{g})$, and [Kéb99, Proposition 4.2] that the Hilbert series of $S_q[G/P_S](f_e^{-1})_0$ coincides with the Hilbert series of a commutative polynomial ring generated by finitely many elements in positive degrees. Hence the Hilbert series of $S_q[G/P_S](f_e^{-1})_0$ satisfies condition (11). ■

As we weren't able to locate a proof of the Gorenstein property of the commutative analogue of $S_q[G/P_S]$ in the literature we give a proof for the minuscule case, following methods of Stanley's paper [Sta78, Corollary 4.7].

Proposition 4.11. *Let $\lambda = \omega_s$ be a minuscule weight and $S = \pi \setminus \{\alpha_s\}$. Then the graded algebra $S_q[G/P_S]$ is AS-Gorenstein.*

Proof: It follows from Proposition 2.6 that the Hilbert series of $S_q[G/P_S]$ for $(\mathfrak{g} = \mathfrak{b}_n, \omega_s = \omega_n)$ coincides with the Hilbert series for $(\mathfrak{g} = \mathfrak{d}_{n+1}, \omega_s = \omega_{n+1})$. Similarly, for $(\mathfrak{g} = \mathfrak{c}_n, \omega_s = \omega_1)$ the Hilbert series of $S_q[G/P_S]$ coincides with the Hilbert series of a polynomial ring. Hence we may restrict to the case where $\mathfrak{g}$ is of type ADE.

Let ρ denote the half sum of all positive roots and define $\Delta^+_S = \{\alpha \in \Delta^+ \,|\, (\omega_s, \alpha) = 0\}$. It follows from the Weyl character formula that the Hilbert series of $S_q[G/P_S]$ is given by

$$H(t) = \sum_{n=0}^{\infty} p(n) t^n = p\big(t\frac{d}{dt}\big)\frac{1}{1-t} \tag{12}$$

where p denotes the polynomial defined by

$$p(n) = \prod_{\alpha \in \Delta^+ \setminus \Delta^+_S} \frac{n + (\rho, \alpha)}{(\rho, \alpha)}.$$

It is straightforward to check that the function H as a rational function of t satisfies the relation

$$H(t^{-1}) = p\big(-t\frac{d}{dt}\big)\frac{1}{1-t^{-1}} = -\sum_{n=1}^{\infty} p(-n) t^n. \tag{13}$$

Define $r = (\rho, \alpha_0)$ where $\alpha_0 \in \Delta^+$ denotes the longest root, and for $k = 1, \dots, r$ set $c_k = \#\{\alpha \in \Delta^+_S \,|\, (\rho, \alpha) = k\}$. Note that $p(-k) = 0$ for all $k = 1, \dots, r$. Moreover, $c_k = c_{r+1-k}$ and hence $p(n) = (-1)^{l(w^S)} p(-n-r-1)$ holds for all $n \in \mathbb{Z}$. In view of (12) and (13) this implies

$$H(t^{-1}) = (-1)^{l(w^S)+1} t^{r+1} H(t)$$

and the AS-Gorenstein property follows again from Stanley's Theorem. ■

4.5 An example

We end this paper with a first glance at what might happen in the the non-minuscule case. We verify the AS-Cohen-Macaulay property for $\mathfrak{g} = \mathfrak{b}_n = \mathfrak{so}_{2n+1}$ and $\lambda = \omega_1$ the first fundamental weight in the conventions of [Hum72, p. 58]. In this case

$$V(\omega_1) \otimes V(\omega_1) = V(2\omega_1) \oplus V(b_n\omega_2) \oplus V(0)$$

where $b_n = 2$ if $n = 2$ and $b_n = 1$ if $n \geq 3$. The quotient of the tensor algebra $T(V(\omega_1)^*)$ by the ideal generated by the subspace $V(b_n\omega_2)^*$ of $T^2(V(\omega_1)^*)$ coincides with the well known quantum Euclidean space $O_q^{2n+1}(\mathbb{K})$ introduced in [FRT89] for $\mathbb{K} = \mathbb{C}$ (cp. also [KS97, 9.3.2]). The algebra $O_q^{2n+1}(\mathbb{K})$ contains a central element z which spans the component $V(0)^* \subset T^2(V(\omega_1)^*)$. By construction one has

$$S_q[G/P_S] \cong O_q^{2n+1}(\mathbb{K})/\langle z \rangle. \tag{14}$$

It follows from the explicit relations [FRT89, Definition 12], [KS97, 9.3.2] that $O_q^{2n+1}(\mathbb{K})$ is a quantum graded algebra with a straightening law on the poset $\{1, 2, \ldots, 2n+1\}$. Hence Theorem 4.6 implies

$$\mathrm{depth}_{O_q^{2n+1}(\mathbb{K})} O_q^{2n+1}(\mathbb{K}) = \mathrm{GKdim}\, O_q^{2n+1}(\mathbb{K}) = \mathrm{depth}_{O_q^{2n+1}(\mathbb{K})^\circ} O_q^{2n+1}(\mathbb{K})^\circ.$$

Note that $\mathrm{GKdim}\, O_q^{2n+1}(\mathbb{K}) = 2n+1$ and $\mathrm{GKdim}\, S_q[G/P_S] = 2n$ by Example 4.1. In view of (14) it now follows from [LR06, Lemma 2.1.7 (iii)] that $S_q[G/P_S]$ is AS-Cohen-Macaulay. A similar argument proves the AS-Cohen-Macaulay property for the corresponding Schubert varieties. The quantum coordinate ring of the big cell is isomorphic to $O_q^{2n-1}(\mathbb{K})$ if $n \geq 3$. Hence quantum big cells and determinantal varieties are also AS-Cohen-Macaulay in this case. AS-Gorensteinness of $S_q[G/P_S]$ and its big cell follows again directly from Stanley's Theorem.

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