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Twisted Multi-Homogeneous Coordinate Rings

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Artin and Van den Bergh have constructed noncommutative analogues of the homogeneous coordinate ring of a projective scheme. We generalize their results to the multiprojective setting. Such multi-homogeneous coordinate rings include tensor products of twisted homogeneous coordinate rings as well as their Rees algebras. Finally, we show that these last two examples are noetherian (assuming ampleness conditions) though the coordinate ring in general is not.

1. INTRODUCTION

Given a projective scheme X over a field k and an ample invertible sheaf L on X, one can construct the homogeneous coordinate ring

$$B = \bigoplus_{i \geq 0} H^0(X, L^{\otimes i}),$$

which reflects the geometry of X in the sense of Serre's theorem (see [FAC, Chap. III, Sect. 2] or [EGA, Sect. 2.7]). In [AV], Artin and Van den Bergh defined a noncommutative version of this ring by using the same formula but replacing the invertible sheaf with its noncommutative analogue, an invertible bimodule (see below for definitions). Furthermore, this twisted homogeneous coordinate ring, denoted by $B(X; L)$, reflects the geometry of X in the same way the commutative coordinate ring does. This tight connection with geometry enabled Artin and Van den Bergh to prove that $B(X; L)$ is noetherian when L is ample.

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Let $B = B(X; L)$ and let $\mathfrak{m} = B_1 \oplus B_2 \oplus \cdots$ denote its augmentation ideal. Then one expects the Rees algebra $B[\mathfrak{m}t]$ to be noetherian since, algebro-geometrically, it corresponds to a blowing up. Similarly, since the tensor product corresponds to the fibre product, one expects the tensor product of twisted homogeneous coordinate rings to be noetherian. The main goal of this paper is to prove these two facts along the lines of [AV] by interpreting these rings from a geometric standpoint.

In both the case of the Rees algebra and the tensor product, the resulting algebra is bigraded, which in commutative algebra corresponds to biprojective geometry. Given two line bundles L, M on the projective scheme X one can form the bihomogeneous coordinate ring,

$$B(X; L, M) = \bigoplus_{i, j \geq 0} H^0\big(X, L^{\otimes i} \otimes M^{\otimes j}\big).$$

We mimic this construction by replacing L and M with invertible bimodules. The only obstacle is that $M \otimes L$ and $L \otimes M$ are not necessarily isomorphic, let alone canonically isomorphic, so it is unclear how to define multiplication. The construction of the twisted bihomogeneous coordinate ring thus involves the additional data of a commutation relation, which is a bimodule isomorphism $\phi: M \otimes L \to L \otimes M$. We in fact construct multi-homogeneous coordinate rings from an arbitrary number of invertible bimodules. It turns out that the Rees algebra and tensor product of twisted homogeneous coordinate rings are indeed examples of such rings. We show that the multi-homogeneous coordinate ring reflects the geometry as per Serre's theorem if the invertible bimodules satisfy some analogue of ampleness. Unfortunately, such rings are not automatically noetherian. However, we do give a criterion for the ACC to hold which includes the case of the Rees algebra and tensor product of twisted homogeneous coordinate rings.

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2. BIMODULE GENERATORS AND RELATIONS

We wish to construct a bimodule algebra analogous to the construction of algebras via generators and relations. Before recalling Van den Bergh's notion of a bimodule algebra we consider algebras in a general setting.

Let $\mathscr{C}$ be a monoidal, abelian category with infinite direct sums. For the definition of a monoidal category we refer the reader to [M1, Chap. 7, Sect. 1]. Let the bifunctor from the monoidal structure be called tensor product and be denoted by $\otimes$. Assume that the tensor product is right exact in each variable and commutes with infinite direct sums. Hence $\mathscr{C}$ is essentially a tensored category in the sense of [M2, Sect. 6] except that we do not insist that the tensor product be commutative, but we do require infinite direct sums and the compatibility of such sums with tensor products. We define the free algebra on $L_1, \dots, L_s \in \mathscr{C}$ to be

$$T = T(L_1, \dots, L_s) := \bigoplus_{i_1, \dots, i_k} L_{i_1} \otimes \cdots \otimes L_{i_k}, \tag{1}$$

where the indices $i_j \in \{1, \dots, s\}$ and $k \in \mathbb{N}$. As usual, when $k = 0$, we get the empty tensor product which we define to be the two-sided identity I, of $\otimes$. In the future, we will omit the tensor symbol if its meaning is clear. Now T is naturally graded by the free semigroup Γ on s generators. For $\nu \in \Gamma$ we will denote the νth graded component by T_ν.

Recall that an *algebra* or *monoid* in $\mathscr{C}$ is an object A in $\mathscr{C}$ equipped with a multiplication morphism μ: $A \otimes A \to A$ and a unit morphism $I \to A$, such that the usual compatibilities hold (see [M1, Chap. 7, Sect. 3] for details). The object $T(L_1, \dots, L_s)$ is naturally an algebra in $\mathscr{C}$. The unit map is precisely the identification of I with T_e, where e is the identity in Γ. Since tensor products commute with infinite direct sums, the multiplication map can be defined by giving morphisms $\mu_{\nu\eta}$: $T_\nu \otimes T_\eta \to T_{\nu\eta}$. We define $\mu_{\nu\eta}$ to be the canonical isomorphism.

Let $\mathscr{C}_{\text{alg}}$ denote the category of algebras in $\mathscr{C}$ and let A be an object in $\mathscr{C}_{\text{alg}}$. Suppose we are given a set of objects $\{U_\alpha\}_{\alpha \in J}$ in $\mathscr{C}$ and sets of morphisms $\{\phi_{\alpha, i}: U_\alpha \to A\}_{i \in J_\alpha}$ in $\mathscr{C}$. Even though the U_α's are only objects in $\mathscr{C}$, we can still speak of the coequalizer in $\mathscr{C}_{\text{alg}}$ of the diagram

$$U_\alpha \xrightarrow{\phi_{\alpha, i}} A,$$

where α runs through J and the $\phi_{\alpha, i}$ run through J_α. The *coequalizer in* $\mathscr{C}_{\text{alg}}$ is an object B in $\mathscr{C}_{\text{alg}}$ equipped with a morphism π: $A \to B$ in $\mathscr{C}_{\text{alg}}$ which, first, satisfies the property that given any two maps $\phi_{\alpha, i}$, $\phi_{\alpha, j}$: $U_\alpha \to A$ in the diagram, we have the equality $\pi \circ \phi_{\alpha, i} = \pi \circ \phi_{\alpha, j}$ in $\mathscr{C}$ and, second, B is the universal object in $\mathscr{C}_{\text{alg}}$ with respect to this property.

We define a *relation* on T to be a morphism of the form ϕ: $U \to T$, where U is a subobject of T (in $\mathscr{C}$).

DEFINITION 2.1. Let $\Phi = \{\phi_\alpha: U_\alpha \to T\}$ be a set of relations on T. We define the algebra with generators L_i and relations Φ to be the coequalizer in $\mathscr{C}_{\text{alg}}$ of the diagram

$$U_\alpha \underset{i_\alpha}{\overset{\phi_\alpha}{\rightrightarrows}} T,$$

where i_α is the canonical inclusion. We denote this algebra by $T(L)/(\Phi)$.

Hence $T(L)/(\Phi)$ is the "largest" quotient algebra of T which identifies the domains of the ϕ_α's with their images.

EXAMPLE 1. We show here how an algebra over a field k defined by generators and relations can be constructed in this setting. Let $\mathscr{C}$ be the category of vector spaces over k and let $\otimes$ be the usual tensor product. The category $\mathscr{C}_{\text{alg}}$ of algebras in $\mathscr{C}$ is precisely the category of k-algebras. Let A be the algebra with generators x_i for $i = 1, \ldots, n$ and relations $r_\alpha = s_\alpha$, where r_α and s_α are noncommutative polynomials in the x_i. Let $L_i = kx_i$, $U_\alpha = kr_\alpha$, and ϕ_α: $U_\alpha \to T$ be the map which sends $r_\alpha \mapsto s_\alpha$. Then $A \simeq T(L)/(\Phi)$.

We will only be interested in relations which are of the form ϕ: $T_\nu \to T$, where $\nu \in \Gamma$ and the image lies in the sum of finitely many graded components of T. We shall call such relations *monic*.

Since $\otimes$ is a bifunctor, we may, for each monic relation ϕ: $T_\nu \to T$, consider the morphism $i_\xi \otimes \phi \otimes i_\eta$: $T_{\xi\nu\eta} = T_\xi \otimes T_\nu \otimes T_\eta \to T \otimes T \otimes T$, where ξ, $\eta \in \Gamma$ and i_ξ, i_η are the natural inclusions. Abusing notation, we will let $i_\xi \otimes \phi \otimes i_\eta$ also denote the composite

$$T_{\xi\nu\eta} \xrightarrow{i_\xi \otimes \phi \otimes i_\eta} T \otimes T \otimes T \xrightarrow{\mu \circ (\mu \otimes 1)} T,$$

where μ is the multiplication map. We extend this is to an endomorphism $1_\xi \otimes \phi \otimes 1_\eta$ of T defined componentwise by

$$\left(1_\xi \otimes \phi \otimes 1_\eta\right)\big|_{T_{\nu'}} = \begin{cases} i_{\nu'} & \text{if } \nu' \neq \xi\nu\eta \\ \mu \circ (\mu \otimes 1) \circ (i_\xi \otimes \phi \otimes i_\eta) & \text{if } \nu' = \xi\nu\eta. \end{cases}$$

Let $\mathscr{D}$ be the smallest subcategory of $\mathscr{C}$ with one object T and containing the morphisms $1_\xi \otimes \phi \otimes 1_\eta$ for all $\xi, \eta \in \Gamma$ and $\phi \in \Phi$.

As usual, it is convenient to have a notion of a Gröbner basis and hence also a version of the diamond lemma. We translate the appropriate notions in [B] to our context. Let $\leq$ be a partial order on Γ. We say that a morphism of the form ϕ: $T_\nu \to T$ is *decreasing* (with respect to $\leq$) if it has image in $\bigoplus_{\nu' < \nu} T_{\nu'}$. We say that the monic relations Φ are *decreasing* if every relation in Φ is decreasing. Given two morphisms f_1, f_2 in $\mathscr{C}$, we say the *categorical confluence* or *diamond condition* holds for them in $\mathscr{D}$ if there exist morphisms g_1, g_2 in $\mathscr{D}$ such that $g_1 f_1 = g_2 f_2$. In the applications, f_1 and f_2 will always have the correct codomain, T, for composition.

Now suppose $\leq$ is a semigroup partial ordering on Γ with the descending chain condition (see [B, p. 180] for definitions) and Φ is decreasing with respect to this ordering. This ensures, in particular, that each $i_\xi \otimes \phi \otimes i_\eta$ is decreasing for $\xi, \eta \in \Gamma$ and $\phi \in \Phi$. Let $\Lambda \subset \Gamma$ be the subset of elements which are not of the form $\xi\lambda\eta$ for any $\xi, \eta, \lambda \in \Gamma$ with T_λ the domain of some relation. Let $\nu \in \Gamma - \Lambda$. Then there are $\xi, \eta \in \Gamma$ and $\phi \in \Phi$ such that T_ν is the domain of $i_\xi \otimes \phi \otimes i_\eta$. We factor $i_\xi \otimes \phi \otimes i_\eta$ into $T_\nu \to$

$\bigoplus_{\mu \in M} T_\mu \to T$, where $M \subset \Gamma$ is chosen to be as small as possible. If the image of $i_\xi \otimes \phi \otimes i_\eta$ does not lie in $\bigoplus_{\lambda \in \Lambda} T_\lambda$, then there is some $\mu \in M$ which is the domain of some $i_{\xi'} \otimes \phi' \otimes i_{\eta'}$, where $\xi', \eta' \in \Gamma$ and $\phi' \in \Phi$. We can ask again whether or not the image of the composite $(1_{\xi'} \otimes \phi' \otimes 1_{\eta'}) \circ (i_\xi \otimes \phi \otimes i_\eta)$ lies in $\bigoplus_{\lambda \in \Lambda} T_\lambda$ and if not, compose it with another map of the form $1_{\xi''} \otimes \phi'' \otimes 1_{\eta''}$ with $\xi'', \eta'' \in \Gamma$ and $\phi'' \in \Phi$. Continuing in this fashion, the descending chain condition ensures the process terminates. We shall call the resulting composite map a *terminal* morphism from $T_\nu \to T$. By definition, the image of a terminal morphism lies in $\bigoplus_{\lambda \in \Lambda} T_\lambda$.

EXAMPLE 2. As an illustration, we shall construct the Jordan affine plane $A = k\langle x, y\rangle/(yx - xy - x^2)$ (k a field) in two different ways. As in Example 1, let $\mathscr{C}$ be the category of vector spaces over k. One possibility for constructing A is to use two generators $L_1 = kx$ and $L_2 = ky$. Then Γ can be taken to be monomials in L_1 and L_2. Let Φ consist of a single monic relation given by ϕ: $L_2 L_1 \to T(L_1, L_2)$: $yx \mapsto xy + x^2$. Then $A \simeq T(L_1, L_2)/(\Phi)$. Furthermore, if we order the set of monomials Γ lexicographically, then Φ is decreasing. Another way to construct A is to have just the one generator $L = kx \oplus ky$. Let Φ consist of the single monic relation ϕ: $LL \to T$ defined by

$$x^2 \mapsto x^2, \qquad xy \mapsto xy, \qquad yx \mapsto xy + x^2, \qquad y^2 \mapsto y^2.$$

We still have $A \simeq T(L)/(\Phi)$, but we cannot order Γ so that ϕ is decreasing.

The desired version of Bergman's diamond lemma is,

THEOREM 2.2. *Let $\leq$ be a semigroup partial ordering on Γ with the descending chain condition. Let $L_1, \ldots, L_s \in \mathscr{C}$ generate the free algebra T and let Φ be a set of decreasing monic relations on T. Let $\mathscr{D}$ denote the category with one object T, defined above. For any ξ, η, $\zeta \in \Gamma$, assume further that the following two conditions hold*:

(OV) *If there are two distinct relations $\phi_1, \phi_2 \in \Phi$ whose domains are, respectively, $T_{\xi\eta}$ and $T_{\eta\zeta}$ then the morphisms $i_\xi \otimes \phi_2$ and $\phi_1 \otimes i_\zeta$ satisfy the confluence condition in $\mathscr{D}$.*

(IN) *If there are two distinct relations $\phi_1, \phi_2 \in \Phi$ whose domains are, respectively, $T_{\xi\eta\zeta}$ and T_η then ϕ_1 and $i_\xi \otimes \phi_2 \otimes i_\zeta$ satisfy the confluence condition in $\mathscr{D}$.*

Then we have the isomorphism in $\mathscr{C}$,

$$T(L)/(\Phi) \cong \bigoplus_{\lambda \in \Lambda} T_\lambda,$$

where Λ is the subset of Γ defined above. Furthermore, the universal morphism π: $T \to T/(\Phi)$ is a projection which, on each graded component T_ν, can be defined by any terminal morphism from $T_\nu \to T$.

Conversely, suppose $\phi_1, \phi_2 \in \Phi$ *have domains* $T_{\xi\eta}$ *and* $T_{\eta\zeta}$ *as in* (OV). *Choose morphisms* $g_1, g_2 \in \mathit{Mor}\,\mathscr{D}$ *so that* $g_1 \circ (\phi_1 \otimes i_\zeta)$ *and* $g_2 \circ (i_\xi \otimes \phi_2)$ *are terminal. If* $g_1 \circ (\phi_1 \otimes i_\zeta) \neq g_2 \circ (i_\xi \otimes \phi_2)$ *then* $T(L)/(\Phi)$ *is a proper quotient of* $\bigoplus_{\lambda\in\Lambda} T_\lambda$. *A similar statement can be made for the case* (IN).

Comment on the Proof. Bergman's diamond lemma [B, Theorem 1.2] is the case where $\mathscr{C}$ is the category of k-modules, k a commutative ring, and the generators are free rank 1 modules. The proof given there is actually set up in the context given above and so generalizes to the desired theorem painlessly.

Remarks. 1. We have seen that there is an algorithm for finding terminal morphisms. Hence, the converse half of the diamond lemma provides a simple procedure for verifying the hypotheses (OV) and (IN). We will refer to this as checking overlaps or inclusions as the case may be.

2. We will call the set of graded components $\{T_\lambda\}_{\lambda\in\Lambda}$ a Gröbner basis for $T(L)/(\Phi)$.

3. In the example of the Jordan affine plane, the diamond lemma only applies to the first construction. In general, we expect that the more we can decompose the generators L_i, the more information we can extract from the diamond lemma.

We return now to the study of bimodule algebras. Fix a noetherian base scheme S. All morphisms and products will be considered over S unless otherwise stated. Let X be an S-scheme of finite type and let $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$, and $\mathrm{pr}_{ij} : X \times X \times X \to X \times X$ for $1 \le i \le j \le 3$ be the canonical projections. Recall,

DEFINITION 2.3 ([V]). An ($\mathscr{O}_S$-central) $\mathscr{O}_X$-bimodule is a quasi-coherent sheaf M on $X \times X$, where pr_1, pr_2 are relatively locally finite, i.e., given any coherent subsheaf L of M, then if Z is the support of L we have $\mathrm{pr}_1|_Z$, $\mathrm{pr}_2|_Z$ finite. Given two such bimodules, L and M, we define the tensor product to be $L \otimes_{\mathscr{O}_X} M := \mathrm{pr}_{13*}\,(\mathrm{pr}_{12}^* L \otimes_{\mathscr{O}_{X^3}} \mathrm{pr}_{23}^* M)$. We define the tensor product of an $\mathscr{O}_X$-module M with a bimodule L by $M \otimes_{\mathscr{O}_X} L = \mathrm{pr}_{2*}(\mathrm{pr}_1^* M \otimes_{\mathscr{O}_X} L)$. An $\mathscr{O}_X$-bimodule algebra is an algebra $\mathscr{B}$ in the category of $\mathscr{O}_X$-bimodules. A (right) $\mathscr{B}$-module is a quasi-coherent sheaf $\mathscr{M}$ on X equipped with a morphism $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{B} \to \mathscr{M}$ satisfying the usual module axioms.

The category of $\mathscr{O}_X$-bimodules is an abelian category with infinite direct sums. Furthermore, the tensor product and the bimodule $\mathscr{O}_X$ furnish the category with a monoidal structure (see [V, Proposition 2.5]). Finally, $\otimes$ is right exact [V, Proposition 2.6] and it clearly commutes with infinite direct sums. Hence, given a set of bimodule generators $L_1, \dots, L_s$ and relations Φ we may form the corresponding $\mathscr{O}_X$-bimodule algebra with relations, $\mathscr{B} =$

$T(L)/(\Phi)$. Since bimodules are in particular sheaves, we can consider their global sections as well as their higher sheaf cohomology. As was noted in [V, page 452], $H^0(\mathcal{B})$ is an $\mathcal{O}(S)$-algebra for any $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$. The multiplication is given by the composition $H^0(\mathcal{B}) \otimes H^0(\mathcal{B}) \to H^0(\mathcal{B} \otimes \mathcal{B}) \stackrel{H^0(\mu)}{\to} H^0(\mathcal{B})$. The unit map is given by $H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_X) \stackrel{H_0(1)}{\to} H^0(\mathcal{B})$. Ring axioms follow from the corresponding axioms for bimodule algebras and functoriality.

3. TWISTED MULTI-HOMOGENEOUS COORDINATE RINGS

We shall assume henceforth that our base scheme S is the spectrum of an algebraically closed field k and that X is projective. Suppose now that the generating bimodules $L_1, \dots, L_s$ are invertible bimodules in the sense of [AV], i.e., they are units in the semigroup of $\mathcal{O}_X$-bimodules. We consider monic relations of the form $\phi_{ij}\colon L_jL_i \stackrel{\sim}{\to} L_iL_j$ for all $1 \leq i < j \leq s$ which are all ($\mathcal{O}_X$-bimodule) isomorphisms. These shall be referred to as commutation relations for the L_i. They are said to be compatible if in $\operatorname{Hom}(L_kL_jL_i, L_iL_jL_k)$ we have,

$$\begin{aligned}(\phi_{ij} \otimes 1_{L_k}) \circ (1_{L_j} \otimes \phi_{ik}) \circ (\phi_{jk} \otimes 1_{L_i})\\ = (1_{L_i} \otimes \phi_{jk}) \circ (\phi_{ik} \otimes 1_{L_j}) \circ (1_{L_k} \otimes \phi_{ij}).\end{aligned}$$

This is of course nothing more than the overlap hypothesis (OV) in Theorem 2.2. Case (IN) of Theorem 2.2 does not arise. We partially order the set Γ lexicographically. This ordering satisfies the hypotheses of Theorem 2.2 (see for example [B, page 186]).

DEFINITION 3.1. The twisted multi-homogeneous coordinate ring on X with respect to the invertible bimodules $L_1, \dots L_s$ and compatible commutation relations $\Phi = \{\phi_{ij}\}$ is defined to be the k-algebra $H^0(T(L_1, \dots, L_s)/(\Phi))$. We denote this algebra by $B(X; L_1, \dots, L_s; \phi_{ij})$ omitting arguments ϕ_{ij}, L_i, and X when they are understood.

One obtains the commutative multi-homogeneous coordinate ring when the L_i's are invertible sheaves and the commutation relations are the canonical morphisms $\phi_{ij}\colon L_jL_i \to L_iL_j$ which map $s \otimes t \mapsto t \otimes s$ for every section s of L_j, and t of L_i.

We apply Theorem 2.2 to $T(L)/(\Phi)$. First note that one can define for each graded component T_ν, the degree in L_i. Hence, we can also define the multi-degree as the s-tuple of degrees in the L_i's. Now, $T(L)/(\Phi) = \bigoplus_{n_1, \dots, n_s \geq 0} \mathcal{B}_{n_1, \dots, n_s}$, where $\mathcal{B}_{n_1, \dots, n_s} = L_1^{n_1} \cdots L_s^{n_s}$. This is a $\mathbb{Z}^s$-graded ring. Note that since the commutation relations ϕ_{ij} are isomorphisms, Theorem 2.2 gives a canonical isomorphism between all graded components of a given multi-degree. Hence, we can take for $\mathcal{B}_{n_1,\dots,n_s}$ any graded component T_ν with multi-degree $(n_1, \dots, n_s)$. It will often be convenient to do this and we shall do so in the future without further comment. Taking global sections we obtain a similar decomposition of $B(X; L_1, \dots, L_s; \phi_{ij})$ into multi-graded components.

As an exercise, we shall unravel the definition of a twisted multi-homogeneous coordinate ring in the case where there are only two invertible bimodules, expressing our result in the pre-[AV] language of say the [ATV] paper. We need some basic facts from [AV, Lemma 2.11]. Recall that every invertible bimodule has the form $L_\sigma := \pi_1^* L$, where L is an invertible sheaf on X, σ is an automorphism of X, and $\pi_1\colon G \to X$ is the first projection of the graph $G \subset X \times X$ of σ to X. Clearly $H^0(L_\sigma) = H^0(L)$. Let L_σ, M_τ be two such invertible bimodules. We have a tensor product formula for invertible bimodules (see [AV, Lemma 2.14]), namely, $L_\sigma M_\tau = (L \otimes \sigma^* M)_{\tau\sigma}$. Hence for there to be a commutation relation between L_σ and M_τ, we must have a sheaf isomorphism $\phi\colon M \otimes \tau^* L \xrightarrow{\sim} L \otimes \sigma^* M$ and $\tau\sigma = \sigma\tau$. There are no overlap checks so we may construct a twisted bihomogeneous coordinate ring from these data. From the previous paragraph we see, using the tensor product formula, that

$$B(X; L_\sigma, M_\tau; \phi) = \bigoplus_{i,\, j \geq 0} H^0\big(L \otimes \sigma^* L \otimes \cdots \sigma^{i-1*} L \otimes \sigma^{i*} M \otimes (\sigma^i \tau)^* M \otimes \cdots \otimes (\sigma^i \tau^{j-1})^* M\big), \qquad (2)$$

where by abuse of notation we have used ϕ to represent the bimodule isomorphism induced by the sheaf isomorphism and we have chosen the Gröbner basis. In [AV, Eq. (1.2)], the multiplication rule was given by the map

$$\begin{aligned} \mu\colon & H^0(L \otimes \cdots \otimes (\sigma^{i_1}\tau^{j_1-1})^* M) \otimes H^0(L \otimes \cdots \otimes (\sigma^{i_2}\tau^{j_2-1})^* M) \\ & \to H^0(L \otimes \cdots \otimes (\sigma^{i_1}\tau^{j_1-1})^* M \otimes (\sigma^{i_1}\tau^{j_1})^* L \otimes \cdots \otimes (\sigma^{i_1+i_2}\tau^{j_1+j_2-1})^* M), \end{aligned}$$

which sends

$$a \otimes b \mapsto a \otimes (\sigma^{i_1}\tau^{j_1})^* b.$$

Of course the right-hand side must be identified with its image in $H^0(T(L_\sigma, M_\tau)/(\phi))$. Alternatively, if one wishes to convert this back into the Gröbner basis as in Eq. (2) then one needs to compose this with the map

$$\begin{aligned} & H^0(L \otimes \cdots \otimes (\sigma^{i_1}\tau^{j_1-1})^* M \otimes \big(\sigma^{i_1}\tau^{j_1}\big)^* L \otimes \cdots \otimes \big(\sigma^{i_1+i_2}\tau^{j_1+j_2-1}\big)^* M) \\ & \xrightarrow{\sim} H^0\big(L_\sigma^{i_1} M_\tau^{j_1} L_\sigma^{i_2} M_\tau^{j_2}\big) \xrightarrow{H^0(\pi)} H^0\big(L_\sigma^{i_1+i_2} M_\tau^{j_1+j_2}\big) \\ & \xrightarrow{\sim} H^0\big(L \otimes \cdots \otimes (\sigma^{i_1+i_2-1})^* L \otimes (\sigma^{i_1+i_2})^* M \otimes \cdots \otimes (\sigma^{i_1+i_2}\tau^{j_1+j_2-1})^* M\big), \end{aligned}$$

where the first and last morphisms are the canonical ones and π is the universal morphism of Theorem 2.2 appropriately restricted. For example, multiplication of the $(2,1)$-graded piece with the $(1,1)$-graded piece is given by $a \otimes b \mapsto (\mathrm{id}_{L\otimes\sigma^* L} \otimes \sigma^{2*}\pi \otimes \mathrm{id}_{(\sigma^3\tau)^* M})(a \otimes (\sigma^2\tau)^* b)$.

One can see from this computation that multiplication in the [ATV] language of invertible sheaves and their twists is considerably more complicated than in the language of invertible bimodules. There are several conceptual advantages in working with invertible bimodules as opposed to invertible sheaves and their twists. For example, the definition of a commutation relation is much more transparent in the bimodule setting and associativity is an immediate consequence of the definition of multiplication.

We now turn our attention to generalizing Serre's theorem along the lines of [AV, Theorem 3.12]. To this end we define,

DEFINITION 3.2. Let $\geq$ denote the partial order on $\mathbb{Z}^s$ defined by taking the product order of the usual order on each $\mathbb{Z}$. The expression "for $i_1, \dots, i_s \in \mathbb{Z}^s$ large enough" will mean for all $(i_1, \dots, i_s) \geq (m_1, \dots, m_s)$ for some fixed $m_1, \dots, m_s$. We say that the generating bimodules $L_1, \dots, L_s$ for a twisted multi-homogeneous ring form a right ample set if given any coherent sheaf F on X, we have $H^q(FL^{i_1} \cdots L^{i_s}) = 0$ for $q > 0$ and $i_1, \dots, i_s$ large enough.

We leave it to the reader to define left ample. Ample will mean left and right ample.

DEFINITION 3.3. We say that a $\mathbb{Z}^s$-graded object M in an abelian category is first quadrant bounded if $M_{i_1 \cdots i_s} = 0$ for $i_1, \dots, i_s$ large enough. Direct limits of such objects are said to be torsion. Let *tors* denote the category of torsion $\mathbb{Z}^s$-graded right B-modules.

Let $\mathcal{A}$ be an $\mathcal{O}_X$-bimodule algebra graded by a group G. If e is the identity of G, assume that $\mathcal{A}_e \simeq \mathcal{O}_X$. We say that $\mathcal{A}$ is *strongly G-graded* if the multiplication map restricts to surjections $\mathcal{A}_g \otimes \mathcal{A}_h \to \mathcal{A}_{gh}$ for all $g, h \in G$.

Let Gr-B denote the category of (multi-) graded right B-modules and let $\mathcal{O}_X$-Mod denote the category of quasi-coherent sheaves on X. Then Serre's theorem in this case is

THEOREM 3.4. *Let B be a twisted homogeneous coordinate ring with respect to a right ample set of invertible bimodules $L_1, \dots, L_s$ and compatible commutation relations $\Phi = \{\phi_{ij}\}$. There exists a strongly $\mathbb{Z}^s$-graded bimodule algebra $\mathcal{B}$ such that $T(L)/(\Phi) = \mathcal{B}_{\geq 0}$. Further, $\mathcal{B}$ has the property that the functors*

$$H^0(\cdot \otimes_{\mathcal{O}_X} \mathcal{B})_{\geq 0} \colon \mathcal{O}_X\text{-Mod} \to \text{Gr-}B$$

$$(\cdot \otimes_B \mathcal{B})_0 \colon \text{Gr-}B \to \mathcal{O}_X\text{-Mod}$$

are adjoint and induce inverse category equivalences between $\mathcal{O}_X$-Mod *and* Gr-B/tors.

Note. For a right B-module M, $M \otimes_B \mathcal{B}$ is defined to be the sheaf associated to the presheaf $U \mapsto M \otimes_B \mathcal{B}(X \times U)$.

Proof. The proof in the $\mathbb{Z}$-graded case given in [AV] carries over readily. The desired category equivalence comes from composing the equivalences

$$\mathcal{O}_X\text{-Mod} \to \text{Gr-}\mathcal{B}\colon F \mapsto F \otimes_{\mathcal{O}_X} \mathcal{B}$$
$$\text{Gr-}\mathcal{B} \to \mathcal{O}_X\text{-Mod}\colon \mathcal{M} \mapsto \mathcal{M}_0$$

and

$$\text{Gr-}\mathcal{B} \to \text{Gr-}B/(\text{tors})\colon \mathcal{M} \mapsto H^0(\mathcal{M})_{\geq 0}$$
$$\text{Gr-}B/(\text{tors}) \to \text{Gr-}\mathcal{B}\colon M \mapsto M \otimes_B \mathcal{B}.$$

We first construct the ring $\mathcal{B}$. Let M_i for $i = 1, \dots, s$ be the inverses of the L_i. We consider a bimodule algebra with generating bimodules L_i, M_i, $i = 1, \dots, s$, and monic relations of the form

$$\begin{array}{ll} \phi_{ij}\colon L_jL_i \xrightarrow{\sim} L_iL_j & \phi_{ij}^{\vee}\colon M_jM_i \xrightarrow{\sim} M_iM_j \\ \phi_i\colon L_iM_i \xrightarrow{\sim} \mathcal{O}_X & \phi_i'\colon M_iL_i \xrightarrow{\sim} \mathcal{O}_X \\ \phi_{ij}'\colon M_jL_i \xrightarrow{\sim} L_iM_j & \phi_{ij}'^{\vee}\colon L_jM_i \xrightarrow{\sim} M_iL_j \end{array}$$

to be defined below.

We need first observe that inverting an invertible bimodule L is a functor on the category of invertible bimodules and isomorphisms. This follows from the tensor product formula, which gives the inverse of L as $L^{\vee} = (\sigma^{-1*}(\mathrm{pr}_{1*} L)^{\vee})_{\sigma^{-1}}$, where σ is the automorphism whose graph is Supp L. We observe also that for arbitrary invertible bimodules L and M, $(LM)^{\vee}$ is canonically isomorphic to $M^{\vee}L^{\vee}$.

We can now give the defining relations. We define $\phi_{ij}^{\vee}$ by applying the inverse functor to ϕ_{ij} and using the canonical isomorphism above. For any invertible bimodule L and open $U \subseteq X$, let $L(U) = L(U \times X)$. Note that $L^{\vee}(\sigma U) = L(U)^{\vee}$ for U small enough and σ the above automorphism. There is a canonical map $\phi_L\colon L \otimes L^{\vee} \xrightarrow{\sim} \mathcal{O}_X$ defined by $s \otimes t \in L(U) \otimes_{\mathcal{O}(\sigma U)} L^{\vee}(\sigma U)$ maps to $t(s) \in \mathcal{O}_X(U)$ for all U sufficiently small. Define $\phi_i = \phi_{L_i}$. Similarly, there is a canonical map $\phi_L'\colon L^{\vee} \otimes L \xrightarrow{\sim} \mathcal{O}_X$ defined by $s \otimes t \in L^{\vee}(U) \otimes_{\mathcal{O}_X(\sigma^{-1}U)} L(\sigma^{-1}U)$ maps to $s(t)^{\sigma^{-1}} \in \mathcal{O}_X(U)$. We set $\phi_i' = \phi_{L_i}'$. We define ϕ_{ij}' to be the composite

$$\phi_{ij}'\colon M_jL_i \xrightarrow{(\phi_i)^{-1}\otimes 1} L_iM_iM_jL_i \xrightarrow{1\otimes(\phi_{ij}^{\vee})^{-1}\otimes 1} L_iM_jM_iL_i \xrightarrow{1\otimes\phi_i'} L_iM_j.$$

We obtain $\phi_{ij}'^{\vee}$ from ϕ_{ij}' by applying the inverse functor.

We wish to show that the diamond lemma holds in this case so that $\mathscr{B}=\bigoplus_{i_1,\dots,i_s\in\mathbb{Z}} L_1^{i_1}\cdots L_s^{i_s}$, where if $i<0$ then L_j^i denotes M_j^{-i}. The case (IN) of Theorem 2.2 does not arise. There are overlap checks for the monomials $L_kL_jL_i$, $L_kL_jM_i$, $L_kL_jM_j$, $L_kM_jL_i$, $L_kM_jL_j$, $L_kM_kL_j$, $L_kM_kL_k$, $M_kL_kL_j$, $M_kL_jL_i$ for $i<j<k$. The other overlap checks are for monomials obtained from the above ones by swapping M with L. Note that the set of defining relations is self-dual in that applying the inverse functor to any defining relation yields another. Hence, checking overlaps for any monomial verifies the overlap check for the monomial obtained by swapping L with M so we need only consider the nine monomials above. The overlap check for $L_kL_jL_i$ was hypothesized while that for $L_jM_jL_j$ is easily verified on sections. We carry out the somewhat tedious verification of the others.

For the monomial $L_kM_kL_j$ we must show the maps

$$L_kM_kL_j \xrightarrow{\phi_k\otimes 1} L_j \tag{3}$$

and

$$\begin{aligned} L_kM_kL_j &\xrightarrow{1\otimes\phi_j^{-1}\otimes 1} L_kL_jM_jM_kL_j \xrightarrow{1\otimes(\phi_{jk}^\vee)^{-1}\otimes 1} L_kL_jM_kM_jL_j \\ &\xrightarrow{\phi_{jk}\otimes 1} L_jL_kM_kM_jL_j \xrightarrow{1\otimes\phi_k\otimes\phi_j'} L_j \end{aligned} \tag{4}$$

are equal. The last map of this composite is equal to

$$L_jL_kM_kM_jL_j \xrightarrow{1\otimes\phi_k\otimes 1} L_jM_jL_j \xrightarrow{\phi_j\otimes 1} L_j$$

by the overlap check for $L_jM_jL_j$. Tracing the maps in (4) we see it suffices to show that the two maps

$$(L_kL_j)(L_kL_j)^\vee \xrightarrow{\phi_{L_kL_j}} \mathscr{O}_X$$

and

$$(L_kL_j)(L_kL_j)^\vee \xrightarrow{\phi_{jk}\otimes(\phi_{jk}^{-1})^\vee} (L_jL_k)(L_jL_k)^\vee \xrightarrow{\phi_{L_jL_k}} \mathscr{O}_X$$

are equal. This can be checked on sections. Checking overlaps for the other monomials where two of the indices are the same is done by a similar (or easier) argument.

We now check the overlap for $M_kL_jL_i$, i.e., equality of the two maps

$$\begin{aligned} \psi_1\colon M_kL_jL_i &\xrightarrow{1\otimes\phi_{ij}} M_kL_iL_j \xrightarrow{\phi_{ik}'\otimes 1} L_iM_kL_j \xrightarrow{1\otimes\phi_{jk}'} L_iL_jM_k \\ \psi_2\colon M_kL_jL_i &\xrightarrow{\phi_{jk}'\otimes 1} L_jM_kL_i \xrightarrow{1\otimes\phi_{ik}'} L_jL_iM_k \xrightarrow{\phi_{ij}\otimes 1} L_iL_jM_k. \end{aligned}$$

It suffices to show that these are equal, respectively, to

$$\begin{aligned}\psi_1'\colon M_kL_jL_i &\longrightarrow M_kL_jL_iL_kM_k \longrightarrow M_kL_iL_jL_kM_k \\ &\longrightarrow M_kL_iL_kL_jM_k \longrightarrow M_kL_kL_iL_jM_k \longrightarrow L_iL_jM_k\end{aligned}$$

and

$$\begin{aligned}\psi_2'\colon M_kL_jL_i &\longrightarrow M_kL_jL_iL_kM_k \longrightarrow M_kL_jL_kL_iM_k \\ &\longrightarrow M_kL_kL_jL_iM_k \longrightarrow M_kL_kL_iL_jM_k \longrightarrow L_iL_jM_k\end{aligned}$$

since $\psi_1' = \psi_2'$ by the overlap check for $L_kL_jL_i$. We first show that the two maps

$$M_kL_iL_jL_k \longrightarrow L_iL_jM_kL_k \longrightarrow L_iL_j \tag{5}$$

and

$$M_kL_iL_jL_k \longrightarrow M_kL_kL_iL_j \longrightarrow L_iL_j \tag{6}$$

are equal. We compare with the two maps

$$M_kL_iL_jL_k \longrightarrow M_kL_iL_kM_kL_jL_k \rightrightarrows L_iL_j, \tag{7}$$

where the top map comes from commuting the L_k's through L_i, L_j and then contracting with ϕ_k', and the bottom map comes from commuting the M_k's. These two maps are equal to (5) and (6), respectively, by the overlap check for $L_kM_kL_k$ and $M_kL_kM_k$. However, the two maps in (7) are equal by the overlap check for $M_kL_kL_i$ and $M_kL_kL_j$.

Now consider the last part of the map ψ_1' consisting of

$$M_kL_iL_jL_kM_k \longrightarrow M_kL_kL_iL_jM_k \longrightarrow L_iL_jM_k.$$

This equals the map

$$M_kL_iL_jL_kM_k \longrightarrow L_iL_jM_kL_kM_k \xrightarrow{1\otimes\phi_k'\otimes 1} L_iL_jM_k$$

by what we just proved. This in turn equals

$$M_kL_iL_jL_kM_k \longrightarrow L_iL_jM_kL_kM_k \xrightarrow{1\otimes\phi_k} L_iL_jM_k$$

by the overlap check for $M_kL_kM_k$. From this we see that $\psi_1 = \psi_1'$. A symmetrical argument shows $\psi_2 = \psi_2'$ as desired. The overlap check for $L_kL_jM_i$ follows by a similar argument comparing the requisite maps with the "two" maps $M_iL_iL_kL_jM_i \rightrightarrows M_iL_jL_kL_iM_i$ and the overlap check for $L_kM_jL_i$ by considering $M_jL_kL_jL_iM_j \rightrightarrows M_jL_iL_jL_kM_j$. Hence, Theorem 2.2 is applicable and we find that $\mathscr{B}$ is a strongly graded algebra with $\mathscr{B}_{\geq 0} = T(L)/(\Phi)$.

We return now to the two category equivalences mentioned at the beginning of the proof. We prove the first of these in

LEMMA 3.5. *We have a natural category equivalence,*

$$\mathcal{O}_X\text{-Mod} \leftrightarrow \text{Gr-}\mathcal{B}\colon F \to F \otimes \mathcal{B}, \qquad \mathcal{M}_0 \leftarrow \mathcal{M}.$$

Proof. This holds true for any $\mathcal{B}$ strongly graded so we shall prove it in this setting. First, it is clear that the composed functor $F \mapsto (F \otimes_{\mathcal{O}_X} \mathcal{B})_0$ is naturally isomorphic to the identity functor on $\mathcal{O}_X$-Mod. For the reverse composition, we have a natural transformation $\chi_{\mathcal{M}}\colon \mathcal{M}_0 \otimes \mathcal{B} \to \mathcal{M}$ from the multiplication map. Note that $\chi_{\mathcal{M}}$ is surjective since the composite $\mathcal{M}_v \otimes \mathcal{B}_{-v} \otimes \mathcal{B}_v \to \mathcal{M}_0 \otimes \mathcal{B}_v \to \mathcal{M}_v$ is for any $v \in \mathbb{Z}^s$. Let $\mathcal{K} = \ker \chi_{\mathcal{M}}$. Then $\mathcal{K}_v = \operatorname{im}(\mathcal{K}_v \otimes \mathcal{B}_{-v} \otimes \mathcal{B}_v \to \mathcal{K}_v)$. However, the latter is zero since $\mathcal{K}_v \otimes \mathcal{B}_{-v} \to \mathcal{K}_0 = 0$. Hence, $\chi_{\mathcal{M}}$ is an isomorphism.

Finally, we leave it to the reader to verify that the proof of the category equivalence Gr-$\mathcal{B} \leftrightarrow$ Gr-B tors given in [AV] carries over without change. In the course of the proof, the following fact was established whose multigraded analogue we will need.

LEMMA 3.6. *Assume the hypotheses of the theorem hold. If F is a coherent $\mathcal{O}_X$-module then $F \otimes \mathcal{B}_v$ is generated by global sections for $v \in \mathbb{Z}^s$ large enough.*

Proof. The proof is as in [AV, Proposition 3.2(ii)].

4. EXAMPLES

In this section we give some examples of naturally occurring twisted bihomogeneous coordinate rings. Throughout this section we shall assume that our base scheme is the spectrum of the algebraically closed field k.

EXAMPLE 4.1. *Some Ore extensions of twisted homogeneous coordinate rings*. Let X be a projective scheme over k and let L be a right ample invertible bimodule on X. Suppose τ is an automorphism of X such that there is a bimodule isomorphism $\phi\colon \mathcal{O}_\tau \otimes L \to L \otimes \mathcal{O}_\tau$. We can then consider the twisted bihomogeneous coordinate ring $B = B(X; L, \mathcal{O}_\tau; \phi)$. The (i, j)th graded component of B is $B_{ij} = H^0(X, L^i \mathcal{O}_\tau^j)$. But every section in $H^0(X, L^i \mathcal{O}_\tau^j)$ has the form $a \otimes 1_{\mathcal{O}_\tau} \otimes \cdots \otimes 1_{\mathcal{O}_\tau}$, where $a \in H^0(X, L^i)$ and $1_{\mathcal{O}_\tau} = \mathrm{pr}_1^* 1_{\mathcal{O}_X} = \mathrm{pr}_2^* 1_{\mathcal{O}_X} \in \mathcal{O}_\tau(X \times X)$. Hence if we write t for the element $1_{\mathcal{O}_\tau} \in B_{01}$ then right multiplication in B by t^j maps B_{i0} isomorphically onto B_{ij}. We see immediately that B is an Ore extension of the twisted homogeneous coordinate ring $B(X; L)$. Note that L, $\mathcal{O}_\tau$ form a right ample set.

EXAMPLE 4.2. *Rees algebras of twisted homogeneous coordinate rings*. Let X be a projective scheme over k and let L be a right ample invertible bimodule on X. Let $R = B(X; L)$. We consider $B = B(X; L, L; \phi)$, where ϕ: $L \otimes L \to L \otimes L$ is the identity map. Then $B_{ij} = H^0(X, L^{i+j})$ so $B = \oplus_{i,\, j\geq 0} R_{i+j} t^j$, where t is just a place marker to distinguish the different graded components. However, unravelling the definition of multiplication in B we see that the equality above is actually a ring isomorphism if, in the right-hand side, t is treated as a central indeterminate. Hence B is the Rees algebra $R[\mathfrak{m}t]$, where $\mathfrak{m}$ is the augmentation ideal $\mathfrak{m} = B_1 \oplus B_2 \oplus \cdots$. Furthermore, the pair L, L is right ample.

EXAMPLE 4.3. *Tensor products of twisted homogeneous coordinate rings*. Let X and Y be projective schemes over k and let L_σ, M_τ be right ample invertible bimodules on X, Y, respectively. Let pr_1, pr_2 be the projections from $X \times Y$ to X and Y, respectively, and let $\tilde{\sigma} = \sigma \times 1_Y$ and $\tilde{\tau} = 1_X \times \tau$. Observe that $\tilde{\sigma}$ and $\tilde{\tau}$ commute. Also, since $\mathrm{pr}_1 = \mathrm{pr}_1 \circ \tilde{\tau}$ and $\mathrm{pr}_2 = \mathrm{pr}_2 \circ \tilde{\sigma}$ there are canonical isomorphisms

$$\begin{aligned} \mathrm{pr}_2^* M \otimes \tilde{\tau}^* \mathrm{pr}_1^* L &\xrightarrow{\sim} \mathrm{pr}_2^* M \otimes \mathrm{pr}_1^* L \xrightarrow{\sim} \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* M \\ &\xrightarrow{\sim} \mathrm{pr}_1^* L \otimes \tilde{\sigma}^* \mathrm{pr}_2^* M. \end{aligned} \tag{8}$$

Thus if we let $\mathrm{pr}_1^*(L_\sigma)$ denote $(\mathrm{pr}_1^* L)_{\tilde{\sigma}}$ and $\mathrm{pr}_2^*(M_\tau)$ denote $(\mathrm{pr}_2^* M)_{\tilde{\tau}}$ then we have a bimodule isomorphism ϕ: $\mathrm{pr}_2^* M_\tau \otimes \mathrm{pr}_1^* L_\sigma \to \mathrm{pr}_2^* M_\tau \otimes \mathrm{pr}_1^* L_\sigma$ and so can form the twisted bihomogeneous coordinate ring $B = B(X \times Y; \mathrm{pr}_1^* L_\sigma, \mathrm{pr}_2^* M_\tau; \phi)$. The (i, j)th graded component of B in this case is

$$H^0\big(X \times Y, \mathrm{pr}_1^* L_\sigma \otimes \cdots \otimes \tilde{\sigma}^{i-1*} \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* M \otimes \cdots \otimes \tilde{\tau}^{j-1*} \mathrm{pr}_2^* M\big),$$

which by the Künneth formula is $B(X; L_\sigma)_i \otimes_k B(Y; M_\tau)_j$. In fact, from (8) we see that we have $B = B(X; L_\sigma) \otimes_k B(Y; M_\tau)$ as algebras.

5. A CRITERION FOR ACC

Throughout this section, X will be a projective scheme over k.

Serre's category equivalence was used in [AV, Theorem 3.14] to give a proof that the twisted homogeneous coordinate ring with respect to a right ample invertible bimodule is right noetherian. It is not surprising that this fails in the multigraded case.

EXAMPLE 5.1. *Non-noetherian $B(X; L, M)$*. We in fact have a commutative example. Let X be a smooth curve of positive genus g. Let L be a line bundle of degree 0 such that no tensor power of L is isomorphic to $\mathcal{O}_X$ and M is a line bundle of degree greater than $g - 1$. Then L, M form an ample set of line bundles. However, the $(i, 0)$-graded component of $B = B(X; L, M)$ for $i > 0$ is zero while all other graded components are non-zero. Let $I = B_{>0}$ be the augmentation ideal of the associated $\mathbb{Z}$-graded algebra. Then I/I^2 contains a copy of the infinite dimensional vector space $\bigoplus_{i>0} B_{i1}$. Thus, B is not finitely generated and hence not noetherian.

The problem here is that L does not define a map into projective space. To eliminate this possibility we consider the following condition on an invertible bimodule L_σ on X;

(*) *There exists a projective scheme Y over k with an automorphism σ and a σ-equivariant morphism $f\colon X \to Y$. There also exists a line bundle L' on Y such that $L = f^*L'$ and such that L'_σ is right ample.*

We have the following sufficient criterion for a twisted bihomogeneous coordinate ring to be right noetherian.

THEOREM 5.2. *Let X be a projective scheme over k and let L_σ, M_τ be a right ample set of invertible bimodules on X with some commutation relation $\phi\colon M_\tau L_\sigma \to L_\sigma M_\tau$. Suppose that L_σ and M_τ satisfy* (*). *Then $B(X; L_\sigma, M_\tau; \phi)$ is right noetherian.*

Proof. We first show

Claim 5.3. *Under the hypotheses of the theorem we have that $\bigoplus_{i>i_0} H^q(X, L^i_\sigma M^j_\tau)$ is a noetherian $B(X; L_\sigma)$-module for all j, q, i_0 and similarly $\bigoplus_{j>j_0} H^q(X, L^i_\sigma M^j_\tau)$ is a noetherian $B(X; M_\tau)$-module for all i, q, j_0.*

Proof. We first observe that there is a natural ring homomorphism $B(Y; L'_\sigma) \to B(X; L_\sigma)$. Recall that $B(Y, L'_\sigma)$ is right noetherian by [AV, Theorem 3.14]. Now $\bigoplus_{i>i_0} H^q(X, L^i_\sigma M^j_\tau)$ is a right $B(X; L_\sigma)$-module by functoriality of cohomology. It suffices to show it is finitely generated over $B(Y; L'_\sigma)$. Let $g = f \circ \tau^j$ and consider the Leray spectral sequence

$$E_2^{pq} = H^p\Bigg(Y, R^q g_*\Bigg(\bigoplus_{i>i_0} M \otimes \cdots \otimes \tau^{j-1*}M \otimes \tau^{j*}L \otimes \cdots \otimes \tau^{j*}\sigma^{i-1*}L\Bigg)\Bigg)$$
$$\Rightarrow \quad H^{p+q}\Bigg(X, \bigoplus_{i>i_0} M \otimes \cdots \otimes \tau^{j-1*}M \otimes \tau^{j*}L \otimes \cdots \otimes \tau^{j*}\sigma^{i-1*}L\Bigg).$$

From the projection formula [EGA, Chap. 0_{III}, Proposition 12.2.3] we see the E_2^{pq} term is

$$H^p\big(Y, \oplus_i R^q g_*(M \otimes \cdots \otimes \tau^{j-1*}M) \otimes L' \otimes \cdots \otimes \sigma^{i-1*}L'\big).$$

Since L'_σ is right ample the spectral sequence degenerates in high degree to give isomorphisms $E_2^{0q} \xrightarrow{\sim} H^q(\bigoplus_i L^i_\sigma M^j_\tau)$ in $(\text{Gr-}B(Y; L'_\sigma))/\text{tors}$ by naturality of the spectral sequence. However, under the category equivalence

$(\text{Gr-}B(Y;L'_\sigma))/\text{tors} \to \mathscr{O}_Y\text{-Mod}$, E_2^{0q} corresponds to the coherent sheaf $R^q g_*(M \otimes \cdots \otimes \tau^{j-1*}M)$ and so must be finitely generated as was to be shown. A symmetrical argument fixing i instead of j completes the claim.

We show now how the proof of the ascending chain condition in [AV, pp. 261–263] pushes through to the bigraded case under the given hypotheses. Let I be a bigraded right ideal of $B = B(X;L_\sigma, M_\tau)$. It suffices to show that I is finitely generated.

Recall from Theorem 3.4 that we have two adjoint functors, $\text{Gr-}B \to \mathscr{O}_X\text{-Mod}$: $M \mapsto (M \otimes_B \mathscr{B})_0$ and $\mathscr{O}_X\text{-Mod} \to \text{Gr-}B$: $F \mapsto H^0(F \otimes \mathscr{B})_{\geq 0}$. Composing the two yields an endofunctor $M \mapsto \overline{M}$ of Gr-B. We have a natural transformation η_M: $M \to \overline{M}$ which is an isomorphism modulo torsion.

Now η_B is an isomorphism so the injection $I \hookrightarrow B$ factors through η_I: $I \to \overline{I}$. Hence, η_I is an injection. Now, η_I has torsion cokernel so it suffices to prove the following two lemmas,

LEMMA 5.4. *$\overline{I}$ is finitely generated.*

LEMMA 5.5. *Given an exact sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *in* Gr-B *with M_2 finitely generated and M_3 torsion, then M_1 is also finitely generated.*

Proof of Lemma 5.4. Since $(\cdot \otimes_B \mathscr{B})_0$: Gr-$B$/tors $\to \mathscr{O}_X$-Mod is one of the inverse equivalences in Theorem 3.4, $(I \otimes_B \mathscr{B})_0$ is a subsheaf of $\mathscr{O}_X$. It thus suffices to show that for any coherent sheaf F on X, $H^q(F \otimes \mathscr{B})_{\geq 0}$ for $q \in \mathbb{N}$ is finitely generated. We achieve this by downward induction on q. The case for large q is clear by Grothendieck's vanishing theorem. By Lemma 3.6, we have a surjective map $\mathscr{O}_X^n \to F \otimes \mathscr{B}_v$ for some $n \in \mathbb{N}$ and some $v \in \mathbb{Z}^2$ sufficiently large. Tensoring by a shift of $\mathscr{B}$ yields the exact sequence

$$0 \to \mathscr{K} \to \mathscr{P} \to F \otimes \mathscr{B} \to 0,$$

where $\mathscr{P}$ is a finite sum of shifts of $\mathscr{B}$. Further, by the equivalence of categories $\mathscr{O}_X\text{-Mod} \leftrightarrow \text{Gr-}\mathscr{B}$, we see that $\mathscr{K} \simeq \mathscr{K}_0 \otimes \mathscr{B}$, where $\mathscr{K}_0$ is a coherent $\mathscr{O}_X$-module. We have an exact sequence $H^q(\mathscr{P})_{\geq 0} \to H^q(F \otimes \mathscr{B})_{\geq 0} \to H^{q+1}(\mathscr{K})_{\geq 0}$. By induction, we can assume the last term to be finitely generated so it remains to observe that $H^q(\mathscr{B})_{\geq w}$ is too, for any $w \in \mathbb{Z}^2$. The following fact was proved in [AV, pp. 261–262] for the case $q = 0$, $s = 1$:

LEMMA 5.6. *For each $u \in \mathbb{Z}^s$ $(s \in \mathbb{N})$, large enough, there exists $v \in \mathbb{Z}^s$ such that the multiplication map $H^q(\mathscr{B})_u \otimes_k H^0(\mathscr{B})_{v'} \to H^q(\mathscr{B})_{u+v'}$ is surjective whenever $v' \geq v$.*

Proof. For $q > 0$ the statement is trivial as ampleness ensures $H^q(\mathcal{B})_u = 0$ for u large enough so we assume $q = 0$. By Lemma 3.6, $\mathcal{B}_u$ is generated by sections for u large enough. This gives an exact sequence

$$0 \to \mathcal{G} \to H^0(\mathcal{B}_u) \otimes_k \mathcal{O}_X \to \mathcal{B}_u \to 0.$$

Choosing v so that $H^q(\mathcal{G} \otimes \mathcal{B}_{v'}) = 0$ whenever $v' \geq v$ and $q > 0$ gives the lemma.

Returning to the proof of Lemma 5.4, fix $u \geq w$ large enough that Lemma 5.6 holds. Then $H^q(\mathcal{B})_{\geq w}$ is generated by the graded components $H^q(\mathcal{B})_{w'}$ where w' is not greater than or equal to $v + u$. But $H^q(\mathcal{B})_{\geq w}/H^q(\mathcal{B})_{\geq v+u}$ is finitely generated since it has a filtration with finitely generated, factors by our claim (5.3). This shows that $H^q(\mathcal{B})_{\geq w}$ is finitely generated, verifying the lemma.

Proof of Lemma 5.5. Since M_3 is finitely generated torsion, we may choose $v \in \mathbb{Z}^2$ large enough so that $M_1 \supset (M_2)_{\geq v}$. We write M_2 as a quotient of a finite sum of shifts of B, say $\bigoplus_{i=1}^n B(v_i)$, where $n \in \mathbb{N}$, $v_i \in \mathbb{Z}^2$. Now, $M_2/(M_2)_{\geq v}$ is a quotient of $\bigoplus B(v_i)/B(v_i)_{\geq v}$ and so is a noetherian module by our claim. It thus suffices to show that $(M_2)_{\geq v}$ is finitely generated or that $B(v_i)_{\geq v} = B_{\geq v_i + v}$ is finitely generated. This has already been verified above. So ends the proof of the theorem.

On applying the theorem to Example 4.1, we recover a special case of Hilbert's basis theorem. The theorem also applies to Example 4.2 giving

COROLLARY 5.7. *Let L be a right ample invertible bimodule on a projective scheme X over k. Then the Rees algebra $B[\mathfrak{m}t]$ of $B = B(X; L)$ is right noetherian.*

COROLLARY 5.8. *Let L_σ and M_τ be right ample invertible bimodules on projective schemes X and Y, respectively. Then the tensor product $B(X; L_\sigma) \otimes_k B(Y; M_\tau)$ of the twisted homogeneous coordinate rings is right noetherian.*

Proof. Using the notation of Example 4.3 we see $\mathrm{pr}_1^* L_\sigma$, $\mathrm{pr}_2^* M_\tau$ satisfy (*) so the hypotheses of the theorem hold provided they form a right ample set. We verify this now. Let $\overline{L} := \mathrm{pr}_1^* L$ and let σ, τ also denote the automorphisms $\sigma \times 1$, $1 \times \tau$ of $X \times Y$. Let F be a coherent sheaf on $X \times Y$. As in Claim 5.3 we consider the Leray spectral sequence

$$\begin{aligned} &H^p\big(Y, R^q \mathrm{pr}_{2*}(F \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}) \otimes M \otimes \cdots \otimes \tau^{j-1*}M\big) \\ &\qquad \Rightarrow H^{p+q}\big(X \times Y, F\overline{L}_\sigma^i \overline{M}_\tau^j\big). \end{aligned}$$

It suffices to show that for i, j large enough, the first term vanishes whenever $p > 0$ or $q > 0$. Pick ample line bundles $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ on X and

Y, respectively, and let $\mathcal{O}(m,n) := \mathrm{pr}_1^*\,\mathcal{O}_X(m) \otimes \mathrm{pr}_2^*\,\mathcal{O}_Y(n)$. We will make use of the exact sequence

$$0 \to K \to \oplus \mathcal{O}(n_l, n_l) \to F \to 0,$$

where the sum in the middle is finite. We first show that for any coherent sheaf F on $X \times Y$, $R^q\,\mathrm{pr}_{2*}(F \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}) = 0$ for $q > 0$ and i large enough, by downward induction on q. Grothendieck's vanishing theorem [EGA, Chap. III, Corollary 4.2.2] again dispenses with the large q case and, assuming inductively the result for $q+1$ and K, we see it suffices to prove the result for $F = \mathcal{O}(n,n)$. But

$$\begin{aligned}
&R^q\,\mathrm{pr}_{2*}\big(\mathcal{O}(n,n) \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}\big)\\
&\quad = \mathcal{O}_Y(n) \otimes R^q\,pr_{2*}\big(\mathcal{O}(n,0) \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}\big)\\
&\quad = \mathcal{O}_Y(n) \otimes H^q\big(X, L \otimes \cdots \otimes \sigma^{i-1*}L(n)\big)
\end{aligned}$$

by the projection formula and the commutativity of cohomology with flat base change (see [EGA, Chap. III, Proposition 1.4.15]). Right ampleness of L_σ now ensures the last term is zero for i large enough.

We now show that

$$E_2^{p0} = H^p\big(Y, \mathrm{pr}_{2*}(F \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}L) \otimes M \otimes \cdots \otimes \tau^{j-1*}M\big) = 0$$

for i, j large enough. We use downward induction on p. By the previous paragraph, for i large enough, we have an exact sequence

$$\begin{aligned}
0 &\to \mathrm{pr}_{2*}\big(K \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}\big)\\
&\to \mathrm{pr}_{2*}\Big(\bigoplus_l \mathcal{O}(n_l, n_l) \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}\Big)\\
&\to \mathrm{pr}_{2*}\big(F \otimes \overline{L} \otimes \cdots \otimes \sigma^{i-1*}\overline{L}\big) \to 0.
\end{aligned}$$

Tensoring this sequence by $M \otimes \cdots \otimes \tau^{j-1*}M$ and considering the long exact sequence in cohomology we are again reduced to the case $F = \mathcal{O}(n,n)$. But then

$$E_2^{p0} = H^p\big(Y, \mathcal{O}_Y(n) \otimes H^0(X, L \otimes \cdots \otimes \sigma^{i-1*}L) \otimes M \otimes \cdots \otimes \tau^{j-1*}M\big),$$

which by ampleness is zero for j large enough independent of i. This completes the proof of the corollary.

COROLLARY 5.9. *The tensor product of three-dimensional Artin–Schelter regular algebras is noetherian.*

Proof. Let A_1, A_2 be Artin–Schelter regular algebras of dimension 3. From [ATV, Proposition 6.7(i) and Theorem 6.8(i)] we know there exist normal homogeneous elements $g_i \in A_i$ of positive degree or 0 for $i = 1, 2$ such that $A_i/(g_i)$ is the twisted homogeneous coordinate ring of a projective scheme with respect to an ample invertible bimodule. Thus Corollary 5.8 together with its left-handed companion show that $(A_1 \otimes A_2)/(g_1 \otimes 1, 1 \otimes g_2)$ is noetherian. Since $g_1 \otimes 1$ and $1 \otimes g_2$ are normal, the result follows from two or fewer applications of [ATV, Lemma 8.2].

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