ON THE SPECTRA OF QUANTUM GROUPS

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Abstract. Joseph and Hodges–Levasseur (in the A case) described the spectra of all quantum function algebras $R_q[G]$ on simple algebraic groups in terms of the centers of certain localizations of quotients of $R_q[G]$ by torus invariant prime ideals, or equivalently in terms of orbits of finite groups. These centers were only known up to finite extensions. We determine the centers explicitly under the general conditions that the deformation parameter is not a root of unity and without any restriction on the characteristic of the ground field. From it we deduce a more explicit description of all prime ideals of $R_q[G]$ than the previously known ones and an explicit parametrization of Spec$R_q[G]$. We combine the latter with a result of Kogan and Zelevinsky to obtain in the complex case a torus equivariant Dixmier type map from the symplectic foliation of the group G to the primitive spectrum of $R_q[G]$. Furthermore, under the general assumptions on the ground field and deformation parameter, we prove a theorem for separation of variables for the De Concini–Kac–Procesi algebras $\mathcal{U}^w_\pm$, and classify the sets of their homogeneous normal elements and primitive elements. We apply those results to obtain explicit formulas for the prime and especially the primitive ideals of $\mathcal{U}^w_\pm$ lying in the Goodearl–Letzter stratum over the $\{0\}$-ideal. The results on $\mathcal{U}^w_\pm$ are in turn used to derive a theorem for separation of variables for all Joseph's localizations of quotients of $R_q[G]$ by torus invariant prime ideals. From it we derive a classification of the maximal spectrum of $R_q[G]$ and use it to resolve a question of Goodearl and Zhang, showing that all maximal ideals of $R_q[G]$ have finite codimension. We prove that $R_q[G]$ possesses a stronger property than that of the classical catenarity: all maximal chains in Spec$R_q[G]$ have the same length equal to GK dim $R_q[G] = \dim G$, i.e. $R_q[G]$ satisfies the first chain condition for prime ideals in the terminology of Nagata.

Contents

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1. Introduction

In the last 20 years the area of quantum groups attracted a lot of attention in ring theory, since it supplied large families of concrete algebras on which general techniques can be developed and tested. One of the most studied family is the one of the quantum function algebras $R_q[G]$ on simple groups. In works from the early 90's Joseph [24, 25] and Hodges–Levasseur [20, 21] (and jointly with Toro [22]) made fundamental contributions to the problem of determining their spectra by describing the spectra set theoretically and the topology of each stratum in a finite stratification of $\mathrm{Spec} R_q[G]$. Despite the fact that a lot of research has been done consequently (e.g. [6, 13, 14, 15, 17, 18, 19, 26, 31, 32, 36]), many ring theoretic questions for the algebras $R_q[G]$ remain open.

In this paper we describe solutions of ring theoretic problems for $R_q[G]$ and the related De Concini–Kac–Procesi algebras $\mathcal{U}^w_\pm$, which range from the older question of determining explicitly the Joseph strata of $\mathrm{Spec} R_q[G]$ and setting up a torus equivariant Dixmier type map between the symplectic foliation of G and $\mathrm{Prim}\, R_q[G]$, to newer ones such as the question raised by Goodearl and Zhang [18] on whether all maximal ideals of $R_q[G]$ have finite codimension and the question of classifying $\mathrm{Max} R_q[G]$.

In order to describe in concrete terms the problems addressed in this paper, we introduce some notation on quantum groups. We start with a quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ of rank r. Throughout the paper our assumption is that the base field $\mathbb{K}$ is arbitrary and the deformation parameter $q \in \mathbb{K}^*$ is not a root of unity (except the small part on the Dixmier map and Poisson geometry where $\mathbb{K} = \mathbb{C}$ and $q \in \mathbb{C}^*$ is not a root of unity). We do not use specialization at any point and thus do not need the ground field to have characteristic 0 and q to be transcendetal over $\mathbb{Q}$. The quantized algebra of functions $R_q[G]$ on the "connected, simply connected" group G is the Hopf subalgebra of the restricted dual $(\mathcal{U}_q(\mathfrak{g}))^\circ$ consisting of the matrix coefficients of all finite dimensional type 1 representations of $\mathcal{U}_q(\mathfrak{g})$. (Here G is just a symbol, since all arguments are carried out over an arbitrary base field $\mathbb{K}$. The only restriction is that $\mathbb{K}$ is not finite, since $q \in \mathbb{K}^*$ should not be a root of unity.)

All finite dimensional type 1 $\mathcal{U}_q(\mathfrak{g})$-modules are completely reducible. The irreducible ones are parametrized by the dominant integral weights P^+ of $\mathfrak{g}$ and have q-weight space decompositions $V(\lambda) = \oplus_{\mu \in P} V(\lambda)_\mu$, where P is the weight lattice of $\mathfrak{g}$. The matrix coefficient of $\xi \in V(\lambda)^*$ and $v \in V(\lambda)$ will be denoted by $c^\lambda_{\xi,v} \in R_q[G]$. The algebra $R_q[G]$ is $P \times P$-graded by

$$R_q[G]_{\nu,\mu} = \mathrm{Span}\{c^\lambda_{\xi,v} \mid \lambda \in P^+, \xi \in (V(\lambda)^*)_\nu, V(\lambda)_\mu\} \tag{1.1}$$

and has two distinguished subalgebras $R^\pm$ which are spanned by matrix coefficients $c^\lambda_{\xi,v}$ for $v \in V(\lambda)_\lambda$ and $v \in V(\lambda)_{w_0\lambda}$, respectively, where w_0 denotes the longest element of the Weyl group W of $\mathfrak{g}$. Joseph proved [24] that $R_q[G] = R^+R^- = R^-R^+$. For $w \in W$ one defines the Demazure modules

$$V_w^+(\lambda) = \mathcal{U}_+ V(\lambda)_{w\lambda} \subseteq V(\lambda) \text{ and } V_w^-(\lambda) = \mathcal{U}_- V(-w_0\lambda)_{-w\lambda} \subseteq V(-w_0\lambda),$$

where $\mathcal{U}_\pm$ are the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by the positive and negative Cartan generators. To $w_\pm \in W$, Joseph associated certain ideals $I^\pm_{w_\pm}$ of $R^\pm$ by considering the span of those matrix coefficients for which the vector ξ is orthogonal to $V^\pm_{w_\pm}(\lambda)$, see §2.3 for details. Those are combined into the following ideals of $R_q[G]$:

$$I_{\mathrm{w}} = I^+_{w_+} R^- + R^+ I^-_{w_-}, \quad \mathrm{w} = (w_+, w_-),$$

which are a key building block in Joseph's analysis of $R_q[G]$. The other part is a set of normal elements of the quotients $R_q[G]/I_{\mathrm{w}}$. Up to an appropriate normalization, for $\lambda \in P^+$ one defines

$$c^+_{w_+,\lambda} = c^\lambda_{-w_+\lambda,\lambda} \in R^+ \text{ and } c^-_{w_-,\lambda} = c^{-w_0\lambda}_{w_-\lambda,-\lambda} \in R^-, \tag{1.2}$$

where vectors in $\mathcal{U}_q(\mathfrak{g})$-modules are substituted with weights for the (one dimensional) weight spaces to which they belong, see §2.4. The multiplicative subsets of $R^\pm$ generated by the first and second kind of elements will be denoted by $E^\pm_{w_\pm}$, and their product by $E_{\mathrm{w}} \subset R_q[G]$. (As it is customary we will denote by the same symbols the images of elements of $R_q[G]$ in its quotients.) The sets $E_{\mathrm{w}} \subset R_q[G]/I_{\mathrm{w}}$ consist of regular normal elements, thus one can localize $R_{\mathrm{w}} := (R_q[G]/I_{\mathrm{w}})[E_{\mathrm{w}}^{-1}]$. Joseph [24, 25] and Hodges–Levasseur (in the A case [21] and jointly with Toro in the multiparameter case [22]) proved that one can break down

$$\mathrm{Spec} R_q[G] = \bigsqcup_{\mathrm{w} \in W \times W} \mathrm{Spec}_{\mathrm{w}} R_q[G], \tag{1.3}$$

in such a way that $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ is homeomorphic to the spectrum of the center $Z_{\mathrm{w}} := Z(R_{\mathrm{w}})$ via:

$$J^0 \in \mathrm{Spec} Z_{\mathrm{w}} \mapsto \iota_{\mathrm{w}}(J^0) \in \mathrm{Spec} R_q[G] \text{ so that } \iota_{\mathrm{w}}(J^0)/I_{\mathrm{w}} = (R_{\mathrm{w}} J^0) \cap (R_q[G]/I_{\mathrm{w}}), \tag{1.4}$$

see Theorem 2.3 for details. Joseph's original formulation of the parametrization of $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ is in slightly different terms using an action of $\mathbb{Z}_2^{\times r}$, see [24, Theorem 8.11], [25, Theorem 10.3.4]. In this form the parametrization of $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ is stated in Hodges–Levasseur–Toro [22, Theorem 4.15].

He proved that for all $\mathrm{w} = (w_+, w_-) \in W \times W$, Z_{w} is a Laurent polynomial ring over $\mathbb{K}$ of dimension $\dim \ker(w_+ - w_-)$, that Z_{w} contains a particular Laurent polynomial ring, and that it is a free module over it of rank at most 2^r. Joseph also proved that the closure of each stratum is a union of strata given in terms of the inverse Bruhat order on $W \times W$, but the nature of the gluing of the strata $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ inside $\mathrm{Spec} R_q[G]$ with the Zariski topology is unknown.

One needs to know the explicit structure of Z_{w} to begin studying the Zariski topology of the space $\mathrm{Spec} R_q[G]$ in the sense of the interaction between the different strata in (1.3). This is also needed for the construction of an equivariant Dixmier type map from the symplectic foliation of the underlying Poisson Lie group G to $\mathrm{Prim}\, R_q[G]$ (when $\mathbb{K} = \mathbb{C}$), since one cannot determine the stabilizers of the primitive ideals of $R_q[G]$ with respect to the natural torus actions (see

(2.31)–(2.32) below) from Joseph's theorem. The centers Z_{w} are explicitly known only for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$ due to Hodges–Levasseur [20] and Goodearl–Lenagan [15]. The first problem that we solve in this paper is the one of the explicit description of Z_{w} in full generality. At the time of the writing of [24, 21] a similar problem existed on the symplectic side. Hodges and Levasseur proved [20] that in the complex case the double Bruhat cells $G^{\mathrm{w}} \subset G$ are torus orbits of symplectic leaves and that certain intersections are finite unions of at most 2^r symplectic leaves, but no results were available on their connected components. Nine years later this problem was solved by Kogan and Zelevinsky [28] by combinatorial methods using the theory of generalized minors. The problem of determining Z_{w} is the ring theoretic counterpart of the problem which they solved.

For an element $\mathrm{w} = (w_+, w_-)$ denote by $\mathcal{S}(\mathrm{w})$ the subset of $\{1, \ldots, r\}$ consisting of all simple reflections s_i which appear either in a reduced expression of w_+ or w_- (one thinks of it as of the support of w). Denote by $\mathcal{I}(\mathrm{w})$ its complement in $\{1, \ldots, r\}$, which is the set of all fundamental weights ω_i fixed by w_+ and w_-. We denote

$$P_{\mathcal{S}(\mathrm{w})} = \oplus_{i \in \mathcal{S}(\mathrm{w})} \mathbb{Z}\omega_i \text{ and } \widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w}) = \ker(w_+ - w_-) \cap P_{\mathcal{S}(\mathrm{w})}.$$

It is easy to see that $\widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w})$ is a lattice of rank $\dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$. One extends the definition (1.2) to $\lambda \in P$ to obtain elements of the localizations R_{w}. We have:

Theorem 1.1. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. For any of the quantum function algebras $R_q[G]$ and $\mathrm{w} = (w_+, w_-) \in W \times W$ the center Z_{w} of Joseph's localization R_{w} coincides with the Laurent polynomial algebra over $\mathbb{K}$ of dimension $\dim \ker(w_+ - w_-)$ with generators*

$$\{c^+_{w_+, \omega_i} \mid i \in \mathcal{I}(\mathrm{w})\} \sqcup \{c^+_{w_+, \lambda^{(j)}} (c^-_{w_-, \lambda^{(j)}})^{-1}\}_{j=1}^k,$$

where $k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$ and $\lambda^{(1)}, \ldots, \lambda^{(k)}$ is a basis of $\widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w})$.

Joseph proved that for all $\lambda \in \widetilde{\mathcal{L}}(\mathrm{w}) = \mathrm{Ker}(w_+ - w_-) \cap P$,

$$c^+_{w_+, \lambda} (c^+_{w_+, \lambda})^{-1} \in Z_{\mathrm{w}}. \tag{1.5}$$

The center Z_{w} is a free module over it of rank equal to $2^{|\mathcal{I}(\mathrm{w})|}$. The difficulty in the proof of the above theorem is not the guess of the exact form of the center Z_{w} (which can be interpreted as taking square roots of some of the elements (1.5)), but the proof of the fact that Z_{w} does not contain additional elements.

The proof of Theorem 1.1 appears in Section 3. We make use of a model of the algebra R_{w} due to Joseph. We expect that this model will play an important role in the future study of $\mathrm{Spec} R_q[G]$. Firstly, Joseph [24, 25] defines the algebras $S^{\pm}_{w_\pm}$ as the 0 components of the localizations $(R^{\pm}/I^{\pm}_{w_\pm})[(E^{\pm}_{w_\pm})^{-1}]$ with respect to the second grading in (1.1) (induced to the localization). He then defines an algebra S_{w} which is a kind of bicrossed product of $S^+_{w_+}$ and $S^-_{w_+}$, and proves that R_{w} is isomorphic to a smash product of a quantum torus and a localization of S_{w} by a set of normal elements. We refer the reader to [25, §§9.1-9.3 and §10.3] and §3.4 for details.

De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}^w_\pm \subseteq \mathcal{U}_\pm \subset \mathcal{U}_q(\mathfrak{g})$, which are parametrized by the elements w of the Weyl group of $\mathfrak{g}$. They can be viewed as deformations of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_\pm \cap w(\mathfrak{n}_\mp))$, where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras. The algebras $\mathcal{U}^w_\pm$ are defined in terms of the Lusztig's root vectors of $\mathcal{U}_q(\mathfrak{g})$. Our approach to determining the center Z_{w} of R_{w} is to make use of a family of isomorphisms $\varphi^\pm_{w_\pm} \colon S^\pm_{w_\pm} \to \mathcal{U}^{w_\pm}_\mp$, which was a main ingredient in our work [43] on the torus invariant spectra of $\mathcal{U}^{w_\pm}_\mp$ (see Theorem 2.6 below). With the help of these isomorphisms we study R_{w} using on one side the De Concini–Kac–Procesi PBW bases of $\mathcal{U}^{w_\pm}_\mp$ and the Levendorskii–Soibelman straightening rule. On the other side we use techniques from quantum function algebras such as producing good supplies of normal elements and having the R-matrix type commutation relations inside the algebras $S^\pm_{w_\pm}$ and between them in the “bicrossed product” S_{w}. The weight lattice P of $\mathfrak{g}$ acts in a natural way on the algebras $S^\pm_{w_\pm}$ and S_{w} by algebra automorphisms. Using the above mentioned techniques we investigate the set of homogeneous P-normal elements of the algebras $S^\pm_{w_\pm}$ and S_{w}, and obtain from that Theorem 1.1 for the center of R_{w}.

There is a natural action of the torus $\mathbb{T}^r \times \mathbb{T}^r = (\mathbb{K}^*)^{\times 2r}$ on $R_q[G]$ by algebra automorphisms, which quantizes the left and right regular actions of the maximal torus of G on the coordinate ring of G, see (4.6). Joseph proved that, if the base field $\mathbb{K}$ is algebraically closed, then the stratum of primitive ideals $\mathrm{Prim}_{\mathrm{w}} R_q[G] = \mathrm{Prim}\, R_q[G] \cap \mathrm{Spec}_{\mathrm{w}} R_q[G]$ is a single $\mathbb{T}^r$-orbits with respect to the action of each component. Each stratum $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ is preserved by $\mathbb{T}^r \times \mathbb{T}^r$. In Section 4 we apply the results of Theorem 1.1 to determine the exact structure of $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ as a $\mathbb{T}^r \times \mathbb{T}^r$-homogeneous space.

Now let us restrict ourselves to the case when $\mathbb{K} = \mathbb{C}$ and $q \in \mathbb{C}^*$ is not a root of unity. The connected, simply connected complex algebraic group G corresponding to $\mathfrak{g}$ is equipped with the so called standard Poisson structure π_G. It follows from the Kogan–Zelevinsky results [28] that all symplectic leaves of π_G are locally closed in the Zariski topology. We denote by $\mathrm{Sympl}(G, \pi_G)$ the symplectic foliation of π_G with the topology induced from the Zariski topology of G. Joseph established that there is a bijection from $\mathrm{Sympl}(G, \pi_G)$ to $\mathrm{Prim}\, R_q[G]$, that maps symplectic leaves in a double Bruhat cell G^{w} to the primitive ideals in $\mathrm{Prim}_{\mathrm{w}} R_q[G]$, thus settling a conjecture of Hodges and Levasseur [20]. In Section 4 we make this picture $\mathbb{T}^r \times \mathbb{T}^r$-equivariant. The torus $\mathbb{T}^r \times \mathbb{T}^r$ acts on (G, π_G) by Poisson maps. (One identifies $\mathbb{T}^r$ with a certain maximal torus of G and uses the left and right regular actions.) This induces a $\mathbb{T}^r \times \mathbb{T}^r$-action on $\mathrm{Sympl}(G, \pi_G)$. Combining Theorem 1.1 with the results of Kogan and Zelevinsky [28], we prove:

Theorem 1.2. *For each connected, simply connected, complex simple algebraic group G and $q \in \mathbb{C}^*$ which is not a root of unity, there exists a $\mathbb{T}^r \times \mathbb{T}^r$-equivariant map*

$$D_G \colon \mathrm{Sympl}(G, \pi_G) \to \mathrm{Prim}\, R_q[G]. \tag{1.6}$$

This map is explicitly constructed in §4.5. The Hodges–Levasseur idea [20] for an orbit method for $R_q[G]$ now can be formulated more concretely by conjecturing that (1.6) is a homeomorphism.

For the rest of this introduction we return to the general assumptions on $\mathbb{K}$ and q. In order to be able to compare prime ideals in different strata $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ for $\mathrm{w} = (w_+, w_-) \in W \times W$, one needs to investigate the maps ι_{w} from (1.4). For this one needs to know the structure of the algebras $R_{\rm w}$ as modules over their subalgebras generated by Joseph's normal sets $E_{\rm w}^{\pm 1}$. Because Joseph's model for $R_{\rm w}$ is based on the algebras $S^{\pm}_{w_\pm}$, one first needs to investigate the module structure of $S^{\pm}_{w_\pm}$ over their subalgebras $N^{\pm}_{w_\pm}$, generated by the following normal elements of $S^{\pm}_{w_\pm}$:

$$d^{\pm}_{w_\pm,\lambda} = (c^{\pm}_{w_\pm,\lambda})^{-1} c^{\pm}_{1,\lambda}, \quad \lambda \in P^+_{\mathcal{S}(w_\pm)}, \tag{1.7}$$

cf. (3.2). The classical theorems for separation of variables of Kostant [30] and Joseph–Letzter [27] prove that $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ are free modules over their centers and establish a number of properties of the related decomposition. In Section 5 we prove a result for separation of variables for the algebras $S^{\pm}_{w_\pm}$. The difference here is that the algebras $S^{\pm}_{w_\pm}$ behave like universal enveloping algebras of nilpotent Lie algebras since they are isomorphic to the De Concini–Kac–Procesi algebras $\mathcal{U}^{w_\pm}_{\mp}$ which are deformations of $\mathcal{U}(\mathfrak{n}_\mp \cap w_\pm(\mathfrak{n}_\pm))$. Because of this, generally they have small centers compared to their localizations by the multiplicative set of scalar multiples of the elements (1.7), see Remark 6.11 for a precise comparison. Because of this and for the ultimate purposes of classifying $\mathrm{Max} R_q[G]$, we consider the structure of $S^{\pm}_{w_\pm}$ as modules over their subalgebras $N^{\pm}_{w_\pm}$.

Theorem 1.3. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and $w_\pm \in W$, the algebras $S^{\pm}_{w_\pm}$ are free left and right modules over their subalgebras $N^{\pm}_{w_\pm}$ (generated by the normal elements* (1.7)*), in which $N^{\pm}_{w_\pm}$ are direct summands.*

Moreover we construct explicit bases of $S^{\pm}_{w_\pm}$ as $N^{\pm}_{w_\pm}$-modules using the PBW bases of $\mathcal{U}^{w_\pm}_{\mp}$, see Theorem 5.5 for details. Theorem 1.3 and some detailed analysis of the normal elements of $S^{\pm}_{w_\pm}$ lead us to the following classification result, proved in Section 6.

Theorem 1.4. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and $w \in W$, the nonzero homogeneous normal elements of $S^{\pm}_{w}$ are precisely the nonzero scalar multiples of the elements* (1.7)*. All such elements are distinct and even more the elements* (1.7) *are linearly independent.*

The De Concini–Kac–Procesi algebras $\mathcal{U}^{w}_{\pm}$ attracted a lot of attention from a ring theoretic perspective in recent years. The reason is that they contain as special cases various important families such as the algebras of quantum matrices. Moreover, they are the largest known family of Cauchon–Goodearl–Letzter extensions which are a kind of iterated skew polynomial extensions for which both the Goodearl–Letzter stratification theory [17] and the Cauchon theory of deleted derivations [9] work. A number of results were obtained for their torus invariant prime ideals. Mériaux and Cauchon classified them as a set [38]. The author described all inclusions between them and obtained an explicit formula for each torus invariant prime ideal [43], using results of Gorelik [19]. However there are no results explicitly describing prime ideals which are not torus invariant, except in some very special cases. Theorems 1.3 and 1.4 have many applications in this direction. Firstly Theorem 1.4 classifies the sets of all homogeneous normal elements of $\mathcal{U}^{\pm}_{w_\pm}$ (only the case when w equals the longest element of the Weyl group of $\mathfrak{g}$ was previously known due to Caldero [8]). In Section 6 we obtain an explicit formula for the prime and especially the primitive ideals in the Goodearl–Letzter stratum of Spec$\mathcal{U}_{\pm}^{w_{\pm}}$ over the $\{0\}$-ideal. A result of Launois, Lenagan and Rigal [31, Theorem 3.7] implies that the algebras $\mathcal{U}_{\pm}^{w_{\pm}}$ are noetherian unique factorization domains (see §6.2 for background). Therefore one is interested in knowing the sets of their prime elements. We classify all prime elements of the algebras $\mathcal{U}_{\pm}^{w_{\pm}}$ in Theorems 6.2 (ii) and 6.16. (For the latter theorem one needs to translate the results from $S_{w_{\pm}}^{\mp}$ to $\mathcal{U}_{\pm}^{w_{\pm}}$ via the isomorphism of Theorem 2.6, which is straightforward and is not stated separately.) As a corollary we obtain explicit formulas for all height one prime ideals of $\mathcal{U}_{\pm}^{w_{\pm}}$.

We return to the problem of describing the structure of the localizations R_{w} as modules over their subalgebras generated by the Joseph sets of normal elements $(E_{\mathrm{w}})^{\pm 1}$ for $\mathrm{w} = (w_+, w_-) \in W \times W$. We denote the latter subalgebras of R_{w} by L_{w}. One cannot deduce the module structure of R_{w} over L_{w} immediately from the one of $S_{w_{\pm}}^{\pm}$ over $N_{w_{\pm}}^{\pm}$ (recall Theorem 1.3), because the former is not a tensor product of the latter in Joseph's model for R_{w}. To overcome this difficulty, in Section 7, we define nontrivial $Q \times Q$ filtrations of the algebras S_{w}, where Q is the root lattice of $\mathfrak{g}$. (Note that the $P \times P$-grading (1.1) of $R_q[G]$ only induces a Q-grading on S_{w}.) The associated graded of the new $Q \times Q$-filtration of S_{w} breaks down in a certain way the "bicrossed product" of $S_{w_+}^{+}$ and $S_{w_-}^{-}$, and then one can apply Theorem 1.3. In Section 7 we prove:

Theorem 1.5. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and $\mathrm{w} \in W \times W$, the algebras L_{w} are quantum tori of dimension $r + |\mathcal{S}(\mathrm{w})|$. Moreover, Joseph's localizations R_{w} are free (left and right) modules over L_{w} in which L_{w} are direct summands.*

In addition, in Theorem 7.13 we construct an explicit L_{w}-basis of each R_{w}. In Theorems 1.3 and 1.5 we do not obtain further representation theoretic properties of the bases for those free modules as Kostant and Joseph–Letzter did in the cases of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$. We think that this is an important problem which deserves future study.

For the purposes of the study of catenarity and homological properties of $R_q[G]$ and its Hopf algebra quotients, Goodearl and Zhang [18] raised the question whether all maximal ideals of $R_q[G]$ have finite codimension. So far this was known in only two cases, $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$ due to Hodges–Levasseur [20] and Goodearl–Lenagan [15]. In Section 8 we classify Max$R_q[G]$ and settle affirmatively the question of Goodearl and Zhang in full generality.

Theorem 1.6. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and a simple Lie algebra $\mathfrak{g}$ the maximal spectrum of $R_q[G]$ is*

$$\mathrm{Max} R_q[G] = \mathrm{Prim}_{(1,1)} R_q[G].$$

The maximal spectrum of $R_q[G]$ is homeomorphic to the maximal spectrum of an r dimensional Laurent polynomial ring. All maximal ideals of $R_q[G]$ have finite codimension.

In addition, Theorem 8.9 provides an explicit formula for all maximal ideals of $R_q[G]$, see also Corollary 8.10 for the case when $\mathbb{K}$ is algebraically closed.

The difficult part of Theorem 1.6 is to show that $\mathrm{Max} R_q[G] \subset \mathrm{Spec} R_q[G]$. Our approach is to consider the projection $\pi_{\mathrm{w}} \colon R_{\mathrm{w}} \to L_{\mathrm{w}}$ along the direct complement from Theorem 1.5. We use the formula for primitive ideals $J \in \mathrm{Prim}_{\mathrm{w}} R_q[G]$ from Section 4 to study the projection $\pi_{\mathrm{w}}(J)$. We compare it to $\pi_{\mathrm{w}}(I_{(1,1)})$, to deduce that for $\mathrm{w} \neq (1,1)$, $J + I_{(1,1)} \neq R_q[G]$.

A ring R satisfies *the first chain condition for prime ideals* if all maximal chains in $\mathrm{Spec} R$ have the same length equal to $\mathrm{GK} \dim R$. This is a stronger property than catenarity. It was introduced by Nagata [39] in the commutative case. Combining Theorem 1.6 and results of Goodearl and Zhang [18], in Section 9 we prove:

Theorem 1.7. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and Hopf ideals I of $R_q[G]$, the quotient $R_q[G]/I$ satisfies the first chain condition for prime ideals and Tauvel's height formula holds.*

In the last stage of the typing of this manuscript we received a draft of a preprint of Zwicknagl [46] announcing a proof that for every quantum cluster algebra A the specialization map establishes a homeomorphism from $\mathrm{Spec} A$ to the Poisson prime spectrum of the semiclassical limit. This is combined with results of Geiß, Leclerc and Schröer [12] to claim such facts for $R_q[SL_n]$ and $\mathcal{U}_+^w$ for simply laced $\mathfrak{g}$. In each of those cases the Poisson prime spectrum is not explicitly known yet.

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2. Previous results on spectra of quantum function algebras

2.1. **Quantized universal enveloping algebras.** In this section we collect background material on quantum groups and the previous work on their spectra, which will be used in the paper.

We fix a base field $\mathbb{K}$ and $q \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ which is not a root of unity. Let $\mathfrak{g}$ be a simple Lie algebra of rank r with Cartan matrix (c_{ij}). Its quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ over $\mathbb{K}$ with deformation parameter q is a Hopf algebra over $\mathbb{K}$ with generators

$$X_i^{\pm}, K_i^{\pm 1}, \ i = 1, \ldots, r$$

and relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \ K_i K_j = K_j K_i,$$
$$K_i X_j^{\pm} K_i^{-1} = q_i^{\pm c_{ij}} X_j^{\pm},$$
$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \ i \neq j,$$

where $q_i = q^{d_i}$ and $\{d_i\}_{i=1}^r$ are the positive relatively prime integers such that $(d_i c_{ij})$ is symmetric. The comultiplication of $\mathcal{U}_q(\mathfrak{g})$ is given by:

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+,$$
$$\Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-.$$

Its antipode and counit are given by:

$$S(K_i) = K_i^{-1},\ S(X_i^+) = -K_i^{-1} X_i^+,\ S(X_i^-) = -X_i^- K_i,$$
$$\epsilon(K_i),\ \epsilon(X_i^\pm) = 0.$$

As usual q-integers, q-factorials, and q-binomial coefficients are denoted by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},\ [n]_q! = [1]_q \dots [n]_q,\ \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q}{[m]_q [n-m]_q},$$

$n, m \in \mathbb{N}$, $m \le n$. We refer to [23, Ch. 4] for more details.

Denote by $\mathcal{U}_\pm$ the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm\}_{i=1}^r$. Let H be the group generated by $\{K_i^{\pm 1}\}_{i=1}^r$, i.e. the group of all group-like elements of $\mathcal{U}_q(\mathfrak{g})$.

2.2. **Type 1 modules and braid group action.** The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$, respectively. Denote by P and P^+ the sets of integral and dominant integral weights of $\mathfrak{g}$. For $\lambda = \sum_i n_i \omega_i \in P$, let

$$\operatorname{Supp} \lambda = \{ i \in \{1, \dots, r\} \mid n_i \neq 0 \}. \tag{2.1}$$

Denote the root lattice $Q = \sum_i \mathbb{Z} \alpha_i$ of $\mathfrak{g}$. Set $Q^+ = \sum_i \mathbb{N} \alpha_i$. Let $Q^\vee$ be the coroot lattice of $\mathfrak{g}$. We will use the following standard partial order on P:

$$\text{for } \lambda_1, \lambda_2 \in P,\ \lambda_1 < \lambda_2, \text{ if and only if } \lambda_2 - \lambda_1 \in Q^+ \backslash \{0\}. \tag{2.2}$$

Denote by Δ^+ and Δ^- the sets of positive and negative roots of $\mathfrak{g}$.

Let $\langle ., . \rangle$ be the nondegenerate bilinear form on the dual of the Cartan subalgebra of $\mathfrak{g}$ defined by

$$\langle \alpha_i, \alpha_j \rangle = d_i c_{ij}. \tag{2.3}$$

The q-weight spaces of an H-module V are defined by

$$V_\mu = \{ v \in V \mid K_i v = q^{\langle \mu, \alpha_i \rangle} v,\ \forall i = 1, \dots, r \},\ \mu \in P.$$

A $\mathcal{U}_q(\mathfrak{g})$-module is called a type 1 module if it is the sum of its q-weight spaces, see [23, Ch. 5] for details. The irreducible finite dimensional type 1 $\mathcal{U}_q(\mathfrak{g})$-modules are parametrized by P^+, [23, Theorem 5.10]. Let $V(\lambda)$ denote the irreducible type 1 $\mathcal{U}_q(\mathfrak{g})$-module of highest weight $\lambda \in P^+$. Let $M(\lambda)$ denote the Verma module of $\mathcal{U}_q(\mathfrak{g})$ with highest weight λ and highest weight vector u_λ. For an arbitrary base field $\mathbb{K}$, and $q \in \mathbb{K}^*$ which is not a root of unity, the Weyl character formula holds for $V(\lambda)$ and $V(\lambda)$ is given as a quotient of $M(\lambda)$ by the standard formula from the classical case (see [1, Corollary 7.7] and [23, p. 126]):

$$V(\lambda) \cong M(\lambda) / \Big(\sum_{i=1}^r \mathcal{U}_q(\mathfrak{g}) (X_i^-)^{\langle \lambda, \alpha_i^\vee \rangle + 1} u_\lambda \Big). \tag{2.4}$$

All duals of finite dimensional $\mathcal{U}_q(\mathfrak{g})$-modules will be considered as left modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$. The category of finite dimensional type 1 $\mathcal{U}_q(\mathfrak{g})$-modules is semisimple [23, Theorem 5.17] (cf. also the remark on p. 85 of [23]) and is closed under taking tensor products and duals.

Denote by W and $\mathcal{B}_\mathfrak{g}$ the Weyl and braid groups associated to $\mathfrak{g}$. The simple reflections of W corresponding to $\alpha_1, \ldots, \alpha_r$ will be denoted by $s_1, \ldots, s_r$. The corresponding generators of $\mathcal{B}_\mathfrak{g}$ will be denoted by $T_1, \ldots, T_r$. For a Weyl group element w, $l(w)$ will denote its length. The Bruhat order on W will be denoted by $\leq$.

Lusztig defined actions of $\mathcal{B}_\mathfrak{g}$ on all finite dimensional type 1 modules and $\mathcal{U}_q(\mathfrak{g})$. On a finite dimensional type 1 module V the generators $T_1, \ldots, T_r$ of $\mathcal{B}_\mathfrak{g}$ act by (see [37, §5.2] and [23, §8.6] for details):

$$T_i(v) = \sum_{l,m,n} (-1)^m q_i^{ln-m} (X_i^+)^{(l)} (X_i^-)^{(m)} (X_i^+)^{(n)} v, \quad v \in V_\mu, \mu \in P, \tag{2.5}$$

where the sum is over $l, m, n \in \mathbb{N}$ such that $-l + m - n = \langle \mu, \alpha_i^\vee \rangle$ and

$$(X_i^\pm)^{(l)} = \frac{(X_i^\pm)^l}{[l]_{q_i}}.$$

The action of the braid group $\mathcal{B}_\mathfrak{g}$ satisfies

$$T_w V(\lambda)_\mu = V(\lambda)_{w(\mu)}, \quad \forall \lambda \in P^+, \mu \in P, w \in W.$$

This implies that $\dim V(\lambda)_{w\lambda} = 1$ for $\lambda \in P^+, w \in W$. The braid group $\mathcal{B}_g$ acts on $\mathcal{U}_q(\mathfrak{g})$ by

$$T_i(X_i^+) = -X_i^- K_i, \; T_i(X_i^-) = -K_i^{-1} X_i^+, \; T_i(K_j) = K_j K_i^{-c_{ij}},$$

$$T_i(X_j^+) = \sum_{k=0}^{-c_{ij}} (-q_i)^{-k} (X_i^+)^{(-c_{ij}-k)} X_j^+ (X_i^+)^{(k)}, \; j \neq i,$$

$$T_i(X_j^-) = \sum_{k=0}^{-c_{ij}} (-q_i)^{k} (X_i^-)^{(k)} X_j^- (X_i^-)^{(-c_{ij}-k)}, \; j \neq i,$$

see [37, §37.1] and [23, 8.14] for details. The two actions on type 1 finite dimensional modules and $\mathcal{U}_q(\mathfrak{g})$ are compatible:

$$T_w(x.v) = (T_w x).(T_w v), \tag{2.6}$$

for all $w \in W$, $x \in \mathcal{U}_q(\mathfrak{g})$, $v \in V(\lambda)$, see [23, 8.13].

2.3. ***H*-prime ideals of Quantum Groups.** Denote by $R_q[G]$ the Hopf subalgebra of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by all matrix coefficients of the modules $V(\lambda)$. For $\xi \in V(\lambda)^*$, $v \in V(\lambda)$ define

$$c^\lambda_{\xi,v} \in R_q[G] \; \text{ by } \; c^\lambda_{\xi,v}(x) = \xi(xv), \; \forall x \in \mathcal{U}_q(\mathfrak{g}).$$

There are two canonical left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $R_q[G]$ given by

$$x \rightharpoonup c = \sum c_{(2)}(x) c_{(1)}, \; c \leftharpoonup x = \sum c_{(1)}(x) c_{(2)}, \; x \in \mathcal{U}_q(\mathfrak{g}), c \in R_q[G] \tag{2.7}$$

and a corresponding $P \times P$-grading on $R_q[G]$

$$R_q[G]_{\nu,\mu} = \{c^\lambda_{\xi,v} \mid \lambda \in P^+, \xi \in V(\lambda)^*_\nu, v \in V(\lambda)_\mu\}, \quad \nu, \mu \in P. \tag{2.8}$$

Define the subalgebras of $R_q[G]$

$$R^+ = \mathrm{Span}\{c^\lambda_{\xi,v} \mid \lambda \in P^+, v \in V(\lambda)_\lambda, \xi \in V(\lambda)^*\},$$
$$R^- = \mathrm{Span}\{c^\lambda_{\xi,v} \mid \lambda \in P^+, v \in V(\lambda)_{w_0\lambda}, \xi \in V(\lambda)^*\},$$

where w_0 denotes the longest element of W. Joseph proved [25, Proposition 9.2.2] that $R_q[G] = R^+R^- = R^-R^+$, see §3.4 below for more details.

Throughout the paper we fix highest weight vectors $v_\lambda \in V(\lambda)_\lambda$, $\lambda \in P^+$. Denote the corresponding lowest weight vectors $v_{-\lambda} = T_{w_0} v_\lambda \in V(-w_0\lambda)_{-\lambda}$. For $\xi \in V(\lambda)^*$ and $\xi' \in V(-w_0\lambda)^*$ denote

$$c^\lambda_{\xi,\lambda} = c^\lambda_{\xi,v_\lambda} \quad \text{and} \quad c^{-w_0\lambda}_{\xi',-\lambda} = c^{-w_0\lambda}_{\xi',v_{-\lambda}}. \tag{2.9}$$

As vector spaces R^+ and R^- can be identified with $\oplus_{\lambda \in P^+} V(\lambda)^*$ by

$$\xi \in V(\lambda)^* \mapsto c^\lambda_{\xi,\lambda} \quad \text{and} \quad \xi' \in V(-w_0\lambda)^* \mapsto c^{-w_0\lambda}_{\xi',-\lambda},$$

respectively. Then the multiplication in $R^\pm$ can be identified with the Cartan multiplication rule (see [25, §9.1.6])

$$V(\lambda_1)^* V(\lambda_2)^* \to V(\lambda_1+\lambda_2)^*, \quad \xi_1.\xi_2 := (\xi_1 \otimes \xi_2)|_{V(\lambda_1+\lambda_2)}, \tag{2.10}$$

where $\lambda_i \in P^+$, $\xi_i \in V(\lambda_i)^*$, $i = 1, 2$. We normalize the embeddings $V(\lambda_1 + \lambda_2) \hookrightarrow V(\lambda_1) \otimes_{\mathbb{K}} V(\lambda_2)$ so that $v_{\lambda_1+\lambda_2} \mapsto v_{\lambda_1} \otimes_{\mathbb{K}} v_{\lambda_2}$. One should note that $T_{w_0} v_{\lambda_1+\lambda_2} \mapsto T_{w_0} v_{\lambda_1} \otimes T_{w_0} v_{\lambda_2}$, because of (2.44). Thus $v_{-\lambda_1-\lambda_2} \mapsto v_{-\lambda_1} \otimes v_{-\lambda_2}$ under $V(-w_0(\lambda_1+\lambda_2)) \hookrightarrow V(-w_0\lambda_1) \otimes_{\mathbb{K}} V(-w_0\lambda_2)$

Recall that for all $w \in W$ the weight spaces $V(\lambda)_{w\lambda}$ are one dimensional, see §2.2. Define the Demazure modules

$$V^+_w(\lambda) = \mathcal{U}_+ V(\lambda)_{w\lambda} \subseteq V(\lambda), \quad V^-_w(\lambda) = \mathcal{U}_- V(-w_0\lambda)_{-w\lambda} \subseteq V(-w_0\lambda), \tag{2.11}$$

for $\lambda \in P^+$, $w \in W$, and the canonical projections

$$g^+_{w_+} : V(\lambda)^* \to (V^+_{w_+}(\lambda))^* \quad \text{and} \quad g^-_{w_-} : V(-w_0\lambda)^* \to (V^-_{w_-}(\lambda))^*. \tag{2.12}$$

Following Joseph [24, 25] and Hodges–Levasseur [20, 21], define

$$I^+_w = \mathrm{Span}\{c^\lambda_{\xi,v} \mid \lambda \in P^+, v \in V(\lambda)_\lambda, \xi \in V^+_w(\lambda)^\perp\} \subset R^+, \tag{2.13}$$

$$I^-_w = \mathrm{Span}\{c^{-w_0\lambda}_{\xi,v} \mid \lambda \in P^+, v \in V(-w_0\lambda)_{-\lambda}, \xi \in V^-_w(\lambda)^\perp\} \subset R^-. \tag{2.14}$$

For $\mathrm{w} = (w_+, w_-) \in W \times W$ define

$$I_{\mathrm{w}} = I^+_{w_+} R^- + R^+ I^-_{w_-} \subset R_q[G]. \tag{2.15}$$

Theorem 2.1. *(Joseph,* [25, Proposition 8.9]*,* [24, Proposition 10.1.8, Proposition 10.3.5]*)*

(i) For each $w \in W$, $I^\pm_w$ is an H-invariant completely prime ideal of $R^\pm$ with respect to the left action (2.7) *of H. All H-primes of $R^\pm$ are of this form.*

(ii) For each $\mathrm{w} \in W \times W$, I_{w} is a an H-invariant completely prime ideal of $R_q[G]$ with respect to the left action of H. All H-primes of $R_q[G]$ are of this form.

In [24] Theorem 2.1 was stated for $\mathbb{K} = \mathbb{C}$, $q \in \mathbb{C}^*$ not a root of unity and in [25] Theorem 2.1 was stated for $\mathbb{K} = k(q)$ for a field k of characteristic 0. It is well known that Joseph's proof works for an arbitrary field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of 1, see [29, §3.3] for a related discussion.

The ideals $I_{\rm w}$ are also H-stable with respect to the right action (2.7). The left and right invariance property of the ideals $I_{\rm w}$ with respect to H can be formulated in terms of invariance with respect to a torus action. See §2.6 for details.

2.4. **Sets of normal elements.** Recall that for $w \in W$ the weight spaces $V(\lambda)_{w\lambda}$ are one dimensional. For $\lambda \in P^+$, $w \in W$, denote by $\xi^+_{w,\lambda} \in V(\lambda)^*_{-w\lambda}$ and $\xi^-_{w,\lambda} \in V(-w_0\lambda)^*_{w\lambda}$ the vectors normalized by

$$\langle \xi^+_{w,\lambda}, T_w v_\lambda \rangle = 1 \;\text{ and }\; \langle T_{w^{-1}} \xi^-_{w,\lambda}, v_{-\lambda} \rangle = 1. \tag{2.16}$$

Define

$$c^+_{w,\lambda} = c^\lambda_{\xi^+_{w,\lambda}, v_\lambda}, \quad c^-_{w,\lambda} = c^{-w_0\lambda}_{\xi^-_{w,-\lambda}, v_{-\lambda}} \tag{2.17}$$

in terms of the highest and lowest weight vectors v_λ and $v_{-\lambda}$, fixed in §2.3. The second normalization condition in (2.16) was chosen to match it with the Kogan–Zelevinsky normalizations [28]. This will ensure a proper alignment of the semiclassical and quantum pictures in Section 4. We discuss this in more detail at the end of Section 4.

The coproduct property (2.44) of the braid group action, reviewed in §2.7, implies:

$$c^+_{w,\lambda_1} c^+_{w,\lambda_2} = c^+_{w,\lambda_1+\lambda_2} \;\text{ and }\; c^-_{w,\lambda_1} c^-_{w,\lambda_2} = c^-_{w,\lambda_1+\lambda_2}, \quad \forall \lambda_1, \lambda_2 \in P^+. \tag{2.18}$$

2.5. **Localizations of quotients of $R_q[G]$ by its H-primes.** The algebra $\mathcal{U}_q(\mathfrak{g})$ is Q-graded by

$$\deg X^\pm_i = \pm\alpha_i, \;\; \deg K_i = 0, \quad i = 1, \ldots, r. \tag{2.19}$$

The homogeneous component of $\mathcal{U}_q(\mathfrak{g})$ corresponding to $\gamma \in Q$ will be denoted by $(\mathcal{U}_q(\mathfrak{g}))_\gamma$.

For $\gamma \in Q^+$, $\gamma \neq 0$ denote $m(\gamma) = \dim(\mathcal{U}_+)_\gamma = \dim(\mathcal{U}_-)_{-\gamma}$ and fix a pair of dual bases $\{u_{\gamma,i}\}_{i=1}^{m(\gamma)}$ and $\{u_{-\gamma,i}\}_{i=1}^{m(\gamma)}$ of $(\mathcal{U}_+)_\gamma$ and $(\mathcal{U}_-)_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [23, Ch. 6] for a discussion of the properties of this form for arbitrary fields $\mathbb{K}$.

The R-matrix commutation relations in $R_q[G]$ (see [6, Theorem I.8.15]) imply:

Lemma 2.2. *Let $\lambda_i \in P^+$, $\nu_i \in P$, $i = 1, 2$ and $\xi_2 \in V(\lambda_2)^*_{-\nu_2}$.*

(i) For all $\mu_1 \in P$, $v_1 \in V(\lambda_1)_{\mu_1}$ and $\xi_1 \in V(\lambda_1)^*_{-\nu_1}$:

$$c^{\lambda_1}_{\xi_1,v_1} c^{\lambda_2}_{\xi_2,\lambda_2} = q^{\langle\mu_1,\lambda_2\rangle - \langle\nu_1,\nu_2\rangle} c^{\lambda_2}_{\xi_2,\lambda_2} c^{\lambda_1}_{\xi_1,v_1} + \sum_{\gamma\in Q^+,\gamma\neq 0} \sum_{i=1}^{m(\gamma)} q^{\langle\mu_1,\lambda_2\rangle - \langle\nu_1+\gamma,\nu_2-\gamma\rangle} c^{\lambda_2}_{S^{-1}(u_{\gamma,i})\xi_2,\lambda_2} c^{\lambda_1}_{S^{-1}(u_{-\gamma,i})\xi_1,v_1}.$$

(ii) For all $\mu_2 \in P$, $v_2 \in V(\lambda_2)_{\mu_2}$ and $\xi_1 \in V(-w_0\lambda_1)^*_{-\nu_1}$:

$$c^{-w_0\lambda_1}_{\xi_1,-\lambda_1} c^{\lambda_2}_{\xi_2,v_2} = q^{-\langle\mu_1,\lambda_2\rangle - \langle\nu_1,\nu_2\rangle} c^{\lambda_2}_{\xi_2,v_2} c^{-w_0\lambda_1}_{\xi_1,-\lambda_1} + \sum_{\gamma\in Q^+,\gamma\neq 0} \sum_{i=1}^{m(\gamma)} q^{-\langle\mu_1,\lambda_2\rangle - \langle\nu_1+\gamma,\nu_2-\gamma\rangle} c^{\lambda_2}_{S^{-1}(u_{\gamma,i})\xi_2,v_2} c^{-w_0\lambda_1}_{S^{-1}(u_{-\gamma,i})\xi_1,-\lambda_1}.$$

Thus for all $\lambda \in P^+$, $w \in W$, $\nu, \mu \in P$ and $c \in R_q[G]_{-\nu,\mu}$

$$c^+_{w,\lambda} c = q^{\langle w\lambda, \nu\rangle - \langle \lambda, \mu\rangle} c c^+_{w,\lambda} \mod I^+_w R^-, \tag{2.20}$$

$$c^-_{w,\lambda} c = q^{\langle w\lambda, \nu\rangle - \langle \lambda, \mu\rangle} c c^-_{w,\lambda} \mod R^+ I^-_w. \tag{2.21}$$

By abuse of notation we will denote the images of $c^\lambda_{\xi,v}$ and $c^\pm_{w,\lambda}$ in $R^\pm/I^\pm_{w\pm}$ and $R_q[G]/I_{\mathrm{w}}$ by the same symbols (recall (2.15)), as it is commonly done in [24, 25, 22]. All $c^\pm_{w,\lambda} \in R/I^\pm_{w_\pm}$ are nonzero normal elements, see (2.20)–(2.21). Their images in $R_q[G]/I_{\mathrm{w}}$ are also nonzero normal elements. Denote the multiplicative subsets of $R^\pm$, $R^\pm/I^\pm_{w^\pm}$ and $R_q[G]/I_{\mathrm{w}}$

$$E^\pm_{w_\pm} = \{c^\pm_{w_\pm,\lambda} \mid \lambda \in P^+\}. \tag{2.22}$$

Denote the multiplicative subset of $R_q[G]$ and $R_q[G]/I_{\mathrm{w}}$

$$E_{\mathrm{w}} = E^+_{w_+} E^-_{w_-}, \tag{2.23}$$

the localization

$$R_{\mathrm{w}} = (R_q[G]/I_{\mathrm{w}})[E_{\mathrm{w}}^{-1}] \tag{2.24}$$

and its center

$$Z_{\mathrm{w}} = Z(R_{\mathrm{w}}). \tag{2.25}$$

Since the ideal I_{w} is homogeneous with respect to the $P \times P$-grading (2.8) of $R_q[G]$, $R_q[G]/I_{\mathrm{w}}$ inherits a $P \times P$-grading. Denote the corresponding components

$$(R_q[G]/I_{\mathrm{w}})_{\nu,\mu} = (R_q[G]_{\nu,\mu} + I_{\mathrm{w}})/I_{\mathrm{w}}, \quad \nu, \mu \in P. \tag{2.26}$$

The elements of E_{w} are $P \times P$-homogeneous. Thus R_{w} also inherits a $P \times P$-grading. Its components will be denoted by $(R_{\mathrm{w}})_{\nu,\mu}$.

Recall that $c^\pm_{w,\lambda_1} c^\pm_{w,\lambda_2} = c^\pm_{w,\lambda_1+\lambda_2}$ for all $\lambda_1, \lambda_2 \in P^+$. Write $\lambda \in P$ as $\lambda = \lambda_1 - \lambda_2$ for some $\lambda_1, \lambda_2 \in P^+$ and define

$$c^\pm_{w,\lambda} = c^\pm_{w,\lambda_+} (c^\pm_{w,\lambda_-})^{-1} \in R_{\mathrm{w}}. \tag{2.27}$$

This definition does not depend on the choice of λ_1 and λ_2, because of the above mentioned property of the elements $c^\pm_{w,\lambda}$. We have:

$$c^\pm_{w,\lambda_1} c^\pm_{w,\lambda_2} = c^\pm_{w,\lambda_1+\lambda_2}, \quad \forall \lambda_1, \lambda_2 \in P. \tag{2.28}$$

Eqs. (2.20)–(2.21) imply that

$$c^\pm_{w,\lambda} c = q^{\langle w\lambda, \nu\rangle - \langle \lambda, \mu\rangle} c c^\pm_{w,\lambda}, \tag{2.29}$$

for all $\lambda, \nu, \mu \in P$ and $c \in (R_{\mathrm{w}})_{-\nu,\mu}$.

2.6. **Spectral decomposition theorem for $R_q[G]$.** Consider the torus $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$ and define

$$t^\mu = \prod_{i=1}^r t_i^{\langle \mu, \alpha_i^\vee \rangle}, \quad t = (t_1, \ldots, t_r) \in \mathbb{T}^r, \mu \in P. \tag{2.30}$$

There are two commuting rational $\mathbb{T}^r$-actions on $R_q[G]$ by $\mathbb{K}$-algebra automorphisms:

$$t \cdot c = t^\mu c, \quad t \in \mathbb{T}^r, c \in R_q[G]_{-\nu,\mu}, \nu, \mu \in P \tag{2.31}$$

and

$$t \cdot c = t^{\nu} c, \quad t \in \mathbb{T}^r, c \in R_q[G]_{-\nu,\mu}, \nu, \mu \in P. \tag{2.32}$$

These actions are extensions of the left and right actions (2.7) of H on $R_q[G]$, respectively, under the embedding $H \hookrightarrow \mathbb{T}^r$ given by $K_i \mapsto (1, \ldots, 1, q_i, 1, \ldots, 1)$, $i = 1, \ldots, r$, where $q_i = q^{d_i}$ is in position i.

Theorem 2.3. *(Joseph [24], Hodges–Levasseur [21]) (i) For each prime ideal J of $R_q[G]$, there exists a unique $\mathrm{w} \in W \times W$ such that $J \supseteq I_{\mathrm{w}}$ and $(J/I_{\mathrm{w}}) \cap E_{\mathrm{w}} = \emptyset$.*

(ii) For each $\mathrm{w} = (w_+, w_-)$, the ring Z_{w} is isomorphic to a Laurent polynomial ring over $\mathbb{K}$ of dimension $\dim \ker(w_+ - w_-)$. Moreover the stratum $\mathrm{Spec}_{\mathrm{w}} R_q[G] \subset \mathrm{Spec} R_q[G]$ of ideals corresponding to w by (i) is homeomorphic to $\mathrm{Spec} Z_{\mathrm{w}}$ via the map $\iota_{\mathrm{w}} \colon \mathrm{Spec} Z_{\mathrm{w}} \to \mathrm{Spec}_{\mathrm{w}} R_q[G]$ defined as follows. For each $J^0 \in \mathrm{Spec} Z_{\mathrm{w}}$, $\iota_{\mathrm{w}}(J^0)$ is the unique ideal of $R_q[G]$ containing I_{w} such that

$$\iota_{\mathrm{w}}(J^0)/I_{\mathrm{w}} = (R_{\mathrm{w}} J^0) \cap (R_q[G]/I_{\mathrm{w}}).$$

(iii) For each $\mathrm{w} \in W \times W$, the set of primitive ideals $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ in the stratum $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ is precisely $\iota_{\mathrm{w}}^{-1}(\mathrm{Max} Z_{\mathrm{w}})$. If the base field $\mathbb{K}$ is algebraically closed, then $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ is the $\mathbb{T}^r$-orbit of a single primitive ideal.

Hodges and Levasseur proved the theorem in the A case in [21]. Joseph gave a proof in the general case [24]. We refer the reader to Joseph's book [25] for a detailed treatment of these and many other related results. A multiparameter version of this result was obtained by Hodges, Levasseur, and Toro in [22]. For part (i) see [24, Corollary 6.4] and [22, Theorem 4.4], and for part (iii) [24, Theorem 9.2] and [22, Theorem 4.16]. Joseph states part (ii) of Theorem 2.3 in terms of orbits of $\mathbb{Z}_2^{\times r}$, see [24, Theorem 8.11], [25, Theorem 10.3.4]. In the above form it is stated in Hodges–Levasseur–Toro [22, Theorem 4.15]. Brown, Goodearl and Letzter [5, 17] observed that the strata of $\mathrm{Spec} R_q[G]$ can be also described by

$$\mathrm{Spec}_{\mathrm{w}} R_q[G] = \{J \in \mathrm{Spec} R_q[G] \mid \cap_{t \in \mathbb{T}^r} t \cdot J = I_{\mathrm{w}}\}$$

(with respect to either (2.31) or (2.32)) and developed this point of view to a general stratification method for the spectra of algebras with torus actions [17, 6, 13]. In [24, 22] Theorem 2.3 is stated for $\mathbb{K} = \mathbb{C}$, $q \in \mathbb{C}^*$ not a root of unity and in [25] for $\mathbb{K} = k(q)$ for a field k of characteristic 0. It is well known that the proofs of Joseph and Hodges–Levasseur–Toro of Theorem 2.3 work for all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, as was noted in a similar context for the results in Theorem 2.1.

Joseph [24, 25] determined the centers Z_{w} up to a finite extension. The next section contains a detailed discussion of this and an explicit description of Z_{w}. It follows from Theorem 2.1 (ii), as well as from Theorem 2.3 (ii), that: (1) The ideals I_{w}, $\mathrm{w} \in W \times W$ are stable under both actions (2.31) and (2.32) of $\mathbb{T}^r$ on $R_q[G]$, and (2) every prime ideal of $R_q[G]$ which is $\mathbb{T}^r$-stable under (2.31) or (2.32) is of this form.

We also note that the algebras R_{w} play an important role in the work of Berenstein and Zelevinsky [4] on quantum cluster algebras. They are quantizations of the coordinate rings of double Bruhat cells in simple Lie groups, which were proved to be upper cluster algebras by Berenstein, Fomin and Zelevinsky [3].

2.7. **The De Concini–Kac–Procesi algebras.** Recall from §2.2 that the braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ acts on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms.

Fix $w \in W$. Let

$$w = s_{i_1} \dots s_{i_l} \tag{2.33}$$

be a reduced expression of w. Recall that the roots in $\Delta^+ \cap w(\Delta^-)$ are given by

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_l = s_{i_1} \dots s_{i_{l-1}}(\alpha_{i_l}). \tag{2.34}$$

Define Lusztig's root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{i_1}(X^\pm_{i_2}), \dots, X^\pm_{\beta_l} = T_{i_1} \dots T_{i_{l-1}}(X^\pm_{i_l}), \tag{2.35}$$

see [37, §39.3] for details. The elements $X^\pm_{\beta_k}$ satisfy the Levendorskii–Soibelman straightening rule [33]:

$$\begin{aligned} X^\pm_{\beta_i} X^\pm_{\beta_j} &- q^{\pm\langle \beta_i, \beta_j \rangle} X^\pm_{\beta_j} X^\pm_{\beta_j} \\ &= \sum_{\mathbf{k}=(k_{i+1},\dots,k_{j-1}) \in \mathbb{N}^{\times (j-i-2)}} p_\mathbf{k} (X^\pm_{\beta_{j-1}})^{k_{j-1}} \dots (X^\pm_{\beta_{i+1}})^{k_{i+1}}, \;\; p_\mathbf{k} \in \mathbb{K}, \end{aligned} \tag{2.36}$$

for $i < j$. We refer to [6, Proposition I.6.10] for the version of $\mathcal{U}_q(\mathfrak{g})$ used in this paper. One can change the order of the monomials in the right hand side of (2.36) by repeatedly commuting the $X^\pm_{\beta_k}$'s in those monomials using (2.36). The result is the following second version of (2.36), which we will need in this paper:

$$\begin{aligned} X^\pm_{\beta_i} X^\pm_{\beta_j} &- q^{\pm\langle \beta_i, \beta_j \rangle} X^\pm_{\beta_j} X^\pm_{\beta_j} \\ &= \sum_{\mathbf{k}=(k_{i+1},\dots,k_{j-1}) \in \mathbb{N}^{\times (j-i-2)}} p_\mathbf{k} (X^\pm_{\beta_{i+1}})^{k_{i+1}} \dots (X^\pm_{\beta_{j-1}})^{k_{j-1}}, \;\; p_\mathbf{k} \in \mathbb{K}, \end{aligned} \tag{2.37}$$

for $i < j$.

De Concini, Kac and Procesi defined [11] the subalgebras $\mathcal{U}^w_\pm$ of $\mathcal{U}_\pm$ generated by $X^\pm_{\beta_j}$, $j = 1, \dots, l$ and proved the following result:

Theorem 2.4. *(De Concini, Kac, Procesi)* [11, Proposition 2.2] *The algebras $\mathcal{U}^w_\pm$ do not depend on the choice of a reduced expression of w and have the PBW basis*

$$(X^\pm_{\beta_1})^{n_1} \dots (X^\pm_{\beta_l})^{n_l}, \;\; n_1, \dots, n_l \in \mathbb{Z}_{\geq 0}. \tag{2.38}$$

Lusztig established independently [37, Proposition 40.2.1] that the space spanned by the monomials (2.38) does not depend on the choice of a reduced expression of w.

Usually Theorem 2.4 is stated in the form that the monomials $(X^\pm_{\beta_l})^{n_l} \dots (X^\pm_{\beta_1})^{n_1}$ form a basis of $\mathcal{U}^w_\pm$. This is proved by repeatedly applying (2.36) to achieve the decreasing order of $X^\pm_{\beta_k}$'s. Similarly, the above result is proved, using (2.37) to achieve the increasing order of $X^\pm_{\beta_k}$'s. We will need this version and not the original one for the proof of the freeness result in Section 5. For the other applications in this paper one can equally use (2.36).

In relation to Theorem 2.4, for $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{N}^{\times l}$ denote the monomial

$$(X^\pm)^\mathbf{n} = (X^\pm_{\beta_1})^{n_1} \dots (X^\pm_{\beta_l})^{n_l}. \tag{2.39}$$

These monomials form a basis of $\mathcal{U}^w_\pm$ over $\mathbb{K}$. We will say that $(X^\pm)^{\mathbf{n}}$ has *degree* $\mathbf{n}$. Introduce the lexicographic order on $\mathbb{N}^{\times l}$:

$$\text{(2.40)} \quad \mathbf{n} = (n_1, \ldots, n_l) < \mathbf{m} = (m_1, \ldots, m_l), \text{ if there exists } j \in \{1, \ldots, l\} \\ \text{such that } n_j < m_j \text{ and } n_{j+1} = m_{j+1}, \ldots, n_l = m_l.$$

We will say that the *highest term* of a nonzero element $u \in \mathcal{U}^w_\pm$ is $p(X^\pm)^{\mathbf{n}}$, where $\mathbf{n} \in \mathbb{N}^{\times l}$ and $p \in \mathbb{K}^*$, if

$$u - p(X^\pm)^{\mathbf{n}} \in \mathrm{Span}\{(X^\pm)^{\mathbf{n}'} \mid \mathbf{n}' \in \mathbb{N}^{\times l}, \mathbf{n}' < \mathbf{n}\}.$$

The Levendorskii–Soibelman straightening rule implies that one obtains an $\mathbb{N}^{\times l}$-filtration on $\mathcal{U}^w_\pm$ by collecting the elements with highest terms of degree $\leq \mathbf{n}$ for $\mathbf{n} \in \mathbb{N}^{\times l}$:

Lemma 2.5. *For all $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^{\times l}$ the highest term of the product $(X^\pm)^{\mathbf{n}}(X^\pm)^{\mathbf{n}'}$ is $q^{m_{\mathbf{n},\mathbf{n}'}}(X^\pm)^{\mathbf{n}+\mathbf{n}'}$, for some $m_{\mathbf{n},\mathbf{n}'} \in \mathbb{Z}$.*

2.8. **A second presentation of $\mathcal{U}^w_\pm$.** The algebras $\mathcal{U}^w_\pm$ are isomorphic to the algebras $S^\mp_w$ which play an important role in Joseph's work [24, 25]. The latter algebras are defined as follows. The quotients $R^\pm/I^\pm_{w_\pm}$ can be canonically identified as vector spaces with $\oplus_{\lambda \in P^+} V^\pm_{w_\pm}(\lambda)^*$ by $c^{\lambda_1}_{\xi_1,\lambda_1} \mapsto g^+_{w_+}(\xi_1)$ and $c^{-w_0\lambda_2}_{\xi_2,-\lambda_2} \mapsto g^-_{w_-}(\xi_2)$, for $\lambda_1, \lambda_2 \in P^+$, $\xi_1 \in V(\lambda_1)^*$, $\xi_2 \in V(-w_0\lambda_1)^*$, where we used the projections (2.12). Recall that $R^\pm_{w_\pm} = (R^\pm/I^\pm_{w_\pm})[E^\pm_{w_\pm}]$. The invariant subalgebras $S^\pm_{w_\pm}$ of $R^\pm_{w_\pm}$ with respect to the left action (2.7) of H will be denoted by $S^\pm_{w_\pm}$. In terms of the above vector space identifications

$$S^\pm_{w_\pm} = \varinjlim_{\lambda \in P^+} (c^\pm_{w_\pm,\lambda})^{-1} V^\pm_w(\lambda)^*. \tag{2.41}$$

Recall the definition (2.12) of the projections $g^\pm_{w_\pm}$. For $\lambda_1, \lambda_2 \in P^+$ the embedding

$$(c^\pm_{w_\pm,\lambda_2})^{-1} V^\pm_{w_\pm}(\lambda_2)^* \hookrightarrow (c^\pm_{w_\pm,\lambda_1+\lambda_2})^{-1} V^\pm_{w_\pm}(\lambda_1+\lambda_2)^*$$

is given by $(c^\pm_{w_\pm,\lambda_2})^{-1}\xi \mapsto (c^\pm_{w_\pm,\lambda_1+\lambda_2})^{-1}(g^\pm_{w_\pm}(\xi^\pm_{w_\pm,\lambda_1}).\xi)$, where $\xi \in V^\pm_w(\lambda_2)^*$. The product in the right hand side is the Cartan multiplication (2.10) and $\xi^\pm_{w_\pm,\lambda_1}$ are the weight vectors, defined in §2.4. The $P \times P$-grading of $R_q[G]$ induces $P \times P$-gradings on $R^\pm/I^\pm_{w_\pm}$, $R^\pm_{w_\pm}$, and $S^\pm_{w_\pm}$, analogously to (2.26). Denote the graded components of the algebra $S^\pm_{w_\pm}$ by $(S^\pm_{w_\pm})_{\nu,\mu}$, $\nu, \mu \in P$. It is clear that $(S^\pm_{w_\pm})_{\nu,\mu} = 0$, if $\nu \notin Q$ or $\mu \neq 0$. Thus, effectively we have a Q-grading on $S^\pm_{w_\pm}$. Eq. (3.22) below describes the nonzero components of this grading.

The Q-grading (2.19) of $\mathcal{U}_q(\mathfrak{g})$ induces a Q-grading of the algebras $\mathcal{U}^w_\pm$, explicitly given by

$$\deg X^\pm_{\beta_j} = \pm\beta_j, \quad j = 1, \ldots, l. \tag{2.42}$$

It is clear that

$$(\mathcal{U}^w_\pm)_\gamma \neq 0 \text{ if and only if } \pm\gamma \in \sum_{j=1}^{l} \mathbb{N}\beta_j. \tag{2.43}$$

The group H acts on $\mathcal{U}_q(\mathfrak{g})$ by conjugation. The subalgebras $\mathcal{U}^w_\pm$ are stable under this action. The eigenspaces for the action are precisely the graded components with respect to the grading (2.42).

For $\gamma \in Q^+$, $\gamma \neq 0$ denote $m_w(\gamma) = \dim(\mathcal{U}^w_+)_\gamma = \dim(\mathcal{U}^w_-)_{-\gamma}$. Fix a pair of dual bases $\{u_{\gamma,i}\}_{i=1}^{m_w(\gamma)}$ and $\{u_{-\gamma,i}\}_{i=1}^{m_w(\gamma)}$ of $(\mathcal{U}^w_+)_\gamma$ and $(\mathcal{U}^w_-)_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [23, Ch. 6]. The quantum R matrix corresponding to w is given by

$$\mathcal{R}^w = \sum_{\gamma \in Q_+} \sum_{i=1}^{m_w(\gamma)} u_{\gamma,i} \otimes u_{-\gamma,i} \in \mathcal{U}_+ \widehat{\otimes} \mathcal{U}_-.$$

Here $\mathcal{U}_+ \widehat{\otimes} \mathcal{U}_-$ denotes the completion of $\mathcal{U}_+ \otimes_{\mathbb{K}} \mathcal{U}_-$ with respect to the descending filtration [37, §4.1.1]. In particular $\mathcal{R}^w - 1$ belongs to the completed tensor product of the augmentation ideals of $\mathcal{U}_+$ and $\mathcal{U}_-$. For each two type 1 modules V_1 and V_2 one has

$$T_{w, V_1 \otimes_{\mathbb{K}} V_2} = (\mathcal{R}^w)^{-1} \left(T_{w,V_1} \otimes T_{w,V_2} \right). \tag{2.44}$$

For $w \in W$, define the maps

$$\varphi^\pm_w \colon S^\pm_w \to \mathcal{U}^w_\mp$$

by

$$\varphi^+_w((c^+_{w,\lambda})^{-1} c^\lambda_{\xi,\lambda}) = (c^\lambda_{\xi, T_w v_\lambda} \otimes \mathrm{id}) \mathcal{R}^w \quad \text{and} \tag{2.45}$$

$$\varphi^-_w((c^-_{w,\lambda})^{-1} c^{-w_0\lambda}_{\xi',-\lambda}) = (\mathrm{id} \otimes c^{-w_0\lambda}_{\xi', (T_{w^{-1}})^{-1} v_{-\lambda}}) \mathcal{R}^w. \tag{2.46}$$

In the right hand sides the elements of $R_q[G]$ are viewed as functionals on $\mathcal{U}_q(\mathfrak{g})$. The choice of $(T_{w^{-1}})^{-1}$ rather than simply T_w matches the second normalization in (2.17).

Theorem 2.6. ([43, Theorem 3.7]) *The maps $\varphi^\pm_w \colon S^\pm_w \to \mathcal{U}^w_\mp$ are well defined graded algebra isomorphisms.*

The property that $\varphi^\pm_w$ are graded isomorphisms means that $\varphi^\pm_w((S^\pm_w)_{\gamma,0}) = (\mathcal{U}^w_\mp)_\gamma$ for all $\gamma \in Q$, recall (2.42). Theorem 3.7 in [43] was stated for a base field field $\mathbb{K}$ of characteristic 0, and q transcendental over $\mathbb{Q}$. However it is valid for an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity, $q \neq 0$, because the proof given in [43] only uses (2.4) and the nondegeneracy of the Rosso–Tanisaki form, see [23, Ch. 6], which are both valid under these general conditions. We also note that [43] proves the plus case, and the minus case is analogous.

3. A description of the centers of Joseph's localizations

3.1. **Statement of main result.** In this section we obtain an explicit description of the centers Z_{w} of Joseph's localizations R_{w}. This is done in Theorem 3.1. It is the building block of the paper. On the one hand, it leads to a more explicit description of the prime ideals of $R_q[G]$, which in particular allows to compute the stabilizers of those ideals under the actions (2.31) and (2.32) of $\mathbb{T}^r$ and to construct a torus equivariant Dixmier type map in the next section. This description of prime ideals eventually leads to a classification of the maximal spectrum of $R_q[G]$, which allows us to settle a question of Goodearl and Zhang

[18], by proving that all maximal ideals of $R_q[G]$ have finite codimension. On the other hand, Theorem 3.1 and the methods developed in its proof play a key role in two "separation of variables" theorems which we prove in Sections 5 and 7 for the De Concini–Kac–Procesi algebras and Joseph's localizations R_{w}. The first is a freeness result for $\mathcal{U}^w_\pm$ as a module over its subalgebra of homogeneous normal elements, and the second is a freeness result for R_{w} over its subalgebra generated by Joseph's set of normal elements $(E_{\mathrm{w}})^{\pm 1}$. The latter supplies the second key ingredient in the classification of $\mathrm{Max} R_q[G]$ in Section 8.

For a subset $I \subset \{1, \dots, r\}$ denote

$$P_I = \bigoplus_{i \in I} \mathbb{Z}\omega_i, \quad P_I^+ = \bigoplus_{i \in I} \mathbb{N}\omega_i, \quad Q_I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad Q_I^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee. \tag{3.1}$$

For $w \in W$ set

$$\mathcal{I}(w) = \{ i = 1, \dots, r \mid w(\omega_i) = \omega_i \} \text{ and } \mathcal{S}(w) = \{1, \dots, r\} \backslash \mathcal{I}(w). \tag{3.2}$$

For $\mathrm{w} = (w_+, w_-) \in W \times W$ set

$$\mathcal{I}(\mathrm{w}) = \mathcal{I}(w_+) \cap \mathcal{I}(w_-) \tag{3.3}$$

and

$$\mathcal{S}(\mathrm{w}) = \mathcal{S}(w_+) \cup \mathcal{S}(w_-) = \{1, \dots, r\} \backslash \mathcal{I}(\mathrm{w}). \tag{3.4}$$

The intersection

$$\widetilde{\mathcal{L}}(\mathrm{w}) = \ker(w_+ - w_-) \cap P \tag{3.5}$$

is a lattice of rank dim $\ker(w_+ - w_-)$. Its reduced version

$$\widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w}) = \ker(w_+ - w_-) \cap P_{\mathcal{S}(\mathrm{w})} \tag{3.6}$$

is a lattice of rank

$$k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|, \tag{3.7}$$

because $P_{\mathcal{I}(\mathrm{w})} \subset \ker(w_+ - w_-)$ and thus

$$\widetilde{\mathcal{L}}(\mathrm{w}) = P_{\mathcal{I}(\mathrm{w})} \oplus \widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w}). \tag{3.8}$$

Choose a basis $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}$ of $\widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w})$. For each $j = 1, \dots, k$ denote

$$a_j = c^+_{w_+, \lambda^{(j)}} (c^-_{w_-, \lambda^{(j)}})^{-1}, \tag{3.9}$$

recall (2.27).

Theorem 3.1. *Assume that $\mathbb{K}$ is an arbitrary base field, and $q \in \mathbb{K}^*$ is not a root of unity. Then for each $\mathrm{w} = (w_+, w_-) \in W \times W$ the center Z_{w} of the algebra R_{w} coincides with the Laurent polynomial algebra over $\mathbb{K}$ of dimension* $\dim \ker(w_+ - w_-)$ *with generators*

$$\{ c^+_{w_+, \omega_i} \mid i \in \mathcal{I}(\mathrm{w}) \} \sqcup \{a_1, \dots, a_k\}. \tag{3.10}$$

Here k and $a_1, \dots, a_k$ are given by (3.7) and (3.9).

Kogan and Zelevinsky [28] proved that similar equations are cutting the symplectic leaves of the standard Poisson structure on the corresponding connected, simply connected, complex, simple Lie group within a double Bruhat cell. Section 4 will establish a connection between the two results.

The cases of $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$ of Theorem 3.1 were obtained by Hodges–Levasseur [20] and Goodearl–Lenagan [15], respectively. Their methods are very different from ours and use in an essential way the low rank of the underlying Lie algebra.

3.2. **Associated root and weight spaces.** Next, we gather some simple facts for the sets $\mathcal{I}(w)$ and $\mathcal{S}(w)$, $w \in W$.

Lemma 3.2. *Fix $w \in W$.*

(i) Then $\mathcal{S}(w) = \{i = 1, \ldots, r \mid s_i \leq w\}$ with respect to the Bruhat order $\leq$ on W, i.e. for each reduced expression $w = s_{i_1} \ldots s_{i_l}$

$$\mathcal{S}(w) = \cup_{j=1}^{l} \{i_j\}.$$

(ii) We have

$$\sum_{\beta \in \Delta^+ \cap w(\Delta^-)} \mathbb{Z}\beta = Q_{\mathcal{S}(w)}, \quad \sum_{\beta \in \Delta^+ \cap w(\Delta^-)} \mathbb{Z}\beta^\vee = Q^\vee_{\mathcal{S}(w)} \tag{3.11}$$

and

$$(\Delta^+ \cap w(\Delta^-))^\perp \cap P = (Q_{\mathcal{S}(w)})^\perp \cap P = P_{\mathcal{I}(w)}. \tag{3.12}$$

Proof. For the reduced expression in (i) denote $S = \cup_{j=1}^{l} \{i_j\}$ and $I = \{1, \ldots, r\} \backslash S$. One has

$$\Delta^+ \cap w(\Delta^-) = \{\beta_j = s_{i_1} \ldots s_{i_{j-1}}(\alpha_{i_j}) \mid j = 1, \ldots, l\},$$

cf. §2.7. Since

$$\beta_j - \alpha_{i_j} \in \sum_{n=1}^{j-1} \mathbb{Z}\alpha_{i_n}, \quad \forall j = 1, \ldots, l,$$

we have

$$\sum_{\beta \in \Delta^+ \cap w(\Delta^-)} \mathbb{Z}\beta = \bigoplus_{i \in S} \mathbb{Z}\alpha_i = Q_S. \tag{3.13}$$

Analogously

$$\sum_{\beta \in \Delta^+ \cap w(\Delta^-)} \mathbb{Z}\beta^\vee = Q^\vee_S. \tag{3.14}$$

Obviously $I \subseteq \mathcal{I}(w)$. If $i \in \mathcal{I}(w)$, then for all $\beta \in \Delta^+ \cap w(\Delta^-)$,

$$0 \leq \langle \omega_i, \beta \rangle = \langle w^{-1}(\omega_i), w^{-1}(\beta) \rangle = \langle \omega_i, w^{-1}(\beta) \rangle \leq 0,$$

thus $\langle \omega_i, \beta \rangle = 0$. Taking (3.13) into account, we obtain that $i \in \mathcal{I}(w)$ implies $\omega_i \in (Q_S)^\perp \cap P = P_I$, i.e. $i \in I$. Therefore $I = \mathcal{I}(w)$ and $S = \mathcal{S}(w)$. Now the second part follows from (3.13) and (3.14). $\square$

3.3. **One side inclusion in Theorem 3.1.** Joseph proved [24] that

$$c^+_{w_+,\lambda}(c^-_{w_-,\lambda})^{-1} \in Z_{\mathrm{w}}, \quad \text{for all } \lambda \in \mathcal{L}(\mathrm{w}). \tag{3.15}$$

This follows from (2.29). In particular, in the setting of §3.1,

$$a_j \in Z_{\mathrm{w}}, \quad \forall j = 1, \ldots, k.$$

The following proposition provides the rest needed to claim that Z_{w} contains all elements in (3.10).

Proposition 3.3. *For all* $\mathrm{w} = (w_+, w_-) \in W \times W$ *and* $i \in \mathcal{I}(\mathrm{w})$*,*

$$c^+_{w_+,\omega_i} \in Z_{\mathrm{w}}.$$

Proof. Fix $i \in \mathcal{I}(\mathrm{w})$. Since $R_q[G] = R^+R^-$, it is sufficient to prove that $c^+_{w_+,\omega_i}$ commutes with the images of R^+ and R^- in R_{w}. We will prove the former. The latter is analogous and is left to the reader. Let $\lambda \in P^+$. Recall that $(V_{w_+}(\lambda))_\nu \neq 0$ implies that $\nu = w_+(\lambda) + \gamma$ for some

$$\gamma \in \sum_{\beta \in \Delta^+ \cap w_+(\Delta^-)} \mathbb{N}\beta \subset Q_{\mathcal{S}(w_+)}, \tag{3.16}$$

cf. Lemma 3.2. The definition of $I^+_{w_+}$ implies that, if the image of $c^\lambda_{\xi,\lambda}$ in R_{w} is nonzero for some $\xi \in V(\lambda)^*_{-\nu}$, then $\nu = w_+(\lambda) + \gamma$ with γ as in (3.16), in particular $\gamma \in Q_{\mathcal{S}(w_+)}$. Using (2.29), we obtain that

$$c^+_{w_+,\omega_i} c^\lambda_{\xi,\lambda} = q^{\langle w\omega_i, w\lambda+\gamma\rangle - \langle \omega_i, \lambda\rangle} c^\lambda_{\xi,\lambda} c^+_{w_+,\omega_i} = q^{\langle \omega_i, \gamma\rangle} c^\lambda_{\xi,\lambda} c^+_{w,\omega_i} = c^\lambda_{\xi,\lambda} c^+_{w,\omega_i}$$

in R_{w}, since $i \in \mathcal{I}(\mathrm{w}) \subseteq \mathcal{I}(w_+)$ implies $q^{\langle \omega_i, \gamma\rangle} = 1$, $\forall \gamma \in Q_{\mathcal{S}(w_+)}$, see Lemma 3.2 (ii). This completes the proof of Proposition 3.3. $\square$

3.4. **Joseph's description of R_{w}.** Our treatment of R_{w} and its center uses a model of Joseph of R_{w}, which represents it as a kind of "bicrossed product" of the algebras $S^\pm_{w_\pm}$, modulo a simple additional localization and a smash product by a quantum torus. We refer the reader to [25, §9.1-9.2 and §10.3] for details. This model and Theorem 2.6 allow the simultaneous application of techniques from quantum function algebras (e.g. $R_q[G]$, its quotients and localizations) and quantized universal enveloping algebras of nilpotent Lie algebras (the algebras $\mathcal{U}^w_\pm$).

First, define $R^+ \circledast R^-$ as the quotient of $R^+ * R^-$ by the following relations (which are analogous to the ones in Lemma 2.2):

$$\begin{aligned} c^{-w_0\lambda_1}_{\xi_1,-\lambda_1} c^{\lambda_2}_{\xi_2,\lambda_2} &= q^{-\langle\lambda_1,\lambda_2\rangle - \langle\nu_1,\nu_2\rangle} c^{\lambda_2}_{\xi_2,\lambda_2} c^{-w_0\lambda_1}_{\xi_1,-\lambda_1} + \\ & \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{i=1}^{m(\gamma)} q^{-\langle\lambda_1,\lambda_2\rangle - \langle\nu_1+\gamma,\nu_2-\gamma\rangle} c^{\lambda_2}_{S^{-1}(u_{\gamma,i})\xi_2,\lambda_2} c^{-w_0\lambda_1}_{S^{-1}(u_{-\gamma,i})\xi_1,-\lambda_1}, \end{aligned} \tag{3.17}$$

for all $\lambda_i \in P^+$, $\nu_i \in P$, $i = 1, 2$, $\xi_1 \in V(-w_0\lambda_1)^*_{-\nu_1}$, $\xi_2 \in V(\lambda_2)^*_{-\nu_2}$. Joseph proved [25, Lemma 9.1.8] that the multiplication map induces the vector space isomorphism

$$R_+ \otimes_{\mathbb{K}} R_- \xrightarrow{\cong} R^+ \circledast R^-. \tag{3.18}$$

He also proved that the multiplication map $R^+ \otimes_{\mathbb{K}} R^- \to R_q[G]$ induces a surjective $\mathbb{K}$-algebra homomorphism $R^+ \circledast R^- \to R_q[G]$ and described its kernel in [25, Corollary 9.2.4].

For the remainder of this section we fix $\mathrm{w} = (w_+, w_-) \in W \times W$. By [25, Corollary 10.1.10]

$$\widehat{I}_{\mathrm{w}} = I^+_{w_+} R^- + R^+ I^-_{w_-}$$

is a completely prime ideal of $R^+ \circledast R^-$. The embeddings $R^\pm \hookrightarrow R^+ \circledast R^-$ induce [25, §10.3.1] embeddings $R^\pm / I^\pm_{w_\pm} \hookrightarrow (R^+ \circledast R^-)/I_{\mathrm{w}}$. The images of $c^\pm_{w_\pm, \lambda}$ are nonzero normal elements in $R^\pm / I^\pm_{w_\pm}$ and $(R^+ \circledast R^-)/I_{\mathrm{w}}$. These images will be denoted by the same symbols. Recall the definition (2.22) of the multiplicative subsets $E^\pm_{w_\pm}$ of $R^\pm / I^\pm_{w_\pm}$. Define the multiplicative subset

$$\widehat{E}_{\mathrm{w}} = E^+_{w_+} E^-_{w_-}$$

of $(R^+ \circledast R^-)/I_{\mathrm{w}}$ and denote the localization

$$\widehat{R}_{\mathrm{w}} = ((R^+ \circledast R^-)/I_{\mathrm{w}})[\widehat{E}_{\mathrm{w}}^{-1}].$$

Recall the definition of the subalgebras $S^\pm_{w_\pm}$ of $R^\pm_{w_\pm}$ from §2.8. The embeddings $R^\pm / I^\pm_{w_\pm} \hookrightarrow (R^+ \circledast R^-)/I_{\mathrm{w}}$ induce embeddings $R^\pm_{w_\pm} \hookrightarrow \widehat{R}_{\mathrm{w}}$. We denote the images of $S^\pm_{w_\pm}$ in $\widehat{R}_{\mathrm{w}}$ by the same symbols. Following Joseph [25, §10.3.2], define

$$S_{\mathrm{w}} = S^+_{w_+} S^-_{w_-}.$$

By (3.17), $S^+_{w_+} S^-_{w_-} = S^-_{w_-} S^+_{w_+}$. More precisely, (2.29) and (3.17) imply the following commutation relation between the elements of $S^+_{w_+}$ and $S^-_{w_-}$. In terms of the identifications (2.41) and the projections $g^\pm_{w_\pm}$ from (2.12), we have

$$\begin{aligned} &\big[(c^-_{w_-, \lambda_1})^{-1} g^-_{w_-}(\xi_1)\big] \big[(c^-_{w_+, \lambda_2})^{-1} g^+_{w_+}(\xi_2)\big] \\ &= q^{-\langle \nu_1 + w_- \lambda_1, \nu_2 - w_+ \lambda_2 \rangle} \big[(c^+_{w_+, \lambda_2})^{-1} g^+_{w_+}(\xi_2)\big] \big[(c^-_{w_-, \lambda_1})^{-1} g^-_{w_-}(\xi_1)\big] \\ &+ \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{i=1}^{m(\gamma)} q^{-\langle \nu_1 + \gamma + w_- \lambda_1, \nu_2 - \gamma - w_+ \lambda_2 \rangle} \big[(c^-_{w_+, \lambda_2})^{-1} g^+_{w_+}(S^{-1}(u_{\gamma, i}) \xi_2)\big] . \\ &\qquad \big[(c^-_{w_-, \lambda_1})^{-1} g^-_{w_-}(S^{-1}(u_{-\gamma, i}) \xi_1)\big], \end{aligned} \tag{3.19}$$

for all $\lambda_i \in P^+$, $\nu_i \in P$, $i = 1, 2$, $\xi_1 \in V(-w_0 \lambda_1)^*_{-\nu_1}$, $\xi_2 \in V(\lambda_2)^*_{-\nu_2}$. It follows from (3.18) that the multiplication in S_{w} induces the vector space isomorphism

$$S^+_{w_+} \otimes_{\mathbb{K}} S^-_{w_-} \xrightarrow{\cong} S_{\mathrm{w}}. \tag{3.20}$$

The algebra $R^+ \circledast R^-$ inherits a canonical $P \times P$-grading from the $P \times P$-gradings (2.8) on $R^\pm$. This induces a $P \times P$-grading on R_{w} and S_{w}. For $\gamma \in P$, there exists $\lambda \in P^+$ such that $(V^\pm_{w_\pm}(\lambda))_{\pm w_\pm(\lambda) + \gamma} \neq 0$, if and only if

$$\pm \gamma \in \sum_{\beta \in \Delta^+ \cap w_\pm(\Delta^-)} \mathbb{N} \beta.$$

For $\xi^\pm \in (V^\pm_{w_\pm}(\lambda))^*_{\mp w_\pm(\lambda) - \gamma}$,

$$(c^\pm_{w_\pm, \lambda})^{-1} \xi^\pm \in (S^\pm_{w_\pm})_{-\gamma, 0} \tag{3.21}$$

in terms of the identifications (2.41). Therefore:

$$(3.22)\quad (S^{\pm}_{w_\pm})_{\gamma,0} \neq 0, \ \ \forall \gamma \in \mp \sum_{\beta \in \Delta^+ \cap w_\pm(\Delta^-)} \mathbb{N}\beta \quad \text{and} \quad (S^{\pm}_{w_\pm})_{\nu,\mu} = 0, \ \ \text{otherwise}.$$

This also follows from (2.43) and the isomorphisms in Theorem 2.6. Eq. (3.22) implies that

$$(3.23)\quad (S_{\mathrm{w}})_{\gamma,0} \neq 0, \ \ \forall \gamma \in - \sum_{\beta \in \Delta^+ \cap w_+(\Delta^-)} \mathbb{N}\beta + \sum_{\beta \in \Delta^+ \cap w_-(\Delta^-)} \mathbb{N}\beta \quad \text{and} \quad (S_{\mathrm{w}})_{\nu,\mu} = 0, \ \ \text{otherwise}.$$

In [44, Theorem 3.6] we proved that the algebras $S^{\pm}_{w_\pm}$ play the role of quantum Schubert cells in relation to the H-spectra of quantum partial flag varieties. In a forthcoming publication we will prove that the algebras $R^+ \circledast R^-$ and S_{w} are closely related to the quantizations of the standard Poisson structure on the double flag variety [42] and its restrictions to double Schubert cells, and will study the spectra of related double versions of the De Concini–Kac–Procesi algebras.

Denote by $\widehat{L}^{\pm}_{w_\pm}$ and $\widehat{L}_{\mathrm{w}}$ the subalgebras of $\widehat{R}_{\mathrm{w}}$ generated by $(E^{\pm}_{w_\pm})^{\pm 1}$ and $(\widehat{E}_{\mathrm{w}})^{\pm 1}$, respectively. The algebras $\widehat{L}^{+}_{w_+}$ and $\widehat{L}^{-}_{w_-}$ are r dimensional Laurent polynomial algebras over $\mathbb{K}$ with generators $(c^{+}_{w_+,\omega_i})^{\pm 1}$, $i = 1, \ldots, r$ and $(c^{-}_{w_-,\omega_i})^{\pm 1}$, $i = 1, \ldots, r$. We have the algebra isomorphism [25, 10.3.2(2)],

$$(3.24)\qquad S_{\mathrm{w}} \# \widehat{L}_{\mathrm{w}} \xrightarrow{\cong} \widehat{R}_{\mathrm{w}},$$

where the smash product is computed using the actions

$$(3.25)\qquad c^{\pm}_{w_\pm,\lambda} \cdot u = q^{\langle w_\pm \lambda, \nu \rangle} x, \quad \text{for} \ \ u \in (S_{\mathrm{w}})_{-\nu,0}, \nu \in Q,$$

because of (2.29) and (3.17).

For $\lambda \in P^+$ choose an identification $V(\lambda)^* \cong V(-w_0\lambda)$ normalized so that $\xi^{+}_{1,\lambda} \mapsto v_{-\lambda}$ in terms of the lowest weight vectors fixed in §2.3 and the vectors $\xi^{+}_{w,\lambda}$ defined in §2.4. Let $\{\xi_i\}$ and $\{v_i\}$ be two sets of dual weight vectors of $V(\lambda)^*$ and $V(\lambda)$. Define

$$(3.26)\qquad x_\lambda = \sum_i c^{\lambda}_{\xi_i,\lambda} c^{-w_0\lambda}_{v_i,-\lambda},$$

where in the second term we used the identification $V(-w_0\lambda)^* \cong V(\lambda)^{**} \cong V(\lambda)$. Then $x_\lambda \in Z(R^+ \circledast R^-)$, see [25, Lemma 9.1.12]. The images of x_λ in $\widehat{R}_{\mathrm{w}}$ will be denoted by the same symbols. Denote by $\widehat{E}$ the multiplicative subset of $\widehat{R}_{\mathrm{w}}$ generated by x_{ω_i}, $i = 1, \ldots, r$ and by $\widehat{L}$ the $\mathbb{K}$-subalgebra of $\widehat{R}_{\mathrm{w}}[\widehat{E}^{-1}]$ generated by $x^{\pm 1}_{\omega_i}$, $i = 1, \ldots, r$. Denote

$$(3.27)\qquad y_{\omega_i} = (c^{+}_{w_+,\omega_i})^{-1} (c^{-}_{w_-,\omega_i})^{-1} x_{\omega_i} \in S_{\mathrm{w}}, \ \ i = 1, \ldots, r.$$

Continuing (3.24), we have [25, 10.3.2(4)],

$$\widehat{L} \otimes_{\mathbb{K}} (S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1, \ldots, r] \# \widehat{L}^{-}_{w_-}) \xrightarrow{\cong} \widehat{R}_{\mathrm{w}}[\widehat{E}^{-1}].$$

Joseph proved [25, §9.2.4] that the evaluation map $x_{\omega_i} \mapsto 1$, $i = 1, \ldots, r$ (i.e. $y_{\omega_i} \mapsto (c^{+}_{w,\omega_i})^{-1} (c^{+}_{w,\omega_i})^{-1}$) induces a surjective homomorphism $\widehat{R}_{\mathrm{w}}[\widehat{E}^{-1}] \to R_{\mathrm{w}}$, from which he obtained the algebra isomorphism [25, 10.3.2(5)],

$$\psi_{\mathrm{w}} \colon S_{\mathrm{w}}[y_{\omega_i}^{-1}, i = 1, \ldots, r] \# \widehat{L}^-_{w_-} \xrightarrow{\cong} R_{\mathrm{w}}. \tag{3.28}$$

3.5. **Homogeneous P-normal elements of the algebras $S^{\pm}_{w_\pm}$.** Our proof of Theorem 3.1 is based upon a study of a special kind of normal elements of the algebras $S^{\pm}_{w_\pm}$ and S_{w}. These normal elements commute with the elements of the algebras $S^{\pm}_{w_\pm}$ and S_{w} up to an automorphism coming from the action (2.32) of $\mathbb{T}^r$, restricted to a subgroup of $\mathbb{T}^r$ isomorphic to the weight lattice P.

Definition 3.4. We say that $z_\pm \in S^{\pm}_{w_\pm}$ is a P-normal element if there exists $\delta_\pm \in P$ such that

$$z_\pm s = q^{\langle \delta_\pm, \gamma \rangle} s z_\pm, \quad \forall s \in (S^{\pm}_{w_\pm})_{-\gamma, 0}, \gamma \in Q.$$

Analogously, we say that $z \in S_{\mathrm{w}}$ is a P-normal element if there exists $\delta \in P$ such that

$$zs = q^{\langle \delta, \gamma \rangle} sz, \quad \forall s \in (S_{\mathrm{w}})_{-\gamma, 0}, \gamma \in Q.$$

The motivation for the above definition is as follows. The abelian group P acts on S_{w} by

$$\mu \cdot s = q^{\langle \mu, \gamma \rangle} s, \ \text{ for } \ s \in (S_{\mathrm{w}})_{-\gamma, 0}, \mu \in P$$

and preserves its subalgebras $S^{\pm}_{w_\pm}$. (It is easy to see that the action (2.32) of $\mathbb{T}^r$ on $R_q[G]$ induces an action on S_{w}. The above action is a restriction of this action to a subgroup of $\mathbb{T}^r$ isomorphic to P.) An element $z \in S_{\mathrm{w}}$ is P-normal, if it is a normal element and it commutes with the elements of S_{w} via an algebra automorphism coming from the P-action:

$$zs = (\delta \cdot s) z, \quad \forall s \in S_{\mathrm{w}},$$

for some $\delta \in P$. The same applies to the subalgebras $S^{\pm}_{w_\pm}$.

Remark 3.5. Lemma 3.2 and (3.22) imply that the $\mathbb{Z}$-span of all roots $\gamma \in Q$ such that $(S^{\pm}_{w_\pm})_{-\gamma,0} \neq 0$ is $Q_{\mathcal{S}(w_\pm)}$. Thus in Definition 3.4 one can assume that $\delta_\pm \in P_{\mathcal{S}(w_\pm)}$. Analogously (3.23) and Lemma 3.2 imply that the $\mathbb{Z}$-span of all $\gamma \in Q$ such that $(S_{\mathrm{w}})_{-\gamma,0} \neq 0$ is $Q_{\mathcal{S}(\mathrm{w})}$. Therefore in Definition 3.4 one can assume that $\delta \in P_{\mathcal{S}(\mathrm{w})}$.

For $\lambda \in P^+$ denote

$$d^{\pm}_{w_\pm, \lambda} = (c^{\pm}_{w_\pm, \lambda})^{-1} c^{\pm}_{1, \lambda} \in (S^{\pm}_{w_\pm})_{\pm (w_\pm - 1)\lambda, 0}. \tag{3.29}$$

These elements are P-normal; applying Lemma 2.2, we obtain

$$d^{\pm}_{w_\pm, \lambda} s = q^{-\langle (w_\pm + 1)\lambda, \gamma \rangle} s d^{\pm}_{w_\pm, \lambda}, \quad \forall s \in (S^{\pm}_{w_\pm})_{-\gamma, 0}. \tag{3.30}$$

For all $\lambda_1, \lambda_2 \in P^+$,

$$d^{\pm}_{w_\pm, \lambda_1} d^{\pm}_{w_\pm, \lambda_2} = q^{\pm \langle \lambda_1, (w_\pm - 1)\lambda_2 \rangle} d^{\pm}_{w_\pm, \lambda_1 + \lambda_2}. \tag{3.31}$$

One verifies this using (2.29) and (2.18):

$$\begin{aligned} & d^{\pm}_{w_\pm, \lambda_1} d^{\pm}_{w_\pm, \lambda_2} = (c^{\pm}_{w_\pm, \lambda_1})^{-1} c^{\pm}_{1, \lambda_1} (c^{\pm}_{w_\pm, \lambda_2})^{-1} c^{\pm}_{1, \lambda_2} \\ =& q^{\pm \langle \lambda_1, (w_\pm - 1)\lambda_2 \rangle} (c^{\pm}_{w_\pm, \lambda_1})^{-1} (c^{\pm}_{w_\pm, \lambda_2})^{-1} c^{\pm}_{1, \lambda_1} c^{\pm}_{1, \lambda_2} \\ =& q^{\langle \pm \lambda_1, (w_\pm - 1)\lambda_2 \rangle} (c^{\pm}_{w_\pm, \lambda_1 + \lambda_2})^{-1} c^{\pm}_{1, \lambda_1 + \lambda_2} = q^{\langle \pm \lambda_1, (w_\pm - 1)\lambda_2 \rangle} d^{\pm}_{w_\pm, \lambda_1 + \lambda_2}. \end{aligned}$$

The following result relates the degrees of the homogeneous P-normal elements of the algebras $S^{\pm}_{w_\pm}$ and the $\delta_\pm$ weights in Definition 3.4.

Theorem 3.6. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. Let $z_\pm \in (S^{\pm}_{w_\pm})_{\nu_\pm,0}$ be a homogeneous P-normal element. Then there exists $\eta_\pm \in P_{\mathcal{S}(w_\pm)}$ such that $\nu_\pm = \pm(w_\pm - 1)\eta_\pm$ and*

$$z_\pm s = q^{-\langle (w_\pm+1)\eta_\pm, \gamma\rangle} s z_\pm, \quad \forall s \in (S^{\pm}_{w_\pm})_{-\gamma,0}, \gamma \in Q.$$

Caldero determined [8] the set of normal elements of $\mathcal{U}_+$ with very different methods, using the Joseph–Letzter results [27]. In the special case of $w_\pm = w_0$ (where w_0 is the longest element of W), Theorem 3.6 follows from [8]. In Section 6, building upon Theorem 3.6 and other results, we will prove that every homogeneous normal element of $S^{\pm}_{w_\pm}$ is P-normal and eventually show that all homogeneous normal elements of $S^{\pm}_{w_\pm}$ are scalar multiples of $d^{\pm}_{w_\pm,\lambda}$ for $\lambda \in P^+$. Those results are postponed to a later section, since they require various intermediate steps.

For the proof of Theorem 3.6 we will need the following lemma.

Lemma 3.7. *Assume that $z_\pm \in (S^{\pm}_{w_\pm})_{\nu_\pm,0}$ is a homogeneous P-normal element such that*

$$z_\pm s = q^{\langle \delta_\pm, \gamma\rangle} s z_\pm, \quad \forall s \in (S^{\pm}_{w_\pm})_{-\gamma,0}, \gamma \in Q, \tag{3.32}$$

for some $\delta_\pm \in P$. Then for all $i \in \mathcal{I}(\mathrm{w})$,

$$\langle \nu_\pm + \delta_\pm, \alpha_i^\vee\rangle \ \textit{and} \ \langle \nu_\pm - \delta_\pm, \alpha_i^\vee\rangle$$

are even integers.

Proof. Fix a reduced expression $w_\pm = s_{j_1} \ldots s_{j_l}$. Denote by $\beta_1, \ldots, \beta_l$ the roots (2.34) and by $X^{\pm}_{\beta_1}, \ldots, X^{\pm}_{\beta_l}$ the root vectors (2.35). Recall the graded isomorphisms $\varphi^{\pm}_w \colon S^{\pm}_w \to \mathcal{U}^w_{\mp}$ from Theorem 2.6. We have

$$\varphi^{\pm}_{w_\pm}(z_\pm) X^{\mp}_{\beta_j} = q^{\langle \delta_\pm, \beta_j\rangle} X^{\mp}_{\beta_j} \varphi^{\pm}_{w_\pm}(z_\pm), \quad \forall j = 1, \ldots l. \tag{3.33}$$

Recall the notation (2.39) and the notion of highest term of a nonzero element of $\mathcal{U}^w_{\pm}$, defined in §2.7. Let $p(X^{\mp})^{\mathbf{n}}$, $p \in \mathbb{K}^*$, $\mathbf{n} \in \mathbb{N}^{\times l}$ be the highest term of $\varphi^{\pm}_{w_\pm}(z_\pm)$. Since $\mathcal{U}^w_{\pm}$ are Q-graded algebras, we have

$$\nu_\pm = \mp \sum_{i=1}^{l} n_i \beta_i. \tag{3.34}$$

Applying (2.37) we obtain that for $j = 1, \ldots, l$

$$\varphi^{\pm}_{w_\pm}(z_\pm) X^{\mp}_{\beta_j} - p q^{\pm\langle \sum_{i=j+1}^{l} n_i\beta_i, \beta_j\rangle} (X^{\mp})^{(n_1, \ldots, n_j+1, \ldots n_l)}$$

and

$$X^{\mp}_{\beta_j} \varphi^{\pm}_{w_\pm}(z_\pm) - p q^{\pm\langle \sum_{i=1}^{j-1} n_i\beta_i, \beta_j\rangle} (X^{\mp})^{(n_1, \ldots, n_j+1, \ldots n_l)}$$

belong to $\mathrm{Span}\{(X^{\pm})^{\mathbf{n}'} \mid \mathbf{n}' < (n_1, \ldots, n_j + 1, \ldots n_l)\}$. Comparing this with (3.33) leads to

$$\langle \delta_\pm, \pm\beta_j\rangle = \mp \sum_{i=1}^{j-1} n_i \langle \beta_i, \beta_j\rangle \pm \sum_{i=j+1}^{l} n_i \langle \beta_i, \beta_j\rangle, \quad j = 1, \ldots, l.$$

Now (3.34) implies

$$\langle \delta_\pm \mp \nu_\pm, \beta_j \rangle = \langle \beta_j, \beta_j \rangle + 2 \sum_{i=j+1}^{l} n_i \langle \beta_i, \beta_j \rangle.$$

Hence

$$\begin{aligned}\langle \delta_\pm \mp \nu_\pm, \beta_j^\vee \rangle &= \langle \beta_j, \beta_j^\vee \rangle + 2 \sum_{i=j+1}^{l} n_i \langle \beta_i, \beta_j^\vee \rangle \\ &= 2 + 2 \sum_{i=j+1}^{l} n_i \langle \beta_i, \beta_j^\vee \rangle.\end{aligned}$$

is even for $j = 1, \dots, l$. Part (ii) of Lemma 3.2 implies that $\langle \delta_\pm \mp \nu_\pm, \alpha_i^\vee \rangle$ is even for all $i \in \mathcal{S}(w_\pm)$. Therefore

$$\langle \delta_\pm \pm \nu_\pm, \alpha_i^\vee \rangle = \langle \delta_\pm \mp \nu_\pm, \alpha_i^\vee \rangle \pm 2 \langle \nu_\pm, \alpha_i^\vee \rangle$$

is also even for all $i \in \mathcal{S}(w_\pm)$. □

Proof of Theorem 3.6. Assume that $\delta_\pm \in P_{\mathcal{S}(w_\pm)}$ is such that (3.32) holds, recall Remark 3.5. Using (3.32) and (3.30) we obtain

$$\begin{aligned} z_\pm d^\pm_{w_\pm, \lambda} &= q^{-\langle \delta_\pm, \pm(w_\pm - 1)\lambda \rangle} d^\pm_{w_\pm,\lambda} z_\pm \\ &= q^{-\langle \nu_\pm, (w_\pm + 1)\lambda \rangle} d^\pm_{w_\pm,\lambda} z_\pm, \quad \forall \lambda \in P^+. \end{aligned}$$

Since $S^\pm_{w_\pm}$ is a domain and $z_\pm, d^\pm_{w_\pm,\lambda} \neq 0$,

$$\langle \delta_\pm, \pm(w_\pm - 1)\lambda \rangle = \langle \nu_\pm, (w_\pm + 1)\lambda \rangle, \quad \forall \lambda \in P^+,$$

i.e.

$$\langle \nu_\pm \mp \delta_\pm, w_\pm(\lambda) \rangle = -\langle \nu_\pm \pm \delta_\pm, \lambda \rangle = -\langle w_\pm(\nu_\pm \pm \delta_\pm), w_\pm(\lambda) \rangle, \quad \forall \lambda \in P^+.$$

Therefore

$$w_\pm(\nu_\pm \pm \delta_\pm) + (\nu_\pm \mp \delta_\pm) = 0.$$

So,

$$(w_\pm + 1)\nu_\pm = \mp(w_\pm - 1)\delta_\pm. \tag{3.35}$$

Decompose

$$\mathfrak{h} = \mathfrak{h}^{(1)}_\pm \oplus \mathfrak{h}^{(-1)}_\pm \oplus \mathfrak{h}^{(c)}_\pm, \tag{3.36}$$

where $\mathfrak{h}^{(1)}_\pm$, $\mathfrak{h}^{(-1)}_\pm$ are the eigenspaces of $w_\pm$ with eigenvalues $1, -1$, and $\mathfrak{h}^{(c)}_\pm$ is the direct sum of the other eigenspaces of $w_\pm$. Denote by $\nu^{(1)}_\pm$, $\nu^{(-1)}_\pm$, $\nu^{(c)}_\pm$ and $\delta^{(1)}_\pm$, $\delta^{(-1)}_\pm$, $\delta^{(c)}_\pm$ the components of $\nu_\pm$ and $\delta_\pm$ in the decomposition (3.36). Then (3.35) implies that $\nu^{(1)}_\pm = 0$, $\delta^{(-1)}_\pm = 0$ and $(w_\pm + 1)\nu^{(c)}_\pm = \mp(w_\pm - 1)\delta^{(c)}_\pm$. Therefore

$$\widetilde{\eta}_\pm = -[\delta^{(1)}_\pm / 2 \pm \nu^{(-1)}_\pm / 2 + (w_\pm + 1)^{-1} \delta^{(c)}_\pm]$$

satisfies

$$\begin{aligned} \nu_\pm &= \pm(w_\pm - 1)\widetilde{\eta}_\pm, \\ \delta_\pm &= -(w_\pm + 1)\widetilde{\eta}_\pm. \end{aligned}$$

We have $\widetilde{\eta}_\pm = -(\delta_\pm \pm \nu_\pm)/2 \in (1/2)P$. Let

$$\widetilde{\eta}_\pm = \eta_\pm + \overline{\eta}_\pm, \quad \text{where } \eta_\pm \in (1/2)P_{\mathcal{S}(w_\pm)}, \overline{\eta}_\pm \in (1/2)P_{\mathcal{I}(w_\pm)}.$$

Lemma 3.7 implies that $\eta_\pm \in P_{\mathcal{S}(w_\pm)}$. Since $\overline{\eta}_\pm \in \ker(w_\pm - 1)$,

$$\nu_\pm = \pm(w_\pm - 1)\eta_\pm. \tag{3.37}$$

Moreover

$$\delta_\pm + (w_\pm + 1)\eta_\pm = -(w_\pm + 1)\overline{\eta}_\pm = -2\overline{\eta}_\pm$$

belongs to $P_{\mathcal{I}(w_\pm)}$ and is thus orthogonal to all $\gamma \in Q$ such that $(S^\pm_{w_\pm})_{\gamma,0} \neq 0$, because of (3.21) and Lemma 3.2. Hence (3.32) implies

$$z_\pm s = q^{\langle -w_\pm(\eta_\pm) - \eta_\pm, \gamma \rangle} s z_\pm, \quad \forall s \in (S^\pm_{w_\pm})_{-\gamma,0}, \gamma \in Q.$$

This equation and (3.37) establish the statement of Theorem 3.6. $\square$

3.6. **Homogeneous P-normal elements of the algebras S_{w}.** We proceed with establishing certain properties of the homogeneous P-normal elements of S_{w}, which are similar to the ones in Theorem 3.6 for the algebras $S^\pm_{w_\pm}$.

Theorem 3.8. *Assume that $\mathbb{K}$ is an arbitrary base field, and $q \in \mathbb{K}^*$ is not a root of unity. Let $z \in (S_{\mathrm{w}})_{\nu,0}$ be a homogeneous P-normal element. Then there exists $\eta \in P_{\mathcal{S}(\mathrm{w})}$ such that $\nu = (w_+ - w_-)\eta$ and*

$$zs = q^{\langle -(w_+ + w_-)\eta, \gamma \rangle} sz, \quad \forall s \in (S_{\mathrm{w}})_{-\gamma,0}, \gamma \in Q_{\mathcal{S}(\mathrm{w})}. \tag{3.38}$$

Proof. Let $\delta \in P$ be such that

$$zs = q^{\langle \delta, \gamma \rangle} sz, \quad \forall s \in (S_{w_\pm})_{-\gamma,0}, \gamma \in Q. \tag{3.39}$$

For $\tau \in Q^+$ denote

$$(S^+_{w_+})_{>-\tau,0} = \bigoplus_{\tau' \in Q^+, \tau' < \tau} (S^+_{w_+})_{-\tau',0} \tag{3.40}$$

and

$$(S^-_{w_-})_{<\tau,0} = \bigoplus_{\tau' \in Q^+, \tau' < \tau} (S^-_{w_-})_{\tau',0} \tag{3.41}$$

in terms of the partial order (2.2). Eq. (3.19) implies that for all $\tau \in Q^+$, $S^+_{w_+}(S^-_{w_-})_{<\tau,0} = (S^-_{w_-})_{<\tau,0} S^+_{w_+}$ and $(S^+_{w_+})_{>-\tau,0} S^-_{w_-} = S^-_{w_-}(S^+_{w_+})_{>-\tau,0}$. We have

$$(S_{\mathrm{w}})_{\nu,0} = \bigoplus_{\tau \in Q^+} (S^+_{w_+})_{-\tau,0}(S^-_{w_-})_{\nu+\tau,0}. \tag{3.42}$$

Denote by $\tau_1, \ldots, \tau_m$ the set of maximal elements of the set consisting of those $\tau \in Q^+$ for which z has a nontrivial component in $(S^+_{w_+})_{-\tau,0}(S^-_{w_-})_{\nu+\tau,0}$, recall (3.20). Denote the component of z in $(S^+_{w_+})_{-\tau_i,0}(S^-_{w_-})_{\nu+\tau_i,0}$ by

$$z_i = \sum_{j=1}^{h(i)} z^+_{ij} z^-_{ij},$$

where $z^+_{ij} \in (S^+_{w_+})_{-\tau_i,0}$, $z^-_{ij} \in (S^-_{w_-})_{\nu+\tau_i,0}$, for all $1 \le i \le m$, $1 \le j \le h(i)$ and for each $1 \le i \le m$

$$z^+_{i1}, \ldots, z^+_{ih(i)} \quad \text{are linearly independent.} \tag{3.43}$$

Fix $s_- \in (S^-_{w_-})_{-\gamma_-,0}$, for some

$$\gamma_- \in \sum_{\beta \in \Delta^+ \cap w_-(\Delta^-)} \mathbb{N}\beta, \tag{3.44}$$

recall (3.22). From (3.19) we obtain

$$s_- z = \sum_{ij} s_- z^+_{ij} z^-_{ij} = \sum_{ij} q^{-\langle \tau_i, \gamma_- \rangle} z^+_{ij} s_- z^-_{ij} \mod \Big(\sum_i (S^+_{w_+})_{>-\tau_i,0} S^-_{w_-} \Big),$$

while (3.39) implies

$$s_- z = q^{-\langle \delta, \gamma_- \rangle} z s_- = q^{-\langle \delta, \gamma_- \rangle} \sum_{ij} z^+_{ij} z^-_{ij} s_- \mod \Big(\sum_i (S^+_{w_+})_{>-\tau_i,0} S^-_{w_-} \Big).$$

Applying (3.42), (3.43) and the fact that the multiplication map $S^+_{w_+} \otimes_{\mathbb{K}} S^-_{w_-} \to S_{\mathrm{w}}$ is a vector space isomorphism, leads to

$$z^-_{ij} s_- = q^{\langle \delta - \tau_i, \gamma_- \rangle} s_- z^-_{ij}, \quad \forall i, j. \tag{3.45}$$

Therefore all z^-_{ij} are homogeneous P-normal elements of $S^-_{w_-}$. Theorem 3.6 implies that there exists $\eta_- \in P_{\mathcal{S}(w_-)}$ such that $z^-_{11} \in (S^-_{w_-})_{-(w_- - 1)\eta_-}$, i.e.

$$\nu + \tau_1 = -(w_- - 1)\eta_- \tag{3.46}$$

and

$$q^{\langle \delta - \tau_i, \gamma_- \rangle} = q^{-\langle (w_- + 1)\eta_-, \gamma_- \rangle},$$

for all γ_- as in (3.44), recall (3.22). Taking into account Lemma 3.2, we obtain

$$\delta - \tau_1 + (w_- + 1)\eta_- \in P_{\mathcal{I}(w_-)}. \tag{3.47}$$

Interchanging the roles of $S^+_{w_+}$ and $S^-_{w_-}$, we represent z_i as in (3.45) with $z^\pm_{ij}$ such that

$$z^-_{i1}, \dots, z^-_{ih(i)} \quad \text{are linearly independent},$$

instead of (3.43). For all $s_+ \in (S^+_{w_+})_{-\gamma_+,0}$ we obtain

$$z s_+ = \sum_{ij} z^+_{ij} z^-_{ij} s_+ = \sum_{ij} q^{\langle \nu + \tau_i, \gamma_+ \rangle} z^+_{ij} s_+ z^-_{ij} \mod \Big(\sum_i S^+_{w_+} (S^-_{w_-})_{<\nu + \tau_i, 0} \Big)$$

and

$$z s_+ = q^{\langle \delta, \gamma_+ \rangle} s_+ z = q^{\langle \delta, \gamma_+ \rangle} \sum_{ij} s_+ z^+_{ij} z^-_{ij} \mod \Big(\sum_i S^+_{w_+} (S^-_{w_-})_{<\nu + \tau_i, 0} \Big),$$

from (3.19) and (3.39), respectively. Therefore all z^-_{ij} are homogeneous P-normal elements of $S^-_{w_-}$ and

$$z^+_{ij} s_+ = q^{\langle \delta - \nu - \tau_i, \gamma_+ \rangle} s_+ z^+_{ij}.$$

Applying Theorem 3.6, we obtain that there exists $\eta_+ \in P_{\mathcal{S}(w_+)}$ such that $z^+_{11} \in (S^+_{w_+})_{(w_+ - 1)\eta_+}$, i.e.

$$-\tau_1 = (w_+ - 1)\eta_+ \tag{3.48}$$

and

$$q^{\langle \delta - \nu - \tau_1, \gamma_+ \rangle} = q^{-\langle (w_+ + 1)\eta_+, \gamma_+ \rangle}$$

for all $\gamma_+ \in Q_{\mathcal{S}(w_+)}$, recall Lemma 3.2 and (3.22). The latter is equivalent to

$$\delta - \nu - \tau_1 + (w_+ + 1)\eta_+ \in P_{\mathcal{I}(w_+)}. \tag{3.49}$$

Adding (3.46) and (3.48) gives

$$\nu = (w_+ - 1)\eta_+ - (w_- - 1)\eta_-. \tag{3.50}$$

Combining (3.47) and (3.48) leads to

$$\delta + (w_+ - 1)\eta_+ + (w_- + 1)\eta_- \in P_{\mathcal{I}(w_-)}. \tag{3.51}$$

Similarly (3.49) and (3.46) imply

$$\delta + (w_+ + 1)\eta_+ + (w_- - 1)\eta_- \in P_{\mathcal{I}(w_+)}. \tag{3.52}$$

Decompose

$$\eta_\pm = \widehat{\eta}_\pm + \widetilde{\eta}_\pm,$$

so that

$$\widehat{\eta}_+ \in P_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}, \;\; \widetilde{\eta}_+ \in P_{\mathcal{S}(w_+)\backslash(\mathcal{S}(w_+)\cap\mathcal{S}(w_-))} = P_{\mathcal{S}(w_+)\cap\mathcal{I}(w_-)}, \tag{3.53}$$

$$\widehat{\eta}_- \in P_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}, \;\; \widetilde{\eta}_- \in P_{\mathcal{S}(w_-)\backslash(\mathcal{S}(w_+)\cap\mathcal{S}(w_-))} = P_{\mathcal{S}(w_-)\cap\mathcal{I}(w_+)}. \tag{3.54}$$

In particular,

$$(w_\pm - 1)\widetilde{\eta}_\mp = 0. \tag{3.55}$$

Subtracting the left hand sides of (3.51) and (3.52), shows that

$$2(\eta_+ - \eta_-) \perp Q_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}.$$

Therefore $\widehat{\eta}_+ = \widehat{\eta}_-$. Denote

$$\eta = \widehat{\eta}_+ + \widetilde{\eta}_+ + \widetilde{\eta}_- = \eta_+ + \widetilde{\eta}_- = \eta_- + \widetilde{\eta}_+.$$

From (3.50) we have

$$\begin{aligned} \nu &= (w_+ - 1)(\eta - \widetilde{\eta}_-) - (w_- - 1)(\eta - \widetilde{\eta}_+) \\ &= (w_+ - 1)\eta - (w_- - 1)\eta = (w_+ - w_-)\eta, \end{aligned} \tag{3.56}$$

because of (3.55). Eqs. (3.51) and (3.55) imply

$$\delta + (w_+ - 1)(\eta - \widetilde{\eta}_-) + (w_- + 1)(\eta - \widetilde{\eta}_+) = \delta + (w_+ + w_-)\eta - 2\widetilde{\eta}_+ \in P_{\mathcal{I}(w_-)},$$

so

$$\delta + (w_+ + w_-)\eta \in P_{\mathcal{I}(w_-)}.$$

Analogously (3.52) and (3.55) imply

$$\delta + (w_+ + w_-)\eta \in P_{\mathcal{I}(w_+)},$$

i.e.

$$\delta + (w_+ + w_-)\eta \in P_{\mathcal{I}(w_+)} \cap P_{\mathcal{I}(w_-)} = P_{\mathcal{I}(\mathrm{w})}.$$

From (3.39) we obtain that η satisfies (3.38). Since it also satisfies (3.56), it provides the needed weight for the theorem. $\square$

3.7. **Proof of Theorem 3.1.** Denote

$$(R_{\rm w})_\mu = \bigoplus_{\nu \in P} (R_{\rm w})_{\nu,\mu}. \tag{3.57}$$

Recall that $Z_{\rm w} = Z(R_{\rm w})$ and denote

$$(Z_{\rm w})_{\nu,\mu} = Z_{\rm w} \cap (R_{\rm w})_{\nu,\mu}, \quad (Z_{\rm w})_\mu = Z_{\rm w} \cap (R_{\rm w})_\mu, \quad \forall \nu, \mu \in P.$$

Obviously

$$Z_{\rm w} = \bigoplus_{\nu,\mu \in P} (Z_{\rm w})_{\nu,\mu}.$$

We will need the following theorem of Joseph and Hodges–Levasseur–Toro.

Theorem 3.9. *(Joseph* [24, Theorem 8.11]*, Hodges-Levasseur–Toro* [22, Theorem 4.14 (3)]*) For all $\mu \in P$,*

$$\dim (Z_{\rm w})_\mu = 0 \ \ or \ \ 1.$$

Similarly to Theorem 2.3 the proof of this result in [24, 22] only uses the assumption that $q \in \mathbb{K}^*$ is not a root of unity, without restrictions on $\mathbb{K}$.

Denote by $A_{\rm w}$ the subalgebra of $R_{\rm w}$ generated by

$$\{(c^+_{w,\omega_i})^{\pm 1} \mid i \in \mathcal{I}({\rm w})\} \cup \{a_1^{\pm 1}, \ldots, (a_k)^{\pm 1}\},$$

recall §3.1. Since each of the generators of $A_{\rm w}$ is $P \times P$ homogeneous,

$$A_{\rm w} = \bigoplus_{\nu,\mu \in P} (A_{\rm w})_{\nu,\mu}, \quad \text{where} \ \ (A_{\rm w})_{\nu,\mu} = A_{\rm w} \cap (R_{\rm w})_{\nu,\mu}.$$

Define

$$(A_{\rm w})_\mu = \bigoplus_{\nu \in P} (A_{\rm w})_{\nu,\mu}.$$

Because $\{\lambda^{(1)}, \ldots, \lambda^{(k)}\} \cup \{\omega_i \mid i \in \mathcal{I}({\rm w})\}$ is a linearly independent set (recall §3.1), the monomials

$$\prod_{i \in \mathcal{I}({\rm w})} (c^+_{w_+,\omega_i})^{n_i} \prod_{j=1}^k a_j^{m_j} \in (A_{\rm w})_\nu, \quad \nu = \sum_{i \in \mathcal{I}({\rm w})} n_i \omega_i + 2 \sum_{j=1}^k m_j \lambda^{(j)}$$

are linearly independent for different $\{n_i \mid i \in \mathcal{I}({\rm w})\} \in \mathbb{Z}^{\times |\mathcal{I}({\rm w})|}$, $(m_1, \ldots, m_k) \in \mathbb{Z}^{\times k}$. Therefore

$$A_{\rm w} \cong \mathbb{K}[(c^+_{w,\omega_i})^{\pm 1}, a_j^{\pm 1}, i \in \mathcal{I}({\rm w}), j = 1, \ldots k]. \tag{3.58}$$

Recall (3.3), (3.4), (3.5) and (3.6), and denote

$$\mathcal{L}({\rm w}) = 2\widetilde{\mathcal{L}}_{\rm red}({\rm w}) \bigoplus \left(\bigoplus_{i \in \mathcal{I}({\rm w})} \mathbb{Z}\omega_i \right) = 2\widetilde{\mathcal{L}}({\rm w}) + P_{\mathcal{I}({\rm w})}. \tag{3.59}$$

Since $P_{\mathcal{I}({\rm w})} \subseteq \mathcal{L}({\rm w})$,

$$2\widetilde{\mathcal{L}}({\rm w}) \subset \mathcal{L}({\rm w}) \subset \widetilde{\mathcal{L}}({\rm w})$$

and

$$\mathcal{L}({\rm w})/2\widetilde{\mathcal{L}}({\rm w}) \cong \mathbb{Z}_2^{\times |\mathcal{I}({\rm w})|}.$$

We have

$$a_j \in (A_{\rm w})_{-(w_+ - w_-)\lambda^{(j)}, 2\lambda^{(j)}}, \quad j = 1, \ldots, k,$$

cf. (3.9) and

$$c^{+}_{w,\omega_j} \in (A_{\mathrm{w}})_{\omega_j,\omega_j}, \quad \forall j \in \mathcal{I}(\mathrm{w}),$$

which leads to:

Lemma 3.10. *For all $\mu \in \mathcal{L}(\mathrm{w})$, $\dim(A_{\mathrm{w}})_\mu = 1$ and for all $\mu \notin \mathcal{L}(\mathrm{w})$, $(A_{\mathrm{w}})_\mu = 0$.*

Remark 3.11. Joseph proved [24] that the set of all $\mu \in P$ such that $(Z_{\mathrm{w}})_\mu \neq 0$ contains $2\widetilde{\mathcal{L}}(\mathrm{w})$ and is contained in $\widetilde{\mathcal{L}}(\mathrm{w})$. Theorem 3.1 determines explicitly this set; it is equal to $\mathcal{L}(\mathrm{w})$.

Proof of Theorem 3.1. By (3.15) and Proposition 3.3, A_{w} is a subalgebra of Z_{w}. We need to prove that $Z_{\mathrm{w}} = A_{\mathrm{w}}$. Let $\nu', \mu \in P$. We will prove that

$$(Z_{\mathrm{w}})_{\nu',\mu} \neq 0 \tag{3.60}$$

forces

$$\mu \in \mathcal{L}(\mathrm{w}). \tag{3.61}$$

Then we can apply Theorem 3.9 and Lemma 3.10 to deduce that $(Z_{\mathrm{w}})_\mu = (A_{\mathrm{w}})_\mu$, $\forall \mu \in P$. Therefore $Z_{\mathrm{w}} = A_{\mathrm{w}}$.

We are left with showing that (3.60) implies (3.61). Fix $\nu', \mu \in P$ and

$$d \in (Z_{\mathrm{w}})_{\nu',\mu}, \ \ d \neq 0.$$

The isomorphism (3.28) and eq. (3.23) imply that

$$d = \psi_{\mathrm{w}}(u \# (c^{-}_{w_-,\mu})^{-1}), \quad \text{for some } \ u \in (S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1, \ldots, r])_{\nu' + w_-(\mu), 0}.$$

For $\lambda = \sum_{i=1}^r n_i \omega_i \in P^+$ write

$$y_\lambda = (y_{\omega_1})^{n_1} \ldots (y_{\omega_r})^{n_r}. \tag{3.62}$$

From (3.26) and (3.27) we have

$$y_\lambda \in (S_{\mathrm{w}})_{(w_+ - w_-)\lambda, 0}. \tag{3.63}$$

Let $u = z y_\lambda^{-1}$ for some $\lambda \in P^+$ and

$$z \in (S_{\mathrm{w}})_{\nu,0}, \ \ z \neq 0, \tag{3.64}$$

where $\nu = \nu' - (w_+ - w_-)(\lambda) + w_-(\mu)$. Thus

$$\psi_{\mathrm{w}}\big((z y_\lambda^{-1}) \# (c^{-}_{w_-,\mu})^{-1}\big) \in Z(R_{\mathrm{w}})_{\nu + (w_+ - w_-)(\lambda) - w_-(\mu), \mu} \ \text{ with } \ z \neq 0. \tag{3.65}$$

In particular, it commutes with $c^{\pm}_{w_\pm,\mu'}$, for all $\mu' \in P$. Using (2.29), we obtain

$$\langle w_\pm \mu', \nu + (w_+ - w_-)\lambda - w_-(\mu) \rangle + \langle w_\pm \mu', w_\pm \mu \rangle = 0, \quad \forall \mu' \in P.$$

Therefore

$$\nu + (w_+ - w_-)\lambda - w_-(\mu) = -w_+(\mu) = -w_-(\mu),$$

i.e.

$$\nu = -(w_+ - w_-)\lambda \ \text{ and } \ \mu \in \ker(w_+ - w_-). \tag{3.66}$$

Since $x_{\omega_i} \in Z(R^+ \circledast R^-)$, (3.62) and (2.29) imply

$$y_\lambda s' = q^{\langle -(w_+ + w_-)\lambda, \gamma' \rangle} s' y_\lambda, \quad \forall s' \in (S_{\mathrm{w}})_{-\gamma', 0}. \tag{3.67}$$

Because of (3.25) one has
(3.68)
$$(1\#c^-_{w_-,\mu})(s'\#1) = q^{\langle w_-(\mu),\gamma'\rangle}(s'\#1)(1\#c^-_{w_-,\mu}), \ \forall s' \in (S_{\mathrm{w}})_{-\gamma',0}, \gamma' \in Q_{\mathcal{S}(\mathrm{w})}.$$

From (3.65), (3.67) and (3.68) it follows that

$$zs' = q^{\langle -(w_+ + w_-)\lambda + w_-(\mu),\gamma'\rangle} s'z, \quad \forall s' \in (S_{\mathrm{w}})_{-\gamma',0}, \gamma' \in Q_{\mathcal{S}(\mathrm{w})}, \tag{3.69}$$

recall (3.23) and Lemma 3.2. In particular, $z \in (S_{\mathrm{w}})_{\nu,0}$ is a homogeneous P-normal element. Theorem 3.8 implies that there exists $\eta \in P_{\mathcal{S}(\mathrm{w})}$ such that

$$\nu = (w_+ - w_-)\eta \tag{3.70}$$

and

$$zs' = q^{\langle -(w_+ + w_-)\eta,\gamma'\rangle} s'z, \quad \forall s' \in (S_{\mathrm{w}})_{-\gamma',0}, \gamma' \in Q_{\mathcal{S}(\mathrm{w})}. \tag{3.71}$$

Comparing (3.66) and (3.70), gives that $\lambda - \eta \in \ker(w_+ - w_-)$. Therefore

$$\lambda - \eta \in \widetilde{\mathcal{L}}(\mathrm{w}). \tag{3.72}$$

Combining (3.69) and (3.71) implies that

$$w_-(\mu) - (w_+ + w_-)(\lambda - \eta) = w_-(\mu) - 2w_-(\lambda - \eta) \in P_{\mathcal{I}(\mathrm{w})}.$$

Thus

$$\mu - 2(\lambda - \eta) \in P_{\mathcal{I}(\mathrm{w})},$$

because each element of $P_{\mathcal{I}(\mathrm{w})}$ is fixed under w_-^{-1}. Finally this, together with (3.72), leads to

$$\mu \in \widetilde{\mathcal{L}}(\mathrm{w}) + P_{\mathcal{I}(\mathrm{w})} = \mathcal{L}(\mathrm{w}).$$

Therefore (3.60) implies (3.61), which completes the proof of Theorem 3.1 □

Theorem 3.1 makes Joseph's description of prime ideals of $R_q[G]$ more explicit. In parts (ii) and (iii) of Theorem 2.3 one can replace Z_{w} with the explicit Laurent polynomial ring A_{w} given by (3.58).

Corollary 3.12. *Assume that $\mathbb{K}$ is an arbitrary base field, and $q \in \mathbb{K}^*$ is not a root of unity. For $\mathrm{w} \in W \times W$ and $J^0 \in \mathrm{Spec} A_{\mathrm{w}}$ define*

$$\iota_{\mathrm{w}}(J^0) = \{r \in R_q[G] \mid (r + I_{\mathrm{w}}) \in R_{\mathrm{w}} J^0\}.$$

Then $\iota_{\mathrm{w}}(J^0) \in \mathrm{Spec}_{\mathrm{w}} R_q[G]$ and

$$\iota_{\mathrm{w}} \colon \mathrm{Spec} A_{\mathrm{w}} \to \mathrm{Spec}_{\mathrm{w}} R_q[G]$$

is a homeomorphism for all $\mathrm{w} \in W \times W$. Moreover ι_{w} restricts to a homeomorphism from $\mathrm{Max} A_{\mathrm{w}}$ to $\mathrm{Prim}_{\mathrm{w}} R_q[G]$.

The application of this result to the primitive spectrum of $R_q[G]$, described in Theorem 4.1, is the starting point for explicitly relating $\mathrm{Prim}\, R_q[G]$ to the symplectic foliation of the underlying Poisson Lie group, discussed in the next section.

4. Primitive ideals of $R_q[G]$ and a Dixmier map for $R_q[G]$

4.1. **A formula for the primitive ideals of $R_q[G]$.** When the base field $\mathbb{K}$ is algebraically closed, the results from the previous section lead to an explicit parametrization of $\operatorname{Prim} R_q[G]$ as well as a more explicit formula for the primitive ideals of $R_q[G]$ than the previously known ones, which is in turn used in Section 8 to classify $\operatorname{Max} R_q[G]$. Based on this formula, we explicitly determine the stabilizers of the primitive ideals of $R_q[G]$ under the $\mathbb{T}^r \times \mathbb{T}^r$-action obtained by combining the actions (2.31) and (2.32). This was not possible with the previously known formulas. In light of Theorem 2.3 (iii), we obtain the exact structure of $\operatorname{Prim}_{\mathrm{w}} R_q[G]$ is a $\mathbb{T}^r \times \mathbb{T}^r$-homogeneous space. For $\mathbb{K} = \mathbb{C}$, we combine this with the Kogan–Zelevinsky results [28] to construct a $\mathbb{T}^r \times \mathbb{T}^r$-equivariant map from the symplectic foliation of the corresponding Poisson Lie group to $\operatorname{Prim} R_q[G]$. In this paper we use the term Dixmier type map in the wide sense, referring to a map from the topological space of the symplectic foliation associated with the semiclassical limit of an algebra R to $\operatorname{Prim} R$, which is expected to be a homeomorphism.

Throughout the section the base field $\mathbb{K}$ will be assumed to be algebraically closed. Recall the setting of §3.1. For $\mathrm{w} = (w_+, w_-)$, we fix a basis $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of $\widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w})$, where $k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$, recall (3.6). Represent

$$\lambda^{(j)} = \lambda^{(j)}_+ - \lambda^{(j)}_-,$$

for some $\lambda^{(j)}_+$ and $\lambda^{(j)}_-$, which belong to P^+ and have disjoint support, cf. (2.1). For $\zeta_j \in \mathbb{K}$ define

$$b_j(\zeta_j) = c^+_{w_+, \lambda^{(j)}_+} c^-_{w_-, \lambda^{(j)}_-} - \zeta_j c^+_{w_+, \lambda^{(j)}_-} c^-_{w_-, \lambda^{(j)}_+}, \quad j = 1, \ldots, k. \tag{4.1}$$

Then

$$\begin{aligned} a_j - \zeta_j &= c^+_{w_+, \lambda^{(j)}} (c^-_{w_-, \lambda^{(j)}})^{-1} - \zeta_j \\ &= (c^+_{w_+, \lambda^{(j)}_-})^{-1} c^+_{w_+, \lambda^{(j)}_+} c^-_{w_-, \lambda^{(j)}_-} (c^-_{w_-, \lambda^{(j)}_+})^{-1} - \zeta_j \\ &= (c^+_{w_+, \lambda^{(j)}_-})^{-1} b_j(\zeta_j) (c^-_{w_-, \lambda^{(j)}_+})^{-1}, \end{aligned} \tag{4.2}$$

recall (3.9). Thus $b_j(\zeta_j) = c^+_{w_+, \lambda^{(j)}_-} (a_j - \zeta_j) c^-_{w_-, \lambda^{(j)}_+}$. Using (2.29) and the fact that $a_j \in R_{\mathrm{w}}$ are central elements, we obtain that $b_j(\zeta_j) \in R_q[G]/I_w$ are normal:
(4.3)
$$b_j(\zeta_j) c = q^{\langle w_+(\lambda^{(j)}_-) + w_-(\lambda^{(j)}_+), \mu \rangle - \langle \lambda^{(j)}_+ + \lambda^{(j)}_-, \nu \rangle} c\, b_j(\zeta_j), \ \ \forall c \in (R_q[G]/I_{\mathrm{w}})_{-\nu, \mu}, \nu, \mu \in P.$$

For $\zeta = (\zeta_1, \ldots, \zeta_k) \in (\mathbb{K}^*)^{\times k}$ and $\theta = \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} \in (\mathbb{K}^*)^{\times |\mathcal{I}(\mathrm{w})|}$ denote

$$J_{\mathrm{w}, \zeta, \theta} = \iota_{\mathrm{w}} \Big(\sum_{j=1}^k R_{\mathrm{w}} (a_j - \zeta_j) + \sum_{i \in \mathcal{I}(\mathrm{w})} R_{\mathrm{w}} (c^+_{w_+, \omega_i} - \theta_i) \Big). \tag{4.4}$$

Eq. (4.3) implies that

$$J_{\mathrm{w},\zeta,\theta} = \Big\{ r \in R_q[G] \;\Big|\; cr \in \sum_{j=1}^{k} R_q[G] b_j(\zeta_j) + \sum_{i \in \mathcal{I}(\mathrm{w})} R_q[G](c^{+}_{w_+,\omega_i} - \theta_i) + I_{\mathrm{w}} \text{ for some } c \in E_{\mathrm{w}} \Big\}, \tag{4.5}$$

recall (2.23). Theorem 2.3 (iii) and Theorem 3.1 lead to the following result, cf. Corollary 3.12.

Theorem 4.1. *Assume that $\mathbb{K}$ is an algebraically closed field and $q \in \mathbb{K}^*$ is not a root of unity. Then for all $\mathrm{w} = (w_+, w_-) \in W \times W$, the stratum of primitive ideals $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ consists of the ideals $J_{\mathrm{w},\zeta,\theta}$ given by (4.5), where $\zeta = (\zeta_1, \ldots, \zeta_k) \in (\mathbb{K}^*)^{\times k}$, $\theta = \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} \in (\mathbb{K}^*)^{\times |\mathcal{I}(\mathrm{w})|}$ and $k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$.*

This result plays a key role in our classification of the maximal ideals of $R_q[G]$ in Section 8.

The cases of $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$ of Theorem 4.1 were obtained by Hodges–Levasseur [20] and Goodearl–Lenagan [15], respectively, who also proved a stronger result without the E_{w} localization in (4.5). Their methods are very different from ours and use in an essential way the low rank of the underlying Lie algebra.

4.2. **Structure of $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ as a $\mathbb{T}^r \times \mathbb{T}^r$-homogeneous space.** The commuting $\mathbb{T}^r$-actions (2.31) and (2.32) on $R_q[G]$ can be combined to the following rational $\mathbb{T}^r \times \mathbb{T}^r$-action by $\mathbb{K}$-algebra automorphisms:

$$(t', t) \cdot c = (t')^{\nu} t^{\mu} c, \quad t', t \in \mathbb{T}^r, c \in R_q[G]_{-\nu,\mu}, \nu, \mu \in P. \tag{4.6}$$

We obtain induced $\mathbb{T}^r \times \mathbb{T}^r$ actions on R_{w}, $Z_{\mathrm{w}} = A_{\mathrm{w}}$, $\mathrm{Spec} A_{\mathrm{w}}$, $\mathrm{Spec}_{\mathrm{w}} R_q[G]$.

Denote by $\mathrm{Stab}_{\mathbb{T}^r}(.)$ and $\mathrm{Stab}_{\mathbb{T}^r \times \mathbb{T}^r}(.)$ the stabilizers with respect to the actions (2.31) and (4.6), respectively. The map $\iota_{\mathrm{w}} \colon \mathrm{Spec} A_{\mathrm{w}} \to \mathrm{Spec}_{\mathrm{w}} R_q[G]$ is $\mathbb{T}^r \times \mathbb{T}^r$-equivariant. In particular,

$$\mathrm{Stab}_{\mathbb{T}^r \times \mathbb{T}^r} \iota_{\mathrm{w}}(J^0) = \mathrm{Stab}_{\mathbb{T}^r \times \mathbb{T}^r}(J^0), \quad \forall J^0 \in \mathrm{Spec} A_{\mathrm{w}}.$$

The equivariance of ι_{w} and (4.4) imply that the $\mathbb{T}^r \times \mathbb{T}^r$-action on $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ is given by

$$(t', t) \cdot J_{\mathrm{w},\zeta,\theta} = J_{\mathrm{w},(\mathbf{t}',\mathbf{t})\cdot\zeta,(\mathbf{t}',\mathbf{t})\cdot\theta}, \tag{4.7}$$

where

$$(t', t) \cdot \{\zeta_j\}_{j=1}^{k} = \{(t')^{-(w_+ - w_-)\lambda^{(j)}} t^{-2\lambda^{(j)}} \zeta_j\}_{j=1}^{k} \text{ and} \tag{4.8}$$

$$(t', t) \cdot \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} = \{(t_i t'_i)^{-1} \theta_i\}_{i \in \mathcal{I}(\mathrm{w})}, \tag{4.9}$$

because

$$(t', t) \cdot a_j = (t')^{(w_+ - w_-)\lambda^{(j)}} t^{2\lambda^{(j)}}, j = 1, \ldots, k, \;\; (t', t) \cdot c^{+}_{w_+,\omega_i} = t'_i t_i c^{+}_{w_+,\omega_i}, i \in \mathcal{I}(\mathrm{w}).$$

This implies the following result describing the stabilizers of $J_{\mathrm{w},\zeta,\theta}$ under the action (4.6) of $\mathbb{T}^r \times \mathbb{T}^r$ and in particular under the action (2.31) of $\mathbb{T}^r$.

Proposition 4.2. *If $\mathbb{K}$ is algebraically closed and $q \in \mathbb{K}^*$ is not a root of unity, then for all* $\mathrm{w} = (w_+, w_-) \in W \times W$, $\zeta = (\zeta_1, \ldots, \zeta_k) \in (\mathbb{K}^*)^{\times k}$, $\theta = \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} \in (\mathbb{K}^*)^{\times |\mathcal{I}(\mathrm{w})|}$*:*

$$\begin{aligned} \text{(4.10)} \quad \mathrm{Stab}_{\mathbb{T}^r \times \mathbb{T}^r} J_{\mathrm{w},\zeta,\theta} = \{(t', t) \in \mathbb{T}^r \times \mathbb{T}^r \mid \\ t^{2\lambda} = (t')^{-(w_+ - w_-)\lambda}, \ \forall \lambda \in \widetilde{\mathcal{L}}_{\mathrm{red}}(\mathrm{w}), \ \ t_i = (t'_i)^{-1}, \ \forall i \in \mathcal{I}(\mathrm{w})\}, \end{aligned}$$

recall (3.6)*. In particular, we have:*

$$\mathrm{Stab}_{\mathbb{T}^r} J_{\mathrm{w},\zeta,\theta} = \{t \in \mathbb{T}^r \mid t_i = 1, \forall i \in \mathcal{I}(\mathrm{w}), \ t^{2\lambda^{(j)}} = 1, \forall j = 1, \ldots, k\} \tag{4.11}$$

$$= \{t \in \mathbb{T}^r \mid t^\lambda = 1, \ \forall \lambda \in \mathcal{L}(\mathrm{w})\}, \tag{4.12}$$

cf. (3.59)*.*

Proof. Eq. (4.10) follows directly from (4.7), (4.8), and (4.9). Eq. (4.11) is the restriction of (4.10). Eq. (4.12) is a consequence of (3.59) and (4.11). $\square$

4.3. **The standard Poisson Lie structure on G and its symplectic leaves.** In the remaining part of this section we assume that the base field is $\mathbb{K} = \mathbb{C}$. The assumption on the deformation parameter $q \in \mathbb{C}^*$ will be that it is not a root of unity, as before. Thus $\mathfrak{g}$ will be a complex simple Lie algebra. We will denote by G the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}$. Let $B_\pm$ be a pair of opposite Borel subgroups of G, and $T = B_+ \cap B_-$ be the corresponding maximal torus of G. One has the isomorphism of complex tori:

$$T \cong \mathbb{T}^r, \quad \exp(\zeta_1 \alpha_1^\vee + \ldots + \zeta_r \alpha_r^\vee) = (\exp \zeta_1, \ldots, \exp \zeta_r). \tag{4.13}$$

Denote $\mathfrak{h} = \mathrm{Lie}\, T$. Let $\langle ., . \rangle$ be the nondegenerate invariant bilinear on $\mathfrak{g}$ which matches the form (2.3) on $\mathfrak{h}^*$. For $\mu \in P$ define the characters t^μ of T by

$$\exp(h)^\mu = \exp(\langle \mu, h \rangle).$$

This matches (2.30) under the isomorphism (4.13). Additionally, denote $t^w = w^{-1} t w$ for $w \in W$, $t \in T$. For $\mathrm{w} = (w_+, w_-) \in W \times W$, set [28, §2.4]

$$T^{\mathrm{w}} = \{(t^{w_+})^{-1} t^{w_-} \mid t \in T\}.$$

Let $\{e_\alpha\}$ and $\{f_\alpha\}$, $\alpha \in \Delta^+$ be sets of dual root vectors of $\mathfrak{g}$, normalized by $\langle e_\alpha, f_\alpha \rangle = 1$. For $x \in \mathfrak{g}$ denote by $L(x)$ and $R(x)$ the left and right invariant vector fields on G. The standard Poisson structure on G is given by

$$\pi_G = \sum_{\alpha \in \Delta^+} L(e_\alpha) \wedge L(f_\alpha) - \sum_{\alpha \in \Delta^+} R(e_\alpha) \wedge R(f_\alpha).$$

For $j = 1, \ldots, r$ choose the representative

$$\overline{s}_j = \exp(e_{\alpha_j}) \exp(-f_{\alpha_j}) \exp(e_{\alpha_j}) \in N_G(T) \tag{4.14}$$

of $s_j \in W$, where $N_G(T)$ denotes the normalizer of T in G. This choice is slightly different from the one of Kogan and Zelevinsky [28], but we need it to match it to the braid group action (2.5). For $w \in W$ choose a reduced expression $w = s_{j_1} \ldots s_{j_l}$ and define

$$\overline{w} = \overline{s}_{j_1} \ldots \overline{s}_{j_l} \in N_G(T).$$

This choice of representative of a Weyl group element in the normalizer of the torus T does not depend on the choice of the reduced expression, because the elements (4.14) satisfy the braid relations analogously to [28].

Denote the unipotent radicals of $B_\pm$ by $U_\pm$. We have $U_- T U_+ \cong U_- \times T \times U_+$ under the group product. For $g \in U_- T U_+$ denote its components in U_-, T and U_+ by $[g]_-$, $[g]_0$ and $[g]_+$, respectively.

The left and right regular actions of T on G, preserve π_G. The T-orbits of symplectic leaves of π_G (under any of those actions) are [20] the double Bruhat cells $G^{\mathrm{w}} = G^{w_+, w_-} = B_+ w_+ B_+ \cap B_- w_- B_-$ of G, $\mathrm{w} = (w_+, w_-) \in W \times W$. The symplectic leaves of (G^{w}, π_G) were determined by Kogan and Zelevinsky [28, Theorem 2.3].

Theorem 4.3. *(Kogan–Zelevinsky [28]) For every* $\mathrm{w} = (w_+, w_-) \in W \times W$*, the set*

$$\begin{aligned}(4.15)\quad \mathcal{SL}_{\mathrm{w}} = \big\{ g \in G^{\mathrm{w}} \mid \big[\overline{w_+}^{\,-1} g\big]_0 . \big(\big[g \overline{w_-^{-1}}\big]\big)^{w_-} \in T^{\mathrm{w}}, \\ \big[\overline{w_+}^{\,-1} g\big]_0^{\omega_i} = 1, \ \forall i \in \mathcal{I}(\mathrm{w}) \big\}\end{aligned}$$

is a symplectic leaf of (G, π_G)*. All symplectic leaves of* G *have the form* $\mathcal{SL}_{\mathrm{w}}.t$ *for some* $t \in T$*,* $\mathrm{w} \in W \times W$*.*

In particular, the double Bruhat cell G^{w} is the T-orbit of the symplectic leaf $\mathcal{SL}_{\mathrm{w}}$ under both the left and right T-actions.

4.4. **Equations for the symplectic leaves of** (G^{w}, π_G)**.** Next, we make a minor reformulation of Theorem 4.3 to match it to Theorem 4.1. We have the following description of T^{w}.

Lemma 4.4. *The torus* T^{w} *is given by*

$$T^{\mathrm{w}} = \{ t \in T \mid t^\mu = 0, \forall \mu \in \widetilde{\mathcal{L}}(\mathrm{w}) \},$$

recall (3.5)*.*

Sketch of the proof. For $\mu \in P$, $w \in W$ one has $(t^w)^\lambda = t^{w\lambda}$. Therefore

$$T^{\mathrm{w}} \subseteq \{ t \in T \mid t^\mu = 0, \forall \mu \in \widetilde{\mathcal{L}}(\mathrm{w}) \},$$

because $((t^{w_+})^{-1} t^{w_-})^\lambda = t^{(w_- - w_+)\lambda} = 1$ for all $\lambda \in \widetilde{\mathcal{L}}(\mathrm{w})$. It is clear that both sides of the above inclusion are algebraic subgroups of T of codimension $\dim \ker(w_+ - w_-)$. One easily checks that they are both connected, thus they coincide. $\square$

For $\lambda \in P^+$ denote by $\widetilde{V}(\lambda)$ the irreducible finite dimensional module of G with highest weight λ. For $v \in \widetilde{V}(\lambda)$ and $\xi \in \widetilde{V}(\lambda)^*$ denote the matrix coefficient

$$\widetilde{c}^{\lambda}_{\xi, v} \in \mathbb{C}[G], \quad \widetilde{c}^{\lambda}_{\xi, v}(g) = \xi(gv), \ g \in G.$$

Let $v_\lambda \in V(\lambda)_\lambda$ and $\xi_\lambda \in V(\lambda)^*_{-\lambda}$, be such that $\xi_\lambda(v_\lambda) = 1$. Similarly let $v_{-\lambda} \in V(-w_0\lambda)_{-\lambda}$ and $\xi_{-\lambda} \in V(-w_0\lambda)^*_\lambda$, be such that $\xi_{-\lambda}(v_{-\lambda}) = 1$. Analogously to the quantum case for $\lambda \in P^+$ and $w \in W$ define

$$(4.16) \qquad \widetilde{c}^{+}_{w,\lambda} = \widetilde{c}^{\lambda}_{\overline{w}\xi_\lambda, v_\lambda}, \quad \widetilde{c}^{-}_{w,\lambda} = \widetilde{c}^{-w_0\lambda}_{\left(\overline{w^{-1}}\right)^{-1} \xi_{-\lambda}, v_{-\lambda}}.$$

Their key property is that

$$\widetilde{c}^{+}_{w,\lambda}(g) = \big(\big[\overline{w_+}^{-1} g\big]_0\big)^{\lambda}, \quad \widetilde{c}^{-}_{w,\lambda}(g) = \big(\big[g\overline{w_-^{-1}}\big]\big)^{-w_-\lambda}, \tag{4.17}$$

which is verified by a direct computation. This property is the reason for the above normalization of $\widetilde{c}^{-}_{w,\lambda}$.

We also have

$$\widetilde{c}^{+}_{w,\lambda_1}\widetilde{c}^{+}_{w,\lambda_2} = \widetilde{c}^{+}_{w,\lambda_1+\lambda_2}, \quad \forall \lambda_1, \lambda_2 \in P^+. \tag{4.18}$$

The functions $\widetilde{c}^{\pm}_{w_\pm,\lambda}$ are regular and nowhere vanishing on G^{w_+,w_-}, for all $\lambda \in P^+$. Fix $\lambda \in P$, represent it as $\lambda = \lambda_1 - \lambda_2$ for some $\lambda_1, \lambda_2 \in P^+$, and define

$$\widetilde{c}^{\pm}_{w_\pm,\lambda} = \widetilde{c}^{\pm}_{w_\pm,\lambda_1}(\widetilde{c}^{\pm}_{w_\pm,\lambda_2})^{-1}, \tag{4.19}$$

considered as a rational function on G and a regular function on G^{w}. The definition (4.19) does not depend on the choice of λ_1 and λ_2, because of (4.18). Eq. (4.17) holds for all $\lambda \in P$.

For $j = 1, \ldots, k$ denote

$$\widetilde{a}_j = \widetilde{c}^{+}_{w_+,\lambda^{(j)}}(\widetilde{c}^{-}_{w_-,\lambda^{(j)}})^{-1}.$$

Corollary 4.5. *Let* $\mathrm{w} = (w_+, w_-) \in W \times W$. *Then the symplectic leaves of* (G, π_G) *inside the double Bruhat cell* G^{w} *are parametrized by* $(\mathbb{C}^*)^{\times \dim \ker(w_+ - w_-)}$. *They are exactly the sets*

$$\mathcal{SL}_{\mathrm{w},\zeta,\theta} = \{g \in G^{\mathrm{w}} \mid \widetilde{a}_j(g) = \zeta_j, j = 1, \ldots, k, \ \widetilde{c}^{+}_{w_+,\omega_i}(g) = \theta_i, i \in \mathcal{I}(\mathrm{w})\}, \tag{4.20}$$

for $\zeta = (\zeta_1, \ldots, \zeta_k) \in (\mathbb{C}^*)^{\times k}$, $\theta = \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} \in (\mathbb{C}^*)^{\times |\mathcal{I}(\mathrm{w})|}$ *and* $k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$.

Proof. Lemma 4.4 and (4.17) imply that $\mathcal{SL}_{\mathrm{w}} = \mathcal{SL}_{\mathrm{w},\zeta,\theta}$ for $\zeta_j = \theta_i = 1$, $\forall j = 1, \ldots, k$, $i \in \mathcal{I}(\mathrm{w})$. Theorem 4.3 now implies the statement using the right regular action of T. □

4.5. **A $\mathbb{T}^r \times \mathbb{T}^r$-equivariant Dixmier map for $R_q[G]$.** Denote by $\mathrm{Sympl}(G, \pi_G)$ the symplectic foliation space of the Poisson structure π_G (i.e the set of symplectic leaves with the induced topology from the Zariski topology on G). Define the Dixmier type map

$$D_G \colon \mathrm{Sympl}(G, \pi_G) \to \mathrm{Prim}\, R_q[G], \quad D_G(\mathcal{SL}_{\mathrm{w},\zeta,\theta}) = J_{\mathrm{w},\zeta,\theta},$$

$\mathrm{w} \in W \times W$, $\theta = \{\theta_i\}_{i \in \mathcal{I}(\mathrm{w})} \in (\mathbb{C}^*)^{\times |\mathcal{I}(\mathrm{w})|}$, $\zeta = (\zeta_1, \ldots, \zeta_k) \in (\mathbb{C}^*)^{\times k}$, where $k = \dim \ker(w_+ - w_-) - |\mathcal{I}(\mathrm{w})|$.

Consider the $T \times T$-action on G coming from the left and right regular actions

$$(t, t') \cdot g = (t')^{-1} g t^{-1}$$

and transfer it to a $\mathbb{T}^r \times \mathbb{T}^r$-action on G via (4.13). This action preserves π_G and thus induces an action on $\mathrm{Sympl}(G, \pi_G)$. (The choice of the inverses is made to match this action with the actions (4.6) and (2.7) in the quantum situation.) Analogously to (4.7) one shows that

$$(t', t) \cdot \mathcal{SL}_{\mathrm{w},\zeta,\theta} = \mathcal{SL}_{\mathrm{w},(\mathbf{t}',\mathbf{t})\cdot\zeta,(\mathbf{t}',\mathbf{t})\cdot\theta}, \tag{4.21}$$

in terms of (4.8) and (4.9). Combining Theorem 4.1, Theorem 4.3, (4.7) and (4.21), we obtain:

Theorem 4.6. *Assume that the base field $\mathbb{K}$ is $\mathbb{C}$ and $q \in \mathbb{C}^*$ is not a root of unity. Then the Dixmier type map $D_G\colon \mathrm{Sympl}(G, \pi_G) \to \mathrm{Prim}\, R_q[G]$ is a $\mathbb{T}^r \times \mathbb{T}^r$-equivariant bijection.*

The original orbit method conjecture [20] of Hodges and Levasseur for $R_q[G]$ can be formulated more precisely as follows:

Conjecture 4.7. Under the above assumptions, the Dixmier map

$$D_G\colon \mathrm{Sympl}(G, \pi_G) \to \mathrm{Prim}\, R_q[G]$$

is a homeomorphism.

Remark 4.8. In the special case when the base field is $\mathbb{K} = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$, one can prove that the elements $c^{\pm}_{w,\lambda}$ defined in (2.17) specialize to the elements $\widetilde{c}^{\pm}_{w,\lambda}$ defined in (4.16) for all $\lambda \in P^+$, $w \in W$, when q is specialized to 1. The elements $\widetilde{c}^{\pm}_{w,\lambda}$ are in turn related to the setting of Kogan and Zelevinsky [28] via (4.17). The normalization of the elements $c^{\pm}_{w,\lambda}$ in §2.4 was made so that our setting matches the latter whenever specialization can be defined (which in general is the case when $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$).

The special case of Theorem 3.1 for base fields $\mathbb{K}$ of characteristic 0 and $q \in \mathbb{K}$ transcendental over $\mathbb{Q}$ can be proved in a faster way using specialization and the Kogan–Zelevinsky result [28]. One should point out though that the results on P-normal elements of the algebras $S^{\pm}_{w_\pm}$ and S_{w} which are the building blocks of the proof of Theorem 3.1 continue playing an important role in the rest of the paper.

A result of [41] proves that in the complex case the Haar functional on $R_q[G]$ is an integral of the traces of the irreducible $*$-representations of $R_q[G]$ classified in [33], see [41, Theorem 5.2] for details. Those representations correspond to particular primitive ideals in $\mathrm{Prim}_{(w,w)}\, R_q[G]$ for $w \in W$. We finish with raising the question whether irreducible representations corresponding to the other primitive ideals of $R_q[G]$ play any (noncommutative) differential geometric role.

5. Separation of variables for the algebras $S^{\pm}_w$

5.1. **Statement of the freeness result.** Recall that Joseph's isomorphism (3.28) represents the localizations R_{w}, $\mathrm{w} = (w_+, w_-) \in W \times W$ in terms of the algebras $S^{\pm}_{w_\pm}$. In this and the next sections we prove a number of results for the algebras $S^{\pm}_{w_\pm}$ which will play a key role in our study of R_{w} and $\mathrm{Max} R_q[G]$ in the following sections. These results also establish important properties of the De Concini–Kac–Procesi algebras via the isomorphisms from Theorem 2.6.

Throughout this section we fix a Weyl group element $w \in W$. Denote by $N^{\pm}_w$ the subalgebras of $S^{\pm}_w$ generated by the normal elements $d^{\pm}_{w,\omega_i}$, $i \in \mathcal{S}(w)$, recall (3.29). In this section we describe the structure of the algebras $S^{\pm}_w$, considered as $N^{\pm}_w$-modules. We apply these results in several directions. In Section 6 we use them to classify all homogeneous normal elements of the algebras $S^{\pm}_w$ and equivalently the De Concini–Kac–Procesi algebras $\mathcal{U}^w_{\pm}$. In fact, we prove that all homogeneous normal elements of the algebras $S^{\pm}_w$ are scalar multiples of $d^{\pm}_{w,\lambda}$, $\lambda \in P^+_{\mathcal{S}(w)}$. As another application in Section 6 we classify all prime elements of the algebras $S^{\pm}_w$. In Section 7 the results of this section are used to describe the structure of $R_{\rm w}$ as a module over its subalgebra generated by the sets of normal elements $E^{\pm 1}_{\rm w}$, recall (2.23). This is then applied to classify the maximal ideals of $R_q[G]$ in Section 8.

We start by noting that (3.29) implies $d^{\pm}_{w,\omega_i} \in \mathbb{K}^*$, for $i \in \mathcal{I}(w)$. Because of this, one only needs to consider $d^{\pm}_{w,\omega_i}$ for $i \in \mathcal{S}(w)$. It follows from (3.30) that

$$d^{\pm}_{w,\lambda_1} d^{\pm}_{w,\lambda_1} = q^{\pm(\langle w_{\pm}(\lambda_1), \lambda_2\rangle - \langle \lambda_1, w_{\pm}(\lambda_2)\rangle)} d^{\pm}_{w,\lambda_2} d^{\pm}_{w,\lambda_1}, \quad \forall \lambda_1, \lambda_2 \in P^+,$$

and in particular

$$d^{\pm}_{w,\omega_i} d^{\pm}_{w,\omega_j} = q^{\pm(\langle w_{\pm}(\omega_i), \omega_j\rangle - \langle \omega_i, w_{\pm}(\omega_j)\rangle)} d^{\pm}_{w,\omega_j} d^{\pm}_{w,\omega_i}, \quad \forall i, j \in \mathcal{S}(w). \tag{5.1}$$

The main result of the section is:

Theorem 5.1. *Let $\mathbb{K}$ be an arbitrary base field, q be an element of $\mathbb{K}^*$ which is not a root of unity, and $w \in W$. Then:*

(i) The algebra $N^{\pm}_w$ is isomorphic to the quantum affine space algebra over $\mathbb{K}$ of dimension $|\mathcal{S}(w)|$ with generators $d^{\pm}_{w,\omega_i}$, $i \in \mathcal{S}(w)$ and relations (5.1).

(ii) The algebra $S^{\pm}_w$ is a free left and right $N^{\pm}_w$-module in which $N^{\pm}_w$ is a direct summand, viewed as a module over itself.

An explicit form of the freeness result in the second part of the theorem is obtained in Theorem 5.5 below. The special case of $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and $w = w_0$ in Theorems 5.1 (ii) and 5.5 is due to Lopes [34].

In the next section we classify the homogeneous normal elements of $S^{\pm}_w$. A consequence of this result is that $N^{\pm}_w$ coincides with the subalgebra of $S^{\pm}_w$ generated by all of its homogeneous normal elements. In particular, $Z(S^{\pm}_w) \subset N^{\pm}_w$. Theorems of the above kind are motivated by the desire to extend the theorems for separation of variables of Kostant [30] and Joseph–Letzter [27] to quantized universal enveloping algebras of nilpotent Lie algebras. Kostant, and Joseph and Letzter proved that $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ are free as modules over their centers and deduced further important properties of the corresponding bases. In our case the centers of $S^{\pm}_w$ are in general too small compared to the centers of $S^{\pm}_w[E^{\pm}_w]$, see Lemma 6.10. Thus one would obtain weaker results by considering the module structure of $S^{\pm}_w$ over their centers $Z(S^{\pm}_w)$ as opposite to the subalgebras generated by the "numerators" and "denominators" of the central elements of $S^{\pm}_w[E^{\pm}_w]$. It is the structure of $S^{\pm}_w$ as a module over the "normal subalgebra" $N^{\pm}_w$ that has applications to the structure of $\mathrm{Spec} R_q[G]$ and $\mathrm{Spec} S^{\pm}_w$. Two more freeness results will be obtained in Section 7 for the algebras $S_{\rm w}$ and $R_{\rm w}$.

We recall that a quantum affine space algebra is an algebra over $\mathbb{K}$ with generators $X_1, \ldots, X_m$ and relations

$$X_i X_j = p_{ij} X_j X_i, \quad i, j = 1, \ldots, m, \tag{5.2}$$

for some $p_{ij} \in \mathbb{K}^*$ such that $p_{ij} p_{ji} = 1$, for all $i \neq j \in \{1, \ldots, m\}$, $p_{ii} = 1$, $i \in \{1, \ldots, m\}$. Such an algebra has GK dimension equal to m. It has a $\mathbb{K}$-basis, consisting of the monomials

$$(X_1)^{n_1} \ldots (X_m)^{n_m}, \quad n_1, \ldots, n_m \in \mathbb{N}. \tag{5.3}$$

On the other hand, if a $\mathbb{K}$-algebra is generated by some elements $X_1, \ldots, X_m$, which satisfy (5.2) and the monomials (5.3) are linearly independent, then the algebra is isomorphic to the above quantum affine space algebra. We also recall that the localization of this algebra by the multiplicative subset generated by $X_1^{-1}, \dots, X_m^{-1}$ is called quantum torus algebra.

Because of (3.31), the first part of Theorem 5.1 essentially claims that the elements $d^{\pm}_{w,\lambda}$ are linearly independent over $\mathbb{K}$ for different $\lambda \in P^+_{\mathcal{S}(w)}$.

5.2. **Leading terms of the normal elements $d^{\pm}_{w,\lambda}$.** For the rest of this section we fix a reduced expression of w

$$w = s_{i_1} \dots s_{i_l}, \tag{5.4}$$

where $l = l(w)$ is the length of w. Denote this reduced expression by $\vec{w}$. For $j \in \mathcal{S}(w)$, let

$$\mathrm{Supp}_j(\vec{w}) = \{k = 1, \dots, l \mid i_k = j\}.$$

Recall the definition of the roots β_k (see (2.34)) and the root vectors $X^{\pm}_{\beta_k}$ (see (2.35)), $k = 1, \dots, l$, associated to the reduced expression $\vec{w}$. Recall the definition (2.39) of the monomials $(X^{\pm})^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^{\times l}$, the notions of leading term of an element of $\mathcal{U}^w_{\pm}$ and degree of a monomial from §2.7.

For $j \in \mathcal{S}(w)$ denote

$$\mathbf{e}(\vec{w})_j = (n_{j1}, \dots, n_{jl}) \in \mathbb{N}^{\times l},$$

where

$$n_{jk} = 1 \;\text{ if }\; k \in \mathrm{Supp}_j(\vec{w}), \;\; n_{jk} = 0 \;\text{ if }\; k \notin \mathrm{Supp}_j(\vec{w}). \tag{5.5}$$

Recall the isomorphisms $\varphi^{\pm}_w \colon S^{\pm}_w \to \mathcal{U}^w_{\mp}$ from Theorem 2.6. The key fact for the proof of Theorem 5.1 is as follows:

Proposition 5.2. *Let $\mathbb{K}$ be an arbitrary base field, q be an element of $\mathbb{K}^*$ which is not a root of unity, and $\vec{w}$ be a reduced expression of $w \in W$. Then for all $\lambda \in P^+$ the leading term of $\varphi^{\pm}_w(d^{\pm}_{w,\lambda})$ has degree*

$$(\langle \lambda, \alpha^\vee_{i_1} \rangle, \dots, \langle \lambda, \alpha^\vee_{i_l} \rangle).$$

In particular, for all $j \in \mathcal{S}(w)$, the leading term of $\varphi^{\pm}_w(d^{\pm}_{w,\omega_j})$ has degree $\mathbf{e}(\vec{w})_j$.

Recall from §2.3 that for $\lambda \in P^+$, v_λ and $v_{-\lambda}$ denote a fixed highest weight vector of $V(\lambda)$ and a fixed lowest weight vector of $V(-w_0\lambda)$, respectively. The following lemma is known. We sketch its proof for completeness.

Lemma 5.3. *Given a reduced expression $\vec{w}$ of $w \in W$, for all $\lambda \in P^+$ we have*

$$T_w(v_{\pm\lambda}) = p (X^{\mp}_{\beta_l})^{\langle \lambda, \alpha^\vee_{i_l} \rangle} \dots (X^{\mp}_{\beta_1})^{\langle \lambda, \alpha^\vee_{i_1} \rangle} v_{\pm\lambda}, \tag{5.6}$$

for some $p \in \mathbb{K}^*$.

Proof. We argue by induction on $l(w)$, the statement being trivial for $w = 1$. Assume the validity of (5.6) for Weyl group elements of length l. Let $w' = s_{i_0} w$ be such that $l(w') = l + 1$, and $s_{i_0} s_{i_1} \dots s_{i_l}$ be a reduced expression of w', which will be denoted by $\vec{w}'$. Denote by β_k and $X^{\mp}_{\beta_k}$, $k = 1, \dots, l$, the roots and root vectors corresponding to the reduced expression $s_{i_1}\dots s_{i_l}$ of w. Using (2.6) and the induction hypothesis, we obtain

$$\begin{aligned}T_{w'}v_{\pm\lambda} = T_{i_0}T_w v_{\pm\lambda} = pT_{i_0}\big((X^{\mp}_{\beta_l})^{\langle\lambda,\alpha^\vee_{i_l}\rangle}\dots(X^{\mp}_{\beta_1})^{\langle\lambda,\alpha^\vee_{i_1}\rangle}v_{\pm\lambda}\big)\\ =p(T_{i_0}(X^{\mp}_{\beta_l}))^{\langle\lambda,\alpha^\vee_{i_l}\rangle}\dots(T_{i_0}(X^{\mp}_{\beta_1}))^{\langle\lambda,\alpha^\vee_{i_1}\rangle}\big(T_{i_0}(v_{\pm\lambda})\big).\end{aligned}$$

Eq. (5.6) for w' now follows from the fact that $T_{i_0}(v_{\pm\lambda}) = p'(X^{\mp}_{i_0})^{\langle\lambda,\alpha^\vee_{i_0}\rangle}v_{\pm\lambda}$ for some $p'\in\mathbb{K}^*$ (which follows from (2.5)) and the fact that the root vectors corresponding to the reduced expression $\vec{w}'$ of w' are $X^{\mp}_{i_0}, T_{i_0}(X^{\mp}_{\beta_1}),\dots,T_{i_0}(X^{\mp}_{\beta_l})$. □

Proof of Proposition 5.2. We will prove the statement in the plus case, using Lemma 5.3. The minus case is analogous. Assume that the reduced expression $\vec{w}$ is given by (5.4). Recall the definition of the vectors $\xi^+_{1,\lambda}\in V(\lambda)^*_{-\lambda}$ from §2.4. In light of the definition (2.45) of the isomorphism $\varphi^w_\pm\colon S^\pm_w\to\mathcal{U}^w_\pm$ we need to prove that

$$\langle\xi^+_{1,\lambda},(X^+_{\beta_1})^{\langle\lambda,\alpha^\vee_{i_1}\rangle}\dots(X^+_{\beta_l})^{\langle\lambda,\alpha^\vee_{i_l}\rangle}T_w v_\lambda\rangle\neq 0 \tag{5.7}$$

and

$$\langle\xi^+_{1,\lambda},(X^+)^{\mathbf{n}}T_w v_\lambda\rangle\neq 0,\ \mathbf{n}\in\mathbb{N}^{\times l}\quad\Rightarrow\quad\mathbf{n}\le(\langle\lambda,\alpha^\vee_{i_1}\rangle,\dots,\langle\lambda,\alpha^\vee_{i_l}\rangle) \tag{5.8}$$

in the lexicographic order from (2.40).

For $k=0,1,\dots,l$ denote $w_{(k)}=s_{i_1}\dots s_{i_k}$. Using Lemma 5.3, we obtain

$$\begin{aligned}&(X^+_{\beta_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}\big(T_{w(k)}v_\lambda\big)\\ =&T_{w(k-1)}(X^+_{i_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}T_{w(k-1)}(X^-_{i_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}\big(T_{w(k-1)}v_\lambda\big)\\ =&T_{w(k-1)}\big((X^+_{i_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}(X^-_{i_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}v_\lambda\big)=p_kT_{w(k-1)}v_\lambda\end{aligned} \tag{5.9}$$

for some $p_k\in\mathbb{K}^*$. Analogously one proves that for $m>0$

$$(X^+_{\beta_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle+m}\big(T_{w(k)}(v_\lambda)\big)=0. \tag{5.10}$$

Using Lemma 5.3 and recursively applying (5.9), one obtains

$$\begin{aligned}&\langle\xi^+_{1,\lambda},(X^+_{\beta_1})^{\langle\lambda,\alpha^\vee_{i_1}\rangle}\dots(X^+_{\beta_l})^{\langle\lambda,\alpha^\vee_{i_l}\rangle}T_w v_\lambda\rangle=\dots\\ =&p_l\dots p_{k+1}\langle\xi^+_{1,\lambda},(X^+_{\beta_1})^{\langle\lambda,\alpha^\vee_{i_1}\rangle}\dots(X^+_{\beta_k})^{\langle\lambda,\alpha^\vee_{i_k}\rangle}T_{w(k)}v_\lambda\rangle=\dots\\ =&p_l\dots p_1\langle\xi^+_{1,\lambda},v_\lambda\rangle\neq 0.\end{aligned}$$

This proves (5.7). Assume that $\mathbf{n}\in\mathbb{N}^{\times l}$ and $\mathbf{n}>(\langle\lambda,\alpha^\vee_{i_1}\rangle,\dots,\langle\lambda,\alpha^\vee_{i_l}\rangle)$. Then there exists $k\in[1,l]$ such that $n_j=\langle\lambda,\alpha^\vee_{i_1}\rangle$ for $j=k+1,\dots,l$ and $n_k>\langle\lambda,\alpha^\vee_{i_k}\rangle$. Using Lemma 5.3, (5.9) and (5.10) one obtains

$$\langle\xi^+_{1,\lambda},(X^+)^{\mathbf{n}}T_w v_\lambda\rangle=\dots=p_l\dots p_{k+1}\langle\xi^+_{1,\lambda},(X^+_{\beta_1})^{n_1}\dots(X^+_{\beta_k})^{n_k}T_{w(k)}v_\lambda\rangle=0.$$

This proves (5.8) and completes the proof of the Proposition. □

5.3. **Proof of Theorem 5.1.** We begin with the proof of the first part of Theorem 5.1. The second part of the theorem requires some additional facts. It is given at the end of the subsection.

Proof of part (i) of Theorem 5.1. If $\lambda_1, \lambda_2 \in P^+_{\mathcal{S}(w)}$ and $\lambda_1 \neq \lambda_2$, then there exists $j \in \mathcal{S}(w)$ such that $\langle \lambda_1, \alpha_j^\vee \rangle \neq \langle \lambda_2, \alpha_j^\vee \rangle$. Proposition 5.2 implies that all elements $\{d^\pm_{w,\lambda}\}_{\lambda \in P^+_{\mathcal{S}(w)}}$ have leading terms of different degrees. Therefore they are linearly independent because of Theorem 2.4, which proves part (i) of Theorem 5.1. □

Denote the following two subsets of $\mathbb{N}^{\times l}$:

$$\Sigma(\vec{w}) = \bigoplus_{j \in \mathcal{S}(w)} \mathbb{N}\mathbf{e}(\vec{w})_j$$

and

(5.11)

$$\Delta(\vec{w}) = \{(n_1, \dots, n_l) \in \mathbb{N}^{\times l} \mid \forall j \in \mathcal{S}(w), \ \exists k \in \mathrm{Supp}_j(\vec{w}) \ \text{ such that } \ n_k = 0\}.$$

According to Proposition 5.2 the first subset consists of the degrees of the leading terms of the elements $d^\pm_{w,\lambda} \in N^\pm_w$, $\lambda \in P^+_{\mathcal{S}(w)}$. The following fact shows that the second subset is complementary to the first one. Its proof is left to the reader.

Lemma 5.4. *Each element of $\mathbb{N}^{\times l}$ is representable in a unique way as the sum of an element of $\Sigma(\vec{w})$ and an element of $\Delta(\vec{w})$.*

The second part of Theorem 5.1 follows directly from the following theorem which provides an explicit presentation of $S^\pm_w$ as a free $N^\pm_w$-module.

Theorem 5.5. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and a reduced expression $\vec{w}$ of $w \in W$:*

$$S^\pm_w = \bigoplus_{\mathbf{n} \in \Delta(\vec{w})} N^\pm_w \cdot (\varphi^\pm_w)^{-1}\big((X^\mp)^{\mathbf{n}}\big) = \bigoplus_{\mathbf{n} \in \Delta(\vec{w})} (\varphi^\pm_w)^{-1}\big((X^\mp)^{\mathbf{n}}\big) \cdot N^\pm_w.$$

Note that $\mathbf{0} \in \Delta(\vec{w})$ and

$$N^\pm_w \cdot (\varphi^\pm_w)^{-1}\big((X^\mp)^{\mathbf{0}}\big) = N^\pm_w.$$

Proof of Theorem 5.5. It is sufficient to prove the first equality since the algebra $N^\pm_w$ is spanned by P-normal elements, which q-commute with $(\varphi^\pm_w)^{-1}\big((X^\pm)^{\mathbf{n}}\big)$, for $\mathbf{n} \in \mathbb{N}^{\times l}$.

The theorem follows from the fact that the associated graded of $\mathcal{U}^w_\pm$ with respect to the filtration induced from the ordering (2.40) is free over the associated graded of $\varphi^\pm_w(N^\pm_w)$, because of Proposition 5.2 and Lemma 5.4. Here are the details: to show

$$S^\pm_w = \sum_{\mathbf{n} \in \Delta(\vec{w})} N^\pm_w \cdot (\varphi^\pm_w)^{-1}\big((X^\pm)^{\mathbf{n}}\big), \tag{5.12}$$

fix $s \in S^\pm_w$, $s \neq 0$. Proposition 5.2, and Lemmas 2.5 and 5.4 imply that there exist $\lambda \in P^+_{\mathcal{S}(w)}$, $\mathbf{n} \in \Delta(\vec{w})$, and $p \in \mathbb{K}^*$, such that either $\varphi^\pm_w(s) - \varphi^\pm_w(d^\pm_{w,\lambda})(X^\mp)^{\mathbf{n}} = 0$, or it is a nonzero element whose leading term has degree strictly less than that of the leading term of $\varphi^{\pm}_w(s)$, recall (2.40). Iterating this, gives (5.12).

The set
$$\{\varphi^{\pm}_w(d^{\pm}_{w,\lambda})(X^{\mp})^{\mathbf{n}} \mid \lambda \in P^+_{\mathcal{S}(w)}, \mathbf{n} \in \Delta(\vec{w})\}$$
is linearly independent over $\mathbb{K}$, because the elements of this set have leading terms of different degrees. This follows from Proposition 5.2 and Lemma 5.4. □

6. A classification of the homogeneous normal elements and prime elements of the De Concini–Kac–Procesi algebras

6.1. **Statement of the classification result.** In this section we develop further the line of argument of §3.5 and obtain a classification of the sets of homogeneous normal elements of all De Concini–Kac–Procesi algebras $\mathcal{U}^w_{\pm}$. Equivalently, this gives a classification of the homogeneous normal elements of the algebras $S^{\pm}_w$. We combine these results with the results from the previous section to obtain an explicit description of the primitive ideals in the Goodearl–Letzter stratum [17] of $\operatorname{Prim} S^{\pm}_w$ over the $\{0\}$ ideal. These results are then applied to obtain a classification of all prime elements of the algebras $S^{\pm}_w$. Our approach to the classification problem for the homogeneous normal elements of the algebras $S^{\pm}_w$, is to prove first that each such element is P-normal, recall Definition 3.4. We then obtain the classification by an argument, which combines Theorem 3.1, Theorem 5.1 on separation of variables for the algebras $S^{\pm}_w$, and a strong rationality result for H-primes of iterated skew polynomial extensions of Goodearl [6].

Throughout this section w will denote a fixed element of the Weyl group W. Recall that the algebras $\mathcal{U}^w_{\pm}$ and $S^{\pm}_w$ are $Q_{\mathcal{S}(w)}$-graded, by (2.43) and (3.22). We call an element of these algebras homogeneous, if it is homogeneous with respect to the corresponding grading.

Recall that an element u of a noetherian domain R is called prime if it is normal and Ru is a height one prime ideal, which is completely prime. Recall the definition (3.29) of the normal elements $d^{\pm}_{w,\lambda} \in (S^{\pm}_w)_{\pm(w-1)\lambda,0}$, cf. (3.30). The following theorem contains our classification result for homogeneous prime and homogeneous normal elements of the algebras $S^{\pm}_w$. In Theorem 6.16 below we obtain a classification of the inhomogeneous prime elements of the algebras $S^{\pm}_w$.

Theorem 6.1. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. Let $w \in W$. Then:*

(i) Every nonzero homogeneous normal element of $S^{\pm}_w$ is equal to an element of the form
$$p d^{\pm}_{w,\lambda} \in (S^{\pm}_w)_{\pm(w-1)\lambda,0} \tag{6.1}$$
for some $p \in \mathbb{K}^*$, $\lambda \in P^+_{\mathcal{S}(w)}$. All such elements are distinct and even more the elements $d^{\pm}_{w,\lambda}$, $\lambda \in P^+_{\mathcal{S}(w)}$ are linearly independent, cf. Theorem 5.1.

(ii) For all $i \in \mathcal{S}(w)$, $d^{\pm}_{w,\omega_i} \in (S^{\pm}_w)_{\pm(w-1)\omega_i,0}$ are pairwise nonproportional prime elements of $S^{\pm}_w$ and all homogeneous prime elements of $S^{\pm}_w$ are nonzero scalar multiples of them.

Here and below, "pairwise nonproportional elements" means not a scalar multiple of each other.

In view of (3.31), another way to formulate Theorem 6.1 is to say that every nonzero homogeneous normal element of $S^{\pm}_w$ is equal to an element of the form

$$p \prod_{i \in \mathcal{S}(w)} (d^{\pm}_{w,\omega_i})^{n_i} \tag{6.2}$$

for some $n_1, \ldots, n_r \in \mathbb{N}$, $p \in \mathbb{K}^*$. The elements $d^{\pm}_{w,\omega_i}$ do not commute. In (6.2) we take the product over i in any fixed order. (Recall that the elements $d^{\pm}_{w,\omega_i}$ q-commute.)

Recall the graded isomorphisms $\varphi^{\pm}_w \colon S^{\pm}_w \to \mathcal{U}^w_{\mp}$ from Theorem 2.6. We have the following reformulation of Theorem 6.1, which provides a classification of the sets of homogeneous prime elements and homogeneous normal elements of the De Concini–Kac–Procesi algebras $\mathcal{U}^w_{\pm}$.

Theorem 6.2. *(i) In the setting of Theorem 6.1, every nonzero homogeneous normal element of $\mathcal{U}^w_{\pm}$ is equal to an element of the form*

$$p\varphi^{\mp}_w(d^{\mp}_{w,\lambda}) \in (\mathcal{U}^w_{\pm})_{\mp(w-1)\lambda}$$

for some $p \in \mathbb{K}^*$, $\lambda \in P^+_{\mathcal{S}(w)}$. All such elements are distinct and even linearly independent for different λ's.

(ii) For all $i \in \mathcal{S}(w)$, $\varphi^{\mp}_w(d^{\mp}_{w,\omega_i}) \in (\mathcal{U}^w_{\pm})_{\mp(w-1)\omega_i}$ are pairwise nonproportional prime elements of $\mathcal{U}^w_{\pm}$ and all homogeneous prime elements of $\mathcal{U}^w_{\pm}$ are nonzero scalar multiples of them.

The case of $w = w_0$ of the first part of the theorem was proved by Caldero [8], using very different methods from ours, based on the Joseph–Letzter results [27]. The case of the second part of the theorem for the algebras of quantum matrices is due to Launois, Lenagan and Rigal [31, Proposition 4.2].

In the case when the characteristic of $\mathbb{K}$ is 0 and q is transcendental over $\mathbb{Q}$, one can deduce part (ii) of Theorem 6.2 from [43, Theorem 1.1 (c)].

6.2. **Homogeneous normal and P-normal elements of $S^{\pm}_w$.** First, we show that each homogeneous normal element of the algebras $S^{\pm}_w$ is P-normal in the sense of Definition 3.4.

Proposition 6.3. *All homogeneous normal elements of $S^{\pm}_w$ are P-normal.*

The proof of this proposition will be given in §6.4. For this proof we will need two results. The first concerns the number of pairwise nonproportional prime elements of the algebras $\mathcal{U}^w_{\pm}$ (proved in this subsection) and the second concerns a special kind of "diagonal" automorphisms of the algebras $\mathcal{U}^w_{\pm}$ (proved in §6.3).

A noetherian domain R is said [10] to be a unique factorization domain, if R has at least one height one prime ideal, and every height one prime ideal is generated by a prime element. Torsion free CGL extensions (for Cauchon–Goodearl–Letzter) are skew polynomial algebras with a rational action of a torus, satisfying certain general conditions, see [31, Definition 3.1]. Launois, Lenagan, and Rigal [31, Theorem 3.7] proved that every CGL extension is a noetherian unique factorization domain. The algebras $\mathcal{U}^w_{\pm}$ are all torsion free CGL extensions see [38]; thus they are all noetherian unique factorization domains.

For $y \in W$, $y \leq w$ define the ideals

(6.3)
$$I_w^+(y) = \mathrm{Span}\{(c_{w,\lambda}^+)^{-1}\xi \mid \lambda \in P, \xi \in (V_w^+(\lambda))^*, \, \xi \perp (V_w^+(\lambda) \cap \mathcal{U}_- V(\lambda)_{y\lambda})\},$$

(6.4)
$$I_w^-(y) = \mathrm{Span}\{(c_{w,\lambda}^-)^{-1}\xi \mid \lambda \in P, \xi \in (V_w^-(\lambda))^*, \, \xi \perp (V_w^-(\lambda) \cap \mathcal{U}_+ V(-w_0\lambda)_{-y\lambda})\}$$

of S_w^+ and S_w^-, respectively, using the identifications (2.41).

In [45], using results of Gorelik [19], we proved that the algebras $\mathcal{U}_-^w$ (and thus S_w^+) are catenary and that the H-invariant height one prime ideals of $\mathcal{U}_-^w$ (with respect to the conjugation action of H) are precisely the ideals $\varphi_w^+(I_w^+(s_i))$ for $i \in \mathcal{S}(w)$. The analogous fact for $\mathcal{U}_+^w$ is proved by interchanging the role of plus and minus generators $X_i^\pm$. Since $\mathcal{U}_\pm^w$ are noetherian unique factorization domains and a normal element of $\mathcal{U}_\pm^w$ is homogeneous, if and only if it generates an H-invariant ideal, we have:

Lemma 6.4. *The number of pairwise nonproportional homogeneous prime elements of $\mathcal{U}_\pm^w$ is equal to $|\mathcal{S}(w)|$.*

6.3. **A lemma on diagonal automorphisms of $\mathcal{U}_\pm^w$.** Let

$$w = s_{i_1} \dots s_{i_l}$$

be a reduced expression of w, $l = l(w)$. Let $\beta_1, \dots, \beta_l$ and $X_{\beta_1}^\pm, \dots, X_{\beta_l}^\pm$ be the roots and root vectors, given by (2.34) and (2.35), respectively.

Lemma 6.5. *If $\psi \in \mathrm{Aut}(\mathcal{U}_\pm^w)$ is such that*

$$\psi(X_{\beta_j}^\pm) = q_{i_j}^{k_j} X_{\beta_j}^\pm$$

for some $k_1, \dots, k_{l(w)} \in \mathbb{Z}$, then there exists $\delta_\pm \in P$ such that $\langle \delta_\pm, \pm\beta_j^\vee \rangle = k_j$, (i.e. $\langle \delta_\pm, \pm\beta_j \rangle = d_j k_j$), for all $j = 1, \dots, l(w)$.

Recall from §2.1 that $q_i = q^{d_i}$, where $(d_1, \dots, d_r)$ is the vector of relatively prime positive integers symmetrizing the Cartan matrix of $\mathfrak{g}$.

Lemma 6.5 (and the statement in Remark 6.6 below) are well known and easy to prove for various special cases, e.g. the algebras of quantum matrices or $w = w_0$. The emphasis here is on the validity of the statements for all $\mathfrak{g}$ and $w \in W$.

Proof of Lemma 6.5. We argue by induction on $l(w)$, the case $l(w) = 0$ being trivial. Assume that the statement of the lemma is true for $w \in W$ of length l. Let $w' \in W$, $l(w') = l + 1$, and $s_{i_1} \dots s_{i_l} s_{i_{l+1}}$ be a reduced expression of w'. Denote $w = s_{i_1} \dots s_{i_l}$. For this reduced expression of w, denote by β_j and $X_{\beta_j}^\pm$, $j = 1, \dots, l$ the roots and root vectors of $\mathcal{U}_\pm^w$ given by (2.34) and (2.35). Denote

$$\beta_{l+1} = s_{i_1} \dots s_{i_l}(\alpha_{i_{l+1}}) \text{ and } X_{\beta_{l+1}}^\pm = T_{i_1} \dots T_{i_l}(X_{i_{l+1}}^\pm) \in \mathcal{U}_\pm^{w'}.$$

Clearly $\mathcal{U}_\pm^w \subset \mathcal{U}_\pm^{w'}$.

Let $\psi \in \mathrm{Aut}(\mathcal{U}_\pm^{w'})$ and

$$\psi(X_{\beta_j}^\pm) = q_{i_j}^{k_j} X_{\beta_j}^\pm, \quad \forall j = 1, \dots, l+1, \tag{6.5}$$

for some $k_1, \ldots, k_{l+1} \in \mathbb{Z}$. Then ψ restricts to an automorphism of $\mathcal{U}^w_\pm$ satisfying the assumptions of the lemma. Applying the inductive assumption, we obtain that there exists $\delta_\pm \in P$ such that

$$\langle \delta_\pm, \pm\beta_j^\vee \rangle = k_j, \text{ i.e. } \langle \delta_\pm, \pm\beta_j \rangle = d_j k_j, \quad \forall j = 1, \ldots, l. \tag{6.6}$$

By Remark 3.5 we can assume that $\delta \in P_{\mathcal{S}(w)}$.

First, consider the case when there exists $j \in \{1, \ldots, l\}$ such that

$$X^\pm_{\beta_j} X^\pm_{\beta_{l+1}} - q^{\pm\langle \beta_j, \beta_{l+1}\rangle} X^\pm_{\beta_{l+1}} X^\pm_{\beta_j} \neq 0.$$

The Levendorskii–Soibelman straightening rule (2.37), the fact that $\mathcal{U}^{w'}_\pm$ is Q-graded by $\deg X^\pm_{\beta_1} = \pm\beta_1, \ldots, \deg X^\pm_{\beta_{l+1}} = \pm\beta_{l+1}$, and (6.6) imply:

$$\psi(X^\pm_{\beta_j} X^\pm_{\beta_{l+1}} - q^{\pm\langle \beta_j, \beta_{l+1}\rangle} X^\pm_{\beta_{l+1}} X^\pm_{\beta_j}) \\ = q^{\langle \delta_-, \pm(\beta_j + \beta_{l+1})\rangle} (X^\pm_{\beta_j} X^\pm_{\beta_{l+1}} - q^{\pm\langle \beta_j, \beta_{l+1}\rangle} X^\pm_{\beta_{l+1}} X^\pm_{\beta_j}).$$

Since $\mathcal{U}^w_\pm$ is a domain, from the above, (6.5) and (6.6), we obtain

$$\langle \delta_\pm, \pm\beta_{l+1}\rangle = d_{l+1} k_{l+1}.$$

Thus the weight $\delta_\pm$ for w also works for w', which proves the statement of the lemma.

Now consider the case when

$$X^-_{\beta_j} X^\pm_{\beta_{l+1}} - q^{-\langle \beta_j, \beta_{l+1}\rangle} X^\pm_{\beta_{l+1}} X^\pm_{\beta_j} = 0, \quad \forall j = 1, \ldots, l. \tag{6.7}$$

Theorem 2.4 implies that $X^\pm_{\beta_{l+1}}$ is a homogeneous prime element of $\mathcal{U}^{w'}_\pm$. Theorem 2.4 and (6.7) also imply that each homogeneous prime element of $\mathcal{U}^w_\pm$ is a prime element of $\mathcal{U}^{w'}_\pm$. Therefore the number of homogeneous prime elements of $\mathcal{U}^{w'}_\pm$ is strictly greater than that of $\mathcal{U}^w_\pm$. Lemma 6.4 and $|\mathcal{S}(w')| \leq |\mathcal{S}(w)| + 1$ imply $|\mathcal{S}(w')| = |\mathcal{S}(w)| + 1$. By part (ii) of Lemma 3.2

$$i_{l+1} \notin \mathcal{S}(w), \text{ i.e. } \alpha^\vee_{i_{l+1}} \notin Q^\vee_{\mathcal{S}(w)}. \tag{6.8}$$

Since $\beta_{l+1} = s_{i_1} \ldots s_{i_l}(\alpha_{i_{l+1}})$,

$$\beta^\vee_{l+1} = \alpha^\vee_{i_{l+1}} + \sum_{i \in \mathcal{S}(w)} m_i \alpha_i^\vee, \tag{6.9}$$

for some $\{m_i \in \mathbb{Z} \mid i \in \mathcal{S}(w)\}$. Set

$$\delta'_\pm = \delta_\pm \pm \Big(k_{l+1} \mp \sum_{i \in \mathcal{S}(w)} m_i \langle \delta_\pm, \alpha_i^\vee \rangle\Big)\omega_{l+1}.$$

Because $\delta_\pm \in P_{\mathcal{S}(w)}$, (6.8) implies that $\langle \delta_\pm, \alpha^\vee_{i_{l+1}}\rangle = 0$. Therefore

$$\langle \delta'_\pm, \pm\alpha^\vee_{i_{l+1}}\rangle = k_{l+1} - \sum_{i \in \mathcal{S}(w)} m_i \langle \delta_\pm, \pm\alpha_i^\vee \rangle. \tag{6.10}$$

From (6.8) we also obtain that

$$\langle \delta'_\pm, \alpha_i^\vee \rangle = \langle \delta_\pm, \alpha_i^\vee \rangle, \quad \forall i \in \mathcal{S}(w). \tag{6.11}$$

The induction hypothesis and Lemma 3.2 (ii) imply:

$$\langle \delta'_\pm, \pm\beta_j^\vee \rangle = \langle \delta_\pm, \pm\beta_j^\vee \rangle = k_j, \quad \forall j = 1, \ldots, l.$$

Combining (6.9), (6.10) and (6.11), we obtain

$$\langle \delta'_\pm, \pm\beta^\vee_{l+1}\rangle = \langle \delta'_\pm, \pm\alpha^\vee_{i_{l+1}}\rangle + \sum_{i\in\mathcal{S}(w)} m_i \langle \delta', \pm\alpha^\vee_i\rangle$$
$$= k_{l+1} - \sum_{i\in\mathcal{S}(w)} m_i \langle \delta_\pm, \pm\alpha^\vee_i\rangle + \sum_{i\in\mathcal{S}(w)} m_i \langle \delta'_\pm, \pm\alpha^\vee_i\rangle = k_{l+1},$$

which completes the proof of the lemma. □

Remark 6.6. Define an action of the torus $\mathbb{T}^{|\mathcal{S}(w)|} = (\mathbb{K}^*)^{\times|\mathcal{S}(w)|}$ on $\mathcal{U}^w_\pm$ by

$$t \cdot X^\pm_{\beta_j} = \Big(\prod_{i\in\mathcal{S}(w)} t_i^{\langle \omega^\vee_i, \beta_j\rangle}\Big) X^\pm_{\beta_j}, \tag{6.12}$$

for $t = (t_i)_{i\in\mathcal{S}(w)} \in \mathbb{T}^{|\mathcal{S}(w)|}$, in terms of the generators (2.35) of $\mathcal{U}^w_\pm$. Here $\omega^\vee_1, \ldots, \omega^\vee_r$ denote the fundamental coweights of $\mathfrak{g}$. This is an action by algebra automorphisms since the algebras $\mathcal{U}^w_\pm$ are $Q_{\mathcal{S}(w)}$-graded by (2.43). Analogously to the proof of Lemma 6.5 one shows:

If $\psi \in \mathrm{Aut}(\mathcal{U}^w_\pm)$ is such that

$$\psi(X^\pm_{\beta_j}) = p_j X^\pm_{\beta_j}, \quad \forall j = 1, \ldots, l,$$

for some $p_j \in \mathbb{K}^*$, then there exists $t \in \mathbb{T}^r$ such that $\psi(x) = t\cdot x$, $\forall x \in \mathcal{U}^w_\pm$.

6.4. **Proof of Proposition 6.3.** Assume that $u \in \mathcal{U}^w_\pm$ is a nonzero homogeneous normal element. We will prove that there exists $\delta_\pm \in P_{\mathcal{S}(w)}$ such that

$$u X^\pm_{\beta_j} = q^{\langle \delta_\pm, \pm\beta_j\rangle} X^\pm_{\beta_j} u, \quad \forall j = 1, \ldots, l.$$

Then applying the graded isomorphism from Theorem 2.6 would imply the statement of the proposition.

Fix $j \in \{1, \ldots, l\}$. Then

$$u X^\pm_{\beta_j} = Y_j u \ \text{ for some } \ Y \in (\mathcal{U}^w_\pm)_{\pm\beta_j}. \tag{6.13}$$

Recall the notation (2.39), and the notions of highest term of a nonzero element of $\mathcal{U}^w_\pm$ and degree of a monomial from §2.7. Assume that the highest term of u is has degree $\mathbf{n}$ for some $\mathbf{n} \in \mathbb{N}^{\times l}$. Denote $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in position j. Then, by Lemma 2.5 the highest term of the left hand side of (6.13) has degree $\mathbf{n} + \mathbf{e}_j$. Again applying Lemma 2.5, we obtain that the highest term of Y_j has degree $\mathbf{e}_j$, i.e. the highest term is a nonzero scalar multiple of $X^\pm_{\beta_j}$. At the same time $Y \in (\mathcal{U}^w_\pm)_{\pm\beta_j}$; that is

$$Y_j \in \mathrm{Span}\{(X^\pm)^{\mathbf{n}'} \mid \mathbf{n}' = (n'_1, \ldots, n'_l) \in \mathbb{N}^{\times l}, n'_1\beta_1 + \ldots + n'_l\beta_l = \beta_j\}. \tag{6.14}$$

It is well known that the ordering of the roots

$$\beta_1, \ldots, \beta_l \tag{6.15}$$

of $\Delta^+ \cap w(\Delta^-)$ is convex, i.e. if a root in (6.15) is equal to the sum of two other roots in (6.15), then it is listed in between. Moreover if a root of $\mathfrak{g}$ is the sum of two roots of $\Delta^+ \cap w(\Delta^-)$, then it belongs to $\Delta^+ \cap w(\Delta^-)$. This implies that if a root in the list (6.15) is a positive integral combination of several roots in (6.15), then it is listed between the leftmost and rightmost ones. This property and (6.14) imply that the highest term of Y_j will not be a nonzero scalar multiple of $X^{\pm}_{\beta_j}$ unless Y_j is itself a scalar multiple of $X^{\pm}_{\beta_j}$. Therefore

$$uX^{\pm}_{\beta_j} = p_j X^{\pm}_{\beta_j} u$$

for some $p_j \in \mathbb{K}^*$. Comparing the highest terms of both sides and using Lemma 2.5, we obtain

$$uX^{\pm}_{\beta_j} = q_{i_j}^{k_j} X^{\pm}_{\beta_j} u, \quad j = 1, \dots, l,$$

for some $k_j \in \mathbb{Z}$. Repeated applications of (2.37) give

$$\begin{aligned} k_j &= \pm \sum_{k=1}^{j-1} \frac{n_k \langle \beta_j, \beta_k \rangle}{d_{i_j}} \mp \sum_{k=j+1}^{l} \frac{n_k \langle \beta_j, \beta_k \rangle}{d_{i_j}} \\ &= \pm \sum_{k=1}^{j-1} n_k \langle \beta_j^\vee, \beta_k \rangle \mp \sum_{k=j+1}^{l} n_k \langle \beta_j^\vee, \beta_k \rangle \in \mathbb{Z}. \end{aligned}$$

Here we used the fact that $\langle \beta_j, \beta_j \rangle = \langle \alpha_{i_j}, \alpha_{i_j} \rangle = d_{i_j}$.

Applying Lemma 6.5, we obtain that there exists $\delta_{\pm} \in P_{\mathcal{S}(w)}$ such that $\langle \delta_{\pm}, \pm\beta_j \rangle = d_{i_j} k_j$ for all $j = 1, \dots, l$; that is

$$uX^{\pm}_{\beta_j} = q^{\langle \delta_{\pm}, \pm\beta_j \rangle} X^{\pm}_{\beta_j} u, \quad \forall j = 1, \dots, l.$$

This completes the proof of Proposition 6.3. □

6.5. **Proof of Theorem 6.1.** Denote by

$$M^{\pm}_w = \{ p d^{\pm}_{w,\lambda} \mid p \in \mathbb{K}^*, \lambda \in P^+_{\mathcal{S}(w)} \} \tag{6.16}$$

the multiplicative subset of all nonzero homogeneous normal elements of $S^{\pm}_w$, cf. Theorem 6.1 (i). We start with a lemma which narrows down the set of homogeneous normal elements of S^+_w.

Lemma 6.7. *The set of homogeneous normal elements of $S^{\pm}_w$ consists of those elements of $S^{\pm}_w[(M^{\pm}_w)^{-1}]$ which have the form*

$$p \prod_{i \in \mathcal{S}(w)} (d^{\pm}_{w,\omega_i})^{n_i}$$

for some $p \in \mathbb{K}$, $n_i \in \mathbb{Z}$ and belong to $S^{\pm}_w$. The product over i is taken in any fixed order as in (6.2).

Each reduced expression $w = s_{i_1} \dots s_{i_l}$ gives rise to a presentation of the algebra $\mathcal{U}^w_{\pm}$ as an iterated skew polynomial algebra

$$\mathbb{K}[X^{\pm}_{\beta_1}][X^{\pm}_{\beta_2}; \tau_2, \theta_2] \dots [X^{\pm}_{\beta_l}; \tau_{l(w)}, \theta_{l(w)}] \tag{6.17}$$

where for $j = 1, \dots, l(w)$, τ_j is an automorphism of $(j-1)$-st algebra in the extension and θ_j is a τ_j-derivation of the same algebra. (One constructs τ_j and θ_j from the Levendorskii–Soibelman straightening rule (2.37), see [38].) Moreover the following conditions are trivially satisfied (and also follow from the the property that $\mathcal{U}^w_{\pm}$ are CGL extensions):

(i) All $X^{\pm}_{\beta_1}, \dots, X^{\pm}_{\beta_l}$ are eigenvalues of H under the conjugation action.

(ii) For $j = 1, \dots, l(w)$ there exist elements of $H_j \in H$ such that $\tau_j(x_k) = H_j x_k H_j^{-1}$ for $j > k$ and such that the H_j eigenvalue of x_j is not a root of unity for all j.

Goodearl proved [6, Theorem 6.4.II] that, if A is an iterated skew polynomial algebra as in (6.17) which satisfies the properties (i)–(ii) above, then every H-prime I of A is strongly rational, i.e. $Z(\operatorname{Fract} A/I)^H = \mathbb{K}$. Strictly speaking we need to use the extension of the conjugation action of H on $\mathcal{U}^w_\pm$ to the torus action (6.12) (the H-invariant ideals being the same as the $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant ideals). Using the isomorphisms $\varphi^\pm_w \colon S^\pm_w \to \mathcal{U}^w_\mp$ (see Theorem 2.6) and applying this result to the $\{0\}$ ideals of the algebras $\mathcal{U}^w_\mp$, we obtain:

$$Z(S^\pm_w[(M^\pm_w)^{-1}])_{0,0} = \mathbb{K}. \tag{6.18}$$

Proof of Lemma 6.7. Assume that u is a nonzero homogeneous normal element of $S^\pm_w$. By Proposition 6.3 it is P-normal. We then apply Theorem 3.6 to obtain that there exists $\eta \in P_{\mathcal{S}(w)}$ such that $u \in (S^\pm_w)_{\pm(w-1)\eta,0}$ and

$$us = q^{\langle -(w+1)\eta, \gamma\rangle} su, \quad \forall s \in (S^\pm_w)_{-\gamma,0}, \gamma \in Q_{\mathcal{S}(w)}. \tag{6.19}$$

Let $\eta = \sum_{i \in \mathcal{S}(w)} n_i \omega_i$ for some $n_i \in \mathbb{Z}$. Denote

$$u' = \prod_{i \in \mathcal{S}(w)} (d^\pm_{w,\omega_i})^{n_i},$$

where the product over i is taken in any order. Then $u' \in (S^\pm_w)_{\pm(w-1)\eta,0}$ and

$$u's = q^{\langle -(w+1)\eta, \gamma\rangle} su', \quad \forall s \in (S^\pm_w)_{-\gamma,0}, \gamma \in Q_{\mathcal{S}(w)}, \tag{6.20}$$

recall (3.30). Eq. (6.19) and (6.20) imply

$$u(u')^{-1} \in Z(S^\pm_w[(M^\pm_w)^{-1}])_{0,0}$$

From (6.18) we obtain that $u(u')^{-1} \in \mathbb{K}^*$, i.e.

$$u = pu' = p \prod_{i \in \mathcal{S}(w)} (d^\pm_{w,\omega_i})^{n_i},$$

for some $p \in \mathbb{K}^*$. □

Proof Theorem 6.1. Part (i): Assume that $u \in S^\pm_w$ is a nonzero homogeneous normal element. Lemma 6.7 implies that it is given by (6.2) for some $p \in \mathbb{K}^*$, $n_i \in \mathbb{Z}$. We claim that $n_i \in \mathbb{N}$ for all $i \in \mathcal{S}(w)$. Assume that this is not the case. Then the element u would be linearly independent from the set $d^\pm_{w,\lambda}$, $\lambda \in P^+_{\mathcal{S}(w)}$. Indeed, if this is not the case, then after multiplying it with $d^\pm_{w,\mu}$ for some $\mu \in P^+_{\mathcal{S}(w)}$ we will get a linear dependence in the set $\{d^\pm_{w,\lambda}\}_{\lambda \in P^+_{\mathcal{S}(w)}}$, which contradicts with the first part of Theorem 5.1. Therefore $u \notin N^\pm_w$ and for some $\mu \in P^+_{\mathcal{S}(w)}$, $d^\pm_{w,\mu} u \in N^\pm_w$. This contradicts with the fact that $S^\pm_w$ is a free (left and right) $N^\pm_w$-module (by Theorem 5.1 (ii)) and completes the proof of part (i) of Theorem 6.1.

Part (ii): By the first part of the theorem each homogeneous normal element of $S^{\pm}_w$ has the form (6.2), for some $p \in \mathbb{K}^*$, $n_i \in \mathbb{N}$. Therefore the set of homogeneous prime elements of $S^{\pm}_w$ is a subset of $\{p d^{\pm}_{w,\omega_i} \mid p \in \mathbb{K}^*, i \in \mathcal{S}(w)\}$. By Lemma 6.4 $S^{\pm}_w$ has at least $|\mathcal{I}(w)|$ pairwise nonproportional homogeneous prime elements. This is only possible if all $d^{\pm}_{w,\omega_i}$ are prime elements of $S^{\pm}_w$, for all $i \in \mathcal{S}(w)$. They are linearly independent because of Theorem 5.1 (i). □

As a corollary of Theorem 6.1 (ii), we obtain explicit formulas and generators for the height one H-primes $I^{\pm}_w(s_i)$, $i \in \mathcal{S}(w)$ of $S^{\pm}_w$, (recall (6.3)–(6.4)) under the general conditions on $\mathbb{K}$ and q.

Proposition 6.8. *For any base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, $w \in W$, and $i \in \mathcal{S}(w)$:*

$$I^{\pm}_w(s_i) = S^{\pm}_w d^{\pm}_{w,\omega_i}. \tag{6.21}$$

Proof. Combining the Launois–Lenagan–Rigal result [31] that $\mathcal{U}^w_{\mp}$ is a unique factorization domain, the fact that the ideals $I^{\pm}_w(s_i)$ are height one prime ideals of $S^{\pm}_w$ and part (ii) of Theorem 6.1, we obtain that for each $i \in \mathcal{S}(w)$ there exists $k \in \mathcal{S}(w)$ such that

$$I^{\pm}_w(s_i) = S^{\pm}_w d^{\pm}_{w,\omega_k}.$$

Since $d^{\pm}_{w,\omega_i} \in I^{\pm}_w(s_i)$, this is only possible if $k = i$, which establishes (6.21). □

6.6. **Prime and primitive ideals in the $\{0\}$-stratum of** $\mathrm{Spec} S^{\pm}_w$**.** As an application of Theorems 5.1 and 6.1, we obtain a formula for the prime (and a more explicit formula for the primitive) ideals of the algebras $S^{\pm}_w$ lying in the Goodearl–Letzter stratum over the $\{0\}$ ideal. As usual, some of the results for primitive ideals require the base field $\mathbb{K}$ to be algebraically closed. There will be no such restriction for the results on prime ideals, which are valid for arbitrary base fields $\mathbb{K}$. Similar arguments are applied in the next subsection to obtain a classification of all prime elements of the algebras $S^{\pm}_w$ (and $\mathcal{U}^w_{\pm}$). In particular, this gives explicit formulas for all height one prime ideals of $S^{\pm}_w$.

Via the isomorphism of Theorem 2.6 these results give similar explicit formulas for the prime/primitive ideals in the $\{0\}$-stratum of $\mathrm{Spec}\mathcal{U}^w_{\mp}$. The restatement is straightforward and will not be formulated separately.

Recall [17] that the $\{0\}$-stratum of $\mathrm{Spec} S^{\pm}_w$ is defined by

$$\mathrm{Spec}_{\{0\}} S^{\pm}_w = \{I \in \mathrm{Spec} S^{\pm}_w \mid \cap_{t \in \mathbb{T}^r} t \cdot I = \{0\},$$

where we use the rational action (2.32) of $\mathbb{T}^r$ on $S^{\pm}_w$. Set $\mathrm{Prim}_{\{0\}} S^{\pm}_w = \mathrm{Prim}\, S^{\pm}_w \cap \mathrm{Spec}_{\{0\}} S^{\pm}_w$.

Recall also that $M^{\pm}_w$ denotes the multiplicative subset (6.16) of all nonzero homogeneous normal elements of $S^{\pm}_w$, see Theorem 6.1 (i). First, we obtain a description of the center $Z(S^{\pm}_w[(M^{\pm}_w)^{-1}])$. Each $\mu \in P_{\mathcal{S}(w)}$ can be represented in a unique way as $\mu = \mu_+ - \mu_-$ for some $\mu_+, \mu_- \in P^+_{\mathcal{S}(w)}$ with disjoint support, see (2.1). For $\mu \in P_{\mathcal{S}(w)}$ define

$$d^{\pm}_{w,\mu} = (d^{\pm}_{w,\mu_-})^{-1} d^{\pm}_{w,\mu_+} \in N^{\pm}_w[(M^{\pm}_w)^{-1}]. \tag{6.22}$$

It follows from (3.30) and (3.31) that for all $\mu, \mu' \in P_{\mathcal{S}(w)}$,

$$d^{\pm}_{w,\mu} d^{\pm}_{w,\mu'} = q^{j(\mu,\mu')} d^{\pm}_{w,\mu+\mu'}, \text{ for some } j(\mu,\mu') \in \mathbb{Z}. \tag{6.23}$$

(Recall that $d^{\pm}_{w,\mu} \in \mathbb{K}^*$ for all $\mu \in P^+_{\mathcal{I}(w)}$. Because of this and (3.31), one does not need to extend the definition (6.22) to $\mu \in P$.) Theorem 5.1 (i) implies that the localization $N^{\pm}_w[(M^{\pm}_w)^{-1}]$ is isomorphic a quantum torus over $\mathbb{K}$ of dimension $|\mathcal{S}(w)|$. In particular,

$$\{d^{\pm}_{w,\mu} \mid \mu \in P_{\mathcal{S}(w)}\} \text{ is a } \mathbb{K}\text{-basis of } N^{\pm}_w[(M^{\pm}_w)^{-1}]. \tag{6.24}$$

Applying (3.30) we also obtain

(6.25)

$$d^{\pm}_{w,\mu} s = q^{-\langle (w+1)\mu, \gamma\rangle} s d^{\pm}_{w,\mu}, \quad \forall \mu \in P_{\mathcal{S}(w)}, s \in (S^{\pm}_w[(M^{\pm}_w)^{-1}])_{-\gamma,0}, \gamma \in Q_{\mathcal{S}(w)}.$$

Define the lattice

$$\begin{aligned} \mathcal{K}(w) &= \{\mu \in P_{\mathcal{S}(w)} \mid (w+1)\mu \in (Q_{\mathcal{S}(w)})^{\perp}\} \\ &= \{\mu \in P_{\mathcal{S}(w)} \mid (w+1)\mu \in P_{\mathcal{I}(w)}\} \\ &= \{\mu \in P_{\mathcal{S}(w)} \mid \exists \nu \in P_{\mathcal{I}(w)} \text{ such that } \mu + \nu/2 \in \ker(w+1)\}. \end{aligned} \tag{6.26}$$

The first equality follows from $(Q_{\mathcal{S}(w)})^{\perp} \cap P = P_{\mathcal{I}(w)}$. The second equality follows from the fact that $w(\nu) = \nu$ for all $\nu \in P_{\mathcal{I}(w)}$, thus for any $\mu \in P$ and $\nu \in P_{\mathcal{I}(w)}$:

$$(w+1)\mu = \nu, \text{ if and only if } (w+1)(\mu - \nu/2) = 0.$$

The lattice $\mathcal{K}(w)$ has rank

$$m(w) := \dim \ker(w+1). \tag{6.27}$$

To see this, denote the projection $\sigma \colon P_{\mathcal{S}(w)} \oplus P_{\mathcal{I}(w)}/2 \to P_{\mathcal{S}(w)}$ along $P_{\mathcal{I}(w)}/2$. The third equality in (6.26) implies that

$$\mathcal{K}(w) = \sigma\big(\ker(w+1) \cap (P_{\mathcal{S}(w)} \oplus P_{\mathcal{I}(w)}/2)\big).$$

The statement follows from the facts that $\ker(w+1) \cap (P_{\mathcal{S}(w)} \oplus P_{\mathcal{I}(w)}/2)$ is a lattice of rank $\dim \ker(w+1)$ and the restriction

$$\sigma \colon \ker(w+1) \cap (P_{\mathcal{S}(w)} \oplus P_{\mathcal{I}(w)}/2) \to \mathcal{K}(w)$$

is bijective.

Fix a basis $\mu^{(1)}, \ldots, \mu^{(m(w))}$ of $\mathcal{K}(w)$. For $\mu = k_1 \mu^{(1)} + \ldots + k_{m(w)} \mu^{(m(w))} \in \mathcal{K}(w)$ define

$$e^{\pm}_{w,\mu} = (d^{\pm}_{w,\mu^{(1)}})^{k_1} \ldots (d^{\pm}_{w,\mu^{(m(w))}})^{k_{m(w)}}.$$

We have $e^{\pm}_{w,\mu} e^{\pm}_{w,\mu'} = e^{\pm}_{w,\mu+\mu'}$, for all $\mu, \mu' \in \mathcal{K}(w)$. By (6.23) for all $\mu \in \mathcal{K}(w)$,

$$e^{\pm}_{w,\mu} = q^{j_\mu} d^{\pm}_{w,\mu}, \quad \text{for some } i_\mu \in \mathbb{Z}. \tag{6.28}$$

Denote by $A^{\pm}_w$ the subalgebra of the quantum torus $N^{\pm}_w[(M^{\pm}_w)^{-1}]$ generated by

$$d^{\pm}_{w,\mu^{(i)}}, (d^{\pm}_{w,\mu^{(i)}})^{-1}, \quad i = 1, \ldots, m(w). \tag{6.29}$$

Theorem 5.1 (i) and (6.25) imply that $A^{\pm}_w$ is a Laurent polynomial algebra over $\mathbb{K}$ of dimension $m(w)$ with generators (6.29). The set

$$\{e^{\pm}_{w,\mu} \mid \mu \in \mathcal{K}(w)\} \tag{6.30}$$

is a $\mathbb{K}$-basis of $A_w^\pm$. Of course, the same is true for the set $\{d^\pm_{w,\mu} \mid \mu \in \mathcal{K}(w)\}$.

Remark 6.9. The one half in (6.26) is needed as is shown by the following simple example. Let $\mathfrak{g} = \mathfrak{sl}_3$, $w = s_1$. Then $\mathcal{S}(s_1) = \{1\}$ and $\mathcal{I}(s_1) = \{2\}$. Moreover, $\mathcal{K}(s_1) = \mathbb{Z}\omega_1$ and

$$\omega_1 - \omega_2/2 \in \ker(s_1 + 1),$$

cf. the third equality in (6.26). In this case $S^\pm_{s_1} = \mathbb{K}[d^\pm_{s_1,\omega_1}]$ is a polynomial ring and $A^\pm_w = \mathbb{K}[d^\pm_{s_1,\omega_1}, (d^\pm_{s_1,\omega_1})^{-1}]$.

Lemma 6.10. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and $w \in W$, the center $Z(S^\pm_w[(M^\pm_w)^{-1}])$ coincides with the algebra $A^\pm_w$.*

The special case of Lemma 6.10 when w is equal to the longest element w_0 of W is due to Caldero [7]. The fact that $\mathrm{Spec}_{\{0\}}\mathcal{U}^w_\pm$ has dimension equal to $\dim\ker(w+1)$ is due to Bell and Launois [2].

Remark 6.11. Theorem 5.1 (i) and Lemma 6.10 imply that the center of the algebra $S^\pm_w$ is the algebra

$$Z(S^\pm_w) = \{d^\pm_{w,\mu} \mid \mu \in \mathcal{K}(w) \cap P^+_{\mathcal{S}(w)}\}.$$

Often this algebra is much smaller than $A^\pm_w$. For instance in many cases it is trivial while $A^\pm_w$ is not. This is why conceptually it was more important to study the structure of $S^\pm_w$ as an $N^\pm_w$-module rather than $Z(S^\pm_w)$-module.

Proof of Lemma 6.10. It follows from of (6.25) and the first equality in (6.26) that $A^\pm_w \subseteq Z(S^\pm_w[(M^\pm_w)^{-1}])$. To show the opposite inclusion, let $(d^\pm_{w,\lambda})^{-1}u \in Z(S^\pm_w[(M^\pm_w)^{-1}])$ for some homogeneous element $u \in S^\pm_w$ and $\lambda \in P^+_{\mathcal{S}(w)}$. Then $u \in S^\pm_w$ should be a homogeneous normal element. Applying Theorem 6.1 (i) and (6.23), we obtain that

$$(d^\pm_{w,\lambda})^{-1}u = p d^\pm_{w,\mu} \tag{6.31}$$

for some $p \in \mathbb{K}^*$, $\mu \in P_{\mathcal{S}(w)}$. Using (6.25), we obtain that $d^\pm_{w,\mu} \in Z(S^\pm_w[(M^\pm_w)^{-1}])$ if and only if $\mu \in \mathcal{K}(w)$, cf. the first equality in (6.26). Therefore $(d^\pm_{w,\lambda})^{-1}u$ is a scalar multiple of one of the elements in (6.30), and thus belongs to $A^\pm_w$. □

In [45] we prove that $\mathrm{Spec}\mathcal{U}^w_\pm$ is normally separated, under the same general assumption on $\mathbb{K}$ and q as the ones in this paper. Using the isomorphism from Theorem 2.6, we obtain that the same is true for the algebras $S^\pm_w$. By [13, Theorems 5.3 and 5.5] every prime ideal in $\mathrm{Spec}_{\{0\}}S^\pm_w$ is of the form

$$(S^\pm_w[(M^\pm_w)^{-1}].J^0) \cap S^\pm_w, \tag{6.32}$$

for some prime ideal J^0 of $Z(S^\pm_w[(M^\pm_w)^{-1}])$. Moreover each primitive ideal in $\mathrm{Prim}_{\{0\}} S^\pm_w$ has the form (6.32) for a maximal ideal J^0 of $Z(S^\pm_w[(M^\pm_w)^{-1}])$. Applying the freeness result Theorem 5.1 (ii) and Lemma 6.10 leads to the following result. Its proof is straightforward and will be omitted.

Proposition 6.12. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, and $w \in W$. Then:*

(i) All prime ideals in $\mathrm{Spec}_{\{0\}} S^{\pm}_w$ *have the form*

$$\big((J^0.N^{\pm}_w[(M^{\pm}_w)^{-1}]) \cap N^{\pm}_w\big).S^{\pm}_w \tag{6.33}$$

for some prime ideal J^0 *of the Laurent polynomial ring* $A^{\pm}_w$*, see Lemma 6.10.*

(ii) The primitive ideals in $\mathrm{Prim}_{\{0\}} S^{\pm}_w$ *are the ideals given by* (6.33) *for maximal ideals* J^0 *of* $A^{\pm}_w$*.*

The point of Proposition 6.12 is that it reduces the possibly complicated intersections from (6.32) in the algebras $S^{\pm}_w$ to the intersections (6.33) inside the quantum affine space algebras $N^{\pm}_w$. The latter intersections are obviously much simpler. Moreover the centers $Z(S^{\pm}_w[(M^{\pm}_w)^{-1}])$ are substituted by the explicit Laurent polynomial algebras $A^{\pm}_w$.

Next, we proceed with making the description from part (ii) of Proposition 6.12 even more explicit. For $\lambda, \lambda' \in P^+_{\mathcal{S}(w)}$, such that

$$\lambda' - \lambda = k_1 \mu^{(1)} + \ldots + k_{m(w)} \mu^{(m(w))} \in \mathcal{K}(w)$$

set

$$n^{\pm}_{\lambda,\lambda'} = \mp 2 \sum_i k_i \langle \mu^{(i)}, \lambda \rangle \mp 2 \sum_{j<i} k_j \langle \mu^{(j)}, \mu^{(i)} \rangle \\ \mp \sum_i |k_i|(|k_i|-1)\langle \mu^{(i)}, \mu^{(i)} \rangle \pm 2 \sum_i k_i \langle \mu^{(i)}, \mu^{(i)}_{-\mathrm{sign}(k_i)} \rangle. \tag{6.34}$$

Applying repeatedly (3.30) and (3.31), and using the fact that $(w-1)\mu^{(i)} = -2\mu^{(i)}$, because $\mu^{(i)} \in \ker(w+1)$, gives

$$d^{\pm}_{w,\lambda} e^{\pm}_{w,\lambda'-\lambda} = q^{n^{\pm}_{\lambda,\lambda'}} d^{\pm}_{w,\lambda'}, \quad \forall \lambda, \lambda' \in P^+_{\mathcal{S}(w)} \text{ such that } \lambda' - \lambda \in \mathcal{K}(w). \tag{6.35}$$

We leave the details of this long but straightforward computation to the reader.

Denote by $J^0_{w,\mathbf{1}}$ the maximal ideal of $A^{\pm}_w$ generated by

$$d^{\pm}_{w,\mu^{(i)}} - 1, i = 1, \ldots, m(w).$$

Fix a set $\Lambda_w \subset P_{\mathcal{S}(w)}$ of representatives of $P_{\mathcal{S}(w)}/\mathcal{K}(w)$, recall (6.26). Let $\Lambda^+_w \subset P^+_{\mathcal{S}(w)}$ be a set of representatives of those cosets in $P_{\mathcal{S}(w)}/\mathcal{K}(w)$ that intersect $P^+_{\mathcal{S}(w)}$ nontrivially.

Denote by $J_{w,\mathbf{1}}$ the subspace of $N^{\pm}_w$, which is the span over $\lambda \in \Lambda^+_w$ of all elements of the form

$$\sum_{\mu \in \mathcal{K}(w) \cap (-\lambda + P^+_{\mathcal{S}(w)})} p_{\lambda,\lambda+\mu} d^{\pm}_{w,\lambda+\mu},$$

for $p_{\lambda,\lambda+\mu} \in \mathbb{K}$ such that

$$\sum_{\mu \in \mathcal{K}(w) \cap (-\lambda + P^+_{\mathcal{S}(w)})} q^{-n^{\pm}_{\lambda,\lambda+\mu}} p_{\lambda,\lambda+\mu} = 0.$$

The next lemma proves that $J_{w,\mathbf{1}}$ is an ideal of $N^{\pm}_w$ and relates it to the setting of Proposition 6.12.

Lemma 6.13. *Let $w \in W$.*

(i) One has

$$J^0_{w,\mathbf{1}}.N^\pm_w[(M^\pm_w)^{-1}] = \Big\{ \sum_{\lambda \in \Lambda_w} \sum_{\mu \in \mathcal{K}(w)} p_{\lambda,\lambda+\mu} d^\pm_{w,\lambda} e^\pm_{w,\mu} \,\Big|\, p_{\lambda,\lambda+\mu} \in \mathbb{K}, \\ \sum_{\mu \in \mathcal{K}(w)} p_{\lambda,\lambda+\mu} = 0, \forall \lambda \in \Lambda_w \Big\}.$$

(ii) The ideal $\big(J^0_{w,\mathbf{1}}.N^\pm_w[(M^\pm_w)^{-1}]\big) \cap N^\pm_w$ of $N^\pm_w$ equals $J_{w,\mathbf{1}}$.

Proof. (i) Since $\{d^\pm_{w,\mu} \mid \mu \in P_{\mathcal{S}(w)}\}$ is a $\mathbb{K}$-basis of $N^\pm_w[(M^\pm_w)^{-1}]$, from (6.23), (6.28) and the definition of Λ_w, we obtain that

$$\{d^\pm_{w,\lambda} e^\pm_{w,\mu} \mid \lambda \in \Lambda_w, \mu \in \mathcal{K}(w)\} \text{ is a } \mathbb{K}\text{-basis of } N^\pm_w[(M^\pm_w)^{-1}]. \tag{6.36}$$

The statement of part (i) now follows from the definition of the ideal $J^0_{w,\mathbf{1}}$, and the facts that $e^\pm_{w,\mu} e^\pm_{w,\mu'} = e^\pm_{w,\mu+\mu'}$, $\forall \mu, \mu' \in \mathcal{K}(w)$ and $e^\pm_{w,\mu^{(i)}} = d^\pm_{w,\mu^{(i)}}$, $\forall i = 1, \dots m(w)$.

(ii) Part (i), (6.28), (6.36) and Theorem 5.1 (i) imply that $\big(J^0_{w,\mathbf{1}}.N^\pm_w[(M^\pm_w)^{-1}]\big) \cap N^\pm_w$ is equal to the ideal of $N^\pm_w$, which is the span over $\lambda \in \Lambda^+_w$ of all elements of the form

$$\sum_{\mu \in \mathcal{K}(w) \cap (-\lambda + P^+_{\mathcal{S}(w)})} p_{\lambda,\lambda+\mu} d^\pm_{w,\lambda} e^\pm_{w,\mu}, \tag{6.37}$$

for $p_{\lambda,\lambda+\mu} \in \mathbb{K}$ such that

$$\sum_{\mu \in \mathcal{K}(w) \cap (-\lambda + P^+_{\mathcal{S}(w)})} p_{\lambda,\lambda+\mu} = 0.$$

It follows from (6.35) that this is exactly the ideal $J_{w,\mathbf{1}}$. $\square$

Theorem 6.14. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, and $w \in W$. Then $J_{w,\mathbf{1}} S^\pm_w$ is a primitive ideal in $\operatorname{Prim}_{\{0\}} S^\pm_w$. If the field $\mathbb{K}$ is algebraically closed, then the primitive ideals of $S^\pm_w$ in the $\{0\}$-stratum of $\operatorname{Prim} S^\pm_w$ are the ideals*

$$t \cdot (J_{w,\mathbf{1}} S^\pm_w)$$

for $t \in \mathbb{T}^r$, with respect to the action (2.32).

In the special case when $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and w is equal to longest element w_0 of W, Theorem 6.14 and Corollary 6.15 below are due to Lopes, [35].

We note that the freeness result of Theorem 5.5 provides an explicit formula for the primitive ideal $J_{w,\mathbf{1}} S^\pm_w$ of $S^\pm_w$. Indeed, we have that for each reduced expression $\vec{w}$ of w:

$$J_{w,\mathbf{1}} S^\pm_w = \bigoplus_{\mathbf{n} \in \Delta(\vec{w})} J_{w,\mathbf{1}} \cdot (\varphi^\pm_w)^{-1}\big((X^\mp)^{\mathbf{n}}\big), \tag{6.38}$$

cf. (5.11).

Proof of Theorem 6.14. The theorem follows from Proposition 6.12 (ii), Lemma 6.13 (i), and the fact that $\operatorname{Prim}_{\{0\}} S^\pm_w$ is a single $\mathbb{T}^r$-orbit under (2.32), which

which is a consequence of [13, Theorem 5.5]. □

The definition of the ideals $J_{w,\mathbf{1}}$ gives immediately efficient generating sets for the ideals $J_{w,\mathbf{1}} S^{\pm}_w$. Represent each $\mu \in \mathcal{K}(w)$ as $\mu = \mu_+ - \mu_-$ for $\mu_+, \mu_- \in P^+_{\mathcal{S}(w)}$ with disjoint support, cf. (2.1). Then (6.35) implies

$$d^{\pm}_{w_\pm, \mu_-}(1 - e^{\pm}_{w_\pm, \mu}) = d^{\pm}_{w_\pm, \mu_-} - q^{n^{\pm}_{\mu_-, \mu_+}} d^{\pm}_{w_\pm, \mu_-} \in J_{w,\mathbf{1}}. \tag{6.39}$$

For all $\mu \in \mathcal{K}(w)$, the above are normal elements of the algebras $N^{\pm}_w$ and $S^{\pm}_w$ since $e^{\pm}_{w_\pm, \mu} \in Z(S^{\pm}_w[(M^{\pm}_w)^{-1}])$. From this and the definition of $J_{w,\mathbf{1}}$, we obtain

$$J_{w,\mathbf{1}} = \sum_{\mu \in \mathcal{K}(w)} (d^{\pm}_{w_\pm, \mu_-} - q^{n^{\pm}_{\mu_-, \mu_+}} d^{\pm}_{w_\pm, \mu_-}) N^{\pm}_w \tag{6.40}$$

and

$$J_{w,\mathbf{1}} S^{\pm}_w = \sum_{\mu \in \mathcal{K}(w)} (d^{\pm}_{w_\pm, \mu_-} - q^{n^{\pm}_{\mu_-, \mu_+}} d^{\pm}_{w_\pm, \mu_-}) S^{\pm}_w. \tag{6.41}$$

In each particular case one easily isolates a finite generating subset in (6.40) (consisting of elements of the form (6.39) for μ in a finite subset of $\mathcal{K}(w)$).Then the same set (of normal elements of $S^{\pm}_w$) generates the ideal $J_{w,\mathbf{1}} S^{\pm}_w$. Here is a simple general example of this.

Corollary 6.15. *Assume that $w \in W$ is such that the lattice $\mathcal{K}(w)$ has a basis $\mu^{(1)}, \ldots \mu^{(m(w))}$, consisting of elements of $P^+_{\mathcal{S}(w)}$ with pairwise disjoint support. Then*

$$J_{w,\mathbf{1}} S^{\pm}_w = \sum_{i=1}^{m(w)} (1 - d^{\pm}_{w, \mu^{(i)}}) S^{\pm}_w. \tag{6.42}$$

Proof. The condition on the element w implies that for every $\lambda \in P^+_{\mathcal{S}(w)}$ there exists $\lambda_{\min} \in P^+_{\mathcal{S}(w)}$ such that

$$(\lambda + \mathcal{K}(w)) \cap P^+_{\mathcal{S}(w)} = \lambda_{\min} + \big(\mathbb{N}\mu^{(1)} \oplus \ldots \oplus \mathbb{N}\mu^{(m(w))}\big).$$

We can then choose Λ^+_w to be the set of all such elements $\lambda_{\min}$. We have

$$P^+_{\mathcal{S}(w)} = \Lambda^+_w \oplus \big(\mathbb{N}\mu^{(1)} \oplus \ldots \oplus \mathbb{N}\mu^{(m(w))}\big).$$

It follows from (6.37) that

$$J_{w,\mathbf{1}} = \sum_{i=1}^{m(w)} (1 - d^{\pm}_{w, \mu^{(i)}}) N^{\pm}_w$$

which implies (6.42). □

6.7. **A classification of the prime elements of $S_w^\pm$.** As another application of Theorem 6.1 we obtain a classification of all inhomogeneous prime elements of the algebras $S_w^\pm$. When this is combined with Theorem 6.1 (ii), it gives a classification of all prime elements of the algebras $S_w^\pm$. The isomorphism from Theorem 2.6 gives an analogous classification of all prime elements of the algebras $\mathcal{U}^w_\pm$. The formulation of the latter is straightforward and will not be stated separately.

For $n \in \mathbb{N}$ denote

$$\mathrm{Irr}^0_n(\mathbb{K}) = \{ f(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n] \mid f(x) \text{ is irreducible and } f(0, \ldots, 0) = 1 \}. \tag{6.43}$$

Recall the definition (6.26) of the lattice $\mathcal{K}(w) \subset P_{\mathcal{S}(w)}$ and the notation $m(w) = \dim \ker(w+1)$. Recall from the previous subsection that $\mu^{(1)}, \ldots, \mu^{(m(w))}$ denotes a fixed basis of $\mathcal{K}(w)$.

For each $f(x_1, \ldots, x_{m(w)}) \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$ there exists a unique $\lambda_f \in P^+_{\mathcal{S}(w)}$ such that

$$d^\pm_{w,\lambda_f} f(d^\pm_{w,\mu^{(1)}}, \ldots, d^\pm_{w,\mu^{(m(w))}}) = \sum_{\lambda' \in P^+_{\mathcal{S}(w)}} p_{\lambda'} d^\pm_{w,\lambda'} \in N^\pm_w$$

and

$$\cap \{ \mathrm{Supp}\, \lambda' \mid \lambda' \in P^+_{\mathcal{S}(w)}, p_{\lambda'} \neq 0 \} = \emptyset, \tag{6.44}$$

recall (2.1). We denote

$$\widehat{f} = d^\pm_{w,\lambda_f} f(d^\pm_{w,\mu^{(1)}}, \ldots, d^\pm_{w,\mu^{(m(w))}}) \in N^\pm_w. \tag{6.45}$$

Since the second factor belongs to the center of $S^\pm_w[(M^\pm_w)^{-1}]$, we have from (3.30) that $\widehat{f} \in S^\pm_w$ is normal and more precisely:

$$\widehat{f} s = q^{-\langle (w+1)\lambda_f, \gamma \rangle} s \widehat{f}, \quad \forall s \in (S^\pm_w)_{-\gamma,0}, \gamma \in Q_{\mathcal{S}(w)}. \tag{6.46}$$

It follows from (6.44) that

$$(d^\pm_{w,\omega_i})^{-1} \widehat{f} \notin S^\pm_w, \quad \forall i \in \mathcal{S}(w), \tag{6.47}$$

because of (6.23), Theorem 5.1 (ii) and (6.24).

The next theorem contains our classification result for the inhomogeneous prime elements of $S^\pm_w$. Equivalently, it provides an explicit description of the height one prime ideals of $S^\pm_w$ which are not $\mathbb{T}^r$-invariant with respect to (2.32). The latter is an example of a case in which the formula from Proposition 6.12 (i) for the prime ideals in $\mathrm{Spec}_{\{0\}} S^\pm_w$ simplifies.

Theorem 6.16. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity and $w \in W$. Then every inhomogeneous prime element of $S^\pm_w$ is of the form $p\widehat{f}$, for some $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$ and $p \in \mathbb{K}^*$, cf. (6.27) and (6.43). All such elements are distinct.*

The height one prime ideals of $S^\pm_w$ which are not $\mathbb{T}^r$-invariant with respect to (2.32) have the form $\widehat{f} S^\pm_w$, for some $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$. All such ideals are distinct.

Proof. All height one prime ideals of $S_w^{\pm}$ are the $\mathbb{T}^r$-invariant height one prime ideals with respect to (2.32) (which as mentioned in §6.2 are the ideals $I_w^{\pm}(s_i)$ for $i \in \mathcal{S}(w)$) and the height one prime ideals in $\mathrm{Spec}_{\{0\}} S_w^{\pm}$. The latter family consists of ideals which are not $\mathbb{T}^r$-invariant with respect to (2.32). By [13, Theorem 5.3] every height one prime ideal in $\mathrm{Spec}_{\{0\}} S_w^{\pm}$ is of the form

$$(S_w^{\pm}[(M_w^{\pm})^{-1}].J^0) \cap S_w^{\pm} \tag{6.48}$$

for some height one prime ideal J^0 of $Z(S_w^{\pm}[(M_w^{\pm})^{-1}]) = A_w^{\pm}$, cf. (6.32), and all such ideals are distinct. We have

$$Z(S_w^{\pm}[(M_w^{\pm})^{-1}]) = A_w^{\pm} \cong \mathbb{K}[x_1^{\pm 1}, \ldots, x_{m(w)}^{\pm 1}], \quad d_{w,\mu^{(i)}}^{\pm} \mapsto x_i, i = 1, \ldots, m(w).$$

Each height one prime ideal of the Laurent polynomial ring $\mathbb{K}[x_1^{\pm 1}, \ldots, x_{m(w)}^{\pm 1}]$ is generated by a prime element. Each prime element of $\mathbb{K}[x_1^{\pm 1}, \ldots, x_{m(w)}^{\pm 1}]$ is uniquely represented as a product $uf(x_1, \ldots, x_{m(w)})$ where u is a unit of the Laurent polynomial ring (i.e. a nonzero Laurent monomial) and $f(x_1, \ldots, x_{m(w)}) \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$. Therefore each height one prime ideal of $S_w^{\pm}$ has the form

$$J(f) = \{s \in S_w^{\pm} \mid \exists \lambda \in P^+_{\mathcal{S}(w)} \text{ such that } d_{w,\lambda}^{\pm} s \in S_w^{\pm} \widehat{f}\}$$

for some $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$, because (6.45) implies $S_w^{\pm}[(M_w^{\pm})^{-1}] f = S_w^{\pm}[(M_w^{\pm})^{-1}] \widehat{f}$. We claim that

$$J(f) = S_w^{\pm} \widehat{f}, \quad \forall f \in \mathrm{Irr}^0_{w(w)}(\mathbb{K}).$$

To prove this, all we need to show is that for $s, s' \in S_w^{\pm}$,

$$d_{w,\omega_i}^{\pm} s = s' \widehat{f} \quad \Rightarrow \quad s \in S_w^{\pm} \widehat{f}. \tag{6.49}$$

Since $d_{w,\omega_i}^{\pm} \in S_w^{\pm}$ is prime (see Theorem 6.1 (ii)), if $d_{w,\omega_i}^{\pm} s = \widehat{f} s'$ then either s' or $\widehat{f}$ is a multiple of $d_{w,\omega_i}^{\pm}$. The second is not possible because of (6.47). Hence $s' = d_{w,\omega_i}^{\pm} s''$ for some $s'' \in S_w^{\pm}$ and thus $s = s'' \widehat{f} \in S_w^{\pm} \widehat{f}$.

Therefore all height one prime ideals have the form

$$J(f) = S_w^{\pm} \widehat{f}$$

for some $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$ and all such ideals are distinct. This implies that every inhomogeneous prime element of $S_w^{\pm}$ is of the form $p\widehat{f}$, for some $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$ and $p \in \mathbb{K}^*$, and these elements are distinct. To show that the ideals $J(f)$ are completely prime for all $f \in \mathrm{Irr}^0_{m(w)}(\mathbb{K})$, one either applies [31, Proposition 4.2] to conclude that all height one prime ideals of $S_w^{\pm} \cong \mathcal{U}^w_{\pm}$ are generated by prime elements, or (6.46) and the fact [16, Theorem 2.1] that all prime ideals of $S_w^{\pm} \cong \mathcal{U}^w_{\mp}$ are completely prime. □

7. Module structure of R_{w} over their subalgebras generated by Joseph's normal elements

7.1. **Statement of the freeness result.** In this section we analyze the structure of R_{w} as a module over its subalgebra generated by the Joseph set of normal elements $E_{\mathrm{w}}^{\pm 1}$, recall (2.23). We prove that S_{w} is a free module over its subalgebra generated by the normal elements y_{ω_i}, $i = 1, \ldots, r$. We use this to prove that

$R_{\rm w}$ is a free module over its subalgebra generated by the set $E_{\rm w}^{\pm 1}$. This result will be the main tool in classifying $\mathrm{Max}R_q[G]$ in the next section, which in turn will be used to answer affirmatively a question of Goodearl and Zhang [18], that all maximal ideals of $R_q[G]$ have finite codimension. The latter will be applied in the last section to prove that $R_q[G]$ satisfies a stronger property than catenarity, namely that all maximal chains of prime ideals in $R_q[G]$ have the same length, equal to $\mathrm{GK}\dim R_q[G]$.

Denote by $L_{\rm w}$ the subalgebra of $R_{\rm w}$ generated by $E_{\rm w}^{\pm 1}$, i.e. the subalgebra of $R_{\rm w}$ spanned by $\{c^+_{w_+,\mu_1} c^-_{w_-,\mu_2} \mid \mu_1, \mu_2 \in P\}$. Then:

Theorem 7.1. *For an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity, the algebra $R_{\rm w}$ is a free left and right $L_{\rm w}$-module in which $L_{\rm w}$ is a direct summand viewed as a module over itself.*

Recall (3.27). Denote by $N'_{\rm w}$ the subalgebra of $S_{\rm w}$, generated by y_{ω_i}, $i \in \{1, \dots, r\}$. We will prove the following result and deduce from it Theorem 7.1:

Theorem 7.2. *For an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity, the algebra $S_{\rm w}$ is a free left and right $N'_{\rm w}$-module in which $N'_{\rm w}$ is a direct summand viewed as a module over itself.*

Explicit versions of the decompositions in Theorems 7.1 and 7.2 will be obtained in Theorems 7.13 and 7.8. These results can be viewed as "separation of variables" theorems for the algebras $S_{\rm w}$ and $R_{\rm w}$ in analogy with the classical results of Kostant [30] and Joseph–Letzter [27]. Similarly to the algebras $S_w^{\pm}$, the centers of $R_{\rm w}$ and $S_{\rm w}$ are much smaller than the subalgebras generated by the homogeneous normal elements of the Joseph's set $E_{\rm w}$ and the multiplicative subset of $S_{\rm w}$ generated by y_{ω_i}, $i \in \{1, \dots, r\}$, respectively. Because of this, one obtains stronger results when considering the module structure of $R_{\rm w}$ and $S_{\rm w}$ over their subalgebras $L_{\rm w}$ and $N'_{\rm w}$, rather than their centers. It is this type of results that are eventually applicable to classify $\mathrm{Max}R_q[G]$.

7.2. **A $Q \times Q$-filtration of $S_{\rm w}$.** The algebra $S_{\rm w}$ is only Q-graded by (3.23). In this subsection we prove that it has a nontrivial $Q \times Q$-filtration which reveals a richer structure than the grading. This and the freeness result of Section 5 will be the main tools in the proofs of Theorems 7.1 and 7.2. For $w \in W$, denote

$$Q_w^+ = \sum_{\beta \in w(\Delta_+) \cap \Delta_-} \mathbb{N}\beta \subset Q^+ \cap Q_{\mathcal{S}(w)}. \tag{7.1}$$

Recall that

$$S_{\rm w} = \bigoplus_{(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}} (S^+_{w_+})_{-\gamma_+, 0} (S^-_{w_-})_{\gamma_-, 0} \tag{7.2}$$

and

$$(S^+_{w_+})_{-\gamma_+, 0} (S^-_{w_-})_{\gamma_-, 0} \cong (S^+_{w_+})_{-\gamma_+, 0} \otimes_{\mathbb{K}} (S^-_{w_-})_{\gamma_-, 0} \tag{7.3}$$

as $\mathbb{K}$-vector spaces (via the multiplication map), see (3.20) and (3.22). For $(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$ denote

$$(S_{\rm w})^{(\gamma_+, \gamma_-)} = (S^+_{w_+})_{-\gamma_+, 0} (S^-_{w_-})_{\gamma_-, 0}. \tag{7.4}$$

Consider the induced partial order on $Q^+_{w_+} \times Q^+_{w_-}$ from the product partial order (2.2) of $Q \times Q$. Thus, for two pairs (γ'_+, γ'_-), $(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$ we set $(\gamma'_+, \gamma'_-) \prec (\gamma_+, \gamma_-)$ if $\gamma'_+ < \gamma_+$ and $\gamma'_- < \gamma_-$, i.e. if there exist $\beta_\pm \in Q^+ \backslash \{0\}$ such that $\gamma_+ = \gamma'_+ + \beta_+$ and $\gamma_- = \gamma'_- + \beta_-$.

For $(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$ define

$$\begin{aligned}(S_{\mathrm{w}})^{\prec(\gamma_+,\gamma_-)} &= \bigoplus_{(\gamma'_+,\gamma'_-)\in Q^+_{w_+}\times Q^+_{w_-},(\gamma'_+,\gamma'_-)\prec(\gamma_+,\gamma_-)} (S_{\mathrm{w}})^{(\gamma_+,\gamma_-)} \\ &= \bigoplus_{\gamma'_\pm \in Q^+_{w_\pm}, \gamma'_\pm < \gamma_\pm} (S^+_{w_+})_{-\gamma'_+,0} (S^-_{w_-})_{\gamma'_-,0}\end{aligned}$$

and

$$(S_{\mathrm{w}})^{\preceq(\gamma_+,\gamma_-)} = (S_{\mathrm{w}})^{(\gamma_+,\gamma_-)} \oplus (S_{\mathrm{w}})^{\prec(\gamma_+,\gamma_-)}. \tag{7.5}$$

Since the algebras $S^\pm_{w_\pm}$ are Q-graded we have

$$s_+ (S_{\mathrm{w}})^{\prec(\gamma'_+,\gamma'_-)} s_- \subseteq (S_{\mathrm{w}})^{\prec(\gamma'_+{}^1+\gamma_+,\gamma'_-+\gamma_-)}, \tag{7.6}$$

for all $s_\pm \in (S^\pm_{w_\pm})_{\gamma_\pm}$, (γ'_+, γ'_-), $(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$.

Consider the $Q^+_{w_+} \times Q^+_{w_-}$ (exhaustive ascending) filtration of the space S_{w} by the subspaces $(S_{\mathrm{w}})^{\preceq(\gamma_+,\gamma_-)}$, $(\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$. The next result proves that this is an algebra filtration and computes the structure of the associate graded algebra.

Proposition 7.3. *For all* $(\gamma_+, \gamma_-), (\gamma'_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$, $s_\pm \in (S^\pm_{w_\pm})_{\mp\gamma_\pm}$ *and* $s'_\pm \in (S^\pm_{w_\pm})_{\mp\gamma'_\pm}$, *we have*

$$\big(s_+ s_-\big).\big(s'_+ s'_-\big) = q^{-\langle \gamma_-, \gamma'_+\rangle} \big((s_+ s'_+)(s_- s'_-)\big) \mod (S_{\mathrm{w}})^{\prec(\gamma_+ + \gamma'_+, \gamma_- + \gamma'_-)}.$$

Note that in the setting of the proposition

$$\begin{aligned}&s_+ s_- \in (S_{\mathrm{w}})^{(\gamma_+,\gamma_-)}, \ \ s'_+ s'_- \in (S_{\mathrm{w}})^{(\gamma'_+,\gamma'_-)} \ \text{ and} \\ &(s_+ s'_+)(s_- s'_-) \in (S_{\mathrm{w}})^{(\gamma_+ + \gamma'_+, \gamma_- + \gamma'_-)}.\end{aligned}$$

We will identify

$$(S_{\mathrm{w}})^{\preceq(\gamma_+,\gamma_-)} / (S_{\mathrm{w}})^{\prec(\gamma_+,\gamma_-)} \cong (S_{\mathrm{w}})^{(\gamma_+,\gamma_-)} \ \text{ for } \ (\gamma_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-},$$

(cf. (7.5)) and

$$\mathrm{gr}\, S_{\mathrm{w}} \cong \bigoplus_{(\gamma_+,\gamma_-)\in Q^+_{w_+}\times Q^+_{w_-}} (S_{\mathrm{w}})^{(\gamma_+,\gamma_-)}, \tag{7.7}$$

(cf. (7.4)). Denote the multiplication in $\mathrm{gr}\, S_{\mathrm{w}}$ by $\odot$.

Corollary 7.4. *Under the identification of* (7.7) *the multiplication in* $\mathrm{gr}\, S_{\mathrm{w}}$ *is given by*

$$\big(s_+ s_-\big) \odot \big(s'_+ s'_-\big) = q^{-\langle \gamma_-, \gamma'_+\rangle} \big((s_+ s'_+)(s_- s'_-)\big),$$

for all $(\gamma_+, \gamma_-), (\gamma'_+, \gamma_-) \in Q^+_{w_+} \times Q^+_{w_-}$, $s_\pm \in (S^\pm_{w_\pm})_{\mp\gamma_\pm}$ *and* $s'_\pm \in (S^\pm_{w_\pm})_{\mp\gamma'_\pm}$.

Proof of Proposition 7.3. It follows from (3.19) that

$$s_- s'_+ = q^{-\langle \gamma_-, \gamma'_+ \rangle} s'_+ s_- + \sum_{i=1}^k (s'_+)^{(i)} (s_-)^{(i)},$$

for some $(s'_+)^{(i)} \in (S^+_{w_+})_{-\gamma^{(i)}_+}$, $(s_-)^{(i)} \in (S^-_{w_-})_{\gamma^{(i)}_-}$, $\gamma^{(i)}_+ \in Q^+_{w_+}$, $\gamma^{(i)}_+ < \gamma'_+$, $\gamma^{(i)}_- \in Q^-_{w_-}$, $\gamma^{(i)}_- < \gamma_-$, $i = 1, \ldots, k$. Therefore

$$s_- s'_+ = q^{-\langle \gamma_-, \gamma'_+ \rangle} s'_+ s_- \mod (S_{\mathrm{w}})^{\prec (\gamma'_+, \gamma_-)}. \tag{7.8}$$

Multiplying (7.8) on the left by s_+ and on the right by s'_-, and using (7.6) implies the statement of the proposition. □

7.3. **The action of** $\mathrm{gr}\, N'_{\mathrm{w}}$ **on** $\mathrm{gr}\, S_{\mathrm{w}}$**.** Next, we apply the results from the previous subsection to the N'_{w}-module structure of S_{w}.

First, observe from (3.26) that for all $i \in \mathcal{I}(\mathrm{w})$ the image of x_{ω_i} in $\widehat{R}_{\mathrm{w}}$ is equal to $c^+_{1,\omega_i} c^-_{1,\omega_i} = c^+_{w_+,\omega_i} c^-_{w_-,\omega_i}$. Applying (2.29), we get

$$y_{\omega_i} = (c^+_{w_+,\omega_i})^{-1} (c^-_{w_-,\omega_i})^{-1} c^+_{w_+,\omega_i} c^-_{w_-,\omega_i} = 1, \quad \forall i \in \mathcal{I}(\mathrm{w}). \tag{7.9}$$

Recall the definition (3.29) of the elements $d^\pm_{w_\pm,\lambda} \in (S^\pm_{w_\pm})_{\pm(w_\pm - 1)\lambda, 0}$. We have

$$d^+_{w_+,\lambda} d^-_{w_-,\lambda} \in (S_{\mathrm{w}})^{((1-w_+)\lambda, (1-w_-)\lambda)}. \tag{7.10}$$

Eqs. (3.26) and (2.29) imply that

(7.11)
$$\begin{aligned} y_{\omega_i} &= (c^+_{w_+,\omega_i})^{-1} (c^-_{w_-,\omega_i})^{-1} c^+_{1,\omega_i} c^-_{1,\omega_i} = q^{\langle \omega_i, (1-w_-)\omega_i \rangle} (c^+_{w,\omega_i})^{-1} c^+_{1,\omega_i} (c^-_{w,\omega_i})^{-1} c^-_{1,\omega_i} \\ &= q^{\langle \omega_i, (1-w_-)\omega_i \rangle} d^+_{w_+,\omega_i} d^-_{w_-,\omega_i} \mod (S_{\mathrm{w}})^{\prec ((1-w_+)\omega_i, (1-w_-)\omega_i)}, \end{aligned}$$

for all $i \in \mathcal{S}(\mathrm{w})$.

Recall the definition (3.62) of the elements $y_\lambda \in N'_{\mathrm{w}}$, $\lambda \in P^+$. Applying repeatedly Proposition 7.3 and using the fact that $d^\pm_{w_\pm,\omega_i} \in S^\pm_{w_\pm}$ are P-normal, we obtain:

Corollary 7.5. *For every $\lambda \in P^+_{\mathcal{S}(\mathrm{w})}$, $s_\pm \in (S^\pm_{w_\pm})_{\mp\gamma_\pm, 0}$ there exists $m \in \mathbb{Z}$ such that*

$$y_\lambda (s_+ s_-) = q^m (s_+ d^+_{w_+,\lambda})(d^-_{w_-,\lambda} s_-) \mod (S_{\mathrm{w}})^{\prec (\gamma_+ + (1-w_+)\lambda, \gamma_- + (1-w_-)\lambda)}.$$

Note that in the setting of Corollary 7.5,

$$(s_+ d^+_{w_+,\lambda})(d^-_{w_-,\lambda} s_-) \in (S_{\mathrm{w}})^{(\gamma_+ + (1-w_+)\lambda, \gamma_- + (1-w_-)\lambda)}.$$

Setting $s_+ = 1$, $s_- = 1$, we obtain that for all $\lambda \in P^+_{\mathcal{S}(\mathrm{w})}$

$$y_\lambda = q^{m_\lambda} d^+_{w_+,\lambda} d^-_{w_-,\lambda} \mod (S_{\mathrm{w}})^{\prec ((1-w_+)\lambda, (1-w_-)\lambda)}, \tag{7.12}$$

for some $m_\lambda \in \mathbb{Z}$ and $d^+_{w_+,\lambda} d^-_{w_-,\lambda} \in (S_{\mathrm{w}})^{((1-w_+)\lambda, (1-w_-)\lambda)}$.

Denote

$$\Gamma_{\mathrm{w}} = \{((1-w_+)\lambda, (1-w_-)\lambda) \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\}.$$

Eq. (7.12) implies

$$\operatorname{gr} N'_{\rm w} \cong \bigoplus_{(\gamma_+,\gamma_-)\in\Gamma_{\rm w}} (N'_{\rm w})^{(\gamma_+,\gamma_-)}, \tag{7.13}$$

where for $\lambda \in P^+_{\mathcal{S}(\rm w)}$

$$\begin{aligned}(N'_{\rm w})^{((1-w_+)\lambda,(1-w_-)\lambda)} &= N'_{\rm w} \cap (S_{\rm w})^{((1-w_+)\lambda,(1-w_-)\lambda)} \\ &= \operatorname{Span}\{y_\mu \mid \mu \in P_{\mathcal{S}(\rm w)}, \mu - \lambda \in \ker(1-w_+) \cap \ker(1-w_-)\}\end{aligned}$$

in the identification (7.7). Denote by $\operatorname{gr} y_\lambda$ the image of y_λ in $\operatorname{gr} N'_{\rm w}$. Eq. (7.12) implies that for each $\lambda \in P^+_{\mathcal{S}(\rm w)}$ there exists $m_\lambda \in \mathbb{Z}$ such that

$$\operatorname{gr} y_\lambda = q^{m_\lambda} d^+_{w_+,\lambda} d^-_{w_-,\lambda} \tag{7.14}$$

in terms of the identification (7.7).

For $s \in (S_{\rm w})^{\preceq(\gamma_+,\gamma_-)}$ denote by $\operatorname{gr} s$ its image in $\operatorname{gr} S_{\rm w}$. Corollary 7.5 implies that for all $\lambda \in P^+_{\mathcal{S}(\rm w)}$, $s_\pm \in (S^\pm_{w_\pm})_{\mp\gamma_\pm,0}$ $\gamma_\pm \in Q^+_{w_\pm}$ there exists $m \in \mathbb{Z}$ such that

$$(\operatorname{gr} y_\lambda) \odot \big(\operatorname{gr}(s_+ s_-)\big) = q^m (s_+ d^+_{w_+,\lambda})(d^-_{w_-,\lambda} s_-), \tag{7.15}$$

where in the right hand side we used the identification (7.7).

7.4. **The structure of the algebra $N'_{\rm w}$ and separation of variables for $S_{\rm w}$.** Recall that for $\mathrm{w} = (w_+, w_-) \in W \times W$

$$\mathcal{S}(\mathrm{w}) = \mathcal{S}(w_+) \cup \mathcal{S}(w_-).$$

We have

$$\mathcal{S}(\mathrm{w}) = \big(\mathcal{S}(w_+) \cap \mathcal{S}(w_-)\big) \bigsqcup \big(\mathcal{S}(w_+) \backslash \mathcal{S}(w_-)\big) \bigsqcup \big(\mathcal{S}(w_-) \backslash \mathcal{S}(w_+)\big)$$

and the corresponding decomposition

$$P^+_{\mathcal{S}(\rm w)} = P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} \bigoplus P^+_{\mathcal{S}(w_+)\backslash\mathcal{S}(w_-)} \bigoplus P^+_{\mathcal{S}(w_-)\backslash\mathcal{S}(w_+)}. \tag{7.16}$$

For $\lambda \in P^+_{\mathcal{S}(\rm w)}$, denote its components

$$(\lambda)_0 \in P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}, \quad (\lambda)_+ \in P^+_{\mathcal{S}(w_+)\backslash\mathcal{S}(w_-)}, \quad (\lambda)_- \in P^+_{\mathcal{S}(w_-)\backslash\mathcal{S}(w_+)} \tag{7.17}$$

in the decomposition (7.16). For $\mu \in P^+_{\mathcal{S}(w_\mp)\backslash\mathcal{S}(w_\pm)}$, $d^\pm_{w_\pm,\mu}$ is a nonzero scalar by (7.9) and (3.31). Using this and one more time (3.31), we obtain that for each $\lambda \in P^+_{\mathcal{S}(\rm w)}$ there exist integers m_λ and m'_λ such that

$$d^+_{w_+,\lambda} = q^{m_\lambda} d^+_{w_+,(\lambda)_0+(\lambda)_+} \tag{7.18}$$

and

$$d^-_{w_-,\lambda} = q^{m'_\lambda} d^-_{w_-,(\lambda)_0+(\lambda)_-}. \tag{7.19}$$

It follows from (3.63) that

$$y_{\omega_i} y_{\omega_j} = q^{\langle w_-\omega_i, w_+\omega_j\rangle - \langle w_+\omega_i, w_-\omega_j\rangle} y_{\omega_j} y_{\omega_i}, \quad i,j = 1,\dots,r. \tag{7.20}$$

The following result describes the structure of the algebra $N'_{\rm w}$.

Proposition 7.6. *For all* $\mathrm{w} \in W \times W$ *the algebra* N'_{w} *is isomorphic to the quantum affine space algebra over* $\mathbb{K}$ *of dimension* $|\mathcal{S}(\mathrm{w})|$ *with generators* y_{ω_i}, $i \in \mathcal{S}(\mathrm{w})$ *and relations* (7.20)*. The set* $\{y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\}$ *is a* $\mathbb{K}$*-basis of* N'_{w}*.*

Proof. By part (i) of Theorem 5.1 the elements $d^{\pm}_{w_\pm,\lambda} \in S^{\pm}_{w_\pm}$, $\lambda \in P^+_{\mathcal{S}(w_\pm)}$ are linearly independent. Taking (7.3) into account, we see that

$$\{d^+_{w_+,\lambda_1} d^-_{w_-,\lambda_2} \mid \lambda_1 \in P^+_{\mathcal{S}(w_+)}, \lambda_2 \in P^+_{\mathcal{S}(w_-)}\} \subset S_{\mathrm{w}} \tag{7.21}$$

is a linearly independent set.

Recall the discussion of quantum affine space algebras from §5.1. Because the relations (7.20) hold, all we need to show is that the ordered monomials in y_{ω_i}, $i \in \mathcal{S}(\mathrm{w})$ are linearly independent. The latter are y_λ, $\lambda \in P^+_{\mathcal{S}(\mathrm{w})}$ up to a nonzero scalar multiple. Applying (7.14) and (7.18)–(7.19), we obtain that there exist integers n_λ, n'_λ for $\lambda \in P^+_{\mathcal{S}(w_\pm)}$ such that

$$\begin{aligned} &\{q^{n_\lambda} \operatorname{gr} y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \\ =&\{q^{n'_\lambda} d^+_{w_+,\lambda} d^-_{w_-,\lambda} \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \\ =&\{d^+_{w_+,(\lambda)_0+(\lambda)_+} d^-_{w_-,(\lambda)_0+(\lambda)_-} \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \end{aligned} \tag{7.22}$$

in the identification (7.7). Since the third set is a subset of the set in (7.21), the elements $\{\operatorname{gr} y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\}$ are linearly independent. Therefore the elements $\{y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\}$ are linearly independent. $\square$

Denote

$$\Omega_{\mathrm{w}} = \{(\mu_1, \mu_2) \in P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} \times P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} \mid \operatorname{Supp}\mu_1 \cap \operatorname{Supp}\mu_2 = \emptyset\},$$

recall (2.1). For a set Y denote by $\operatorname{Diag}(Y)$ the diagonal subset of $Y \times Y$.

Lemma 7.7. *Let* $\mathrm{w} = (w_+, w_-) \in W \times W$*. Then:*

(i) Each element of $P^+_{\mathcal{S}(w_+)} \times P^+_{\mathcal{S}(w_-)}$ *can be uniquely represented as a sum of an element of* Ω_{w} *and an element of the set*

$$\begin{aligned} &\{(\lambda)_0 + (\lambda)_+, (\lambda)_0 + (\lambda)_-) \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \\ =& \operatorname{Diag}\big(P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}\big) \bigoplus \big(P^+_{\mathcal{S}(w_+)\setminus\mathcal{S}(w_-)} \times P^+_{\mathcal{S}(w_-)\setminus\mathcal{S}(w_+)}\big), \end{aligned}$$

cf. (7.16)*.*

(ii) There exist integers $\{m_{\lambda_1,\lambda_2} \mid (\lambda_1, \lambda_2) \in P^+_{\mathcal{S}(w_+)} \times P^+_{\mathcal{S}(w_-)}\}$ *such that the set*

$$\{q^{m_{\lambda_1,\lambda_2}} d^+_{w_+,\lambda_1} d^-_{w_-,\lambda_2} \mid (\lambda_1, \lambda_2) \in P^+_{\mathcal{S}(w_+)} \times P^+_{\mathcal{S}(w_-)}\}$$

coincides with the set

$$\big\{d^+_{w_+,\mu_1}\big(d^+_{w_+,\lambda} d^-_{w_-,\lambda}\big) d^-_{w_-,\mu_2} \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}, (\mu_1, \mu_2) \in \Omega_{\mathrm{w}}\big\}.$$

Proof. (i) We have

$$P^+_{\mathcal{S}(w_\pm)} = P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} \bigoplus P^+_{\mathcal{S}(w_\pm)\setminus\mathcal{S}(w_\mp)}.$$

Because of this, the statement of the first part is equivalent to

$$P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} \times P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)} = \Omega_{\mathrm{w}} \bigoplus \operatorname{Diag}\big(P^+_{\mathcal{S}(w_+)\cap\mathcal{S}(w_-)}\big).$$

This fact is easy to verify and is left to the reader.

(ii) Eq. (3.31) and the second equality in (7.22) imply that there exist integers m_{λ,μ_1,μ_2}, $\lambda \in P^+_{\mathcal{S}(\mathrm{w})}$, $(\mu_1,\mu_2) \in \Omega_{\mathrm{w}}$ such that

$$\begin{aligned}&\big\{d^+_{w_+,\mu_1}\big(d^+_{w_+,\lambda}d^-_{w_-,\lambda}\big)d^-_{w_-,\mu_2} \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}, (\mu_1,\mu_2) \in \Omega_{\mathrm{w}}\big\}\\ =&\{q^{m_{\lambda,\mu_1,\mu_2}}d^+_{w_+,(\lambda)_0+(\lambda)_++\mu_1}d^-_{w_-,(\lambda)_0+(\lambda)_-+\mu_2} \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}, (\mu_1,\mu_2) \in \Omega_{\mathrm{w}}\}.\end{aligned}$$

Now the second part follows from the first one. $\square$

Denote $l_\pm = l(w_\pm)$. Fix reduced expressions $\vec{w}_\pm$ of $w_\pm$. Recall the definition (5.11) of the sets $\Delta(\vec{w}_\pm) \subseteq \mathbb{N}^{l_\pm}$. Denote

$$B'_{\mathrm{w}} = \big\{(\varphi^+_{w_+})^{-1}\big((X^-)^{\mathbf{n}_+}\big)d^+_{w_+,\mu_1}d^-_{w_-,\mu_2}(\varphi^-_{w_-})^{-1}\big((X^+)^{\mathbf{n}_-}\big) \mid \mathbf{n}_+ \in \Delta(\vec{w}_+), \mathbf{n}_- \in \Delta(\vec{w}_-), (\mu_1,\mu_2) \in \Omega_{\mathrm{w}}\big\} \subset S_{\mathrm{w}} \tag{7.23}$$

and

$$D'_{\mathrm{w}} = \mathrm{Span}\, B'_{\mathrm{w}}. \tag{7.24}$$

The following theorem is our explicit freeness result for the module structure of S_{w} over their subalgebras N'_{w}. Theorem 7.1 follows immediately from it.

Theorem 7.8. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathrm{w} = (w_+, w_-) \in W \times W$, and that $\vec{w}_\pm$ are reduced expressions of $w_\pm$. Then we have the following freeness of S_{w} as a left and right N'_{w} module:*

$$S_{\mathrm{w}} \cong N'_{\mathrm{w}} \bigotimes_{\mathbb{K}} D'_{\mathrm{w}} \cong D'_{\mathrm{w}} \bigotimes_{\mathbb{K}} N'_{\mathrm{w}}.$$

To prove Theorem 7.8 it is sufficient to prove the corresponding statement at the level of associated graded modules, which is established by the following result.

Proposition 7.9. *In the setting of Theorem 7.8, the sets*

$$\{\mathrm{gr}\, y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \odot \mathrm{gr}\, B'_{\mathrm{w}} \quad \textit{and} \quad \mathrm{gr}\, B'_{\mathrm{w}} \odot \{\mathrm{gr}\, y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\}$$

are bases of $\mathrm{gr}\, S_{\mathrm{w}}$. In other words

$$\mathrm{gr}\, S_{\mathrm{w}} \cong \mathrm{gr}\, N'_{\mathrm{w}} \bigotimes_{\mathbb{K}} \mathrm{gr}\, D'_{\mathrm{w}} \cong \mathrm{gr}\, D'_{\mathrm{w}} \bigotimes_{\mathbb{K}} \mathrm{gr}\, N'_{\mathrm{w}}$$

by using the multiplication $\odot$ in $\mathrm{gr}\, S_{\mathrm{w}}$.

Proof. Since the elements $\mathrm{gr}\, y_\lambda$ normalize each element of $\mathrm{gr}\, B'_{\mathrm{w}}$, it suffices to prove that $\{\mathrm{gr}\, y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \odot \mathrm{gr}\, B'_{\mathrm{w}}$ is a basis of S_{w}.

Applying (7.15), we obtain that for each $\lambda \in P^+_{\mathcal{S}(\mathrm{w})}$, $(\mu_1,\mu_2) \in \Omega_{\mathrm{w}}$, $\mathbf{n}_\pm \in \Delta(\vec{w}_\pm)$ there exists an integer $m_{\lambda,\mu_1,\mu_2,\mathbf{n}_+,\mathbf{n}_-}$ such that

$$\begin{aligned}&\big(\mathrm{gr}\, y_\lambda\big) \odot \mathrm{gr}\,\big[(\varphi^+_{w_+})^{-1}\big((X^-)^{\mathbf{n}_+}\big)d^+_{w_+,\mu_1}d^-_{w_-,\mu_2}(\varphi^-_{w_-})^{-1}\big((X^+)^{\mathbf{n}_-}\big)\big]\\ &= q^{m_{\mu_1,\mu_2,\mathbf{n}_+,\mathbf{n}_-}}(\varphi^+_{w_+})^{-1}\big((X^-)^{\mathbf{n}_+}\big)d^+_{w_+,\mu_1}\big(d^+_{w_+,\lambda}d^-_{w_-,\lambda}\big)d^-_{w_-,\mu_2}(\varphi^-_{w_-})^{-1}\big((X^+)^{\mathbf{n}_-}\big),\end{aligned}$$

where in the right hand side we used the identification (7.7). Lemma 7.7 (ii) now implies that for some integers $m_{\lambda_1,\lambda_2,\mathbf{n}_+,\mathbf{n}_-}$, $(\lambda_1,\lambda_2) \in P^+_{\mathcal{S}(w_+)} \times P^+_{\mathcal{S}(w_-)}$, $\mathbf{n}_\pm \in \Delta(\vec{w}_\pm)$,

$$\begin{aligned}&\{\operatorname{gr} y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \odot \operatorname{gr} B'_{\mathrm{w}} \\ =&\big\{q^{m_{\lambda,\lambda_1,\lambda_2,\mathbf{n}_+,\mathbf{n}_-}}(\varphi^+_{w_+})^{-1}\big((X^-)^{\mathbf{n}_+}\big)d^+_{w_+,\lambda_1}d^-_{w_-,\lambda_2}(\varphi^-_{w_-})^{-1}\big((X^+)^{\mathbf{n}_-}\big) \mid \\ &(\lambda_1,\lambda_2) \in P^+_{\mathcal{S}(w_+)} \times P^+_{\mathcal{S}(w_-)}, \mathbf{n}_\pm \in \Delta(\vec{w}_\pm)\big\},\end{aligned}$$

where again the right hand side uses the identification (7.7). By Theorem 5.1 (ii)

$$\big\{(\varphi^+_{w_+})^{-1}\big((X^-)^{\mathbf{n}_+}\big)d^+_{w_+,\lambda_1} \,\big|\, \mathbf{n}_+ \in \Delta(\vec{w}_+), \lambda_1 \in P^+_{\mathcal{S}(w_+)}\big\}$$

and

$$\big\{d^-_{w_-,\lambda_2}(\varphi^-_{w_-})^{-1}\big((X^+)^{\mathbf{n}_-}\big) \,\big|\, \mathbf{n}_- \in \Delta(\vec{w}_-), \lambda_2 \in P^+_{\mathcal{S}(w_-)}\big\}$$

are bases of $S^+_{w_+}$ and $S^-_{w_-}$ respectively. Since $S_{\mathrm{w}} \cong S^+_{w_+} \otimes_{\mathbb{K}} S^-_{w_-}$ under the multiplication map, we obtain that $\{\operatorname{gr} y_\lambda \mid \lambda \in P^+_{\mathcal{S}(\mathrm{w})}\} \odot \operatorname{gr} B'_{\mathrm{w}}$ is a basis of $\operatorname{gr} S_{\mathrm{w}}$. □

7.5. **The structure of algebras L_{w} and freeness of R_{w} over L_{w}.** In this subsection we obtain an explicit version of the freeness result in Theorem 7.1 and describe the structure of the algebra L_{w}.

We begin with some implications of the results from the previous subsection to the structure of $S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1,\dots,r]$. Denote by L'_{w} the subalgebra of $S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1,\dots,r]$, generated by $y^{\pm1}_{\omega_i}$, $i = 1,\dots,r$. Theorem 7.8 immediately implies:

Corollary 7.10. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathrm{w} \in W \times W$, and $\vec{w}_\pm$ are reduced expressions of $w_\pm$. Then:*

(i) L'_{w} is isomorphic to the quantum torus algebra over $\mathbb{K}$ of dimension $|\mathcal{S}(\mathrm{w})|$ with generators $(y_{\omega_i})^{\pm1}$, $i \in \mathcal{S}(\mathrm{w})$ and relations (7.20).

(ii) The ring $S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1,\dots,r]$ is a free left and right L'_{w}-module and more precisely:

$$S_{\mathrm{w}}[y^{-1}_{\omega_i}, i = 1,\dots,r] \cong L_{\mathrm{w}} \bigotimes_{\mathbb{K}} D'_{\mathrm{w}} \cong D'_{\mathrm{w}} \bigotimes_{\mathbb{K}} L_{\mathrm{w}}.$$

Recall from §7.1 that L_{w} denotes the subalgebra of R_{w} which is generated by $c^+_{w_+,\lambda}$, $c^-_{w_-,\lambda}$, $\lambda \in P$. Its structure is described in the following result.

Proposition 7.11. *The algebra L_{w} is a quantum torus algebra over $\mathbb{K}$ of dimension $r + |\mathcal{S}(\mathrm{w})|$ with generators $(c^+_{w_+,\omega_i})^{\pm1}$, $i \in \mathcal{S}(\mathrm{w})$ and $(c^-_{w_-,\omega_j})^{\pm1}$, $j = 1,\dots,r$, and relations*

$$\begin{aligned} c^+_{w_+,\omega_{i_1}}c^+_{w_+,\omega_{i_2}} &= c^+_{w_+,\omega_{i_2}}c^+_{w_+,\omega_{i_1}}, \quad i_1, i_2 \in \mathcal{S}(\mathrm{w}), \\ c^-_{w_-,\omega_{j_1}}c^-_{w_-,\omega_{j_2}} &= c^-_{w_-,\omega_{j_2}}c^-_{w_-,\omega_{j_1}}, \quad j_1, j_2 = 1,\dots,r, \\ c^+_{w_+,\omega_i}c^-_{w_-,\omega_j} &= q^{-\langle w_+\omega_i, w_-\omega_j\rangle}c^-_{w_+,\omega_j}c^+_{w_+,\omega_i}, \quad i \in \mathcal{S}(\mathrm{w}), j = 1,\dots,r. \end{aligned}$$

Proof. The inverse of the isomorphism (3.28) restricts to an algebra isomorphism

$$\psi^{-1}_{\mathrm{w}} \colon L_{\mathrm{w}} \to L'_{\mathrm{w}} \# \widehat{L}^-_{w_-},$$

where

$$\psi_{\mathrm{w}}^{-1}(c^-_{w_-,\omega_j}) = c^-_{w_-,\omega_j}, \ \ j = 1, \ldots, r, \tag{7.25}$$

$$\psi_{\mathrm{w}}^{-1}(c^+_{w_+,\omega_i}) = q^{-\langle w_+\omega_i, w_-\omega_i\rangle+1}(y_{\omega_i})^{-1}\#(c^-_{w_-,\omega_j})^{-1}, \ \ i = 1, \ldots, r. \tag{7.26}$$

The second equality follows from (3.27) and (2.29). By Corollary 7.10, L'_{w} is a quantum torus algebra over $\mathbb{K}$ of dimension $\mathcal{S}(\mathrm{w})$ with generators $(y_{\omega_i})^{\pm 1}$, $i \in \mathcal{S}(\mathrm{w})$ and by (7.9), $y_{\omega_i} = 1$ for all $i \in \mathcal{I}(\mathrm{w})$. Recall from §3.4 that $\widehat{L}^-_{w_-}$ is an r dimensional Laurent polynomial algebra over $\mathbb{K}$ with generators $(c^-_{w_-,\omega_j})^{\pm 1}$, $j = 1, \ldots, r$. The commutation relation (3.25) implies that $L'_{\mathrm{w}}\#\widehat{L}^-_{w_-}$ is a quantum torus algebra over $\mathbb{K}$ of dimension $r + |\mathcal{S}(\mathrm{w})|$ with generators $(y_{\omega_i}\#1)^{\pm 1}$, $i \in \mathcal{S}(\mathrm{w})$ and $(c^-_{w_-,\omega_j})^{\pm 1}$, $j = 1, \ldots, r$. Therefore $L'_{\mathrm{w}}\#\widehat{L}^-_{w_-}$ is also a quantum torus algebra with generators $((y_{\omega_i})^{-1}\#(c^-_{w_-,\omega_j})^{-1})^{\pm 1}$, $i \in \mathcal{S}(\mathrm{w})$ and $(c^-_{w_-,\omega_j})^{\pm 1}$, $j = 1, \ldots, r$. It follows from (7.25)–(7.26) that the algebra L_{w} is isomorphic to the quantum torus algebra over $\mathbb{K}$ of dimension $r + |\mathcal{S}(\mathrm{w})|$ with generators $(c^+_{w_+,\omega_i})^{\pm 1}$, $i \in \mathcal{S}(\mathrm{w})$ and $(c^-_{w_-,\omega_j})^{\pm 1}$, $j = 1, \ldots, r$. The commutation relations between them are derived from (2.18) and (2.29). □

Corollary 7.12. *For all* $\mathrm{w} = (w_+, w_-) \in W \times W$*, the algebra* L_{w} *has a basis consisting of*

$$c^+_{w_+,\mu_1}c^-_{w_-,\mu_2} \tag{7.27}$$

for $\mu_1 \in P_{\mathcal{S}(\mathrm{w})}$*,* $\mu_2 \in P$*. We have*

$$c^+_{w_+,\mu}c^-_{w_-,\mu} \in \mathbb{K}^*, \ \ \forall \mu \in P_{\mathcal{I}(\mathrm{w})}. \tag{7.28}$$

In particular, the set (7.27) *for* $\mu_1 \in P$*,* $\mu_2 \in P_{\mathcal{S}(\mathrm{w})}$ *is also a basis of* L_{w}*.*

Proof. The corollary follows from Proposition 7.11 and (2.28). □

Recall (3.28), (7.23) and (7.24), and denote

$$B_{\mathrm{w}} = B'_{\mathrm{w}}\#1 = \{b\#1 \mid b \in B'_{\mathrm{w}}\} \subset S_{\mathrm{w}}[y^{-1}_{\omega_1}, \ldots, y^{-1}_{\omega_r}]\#\widehat{L}^-_{w_-}, \quad D_{\mathrm{w}} = \operatorname{Span} B_{\mathrm{w}}.$$

The next theorem provides an explicit form of the freeness result from Theorem 7.1.

Theorem 7.13. *Assume that* $\mathbb{K}$ *is an arbitrary base field,* $q \in \mathbb{K}^*$ *is not a root of unity,* $\mathrm{w} \in W \times W$*, and* $\vec{w}_\pm$ *are reduced expressions of* $w_\pm$*. Then the algebra* R_{w} *is a free left and right* L_{w}*-module via*

$$R_{\mathrm{w}} \cong L_{\mathrm{w}} \bigotimes_{\mathbb{K}} (\psi_{\mathrm{w}})^{-1}(D_{\mathrm{w}}) \cong (\psi_{\mathrm{w}})^{-1}(D_{\mathrm{w}}) \bigotimes_{\mathbb{K}} L_{\mathrm{w}}.$$

Proof. The isomorphism (3.28), Corollary 7.10 (ii) and the fact that ψ_{w} restricts to an algebra isomorphism $L'_{\mathrm{w}}\#\widehat{L}^-_{w_-} \to L_{\mathrm{w}}$ imply

$$R_{\mathrm{w}} \cong (\psi_{\mathrm{w}})^{-1}(D_{\mathrm{w}}) \bigotimes_{\mathbb{K}} L_{\mathrm{w}}.$$

The equality

$$L_{\mathrm{w}} \bigotimes_{\mathbb{K}} (\psi_{\mathrm{w}})^{-1}(D_{\mathrm{w}}) \cong (\psi_{\mathrm{w}})^{-1}(D_{\mathrm{w}}) \bigotimes_{\mathbb{K}} L_{\mathrm{w}}$$

follows from (3.25). □

8. A classification of maximal ideals of $R_q[G]$ and a question of Goodearl and Zhang

8.1. **A projection property of the ideal $I_{(1,1)}$.** In this section we classify all maximal ideals of $R_q[G]$ and derive an explicit formula for each of them. We apply this result to resolve a question of Goodearl and Zhang [18] by showing that all maximal ideals of $R_q[G]$ have finite codimension. In the next section we use this result to prove that $R_q[G]$ has the property that all maximal chains of prime ideals of it have the same length. The main step in the proof of the classification theorem is to prove that only the highest stratum of the decomposition of $\mathrm{Spec} R_q[G]$ in Theorem 2.3 contains maximal ideals. To obtain this, we combine the methods from Section 4 giving formulas for the primitive ideals of $R_q[G]$ and the "separation of variables" theorem for the algebras R_{w} from Section 7. We analyze the images of the primitive ideals and the ideal $I_{(1,1)}$ in the direct sum decomposition from Theorem 7.13 and eventually deduce that none of the primitive ideals in $\mathrm{Prim}_{\mathrm{w}} R_q[G]$ is maximal for $\mathrm{w} \neq (1,1)$.

We start with the statement of the difficult step of the classification result:

Theorem 8.1. *For an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity, all maximal ideals of $R_q[G]$ belong to $\mathrm{Spec}_{(1,1)} R_q[G]$, i.e.*

$$\mathrm{Max} R_q[G] \subset \mathrm{Spec}_{(1,1)} R_q[G].$$

In the setting of §7.4–7.5 denote

$$B^\circ_{\mathrm{w}} = B_{\mathrm{w}} \backslash \{1\}$$

and

$$R^\circ_{\mathrm{w}} = (\psi_{\mathrm{w}})^{-1}(\mathrm{Span}\, B^\circ_{\mathrm{w}}) \bigotimes_{\mathbb{K}} L_{\mathrm{w}} = L_{\mathrm{w}} \bigotimes_{\mathbb{K}} (\psi_{\mathrm{w}})^{-1}(\mathrm{Span}\, B^\circ_{\mathrm{w}}).$$

By Theorem 7.8 we have the direct sum decomposition of L_{w}-bimodules

$$R_{\mathrm{w}} = L_{\mathrm{w}} \oplus R^\circ_{\mathrm{w}}.$$

We denote by

$$\pi_{\mathrm{w}} \colon R_{\mathrm{w}} \to L_{\mathrm{w}}$$

the projection onto the first component (which is a homomorphism of L_{w}-bimodules). Denote by N_{w} the subalgebra of L_{w}, generated by $c^+_{w_+,\lambda}$ and $c^-_{w_-,\lambda}$ for $\lambda \in P^+$. We will need two N_{w}-(bi)submodules of L_{w}

$$M^{++}_{\mathrm{w}} \subset M^+_{\mathrm{w}} \subset L_{\mathrm{w}},$$

defined as follows. Denote the submonoids

$$Y^{++}_1 = \{\mu \in P \mid (1 - w_+)\mu > 0\},$$
$$Y^{++}_2 = \{\mu \in P_{\mathcal{S}(\mathrm{w})} \mid (1 - w_-)\mu > 0\},$$

where the inequalities are in terms of the partial order (2.2). Denote also the following two submonoids of $P \times P_{\mathcal{S}(\mathrm{w})}$:

$$Y^{++}_{\mathrm{w}} = \big(Y^{++}_1 \times Y^{++}_2\big) \bigsqcup \big(Y^{++}_1 \times P_{\mathcal{S}(\mathrm{w}) \cap \mathcal{I}(w_-)}\big) \bigsqcup \big(P_{\mathcal{I}(w_+)} \times Y^{++}_2\big) \tag{8.1}$$

and

$$Y^+_{\mathrm{w}} = Y^{++}_{\mathrm{w}} \bigsqcup \big(P_{\mathcal{I}(w_+)} \times P_{\mathcal{S}(\mathrm{w}) \cap \mathcal{I}(w_-)}\big). \tag{8.2}$$

We have disjoint unions in (8.1)-(8.2), because

$$\mathcal{I}(w_+) = \{i = 1, \dots, r \mid (1-w_+)\omega_i = 0\},$$
$$\mathcal{S}(\mathrm{w}) \cap \mathcal{I}(w_-) = \mathcal{S}(w_+) \cap \mathcal{I}(w_-) = \{i \in \mathcal{S}(\mathrm{w}) \mid (1-w_-)\omega_i = 0\}.$$

Remark 8.2. Note that in general Y^{++}_{w} and Y^{+}_{w} are strictly contained in the sets

$$\{(\mu_1, \mu_2) \in P \times P_{\mathcal{S}(\mathrm{w})} \mid (1-w_+)\mu_1 \geq 0, (1-w_-)\mu_2 \geq 0 \quad \text{and at lest one inequality is strict } \}$$

and

$$\{(\mu_1, \mu_2) \in P \times P_{\mathcal{S}(\mathrm{w})} \mid (1-w_+)\mu_1 \geq 0, (1-w_-)\mu_2 \geq 0\},$$

respectively. This is so, because $\ker(1-w_\pm) \cap P$ are generally larger than $P_{\mathcal{I}(w_\pm)}$.

Let

$$M^{++}_{\mathrm{w}} = \{c^+_{w_+,\mu_1} c^-_{w_-,\mu_2} \mid (\mu_1, \mu_2) \in Y^{++}_{\mathrm{w}}\}, \tag{8.3}$$

$$M^{+}_{\mathrm{w}} = \{c^+_{w_+,\mu_1} c^-_{w_-,\mu_2} \mid (\mu_1, \mu_2) \in Y^{+}_{\mathrm{w}}\}. \tag{8.4}$$

Since $(1-w_\pm)\lambda > 0$ for all $\lambda \in P^+_{\mathcal{S}(w_\pm)}$, using (2.28) we obtain that $M^{++}_{\mathrm{w}} \subset M^+_{\mathrm{w}}$ are N_{w}-(bi)submodules of L_{w}. Although we will not need this below, we note that $N_{\mathrm{w}} \subset M^+_{\mathrm{w}}$, which follows from (7.28) and the fact that $(1-w_\pm)\lambda > 0$ for all $\lambda \in P^+_{\mathcal{S}(w_\pm)} \backslash \{0\}$.

Corollary 7.12 implies:

Lemma 8.3. *For all* $\mathrm{w} = (w_+, w_-) \in W \times W$*:*

(i) The algebra N_{w} *has a* $\mathbb{K}$*-basis consisting of*

$$c^+_{w_+,\mu} c^-_{w_-,\lambda}, \ \ \mu \in P^+_{\mathcal{S}(\mathrm{w})} \oplus P_{\mathcal{I}(\mathrm{w})}, \lambda \in P^+_{\mathcal{S}(\mathrm{w})}.$$

(ii) The spanning sets in (8.3) *and* (8.4) *are* $\mathbb{K}$*-bases of the* N_{w}*-modules* M^{++}_{w} *and* M^+_{w}*, respectively.*

The following proposition contains the main tool for the proof of Theorem 8.1.

Proposition 8.4. *For an arbitrary base field* $\mathbb{K}$*,* $q \in \mathbb{K}^*$ *which is not a root of unity and* $\mathrm{w} \in W \times W$*, we have*

$$\pi_{\mathrm{w}}(R_q[G]/I_{\mathrm{w}}) \subset M^+_{\mathrm{w}} \tag{8.5}$$

and

$$\pi_{\mathrm{w}}(I_{(1,1)}/I_{\mathrm{w}}) \subset M^{++}_{\mathrm{w}}. \tag{8.6}$$

8.2. **Proof of Proposition 8.4.** Theorem 7.8 implies the direct sum decomposition of N'_{w}-(bi)modules

$$S_{\mathrm{w}} = N'_{\mathrm{w}} \oplus S^\circ_{\mathrm{w}}, \ \ \text{where} \ \ S^\circ_{\mathrm{w}} = N'_{\mathrm{w}} \bigotimes_{\mathbb{K}} \mathrm{Span}(B'_{\mathrm{w}} \backslash \{1\}).$$

Denote by

$$\pi'_{\mathrm{w}} \colon S_{\mathrm{w}} \to N'_{\mathrm{w}}$$

the corresponding projection into the first summand. Recall the isomorphism (3.28) and eqs. (7.25)–(7.26). Clearly we have

$$\psi_{\mathrm{w}}\big((y_\lambda^{-1}\pi'_{\mathrm{w}}(s))\# c^-_{w_-,\mu}\big) = \pi_{\mathrm{w}}\big(\psi_{\mathrm{w}}((y_\lambda^{-1}s)\# c^-_{w_-,\mu})\big) \tag{8.7}$$

for all $s \in S_{\mathrm{w}}$, $\lambda \in P^+$ and $\mu \in P$.

For simplicity of the exposition we will split the proof of Proposition 8.4 into two parts: the proofs of (8.6) and (8.5). First note that

$$\begin{aligned} I_{(1,1)} = \mathrm{Span}\{c^{\lambda_1}_{\xi_1,\lambda_1} c^{-w_0\lambda_2}_{\xi_2,-\lambda_2} \mid \lambda_1, \lambda_2 \in P^+;\ \xi_1 \in (V(\lambda_1)^*)_{-\mu_1}, \\ \xi_2 \in (V(-w_0\lambda_2)^*)_{\mu_2};\ \ \mu_1, \mu_2 \in P,\ \mu_1 < \lambda_1 \ \text{ or } \ \mu_2 < \lambda_2\} \end{aligned} \tag{8.8}$$

and

$$R_q[G] = I_{(1,1)} \oplus \mathrm{Span}\{c^+_{1,\lambda_1} c^-_{1,\lambda_2} \mid \lambda_1, \lambda_2 \in P^+\}. \tag{8.9}$$

For two elements a and b of a $\mathbb{K}$-algebra R we denote

$$a \approx b, \quad \text{if} \quad a = q^m b \ \text{ for some } \ m \in \mathbb{Z}. \tag{8.10}$$

Proof of (8.6) *in Proposition 8.4.* Recall that the images of the elements $c^\lambda_{\xi,v} \in R_q[G]$ in $R_q[G]/I_{\mathrm{w}}$ are denoted by the same symbols.

Fix $\lambda_1, \lambda_2 \in P^+$, $\mu_1, \mu_2 \in P$, and $\xi_1 \in (V(\lambda_1)^*)_{-\mu_1}$, $\xi_2 \in (V(-w_0\lambda_2)^*)_{\mu_2}$. In view of (8.8) we need to prove that

$$\pi_{\mathrm{w}}(c^{\lambda_1}_{\xi_1,\lambda_1} c^{-w_0\lambda_2}_{\xi_2,-\lambda_2}) \in M^{++}_{\mathrm{w}} \tag{8.11}$$

in the following three cases: Case (1) $\mu_1 < \lambda_1$ and $\mu_2 < \lambda_2$; Case (2) $\mu_1 < \lambda_1$ and $\mu_2 = \lambda_2$; Case (3) $\mu_1 = \lambda_1$ and $\mu_2 > \lambda_2$. We will prove (8.11) in cases (1) and (2). Case (3) is analogous to (2) and is left to the reader.

Recall the definition (2.12) of the projections $g^\pm_{w_\pm}$. Denote for brevity

$$c = c^{\lambda_1}_{\xi_1,\lambda_1} c^{-w_0\lambda_2}_{\xi_2,-\lambda_2}. \tag{8.12}$$

Using the identification (2.41) and eq. (2.29) we obtain

$$c \approx c^+_{w_+,\lambda_1} c^-_{w_-,\lambda_2}.\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big)\big((c^-_{w_-,\lambda_2})^{-1} g^-_{w_-}(\xi_2)\big),$$

cf. (8.10). It follows form (7.25)–(7.26) and (2.28) that

$$(\psi_{\mathrm{w}})^{-1}(c) \approx \big[y^{-1}_{\lambda_1}\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big)\big((c^-_{w_-,\lambda_2})^{-1} g^-_{w_-}(\xi_2)\big)\big] \# c^-_{w_-,\lambda_2-\lambda_1}. \tag{8.13}$$

Case (1): Since $\mu_1 < \lambda_1$ and $\mu_2 < \lambda_2$ we have

$$\begin{aligned} \big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big)\big((c^-_{w_-,\lambda_2})^{-1} g^-_{w_-}(\xi_2)\big) \in (S_{\mathrm{w}})^{\mu_1 - w_+\lambda_1, \mu_2 - w_-\lambda_2} \\ \subset (S_{\mathrm{w}})^{\prec((1-w_+)\lambda_1, (1-w_-)\lambda_2)} \end{aligned}$$

in terms of the partial order $\prec$ on $Q \times Q$ from §7.2. Proposition 7.9 and (7.14) imply

$$\begin{aligned} &\pi'_{\mathrm{w}}\big[\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big)\big((c^-_{w_-,\lambda_2})^{-1} g^-_{w_-}(\xi_2)\big)\big] \\ &\quad \in \mathrm{Span}\{y_\lambda \mid \lambda \in P^+, (1-w_+)\lambda < (1-w_+)\lambda_1, (1-w_-)\lambda < (1-w_-)\lambda_2\}. \end{aligned}$$

It follows from (8.7) that

$$\pi_{\rm w}(c) \in \mathrm{Span}\{\psi_{\rm w}((y_{\lambda_1})^{-1} y_\lambda \# c^-_{w_-,\lambda_2-\lambda_1}) \mid \lambda \in P^+,\\ (1-w_+)(\lambda_1-\lambda) > 0, (1-w_-)(\lambda_2-\lambda) > 0\}.$$

Taking into account (7.25)–(7.26) and (7.28), we obtain that $\pi_{\rm w}(c)$ belongs to

$$\mathrm{Span}\{c^+_{w_+,\lambda_1-\lambda} c^-_{w_-,\lambda_2-\lambda} \mid \lambda \in P^+, (1-w_+)(\lambda_1-\lambda) > 0, (1-w_-)(\lambda_2-\lambda) > 0\}\\ \subseteq \mathrm{Span}\{c^+_{w_+,\nu_1} c^-_{w_-,\nu_2} \mid \nu_1, \nu_2 \in P, (1-w_+)\nu_1 > 0, (1-w_-)\nu_2 > 0\}\\ = \mathrm{Span}\{c^+_{w_+,\nu_1} c^-_{w_-,\nu_2} \mid (\nu_1,\nu_2) \in Y_1^{++} \times Y_2^{++}\} \subset M_{\rm w}^{++}.$$

Case (2): In this case $c^{-w_0\lambda_2}_{\xi_2,-\lambda_2}$ is a scalar multiple of $c^-_{w_-,\lambda_2}$ and after rescaling we can assume that

$$c = c^{\lambda_1}_{\xi_1,\lambda_1} c^-_{1,\lambda_2},$$

cf. (8.12). It follows from (8.13) that

$$(\psi_{\rm w})^{-1}(c) \approx \big[y^{-1}_{\lambda_1}\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big) d^-_{w_-,\lambda_2}\big] \# c^-_{w_-,\lambda_2-\lambda_1},$$

recall (3.29). Then

$$\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big) d^-_{w_-,\lambda_2} \in (S_{\rm w})^{(\mu_1 - w_+\lambda_1, (1-w_-)\lambda_2)}.$$

Proposition 7.9, (7.23), (7.14) and the assumption $\mu_1 < \lambda_1$ imply that

$$\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big) d^-_{w_-,\lambda_2} \in \big(S^\circ_{\rm w} \oplus \mathrm{Span}\{y_\lambda \mid \lambda \in P^+, \lambda - \lambda_2 \in P_{\mathcal{I}(w_-)},\\ (1-w_+)\lambda < (1-w_+)\lambda_1\}\big) + (S_{\rm w})^{\prec((1-w_+)\lambda_1,(1-w_-)\lambda_2)}.$$

Therefore

$$\pi'_{\rm w}\big[\big((c^+_{w_+,\lambda_1})^{-1} g^+_{w_+}(\xi_1)\big) d^-_{w_-,\lambda_2}\big] \in\\ \mathrm{Span}\{y_\lambda \mid \lambda \in P^+, (1-w_+)\lambda < (1-w_+)\lambda_1, \lambda-\lambda_2 \in P_{\mathcal{I}(w_-)}\}\\ \oplus \mathrm{Span}\{y_\lambda \mid \lambda \in P^+, (1-w_+)\lambda < (1-w_+)\lambda_1, (1-w_-)\lambda < (1-w_-)\lambda_2\}.$$

As in case (1), (8.7) implies that $\pi_{\rm w}(c)$ belongs to the span of the elements $\psi_{\rm w}((y_{\lambda_1})^{-1} y_\lambda \# c^-_{w_-,\lambda_2-\lambda_1})$ where $\lambda \in P^+$ and either

$$(1-w_+)(\lambda_1-\lambda) > 0, \lambda - \lambda_2 \in P_{\mathcal{I}(w_-)}, \tag{8.14}$$

or

$$(1-w_+)(\lambda_1-\lambda) > 0, (1-w_-)(\lambda_2-\lambda) > 0. \tag{8.15}$$

Eqs. (7.25), (7.26), and (7.28) imply that the span of these elements is the space

$$\mathrm{Span}\{c^+_{w_+,\lambda_1-\lambda} c^-_{w_-,\lambda_2-\lambda} \mid \lambda \in P^+ \text{ satisfies either (8.14) or (8.15)}\}\\ \subseteq \mathrm{Span}\{c^+_{w_+,\nu_1} c^-_{w_-,\nu_2} \mid (\nu_1,\nu_2) \in (Y_1^{++} \times P_{\mathcal{S}({\rm w}) \cap \mathcal{I}(w_-)}) \sqcup (Y_1^{++} \times Y_2^{++}), \} \subset M_{\rm w}^{++}$$

which completes the proof of (8.5). □

Recall the notation (7.17). We have the decomposition

$$P = P_{\mathcal{S}({\rm w})} \oplus P_{\mathcal{I}({\rm w})}.$$

For $\lambda \in P$ denote its components

$$\overline{\lambda} \in P_{\mathcal{S}(\mathrm{w})}, \quad \overline{\overline{\lambda}} \in P_{\mathcal{I}(\mathrm{w})} \tag{8.16}$$

with respect to this decomposition. Denote the delta function on P^+: for $\lambda_1, \lambda_2 \in P^+$

$$\delta_{\lambda_1,\lambda_2} = 1, \text{ if } \lambda_1 = \lambda_2; \ \delta_{\lambda_1,\lambda_2} = 0, \text{ otherwise.}$$

Eq. (8.5) in Proposition 8.4 follows from (8.6) and the following lemma.

Lemma 8.5. *For all $\lambda_1, \lambda_2 \in P^+$,*

$$\pi_{\mathrm{w}}(c^+_{1,\lambda_1} c^-_{1,\lambda_2}) \in \mathrm{Span}\{c^+_{w_+,\mu_1} c^-_{w_-,\mu_2} \mid (\mu_1,\mu_2) \in (P_{\mathcal{I}(w_+)} \times P_{\mathcal{S}(\mathrm{w})\cap\mathcal{I}(w_-)}) \sqcup (Y_1^{++} \times Y_2^{++})\} \subset M^+_{\mathrm{w}},$$

cf. (8.2) *and* (8.4)*.*

Proof. Set

$$c = c^+_{1,\lambda_1} c^-_{1,\lambda_2}.$$

Applying (8.13), (7.9), (7.18) and (7.19), we obtain

$$\begin{aligned}(\psi_{\mathrm{w}})^{-1}(c) &\approx \big(y^{-1}_{\lambda_1} d^+_{w_+,\lambda_1} d^-_{w_-,\lambda_2}\big) \# c^-_{w_-,\lambda_2-\lambda_1} \\ &\approx \big(y^{-1}_{\overline{\lambda_1}} d^+_{w_+,(\overline{\lambda_1})_0+(\overline{\lambda_1})_+} d^-_{w_-,(\overline{\lambda_2})_0+(\overline{\lambda_2})_-}\big) \# c^-_{w_-,\lambda_2-\lambda_1}.\end{aligned}$$

Proposition 7.9, and eqs. (7.14) and (7.23) imply that

$$d^+_{w_+,(\overline{\lambda_1})_0+(\overline{\lambda_1})_+} d^-_{w_-,(\overline{\lambda_2})_0+(\overline{\lambda_2})_-} \in \mathbb{K} y_{(\overline{\lambda_1})_0+(\overline{\lambda_1})_+ + (\overline{\lambda_2})_-} + (S_{\mathrm{w}})^{\prec((1-w_+)\lambda_1,(1-w_-)\lambda_2)},$$

if $(\overline{\lambda_1})_0 = (\overline{\lambda_2})_0$ and

$$d^+_{w_+,(\overline{\lambda_1})_0+(\overline{\lambda_1})_+} d^-_{w_-,(\overline{\lambda_2})_0+(\overline{\lambda_2})_-} \in S^\circ_{\mathrm{w}} + (S_{\mathrm{w}})^{\prec((1-w_+)\lambda_1,(1-w_-)\lambda_2)},$$

otherwise. Thus

$$\begin{aligned}&\pi'_{\mathrm{w}}(d^+_{w_+,(\overline{\lambda_1})_0+(\overline{\lambda_1})_+} d^-_{w_-,(\overline{\lambda_2})_0+(\overline{\lambda_2})_-}) \in \delta_{(\overline{\lambda_1})_0,(\overline{\lambda_2})_0} \mathbb{K} y_{(\overline{\lambda_1})_0+(\overline{\lambda_1})_+ + (\overline{\lambda_2})_-} \\ &\oplus \mathrm{Span}\{y_\lambda \mid \lambda \in P^+, (1-w_+)\lambda < (1-w_+)\lambda_1, (1-w_-)\lambda < (1-w_-)\lambda_2\}.\end{aligned}$$

As in cases (1) and (2) of the proof of (8.5), using (8.7), (7.25)–(7.26) and (2.28) we obtain:

$$\begin{aligned}\pi(c) \in &\delta_{(\overline{\lambda_1})_0,(\overline{\lambda_2})_0} \mathbb{K} c^+_{w_+,(\overline{\lambda_1})_- - (\overline{\lambda_2})_- + \overline{\overline{\lambda_1}}} c^-_{w_-,(\overline{\lambda_2})_+ - (\overline{\lambda_2})_+ + \overline{\overline{\lambda_2}}} \\ &\oplus \mathrm{Span}\{c^+_{w_+,\mu_1} c^-_{w_-,\mu_2} \mid (\mu_1,\mu_2) \in P_{\mathcal{I}(w_+)} \times P_{\mathcal{S}(\mathrm{w})\cap\mathcal{I}(w_-)}\}.\end{aligned}$$

Since (7.28) implies

$$c^+_{w_+,(\overline{\lambda_1})_- - (\overline{\lambda_2})_- + \overline{\overline{\lambda_1}}} c^-_{w_-,(\overline{\lambda_2})_+ - (\overline{\lambda_2})_+ + \overline{\overline{\lambda_2}}} \approx c^+_{w_+,(\overline{\lambda_1})_- - (\overline{\lambda_2})_- + \overline{\overline{\lambda_1}} - \overline{\overline{\lambda_2}}} c^-_{w_-,(\overline{\lambda_2})_+ - (\overline{\lambda_2})_+}$$

and in addition

$$\big((\overline{\lambda_1})_- - (\overline{\lambda_2})_- + \overline{\overline{\lambda_1}} - \overline{\overline{\lambda_2}}, (\overline{\lambda_2})_+ - (\overline{\lambda_2})_+\big) \in P_{\mathcal{I}(w_+)} \times P_{\mathcal{S}(\mathrm{w})\cap\mathcal{I}(w_-)},$$

we obtain the statement of the lemma. $\square$

8.3. **Proof of Theorem 8.1.** We first analyze the projections of the ideals $\pi_{\rm w}(J_{{\rm w},\zeta,\theta}/I_{\rm w})$ for the primitive ideals defined in §4.1. Our proof of Theorem 8.1 relies on a combination of this with Proposition 8.4.

Fix ${\rm w} = (w_+, w_-) \in W \times W$. Recall the definitions (3.5) and (3.6) of the lattices $\widetilde{\mathcal{L}}({\rm w})$ and $\widetilde{\mathcal{L}}({\rm w})_{\rm red}$. Recall from §4.1 that $\{\lambda^{(1)}, \ldots, \lambda^{(k)}\}$ is a basis of $\widetilde{\mathcal{L}}({\rm w})$. Denote by $J_{{\rm w},\mathbf{1},\mathbf{1}}$ the ideal (4.5) corresponding to $\zeta_j = 1$ for $j = 1, \ldots, k$ and $\theta_i = 1$ for $i \in \mathcal{I}({\rm w})$. Denote

$$J^0_{{\rm w},\mathbf{1},\mathbf{1}} = \sum_{j=1}^k R_q[G] b_j(1) + \sum_{i \in \mathcal{I}({\rm w})} R_q[G](c^+_{w_+,\omega_i} - 1) + I_{\rm w}.$$

Then

$$J_{{\rm w},\mathbf{1},\mathbf{1}} = \{ r \in R_q[G] \mid cr \in J_{{\rm w},\mathbf{1},\mathbf{1}} \ \text{ for some } \ c \in E_{\rm w} \},$$

cf. (2.23). Recall from Lemma 8.3 (ii) that $M^+_{\rm w}$ has a basis comprised of the elements in (8.4). Denote by $(M^+_{\rm w})_{\mathbf{1}}$ the subspace of $M^+_{\rm w}$ which consists of those elements

$$\sum_{(\mu_1,\mu_2) \in Y^+_{\rm w}} p_{\mu_1,\mu_2} c^+_{w_+,\mu_1} c^-_{w_-,\mu_2}, \quad p_{\mu_1,\mu_2} \in \mathbb{K},$$

which have the property that for all $(\mu_1, \mu_2) \in Y^+_{\rm w}$

$$\sum_{\lambda \in \widetilde{\mathcal{L}}({\rm w})} p_{\mu_1 + \lambda, \mu_2 - \overline{\lambda}} = 0,$$

recall (8.16). The subspace $(M^+_{\rm w})_{\mathbf{1}}$ is an $N_{\rm w}$ sub-bimodule of $M^+_{\rm w}$ by the following lemma.

Lemma 8.6. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in P$ and $\{p_\lambda \in \mathbb{K} \mid \lambda \in \widetilde{\mathcal{L}}({\rm w})\}$ be a collection of scalars of which only finitely many are nozero. Then:*

$$\begin{aligned} (8.17) \quad & c^+_{w_+,\nu_1} c^-_{w_-,\nu_2} \left(\sum_{\lambda \in \widetilde{\mathcal{L}}({\rm w})} p_\lambda c^+_{w_+,\mu_1+\lambda} c^-_{w_-,\mu_2 - \overline{\lambda}} \right) \\ & = q^{\langle w_-\nu_2, (w_+ - w_-)\mu_1 \rangle} \left(\sum_{\lambda \in \widetilde{\mathcal{L}}({\rm w})} p_\lambda c^+_{w_+,\mu_1+\nu_1+\lambda} c^-_{w_-,\mu_2+\nu_2-\overline{\lambda}} \right) \end{aligned}$$

and

$$\begin{aligned} (8.18) \quad & \left(\sum_{\lambda \in \widetilde{\mathcal{L}}({\rm w})} p_\lambda c^+_{w_+,\mu_1+\lambda} c^-_{w_-,\mu_2 - \overline{\lambda}} \right) c^+_{w_+,\nu_1} c^-_{w_-,\nu_2} \\ & = q^{\langle w_+\nu_1, (w_- - w_+)\mu_2 \rangle} \left(\sum_{\lambda \in \widetilde{\mathcal{L}}({\rm w})} p_\lambda c^+_{w_+,\mu_1+\nu_1+\lambda} c^-_{w_-,\mu_2+\nu_2-\overline{\lambda}} \right). \end{aligned}$$

Proof. Since $\widetilde{\mathcal{L}}({\rm w}) \subset \ker(w_+ - w_-)$, for $\lambda \in \widetilde{\mathcal{L}}({\rm w})$ we have

$$\begin{aligned} & \langle w_-\nu_2, w_+(\mu_1 + \lambda) \rangle - \langle \nu_2, \mu_1 + \lambda \rangle = \\ = & \langle w_-\nu_2, w_+(\mu_1 + \lambda) \rangle - \langle w_-\nu_2, w_-(\mu_1 + \lambda) \rangle = \langle w_-\nu_2, (w_+ - w_-)\mu_1 \rangle. \end{aligned}$$

Eq. (2.29) implies

$$c^-_{w_-,\nu_2} c^+_{w_+,\mu_1+\lambda} = q^{\langle w_-\nu_2, w_+(\mu_1+\lambda)\rangle - \langle \nu_2, \mu_1+\lambda\rangle} c^+_{w_+,\mu_1+\lambda} c^-_{w_-,\nu_2}$$
$$= q^{\langle w_-\nu_2, (w_+-w_-)\mu_1\rangle} c^+_{w_+,\mu_1+\lambda} c^-_{w_-,\nu_2}.$$

Now (8.17) follows from (2.28). Eq. (8.18) is proved in an analogous way using the fact that for $\lambda \in \widetilde{\mathcal{L}}(\mathrm{w})$, $\overline{\lambda} \in \widetilde{\mathcal{L}}(\mathrm{w})_{\mathrm{red}}$, cf. (3.6) and (3.8). □

The following result relates the image of the π_{w}-projection of $J_{\mathrm{w},\mathbf{1},\mathbf{1}}/I_{\mathrm{w}}$ and the above defined subspace of M^+_{w}.

Proposition 8.7. *For an arbitrary base field* $\mathbb{K}$, $q \in \mathbb{K}^*$ *which is not a root of unity, and* $\mathrm{w} = (w_+, w_-) \in W \times W$, *we have*

$$\pi_{\mathrm{w}}(J_{\mathrm{w},\mathbf{1},\mathbf{1}}/I_{\mathrm{w}}) \subseteq (M^+_{\mathrm{w}})_{\mathbf{1}}.$$

Proof. First we will prove that

$$\pi_{\mathrm{w}}(J^0_{\mathrm{w},\mathbf{1},\mathbf{1}}/I_{\mathrm{w}}) \subseteq (M^+_{\mathrm{w}})_{\mathbf{1}}. \tag{8.19}$$

Using Proposition 8.4, we see that to prove (8.19) it is sufficient to prove

$$M^+_{\mathrm{w}} b_j(1) \subseteq (M^+_{\mathrm{w}})_{\mathbf{1}}, \quad M^+_{\mathrm{w}}(c^+_{w_+,\omega_i} - 1) \subseteq (M^+_{\mathrm{w}})_{\mathbf{1}}, \tag{8.20}$$

for $j = 1, \dots, k$, $i \in \mathcal{I}(\mathrm{w})$. From (8.17) it follows that

$$c^+_{w_+,\lambda_1} c^-_{w_-,\lambda_2} \Big(c^+_{w_+,\lambda^{(j)}_+} c^-_{w_-,\lambda^{(j)}_-} - c^+_{w_+,\lambda^{(j)}_-} c^-_{w_-,\lambda^{(j)}_+} \Big) \in (M^+_{\mathrm{w}})_{\mathbf{1}},$$
$$c^+_{w_+,\lambda_1} c^-_{w_-,\lambda_2} (c^+_{w_+,\omega_i} - 1) \in (M^+_{\mathrm{w}})_{\mathbf{1}}, \quad \forall j = 1, \dots, k, i \in \mathcal{I}(\mathrm{w}), (\lambda_1, \lambda_2) \in Y^{++}_{\mathrm{w}},$$

recall (4.1). This proves (8.20) and thus (8.19).

Lemma 8.6 implies

$$\big((c^+_{w_+,\lambda_1} c^-_{w_-,\lambda_2})^{-1} (M^+_{\mathrm{w}})_{\mathbf{1}} \big) \cap M^+_{\mathrm{w}} \subseteq (M^+_{\mathrm{w}})_{\mathbf{1}}, \quad \forall \lambda_1, \lambda_2 \in P^+,$$

where the intersection in the left hand side is taken inside L_{w}. Since

$$\pi_{\mathrm{w}}(J_{\mathrm{w},\mathbf{1},\mathbf{1}}/I_{\mathrm{w}}) \subseteq \bigcup_{\lambda_1,\lambda_2 \in P^+} \Big(\big((c^+_{w_+,\lambda_1} c^-_{w_-,\lambda_2})^{-1} \pi_{\mathrm{w}}(J^0_{\mathrm{w},\mathbf{1},\mathbf{1}}/I_{\mathrm{w}}) \big) \cap M^+_{\mathrm{w}} \Big),$$

the proposition follows from (8.19). □

Next, we proceed with the proof of Theorem 8.1.

Proof of Theorem 8.1. First we establish the validity of the theorem for algebraically closed fields $\mathbb{K}$. Assume that the statement of the theorem is not correct, i.e. there exists $\mathrm{w} \in W \times W$, $\mathrm{w} \neq (1,1)$ such that

$$\mathrm{Max} R_q[G] \cap \mathrm{Spec}_{\mathrm{w}} R_q[G] \neq \emptyset. \tag{8.21}$$

Let $J \in \mathrm{Spec}_{\mathrm{w}} R_q[G]$ be a maximal ideal of $R_q[G]$. Theorem 2.3 (iii) implies that there exits $t \in \mathbb{T}^r$ such that

$$J \subseteq t.J_{\mathrm{w},\mathbf{1},\mathbf{1}}, \tag{8.22}$$

where in the right hand side we use the action (2.31). Since J is a maximal ideal, we have an equality in (8.22). Then

$$J_{\mathrm{w},\mathbf{1},\mathbf{1}} = t^{-1}.J$$

is also a maximal ideal of $R_q[G]$ since the $\mathbb{T}^r$-action (2.31) is by algebra automorphisms. Because $J_{\mathrm{w},\mathbf{1},\mathbf{1}} \in \mathrm{Spec}_{\mathrm{w}} R_q[G]$ and $\mathrm{w} \neq (1,1)$, we have $I_{(1,1)} \not\subseteq J_{\mathrm{w},\mathbf{1},\mathbf{1}}$. Therefore

$$J_{\mathrm{w},\mathbf{1},\mathbf{1}} + I_{(1,1)} = R_q[G]. \tag{8.23}$$

Thus there exits

$$c \in J_{\mathrm{w},\mathbf{1},\mathbf{1}} \text{ such that } c - 1 \in I_{(1,1)}.$$

Let

$$\pi_{\mathrm{w}}(c) = \sum_{(\mu_1,\mu_2)\in Y_{\mathrm{w}}^+} p_{\mu_1,\mu_2} c^+_{w_+,\mu_1} c^-_{w_-,\mu_2},$$

for some $p_{\mu_1,\mu_2} \in \mathbb{K}$. Observe that

$$(\lambda, -\overline{\lambda}) \notin Y_{\mathrm{w}}^{++}, \ \ \forall \lambda \in \widetilde{\mathcal{L}}(\mathrm{w}).$$

Indeed, for all $\lambda \in \widetilde{\mathcal{L}}(\mathrm{w})$,

$$(1-w_+)\lambda + (1-w_-)(-\overline{\lambda}) = (1-w_+)\lambda + (1-w_-)(-\lambda) = (w_- - w_+)\lambda = 0,$$

while every pair $(\mu_1, \mu_2) \in Y_{\mathrm{w}}^{++}$ has the property that

$$(1-w_+)\mu_1 \geq 0, \ (1-w_-)\mu_2 \geq 0, \text{ and at least one of the inequalities is strict,}$$

see Remark 8.2. Since $c-1 \in I_{(1,1)}$, applying Proposition 8.4 and Corollary 7.12, we obtain

$$p_{0,0} = 1 \text{ and } p_{\lambda,-\overline{\lambda}} = 0, \ \forall \lambda \in \widetilde{\mathcal{L}}(\mathrm{w}), \lambda \neq 0.$$

Therefore

$$\sum_{\lambda \in \widetilde{\mathcal{L}}(\mathrm{w})} p_{\lambda,-\overline{\lambda}} = 1,$$

which contradicts with $c \in J_{\mathrm{w},\mathbf{1},\mathbf{1}}$, see Proposition 8.7. This completes the proof of the proposition in the case when $\mathbb{K}$ is algebraically closed.

The general case of the theorem is obtained by a base change argument. Now assume that $\mathbb{K}$ is an arbitrary base field. Denote by $\overline{\mathbb{K}}$ its algebraic closure. For a $\mathbb{K}$-algebra R, denote $R_{\overline{\mathbb{K}}} = R \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. The algebra $(R_q[G])_{\overline{\mathbb{K}}}$ is isomorphic to the analog of the algebra $R_q[G]$ defined over the base field $\overline{\mathbb{K}}$. It is well known and easy to verify that the counterparts of I_{w} and $Z(R_{\mathrm{w}})$ for $(R_q[G])_{\overline{\mathbb{K}}}$ are $(I_{\mathrm{w}})_{\overline{\mathbb{K}}}$ and $Z((R_{\mathrm{w}})_{\overline{\mathbb{K}}})$. Denote by $\overline{\iota}_{\mathrm{w}} \colon Z((R_{\mathrm{w}})_{\overline{\mathbb{K}}}) \to \mathrm{Spec}_{\mathrm{w}}(R_q[G])_{\overline{\mathbb{K}}}$ the counterpart of ι_{w}.

Let $\mathrm{w} \in W \times W$, $\mathrm{w} \neq (1,1)$. If $J \in \mathrm{Spec}_{\mathrm{w}} R_q[G]$, then by Theorem 2.3 (ii)

$$J = \iota_{\mathrm{w}}(J^0), \ \text{ for some } \ J^0 \in \mathrm{Spec} Z(R_{\mathrm{w}}).$$

Moreover $(J^0)_{\overline{\mathbb{K}}}$ is a proper two sided ideal of $Z((R_{\mathrm{w}})_{\overline{\mathbb{K}}})$. Thus there exits a maximal ideal $\overline{J}^0$ of $Z((R_{\mathrm{w}})_{\overline{\mathbb{K}}})$, containing $(J^0)_{\overline{\mathbb{K}}}$. By Theorem 2.3 (ii)

$$\overline{\iota}_{\mathrm{w}}(\overline{J}^0) \in \mathrm{Spec}_{\mathrm{w}}(R_q[G])_{\overline{\mathbb{K}}}$$

and by Theorem 8.1 for algebraically closed base fields

$$\overline{\iota}_{\mathrm{w}}(\overline{J}^0) + (I_{(1,1)})_{\overline{\mathbb{K}}} \subsetneq (R_q[G])_{\overline{\mathbb{K}}},$$

because $\overline{\iota}_{\mathrm{w}}(\overline{J}^0)$ is contained in a maximal ideal which is necessarily in the stratum $\mathrm{Spec}_{(1,1)}(R_q[G])_{\overline{\mathbb{K}}}$. Therefore

$$(J + I_{(1,1)}) \otimes_{\mathbb{K}} \overline{\mathbb{K}} = \iota_{\mathrm{w}}(J^0) \otimes_{\mathbb{K}} \overline{\mathbb{K}} + (I_{(1,1)})_{\overline{\mathbb{K}}} \subseteq \overline{\iota}_{\mathrm{w}}(\overline{J}^0) + (I_{(1,1)})_{\overline{\mathbb{K}}} \subsetneq (R_q[G])_{\overline{\mathbb{K}}},$$

so

$$J + I_{(1,1)} \subsetneq R_q[G].$$

Consequently $J \notin \mathrm{Max} R_q[G]$, since $J + I_{(1,1)}$ is a proper two sided ideal of $R_q[G]$, properly containing J. The latter holds because all ideals in $\mathrm{Spec}_{\mathrm{w}} R_q[G]$ do not contain $I_{(1,1)}$. We obtain that $\mathrm{Max} R_q[G] \cap \mathrm{Spec}_{\mathrm{w}} R_q[G] = \emptyset$ for all $\mathrm{w} \in W \times W$, $\mathrm{w} \neq (1,1)$, which proves the theorem for general base fields $\mathbb{K}$. □

8.4. **Classification of** $\mathrm{Max} R_q[G]$ **and a question of Goodearl and Zhang.** By Theorem 8.1 each maximal ideal of $R_q[G]$ contains the ideal $I_{(1,1)}$. The structure of $R_q[G]/I_{(1,1)}$ is easy to describe; it is isomorphic to an r dimensional Laurent polynomial algebra over $\mathbb{K}$. From this we obtain an explicit classification of $\mathrm{Max} R_q[G]$ and an explicit formula for all maximal ideals of $R_q[G]$.

The quotient $R_q[G]/I_{(1,1)}$ is spanned by elements of the form $c^+_{1,\lambda_1} c^-_{1,\lambda_2}$, $\lambda_1, \lambda_2 \in P^+$. Since $\mathcal{I}(1,1) = \{1, \ldots, r\}$, Corollary 7.12 implies that

$$c^+_{1,\omega_i} c^-_{1,\omega_i} \in \mathbb{K}^*, \quad \forall i = 1, \ldots, r.$$

In fact

$$c^+_{1,\omega_i} c^-_{1,\omega_i} = 1, \quad \forall i = 1, \ldots, r,$$

because $x_{\omega_i} = c^+_{1,\omega_i} c^-_{1,\omega_i}$ in $\widehat{R}_{(1,1)}$ (cf. (3.26)), and under the canonical homomorphism $\widehat{R}_{(1,1)} \to R_{(1,1)}$, $x_{\omega_i} \mapsto 1$ (see §3.4 for details). Define the algebra homomorphism

$$\kappa \colon \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \to R_q[G]/I_{(1,1)}$$

by

$$\kappa(x_i) = c^+_{1,\omega_i}, \ \text{ i.e. } \ \kappa(x_i^{-1}) = c^-_{1,\omega_i}, \ \ i = 1, \ldots, r.$$

Lemma 8.8. *In the above setting, the map* $\kappa \colon \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \to R_q[G]/I_{(1,1)}$ *is an algebra isomorphism.*

Proof. From the above discussion we have that κ is surjective. It is injective by Proposition 7.11, which for $\mathrm{w} = (1,1)$ states that $L_{(1,1)}$ is an r dimensional Laurent polynomial algebra over $\mathbb{K}$ with generators $c^+_{w_+,\omega_i}$, $i = 1, \ldots, r$. □

Although we will not need this below, we note that the above arguments establish that

$$L_{(1,1)} = N_{(1,1)} = R_q[G]/I_{(1,1)}.$$

Denote the canonical projection

$$\Delta_{(1,1)} \colon R_q[G] \to R_q[G]/I_{(1,1)}.$$

The following result describes explicitly the maximal spectrum of $R_q[G]$ and provides an explicit formula for each maximal ideal.

Theorem 8.9. *Assume that* $\mathbb{K}$ *is an arbitrary base field,* $q \in \mathbb{K}^*$ *is not a root of unity. Then for each quantum group* $R_q[G]$ *we have the homeomorphism*

$$\mathrm{Max} R_q[G] \cong \mathrm{Max} \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}],$$

where both spaces are equipped with the corresponding Zariski topologies. Moreover the maximal ideal of $R_q[G]$ *corresponding to* $J' \in \mathrm{Max} \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ *is*

$$\Delta^{-1}_{(1,1)}(\kappa(J')).$$

Proof. The isomorphism κ induces a homeomorphism

$$\kappa\colon \mathrm{Max}\mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \xrightarrow{\cong} \mathrm{Max}(R_q[G]/I_{(1,1)}).$$

The theorem now follows from the fact that each maximal ideal of $R_q[G]$ contains $I_{(1,1)}$ by Theorem 8.1. □

The statement of Theorem 8.9 is even more explicit in the case of algebraically closed base fields $\mathbb{K}$.

Corollary 8.10. *If the base field $\mathbb{K}$ is algebraically closed and $q \in \mathbb{K}^*$ is not a root of unity, then each maximal ideal of $R_q[G]$ has the form*

$$I_{(1,1)} + (c^+_{1,\omega_1} - p_1)R_q[G] + \ldots + (c^+_{1,\omega_r} - p_r)R_q[G], \tag{8.24}$$

for some $p_1, \ldots, p_r \in \mathbb{K}^*$.

Note that in (8.24) one only needs to multiply the terms $(c^+_{1,\omega_i} - p_i)$, $i = 1, \ldots, r$ by polynomials in c^+_{1,ω_i}, c^-_{1,ω_i}, $i = 1, \ldots, r$, because the rest is absorbed by $I_{(1,1)}$, see Lemma 8.8.

Another consequence of Theorem 8.1 is the following result.

Corollary 8.11. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ which is not a root of unity and for each quantum group $R_q[G]$ we have*

$$\mathrm{Max}R_q[G] = \mathrm{Prim}_{(1,1)}\, R_q[G].$$

Finally using Theorem 8.9, we settle a question of Goodearl and Zhang [18]. The next section contains a detailed discussion of the implications of this question.

Corollary 8.12. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. Then all maximal ideals of the quantum function algebras $R_q[G]$ have finite codimension. If the base field $\mathbb{K}$ is algebraically closed, then all maximal ideals of $R_q[G]$ have codimension one.*

Proof. By Theorem 8.9, if $J \in \mathrm{Max}R_q[G]$, then there exists

$$J' \in \mathrm{Max}\mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \text{ such that } J = \Delta^{-1}_{(1,1)}\kappa(J').$$

Clearly

$$R_q[G]/J \cong \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]/J'.$$

Since the latter is a quotient of a commutative algebra by a maximal ideal, it is finite dimensional. Thus $\dim R_q[G]/J < \infty$. □

9. Chain properties and homological applications

9.1. **Applications.** This section contains applications of the results from the previous sections to chain properties of ideals and homological properties of $R_q[G]$.

We start by recalling two theorems of Goodearl and Zhang, and Lu, Wu and Zhang.

Theorem 9.1. *(Goodearl–Zhang* [18, Theorem 01.]*) Assume that A is a Hopf algebra over a field $\mathbb{K}$ which satisfies the following three conditions:*

(H1) A is noetherian and Spec$A$ *is normally separated, i.e. for each two prime ideals $J_1 \subseteq J_2$, there exists a normal regular element in the ideal $J_2/J_1 \subseteq A/J_1$.*

(H2) A has an exhaustive ascending $\mathbb{N}$-filtration such that the associated graded algebra $\operatorname{gr} A$ *is connected graded noetherian with enough normal elements, i.e. every simple graded prime factor algebra of* $\operatorname{gr} A$ *contains a homogeneous normal element of positive degree.*

(H3) Every maximal ideal of A is of finite codimension over $\mathbb{K}$.

Then A is Auslander–Gorenstein and Cohen–Macauley, and has a quasi-Frobenius classical quotient ring. Furthermore, Spec$A$ *is catenary and Tauvel's height formula holds.*

Theorem 9.2. *(Lu–Wu–Zhang* [36, Theorem 0.4]*) Assume that A is a noetherian Hopf algebra which satisfies the condition (H2). Then A is Auslander–Gorenstein and Cohen–Macauley.*

Lu, Wu and Zhang also proved several other properties of noetherian Hopf algebras with the property (H2). We refer the reader to [36, Theorem 0.4] for details.

We recall that a ring R is *catenary* if each two chains of prime ideals between two prime ideals of R have the same length. *Tauvel's height formula* holds for R, if for each prime ideal I of R its height is equal to

$$\operatorname{GK}\dim R - \operatorname{GK}\dim(R/I).$$

Recall that a ring R satisfies the first chain condition for prime ideals if all maximal chains in Spec$R$ have the same length equal to $\operatorname{GK}\dim R$. Such a ring is necessarily catenary. this notion was introduced by Nagata [39] in the commutative case. We refer the reader to Ratliff's book [40] for an exposition of chain conditions for prime ideals.

Corollary 9.3. *If A is a Hopf algebra over the field $\mathbb{K}$ which satisfies the conditions (H1),(H2) and (H3), then A satisfies the first chain condition for prime ideals.*

Proof. If $I \in \operatorname{Max}A$, then its height is equal to

$$\operatorname{GK}\dim R - \operatorname{GK}\dim(R/I) = \operatorname{GK}\dim R.$$

Here $\operatorname{GK}\dim(R/I) = 0$ since I has finite codimension. □

We now turn to applications to the quantum function algebras $R_q[G]$ and their Hopf algebra quotients. Among the three conditions (H1), (H2) and (H3) for $R_q[G]$, the third turned out to be the hardest to prove. It was shown by Hodges and Levasseur [20] for $\mathfrak{g} = \mathfrak{sl}_2$, and Goodearl and Lenagan [15] for $\mathfrak{g} = \mathfrak{sl}_3$, but was unknown for any other simple Lie algebra $\mathfrak{g}$. Corollary 8.12 establishes the validity of (H3) for $R_q[G]$ in full generality.

Regarding condition (H1) for $R_q[G]$, Joseph proved [24, 25] that $R_q[G]$ is noetherian, and Brown and Goodearl [5] proved that Spec$R_q[G]$ is normally separated. These facts are true for an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity. In this generality, the noetherianity of $R_q[G]$ was proved in the book of Brown and Goodearl [6, Theorem I.8.18]. The proof of [5] of the normal separation of Spec$R_q[G]$ works in this generality. We briefly sketch a well known proof of this. Consider the action (2.31) of $\mathbb{T}^r$ on $R_q[G]$. Then $\mathbb{T}^r - \mathrm{Spec}R_q[G]$ is $\mathbb{T}^r$-normally separated. Indeed, assume that I' is a $\mathbb{T}^r$-stable prime ideal of $R_q[G]$ containing the ideal I_{w} for some $\mathrm{w} \in W \times W$ (i.e. I' is equal to one of the ideals $I_{\mathrm{w}'}$ for some $\mathrm{w}' \in W \times W$). Then $E_{\mathrm{w}} \cap I' \neq \emptyset$ by Theorem 2.3 (i). Any element of $E_{\mathrm{w}} \cap I'$ provides the $\mathbb{T}^r$-normal separation of I_{w} and I'. The normal separation of Spec$R_q[G]$ follows from the $\mathbb{T}^r$-normal separation of $\mathbb{T}^r - \mathrm{Spec}R_q[G]$ by [13, Corollary 4.6]. The existence of separating regular normal elements follows from Joseph's theorem [24, Theorem 11.4] that all prime ideals of $R_q[G]$ are completely prime. His proof works for an arbitrary base field $\mathbb{K}$ (and $q \in \mathbb{K}^*$ not a root of unity). One can also prove this invoking two results of Joseph (which are slighly shorter to verify under the weaker assumptions on $\mathbb{K}$) and a result of Goodearl and Letzter. If $J \in \mathrm{Spec}R_q[G]$, then by [25, Corollary 9.3.9] $J \cap E_1^+ = \emptyset$, recall (2.22). By [25, Lemma 9.2.13, Proposition 9.2.14] $R_q[G]\,[(E_1^+)^{-1}]$ is an iterated skew polynomial/Laurent extension of $\mathbb{K}$. Applying [16, Theorem 2.1], one obtains that the quotient

$$(R_q[G]/J)[(E_1^+)^{-1}] \cong (R_q[G]\,[(E_1^+)^{-1}])/(J[(E_1^+)^{-1}])$$

is a domain, where by abuse of notation we denote the image of E_1^+ in $R_q[G]/J$ by the same symbol. Thus $R_q[G]/J$ is a domain and J is a completely prime ideal.

The condition (H2) for $R_q[G]$ was proved by Goodearl and Zhang [18] under the assumptions that $\mathbb{K} = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$. One can show that their proof works in the general situation by using the fact that all H-primes of the De Concini–Kac–Procesi algebras are polynormal, proved in [45]. Instead, we offer a new and elementary proof of this in the next subsection.

Thus $R_q[G]$ satisfies all three conditions (H1), (H2) and (H3). This implies that any quotient of $R_q[G]$ also satisfies them. Applying Theorem 9.1 we obtain:

Theorem 9.4. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. Let I be a Hopf algebra ideal of any of the quantized function algebras $R_q[G]$. Then $R_q[G]/I$ satisfies the first chain condition for prime ideals and Tauvel's height formula holds. In addition, it is Auslander–Gorenstein and Cohen–Macauley, and has a quasi-Frobenius classical quotient ring.*

The Gelfand–Kirillov dimension of $R_q[G]$ is equal to $\dim \mathfrak{g}$. The fact that $R_q[G]$ has the property that all maximal chains of prime ideals of $R_q[G]$ have the same length equal to $\dim \mathfrak{g}$ is new for all $\mathfrak{g} \neq \mathfrak{sl}_2, \mathfrak{sl}_3$ (the two special cases are in [20, 17] combined with [18]). Previously Goodearl and Lenagan [14] proved that $R_q[SL_n]$ is catenary and Tauvel's height formula holds. For $\mathbb{K} = \mathbb{C}$ and q transcendental over $\mathbb{Q}$, Goodearl and Zhang proved that $R_q[G]$ is catenary and Tauvel's height formula holds. The Auslander–Gorenstein and Cohen–Macauley properties of $R_q[SL_n]$ were established by Levasseur and Stafford [32]. In the case when $\mathbb{K} = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$, those properties of $R_q[G]$ were proved by Goodearl and Zhang, and for all Hopf algebra quotients $R_q[G]/I$ by Lu, Wu and Zhang, based on Theorems 9.1 and 9.2, respectively.

9.2. $R^+ \circledast R^-$ **is an algebra with enough normal elements.** The algebra $R^+ \circledast R^-$ has a canonical $\mathbb{N}$-filtration with respect to which it is connected. We prove that its augmentation ideal is polynormal from which we deduce that $R^+ \circledast R^-$ is an algebra with enough normal elements. We use this to give an elementary proof of the fact that $R_q[G]$ satisfies the condition (H2) from the previous subsection, under the assumption that $q \in \mathbb{K}^*$ is not a root of unity and without any restrictions on the characteristic of $\mathbb{K}$.

Recall the definition of $R^+ \circledast R^-$ from §3.4. It is a noetherian algebra, see [25, Proposition 9.1.11]. One can also prove this analogously to [6, Theorem I.8.18]. For $\lambda \in P$ denote

$$\mathrm{ht}(\lambda) = \langle \lambda, \alpha_1^\vee + \ldots + \alpha_r^\vee \rangle.$$

Because of (2.10), the algebras $R^\pm$ are connected $\mathbb{N}$-graded by imposing

$$\deg c^{\lambda_1}_{\xi_1, \lambda_1} = \mathrm{ht}(\lambda_1), \ \ \xi_1 \in V(\lambda_1)^*, \quad \deg c^{-w_0\lambda_2}_{\xi_2, -\lambda_2} = \mathrm{ht}(\lambda_2), \ \ \xi_2 \in V(-w_0\lambda_2)^*,$$

for all $\lambda_1, \lambda_2 \in P^+$, recall (2.9). It follows from (3.17) and (3.18) that

$$\deg(c^{\lambda_1}_{\xi_1,\lambda_1} c^{-w_0\lambda_2}_{\xi_2,-\lambda_2}) = \mathrm{ht}(\lambda_1) + \mathrm{ht}(\lambda_2), \ \ \lambda_1, \lambda_2 \in P^+, \xi_1 \in V(\lambda)^*, \xi_2 \in V(-w_0\lambda)^* \tag{9.1}$$

makes $R^+ \circledast R^-$ a connected $\mathbb{N}$-graded algebra. Denote by I_{++} its augmentation ideal, spanned by elements of positive degree.

Recall that an ideal J of a ring R is called polynormal if it has a sequence of generators $c_1, \ldots, c_n$ such that c_i is normal modulo the ideal generated by $c_1, \ldots, c_{i-1}$, for $i = 1, \ldots, n$.

For $i = 1, \ldots, r$ fix a basis B_i of $V(\omega_i)^*$ consisting of weight vectors. Let $B = B_1 \sqcup \ldots \sqcup B_r$ and $\overline{B} = B \times \{+, -\}$, where the second term is the set with two elements $+$ and $-$. For $\eta = \{\xi, s\} \in \overline{B}$, $\xi \in V(\omega_i)^*_\mu$, denote $[\eta]_1 = i$, $[\eta]_2 = \mu$, $[\eta]_3 = s$ and

$$c(\eta) = c^{\omega_i}_{\xi, \omega_i}, \quad \text{if } \ s = +,$$
$$c(\eta) = c^{\omega_i}_{\xi, -w_0\omega_i}, \quad \text{if } \ s = -.$$

(We recall that for each $i = 1, \ldots, r$, there exists $j = 1, \ldots, r$ such that $-w_0(\omega_i) = \omega_j$.) It is well known that $\{c(\eta) \mid \eta \in \overline{B}\}$ is a generating set of the algebra $R^+ \circledast R^-$. In particular, this set generates the augmentation ideal I_{++}. Fix any linear order on $\overline{B}$ with the properties that:

$$\text{if } \ \eta, \eta' \in B \ \text{ and } \ [\eta]_1 = [\eta']_1, [\eta]_2 > [\eta']_2, [\eta]_3 = [\eta']_3 = +, \quad \text{then } \ \eta < \eta', \tag{9.2}$$

$$\text{if } \ \eta, \eta' \in B \ \text{ and } \ [\eta]_1 = [\eta']_1, [\eta]_2 < [\eta']_2, [\eta]_3 = [\eta']_3 = -, \quad \text{then } \ \eta < \eta', \tag{9.3}$$

where we use the order (2.2) on P.

Theorem 9.5. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. If $\{\eta_1 < \ldots < \eta_N\}$ is a linear order on $\overline{B}$ satisfying* (9.2)–(9.3)*, then*

$$c(\eta_1), \ldots, c(\eta_N)$$

is a polynormal generating sequence of I_{++}.

Proof. Fix $\eta_k = (\xi, s)$. We will prove that $c(\eta_k)$ is normal modulo the ideal generated by $c(\eta_1), \ldots, c(\eta_{k-1})$ in the case $s = +$. The case $s = -$ is treated analogously.

We have $\xi \in V(\omega_i)^*_\mu$, where $i = [\eta_k]_1$ and $\mu = [\eta_k]_2$. It follows from (9.2) that there exists a subset $\{j_1, \ldots, j_l\} \subseteq \{1, \ldots, k-1\}$ such that

$$\eta_{j_m} = (\xi_{j_m}, +), m = 1, \ldots, l \text{ for some } \xi_{j_m} \in B_i$$

with the property that

$$\{\xi_{j_1}, \ldots, \xi_{j_l}\} \text{ is a basis of } \bigoplus_{\mu' \in P, \mu' > \mu} V(\omega_i)^*_{\mu'}.$$

Lemma 2.2 (i) and (3.17) imply that for all $a \in (R^\pm)_{\nu,\lambda} \subset R^+ \circledast R^-$, $\nu, \lambda \in P$:

$$ac(\eta_k) = q^{\langle \omega_i, \lambda \rangle - \langle \mu, \nu \rangle} c(\eta_k) a \mod \sum_{m=1}^{l} c(\eta_{j_m})(R^+ \circledast R^-).$$

This completes the proof of the theorem. □

The second part of the following corollary proves that $R_q[G]$ satisfies the property (H2) from the previous subsection.

Corollary 9.6. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity.*

(i) Then the algebra $R^+ \circledast R^-$ is a connected $\mathbb{N}$-graded noetherian algebra with respect to the grading (9.1) *with enough normal elements.*

(ii) Consider the induced ascending $\mathbb{N}$-filtration on $R^+ \circledast R^-$ from the grading (9.1) *and the induced $\mathbb{N}$-filtration on $R_q[G]$ from the canonical surjective homomorphism $R^+ \circledast R^- \to R_q[G]$, recall §3.4. Then the associated graded $\operatorname{gr} R_q[G]$ is a connected $\mathbb{N}$-graded noetherian algebra with enough normal elements.*

Proof. (i) Let J be a graded ideal of $R^+ \circledast R^-$ of codimension strictly greater than 1. Then I_{++} is not contained in J. Let $c_1, \ldots, c_n$ be a polynormal generating sequence of I_{++}. If c_k is the first element in the sequence which has a nonzero image in $(R^+ \circledast R^-)/J$, then this image is a nonzero normal element of the quotient. This proves (i).

Part (ii) follows from part (i), because $\operatorname{gr} R_q[G]$ is a graded quotient of $\operatorname{gr}(R^+ \circledast R^-) \cong R^+ \circledast R^-$. □

References

[1] H. H. Andersen, P. Polo and K. X. Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), 1–59.

[2] J. P. Bell, and S. Launois, *On the dimension of H-strata in quantum algebras*, Algebra Number Theory **4** (2010), 175–200.

[3] A. Berenstein, S. Fomin and A. Zelevinsky, *Cluster algebras III, Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[4] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math **155** (2005), 405–455.

[5] K. A. Brown and K. R. Goodearl, *Prime spectra of quantum semisimple groups*, Trans. Amer. Math. Soc. **348** (1996), 2465–2502.

[6] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[7] Ph. Caldero, *Sur le centre de $U_q(\mathfrak{n}^+$*, Beitr"age Algebra Geom. **35** (1994) 13–24. Festschrift on the occasion of the 65th birthday of Otto Krötenheerdt.

[8] Ph. Caldero, *Sur le centre de $U_q(\mathfrak{n}^+)$*. Bieträge Algebra Geom. **35** (1994), 13–24.

[9] G. Cauchon, *Effacement des dérivations et spectres premiers d'algébres quantiques,* J. Algebra 260 (2003) 476–518.

[10] A. W. Chatters, *Non-commutative unique factorisation domains*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 49–54.

[11] C. De Concini, V. Kac and C. Procesi, *Some quantum analogues of solvable Lie groups,* In: Geometry and analysis (Bombay, 1992), pp. 41-65, Tata Inst. Fund. Res., Bombay, 1995.

[12] C. Geiß, B. Leclerc and J. Schröer, *Cluster structures on quantized coordinate rings*, preprint arXiv:1104.0531.

[13] K. R. Goodearl, *Prime spectra of quantized coordinate rings*, In: Interactions between ring theory and representations of algebras (Murcia), pp. 205–237, Lect. Notes in Pure and Appl. Math. 210, Dekker, New York, 2000.

[14] K. R. Goodearl and T. H. Lenagan, *Catenarity in quantum algebras,* J. Pure Appl. Algebra **111** (1996), no. 1-3, 123–142.

[15] K. R. Goodearl and T. H. Lenagan, *Primitive ideals in quantum SL_3 and GL_3*, preprint arXiv:1008.4462.

[16] K. R. Goodearl and E. S. Letzter, *Prime factor algebras of the coordinate ring of quantum matrices*, Proc. Amer. Math. Soc. **121**(1994), 1017–1025.

[17] K. R. Goodearl and E. S. Letzter, *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras,* Trans. Amer. Math. Soc. **352** (2000), 1381–1403.

[18] K. R. Goodearl and J. J. Zhang, *Homological properties of quantized coordinate rings of semisimple groups*, Proc. Lond. Math. Soc. (3) **94** (2007), 647–671.

[19] M. Gorelik, *The prime and the primitive spectra of a quantum Bruhat cell translate,* J. Algebra **227** (2000), 211–253.

[20] T. J. Hodges and T. Levasseur, *Primitive ideals of* $\mathbb{C}_q[\mathrm{SL}(3)]$, Comm. Math. Phys. **156** (1993), 581–605.

[21] T. J. Hodges and T. Levasseur, *Primitive ideals of* $\mathbb{C}_q[\mathrm{SL}(n)]$, J. Algebra **168** (1994), 455–468.

[22] T. J. Hodges, T. Levasseur and M. Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. **126** (1997), 52–92.

[23] J. C. Jantzen, *Lectures on quantum groups,* Grad. Studies in Math., 6. Amer. Math. Soc., Providence, RI, 1996.

[24] A. Joseph, *On the prime and primitive spectra of the algebra of functions on a quantum group*, J. Algebra **169** (1994), 441–511.

[25] A. Joseph, *Quantum groups and their primitive ideals,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1995.

[26] A. Joseph, *Sur les idéaux génériques de l'agèbre des fonctions sur un groupe quantique*, C. R. Acad. Sci. Paris Sér. I. Math. **321** (1995), 135–140.

[27] A. Joseph and G. Letzter, *Separation of variables for quantized enveloping algebras*, Amer. J. Math. **116** (1994), 127–177.

[28] M. Kogan and A. Zelevinsky, *On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups*, Int. Math. Res. Not. **2002**, no. 32, 1685–1702.

[29] S. Kolb, *The AS-Cohen-Macaulay property for quantum flag manifolds of minuscule weight*, J. Algebra **319** (2008), 3518–3534.

[30] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[31] S. Launois, T. H. Lenagan and L. Rigal, *Quantum unique factorisation domains*, J. London Math. Soc. (2) **74** (2006), 321–340.

[32] T. Levasseur and J. T. Stafford, *The quantum coordinate ring of the special linear group,* J. Pure Appl. Algebra **86** (1993), no. 2, 181–186.

[33] S. Z. Levendorskii and Ya. S. Soibelman, *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Comm. Math. Phys. *139* (1991), 141–170.

[34] S. A. Lopes, *Separation of variables for* $U_q(\mathfrak{sl}_{n+1})^+$, Canad. Math. Bull. **48** (2005), 587–600.

[35] S. A. Lopes, *Primitive ideals of* $U_q(\mathfrak{sl}_n^+)$, Comm. Alg. **34** (2006), 4523–4550.
[36] D.-M. Lu, Q.-S. Wu and J. J. Zhang, *Hopf algebras with rigid dualizing complexes*, Israel J. Math. **169** (2009), 89–108.
[37] G. Lusztig, *Introduction to quantum groups*, Progr. Math. 110, Birkhäuser, 1993.
[38] A. Mériaux and G. Cauchon, *Admissible diagrams in quantum nilpotent algebras and combinatoric properties of Weyl groups,* Represent. Theory **14** (2010), 645–687.
[39] M. Nagata, *On the chain problem of prime ideals*, Nagoya Math. J. **10** (1956), 51–64.
[40] L. J. Ratliff, Jr., *Chain conjectures in ring theory. An exposition of conjectures on catenary chains*, Lect. Notes in Math. 647, Springer, Berlin, 1978.
[41] N. Reshetikhin and M. Yakimov, *Quantum invariant measures*, Comm. Math. Phys. **224** (2001), 399–426.
[42] B. Webster and M. Yakimov, *A Deodhar-type stratification on the double flag variety*, Transform. Groups **12** (2007), 769–785.
[43] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. London Math. Soc. (3) **101** (2010), no. 2, 454–476,
[44] M. Yakimov, *A classification of* H*-primes of quantum partial flag varieties*, Proc. Amer. Math. Soc. **138** (2010), 1249–1261.
[45] M. Yakimov, *A proof of the Goodearl–Lenagan polynormality conjecture*, in preparation.
[46] S. Zwicknagl, *Poisson ideals in cluster algebras and the spectra of quantized coordinate rings*, preprint.

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