

CRYSTAL BASES OF MODIFIED QUANTIZED
ENVELOPING ALGEBRA

MASAKI KASHIWARA

0. Introduction.

0.1. G. Lusztig gives the crystal base on the modified quantized enveloping algebra $\tilde{U}_q(\mathfrak{g})$ in [L2]. The algebra $\tilde{U}_q(\mathfrak{g})$ is obtained from the quantized universal enveloping algebra $U_q(\mathfrak{g})$ by modifying the torus part $\bigoplus_{h \in P^*} \mathbb{Q}(q)q^h$ to $\bigoplus_{\lambda} \mathbb{Q}(q)a_{\lambda}$, where a_{λ} is the projector to the weight space of weight λ (see §1.2). He gives also several conjectures on its properties in [L3]. The purpose of this paper is to study the structure of crystal bases $B(\tilde{U}_q(\mathfrak{g}))$ of $\tilde{U}_q(\mathfrak{g})$ and to give an affirmative answer to some of his conjectures.

0.2. Let us explain the results obtained here more precisely. We establish that the crystal structure of $\tilde{U}_q(\mathfrak{g})$ is described by those of $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$. Namely, let $B(\infty)$ be the crystal base of $U_q^-(\mathfrak{g})$ and $B(-\infty)$ the one of $U_q^+(\mathfrak{g})$. Let T_{λ} be the crystal consisting of a single element of weight λ . Then the crystal base of $\tilde{U}_q(\mathfrak{g})$ is isomorphic to the direct sum of $B(\infty) \otimes T_{\lambda} \otimes B(-\infty)$ (Theorem 3.1.1). This fact is a reflection of $\tilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda} U_q^-(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) \otimes \mathbb{Q}(q)a_{\lambda}$. The algebra $\tilde{U}_q(\mathfrak{g})$ has the antiautomorphism $*$ that sends e_i, f_i to themselves and a_{λ} to $a_{-\lambda}$. We prove that the crystal base is stable by $*$ (Theorem 4.3.2). This is one of the conjectures of Lusztig [L3]. This automorphism sends $b_1 \otimes t_{\lambda} \otimes b_2 \in B(\infty) \otimes T_{\lambda} \otimes B(-\infty) \subset B(\tilde{U}_q(\mathfrak{g}))$ to $b_1^* \otimes t_{-\lambda - wt b_1 - wt b_2} \otimes b_2^*$. By this automorphism, $B(\tilde{U}_q(\mathfrak{g}))$ has another crystal structure. These two structures are compatible (see §5), and $B(\tilde{U}_q(\mathfrak{g}))$ may be regarded as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$. This is a reflection of the $U_q(\mathfrak{g})$ -bimodule structure of $\tilde{U}_q(\mathfrak{g})$.

0.3. In [K2], the author introduces "the dual algebra" $A_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ and its crystal base $B(A_q(\mathfrak{g}))$. This algebra has the Peter-Weyl-type decomposition

$$A_q(\mathfrak{g}) = \bigoplus_{\lambda \in P_+} V^*(\lambda) \otimes V(\lambda).$$

Here P_+ is the set of dominant integral weights and $V(\lambda)$ and $V^*(\lambda)$ are the left and right highest-weight module with highest weight λ . Accordingly $B(A_q(\mathfrak{g}))$ has the crystal structure

$$B(A_q(\mathfrak{g})) = \bigoplus_{\lambda \in P_+} B(\lambda) \otimes B(\lambda).$$

Received 25 March 1993.

When \mathfrak{g} is finite-dimensional, $B(\tilde{U}_q(\mathfrak{g}))$ is isomorphic to $B(A_q(\mathfrak{g}))$. When \mathfrak{g} is affine, $\tilde{U}_q(\mathfrak{g})$ is the direct sum of three algebras $\tilde{U}_q(\mathfrak{g})_+$, $\tilde{U}_q(\mathfrak{g})_0$, and $\tilde{U}_q(\mathfrak{g})_-$. They consist of elements of positive level, level 0, and negative level, respectively. Accordingly $B(\tilde{U}_q(\mathfrak{g}))$ is the direct sum of $B(\tilde{U}_q(\mathfrak{g})_+)$, $B(\tilde{U}_q(\mathfrak{g})_0)$, and $B(\tilde{U}_q(\mathfrak{g})_-)$. The crystal structure of $B(\tilde{U}_q(\mathfrak{g})_+)$ is rather simple. Namely, $B(\tilde{U}_q(\mathfrak{g})_+) \oplus B(0)$ is isomorphic to $B(A_q(\mathfrak{g}))$. Here $B(0)$ is the crystal base of the trivial representation. However, the author does not know much about the structure of $B(\tilde{U}_q(\mathfrak{g})_0)$. The result of [IIMNT] suggests that $B(\tilde{U}_q(\mathfrak{g})_0)$ is a direct sum of the affinization of the crystals of finite-dimensional $\tilde{U}_q(\mathfrak{g})$ -modules. We show here one property: for any connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, $\{wt(b), wt(b); b \in B'\}$ is bounded from above (see §9.3).

0.4. We show also that the Weyl group operates on $B(\tilde{U}_q(\mathfrak{g}))$ or more generally on the crystal base of integrable $U_q(\mathfrak{g})$ -modules. The action S_i of simple reflection is given by

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, wt(b) \rangle} b & \text{if } \langle h_i, wt(b) \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, wt(b) \rangle} b & \text{if } \langle h_i, wt(b) \rangle \leq 0. \end{cases}$$

We show (Theorem 7.2.2) that $\{S_i\}$ satisfies the braid relation. We call a crystal base b an extremal vector if, for any $w \in W$ and $i \in I$, $S_w b$ is killed by either \tilde{e}_i or \tilde{f}_i . We show another remarkable property of $B(\tilde{U}_q(\mathfrak{g}))$. Let B' be a connected component of $B(\tilde{U}_q(\mathfrak{g}))$. Then B' may not contain either a highest-weight vector or a lowest-weight vector, but it always contains an extremal vector.

1. Notations.

1.1. Definition of quantized enveloping algebra. Let us recall the definition of $\tilde{U}_q(\mathfrak{g})$ (cf. [K]). We prepare the following data:

- (1.1.1) a free \mathbb{Z} -module P (weight lattice),
- (1.1.2) an index set I and $\alpha_i \in P$ and $h_i \in P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ for $i \in I$,
- (1.1.3) a \mathbb{Q} -valued symmetric bilinear form $(\ , \)$ on P .

We assume for the sake of simplicity that there exist $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for any $j \in I$. We call Λ_i the fundamental weight. We assume further that $\{\alpha_i\}_{i \in I}$ is linearly independent. Many of the results in this paper still hold without these assumptions. Assume that they satisfy the conditions

$$(1.1.4) \quad (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \quad \text{for } i \in I,$$

$$(1.1.5) \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for } i \in I \text{ and } \lambda \in P,$$

$$(1.1.6) \quad (\alpha_i, \alpha_j) \leq 0 \quad \text{for } i \neq j \in I.$$

The quantized enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ -algebra generated by e_i, f_i ($i \in I$), and q^h ($h \in P^*$) satisfying the defining relations

$$(1.1.7) \quad q^h = 1 \quad \text{for } h = 0;$$

$$(1.1.8) \quad q^{h_1} q^{h_2} = q^{h_1 + h_2} \quad \text{for } h_1, h_2 \in P^*;$$

$$(1.1.9) \quad q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \quad \text{and}$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i;$$

$$(1.1.10) \quad [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$$

$$\text{where } q_i = q^{(\alpha_i, \alpha_i)/2} \quad \text{and} \quad t_i = q^{(\alpha_i, \alpha_i)h_i/2};$$

$$(1.1.11) \quad \text{for } i \neq j \in I, \quad \text{setting } c = 1 - \langle h_i, \alpha_j \rangle,$$

$$\sum_{n=0}^c (-)^n e_i^{(n)} e_j e_i^{(c-n)} = \sum (-)^n f_i^{(n)} f_j f_i^{(c-n)} = 0.$$

Here $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, and $f_i^{(n)} = f_i^n/[n]_i!$.

We set $\{x\} = (x - x^{-1})/(q - q^{-1})$ and $\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \prod_{k=1}^n \frac{\{q^{1-k}x\}}{[k]}$. We denote by $U_q^{\mathbb{Z}}(\mathfrak{g})$ the

$\mathbb{Z}[q, q^{-1}]$ -algebra generated by $e_i^{(n)}, f_i^{(n)}, q^h, \left\{ \begin{matrix} q^h \\ n \end{matrix} \right\}$ ($i \in I, h \in P^*, n \in \mathbb{Z}_{\geq 0}$).

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the $\mathbb{Q}(q)$ -subalgebra of $U_q(\mathfrak{g})$ generated by the e_i 's (resp. the f_i 's) and $U_q^+(\mathfrak{g})_{\mathbb{Z}}$ (resp. $U_q^-(\mathfrak{g})_{\mathbb{Z}}$) the $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by the $e_i^{(n)}$'s (resp. the $f_i^{(n)}$'s). We set $U_q^{\mathbb{Q}}(\mathfrak{g}) = \mathbb{Q}[q, q^{-1}] \otimes_{\mathbb{Z}[\mathfrak{g}]} U_q^{\mathbb{Z}}(\mathfrak{g})$, etc. For $\xi \in \mathcal{Q} = \bigoplus \mathbb{Z}\alpha_i$, we set

$$U_q(\mathfrak{g})_{\xi} = \{P \in U_q(\mathfrak{g}); q^h P q^{-h} = q^{\langle h, \xi \rangle} P \text{ for any } h \in P^*\},$$

$$U_q^{\pm}(\mathfrak{g})_{\xi} = U_q^{\pm}(\mathfrak{g}) \cap U_q(\mathfrak{g})_{\xi}.$$

We set $|\xi| = \sum |n_i|$ for $\xi = \sum n_i \alpha_i \in \mathcal{Q}$. We define the filtration F of $U_q^{\pm}(\mathfrak{g})$ by $F_n(U_q^{\pm}(\mathfrak{g})) = \bigoplus_{|\xi| \leq n} U_q^{\pm}(\mathfrak{g})_{\xi}$.

We say $\lambda \in P$ is dominant (resp. antidominant) if $\langle h_i, \lambda \rangle \geq 0$ (resp. $\langle h_i, \lambda \rangle \leq 0$) for any i . We write P_{\pm} for the set of dominant (resp. antidominant) integral weights.

1.2. Definition of modified quantized enveloping algebra. Let $\text{Mod}(\mathfrak{g}, P)$ denote the category of left $U_q(\mathfrak{g})$ -modules M with the weight decomposition

$$M = \bigoplus_{\lambda \in P} M_{\lambda}$$

where

$$M_\lambda = \{u \in M; q^h u = q^{\langle h, \lambda \rangle} u \text{ for any } h \in P^*\}.$$

Let (forget) be the functor from $\text{Mod}(g, P)$ to the category of vector spaces over $\mathbb{Q}(q)$, forgetting the $U_q(g)$ -module structure. Let R denote the endomorphism ring of (forget). Hence to give an element of R is to associate an endomorphism $\varphi(M)$ of M with each M in $\text{Mod}(g, P)$ such that, for any morphism $f: M \rightarrow M'$ in $\text{Mod}(g, P)$, $\varphi(M') \circ f = f \circ \varphi(M)$ holds. Note that R contains $U_q(g)$. For $\lambda \in P$ let $a_\lambda \in R$ denote the projector $M \rightarrow M_\lambda$ to the weight space. Then the defining relation of a_λ (as a left $U_q(g)$ -module) is

$$(1.2.1) \quad q^h a_\lambda = q^{\langle h, \lambda \rangle} a_\lambda.$$

We have

$$(1.2.2) \quad a_\lambda P = P a_{\lambda - \xi} \quad \text{for } \xi \in Q \text{ and } P \in U_q(g)_\xi,$$

$$(1.2.3) \quad a_\lambda a_\mu = \delta_{\lambda, \mu} a_\lambda.$$

Then R is isomorphic to the direct product $\prod_{\lambda \in P} U_q(g) a_\lambda$. We set

$$\tilde{U}_q(g) = \bigoplus_{\lambda \in P} U_q(g) a_\lambda.$$

Then $\tilde{U}_q(g)$ is a subring of R by (1.2.2) and (1.2.3). Hence any object M in $\text{Mod}(g, P)$ may be regarded as a left $\tilde{U}_q(g)$ -module. We set

$$(1.2.4) \quad \tilde{U}_q^Z(g) = \bigoplus_{\lambda} U_q^Z(g) a_\lambda,$$

$$(1.2.5) \quad \tilde{U}_q^\pm(g) = \bigoplus_{\lambda} U_q^\pm(g) a_\lambda = \bigoplus_{\lambda} a_\lambda U_q^\pm(g).$$

They are subrings of $\tilde{U}_q(g)$.

1.3. The automorphisms of $U_q(g)$. Let $*$ denote the antiautomorphism of $U_q(g)$ given by

$$(1.3.1) \quad q^* = q, \quad (q^h)^* = q^{-h}, \quad e_i^* = e_i, \quad f_i^* = f_i.$$

Let φ be the antiautomorphism of $U_q(g)$ given by

$$(1.3.2) \quad \varphi(q) = q, \quad \varphi(q^h) = q^h, \quad \varphi(f_i) = f_i, \quad \varphi(e_i) = e_i.$$

Let us denote by \vee the automorphism $\varphi \circ * = * \circ \varphi$ of $U_q(g)$. Hence we have

$$(1.3.3) \quad q^\vee = q, \quad (q^h)^\vee = q^{-h}, \quad e_i^\vee = f_i, \quad f_i^\vee = e_i.$$

Let $-$ be the automorphism of $U_q(g)$ given by

$$(1.3.4) \quad \bar{q} = q^{-1}, \quad (q^h)^- = q^{-h}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i.$$

They commute with each other.

These automorphisms and antiautomorphisms are extended to those of $\tilde{U}_q(g)$. We shall denote them by the same letter. We have

$$(1.3.5) \quad a_\lambda^* = a_{-\lambda}, \quad \varphi(a_\lambda) = a_\lambda, \quad a_\lambda^\vee = a_{-\lambda}, \quad \bar{a}_\lambda = a_\lambda.$$

1.4. Tensor product. In this article, we take the comultiplication Δ of $U_q(g)$ given by

$$\Delta q^h = q^h \otimes q^h,$$

$$(1.4.1) \quad \Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$

$$\Delta f_i = f_i \otimes 1 + t_i \otimes f_i.$$

By this comultiplication, the tensor product of $U_q(g)$ -modules has a structure of $U_q(g)$ -module.

For an automorphism g of $U_q(g)$ and a $U_q(g)$ -module M , let us denote by M^g the $U_q(g)$ -module $\{u^g; u \in M\}$ with $Pu^g = (g(P)u)^g$ for $u \in M$ and $P \in U_q(g)$. With this notation we have

$$(1.4.2) \quad (M \otimes N)^\wedge \cong N^\wedge \otimes M^\wedge$$

by $(u \otimes v)^\wedge \leftrightarrow v^\wedge \otimes u^\wedge$ for $u \in M, v \in N$.

1.5. Crystals. Let us recall the definition of crystals (cf. [K3]).

Definition 1.5.1. A crystal B is a set with the following data:

$$(1.5.1) \quad \text{a map } \text{wt}: B \rightarrow P,$$

$$(1.5.2) \quad \varepsilon_i: B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i: B \rightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I,$$

$$(1.5.3) \quad \tilde{e}_i: B \rightarrow B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i: B \rightarrow B \sqcup \{0\} \quad \text{for } i \in I.$$

They satisfy the following axioms.

$$(1.5.4) \quad \text{For } b \in B, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle.$$

$$(1.5.5) \quad \text{For } b \in B \text{ with } \tilde{e}_i b \in B,$$

$$\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i.$$

$$(1.5.6) \quad \text{For } b \in B \text{ with } \tilde{f}_i b \in B,$$

$$\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i.$$

$$(1.5.7) \quad \text{For } b_1, b_2 \in B, \quad \tilde{f}_i b_2 = b_1 \quad \text{if and only if } \tilde{e}_i b_1 = b_2.$$

$$(1.5.8) \quad \text{If } \varepsilon_i(b) = -\infty \quad \text{then } \tilde{e}_i b = \tilde{f}_i b = 0.$$

Definition 1.5.2. A morphism $\psi: B_1 \rightarrow B_2$ from a crystal B_1 to a crystal B_2 is a map $\psi: B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ satisfying the following axioms:

$$(1.5.9) \quad \psi(0) = 0;$$

$$(1.5.10) \quad \text{if } b \in B_1 \text{ and } \psi(b) \in B_2, \text{ then}$$

$$\text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b) \quad \text{and} \quad \varphi_i(\psi(b)) = \varphi_i(b);$$

$$(1.5.11) \quad \text{if } b \in B_1 \text{ satisfies } \psi(\tilde{e}_i b) \neq 0 \text{ and } \psi(b) \neq 0, \text{ then } \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b);$$

$$(1.5.12) \quad \text{if } b \in B_1 \text{ satisfies } \psi(\tilde{f}_i b) \neq 0 \text{ and } \psi(b) \neq 0, \text{ then } \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b).$$

The definition of morphisms is slightly different from [K3].

Let $\mathcal{C}(I, P)$ denote the category of crystals.

A morphism $\psi: B_1 \rightarrow B_2$ of crystals is called *strict* if the associated map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$ commutes with all \tilde{e}_i and \tilde{f}_i . If the associated map is injective, then ψ is called *embedding*.

A crystal B is called *seminormal* if, for any $b \in B$ and $i \in I$, $\varepsilon_i(b)$ and $\varphi_i(b)$ are nonnegative integers and

$$\varepsilon_i(b) = \max \{n \geq 0; \tilde{e}_i^n b \in B\},$$

$$\varphi_i(b) = \max \{n \geq 0; \tilde{f}_i^n b \in B\}.$$

In such a case, we set

$$\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b \quad \text{and}$$

$$\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b.$$

A crystal B is called *normal* if, for any subset J of I such that $\{(\alpha_i, \alpha_j)\}_{i,j \in J}$ is a positive-definite symmetric matrix, B is isomorphic (in $\mathcal{C}(J, P)$) to a crystal base of an integrable $U_q(\mathfrak{g}_J)$ -module. Here $U_q(\mathfrak{g}_J)$ is the quantized universal enveloping algebra generated by e_j, f_j ($j \in J$), and q^h ($h \in P^*$).

For crystals B_1 and B_2 , let us define their tensor product $B_1 \otimes B_2$ by

$$(1.5.13) \quad B_1 \otimes B_2 = \{b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2\},$$

$$(1.5.14) \quad \text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$(1.5.15) \quad \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle),$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle),$$

$$(1.5.16) \quad \tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. Then \otimes is a functor from $\mathcal{C}(I, P) \times \mathcal{C}(I, P)$ to $\mathcal{C}(I, P)$ and satisfies the associative law: $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \mapsto b_1 \otimes (b_2 \otimes b_3)$.

For a crystal B , let us denote by B^\vee the crystal defined by

$$(1.5.17) \quad B^\vee = \{b^\vee; b \in B\},$$

$$(1.5.18) \quad \text{wt}(b^\vee) = -\text{wt}(b), \quad \varepsilon_i(b^\vee) = \varphi_i(b), \quad \varphi_i(b^\vee) = \varepsilon_i(b),$$

$$\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee \quad \text{and} \quad \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee.$$

Then we have (cf. (1.4.2))

$$(1.5.19) \quad (B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee \quad \text{by} \quad (b_1 \otimes b_2)^\vee \mapsto b_2^\vee \otimes b_1^\vee.$$

Example 1.5.3.

1. $C = \{c\}$ with

$$\text{wt}(c) = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0,$$

$$\tilde{e}_i c = \tilde{f}_i c = 0.$$

For any seminormal crystal B , $B \otimes C$ and $C \otimes B$ are isomorphic to B .

2. For $\lambda \in P$, $T_\lambda = \{t_\lambda\}$ with

$$\text{wt}(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty,$$

$$\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0.$$

We have $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$ and $B \otimes T_0 \cong T_0 \otimes B \cong B$ for any crystal B .

3. For $i \in I$, $B_i = \{b_i(n); n \in \mathbb{Z}\}$ with $\text{wt}(b_i(n)) = n\alpha_i$,

$$\varepsilon_i(b_i(n)) = -n, \quad \varphi_i(b_i(n)) = n,$$

$$\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty \quad \text{for } j \neq i,$$

$$\text{and} \quad \tilde{e}_i b_i(n) = b_i(n+1), \quad \tilde{f}_i b_i(n) = b_i(n-1),$$

$$\tilde{e}_j b_i(n) = \tilde{f}_j b_i(n) = 0 \quad \text{for } j \neq i.$$

We write b_i for $b_i(0)$.

4. $B(\infty)$ denotes the crystal associated with $U_q^-(\mathfrak{g})$. We denote by u_∞ the vector of weight 0.

5. $B(-\infty) = B(\infty)^\vee$. This is regarded as the crystal associated with $U_q^+(\mathfrak{g})$. We set $u_{-\infty} = u_\infty^\vee$.

6. For $\lambda \in P_+$, let $B(\lambda)$ denote the crystal associated with the irreducible module $V(\lambda)$ of highest weight λ . Set $B(-\lambda) = B(\lambda)^\vee$. Then $B(-\lambda)$ is isomorphic to the crystal associated with the irreducible module $V(-\lambda)$ of lowest weight $-\lambda$. Then C is isomorphic to $B(0)$.

Let us recall that the automorphism $*$ of $U_q^-(\mathfrak{g})$ induces the automorphism of $B(\infty)$ (cf. [K3]). We shall also denote it by $*$. We set $e_i^*(b) = \varepsilon_i(b^*)$, $\varphi_i^*(b) = \varphi_i(b^*)$, $\tilde{e}_i^* b = (\tilde{e}_i b^*)^*$, and $\tilde{f}_i^* b = (\tilde{f}_i b^*)^*$. For $\lambda \in P_+$, there exists a unique embedding $B(\lambda)$ into $B(\infty) \otimes T_\lambda$, whose image is $\{b \otimes t_\lambda \in B(\infty) \otimes T_\lambda; \varepsilon_i^*(b) \leq \langle h_i, \lambda \rangle\}$. Similarly, we define $*$, e_i^* , etc., for $B(-\infty)$. Then, for $\lambda \in P_-$, $B(\lambda)$ is isomorphic to the subcrystal $\{t_\lambda \otimes b \in T_\lambda \otimes B(-\infty); \varphi_i^*(b) \leq -\langle h_i, \lambda \rangle\}$ of $T_\lambda \otimes B(-\infty)$.

Let us also recall that, for any $i \in I$, there is a unique strict embedding

$$\Phi_i: B(\infty) \rightarrow B(\infty) \otimes B_i$$

such that $\Phi_i(u_\infty) = u_\infty \otimes b_i$. We have

$$\Phi_i(b) = \tilde{e}_i^{*\varepsilon_i^*(b)} b \otimes \tilde{f}_i^{\varepsilon_i^*(b)} b_i.$$

Also we have

$$(1.5.20) \quad B(\infty) \otimes B_i \cong \bigoplus_{n \geq 0} B(\infty) \otimes T_{n\alpha_i}.$$

Taking \vee we obtain

$$\Phi_i^\vee: B(-\infty) \rightarrow B_i \otimes B(-\infty) \quad \text{and}$$

$$B_i \otimes B(-\infty) \cong \bigoplus_{n \geq 0} T_{-n\alpha_i} \otimes B(-\infty).$$

1.6. *Balanced triples.* Let us recall the definition of balanced triple.

Let V be a vector space over $\mathbb{Q}(q)$. For a subring B of $\mathbb{Q}(q)$, a B -lattice of V is a B -submodule M of V such that $V \cong \mathbb{Q}(q) \otimes_B M$.

Let A (resp. \bar{A}) be the subring of $\mathbb{Q}(q)$ consisting of functions regular at $q = 0$ (resp. $q = \infty$). Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}[q, q^{-1}]$ -lattice of V , L an A -lattice of V , and \bar{L} an \bar{A} -lattice of V . Then we have the following lemma.

LEMMA 1.6.1 [K1]. Set $E = V_{\mathbb{Z}} \cap L \cap \bar{L}$. Then the following conditions are equivalent.

(i) $E \rightarrow V_{\mathbb{Z}} \cap L / V_{\mathbb{Z}} \cap qL$ is an isomorphism.

(ii) $E \rightarrow V_{\mathbb{Z}} \cap \bar{L} / V_{\mathbb{Z}} \cap q^{-1}\bar{L}$ is an isomorphism.

(iii) $(V_{\mathbb{Z}} \cap qL) \oplus (V_{\mathbb{Z}} \cap \bar{L}) \rightarrow V_{\mathbb{Z}}$ is an isomorphism.

(iv) $A \otimes_{\mathbb{Z}} E \rightarrow L$, $\bar{A} \otimes_{\mathbb{Z}} E \rightarrow \bar{L}$, $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} E \rightarrow V_{\mathbb{Z}}$, $\mathbb{Q}(q) \otimes_{\mathbb{Z}} E \rightarrow V$ are isomorphisms.

We call $(L, \bar{L}, V_{\mathbb{Z}})$ *balanced* if these equivalent conditions are satisfied. Let us denote by G the inverse of the isomorphism $E \rightarrow V_{\mathbb{Z}} \cap L / V_{\mathbb{Z}} \cap qL$. If B is a base of $V_{\mathbb{Z}} \cap L / V_{\mathbb{Z}} \cap qL$, then $\{G(b); b \in B\}$ is a base of V . The following proposition is easily proven (e.g., by (iii)).

PROPOSITION 1.6.2 (Triangular property). Let $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ be an exact sequence of vector spaces over $\mathbb{Q}(q)$. Let $V_{i\mathbb{Z}}$ (resp. L_i , \bar{L}_i) be a $\mathbb{Z}[q, q^{-1}]$ -lattice (resp. A -lattice, \bar{A} -lattice) of V_i ($i = 1, 2, 3$). Assume that

$$0 \rightarrow V_{1\mathbb{Z}} \rightarrow V_{2\mathbb{Z}} \rightarrow V_{3\mathbb{Z}} \rightarrow 0,$$

$$0 \rightarrow V_{1\mathbb{Z}} \cap L_1 \rightarrow V_{2\mathbb{Z}} \cap L_2 \rightarrow V_{3\mathbb{Z}} \cap L_3 \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow V_{1\mathbb{Z}} \cap \bar{L}_1 \rightarrow V_{2\mathbb{Z}} \cap \bar{L}_2 \rightarrow V_{3\mathbb{Z}} \cap \bar{L}_3 \rightarrow 0$$

are exact. If two of $(V_{i\mathbb{Z}}, L_i, \bar{L}_i)$ are balanced, then so is the other.

2. Result of Lusztig [L2].

2.1. *Global base of $\tilde{U}_q(\mathfrak{g})$.* Let us recall the result of Lusztig on the crystal base of $\tilde{U}_q(\mathfrak{g})$. For a dominant integral weight $\lambda \in P_+$, let us denote by $V(\lambda)$ (resp. $V(-\lambda)$) the irreducible module with highest- (resp. lowest-) weight λ (resp. $-\lambda$). Let u_λ (resp. $u_{-\lambda}$) be the highest- (resp. lowest-) weight vector. For $\lambda \in P_+$, $\mu \in P_-$, we set $V(\lambda, \mu) = V(\lambda) \otimes V(\mu)$. Then $V(\lambda, \mu)$ is generated by $u_\lambda \otimes u_\mu$ as a $U_q(\mathfrak{g})$ -module, and the

defining relation of $u_\lambda \otimes u_\mu$ is

$$(2.1.1) \quad q^h(u_\lambda \otimes u_\mu) = q^{\langle h, \lambda + \mu \rangle}(u_\lambda \otimes u_\mu),$$

$$(2.1.2) \quad e_i^{1 - \langle h_i, \mu \rangle}(u_\lambda \otimes u_\mu) = 0,$$

$$f_i^{1 + \langle h_i, \lambda \rangle}(u_\lambda \otimes u_\mu) = 0.$$

Let us define the automorphism τ of $V(\lambda, \mu)$ by

$$(2.1.3) \quad (P(u_\lambda \otimes u_\mu))^- = \bar{P}(u_\lambda \otimes u_\mu) \quad \text{for } P \in U_q(\mathfrak{g}).$$

We set $L(\lambda, \mu) = L(\lambda) \otimes_A L(\mu)$ and $B(\lambda, \mu) = B(\lambda) \otimes B(\mu) \subset L(\lambda, \mu)/qL(\lambda, \mu)$. Then $(L(\lambda, \mu), B(\lambda, \mu))$ is a crystal base of $V(\lambda, \mu)$.

Set $V_Z(\lambda, \mu) = U_q^Z(\mathfrak{g})(u_\lambda \otimes u_\mu) = U_q^Z(\mathfrak{g})u_\lambda \otimes U_q^Z(\mathfrak{g})u_\mu$ and $V_Q(\lambda, \mu) = U_q^Q(\mathfrak{g})(u_\lambda \otimes u_\mu)$. The following results are due to Lusztig.

PROPOSITION 2.1.1 [L2]. $(L(\lambda, \mu), L(\lambda, \mu)^-, V_Z(\lambda, \mu))$ is balanced.

Let G be the inverse of the isomorphism $L(\lambda, \mu) \cap L(\lambda, \mu)^- \cap V_Q(\lambda, \mu) \rightarrow L(\lambda, \mu)/qL(\lambda, \mu)$. Then $V_Z(\lambda, \mu) = \bigoplus_{b \in B(\lambda, \mu)} \mathbb{Z}[q, q^{-1}]G(b)$.

THEOREM 2.1.2 [L2]. There exist a unique A -lattice $L(\tilde{U}_q(\mathfrak{g}))$ of $\tilde{U}_q(\mathfrak{g})$ and a unique base $B(\tilde{U}_q(\mathfrak{g}))$ of $L(\tilde{U}_q(\mathfrak{g}))/qL(\tilde{U}_q(\mathfrak{g}))$ satisfying the following properties.

- (i) $(L(\tilde{U}_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g}))^-, \tilde{U}_q^Q(\mathfrak{g}))$ is balanced.
- (ii) Let G denote the inverse of $L(\tilde{U}_q(\mathfrak{g})) \cap L(\tilde{U}_q(\mathfrak{g}))^- \cap \tilde{U}_q^Q(\mathfrak{g}) \xrightarrow{\sim} L(\tilde{U}_q(\mathfrak{g}))/qL(\tilde{U}_q(\mathfrak{g}))$. $L(\tilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda \in P} L(\tilde{U}_q(\mathfrak{g})a_\lambda)$ and $B(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in P} B(\tilde{U}_q(\mathfrak{g})a_\lambda)$ where $L(\tilde{U}_q(\mathfrak{g})a_\lambda) = L(\tilde{U}_q(\mathfrak{g})) \cap U_q(\mathfrak{g})a_\lambda$ and $B(\tilde{U}_q(\mathfrak{g})a_\lambda) = B(\tilde{U}_q(\mathfrak{g})) \cap (L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda))$.
- (iii) For any $\xi \in P_+$ and $\eta \in P_-$, let $\Phi(\xi, \eta)$ denote the $U_q(\mathfrak{g})$ -linear map $U_q(\mathfrak{g})a_{\xi+\eta} \rightarrow V(\xi) \otimes V(\eta)$ sending $a_{\xi+\eta}$ to $u_\xi \otimes u_\eta$. Then $\Phi(\xi, \eta)L(U_q(\mathfrak{g})a_{\xi+\eta}) = L(\xi) \otimes L(\eta)$.
- (iv) Let $\bar{\Phi}(\xi, \eta)$ be the induced homomorphism $L(U_q(\mathfrak{g})a_{\xi+\eta})/qL(U_q(\mathfrak{g})a_{\xi+\eta})$ to $L(\xi) \otimes L(\eta)/qL(\xi) \otimes L(\eta)$. Then $\{b \in B(U_q(\mathfrak{g})a_{\xi+\eta}); \bar{\Phi}(\xi, \eta)b \neq 0\} \xrightarrow{\sim} B(\xi) \otimes B(\eta)$ and $\Phi(\xi, \eta)G(b) = G(\bar{\Phi}(\xi, \eta)b)$ for any $b \in B(U_q(\mathfrak{g})a_{\xi+\eta})$.
- (v) $B(\tilde{U}_q(\mathfrak{g}))$ has a structure of crystal such that

$$B(\xi) \otimes B(\eta) \rightarrow B(U_q(\mathfrak{g})a_{\xi+\eta}) \subset B(\tilde{U}_q(\mathfrak{g}))$$

is a strict embedding.

The crystal $B(\tilde{U}_q(\mathfrak{g}))$ is therefore a normal crystal.

2.2. Arguments of Lusztig. Since we use the arguments of Lusztig later, we shall review his argument briefly. The following lemma is easily checked.

LEMMA 2.2.1. Let $l_+, l_- \in \mathbb{Z}_{\geq 0}$ and $\lambda \in P_+$, $\mu \in P_-$.

- (i) $F_{l_+}(U_q^-(\mathfrak{g}))F_{l_-}(U_q^+(\mathfrak{g}))(u_\lambda \otimes u_\mu) = F_{l_+}(U_q^+(\mathfrak{g}))F_{l_-}(U_q^-(\mathfrak{g}))(u_\lambda \otimes u_\mu) = (F_{l_+}(U_q^-(\mathfrak{g}))u_\lambda) \otimes (F_{l_-}(U_q^+(\mathfrak{g}))u_\mu)$.

(ii) For $P_\pm \in F_{l_\pm}(U_q^\pm(\mathfrak{g}))$, we have

$$(P_-u_\lambda) \otimes (P_+u_\mu) = P_-P_+(u_\lambda \otimes u_\mu) = P_+P_-(u_\lambda \otimes u_\mu) \mod F_{l_+-1}(U_q^-(\mathfrak{g}))u_\lambda \otimes F_{l_+-1}(U_q^+(\mathfrak{g}))u_\mu.$$

$$(iii) \quad L(\lambda, \mu)^- \cap F_{l_-}(U_q^-(\mathfrak{g}))u_\lambda \otimes F_{l_+}(U_q^+(\mathfrak{g}))u_\mu \\ = (\bar{L}(\lambda) \cap F_{l_-}(U_q^-(\mathfrak{g}))u_\lambda) \otimes (\bar{L}(\mu) \cap F_{l_+}(U_q^+(\mathfrak{g}))u_\mu) \mod F_{l_+-1}(\mathfrak{g})u_\lambda \otimes F_{l_+-1}(\mathfrak{g})u_\mu.$$

(iv) $\bar{u} \otimes \bar{v} - \bar{u} \otimes \bar{v} \in (U_q^+(\mathfrak{g})_{>0} \bar{u}) \otimes (U_q^-(\mathfrak{g})_{>0} \bar{v})$. Here we set $U_q^+(\mathfrak{g})_{>0} = \sum_i U_q^+(\mathfrak{g})e_i$ and $U_q^-(\mathfrak{g})_{>0} = \sum_i U_q^-(\mathfrak{g})f_i$.

The property (iv) follows from the existence of the universal R -matrix (cf. [L2]).

PROPOSITION 2.2.2. Let $\lambda \in P_+$ and $\mu \in P_-$, and let N be a $U_q^+(\mathfrak{g})$ -submodule of $V(\lambda)$ and N' a $U_q^-(\mathfrak{g})$ -submodule of $V(\mu)$. Assume that N and N' are generated by global bases, i.e., $N = \bigoplus_{b \in B_N} \mathbb{Q}(q)G(b)$ and $N' = \bigoplus_{b \in B_{N'}} \mathbb{Q}(q)G(b')$ for some $B_N \subset B(\lambda)$ and $B_{N'} \subset B(\mu)$. Then $(L(\lambda, \mu) \cap N \otimes N', \bar{L}(\lambda, \mu) \cap N \otimes N', N_Z \otimes N'_Z)$ is balanced. Here $N_Z = \bigoplus_{b \in B_N} \mathbb{Z}[q, q^{-1}]G(b)$, etc.

Proof. Set $F_l(N) = N \cap F_l(U_q^-(\mathfrak{g}))u_\lambda$ and $F_l(N') = N' \cap F_l(U_q^+(\mathfrak{g}))u_\mu$. Then $F_l(N)$ (resp. $F_l(N')$) is a $U_q^+(\mathfrak{g})$ - (resp. $U_q^-(\mathfrak{g})$ -) submodule and is generated by global bases. Hence it is enough to show

$$(2.2.1)_l \quad (L(\lambda, \mu) \cap F_l(N) \otimes F_l(N'), L(\lambda, \mu)^- \cap F_l(N) \otimes F_l(N'), F_l(N_Z) \otimes F_l(N'_Z))$$

is balanced. We prove this by the induction on l . Assume that $(2.2.1)_{l-1}$ is satisfied. Let $Gr_l N$, etc., be the gradation with respect to the filtration F . Then Lemma 2.2.1 implies

$$Gr_l(L(\lambda, \mu) \cap N \otimes N') = Gr_l(L(\lambda) \cap N) \otimes Gr_l(L(\mu) \cap N')$$

$$Gr_l(\bar{L}(\lambda, \mu) \cap N \otimes N') = Gr_l(\bar{L}(\lambda) \cap N) \otimes Gr_l(\bar{L}(\mu) \cap N').$$

Hence $(Gr_l(L(\lambda, \mu) \cap N \otimes N'), Gr_l(\bar{L}(\lambda, \mu) \cap N \otimes N'), Gr_l(N_Z \otimes N'_Z))$ is balanced. Then we obtain $(2.2.1)_l$ by the triangular property of balancedness (Proposition 1.6.2). Q.E.D.

Applying this lemma to $N = V(\lambda)$ and $N' = V(\mu)$ we obtain Proposition 2.1.1. We also obtain the following lemma.

LEMMA 2.2.3. Let $\lambda \in P_+$ and $\mu \in P_-$. Then for $b \in B(\lambda)$, $b' \in B(\mu)$

$$G(b \otimes b') \equiv G(b) \otimes G(b') \mod \bigoplus_{\xi \in Q_+ \setminus \{0\}} V(\lambda)_{\text{wt}(b)+\xi} \otimes V(\mu)_{\text{wt}(b')-\xi}.$$

Here Q_+ is the set of linear combinations of simple roots with nonnegative integer coefficients.

3. Description of $B(U_q(\mathfrak{g})a_\lambda)$.

3.1. *Relation with $B(\infty)$ and $B(-\infty)$.* Let λ be an integral weight. For $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$, $B(\xi) \otimes B(\eta)$ is embedded into $B(U_q(\mathfrak{g})a_\lambda)$. Since we have $B(\xi) \subset B(\infty) \otimes T_\xi$ and $B(\eta) \subset T_\eta \otimes B(-\infty)$, $B(\xi) \otimes B(\eta)$ is embedded into the crystal $B(\infty) \otimes T_\lambda \otimes B(-\infty)$ through $T_\lambda \cong T_\xi \otimes T_\eta$. Now take $\zeta \in P_+$. Then it is easy to see that

$$\begin{array}{ccc} B(\xi + \zeta) \otimes B(\eta - \zeta) & \hookrightarrow & B(\infty) \otimes T_\lambda \otimes B(-\infty) \\ \uparrow & \nearrow & \\ B(\xi) \otimes B(\eta) & & \end{array}$$

commutes. Thus we obtain the following theorem.

THEOREM 3.1.1. $B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$.

Note that $B(\xi) \otimes B(\eta)$ is a strict subcrystal of $B(\infty) \otimes T_\lambda \otimes B(-\infty)$.

By Lemma 2.2.1 (ii) and Lemma 2.2.3, we have, for $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$,

$$(3.1.1) \quad G(b_1 \otimes t_\lambda \otimes b_2) \equiv G(b_1)G(b_2)a_\lambda \pmod{F_{|wtb_1|-1}(U_q^-(\mathfrak{g}))F_{|wtb_2|-1}(U_q^+(\mathfrak{g}))a_\lambda}.$$

3.2. *Filtration by Bruhat order.* Let W be the Weyl group. For $w \in W$, let us take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. We define a subset $B_w(\infty)$ of $B(\infty)$ by

$$(3.2.1) \quad B_w(\infty) = \{ \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_l}^{a_l} u_\infty; a_1, \dots, a_l \geq 0 \}.$$

Then $B_w(\infty)$ does not depend on the choice of reduced expression (see [K3]), and

$$(3.2.2) \quad \bigoplus_{b \in B_w(\infty)} \mathbb{Z}[q, q^{-1}]G(b) = \sum_{a_1, \dots, a_l \in \mathbb{Z}_{\geq 0}} \mathbb{Z}[q, q^{-1}]f_{i_1}^{(a_1)} \cdots f_{i_l}^{(a_l)}.$$

Set $B_w(-\infty) = B_w(\infty)^\vee \subset B(-\infty)$. Then we have

$$(3.2.3) \quad \bigoplus_{b \in B_w(-\infty)} \mathbb{Z}[q, q^{-1}]G(b) = \sum \mathbb{Z}[q, q^{-1}]e_{i_1}^{(a_1)} \cdots e_{i_l}^{(a_l)}.$$

We set for any $i \in I$

$$(3.2.4) \quad \begin{aligned} \tilde{U}_q^+(\mathfrak{g})_i^{\mathbb{Z}} &= \bigoplus \mathbb{Z}[q, q^{-1}]e_i^{(n)}a_\lambda \quad \text{and} \\ \tilde{U}_q^-(\mathfrak{g})_i^{\mathbb{Z}} &= \bigoplus \mathbb{Z}[q, q^{-1}]f_i^{(n)}a_\lambda. \end{aligned}$$

THEOREM 3.2.1. Let $w, w' \in W$, and let $w = s_{i_1} \cdots s_{i_l}$, $w' = s_{j_1} \cdots s_{j_{l'}}$ be their reduced expressions. Set $B_{w, w'}(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_\lambda B_w(\infty) \otimes T_\lambda \otimes B_{w'}(-\infty) \subset B(\tilde{U}_q(\mathfrak{g}))$. Then

$$\bigoplus_{b \in B_{w, w'}(\tilde{U}_q(\mathfrak{g}))} \mathbb{Z}[q, q^{-1}]G(b)$$

is equal to the $\mathbb{Z}[q, q^{-1}]$ -module $\tilde{U}_q^-(\mathfrak{g})_{i_1}^{\mathbb{Z}} \cdots \tilde{U}_q^-(\mathfrak{g})_{i_l}^{\mathbb{Z}} \tilde{U}_q^+(\mathfrak{g})_{j_1}^{\mathbb{Z}} \cdots \tilde{U}_q^+(\mathfrak{g})_{j_{l'}}^{\mathbb{Z}}$.

This follows immediately from Proposition 2.2.2. by taking

$$\lambda = \xi + \eta \text{ with } \xi, -\eta \in P_+ \text{ and}$$

$$N = \bigoplus_{b \in B_w(\infty)} \mathbb{Q}(q)G(b)u_\xi \quad \text{and} \quad N' = \bigoplus_{b \in B_{w'}(-\infty)} \mathbb{Q}(q)G(b)u_\eta.$$

This theorem shows an affirmative answer to a conjecture of Lusztig [L3].

4. A metric of $\tilde{U}_q(\mathfrak{g})$.

4.1. *General facts.* Let us define a metric on $\tilde{U}_q(\mathfrak{g})$ that behaves well with crystal bases. Let ψ be the antiautomorphism of the $\mathbb{Q}(q)$ -algebra $U_q(\mathfrak{g})$ defined by

$$(4.1.1) \quad \begin{aligned} \psi(e_i) &= q_i^{-1}t_i^{-1}f_i, \\ \psi(f_i) &= q_i^{-1}t_i e_i, \\ \psi(q^h) &= q^h. \end{aligned}$$

Let M_1 and M_2 be $U_q(\mathfrak{g})$ -modules, and let $(\ , \)_{M_1}$ and $(\ , \)_{M_2}$ be a symmetric form on M_1 and M_2 satisfying

$$(Pu, v)_{M_1} = (u, \psi(P)v)_{M_1} \quad \text{for } u, v \in M_1, \quad P \in U_q(\mathfrak{g}).$$

We define the symmetric form $(\ , \)$ on $M = M_1 \otimes M_2$ by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_{M_1} (u_2, v_2)_{M_2} \quad \text{for } u_i, v_i \in M_i.$$

Then it is known (cf. [K1]) that $(\ , \)$ satisfies

$$(4.1.1) \quad (Pu, v) = (u, \psi(P)v) \quad \text{for } u, v \in M.$$

Let us call this metric the tensor product of $(\ , \)_{M_1}$ and $(\ , \)_{M_2}$.

4.2. *Definition of a metric on $\tilde{U}_q(\mathfrak{g})$.* For $\lambda \in P_+ \cup (-P_+)$, there exists a unique nondegenerate symmetric bilinear form $(\ , \)$ on $V(\lambda)$ such that

$$(4.2.1) \quad \begin{aligned} (u_\lambda, u_\lambda) &= 1 \\ (Pu, v) &= (P, \psi(P)v) \quad \text{for any } u, v \in V(\lambda) \quad \text{and} \quad P \in U_q(\mathfrak{g}). \end{aligned}$$

Therefore, for $\lambda, \mu \in P_+$, the tensor product of those metrics gives a metric on $V(\lambda) \otimes V(-\mu)$. Let us take an arbitrary $\lambda \in P$. For $\xi \in P_+ \cap (\lambda + P_+)$, $V(\xi) \otimes V(\lambda - \xi)$ has a metric.

LEMMA 4.2.2. For any $P, Q \in U_q(\mathfrak{g})$, there exists a unique polynomial $f(x)$ in $x = (x_i)_{i \in I}$ such that, for any $\xi \in P_+ \cap (\lambda + P_+)$,

$$(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = f(x) \quad \text{with } x_i = q_i^{2\langle h_i, \xi \rangle}.$$

Proof. Let us take $P_\pm^{(v)} \in U_q^\pm(\mathfrak{g}) \otimes Q(q)[q^h; h \in P^*]$ such that $\psi(Q)P = \sum P_\pm^{(v)}P_\pm^{(v)}$. Then

$$(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = \sum (P_\pm^{(v)}(u_\xi \otimes u_{\lambda-\xi}), \psi(P_\pm^{(v)})(u_\xi \otimes u_{\lambda-\xi})).$$

Hence we may assume that P and Q belong to $U_q^-(\mathfrak{g}) \otimes Q(q)[q^h; h \in P^*]$. Since $q^h(u_\xi \otimes u_{\lambda-\xi}) = q^{\langle \lambda, h \rangle}(u_\xi \otimes u_{\lambda-\xi})$, we may assume $P, Q \in U_q^-(\mathfrak{g})$. In this case $P(u_\xi \otimes u_{\lambda-\xi}) = Pu_\xi \otimes u_{\lambda-\xi}$ and hence $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi})) = (Pu_\xi, Qu_\xi)$. Then the result follows from [K1, Lemma 4.7.1]. Q.E.D.

We define a metric on $U_q(\mathfrak{g})a_\lambda$ by

$$(Pa_\lambda, Qa_\lambda) = f(0)$$

where f is the polynomial given in Lemma 4.2.2. Hence (Pa_λ, Qa_λ) is the limit of $(P(u_\xi \otimes u_{\lambda-\xi}), Q(u_\xi \otimes u_{\lambda-\xi}))$ when all $\langle h_i, \xi \rangle$ tend to infinity. Here we regard $|q| < 1$. We extend this metric to the metric of $\tilde{U}_q(\mathfrak{g})$ such that $U_q(\mathfrak{g})a_\lambda$ and $U_q(\mathfrak{g})a_\mu$ are orthogonal for different λ, μ .

In [K1], we define a metric on $U_q^-(\mathfrak{g})$. The relation is given by the following formula (4.2.3).

$$(4.2.3) \quad (Pa_\lambda, Qa_\lambda) = \prod_i (1 - q_i^2)^{n_i} (P, Q) \quad \text{for } P, Q \in U_q^-(\mathfrak{g})_\xi \quad \text{with } \xi = \sum n_i \alpha_i.$$

Hence, we can apply the result of [K1]. The relation $(P, Q) = (P^*, Q^*)$ [K1, Proposition 5.2.1] implies

$$(4.2.4) \quad (u, v) = (u^*, v^*) \quad \text{for } u, v \in \tilde{U}_q^-(\mathfrak{g}) = \bigoplus_\lambda U_q^-(\mathfrak{g})a_\lambda.$$

LEMMA 4.2.3. For $u \in \tilde{U}_q^-(\mathfrak{g})$ and $v \in \tilde{U}_q^-(\mathfrak{g})$, $(u, v) = (uq_i t_i^{-1} e_i, v)$.

Proof. Set $u = Pa_{\lambda+\alpha_i}$, $v = Qa_\lambda$ for $\lambda \in P$, $P, Q \in U_q^-(\mathfrak{g})$. Take $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$. Then $Pe_i(u_\xi \otimes u_\eta) = P(u_\xi \otimes e_i u_\eta)$. On the other hand, we have

$$\Delta P = P \otimes 1 + t_i e_i'' P \otimes f_i \text{ mod } \bigoplus_{\xi \neq 0, -\alpha_i} U_q^-(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})_\xi.$$

For the definition of e_i'' , see [K1]. Hence we obtain with $\xi_i = \langle h_i, \xi \rangle$, etc.

$$\begin{aligned} Pe_i(u_\xi \otimes u_\eta) &= Pu_\xi \otimes u_\eta + t_i e_i'' Pu_\xi \otimes f_i e_i u_\eta \\ &= Pu_\xi \otimes u_\eta + q_i^{\xi_i} [-\eta_i]_i (Ad t_i e_i'' P) u_\xi \otimes u_\eta. \end{aligned}$$

Thus

$$(Pe_i(u_\xi \otimes u_\eta), Q(u_\xi \otimes u_\eta)) = \frac{q_i^{\xi_i + \eta_i + 1} - q_i^{\xi_i - \eta_i + 1}}{1 - q_i^2} ((Ad t_i e_i'' P) u_\xi, Qu_\eta)$$

Letting ξ go to infinity, we obtain

$$(Pe_i a_\lambda, Qa_\lambda) = \frac{q_i^{\lambda_i + 1}}{1 - q_i^2} a(q) ((Ad t_i) e_i'' P, Q).$$

Here $a(q) = \prod_i (1 - q_i^2)^{n_i}$ and $\text{wt} Q = \sum n_i \alpha_i$. By [K1, Lemma 5.2.2], we have

$$(4.2.5) \quad ((Ad t_i) e_i'' P, Q) = (P, Qf_i).$$

Finally we obtain

$$\begin{aligned} (Pa_{\lambda+\alpha_i} q_i t_i^{-1} e_i, Qa_\lambda) &= q_i^{-1-\lambda_i} (Pe_i a_\lambda, Qa_\lambda) \\ &= (Pa_{\lambda+\alpha_i}, Qf_i a_{\lambda+\alpha_i}). \end{aligned} \quad \text{Q.E.D.}$$

LEMMA 4.2.4. $(u, v) = (uq_i^{-1} t_i e_i, v)$ for any $u, v \in \tilde{U}_q(\mathfrak{g})$.

Proof. Assume first $u \in \tilde{U}_q^-(\mathfrak{g})$. Let us write $v = Pw$ for $P \in U_q^+(\mathfrak{g})$, $w \in \tilde{U}_q^-(\mathfrak{g})$. Then $(ue_i, v) = (\psi(P)ue_i, w) = (\psi(P)u, wq_i t_i f_i) = (u, Pwq_i t_i f_i) = (u, vq_i t_i f_i)$ by the last lemma. Hence the lemma is true if $u \in \tilde{U}_q^-(\mathfrak{g})$. In general case, writing $u = Pw$ with $P \in U_q^+(\mathfrak{g})$ and $w \in \tilde{U}_q^-(\mathfrak{g})$, we can argue similarly. Q.E.D.

Set $\psi^*(P) = (\psi(P^*))^*$. Then $\psi^*(f_i) = q_i t_i^{-1} e_i$. Hence Lemma 4.2.3 implies

$$(uf_i, v) = (u, v\psi^*(f_i)).$$

This implies easily the following lemma.

LEMMA 4.2.6. $(uP, u) = (u, v\psi^*(P))$ for any $u, v \in \tilde{U}_q(\mathfrak{g})$ and $P \in U_q(\mathfrak{g})$.

PROPOSITION 4.2.5. $(u, v) = (u^*, v^*)$ for any $u, v \in \tilde{U}_q(\mathfrak{g})$.

Proof. This is already shown for $u, v \in \tilde{U}_q^-(\mathfrak{g})$. If $u \in U_q^-(\mathfrak{g})$, then, writing $v = Pw$ with $P \in U_q^+(\mathfrak{g})$ and $w \in \tilde{U}_q^-(\mathfrak{g})$, we have

$$(u, v) = (\psi(P)u, w) = (u^* \psi(P)^*, w^*) = (u^*, w^* P^*) = (u^*, v^*).$$

The general case can be argued similarly.

Q.E.D.

4.3. *Metric and crystal base.* The relation of the metric of $V(\lambda)$ and its crystal base implies the properties

$$(4.3.1) \quad (L(\tilde{U}_q(g)), L(\tilde{U}_q(g))) \subset A,$$

$$(4.3.2) \quad (G(b), G(b')) \equiv \delta_{b,b'} \pmod{qA}.$$

Thus (\cdot, \cdot) is a nondegenerate metric on $\tilde{U}_q(g)$. Let A_Z be the subring of $\mathbb{Q}(q)$ generated by $q, (1 - q^n)^{-1} (n > 0)$. Then one can see easily

$$(4.3.3) \quad (\tilde{U}_q^Z(g), \tilde{U}_q^Z(g)) \subset A_Z[q^{-1}].$$

Hence we can apply the similar arguments as in [K2, §6.1]. Thus we obtain the following.

PROPOSITION 4.3.1. (i) $L(\tilde{U}_q(g)) = \{u \in \tilde{U}_q(g); (u, u) \in A\}$.
(ii) If $u \in \tilde{U}_q^Z(g)$ and $(u, u) \in 1 + qA$, then $u \equiv G(b) \pmod{qL(\tilde{U}_q(g))}$ for some $b \in B((\tilde{U}_q(g)) \sqcup -B(\tilde{U}_q(g)))$.

We have Theorem 4.3.2 below as a corollary.

THEOREM 4.3.2. (i) $L(\tilde{U}_q(g))$ is invariant by $*$.
(ii) $B(\tilde{U}_q(g))^* = B(\tilde{U}_q(g))$.
(iii) $G(b^*) = G(b)^*$ for $b \in B(\tilde{U}_q(g))$.

The proofs of (i) and (iii) are similar to [K1]; we will prove only (ii). For $b \in B(\tilde{U}_q(g))$, $b^* \in B(\tilde{U}_q(g)) \sqcup (-B(\tilde{U}_q(g)))$. Write $b = b_1 \otimes t_\lambda \otimes b_2$, $l_v = |wtb_v|$. Then

$$G(b) \equiv G(b_1)G(b_2)a_\lambda \pmod{F_{l_1-1}(U_q^-(g))F_{l_2-1}(U_q^+(g))a_\lambda}.$$

Hence we obtain

$$\begin{aligned} G(b)^* &\equiv a_{-\lambda}G(b_2)^*G(b_1)^* = a_{-\lambda}G(b_2^*)G(b_1^*) = G(b_2^*)G(b_1^*)a_\mu \\ &\equiv G(b_1^*)G(b_2^*)a_\mu \pmod{F_{l_1-1}(U_q^-(g))F_{l_2-1}(U_q^+(g))a_\mu} \end{aligned}$$

with

$$\mu = -\lambda - wtb_1 - wtb_2.$$

Since $G(b)^* = G(b')$ for some $b' \in B(\tilde{U}_q(g)) \sqcup -B(\tilde{U}_q(g))$, b' must be $b_1^* \otimes t_\mu \otimes b_2^*$. (See (3.1.1).)

As seen in the course of the proof above, we have the following.

COROLLARY 4.3.3. For $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, we have

$$(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda - wtb_1 - wtb_2} \otimes b_2^*.$$

Theorem 4.3.2 is conjectured by Lusztig [L3].

5. Right structure.

5.1. *Two crystal structures on $B(\tilde{U}_q(g))$.* We define for $b \in B(\tilde{U}_q(g))$

$$\begin{aligned} (5.1.1) \quad \varepsilon_i^*(b) &= \varepsilon_i(b^*), \\ \varphi_i^*(b) &= \varphi_i(b^*), \\ \tilde{e}_i^*b &= (\tilde{e}_ib^*)^* \quad \text{and} \\ \tilde{f}_i^*b &= (\tilde{f}_ib^*)^*. \end{aligned}$$

Then this defines another crystal structure on $B(\tilde{U}_q(g))$. By Corollary 4.3.3 we obtain easily the following formula for $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, $\lambda \in P$:

$$\begin{aligned} (5.1.2) \quad \varepsilon_i^*(b_1 \otimes t_\lambda \otimes b_2) &= \max(\varepsilon_i^*(b_1), \varphi_i^*(b_2) + \langle h_i, \lambda \rangle), \\ \varphi_i^*(b_1 \otimes t_\lambda \otimes b_2) &= \max(\varepsilon_i^*(b_1) - \langle h_i, \lambda \rangle, \varphi_i^*(b_2)), \\ \varphi_i^*(b_1 \otimes t_\lambda \otimes b_2) - \varepsilon_i^*(b_1 \otimes t_\lambda \otimes b_2) &= -\langle h_i, \lambda \rangle, \end{aligned}$$

$$(5.1.3) \quad \tilde{e}_i^*(b_1 \otimes t_\lambda \otimes b_2) = \begin{cases} \tilde{e}_i^*b_1 \otimes t_{\lambda - \alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) \geq \varphi_i^*(b_2) + \langle h_i, \lambda \rangle \\ b_1 \otimes t_{\lambda - \alpha_i} \otimes \tilde{e}_i^*b_2 & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \langle h_i, \lambda \rangle, \end{cases}$$

$$(5.1.4) \quad \tilde{f}_i^*(b_1 \otimes t_\lambda \otimes b_2) = \begin{cases} \tilde{f}_i^*b_1 \otimes t_{\lambda + \alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \langle h_i, \lambda \rangle \\ b_1 \otimes t_{\lambda + \alpha_i} \otimes \tilde{f}_i^*b_2 & \text{if } \varepsilon_i^*(b_1) \leq \varphi_i^*(b_2) + \langle h_i, \lambda \rangle. \end{cases}$$

We prove that these two structures of crystal may be regarded as a crystal structure over $\mathfrak{g} \oplus \mathfrak{g}$. This is compared with the fact that the bimodule structure of $\tilde{U}_q(g)$ may be regarded as a left $U_q(\mathfrak{g} \oplus \mathfrak{g})$ -module structure.

In order to see this, it is enough to show the following theorem.

THEOREM 5.1.1. \tilde{e}_i^* and \tilde{f}_i^* are strict morphisms of crystals (with respect to \tilde{e}_i and \tilde{f}_i).

In order to prove this theorem, we use the following lemma. The proof being straightforward, we omit it.

LEMMA 5.1.2. For any i and $\lambda \in P$, let us define the map $E_i: B_i \otimes T_\lambda \otimes B_i \rightarrow B_i \otimes T_{\lambda - \alpha_i} \otimes B_i$ by

$$E_i(b_i(n) \otimes t_\lambda \otimes b_i(m)) = \begin{cases} b_i(n+1) \otimes t_{\lambda - \alpha_i} \otimes b_i(m) & \text{if } n+m+\langle h_i, \lambda \rangle \leq 0, \\ b_i(n) \otimes t_{\lambda - \alpha_i} \otimes b_i(m+1) & \text{if } n+m+\langle h_i, \lambda \rangle > 0. \end{cases}$$

Then E_i is a strict morphism of crystal.

Now, there are strict embeddings from $B(\infty)$ into $B(\infty) \otimes B_i$ and from $B(-\infty)$ into $B_i \otimes B(-\infty)$. There are also strict morphisms from $B(\infty) \otimes B_i$ to $B(\infty)$ and from $B_i \otimes B(-\infty)$ to $B(-\infty)$ (cf. (1.5.20)). Thus we obtain a chain of strict morphisms of crystals

$$(5.1.5) \quad B(\infty) \otimes T_\lambda \otimes B(-\infty) \rightarrow B(\infty) \otimes B_i \otimes T_\lambda \otimes B_i \otimes B(-\infty) \\ \xrightarrow{E_i} B(\infty) \otimes B_i \otimes T_{\lambda-\alpha_i} \otimes B_i \otimes B(-\infty) \\ \rightarrow B(\infty) \otimes T_{\lambda-\alpha_i} \otimes B(-\infty).$$

LEMMA 5.1.3. *The composition of the morphisms (5.1.5) coincides with \tilde{e}_i^* .*

This follows immediately from the formulas (5.1.2) and (5.1.3). Thus \tilde{e}_i^* is a strict morphism of crystal. Similarly \tilde{f}_i^* is a strict morphism of crystals.

6. Properties of global bases.

6.1. *Preliminary.* We study in [K2] the properties of global bases of integrable modules. They give the properties of the global bases of $\tilde{U}_q(\mathfrak{g})$ reducing to those of $V(\xi) \otimes V(\eta)$. For $N \geq 0$, we set

$$I_N = \sum_i \tilde{U}_q(\mathfrak{g}) e_i^N + \sum_i \tilde{U}_q(\mathfrak{g}) f_i^N.$$

PROPOSITION 6.1.1. *Let n be a nonnegative integer and $i \in I$. Then $u \in \tilde{U}_q(\mathfrak{g})$ satisfies $u \in f_i^n \tilde{U}_q(\mathfrak{g}) + I_N$ (resp. $u \in e_i^n \tilde{U}_q(\mathfrak{g}) + I_N$) for any N if and only if u is a linear combination of $G(b)$ with $\varepsilon_i(b) \geq n$.*

Proof. Assuming $u \in U_q(\mathfrak{g}) a_\lambda$, let us take $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$. Then $\Phi_{\xi, \eta}(u) \in f_i^n(V(\xi) \otimes V(\eta))$ if and only if $u \in f_i^n \tilde{U}_q(\mathfrak{g}) + \sum_j \tilde{U}_q(\mathfrak{g}) e_j^{1+\langle h_j, \xi \rangle} + \sum_j \tilde{U}_q(\mathfrak{g}) f_j^{1+\langle h_j, \eta \rangle}$. Hence taking ξ such that $\langle h_j, \xi \rangle \gg 0$ for all j , we obtain the lemma by the corresponding result of global bases of $V(\xi) \otimes V(\eta)$. Q.E.D.

6.2. *Definition of $V_i(\lambda)$.* To get more precise results than Proposition 6.1.1, let us generalize the results of Lusztig little bit.

For $i \in I$ and $\lambda \in P_+$, let $V_i(\lambda)$ be the $U_q(\mathfrak{g})$ -module generated by $u_{i, \lambda}$ with the defining relation $q^h u_{i, \lambda} = q^{\langle h, \lambda \rangle} u_{i, \lambda}$, $e_j u_{i, \lambda} = 0$ for any $j \in I$ and $f_i^{1+\langle h_i, \lambda \rangle} u_{i, \lambda} = 0$. Similarly for $\mu \in P_-$, let $V_i(\mu)$ be the $U_q(\mathfrak{g})$ -module generated by $u_{i, \mu}$ with the defining relation $q^h u_{i, \mu} = q^{\langle h, \mu \rangle} u_{i, \mu}$, $f_j u_{i, \mu} = 0$ for any $j \in I$ and $e_i^{1-\langle h_i, \mu \rangle} u_{i, \mu} = 0$. Then

$$(6.2.1) \quad V_i(\lambda) \cong U_q^-(\mathfrak{g})/U_q^-(\mathfrak{g}) f_i^{1+\langle h_i, \lambda \rangle},$$

$$(6.2.2) \quad V_i(\mu) \cong U_q^+(\mathfrak{g})/U_q^+(\mathfrak{g}) e_i^{1-\langle h_i, \mu \rangle}.$$

Now consider the $U_q(\mathfrak{g})$ -module $V_i(\lambda) \otimes V_i(\mu)$. One can see easily that this is a $U_q(\mathfrak{g})$ -

module generated by $u_{i, \lambda} \otimes u_{i, \mu}$ with the defining relation

$$q^h(u_{i, \lambda} \otimes u_{i, \mu}) = q^{\langle h, \lambda + \mu \rangle} (u_{i, \lambda} \otimes u_{i, \mu}) \quad \text{for any } h \in P^*,$$

$$f_i^{1+\langle h_i, \lambda \rangle} (u_{i, \lambda} \otimes u_{i, \mu}) = 0 \quad \text{and} \quad e_i^{1-\langle h_i, \mu \rangle} (u_{i, \lambda} \otimes u_{i, \mu}) = 0.$$

Hence there is a chain of surjective homomorphisms

$$U_q(\mathfrak{g}) a_{\lambda+\mu} \xrightarrow{\Phi_{\lambda, \mu}^i} V_i(\lambda) \otimes V_i(\mu) \rightarrow V(\lambda) \otimes V(\mu).$$

6.3. *Refinement.* For $\xi \in Q_\pm$, let us denote

$$F_\xi U_q^\pm(\mathfrak{g}) = \bigoplus_{\eta \in \xi + Q_\pm} U_q^\pm(\mathfrak{g})_\eta.$$

For $\xi_\pm \in P_\pm$, we set

$$F_{\xi_+, \xi_-} U_q(\mathfrak{g}) = F_{\xi_+}(U_q^+(\mathfrak{g})) F_{\xi_-}(U_q^-(\mathfrak{g})) \mathcal{T} \\ = F_{\xi_-}(U_q^-(\mathfrak{g})) F_{\xi_+}(U_q^+(\mathfrak{g})) \mathcal{T},$$

where $\mathcal{T} = \bigoplus_{h \in P^*} \mathbf{Q}(q) q^h$. For $\xi \in Q_+$, we set $F_\xi U_q(\mathfrak{g}) = F_{\xi, -\xi} U_q(\mathfrak{g})$. Then, by Proposition 2.2.2, $F_{\xi_+, \xi_-} U_q(\mathfrak{g}) a_{\lambda+\mu}$ is generated by global bases. The purpose of this section is to prove the following proposition.

PROPOSITION 6.3.1. *Set $L(V_i(\lambda) \otimes V_i(\mu)) = \Phi_{\lambda, \mu}^i(L(U_q(\mathfrak{g}) a_{\lambda+\mu}))$. Let $\bar{\Phi}_{\lambda, \mu}^i$ be the induced morphism $L(U_q(\mathfrak{g}) a_{\lambda+\mu})/qL(U_q(\mathfrak{g}) a_{\lambda+\mu}) \rightarrow L(V_i(\lambda) \otimes V_i(\mu))/qL(V_i(\lambda) \otimes V_i(\mu))$. Set $B(V_i(\lambda) \otimes V_i(\mu)) = \{\bar{\Phi}_{\lambda, \mu}^i(b); b \in B(U_q(\mathfrak{g}) a_{\lambda+\mu})\} \setminus \{0\}$. Then:*

(i) *$(L(V_i(\lambda) \otimes V_i(\mu)), B(V_i(\lambda) \otimes V_i(\mu)))$ is a crystal base of the integrable $U_q(\mathfrak{g}_i)$ -module $V_i(\lambda) \otimes V_i(\mu)$. Here $U_q(\mathfrak{g}_i)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i , f_i , and q^h ($h \in P^*$).*

(ii) *For $b \in B(U_q(\mathfrak{g}) a_{\lambda+\mu})$, $\bar{\Phi}_{\lambda, \mu}^i(b) \neq 0$ if and only if $\varepsilon_i^*(b) \leq \langle h_i, \lambda \rangle$.*

(iii) *$\{b \in B(U_q(\mathfrak{g}) a_{\lambda+\mu}); \bar{\Phi}_{\lambda, \mu}^i(b) \neq 0\}$ is isomorphic to $B(V_i(\lambda) \otimes V_i(\mu))$.*

(iv) *$\{G(b)(u_{i, \lambda} \otimes u_{i, \mu})\}$ forms a base of $V_i(\lambda) \otimes V_i(\mu)$ where b ranges over $\{b \in B(U_q(\mathfrak{g}) a_{\lambda+\mu}); \bar{\Phi}_{\lambda, \mu}^i(b) \neq 0\}$.*

(v) *$\bar{\Phi}_{\lambda, \mu}^i$ is a morphism (in $\mathcal{C}(\{i\}, P)$) from $B(U_q(\mathfrak{g}) a_{\lambda+\mu})$ to $B(V_i(\lambda) \otimes V_i(\mu))$.*

Proof. Let us take $\xi \in Q_+$. Since $V_i(\lambda) \otimes V_i(\mu)$ is integrable as $U_q(\mathfrak{g}_i)$ -module, $F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i, \lambda} \otimes u_{i, \mu})$ is stationary when n increases. Take N such that $F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i, \lambda} \otimes u_{i, \mu}) = F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i, \lambda} \otimes u_{i, \mu})$ for $n \geq N$. Then $F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i, \lambda} \otimes u_{i, \mu})$ is a finite-dimensional $U_q(\mathfrak{g}_i)$ -module. Taking $\eta \in P_+$ such that $\langle h_j, \eta \rangle \gg 0$ for $j \neq i$ and $\langle h_i, \eta \rangle = 0$,

$$F_{\xi+N\alpha_i} U_q(\mathfrak{g})(u_{i, \lambda} \otimes u_{i, \mu}) \rightarrow V(\lambda + \eta) \otimes V(\mu - \eta)$$

is injective. Now consider the chain of homomorphisms

$$U_q(\mathfrak{g})a_{\lambda+\mu} \xrightarrow{\Phi_{\lambda,\mu}^i} V_i(\lambda) \otimes V_i(\mu) \xrightarrow{p} V(\lambda + \eta) \otimes V(\mu - \eta).$$

Then $p\Phi_{\lambda,\mu}^i(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})) = F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda + \eta) \otimes L(\mu - \eta)$. Hence $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu))$ is contained in $p^{-1}p\Phi_{\lambda,\mu}^i(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})) \subset p^{-1}(0) + \Phi_{\lambda,\mu}^i(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu}))$. Since $p^{-1}(0) \cap F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) = 0$, we obtain

$$F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu)) = \Phi_{\lambda,\mu}^i(F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})).$$

Thus, we obtain

$$\begin{aligned} & (F_{\xi+N\alpha_i}U_q(\mathfrak{g})a_{\lambda+\mu} \cap L(U_q(\mathfrak{g})a_{\lambda+\mu})) \\ & \rightarrow (F_{\xi+N\alpha_i}U_q(\mathfrak{g}))(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu)) \\ & \xrightarrow{\sim} (F_{\xi+N\alpha_i}U_q(\mathfrak{g}))(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(\lambda + \eta) \otimes L(\mu - \eta). \end{aligned}$$

Now, let us prove Proposition 6.2.1. For example, let us show that $L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice. Since $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta}) \cap L(V(\lambda + \eta) \otimes V(\mu - \eta))$ is a crystal lattice of the integrable $U_q(\mathfrak{g})$ -module $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{\lambda+\eta} \otimes u_{\mu-\eta})$, $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu}) \cap L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice of $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$. Since $V_i(\lambda) \otimes V_i(\mu)$ is a union of $F_{\xi+N\alpha_i}U_q(\mathfrak{g})(u_{i\lambda} \otimes u_{i\mu})$, $L(V_i(\lambda) \otimes V_i(\mu))$ is a crystal lattice. The other statements can be proven similarly. Q.E.D.

6.4. Interpretation of ε and φ . By using Proposition 6.3.1, we can sharpen Proposition 6.1.1. Set

$$I_N^i = \tilde{U}_q(\mathfrak{g})e_i^N + \tilde{U}_q(\mathfrak{g})f_i^N.$$

Then the proof of the following proposition is similar to the one of Proposition 6.1.1.

PROPOSITION 6.4.1. Let n be a nonnegative integer and $i \in I$. Then $u \in \tilde{U}_q(\mathfrak{g})$ satisfies $u \in \tilde{f}_i^n \tilde{U}_q(\mathfrak{g}) + I_N^i$ (resp. $u \in e_i^n \tilde{U}_q(\mathfrak{g}) + I_N^i$) for any N if and only if u is a linear combination of $G(b)$ with $\varepsilon_i(b) \geq n$.

PROPOSITION 6.4.2. For $a, c \in \mathbb{Z}_{\geq 0}$ and $i \in I$,

$$(6.4.1) \quad e_i^a \tilde{U}_q(\mathfrak{g}) + f_i^c \tilde{U}_q(\mathfrak{g}) = \bigoplus_b \mathbb{Q}(q)G(b).$$

Here b ranges over $\{b \in B(\tilde{U}_q(\mathfrak{g})); \varphi_i(b) \geq a \text{ or } \varepsilon_i(b) \geq c\}$.

Proof. Taking $*$, it is enough to show

$$(6.4.2) \quad U_q(\mathfrak{g})e_i^a a_\lambda + U_q(\mathfrak{g})f_i^c a_\lambda = \bigoplus_b \mathbb{Q}(q)G(b).$$

Here b ranges over $\{b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda), \varphi_i^*(b) \geq a \text{ or } \varepsilon_i^*(b) \geq c\}$.

(Step I) The case where $\langle h_i, \lambda \rangle = c - a$. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that

$$\lambda = \xi + \eta, \quad \langle h_i, \xi \rangle = 1 + c, \quad \langle h_i, \eta \rangle = -1 - a.$$

Then the left-hand side of (6.4.2) coincides with the kernel of $\Phi^i(\xi, \eta)$. Since the kernel of $\Phi^i(\xi, \eta)$ is generated by $G(b)$ with $b \in B(\tilde{U}_q(\mathfrak{g})a_\lambda) \setminus B(\xi) \otimes B(\eta) = \{b_1 \otimes t_\lambda \otimes b_2; \varepsilon_i^*(b_1) \leq c, \varphi_i^*(b_2) \leq a\}$, we obtain the desired result.

(Step II) General case. We define \tilde{a}, \tilde{c} by

$$\tilde{c} = c, \quad \tilde{a} = c - \langle h_i, \lambda \rangle \quad \text{if } \langle h_i, \lambda \rangle \leq c - a,$$

$$\tilde{a} = a, \quad \tilde{c} = \langle h_i, \lambda \rangle + a \quad \text{if } \langle h_i, \lambda \rangle \geq c - a.$$

Then $\tilde{c} \geq c$ and $\tilde{a} \geq a$. On the other hand, $\varepsilon_i^*(b) - \varphi_i^*(b) = \langle h_i, \lambda \rangle$ implies that the condition $\varphi_i^*(b) \geq a$ or $\varepsilon_i^*(b) \geq c$ is equivalent to $\varphi_i^*(b) \geq \tilde{a}$ or $\varepsilon_i^*(b) \geq \tilde{c}$. Hence the right-hand side of (6.4.2) is equal to $\tilde{U}_q(\mathfrak{g})e_i^{\tilde{a}} a_\lambda + U_q(\mathfrak{g})f_i^{\tilde{c}} a_\lambda$. Then it is enough to apply

$$U_q(\mathfrak{g})e_i^{\tilde{a}} a_\lambda + U_q(\mathfrak{g})f_i^{\tilde{c}} a_\lambda = U_q(\mathfrak{g})e_i^a a_\lambda + U_q(\mathfrak{g})f_i^c a_\lambda. \quad \text{Q.E.D.}$$

Let us remark that $\varepsilon_i(b) \geq n$ does not imply $G(b) \in f_i^n \tilde{U}_q(\mathfrak{g})$ in general. This is only true modulo I_N^i or modulo $e_i^n \tilde{U}_q(\mathfrak{g})$.

PROPOSITION 6.4.3. For $b \in B(\tilde{U}_q(\mathfrak{g}))$,

$$(6.4.3) \quad f_i^{(n)} G(b) = \left[\begin{matrix} \varepsilon_i(b) + n \\ n \end{matrix} \right]_i G(\tilde{f}_i^n b) + \sum_{b'} F_{b,b'}^i(q) G(b').$$

Here b' ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\varepsilon_i(b') > \varepsilon_i(b) + n$ and $F_{b,b'}^i(q) \in \mathbb{Z}[q, q^{-1}]$. In addition,

$$(6.4.4) \quad e_i^{(n)} G(b) = \left[\begin{matrix} \varphi_i(b) + n \\ n \end{matrix} \right]_i G(\tilde{e}_i^n b) + \sum_{b'} E_{b,b'}^i(q) G(b').$$

Here b' ranges over the set of $B(\tilde{U}_q(\mathfrak{g}))$ such that $\varphi_i(b') > \varphi_i(b) + n$.

Formulas (6.4.3) and (6.4.4) follow immediately from the corresponding results on global bases of integrable modules (see [K2]).

6.5. *Further property.* The following results are used later.

LEMMA 6.5.1. *If $\langle h_i, \lambda \rangle \geq k$, then*

$$L(U_q(\mathfrak{g})a_{\lambda - k\alpha_i})f_i^{(k)} = q_i^{k(k - \langle h_i, \lambda \rangle)}L(\tilde{U}_q(\mathfrak{g})) + \tilde{U}_q(\mathfrak{g})e_i + \tilde{U}_q(\mathfrak{g})f_i^{1 + \langle h_i, \lambda \rangle}.$$

Proof. Taking $*$ and changing λ with $-\lambda$, it is enough to show

$$a_{\lambda}f_i^{(k)}L(\tilde{U}_q(\mathfrak{g})) = q_i^{k(k + \langle h_i, \lambda \rangle)}L(\tilde{U}(\mathfrak{g})) + e_i\tilde{U}_q(\mathfrak{g}) + f_i^{1 - \langle h_i, \lambda \rangle}\tilde{U}_q(\mathfrak{g})$$

$$\text{when } \langle h_i, \lambda \rangle + k \leq 0,$$

or for $b \in B(\tilde{U}_q(\mathfrak{g}))$ of weight $\lambda + k\alpha_i$, writing $f_i^{(k)}G(b) = \sum c_{b'}G(b')$,

$$(6.5.1) \quad \varphi_i(b') = 0 \quad \text{implies } c_{b'} \in q_i^{k(k + \langle h_i, \lambda \rangle)}.$$

This reduces to the same statement for $b \in B(V(\xi) \otimes V(\eta))$ for $\xi \in P_+$ and $\eta \in P_-$. Write $G(b) = \sum f_i^{(n)}u_n$ with $e_i u_n = 0$ and $\text{wt}(u_n) = \lambda + (n + k)\alpha_i$. Then $u_n \in L(\xi) \otimes L(\eta)$ and $f_i^{(k)}G(b) = \sum \begin{bmatrix} n+k \\ k \end{bmatrix} f_i^{(k+n)}u_n$. If $k + n \neq \langle h_i, \lambda + (n + k)\alpha_i \rangle$, then $f_i^{(k+n)}u_n$ belongs to $e_i(V(\xi) \otimes V(\eta))$. If $k + n = \langle h_i, \lambda + (n + k)\alpha_i \rangle$, then $\begin{bmatrix} n+k \\ k \end{bmatrix} f_i^{(k+n)}u_n$ belongs to $q_i^{-nk}L(\xi) \otimes L(\eta) = q_i^{k(\langle h_i, \lambda \rangle + k)}L(\xi) \otimes L(\eta)$. Hence $f_i^{(k)}G(b) \in q_i^{k(\langle h_i, \lambda \rangle + k)}L(\xi) \otimes L(\eta) + e_i(V(\xi) \otimes V(\eta))$. This implies (6.5.1). Q.E.D.

7. The Weyl group action on crystal bases.

7.1. *Action of simple reflections.* Let B be a normal crystal. We define the action of the Weyl group W on the underlying set B . For $i \in I$ and $b \in B$, we set

$$(7.1.1) \quad S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \leq 0. \end{cases}$$

Then we have the obvious relation

$$(7.1.2) \quad \text{wt}(S_i b) = s_i(\text{wt}(b))$$

where $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ is the simple reflection,

$$(7.1.3) \quad S_i^2 = id,$$

$$(7.1.4) \quad S_i \tilde{e}_i S_i^{-1} = \tilde{f}_i.$$

We show that this extends to the action of the Weyl group. In order to see this, it is enough to check the braid relation for a finite-dimensional \mathfrak{g} of rank 2.

7.2. *Braid relation.* Set $I = \{1, 2\}$ and assume \mathfrak{g} is finite-dimensional. Let $w_0 = s_{i_1} \cdots s_{i_l}$ be a reduced expression of the longest element of W . There are two choices. We show that

$$(7.2.1) \quad S_{i_1} \cdots S_{i_l} b \text{ does not depend on the choice of reduced expression.}$$

In order to see this we may assume that $\text{wt}(b)$ is dominant. If $\text{wt}(b)$ is not regular, (7.2.1) is trivial. Hence we may assume $\text{wt}(b)$ is regular and dominant.

For any normal crystal B , set $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$. Then we have that

$$(7.2.2) \quad \tilde{f}_i^{\max} \cdots \tilde{f}_i^{\max} b \text{ does not depend on the choice of reduced expression.}$$

In fact this vector is the unique lowest-weight vector in the connected component containing b (cf. [K3]).

Now we remark

$$(7.2.3) \quad \tilde{f}_i^{\max}(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{\varphi_i(b_1) - \varepsilon_i(b_2)} b_1 \otimes \tilde{f}_i^{\max} b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i^{\max} b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

LEMMA 7.2.1. *Let b be a vector with $\langle h_i, \text{wt}(b) \rangle > 0$. Then, for any b' in any crystal B' , there exists an integer m and $b'' \in B^{\otimes m} \otimes B'$ such that for any $n \geq m$,*

$$\tilde{f}_i^{\max}(b^{\otimes n} \otimes b') = (S_i b)^{\otimes(n-m)} \otimes b''.$$

Proof. We have for $1 \leq v$

$$\begin{aligned} \varepsilon_i(b^{\otimes v} \otimes b') &= \max\{\varepsilon_i(b) - k\langle h_i, \text{wt}(b) \rangle \mid 0 \leq k < v\}, \varepsilon_i(b') - v\langle h_i, \text{wt}(b) \rangle\} \\ &= \max(\varepsilon_i(b), \varepsilon_i(b') - v\langle h_i, \text{wt}(b) \rangle). \end{aligned}$$

Hence if $\varepsilon_i(b') - v\langle h_i, \text{wt}(b) \rangle \leq \varepsilon_i(b)$, (7.2.3) and $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \text{wt}(b) \rangle$ imply $\tilde{f}_i^{\max}(b^{\otimes(v+1)} \otimes b') = S_i b \otimes \tilde{f}_i^{\max}(b^{\otimes v} \otimes b')$. Thus we obtain the desired result. Q.E.D.

This lemma implies that $\tilde{f}_i^{\max} \cdots \tilde{f}_i^{\max}(b^{\otimes n}) = S_{i_1} \cdots S_{i_l} b \otimes b'$ for some $b' \in B^{\otimes(n-1)}$ if $n \gg 0$. Thus (7.2.2) implies the following result.

THEOREM 7.2.2. $\{S_i\}$ satisfies the braid relation.

7.3. *Application.* For $\lambda \in P$ and $w \in W$, take a reduced expression $w = s_{i_1} \cdots s_{i_l}$ of w . Then the condition $\langle h_{i_1}, s_{i_{l-1}} \cdots s_{i_1} \lambda \rangle \geq 0, \dots, \langle h_{i_l}, \lambda \rangle \geq 0$ does not depend on the choice of reduced expression. If this condition is satisfied, we say that λ is w -dominant. If λ is w -dominant, we set $\tilde{f}_{w,\lambda} = \tilde{f}_{i_1}^{\langle h_{i_1}, s_{i_{l-1}} \cdots s_{i_1} \lambda \rangle} \cdots \tilde{f}_{i_l}^{\langle h_{i_l}, \lambda \rangle}$.

PROPOSITION 7.4.1. *If $\lambda \in P$ is w -dominant, the definition of $\tilde{f}_{w,\lambda}$ does not depend on the choice of reduced expression of w . Here we regard $\tilde{f}_{w,\lambda}$ as an operator on normal crystals, $B(\infty)$ or $B(-\infty)$.*

Proof. It is enough to show this on $B(\infty)$. Then it follows from

$$(7.4.2) \quad S_{i_1} \cdots S_{i_l}(b \otimes t_{\lambda - w(b)} \otimes u_{-\infty}) = \tilde{f}_{w, \lambda} b \otimes t_{\lambda - w(b)} \otimes u_{-\infty}. \quad \text{Q.E.D.}$$

Remark 7.4.2. $w\alpha_i = \alpha_j$ does not imply $S_w \tilde{e}_i S_w^{-1} = \tilde{e}_j$ even in the A_2 -case.

8. Extremal vectors.

8.1. Definition of extremal vectors. Let M be an integral $U_q(\mathfrak{g})$ -module. A weight vector u of weight $\lambda \in P$ of M is called i -extremal if $e_i u = 0$ or $f_i u = 0$. In this case, we set $S_i u = f_i^{\langle h_i, \lambda \rangle} u$ or $e_i^{\langle -h_i, \lambda \rangle} u$, respectively.

Definition 8.1.1. A weight vector u is called extremal if, for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} u$ is i -extremal for any $i, i_1, \dots, i_l \in I$.

This notion generalizes that of highest-weight vector. A similar definition is possible for an element of a normal crystal. An element b of a normal crystal B is called i -extremal if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$.

Definition 8.1.2. An element b of B is called extremal if, for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} b$ is i -extremal for any $i, i_1, \dots, i_l \in I$.

8.2. Modules generated by extremal vectors. Let $\lambda \in P$ be an integral weight. Set $\lambda_+ = \sum_{\langle h_i, \lambda \rangle \geq 0} \langle h_i, \lambda \rangle \Lambda_i$ and $\lambda_- = \lambda_+ - \lambda \in P_+$. Then $V'(\lambda) = V(\lambda_+) \otimes V(-\lambda_-)$ is isomorphic to

$$U_q(\mathfrak{g})a_\lambda / \sum_{\langle h_i, \lambda \rangle \geq 0} U_q(\mathfrak{g})e_i^{1-\langle h_i, \lambda \rangle} + \sum_{\langle h_i, \lambda \rangle \leq 0} U_q(\mathfrak{g})f_i^{1+\langle h_i, \lambda \rangle},$$

and $V'(\lambda)$ has a global base: $V'(\lambda) = \bigoplus_{e_i^*(b) = \langle h_i, \lambda \rangle_+} \mathbb{Q}(q)G(b)$. Let $p_\lambda: U_q(\mathfrak{g})a_\lambda \rightarrow V'(\lambda)$ be the projection.

LEMMA 8.2.1. For $i \in I$ and $\lambda \in P$, consider the commutative diagram

$$\begin{array}{ccc} U_q(\mathfrak{g})a_\lambda & \xrightarrow{\psi_1} & U_q(\mathfrak{g})a_{s_i \lambda} \\ & \searrow \psi & \downarrow p_{s_i \lambda} \\ & & V'(s_i \lambda). \end{array}$$

Here ψ_1 is given by

$$\psi_1(a_\lambda) = \begin{cases} e_i^{\langle h_i, \lambda \rangle} a_{s_i \lambda} & \text{if } \langle h_i, \lambda \rangle \geq 0, \\ f_i^{\langle -h_i, \lambda \rangle} a_{s_i \lambda} & \text{if } \langle h_i, \lambda \rangle \leq 0. \end{cases}$$

Then for $b \in B(U_q(\mathfrak{g})a_\lambda)$ we have $\psi(G(b)) = p_{s_i \lambda} G(S_i^* b)$. Here $S_i^* b = (S_i(b^*))^*$.

Proof. The other case being similarly proved, let us assume $\langle h_i, \lambda \rangle \geq 0$. Then $\psi(G(b)) = p_{s_i \lambda} G(b) e_i^{\langle h_i, \lambda \rangle} a_{s_i \lambda}$. If $e_i^*(b) > 0$, then $G(b) \in \tilde{U}_q(\mathfrak{g})e_i + I_N$ for any $N \geq 0$

and hence $p_{s_i \lambda}(e_i^{\langle h_i, \lambda \rangle + 1} a_{s_i \lambda}) = 0$ implies $\psi(G(b)) = 0$. If $e_i^*(b) = 0$ then

$$G(b)e_i^{\langle h_i, \lambda \rangle} \equiv G(\tilde{e}_i^{\langle h_i, \lambda \rangle} b) \bmod \tilde{U}_q(\mathfrak{g})e_i^{\langle h_i, \lambda \rangle + 1} + I_N.$$

Therefore we have

$$\psi(G(b)) = p_{s_i \lambda}(G(\tilde{e}_i^{\langle h_i, \lambda \rangle} b) a_{s_i \lambda}). \quad \text{Q.E.D.}$$

Thus, repeating this procedure, we obtain the following.

PROPOSITION 8.2.2. For $\lambda \in P$, set $B^{\max}(\lambda) = \{b \in B(U_q(\mathfrak{g})a_\lambda); b^* \text{ is extremal}\}$ and

$$I_\lambda = \bigoplus_{b \in B(U_q(\mathfrak{g})a_\lambda) \setminus B^{\max}(\lambda)} \mathbb{Q}(q)G(b).$$

Then we have:

- (i) I_λ is a left $U_q(\mathfrak{g})$ -submodule of $U_q(\mathfrak{g})a_\lambda$.
- (ii) $V^{\max}(\lambda) = U_q(\mathfrak{g})a_\lambda / I_\lambda$ is an integrable $U_q(\mathfrak{g})$ -module.
- (iii) Let $p_\lambda: U_q(\mathfrak{g})a_\lambda \rightarrow V^{\max}(\lambda)$ be the projection and $u_\lambda = p_\lambda(a_\lambda)$. Then u_λ is an extremal vector of weight λ .
- (iv) For any $i \in I$, we have an isomorphism

$$V^{\max}(\lambda) \simeq V^{\max}(s_i \lambda)$$

$$\text{by } u_\lambda \mapsto S_i u_{s_i \lambda}.$$

Moreover this isomorphism sends the global base to the global base.

- (v) For any i , S_i^* gives an isomorphism $B^{\max}(\lambda) \simeq B^{\max}(s_i \lambda)$.

We have

$$(8.2.2) \quad V^{\max}(\lambda) \cong \bigoplus_{b \in B^{\max}(\lambda)} \mathbb{Q}(q)G(b).$$

Thus $B^{\max}(\lambda)$ is a crystal base of $V^{\max}(\lambda)$.

If λ is dominant (resp. antidominant), then $V^{\max}(\lambda)$ is the irreducible $U_q(\mathfrak{g})$ -module with highest- (resp. lowest-) weight λ .

8.3. A proof of the Parthasarathy-Varadarajan-Rao conjecture. For $\lambda \in P$, let us denote by $B(\lambda)$ the connected component of $B(U_q(\mathfrak{g})a_\lambda)$ containing a_λ . Let us write $u_\lambda \in B(\lambda)$ for a_λ . Then $S_w^* = *S_w^*$ gives an isomorphism

$$(8.3.1) \quad B(\lambda) \cong B(w\lambda) \quad \text{for any } w \in W.$$

If λ is dominant (resp. antidominant), $B(\lambda)$ and $B^{\max}(\lambda)$ coincide with the sixth example in Example 1.5.3. (cf. the proof of Lemma 10.2.1.)

LEMMA 8.3.1. For $\lambda \in P_+$ and $\mu \in P$, $B(\lambda + \mu)$ is isomorphic to the connected component of $B(\lambda) \otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$.

Proof. This follows immediately from the following chain of morphisms $B(\lambda + \mu) \hookrightarrow B(\lambda + \mu_+) \otimes B(-\mu_-) \hookrightarrow B(\lambda) \otimes B(\mu_+) \otimes B(-\mu_-) \hookrightarrow B(\lambda) \otimes B(\mu)$.

Q.E.D.

As an application, we can obtain a new proof of the Parthasarathy-Varadarajan-Rao conjecture:

(8.3.2) For $\lambda, \mu \in P_+$, $w, w' \in W$, if $w'(\lambda + w\mu) \in P_+$, then $V(w'(\lambda + w\mu))$ appears in $V(\lambda) \otimes V(\mu)$.

In fact, it is enough to show that $B(w'(\lambda + w\mu))$ appears in $B(\lambda) \otimes B(\mu)$. However this follows from $B(w'(\lambda + w\mu)) \cong B(\lambda + w\mu) \subset B(\lambda) \otimes B(w\mu) \cong B(\lambda) \otimes B(\mu)$.

9. A property of $L(\tilde{U}_q(\mathfrak{g}))$.

9.1. Property of $L(\tilde{U}_q(\mathfrak{g}))u$. Let us investigate properties of $L(\tilde{U}_q(\mathfrak{g}))u$ for an element u of an integrable $U_q(\mathfrak{g})$ -module.

PROPOSITION 9.1.1. Let M be an integrable $U_q(\mathfrak{g})$ -module, $\lambda \in P$, and u an element of M_λ . Then $L(\tilde{U}_q(\mathfrak{g})a_\lambda)u$ is invariant by \tilde{e}_i and \tilde{f}_i .

Proof. Take N such that $e_i^N u = f_i^N u = 0$ for any i . Then take $\xi \in P_+$, $\eta \in P_-$ such that $\lambda = \xi + \eta$ and $\langle h_i, \xi \rangle > N$, $-\langle h_i, \eta \rangle > N$. Then the morphism $\tilde{U}_q(\mathfrak{g})a_\lambda \rightarrow M$ splits through $V(\xi) \otimes V(\eta)$. The result follows from the fact that any $U_q(\mathfrak{g})$ -linear homomorphism commutes with \tilde{e}_i and \tilde{f}_i . Q.E.D.

PROPOSITION 9.1.2. Let L be a crystal base of an integrable $U_q(\mathfrak{g})$ -module M . For $\lambda \in P$, set

$$N = \{u \in M; L(\tilde{U}_q(\mathfrak{g}))u \subset q^\lambda L \text{ for some } \lambda\}.$$

Then N is a $U_q(\mathfrak{g})$ -module.

Proof. It is trivial that N is a $\mathbb{Q}(q)$ -vector space. Hence it is enough to show that N is invariant by e_i and f_i . Thus the proposition is reduced to the following statement.

(9.1.1) If $u \in M_\lambda$ satisfies $L(U_q(\mathfrak{g})a_\lambda)u \subset L$, then $L(U_q(\mathfrak{g})a_{\lambda-\alpha_i})f_i u \subset q^\lambda L$ for some c .

Let us take N such that $f_i^{N+1}u = 0$. We shall show

$$(9.1.2) \quad L(\infty)u \subset q_i^{-N}L.$$

For any $P \in L(\infty)$, there exists $Q_n \in L(\infty) \cap \text{Ker } e_i^n$ such that $P = \sum Q_n f_i^{(n)}$ (cf. [K1]).

Then

$$\begin{aligned} Pu &= \sum Q_n f_i^{(n)} f_i u \\ &= \sum [n+1]_i Q_n f_i^{(n+1)} u. \end{aligned}$$

Since $f_i^{N+1}u = 0$, we may assume that n ranges over the integers $n \leq N$. In this case $[n+1]_i Q_n f_i^{(n+1)} u \subset q_i^{-n} L(U_q(\mathfrak{g})a_\lambda)u \subset q_i^{-N} L$. Thus we obtain (9.1.2). To complete the proof, it is enough to apply the following proposition. Q.E.D.

PROPOSITION 9.1.3. Let M be an integrable $U_q(\mathfrak{g})$ -module and L an A submodule of M invariant by \tilde{e}_i and \tilde{f}_i . Let $\psi: U_q(\mathfrak{g})a_\lambda \rightarrow M$ be a $U_q(\mathfrak{g})$ -linear homomorphism such that $\psi(L(\infty)a_\lambda) \subset L$. Then $\psi(\tilde{U}_q(\mathfrak{g})a_\lambda) \subset L$.

Proof. Since ψ splits $U_q(\mathfrak{g})a_\lambda \rightarrow V(\xi) \otimes V(\eta)$ for some $\xi, -\eta \in P_+$ with $\lambda = \xi + \eta$, the result follows from Lemma 9.1.4. below.

LEMMA 9.1.4. Let $\xi \in P_+$ and $\eta \in P_-$. Then $L(\xi) \otimes L(\eta)$ is the smallest A -module of $V(\xi) \otimes V(\eta)$ that is invariant by \tilde{e}_i and \tilde{f}_i and that contains $L(\lambda) \otimes u_\eta$.

Proof. Let L be an A -submodule of $V(\xi) \otimes V(\eta)$ invariant by \tilde{e}_i and \tilde{f}_i . For $w \in W$, with reduced expression $w = s_{i_1} \cdots s_{i_r}$, set $V_w(\lambda) = \sum \mathbb{Q}(q) f_{i_1}^{(a_1)} \cdots f_{i_r}^{(a_r)} u_\lambda$ and $V_w(\eta) = \sum \mathbb{Q}(q) l_{i_1}^{(a_1)} \cdots l_{i_r}^{(a_r)} u_\eta$. Then $V_w(\lambda)$ and $V_w(\eta)$ are finite-dimensional submodules. Set $L_w(\lambda) = L(\lambda) \cap V_w(\lambda)$ and $L_w(\eta) = L(\eta) \cap V_w(\eta)$.

In order to prove the lemma it is enough to show that,

$$(9.1.3) \quad \text{if } s_i w > w, s_i w' < w' \text{ and if } L \supset L_w(\xi) \otimes (L(\eta) \cap V_w(\eta)), \text{ then } L \supset L_{w'}(\xi) \otimes (L(\eta) \cap V_{s_i w}(\eta)).$$

Let us take $b_1 \in B_w(\xi)$ and $b_2 \in B_{s_i w}(\eta)$. Then $\tilde{f}_i^{\max} b_2 \in B_w(\eta)$, and there is c such that

$$\tilde{f}_i^{\max}(b_1 \otimes b_2) = \tilde{f}_i^c b_1 \otimes \tilde{f}_i^{\max} b_2.$$

Hence $\tilde{f}_i^c b_1 \otimes \tilde{f}_i^{\max} b_2$ belongs to $L_{w'}(\xi) \otimes L_w(\eta) \bmod qL_{w'}(\xi) \otimes L_{s_i w}(\eta)$. Hence $b_1 \otimes b_2$ belongs to $L \bmod qL_{w'}(\xi) \otimes L_{s_i w}(\eta)$. This shows that

$$L_{w'}(\xi) \otimes L_{s_i w}(\eta) \subset L + qL_{w'}(\xi) \otimes L_{s_i w}(\eta).$$

Nakayama's lemma implies the desired result: $L_{w'}(\xi) \otimes L_{s_i w}(\eta) \subset L$. Q.E.D.

9.2. Crystal lattice and $L(\tilde{U}(\mathfrak{g}))$.

THEOREM 9.2.1. Let M be an integrable $U_q(\mathfrak{g})$ -module and L a lower crystal lattice of M . Then

$$L' = \bigoplus_{\lambda} \{u \in M_\lambda; L(U_q(\mathfrak{g})a_\lambda)u \subset L\}$$

is invariant by \tilde{e}_i^{up} and \tilde{f}_i^{up} . (For \tilde{e}_i^{up} and \tilde{f}_i^{up} see [K2]).

We remark that, by the relation of upper and lower crystal bases, the statement above is equivalent to the statement that

$$\bigoplus_{\lambda} \{u \in M_{\lambda}; L(U_q(\mathfrak{g})a_{\lambda})u \subset q^{1/2(\|\lambda\|^2 - c)}L\}$$

is invariant by \tilde{e}_i and \tilde{f}_i . Here c is a number such that $\|\lambda\|^2 - c \in 2\mathbb{Z}$ for any weight λ of M . Here $\|\lambda\|^2 = (\lambda, \lambda)$. Note that, if $\lambda, \mu \in P$ satisfy $\lambda - \mu \in \sum \mathbb{Z}\alpha_i$, then $\|\lambda\|^2 - \|\mu\|^2 \in 2\mathbb{Z}$.

We shall prove the theorem under the last form. Set $L' = \bigoplus_{\lambda} \{u \in M_{\lambda}; L(U_q(\mathfrak{g})a_{\lambda})u \in q^{1/2(\|\lambda\|^2 - c)}L\}$

LEMMA 9.2.2. If $e_i u = 0$ and if $u \in L'_{\lambda}$ then $f_i^{(k)}u \in L'$.

Proof. We may assume $k \leq \langle h_i, \lambda \rangle$. Since $\|\lambda - k\alpha_i\|^2 = \|\lambda\|^2 + k(\alpha_i, \alpha_i) \cdot (k - \langle h_i, \lambda \rangle)$, it is enough to show

$$L(U_q(\mathfrak{g})a_{\lambda - k\alpha_i})f_i^{(k)}u \subset q_i^{k(k - \langle h_i, \lambda \rangle)}L(U_q(\mathfrak{g})a_{\lambda})u.$$

This is reduced to the statement

$$(9.2.1) \quad L(U_q(\mathfrak{g})a_{\lambda - k\alpha_i})f_i^{(k)} \subset q_i^{k(k - \langle h_i, \lambda \rangle)}L(U_q(\mathfrak{g})a_{\lambda}) + \tilde{U}_q(\mathfrak{g})e_i + \tilde{U}_q(\mathfrak{g})f_i^{1 + \langle h_i, \lambda \rangle}$$

This follows from Lemma 6.5.1.

Q.E.D.

Proof of Theorem 9.2.1. Let us take an element u in L' of weight λ . We write $u = \sum_{n \leq N} f_i^{(n)}u_n$ with $e_i u_n = 0$ and $\text{wt}(u_n) = \lambda + n\alpha_i$. By the preceding lemma, it is enough to show $u_n \in L'$. We begin with induction on N . We have

$$e_i^{(N)}u = e_i^{(N)}f_i^{(N)}u_N = \begin{bmatrix} \langle h_i, \lambda + N\alpha_i \rangle \\ N \end{bmatrix} u_N$$

$$\text{and} \quad e_i^{N+1}u = 0.$$

The crystal lattice of $\tilde{U}(\mathfrak{g})$ has the property

$$(9.2.2) \quad L(U_q(\mathfrak{g})a_{\lambda + N\alpha_i})e_i^{(N)} \subset L(U_q(\mathfrak{g})a_{\lambda}) + \tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m \quad \text{for any } m.$$

Admitting this, let us finish the proof of Theorem 9.2.1. We have

$$\begin{aligned} L(U_q(\mathfrak{g})a_{\lambda + N\alpha_i})u_N &= q_i^{N(\langle h_i, \lambda \rangle + N)}L(U_q(\mathfrak{g})a_{\lambda + N\alpha_i})e_i^{(N)}u \\ &\subset q_i^{N(\langle h_i, \lambda \rangle + N)}L(U_q(\mathfrak{g})a_{\lambda})u \subset q_i^{N(\langle h_i, \lambda \rangle + N)}q^{1/2(\|\lambda\|^2 - c)}L \\ &= q^{1/2(\|\lambda + N\alpha_i\|^2 - c)}L. \end{aligned}$$

Thus u_N belongs to L' . This implies $f_i^{(N)}u_N \in L'$ and hence $\sum_{n \leq N} f_i^{(n)}u_n$ also belongs to L' . Thus the induction proceeds.

Now it remains to prove (9.2.2). For $b \in B(\tilde{U}_q(\mathfrak{g}))$, if $\psi_i^*(b) > 0$, then $G(b)e_i^{(N)} \subset \tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m$ for any m by Proposition 6.4.2. If $\psi_i^*(b) = 0$, then Proposition 6.4.4 implies $G(b)e_i^{(N)} \equiv G(\tilde{e}_i^{*N}b) \pmod{\tilde{U}_q(\mathfrak{g})e_i^{N+1} + I_m}$. This completes the proof of Theorem 9.2.1.

9.3. Applications. Let us give applications of Theorem 9.2.1.

PROPOSITION 9.3.1. For $\lambda \in P_+$, let $L^{up}(\lambda)$ be the upper crystal lattice of $V(\lambda)$. Then $L(\tilde{U}_q(\mathfrak{g}))L^{up}(\lambda) \subset L(\lambda)$.

Proof. Set $L' = \{u \in V(\lambda); L(\tilde{U}_q(\mathfrak{g}))u \subset L(\lambda)\}$. Then $L'_{\lambda} = Au_{\lambda}$, and L' is invariant by \tilde{e}_i^{up} and \tilde{f}_i^{up} . Hence $L' = L^{up}(\lambda)$. Q.E.D.

PROPOSITION 9.3.2. For any connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, $\{\|wt(b)\|^2; b \in B'\}$ is bounded from above.

Proof. Let us take $\xi \in P_+$ and $\eta \in P_-$ such that $B' \subset B(\xi) \otimes B(\eta)$. Let us take $b_0 \in B'$ and let λ_0 be the weight of b_0 . Then by Lemma 9.1.2 there exists $c \in \|\lambda_0\|^2 + 2\mathbb{Z}$ such that

$$L(\tilde{U}_q(\mathfrak{g}))G(b_0) \subset q^{1/2(\|\lambda_0\|^2 - c)}L(\xi) \otimes L(\eta).$$

Set

$$L' = \bigoplus_{\lambda} \{u \in (L(\xi) \otimes L(\eta))_{\lambda}; L(U_q(\mathfrak{g})a_{\lambda})u \subset q^{1/2(\|\lambda\|^2 - c)}L(\xi) \otimes L(\eta)\}.$$

Then L' is invariant by \tilde{e}_i and \tilde{f}_i by Theorem 9.2.1. Let ψ be the map $L' \rightarrow L(\lambda) \otimes L(\eta)/qL(\xi) \otimes L(\eta)$. Then ψ is invariant by \tilde{e}_i and \tilde{f}_i , and hence the image of ψ contains B' . For any $b \in B'$, let us take $v \in L'$ such that $\psi(v) = b$. Then $v \notin qL(\xi) \otimes L(\eta)$ and $v \in L(\tilde{U}_q(\mathfrak{g}))v \subset q^{1/2(\|wt(b)\|^2 - c)}L(\xi) \otimes L(\eta)$. They imply $\|wt(b)\|^2 \leq c$. Q.E.D.

For a connected component B' of $B(\tilde{U}_q(\mathfrak{g}))$, an element $b \in B'$ is an extremal vector if $\|wt(b)\|^2$ is maximal. Hence we obtain the following corollary.

COROLLARY 9.3.3. Any connected component of $B(\tilde{U}_q(\mathfrak{g}))$ contains an extremal vector.

We obtain Corollary 9.3.4 by applying this to b^* .

COROLLARY 9.3.4. Any connected component of $B(\tilde{U}_q(\mathfrak{g}))$ can be embedded into some $B^{\max}(\lambda)$.

In the course of the proof of Proposition 9.3.2, if B' is the connected component $B(\lambda)$ of $B(\tilde{U}_q(\mathfrak{g}))$ containing a_{λ} , we can take $b_0 = a_{\lambda}$ and $c = 0$. Thus we obtain the following.

PROPOSITION 9.3.5. For any $b \in B(\lambda)$, $\|wt(b)\|^2 \leq \|\lambda\|^2$.

Remark. The result of Proposition 9.3.2 gives a strong constraint on the crystal structure of $B(\tilde{U}_q(\mathfrak{g}))$. For example, for $\lambda \in P_-$ and $\mu \in P_+$, the connected component B of $B(\lambda) \otimes B(\mu)$ containing $u_\lambda \otimes u_\mu$ does not satisfy the bounded condition in Proposition 9.3.2. In fact, taking \tilde{e}_i^{\max} successively, B contains $u_{w\lambda} \otimes u_\mu$ for any $w \in W$. However $\{\|w\lambda + \mu\|^2; w \in W\}$ is not bounded from above even in the affine case (if λ, μ are regular).

10. Comparison with the result of [K2].

10.1. *Relation of $A_q(\mathfrak{g})$ and $\tilde{U}_q(\mathfrak{g})$.* In [K2], we define the crystal base of $A_q(\mathfrak{g})$. Let us recall that

$$A_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} \{u \in (U_q(\mathfrak{g})^*)_\lambda; \text{ there exists } l \geq 0 \text{ such that } e_{i_1} \cdots e_{i_l} u = u f_{i_1} \cdots f_{i_l} = 0\}$$

for any $i_1, \dots, i_l \in I$.

$A_q(\mathfrak{g}) \cong \bigoplus_{\lambda \in P_+} V^r(\lambda) \otimes V(\lambda)$, and $A_q(\mathfrak{g})$ has an upper global base. Here $V^r(\lambda)$ is the irreducible right highest-weight $U_q(\mathfrak{g})$ -module generated by the highest-weight vector v_λ . We have $B(A_q(\mathfrak{g})) = \bigoplus_\lambda B^r(\lambda) \otimes B(\lambda)$. There exists a canonical coupling

$$(10.1.1) \quad \langle \cdot, \cdot \rangle: A_q(\mathfrak{g}) \otimes \tilde{U}_q(\mathfrak{g}) \rightarrow \mathbf{Q}(q).$$

Set $A_q^Z(\mathfrak{g}) = \{u \in A_q(\mathfrak{g}); \langle u, U_q^Z(\mathfrak{g}) \rangle \subset \mathbf{Z}[q, q^{-1}]\}$. Then we can see easily

$$\langle A_q^Z(\mathfrak{g}), \tilde{U}_q(\mathfrak{g})_Z \rangle \subset \mathbf{Q}[q, q^{-1}].$$

THEOREM 10.1.1. *There exists a unique embedding $\psi: B(A_q(\mathfrak{g})) \subset B(\tilde{U}_q(\mathfrak{g}))$ such that $\langle G(b), G(b') \rangle = \delta_{\psi(b), b'}$ for any $b \in B(A_q(\mathfrak{g}))$ and $b' \in B(\tilde{U}_q(\mathfrak{g}))$.*

Proof. There exists a unique embedding of crystals over $\mathfrak{g} \oplus \mathfrak{g}$

$$B^r(\lambda) \otimes B(\lambda) \rightarrow B(\tilde{U}_q(\mathfrak{g}))$$

that sends $v_\lambda \otimes u_\lambda$ to a_λ for $\lambda \in P_+$. This gives an embedding $B(A_q(\mathfrak{g})) \subset B(\tilde{U}_q(\mathfrak{g}))$. In order to see that this satisfies the required property, let us remark the following lemma.

LEMMA 10.1.2. $\langle L(A_q(\mathfrak{g})), L(\tilde{U}_q(\mathfrak{g})) \rangle \subset A$.

Proof. By the definition, it is enough to show $\langle L^{up}(V^r(\lambda)), L(\tilde{U}_q(\mathfrak{g}))L^{up}(\lambda) \rangle \subset A$. Since $\{u \in V(\lambda); \langle L^{up}(\lambda), u \rangle \subset A\} = L^{low}(\lambda)$, this follows from Proposition 9.3.1.

Q.E.D.

Thus we obtain $\langle G(b'), G(b) \rangle \subset A \cap \bar{A} \cap \mathbf{Q}[q, q^{-1}] = \mathbf{Q}$ for $b \in B(A_q(\mathfrak{g}))$, $b' \in B(\tilde{U}_q(\mathfrak{g}))$. Since $\langle \cdot, \cdot \rangle$ is invariant by $\tilde{e}_i, \tilde{f}_i, \tilde{e}_i^*$ and \tilde{f}_i^* , we obtain Theorem 10.1.1, because $\langle G(v_\lambda \otimes u_\lambda), a_\lambda \rangle = \langle v_\lambda, u_\lambda \rangle = 1$.

10.2. *Finite-dimensional and affine case.* Let us regard $B(A_q(\mathfrak{g}))$ as a subset of $B(\tilde{U}_q(\mathfrak{g}))$. Then $B(A_q(\mathfrak{g}))$ is the smallest subcrystal (with respect $\mathfrak{g} \oplus \mathfrak{g}$) of $B(\tilde{U}_q(\mathfrak{g}))$ that contains all $a_\lambda (\lambda \in P)$.

Let us denote by T the Tits cone; i.e., $T = \bigcup_{w \in W} wP_+$.

LEMMA 10.2.1. *If $\lambda \in T$ then $B^{\max}(\lambda) \subset B(A_q(\mathfrak{g}))$.*

Proof. For $w \in W$, S_w^* sends $B^{\max}(\lambda)$ onto $B^{\max}(w\lambda)$. Hence we may assume $\lambda \in P_+$ from the beginning. If $b = b_1 \otimes t_\lambda \otimes b_2 \in B^{\max}(\lambda)$, then $\varphi_i^*(b) = \max(\varphi_i^*(b_2), \varepsilon_i^*(b_1) - \langle h_i, \lambda \rangle) = 0$. Hence $\varphi_i^*(b_2) = 0$ for all i and hence $b_2 = u_{-\infty}$ and $\varepsilon_i^*(b_1) \leq \langle h_i, \lambda \rangle$. This shows $B^{\max}(\lambda) = B(\lambda) \otimes t_\lambda \otimes u_{-\infty}$. Then the desired result follows from the connectedness of $B(\lambda)$. Combining this with Corollary 9.3.3 and 9.3.4 we obtain the following result.

PROPOSITION 10.2.2. (i) *If \mathfrak{g} is finite dimensional, $B(A_q(\mathfrak{g})) = B(\tilde{U}_q(\mathfrak{g}))$.*

(ii) *If \mathfrak{g} is affine, let $B(\tilde{U}_q(\mathfrak{g})_+)$ be the subcrystal of $B(\tilde{U}_q(\mathfrak{g}))$ consisting of vectors with positive level. Then $B(A_q(\mathfrak{g})) = B(\tilde{U}_q(\mathfrak{g})_+) \sqcup \{a_0\}$.*

Recall that when \mathfrak{g} is affine we take $c \in \sum \mathbf{Z}_{>0} h_i$ with $\langle c, \alpha_i \rangle = 0$ for any i . Then $\langle c, \lambda \rangle$ is called the level of $\lambda \in P$.

REFERENCES

- [IIMNT] M. IDZUMI, K. IOHARA, M. JIMBO, T. MIWA, T. NAKASHIMA, AND T. TOKIHIRO, *quantum affine symmetry in Vertex models*, to appear in Internat. J. Modern Phys. A.
- [K1] M. KASHIWARA, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [K2] ———, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993) 455–485.
- [K3] ———, *Crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993) 839–858.
- [L1] G. LUSZTIG, *Canonical bases arising from quantized enveloping algebra*, J. Amer. Math. Soc. **3** (1990), 447–498.
- [L2] ———, *Canonical bases in tensor product*, Proc. Nat. Acad. Sci. U.S.A. **89** (1992), 8177–8179.
- [L3] ———, *Problems on canonical bases*, preprint.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606, JAPAN