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Quantum Flag and Schubert Schemes

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Abstract: For a semi-simple algebraic group G, we construct a Hopf k(q)-algebra $k_q[G]$ as a quantization of k[G] and we develop a standard monomial theory for $k_q[G]$. The quantum flag variety $k_q[G/B]$ is constructed as a certain subalgebra of $k_q[G]$. It is shown that $k_q[G/B]$ has a canonical $\mathbb{Z}^\ell$-gradation (ℓ = rank G) and a canonical left $k_q[G]$-comodule structure. We also construct the algebra $k_q[X(w)]$, $w \in W$, the Weyl group, as quantization of k[X(w)], the multigraded homogeneous co-ordinate ring of X(w). The algebra $k_q[X(w)]$ also has a canonical $\mathbb{Z}^\ell$-gradation and a canonical left $k_q[B]$-comodule structure. We also give a presentation for $k_q[G/B]$.

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Introduction.

In this paper, we prove the results announced in $[LR]_1$. In [H], [H-P], Hodge constructed canonical bases for the homogeneous co-ordinate rings of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of "Standard Monomials" in the Plücker coordinates. We generalized this result of Hodge to a semi-simple algebraic group G by developing a Standard Monomial Theory for G (cf $[LS]_1$, [L-Ra], $[L]_1$, $[L]_2$). In this paper, we develop a Standard Monomial Theory for Quantum groups.

Let G be a semi-simple algebraic group split over k. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and $U(\mathfrak{g})$ its universal enveloping algebra. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra (over $\mathbb{Q}$) as constructed in $[D]_1$, [J], and $U_{\mathcal{A}}$ the Kostant-Lusztig $\mathcal{A}$-form of $U_q(\mathfrak{g})$, where $\mathcal{A} = \mathbb{Z}\,[q,q^{-1}]$ (cf $[Lu]_2$). We construct $\mathbb{Z}_q[G]$ as a "$\mathcal{A}$-dual" to $U_{\mathcal{A}}$ and for any field k, we define $k_q[G]$ as $\mathbb{Z}_q[G] \otimes_{\mathcal{A}} k(q)$.

Let P_d be a maximal parabolic subgroup of G with associated fundamental weight ω_d. We construct $\{x_i^{\omega_d},\ i \in I)$ (the indexing set being as in the Standard Monomial Theory (cf $[LS]_1$, [L-Ra], $[L]_1$, $[L]_2$)) as certain elements of $\mathbb{Z}_q[G]$, and we define $\mathbb{Z}_q[G/P_d]$ (resp.

$\mathbb{Z}_q[G/B]$) as the $\mathcal{A}$-sub algebra of $\mathbb{Z}_q[G]$ generated by $\{x_i^{\omega_d},\ 1 \le i \le N_d(= \dim V^{\omega_d}),\ 1 \le d \le \ell\}$ (here V^{ω_i} is the irreducible $U_q(\mathfrak{g})$-module with highest weight ω_i). For $w \in W$, we define $\mathbb{Z}_q[X(w)]$ as a certain quotient of $\mathbb{Z}_q[G/B]$, and $k_q[X(w)]$ as $\mathbb{Z}_q[X(w)] \otimes_{\mathcal{A}} k(q)$. Let $\underline{a} = (a_1, \ldots, a_\ell) \in (\mathbb{Z}^+)^\ell$. Following $[LS]_1$, $[L]_1$, $[L]_2$ we define the notion of a monomial in the $x_i^{\omega_d}$'s of type $\underline{a}$ (or multidegree $\underline{a}$) being standard on $X(w)$ (cf §3, §4). Let $(k_q[X(w)])_{\underline{a}}$ be the $k(q)$-span of all monomials f (in $k_q[X(w)]$) such that f has a_i factors $x_{ij}^{\omega_i}$, $1 \le j \le a_i$, $1 \le i \le \ell$. We prove (cf Theorems 3.10 and 4.10).

Theorem (a) Standard monomials on $X(w)$ of type $\underline{a}$ form a $k(q)$-basis for $(k_q[X(w)])_{\underline{a}}$

(b) $k_q[X(w)] = \bigoplus_{\underline{a}} (k_q[X(w)])_{\underline{a}}$

(c) $(k_q[X(w)])_{\underline{a}}$ has a left $k_q[B]$-comodule structure and $\dim_{k(q)}(k_q[X(w)])_{\underline{a}} = s_{\underline{a}}(w)$ (= # { Standard monomials on $X(w)$ of type $\underline{a}$} (here $k_q[B]$ is the quantum Borel subgroup)

(d) For $X(w) = G/B$, $(k_q[G/B])_{\underline{a}}$ has a left $k_q[G]$-comodule structure.

Outline of proof: The philosophy is same as in $[L\text{-}R]_2$. Linear independence of standard monomials on $X(w)$ in arbitrary characteristic is obtained as a consequence of the linear independence of standard monomials for the case $q = 1$ (cf. $[L\text{-}S]_1$). In view of linear independence of standard monomials in arbitrary characteristic, it suffices to prove generation by standard monomials when $k = \mathbb{Q}$. Generation by standard monomials for the case $k = \mathbb{Q}$ is proved by considering the Clebsch-Gordan coefficient matrix giving the projection $V^{\lambda} \otimes V^{\mu} \longrightarrow V^{\nu}$, where V^{ν} is a factor in the expression for $V^{\lambda} \otimes V^{\mu}$ as a direct sum of irreducible $U_q(\mathfrak{g})$-modules (here, for a dominant integral weight λ, V^{λ} denotes the corresponding irreducible $U_q(\mathfrak{g})$-module. If $q^r = 1$, then r is supposed to be sufficiently large).

The paper is organized as follows: In §1, we recall results from Standard Monomial Theory. In §2, we construct $\mathbb{Z}_q[G]$, $k_q[G]$, and prove some Lemmas relating to Quantum Clebsch-Gordan coefficients. In §3, we present results on quantum flag schemes. In §4, we present results on quantum Schubert schemes. In §5, we give a presentation for $k_q[G/B]$.

§1 Brief review of Standard Monomial Theory

Let G be a semi-simple algebraic group split over k. Let T be a maximal k-split torus, B a Borel subgroup, $B \supset T$. Let W be the Weyl group of G. For $w \in W$, let

$X(w) = \overline{BwB}$ (mod B) be the associated Schubert variety in G/B. Let $\ell = \operatorname{rank}(G)$. Let P_d be a maximal parabolic subgroup of G with associated fundamental weight ω_d, W_{P_d} the Weyl group of P, and W^{P_d}, the set of minimal representatives in W of W/W_{P_d}. Let L_d be the ample generator of $\operatorname{Pic}(G/P_d)$. A nice basis $\{p_i^d\}$ for $H^0(G/P_d, L_d)$ (as well as $H^0(X(w), L_d)$, $X(w) \subset G/P_d$) has been constructed in [LS]$_1$, [L-Ra], [L]$_1$, [L]$_2$. The indexing set consists of certain pairs of elements of W^{P_d} together with certain sequences of numbers. A notion of monomials in the p_i^d's, $1 \le i \le N_d$ $(= \dim H^0(G/P_d, L_d))$ being standard on a Schubert variety $X(w)$ is defined and the following Theorem is proved.

<u>Theorem.</u> Let $\underline{a} = (a_1, \ldots, a_\ell) \in (\mathbb{Z}^+)^\ell$. Let $L_{\underline{a}} = \bigotimes_{i=1}^{\ell} L_i^{\otimes a_i}$. Let $s_{\underline{a}}(w) = \#\{$standard monomials on $X(w)$ of multi-degree $\underline{a}\}$. Let $(R(w))_{\underline{a}}$ be the span of

$$\left\{ f \;\middle|\; \begin{array}{l} (1)\ f \text{ is a monomial of multi-degree } \underline{a} \\ (2)\ f|_{X(w)} \neq 0 \end{array} \right\}.$$

(1) Standard monomials on $X(w)$ of multi-degree $\underline{a}$ form a k-basis for $(R(w))_{\underline{a}}$

(2) $H^0(X(w), L_{\underline{a}}) = (R(w))_{\underline{a}}$

Let $k[X(w)]$ be the $\mathbb{Z}^\ell$-graded co-ordinate ring of $X(w)$.

Then we have (in view of the above Theorem) $k[X(w)] = \underset{\underline{a}}{\oplus} H^0(X(w),L_a)$.

§2 The Hopf algebra $k_q[G]$

Let k be the base field and q an indeterminate taking values in k^*. If $q^r = 1$, then we shall suppose that $r >> 0$.

2.1 The Hopf algebra $U_{\mathcal{A}}$ (cf $[Lu]_2$)

Let $\mathfrak{g}$ be a split semi-simple Lie-algebra over $\mathbb{Q}$ and let $\ell = \mathrm{rank}(\mathfrak{g})$. Let $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$ and $\mathcal{A}' = \mathbb{Q}(q)$. Let $U_{\mathcal{A}'}$ be the $\mathcal{A}'$-algebra generated by $\{E_i, F_i, K_i, K_i^{-1}\}$ and relations as in $[Lu]_2$. Let $U_{\mathcal{A}}$ be the $\mathcal{A}$-sub algebra of $U_{\mathcal{A}'}$ generated by $E_i^{(N)}, F_i^{(N)}, K_i, K_i^{-1}$, $1 \le i \le \ell$, $N \ge 0$ where $E_i^{(N)} = E_i^N/ [N]_q!$, $F_i^{(N)} = F_i^N/ [N]_q!$, $[N]_q! = [N]_q \ldots [1]_q$, $[m]_q = \dfrac{q^m - q^{-m}}{q - q^{-1}}$. We have (cf $[Lu]_2$), $U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}' = U_{\mathcal{A}'}$. Let $U_{\mathcal{A}}^+$ (resp. $U_{\mathcal{A}}^-$) be the $\mathcal{A}$-sub algebra of $U_{\mathcal{A}}$ generated by $E_i^{(N)}$ (resp. $F_i^{(N)}$), $1 \le i \le \ell$, $N \ge 0$. We have (cf $[Lu]_2$), $U_{\mathcal{A}}$ is an $\mathcal{A}$-algebra; further, the comultiplication on $U_{\mathcal{A}'}$ induces a comultiplication on $U_{\mathcal{A}}$ giving a Hopf $\mathcal{A}$-algebra structure to $U_{\mathcal{A}}$.

2.2 Highest Weight $U_{\mathcal{A}'}$-modules

The theories of finite dimensional respresentations of $U_{\mathcal{A}'}$ and $\mathfrak{g}$ are quite parallel (cf $[Lu]_1$, $[Lu]_2$, [Ro]).

In particular we have

(1) The finite dimensional representations of $U_{\mathcal{A}'}$ are completely reducible.

(2) The finite dimensional irreducible representations of $U_{\mathcal{A}'}$ are parametrized by the dominant, integral weights of $\mathfrak{g}$.

(3) Given a dominant integral weight λ, let V^{λ} be the corresponding irreducible $U_{\mathcal{A}'}$-module. Let $V^{\lambda} = \underset{\mu}{\oplus} V^{\lambda}(\mu)$ (direct sum of weight spaces). The dimensions of $V^{\lambda}(\mu)$'s are the same as those of the corresponding weight spaces of the irreducible $\mathfrak{g}$-module with highest weight λ.

2.3 $\mathcal{A}$-lattices

Let V^{λ} be the irreducible $U_{\mathcal{A}'}$-module with highest weight λ (λ being a dominant integral weight of $\mathfrak{g}$). Let us fix a highest weight vector e_{λ} in V^{λ}. Let $V^{\lambda}_{\mathcal{A}} = U^{-}_{\mathcal{A}} e_{\lambda}$. Then we have (cf $[\mathrm{Lu}]_1$, $[\mathrm{Lu}]_2$).

1. $V^{\lambda}_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-submodule of V^{λ}
2. The natural map $V^{\lambda}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}' \longrightarrow V^{\lambda}$ is an isomorphism of $\mathcal{A}'$-vector spaces.
3. $V^{\lambda}_{\mathcal{A}}$ is a direct sum of its intersections with the weight spaces of V^{λ}.
4. Each intersection in (3) is a finitely generated free $\mathcal{A}$-module of finite rank.

2.4 A nice basis for fundamental representations

Let ω_d be a fundamental weight of $\mathfrak{g}$ and let $\overline{V}^{\omega_d}$ (resp. V^{ω_d}) be the corresponding irreducible $\mathfrak{g}$-module (resp. $U_{\mathcal{A}'}$-module). Let P_d be the maximal parabolic subgroup of G corresponding to ω_d. A nice basis for $\overline{V}^{\omega_d}$ has been constructed in [LS]$_1$, [L-Ra], [L]$_1$, [L]$_2$, the indexing set being certain pairs of elements of W^{P_d} together with certain sequences of numbers. Adopting the same procedure, as in the papers cited above, we construct a (similar) basis for V^{ω_d}, which we describe below. For simplicity of exposition, we shall suppose ω_d to be of classical type (cf [LS]$_1$) in the discussion below. (For a non-classical ω, the construction is given in the Appendix).

We have (cf [LS]$_1$) dim $\overline{V}^{\omega_d}$ (= dim V^{ω_d}) = #{admissible pairs in W^{P_d}} (we recall (cf [LS]$_1$) that a pair (τ,φ) in W^{P_d} is <u>admissible</u> if either $\tau = \varphi$ (in which case, we call it a trivial admissible pair) or there exists a sequence $\{\tau_i\}$, $\tau_0 = \tau > \tau_1 > \ldots > \tau_r = \varphi$, $\ell(\tau_{i-1}) = \ell(\tau_i) + 1$, such that $(\tau_i(\omega_d), \beta_i^*) = 2$, where β_i is the positive root such that $\tau_{i-1}=s_{\beta_i}\tau_i$, $1 \le i \le r$). Let us fix a highest weight vector e in V^{ω_d}. We first construct the extremal weight vectors e_φ, $\varphi \in W^{P_d}$, as follows. Let $\varphi = s_{\gamma_s} \dots s_{\gamma_1}$ be a reduced expression for φ, where γ_i's are simple. Further, let us denote $\varphi_0 = \mathrm{Id}$, $\varphi_i = s_{\gamma_i} \dots s_{\gamma_1}$, $1 \le i \le s$, and $m_i = (\varphi_i(\omega_d), \gamma_{i+1}^*)$, $0 \le i \le s-1$ (note that $m_i = 1$ or 2). Then $F_{\gamma_s}^{[m_s]} F_{\gamma_{s-1}}^{[m_{s-1}]} \dots F_{\gamma_1}^{[m_1]} e$ is an extremal weight vector of weight $\varphi(\omega_d)$ and we denote it by e_φ.

The non-extremal weight vectors are constructed as follows: Let (τ, φ) be a (non-trivial) admissible pair. Let us fix <u>any</u> sequence $\{\tau_i\}$, $\tau_0 = \tau > \tau_1 > \dots > \tau_r = \varphi$ and $\ell(\tau_i) = \ell(\tau_{i+1}) + 1$, $0 \le i \le r-1$. Let β_i be the positive root such that $\tau_{i-1} = s_{\beta_i} \tau_i$. Then we have (cf $[LS]_1$)

(1) β_i is simple $1 \le i \le r$

(2) $(\tau_i(\omega_d), \beta_i^*) = 2$

We set

$$Q_{\tau,\varphi} = F_{\beta_1} \dots F_{\beta_r} e_\varphi$$

Remark 2.5 As in $[LS]_1$, it can be checked easily (using the commutation relations in $U_{\mathcal{A}}$ (cf $[Lu]_1$, $[Lu]_2$)) that the $Q_{\tau,\varphi}$ as constructed above is uniquely determined by the admissible pair (τ, φ) (and does not depend on the path from $X(\varphi)$ to $X(\tau)$), once a choice of e_φ has been made.

Proposition 2.6 The set $\{Q_{\tau,\varphi},\ (\tau,\varphi)$ an admissible pair in $W^{P_d}\}$ is a $\mathbb{Q}(q)$-basis for V^{ω_d}.

Proof: We have (from above) $\#\{Q_{\tau,\varphi}\} = \dim V^{w_d}$. We claim: $\{Q_{\tau,\varphi}\}$ is linearly independent over $\mathbb{Q}(q)$. Assume that the claim is not true. Let $\Sigma a_{\tau,\varphi} Q_{\tau,\varphi} = 0$ be a non-trivial linear relation, where we may suppose (after clearing the denominator) that $a_{\tau,\varphi} \in \mathbb{Q}[q]$. Cancelling the maximum power of (q-1) that occurs as a factor in all the $a_{\tau,\varphi}$'s, we obtain a non-trivial relation (for the case q = 1) which is not possible (in view of linear independence of $\{Q_{\tau,\varphi}\}$ for q = 1 (cf $[LS]_1$)).

Remark 2.7 Using the commutation relations in $U_{\mathcal{A}}$(cf $[Lu]_1$, $[Lu]_2$) it can be checked (in the same spirit as in $[LS]_1$) that $\{Q_{\tau,\varphi},\ (\tau,\varphi)$ an admissible pair in $W^{P_d}\}$ is $U_{\mathcal{A}}$-stable and thus gives an $\mathcal{A}$-basis for $V_{\mathcal{A}}^{\omega_d}$.

2.8 The Hopf algebra $k_q[G]$

For each dominant, integral weight λ of $\mathfrak{g}$, we fix a $\mathcal{A}$-basis $\{e_i^{\lambda}\}$ for $V_{\mathcal{A}}^{\lambda}$ consisting of weight vectors. Consider the free $\mathcal{A}$-module $G_{\mathcal{A}}$ on $\{T_{ij}^{\lambda}, \lambda$ a dominant integral weight, $1 \le i,j \le \dim V^{\lambda}\}$. We define a pairing $\langle\ ,\ \rangle$ on $(G_{\mathcal{A}} \times U_{\mathcal{A}})$ by

$$< T^{\lambda}_{ij}, \ b > = T^{\lambda}_{ij}(b)$$

where $b \in U_{\mathcal{A}}$ and $(T^{\lambda}_{ij}(b))$ is the matrix giving the action of b on $V^{\lambda}_{\mathcal{A}}$ with respect to the basis $\{e^{\lambda}_{i}\}$. We set

$$\mathbb{Z}_q[G] = G_{\mathcal{A}}$$

The Hopf $\mathcal{A}$-algebra structure on $U_{\mathcal{A}}$ induces a Hopf $\mathcal{A}$-algebra structure on $\mathbb{Z}_q[G]$. In particular, the comultiplication on $\mathbb{Z}_q[G]$ is given by

$$\Delta(T^{\lambda}_{ij}) = \sum_{r} T^{\lambda}_{ir} \otimes T^{\lambda}{}_{rj}$$

For any field k, we set

$$k_q[G] = \mathbb{Z}_q[G] \otimes_{\mathcal{A}} k(q)$$

We shall denote $T^{\lambda}_{ij} \otimes 1$ by just T^{λ}_{ij}.

2.9 $\mathbb{Z}_q[G]$-comodule structure for $V^{\lambda}_{\mathcal{A}}$

From the definition of $\mathbb{Z}_q[G]$, it follows that $V^{\lambda}_{\mathcal{A}}$ (notation as above) has a left $\mathbb{Z}_q[G]$-comodule structure given by

$$\delta\colon\ V^{\lambda}_{\mathcal{A}} \longrightarrow \mathbb{Z}_q[G] \otimes_{\mathcal{A}} V^{\lambda}_{\mathcal{A}}$$

$$\delta(e^{\lambda}_{j}) = \sum_{i} T^{\lambda}_{ij} \otimes e^{\lambda}_{i} \quad (= T^{\lambda} \otimes e^{\lambda}_{j})$$

2.10 The Hopf algebra $k_q(B)$

Let $U_{\mathcal{A}}(b^+)$ be the $\mathcal{A}$-sub algebra of $U_{\mathcal{A}}$, generated by $E_i^{(N)}$, K_i, K_i^{-1}, $1 \le i \le \ell$, $N \ge 0$. Proceeding as in 2.8 let us consider the free $\mathcal{A}$-module $B_{\mathcal{A}}$ on $\{T^{\lambda}_{ij}$, $i \le j$, λ a dominant integral weight, $1 \le i$, $j \le \dim V^{\lambda}\}$. We define a pairing $<\ ,\ >$ on $B_{\mathcal{A}} \times U_{\mathcal{A}}(b^+)$ by $<T^{\lambda}_{ij}, f> = T^{\lambda}_{ij}(f)$ where $f \in U_{\mathcal{A}}(b^+)$ and $(T^{\lambda}_{ij}(f))$ is the upper triangular matrix giving the action of f on $V^{\lambda}_{\mathcal{A}}$ with respect to the basis $\{e^{\lambda}_i\}$. We set

$$\mathbb{Z}_q[B] = B_{\mathcal{A}}$$

The Hopf $\mathcal{A}$-algebra structure on $U_{A}(b^+)$ induces a Hopf $\mathcal{A}$-algebra structure on $\mathbb{Z}_q[B]$. For any field k, we set

$$k_q[B] = \mathbb{Z}_q[B] \otimes_{\mathcal{A}} k(q)$$

2.11 The elements $x_i^{\omega_d}$

Let $1 \le d \le \ell$. For $1 \le i \le \dim V^{\omega_d}$, we set $x_i^{\omega_d} = T_{i1}^{\omega_d}$ $(\in \mathbb{Z}_q[G])$ where we suppose that $e_1^{\omega_d}$ $(\in V_{\mathcal{A}}^{\omega_d})$ is the highest weight vector. For any field k, we shall denote the image of $x_i^{\omega_d}$ in $k_q[G]$, under the canonical map

$$\mathbb{Z}_q[G] \longrightarrow \mathbb{Z}_q[G] \otimes_{\mathcal{A}} k(q),\ x \longrightarrow x \otimes 1,$$

by just $x_i^{\omega_d}$.

2.12 Quantum Clebsch-Gordan Coefficients

Let $X^{\omega_d} = \sum_{i=1}^{N_d} x_i^{\omega_d} \otimes e_i^{\omega_d}$ $(= T^{\omega_d} \otimes e_1^{\omega_d})$. Note that $X^{\omega_d} \in \mathbb{Z}_q[G] \otimes V_{\mathcal{A}}^{\omega_d}$. For $\lambda = \sum_{i=1}^{\ell} a_i\omega_i$, $a_i \in \mathbb{Z}^+$, let $X^\lambda = \otimes_{d=1}^{\ell} (X^{\omega_d})^{\otimes a_d}$. (Note that $X^\lambda \in \mathbb{Z}_q[G] \otimes (\otimes_{d=1}^{\ell} (V_{\mathcal{A}}^{\omega_d})^{\otimes a_d}$.)

Lemma 2.13 Let R be the universal R-matrix in $U_{\overline{\mathcal{A}}} \otimes_{\overline{\mathcal{A}}} U_{\overline{\mathcal{A}}}$, where $\overline{\mathcal{A}} = \mathbb{Q}[[q-1]]$ and $U_{\overline{\mathcal{A}}}$ is the quasi-triangular Hopf algebra $U_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}}$. For $1 \le d, d' \le \ell$, let $R^{\omega_d \omega_{d'}} = (\rho^{\omega_d} \otimes \rho^{\omega_{d'}})(R)$, where ρ^{ω_d} is the map $\rho^{\omega_d}: U_{\overline{\mathcal{A}}} \longrightarrow \mathrm{End}\,(V^{\omega_d} \otimes_{\mathcal{A}'} \overline{\mathcal{A}})$ (and $\rho^{\omega_{d'}}$ has a similar discription). Then

$$R^{\omega_d \omega_{d'}} X_1^{\omega_d} X_2^{\omega_{d'}} = q^{2(\omega_d, \omega_{d'})} X_2^{\omega_{d'}} X_1^{\omega_d},$$

where

$$X_1^{\omega_d} X_2^{\omega_{d'}} = \sum x_i^{\omega_d} x_j^{\omega_{d'}} \otimes e_i^{\omega_d} \otimes e_j^{\omega_{d'}},$$

$$X_2^{\omega_{d'}} X_1^{\omega_d} = \sum x_j^{\omega_{d'}} x_i^{\omega_d} \otimes e_i^{\omega_d} \otimes e_j^{\omega_{d'}}$$

(here $(\ , \ )$ is a W-invariant inner product on $\mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan sub algebra of $\mathfrak{g}$).

Proof. We have $X^{\omega_d} = T^{\omega_d} e^{\omega_d}$, where e^{ω_d} is the highest weight vector in V^{ω_d} (by the definition of X^{ω_d}). Hence

$$R^{\omega_d \omega_{d'}} X_1^{\omega_d} X_2^{\omega_{d'}} = R^{\omega_d \, \omega_{d'}} T_1^{\omega_d} T_2^{\omega_{d'}} (e^{\omega_d} \otimes e^{\omega_{d'}}),$$

where $T_1^{\omega_d} = T^{\omega_d} \otimes \mathrm{Id}$, $T_2^{\omega_{d'}} = \mathrm{Id} \otimes T^{\omega_{d'}}$. Also, by the property of universal R-matrices (cf $[D]_1$), we have

$$R^{\omega_d \omega_{d'}} T_1^{\omega_d} T_2^{\omega_{d'}} = T_2^{\omega_{d'}} T_1^{\omega_d} R^{\omega_d \omega_{d'}}.$$

Further by [R] we have,

$$R^{\omega_d \omega_{d'}} e^{\omega_d} \otimes e^{\omega_{d'}} = q^{2(\omega_d, \omega_{d'})} e^{\omega_d} \otimes e^{\omega_{d'}}$$

(since muliplicity of $V^{\omega_d + \omega_{d'}}$ in $V^{\omega_d} \otimes V^{\omega_{d'}}$ is 1).

Hence we obtain

$$R^{\omega_d \omega_{d'}} X_1^{\omega_d} X_2^{\omega_{d'}} = q^{2(\omega_d, \omega_{d'})} T_2^{\omega_{d'}} T_1^{\omega_d} (e^{\omega_d} \otimes e^{\omega_{d'}}) =$$

$$q^{2(\omega_d, \omega_{d'})} X_2^{\omega_{d'}} X_1^{\omega_d}.$$

Corollary 2.14 With notation as in Lemma 2.13, we have, for a dominant integral weight δ,

$$R^{\delta \omega_d} X_1^{\delta} X_2^{\omega_d} = q^{2(\delta, \omega_d)} X_2^{\omega_d} X_1^{\delta}.$$

Proof: Let $\delta = \sum_{i=1}^{\ell} a_i \omega_i$. We prove the result by induction on $n(\delta) = \Sigma a_i$. When $n(\delta) = 1$, the result follows from Lemma 2.13. Let us write $\delta = \lambda + \mu$, where $n(\lambda)$ and $n(\mu)$ are both $< n(\delta)$. Now we have (by induction hypothesis),

$$R_{13}^{\lambda \omega_d} R_{23}^{\mu \omega_d} X_1^{\lambda} X_2^{\mu} X_3^{\omega_d} = q^{2(\lambda+\mu,\ \omega_d)} (X_3^{\omega_d} X_1^{\lambda} X_2^{\mu})$$

Also, by quasi-triangularity (cf $[D]_1$), we have,

$$K_{12}^{\lambda+\mu}\left(R_{13}^{\lambda\omega_d}\, R_{23}^{\mu\omega_d}\right) = R_{(12),3}^{\lambda+\mu,\,\omega_d}\, K_{12}^{\lambda+\mu},$$

where $K_{12}^{\lambda+\mu}$ is the projection $V^{\lambda}\otimes V^{\mu}\otimes V^{\omega_d} \longrightarrow V^{\lambda+\mu}\otimes V^{\omega_d}$.

Hence,

$$R_{(12),3}^{\lambda+\mu,\,\omega_d}\, K_{12}^{\lambda+\mu}\left(X_1^{\lambda}\, X_2^{\mu}\, X_3^{\omega_d}\right) = q^{2(\lambda+\mu,\ \omega_d)}\, K_{12}^{\lambda+\mu}\left(X_3^{\omega_d}\, X_1^{\lambda}\, X_2^{\mu}\right)$$

$$= q^{2(\lambda+\mu,\omega_d)}\, X_3^{\omega_d}\, X_{12}^{\lambda+\mu}, \text{ where } X_{(12)}^{\lambda+\mu} = K_{12}^{\lambda+\mu}\left(X_1^{\lambda}\, X_2^{\mu}\right).$$

Hence,

$$R_{(12),3}^{\lambda+\mu}\, X_{(12)}^{\lambda+\mu} X_3^{\omega_d} = q^{2(\lambda+\mu,\ \omega_d)}\, X_3^{\omega_d}\, X_{(12)}^{\lambda+\mu}$$

This proves the result for δ.

Corollary 2.15 Let λ,μ be dominant and integral. Then

$$R^{\lambda\mu}\, X_1^{\lambda}\, X_2^{\mu} = q^{2(\lambda,\mu)}\, X_2^{\mu}\, X_1^{\lambda}.$$

Proof: Writing $\mu = \Sigma a_i\omega_i$ and denoting $n(\mu)= \Sigma a_i$, we obtain the result by induction on $n(\mu)$, the starting point of induction, namely $n(\mu) = 1$ being true by Corollary 2.14.

Lemma 2.16 Let λ,μ be dominant and integral. Let $V^{\lambda}\otimes V^{\mu} = \underset{\nu}{\oplus}\, W_{\nu}\otimes V^{\nu}$, where W_{ν} is the space of multiplicites of

V^{ν}. Let $\tilde{P}_{\nu}$ be the projection $V^{\lambda} \otimes V^{\mu} \longrightarrow W_{\nu} \otimes V^{\nu}$. Then $R_{21}^{\mu\lambda} R_{12}^{\lambda\mu} = \sum_{\nu} q^{2(c(\nu)-c(\lambda)-c(\mu))} \tilde{P}_{\nu}$ (here c is the Casimir operator $c(\lambda) = (\lambda,\lambda) + 2(\rho,\lambda)$, ρ being 1/2 sum of positive roots).

Proof: Let $R = \Sigma\alpha_i \otimes \beta_i$ and let $u = \Sigma\, S(\beta_i)\alpha_i$ (where S is the antopode). Let $v = uq^{-\rho}$. Then $v \in$ center of $U_{\bar{\mathcal{A}}}$ and v acts on V^{λ} by $q^{2c(\lambda)}$ (cf $[D]_2$). Also, by quasi-triangularity, we have $\Delta u = R_{21}R_{12}(u\otimes u)$. Hence we obtain,

$$\Delta v = R_{21}R_{12}(v\otimes v), \text{ i.e., } R_{21}R_{12} = v^{-1}\otimes v^{-1}\Delta(v).$$

Hence,

$$R_{21}^{\mu\lambda} \; R_{12}^{\lambda\mu} \, (V^{\lambda}\otimes V^{\mu}) = (v^{-1}\otimes v^{-1})\Delta(v)(V^{\lambda} \otimes V^{\mu}) =$$

$$(v^{-1}\otimes v^{-1})\Delta(v)(\sum_{\nu} W_{\nu}\otimes V^{\nu}) = \sum_{\upsilon} q^{2(c(\upsilon) - c(\lambda)-c(\mu))} W_{\upsilon}\otimes V^{\upsilon}.$$

The required result follows from this.

Lemma 2.17 Let λ be dominant and integral, say $\lambda = \Sigma a_i\omega_i$. Let $(V^{\omega_1})^{\otimes a_1} \otimes \ldots \otimes (V^{\omega_\ell})^{\otimes a_\ell} = \bigoplus_{\nu} W_{\nu} \otimes V^{\upsilon}$, where W_{υ} is the space of multiplicities of V^{υ}. Let $\tilde{P}_{\upsilon}$ be the projection $\bigotimes_{i=1}^{\ell} (V^{\omega_i})^{\otimes a_i} \longrightarrow W_{\upsilon} \otimes V^{\upsilon}$. Then

$$\tilde{P}_{\upsilon}(X^{\lambda}) = 0, \;\; \upsilon \neq \lambda$$

(where recall that $X^{\lambda} = \overset{\ell}{\underset{i=1}{\otimes}} X_i^{\otimes a_i}$).

Proof: Let us write $\lambda = \lambda' + \omega_d$, for some d such that $a_d > 0$. In view of the facts $P \circ R_{12}: V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ and

$$R^{\omega_{d_i} \omega_{d_{i+1}}}(\ldots X_i^{d_i} X_{i+1}^{d_{i+1}} \ldots) = q^{2(\omega_{d_i}, \omega_{d_{i+1}})}(\ldots X_{i+1}^{d_{i+1}} X_i^{d_i} \ldots)$$

(cf Lemma 2.13), we have, $\tilde{P}_{\upsilon}(X^{\lambda}) = 0$ if and only if $\tilde{P}_{\upsilon}(X^{\lambda'} \otimes X^{\omega_d}) = 0$ (here $P: V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$ is the map $P(v_1 \otimes v_2) = v_2 \otimes v_1$). We have (cf Lemma 2.16),

$$(1) \quad R_{21}^{\omega_d \lambda'} R_{12}^{\lambda' \omega_d} = \sum_{\upsilon} q^{2(c(\upsilon) - c(\lambda') - c(\omega_d))} \tilde{P}_{\upsilon}.$$

Also, by Corollary 2.14, we have

$$(2) \quad R_{21}^{\omega_d \lambda'} R_{12}^{\lambda' \omega_d}(X^{\lambda'} \otimes X^{\omega_d}) = q^{4(\lambda', \omega_d)} X^{\lambda'} \otimes X^{\omega_d} =$$

$$q^{4(\lambda', \omega_d)} \sum_{\upsilon} \tilde{P}_{\upsilon}(X^{\lambda'} \otimes X^{\omega_d}).$$

From (1) and (2), we obtain,

$$\sum_{\upsilon \neq \lambda' + \omega_d} (q^{2c(\upsilon)} - q^{2c(\lambda' + \omega_d)}) \tilde{P}_{\upsilon}(X^{\lambda'} \otimes X^{\omega_d}) = 0$$

We <u>claim:</u> $c(\upsilon) \neq c(\lambda' + \omega_d)$, $\upsilon \neq \lambda' + \omega_d$ (note that the required result follows from the claim).

Proof of claim: By [Li], any dominant weight υ such that V^{υ} occurs in $V^{\lambda'} \otimes V^{\omega_d}$ has the form $\upsilon = \lambda' + \mu$, where $\mu \leq \omega_d$. Hence if $\upsilon \neq \lambda' + \omega_d$, then we can write $\omega_d = \mu + \sum_{i=1}^{\ell} c_i \alpha_i$, where at least one c_i is non-zero. Hence, $c(\lambda'+\omega_d) =$

$c(\lambda')+ c(\mu) + c(\Sigma c_i\alpha_i) + 2(\lambda',\mu) + 2(\mu,\Sigma c_i\alpha_i)+2(\lambda',\Sigma c_i\alpha_i)$

while $c(\lambda'+\mu) = c(\lambda') + c(\mu) + 2(\lambda',\mu)$. Hence

$$c(\lambda'+ \omega_d) - c(\upsilon) = c(\Sigma c_i\alpha_i) + 2(\lambda'+\mu,\Sigma c_i\alpha_i)$$

$$= c(\Sigma c_i\alpha_i) + 2(\upsilon, \Sigma c_i\alpha_i) > 0$$

(note that if $\delta = \Sigma c_i\alpha_i$, then $c(\delta) = (\delta,\delta) + 2(\rho,\Sigma c_i\alpha_i) > 0$; also $(\upsilon,\Sigma c_i\alpha_i) \geq 0$, since υ is dominant).

This completes the proof of the claim and hence of Lemma 2.17.

Lemma 2.18 With notation as in Lemma 2.17, let

$\bigotimes_{d=1}^{\ell} (V^{\omega_d})^{\otimes a_d} = \bigoplus_{\upsilon} W_\upsilon \otimes V^\upsilon$, where W_υ = the space of multiplicities for V^υ. Let $\{e^\upsilon_{i,t}, 1 \leq t \leq m_\upsilon\}$ be a basis for $W_\upsilon \otimes V^\upsilon$ (here $m_\upsilon = \dim W_\upsilon$) consisting of weight vectors. Let

$$e^\upsilon_{i,t} = \sum_{J_{\underline{a}}} c^\upsilon_{i,t}(J_{\underline{a}})\, v_{J_{\underline{a}}}, \qquad c^\upsilon_{i,t}(J_{\underline{a}}) \in \mathbb{Q}(q)$$

where $J_{\underline{a}} = \{J_{mj}, 1 \leq J_{mj} \leq N_m, 1 \leq j \leq a_m, 1 \leq m \leq \ell\}$ and

$v_{J_{\underline{a}}} = \bigotimes_{m=1}^{\ell} \bigotimes_{j=1}^{a_m} v^{\omega_m}_{J_{mj}}$ (here $\{v^{\omega_m}_i, 1 \leq i \leq N_m\}$ is the basis $\{Q_{\tau,\varphi}\}$ for V^{ω_m} as constructed in 2.4 above. Then

$$\sum_{J_a} c^\upsilon_{i,t}(J_{\underline{a}})\, X(J_{\underline{a}}) = 0$$

where

$$X(J_{\underline{a}}) = \prod_{m=1}^{\ell} \prod_{j=1}^{a_m} x_{J_{mj}}^{\omega_m}.$$

Proof: This is immediate from Lemma 2.17.

§3 Quantum G/B

3.1 The algebras $k_q[G/P_d]$ and $k_q[G/B]$

We preserve the notation of §2. Further, If $\lambda = \omega_d$, then we take for $\{e_i^\lambda\}$, the $\mathcal{A}$-basis $\{Q_{\tau,\varphi}\}$ as constructed in 2.4 above.

We define $\mathbb{Z}_q[G/P_d]$, $1 \le d \le \ell$ as the $\mathcal{A}$-sub algebra of $\mathbb{Z}_q[G]$ generated by $\{x_i^{\omega}d,\ 1 \le i \le N_d\}$, and $\mathbb{Z}_q[G/B]$ as the $\mathcal{A}$-sub algebra of $\mathbb{Z}_q[G]$ generated by $\{x_i^{\omega}d,\ 1 \le i \le N_d,$ $1 \le d \le \ell\}$. For any field k, we set

$$k_q[G/P_d] = \mathbb{Z}_q[G/P_d] \otimes_{\mathcal{A}} k(q)$$

and

$$k_q[G/B] = \mathbb{Z}_q[G/B] \otimes_{\mathcal{A}} k(q)$$

3.2 Standard Monomials

Recall (cf §2) that $\{x_i^{\omega_d}\}$ has an indexing $I^d = \{(\tau,\ \varphi)_N\}$ by certain pairs of elements of W^{P_d}, together with certain sequences of numbers. In the sequel, we shall denote $x_i^{\omega}d$ by just x_i. Further, if i corresponds to $(\tau,\varphi)_N$, then we shall denote x_i by $x_{(\tau,\varphi)_N}$ also.

Definition 3.3 Let $\underline{a} = (a_1, \ldots, a_\ell)$, $a_i \in \mathbb{Z}^+$. A monomial f in $k_q[G/B]$ is said to be standard of type $\underline{a}$ (or multi-degree $\underline{a}$), if

(a) $f = \prod_i \prod_j x_{ij}$, $1 \le j \le a_i$, $1 \le i \le \ell$

(b) Let x_{ij} correspond to $(\tau_{ij}, \varphi_{ij})_N$ (where note that $\tau_{ij}, \varphi_{ij} \in W^{P_i}$). There exists a sequence $\{\theta_{ij}, \delta_{ij}, 1 \le j \le a_i, 1 \le i \le \ell\}$ in W such that

(1) $\Pi_i(X(\theta_{ij})) = X(\tau_{ij})$, $\Pi_i(X(\delta_{ij})) = X(\varphi_{ij})$ under Π_i: $G/B \longrightarrow G/P_i$

(2) $X(\theta_{11}) \supseteq X(\delta_{11}) \supseteq X(\theta_{12}) \supseteq .. \supseteq X(\delta_{1a_1}) \supseteq X(\theta_{21}) \supseteq .. \supseteq X(\delta_{\ell a_\ell})$ (in G/B)

Proposition 3.4 Standard monomials are linearly independent over k(q).

Proof: Let $\sum a_i f_i = 0$, $a_i \in k(q)^*$ be a non-trivial linear relation among standard monomials. Clearing the denominators, we may suppose that $a_i \in k[q]$. Let r be the largest integer such that $(q - 1)^r$ divides all the a_i's. Cancelling $(q - 1)^r$ and going modulo the ideal $(q - 1)$, we obtain a non-trivial relation among standard monomials (for q = 1), which is not possible (cf$[LS]_1$, [L-Ra], $[L]_1$, $[L]_2$).

Remark 3.5 Below, we shall show that standard monomials generate the k(q)-vector space $k_q[G/B]$. In view of linear independence of standard monomials in arbitrary characteristic, to prove generation by standard monomials over $k(q)$, k being an arbitrary field, it suffices to prove generation for the case $k = \mathbb{Q}$.

3.6 Generation by standard monomials for the case $k = \mathbb{Q}$

Let $(k_q[G/B])_{\underline{a}}$ be the $k(q)$-span of $\{f \mid f$ is a monomial in $x_i^{\omega_d}$'s of multi-degree $\underline{a}$ having a_i linear factors $x_{ij}^{\omega_i}$, $1 \le j \le a_i$, $1 \le i \le \ell$ (the factors appearing in some order)$\}$. Let $N_a = \#\{$monomials of multi-degree $\underline{a}\}$. By Lemma 2.18, we have

I: $\sum_{J_{\underline{a}}} c_{it}^{\nu}(J_{\underline{a}})\, X(J_{\underline{a}}) = 0$, $c_{it}^{\nu}(J_{\underline{a}}) \in \mathbb{Q}(q)$ (Notation being as in Lemma 2.18). These give $N_{\underline{a}} - s_{\underline{a}}$ linear equations among the $N_{\underline{a}}$ monomials of type $\underline{a}$, where $s_{\underline{a}} = \#\{$standard monomials of type $\underline{a}\}$ ($= \dim V^{\lambda}$, $\lambda = \sum_{i=1}^{\ell} a_i\omega_i$). Further, the coefficient matrix of I has maximal rank ($= N_{\underline{a}} - s_{\underline{a}}$) in view of linear independence of $\{e_{i,t}^{\nu},\ 1 \le t \le m_{\nu}\}_{i,\nu}$. Hence taking the standard monomials of type $\underline{a}$ as the free variables of I (in view of linear independence of standard monomials), we obtain that each non-standard monomial of type $\underline{a}$ has an expression as a linear combination (over $\mathbb{Q}(q)$) of standard monomials of type $\underline{a}$.

Thus we obtain

Proposition 3.7 Standard monomials of type $\underline{a}$ generate $(k_q[G/B])_{\underline{a}}$

Combining Propositions 3.4 and 3.7, we obtain

Theorem 3.8. Standard monomials of type $\underline{a}$ form a $k(q)$-basis for $(k_q[G/B])_{\underline{a}}$ (k being an arbitrary field).

3.9 $\mathbb{Z}^{\ell}$-gradation and $k_q[G]$-comodule structure

In view of Proposition 3.4 and Theorem 3.8, we obtain a natural $\mathbb{Z}^{\ell}$-gradation for $k_q[G/B]$ given by

$$k_q[G/B] = \oplus_{\underline{a}}(k_q[G/B])_{\underline{a}}\ ,\ \underline{a} \in (\mathbb{Z}^+)^{\ell}.$$

Now the comultiplication $\Delta: k_q[G] \longrightarrow k_q[G] \otimes k_q[G]$, $\Delta(T^{\lambda}_{ij}) = \sum_r T^{\lambda}_{ir} \otimes T^{\lambda}_{rj}$, induces a left $k_q[G]$-comodule structure on $(k_q[G/B]_{\underline{a}}$ given by $\Delta: (k_q[G])_{\underline{a}} \longrightarrow k_q[G] \otimes (k_q[G/B])_{\underline{a}}$,

$\Delta(x_i^{\omega_d})(= \Delta T^{\omega_d}_{i1})) = \sum_r T^{\omega_d}_{ir} \otimes T^{\omega_d}_{r1} = \sum_r T^{\omega_d}_{ir} \otimes x_r^{\omega_d}$. Thus we obtain

Theorem 3.10 (a) $k_q[G/B] = \oplus_{\underline{a}} (k_q[G/B])_{\underline{a}}$

(b) $(k_q[G/B])_{\underline{a}}$ is a left $k_q[G]$-comodule and

$\dim_{k(q)}(k_q[G/B])_{\underline{a}} = s_{\underline{a}}$

(c) $k_q[G/B]$ has a canonical left $k_q[G]$-comodule structure.

Remark 3.11 In view of Theorems 3.8 and 3.10(a), we infer that all relations (among $x_i^{\omega_d}$'s) are consequences of the relations expressing non-standard monomials as sums of standard monomials, which in turn are consequences of relations given in Lemma 2.18.

§4 Quantum Schubert Schemes

4.1 The algebra $k_q[X(w)]$

Let $w \in W$ and let I_w be the two-sided ideal in $\mathbb{Z}_q[G/B]$ generated by $\{x^{\omega_d}_{(\tau,\varphi)_N}, 1 \le d \le \ell | w \not\ge \tau\}$. We define

$$\mathbb{Z}_q[X(w)] = \mathbb{Z}_q[G/B]/I_w$$

For any field k, we set $k_q[X(w)] = \mathbb{Z}_q[X(w)] \otimes_{\mathcal{A}} k(q)$. In the sequel, we shall denote $k_q[X(w)]$ by just $R_q(w)$.

4.2 Standard monomials on X(w)

Definition 4.3 A monomial f as in Definition 3.3 is said to be <u>standard on X(w)</u> (or in $R_q(w)$) of type $\underline{a}$, if in addition to the conditions (a) and (b) in Definition 3.3, we also have, $X(w) \ge X(\theta_{11})$.

Propositon 4.4 Standard monomials in $R_q(w)$ of type $\underline{a}$ are k(q)-linearly independent.

The proof is similar to that of Proposition 3.4 (using the linear independence for q = 1 (cf $[L\text{-}S]_1$, [L-Ra], $[L]_1$, $[L]_2$)).

4.5 Generation by standard monomials

Let $(R_q(w))_{\underline{a}}$ be the k(q)-span (in $R_q(w)$) of monomials of type $\underline{a}$. We shall show that standard monomials on X(w) of type $\underline{a}$ generate the k(q)-vector space $(R_q(w))_{\underline{a}}$. We first observe that all relations in $R_q(w)$ are consequences of relations of the following type. Let f be a non-standard monomial of type $\underline{a}$ in $k_q[G/B]$. Further let

(*) $f = \sum a_i f_i$, $a_i \in k(q)$, where f_i are standard monomials of type $\underline{a}$ in $k_q[G/B]$. Now going modulo I_w, some of the f_i's on the R.H.S. of (*) may not be standard in $R_q(w)$, while the L.H.S. of (*) is non-zero or zero in $R_q(w)$ according as f does not or does contain a factor $x^{\omega_d}_{(\tau,\varphi)_N}$ (for some d, $1 \le d \le \ell$) such that $w \not\geq \tau$. When $q = 1$, these relations in k[X(w)] give rise to expressions for non-standard monomials of type $\underline{a}$ on X(w) as sums of standard monomials on X(w) of type $\underline{a}$ (cf $[LS]_1$, [L-Ra], $[L]_1$, $[L]_2$). From this, it follows that a non-standard monomial (in $R_q(w)$) of type $\underline{a}$ has an expression as a sum of standard monomials on X(w) of type $\underline{a}$ in $R_q(w)$. To make it more precise, considering all the above relations (in $R_q(w)$) as a linear system of equations in monomials of type $\underline{a}$ in $R_q(w)$, let us denote the corresponding coefficient matrix by $A_q(w)$. Denoting $s_{\underline{a}}(w) = \#\{$standard monomials of type $\underline{a}$ in $R_q(w)\}$, we have (in view of linear independence of standard monomials of in $R_q(w)$),

$$s_{\underline{a}}(w) \leq \# \{\text{free variables of the above system}\}$$
$$\leq \# \{\text{free variables of the system for } q = 1\}$$
$$= s_{\underline{a}}(w).$$

Hence we obtain that # {free variables of the above system} = $s_{\underline{a}}(w)$. This together with the linear independence of standard monomials in $R_q(w)$ implies the following

Proposition 4.6 $(R_q(w))_{\underline{a}}$ is spanned by standard monomials of type $\underline{a}$ in $R_q(w)$.

Combining Propositions 4.4 and 4.6, we obtain

Theorem 4.7 Standard monomials of type $\underline{a}$ in $R_q(w)$ form a k(q)-basis for $(R_q(w))_{\underline{a}}$.

4.8 $\mathbb{Z}^{\ell}$-gradation

In view of Proposition 4.4 and Theorem 4.7, we obtain a natural $\mathbb{Z}^{\ell}$-gradation for $R_q(w)$ given by $R_q(w) = \underset{\underline{a}}{\oplus} (R_q(w))_{\underline{a}}$, $\underline{a} \in (\mathbb{Z}^+)^{\ell}$.

4.9 $U_{\mathscr{A}}(b^+)$-stability for I_w

Let $x_i^{\omega_d} \in I_w$. For the sake of simplicity of our discussion, we shall suppose that ω_d is a fundamental weight of classical type. Let then $x_i^{\omega_d} = x_{\tau,\varphi}$ where

$w \not\geq \tau$. Let α be a positive root such that $E_\alpha x_{\tau,\varphi} \neq 0$. Let $E_\alpha x_{\tau,\varphi} = \sum c_{\theta,\delta} x_{\theta,\delta}$, $c_{\theta,\delta} \in \mathbb{Q}(q)^*$. This implies that for each (θ,δ) on the R.H.S., the vector $E_\alpha Q_{\theta,\delta}$ is nonzero, and in the expression for $E_\alpha Q_{\theta,\delta}$ as a linear combination of the $Q_{\xi,\eta}$'s, the vector $Q_{\tau,\varphi}$ occurs with a nonzero coefficient.

Claim: Given an admissible pair (θ,δ), let α be a positive root such that $E_\alpha Q_{\theta,\delta} \neq 0$. Then in the expression $E_\alpha Q_{\theta,\delta} = \sum b_{\xi,\eta} Q_{\xi,\eta}$, $b_{\xi,\eta} \in k(q)^*$, each ξ on the R.H.S. is $\leq \theta$.

Proof of the claim: Clearly it suffices to prove the claim for α simple. Let $\{\beta_i\}$, $1 \leq i \leq r$ be simple roots such that if $\delta_t = s_{\beta_t} \ldots s_{\beta_1} \delta$, $1 \leq t \leq r$ then

(1) $\theta = \delta_r$

(2) $(\delta_{t-1}, \beta_t^*) = 2$, $1 \leq t \leq r$.

(here $\delta_0 = \delta$). We have $E_\alpha Q_{\theta,\delta} = F_{\beta_r} \ldots F_{\beta_1} E_\alpha Q_\delta$.

(Using the commutation relation $[E_\alpha, F_\alpha] = \sin(\frac{1}{2} dH_\alpha h)/\sinh(\frac{1}{2} dh)$ (here d = length of α) and induction on r, we may assume $\alpha \neq \beta_i$, $1 \leq i \leq r$.) The hypothesis that $E_\alpha Q_{\theta,\delta}$ is nonzero implies that $E_\alpha Q_\delta \neq 0$. Hence we obtain $(\delta(\omega), \alpha^*) < 0$. We now distinguish the following two cases:

Case 1: $(\delta(\omega),\alpha^*) = -1$. In this case we have (cf. $[LS]_1$), $(\delta_t(\omega),\alpha^*) = -1$, $1 \le t \le r$ and $(s_\alpha\delta_{t-1},\beta_t^*) = 2$, $1 \le t \le r$. Also $E_\alpha Q_\delta = Q_{s_\alpha\delta}$ (since $(\delta(\omega),\alpha^*) = -1$). Hence $E_\alpha Q_{\theta,\delta} = Q_{s_\alpha\theta,s_\alpha\delta}$. From this, claim follows in this case (note that $s_\alpha\delta < \delta$, since $(\delta(\omega),\alpha^*) < 0$).

Case 2: $(\delta(\omega),\alpha^*) = -2$. In this case we have $E_\alpha Q_\delta = Q_{\delta,s_\alpha\delta}$ (cf. Remark 2.5). Hence $E_\alpha Q_{\theta,\delta} = Q_{\theta,s_\alpha\delta}$ and the claim follows from this.

Now claim implies that in $E_\alpha x_{\tau,\varphi} = \sum c_{\theta,\delta} x_{\theta,\delta}$, for each non-zero $c_{\theta,\delta}$, $\theta \ge \tau$. Hence if $x_{\tau,\varphi} \in I_w$, then so does $x_{\theta,\delta}$ (note that $\tau \not\le w$ implies $\theta \not\le w$). Now the fact that I_w is $U_{\mathcal{A}}(b^+)$-stable implies that $R_q(w)$ is $U_q(b^+)$-stable. The pairing between $U_{\mathcal{A}}(b^+)$ and $\mathbb{Z}_q[B]$ (cf. 2.10) induces a $k_q[B]$-comodule structure on $R_q(w)$. Thus we obtain (in view of 4.8 and 4.9)

Theorem 4.10

(a) $R_q(w)$ has a canonical $\mathbb{Z}^\ell$-gradation given by $R_q(w) = \bigoplus_{\underline{a}} (R_q(w))_{\underline{a}}$, $\underline{a} \in (\mathbb{Z}^+)^\ell$.

(b) $R_q(w)$ has a canonical left $k_q[B]$-comodule structure.

(c) $(R_q(w))_{\underline{a}}$ is a left $k_q[B]$-comodule and $\dim_{k(q)} R_q(w)_{\underline{a}} = s_{\underline{a}}(w)$ (= # {standard monomials of type $\underline{a}$ on $X(w)$).

§5 A Presentation for $k_q[G/B]$

5.1 The $\mathcal{A}$-algebra $\mathcal{A}(G/B)$

Let $\mathcal{A}(G/B)$ be the associative $\mathcal{A}$-algebra with generators $\{y_i^{\omega_d}, 1 \le i \le N_d, 1 \le d \le \ell\}$ and relations

$$R^{\omega_d \omega_{d'}} Y_1^{\omega_d} Y_2^{\omega_{d'}} = q^{2(\omega_d, \omega_{d'})} Y_2^{\omega_{d'}} Y_1^{\omega_d}$$

where $Y^{\omega_d} = \sum_i y_i^{\omega_d} \otimes e_i^{\omega_d}$ (and a similar description for $Y^{\omega_{d'}}$), $\{e_i^{\omega_d}\}$ being the basis $\{Q_{\tau,\varphi}\}$ for $V_{\mathcal{A}}^{\omega_d}$ as constructed in §2. (Note that $Y^{\omega_d} \in \mathcal{A}(G/B) \otimes_{\mathcal{A}} V_{\mathcal{A}}^{\omega_d}$). For any field k, we set

$$\mathcal{A}_k(G/B) = \mathcal{A}(G/B) \otimes_{\mathcal{A}} k(q)$$

5.2 $\mathbb{Z}_q[G]$-comodule structure

The map Δ: $\mathcal{A}(G/B) \longrightarrow \mathbb{Z}_q[G] \otimes_{\mathcal{A}} \mathcal{A}(G/B)$, $\Delta(y_i^{\omega_d}) = \sum_r t_{ir}^{\omega_d} \otimes y_r^{\omega_d}$ defines a canonical left $\mathbb{Z}_q[G]$-comodule structure on $\mathcal{A}(G/B)$.

5.3 The map θ

Define θ: $\mathcal{A}(G/B) \longrightarrow \mathbb{Z}_q[G/B]$, $\theta(y_i^{\omega_d}) = x_i^{\omega_d}$. Note that θ is well-defined (in view of Lemma 2.13) and that θ is an $\mathcal{A}$-algebra homomorphism.

5.4 Standard monomials

Let us define a monomial in $y_i^{\omega_d}$'s to be standard similar to Definition 3.3.

We have

Proposition 5.5 Standard monomials are linearly independent over k(q), k being an arbitrary field.

Proof: The result follows by considering the k(q)-algebra homomorphism $\theta : \mathcal{A}_k(G/B) \longrightarrow k_q[G/B]$, $\theta(y_i^{\omega_d}) = x_i^{\omega_d}$ and using Proposition 3.4.

Proposition 5.6 For a dominant integral weight $\lambda = \sum_{i=1}^{\ell} a_i\omega_i$, let us define $Y^\lambda = \otimes_{i=1}^{\ell}(Y_i^{\omega_i})^{\otimes a_i}$. With notation as in Lemma 2.17, we have $\tilde{P}_\nu(Y^\lambda) = 0$, $\upsilon \neq \lambda$.

Proof: Observe that Lemma 2.17 was proved as a direct consequence of Lemma 2.13, and that the relation stated in Lemma 2.13 hold by replacing X^{ω_d}, $X^{\omega_{d'}}$ respectively by Y^{ω_d}, $Y^{\omega_{d'}}$ (by the very definition of $\mathcal{A}(G/B)$). Hence the result follows.

As a consequence of Proposition 5.6, we have (similar to Lemma 2.18).

Lemma 5.7 With notation as in Lemma 2.18, we have $\sum c_{i,t}^{\nu}(J_{\underline{a}})Y(J_{\underline{a}}) = 0$, where $Y(J_{\underline{a}}) = \prod_{m=1}^{\ell}\prod_{j=1}^{a_m} y_{J_{mj}}^{\omega_m}$.

Proposition 5.8 Let $\underline{a} = \sum_{i=1}^{\ell} a_i\omega_i$ and let $(\mathcal{A}_k(G/B))_{\underline{a}}$ be the k(q)-span of $\{f \mid f$ is a monomial in $y_t^{\omega_d}$'s of multi-degree $\underline{a}$ having a_i linear factors $y_{ij}^{\omega_i}$, $1 \le j \le a_i$, $1 \le i \le \ell\}$ (the factors appearing in some order). Then standard monomials of type $\underline{a}$ generate $(\mathcal{A}_k(G/B))_{\underline{a}}$.

Proof: The result follows by the same reasoning as in 3.6.

Combining Propositions 5.5 and 5.8, we obtain

Theorem 5.9 Standard monomials of type $\underline{a}$ form a $k(q)$-basis for $(\mathcal{A}_k(G/B))_{\underline{a}}$.

5.10 $\mathbb{Z}^\ell$-gradation

In view of Proposition 5.5 and Theorem 5.9, we obtain a $\mathbb{Z}^\ell$-gradation for $\mathcal{A}_k(G/B)$ given by $\mathcal{A}_k(G/B) = \oplus_{\underline{a}} \mathcal{A}_k(G/B))_{\underline{a}}$, $\underline{a} \in (\mathbb{Z}^+)^\ell$. Further Δ: $\mathcal{A}_k(G/B) \longrightarrow k_q[G] \otimes \mathcal{A}_k(G/B)$, $\Delta(y_i^{\omega_d}) = \sum_r T_{ir}^{\omega_d} \otimes y_r^{\omega_d}$ induces a left $k_q[G]$-comodule structure on $(\mathcal{A}_k(G/B))_{\underline{a}}$. Thus we obtain

Theorem 5.11 (a) $\mathcal{A}_k(G/B) = \oplus_{\underline{a}} (\mathcal{A}_k(G/B))_{\underline{a}}$, $\underline{a} \in (\mathbb{Z}^+)^\ell$

(b) $(\mathcal{A}_k(G/B))_{\underline{a}}$ is a left $k_q[G]$-comodule and $\dim_{k(q)}(\mathcal{A}_k(G/B))_{\underline{a}} = s_{\underline{a}}$

(c) $\mathcal{A}_k(G/B)$ has a canonical left $k_q[G]$-comodule structure.

Theorem 5.12 The map θ: $\mathcal{A}_k(G/B) \longrightarrow k_q(G/B)$, $\theta(y_i^{\omega_d}) =$

$x_i^{\omega_d}$ is a (degree zero) graded k(q)-algebra isomorphism, preserving the left $k_q[G]$-comodule structures of the respective graded pieces. In particular, $k_q[G/B]$ is a quadratic algebra.

Proof: The result follows from Theorems 3.10, 5.11, Remark 3.11 and Lemma 5.7.

Remark 5.13: One can give a similar presentation for $R_q(w)$ and deduce that $R_q(w)$ is again a quadratic algebra.

Appendix A

A nice basis for V^{ω_d} for non-classical ω_d's

With notation as in §2, let ω_d be a fundamental weight of non-classical type, i.e., there exists a positive root β such that $(\omega_d, \beta^*) > 2$. We first construct the extremal weight vectors in V^{ω_d} in exactly the same way as in §2. To construct the non-extremal weight vectors, let us use the indexing I′ as in $[L]_2$. (see also $[LS]_2$) We recall the set I′. The set I′ consists of $\{(\tau,\mu)_N,\ \tau,\mu \in W^{P_d}\}$ where τ,μ and N are given as follows:

(a) There exists a sequence $\{\mu_i \in W^{P_d}, 0 \le i \le r+1\}$ such that

$$\tau = \mu_0 > \mu_1 > \dots > \mu_{r+1} = \mu,\ \ell(\mu_i) = \ell(\mu_{i+1}) + 1$$

(b) Let $\mu_i = s_{\beta_i}\ \mu_{i+1}$ (where β_i is positive), and $m_i = (\mu_{i+1}(\omega_d), \beta_i^*)$. There exist positive integers n_i, $0 \le i \le r$, such that

$$1 > \frac{n_r}{m_r} \geq \dots \geq \frac{n_0}{m_0} > 0$$

(in particular, note that $m_i > 1$)

(c) Let $\frac{p_{i_t}}{q_{i_t}} > \dots > \frac{p_{i_1}}{q_{i_1}}$ be all the distinct numbers in $\{\frac{n_r}{m_r},\dots,\frac{n_0}{m_0}\}$. Then $N = (\frac{P_{i_t}}{q_{i_t}},\dots,\frac{P_{i_1}}{q_{i_1}})$. To a $(\tau,\mu)_N$, we associate the vector $F_{\beta_0}^{[n_0]} F_{\beta_1}^{[n_1]} \dots F_{\beta_r}^{[n_r]} e_\mu$ (here e_μ, as in §2, is the extremal weight vector of weight $\mu(\omega_d)$; also, for β non-simple, F_β is to the understood as in $[Lu]_1$)

Appendix B

Relationship between $\mathbb{C}_q[G]$ and the Hopf algebra $A_q(G)$ of [FRT]

Type A_ℓ: $A_q(G)$ is the associative algebra with 1 over $\mathbb{C}(q)$ generated by $t_{ij}^{\omega_1}$ (or just t_{ij}), $1 \leq i.\ j \leq n$ $(=\ell+1)$, the relations being

$$R\ T_1T_2 = T_2T_1R \tag{1}$$

and

$$\sum_{\sigma\in S_n} (-q)^{-\ell(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)} = 1 \tag{2}$$

where

$$R = \sum_{\substack{i \neq j \\ i,j=1}}^{n} e_{ii} \otimes e_{jj} + q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + (q-q^{-1}) \sum_{1 \le j \le i \le n} e_{ij} \otimes e_{ji}$$

(e_{ij}'s being the elementary matrices), $T_1 = T \otimes \mathrm{Id}$, $T_2 = \mathrm{Id} \otimes T$, $T = (t_{ij})$. It can be seen easily that map $\theta : A_q(G) \longrightarrow \mathbb{C}_q[G]$, $t_{ij} \longmapsto T_{ij}^{\omega_1}$ defines an isomorphism of Hopf algebras.

Type C_n: $A_q(G)$ is the associative $\mathbb{C}(q)$ algebra with 1 generated by $t_{ij}^{\omega_1}$ or just t_{ij}, $1 \le i \le 2n$, the relations being

$$R\, T_1 T_2 = T_2 T_1\, R \tag{1}$$

and

$$T\, C\, {}^tT\, C^{-1} = C\, {}^tT\, C^{-1}\, T = I \tag{2}$$

where

$$R = q \sum_{i=1}^{2n} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j, j'}}^{2n} e_{ii} \otimes e_{jj}$$

$$+ q^{-1} \sum_{i=1}^{2n} e_{ii} \otimes e_{i'i'} + (q-q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{2n} e_{ij} \otimes e_{ji}$$

$$+ (q-q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{2n} \varepsilon_i \varepsilon_j e_{ij} \otimes e_{i'j'}$$

where $C = \text{anti-diag}\,(q^n, \ldots, q, -q^{-1}, \ldots, -q^{-n})$,

$i'=2n+1-i$, $\varepsilon_i = 1$ or -1 according as $i \le n$ or $i > n$. As in type A_n, the map $\theta: A_q(G) \longrightarrow \mathbb{C}_q[G]$, $\theta(t_{ij}) = T_{ij}^{\omega_1}$ induces a Hopf algebra isomorphism.

Type B_n, D_n: The algebra $A_q(G)$ is generated by $t_{ij}^{\omega_n}$, $1 \le i, j \le \dim V^{\omega_n}$ for type B_n and by $t_{ij}^{\omega_n}$, $t_{ij}^{\omega_{n-1}}$, $1 \le i, j \le \dim V^{\omega_n}$ ($= \dim V^{\omega_{n-1}}$) for type D_n. The generators satisfy similar relations as (1) above and some extra relations. These extra relations could also be written explicity, but they are more complicated and can be deduced from [R]. As in Type A_n and C_n, we have $A_q(G) \approx \mathbb{C}_q[G]$ (as Hopf algebras).

Appendix C

The algebra $A_q(G/B)$

Again for simplicity of discussion, let us suppose that G is classical. Let $1 \le d \le \ell$ (= rank G); let $d \ne n$, if G is of type B_n, and $d \ne n-1, n$, if G is of type D_n. Observe that V^{ω_d} occurs as an irreducible factor in $(V^{\omega_1})^{\otimes d}$. Let

$$K^{\omega_d} = \text{the projection } (V^{\omega_1})^{\otimes d} \longrightarrow V^{\omega_d}$$

$$\bar{K}^{\omega_d} = \text{the inclusion } V^{\omega_d} \hookrightarrow (V^{\omega_1})^{\otimes d}, \text{such that } K^{\omega_d}\bar{K}^{\omega_d} = \mathrm{Id}_{V^{\omega_d}}$$

$$T^{\omega_d} = K^{\omega_d}(T^{\omega_1})^{\otimes d}\bar{K}^{\omega_d}$$

(Observe that

$$T^{\omega_1} = K^{\omega_n\omega_n}_{\omega_1}(T^{\omega_n}\otimes T^{\omega_n})\bar{K}^{\omega_n\omega_n}_{\omega_1}, \text{ if G is of type } B_n, \text{ and}$$

$$T^{\omega_1} = K^{\bar{\omega}_n\omega_n}_{\omega_1}(T^{\bar{\omega}_n}\otimes T^{\omega_n})\bar{K}^{\bar{\omega}_n\omega_n}_{\omega_1}, \text{ if G is of type } D_n, \text{ where } \bar{\omega}_n =$$

ω_n (resp. ω_{n-1}) if n is even (resp. odd), $K^{\lambda\mu}_{\upsilon}$ is the projection $V^{\lambda}\otimes V^{\mu} \longrightarrow V^{\upsilon}$, and $\bar{K}^{\lambda\mu}_{\upsilon}$ is the inclusiion $V^{\upsilon} \hookrightarrow V^{\lambda}\otimes V^{\mu}$ such that $K^{\lambda\mu}_{\nu}\ \bar{K}^{\lambda\mu}_{\nu'} = \mathrm{Id}_{V_\upsilon}\ \delta_{\upsilon\upsilon'}$. Thus t_{ij} (= $t^{\omega_1}_{ij}$) is a quadratic expression in $t^{\omega_n}_{ij}$, if G is of type B_n, and is a bilinear expression in $t^{\omega_n}_{ij}$, $t^{\bar{\omega}_n}_{ij}$, if G is of type D_n. For $1 \le d \le \ell$, let us write $T^{\omega_d} = (t^{\omega_d}_{ij})$, $1 \le i, j \le N_d$ (= dim V^{ω_d}), and set $\bar{x}^{\omega_d}_i = t^{\omega_d}_{i1}$. Note that $\bar{x}^{\omega_d}_i$'s are polynomials in $t^{\omega_1}_{ij}$'s, if G is of type A_n or C_n. For type B_n, $\bar{x}^{\omega_d}_i$'s are polnomials in $t^{\omega_n}_{ij}$'s and for type D_n, $\bar{x}^{\omega_d}_i$'s are polnomials in $t^{\omega_n}_{ij}$'s and $t^{\bar{\omega}_n}_{ij}$'s.

Define $A_q(G/B)$ as the sub algebra of $A_q(G)$ generated by $\{\bar{x}^{\omega_d}_i,\ 1 \le i \le N_d,\ 1 \le d \le \ell\}$. Then it is easily seen that the map $\theta: A_q(G/B) \longrightarrow \mathbb{C}_q[G/B]$, $\theta(\bar{x}^{\omega_d}_i) = x^{\omega_d}_i$ induces an algebra isomorphism. Similarly, we have

$A_q(G/P_d) \simeq \mathbb{C}_q[G/P_d]$.

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