Koszul and Gorenstein properties for homogeneous algebras

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Abstract

Koszul property was generalized to homogeneous algebras of degree $N > 2$ in [5], and related to N-complexes in [7]. We show that if the N-homogeneous algebra A is generalized Koszul, AS-Gorenstein and of finite global dimension, then one can apply the Van den Bergh duality theorem [23] to A, i.e., there is a Poincaré duality between Hochschild homology and cohomology of A, as for $N = 2$.

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1 Introduction

The appearance of noncommutative projective geometry has given a new impulse to the study of (noncommutative) graded algebras [20]. Some interesting classes of graded algebras have been investigated, for example, the class of the AS-regular algebras [1]. Those algebras are often defined as graded algebras of finite global dimension and having AS-Gorenstein property (the polynomial growth imposed by Artin and Schelter is often removed and in fact, it is not necessary for our purpose). Generalized Koszulity is another property which is natural to assume for getting noncommutative analogues of polynomial algebras [5, 7]. This article examines the class of AS-regular algebras which are generalized Koszul. Our main result is that the Poincaré duality holds in this class. Besides the quadratic case already known by Michel Van den Bergh [23], the first known Poincaré duality in a nonquadratic homogeneous situation has been obtained by the second author [16] for the generic cubic AS-regular algebras (with polynomial growth) of global dimension three and of type A.

The graded algebras of our class are N-homogeneous where N is an integer ≥ 2, that is, they are generated in degree one and all their relations are homogeneous of the same degree N. Koszul property is well-known for quadratic algebras (i.e., for $N = 2$) [18], and it was recently generalized by the first author [5] for $N > 2$. This generalization follows along the definition given by Beilinson, Ginzburg, Soergel [3] for $N = 2$: the minimal projective resolution of the trivial module has to be pure in the lowest degrees. Here the minimal projective resolution makes sense in the category of graded modules (with morphisms of degree zero), so the definition is very natural as far as graded algebras are concerned. On the other hand, following Manin's monograph on quadratic algebras [15], Michel Dubois-Violette, Marc Wambst and the first author have given an alternative set-up of generalized Koszul property based on N-complexes [7]. Actually, if one follows the categorical point of view of Manin, the Koszul complex of the quadratic case becomes naturally an N-complex in the N-case.

What is an N-complex ? It is just a complex for which the condition $d^2 = 0$ on the differential d is replaced by $d^N = 0$. Thus a usual complex is a 2-complex. There exist some reserved feelings concerning the interest of N-complexes when $N > 2$. It lies on the fact that N-homologies in various classical contexts (simplicial, Hochschild, cyclic) do not bring new things [11, 13, 24]. It perhaps comes from the fact that acyclicity of N-complexes is too strong. For example, acyclicity of the Koszul N-complex implies that the algebra is trivial. As shown in [7], there is one (and only one) adequate contraction of the Koszul N-complex whose acyclicity provides the (highly nontrivial !) Koszul property. Similarly, AS-Gorenstein property of generalized Koszul algebras coincides with acyclicity of one contraction of the dual Koszul N-complex.

So the Koszul 2-complexes of N-homogeneous algebras are adequate contractions of the Koszul N-complexes. Of course, the Koszul 2-complexes can be defined without N-complexes, and it was doing so in [5]. Why is it interesting to have N-complexes behind the Koszul 2-complexes ? A striking answer is the following. If we want to define the *bimodule* Koszul 2-complex of an N-homogeneous algebra for $N > 2$, the knowledge of the left and right Koszul 2-complexes is not sufficient. Actually, it is impossible to avoid N-complexes in defining the differential of the bimodule Koszul 2-complex [6]. The precise reason is that the left and right N-differentials commute, but not the left and right 2-differentials.

The usefulness of N-complexes can also be demonstrated from constructions of Theoretical Physics (see [7, 12] and references therein). It is worth noticing here that Yang-Mills algebras studied recently by Connes and Dubois-Violette [10] provide new examples of cubic Koszul AS-regular algebras.

Let us present the plan and some results of this paper. In Section 2, generalized Koszulity is introduced as in [5], including several points which were omitted in [5]. Koszulity of Manin products is discussed.

In Section 3, the link with N-complexes is explicited. By shifting the Koszul N-complex, the Yoneda algebra of a generalized Koszul algebra is computed, with particular attention to the signs in the Yoneda product.

In Section 4, the bimodule Koszul 2-complex is drawn from the bimodule Koszul N-complex, and is used to get a bimodule characterization of Koszulity. A consequence is that Hochschild dimension and global dimension of generalized Koszul algebras are the same.

In Section 5, AS-Gorenstein property is recalled. If A is generalized Koszul over the ground field k, with finite global dimension D such that $\mathrm{Ext}_A^D(k, A) = k$, then an explicit N-complexes morphism Φ is constructed from the dual Koszul N-complex to the Koszul N-complex. From Φ, we deduce a convenient criterion for A to be AS-Gorenstein (Theorem 5.4). This criterion has the following consequences.

Theorem 1.1 *For $n \geq N \geq 2$, consider the N-homogeneous algebra A over k of characteristic 0, with generators $x_1, \ldots, x_n$. The relations of A are the antisymmetrizers of degree N, i.e., are the following*

$$\sum_{\sigma \in \mathbf{S}_N} \mathrm{sgn}(\sigma)\ x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(N)}} = 0\,,\ 1 \leq i_1 < \cdots < i_N \leq n\,,$$

where $\mathbf{S}_N$ denotes the permutation group of $\{1, \ldots, N\}$. We know [5] that A is generalized Koszul, of finite global dimension. Then A is AS-Gorenstein if and only if either $N = 2$ or ($N > 2$ and $n = Nq + 1$ for some integer $q \geq 1$).

Theorem 1.2 *Let A be N-homogeneous, generalized Koszul, of finite global dimension. Denote by $E(A)$ the Yoneda algebra of A. Then A is AS-Gorenstein if and only if the finite dimensional algebra $E(A)$ is Frobenius.*

The latter result is due to Smith [19] when $N = 2$. Our proof follows along the same lines, but the explicit Φ is not in [19]. The situation $N > 2$ is a bit more involved because the jump map defining the adequate contraction of the Koszul N-complex is not additive. Furthermore we have to pay some attention to the signs in the Yoneda product. Actually, Theorem 1.2 is a straightforward consequence of a general theorem by Lu, Palmieri, Wu and Zhang [14], which asserts that if A is any connected graded algebra, then A is AS-regular if and only if the Yoneda algebra of A is Frobenius. Their proof uses A_∞-algebras. Under generalized Koszulity assumption, A_∞-algebras are not necessary. But the N-complexes machinery is required if we do not want to use the general theorem. We thank Bernhard Keller to have pointed out to us the article [14].

In Section 6, we prove the assumptions of the Van den Bergh duality theorem [23] for our class of algebras. An essential ingredient is the adaptation to the N-case of a computation used by Van den Bergh in the situation of the quadratic Koszul noetherian AS-regular algebras (Theorem 9.2 in [22]). Note that our algebras may be non noetherian (those of Theorem 1.1 are not noetherian if $N > 2$), so that the Yekutieli balanced dualizing complex [26] cannot be used. Then we get our main result (HH denotes Hochschild homology or cohomology) :

Theorem 1.3 *Let A be an N-homogeneous algebra which is generalized Koszul, AS-Gorenstein and with finite global dimension D. For any bimodule M, we have*

$$HH^i(A, M) \cong HH_{D-i}(A, {}_{\varepsilon^{D+1}\phi}M).$$

For $N = 2$, this is Van den Bergh's Proposition 2 [23]. The meaning of ε is the same as for $N = 2$. Let ν be the automorphism of the Frobenius algebra $E(A)$ (Theorem 1.2 above) such that the bimodule $E(A)^*$ is canonically $E(A)_\nu$ (usual action on the left and action twisted by ν on the right). The automorphism ϕ of A is such that the homogeneous component ϕ_1 is the transposed linear map of ν_1, as for $N = 2$. It is important to note that, if $N > 2$, $E(A)$ is not generated in degree 1, and ν is not determined by ν_1 (but $E(A)$ is generated in degree 1 and 2, and ν is determined by ν_1 and ν_2). The end of Section 6 is devoted to give direct proofs of Theorem 1.3 for $M = A$, and specially to recover the proof used by the second author for the generic cubic AS-regular algebras (with polynomial growth) of global dimension three and of type A [16]. This proof works well for any A such that there is no twist on $M = A$, i.e., such that $\phi = \varepsilon^{D+1}$. On the other hand, as for $N = 2$, the automorphisms ν and ϕ are interpreted in terms of the Artin-Schelter matrix Q [1] when A is a cubic AS-regular algebra (with polynomial growth) of global dimension three.

2 Generalized Koszul algebras

Some material about generalized Koszul property can be found in [5, 6, 7, 10, 12, 16]. It is completed here on several particular points. Throughout this paper, k is a field and V is a finite-dimensional k-vector space which is considered as graded, with its gradation concentrated in degree 1. We fix a natural number $N \geq 2$ and a graded subspace R of the tensor algebra $\mathrm{Tens}(V)$, which is concentrated in degree N. In other words R is a subspace of $V^{\otimes N}$. The two-sided ideal $I(R) = I$ generated by R in $\mathrm{Tens}(V)$ is graded by the subspaces I_n given by $I_n = 0$, $0 \leq n \leq N-1$, and

$$I_n = \sum_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\,,\ n \geq N.$$

The algebra A=$\mathrm{Tens}(V)/I$ is called an *N-homogeneous algebra* on V, with R as space of relations. The algebra A is graded by the subspaces $A_n = V^{\otimes n}/I_n$, and generated by V (hence in degree 1). Clearly, $A_n = V^{\otimes n}$ for $0 \leq n \leq N-1$. Introduce the jump map $n : \mathbf{N} \to \mathbf{N}$ by

$$n(2q) = qN,\ n(2q+1) = qN+1,\ q \text{ integer} \geq 0.$$

The jump map n is additive if and only if $N = 2$. If $N > 2$, $n(i+j) = n(i)+n(j)$ holds unless if i and j are both odd. We begin to generalize a result of [3].

Proposition 2.1 *Let A be an N-homogeneous algebra. The graded vector space $\mathrm{Tor}_i^A(k,k)$ lives in degrees $\geq n(i)$ for $i \geq 0$.*

Proof. We prove the result under the more general assumption that R lives in degrees $\geq N$. Let $P \to k$ be a minimal projective resolution of the trivial module k in the category A-grMod of the graded left A-modules (with morphisms of degree 0). We know that $P_i = A \otimes E_i$, where E_i is a graded vector subspace of P_{i-1}, and the differential $d_i : P_i \to P_{i-1}$ is defined by this inclusion. Since the graded vector space $\mathrm{Tor}_i^A(k,k)$ is isomorphic to $k \otimes_A P_i = E_i$, it suffices to prove that E_i lives in degree $\geq n(i)$ for $i \geq 0$. Proceed by induction on i. We can choose $E_0 = k$, $E_1 = V$, $E_2 = R$ with obvious inclusions, so that the property holds for $i = 0$, 1, 2. Assume $i \geq 3$ and the property true for $i-1$. First assume $i = 2q+1$. Since P_{i-1} lives in degrees $\geq qN$, E_i lives in degrees $\geq qN$. But $P_{i-1,qN} = E_{i-1,qN}$, so that $\ker(d_{i-1})$ vanishes in degree qN. Hence, $E_{i,qN} = 0$ and E_i lives in degrees $\geq qN+1$. Assume now $i = 2q$. Since P_{i-1} lives in degrees $\geq (q-1)N+1$, E_i lives in degrees $\geq (q-1)N+1$. Fix u with $1 \leq u \leq N-1$. We have

$$P_{i-1,(q-1)N+u} = \bigoplus_{1 \leq v \leq u} A_{u-v} \otimes E_{i-1,(q-1)N+v}\,.$$

For proving that E_i lives in degrees $\geq qN$, it suffices to prove that d_{i-1} is injective on each $A_{u-v} \otimes E_{i-1,(q-1)N+v}$. But d_{i-1} restricted to $E_{i-1,(q-1)N+v}$ is the inclusion into the direct sum

$$\bigoplus_{1 \leq w \leq v} A_{v-w} \otimes E_{i-2,(q-1)N+w}\,.$$

Therefore, d_{i-1} sends $A_{u-v} \otimes E_{i-1,(q-1)N+v}$ into the direct sum

$$\bigoplus_{1 \leq w \leq v} A_{u-w} \otimes E_{i-2,(q-1)N+w} .$$

Since $u - w < N$, $A_{u-w} = V^{\otimes(u-w)}$ which is not a quotient, hence the result. ■

Definition 2.2 *An N-homogeneous algebra A is said to be (generalized) Koszul if for each $i \geq 3$, the graded vector space $\mathrm{Tor}_i^A(k,k)$ is concentrated in degree $n(i)$ (or, equivalently, $\underline{\mathrm{Ext}}_A^i(k,k)$ is concentrated in degree $-n(i)$).*

When $N = 2$, it is exactly Priddy's definition [18]. Here, $\underline{\mathrm{Ext}}$ denotes the derived functor of the functor $\underline{\mathrm{Hom}}$ of A-grMod. Let us denote by hom the functor Hom of A-grMod (in which the morphisms are of degree 0), and ext its derived functor. For any objects M and M' of A-grMod, we have $\underline{\mathrm{Ext}}_A^{i,j}(M, M') = \mathrm{ext}_A^i(M, M'(j))$. The following proposition generalizes a result of [3]. Recall that the gradation of the shift $M(l)$ of a graded A-module M is defined by $M(l)_n = M_{n+l}$.

Proposition 2.3 *Let A be a connected graded k-algebra. Fix $N \geq 2$. The following are equivalent.*

(i) A is N-homogeneous and Koszul.

(ii) $\mathrm{ext}_A^i(k, k(-n)) = 0$ whenever $n \neq n(i)$.

(iii) For any objects M and N of A-grMod, respectively concentrated in degrees m and n, $\mathrm{ext}_A^i(M, N) = 0$ whenever $n \neq m + n(i)$.

Proof. Clearly, $\underline{\mathrm{Ext}}_A^{i,-n}(k,k) = \mathrm{ext}_A^i(k, k(-n))$, hence (i)⇔(ii). (iii)⇒(i) is obvious. Assume (i). Let P be a minimal projective resolution of k, $P_i = A \otimes E_i$ with E_i concentrated in degree $n(i)$. As $\mathrm{ext}_A^i(M,N) = \mathrm{ext}_A^i(M(-m), N(-m))$, we assume $m = 0$. As M is a direct sum of copies of k, we assume $M = k$. Since $\mathrm{hom}_A(P_j, N) = \mathrm{hom}_k(E_j, N)$ vanishes whenever $n \neq n(j)$, each component of the complex $\mathrm{hom}_A(P, N)$ vanishes unless perhaps one. Therefore, $\mathrm{ext}_A^i(k, N) = \mathrm{hom}_k(E_i, N)$, which vanishes whenever $n \neq n(i)$. ■

As for $N = 2$, there exists a remarkable complex whose acyclicity is equivalent to Koszulity of A. For $n \geq 0$, introduce the subspace of $V^{\otimes n}$ by

$$W_n = \bigcap_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

If $n < N$, $W_n = V^{\otimes n}$. As defined in [5], the *Koszul 2-complex* K of A is the following complex of A-grMod

$$\cdots \longrightarrow K_i \xrightarrow{\delta_i} K_{i-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{\delta_1} K_0 \longrightarrow 0\,, \qquad (2.1)$$

in which $K_i = A \otimes W_{n(i)}$, and the differential δ_i is defined by the inclusion of $W_{n(i)}$ in K_{i-1}. The homology of K is k in degree 0, and 0 in degree 1.

Theorem 2.4 *Let A be N-homogeneous. Then A is Koszul if and only if the Koszul 2-complex K is exact in degrees ≥ 2, i.e., is a resolution (necessarily minimal projective in A-grMod) of k.*

Proof. If K is a resolution of k, then $\mathrm{Tor}_i^A(k,k) = W_{n(i)}$, which is concentrated in degree $n(i)$, hence A is Koszul. Conversely, assume A Koszul. We want to prove $H_i(K) = 0$ by induction on $i \geq 1$. It holds for $i = 1$. Let $i \geq 2$ be such that it holds for any integer $< i$. Set $Z_i = \ker \delta_i$. By dimension shifting ([25] p.47), one has

$$\mathrm{Tor}_{i+1}^A(k,k) = \ker(k \otimes_A Z_i \to k \otimes_A K_i).$$

But $k \otimes_A Z_i$ lives in degree $> n(i)$, whereas $k \otimes_A K_i$ lives in degree $n(i)$. Therefore, $\mathrm{Tor}_{i+1}^A(k,k) = k \otimes_A Z_i$, and $k \otimes_A Z_i$ is concentrated in degree $n(i+1)$. Thus the projective cover $A \otimes_k (k \otimes_A Z_i)$ of Z_i is generated in degree $n(i+1)$, and it is the same for Z_i itself.

To conclude that $Z_i = \mathrm{im}\, \delta_{i+1}$, it suffices to prove this equality in degree $n(i+1)$. But $K_{i,n(i+1)}$ is $V \otimes W_{n(i)}$ or $V^{\otimes(N-1)} \otimes W_{n(i)}$, according to i is even or odd. Consequently $\delta_{i+1}(W_{n(i+1)})$ is the natural inclusion of $W_{n(i+1)}$ into $V \otimes W_{n(i)}$ or $V^{\otimes(N-1)} \otimes W_{n(i)}$. On the other hand, δ_i sends naturally $V \otimes W_{n(i)}$ or $V^{\otimes(N-1)} \otimes W_{n(i)}$ into $A_N \otimes W_{n(i-1)}$. If i is even, we have

$$(\ker \delta_i)_{n(i+1)} = (V \otimes W_{n(i)}) \cap (R \otimes W_{n(i-1)}),$$

which is $W_{n(i+1)}$, as expected. If i is odd, we have

$$(\ker \delta_i)_{n(i+1)} = (V^{\otimes(N-1)} \otimes W_{n(i)}) \cap (R \otimes W_{n(i-1)}).$$

If $N = 2$, it is $W_{n(i+1)}$, as expected. It is still $W_{n(i+1)}$ if $N > 2$, because, according to the lemma just below, Koszulity of A implies the following

$$(V^{\otimes(N-1)} \otimes R) \cap (R \otimes V^{\otimes(N-1)}) = W_{2N-1}. \blacksquare$$

Lemma 2.5 *If A is Koszul N-homogeneous, one has for $m = 1, \ldots, N-1$,*

$$(V^{\otimes m} \otimes R) \cap (R \otimes V^{\otimes m}) = W_{N+m}. \tag{2.2}$$

Proof. In the previous induction, we can take the equality $H_i(K) = 0$ for granted when $i = 2$. So $(\ker \delta_2)_{N+m} = V^{\otimes(m-1)} \otimes W_{N+1}$. Since δ_2 is the natural map from $V^{\otimes m} \otimes R$ to $A_{N+m-1} \otimes V$, we get

$$(V^{\otimes m} \otimes R) \cap (R \otimes V^{\otimes m} + \cdots + V^{\otimes(m-1)} \otimes R \otimes V) = V^{\otimes(m-1)} \otimes W_{N+1} \tag{2.3}$$

which implies the inclusion

$$(V^{\otimes m} \otimes R) \cap (R \otimes V^{\otimes m}) \subseteq V^{\otimes (m-1)} \otimes R \otimes V, \tag{2.4}$$

and this inclusion gives (2.2) for $m = 2$. Assume now (2.2) for m. Then (2.4) for $m+1$ shows that $(V^{\otimes (m+1)} \otimes R) \cap (R \otimes V^{\otimes (m+1)})$ is included into

$$(V^{\otimes (m+1)} \otimes R) \cap (R \otimes V^{\otimes (m+1)}) \cap (V^{\otimes m} \otimes R \otimes V).$$

Using induction hypothesis, the latter subspace is

$$(V^{\otimes (m+1)} \otimes R) \cap (W_{N+m} \otimes V) = W_{N+m+1}. \ \blacksquare$$

Corollary 2.6 *Let A be Koszul N-homogeneous. Knowing that the resolution K is minimal in A-grMod and setting $W_{n(\infty)} = 0$, we have*

$$\mathrm{gl.dim}(A) = \sup\{ i \in \mathbf{N} \cup \{\infty\} \, ; \ W_{n(i)} \neq 0 \}.$$

Relations (2.3) for $m = 2, \ldots, N-1$ are put together in order to define the so-called *extra condition* (actually, the relation (2.3) for $m = N-1$ implies the other ones). If the extra condition is added to a certain infinite sequence of distributivity relations (generalized Backelin's relations), one gets an explicit characterization for A to be Koszul ([5], Theorem 2.11). The extra condition is void if $N = 2$, and it is a consequence of the jump of degrees if $N > 2$. Generalized Backelin's relations are satisfied if A is *distributive*. Recall this definition [5]. Denote by $\mathcal{L}(V^{\otimes n})$ the lattice (for inclusion) of the vector subspaces of $V^{\otimes n}$. Infimum and supremum in this lattice are respectively intersection and sum of subspaces.

Definition 2.7 *Let A be N-homogeneous. One says that A is distributive if for any $n \geq N$ ($n \geq N+2$ suffices), the sublattice of $\mathcal{L}(V^{\otimes n})$ generated by the subspaces $V^{\otimes i} \otimes R \otimes V^{\otimes j}$, $i + j + N = n$, is distributive.*

Note that if A is distributive, Koszulity (i.e., the extra condition) is equivalent to relations (2.2) (or to relations (2.4)). A stronger condition (very useful in practice) than distributivity is Bergman's confluence relatively to a basis X of V [5]. For example, the algebras in Theorem 1.1 (Section 1) are confluent for any basis of V. Another example of X-confluence is the case where the relations of A are monomials in X. In this case, the extra condition has a remarkable combinatorial characterization ([5], Proposition 3.8).

Let us discuss Koszulity of Manin products, which is well-known for $N = 2$ [2]. Let A and A' be N-homogeneous algebras on V and V', with spaces of relations R and R', respectively. Their Manin products [7] are the N-homogeneous algebras $A \circ A'$ and $A \bullet A'$, both on $V \otimes V'$, with respective spaces of relations $R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R'$ and $R \otimes R'$. The spaces of relations are in $(V \otimes V')^{\otimes N}$ by the canonical isomorphism $V^{\otimes N} \otimes V'^{\otimes N} \cong (V \otimes V')^{\otimes N}$. Show that if A and A' are distributive, then $A \circ A'$ and $A \bullet A'$ are distributive. In fact (we omit tensor products to simplify), if B (resp. B') is a basis of V^n (resp. $V'^{n'}$) distributing the family V^iRV^j, $i+j+N=n$ (resp. $V'^{i'}R'V'^{j'}$, $i'+j'+N=n'$), BB' is a basis of $V^nV'^{n'}$ distributing the family $V^iRV^jV'^{i'}R'V'^{j'}$, $V^iRV^jV'^{n'}$, $V^nV'^{i'}R'V'^{j'}$, $i+j+N=n$, $i'+j'+N=n'$.

Proposition 2.8 *Let A and A' be N-homogeneous and distributive. If A and A' are Koszul, then $A \bullet A'$ is Koszul.*

Proof. Set $\mathcal{V} = V \otimes V'$. Consider $\mathcal{R}$ as being $R \otimes R'$ naturally viewed in $\mathcal{V}^{\otimes N}$. Omit again tensor products in the notations. It suffices to prove the inclusion

$$(\mathcal{V}^{N-1}\mathcal{R}) \cap (\sum_{i+j=N-1, j\geq 1} \mathcal{V}^i\mathcal{R}\mathcal{V}^j) \subseteq \mathcal{V}^{N-2}\mathcal{R}\mathcal{V}.$$

Identifying $\mathcal{V}^i\mathcal{R}\mathcal{V}^j$ to $V^iRV^jV'^iR'V'^j$ and using distributivity, we see that the left-hand side of the inclusion coincides with

$$\sum_{i+j=N-1, j\geq 1} (V^{N-1}R \cap V^iRV^j)(V'^{N-1}R' \cap V'^iR'V'^j),$$

which is included into

$$(\sum_{i+j=N-1, j\geq 1} V^{N-1}R \cap V^iRV^j)(\sum_{i'+j'=N-1, j'\geq 1} V'^{N-1}R' \cap V'^{i'}R'V'^{j'}).$$

By Koszulity of A and A', the latter space is included into $V^{N-2}RVV'^{N-2}R'V'$ which is identified to $\mathcal{V}^{N-2}\mathcal{R}\mathcal{V}$. ■

For $N=2$, if A is Koszul, then the dual algebra $A^!$ is Koszul. But it is no longer true for $N>2$ [5]. The same phenomenon occurs between the dual Manin products $\circ$ and $\bullet$.

Example 2.9 Here $N=3$, A (resp. A') has two generators x_1, x_2 (resp. x'_1, x'_2) with the relation $x_1^3 = 0$ (resp. $x_1'^3 = 0$). Then A and A' are monomial, and the combinatorial condition for Koszulity holds for them. So A and A' are Koszul. But $A \circ A'$ is not Koszul. In fact $A \circ A'$ is monomial, with set E of monomials formed of the x_1^3m', m' word of length 3, and of the $mx_1'^3$, m word of length 3. However the combinatorial condition for Koszulity does not hold for E: $x_1^3x_2^2x_2'^2x_1'^3$ is such that the first and the last factor of length 3 are in E, but the central factor $x_1^2x_2x'_2x_1'^2$ is not in E.

3 Koszul N-complexes

Throughout this section, A is an N-homogeneous algebra on V, with R as space of relations. Let us define the left Koszul N-complex $K_l(A)$ of A in an elementary way (for a more conceptual way, see [7]), which will be enough for us. The right Koszul N-complex $K_r(A)$ would be defined analogously. Recall that $A^!$ is the N-homogeneous algebra on the dual vector space V^*, with $R^\perp$ as space of relations. Introduce the canonical element $\xi_l = \sum_i e_i \otimes e_i^*$, where (e_i) is any basis of V. Then $K_l(A)$ is the N-complex

$$\cdots \to A \otimes A_n^{!*} \to A \otimes A_{n-1}^{!*} \to \cdots \to A \otimes A_2^{!*} \to A \otimes V \to A\,, \tag{3.1}$$

where the differential $d : A \otimes A_n^{!*} \to A \otimes A_{n-1}^{!*}$ is defined by

$$d(a \otimes \alpha) = (a \otimes \alpha).\xi_l,\ a \in A,\ \alpha \in A_n^{!*}.$$

The action of ξ_l on $A \otimes A_n^{!*}$ is defined by

$$(a \otimes \alpha).\xi_l = \sum_i ae_i \otimes \alpha.e_i^*,$$

and $\alpha.e_i^*$ is the element of $A_{n-1}^{!*}$ defined by $\langle \alpha.e_i^*, f\rangle = \langle \alpha, e_i^* f\rangle$, f in $A_{n-1}^!$. Here ae_i is the product in A, $e_i^* f$ is the product in $A^!$, and $\langle\ ,\ \rangle$ is the natural pairing between a vector space and its dual. More generally, the algebra $A \otimes A^!$ acts on $A \otimes A^{!*}$, where $A^{!*}$ is the graded dual. We have $d^N = 0$ because $\xi_l^N = 0$ in the algebra $A \otimes A^!$.

There are several manners to contract (3.1) into 2-complexes, by putting together alternately p and $N-p$ arrows d, $1 \leq p \leq N-1$ [7]. The adequate contraction, denoted by $K_l'(A)$, is obtained from (3.1) by keeping the arrow at the far right, putting together the $N-1$ previous ones, and continuing alternately :

$$\cdots \to A \otimes A_{2N}^{!*} \to A \otimes A_{N+1}^{!*} \to A \otimes A_N^{!*} \to A \otimes V \to A\,. \tag{3.2}$$

It is easily proved that $K_l'(A)$ is isomorphic to the Koszul 2-complex K introduced in Section 2.

From now on, A is supposed to be generalized Koszul. Our aim is to make explicit the Yoneda algebra (or Ext-algebra) $E(A) = \mathrm{Ext}_A^*(k,k)$. This algebra is graded by the $E(A)_i = \mathrm{Ext}_A^i(k,k)$, $i \geq 0$. In general, its product (the Yoneda product) $\rho : E(A) \times E(A) \to E(A)$ can be defined as follows (2.7 in [4], paragraph 7 in [8]). Let $(P, \delta) \xrightarrow{\epsilon} k$ be a projective resolution of k in A-grMod. An element of $E(A)$ is the class $[f]$ of a cycle f of the complex

$\mathrm{Hom}_A(P,k)$. Then $\rho([f],[g]) = [f \circ \tilde{g}]$, where $\tilde{g}$ is any cycle of the complex $\mathrm{Hom}_A(P,P)$ such that $\epsilon \circ \tilde{g} = g$. Recall also that $\mathrm{Hom}_A(P,P)$ is $\mathbf{Z}$-graded by

$$\mathrm{Hom}_A(P,P)_n = \bigoplus_i \mathrm{Hom}_A(P_i, P_{i+n}),$$

with differential $\tilde{\delta}_n : \mathrm{Hom}_A(P,P)_n \to \mathrm{Hom}_A(P,P)_{n-1}$ defined by the graded commutator $\tilde{\delta}\varphi = [\delta, \varphi]$. In other words, one has

$$\tilde{\delta}_n\varphi(x) = \delta_{i+n}(\varphi(x)) - (-1)^n\varphi(\delta_i(x)),\ \varphi \in \mathrm{Hom}_A(P_i, P_{i+n}),\ x \in P_i.$$

In our situation, take for $P \xrightarrow{\epsilon} k$ the Koszul resolution $K'_l(A)$ with the usual $\epsilon : A \to k$. So $P_i = A \otimes A^{!*}_{n(i)}$, $i \geq 0$. Since the graded A-module P_i is generated in degree $n(i)$, the vector space $\mathrm{Hom}_A(P_i, k)$ is graded and concentrated in degree $-n(i)$ (k is concentrated in degree 0). Therefore $\mathrm{Hom}_A(P,k)$ is a complex of k-grMod whose differential vanishes. Using the canonical isomorphisms

$$\mathrm{Hom}_A(P_i, k) \cong \mathrm{Hom}_k(A^{!*}_{n(i)}, k) \cong A^!_{n(i)},$$

we identify $\mathrm{Hom}_A(P_i, k)$ to $A^!_{n(i)}$ concentrated in degree $-n(i)$. Thus we get

$$E(A)_i \cong A^!_{n(i)},\ i \geq 0.$$

It remains to determine the Yoneda product in this identification. Fix $f \in A^!_{n(i)}$ and $g \in A^!_{n(j)}$. Denote their Yoneda product by $f \bullet g \in A^!_{n(i+j)}$ which has not to be confused with their product fg in the algebra $A^!$. In order to apply the above definition of the Yoneda product, come back to f in $\mathrm{Hom}_A(P_i, k)$ and g in $\mathrm{Hom}_A(P_j, k)$. We have

$$(\mathrm{im}(P_{j+1} \xrightarrow{\delta_{j+1}} P_j))_{n(j+1)} \subseteq (A_1 \text{ or } A_{N-1}) \otimes A^{!*}_{n(j)},$$

according to $n(j+1)$ is $n(j)+1$ or $n(j)+N-1$. For any v in $A^{!*}_{n(j)}$, $g(a \otimes v) = \epsilon(a)g(v)$ vanishes if a is in A_1 or A_{N-1}. Therefore $g \circ \delta_{j+1} = 0$. Denote by Q' the following projective complex of A-grMod :

$$\cdots \xrightarrow{(-1)^j\delta} P_{j+2}(n(j)) \xrightarrow{(-1)^j\delta} P_{j+1}(n(j)) \xrightarrow{(-1)^j\delta} P_j(n(j)) \xrightarrow{(-1)^j\delta} \cdots,$$

in which the last written term is the 0-degree term of the complex. The shift by $n(j)$ makes $g : P_j(n(j)) \to k$ of degree 0, so that we can consider the morphism of complexes $Q' \xrightarrow{g} k$ in A-grMod where k is considered as a complex concentrated in degree 0. By comparison of this morphism with the Koszul resolution $P \xrightarrow{\epsilon} k$, there exists a morphism of complexes $\tilde{g} : Q' \to P$ of A-grMod such that $\epsilon \circ \tilde{g} = g$. Clearly $\tilde{g}$ may be viewed in $\mathrm{Hom}_A(P,P)$, of degree $-j$. Note that $\tilde{g}_{-1} = \tilde{g}_{-2} = \cdots = 0$. On the other hand, the sign $(-1)^j$ in Q' implies that $\tilde{g}$ is a *cycle* of $\mathrm{Hom}_A(P, P)$. So $f \bullet g$ is identified to the element $f \circ \tilde{g}_i$ of $\mathrm{Hom}_A(P_{i+j}, k)$.

A first consequence is the following. The map $\tilde{g}_i : P_{i+j}(n(j)) \to P_i$ is of degree 0. Since $P_{i+j}(n(j))$ is generated in degree $n(i+j) - n(j)$ and P_i in degree $n(i)$, we see that $f \circ \tilde{g}_i = 0$ if $n(i+j) \neq n(i) + n(j)$, i.e., if $N > 2$ with i and j odd, so that $f \bullet g = 0$ in this case. That is really different from fg !

Assume now $n(i+j) = n(i)+n(j)$, i.e., $N = 2$ or ($N > 2$ with i or j even). We are going to give an explicit $\tilde{g}$ by contraction of a natural N-complex morphism. Let Q be the following N-complex whose Q' is a contraction (in order to simplify the notations, the internal shift by $n(j)$ in each term is omitted) :

$$\xrightarrow{(-1)^j d} A\otimes A^{!*}_{n(j)+N} \xrightarrow{(-1)^j d} A\otimes A^{!*}_{n(j)+N-1} \xrightarrow{d} \cdots \xrightarrow{d} A\otimes A^{!*}_{n(j)+1} \xrightarrow{(-1)^j d} A\otimes A^{!*}_{n(j)} \xrightarrow{(-1)^j d}$$

in which the last written term is the 0-degree term of the N-complex. More explicitly, each term $A^{!*}_{n(j)+m}$ for $m \equiv 0 \,(\mathrm{mod} N)$ is surrounded with arrows $(-1)^j d$, and the other arrows are d.

Let us define an N-complex morphism $\tilde{G} : Q \to K_l(A)$ in A-grMod as follows. For each $m \geq 0$, $\tilde{G}_m = 1_A \otimes G_m$ where $G_m : A^{!*}_{n(j)+m} \to A^{!*}_m$ is defined, for any $\alpha \in A^{!*}_{n(j)+m}$, by

$$\begin{aligned} G_m(\alpha) &= g.\alpha && \text{if } m \equiv 0 \ (\mathrm{mod} N) \\ &= (-1)^j g.\alpha && \text{otherwise.} \end{aligned}$$

Here g is seen in $A^!_{n(j)}$. The action of g on $A^{!*}_{n(j)+m}$ is defined by

$$\langle g.\alpha, h\rangle = \langle \alpha, hg\rangle, \ \ h \in A^!_m,$$

where hg is the product in $A^!$. The commutativity of the diagram

$$\begin{array}{ccc} A^{!*}_{n(j)+m+1} & \xrightarrow{\bar{d}} & V \otimes A^{!*}_{n(j)+m} \\ \downarrow G_{m+1} & & \downarrow 1_V \otimes G_m \\ A^{!*}_{m+1} & \xrightarrow{d} & V \otimes A^{!*}_m \end{array} \tag{3.3}$$

in which $\bar{d}$ is d or $(-1)^j d$, is easy to check. In fact, for any α in $A^{!*}_{n(j)+m+1}$, we have the equalities (with the *same* sign $\pm$)

$$d \circ G_{m+1}(\alpha) = \pm \sum_i e_i \otimes ((g.\alpha).e_i^*),$$

$$(1_V \otimes G_m) \circ \bar{d}\,(\alpha) = \pm \sum_i e_i \otimes (g.(\alpha.e_i^*)),$$

and it is immediate that $(g.\alpha).e_i^* = g.(\alpha.e_i^*)$.

As $G_0 = g$, we have $\epsilon \circ \tilde{G} = g$. Next, $\tilde{g}_\ell : P_{\ell+j} \to P_\ell$ is defined for any $\ell \geq 0$ by

$$\begin{aligned} \tilde{g}_\ell &= \tilde{G}_{n(\ell)} && \text{if } \ell \text{ or } j \text{ even} \\ &= \tilde{G}_{n(\ell)} \circ d^{N-2} && \text{if } \ell \text{ and } j \text{ odd.} \end{aligned}$$

Note that $\tilde{g}_\ell$ is well-defined in both cases. In the first case, it is clear because $n(\ell+j) = n(\ell)+n(j)$. In the second case, since $n(\ell+j) = n(\ell)+n(j)+N-2$, $d^{N-2} : P_{\ell+j} \to A \otimes A^{!*}_{n(\ell)+n(j)}$ is followed by $\tilde{G}_{n(\ell)} : A \otimes A^{!*}_{n(\ell)+n(j)} \to P_\ell$.

Clearly $\tilde{g} : Q' \to P$ is a morphism of complexes of A-grMod such that $\epsilon \circ \tilde{g} = g$. So $f \bullet g$ viewed in $A^!_{n(i+j)}$ coincides with $f \circ G_{n(i)}$. Since $n(i) \equiv 0 \ (\mathrm{mod} N)$ if and only if i is even, one has for any α in $A^{!*}_{n(i+j)}$,

$$f \circ G_{n(i)}(\alpha) = f((-1)^{ij} g.\alpha) = (-1)^{ij} \langle fg, \alpha \rangle,$$

hence $f \bullet g = (-1)^{ij} fg$. We have obtained :

Proposition 3.1 *Let f be in $E(A)_i = A^!_{n(i)}$ and g in $E(A)_j = A^!_{n(j)}$. Denote by $f \bullet g$ the Yoneda product, and fg the product in $A^!$. Then we have*

$$\begin{aligned} f \bullet g &= (-1)^{ij} fg && \text{if } N = 2 \text{ or } (N > 2 \text{ with } i \text{ or } j \text{ even}) \\ &= 0 && \text{if } N > 2 \text{ with } i \text{ and } j \text{ odd.} \end{aligned}$$

If $N = 2$, we find $f \bullet g = (-1)^{ij} fg$. Note that if $N > 2$, the factor $(-1)^{ij}$ is always $+1$. Show how this factor agrees in any case with the Koszul-Quillen sign rule. The canonical injection can : $W_m \to V^{\otimes m}$ has a surjective transposed map $\mathrm{can}^* : (V^{\otimes m})^* \to W_m^*$ whose kernel is $I(R^\perp)_m$, once $(V^{\otimes m})^*$ is identified to $(V^*)^{\otimes m}$. Write down this identification for $m = 2$ (the general case will then be clear) : if α, β are in V^*, the element $\alpha \otimes \beta$ of $V^* \otimes V^*$ is identified to the element $a \otimes b \mapsto \alpha(a)\beta(b)$ of $(V \otimes V)^*$. It is worth noticing that Beilinson, Ginzburg, Soergel [3] adopt another convention by taking $\beta(a)\alpha(b)$ instead of $\alpha(a)\beta(b)$. But to recover the Koszul-Quillen tensor product of the linear forms as below with this convention, one should set $E(A) = \mathrm{Ext}^*_{A^\circ}(k, k)$ where A° is the opposite algebra of A.

From can^*, we deduce the isomorphism $A^!_m \cong W_m^*$, which sends the class $\bar{u}$ of an element u of $(V^{\otimes m})^*$ to its restriction $u|_{W_m}$ to W_m. Then the product in the algebra $A^!$ of $\bar{u} \in A^!_m$ and $\bar{v} \in A^!_n$ is sent to $(u \otimes v)|_{W_{m+n}}$, with our convention on $u \otimes v$ considered in $(V^{\otimes(m+n)})^*$. If $m = n(i)$ and $n = n(j)$, the Yoneda product of $\bar{u} \in E(A)_i$ and $\bar{v} \in E(A)_j$ is sent to $(-1)^{ij}(u \otimes v)|_{W_{m+n}}$ when $n(i+j) = n(i) + n(j)$. But the Koszul-Quillen tensor product of the linear forms u and v is also given by

$$\langle u \overset{K-Q}{\otimes} v, a \otimes b \rangle = (-1)^{ij} \langle u, a \rangle \langle v, b \rangle, \ a \in W_{n(i)}, \ b \in W_{n(j)},$$

since v has degree j, and a has degree i.

4 Bimodule Koszul N-complexes

Throughout this section, A is an N-homogeneous algebra on V, with R as space of relations. We need the two canonical elements $\xi_l = \sum_i e_i \otimes e_i^*$ and $\xi_r = \sum_i e_i^* \otimes e_i$. In the category A-grMod-A of graded A-A-bimodules (with morphisms of degree 0), define the following N-differentials :

$$d_l,\ d_r : \quad A \otimes A_n^{!*} \otimes A \to A \otimes A_{n-1}^{!*} \otimes A,\ n \geq 0,$$

$$d_l(a \otimes \alpha \otimes b) = ((a \otimes \alpha).\xi_l) \otimes b = \sum_i ae_i \otimes \alpha.e_i^* \otimes b,$$

$$d_r(a \otimes \alpha \otimes b) = a \otimes (\xi_r.(\alpha \otimes b)) = \sum_i a \otimes e_i^*.\alpha \otimes e_i b.$$

The left and right actions $\alpha.e_i^*$, $e_i^*.\alpha$ have already been defined. Since these actions commute, d_l and d_r commute. Fix a primitive N-root of unity q (we enlarge the ground field k if necessary). Define $d :\ A \otimes A_n^{!*} \otimes A \to A \otimes A_{n-1}^{!*} \otimes A$ by $d = d_l - q^{n-1} d_r$. Explicitly we have

$$\cdots \xrightarrow{d_l - d_r} A \otimes A_N^{!*} \otimes A \xrightarrow{d_l - q^{N-1} d_r} \cdots \xrightarrow{d_l - q d_r} A \otimes V \otimes A \xrightarrow{d_l - d_r} A \otimes A. \qquad (4.1)$$

Since $\prod_{n=0}^{n=N-1}(d_l - q^n d_r) = d_l^N - d_r^N = 0$, (4.1) is an N-complex of A-grMod-A, called the *bimodule Koszul N-complex* of A and denoted by $K_{l-r}(A)$.

The *bimodule Koszul 2-complex* $K'_{l-r}(A)$ (see the reference [6] which puts right the faulty definition in [5]) is the adequate contraction of $K_{l-r}(A)$, defined similarly from (4.1) by keeping the arrow at the far right, putting together the $N-1$ previous ones, and continuing alternately :

$$\cdots \xrightarrow{d^{N-1}} A \otimes A_{N+1}^{!*} \otimes A \xrightarrow{d} A \otimes A_N^{!*} \otimes A \xrightarrow{d^{N-1}} A \otimes V \otimes A \xrightarrow{d} A \otimes A. \qquad (4.2)$$

Here $d = d_l - d_r$ and $d^{N-1} = d_l^{N-1} + d_l^{N-2} d_r + \cdots + d_l d_r^{N-2} + d_r^{N-1}$, so that $K'_{l-r}(A)$ makes sense on any ground field.

Obviously, $K'_{l-r}(A) \otimes_A k \cong K'_l(A)$ (as 2-complexes). In order to transfer acyclicity from $K'_l(A)$ to $K'_{l-r}(A)$, we need the following result (which also corrects a faulty version in [5]). As we shall see, the proof of this result lies upon well-known "filtration-gradation" techniques [17].

Proposition 4.1 *Let A be a connected graded algebra. Assume that the complex of A-grMod*

$$L \xrightarrow{f} M \xrightarrow{g} N \qquad (4.3)$$

is formed of graded-free modules, with L bounded below. Then this complex is exact if the following is exact :

$$k \otimes_A L \xrightarrow{1_k \otimes_A f} k \otimes_A M \xrightarrow{1_k \otimes_A g} k \otimes_A N. \qquad (4.4)$$

Proof. Write down the graded-free modules of (4.3) as $L = A \otimes E$, $M = A \otimes F$, $N = A \otimes G$, where E, F, G are graded k-vector spaces (tensor products over k are denoted without subscript). As L is bounded below, there exists an integer s such that $E_i = 0$ if $i < s$. Consider A as a filtered k-algebra by the

$$A_{\geq p} = A_p \oplus A_{p+1} \oplus \cdots, \; p \geq 0.$$

The filtration is non-increasing, exhaustive, separated. Then M is a filtered A-module by the $A_{\geq p} \otimes F$, $p \geq 0$, and this filtration is non-increasing, exhaustive, separated. Idem for L and N. Since $A_{\geq p} = \bigoplus_n (A_{\geq p} \cap A_n)$, the filtration of A is compatible with its gradation. It is straightforward to verify that

$$\bigoplus_{n \in \mathbf{Z}} (A_{\geq p} \otimes F) \cap (A \otimes F)_n = A_{\geq p} \otimes F,$$

which means that the filtration of M is compatible with its gradation. We have the same with L and N. Note that $(A_{\geq p} \otimes E) \cap (A \otimes E)_n = 0$ if $p > n - s$. Therefore the filtration of L induces on each homogeneous component L_n a *finite* (hence complete) filtration. We can say that the filtration of L is graded-complete.

The complex (4.4) is in k-grMod. Identifying naturally $k \otimes_A L$ to E, ..., one sees that (4.4) is identified to the k-grMod complex

$$E \xrightarrow{\tilde{f}} F \xrightarrow{\tilde{g}} G. \tag{4.5}$$

Here $\tilde{f}$ is defined as the composite $E \hookrightarrow L \xrightarrow{f} M \twoheadrightarrow F$, where $E \hookrightarrow L$ (resp. $M \twoheadrightarrow F$) is the canonical injection (resp. projection). Idem for $\tilde{g}$.

Since A is $\mathbf{N}$-graded, f sends E to $A_{\geq 0} \otimes F$. Thus by linearity, f sends $A_{\geq p} \otimes E$ to $A_{\geq p} \otimes F$. So f is compatible with the filtrations of L and M. Similarly g is compatible with the filtrations of M and N. By $A_{\geq p}/A_{\geq p+1} \cong A_p$, the associated graded algebra $\mathrm{gr}(A)$ is naturally isomorphic to A, and $\mathrm{gr}(L) \cong A \otimes E$, $\mathrm{gr}(M) \cong A \otimes F$, $\mathrm{gr}(N) \cong A \otimes G$. Moreover, $\mathrm{gr}(f)$ is identified to $1_A \otimes \tilde{f} : A \otimes E \to A \otimes F$. Similarly, $\mathrm{gr}(g)$ is identified to $1_A \otimes \tilde{g} : A \otimes F \to A \otimes G$.

Assume (4.4) exact. Applying the exact functor $A \otimes -$ to the exact complex (4.5) and using the above identifications, one sees that the complex

$$\mathrm{gr}(L) \xrightarrow{\mathrm{gr}(f)} \mathrm{gr}(M) \xrightarrow{\mathrm{gr}(g)} \mathrm{gr}(N)$$

is exact. One concludes thanks to the following proposition, which is the graded analogue of a classical result (Theorem III.3 in [17]) and whose proof is left to the reader. ■

Proposition 4.2 *Let R be a k-algebra endowed with a non-increasing $\mathbf{Z}$-filtration F. Let L, M, N be three filtered R-modules. The filtration of these modules is also denoted by F. Assume that the k-algebra R is $\mathbf{Z}$-graded and that the R-modules L, M, N are $\mathbf{Z}$-graded. Assume that the filtrations and the gradations are compatible. Consider the following complex of filtered R-modules*

$$L \xrightarrow{f} M \xrightarrow{g} N, \tag{4.6}$$

such that f and g are homogeneous of degree 0 (for the gradations). If the filtration of L is complete in R-grMod (i.e., graded-complete) and if the filtration of M is exhaustive and separated, the exactness of the $gr_F(R)$-grMod complex

$$\mathrm{gr}_F(L) \xrightarrow{\mathrm{gr}_F(f)} \mathrm{gr}_F(M) \xrightarrow{\mathrm{gr}_F(g)} \mathrm{gr}_F(N)$$

implies the exactness of (4.6).

Denote by $\mu : A \otimes A \to A$ the product of A. For $\alpha \in V$, we have

$$d_l(a \otimes \alpha \otimes b) = \sum_i \langle \alpha, e_i^* \rangle a e_i \otimes b,$$

$$d_r(a \otimes \alpha \otimes b) = \sum_i \langle \alpha, e_i^* \rangle a \otimes e_i b,$$

hence $\mu \circ (d_l - d_r) = 0$. Since the beginning $A \otimes R \to A \otimes V \to A \xrightarrow{\epsilon} k \to 0$ of the Koszul 2-complex (with the augmentation ϵ) is always exact, Proposition 4.1 (or rather, its right modules version) implies immediately :

Proposition 4.3 *The following complex is exact*

$$A \otimes A_N^{!*} \otimes A \xrightarrow{d^{N-1}} A \otimes V \otimes A \xrightarrow{d} A \otimes A \xrightarrow{\mu} A \to 0.$$

We can now state the bimodule characterization for generalized Koszulity.

Theorem 4.4 *The N-homogeneous algebra A is Koszul if and only if the following complex is exact :*

$$K'_{l-r}(A) \xrightarrow{\mu} A \to 0. \tag{4.7}$$

Proof. Assume A Koszul. Applying the functor $- \otimes_A k$ to (4.7), we get the Koszul 2-resolution of the trivial module k in A-grMod, and we conclude by Proposition 4.1. Conversely, assume (4.7) exact. Viewing (4.7) as a projective resolution of A in grMod-A and comparing it with the identity map of A, one sees that there are in grMod-A two morphisms of resolutions $f : K'_{l-r}(A) \to A$

and $g : A \to K'_{l-r}(A)$ (where A is considered as a complex concentrated in degree 0), and a chain homotopy $s : K'_{l-r}(A) \to K'_{l-r}(A)$ such that

$$1_{K'_{l-r}(A)} - gf = sd + ds.$$

Clearly, $1_{K'_{l-r}(A)} = sd + ds$ in degree > 0. Applying $- \otimes_A k$, we draw

$$1_{K'_l(A)} = (s \otimes_A 1_k)d_l + d_l(s \otimes_A 1_k)$$

in degree > 0, which implies that $K'_l(A)$ is exact in degree > 0. ■

Assume A Koszul. Each bimodule of the projective resolution (4.7) is generated in only one degree, and these degrees are increasing. Thus the resolution is minimal in A-grMod-A. It is called the *bimodule Koszul resolution* of A. By minimality, its length (in $\mathbf{N} \cup \{\infty\}$) is the Hochschild dimension of A (the Hochschild dimension of A is defined as being the projective dimension of A in A-grMod-A, see just below). Joining this with Corollary 2.6, we get :

Theorem 4.5 *Assume that the N-homogeneous algebra A is Koszul. The Hochschild dimension of A coincides with the global dimension of A.*

For the convenience of the reader, let us recall some basic facts on Hochschild dimension (simply called dimension by Cartan and Eilenberg [9]). Let k be a commutative ring and let A be an associative k-algebra with unit. Set $A^e = A \otimes A^\circ$ (tensor product over k). Each A-A-bimodule M is a left (right) A^e-module M_l (M_r) for the action $(a \otimes b).m = amb$ ($m.(a \otimes b) = bma$). Since $(A^e)^\circ$ is isomorphic to A^e by the flip, it is clear that $\mathrm{pd}_{A^e} M_l = \mathrm{pd}_{A^e} M_r$, which will be denoted by $\mathrm{pd}_{A^e} M$.

Definition 4.6 $\mathrm{pd}_{A^e} A$ *is called the Hochschild dimension of A.*

Using $Ext^i_{A^e}(A, M) = HH^i(A, M)$, the terminology comes from

$$\mathrm{pd}_{A^e} A = \sup\{i \in \mathbf{N} \cup \{\infty\} \,; \text{there exists } M \text{ such that } HH^i(A, M) \neq 0\,\},$$

with the convention $HH^\infty(A, M) = 0$. If k is a field, one has inequalities

$$\mathrm{l.gl.\,dim}\, A \leq \mathrm{pd}_{A^e} A, \quad \mathrm{r.gl.\,dim}\, A \leq \mathrm{pd}_{A^e} A,$$

which turn out to be equalities in some important circumstances, as group algebras, or enveloping algebras of Lie algebras [9]. So Theorem 4.5 provides another circumstance for equalities. On the other hand, suppose that k is a field and A is connected graded. For any bounded below left A-module M, $\mathrm{pd}_A M$ coincides with $\mathrm{gr.pd}_A M$ (i.e., the projective dimension of M computed in the category A-grMod), and thus coincides with the length of a minimal projective resolution of M in A-grMod [17]. In particular, this holds for the algebra A^e instead of A, and the bimodule A instead of M. We have also used the well-known fact that the (left and right) global dimension of A is equal to $\mathrm{pd}_A({}_A k)$ [17].

5 AS-Gorenstein algebras

Definition 5.1 *Let A be a connected graded k-algebra of finite global dimension D. One says that A is AS-Gorenstein if* $\mathrm{Ext}^i_A(k, A) = 0$ *for $i \neq D$, and* $\mathrm{Ext}^D_A(k, A) = k$.

In this text, Ext will be always understood as a functor of the category A-Mod of left A-modules. So AS-Gorenstein means left AS-Gorenstein in this definition. Actually, dealing with finite global dimension algebras, right AS-Gorenstein is equivalent to left AS-Gorenstein ([21], Proposition 3.1). We have a first link with Koszul property in low dimension.

Proposition 5.2 *Let A be a connected graded k-algebra. The space V of generators is concentrated in degree 1, and the space R of relations lives in degrees ≥ 2. Assume that the global dimension D of A is 2 or 3, and that A is AS-Gorenstein. Then A is N-homogeneous and Koszul, with $N = 2$ if $D = 2$, and $N \geq 2$ if $D = 3$.*

Proof. We know that $\mathrm{Ext}^0_A(k, k) = k$ is concentrated in degree 0, $\mathrm{Ext}^1_A(k; k) = V^*$ is concentrated in degree -1, $\mathrm{Ext}^2_A(k, k) = R^*$ lives in degrees ≤ -2. Moreover $\mathrm{Ext}^D_A(k, k) = k$ is necessarily concentrated in one degree, which will be denoted by $-e$. Proposition 2.1 shows that $e \geq D$. AS-Gorenstein property implies that there exist isomorphisms of graded k-vector spaces [19]

$$\mathrm{Ext}^i_A(k, k) \cong (\mathrm{Ext}^{D-i}_A(k(e), k))^*, \ 0 \leq i \leq D. \tag{5.1}$$

Assume $D = 2$. Then (5.1) for $i = 1$ leads to $e = 2$, so that R is concentrated in degree 2 with $\dim R = 1$. Koszulity is clear. Assume now $D = 3$. Then (5.1) for $i = 1$ implies that R is concentrated in degree $e - 1$, with $\dim R = \dim V$. Hence A is N-homogeneous, $N = e - 1$. Since $\mathrm{Ext}^3_A(k, k)$ is concentrated in degree $-N - 1$, A is Koszul. ■

This result is implicit in the paper of Artin and Schelter [1], and it is a starting point for their classification. Examples for $D = 2$ or 3 with $N = 2$, and for $D = N = 3$, are in [1]. Examples in characteristic 0 for $D = 3$ with any $N > 2$ are given by Theorem 1.1 when $q = 1$. Note that if $D = 4$ in the hypotheses of the proposition, A can be non-homogeneous ([1], Proposition 1.20), but if A is N-homogeneous, then A is Koszul (use again (5.1)) and $N = 2$ by Proposition 5.3 below.

We want to investigate AS-Gorenstein property for generalized Koszul algebras. If A is a Koszul N-homogeneous algebra, then the right A-module $\mathrm{Ext}^i_A(k, A)$ is naturally identified to the right A-module $H^i(\mathrm{Hom}_A(K'_l(A), A))$, $i \geq 0$. As Manin showed it in the quadratic case [15], there is a nice way to express the grMod-A complex $\mathrm{Hom}_A(K'_l(A), A)$ [7]. Conformally to our point of view, we begin to describe an N-complex. For any N-homogeneous algebra A, we introduce the N-complex $L_r(A)$ as being $A^! \otimes A$ graded by the $A^!_n \otimes A$, $n \geq 0$, and endowed with the N-differential $\xi_r \cdot$ (left multiplication by $\xi_r = \sum_i e_i^* \otimes e_i$). Viewing $A^!_n$ as concentrated in degree $-n$, $L_r(A)$ is a cochain N-complex of grMod-A. Then, it is easy to find a natural isomorphism of cochain N-complexes of grMod-A between $L_r(A)$ and $\mathrm{Hom}_A(K_l(A), A)$ [7]. The N-complex $L_l(A)$ is defined analogously.

Let us write down $L_r(A)$ more explicitly :

$$A \xrightarrow{\xi_r \cdot} V^* \otimes A \xrightarrow{\xi_r \cdot} \cdots \xrightarrow{\xi_r \cdot} A^!_n \otimes A \xrightarrow{\xi_r \cdot} A^!_{n+1} \otimes A \xrightarrow{\xi_r \cdot} \cdots \tag{5.2}$$

If we keep the first arrow, put together the $N-1$ following ones, and continue alternately, we define the adequate contraction $L'_r(A)$:

$$A \xrightarrow{\xi_r \cdot} V^* \otimes A \xrightarrow{\xi_r^{N-1} \cdot} A^!_N \otimes A \xrightarrow{\xi_r \cdot} A^!_{N+1} \otimes A \xrightarrow{\xi_r^{N-1} \cdot} \cdots \tag{5.3}$$

Then the grMod-A 2-complexes $L'_r(A)$ and $\mathrm{Hom}_A(K'_l(A), A)$ are naturally isomorphic. Comparing $L'_r(A)$ with $K'_r(A)$, we are going to give a convenient criterion for generalized Koszul algebras to be AS-Gorenstein.

We work in an intermediate class of algebras formed of the Koszul N-homogeneous algebras A with global dimension $D < \infty$ and $\mathrm{Ext}^D_A(k, A) = k$. Let A be such an algebra. Since $\mathrm{Ext}^D_A(k, A) = H^D(L'_r(A))$, there is a linear map $\epsilon' : A^!_{n(D)} \otimes A \to k$ such that the small complex

$$\longrightarrow A^!_{n(D)} \otimes A \xrightarrow{\epsilon'} k \longrightarrow 0,$$

in which the first arrow is $\xi_r \cdot$ or $\xi_r^{N-1} \cdot$, is exact. The image of the first arrow is contained in $A^!_{n(D)} \otimes A_{\geq 1}$ or $A^!_{n(D)} \otimes A_{\geq N-1}$, so ϵ' is injective on $A^!_{n(D)} \otimes A_0$ or $A^!_{n(D)} \otimes (A_0 \oplus \cdots \oplus A_{N-2})$. Therefore, $\dim(A^!_{n(D)}) = 1$ and, if $N > 2$, the second case does not occur i.e., D is odd. We have proved :

Proposition 5.3 *Assume that A is a Koszul N-homogeneous algebra with global dimension $D < \infty$ and $\mathrm{Ext}^D_A(k, A) = k$. Then $\dim(A^!_{n(D)}) = 1$ and, if $N > 2$, D is odd. Moreover the following additivity holds*

$$n(i) + n(D-i) = n(D),\ 0 \leq i \leq D.$$

Let us keep A in our intermediate class of algebras. Since A is Koszul of global dimension D, we have $A^{!*}_{n(D)+N-1} = 0$, hence $A^!_i = 0$ for $i \geq n(D)+N-1$ (Warning! $A^!_i \neq 0$ can occur if $n(D) < i < n(D)+N-1$: take $N > 2$, $R = 0$ and $\dim V = 1$). So the cochain complex $L'_r(A)$ is the following

$$A \xrightarrow{\xi_r \cdot} V^* \otimes A \xrightarrow{\xi_r^{N-1} \cdot} A^!_N \otimes A \xrightarrow{\xi_r \cdot} \cdots \xrightarrow{\xi_r^{N-1} \cdot} A^!_{n(D-1)} \otimes A \xrightarrow{\xi_r \cdot} A^!_{n(D)} \otimes A \to 0. \tag{5.4}$$

Denote by $L'_r(A)^{ch}$ the naturally associated chain complex, i.e., in (5.4) the complex degrees are successively $0, -1, -2, \ldots, -D+1, -D$. For comparing with the chain complex $K'_r(A)$, we need the chain complex $L'_r(A)^{ch}[-D]$ where the successive complex degrees are $D, D-1, D-2, \ldots, 1, 0$. It is clear that $L'_r(A)^{ch}[-D]$ is also the adequate contraction $L_r(A)^{ch}[-n(D)]'$. Here [ ] denotes the shift of N-complexes.

Fix a generator u of $A^!_{n(D)}$ and choose $\epsilon' = \epsilon_u$, where ϵ_u is defined by $\epsilon_u(u \otimes 1) = 1$. Denoting as usual by ( ) the shift in grMod-A, ϵ_u is a morphism of grMod-A complexes of $L_r(A)^{ch}[-n(D)](-n(D))'$ to k (recall that $A^!_i$ is concentrated in degree $-i$). We can also consider ϵ_u as a morphism of grMod-A N-complexes from $L_r(A)^{ch}[-n(D)](-n(D))$ to k.

Introduce now the left $A^! \otimes A$-linear map $\Phi : A^! \otimes A \to A^{!*} \otimes A$ by $\Phi = \bar{\Phi} \otimes 1_A$, $\bar{\Phi} : A^! \to A^{!*}$, and $\bar{\Phi}(1) = u^*$. Here u^* is the basis of $A^{!*}_{n(D)}$ dual to the basis u of $A^!_{n(D)}$. Clearly, Φ is graded by the $\Phi_i = \bar{\Phi}_i \otimes 1_A$, $\bar{\Phi}_i : A^!_{n(D)-i} \to A^{!*}_i$. The N-differentials of $L_r(A)$ and $K_r(A)$ are both left multiplications by ξ_r. Therefore, taking into account the degrees, one sees that

$$\Phi : L_r(A)^{ch}[-n(D)](-n(D)) \to K_r(A)$$

is a morphism of grMod-A N-complexes. Since $\bar{\Phi}_0 : A^!_{n(D)} \to k$ coincides with u^*, one has $\epsilon \circ \Phi = \epsilon_u$. The adequate contraction Φ' of Φ is the following morphism of grMod-A complexes:

$$\Phi' : L_r(A)^{ch}[-n(D)](-n(D))' \to K'_r(A),$$

formed by $\Phi_0, \Phi_1, \Phi_N, \Phi_{N+1}, \ldots, \Phi_{n(D)-1}, \Phi_{n(D)}$. One has $\epsilon \circ \Phi' = \epsilon_u$.

Then A is AS-Gorenstein if and only if $H^i(L'_r(A)) = 0$ for any $i > 0$, i.e., $\epsilon_u : L_r(A)^{ch}[-n(D)](-n(D))' \to k$ is a resolution, necessarily minimal projective in grMod-A (minimality comes from the fact that each module of the projective resolution is generated in only one degree, and the degrees are increasing). Knowing that a morphism between two minimal projective resolutions is an isomorphism, we have obtained the following criterion.

Theorem 5.4 *Assume that A is a Koszul N-homogeneous algebra with global dimension $D < \infty$ and $\mathrm{Ext}^D_A(k, A) = k$. Fix a generator u of the vector space $A^!_{n(D)}$. For $0 \leq i \leq D$, define the linear maps $\bar{\Phi}_{n(i)} : A^!_{n(D-i)} \to A^{!*}_{n(i)}$ by $f \mapsto f.u^*$. Then A is AS-Gorenstein if and only if these maps are bijective.*

Remark 5.5 $\bar{\Phi}_0 : u \mapsto 1$ and $\bar{\Phi}_{n(D)} : 1 \mapsto u^*$ are bijective. On the other hand, the commutative diagram with exact lines

$$\begin{array}{ccccc} A^!_{n(D-1)} \otimes A & \xrightarrow{\xi\cdot} & A^!_{n(D)} \otimes A & \xrightarrow{\epsilon_u} & k \\ \downarrow \Phi_1 & & \downarrow \Phi_0 & & \downarrow 1_k \\ V \otimes A & \xrightarrow{d} & A & \xrightarrow{\epsilon} & k \end{array} \tag{5.5}$$

shows that $d(\Phi_1(A^!_{n(D-1)} \otimes A)) = A_{\geq 1} = d(V \otimes A)$. As the map $d : V \otimes A \to d(V \otimes A)$ is essential, we have $\Phi_1(A^!_{n(D-1)} \otimes A) = V \otimes A$. Thus $\bar{\Phi}_1$ is surjective.

Remark 5.6 Since $\Phi_{n(D)}$ and $d : A^{!*}_{n(D)} \otimes A \to A^{!*}_{n(D-1)} \otimes A$ are injective (the second one by Koszulity), $\xi\cdot : A \to A^!_1 \otimes A$ is injective. Thus $\mathrm{Hom}_A(k, A) = 0$.

Remark 5.7 $\bar{\Phi} : A^! \to A^{!*}$ is bijective if and only if the finite dimensional algebra $A^!$ is Frobenius and the highest i such that $A^!_i \neq 0$ is $i = n(D)$ (see [19] for a detailed account on graded Frobenius algebras). If we take $N > 2$, $\dim V = 1$, $R = 0$, then A is N-homogeneous, Koszul, AS-Gorenstein, of global dimension 1, and $A^!$ is Frobenius with $\bar{\Phi}$ non bijective. Note also that if $A^!$ is Frobenius and $i > n(D)$, then $N > 2$, $\dim V = 1$ and $R = 0$.

Corollary 5.8 *Assume that A is a Koszul N-homogeneous algebra with global dimension 2. If $\mathrm{Ext}^2_A(k, A) = k$, then A is AS-Gorenstein.*

Proof. Immediate from Theorem 5.4 and Remark 5.5 since $\bar{\Phi}_0$, $\bar{\Phi}_N$ are bijective, and $\bar{\Phi}_1 : V^* \to V$ is surjective. ■

Example 5.9 A has $n \geq 2$ generators $x_1, \ldots, x_n$, and one relation

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n - x_nx_1 = 0.$$

For $n = 2$, $A = k[x_1, x_2]$. A is X-confluent for the basis $X : x_1 < x_2 < \cdots < x_n$ since there is no ambiguous word of length 3 (the only nonreduced word of length 2 is x_nx_1). So A is Koszul. A basis of A is formed of words not containing x_nx_1 as a factor. A basis of $R^\perp$ is formed of words $x_i^*x_j^*$ such that x_ix_j is not in the relation of A, and of binomials $x_i^*x_{i+1}^* + x_n^*x_1^*$, $1 \leq i \leq n-1$. The quadratic algebra $A^!$ is X^*-confluent for $X^* : x_1^* > x_2^* > \cdots > x_n^*$. A basis of $A^!_2$ is $x_n^*x_1^*$, and $A^!_i = 0$ for $i > 2$. Thus $\mathrm{gl.dim}\, A = 2$, and the Hilbert series of the graded algebra A is the following:

$$h_A(t) = \frac{1}{1 - nt + t^2}\,.$$

For $n > 2$, the latter has a pole between 0 and 1, so $\mathrm{GK.dim}\, A = \infty$. A theorem by Stephenson and Zhang [21] implies that A is not (left or right) noetherian in this case. Let us show that A is AS-Gorenstein by using Corollary 5.8. It suffices to check that the complex

$$A^!_1 \otimes A \xrightarrow{\xi\cdot} A^!_2 \otimes A \xrightarrow{\epsilon_u} k$$

is exact. Here $\xi = \sum_{i=1}^n x_i^* \otimes x_i$ and $u = x_n^*x_1^*$. From the relations of $A^!$, it is easy to prove that $\mathrm{im}(\xi\cdot)$ is the set of the elements

$$x_n^*x_1^* \otimes (x_na_1 - x_1a_2 - \cdots - x_{n-1}a_n),\ a_1, \ldots, a_n \text{ in } A.$$

Hence $\mathrm{im}(\xi\cdot) = A^!_2 \otimes A_{\geq 1}$ which is $\ker(\epsilon_u)$.

The next corollary is Theorem 1.1 of the introduction. It would be much harder to prove it by showing directly that $L'_r(A)$ is exact. Even if $n = N = 3$, the computation of the $H^i(L'_r(A))$ is rather involved.

Corollary 5.10 *For $n \geq N \geq 2$, consider the N-homogeneous algebra A over k of characteristic 0, with generators $x_1, \ldots, x_n$. The relations of A are the antisymmetrizers of degree N, i.e., are the following*

$$\sum_{\sigma \in \mathbf{S}_N} \mathrm{sgn}(\sigma)\ x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(N)}} = 0\,,\ \ 1 \leq i_1 < \cdots < i_N \leq n\,,$$

where $\mathbf{S}_N$ denotes the permutation group of $\{1, \ldots, N\}$. We know [5] that A is generalized Koszul, of finite global dimension. Then A is AS-Gorenstein if and only if either $N = 2$ or ($N > 2$ and $n = Nq + 1$ for some integer $q \geq 1$).

Proof. Assume that A is AS-Gorenstein with $N > 2$. Using Proposition 5.3, D is odd. Write down $D = 2q + 1$. According to [5], $n = Nq + r$ with $1 \leq r \leq N - 1$. Theorem 5.4 shows that $A^!_{n(D-i)}$ has the same dimension as $A^!_{n(i)}$, $0 \leq i \leq D$. Recall [5] that for $N \leq m \leq n$, $A^!_m$ has a basis formed of the words $x^*_{i_1} \cdots x^*_{i_m}$ with $i_1 > \cdots > i_m$ (and $A^!_m = 0$ if $m > n$). Then

$$\left(\begin{array}{c} n \\ n(D) - n(i) \end{array} \right) = \left(\begin{array}{c} n \\ n(i) \end{array} \right),$$

which implies $n(D) = 2n(i)$ or $n(D) = n$. The first equality does not hold for any i, so $n(D) = n$. As $n(D) = Nq + 1$, one has $r = 1$.

Conversely, assume $n = Nq + 1$ for some integer $q \geq 1$, and $N \geq 2$ (if $N = 2$, the following proof works for n arbitrary, but A is just a polynomial algebra in this case!). By [5], $D = 2q + 1$, hence $n(D) = n$. Recall also that a basis of $R^\perp$ is formed of words $x^*_{j_1} \ldots x^*_{j_N}$ having two identical letters, and of binomials $x^*_{j_1} \ldots x^*_{j_N} - s\, x^*_{i_1} \ldots x^*_{i_N}$ where $j_1, \ldots, j_N$ are distinct, not in the decreasing order, $\{i_1, \ldots, i_N\} = \{j_1, \ldots, j_N\}$ with $i_1 > \cdots > i_N$, and s is the sign of the permutation $i_1 \mapsto j_1, \ldots, i_N \mapsto j_N$. First prove $\mathrm{Ext}^D_A(k, A) = k$, i.e., the complex

$$A^!_{n-1} \otimes A \xrightarrow{\xi\cdot} A^!_n \otimes A \xrightarrow{\epsilon_u} k$$

is exact. Here $\xi = \sum_{i=1}^n x^*_i \otimes x_i$ and $u = x^*_n \cdots x^*_1$. Clearly $\ker(\epsilon_u) = A^!_n \otimes A_{\geq 1}$. On the other hand, for $1 \leq i \leq n$, set $u_i = x^*_n \cdots \vee_i \cdots x^*_1$ where $\vee_i$ means that x^*_i is removed. Then the u_i form the basis of $A^!_{n-1}$, and we have $(\xi\cdot)(u_i \otimes 1) = \pm u \otimes x_i$. Therefore $(\xi\cdot)(A^!_{n-1} \otimes A_0) = A^!_n \otimes A_1$. Since $A_1.A = A_{\geq 1}$, we conclude that $(\xi\cdot)(A^!_{n-1} \otimes A) = A^!_n \otimes A_{\geq 1}$.

Using Theorem 5.4, it remains to prove that $\bar{\Phi}_{n(i)}$ is bijective for $0 \leq i \leq D$. Actually, we shall prove that $\bar{\Phi}_i$ is bijective for $0 \leq i \leq n$ such that i and $n-i$ are different from $2, 3, \ldots, N-1$. In this case, one has already $\dim A^!_{n-i} = \dim A^{!*}_i$. For $j_1 > \cdots > j_i$, the words $f_{j_1 \ldots j_i} = x^*_n \cdots \vee_{j_1} \cdots \vee_{j_2} \cdots \vee_{j_i} \cdots x^*_1$, in which the letters $x^*_{j_1}, \ldots, x^*_{j_i}$ are removed, form the basis of $A^!_{n-i}$. For $k_1 > \cdots > k_i$, one has

$$\langle x^*_{k_1} \ldots x^*_{k_i}\, ,\, \bar{\Phi}_i(f_{j_1 \ldots j_i}) \rangle = \langle x^*_{k_1} \ldots x^*_{k_i}\, f_{j_1 \ldots j_i}\, ,\, u^* \rangle,$$

which amounts to ± 1 if $\{k_1, \ldots, k_i\} = \{j_1, \ldots, j_i\}$, and 0 otherwise. Therefore $\bar{\Phi}_i(f_{j_1 \ldots j_i})$ coincides up to a sign with the element $(x^*_{j_1} \ldots x^*_{j_i})^*$ of the basis which is dual to the basis of $A^!_i$. Thus $\bar{\Phi}_i$ is bijective. ■

Remark 5.11 If in the previous proof, one has $N > 2$, $n = Nq + 1$ and $2 \leq i \leq N-1$, then $\bar{\Phi}_i$ is not bijective. Hence $A^!$ is not Frobenius (Remark 5.7). Note also that for $N > 2$, an algebra A as in Corollary 5.10 contains the free algebra in x_1 and x_2, so GK.dim$A = \infty$ and A is not (left or right) noetherian.

The next corollary is Theorem 1.2 of the introduction, generalizing a result by Smith. Our proof uses the Yoneda product computed in Section 3.

Corollary 5.12 *Let A be N-homogeneous, generalized Koszul, of finite global dimension D. Denote by $E(A)$ the Yoneda algebra of A. Then A is AS-Gorenstein if and only if the finite dimensional algebra $E(A)$ is Frobenius.*

Proof. Assume that A is AS-Gorenstein. Choose a generator u of $A^!_{n(D)}$. First we define an arrow $\varphi : E(A) \to E(A)^*(-D)$ in $E(A)$-grMod as follows. For $0 \leq i \leq D$, $E(A)_i = A^!_{n(i)}$. The maps $\varphi_i : E(A)_i \to (E(A)_{D-i})^* = E(A)^*(-D)_i$ defined by

$$\varphi_i = (-1)^{i.(D-i)} \bar{\Phi}_{n(D-i)}$$

provide an arrow $\varphi : E(A) \to E(A)^*(-D)$ in k-grMod. It remains to prove the left $E(A)$-linearity. Here $E(A)^*$ is naturally a left $E(A)$-module. Denote this action by $\bullet$ as the Yoneda product. Take $f \in E(A)_i$ and $\alpha \in E(A)^*_{i+j}$. Proposition 3.1 shows that $f \bullet \alpha$ vanishes if $N > 2$ with i and j odd, and otherwise, $f \bullet \alpha = (-1)^{i.j} f \cdot \alpha$ where $f \cdot \alpha$ denotes the natural left action of $A^!$ on $A^{!*}$. For $f \in E(A)_i$ and $g \in E(A)_j$, one has to check the equality $\varphi(f \bullet g) = f \bullet \varphi(g)$. It is clear if $N > 2$ with i and j odd, since $f \bullet g = 0 = f \bullet \varphi(g)$ in this case (remember that D is odd!). Assume now $n(i+j) = n(i) + n(j)$. Using Proposition 3.1 and the left $A^!$-linearity of $\bar{\Phi}$, we have

$$\varphi(f \bullet g) = (-1)^{ij + (i+j).(D-i-j)} f \cdot \bar{\Phi}(g).$$

We have on the other hand

$$f \bullet \varphi(g) = (-1)^{i.(D-i-j) + j.(D-j)} f \cdot \bar{\Phi}(g).$$

The sign is the same as the previous one, hence the equality (note that all the signs are $+1$ if $N > 2$). Then Theorem 5.4 shows that φ is an isomorphism, so that $E(A)$ is Frobenius.

Conversely, assume that the finite dimensional graded algebra $E(A)$ is Frobenius. The highest i such that $E(A)_i \neq 0$ is D, since A is Koszul of global dimension D. According to Lemma 3.2 of [19], there is an isomorphism $\psi : E(A) \to E(A)^*(-D)$ in $E(A)$-grMod. It is worth noticing that D is odd if $N > 2$. In fact, $\psi(1)$ belongs to $(E(A)_{1+(D-1)})^*$, and $\psi(e_i^*) = e_i^* \bullet \psi(1)$ vanishes if D is even. Accordingly ψ_i sends $A^!_{n(i)}$ to $A^{!*}_{n(D)-n(i)}$. So $\psi \otimes 1_A$ is a right A-linear 0-graded map from the cochain complex $L'_r(A)$ to the cochain complex $K'_r(A)^{coch}[-D]$. The next lemma shows that $\psi \otimes 1_A$ is a morphism of complexes unless if $N = 2$. Checking the lemma is easier if $N > 2$ (no sign occurs!), but we prefer a set-up including the quadratic case.

Lemma 5.13 *Setting $\tilde{\psi}_i = (-1)^{i.(D-i)}\psi_i$ for $0 \leq i \leq D$, $\tilde{\psi} \otimes 1_A$ is a morphism of complexes.*

Assume that the lemma holds. Then $\tilde{\psi} \otimes 1_A$ realizes an isomorphism from the complex $L'_r(A) \xrightarrow{\tilde{\psi}_D \otimes \epsilon} k \to 0$ to the complex $K'_r(A)^{coch}[-D] \xrightarrow{\epsilon} k \to 0$. The latter is exact, thus the former is exact. As $L'_r(A)$ computes the $\mathrm{Ext}^i_A(k, A)$, A is AS-Gorenstein. It remains to prove the lemma, i.e., the commutativity of the diagram

$$\begin{array}{ccc} E(A)_i \otimes A & \longrightarrow & E(A)_{i+1} \otimes A \\ \downarrow \tilde{\psi}_i \otimes 1_A & & \downarrow \tilde{\psi}_{i+1} \otimes 1_A \\ E(A)^*_{D-i} \otimes A & \longrightarrow & E(A)^*_{D-i-1} \otimes A, \end{array} \tag{5.6}$$

where the differentials are both $\xi\cdot$ or $\xi^{N-1}\cdot$. Fix f in $E(A)_i$. Assume firstly $N = 2$ or ($N > 2$ and i even). The differentials are both $\xi\cdot$. Expressing the action $\cdot$ into the action $\bullet$, we have

$$(\xi\cdot) \circ (\tilde{\psi}_i \otimes 1_A)(f \otimes 1) = (-1)^{i.(D-i)+D-i-1} \sum_j \psi(e_j^* \bullet f) \otimes e_j.$$

On the other hand, we have

$$(\tilde{\psi}_{i+1} \otimes 1_A) \circ (\xi\cdot)(f \otimes 1) = (-1)^{(i+1).(D-i-1)+i} \sum_j \psi(e_j^* \bullet f) \otimes e_j.$$

Hence the result. Assume now $N > 2$ and i odd. The differentials are both $\xi^{N-1}\cdot$. Since $A^!$ is strongly graded, write down $f = \sum_j e_j^* f_j$ with f_j in $E(A)_{i-1}$ (note that $n(i) - 1 = n(i-1)$). Starting with

$$(\xi^{N-1}\cdot)\circ(\tilde{\psi}_i \otimes 1_A)(f \otimes 1) = (-1)^{i.(D-i)} \sum_{j_1 \dots j_{N-1}, j} ((e^*_{j_1} \dots e^*_{j_{N-1}}) \cdot \psi(e_j^* f_j)) \otimes e_{j_1} \dots e_{j_{N-1}}$$

the right-hand side is next reduced to

$$(-1)^{i.(D-i)+i-1+D-i} \sum_{j_1 \cdots j_{N-1}} \psi(e^*_{j_1} \dots e^*_{j_{N-1}} f) \otimes e_{j_1} \dots e_{j_{N-1}}.$$

On the other hand, $(\tilde{\psi}_{i+1} \otimes 1_A) \circ (\xi^{N-1} \cdot)(f \otimes 1)$ is equal to

$$(-1)^{(i+1).(D-i-1)} \sum_{j_1 \cdots j_{N-1}} \psi(e^*_{j_1} \dots e^*_{j_{N-1}} f) \otimes e_{j_1} \dots e_{j_{N-1}}.$$

Hence the result again. ■

6 Poincaré duality

In this section, A is an N-homogeneous algebra which is generalized Koszul, AS-Gorenstein and of finite global dimension D. We want to apply the Van den Bergh duality theorem [23] to A. Since A has a finite Hochschild dimension (Theorem 4.5), it remains to prove that $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq D$, and that $U = \mathrm{Ext}^D_{A^e}(A, A^e)$ is an invertible bimodule. By Theorem 4.4, one has $\mathrm{Ext}^i_{A^e}(A, A^e) = H^i(\mathrm{Hom}_{A^e}(K'_{l-r}(A), A^e))$, $i \geq 0$. We are going to describe the A-grMod-A complex $\mathrm{Hom}_{A^e}(K'_{l-r}(A), A^e))$ by the bimodule version $L'_{r-l}(A)$ of $L'_r(A)$. Here again, we begin to define the N-complex $L_{r-l}(A)$.

For any N-homogeneous algebra A, set $L_{r-l}(A) = A \otimes A^! \otimes A$, graded by the $A \otimes A^!_n \otimes A$, $n \geq 0$, and endowed with the cochain N-differentials $\partial_r = 1_A \otimes (\xi_r \cdot)$ and $\partial_l = (\cdot \xi_l) \otimes 1_A$. These N-differentials commute. Fix a primitive N-root of unity q (and enlarge k if necessary). Define $\partial :\ A \otimes A^{!*}_n \otimes A \to A \otimes A^{!*}_{n+1} \otimes A$ by $\partial = \partial_r - q^n \partial_l$. Explicitly we have

$$A \otimes A \xrightarrow{\partial_r - \partial_l} A \otimes V^* \otimes A \xrightarrow{\partial_r - q\partial_l} \cdots \xrightarrow{\partial_r - q^{N-1}\partial_l} A \otimes A^!_N \otimes A \xrightarrow{\partial_r - \partial_l} \cdots . \tag{6.1}$$

Viewing $A^!_n$ as concentrated in degree $-n$, $(L_{r-l}(A), \partial)$ is a cochain N-complex of A-grMod-A. Using the canonical grMod-A^e isomorphisms (where E is a finite dimensional graded k-vector space)

$$\mathrm{Hom}_{A^e}(A^e \otimes E, A^e) \cong \mathrm{Hom}_k(E, A^e) \cong E^* \otimes A^e,$$

and the identifications $A \otimes E \otimes A \cong A^e \otimes E$, $E^* \otimes A^e \cong A \otimes E^* \otimes A$, one constructs easily an isomorphism of A-grMod-A cochain N-complexes from $\mathrm{Hom}_{A^e}(K_{l-r}(A), A^e))$ to $L_{r-l}(A)$.

If we keep in (6.1) the first arrow, put together the $N-1$ following ones, and continue alternately, we define the adequate contraction $L'_{r-l}(A)$:

$$A \otimes A \xrightarrow{\partial} A \otimes V^* \otimes A \xrightarrow{\partial^{N-1}} A \otimes A^!_N \otimes A \xrightarrow{\partial} A \otimes A^!_{N+1} \otimes A \xrightarrow{\partial^{N-1}} \cdots \tag{6.2}$$

Here $\partial = \partial_r - \partial_l$ and $\partial^{N-1} = \partial_r^{N-1} + \partial_r^{N-2}\partial_l + \cdots + \partial_r\partial_l^{N-2} + \partial_l^{N-1}$, so that $L'_{r-l}(A)$ makes sense on any ground field. Then the A-grMod-A 2-complexes $\mathrm{Hom}_{A^e}(K'_{l-r}(A), A^e)$ and $L'_{r-l}(A)$ are isomorphic. Returning to the assumptions of the beginning of the section, we have obtained:

$$\mathrm{Ext}^i_{A^e}(A, A^e) = H^i(L'_{r-l}(A)),\ i \geq 0.$$

Since A is AS-Gorenstein, $H^i(L'_r(A)) = 0$ if $i \neq D$. But $L'_r(A) = k \otimes_A L'_{r-l}(A)$. Therefore, Proposition 4.1 shows that $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq D$.

Now recall that the Frobenius algebra $E(A)$ (Corollary 5.12) is defined by the $E(A)$-grMod isomorphism $\varphi : E(A) \to E(A)^*(-D)$ such that $\varphi(1) = u^*$. The associated Frobenius pairing $(-,-)$ [19] is defined by

$$(x, y) = \langle x,\ \varphi(y)\rangle = \langle x \bullet y,\ u^*\rangle,\ x \in E(A),\ y \in E(A).$$

For $x \in E(A)_i$ and $y \in E(A)_j$, $(x, y) = 0$ if $i + j \neq D$. Assume $i + j = D$. Then Proposition 3.1 shows that

$$(x, y) = (-1)^{i.(D-i)}\langle xy,\ u^*\rangle,$$

where xy is the product in $A^!$. Let us be more explicit in this case. Write down $u = \bar{\mathtt{u}}$, $\mathtt{u} \in (V^{\otimes n(D)})^*$. The canonical isomorphism $A^!_{n(D)} \cong W^*_{n(D)}$ (see end of Section 3) maps u to $\mathtt{u}|_{W_{n(D)}}$. By the transposed isomorphism, u^* is the image of an element $w \in W_{n(D)} \subseteq V^{\otimes n(D)}$. Considering w as a linear form on $(V^{\otimes n(D)})^*$, w is the unique element of $W_{n(D)}$ which maps $\mathtt{u}$ to 1. Then if $x = \bar{f}$, $y = \bar{g}$, $f \in (V^{\otimes n(i)})^*$, $g \in (V^{\otimes n(D-i)})^*$, we have

$$(x, y) = (-1)^{i.(D-i)}\langle f \otimes g,\ w\rangle, \tag{6.3}$$

where $f \otimes g$ is the usual tensor product of the linear forms f and g. If $N > 2$, the sign in (6.3) is $+1$.

Recall that the graded Frobenius algebra $E(A)$ is said to be *symmetric* (resp. *graded symmetric*) if for any $x \in E(A)_i$ and $y \in E(A)_{D-i}$, $(y, x) = (x, y)$ (resp. $(y, x) = (-1)^{i.(D-i)}(x, y)$). With the above notations, it is equivalent to saying that $\langle g \otimes f, w\rangle = \langle f \otimes g, w\rangle$ (resp. $\langle g \otimes f, w\rangle = (-1)^{i.(D+1)}\langle f \otimes g, w\rangle$) for any f and g. If $N > 2$, both conditions are equivalent. In the examples, it will be important to compute w.

Let us define the automorphism ν of the graded algebra $E(A)$ by

$$(x, y) = (y, \nu(x)),\ x \in E(A),\ y \in E(A).$$

In particular, one has

$$x \bullet y = y \bullet \nu(x),\ x \in E(A)_i,\ y \in E(A)_{D-i}. \tag{6.4}$$

Clearly φ is an isomorphism from the bimodule $E(A)_\nu$ to the bimodule $E(A)^*$. The notation $E(A)_\nu$ means that the right action is twisted by ν (and the left action is the usual one). In other words, for a, b, x in $E(A)$, the action on x in this bimodule is $(a,b) \mapsto a \bullet x \bullet \nu(b)$. Let $\nu_1 : V^* \to V^*$ be the component of degree 1 of ν.

Lemma 6.1 *$\nu_1^{\otimes N}(R^\perp)$ is included into $R^\perp$.*

Proof. From (6.4), we get the formula

$$e^*_{i_1} \cdots e^*_{i_{n(D)}} = \pm\, e^*_{i_{N+1}} \cdots e^*_{i_{n(D)}} \nu_1(e^*_{i_1}) \cdots \nu_1(e^*_{i_N}).$$

Let $f = \sum \lambda_{i_1,\ldots,i_N} e^*_{i_1} \otimes \cdots \otimes e^*_{i_N}$ be any element of $R^\perp$. The previous formula shows that

$$x \bullet \overline{\nu_1^{\otimes N}(f)} = 0$$

for any x in $E(A)_{D-2}$. Thus $\overline{\nu_1^{\otimes N}(f)} = 0$ since the Frobenius pairing is non-degenerate. ■

Define now $\phi_1 : V \to V$ as being the transposed linear map of ν_1. Lemma 6.1 implies that $\phi_1^{\otimes N}(R)$ is included into R. So $\mathrm{Tens}(\phi_1)$ defines an automorphism ϕ of the graded algebra A, homogeneous of degree 0. Our aim is to compute the bimodule $U = H^D(L'_{r-l}(A))$, i.e., the cokernel of

$$\partial = \partial_r + (-1)^D \partial_l : A \otimes A^!_{n(D)-1} \otimes A \longrightarrow A \otimes A^!_{n(D)} \otimes A. \qquad (6.5)$$

Let ε be the automorphism of A which is the multiplication by $(-1)^m$ on A_m. Define $\mu^\phi_u : A \otimes A^!_{n(D)} \otimes A \to A$ by $\mu^\phi_u(a \otimes u \otimes b) = (\varepsilon^{D+1}\phi)(a)b$. Adapting the computation made in the proof of Theorem 9.2 of [22], we prove the following.

Lemma 6.2 $\mu^\phi_u \circ \partial = 0$.

Proof. Introduce $(\zeta^*_i)_i$ the left dual basis to $(\zeta_i = e^*_i)_i$ for the Frobenius pairing. In other words, $\zeta^*_i \in E(A)_{D-1}$ with $(\zeta^*_i, e^*_j) = \delta_{ij}$. One has

$$\partial(1 \otimes \zeta^*_i \otimes 1) = \sum_j 1 \otimes e^*_j \zeta^*_i \otimes e_j + (-1)^D e_j \otimes \zeta^*_i e^*_j \otimes 1.$$

Proposition 3.1 gives $\zeta^*_i e^*_j = (-1)^{D-1} \zeta^*_i \bullet e^*_j = (-1)^{D-1}\delta_{ij} u$. Moreover

$$e^*_j \zeta^*_i = (-1)^{D-1} e^*_j \bullet \zeta^*_i = (-1)^{D-1} \zeta^*_i \bullet \nu(e^*_j) = (-1)^{D-1}(\zeta^*_i,\, \nu(e^*_j))u.$$

But the linear forms $x \mapsto (\zeta^*_i, x)$ and $x \mapsto \langle e_i, x\rangle$ on V^* are the same. Therefore $e^*_j \zeta^*_i = (-1)^{D-1}\langle \phi(e_i), e^*_j\rangle u$. So we get

$$\partial(1 \otimes \zeta^*_i \otimes 1) = (-1)^{D-1}[1 \otimes u \otimes \phi(e_i) + (-1)^D e_i \otimes u \otimes 1].$$

We conclude that

$$\mu_u^\phi \circ \partial(1 \otimes \zeta_i^* \otimes 1) = (-1)^{D-1}[\phi(e_i) + (-1)^D(-1)^{D+1}\phi(e_i)] = 0. \blacksquare$$

Applying the functor $k \otimes_A -$ to the A-grMod-A complex

$$A \otimes A^!_{n(D)-1} \otimes A \xrightarrow{\partial} A \otimes A^!_{n(D)} \otimes A \xrightarrow{\mu_u^\phi} {}_{\varepsilon^{D+1}\phi} A(n(D)) \to 0, \tag{6.6}$$

we get the exact grMod-A complex

$$A^!_{n(D)-1} \otimes A \xrightarrow{\xi_r \cdot} A^!_{n(D)} \otimes A \xrightarrow{\epsilon_u} k(n(D)) \to 0.$$

Therefore, Proposition 4.1 shows that (6.6) is exact, and we have

$$U = {}_{\varepsilon^{D+1}\phi} A(n(D)).$$

Thus the bimodule U is invertible and $U \otimes_A M \cong {}_{\varepsilon^{D+1}\phi} M$. So we obtain the following N-generalization of Van den Bergh's Proposition 2 [23].

Theorem 6.3 *Assume that A is an N-homogeneous algebra which is generalized Koszul, AS-Gorenstein and with finite global dimension D. Let ε be the automorphism of A which is the multiplication by $(-1)^m$ on A_m. Let ν be the automorphism of the Frobenius algebra $E(A)$ such that the bimodule $E(A)^*$ is canonically $E(A)_\nu$. Let ϕ be the automorphism of A such that the homogeneous component ϕ_1 is the transposed linear map of ν_1. Then for any A-A-bimodule M, we have*

$$HH^i(A, M) \cong HH_{D-i}(A, {}_{\varepsilon^{D+1}\phi}M).$$

Let us examine the condition for having no twist on M, i.e., $\phi = \varepsilon^{D+1}$. This condition is related to the existence of a bimodule version of the left $A^! \otimes A$-linear map $\Phi : A^! \otimes A \to A^{!*} \otimes A$ defined just before Theorem 5.4. Setting $\Phi = \Phi_r$, recall that it is defined by $\Phi_r(1 \otimes 1) = u^* \otimes 1$. Introduce the right $A \otimes A^!$-linear map $\Phi_l : A \otimes A^! \to A \otimes A^{!*}$ by $\Phi_l(1 \otimes 1) = 1 \otimes u^*$. Then $1_A \otimes \Phi_r$ and $\Phi_l \otimes 1_A$ are A-A-bimodule morphisms from $A \otimes A^! \otimes A$ to $A \otimes A^{!*} \otimes A$. Let λ be the graded bimodule automorphism of $A \otimes A^! \otimes A$ which is the multiplication by $(-1)^{m.(D+1)}$ on $A^!_m$. Notice that λ is the identity automorphism when $N > 2$.

Proposition 6.4 *Keep the assumptions of Theorem 6.3. Then $\phi = \varepsilon^{D+1}$ if and only if*

$$1_A \otimes \Phi_r = (\Phi_l \otimes 1_A) \circ \lambda. \tag{6.7}$$

Proof. (6.7) is equivalent to $f.u^* = (-1)^{m.(D+1)}u^*.f$ for any f in $A^!_m$, i.e., $e_i^*.u^* = (-1)^{D+1}u^*.e_i^*$ for any i. Recall that for $f \in E(A)_i$ and $\alpha \in E(A)^*_{i+j}$ with $n(i+j) = n(i) + n(j)$, $f \bullet \alpha = (-1)^{ij} f \cdot \alpha$ and $\alpha \bullet f = (-1)^{ij}\alpha \cdot f$. Using (6.4), it is easy to check that $\alpha \bullet f = \nu(f) \bullet \alpha$. So (6.7) is equivalent to

$$e_i^* \bullet u^* = (-1)^{D+1}\nu(e_i^*) \bullet u^*$$

for any i, that is to $\nu_1 = (-1)^{D+1}1_{V^*}$. ■

Assume that $\phi = \varepsilon^{D+1}$. Then $\Phi_{r-l} = 1_A \otimes \Phi_r = (\Phi_l \otimes 1_A) \circ \lambda$ is an N-complex morphism from $L_{r-l}(A)^{ch}[-n(D)]$ to $K_{r-l}(A)$. After the adequate contraction, we get a 2-complex isomorphism Φ'_{r-l} from $L'_{r-l}(A)^{ch}[-D]$ to $K'_{r-l}(A)$. It is an isomorphim of *resolutions* from $L'_{r-l}(A)^{ch}[-D] \xrightarrow{\mu_u^\phi} A$ to $K'_{r-l}(A) \xrightarrow{\mu} A$, since $\mu_u^\phi(1 \otimes u \otimes 1) = 1 = \mu(1 \otimes 1)$. That isomorphism has been already obtained by the second author for the generic cubic AS-regular algebras of global dimension three and of type A, allowing him to prove the Poincaré duality for these algebras when $M = A$ [16].

Now let us give another proof of Theorem 6.3, without using the Van den Bergh duality theorem, but for special bimodules M. Let E be a k-vector space. Then $M = E \otimes A$ (tensor product over k) is naturally a bimodule. Set $L'_{r-l}(M) = E \otimes L'_{r-l}(A)$. Applying the exact functor $E \otimes -$ to the resolution

$$L'_{r-l}(A) \xrightarrow{\mu_u^\phi} {}_{\varepsilon^{D+1}\phi} A \to 0,$$

we see that the chain complex $L'_{r-l}(M)^{ch}[-D]$ is a resolution of the bimodule ${}_{\varepsilon^{D+1}\phi}M$. Therefore

$$HH_{D-i}(A, {}_{\varepsilon^{D+1}\phi}M) \cong H_{D-i}(L'_{r-l}(M)^{ch}[-D] \otimes_{A^e} A) = H^i(L'_{r-l}(M) \otimes_{A^e} A).$$

For any A-A-bimodules N and P, the canonical isomorphisms

$$(E \otimes N) \otimes_{A^e} P \cong E \otimes (N \otimes_{A^e} P) \cong (N \otimes_{A^e} P) \otimes E \cong N \otimes_{A^e} (P \otimes E)$$

show that the complex $L'_{r-l}(M) \otimes_{A^e} A$ is isomorphic to $L'_{r-l}(A) \otimes_{A^e} M$. Thus

$$H^i(L'_{r-l}(M) \otimes_{A^e} A) \cong H^i(\mathrm{Hom}_{A^e}(K'_{l-r}(A), A^e) \otimes_{A^e} M)$$

$$\cong H^i(\mathrm{Hom}_{A^e}(K'_{l-r}(A), M)) \cong HH^i(A, M).$$

As shown by Van den Bergh in the quadratic case ([22], Corollary 9.3), the automorphism ϕ is related to the Artin-Schelter matrix Q [1] in the cubic case. Let A be an AS-regular algebra (with polynomial growth) of global dimension 3, with cubic relations. Then A is Koszul (Proposition 5.2). The basis of

V is (x, y), and the basis of R is (f_1, f_2). The nonvanishing components of $E(A)$ are: $E(A)_0 = k$, $E(A)_1 = V^*$ with basis (x^*, y^*), $E(A)_2 = R^*$ with basis (f_1^*, f_2^*), $E(A)_3 = W_4^*$ with basis w^*. We have

$$w = xf_1 + yf_2 = g_1 x + g_2 y, \tag{6.8}$$

where (g_1, g_2) is the basis of R defined by the matrix product

$$(g_1 \ g_2) = (f_1 \ f_2)Q^t.$$

Here $Q = (q_{ij})$ is some invertible 2×2 matrix with scalar entries. Using (6.3) and (6.8), it is easy to compute the Frobenius pairing. We find

$$(x^*, f_1^*) = (y^*, f_2^*) = 1, \ (x^*, f_2^*) = (y^*, f_1^*) = 0,$$

$$(f_1^*, x^*) = q_{11}, \ (f_2^*, x^*) = q_{12}, \ (f_1^*, y^*) = q_{21}, \ (f_2^*, y^*) = q_{22}.$$

We draw $\nu^{-1}(x^*) \bullet f_1^* = f_1^* \bullet x^* = q_{11}w^*$, and next, $\nu^{-1}(x^*) \bullet f_2^* = q_{12}w^*$. Thus $\nu^{-1}(x^*) = q_{11}x^* + q_{12}y^*$. Similarly, $\nu^{-1}(y^*) = q_{21}x^* + q_{22}y^*$. Denoting by X^* the column vector with two entries x^* and y^*, the isomorphism ν_1 has matrix form $X^* \mapsto Q^{-1}X^*$. Denoting by F^* the column vector with two entries f_1^* and f_2^*, a similar computation shows that the isomorphism ν_2 has matrix form $F^* \mapsto Q^t F^*$. Thus the Frobenius algebra $E(A)$ is symmetric if and only if A is of type A. We have also obtained the following.

Proposition 6.5 *Assume that the algebra A is AS-regular (with polynomial growth), of global dimension* 3, *with cubic relations. The isomorphism ϕ_1 has matrix form $X \mapsto (Q^{-1})^t X$. In particular, $\varepsilon^{D+1}\phi = 1_A$ if and only if A is of type* A.

We finish by another class of examples. Consider A as in Corollary 5.10 with $N = 2$ or ($N > 2$ and $n = Nq + 1$). Then $D = n$ or $D = 2q + 1$, respectively. In both cases, $n(D) = n$. Recall that $u = x_n^* \cdots x_1^*$. A straightforward computation provides

$$w = \sum_{\sigma \in \mathbf{S}_n} \mathrm{sgn}(\sigma) \ x_{\sigma(n)} \otimes \cdots \otimes x_{\sigma(1)}$$

where $\mathbf{S}_n$ denotes the permutation group of $\{1, \ldots, n\}$. We leave to the reader the computation of the Frobenius pairing by using (6.3). For $j_1 > \cdots > j_{n(i)}$, $0 \leq i \leq D$, one has

$$\nu(x_{j_1}^* \cdots x_{j_{n(i)}}^*) = (-1)^{n(i).(n+1)} x_{j_1}^* \cdots x_{j_{n(i)}}^*.$$

Thus $E(A)$ is symmetric if and only if n is odd, and for $N = 2$, it is always graded symmetric. On the other hand, $\phi = \varepsilon^{n+1}$, so that the equality $\varepsilon^{D+1}\phi = 1_A$ holds for $N = 2$, and this equality holds for $N > 2$ if and only if n is odd.

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