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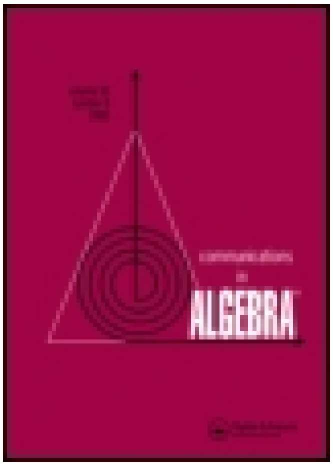

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Some remarks on exact sequences of quantum groups

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SOME REMARKS ON EXACT SEQUENCES OF QUANTUM GROUPS

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Introduction

In this paper, as in [Dr], [Ma] and [PW], the dual category of the category of Hopf algebras over a field k is called the category of *quantum groups*. Parshall and Wang [PW] introduce *short exact sequences of quantum groups* as sequences whose dual sequences of Hopf algebras are of the form

$$B \overset{i}{\subset} A \xrightarrow{p} A/I ,$$

where A is a Hopf algebra, B a Hopf subalgebra and I a Hopf ideal of A such that the canonical surjective map p is the cokernel of the inclusion map i in the category of Hopf algebras.

This definition corresponds to the usual notion of exact sequences of affine group schemes if A is commutative.

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Formal group schemes are equivalent to cocommutative Hopf algebras [Ga]. However, exact sequences of cocommutative Hopf algebras in the sense of [PW] do not correspond to exact sequences of formal group schemes. Furthermore, in the general case of an exact sequence of Hopf algebras, B need not be a normal Hopf subalgebra and I need not be a normal Hopf ideal as shown by an example in 1.1 contradicting [PW,(1.5.1) (2)].

The natural way to generalize exact sequences of commutative or cocommutative Hopf algebras seems to be the following stricter definition:

Strictly exact sequences of Hopf algebras are defined by the following equivalent conditions (1) and (2):

(1) (a) B is a normal Hopf subalgebra,
 (b) A is right faithfully flat over B,
 (c) p is the cokernel of i in the category of Hopf algebras.

(2) (a) I is a normal Hopf ideal,
 (b) A is right faithfully coflat over A/I,
 (c) i is the kernel of p in the category of Hopf algebras.

The equivalence of (1) and (2) is shown in 1.4 depending on results of Takeuchi in [T2]. For the definition of normal Hopf subalgebras and normal Hopf ideals see the first section of this paper.

Exact sequences of commutative Hopf algebras in the sense of [PW] are strictly exact by the quotient theory of affine group schemes [DG], [T1]. Strictly exact sequences of cocommutative Hopf algebras correspond to the usual notion of exact sequences of formal group schemes [Ga]. Also coverings of quantum groups in the sense of [PW] give rise to strictly exact sequences.

It is an open question as to whether (b) follows from (a) in (1) and (2) of the above definition. In the case of commutative or cocommutative Hopf algebras, (b) does follow from (a) ([DG], [Ga], [T1]).

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In the general case, a positive answer is given in 2.1 under finiteness assumptions:

If I is a *normal* Hopf ideal and A/I is *finite dimensional*, then A is right and left faithfully coflat over A.

If B is a *finite dimensional normal* Hopf subalgebra of the Hopf algebra A, then A is a right and left free B-module.

The last result was proved by Radford [Ra, Th.1] in the case when A is commutative. A finite dimensional Hopf algebra A is free over any (not necessarily normal) Hopf subalgebra by the basic theorem of Nichols and Zoeller [NZ, Th. 8].

More generally, in 2.1 (1), an exactness criterion for (not necessarily normal) subgroups is formulated covering exactness results of some special quantum groups in [PW].

Finally, in 3.3 it is shown (by a variation of the new proof of Masuoka and Wigner in the commutative case [MW]) that a left or right noetherian Hopf algebra is faithfully flat over *central* Hopf subalgebras.

Hence, in the definition of coverings $\varphi : G \longrightarrow H$ of quantum groups in [PW (1.8)], the faithful flatness condition (C2) follows from the first condition (C1) when the dual algebra of G is left or right noetherian.

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1. Exact sequences

Let k be a fixed field, $\otimes = \otimes_k$ and $\mathscr{H}$ the category of Hopf algebras over k (see [Sw]). If H is a Hopf algebra with augmentation $\varepsilon : H \longrightarrow k$ and comultiplication map $\Delta : H \longrightarrow H \otimes H$, then $H^+ := \mathrm{ke}(\varepsilon)$ is the augmentation ideal of H and the following standard notation will be used : $\Delta_1(x) := \Delta(x) =: \sum x_1 \otimes x_2$, $\Delta_n(x) := (\mathrm{id} \otimes \Delta)\, \Delta_{n-1}(x) =: \sum x_1 \otimes \cdots \otimes x_{n+1}$. S will denote the antipode of H.

The following description of kernels and cokernels in $\mathscr{H}$ can be easily checked: In the category $\mathscr{H}$, *kernel and cokernel of a pair of morphisms*

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$f, g : X \longrightarrow Y$ exist and are described as follows:

$\mathcal{H}-\mathrm{ke}(f,g) := \{x \in X \mid \sum x_1 \otimes f(x_2) \otimes x_3 = \sum x_1 \otimes g(x_2) \otimes x_3 \text{ for all } x \in X\}$ is a Hopf subalgebra of X and the inclusion map $i : \mathcal{H}-\mathrm{ke}(f,g) \longrightarrow X$ is the kernel of f, g in $\mathcal{H}$ (and in the category of coalgebras).

$\mathcal{H}-\mathrm{coke}(f,g) := Y/Y\{f(x) - g(x) \mid x \in X\}Y$ is a quotient Hopf algebra of Y and the cokernel of f, g in $\mathcal{H}$ (and in the category of algebras).

In particular, for any Hopf algebra homomorphism $f : X \longrightarrow Y$ let $\mathcal{H}-\mathrm{ke}(f) := \mathcal{H}-\mathrm{ke}(f,\eta\varepsilon)$ be its *kernel*, and $\mathcal{H}-\mathrm{coke}(f) := \mathcal{H}-\mathrm{coke}(f,\eta\varepsilon)$ its *cokernel* in $\mathcal{H}$. Here, $\eta\varepsilon : X \longrightarrow Y$, $x \longmapsto \varepsilon(x)1$, is the unity map.

Hopf algebras generalize groups (group algebras). The notion of a normal subgroup is generalized in the following way. As always, in $\mathcal{H}$ there is also the dual notion.

DEFINITION 1.1. Let $f : X \longrightarrow Y$ be a Hopf algebra homomorphism. f is called *normal* if for all $x \in X$, $y \in Y$

$$\sum y_1 f(x) S(y_2) \in f(X) \text{ and } \sum S(y_1) f(x) y_2 \in f(X).$$

f is called *conormal* if for all $x \in \mathrm{ke}(f)$ $(:= f^{-1}(0))$

$$\sum x_2 \otimes S(x_1) x_3 \in \mathrm{ke}(f) \otimes X \text{ and } \sum x_2 \otimes x_1 S(x_3) \in \mathrm{ke}(f) \otimes X.$$

Let A be a Hopf algebra. A *Hopf subalgebra* $B \subset A$ is called *normal* if the inclusion map $B \subset A$ is normal. A *Hopf ideal* $I \subset A$ is called *normal* if the canonical surjection $A \longrightarrow A/I$ is conormal.

For example, if G is a group with group algebra $k[G]$ and N is a subgroup of G, then $k[N]$ is a normal Hopf subalgebra of $k[G]$ iff N is a normal subgroup of G. If G is an affine group scheme (i.e., a functor from commutative k-algebras to the category of groups which is representable as a set-valued functor), $N \subset G$ a closed subgroup scheme of G represented by the surjective map $p : A \longrightarrow A/I$ of commutative Hopf algebras, then N is a normal subgroup scheme (i.e., $N(R)$ is a normal subgroup of $G(R)$ for all commutative k-algebras R) iff $I \subset A$ is a normal Hopf ideal ([DG], [T1]).

Of course, if Y is commutative resp. X is cocommutative, then f is always normal resp. conormal. In the general case both notions have to be used simultaneously.

In [PW, (1.6.1) (1), (1.5.1) (2)] it is claimed that cokernels in $\mathscr{H}$ are conormal. However, for arbitrary Hopf algebra maps cokernels need not be conormal as the following example of a Hopf subalgebra in Taft's Hopf algebra [Ta] shows.

EXAMPLE 1.2. Let n be an integer ≥ 2 and let ζ be a primitive n-th root of unity in k. Let $k\langle T,X\rangle$ be the free algebra in non-commutative variables T,X and I the ideal generated by $T^n - 1$, X^n and $XT - \zeta TX$. Then the quotient algebra $A := k\langle T,X\rangle/I$, $t := T + I$, $x := X + I$, has k-basis $t^i x^j$, $0 \leq i, j \leq n-1$. A is a Hopf algebra where $\Delta(t) = t \otimes t$, $\varepsilon(t) = 1$, $\Delta(x) = x \otimes t + 1 \otimes x$, $\varepsilon(x) = 0$, $S(x) = -x$ [Ta].

Now assume $n = rs$ for integers r, s > 1. Then $B := k[t^r]$, the subalgebra generated by the group-like element t^r, is a Hopf subalgebra. But the Hopf ideal AB^+A is not normal in A. In other words, the cokernel of the inclusion map $B \subset A$ is not conormal.

PROOF. Let $\langle u\rangle$ be a cyclic group of order r and $k[u]$ the group algebra of $\langle u\rangle$. Let J be the kernel of the surjective Hopf algebra map

$$f : A \longrightarrow k[u],\ f(t) := u,\ f(x) := 0.$$

This map is well-defined since $u^n = u^{rs} = 1$. The Hopf ideal J is not normal in A, since $x \in J$ and

$$\begin{aligned}\sum f(x_2) \otimes x_1 S(x_3) &= f(t) \otimes x S(t) + f(x) \otimes S(t) + f(1) \otimes S(x)\\ &= u \otimes x t^{-1} + 1 \otimes (-x) \neq 0 \quad \text{in} \quad k[u] \otimes A.\end{aligned}$$

It remains to show that $J = AB^+A$. Obviously $AB^+A \subset J$, since B^+ is generated by $t^r - 1 \in J$. To prove the other inclusion, write an element $y \in J$ as a sum $y = z + ax$, for some $z \in k[t]$ and $a \in A$. Then $f(z) = 0$, and therefore $z \in k[t]k[t^r]^+ \subset AB^+A$. But also $x \in AB^+A$ by the following argument: $xt^r = \zeta^r t^r x$, hence $x(t^r - 1) = \zeta^r(t^r - 1)x + (\zeta^r - 1)x$. Therefore, $(\zeta^r - 1)x \in AB^+A$ and then $x \in AB^+A$ since $\zeta^r \neq 1$. ∎

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The categorical notions of kernel resp. cokernel in $\mathcal{H}$ seem to be most useful in the case of conormal resp. normal morphisms. In this case, kernels resp. cokernels *are* normal resp. conormal.

LEMMA 1.3. *Let $f : X \longrightarrow Y$ be a Hopf algebra homomorphism.*

(1) If f is conormal, then

$$\begin{aligned}\mathcal{H}\text{-}ke(f) = X^{coY} &:= \{x \in X \mid \sum x_1 \otimes f(x_2) = x \otimes 1 \text{ for all } x \in X\} \\ = {}^{coY}X &:= \{x \in X \mid \sum f(x_1) \otimes x_2 = 1 \otimes x \text{ for all } x \in X\}\end{aligned}$$

is a normal Hopf subalgebra of X.

(2) If f is normal, then $Yf(X)^+Y = Yf(X)^+ = f(X)^+Y$ is a normal Hopf ideal in Y and $Y \longrightarrow \mathcal{H}\text{-}coke(f)$ is conormal.

PROOF. (1) By assumption, the Hopf ideal $I := ke(f)$ is normal in X. First it will be shown that $X^{coY} = {}^{coY}X$. If $x \in X^{coY}$ then

$$\sum x_1 \otimes x_2 - x \otimes 1 \in X \otimes I, \text{ hence}$$
$$\sum x_1 \otimes x_3 \otimes S(x_2)x_4 - x \otimes 1 \otimes 1 \in X \otimes I \otimes X,$$

since I is normal (first part of the definition). Multiplying the first and third factors gives

$$\sum x_1 S(x_2) x_4 \otimes x_3 - x \otimes 1 \in X \otimes I.$$

Thus $x \in {}^{coY}X$, i.e. $\sum x_1 \otimes x_2 - 1 \otimes x \in I \otimes X$, since

$$\sum x_1 S(x_2) x_4 \otimes x_3 = \sum x_2 \otimes x_1.$$

The other inclusion is proved similarly using the second part of the definition of normal Hopf ideals.

To prove the equality $\mathcal{H}\text{-}ke(f) = X^{coY} (= {}^{coY}X)$ note that $\mathcal{H}\text{-}ke(f)$ is contained in X^{coY} by applying $id \otimes id \otimes \varepsilon$. If $x \in X^{coY}$, then

$$\sum x_1 \otimes x_2 - x \otimes 1 \in X \otimes I. \text{ hence}$$
$$\sum x_1 \otimes x_2 \otimes x_3 - \sum x_1 \otimes x_2 \otimes 1 \in X \otimes X \otimes I.$$

Therefore, $\sum x_1 \otimes x_2 \in X \otimes X^{coY} = X \otimes {}^{coY}X$. and

$$\sum x_1 \otimes x_2 \otimes x_3 - \sum x_1 \otimes 1 \otimes x_2 \in X \otimes I \otimes X.$$

i.e., $x \in \mathcal{H}\text{-}ke(f)$.

It remains to show that the Hopf subalgebra $\mathcal{H}\text{-}ke(f)$ is normal in X. Let $x \in X$, $b \in \mathcal{H}\text{-}ke(f) = X^{coY}$ and define $c := \sum x_1 b S(x_2)$. Then

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$$\begin{aligned}\sum c_1 \otimes f(c_2) &= \sum x_1 b_1 S(x_4) \otimes f(x_2) f(b_2) f(S(x_3)) \\ &= \sum x_1 b S(x_4) \otimes f(x_2) f(S(x_3)), \quad \text{since } b \in X^{\mathrm{co}Y} \\ &= \sum x_1 b S(x_2) \otimes 1.\end{aligned}$$

Hence $c \in X^{\mathrm{co}Y} = \mathscr{H}\text{-}\mathrm{ke}(f)$. Similarly, $\sum S(x_1) b x_2 \in \mathscr{H}\text{-}\mathrm{ke}(f)$.

(2) First it will be shown that $Y f(X)^+ = f(X)^+ Y$. Let $y \in Y$ and $x \in X$, $\varepsilon(f(x)) = 0$. Then $y f(x) = \sum y_1 f(x) S(y_2) y_3 \in f(X)^+ Y$, since f is normal (first part of the definition).

Similarly, the other inclusion follows from the second part of the definition of normal.

Then obviously, $Y f(X)^+ Y = Y f(X)^+ = f(X)^+ Y$.

It remains to show that the Hopf ideal $I := Y f(X)^+ Y$ is normal in Y. Let $y \in I$. To show that $\sum y_2 \otimes S(y_1) y_3 \in I \otimes Y$, it suffices to assume that $y = f(x) z$ for some $x \in X$, $\varepsilon(f(x)) = 0$, and $z \in Y$. Then mod $I \otimes Y$

$$\begin{aligned}\sum y_2 \otimes S(y_1) y_3 &= \sum f(x_2) z_2 \otimes S(z_1) S(f(x_1)) f(x_3) z_3 \\ &\equiv \sum \varepsilon(f(x_2)) z_2 \otimes S(z_1) S(f(x_1)) f(x_3) z_3, \quad \text{since } I = f(X)^+ Y \\ &= \sum z_2 \otimes S(z_1) \sum S(f(x_1)) f(x_2) z_3 \\ &= 0, \qquad \text{since } \sum S(f(x_1)) f(x_2) = \varepsilon(f(x)) = 0.\end{aligned}$$

Similarly, $\sum y_2 \otimes y_1 S(y_3) \in I \otimes Y$. ∎

Let C be a coalgebra, V a right and W a left C-comodule with structure maps $\Delta_V : V \longrightarrow V \otimes C$ and $\Delta_W : W \longrightarrow C \otimes W$. Then the *cotensor-product* $V \square_C W$ is defined as the kernel of the linear map

$$V \otimes W \longrightarrow V \otimes C \otimes W, \; v \otimes w \longmapsto \Delta_V(v) \otimes w - v \otimes \Delta_W(w).$$

V is called *(faithfully) coflat* if the functor $W \longmapsto V \square_C W$ from left C-comodules to vectorspaces preserves (and reflects) exact sequences.

If X, Y are Hopf algebras and $f : X \longrightarrow Y$ is a homomorphism of Hopf algebras, then f is called ***right (faithfully) coflat*** if X is a right (faithfully) coflat Y-comodule with comodule structure $x \longmapsto \sum x_1 \otimes f(x_2)$. f is called ***right (faithfully) flat*** if Y is a right (faithfully) flat X-module with module structure $y \otimes x \longmapsto y f(x)$.

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Using lemma 1.3, [T2, Th. 1 and 2] yields the following bijective correspondence between Hopf subalgebras and Hopf ideals.

THEOREM 1.4. *Let A be a Hopf algebra. Let $\mathcal{S}(A)$ be the set of all normal Hopf subalgebras B such that A is right faithfully flat over B. Let $I(A)$ be the set of all normal Hopf ideals I such that A is right faithfully coflat over A/I. Then*

$$\mathcal{S}(A) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} I(A),$$

$$\Phi(B) := AB^+ \;(= B^+A),\; \Psi(I) := A^{\,coA/I} (= {}^{coA/I}A).$$

are inverse bijections.

PROOF. Let $B \in \mathcal{S}(A)$. Then $I := AB^+ = B^+A$ is a normal Hopf ideal by 1.3. By [T2, Th. 1] (for left modules and comodules), $A \longrightarrow A/I$ is right faithfully coflat and $B = A^{coA/I}$.

Let $I \in I(A)$. Then $B := A^{coA/I} = {}^{coA/I}A$ is a normal Hopf subalgebra by 1.3. By [T2, Th. 2], A is right faithfully flat over B and $I = AB^+$. ∎

Theorem 1.4 suggests the

DEFINITION 1.5. A sequence of quantum groups

$$\mathcal{E}: \qquad 1 \longrightarrow N \xrightarrow{\eta} G \xrightarrow{\pi} H \longrightarrow 1$$

is called *strictly exact* iff

the comorphism of π is normal and right faithfully flat,

and η is the kernel of π in the category of quantum groups.

or iff (by th. 1.4)

the comorphism of η is conormal and right faithfully coflat,

and π is the cokernel of η in the category of quantum groups.

Note that strictly exact sequences of quantum groups and of Hopf algebras (as defined in the introduction) are dual notions. This is clear from the definition since right faithfully flat resp. coflat morphisms of Hopf algebras are injective resp. surjective.

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In the terminology of [PW], an exact sequence $\mathcal{E}$ of quantum groups is strictly exact iff N is an exact and normal subgroup of G (with embedding η).

REMARK 1.6. *Let A be a Hopf algebra.*

(1) If $B \subset A$ is a normal Hopf subalgebra, then the canonical map

$$A \otimes_B A \longrightarrow A \otimes A/AB^+ ,\ x \otimes y \longmapsto \sum xy_1 \otimes \overline{y}_2 ,$$

is bijective with inverse $x \otimes \overline{y} \longmapsto \sum xS(y_1) \otimes y_2$.

(2) If $I \subset A$ is a normal Hopf ideal and $B := A^{coA/I}$, then the canonical map

$$A \otimes B \longrightarrow A \square_{A/I} A ,\ x \otimes y \longmapsto \sum x_1 \otimes x_2 y ,$$

is bijective with inverse $\sum x_i \otimes y_i \longmapsto \sum x_{i1} \otimes S(x_{i2}) y_i$.

PROOF. This is easily checked. ■

2. Finite kernels and cokernels

It does not seem to be known whether or not arbitrary Hopf algebras are faithfully flat over normal Hopf subalgebras or surjective conormal Hopf algebra maps are faithfully coflat. In this section, positive results are obtained under finiteness assumptions.

The following depends on two basic results on finite dimensional Hopf algebras H: Kreimer and Takeuchi's characterization of H-Galois extensions [KT (1.7)] and Nichols and Zoeller's theorem on the freeness of H over any Hopf subalgebra [NZ1, Th. 8].

THEOREM 2.1. *Let A be a Hopf algebra.*

(1) Let $I \subset A$ be a Hopf ideal and assume there is a normal Hopf ideal $J \subset A$ of finite codimension such that the $\mathcal{K}$-kernel of the canonical map $A/I \longrightarrow A/(I + J)$ is a cosemisimple coalgebra. For example, I could be a normal Hopf ideal of finite codimension. Then A is left and right faithfully coflat over A/I.

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(2) Let $B \subset A$ be a Hopf subalgebra and assume there is a finite dimensional normal Hopf subalgebra $C \subset A$ such that $B/B(B \cap C)^+$ is a semisimple algebra. For example, B could be a finite dimensional normal Hopf subalgebra. Then A is free as a left and right module over B and B is finite dimensional.

In the language of quantum groups [PW], 2.1 (1) could be expressed as follows: Let G' be a closed subgroup of a quantum group G. Assume there is a finite normal closed subgroup $N \subset G$ such that $G'/G' \cap N$ is linearly reductive, i.e. its dual Hopf algebra is cosemisimple. Then G' is an exact subgroup of G.

[PW, 7.5.2] and related results (proved in [PW] by a spectral sequence argument) follow from this exactness criterion. 2.1 (2) generalizes a theorem of Radford [Ra, Th. 1] who showed that commutative Hopf algebras are free over finite dimensional Hopf subalgebras.

The proof of 2.1 requires some preparations. Let H be a Hopf algebra. A right *H-comodule algebra* A is an algebra A with a right H-comodule structure $\Delta_A : A \longrightarrow A \otimes H$, $\Delta_A(a) =: \sum a_0 \otimes a_1$, $a \in A$, such that Δ_A is an algebra map. A right *H-module coalgebra* C is a coalgebra C with a right H-module structure $C \otimes H \longrightarrow C$, $c \otimes h \longmapsto ch$, such that $\Delta(ch) = \sum c_1h_1 \otimes c_2h_2$ and $\varepsilon(ch) = \varepsilon(c)\varepsilon(h)$ for all $c \in C$ and $h \in H$.

Let A be a right H-comodule algebra and let

$$B := A^{coH} = \{a \in A \mid \sum a_0 \otimes a_1 = a \otimes 1\}$$

be the algebra of coinvariant elements. Then A is called *H-Galois* if the canonical map

$$\mathrm{can} : A \otimes_B A \longrightarrow A \otimes H,\ x \otimes y \longmapsto \sum xy_1 \otimes y_2 .$$

is bijective [KT].

Let C be a right H-module coalgebra and let $D := C/CH^+$ be the quotient coalgebra. Then C is called *H-cogalois* if the canonical map

$$\mathrm{can} : C \otimes H \longrightarrow C \,\square_D\, C,\ c \otimes h \longmapsto \sum c_1 \otimes c_2 h.$$

is bijective.

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LEMMA 2.2. *Let H be a Hopf algebra, A a right H-comodule algebra, $B := A^{coH}$ and C a right H-module coalgebra, $D := C/CH^{+}$.*

(1) If A is H-Galois and H is a semisimple algebra, then $B \subset A$ is a separable algebra extension, i.e. the multiplication map $A \otimes_B A \longrightarrow A$ splits as a map of left and right A-modules.
In particular, if P is a left or right A-module which is projective over B, then P is a projective A-module.

(2) If C is H-cogalois and H is a cosemisimple coalgebra, then the canonical surjection $C \longrightarrow D$ is a coseparable map of coalgebras, i.e. there is a left and right C-colinear map $\rho : C \square_D C \longrightarrow C$ such that $\rho(\sum c_1 \otimes c_2) = c$ for all $c \in C$.
In particular, if V is a left or right C-comodule which is coflat over D, then V is a coflat C-comodule.

PROOF. (1) is shown by the well-known argument of Maschke's theorem ([LS, Th. 3], [DT, 3.14], [BM, 2.6]). In particular, if $U \subset V$ is an inclusion of left or right A-modules which splits over B, then U is an A-direct summand in V.

Let P be a right A-module which is projective over B. Then the multiplication map $P \otimes A \longrightarrow P$ splits over B. Hence P is an A-direct summand in $P \otimes A$ and therefore projective over A.

(2) is proved dually to (1): Since H is cosemisimple, there is a right H-colinear map $\lambda : H \longrightarrow k$ such that $\lambda(1) = 1$ [Sw, 14.0.3]. By assumption,

$$\mathrm{can} : C \otimes H \longrightarrow C \square_D C, \; c \otimes h \longmapsto \sum c_1 \otimes c_2 h,$$

is bijective. Note that can is left and right C-colinear where the left resp. right C- comodule structures on $C \otimes H$ are defined by

$$c \otimes h \longmapsto \sum c_1 \otimes c_2 \otimes h \text{ resp. } c \otimes h \longmapsto \sum c_1 \otimes h_1 \otimes c_2 h_2 .$$

Hence $\rho := (\mathrm{id} \otimes \lambda)\mathrm{can}^{-1}$ is left and right C-colinear and for all $c \in C$,

$$\rho(\sum c_1 \otimes c_2) = (\mathrm{id} \otimes \lambda)(c \otimes 1) = c\,\lambda(1) = c.$$

Hence, if V is a C-comodule, $U \subset V$ a C-subcomodule and $f : V \longrightarrow U$ a map of D-comodules such that $f \mid U = \mathrm{id}$, then

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$V \xrightarrow{g} U \,\square_C C \square_D C \xrightarrow{U \square \rho} U \,\square_C C \cong U$, $g(v) := \sum \Delta_U(f(v_0)) \otimes v_1$,
is a C-colinear retraction of the inclusion map $U \subset V$.

In particular, if Q is a right C-comodule which is coflat (= injective) over D, then the C-comodule structure map $Q \longrightarrow Q \otimes C$ splits as a map of D-comodules hence also of C-comodules. Hence Q is C-coflat. ■

Next, the dual of [KT (1.7)] will be proved.

THEOREM 2.3. *Let H be a finite dimensional Hopf algebra. C a right H-module coalgebra and* $D := C/CH^+$ *the quotient coalgebra. Assume*

$$can : C \otimes H \longrightarrow C \,\square_D C\ ,\ c \otimes h \longmapsto \sum c_1 \otimes c_2 h\ ,$$

is injective. Then can is bijective. i.e. C is H-cogalois and C is left and right D-coflat.

PROOF. The direct proof of [KT. (1.7)] in [S2] can be dualized as follows.

(1) The antipode S of H is bijective by [LS. Prop. 2]. since H is finite dimensional. Hence $C \otimes H \longrightarrow C \otimes H$, $c \otimes h \longmapsto \sum c h_1 \otimes S(h_2)$. is bijective with inverse $c \otimes h \longmapsto \sum c h_2 \otimes S^{-1}(h_1)$. Let

$$can' : C \otimes H \longrightarrow C \,\square_D C\ ,\ c \otimes h \longmapsto \sum c_1 h \otimes c_2\ ,$$

be the composition of can with this isomorphism. Then can' is injective by assumption.

By [LS. Prop. 7] there are elements Λ in H and λ in the dual $H^* = \mathrm{Hom}(H,k)$ such that

$$h\Lambda = \varepsilon(h)\Lambda,\ \sum h_1 \lambda(h_2) = \lambda(h)1 \text{ for all } h \in H,$$
$$\lambda(\Lambda) = 1\ ,\ \lambda(S^{-1}(\Lambda)) = 1\ .$$

Choose a linear map $\varphi : C \otimes C \longrightarrow k$ such that

$$\sum \varphi(c_1 h \otimes c_2) = \varepsilon(c)\,\lambda(h) \text{ for all } c \in C,\ h \in H.$$

Such a map exists since $C \otimes H \longrightarrow C \otimes C$, $c \otimes h \longmapsto \sum c_1 h \otimes c_2$, is injective as shown before. The map

$$f : D = C/CH^+ \longrightarrow C,\ f(\overline{c}) := c\Lambda\ ,$$

is well-defined (since $ch\Lambda = c\,\varepsilon(h)\Lambda = 0$ for all $c \in C$ and $h \in H^+$) and left and right D-colinear.

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(2) For all $c \in C$, $\sum c_1 \Lambda_1 \varphi(c_2 \Lambda_2 \otimes c_3) = c$, since

$$\begin{aligned}\sum c_1 \Lambda_1 \varphi(c_2 \Lambda_2 \otimes c_3) &= \sum c_1 \Lambda_1 \varphi(\mathrm{can}'(c_2 \otimes \Lambda_2)) \\ &= \sum c_1 \Lambda_1 \varepsilon(c_2) \lambda(\Lambda_2), \text{ by construction of } \varphi \\ &= c \sum \Lambda_1 \lambda(\Lambda_2) \\ &= c, \qquad \text{by the choice of } \Lambda \text{ and } \lambda.\end{aligned}$$

(3) Using the identity in (2), it can now be shown that can is surjective: Let $\sum x_i \otimes y_i$ be an element of $C \square_D C$ where x_i, y_i are finitely many elements in C. Then by definition of the cotensorproduct

$$\sum x_{i1} \otimes \overline{x_{i2}} \otimes y_i = \sum x_i \otimes \overline{y_{i1}} \otimes y_{i2}, \text{ and}$$

$$(*) \qquad \sum x_{i1} \otimes x_{i2}\Lambda \otimes y_i = \sum x_i \otimes y_{i1}\Lambda \otimes y_{i2}, \text{ since } f \text{ is well-defined.}$$

Hence

$$\begin{aligned}\mathrm{can}(\sum x_{i1} \otimes \Lambda_1 \varphi(x_{i2}\Lambda_2 \otimes y_i)) &= \\ &= \sum x_{i1} \otimes x_{i2}\Lambda_1 \varphi(x_{i3}\Lambda_2 \otimes y_i) \\ &= \sum x_i \otimes y_{i1}\Lambda_1 \varphi(y_{i2}\Lambda_2 \otimes y_{i3}), \qquad \text{by } (*) \\ &= \sum x_i \otimes y_i, \qquad \text{by (2).}\end{aligned}$$

(4) The left D-comodule C is coflat or equivalently injective iff there is a left D- colinear retraction Φ of the comodule structure map. Define $\Phi : D \otimes C \longrightarrow C$ by $\Phi := (\mathrm{id} \otimes \varphi)(\Delta \otimes \mathrm{id})(f \otimes \mathrm{id})$. Since f is left D-colinear, so is Φ. Here, the D-left comodule structure of $D \otimes C$ is $\Delta_D \otimes \mathrm{id}$. By definition of Φ, for all $c \in C$,

$$\begin{aligned}\Phi(\sum \overline{c}_1 \otimes c_2) &= \sum c_1 \Lambda_1 \varphi(c_2 \Lambda_2 \otimes c_3) \\ &= c, \qquad \text{by (2).}\end{aligned}$$

Hence C is left D-coflat.

(5) To show that C is right D-coflat one can proceed as in (4) using the identity

$$\sum \varphi(c_1 \otimes c_2 \Lambda_1) c_3 \Lambda_2 = c \text{ for all } c \in C$$

instead of (2) (since $\lambda(S^{-1}(\Lambda)) = 1$) and the map $(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\mathrm{id} \otimes f)$ instead of Φ. ∎

Finally, the proof of the freeness result in 2.1 (2) requires the following generalization of [NZ2, Th. 4].

If A is a Hopf algebra and $B \subset A$ a Hopf subalgebra, a left *(A,B)-Hopf module* is a left B-module M which is also a left A-comodule such that

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the comodule structure map $M \longrightarrow A \otimes M$ is B-linear (B operating diagonally on $A \otimes M$).

THEOREM 2.4. *Let A be a Hopf algebra and $B \subset A$ a finite dimensional Hopf subalgebra. If A is projective as a left B-module, then A is a free B-module. Moreover, every infinite dimensional B-projective left (A,B)-Hopf module is free as a left B-module.*

PROOF (generalizing the proof in [NZ2], where B is supposed to be semisimple). A is a left (A,B)-Hopf module in the obvious way. If A is finite dimensional, then A is free over B by [NZ1]. Hence it suffices to prove the statement on Hopf modules.

Write

$$B \cong P_1^{e_1} \oplus \cdots \oplus P_n^{e_n}, \; e_i \geq 1 \text{ for all } i,$$

where the P_i are the indecomposable projective left B-modules, P_i not isomorphic to P_j if $i \neq j$. Let M be an infinite dimensional B-projective left (A,B)-Hopf module. Since B is finite dimensional, there are index sets $I_1, \cdots, I_n$ such that

$$M \cong P_1^{(I_1)} \oplus \cdots \oplus P_n^{(I_n)} \text{ (cf. [AF, 27.11]).}$$

It is enough to show that for all i, $P_i^{(I_i)}$ and M have the same k-dimension.

Assume there is some i such that $\dim(P_i^{(I_i)})$ is less than $\dim(M)$. Let $W \subset M$ be the image of $P_i^{(I_i)}$ in M. Then $N := W \leftarrow A^*$, the (A,B)-Hopf submodule of M generated by W, also has dimension less than $\dim(M)$. Hence M/N is a non-zero Hopf module.

By [NZ2, Prop. 3], M/N is faithful as a left B-module. Therefore also M/W is faithful over the Frobenius algebra B. But then M/W must contain a copy of P_i as a direct summand by [CR, (59.3)] contradicting the construction of N by the theorem of Krull - Schmidt - Azumaya. ∎

PROOF of 2.1.

(1) (a) First assume I is a normal Hopf ideal of finite codimension and define $B := A^{\mathrm{co}A/I}$. Then the canonical map

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$$\mathrm{can}\ :\ A \otimes_B A \longrightarrow A \otimes A/I,\ x \otimes y \longmapsto \sum x y_1 \otimes \overline{y}_2,$$

is surjective. Since A/I is a finite dimensional Hopf algebra, can is bijective, i.e. A is A/I-Galois, and A is finitely generated and projective as a right or left B-module by [KT, (1.7)].

By 1.3, B is a (normal) Hopf subalgebra of A. By [D, Th. 4 and Cor. 1], the Hopf module A/B is projective as a right B-module. Hence B is a B-direct summand in A and A is right faithfully flat over B. Then A is faithfully coflat as a right A/I-comodule by the argument in [T3, 1.5].

In the same way (viewing A as a left A/I-comodule algebra) one shows that A is also left faithfully coflat over A/I.

(b) In the general case, the canonical map $p : A/I \longrightarrow A/(I + J)$ is a conormal map of Hopf algebras and $A/(I + J)$ is finite dimensional. Hence by (a), A/I is right and left faithfully coflat over $A/(I + J)$. Let H be the $\mathcal{H}$-kernel of p. By 1.4 and 1.6 (2), $(I + J)/I = (A/I)H^{+}$ and A/I is H-cogalois.

A is coflat over A/J by (a) and A/J is coflat over $A/(I + J)$ by [NZ1], since $A/J \longrightarrow A/(I + J)$ is a surjective map of finite dimensional Hopf algebras (the dual algebra $(A/J)^*$ is free over $(A/(I + J)^*)$. Hence A is coflat over $A/(I + J)$. By 2.2 (2), A is also coflat over A/I since H is cosemisimple by assumption. Hence, by [D1, Th. 2], A is faithfully coflat as a left or right A/I-comodule.

(2) (a) First assume B is a finite dimensional normal Hopf subalgebra and define $I := AB^{+}$. Then A is a right B-module coalgebra by multiplication in A, and

$$\mathrm{can}\ :\ A \otimes B \longrightarrow A \,\square_{A/I} A\ ,\ a \otimes b \longmapsto \sum a_1 \otimes a_2 b\ .$$

is an injective map. Hence, by 2.3, A is B-cogalois and A is right A/I-coflat. By [D1, Th. 2], A is faithfully A/I-coflat, since A/I is a Hopf algebra by 1.3. Then A is a projective right B-module (cf. [S1, Th. II]). Hence, by 2.4, A is free as a right B-module.

In the same way one shows that A is free as a left B-module.

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(b) In the general case, $B \cap C \subset B$ is a finite dimensional normal Hopf subalgebra. Hence by (a), B is free over $B \cap C$. By 1.4 and 1.6 (1), $B \cap C \subset B$ is a right $B/B(B \cap C)^+$-Galois extension.

By assumption, the Hopf algebra $B/B(B \cap C)^+$ is a semisimple algebra, hence finite dimensional by [Sw, V, Ex. 4]. Then B is finitely generated over $B \cap C$ by [KT, (1.7)] and B is finite dimensional. By 2.4, it suffices to show that A is a projective (left or right) B-module. But A is free over C by (a) and C is free over $B \cap C$ by [NZ1]. Hence A is free over $B \cap C$. Therefore, by 2.2 (1), A is projective over B. ■

REMARK 2.5. (1) The conclusions of 2.1 also hold under the following commutativity resp. cocommutativity assumptions:

Let A be a Hopf algebra.

(a) Let $I \subset A$ be a Hopf ideal of finite codimension and assume $A^{coA/I}$ is commutative. Then A is left and right faithfully coflat over A/I.

(b) Let $B \subset A$ be a finite dimensional Hopf subalgebra and assume A/AB^+ is cocommutative. Then A is free as a left and right B-module.

PROOF. (a) This is shown as in the proof of 2.1 (1) (a) since by [KT (1,10)] A is faithfully flat over $B := A^{coA/I}$.

(b) By 2.3 and the following remark (2) on coalgebras, A is right faithfully coflat over A/AB^+. Hence A is free over B as in the proof of 2.1 (2) (a). ■

(2) *Let C, D be coalgebras and $p : C \longrightarrow D$ a surjective coalgebra map such that C is right D-coflat via p. Assume there is a field extension l of k such that $D \otimes l$ is pointed over l and the direct sum of its irreducible components. For example, D could be cocommutative [Sw, 8.0.5]. Then C is right faithfully D-coflat.*

PROOF. After ground-field extension one can assume $l = k$. Let G be the set of all group-like elements of D. Then $k[G] = D_0$ is the co-

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radical of D. For all $g \in G$ let D^g be the irreducible component of D containing g. By assumption, D is the direct sum of all D^g, $g \in G$. Hence $C \cong C \square_D D$ is isomorphic to the direct sum of all $C \square_D D^g$, $g \in G$. For all $g \in G$ let $C^g := \{c \in C \mid \sum c_1 \otimes p(c_2) \in C \otimes D^g\}$. Then

$$C^g \cong C \square_D D^g \neq 0 \text{ for all } g \in G,$$

since $p : C = \bigoplus_{g \in G} C^g \longrightarrow D = \bigoplus_{g \in G} D^g$ is surjective and the image of C^g lies in D^g for all g.

By [S1, 1.3], C is right faithfully coflat iff $\varepsilon \otimes 1 : C \square_D kg \longrightarrow kg$ is surjective for all $g \in G$.

Let g be a group-like element in G. Assume $\varepsilon \otimes 1 : C \square_D kg \longrightarrow kg$ is the zero map. Then for all $c \in C_g := \{c \in C \mid \sum c_1 \otimes p(c_2) = c \otimes g\}$, $p(c) = \varepsilon(c) g = 0$. But $\Delta(C_g) \subset C \otimes C_g$. Hence $\sum c_1 \otimes p(c_2) = 0$ for all $c \in C_g$ and $C_g = 0$. However, $C_g \cong C^g \square_D kg$ is the socle of the non-zero D^g-comodule C^g, hence $C_g \neq 0$. ∎

3. Central Hopf subalgebras

A subalgebra B of the Hopf algebra A is called a *right coideal subalgebra* if $\Delta(B) \subset B \otimes A$. In this situation, a (right) (A,B)-Hopf module M is by definition a right B-module M which is a right A-comodule such that the comodule structure map Δ_M is B-linear, i.e., $\Delta_M(mb) = \sum m_0 b_1 \otimes m_1 b_2$ for all $m \in M$, $b \in B$.

Recently, Masuoka and Wigner [MW] proved that a commutative Hopf algebra is flat over any right coideal subalgebra. A variation of their method of proof yields the following faithful flatness result for central Hopf subalgebras.

Here, a ring R is called *weakly finite* [Ro, 1.3.30] if for all matrices X, Y in $M_n(R)$, $n \geq 1$, $XY = E$ implies $YX = E$, or equivalently, if any surjective endomorphism of any finitely generated free left or right R-module is bijective. In particular, commutative rings and subrings of one-sided noetherian rings are weakly finite.

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LEMMA 3.1. *Let A be a Hopf algebra and $B \subset A$ a central right coideal subalgebra. If A_{B^+} (central localization of A at the augmentation ideal B^+ of B) is weakly finite, in particular, if A is left or right noetherian, then for all (A,B)-Hopf modules M the localization M_{B^+} is a flat B-module.*

PROOF (as in [MW, 3.3], where A is commutative). For completeness, the proof of [MW] will be repeated in the situation of 3.1. Localization at the maximal ideal B^+ will be written as $(-)' = (-)_{B^+}$.

Let M be an (A,B)-Hopf module. Since M is a filtered union of B-finitely generated Hopf modules [T1, 2.3], one can assume that M is a finitely generated B-module. Let r be the rank of M/MB^+ over k. Then it will be shown that $M' = M_{B^+}$ is free of rank r over $B' = B_{B^+}$, hence flat over B.

Since M is a Hopf module,

$$M \otimes_B A \longrightarrow M/MB^+ \otimes A,\ m \otimes a \longmapsto \sum \overline{m}_0 \otimes m_1 a,$$

is bijective (the inverse mapping is given by $\overline{m} \otimes a \longmapsto \sum m_0 \otimes S(m_1)a$). Hence $M \otimes_B A$ is free of rank r as a right A-module.

By Nakayama's lemma, there is a surjective B'-linear map

$$f : B'^r \longrightarrow M'$$

since $M'/M'B^+ \cong M/MB^+$ has rank r. To show that f is injective it suffices to find a ring extension $B' \subset T$ such that $f \otimes_{B'} T$ is injective (If $i(X) : X \longrightarrow X \otimes_{B'} T$, X any B'-module, is the natural transformation mapping $x \in X$ onto $x \otimes 1$, then $i(M')f = (f \otimes_{B'} T)\, i(B'^r)$ and $i(B'^r)$ is injective since B'^r is flat).

Consider the ring extension $B' \subset A' = A \otimes_B B'$ (central localization). Then $f \otimes_{B'} A' : B'^r \otimes_{B'} A' \longrightarrow M' \otimes_{B'} A'$ is a surjective right A'-linear map. Now $B'^r \otimes_{B'} A' \cong A'^r$, and also $M' \otimes_{B'} A' \cong (M \otimes_B A)' \cong A'^r$ as right A'-modules. By assumption A' is weakly finite. Hence $f \otimes_{B'} A'$ is bijective. ∎

LEMMA 3.2. *Let A be a Hopf algebra and $B \subset A$ a commutative Hopf subalgebra. Assume B is a finitely generated k-algebra and M_{B^+}*

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is flat over B for all (A,B)-Hopf modules M. Then any (A,B)-Hopf module is flat over B.

PROOF. Clearly k can be assumed to be algebraically closed. Let P be a maximal ideal of B. Then $B/P \cong k$ by the Nullstellensatz.

Let $\pi : B \twoheadrightarrow B/P \cong k$ be the canonical epimorphism and

$$\beta : B \longrightarrow B,\ \beta(b) := \sum \pi(S(b_1))\, b_2,$$

the induced automorphism of B mapping B^+ onto P.

If X is any right B-module and γ any algebra automorphism of B, define X_γ to be the B-module whose underlying k-module is X and whose B-module structure $\circ$ is given by $x \circ b := x\gamma(b)$, $x \in X$, $b \in B$. For any B-module Y, $X_\gamma \otimes_B Y \cong X \otimes_B Y_{\gamma^{-1}}$, $x \otimes y \mapsto x \otimes y$. Hence X_γ is B-flat if X is B-flat.

Let M be any (A,B)-Hopf module. Then for all $m \in M$ and $b \in B$

$$\Delta_M(m\beta(b)) = \sum m_0 \pi(S(b_1))\, b_2 \otimes m_1 b_3 = \sum m_0 \beta(b_1) \otimes m_1 b_2 .$$

Hence M_β is an (A,B)-Hopf module where M_β is the twisted B-module as described before and $M_\beta = M$ as A-comodules.

By assumption $(M_\beta)_{B^+}$ is B-flat. Since

$$(M_\beta)_{B^+} \cong (M_{\beta(B^+)})_\beta\ ,\ \frac{m}{s} \longmapsto \frac{m}{\beta(s)}\ ,$$

is an isomorphism of B-modules and $\beta(B^+) = P$, $(M_P)_\beta$ is B-flat. Hence $M_P = ((M_P)_\beta)_{\beta^{-1}}$ is B-flat. This holds for all maximal ideals P. Hence M is B-flat. ∎

THEOREM 3.3. *Let A be a Hopf algebra and $B \subset A$ a central Hopf subalgebra. If A is left or right noetherian, then any (A,B)-Hopf module is flat over B, and A is a faithfully flat B-module.*

PROOF. As in [T1, 2.4] it suffices to show that (A,B)-Hopf modules are B-flat (since A/B is a Hopf module).

Let M be an (A,B)-Hopf module. The commutative Hopf algebra B is a filtered union of Hopf algebras B_i which are finitely generated as k-algebras. Then M is an (A,B_i)-Hopf module for all i by restriction. By 3.1 and 3.2, M is B_i-flat for all i. This implies that M is B-flat. ∎

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