NOTES ON PERVERSE SHEAVES AND VANISHING CYCLES

David B. Massey

§0. **Introduction to Version 3-10**

These notes are my continuing effort to provide a sort of working mathematician's guide to the derived category and perverse sheaves. They began in 1991 as handwritten notes for David Mond and his students, and then I decided to type them for possible inclusion as an appendix in a paper or book. Now, however, these notes represent more of a journal of my understanding of this machinery; whenever I believe that I have understood a new, significant chunk of the subject, I put it here – so that the next time that I want to understand or use that chunk, I will only have to look in one place. Moreover, the place that I have to look will be written in a manner that I can follow easily (and that I can carry around easily). The version number indicates the month and year when I last added to this effort.

The only results of my own that appear here are the Sebastiani-Thom Isomorphism and the fact that Verdier dualizing commutes with the shifted nearby and vanishing cycles up to natural isomorphisms. For the most part, I have merely attempted to pull together some results from a number of sources. Primary sources are [**BBD**], [**G-M3**], [**H**], [**I**], [**K-S**], [**Mac1**], [**M-V**], and [**V**]. There are no proofs given in these notes, though many of the results follow easily from earlier statements. In addition, there are two recent books of Dimca [**Di**] and Schürmann [**Sch**] which are very nice; the book of Dimca is relatively expository, while the book of Schürmann is highly technical.

While the results described here may seem very formal, in fact, the treatment here is fairly informal. If one wishes to avoid the formality of the derived category altogether and, yet, still understand perverse sheaves, there are the AMS notes of MacPherson [**Mac2**] which describe intersection homology and perverse sheaves via Eilenberg-Steenrod type axioms, and MacPherson's more recent treatment of perverse sheaves on regular cell complexes.

I have made some attempt to note results that conflict with the statements that appear in other places. I note these not to emphasize the mistakes in those papers, but rather to let the reader know that I am aware of the discrepancy and believe that the statement that I give is the correct one.

Of course, this is not to say that there may not be mistakes in these notes – in fact, I find typographical mistakes constantly. There may also be mistakes of others that I have copied, or mistakes resulting from my own lack of understanding. For all of these mistakes, I apologize.

Many find it hard to believe that all the machinery in these notes is necessary – or even very useful – for investigating problems in the topology of singularities. So, in the rest of the introduction, I give my own initial motivation for learning this material.

Suppose that $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a polynomial with a critical point at the origin. We wish to discuss the case where this critical point is non-isolated; so, let Σf denote the critical locus of f and let s denote $\dim_{\mathbf{0}}\Sigma f$.

The Milnor fibration for f at the origin exists even for non-isolated singularities. The Milnor fibre of f at the origin has possibly non-trivial cohomology only in dimensions between $n - s$ and n (inclusive).

If $s \geqslant 1$, the origin is not an isolated point in Σf and so, at points $\mathbf{p} \in \Sigma f$ arbitrarily close to $\mathbf{0}$, we may talk about the Milnor fibre of f **at** $\mathbf{p}$. Thus, we have a collection of local data at each point of Σf, and sheaf theory encodes how all this local data patches together.

This is fine. So why does one need a derived category of complexes of sheaves, instead of just plain old normal everyday sheaves? The problem is: at each point of Σf, we wish to have the information about the cohomology of the Milnor fibre at that point. This means that **after we look at the stalk at a point** $\mathbf{p} \in \Sigma f$, we still wish to have cohomology groups in all dimensions at our disposal. It does not take long to realize that what you need is a complex of sheaves. But, really, one frequently only cares about this complex of sheaves up to cohomology. Very loosely speaking, this is what the derived category gives you.

In this example, the two complexes of sheaves that one associates with the Milnor fibre data correspond to the cohomology and reduced cohomology of the Milnor fibre – they are the complexes of sheaves of nearby and vanishing cycles, respectively. The earlier statement that the Milnor fibre has possibly non-trivial cohomology only in dimensions between $n-s$ and n is a type of vanishing condition on these two complexes of sheaves, a vanishing condition which goes by the name "perverse". The general results on perverse sheaves are what give so much power to the machinery of the derived category.

Currently, this paper is organized as follows:

§1. **Constructible Complexes** – This section contains general results on bounded, constructible complexes of sheaves and the derived category.

§2. **Perverse Sheaves** – This section contains the definition and basic results on perverse sheaves. Here, we also give the axiomatic characterization of the intersection cohomology complex. Finally in this section, we also give some results on the category of perverse sheaves. This categorical information is augmented by that in section 5.

§3. **Nearby and Vanishing Cycles** – In this section, we define and examine the complexes of sheaves of nearby and vanishing cycles of an analytic function. These complexes contain hypercohomological information on the Milnor fibre of the function under consideration.

§4. **Some Quick Applications** – In this section, we give three easy examples of results on Milnor fibres which follow from the machinery described in the previous three sections.

§5. **Truncation and Perverse Cohomology** – This section contains an informal discussion on t-structures. This enables us to describe truncation functors and the perverse cohomology of a complex. It also sheds some light on our earlier discussion of the categorical structure of perverse sheaves.

§1. **Constructible Complexes**

Much of this section is lifted from Goresky and MacPherson's "Intersection Homology II" [**G-M3**].

In these notes, we are primarily interested in sheaves on complex analytic spaces, and we make an effort to state most results in this context. However, as one frequently wishes to do such things as intersect with a closed ball, one really needs to consider at least the real semi-analytic case (that is, spaces locally defined by finitely many real analytic inequalities). In fact, one can treat the subanalytic case. Generally, when we leave the analytic category we shall do so without comment, assuming the natural generalizations of any needed results. However, the precise statements in the subanalytic case can be found in [**G-M2**], [**G-M3**], and [**K-S**].

Let R be a regular Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}, \mathbb{Q}, \text{or } \mathbb{C}$). A complex $(\mathbf{A}^\bullet, d^\bullet)$

(usually denoted simply by $\mathbf{A}^\bullet$ if the differentials are clear or arbitrary)

$$\cdots \to \mathbf{A}^{-1} \xrightarrow{d^{-1}} \mathbf{A}^0 \xrightarrow{d^0} \mathbf{A}^1 \xrightarrow{d^1} \mathbf{A}^2 \xrightarrow{d^2} \cdots$$

of sheaves of R-modules on a complex analytic space, X, is *bounded* if $\mathbf{A}^p = 0$ for $|p|$ large.

The cohomology sheaves $\mathbf{H}^p(\mathbf{A}^\bullet)$ arise by taking the (sheaf-theoretic) cohomology of the complex. The stalk of $\mathbf{H}^p(\mathbf{A}^\bullet)$ at a point x is written $\mathbf{H}^p(\mathbf{A}^\bullet)_x$ and is isomorphic to what one gets by first taking stalks and then taking cohomology, i.e., $H^p(\mathbf{A}^\bullet_x)$. The *support of* $\mathbf{A}^\bullet$, $\operatorname{supp}\mathbf{A}^\bullet$, is the closure of the set of points where $\mathbf{A}^\bullet$ has non-zero stalk cohomology, i.e.,

$$\operatorname{supp}\mathbf{A}^\bullet \;=\; \overline{\{x \in X \mid H^*(\mathbf{A}^\bullet)_x \neq 0\}}.$$

The complex $\mathbf{A}^\bullet$ is *constructible* with respect to a complex analytic stratification, $\mathcal{S} = \{S_\alpha\}$, of X provided that, for all α and i, the cohomology sheaves $\mathbf{H}^i(\mathbf{A}^\bullet)_{|S_\alpha}$ are locally constant and have finitely-generated stalks; we write $\mathbf{A}^\bullet \in \mathbf{D}_{\mathcal{S}}(X)$. If $\mathbf{A}^\bullet \in \mathbf{D}_{\mathcal{S}}(X)$ and $\mathbf{A}^\bullet$ is bounded, we write $\mathbf{A}^\bullet \in \mathbf{D}^b_{\mathcal{S}}(X)$.

If $\mathbf{A}^\bullet \in \mathbf{D}^b_{\mathcal{S}}(X)$ for some stratification (and, hence, for any refinement of $\mathcal{S}$) we say that $\mathbf{A}^\bullet$ is a bounded, constructible complex and write $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$. (Note, however, that $\mathbf{D}^b_c(X)$ actually denotes the *derived* category and, while the objects of this category are, in fact, the bounded, constructible complexes, the morphisms are not merely maps between complexes. We shall return to this.)

When it is important to indicate the base ring in the notation, we write $\mathbf{D}_{\mathcal{S}}(R_X)$, $\mathbf{D}^b_{\mathcal{S}}(R_X)$, and $\mathbf{D}^b_c(R_X)$.

A single sheaf $\mathbf{A}$ on X is considered a complex, $\mathbf{A}^\bullet$, on X by letting $\mathbf{A}^0 = \mathbf{A}$ and $\mathbf{A}^i = 0$ for $i \neq 0$; thus, $\mathbf{R}^\bullet_X$ denotes the constant sheaf on X.

The shifted complex $\mathbf{A}^\bullet[n]$ is defined by $(\mathbf{A}^\bullet[n])^k = \mathbf{A}^{n+k}$ and differential $d^k_{[n]} = (-1)^n d^{k+n}$.

A map of complexes is a graded collection of sheaf maps $\phi^\bullet : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ which commute with the differentials. The shifted sheaf map $\phi^\bullet_{[n]} : \mathbf{A}^\bullet[n] \to \mathbf{B}^\bullet[n]$ is defined by $\phi^k_{[n]} := \phi^{k+n}$ (note the lack of a $(-1)^n$). A map of complexes is a *quasi-isomorphism* provided that the induced maps

$$\mathbf{H}^p(\phi^\bullet) : \mathbf{H}^p(\mathbf{A}^\bullet) \to \mathbf{H}^p(\mathbf{B}^\bullet)$$

are isomorphisms for all p. We use the term "quasi-isomorphic" to mean the equivalence relation generated by "existence of a quasi-isomorphism"; this is sometimes refered to as "generalized" quasi-isomorphic.

If $\phi^\bullet : \mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is a quasi-isomorphism and each $\mathbf{I}^p$ is injective, then $\mathbf{I}^\bullet$ is called an *injective resolution* of $\mathbf{A}^\bullet$. Injective resolutions always exist (in our setting), and are unique up to chain homotopy. However, it is sometimes important to associate one particular resolution to a complex, so it is important that there is a *canonical injective resolution* which can be associated to any complex (we shall not describe the canonical resolution here).

If $\mathbf{A}^\bullet$ is a complex on X, then the *hypercohomology module*, $\mathbb{H}^p(X;\mathbf{A}^\bullet)$, is defined to be the p-th cohomology of the global section functor applied to the canonical injective resolution of $\mathbf{A}^\bullet$.

Note that if $\mathbf{A}$ is a single sheaf on X and we form $\mathbf{A}^\bullet$, then $\mathbb{H}^p(X;\mathbf{A}^\bullet) = H^p(X;\mathbf{A})$ = ordinary sheaf cohomology. In particular, $\mathbb{H}^p(X;\mathbf{R}^\bullet_X) = H^p(X;R)$.

Note also that if $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are quasi-isomorphic, then $\mathbb{H}^*(X;\mathbf{A}^\bullet) \cong \mathbb{H}^*(X;\mathbf{B}^\bullet)$.

If Y is a subspace of X and $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, then one usually writes $\mathbb{H}^*(Y;\mathbf{A}^\bullet)$ in place of $\mathbb{H}^*(Y;\mathbf{A}^\bullet{}_{|Y})$.

The usual Mayer-Vietoris sequence is valid for hypercohomology; that is, if U and V form an open cover of X and $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, then there is an exact sequence

$$\cdots \to \mathbb{H}^i(X;\mathbf{A}^\bullet) \to \mathbb{H}^i(U;\mathbf{A}^\bullet) \oplus \mathbb{H}^i(V;\mathbf{A}^\bullet) \to \mathbb{H}^i(U\cap V;\mathbf{A}^\bullet) \to \mathbb{H}^{i+1}(X;\mathbf{A}^\bullet) \to \dots .$$

Of course, hypercohomology is not a homotopy invariant. However, it is true that: if $\mathcal{S}$ is a real analytic Whitney stratification of X, $\mathbf{A}^\bullet \in \mathbf{D}^b_{\mathcal{S}}(X)$, and $r : X \to [0,1)$ is a proper real analytic map such that, for all $S \in \mathcal{S}$, $r_{|_S}$ has no critical values in $(0,1)$, then the inclusion $r^{-1}(0) \hookrightarrow X$ induces an isomorphism

$$\mathbb{H}^i(X;\mathbf{A}^\bullet) \cong \mathbb{H}^i(r^{-1}(0);\mathbf{A}^\bullet).$$

If R is an integral domain, we may talk about the rank of a finitely-generated R-module. In this case, if $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, then the Euler characteristic, χ, of the stalk cohomology is defined as the alternating sum of the ranks of the cohomology modules, i.e., $\chi(\mathbf{A}^\bullet)_x = \sum(-1)^i \operatorname{rank} \mathbf{H}^i(\mathbf{A}^\bullet)_x$. If the hypercohomology modules are finitely-generated – for instance, if $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$ and X is compact – then the Euler characteristic $\chi\big(\mathbb{H}^*(X;\mathbf{A}^\bullet)\big)$ is defined analogously.

If $\mathbf{H}^i(\mathbf{A}^\bullet) = 0$ for all but, possibly, one value of i - say, $i = p$, then $\mathbf{A}^\bullet$ is quasi-isomorphic to the complex that has $\mathbf{H}^p(\mathbf{A}^\bullet)$ in degree p and zero elsewhere. We reserve the term *local system* for a locally constant single sheaf or a complex which is concentrated in degree zero and is locally constant. If M is the stalk of a local system $\mathcal{L}$ on a path-connected space X, then $\mathcal{L}$ is determined up to isomorphism by a monodromy representation $\pi_1(X,\mathbf{x}) \to Aut(M)$, where x is a fixed point in X.

For any $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, there is an E_2 cohomological spectral sequence:

$$E_2^{p,q} = H^p(X;\mathbf{H}^q(\mathbf{A}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X;\mathbf{A}^\bullet).$$

If $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, $x \in X$, and (X,x) is locally embedded in some $\mathbb{C}^n$, then for all $\epsilon > 0$ small, the restriction map $\mathbb{H}^q(\overset{\circ}{B}_\epsilon(x);\mathbf{A}^\bullet) \to \mathbf{H}^q(\mathbf{A}^\bullet)_{\mathbf{x}}$ is an isomorphism $\big($here, $\overset{\circ}{B}_\epsilon(x) = \big\{\mathbf{z} \in \mathbb{C}^n \,\big|\, |\mathbf{z}-\mathbf{x}| < \epsilon\big\}$ $\big)$. If, in addition, R is a principal ideal domain, the Euler characteristic $\chi\big(\mathbb{H}^*(\overset{\circ}{B}_\epsilon(x) - x;\mathbf{A}^\bullet)\big)$ is defined and

$$\chi\big(\mathbb{H}^*(\overset{\circ}{B}_\epsilon(x) - x;\mathbf{A}^\bullet)\big) = \chi\big(\mathbb{H}^*(S_{\epsilon'}(x);\mathbf{A}^\bullet)\big) = 0,$$

where $0 < \epsilon' < \epsilon$ and $S_{\epsilon'}(x)$ denotes the sphere of radius ϵ' centered at x.

We now wish to say a little about the morphisms in the derived category $\mathbf{D}^b_c(X)$. The derived category is obtained by formally inverting the quasi-isomorphisms so that they become isomorphisms in $\mathbf{D}^b_c(X)$. Thus, $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are isomorphic in $\mathbf{D}^b_c(X)$ provided that there exists a complex $\mathbf{C}^\bullet$ and quasi-isomorphisms $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet \to \mathbf{B}^\bullet$; $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are then said to be *incarnations* of the same isomorphism class in $\mathbf{D}^b_c(X)$.

More generally, a morphism in $\mathbf{D}^b_c(X)$ from $\mathbf{A}^\bullet$ to $\mathbf{B}^\bullet$ is an equivalence class of diagrams of maps of complexes $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet \to \mathbf{B}^\bullet$ where $\mathbf{A}^\bullet \leftarrow \mathbf{C}^\bullet$ is a quasi-isomorphism. Two such diagrams,

$$\mathbf{A}^\bullet \xleftarrow{f_1} \mathbf{C}_1^\bullet \xrightarrow{g_1} \mathbf{B}^\bullet, \qquad \mathbf{A}^\bullet \xleftarrow{f_2} \mathbf{C}_2^\bullet \xrightarrow{g_2} \mathbf{B}^\bullet$$

are equivalent provided that there exists a third such diagram $\mathbf{A}^\bullet \xleftarrow{f} \mathbf{C}^\bullet \xrightarrow{g} \mathbf{B}^\bullet$ and a diagram

$$\begin{tikzcd} & \mathbf{C}_1^\bullet \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ \mathbf{A}^\bullet & \mathbf{C}^\bullet \arrow[l, "f"'] \arrow[r, "g"] \arrow[u] \arrow[d] & \mathbf{B}^\bullet \\ & \mathbf{C}_2^\bullet \arrow[ul, "f_2"] \arrow[ur, "g_2"'] & \end{tikzcd}$$

which commutes up to (chain) homotopy.

Composition of morphisms in $\mathbf{D}^b_c(X)$ is not difficult to describe. If we have two representatives of morphisms, from $\mathbf{A}^\bullet$ to $\mathbf{B}^\bullet$ and from $\mathbf{B}^\bullet$ to $\mathbf{D}^\bullet$, respectively,

$$\mathbf{A}^\bullet \xleftarrow{f_1} \mathbf{C}^\bullet_1 \xrightarrow{g_1} \mathbf{B}^\bullet, \qquad \mathbf{B}^\bullet \xleftarrow{f_2} \mathbf{C}^\bullet_2 \xrightarrow{g_2} \mathbf{D}^\bullet$$

then we consider the pull-back $\mathbf{C}^\bullet_1 \times_{\mathbf{B}^\bullet} \mathbf{C}^\bullet_2$ (in the category of chain complexes) and the projections π_1 and π_2 to $\mathbf{C}^\bullet_1$ and $\mathbf{C}^\bullet_2$, respectively. As f_2 is a quasi-isomorphism, so is π_1, and the composed morphism from $\mathbf{A}^\bullet$ to $\mathbf{D}^\bullet$ is represented by $\mathbf{A}^\bullet \xleftarrow{f_1\circ\pi_1} \mathbf{C}^\bullet_1 \times_{\mathbf{B}^\bullet} \mathbf{C}^\bullet_2 \xrightarrow{g_2\circ\pi_2} \mathbf{D}^\bullet$.

If we restrict ourselves to considering only injective complexes, by associating to any complex its canonical injective resolution, then morphisms in the derived category become easy to describe – they are chain-homotopy classes of maps between the injective complexes.

The moral is: in $\mathbf{D}^b_c(X)$, we essentially only care about complexes up to quasi-isomorphism. Note, however, that the objects of $\mathbf{D}^b_c(X)$ are **not** equivalence classes – this is one reason why it is important that to each complex we can associate a **canonical** injective resolution. It allows us to talk about certain functors in $\mathbf{D}^b_c(X)$ being **naturally** isomorphic. When we write $\mathbf{A}^\bullet \cong \mathbf{B}^\bullet$, we mean in $\mathbf{D}^b_c(X)$. As we shall discuss later, $\mathbf{D}^b_c(X)$ is an additive category, but is not Abelian.

Warning: While morphisms of complexes which induce isomorphisms on cohomology sheaves become isomorphisms in the derived category, there are morphisms of complexes which induce the zero map on cohomology sheaves but are not zero in the derived category. The easiest example of such a morphism is given by the following.

Let X be a space consisting of two complex lines L_1 and L_2 which intersect in a single point $\mathbf{p}$. For $i = 1, 2$, let $\widetilde{\mathbb{C}}_{L_i}$ denote the $\mathbb{C}$-constant sheaf on L_i extended by zero to all of X. There is a canonical map, α, from the sheaf $\mathbb{C}_X$ to the direct sum of sheaves $\widetilde{\mathbb{C}}_{L_1} \oplus \widetilde{\mathbb{C}}_{L_2}$, which on $L_1 - \mathbf{p}$ is $\mathrm{id}\oplus 0$, on $L_2 - \mathbf{p}$ is $0 \oplus \mathrm{id}$, and is the diagonal map on the stalk at $\mathbf{p}$. Consider the complex, $\mathbf{A}^\bullet$, which has $\mathbb{C}_X$ in degree 0, $\widetilde{\mathbb{C}}_{L_1} \oplus \widetilde{\mathbb{C}}_{L_2}$ in degree 1, zeroes elsewhere, and the coboundary map from degree 0 to degree 1 is α. This complex has cohomology only in degree 1. Nonetheless, the morphism of complexes from $\mathbf{A}^\bullet$ to $\mathbb{C}^\bullet_X$ which is the identity in degree 0 and is zero elsewhere determines a non-zero morphism in the derived category.

We now wish to describe *derived functors*; for this, we will need the derived category of an arbitrary Abelian category $\mathcal{C}$.

Let $\mathcal{C}$ be an Abelian category. Then, the derived category of bounded complexes in $\mathcal{C}$ is the category whose objects consist of bounded differential complexes of objects of $\mathcal{C}$, and where the morphisms are obtained exactly as in the case of $\mathbf{D}^b_c(X)$ – namely, by inverting the quasi-isomorphisms as we did above. Naturally, we denote this derived category by $\mathbf{D}^b(\mathcal{C})$.

We need some more general notions before we come back to complexes of sheaves. If $\mathcal{C}$ is an Abelian category, then we let $\mathbf{K}^b(\mathcal{C})$ denote the category whose objects are again bounded differential complexes of objects of $\mathcal{C}$, but where the morphisms are chain-homotopy classes of maps of differential complexes. A *triangle* in $\mathbf{K}^b(\mathcal{C})$ is a sequence of morphisms $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to \mathbf{A}^\bullet[1]$, which is usually written in the more"triangular" form

$$\begin{tikzcd} \mathbf{A}^\bullet \arrow[rr] & & \mathbf{B}^\bullet \arrow[dl] \\ & \mathbf{C}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

A triangle in $\mathbf{K}^b(\mathcal{C})$ is called *distinguished* if it is isomorphic in $\mathbf{K}^b(\mathcal{C})$ to a diagram of maps of complexes

$$\begin{array}{ccc} \widetilde{\mathbf{A}}^\bullet & \xrightarrow{\phi} & \widetilde{\mathbf{B}}^\bullet \\ {}_{[1]}\nwarrow & & \swarrow \\ & \mathbf{M}^\bullet & \end{array}$$

where $\mathbf{M}^\bullet$ is the algebraic mapping cone of ϕ and $\widetilde{\mathbf{B}}^\bullet \to \mathbf{M}^\bullet \to \widetilde{\mathbf{A}}^\bullet[1]$ are the canonical maps. (Recall that the algebraic mapping cone is defined by

$$\mathbf{M}^k := \widetilde{\mathbf{A}}^{k+1} \oplus \ \widetilde{\mathbf{B}}^k \longrightarrow \widetilde{\mathbf{A}}^{k+2} \oplus \ \widetilde{\mathbf{B}}^{k+1} =: \mathbf{M}^{k+1}$$

$$(a, b) \longmapsto (-\partial a, \phi a + \delta b)$$

where ∂ and δ are the differentials of $\widetilde{\mathbf{A}}^\bullet$ and $\widetilde{\mathbf{B}}^\bullet$ respectively.) Note that if $\phi = 0$, then we have an equality $\mathbf{M}^\bullet = \mathbf{A}^\bullet[1] \oplus \mathbf{B}^\bullet$ (recall that the shifted complex $\mathbf{A}^\bullet[1]$ has as its differential the negated, shifted differential of $\mathbf{A}^\bullet$).

Now we can define derived functors. Let $\mathcal{C}$ denote the Abelian category of sheaves of R-modules on an analytic space X, and let $\mathcal{C}'$ be another Abelian category. Suppose that F is an additive, covariant functor from $\mathbf{K}^b(\mathcal{C})$ to $\mathbf{K}^b(\mathcal{C}')$ such that $F \circ [1] = [1] \circ F$ and such that F takes distinguished triangles to distinguished triangles (such an F is called a *functor of triangulated categories*). Suppose also that, for all complexes of injective sheaves $\mathbf{I}^\bullet \in \mathbf{K}^b(\mathcal{C})$ which are quasi-isomorphic to 0, $F(\mathbf{I}^\bullet)$ is also quasi-isomorphic to 0.

Then, F induces a morphism RF – the *right derived functor of* F – from $\mathbf{D}^b(X)$ to $\mathbf{D}^b(\mathcal{C}')$; for any $\mathbf{A}^\bullet \in \mathbf{D}^b(X)$, let $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ denote the canonical injective resolution of $\mathbf{A}^\bullet$, and define $RF(\mathbf{A}^\bullet) := F(\mathbf{I}^\bullet)$. The action of RF on the morphisms is the obvious associated one.

A morphism $F : \mathbf{K}^b(\mathcal{C}) \to \mathbf{K}^b(\mathcal{C}')$ as described above is frequently obtained by starting with a left-exact functor $T : \mathcal{C} \to \mathcal{C}'$ and then extending T in a term-wise fashion to be a functor from $\mathbf{K}^b(\mathcal{C})$ to $\mathbf{K}^b(\mathcal{C}')$. In this case, we naturally write RT for the derived functor.

This is the process which is applied to:

$\Gamma(X; \cdot)$ (global sections);

$\Gamma_c(X; \cdot)$ (global sections with compact support);

f_* (direct image);

$f_!$ (direct image with proper supports); and

f^* (pull-back or inverse image),

where $f : X \to Y$ is a continuous map (actually, in these notes, we would need an analytically constructible map; e.g., an analytic map).

If the functor T is an exact functor from sheaves to sheaves, then $RT(\mathbf{A}^\bullet) \cong T(\mathbf{A}^\bullet)$; in this case, we normally suppress the R. Hence, if $f : X \to Y$, $\mathbf{A}^\bullet \in \mathbf{D}^b(X)$, and $\mathbf{B}^\bullet \in \mathbf{D}^b(Y)$, we write:

$f^*\mathbf{B}^\bullet$;

$f_!\mathbf{A}^\bullet$, if f is the inclusion of a locally closed subset and, hence, $f_!$ is extension by zero;

$f_*\mathbf{A}^\bullet$, if f is the inclusion of a closed subspace.

Note that hypercohomology is just the cohomology of the derived global section functor, i.e., $\mathbb{H}^*(X; \cdot) = H^* \circ R\Gamma(X; \cdot)$. The cohomology of the derived functor of global sections with compact support is the *compactly supported hypercohomology* and is denoted $\mathbb{H}^*_c(X; \mathbf{A}^\bullet)$.

If $f : X \to Y$ is the inclusion of a subset and $\mathbf{B}^\bullet \in \mathbf{D}^b(Y)$, then the *restriction of* $\mathbf{B}^\bullet$ to X is defined to be $f^*(\mathbf{B}^\bullet)$, and is usually denoted by $\mathbf{B}^\bullet{}_{|X}$.

If $f : X \to Y$ is continuous and $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, there is a natural map

$$Rf_!\mathbf{A}^\bullet \to Rf_*\mathbf{A}^\bullet.$$

For $f : X \to Y$ continuous, there are canonical isomorphisms

$$R\Gamma(X;\mathbf{A}^\bullet) \cong R\Gamma(Y;Rf_*\mathbf{A}^\bullet) \text{ and } R\Gamma_c(X;\mathbf{A}^\bullet) \cong R\Gamma_c(Y;Rf_!\mathbf{A}^\bullet)$$

which lead to canonical isomorphisms

$$\mathbb{H}^*(X;\mathbf{A}^\bullet) \cong \mathbb{H}^*(Y;Rf_*\mathbf{A}^\bullet) \quad\text{and}\quad \mathbb{H}_c^*(X;\mathbf{A}^\bullet) \cong \mathbb{H}_c^*(Y;Rf_!\mathbf{A}^\bullet)$$

for all $\mathbf{A}^\bullet$ in $\mathbf{D}_c^b(X)$.

If $f : X \to Y$ is continuous, $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, and $\mathbf{B}^\bullet \in \mathbf{D}_c^b(Y)$, there are natural maps induced by restriction of sections

$$\mathbf{B}^\bullet \to Rf_*f^*\mathbf{B}^\bullet \quad\text{and}\quad f^*Rf_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet.$$

If f is the inclusion of a locally closed subset , then $f^*Rf_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet$ is an isomorphism.

If $\{S_\alpha\}$ is a stratification of X, $\mathbf{A}^\bullet \in \mathbf{D}^b_{\{S_\alpha\times\mathbb{C}^k\}}(X\times\mathbb{C}^k)$, and $\pi : X\times\mathbb{C}^k \to X$ is the projection, then restriction of sections induces a quasi-isomorphism $\pi^*R\pi_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet$.

It follows easily that if $j : X \hookrightarrow X\times\mathbb{C}^k$ is the zero section, then $\pi^*j^*\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$. This says exactly what one expects: the complex $\mathbf{A}^\bullet$ has a product structure in the $\mathbb{C}^k$ directions.

An important consequence of this is the following: let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of X and let $\mathbf{A}^\bullet \in \mathbf{D}^b_{\mathcal{S}}(X)$. Let $x \in S_\alpha \subseteq X$. As S_α is a Whitney stratum, X has a product structure along S_α near x. By the above, $\mathbf{A}^\bullet$ itself also has a product structure along S_α. Hence, by taking a normal slice, many problems concerning the complex $\mathbf{A}^\bullet$ can be reduced to considering a zero-dimensional stratum.

Let $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in \mathbf{D}_c^b(X)$. Define $\mathbf{A}^\bullet \otimes \mathbf{B}^\bullet$ to be the single complex which is associated to the double complex $\mathbf{A}^p \otimes \mathbf{B}^q$. The left derived functor $\mathbf{A}^\bullet \overset{L}{\otimes} *$ is defined by

$$\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet = \mathbf{A}^\bullet \otimes \mathbf{J}^\bullet,$$

where $\mathbf{J}^\bullet$ is a flat resolution of $\mathbf{B}^\bullet$, i.e., the stalks of $\mathbf{J}^\bullet$ are flat R-modules and there exists a quasi-isomorphism $\mathbf{J}^\bullet \to \mathbf{B}^\bullet$.

For all $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in \mathbf{D}_c^b(X)$, there is an isomorphism $\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet \cong \mathbf{B}^\bullet \overset{L}{\otimes} \mathbf{A}^\bullet$.

For any map $f : X \to Y$ and any $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in \mathbf{D}_c^b(Y)$,

$$f^*(\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet) \cong f^*\mathbf{A}^\bullet \overset{L}{\otimes} f^*\mathbf{B}^\bullet.$$

Fix a complex $\mathbf{B}^\bullet$ on X. There are two covariant functors which we wish to consider: the functor $\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$ from the category of complexes of sheaves to complexes of sheaves and the functor

$Hom^\bullet(\mathbf{B}^\bullet, *)$ from the category of complexes of sheaves to the category of complexes of R-modules. These functors are given by

$$(\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))^n = \prod_{p\in\mathbb{Z}} \mathbf{Hom}(\mathbf{B}^p, \mathbf{A}^{n+p})$$

and

$$(Hom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))^n = \prod_{p\in\mathbb{Z}} Hom(\mathbf{B}^p, \mathbf{A}^{n+p})$$

with differential given by

$$[\partial^n f]^p = \partial^{n+p} f^p + (-1)^{n+1} f^{p+1}\partial^p$$

(there is an indexing error in [**I**, 12.4]). The associated derived functors are, respectively, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$ and $RHom^\bullet(\mathbf{B}^\bullet, *)$.

If $\mathbf{P}^\bullet \to \mathbf{B}^\bullet$ is a projective resolution of $\mathbf{B}^\bullet$, then, in $\mathbf{D}^b_c(X)$, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)$ is isomorphic to $\mathbf{Hom}^\bullet(\mathbf{P}^\bullet, \mathbf{A}^\bullet)$. For all k, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet[k]) = R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)[k]$.

The functor $RHom^\bullet(\mathbf{B}^\bullet, *)$ is naturally isomorphic to the derived global sections functor applied to $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$, i.e., for any $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$,

$$RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong R\Gamma\left(X; R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)\right).$$

$H^0(RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))$ is naturally isomorphic as an R-module to the derived category homomorphisms from $\mathbf{B}^\bullet$ to $\mathbf{A}^\bullet$, i.e.,

$$H^0(RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)) \cong Hom_{\mathbf{D}^b_c(X)}(\mathbf{B}^\bullet, \mathbf{A}^\bullet).$$

If $\mathbf{B}^\bullet$ and $\mathbf{A}^\bullet$ have locally constant cohomology sheaves on X then, for all $x \in X$, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)_x$ is naturally isomorphic to $RHom^\bullet(\mathbf{B}^\bullet_x, \mathbf{A}^\bullet_x)$.

For all $\mathbf{A}^\bullet, \mathbf{B}^\bullet, \mathbf{C}^\bullet \in \mathbf{D}^b_c(X)$, there is a natural isomorphism

$$R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet)).$$

Moreover, if $\mathbf{C}^\bullet$ has locally constant cohomology sheaves, then there is an isomorphism

$$R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet \overset{L}{\otimes} \mathbf{C}^\bullet) \cong R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \overset{L}{\otimes} \mathbf{C}^\bullet.$$

For all j, we define $\mathbf{Ext}^j(\mathbf{B}^\bullet, \mathbf{A}^\bullet) := \mathbf{H}^j(R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))$ and define

$$Ext^j(\mathbf{B}^\bullet, \mathbf{A}^\bullet) := H^j(RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)).$$

It is immediate that we have isomorphisms of R-modules

$$Ext^j(\mathbf{B}^\bullet, \mathbf{A}^\bullet) = H^0(RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet[j])) \cong Hom_{\mathbf{D}^b_c(X)}(\mathbf{B}^\bullet, \mathbf{A}^\bullet[j]).$$

If $X = point$, $\mathbf{B}^\bullet \in \mathbf{D}^b_c(X)$, and the base ring is a PID, then $\mathbf{B}^\bullet \cong \bigoplus_k \mathbf{H}^k(\mathbf{B}^\bullet)[-k]$ in $\mathbf{D}^b_c(X)$; if we also have $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, then

$$\mathbf{H}^i(\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet) \cong \Big(\bigoplus_{p+q=i} \mathbf{H}^p(\mathbf{A}^\bullet) \otimes \mathbf{H}^q(\mathbf{B}^\bullet)\Big) \ \oplus\ \Big(\bigoplus_{r+s=i+1} \mathrm{Tor}(\mathbf{H}^r(\mathbf{A}^\bullet), \mathbf{H}^s(\mathbf{B}^\bullet))\Big).$$

If, in addition, the cohomology modules of $\mathbf{A}^\bullet$ are projective (hence, free), then

$$Hom_{\mathbf{D}_c^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \cong \bigoplus_k Hom\left(\mathbf{H}^k(\mathbf{A}^\bullet), \mathbf{H}^k(\mathbf{B}^\bullet)\right).$$

If we have a map $f : X \to Y$, then the functors f^* and Rf_* are adjoints of each other in the derived category. In fact, for all $\mathbf{A}^\bullet$ on X and $\mathbf{B}^\bullet$ on Y, there is a canonical isomorphism in $\mathbf{D}_c^b(Y)$

$$R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, Rf_*\mathbf{A}^\bullet) \cong Rf_* R\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, \mathbf{A}^\bullet)$$

and so

$$Hom_{\mathbf{D}_c^b(Y)}(\mathbf{B}^\bullet, Rf_*\mathbf{A}^\bullet) \cong H^0\left(RHom^\bullet(\mathbf{B}^\bullet, Rf_*\mathbf{A}^\bullet)\right) \cong \mathbb{H}^0\left(Y; R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, Rf_*\mathbf{A}^\bullet)\right) \cong$$

$$\mathbb{H}^0\left(X; R\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, \mathbf{A}^\bullet)\right) \cong H^0\left(RHom^\bullet(f^*\mathbf{B}^\bullet, \mathbf{A}^\bullet)\right) \cong Hom_{\mathbf{D}_c^b(X)}(f^*\mathbf{B}^\bullet, \mathbf{A}^\bullet).$$

We wish now to describe an analogous adjoint for $Rf_!$

Let $\mathbf{I}^\bullet$ be a complex of injective sheaves on Y. Then, $f^!(\mathbf{I}^\bullet)$ is defined to be the sheaf associated to the presheaf given by

$$\Gamma(U; f^!\mathbf{I}^\bullet) = Hom^\bullet(f_!\mathbf{K}_U^\bullet, \mathbf{I}^\bullet),$$

for any open $U \subseteq X$, where $\mathbf{K}_U^\bullet$ denotes the canonical injective resolution of the constant sheaf $\mathbf{R}_U^\bullet$. For any $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, define $f^!\mathbf{A}^\bullet$ to be $f^!\mathbf{I}^\bullet$, where $\mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$.

Now that we have this definition, we may state:

(Verdier Duality) If $f : X \to Y$, $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, and $\mathbf{B}^\bullet \in \mathbf{D}_c^b(Y)$, then there is a canonical isomorphism in $\mathbf{D}_c^b(Y)$:

$$Rf_* R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong R\mathbf{Hom}^\bullet(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet)$$

and so

$$Hom_{\mathbf{D}_c^b(X)}(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong Hom_{\mathbf{D}_c^b(Y)}(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet).$$

If $\mathbf{B}^\bullet$ and $\mathbf{C}^\bullet$ are in $\mathbf{D}_c^b(Y)$, then we have an isomorphism

$$f^! R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong R\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, f^!\mathbf{C}^\bullet).$$

Let $f : X \to$ *point*. Then, the *dualizing complex*, $\mathbb{D}_X^\bullet$, is $f^!$ applied to the constant sheaf, i.e., $\mathbb{D}_X^\bullet = f^!\mathbf{R}_{pt}^\bullet$. For any complex $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, the *Verdier dual* (or, simply, *the dual*) of $\mathbf{A}^\bullet$ is $R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}_X^\bullet)$ and is denoted by $\mathcal{D}_X\mathbf{A}^\bullet$ (or just $\mathcal{D}\mathbf{A}^\bullet$). There is a canonical isomorphism between $\mathbb{D}_X^\bullet$ and the dual of the constant sheaf on X, i.e., $\mathbb{D}_X^\bullet \cong \mathcal{D}\mathbf{R}_X^\bullet$.

Let $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$. If the base ring R is a field or a Dedekind domain (e.g., a PID), then $\mathcal{D}\mathbf{A}^\bullet$ is well-defined up to quasi-isomorphism by:

for any open $U \subseteq X$, there is a natural split exact sequence:

$$0 \to Ext(\mathbb{H}_c^{q+1}(U; \mathbf{A}^\bullet), R) \to \mathbb{H}^{-q}(U; \mathcal{D}\mathbf{A}^\bullet) \to Hom(\mathbb{H}_c^q(U; \mathbf{A}^\bullet), R) \to 0.$$

In particular, if R is a field, then $\mathbb{H}^{-q}(U;\mathcal{D}\mathbf{A}^\bullet)\cong\mathbb{H}^q_c(U;\mathbf{A}^\bullet)$, and so

$$\mathbf{H}^q(\mathcal{D}\mathbf{A}^\bullet)_x\cong\mathbb{H}^q(\overset{\circ}{B}_\epsilon(x);\mathcal{D}\mathbf{A}^\bullet)\cong\mathbb{H}^{-q}_c(\overset{\circ}{B}_\epsilon(x);\mathbf{A}^\bullet).$$

If, in addition, X is compact, $\mathbb{H}^{-q}(X;\mathcal{D}\mathbf{A}^\bullet)\cong\mathbb{H}^q(X;\mathbf{A}^\bullet)$.

Dualizing is a local operation, i.e., if $i:U\hookrightarrow X$ is the inclusion of an open subset and $\mathbf{A}^\bullet\in\mathbf{D}^b_c(X)$, then $i^*\mathcal{D}\mathbf{A}^\bullet\cong\mathcal{D}i^*\mathbf{A}^\bullet$.

If $\mathcal{L}$ is a local system on a connected real m-manifold, N, then $(\mathcal{D}\mathcal{L}^\bullet)[-m]$ is a local system; if, in addition, N is oriented, and $\mathcal{L}$ is actually locally free with stalks R^a and monodromy representation $\eta:\pi_1(N,\mathbf{p})\to Aut(R^a)$, then $\mathcal{D}\mathcal{L}^\bullet[-m]$ is a local system with stalks equal to the dual, $(R^a)^\vee:=Hom_R(R^a,R)\cong R^a$, of R^a and monodromy ${}^\vee\eta:\pi_1(N,\mathbf{p})\to Aut((R^a)^\vee)$, where ${}^\vee\eta(\alpha)$ is the dual of $(\eta(\alpha))^{-1}$, i.e., $\big({}^\vee\eta(\alpha)\big)(f):=f\circ(\eta(\alpha))^{-1}=f\circ(\eta(\alpha^{-1}))$.

If $\mathbf{A}^\bullet\in\mathbf{D}^b_c(X)$, then $\mathcal{D}(\mathbf{A}^\bullet[n])=(\mathcal{D}\mathbf{A}^\bullet)[-n]$.

If $\pi:X\times\mathbb{C}^n\to X$ is projection, then $\mathcal{D}(\pi^*\mathbf{A}^\bullet)[-n]\cong\pi^*(\mathcal{D}\mathbf{A}^\bullet)[n]$.

The dualizing complex, $\mathbb{D}_X$, is quasi-isomorphic to the complex of sheaves of singular chains on X which is associated to the complex of presheaves, $\mathbf{C}^\bullet$, given by $\Gamma(U;\mathbf{C}^{-p}):=C_p(X,X-U;R)$.

The cohomology sheaves of $\mathbb{D}^\bullet_X$ are non-zero in negative degrees only, with stalks $\mathbf{H}^{-p}(\mathbb{D}^\bullet_X)_x=H_p(X,X-x;R)$.

If X is an oriented, real m-manifold, then $\mathbb{D}^\bullet_X[-m]$ is quasi-isomorphic to $\mathbf{R}^\bullet_X$, and so $\mathcal{D}\mathbf{R}^\bullet_X\cong\mathbf{R}^\bullet_X[m]$ and, for all $\mathbf{A}^\bullet\in\mathbf{D}^b_c(X)$,

$$\mathcal{D}\mathbf{A}^\bullet\cong R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{R}^\bullet_X[m])=\big(R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{R}^\bullet_X)\big)[m].$$

Note that if X is an even-dimensional, oriented, real m-manifold, with $m=2n$, then $\mathcal{D}(\mathbf{R}^\bullet_X[n])\cong\mathbf{R}^\bullet_X[n]$, i.e., $\mathbf{R}^\bullet_X[n]$ is self-dual.

$\mathbb{D}^\bullet_{V\times W}$ is naturally isomorphic to $\pi_1^*\mathbb{D}^\bullet_V\overset{L}{\otimes}\pi_2^*\mathbb{D}^\bullet_W$, where π_1 and π_2 are the projections onto V and W, respectively.

$\mathbb{H}^*(X;\mathbb{D}^\bullet_X)\cong$ homology with closed supports = Borel-Moore homology.

If X is a real, smooth, oriented m-manifold and $R=\mathbb{R}$, then $\mathbb{D}^\bullet_X[-m]$ is naturally isomorphic to the complex of real differential forms on X.

$\mathbb{D}^\bullet_X$ is constructible with respect to any Whitney stratification of X. It follows that if $\mathcal{S}$ is a Whitney stratification of X, then $\mathbf{A}^\bullet\in\mathbf{D}^b_{\mathcal{S}}(X)$ if and only if $\mathcal{D}\mathbf{A}^\bullet\in\mathbf{D}^b_{\mathcal{S}}(X)$.

The functor $\mathcal{D}$ from $\mathbf{D}^b_c(X)$ to $\mathbf{D}^b_c(X)$ is contravariant, and $\mathcal{D}\mathcal{D}$ is naturally isomorphic to the identity. For all $\mathbf{A}^\bullet,\mathbf{B}^\bullet\in\mathbf{D}^b_c(X)$, we have isomorphisms

$$R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{B}^\bullet)\cong R\mathbf{Hom}^\bullet(\mathcal{D}\mathbf{B}^\bullet,\mathcal{D}\mathbf{A}^\bullet)\cong\mathcal{D}\left(\mathcal{D}\mathbf{B}^\bullet\overset{L}{\otimes}\mathbf{A}^\bullet\right).$$

If $f:X\to Y$ is continuous, then we have natural isomorphisms

$$Rf_!\cong\mathcal{D}Rf_*\mathcal{D}\quad\text{and}\quad f^!\cong\mathcal{D}f^*\mathcal{D}.$$

If $Y \subseteq X$ and $f : X - Y \hookrightarrow X$ is the inclusion, we define

$$\mathbb{H}^k(X, Y; \mathbf{A}^\bullet) := \mathbb{H}^k(X; f_! f^! \mathbf{A}^\bullet).$$

Suppose that X is compact, U is an open subset of X, and $Y := X - U$. Then,

$$\mathbb{H}^k_c(U; \mathbf{A}^\bullet) \cong \mathbb{H}^k(X, Y; \mathbf{A}^\bullet).$$

If $x \in X$ and $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, then for all $\epsilon > 0$ sufficiently small,

$$\mathbb{H}^q_c(\overset{\circ}{B}_\epsilon(x); \mathbf{A}^\bullet) \cong \mathbb{H}^q(\overset{\circ}{B}_\epsilon(x), \overset{\circ}{B}_\epsilon(x) - x; \mathbf{A}^\bullet) \cong \mathbb{H}^q(X, X - x; \mathbf{A}^\bullet)$$

and so, if R is a field,

$$\mathbf{H}^{-q}(\mathcal{D}\mathbf{A}^\bullet)_x \cong \mathbb{H}^q(\overset{\circ}{B}_\epsilon(x), \overset{\circ}{B}_\epsilon(x) - x; \mathbf{A}^\bullet) \cong \mathbb{H}^q(X, X - x; \mathbf{A}^\bullet).$$

Suppose that the base ring R is a field, that X is a compact space, and that $\mathbf{A}^\bullet$ is a self-dual complex on X, i.e., there exists an isomorphism $\alpha : \mathbf{A}^\bullet \to \mathcal{D}\mathbf{A}^\bullet$. Then, for all k, there are isomorphisms

$$\mathbb{H}^{-k}(X; \mathbf{A}^\bullet) \cong \mathbb{H}^{-k}(X; \mathcal{D}\mathbf{A}^\bullet) \cong Hom(\mathbb{H}^k_c(X; \mathbf{A}^\bullet), R) \cong Hom(\mathbb{H}^k(X; \mathbf{A}^\bullet), R)$$

and so, there is an induced bilinear map:

$$\mathbb{H}^k(X; \mathbf{A}^\bullet) \times \mathbb{H}^{-k}(X; \mathbf{A}^\bullet) \to R.$$

When $k = 0$, the map above is generally referred to as the *intersection pairing* or *intersection form*. This agrees with the standard usage of the term for a compact, even-dimensional, oriented, real $2n$-manifold, where $\mathbf{A}^\bullet = \mathbf{R}^\bullet_X[n]$, for then $\mathbb{H}^0(X; \mathbf{A}^\bullet) = \mathbb{H}^0(X; \mathbf{R}^\bullet_X[n]) \cong H^n(X; R)$.

If $f : X \to Y$ and $g : Y \to Z$, then there are natural isomorphisms

$$R(g \circ f)_* \cong Rg_* \circ Rf_* \qquad R(g \circ f)_! \cong Rg_! \circ Rf_!$$

and

$$(g \circ f)^* \cong f^* \circ g^* \qquad (g \circ f)^! \cong f^! \circ g^!.$$

Suppose that $f : Y \hookrightarrow X$ is inclusion of a subset. Then, if Y is open, there is a natural isomorphism $f^! \cong f^*$. More generally, if $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$ and $Y \cap \operatorname{supp} \mathbf{A}^\bullet$ is open in $\operatorname{supp} \mathbf{A}^\bullet$, then $f^!\mathbf{A}^\bullet \cong f^*\mathbf{A}^\bullet$. If Y is closed, then there are natural isomorphisms $Rf_! \cong f_! \cong f_* \cong Rf_*$.

Excision has the following form: if $Y \subseteq U \subseteq X$, where U is open in X and Y is closed in X, then

$$\mathbb{H}^k(X, X - Y; \mathbf{A}^\bullet) \cong \mathbb{H}^k(U, U - Y; \mathbf{A}^\bullet).$$

This isomorphism on hypercohomology actually follows from a natural isomorphism of functors. Let j denote the inclusion of Y into U, and let i denote the inclusion of U into X. Then, there is a natural isomorphism $(ij)_!(ij)^! \cong i_* j_! j^! i^*$.

Suppose that $f : X \to Y$ is continuous. One might hope that, as an analog to the natural map $Rf_! \to Rf_*$, there would be a natural map $f^* \to f^!$; however, the situation is more complicated than that. If $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are in $\mathbf{D}^b_c(Y)$, then there is a natural map

$$f^!\mathbf{A}^\bullet \overset{L}{\otimes} f^*\mathbf{B}^\bullet \to f^!\big(\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet\big).$$

In particular, taking $\mathbf{A}^\bullet$ to be the constant sheaf $\mathbf{R}^\bullet_Y$, there is a natural map

$$f^!\mathbf{R}^\bullet_Y \overset{L}{\otimes} f^* \to f^!.$$

The complex $f^!\mathbf{R}^\bullet_Y$ is referred to as the *relative dualizing complex*, and is denoted by $\omega_{X/Y}$ (when the map f is clear).

The map $f^!\mathbf{R}^\bullet_Y \overset{L}{\otimes} f^* \to f^!$ is an isomorphism if f is a *topological submersion* of some dimension r (we use r here because it is a **real** dimension), i.e., if every point in X possesses an open neighborhood $\mathcal{U}$ such that $\mathcal{V} := f(\mathcal{U})$ is open and such that the restriction $f : \mathcal{U} \to \mathcal{V}$ is homeomorphic to the projection $\mathcal{V} \times \mathbb{R}^r \to \mathcal{V}$. If $f : X \to Y$ is a topological submersion of dimension r, and X and Y are orientable manifolds, then $\omega_{X/Y} \cong \mathbf{R}^\bullet_X[r]$ and so, for all $\mathbf{B}^\bullet \in \mathbf{D}^b_c(Y)$,

$$f^*\mathbf{B}^\bullet[r] \cong f^!\mathbf{B}^\bullet.$$

If $g : Y \hookrightarrow X$ is the inclusion of an orientable submanifold into another orientable manifold, and r is the real codimension of Y in X, and $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$ has locally constant cohomology on X, then $g^!\mathbf{F}^\bullet$ has locally constant cohomology on Y and

$$g^*\mathbf{F}^\bullet[-r] \cong g^!\mathbf{F}^\bullet.$$

(There is an error here in [**G-M3**]; they have the negation of the correct shift.)

If $f : X \to Y$ is continuous, $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, and $\mathbf{B}^\bullet \in \mathbf{D}^b_c(Y)$, then dual to the canonical maps

$$\mathbf{B}^\bullet \to Rf_*f^*\mathbf{B}^\bullet \quad\text{and}\quad f^*Rf_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet$$

are the canonical maps

$$Rf_!f^!\mathbf{B}^\bullet \to \mathbf{B}^\bullet \quad\text{and}\quad \mathbf{A}^\bullet \to f^!Rf_!\mathbf{A}^\bullet,$$

and, if f is the inclusion of a locally closed subset, $\mathbf{A}^\bullet \to f^!Rf_!\mathbf{A}^\bullet$ is an isomorphism.

If

$$\begin{array}{ccc} Z & \xrightarrow{\hat{f}} & W \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f} & S \end{array}$$

is a pull-back diagram (fibre square, Cartesian diagram), then for all $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, $R\hat{f}_!\hat{\pi}^*\mathbf{A}^\bullet \cong \pi^*Rf_!\mathbf{A}^\bullet$ (there is an error in [**G-M3**]; they have lower $*$'s, not lower !'s, but see below for when these agree) and, dually, $R\hat{f}_*\hat{\pi}^!\mathbf{A}^\bullet \cong \pi^!Rf_*\mathbf{A}^\bullet$. In particular, if f is proper (and, hence, $\hat{f}$ is proper) or π is the inclusion of an open subset (and, hence, so is $\hat{\pi}$, up to homeomorphism), then $R\hat{f}_*\hat{\pi}^*\mathbf{A}^\bullet \cong \pi^*Rf_*\mathbf{A}^\bullet$; this is also true if $W = S \times \mathbb{C}^n$ and $\pi : W \to S$ is projection (and, hence, up to homeomorphism, $\hat{\pi}$ is projection from $X \times \mathbb{C}^n$ to X).

Still looking at the pull-back diagram above, we find that the natural map $Rf_!f^! \to \mathrm{id}$ yields a natural map $R\hat{f}_!\hat{\pi}^*f^! \cong \pi^*Rf_!f^! \to \pi^*$, which in turn yields natural maps

$$\hat{\pi}^*f^! \to \hat{f}^!R\hat{f}_!\hat{\pi}^*f^! \to \hat{f}^!\pi^*.$$

If we have $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$ and $\mathbf{B}^\bullet \in \mathbf{D}_c^b(W)$, then we let $\mathbf{A}^\bullet \overset{L}{\boxtimes}_S \mathbf{B}^\bullet := \hat{\pi}^*\mathbf{A}^\bullet \overset{L}{\otimes} \hat{f}^*\mathbf{B}^\bullet$, assuming that the maps $\hat{\pi}$ and $\hat{f}$ are clear. If S is a point, so that $Z \cong X \times W$, then we omit the S in the notation and write simply $\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet$.

There is a Künneth formula, which we now state in its most general form, in terms of maps over a base space S. Suppose that we have two maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ over S, i.e., there are commutative diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ r_1 \searrow & & \swarrow t_1 \\ & \mathrm{S} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_2 & \xrightarrow{f_2} & Y_2 \\ r_2 \searrow & & \swarrow t_2 \\ & \mathrm{S} & \end{array}.$$

Then, there is an induced map $f = f_1 \times_S f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$. If $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X_1)$ and $\mathbf{B}^\bullet \in \mathbf{D}_c^b(X_2)$, there is the **Künneth isomorphism**

$$Rf_!(\mathbf{A}^\bullet \overset{L}{\boxtimes}_S \mathbf{B}^\bullet) \cong Rf_{1!}\mathbf{A}^\bullet \overset{L}{\boxtimes}_S Rf_{2!}\mathbf{B}^\bullet.$$

Using the above notation, if S is a point and $\mathbf{F}^\bullet \in \mathbf{D}_c^b(Y_1)$ and $\mathbf{G}^\bullet \in \mathbf{D}_c^b(Y_2)$, then there is a natural isomorphism (the **adjoint Künneth isomorphism**)

$$f^!(\mathbf{F}^\bullet \overset{L}{\boxtimes} \mathbf{G}^\bullet) \cong {f_1}^!\mathbf{F}^\bullet \overset{L}{\boxtimes} {f_2}^!\mathbf{G}^\bullet.$$

If we let q_1 and q_2 denote the projections from $Y_1 \times Y_2$ onto Y_1 and Y_2, respectively, then the adjoint Künneth formula can be proved by using the following natural isomorphism twice

$$\mathcal{D}\mathbf{F}^\bullet \overset{L}{\boxtimes} \mathbf{G}^\bullet \cong R\mathbf{Hom}^\bullet(q_1^*\mathbf{F}^\bullet, q_2^!\mathbf{G}^\bullet).$$

Let Z be a locally closed subset of an analytic space X. There are two derived functors, associated to Z, that we wish to describe: the derived functors of restricting-extending to Z, and of taking the sections supported on Z. Let i denote the inclusion of Z into X.

If $\mathbf{A}$ is a (single) sheaf on X, then the restriction-extension of $\mathbf{A}$ to Z, $(\mathbf{A})_Z$, is given by $i_!i^*(\mathbf{A})$. Thus, up to isomorphism, $(\mathbf{A})_Z$ is characterized by $((\mathbf{A})_Z)_{|Z} \cong \mathbf{A}_{|Z}$ and $((\mathbf{A})_Z)_{|X-Z} = 0$. This functor is exact, and so we also denote the derived functor by $()_Z$.

Now, we want to define the sheaf of sections of $\mathbf{A}$ supported by Z, $\Gamma_Z(\mathbf{A})$. If $\mathcal{U}$ is an open subset of X which contains Z as a closed subset (of $\mathcal{U}$), then we define

$$\Gamma_Z(\mathcal{U}; \mathbf{A}) := \ker\left\{\Gamma(\mathcal{U}; \mathbf{A}) \to \Gamma(\mathcal{U} - Z; \mathbf{A})\right\}.$$

Up to isomorphism, $\Gamma_Z(\mathcal{U}; \mathbf{A})$ is independent of the open set $\mathcal{U}$ (this uses that $\mathbf{A}$ is a sheaf, not just a presheaf). The sheaf $\Gamma_Z(\mathbf{A})$ is defined by, for all open $\mathcal{U} \subseteq X$, $\Gamma(\mathcal{U}; \Gamma_Z(\mathbf{A})) := \Gamma_{\mathcal{U} \cap Z}(\mathcal{U}; \mathbf{A})$.

One easily sees that $\operatorname{supp}\Gamma_Z(\mathbf{A}) \subseteq \overline{Z}$. It is also easy to see that, if Z is open, then $\Gamma_Z(\mathbf{A}) = i_*i^*(\mathbf{A})$. The functor $\Gamma_Z()$ is left exact; of course, we denote the right derived functor by $R\Gamma_Z()$. There is a canonical isomorphism $i^! \cong i^* \circ R\Gamma_Z$. It follows that, if Z is closed, then $R\Gamma_Z \cong i_!i^!$. In addition, if Z is open, then $R\Gamma_Z \cong Ri_*i^*$.

Avoiding Injective Resolutions:

To calculate right derived functors from the definition, one must use injective resolutions. However, this is inconvenient in many proofs if some functor involved in the proof does not take injective complexes to injective complexes. There are (at least) four "devices" which come to our aid, and enable one to prove many of the isomorphisms described earlier; these devices are **fine resolutions**, **flabby resolutions**, **c-soft resolutions**, and **injective subcategories with respect to a functor**.

If T is a left-exact functor on the category of sheaves on X, then the right derived functor RT is defined by applying T term-wise to the sheaves in a canonical injective resolution. The importance of saying that a certain subcategory of the category of sheaves on X is injective with respect to T is that one may take a resolution in which the individual sheaves are in the given subcategory, then apply T term-wise, and end up with a complex which is canonically isomorphic to that produced by RT.

Recall that a single sheaf $\mathbf{A}$ on X is:

- fine, if partitions of unity of $\mathbf{A}$ subordinate to any given locally finite open cover of X exist;

- flabby, if for every open subset $\mathcal{U} \subseteq X$, the restriction homomorphism $\Gamma(X;\mathbf{A}) \to \Gamma(\mathcal{U};\mathbf{A})$ is a surjection;

- c-soft, if for every compact subset $\mathcal{K} \subseteq X$, the restriction homomorphism $\Gamma(X;\mathbf{A}) \to \Gamma(\mathcal{K};\mathbf{A})$ is a surjection;

Injective sheaves are flabby, and flabby sheaves are c-soft. In addition, fine sheaves are c-soft.

The subcategory of c-soft sheaves is injective with respect to the functors $\Gamma(X;*)$, $\Gamma_c(X;*)$, and $f_!$. The subcategory of flabby sheaves is injective with respect to the functor f_*.

If $\mathbf{A}^\bullet$ is a bounded complex of sheaves, then a bounded c-soft resolution of $\mathbf{A}^\bullet$ is given by $\mathbf{A}^\bullet \to \mathbf{A}^\bullet \otimes \mathbf{S}^\bullet$, where $\mathbf{S}^\bullet$ is a c-soft, bounded above, resolution of the base ring (which always exists in our context).

Triangles:

$\mathbf{D}_c^b(X)$ is an additive category, but is not an Abelian category. In place of short exact sequences, one has distinguished triangles, just as we did in $\mathbf{K}^b(\mathcal{C})$. A triangle of morphisms in $\mathbf{D}_c^b(X)$

$$\begin{tikzcd} \mathbf{A}^\bullet \arrow[rr] & & \mathbf{B}^\bullet \arrow[dl] \\ & \mathbf{C}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

(the [1] indicates a morphism shifted by one, i.e., a morphism $\mathbf{C}^\bullet \to \mathbf{A}^\bullet[1]$) is called *distinguished* if it is isomorphic in $\mathbf{D}_c^b(X)$ to a diagram of sheaf maps

$$\begin{tikzcd} \tilde{\mathbf{A}}^\bullet \arrow[rr, "\phi"] & & \tilde{\mathbf{B}}^\bullet \arrow[dl] \\ & \mathbf{M}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $\mathbf{M}^\bullet$ is the algebraic mapping cone of ϕ and $\mathbf{B}^\bullet \to \mathbf{M}^\bullet \to \mathbf{A}^\bullet[1]$ are the canonical maps. The "in-line" notation for a triangle is $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to \mathbf{A}^\bullet[1]$ or $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \xrightarrow{[1]}$.

Any short exact sequence of complexes becomes a distinguished triangle in $\mathbf{D}_c^b(X)$. Any edge of a distinguished triangle determines the triangle up to (non-canonical) isomorphism in $\mathbf{D}_c^b(X)$; more specifically, we can "turn" the distinguished triangle: $\mathbf{A}^\bullet \xrightarrow{\alpha} \mathbf{B}^\bullet \xrightarrow{\beta} \mathbf{C}^\bullet \xrightarrow{\gamma} \mathbf{A}^\bullet[1]$ is a distinguished triangle if and only if

$$\mathbf{B}^\bullet \xrightarrow{\beta} \mathbf{C}^\bullet \xrightarrow{\gamma} \mathbf{A}^\bullet[1] \xrightarrow{-\alpha[1]} \mathbf{B}^\bullet[1]$$

is a distinguished triangle.

Given two distinguished triangles and maps u and v which make the left-hand square of the following diagram commute

$$\begin{array}{llll}
\mathbf{A}^\bullet \to & \mathbf{B}^\bullet \to & \mathbf{C}^\bullet \to & \mathbf{A}^\bullet[1] \\
\downarrow u & \downarrow v & & \downarrow u[1] \\
\widetilde{\mathbf{A}^\bullet} \to & \widetilde{\mathbf{B}^\bullet} \to & \widetilde{\mathbf{C}^\bullet} \to & \widetilde{\mathbf{A}^\bullet}[1],
\end{array}$$

there exists a (not necessarily unique) $w : \mathbf{C}^\bullet \to \widetilde{\mathbf{C}^\bullet}$ such that

$$\begin{array}{llll}
\mathbf{A}^\bullet \to & \mathbf{B}^\bullet \to & \mathbf{C}^\bullet \to & \mathbf{A}^\bullet[1] \\
\downarrow u & \downarrow v & \downarrow w & \downarrow u[1] \\
\widetilde{\mathbf{A}^\bullet} \to & \widetilde{\mathbf{B}^\bullet} \to & \widetilde{\mathbf{C}^\bullet} \to & \widetilde{\mathbf{A}^\bullet}[1]
\end{array}$$

also commutes. We say that the original commutative square embeds in a morphism of distinguished triangles.

We will now give the *octahedral lemma*, which allows one to realize an isomorphism between mapping cones of two composed maps.

Suppose that we have two distinguished triangles $\mathbf{A}^\bullet \xrightarrow{f} \mathbf{B}^\bullet \xrightarrow{g} \mathbf{C}^\bullet \xrightarrow{h} \mathbf{A}^\bullet[1]$ and $\mathbf{B}^\bullet \xrightarrow{\beta} \mathbf{E}^\bullet \xrightarrow{\gamma} \mathbf{F}^\bullet \xrightarrow{\delta} \mathbf{B}^\bullet[1]$. Then, there exists a complex $\mathbf{M}^\bullet$ and two distinguished triangles

$$\mathbf{A}^\bullet \xrightarrow{\beta\circ f} \mathbf{E}^\bullet \xrightarrow{\tau} \mathbf{M}^\bullet \xrightarrow{\omega} \mathbf{A}^\bullet[1]$$

and

$$\mathbf{C}^\bullet \xrightarrow{\sigma} \mathbf{M}^\bullet \xrightarrow{\nu} \mathbf{F}^\bullet \xrightarrow{g[1]\circ\delta} \mathbf{C}^\bullet[1].$$

such that the following diagram commutes

$$\begin{array}{cccccccc}
\mathbf{A}^\bullet & \xrightarrow{f} & \mathbf{B}^\bullet & \xrightarrow{g} & \mathbf{C}^\bullet & \xrightarrow{h} & \mathbf{A}^\bullet[1] \\
\mathrm{id}\downarrow & & \beta\downarrow & & \sigma\downarrow & & \mathrm{id}\downarrow \\
\mathbf{A}^\bullet & \xrightarrow{\beta\circ f} & \mathbf{E}^\bullet & \xrightarrow{\tau} & \mathbf{M}^\bullet & \xrightarrow{\omega} & \mathbf{A}^\bullet[1] \\
f\downarrow & & \mathrm{id}\downarrow & & \nu\downarrow & & f[1]\downarrow \\
\mathbf{B}^\bullet & \xrightarrow{\beta} & \mathbf{E}^\bullet & \xrightarrow{\gamma} & \mathbf{F}^\bullet & \xrightarrow{\delta} & \mathbf{B}^\bullet[1] \\
g\downarrow & & \tau\downarrow & & \mathrm{id}\downarrow & & g[1]\downarrow \\
\mathbf{C}^\bullet & \xrightarrow{\sigma} & \mathbf{M}^\bullet & \xrightarrow{\nu} & \mathbf{F}^\bullet & \xrightarrow{g[1]\circ\delta} & \mathbf{C}^\bullet[1].
\end{array}$$

It is somewhat difficult to draw this in its octahedral form (and worse to type it); moreover, it is no easier to read the relations from the octahedron. However, the interested reader can give it a try: the octahedron is formed by gluing together two pyramids along their square bases. One pyramid has $\mathbf{B}^\bullet$ at its top vertex, with $\mathbf{A}^\bullet$, $\mathbf{C}^\bullet$, $\mathbf{E}^\bullet$, and $\mathbf{F}^\bullet$ at the vertices of its base, and has the original two distinguished triangles as opposite faces. The other pyramid has $\mathbf{M}^\bullet$ at its top vertex, with $\mathbf{A}^\bullet$, $\mathbf{C}^\bullet$, $\mathbf{E}^\bullet$, and $\mathbf{F}^\bullet$ at the vertices of its base, and has the other two distinguished triangles (whose existence is asserted in the lemma) as opposite faces. The two pyramids are joined together by matching the vertices of the two bases, forming an octahedron in which the faces are alternately distinguished and commuting.

A distinguished triangle determines long exact sequences on cohomology and hypercohomology:

$$\cdots \to \mathbf{H}^p(\mathbf{A}^\bullet) \to \mathbf{H}^p(\mathbf{B}^\bullet) \to \mathbf{H}^p(\mathbf{C}^\bullet) \to \mathbf{H}^{p+1}(\mathbf{A}^\bullet) \to \cdots$$

$$\cdots \to \mathbb{H}^p(X;\mathbf{A}^\bullet) \to \mathbb{H}^p(X;\mathbf{B}^\bullet) \to \mathbb{H}^p(X;\mathbf{C}^\bullet) \to \mathbb{H}^{p+1}(X;\mathbf{A}^\bullet) \to \cdots .$$

If $f : X \to Y$ and $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then the functors $Rf_*, Rf_!, f^*,\ f^!,$ and $\mathbf{F}^\bullet \overset{L}{\otimes} *$ all take distinguished triangles to distinguished triangles (with all arrows in the same direction and the shift in the same place).

As for $R\mathbf{Hom}^\bullet$, if $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$ and $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to \mathbf{A}^\bullet[1]$ is a distinguished triangle in $\mathbf{D}^b_c(X)$, then we have distinguished triangles

$$\begin{tikzcd} R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet,\mathbf{A}^\bullet) \arrow[rr] & & R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet,\mathbf{B}^\bullet) \arrow[dl] \\ & R\mathbf{Hom}^\bullet(\mathbf{F}^\bullet,\mathbf{C}^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet,\mathbf{F}^\bullet) \arrow[dr, "{[1]}"'] & & R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet,\mathbf{F}^\bullet) \arrow[ll] \\ & R\mathbf{Hom}^\bullet(\mathbf{C}^\bullet,\mathbf{F}^\bullet). \arrow[ur] & \end{tikzcd}$$

By applying the right-hand triangle above to the special case where $\mathbf{F}^\bullet = \mathbb{D}^\bullet_X$, we find that the dualizing functor $\mathcal{D}$ also takes distinguished triangles to distinguished triangles, but with a reversal of arrows, i.e., if we have a distinguished triangle $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to \mathbf{A}^\bullet[1]$ in $\mathbf{D}^b_c(X)$, then, by dualizing, we have distinguished triangles

$$\begin{tikzcd} \mathcal{D}\mathbf{A}^\bullet \arrow[dr, "{[1]}"'] & & \mathcal{D}\mathbf{B}^\bullet \arrow[ll] \\ & \mathcal{D}\mathbf{C}^\bullet \arrow[ur] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathcal{D}\mathbf{C}^\bullet \arrow[rr] & & \mathcal{D}\mathbf{B}^\bullet \arrow[dl] \\ & \mathcal{D}\mathbf{A}^\bullet. \arrow[ul, "{[1]}"] & \end{tikzcd}$$

There are (at least) six distinguished triangles associated to the functors $()_Z$ and $R\Gamma_Z$. Let $\mathbf{F}^\bullet$ be in $\mathbf{D}^b_c(X)$, $\mathcal{U}_1$ and $\mathcal{U}_2$ be open subsets of X, Z_1 and Z_2 be closed subsets of X, Z be locally closed in X, and Z' be closed in Z. Then, we have the following distinguished triangles:

$$R\Gamma_{\mathcal{U}_1\cup\mathcal{U}_2}(\mathbf{F}^\bullet) \to R\Gamma_{\mathcal{U}_1}(\mathbf{F}^\bullet) \oplus R\Gamma_{\mathcal{U}_2}(\mathbf{F}^\bullet) \to R\Gamma_{\mathcal{U}_1\cap\mathcal{U}_2}(\mathbf{F}^\bullet) \xrightarrow{[1]}$$

$$R\Gamma_{Z_1\cap Z_2}(\mathbf{F}^\bullet) \to R\Gamma_{Z_1}(\mathbf{F}^\bullet) \oplus R\Gamma_{Z_2}(\mathbf{F}^\bullet) \to R\Gamma_{Z_1\cup Z_2}(\mathbf{F}^\bullet) \xrightarrow{[1]}$$

$$(\mathbf{F}^\bullet)_{\mathcal{U}_1\cap\mathcal{U}_2} \to (\mathbf{F}^\bullet)_{\mathcal{U}_1} \oplus (\mathbf{F}^\bullet)_{\mathcal{U}_2} \to (\mathbf{F}^\bullet)_{\mathcal{U}_1\cup\mathcal{U}_2} \xrightarrow{[1]}$$

$$(\mathbf{F}^\bullet)_{Z_1\cup Z_2} \to (\mathbf{F}^\bullet)_{Z_1} \oplus (\mathbf{F}^\bullet)_{Z_2} \to (\mathbf{F}^\bullet)_{Z_1\cap Z_2} \xrightarrow{[1]}$$

$$R\Gamma_{Z'}(\mathbf{F}^\bullet) \to R\Gamma_Z(\mathbf{F}^\bullet) \to R\Gamma_{Z-Z'}(\mathbf{F}^\bullet) \xrightarrow{[1]}$$

$$(\mathbf{F}^\bullet)_{Z-Z'} \to (\mathbf{F}^\bullet)_Z \to (\mathbf{F}^\bullet)_{Z'} \xrightarrow{[1]} .$$

If $j : Y \hookrightarrow X$ is the inclusion of a closed subspace and $i : U \hookrightarrow X$ the inclusion of the open complement, then for all $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$, the last two triangles above give us distinguished triangles

$$\begin{tikzcd} Ri_!i^!\mathbf{A}^\bullet \arrow[r] & \mathbf{A}^\bullet \arrow[dl] \\ Rj_*j^*\mathbf{A}^\bullet \arrow[u, "{[1]}"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Rj_!j^!\mathbf{A}^\bullet \arrow[r] & \mathbf{A}^\bullet \arrow[dl] \\ Ri_*i^*\mathbf{A}^\bullet, \arrow[u, "{[1]}"] & \end{tikzcd}$$

where the second triangle can be obtained from the first by dualizing. (Note that $Ri_! = i_!$, $Rj_* = j_* = j_! = Rj_!$, and $i^! = i^*$.) The associated long exact sequences on hypercohomology are those for the pairs $\mathbb{H}^*(X, Y; \mathbf{A}^\bullet)$ and $\mathbb{H}^*(X, U; \mathbf{A}^\bullet)$, respectively. From either of these triangles, and the fact that $j^*i_! = 0$, one easily obtains a natural isomorphism $j^!Ri_![1] \cong j^*Ri_*$.

It is worth noting that the natural map $Rj_*j^*[-1]\mathbf{A}^\bullet \to Ri_!i^!\mathbf{A}^\bullet$ induces the zero map on stalk cohomology, but would usually **not** be the zero map in the derived category.

As in our earlier discussion of the octahedral lemma, all of the morphisms of the last two paragraphs fit into the *fundamental octahedron of the pair* (X, Y). The four distinguished triangles making up the fundamental octahedron are the top pair

$$Ri_!i^!\mathbf{A}^\bullet \to \mathbf{A}^\bullet \to Rj_*j^*\mathbf{A}^\bullet \to Ri_!i^!\mathbf{A}^\bullet[1]$$

and

$$\mathbf{A}^\bullet \to Ri_*i^*\mathbf{A}^\bullet \to Rj_!j^!\mathbf{A}^\bullet[1] \to \mathbf{A}^\bullet[1]$$

and the bottom pair

$$Ri_!i^!\mathbf{A}^\bullet \to Ri_*i^*\mathbf{A}^\bullet \to \mathbf{M}^\bullet \to Ri_!i^!\mathbf{A}^\bullet[1]$$

and

$$Rj_*j^*\mathbf{A}^\bullet \to \mathbf{M}^\bullet \to Rj_!j^!\mathbf{A}^\bullet[1] \to Rj_*j^*\mathbf{A}^\bullet[1],$$

where $\mathbf{M}^\bullet \cong Rj_!j^!Ri_!i^!\mathbf{A}^\bullet[1] \cong Rj_*j^*Ri_*i^*\mathbf{A}^\bullet$.

Consider again the situation in which we have a pull-back diagram

$$\begin{tikzcd} Z \arrow[r, "\hat{f}"] \arrow[d, "\hat{\pi}"'] & W \arrow[d, "\pi"] \\ X \arrow[r, "f"] & S, \end{tikzcd}$$

where we saw that the natural map $Rf_!f^! \to \mathrm{id}$ yields a natural map $R\hat{f}_!\hat{\pi}^*f^! \cong \pi^*Rf_!f^! \to \pi^*$, which in turn yields natural maps

$$\hat{\pi}^*f^! \to \hat{f}^!R\hat{f}_!\hat{\pi}^*f^! \to \hat{f}^!\pi^*.$$

If f is the inclusion of an open or closed subset, and $l: S - X \to S$ is the inclusion of the complement, then, for all $\mathbf{A}^\bullet \in \mathbf{D}_c^b(S)$, we have the distinguished triangle

$$Rf_!f^!\mathbf{A}^\bullet \to \mathbf{A}^\bullet \to Rl_*l^*\mathbf{A}^\bullet \to Rf_!f^!\mathbf{A}^\bullet[1],$$

and so, have the induced triangle

$$\hat{f}^!\pi^*Rf_!f^!\mathbf{A}^\bullet \to \hat{f}^!\pi^*\mathbf{A}^\bullet \to \hat{f}^!\pi^*Rl_*l^*\mathbf{A}^\bullet \to \hat{f}^!\pi^*Rf_!f^!\mathbf{A}^\bullet[1],$$

which, up to isomorphism, is

$$\hat{\pi}^*f^!\mathbf{A}^\bullet \to \hat{f}^!\pi^*\mathbf{A}^\bullet \to \hat{f}^!\pi^*Rl_*l^*\mathbf{A}^\bullet \to \hat{\pi}^*f^!\mathbf{A}^\bullet[1];$$

it follows that, if $\pi^*Rl_*l^*\mathbf{A}^\bullet \cong \pi^!Rl_*l^*\mathbf{A}^\bullet$, then $\hat{f}^!\pi^*Rl_*l^*\mathbf{A}^\bullet = 0$, and so $\hat{\pi}^*f^!\mathbf{A}^\bullet \to \hat{f}^!\pi^*\mathbf{A}^\bullet$ is an isomorphism. This would be the case, for instance, if f and π are inclusions of closed subsets, and $S = X \cup W$.

§2. **Perverse Sheaves**

Let X be a complex analytic space, and for each $\mathbf{x} \in X$, let $j_{\mathbf{x}}: \mathbf{x} \hookrightarrow X$ denote the inclusion.

If $\mathbf{F}^\bullet \in \mathbf{D}_c^b(X)$, then the *support of* $\mathbf{H}^i(\mathbf{F}^\bullet)$ is the closure in X of

$$\{\mathbf{x} \in X \mid \mathbf{H}^i(\mathbf{F}^\bullet)_{\mathbf{x}} \neq 0\} = \{\mathbf{x} \in X \mid \mathbf{H}^i(j_{\mathbf{x}}^*\mathbf{F}^\bullet) \neq 0\};$$

we denote this by $\operatorname{supp}^{\mathrm{i}} \mathbf{F}^\bullet$.

The *i-th cosupport of* $\mathbf{F}^\bullet$ is the closure in X of

$$\{\mathbf{x} \in X \mid \mathbf{H}^i(j_{\mathbf{x}}^!\mathbf{F}^\bullet) \neq 0\} \ = \ \{\mathbf{x} \in X \mid \mathbb{H}^i(\mathring{B}_\epsilon(x), \mathring{B}_\epsilon(x) - x;\ \mathbf{F}^\bullet) \neq 0\};$$

we denote this by $\operatorname{cosupp}^{\mathrm{i}} \mathbf{F}^\bullet$.

If the base ring, R, is a field, then $\operatorname{cosupp}^{\mathrm{i}} \mathbf{F}^\bullet = \operatorname{supp}^{\text{-i}} \mathcal{D}\mathbf{F}^\bullet$.

Definition: Let X be a complex analytic space (not necessarily pure dimensional). Then, $\mathbf{P}^\bullet \in \mathbf{D}_c^b(X)$ is *perverse* provided that for all i:

(support) $\quad \dim(\operatorname{supp}^{\text{-i}} \mathbf{P}^\bullet) \leqslant i;$

(cosupport) $\quad \dim(\operatorname{cosupp}^{\mathrm{i}} \mathbf{P}^\bullet) \leqslant i,$

where we set the dimension of the empty set to be $-\infty$.

This definition is equivalent to: let $\{S_\alpha\}$ be any stratification of X with respect to which $\mathbf{P}^\bullet$ is constructible, and let $s_\alpha: S_\alpha \hookrightarrow X$ denote the inclusion. Then,

(support) $\quad \mathbf{H}^k(s_\alpha^*\mathbf{P}^\bullet) = 0 \quad$ for $k > -\dim_{\mathbb{C}} S_\alpha$;

(cosupport) $\quad \mathbf{H}^k(s_\alpha^!\mathbf{P}^\bullet) = 0 \quad$ for $k < -\dim_{\mathbb{C}} S_\alpha$.

(There is a missing minus sign in [**G-M2**, 6.A.5].)

From the definition, it is clear that being perverse is a local property.

If the base ring R is, in fact, a field, then the support and cosupport conditions can be written in the following form, which is symmetric with respect to dualizing:

(support) $\quad \dim(\mathrm{supp}^{-i}\,\mathbf{P}^\bullet) \leqslant i;$

(cosupport) $\quad \dim(\mathrm{supp}^{-i}\,\mathcal{D}\mathbf{P}^\bullet) \leqslant i.$

Suppose that $\mathbf{P}^\bullet$ is perverse on X, (X,x) is locally embedded in $\mathbb{C}^n$, S is a stratum of a Whitney stratification with respect to which $\mathbf{P}^\bullet$ is constructible, and $x \in S$. Let M be a normal slice of X at x; that is, let M be a smooth submanifold of $\mathbb{C}^n$ of dimension $n - \dim S$ which transversely intersects S at x. Then, for some open neighborhood U of x in X, $\mathbf{P}^\bullet_{|_{X\cap M\cap U}}[-\dim S]$ is perverse on $X \cap M \cap U$.

Let $\mathbf{P}^\bullet \in \mathbf{D}^b_c(X)$; one can use this normal slicing proposition to prove that, if $\mathbf{P}^\bullet$ is perverse, then $\mathbf{H}^i(\mathbf{P}^\bullet) = 0$ for all $i < -\dim X$.

A converse to the normal slicing proposition is: if $\pi : X \times \mathbb{C}^s \to X$ is projection and $\mathbf{P}^\bullet$ is perverse on X, then $\pi^*\mathbf{P}^\bullet[s]$ is perverse.

Suppose $\mathbf{P}^\bullet \in \mathbf{D}^b_c(X^n)$. Let $\Sigma = \mathrm{supp}\ \mathbf{P}^\bullet$. Then, $\mathbf{P}^\bullet$ is perverse on X if and only if $\mathbf{P}^\bullet_{|_\Sigma}$ is perverse on Σ. Another way of saying this is: if $j : \Sigma \to X$ is the inclusion of a closed subspace, then $\mathbf{Q}^\bullet$ is perverse on Σ if and only if $j_!\mathbf{Q}^\bullet$ is perverse on X. It follows that if $\mathbf{P}^\bullet$ is perverse, then $\mathbf{H}^i(\mathbf{P}^\bullet) = 0$ unless $-\dim \Sigma \leqslant i \leqslant 0$.

If X is (not necessarily purely) n-dimensional, then it is trivial to show that the shifted constant sheaf $\mathbf{R}^\bullet_X[n]$ satisfies the support condition. If X is a curve, then the constant sheaf $\mathbf{R}^\bullet_X[1]$ is perverse. If X is a surface, then $\mathbf{R}^\bullet_X[2]$ is perverse if and only if, for all $p \in X$, for all sufficiently small open balls B (in some Riemannian metric), centered at p, $B - \{p\}$ is connected; this would, for instance, be the case if X were a surface, with isolated singularities, which is locally irreducible everywhere. If X is a purely n-dimensional local complete intersection, then $\mathbf{R}^\bullet_X[n]$ is perverse.

The other basic example of a perverse sheaf that we wish to give is that of intersection cohomology with local coefficients (with the perverse indexing, i.e., cohomology in degrees less than or equal to zero). Note that the definition below is shifted by $-\dim_{\mathbb{C}} X$ from the definition in [**G-M3**], and yields a perverse sheaf which has possibly non-zero stalk cohomology only in degrees between $-\dim_{\mathbb{C}} X$ and -1, inclusive (unless the space has isolated points).

Let X be a complex analytic set, let $\overset{\circ}{X}$ be an open dense subset of the smooth part of X, and let $\mathcal{L}^\bullet$ be a perverse sheaf on $\overset{\circ}{X}$, such that the restriction of $\mathcal{L}^\bullet$ to each connected component C of $\overset{\circ}{X}$ is a local system on C, shifted by $\dim C$ (i.e., shifted into degree $-\dim C$). Let $Y := X - \overset{\circ}{X}$ and let $j : Y \hookrightarrow X$ denote the (closed) inclusion.

Then, in $\mathbf{D}^b_c(X)$, there is an object, $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)$, called the *intersection cohomology with coefficients in* $\mathcal{L}$ which is uniquely determined up to quasi-isomorphism by:

1) $\quad \mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)_{|_{\overset{\circ}{X}}} \;=\; \mathcal{L}^\bullet;$

for all i,

2) $\quad \dim \mathrm{supp}^{-i}\left(j^*[-1]\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)\right) \;=\; \dim \overline{\left\{\mathbf{x} \in Y \mid H^{-(i+1)}\big(\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)\big)_x \neq 0\right\}} \;\leq i;$

3) $\dim \operatorname{cosupp}^i\left(j^![1]\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)\right) \;=\; \dim \overline{\left\{\mathbf{x}\in Y \mid \mathbb{H}^{i+1}\left(\overset{\circ}{B}_\epsilon(x), \overset{\circ}{B}_\epsilon(x)-x;\ \mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)\right)\neq 0\right\}} \;\leq i.$

$\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)$ is a perverse sheaf. The uniqueness assertion implies that $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)$ is the direct sum of the extension by zero of the intersection cohomology complex of each irreducible component of X using $\mathcal{L}^\bullet$ restricted to an open dense set of that component. If X is purely n-dimensional, and $\mathcal{L}$ is a local system (in degree 0) on an open dense set subset of the smooth part of X, then one usually writes simply $\mathbf{IC}^\bullet_X(\mathcal{L})$ in place of $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet[n])$. See section 5 for more on $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet)$.

The Category of Perverse Sheaves (see, also, section 5)

The *category of perverse sheaves* on X, $Perv(X)$, is the full subcategory of $\mathbf{D}^b_c(X)$ whose objects are the perverse sheaves. Given a Whitney stratification, $\mathcal{S}$, of X, it is also useful to consider the category $Perv_{\mathcal{S}}(X) := Perv(X)\cap \mathbf{D}^b_{\mathcal{S}}(X)$ of perverse sheaves which are constructible with respect to $\mathcal{S}$.

$Perv(X)$ and $Perv_{\mathcal{S}}(X)$ are both Abelian categories in which the short exact sequences

$$0\rightarrow \mathbf{A}^\bullet\rightarrow \mathbf{B}^\bullet\rightarrow \mathbf{C}^\bullet\rightarrow 0$$

are precisely the distinguished triangles

$$\begin{tikzcd} \mathbf{A}^\bullet \arrow[rr] & & \mathbf{B}^\bullet \arrow[dl] \\ & \mathbf{C}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}.$$

If we have complexes $\mathbf{A}^\bullet, \mathbf{B}^\bullet$, and $\mathbf{C}^\bullet$ in $\mathbf{D}^b_c(X)$ (resp. $\mathbf{D}^b_{\mathcal{S}}(X)$), a distinguished triangle $\mathbf{A}^\bullet\rightarrow\mathbf{B}^\bullet\rightarrow \mathbf{C}^\bullet\rightarrow\mathbf{A}^\bullet[1]$, and $\mathbf{A}^\bullet$ and $\mathbf{C}^\bullet$ are perverse, then $\mathbf{B}^\bullet$ is also in $Perv(X)$ (resp. $Perv_{\mathcal{S}}(X)$).

Suppose that the base ring, R, is a field. Then, $Perv(X)$ is Noetherian and locally Artinian; in particular, locally, each perverse sheaf has a finite composition series in $Perv(X)$ with uniquely determined simple subquotients. If the Whitney stratification $\mathcal{S}$ has a finite number of strata, then $Perv_{\mathcal{S}}(X)$ is Artinian and Noetherian; if X is compact, then $Perv(X)$ is also Artinian. When $\mathbf{P}^\bullet$ is a perverse sheaf with a composition series, we let $[\mathbf{P}^\bullet]$ denote the formal sum of the simple subquotients of the composition series, counted with multiplicities, i.e., we consider $\mathbf{P}^\bullet$ in the Grothendieck group of perverse sheaves. If T is a morphism from a perverse sheaf $\mathbf{P}^\bullet$ to itself, then (locally, at least) $[\ker T]=[\operatorname{coker} T]$; in particular, $\operatorname{supp}(\ker T)=\operatorname{supp}(\operatorname{coker} T)$.

Continue to assume that the base ring is a field. Then, the simple objects in $Perv(X)$ (resp. $Perv_{\mathcal{S}}(X)$) are extensions by zero of intersection cohomology sheaves on irreducible analytic subvarieties (resp. connected components of strata) of X with coefficients in irreducible local systems. To be precise, let M be a d-dimensional connected analytic submanifold (resp. a connected component of a stratum) of X and let $\mathcal{L}_M$ be an irreducible local system on M; then, the pair $(\overline{M}, \mathcal{L}_M)$ is called an *irreducible enriched subvariety* of X (where $\overline{M}$ denotes the closure of M). Let $j:\overline{M}\hookrightarrow X$ denote the inclusion. Then, the simple objects of $Perv(X)$ (resp. $Perv_{\mathcal{S}}(X)$) are those of the form $j_!\mathbf{IC}^\bullet_{\overline{M}}(\mathcal{L}^\bullet_M[d])$, where $(\overline{M}, \mathcal{L}_M)$ is an irreducible enriched subvariety (again, we are indexing intersection cohomology so that it is non-zero only in non-positive dimensions).

Warning: The stalk cohomology of kernels, cokernels, and images in the perverse category do **not** correspond in any simple way to kernels, cokernels, and images of the stalk cohomology. In fact, the easiest example of this phenomenon is essentially our example from the warning in Section 1.

Recall that we let X be a space consisting of two complex lines L_1 and L_2 which intersect in a single point $\mathbf{p}$. For $i=1,2$, let $\widetilde{\mathbb{C}}_{L_i}$ denote the $\mathbb{C}$-constant sheaf on L_i extended by zero to all of X. Now, this time, we shift to obtain perverse sheaves.

There is a canonical map, α, from the perverse sheaf $\mathbf{B}^\bullet := \mathbb{C}^\bullet_X[-1]$ to the perverse sheaf which is the direct sum of sheaves $\mathbf{C}^\bullet := \widetilde{\mathbb{C}}^\bullet_{L_1}[-1] \oplus \widetilde{\mathbb{C}}^\bullet_{L_2}[-1]$, such that, on $L_1 - \mathbf{p}$, α is $\mathrm{id} \oplus 0$, on $L_2 - \mathbf{p}$ is $0 \oplus \mathrm{id}$, and is the diagonal map on the stalk at $\mathbf{p}$. In fact, this map α is the canonical map from the constant sheaf to the intersection cohomology sheaf (with constant coefficients); see Section 5. As before (essentially), consider the complex, $\mathbf{A}^\bullet$, which has $\mathbb{C}_X$ in degree -1, $\widetilde{\mathbb{C}}_{L_1} \oplus \widetilde{\mathbb{C}}_{L_2}$ in degree 0, zeroes elsewhere, and the coboundary map from degree 0 to degree 1 is α. This complex is isomorphic in $Perv(X)$ to the extension by 0 of the constant sheaf on the point $\mathbf{p}$. Let γ be the morphism of complexes from $\mathbf{A}^\bullet$ to $\mathbf{B}^\bullet$ which is the identity in degree -1 and is zero elsewhere. Then,

$$0 \to \mathbf{A}^\bullet \xrightarrow{\gamma} \mathbf{B}^\bullet \xrightarrow{\alpha} \mathbf{C}^\bullet \to 0$$

is a short exact sequence in $Perv(X)$.

However, the induced maps, γ^k_x and α^k_x, on the degree k stalk cohomology, at a point $x \in X$, are zero maps and injections, respectively.

However, despite the above warning, we do have the following:

Theorem: Suppose that the base ring is a field, and that T is a morphism from a perverse sheaf $\mathbf{P}^\bullet$ on X to itself.

Then, $\operatorname{supp}(\ker T) = \overline{\{x \in X \mid \ker T^*_x \neq 0\}}$ and, for all $p \in \operatorname{supp}(\ker T)$, if $s := \dim_p \operatorname{supp}(\ker T)$, then $H^{-s}(\ker T)_p \cong \ker T^{-s}_p$; in particular, for generic $p \in X$, $\ker T^k_p$ is non-zero only in degree $-s$.

The same statement, with each occurrence of ker replaced by coker is also true.

Proof. We shall prove the kernel statement, and leave the cokernel statement as an exercise. Let $S := \overline{\{x \in X \mid \ker T^*_x \neq 0\}}$.

If $p \notin \operatorname{supp}(\ker T)$, then, in a neighborhood of p, both $\ker T$ and $\operatorname{coker} T$ are zero, i.e., T is an isomorphism in a neighborhood of p. Thus, in a neighborhood of p, the induced maps on stalk cohomology are isomorphisms, and so $p \notin S$. Therefore, $S \subseteq \operatorname{supp}(\ker T)$. We will show the reverse inclusion in the midst of the argument below.

Fix $p \in \operatorname{supp}(\ker T)$. Let $s := \dim_p \operatorname{supp}(\ker T) = \dim_p \operatorname{supp}(\operatorname{coker} T)$. Consider the canonical short exact sequences $0 \to \ker T \to \mathbf{P}^\bullet \xrightarrow{\alpha} \operatorname{im} T \to 0$ and $0 \to \operatorname{im} T \xrightarrow{\beta} \mathbf{P}^\bullet \to \operatorname{coker} T \to 0$, where $\beta \circ \alpha = T$. One has the following portions of the corresponding long exact sequences on stalk cohomology

$(*)$

$$\cdots \to 0 \to H^{-s-1}(\mathbf{P}^\bullet)_p \xrightarrow{\alpha_p^{-s-1}} H^{-s-1}(\operatorname{im} T)_p \to H^{-s}(\ker T)_p \to H^{-s}(\mathbf{P}^\bullet)_p \xrightarrow{\alpha_p^{-s}} H^{-s}(\operatorname{im} T)_p \to \dots$$

and

$$(\dagger) \qquad \cdots \to 0 \to H^{-s-1}(\operatorname{im} T)_p \xrightarrow{\beta_p^{-s-1}} H^{-s-1}(\mathbf{P}^\bullet)_p \to 0 \to H^{-s}(\operatorname{im} T)_p \xrightarrow{\beta_p^{-s}} H^{-s}(\mathbf{P}^\bullet)_p \to \dots.$$

By $(\dagger)$, $H^{-s-1}(\operatorname{im} T)_p$ is isomorphic to $H^{-s-1}(\mathbf{P}^\bullet)_p$. Thus, as our base ring is a field, the inclusion in $(*)$, $H^{-s-1}(\mathbf{P}^\bullet)_p \xrightarrow{\alpha_p^{-s-1}} H^{-s-1}(\operatorname{im} T)_p$, must be an isomorphism. Thus, in $(*)$, we see that $H^{-s}(\ker T)_p$ is isomorphic to the kernel of α_p^{-s}. Looking again at $(\dagger)$, we see that β_p^{-s} is an injection. Hence, the kernel of α_p^{-s} equals the kernel of $\beta_p^{-s} \circ \alpha_p^{-s} = T_p^{-s}$, and we are finished. $\square$

Finally, we wish to state the *decomposition theorem* of Beilinson, Bernstein, Deligne, and Gabber. For this statement, we must restrict ourselves to $R = \mathbb{Q}$. We give the statement as it appears in [**Mac1**], except that in [**Mac1**] intersection cohomology is defined as a shifted perverse sheaf, and we must adjust by shifting back. Note that, in [**Mac1**], the setting is algebraic; the analytic version appears in [**Sa**].

An algebraic map $f : X \to Y$ is called *projective* if it can be factored as an embedding $X \hookrightarrow Y \times \mathbb{P}^m$ (for some m) followed by projection $Y \times \mathbb{P}^m \to Y$.

The Decomposition Theorem [BBD, 6.2.5]: If X is purely d-dimensional and $f : X \to Y$ is proper, then there exists a unique set of irreducible enriched subvarieties $\{(\overline{M}_\alpha, \mathcal{L}_\alpha)\}$ in Y and Laurent polynomials $\{\phi^\alpha = \cdots + \phi^\alpha_{-2}t^{-2} + \phi^\alpha_{-1}t^{-1} + \phi^\alpha_0 + \phi^\alpha_1 t + \phi^\alpha_2 t^2 + \dots\}$ such that there is a quasi-isomorphism

$$Rf_*\mathbf{IC}^\bullet_X(\mathbb{Q}^\bullet_{\overset{\circ}{X}}[d]) \;\cong\; \bigoplus_{\alpha, i} \mathbf{IC}^\bullet_{\overline{M}_\alpha}(\mathcal{L}^\bullet_\alpha[d_\alpha])[-i] \otimes \mathbb{Q}^{\phi^\alpha_i},$$

(here, $\mathbf{IC}^\bullet_{\overline{M}_\alpha}(\mathcal{L}^\bullet_\alpha[d_\alpha])$ actually equals $j_{\alpha !}\mathbf{IC}^\bullet_{\overline{M}_\alpha}(\mathcal{L}^\bullet_\alpha[d_\alpha])$, where $j_\alpha : \overline{M}_\alpha \hookrightarrow Y$ is the inclusion and, of course, d_α is the dimension of M_α).

Moreover, if f is projective, then the coefficients of ϕ^α are palindromic around 0 (i.e., $\phi^\alpha(t^{-1}) = \phi^\alpha(t)$) and the even and odd terms are separately unimodal (i.e., if $i \leqslant 0$, then $\phi^\alpha_{i-2} \leqslant \phi^\alpha_i$).

Applying hypercohomology to each side, we obtain:

$$IH^k(X; \mathbb{Q}) = \bigoplus_{\alpha, i} \left(IH^{k-i}(\overline{M}_\alpha; \mathcal{L}^\bullet_\alpha[d_\alpha])\right)^{\phi^\alpha_i}.$$

We now wish to describe the category of perverse sheaves on a one-dimensional space; this is a particularly nice case of the results obtained in [**M-V**]. Unfortunately, we will use the notions of vanishing cycles and nearby cycles, which are not covered until the next section. Nonetheless, it seems appropriate to place this material here.

We actually wish to consider perverse sheaves on the germ of a complex analytic space X at a point x. Hence, we assume that X is a one-dimensional complex analytic space with irreducible analytic components $X_1, \dots, X_d$ which all contain x, such that X_i is homeomorphic to a complex line and $X_i - \{x\}$ is smooth for all i. We wish to describe the category, $\mathcal{C}$, of perverse sheaves on X with complex coefficients which are constructible with respect to the stratification $\{X_1 - \{x\}, \dots, X_d - \{x\}, \{x\}\}$.

Since perverse sheaves are topological in nature, we may reduce ourselves to considering exactly the case where X consists of d complex lines through the origin in some $\mathbb{C}^N$. Let L denote a linear form on $\mathbb{C}^N$ such that $X \cap L^{-1}(0) = \{\mathbf{0}\}$.

Suppose now that $\mathbf{P}^\bullet$ is in $\mathcal{C}$, i.e., $\mathbf{P}^\bullet$ is perverse on X and constructible with respect to the stratification which has $\{\mathbf{0}\}$ as the only zero-dimensional stratum. Then $\mathbf{P}^\bullet_{|X - \{\mathbf{0}\}}$ consists of a collection of local systems, $\mathcal{L}_1, \dots \mathcal{L}_d$, in degree -1. These local systems are completely determined by monodromy isomorphisms $h_i : \mathbb{C}^{r_i} \to \mathbb{C}^{r_i}$ representing looping once around the origin in X_i. In terms of nearby cycles, the monodromy automorphism on $H^0(\psi_L\mathbf{P}^\bullet[-1])_\mathbf{0} \cong \bigoplus_i \mathbb{C}^{r_i}$ is given by $\bigoplus_i h_i$.

The vanishing cycles $\phi_L\mathbf{P}^\bullet[-1]$ are a perverse sheaf on a point, and so have possibly non-zero cohomology only in degree 0; say, $H^0(\phi_L\mathbf{P}^\bullet[-1])_\mathbf{0} \cong \mathbb{C}^\lambda$. We have the canonical map

$$r : H^0(\psi_L\mathbf{P}^\bullet[-1])_\mathbf{0} \to H^0(\phi_L\mathbf{P}^\bullet[-1])_\mathbf{0}$$

and the variation map

$$\mathrm{var} : H^0(\phi_L\mathbf{P}^\bullet[-1])_\mathbf{0} \to H^0(\psi_L\mathbf{P}^\bullet[-1])_\mathbf{0},$$

and $\mathrm{var} \circ r = \mathrm{id} - \bigoplus_i h_i$.

Thus, an object in $\mathcal{C}$ determines a vector space $W := H^0(\phi_L\mathbf{P}^\bullet[-1])_\mathbf{0}$, a vector space $V_i := \mathbb{C}^{r_i}$ for each irreducible component X_i, an automorphism h_i on V_i, and two linear maps $\alpha : \bigoplus_i V_i \to W$ and $\beta : W \to \bigoplus_i V_i$ such that $\beta \circ \alpha = \mathrm{id} - \bigoplus_i h_i$. This situation is nicely represented by a commutative triangle

$$\begin{tikzcd} \oplus_i V_i \arrow[rr, "\mathrm{id} - \bigoplus_i h_i"] \arrow[dr, "\alpha"'] & & \oplus_i V_i \\ & W \arrow[ur, "\beta"'] & \end{tikzcd}.$$

The category $\mathcal{C}$ is equivalent to the category of such triangles, where a morphism of triangles is defined in the obvious way: a morphism is determined by linear maps $\tau_i : V_i \to V_i'$ and $\eta : W \to W'$ such that

$$\begin{array}{ccccc} \oplus_i V_i & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & \oplus_i V_i \\ \oplus_i \tau_i \downarrow & & \eta \downarrow & & \oplus_i \tau_i \downarrow \\ \oplus_i V_i' & \xrightarrow{\alpha'} & W' & \xrightarrow{\beta'} & \oplus_i V_i' \end{array}$$

commutes.

§3. **Nearby and Vanishing Cycles**

Historically, there has been some confusion surrounding the terminology *nearby (or neighboring) cycles* and *vanishing cycles*; now, however, the terminology seems to have stabilized. In the past, the term "vanishing cycles" was sometimes used to describe what are now called the nearby cycles (this is true, for instance, in [**A'C**], [**BBD**], and [**G-M1**].)

The two different indexing schemes for perverse sheaves also add to this confusion in statements such as "the nearby cycles of a perverse sheaf are perverse". Finally, a new piece of confusion has been added in [**K-S**], where the sheaf of vanishing cycles is shifted by one from the usual definition (we will **not** use this new, shifted definition).

The point is: one should be very careful when reading works on nearby and vanishing cycles.

We will first give the traditional, formal definitions, and then give a different presentation, which will hopefully illuminate what is going on.

Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of X and suppose $\mathbf{F}^\bullet \in \mathbf{D}^b_{\mathcal{S}}(X)$. Given an analytic map $f : X \to \mathbb{C}$, define a *(stratified) critical point* of f (with respect to $\mathcal{S}$) to be a point $x \in S_\alpha \subseteq X$ such that $f_{|S_\alpha}$ has a critical point at x; we denote the set of such critical points by $\Sigma_{\mathcal{S}} f$.

We wish to investigate how the cohomology of the level sets of f with coefficients in $\mathbf{F}^\bullet$ changes at a critical point (which we normally assume lies in $f^{-1}(0)$).

Consider the diagram

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{\mathbb{C}^*} \\ \hat{\pi} \big\downarrow & & \big\downarrow \pi \\ X - f^{-1}(0) & \xrightarrow{\hat{f}} & \mathbb{C}^* \\ i \downarrow & & \\ f^{-1}(0) \underset{j}{\hookrightarrow} X & & \end{array}$$

where:

$j : f^{-1}(0) \hookrightarrow X$ is inclusion;

$i : X - f^{-1}(0) \hookrightarrow X$ is inclusion;

$\hat{f}$ = restriction of f;

$\widetilde{\mathbb{C}^*}$ = cyclic (universal) cover of $\mathbb{C}^*$;

and E denotes the pull-back.

The *nearby (or neighboring) cycles* of $\mathbf{F}^\bullet$ along f are defined to be

$$\psi_f \mathbf{F}^\bullet := j^* R(i \circ \hat{\pi})_* (i \circ \hat{\pi})^* \mathbf{F}^\bullet.$$

Note that this is a sheaf on $f^{-1}(0)$.

As $\psi_f\big(\mathbf{F}^\bullet[k]\big) = \big(\psi_f \mathbf{F}^\bullet\big)[k]$, we may write $\psi_f \mathbf{F}^\bullet[k]$ unambiguously. In fact, it is frequently useful to consider the functor where one first shifts the complex by k and then takes the nearby cycles; thus, we introduce the notation $\psi_f[k]$ to be the functor such that $\psi_f[k]\mathbf{F}^\bullet = \psi_f \mathbf{F}^\bullet[k]$ (and which has the corresponding action on morphisms). The functor ψ_f takes distinguished triangles to distinguished triangles.

If $\mathbf{P}^\bullet$ is a perverse sheaf on X, then $\psi_f[-1]\mathbf{P}^\bullet$ is perverse on $f^{-1}(0)$. (Actually, to conclude that $\psi_f[-1]\mathbf{P}^\bullet$ is perverse, we only need to assume that $\mathbf{P}^\bullet|_{X-f^{-1}(0)}$ is perverse.) Because $\psi_f[-1]$ takes perverse sheaves to perverse sheaves, it is useful to include the shift by -1 in many statements about ψ_f. Consequently, we also want to shift $j^*\mathbf{F}^\bullet$ by -1 in many statements, and so we write $j^*[-1]$ for the functor which first shifts by -1 and then pulls-back by j.

As there is a canonical map $\mathbf{F}^\bullet \to Rg_* g^* \mathbf{F}^\bullet$ for any map $g : Z \to X$, there is a map

$$\mathbf{F}^\bullet \to R(i \circ \hat{\pi})_* (i \circ \hat{\pi})^* \mathbf{F}^\bullet$$

and, hence, a natural map, $\mathrm{comp} = \mathrm{comp}_f$, called the *comparison map*:

$$j^*[-1]\mathbf{F}^\bullet \xrightarrow{\ \mathrm{comp}\ } j^*[-1]R(i \circ \hat{\pi})_* (i \circ \hat{\pi})^* \mathbf{F}^\bullet = \psi_f[-1]\mathbf{F}^\bullet.$$

For $x \in f^{-1}(0)$, the stalk cohomology of $\psi_f \mathbf{F}^\bullet$ at x is the cohomology of the Milnor fibre of f at x with coefficients in $\mathbf{F}^\bullet$, i.e., for all $\epsilon > 0$ small and all $\xi \in \mathbb{C}^*$ with $|\xi| << \epsilon$,

$$\mathbf{H}^i(\psi_f \mathbf{F}^\bullet)_x \cong \mathbb{H}^i(\overset{\circ}{B}_\epsilon(x) \cap X \cap f^{-1}(\xi); \mathbf{F}^\bullet),$$

where the open ball $\overset{\circ}{B}_\epsilon(x)$ is taken inside any local embedding of (X, x) in affine space. The sheaf $\psi_f \mathbf{F}^\bullet$ depends only on f and $\mathbf{F}^\bullet|_{X-f^{-1}(0)}$. One may also use a closed ball in place of the open ball.

There is an automorphism, the *monodromy automorphism* $T_f : \psi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$, which comes from the deck transformation obtained in our definition of $\psi_f \mathbf{F}^\bullet$ (and, hence, $\psi_f[-1]\mathbf{F}^\bullet$) by traveling once around the origin in $\mathbb{C}$. In fact, T_f is a natural isomorphism from the functor $\psi_f[-1]$ to itself; thus, strictly speaking, when we write $T_f : \psi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$, we should include $\mathbf{F}^\bullet$ in the notation for T_f – however, we shall normally omit the explicit reference to $\mathbf{F}^\bullet$ if the complex is clear. The comparison map $j^*[-1]\mathbf{F}^\bullet \xrightarrow{\ \mathrm{comp}\ } \psi_f[-1]\mathbf{F}^\bullet$ is T_f-equivariant, i.e., $\mathrm{comp} = T_f \circ \mathrm{comp}$.

There is a natural distinguished triangle

$$\begin{array}{ccc} j^*[-1]Ri_* i^* \mathbf{F}^\bullet & \longrightarrow & \psi_f[-1]\mathbf{F}^\bullet \\ {\scriptstyle [1]} \nwarrow & & \swarrow \ \mathrm{id} - T_f \\ & \psi_f[-1]\mathbf{F}^\bullet. & \end{array}$$

The associated long exact sequences on stalk cohomology are the Wang sequences.

Since we have a map comp[1] : $j^*\mathbf{F}^\bullet \to \psi_f\mathbf{F}^\bullet$, the third vertex of a distinguished triangle is defined up to quasi-isomorphism. We define the sheaf of *vanishing cycles*, $\phi_f\mathbf{F}^\bullet$, of $\mathbf{F}^\bullet$ along f to be this third vertex (the choice can be made in a canonical manner), and so, there is a distinguished triangle

$$\begin{tikzcd} j^*\mathbf{F}^\bullet \arrow[r] & \psi_f\mathbf{F}^\bullet \arrow[dl] \\ \phi_f\mathbf{F}^\bullet . \arrow[u, "{[1]}"] & \end{tikzcd}$$

Letting $\phi_f[-1]$ denote the functor which first shifts by -1 and then applies ϕ_f, we can write the triangle above as

$$\begin{tikzcd} j^*[-1]\mathbf{F}^\bullet \arrow[r, "\mathrm{comp}"] & \psi_f[-1]\mathbf{F}^\bullet \arrow[dl] \\ \phi_f[-1]\mathbf{F}^\bullet . \arrow[u, "{[1]}"] & \end{tikzcd}$$

We refer to this (or the natural triangle of functors) as the *Milnor triangle*. Note that this is a triangle of sheaves on $f^{-1}(0)$. The map from $\psi_f[-1]\mathbf{F}^\bullet$ to $\phi_f[-1]\mathbf{F}^\bullet$ is usually referred to as the *canonical map*, and is denoted can $=$ can$_f$.

For $x \in f^{-1}(0)$, the stalk cohomology of $\phi_f\mathbf{F}^\bullet$ at x is the relative cohomology of the Milnor fibre of f at x with coefficients in $\mathbf{F}^\bullet$ and with a shift by one, i.e., for all $\epsilon > 0$ small and all $\xi \in \mathbb{C}^*$ with $|\xi| << \epsilon$,

$$\mathbf{H}^i(\phi_f\mathbf{F}^\bullet)_x \cong \mathbb{H}^{i+1}(\overset{\circ}{B}_\epsilon(x) \cap X, \overset{\circ}{B}_\epsilon(x) \cap X \cap f^{-1}(\xi); \mathbf{F}^\bullet),$$

where, again, one may use a closed ball in place of the open ball.

As an example, if $X = \mathbb{C}^{n+1}$ and $\mathbf{F}^\bullet = \mathbb{C}^\bullet_X$, then for all $x \in f^{-1}(0)$, $H^i(\psi_f\mathbb{C}^\bullet_X)_x = i$-th cohomology of the Milnor fibre of f at x (with $\mathbb{C}$ coefficients) $= H^i(F_{f,x};\mathbb{C})$, while $H^i(\phi_f\mathbb{C}^\bullet_X)_x =$ **reduced** i-th cohomology of the Milnor fibre of f at x $= \widetilde{H}^i(F_{f,x};\mathbb{C})$.

From the constructions, it is clear that taking nearby cycles and vanishing cycles are local operations. That is, if $l : W \hookrightarrow X$ is the inclusion of an open set, $\hat{l} : W \cap f^{-1}(0) \hookrightarrow f^{-1}(0)$ is the corresponding inclusion, and $\hat{f} := f_{|_W}$, then there are natural isomorphisms

$$\hat{l}^*\psi_f \ \cong\ \psi_{\hat{f}}\, l^* \quad \text{and} \quad \hat{l}^*\phi_f \ \cong\ \phi_{\hat{f}}\, l^*.$$

For any Whitney stratification, $\mathcal{S}$, with respect to which $\mathbf{F}^\bullet$ is constructible, the support of $\mathbf{H}^*(\phi_f\mathbf{F}^\bullet)$ is contained in the stratified critical locus of f, $\Sigma_{\mathcal{S}}f$. In addition, if $\mathcal{S}$ is a Whitney stratification with respect to which $\mathbf{F}^\bullet$ is constructible and such that $f^{-1}(0)$ is a union of strata, then – by [**BMM**] and [**P**] – it follows that $\mathcal{S}$ also satisfies Thom's a_f condition; by Thom's second isotopy lemma, this implies that the entire situation locally trivializes over strata, and hence both $\psi_f\mathbf{F}^\bullet$ and $\phi_f\mathbf{F}^\bullet$ are constructible with respect to $\{S \in \mathcal{S} \mid S \subseteq f^{-1}(0)\}$.

The functor ϕ_f takes distinguished triangles to distinguished triangles, and if $\mathbf{P}^\bullet$ is a perverse sheaf on X, then $\phi_f[-1]\mathbf{P}^\bullet$ is a perverse sheaf on $f^{-1}(0)$.

There is a monodromy automorphism $\widetilde{T}_f : \phi_f[-1]\mathbf{F}^\bullet \to \phi_f[-1]\mathbf{F}^\bullet$ which, again, is actually comes from a natural isomorphism $\widetilde{T}_f : \phi_f[-1] \to \phi_f[-1]$. This monodromy and that of the nearby cycles fit into a monodromy automorphism on the distinguished triangle

$$j^*[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{comp}} \psi_f[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{can}} \phi_f[-1]\mathbf{F}^\bullet \longrightarrow j^*\mathbf{F}^\bullet$$

given by $(\mathrm{id}, T_f, \widetilde{T}_f)$, i.e., a commutative diagram

$$\begin{array}{ccccccc}
j^*[-1]\mathbf{F}^\bullet & \xrightarrow{\text{comp}} & \psi_f[-1]\mathbf{F}^\bullet & \xrightarrow{\text{can}} & \phi_f[-1]\mathbf{F}^\bullet & \longrightarrow & j^*\mathbf{F}^\bullet \\
\mathrm{id}\downarrow & & T_f\downarrow & & \widetilde{T}_f\downarrow & & \mathrm{id}\downarrow \\
j^*[-1]\mathbf{F}^\bullet & \xrightarrow{\text{comp}} & \psi_f[-1]\mathbf{F}^\bullet & \xrightarrow{\text{can}} & \phi_f[-1]\mathbf{F}^\bullet & \longrightarrow & j^*\mathbf{F}^\bullet.
\end{array}$$

From this, it follows formally that there exists a *variation morphism*,

$$\mathrm{var} = \mathrm{var}_f : \phi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$$

such that $\mathrm{can} \circ \mathrm{var} = \mathrm{id} - \widetilde{T}_f$ and $\mathrm{var} \circ \mathrm{can} = \mathrm{id} - T_f$. (Note that, if we are not using field coefficients, then the variation morphism does not necessarily exist on the level of chain complexes – the derived category structure is necessary here.)

In addition to the fact that the monodromy isomorphisms T_f and $\widetilde{T}_f$ are natural automorphisms of the (shifted) nearby cycle and vanishing cycle functors, respectively, we should emphasize that the maps comp, can, and var above are all natural maps.

Note also that, by replacing $\mathbf{F}^\bullet$ with $i_!i^!\mathbf{F}^\bullet$ in the Milnor triangle, we conclude that there is a natural isomorphism $\alpha_f : \psi_f[-1] \xrightarrow{\cong} \phi_f[-1]i_!i^!$. There is another natural isomorphism $\beta_f : \phi_f[-1]Ri_*i^* \xrightarrow{\cong} \psi_f[-1]$ which we will describe on the stalk level, once we have described the vanishing cycles in a more "concrete" way below.

Applying the shifted vanishing cycle functor to the canonical morphisms $i_!i^! \to \mathrm{id}$ and $\mathrm{id} \to Ri_*i^*$, we obtain

$$\psi_f[-1]\mathbf{F}^\bullet \xrightarrow[\cong]{\alpha_f} \phi_f[-1]i_!i^!\mathbf{F}^\bullet \xrightarrow{\rho_f} \phi_f[-1]\mathbf{F}^\bullet \xrightarrow{\tau_f} \phi_f[-1]Ri_*i^*\mathbf{F}^\bullet \xrightarrow[\cong]{\beta_f} \psi_f[-1]\mathbf{F}^\bullet,$$

where $\mathrm{can}_f = \rho_f \circ \alpha_f$ and $\mathrm{var}_f = \beta_f \circ \tau_f$.

Applying the shifted vanishing cycle functor to the distinguished triangle

$$\begin{array}{ccc}
j_!j^!\mathbf{F}^\bullet & \longrightarrow & \mathbf{F}^\bullet \\
{\scriptstyle [1]}\nwarrow & & \swarrow \\
& Ri_*i^*\mathbf{F}^\bullet, &
\end{array}$$

noting that $\phi_f[-1](j_!j^!\mathbf{F}^\bullet) \cong j^!\mathbf{F}^\bullet$, and using the natural isomorphism

$$\beta_f : \phi_f[-1](Ri_*i^*\mathbf{F}^\bullet) \xrightarrow{\cong} \psi_f[-1]\mathbf{F}^\bullet,$$

we obtain the distinguished triangle

$$\begin{array}{ccc}
j^!\mathbf{F}^\bullet & \longrightarrow & \phi_f[-1]\mathbf{F}^\bullet \\
{\scriptstyle [1]}\nwarrow & & \swarrow\ \mathrm{var} \\
& \psi_f[-1]\mathbf{F}^\bullet. &
\end{array}$$

Using our in-line notation and turning the triangle, this gives us a distinguished triangle

$$\phi_f[-1]\mathbf{F}^\bullet \xrightarrow{\text{var}} \psi_f[-1]\mathbf{F}^\bullet \xrightarrow{\text{pmoc}} j^![1]\mathbf{F}^\bullet \longrightarrow \phi_f\mathbf{F}^\bullet,$$

which we refer to as the *variation triangle* or the *dual Milnor triangle*.

The reader may wonder how the above variation morphism is related to the classical variation isomorphism for isolated affine hypersurface singularities (see, for instance [**Lo**]). Let m_x denote the inclusion of x into $f^{-1}(0)$, and let l_x denote the inclusion of x into X, so that $l_x = j \circ m_x$. Then, the variation morphism yields a natural map from $m_x^!\phi_f[-1]\mathbf{F}^\bullet$ to $m_x^!\psi_f[-1]\mathbf{F}^\bullet$, and so gives corresponding maps on cohomology; these maps are, in a sense, the generalization of the classical variation maps.

In particular, suppose that x is an isolated point in the support of $\phi_f[-1]\mathbf{F}^\bullet$. Then, there is a natural isomorphism $\gamma_x : m_x^*\phi_f[-1]\mathbf{F}^\bullet \xrightarrow{\cong} m_x^!\phi_f[-1]\mathbf{F}^\bullet$. By composing with

$$m_x^*(\mathrm{can}) : m_x^*\psi_f[-1]\mathbf{F}^\bullet \to m_x^*\phi_f[-1]\mathbf{F}^\bullet$$

and

$$m_x^!(\mathrm{var}) : m_x^!\phi_f[-1]\mathbf{F}^\bullet \to m_x^!\psi_f[-1]\mathbf{F}^\bullet,$$

we obtain a morphism

$$v_x := m_x^!(\mathrm{var}) \circ \gamma_x \circ m_x^*(\mathrm{can}) : m_x^*\psi_f[-1]\mathbf{F}^\bullet \to m_x^!\psi_f[-1]\mathbf{F}^\bullet.$$

Using a local embedding into affine space, a closed ball $B_\epsilon(x)$, centered at x, of radius ϵ, and assuming $0 \ll |\xi| \ll \epsilon \lll 1$, we obtain induced maps on cohomology

$$\mathbb{H}^{i-1}(B_\epsilon(x) \cap X \cap f^{-1}(\xi); \mathbf{F}^\bullet) \xrightarrow{v_x^i} \mathbb{H}^{i-1}(B_\epsilon(x) \cap X \cap f^{-1}(\xi), \partial B_\epsilon(x) \cap X \cap f^{-1}(\xi); \mathbf{F}^\bullet).$$

Writing $F_{f,x}$ for the (compact) Milnor fiber $B_\epsilon(x) \cap X \cap f^{-1}(\xi)$, this gives us

$$\mathbb{H}^{i-1}(F_{f,x}; \mathbf{F}^\bullet) \xrightarrow{v_x^i} \mathbb{H}^{i-1}(F_{f,x}, \partial F_{f,x}; \mathbf{F}^\bullet).$$

If $l_x^![1]\mathbf{F}^\bullet \cong m_x^! j^![1]\mathbf{F}^\bullet$ is zero in degree i and degree $i-1$, then $m_x^!(\mathrm{var})$ is an isomorphism in degree i. If $l_x^*[-1]\mathbf{F}^\bullet \cong m_x^* j^*[-1]\mathbf{F}^\bullet$ is zero in degree i and $i+1$, then $m_x^*(\mathrm{can})$ is an isomorphism in degree i. Thus, when the conditions of the previous two sentences are satisfied, v_x^i is an isomorphism.

In the classical case, X is an open subset of $\mathbb{C}^{n+1}$, $\mathbf{F}^\bullet$ is the constant sheaf $\mathbb{Z}_X$, and one looks v_x^{n+1}, where $n \geq 1$. In this case, all of the maps composed to yield $v_x := m_x^!(\mathrm{var}) \circ \gamma_x \circ m_x^*(\mathrm{can})$ are isomorphisms in degree $n+1$. v_x^{n+1} is the classical variation isomorphism on cohomology.

Starting with the two distinguished triangles

$$\phi_f[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{var}} \psi_f[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{pmoc}} j^![1]\mathbf{F}^\bullet \longrightarrow \phi_f\mathbf{F}^\bullet$$

and

$$\psi_f[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{can}} \phi_f[-1]\mathbf{F}^\bullet \longrightarrow j^*\mathbf{F}^\bullet \xrightarrow{\mathrm{comp}} \psi_f\mathbf{F}^\bullet,$$

we may apply the octahedral lemma to conclude that there exists a complex $w_f\mathbf{F}^\bullet$ and two distinguished triangles

$$\phi_f[-1]\mathbf{F}^\bullet \xrightarrow{\mathrm{id} - \widetilde{T}_f} \phi_f[-1]\mathbf{F}^\bullet \longrightarrow w_f\mathbf{F}^\bullet \longrightarrow \phi_f\mathbf{F}^\bullet$$

and

$$j^![1]\mathbf{F}^\bullet \longrightarrow w_f\mathbf{F}^\bullet \longrightarrow j^*\mathbf{F}^\bullet \xrightarrow{\tau} j^![2]\mathbf{F}^\bullet.$$

We refer to the morphism $\omega_f := \tau[-1] = \mathrm{pmoc} \circ \mathrm{comp}$ from $j^*[-1]\mathbf{F}^\bullet$ to $j^![1]\mathbf{F}^\bullet$ as the *Wang morphism of f*. The application of the octahedral lemma above tells us that the mapping cone of $\mathrm{id} - \widetilde{T}_f$ is isomorphic to the mapping cone of ω_f. Note that, while $j^*[-1]\mathbf{F}^\bullet$ and $j^![1]\mathbf{F}^\bullet$ depend only on $f^{-1}(0)$ (and $\mathbf{F}^\bullet$), ω_f may change if f (or some factor of f) is raised to a power.

Note that the Wang morphism is **not** (usually) the natural morphism from $j^*[-1]\mathbf{F}^\bullet$ to $j^![1]\mathbf{F}^\bullet$ obtained from the composition of natural maps

$$j^*[-1]\mathbf{F}^\bullet \longrightarrow Ri_!i^!\mathbf{F}^\bullet \longrightarrow Ri_*i^*\mathbf{F}^\bullet \longrightarrow j^![1]\mathbf{F}^\bullet.$$

The map above induces the zero map on stalk cohomology; the Wang morphism induces isomorphisms on the stalk cohomology at all points of $f^{-1}(0) - \operatorname{supp}\phi_f[-1]\mathbf{F}^\bullet$.

Suppose we have $X \xrightarrow{\pi} Y \xrightarrow{f} \mathbb{C}$, where π is proper and $\hat{\pi} : \pi^{-1}f^{-1}(0) \to f^{-1}(0)$ is the restriction of π. Then, for all $\mathbf{A}^\bullet \in \mathbf{D}^b_c(X)$, there are natural isomorphisms

$$R\hat{\pi}_*(\psi_{f\circ\pi}\mathbf{A}^\bullet) \cong \psi_f(R\pi_*\mathbf{A}^\bullet) \quad \text{and} \quad R\hat{\pi}_*(\phi_{f\circ\pi}\mathbf{A}^\bullet) \cong \phi_f(R\pi_*\mathbf{A}^\bullet).$$

If $\mathbf{Q}^\bullet$ is perverse on $X - f^{-1}(0)$ and $i : X - f^{-1}(0) \to X$ is the inclusion, then it is easy to see that $Ri_*\mathbf{Q}^\bullet$ satisfies the cosupport condition; moreover, by combining the fact that $\psi_f(Ri_*\mathbf{Q}^\bullet)[-1]$ is perverse on $f^{-1}(0)$ with the Wang sequences on stalk cohomology, one can prove that $Ri_*\mathbf{Q}^\bullet$ also satisfies the support condition - hence, $Ri_*\mathbf{Q}^\bullet$ is perverse. In an analogous fashion, one obtains that $Ri_!\mathbf{Q}^\bullet$ is perverse (if the base ring is a field, this can be obtained by dualizing).

The functors $\psi_f[-1]$ and $\mathcal{D}$ commute, up to natural isomorphism, as do $\phi_f[-1]$ and $\mathcal{D}$; i.e.,

$$\mathcal{D}(\psi_f\mathbf{A}^\bullet[-1]) \ \cong \ \psi_f(\mathcal{D}\mathbf{A}^\bullet)[-1] \quad \text{and} \quad \mathcal{D}(\phi_f\mathbf{A}^\bullet[-1]) \ \cong \ \phi_f(\mathcal{D}\mathbf{A}^\bullet)[-1].$$

The Unipotent and Complementary Nearby and Vanishing Cycles

Suppose that our base ring is a field, that $\mathbf{P}^\bullet$ is a perverse sheaf on X, and that we have an analytic function $f : X \to \mathbb{C}$. Then, (globally, in the algebraic setting; locally, in the analytic setting) there exists n such that, in $Perv(f^{-1}(0))$, for all $i \geq n$,

$$\ker(\operatorname{id} - T_f)^n \ = \ \ker(\operatorname{id} - T_f)^i \ \subseteq \ \psi_f[-1]\mathbf{P}^\bullet$$

and

$$\ker\left(\operatorname{id} - \widetilde{T}_f\right)^n = \ker\left(\operatorname{id} - \widetilde{T}_f\right)^i \ \subseteq \ \phi_f[-1]\mathbf{P}^\bullet.$$

The perverse sheaves $\ker(\operatorname{id} - T_f)^n$ and $\ker\left(\operatorname{id} - \widetilde{T}_f\right)^n$ are maximal perverse subsheaves of $\psi_f[-1]\mathbf{P}^\bullet$ and $\phi_f[-1]\mathbf{P}^\bullet$, respectively, on which T_f and $\widetilde{T}_f$, respectively, act unipotently; they are denoted by $\psi^u_f[-1]\mathbf{P}^\bullet$ and $\phi^u_f[-1]\mathbf{P}^\bullet$, and are called the *unipotent* nearby and vanishing cycles, respectively.

We refer to the images $\psi^\perp_f[-1]\mathbf{P}^\bullet := \operatorname{im}(\operatorname{id} - T_f)^n$ and $\phi^\perp_f[-1]\mathbf{P}^\bullet := \operatorname{im}\left(\operatorname{id} - \widetilde{T}_f\right)^n$ as the *complementary* nearby and vanishing cycles, respectively.

It is easy to show that the maps $\operatorname{can} : \psi_f[-1]\mathbf{P}^\bullet \to \phi_f[-1]\mathbf{P}^\bullet$ and $\operatorname{var} : \phi_f[-1]\mathbf{P}^\bullet \to \psi_f[-1]\mathbf{P}^\bullet$ restrict to isomorphisms

$$\psi^\perp_f[-1]\mathbf{P}^\bullet \xrightarrow[\cong]{\operatorname{can}^\perp} \phi^\perp_f[-1]\mathbf{P}^\bullet \qquad \text{and} \qquad \phi^\perp_f[-1]\mathbf{P}^\bullet \xrightarrow[\cong]{\operatorname{var}^\perp} \psi^\perp_f[-1]\mathbf{P}^\bullet$$

and, hence, that $\operatorname{id} - T_f$ and $\operatorname{id} - \widetilde{T}_f$ restrict to isomorphisms on $\psi^\perp_f[-1]\mathbf{P}^\bullet$ and $\phi^\perp_f[-1]\mathbf{P}^\bullet$, respectively.

Furthermore, letting $j : f^{-1}(0) \to X$ denote the inclusion, restriction yields distinguished triangles

$$j^*[-1]\mathbf{P}^\bullet \xrightarrow{\operatorname{comp}^u} \psi^u_f[-1]\mathbf{P}^\bullet \xrightarrow{\operatorname{can}^u} \phi^u_f[-1]\mathbf{P}^\bullet \longrightarrow j^*\mathbf{P}^\bullet$$

and

$$\phi_f^u[-1]\mathbf{P}^\bullet \xrightarrow{\mathrm{var}^u} \psi_f^u[-1]\mathbf{P}^\bullet \xrightarrow{\mathrm{pmoc}^u} j^![1]\mathbf{P}^\bullet \longrightarrow \phi_f^u\mathbf{P}^\bullet.$$

Finally, $\psi_f[-1]\mathbf{P}^\bullet \cong \psi_f^u[-1]\mathbf{P}^\bullet \oplus \psi_f^\perp[-1]\mathbf{P}^\bullet$, $\phi_f[-1]\mathbf{P}^\bullet \cong \phi_f^u[-1]\mathbf{P}^\bullet \oplus \phi_f^\perp[-1]\mathbf{P}^\bullet$ and, via these isomorphisms, $\mathrm{can} = \big(\mathrm{can}^u, \mathrm{can}^\perp\big)$ and $\mathrm{var} = \big(\mathrm{var}^u, \mathrm{var}^\perp\big)$.

The Sebastiani-Thom Isomorphism

Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be complex analytic functions. Let π_1 and π_2 denote the projections of $X \times Y$ onto X and Y, respectively. Let $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ be bounded, constructible complexes of sheaves of R-modules on X and Y, respectively. In this situation, $\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet := \pi_1^*\mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{B}^\bullet$. Let us adopt the similar notation $f \boxplus g := f \circ \pi_1 + g \circ \pi_2$.

Let p_1 and p_2 denote the projections of $V(f) \times V(g)$ onto $V(f)$ and $V(g)$, respectively, and let k denote the inclusion of $V(f) \times V(g)$ into $V(f \boxplus g)$.

Theorem (Sebastiani-Thom Isomorphism): *There is a natural isomorphism*

$$k^*\phi_{f\boxplus g}[-1]\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big) \;\cong\; \phi_f[-1]\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g[-1]\mathbf{B}^\bullet,$$

and this isomorphism commutes with the corresponding monodromies.

Moreover, if we let $\mathbf{p} := (\mathbf{x}, \mathbf{y}) \in X \times Y$ *be such that* $f(\mathbf{x}) = 0$ *and* $g(\mathbf{y}) = 0$*, then, in an open neighborhood of* $\mathbf{p}$*, the complex* $\phi_{f\boxplus g}[-1]\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big)$ *has support contained in* $V(f) \times V(g)$*, and, in any open set in which we have this containment, there are natural isomorphisms*

$$\phi_{f\boxplus g}[-1]\big(\mathbf{A}^\bullet \overset{L}{\boxtimes} \mathbf{B}^\bullet\big) \;\cong\; k_!(\phi_f[-1]\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g[-1]\mathbf{B}^\bullet) \;\cong\; k_*(\phi_f[-1]\mathbf{A}^\bullet \overset{L}{\boxtimes} \phi_g[-1]\mathbf{B}^\bullet).$$

A'Campo's Theorem: Suppose that the base ring is a field; if m_x denotes the maximal ideal of X at x and $f \in m_x^2$, then the Lefschetz number of the map $\mathbf{H}^*(\psi_f[-1]\mathbf{A}^\bullet)_x \xrightarrow{T_f} \mathbf{H}^*(\psi_f[-1]\mathbf{A}^\bullet)_x$ equals 0, i.e.,

$$\sum_i (-1)^i \;\; \mathrm{Trace}\{\mathbf{H}^i(\psi_f[-1]\mathbf{A}^\bullet)_x \xrightarrow{T_f} \mathbf{H}^i(\psi_f[-1]\mathbf{A}^\bullet)_x\} \;=\; 0.$$

The Monodromy Theorem: Let $\mathbf{A}^\bullet \in \mathbf{D}_c^b(X)$ and $f : X \to \mathbb{C}$. The monodromy automorphism $\psi_f[-1]\mathbf{A}^\bullet \xrightarrow{T_f} \psi_f[-1]\mathbf{A}^\bullet$ induces a map on stalk cohomology which is quasi-unipotent, i.e., letting T_f also denote the map on stalk cohomology, this means that there exist integers k and j such that $(\mathrm{id} - T_f^k)^j = 0$. Over $\mathbb{C}$, this is equivalent to all of the eigenvalues of T_f being roots of unity.

If $\mathbf{P}^\bullet$ is a perverse sheaf, then we may use the Abelian structure of the category $Perv(X)$ to investigate the map $\psi_f[-1]\mathbf{P}^\bullet \xrightarrow{T_f} \psi_f[-1]\mathbf{P}^\bullet$. The Monodromy Theorem implies that this morphism can be factored (globally, in the algebraic setting; locally, in the analytic setting) into $T_f = F \cdot (1 + N)$, where F has finite order, N is nilpotent, and F and N commute.

Nearby and Vanishing Cycles at Fixed Angles

In the above discussion, we used definitions of the nearby cycles and vanishing cycles which treat all angular directions equally, it is illuminating to fix an angle θ and define the nearby cycles and vanishing cycles at the given angle. We continue to let j denote the inclusion of $f^{-1}(0)$ into X.

Let
$$B_\theta := f^{-1}\left(e^{i\theta}\{v \in \mathbb{C} \mid \operatorname{Re} v \geq 0\}\right),$$
and let p_θ denote the inclusion of B_θ into X. Let
$$A_\theta := f^{-1}\left(e^{i\theta}\{v \in \mathbb{C} \mid \operatorname{Re} v > 0\}\right),$$
and let q_θ denote the inclusion of A_θ into X. Note that $A_\theta = X - B_{\theta+\pi}$.

Let
$$L_\theta := f^{-1}\left(e^{i\theta}[0, \infty)\right),$$
and let $l_\theta : f^{-1}(0) \hookrightarrow L_\theta$, $m_\theta : L_\theta \hookrightarrow X$, and $n_\theta : L_\theta - f^{-1}(0) \hookrightarrow L_\theta$ denote the inclusions. Note that $j = m_\theta l_\theta$.

Then, the shifted *nearby cycles and vanishing cycles of* $\mathbf{F}^\bullet$ *along* f *at angle* θ are defined by
$$\psi_f^\theta[-1]\mathbf{F}^\bullet \ := \ j^* q_{\theta+\pi *} q_{\theta+\pi}^* \mathbf{F}^\bullet[-1] \ \cong \ l_\theta^* n_{\theta *} n_\theta^* m_\theta^* \mathbf{F}^\bullet[-1]$$
and
$$\phi_f^\theta[-1]\mathbf{F}^\bullet \ := \ j^* p_{\theta+\pi !} p_{\theta+\pi}^! \mathbf{F}^\bullet \ \cong \ \left(R\Gamma_{B_{\theta+\pi}}(\mathbf{F}^\bullet)\right)_{|_{f^{-1}(0)}} \ \cong \ l_\theta^* l_{\theta !} l_\theta^! m_\theta^* \mathbf{F}^\bullet \ \cong \ l_\theta^! m_\theta^* \mathbf{F}^\bullet.$$
respectively.

Then, one can define $\psi_f[-1]\mathbf{F}^\bullet := \psi_f^0[-1]\mathbf{F}^\bullet$ and $\phi_f[-1]\mathbf{F}^\bullet := \phi_f^0[-1]\mathbf{F}^\bullet$, and consider the natural isomorphisms $T_f^\theta : \psi_f[-1]\mathbf{F}^\bullet \to \psi_f^\theta[-1]\mathbf{F}^\bullet$ and $\widetilde{T}_f^\theta : \phi_f[-1]\mathbf{F}^\bullet \to \phi_f^\theta[-1]\mathbf{F}^\bullet$ which correspond to "rotating" B_θ or L_θ an angle of θ counterclockwise around the origin over $\mathbb{C}$. Then, $T_f^{2\pi}$ and $\widetilde{T}_f^{2\pi}$ are the usual monodromy actions on the shifted nearby and vanishing cycles.

The canonical map can : $\psi_f[-1]\mathbf{F}^\bullet \to \phi_f[-1]\mathbf{F}^\bullet$ is the map induced by the natural maps $q_{\pi *} q_\pi^*[-1] \to p_{\pi !} p_\pi^!$ and/or $n_{\theta *} n_\theta^*[-1] \to l_{\theta !} l_\theta^!$. The variation map var : $\phi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$ is essentially given by the morphism $\phi_f[-1]\mathbf{F}^\bullet \to \phi_f[-1] R i_* i^* \mathbf{F}^\bullet$, induced by the natural map $\mathbf{F}^\bullet \to R i_* i^* \mathbf{F}^\bullet$. We wrote "essentially" because, as we mentioned earlier, there is a natural isomorphism $\beta_f : \phi_f[-1] R i_* i^* \mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$. We wish, finally, to look at this isomorphism on the level of stalk cohomology.

Let $x \in f^{-1}(0)$, let $\overset{\circ}{B}_\epsilon(x)$ be an open ball (using some local embedding into affine space), centered at x, of radius ϵ, and let $\overset{\circ}{\mathbb{D}}_\delta$ be an open disk, of radius δ, around the origin in $\mathbb{C}$. Suppose that $0 \ll \delta \ll \epsilon \ll 1$, and let $\mathcal{W} := \overset{\circ}{B}_\epsilon(x) \cap X \cap f^{-1}(\overset{\circ}{\mathbb{D}}_\delta)$. Then,
$$H^k(\phi_f[-1]\mathbf{F}^\bullet)_x \ \cong \ \mathbb{H}^k(\mathcal{W} - f^{-1}(0), \mathcal{W} \cap A_0; \mathbf{F}^\bullet).$$
Using the local triviality of the Milnor fibration, this last hypercohomology module is isomorphic to
$$\mathbb{H}^k\big(\mathcal{W} \cap f^{-1}(\partial \mathbb{D}_{\delta'}),\ \mathcal{W} \cap f^{-1}(\partial \mathbb{D}_{\delta'}) \cap A_0; \mathbf{F}^\bullet\big),$$
where $0 < \delta' < \delta$. Using the local triviality again, this is isomorphic to
$$M^k := \mathbb{H}^k\big(F_{f,x} \times [0,1],\ (F_{f,x} \times \{0\}) \cup (F_{f,x} \times \{1\}); \mathbf{F}^\bullet\big),$$
where $F_{f,x} \times \{0\}$ embeds into $F_{f,x} \times [0,1]$ in the trivial manner $(y, 0) \mapsto (y, 0)$, and $F_{f,x} \times \{1\}$ embeds into $F_{f,x} \times [0,1]$ by $(y, 1) \mapsto (h(y), 1)$, where h is a characteristic homeomorphism of the Milnor fibration. Thus, M^k fits into the long exact sequence
$$\cdots \to M^k \to H^k(F_{f,x}; \mathbf{F}^\bullet) \xrightarrow{\mathrm{id} \oplus T_{f,x}} H^k(F_{f,x}; \mathbf{F}^\bullet) \oplus H^k(F_{f,x}; \mathbf{F}^\bullet) \to M^{k+1} \to \cdots,$$

or, equivalently,

$$\cdots \to M^k \to H^{k+1}(\psi_f[-1]\mathbf{F}^\bullet)_x \xrightarrow{\mathrm{id}\oplus T^{k+1}_{f,x}} H^{k+1}(\psi_f[-1]\mathbf{F}^\bullet)_x \oplus H^{k+1}(\psi_f[-1]\mathbf{F}^\bullet)_x \to M^{k+1} \to \cdots .$$

It follows that

$$M^k \;\cong\; \operatorname{coker}\{\mathrm{id}\oplus T^k_{f,x}\} \;\cong\; H^k(\psi_f[-1]\mathbf{F}^\bullet)_x.$$

From this discussion, one also sees, on the stalk level, why $\mathrm{var}\circ\mathrm{can} = \mathrm{id} - T_f$.

§4. **Some Quick Applications**

The applications of perverse sheaves are widespread and are frequently quite deep - particularly for those applications which rely on the decomposition theorem. For beautiful discussions of these applications, we highly recommend [**Mac1**] and [**Mac2**]. We shall not describe any of these applications here; rather we shall give some fairly easy results on general Milnor fibres. These results are "easy" now that we have all the machinery of the first three sections at our disposal. While the applications below could undoubtedly be proved without the general theory of perverse sheaves, with this theory in hand, the results and their proofs can be presented in a unified manner and, what is more, the proofs become mere exercises.

Consider the classical case of the Milnor fibre of a non-zero map $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$. Let $X = \mathbb{C}^{n+1}$ and let $s = \dim \Sigma f$. Then, as X is a manifold, $\mathbb{Z}^\bullet_X[n+1]$ is a perverse sheaf and so $\phi_f\mathbb{Z}^\bullet_X[n]$ is perverse on $f^{-1}(0)$ with support only on Σf. It follows that the stalk cohomology of $\phi_f\mathbb{Z}^\bullet_X[n]$ is non-zero only for degrees i with $-s \leqslant i \leqslant 0$; that is, we recover the well-known result that the reduced cohomology of the Milnor fibre can be non-zero only in degrees i such that $n - s \leqslant i \leqslant n$.

A much more general case is just as easy to derive from the machinery that we have. Suppose that X is a purely $(n+1)$-dimensional local complete intersection with arbitrary singularities. Let $\mathcal{S}$ be a Whitney stratification of X. Let $\mathbf{p} \in X$ be such that $\dim_{\mathbf{p}} f^{-1}(0) = n$, and let $F_{f,\mathbf{p}}$ denote the Milnor fibre of f at $\mathbf{p}$. Then, as X is a local complete intersection, $\mathbb{Z}^\bullet_X[n+1]$ is a perverse sheaf and so $\phi_f\mathbb{Z}^\bullet_X[n]$ is perverse on $f^{-1}(0)$ with support only on $\Sigma_{\mathcal{S}} f$. It follows that the stalk cohomology of $\phi_f\mathbb{Z}^\bullet_X[n]$ is non-zero only for dimensions i with $-\dim_{\mathbf{p}}\Sigma_{\mathcal{S}} f \leqslant i \leqslant 0$. Hence, the reduced cohomology of $F_{f,\mathbf{p}}$ can be non-zero only in degrees i such that $n - \dim_{\mathbf{p}}\Sigma_{\mathcal{S}} f \leqslant i \leqslant n$

While this general statement could no doubt be proved by induction on hyperplane sections, the above proof via general techniques avoids the re-working of many technical lemmas on privileged neighborhoods and generic slices.

Another application relates to the homotopy-type of the complex link of a space at a point; for instance, for an s-dimensional local complete intersection, the complex link has the homotopy-type of a bouquet of spheres of real dimension $s-1$. In terms of vanishing cycles and perverse sheaves, we only obtain this result up to cohomology: let (X, x) be a germ of an analytic space embedded in some $\mathbb{C}^n$, and assume $s := \dim X = \dim_x X$. Suppose that we have a perverse sheaf, $\mathbf{P}^\bullet$, on X (e.g., the shifted constant sheaf, if X is a local complete intersection). Let l be a generic linear form, and consider $\phi_{l-l(x)}[-1]\mathbf{P}^\bullet$; this is a perverse sheaf on an $s-1$ dimensional space and, as l is generic, it is supported at the single point x (because the hyperplane slice $l = l(x)$ can be chosen to transversely intersect all the strata of any stratification with respect to which $\mathbf{P}^\bullet$ is constructible - except, possibly, the point-stratum x itself). Hence, $H^*(\phi_{l-l(x)}[-1]\mathbf{P}^\bullet)_x$ is (possibly) non-zero only in degree 0. In the case of the shifted constant sheaf $\mathbb{Z}^\bullet_X[s]$ on a local complete intersection, this gives the desired result.

For our final application, we wish to investigate functions with one-dimensional critical loci; we must first set up some notation.

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$ and suppose that $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ has a one-dimensional critical locus at the origin, i.e., $\dim_{\mathbf{0}}\Sigma f = 1$. The reduced cohomology of the Milnor fibre, $F_{f,\mathbf{0}}$, of f at the origin is possibly non-zero only in dimensions $n-1$ and n. We wish to show that the $(n-1)$-st cohomology group embeds inside another group which is fairly easy to describe; thus, we obtain a bound on the $(n-1)$-st Betti number of the Milnor fibre of f.

For each component ν of Σf, one may consider a generic hyperplane slice, H, at points $\mathbf{p} \in \nu - \mathbf{0}$ close to the origin; then, the restricted function, $f_{|_H}$, will have an isolated critical point at $\mathbf{p}$. By shrinking the neighborhood $\mathcal{U}$ if necessary, we may assume that the Milnor number of this isolated singularity of $f_{|_H}$ at $\mathbf{p}$ is independent of the point $\mathbf{p} \in \nu - \mathbf{0}$; denote this value by $\mathring{\mu}_\nu$. As $\nu - \mathbf{0}$ is homotopy-equivalent to a circle, there is a monodromy map from the Milnor fibre of $f_{|_H}$ at $\mathbf{p} \in \nu - \mathbf{0}$ to itself, which induces a map on the middle dimensional cohomology, i.e., a map $h_\nu : \mathbb{Z}^{\mathring{\mu}_\nu} \to \mathbb{Z}^{\mathring{\mu}_\nu}$. We wish to show that $H^{n-1}(F_{f,\mathbf{0}})$ (with integer coefficients) injects into $\oplus_\nu \ker(id - h_\nu)$.

Let j denote the inclusion of the origin into $X = V(f)$, let i denote the inclusion of $X - \mathbf{0}$ into X, and let $\mathbf{K}^\bullet$ denote $\phi_f \mathbb{Z}^\bullet_{\mathcal{U}}[n]$. As $\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ is perverse, $\phi_f \mathbb{Z}^\bullet_{\mathcal{U}}[n]$ is perverse with one-dimensional support (as we are assuming a one-dimensional critical locus). Also, we always have the distinguished triangle

$$\begin{tikzcd} j_!j^!\mathbf{K}^\bullet \arrow[r] & \mathbf{K}^\bullet \arrow[dl] \\ Ri_*i^*\mathbf{K}^\bullet \arrow[u, "{[1]}"] & \end{tikzcd}$$

We wish to examine the associated stalk cohomology exact sequence at the origin.

First, we have that $H^{-1}((j_!j^!\mathbf{K}^\bullet)_{\mathbf{0}}) = H^{-1}(j^!\mathbf{K}^\bullet)$ and so, by the cosupport condition for perverse sheaves, $H^{-1}((j_!j^!\mathbf{K}^\bullet)_{\mathbf{0}}) = 0$.

Now, we need to look more closely at the sheaf $Ri_*i^*\mathbf{K}^\bullet$. $i^*\mathbf{K}^\bullet$ is the restriction of $\mathbf{K}^\bullet$ to $X - \mathbf{0}$; near the origin, this sheaf has cohomology only in degree -1 with support on $\Sigma f - \mathbf{0}$. Moreover, the cohomology sheaf $\mathbf{H}^{-1}(i^*\mathbf{K}^\bullet)$ is locally constant when restricted to $\Sigma f - \mathbf{0}$. It follows that $i^*\mathbf{K}^\bullet$ is naturally isomorphic in the derived category to the extension by zero of a local system of coefficients in degree -1 on $\Sigma - \mathbf{0}$.

To be more precise, let p denote the inclusion of the closed subset $\Sigma f - \mathbf{0}$ into $X - \mathbf{0}$. Then, there exists a locally constant (single) sheaf, $\mathcal{L}$, on $\Sigma f - \mathbf{0}$ such that when $\mathcal{L}$ is considered as a complex, $\mathcal{L}^\bullet$, we have that $p_!\mathcal{L}^\bullet[1] \cong p_*\mathcal{L}^\bullet[1]$ is naturally isomorphic to $i^*\mathbf{K}^\bullet$. For each component ν of Σf, the restriction of $\mathcal{L}$ to $\nu - \mathbf{0}$ is a local system with stalks $\mathbb{Z}^{\mathring{\mu}_\nu}$ which is completely determined by the monodromy map $h_\nu : \mathbb{Z}^{\mathring{\mu}_\nu} \to \mathbb{Z}^{\mathring{\mu}_\nu}$.

Therefore, inside a small open ball $\mathring{B}$,

$$H^0((Ri_*i^*\mathcal{L}^\bullet)_{\mathbf{0}}) \cong \oplus_\nu \mathbb{H}^0(\mathring{B} \cap (\nu - \mathbf{0}); \mathcal{L})$$

and these global sections are well-known to be given by $\ker(id - h_\nu)$. It follows that

$$H^{-1}((Ri_*i^*\mathbf{K}^\bullet)_{\mathbf{0}}) \cong \oplus_\nu \ker(id - h_\nu).$$

Thus, when we consider the long exact sequence on stalk cohomology associated to our distinguished triangle, we find – starting in dimension $n-1$ – that it begins

$$0 \to \widetilde{H}^{n-1}(F_{f,\mathbf{0}}) \to \oplus_\nu \ker(id - h_\nu) \to \dots .$$

The desired conclusion follows.

§5. **Truncation and Perverse Cohomology**

This section is taken entirely from [**BBD**], [**G-M3**], and [**K-S**].

There are (at least) two forms of truncation associated to an object $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$ – one form of truncation is related to the ordinary cohomology of the complex, while the other form leads to something called the *perverse cohomology* or *perverse projection.* These two types of truncation bear little resemblance to each other, except in the general framework of a t-structure on $\mathbf{D}^b_c(X)$.

Loosely speaking, a t-structure on $\mathbf{D}^b_c(X)$ consists of two full subcategories, denoted $\mathbf{D}^{\leqslant 0}(X)$ and $\mathbf{D}^{\geqslant 0}(X)$, such that for any $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, there exist $\mathbf{E}^\bullet \in \mathbf{D}^{\leqslant 0}(X)$, $\mathbf{G}^\bullet \in \mathbf{D}^{\geqslant 0}(X)$, and a distinguished triangle

$$\begin{tikzcd} \mathbf{E}^\bullet \arrow[rr] & & \mathbf{F}^\bullet \arrow[dl] \\ & \mathbf{G}^\bullet[-1] \arrow[ul, "{[1]}"] & \end{tikzcd} ;$$

moreover, such $\mathbf{E}^\bullet$ and $\mathbf{G}^\bullet$ are required to be unique up to isomorphism in $\mathbf{D}^b_c(X)$.

Given a t-structure as above, and using the same notation, we write $\mathbf{E}^\bullet = \tau_{\leqslant 0}\mathbf{F}^\bullet$ (the *truncation of* $\mathbf{F}^\bullet$ *below* 0) and $\mathbf{G}^\bullet = \tau^{\geqslant 0}(\mathbf{F}^\bullet[1])$ (the *truncation of* $\mathbf{F}^\bullet[1]$ *above* 0); these are the basic truncation functors associated to the t-structure.

In addition, we write $\mathbf{D}^{\leqslant n}(X)$ for

$$\mathbf{D}^{\leqslant 0}(X)[-n] := \left\{ \mathbf{F}^\bullet[-n] \mid \mathbf{F}^\bullet \in \mathbf{D}^{\leqslant 0}(X) \right\},$$

and we analogously write $\mathbf{D}^{\geqslant n}(X)$ for $\mathbf{D}^{\geqslant 0}(X)[-n]$.

Also, we define $\tau_{\leqslant n}\mathbf{F}^\bullet$ by

$$\tau_{\leqslant n}\mathbf{F}^\bullet \;=\; \left(\tau_{\leqslant 0}(\mathbf{F}^\bullet[n])\right)[-n] \;=\; ([-n]\circ\tau_{\leqslant 0}\circ[n])\mathbf{F}^\bullet,$$

and we analogously define $\tau^{\geqslant n}\mathbf{F}^\bullet$ as $([-n]\circ\tau^{\geqslant 0}\circ[n])\mathbf{F}^\bullet$.

Note that $\tau_{\leqslant n}\mathbf{F}^\bullet \in \mathbf{D}^{\leqslant n}(X)$, $\tau^{\geqslant n}\mathbf{F}^\bullet \in \mathbf{D}^{\geqslant n}(X)$ and, for all n, we have a distinguished triangle

$$\begin{tikzcd} \tau_{\leqslant n}\mathbf{F}^\bullet \arrow[rr] & & \mathbf{F}^\bullet \arrow[dl] \\ & \tau^{\geqslant n+1}\mathbf{F}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd} .$$

Writing $\simeq$ to denote natural isomorphisms between functors: for all a and b,

$$\tau_{\leqslant b}\circ\tau^{\geqslant a} \simeq \tau^{\geqslant a}\circ\tau_{\leqslant b},$$

$$\tau_{\leqslant b}\circ\tau_{\leqslant a} \simeq \tau_{\leqslant a}\circ\tau_{\leqslant b},$$

and

$$\tau^{\geqslant b}\circ\tau^{\geqslant a} \simeq \tau^{\geqslant a}\circ\tau^{\geqslant b}.$$

If $a \geqslant b$, then

$$\tau_{\leqslant b}\circ\tau_{\leqslant a} \simeq \tau_{\leqslant b},$$

and

$$\tau^{\geqslant a} \circ \tau^{\geqslant b} \simeq \tau^{\geqslant a}.$$

Also, if $a > b$, then

$$\tau_{\leqslant b} \circ \tau^{\geqslant a} = \tau^{\geqslant a} \circ \tau_{\leqslant b} = 0.$$

The *heart* of the t-structure is defined to be the full subcategory $\mathcal{C} := \mathbf{D}^{\leqslant 0}(X) \cap \mathbf{D}^{\geqslant 0}(X)$; this is always an Abelian category. We wish to describe the kernels and cokernels in this category.

Let $\mathbf{E}^\bullet, \mathbf{F}^\bullet \in \mathcal{C}$ and let f be a morphism from $\mathbf{E}^\bullet$ to $\mathbf{F}^\bullet$. We can form a distinguished triangle in $\mathbf{D}^b_c(X)$

$$\begin{tikzcd} \mathbf{E}^\bullet \arrow[rr, "f"] & & \mathbf{F}^\bullet \arrow[dl] \\ & \mathbf{G}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd},$$

where $\mathbf{G}^\bullet$ need not be in $\mathcal{C}$. Then, up to natural isomorphism,

$$\operatorname{coker} f = \tau^{\geqslant 0}\mathbf{G}^\bullet \text{ and } \ker f = \tau_{\leqslant 0}(\mathbf{G}^\bullet[-1]).$$

We define cohomology associated to a t-structure as follows. Define ${}^tH^0(\mathbf{F}^\bullet)$ to be $\tau^{\geqslant 0}\tau_{\leqslant 0}\mathbf{F}^\bullet$; this is naturally isomorphic to $\tau_{\leqslant 0}\tau^{\geqslant 0}\mathbf{F}^\bullet$. Now, define ${}^tH^n(\mathbf{F}^\bullet)$ to be

$${}^tH^0(\mathbf{F}^\bullet[n]) = \left(\tau^{\geqslant n}\tau_{\leqslant n}\mathbf{F}^\bullet\right)[n].$$

Note that this cohomology does not give back modules or even sheaves of modules, but rather gives back complexes which are objects in the heart of the t-structure.

If $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then the following are equivalent:

1) $\mathbf{F}^\bullet \in \mathbf{D}^{\leqslant 0}(X)$ (resp. $\mathbf{D}^{\geqslant 0}(X)$);

2) the morphism $\tau_{\leqslant 0}\mathbf{F}^\bullet \to \mathbf{F}^\bullet$ is an isomorphism (resp. the morphism $\mathbf{F}^\bullet \to \tau^{\geqslant 0}\mathbf{F}^\bullet$ is an isomorphism);

3) $\tau^{\geqslant 1}\mathbf{F}^\bullet = 0$ (resp. $\tau_{\leqslant -1}\mathbf{F}^\bullet = 0$));

4) $\tau^{\geqslant i}\mathbf{F}^\bullet = 0$ for all $i \geqslant 1$ (resp. $\tau_{\leqslant i}\mathbf{F}^\bullet = 0$ for all $i \leqslant -1$);

5) there exists a such that $\mathbf{F}^\bullet \in \mathbf{D}^{\leqslant a}(X)$ and ${}^tH^i(\mathbf{F}^\bullet) = 0$ for all $i \geqslant 1$ (resp. there exists a such that $\mathbf{F}^\bullet \in \mathbf{D}^{\geqslant a}(X)$ and ${}^tH^i(\mathbf{F}^\bullet) = 0$ for all $i \leqslant -1$).

It follows that, if $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then the following are equivalent:

1) $\mathbf{F}^\bullet \in \mathcal{C}$;

2) ${}^tH^0(\mathbf{F}^\bullet)$ is isomorphic to $\mathbf{F}^\bullet$;

3) there exist a and b such that $\mathbf{F}^\bullet \in \mathbf{D}^{\geqslant a}(X)$, $\mathbf{F}^\bullet \in \mathbf{D}^{\leqslant b}(X)$, and ${}^tH^n(\mathbf{F}^\bullet) = 0$ for all $n \neq 0$.

As the heart is an Abelian category, we may talk about exact sequences in $\mathcal{C}$. Any distinguished triangle in $\mathbf{D}^b_c(X)$ determines a long exact sequence of objects in the heart of the t-structure; if

$$\begin{tikzcd} \mathbf{E}^\bullet \arrow[rr] & & \mathbf{F}^\bullet \arrow[dl] \\ & \mathbf{G}^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is a distinguished triangle in $\mathbf{D}^b_c(X)$, then the associated long exact sequence in $\mathcal{C}$ is

$$\cdots \to {}^tH^{-1}(\mathbf{G}^\bullet) \to {}^tH^0(\mathbf{E}^\bullet) \to {}^tH^0(\mathbf{F}^\bullet) \to {}^tH^0(\mathbf{G}^\bullet) \to {}^tH^1(\mathbf{E}^\bullet) \to \dots .$$

We are finished now with our generalities on t-structures and wish to, at last, give our two primary examples.

The "ordinary" t-structure

The "ordinary" t-structure on $\mathbf{D}^b_c(X)$ is given by

$$\mathbf{D}^{\leqslant 0}(X) = \{\mathbf{F}^\bullet \in \mathbf{D}^b_c(X) \mid \mathbf{H}^i(\mathbf{F}^\bullet) = 0 \text{ for all } i > 0\}$$

and

$$\mathbf{D}^{\geqslant 0}(X) = \{\mathbf{F}^\bullet \in \mathbf{D}^b_c(X) \mid \mathbf{H}^i(\mathbf{F}^\bullet) = 0 \text{ for all } i < 0\}.$$

The associated truncation functors are the ordinary ones described in [**G-M3**]. If $\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then

$$(\tau_{\leqslant p}\mathbf{F}^\bullet)^n = \begin{cases} \mathbf{F}^n & \text{if } n < p \\ \mathbf{ker}\ d^p & \text{if } n = p \\ 0 & \text{if } n > p \end{cases}$$

and

$$\left(\tau^{\geqslant p}\mathbf{F}^\bullet\right)^n = \begin{cases} 0 & \text{if } n < p \\ \mathbf{coker}\ d^{p-1} & \text{if } n = p \\ \mathbf{F}^n & \text{if } n > p. \end{cases}$$

These truncated complexes are naturally quasi-isomorphic to the complexes

$$(\tilde{\tau}_{\leqslant p}\mathbf{F}^\bullet)^n = \begin{cases} \mathbf{F}^n & \text{if } n \leqslant p \\ \mathbf{Im}\ d^p & \text{if } n = p+1 \\ 0 & \text{if } n > p+1 \end{cases}$$

and

$$\left(\tilde{\tau}^{\geqslant p}\mathbf{F}^\bullet\right)^n = \begin{cases} 0 & \text{if } n < p-1 \\ \mathbf{Im}\ d^{p-1} & \text{if } n = p-1 \\ \mathbf{F}^n & \text{if } n \geqslant p. \end{cases}$$

If $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in \mathbf{D}_c^b(X)$, then

1. $(\tau_{\leqslant p}\mathbf{A}^\bullet)_x = \tau_{\leqslant p}(\mathbf{A}^\bullet_x)$;

2. $\mathbf{H}^k(\tau_{\leqslant p}\mathbf{A}^\bullet)_x = \begin{cases} \mathbf{H}^k(\mathbf{A}^\bullet)_x & \text{if } k \leqslant p \\ 0 & \text{for } k > p. \end{cases}$

3. If $\phi : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ is a morphism of complexes of sheaves which induces isomorphisms on the associated cohomology sheaves

$$\phi^* : \mathbf{H}^n(\mathbf{A}^\bullet) \cong \mathbf{H}^n(\mathbf{B}^\bullet) \text{ for all } n \leqslant p,$$

then $\tau_{\leqslant p}\phi : \tau_{\leqslant p}\mathbf{A}^\bullet \to \tau_{\leqslant p}\mathbf{B}^\bullet$ is a quasi-isomorphism.

4. If $f : X \to Y$ is a continuous map and $\mathbf{C}^\bullet$ is a complex of sheaves on Y, then

$$\tau_{\leqslant p} f^*(\mathbf{C}^\bullet) \cong f^*\tau_{\leqslant p}(\mathbf{C}^\bullet).$$

5. If R is a field and $\mathbf{A}^\bullet$ is a complex of sheaves of R-modules on X with locally constant cohomology sheaves, then there are natural quasi-isomorphisms

$$\tau^{\geqslant -p} R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{R}^\bullet_X) \to \tau^{\geqslant -p} R\mathbf{Hom}^\bullet(\tau_{\leqslant p}\mathbf{A}^\bullet, \mathbf{R}^\bullet_X) \leftarrow R\mathbf{Hom}^\bullet(\tau_{\leqslant p}\mathbf{A}^\bullet, \mathbf{R}^\bullet_X).$$

The heart of this t-structure consists of those complexes which have non-zero cohomology sheaves only in degree 0; such complexes are quasi-isomorphic to complexes which are non-zero only in degree 0.

The t-structure cohomology of a complex $\mathbf{F}^\bullet$ is essentially the sheaf cohomology of $\mathbf{F}^\bullet$; ${}^tH^n(\mathbf{F}^\bullet)$ is quasi-isomorphic to a complex which has $\mathbf{H}^n(\mathbf{F}^\bullet)$ in degree 0 and is zero in all other degrees. With this identification, the t-structure long exact sequence associated to a distinguished triangle is merely the usual long exact sequence on sheaf cohomology.

We are now going to give the construction of the intersection cohomology complexes as it is presented in [**G-M3**]. Our indexing will look different from that of [**G-M3**] for several reasons.

First, we are dealing only with complex analytic spaces, X, and we are using only middle perversity; this accounts for some of the indexing differences. In addition, in this setting, the intersection cohomology complex defined in [**G-M3**] would have possibly non-zero cohomology only in degrees between $-2\dim_{\mathbb{C}} X$ and $-(\dim_{\mathbb{C}} X)-1$, inclusive. The definition below is shifted by $-\dim_{\mathbb{C}} X$ from the [**G-M3**] definition, and yields a perverse sheaf which has possibly non-zero stalk cohomology only in degrees between $-\dim_{\mathbb{C}} X$ and -1, inclusive (assuming the space has no isolated points).

Let X be a purely n-dimensional complex analytic space with a complex analytic Whitney stratification $\mathcal{S} = \{S_\alpha\}$.

For all k, let X^k denote the union of the strata of dimension less than or equal to k. By convention, we set $X^{-1} = \emptyset$. Hence, we have a filtration

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^{n-1} \subseteq X^n = X.$$

For all k, let $\mathcal{U}_k := X - X^{n-k}$, and let i_k denote the inclusion $\mathcal{U}_k \hookrightarrow \mathcal{U}_{k+1}$. Let $\mathcal{L}^\bullet_{\mathcal{U}_1}$ be a local system on the top-dimensional strata.

Then, the intersection cohomology complex on X with coefficients in $\mathcal{L}^\bullet_{\mathcal{U}_1}$, as described in section 2, is given by

$$\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet_{\mathcal{U}_1}) := \tau_{\leqslant -1}Ri_{n*}\dots\tau_{\leqslant 1-n}Ri_{2*}\tau_{\leqslant -n}Ri_{1*}\big(\mathcal{L}^\bullet_{\mathcal{U}_1}[n]\big).$$

Up to quasi-isomorphism, this complex is independent of the stratification. Note that the cohomology sheaves of $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet_{\mathcal{U}_1})$ are supported only in degrees k for which $-n \leqslant k \leqslant -1$ (unless X is 0-dimensional, and then $\mathbf{IC}^\bullet_X(\mathcal{L}^\bullet_{\mathcal{U}_1}) \cong \mathcal{L}^\bullet_{\mathcal{U}_1}$).

Also note that it follows from the construction that there is always a canonical map from the shifted constant sheaf $\mathbf{R}^\bullet_X[n]$ to $\mathbf{IC}^\bullet_X(\mathbf{R}^\bullet_{\mathcal{U}_1})$ which induces an isomorphism when restricted to $\mathcal{U}_1$.

To see this, consider the canonical morphism $\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n] \to Ri_{k*}i_k^*\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n]$ for each $k \geqslant 1$. As $i_k^*\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n] \cong \mathbf{R}^\bullet_{\mathcal{U}_k}[n]$, we have a canonical map $\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n] \to Ri_{k*}\mathbf{R}^\bullet_{\mathcal{U}_k}[n]$ and, hence, a canonical map between the truncations $\tau_{\leqslant k-n-1}\big(\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n]\big) \to \tau_{\leqslant k-n-1}Ri_{k*}\big(\mathbf{R}^\bullet_{\mathcal{U}_k}[n]\big)$. But,

$$\tau_{\leqslant k-n-1}\big(\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n]\big) \cong \mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n]$$

and so we have a canonical map $\mathbf{R}^\bullet_{\mathcal{U}_{k+1}}[n] \to \tau_{\leqslant k-n-1}Ri_{k*}\big(\mathbf{R}^\bullet_{\mathcal{U}_k}[n]\big)$. By piecing all of these maps together, one obtains the desired morphism.

The perverse t-structure

The *perverse* t-structure (with middle perversity μ) on $\mathbf{D}^b_c(X)$ is given by

$${}^\mu\mathbf{D}^{\leqslant 0}(X) = \{\mathbf{F}^\bullet \in \mathbf{D}^b_c(X) \mid \dim \operatorname{supp}^{\text{-j}} \mathbf{F}^\bullet \leqslant j \text{ for all } j\}$$

and

$${}^\mu\mathbf{D}^{\geqslant 0}(X) = \{\mathbf{F}^\bullet \in \mathbf{D}^b_c(X) \mid \dim \operatorname{cosupp}^{\text{j}} \mathbf{F}^\bullet \leqslant j \text{ for all } j\}.$$

Note that the heart of this t-structure is precisely $Perv(X)$. Thus, every distinguished triangle in $\mathbf{D}^b_c(X)$ determines a long exact sequence in the Abelian category $Perv(X)$.

We naturally call the t-structure cohomology associated to the perverse t-structure the *perverse cohomology* or *perverse projection* and denote it in degree n by ${}^\mu H^n(\mathbf{F}^\bullet)$.

Let d be an integer, and let $f : Y \to X$ be a morphism of complex spaces such that $\dim f^{-1}(x) \leqslant d$, for all $\mathbf{x} \in X$. Let $\dim Y/X := \dim Y - \dim X$. Then,

1) f^* sends ${}^\mu\mathbf{D}^{\leqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\leqslant d}(Y)$, and sends ${}^\mu\mathbf{D}^{\geqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\geqslant \dim Y/X}(Y)$;

2) $f^!$ sends ${}^\mu\mathbf{D}^{\geqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\geqslant -d}(Y)$, and sends ${}^\mu\mathbf{D}^{\leqslant 0}(X)$ to ${}^\mu\mathbf{D}^{\leqslant -\dim Y/X}(Y)$;

3) if $\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(Y)$ and $Rf_!\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then $Rf_!\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\leqslant d}(X)$;

4) if $\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\geqslant 0}(Y)$ and $Rf_*\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then $Rf_*\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\geqslant -d}(X)$.

Let $f : Y \to X$ be a morphism of complex spaces such that each point in X has an open neighborhood $\mathcal{U}$ such that $f^{-1}(\mathcal{U})$ is a Stein space (e.g., an affine map between algebraic varieties). Then,

a) if $\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(Y)$ and $Rf_*\mathbf{F}^\bullet \in \mathbf{D}^b_c(X)$, then $Rf_*\mathbf{F}^\bullet \in {}^\mu\mathbf{D}^{\leqslant 0}(X)$;

b) if $\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(Y)$ and $Rf_!\mathbf{F}^\bullet \in \mathbf{D}_c^b(X)$, then $Rf_!\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(X)$.

If $f : X \to \mathbb{C}$ is an analytic map, then the functors $\psi_f[-1]$ and $\phi_f[-1]$ are *t-exact* with respect to the perverse t-structures; this means that if $\mathbf{E}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ and $\mathbf{F}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(X)$, then $\psi_f\mathbf{E}^\bullet[-1]$ and $\phi_f\mathbf{E}^\bullet[-1]$ are in ${}^{\mu}\mathbf{D}^{\leqslant 0}(f^{-1}(0))$, and $\psi_f\mathbf{F}^\bullet[-1]$ and $\phi_f\mathbf{F}^\bullet[-1]$ are in ${}^{\mu}\mathbf{D}^{\geqslant 0}(f^{-1}(0))$.

In particular, $\psi_f[-1]$ and $\phi_f[-1]$ take perverse sheaves to perverse sheaves and, for any $\mathbf{F}^\bullet \in \mathbf{D}_c^b(X)$,

$$
{}^{\mu}H^n(\psi_f\mathbf{F}^\bullet[-1]) \cong \psi_f {}^{\mu}H^n(\mathbf{F}^\bullet)[-1] \quad \text{and} \quad {}^{\mu}H^n(\phi_f\mathbf{F}^\bullet[-1]) \cong \phi_f {}^{\mu}H^n(\mathbf{F}^\bullet)[-1].
$$

While there are several proofs in the literature of the fact that $\psi_f[-1]$ is a perverse functor, i.e., takes perverse sheaves to perverse sheaves, it is not so easy to find proofs that $\phi_f[-1]$ is a perverse functor. Let j denote the inclusion of $f^{-1}(0)$ into X. Let $\mathbf{P}^\bullet$ be a perverse sheaf on X. By 1) and 2) above, $j^*\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant 0}(X)$ and $j^!\mathbf{P}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(X)$. Knowing that $\psi_f[-1]\mathbf{P}^\bullet$ is perverse, and applying perverse cohomology to the distinguished triangles

$$
\phi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{var}} \psi_f[-1]\mathbf{P}^\bullet \longrightarrow j^![1]\mathbf{P}^\bullet \to \phi_f\mathbf{P}^\bullet
$$

and

$$
j^*[-1]\mathbf{P}^\bullet \to \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{r} \phi_f[-1]\mathbf{P}^\bullet \to j^*\mathbf{P}^\bullet,
$$

one finds, respectively, that ${}^{\mu}H^i(\phi_f[-1]\mathbf{P}^\bullet)$ is zero if $i \leqslant -1$ and if $i \geqslant 1$, i.e., $\phi_f[-1]\mathbf{P}^\bullet$ is perverse.

If the base ring is a field, then the functor ${}^{\mu}H^0$ also commutes with Verdier dualizing; that is, there is a natural isomorphism

$$
\mathcal{D} \circ {}^{\mu}H^0 \cong {}^{\mu}H^0 \circ \mathcal{D}.
$$

Intersection cohomology has a functorial definition related to ${}^{\mu}H^0$. Let X be a purely n-dimensional complex analytic set, and let $\mathcal{L}$ be a local system (in degree 0) on a smooth, open dense subset, $\mathcal{U}$, of the smooth part of X. Let i denote the inclusion of $\mathcal{U}$ into X. Then, by applying the functor ${}^{\mu}H^0$ to the canonical map $i_!\mathcal{L}^\bullet[n] \to Ri_*\mathcal{L}^\bullet[n]$, one obtains a morphism $\omega : {}^{\mu}H^0\big(i_!\mathcal{L}^\bullet[n]\big) \to {}^{\mu}H^0\big(Ri_*\mathcal{L}^\bullet[n]\big)$ in the category of perverse sheaves on X. The image of ω in this Abelian category is naturally isomorphic to the intersection cohomology complex $\mathbf{IC}_X^\bullet(\mathcal{L})$.

If X is not pure-dimensional, one can still use essentially the construction above, except that $\mathcal{L}^\bullet[n]$ needs to be replaced by a "shifted piecewise local system", i.e., one takes the image of $\omega : {}^{\mu}H^0\big(i_!\mathbf{P}^\bullet\big) \to {}^{\mu}H^0\big(Ri_*\mathbf{P}^\bullet\big)$, where $\mathbf{P}^\bullet$ is a complex on $\mathcal{U}$, whose restriction to each connected component C of $\mathcal{U}$ is a local system on C, shifted by $\dim C$ (i.e., shifted into degree $-\dim C$); note that such a shifted piecewise local system $\mathbf{P}^\bullet$ is a perverse sheaf on $\mathcal{U}$.

More generally, suppose that X is a complex analytic space, that Y is a closed complex analytic subspace of X, and that $j : Y \to X$ and $i : X - Y \to X$ are the inclusions. Then, the functor from $Perv(X - Y)$ to $Perv(X)$ which takes $\mathbf{P}^\bullet$ to the image (in $Perv(X)$) of the canonical map ${}^{\mu}H^0(i_!\mathbf{P}^\bullet) \to {}^{\mu}H^0(Ri_*\mathbf{P}^\bullet)$ is denoted by $i_{!*}$, and is called the **intermediate extension** functor. The perverse sheaf $i_{!*}\mathbf{P}^\bullet$ is the unique (up to isomorphism) perverse extension $\mathbf{A}^\bullet$ of $\mathbf{P}^\bullet$ from $X - Y$ to X satisfying either/both of the two equivalent conditions:

1) $j^*[-1]\mathbf{A}^\bullet \in {}^{\mu}\mathbf{D}^{\leqslant 0}(Y)$ and $j^![1]\mathbf{A}^\bullet \in {}^{\mu}\mathbf{D}^{\geqslant 0}(Y)$;

2) $\mathbf{A}^\bullet$ has no non-trivial subobjects or quotient objects in $Perv(X)$ whose support is contained in Y.

If $X - Y$ is dense in X, then the intermediate extension takes intersection cohomology, with possibly local system coefficients, on $X - Y$ to intersection cohomology on X, with the same local system coefficients.

Suppose now that $\mathbf{I}^\bullet$ is an intersection cohomology complex on X, with possibly local system coefficients, and that $f : X \to \mathbb{C}$ is such that f does not vanish on any irreducible component of $\operatorname{supp}\mathbf{I}^\bullet$. Let i denote the (open) inclusion of $X - f^{-1}(0)$ into X, and let j denote the (closed) inclusion of $f^{-1}(0)$ into X. As $Ri_*i^*\mathbf{I}^\bullet$ is perverse, the canonical map $\mathbf{I}^\bullet \to Ri_*i^*\mathbf{I}^\bullet$ is a morphism of perverse sheaves, with kernel and cokernel given by ${}^\mu H^{-1}\big(j_!j^![1]\mathbf{I}^\bullet\big)$ and ${}^\mu H^0\big(j_!j^![1]\mathbf{I}^\bullet\big)$, respectively, and ${}^\mu H^i\big(j_!j^![1]\mathbf{I}^\bullet\big) = 0$ for $i \neq -1, 0$. As $\mathbf{I}^\bullet$ has no non-zero perverse subobjects whose support is contained in a nowhere dense subset of $\operatorname{supp}\mathbf{I}^\bullet$, we conclude that ${}^\mu H^{-1}\big(j_!j^![1]\mathbf{I}^\bullet\big) = 0$ and, hence, that $j_!j^![1]\mathbf{I}^\bullet$ is perverse. The dual argument implies that $j_*j^*[-1]\mathbf{I}^\bullet$ is perverse.

Continuing with the assumptions of the previous paragraph, we obtain that the two canonical distinguished triangles yield short exact sequences in $Perv(X)$:

$$0 \to \phi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{var}} \psi_f[-1]\mathbf{P}^\bullet \longrightarrow j^![1]\mathbf{P}^\bullet \longrightarrow 0$$

and

$$0 \to j^*[-1]\mathbf{P}^\bullet \to \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\text{can}} \phi_f[-1]\mathbf{P}^\bullet \to 0.$$

Thus, the image, in $Perv(X)$, of the endomorphism $\text{var} \circ \text{can} = \text{id} - T_f$ on $\psi_f[-1]\mathbf{P}^\bullet$ is isomorphic to $\phi_f[-1]\mathbf{P}^\bullet$.

Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves on X which is constructible with respect to a connected Whitney stratification $\{S_\alpha\}$ of X, and let $d_\alpha := \dim S_\alpha$. Then, ${}^\mu H^0(\mathbf{F}^\bullet)$ is also constructible with respect to $\mathcal{S}$, and $\big({}^\mu H^0(\mathbf{F}^\bullet)\big)_{|_{\mathbb{N}_\alpha}}[-d_\alpha]$ is naturally isomorphic to ${}^\mu H^0(\mathbf{F}^\bullet_{|_{\mathbb{N}_\alpha}}[-d_\alpha])$, where $\mathbb{N}_\alpha$ denotes a normal slice to S_α.

Let $S_{\max}$ be a maximal stratum contained in the support of $\mathbf{F}^\bullet$, and let $m = \dim S_{\max}$. Then, $\big({}^\mu H^0(\mathbf{F}^\bullet)\big)_{|_{S_{\max}}}$ is isomorphic (in the derived category) to the complex which has $(\mathbf{H}^{-m}(\mathbf{F}^\bullet))_{|_{S_{\max}}}$ in degree $-m$ and zero in all other degrees.

In particular, $\operatorname{supp}\mathbf{F}^\bullet = \bigcup_i \operatorname{supp}{}^\mu H^i(\mathbf{F}^\bullet)$, and if $\mathbf{F}^\bullet$ is supported on an isolated point, $\mathbf{q}$, then $H^0({}^\mu H^0(\mathbf{F}^\bullet))_{\mathbf{q}} \cong H^0(\mathbf{F}^\bullet)_{\mathbf{q}}$. From this, and the fact that perverse cohomology commutes with nearby and vanishing cycles shifted by -1, one easily concludes that, at all points $\mathbf{x} \in X$,

$$\chi(\mathbf{F}^\bullet)_{\mathbf{x}} = \sum_k (-1)^k \chi\big({}^\mu H^k(\mathbf{F}^\bullet)\big)_{\mathbf{x}}.$$

Switching Coefficients

Suppose that the base ring R is a p.i.d. For each prime ideal $\mathfrak{p}$ of R, let $k_{\mathfrak{p}}$ denote the field of fractions of $R/\mathfrak{p}$, i.e., k_0 is the field of fractions of R, and for $\mathfrak{p} \neq 0$, $k_{\mathfrak{p}} = R/\mathfrak{p}$. There are the obvious functors $\delta_{\mathfrak{p}} : \mathbf{D}^b_c(R_X) \to \mathbf{D}^b_c((k_{\mathfrak{p}})_X)$, which sends $\mathbf{F}^\bullet$ to $\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X$, and $\epsilon_{\mathfrak{p}} : \mathbf{D}^b_c((k_{\mathfrak{p}})_X) \to \mathbf{D}^b_c(R_X)$, which considers $k_{\mathfrak{p}}$-vector spaces as R-modules.

If $\mathbf{A}^\bullet$ is a complex of $k_{\mathfrak{p}}$-vector spaces, we may consider the perverse cohomology of $\mathbf{A}^\bullet$, ${}^\mu H^i_{k_{\mathfrak{p}}}(\mathbf{A}^\bullet)$, or the perverse cohomology of $\epsilon_{\mathfrak{p}}(\mathbf{A}^\bullet)$, which we denote by ${}^\mu H^i_R(\mathbf{A}^\bullet)$. If $\mathbf{A}^\bullet \in \mathbf{D}^b_c((k_{\mathfrak{p}})_X)$ and $S_{\max}$ is a maximal stratum contained in the support of $\mathbf{A}^\bullet$, then there is a canonical isomorphism

$$\epsilon_{\mathfrak{p}}\big(({}^\mu H^i_{k_{\mathfrak{p}}}(\mathbf{A}^\bullet))_{|_{S_\alpha}}\big) \cong ({}^\mu H^i_R(\mathbf{A}^\bullet))_{|_{S_\alpha}};$$

in particular, $\operatorname{supp}{}^\mu H^i_{k_{\mathfrak{p}}}(\mathbf{A}^\bullet) = \operatorname{supp}{}^\mu H^i_R(\mathbf{A}^\bullet)$.

If $\mathbf{F}^\bullet \in \mathbf{D}^b_c(R_X)$, $S_{\max}$ is a maximal stratum contained in the support of $\mathbf{F}^\bullet$, and $\mathbf{x} \in S_{\max}$, then for some prime ideal $\mathfrak{p} \subset R$ and for some integer i, $H^i(\mathbf{F}^\bullet)_{\mathbf{x}} \otimes k_{\mathfrak{p}} \neq 0$; it follows that $S_{\max}$ is also a maximal stratum in the support of $\mathbf{F}^\bullet \overset{L}{\otimes} (k_\mathfrak{p})^\bullet_X$. Thus,

$$\operatorname{supp}\mathbf{F}^\bullet = \bigcup_{\mathfrak{p}} \operatorname{supp}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_\mathfrak{p})^\bullet_X)$$

and so

$$\operatorname{supp}\mathbf{F}^\bullet = \bigcup_{i,\mathfrak{p}} \operatorname{supp} {}^\mu H^i_{k_\mathfrak{p}}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_\mathfrak{p})^\bullet_X),$$

where the boundedness and constructibility of $\mathbf{F}^\bullet$ imply that this union is locally finite.

References

[**A'C**] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Proc. Kon. Ned. Akad. Wet., Series A **76** (1973), 113–118.

[**BBD**] A. Beilinson, J. Berstein, and P. Deligne, *Faisceaux Pervers*, Astérisque 100, Soc. Math de France, 1983.

[**Bo**] A. Borel et al., *Intersection Cohomology*, Progress in Math. 50, Birkhäuser, 1984.

[**BMM**] J. Briançon, P. Maisonobe, and M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. **117** (1994), 531–550.

[**Br**] J. Brylinski, *Transformations canoniques, Dualité projective, Théorie de Lefschetz, Transformations de Fourier et sommes trigonométriques*, Soc. Math de France, Astérisque **140** (1986).

[**De**] P. Deligne, *Comparaison avec la théorie transcendante*, Séminaire de géométrie algébrique du Bois-Marie, SGA 7 II, Springer Lect. Notes **340** (1973).

[**Di**] A. Dimca, *Sheaves in Topology*, Universitext, Springer-Verlag, 2004.

[**Gi**] V. Ginsburg, *Characteristic Varieties and Vanishing Cycles*, Inv. Math. **84** (1986), 327–403.

[**G-M1**] M. Goresky and R. MacPherson, *Morse Theory and Intersection Homology*, Analyse et Topologie sur les Espaces Singuliers. Astérisque **101** (1983), Soc. Math. France, 135–192.

[**G-M2**] ———, *Stratified Morse Theory*, Ergebnisse der Math. 14, Springer-Verlag, Berlin, 1988.

[**G-M3**] ———, *Intersection homology II*, Inv. Math **71** (1983), 77–129.

[**G-M4**] ———, *Intersection homology theory*, Topology **19** (1980), 135–162.

[**Gr**] P. P. Grivel, *Les foncteurs de la categorie des faisceaux associes a une application continue*, Intersection Cohomology, Prog. in Math. **50** (1984), 183–207.

[**H**] R. Hartshorne, *Residues and Duality*, Springer Lecture Notes 20, Springer-Verlag, 1966.

[**I**] B. Iverson, *Cohomology of Sheaves*, Springer-Verlag, 1986.

[**K-S**] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grund. der math. Wiss. 292, Springer - Verlag, 1990.

[**Lê1**] Lê D. T., *Morsification of* **D**-*Modules*, Bol. Soc. Mat. Mexicana (3) **4** (1998), 229–248.

[**Lê2**] ———, *Sur les cycles évanouissants des espaces analytiques*, C.R. Acad. Sci. Paris, Ser. A **288** (1979), 283–285.

[**Lo**] E. Looijenga, *Isolated Singular Points on Complete Intersections*, Cambridge. Univ. Press, 1984.

[**Mac1**] R. MacPherson, *Global Questions in the Topology of Singular Spaces*, Proc. Internat. Congress of Math., Warsaw (1983), 213–235.

[**Mac2**] ———, *Intersection Homology and Perverse Sheaves*, unpublished AMS notes (1990), 1–161.

[**M-V**] R. MacPherson and K. Vilonen, *Elementary construction of perverse sheaves*, Invent. Math. **84** (1986), 403–435.

[**P**] A. Parusiński, *Limits of Tangent Spaces to Fibres and the w_f Condition*, Duke Math. J. **72** (1993), 99–108.

[**Sa**] M. Saito, *Modules de Hodge polarisables*, Publ. RIMS, Kyoto Univ. **24** (1988), 849–995.

[**Sch**] J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, Monografie Matematyczne, vol. 63, Birkhauser, 2003.

[**V**] J. L. Verdier, *Catégories dérivées*, Etat 0, SGA $4\frac{1}{2}$, Lecture Notes in Math. **569** (1977), 262–311.

David B. Massey, Dept. of Mathematics, Northeastern University, Boston, MA, 02115, USA
E-mail address: DMASSEY@NEU.edu