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SHELLABLE AND COHEN-MACAULAY PARTIALLY ORDERED SETS

BY
ANDERS BJÖRNER

ABSTRACT. In this paper we study shellable posets (partially ordered sets), that is, finite posets such that the simplicial complex of chains is shellable. It is shown that all admissible lattices (including all finite semimodular and supersolvable lattices) and all bounded locally semimodular finite posets are shellable. A technique for labeling the edges of the Hasse diagram of certain lattices, due to R. Stanley, is generalized to posets and shown to imply shellability, while Stanley's main theorem on the Jordan-Hölder sequences of such labelings remains valid. Further, we show a number of ways in which shellable posets can be constructed from other shellable posets and complexes. These results give rise to several new examples of Cohen-Macaulay posets. For instance, the lattice of subgroups of a finite group G is Cohen-Macaulay (in fact shellable) if and only if G is supersolvable. Finally, it is shown that all the higher order complexes of a finite planar distributive lattice are shellable.

Introduction. A pure finite simplicial complex Δ is said to be shellable if its maximal faces can be ordered F_1, F_2, \dots, F_n in such a way that $F_k \cap (\cup_{i=1}^{k-1} F_i)$ is a nonempty union of maximal proper faces of F_k for $k = 2, 3, \dots, n$. It is known that a shellable complex Δ must be Cohen-Macaulay, that is, a certain commutative ring associated with Δ is a Cohen-Macaulay ring (see the appendix for details). The notion of shellability, which originated in polyhedral theory, is emerging as a useful concept also in combinatorics with applications in matroid theory and order theory.

In this paper we study shellable posets (partially ordered sets), that is, finite posets for which the order complex consisting of all chains $x_1 < x_2 < \dots < x_k$ is shellable. The material is organized as follows.

After some preliminary remarks in §1, we discuss in §2 a certain type of labeling of the edges of the Hasse diagram of finite posets. We call posets which admit such labeling *lexicographically shellable*, and we prove that lexicographically shellable posets are indeed shellable. In lexicographically shellable posets the Möbius function can be interpreted as counting certain distinctly labeled maximal chains. We elaborate somewhat on this principle, point out its natural connection with shellability, and exemplify its use.

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In §3 we show that all admissible lattices (including all finite semimodular and supersolvable lattices) are lexicographically shellable. That such lattices are Cohen-Macaulay was conjectured by R. Stanley. In particular, the lattice of subgroups of a finite group G is Cohen-Macaulay if and only if G is supersolvable.

In §4 we show that (lexicographic) shellability is preserved under several of the common poset constructions. We consider rank-selected subposets, direct products, ordinal sums, cardinal powers and interval posets.

The fifth section contains a proof that shellability is preserved under barycentric subdivision of simplicial complexes. We also point out that face-lattices of convex polytopes are shellable.

In §6 we prove that bounded locally semimodular finite posets are shellable. Also, a certain inequality, known in polyhedral theory as the Hirsch conjecture, is shown to hold for such posets.

The order complex of a finite poset P is the first member of a family of associated simplicial complexes $\Delta_k(P)$, $k = 1, 2, \dots$. In §7 we prove that all these higher order complexes $\Delta_k(P)$ are shellable when P is a finite planar distributive lattice.

Finally, in an appendix we briefly review the definitions and basic properties of shellable and Cohen-Macaulay complexes.

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1. Preliminaries. For poset terminology not otherwise explained we refer to [2]. A poset is said to be *bounded* if it has a least element and a greatest element. These will always be denoted by $\hat{0}$ and $\hat{1}$ respectively. For a bounded poset P we let \bar{P} denote the subposet $P - \{\hat{0}, \hat{1}\}$. For a poset Q we let \hat{Q} denote the (essentially) unique bounded poset P such that $\bar{P} = Q$. A finite poset is said to be *pure* if all maximal chains have the same length. A pure poset satisfies the Jordan-Dedekind condition: all unrefinable chains between two comparable elements have the same length. We shall call a poset *graded* if it is finite, bounded and pure. Any element x of a graded poset P has a well-defined *rank* $\rho(x)$ equal to the common length of all unrefinable chains from $\hat{0}$ to x in P . The symbol “ $<$ ” denotes the covering relation: $x < y$ means that $x < y$ and $x < z < y$ for no z . The notation “ $\mathbf{c}: x_0 < x_1 < \dots < x_n$ ” will be frequently used for denoting a chain $\mathbf{c} = \{x_0, x_1, \dots, x_n\}$ such that $x_0 < x_1 < \dots < x_n$.

The *order complex* $\Delta(P)$ of a finite poset P is the simplicial complex of all chains of P . We shall say that a finite pure poset P is *shellable* if its order complex $\Delta(P)$ is shellable. Similarly, P will be called *Cohen-Macaulay* if $\Delta(P)$ is a Cohen-Macaulay complex. These properties of simplicial complexes are defined and commented on in an appendix at the end of this paper. Note in particular that a shellable poset must be Cohen-Macaulay, and that a Cohen-Macaulay poset must be pure. For convenience, when we write simply “Cohen-Macaulay” (for instance, in Theorem 3.3) the intention is that any one of the slightly different concepts of “Cohen-Macaulay-ness” defined in the appendix may be applied. Note that a finite poset P

is shellable (Cohen-Macaulay) if and only if \hat{P} is shellable (Cohen-Macaulay).

The cardinality of a finite set S will be written $|S|$. For a positive integer n we let $[n]$ denote the set $\{1, 2, \dots, n\}$.

2. Lexicographically shellable posets. For any finite poset P we let $C(P)$ denote its covering relation, $C(P) = \{(x, y) \in P \times P \mid x < y\}$. An *edge-labeling* of P is a map $\lambda: C(P) \rightarrow \Lambda$, where Λ is some poset. An edge-labeling therefore corresponds to an assignment of elements of Λ to the edges of the Hasse diagram of P . An unrefinable chain $x_0 < x_1 < \dots < x_n$ in a poset with an edge-labeling λ will be called *rising* if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \dots \leq \lambda(x_{n-1}, x_n)$.

DEFINITION 2.1. Let $\lambda: C(P) \rightarrow \Lambda$ be an edge-labeling of a graded poset P . λ is said to be an *R-labeling* if in every interval $[x, y]$ of P there is a unique rising unrefinable chain $x = x_0 < x_1 < \dots < x_n = y$. λ is said to be an *L-labeling* in case (i) λ is an R-labeling and (ii) for every interval $[x, y]$ of P if $x = x_0 < x_1 < \dots < x_n = y$ is the unique rising unrefinable chain and $x < z \leq y$, $z \neq x_1$, then $\lambda(x, x_1) < \lambda(x, z)$.

DEFINITION 2.2. A poset is *lexicographically shellable* (or *L-shellable*) if it is graded and admits an L-labeling.

The following result is fundamental for this paper.

THEOREM 2.3. *Let P be a lexicographically shellable poset. Then P is shellable.*

The proof will be given after some useful technical properties of L-labelings have been derived.

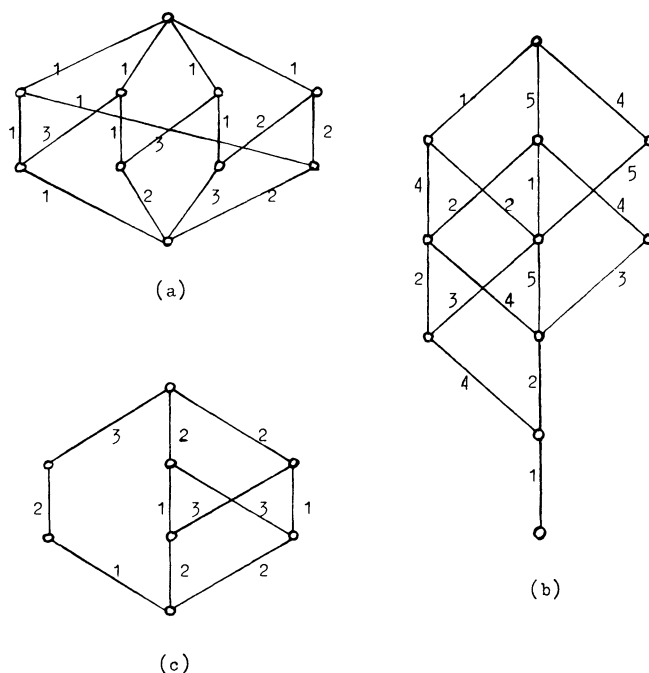


FIGURE 1

EXAMPLE 2.4. To illustrate the above definitions we exhibit L -labelings ($\Lambda = \mathbf{Z}$) of the face-lattice of a square (Figure 1a) and of the poset of partitions of the integer 6 ordered by refinement (Figure 1b). The poset P of Figure 1c has been given an R -labeling which is not an L -labeling. In fact, by the above theorem there cannot exist any L -labeling of P , since P is clearly not shellable.

With every unrefinable n -chain $\mathbf{c}: x_0 < x_1 < \cdots < x_n$ of a poset P having an edge-labeling $\lambda: C(P) \rightarrow \Lambda$ we associate the n -tuple

$$\pi(\mathbf{c}) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{n-1}, x_n)) \in \Lambda^n.$$

Following Stanley [19] we call $\pi(\mathbf{c})$ the *Jordan-Hölder sequence* of \mathbf{c} . For any poset Λ the cartesian product Λ^n is partially ordered by the *lexicographic order*: $(a_1, a_2, \dots, a_n) \in \Lambda^n$ precedes $(b_1, b_2, \dots, b_n) \in \Lambda^n$ if and only if $a_i < b_i$ in the first coordinate where they differ.

When dealing with a poset P having an R -labeling we shall for the rest of this paper let $\mathbf{c}(x, y)$ denote the unique rising unrefinable chain from x to y in P . The Jordan-Hölder sequences of L -labelings have the following important property.

PROPOSITION 2.5. *Let $\lambda: C(P) \rightarrow \Lambda$ be an R -labeling of a graded poset P . Then (i) and (ii) below are equivalent conditions, and both are implied by (iii):*

- (i) λ is an L -labeling,
- (ii) for every interval $[x, y]$ in P and every unrefinable chain \mathbf{d} from x to y other than $\mathbf{c}(x, y)$ it is true that $\pi(\mathbf{c}(x, y)) < \pi(\mathbf{d})$ in the lexicographic order,
- (iii) for every interval $[x, y]$ in P , if $x < x_1$ is the first link of $\mathbf{c}(x, y)$: $x = x_0 < x_1 < \cdots < x_n = y$ then $\lambda(x, x_1) \leq \lambda(u, v)$ for all $x \leq u < v \leq y$.

PROOF. (i) \Rightarrow (ii): Let $\mathbf{d}: x = y_0 < y_1 < \cdots < y_n = y$ be an unrefinable chain different from $\mathbf{c}(x, y)$: $x = x_0 < x_1 < \cdots < x_n = y$. If k is the least index such that $x_k \neq y_k$, we know that $\lambda(x_{k-1}, x_k) < \lambda(x_{k-1}, y_k) = \lambda(y_{k-1}, y_k)$, since λ is an L -labeling and $x_{k-1} < x_k < \cdots < x_n = y$ must be the unique rising chain of the interval $[x_{k-1}, y]$. Hence, $\pi(\mathbf{c}(x, y)) < \pi(\mathbf{d})$.

(ii) \Rightarrow (i): Let $x < z \leq y$, $z \neq x_1$, where $\mathbf{c}(x, y)$: $x = x_0 < x_1 < \cdots < x_n = y$. Also, let $\mathbf{c}(z, y)$: $z = z_0 < z_1 < \cdots < z_{n-1} = y$. Now, $\pi(\mathbf{c}(x, y)) < \pi(\mathbf{d})$, where \mathbf{d} : $x < z_0 < z_1 < \cdots < z_{n-1} = y$. Hence, $\lambda(x, x_1) \leq \lambda(x, z_0)$. If $\lambda(x, x_1) = \lambda(x, z_0)$ then $\lambda(x, z_0) = \lambda(x, x_1) \leq \lambda(x_1, x_2) \leq \lambda(z_0, z_1) \leq \lambda(z_1, z_2) \leq \cdots \leq \lambda(z_{n-2}, y)$ so that \mathbf{d} is also rising from x to y , which contradicts the uniqueness of $\mathbf{c}(x, y)$. Consequently, $\lambda(x, x_1) < \lambda(x, z)$.

(iii) \Rightarrow (i): Preserve the meaning of x, y, z , $\mathbf{c}(x, y)$, $\mathbf{c}(z, y)$ and \mathbf{d} from the preceding paragraph. By assumption, $\lambda(x, x_1) \leq \lambda(x, z)$ and $\lambda(x, x_1) \leq \lambda(z, z_1)$. Hence, if $\lambda(x, x_1) = \lambda(x, z)$ then \mathbf{d} would be rising. So, $\lambda(x, x_1) < \lambda(x, z)$.

(i) does not imply (iii) as simple counterexamples show. \square

Let us say that a sequence $\pi = (a_1, a_2, \dots, a_n)$ of elements from a poset Λ has a *descent* at e , $e \in [n-1]$, if $a_e \not\leq a_{e+1}$. The sequence π is said to have *descent set* $D(\pi) = \{e \in [n-1] \mid a_e \not\leq a_{e+1}\}$.

LEMMA 2.6. Let $\lambda: C(P) \rightarrow \Lambda$ be an L -labeling of a graded poset P of length n , and suppose that $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ is a maximal chain of P . Then $\pi(\mathbf{m})$ has a descent at $e \in [n-1]$ if and only if $\mathbf{h} \cap \mathbf{m} = \mathbf{m} - \{x_e\}$ for some maximal chain \mathbf{h} such that $\pi(\mathbf{h}) < \pi(\mathbf{m})$ in the lexicographic order on Λ^n .

PROOF. Let $\mathbf{c}(x_{e-1}, x_{e+1}): x_{e-1} < y < x_{e+1}$ be the unique rising chain from x_{e-1} to x_{e+1} and define $\mathbf{d}: \hat{0} = x_0 < x_1 < \cdots < x_{e-1} < y < x_{e+1} < x_{e+2} < \cdots < x_n = \hat{1}$. If $\pi(\mathbf{m})$ has a descent at e , then $y \neq x_e$ and $\mathbf{h} = \mathbf{d}$ has the required properties. Conversely, if $\pi(\mathbf{m})$ does not have a descent at e , then $\mathbf{m} = \mathbf{d}$ and since λ is an L -labeling it follows that $\pi(\mathbf{m}) < \pi(\mathbf{h})$ for all maximal chains \mathbf{h} such that $\mathbf{h} \cap \mathbf{m} = \mathbf{m} - \{x_e\}$. \square

PROOF OF THEOREM 2.3. Let $\lambda: C(P) \rightarrow \Lambda$ be an L -labeling of the graded poset P . We must prove that the set \mathfrak{M} of maximal chains of P has a shelling, that is, a linear order Ω such that

if $\mathbf{k} <^\Omega \mathbf{m}$ for $\mathbf{k}, \mathbf{m} \in \mathfrak{M}$ then there is an $\mathbf{h} \in \mathfrak{M}$ with $\mathbf{h} <^\Omega \mathbf{m}$ such that $(\mathbf{k} \cap \mathbf{m}) \subseteq (\mathbf{h} \cap \mathbf{m})$ and $|\mathbf{h} \cap \mathbf{m}| = |\mathbf{m}| - 1$.

We shall show that, in fact, every linear order of the set \mathfrak{M} that is compatible with the lexicographic order of the associated Jordan-Hölder sequences is a shelling.

Assign a linear order Ω to the set \mathfrak{M} in such a way that if $\pi(\mathbf{m}) < \pi(\mathbf{m}')$ in the lexicographic order on Λ^n then $\mathbf{m} <^\Omega \mathbf{m}'$. This is clearly always possible. Now, consider two maximal chains of P $\mathbf{k}: \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$ and $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$, and suppose that $\mathbf{k} <^\Omega \mathbf{m}$. Let d be the greatest integer such that $x_i = y_i$ for $i = 0, 1, \dots, d$, and let g be the least integer greater than d such that $x_g = y_g$. Then $g - d \geq 2$ and $d < i < g$ implies that $x_i \neq y_i$. The chain $x_d < x_{d+1} < \cdots < x_g$ cannot be the unique rising chain in the interval $[x_d, x_g]$, because in that case $\pi(\mathbf{m})$ would precede $\pi(\mathbf{k})$ in the lexicographic order, which contradicts $\mathbf{k} <^\Omega \mathbf{m}$. Therefore, $\pi(\mathbf{m})$ must have a descent at some e such that $d < e < g$. Consequently, by Lemma 2.6 there is a $\mathbf{h} \in \mathfrak{M}$ with $\pi(\mathbf{h}) < \pi(\mathbf{m})$, hence $\mathbf{h} <^\Omega \mathbf{m}$, such that $\mathbf{h} \cap \mathbf{m} = \mathbf{m} - \{x_e\} \supseteq \mathbf{k} \cap \mathbf{m}$. This completes the proof. \square

Let P be a graded poset of length n . Then $\rho(x) \in [n-1]$ for all $x \in \bar{P}$. For any subset $S \subseteq [n-1]$ we define the *rank-selected subposet* P_S by

$$P_S = \{x \in P \mid \rho(x) \in S \cup \{0, n\}\}.$$

Thus, P_S is also graded and its length equals $|S| + 1$. For the definition and fundamental properties of the *Möbius function* μ_Q of a finite poset Q we refer to [17]. It will be convenient to write $\mu(Q)$ instead of $\mu_Q(\hat{0}, \hat{1})$ for a finite bounded poset Q .

THEOREM 2.7 (R. STANLEY). Let P be a graded poset of length n , and suppose that an R -labeling of P is given. If $S \subseteq [n-1]$, then $(-1)^{|S|+1} \mu(P_S)$ is equal to the number of maximal chains \mathbf{m} in P with Jordan-Hölder sequences $\pi(\mathbf{m})$ having descent set $D(\pi(\mathbf{m})) = S$.

This theorem, formulated for admissible lattices, plays a fundamental role in [18] and [19]. The proof below is modelled on Stanley's proof of Theorem 1.2 in [18, p. 206].

PROOF. Let \mathfrak{N} denote the set of maximal chains of P , and similarly for $S \subseteq [n-1]$ let \mathfrak{N}_S be the set of maximal chains of P_S . The R -labeling of P determines a map $\theta: \mathfrak{N}_S \rightarrow \mathfrak{N}$ defined by "filling in the gaps with rising chains". More precisely, if $\mathbf{m} \in \mathfrak{N}_S$, $\mathbf{m}: \hat{0} < x_{i_1} < x_{i_2} < \cdots < x_{i_s} < \hat{1}$ where $s = |S|$, $\rho(x_{i_k}) = i_k$ and $i_k \in S$ for $k = 1, 2, \dots, s$, then $\theta(\mathbf{m}) = \mathbf{c}(\hat{0}, x_{i_1}) * \mathbf{c}(x_{i_1}, x_{i_2}) * \cdots * \mathbf{c}(x_{i_s}, \hat{1})$, where "*" denotes the operation of concatenating chains. We claim that θ is a bijection between \mathfrak{N}_S and the set $\{m \in \mathfrak{N} \mid D(\pi(\mathbf{m})) \subseteq S\}$. By construction, $D(\pi(\theta(\mathbf{m}))) \subseteq S$ for all $\mathbf{m} \in \mathfrak{N}_S$. Since $\mathbf{m} = \theta(\mathbf{m}) \cap P_S$, θ is injective. Since for every $\mathbf{m}' \in \{m \in \mathfrak{N} \mid D(\pi(\mathbf{m})) \subseteq S\}$ it is true that $\mathbf{m}' = \theta(\mathbf{m}' \cap P_S)$, θ is surjective onto that set.

Now, let $\delta(S)$ be the number of maximal chains $\mathbf{m} \in \mathfrak{N}$ such that $D(\pi(\mathbf{m})) = S$. We have just proved that $|\mathfrak{N}_S| = \sum_{T \subseteq S} \delta(T)$. Therefore,

$$\delta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} |\mathfrak{N}_T| = (-1)^{|S|+1} \sum_{T \subseteq S} (-1)^{|T|+1} |\mathfrak{N}_T| = (-1)^{|S|+1} \mu(P_S),$$

where the first equality is obtained by Möbius inversion [17, p. 344] and the last is provided by the well-known theorem of P. Hall's [17, Proposition 6, p. 346]. \square

Let P be a graded poset of length n and suppose that there is an L -labeling $\lambda: C(P) \rightarrow \Lambda$ of P . We know from Theorem 2.3 that P is shellable and from Theorem 2.7 that $(-1)^n \mu(P)$ equals the number of maximal chains $x_0 < x_1 < \cdots < x_n$ of P such that

$$\lambda(x_0, x_1) \nless \lambda(x_1, x_2) \nless \cdots \nless \lambda(x_{n-1}, x_n). \quad (*)$$

Judging from the above proofs these results may appear totally unrelated. There is, however, a close connection. As we shall informally indicate, this connection can be seen as a special case of a more widely applicable principle.

Let Δ be a shellable simplicial complex and let F_1, F_2, \dots, F_t be a shelling of Δ (cf. the appendix). Each facet F_j then has a unique minimal face \mathring{F}_j that is not contained in F_i for $i < j$. It is easy to see that

$$(\alpha) \ x \in \mathring{F}_j \text{ if and only if } F_j - \{x\} \subseteq F_i \text{ for some } i < j.$$

Also, using the elementary formula $\tilde{\chi}(A) + \tilde{\chi}(B) = \tilde{\chi}(A \cup B) + \tilde{\chi}(A \cap B)$, it becomes immediately clear that

(β) the number of facets F_j such that $\mathring{F}_j = F_j$ equals $(-1)^{\dim \Delta} \tilde{\chi}(\Delta)$, where $\tilde{\chi}$ denotes the Euler characteristic in reduced simplicial homology.

After these general considerations, let us return to the graded poset P with its L -labeling λ . In the proof of Theorem 2.3 we showed how to derive from λ a shelling of the order complex $\Delta(P)$. If $\hat{0}$ and $\hat{1}$ are removed from the maximal chains of P we obtain a shelling of $\Delta(\bar{P})$. For any maximal chain \mathbf{m} of \bar{P} , Lemma 2.6 tells us, in view of statement (α) above, precisely that $\hat{\mathbf{m}} = \mathbf{m} \cap P_S$, where $S = D(\pi(\hat{\mathbf{m}}))$ and $\hat{\mathbf{m}} = \mathbf{m} \cup \{\hat{0}, \hat{1}\}$. Therefore, the facets \mathbf{m} of $\Delta(\bar{P})$ such that $\hat{\mathbf{m}} = \mathbf{m}$ are those maximal chains \mathbf{m} of \bar{P} such that $D(\pi(\hat{\mathbf{m}})) = [n-1]$, or, in other words, such that $\hat{\mathbf{m}}: x_0 < x_1 < \cdots < x_n$ satisfies (*). Since $\mu(P) = \tilde{\chi}(\Delta(\bar{P}))$ by P.

Hall's theorem [17, Proposition 6, p. 346], we find that statement (β) can be rephrased, for our particular shelling, as saying that the number of maximal chains of P which satisfy $(*)$ equals $(-1)^n \mu(P)$.

The considerations of the preceding paragraph can be fully generalized to rank-selected subposets P_S of P . It will later be proved (Theorem 4.1) that any shelling of P induces shellings of all P_S , $S \subseteq [n - 1]$. Still assuming that we have an L -labeling of P , it can be shown that the conclusion of Theorem 2.7 is just a reformulation of statement (β) above for the shelling which is induced on \bar{P}_S by the lexicographic shelling of P . We omit further details.

In Examples 2.9 and 3.8 below we present some interesting applications of the following general principle.

PROPOSITION 2.8. *Let P be a graded poset and $\lambda: C(P) \rightarrow \Lambda$ an L -labeling of P . Suppose that Q is a subset of P such that*

- (i) $\hat{0}, \hat{1} \in Q$, and
- (ii) *if $x, y \in Q$ and $x < y$ in P then $c(x, y) \subseteq Q$, where $c(x, y)$ is the unique rising chain from x to y in P .*

Then Q is a graded poset under the inherited order, $C(Q) \subseteq C(P)$, and the restriction of λ to $C(Q)$ is an L -labeling of Q .

PROOF. The verification is straightforward. \square

EXAMPLE 2.9. Let Π_n denote the lattice of partitions of the set $[n]$ ordered by refinement. A covering relation $\pi < \pi'$ in Π_n corresponds to a merging of two distinct blocks B_1 and B_2 of π into one block $B_1 \cup B_2$ of π' . Let $\lambda(\pi < \pi') = \max\{\min B_1, \min B_2\}$. It is not hard to see that this edge-labeling λ of Π_n is an L -labeling. λ was first suggested to the author by I. Gessel. λ can also be obtained as the induced edge-labeling of an admissible map by the standard construction for supersolvable lattices (cf. the following section and [19, Proposition 2.4, p. 363]). The Jordan-Hölder sequences of maximal chains of Π_n under the labeling λ are permutations of the set $\{2, 3, \dots, n\}$. It is an easy exercise to verify that there are $(n - 1)!$ maximal chains with strictly decreasing Jordan-Hölder sequences. Hence, by Theorem 2.7, $\mu(\Pi_n) = (-1)^{n-1} (n - 1)!$.

Following G. Kreweras [10] we shall call a partition $\pi \in \Pi_n$ *noncrossing* if for any blocks B_1 and B_2 of π the conditions $x_1, x_3 \in B_1$, $x_2, x_4 \in B_2$ and $x_1 < x_2 < x_3 < x_4$ imply $B_1 = B_2$. It was observed jointly by P. Edelman and the author that the set \mathfrak{T}_n of all noncrossing partitions of $[n]$ satisfies condition 2.8(i) and (ii) above. Hence, λ restricts to an L -labeling of \mathfrak{T}_n , which in fact is a lattice under the inherited refinement order [10, Théorèmes 2 et 3, p. 335]. Consequently, \mathfrak{T}_n is a shellable lattice. Let us now count the number of maximal chains in \mathfrak{T}_n which have strictly decreasing Jordan-Hölder sequences.

Suppose that we select n elements x_1, x_2, \dots, x_n from a set in which a nonassociative, noncommutative binary composition is defined. A *correct bracketing* of the sequence $x_1 x_2 \dots x_n$ is an assignment of left and right bracket symbols “(“ and “)” to the sequence in such a way that the resulting expression is well-formed with respect to the binary composition. Given a correct bracketing β of the sequence

$x_1 x_2 \dots x_n$ we define a maximal chain $f(\beta): \pi_0 < \pi_1 < \dots < \pi_{n-1}$ in Π_n by the following rule:

(1) π_0 is the discrete partition with only singleton blocks,

(2) π_{i+1} is obtained from π_i by joining the block to which $n - i$ belongs with the block to which $\varphi(n - i)$ belongs, where $\varphi: ([n] - \{1\}) \rightarrow [n]$ is defined as follows:

(i) If there are no brackets or only left brackets between x_{k-1} and x_k then $\varphi(k) = k - 1$.

(ii) If there are right brackets between x_{k-1} and x_k , let \rangle be that right bracket which stands closest to x_k and let \langle be the left “mate” of \rangle (in the obvious sense). Then $\varphi(k)$ is that integer for which $x_{\varphi(k)}$ is the first element to the right of \langle .

For example, if β is the correct bracketing

$$\beta: (x_1 x_2)((x_3(x_4 x_5))x_6),$$

then $f(\beta)$ is the maximal chain $\pi_0 < \pi_1 < \dots < \pi_5$ in Π_6 given by

$$\pi_0: 1 - 2 - 3 - 4 - 5 - 6,$$

$$\pi_1: 1 - 2 - 36 - 4 - 5,$$

$$\pi_2: 1 - 2 - 36 - 45,$$

$$\pi_3: 1 - 2 - 3456,$$

$$\pi_4: 13456 - 2,$$

$$\pi_5: 123456,$$

and the Jordan-Hölder sequence of $f(\beta)$ under λ is $(6, 5, 4, 3, 2)$. It is not hard to prove for the general case that the elements of $f(\beta)$ have to be noncrossing partitions and that the Jordan-Hölder sequence of $f(\beta)$ must be strictly decreasing. In fact, one can show that f defines a bijection between the set of correct bracketings of the sequence $x_1 x_2 \dots x_n$ and the set of maximal chains of \mathfrak{T}_n with strictly decreasing Jordan-Hölder sequences. It is well known in combinatorics that the number of correct bracketings of $x_1 x_2 \dots x_n$ is equal to the Catalan number

$$C_n = \frac{1}{n} \binom{2n-2}{n-1}.$$

Hence, we find using Theorem 2.7 that

$$\mu(\mathfrak{T}_n) = (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1}.$$

This formula was first obtained by Kreweras [10, Théorème 6, p. 348].

3. Admissible lattices. Let L be a finite lattice and $\omega: I(L) \rightarrow \mathbf{P}$ a map from the set $I(L)$ of join-irreducibles of L to the set \mathbf{P} of positive integers. Such a map ω induces an edge-labeling $\gamma: C(L) \rightarrow \mathbf{P}$ of L by the rule

$$\gamma(x < y) = \min\{\omega(z) | z \in I(L), x < x \vee z = y\}.$$

If γ is an R -labeling then ω is called an *admissible map*. A finite lattice L is said to be *admissible* if it is graded and there exists an admissible map $\omega: I(L) \rightarrow \mathbf{P}$. These definitions are due to R. Stanley [19], who also showed that all upper-semimodular

and all supersolvable finite lattices are admissible. The class of admissible lattices is larger, but no other major subclass of interest seems to be known.

THEOREM 3.1. *An admissible lattice is lexicographically shellable.*

PROOF. Let $\omega: I(L) \rightarrow \mathbf{P}$ be an admissible map of a graded lattice L . To show that the induced edge-labeling γ is an L -labeling it will suffice to verify condition (iii) of Proposition 2.5 for $\lambda = \gamma$. Assume that $\mathbf{c}(x, y): x = x_0 < x_1 < \cdots < x_n = y$ is the unique maximal chain in the interval $[x, y]$ which is rising under the R -labeling γ . Let $I' = \{z \in I(L) | x < x \vee z \leq y\}$, $a = \min\{\omega(z) | z \in I'\}$ and $\omega(z') = a$, $z' \in I'$. Clearly, $a \leq \gamma(u, v)$ for all $x \leq u < v \leq y$. Let i be the least integer such that $z' \leq x_i$. Then $a \leq \gamma(x, x_1) \leq \gamma(x_{i-1}, x_i) = a$. Hence, $\gamma(x, x_1) = a \leq \gamma(u, v)$ for all $x \leq u < v \leq y$. \square

COROLLARY 3.2. *An admissible lattice is Cohen-Macaulay.* \square

Corollary 3.2 was conjectured by R. Stanley in [20, p. 60]. It was previously known that finite upper-semimodular lattices are Cohen-Macaulay and that finite distributive lattices are shellable. These results are due to J. Folkman [7] and S. Provan [12] respectively.

Our results contain the remaining pieces of information needed to fully settle the question: for which finite groups G is the lattice of subgroups $L(G)$ Cohen-Macaulay? The Cohen-Macaulay property has been studied for certain subposets of $L(G)$ by D. Quillen [14].

THEOREM 3.3. *Let G be a finite group and $L(G)$ its lattice of subgroups. Then the following conditions are equivalent:*

- (i) G is supersolvable,
- (ii) $L(G)$ is Cohen-Macaulay,
- (iii) $L(G)$ is lexicographically shellable.

PROOF. We have the following loop of implications: G is supersolvable $\overset{A}{\Rightarrow} L(G)$ is admissible $\overset{B}{\Rightarrow} L(G)$ is lexicographically shellable $\overset{C}{\Rightarrow} L(G)$ is shellable $\overset{D}{\Rightarrow} L(G)$ is Cohen-Macaulay $\overset{E}{\Rightarrow} L(G)$ is pure $\overset{F}{\Rightarrow} G$ is supersolvable. A is due to R. Stanley ([18, Example 2.5, p. 204] and [19, Proposition 2.4, p. 363]). B and C are provided by Theorems 3.1 and 2.3 respectively. D and E are well known (cf. the appendix). F was proved by K. Iwasawa in 1941 (see [22, Theorem 9, p. 9] or [2, Theorem 21, p. 177]). \square

Some of the important examples of L -shellable posets admit L -labelings with the following stronger property.

DEFINITION 3.4. An SL -labeling λ of a graded poset P is an L -labeling $\lambda: C(P) \rightarrow \Lambda$ such that if $x = x_0 < x_1 < \cdots < x_n = y$ is the unique rising maximal chain in the interval $[x, y]$ then $\lambda(x_{n-1}, y) > \lambda(z, y)$ for all $z \neq x_{n-1}$ such that $x \leq z < y$. A poset is said to be *strongly lexicographically shellable* (or *SL-shellable*) if it is graded and admits an SL -labeling.

Figure 1b above (Example 2.4) shows an SL -labeling, while Figure 1a shows an L -labeling which is not SL .

PROPOSITION 3.5. *If P is an SL -shellable poset, then its order dual P^* is also SL -shellable.*

PROOF. If $\lambda: C(P) \rightarrow \Lambda$ is an SL -labeling of P , then $\lambda^*(x <^* y) = \lambda(y < x)$ defines an SL -labeling $\lambda^*: C(P^*) \rightarrow \Lambda^*$ of P^* . \square

PROPOSITION 3.6. *Let L be a graded lattice with join-irreducibles $I(L)$. Then conditions (i) and (ii) below are equivalent and both imply (iii):*

- (i) *there is an injective admissible map $\omega: I(L) \rightarrow \mathbf{P}$,*
- (ii) *there is an admissible map $\omega: I(L) \rightarrow \mathbf{P}$ such that under the induced edge-labeling γ of L all rising chains $x_0 < x_1 < \cdots < x_n$ have strictly increasing Jordan-Hölder sequences $\gamma(x_0, x_1) < \gamma(x_1, x_2) < \cdots < \gamma(x_{n-1}, x_n)$.*
- (iii) *L is SL -shellable.*

PROOF. (i) implies (ii): The same map ω will do.

(ii) implies (i): The map $\omega: I(L) \rightarrow \mathbf{P}$ induces a linearly ordered partition of $I(L)$ with blocks $\omega^{-1}(n)$, $n \in \text{range } \omega$. Let $\omega': I(L) \rightarrow \mathbf{P}$ be an injective map such that $\omega'(t_1) < \omega'(t_2)$ if $\omega(t_1) < \omega(t_2)$, $t_1, t_2 \in I(L)$. Such a map ω' clearly exists. If $x < z < y$ in L then, by assumption either $\gamma(x, z) < \gamma(z, y)$ or $\gamma(x, z) > \gamma(z, y)$. It is easy to check that $\gamma(x, z) < \gamma(z, y)$ if and only if $\gamma'(x, z) < \gamma'(z, y)$, where γ' is the edge-labeling of L which ω' induces. Since γ is an R -labeling, we must conclude that γ' is also an R -labeling. Hence, ω' is admissible.

(ii) implies (iii): It is already known that the induced edge-labeling γ of L is an L -labeling (Theorem 3.1), and we shall verify that under condition (ii) it is an SL -labeling by induction on the length of intervals of L . If $[x, y]$ is an interval of length one there is nothing to prove. Assume that γ is an SL -labeling of all intervals of length $\leq n-1$, $n \geq 2$, and let $[x, y]$ be an interval of length n with unique rising chain $x = x_0 < x_1 < \cdots < x_n = y$. Suppose that $z \neq x_{n-1}$ and $x \leq z < y$. Also, let $t \in I(L)$ such that $x < t \vee x = x_1$ and $\gamma(x, x_1) = \omega(t)$. If $t \leq z$, then $x_1 = t \vee x \leq z$, and since $[x_1, y]$ is an interval of length $n-1$ and $x_1 < x_2 < \cdots < x_n = y$ must be its unique rising chain, the induction assumption forces $\gamma(z, y) < \gamma(x_{n-1}, y)$. If $t \not\leq z$, then $\gamma(z, y) \leq \omega(t)$ since $t \leq y$. Hence, $\gamma(z, y) \leq \omega(t) = \gamma(x, x_1) < \gamma(x_{n-1}, y)$, where the final inequality is forced by condition (ii). \square

The implication (ii) \Rightarrow (iii) of the preceding proposition can be made more precise: Let L be a graded lattice, $\omega: I(L) \rightarrow \mathbf{P}$ an admissible map and γ the induced edge-labeling of L . Then γ is an SL -labeling if and only if whenever $x_0 < x_1 < \cdots < x_n$ and $\gamma(x_0, x_1) = \gamma(x_1, x_2) = \cdots = \gamma(x_{n-1}, x_n)$ then $[x_0, x_n]$ is a chain. We omit the proof.

The induced edge-labeling of an admissible map does not have to be an SL -labeling as the following counterexample shows (Figure 2).

We do not know whether *all* admissible lattices are SL -shellable, but we can show that the important examples are.

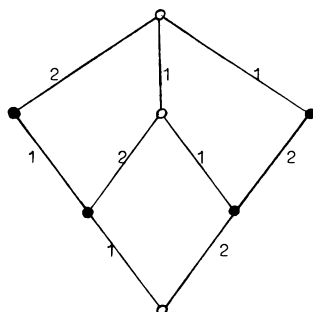


FIGURE 2

THEOREM 3.7. *Finite upper-semimodular, lower-semimodular and supersolvable lattices are SL -shellable.*

PROOF. R. Stanley has shown that for every finite supersolvable lattice L there is an admissible map ω which satisfies condition (ii) of Proposition 3.6 [19, Proposition 2.4, p. 363]. He has also proved that any injective order-preserving map $\omega: I(L) \rightarrow \mathbf{P}$ is admissible when L is finite and upper-semimodular [19, Proposition 2.2, p. 362]. Hence, Propositions 3.5 and 3.6 imply the present result. \square

REMARK. Extending Stanley's above-mentioned result I. Rival has shown [16, Theorem 2, p. 102] that a finite lattice L is upper-semimodular if and only if every injective order-preserving map $\omega: I(L) \rightarrow \mathbf{P}$ is admissible. As the following example of a modular lattice shows (Figure 3), an admissible map does not have to be order-preserving. Hence, Proposition 2.1 in [19, p. 362] is false.

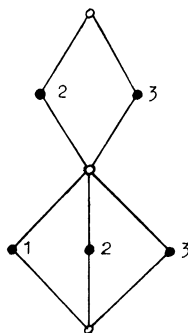


FIGURE 3

EXAMPLE 3.8. Let G be a finite geometric lattice. K. Baclawski has observed [23, Corollary 4.3] that $G - \{x\}$ is a shellable poset for every $x \in G$. We will show that in fact all chains and certain antichains can be removed from G without losing shellability. The case of chains has independently been considered and further generalized by Baclawski [24].

Suppose that $\mathbf{c}: x_1 < x_2 < \cdots < x_k$ is a chain in \bar{G} . Let $I(G)$ be the set of join-irreducibles (atoms) of G , and let $\omega: I(G) \rightarrow [n]$ be a bijection such that $p \leq x_i < q \vee x_i$ implies $\omega(q) < \omega(p)$ for all $p, q \in I(G)$ and $x_i \in \mathbf{c}$. Since G is upper-semimodular we know that the map ω is admissible and hence induces an

L -labeling γ of G in the usual way. It is easy to verify that if $y < x_i < z$ in G with $x_i \in \mathbf{c}$ then $\gamma(y, x_i) > \gamma(x_i, z)$. Hence, by Proposition 2.8 the subposet $G - \mathbf{c}$ is L -shellable under the restriction of γ . Now, a maximal chain of G with strictly decreasing Jordan-Hölder sequence either is contained in $G - \mathbf{c}$ or includes an element of \mathbf{c} . Counting the chains of the latter type by the principle of inclusion-exclusion and using Theorem 2.7 we are led to the following formula:

$$\mu(G - \mathbf{c}) = \mu(G) + \sum_{e=1}^k (-1)^e \sum_{1 < i_1 < i_2 < \dots < i_e < k} \mu(\hat{0}, x_{i_1}) \mu(x_{i_1}, x_{i_2}) \dots \mu(x_{i_{e-1}}, x_{i_e}) \mu(x_{i_e}, \hat{1}). \quad (3.9)$$

Also certain antichains can be deleted from geometric lattices without losing shellability. Apart from the case of full rank levels (cf. Theorem 4.1 below) we have the following. Let G be a finite geometric lattice, $S = \{x_1, x_2, \dots, x_s\}$ a subset of \bar{G} , and let A be the set of atoms under S , that is, $A = \{p \in I(G) \mid p \leq x \text{ for some } x \in S\}$. Assume that S has the property that if $x < z$ in G and $x \in S$ then $a \leq z$ for some atom $a \in I(G) - A$. In particular, S must be an antichain. Now, choose a bijection $\omega: I(G) \rightarrow [n]$ such that $\omega(I(G) - A) = [n - |A|]$. As before, it is easy to check that the induced L -labeling γ of G has the property that $\gamma(y, x) > \gamma(x, z)$ when $y < x < z$ in G and $x \in S$. So again by Proposition 2.8 we find that the subposet $G - S$ is L -shellable. Counting maximal chains with strictly decreasing Jordan-Hölder sequences in G and $G - S$ we obtain the formula:

$$\mu(G - S) = \mu(G) - \sum_{i=1}^s \mu(\hat{0}, x_i) \mu(x_i, \hat{1}). \quad (3.10)$$

Examples of antichains S which satisfy the above requirement include the set of maximal complements of a fixed element in G . In particular, if S is the set of all complements of a modular element in G , then (3.10) in combination with H. Crapo's complementation theorem shows that $\mu(G - S) = 0$, so $G - S$ is acyclic.

4. Constructing shellable posets. This section is devoted to showing a number of ways in which shellable posets can be constructed from other shellable or L -shellable posets.

Let P be a finite pure poset of length $r - 1$. Then \hat{P} is graded with rank function ρ , and $\rho(x) \in [r]$ for all $x \in P$. For any subset $S \subseteq [r]$ we define $P_S = \{x \in P \mid \rho(x) \in S\}$. This definition is slightly more general than the definition of rank-selected subposets given in §2, where only bounded posets were considered.

THEOREM 4.1. *If P is a shellable poset of length $r - 1$, then P_S is shellable for all $S \subseteq [r]$.*

PROOF. Fix a subset $S \subseteq [r]$ and let \mathfrak{N} and \mathfrak{N}_S denote the sets of maximal chains of P and P_S respectively. Assume that a certain linear order Ω of the set \mathfrak{N} is a shelling. For every $\mathbf{c} \in \mathfrak{N}_S$ define $\theta(\mathbf{c})$ to be the least element under Ω of the set $\{\mathbf{m} \in \mathfrak{N} \mid \mathbf{c} \subseteq \mathbf{m}\}$. This determines an injective map $\theta: \mathfrak{N}_S \rightarrow \mathfrak{N}$, and we derive a linear order Ω' of the set \mathfrak{N}_S by the rule: $\mathbf{c} <^{\Omega'} \mathbf{d}$ if and only if

$\theta(\mathbf{c}) <^{\Omega} \theta(\mathbf{d})$, $\mathbf{c}, \mathbf{d} \in \mathfrak{N}_S$. Now, suppose that $\mathbf{c} <^{\Omega'} \mathbf{d}$ for $\mathbf{c}, \mathbf{d} \in \mathfrak{N}_S$. Since $\theta(\mathbf{c}) <^{\Omega} \theta(\mathbf{d})$ and Ω is a shelling, we know that there exists a maximal chain $\mathbf{m} \in \mathfrak{N}$ such that $\mathbf{m} <^{\Omega} \theta(\mathbf{d})$ and

$$\theta(\mathbf{c}) \cap \theta(\mathbf{d}) \subseteq \mathbf{m} \cap \theta(\mathbf{d}) = \theta(\mathbf{d}) - \{x\} \quad \text{for some } x \in \theta(\mathbf{d}). \quad (*)$$

In case $x \notin \mathbf{d}$, then $\mathbf{d} \subseteq \theta(\mathbf{d}) - \{x\} \subseteq \mathbf{m}$, which contradicts the definition of $\theta(\mathbf{d})$ since $\mathbf{m} <^{\Omega} \theta(\mathbf{d})$. Hence $x \in \mathbf{d}$. Let $\mathbf{b} = \mathbf{m} \cap P_S$. Then $\mathbf{b} \in \mathfrak{N}_S$ and $\theta(\mathbf{b}) \leq^{\Omega} \mathbf{m} < \theta(\mathbf{d})$ so $\mathbf{b} <^{\Omega'} \mathbf{d}$. Applying the operator $P_S \cap \cdot$ to the formula $(*)$ we get $\mathbf{c} \cap \mathbf{d} \subseteq \mathbf{b} \cap \mathbf{d} = \mathbf{d} - \{x\}$. Hence, Ω' is a shelling of the set \mathfrak{N}_S . \square

PROPOSITION 4.2. *If P is an $(L-)$ shellable poset then all intervals of P are $(L-)$ shellable.*

PROOF. Suppose that P is shellable and that $[x, y]$ is an interval of P . Let \mathbf{c} : $x_1 < x_2 < \cdots < x_g = x$ and \mathbf{d} : $y = y_1 < y_2 < \cdots < y_h$ be two unrefinable chains in P such that x_1 is a minimal and y_h is a maximal element. Let $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_l$ be the maximal chains in P which contain $\mathbf{c} \cup \mathbf{d}$, and assume that they are listed in the order in which they appear in the shelling of P . It is straightforward to verify that $((\mathbf{m}_i - (\mathbf{c} \cup \mathbf{d})) \cup \{x, y\})'_{i=1}$ is a shelling of $[x, y]$.

In case P is L -shellable the conclusion is immediate from Definition 2.1. \square

Let P and Q be two posets. The *direct product* $P \times Q$ is the poset defined on the product set by $(x, y) \leq (x', y')$ if and only if $x \leq x'$ in P and $y \leq y'$ in Q . The *ordinal sum* $P \oplus Q$ is the poset on the disjoint union of P and Q defined by the rule: $x \leq y$ in $P \oplus Q$ if and only if (i) $x, y \in P$ and $x \leq y$ in P , or (ii) $x, y \in Q$ and $x \leq y$ in Q , or (iii) $x \in P$ and $y \in Q$.

THEOREM 4.3. *$P \times Q$ is L -shellable if and only if both P and Q are L -shellable.*

PROOF. $P \times Q$ is clearly a graded poset if and only if both P and Q are. Let λ : $C(P) \rightarrow \Lambda$ and λ' : $C(Q) \rightarrow \Lambda'$ be L -labelings of the graded posets P and Q . A covering relation $(x, y) < (x', y')$ occurs in $P \times Q$ if and only if $x = x'$ and $y < y'$ or $x < x'$ and $y = y'$. Define an edge-labeling λ'' : $C(P \times Q) \rightarrow \Lambda \oplus \Lambda'$ by $\lambda''((x, y) < (x', y')) = \lambda(x < x')$ if $y = y'$ and $\lambda''((x, y) < (x', y')) = \lambda'(y < y')$ if $x = x'$. It is straightforward to verify that λ'' is an L -labeling of $P \times Q$.

If $P \times Q$ is L -shellable then, by Proposition 4.2, so are $P \simeq [(\hat{0}, \hat{0}), (\hat{1}, \hat{0})]$ and $Q \simeq [(\hat{0}, \hat{0}), (\hat{0}, \hat{1})]$. \square

THEOREM 4.4. *Let P and Q be two posets such that \hat{P} and \hat{Q} are L -shellable. Then \hat{X} is L -shellable, where $X = P \oplus Q$. Also, $P \oplus Q$ is shellable if and only if both P and Q are.*

PROOF. Suppose that λ : $C(\hat{P}) \rightarrow \Lambda$ and λ' : $C(\hat{Q}) \rightarrow \Lambda'$ are L -labelings of the graded posets \hat{P} and \hat{Q} . Define an edge-labeling λ'' : $C(\hat{X}) \rightarrow (\Lambda \oplus \Lambda') \times (\Lambda \oplus \Lambda')$ by (i) $\lambda''(x < y) = (\lambda(x < y), \lambda(x < y))$ if $x < y$ and $y \neq \hat{1}$ in \hat{P} , (ii) $\lambda''(x < y) = (\lambda'(x < y), \lambda'(x < y))$ if $x < y$ and $x \neq \hat{0}$ in \hat{Q} , and (iii) $\lambda''(x < y) = (\lambda(x < \hat{1}), \lambda'(\hat{0} < y))$ if $x < \hat{1}$ in \hat{P} and $\hat{0} < y$ in \hat{Q} . λ'' is an L -labeling of \hat{X} . We omit the easy verification.

Suppose that $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_s$ is a shelling of P and that $\mathbf{m}'_1, \mathbf{m}'_2, \dots, \mathbf{m}'_t$ is a shelling of Q . Then $\{\mathbf{c}_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq t}$, where $\mathbf{c}_{ij} = \mathbf{m}_i \cup \mathbf{m}'_j$, is the set of maximal chains of $P \oplus Q$, and it is easy to check that the lexicographic order of the indices determines a shelling of $P \oplus Q$. If on the other hand $P \oplus Q$ is shellable then so are both P and Q by Theorem 4.1. \square

The *cardinal power* Q^P of two posets P and Q is the set of order-preserving maps $f: P \rightarrow Q$ partially ordered by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$.

THEOREM 4.5. *Let P and Q be two finite posets. Then Q^P is L -shellable if and only if Q is L -shellable.*

PROOF. First note that Q^P is bounded if and only if Q is bounded. Also, if Q is bounded then $Q \simeq [f, g]$ in Q^P , where $f(x) = \hat{0}$ for all $x \in P$ and $g(x) = \hat{1}$ for all $x \in P$ with the exception of a maximal element $m \in P$ for which $g(m) = \hat{1}$. Hence, by Proposition 4.2 the “only if” part is done.

Let $\lambda: C(Q) \rightarrow \Lambda$ be an L -labeling of the graded poset Q . It is easy to see that $f < g$ in Q^P if and only if $f(x) = g(x)$ for all $x \in P$ with one exception $p \in P$ for which $f(p) < g(p)$. Hence, in particular, Q^P is also graded. Now, let $\sigma: P \rightarrow \mathbf{Z}$ be an order-reversing injective map from P to the integers. Define an edge-labeling $\lambda': C(Q^P) \rightarrow \mathbf{Z} \otimes \Lambda$ by $\lambda'(f < g) = (\sigma(p), \lambda(f(p) < g(p)))$ where p is the unique element of P such that $f(p) < g(p)$. Here $\mathbf{Z} \otimes \Lambda$ denotes the *ordinal product*, which is defined by the lexicographic order on the product set: $(n, \lambda) \leq (n', \lambda')$ if and only if $n < n'$ or $n = n'$ and $\lambda \leq \lambda'$. Again, we leave the verification that λ' is an L -labeling to the reader. \square

The preceding result reconfirms that finite distributive lattices are L -shellable, since by a theorem of G. Birkhoff [2, Theorem 3, p. 59] $L \simeq 2^{I(L)}$, where L is a finite distributive lattice and $I(L)$ its subposet of join-irreducibles.

The *interval poset* $\text{Int}(P)$ of a poset P is the set of all intervals of P ordered by containment. Thus $[x, y] \leq [u, v]$ in $\text{Int}(P)$ if and only if $u \leq x \leq y \leq v$ in P . By convention we also adjoin the empty interval \emptyset to $\text{Int}(P)$, so that $\text{Int}(P)$ is bounded if and only if P is bounded.

THEOREM 4.6. *$\text{Int}(P)$ is L -shellable if and only if both P and its order dual P^* are L -shellable.*

PROOF. Suppose that P is a graded poset and that $\lambda: C(P) \rightarrow \Lambda$ and $\lambda': C(P^*) \rightarrow \Lambda'$ are L -labelings. Define an edge-labeling $\lambda'': C(\text{Int}(P)) \rightarrow \Lambda' \oplus \mathbf{Z} \oplus \Lambda$ by (i) $\lambda''([x, y] < [u, v]) = \lambda(y < v)$ if $x = u$, (ii) $\lambda''([x, y] < [u, v]) = \lambda'(x <^* u)$ if $y = v$, and (iii) $\lambda''(\emptyset < [x, x]) = \rho(x)$, where ρ denotes the rank function in P . It is not hard to verify that λ'' is an L -labeling of $\text{Int}(P)$.

If P is a bounded poset then clearly $P \simeq [[\hat{0}, \hat{0}], [\hat{0}, \hat{1}]]$ and $P^* \simeq [[\hat{1}, \hat{1}], [\hat{0}, \hat{1}]]$ in $\text{Int}(P)$. Hence, if $\text{Int}(P)$ is L -shellable then by Proposition 4.2 so are P and P^* . \square

The preceding Theorems 4.2–4.6 remain true if “ L -shellable” is everywhere replaced by “ SL -shellable” (cf. Definition 3.4). In particular $\text{Int}(P)$ is SL -shellable if and only if P is SL -shellable.

5. Face-lattices of complexes. If we order the faces of a simplicial or polyhedral complex K by inclusion, we get a poset P_K . Adjoining a greatest element $\hat{1}$ to P_K we obtain a lattice L_K , the *face-lattice* of K . The order complex $\Delta(\bar{L}_K)$ is known in topology as the *barycentric subdivision* $\text{sd } K$ of K . We shall in this section discuss some cases for which the shellability of K is inherited by P_K (or, equivalently, L_K or \bar{L}_K):

THEOREM 5.1. *Let K be a shellable simplicial complex. Then the barycentric subdivision $\text{sd } K$ is shellable.*

PROOF. Suppose that f_1, f_2, \dots, f_t are the facets of K arranged in a shelling order. The poset P_K of faces of K is a pure poset of length $= n = \dim K + 1$, with maximal elements f_1, f_2, \dots, f_t and least element $\hat{0} = \emptyset$. Since $\text{sd } K = \Delta(P_K - \{\hat{0}\})$ the theorem will follow if we construct a shelling order Ω of the set \mathfrak{N} of maximal chains of P_K .

Let \mathfrak{N}_i be the set of all maximal chains of P_K which contain f_i , so that $\mathfrak{N} = \bigcup_{i=1}^t \mathfrak{N}_i$. For every $i, i = 1, 2, \dots, t$, $[\hat{0}, f_i]$ is a finite Boolean algebra on n atoms. Suppose that the elements which are covered by f_i are labeled $c_{i1}, c_{i2}, \dots, c_{in}$ in such a way that for $j = 1, 2, \dots, k_i$ the element c_{ij} is also covered by f_e for some $e < i$, but for $j = k_i + 1, k_i + 2, \dots, n$ this is not the case. Since K is shellable $k_i \geq 1$ when $i > 1$. The map $\omega: c_{ij} \mapsto j$ is an admissible map from the join-irreducibles of the dual lattice $[\hat{0}, f_i]^*$, and therefore it induces an L -labeling, which in turn lexicographically generates a shelling order of \mathfrak{N}_i (cf. Theorems 2.3 and 3.1). Let the elements of \mathfrak{N}_i be called $\mathbf{m}_{i1}, \mathbf{m}_{i2}, \dots, \mathbf{m}_{in_i}$, where the second index is compatible with the shelling order we have obtained. The important feature of this order, except that it shells \mathfrak{N}_i , is that whenever $c_{ie_1} \in \mathbf{m}_{ij_1}$, $c_{ie_2} \in \mathbf{m}_{ij_2}$ and $e_1 < k_i < e_2$, then $j_1 < j_2$. Suppose the above process has been carried out for each $i, 1 \leq i \leq t$, so that $\mathfrak{N} = \{\mathbf{m}_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq n_i}$. The lexicographic order of the indices defines a linear order Ω on \mathfrak{N} .

To verify that Ω is a shelling, assume given $\mathbf{m}_{i_0j_0} \in \mathfrak{N}$. If $\mathbf{m}_{i_1j_1} <^\Omega \mathbf{m}_{i_0j_0}$ then either (a) $i_1 = i_0$ and $j_1 < j_0$ or (b) $i_1 < i_0$. In case (a), $\mathbf{m}_{i_1j_1}$ and $\mathbf{m}_{i_0j_0}$ both belong to \mathfrak{N}_{i_0} and since the second index determines a shelling of \mathfrak{N}_{i_0} we may conclude the existence of a $\mathbf{m}_{i_0j_2} \in \mathfrak{N}_{i_0}$ with $j_2 < j_0$ and $\mathbf{m}_{i_1j_1} \cap \mathbf{m}_{i_0j_0} \subseteq \mathbf{m}_{i_0j_2} \cap \mathbf{m}_{i_0j_0} = \mathbf{m}_{i_0j_0} - \{x\}$ for some $x \in \mathbf{m}_{i_0j_0}$. Case (b) must be further subdivided into two subcases as follows. Suppose that $\mathbf{m}_{i_0j_0}: \hat{0} = x_0 < x_1 < \dots < x_n = f_{i_0}$ and that $x_{n-1} = c_{i_0e}$. In case $e \leq k_{i_0}$ then $x_{n-1} < f_{i_2}$ for some $i_2 < i_0$, so $\mathbf{m}_{i_2j_2}: x_0 < x_1 < \dots < x_{n-1} < f_{i_2}$, which precedes $\mathbf{m}_{i_0j_0}$ under Ω , satisfies $\mathbf{m}_{i_1j_1} \cap \mathbf{m}_{i_0j_0} \subseteq \mathbf{m}_{i_2j_2} \cap \mathbf{m}_{i_0j_0} = \mathbf{m}_{i_0j_0} - \{f_{i_0}\}$. Assume next that $e > k_{i_0}$. Let $g = \max\{h | x_h \in \mathbf{m}_{i_1j_1} \cap \mathbf{m}_{i_0j_0}\}$. Then $0 < g \leq n - 2$. Since f_1, f_2, \dots, f_t is a shelling of K we know that $x_g < c_{i_0d} < f_{i_0}$ for some $d \leq k_{i_0}$. Let $x_g = y_0 < y_1 < \dots < y_{n-g-1} = c_{i_0d}$ be an arbitrary unrefinable chain from x_g to c_{i_0d} . Then $\mathbf{m}_{i_0j_2}: x_0 < x_1 < \dots < x_g = y_0 < y_1 < \dots < y_{n-g-1} = c_{i_0d} < f_{i_0}$ precedes $\mathbf{m}_{i_0j_0}$ in the shelling order of \mathfrak{N}_{i_0} . Hence, there exists a $\mathbf{m}_{i_0j_3} \in \mathfrak{N}_{i_0}$ with $j_3 < j_0$ such that $\mathbf{m}_{i_0j_2} \cap \mathbf{m}_{i_0j_0} \subseteq \mathbf{m}_{i_0j_3} \cap \mathbf{m}_{i_0j_0} = \mathbf{m}_{i_0j_0} - \{x\}$ for some $x \in \mathbf{m}_{i_0j_0}$. Since $\mathbf{m}_{i_1j_1} \cap \mathbf{m}_{i_0j_0} \subseteq \{x_0, x_1, \dots, x_g\} \subseteq \mathbf{m}_{i_0j_2} \cap \mathbf{m}_{i_0j_0}$, this concludes the proof. \square

of P are semimodular. As was apparently first noticed by O. Ore, bounded semimodular finite posets are pure. However, such posets need not be Cohen-Macaulay. It was shown independently by K. Baclawski [1] and F. Farmer [6] that bounded locally semimodular finite posets are Cohen-Macaulay over \mathbf{Z} . Notice that a lattice is locally semimodular as a poset if and only if it is upper-semimodular in the usual sense.

THEOREM 6.1. *Suppose that a finite poset P is bounded and locally semimodular. Then P is shellable.*

PROOF. Let us say that a linear order Ω of the set of maximal chains \mathfrak{N} of a graded poset Q has *property Ψ* if whenever $\mathbf{m}, \mathbf{m}' \in \mathfrak{N}$, $\mathbf{m}': \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$, $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$, $x_i = y_i$ for $i = 0, 1, \dots, e$ and $x_{e+1} \neq y_{e+1}$, then

(α) if $\{y_0, y_1, \dots, y_{e+1}\} \subseteq \mathbf{m}'' \in \mathfrak{N}$ and $\mathbf{m}' <^\Omega \mathbf{m}$ then $\mathbf{m}'' <^\Omega \mathbf{m}$, and

(β) if $\mathbf{m}' - \{y_e\} \subseteq \mathbf{m}''$ for some $\mathbf{m}'' \in \mathfrak{N}$ such that $\mathbf{m}'' <^\Omega \mathbf{m}$ but $\mathbf{m} - \{x_e\} \not\subseteq \mathbf{m}'''$ for all $\mathbf{m}''' \in \mathfrak{N}$ such that $\mathbf{m}''' <^\Omega \mathbf{m}$ then $\mathbf{m}' <^\Omega \mathbf{m}$.

We shall prove for every graded poset that the set of maximal chains can be given a linear order having property Ψ . Since for posets of length 2 every linear order trivially possesses property Ψ , we may proceed by induction on length.

Suppose that Q is a graded poset of length n , $n \geq 3$, and let Q' denote the rank-selected subposet $Q_{[n-2]} = \{x \in Q \mid \rho(x) \neq n-1\}$. Q' is graded and of length $n-1$, so by the induction assumption there is a linear order Ω' with property Ψ of the set \mathfrak{N}' of maximal chains of Q' . Label the elements of \mathfrak{N}' , $\mathbf{m}'_1, \mathbf{m}'_2, \dots, \mathbf{m}'_s$, in such a way that the natural order of the indices expresses the order Ω' . For $\mathbf{m}'_i \in \mathfrak{N}'$, $\mathbf{m}'_i: \hat{0} = x_0 < x_1 < \cdots < x_{n-2} < \hat{1}$, let $A_i = \{z \in Q \mid x_{n-2} < z < \hat{1}\}$, $B_i = \{z \in A_i \mid \text{there is a } y \in Q \text{ such that } x_{n-3} < y < z \text{ and } (\mathbf{m}'_i - \{x_{n-2}\}) \cup \{y\} <^{\Omega'} \mathbf{m}'_i\}$, and $C_i = A_i - B_i$. Label the elements of A_i , $z_{i1}, z_{i2}, \dots, z_{ia_i}$, $a_i = |A_i|$, in such a way that if $z_{ij} \in B_i$ and $z_{ik} \in C_i$ then $j < k$. Let $\mathbf{m}_{ij} = \mathbf{m}'_i \cup \{z_{ij}\}$, for $i = 1, 2, \dots, s$ and $j = 1, 2, \dots, a_i$. Then the lexicographic order of the indices determines a linear order Ω of the set $\mathfrak{N} = \{\mathbf{m}_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq a_i\}$ of maximal chains of Q . We claim that Ω has property Ψ .

To validate this claim, let us suppose that $\mathbf{m}, \mathbf{m}' \in \mathfrak{N}$, $\mathbf{m}': \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$, $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$, $x_i = y_i$ for $i = 0, 1, \dots, e$ and $x_{e+1} \neq y_{e+1}$. Then $0 \leq e \leq n-2$. First consider the case $e = n-2$. Then $\mathbf{m}' = \mathbf{m}_{ij}$ and $\mathbf{m} = \mathbf{m}_{ik}$ for some i, j, k , $1 \leq i \leq s$, $1 \leq j, k \leq a_i$. Condition (α) above is trivially satisfied since \mathbf{m}'' must equal \mathbf{m}' . The hypothesis of condition (β) is equivalent to $y_{n-1} \in B_i$ and $x_{n-1} \in C_i$. By construction therefore $j < k$, that is, $\mathbf{m}' <^\Omega \mathbf{m}$. Next, consider the case $0 \leq e \leq n-3$. Then $\mathbf{m}' <^\Omega \mathbf{m}$ if and only if $(\mathbf{m}' - \{y_{n-1}\}) <^{\Omega'} (\mathbf{m} - \{x_{n-1}\})$. If $\{y_0, y_1, \dots, y_{e+1}\} \subseteq \mathbf{m}'' \in \mathfrak{N}$, then $\{y_0, y_1, \dots, y_{e+1}\} \subseteq (\mathbf{m}'' - \{z\}) \in \mathfrak{N}'$ where $z \in \mathbf{m}''$ has rank $\rho(z) = n-1$. Hence, since the order Ω' satisfies condition (α), so does Ω . In a similar manner one can verify that property (β) is passed along from Ω' to Ω . Consequently, the order Ω has property Ψ , and the induction proof is complete.

Now, suppose that P is a finite poset which is bounded and locally semimodular.

Then P is pure [2, Theorem 14, p. 40], and hence graded. As we have shown, it is possible to assign a linear order Ω having property Ψ to the set \mathfrak{N} of maximal chains of P . It remains to be shown that such an order Ω is a shelling. Let $\mathbf{m}, \mathbf{m}' \in \mathfrak{N}$, $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$, $\mathbf{m}': \hat{0} = y_0 < y_1 < \cdots < y_n = \hat{1}$, and suppose that $\mathbf{m}' <^\Omega \mathbf{m}$. Let d be the greatest integer such that $x_i = y_i$ for $i = 0, 1, \dots, d$, and let g be the least integer for which $y_{d+1} < x_g$. Since P is locally semimodular, there is an element z_{d+2} which covers both y_{d+1} and x_{d+1} in the interval $[x_d, x_g]$. Again, if $g > d + 2$ there is an element z_{d+3} which covers z_{d+2} and x_{d+2} in $[x_d, x_g]$; and so on until the process ends with $z_g = x_g$ (see Figure 5). Also, let $z_{d+1} = y_{d+1}$.

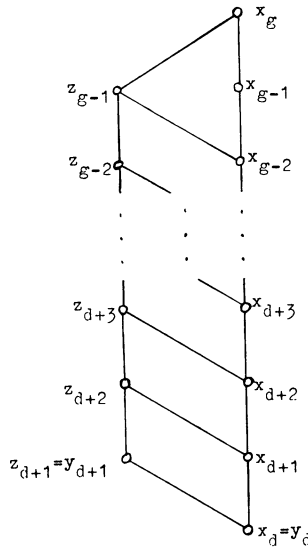


FIGURE 5

Note that, by choice of g , $d + 1 \leq e \leq g - 1$ implies $y_e \neq x_e$ and $z_e \neq x_e$. For $i = d + 1, d + 2, \dots, g - 1$ define \mathbf{m}_i by $\mathbf{m}_i: \hat{0} = x_0 < x_1 < \cdots < x_{i-1} < z_i < z_{i+1} < \cdots < z_{g-1} < x_g < x_{g+1} < \cdots < x_n = \hat{1}$. We have assumed that $\mathbf{m}' <^\Omega \mathbf{m}$. Hence, using property (α) above we find that $\mathbf{m}_{d+1} <^\Omega \mathbf{m}$. If $\mathbf{m} - \{x_{d+1}\} \not\subseteq \mathbf{m}''$ for all $\mathbf{m}'' \in \mathfrak{N}$ such that $\mathbf{m}'' <^\Omega \mathbf{m}$, then property (β) tells us that $\mathbf{m}_{d+2} <^\Omega \mathbf{m}$. In that case, if $\mathbf{m} - \{x_{d+2}\}$ is not included in any maximal chain which precedes \mathbf{m} , then $\mathbf{m}_{d+3} <^\Omega \mathbf{m}$. Continuing this argument based on property (β) we find that either $\mathbf{m} - \{x_e\}$ is contained in an earlier maximal chain for some e , $d + 1 \leq e \leq g - 2$, or $\mathbf{m}_{g-1} <^\Omega \mathbf{m}$. But in the latter case $\mathbf{m} - \{x_{g-1}\}$ is contained in an earlier maximal chain, namely \mathbf{m}_{g-1} . Hence, there is a maximal chain $\mathbf{m}'' \in \mathfrak{N}$ and an e , $d + 1 \leq e \leq g - 1$, such that $\mathbf{m}'' <^\Omega \mathbf{m}$ and $\mathbf{m}' \cap \mathbf{m} \subseteq \mathbf{m}'' \cap \mathbf{m} = \mathbf{m} - \{x_e\}$. The proof is now complete. \square

EXAMPLE 6.2. Let Π_n denote the poset of partitions of the integer n ordered by refinement (see [2, Example 10, p. 16], Π_6 is depicted in Figure 1b above). These posets Π_n as well as their order duals Π_n^* are bounded and semimodular. For $n < 7$

they are even locally semimodular. This is not true for $n \geq 8$. For instance, the intervals $[(3, 2, 1, 1, 1), (5, 3)]$ in Π_8 and $[(6, 2), (2, 2, 2, 1, 1)]$ in Π_8^* are not semimodular. Nevertheless, Π_8 is shellable. For $n \geq 9$ it seems to be unknown even if Π_n is Cohen-Macaulay. \square

With any pure simplicial complex Δ is associated the graph Φ_Δ defined on the set of all facets of Δ by letting two facets be adjacent if their intersection is a maximal proper face of each. Let us call Δ *facet-connected* if Φ_Δ is a connected graph. All Cohen-Macaulay complexes are facet-connected. Following Provan [12, p. 25] we shall say that a facet-connected δ -dimensional simplicial complex Δ with ν vertices satisfies the *Hirsch conjecture* if

$$\text{diam } \Phi_\Delta \leq \nu - \delta - 1, \quad (6.3)$$

where by $\text{diam } \Phi_\Delta$ we understand the diameter, that is, the maximal distance between two vertices of Φ_Δ in the usual graph-theoretic sense. This notion, which stems from polyhedral theory, can be applied to posets by tacitly referring to order complexes. This was also done by Provan, who showed that all finite distributive lattices satisfy the Hirsch conjecture [12, p. 62]; a result which we shall now extend.

THEOREM 6.4. *Let P be a bounded, locally semimodular finite poset. Then P satisfies the Hirsch conjecture.*

PROOF. The inequality (6.3) is trivially satisfied if P has length equal to one. We continue by induction on the length of P , which of course equals the dimension of the order complex $\Delta(P)$. Assume that $\text{length}(P) = \delta \geq 2$ and let $\mathbf{m}: \hat{0} = x_0 < x_1 < \cdots < x_\delta = \hat{1}$ and $\mathbf{m}': \hat{0} = y_0 < y_1 < \cdots < y_\delta = \hat{1}$ be two distinct maximal chains in P . Also, let d be the greatest integer such that $x_i = y_i$ for $i = 0, 1, \dots, d$, and let g be the least integer for which $y_{d+1} < x_g$. Using local semimodularity, select elements $z_{d+2}, z_{d+3}, \dots, z_{g-1}$ in P as illustrated in Figure 5 above. The interval $[y_{d+1}, \hat{1}]$ in P is locally semimodular and of length $\delta - d - 1 < \delta$. By the induction assumption, therefore, there is a path in $\Phi_{\Delta([y_{d+1}, \hat{1}])}$ of length at most $\nu' - (\delta - d - 1) - 1$, $\nu' = |[y_{d+1}, \hat{1}]|$, consisting of maximal chains of $[y_{d+1}, \hat{1}]$, which connects $y_{d+1} < y_{d+2} < \cdots < y_\delta = \hat{1}$ to $y_{d+1} < z_{d+2} < z_{d+3} < \cdots < z_{g-1} < x_g < x_{g+1} < \cdots < x_\delta = \hat{1}$. If these chains are concatenated with $\hat{0} = y_0 < y_1 < \cdots < y_d$ we get a path in $\Phi_{\Delta(P)}$ of length $\leq \nu' + d - \delta$ connecting \mathbf{m}' to \mathbf{m}'' : $\hat{0} = y_0 < y_1 < \cdots < y_{d+1} < z_{d+2} < z_{d+3} < \cdots < z_{g-1} < x_g < x_{g+1} < \cdots < x_\delta = \hat{1}$. It is clear upon inspection of Figure 5 that there is a path in $\Phi_{\Delta(P)}$ of length $g - d - 1$ connecting \mathbf{m}'' to \mathbf{m} . Hence, there is a path in $\Phi_{\Delta(P)}$ of length $\leq (\nu' + d - \delta) + (g - d - 1) = \nu' + g - \delta - 1$ connecting \mathbf{m}' to \mathbf{m} . Since $x_i \notin [y_{d+1}, \hat{1}]$ for $i = 0, 1, \dots, g - 1$ we must conclude that $\nu' + g \leq \nu = |P|$. Consequently, $\text{diam } \Phi_{\Delta(P)} \leq \nu' + g - \delta - 1 \leq \nu - \delta - 1$ as required. \square

A bounded semimodular poset need not satisfy the Hirsch conjecture, as the poset P of Figure 6a shows. P is semimodular but not Cohen-Macaulay and the distance between the two maximal chains through the point x is 6. However, it is a simple exercise to show that any bounded semimodular poset P of finite length δ satisfies $\text{diam } \Phi_{\Delta(P)} \leq \delta(\delta - 1)/2$. Hence, we can improve Theorem 6.4 by asserting that for any bounded, locally semimodular finite poset

$$\text{diam } \Phi_{\Delta(P)} \leq \min\left(\nu - \delta - 1, \frac{\delta(\delta - 1)}{2}\right). \quad (6.5)$$

This upper bound is, in fact, sharp. For any $\delta \geq 1$ and $\nu \geq \delta + 1$ it is possible, as suggested in Figure 6b, to construct an upper-semimodular lattice of length δ and cardinality ν which achieves equality in (6.5).

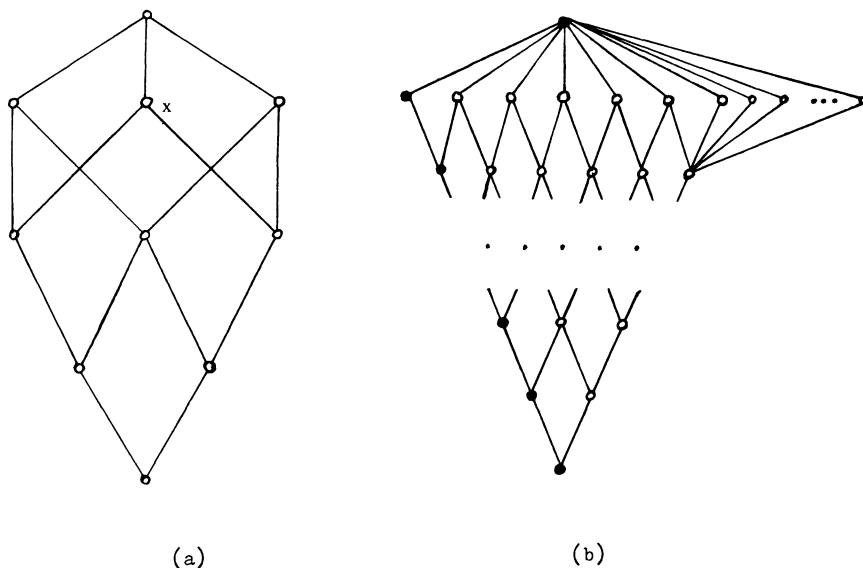


FIGURE 6

7. Higher order complexes. For a finite poset P let $\Delta_k(P)$, $k \geq 1$, denote the simplicial complex of all subsets of P which contain no $(k + 1)$ -element antichain (totally unordered subset). Thus, for $k = 1$, this definition gives us the usual order complex $\Delta(P)$. The higher order complexes $\Delta_k(P)$ were suggested to the author by R. Stanley in private correspondence. Guided by analogy with results in the Schubert calculus, Stanley was led to conjecture that the complexes $\Delta_k(L)$ are Cohen-Macaulay for all $k \geq 1$ when L is a finite planar distributive lattice. In this section we prove a slightly stronger result.

THEOREM 7.1. *Let L be a finite planar distributive lattice. Then $\Delta_k(L)$ is shellable for all $k \geq 1$.*

PROOF. Let P denote the poset of join-irreducibles of such a lattice L and for $x \in L$ let $I^x = \{p \in P \mid p \leq x\}$. According to Birkhoff's representation theorem for finite distributive lattices [2, Theorem 3, p. 59] the map $x \mapsto I^x$ is an isomorphism between L and the poset of order ideals of P ordered by inclusion. Since L is planar, P cannot contain a 3-element antichain. Hence, by Dilworth's well-known decomposition theorem P can be partitioned into two chains. Now, let Q and R be two chains in P such that $P = Q \cup R$, $Q \cap R = \emptyset$, and for every order ideal I in P write $I_Q = I \cap Q$ and $I_R = I \cap R$. We can then define a map $d: L \rightarrow \mathbb{R}^2$ by $d(x) = (|I_R^x| - |I_Q^x|, |I^x|)$. Connect points $d(x)$ and $d(y)$ such that x covers y by straight line segments. It is easy to verify that the graph \mathcal{Q} thus drawn in the

Euclidean plane \mathbf{R}^2 is a planar embedding of the Hasse diagram of L (we think of the x -axis as being "horizontal" and the y -axis as pointing "upward"). Two elements x and y of L satisfy $x \leq y$ if and only if $I_Q^x \subseteq I_Q^y$ and $I_R^x \subseteq I_R^y$. Let us define another partial order " \perp " on L by $x \leq^\perp y$ if and only if $I_Q^x \supseteq I_Q^y$ and $I_R^x \subseteq I_R^y$. It is useful to observe that any pair of elements of L is comparable either under " \leq " or under " \leq^\perp ". Thus, an antichain under " \leq " is a chain under " \leq^\perp ". Further, it is evident that the \perp -order is represented by the diagram \mathfrak{D}^\perp which we get by rotating \mathfrak{D} 90 degrees counterclockwise in \mathbf{R}^2 . Hence, it makes sense to say that x is "to the left of" y or y is "to the right of" x if and only if $x \leq^\perp y$.

Denote the rank-levels of L by L_j , $j = 0, 1, \dots, n = |P|$. Thus, $L_j = \{x \in L \mid \rho(x) = j\} = \{x \in L \mid |I^x| = j\}$ and so the elements of L_j lie on the line $y = j$ in the diagram $\mathfrak{D} \subseteq \mathbf{R}^2$. Let us say that a sequence $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ of maximal chains of L is an *almost disjoint k -sequence* of maximal chains (a.d. k -sequence, for short) if for every $j = 0, 1, \dots, n$,

- (i) $\mathbf{m}_i \cap L_j$ is strictly to the left of $\mathbf{m}_{i+1} \cap L_j$ for $i = 1, 2, \dots, |L_j| - 1$, and
- (ii) $\mathbf{m}_i \cap L_j = \mathbf{m}_{i+1} \cap L_j$ for $i = |L_j|, |L_j| + 1, \dots, k - 1$.

Note that the definitions of the diagram \mathfrak{D} , the order " \perp " and, hence, of an a.d. k -sequence depend on the chosen decomposition of P into chains Q and R . However, for any such choice we can now identify the facets of $\Delta_k(L)$.

LEMMA. *Let L be a finite planar distributive lattice and decompose the join-irreducibles P into chains Q and R .*

(a) *If $F \subseteq L$ does not contain a $(k + 1)$ -element antichain then F is contained in the union of an almost disjoint k -sequence of maximal chains of L .*

(b) *The map $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k) \mapsto \bigcup_{i=1}^k \mathbf{m}_i$ is a bijection between the set of a.d. k -sequences of maximal chains of L and the set of facets of $\Delta_k(L)$.*

PROOF. The lemma holds trivially for $k = 1$. We shall prove part (a) for $k \geq 2$ by induction on the order $|L|$. The argument is based on an idea of J. Backelin's. Statement (a) is trivially true for all $k \geq 1$ when $|L| = 2$. Assume that (a) has been proven for all $k \geq 1$ and for all lattices of order less than that of a given lattice L . If there is a singleton rank-level $L_j = \{x\}$ with $0 < j < n = |P|$, then the induction hypothesis can be applied separately in the two intervals $[\hat{0}, x]$ and $[x, \hat{1}]$, and respective members of the two a.d. k -sequences can be glued together at x to form an a.d. k -sequence in L . We may therefore assume that $|L_j| \geq 2$ for $j = 1, 2, \dots, n - 1$. Denote by A the set of elements in L which are join-irreducible and meet-irreducible and lie on the left boundary of the diagram \mathfrak{D} . It is easy to see that A is not empty, in fact A coincides with the set of minimal elements of the diagram \mathfrak{D}^\perp . Now, consider a subset F of L which does not contain a $(k + 1)$ -antichain, $k \geq 2$. In case $A \not\subseteq F$ then $F \subseteq L - \{a\}$ for some $a \in A$. Since a is doubly irreducible, $L - \{a\}$ is a sublattice of L , hence distributive, and the graph $\mathfrak{D} - d(a)$ is a planar embedding of the Hasse diagram of $L - \{a\}$ in the prescribed sense. So by the induction assumption $F \subseteq \bigcup_{i=1}^k \mathbf{m}_i$, where $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ is an a.d. k -sequence of maximal chains of $L - \{a\}$. Since the

rank-levels L_{j-1} , L_j and L_{j+1} , where $a \in L_j$, are related as illustrated in Figure 7, it is obvious how to derive from $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ an a.d. k -sequence of maximal chains of L which covers F .

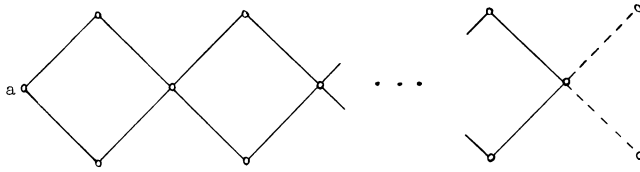


FIGURE 7

Next we must consider the case when $A \subseteq F$. Let \mathbf{m}_0 denote the maximal chain of L which lies on the left boundary of \mathcal{D} . If $B = \mathbf{m}_0 - \{\hat{0}, \hat{1}\}$, then $A \subseteq B$, and $L - B$ is a sublattice of L . Arguing as above, it is clear that the induction assumption applies to $L - B$. Suppose that $F - B$ contains a k -element antichain C . As observed earlier, C is then a chain under the \perp -order of L . Let x be the \perp -least element of C , and let y be the \perp -least (or furthest left) element of L_j , where $x \in L_j$. Then $y \in B$ and since $x \in F - B$ we have $y <^\perp x$. Let z be a \perp -minimal element of L with $z <^\perp y$. Then $z \in A$, and $C \cup \{z\}$ is a $(k + 1)$ -element antichain which is contained in $C \cup A \subseteq F$. This contradiction shows that $F - B$ cannot contain a k -element antichain. By the induction assumption there is therefore an a.d. $(k - 1)$ -sequence $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_{k-1})$ of maximal chains of $L - B$ such that $F - B \subseteq \bigcup_{i=1}^{k-1} \mathbf{m}_i$. But then $(\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_{k-1})$ is an a.d. k -sequence of maximal chains of L such that $F \subseteq \bigcup_{i=0}^{k-1} \mathbf{m}_i$. Thus, part (a) is proved.

Consider the map $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k) \mapsto \bigcup_{i=1}^k \mathbf{m}_i$ of part (b). A subset of L of the form $\bigcup_{i=1}^k \mathbf{m}_i$ obviously cannot contain a $(k + 1)$ -element antichain. Also, it is maximal with this property, because if $x \in L - S$ where $S = \bigcup_{i=1}^k \mathbf{m}_i$ and $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ is an a.d. k -sequence, then $x \in L_j$ for a j such that $|L_j| > k$, and $(S \cap L_j) \cup \{x\}$ is a $(k + 1)$ -element antichain contained in $S \cup \{x\}$. The map φ is clearly injective, and surjectivity was established in part (a) above. \square

Having thus characterized the facets of $\Delta_k(L)$, we are now in a position to conclude the proof of Theorem 7.1 by assigning a shelling order. For a given finite planar distributive lattice L , decompose its poset of join-irreducibles P into chains Q and R and draw the corresponding diagram \mathcal{D} in \mathbb{R}^2 as explained above. Let $\omega: P \rightarrow [n]$, $n = |P|$, be the unique order-preserving bijection which for all $q \in Q$ and $r \in R$ satisfies $\omega(q) > \omega(r)$ only if $q > r$. Then ω is an admissible map and induces an L -labeling γ of L (cf. §3). The L -labeling γ has the distinguishing feature that the unique rising unrefinable chain in any interval $[x, y]$ follows the left boundary of $[x, y]$ as drawn in \mathcal{D} . If F is a facet of $\Delta_k(L)$ and $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ is the unique a.d. k -sequence of maximal chains such that $F = \bigcup_{i=1}^k \mathbf{m}_i$, then associate with F the nk -tuple $\sigma(F) = (\pi(\mathbf{m}_1), \pi(\mathbf{m}_2), \dots, \pi(\mathbf{m}_k)) \in \mathbf{P}^{nk}$, where $\pi(\mathbf{m}_i)$ is the usual Jordan-Hölder sequence of the maximal chain \mathbf{m}_i under the edge-labeling γ (cf. §1). $F \neq F'$ implies that $\sigma(F) \neq \sigma(F')$, and the lexicographic order of the nk -tuples $\sigma(F)$ in \mathbf{P}^{nk} determines a linear order Ω of the facets of $\Delta_k(L)$. It remains to verify that Ω is a shelling.

Let $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k)$ and $(\mathbf{m}'_1, \mathbf{m}'_2, \dots, \mathbf{m}'_k)$ be two a.d. k -sequences of maximal chains of L and let $F = \bigcup_{i=1}^k \mathbf{m}_i$ and $F' = \bigcup_{i=1}^k \mathbf{m}'_i$. Suppose that $F' <^\Omega F$. We may then assume that $\mathbf{m}_i = \mathbf{m}'_i$ for $i = 1, 2, \dots, e-1$, and $\mathbf{m}_e \neq \mathbf{m}'_e$. Let $\mathbf{m}_e: \hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$, $\mathbf{m}'_e: \hat{0} = y_0 < y_1 < \dots < y_n = \hat{1}$ and suppose that $x_i = y_i$ for $i = 0, 1, \dots, g$, $x_i \neq y_i$ for $i = g+1, g+2, \dots, h-1$, and $x_h = y_h$. Then, since $F' <^\Omega F$, by construction y_{g+1} lies to the left of x_{g+1} in \mathcal{D} , and the planarity of \mathcal{D} forces y_i to be to the left of x_i for $i = g+1, g+2, \dots, h-1$. The relevant parts of \mathcal{D} are illustrated in Figure 8.

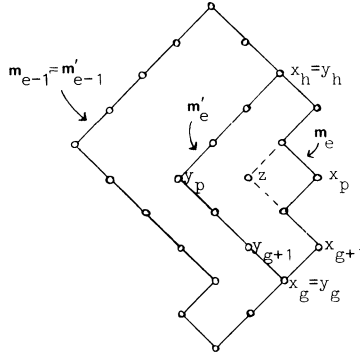


FIGURE 8

The chain $x_g < x_{g+1} < \dots < x_h$ cannot be rising, since then it would by construction trace the left boundary of the interval $[x_g, x_h]$. Say there is a descent $\gamma(x_{p-1}, x_p) > \gamma(x_p, x_{p+1})$ at p , $g < p < h$. Then there exists a unique $z \in L$ such that $x_{p-1} < z < x_{p+1}$ is rising. By our construction z lies immediately to the left of x_p in the rank-level L_p , hence to the right (not necessarily strictly) of y_p . Let \mathbf{m}_z be the maximal chain which we get by replacing x_p by z in \mathbf{m}_e . It is clear from the geometric situation (with z in between y_p and x_p in L_p) that $(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_{e-1}, \mathbf{m}_z, \mathbf{m}_{e+1}, \mathbf{m}_{e+2}, \dots, \mathbf{m}_k)$ is an a.d. k -sequence of maximal chains. Let F'' be the corresponding facet of $\Delta_k(L)$. Then $F'' <^\Omega F$ and $F' \cap F \subseteq F'' \cap F = F - \{x_p\}$. Hence, Ω is a shelling. \square

There seems to be no immediate generalization to the above theorem. For instance, consider the planar modular lattice M and the 3-dimensional distributive lattice D of Figure 9.

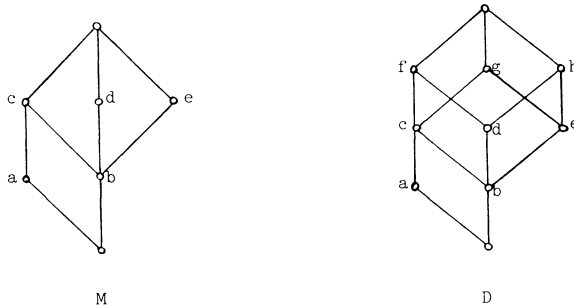


FIGURE 9

The complexes $\Delta_2(M)$ and $\Delta_2(D)$ are not Cohen-Macaulay, in fact, they are not even pure (e.g. $\{\hat{0}, a, b, c, d, \hat{1}\}$ and $\{\hat{0}, b, d, e, \hat{1}\}$ are facets of $\Delta_2(M)$ and $\{\hat{0}, a, b, c, d, g, h, \hat{1}\}$ and $\{\hat{0}, b, d, e, f, g, \hat{1}\}$ are facets of $\Delta_2(D)$).

Appendix. We shall briefly review the definitions of shellable and Cohen-Macaulay complexes and also comment on the logical relationship of these and related concepts.

A finite *simplicial complex* Δ is by definition a nonvoid family of subsets, called *simplices* or *faces*, of a finite set V , called the *vertex-set*, such that $v \in V$ implies $\{v\} \in \Delta$ and $F \subseteq G \in \Delta$ implies $F \in \Delta$. In particular, $\emptyset \in \Delta$. The *dimension* of a face F , $\dim F$, is $|F| - 1$, and the *dimension* of Δ , $\dim \Delta$, is $\max\{\dim F | F \in \Delta\}$. The faces which are maximal under inclusion are called *facets*. A complex is *pure* if all its facets are equicardinal.

Let Δ be a finite simplicial complex. We say that Δ is *shellable* if Δ is pure and the facets of Δ can be given a linear order F_1, F_2, \dots, F_t in such a way that:

if $1 \leq i < k \leq t$ then there is a j , $1 \leq j < k$, and an $x \in F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k - \{x\}$.

In other words, the facet F_k is required to intersect the complex $\bigcup_{i=1}^{k-1} F_i$ in a nonempty union of maximal proper faces of F_k . A linear order of the facets which satisfies this requirement is called a *shelling*. Shellability has been most intensively investigated for complexes which triangulate spheres and balls (see the survey [4]).

For a simplex F in a complex Δ the *link* of F is the subcomplex $\text{lk } F = \{G \in \Delta | G \cup F \in \Delta, G \cap F = \emptyset\}$. Let R denote the ring of rational integers \mathbf{Z} or a field k , and let $\tilde{H}_*(\Delta, R)$ denote reduced simplicial homology of Δ with coefficients in R . A simplicial complex Δ is said to be *Cohen-Macaulay over* R if $\tilde{H}_i(\text{lk } F, R) = 0$ for all $F \in \Delta$ and $i < \dim(\text{lk } F)$. The motivation for this terminology is to be found in a theorem of G. Reisner [15]. Let $V = \{x_1, x_2, \dots, x_n\}$ be the vertex-set of Δ and define I_Δ to be the ideal in the polynomial ring $R[x_1, x_2, \dots, x_n]$ generated by all square-free monomials $x_{i_1}x_{i_2} \cdots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \notin \Delta$. Reisner's theorem states that the complex Δ is Cohen-Macaulay over R if and only if $R[x_1, x_2, \dots, x_n]/I_\Delta$ is a Cohen-Macaulay ring. Using the Universal Coefficient Theorem one can verify that Δ is Cohen-Macaulay over \mathbf{Z} if and only if Δ is Cohen-Macaulay over all fields, and if Δ is Cohen-Macaulay over some field then Δ is Cohen-Macaulay over the rationals \mathbf{Q} . Also, it is easily shown that a Cohen-Macaulay complex (over any R) must be pure. For more information about Cohen-Macaulay complexes see the surveys [9] and [20], and also [21].

The following variation of the Cohen-Macaulay concept was introduced by D. Quillen in [14]: a pure complex Δ is said to be *homotopy Cohen-Macaulay* if the homotopy groups $\pi_i(\text{lk } F)$ are trivial for all $F \in \Delta$ and $i < \dim(\text{lk } F)$. In view of the Hurewicz isomorphism theorem it is clear that Δ is homotopy Cohen-Macaulay if and only if Δ is Cohen-Macaulay over \mathbf{Z} and in addition the links $\text{lk } F$ are simply-connected for all faces $F \in \Delta$ such that $\dim F < \dim \Delta - 3$.

Following Hochster [8] and Stanley [20] we call a complex *constructible* if it belongs to the class of complexes defined recursively by: (1) a simplex is construct-

ible, and (2) if Δ_1 , Δ_2 and $\Delta_1 \cap \Delta_2$ are constructible and $\dim \Delta_1 = \dim \Delta_2 = \dim(\Delta_1 \cap \Delta_2) + 1$, then $\Delta_1 \cup \Delta_2$ is constructible. Clearly, a shellable complex is constructible. Also, using the Mayer-Vietoris exact sequence for homology and the van Kampen theorem for fundamental groups one can verify that a constructible complex must be homotopy Cohen-Macaulay (cf. [8, Remark 8]).

Summarizing, we have seen that among the properties "shellable", "constructible", "homotopy Cohen-Macaulay" and "Cohen-Macaulay over \mathbf{Z} " of a simplicial complex, each implies its successor. Counterexamples are known to two of the three converse implications. R. D. Edwards' 5-sphere, which is the double suspension of a non-simply-connected homology 3-sphere (see [4, p. 41]), is Cohen-Macaulay over \mathbf{Z} but not homotopy Cohen-Macaulay. Also, M. E. Rudin's unshellable 3-ball (see [4, p. 40]) has been proved constructible by S. Provan [13]. We do not know of any complex which is homotopy Cohen-Macaulay but not constructible, although it seems likely that such examples should exist.

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