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Noncommutative Homology of Some Three-Dimensional Quantum Spaces

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Abstract. We compute the Hochschild and cyclic homology of certain three-dimensional quantum spaces (type A algebras), introduced by Artin and Schelter. We show that the Hochschild homology is determined by the quasi-classical limit.

Key words. Hochschild cohomology, cyclic cohomology, quantum spaces.

1. Introduction

In the sequel k will be a field of characteristic zero.

Recall that in Manin's terminology a quantum (affine) space is a finitely generated quadratic algebra [11]. More traditionally, quantum n-space is usually taken to be the algebra

$$k\langle x_1,\dots,x_n\rangle/(x_jx_i - \lambda x_ix_j;\ j > i) \tag{1}$$

with $\lambda \in k$, or its multi-parameter version

$$k\langle x_1,\dots,x_n\rangle/(x_jx_i - \lambda_{ij}x_ix_j;\ j > i) \tag{2}$$

again with $(\lambda_{ij})_{i,j} \in k$.

The Hochschild homology of quantum n-space was computed in [16] and in [18] for the multi-parameter version. It should be noted that the results depend in a sensitive way upon the values of the parameters.

Whereas in dimension two, practically all quantum spaces (with the correct Hilbert series) have the form (2), this is far from the case in higher dimensions.

A classification of three-dimensional quantum spaces was undertaken in [1]. One obtains various families, of which the most nonclassical one is given by the so-called type A algebras. That is, algebras of the form

$$k\langle x, y, z\rangle/(f_1, f_2, f_3) \tag{3}$$

where

$$f_1 = ayz + bzy + cx^2,$$

$$f_2 = azx + bxz + cy^2,$$

$$f_3 = axy + byx + cz^2.$$

Such algebras were further studied in [2, 3]. It was shown that they may be obtained from couples (E, τ), where E is an elliptic curve and τ is a translation.

Independently, generalizations in higher dimensions of type A algebras were discussed in [14, 15] (see also [17]).

Calculation of the Hochschild homology of (1), (2) is somewhat simplified by the fact that these algebras are 'toric'. That is, they carry an action of an n-dimensional torus, with one-dimensional weight spaces. In contrast, type A algebras do not enjoy this property.

In this note, we calculate the Hochschild and cyclic homology of (3) when a, b, c are generic. Our treatment is in the spirit [7, 9]. That is, we write (3) as a deformation of a polynomial algebra with a Poisson bracket. Then we show that the associated Brylinski spectral sequence degenerates.

In somewhat more fancy language, this means that the Hochschild homology of the quantum space (3) is determined by the 'quasi-classical limit', i.e. the homology of the associated Poisson variety. This a phenomenon generally observed for 'quantum type spaces' [7], but as yet unexplained.

The main result of these notes is Theorem 4.1 where we compute the Hochschild homology of (3). From this one can then easily deduce the cyclic homology, the De Rham cohomology, and the periodic cyclic homology of type A algebras. It is perhaps noteworthy that the latter two are the same as for affine three-space.

We now give an overview of the paper, section by section.

In Section 2, we discuss some basic notions which are taken from [9]. Any one who is at least vaguely familiar with Hochschild and cyclic homology should skip this section.

In Section 3, we construct a 'small' complex that computes the Hochschild homology of a Koszul algebra. Its use could have been avoided, but it does perhaps have some independent interest. Note that the material in this section nicely complements [18], where the case of a symmetric algebra, associated to a solution of the Yang–Baxter equation, is treated.

In Section 4, we introduce type A algebras and we state the main result (Theorem 4.1 and its corollary).

In Section 5, we give a proof of Theorem 4.1. As was noted above, the essential point is to show that the Brylinski spectral sequence degenerates. The most natural way to do this, which works for enveloping algebras [9], is to show that (the completion of) its E^1-term is quasi-isomorphic with the small complex introduced in Section 3. Unfortunately, I don't see how to do this.

On the other hand, it is possible to lift homology classes in E^2. This is done by lifting two special elements explicitly, and then using the compatibility of Hochschild homology with the Connes differential and with the action of a certain central element of degree 3 in (3). It is clear that this implies that the Brylinski spectral sequence degenerates.

One other point should be mentioned. The Poisson bracket we obtain from (3) has the form $\{y, z\} = g_1$, $\{z, x\} = g_2$, $\{x, y\} = g_3$ with

$$\nabla \times (g_1, g_2, g_3) = 0. \tag{4}$$

The intrinsic meaning of this last property is a mystery to me, but it is what makes possible the computation of the homology of the corresponding Poisson variety, i.e. it seems to me that without (4), the computations would be much harder.

2. Mixed Complexes

In this section we follow Kassel [9] for some basic definitions.

A mixed complex is a triple (M, b, B), where M is a $\mathbb{N}$-graded k-vector space and b, B are differentials of degrees -1, $+1$, respectively, satisfying

$$b^2 = B^2 = bB + Bb = 0.$$

Let $H_*(M) = H_*(M, b)$ be the homology of the underlying chain complex (M, b). B introduces a differential of degree 1 on $H_*(M)$ and the homology of the complex

$$0 \to H_0(M) \xrightarrow{B} H_1(M) \xrightarrow{B} H_2(M) \xrightarrow{B} \cdots$$

is called the De Rham cohomology of (M, b, B) and is denoted by $H^*_{\mathrm{DR}}(M)$.

The homology of the total complex, associated to the bicomplex,

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & \\
M_2 \arrow[d, "b"] & M_1 \arrow[l, "B"'] \arrow[d, "b"] & M_0 \arrow[l, "B"'] \arrow[d] & 0 \arrow[l] \\
M_1 \arrow[d, "b"] & M_0 \arrow[l, "B"'] \arrow[d] & 0 \arrow[l] & \\
M_0 \arrow[d] & 0 \arrow[l] & & \\
0 & & &
\end{tikzcd} \tag{5}$$

is called the cyclic homology of (M, b, B) and is denoted by $\mathrm{HC}_*(M)$.

The periodic complex associated to (M, b, B) is defined as

$$\to \prod_{i \geqslant 0} M_{2i} \xrightarrow{b+B} \prod_{i \geqslant 0} M_{2i+1} \xrightarrow{b+B} \prod_{i \geqslant 0} M_{2i} \xrightarrow{b+B} \cdots.$$

Its homology is called the periodic cyclic homology of (M, b, B) and is denoted by $\mathrm{HC}_*^{\mathrm{per}}(M)$, where $*$ = 'even' or 'odd'.

Quotienting out the first column in (5) yields the long exact sequence of Connes

$$\cdots \to H_n(M) \xrightarrow{I} \mathrm{HC}_n(M) \xrightarrow{S} \mathrm{HC}_{n-2}(M) \xrightarrow{B} H_{n-1}(M) \to \cdots.$$

If the inverse system

$$\cdots \xrightarrow{S} \mathrm{HC}_n(M) \xrightarrow{S} \mathrm{HC}_{n-2}(M) \xrightarrow{S} \cdots$$

satisfies the Mitag–Leffler condition, then its inverse limit is equal to $\mathrm{HC}_*^{\mathrm{per}}(M)$ where $*$ is 'even' or 'odd', depending on whether n is even or odd.

Filtering (5) by columns yields a spectral sequence

$$E^1\colon H_*(M) \Rightarrow \mathrm{HC}_*(M)$$

which is the spectral sequence of Connes.

If A is an associative k-algebra then there is a mixed complex $(C(A), b, B)$ given by $C(A)_n = A^{\otimes(n+1)}$,

$$\begin{aligned} &b(r_0 \otimes \cdots \otimes r_n) \\ &\quad = (-1)^n r_n r_0 \otimes \cdots \otimes r_{n-1} + \sum_{i=0}^{n-1} (-)^i r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \end{aligned}$$

and

$$\begin{aligned} &B(r_0 \otimes \cdots \otimes r_n) \\ &\quad = \sum_{i=0}^{n} (-)^{ni} 1 \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} + \\ &\qquad + (-)^n \sum_{i=0}^{n} (-)^{ni} r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1} \otimes 1. \end{aligned}$$

One defines

$$\mathrm{HH}_*(A) = H_*(C(A)), \qquad H^*_{\mathrm{DR}}(A) = H^*_{\mathrm{DR}}(C(A)),$$

$$\mathrm{HC}_*(A) = \mathrm{HC}_*(C(A)), \qquad \mathrm{HC}_*^{\mathrm{per}}(A) = \mathrm{HC}_*^{\mathrm{per}}(C(A)).$$

These are, respectively, the Hochschild homology, the De Rham cohomology, the cyclic homology and the periodic cyclic homology of A.

It is known that

$$\mathrm{HH}_*(A) = \mathrm{Tor}^{A \otimes A^0}(A, A)$$

If A is commutative then the shuffle product (see, e.g., [10]) makes $\mathrm{HH}_*(A)$ into a graded commutative differentially graded algebra. More generally, if $R \subset A$ is a central k-subalgebra, then $\mathrm{HH}_*(A)$ is a differentially graded $\mathrm{HH}_*(R)$-module.

In the sequel, we will only use the $R = \mathrm{HH}_0(R)$-module structure of $\mathrm{HH}_*(A)$. This is obtained from left multiplication on the complex $C(A)$.

If A is commutative and smooth then

$$r_0 \otimes r_1 \otimes \cdots \otimes r_n \mapsto \frac{1}{n!} r_0 \, \mathrm{d}r_1 \ldots \mathrm{d}r_n$$

defines a quasi-isomorphism between $C(A)$ and Ω_A^*, where Ω_A^* is the exterior algebra of differential forms over A, with zero differential [8].

Hence, $\mathrm{HH}_*(A) = \Omega_A^*$. Under this quasi-isomorphism, the map B on $C(A)$ corresponds to the natural differential d on differential forms.

3. Some Generalities about Koszul Algebras

Let V be a finite-dimensional k-vector space and let TV be the tensor algebra of V over k. Suppose that $A = TV/(R)$, where $R \subset V \otimes V$, i.e. A is a quadratic k-algebra. The dual algebra of A, denoted by $A^!$, is the quadratic algebra $TV^*/(R^\perp)$.

Let $(x_i)_{i=1,\ldots,n}$ be a basis of V and let $(\zeta_i)_{i=1,\ldots,n}$ be the dual basis of V^*. Then

$$e = \sum_{i=1}^{n} x_i \otimes \zeta_i \in A \otimes A^!$$

has the property that $e^2 = 0$.

Right multiplication by e defines a complex

$$A \otimes (A_0^!)^* \leftarrow A \otimes (A_1^!)^* \leftarrow \cdots \tag{6}$$

A is said to be Koszul if (6) is a resolution of ${}_A k$.

Now let $K'(A) = A \otimes (A^!)^* \otimes A$ be graded by $K'(A)_m = A \otimes (A_m^!)^* \otimes A$. On $K'(A)$ we define two maps of degree -1 as follows. Let $r \otimes f \otimes s \in K'(A)_m$. Then

$$b'_L(r \otimes f \otimes s) = \sum_i r x_i \otimes f \zeta_i \otimes s,$$

$$b'_R(r \otimes f \otimes s) = \sum_i r \otimes x_i f \otimes \zeta_i s.$$

Clearly

$$b_L'^2 = 0, \qquad b_R'^2 = 0 \quad \text{and} \quad b'_L b'_R = b'_R b'_L.$$

Hence, if we put $b' = b'_L + (-)^m b'_R$, then $b'^2 = 0$.

Note that this differential is a mild generalization of [18], where the case of a symmetric algebra, associated to the solution of the Yang–Baxter equation, is treated.

Since $A_0^! = k$, $K'(A)_0$ may be identified with $A \otimes A$. Furthermore, $A_1^! = V^*$ and, hence, $K'(A)_1 = A \otimes V \otimes A$. If $r \otimes v \otimes s \in K'(A)_1$, then

$$b'(r \otimes v \otimes s) = rv \otimes s - r \otimes vs.$$

If we define

$$\varepsilon\colon K'(A)_0 \to A\colon r \otimes s \mapsto rs,$$

then $\varepsilon b' = 0$.

PROPOSITION 3.1. *Suppose that A is Koszul. Then the complex*

$$0 \leftarrow A \xleftarrow{\varepsilon} K'(A)_0 \xleftarrow{b'} K'(A)_1 \xleftarrow{b'} \cdots \tag{7}$$

is exact and, hence, $(K'(A), b')$ defines a minimal free resolution of A as A-A-bimodule.

Proof. (7) is a comlex of free right A-modules. Tensoring with $\otimes_A k$ yields (6) (completed by k on the left). Since (6) is exact by hypothesis, the graded version of Nakayama's lemma [12] implies that (7) is also exact. □

Let $K(A)$ be the graded k-vector space $A \otimes (A^!)^*$, graded by $K(A)_m = A \otimes (A^!_m)^*$. Define differentials as follows. If $r \otimes f \in K(A)_m$ then

$$b_L(r \otimes f) = \sum_i rx_i \otimes f\zeta_i = (r \otimes f)e \quad \text{and}$$

$$b_R(r \otimes f) = \sum_i x_i r \otimes \zeta_i f = e(r \otimes f).$$

Again

$$b_L^2 = b_R^2 = 0 \quad \text{and} \quad b_L b_R = b_R b_L.$$

Therefore, if $b = b_L + (-)^m b_R$, then $b^2 = 0$. Note that if $\phi \in K(A)_m$ then

$$b\phi = \begin{cases} e\phi - \phi e, & \text{if } m \text{ is odd,} \\ e\phi + \phi e, & \text{if } m \text{ is even.} \end{cases}$$

PROPOSITION 3.2. *Suppose that A is Koszul. Then*

$$\mathrm{HH}_*(A) = H_*(K(A), b).$$

Proof. It is easily verified that

$$K(A) = A \otimes_{A \otimes A^\circ} K'(A).$$ □

We now have two complexes that compute the Hochschild homology of A. Namely, $K(A)$ and $C(A)$. It is possible to construct an explicit quasi-isomorphism between them.

We have

$$K(A)_m = A \otimes (A^!_m)^* \quad \text{and} \quad C(A)_m = A \otimes A^{\otimes m}.$$

Now $A^!_m$ is a quotient of $(V^*)^{\otimes m}$ and, hence, $(A^!_m)^* \subset (V^*)^{\otimes m}$. In fact, [4, § 2.4]

$$(A^!_m)^* = \bigcap_{i+j=m-2} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

We now define $q\colon A \otimes (A^!_m)^* \to A \otimes A^{\otimes m}$ as being the restriction of the natural inclusion $A \otimes V^{\otimes m} \hookrightarrow A \otimes A^{\otimes m}$.

PROPOSITION 3.3. *$q\colon K(A) \to C(A)$ is a quasi-isomorphism.*

Proof. It is well-known that

$$(C(A), b) = A \otimes_{A \otimes A^0} (C'(A), b'),$$

where $C'(A)$ is the following complex: $C'(A)_m = A^{\otimes(m+2)}$ and

$$b'(r_0 \otimes \cdots \otimes r_{m+1}) = \sum_{i=0}^{m} (-)^i r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{m+1}.$$

$(C'(A), b')$ is a free resolution of A as A-A-bimodule, where the augmentation map $\varepsilon\colon C'(A)_0 \to A$ is given by $a \otimes b \mapsto ab$.

We define a graded map $q'\colon K'(A) \to C'(A)$ which, in degree m, is the restriction of the natural inclusion of $A \otimes V^{\otimes m} \otimes A \hookrightarrow A \otimes A^{\otimes m} \otimes A$. Clearly, $A \otimes_{A \otimes A^0} q' = q$.

It is now sufficient to show that q' is compatible with b and ε. Elementary linear algebra shows that

$$b'\colon A \otimes (A^!_m)^* \otimes A \to A \otimes (A^!_{m-1})^* \otimes A$$

is the restriction of the map

$$b'\colon A \otimes V^{\otimes m} \otimes A \to A \otimes V^{\otimes(m-1)} \otimes A$$

given by

$$r \otimes v_1 \otimes \cdots \otimes v_m \otimes s \mapsto r v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes$$
$$\otimes s + (-)^m r \otimes v_1 \otimes \cdots \otimes v_{m-1} \otimes v_m s$$

Let $r, s \in (A^!)^*$, $f \in (A^!_m)^*$, i.e. there is a decomposition

$$f = \sum_j v_1^{(j)} \otimes \cdots \otimes v_m^{(j)},$$

where $(v_i^{(j)})_{i,j} \in V$.

Then

$$qb(r \otimes f \otimes s)$$
$$= \sum_j r v_1^{(j)} \otimes v_2^{(j)} \otimes \cdots \otimes v_m^{(j)} \otimes s + (-)^m \sum_j r \otimes v_1^{(j)} \otimes \cdots \otimes v_m^{(j-1)} \otimes v_m^{(j)} s$$

and

$$bq(r \otimes f \otimes s)$$
$$= \sum_j r v_1^{(j)} \otimes \cdots \otimes v_m^{(j)} \otimes s + (-)^m \sum_j r \otimes v_1^{(j)} \otimes \cdots \otimes v_m^{(j)} s +$$
$$+ \sum_{i=1}^{m-1} (-)^i \sum_j r \otimes v_1^{(j)} \otimes \cdots \otimes \otimes v_i^{(j)} v_{i+1}^{(j)} \otimes \cdots \otimes v_m^{(j)} \otimes s.$$

Now we have to remember that

$$f \in V^{\otimes(i-1)} \otimes R \otimes V^{\otimes(m-i+1)}$$

and, hence, all sums of the form

$$\sum_j v_1^{(j)} \otimes \cdots \otimes \otimes v_i^{(j)} v_{i+1}^{(j)} \otimes \cdots \otimes v_m^{(j)}$$

are zero in $A^{\otimes(m-1)}$. This shows that $qb = bq$.

The verification that $q\varepsilon = q$ is trivial and is left to the reader. □

Remark 3.4. It would be interesting if we could make $K(A)$ into a mixed complex, that computes the cyclic homology of A. Unfortunately, this does not seem to be possible in general, at least not in a canonical way.

4. Type A Algebras

Recall than in [1] and subsequently in [2, 3] certain Koszul algebras of dimension 3 were studied, i.e. those graded algebras $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ of global dimension 3 having Hilbert function $\dim A_l = \binom{l+2}{l}$ and the property that $\mathrm{Ext}_A^i(k, A) = \delta_{i,3}k$, where $\delta_{i,j}$ is the Kronecker delta.

Such algebras come in various families of which the most beautiful one, to some tastes, is given by the so-called type A algebras. A type A algebra is of the form $A = k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$ where

$$\begin{aligned} f_1 &= ax_2x_3 + bx_3x_2 + cx_1^2, \\ f_2 &= ax_3x_1 + bx_1x_3 + cx_2^2, \\ f_3 &= ax_1x_2 + bx_2x_1 + cx_3^2, \end{aligned} \tag{8}$$

with $(a, b, c) \in k^3$ (it is necessary to exclude a finite subset of k^3).

Type A algebras may also be obtained from pairs (E, τ) consisting of a smooth elliptic curve and a translation [2, 15].

In the rest of this section A will be a type A algebra. Then A has a central element in degree 3 of the form [1, eq. (10.17)]

$$C_3 = c(c^3 - b^3)x_2^3 + b(c^3 - a^3)x_2x_1x_3 + a(b^3 - c^3)x_1x_2x_3 + c(a^3 - c^3)x_1^3.$$

The complex $K(A)$ may, in this special case, be explicitly described. Let

$$\bar{x} = (x_1, x_2, x_3), \bar{f}^t = (f_1, f_2, f_3).$$

Then $\bar{f} = M\bar{x}^t$, where

$$M = \begin{pmatrix} cx_1 & bx_3 & ax_2 \\ ax_3 & cx_2 & bx_1 \\ bx_2 & ax_1 & cx_3 \end{pmatrix}.$$

Note that also $\bar{f}^t = \bar{x}M$.

$K(A)$ is now given by

$$0 \to A(-3) \xrightarrow{\bar{x}-\bar{x}^0} A(-2)^3 \xrightarrow{M+M^{0t}} A(-1)^3 \xrightarrow{\bar{x}^t-\bar{x}^{0t}} A.$$

Here $(\,)^0$ means using left multiplication on A, i.e. $xy^0 = yx$.

Now we describe some objects in $K(A)$ that are needed to describe the Hochschild homology of A. We denote by Δ the element $1 \in K(A)_3$ and by Π the element $(x_1, x_2, x_3) \in K(A)_2$. Clearly, $b\Delta = b\Pi = 0$. Consider f_1, f_2, f_3 as elements of $V \otimes V \subset A^{\otimes 2}$. Then

$$q(\Pi) = x_1 \otimes f_1 + x_2 \otimes f_2 + x_3 \otimes f_3,$$
$$q(\Delta) = 1 \otimes q(\Pi).$$

Note that $q(\Pi) \in V^{\otimes 3}$ is nothing but the element w, as defined in [1, eq. (2.3)].

Π, Δ define elements of $\mathrm{HH}_{2,3}(A)$ that will be denoted by the same letters.

THEOREM 4.1. *Let A be a generic type A algebra, i.e.*

$$A = k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3),$$

where $(f_i)_i$ are given by (8) *and a, b, c are algebraically independent over $\mathbb{Q} \subset k$. Then $(\mathrm{HH}_i(A))_{i=0,\dots,3}$ are free $k[C_3]$-modules of ranks* 8, 8, 1, 1. *$\mathrm{HH}_2(A)$ is generated by Π and $\mathrm{HH}_3(A)$ is generated by Δ.*

$\mathrm{HH}_{0,1}(A)$ have Hilbert series as follows:

$$H(\mathrm{HH}_0(A), t) = \frac{(1+t)^3}{1-t^3},$$

$$H(\mathrm{HH}_1(A), t) = \frac{(1+t)^3}{1-t^3} - 1.$$

B defines a surjection $\mathrm{HH}_0(A) \to \mathrm{HH}_1(A)$ and an isomorphism between $\mathrm{HH}_2(A)$ and $\mathrm{HH}_3(A)$.

A proof of this theorem will be given at the end of Section 5.

COROLLARY 4.2

(1) $$H^i_{\mathrm{DR}}(A) = \begin{cases} k, & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $$\mathrm{HC}_i(A) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ \mathrm{HH}_0(A), & \text{if } i = 0, \\ k \oplus \mathrm{HH}_2(A), & \text{if } i = 2, \\ k, & \text{if } i \text{ is even and } \geqslant 4. \end{cases}$$

(3) *$S\colon \mathrm{HC}_i(A) \to \mathrm{HC}_{i-2}(A)$ is given by the identity on k if $i \geqslant 6$ and by the inclusion of k as the first factor of $k \oplus \mathrm{HH}_2(A)$ if $i = 4$. If $i = 2$ then $S\colon k \oplus \mathrm{HH}_2(A) \to \mathrm{HH}_0(A)$ is zero on $\mathrm{HH}_2(A)$, and the canonical inclusion $k \hookrightarrow \mathrm{HH}_0(A)$ on k.*

(4) $\mathrm{HC}_*^{\mathrm{per}}(A) = \begin{cases} k, & \textit{if} * \textit{is even}, \\ 0, & \textit{if} * \textit{if odd}. \end{cases}$

Proof.

(1) This is clear.

(2) The E^2-term of the Connes spectral sequence has the form

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & k & \\ \mathrm{HH}_2(A) & 0 & k & & \\ 0 & k & & & \\ \mathrm{HH}_0(A) & & & & \end{array}$$

Since the higher differentials will have degrees $(-t, t-1)$, this spectral sequence degenerates. This yields (2) except that a priori, we only have $\mathrm{gr}\,\mathrm{HC}_2(A) = k \oplus \mathrm{HH}_2(A)$.

However, here k lives in degree 0 and $\mathrm{HH}_2(A)$ in degree $\geqslant 3$ (for the grading on A). Therefore, we may legitimately write $\mathrm{HC}_2(A) = k \oplus \mathrm{HH}_2(A)$.

(3) To compute S one uses the fact that S is obtained from the quotient map of (5) to (5) minus the first column. This gives a map between the E^2-terms of the corresponding Connes spectral sequences.
Making this explicit shows that $\mathrm{gr}\, S$ is the map we have defined. However, since the filtration on $\mathrm{HC}_*(A)$ in each degree (for the grading on A) is concentrated in one place, one has $S = \mathrm{gr}\, S$.

(4) The inverse systems $(\mathrm{HC}_{2i}(A), S)$ and $(\mathrm{HC}_{2i+1}(A), S)$ satisfy the Mittag–Leffler conditions and, hence, $\mathrm{HC}_*^{\mathrm{per}}(A)$ is obtained as an inverse limit. This yields the desired result. □

Remark 4.3.

(1) A has the same De Rham cohomology and periodic cyclic homology as affine three-space.

(2) We have refrained from giving explicit generators for $\mathrm{HH}_0(A)$ and $\mathrm{HH}_1(A)$ since there seems to be no canonical choice.

(3) It would be interesting to make explicit the structure of $\mathrm{HH}_*(A)$ as a differentially graded $\mathrm{HH}_*(k[C_3])$-module.

CONJECTURE 4.4. *The conclusion of Theorem* 4.1 *is valid iff, in the pair* (E, τ) *defining* A, τ *is of finite order.*

This conjecture is very likely true, but our methods are far from sufficient to prove it. Note that (when $k = \mathbb{C}$) those (a, b, c) for which τ is of finite order, is an analytically dense subset of k^3 which is rather hard to describe.

5. Proof of Theorem 4.1

We keep the same notations as in the previous sections. We recall that $A = k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$ where $(f_i)_i$ are given by (8), with a, b, c generic.

By multiplying the relations with a scalar factor we may normalize in such way that $a - b = 2$. Then $(f_i)_i$ may be written as

$$f_1 = [x_2, x_3] - p(x_2x_3 + x_3x_1) - qx_1^2,$$

$$f_2 = [x_3, x_1] - p(x_3x_1 + x_1x_3) - qx_2^2,$$

$$f_3 = [x_1, x_2] - p(x_1x_2 + x_2x_1) - qx_3^2,$$

where

$$a = 1 - p, \qquad b = -1 - p, \qquad c = -q.$$

Clearly, p, q are still algebraically independent over $\mathbb{Q}$.

To prove Theorem 4.1, we may enlarge the base field, and we may therefore assume that there is a field k_0 such that $k = k_0((h))$ and

$$p = p_1h + p_2h^2 + \cdots$$

$$q = q_1h + q_2h^2 + \cdots$$

Let

$$\mathscr{A} = k_0[[h]]\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3).$$

Then

$$A_0 = \mathscr{A} \otimes_{k_0[[h]]} k_0 = k_0[x_1, x_2, x_3]$$

has the same Hilbert series as $A = \mathscr{A} \otimes_{k_0[[h]]} k_0((h))$ and, hence, $\mathscr{A}$ is a flat $k_0[[h]]$-module. In particular, h is a nonzero divisor in $\mathscr{A}$.

The (h)-adic filtration on $\mathscr{A}$ may be extended to a filtration F on A such that $\mathrm{gr}_F A = A_0[h, h^{-1}]$. Note that A is not complete for F. However, each homogenous component of A is complete, i.e. A is complete in the category of *graded* A-modules. F may be further extended to a filtration on $C(A)$ such that

$$\mathrm{gr}\, C(A) = C(A_0)[h, h^{-1}].$$

The commutator operator on A defines a Poisson bracket on A_0 given by

$$\{x_2, x_3\} = 2p_1x_2x_3 - q_1x_1^2 = g_1,$$

$$\{x_3, x_1\} = 2p_1x_1x_3 - q_1x_2^2 = g_2,$$

$$\{x_1, x_2\} = 2p_1x_1x_2 - q_1x_3^2 = g_3.$$

The filtration F on $C(A)$ gives rise to a spectral sequence

$$E^1\colon \Omega_{A_0}^* \otimes k_0[h, h^{-1}] \to \mathrm{HH}_*(A). \tag{9}$$

This spectral sequence is convergent, since in each degree (for the natural grading on A) $C(A)$ is a complex of finite-dimensional k-vector spaces, complete for the h-adic filtration.

The differential in (9) has been computed by Brylinski [5]. It is obtained from a differential ∂ on $\Omega^*_{A_0}$ of degree -1, given by

$$\begin{aligned}\partial(r_0\,\mathrm{d}r_1\ldots\mathrm{d}r_m) &= \sum_{i=1}^{m}(-)^{i+1}\{r_0,r_i\}\,\mathrm{d}r_1\ldots\widehat{\mathrm{d}r_i}\ldots\mathrm{d}r_m + \\ &+ \sum_{1\leqslant i<j\leqslant m}(-)^{i+j}r_0\,\mathrm{d}\{r_i,r_j\}\,\mathrm{d}r_1\ldots\widehat{\mathrm{d}r_i}\ldots\widehat{\mathrm{d}r_j}\ldots\mathrm{d}r_m.\end{aligned}\tag{10}$$

The differential in (9) is then $h\partial$.

Multiplication with C_3 defines a filtered homomorphism $C(A)\to C(A)$ and, hence, a map on (9). It will be more convenient for us to work with $\Phi = -(1/3h)C_3$. Then multiplication with Φ corresponds on E^1 to multiplication with the dominant term ϕ of Φ. One computes

$$\phi = \frac{q_1}{3}(x_1^3 + x_2^3 + x_3^3) + 2p_1x_1x_2x_3.$$

The fact that Φ is central corresponds to $\{x_i, \phi\} = 0$ for $i = 1, 2, 3$. From this we immediately deduce that ∂ is ϕ-linear (as it should).

From the discussion in Section 2 it follows that the map $B\colon C(A)\to C(A)$ corresponds in (9) to the differential d on $\Omega^*_{A_0}$.

$q(\Pi)$ and $q(\Delta)$ give rise to elements of E^1 in (9) which we denote by π and δ respectively. An easy computation shows that

$$\pi = x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + x_2\,\mathrm{d}x_3\,\mathrm{d}x_1 + x_3\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

$$\delta = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.$$

Theorem 4.1 will now follow from the fact that (9) degenerates (yet to be shown) and the following result.

THEOREM 5.1. *$(H_i(\Omega^*_{A_0}, \partial))_i$ are free $k_0[\phi]$-modules of ranks* 8, 8, 1, 1. *$H_2(\Omega^*_{A_0}, \partial)$ is generated by π and $H_3(\Omega^*_{A_0}, \partial)$ is generated by δ. $H_{0,1}(\Omega^*_{A_0}, \partial)$ have Hilbert series as follows:*

$$H(H_0(\Omega^*_{A_0}, \partial), t) = \frac{(1+t)^3}{1-t^3},$$

$$H(H_1(\Omega^*_{A_0}, \partial), t) = \frac{(1+t)^3}{1-t^3} - 1$$

d defines surjections

$$H_0(\Omega^*_{A_0}, \partial) \to H_1(\Omega^*_{A_0}, \partial) \quad and \quad H_2(\Omega^*_{A_0}, \partial) \to H_3(\Omega^*_{A_0}, \partial).$$

Proof. To prove this result, we need to do similar computations as P. Nuss in [13].

We will use notations from vector calculus. If u is a symbol then $\bar{u}$ stands for the vector (u_1, u_2, u_3). $\times$ is the exterior product and $\cdot$ is the inner product. ∇ stands for the operator

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

We have the very important property $\nabla\phi = \bar{g}$ and, hence, $\nabla \times \bar{g} = 0$.

Denote by ∂_i the map $\partial\colon \Omega^i_{A_0} \to \Omega^{i+1}_{A_0}$.

Step 1. First we compute explicitly ∂_1, ∂_2, ∂_3. The results are as follows:

$$\partial_1(X_1\,\mathrm{d}x_1 + X_2\,\mathrm{d}x_2 + X_3\,\mathrm{d}x_3) = (\nabla \times \bar{X})\cdot\nabla\phi,$$

$$\partial_2(X_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + X_2\,\mathrm{d}x_3\,\mathrm{d}x_1 + X_3\,\mathrm{d}x_1\,\mathrm{d}x_2) = (\nabla\cdot\bar{X})\,\mathrm{d}\phi - \mathrm{d}(\bar{X}\cdot\nabla\phi),$$

$$\partial_3(U\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3) = -\mathrm{d}U \wedge \mathrm{d}\phi$$

Step 2. Now we compute the kernels $(\ker \partial_i)_{i=1,2,3}$. We first list the results.

$$\ker \partial_1 = \{\mathrm{d}\psi + \theta\,\mathrm{d}\phi \mid \psi, \theta \in A_0\},$$

$$\ker \partial_2 = \{f(\phi)\pi + \mathrm{d}\theta \wedge \mathrm{d}\phi \mid f \in k_0[\phi], \theta \in A_0\},$$

$$\ker \partial_3 = \{f(\theta)\delta \mid f \in k_0[\theta]\}.$$

ker ∂_1. Suppose that $(\nabla \times \bar{X})\cdot\nabla\phi = 0$. We put $\bar{X}^{(0)} = \bar{X}$.

Since (g_1, g_2, g_3) is a regular sequence there must exist $\bar{X}^{(1)} \in A_0^3$ such that $\nabla \times \bar{X}^{(0)} = \bar{X}^{(1)} \times \bar{g}$. Hence, $\nabla\cdot(\bar{X}^{(1)} \times \bar{g}) = 0$. Since $\nabla \times \bar{g} = 0$, we obtain $(\nabla \times \bar{X}^{(1)})\cdot\bar{g} = 0$.

We can continue this procedure by defining $\bar{X}^{(i+1)} \in A_0^3$ in such a way that $\nabla \times \bar{X}^{(i)} = \bar{X}^{(i+1)} \times \bar{g}$. Eventually $\nabla \times \bar{X}^{(m)} = 0$ and, hence, by the Poincaré lemma there exist $\psi_m \in A_0$ such that $\bar{X}^{(m)} = \nabla\psi_m$.

Since $(\nabla\psi_m) \times \bar{g} = \nabla \times (\psi_m\bar{g})$, we obtain that $\nabla \times (\bar{X}^{(m-1)} - \psi_m\bar{g}) = 0$. Again by the Poincaré lemma there is a $\psi_{m-1} \in A_0$ such that $\bar{X}^{(m-1)} = \nabla\psi_{m-1} + \psi_m\bar{g}$.

Continuing in this way yields the existence of elements $(\psi_i)_i$ in A_0 such that $\bar{X}^{(i)} = \nabla\psi_i + \psi_{i+1}\bar{g}$. Put $\psi = \psi_0$, $\theta = \psi_1$. Then

$$\bar{X} = \nabla\psi + \theta\bar{g}$$

and, hence,

$$X_1\,\mathrm{d}x_1 + X_2\,\mathrm{d}x_2 + X_3\mathrm{d}x_3 = \mathrm{d}\psi + \theta\,\mathrm{d}\phi.$$

$\ker \partial_2$. Here and below we need the following fact. If $U \in A_0$ such that $\mathrm{d}U \wedge \mathrm{d}\phi = 0$ then $U = f(\phi), f \in k_0[\phi]$.

For completeness, we give a proof. Let $U^{(0)} = U$, $\mathrm{d}U_0 \wedge \mathrm{d}\phi = 0$ yields that $\nabla U^{(0)}$ and $\bar{g}$ are colinear. Since g_1, g_2, g_3 have no factors in common, this is only possible it there exists $U^{(1)} \in A_0$ such that $\nabla U^{(0)} = U^{(1)}(g_1, g_2, g_3)$, or, equivalently, $\mathrm{d}U^{(0)} = U^{(1)}\,\mathrm{d}\phi$. Hence, $\mathrm{d}U^{(1)} \wedge \mathrm{d}\phi = 0$. Eventually, $\mathrm{d}U^{(m)} = 0$ and, therefore, $U^{(m)} = \lambda_m$, $\lambda_m \in k_0$. Then

$$U^{(m-1)} = \lambda_m \phi + \lambda_{m-1}, \qquad U^{(m-2)} = \tfrac{1}{2}\lambda_m \phi^2 + \lambda_{m-1}\phi + \lambda_{m-2}, \text{ etc...},$$

for $(\lambda_i)_i \in k_0$. Finally, $U = U^{(0)} = f(\phi)$ with $f \in k_0[\phi]$.

Suppose now that

$$(\nabla \cdot \bar{X})\,\mathrm{d}\phi - \mathrm{d}(\bar{X} \cdot \nabla\phi) = 0.$$

Then $\mathrm{d}(\nabla \cdot \bar{X}) \wedge \mathrm{d}\phi = 0$ and, hence,

$$\nabla \cdot \bar{X} = u'(\phi), \qquad \bar{X} \cdot \nabla\phi = u(\phi), \tag{11}$$

with $u(\phi) \in \phi k_0[\phi]$.

A particular solution to (11) is given by

$$\bar{X}^p = \frac{u(\phi)}{3\phi}\bar{x}.$$

If we put $\bar{X}^h = \bar{X} - \bar{X}^p$, then $\bar{X}^h$ satisfies the homogeneous system

$$\nabla \cdot \bar{X} = 0, \qquad \bar{X} \cdot \nabla\phi = 0.$$

Hence, there exist $\bar{Y} \in A_0^3$ with

$$\bar{X}^h = \nabla \times \bar{Y} \quad \text{and} \quad (\nabla \times \bar{Y}) \cdot \nabla\phi = 0.$$

Such an equation was solved during the computation of $\ker \partial_1$. We obtain $\bar{Y} = \theta\nabla\phi + \nabla\psi$, for $\theta, \psi \in A_0$. Hence,

$$\bar{X}^h = \nabla\theta \times \nabla\phi.$$

Put

$$f(\theta) = \frac{u(\theta)}{3\theta}.$$

Since $\bar{X} = \bar{X}^h + \bar{X}^p$, we obtain

$$\bar{X} = \nabla\theta \times \nabla\phi + f(\phi)\bar{x}$$

from which we deduce

$$X_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + X_2\,\mathrm{d}x_3\,\mathrm{d}x_1 + X_3\,\mathrm{d}x_1\,\mathrm{d}x_2 = \mathrm{d}\theta \wedge \mathrm{d}\phi + f(\phi)\pi.$$

ker ∂_3. If $\mathrm{d}U \wedge \mathrm{d}\phi = 0$ then we have shown in the beginning of the computation of $\ker \partial_2$ that $U = f(\phi), f \in k_0[\phi]$.

Step 3. To be able to compute the Hilbert series of $\ker \partial_1$, $\ker \partial_2$, we now construct some exact sequences.

ker ∂_1. It is a simple verification that there is an exact sequence

$$0 \to k_0[\phi] \xrightarrow{\alpha} A_0 \oplus A_0(-3) \xrightarrow{\beta} \ker \partial_1 \to 0,$$

where

$$\beta(\psi, \theta) = \mathrm{d}\psi + \theta\, \mathrm{d}\phi,$$
$$\alpha(u(\phi)) = (u(\phi), -u'(\phi)).$$

ker ∂_2. There is an exact sequence

$$0 \to k_0[\phi](-3) \xrightarrow{\gamma} k_0[\phi](-3) \oplus A_0(-3) \xrightarrow{\varepsilon} \ker \partial_2 \to 0,$$

where

$$\varepsilon(f(\phi), \theta) = f(\phi)\pi + \mathrm{d}\theta \wedge \mathrm{d}\phi,$$
$$\gamma(g(\phi)) = (0, g(\phi)).$$

Suppose that $\varepsilon(f(\phi), \theta) = 0$. Then $\mathrm{d}(f(\phi)\pi) = 0$. But a computation shows that

$$\mathrm{d}(f(\phi)\pi) = 3(f(\phi) + f'(\phi)\phi)\delta$$

which is different from zero if $f \neq 0$. Hence, $f = 0$ and $\mathrm{d}\theta \wedge \mathrm{d}\phi = 0$ which implies, as before, $\theta = g(\phi)$, $g \in k_0[\phi]$. As a result $(f, \phi) \in \operatorname{im} \varepsilon$.

Step 4. From the long exact sequences

$$0 \to \ker \partial_{i+1} \to \Omega_{A_0}^{i+1} \to \ker \partial_i \to H_i(\Omega_{A_0}^*, \partial) \to 0$$

together with the exact sequences we have constructed in step 3, we can now compute the Hilbert series of $H_i(\Omega_{A_0}^*, \partial)$. The results are as they should. In particular, the Hilbert series for $H_{2,3}(\Omega_{A_0}^*, \partial)$ are $t^3/(1 - t^3)$.

Step 5. Using the description of $(\ker \partial_i)_i$ given in step 2 and the explicit form of $(\partial_i)_i$ given in step 1, we find that

(1) $H_1(\Omega_{A_0}^*, \partial)$ is represented by elements of the form $\mathrm{d}\psi$, $\psi \in A_0$,
(2) $H_2(\Omega_{A_0}^*, \partial)$ is generated by π as $k_0[\phi]$-module,
(3) $H_3(\Omega_{A_0}^*, \partial)$ is generated by δ as $k_0[\phi]$-module.

From the values of the Hilbert series of $H_{2,3}(\Omega_{A_0}^*, \partial)$ we deduce that these are free $k_0[\phi]$-modules of rank 1.

Furthermore $\mathrm{d}\colon H_0(\Omega_{A_0}^*, \partial) \to H_1(\Omega_{A_0}^*, \partial)$ is clearly surjective.

Since

$$\mathrm{d}(f(\phi)\pi) = 3(f(\phi) + f'(\phi)\phi)\delta$$

and $f \mapsto f + \phi f'$ is a surjective map from $k_0[\phi]$ to itself, we see that d defines a surjection between $H_2(\Omega_{A_0}^*, \partial)$ and $H_3(\Omega_{A_0}^*, \partial)$.

Step 6. The only thing left is to show that $H_{0,1}(\Omega^*_{A_0}, \partial)$ are free $k_0[\phi]$-modules. To this end, it is sufficient to show that multiplication with ϕ is injective. We do this now.

$H_0(\Omega^*_{A_0}, \partial)$. From the description of ∂, it is clear that

$$\operatorname{im} \partial_1 = \{\bar{u}\cdot\bar{g} \mid \bar{u} \in A_0^3 \text{ and } \nabla\cdot\bar{u} = 0\}.$$

Let p be in the ideal (g_1, g_2, g_3), i.e. $p = \bar{v}\cdot\bar{g}, \bar{v} \in A_0^3$. p will be in $\operatorname{im} \partial_1$ iff there exist $\bar{r} \in A_0^3$ such that $\nabla\cdot(\bar{v} - \bar{r} \times \bar{g}) = 0$ which is equivalent with $\nabla\cdot\bar{v} = (\nabla \times \bar{r})\cdot\bar{g}$, i.e. p is in $\operatorname{im} \partial_1$ iff $\nabla\cdot\bar{v}$ is in $\operatorname{im} \partial_1$.

Let ψ now be a homogeneous element of A_0, and suppose that $\phi\psi \in \operatorname{im} \partial_1$. By Euler's identity $\phi = \frac{1}{3}\bar{x}\cdot\bar{g}$ and, hence, $\phi\psi = \frac{1}{3}(\psi\bar{x})\cdot\bar{g}$. By the previous paragraph $\nabla\cdot(\psi\bar{x}) \in \operatorname{im} \partial_1$. But $\nabla\cdot(\psi\bar{x}) = (3 + \deg \psi)\psi$ and, hence, $\psi \in \operatorname{im} \partial_1$.

$H_1(\Omega^*_{A_0}, \partial)$. Suppose that there is a homogeneous $\mathrm{d}\theta \in \Omega^1_{A_0}$ such that $\phi\, \mathrm{d}\theta \in \operatorname{im} \partial_2$. Then there must exist a homogeneous $\bar{X} \in A_0^3$ such that

$$-\mathrm{d}(\theta\phi) + \theta\, \mathrm{d}\phi = (\nabla\cdot\bar{X})\, \mathrm{d}\phi - \mathrm{d}(\bar{X}\cdot\nabla\phi)$$

and, hence,

$$\mathrm{d}(\nabla\cdot\bar{X} - \theta) \wedge \mathrm{d}\phi = 0$$

from which we deduce the existence of $u(\phi) \in \phi k_0[\phi]$ such that

$$\nabla\cdot\bar{X} = \theta + u'(\theta), \tag{12}$$

$$\bar{X}\cdot\nabla\phi = \phi\theta + u(\phi). \tag{13}$$

By changing $\bar{X}$, as we did in the computation of $\ker \partial_2$ (see Equation (11)), we may suppose $u = 0$.

From (13) we obtain

$$\nabla\phi\cdot(\bar{X} - \tfrac{1}{3}\theta\bar{x}) = 0.$$

Hence, there exist homogeneous $\bar{r} \in A_0^3$ such that

$$\bar{X} = \tfrac{1}{3}\theta\bar{x} + \bar{r} \times \nabla\phi$$

Plugging this in (12) yields

$$\tfrac{1}{3}(3 + \deg \theta)\theta + (\nabla \times \bar{r})\cdot\nabla\phi = \theta$$

or

$$\theta = -\frac{3}{\deg \theta}(\nabla \times \bar{r})\cdot\nabla\phi.$$

If we put

$$\bar{Y} = \frac{3}{\deg \theta}\nabla \times \bar{r},$$

then

$$\mathrm{d}\theta = (\nabla\cdot\bar{Y})\,\mathrm{d}\phi - \mathrm{d}(\bar{Y}\cdot\nabla\phi)$$

and, hence, $\mathrm{d}\theta \in \operatorname{im}\partial_2$.

This finishes the proof of Theorem 5.1. □

Suppose now that $(G^{\cdot}, d)$ is a cochain complex with an ascending filtration

$$\cdots \subset F_pG^{\cdot} \subset F_{p+1}G^{\cdot} \subset \cdots,$$

then there is a spectral sequence [6, Chapt. XV]

$$E^1\colon H^*(\operatorname{gr} G^{\cdot}) \Rightarrow H^*(G^{\cdot}). \tag{14}$$

Let E_p^r be the pth row of the rth term of this spectral sequence. Then $E_p^r = Z_p^r/B_r^p$, where

$$Z_p^r = \operatorname{im}(H^*(F_pG^{\cdot}/F_{p-r}G^{\cdot}) \to H^*(\operatorname{gr} G^{\cdot})_p),$$

$$B_p^r = \operatorname{im}(H^*(F_{p+r-1}G^{\cdot}/F_pG^{\cdot}) \to H^*(\operatorname{gr} G^{\cdot})_p).$$

In addition, one defines

$$Z_p^{\infty} = \operatorname{im}(H^*(F_pG^{\cdot}) \to H^*(\operatorname{gr} G^{\cdot})_p),$$

$$B_p^{\infty} = \operatorname{im}(H^*(G^{\cdot}/F_pG^{\cdot}) \to H^*(\operatorname{gr} G^{\cdot})_p)$$

which yields inclusions

$$\cdots \subset B_p^r \subset B_p^{r+1} \subset \cdots \subset B_p^{\infty} \subset Z_p^{\infty} \subset \cdots \subset Z_p^{r+1} \subset Z_p^r \subset \cdots$$

and isomorphisms

$$Z_p^r/Z_p^{r+1} \cong B_{p-r}^{r+1}/B_{p-r}^r \tag{15}$$

One has $B_p^{\infty} = \cup_r B_p^r$, but not in general $Z_p^{\infty} = \cap_r Z_p^r$.

Composition with $H^*(F_pG^{\cdot}) \to H^*(F_pG^{\cdot}/F_{p-r}G^{\cdot})$ gives a map

$$\phi_p^r\colon H^*(F_pG^{\cdot}) \to E_p^r.$$

The following is a standard criterion for degeneration of spectral sequence.

LEMMA 5.2. *If, for a given r, $\oplus_p \phi_p^r$ is surjective then* (14) *degenerates at* E^r.

Proof. $\oplus_p \phi_p^r$ being surjective means exactly that for all p

$$Z_p^r = Z_p^{r+1} = \cdots = Z_p^{\infty}$$

and, hence, by (15).

$$B_p^r = B_p^{r+1} = \cdots = B_p^{\infty}.$$

Therefore $E^r = E^{r+1} = \cdots = E^{\infty}$. □

Proof of Theorem 4.1. By Theorem 5.1 it is sufficient to show that the spectral sequence (9) degenerates. To this end we apply Lemma 5.2.

It is clearly sufficient to show that any element of $H_*(\Omega^*_{A_0}, \partial)$ may be lifted to an element of $H_*(F_0(C(A)))$.

$H_0(\Omega^*_{A_0}, \partial)$ is a quotient of A_0, so this case is trivial.

If $u \in H_1(\Omega^*_{A_0}, \partial)$, then $u = d\psi$, $\psi \in A_0$. We lift ψ to an element Ψ of F_0A and $U = B\Psi \in H_1(F_0(C(A)))$. Then $U \in \ker b$ and U defines a lifting of u.

If $u \in H_2(\Omega^*_{A_0}, \partial)$ then $u = f(\psi)\pi$ with $f \in k_0[\phi]$. Hence $U = f(\Phi)\Pi$ is a lifting of u.

Similarly, if $u \in H_3(\Omega^*_{A_0}, \partial)$ then $u = f(\phi)\delta$ and we take $U = f(\Phi)\Delta$. □

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