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# Modules Graded by $G$-sets

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## 0. Introduction

Methods of module theory have proven to be successful tools in the structure theory for noncommutative rings, so when considering a ring $R$ graded by a group, $G$ say, it is natural to consider $G$-graded $R$-modules $M=\bigoplus_{\sigma\in G} M_\sigma$, where $R=\bigoplus_{\sigma\in G} R_\sigma$ and $R_\sigma M_\tau \subset M_{\sigma\tau}$ for all $\sigma$, $\tau\in G$. However it would be even more natural to allow $R$-modules $N$ with a decomposition $N=\bigoplus_{x\in A} N_x$ where $A$ is a $G$-set, that is a set with an action of the group $G$, given by $\sigma.\ x=\sigma x$ for $\sigma\in G$, $x\in A$, and where $R_\sigma N_x \subset N_{\sigma x}$ for all $\sigma\in G$, $x\in A$. Such "graded" modules appear naturally when one views $G$-graded modules over a group algebra $kG$ by restriction of scalars as modules over $kH$ for some subgroup $H$ of $G$. At once this points out the most important example of a $G$-set encountered in the applications, namely $G/H$ the set of left $H$-cosets in $G$ with the usual $G$-action on it defined by translation. Our inspiration for this paper stems from E. Dade's paper [2] where some particular cases of $G$-set gradations are being applied to the Clifford theory for graded rings.

In Sect. 2 we provide the basic theory of $(G, A, R)$-gr, the category consisting of the left $R$-modules graded by a $G$-set $A$ with degree preserving $R$-linear maps for the morphisms. A first result of importance is Theorem 2.8 stating that the category $(G, A, R)$-gr is a Grothendieck category. For finite $G$-sets we introduce a generalization of the notion of the smash product; the smash product $R\# A$ with respect to the finite $G$-set $A$ allows to view $(G, A, R)$-gr as a module category isomorphic to $R\# A$-mod (see Theorem 2.13). We round off Sect. 2 by providing a characterization of smash products in terms of matrix rings and we derive some interesting corollaries, e.g. $R\# G/H$ is Morita anti-equivalent to the ring $(R\# G)*H$, the skew group ring of $H$ over the ring $R\# G$.

In Sect. 3 we extend the applicability of some functors, introduced by E. Dade in [2] for $G$-sets of the form $G/H$, to arbitrary $G$-sets. To a morphism $\varphi\colon A\to A'$ of $G$-sets we associate the functor $T_\varphi\colon (G, A, R)\text{-gr}\to(G, A', R)\text{-gr}$; this functor has an exact right adjoint denoted by $S^\varphi$. We observe that $S^\varphi$ will be a left adjoint for $T_\varphi$ too if for all $x' \in A'$ the set $\varphi^{-1}(x')$ is finite. This simple observation has some interesting consequences, e.g. if $A$ is finite and $Q \in (G, A, R)$-gr is an injective object then $Q$ is injective in $R$-mod.

If $A$ is a $G$-set and $B$ is a subset of $A$ fixed by some subgroup $H$ of $G$, then we may define a functor $T^B$: $(G, A, R)\text{-gr} \to (H, B, R^{(H)})\text{-gr}$, where $R^{(H)} = \bigoplus_{\sigma \in H} R_\sigma$, by putting $T^B(M) = \bigoplus_{x \in B} M_x$. In Theorem 3.7 we establish that $T^B$ has a left adjoint, denoted by $S^B$; the case $A = G, B = H \subset G$ reduces to the case considered by E. Dade in [2]. The same theorem yields that $T^B$ has a right adjoint too, it will be denoted by $S_B$ and we refer to $S^B$ and $S_B$ as the induction- respectively the coinduction-functor. Particular cases of these induction- and coinduction-functors have been used in the literature before. For strongly graded rings induction and coinduction are isomorphic. The case $B = \{x\}$ and $H = G_x$ is the stabilizer subgroup for $x$ leads to the functors:

$$T^x\colon\ (G, A, R)\text{-gr} \to R^{(G_x)}\text{-mod}, \qquad T^x(M) = M_x$$

$$S^x\colon\ R^{(G_x)}\text{-mod} \to (G, A, R)\text{-gr}, \qquad S^x(N) = R \otimes_{R^{(G_x)}} N.$$

If $R$ is a strongly $G$-graded ring and $A$ is a transitive $G$-set then $(G, A, R)$-gr is equivalent to $R^{(G_x)}$-mod and the equivalence is given by $T^x$ and $S^x$ (or $S_x$), moreover for any subgroup $H$ of $G$, $R^{(H)}$-mod is equivalent to $(G, G/H, R)$-gr. We apply induction and coinduction to prove (Theorem 3.15) that if $R$ is a $G$-graded ring with finite support and $Q \in (G, A, R)$-gr is an injective object of finite support then $Q$ is injective in $R$-mod.

Throughout module means left module and all rings are associative with unit. Our basis reference for detail on graded rings and modules is C. Năstăsescu and F. Van Oystaeyen [5].

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## 1. Preliminaries

Throughout $G$ will be a multiplicative group with identity element 1. A ring $R$ is $G$-graded if $R = \bigoplus_{\sigma \in G} R_\sigma$ where each $R_\sigma$ is an additive subgroup of $R$ and $R_\sigma R_\tau \subset R_{\sigma\tau}$ for all $\sigma$, $\tau \in G$. We write $R$-mod, resp. mod-$R$ for the category of left, resp. right $R$-modules, and $R$-gr for the category of graded (left) $R$-modules. A $G$-set is a non-empty set, $A$ say, together with a left action of $G$ on $A$ given by $G \times A \to A$, $(\sigma, x) \mapsto \sigma x$, such that $1x = x$ for all $x \in A$ and $(\sigma\tau)x = \sigma(\tau x)$ for all $\sigma, \tau \in G$, $x \in A$. A $G$-set morphism $\varphi$: $A \to A'$ is a map such that $\varphi(\sigma x) = \sigma\varphi(x)$ for all $\sigma \in G, x \in A$. If $H$ is a subgroup of $G$ then the set of left cosets $G/H = \{\sigma H, \sigma \in G\}$ with $G$-action given by $\tau(\sigma H) = \tau\sigma H$, for $\tau \in G$, all $\sigma \in G$, is a $G$-set and there is a $G$-set isomorphism $\psi$: $G/H \to G/\sigma H \sigma^{-1}$, $\tau H \mapsto (\tau\sigma^{-1})\sigma H \sigma^{-1}$. For $x \in A$ the $G$-orbit of $x$ is the set $Gx$ and the isotropy subgroup of $G$ at $x$ (the stabilizer of $x$) is $G_x = \{\sigma \in G, \sigma x = x\}$. There is an isomorphism of $G$-sets, $\varphi_x$: $G/G_x \to Gx$, $\sigma G_x \to \sigma x$. If $G_x = \{1\}$ for all $x \in A$ then $A$ is said to be a free $G$-set.

If $Gx=A$ for some $x \in A$ then $G$ acts transitively on $A$, and we then say that $A$ is a transitive $G$-set. If $A$ is a transitive $G$-set, then for any $x$, $y \in A$ there is a $\sigma \in G$ such that $y=\sigma x$. By $G$-SET we denote the category of left $G$-sets with morphisms of $G$-sets as introduced above. In $G$-SET the direct sum of a family $(A_i)_{i \in I}$ by considering the direct sum $\coprod_{i \in I} A_i$ (disjoint union) of the sets $A_i$ with the obvious $G$-action extending the $G$-action on each $A_i$, $i \in I$. Any $G$-set $A$ is a direct sum in $G$-SET of its $G$-orbits and hence $A$ is isomorphic to the direct sum $\coprod_x G/G_x$ where $x$ varies through a set of representatives for the $G$-orbits of $A$.

If $G_1$ and $G_2$ are groups then a $G_1-G_2$-set is a non-empty set $A$ with a left action of $G_1$ and a right action of $G_2$ that are compatible in the following sense: $(\sigma_1 x)\sigma_2 = \sigma_1(x\sigma_2)$ for $\sigma_1 \in G_1$, $\sigma_2 \in G_2$, $x \in A$. On a $G_1-G_2$-set $A$ we may define an equivalence relation $x \sim y$ if and only if $y=\sigma_1 x \sigma_2$ for some $\sigma_1 \in G_1$, $\sigma_2 \in G_2$. The $G_1-G_2$-orbit of $x \in A$ is the equivalence class of $x$ for the relation just introduced, so it is the double coset $G_1 x G_2 = \{\sigma_1 x \sigma_2, \sigma \in G_1, \sigma_2 \in G_2\}$.

In the theory of graded rings and modules, $G$-set gradation appears in a natural way by restricting the gradation to a subgroup $H$ of $G$, viewing the $G$-graded object as an object with a "$G/H$-grading". Of particular interest for applications is the case of strongly graded rings. Recall that a $G$-graded ring $R$ is said to be strongly graded if $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma$, $\tau \in G$.

The basic facts concerning modules graded by a $G$-set are contained in the consequent section; the notions concerning graded modules necessary for the reading of this paper may be distilled from this section because we have to reintroduce these notions in the context of modules graded by $G$-sets.

## 2. Modules Graded by a G-set

Let there be given: a group $G$, a $G$-graded ring $R$ and a $G$-set $A$. A (left) graded $R$-module of type $A$ is a (left) $R$-module $M$ such that $M=\bigoplus_{x \in A} M_x$ where each $M_x$ is an additive subgroup of $M$ and for all $\sigma \in G$, $x \in A$ we have $R_\sigma M_x \subset M_{\sigma x}$. If $M=\bigoplus_{x \in A} M_x$ and $N=\bigoplus_{x \in A} N_x$ are (left) graded $R$-modules of type $A$, then a morphism $f\colon M \to N$ is an $R$-linear map such that $f(M_x) \subset N_x$ for all $x \in A$. The category $(G, A, R)$-gr consists of the (left) graded $R$-modules of type $A$ and the morphisms are the ones just defined. We abuse category theoretical language as usual.

Consider $M=\bigoplus_{x \in A} M_x$ in $(G, A, R)$-gr and a non-zero element $m$ in $M$. Then $m$ has a unique decomposition $m=\sum_{x \in A} m_x$, with $m_x \in M_x$, and the non-zero $m_x$ appearing in this decomposition are called the homogeneous components of $m$. A submodule $N$ of $M$ is said to be a graded submodule (of type $A$) if for any $n \in N$, each homogeneous component of $n$ also belongs to $N$, i.e. $N=\bigoplus_{x \in A}(N \cap M_x)$. If $N$ is a graded submodule of $M$, then $M/N$ is a graded module of type $A$ by putting $(M/N)_x=(M_x+N)/N$ for all $x\in A$. In the category $(G, A, R)$-gr direct sums and products exist; indeed, if $(M_i)_{i\in I}$ is a family of objects of $(G, A, R)$-gr, then the module $\bigoplus_{x\in A}(\bigoplus_{i\in I}(M_i)_x)$, resp. $\bigoplus_{x\in A}(\prod_{i\in I}(M_i)_x)$, is a direct sum, resp. product, of this family in the category $(G, A, R)$-gr. For $M$, $N$ in $(G, A, R)$-gr the set $\mathrm{Hom}_{(G,A,R)\text{-gr}}(M, N)$ is a subgroup of $\mathrm{Hom}_R(M, N)$. Since for any morphism $f$ in $\mathrm{Hom}_{(G,A,R)\text{-gr}}(M, N)$ both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are objects of $(G, A, R)$-gr, it follows that $(G, A, R)$-gr is an abelian category.

If $G=A$ with the natural left action of $G$ on itself, then the category $(G, G, R)$-gr is merely the category $R$-gr. If $H$ is a subgroup of $G$, and $A=G/H$, where $G$ acts by left translation, then we write $(G/H, R)$-gr for the category $(G, G/H, R)$-gr. When $H$ is a normal subgroup of $G$, then $(G/H, R)$-gr may be identified with $R\text{-gr}_{G/H}$, the category of all graded left $R$-modules of type $G/H$ (here $R$ is considered as a $G/H$-graded ring).

If $A$ is a singleton with $G$ acting trivially on it, then the category $(G, A, R)$-gr is nothing but $R$-mod. The elementary properties of $(G, A, R)$-gr we now mention are similar to those of $R$-gr and they may be proved in formally the same way.

**2.1. Proposition** *Let $M$, $N$, $P\in(G, A, R)$-gr fit in the following commutative diagram of $R$-linear maps*

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "f"'] & & N \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

*where $f\in\mathrm{Hom}_{(g,A,R)\text{-gr}}(M, P)$. If $g$, resp. $h$, is a morphism in $(G, A, R)$-gr, then there exists a morphism $h'$, resp. $g'$ in this category, such that $f=g\circ h'$ resp. $f=g'\circ h$. In particular, if $M\in(G, A, R)$-gr is projective, resp. injective in $R$-mod, then $M$ is projective, resp. injective, in $(G, A, R)$-gr.*

*Proof.* Same as Lemma I.2.1 of [5].

**2.2. Proposition.** *Let $A$ be a $G$-set such that each stabilizer $G_x$ for $x\in A$ is normal in $G$ (this condition holds, e.g. if $A$ is a free $G$-set or $G$ is abelian). If $N$ is a graded submodule of $M\in(G, A, R)$-gr, then $N$ is essential in $M$ as an object of $(G, A, R)$-gr, if and only if $N$ is essential in $M$ as an object of $R$-mod.*

*Proof.* Since $(G, A, R)$-gr is equivalent to a product of categories of the type $(G/G_x, R)$-gr (we will prove this in Corollary 2.7), we may restrict ourselves to the case $A=G/H$, where $H$ is a normal subgroup of $G$. In this case, as remarked before, $(G/H, R)$-gr is a category of graded modules in the classical sense, and so, we may apply Lemma I.2.8 of [5].

The referee pointed out the following example in order to show that Proposition 2.2 is not true in general:

*2.3. Example.* Let $G$ be $SL(2, 3)$, which is the semi-direct product $P\times Q$ of a cyclic group $P=\langle\Pi\rangle$ of order 3 with a normal quaternion subgroup $Q$. The center $Z(Q)=\langle\xi\rangle$ of $Q$ has order 2 and is also the center $Z(G)$ of $G$ as well as the centralizer $C_Q(P)$ of $P$ in $Q$. For $R$ we take the group algebra $k[G]$

of $G$ over a field $k$ of characteristic 3. Then $R=k[G]$ is naturally a strongly $G$-graded ring. For $A$ we take the $G$-set $G/H$, where $H$ is the subgroup $P \times Z(Q)$ of $G$. As we shall see in Corollary 3.12, there is a natural equivalence between the categories $(G, A, R)$-gr and $R^{(H)}$-mod, sending any $G/H$-graded $R$-module $M$ to its $H$-component $M_H \in R^{(H)}$-mod, and any $R^{(H)}$-module $L$ to the induced module $R \otimes_{R(H)} L \in (G, A, R)$-gr. In particular, a graded submodule $N$ of any $M \in (G, A, R)$-gr is essential in $M$ as an object of $(G, A, R)$-gr if and only if and only if $N_H$ is an essential $R^{(H)}$-submodule of $M_H$.

The idempotent $e=(1-\xi)/2$ of $k[Z(Q)]$ is primitive central in both $k[H]$ and $k[G]$. The algebras $ek[H]$ and $ek[G]$ are remarkably similar in structure. The former is a truncated polynomial ring $ek[x] \simeq k[X]/(X^3)$ generated over $ek \simeq k$ by the element $x=e(\pi-1)$, which is subjected only to the condition that $x^3=0$. The latter is the $2 \times 2$ matrix ring $M_2(ek[y])$ over the truncated polynomial ring $ek[y] \simeq k[Y]/(Y^3)$ generated over $ek \simeq k$ by the element $y=e(K_\pi-1)$, where $K_\pi$ is the sum of the four $G$-conjugates of $\pi$. The element $eK_\pi$ satisfies $(eK_\pi)^3=-8e \equiv e \pmod 3$. This follows from the character table of $G$. Hence $y$ is subjected only to the condition that $y^3=0$. Thus the algebras $ek[H]$ and $ek[G]$ are Morita equivalent. There are only three isomorphism classes of indecomposable $ek[H]$-module having as representatives $S^1=ek[x]/(x)$, $S^2=ek[x]/(x^2)$ and $S^3=ek[x]/(x^3)$ of $k$-dimensions 1, 2 and 3, respectively. Similarly, there are three isomorphism classes of indecomposable $ek[G]$-modules having as representatives

$$T^1=M_{2\times 1}(ek[y]/(y)], \qquad T^2=M_{2\times 1}[ek[y]/(y^2)] \quad \text{and}$$
$$T^3=M_{2\times 1}[ek[y]/(y^3)]$$

of $k$-dimensions 2, 4 and 6 respectively. We know from the Green correspondence that $R \otimes_{R(H)} S^i \simeq_R I_i \oplus (T^3)^{(m_i)}$, for $i=1, 2, 3$, where $m_i \geqq 0$ and $I_i$ is one of $T^1$, $T^2$, or $T^3$. A check of dimensions tells us that

$$R \otimes_{R(H)} S \simeq T^2 \quad \text{and} \quad R \otimes_{R(H)} S^2 \simeq T^1 \oplus T^3.$$

Now let $M=R \otimes_{R(H)} S^2 \in (G, A, R)$-gr. The socle $\mathrm{soc}(S^2)$ of $S^2$ is $k[H]$-essential in $S^2$ and is isomorphic to $S^1$. It follows that $N=R \otimes_{R(H)}(\mathrm{soc}(S^2))$ is an essential subobject of $M$ in $(G, A, R)$-gr. However $N$ is $R$-isomorphic to $T^2$ and hence cannot be essential in $M \cong T^1 \oplus T^3$. Thus the equivalence of Proposition 2.2 does not hold for this $G$, $A$ and $R$.

Let $A$ and $A'$ be $G$-sets. To a morphism of $G$-sets $\varphi: A \to A'$ we may associate a canonical covariant functor $T_\varphi$: $(G, A, R)$-gr $\to (G, A', R)$-gr, defined as follows: $T_\varphi(M)$ is the $R$-module $M$ with an $A'$-gradation defined by

$$M_{x'}=\bigoplus\{M_x | x \in A, \varphi(x)=x'\} \qquad \text{for } x' \in A',$$

where we put $M_{x'}=0$ if $x' \notin \varphi(A)$. It is clear that, for any $\lambda_\sigma \in R_\sigma$, $\lambda_\sigma M_{x'} =\lambda_\sigma(\bigoplus\{M_x | x \in \varphi^{-1}(x')\}) \subset \bigoplus\{M_{\sigma x} | x \in \varphi^1(x')\}=M_{\sigma x'}$; also $M=\bigoplus_{x' \in A'} M_{x'}$ is obvious. For a morphism $f \in \mathrm{Hom}_{(G, a, R)\text{-gr}}(M, N)$, we put $T_\varphi(f)=f$. Observe that $T_\varphi$ an exact functor.

**2.4. Proposition.** *Consider the morphism of G-sets:* $A \xrightarrow[\varphi]{} A' \xrightarrow[\varphi']{} A''$.

1. $T_{\varphi' \circ \varphi} = T_{\varphi'} \circ T_{\varphi}$.
2. *If* $\varphi$ *is an isomorphism of G-sets, then* $T_{\varphi} \circ T_{\varphi^{-1}} = T_{\varphi^{-1}} \circ T_{\varphi} = \mathrm{Id}$, *in particular* $T_{\varphi}$ *is an isomorphism of categories.*

*Proof.* Straightforward.

Let $G$ and $G'$ be groups and $A$ a $G-G'$-set. For $\tau \in G'$ we may define a morphism of $G$-sets: $\varphi_\tau$: $A \to A, x \mapsto x\tau$ for $x \in A$. Because $\varphi_\tau$ is an isomorphism of $G$-sets, the functor $T_\tau = T_{\varphi_\tau}$ is an isomorphism of categories. In a straightforward way one verifies the following equalities:

i. $T_1 = \mathrm{Id}$.
ii. $T_\tau \circ T_{\tau'} = T_{\tau'\tau}$ for all $\tau, \tau' \in G'$.
iii. $T_\tau \circ T_{\tau^{-1}} = \mathrm{Id}$ for all $\tau \in G'$.

If $M \in (G, A, R)$-gr, then we denote $T_\tau(M)$ by $M(\tau)$ for each $\tau \in G'$, so $M(\tau) = M$ as $R$-modules and for each $x \in A$ we have $M(\tau)_x = M_{x\tau}$. The object $M(\tau)$ is called the $\tau$-suspension of $M$.

*2.5. Examples.* 1. Let $\mathrm{Aut}_G(A)$ be the set of all $G$-automorphism of the left $G$-set $A$. Clearly $\mathrm{Aut}_G(A)$ is a group and $A$ may be viewed as a $G$-$\mathrm{Aut}_G(A)$-set by putting $x\varphi = \varphi^{-1}(x)$ for all $\varphi \in \mathrm{Aut}_G(A)$ and $x \in A$. Indeed, since $\varphi^{-1}(gx) = g\varphi^{-1}(x) = g(x\varphi)$ we have $(gx)\varphi = g(x\varphi)$ and hence $A$ is a $G$-$\mathrm{Aut}_G(A)$-set, as claimed.

2. Let $H$ be a subgroup of $G$ and consider the left $G$-set $A = G/H$. Then $\mathrm{Aut}_G(G/H) \simeq N(H)/H$, where $N(H)$ is the normaliser of $H$ in $G$, $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Indeed, we define $\alpha$: $\mathrm{Aut}_G(G/H) \to N(H)/H$ in the following way: if $\varphi \in \mathrm{Aut}_G(G/H)$, then $\varphi(H) = \sigma_0 H$ for some $\sigma_0 \in G$. We put $\alpha(\varphi) = \varphi^{-1}(H)$. Now $H = \varphi^{-1}(\sigma_0 H) = \sigma_0 \varphi^{-1}(H)$, so $\alpha(\varphi) = \sigma_0^{-1} H$. In order to prove that the definition is correct, we must show that $\sigma_0 \in N(H)$. Indeed, if $\bar{\sigma} = \sigma H$ is an element of $G/H$, then $\varphi(\bar{\sigma}) = \varphi(\sigma H) = \sigma\varphi(H) = \sigma\sigma_0 H$. In particular, if $h \in H$, then $\varphi(H) = \varphi(hH) = h\varphi(H) = h\sigma_0 H$, and thus $\sigma_0 H = h\sigma_0 H$. Hence $h\sigma_0 = \sigma_0 h'$ for some $h' \in H$, and so $\sigma_0^{-1} h\sigma_0 \in H$. It follows that $\sigma_0^{-1} H\sigma_0 \subseteq H$. Similarly, from $\varphi^{-1}(H) = \sigma_0^{-1} H$ it follows that $\sigma_0 H\sigma_0^{-1} \subseteq H$, hence $H \subseteq \sigma_0^{-1} H\sigma_0$. Thus, $H = \sigma_0^{-1} H\sigma_0$ and $\sigma_0 \in N(H)$ follows.

Now, if $\psi \in \mathrm{Aut}_G(G/H)$, with $\psi(H) = \tau_0 H$, $\tau_0 \in G$, we have that $\alpha(\psi) = \tau_0^{-1} H$ and

$$\begin{aligned}\alpha(\varphi\psi) &= (\varphi\psi)^{-1}(H) = \psi^{-1}(\varphi^{-1}(H)) = \psi^{-1}(\sigma_0^{-1} H) = \sigma_0^{-1}\psi^{-1}(H) \\ &= \sigma_0^{-1}\tau_0^{-1} H = \alpha(\varphi)\alpha(\psi)\end{aligned}$$

so $\alpha$ is a group morphism. It is clear that $\alpha$ is an isomorphism.

Let $A$ be a $G-G'$-set. An $R$-linear map $f$: $M \to N$, where $M, N \in (G, A, R)$-gr is said to be a graded morphism of degree $\tau \in G'$ if $f(M_x) \subset N_{x\tau}$ for all $x \in A$. Graded morphisms of degree $\tau \in G'$ form an additive subgroup $\mathrm{HOM}_R(M, N)_\tau$ of $\mathrm{Hom}_R(M, N)$. We denote by $\mathrm{HOM}_R(M, N)$ the subgroup of $\mathrm{Hom}_R(M, N)$ generated by all $\mathrm{HOM}_R(M, N)_\tau$, $\tau \in G'$. It is clear that $\mathrm{HOM}_R(M, N) = \bigoplus_{\tau \in G'} \mathrm{HOM}_R(M, N)_\tau$ is a graded abelian group of type $G'$.

If $f\colon M\to N$ and $g\colon N\to P$ are graded morphisms of degrees $\tau, \tau'\in G'$ resp., then $g\circ f$ has degree $\tau\tau'$. Consequently, we may view $\mathrm{END}_R(M)=\mathrm{HOM}_R(M, M)$ as a graded ring of type $G'$ if we define the multiplication by $f\cdot g=g\circ f$.

**2.6. Proposition.** *Consider a $G$-graded ring $R$ and a family of $G$-sets $(A_i)_{i\in I}$. There the category $(G, \coprod_{i\in I} A_i, R)$-gr is equivalent to the product $\prod_{i\in I}(G, A_i, R)$-gr.*

*Proof.* It is harmless to assume that the sets $A_i$ are mutually disjoint; in this case, $A=\bigcup_{i\in I} A_i$ is a direct sum. Define the functor $T, T\colon (G, A, R)$-gr $\to \prod_{i\in I}(G, A_i, R)$-gr by $T(M)=(M_i)_{i\in I}$, where $M_i=\bigoplus_{x\in A_i} M_x$, and for a morphism in $(G, A, R)$-gr, $f\colon M\to N$ say, we put $T(f)=(f_i)_{i\in I}$, where $f_i$ is the restriction of $f$ to $M_i$. Now we define a functor $S, S\colon \prod_{i\in I}(G, A_i, R)$-gr $\to(G, A, R)$-gr, by $S((M_i)_{i\in I})=\bigoplus_{i\in I} M_i=M$ as $R$-modules, and for $x\in A$, $M_x=(M_i)_x$, where $i$ is such that $A_i$ is the unique one containing $x$ and so $M$ is clearly a graded $R$-module of type $A$. Furthermore, if $(f_i)_{i\in I}$ is a morphism in $\prod_{i\in I}(G, A_i, R)$-gr, then $S((f_i)_{i\in I})=f$, where $f=\bigoplus_{i\in I} f_i$. It is obvious that $S\circ T$ is the identity of $(G, A, R)$-gr and $T\circ S$ is the identity of $\prod_{i\in I}(G, A_i, R)$-gr.

**2.7 Corollary.** *If $R=\bigoplus_{\sigma\in G} R_\sigma$ is a $G$-graded ring and $A$ is a $G$-set, then $(G, A, R)$-gr is equivalent to the product of the $(G/G_x, R)$-gr, where $x$ varies over a set of representatives for the $G$-orbits in $A$. In particular, if $A$ is a free $G$-set, then $(G, A, R)$-gr is a product of copies of $R$-gr over the $G$-orbits in $A$.*

*Proof.* We have $A=\bigcup G_x$ and $G_x\simeq G/G_x$, so the assertion follows from the foregoing proposition.$^x$

Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring and $A$ and $G$-set. For each $x\in A$ we define the $x$-suspension $R(x)$ of $R$ to be the object of $(G, A, R)$-gr which coincides with $R$ as an $R$-module, but with gradation defined by

$$R(x)_y=\bigoplus\{R_\sigma \mid \sigma\in G, \sigma x=y\} \quad \text{for } y\in A.$$

**2.8. Theorem.** *The object $V=\bigoplus_{x\in A} R(x)$ is a projective generator of the category $(G, A, R)$-gr and it is free of rank $|A|$ as an $R$-module. In particular, $(G, A, R)$-gr is a Grothendieck category.*

*Proof.* It is clear that $V$ is a free $R$-module, so we may fix a canonical basis $\{e_x \mid x\in A\}$. Consider $N\subset M$ in $(G, A, R)$-gr such that $N\neq M$, say $M=\bigoplus_{x\in A} M_x\, N=\bigoplus_{x\in A} N_x$. Then there exist $x\in A$, $m_x\in M_x$, $m_x\notin N_x$. Now we may define a morphism in $(G, A, R)$-gr by putting $f(e_x)=m_x$, $f(e_{x'})=0$ for $x'\neq x$ in $A$, and it is clear that $\mathrm{Im}(f)\not\subset N$. Consequently, $V$ is a generator for $(G, A, R)$-gr and it is a projective object of $(G, A, R)$-gr by Proposition 2.1.

**2.9. Corollary.** *An $M\in(G, A, R)$-gr is a projective object if and only if $M$ is a projective left $R$-module.*

*Proof.* The "if"-part follows from Proposition 1.1. For the converse, let $M$ be projective in $(G, A, R)$-gr. Theorem 2.7 implies that $M$ is isomorphic to a direct summand of some direct sum of copies of $V$, which is a projective generator in $(G, A, R)$-gr, hence the assertion follows.

The projective dimension of an $M$ in $(G, A, R)$-gr will be denoted by gr-$p\dim(M)$. As a consequence of Corollary 2.9 we also obtain:

**2.10. Corollary.** *For any $M$ in $(G, A, R)$-gr we have* gr-$p\dim(M)=p\dim(M)$.

In the theory of group actions and group-gradings alike, the notion of the smash-product has recently been used extensively in order to obtain duality results and category equivalences that provided a neat approach to categorical properties of the modules studied. We introduce here a generalization of the notion of smash-product for groups to the $G$-set setting. For this purpose we restrict attention to finite $G$-sets.

Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring, $A$ a finite $G$-set, and define $R\# A$ as follows. It is the free left $R$-module with basis $\{p_x|x\in A\}$ and we define the multiplication by

$$\begin{aligned}(a_\sigma p_x)(b_\tau p_y)&=(a_\sigma b_\tau)p_y &&\text{if } \tau y=x\\ &=0 &&\text{if } \tau y\neq x\end{aligned}\qquad (*)$$

for any $\sigma$, $\tau\in G$, $a_\sigma\in R_\sigma$, $b_\tau\in R_\tau$, $x$, $y\in A$. This may be extended by $\mathbb{Z}$-bilinearity to a product on all of $R\# A=\bigoplus\{R_\sigma p_x|\sigma\in G, x\in A\}$.

**2.11. Proposition.** *The multiplication defined by $(*)$ makes $R\# A$ into a ring with identiy $1=\sum_{x\in A} p_x$ and $\{p_x|x\in A\}$ is a set of orthogonal idempotents. The following properties hold:*

1. *The map $\eta$: $R\to R\# A$, $a\mapsto a\cdot 1=\sum_{x\in A} ap_x$ is an injective ring morphism.*

2. *For all $a_\sigma\in R_\sigma$, $x\in A$: $p_x a_\sigma=a_\sigma p_{\sigma^{-1}x}$ (here $p_x a_\sigma$ denotes the product $P_x\eta(a_\sigma)$ in the ring $R\# A$).*

3. *The set $\{p_x|x\in A\}$ is a basis for $R\# A$ as a right $R$-module.*

4. *For $x\in A$, $p_x$ centralizes $R^{(G_x)}=\bigoplus_{\gamma\in G_x} R_\gamma$.*

5. *Put $H=\bigcap\{G_x|x\in A\}$. Then $H$ is a normal subgroup of $G$ such that for any $r\in R^{(H)}=\bigoplus_{h\in H} R_h$, $x\in A$, we have $rp_x=p_x r$.*

6. $\mathrm{Aut}_G(A)$ *acts on the ring $R\# A$ by $(ap_x)^\alpha=ap_{\alpha(x)}^{-1}$, for $\alpha\in\mathrm{Aut}_G(A)$, $a\in R$, $x\in A$.*

*Proof.* In checking the first claim, the associativity of (*) is the only delicate point. Pick $\rho$, $\sigma$, $\tau \in G$, $a_\rho \in R_\rho$, $b_\sigma \in R_\sigma$, $c_\tau \in R_\tau$ and $x$, $y$, $z \in A$. Then we have

$$\begin{aligned} ((a_\rho p_x)(b_\sigma p_y))(c_\tau p_z) &= (a_\rho b_\sigma c_\tau) p_z \quad \text{if } \sigma y = x \text{ and } \tau z = y \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} (a_\rho p_x)((b_\sigma p_y)(c_\tau p_z)) &= (a_\rho b_\sigma c_\tau) p_z \quad \text{if } \tau z = y \text{ and } \sigma\tau z = xr \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If $\tau z = y$, then the equations $\sigma y = x$ and $\sigma\tau z = x$ are equivalent, hence associativity follows. Now let us verify that $\sum\limits_{x\in A} p_x$ (is the identity of $R\#A$). From $a p_x(\sum\limits_{y\in A} p_y) = \sum\limits_{y\in A} (a p_x) p_y = a p_x$ and $(\sum\limits_{x\in A} p_x) b p_y = \sum\limits_{x\in A} p_x (b p_y) = (\sum\limits_{x\in A} (\sum\limits_{\sigma y = x} b_\sigma)) p_y = b p_y$ (since $b = \sum\limits_{x\in A} (\sum\limits_{\sigma y = x} b_\sigma)$) the claim follows.

The statements 1 and 2 are obvious, also 4 and 5 are easily checked, so let us establish 3 and 6.

3. It is clear from 2. that $R_\sigma p_x = p_{\sigma x} R_\sigma \simeq R_\sigma$. We have

$$\begin{aligned} R\#A &= \bigoplus\{R_\sigma p_x \mid \sigma \in G, x \in A\} = \bigoplus\{p_{\sigma x} R_\sigma \mid \sigma\in G, x\in A\} \\ &= \bigoplus\{p_y R_\sigma \mid \sigma \in G, y\in A\} = \{p_y R \mid y \in A\} \end{aligned}$$

with each $p_y R = \bigoplus\{p_y R_\sigma \mid \sigma\in G\} \simeq \bigoplus\{R_\sigma \mid \sigma \in G\} = R$, so 3. holds.

6. Let $\alpha \in \mathrm{Aut}_G(A)$. Then we calculate for $a_\sigma \in R_\sigma$, $b_\tau \in R_\tau$, $x$, $y \in A$:

$$\begin{aligned} ((a_\sigma p_x)(b_\tau p_y))^\alpha = ((a_\sigma b_\tau) p_y)^\alpha &= (a_\sigma b_\tau) p_{\alpha^{-1}(y)} \quad \text{if } \tau y = x \\ &= 0 \quad \text{if } \tau y \neq 0 \end{aligned}$$

and

$$\begin{aligned} (a_\sigma p_{\alpha^{-1}(x)}(b_\tau p_{\alpha^{-1}(y)}) &= (a_\sigma b_\tau) p\alpha^{-1}(y) \quad \text{if } \tau\alpha^{-1}(y) = \alpha^{-1}(x) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Since $\tau y = x$ if and only if $\tau\alpha^{-1}(y) = \alpha^{-1}(x)$, it follows that $((a_\sigma p_x)(b_\tau p_y))^\alpha = (a_\sigma p_x)^\alpha (b_\tau p_y)^\alpha$. In view of the fact that $a p_x \mapsto (a p_x)^\alpha = a p_{\alpha^{-1}(x)}$ defines a bijective map, it follows that $\mathrm{Aut}_G(A)$ acts on the ring $R\#A$ as described.

*2.12. Remarks.* 1. The ring $R\#A$ is called the smash product of $R$ by the $G$-set $A$. If $A = G$ and $G$ acts on $A$ by left translation, then the multiplication in $R\#A$ becomes $(a p_x)(b p_y) = (a b_{xy^{-1}}) p y$ for $a$, $b \in R$, $x$, $y \in G$. In this case $R\#G$ is exactly the smash product defined by M. Cohen, S. Montgomery in [1].

2. If $A = \coprod\limits_{i\in I} A_i$ is a finite direct sum of finite $G$-sets, then then ring $R\#A$ is isomorphic to the direct product $\prod\limits_{i\in I} R\#A_i$. Perhaps the major observation in this section is the following result, which extends Theorem 2.2 of [1] to the case of arbitrary finite $G$-sets.

**2.13. Theorem.** *If* $R=\bigoplus_{\sigma\in G} R_\sigma$ *is a G-graded ring and A is a finite G-set, then the category* $(G, A, R)$-gr *is isomorphic to the category* $R\# A$-mod.

*Proof.* First consider $M\in R\# A$-mod. Since $R$ may be viewed as a subring of $R\# A$ via the morphism $\eta$ of Proposition 2.11.1, $M$ has a left $R$-module structure by restriction of scalars. For $x\in A$ put $M_x=p_x M$, since $1=\sum_{x\in A} p_x$ and $\{p_x | x\in A\}$ is a family of orthogonal idempotents we obtain that $M=\bigoplus_{x\in A} M_x$. Furthermore, if $a_\sigma\in R_\sigma$ then $a_\sigma M_x=a_\sigma p_x M=(p_{\sigma x} a_\sigma) M$ is contained in $p_{\sigma x} M=M_{\sigma x}$, therefore $M$ is an object of $(G, A, R)$-gr. For a morphism $f\colon M\to N$ in $R\# A$-mod we have $f(M_x)=f(p_x M)=p_x f(M)\subset p_x N=N_x$. In this way we obtain a functor, $(\,)_{\mathrm{gr}}\colon R\# A\text{-mod}\to (G, A, R)$-gr, where $(M)_{\mathrm{gr}}$ is the graded structure of type $A$ defined on $M$ as described above, and $(f)_{\mathrm{gr}}=f$. Conversely, let us now start from an object $M$ of $(G, A, R)$-gr. For $a_\sigma\in R_\sigma$, $x\in A$, put $(a_\sigma p_x) m=a_\sigma m_x$, where $m\in M$ is given by $m=\sum_{x\in A} m_x$. Because $1=\sum_{x\in A} p_x$ we get that $1\cdot m=(\sum_{x\in A} p_x) m = \sum_{x\in A} m_x=m$. If $\sigma, \tau\in G$, $a_\sigma\in R_\sigma$, $b_\tau\in R_\tau$, $x, y\in A$ then we calculate.

$$\begin{aligned} ((a_\sigma p_x)(b_\tau p_y))\, m &= ((a_\sigma b_\tau) p_y)\, m = a_\sigma b_\tau m_y && \text{if } \tau y = x \\ &= 0 && \text{if } \tau y \neq x. \end{aligned}$$

On the other hand:

$$\begin{aligned} (a_\sigma p_x)((b_\tau p_y)\, m) &= (a_\sigma p_x)(b_\tau m_y) = a_\sigma b_\tau m_y && \text{if } \tau y = x \\ &= 0 && \text{if } \tau y \neq x. \end{aligned}$$

It follows that we can consider $M$ as an $R\# A$-module. If $f\colon M\to N$ is a morphism in $(G, A, R)$-gr, then $f((a p_x)\, m)=f(a m_x)=a f(m_x)=a p_x f(m)$, because $f(M_x)\subset N_x$. Thus we arrive at a functor $(-)^{\#}\colon (G, A, R)\text{-gr}\to R\# A$-mod, where $M^{\#}$ is the $R$-module $M$ equipped with the structure of $R\# A$-module defined above, and $f^{\#}=f$ for each morphism in the category $(G, A, R)$-gr. If $M\in (G, A, R)$-gr, then $am=(a\cdot 1)\, m=(\sum_{x\in A} a p_x)\, m=\sum_{x\in A} a m_x=a(\sum_{x\in A} m_x)=am$ holds in $M^{\#}$. Therefore $(-)_{\mathrm{gr}}\circ(-)^{\#}$ is the identity. Conversely, if $M\in R\# A$-mod is considered as an object of $(G, A, R)$-gr with the grading $M\bigoplus_{x\in A} M_x$, where $M_x=p_x M$, then $(a p_x)\, m=a m_x$ holds for $a\in R$, $x\in A$, $m\in M$. Since $m=\sum_{x\in A} p_x m$, $m_x=p_x m$, and therefore $(a p_x)\, m=a(p_x m)$. It follows that $(-)^{\#}\circ(-)$-gr is the identity functor.

The second part of this section is devoted to obtaining a characterization of smash products in terms of matrix rings, and properties deriving from this description.

Let $A$ be a finite $G$-set. Then by Theorem 2.8 $V=\bigoplus_{x\in A} R(x)$ is a projective generator of $(G, A, R)$-gr. A result of B. Mitchell, cf. [8], then states that $(G, A, R)$-gr is equivalent to the category mod-$S$ of right $S$-modules, where $S$

$=\mathrm{End}_{(G,A,R)\text{-gr}}(V)$. On the other hand, by Theorem 2.13, the category $(G, A, R)$-gr is equivalent to the category $R\#A$-mod. It follows that $S$ is Morita equivalent to the opposite ring $(R\#A)^{\mathrm{opp}}$. However, we can prove a stronger result:

**2.14. Theorem.** *Let $R$ be a $G$-graded ring and $A=\{x_1, \ldots, x_s\}$ a finite $G$-set. The rings $(R\#A)^{\mathrm{opp}}$, $S=\mathrm{End}_{(G,A,R)-\mathrm{gr}}(V)$ and $T$, as defined below, are isomorphic*

$$T=\begin{pmatrix} R(x_1)_{x_1} & R(x_1)_{x_2} & \cdots & R(x_1)_{x_s} \\ R(x_2)_{x_1} & R(x_2)_{x_2} & \cdots & R(x_2)_{x_s} \\ \vdots & \vdots & & \vdots \\ R(x_s)_{x_1} & R(x_s)_{x_2} & \cdots & R(x_s)_{x_s} \end{pmatrix}.$$

*Proof.* We recall that for each $x\in A$, $R(x)$ is the object of $(G, A, R)$-gr which is equal to $R$ as an $R$-module and has gradation given by $R(x)_y=\bigoplus\{R_\sigma \mid \sigma\in G, \sigma x=y\}$. Theorem 2.13 establishes an equivalence $R\#A\text{-mod}\approx(G, A, R)$-gr. We apply this equivalence to the regular $R\#A$-module $R\#A$. As a left $R\#A$-module this is the direct sum of its submodules $(R\#A)\,p_x=Rp_x$, each of which is a regular left $R$-module. The $A$-grading of $(R\#A)\,p_x$ is given by:

$$((R\#A)\,p_x)_y=p_y(R\#A)\,p_x=p_x=p_y\,R\,p_x=\sum_{\sigma\in G} p_y\,R_\sigma\,p_x$$
$$=\bigoplus\{R_\sigma\,p_x \mid \sigma\in G, \sigma x=y\}.$$

Thus $(R\#A)\,p_x\simeq R(x)$ as objects of $(G, A, R)$-gr. Hence $R\#A=\bigoplus_{x\in A}(R\#A)\,p_x \simeq\bigoplus_{x\in A} R(x)=V$ as objects of $(G, A, R)$-gr. In view of Theorem 2.13, this implies that

$$S=\mathrm{End}_{(G,A,R)\text{-gr}}(V)\simeq\mathrm{End}_{(G,A,R)\text{-gr}}(R\#A)\simeq\mathrm{End}_{R\#A}(R\#A)\simeq(R\#A)^{\mathrm{opp}}.$$

Now if $M\in(G, A, R)$-gr, $M=\bigoplus_{i=1}^{n} M_i$, then

$$\mathrm{End}_{G,A,R)\text{-gr}}(M)$$
$$\simeq\begin{pmatrix} \mathrm{Hom}_{(G,A,R)\text{-gr}}(M_1, M_2) & \mathrm{Hom}_{(G,A,R)\text{-gr}} & \cdots\ \mathrm{Hom}_{(G,A,R)\text{-gr}}(M_n, M_1) \\ \vdots & \vdots & \vdots \\ \mathrm{Hom}_{(G,A,R)\text{-gr}}(M_1, M_2) & \mathrm{Hom}_{(G,A,R)\text{-gr}}(M_2, M_n) & \cdots\ \mathrm{Hom}_{(G,A,R)\text{-gr}}(M_n, M_n) \end{pmatrix}$$

the isomorphism sending each $u\in\mathrm{End}_{(G,A,R)\text{-gr}}(M)$ to the matrix $(u_{ij})_{1\leqq i,j\leqq n}$, $u_{ij}=q_i\circ u\circ p_j$, where $q_i$: $M\to M_i$ and $p_j=M_j\to M$ denote the canonical projections and injections, respectively. Since it may be easily checked that

$$\mathrm{Hom}_{(G,A,R)\text{-gr}}(R(x_i), R(x_j))\simeq R(x_j)_{x_i}$$

the desired isomorphism between $S$ and $T$ follows.

In order to give some applications of Theorem 2.14, we fix notations as follows: we let $A=G/H$, where $H$ is a subgroup of finite index. To the canonical $G$-set morphism $\varphi\colon G\to G/H$ we correspond the canonical functor $T_\varphi$: $R$-gr $\to(G/H, R)$-gr, as defined earlier. If $M\in R$-gr then we put $\mathrm{END}_R(M)=\bigoplus_{\sigma\in G}\mathrm{END}_R(M)_\sigma$, where $\mathrm{END}_R(M)_\sigma=\{f\in\mathrm{Hom}_R(M, M)\mid f(M_\lambda)\subset M_{\lambda\sigma}$ for all $\lambda\in G\}$. Clearly, $\mathrm{END}_R(M)$ is a $G$-graded ring and its multiplication is given by $f\cdot g=g\circ f$ for $f, g\in\mathrm{END}_R(M)$. Put $M^*=T_\varphi(M)$. Then $M^*$ is an object of $(G/H, R)$-gr which is equal to $M$ as an $R$-module but graded as follows: $M^*=\bigoplus_{c\in G/H}M_c$, where $M_c=\bigoplus_{\sigma\in c}M_\sigma$. We have the following:

**2.15. Proposition.** *Suppose either that $M\in R$-gr is a finitely generated object or that $G$ is finite; then $\mathrm{END}_R(M)^{(H)}=\mathrm{End}_{(G/H, R)\text{-gr}}(M^*)$.*

*Proof.* For $f\in\mathrm{END}_R(M)_\sigma$, $\sigma\in H$, and any $c\in G/H$:

$$f(M_c)=f\Big(\bigoplus_{\lambda\in C}M_\lambda\Big)\subset\bigoplus_{\lambda\in c}M_{\lambda\sigma}=M_c.$$

Hence we obtain:

$$\mathrm{END}_R(M)^{(H)}\subset\mathrm{End}_{(G/H, R)\text{-gr}}(M^*).$$

Now consider $f\in\mathrm{End}_{(G/H, R)\text{-gr}}(M^*)$. In the cases considered here, $\mathrm{END}_R(M)=\mathrm{End}_R(M)$ (cf. Corollary I.2.11 of [5]) hence $\mathrm{End}_{(G/H, R)\text{-gr}}(M^*)\subset\mathrm{END}_R(M)$.

So we may decompose $f=f_{\sigma_1}+f_{\sigma_2}+\dots+f_{\sigma_s}$, where $f_{\sigma_i}$ is a non-zero morphism of degree $\sigma_i$. We claim that $\sigma_1,\dots,\sigma_s\in H$. Indeed, since $f_{\sigma_1}\neq 0$, there is a $c\in G/H$ such that $f_{\sigma_1}(M_c)\neq 0$. Put $c=\sigma H$. There exists an element $m\in M_{\sigma h}, h\in H$, $m\neq 0$, such that $f_{\sigma_1}(m)\neq 0$. But $f(m)=f_{\sigma_1}(m)+\dots+f_{\sigma_s}(m)$, and $f_{\sigma_i}(m)\in M_{\sigma h\sigma_i}$ for $i=1,\dots,s$. Since $\sigma h\sigma_i\neq\sigma h\sigma_j$ for $i\neq j$ and $f(m)\in f(M_c)\subset M_c$ then $f_{\sigma_1}(m)\in M_{\sigma h'}$ for some $h'\in H$. Thus $\sigma h\sigma_1=\sigma h'$ and so $\sigma_1=h^{-1}h'\in H$. Similarly $\sigma_2,\dots,\sigma_s\in H$, and finally we arrive at $f\in\mathrm{END}_R(M)^{(H)}$.

If $H<G$ is a subgroup of finite index $s, s<\infty$, we let $\{\sigma_1,\dots,\sigma_s\}$ be a set of representatives for the left cosets of $H$ in $G$. If we put $V=\bigoplus_{i=1}^{s}R(\sigma_i H)$, then $R(\sigma_i H)_{\sigma_j H}=\bigoplus\{R_\sigma\mid\sigma\in G,\ \sigma\sigma_i H=\sigma_j H\}=R^{(\sigma_j H\sigma_i^{-1})}=R(\sigma_i^{-1})^{(\sigma_j H)}$. It follows that $R(\sigma_i H)=\bigoplus_{j=1}^{s}R(\sigma_i^{-1})^{(\sigma_j H)}=T_\varphi(R(\sigma_i^{-1}))$, and so $V\simeq T_\varphi\left(\bigoplus_{i=1}^{s}R(\sigma_i^{-1})\right)$ is a small projective generator of $(G/H, R)$-gr, by Theorem 2.8. In this case, Theorem 2.14 becomes.

**2.16. Corollary.** *With notations as above, the rings $(R\# G/H)^{\mathrm{opp}}$, $S=\mathrm{End}_{(G/H, R)\text{-gr}}(V)$ and $T$ as defined below*

$$T=\begin{pmatrix} R^{(\sigma_1 H\sigma_1^{-1})} & R^{(\sigma_2 H\sigma_1^{-1})} & \dots & R^{(\sigma_1 H\sigma_1^{-1})}\\ R^{(\sigma_1 H\sigma_2^{-1})} & R^{(\sigma_2 H\sigma_2^{-1})} & \dots & R^{(\sigma_s H\sigma_2^{-1})}\\ \vdots & \vdots & & \vdots\\ R^{(\sigma_1 H\sigma_s^{-1})} & R^{(\sigma_2 H\sigma_s^{-1})} & \dots & R^{(\sigma_s H\sigma_s^{-1})}\end{pmatrix}$$

*are isomorphic.*

If $G$ is a finite group, then if we put $U=\bigoplus_{\sigma\in G} R(\sigma)$, we have that $T_\varphi(U)$ is another small projective generator of $(G/H, R)$-gr. Thus, by the Theorem of Mitchell referred to before, $(G/H, R)$-gr is equivalent to mod-$S'$, where $S' = \mathrm{End}_{(G/H,R)\text{-gr}}(T_\varphi(U))$. But by Proposition 2.15, $S'=\mathrm{END}_R(U)^{(H)}$, and by a result of Năstăsescu, Rodino, cf. [6], $\mathrm{END}_R(U)\simeq(R\# G)*G$, where $R\# G$ is the usual smash product (see also [1, 7]). Thus $\mathrm{END}_R(U)^{(H)}\simeq(R\# G)\,H$, the latter denoting the skew group ring of $H$ over the ring $R\# G$. We have established:

**2.17. Corollary.** *If $G$ is a finite group and $H$ is a subgroup, then the rings $(R\# G/H)^{\mathrm{opp}}$ and $(R\# G)*H$ are Morita equivalent. Further information on the ring $(R\# G)*H$ is given by*:

**2.18. Corollary.** *If $G$ is a finite group and $H$ is a subgroup, then*

$$(R\# G)*H\simeq M_{(H)}(R\# G/H).$$

*Proof.* With notation as above, we observe that $T_\varphi(R(\sigma))\simeq T_\varphi(R(\tau))$ whenever $\sigma,\tau\in H$ lie in the same coset $\sigma H=\tau H$ of $H$. It follows that $T_\varphi(U)$ is isomorphic to a direct sum of $|H|$ copies of $V=T_\varphi\left(\bigoplus_{i=1}^{s} R(\sigma_i^{-1})\right)$. Hence $(R\# G)*H$ is anti-isomorphic to $\mathrm{END}_R(U)$, which is isomorphic to $M_{|H|}\left(\mathrm{END}_R\left(\bigoplus_{i=1}^{s} R(\sigma_i^{-1})^{(H)}\right)\right)$. But by Theorem 2.14 and Proposition 2.15 we get that this last ring is isomorphic to $M_{|H|}(R\# G/H)$.

**2.19. Corollary.** *If $R=\bigoplus_{\sigma\in G} R_\sigma$ is strongly $G$-graded and $H$ is a subgroup of finite index in $G$, then $R\# G/H$ and $R$ are Morita equivalent.*

*Proof.* Since $R$ is strongly graded, it follows that ${}_RR$ is a generator in $R$-gr (this follows e.g. from Theorem I.3.4.c of [5]). So $T_\varphi({}_RR)$ is a projective generator in $(G/H, R)$-gr. Consequently, $(G/H, R)$-gr is equivalent to mod-$S$, where $S = \mathrm{End}_{(G/H,R)\text{-gr}}(T_\varphi({}_RR))=\mathrm{END}_R({}_RR)^{(H)}$. Since $\mathrm{END}_R({}_RR)$ is anti-isomorphic to $R$ as graded rings, it follows that $S$ is anti-isomorphic to $R^{(H)}$, and thus $R\# G/H$ is Morita equivalent to $R^{(H)}$.

**2.20. Corollary.** *If $R$ a crossed product and $H$ has finite index $s=[G:H]<\infty$, then*

$$R\# G/H\simeq M_s(R^{(H)}).$$

*Proof.* Let $\{\sigma_1, \dots, \sigma_s\}$ be a left transversal for $H$ in $G$. By Theorem 2.14 we have that $(R\# G/H)^{(\mathrm{opp})}\simeq T$, $T\simeq\mathrm{End}_{(G/H,R)\text{-gr}}(V)$, $V=T_\varphi\left(\bigoplus_{i=1}^{s} R(\sigma_i^{-1})\right)$. Since $R$ is a crossed product, we have that $R=R(\sigma)$ in $R$-gr for all $\sigma\in G$. Therefore $\bigoplus_{i=1}^{s} R(\sigma_i^{-1})\simeq R^s$, hence $V\simeq(T_\varphi({}_RR))^s$ and it follows also that

$$T\simeq\mathrm{End}_{(G/H,R)\text{-gr}}((T_\varphi({}_RR))^s)\simeq M_s(\mathrm{End}_{(G/H,R)\text{-gr}}(T_\varphi({}_RR))).$$

But $\mathrm{End}_{(G/H,R)\text{-gr}}(T_\varphi({}_RR))=\mathrm{End}_R(R)^H$ is anti-isomorphic to $R^{(H)}$ and so $M_s(R^{(H)})$ is anti-isomorphic to $T$. It follows that $R\# G/H\simeq M_s(R^{(H)})$.

## 3. Some Functors

In [2], E. Dade constructed several functors associated to a $G$-set $A$ of the form $G/H$, where $H$ is a subgroup of $G$. Using these ideas and combining them with the idea of considering the functor ${}^{G}(-)$ (see [5], p. 4) the construction of these functors may be carried out for arbitrary $G$-sets.

Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring and let $\varphi$: $A\to A'$ be a morphism of $G$-sets. As in the foregoing section we consider the canonical functor $T_\varphi$: $(G, A, R)$-gr $\to (G, A', R)$-gr.

**3.1. Theorem.** *The functor $T_\varphi$ has a right adjoint $S^\varphi$ and the latter is an exact functor. Moreover, if $\varphi^{-1}(x')$ is a finite set for all $x'\in A'$, then $S^\varphi$ is also a left adjoint for $T_\varphi$.*

*Proof.* For any $R$-module $N$ we denote by $N[A]$ the additive group which is the direct sum $\bigoplus_{x\in A} {}^{x}N$ of copies ${}^{x}N$ of the additive group of $N$. If $x\in A$ and $n\in N$, then ${}^{x}n$ will denote the natural image of $n$ in the subgroup ${}^{x}N$ of $N[A]$. We make $N[A]$ into an $A$-graded $R$-module by setting, $N[A]_x={}^{x}N$ for all $x\in A$ and $\lambda_\sigma\cdot{}^{x}n={}^{\sigma x}(\lambda_\sigma n)$ for all $x\in A$, $\sigma\in G$, $\lambda_\sigma\in R_\sigma$ and $n\in N$. Suppose that $N\in(G, A', R)$-gr. Then $N$ is also an $R$-module and $N[A]\in(G, A, R)$-gr. We define an additive subgroup $S^\varphi(N)$ of $N[A]$ by $S^\varphi(N)=\bigoplus_{x\in A} {}^{x}N_{\varphi(x)}$.

In fact, $S^\varphi(N)$ is a subobject of $N[A]$ in $(G, A, R)$-gr. In particular, $S^\varphi(N)\in (G, A, R)$-gr with $S^\varphi(N)_x={}^{x}(N_{\varphi(x)})$, for all $x\in A$. To check this, look at $\lambda_\sigma\in R_\sigma$, $n\in N_{\varphi(x)}$ and calculate $\lambda_\sigma\cdot{}^{x}n={}^{\sigma x}(\lambda_\sigma n)$. Since $\lambda_\sigma x\in\lambda_\sigma N_{\varphi(x)}\subset N_{\sigma\varphi(x)}=N_{\varphi(\sigma x)}$, we obtain that ${}^{\sigma x}(\lambda_\sigma x)=\lambda_\sigma\cdot{}^{x}n\in{}^{\sigma x}(N_{\varphi(\sigma x)})$, so $S^\varphi(N)$ is a subobject of $N[A]$ in $(G, A, R)$-gr, as claimed. If $f$: $N\to N'$ is a morphism in $(G, A', R)$-gr, then we define $f[A]$: $N[A]\to N'[A]$, ${}^{x}n\to{}^{x}(f(n))$, and the latter is clearly a morphism in $(G, A, R)$-gr. From $f(N_{\varphi(x)})\subset N'_{\varphi(x)}$ it follows that $f[A]\,(S^\varphi(N))\subset S^\varphi(N')$ and so we may define $S^\varphi(f)$ to be the restriction of $f[A]$ to $S^\varphi(N)$. Exactness of the functor $S^\varphi$ is clear. Let us establish that $S^\varphi$ is a right adjoint for $T_\varphi$. Consider $M\in(G, A, R)$-gr, $N\in(G, A', R)$-gr, and define the canonical morphism

$$\alpha\colon \mathrm{Hom}_{(G, A', R)\text{-gr}}(T_\varphi(M), N)\to\mathrm{Hom}_{(G, A, R)\text{-gr}}(M, S^\varphi(N))$$

in the following way:

For $u$: $T_\varphi(M)\to N$ we have $u(T_\varphi(M)_{x'})\subset N_{x'}$ for all $x'\in A'$, hence $u(\bigoplus\{M_x|\varphi(x)=x'\})\subset N_{\varphi(x)}$ or $u(M_x)\subset N_{\varphi(x)}$ for all $x\in A$. So we may define $\alpha(u)\,(m)=\sum_{x\in A} {}^{x}u(m_x)$ in $S^\varphi(N)$, where $m=\sum_{x\in A} m_x\in M$.

Conversely, we let $\psi$: $N[A]\to N$ denote the natural $R$-morphism sending ${}^{x}n$ to $n$ for any $n\in N$, and we define the canonical morphism:

$$\beta\colon \mathrm{Hom}_{(G, A, R)\text{-gr}}(M, S^\varphi(N))\to\mathrm{Hom}_{(G, A', R)\text{-gr}}(T_\varphi(M), N)$$

as follows: for $v\in\mathrm{Hom}_{(G, A, R)\text{-gr}}(M, S^\varphi(N))$ we have $v(M_x)\subset(S^\varphi(N))_x={}^{x}N_{\varphi(x)}$, therefore $\psi v(M_{x'})=\psi v(\bigoplus\{M_x|\varphi(x)=x'\})\subset N_{\varphi(x)}=N_{x'}$. Thus we put $\beta(v)=\psi\circ v$. That $\alpha$ and $\beta$ are inverse to each other may be easily verified.

Suppose now that $\varphi^{-1}(x')$ is a finite set for all $x' \in A'$, and let us show that in this case $S^\varphi$ is also a left adjoint for $T_\varphi$. Let $M \in (G, A, R)$-gr, and $N \in (G, A', R)$-gr, $M = \bigoplus_{x \in A} M_x$, $N = \bigoplus_{x' \in A'} N_{x'}$. We can define the morphisms:

$$\gamma\colon \operatorname{Hom}_{(G,A,R)\text{-gr}}(S^\varphi(N), M) \to \operatorname{Hom}_{(G,A',R)\text{-gr}}(N, T_\varphi(M))$$

by butting for each $u \in \operatorname{Hom}_{(G,A,R)\text{-gr}}(S^\varphi(N), M)$ and $n_{x'} \in N_{x'}$

$$\gamma(u)(n_{x'}) = u\Big(\sum_{\substack{x \in A \\ \varphi(x) = x'}} {}^x n_{x'}\Big)$$

and

$$\delta\colon \operatorname{Hom}_{(G,A',R)\text{-gr}}(N, T_\varphi(M)) \to \operatorname{Hom}_{(G,A,R)\text{-gr}}(S^\varphi(N), M)$$

by putting for each $v \in \operatorname{Hom}_{(G,A',R)\text{-gr}}(N, T_\varphi(M))$ and ${}^x n_{\varphi(x)} \in (S^\varphi(N))_x$, $\delta(v)({}^x n_{\varphi(x)}) = v(n_{\varphi(x)})_x$. Let now $v \in \operatorname{Hom}_{(G,A',R)\text{-gr}}(N, T_\varphi(M))$ and $n_{x'} \in N_{x'}$. Then

$$\begin{aligned}(\gamma \circ \delta)(v)(n_{x'}) &= \gamma(\delta(v))(n_{x'}) = \delta(v)\Big(\sum_{\substack{x \in A \\ \varphi(x) = x'}} {}^x n_{x'}\Big) = \sum_{\substack{x \in A \\ \varphi(x) = x'}} \delta(v)({}^x n_{x'}) \\ &= \sum_{\substack{x \in A \\ \varphi(x) = x'}} v(n_{x'})_x = v(n_{x'}).\end{aligned}$$

So

$$\gamma \circ \delta = 1_{\operatorname{Hom}_{(G,A',R)\text{-gr}}}(N, T_\varphi(M)).$$

Let $u \in \operatorname{Hom}_{(G,A,R)\text{-gr}}(S^\varphi(N), M)$ and ${}^x n_{\varphi(x)} \in {}^x N_{\varphi(x)} = (S^\varphi(N))_x$. Then

$$(\delta \circ \gamma)(u)({}^x n_{\varphi(x)}) = (\gamma(u)(n_{\varphi(x)}))_x = \Big(\sum_{\substack{y \in A \\ \varphi(y) = \varphi(x)}} u({}^y n_{\varphi(x)})\Big)_x = u({}^x n_{\varphi(x)}),$$

so $\delta \circ \gamma = 1_{\operatorname{Hom}_{(G,A,R)\text{-gr}}}(S^\varphi(N), M)$.

*3.2. Remarks.* 1. If $\varphi\colon A \to A'$ has the property that $\varphi^{-1}(x')$ is a finite set for all $x' \in A'$ then there is a canonical $R$-morphism $\alpha\colon M \to T_\varphi(S^\varphi(N))$, such that $\alpha(N_{x'}) \subset \bigoplus \{{}^x N_{x'} \mid \varphi(x) = x'\}$. Indeed, if $n_{x'} \in N_{x'}$, put $\alpha(n_{x'}) = \sum_{\substack{x \in A \\ \varphi(x) = x'}} {}^x n_{x'}$.

2. If $K \subset H \subset G$ are subgroups of $G$ and $\varphi\colon G/K \to G/H$, $\sigma K \mapsto \sigma H$, is the canonical morphism of $G$-sets, then the above construction of $S^\varphi$ reduces to E. Dade's construction given in [2], where it is shown that $S^\varphi$ is a right adjoint for $T_\varphi$ in this particular case.

**3.3. Corollary.** *If $A$ is a finite $G$-set, then $Q \in (G, A, R)$-gr is an injective object in this category if and only if $Q$ is an injective $R$-module.*

*Proof.* The "if"-part follows from Proposition 2.1. For the converse, let $x \in A$ and $\varphi\colon A \to \{x\}$, $\varphi(y) = x$ for all $y \in A$. Then $T_\varphi\colon (G, A, R)\text{-gr} \to (G, \{x\}, R)\text{-gr} \approx R\text{-mod}$ is the functor which "forgets" the graded structure. By Theorem 3.1 this functor has an exact left adjoint, and so it preserves injectivity.

**3.4. Corollary.** [9] *If $G$ is a finite group, then $Q \in R$-gr is gr-injective if and only if $Q$ is injective in $R$-mod.*

**3.5. Corollary.** *If $H \subset G$ is a subgroup of finite index, then $Q \in (G/H, R)$-gr is injective in this category if and only if $Q$ is an injective $R$-module.*

**3.6. Corollary.** *If $K \subset H \subset G$ are subgroups of $G$, and $\varphi\colon G/K \to G/H$, $\sigma K \mapsto \sigma H$, is the canonical morphism of $G$-sets, $K$ has finite index in $H$, and $Q \in (G/K, R)$-gr is an injective object, then $T_\varphi(Q)$ is an injective object in the category $(G/H, R)$-gr.*

Let $A$ be a $G$-set, $H$ a subgroup of $G$. Assume that $B$ is a subset of $A$ such that $\sigma B \subset B$ for all $\sigma \in H$. Then define the functor $T^B$: $(G, A, R)$-gr $\to (H, B, R^{(H)})$-gr, by putting $T^B(M) = M^{(B)} = \bigoplus_{x \in B} M_x$. If $f\colon M \to N$ is a morphism in $(G, A, R)$-gr, we put $T^B(f) = f^{(B)} = f|_{M^{(B)}}$.

**3.7. Theorem.** *The functor $T^B$ has a left adjoint $S^B$ and a right adjoint $S_B$. If $B \subset A$ and $H$ is a subgroup of $G$ such that $\sigma x = y$ with $x, y \in B$ implies that $\sigma \in H$, then $T^B \circ S^B = T^B \circ S_B =$ the identity of $(H, B, R^{(H)})$-gr.*

*Proof.* We show first the existence of a left adjoint for $T^B$. To $N$ in $(N, B, R^{(H)})$-gr we correspond $R \otimes_{\mathbf{Z}} N \in R$-mod where the $R$-module structure is given by: $r(\lambda \otimes x) = r\lambda \otimes n$, for $r \in R$, $\lambda \in R$, $n \in N$. As additive groups: $R \otimes_{\mathbf{Z}} N \simeq \bigoplus_{\sigma \in G} \bigoplus_{y \in B} R_\sigma \otimes N_y$. Put $(R \otimes_{\mathbf{Z}} N)_x = \bigoplus \{R_\sigma \bigotimes N_y \mid \sigma \in G, \gamma \in B, \sigma y = x\}$ (convention: direct sum of an empty family is zero). Clearly, $R \otimes_{\mathbf{Z}} N = \bigoplus_{x \in A} (R \otimes_{\mathbf{Z}} N)_x$ as additive groups. If $\lambda_\tau \otimes n_y$ is in $(R \otimes_{\mathbf{Z}} N)_x$, then $\tau y = x$. If $r_\sigma \in R_\sigma$, then we have $r_\sigma(\lambda_\tau \otimes n_y) = r_\sigma \lambda_\tau \otimes n_y$ and $(\sigma\tau) y = \sigma(\tau y) = \sigma x$. Therefore $R_\sigma(R \otimes_{\mathbf{Z}} N)_x \subset (R \otimes_{\mathbf{Z}} N)_{\sigma x}$ and hence $R \otimes_{\mathbf{Z}} N$ is an object of $(G, A, R)$-gr. Next consider the natural epimorphism $\phi\colon R \otimes_{\mathbf{Z}} N \to R \otimes_{R^{(H)}} N$, where $K = \operatorname{Ker}(\phi)$ is the $R$-submodule generated by the elements $\{a\lambda \otimes n - a \otimes \lambda n \mid a \in R, \lambda \in R^{(H)}, n \in N\}$. Each such generator may be decomposed as a sum of elements of the same form, but with $a$, $\lambda$, $n$ being homogeneous; so $K$ is a graded $R$-submodule of $R \otimes_{\mathbf{Z}} N$ and therefore $R \otimes_{R^{(H)}} N$ is $A$-graded by putting $(R \otimes_{R^{(H)}} N)_x = \phi((R \otimes_{\mathbf{Z}} N)_x)$. It now makes sense to define $S^B\colon (H, B, R^{(H)})$-gr $\to (G, A, R)$-gr, by $S^B(N) = R \otimes_{R^{(H)}} N$. To a morphism $f\colon N \to N'$ in $(H, B, R^{(H)})$-gr we correspond the $R$-morphism $1 \otimes f$ and since the latter acts well on the generators of $K$ it yields a unique morphism: $S^B(f)\colon R \otimes_{R^{(H)}} N \to R \otimes_{R^{(H)}} N$, such that $S^B(f)(\lambda \otimes n) = \lambda \otimes f(n)$. In order to establish that $S^B$ is a left adjoint for $T^B$, consider $M \in (G, A, R)$-gr and $N \in (H, B, R^{(H)})$-gr, and define $\alpha$, $\beta$

$$\operatorname{Hom}_{(G,A,R)}\text{-gr}(S^B(N), M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \operatorname{Hom}_{(H,B,R^{(H)})\text{-gr}}(N, T^B(M))$$

in the following way: to $u \in \operatorname{Hom}_{(G,A,R)\text{-gr}}(S^B(N), M)$ we correspond $\alpha(u)$ given by $\alpha(u)(n) = u(1 \otimes n)$.

If $\lambda \in R^{(H)}$, we have $\alpha(u)(\lambda n) = u(1 \otimes \lambda n) = u(\lambda \otimes n) = u(\lambda(1 \otimes n)) = \lambda u(1 \otimes n) = \lambda\alpha(u)(n)$. On the other hand, if $n \in N_y$, $y \in B$, then $1 \otimes n \in (R \otimes_{R^{(H)}} N)_y$ and hence $\alpha(u)(n) \in M_y$, i.e. $\alpha(u)(Ny) \subset T^B(M)_y$. Thus $\alpha$ is well-defined. To $v \in \operatorname{Hom}_{(H,B,R^{(H)})\text{-gr}}(N, T^B(M))$ we correspond $\beta(v)\colon R \otimes_{R^{(H)}} N \to M$ given by

$\beta(v)(\lambda \otimes n) = \lambda v(N)$. If $x \in A$ and there are no $y \in B$, $\sigma \in G$ such that $\sigma y = x$, then $(R \otimes_{R^{(H)}} N)_x = 0$. Otherwise, let $y \in B$, $\sigma \in G$ be such that $\sigma y = x$ and let $\lambda \in R_\sigma$, $n \in N_y$. Then $v(n) \in T^B(M)_y = M_y$, and $\lambda v(n) \in R_\sigma M_y \subset M_{\sigma y} = M_x$. Hence $\beta$ is well-defined too. Now if $u \in \mathrm{Hom}_{(G,A,R)\text{-gr}}(S^B(N), M)$ then we have: $\beta(\alpha(u))(\lambda \otimes n) = \lambda \alpha(u)(n) = \lambda u(1 \otimes n) = u(\lambda \otimes n)$ hence $\beta \circ \alpha$ is the identity on $\mathrm{Hom}_{(G,A,R)\text{-gr}}(S^B(N), M)$. If $v \in \mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(N, T^B(M))$ then we obtain

$$\alpha(\beta(v))(n) = \beta(v)(1 \otimes n) = v(n),$$

hence $\alpha \circ \beta$ is the identity of $\mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(N, T^B(M))$. Assume now that $H$ is as in the last part of the statement and let $N \in (H, B, R^{(H)})$-gr. We have the functorial morphism $\psi(N)$: $N \to (T^B \circ S^B)(N)$, $\Psi(N(n) = \alpha(1_{S^B(N)})(n) = 1 \otimes n$. But $(T^B \circ S^B)(N) = (R \otimes_{R^{(H)}} N)^{(B)} = \bigoplus_{x \in B} (R \otimes_{R^{(H)}} N)_x$. Since $(R \otimes_{R^{(H)}} N)_x$ is an image of $(R \otimes_{\mathbb{Z}} N)_x$, the former is generated by elements of the form $\lambda \otimes n$, where $\lambda \in R_\sigma$, $n \in N y$, and $\sigma y = x$. By our hypothesis $\sigma \in H$ follows, and therefore $(R \otimes_{R^{(H)}} N)_x \subset R^{(H)} \otimes_{R^{(H)}} N$. Evidently, this yields that $(T^B \circ S^B)(N)$ is isomorphic to $N$, and $\Psi(N)$ is an isomorphism.

We show now the existence of a right adjoint for $T^B$. Let $N \in (H, B, R^{(H)})$-gr, $N = \bigoplus_{y \in B} N_y$. We define for each $x \in A$, $S_B(N)_x = \{f \in \mathrm{Hom}_{R^{(H)}}(R, N) \mid f(R_\theta) = 0$ if there are no $h \in H$, $\sigma \in G$ and $y \in B$ such that $\sigma y = x$ and $\theta = h\sigma^{-1}$, and $f(R_{h\sigma^{-1}}) \subseteq N_{hy}$ for all $h \in H$ if there exist $\sigma \in G$ and $y \in B$ such that $\sigma y = x\}$.

We put $S_B(N) = \sum_{x \in A} S_B(N)_x$. Let us show that the sum is direct. Let $f \in S_B(N)_x \cap \sum_{x' \neq x} S_B(N)_{x'}$. Then $f = \sum_{i=1}^{k} f_{x_i}$, $x_i \neq x$, $i = 1, \dots, k$. Let $r_\theta \in R_\theta$. If there are no $h \in H$, $y \in B$, $\sigma \in G$ such that $\sigma y = x$ and $\theta = h\sigma^{-1}$, then $f(r_\theta) = 0$. Suppose there exist $h \in H$, $\sigma \in G$, $y \in B$ such that $\sigma y = x$ and $\theta = h\sigma^{-1}$, and $h_i \in H$, $\sigma_i \in G$, $y_i \in G$, $y_i \in B$ such that $\sigma_i y_i = x_i$, $\theta = h_i \sigma_i^{-1}$, $i = 1, \dots, l$. We have that $f(r_\theta) = f(r_{h\sigma^{-1}}) \in N_{hy}$. On the other hand, $f(r_\theta) = \sum_{i=1}^{l} f_{x_i}(r_\theta) = \sum_{i=1}^{l} f_{x_i}(r_{h_i\sigma_i^{-1}}) \in \sum_{i=1}^{l} N_{h_i y_i}$. Suppose that there exists $i$ such that $hy = h_i y_i$. Then $x = \sigma y$, so $\sigma^{-1} x = y$ hence $h\sigma^{-1} x = hy$, thus $h_i \sigma_i^{-1} x = h_i y_i$, and so $\sigma_i^{-1} x = y_i$, i.e. $x = \sigma_i y_i = x_i$, a contradiction. It follows that $f(r_\theta) \in N_{hy} \cap \sum_{i=1}^{l} N_{h_i y_i} = 0$, i.e. $f = 0$.

Let us check now that $S_B(N) = \bigoplus_{x \in A} S_B(N)_x \in (G, A, R)$-gr. Let $r_\lambda \in R_\lambda$, $f \in S_B(N)_x$. We put $(rf)(a) = f(ar)$ for all $a, r \in R$, and we show that $r_\lambda f \in S_B(N)_{\lambda x}$. Let $r_\theta \in R_\theta$. We have $(r_\lambda f)(r_\theta) = f(r_\theta r_\lambda)$. If there exist $h \in H$, $\sigma \in G$, $y \in B$ such that $\lambda \sigma y = \lambda x$, and $\theta = h\sigma^{-1}\lambda^{-1}$, then $r_\theta r_\lambda \in R_{h\sigma^{-1}}$ and $\sigma y = x$, so $(r_\lambda f)(r_\theta) \in N_{hy}$. If not, then $(r_\lambda f)(r_\theta) = 0$. Thus $S_B(N) \in (G, A, R)$-gr.

If $N$, $N' \in (H, B, R^{(H)})$-gr, and $\varphi$: $\mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(N, N')$, then we put $S_B(\varphi)$: $S_B(N) \to S_B(N')$, $S_B(\varphi)(f) = \varphi \circ f$. It may easily be checked that

$$S_B(\varphi) \in \mathrm{Hom}_{(G,A,R)\text{-gr}}(S_B(N)_0, S_B(N')).$$

Thus we have defined a functor

$$S_B(-)\colon\ (H, B, R^{(H)})\text{-gr} \to (G, A, R)\text{-gr}.$$

Let us show that $S_B(-)$ is a right adjoint for $T^B(-)$. We define for all $M \in (G, A, R)$-gr, $N \in (H, B, R^{(H)})$-gr, $M = \bigoplus_{x\in A} M_x$ $N = \bigoplus_{y\in B} N_y$, the morphisms $\gamma$, $\delta$

$$\mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(T^B(M), N) \underset{\delta}{\overset{\gamma}{\rightleftarrows}} \mathrm{Hom}_{(G,A,R)\text{-gr}}(M, S_B(N))$$

as follows: to $u \in \mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(T^B(M), N)$ we correspond $\gamma(u)$ given by $\gamma(u)(m_x)(a) = u\ (\sum_{\substack{h\in H,\sigma\in G\\ y\in B,\sigma y=x}} a_{h\sigma^{-1}} m_x)$, $m_x \in M_x$, $a \in R$ using, where necessary, the convention that a sum indexed by an empty family is zero. It may easily be checked that $\gamma(u)(m_x) \in S_B(N)_x$, and so the definition is correct. To $v \in \mathrm{Hom}_{(G,A,R)\text{-gr}}(M, S_B(N))$ we correspond $\delta(v)$ given by $\delta(v)(m_y) = v(m_y)(1)$, $y \in B$, $m_y \in M_y$. Since $v(m_y) \in S_B(N)_y$, we take $h \in H$, $\sigma = h$, $h^{-1}y \in B$ and we have $h(h^{-1}y) = y$, so $v(my)(R_1) = v(m_y)(R_{hh^{-1}}) \subset N_{h(h^{-1}y)} = N_y$, and so $\delta$ is well-defined too.

Let now $v \in \mathrm{Hom}_{(G,A,R)\text{-gr}}(M, S_B(N))$, $m_x \in M_x$, $a \in R$. We have

$$\begin{aligned}\gamma(\delta(v))(m_x)(a) &= \delta(v)(\sum_{\substack{h\in H,\sigma\in G\\ y\in B,\sigma y=x}} a_{h\sigma^{-1}} m_x) = v(\sum_{\substack{h\in H,\sigma\in G\\ y\in B,\sigma y=x}} a_{h\sigma^{-1}} m_x)(1)\\ &= \sum_{\substack{h\in H,\sigma\in G\\ y\in B,\sigma y=x}} a_{h\sigma^{-1}} v(m_x)(1) = v(m_x)(\sum_{\substack{h\in H,\sigma\in G\\ y\in B,\sigma y=x}} a_{h\sigma^{-1}}) = v(m_x)(a),\end{aligned}$$

since $v(m_x) \in S_B(N)_x$, and hence $v(m_x)(a_\lambda) = 0$ if there are no $h \in H$, $\sigma \in G$, $y \in B$ such that $\sigma y = x$ and $\lambda = h\sigma^{-1}$. So $\gamma \circ \delta = \mathrm{Id}$. Conversely, let

$$u \in \mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(T^B(M), N), \quad m_y \in M_y \quad \text{and} \quad y \in B.$$

We have $\delta(\gamma(u))(m_y) = \gamma(u)(m_y)(1) = u(m_y)$, so $\delta \circ \gamma = \mathrm{Id}$.

Now suppose that $H$ is again as in the last part of the statement, and let $N \in (H, B, R^{(H)})$-gr. Then $T^B(S_B(N)) = \bigoplus_{y\in B} S_B(N)_y$. We have the canonical morphism $\phi(N)$: $\bigoplus_{y\in B} S_B(N)_y \to N = \bigoplus_{y\in B} N_y$ defined by $\phi(N)(f) = f(1) \in N_y$ for each $f \in S_B(N)_y$. From the condition $\sigma y = y'$, $y, y' \in B$ implies $\sigma \in H$, it follows that for all $y \in B$ and each $f \in S_B(N)_y$, we have that $f$ is zero outside $R^{(H)}$. It is clear that $\phi(N)$ is an isomorphism in $(H, B, R^{(H)})$-gr.

We will call $S^B$, the left adjoint of $T^B$, the functor induced by $B$ and $H$, and $S_B$, the right adjoint of $T^B$ the functor co-induced by $B$ and $H$.

**3.8. Corollary** (*see Corollary* 2.1 *of* [4]). 1. *If $N$ is injective in $(H, B, R^{(H)})$-gr, then $S_B(N)$ is injective in $(G, A, R)$-gr.*

2. *If $R$ is flat as a right $R^{(H)}$-module, then for every injective object $M \in (G, A, R)$-gr, we have that $T^B(M)$ is injective in $(H, B, R^{(H)})$-gr.*

*3.9. Remarks.* 1. The conditions in the statement of Theorem 3.7 hold in the case $A=G$, $B=H\subset G$. In this case, the construction of the induced functor reduces to E. Dade's construction, as given in [2], Theorem 4.3. The coinduced functor, which is the right adjoint of $T^H(-)$: $R$-gr $\to R^{(H)}$-gr, will be called in this case the graded coinduced functor with respect to $H$ and will be denoted by Gr-$\mathrm{Coind}_H^G(-)$. If $N\in R^{(H)}$-gr and $\sigma\in G$, then Gr-$\mathrm{Coind}_H^G(N)_\sigma$ $=\{f\in \mathrm{Hom}_{R^{(H)}}(R, N) \mid f|_{R(\sigma^{-1})^{(H)}}\in \mathrm{Hom}_{R^H\text{-gr}}(R(\sigma^{-1})^{(H)}, N)$, and $f=0$ elsewhere$\}$. If $\{\sigma_i \mid i\in I\}$ is a left transversal for $H$ in $G$, then we have: Gr-$\mathrm{Coind}_H^G(N)_\sigma$ $=\{f\in \mathrm{HOM}_{R^{(H)}}(\bigoplus_{i\in I} R(\sigma_i^{-1})^{(H)}, N) \mid f|_{R(\sigma_i^{-1})^{(H)}}\in \mathrm{HOM}_{R^{(H)}}(R(\sigma_i^{-1})^{(H)}, N)_{\sigma_i^{-1}\sigma}$ if $\sigma H$ $=\sigma_i H$, and $f(R(\sigma_j^{-1})^{(H)})=f(R^{(H\sigma_j^{-1})})=0$ for $j\neq i\}$. If $H$ has finite index in $G$, then:

$$\text{Gr-Coin}_H^G(N)=\mathrm{HOM}_{R^{(H)}}(R, N)=\mathrm{HOM}_{R^{(H)}}(\bigoplus_{i\in I} R(\sigma_i^{-1})^{(H)}, N).$$

2. If $A=G$, $B=H=\{1\}$, then the left and right adjoints of the functor $T^{\{1\}}$ $=(-)_1$: $R$-gr $\to R_1$-mod, $M\mapsto M_1$, are the induced and coinduced functors as introduced in [4]. In this case, the coinduced functor is denoted by $\mathrm{Coind}(-)$ and has the following form: for each $N\in R_1$-mod and $\sigma\in G$,

$$\mathrm{Coind}(N)_\sigma=\{f\in \mathrm{Hom}_{R_1}(R, N) \mid f(R_\theta)=0, \forall\, \theta\neq\sigma^{-1}\}.$$

3. If $A$ is a $G$-set and $x\in A$ is fixed, then the conditions of Theorem 3.7 hold if we put $H=G_x$ and $B=\{x\}$. In this case, $(G_x, \{x\}, R^{(G_x)})$-gr is just $R^{(G_x)}$-mod, and we obtain the functors:

$$T^x\colon (G, A, R)\text{-gr} \to R^{(G_x)}, \qquad T^x(M)=M_x$$
$$S^x\colon R^{(G_x)}\text{-mod} \to (G, A, R)\text{-gr}, \qquad S^x(N)=R\otimes_{R^{(G_x)}} N$$

(where $S^x(N)$ is $A$-graded as follows: $S^x(N)_y$, $y\in A$ is generated by all elements $\lambda_\sigma\otimes n$ with $\sigma x=y$).

$$S_x\colon R^{(G_x)}\text{-mod} \to (G, A, R)\text{-gr}$$

$$S_x(N)_y=\{f\in \mathrm{Hom}_{R^{(G_x)}}(R, N) \mid f(R_\theta)=0$$
$$\text{if there are no } h\in G_x, \sigma\in G \text{ such that } \sigma x=y, \theta=h\sigma^{-1}\}.$$

By Theorem 3.7 we have that $T^x\circ S^x=T^x\circ S_x=\mathrm{Id}$. If $R$ is a graded ring of finite support, i.e. $R_\sigma=0$ for all but a finite number of $\sigma\in G$, then we have that $S_x(N)$ $=\mathrm{Hom}_{R^{(G_x)}}(R, N)$. Indeed, we have that $S_x(N)_y$ is 0 if $\sigma x\neq y$, $\forall \sigma\in G$ and $S_x(N)_y$ is $\{f\in \mathrm{Hom}_{R^{(G_x)}}(R, N) \mid f=0$ outside $R^{(G_x\sigma^{-1})}\}$ if there exists $\sigma_y\in G\, \sigma_y x=y$.

It may be checked that the definition does not depend on the choice of the element $\sigma_y$. Moreover, $G$ may be written as the direct union $\bigcup_{\substack{y\in A\\ \sigma_y x=y}} G_x\sigma_y^{-1}=G$.

We have that $R=\bigoplus_{\substack{y\in A\\ \sigma_y x=y}} R^{(G_x\sigma^{-1})}$, and this is a finite direct sum whenever $R$ has finite support.

4. If $H \subset G$ is a subgroup, $A = G/H$, $B = \{H\}$, then the right adjoint $S_B(-)$ of the functor $T^B = (-)_H$: $(G/H, R)$-gr $\to R^{(H)}$-mod, $M \mapsto M_H$, will be called the coinduced functor with respect to the subgroup $H$, and will be denoted by $\mathrm{Coind}_H^G(-)$. If $\{\sigma_i \mid i \in I\}$ is a left transversal for $H$ in $G$, then for $N \in R^{(H)}$-mod we have:

$$\mathrm{Coind}_H^G(N)_{\sigma_i H} = \{f \in \mathrm{Hom}_{R^{(H)}}(R, N) \mid f(R^{(H\sigma_j^{-1})}) = 0 \text{ for } j \neq i\}.$$

If $H$ has finite index, then $\mathrm{Coind}_H^G(N) = \mathrm{Hom}_{R^{(H)}}(R, N)$.

**3.10. Proposition.** *If $R$ is a strongly $G$-graded ring, then with the notation and hypotheses of Theorem* 3.7, *the induced and coinduced functors are isomorphic.*

*Proof.* We define a functorial morphism:

$$\alpha(-)\colon\ S^B(-) \to S_B(-)$$

in the following way: if $N \in (H, B, R^{(H)})$-gr, then

$$\alpha(N)\colon\ S^B(N) \to S_B(N)$$

is given by $\alpha(N)(r \otimes n)(s) = srn$ for $r$, $s \in R$, $n \in N$. Clearly $\alpha(N)$ is a morphism in $(G, A, R)$-gr. We show now that if $R$ is strongly $G$-graded, those $\alpha(N)$ are isomorphisms giving the desired functorial isomorphism.

We show first that $\alpha(N)$ is injective. To see this, let $r_\sigma \in R_\sigma$ and $n_y \in N_y$ such that $\alpha(N)(r_\sigma \otimes n_y) = 0$. Let $1 = \sum_{i=1}^{n} a_i b_i$, $a_i \in R_\sigma$, $b_i \in R_{\sigma^{-1}}$, $i = 1, \ldots, k$. Then we have $\alpha(N)(r_\sigma \otimes n_y)(b_i) = b_i r_\sigma n_y = 0$ for all $i = 1, \ldots, k$, and $r_\sigma \otimes n_y = 1 \cdot r_\sigma \otimes n_y = \left(\sum_{i=1}^{n} a_i b_i r_\sigma\right) \otimes n_y = \sum_{i=1}^{n} a_i \otimes b_i r_\sigma n_y = 0$, so $\alpha(N)$ is injective.

Now if $f \in S_B(N)_x$, suppose there exist $\sigma \in G$ and $y \in B$ such that $\sigma y = x$. Then let $1 = \sum_{i=1}^{k} a_i b_i$, $a_i \in R_\sigma$, $b_i \in R_{\sigma^{-1}}$, $i = 1, \ldots, k$, and compute

$$\begin{aligned}\alpha(N)\left(\sum_{i=1}^{i} \alpha_i \otimes f(b_i)\right)(s_{h\sigma^{-1}}) &= \sum_{i=1}^{k} s_{h\sigma^{-1}} a_i f(b_i) = \sum_{i=1}^{k} f(s_{h\sigma^{-1}} a_i b_i)\\ &= f\left(s_{h\sigma^1}\left(\sum_{i=1}^{k} a_i b_i\right)\right) = f(s_{h\sigma^1}),\end{aligned}$$

so $\alpha(N)$ is also surjective.

**3.11. Corollary.** *If $R$ is a strongly $G$-graded ring and $A$ is a transitive $G$-set, then the categories* $(G, A, R)$-gr *and* $R^{(G_x)}$-mod *are equivalent, and this equivalence is given by the functors $T^x$ and $S^x$ (or $S_x$) (see Remark* 3.9.3).

*Proof.* By Theorem 3.7 it will suffice to establish $S^x \circ T^x = \mathrm{Id}$. For $M \in (G, A, R)$-gr, consider the functorial morphism.

$$\alpha(M)\colon\ (S^x \circ T^x)(M) \to M, \qquad \lambda \otimes m \mapsto \lambda_m, \qquad \lambda \in R,\ m \in M_x.$$

Denote $K = \mathrm{Ker}(\alpha(M))$, which is a subobject of $(S^x \circ T^x)(M) = R \otimes_{R(G_x)} M_x$ in $(G, A, R)$-gr, and pick $Z \in K_x$. So $Z = \sum_{i=1}^{k} \lambda_i \otimes m_i$ with $\lambda_i \in R^{(G_x)}$, and $m_i \in M_x$. We have $Z = \sum_{i=1}^{k} 1 \otimes \lambda_i m_i = 1 \otimes \sum_{i=1}^{k} \lambda_i m_i = 0$, since $\alpha(M)(Z) = \sum_{i=1}^{k} \lambda_i m_i = 0$. It follows that $k_x = 0$. Since $R$ is strongly $G$-graded, $R_\sigma K_x = K_{\sigma x}$ for all $\sigma \in G$, and therefore the transivity of $A$ allows to conclude that $\alpha(M)$ is injective. Surjectivity of $\alpha(M)$ is obvious from the definition.

**3.12. Corollary.** *Let $R$ be a strongly $G$-graded ring and $H$ a subgroup of $G$. Then $(G/H, R)$-gr and $R^{(H)}$-mod are equivalent.*

**3.13. Proposition.** *With the notation and hypothesis of Theorem 3.7 let $M \in (G, A, R)$-gr. We define:*

$$v\colon\ M \to S_B(T^B(M))$$

by $v(m)(a) = \sum_{x \in A} \sum_{\substack{h \in H, \sigma \in G \\ y \in B, \sigma y = x}} a_{h\sigma^{-1}} m_x$ where $a \in R$, $a = \sum_{\tau \in G} a_\tau$, $m \in M$, $m = \sum_{x \in A} m_x$.

*Then $v$ is clearly a morphism in $(G, A, R)$-gr, and $v(M)$ is essential in $S_B(T^B(M))$ as an $R$-submodule.*

*Proof.* Let $f \in S_B(T^B(M))$ be nonzero. We denote by $t(f)$ the number of $x \in A$ such that $f_x \notin v(M)$. If $t(f) = 0$, then $f = \sum_{x \in A} f_x \in v(M)$ and there is nothing to prove. So we may assume that $t(f) > 0$ and may fix $x \in A$ with $f_x \notin v(M)$. Clearly $f_x$ is a nonzero element of $S_B(T^B(M))_x$. It follows there exist $h \in H$, $\sigma \in G$, $y \in B$ and $a_{h\sigma^{-1}} \in R_{h\sigma^{-1}}$ such that $\sigma y = x$ and $f_x(a_{h\sigma^{-1}}) \neq 0$, $f_x(a_{h\sigma^{-1}}) \in M_{hy}$. Since $f_x(a_{h\sigma^{-1}}) = (a_{h\sigma^{-1}} f_x)(1_R)$, we conclude that the $hy$-component $a_{h\sigma^{-1}} f_x$ of $a_{h\sigma^{-1}} f$ is nonzero. Hence $a_{h\sigma^{-1}} f$ is nonzero. Now apply $v$ to the element $f_x(a_{h\sigma^{-1}}) \in M_{hy}$. If $b \in R$, then $v(f_x(a_{h\sigma^{-1}}))(b) = \sum_{\substack{h' \in H, \sigma' \in G \\ y' \in B, \sigma' y' = hy}} b_{h'\sigma'^{-1}} f_x(a_{h\sigma^{-1}}) = \sum_{h' \in H} b_{h'\sigma'^{-1}} f_x(a_{h\sigma^{-1}}) = f_x(\sum_{h' \in H} b_{h'\sigma^{-1}}, a_{h\sigma^{-1}})$. Remember that from $\sigma x = y$, $x, y \in B$, it follows that $\sigma \in H$.

On the other hand, the element $a_{h\sigma^{-1}} f_x \in S_B(T^B(M))_{hy}$ satisfies $(a_{h\sigma^{-1}} f_x)(b) = (a_{h\sigma^{-1}} f_x)(\sum_{h' \in H} b_{h'}) = f_x(\sum_{h' \in H} b_h, a_{h\sigma^{-1}})$. We conclude that the $hy$-component $a_{h\sigma^{-1}} f_x$ of $a_{h\sigma^{-1}} f$ is equal to $v(f_x(a_{h\sigma^{-1}}))$ and hence lies in $v(M)$. Any $x'$-component $f_{x'}$ of $f$ lying in $v(M)$ leads to a $h\sigma^{-1}x'$-component $a_{h\sigma^{-1}} f_{x'}$ of $a_{h\sigma^{-1}} f$ lying in $v(M)$. It follows that the number $t(a_{h\sigma^{-1}} f)$ of components of $a_{h\sigma^{-1}} f$ not lying in $v(M)$ satisfies $t(a_{h\sigma^{-1}} f) < t(f)$. The proof of the proposition is now completes by infinite descent.

*3.14. Remark.* Proposition 3.13 learns that the coinduced functor, under certain assumptions may provide information on the structure of injective objects of $(G, A, R)$-gr. Indeed, if $Q \in (G, A, R)$-gr is an injective object which is faithful (this means that for all $x \in A$ and $0 \neq m_y \in Q_y$, $y \in B$, we have

$\sum_{h\in H, \sigma\in G, \sigma y=x} R_{h\sigma^{-1}} m_y \neq 0)$ than $v$: $Q\to S_B(T^B(Q))$ is injective, and hence an isomorphism (see also [4]).

**3.15. Theorem.** *If $R$ is a $G$-graded ring of finite support, $A$ is a $G$-set and $Q\in(G, A, R)$-gr is an injective object in this category and has finite support too (i.e. $Q=\bigoplus_{x\in A} Q_x$ and $Q_x=0$ for all but a finite number of $x\in A$) then $Q$ is injective in $R$-mod.*

*Proof.* For each $x\in A$, we consider the functors $T^x$, $S_x$ (see Remark 3.9.3) and the graded morphisms defined in Proposition 3.13: $v_x$: $Q\to S_x(T^x(Q))=S_x(Q_x)$.

$$v_x(m)(a)=\sum_{y\in A}\sum_{h\in G_x, \sigma\in G, \sigma x=y} a_{h\sigma^{-1}} m_y, \qquad m=\sum_{y\in A} m_y, \qquad a=\sum_{\tau\in G} a_\tau \in R.$$

Since $Q$ has finite support, we can define

$$v\colon\ Q\to\bigoplus_{x\in A} S_x(Q_x)$$

by $v(m)=\sum_{x\in A} v_x(m)$. We have that $v$ is injective. Indeed, if $m_y\in Q_y$ is such that $v(m_y)=0$, then for any $a\in R$ we calculate

$$\begin{aligned} v(m_y)(a)&=\sum_{x\in A} v_x(m_y)(a)=\sum_{x\in A}\sum_{\substack{h\in G_x, \sigma\in G\\ \sigma x=y}} a_{h\sigma^{-1}} m_y\\ &=\Big(\sum_{x\in A}\sum_{\substack{h\in G_x, \sigma\in G\\ \sigma x=y}} a_{h\sigma^{-1}}\Big) m_g=am_y=0 \end{aligned}$$

since for each $\theta\in G$, $\theta=1\cdot(\theta^{-1})^{-1}$, we take $x=\theta y$, $h=1$, $\sigma=\theta^{-1}$. In particular, $v(m_y)(1)=m_y=0$, so $v$ is injective. Now we denote by $E_x(Q_x)$ the injective evelope of $Q_x$ in $R^{(G_x)}$-mod, for all $x\in A$. Since $S_x(-)$ is a left exact functor, we have the monomorphisms $S_x(Q_x)\subset S_x(E_x(Q_x))$ for all $x\in A$. So we obtain

$$Q\hookrightarrow\bigoplus_{x\in A} S_x(Q_x)\hookrightarrow\bigoplus_{x\in A} S_x(E_x(Q_x)).$$

Now because $R$ has finite support, we have that:

$$S_x(E_x(Q_x))=\mathrm{Hom}_{R^{(G_x)}}(R, E_x(Q_x)) \qquad \text{(see Remark 3.9.3)}$$

and it is well-known that the latter is an injective $R$-module. Hence $Q$ is a direct summand in $(G, A, R)$-gr of an object which is injective in $R$-mod. Thus $Q$ is injective in $R$-mod too.

**3.16. Corollary** [4]. *If $R$ is a $G$-graded ring with finite support, and $Q\in R$-gr is an injective object which has finite support too, then $Q$ is injective in $R$-mod.*

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