DIFFERENTIAL FORMS ON REGULAR AFFINE ALGEBRAS

BY

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1. **Introduction.** The formal apparatus of the algebra of differential forms appears as a rather special amalgam of multilinear and homological algebra, which has not been satisfactorily absorbed in the general theory of derived functors. It is our main purpose here to identify the exterior algebra of differential forms as a certain canonical graded algebra based on the Tor functor and to obtain the cohomology of differential forms from the Ext functor of a universal algebra of differential operators similar to the universal enveloping algebra of a Lie algebra.

Let K be a field, R a commutative K-algebra, T_R the R-module of all K-derivations of R, D_R the R-module of the *formal K-differentials* (see §4) on R. It is an immediate consequence of the definitions that T_R may be identified with $\mathrm{Hom}_R(D_R, R)$. However, in general, D_R is not identifiable with $\mathrm{Hom}_R(T_R, R)$. The *algebra of the formal differentials* is the exterior R-algebra $E(D_R)$ built over the R-module D_R. The *algebra of the differential forms* is the R-algebra $\mathrm{Hom}_R(E(T_R), R)$, where $E(T_R)$ is the exterior R-algebra built over T_R and where the product is the usual "shuffle" product of alternating multilinear maps.

The point of departure of our investigation lies in the well-known and elementary observation that T_R and D_R are naturally isomorphic with $\mathrm{Ext}^1_{R^e}(R, R)$ and $\mathrm{Tor}_1^{R^e}(R, R)$, respectively, where $R^e = R \otimes_K R$. Moreover, both $\mathrm{Ext}_{R^e}(R, R)$ and $\mathrm{Tor}^{R^e}(R, R)$ can be equipped in a natural fashion with the structure of a graded skew-commutative R-algebra, and there is a natural duality homomorphism h: $\mathrm{Ext}_{R^e}(R, R) \rightarrow \mathrm{Hom}_R(\mathrm{Tor}^{R^e}(R, R), R)$, which extends the natural isomorphism of T_R onto $\mathrm{Hom}_R(D_R, R)$.

We concentrate our attention chiefly on a regular affine K-algebra R (cf. §2), where K is a perfect field. Our first main result is that then the algebra $\mathrm{Tor}^{R^e}(R, R)$ coincides with the algebra $E(D_R)$ of the formal differentials, $\mathrm{Ext}_{R^e}(R, R)$ coincides with $E(T_R)$, and the above duality homomorphism h is an isomorphism dualizing into an isomorphism of the algebra $E(D_R)$ of the formal differentials onto the algebra $\mathrm{Hom}_R(E(T_R), R)$ of the differential forms.

In order to identify the cohomology of differential forms with an Ext functor, we construct a universal "algebra of differential operators," V_R,

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which is the universal associative algebra for the representations of the K-Lie algebra T_R on R-modules in which the R-module structure and the T_R-module structure are tied together in the natural fashion. After establishing a number of results on the structure and representation theory of V_R, we show that, under suitable assumptions on the K-algebra R and, in particular, if R is a regular affine K-algebra where K is a perfect field, the cohomology K-algebra derived from the differential forms may be identified with $\mathrm{Ext}_{V_R}(R, R)$.

In §2, we show that the tensor product of two regular affine algebras over a perfect field is a regular ring, and we prove a similar result for tensor products of fields. §§3, 4 and 5 include, besides the proof of the first main result, a study of the formal properties of the Tor and Ext algebras and the pairing between them, for general commutative algebras. In the remainder of this paper, we deal with the universal algebra V_R of differential operators. In particular, we prove an analogue of the Poincaré-Birkhoff-Witt Theorem, which is needed for obtaining an explicit projective resolution of R as a V_R-module. Also, we discuss the homological dimensions connected with V_R.

We have had advice from M. Rosenlicht on several points of an algebraic geometric nature, and we take this opportunity to express our thanks to him.

2. **Regular rings.** Let R be a commutative ring and let P be a prime ideal of R. We denote the corresponding ring of quotients by R_P. The elements of R_P are the equivalence classes of the pairs (x, y), where x and y are elements of R, and y does not lie in P, and where two pairs (x_1, y_1) and (x_2, y_2) are called equivalent if there is an element z in R such that z does not lie in P and $z(x_1y_2-x_2y_1)=0$.

By the *Krull dimension* of R is meant the largest non-negative integer k (or ∞, if there is no largest one) for which there is a chain of prime ideals, with proper inclusions, $P_0 \subset \cdots \subset P_k \subset R$. A Noetherian local ring always has finite Krull dimension, and it is called a *regular local ring* if its maximal ideal can be generated by k elements, where k is the Krull dimension. A commutative Noetherian ring R with identity element is said to be *regular* if, for every maximal ideal P of R, the corresponding ring of quotients R_P is a regular local ring [2, §4].

It is well known that a regular local ring is an integrally closed integral domain [14, Cor. 1, p. 302]. It follows that a regular integral domain R is integrally closed; for, if x is an element of the field of quotients of R that is integral over R then $x \in R_P$, for every maximal ideal P of R, which evidently implies that $x \in R$.

Let K be a field. By an *affine K-algebra* is meant an integral domain R containing K and finitely ring-generated over K. An affine K-algebra is Noetherian, and its Krull dimension is equal to the transcendence degree of its field of quotients over K, and the same holds for the Krull dimension of every one of its rings of quotients with respect to maximal ideals [14, Ch. VII, §7].

THEOREM 2.1([2]). *Let K be a perfect field, and let R and S be regular affine K-algebras. Then $R\otimes_K S$ is regular.*

Proof. Suppose first that $R\otimes_K S$ is an integral domain, and let M be one of its maximal ideals. Put $M_1=(M\cap R)\otimes_K S+R\otimes_K(M\cap S)$. Then M_1 is an ideal of $R\otimes_K S$ that is contained in M, and we have $(R\otimes_K S)/M_1 =(R/(M\cap R))\otimes_K(S/(M\cap S))$. Now $R/(R\cap M)$ and $S/(M\cap S)$ are subrings of $(R\otimes_K S)/M$ containing K. Since $(R\otimes_K S)/M$ is a finite algebraic extension field of K, the same is therefore true for $R/(M\cap R)$ and $S/(M\cap S)$. Since K is perfect, it follows that we have a direct K-algebra decomposition $(R\otimes_K S)/M_1=U+M/M_1$. Let z be a representative in $R\otimes_K S$ of a nonzero element of U. Then z does not belong to M, and $zM\subset M_1$. Hence it is clear that $M(R\otimes_K S)_M=M_1(R\otimes_K S)_M$.

Since R is regular, the maximal ideal $(M\cap R)R_{M\cap R}$ of the local ring $R_{M\cap R}$ is generated by d_R elements, where d_R is the degree of transcendence of the field of quotients of R over K. Similarly, $(M\cap S)S_{M\cap S}$ is generated by d_S elements, where d_S is the degree of transcendence of the quotient field of S over K. These d_R+d_S elements may be regarded as elements of $(R\otimes_K S)_M$ and evidently generate the ideal $M_1(R\otimes_K S)_M$. Hence we conclude that the maximal ideal of $(R\otimes_K S)_M$ can be generated by d_R+d_S elements. Since the degree of transcendence of the quotient field of $R\otimes_K S$ over K is equal to d_R+d_S, this means that $(R\otimes_K S)_M$ is a regular local ring. Thus $R\otimes_K S$ is regular.

Now let us consider the general case. Let $Q(R)$ and $Q(S)$ denote the fields of quotients of R and S. Let K^R and K^S be the algebraic closures of K in $Q(R)$ and in $Q(S)$, respectively. Since R and S are integrally closed, we have $K^R\subset R$ and $K^S\subset S$. Since $Q(R)$ and $Q(S)$ are finitely generated extension fields of K, so are K^R and K^S. Thus K^R and K^S are finite algebraic extensions of K.

Let M be a maximal ideal of $R\otimes_K S$. Since K is perfect, we have a direct K-algebra decomposition $K^R\otimes_K K^S=U+M_1$, where $M_1=M\cap(K^R\otimes_K K^S)$. Hence we have

$$R\otimes_K S = R\otimes_{K^R}(K^R\otimes_K K^S)\otimes_{K^S} S = R\otimes_{K^R} U\otimes_{K^S} S + M_2,$$

where the last sum is a direct K-algebra sum, and $M_2=R\otimes_{K^R} M_1\otimes_{K^S} S\subset M$.

Evidently, U may be identified with a subring of the field $(R\otimes_K S)/M$ containing K. Hence U is a finite algebraic extension field of K. Identifying K^R and K^S with their images in U, we may also regard U as a finite algebraic extension field of K^R or K^S. Since K is perfect, U is generated by a single element over K^R or over K^S. The minimum polynomial of this element over K^R or over K^S remains irreducible in $Q(R)[x]$ or in $Q(S)[x]$, because K^R is algebraically closed in $Q(R)$ and K^S is algebraically closed in $Q(S)$. Hence

([2]) The referee informs us that this result is an immediate consequence of cohomology results obtained by D. K. Harrison in a paper on *Commutative algebras and cohomology*, to appear in these Transactions.

$R\otimes_{K^R} U$ and $U\otimes_{K^S} S$ are integral domains. Moreover, by the part of the theorem we have already proved, they are regular.

Let T denote the field $U\otimes_{K^S} Q(S)$. This is a finitely generated extension field of the perfect field K^R. Let $(t_1, \cdots, t_n)$ be a separating transcendence base for T over K^R, and put $T_0 = K^R(t_1, \cdots, t_n)$. We have $Q(R)\otimes_{K^R} T = (Q(R)\otimes_{K^R} T_0)\otimes_{T_0} T$, and we may identify $Q(R)\otimes_{K^R} T_0$ with a subring of $Q(R)(t_1, \cdots, t_n)$, with $(t_1, \cdots, t_n)$ algebraically free over $Q(R)$. Since K^R is algebraically closed in $Q(R)$, it follows that $K^R(t_1, \cdots, t_n)$ is algebraically closed in $Q(R)(t_1, \cdots, t_n)$ [6, Lemma 2, p. 83]. Now it follows by the argument we made above that $R\otimes_{K^R} T$ is an integral domain, so that $R\otimes_{K^R} U\otimes_{K^S} S$ is an integral domain. On the other hand, this is the tensor product, relative to the perfect field U, of the regular affine U-algebras $R\otimes_{K^R} U$ and $U\otimes_{K^S} S$. Hence we may conclude from what we have already proved that $R\otimes_{K^R} U\otimes_{K^S} S$ is regular.

Now consider the direct K-algebra decomposition

$$R\otimes_K S = R\otimes_{K^R} U\otimes_{K^S} S + M_2.$$

Since $M_2\subset M$, the corresponding projection epimorphism $R\otimes_K S \rightarrow R\otimes_{K^R} U\otimes_{K^S} S$ sends the complement of M in $R\otimes_K S$ onto the complement of $M\cap(R\otimes_{K^R} U\otimes_{K^S} S)$ in $R\otimes_{K^R} U\otimes_{K^S} S$. Moreover, there is an element z in the complement of M such that $zM_2=(0)$. Hence it is clear that the projection epimorphism yields an isomorphism of $(R\otimes_K S)_M$ onto the local ring over $R\otimes_{K^R} U\otimes_{K^S} S$ that corresponds to the maximal ideal $M\cap(R\otimes_{K^R} U\otimes_{K^S} S)$. Hence $(R\otimes_K S)_M$ is a regular local ring, and Theorem 2.1 is proved.

THEOREM 2.2. *Let K be an arbitrary field, let F be a finitely and separably generated extension field of K, and let L be an arbitrary field containing K. Then $F\otimes_K L$ is a regular ring.*

Proof. It is known that the (homological) algebra dimension $\dim(F)$, i.e., the projective dimension of F as an $F\otimes_K F$-module is finite; in fact, it is equal to the transcendence degree of F over K [11, Th. 10]. Since $\dim(F\otimes_K L) = \dim(F)$, where $F\otimes_K L$ is regarded as an L-algebra [4, Cor. 7.2, p. 177] we have that $\dim(F\otimes_K L)$ is finite. Since L is a field, this implies that the global homological dimension $d(F\otimes_K L)$ is also finite [4, Prop. 7.6, p. 179]. Since $F\otimes_K L$ is a commutative Noetherian ring, we have, for every maximal ideal M of $F\otimes_K L$, $d((F\otimes_K L)_M)\leq d(F\otimes_K L)$ [4, Ex. 11, p. 142; 1, Th. 1]. Thus each local ring $(F\otimes_K L)_M$ is of finite global homological dimension. By a well-known result of Serre's [12, Th. 3], this implies that $(F\otimes_K L)_M$ is a regular local ring. Hence $F\otimes_K L$ is a regular ring.

Note. Actually, we shall later appeal only to the following special consequence of Theorem 2.2: let F be a finitely separably generated extension field of K; let J be the kernel of the natural epimorphism $F\otimes_K F\rightarrow F$; then the local ring $(F\otimes_K F)_J$ is regular. This special result can be proved much more easily and directly along the lines of our proof of Theorem 2.1. On the other hand, Theorem 2.1 can be derived more quickly, though less elementarily, from the result of Serre used above.

3. The Tor-**algebra for regular rings.** Let R and S be commutative rings with identity elements, and let ϕ be a unitary ring epimorphism $S\to R$. We regard R as a right or left S-module via ϕ, in the usual way, and we consider $\mathrm{Tor}^S(R, R)$.

Since S is commutative, every left S-module may also be regarded as a right S-module, and we shall do so whenever this is convenient. Let H stand for the homology functor on complexes of S-modules, and let U and V be any two S-module complexes. There is an evident canonical homomorphism of $H(U)\otimes_S H(V)$ into $H(U\otimes_S V)$, which gives rise to an algebra structure on $\mathrm{Tor}^S(R, R)$, as follows. Let X be an S-projective resolution of R. With $U=V=R\otimes_S X$, the canonical homomorphism becomes a homomorphism

$$\mathrm{Tor}^S(R, R) \otimes_S \mathrm{Tor}^S(R, R) \to H((R \otimes_S X) \otimes_S (R \otimes_S X)).$$

Evidently, $(R\otimes_S X)\otimes_S(R\otimes_S X)$ may be identified with $(R\otimes_S R)\otimes_S(X\otimes_S X)$, and hence with $R\otimes_S (X\otimes_S X)$. Now $X\otimes_S X$ is an S-projective complex over $R\otimes_S R=R$, whence we have the natural homomorphism

$$H(R \otimes_S (X \otimes_S X)) \to \mathrm{Tor}^S(R, R).$$

Composing this with the homomorphism above, we obtain an S-module homomorphism

$$\mathrm{Tor}^S (R, R) \otimes_S \mathrm{Tor}^S(R, R) \to \mathrm{Tor}^S(R, R).$$

This is the product $\pitchfork$ of [4, p. 211] and it is independent of the choice of the resolution X. Standard arguments on tensor products of complexes and resolutions show that this product is associative and skew-commutative in the sense that $\alpha\beta=(-1)^{pq}\beta\alpha$ when α is homogeneous of degree p and β is homogeneous of degree q. In principle, this product is a product of S-algebras. However, S operates on $\mathrm{Tor}^S(R, R)$ through ϕ: $S\to R$, and we shall accordingly regard $\mathrm{Tor}^S(R, R)$ as an R-algebra.

THEOREM 3.1. *Let S and R be Noetherian commutative rings with identity elements, and let ϕ be a ring epimorphism $S\to R$ with kernel I. Assume that R is a regular ring and that, for every maximal ideal M of S that contains I, the local ring S_M is regular. Then* $\mathrm{Tor}^S(R, R)$ *is finitely generated and projective as an R-module and is naturally isomorphic with the exterior R-algebra constructed over* $\mathrm{Tor}_1^S(R, R)$.

Proof. Let T denote the tensor algebra constructed over the R-module $\mathrm{Tor}_1^S(R, R)$, let P denote the kernel of the canonical R-algebra homomorphism

$\psi: T \to \mathrm{Tor}^S(R, R)$, and put $Q = \mathrm{Tor}^S(R, R)/\psi(T)$. Let U denote the 2-sided ideal of T that is generated by the squares of the elements of $\mathrm{Tor}_1^S(R, R)$. The last assertion of our theorem means that $Q = (0)$ and $P = U$. The statement $Q = (0)$ is equivalent to the statement $R_N \otimes_R Q = (0)$, for all maximal ideals N of R. The statement $P = U$ is equivalent to the statement $(P+U)/P = (0)$ and $(P+U)/U = (0)$, or to the statement $R_N \otimes_R (P+U)/P = (0)$ and $R_N \otimes_R (P+U)/U = (0)$, for all maximal ideals N of R. This, in turn, is equivalent to the statement that the images of $R_N \otimes_R P$ and $R_N \otimes_R U$ in $R_N \otimes_R (P+U)$ coincide with $R_N \otimes_R (P+U)$. Since R_N is R-flat, these tensor products may be identified with their canonical images in $R_N \otimes_R T$; and $R_N \otimes_R P$ is thereby identified with the kernel of the homomorphism of $R_N \otimes_R T$ into $R_N \otimes_R \mathrm{Tor}^S(R, R)$ that is induced by ψ. Hence it is clear that the statement $Q = (0)$ and $P = U$ is equivalent to the statement that the homomorphism of $R_N \otimes_R T$ into $R_N \otimes_R \mathrm{Tor}^S(R, R)$ that is induced by ψ is an epimorphism with kernel $R_N \otimes_R U$, for every maximal ideal N of R.

Let M be the maximal ideal of S that contains I and is such that $M/I = N$. Clearly, the epimorphism ϕ induces an epimorphism $S_M \to R_N$ with kernel IS_M in the natural fashion.

Now let X be an S-projective resolution of R. Since S_M is S-flat, $S_M \otimes_S X$ is then an S_M-projective resolution of $S_M \otimes_S R = S_M/IS_M = R_N$. Hence we have

$$\mathrm{Tor}^{S_M}(S_M/IS_M, S_M/IS_M) = H((S_M/IS_M) \otimes_{S_M} (S_M \otimes_S X)).$$

On the other hand,

$$(S_M/IS_M) \otimes_{S_M} (S_M \otimes_S X) = (S_M/IS_M) \otimes_S X = R_N \otimes_S X = R_N \otimes_R (R \otimes_S X).$$

Since R_N is R-flat, we have $H(R_N \otimes_R (R \otimes_S X)) = R_N \otimes_R \mathrm{Tor}^S(R, R)$. Thus $R_N \otimes_R \mathrm{Tor}^S(R, R)$ is naturally isomorphic with $\mathrm{Tor}^{S_M}(S_M/IS_M, S_M/IS_M)$. Similarly, we see that $R_N \otimes_R T$ is naturally isomorphic with the tensor algebra constructed over the R_N-module $\mathrm{Tor}_1^{S_M}(S_M/IS_M, S_M/IS_M)$. Moreover, it is easily seen that these isomorphisms transport our homomorphism $R_N \otimes_R T \to R_N \otimes_R \mathrm{Tor}^S(R, R)$ into the canonical homomorphism of the tensor algebra over $\mathrm{Tor}_1^{S_M}(S_M/IS_M, S_M/IS_M)$ into $\mathrm{Tor}^{S_M}(S_M/IS_M, S_M/IS_M)$.

Each $\mathrm{Tor}_p^S(R, R)$ is finitely generated as an S-module, and hence also as an R-module. Hence if we show that $R_N \otimes_R \mathrm{Tor}_p^S(R, R)$ is a free R_N-module, for every maximal ideal N of R, we shall be able to conclude from a standard result [4, Ex. 11, p. 142] that $\mathrm{Tor}_p^S(R, R)$ is a finitely generated projective R-module. In particular, if $\mathrm{Tor}_1^S(R, R)$ is a finitely generated projective R-module, we imbed it as a direct R-module summand in a finitely generated free R-module to show that the exterior algebra constructed over it has non-zero components only up to a certain degree and is a finitely generated projective R-module.

From this preparation, it is clear that it suffices to adduce the following result(3): let $L(=S_M)$ be a regular local ring and let $J(=IS_M)$ be a prime ideal of L such that the local ring L/J is regular. Then $\mathrm{Tor}_1^L(L/J, L/J)$ is a finitely generated free L/J-module, and $\mathrm{Tor}^L(L/J, L/J)$ is naturally isomorphic, as an L/J-algebra, with the exterior algebra constructed over $\mathrm{Tor}_1^L(L/J, L/J)$.

To prove this, note first that the assumptions imply that the ideal J can be generated by an L-sequence $(a_1, \cdots, a_j)$ of elements of L, i.e., by a system with the property that each a_k is not a zero-divisor mod the ideal generated by $a_1, \cdots, a_{k-1}$ [14, Th. 26, p. 303 and Cor. 1, p. 302]. If X is the Koszul resolution of L/J as an L-module [4, pp. 151–153], constructed with the use of this L-sequence, then X has the structure of an exterior L-algebra over a free L-module of rank j, this algebra structure being compatible with the boundary map, so that it induces the algebra structure on $\mathrm{Tor}^L(L/J, L/J)$ via $(L/J)\otimes_L X$. Moreover, the boundary map on $(L/J)\otimes_L X$ is the zero map. Hence it follows immediately that $\mathrm{Tor}_1^L(L/J, L/J)$ is a free L/J-module of rank j and that $\mathrm{Tor}^L(L/J, L/J)$ is the exterior algebra over this module. This completes the proof of Theorem 3.1.

4. **Duality between** Tor **and** Ext. Let R and S be commutative rings with identity elements, and let ϕ be a ring epimorphism of S onto R. As before, all R-modules are regarded as S-modules via ϕ. Let X be an S-projective resolution of R, and let A be an R-module. Then $\mathrm{Ext}_S(R, A)=H(\mathrm{Hom}_S(X, A))$. Clearly, we may identify $\mathrm{Hom}_S(X, A)$ with $\mathrm{Hom}_R(R\otimes_S X, A)$, so that we may write $\mathrm{Ext}_S(R, A)=H(\mathrm{Hom}_R(R\otimes_S X, A))$. Now there is a natural map (a specialization of [4, p. 119, last line])

$$h\colon H(\mathrm{Hom}_R(R\otimes_S X, A))\to \mathrm{Hom}_R(\mathrm{Tor}^S(R, R), A)$$

defined as follows. Let ρ be an element of $H(\mathrm{Hom}_R(R\otimes_S X, A))$. Then ρ is represented by an element $u\in\mathrm{Hom}_R(R\otimes_S X, A)$ that annihilates $d(R\otimes_S X)$, where d is the boundary map in the complex $R\otimes_S X$. Hence, by restriction to the cycles of $R\otimes_S X$, u yields an element of $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), A)$, and it is seen immediately that this element depends only on ρ and not on the particular choice of the representative u. Now $h(\rho)$ is defined to be this element of $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), A)$.

Clearly, h is an R-module homomorphism of $\mathrm{Ext}_S(R, A)$ into $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), A)$. In degree 0, we have $\mathrm{Tor}_0^S(R, R)=R\otimes_S R=R$, and $\mathrm{Ext}_S^0(R, A)=\mathrm{Hom}_S(R, A)=\mathrm{Hom}_R(R, A)$, and this last identification transports h into the identity map. Thus h is an isomorphism in degree 0. Note that $\mathrm{Tor}_0^S(R, R)=R$ is projective as an R-module, whence the following lemma implies, in particular, that h is an isomorphism also in degree 1.

LEMMA 4.1. *Let $\phi\colon S\to R$ be an epimorphism of commutative rings with identity elements, and regard R-modules as S-modules via ϕ. Let A be an R-*

(3) This is a special case of [13, Th. 4, etc.], which gave the suggestion for our proof of Theorem 3.1.

module, and let k be a positive integer. Assume that $\mathrm{Tor}_i^S(R, R)$ *is R-projective for all $i<k$. Then the map*

$$h_i\colon \mathrm{Ext}_S^i(R, A) \to \mathrm{Hom}_R(\mathrm{Tor}_i^S(R, R), A),$$

obtained by restriction of the map h defined above, is an isomorphism, for all $i \leqq k$.

Proof. Let Z_i denote the kernel of d in $R\otimes_S X_i$, and put $B_i=d(R\otimes_S X_{i+1})$, $C_i=R\otimes_S X_i$. We have $Z_0=C_0$. Suppose that we have already shown, for some $i<k$, that Z_i is R-projective. Since $\mathrm{Tor}_i^S(R, R)$ is R-projective, the exact sequence $0\to B_i\to Z_i\to \mathrm{Tor}_i^S(R, R)\to 0$ shows that B_i is a direct module summand in Z_i and hence is R-projective. Hence the exact sequence $0\to Z_{i+1}\to C_{i+1}\to^d B_i\to 0$ shows that Z_{i+1} is a direct module summand in C_{i+1} and hence is R-projective. Hence, starting at $i=0$, we conclude that B_i is a direct R-module summand of C_i for all $i<k$, and that Z_i is a direct R-module summand of C_i for all $i\leqq k$.

Now let $i\leqq k$, and let ρ be an element of $\mathrm{Ext}_S^i(R, A)$ such that $h_i(\rho)=0$. Let u be a representative of ρ in $\mathrm{Hom}_R(C_i, A)$. Then u vanishes on Z_i, so that it induces an element $v\in \mathrm{Hom}_R(B_{i-1}, A)$ such that $v\circ d=u$. Since B_{i-1} is a direct R-module summand of C_{i-1}, v can be extended (trivially) to an element $w\in\mathrm{Hom}_R(C_{i-1}, A)$. Now $u=w\circ d$, which means that $\rho=0$. Thus h_i is a monomorphism.

Now let $\gamma\in\mathrm{Hom}_R(\mathrm{Tor}_i^S(R, R), A)$. We may regard γ as an element of $\mathrm{Hom}_R(Z_i, A)$. Since Z_i is a direct R-module summand of C_i, γ can be extended to an element of $\mathrm{Hom}_R(C_i, A)$, which represents an element $\rho\in\mathrm{Ext}_S^i(R, A)$ such that $h_i(\rho)=\gamma$. Thus h_i is an epimorphism. This completes the proof of Lemma 4.1.

In degree 1, we have $\mathrm{Tor}_1^S(R, R)=R\otimes_S I$, where I is the kernel of ϕ, and $\mathrm{Ext}_S^1(R, A)=\mathrm{Hom}_S(I, A)=\mathrm{Hom}_R(R\otimes_S I, A)$, this identification being precisely the one obtained from the isomorphism h_1. Alternatively, we may identify $R\otimes_S I$ with I/I^2 (whose R-module structure is naturally induced from its S-module structure), and thus we may write, compatibly with h_1, $\mathrm{Tor}_1^S(R, R)=I/I^2$, and $\mathrm{Ext}_S^1(R, A)=\mathrm{Hom}_R(I/I^2, A)$.

We are particularly interested in the following special case. Let K be a commutative ring with identity, and let R be a commutative K-algebra with identity. Let S be the K-algebra $R\otimes_K R$, and let ϕ be the epimorphism $S\to R$ that sends $x\otimes y$ onto xy. Let $T_R(A)$ denote the R-module of all K-linear derivations of R into the R-module A. We recall the well-known fact that $T_R(A)$ is naturally isomorphic with $\mathrm{Ext}_S^1(R, A)$. Indeed, if $\zeta\in T_R(A)$ then ζ yields an element $\zeta^*\in\mathrm{Hom}_S(I, A)$ such that $\zeta^*(\sum x\otimes y)=\sum x\cdot\zeta(y) = -\sum y\cdot\zeta(x)$ (since $\sum xy=0$). Clearly, the map $\zeta\to\zeta^*$ is an R-module homomorphism. If $\zeta^*=0$, we have, for every $x\in R$, $\zeta(x)=\zeta^*(1\otimes x-x\otimes 1)=0$. Thus the map $\zeta\to\zeta^*$ is a monomorphism. If $\gamma\in\mathrm{Hom}_S(I, A)$, we define a map ζ of R into A by putting $\zeta(x)=\gamma(1\otimes x-x\otimes 1)$. One checks easily that

$\zeta \in T_R(A)$ and $\zeta^* = \gamma$. Thus the map $\zeta \rightarrow \zeta^*$ is an R-module isomorphism of $T_R(A)$ onto $\mathrm{Ext}^1_S(R, A)$.

Regard S as an R-module such that $x \cdot (y \otimes z) = (xy) \otimes z$, and let J be the R-submodule of S that is generated by the elements of the form $1 \otimes (xy) - x \otimes y - y \otimes x$. The factor module S/J is the R-module of the *formal differentials* of R. The map $x \rightarrow dx =$ the coset of $1 \otimes x$ mod J is the usual derivation of R into the R-module of the formal differentials and, in fact, the definition of these amounts simply to enforcing the rule $d(xy) = xdy + ydx$. Let D_R denote the R-module of these formal differentials. It is easily verified that the map $S \rightarrow I$ that sends $x \otimes y$ onto $x \otimes y - (xy) \otimes 1$ induces in the natural way an isomorphism of the R-module D_R onto $I/I^2 = \mathrm{Tor}^S_1(R, R)$ [3, Exp. 13]. Tracing through our above definitions and identifications, we see immediately that the duality isomorphism h_1: $\mathrm{Ext}^1_S(R, A) \rightarrow \mathrm{Hom}_R(\mathrm{Tor}^S_1(R, R), A)$ is transported into the map $T_R(A) \rightarrow \mathrm{Hom}_R(D_R, A)$ attaching to $\zeta \in T_R(A)$ the element ζ' of $\mathrm{Hom}_R(D_R, R)$ given by $\zeta'(\sum xdy) = \sum x \cdot \zeta(y)$.

Let U be a multiplicatively closed subset of nonzero elements of R containing the identity element. Let R_U denote the corresponding ring of quotients. This is still a K-algebra, and we write S_U for $R_U \otimes_K R_U$. By an argument almost identical with the localization argument of the proof of Theorem 3.1, we see that $R_U \otimes_R \mathrm{Tor}^S(R, R)$ is naturally isomorphic with $\mathrm{Tor}^{S_U}(R_U, R_U)$. In particular, we have $R_U \otimes_R D_R$ naturally isomorphic with D_{R_U}.

It is immediate from Theorems 2.1 and 3.1 that, if K is a perfect field and R is a regular affine K-algebra, then $\mathrm{Tor}^S(R, R)$ is a finitely generated projective R-module. We can use the above results to prove the following converse.

THEOREM 4.1(4). *Let K be a perfect field and let R be an affine K-algebra. Put $S = R \otimes_K R$ and suppose that $\mathrm{Tor}^S_1(R, R)$ is R-projective. Then R is a regular ring.*

Proof. Let Q be the field of quotients of R, and let N be a maximal ideal of R. Since S is Noetherian, $D_R (= I/I^2)$ is finitely generated, and, by assumption, it is R-projective. Hence $R_N \otimes_R D_R$ is a finitely generated projective, and hence free R_N-module. Thus D_{R_N} is a finitely generated free R_N-module. Now $Q \otimes_{R_N} D_{R_N}$ is isomorphic, as a Q-space, with D_Q. We have $\mathrm{Hom}_Q(D_Q, Q)$ isomorphic with $T_Q(Q)$, and, since Q is a finitely generated separable extension field of K, $T_Q(Q)$ is of dimension t over Q, where t is the transcendence degree of Q over K. Hence the dimension of D_Q over Q is equal to t, whence we conclude that D_{R_N} is of rank t over R_N.

Write L for R_N and M for NR_N. Since D_L is a free L-module of rank t, the L/M-space $\mathrm{Hom}_L(D_L, L/M)$ is of dimension t over L/M. We know from the

(4) The essential, local, part of this result is contained in [10, Folgerung, p. 177]. Our proof of the local part is adapted from [3, Exp. 17, Th. 5]. The global theorem has also been obtained by Y. Nakai, *On the theory of differentials in commutative rings*, J. Math. Soc. Japan 13 (1961), 63–84.

above that $\mathrm{Hom}_L(D_L, L/M)$ is isomorphic with $T_L(L/M)$. Hence $T_L(L/M)$ is of dimension t over L/M. Now a derivation of L into L/M must annihilate M^2 and hence induces a derivation of L/M^2 into L/M. Moreover, every derivation of L/M^2 into L/M can evidently be lifted to give a derivation of L into L/M. Hence $T_{L/M^2}(L/M)$ is isomorphic, as an L/M-space with $T_L(L/M)$, and thus is of dimension t.

Now L/M^2 is a finite dimensional algebra over the perfect field K with radical M/M^2. Hence we can write L/M^2 as a semidirect sum $L/M^2=V+M/M^2$, where V is a subalgebra isomorphic with the field L/M. Since K is perfect, every K-linear derivation of V into L/M must therefore be 0. Hence it is clear that the restriction of the elements of $T_{L/M^2}(L/M)$ to M/M^2 yields an isomorphism of $T_{L/M^2}(L/M)$ onto $\mathrm{Hom}_{L/M}(M/M^2, L/M)$. Hence we conclude that the dimension of M/M^2 over L/M is equal to t. It follows by a standard argument from this that M can be generated by t elements. Since t is the Krull dimension of L, this shows that L, i.e., R_N, is a regular local ring. This completes the proof.

5. **Explicit multiplication.** Let K be a commutative ring with identity, and let R be a commutative K-projective K-algebra. As before, let $S=R\otimes_K R$, and let ϕ be the natural epimorphism $S\rightarrow R$. If A and B are S-modules we regard them as two sided R-modules in the usual way, and we form $A\otimes_R B$. This is again a two sided R-module, and hence an S-module. Let X be an S-projective resolution of R. Using that R is K-projective, we see that $X\otimes_R X$ is S-projective; essentially, this follows from the fact that $S\otimes_R S=R\otimes_K R\otimes_K R$, with the two sided R-module structure in which $a\cdot(u\otimes v\otimes w)=(au)\otimes v\otimes w$ and $(u\otimes v\otimes w)\cdot a=u\otimes v\otimes(wa)$, so that $S\otimes_R S$ is S-projective whenever R is K-projective. Moreover, S is R-projective as a left or right R-module, so that X is an R-projective resolution of R. Hence $H(X\otimes_R X)=\mathrm{Tor}^R(R, R)$ and therefore has its components of positive degree equal to 0, so that $X\otimes_R X$ is still an S-projective resolution of R. For two sided R-modules U and V, regard $\mathrm{Hom}_S(U, V)$ as a two sided R-module such that $(r\cdot f)(u)=r\cdot f(u)(=f(r\cdot u))$ and $(f\cdot r)(u)=f(u)\cdot r(=f(u\cdot r))$.

Now the standard S-module homomorphism

$$\psi\colon \mathrm{Hom}_S(X, A)\otimes_R \mathrm{Hom}_S(X, B)\rightarrow \mathrm{Hom}_S(X\otimes_R X, A\otimes_R B),$$

where $\psi(f\otimes g)(u\otimes v)=f(u)\otimes g(v)$, induces an S-module homomorphism

$$\mathrm{Ext}_S(R, A)\otimes_R \mathrm{Ext}_S(R, B)\rightarrow \mathrm{Ext}_S(R, A\otimes_R B).$$

This is the product $\vee$, as given in [4, Ex. 2, p. 229], and it is independent of the choice of the resolution X. In particular, for $A=B=R$, this defines the structure of an associative and skew-commutative R-algebra on $\mathrm{Ext}_S(R, R)$.

In order to make the algebra structures on $\mathrm{Tor}^S(R, R)$ and $\mathrm{Ext}_S(R, R)$ explicit, we use the following well-known resolution Y of R as an S-module. We put $Y_0=S$ and we let $\phi\colon S\rightarrow R$ be the augmentation. Generally, let Y_n be the tensor product, relative to K, of $n+2$ copies of R. The S-module structure of Y_n is defined so that

$$(a \otimes b)\cdot(x_0 \otimes \cdots \otimes x_{n+1}) = (ax_0) \otimes x_1 \otimes \cdots \otimes x_n \otimes (x_{n+1}b).$$

The boundary map d on Y is given by

$$d(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes (x_i x_{i+1}) \otimes \cdots \otimes x_{n+1}.$$

This complex is not only acyclic but it has actually a right R-module homotopy h, where $h(x_1 \otimes \cdots \otimes x_n) = 1 \otimes x_1 \otimes \cdots \otimes x_n$. Since R is K-projective, it follows as above for $X \otimes_R X$ that Y is S-projective. Thus Y is an S-projective resolution of R.

The complex Y can be given the structure of an associative skew-commutative S-algebra with respect to which d is an antiderivation, as follows. If $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ are elements of R let $[x_1, \cdots, x_p; y_1, \cdots, y_q]$ stand for the sum, in the tensor product over K of $p+q$ copies of R, of all terms of the form $\pm z_1 \otimes \cdots \otimes z_{p+q}$, where $z_{i_k} = x_k$ for some ordered subset $(i_1, \cdots, i_p)$ of $(1, \cdots, p+q)$, and $z_{j_k} = y_k$ for the ordered complement $(j_1, \cdots, j_q)$, and where the sign is $+$ or $-$ according to whether the permutation $(i_1, \cdots, i_p, j_1, \cdots, j_q)$ of $(1, \cdots, p+q)$ is even or odd. Then the product in Y is given by the maps $Y_p \otimes_S Y_q \rightarrow Y_{p+q}$ that send

$$(x_0 \otimes \cdots \otimes x_{p+1}) \otimes_S (y_0 \otimes \cdots \otimes y_{q+1})$$

onto

$$(x_0 y_0) \otimes [x_1, \cdots, x_p; y_1, \cdots, y_q] \otimes (y_{q+1} x_{p+1}).$$

It can be verified directly that this is indeed an associative and skew-commutative product and, if α is homogeneous of degree p and β arbitrary, one has $d(\alpha\beta) = d(\alpha)\beta + (-1)^p \alpha d(\beta)$; cf. [4, pp. 218–219].

This product evidently induces a product in $R \otimes_S Y$, and hence in $\mathrm{Tor}^S(R, R)$. By the nature of the definition of the product on $\mathrm{Tor}^S(R, R)$, as given earlier in the general case, the product induced from that on Y is the standard product $\pitchfork$ on $\mathrm{Tor}^S(R, R)$.

Next we shall define a map of the complex Y into the complex $Y \otimes_R Y$ which will serve to make the product on $\mathrm{Ext}_S(R, R)$ explicit. We have $(Y \otimes_R Y)_p = \sum_{r=0}^{p} Y_r \otimes_R Y_{p-r}$. As an S-module, each $Y_r \otimes_R Y_{p-r}$ may be identified with the tensor product, relative to K, of $p+3$ copies of R, i.e., with Y_{p+1}. With this understanding, we define an S-module homomorphism γ_r: $Y_p \rightarrow Y_r \otimes_R Y_{p-r}$ such that

$$\gamma_r(x_0 \otimes \cdots \otimes x_{p+1}) = x_0 \otimes \cdots \otimes x_r \otimes 1 \otimes x_{r+1} \otimes \cdots \otimes x_{p+1}.$$

Now the desired map γ: $Y \rightarrow Y \otimes_R Y$ is defined so that, for $u \in Y_p$, the component of $\gamma(u)$ in $Y_r \otimes_R Y_{p-r}$ is $\gamma_r(u)$. It is somewhat lengthy, but not difficult, to verify that γ is compatible with the boundary maps on Y and on

$Y\otimes_R Y$. The product $\vee$ on $\mathrm{Ext}_S(R, R)$ is induced by the product on $\mathrm{Hom}_S(Y, R)$ induced by γ. In particular, with $\alpha\in\mathrm{Hom}_S(Y_p, R)$ and $\beta\in\mathrm{Hom}_S(Y_q, R)$, we have(5)

$$\begin{aligned}(\alpha\beta)(x_0 \otimes \cdots \otimes x_{p+q+1})\\ &= \alpha(x_0\otimes\cdots\otimes x_p\otimes 1)\beta(1\otimes x_{p+1}\otimes\cdots\otimes x_{p+q+1})\\ &= x_0\alpha(1\otimes x_1\otimes\cdots\otimes x_p\otimes 1)\beta(1\otimes x_{p+1}\otimes\cdots\otimes x_{p+q}\otimes 1)x_{p+q+1}.\end{aligned}$$

Consider a formal differential $\sum xdy\in D_R$. It is easily verified that the corresponding element of $\mathrm{Tor}_1^S(R, R)$ is represented in $R\otimes_S Y_1$ by the element $\sum x\otimes_S(1\otimes y\otimes 1)$. On the other hand, let $\zeta\in T_R(R)=T_R$ (say). Then it is easily seen that its image $\zeta^*\in\mathrm{Ext}_S^1(R, R)$ is represented in $\mathrm{Hom}_S(Y_1, R)$ by the element ζ', where $\zeta'(x_0\otimes x_1\otimes x_2)=x_0\zeta(x_1)x_2$.

Now let $\zeta_1, \cdots, \zeta_n$ be elements of T_R, and let $\zeta_1^*\cdots\zeta_n^*$ denote the product in $\mathrm{Ext}_S^n(R, R)$ of their canonical images ζ_i^* in $\mathrm{Ext}_S^1(R, R)$. Then $\zeta_1^*\cdots\zeta_n^*$ is represented in $\mathrm{Hom}_S(Y_n, R)$ by the product $\zeta_1'\cdots\zeta_n'$, as induced from the above map γ. One sees immediately from the formula written above that

$$(\zeta_1'\cdots\zeta_n')(x_0\otimes\cdots\otimes x_{n+1}) = x_0\zeta_1(x_1)\cdots\zeta_n(x_n)x_{n+1}.$$

Now let $\alpha\in\mathrm{Tor}_p^S(R, R)$, $\beta\in\mathrm{Tor}_q^S(R, R)$, and let us compute $h(\zeta_1^*\cdots\zeta_{p+q}^*)(\alpha\beta)$. Choose representatives $a\in R\otimes_S Y_p$ and $b\in R\otimes_S Y_q$ of α and β, respectively. Then $\alpha\beta$ is represented in $R\otimes_S Y_{p+q}$ by the product ab. We obtain $h(\zeta_1^*\cdots\zeta_{p+q}^*)(\alpha\beta)$ by applying the element of $\mathrm{Hom}_R(R\otimes_S Y_{p+q}, R)$ that corresponds naturally to $\zeta_1'\cdots\zeta_{p+q}'$ to ab. Clearly, the result so obtained is the same as the result one would obtain by performing the shuffling involved in forming ab on the sequence $\zeta_1', \cdots, \zeta_{p+q}'$ rather than on the arguments x_i and y_j in the product formula for ab. Hence we have

$$h(\zeta_1^*\cdots\zeta_{p+q}^*)(\alpha\beta) = \sum_t \sigma(t)h(\zeta_{t(1)}^*\cdots\zeta_{t(p)}^*)(\alpha)h(\zeta_{t(p+1)}^*\cdots\zeta_{t(p+q)}^*)(\beta),$$

where the summation goes over all those permutations t of $(1, \cdots, p+q)$ for which $t(1)<\cdots<t(p)$ and $t(p+1)<\cdots<t(p+q)$, and where $\sigma(t)$ is the signature of t.

Let $A^p(T_R)$ denote the R-module of all alternating (R, p)-linear maps f of p-tuples of elements of T_R into R, where "alternating" is to mean that $f(\zeta_{t(1)}, \cdots, \zeta_{t(p)})=\sigma(t)f(\zeta_1, \cdots, \zeta_p)$, for every permutation t of the set $(1, \cdots, p)$. Then the duality map h: $\mathrm{Ext}_S^p(R, R)\to\mathrm{Hom}_R(\mathrm{Tor}_p^S(R, R), R)$ yields an R-module homomorphism h^* of $\mathrm{Tor}_p^S(R, R)$ into $A^p(T_R)$, where

$$h^*(\alpha)(\zeta_1, \cdots, \zeta_p) = h(\zeta_1^*\cdots\zeta_p^*)(\alpha).$$

Put $A(T_R)=\sum_p A^p(T_R)$. Then $A(T_R)$ is the usual *algebra of the differen-*

(5) This is the value-wise product of cochains, such as was used in [11, Th. 6].

tial forms on R, where the multiplication is defined by means of the above shuffling of the arguments ζ_i and summing with the appropriate signatures. Our result is therefore the following.

THEOREM 5.1. *Let K be a commutative ring with identity, and let R be a K-projective commutative K-algebra with identity. Let $S=R\otimes_K R$. Then the duality map h of* $\mathrm{Ext}_S(R, R)$ *into* $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), R)$ *and the multiplication in* $\mathrm{Ext}_S(R, R)$ *yield, in the natural fashion, an R-algebra homomorphism h^* of* $\mathrm{Tor}^S(R, R)$ *into the R-algebra $A(T_R)$ of the differential forms on R.*

In particular, suppose that R is a regular affine K-algebra, where K is a perfect field. Then, in virtue of Theorem 2.1, the assumptions of Theorem 3.1 are satisfied, and we conclude that $\mathrm{Tor}^S(R, R)$ is finitely generated and projective as an R-module, and may be identified with the exterior R-algebra constructed over $\mathrm{Tor}_1^S(R, R)$. By imbedding $\mathrm{Tor}_1^S(R, R)$ as a direct R-module summand in a finitely generated free R-module, we see that $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), R)$ is isomorphic in the standard fashion (dual of exterior algebra $\approx$ exterior algebra over dual) with the exterior R-algebra constructed over $\mathrm{Hom}_R(\mathrm{Tor}_1^S(R, R), R)$.

Let $E(T_R)$ denote the exterior R-algebra constructed over T_R. The map $\zeta\rightarrow h(\zeta^*)$ is an isomorphism of T_R onto $\mathrm{Hom}_R(\mathrm{Tor}_1^S(R, R), R)$. By what we have just remarked, this extends in the standard fashion to an isomorphism ρ of $E(T_R)$ onto $\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), R)$. Let $\zeta_1, \cdots, \zeta_p$ be elements of T_R, and let $\zeta_1 \cdots \zeta_p$ stand for their product in $E(T_R)$. Then we see from Theorem 5.1 that $h(\zeta_1^* \cdots \zeta_p^*)=\rho(\zeta_1 \cdots \zeta_p)$. By Lemma 4.1, h is an isomorphism. Hence we conclude that the map $\zeta\rightarrow\zeta^*$ extends in the natural fashion to an R-algebra isomorphism of $E(T_R)$ onto $\mathrm{Ext}_S(R, R)$. Moreover, it is clear that, in the present case, h^* is a monomorphism sending $\mathrm{Tor}^S(R, R)$ onto the subalgebra of $A(T_R)$ that is generated by the *strongly alternating* maps, i.e., by the maps that vanish whenever two of the arguments are equal. We may summarize these results as follows.

THEOREM 5.2. *Let R be a regular affine K-algebra, where K is a perfect field. Then* $\mathrm{Tor}^S(R, R)$ *is naturally isomorphic with the exterior algebra $E(D_R)$ constructed over the R-module D_R of the formal differentials, and* $\mathrm{Ext}_S(R, R)$ *is naturally isomorphic with the exterior algebra $E(T_R)$ constructed over the R-module T_R of the K-derivations of R. These isomorphisms transport the duality map h:* $\mathrm{Ext}_S(R, R)\rightarrow\mathrm{Hom}_R(\mathrm{Tor}^S(R, R), R)$ *into the canonical homomorphism* $E(T_R)\rightarrow\mathrm{Hom}_R(E(D_R), R)$*, which is an isomorphism because D_R is finitely generated and projective as an R-module. The homomorphism h^* of Theorem* 5.1 *becomes an isomorphism of $E(D_R)$ onto the R-algebra of the strongly alternating differential forms.*

Now suppose that K is an arbitrary field, and that F is a finitely generated separable extension field of K. Then everything we have said above concerning the regular affine K-algebra R holds equally for F, the only change in the proof being that the appeal to Theorem 2.1 is now replaced with an appeal to Theorem 2.2. Thus we have the following result.

THEOREM 5.3. *The conclusions of Theorem 5.2 hold also when K is an arbitrary field and R is a finitely generated separable extension field of K.*

In his thesis (Chicago, 1956), W. Ballard has obtained a part of Theorem 5.3, namely: $\mathrm{Ext}_S(R, R) \approx E(T_R)$.

It is of interest to observe, in connection with Theorem 5.1, that the weak definition of "alternating" used in describing $A(T_R)$, rather than the usual stronger requirement on "alternating" maps, which demands that they vanish whenever two of the arguments are equal, is appropriate, in general. This is shown by the following example. Let K be a field of characteristic 2 and let $R=K[a]$, with $a^2=u\in K$, but $a\notin K$. Consider the element $1\otimes_S(1\otimes a\otimes a\otimes a)+1\otimes_S(1\otimes u\otimes a\otimes 1)\in R\otimes_S Y_2$. One checks immediately that this element is a cycle and thus represents an element $\alpha\in\mathrm{Tor}_2^S(R, R)$. We have $T_R=R\zeta$, where $\zeta(a)=1$. Now one verifies immediately that $h^*(\alpha)(\zeta, \zeta)=a$. Thus $h^*(\alpha)$ is not alternating in the strong sense. Of course, it is clear from Theorem 5.2 that this phenomenon cannot arise when R is a regular affine algebra over a perfect field. Note that in our present example the element α does not belong to the subalgebra of $\mathrm{Tor}^S(R, R)$ that is generated by $\mathrm{Tor}_1^S(R, R)$; indeed, h^* must evidently vanish on $\mathrm{Tor}_1^S(R, R)\mathrm{Tor}_1^S(R, R)$, in the present case.

6. **The algebra of differential operators.** Let K be a commutative ring with identity, and let R be a commutative K-algebra with identity. Let T_R denote the R-module of all K-derivations of R. Clearly, T_R has naturally the structure of a Lie algebra over K. We make the direct R-module sum $R+T_R$ into a Lie algebra over K, defining the commutators by the formula $[r_1+\tau_1, r_2+\tau_2]=(\tau_1(r_2)-\tau_2(r_1))+[\tau_1, \tau_2]$, where $r_i\in R$, $\tau_i\in T_R$, and $[\tau_1, \tau_2]$ is the ordinary commutator $\tau_1\tau_2-\tau_2\tau_1$ of the derivations τ_1 and τ_2.

Let U denote the universal enveloping algebra of the K-Lie algebra $R+T_R$, defined as the appropriate homomorphic image of the tensor K-algebra constructed over the K-module $R+T_R$. Denote the canonical K-module homomorphism of $R+T_R$ into U by $z\to z'$. Let U^+ denote the subalgebra of U that is generated by these elements z'. Let P denote the two sided ideal of U^+ that is generated by the elements of the form $r'z'-(r\cdot z)'$, where r ranges over R, z ranges over $R+T_R$, and $r\cdot z$ is the r-multiple of z for the R-module structure of $R+T_R$. We define V_R as the factor K-algebra U^+/P.

It is clear from this definition that the unitary V_R-modules are precisely those Lie algebra modules M for the Lie algebra $R+T_R$ on which we have $r\cdot(z\cdot m)=(r\cdot z)\cdot m$, for all $r\in R$, $z\in R+T_R$, $m\in M$, and on which $1\cdot m=m$, where 1 is the identity element of R. We shall call such modules *regular* $(R+T_R)$-modules. In particular, it is clear that R is a regular $(R+T_R)$-module in the natural way, and one sees easily that the representation of $R+T_R$ on R is faithful. Hence the canonical homomorphism of $R+T_R$ into

V_R is actually a monomorphism, and we shall accordingly identify $R+T_R$ with its image in V_R.

Let A and B be any two regular $(R+T_R)$-modules. We define the structure of a regular $(R+T_R)$-module on $\mathrm{Hom}_R(A, B)$ such that, with $r\in R$, $\tau\in T_R$, $a\in A$, $h\in\mathrm{Hom}_R(A, B)$,

$$(r\cdot h)(a) = r\cdot h(a) \quad\text{and}\quad (\tau\cdot h)(a) = \tau\cdot h(a) - h(\tau\cdot a).$$

The verification that the above conditions for regularity are satisfied presents no difficulties. Thus, if A and B are any two unitary V_R-modules, this defines the structure of a unitary V_R-module on $\mathrm{Hom}_R(A, B)$.

LEMMA 6.1. *Let B be a unitary V_R-module, and regard V_R as a V_R-module in the natural fashion (the operators being the left multiplications). Then the V_R-module* $\mathrm{Hom}_R(V_R, B)$, *with the structure defined above, is isomorphic with the V_R-module* $\mathrm{Hom}_R(V_R, B)$ *in which the module structure is defined in the usual way from the right multiplications in V_R,* i.e., *in which $(u\cdot h)(v)=h(vu)$, for all $h\in\mathrm{Hom}_R(V_R, B)$ and all u, v in V_R.*

Proof. For every $h\in\mathrm{Hom}_R(V_R, B)$, define $h^*\in\mathrm{Hom}_R(V_R, B)$ by $h^*(u) =(u\cdot h)(1)$, where $u\cdot h$ denotes the transform of h by the element u of V_R, for the first V_R-module structure of $\mathrm{Hom}_R(V_R, B)$. We claim that, for all u, v in V_R, $(u\cdot h^*)(v)=(v\cdot h)(u)$. This is immediately verified from the definitions when $u\in R$. Now suppose that the result has already been established for some u and all v. Let $\tau\in T_R$. Then we have

$$(\tau u\cdot h^*)(v) = \tau\cdot((u\cdot h^*)(v)) - (u\cdot h^*)(\tau v) = \tau\cdot((v\cdot h)(u)) - (\tau v\cdot h)(u) = (v\cdot h)(\tau u).$$

Thus our claim follows inductively for all right R-multiples u of monomials in elements of T_R and hence generally for all $u\in V_R$.

In particular, $(u\cdot h^*)(1)=h(u)$, so that $h^{**}=h$. Thus our map $h\to h^*$ is an additive (actually K-linear) involution of $\mathrm{Hom}_R(V_R, B)$. Now we have $(v\cdot h)^*(u)=(uv\cdot h)(1)=h^*(uv)$. This means that our involution $h\to h^*$ transports the V_R-module structure defined originally into the V_R-module structure given by $(v\cdot h)(u)=h(uv)$. This completes the proof of Lemma 6.1.

Let A and B be unitary V_R-modules. Using the induced R-module structures on A and B (written on the right or on the left, as the notation requires), we may form the tensor product $A\otimes_R B$. We define the structure of a regular $(R+T_R)$-module, and hence that of a unitary V_R-module, on $A\otimes_R B$ such that, for $r\in R$, $a\in A$, $b\in B$, $\tau\in T_R$, $r\cdot(a\otimes b)=(r\cdot a)\otimes b(=a\otimes(r\cdot b))$ and $\tau\cdot(a\otimes b)=(\tau\cdot a)\otimes b+a\otimes(\tau\cdot b)$. It is verified in a straightforward way that these definitions indeed satisfy the conditions for a regular $(R+T_R)$-module.

LEMMA 6.2. *Let B be a unitary V_R-module, and regard V_R as a V_R-module by left multiplication and as a right R-module by right multiplication. Then the V_R-module $B\otimes_R V_R$, with the module structure defined as above, is isomorphic* *with the V_R-module $V_R \otimes_R B$ in which the module structure is defined from that of V_R alone, i.e., is such that $u \cdot (v \otimes b) = (uv) \otimes b$.*

Proof. There is an evident V_R-module homomorphism ψ: $V_R \otimes_R B \to B \otimes_R V_R$ such that $\psi(u \otimes b) = u \cdot (b \otimes 1)$. We shall show that ψ is actually an isomorphism by exhibiting an inverse.

For this purpose, we must momentarily return to the definition of V_R as the factor algebra U^+/P. The map $r + \tau \to r - \tau$ is evidently an anti-automorphism of order 2 of the K-Lie algebra $R + T_R$ and induces, in the natural way, a K-linear involution $u \to u^*$ of U^+ such that, for $r \in R$ and $\tau \in T_R$, $(r')^* = r'$ and $(\tau')^* = -\tau'$, and, for arbitrary elements u and v of U^+, $(uv)^* = v^* u^*$. Our return to U^+ is necessitated by the fact that this involution does not send the ideal P into itself, so that it does not induce an involution of V_R. Let P_1 be the two sided ideal of U^+ that is generated by the elements of the form $r_1' r_2' - (r_1 r_2)'$, with r_1 and r_2 ranging over R. Write W for U^+/P_1 and P' for P/P_1. Then the map $r \to r'$, followed by the canonical epimorphism $U^+ \to W$, is a homomorphism of R into W and yields the structure of a two sided R-module on W in the natural fashion. Our involution $u \to u^*$ of U^+ sends P_1 into itself and hence induces an involution of W which we still denote by $w \to w^*$. If z is an element of $R + T_R$ we shall write z' also to denote the canonical image of z in W. By copying the above definitions of the V_R-modules $B \otimes_R V_R$ and $V_R \otimes_R B$ with W in the place of V_R, we define the W-modules $B \otimes_R W$ and $W \otimes_R B$. There is evidently a K-module homomorphism $x \to x^*$ of $B \otimes_R W$ into $W \otimes_R B$ such that $(b \otimes u)^* = u^* \otimes b$. Now define the map ϕ: $B \otimes_R W \to W \otimes_R B$ such that $\phi(b \otimes u) = (u^* \cdot (b \otimes 1'))^*$.

We claim that ϕ is a W-module homomorphism. Let $r \in R$. Then we have

$$\begin{aligned}\phi(r' \cdot (b \otimes u)) &= \phi((r \cdot b) \otimes u) = (u^* \cdot ((r \cdot b) \otimes 1'))^* \\ &= (u^* \cdot (b \otimes r'1'))^* = (u^* \cdot (b \otimes 1'r'))^* \\ &= r' \cdot (u^* \cdot (b \otimes 1'))^* = r' \cdot \phi(b \otimes u).\end{aligned}$$

Let $\tau \in T_R$. Then we have

$$\begin{aligned}\phi(\tau' \cdot (b \otimes u)) &= \phi((\tau \cdot b) \otimes u) + \phi(b \otimes (\tau' u)) \\ &= (u^* \cdot ((\tau \cdot b) \otimes 1'))^* - ((u^* \tau') \cdot (b \otimes 1'))^* = -(u^* \cdot (b \otimes \tau' 1'))^* \\ &= -(u^* \cdot (b \otimes 1'\tau'))^* \\ &= \tau' \cdot (u^* \cdot (b \otimes 1'))^* = \tau' \cdot \phi(b \otimes u).\end{aligned}$$

This suffices to establish our claim.

Now we shall show that ϕ maps the canonical image of $B \otimes_R P'$ in $B \otimes_R W$ into the canonical image of $P' \otimes_R B$ in $W \otimes_R B$. We recall that P is the two sided ideal of U^+ that is generated by the elements of the form $r'z' - (rz)'$, where $r \in R$ and $z \in R + T_R$. Actually, P coincides with the right ideal that is generated by these elements. In order to see this, it suffices to show that the

K-subspace spanned by the elements of the form $r'z'-(rz)'$ is stable under commutation with the elements of $R+T_R$. Now let $z_1 \in R+T_R$. Then we have

$$\begin{aligned}[z_1', r'z' - (rz)'] &= [z_1', r']z' + r'[z_1', z'] - [z_1', (rz)'] \\ &= [z_1, r]'z' + r'[z_1, z]' - [z_1, rz]' \\ &= ([z_1, r]'z' - ([z_1, r]z)') + (r'[z_1, z]' - (r[z_1, z])'),\end{aligned}$$

and this is indeed of the required form. Next we observe that $(r'z'-(rz)') \cdot (b\otimes u) = b \otimes ((r'z'-(rz)')u)$, as is easily verified directly from the definition of the W-module structure on $B\otimes_R W$. Since ϕ is a W-module homomorphism, this shows that ϕ maps the element on the right into the canonical image of $P'\otimes_R B$ in $W\otimes_R B$. Hence we may now conclude that ϕ maps the image of $B\otimes_R P'$ in $B\otimes_R W$ into the image of $P'\otimes_R B$ in $W\otimes_R B$. Hence ϕ induces a map γ of $B\otimes_R V_R$ into $V_R\otimes_R B$. Since ϕ is a W-module homomorphism, it is clear that γ is a V_R-module homomorphism. It follows immediately from this that $\gamma\circ\psi$ is the identity map on $V_R\otimes_R B$.

There remains to prove that $\psi\circ\gamma$ is the identity map on $B\otimes_R V_R$. It is immediate that $(\psi\circ\gamma)(b\otimes u) = b\otimes u$ whenever $u\in R$. Suppose that we have already shown that this holds for some $u\in V_R$ and all $b\in B$. Then we have, with $\tau\in T_R$,

$$\begin{aligned}(\psi\circ\gamma)(b\otimes(\tau u)) &= (\psi\circ\gamma)(\tau\cdot(b\otimes u) - (\tau\cdot b)\otimes u) \\ &= \tau\cdot(\psi\circ\gamma)(b\otimes u) - (\tau\cdot b)\otimes u \\ &= \tau\cdot(b\otimes u) - (\tau\cdot b)\otimes u = b\otimes(\tau u).\end{aligned}$$

Hence it follows by an evident induction on the "degree" of u, written as a polynomial in elements of T_R, that $\psi\circ\gamma$ is indeed the identity map. This completes the proof of Lemma 6.2.

Now let us assume that K is a field and that R is an affine K-algebra. Let Q denote the field of quotients of R. Since every K-derivation of R extends uniquely to a K-derivation of Q, we have a natural injection $T_R\rightarrow T_Q$, and hence a canonical Q-linear map $Q\otimes_R T_R\rightarrow T_Q$. Since R is finitely ring-generated over K, we see immediately that this map is an epimorphism. Since Q is R-flat, the map $Q\otimes_R T_R\rightarrow Q\otimes_R T_Q$ induced by the injection $T_R\rightarrow T_Q$ is a monomorphism. Evidently, the canonical map $Q\otimes_R T_Q\rightarrow T_Q$ is an isomorphism. Thus the canonical map $Q\otimes_R T_R\rightarrow T_Q$ is an isomorphism. We shall identify $Q\otimes_R T_R$ with T_Q whenever convenient.

The injection $R+T_R\rightarrow Q+T_Q$ is both a Lie algebra homomorphism and an R-module homomorphism, and hence extends uniquely to a K-algebra homomorphism $V_R\rightarrow V_Q$. This induces a canonical Q-linear epimorphism $Q\otimes_R V_R\rightarrow V_Q$. Now we note that Q is a unitary V_R-module in the natural fashion, so that we may equip $Q\otimes_R V_R$ with the structure of a unitary V_R-module, in the manner explained above. The induced R-module structure coincides with the R-module structure induced by the natural Q-module structure of $Q\otimes_R V_R$. We define a product on $Q\otimes_R V_R$ by means of this V_R-module structure and the natural Q-module structure, the definition being such that

$$(q_1 \otimes v_1)(q_2 \otimes v_2) = q_1\cdot(v_1\cdot(q_2 \otimes v_2)).$$

It is not difficult to verify, by the same kind of induction we used in the proof of Lemma 6.2, that this gives the structure of an associative K-algebra on $Q\otimes_R V_R$. Moreover, it is then clear that our Q-linear epimorphism $Q\otimes_R V_R \rightarrow V_Q$ is actually a K-algebra epimorphism.

For every non-negative integer m, let V_R^m denote the R-submodule of V_R consisting of the elements that can be written as sums of products of elements of R and T_R, where each product has at most m factors from T_R. If m is a negative integer, let $V_R^m=(0)$. Let $G(V_R)$ be the graded R-algebra $\sum_m V_R^m/V_R^{m-1}$ obtained from this filtration of V_R; $G(V_R)$ is indeed an R-algebra, and not only a K-algebra, because the commutation with an element of R sends each V_R^m into V_R^{m-1}. Let $S(T_R)$ denote the symmetric R-algebra built over the R-module T_R. We have an evident natural R-algebra epimorphism $S(T_R) \rightarrow G(V_R)$. In the same way, we define $G(V_Q)$, $S(T_Q)$, and we have the natural Q-algebra epimorphism $S(T_Q)\rightarrow G(V_Q)$.

THEOREM 6.1. *Let R be an affine K-algebra, where K is an arbitrary field, and let Q be the field of quotients of R. Let F be an arbitrary extension field of K, regarded as a K-algebra. Then the canonical epimorphisms $S(T_F)\rightarrow G(V_F)$ and $Q\otimes_R V_R\rightarrow V_Q$ are isomorphisms.*

Proof. The first part of the theorem is analogous to the Poincaré-Birkhoff-Witt Theorem for universal enveloping algebras of Lie algebras and can be proved by the method in [4, Lemma 3.5, p. 272]: if (τ_i) is an ordered F-basis for T_F, it is clear from the definition of V_F that every element of V_F can be written as an F-linear combination of ordered monomials $\tau_{i_1}\cdots\tau_{i_n}$; $i_1\leqq \cdots \leqq i_n$. Now one shows inductively that $S(T_F)$ can be equipped with the structure of a regular $(F+T_F)$-module such that the transform of $1\in S(T_F)$ by $v\in V_F$ is precisely the element of $S(T_F)$ that is represented by the above standard expression for v. This evidently implies that the representation of the elements of V_F in this standard form is unique and that the canonical map $S(T_F)\rightarrow G(V_F)$ is an isomorphism.

In order to prove the second part, let us observe first that the isomorphism $Q\otimes_R T_R\rightarrow T_Q$ extends canonically to a Q-algebra isomorphism $Q\otimes_R S(T_R) \rightarrow S(T_Q)$, because $Q\otimes_R S(T_R)$ may be identified with $S(Q\otimes_R T_R)$. The natural homomorphism $V_R\rightarrow V_Q$ induces an R-algebra homomorphism $G(V_R) \rightarrow G(V_Q)$ which, in turn, induces a Q-algebra homomorphism $Q\otimes_R G(V_R) \rightarrow G(V_Q)$. Thus we have an exact and commutative diagram of homomorphisms

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & S(T_Q) & \longrightarrow & G(V_Q) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & Q \otimes_R S(T_R) & \rightarrow & Q \otimes_R G(V_R) & \rightarrow & 0 \\ & & \uparrow & & & & \\ & & 0 & & & & \end{array}$$

which shows that the homomorphism $Q \otimes_R G(V_R) \to G(V_Q)$ is an isomorphism.

Now let x be an element of $Q \otimes_R V_R$ such that the image of x in V_Q is 0. If $x \neq 0$, let m be the lowest non-negative integer such that x belongs to the canonical image of $Q \otimes_R V_R^m$ in $Q \otimes_R V_R$. Then x represents a nonzero element of $Q \otimes_R (V_R^m / V_R^{m-1}) \subset Q \otimes_R G(V_R)$. Since the map $Q \otimes_R G(V_R) \to G(V_Q)$ is a monomorphism, this contradicts the assumption that the image of x in V_Q is 0. Hence the epimorphism $Q \otimes_R V_R \to V_Q$ is indeed an isomorphism, and Theorem 6.1 is proved.

THEOREM 6.2. *In the notation of Theorem* 6.1, *assume that T_R is projective as an R-module. Then the natural homomorphism $V_R \to V_Q$ is a monomorphism, and the canonical epimorphism $S(T_R) \to G(V_R)$ is an isomorphism.*

Proof. Since T_R is R-projective, so is $S(T_R)$, as is seen by imbedding T_R as a direct R-module summand in a free R-module. Hence the natural homomorphism $S(T_R) \to Q \otimes_R S(T_R) = S(T_Q)$ is a monomorphism. Now we consider the exact and commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \rightarrow & S(T_Q) & \rightarrow & G(V_Q) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & S(T_R) & \rightarrow & G(V_R) & \rightarrow & 0 \\ & & \uparrow & & & & \\ & & 0 & & & & \end{array}$$

This shows immediately that the epimorphism $S(T_R) \to G(V_R)$ is an isomorphism and that the homomorphism $G(V_R) \to G(V_Q)$ is a monomorphism. The last fact implies as above that the homomorphism $V_R \to V_Q$ is a monomorphism, so that Theorem 6.2 is proved.

7. **The** Ext **functor for the algebra of differential operators.** If M is any module for a commutative ring, we denote by $E(M) = \sum_p E^p(M)$ the exterior algebra constructed over M. We deal with an affine algebra R over a field K, and we let Q denote the field of quotients of R. Consider the natural V_Q-module $V_Q \otimes_Q E(T_Q)$, where $u \cdot (v \otimes e) = (uv) \otimes e$. This is graded by the submodules $V_Q \otimes_Q E^p(T_Q)$. Exactly as for the analogous situation of the universal

enveloping algebra of a Lie algebra [4, Th. 7.1, p. 280], we define a homogeneous V_Q-endomorphism d of degree -1 on this module such that $d^2=0$. This endomorphism is given by the formula

$$d(v \otimes \tau_0 \cdots \tau_p) = \sum_{i=0}^{p} (-1)^i (v\tau_i) \otimes \tau_0 \cdots \hat{\tau}_i \cdots \tau_p$$
$$+ \sum_{r<s} (-1)^{r+s} v \otimes [\tau_r, \tau_s]\tau_0 \cdots \hat{\tau}_r \cdots \hat{\tau}_s \cdots \tau_p,$$

where the τ_i are arbitrary elements of T_Q. The definition enforces this formula in the case where the τ_i belong to a given Q-basis of T_Q, and then one verifies easily that the formula holds generally. We augment this complex by the natural V_Q-module epimorphism $V_Q \otimes_Q E^0(T_Q) = V_Q \rightarrow Q$, where the image of $v \in V_Q$ in Q is the transform $v \cdot 1$ of $1 \in Q$ by v, according to the natural V_Q-module structure of Q. Since, by Theorem 6.1, $G(V_Q)$ is isomorphic with $S(T_Q)$, the usual filtration argument [4, pp. 281–282] shows that the augmented complex $V_Q \otimes_Q E(T_Q)$ is acyclic and is therefore a V_Q-free resolution of the V_Q-module Q; a sketch of this argument is included in what follows. Note that this conclusion holds more generally for an arbitrary extension field of K in the place of Q.

Now let us assume that T_R is R-projective. Then the same holds for $E(T_R)$. By Theorem 6.2, the natural homomorphism $V_R \rightarrow V_Q$ is a monomorphism. Hence the induced homomorphism $V_R \otimes_R E(T_R) \rightarrow V_Q \otimes_R E(T_R)$ is a monomorphism. Now $V_Q \otimes_R E(T_R) \equiv V_Q \otimes_Q (Q \otimes_R E(T_R)) = V_Q \otimes_Q E(Q \otimes_R T_R) = V_Q \otimes_Q E(T_Q)$. Taking these identifications into due account, we see that our result means that the map $V_R \otimes_R E(T_R) \rightarrow V_Q \otimes_Q E(T_Q)$ that is induced by the natural maps $V_R \rightarrow V_Q$ and $E(T_R) \rightarrow E(T_Q)$ is a monomorphism.

It is seen immediately from the explicit formula for the boundary map d on $V_Q \otimes_Q E(T_Q)$ that the image of $V_R \otimes_R E(T_R)$ is stable under d. Hence d induces a boundary map (still denoted d) on $V_R \otimes_R E(T_R)$, which satisfies the same explicit formula. We have a filtration of this complex by the R-subcomplexes $\sum_q V_R^{p-q} \otimes_R E^q(T_R)$ (which, since $E(T_R)$ is R-projective, may be identified with their canonical images in $V_R \otimes_R E(T_R)$). Now the associated graded complex may evidently be identified with the complex $G(V_R) \otimes_R E(T_R)$ and hence, using Theorem 6.2, with the complex $S(T_R) \otimes_R E(T_R)$. The boundary operator induced by d (still denoted d) is given by the formula

$$d(u \otimes \tau_0 \cdots \tau_p) = \sum_{i=0}^{p} (-1)^i (u\tau_i) \otimes \tau_0 \cdots \hat{\tau}_i \cdots \tau_p.$$

Now let F be a free R-module containing T_R as a direct R-module summand. Then our complex $S(T_R) \otimes_R E(T_R)$ is a direct R-complex summand of the usual Koszul complex $S(F) \otimes_R E(F)$. We shall prove from this that the augmented complex $V_R \otimes_R E(T_R)$ has an R-homotopy.

As is well known, the augmented complex $S(F)\otimes_R E(F)$ has an R-homotopy sending R onto $S^0(F)\otimes_R E^0(F)$ and each $S^{p-q}(F)\otimes_R E^q(F)$ into $S^{p-q-1}(F)\otimes_R E^{q+1}(F)$, [8, p. 259]. Since $S(T_R)\otimes_R E(T_R)$ is a direct R-complex summand of $S(F)\otimes_R E(F)$, the augmented complex $S(T_R)\otimes_R E(T_R)$ has an induced R-homotopy h sending R onto $S^0(T_R)\otimes_R E^0(T_R)$ and each $S^{p-q}(T_R)\otimes_R E^q(T_R)$ into $S^{p-q-1}(T_R)\otimes_R E^{q+1}(T_R)$. Let X stand for the augmented complex $V_R\otimes_R E(T_R)$. For $p\geqq 0$, let X_p stand for the augmented subcomplex $\sum_q V_R^{p-q}\otimes_R E^q(T_R)$, and put $X_p=(0)$, for $p<0$. Then the augmented complex $S(T_R)\otimes_R E(T_R)$ is the associated graded complex

$$\sum_p X_p/X_{p-1} = \sum_p G^p(X) = G(X).$$

Thus we may regard h as an R-homotopy of $G(X)$ under which each $G^p(X)$ is stable.

Now consider the natural R-module epimorphism $X_p\to X_p/X_{p-1}=G^p(X)$. Since X_p/X_{p-1} is isomorphic, as an R-module, with $\sum_q S^{p-q}(T_R)\otimes_R E^q(T_R)$, for $p>0$, and since X_0 is isomorphic, as an R-module, with $S^0(T_R)\otimes_R E^0(T_R)+R=R+R$, we know that each X_p/X_{p-1} is R-projective. Hence we can make a direct R-module decomposition $X_p=X_{p-1}+Y_p$, and X_p is the direct R-module sum $\sum_{q\leqq p} Y_q$. If α_p denotes the natural R-module epimorphism $X_p\to X_p/X_{p-1}$ then the restriction of α_p to Y_p is an isomorphism, and we may define an R-module isomorphism $\alpha: X\to G(X)$ by making $\alpha=\alpha_p$ on the component Y_p. In particular, it follows that *X is projective as an R-module.*

We have $\alpha_p d=d\alpha_p$. Moreover, $\alpha-\alpha_p$ evidently sends X_p into $\sum_{q<p} G^q(X)$. Hence we have $(\alpha d-d\alpha)(X_p)\subset\sum_{q<p} G^q(X)$. Since $G^p(X)\subset\alpha(X_p)$, this implies that $(d\alpha^{-1}-\alpha^{-1}d)(G^p(X))\subset X_{p-1}$. Put $\gamma=\alpha^{-1}h\alpha$. Then, writing 1 for the identity map, we have

$$\gamma d + d\gamma - 1 = \alpha^{-1}h(\alpha d - d\alpha) + (d\alpha^{-1} - \alpha^{-1}d)h\alpha.$$

Hence we find that $(\gamma d+d\gamma-1)(X_p)\subset X_{p-1}$. Furthermore, each X_p is stable under γ.

Now we have $-(\gamma d+d\gamma-1)^2=\gamma' d+d\gamma'-1$, where $\gamma'=2\gamma-\gamma d\gamma-d\gamma^2$. Hence $(\gamma' d+d\gamma'-1)(X_p)\subset X_{p-2}$, and $(\gamma'-\gamma)(X_p)\subset X_{p-1}$. Iteration of this process leads to a sequence of R-endomorphisms γ_k of X such that $(\gamma_k d+d\gamma_k-1)(X_p)\subset X_{p-2^k}$, and $(\gamma_{k+1}-\gamma_k)(X_p)\subset X_{p-2^k}$. Since $X_q=(0)$, for $q<0$, $\gamma_{k+r}-\gamma_k$ annihilates X_{2^k-1}, for all $r\geqq 0$. Hence there is an R-endomorphism ζ on X such that ζ coincides with γ_k on X_{2^k-1}, for each k, and we have $\zeta d+d\zeta=1$, i.e., ζ is an R-homotopy of X. Thus we have the following result.

THEOREM 7.1. *Let R be an affine algebra over a field K, and suppose that T_R is R-projective. Then the complex $V_R\otimes_R E(T_R)$, as defined above, is a V_R-projective resolution of the V_R-module R and has an R-homotopy. If F is an arbitrary extension field of K, regarded as a K-algebra, then $V_F\otimes_F E(T_F)$ is a V_F-free resolution of the V_F-module F.*

Note. It is clear from our proof that Theorem 7.1 holds more generally for any integral domain R over K such that T_R is R-projective; one must merely replace T_Q with the canonical image of $Q \otimes_R T_R$ in T_Q. As was pointed out to us by A. Shapiro, further generalizations, covering cases where R is not an integral domain (e.g. the algebra of all differentiable functions on a differentiable manifold), are easily obtainable by simultaneous use of a suitably large family of localizations of R. Naturally, these generalizations extend to Corollary 7.1 below.

Let $P=R$, or $P=F$, as in Theorem 7.1, and let A be a unitary V_P-module. By Theorem 7.1, we have $\mathrm{Ext}_{V_P}(P, A) = H(\mathrm{Hom}_{V_P}(V_P \otimes_P E(T_P), A))$. We may identify $\mathrm{Hom}_{V_P}(V_P \otimes_P E(T_P), A)$ with $\mathrm{Hom}_P(E(T_P), A)$. Let δ be the coboundary map in $\mathrm{Hom}_P(E(T_P), A)$ that is induced by the boundary map d on $V_P \otimes_P E(T_P)$. Then, writing the elements of $\mathrm{Hom}_P(E(T_P), A)$ as strongly alternating maps with arguments in T_P and values in A, we have

$$
\begin{aligned}
(\delta f)(\tau_0, \cdots, \tau_p) = & \sum_{i=0}^{p} (-1)^i \tau_i (f(\tau_0, \cdots, \hat{\tau}_i, \cdots, \tau_p)) \\
& + \sum_{r<s} (-1)^{r+s} f([\tau_r, \tau_s], \tau_0, \cdots, \hat{\tau}_r, \cdots, \hat{\tau}_s, \cdots, \tau_p),
\end{aligned}
$$

if $f \in \mathrm{Hom}_P(E^p(T_P), A)$. Thus we see that $\mathrm{Ext}_{V_P}(P, A)$ is naturally isomorphic with the usual cohomology space based on the strongly alternating A-valued differential forms. We state this formally, for reference.

COROLLARY 7.1. *Let $P=R$, or $P=F$, as in Theorem 7.1. Then the cohomology K-space based on the strongly alternating differential forms with values in a unitary V_P-module A may be identified with* $\mathrm{Ext}_{V_P}(P, A)$.

If we assume that P is either a regular affine K-algebra, where K is a perfect field, or that K is an arbitrary field and P is a finitely generated separable extension field of K, we may appeal to Theorem 5.2, or to Theorem 5.3, respectively, to see that the complex of the strongly alternating P-valued differential forms may be identified with the complex $E(D_P)$ of the formal differentials, whose differential operator is the canonical extension of the map $x \to dx$ of P into D_P. Hence, in this case, *the cohomology K-space based on the formal differentials of the K-algebra P may be identified*, by Corollary 7.1, *with* $\mathrm{Ext}_{V_P}(P, P)$ (6).

8. The homological dimension of the algebra of differential operators.

THEOREM 8.1. *Let F be a finitely generated extension field of an arbitrary field K. Then the global homological dimension $d(V_F)$ of V_F is equal to the*

(6) It may be of interest to point out that, in a rather different situation, namely when P is a finitely generated purely inseparable extension field of exponent 1 of K, the cohomology K-algebra of the strongly alternating P-valued differential forms has been determined explicitly by P. Cartier in [5].

dimension over F of the space T_F of all K-derivations of F, and is equal to the projective dimension $d_{V_F}(F)$ of the V_F-module F.

Proof. Let M be any unitary V_F-module. Then the resolution $V_F \otimes_F E(T_F)$ of F, as in Theorem 7.1, dualizes into an exact sequence of V_F-modules and V_F-homomorphisms

$$(0) \to M = \mathrm{Hom}_F(F, M) \to \mathrm{Hom}_F(V_F \otimes_F E^0(T_F), M) \to \mathrm{Hom}_F(V_F \otimes_F E^1(T_F), M) \to \cdots.$$

As a V_F-module, each $\mathrm{Hom}_F(V_F \otimes_F E^p(T_F), M)$ is isomorphic with a direct sum of a finite number of copies of $\mathrm{Hom}_F(V_F, M)$, with the module structure defined just above the statement of Lemma 6.1. However, by Lemma 6.1, this V_F-module is isomorphic with the V_F-module $\mathrm{Hom}_F(V_F, M)$ with the module structure given by $(u\cdot h)(v) = h(vu)$. As is well known, and easy to show directly (using that F is a field), this module is V_F-injective. Hence we conclude that the above exact sequence is a V_F-injective resolution of M. Since $E^p(T_F) = (0)$ when p exceeds the F-dimension of T_F, it follows that $d(V_F)$ does not exceed the F-dimension of T_F.

Now let n denote the F-dimension of T_F and let A be any unitary V_F-module. There is a basis $\zeta_1, \cdots, \zeta_n$ for T_F over F and elements $x_1, \cdots, x_n$ in F such that $\zeta_i(x_j) = \delta_{ij}$ [7, Lemma 2.1]. In particular, we have $[\zeta_i, \zeta_j] = 0$, for all i and j. If we use the resolution $V_F \otimes_F E(T_F)$ of F, the K-space $\mathrm{Ext}^n_{V_F}(F, A)$ appears as the factor space

$$\mathrm{Hom}_F\,(E^n(T_F), A)/\delta(\mathrm{Hom}_F\,(E^{n-1}(T_F), A).$$

Using the explicit formula for δ, as given below the statement of Theorem 7.1, and taking account of the fact that $[\zeta_i, \zeta_j] = 0$, we see that $\delta(\mathrm{Hom}_F(E^{n-1}(T_F), A) \approx T_F\cdot A$. On the other hand, $\mathrm{Hom}_F(E^n(T_F), A) \approx A$. Hence we see that $\mathrm{Ext}^n_{V_F}(F, A)$ is isomorphic with $A/T_F\cdot A$. Taking $A = V_F$, we see from this that $\mathrm{Ext}^n_{V_F}(F, V_F) \approx F \neq (0)$. Hence we have indeed $d(V_F) = d_{V_F}(F) = n$.

Now let R be an affine K-algebra such that T_R is R-projective. In this case, it appears that the relative global homological dimension $d(V_R, R)$ is more easily accessible than $d(V_R)$. We refer to [8] for the requisite notions of relative homological algebra, but we recall that the relative homological notions for (V_R, R) are obtained from the corresponding notions for V_R simply by replacing "exact sequence of V_R-homomorphisms" with "R-split sequence of V_R-homomorphisms" throughout. The ordinary projective dimension of a V_R-module M will be denoted by $d_{V_R}(M)$, and the relative projective dimension of M will be denoted by $d_{(V_R,R)}(M)$. We shall prove the following result.

THEOREM 8.2. *Let K be a field and let R be an affine K-algebra such that T_R is R-projective. Let Q be the field of quotients of R, and let n be the Q-dimension of T_Q. Then, for every unitary V_R-module A, the canonical map (induced* *from maps of the appropriate resolutions) of* $\mathrm{Ext}_{(V_R,R)}(R, A)$ *into* $\mathrm{Ext}_{V_R}(R, A)$ *is an isomorphism. We have* $d_{V_R}(R)=d_{(V_R,R)}(R)=d(V_R, R)=n$. *If* R *is regular then* $d(V_R)\leqq n+t$, *where* t *is the transcendence degree of* Q *over* K.

Proof. By Theorem 7.1, $V_R\otimes_R E(T_R)$ is both a V_R-projective resolution of R and a (V_R, R)-projective resolution with R-homotopy. Hence both $\mathrm{Ext}_{V_R}(R, A)$ and $\mathrm{Ext}_{(V_R,R)}(R, A)$ can be computed as

$$H(\mathrm{Hom}_{V_R}(V_R\otimes_R E(T_R), A)),$$

which establishes the first assertion of our theorem.

We know that $Q\otimes_R E(T_R)$ is isomorphic with $E(T_Q)$. Since $E(T_R)$ is R-projective, the natural map $E(T_R)\rightarrow Q\otimes_R E(T_R)$ is a monomorphism. Hence we have $E^p(T_R)=(0)$ for $p>n$. Hence it is clear from the resolution $V_R\otimes_R E(T_R)$ of R that $d_{V_R}(R)=d_{(V_R,R)}(R)\leqq n$.

Now let us consider the same sequence we used at the beginning of our proof of Theorem 8.1:

$$(0)\rightarrow M=\mathrm{Hom}_R(R, M)\rightarrow \mathrm{Hom}_R(V_R\otimes_R E^0(T_R), M)$$
$$\rightarrow \mathrm{Hom}_R(V_R\otimes_R E^1(T_R), M)\rightarrow\cdots.$$

By Theorem 7.1, the resolution $V_R\otimes_R E(T_R)$ of R has an R-homotopy, which evidently induces an R-homotopy of the above dual sequence. On the other hand, we see from Lemma 6.1, as in the proof of Theorem 8.1, that each $\mathrm{Hom}_R(V_R\otimes_R E^p(T_R), M)$ is isomorphic with a direct V_R-module summand of a direct sum of V_R-modules $\mathrm{Hom}_R(V_R, M)$ with the module structure given by $(v\cdot h)(u)=h(uv)$. By [8, Lemma 1], this last V_R-module is (V_R, R)-injective. Hence the above sequence is a (V_R, R)-injective resolution with R-homotopy of M. Hence it is clear that $d(V_R, R)\leqq n$.

Now we claim that, *for every* V_Q-*module* M, $\mathrm{Ext}_{V_R}(R, M)$ *is isomorphic with* $\mathrm{Ext}_{V_Q}(Q, M)$. By Theorem 6.1, V_Q is isomorphic with $Q\otimes_R V_R$. It is clear from the definition of the algebra structure of $Q\otimes_R V_R$ that this isomorphism transports the right V_R-module structure of $Q\otimes_R V_R$ into the right V_R-module structure of V_Q obtained from the natural map of V_R into V_Q. Since Q is R-flat, this implies that, as a right V_R-module, V_Q is V_R-flat. Hence, if X is any V_R-projective resolution of R, $V_Q\otimes_{V_R} X$ is a V_Q-projective resolution of $V_Q\otimes_{V_R} R$. Using the natural V_Q-module structure of Q, we obtain a V_Q-module epimorphism $V_Q\otimes_{V_R} R\rightarrow Q$ sending $v\otimes r$ onto $v\cdot r\in Q$. It is easy to verify that this is actually an isomorphism, so that we may identify $V_Q\otimes_{V_R} R$ with Q, as a V_Q-module. Thus $V_Q\otimes_{V_R} X$ is a V_Q-projective resolution of Q. Since $\mathrm{Hom}_{V_Q}(V_Q\otimes_{V_R} X, M)$ may be identified with $\mathrm{Hom}_{V_R}(X, M)$, this establishes our claim. Now it is clear from Theorem 8.1 (applied to Q) that $d_{V_R}(R)\geqq n$. We have shown that $n\leqq d_{V_R}(R)=d_{(V_R,R)}(R)\leqq d(V_R, R)\leqq n$, so that all but the last statement of Theorem 8.2 is proved.

As a by-product of our proof of the existence of an R-homotopy in

$V_R \otimes_R E(T_R)$, we had obtained the result that this complex is R-projective. In particular, V_R is R-projective as a left R-module; in fact, as an R-module, V_R is isomorphic with $G(V_R) \approx S(T_R)$. Similarly, V_R is R-projective as a right R-module. Hence we may apply [9, Th. 1] to conclude that, for every unitary V_R-module N,

$$d_{V_R}(N) \leqq d_{(V_R, R)}(N) + d_R(N).$$

If R is a regular affine K-algebra, we have $d(R_M) = t$, for every maximal ideal M of R, by [2, Ths. 1.9, 1.10]. Since $d(R) = \max_M(d(R_M))$ [4, Ex. 11, p. 142; 1, Th. 1], we have therefore $d(R) = t$. Hence the above results give $d(V_R) \leqq d(V_R, R) + d(R) = n + t$. This completes the proof of Theorem 8.2.

9. **The product for** $\mathrm{Ext}_{V_P}(P, *)$. Let K be a field, and let P be a K-algebra which is either an affine K-algebra with T_P P-projective or an arbitrary extension field of K. If A and B are unitary V_P-modules there is a product

$$\mathrm{Ext}_{V_P}(P, A) \otimes_K \mathrm{Ext}_{V_P}(P, B) \rightarrow \mathrm{Ext}_{V_P}(P, A \otimes_P B)$$

which is defined as follows. Let X be any V_P-projective resolution of P. Noting that V_P is P-projective, so that X is P-projective, and appealing to Lemma 6.2, we see that the V_P-module $X \otimes_P X$ is still V_P-projective. Moreover, since X is also a P-projective resolution of P, we have $H(X \otimes_P X) = \mathrm{Tor}^P(P, P)$, whence $H(X \otimes_P X)$ has its components of positive degree equal to (0). Hence $X \otimes_P X$ is still a V_P-projective resolution of $P \otimes_P P = P$. Hence the natural K-space homomorphism

$$\phi\colon \mathrm{Hom}_{V_P}(X, A) \otimes_K \mathrm{Hom}_{V_P}(X, B) \rightarrow \mathrm{Hom}_{V_P}(X \otimes_P X, A \otimes_P B),$$

where $\phi(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$, induces a product for $\mathrm{Ext}_{V_P}(P, *)$, as indicated above. It is seen as usual that this product is associative and skew-commutative. In order to make this product explicit, we require a map of the complex X into the complex $X \otimes_P X$, when $X = V_P \otimes_P E(T_P)$. By imbedding T_P as a direct P-module summand in a free P-module, we see easily that there is a P-module homomorphism $E^n(T_P) \rightarrow E^p(T_P) \otimes_P E^{n-p}(T_P)$ sending each product $\zeta_1 \cdots \zeta_n$ of elements of T_P onto

$$\sum_t \sigma(t) \zeta_{t(1)} \cdots \zeta_{t(p)} \otimes \zeta_{t(p+1)} \cdots \zeta_{t(n)},$$

where the summation goes over all permutations t of $(1, \cdots, n)$ for which $t(1) < \cdots < t(p)$ and $t(p+1) < \cdots < t(n)$, and where $\sigma(t)$ is the signature of t. Hence there is a map $X \rightarrow X \otimes_P X$ sending $v \otimes \zeta_1 \cdots \zeta_n \in V_P \otimes_P E^n(T_P)$ onto

$$v \cdot \left(\sum_{p=0}^{n} \left(\sum_t \sigma(t)(1 \otimes \zeta_{t(1)} \cdots \zeta_{t(p)}) \otimes (1 \otimes \zeta_{t(p+1)} \cdots \zeta_{t(n)}) \right) \right).$$

This map is evidently a V_P-module homomorphism. It is rather tedious to verify that it commutes with the boundary maps on X and $X \otimes_P X$, but no difficulties other than those of notation are encountered in carrying out this verification by induction on the degree in X. On the other hand, it is clear from the definition of this map that our product for $\mathrm{Ext}_{V_P}(P, *)$ is the same product as that obtained from the usual shuffle product of alternating differential forms, via the identification of Corollary 7.1.

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