STANDARD MONOMIAL BASES & GEOMETRIC CONSEQUENCES FOR CERTAIN RINGS OF INVARIANTS

V. LAKSHMIBAI[†] AND P. SHUKLA

ABSTRACT. Consider the diagonal action of $SL_n(K)$ on the affine space $X = V^{\oplus m} \oplus (V^*)^{\oplus q}$ where $V = K^n$, K an algebraically closed field of arbitrary characteristic and $m, q > n$. We construct a "standard monomial" basis for the ring of invariants $K[X]^{SL_n(K)}$. As a consequence, we deduce that $K[X]^{SL_n(K)}$ is Cohen-Macaulay. We also present the first and second fundamental theorems for $SL_n(K)$-actions.

INTRODUCTION

In [6], DeConcini-Procesi constructed a characteristic-free basis for the ring of invariants appearing in classical invariant theory (cf. [25]) for the action of the general linear, symplectic and orthogonal groups. In [6], the authors also considered the $SL_n(K)$-action on $X = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{q \text{ copies}}$, $V = K^n$, K an algebraically closed field of arbitrary characteristic and $m, q > n$, and described a set of algebra generators for $K[X]^{SL_n(K)}$.

The main goal of this paper is to prove the Cohen-Macaulayness of $K[X]^{SL_n(K)}$ (note that the Cohen-Macaulayness of $K[X]^{GL_n(K)}$ follows from the fact that $Spec\,(K[X]^{GL_n(K)})$ is a certain determinantal variety inside $M_{m,q}$, the space of $m \times q$ matrices; note also that in characteristic 0, the Cohen-Macaulayness of $K[X]^{SL_n(K)}$ follows from [1]). In recent times, among the several techniques of proving the Cohen-Macaulayness of algebraic varieties, two techniques have proven to be quite effective, namely, Frobenius-splitting technique and deformation technique. Frobenius-splitting technique is used in [22], for example, for proving the (arithmetic) Cohen-Macaulayness of Schubert varieties. Frobenius-splitting technique is also used in [17, 18, 19] for proving the Cohen-Macaulayness of certain varieties. The deformation technique consists in constructing a flat family over $\mathbb{A}^1$, with the given variety as the generic fiber (corresponding to $t \in K$ invertible). If the special fiber (corresponding to $t = 0$) is Cohen-Macaulay, then one may conclude the Cohen-Macaulayness of the given variety. Hodge algebras (cf. [4]) are typical examples where the deformation technique affords itself very well. Deformation technique is also used in [5, 14, 9, 3, 2]. The philosophy behind these works is that if there is a "standard monomial basis"

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for the co-ordinate ring of the given variety, then the deformation technique will work well in general (using the "straightening relations"). It is this philosophy that we adopt in this paper in proving the Cohen-Macaulayness of $K[X]^{SL_n(K)}$. To be more precise, the proof of the Cohen-Macaulayness of $K[X]^{SL_n(K)}$ is accomplished in the following steps:

- We first construct a K-subalgebra S of $K[X]^{SL_n(K)}$ by prescribing a set of algebra generators $\{f_\alpha, \alpha \in H\}, H$ being a finite partially ordered set and $f_\alpha \in K[X]^{SL_n(K)}$.
- We construct a "standard monomial" basis for S by

(i) defining "standard monomials" in the f_α's (cf. Definition 4.0.1)

(ii) writing down the straightening relation for a non-standard (degree 2) monomial $f_\alpha f_\beta$ (cf. Theorem 4.1.1)

(iii) proving linear independence of standard monomials (by relating the generators of S to certain determinantal varieties) (cf. §4.2)

(iv) proving the generation of S (as a vector space) by standard monomials (using (ii)). In fact, to prove the generation for S, we first prove generation for a "graded version" $R(D)$ of S, where D is a distributive lattice obtained by adjoining $\mathbf{1}, \mathbf{0}$ (the largest and the smallest elements of D) to H. We then deduce the generation for S. In fact, we construct a "standard monomial" basis for $R(D)$. While the generation by standard monomials for S is deduced from the generation by standard monomials for $R(D)$, the linear independence of standard monomials in $R(D)$ is deduced from the linear independence of standard monomials in S (cf. (iii) above).

- We give a presentation for S as a K-algebra (cf. Theorem 4.5.5)
- We prove the normality and Cohen-Macaulayness of $R(D)$ by showing that $\operatorname{Spec} R(D)$ flatly degenerates to the toric variety associated to the distributive lattice D (cf. Theorem 5.4.3).
- We deduce the normality and Cohen-Macaulayness of S from the normality and Cohen-Macaulayness of $R(D)$ (cf. Theorem 5.4.4).
- Using the normality of S and a crucial Lemma concerning GIT (cf. Lemma 2.0.4 which gives a set of sufficient conditions for a <u>normal</u> sub algebra of $K[X]^{SL_n(K)}$ to equal $K[X]^{SL_n(K)}$), we show that S is in fact $K[X]^{SL_n(K)}$, and hence conclude that $K[X]^{SL_n(K)}$ is Cohen-Macaulay.

As a consequence, we present (Theorem 6.0.6)

- **First fundamental Theorem for $SL_n(K)$-invariants**, i.e., describing algebra generators for $K[X]^{SL_n(K)}$.
- **Second fundamental Theorem for $SL_n(K)$-invariants**, i.e., describing generators for the ideal of relations among these algebra generators for $K[X]^{SL_n(K)}$.
- **A standard monomial basis for $K[X]^{SL_n(K)}$**

As a by-product of our main results, we recover Theorem 3.3 of [6] (which describes a set of algebra generators for $K[X]^{SL_n(K)}$). It should be pointed out that in [6], the authors remark (cf. [6], Remark (ii) following Theorem 3.3) "We have in fact explicit bases for the rings $K[X]^{SL_n(K)}, K[X]^{GL_n(K)}$". Of course, combining Theorems 1.2 & 3.1 of [6], one does obtain a basis for $K[X]^{GL_n(K)}$; nevertheless, there are no details given in [6] regarding the basis for $K[X]^{SL_n(K)}$ (probably, the authors had in their minds the same basis for $K[X]^{SL_n(K)}$ as the one constructed in this paper). Our main goal in this paper is to prove the Cohen-Macaulayness of $K[X]^{SL_n(K)}$; as mentioned above, this is accomplished by first constructing a "standard monomial" basis for the subalgebra S of $K[X]^{SL_n(K)}$, deducing Cohen-Macaulayness of S, and then proving that S in fact equals $K[X]^{SL_n(K)}$. Thus we do not use the results of [6] (especially, Theorem 3.3 of [6]), we rather give a different proof of Theorem 3.3 of [6]. Further, using Lemma 2.0.4, we get a GIT-theoretic proof (cf.[23]) of the first and second fundamental theorems for the $GL_n(K)$-action in arbitrary characteristics which we have included in §2.2. (The GIT-theoretic proof as it appears in [23] calls for a mild modification. Further, for the discussions in §3 we need the results on the ring of invariants for the $GL_n(K)$-action - specifically, first and second fundamental theorems for the $GL_n(K)$-action.)

The sections are organized as follows. In §1, after recalling some results (pertaining to standard monomial basis) for Schubert varieties (in the Grassmannian) and determinantal varieties, we derive the straightening relations for certain degree 2 non-standard monomials. In §2, we first derive some lemmas concerning quotients leading to the main Lemma 2.0.4; we then give a GIT-theoretic proof of the first and second fundamental theorems for the $GL_n(K)$-action in arbitrary characteristics. In §3, we define the algebra S. In §4, we construct a standard monomial basis for S; we also introduce the algebra $R(D)$, and construct a standard monomial basis for $R(D)$. In §5, we first prove the normality and Cohen-Macaulayness of $R(D)$, and then deduce the normality and Cohen-Macaulayness of S. In §6, we show that S is in fact $K[X]^{SL_n(K)}$ (using the crucial Lemma 2.0.4) and deduce the Cohen-Macaulayness of $K[X]^{SL_n(K)}$; we also present the first and second fundamental theorems for $SL_n(K)$-actions.

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1. Preliminaries

In this section, we recollect some basic results on determinantal varieties, mainly the standard monomial basis for the co-ordinate rings of determinantal varieties in terms of double standard tableaux. Since the results of §4 rely on an explicit description of the straightening relations (of a degree 2 non-standard monomial) on a determinantal variety, in this section we derive such straightening relations (cf. Proposition 1.6.3) by relating determinantal varieties to Schubert varieties in the Grassmannian. We first recall some results on Schubert varieties in the Grassmannian, mainly the standard monomial basis for the homogeneous co-ordinate rings (for the Plücker embedding) of

Schubert varieties. We then recall results for determinantal varieties (by identifying them as open subsets of suitable Schubert varieties in suitable Grassmannians). We then derive the desired straightening relations.

1.1. **The Grassmannian Variety $G_{d,n}$.** Let us fix the integers $1 \leq d < n$ and let $V = K^n$, K being the base field which we suppose to be algebraically closed of arbitrary characteristic. Let $G_{d,n}$ be the *Grassmannian variety* consisting of d-dimensional subspaces of V.

Let $\rho_d : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$ be the *Plücker* embedding.
Let $I(d,n) := \{\underline{i} = (i_1, \ldots, i_d) | 1 \leq i_1 < \cdots < i_d \leq n\}$. We have a partial order $\geq$ on $I(d,n)$, namely, $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t, \forall t$. Let $N = \# I(d,n)$ (note that $N = \binom{n}{d}$); we shall denote the projective coordinates of $\mathbb{P}(\wedge^d V)$ as $p_{\underline{i}}, \underline{i} \in I(d,n)$, and refer to them as the *Plücker coordinates.*

For $1 \leq t \leq n$, let V_t be the subspace of V spanned by $\{e_1, \ldots, e_t\}$. For $\underline{i} \in I(d,n)$, let $X(\underline{i})$ be the *Schubert variety associated to* $\underline{i}$:

$$X(\underline{i}) = \{U \in G_{d,n} \mid \dim(U \cap V_{i_t}) \geq t \ , \ 1 \leq t \leq d\}.$$

Remark 1.1.1. Note that under the set-theoretic bijection between the set of Schubert varieties and the set $I(d,n)$, the partial order on the set of Schubert varieties given by inclusion induces the partial order $\geq$ on $I(d,n)$.

Let R be the homogeneous co-ordinate ring of $G_{d,n}$ for the Plücker embedding, and for $w \in I(d,n)$, let $R(w)$ be the homogeneous co-ordinate ring of the Schubert variety $X(w)$.

Definition 1.1.2. A monomial $f = p_{\tau_1} \cdots p_{\tau_m}$ is said to be *standard* if

$$(*) \qquad \tau_1 \geq \cdots \geq \tau_m.$$

Such a monomial is said to be *standard on* $X(w)$, if in addition to condition (*), we have $w \geq \tau_1$.

We recall the following fundamental result: (cf. [12, 13]; see also [21])

Theorem 1.1.3. *Standard monomials on $X(w)$ of degree m give a basis for $R(w)_m$.*

As a consequence, we have a qualitative description of a typical quadratic relation on a Schubert variety $X(w)$ as given by the following Proposition. First one definition:

Definition 1.1.4. Given $\tau, \phi \in I(d,n)$, say, $\tau = (a_1, \cdots, a_d), \phi = (b_1, \cdots, b_d), \tau \vee \phi := (c_1, \cdots, c_d), \tau \wedge \phi := (e_1, \cdots, e_d)$, where $c_i = \max\{a_i, b_i\}, e_i = \min\{a_i, b_i\}, \forall i$ are called the *join* and *meet* of τ and ϕ respectively. Note that $\tau \vee \phi$ (resp. $\tau \wedge \phi$) is the smallest (resp. largest) element of $I(d,n)$ which is $>$ (resp. $<$) both τ and ϕ.

Proposition 1.1.5. *Let $w, \tau, \phi \in I(d,n)$, $w > \tau, \phi$. Further let τ, ϕ be non-comparable (so that $p_\tau p_\phi$ is a non-standard degree 2 monomial on $X(w)$). Let*

$$(*) \qquad p_\tau p_\phi = \sum_i c_i p_{\alpha_i} p_{\beta_i},\ c_i \in K^*$$

be the expression for $p_\tau p_\phi$ as a sum of standard monomials on $X(w)$. Then

(1) *for every (α, β) on the R.H.S. we have, $\alpha >$ both τ and ϕ, $\beta <$ both τ and ϕ.*
(2) *for every (α, β) on the right-hand side of (*), we have $\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$ (here $\dot{\cup}$ denotes a disjoint union)*
(3) *the term $p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs on the right-hand side of (*) with coefficient* 1.

Such a relation as in (*) is called a *straightening relation*.

Proof. (1): Pick a minimal element in $\{\alpha_i\}$, call it α_1. Restrict (*) to $X(\alpha_1)$. Then R.H.S. is a non-zero sum of standard monomials on $X(\alpha_1)$. Hence linear independence of standard monomials on $X(\alpha_1)$ implies that the restriction of L.H.S. to $X(\alpha_1)$ is non-zero. Hence it follows that $\alpha_1 \geq$ both τ and ϕ (note that restriction of p_θ to $X(\alpha_1)$ is non-zero if and only if $\alpha_1 \geq \theta$); we have in fact $\alpha > \tau, \phi$, for, if α equals one of $\{\tau, \phi\}$, say $\alpha = \tau$, then $p_\tau p_\phi = p_\alpha p_\phi$ would be standard, a contradiction. The assertion on α follows from this. The assertion on β is proved similarly by working with $w_0\tau, w_0\phi$ (in the place of τ, ϕ), w_0 being the element of largest length in the Weyl group.
(2) follows from weight considerations (note that $p_\tau, \tau \in I(d,n)$ - say, $\tau = (a_1, \cdots, a_d)$ - is a weight vector (for the T-action, T being the maximal torus of diagonal matrices in $GL_n(K)$) of weight $-(\epsilon_{a_1} + \cdots + \epsilon_{a_d})$).
For a proof of (3), refer to [9], Proposition 7.33. □

A presentation for $R(w)$. Let $Z_w = \{\tau \in I(d,n) \mid w \geq \tau\}$. Consider the polynomial algebra $K[x_\tau, \tau \in Z_w]$. For a pair τ, ϕ in Z_w such that τ, ϕ are not comparable, denote $F_{\tau,\phi} = x_\tau x_\phi - \sum_i c_i x_{\alpha_i} x_{\beta_i}, \alpha_i, \beta_i, c_i$ being as in Proposition 1.1.5. Let I_w be the ideal in $K[x_\tau, \tau \in Z_w]$ generated by $\{F_{\tau,\phi}, \tau, \phi \text{ non}-\text{comparable}\}$. Consider the surjective map $f_w : K[x_\tau, \tau \in Z_w] \to R(w), x_\tau \mapsto p_\tau$. We have

Proposition 1.1.6. *f_w induces an isomorphism $K[x_\tau, \tau \in Z_w]/I_w \cong R(w)$.*

See [15, 21] for a proof.

1.2. **The opposite big cell in $G_{d,n}$.** Let P_d be the parabolic subgroup of $G(= GL_n(K))$ consisting of all matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where the 0-matrix is of size $n-d\times d$. Then we have an identification $\varphi_d : G/P_d \cong G_{d,n}$. Denote by O^- the sub group of G consisting of matrices of the form

$$\begin{pmatrix} I_d & 0_{d\times(n-d)} \\ A_{(n-d)\times d} & I_{n-d} \end{pmatrix}$$

where I_d (resp. I_{n-d}) is the $d \times d$ (resp. $n-d\times n-d$) identity matrix. We have that the restriction of the canonical morphism $\theta_d : G \to G/P_d$ to O^- is an open immersion, and $\theta_d(O^-) = B^- e_{id}$, where e_{id} is the coset P_d of G/P_d, and B^- is the Borel sub group of lower triangular matrices in G; also, $\varphi_d(B^- e_{id})$ is the *opposite big cell* in $G_{d,n}$. Thus the opposite big cell in $G_{d,n}$ gets identified with O^-, and in the sequel we shall denote the opposite big cell by just O^-. Note that $O^- \cong \mathbb{A}^{d(n-d)}$.

1.3. **The functions $f_{\underline{j}}$ on O^-:** Consider the morphism $\xi_d : G \to \mathbb{P}(\wedge^d V)$, where $\xi_d = \rho_d \circ \varphi_d \circ \theta_d$, $\rho_d, \varphi_d, \theta_d$ being as above. Then $p_{\underline{j}}(\xi_d(g))$ is simply the d-minor of g consisting of the first d columns and rows given by $j_1, \dots, j_d$. For $\underline{j} \in I(d,n)$, we shall denote by $f_{\underline{j}}$ the restriction of $p_{\underline{j}}$ to O^-. Under the identification

$$O^- = \left\{ \begin{pmatrix} I_d \\ A \end{pmatrix},\, A \in M_{n-d\,,\,d} \right\}$$

we have for $z = \begin{pmatrix} I_d \\ A \end{pmatrix} \in O^-$, $f_{\underline{j}}(z)$ is simply a certain minor of A, which may be explicitly described as follows. Let $\underline{j} = (j_1, \dots, j_d)$, and let j_r be the largest entry $\leq d$. Let $\{k_1, \dots, k_{d-r}\}$ be the complement of $\{j_1, \dots, j_r\}$ in $\{1, \dots, d\}$. Then $f_{\underline{j}}(z)$ is the $(d-r)$-minor of A with column indices $k_1, \dots k_{d-r}$, and row indices $j_{r+1}, \dots, j_d$ (here the rows of A are indexed as $d+1, \dots, n$). Conversely, given a minor of A, say, with column indices $b_1, \dots, b_s$, and row indices $j_{d-s+1}, \dots, j_d$ (again, the rows of A are indexed as $d+1, \dots, n$), it is $f_{\underline{j}}(z)$, where $\underline{j} = (j_1, \dots, j_d)$ is given as follows: $\{j_1, \dots, j_{d-s}\}$ is the complement of $\{b_1, \dots, b_s\}$ in $\{1, \dots, d\}$, and $j_{d-s+1}, \dots, j_d$ are simply the row indices.

Convention. If $\underline{j} = (1, \dots, d)$, then $f_{\underline{j}}$ evaluated at z is 1; we shall make it correspond to the minor of A with row indices (and column indices) given by the empty set.

1.4. **The opposite cell in $X(w)$.** For a Schubert variety $X(w)$ in $G_{d,n}$, let us denote $O^- \cap X(w)$ by $Y(w)$; we refer to $Y(w)$ as the *opposite cell in* $X(w)$. We consider $Y(w)$ as a closed subvariety of O^-. In view of Proposition 1.1.6, we obtain that the ideal defining $Y(w)$ in O^- is generated by

$$\{f_{\underline{i}} \mid \underline{i} \in I(d,n),\ w \not\geq \underline{i}\}.$$

1.5. **Determinantal Varieties.** Let $Z = M_{r,d}(K)$, the space of all $r \times d$ matrices with entries in K. We shall identify Z with $\mathbb{A}^{rd}$. We have $K[Z] = K[x_{i,j},\ 1 \le i \le r,\ 1 \le j \le d]$.

The variety D_t. Let $X = (x_{ij})$, $1 \le i \le r$, $1 \le j \le d$ be a $r \times d$ matrix of indeterminates. Let $A \subset \{1, \cdots, r\}$, $B \subset \{1, \cdots, d\}$, $\#A = \#B = s$, where $s \le \min\{r, d\}$. We shall denote by $p(A, B)$ the s-minor of X with row indices given by A, and column indices given by B. For t, $1 \le t \le \min\{r, d\}$, let $I_t(X)$ be the ideal in $K[x_{i,j}]$ generated by $\{p(A, B),\ \ A \subset \{1, \cdots, r\},\ B \subset \{1, \cdots, d\},\ \#A = \#B = t\}$. Let $D_t(M_{r,d})$ (or just D_t) be the *determinantal variety* (a closed subvariety of Z), with $I_t(X)$ as the defining ideal. In the discussion below, we also allow $t = d + 1$ in which case $D_t = Z$

Identification of D_t with Y_ϕ. Let $G = GL_n(K)$. Let r, d be such that $r + d = n$. Let X be a $r \times d$ matrix of indeterminates. As in §1.2, let us identify the opposite cell O^- in $G/P_d(\cong G_{d,n})$ as

$$O^- = \left\{ \begin{pmatrix} I_d \\ X \end{pmatrix} \right\}.$$

As seen above (cf. §1.3), we have a bijection between $\{f_{\underline{i}},\ \underline{i} \in I(d, n)\}$ and $\{$minors of $X\}$ (note that as seen in §1.3, if $\underline{i} = (1, 2, \cdots, d)$, then $f_{\underline{i}}$ = the constant function 1 considered as the minor of X with row indices (and column indices) given by the empty set).

For example, take $r = 3 = d$. We have,

$$O^- = \left\{ \begin{pmatrix} I_3 \\ X_{3\times3} \end{pmatrix} \right\}.$$

We have, $f_{(1,2,4)} = p(\{1\}, \{3\}),\ f_{(2,4,6)} = p(\{1, 3\}, \{1, 3\})$.

Let ϕ be the d-tuple, $\phi = (t, t + 1, \cdots, d, n + 2 - t, n + 3 - t, \cdots, n)$ (note that ϕ consists of the two blocks $[t, d]$, $[n+2-t, n]$ of consecutive integers - here, for $i < j$, $[i, j]$ denotes the set $\{i, i+1, \cdots, j\}$). If $t = d+1$, then we set $\phi = (n+1-d, n+2-d, \cdots, n)$ (note then that $Y_\phi = O^-(\cong M_{r,d}(K))$).

Theorem 1.5.1. *(cf.[15, 16]) $D_t \cong Y_\phi$.*

Corollary 1.5.2. *$K[D_t] \cong R(\phi)_{(p_{id})}$, the homogeneous localization of $R(\phi)$ at p_{id}.*

1.6. **The partially ordered set $H_{r,d}$.** Let

$$H_{r,d} = \bigcup_{0 \le s \le \min\{r,d\}} I(s, r) \times I(s, d)$$

where our convention is that $(\emptyset, \emptyset)$ is the element of $H_{r,d}$ corresponding to $s = 0$. We define a partial order $\succeq$ on $H_{r,d}$ as follows:

- We declare $(\emptyset, \emptyset)$ as the largest element of $H_{r,d}$.

- For $(A,B),(A',B')$ in $H_{r,d}$, say, $A=\{a_1,\cdots,a_s\}$, $B=\{b_1,\cdots,b_s\}$, $A'=\{a'_1,\cdots,a'_{s'}\}$, $B'=\{b'_1,\cdots,b'_{s'}\}$ for some $s,s'\geq 1$, we define $(A,B)\succeq(A',B')$ if $s\leq s'$, $a_j\geq a'_j$, $b_j\geq b'_j$, $1\leq j\leq s$.

The bijection θ: As above, let $n=r+d$. Then $\succeq$ induces a partial order $\succeq$ on the set of all minors of X, namely, $p(A,B)\succeq p(A',B')$ if $(A,B)\succeq(A',B')$. Given $\underline{i}\in I(d,n)$, let m be such that $i_m\leq d$, $i_{m+1}>d$. Set

$$A_{\underline{i}}=\{n+1-i_d,n+1-i_{d-1},\cdots,n+1-i_{m+1}\},$$

$$B_{\underline{i}}=\text{ the complement of }\{i_1,i_2\cdots,i_m\}\text{ in }\{1,2,\cdots,d\}.$$

Define $\theta: I(d,n)\to\{\text{all minors of }X\}$ by setting $\theta(\underline{i})=p(A_{\underline{i}},B_{\underline{i}})$ (here, the constant function 1 is considered as the minor of X with row indices (and column indices) given by the empty set). Clearly θ is a bijection. Note that θ reverses the respective partial orders, i.e., given $\underline{i},\underline{i}'\in I(d,n)$, we have, $\underline{i}\leq\underline{i}'\iff\theta(\underline{i})\succeq\theta(\underline{i}')$. Using the partial order $\succeq$, we define *standard monomials* in $p(A,B)$'s:

Definition 1.6.1. A monomial $p(A_1,B_1)\cdots p(A_s,B_s)$, $s\in\mathbb{N}$ is standard if $p(A_1,B_1)\succeq\cdots\succeq p(A_s,B_s)$.

In view of Theorem 1.1.3, Theorem 1.5.1, and §1.4, we obtain

Theorem 1.6.2. *Standard monomials in $p(A,B)$'s with $\#\,A\leq t-1$ form a basis for $K[D_t]$, the algebra of regular functions on D_t.*

As a direct consequence of Proposition 1.1.5, we obtain

Proposition 1.6.3. *Let $p(A_1,A_2),p(B_1,B_2)$ (in $K[D_t]$) be not comparable. Let*

$$p(A_1,A_2)p(B_1,B_2)=\sum a_ip(C_{i1},C_{i2})p(D_{i1},D_{i2}),a_i\in K^* \qquad (*)$$

be the straightening relation in $K[D_t]$. Then for every i, $C_{i1},C_{i2},D_{i1},D_{i2}$ have cardinalities $\leq t-1$; further,

1. *$C_{i1}\geq$ both A_1 and B_1; $D_{i1}\leq$ both A_1 and B_1.*
2. *$C_{i2}\geq$ both A_2 and B_2; $D_{i2}\leq$ both A_2 and B_2.*
3. *The term $p((A_1,A_2)\vee(B_1,B_2))p((A_1,A_2)\wedge(B_1,B_2))$ occurs in (*) with coefficient 1.*

Note that via the bijection θ (defined as above), join and meet (cf. Definition 1.1.4) of two non-comparable elements $(A_1,A_2),(B_1,B_2)$ of $H_{r,d}$ exist, and in fact are given by $(A_1,A_2)\vee(B_1,B_2)=(A_1\vee B_1,A_2\vee B_2),(A_1,A_2)\wedge(B_1,B_2)=(A_1\wedge B_1,A_2\wedge B_2)$.

Remark 1.6.4. On the R.H.S. of (*), C_{i1},C_{i2} could both be the empty set (in which case $p(C_{i1},C_{i2})$ is understood as 1). For example, with X being a 2×2 matrix of indeterminates, we have

$$p_{1,2}p_{2,1}=p_{2,2}p_{1,1}-p_{\emptyset,\emptyset}p_{12,12}$$

Remark 1.6.5. In the sequel, while writing a straightening relation as in Proposition 1.6.3, if for some i, C_{i1}, C_{i2} are both the empty set, we keep the corresponding $p(C_{i1}, C_{i2})$ on the right hand side of the straightening relation (even though its value is 1) in order to have homogeneity in the relation.

Taking $t = d+1$ (in which case $D_t = Z = M_{r,d}(K)$) in Theorem 1.6.2 and Proposition 1.6.3, we obtain

Theorem 1.6.6. (1) *Standard monomials in $p(A,B)$'s form a basis for $K[Z](\cong K[x_{ij}, 1 \le i \le r, 1 \le j \le d])$.*
(2) *Relations similar to those in Proposition 1.6.3 hold on Z.*

Remark 1.6.7. Note that Theorem 1.6.6,(1) recovers the result of Doubleit-Rota-Stein (cf. [7], Theorem 2):

Remark 1.6.8. Theorem 1.6.2 is also proved in [6] (Theorem 1.2 in [6]). But we had taken the above approach of deducing Theorem 1.6.2 from Theorems 1.1.3, 1.5.1 in order to derive the straightening relations as given by Proposition 1.6.3(which are crucial for the discussion in §4).

A presentation for $K[(D_t)]$. Let $Z_t = \{(A,B) \ne (\emptyset, \emptyset), (A,B) \in H_{r,d},\ \#A = \#B \le t-1\}$.

Consider the polynomial algebra $K[x(A,B), (A,B) \in H_{r,d}, \#A = \#B \le t-1]$. For two non-comparable pairs (under $\succ$ (cf. §1.6)) $(A_1, A_2), (B_1, B_2)$ in Z_t, denote $F((A_1,A_2);(B_1,B_2)) = x(A_1,A_2)(B_1,B_2) - \sum a_i x(C_{i1}, C_{i2}) x(D_{i1}, D_{i2})$, where $C_{i1}, C_{i2}, D_{i1}, D_{i2}, a_i$ are as in Proposition 1.6.3. Let I_t be the ideal generated by

$$\{F((A_1,A_2);(B_1,B_2)), (A_1,A_2), (B_1,B_2) \text{ non}-\text{comparable}\}$$

Consider the surjective map $f_t : K[x(A,B), (A,B) \in Z_t] \to K[D_t], x(A,B) \mapsto p(A,B)$. Then in view of Proposition 1.1.6 and Theorem 1.5.1, we obtain

Proposition 1.6.9. *(**A presentation for** $K[D_t]$) f_t induces an isomorphism $K[x(A,B),(A,B) \in Z_t]/I_t \cong K[D_t]$.*

2. $GL_n(K)$-action

In this section, we first prove some Lemmas concerning quotients, to be applied to the following situation:

Suppose, we have an action of a reductive group G on an affine variety $X = SpecR$. Suppose that S is a subalgebra of R^G. We give below (cf. Lemma 2.0.4) a set of sufficient conditions for the equality $S = R^G$. We start with recalling

Theorem 2.0.1. *(**Zariski Main Theorem,** [20],**III.9**) Let $\varphi : X \to Y$ be a morphism such that*

(1) *φ is surjective*

(2) *fibers of φ are finite*
(3) *φ is birational*
(4) *Y is normal*

Then φ is an isomorphism.

Let $X = \operatorname{Spec} R$ and a reductive group G act linearly on X, i.e., we have a linear action of G on an affine space $\mathbb{A}^r$ and we have a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}^r$. Further, let R be a graded K-algebra. Let X^{ss} be the set of semi-stable points of X (i.e., points x such that $0 \notin \overline{G \cdot x}$). Let $X_1 = Proj\, R, X_1^{ss}$, the set of semi-stable points of X_1 (i.e., points $y \in X_1$ such that if $\hat{x}$ is any point in $K^{n+1} \setminus 0$ lying over y, then $\hat{x}$ is in X^{ss}). Let $f_1, \cdots, f_N$ be homogeneous G-invariant elements in R. Let $S = K[f_1, \cdots, f_N]$. Then for the morphism $SpecR^G \to SpecS$, the hypothesis (2) in Theorem 2.0.1 may be concluded if $\{f_1, \cdots, f_N\}$ is base-point free on X_1^{ss} as given by the following

Lemma 2.0.2. *Suppose $f_1, \cdots, f_N$ are homogeneous G-invariant elements in R such that for any $x \in X^{ss}$, $f_i(x) \neq 0$, for at least one i. Then $SpecR^G \to SpecS$ has finite fibers.*

Proof. **Case 1:** Let $f_1, \cdots, f_N$ be of the same degree, say, d. Let $Y = SpecR^G (= X /\!\!/ G$, the categorical quotient) and $\varphi : X \to Y$ be the canonical quotient map. Let $X_1 = ProjR$, X_1^{ss} the set of semi-stable points of X_1. Let $Y_1 = ProjR^G (= X_1^{ss} /\!\!/ G)$, and $\varphi_1 : X_1^{ss} \to Y_1$ be the canonical quotient map. Consider $\psi : X \to \mathbb{A}^N, x \mapsto (f_1(x), \cdots, f_N(x))$. This induces a map $\rho : Y \to \mathbb{A}^N$ (since $f_1, \cdots, f_N$ are G-invariant). The commutative diagram

$$\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[dr, "\psi"] & \\ Y \arrow[r, "\rho"'] & \mathbb{A}^N \end{tikzcd}$$

induces the commutative diagram

$$\begin{tikzcd} X_1^{\mathrm{ss}} \arrow[d, "\varphi_1"'] \arrow[dr, "\psi_1"] & \\ Y_1 \arrow[r, "\rho_1"'] & \mathbb{P}^{N-1} \end{tikzcd}$$

Note that $\psi_1 : X_1^{ss} \to \mathbb{P}^{N-1}$ is defined in view of the hypothesis that for any $x \in X^{ss}$, $f_i(x) \neq 0$, for at least one i. Note also that $f_1, \ldots, f_N$ are sections of the ample line bundle $\mathcal{O}_{X_1}(d)$ as well as the basic fact from GIT that this line bundle descends to an ample line bundle on Y_1, which we denote by $\mathcal{O}_{Y_1}(d)$.

Claim 1: ρ_1 is a finite morphism.

Proof of Claim 1: Since $f_1, \cdots, f_N$ are G-invariant, we get that $f_i \in H^0(Y_1, \mathcal{O}_{Y_1}(d))$. Hence we obtain that

$$\rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)) = \mathcal{O}_{Y_1}(d)$$

Thus, $\rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1))$ is ample, and hence ρ_1 is finite (over any fiber $(\rho_1)_z, z \in \mathbb{P}^{N-1}$, $\rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(1)) \,|_{\,(\rho_1)_z}$ is both ample and trivial, and hence $\dim(\rho_1)_z$ is zero), and Claim 1 follows.

Claim 2: ρ is a finite morphism.

Proof of Claim 2: Let $S' = R^G$. Let $S'^{(d)} = \oplus_n S'_{nd}$. We have $\mathbb{P}^{N-1} = \mathrm{Proj}\, K[x_1, \ldots, x_N]$. Since ρ_1 is finite, $\mathcal{O}_{Y_1}$ is a coherent $\mathcal{O}_{\mathbb{P}^{N-1}}$-module.

We see that

$$H^0(\mathbb{P}^{N-1}, \mathcal{O}_{Y_1} \otimes \mathcal{O}_{\mathbb{P}^{N-1}}(n)) \simeq H^0(Y_1, \rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(n))$$

since the direct image of $\rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(n))$ by ρ_1 is $\mathcal{O}_{Y_1} \otimes \mathcal{O}_{\mathbb{P}^{N-1}}(n)$ and ρ_1 is a finite morphism. On the other hand we have

$$\rho_1^*(\mathcal{O}_{\mathbb{P}^{N-1}}(n)) \simeq \mathcal{O}_{Y_1}(nd).$$

Thus we have

$$H^0(\mathbb{P}^{N-1}, \mathcal{O}_{Y_1} \otimes \mathcal{O}_{\mathbb{P}^{N-1}}(n)) \simeq H^0(Y_1, \mathcal{O}_{Y_1}(nd)) \simeq S'_{nd}.$$

Thus the graded $K[x_1, \ldots, x_N]$-module associated to the coherent sheaf $\mathcal{O}_{Y_1}$ on $\mathbb{P}^{N-1}$ is $S'^{(d)}$ and by the basic theorems of Serre, $S'^{(d)}$ is of finite type over $K[x_1, \ldots, x_N]$. Now a d-th power of any homogeneous element of S' is in $S'^{(d)}$ and thus S' is integral over $K[x_1, \ldots, x_N]$, which proves that ρ is finite. Claim 2 and hence the required result follows from this.

Case 2: Let $f_1, \cdots, f_N$ be homogeneous possibly of different degrees, say, $\deg f_i = d_i$. Let $d = l.c.m.\{d_i\}, e_i = \frac{d}{d_i}$. Set $g_i = f_i^{e_i}, 1 \leq i \leq N$. Then $\{g_1, \cdots, g_N\}$ is again base-point free on $(Proj R)^{ss}$. Hence by Case 1, we have that
R^G is a finite $K[g_1, \cdots, g_N]$-module, and hence a finite $K[f_1, \cdots, f_N]$-module
(note that $K[g_1, \cdots, g_N] \hookrightarrow K[f_1, \cdots, f_N] \hookrightarrow R^G$).

$\square$

In the Lemma below, we describe a set of sufficient conditions for (3) of Lemma 2.0.1, namely, birationality.

Lemma 2.0.3. *Suppose $F : X \to Y$ is a surjective morphism of (irreducible) algebraic varieties, and U is an open subset of X such that*

(1) *$F\,|_{\,U} : U \to Y$ is an immersion*
(2) *$dim U = dim Y$.*

Then F is birational.

Proof. Hypothesis (1) implies that $F(U)$ is locally closed in Y. This fact together with Hypothesis (2) implies that $F(U)$ is open in Y, and the result follows. $\square$

We now return to the situation of a linear action of a reductive group G on an affine variety $X = SpecR$ with R a graded K-algebra. Let $f_1, \cdots, f_N$ be homogeneous G-invariant elements in R. Let $S = K[f_1, \cdots, f_N]$. Combining Lemmas 2.0.1, 2.0.2, 2.0.3, we arrive at the following Lemma which gives a set of sufficient conditions for the equality $S = R^G$. Before stating the lemma, let us observe the following. Suppose that U is a non-empty G-stable open subset in X. Since $\varphi : X \longrightarrow Spec\, R^G$ is surjective, $\varphi(U)$ contains a non-empty open subset. Hence by shrinking U, if necessary, we can suppose that $\varphi(U)$ is open. We suppose that this is the case and denote it by $U/\!\!/G$.

Lemma 2.0.4. *Let notation be as above. Let $\psi : X \to \mathbb{A}^N$ be the map, $x \mapsto (f_1(x), \cdots, f_N(x))$. Denote $D = SpecS$. Then D is the categorical quotient of X by G and $\psi : X \to D$ is the canonical quotient map, provided the following conditions are satisfied:*

(i) For $x \in X^{\mathrm{ss}}$, $\psi(x) \neq (0)$.

(ii) There is a G-stable open subset U of X such that $\psi\,|_{U/\!\!/G} : U/\!\!/G \to D$ is an immersion.

(iii) $\dim D = \dim U/\!\!/G$.

(iv) D is normal.

Remark 2.0.5. Suppose that U is a (non-empty) G-stable open subset of X, G operates freely with U/G as quotient, and ψ induces an immersion of U/G in A^N. Then (ii) is satisfied:

This assertion is immediately seen, for we have

$$U/G \longrightarrow U/\!\!/G \longrightarrow \mathbb{A}^N$$

and the fact that $U/G \longrightarrow A^N$ is an immersion implies that $U/\!\!/G \longrightarrow \mathbb{A}^N$ immersion.

In the following subsection, using Lemma 2.0.4, we give a GIT-theoretic proof of the first and second fundamental theorems for the $GL_n(K)$-action in arbitrary characteristics.

2.1. **Classical Invariant Theory:** Let $V = K^n$, $X = \underbrace{V \oplus \cdots \oplus V}_{m\,\text{copies}} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{q\,\text{copies}}$, where $m, q > n$.

The $GL(V)$-action on X: Writing $\underline{u} = (u_1, u_2, ..., u_m)$ with $u_i \in V$ and $\underline{\xi} = (\xi_1, \xi_2, ..., \xi_q)$ with $\xi_i \in V^*$, we shall denote the elements of X by $(\underline{u}, \underline{\xi})$. The (natural) action of $GL(V)$ on V induces an action of $GL(V)$ on V^*, namely, for $\xi \in V^*, g \in GL(V)$, denoting $g \cdot \xi$ by ξ^g, we have

$$\xi^g(v) = \xi(g^{-1}v), v \in V$$

The diagonal action of $GL(V)$ on X is given by

$$g \cdot (\underline{u}, \underline{\xi}) = (g\underline{u}, \underline{\xi}^g) = (gu_1, gu_2, ..., gu_m, \xi_1^g, \xi_2^g, ..., \xi_q^g),\ g \in G, (\underline{u}, \underline{\xi}) \in X$$

The induced action on $K[X]$ is given by

$$(g \cdot f)(\underline{u}, \underline{\xi}) = f(g^{-1}(\underline{u}, \underline{\xi})),\ f \in K[X],\ g \in GL(V)$$

Consider the functions $\varphi_{ij} : X \longrightarrow K$ defined by $\varphi_{ij}((\underline{u}, \underline{\xi})) = \xi_j(u_i)$, $1 \leq i \leq m, 1 \leq j \leq q$. Each φ_{ij} is a $GL(V)$-invariant: For $g \in GL(V)$, we have,

$$\begin{aligned}
(g \cdot \varphi_{ij})((\underline{u}, \underline{\xi})) &= \varphi_{ij}(g^{-1}(\underline{u}, \underline{\xi})) \\
&= \varphi_{ij}((g^{-1}u, \xi^{g^{-1}}) \\
&= \xi_j^{g^{-1}}(g^{-1}u_i) \\
&= \xi_j(u_i) \\
&= \varphi_{ij}((\underline{u}, \underline{\xi}))
\end{aligned}$$

It is convenient to have a description of the above action in terms of coordinates. So with respect to a fixed basis, we write the elements of V as row vectors and those of V^* as column vectors. Thus denoting by $M_{a,b}$ the space of $a \times b$ matrices with entries in K, X can be identified with the affine space $M_{m,n} \times M_{n,q}$. The action of $GL_n(K)\,(= GL(V))$ on X is then given by

$$A \cdot (U, W) = (UA, A^{-1}W),\ A \in GL_n(K),\ U \in M_{m,n},\ W \in M_{n,q}$$

And the action of $GL_n(K)$ on $K[X]$ is given by

$$(A \cdot f)(U, W) = f\left(A^{-1}(U, W)\right) = f\left(UA^{-1}, AW\right),\ f \in K[X]$$

Writing $U = (u_{ij})$ and $W = (\xi_{kl})$ we denote the coordinate functions on X, by u_{ij} and ξ_{kl}. Further, if u_i denotes the i-th row of U and ξ_j the j-th column of W, the invariants φ_{ij} described above are nothing but the entries $\langle u_i, \xi_j \rangle$ $(= \xi_j(u_i))$ of the product UW.

In the sequel, we shall denote $\varphi_{ij}(\underline{u}, \underline{\xi})$ also by $\langle u_i, \xi_j \rangle$.

The function $p(A, B)$**:** For $A \in I(r, m), B \in I(r, q), 1 \leq r \leq n$, let $p(A, B)$ be the regular function on X: $p(A, B)((\underline{u}, \underline{\xi}))$ is the determinant of the $r \times r$-matrix $(\langle u_i, \xi_j \rangle)_{i \in A,\, j \in B}$. Let S be the subalgebra of R^G generated by $\{p(A, B)\}$. We shall now show (using Lemma 2.0.4) that S is in fact equal to R^G.

2.2. The first and second fundamental Theorems of classical invariant theory (cf. [25]) for the action of $GL_n(K)$:

Theorem 2.2.1. *Let $G = GL_n(K)$. Let X be as above. The morphism $\psi : X \to M_{m,q}$, $(\underline{u}, \underline{\xi}) \mapsto \left(\varphi_{ij}(\underline{u}, \underline{\xi})\right) (= (\langle u_i, \xi_j \rangle))$ maps X into the determinantal variety $D_{n+1}(M_{m,q})$, and the induced homomorphism $\psi^* : K[D_{n+1}(M_{m,q})] \to K[X]$ between the coordiante rings induces an isomorphism $\psi^* : K[D_{n+1}(M_{m,q})] \to K[X]^G$, i.e. the determinantal variety $D_{n+1}(M_{m,q})$ is the categorical quotient of X by G.*

Proof. Clearly, $\psi(X) \subseteq D_{n+1}(M_{m,q})$ (since, $\psi(X) = Spec\, S$, and clearly $Spec\, S \subseteq D_{n+1}(M_{m,q})$ (since any $n+1$ vectors in V are linearly independent)). We shall prove the result using Lemma 2.0.4. To be very precise, we shall first check the conditions (i)-(iii) of Lemma 2.0.4 for $\psi : X \to M_{m,q}$, deduce that the inclusion $Spec\, S \subseteq D_{n+1}(M_{m,q})$ is in fact an equality, and hence conclude the normality of $Spec\, S$ (condition (iv) of Lemma 2.0.4).

(i) Let $x = (\underline{u}, \underline{\xi}) = (u_1, \ldots, u_m, \xi_1, \ldots, \xi_q) \in X^{\mathrm{ss}}$. Let $W_{\underline{u}}$ be the subspace of V spanned by x_i's and $W_{\underline{\xi}}$ the subspace of V^* spanned by ξ_j's. Assume if possible that $\psi((\underline{u}, \underline{\xi})) = 0$, i.e. $\langle u_i, \xi_j \rangle = 0$ for all i, j.

Case (a): $W_{\underline{\xi}} = 0$, i.e., $\xi_j = 0$ for all j.

Consider the one parameter subgroup $\Gamma = \{g_t, t \neq 0\}$ of $GL(V)$, where $g_t = tI_n$, I_n being the $n \times n$ identity matrix. Then $g_t \cdot x = g_t \cdot (\underline{u}, \underline{0}) = (t\underline{u}, \underline{0})$, so that $g_t \cdot x \to (0)$ as $t \to 0$. Thus the origin 0 is in the closure of $G \cdot x$, and consequently x is not semi-stable, which is a contradiction.

Case (b): $W_{\underline{\xi}} \neq 0$.

Since the case $W_{\underline{u}} = 0$ is similar to Case (a), we may assume that $W_{\underline{u}} \neq 0$. Also the fact that $W_{\underline{\xi}} \neq 0$ together with the assumption that $\langle x_i, \xi_j \rangle = 0$ for all i, j implies that $\dim W_{\underline{u}} < n$. Let $r = dim\, W_{\underline{u}}$ so that we have $0 < r < n$. Hence, we can choose a basis $\{e_1, \ldots, e_n\}$ of V such that $W_{\underline{u}} = \langle e_1, \ldots, e_r \rangle$, $r < n$, and $W_{\underline{\xi}} \subset \langle e^*_{r+1}, \ldots, e^*_n \rangle$, where $\{e^*_1, \ldots, e^*_n\}$ is the dual basis in V^*. Consider the one parameter subgroup $\Gamma = \{g_t, t \neq 0\}$ of $GL(V)$, where

$$g_t = \begin{pmatrix} tI_r & 0 \\ 0 & t^{-1}I_{n-r} \end{pmatrix}.$$

We have $g_t \cdot (\underline{u}, \underline{\xi}) = (t\underline{u}, t\underline{\xi}) \to 0$ as $t \to 0$. Thus, by the same reasoning as in Case (a), the point $(\underline{u}, \underline{\xi})$ is not semi-stable, which leads to a contradiction. Hence we obtain $\psi((\underline{u}, \underline{\xi})) \neq 0$.

(ii) Let

$$U = \{(\underline{u}, \underline{\xi}) \in X \mid \{u_1, \ldots, u_n\},\ \{\xi_1, \ldots, \xi_n\} \text{ are linearly independent}\}$$

Clearly, U is a G-stable open subset of X.

Claim : G operates freely on U, $U \to U\, mod\, G$ is a G-principal fiber space, and ψ induces an immersion $U/G \to M_{m,q}$.

Proof of Claim: We have a G-equivariant identification

$$(\dagger) \qquad U \cong G \times G \times \underbrace{V \times \cdots \times V}_{(m-n)\text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{(q-n)\text{ copies}}$$

from which it is clear that and G operates freely on U. Further, we see that $U\, mod\, G$ may be identified with the fiber space with base $(G \times G)\, mod\, G$ (G acting on $G \times G$ as $g \cdot (g_1, g_2) = (g_1 g, g^{-1} g_2)$, $g, g_1, g_2 \in G$), and fiber $\underbrace{V \times \cdots \times V}_{(m-n)\text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{(q-n)\text{ copies}}$

associated to the principal fiber space $G \times G \to (G \times G)\,/\,G$. It remains to show that ψ induces an immersion $U/G \to \mathbb{A}^N$, i.e., to show that the map $\psi : U/G \to M_{m,q}$ and its differential $d\psi$ are both injective. We first prove that $\psi : U/G \to M_{m,q}$ is injective. Let x, x' in U/G be such that $\psi(x) = \psi(x')$. Let $\eta, \eta' \in U$ be lifts for x, x' respectively. Using the identification (†) above, we may write

$$\eta = (A, u_{n+1}, \cdots, u_m, B, \xi_{n+1}, \cdots, \xi_q),\ A, B \in G$$
$$\eta' = (A', u'_{n+1}, \cdots, u'_m, B', \xi'_{n+1}, \cdots, \xi'_q),\ A', B' \in G$$

(here, $u_i, 1 \leq i \leq n$ are given by the rows of A, while $\xi_i, 1 \leq i \leq n$ are given by the columns of B; similar remarks on u'_i, ξ'_i). The hypothesis that $\psi(x) = \psi(x')$ implies in particular that

$$\langle u_i, \xi_j \rangle = \langle u'_i, \xi'_j \rangle\ , 1 \leq i, j \leq n$$

which may be written as

$$AB = A'B'$$

This implies that $A' = A \cdot g$, where $g = BB'^{-1}$. Hence on U/G, we may suppose that

$$x = (u_1, \cdots, u_n, u_{n+1}, \cdots, u_m, \xi_1, \cdots, \xi_q)$$
$$x' = (u_1, \cdots, u_n, u'_{n+1}, \cdots, u'_m, \xi'_1, \cdots, \xi'_q)$$

where $\{u_1, \ldots, u_n\}$ is linearly independent.

For a given j, we have,

$$\langle u_i, \xi_j \rangle = \langle u_i, \xi'_j \rangle\ , 1 \leq i \leq n, \text{ implies } \xi_j = \xi'_j$$

(since $\{u_1, \ldots, u_n\}$ is linearly independent). Thus we obtain

$$(*) \qquad \xi_j = \xi'_j, \text{ for all } j$$

On the other hand, we have (by definition of U) that $\{\xi_1, \ldots, \xi_n\}$ is linearly independent. Hence fixing an $i, n+1 \leq i \leq m$, we get

$$\langle u_i, \xi_j \rangle = \langle u'_i, \xi_j \rangle (= \langle u'_i, \xi'_j \rangle)\ , 1 \leq j \leq n, \text{ implies } u_i = u'_i$$

Thus we obtain

$$(**) \qquad u_i = u'_i, \text{ for all } i$$

The injectivity of $\psi : U/G \to M_{m,q}$ follows from (*) and (**).

To prove that the differential dψ is injective, we merely note that the above argument remains valid for the points over $K[\epsilon]$, the algebra of dual numbers ($= K \oplus K\epsilon$, the K-algebra with one generator ϵ, and one relation $\epsilon^2 = 0$), i.e., it remains valid if we replace K by $K[\epsilon]$, or in fact by any K-algebra.

(iii) We have

$$\dim U/G = \dim U - \dim G = (m+q)n - n^2 = \dim D_{n+1}(M_{m,q}).$$

The immersion $U/G \hookrightarrow Spec\, S (\subseteq D_{n+1}(M_{m,q}))$ together with the fact above that $\dim U/G = \dim D_{n+1}(M_{m,q})$ implies that $Spec\, S$ in fact equals $\dim D_{n+1}(M_{m,q})$.

(iv) The normality of $Spec\,S(= D_{n+1}(M_{m,q}))$ follows from Theorem 1.5.1 (and the normality of Schubert varieties). □

Combining the above Theorem with Theorem 1.6.2, we obtain the following

Corollary 2.2.2. *Let X and G be as above. Let φ_{ij} denote the regular function $(\underline{u}, \underline{\xi}) \mapsto \langle u_i, \xi_j \rangle$ on X, $1 \leq i \leq m$, $1 \leq j \leq q$, and let f denote the $m \times q$ matrix (φ_{ij}). The ring of invariants $K[X]^G$ has a basis consisting of standard monomials in the regular functions $p_{\lambda,\mu}(f)$ with $\#\lambda \leq n$, where $\#\lambda = t$ is the number of elements in the sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $p_{\lambda,,\mu}(f)$ is the t-minor with row indices $\lambda_1, \ldots, \lambda_t$ and column indices $\mu_1, \ldots, \mu_t$.*

As a consequence of the above Theorem, we obtain the first and second fundamental Theorems of classical invariant theory (cf. [25]). Let notation be as above.

Theorem 2.2.3. (1) **First fundamental theorem**
The ring of invariants $K[X]^{GL(V)}$ is generated by φ_{ij}, $1 \leq i \leq m, 1 \leq j \leq q$.
(2) **Second fundamental theorem**
The ideal of relations among the generators in (1) is generated by the $(n+1)$-minors of the $m \times q$-matrix (φ_{ij}).

Further, we have (cf. Corollary 2.2.2) :

Theorem 2.2.4. A standard monomial basis for the ring of invariants: *The ring of invariants $K[X]^{GL(V)}$ has a basis consisting of standard monomials in the regular functions $p_(A, B)$, $A \in I(r, m), B \in I(r, q)$, $r \leq n$.*

3. The K-algebra S

Let X be as above. We shall denote $K[X]$ by R so that $R = K[u_{ij}, \xi_{kl}\, 1 \leq i \leq m,\, 1 \leq j, k \leq n, 1 \leq l \leq q]$.

The functions $u(I), \xi(J)$: As above, let
$U = (u_{ij})_{1\leq i \leq m,\, 1 \leq j \leq n}$ and $W = (\xi_{kl})_{1 \leq k \leq n,\, 1 \leq l \leq q}$. For $I \in I(n, m), J \in I(n, q)$, let $u(I), \xi(J))$ denote the following regular functions on X:
$u(I)((\underline{u}, \underline{\xi}))$ = the n-minor of U with row indices given by I.
$\xi(J)((\underline{u}, \underline{\xi}))$ = the n-minor of W with column indices given by J.

Note that for the diagonal action of $SL_n(K)\,(= SL(V))$ on X, we have, $u(I), \xi(J)$ are in $R^{SL_n(K)}$.

The K-algebra S: Let S be the K-subalgebra of R generated by
$\{u(I), \xi(J), p(A, B), I \in I(n, m), J \in I(n, q), A \in I(r, m), B \in I(r, q), 1 \leq r \leq n\}$. We shall denote the set $I(n, m)$ indexing the $u(I)$'s by H_u and the set $I(n, q)$ indexing the

$\xi(J)$'s by H_ξ. Also, we shall denote $H_p := \underset{1\le r\le n}{\cup}(I(r,m)\times I(r,q))$, and set

$$\begin{aligned} H &= H_u \dot{\cup} H_\xi \cup H_p \\ &= I(n,m)\dot{\cup} I(n,q) \cup \underset{1\le r\le n}{\cup}(I(r,m)\times I(r,q)), \end{aligned}$$

where $\dot{\cup}$ denotes a disjoint union. (If $m=q$, then H_u, H_ξ are to be considered as two disjoint copies of $I(n,m)$.) Then the algebra generators
$\{u(I), \xi(J), p(A,B), I \in I(n,m), J \in I(n,q), A \in I(r,m), B \in I(r,q), 1 \le r \le n\}$ of S are indexed by the set H. Clearly $S \subseteq R^{SL(V)}$.

Remark 3.0.5. The K-algebra S could have been simply defined as the K-subalgebra of R^G generated by $\{\langle u_i, \xi_j\rangle\}$ (i.e., by $\{p(A,B), \#A = \#B = 1\}$) and $\{u(I), \xi(J)\}$. But we have a purpose in defining it as above, namely, the standard monomials (in S) will be built out of the $p(A,B)$'s with $\#A \le n$, the $u(I)$'s and $\xi(J)$'s (cf. Definition 4.0.1).

Our goal is to show that S equals $R^{SL(V)}$.

A partial order on H: Define a partial order on H as follows:

(1) The partial order on H_p is as in §1.6 (note that $H_p \subset H_{m,q}$)
(2) The partial order on H_u and H_ξ are as in §1.1.
(3) Any element of H_u and any element of H_ξ are not comparable.
(4) No element of H_u, H_ξ is greater than any element of H_p.
(5) For $I \in H_u$ and $(A,B) \in H_p$, we define $I \le (A,B)$ if $I \le A$ (the partial order being as in §1.6). Similarly, for $J \in H_\xi$ and $(A,B) \in H_p$, we define $J \le (A,B)$ if $J \le B$.

Lemma 3.0.6. *H is a ranked poset of rank $d := (m+q)n - n^2$, i.e., all maximal chains in H have the same cardinality $= (m+q)n - n^2 + 1$.*

Proof. Clearly, H is a ranked poset (since it is composed of ranked posets). To compute the rank of H, we consider the maximal chain consisting of $\tau_1, \cdots, \tau_N$, where the first q of them are given by $(m,q), (m,q-1), \cdots, (m,1)$ (of H_p),
the next $(m-1)$ of them are given by $(m-1,1), (m-2,1), \cdots, (1,1)$ (of H_p).
(thus contributing $m+q-1$ to the cardinality of the chain).
This is now followed by the $q-1$ elements of H_p :
$((1,m),(1,q)), ((1,m),(1,q-1)), \cdots, ((1,m),(1,2))$,
followed by the $m-2$ elements of H_p :
$((1,m-1),(1,2)), ((1,m-2),(1,2)), \cdots, ((1,2),(1,2))$
(thus contributing $m+q-3$ to the cardinality of the chain).
Thus proceeding, finally, we end up with $((1,2,\cdots,n),(1,2,\cdots,n))$ (in H_p). This is now followed by either $(1,2,\cdots,n)$ of H_u or $(1,2,\cdots,n)$ of H_ξ.
The number of elements in the above chain equals

$$[m+q-1+(m+q-3)+\cdots+m+q-(2n-1)]+1=(m+q)n-n^2+1 \qquad \square$$

4. Standard monomials in the K-algebra S

Definition 4.0.1. A monomial F in the $p(A,B)$'s, $u(I)$'s, and $\xi(J)$'s, is said to be *standard* if F satisfies the following conditions:

(1) If F involves $u(I)$, for some I (resp. $\xi(J)$ for some J), then F does not involve $\xi(J')$ for any J' (resp. $u(I')$, for any I').

(2) If $F=p(A_1,B_1)\cdots p(A_r,B_r)u(I_1)\cdots u(I_s)$ (resp. $p(A_1,B_1)\cdots p(A_r,B_r)\xi(J_1)\cdots\xi(J_t)$), where r,s,t are integers ≥ 0, then

$$A_1\geq\cdots\geq A_r\geq I_1\geq\cdots\geq I_s \text{ (resp. } B_1\geq\cdots\geq B_r\geq J_1\geq\cdots\geq J_t)$$

4.1. **Quadratic relations.** In this subsection, we describe certain straightening relations to be used while proving the linear independence of standard monomials and generation (of S as a K-vector space) by standard monomials.

Theorem 4.1.1. (1) *Let $I\in H_u, J\in H_\xi$. We have*

$$u(I)\xi(J)=p(I,J)$$

(2) *Let $I,I'\in H_u$ be not comparable. We have,*

$$u(I)u(I')=\sum_r b_r u(I_r)u(I'_r),\ b_r\in K^*$$

where for all r, $I_r\geq$ both I,I', and $I'_r\leq$ both I,I'.

(3) *Let $J,J'\in H_\xi$ be not comparable. We have,*

$$\xi(J)\xi(J')=\sum_s c_s\xi(J_s)\xi(J'_s),\ c_s\in K^*$$

where for all s, $J_s\geq$ both J,J', and $J'_s\leq$ both J,J'.

(4) *Let $(A_1,A_2),(B_1,B_2)\in H_p$ be not comparable. Then we have*

$$p(A_1,A_2)p(B_1,B_2)=\sum_i a_i p(C_{i1},C_{i2})p(D_{i1},D_{i2}), a_i\in K^*,$$

where $(C_{i1},C_{i2}),(D_{i1},D_{i2})$ belong to H_p; further, for every i, we have

(a) *$C_{i1}\geq$ both A_1 and B_1; $D_{i1}\leq$ both A_1 and B_1.*

(b) *$C_{i2}\geq$ both A_2 and B_2; $D_{i2}\leq$ both A_2 and B_2.*

(5) *Let $I\in H_u,(A,B)\in H_p$ be such that $A\not\geq I$. We have,*

$$p(A,B)u(I)=\sum_t d_t p(A_t,B_t)u(I_t),\ d_t\in K^*$$

where for every t, we have, $A_t\geq$ (resp. $I_t\leq$) both A and I, and $B_t\geq B$.

(6) *Let $J \in H_\xi, (A,B) \in H_p$ be such that $B \not\geq J$. We have,*

$$p(A,B)\xi(J) = \sum_l e_l p(A_l, B_l)\xi(J_l),\ e_l \in K^*$$

where for every l, we have, $A_l \geq A$, and $B_l \geq$ (resp. $J_l \leq$) both B and J.

Proof. In the course of the proof, we will be repeatedly using the fact that the subalgebra generated by $\{p(A,B),\ A \in I(r,m), B \in I(r,q)\, 1 \leq r \leq n\}$ being $R^{GL(V)}$ (cf. Theorem 2.2.3,(1)), the results given in Theorem 2.2.3,(1), Theorem 2.2.4 apply to this subalgebra.
(1) is clear from the definitions of $u(I), \xi(J)$ and $p(I,J)$.
(2). We shall denote a minor of $U(=(u_{ij})_{1\leq i\leq m,\, 1\leq j\leq n})$ with rows and columns given by I, J (where $I, J \in I(r,m)$ for some $r \leq n$) by $\Delta(I,J)$. Observe that if $\#I = n$, then $J = (1, 2, \cdots, n)$ necessarily (since U has size $m \times n$). Thus for $I \in H_u$, we have that $u(I) = \Delta(I, I_n), u(I') = \Delta(I', I_n)$ (as minors of U), where $I_n = (1, 2, \cdots, n)$, we have, in view of Theorem 1.6.6, (2),

$$u(I)u(I') = \Delta(I, I_n)\Delta(I', I_n) = \sum_i b_i \Delta(C_{i1}, C_{i2})\Delta(D_{i1}, D_{i2}), a_i \in K^*,$$

where we have for every i, $C_{i1} \geq$ both I and I'; $D_{i1} \leq$ both I and I'; $C_{i2} \geq I_n$; $D_{i2} \leq I_n$ which forces $\#D_{i2} = n$ (in view of the partial order (cf. §1.6); note that D_{i2} being the column indices of a minor of the $m \times n$ matrix U, we have that $\#D_{i2} \leq n$). Hence we obtain that $D_{i2} = I_n$, for all i. In particular, we obtain that $\#D_{i1}(= \#D_{i2}) = n$. This in turn implies (by consideration of the degrees in u_{ij}'s of the terms in the above sum) that $\#C_{i1} = \#C_{i2} = n$. Hence $C_{i2} = I_n$ (again note that C_{i2} gives the column indices of the n-minor $\Delta(C_{i1}, C_{i2})$ of the $m \times n$ matrix U). Thus the above relation becomes

$$u(I)u(I') = \sum_i b_i u(C_{i1})u(D_{i1}),$$

with $C_{i1} \geq$ both I and I'; $D_{i1} \leq$ both I and I'. This proves (2).
Proof of (3) is similar to that of (2).
(4) is a direct consequence of Theorem 2.2.3,(2) and Proposition 1.6.3.
(5). If $\#A = n = \#B$, then $p(A,B)u(I) = u(A)u(I)\xi(B)$. By (2),

$$u(A)u(I) = \sum_i d_i u(C_i)u(D_i),\ d_i \in K^*$$

where $C_i \geq$ both A, I, and $D_i \leq$ both A, I. Hence

$$p(A,B)u(I) = \sum_i d_i u(C_i)u(D_i)\xi(B) = \sum_i d_i p(C_i, B)u(D_i)$$

where $C_i \geq$ both A, I, and $D_i \leq$ both A, I, and the result follows.

Let then $\#A < n$. By (1), we have $u(I)\xi(I_n) = p(I, I_n)$. Hence, $p(A, B)u(I)\xi(I_n) = p(A, B)p(I, I_n)$. The hypothesis that $A \not\geq I$ implies that $p(A, B)p(I, I_n)$ is not standard (note that the facts that $\#A < n, \#I = n$ implies that $I \not\geq A$). Hence (4) implies that

$$p(A, B)p(I, I_n) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), a_i \in K^*,$$

where $(C_{i1}, C_{i2}), (D_{i1}, D_{i2})$ belong to H_p; further, for every i, $C_{i1} \geq$ both A and I; $D_{i1} \leq$ both A and I; $C_{i2} \geq$ both B and I_n; $D_{i2} \leq$ both B and I_n which forces $D_{i2} = I_n$ (note that in view of Theorem 2.2.4, all minors in the above relation have size $\leq n$); and hence $\#D_{i1} = n$, for all i. Hence $p(D_{i1}, D_{i2}) = u(D_{i1})\xi(I_n)$, for all i. Hence cancelling $\xi(I_n)$, we obtain

$$p(A, B)u(I) = \sum a_i p(C_{i1}, C_{i2})u(D_{i1}),$$

where $C_{i1} \geq$ both A and I, $D_{i1} \leq$ both A and I, and $C_{i2} \geq B$. This proves (5).
Proof of (6) is similar to that of (5).

□

4.2. **Linear independence of standard monomials:** In this subsection, we prove the linear independence of standard monomials.

Lemma 4.2.1. *Let $(A, B) \in H_p, I \in H_u, J \in H_\xi$.*

1. *The set of standard monomials in the $p(A, B)$'s is linearly independent.*
2. *The set of standard monomials in the $u(I)$'s is linearly independent.*
3. *The set of standard monomials in the $\xi(J)$'s is linearly independent.*

Proof. (1) follows from Theorem 2.2.4.
(2), (3) follow from Theorem 1.6.6,(1) applied to $K[u_{ij}, 1 \leq i \leq m, 1 \leq j \leq n]$, $K[\xi_{kl}, 1 \leq k \leq n, 1 \leq l \leq q]$ respectively. □

Proposition 4.2.2. *Standard monomials are linearly independent.*

Proof. For a monomial M, by *u-degree* (resp. *ξ-degree*) of M, we shall mean the degree of M in the variables u_{ij}'s (resp. ξ_{kl}'s). We have

$$u\text{−degree of } p(A_1, B_1)\cdots p(A_r, B_r) = \xi\text{−degree of } p(A_1, B_1)\cdots p(A_r, B_r) = \sum_i \#A_i$$

$$u\text{−degree of } u(I_1)\cdots u(I_s) = ns,\ \xi\text{−degree of } u(I_1)\cdots u(I_s) = 0$$

$$\xi\text{−degree of } \xi(J_1)\cdots\xi(J_t) = nt,\ u\text{−degree of } \xi(J_1)\cdots\xi(J_t) = 0$$

By considering the u-degree and the ξ-degree, and using Lemma 4.2.1 we see that $\{p(A_1, B_1)\cdots p(A_r, B_r), u(I_1)\cdots u(I_s), \xi(J_1)\cdots\xi(J_t), r, s, t \geq 0\}$ is linearly independent.

Let

$$(*) \qquad F := R + S = 0$$

be a relation among standard monomials, where $R = \sum a_i M_i$, $S = \sum b_i N_i$ such that each M_i (resp. N_i) is a standard monomial of the form $p(A_1, B_1) \cdots p(A_{r_i}, B_{r_i})$ (resp. $p(A_1, B_1) \cdots p(A_{q_i}, B_{q_i}) u(I_1) \cdots u(I_{s_i}) \xi(J_1) \cdots \xi(J_{t_i})$, $q_i \geq 0$, and at least one of $\{s_i, t_i\} > 0$). If g is in $GL_n(K)$, with $\det g \neq$ a root of unity, then using the facts that $g \cdot p(A, B) = p(A, B), g \cdot u(I) = (det\, g) u(I), g \cdot \xi(J) = (det\, g) \xi(J)$, we have, $F - gF = \sum b_i (1 - (detg)^{s_i + t_i}) N_i = 0$. Hence if we show that the N_i's are linearly independent, then (in view of Lemma 4.2.1,(1)), we would obtain that (*) is the trivial relation. Thus we may suppose that

$$(**) \qquad F = \sum b_i N_i = 0,$$

where each N_i is a standard monomial of the form

$$p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s) \xi(J_1) \cdots \xi(J_t)$$

where $r \geq 0$ and at least one of $\{s, t\} > 0$; in fact, N_i's being standard, in any N_i, precisely one of $\{s_i, t_i\}$ is non-zero.

We first multiply (**) by $u(I_n)^N$ (I_n being $(1, 2, \cdots, n)$), for a sufficiently large N (N could be taken to be any integer greater than all of the t's, appearing in the $\xi(J_1) \cdots \xi(J_t)$'s); we then replace a $\xi(J) u(I_n)$ by $p(I_n, J)$ (cf. Theorem 4.1.1, (1)). Then in the resulting sum, any monomial will involve only the $p(A, B)$'s and the $u(I)$'s. Thus we may suppose that (**) is of the form

$$(***) \qquad G := \sum c_i G_i = 0$$

where each G_i is of the form $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$. Note that for each standard monomial M=$p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ (resp. $p(A_1, B_1) \cdots p(A_r, B_r) \xi(J_1) \cdots \xi(J_t)$) appearing in (**), $M\, u(I_n)^N$ is again standard. Again, considering $G - gG, g \in GL_n(K)$, with $\det g \neq$ a root of unity, as above, we may suppose that in each monomial $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ appearing in (***), $s > 0$. Further, in view of Lemma 4.2.1,(2), we may suppose that for at least one monomial $r > 0$. Now considering the ξ-degree of the monomials, we may suppose (in view of Lemma 4.2.1,(2)) that in each monomial $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ appearing in (***), $r > 0$.

Thus, for each monomial $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ appearing in (***), we have, $r, s > 0$. Now the ξ-degree (as well as the u-degree) being the same for all of the monomials in (***), for any two monomials $G_i, G_{i'}$, say

$$G_i = p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s), G_{i'} = p(A'_1, B'_1) \cdots p(A'_{r'}, B'_{r'}) u(I'_1) \cdots u(I'_{s'})$$

we have $\sum\limits_{1 \leq i \leq r} \# A_i = \sum\limits_{1 \leq i \leq r'} \# A'_i$. This together with the fact that the u-degree is the same for all of the terms G_k's in (***) implies that $s = s'$. Thus we obtain that in all of the monomials $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ in (***), the integer s is the same (and $s > 0$). Now we multiply (***) through out by $\xi(I_n)^s$ (where $I_n = (1, 2, \cdots, n)$) to arrive at a linear sum

$$\sum d_i H_i = 0$$

where each H_i is a standard monomial in the $p(A, B)$'s (note that $H_i = p(A_1, B_1) \cdots p(A_r, B_r) p(I_1, I_n) \cdots p(I_s, I_n)$ is standard). Now the required result follows from the linear independence of $p(A, B)$'s (cf. Lemma 4.2.1,(1)).

□

4.3. **The algebra** $S(D)$**.** To prove the generation of S (as a K-vector space) by standard monomials, we define a K-algebra $S(D)$, construct a standard monomial basis for $S(D)$ and deduce the results for S (in fact, it will turn out that $S(D) \cong S$). We first define the K-algebra $R(D)$ as follows:

Let

$$D = H \cup \{\mathbf{1}\} \cup \{\mathbf{0}\}$$

H being as in the beginning of §3. Extend the partial order on H to D by declaring $\{\mathbf{1}\}$ (resp. $\{\mathbf{0}\}$) as the largest (resp. smallest) element. Let $P(D)$ be the polynomial algebra

$$P(D) := K[X(A, B), Y(I), Z(J), X(\mathbf{1}), X(\mathbf{0}), (A, B) \in H_p, I \in H_u, J \in H_\xi]$$

Let $\mathfrak{a}(D)$ be the homogeneous ideal in the polynomial algebra $P(D)$ generated by the six relations of Theorem 4.1.1 ($X(A, B), Y(I), Z(J)$ replacing $p(A, B), u(I), \xi(J)$ respectively), with relations (1) and (4) homogenized as follows: (1) is homogenized as

$$(*) \qquad X(I)Y(J) = X(I, J)X(\mathbf{0})$$

while (4) is homogenized as

$$X(A_1, A_2)X(B_1, B_2) = \sum a_i X(C_{i1}, C_{i2}) X(D_{i1}, D_{i2})$$

where $X(C_{i1}, C_{i2})$ is to be understood as $X(\mathbf{1})$ if both C_{i1}, C_{i2} equal the empty set (cf. Remark 1.6.5). Let

$$R(D) = P(D)/\mathfrak{a}(D)$$

We shall denote the classes of $X(A, B), Y(I), Z(J), X(\mathbf{1}), X(\mathbf{0})$ in $R(D)$ by $x(A, B), y(I), z(J), x(\mathbf{1}), x(\mathbf{0})$ respectively.

The algebra $M(D)$**:** Set $M(D) = R(D)_{(x(\mathbf{0}))}$, the homogeneous localization of $R(D)$ at $x(\mathbf{0})$. We shall denote $\frac{x(\mathbf{1})}{x(\mathbf{0})}, \frac{x(A,B)}{x(\mathbf{0})}, \frac{y(I)}{x(\mathbf{0})}, \frac{z(J)}{x(\mathbf{0})}$ (in $M(D)$) by $q(\mathbf{1}), r(A, B), s(I), t(J)$ respectively.

A grading for $M(D)$**:** We give a grading for $M(D)$ by assigning degree one to $s(I), t(J)$, and degree 2 to $q(\mathbf{1}), r(A, B)$, where as above $I \in H_u, J \in H_\xi, (A, B) \in H_p$.

The algebra $S(D)$: Set $S(D) = M(D)_{(q(\mathbf{1}))}$, the homogeneous localization of $M(D)$ at $q(\mathbf{1})$. We shall denote $\frac{r(A,B)}{q(\mathbf{1})}, \frac{s(I)}{q(\mathbf{1})}, \frac{t(J)}{q(\mathbf{1})}$ (in $S(D)$) by $c(A,B), d(I), e(J)$ respectively.

Let $\varphi_D : S(D) \to S$ be the map, $\varphi_D(c(A,B)) = p(A,B), \varphi_D(d(I)) = u(I), \varphi_D(e(J)) = \xi(J)$. Consider the canonical maps

$$\theta_D : R(D) \to M(D), \delta_D : M(D) \to S(D)$$

Denote $\gamma_D : R(D) \to S$ as the composite $\gamma_D = \varphi_D \circ \delta_D \circ \theta_D$.

4.4. **A standard monomial basis for $R(D)$:** We define a monomial in $x(A,B), y(I), z(J), x(\mathbf{1}), x(\mathbf{0})$ (in $R(D)$) to be standard in exactly the same way as in Definition 4.0.1 (we declare $x(\mathbf{1})$ (resp. $x(\mathbf{0})$) as the largest (resp. smallest)).

Proposition 4.4.1. *The standard monomials in the $x(A,B)$'s, $y(I)$'s, $z(J)$'s, $x(\mathbf{1})$'s, $x(\mathbf{0})$'s are linearly independent.*

Proof. The result follows by considering $\gamma_D : R(D) \to S$, and using the linear independence of standard monomials in S (cf. Proposition 4.2.2). □

Generation of $R(D)$ by standard monomials: We shall now show that any non-standard monomial F in $R(D)$ is a linear sum of standard monomials. Observe that if M is a standard monomial, then $x(\mathbf{1})^l M$ (resp. $Mx(\mathbf{0})^l$) is again standard; hence we may suppose F to be:

$$F = x(A_1, B_1) \cdots x(A_r, B_r) y(I_1) \cdots y(I_s) z(J_1) \cdots z(J_t)$$

Using the relations $y(I)z(J) = x(I,J)x(\mathbf{0})$, we may suppose that $F = x(A_1, B_1) \cdots x(A_r, B_r) y(I_1) \cdots y(I_s)$ or $F = x(A_1, B_1) \cdots x(A_r, B_r) z(J_1) \cdots z(J_t)$, say, $F = x(A_1, B_1) \cdots x(A_r, B_r) y(I_1) \cdots y(I_s)$.

Fix an integer N sufficiently large. To each element $A \in \cup_{r=1}^{n} I(r,m)$, we associate an $(n+1)$-tuple as follows: Let $A \in I(r,m)$, for some r, say, $A = (a_1, \cdots, a_r)$. To A, we associate the $n+1$-tuple

$$\overline{A} := (a_1, \cdots, a_r, m, m, \cdots, m, 1)$$

Similarly, for $B \in \cup_{r=1}^{n} I(r,q)$, say, $B = (b_1, \cdots, b_r)$, we associate the $n+1$-tuple

$$\overline{B} := (b_1, \cdots, b_r, q, q, \cdots, q, 1)$$

To F, we associate the integer n_F (and call it the *weight of* F) which has the entries of $\overline{A_1}, \overline{B_1}, \overline{A_2}, \overline{B_2}, \cdots, \overline{A_r}, \overline{B_r}, \overline{I_1}, \cdots, \overline{I_s}$ as digits (in the N-ary presentation). The hypothesis that F is non-standard implies that either $x(A_i, B_i)x(A_{i+1}, B_{i+1})$ is non-standard for some $i \leq r-1$, or, $x(A_r, B_r)y(I_1)$ is non-standard or $y(I_j)y(I_{j+1})$ is non-standard for some $j \leq s-1$. Straightening these using Theorem 4.1.1, we obtain that $F = \sum a_i F_i$ where $n_{F_i} > n_F, \forall i$, and the result follows by decreasing induction on n_F (note that while straightening a degree 2 relation using Theorem 4.1.1, (4), if $x(\mathbf{1})$ occurs in a monomial G, then the digits in n_G corresponding to $x(\mathbf{1})$ are taken to be $(\underbrace{m, m \cdots, m}_{n+1 \text{ times}}, \underbrace{q, q \cdots, q}_{n+1 \text{ times}})$. Also note that the largest F of degree r in $x(A,B)$'s and degree s in the $y(I)$'s is $x(\{m\},\{q\})^r u(I_0)^s$ (where I_0 is the n-tuple $(m+1-n, m+2-n, \cdots, m)$) which is clearly standard (the starting point of the decreasing induction).

Hence we obtain

Proposition 4.4.2. *Standard monomials in $x(A,B), y(I), z(J), x(\mathbf{1}), x(\mathbf{0})$ generate $R(D)$ as a K-vector space.*

Combining Propositions 4.4.1, 4.4.2, we obtain

Theorem 4.4.3. *Standard monomials in $x(A,B), y(I), z(J), x(\mathbf{1}), x(\mathbf{0})$ give a basis for the K-vector space $R(D)$.*

4.5. **Standard monomial bases for $M(D), S(D)$.** Standard monomials in $r(A,B), s(I), t(J), q(\mathbf{1}))$ in $M(D)$ (resp. $c(A,B), d(I), e(J))$ in $S(D))$ are defined in exactly the same way as in Definition 4.0.1.

Proposition 4.5.1. *Standard monomials in $r(A,B), s(I), t(J), q(\mathbf{1})$ give a basis for the K-vector space $M(D)$.*

Proof. The linear independence of standard monomials follows as in the proof of Prop 4.4.1 by considering $\varphi_D \circ \delta_D : M(D) \to S$, and using the linear independence of standard monomials in S (cf. Proposition 4.2.2).
To see the generation of $M(D)$ by standard monomials, consider a non-standard monomial F in $M(D)$. Since $q(\mathbf{1})^l$ is the largest monomial of a given degree l, we may suppose F to be:

$$F = r(A_1, B_1) \cdots r(A_i, B_i) s(I_1) \cdots s(I_k) t(J_1) \cdots t(J_l)$$

In view of Theorem 4.1.1, (1), we may suppose that
$F = r(A_1,B_1) \cdots r(A_i,B_i) s(I_1) \cdots s(I_k)$ or $r(A_1,B_1) \cdots r(A_i,B_i) t(J_1) \cdots t(J_l)$, say, $F = r(A_1,B_1) \cdots r(A_i,B_i) s(I_1) \cdots s(I_k)$. Then $F = \theta_D(H)$, where
$H = x(A_1,B_1) \cdots x(A_i,B_i) y(I_1) \cdots y(I_k)$. The required result follows from Proposition 4.4.2. $\square$

Proposition 4.5.2. *Standard monomials in $c(A,B), d(I), e(J)$ give a basis for the K-vector space $S(D)$.*

The proof is completely analogous to that of Proposition 4.5.1 (in view of the fact that $S(D) = M(D)_{(q(\mathbf{1}))}$).

Theorem 4.5.3. *Standard monomials in $p(A,B), u(I), \xi(J)$ form a basis for the K-vector space S.*

Proof. We already have established the linear independence of standard monomials (cf. Proposition 4.2.2). The generation by standard monomials follows by considering the surjective map $\varphi_D : S(D) \to S$ and using the generation of $S(D)$ by standard monomials (cf. Theorem 4.5.2). □

Theorem 4.5.4. *The map $\varphi_D : S(D) \to S$ is an isomorphism of K-algebras.*

Proof. Under φ_D, the standard monomials in $S(D)$ are mapped bijectively onto the standard monomials in S. The result follows from Proposition 4.5.2 and Theorem 4.5.3. □

Theorem 4.5.5. ***A presentation for** S**:***

(1) *The K-algebra S is generated by $\{p(A,B), u(I), \xi(J), (A,B) \in H_p, I \in H_u, J \in H_\xi\}$.*
(2) *The ideal of relations among the generators $\{p(A,B), u(I), \xi(J), (A,B) \in H_p, I \in H_u, J \in H_\xi\}$ is generated by the six type of relations as given by Theorem 4.1.1.*

Proof. The result follows from Theorem 4.5.4, Proposition 4.5.2 (and the definition of $S(D)$) □

5. Normality and Cohen-Macaulayness of the K-algebra S

In this section, we prove the normality and Cohen-Macaulayness of $Spec\, S$ by relating it to a toric variety. From §3, §4, we have

- $\{u(I), \xi(J), p(A,B), I \in H_u, J \in H_\xi, (A,B) \in H_p\}$ generates S as a K-algebra.
- Standard monomials in $\{u(I), \xi(J), p(A,B), I \in H_u, J \in H_\xi, (A,B) \in H_p\}$ form a K-basis for S.
- Considering S as a quotient of the polynomial algebra

$$K[X(A,B), Y(I), Z(J), (A,B) \in H_p, I \in H_u, J \in H_\xi]$$

the ideal $\mathfrak{a}$ of relations is generated by the six kinds of quadratic relations as given in Theorem 4.1.1.

5.1. The algebra associated to a distributive lattice.

Definition 5.1.1. A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$, $x \wedge y$, called the *join*, respectively the *meet* of x and y, satisfying:

$$x \vee y \geq x,\ x \vee y \geq y, \text{ and if } z \geq x \text{ and } z \geq y, \text{ then } z \geq x \vee y,$$
$$x \wedge y \leq x,\ x \wedge y \leq y, \text{ and if } z \leq x \text{ and } z \leq y, \text{ then } z \leq x \wedge y.$$

Definition 5.1.2. A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Definition 5.1.3. Given a finite lattice $\mathcal{L}$, the *ideal associated to* $\mathcal{L}$, denoted by $I(\mathcal{L})$, is the ideal of the polynomial algebra $K[\mathcal{L}](= K[x_\alpha, \alpha \in \mathcal{L}])$ generated by the set of binomials

$$\mathcal{G}_{\mathcal{L}} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{L} \text{ non-comparable}\}.$$

Set $A(\mathcal{L}) = K[\mathcal{L}]/I(\mathcal{L})$, *the algebra associated to* $\mathcal{L}$.

The chain lattice $\mathcal{C}(n_1, \ldots, n_d)$: Given an integer $n \geq 1$, let $\mathcal{C}(n)$ denote the chain $\{1 < \cdots < n\}$, and for $n_1, \ldots, n_d > 1$, let $\mathcal{C}(n_1, \ldots, n_d)$ denote the chain product lattice $\mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_d)$ consisting of all d-tuples $(i_1, \ldots, i_d)$, with $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d$. For $(i_1, \ldots, i_d)$, $(j_1, \ldots, j_d)$ in $\mathcal{C}(n_1, \ldots, n_d)$, we define

$$(i_1, \ldots, i_d) \leq (j_1, \ldots, j_d) \iff i_1 \leq j_1, \ldots, i_d \leq j_d\ .$$

We have

$$(i_1, \ldots, i_d) \vee (j_1, \ldots, j_d) = (\max\{i_1, j_1\}, \ldots, \max\{i_d, j_d\})$$
$$(i_1, \ldots, i_d) \wedge (j_1, \ldots, j_d) = (\min\{i_1, j_1\}, \ldots, \min\{i_d, j_d\}).$$

Clearly, $\mathcal{C}(n_1, \ldots, n_d)$ is a finite distributive lattice.

5.2. **Flat degenerations of certain K-algebras:** Let $\mathcal{L}$ be a finite lattice, and R a K-algebra with generators $\{p_\alpha \mid \alpha \in \mathcal{L}\}$.

Definition 5.2.1. A monomial $p_{\alpha_1} \ldots p_{\alpha_r}$ is said to be standard if $\alpha_1 \geq \cdots \geq \alpha_r$.

Suppose that the standard monomials form a K-basis for R. Given any nonstandard monomial F, the expression

$$F = \sum c_i F_i, \qquad c_i \in K^*$$

for F as a sum of standard monomials will be referred to as a *straightening relation*. Consider the surjective map

$$\pi : K[\mathcal{L}] \to R, \qquad x_\alpha \mapsto p_\alpha.$$

Let us denote $\ker \pi$ by I.

For $\alpha, \beta \in H$ with $\alpha > \beta$, we set

$$]\beta, \alpha[= \{\gamma \in \mathcal{L} \mid \alpha > \gamma > \beta\}.$$

Recall the following theorem (cf.[9], Theorem 5.2)

Theorem 5.2.2. *Let $\mathcal{L}, R, I$ be as above. Suppose that there exists a lattice embedding $\mathcal{L} \hookrightarrow \mathcal{C}$, where $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$ for some $n_1, \ldots, n_d \geq 1$, such that the entries of the d-tuple $(\theta_1, \ldots, \theta_d)$ representing an element θ of $\mathcal{L}$ form a non-decreasing sequence, i.e. $\theta_1 \leq \cdots \leq \theta_d$. Suppose that I is generated as an ideal by elements of the form $x_\tau x_\varphi - \sum c_{\alpha\beta} x_\alpha x_\beta$(where τ, φ are non-comparable, and $\alpha \geq \beta$). Further suppose that in the straightening relation*

$$(*) \qquad p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta,$$

the following hold:

(a) $p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs on the right-hand side of (*) with coefficient 1.

(b) $\tau, \phi \in]\beta, \alpha[$, for every pair (α, β) appearing on the right-hand side of (*).

(c) Under the embedding $\mathcal{L} \hookrightarrow \mathcal{C}$, we have $\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$, for every (α, β) on the right-hand side of (*).

Then there exists a flat family over Spec $K[t]$ whose special fiber ($t = 0$) is Spec $A(\mathcal{L})$ and general fiber (t invertible) is Spec R.

Corollary 5.2.3. *Spec R flatly degenerates to a (normal) toric variety. In particular, Spec R is normal and Cohen-Macaulay.*

Proof. We have (cf. [11]) that $A(\mathcal{L})$ is a normal domain. Hence we obtain that $I(\mathcal{L})$ is a binomial prime ideal. On the other hand, we have (cf. [8]) that a binomial prime ideal is a toric ideal (in the sense of [24]). It follows that *Spec* $A(\mathcal{L})$ is a (normal) toric variety and we obtain the first assertion. The first assertion together with Theorem 5.2.2 and the fact that a toric variety is Cohen-Macaulay implies that *Spec* R is normal and Cohen-Macaulay. ☐

5.3. **The distributive lattice D:** Consider the partially ordered set

$$D = H \cup \{\mathbf{1}\} \cup \{\mathbf{0}\}$$

defined in §4.3. We equip D with the structure of a distributive lattice by embedding it inside the chain lattice $\mathcal{C}(\underline{m}, \underline{q}) := \mathcal{C}(\underbrace{m, m \cdots, m}_{n+1 \text{ times}}, \underbrace{q, q \cdots, q}_{n+1 \text{ times}})$, as follows:

To each element of D, we associate a $2n+2$-tuple:

For $A = (a_1, \cdots, a_r) \in I(r, m), B = (b_1, \cdots, b_r) \in I(r, q)$, let $\overline{A}, \overline{B}$ denote the $n+1$-tuples:

$$\overline{A} := (a_1, \cdots, a_r, m, m, \cdots, m, 1), \overline{B} := (b_1, \cdots, b_r, q, q, \cdots, q, 1)$$

(i) Let $(A, B) \in H_p$, say, $A \in I(r, m), B \in I(r, q)$, for some $r, 1 \leq r \leq n$. We let $\overline{(A, B)}$ be the $(2n+2)$-tuple : $\overline{(A, B)} = (\overline{A}, \overline{B})$.

(ii) Let $I \in H_u$, say, $I = (i_1, \cdots, i_n) \, (\in I(n, m))$. We let $\tilde{I}$ be the $(2n+2)$-tuple : $\tilde{I} = (i_1, \cdots, i_n, 1, \underbrace{1, \cdots, 1}_{n+1 \text{ times}})$

(iii) Let $\xi \in H_\xi$, say, $J = (j_1, \cdots, j_n) \, (\in I(n,m)))$, we let $\tilde{J}$ be the $(2n+2)$-tuple : $\tilde{J} = (\underbrace{1, \cdots, 1}_{n+1 \text{ times}}, j_1, \cdots, j_n, 1)$.

(iv) Corresponding to $\mathbf{1}, \mathbf{0}$, we let $\tilde{\mathbf{1}}, \tilde{\mathbf{0}}$ be the $(2n+2)$-tuples :

$$\tilde{\mathbf{1}} = (\underbrace{m, m \cdots, m}_{n+1 \text{ times}}, \underbrace{q, q \cdots, q}_{n+1 \text{ times}}), \;\; \tilde{\mathbf{0}} = (\underbrace{1, \cdots, 1}_{2n+2 \text{ times}})$$

This induces a canonical embedding of D inside the chain lattice $\mathcal{C}(\underbrace{m, m \cdots, m}_{n+1 \text{ times}}, \underbrace{q, q \cdots, q}_{n+1 \text{ times}})$.

Lemma 5.3.1. *Let $\tau_1, \tau_2 \in \mathcal{C}(\underline{m}, \underline{q})$. Suppose $\tau_1, \tau_2 \in D$. Then $\tau_1 \vee \tau_2, \tau_1 \wedge \tau_2$ are also in D. Thus D acquires the structure of a distributive lattice.*

Proof. Clearly the Lemma requires a proof only when τ_1, τ_2 are non-comparable. We consider the following cases. For two s-tuples $E = \{e_1, \cdots, e_s\}, F = \{f_1, \cdots, f_s\}$, we shall denote

$$E \vee F := (\max\{e_1, f_1\}, \ldots, \max\{e_s, f_s\})$$
$$E \wedge F := (\min\{e_1, f_1\}, \ldots, \min\{e_s, f_s\}).$$

Case 1: $\tau_1, \tau_2 \in H_p$, say $\tau_1 = (\overline{A_1}, \overline{B_1}), \tau_2 = (\overline{A_2}, \overline{B_2})$. We have

$$\tau_1 \vee \tau_2 = (\overline{A_1} \vee \overline{A_2}, \overline{B_1} \vee \overline{B_2}), \;\; \tau_1 \wedge \tau_2 = (\overline{A_1} \wedge \overline{A_2}, \overline{B_1} \wedge \overline{B_2})$$

Clearly $\tau_1 \vee \tau_2, \tau_1 \wedge \tau_2$ are in H_p, and hence in D.

Case 2: $\tau_1 \in H_p, \tau_2 \in H_u$, say $\tau_1 = (\overline{A}, \overline{B}), \tau_2 = \tilde{I}$ (for some $I \in H_u$). Let $\overline{I}$ be the $n+1$-tuple $(I, 1)$ (entries of I followed by 1). We have

$$\tau_1 \vee \tau_2 = (\overline{A} \vee \overline{I}, \overline{B}), \;\; \tau_1 \wedge \tau_2 = (\overline{A} \wedge \overline{I}, (\underbrace{1, \cdots, 1}_{n+1 \text{ times}}))$$

Clearly $\tau_1 \vee \tau_2 \in H_p, \tau_1 \wedge \tau_2 \in H_u$.

Case 3: $\tau_1 \in H_p, \tau_2 \in H_\xi$, say $\tau_1 = (\overline{A}, \overline{B}), \tau_2 = \tilde{J}$ (for some $J \in H_\xi$). Let $\overline{J}$ be the $n+1$-tuple $(J, 1)$ (entries of I followed by 1). We have

$$\tau_1 \vee \tau_2 = (\overline{A}, \overline{B} \vee \overline{J}), \;\; \tau_1 \wedge \tau_2 = (\underbrace{1, \cdots, 1}_{n+1 \text{ times}}, \overline{B} \wedge \overline{J})$$

Clearly $\tau_1 \vee \tau_2 \in H_p, \tau_1 \wedge \tau_2 \in H_\xi$.

Case 4: $\tau_1, \tau_2 \in H_u$, say $\tau_1 = \tilde{I}_1, \tau_2 = \tilde{I}_2$ (for some $I_1, I_2 \in H_u$). We have

$$\tau_1 \vee \tau_2 = \widetilde{I_1 \vee I_2}, \;\; \tau_1 \wedge \tau_2 = \widetilde{I_1 \wedge I_2}$$

Clearly $\tau_1 \vee \tau_2, \tau_1 \wedge \tau_2$ are in H_u.

Case 5: $\tau_1, \tau_2 \in H_\xi$.

This case is similar to Case 4.

Case 6: $\tau_1 \in H_u, \tau_2 \in H_\xi$, say $\tau_1 = \tilde{I}, \tau_2 = \tilde{J}$ (for some I, J in H_u, H_ξ respectively). We have

$$\tau_1 \vee \tau_2 = (\overline{I}, \overline{J}),\ \tau_1 \wedge \tau_2 = \tilde{0}$$

Clearly $\tau_1 \vee \tau_2 \in H_p$, $\tau_1 \wedge \tau_2 \in D$.

□

Lemma 5.3.2. *We have rank$(D) = (m+q)n - n^2 + 2\,(= d+2,$ where $d = (m+q)n - n^2)$. In particular, $\dim A(D) = d+3$*

This is immediate from Lemma 3.0.6.

5.4. **Flat degeneration of Spec $R(D)$ to the toric variety Spec $A(D)$.** In this subsection, we show that Spec $R(D)$ flatly degenerates to the toric variety Spec $A(D)$ by showing that $R(D)$ satisfies the hypotheses of Lemma 5.2.2. We first prove some preparatory Lemmas.

Lemma 5.4.1. *Let τ, ϕ be two non-comparable elements of H. Then in the straightening relation for $p_\tau p_\phi$ as given by Theorem 4.1.1, $p_{\tau\vee\phi}p_{\tau\wedge\phi}$ occurs with coefficient* 1 *(here for an element φ of H, p_φ stands for $p(A,B), u(I)$ or $\xi(J)$ according as $\varphi = (A,B) \in H_p, I \in H_u$ or $J \in H_\xi$).*

Proof. The assertion is clear if the relation is of the type (1) of Theorem 4.1.1.
If the relation is of the type (4) of Theorem 4.1.1, then the result follows from Proposition 1.1.5,(3) (one uses the identification - as described in §1.5, §1.6 - of $\{p(A,B), (A,B) \in H_p\}$ with the Plücker co-ordinates $\{p_\tau, \tau \in I(q, m+q)\}$ restricted to the opposite cell in $G_{q,m+q}$).

Similarly, if the relation is of the type (2) (resp. (3)) of Theorem 4.1.1, by identifying $M_{m,n}$ (resp. $M_{n,q}$ with the opposite cell in $G_{n,m+n}$ (resp. $G_{q,n+q}$) (and using the identifications as described in §1.5, §1.6), the result follows as above (in view of Proposition 1.1.5,(3))
Let then the relation be of the type (5) or (6) of Theorem 4.1.1, say of type (5) (the proof is similar if it is of type (6)):

$$(*) \qquad p(A,B)u(I) = \sum_t c_t p(A_t, B_t)u(I_t)$$

where $I \in I(n,m), (A,B) \in H_p$, and $A \not\geq I$. As in the proof of Theorem 4.1.1, (5), we multiply through out by $\xi(I_n)$ to arrive at

$$(**) \qquad p(A,B)p(I,I_n) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), a_i \in K^*$$

where $(C_{i1}, C_{i2}), (D_{i1}, D_{i2})$ belong to H_p. As above, using Proposition 1.1.5,(3), we obtain that $p((A,B)\vee(I,I_n))p((A,B)\wedge(I,I_n))$ occurs in (**) with coefficient 1. We have (in view of Lemma 5.3.1, rather its proof),

$$p((A,B)\vee(I,I_n))p((A,B)\wedge(I,I_n)) = p(\overline{A}\vee\overline{I},\overline{B})p(\overline{A}\wedge\overline{I},\overline{I_n}) = p(\overline{A}\vee\overline{I},\overline{B})u(\overline{A}\wedge\overline{I})\xi(\overline{I_n})$$

Also from the proof of Theorem 4.1.1, (5), we have, for every $i, D_{i2} = I_n$ (in (**)). Hence writing $p(D_{i1}, D_{i2}) = u(D_{i1})\xi(I_n)$, cancelling out $\xi(I_n)$ (note that L.H.S. of (**)$= p(A,B)u(I)\xi(I_n)$), we obtain that $p(\overline{A} \vee \overline{I}, \overline{B})u(\overline{A} \wedge \overline{I})$ occurs in (*) with coefficient 1 (note that by Case 2 in the proof of Lemma 5.3.1, we have $(A,B) \vee I = (\overline{A} \vee \overline{I}, \overline{B}),\ (A,B) \wedge I = (\overline{A} \wedge \overline{I}, (\underbrace{1, \cdots, 1}_{n+1 \text{ times}})))$.

Thus the result follows if the relation is of the type (5) (or (6)) of Theorem 4.1.1.

□

Lemma 5.4.2. *Let τ, ϕ be two non-comparable elements of D. Then for every (α, β) on the right-hand side of the straightening relation (in $R(D)$, as given by Theorem 4.1.1), we have*

(1) $\tau, \phi \in]\beta, \alpha[$,
(2) $\tau \dot\cup \phi = \alpha \dot\cup \beta$
(here, $\dot\cup$ denotes a disjoint union).

Proof. The assertions follow from Theorem 4.1.1 (and the identification of D as a sublattice of $\mathcal{C}(\underline{m}, \underline{q})$).

□

Theorem 5.4.3. *There exists a flat family over $\mathbb{A}^1$, with Spec $R(D)$ as the generic fiber and Spec $A(D)$ as the special fiber. In particular, $R(D)$ is a normal Cohen-Macaulay ring of dimension $d+3$ (where $d = (m+q)n - n^2$).*

Proof. In view of Theorem 5.2.2, and Corollary 5.2.3, it suffices to show that (a)- (c) of Theorem 5.2.2 hold for R_D.

(a) follows from Lemma 5.4.1; (b) and (c) follow from Lemma 5.4.2.

Clearly $R(D)$ has dim $d+3$ (since $\dim A(D) = d+3$ (cf. Lemma 5.3.2)). □

Theorem 5.4.4. *The K-algebra S is normal, Cohen-Macaulay of dimension $(m+q)n - n^2 + 1$.*

Proof. The algebra $M(D)(= R(D)_{(x(\mathbf{0}))})$ being a homogeneous localization of the normal, Cohen-Macaulay ring $R(D)$, is a normal, Cohen-Macaulay ring of $\dim d+2$. Considering $M(D)$ as a graded ring (cf. §4.3), we have $S(D) = M(D)_{(x(\mathbf{1}))}$. Hence $S(D)$ being a homogeneous localization of the normal, Cohen-Macaulay ring $M(D)$, is a normal, Cohen-Macaulay ring of dimension $d+1$. This together with Theorem 4.5.4 implies that S is a normal, Cohen-Macaulay ring of dimension $d+1$ (note that $d = (m+q)n - n^2$). □

6. The ring of invariants $K[X]^{SL_n(K)}$

We preserve the notation of §3, §4. In this section, we shall show that the inclusion $S \subseteq R^{SL_n(K)}$ is in fact an equality, i.e., $S = R^{SL_n(K)}$.

We now apply Lemma 2.0.4 to our situation. Let $G = SL_n(K)$. Consider

$$X = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{q \text{ copies}} = Spec\, R, \mathbb{A}^N = M_{m,q}(K) \times K^{\binom{m}{n}} \times K^{\binom{q}{n}}$$

Let $\{\langle u_i, \xi_j \rangle), 1 \leq i \leq m, 1 \leq j \leq q, u(I), \xi(J),\ I \in H_u, J \in H_\xi\}$ be denoted by $\{f_1, \cdots, f_N\}$ (note that $f_1, \cdots, f_N$ are G-invariant elements in R). Let $x = (\underline{u}, \underline{\xi}) \in X$. Let $\psi : X \to \mathbb{A}^N$ be the map, $\psi(x) = (f_1(x), \cdots, f_N(x))$. Clearly $\psi(X) = Spec\, S$. Let us denote $Y = Spec\, S$.

Proposition 6.0.5. *With $X, \mathbb{A}^N, \psi, Y$ as above, the hypotheses of Lemma 2.0.4 are satisfied.*

Proof. (i) Let $x \in X^{ss}$. We need to show that $\psi(x) \neq 0$. If possible, let us assume that $\psi(x) = 0$. Let $x = (\underline{u}, \underline{\xi})$. Let W_u (resp. W_ξ) be the span of $\{u_1, \cdots, u_m\}$ (resp.$\{\xi_1, \cdots, \xi_q\}$). Further, let dim $W_u = r$, dim $W_\xi = s$. The assumption that $\psi(x) = 0$ implies in particular that $u(I)(x) = 0, \forall I \in I(n,m), \xi(J)(x) = 0, \forall J \in I(n,q)$. Hence, W_u (resp. W_ξ) is not equal to V (resp.V^*). Therefore, we get $r < n, s < n$. Also at least one of $\{r, s\}$ is non-zero; otherwise, $r = 0 = s$ would imply $u_i = 0, \forall i, \xi_j = 0, \forall j$, i.e., $x = 0$ which is not possible, since $x \in X^{ss}$. Let us suppose that $r \neq 0$. (The proof is similar if $s \neq 0$.) The assumption that $\psi(x) = 0$ implies in particular that $\langle u_i, \xi_j \rangle = 0$, for all i, j; hence, $W_\xi \subseteq W_u^\perp$. Therefore, $s \leq n - r$. Hence we can choose a basis $\{e_1, \cdots, e_n\}$ of V such that $W_u =$ the K-span of $\{e_1, \cdots, e_r\}$, and $W_\xi \subseteq$ the K-span of $\{e^*_{r+1}, \cdots, e^*_n\}$. Writing each vector u_i as a row vector (with respect to this basis), we may represent the u's by the $m \times n$ matrix $\mathcal{U}$ given by

$$\mathcal{U} := \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} & 0 & \dots & 0 \\ u_{21} & u_{22} & \dots & u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mr} & 0 & \dots & 0 \end{pmatrix}$$

Similarly, writing each vector ξ_j as a column vector (with respect to the above basis), we may represent ξ's by the $n \times q$ matrix Λ given by

$$\Lambda := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \xi_{r+1\,1} & \xi_{r+1\,2} & \dots & \xi_{r+1\,q} \\ \vdots & \vdots & & \vdots \\ \xi_{n\,1} & \xi_{n\,2} & \dots & \xi_{n\,q} \end{pmatrix}$$

Choose integers $a_1, \cdots, a_r, b_{r+1}, \cdots, b_n$, all of them > 0 so that $\sum a_i = \sum b_j$.

Let g_t be the diagonal matrix in $G(= SL_n(K)), g_t = diag(t^{a_1}, \cdots t^{a_r}, t^{-b_{r+1}}, \cdots, t^{-b_n})$. We have, $g_t x = g \cdot (\mathcal{U}, \Lambda) = (\mathcal{U} g_t, g_t^{-1} \Lambda)$ (cf. §2.1) $= (\mathcal{U}_t, \Lambda_t)$, where

$$\mathcal{U}_t = \begin{pmatrix} t^{a_1} u_{11} & t^{a_2} u_{12} & \dots & t^{a_r} u_{1r} & 0 & \dots & 0 \\ t^{a_1} u_{21} & t^{a_2} u_{22} & \dots & t^{a_r} u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ t^{a_1} u_{m1} & t^{a_2} u_{m2} & \dots & t^{a_r} u_{mr} & 0 & \dots & 0 \end{pmatrix}$$

and

$$\Lambda_t = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ t^{b_{r+1}} \xi_{r+1\,1} & t^{b_{r+1}} \xi_{r+1\,2} & \dots & t^{b_{r+1}} \xi_{r+1\,q} \\ \vdots & \vdots & & \vdots \\ t^{b_n} \xi_{n\,1} & t^{b_n} \xi_{n\,2} & \dots & t^{b_n} \xi_{n\,q} \end{pmatrix}$$

Hence $g_t x \to 0$ as $t \to 0$, and this implies that $0 \in \overline{G \cdot x}$ which is a contradiction to the hypothesis that x is semi-stable. Therefore our assumption that $\psi(x) = 0$ is wrong and (i) of Lemma 2.0.4 is satisfied.
(ii) Let

$$U = \{(\underline{u}, \underline{\xi}) \in X \mid \{u_1, \dots, u_n\},\ \{\xi_1, \dots, \xi_n\} \text{ are linearly independent}\}$$

Clearly, U is a G-stable open subset of X.
Claim : G operates freely on U, $U \to U\, mod\, G$ is a G-principal fiber space, and F induces an immersion $U/G \to \mathbb{A}^N$.
Proof of Claim: Let $H = GL_n(K)$. We have a G-equivariant identification

$$(*) \qquad U \cong H \times H \times \underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{(q-n) \text{ copies}} = E \times F, \text{ say}$$

where $E = H \times H, F = \underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{(q-n) \text{ copies}}$. From this it is clear that G operates freely on U. Further, we see that $U\, mod\, G$ may be identified with the fiber space with base $(H \times H)\, mod\, G$ (G acting on $H \times H$ as $g \cdot (h_1, h_2) = (h_1 g, g^{-1} h_2), g \in G, h_1, h_2 \in H$), and fiber $\underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{(q-n) \text{ copies}}$ associated to the principal fiber space $H \times H \to (H \times H)\,/\,G$. It remains to show that ψ induces an immersion $U/G \to \mathbb{A}^N$, i.e., to show that the map $\psi : U/G \to \mathbb{A}^N$ and its differential $d\psi$ are both injective. We first prove the injectivity of $\psi : U/G \to \mathbb{A}^N$. Let x, x' in U/G be such that $\psi(x) = \psi(x')$. Let $\eta, \eta' \in U$ be lifts for x, x' respectively. Using the identification

(*) above, we may write

$$\eta = (A, u_{n+1}, \cdots, u_m, B, \xi_{n+1}, \cdots, \xi_q),\ A, B \in H$$
$$\eta' = (A', u'_{n+1}, \cdots, u'_m, B', \xi'_{n+1}, \cdots, \xi'_{q'}),\ A', B' \in H$$

(here, $u_i, 1 \leq i \leq n$ are given by the rows of A, while $\xi_i, 1 \leq i \leq n$ are given by the columns of B; similar remarks on u'_i, ξ'_i). The hypothesis that $\psi(x) = \psi(x')$ implies in particular that

$$\langle u_i, \xi_j \rangle = \langle u'_i, \xi'_j \rangle\ , 1 \leq i, j \leq n$$

which may be written as $AB = A'B'$. This implies that

$$(**) \qquad A' = A \cdot g,$$

where $g = BB'^{-1}(\in H)$. Further, the hypothesis that $u(I)(x) = u(I)(x'), \forall I$, implies in particular that $u(I_n)(x) = u(I_n)(x')$ (where $I_n = (1, 2, \cdots, n)$). Hence we obtain

$$(***) \qquad detA = detA'$$

Now (**) and (***) imply that g in fact belongs to $G(= SL_n(K))$. Hence on U/G, we may suppose that

$$x = (u_1, \cdots, u_n, u_{n+1}, \cdots, u_m, \xi_1, \cdots, \xi_q)$$
$$x' = (u_1, \cdots, u_n, u'_{n+1}, \cdots, u'_m, \xi'_1, \cdots, \xi'_q)$$

where $\{u_1, \cdots, u_n\}$ is linearly independent.

For a given j, we have,

$$\langle u_i, \xi_j \rangle = \langle u_i, \xi'_j \rangle\ , 1 \leq i \leq n,\ \text{implies}, \xi_j = \xi'_j$$

(since, $\{u_1, \cdots, u_n\}$ is linearly independent). Thus we obtain

$$(\dagger) \qquad \xi_j = \xi'_j, \text{ for all } j$$

On the other hand, we have (by definition of U) that $\{\xi_1, \ldots, \xi_n\}$ is linearly independent. Hence fixing an $i, n+1 \leq i \leq m$, we get

$$\langle u_i, \xi_j \rangle = \langle u'_i, \xi_j \rangle (= \langle u'_i, \xi'_j \rangle)\ , 1 \leq j \leq n \text{ implies}, u_i = u'_i.$$

Thus we obtain

$$(\dagger\dagger) \qquad u_i = u'_i, \text{ for all } i$$

The injectivity of $\psi : U/G \to \mathbb{A}^N$ follows from (†),(††).

To prove that the differential dψ is injective, we merely note that the above argument remains valid for the points over $K[\epsilon]$, the algebra of dual numbers ($= K \oplus K\epsilon$, the K-algebra with one generator ϵ, and one relation $\epsilon^2 = 0$), i.e., it remains valid if we replace K by $K[\epsilon]$, or in fact by any K-algebra.

(iii) The above Claim implies in particular that $\dim U/G = \dim U$ - $\dim G =$ $(m + q)n - (n^2 - 1) = \dim Spec\, S$ (cf. Theorem 5.4.4).

The condition (iv) of Lemma 2.0.4 follows from Theorem 5.4.4.

□

Theorem 6.0.6. *Let* $V = K^n$, $X = \underbrace{V \oplus \cdots \oplus V}_{m\,\text{copies}} \times \underbrace{V^* \oplus \cdots \oplus V^*}_{q\,\text{copies}}$, *where* $m, q > n$. *Then for the diagonal action of* $G := SL_n(K)$, *we have*

(1) ***First Fundamental Theorem for*** $SL_n(K)$***-invariants:*** $K[X]^G$ *is generated by* $\{p(A,B), u(I), \xi(J), (A,B) \in H_p, I \in H_u, J \in H_\xi\}$.
(2) ***Second Fundamental Theorem for*** $SL_n(K)$***-invariants:*** *The ideal of relations among the generators* $\{p(A,B), u(I), \xi(J), (A,B) \in H_p, I \in H_u, J \in H_\xi\}$ *is generated by the six type of relations as given by Theorem 4.1.1.*
(3) ***A standard monomial basis for*** $SL_n(K)$***-invariants:*** *Standard monomials in* $\{p(A,B), u(I), \xi(J), (A,B) \in H_p, I \in H_u, J \in H_\xi\}$ *form a* K*-basis for* $K[X]^G$.
(4) $K[X]^G$ *is Cohen-Macaulay.*

Proof. Lemma 2.0.4 implies that $Spec\, S$ is the categorical quotient of X by G and $\psi : X \to Spec\, S$ is the canonical quotient map. Assertion (1) follows from this. Assertion (2) follows from Theorem 4.5.5. Assertion (3) follows from Theorem 4.5.3. Assertion (4) follows from Theorem 5.4.4 □

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Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: `lakshmibai@neu.edu`

Department of Mathematics, Suffolk University, Boston, MA 02114
E-mail address: `shukla@mcs.suffolk.edu`