Algebras associated to set-theoretical solutions of YBE

1 Generalities

Given $n, m \in \mathbb{Z}$ we set $[\![n, m]\!] = \{k \in \mathbb{Z} \mid n \leq k \leq m\}$, and if $n \in \mathbb{N}$ we will also write $[\![n]\!] = [\![1, n]\!]$. We work over $\mathbb{C}$, so all vector spaces, tensor products, etc. are over this field.

We denote by ISyb the category of all non-degenerate and involutive set theoretical solutions to the Yang-Baxter equation, and by YB the category of all finite dimensional solutions over the complex numbers.

1.1. Let $n \in \mathbb{N}$. Given $i, j \in [\![n]\!]$ we denote by $e_i \in \mathbb{C}^n$ the i-th element of the canonical basis, and by $E^i_j \in \mathsf{Mat}_n(\mathbb{C})$ the matrix having a 1 as its (i, j)-th entry, all other entries equal to zero. We identify $\mathbb{C}^n \otimes \mathbb{C}^n$ with $\mathsf{Mat}_n(\mathbb{C})$ through the linear correspondence $e_i \otimes e_j \mapsto E^i_j$.

Define the map $i : \mathsf{Mat}_n(\mathbb{C}) \otimes \mathsf{Mat}_n(\mathbb{C}) \longrightarrow \mathsf{End}(\mathsf{Mat}_n(\mathbb{C}))$ where for each $A, B, C \in \mathsf{Mat}_n(\mathbb{C})$ we have $i(A \otimes B)(C) = AC^tB$. If $F \in \mathsf{End}(\mathsf{Mat}_n(\mathbb{C}))$ and $F(E^i_j) = \sum_{k,l} \alpha^{j,l}_{i,k} E^k_l$ then setting $\overline{F} = \sum_{i,j,k,l} \alpha^{j,l}_{i,k} E^k_j \otimes E^i_l$ we obtain $i(\overline{F}) = F$. Thus i is an epimorphism, and since the dimensions of domain and codomain are the same it is an isomorphism, with inverse $F \mapsto \overline{F}$.

Denote by $\tau \in \mathsf{End}(\mathsf{Mat}_n(\mathbb{C}))$ the transposition map. Then we have

$$\overline{F \circ \tau} = \sum_{i,j,k,l} \alpha^{i,l}_{j,k} E^k_j \otimes E^i_l = \sum_{i,j,k,l} \alpha^{j,l}_{i,k} E^k_i \otimes E^j_l;$$
$$\overline{\tau \circ F} = \sum_{i,j,k,l} \alpha^{j,l}_{i,k} E^l_j \otimes E^i_k = \sum_{i,j,k,l} \alpha^{j,k}_{i,l} E^k_j \otimes E^i_l.$$

Denote by $\langle -, - \rangle$ the only inner product on $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $\{e_i \otimes e_j \mid i, j \in [\![n]\!]\}$ forms an orthogonal basis. This is the pullback of the usual inner product on $\mathsf{Mat}_n(\mathbb{C})$ for which the basis $\{E^i_j \mid i, j \in [\![n]\!]\}$ is an orthogonal basis. We denote by F^{t_1} the only map such that $\langle F^{t_1}(E^k_j), E^i_l \rangle = \langle F(E^i_j), E^k_l \rangle$. In other words $F^{t_1}(E^k_j) = \sum_{i,l} \alpha^{j,l}_{i,k} E^i_l$. We define F^{t_2} analogously, so $F^{t_2}(E^i_l) = \sum_{k,j} \alpha^{j,l}_{i,k} E^k_j$. Thus we have

$$\overline{F^{t_1}} = \sum_{i,j,k,l} \alpha^{j,l}_{k,i} E^k_j \otimes E^i_l = \sum_{i,j,k,l} \alpha^{j,l}_{i,k} E^i_j \otimes E^k_l$$
$$\overline{F^{t_2}} = \sum_{i,j,k,l} \alpha^{l,j}_{i,k} E^k_j \otimes E^i_l = \sum_{i,j,k,l} \alpha^{j,l}_{i,k} E^k_i \otimes E^i_j.$$

1.2. Let $n \in \mathbb{N}$ and let $(\llbracket n \rrbracket, S)$ be an symmetric set, and for each $i, j \in \llbracket n \rrbracket$ put $S(i,j) = (g_i(j), f_j(i))$. Then S induces a classical solution to the YBE by setting $S(e_i \otimes e_j) = e_{g_i(j)} \otimes e_{f_j(i)}$. Using the notation from the previous paragraph

$$\overline{S} = \sum_{i,j} E_j^{g_i(j)} \otimes E_{f_j(i)}^i; \qquad \overline{\tau \circ S} = \sum_{i,j} E_j^{f_j(i)} \otimes E_{g_i(j)}^i;$$

$$\overline{S^{t_1}} = \sum_{i,j} E_j^i \otimes E_{f_j(i)}^{g_i(j)}; \qquad \overline{S^{t_2}} = \sum_{i,j} E_{f_j(i)}^{g_i(j)} \otimes E_j^i.$$

1.3. In [ESS99, §3.2] Etingof, Schedler and Soloviev introduced the *retraction* of an involutive solution (X, r). Writing $i \equiv j$ if and only if $g_i = g_j$ for $i, j \in X$, they show that $i \equiv j$ also implies that $f_i = f_j$. Setting $Y = X/\equiv$ and writing $[i]$ for the class of i in Y, they also show that $s([i],[j]) = ([g_i(j)], [f_j(i)])$ is an involutive solution with underlying set Y.

References

[ESS99] P. Etingof, T. Schedler, and A. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), no. 2, 169–209.