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QUANTUM DEFORMATION AND TABLEAUX REALIZATION OF SIMPLE DENSE $\mathfrak{gl}(n, \mathbb{C})$-MODULES

VOLODYMYR MAZORCHUK

Department of Mathematics, Uppsala University, Box 480
SE-75106, Uppsala, Sweden
mazor@math.uu.se

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We show that all simple dense $\mathfrak{gl}(n, \mathbb{C})$-modules with finite-dimensional weight spaces admit quantum deformation to the corresponding Drinfeld–Jimbo quantum group. We also realize almost all such modules via Gelfand–Zetlin construction, presenting them by a basis and precise formulae for the action of generators of $\mathfrak{gl}(n, \mathbb{C})$. This construction has a direct analogue on the quantum level, hence for such modules the main result becomes very easy.

Keywords: Weight module; Gelfand–Zetlin method; quantum deformation.

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1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. A $\mathfrak{g}$-module, M, is called *weight* if the action of $\mathfrak{h}$ on M is diagonalizable. If M is a weight module and $\lambda \in \mathfrak{h}^*$, then the *weight space* M_λ of M is the collection of all $v \in M$, on which all $h \in \mathfrak{h}$ act via $\lambda(h)$. The set of all λ such that M_λ is non-zero is called the *support* of M and is denoted by $\mathrm{supp}(M)$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Q be the abelian subgroup of $\mathfrak{h}^*$ generated by Δ. A weight module, M, is called *dense* provided that $\mathrm{supp}(M)$ equals a coset from $\mathfrak{h}^*/Q$. According to Fernando–Futorny's Theorem, [3, 4], any simple weight $\mathfrak{g}$-module is either dense or induced from a dense module over a parabolic subalgebra of $\mathfrak{g}$. This, in particular, reduces the classification of simple weight modules to that of dense modules. Under the assumption that the weight spaces of M are finite dimensional, all simple dense modules were recently classified by Mathieu [12] (the case of modules with one-dimensional weight spaces has been earlier completed by Benkart, Britten and Lemire [1]). It happens that such modules exist only if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ or $\mathfrak{g}$ is of type C_n (this result itself belongs to Fernando [3], but in [12] the reader can find short and easy proof). As in this paper we will work mostly with weight modules having finite-dimensional weight spaces, in what follows by weight we will mean weight with finite-dimensional weight spaces.

The aim of this paper is to show that all $\mathfrak{sl}(n,\mathbb{C})$-modules, constructed by Mathieu, admit true quantum deformations to the corresponding Drinfeld–Jimbo quantum group. The way Mathieu constructed these modules will allow us to avoid the usual $\mathbb{A}$-form technique used for such kind of problems, see [8, 9, 11], which also usually involves some extra integrality conditions. Moreover, it happens that almost all simple weight modules can be constructed via Gelfand–Zetlin method, see [13] for the overview and references therein for the original results, which provides a special basis in the module and gives precise formulae for the action of generators of $\mathfrak{g}$. This is similar to tableaux realization of weight modules in the sense of [14]. This construction has a straightforward quantum analogue by [16] and thus gives us an easy way to deform corresponding simple weight modules. We also note that this approach was used by Jimbo [6], to deform finite-dimensional $\mathfrak{sl}(n,\mathbb{C})$-module before the general result of Lusztig [11].

Our realization of simple dense modules via Gelfand–Zetlin construction is very natural from the point of view of parameterization of simple dense modules. The series of such modules (called coherent families in [12]) over $\mathfrak{sl}(n,\mathbb{C})$ are parameterized by special central characters, which are realized as $n+1$-tuples of complex parameters. In an obvious way each tuple defines a Gelfand–Zetlin tableau, i.e. an element from the distinguished basis, which extends to a complete basis of the module. However, in each coherent family there are several modules (the corresponding parameters are defined by some non-trivial linear equations), such that functions, appearing in Gelfand–Zetlin formulae, have singularities on certain basis vectors of such modules. This is precisely the case, when we do not know how to realize the corresponding module via Gelfand–Zetlin construction. But, as we have mentioned, the parameters of such modules satisfy some non-trivial linear equations and hence they really form a small set (countable union of hyperplanes in the variety of all parameters), so we will call them *degenerate* to emphasize the fact the Gelfand–Zetlin formulae degenerate on these modules. However, for *pointed* coherent families, i.e. those with one dimensional weight subspaces, the functions in Gelfand–Zetlin formulae have removable singularities and thus our construction can be extended even to degenerate modules. Here we have to remark that for coherent families the notions of pointed, i.e. having a one-dimensional weight space, and *completely pointed*, i.e. whose all weight spaces are finite-dimensional, coincide. So, in what follows we will use the first one, which is shorter.

In case of *degenerate* simple dense $\mathfrak{sl}(n,\mathbb{C})$-modules (i.e. those modules, which can not be constructed via Gelfand–Zetlin method) we adjust Mathieu's technique of localization of the universal enveloping algebra for the quantum group case and use it to construct simple dense modules starting from another simple dense module. In this way we do not have a nice realization of obtained simple dense modules, but comparison of parameters allows us to claim that they tend to the necessary classical simple dense modules under the classical limit. It happens that the set of simple dense modules, which can be realized via Gelfand–Zetlin construction, is big enough to obtain all other simple dense modules applying some twists in the localized algebra. This technique automatically guarantees that the dimension of the weight spaces is preserved during the deformation process, and hence the obtained modules are true quantum deformations of the classical modules we started with.

As a byproduct of our Gelfand–Zetlin realization, we construct several new families of simple weight $\mathfrak{sl}(n,\mathbb{C})$-modules with infinite-dimensional weight spaces, in particular, new simple Harish–Chandra module with respect to a well-embedded $\mathfrak{sl}(k,\mathbb{C})$-subalgebra.

Since the Gelfand–Zetlin method is better adjusted to the reductive algebra $\mathfrak{gl}(n,\mathbb{C})$ we will work with it. Using the canonical inclusion $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ one easily translates all the results to $\mathfrak{sl}(n,\mathbb{C})$-case.

In the case of Lie algebras of type C_n the situation with Gelfand–Zetlin approach is more complicated. The case of finite-dimensional modules has been recently handled by Molev [17]. However, at the moment it is not clear how to extend his method to infinite-dimensional modules. Indeed, the localization technique can be easily applied also in this case. However, it is not apparent how to construct quantum deformation for a big enough class of modules (the first step), which in the A_n case can be done using the Gelfand–Zetlin method. This is the reason why this case will be treated in a separate paper.

Let us briefly describe the structure of the paper. In the next section we collect necessary preliminaries about $U_q(\mathfrak{gl}(n,\mathbb{C}))$, Gelfand–Zetlin construction and Mathieu's classification. In Sec. 3 we construct tableaux realization for simple dense modules over $\mathfrak{gl}(n,\mathbb{C})$. This works for almost all modules, but, in the case of pointed modules, a variation of our construction works always. This is the content of Sec. 4. Section 5 is a quantum version of two preceding sections. In Sec. 6 we show how one can use Gelfand–Zetlin tableaux to construct new simple modules over $\mathfrak{gl}(n,\mathbb{C})$ and $U_q(\mathfrak{gl}(n,\mathbb{C}))$. Finally, in Sec. 7 we complete the picture with constructing quantum deformations for all simple dense $\mathfrak{gl}(n,\mathbb{C})$-modules.

2. Preliminaries

2.1. *Quantum algebra* $U_q(\mathfrak{gl}(n,\mathbb{C}))$

Here we give the basic definitions for quantum groups and refer the reader to [5, 7, 10] for details.

Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{gl}(n,\mathbb{C})$, consisting of all diagonal matrices, and Δ be the corresponding root system of $\mathfrak{g}$. We fix the standard basis π of Δ and denote by α_i the root corresponding to the matrix unit $e_{i,i+1}$, $i = 1,\dots,n-1$. Then W is the Weyl group and $(\cdot,\cdot)$ is the standard W-invariant form on $\mathfrak{h}^*$. Choose $q \in \mathbb{C}$ such that $q \neq 0$ and $q^l \neq 1$, $l \in \mathbb{N}$. Fix some $h \in \mathbb{C}$ such that $q = \exp(h)$ and for $x \in \mathbb{C}$ set $q^x = \exp(hx)$. Denote by $1(q)$ the set of all $x \in \mathbb{C}$ such that $q^x = 1$. For $x \in \mathbb{C}$ we put $[x]_q = \frac{q^x - q^{x-1}}{q - q^{-1}}$. Clearly, $[x]_q = 0$ if and only if $2x \in 1(q)$.

The quantized universal enveloping algebra $U_q = U_q(\mathfrak{gl}(n,\mathbb{C}))$ is defined as the unital associative $\mathbb{C}$-algebra with generators E_i, F_i, $i = 1,\dots,n-1$; K_i, K_i^{-1}, $i = 1,\dots,n$, subject to the following relations (here indices of K run through $\{1,\dots,n\}$ and indices of E, F run through $\{1,\dots,n-1\}$):

$$K_iK_i^{-1} = K_i^{-1}K_i = 1\,, \quad K_i^{\pm 1}K_j^{\pm 1} = K_j^{\pm 1}K_i^{\pm 1}\,,$$

$$K_iE_j = q^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i\,, \quad K_iF_j = q^{-\delta_{i,j}+\delta_{i,j+1}}F_jK_i\,,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q-q^{-1}}\,, \quad [E_i,E_j] = [F_i,F_j] = 0\,, \quad |i-j| \geq 2\,,$$

$$E_i^2E_{i\pm 1} - (q+q^{-1})E_iE_{i\pm 1}E_i + E_{i\pm 1}E_i^2 = 0\,,$$

$$F_i^2F_{i\pm 1} - (q+q^{-1})F_iF_{i\pm 1}F_i + F_{i\pm 1}F_i^2 = 0\,.$$

A *weight U_q-module* is a module, on which the action of the subalgebra $\mathcal{K}$, generated by all K_i, is diagonalizable. Let V be a weight $U(\mathfrak{g})$-module. There is a natural correspondence between weight spaces with respect to $\mathfrak{h}$ and weight spaces with respect to $\mathcal{K}$ given by $\mathfrak{h}^* \ni \lambda \mapsto q^{(\lambda,\cdot)}$, the last being evaluated on $K_\alpha^{\pm 1}$ as $q^{\pm(\lambda,\alpha)}$. However, the inequality $\lambda \neq \mu \in \mathfrak{h}^*$ does not guarantee $q^{\pm(\lambda,\alpha)} \neq q^{\pm(\mu,\alpha)}$ for some $\alpha \in \pi$. At the same time, as q is a non-zero non-root of unity, for any non-zero $\mu \in Q$ there always exists $\alpha \in \pi$ such that $q^{\pm(\lambda,\alpha)} \neq q^{\pm(\lambda+\mu,\alpha)}$. Then a *true quantum deformation* of a weight $\mathfrak{g}$-module, V, is a module, V_q, over U_q, such that V is obtained from V_q by taking the limit $q \to 1$ and the dimensions of the corresponding weight spaces in V and V_q are the same.

2.2. *Tableaux and Gelfand–Zetlin construction for $\mathfrak{gl}(n,\mathbb{C})$ and $U_q(\mathfrak{gl}(n,\mathbb{C}))$*

The Gelfand–Zetlin method of constructing simple finite-dimensional $\mathfrak{sl}(n,\mathbb{C})$-modules is, in fact, a restriction from the $\mathfrak{gl}(n,\mathbb{C})$ case. Denote by $e_{i,j}$, $1 \leq i$, $j \leq n$, the matrix units. By a *tableau* we will mean a doubly-indexed complex vector, $[l] = (l_{i,j})_{i=1,\dots,n}^{j=1,\dots,i}$. Let $[\delta^{i,j}]$ denote the Kronecker tableau, i.e. $\delta_{i,j}^{i,j} = 1$ and $\delta_{k,l}^{i,j} = 0$ if $k \neq i$ or $l \neq j$.

Simple finite-dimensional $\mathfrak{gl}(n,\mathbb{C})$ modules are parameterized by complex vectors, $\mathbf{m} = (m_1,\dots,m_n)$, satisfying $m_i - m_{i+1} \in \mathbb{N}$. The corresponding module $V(\mathbf{m})$ can be realized as a space with the basis

$$B(\mathbf{m}) = \{[l] | l_{n,i} = m_i;\ l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+;\ l_{i-1,j} - l_{i,j+1} \in \mathbb{N},\ 1 \leq j \leq i \leq n\}$$

on which the action of generators $e_{i,i}$, $i = 1,\dots,n$ and $e_{i,i+1}$, $e_{i+1,i}$, $i = 1,\dots,n-1$, is given by the following *Gelfand–Zetlin formulae*:

$$e_{i,i+1}[l] = -\sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1}(l_{i,j} - l_{i+1,k})}{\prod_{k\neq j}(l_{i,j} - l_{i,k})}\,[l+\delta^{i,j}]\,,$$

$$e_{i+1,i}[l] = \sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1}(l_{i,j} - l_{i-1,k})}{\prod_{k\neq j}(l_{i,j} - l_{i,k})} [l - \delta^{i,j}] ,$$

$$e_{i,i}[l] = \left(\sum_{j=1}^{i} l_{i,j} - \sum_{j=1}^{i-1} l_{i-1,j} \right) [l] .$$

Restricting $V(\mathbf{m})$ to the canonical copy of $\mathfrak{sl}(n,\mathbb{C})$ inside $\mathfrak{gl}(n,\mathbb{C})$ we get a simple $\mathfrak{sl}(n,\mathbb{C})$-module, which we will denote also by $V(\mathbf{m})$. Moreover, all simple finite-dimensional $\mathfrak{sl}(n,\mathbb{C})$-modules are obtained in this way. Two modules $V(\mathbf{m}^{(1)})$ and $V(\mathbf{m}^{(2)})$ are isomorphic as $\mathfrak{gl}(n,\mathbb{C})$-modules if and only if $\mathbf{m}^{(1)} = \mathbf{m}^{(2)}$, and are isomorphic as $\mathfrak{sl}(n,\mathbb{C})$-modules if and only if there exists $c \in \mathbb{C}$ such that $\mathbf{m}_i^{(1)} = \mathbf{m}_i^{(2)} + c$ for all i. In particular, for finite-dimensional modules over $\mathfrak{sl}(n,\mathbb{C})$ it is always sufficient to assume that all entries of $\mathbf{m}$ are integers.

In [6] it is noticed that this construction has a straightforward analogue for $U_q(\mathfrak{sl}(n,\mathbb{C}))$. Indeed, for $\mathfrak{sl}(n,\mathbb{C})$ we can assume that all entries of $\mathbf{m}$ are integers. Then the quantum deformation $V_q(\mathbf{m})$ of $V(\mathbf{m})$ has the same basis $B(\mathbf{m})$ and the action of generators of $U_q(\mathfrak{gl}(n,\mathbb{C}))$ is given by the following *quantum Gelfand–Zetlin formulae*.

$$E_i[l] = -\sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1}[l_{i,j} - l_{i+1,k}]_q}{\prod_{k\neq j}[l_{i,j} - l_{i,k}]_q} [l + \delta^{i,j}] ,$$

$$F_i[l] = \sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1}[l_{i,j} - l_{i-1,k}]_q}{\prod_{k\neq j}[l_{i,j} - l_{i,k}]_q} [l - \delta^{i,j}] ,$$

$$K_i[l] = q^{\left(\sum_{j=1}^{i} l_{i,j} - \sum_{j=1}^{i-1} l_{i-1,j}\right)}[l] .$$

It is obvious that quantum Gelfand–Zetlin formulae tend to usual Gelfand–Zetlin formulae if $q \to 1$ and thus the above construction defines a true quantum deformation of simple finite-dimensional $\mathfrak{gl}(n,\mathbb{C})$-modules (and hence $\mathfrak{sl}(n,\mathbb{C})$-modules as well).

The idea how one can use these formulae to construct new (infinite-dimensional) weight $\mathfrak{gl}(n,\mathbb{C})$- (resp. $U_q(\mathfrak{gl}(n,\mathbb{C}))$-) modules goes back to Drozd, Futorny and Ovsienko, see [2] and references therein (resp. [16]). For this we start with a tableau, $[l]$, satisfying $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all i and all $j \neq k$. Set $B([l]) = \{[t] | l_{i,j} - t_{i,j} \in \mathbb{Z}, l_{n,i} = t_{n,i}, 1 \leq j \leq i \leq n\}$. Then (quantum) Gelfand–Zetlin formulae define on the linear span $V([l])$ of $B([l])$ the structure of a $\mathfrak{gl}(n,\mathbb{C})$- (resp. $U_q(\mathfrak{gl}(n,\mathbb{C}))$-) module of finite length.

Sometimes it is also useful to know that there is a special subalgebra, Γ, of $U(\mathfrak{g})$ which separates the elements of the Gelfand–Zetlin basis $B(\mathbf{m})$, resp. $B([l])$. This one is constructed as follows (see [2]): we fix the standard inclusions of $\mathfrak{gl}(i,\mathbb{C})$ to $\mathfrak{gl}(i+1,\mathbb{C})$ with respect to the upper left corner. Then the commutative subalgebra Γ of $U(\mathfrak{gl}(n,\mathbb{C}))$, generated by the centers of all $U(\mathfrak{gl}(i,\mathbb{C}))$, $i=1,\dots,n$, is called the *Gelfand–Zetlin subalgebra* of $U(\mathfrak{gl}(n,\mathbb{C}))$. All elements from $B(\mathbf{m})$ and $B([l])$ above are eigenvectors with respect to Γ, moreover, for $1\leq i\leq n$ there is a basis in the center of $U(\mathfrak{gl}(i,\mathbb{C}))$, such that the eigenvalues of the action of these basis elements on $[t]$ are precisely the elementary symmetric functions in $t_{i,j}$, $j=1,\dots,i$. In particular, if $[t]$ and $[s]$ are tableaux and there exists i such that $(t_{i,1},\dots,t_{i,i})$ can not be obtained from $(s_{i,1},\dots,s_{i,i})$ by a permutation of components, then the action of Γ separates $[t]$ and $[s]$.

2.3. *Mathieu's classification of simple dense modules*

The Mathieu's approach to the classification of simple dense modules is based on the notion of coherent family. Denote by $U(\mathfrak{g})_0$ the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then a *coherent family* is a weight $\mathfrak{g}$-module, $\mathcal{M}$, satisfying the following two conditions:

(1) $\dim(\mathcal{M}_\lambda)=\dim(\mathcal{M}_\mu)$ for all λ, $\mu\in\mathfrak{h}^*$;
(2) the function $\lambda\mapsto \mathrm{Tr}(u)|_\lambda$ is polynomial in λ for all $u\in U(\mathfrak{g})_0$.

One of the main steps in the Mathieu's classification is that each simple dense module is a direct summand of some coherent family. The next step of reduction is that each coherent family contains an admissible simple highest weight module as a subquotient. Here by *admissible* we mean a module, whose dimensions of weight spaces are uniformly bounded. So the irreducible coherent families can be classified by admissible simple highest weight module they contain, which is a relatively easy technical problem. It turns out that infinite-dimensional admissible simple highest weight modules, and hence coherent families, exist only if $\mathfrak{g}$ is of type A_n or C_n. Moreover, if $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{C})$, then irreducible coherent families are in bijection with tuples $(m_1,\dots,m_n)$ satisfying $\sum_{i=1}^n m_i=0$, and $m_i-m_{i+1}\in\mathbb{N}$, $i=2,\dots,n-1$. A certain permutation of entries in $(m_1,\dots,m_n)$ defines the (shifted) highest weight of a simple admissible highest weight module, occurring in this coherent family. However, the central character of this module is computed as certain invariant function in entries of $\mathbf{m}$ and hence does not change under permutations.

We will need (a restricted version of) one more technical tool used in [12], which we will call *Mathieu's localization of* $U(\mathfrak{g})$. Fix any Weyl–Chevalley basis, X_α, $\alpha\in\Delta$, and H_α, $\alpha\in\pi$, in $\mathfrak{g}$. Let $\alpha\in\Delta$. As $X_{-\alpha}$ is locally *ad*-nilpotent on $U(\mathfrak{g})$, the multiplicative set $\{X_{-\alpha}^i|i\in\mathbb{N}\}$ satisfies Ore's condition for localizability in $U(\mathfrak{g})$ and we can form the corresponding localized algebra U_α. On U_α there exists a unique 1-parameter family of automorphisms $\theta_x: U_\alpha\to U_\alpha$ satisfying

(1) $\theta_x(u) = X_{-\alpha}^{x} u X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$;
(2) the map $x \mapsto \theta_a(u)$ is polynomial in x for all $u \in U(\mathfrak{g})$.

We refer the reader to [12] for details.

3. Tableaux Realization of Dense $\mathfrak{gl}(n,\mathbb{C})$-Modules

In this section we realize almost all simple dense $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$-modules via Gelfand–Zetlin construction and then transfer this to the quantum case. As quantum Gelfand–Zetlin formulae tend to classical Gelfand–Zetlin formulae in the classical limit, we automatically get that our construction gives us a true quantum deformation of simple dense modules we consider (see Sec. 5).

Let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$ be a tuple satisfying the following condition:

$$m_i - m_{i+1} \in \mathbb{N}, \quad i = 2, \ldots, n-1 .$$

Choose $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$ such that $x_i - m_2 \notin \mathbb{Z}$, $i = 1, \ldots, n-1$, and consider the set $B(\mathbf{m}, x)$ which consists of all tableaux $[l]$ satisfying the following conditions:

(1) $l_{n,j} = m_j$, $j = 1, \ldots, n$;
(2) $l_{i,1} - x_i \in \mathbb{Z}$, $i = 1, \ldots, n-1$;
(3) $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$, $i = 3, \ldots, n$, $j = 2, \ldots, i$;
(4) $l_{i-1,j} - l_{i,j+1} \in \mathbb{N}$, $i = 3, \ldots, n$, $j = 2, \ldots, i$.

Denote by $M(\mathbf{m}, x)$ the $\mathbb{C}$-vector space, spanned by $B(\mathbf{m}, x)$.

Lemma 3.1. *The Gelfand–Zetlin formulae define on $M(\mathbf{m}, x)$ the structure of a $\mathfrak{g}$-module.*

Proof. Let $u = 0$ be a relation in $U(\mathfrak{g})$. It is sufficient to prove that $u[l] = 0$ for any $[l] \in B(\mathbf{m}, x)$. We use Gelfand–Zetlin formulae and write

$$u[l] = \sum_{[t] \in I(u,[l])} f(u, [l], [t])[l+t] .$$

The coefficients $f(u, [l], [t])$ can be considered as rational functions in entries of $[l]$. Write u in $U(\mathfrak{g})$ as a linear combination of monomials in $\{e_{i,i+1}, e_{i,i}\}$ and let N be the length of the longest monomial occurring (it may be different for different decomposition of u, but we fix one of them). Define $\hat{\mathbf{m}}$ as follows: $\hat{\mathbf{m}}_j = m_j$, $j > 1$, and $\hat{\mathbf{m}}_1 - \hat{\mathbf{m}}_2$ is a very big positive integer, say is bigger than $3N^n + 1$, and consider the finite-dimensional simple $\mathfrak{g}$-module $V(\hat{\mathbf{m}})$. Take any tableau, $[l']$, from the Gelfand–Zetlin basis for $V(\hat{\mathbf{m}})$ satisfying $l'_{i,j} = l_{i,j}$, $i = 2, \ldots, n$, $j = 2, \ldots, i$; $l'_{1,1} > l_{n,2} + N$ and $l'_{i,1} > l'_{i-1,1} + N$, $i = 2, \ldots, n$. Then our choice of $\hat{\mathbf{m}}_1$ guarantees that

$$u[l'] = \sum_{[t] \in I(u,[l'])} f(u, [l'], [t])[l+t] \quad \text{and} \quad I(u, [l']) = I(u, [l]) .$$

In particular, we get $f(u, [l'], [t]) = 0$ as rational functions in $l'_{i,j}$, evaluated in entries of $[l']$. As there exists infinitely many independent ways to choose $l'_{i,1}$, $i = 1, \dots, n$, and $f(u, [l'], [t])$ are rational functions, we derive that $f(u, [l'], [t])$ does not depend on $l'_{i,1}$ and hence $f(u, [l], [t])$ does not depend on $l_{i,1}$. In particular, $f(u, [l], [t]) = f(u, [l'], [t]) = 0$ since $l'_{i,j} = l_{i,j}$, i, $j > 1$. This implies $u[l] = 0$ and completes the proof. $\square$

Define $\phi(\mathbf{m}) = (m_2, \dots, m_n)$. From our assumptions about $\mathbf{m}$ it follows that $\phi(\mathbf{m})$ is a parameter of some simple finite-dimensional $\mathfrak{gl}(n-1, \mathbb{C})$-module.

Lemma 3.2. *$M(\mathbf{m}, x)$ is a weight, dense $\mathfrak{gl}(n, \mathbb{C})$-module and all non-trivial weight subspaces of $M(\mathbf{m}, x)$ have dimension $\dim(V(\phi(\mathbf{m})))$ (the last is the dimension of the simple finite-dimensional $\mathfrak{gl}(n-1, \mathbb{C})$-module $V(\phi(\mathbf{m}))$).*

Proof. The first assertion ($M(\mathbf{m}, x)$ is weight and dense) is obvious by construction of $M(\mathbf{m}, x)$. To prove the second we fix $[l] \in B(\mathbf{m}, x)$. By Gelfand–Zetlin formulae, $[l] \in M(\mathbf{m}, x)_\lambda$, for some $\lambda \in \mathfrak{h}^*$ (it can be precisely computed, but we do not need this). But from Gelfand–Zetlin formulae it also follows that another element, $[t] \in M(\mathbf{m}, x)_\lambda$, belongs to $M(\mathbf{m}, x)_\lambda$ if and only if for any $i = 1, \dots, n-1$ there holds $\sum_{j=1}^{i} l_{i,j} = \sum_{j=1}^{i} t_{i,j}$. Now it is easy to construct a bijection between Gelfand–Zetlin basis of $V(\phi(\mathbf{m}))$ and the subset S of $B(\mathbf{m}, x)$, which consists of all tableaus, lying in $M(\mathbf{m}, x)_\lambda$. If $[s] \in B(\phi(\mathbf{m}))$ we define $[t^s] \in S$ as follows:

$$t^s_{i,j} = s_{i-1,j-1}\,, \quad i = 2, \dots, n\,, \quad j = 2, \dots, i\,;$$

$$t^s_{i,1} = \sum_{j=1}^{i} l_{i,j} - \sum_{j=2}^{i} t_{i,j}\,, \quad i = 1, \dots, n\,.$$

Hence $\dim(M(\mathbf{m}, x)_\lambda) = \dim(V(\phi(\mathbf{m})))$. $\square$

Corollary 3.1. *$M(\mathbf{m}, x)$ has finite length.*

Proof. Follows from Lemma 3.2 and [3, Section 4]. $\square$

We note that from the definition of $B(\mathbf{m}, x)$ it follows that Γ separates the elements in $B(\mathbf{m}, x)$.

Lemma 3.3. *$M(\mathbf{m}, x)$ is simple if and only if $x_i - x_{i+1} \notin \mathbb{Z}$, $i = 1, \dots, n-2$, and $x_{n-1} - m_1 \notin \mathbb{Z}$.*

Proof. By [3, Section 4], the simplicity of $M(\mathbf{m}, x)$ is equivalent to the fact that all $e_{i,i+1}$ and all $e_{i+1,i}$ act injectively (hence bijectively) on $M(\mathbf{m}, x)$. Fix i and let $v = \sum a_{[l]}[l] \in M(\mathbf{m}, x)$. Then Gelfand–Zetlin formulae and the fact that Γ separates the elements of $B(\mathbf{m}, x)$ imply that $e_{i,i+1}v = 0$ (resp. $e_{i+1,i}v = 0$) is equivalent to $e_{i,i+1}[l] = 0$ (resp. $e_{i+1,i}[l] = 0$) for all $[l]$ such that $a_{[l]} \neq 0$. Therefore the injectivity of $e_{i,i+1}$ (resp. $e_{i+1,i}$) should be checked only on elements from $B(\mathbf{m}, x)$. But, by

Gelfand–Zetlin formulae, $e_{i,i+1}[l]=0$ if and only if $l_{i,j}=l_{i+1,j}$ for all $j=1,\dots,i$ and $e_{i+1,i}[l]=0$ if and only if $l_{i,j}=l_{i-1,j}$ for all $j=1,\dots,i-1$. According to the definition of $B(\mathbf{m},x)$, the non-existence of such $[l]$ is obviously equivalent to the conditions of the lemma. $\square$

Our choice of $\mathbf{m}$ above uniquely defines a coherent family, $\mathcal{M}=\mathcal{M}(\mathbf{m})$. All simple dense modules in $\mathcal{M}$ have form $\mathcal{M}(\lambda)=\oplus_{\mu\in\lambda+Q}\mathcal{M}_\mu$ for some $\lambda\in\mathfrak{h}^*$ (we remark that not all $\lambda\in\mathfrak{h}^*$ define simple modules). So, now we can formulate our main result about the realization of $\mathcal{M}(\lambda)$ via Gelfand–Zetlin construction. Take some $\mu\in\mathfrak{h}^*$ and consider the following system of linear equations with indeterminates y_i, $i=1,\dots,n-1$:

$$\begin{cases} y_1 = \mu(e_{1,1}), \\ -y_1+y_2 = \mu(e_{2,2})-m_2, \\ \cdots \quad \cdots \quad \cdots \\ -y_{i-1}+y_i = \mu(e_{i,i})-m_i, \\ \cdots \quad \cdots \quad \cdots \\ -y_{n-2}+y_{n-1} = \mu(e_{n-1,n-1})-m_{n-1}, \\ -y_{n-1} = \mu(e_{n,n})-m_1-m_n. \end{cases} \tag{$*$}$$

Clearly, $(*)$ has a (unique) solution if and only if $\mathcal{M}_\mu\neq 0$. So, we assume that $\mathcal{M}_\mu\neq 0$ and denote by $y(\mu)$ the solution of $(*)$. It follows from Gelfand–Zetlin formulae that, taking $x=y(\mu)$, the Gelfand–Zetlin tableau $[l]$, defined by $l_{i,1}=x_i$, $i=1,\dots,n-1$; $l_{n,1}=m_1$ and $l_{i,j}=m_j$, $i,\,j>1$, represents a vector of weight μ.

Theorem 3.1. *Let $\lambda\in\mathfrak{h}^*$ and $x=y(\lambda)$ does exist. Then the module $\mathcal{M}(\lambda)$ is simple if and only if $M(\mathbf{m},x)$ is simple. Moreover, if $\mathcal{M}(\lambda)$ is simple then $\mathcal{M}(\lambda)\simeq M(\mathbf{m},x)$.*

Proof. After the previous results about $M(\mathbf{m},x)$ this is an easy corollary of Mathieu's classification. Indeed, $M(\mathbf{m},x)$ is a dense module with finite-dimensional weight spaces and has, by construction, the same central character as $\mathcal{M}$. Moreover, $M(\mathbf{m},x)_\lambda\neq 0$ by the choice of λ. As the coherent family, corresponding to $\mathbf{m}$ is unique, the one, generated by $M(\mathbf{m},x)$ is precisely $\mathcal{M}$ (since, after the applications of the Mathieu's twisting functor, we will come to the same simple admissible highest weight module, which correspond to $\mathbf{m}$). Therefore the simple subquotients of finite length modules $M(\mathbf{m},x)$ and $\mathcal{M}_\lambda$ coincide, in particular, these modules are isomorphic as soon as one of them is simple. This completes the proof. $\square$

This theorem defines almost all simple $\mathcal{M}_\lambda$ via its tableaux realization. Those, which can not be constructed in this way correspond to λ such that there exist $i\in\{1,\dots,n-1\}$ having the property that $y_i-m_2\in\mathbb{Z}$. So these modules are defined by some linear equations in the variety of all parameters. It the next section we show that in the case $\dim(V(\phi(\mathbf{m}))) = 1$ even these degenerate cases can be covered.

4. Tableaux Realization of Pointed $\mathfrak{gl}(n, \mathbb{C})$-Modules

In this section we extend the above realization of simple dense $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$-modules to all pointed modules, i.e. ones with one-dimensional weight spaces.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$ be a tuple satisfying the following condition:

$$m_i = m_{i+1} = c\,, \quad i = 2, \ldots, n-1 \quad \text{and} \quad x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$$

(we note that the parameterization we have chosen here differs from the one used by Mathieu by a constant shift of parameters). Consider the set $B(\mathbf{m}, x)$ which consists of all tableaux $[l]$ satisfying the following conditions:

(1) $l_{n,1} = m_1$;
(2) $l_{i,1} - x_i \in \mathbb{Z}$, $i = 1, \ldots, n-1$;
(3) $l_{i,j} = m_j = c$, $i = 2, \ldots, n$, $j = 2, \ldots, i$.

Denote by $M(\mathbf{m}, x)$ the $\mathbb{C}$-vector space, spanned by $B(\mathbf{m}, x)$. For $[l] \in B(\mathbf{m}, x)$ define the action of the generators of $\mathfrak{gl}(n, \mathbb{C})$ on $[l]$ by the following *pointed Gelfand–Zetlin formulae*:

$$\begin{aligned}
e_{i,i+1}[l] &= -(l_{i,1} - l_{i+1,1})[l + \delta^{i,1}]\,, \quad i > 1\,,\\
e_{i+1,i}[l] &= (l_{i,1} - l_{i-1,1})[l - \delta^{i,1}]\,, \quad i > 1\,,\\
e_{1,2}[l] &= -(l_{1,1} - l_{2,1})(l_{1,1} - c)[l + \delta^{1,1}]\,,\\
e_{2,1}[l] &= [l - \delta^{1,1}]\,,\\
e_{i,i}[l] &= (l_{i,1} - l_{i-1,1} + c)[l]\,.
\end{aligned}$$

Let $M(\mathbf{m}, x)$ denote the vector space spanned by $B(\mathbf{m}, x)$.

Lemma 4.1. *The pointed Gelfand–Zetlin formulae define on $M(\mathbf{m}, x)$ the structure of pointed $\mathfrak{gl}(n, \mathbb{C})$-module. Moreover, $M(\mathbf{m}, x)$ is simple if and only if $x_1 - c \notin \mathbb{Z}$, $x_{n-1} - m_1 \notin \mathbb{Z}$ and $x_i - x_{i+1} \notin \mathbb{Z}$ for all $i = 1, \ldots, n-2$.*

One can prove this statement by a careful checking of Serre's relations for $\mathfrak{gl}(n, \mathbb{C})$ on $M(\mathbf{m}, x)$. However, the referee of the paper has suggested the following shorter and more elegant proof:

Proof. It is well known and easy to show that

$$N(a_1, \ldots, a_n) = \operatorname{span}_{\mathbb{C}}\{x_1^{a_1+k_1} x_2^{a_2+k_2-k_1} \cdots x_n^{a_n-k_{n-1}} \mid k_i \in \mathbb{Z}\}$$

is a $\mathfrak{gl}(n,\mathbb{C})$-modules where the action is defined by $e_{i,j} \mapsto x_i\partial_j$. One can set up a module isomorphism. Select a tableau, $[l]$, so that $l_{1,1} = a_1$ is not a negative integer and then define the map:

$$[l] = \begin{pmatrix} a_1 + a_2 + \cdots + a_n & & & c & c & & \cdots & c \\ & a_1 + a_2 + \cdots + k_{n-1} & c & \cdots & & c & \cdots & c \\ & & & & \vdots & & & \\ & & & & a_1 + k_1 & & & \end{pmatrix}$$

$$\mapsto \begin{cases} \displaystyle\prod_{p=k_1+1}^{0} (a_1+p) x_1^{a_1+k_1} x_2^{a_2+k_2-k_1} \cdots x_n^{a_n-k_{n-1}} & \text{if } k_1 < 0\,, \\ x_1^{a_1} x_2^{a_2+k_2} \cdots x_n^{a_n-k_{n-1}} & \text{if } k_1 = 0\,, \\ \left(\displaystyle\prod_{p=1}^{k_1} (a_1+p)\right)^{-1} x_1^{a_1+k_1} x_2^{a_2+k_2-k_1} \cdots x_n^{a_n-k_{n-1}} & \text{if } k_1 > 0\,. \end{cases}$$

The pointed Gelfand–Zetlin formulae are now easily checked for the generators $e_{i,j}$. $\square$

Let $\mathcal{M} = \mathcal{M}(\mathbf{m})$ be the coherent family defined by our choice of $\mathbf{m}$. Take some $\mu \in \mathfrak{h}^*$ such that $\mathcal{M}_\mu \neq 0$ and consider the system $(*)$ of linear equations with indeterminates y_i, $i = 1, \ldots, n-1$. It is easy to see that this system has a unique solution, which we will denote by $\hat{y}(\mu)$.

Theorem 4.1. *Let* $\lambda \in \mathfrak{h}^*$ *and* $x = \hat{y}(\lambda)$. *Then the module* $\mathcal{M}(\lambda)$ *is simple if and only if* $M(\mathbf{m}, x)$ *is simple. Moreover, if* $\mathcal{M}(\lambda)$ *is simple then* $\mathcal{M}(\lambda) \simeq M(\mathbf{m}, x)$.

Proof. The same as for Theorem 3.1. $\square$

As now we did not have any restriction on x, Theorem 4.1 gives a realization for all pointed simple dense $\mathfrak{sl}(n,\mathbb{C})$-modules.

5. Quantum Deformation of Dense Modules with Tableaux Realization

Theorem 5.1. *Let* $\mathbf{m}$, x, $B(\mathbf{m}, x)$ *and* $M(\mathbf{m}, x)$ *be as in Sec.* 3 *and, additionally,* $2(x_i - m_2) \notin 1(q) + 2\mathbb{Z}$, $i = 2, \ldots, n-1$. *Then quantum Gelfand–Zetlin formulae define on* $M(\mathbf{m}, x)$ *the structure of an* $U_q(\mathfrak{gl}(n,\mathbb{C}))$*-module, which we will denote by* $M_q(\mathbf{m}, x)$. *Moreover,* $M_q(\mathbf{m}, x)$ *is a true quantum deformation of* $M(\mathbf{m}, x)$.

Proof. The condition $2(x_i - m_2) \notin 1(q) + 2\mathbb{Z}$, $i = 2, \ldots, n-1$, guarantees that all coefficients in quantum Gelfand–Zetlin formulae are well-defined. Now the proof for the first part is the same as that of Lemma 3.1, with substitution of $l_{i,j}$ by $q^{l_{i,j}}$, see also proof of [16, Theorem 2].

Comparing the classical and quantum Gelfand–Zetlin formulae one gets that if an element, $[l] \in B(\mathbf{m}, x)$, has $\mathfrak{h}$-weight λ in $M(\mathbf{m}, x)$ then it has $\mathcal{K}$-weight $q^{(\lambda,\cdot)}$ in $M_q(\mathbf{m}, x)$. As both $M(\mathbf{m}, x)$ and $M_q(\mathbf{m}, x)$ have $B(\mathbf{m}, x)$ as a basis, we get that the dimensions of the corresponding weight spaces are the same. And, finally, the remark that quantum Gelfand–Zetlin formulae tend to classical ones under $q \to 1$ completes the proof. □

Now we deform the pointed modules. Let $\mathbf{m}$ and x be as in Sec. 4. For $[l] \in B(\mathbf{m}, x)$ define the action of the generators of $U_q(\mathfrak{gl}(n, \mathbb{C}))$ on $[l]$ by the following *quantum pointed Gelfand–Zetlin formulae*:

$$
\begin{aligned}
E_i[l] &= -[l_{i,1} - l_{i+1,1}]_q [l + \delta^{i,1}]\,, \quad i > 1 \\
F_i[l] &= [l_{i,1} - l_{i-1,1}]_q [l - \delta^{i,1}]\,, \quad i > 1 \\
E_1[l] &= -[l_{1,1} - l_{2,1}]_q [l_{1,1} - c]_q [l + \delta^{1,1}]\,, \\
F_1[l] &= [l - \delta^{1,1}]\,, \\
K_i[l] &= q^{(l_{i,1} - l_{i-1,1} + c)}[l]\,.
\end{aligned}
$$

Theorem 5.2. *Let* $\mathbf{m}$, x, $B(\mathbf{m}, x)$ *and* $M(\mathbf{m}, x)$ *be as in Sec.* 4. *Then quantum pointed Gelfand–Zetlin formulae define on* $M(\mathbf{m}, x)$ *the structure of an* $U_q(\mathfrak{gl}(n, \mathbb{C}))$*-module, which we will denote by* $M_q(\mathbf{m}, x)$. *Moreover,* $M_q(\mathbf{m}, x)$ *is a true quantum deformation of* $M(\mathbf{m}, x)$.

Proof. Analogous to that of Theorem 5.1. □

There is one interesting effect appearing in the above theorems in the case of non-integral weights. It happens that for a fixed q the module $M_q(\mathbf{m}, x)$ may be reducible even if the original module $M(\mathbf{m}, x)$ was simple. Why this may happen is explained in the following simplicity criterion for $M_q(\mathbf{m}, x)$.

Lemma 5.1. (1) *Let* $\mathbf{m}$, x, *and* $B(\mathbf{m}, x)$ *be as in Sec.* 3. *Then* $M_q(\mathbf{m}, x)$ *is simple if and only if* $2(x_i - x_{i+1}) \notin 1(q) + 2\mathbb{Z}$, $i = 1, \ldots, n-1$, *and* $2(x_{n-1} - m_1) \notin 1(q) + 2\mathbb{Z}$.

(2) *Let* $\mathbf{m}$, x, c, *and* $B(\mathbf{m}, x)$ *be as in Sec.* 4. *Then* $M_q(\mathbf{m}, x)$ *is simple if and only if* $2(x_1 - c) \notin 1(q) + 2\mathbb{Z}$, $2(x_{n-1} - m_1) \notin 1(q) + 2\mathbb{Z}$ *and* $2(x_i - x_{i+1}) \notin 1(q) + 2\mathbb{Z}$ *for all* $i = 1, \ldots, n-2$.

Proof. We prove the first statement and the second one can by done by analogous arguments. Using the quantum Gelfand–Zetlin formulae and the quantum analogue of Γ, [16, Section 3], by arguments, analogous to that of Lemma 3.3, we get that $M_q(\mathbf{m}, x)$ is not simple if and only if there exists $[l] \in B(\mathbf{m}, x)$, which is annihilated by some E_α or F_α. The last condition is equivalent to the fact that certain coefficients in quantum Gelfand–Zetlin formulae are zero. In other words that $[x_i - x_{i+1}]_q = 0$ or $[x_{n-1} - m_1]_q = 0$. But $q^x - q^{-x} = 0$ is equivalent to $2x \in 1(q)$. So, the reducibility of $M_q(\mathbf{m}, x)$ is equivalent to the failure of the conditions of our lemma. $\square$

6. New Simple $U(\mathfrak{gl}(n,\mathbb{C}))$- and $U_q(\mathfrak{gl}(n,\mathbb{C}))$-Modules

The above technique of constructing simple dense modules suggests the following two general constructions, which give new classes of simple modules over $U(\mathfrak{gl}(n,\mathbb{C}))$ $(U(\mathfrak{sl}(n,\mathbb{C})))$ and their quantum analogs. In these cases it is not clear what a true quantum deformation is, because these modules, although being weight, have infinite-dimensional weight spaces.

The first construction is a more or less direct generalization of the one from Sec. 3. Fix $1 < k < n$ and a tableau, $[t]$, satisfying the following conditions:

(1) $t_{n,j} - t_{n,j+1} \in \mathbb{N}$, $k \le j < n$;
(2) $t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+$, $k \le j \le i$;
(3) $t_{i-1,j} - t_{i,j+1} \in \mathbb{N}$, $k \le j \le i$;
(4) $t_{i,j} - t_{i,s} \notin \mathbb{Z}$, $2 \le i \le n-1$, $1 \le j < k$, $1 \le s \le n$.

Consider the set $B([t])$, which consists of all tableaux $[l]$, satisfying the following conditions:

(1) $l_{n,j} = t_{n,j}$, $1 \le j \le n$;
(2) $t_{i,j} - l_{i,j} \in \mathbb{Z}$, $1 \le j \le i \le n-1$;
(3) $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$, $k \le j \le i$;
(4) $l_{i-1,j} - l_{i,j+1} \in \mathbb{N}$, $k \le j \le i$.

Let $M([t])$ denote the $\mathbb{C}$-vector space with $B([t])$ as a basis. We note that the case $k = 2$ coincides with the situation considered in Sec. 3.

Theorem 6.1. *Gelfand–Zetlin formulae define on $M([t])$ the structure of an $U(\mathfrak{gl}(n,\mathbb{C}))$-module, which is weight and has finite length. $M([t])$ is simple if and only if $t_{i,j} - t_{i+1,s} \notin \mathbb{Z}$ for all $i = 1, \dots, n-1$ and $j,\ s < k$. If $k > 2$, all weight spaces of $M([t])$ are infinite-dimensional.*

Proof. The $U(\mathfrak{gl}(n,\mathbb{C}))$-module structure follows by the same arguments as Lemma 3.1. The simplicity criterion follows by the same arguments as Lemma 3.3. The infinite-dimensionality of the weight spaces is obtained by the same arguments as Lemma 3.2. The finite length arguments are the same as in [15, Section 7.4] and can be sketched as follows: The algebra Γ separates the elements of $B([t])$ by construction of $B([t])$, so any simple subquotient, N, of $M([t])$ comes with a subset of $B([t])$, which is a basis of N. Now we can draw a non-oriented graph G with vertices $B([t])$ and two vertices, $[l]$ and $[l + \delta^{i,j}]$, are connected if and only if $[l + \delta^{i,j}]$ (resp. $[l]$) appears with non-zero coefficient in decomposition of $e_{i,i+1}[l]$

(resp. $e_{i+1,i}[l+\delta^{i,j}]$). It is obvious that simple subquotients of $B([t])$ correspond to connected components of G and now one easily derives from Gelfand–Zetlin formulae that the number of such components is finite. □

As in Sec. 5 the modules above admit a straightforward quantum deformation, which can be considered as a true quantum deformation in the sense that both $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ actually act on the same space $M([t])$ the second action tends to the first one under $q \to 1$.

Theorem 6.2. *Assume that* $2(t_{i,j}-t_{i,s}) \notin 1(q)+2\mathbb{Z}$, $2 \le i \le n-1$, $1 \le j < k$, $1 \le s \le n$. *Then quantum Gelfand–Zetlin formulae define on* $M([t])$ *the structure of a* $U_q(\mathfrak{gl}(n,\mathbb{C}))$*-module, which is weight and has finite length. We will denote this module by* $M_q([t])$. $M_q([t])$ *is simple if and only if* $2(t_{i,j}-t_{i+1,s}) \notin 1(q)+2\mathbb{Z}$ *for all* $i = 1,\dots,n-1$ *and* $j, s < k$. *If* $k > 2$, *all weight spaces of* $M([t])$ *are infinite-dimensional.*

Proof. Everything but finite-length is analogous to Theorem 5.1 and Lemma 5.1. The finite length arguments are the same as in [16, Theorem 2]. □

The second construction, which follows, was inspired by the first one during some discussions with L. Turowska. We fix $k < n$. Choose a tableaux, $[t]$, satisfying the conditions

(1) $t_{i,j} - t_{i,s} \notin \mathbb{Z}$, $k < i < n$, $1 \le j < s \le i$;
(2) $t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+$, $2 \le i \le k$, $1 \le i \le i-1$;
(3) $t_{i-1,j} - t_{i,j+1} \in \mathbb{N}$, $2 \le i \le k$, $1 \le i \le i-1$.

Denote by $C([t])$ the set of all tableaux $[l]$ satisfying the following conditions:

(1) $l_{n,j} = t_{n,j}$, $1 \le j \le n$;
(2) $l_{i,j} - t_{i,j} \in \mathbb{Z}$, $1 \le j \le i \le n-1$;
(3) $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$, $2 \le i \le k$, $1 \le i \le i-1$;
(4) $l_{i-1,j} - l_{i,j+1} \in \mathbb{N}$, $2 \le i \le k$, $1 \le i \le i-1$.

Let $N([t])$ be the $\mathbb{C}$-vector space with $C([t])$ as a basis.

Theorem 6.3. *Gelfand–Zetlin formulae define on* $N([t])$ *the structure of an* $U(\mathfrak{gl}(n,\mathbb{C}))$*-module, which is weight and has finite length.* $N([t])$ *is simple if and only if* $t_{i,j} - t_{i+1,s} \notin \mathbb{Z}$ *for all* $i = k,\dots,n-1$ *and all* j, s. *With respect to the subalgebra* $\mathfrak{gl}(k,\mathbb{C})$, *which is embedded with respect to the upper left corner,* $N([t])$ *is a direct sum of simple finite-dimensional modules.*

Proof. First we show how to adjust arguments from Lemma 3.1 to this case. We write $u[l] = \sum_{[t]\in I(u,[l])} f(u,[l],[t])[l+t]$ for some $[l] \in C([t])$ and some relation $u = 0$ in $U(\mathfrak{g})$. Now we apply u to all tableaux $[s]$ in all simple finite-dimensional $U(\mathfrak{g})$-modules, which satisfy $s_{i,j} = l_{i,j}$, $1 \le j \le i \le k$. It is easy to see that this set of tableaux is big enough to conclude that all $f(u,[l],[t]) = 0$ as in Lemma 3.1. This gives us the $U(\mathfrak{gl}(n,\mathbb{C}))$-module structure on $N([t])$ and it is obvious that $N([t])$ is weight. The finite length arguments are the same as in Theorem 6.1. Moreover, these arguments also immediately imply the simplicity criterion for $N([t])$.

Consider the subalgebra $\mathfrak{a}_k = \mathfrak{gl}(k,\mathbb{C})$ of $\mathfrak{gl}(n,\mathbb{C})$, embedded with respect to the upper left corner and fix $[l] \in C([t])$. Denote by $D([l])$ the set of all $[s] \in C([t])$ satisfying $s_{i,j} = l_{i,j}$, $k \le i \le n$, $1 \le j \le i$. By Gelfand–Zetlin formulae and definition of $N([t])$, the linear span T_1 of all $[s] \in D([l])$ and the linear span T_2 of all other tableaux are closed under the $\mathfrak{a}_k$-action, i.e. $N([t]) = T_1 \oplus T_2$ as $\mathfrak{a}_k$-module. Moreover, by our restrictions on elements in $C([t])$, the set $D([t])$ is finite, and hence T_1 is finite-dimensional. This completes the proof. □

The easiest case is $k = n-1$, in which it is easy to see that $N([t])$ is a direct sum of non-isomorphic simple finite-dimensional modules. Again, by out general philosophy of quantum deformation of Gelfand–Zetlin formulae we immediately get the following:

Theorem 6.4. *Assume that* $2(t_{i,j} - t_{i,s}) \notin 1(q) + 2\mathbb{Z}$, $k < i < n$, $1 \le j \le i$, $1 \le s \le i+1$. *Then quantum Gelfand–Zetlin formulae define on* $N([t])$ *the structure of an* $U_q(\mathfrak{gl}(n,\mathbb{C}))$*-module, which is weight and has finite length. We will denote this module by* $N_q([t])$. $N_q([t])$ *is simple if and only if* $2(t_{i,j} - t_{i+1,s}) \notin 1(q) + 2\mathbb{Z}$ *for all* $i = k, \dots, n-1$ *and all* j, s. *With respect to the canonical subalgebra* $U_q(\mathfrak{gl}(k,\mathbb{C}))$, *the module* $N_q([t])$ *is a direct sum of simple finite-dimensional modules.*

Proof. Is the quantum version of that of Theorem 6.3. □

We also remark that in the case $k = n-1$ the additional condition of Theorem 6.4 disappears and hence in this case all $N([t])$ admit a quantum deformation.

As we have mentioned in the proof of Theorem 6.1, all simple subquotients of all modules $M(\mathbf{m}, x)$, $M([t])$ and $N([t])$ considered above correspond to particular subsets in $B(\mathbf{m}, x)$, $B([t])$ and $C([t])$ respectively. Hence analogous quantum construction for them give us quantum deformation of all these modules, in particular, of all simple $\mathfrak{gl}(k,\mathbb{C}) - \mathfrak{gl}(n,\mathbb{C})$ Harish–Chandra modules, which can be realized as simple subquotients of $N([t])$ constructed above.

7. Quantum Deformation of Simple Dense Modules in the General Case

In this section we retain all notation from Secs. 1 and 2. Let $\mathcal{M}$ be an irreducible coherent family and $\lambda \in \mathfrak{h}^*$ be such that $\mathcal{M}(\lambda)$ is simple (hence dense) and has a realization via Gelfand–Zetlin construction. As we have already mentioned in Sec. 3, most of λ satisfy this condition. We fix one of them. Let $\mu \in \mathfrak{h}^*$ be such that the module $\mathcal{M}(\mu) = \bigoplus_{\nu \in \mu + Q} \mathcal{M}_\nu$ is simple. Then $\mu - \lambda = \sum_{i=1}^{n-1} a_i \alpha_i$ for some uniquely defined $a_i \in \mathbb{C}$. For $k \in \{1, \dots, n-1\}$ and $x \in \mathbb{C}$ denote by $\mathcal{F}_{k,x}$ the composition of the following functors: $U_{\alpha_k} \bigoplus_{U(\mathfrak{g})} -$, followed by the θ_x-twist, followed by the restriction to $U(\mathfrak{g})$. It is a direct corollary from the definitions of θ_x

and coherent families (see also [12, Proposition 4.8]) that all simple subquotients of $\mathcal{F}_{k,x}(\mathcal{M})$ belong to $\mathcal{M}$. The following two lemmas give us a direct way to represent $\mathcal{M}(\mu)$ in terms of $\mathcal{M}(\lambda)$ and $\mathcal{F}_{k,x}$.

Lemma 7.1. *Let $\nu \in \mathfrak{h}^*$ and $x \in \mathbb{C}$. Then the modules $\mathcal{M}(\nu+x\alpha_k)$ and $\mathcal{F}_{k,x}(\mathcal{M}(\nu))$ have the same composition subquotients with the same multiplicity.*

Proof. By polynomiality properties of $\mathcal{F}_{k,x}$ and coherent families, $\mathcal{F}_{k,x}$ preserves $\mathcal{M}$, i.e. sends any module from $\mathcal{M}$ to a module, all simple subquotients of which are simple subquotients of $\mathcal{M}$. Hence it is enough to show that the characters of $\mathcal{M}(\nu + x\alpha_k)$ and $\mathcal{F}_{k,x}(\mathcal{M}(\nu))$ coincide. But the first step in $\mathcal{F}_{k,x}$ is the induction to U_{α_k}, which means that for any weight τ of $\mathcal{M}(\nu)$ it is enough to find some $N \in \mathbb{N}$ such that $X_{-\alpha_k}$ acts injectively on $\bigoplus_{i>N} \mathcal{M}(\nu)_{\tau-i\alpha_k}$. Now we recall that $\bigoplus_{i\in\mathbb{Z}} \mathcal{M}(\nu)_{\tau-i\alpha_k}$ is an admissible dense $\mathfrak{sl}(2,\mathbb{C})$-module. Hence the kernel of $X_{-\alpha_k}$ is finite-dimensional on it. This completes the proof. □

Lemma 7.2. $M = \mathcal{F}_{n-1,a_{n-1}} \circ \cdots \circ \mathcal{F}_{1,a_1}(\mathcal{M}(\lambda)) \simeq \mathcal{M}(\mu)$.

Proof. If $M \neq 0$ then it has finite length by [3, Section 4], all its simple subquotients belong to $\mathcal{M}$ and $\mathrm{supp}(M) \subset \lambda + \sum_{i=1}^{n} a_i\alpha_i + Q = \mu + Q$. Hence $M \simeq \mathcal{M}(\mu)$ as $\mathcal{M}(\mu)$ is the only simple subquotient of $\mathcal{M}$, whose support intersects $\mu + Q$ and the dimension of weight subspaces of M is less than or equal to that of $\mathcal{M}(\mu)$. So, it is enough to prove that $M \neq 0$. But the last follows directly from Lemma 7.1. □

Consider now $U_q(\mathfrak{g})$. By [7, 4.4.12], the set $F = \{F_k^l | l \in \mathbb{N}\}$ is an Ore set in $U_q(\mathfrak{g})$ and we denote by $U_q^{(k)}$ the localization of $U_q(\mathfrak{g})$ with respect to F. Now we need certain analogs of θ_x for $U_q^{(k)}$.

Lemma 7.3. *Let $u \in U_q^{(k)}$. Then the vector space $T(u) \subset U_q^{(k)}$, spanned by all $F_k^l u F_k^{-l}$, $l \in \mathbb{Z}_+$, is finite-dimensional.*

Proof. Clearly, we can assume that u is a monomial in $U_q^{(k)}$, say $u = Z_1 \cdots Z_m$, where each Z_j equals one of F_i, E_i, $K_i^{\pm 1}$, $K_i^{\pm 1}F_k^{-1}$, F_k^{-1}, or $F_kF_iF_k^{-1}$. We will say that another monomial, $u' = Z_1' \cdots Z_m'$, is obtained from u by *admissible substitution* if for all $j = 1, \ldots, m$ the pair (Z_j, Z_j') belongs to the following list:

Z_j:	$K_i^{\pm 1}$	$K_k^{\pm 1}K_{k+1}^{\mp 1}F_k^{-1}$	E_k	E_i, $i \neq k$	F_i	$F_kF_iF_k^{-1}$	F_k^{-1}
Z_j':	$K_i^{\pm 1}$	$K_k^{\pm 1}K_{k+1}^{\mp 1}F_k^{-1}$	E_k $K_kK_{k+1}^{-1}F_k^{-1}$ $K_k^{-1}K_{k+1}F_k^{-1}$	E_i	F_i $F_kF_iF_k^{-1}$	F_i $F_kF_iF_k^{-1}$	F_k^{-1}

It is obvious that, for fixed u, the set $S(u)$ of monomials, obtained from u by admissible substitutions, is finite. Now we claim that if u' is obtained from u by admissible substitution, then $F_k u' F_k^{-1}$ is a linear combination of monomials from $S(u)$. We write

$$F_k u' F_k^{-1} = F_k Z_1' F_k^{-1} F_k Z_2' F_k^{-1} \cdots F_k Z_m' F_k^{-1}$$

and thus reduce the problem to the case $m = 1$, in which it follows from the defining relations for $U_q(\mathfrak{g})$, see Sec. 2.1. Indeed, from these relations we have:

$$F_k K_i F_k^{-1} = q^{\delta_{i,k}-\delta_{i,k+1}} K_i\,; \quad F_k K_i^{-1} F_k^{-1} = q^{-\delta_{i,k}+\delta_{i,k+1}} K_i^{-1}\,;$$

$$F_k E_k F_k^{-1} = E_k - (q-q^{-1})^{-1} K_k K_{k+1}^{-1} F_k^{-1} + (q-q^{-1})^{-1} K_k^{-1} K_{k+1} F_k^{-1}\,;$$

$$F_k E_i F_k^{-1} = E_i\,, \quad i \neq k\,; \quad F_k F_k^{-1} F_k^{-1} = F_k^{-1}\,;$$

$$F_k F_i F_k^{-1} = F_i\,, \quad |i-j| > 1\,.$$

Finally, from the quantum Serre's relations we get

$$F_k^2 F_{k\pm1} F_k^{-2} = -F_{k\pm1} + (q+q^{-1}) F_k F_{k\pm1} F_k^{-1}\,.$$

This completes the proof. □

Lemma 7.4. *There exists a unique family, ϑ_x^q, of automorphisms of $U_q^{(k)}$ satisfying the following two conditions*:

(1) $\vartheta_x^q(u) = F_\alpha^x u F_\alpha^{-x}$, $x \in \mathbb{Z}$;
(2) *the map $\mathbb{C} \ni x \mapsto \vartheta_x^q(u)$ is polynomial in $q^{\pm x/2}$ for any $u \in U_q^{(k)}$.*

Proof. Let $u \in U_q^{(k)}$ and u_i, $i \in I$, be the list of all monomials in u. Set $S(u) = \cup_{i\in I} S(u_i)$. Then $S(u)$ is a finite union of finite sets defined in Lemma 7.3. So, the vector subspace $T \subset U_q^{(k)}$, spanned by $S(u)$, is finite-dimensional and contains all $F_\alpha^k u F_\alpha^{-k}$, $k \in \mathbb{N}$. From defining relations in $U_q(\mathfrak{g})$ and construction of $S(u_i)$ in Lemma 7.3 it follows immediately that the map $u \mapsto F_\alpha^k u F_\alpha^{-k}$, $k \in \mathbb{N}$, is polynomial in $q^{\pm k/2}$ on T, hence admits a unique extension for all $k \in \mathbb{C}$. This completes the proof. □

Corollary 7.1. *The automorphisms ϑ_x^q tend to θ_x under $q \to 1$.*

Proof. Under $q \to 1$ the relations of $U_q(\mathfrak{g})$ tend to the relations of $U(\mathfrak{g})$, which are polynomial in x. Hence, by construction of ϑ_x^q, its limit will be a family, t_x, of automorphisms of $U(\mathfrak{g})$, such that $t_x(u) = X_{-\alpha}^x u X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$, and the map $u \mapsto t_x(u)$ is polynomial in x. But then $t_x = \theta_x$ as θ_x is the unique family with this property by [12, Lemma 4.3]. □

For $k \in \{1,\ldots,n-1\}$ and $x \in \mathbb{C}$ we denote by $\mathcal{F}_{k,x}^q$ the composition of the following three functors: $U_q^{(k)} \bigoplus_{U_q(\mathfrak{g})} -$, followed by ϑ_x^q-twist, followed by restriction to $U_q(\mathfrak{g})$. Now we can state our main result.

Theorem 7.1. (1) *Let M_q be a true quantum deformation of a weight module, M, $k \in \{1, \dots, n-1\}$ and $x \in \mathbb{C}$. If the dimensions of the weight spaces of M are uniformly bounded, then $\mathcal{F}^q_{k,x}(M)$ is a true quantum deformation of $\mathcal{F}_{k,x}(M)$.*

(2) *Let λ, μ and $\{a_i\}$ be as in the first paragraph of this section. Denote by $\mathcal{M}_q(\lambda)$ the true quantum deformation of $\mathcal{M}(\lambda)$, constructed in Sec. 5. Then the module $\mathcal{M}_q(\mu) = \mathcal{F}^q_{n-1,a_{n-1}} \circ \cdots \circ \mathcal{F}^q_{1,a_1}(\mathcal{M}_q(\lambda))$ is a true quantum deformation of $\mathcal{M}(\mu)$.*

Proof. The second statement follows in an obvious way from the first one and Lemma 7.2. So we will prove only the first statement.

Because of the definitions of $\mathcal{F}^q_{k,x}$ and $\mathcal{F}_{k,x}$, which are based on a single root, $\alpha_k \in \pi$, the first statement naturally reduces to the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. The module M still has uniformly bounded dimensions of the weight spaces and hence is a module from the BGG-category $\mathcal{O}$, [7, 4.1.4]. The list of indecomposable modules in $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$ is very short and contains simple finite-dimensional modules, Verma module, dual Verma modules and the so-called big projective modules, which can be obtained from a Verma module by tensoring with a finite-dimensional modules. The main point is that all Verma modules possess a true quantum deformation in a natural way, [7, Chapter 4], [5, Chapter 5]. And hence, tensoring with finite-dimensional modules over $U_q(\mathfrak{sl}(2, \mathbb{C}))$ one obtains true quantum deformations of the big projective modules as well. All finite-dimensional modules are, certainly, killed by both $\mathcal{F}^q_{k,x}$ and $\mathcal{F}_{k,x}$, and the part of a dual Verma module, not killed but these functors, is a Verma module. Hence, over $\mathfrak{sl}(2, \mathbb{C})$ the question reduces to calculations with Verma modules and big projective modules, which are filtered by Verma modules such that this Verma filtration is unique and has length two.

First we note that, if M is a Verma module, the necessary statement follows immediately from Sec. 5, since all Verma modules over $\mathfrak{sl}(2, \mathbb{C})$ can be realized via Gelfand–Zetlin construction. This is enough to prove the second statement for $\mathfrak{sl}(2, \mathbb{C})$, but not enough to prove the first one in the general case, as the restriction of M to $\mathfrak{sl}(2, \mathbb{C})$ can also have big projective modules as direct summands. So, to complete the proof we have to prove our statement for big projective modules.

Let $0 \to M_1 \to N \to M_2 \to 0$ be an exact sequence, where M_i, $i = 1, 2$, are Verma modules and N is a big projective modules. Then there is a corresponding sequence, $0 \to M_1^q \to N^q \to M_2^q \to 0$, for quantum deformations. As $X_{-\alpha_k}$ (resp. F_k) acts injectively on all modules in the sequence, the definition of $\mathcal{F}_{k,x}$ (resp. $\mathcal{F}^q_{k,x}$) implies that the sequence

$$0 \to \mathcal{F}_{k,x}(M_1) \to \mathcal{F}_{k,x}(N) \to \mathcal{F}_{k,x}(M_2) \to 0$$

$$(\text{resp. } 0 \to \mathcal{F}^q_{k,x}(M_1^q) \to \mathcal{F}^q_{k,x}(N^q) \to \mathcal{F}^q_{k,x}(M_2^q) \to 0)$$

is also exact. Hence the corresponding weight spaces of $\mathcal{F}_{k,x}(N)$ and $\mathcal{F}^q_{k,x}(N^q)$ have the same dimension (in fact, it is either 2 or 0). Now, as $\mathcal{F}^q_{k,x}(N^q)$ is a weight module with finite-dimensional weight spaces, we can apply Corollary 7.1 and get that the action of $U_q^{(k)}$ on $\mathcal{F}^q_{k,x}(N^q)$ tends to the action of U_α on $\mathcal{F}_{k,x}(N)$ and hence the action of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $\mathcal{F}^q_{k,x}(N^q)$ tends to the action of $U(\mathfrak{sl}(2,\mathbb{C}))$ on $\mathcal{F}_{k,x}(N)$, when $q \to 1$. So, $\mathcal{F}^q_{k,x}(N^q)$ is a true quantum deformation of $\mathcal{F}_{k,x}(N)$ and the proof is complete. □

We remark that the first statement of Theorem 7.1 can be generalized to weight modules with finite-dimensional weight spaces with respect to the Gelfand–Zetlin subalgebra.

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