An Introduction to Noncommutative Noetherian Rings

Second Edition

K. R. Goodearl and R. B. Warfield, Jr.

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Second Edition

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Deceased

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Introduction to the Second Edition

Dedicated to my friend and coauthor,

Robert Breckenridge Warfield, Jr. (1940–1989)

Since the publication of the first edition in 1989, this book has been used by several generations of graduate students. From the accumulated comments, it became clear that a number of changes in the presentation of the material would make the book more accessible, particularly to students reading the text on their own. During this same period, the explosive growth of the area of quantum groups provided a large new crop of noetherian rings to be analyzed, and thus gave major impetus to research in noetherian ring theory. While a general development of the theory of quantum groups would not fit into a book of the present scope, many of the basic types of quantum groups are ideally suited as examples on which the concepts and tools developed in the text can be tested. Finally, readers of the first edition found a substantial list of typographical and other minor errors. This revised edition is designed to address all these points. Undoubtedly, however, the retyping of the text in TeX has introduced a new supply of typos for readers' entertainment.

Here is more detail:

Changes to the order and emphasis of topics were based, as mentioned, on the combined experience and comments of numerous students and professors who used the first edition over the past 14 years. In particular, more examples and additional manipulations with specific rings – especially in the early part of the book – were requested. In response to both requests, further examples of the types discussed in the first edition are worked out, and new examples from quantum groups (as well as a few from the representation theory of Lie algebras) have been inserted throughout. The discussion of skew polynomial rings, which many students initially found difficult to digest in full generality, has been expanded considerably. The present development keeps to the case of twists by automorphisms in Chapter 1 and begins Chapter 2 by outlining the case of twists by derivations; thus, readers have a chance to familiarize themselves with these more basic types of skew polynomial rings before moving on to the general situation. In addition, the universal properties of these rings are now emphasized, as are presentations by generators and relations.

It was also brought to my attention that most students have a strong preference for following the given sequence of topics in a text, as opposed to skipping certain sections and returning to them later. Thus, some topics which were presented in the early chapters of the first edition, but were not essential until later, have been moved. For example, the development of affiliated primes and affiliated series, which originally occurred in the introductory chapter on prime ideals, has now been shifted to Chapter 8. To address a few points in earlier chapters where affiliated series had provided motivation, the (simpler) concept of a prime series is introduced in Chapter 3.

When the first edition was being written, students' opinions on two possible approaches to Goldie's Theorem (the classical construction of rings of fractions versus ring structures on injective hulls) were evenly split among those polled. The injective hull approach was chosen mainly for the sake of variety (to contrast with presentations in other sources). In the meantime, however, opinion has swung overwhelmingly in favor of the classical approach. Consequently, the development of ring structures on injective hulls has been removed. The accompanying material on nonsingular modules has been replaced by a discussion of torsionfree modules with respect to Ore sets. To accommodate the classical approach, a basic construction of rings of fractions (with respect to Ore sets of regular elements) is now given at the beginning of Chapter 6; the general case (in Chapter 10) is tackled by reduction to this basic case. In keeping with one of the main themes of the book, rings of fractions are obtained as rings of endomorphisms of appropriate modules, thus avoiding tedious computations with equivalence classes of ordered pairs.

The topic of quantum groups is a tricky one for an introductory book. Certainly, the algebraic side of that area has provided fertile ground for applications of noetherian ring theory. However, on one hand, the subject – like those of group algebras and enveloping algebras – has given rise to such an extensive theory of its own that a general treatment would completely overbalance the present book. On the other hand, the theory of quantum groups is still evolving rapidly even though its foundations are not yet settled; in fact, there is still no axiomatic definition of a "quantum group" at present, only a list of examples which have been so baptized by general consensus. For these reasons, it did not appear useful, at this point, to attempt an introductory account of the topic trimmed to the length of a chapter or two. Thus, in place of a systematic treatment, quantum groups have been integrated into the general flow of the book to illustrate the theory. Moreover, a sketch of the philosophy behind the concept of a quantum group has been added to the Prologue, to accompany the previous sketches of other areas of application of noetherian ring theory. A selection of easily accessible examples, constructible from iterated skew polynomial rings, is introduced at that point. These examples are analyzed in detail in the first two chapters (in both text and exercises) and are used repeatedly in later chapters to test new concepts and methods.

For many helpful comments and suggestions, most of which I have tried to

incorporate into this revised edition, I would like to thank Allen Bell, Gary Brookfield, Ken Brown, R. N. Gupta, Charu Hajarnavis, Heidi Haynal, Karen Horton, Brian Jue, Dennis Keeler, Tom Lenagan, Ed Letzter, Ian Musson, Kim Retert, Dan Rogalski, Lance Small, Paul Smith, Toby Stafford, Peter Thompson, and Scot Woodward.

Ken Goodearl
July 2003

Introduction to the First Edition

Noncommutative noetherian rings are presently the subject of very active research. Recently the theory has attracted particular interest due to its applications in related areas, especially the representation theories of groups and Lie algebras. We find the subject of noetherian rings an exciting one, for its own sake as well as for its applications, and our primary purpose in writing this volume was to attract more participants into the area.

This book is an introduction to the subject intended for anyone who is potentially interested, but primarily for students who are at the level which in the United States corresponds to having completed one year of graduate study. Since the topics included in an American first year graduate course vary considerably, and since those in analogous courses in other countries (e.g., third year undergraduate or M.Sc. courses in Britain) vary even more, we have attempted to minimize the actual prerequisites in terms of material, by reviewing some topics that many readers may already have in their repertoires. More importantly, we have concentrated on developing the basic tools of the subject, in order to familiarize the student with current methodology. Thus we focus on results which can be proved from a common point of view and steer away from miraculous arguments which can be used only once. In this spirit, our treatment is deliberately not encyclopedic, but is rather aimed at what we see as the major threads and key topics of current interest.

It is our hope that this book can be read by a student without the benefit of a course or an instructor. To encourage this possibility, we have tried to include details when they might have been omitted, and to discuss the motivations for proceeding as we do. Moreover, we have woven an extensive selection of exercises into the text. These exercises are particularly designed to give the novice some experience and familiarity with both the material and the tools being developed.

One of the fundamental differences between the theories of commutative and noncommutative rings is that the former arise naturally as rings of functions, whereas the latter arise naturally as rings of operators. For example, early in the twentieth century, some of the first noncommutative rings that received serious study were certain rings of differential operators. More generally, given any set of linear transformations of a vector space, we can form the algebra of linear transformations generated by this set, and many problems of interest concerning the original transformations become module-theoretic questions, where we view the original vector space as a module over the algebra we have created. In many modern applications, in turn, it is essential to regard noncommutative rings as rings of transformations or operators of various kinds. We are partial to this point of view. This has led us to emphasize the role of modules when studying a ring, for modules are simply ways of representing the ring at issue in terms of endomorphisms of abelian groups. Also, when defining a ring we have tended to present it as a ring of operators of some sort rather than by taking a more formal approach, such as giving generators and relations. For example, when constructing rings of fractions, we have preferred to find them as rings of endomorphisms rather than as sets of equivalence classes of ordered pairs of elements.

Although the noetherian condition is very natural in commutative ring theory, since it holds for the rings of integers in algebraic number fields and for the coordinate rings crucial to algebraic geometry, it was originally less clear that this condition would be useful in the noncommutative setting. For instance, Jacobson's definitive book of 1956 makes only minimal mention of noetherian rings. Similarly, prime ideals, essential in the commutative theory, seemed to have relatively less importance for noncommutative rings; in fact, because of the fundamental role of representation-theoretic ideas in the development of the noncommutative theory, the initial emphasis in the subject was almost exclusively on irreducible representations (i.e., simple modules) and primitive ideals (i.e., annihilators of irreducible representations). In the meantime, however, it has turned out that various important types of noncommutative rings – in particular, certain infinite group rings and the enveloping algebras of finite dimensional Lie algebras – are in fact noetherian. This has been used to good effect in recent work on the representation theory of the corresponding groups and Lie algebras, just as the theory of finite dimensional algebras and artinian rings has played a key role in research on the representations of finite groups. Also, as soon as noetherian rings and their modules received serious attention, prime ideals forced themselves into the picture, even in contexts where the original interest had been entirely in primitive ideals. As a consequence, we have made prime ideals a major theme in our text.

The first important result in the theory of noncommutative noetherian rings was proved relatively recently, in 1958. This was Goldie's Theorem, which gives an analog of a field of fractions for factor rings R/P where R is a noetherian ring and P a prime ideal of R. Once this milestone had been reached, noetherian ring theory proceeded apace, partly from its own impetus and partly through feedback from neighboring areas in which noetherian ideas found applications. One of our aims in this book has been to develop those aspects of the theory of noetherian rings which have the strongest connections with the representation-theoretic areas to which we have alluded. However, as these areas have their own extensive theories, it was impossible to treat them in any generality in this volume. Instead, we present a brief discussion in the prologue, giving some representative examples to which the theory in the text can be applied relatively directly, without extensive side trips into technical intricacies.

To give the reader an idea of the historical sources of the theory, we have included some bibliographical notes at the end of each chapter. We have sought to make these notes as accurate as possible, but as with any evolving theory complete precision is difficult to attain, especially since in many research papers sources are not well documented. Some inaccuracies are thus probably inevitable, and we apologize in advance for any that may have occurred.

In an appendix we discuss some open problems in noetherian ring theory; we hope that our readers will be stimulated to solve them.

For helpful comments on various drafts of the book, we would like to thank A. D. Bell, K. A. Brown, D. A. Jordan, T. H. Lenagan, P. Perkins, L. W. Small, and J. T. Stafford. We would also like to thank our competitors J. C. McConnell and J. C. Robson for letting us see early drafts of various chapters from their noetherian rings book [1987].

Prologue

Since much of the current interest in noncommutative noetherian rings stems from applications of the general theory to several specific types, we present here a very sketchy introduction to some major areas of application: polynomial identity rings, group algebras, rings of differential operators, enveloping algebras, and quantum groups. Each of these areas has a very extensive theory of its own, far too voluminous to be incorporated into a book of this size. (See for instance Rowen [1980], Passman [1985], McConnell-Robson [2001], and Brown-Goodearl [2002]). Instead, we shall concentrate on surrogates – some classes of rings that are either simple prototypes or analogs of the major types just mentioned – which we can investigate by relatively direct methods while still exhibiting the flavor of the areas they represent. These surrogates are module-finite algebras over commutative rings (for polynomial identity rings), skew-Laurent rings (for group algebras), formal differential operator rings (for rings of differential operators and some enveloping algebras), and general skew polynomial rings (for some enveloping algebras and quantum groups). They will be introduced below and studied in greater detail in the following two chapters.

We will conclude the Prologue with a few comments about our notation and terminology.

• POLYNOMIAL IDENTITY RINGS •

Commutativity in a ring may be phrased in terms of a relation that holds identically, namely $xy - yx = 0$ for all choices of x and y from the ring. More complicated identities sometimes also hold in noncommutative rings. For example, if x and y are any 2×2 matrices over a commutative ring S, then the trace of $xy - yx$ is zero, and so it follows from the Cayley-Hamilton Theorem that $(xy - yx)^2$ is a scalar matrix. Consequently, $(xy - yx)^2$ commutes with every 2×2 matrix z, and hence the relation

$$(xy - yx)^2 z - z(xy - yx)^2 = 0$$

holds for all choices of x, y, z from the ring $M_2(S)$ (the ring of all 2×2 matrices over S). A much deeper result, the Amitsur-Levitzki Theorem, asserts that, for all choices of $2n$ matrices $x_1, \dots, x_{2n}$ from the $n \times n$ matrix ring $M_n(S)$,

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)} = 0,$$

where S_{2n} is the symmetric group on $\{1, 2, \dots, 2n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of a permutation σ (namely $+1$ or -1, depending on whether σ is even or odd).

Such an “identical relation” on a ring may be thought of as saying that a certain polynomial – with noncommuting variables! – vanishes identically on the ring. In this context, the polynomials are usually restricted to having integer coefficients. Thus a *polynomial identity* on a ring R is a polynomial $p(x_1, \dots, x_n)$ in noncommuting variables $x_1, \dots, x_n$ with coefficients from $\mathbb{Z}$ such that $p(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in R$. A *polynomial identity ring*, or *P.I. ring* for short, is a ring R which satisfies some *monic* polynomial identity $p(x_1, \dots, x_n)$ (that is, among the monomials of highest total degree which appear in p, at least one has coefficient 1).

The Amitsur-Levitzki Theorem implies that every matrix ring over a commutative ring is a P.I. ring, and consequently so is every factor ring of a subring of such a matrix ring. For example, the endomorphism ring of a finitely generated module A over a commutative ring S has this form. To see that, identify A with S^n/K for some $n \in \mathbb{N}$ and some submodule K of S^n, and identify the matrix ring $M_n(S)$ with the endomorphism ring of S^n. Then the set

$$T = \{f \in M_n(S) \mid f(K) \subseteq K\}$$

is a subring of $M_n(S)$, the set $I = \{f \in M_n(S) \mid f(S^n) \subseteq K\}$ is an ideal of T, and $T/I \cong \operatorname{End}_S(A)$. Therefore $\operatorname{End}_S(A)$ is a P.I. ring.

Certain algebras over commutative rings fit naturally into this context. Recall that an *algebra* over a commutative ring S is just a ring R equipped with a specified ring homomorphism ϕ from S to the center of R. (The map ϕ is not assumed to be injective.) Then ϕ is used to define products of elements of S with elements of R: For $s \in S$ and $r \in R$, we set sr and rs equal to $\phi(s)r$ (or $r\phi(s)$, which is the same because $\phi(s)$ is in the center of R). Using this product, we can view R as an S-module. We say that R is a *module-finite* S-algebra if R is a finitely generated S-module. Note that $R \cong \operatorname{End}_R(R_R) \subseteq \operatorname{End}_S(R)$ as rings, and so any polynomial identity satisfied in $\operatorname{End}_S(R)$ will also be satisfied in R. Taking the preceding paragraph into account, we conclude that any module-finite algebra over a commutative ring is a P.I. ring.

The class of module-finite algebras over commutative noetherian rings provides us with a supply of prototypical examples of noetherian P.I. rings. To illustrate some applications of the noetherian theory to P.I. rings, we shall at times work out consequences of the former for our class of examples. In this setting, we will be able to replace P.I. theory by some much more direct methods from commutative ring theory.

• GROUP ALGEBRAS •

One of the earliest stimuli to the modern development of noncommutative ring theory came from the study of *group representations*. The key idea was to study a group G by "representing" it in terms of linear transformations on a vector space V, namely, by studying a group homomorphism ϕ from G to the group of invertible linear transformations on V. Linear algebra can then be used to study the group $\phi(G)$, and the information gleaned can be pulled back to G via the *representation* ϕ. Using ϕ, there is an "action" of G on V, namely, a product $G \times V \to V$ given by the rule $g \cdot v = \phi(g)(v)$, and since ϕ is a homomorphism, $(gh) \cdot v = g \cdot (h \cdot v)$ for all $g, h \in G$ and $v \in V$. This looks a lot like module multiplication, if we ignore the lack of an addition for elements of G, and in fact V is called a *G-module* in this situation.

To make V into an actual module over a ring, we build G and its multiplication into a ring, along with whichever field k we are using for scalars. Just make up a vector space with a basis which is in one-to-one correspondence with the elements of G, identify each element of G with the corresponding basis element, and then extend the multiplication from G to this vector space linearly:

$$\Big(\sum_{g \in G} \alpha_g g\Big)\Big(\sum_{h \in G} \beta_h h\Big) = \sum_{g,h \in G} (\alpha_g \beta_h)(gh).$$

The result is a k-algebra called the *group algebra* of G over k, denoted $k[G]$ or just kG. Except for the obvious changes in terminology, $k[G]$-modules are the same as representations of G on vector spaces over k.

In the case of a finite group G, the group algebra $k[G]$ is finite dimensional, and the theory of finite dimensional algebras has much to say about representations of G. A noetherian group algebra is known to occur when G is *polycyclic-by-finite*, that is, when G has a series of subgroups

$$G_0 = (1) \subset G_1 \subset \cdots \subset G_n \subseteq G_{n+1} = G$$

such that each G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is infinite cyclic for $i = 1, \dots, n$, while G/G_n is finite. (It is an open problem whether $k[G]$ is noetherian only when G is polycyclic-by-finite.) One of the simplest infinite non-abelian examples is the group G with two generators x, y and the sole relation $yxy^{-1} = x^{-1}$. In this case, elements of $k[G]$ can all be put in the form $\sum_{i=-n}^{n} p_i(x) y^i$, where each $p_i(x)$ is a Laurent polynomial (i.e., a polynomial in x and x^{-1}). From the relation $yxy^{-1} = x^{-1}$ it follows that $yp(x)y^{-1} = p(x^{-1})$ for all Laurent polynomials $p(x)$. Hence, the Laurent polynomial ring $k[x, x^{-1}]$ is sent into itself by the map $p(x) \mapsto yp(x)y^{-1}$, and this map coincides with the map $p(x) \mapsto p(x^{-1})$, which is an automorphism of $k[x, x^{-1}]$.

The pattern of this example suggests a construction that starts with a ring R and an automorphism α of R, and then builds a ring T whose elements look like Laurent polynomials over R in a new indeterminate y, except that instead of commuting with y, elements $r \in R$ satisfy the relation $yry^{-1} = \alpha(r)$, or $yr = \alpha(r)y$. Since the usual multiplication of polynomials has been "skewed" through α, the ring T is called a *skew-Laurent ring*. Thus the group algebra of the previously discussed group with the relation $yxy^{-1} = x^{-1}$ may be viewed as a skew-Laurent ring with coefficient ring $k[x, x^{-1}]$.

We shall see that any skew-Laurent ring with a noetherian coefficient ring is itself noetherian. This fact actually provides the method used to show that the group algebra of any polycyclic-by-finite group G is noetherian. Namely, if

$$G_0 = (1) \subset G_1 \subset \cdots \subset G_n \subseteq G_{n+1} = G$$

is the series of subgroups of G occurring in the definition of "polycyclic-by-finite," it can be shown that for $i = 1, \dots, n$ the group algebra $k[G_i]$ is isomorphic to a skew-Laurent ring whose coefficient ring is $k[G_{i-1}]$. Starting at the bottom with $k[G_0] = k$, it follows immediately by induction that $k[G_n]$ is noetherian. It then just remains to observe that $k[G]$ is a finitely generated right or left module over $k[G_n]$ to conclude that $k[G]$ itself is noetherian. In particular, we see from this discussion that (iterated) skew-Laurent rings are a better match for group algebras of polycyclic-by-finite groups than might have been suggested by the very special example given above.

• RINGS OF DIFFERENTIAL OPERATORS •

Another early stimulus to noncommutative ring theory came from the study of differential equations. Late in the nineteenth century, it was realized that, just as polynomial functions provide a useful means of dealing with algebraic equations, "differential operators" are convenient for handling linear differential equations. For example, a homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

can be rewritten very compactly as $d(y) = 0$, where d denotes the *linear differential operator*

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x).$$

From this viewpoint, d is a linear transformation on some vector space of functions, and the solution space of the original differential equation is just the null space of d.

To be a bit more specific, let us consider the special case in which coefficients and solutions are real-valued rational functions. Then our differential operators are $\mathbb{R}$-linear transformations on the field $\mathbb{R}(x)$. The composition of two differential operators is certainly a linear transformation, but it takes a minute to see that such a composition is actually another differential operator. In order to make the notation more convenient, we use the symbol D

to denote the operator d/dx. If we form the operator composition Da, which means "first multiply by the function $a(x)$ and then differentiate," we see that

$$(Da)(y) = (ay)' = ay' + a'y = aD(y) + a'y$$

for any function y, and so $Da = aD + a'$. Iterated use of this identity then allows us to write the composition of any two differential operators in the standard form of a differential operator, i.e., as a sum of terms a_iD^i, where $a_i \in \mathbb{R}(x)$. Thus the collection of differential operators on $\mathbb{R}(x)$ forms a ring, which is sometimes denoted $B_1(\mathbb{R})$. We may think of $B_1(\mathbb{R})$ as a polynomial ring $\mathbb{R}(x)[D]$ in which, however, the multiplication is twisted to make a noncommutative ring. This ring attracted particular attention early in the twentieth century, when it was proved that it is a principal ideal domain (that is, all left and right ideals are principal) and that it satisfies a form of unique factorization.

We can of course proceed in the same way using for coefficients other rings of functions that are closed under differentiation. For example, if we start with the ring $\mathbb{C}[x]$ of complex polynomials, the ring of differential operators we obtain looks like a twisted polynomial ring in two variables, $\mathbb{C}[x][D]$. This ring is called the *first complex Weyl algebra* and is denoted $A_1(\mathbb{C})$. More generally, we may start with a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ in several variables and build differential operators using the partial derivatives $\partial/\partial x_i$, abbreviated D_i. This results in a twisted polynomial ring in $2n$ variables, $\mathbb{C}[x_1, \ldots, x_n][D_1, \ldots, D_n]$, which is called the *$n$-th complex Weyl algebra* and is denoted $A_n(\mathbb{C})$.

Examples such as $B_1(\mathbb{R})$ and $A_n(\mathbb{C})$, which will often recur in the text, can be taken as representative of a more general class that has assumed some importance in recent years: rings of differential operators on algebraic varieties. We cannot discuss these in detail but will content ourselves with indicating how they can be described. We recall that a *complex affine algebraic variety* is a subset V of $\mathbb{C}^n$ which is the set of common zeroes of some collection I of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. If I contains all the polynomials that vanish on V, then I is an ideal in the polynomial ring, and the factor ring $R = \mathbb{C}[x_1, \ldots, x_n]/I$ is the *coordinate ring of V*. The *ring of differential operators on V*, denoted $\mathcal{D}(V)$, consists of those differential operators on $\mathbb{C}[x_1, \ldots, x_n]$ that induce operators on R, modulo those that induce the zero operator on R. More precisely, the set

$$S = \{s \in A_n(\mathbb{C}) \mid s(I) \subseteq I\}$$

is a subring of $A_n(\mathbb{C})$, the set

$$J = \{s \in A_n(\mathbb{C}) \mid s(\mathbb{C}[x_1, \ldots, x_n]) \subseteq I\}$$

is an ideal of S, and $\mathcal{D}(V) = S/J$. It has been proved that $\mathcal{D}(V)$ is noetherian in case V has no singularities and in case V is a curve, but it appears that for higher dimensional varieties with singularities $\mathcal{D}(V)$ is usually not noetherian.

• ENVELOPING ALGEBRAS •

A *Lie algebra* over a field k is a vector space L over k equipped with a nonassociative product $[\cdot\cdot]$ satisfying the usual bilinear and distributive laws as well as the rules

$$[xx] = 0 \qquad \text{and} \qquad [x[yz]] + [y[zx]] + [z[xy]] = 0$$

for all $x, y, z \in L$. For example, $\mathbb{R}^3$ equipped with the usual vector cross product is a real Lie algebra. The standard model for the product in a Lie algebra is the *additive commutator* operation $[x, y] = xy - yx$ in an associative ring (more precisely, any associative k-algebra when equipped with the operation $[\cdot, \cdot]$ becomes a Lie algebra over k). Conversely, starting with a Lie algebra L, one can build an associative k-algebra $U(L)$ using the elements of L as generators, together with relations $xy - yx = [xy]$ for all $x, y \in L$. The algebra $U(L)$ is called the *(universal) enveloping algebra* of L, and it is known to be noetherian in case L is finite dimensional. (Whether it is possible for the enveloping algebra of an infinite dimensional Lie algebra to be noetherian is an open problem.)

The simplest Lie algebra L with a nonzero product is 2-dimensional, with a basis $\{x, y\}$ such that $[yx] = x$. Elements of the enveloping algebra $U(L)$ can in that case all be put into the form $\sum_{i=0}^{n} p_i(x)y^i$, where each $p_i(x)$ is an ordinary polynomial in the variable x. In $U(L)$, the relation $[yx] = x$ becomes $[y, x] = x$, and from this it follows easily that $[y, p(x)] = x\frac{d}{dx}(p(x))$ for all polynomials $p(x)$. In other words, $[y, -]$ maps the polynomial ring $k[x]$ into itself, and its action on polynomials is given by the operator $x\frac{d}{dx}$. The reader should note that this is very similar to the ring $A_1(\mathbb{C})$ discussed above, the difference being that in $A_1(\mathbb{C})$ we have the relation $[D, p(x)] = \frac{d}{dx}(p(x))$. (In fact, $U(L)$ in our example is isomorphic to the subalgebra of $A_1(k)$ generated by x and xD.)

Abstracting this pattern, we may start with a ring R and a map $\delta : R \to R$ which is a *derivation* (that is, δ is additive and satisfies the usual product rule for derivatives) and then build a larger ring T using an indeterminate y such that $[y, r] = \delta(r)$ for all $r \in R$. The elements of T look like differential operators $\sum r_i\delta^i$ on R, except that it may be possible for $\sum r_i\delta^i$ to be the zero operator without all the coefficients r_i being zero. Thus, the elements $\sum r_i y^i$ in T are called *formal differential operators*, and T is called a *(formal) differential operator ring*.

We shall see that all formal differential operator rings with noetherian coefficient rings are themselves noetherian, and we shall view them as representative analogs of enveloping algebras. The analogy is actually a little better than one might think, knowing only the single example mentioned above. Namely, if L is a finite dimensional Lie algebra which can be realized as a Lie algebra of upper triangular matrices over k (using $[\cdot, \cdot]$ for the Lie product), then $U(L)$ can be built as an *iterated differential operator ring* through a series of extensions

$$T_0 = k \subset T_1 \subset \cdots \subset T_m = U(L),$$

where each T_i is isomorphic to a differential operator ring with coefficients from T_{i-1}. (Over $\mathbb{C}$, the finite dimensional Lie algebras that can be realized as upper triangular matrices are precisely the *solvable* Lie algebras.)

Among the most important Lie algebras are the *special linear* Lie algebras $\mathfrak{sl}_n(k)$, which consist of $n \times n$ matrices over k having trace 0 (again with Lie product $[\cdot,\cdot]$). In particular, $\mathfrak{sl}_2(k)$ is 3-dimensional, and one typically chooses the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis. The Lie products (commutators) among e, f, and h are given by

$$[ef] = h \qquad [he] = 2e \qquad [hf] = -2f.$$

In the enveloping algebra $U(\mathfrak{sl}_2(k))$, the Lie product relation $[he] = 2e$ becomes $he - eh = 2e$, or $eh = (h-2)e$. It follows that $ep(h) = p(h-2)e$ for any polynomial $p(h) \in k[h]$. This allows us to think of the k-algebra generated by e and h as a twisted polynomial ring in two variables, $k[h][e]$, where the twist arises from the map $p(h) \mapsto p(h-2)$. The latter map being an automorphism of $k[h]$, we thus see that $k[h][e]$ is a polynomial version of the skew-Laurent ring construction discussed above.

When the element f is added to the picture, we have to deal with the relations $ef - fe = h$ and $hf - fh = -2f$, or $fe = ef - h$ and $fh = (h+2)f$. The last equation is reminiscent of (the inverse of) the automorphism $p(h) \mapsto p(h-2)$ above, and indeed there is an automorphism α of the ring $k[h][e]$ such that $\alpha(h) = h + 2$ and $\alpha(e) = e$. The relation $fe = ef - h$ turns out to be accounted for by a linear map δ on $k[h][e]$ such that $\delta(e) = -h$ and $\delta(h) = 0$, the end result being $fr = \alpha(r)f + \delta(r)$ for all $r \in k[h][e]$. (The map δ is similar to a derivation – it satisfies a "skew product rule" $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ and is called a *skew derivation*.) We thus view $U(\mathfrak{sl}_2(k))$ as a twisted polynomial ring in three variables, $k[h][e][f]$, where the final twist involves both an automorphism and a skew derivation. Each of the steps $k \rightsquigarrow k[h] \rightsquigarrow k[h][e] \rightsquigarrow k[h][e][f]$ is a type of *skew polynomial ring*, and in summary we say that $U(\mathfrak{sl}_2(k))$ is an *iterated skew polynomial ring*. For our purposes, this structure is a means to let us see that $U(\mathfrak{sl}_2(k))$ is a noetherian domain. (While $U(\mathfrak{sl}_n(k))$ is a noetherian domain for any n, other methods are needed to prove that, since $U(\mathfrak{sl}_n(k))$ is not an iterated skew polynomial ring when $n \geq 3$.)

• QUANTUM GROUPS •

Of all the areas mentioned in this Prologue, that of quantum groups is the most difficult to introduce in a few sentences. First of all, quantum groups are not groups at all, but certain algebras that arose in the 1980s in connection with research on some problems in quantum statistical mechanics. A phrase that captures the philosophical viewpoint of the subject is this: *Quantum groups are algebras of functions on nonexistent groups*! Let us try to explain the flavor of that statement as it relates to the *special linear group* $SL_2(k)$, the group of all 2×2 matrices over a field k having determinant 1. First of all, $SL_2(k)$ lies inside the 4-dimensional vector space $M_2(k)$, where it can be described as the set of zeroes of the polynomial $X_{11}X_{22} - X_{12}X_{21} - 1$; here X_{11}, X_{12}, X_{21}, X_{22} are just four independent indeterminates, conveniently labelled for application to the entries of 2×2 matrices. Thus, $SL_2(k)$ is an affine algebraic variety over k, and its coordinate ring is the algebra

$$\mathcal{O}(SL_2(k)) = k[X_{11}, X_{12}, X_{21}, X_{22}]/\langle X_{11}X_{22} - X_{12}X_{21} - 1\rangle.$$

The coordinate ring $\mathcal{O}(SL_2(k))$ effectively encodes the geometry of the variety $SL_2(k)$, but that is only the structure of the set of points in $SL_2(k)$. The group structure is encoded in certain algebra homomorphisms. In particular, the group operation, viewed as a map $SL_2(k) \times SL_2(k) \to SL_2(k)$, induces (by composition of functions) a k-algebra homomorphism

$$\Delta : \mathcal{O}(SL_2(k)) \longrightarrow \mathcal{O}\big(SL_2(k) \times SL_2(k)\big) \xrightarrow{\cong} \mathcal{O}(SL_2(k)) \otimes_k \mathcal{O}(SL_2(k))$$

called *comultiplication*. There are also k-algebra homomorphisms

$$\epsilon : \mathcal{O}(SL_2(k)) \longrightarrow k \qquad\qquad S : \mathcal{O}(SL_2(k)) \longrightarrow \mathcal{O}(SL_2(k))$$

corresponding to the identity and taking inverses in the group, and the group axioms imply certain relations among these maps. The algebra $\mathcal{O}(SL_2(k))$ together with the three maps Δ, ϵ, S forms a structure called a *Hopf algebra*, which we will not define here.

Among the algebras that arose in the theoretical physics research mentioned above was one that bears a striking resemblance to $\mathcal{O}(SL_2(k))$ – it is a Hopf algebra, and it has four generators that satisfy a relation very similar to the equation "determinant = 1" which characterizes $SL_2(k)$. The only drawback is that this new algebra is not commutative, and so it cannot be an algebra of k-valued functions on anything. Nonetheless, thinking of this algebra *as if* it consisted of functions proved useful in investigating it, and it became known as the *coordinate ring of quantum* $SL_2(k)$. Thus, there is no "quantum group" $SL_2(k)$ per se; the group has disappeared, and only the algebra of "functions" on it remains.

Regardless (or because of) its origins, this new algebra is an interesting object of study, and among many other properties it turns out to be noetherian. Here is a brief description of the construction. First, pick a nonzero scalar q (the "quantum parameter") in k. (Originally, q was taken to be $e^{\hbar}$, where $\hbar$ is Planck's constant, so that q was a real number very close to 1.) Next, one forms a k-algebra with four generators x_{11}, x_{12}, x_{21}, x_{22} and six relations that we will give in full later; for instance, $x_{11}x_{12} = qx_{12}x_{11}$ and $x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}$. This algebra is the *coordinate ring of quantum* 2×2 *matrices*, $\mathcal{O}_q(M_2(k))$. The role of the determinant is taken over by the element $D_q := x_{11}x_{22} - qx_{12}x_{21}$, the *quantum determinant*, which lies in the center of $\mathcal{O}_q(M_2(k))$. Finally, the coordinate ring of quantum $SL_2(k)$ is the algebra $\mathcal{O}_q(SL_2(k)) = \mathcal{O}_q(M_2(k))/\langle D_q - 1\rangle$.

We shall see that $\mathcal{O}_q(M_2(k))$ is an iterated skew polynomial ring, where – as in the case of $U(\mathfrak{sl}_2(k))$ – twists involving both automorphisms and skew derivations are needed. Consequently, $\mathcal{O}_q(M_2(k))$ is noetherian, and therefore $\mathcal{O}_q(SL_2(k))$ is noetherian as well.

A fascinating aspect of quantum groups is that quite a number of classical facts about algebro-geometric groups like $SL_2(k)$ can be translated very neatly into the quantum setting, once they have been suitably rephrased in terms of coordinate rings. For a very simple example, multiplication of column vectors on the left by matrices gives a map $M_2(k) \times k^2 \to k^2$; composition of this map with polynomial functions on k^2 gives a k-algebra homomorphism $\lambda : \mathcal{O}(k^2) \to \mathcal{O}(M_2(k)) \otimes_k \mathcal{O}(k^2)$ such that $\lambda(x_i) = x_{i1} \otimes x_1 + x_{i2} \otimes x_2$ for $i = 1, 2$. Now $\mathcal{O}(k^2)$ is just a polynomial ring in two variables; the natural quantum analog, $\mathcal{O}_q(k^2)$, is a skew polynomial ring $k[x_1][x_2]$ in which $x_1x_2 = qx_2x_1$. The map λ carries over to a k-algebra homomorphism $\mathcal{O}_q(k^2) \to \mathcal{O}_q(M_2(k)) \otimes_k \mathcal{O}_q(k^2)$ which behaves exactly like λ on x_1 and x_2. In fact, the existence of this map and its right-hand analog are sufficient to pin down the relations in $\mathcal{O}_q(M_2(k))$, as we shall see later.

It would take us too far afield to discuss more of the background of quantum algebras. Let us mention here, though, that other examples, including quantum versions of the Weyl algebra $A_1(\mathbb{C})$ and of the enveloping algebra $U(\mathfrak{sl}_2(k))$, will appear in the text later.

• NOTATION AND TERMINOLOGY •

The background needed for this book is fairly standard and may be found in most graduate-level texts on algebra, such as Cohn [1982, 1989, 1991], Hungerford [1989], or Jacobson [1985, 1989]. The following short lists, giving some reference sources and some notation, are not meant to be exhaustive but to help keep the reader on track. We emphasize one convention: Our rings, modules, and ring homomorphisms are assumed to be unital except in a few rare, specified cases. Also, all our homomorphisms and other functions are written on the left of their arguments, i.e., in the form $f(x)$.

The following are some references for frequently used notions in this book. These references contain more information than we actually need, and the reader to whom the italicized terms are all familiar should not feel it necessary to read through these references unless a particular problem arises. *Algebras* – Cohn [1982, §4.5; 1989, §5.1], Hungerford [1989, §4.7], Jacobson [1985, §7.1; 1989, §3.9]. *Direct Sums and Products* – Cohn [1982, §10.3; 1989, §4.1], Hungerford [1989, §§3.2, 4.1], Jacobson [1985, §3.5; 1989, §3.4]. *Domains* – Cohn [1982, §6.1], Jacobson [1985, §2.2]. *Epimorphisms, Monomorphisms, and Isomorphisms* – Cohn [1991, §3.1], Hungerford [1989, §4.1], Jacobson [1989, §1.2]. *Free Algebras* – Cohn [1989, §11.5; 1991, §2.2]. *Free and Projective Modules* – Cohn [1982, §10.4; 1989, §§4.4, 4.5], Hungerford [1989, §§4.2, 4.3], Jacobson [1985, §3.4; 1989, §§1.7, 3.10]. *Indecomposable Rings and Modules* – Cohn [1989, §5.2], Jacobson [1989, §3.4]. *Independent Families of Submodules* – Cohn [1982, §10.3], Hungerford [1989, §9.4], Jacobson [1985, §3.5; 1989, §3.5]. *Opposite Rings* – Cohn [1982, §10.2], Hungerford [1989, §7.1], Jacobson [1985, §2.8]. *Ring Homomorphisms* – Cohn [1982, §10.1], Hungerford [1989, §3.1], Jacobson [1985, §2.7]. *Sums of Submodules* – Cohn [1982, §10.3; 1989, §4.1], Hungerford [1989, §4.1], Jacobson [1985, §3.3; 1989, §3.5]. *Tensor Products* – Cohn [1989, §§4.7, 5.5], Hungerford [1989, §4.5], Jacobson [1989, §§3.7, 3.9]. *Zorn's Lemma* – Cohn [1989, §1.2], Hungerford [1989, §0.7], Jacobson [1989, §0.1].

Finally, we list some standard notation:

$\subset$ and $\supset$	Proper inclusions.
$\sqcup$ or $\bigsqcup$	A disjoint union.
$\mathbb{N}$	The set of natural numbers (i.e., positive integers).
$\mathbb{Z}^+$	The set of nonnegative integers.
$\mathbb{Z}$	The ring of integers.
$\mathbb{Q}$	The field of rational numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{H}$	The division ring of real quaternions.
A_R	A right module A over a ring R. (In case A can be considered as a module over several different rings, the notation A_R is used to indicate that A is being viewed as an R-module.)
${}_RA$	A left module A over a ring R.
R_R	A ring R viewed as a right module over itself.
${}_RR$	A ring R viewed as a left module over itself.
$\mathrm{Hom}_R(A, B)$	The abelian group of all R-module homomorphisms from an R-module A to an R-module B.

$\mathrm{End}_R(A)$	The ring of all R-module endomorphisms of an R-module A.
A^n or $\oplus^n A$	The direct sum of n copies of a module A. (For a right or left ideal A in a ring, $\oplus^n A$ is used to avoid confusion with multiplicative powers of A.)
I^n	The n-th multiplicative power of a right or left ideal I in a ring (i.e., the set of all sums of n-fold products $i_1 i_2 \cdots i_n$ where $i_1, \dots, i_n \in I$).
IJ	The multiplicative product of right or left ideals I and J in a ring (i.e., the set of all sums of products ij where $i \in I$ and $j \in J$).
$\langle a, b, c, \dots \rangle$	The ideal generated by elements $a, b, c, \dots$ in a ring.
$k^\times$	The multiplicative group of nonzero elements in a field k.

1. A Few Noetherian Rings

After a review of the definition and basic properties of noetherian modules and rings, we introduce a few classes of examples of noetherian rings, which will serve to illustrate and support the later theory. We concentrate particularly on some of the "surrogate" examples outlined in the Prologue, namely, module-finite algebras over commutative rings, skew-Laurent rings, and the corresponding skew polynomial rings twisted by automorphisms. The general theory of skew polynomial rings will be addressed in the following chapter, where we study the Weyl algebras, formal differential operator rings, and other examples from the Prologue.

• THE NOETHERIAN CONDITION •

We begin with several basic equivalent conditions which are abbreviated by the adjective "noetherian," honoring E. Noether, who first demonstrated the importance and usefulness of these conditions. Recall that a collection $\mathcal{A}$ of subsets of a set A satisfies the *ascending chain condition* (or *ACC*) if there does not exist a properly ascending infinite chain $A_1 \subset A_2 \subset \cdots$ of subsets from $\mathcal{A}$. Recall also that a subset $B \in \mathcal{A}$ is a *maximal element* of $\mathcal{A}$ if there does not exist a subset in $\mathcal{A}$ that properly contains B. To emplasize the order-theoretic nature of these considerations, we often use the notation of inequalities ($\leq$, $<$, $\not\leq$, etc.) for inclusions among submodules and/or ideals. In particular, if A is a module, the notation $B \leq A$ means that B is a submodule of A, and the notation $B < A$ (or $A > B$) means that B is a proper submodule of A.

Proposition 1.1. *For a module A, the following conditions are equivalent:*

(a) *A has the ACC on submodules.*

(b) *Every nonempty family of submodules of A has a maximal element.*

(c) *Every submodule of A is finitely generated.*

Proof. (a) $\Longrightarrow$ (b): Suppose that $\mathcal{A}$ is a nonempty family of submodules of A without a maximal element. Choose $A_1 \in \mathcal{A}$. Since A_1 is not maximal, there exists $A_2 \in \mathcal{A}$ such that $A_2 > A_1$. Continuing in this manner, we obtain a properly ascending infinite chain $A_1 < A_2 < A_3 < \cdots$ of submodules of A, contradicting the ACC.

(b) $\Longrightarrow$ (c): Let B be a submodule of A, and let $\mathcal{B}$ be the family of all finitely generated submodules of B. Note that $\mathcal{B}$ contains 0 and so is nonempty. By (b), there exists a maximal element $C \in \mathcal{B}$. If $C \neq B$, choose an element $x \in B \setminus C$, and let C' be the submodule of B generated by C and x. Then $C' \in \mathcal{B}$ and $C' > C$, contradicting the maximality of C. Thus $C = B$, whence B is finitely generated.

(c) $\Longrightarrow$ (a): Let $B_1 \leq B_2 \leq \cdots$ be an ascending chain of submodules of A. Let B be the union of the B_n. By (c), there exists a finite set X of generators for B. Since X is finite, it is contained in some B_n, whence $B_n = B$. Thus $B_m = B_n$ for all $m \geq n$, establishing the ACC for submodules of A. $\square$

Definition. A module A is *noetherian* if and only if the equivalent conditions of Proposition 1.1 are satisfied. As follows from the proof of (b) $\Longrightarrow$ (c), a further equivalent condition is that A have the ACC on *finitely generated* submodules.

For example, any finite dimensional vector space V over a field k is a noetherian k-module, since a properly ascending chain of submodules (subspaces) of V cannot contain more than $\dim_k(V) + 1$ terms.

Definition. A ring R is *right (left) noetherian* if and only if the right module R_R (left module ${}_RR$) is noetherian. If both conditions hold, R is called a *noetherian ring.*

Rephrasing Proposition 1.1 for the ring itself, we see that a ring R is right (left) noetherian if and only if R has the ACC on right (left) ideals, if and only if all right (left) ideals of R are finitely generated. For example, $\mathbb{Z}$ is a noetherian ring because all its ideals are principal (singly generated). The same is true of a polynomial ring $k[x]$ in one indeterminate over a field k.

Exercise 1A. (a) Show that the 2×2 matrices over $\mathbb{Q}$ of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ make a ring which is right noetherian but not left noetherian.

(b) Show that any finite direct product of right (left) noetherian rings is right (left) noetherian. $\square$

Proposition 1.2. *Let B be a submodule of a module A. Then A is noetherian if and only if B and A/B are both noetherian.*

Proof. First assume that A is noetherian. Since any ascending chain of submodules of B is also an ascending chain of submodules of A, it is immediate that B is noetherian. If $C_1 \leq C_2 \leq \cdots$ is an ascending chain of submodules of A/B, each C_i is of the form A_i/B for some submodule A_i of A that contains B, and $A_1 \leq A_2 \leq \cdots$. Since A is noetherian, there is some n such that $A_i = A_n$ for all $i \geq n$, and then $C_i = C_n$ for all $i \geq n$. Thus A/B is noetherian.

Conversely, assume that B and A/B are noetherian, and let $A_1 \leq A_2 \leq \cdots$ be an ascending chain of submodules of A. There are ascending chains of submodules

$$A_1 \cap B \leq A_2 \cap B \leq \cdots$$
$$(A_1 + B)/B \leq (A_2 + B)/B) \leq \cdots$$

in B and in A/B. Hence, there is some n such that $A_i \cap B = A_n \cap B$ and $(A_i + B)/B = (A_n + B)/B$ for all $i \geq n$, and the latter equation yields $A_i + B = A_n + B$. For all $i \geq n$, we conclude that

$$A_i = A_i \cap (A_i + B) = A_i \cap (A_n + B) = A_n + (A_i \cap B) = A_n + (A_n \cap B) = A_n$$

(using the modular law for the third equality). Therefore A is noetherian. □

In particular, Proposition 1.2 shows that any factor ring of a right noetherian ring is right noetherian. (Note that if I is an ideal of a ring R, then the right ideals of R/I are the same as the right R-submodules.)

Corollary 1.3. *Any finite direct sum of noetherian modules is noetherian.*

Proof. It suffices to prove that the direct sum of any two noetherian modules A_1 and A_2 is noetherian. The module $A = A_1 \oplus A_2$ has a submodule $B = A_1 \oplus 0$ such that $B \cong A_1$ and $A/B \cong A_2$. Then B and A/B are noetherian, whence A is noetherian by Proposition 1.2. □

Corollary 1.4. *If R is a right noetherian ring, all finitely generated right R-modules are noetherian.*

Proof. If A is a finitely generated right R-module, then $A \cong F/K$ for some finitely generated free right R-module F and some submodule $K \leq F$. Since F is isomorphic to a finite direct sum of copies of the noetherian module R_R, it is noetherian by Corollary 1.3. Then, by Proposition 1.2, A must be noetherian. □

Corollary 1.5. *Let S be a subring of a ring R. If S is right noetherian and R is finitely generated as a right S-module, then R is right noetherian.*

Proof. By Corollary 1.4, R is noetherian as a right S-module. Since all right ideals of R are also right S-submodules, the ACC on right ideals follows. □

Using Corollary 1.5, we obtain some easy examples of noncommutative noetherian rings.

Proposition 1.6. *If R is a module-finite algebra over a commutative noetherian ring S, then R is a noetherian ring.*

Proof. The image of S in R is a noetherian subring S' of the center of R such that R is a finitely generated (right or left) S'-module. Apply Corollary 1.5. □

For instance, let $S = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\mathbf{k}$, a subring of the division ring $\mathbb{H}$. Since S is a finitely generated module over the noetherian ring $\mathbb{Z}$, Proposition 1.6 shows that S is a noetherian ring. For another example, Proposition 1.6 shows that, for any positive integer n, the ring of all $n \times n$ matrices over a commutative noetherian ring is noetherian. This also holds for matrix rings over noncommutative noetherian rings, as follows.

Definition. Given a ring R and a positive integer n, we use $M_n(R)$ to denote the ring of all $n \times n$ matrices over R. The *standard $n \times n$ matrix units* in $M_n(R)$ are the matrices e_{ij} (for $i, j = 1, \dots, n$) such that e_{ij} has 1 for the i,j-entry and 0 for all other entries.

Proposition 1.7. *Let R be a right noetherian ring and S a subring of a matrix ring $M_n(R)$. If S contains the subring*

$$R' = \left\{ \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} \;\middle|\; r \in R \right\}$$

of all "scalar matrices," then S is right noetherian. In particular, $M_n(R)$ is a right noetherian ring.

Proof. Clearly $R' \cong R$, whence R' is a right noetherian ring. Observe that $M_n(R)$ is generated as a right R'-module by the standard $n \times n$ matrix units. Hence, Corollary 1.4 implies that $M_n(R)$ is a noetherian right R'-module. As all right ideals of S are also right R'-submodules of $M_n(R)$, we conclude that S is right noetherian. $\square$

• FORMAL TRIANGULAR MATRIX RINGS •

One way to construct rings to which Corollary 1.5 and Proposition 1.7 apply is to take an upper (or lower) triangular matrix ring over a known ring, or to take a subring of a triangular matrix ring. For instance, if S and T are subrings of a ring B, the set R of all matrices of the form $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ (for $s \in S$, $b \in B$, $t \in T$) is a subring of $M_2(B)$. (If S and T are right noetherian, and B_T is finitely generated, it follows easily from Corollary 1.5 that R is right noetherian.) Note that B need not be a ring itself in order for R to be a ring – rather, B must be closed under addition, left multiplication by elements of S, and right multiplication by elements of T. More formally, the symbols $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ will form a ring under matrix addition and multiplication provided only that B is simultaneously a left S-module and a right T-module satisfying an associative law connecting its left and right module structures. We focus on this ring construction because it provides a convenient source for any number of interesting examples. Later, we shall see such left/right modules as B appearing for their own sake in noetherian ring theory.

Definition. Let S and T be rings. An (S,T)*-bimodule* is an abelian group B equipped with a left S-module structure and a right T-module structure (both utilizing the given addition) such that $s(bt) = (sb)t$ for all $s \in S$, $b \in B$, $t \in T$. The symbol ${}_S B_T$ is used to denote this situation. An (S,T)*-sub-bimodule* of B (or just a *sub-bimodule*, if S and T are clear from the context) is any subgroup of B which is both a left S-submodule and a right T-submodule. Note that if C is a sub-bimodule of B, the factor group B/C is a bimodule in the obvious manner.

For instance, if S is a ring and T is a subring, then S itself (or an ideal of S) can be regarded as an (S,T)-bimodule (or as a (T,S)-bimodule). For another example, if B is a right module over a ring T and S is a subring of $\operatorname{End}_T(B)$, then B is an (S,T)-bimodule. Perhaps most importantly, if $I \subseteq J$ are ideals in a ring S, then J/I is an (S,S)-bimodule. The next exercise shows that in a sense every bimodule appears this way, as an ideal of a formal triangular matrix ring.

Exercise 1B. Let ${}_S B_T$ be a bimodule, and write $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ for the abelian group $S \oplus B \oplus T$, where triples (s,b,t) from $S \oplus B \oplus T$ are written as formal 2×2 matrices $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$.

(a) Show that formal matrix addition and multiplication make sense in $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$, and that by using those operations $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ becomes a ring.

(b) Show that there is also a ring $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix}$ of formal lower triangular matrices, and that $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix} \cong \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$.

(c) Observe that the set $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ of matrices $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is an ideal of $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$, and that, under the obvious abelian group isomorphism of B onto $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, left S-submodules (right T-submodules, (S,T)-sub-bimodules) of B correspond precisely to left ideals (right ideals, two-sided ideals) of $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ contained in $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. $\square$

Definition. A *formal triangular matrix ring* is any ring of the form $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ or $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix}$ as described in Exercise 1B. By way of abbreviation, we write "let $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring" in place of "let S and T be rings, let B be an (S,T)-bimodule, and let $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be the corresponding formal triangular matrix ring."

Observe that if S and T are subrings of a ring U, and B is an (S,T)-sub-bimodule of U, the formal triangular matrix ring $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ is isomorphic to the subring of $M_2(U)$ consisting of all honest matrices of the form $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ with $s \in S$, $b \in B$, $t \in T$.

Proposition 1.8. *Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring. Then R is right noetherian if and only if S and T are right noetherian and B_T is finitely generated. Similarly, R is left noetherian if and only if S and T are left noetherian and ${}_SB$ is finitely generated.*

Proof. Assume first that S and T are right noetherian and B_T is finitely generated. Observe that the diagonal subring $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is isomorphic to $S \times T$ and so is right noetherian. Observe also that if elements $b_1, \dots, b_n$ generate B as a right T-module, then the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}$$

generate R as a right $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$-module. Consequently, Corollary 1.5 shows that R is right noetherian.

Conversely, assume that R is right noetherian. Observing that the projection maps $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \mapsto s$ and $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \mapsto t$ are ring homomorphisms of R onto S and of R onto T, we see that S and T must be right noetherian. Moreover, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is a right ideal of R and must have a finite list of generators

$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix},$$

from which we infer that the elements $b_1, \dots, b_n$ generate B_T.

The left noetherian analog is proved in the same manner. □

For example, it is immediate from Proposition 1.8 that the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian (Exercise 1A(a)).

Exercise 1C. Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring. The purpose of this exercise is to give a description of all right R-modules in terms of right S-modules and T-modules.

(a) Let A be a right S-module, C a right T-module, and f a homomorphism in $\mathrm{Hom}_T(A \otimes_S B, C)$. For $(a, c) \in A \oplus C$ and $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \in R$, define

$$(a, c)\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} = (as,\ f(a \otimes b) + ct).$$

Show that, using this multiplication rule, $A \oplus C$ is a right R-module.

(b) Show that the R-module $A \oplus C$ in (a) is finitely generated if and only if A is a finitely generated S-module and $C/f(A \otimes_S B)$ is a finitely generated T-module.

(c) Show that every right R-module is isomorphic to one of the type $A \oplus C$ constructed in (a). □

Exercise 1D. Let ${}_SB_T$ be a bimodule, and form the ring $R = S^{\mathrm{op}} \otimes_{\mathbb{Z}} T$, where S^{op} denotes the *opposite ring* of S. (That is, S^{op} is the same abelian group as S, but with the opposite multiplication: The product of s_1 and s_2 in S^{op} is s_2s_1.) Show that B can be made into a right R-module where $b(s \otimes t) = sbt$ for all $s \in S$, $t \in T$, $b \in B$, and that the right R-submodules of B are precisely its (S, T)-sub-bimodules. Conversely, show that every right R-module can be made into an (S, T)-bimodule. $\square$

• THE HILBERT BASIS THEOREM •

A large class of examples of noetherian rings (particularly, commutative ones) is revealed by this famous theorem. There are several different proofs available; we sketch one that we shall adapt later for skew polynomial rings.

Theorem 1.9. [Hilbert's Basis Theorem] *Let $S = R[x]$ be a polynomial ring in one indeterminate. If the coefficient ring R is right (left) noetherian, then so is S.*

Proof. The two cases are symmetric; let us assume that R is right noetherian and prove that any right ideal I of S is finitely generated. We need only consider the case when $I \neq 0$.

Step 1. Let J be the set of leading coefficients of elements of I, together with 0. More precisely,

$$J = \{r \in R \mid rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I \text{ for some } r_{d-1}, \ldots, r_0 \in R\}.$$

Then check that J is a right ideal of R. (Note that if $r, r' \in J$ are leading coefficients of elements $s, s' \in I$ with degrees d, d', then, after replacing s and s' by $sx^{d'}$ and $s'x^d$, we may assume that s and s' have the same degree.)

Step 2. Since R is right noetherian, J is finitely generated. Let $r_1, \ldots, r_k$ be a finite list of generators for J; we may assume that they are all nonzero. Each r_i occurs as the leading coefficient of a polynomial $p_i \in I$ of some degree n_i. Set $n = \max\{n_1, \ldots, n_k\}$ and replace each p_i by $p_ix^{n-n_i}$. Thus, there is no loss of generality in assuming that all the p_i have the same degree n.

Step 3. Set $N = R + Rx + \cdots + Rx^{n-1} = R + xR + \cdots + x^{n-1}R$, the set of elements of S with degree less than n. This is not an ideal of S, but it is a left and right R-submodule. Viewed as a right R-module, N is finitely generated, and so it is noetherian by Corollary 1.4. Now $I \cap N$ is a right R-submodule of N, and consequently it must be finitely generated. Let $q_1, \ldots, q_t$ be a finite list of right R-module generators for $I \cap N$.

Step 4. We claim that $p_1, \ldots, p_k, q_1, \ldots, q_t$ generate I. Let I_0 denote the right ideal of S generated by these polynomials; then $I_0 \subseteq I$ and it remains to show that any polynomial $p \in I$ actually lies in I_0. This is easy if p has degree less than n, since in that case $p \in I \cap N$ and $p = q_1a_1 + \cdots + q_ta_t$ for some $a_j \in R$.

Step 5. Suppose that $p \in I$ has degree $m \geq n$ and that I_0 contains all elements of I with degree less than m. Let r be the leading coefficient of p. Then $r \in J$, and so $r = r_1a_1 + \cdots + r_ka_k$ for some $a_i \in R$. Set $q = (p_1a_1 + \cdots + p_ka_k)x^{m-n}$, an element of I_0 with degree m and leading coefficient r. Now $p - q$ is an element of I with degree less than m. By the induction hypothesis, $p - q \in I_0$, and thus $p \in I_0$.

Therefore $I = I_0$ and we are done. $\square$

It immediately follows that any polynomial ring $R[x_1, \ldots, x_n]$ in a finite number of indeterminates over a right (left) noetherian ring R is right (left) noetherian, since we may view $R[x_1, \ldots, x_n]$ as a polynomial ring in the single indeterminate x_n with coefficients from the ring $R[x_1, \ldots, x_{n-1}]$.

Corollary 1.10. *Let R be an algebra over a field k. If R is commutative and finitely generated as a k-algebra, then R is noetherian.*

Proof. Let $x_1, \ldots, x_n$ generate R as a k-algebra, and let $S = k[y_1, \ldots, y_n]$ be a polynomial ring over k in n independent indeterminates. Since R is commutative, there exists a k-algebra map $\phi : S \to R$ such that $\phi(y_i) = x_i$ for each i, and ϕ is surjective because the x_i generate R. Hence, $R \cong S/\ker(\phi)$. By the Hilbert Basis Theorem, S is a noetherian ring, and therefore R is noetherian. $\square$

Noncommutative finitely generated algebras need not be noetherian, as the following examples show.

Exercise 1E. Let k be a field.

(a) Let V be a countably infinite dimensional vector space over k with a basis $\{v_1, v_2, \ldots\}$. Define $s, t \in \operatorname{End}_k(V)$ so that $s(v_i) = v_{i+1}$ for all i while $t(v_i) = v_{i-1}$ for all $i > 1$ and $t(v_1) = 0$, and let R be the k-subalgebra of $\operatorname{End}_k(V)$ generated by s and t. Show that R is neither right nor left noetherian. [Hint: Define $e_1, e_2, \ldots$ in $\operatorname{End}_k(V)$ so that $e_i(v_i) = v_i$ for all i while $e_i(v_j) = 0$ for all $i \neq j$, and show that each $e_i \in R$. Then show that $\sum_i e_iR$ and $\sum_i Re_i$ are not finitely generated.]

(b) If F is the free k-algebra on letters X and Y, there is a unique k-algebra homomorphism $\phi : F \to R$ such that $\phi(X) = s$ and $\phi(Y) = t$. Since ϕ is surjective (by definition of R), we have $R \cong F/\ker(\phi)$, and so it is clear from part (a) that F cannot be right or left noetherian. Give a direct proof of this fact. [For instance, show that $\sum_i X^iYF$ and $\sum_i FXY^i$ are not finitely generated.] $\square$

Exercise 1F. Let R be an algebra over a field k, and suppose that R is generated by two elements x and y such that $xy = -yx$. Show that x^2 and y^2 are in the center of R, and that R is a finitely generated module over the subalgebra S generated by x^2 and y^2. [Hint: Use $1, x, y, xy$ to generate R.] Then apply Corollary 1.10 and Proposition 1.6 to conclude that R is noetherian.

Now suppose that, instead of $xy = -yx$, we have $xy = \xi yx$ for some scalar $\xi \in k^\times$ which is a root of unity, that is, $\xi^n = 1$ for some positive integer n. Modify the steps above to show that R is also noetherian in this case. □

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• SKEW POLYNOMIAL RINGS TWISTED BY AUTOMORPHISMS •

In the Prologue we saw several examples of rings that look like polynomial rings in one indeterminate but in which the indeterminate does not commute with the coefficients – rather, multiplication by the indeterminate has been "skewed" or "twisted" by means of an automorphism of the coefficient ring, or a derivation, or a combination of such maps. To help the reader get used to constructing and working with such twisted polynomial rings, we begin here by concentrating on the case where the twisting is done by an automorphism. In Chapter 2, we move on to twists by derivations and then to general skew polynomial rings.

Thus, let R be a ring, α an automorphism of R, and x an indeterminate. Let S be the set of all formal expressions $a_0 + a_1x + \cdots + a_nx^n$, where n is a nonnegative integer and the $a_i \in R$. It is often convenient to write such an expression as a sum $\sum_i a_ix^i$, leaving it understood that the summation runs over a finite sequence of nonnegative integers i, or by thinking of it as an infinite sum in which almost all of the coefficients a_i are zero. We define an addition operation in S in the usual way:

$$\Big(\sum_i a_ix^i\Big) + \Big(\sum_i b_ix^i\Big) = \sum_i (a_i + b_i)x^i.$$

As for multiplication, we would like the coefficients to multiply together as they do in R, and we would like the powers of x to multiply following the usual rules for exponents. We take the product of an element $a \in R$ with a power x^i (in that order) to be the single-term sum ax^i. It is in a product of the form x^ia that the twist enters. We define xa to be $\alpha(a)x$ and iterate that rule to obtain $x^ia = \alpha^i(a)x$. This leads us to define the following multiplication rule in S:

$$\Big(\sum_i a_ix^i\Big)\Big(\sum_j b_jx^j\Big) = \sum_{i,j} a_i\alpha^i(b_j)x^{i+j} = \sum_k \Big(\sum_{i+j=k} a_i\alpha^i(b_j)\Big)x^k.$$

Exercise 1G. Verify that the set S together with the operations defined above is a ring, and that when R is identified with the set of elements of S involving no positive powers of x, it becomes a subring of S. □

Exercise 1H. Here is a more formal description of S, in which the symbol x does not make an a priori appearance.

Let $\overline{S}$ denote the set of those infinite sequences $a = (a_0, a_1, a_2, \dots)$ of elements of R in which $a_i = 0$ for all but finitely many indices i. For any $a, b \in \overline{S}$, define $a + b$ and ab to be the sequences in $\overline{S}$ with entries

$$(a+b)_i = a_i + b_i \qquad\qquad (ab)_k = \sum_{i+j=k} a_i \alpha^i(b_j)$$

for all i and k. Show that $\overline{S}$ with these operations is a ring, and that $\overline{S} \cong S$ via the rule $a \mapsto \sum_i a_i x^i$. This isomorphism makes it clear that x is just a name for a particular special element of S, corresponding to the sequence $(0, 1, 0, 0, 0, \dots)$ in $\overline{S}$. $\square$

We have glossed over an important point in our discussion of S – the question of when two formal expressions define the same element of S. Namely, we have taken it as understood that two elements of S are the same only if their coefficients are the same, that is, $\sum_i a_i x^i = \sum_i b_i x^i$ if and only if $a_i = b_i$ for all i. Missing coefficients are understood to be zero: In case the equation concerns finite sums and an index i occurs in the first sum but not in the second, equality of coefficients means that $a_i = 0$. Using the language of linear algebra, we can thus say that the elements $1, x, x^2, \dots$ in S are linearly independent over R. Since every element of S is a linear combination of these powers, S is thus a free left R-module with the powers of x forming a basis. This leads us to the following definition.

Definition. Let R be a ring and α an automorphism of R. We write

$$S = R[x; \alpha]$$

(where S and x may or may not already occur in the discussion) to mean that

(a) S is a ring, containing R as a subring;
(b) x is an element of S;
(c) S is a free left R-module with basis $\{1, x, x^2, \dots\}$;
(d) $xr = \alpha(r)x$ for all $r \in R$.

Thus, the expression $S = R[x; \alpha]$ can be used either to introduce a new ring S (constructed as above) or to say that a given ring S and element x satisfy conditions (a)–(d). Whenever $S = R[x; \alpha]$, we say that S is a *skew polynomial ring over R*.

In many algebra texts, rings of polynomials are introduced as specific rings resulting from special constructions. Note that the definition above is of a different type, since $R[x; \alpha]$ is defined to be any ring extension of R satisfying certain properties, rather than as any specific ring (although some construction is needed to guarantee that such skew polynomial rings exist). In particular, the element x in $R[x;\alpha]$ is just a ring element with certain special properties, not a mysterious "indeterminate."

The advantage of the type of definition just given is that, in many situations, we will be able to say that some ring *equals* a skew polynomial ring $R[x;\alpha]$, rather than having to say that it is *isomorphic to* $R[x;\alpha]$. This is useful even in the context of ordinary polynomial rings. For example, we can say that the subring of $\mathbb{R}$ generated by $\mathbb{Q}$ and π *is* a polynomial ring $\mathbb{Q}[\pi]$, instead of having to name an indeterminate x and a $\mathbb{Q}$-algebra isomorphism of $\mathbb{Q}[x]$ onto $\mathbb{Q}[\pi]$.

The discussion above shows that, given R and α, a skew polynomial ring $S = R[x;\alpha]$ does exist. As is the case for ordinary polynomial rings, we would like S to be unique, up to appropriate isomorphisms. We prove this with the help of the following *universal mapping property*, in which the map ψ may be thought of as an analog of an evaluation map on ordinary polynomials in the commutative theory. The main ingredients of the lemma may be displayed as in the following diagram.

$$\begin{tikzcd} R[x;\alpha] \arrow[dr, dashed, "\exists!\ \psi"] & \\ R \arrow[u, hook, "\subset"] \arrow[r, "\phi"'] & T \end{tikzcd}$$

Lemma 1.11. *Let R be a ring, α an automorphism of R, and $S = R[x;\alpha]$. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proof. Clearly any such map would have to be given by the rule

$$\psi\Big(\sum_i a_i x^i\Big) = \sum_i \phi(a_i) y^i,$$

and so there is at most one possibility for ψ. This rule does give a well-defined function $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$, and so we just need to show that ψ is a ring homomorphism. It is clear that ψ is additive and that $\psi(1) = 1$. The rule $y\phi(r) = \phi\alpha(r)y$ implies (by induction) that $y^i\phi(r) = \phi\alpha^i(r)y^i$ for all $i \in \mathbb{Z}^+$ and $r \in R$. Hence,

$$\begin{aligned}\Big[\psi\Big(\sum_i a_i x^i\Big)\Big]\Big[\psi\Big(\sum_j b_j x^j\Big)\Big] &= \Big[\sum_i \phi(a_i)y^i\Big]\Big[\sum_j \phi(b_j)y^j\Big] \\ &= \sum_{i,j} \phi(a_i)\phi\alpha^i(b_j)y^{i+j} = \sum_k \Big(\sum_{i+j=k} \phi(a_i)\phi\alpha^i(b_j)\Big)y^k \\ &= \psi\Big[\sum_k \Big(\sum_{i+j=k} a_i\alpha^i(b_j)\Big)x^k\Big] = \psi\Big[\Big(\sum_i a^i x^i\Big)\Big(\sum_j b_j x^j\Big)\Big]\end{aligned}$$

for all elements $\sum_i a^i x^i$ and $\sum_j b_j x^j$ in S. Therefore ψ is a ring homomorphism, as required. $\square$

Corollary 1.12. *Let R be a ring and α an automorphism of R. Suppose that $S = R[x;\alpha]$ and $S' = R[x';\alpha]$. Then there is a unique ring isomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R$ is the identity on R.*

Proof. First, apply Lemma 1.11 with $\phi : R \to S'$ being the inclusion map; we obtain a unique ring homomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R = \phi$. We may rephrase the last property by saying that $\psi|_R$ is the identity on R. By symmetry, Lemma 1.11 also provides a ring homomorphism $\psi' : S' \to S$ such that $\psi'(x') = x$ and $\psi'|_R$ is the identity on R.

Now $\psi'\psi : S \to S$ is a ring homomorphism such that $(\psi'\psi)(x) = x$ and $(\psi'\psi)|_R$ is the identity on R. The identity map on S enjoys the same properties. Hence, the uniqueness part of Lemma 1.11 (where now $T = S$ and $y = x$) implies that $\psi'\psi$ equals the identity map on S. Similarly, $\psi\psi'$ equals the identity map on S'.

Therefore ψ and ψ' are mutually inverse isomorphisms. $\square$

The proof of Corollary 1.12 illustrates a general principle, that objects with universal mapping properties are unique up to isomorphism. We shall see this principle in action a number of times later.

It is time to consider a specific example. Let $R = k[y]$ be an ordinary polynomial ring over a field k. Given a nonzero scalar $q \in k$, we can define a k-algebra automorphism α on R such that $\alpha(y) = qy$. (In function notation, $\alpha\big(p(y)\big) = p(qy)$ for $p(y) \in k[y]$.) Now let $S = R[x;\alpha]$. Then $xy = \alpha(y)x = qyx$, the basic "commutation rule" in S. Since the polynomials in R are just k-linear combinations of powers of y, elements of S can be written in the form $\sum_{i,j} \lambda_{ij} y^i x^j$ for scalars λ_{ij} (all but finitely many of which are zero), and multiplication in S follows the rule

$$\begin{aligned}\Big(\sum_{i,j} \lambda_{ij} y^i x^j\Big)\Big(\sum_{s,t} \mu_{st} y^s x^t\Big) &= \sum_{i,j,s,t} \lambda_{ij}\mu_{st} q^{js} y^{i+s} x^{j+t} \\ &= \sum_{l,m}\Big(\sum_{\substack{i+s=l \\ j+t=m}} \lambda_{ij}\mu_{st} q^{js}\Big) y^l x^m.\end{aligned}$$

This example looks very much like one from the Prologue, which we now recall.

Definition. Let k be a field and $q \in k^\times$. The *quantized coordinate ring of* k^2 (corresponding to the choice of q) is a k-algebra, denoted $\mathcal{O}_q(k^2)$, presented by two generators x and y and the relation $xy = qyx$. In short,

$$\mathcal{O}_q(k^2) = k\langle x, y \mid xy = qyx\rangle.$$

In algebraic geometry, k^2 is the affine plane over k. Hence, $\mathcal{O}_q(k^2)$ is also known as a *coordinate ring of a quantum plane* (over k), or just as – the handiest abbreviation – a *quantum plane*.

In the example discussed just prior to the definition, S is a k-algebra, and it is generated by elements called x and y, which satisfy the relation $xy = qyx$. Does this mean that $\mathcal{O}_q(k^2)$ and S are the same? To answer this question, we must make clear exactly what is meant by the definition we have given for $\mathcal{O}_q(k^2)$. We do *not* mean "any algebra generated by two elements satisfying the given relation," since there are too many possibilities. For instance, the polynomial ring $k[x]$ is generated as a k-algebra by x and 0, and certainly $x0 = q0x$. Even more extreme, the base field k itself is generated as a k-algebra by 1 and 0, and $1 \cdot 0 = q0 \cdot 1$. These algebras have "unnecessary" relations – for instance, the second generator is zero in these algebras but not in S.

What is tacitly assumed in the definition of $\mathcal{O}_q(k^2)$ (and is encoded by using the term "presented") is that x and y satisfy no "extra" relations, i.e., no relations beyond those consequences of the given relation $xy = qyx$ forced by the axioms for a k-algebra (such as $xy^3 = q^3y^3x$). The way to make the idea of "no extra relations" precise is to start with a free algebra and factor out the minimum required to achieve the desired relations. Thus, if $k\langle X, Y\rangle$ is the free algebra on two letters X and Y (which satisfy no relations at all), and $\langle XY - qYX\rangle$ denotes the ideal of $k\langle X, Y\rangle$ generated by $XY - qYX$, we are declaring that

$$\mathcal{O}_q(k^2) = k\langle X, Y\rangle / \langle XY - qYX\rangle.$$

The elements x and y in the definition of $\mathcal{O}_q(k^2)$ are then the cosets of X and Y. It follows easily from this description that $\mathcal{O}_q(k^2)$ satisfies a universal mapping property and is therefore uniquely determined up to isomorphism of k-algebras, as follows.

Exercise 1I. Let k be a field, $q \in k^\times$, and T a k-algebra. Suppose there are elements $u, v \in T$ satisfying the equation $uv = qvu$. Show that there is a unique k-algebra homomorphism $\phi : \mathcal{O}_q(k^2) \to T$ such that $\phi(x) = u$ and $\phi(y) = v$.

Conclude that if $\mathcal{O}_q(k^2)'$ is a k-algebra presented by two generators x' and y' and one relation $x'y' = qy'x'$, then $\mathcal{O}_q(k^2)' \cong \mathcal{O}_q(k^2)$. $\square$

To continue our discussion above, let us keep the symbols x and y as in the definition of $\mathcal{O}_q(k^2)$ but use new symbols $\hat{x}$ and $\hat{y}$ to rename the indeterminates in the skew polynomial ring S. By Exercise 1I, there is a unique k-algebra homomorphism $\phi : \mathcal{O}_q(k^2) \to S$ such that $\phi(x) = \hat{x}$ and $\phi(y) = \hat{y}$. Observe that ϕ is at least surjective, since $\hat{x}$ and $\hat{y}$ generate S. There are several ways to see that ϕ is actually an isomorphism; here are two.

Exercise 1J. (a) Use the relation $xy = qyx$ to show that every element of $\mathcal{O}_q(k^2)$ is a k-linear combination of the monomials y^ix^j. Then show that the monomials $\hat{y}^i\hat{x}^j$ in S are linearly independent over k. Since $\phi(y^ix^j) = \hat{y}^i\hat{x}^j$ for all i,j, conclude that the monomials y^ix^j are linearly independent and thus that ϕ is an isomorphism.

(b) Since $R = k[\hat{y}]$ is a polynomial ring over k, there is a unique k-algebra homomorphism $\eta : R \to \mathcal{O}_q(k^2)$ such that $\eta(\hat{y}) = y$. Show that $x\eta(r) = \eta\alpha(r)x$ for all $r \in R$, and conclude from Lemma 1.11 that η extends uniquely to a ring homomorphism $\psi : S \to \mathcal{O}_q(k^2)$ such that $\psi(\hat{x}) = x$. Finally, show that ϕ and ψ are inverses of each other. $\square$

Now that we have $\mathcal{O}_q(k^2) \cong S$, we can say that $\mathcal{O}_q(k^2)$ is a skew polynomial ring. Let us record this information in the following form.

Proposition 1.13. *Let k be a field and $q \in k^\times$. Then $\mathcal{O}_q(k^2) = k[y][x;\alpha]$, where $k[y]$ is a polynomial ring and α is the k-algebra automorphism of $k[y]$ such that $\alpha(y) = qy$.* $\square$

Of course, all this can be done with the variables in the reverse order. Thus,

$$\mathcal{O}_q(k^2) = k[x][y;\beta],$$

where β is the k-algebra automorphism of the polynomial ring $k[x]$ such that $\beta(x) = q^{-1}x$. We can also adapt the above discussion to any number of variables, as follows.

Definition. Let k be a field. A *multiplicatively antisymmetric matrix* over k is an $n \times n$ matrix $\mathbf{q} = (q_{ij})$ with entries $q_{ij} \in k^\times$ such that $q_{ii} = 1$ for all i and $q_{ji} = q_{ij}^{-1}$ for all i,j. Given such a matrix, the corresponding *multiparameter quantized coordinate ring of affine n-space*, or just *multiparameter quantum n-space*, is the k-algebra $\mathcal{O}_{\mathbf{q}}(k^n)$ presented by generators $x_1, \dots, x_n$ and relations $x_ix_j = q_{ij}x_jx_i$ for all i,j. For short, we write

$$\mathcal{O}_{\mathbf{q}}(k^n) = k\langle x_1, \dots, x_n \mid x_ix_j = q_{ij}x_jx_i \text{ for } 1 \le i,j \le n\rangle.$$

(The assumptions on $\mathbf{q}$ mean that the relation $x_ix_i = q_{ii}x_ix_i$ is trivial and that the relation $x_jx_i = q_{ji}x_ix_j$ duplicates the relation $x_ix_j = q_{ij}x_jx_i$. This prevents undesired relations, such as $x_i^2 = 0$, from occurring.)

As a special case, fix $q \in k^\times$ and let $\mathbf{q}$ be the unique multiplicatively antisymmetric $n \times n$ matrix with $q_{ij} = q$ for all $i < j$. In this case, we use the subscript q in place of $\mathbf{q}$. Thus, $\mathcal{O}_q(k^n)$ is the k-algebra with generators $x_1, \dots, x_n$ and relations $x_ix_j = qx_jx_i$ for all $i < j$. It is called a *single parameter quantum n-space*.

Exercise 1K. Show that any quantum n-space can be expressed as an *iterated skew polynomial ring*, that is,

$$\mathcal{O}_{\mathbf{q}}(k^n) = k[x_1][x_2;\alpha_2][x_3;\alpha_3]\cdots[x_n;\alpha_n],$$

where $k[x_1]$ is an ordinary polynomial ring and α_i (for $i = 2, \dots, n$) is a k-algebra automorphism of $k[x_1][x_2; \alpha_2] \cdots [x_{i-1}; \alpha_{i-1}]$. $\square$

The simplest example of an enveloping algebra discussed in the Prologue arose from a 2-dimensional Lie algebra L with a basis $\{x, y\}$ such that $[yx] = x$. These elements generate the enveloping algebra $U(L)$, where the basic relation becomes $yx - xy = x$. This enveloping algebra can be exhibited as a skew polynomial ring in the following way.

Exercise 1L. Let A be the algebra over a field k presented by two elements x and y and the relation $yx-xy = x$. Show that $A = k[y][x; \alpha]$, where α is the k-algebra automorphism of the polynomial ring $k[y]$ such that $\alpha(y) = y-1$. $\square$

• SKEW-LAURENT RINGS •

The discusion of group algebras in the Prologue led us to the idea of a twisted version of a Laurent polynomial ring. Such a ring would look very much like the skew polynomial rings we have just developed, except that the indeterminate would now be invertible, i.e., negative as well as positive powers would occur. Making the obvious modifications to our definition of skew polynomial rings, we now define skew-Laurent (polynomial) rings.

Definition. Let R be a ring and α an automorphism of R. We write

$$T = R[x^{\pm 1}; \alpha]$$

to mean that

(a) T is a ring, containing R as a subring;
(b) x is an invertible element of T;
(c) T is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \dots\}$;
(d) $xr = \alpha(r)x$ for all $r \in R$.

When $T = R[x^{\pm 1}; \alpha]$, we say that S is a *skew-Laurent ring over* R, or a *skew-Laurent extension of* R.

Exercise 1M. Let α be an automorphism of a ring R.
(a) Show that a skew-Laurent ring $R[x^{\pm 1}; \alpha]$ exists.
(b) If $T = R[x^{\pm 1}; \alpha]$ and $S = \sum_{i=0}^{\infty} Rx^i \subseteq T$, show that S is a subring of T and that $S = R[x; \alpha]$. $\square$

Skew-Laurent rings satisfy a universal mapping property and are consequently unique up to isomorphism, as follows.

Exercise 1N. Let α be an automorphism of a ring R and $T = R[x^{\pm 1}; \alpha]$.
(a) Suppose that we have a ring U, a ring homomorphism $\phi : R \to U$, and a unit $y \in U$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Show that there is a unique ring homomorphism $\psi : T \to U$ such that $\psi|_R = \phi$ and $\psi(x) = y$.
(b) If $U = R[y^{\pm 1}; \alpha]$, show that there is a unique ring isomorphism $\psi : T \to U$ such that $\psi(x) = y$ and $\psi|_R$ is the identity map on R.

[Hint: Modify the proofs of Lemma 1.11 and Corollary 1.12.] □

To take an example from group algebras, let k be a field and let H be the *Heisenberg group*, which is presented by three generators x, y, z and the relations

$$xyx^{-1}y^{-1} = z \qquad\qquad xz = zx \qquad\qquad yz = zy.$$

Elements of H can be uniquely written as products $z^i y^j x^m$ for integers i, j, m, and so these products form a basis for the group algebra $k[H]$. Since y and z commute, the subalgebra of $k[H]$ generated by $y^{\pm 1}$ and $z^{\pm 1}$ is an ordinary Laurent polynomial ring, $k[y^{\pm 1}, z^{\pm 1}]$. Observe that since $xyx^{-1} = zy$ and $xzx^{-1} = z$, we have $x\big(k[y^{\pm 1}, z^{\pm 1}]\big)x^{-1} = k[y^{\pm 1}, z^{\pm 1}]$. In fact, conjugating elements of $k[y^{\pm 1}, z^{\pm 1}]$ by x has the same effect as applying the k-algebra automorphism α such that $\alpha(y) = zy$ and $\alpha(z) = z$, that is, $xr = \alpha(r)x$ for all $r \in k[y^{\pm 1}, z^{\pm 1}]$. Since the products $z^i y^j x^m$ form a basis for $k[H]$ over k, the powers x^m form a basis for $k[H]$ as a free left module over $k[y^{\pm 1}, z^{\pm 1}]$. Therefore we conclude that

$$k[H] = k[y^{\pm 1}, z^{\pm 1}][x^{\pm 1}; \alpha],$$

a skew-Laurent extension of a Laurent polynomial ring.

Definition. Let k be a field and $q \in k^\times$. The *quantized coordinate ring of* $(k^\times)^2$ (corresponding to the choice of q) is the k-algebra $\mathcal{O}_q((k^\times)^2)$ presented by generators x, x', y, y' and relations

$$xx' = x'x = yy' = y'y = 1 \qquad\qquad xy = qyx.$$

In brief, we may say that $\mathcal{O}_q((k^\times)^2)$ is presented by generators $x^{\pm 1}$ and $y^{\pm 1}$ satisfying $xy = qyx$. In algebraic geometry, $(k^\times)^2$ is known as an *algebraic torus* (of rank 2), and hence $\mathcal{O}_q((k^\times)^2)$ picks up the nickname *quantum torus*.

Exercise 1O. For k a field and $q \in k^\times$, show that $\mathcal{O}_q((k^\times)^2) = k[y^{\pm 1}][x^{\pm 1}; \alpha]$, where $k[y^{\pm 1}]$ is an ordinary Laurent polynomial ring and α is the k-algebra automorphism of $k[y^{\pm 1}]$ such that $\alpha(y) = qy$. Conclude, in particular, that the subalgebra of $\mathcal{O}_q((k^\times)^2)$ generated by x and y coincides with $\mathcal{O}_q(k^2)$ (rather than just being a homomorphic image of it). □

Definition. Let k be a field and $\mathbf{q} = (q_{ij})$ a multiplicatively antisymmetric $n \times n$ matrix over k. The corresponding *multiparameter quantum torus* is the k-algebra $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ presented by generators $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ and relations $x_i x_j = q_{ij} x_j x_i$ for all i, j. The single parameter version $\mathcal{O}_q((k^\times)^n)$, for $q \in k^\times$, is the special case when $q_{ij} = q$ for all $i < j$.

Exercise 1P. Show that any quantum torus is an *iterated skew-Laurent ring* in parallel with Exercise 1K. □

• A SKEW HILBERT BASIS THEOREM •

We derive a version of the Hilbert Basis Theorem for the skew polynomial rings $R[x;\alpha]$ discussed above; an analogous result for skew-Laurent rings will follow as a corollary.

Theorem 1.14. *Let α be an automorphism of a ring R and $S = R[x;\alpha]$. If R is right (left) noetherian, then so is S.*

Proof. **Case I**. Let us first assume that R is right noetherian and prove that any nonzero right ideal I of S is finitely generated. We follow the steps in the proof of Theorem 1.9, but some details require extra care.

Step 1. Let J be the set of leading coefficients of elements of I, together with 0:

$$J = \{r \in R \mid rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I \text{ for some } r_{d-1}, \ldots, r_0 \in R\}.$$

As before, it is easy to see that J is an additive subgroup of R. Now consider elements $r \in J$ and $a \in R$; we need to show that $ra \in J$. There is some skew polynomial of the form $p = rx^d + [\text{lower terms}]$ in I. While $pa \in I$, this does not help us, since $pa = r\alpha^d(a)x^d + [\text{lower terms}]$, which only yields $r\alpha^d(a) \in J$. To obtain ra instead, we should replace a by $\alpha^{-d}(a)$. More precisely, we have $p\alpha^{-d}(a) \in I$ and

$$p\alpha^{-d}(a) = rax^d + [\text{lower terms}],$$

whence $ra \in J$. This shows that J is a right ideal of R.

Step 2. Since R is right noetherian, J is finitely generated; say $r_1, \ldots, r_k$ is a finite list of nonzero generators for J. There exist $p_1, \ldots, p_k \in I$ such that p_i has leading coefficient r_i and some degree n_i. Set $n = \max\{n_1, \ldots, n_k\}$, and note that $p_i x^{n-n_i}$ is an element of I with leading coefficient r_i but with degree n. Thus, there is no loss of generality in assuming that all the p_i have the same degree n, that is,

$$p_i = r_i x^n + [\text{lower terms}].$$

Step 3. Set $N = R + Rx + \cdots + Rx^{n-1}$, the set of elements of S with degree less than n. Observe that $N = R + xR + \cdots + x^{n-1}R$, since

$$\begin{aligned} b_0 + b_1x + \cdots + b_{n-1}x^{n-1} &= b_0 + x\alpha^{-1}(b_1) + \cdots + x^{n-1}\alpha^{1-n}(b_{n-1}) \\ c_0 + xc_1 + \cdots + x^{n-1}c_{n-1} &= c_0 + \alpha(c_1)x + \cdots + \alpha^{n-1}(c_{n-1})x^{n-1} \end{aligned}$$

for all $b_j, c_j \in R$. Consequently, N is a right (as well as left) R-submodule of S. Viewed as a right R-module, N is finitely generated, and so it is noetherian by Corollary 1.4. Hence, its submodule $I \cap N$ is a finitely generated right R-module; say $q_1, \ldots, q_t$ generate $I \cap N$.

Step 4. Let I_0 be the right ideal of S generated by $p_1, \ldots, p_k, q_1, \ldots, q_t$. Then $I_0 \subseteq I$, and we claim that they are equal. If $p \in I$ with degree less than n, then $p \in I \cap N$ and $p = q_1 a_1 + \cdots + q_t a_t$ for some $a_j \in R$, whence $p \in I_0$.

Step 5. Now consider some $p \in I$ with degree $m \geq n$, and suppose that all elements of I with degree less than m lie in I_0. Let r be the leading coefficient of p; thus

$$p = rx^m + [\text{lower terms}].$$

Since $p \in I$, its leading coefficient r is in J, and so $r = r_1 a_1 + \cdots + r_k a_k$ for some $a_i \in R$. We wish to construct an element of I_0 which also has degree m and leading coefficient r, but the combination $(p_1 a_1 + \cdots + p_k a_k) x^{m-n}$ that we used in the proof of Theorem 1.9 no longer works. The problem and solution are the same as in Step 1 – we should apply appropriate negative powers of α to the a_i. More precisely, observe that

$$p_i \alpha^{-n}(a_i) = r_i a_i x^n + [\text{lower terms}]$$

for all i. Consequently, if $q = \big(p_1 \alpha^{-n}(a_1) + \cdots + p_k \alpha^{-n}(a_k)\big) x^{m-n}$, then $q \in I_0$ and

$$q = rx^m + [\text{lower terms}].$$

Now $p - q$ is an element of I with degree less than m. By the induction hypothesis, $p - q \in I_0$, and thus $p \in I_0$.

This induction has shown that $I = I_0$, so that I is finitely generated. Therefore, S is right noetherian.

Case II. Assume now that R is left noetherian, and let I be an arbitrary nonzero left ideal of S. Here one should try to follow the line of Case I just enough to understand the difficulties. There is a pitfall right at the beginning – in Case II, the set J as defined in Step 1 need not be closed under addition. (The problem is that, since we are only allowed to multiply elements of I on the *left* by powers of x, we cannot guarantee that an element of J which occurs as the leading coefficient of some element of I with degree d will also occur as a leading coefficient for elements of I with degrees greater than d.)

The way around such difficulties is to reverse the order of multiplication in *all* our expressions – including those that just display coefficients of skew polynomials. In other words, for the duration of the proof of Case II, all elements of S should be written with *right-hand coefficients* (that this is always possible is shown by equations like those displayed in Step 3). Note that this changes the definition of "leading coefficient" (but not that of "degree") for elements of S. With this change, analogs of Steps 1–5 are easily carried out; we leave the details to the reader. A more efficient way to deal with the switch from left-hand to right-hand coefficients is to work with opposite rings – see Exercise 1Q. $\square$

Immediate consequences of Theorem 1.14 are that the quantum planes $\mathcal{O}_q(k^2)$ are noetherian and (by induction) the quantum n-spaces $\mathcal{O}_{\mathbf{q}}(k^n)$ are noetherian.

Exercise 1Q. Let α be an automorphism of a ring R. Show that α^{-1} is an automorphism of the opposite ring R^{op} and that $R[x;\alpha]^{\mathrm{op}} = R^{\mathrm{op}}[x;\alpha^{-1}]$. Use this result to show that the left noetherian case of Theorem 1.14 follows immediately from the right noetherian case. $\square$

Corollary 1.15. *Let α be an automorphism of a ring R and $T = R[x^{\pm 1};\alpha]$. If R is right (left) noetherian, then so is T.*

Proof. Set $S = R[x;\alpha]$ and remember that S is a subring of T. We proceed by relating the right ideals of T to those of S, as follows.

Claim: If I is a right ideal of T, then $I \cap S$ is a right ideal of S and $I = (I \cap S)T$.

It is clear that $I \cap S$ is a right ideal of S and that $(I \cap S)T \subseteq I$. If $p \in I$, then

$$p = a_m x^m + a_{m+1}x^{m+1} + \cdots + a_n x^n$$

for some integers $m \le n$ and coefficients $a_i \in R$. Since $px^{-m} \in I \cap S$ and $p = (px^{-m})x^m$, we see that $p \in (I \cap S)T$, and the claim is proved.

Now suppose that R is right noetherian, and let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of right ideals of T. Then $I_1 \cap S \subseteq I_2 \cap S \subseteq \cdots$ is an ascending chain of right ideals of S. Since S is right noetherian by Theorem 1.14, there is an index n such that $I_m \cap S = I_n \cap S$ for all $m \ge n$. Thus

$$I_m = (I_m \cap S)T = (I_n \cap S)T = I_n$$

for all $m \ge n$, which establishes the ACC for right ideals of T. Therefore T is right noetherian.

The left noetherian case is proved symmetrically. $\square$

From this corollary we immediately obtain that all quantum tori $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ are noetherian.

Exercise 1R. Here is another way to obtain Corollary 1.15 from Theorem 1.14. Let $S = R[x;\alpha]$ and $T = R[x^{\pm 1};\alpha]$, where α is an automorphism of R. Show that the rule $\beta(s) = x^{-1}sx$ defines an automorphism β of S, and then use the universal mapping property of $S[y;\beta]$ to show that the inclusion map $S \to T$ extends to a ring homomorphism $\phi : S[y;\beta] \to T$ such that $\phi(y) = x^{-1}$. Conclude that $T \cong S[y;\beta]/\ker(\phi)$. Thus, if R is, say, right noetherian, then two applications of Theorem 1.14 show that $S[y;\beta]$ is right noetherian, and therefore T is right noetherian. $\square$

With the help of Corollary 1.15, we can fill in the details of the theorem on group algebras of polycyclic-by-finite groups announced in the Prologue.

Theorem 1.16. [Hall] *If k is a field and G a polycyclic-by-finite group, then the group algebra $k[G]$ is a noetherian ring.*

Proof. By assumption, there exist subgroups

$$G_0 = (1) \subset G_1 \subset \cdots \subset G_n \subseteq G_{n+1} = G$$

such that each G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is infinite cyclic for $i = 1, \dots, n$, while G/G_n is finite. There is a corresponding ascending sequence of subalgebras

$$k[G_0] = k \subset k[G_1] \subset \cdots \subset k[G_n] \subseteq k[G],$$

and we shall prove that each $k[G_i]$ is noetherian. This is clear for $i = 0$.

Now let $1 \le i \le n$ and assume that $k[G_{i-1}]$ is noetherian. Choose a coset $G_{i-1}x$ which generates the infinite cyclic group G_i/G_{i-1}. Then G_i is the disjoint union of the cosets $G_{i-1}x^j$ for $j \in \mathbb{Z}$, and so the rule $(g, j) \mapsto gx^j$ gives a bijection $G_{i-1} \times \mathbb{Z} \to G_i$. Consequently,

$$k[G_i] = \bigoplus_{j\in\mathbb{Z}} \bigoplus_{g\in G_{i-1}} kgx^j = \bigoplus_{j\in\mathbb{Z}} \Big(\bigoplus_{g\in G_{i-1}} kg \Big) x^j = \bigoplus_{j\in\mathbb{Z}} k[G_{i-1}]x^j,$$

that is, $k[G_i]$ is a free left module over $k[G_{i-1}]$ with basis $\{x^j \mid j \in \mathbb{Z}\}$. Since G_{i-1} is a normal subgroup of G_i, we have $xG_{i-1}x^{-1} = G_{i-1}$, and hence $x(k[G_{i-1}])x^{-1} = k[G_{i-1}]$. As a result, the rule $\alpha(r) = xrx^{-1}$ defines an automorphism α of $k[G_{i-1}]$. By definition of α, we have $xr = \alpha(r)x$ for all $r \in k[G_{i-1}]$, and thus $k[G_i] = k[G_{i-1}][x^{\pm 1}; \alpha]$. Corollary 1.15 now shows that $k[G_i]$ is noetherian.

Thus, by induction, we conclude that $k[G_n]$ is noetherian. Now G is a finite union of cosets $G_ny_1, \dots, G_ny_t$, and so $k[G] = \sum_{j=1}^t \sum_{g\in G_n} kgy_j = \sum_{j=1}^t k[G_n]y_j$, that is, $k[G]$ is finitely generated as a left $k[G_n]$-module. Therefore $k[G]$ is left noetherian by Corollary 1.5, and by symmetry it is right noetherian as well. $\square$

• SIMPLICITY IN SKEW-LAURENT RINGS •

We conclude the chapter with a few considerations about (two-sided) ideals in skew polynomial and skew-Laurent rings. In the theory of commutative noetherian rings, ideals are ubiquitous, but this is no longer true in the noncommutative theory, because noncommutative noetherian rings need not have very many ideals. In the extreme case, there may be no ideals other than the two obvious ones.

Definition. A *simple ring* is any nonzero ring R such that the only ideals of R are 0 and R. (This terminology is only supposed to suggest that the ideal theory of R is simple, not that the structure of R is necessarily simple in any other respect.)

The only commutative simple rings are fields, but we shall soon find noncommutative noetherian simple rings which are not division rings. The examples most immediately accessible to us at this point are skew-Laurent rings of the following type.

Exercise 1S. Let $T = K[x^{\pm 1}; \alpha]$, where K is a field and α is an automorphism of K with infinite order (i.e., no nonzero power of α is the identity). Show that T is a simple ring as follows.

Let I be a nonzero ideal of T, and pick a nonzero $p \in I \cap K[x; \alpha]$ of minimal degree, say degree n. Note that the constant term of p is nonzero (for otherwise p could be replaced by px^{-1}). Observe that for any $r \in K$, the difference $pr - \alpha^n(r)p$ is an element of $I \cap K[x; \alpha]$ with degree at most $n - 1$, and so $pr - \alpha^n(r)p = 0$. Compare constant terms and conclude that $\alpha^n(r) = r$ for all $r \in K$. Consequently, $n = 0$, whence p is a scalar, and therefore $I = T$. $\square$

Definition. Let α be an automorphism of a ring R. An *α-ideal* of R is any ideal I of R that is stable under α, that is, $\alpha(I) = I$. The ring R is said to be *α-simple* provided R is nonzero and its only α-ideals are 0 and R.

Exercise 1T. Let $S = R[x; \alpha]$ and $T = R[x^{\pm 1}; \alpha]$, where α is an automorphism of R. Let I be an α-ideal of R and $\hat{\alpha}$ the automorphism of R/I induced by α.

(a) Show that $IS = SI$ and $IT = TI$. Thus, IS is an ideal of S and IT is an ideal of T. Show also that $IS \cap R = IT \cap R = I$, and conclude that the rings $(R + IS)/IS \subseteq S/IS$ and $(R + IT)/IT \subseteq T/IT$ are isomorphic to R/I.

(b) Show that $S/IS \cong (R/I)[\hat{x}; \hat{\alpha}]$ and $T/IT \cong (R/I)[\hat{x}^{\pm 1}; \hat{\alpha}]$. $\square$

Exercise 1U. Let $T = R[x^{\pm 1}; \alpha]$, where α is an automorphism of R. Suppose that some positive power α^n is an inner automorphism of R; say there is a unit $u \in R$ such that $\alpha^n(r) = uru^{-1}$ for all $r \in R$. Set $v = u\alpha(u)\alpha^2(u) \cdots \alpha^{n-1}(u)$, and show that $\alpha(v) = v$. [Hint: Check that $\alpha^i(u)r\alpha^i(u)^{-1} = \alpha^n(r)$ for all $i \in \mathbb{Z}$ and $r \in R$, and show that u commutes with all $\alpha^i(u)$.] Next, show that $v^{-1}x^{n^2}$ lies in the center of T. [At first glance, one might expect to use $u^{-1}x^n$, which commutes with all elements of R; however, $u^{-1}x^n$ need not commute with x.] Finally, show that $1 + v^{-1}x^{n^2}$ is not invertible in T, and conclude that $T(1 + v^{-1}x^{n^2})$ is a proper, nonzero (two-sided) ideal of T. $\square$

A skew polynomial ring $S = R[x; \alpha]$ has no chance to be simple, since Sx is always a nontrivial ideal of S (as are $Sx^2, Sx^3, \dots$). If a skew-Laurent ring $T = R[x^{\pm 1}; \alpha]$ is to be simple, Exercise 1T shows that R needs to be α-simple, and Exercise 1U shows that no positive power (equivalently, no nonzero power) of α can be inner. These are the only conditions that need to be satisfied, as the next theorem shows.

Theorem 1.17. *Let $T = R[x^{\pm 1}; \alpha]$, where α is an automorphism of R. Then T is a simple ring if and only if the following hold:*

(a) *R is an α-simple ring.*

(b) *No positive power of α is an inner automorphism of R.*

Proof. As noted above, Exercises 1T and 1U show the necessity of conditions (a) and (b). Conversely, assume that (a) and (b) hold.

Let I be a nonzero ideal of T; we must show that $I = T$. Set $S = R[x;\alpha]$ and recall from the proof of Corollary 1.15 that $I = (I \cap S)T$. Thus, $I \cap S \neq 0$. Since I is an ideal in T, we are allowed to multiply it by either x or x^{-1}. Thus $xIx^{-1} \subseteq I$ and $x^{-1}Ix \subseteq I$, whence $xIx^{-1} = I$. We also have $xSx^{-1} = S$ (cf. Exercise 1R), and therefore $x(I \cap S)x^{-1} = I \cap S$.

Let n be the least degree that occurs for nonzero elements of $I \cap S$, and set

$$J = \{r \in R \mid rx^n + r_{n-1}x^{n-1} + \cdots + r_0 \in I \cap S \text{ for some } r_{n-1}, \ldots, r_0 \in R\}.$$

As in Step 1 of Theorem 1.14, we check that J is an ideal of R, nonzero by choice of n. Given $r \in J$, there is a skew polynomial $p \in I \cap S$ of the form $p = rx^n + [\text{lower terms}]$. The skew polynomial $xpx^{-1} = \alpha(r)x^n + [\text{lower terms}]$ also lies in $I \cap S$, whence $\alpha(r) \in J$. Hence, $\alpha(J) \subseteq J$, and a similar argument shows that $\alpha^{-1}(J) \subseteq J$. Thus, $\alpha(J) = J$.

Now J is a nonzero α-ideal of R. Since R is α-simple, we must have $J = R$, whence $1 \in J$. Therefore there is an element $p \in I \cap S$ of the form

$$p = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with the $a_i \in R$. If $a_0 = 0$, then $px^{-1} = x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$ would be a nonzero element of $I \cap S$ with degree $n-1$, contradicting the minimality of n. Hence, $a_0 \neq 0$. Observe that

$$xpx^{-1} = x^n + \alpha(a_{n-1})x^{n-1} + \cdots + \alpha(a_0),$$

and so $xpx^{-1} - p$ is an element of $I \cap S$ with degree at most $n-1$. The minimality of n implies that $xpx^{-1} - p = 0$, and thus $\alpha(a_i) = a_i$ for all i.

Next, consider an arbitrary element $r \in R$, and note that

$$\begin{aligned} pr &= \alpha^n(r)x^n + a_{n-1}\alpha^{n-1}(r)x^{n-1} + \cdots + a_0r \\ \alpha^n(r)p &= \alpha^n(r)x^n + \alpha^n(r)a_{n-1}x^{n-1} + \cdots + \alpha^n(r)a_0. \end{aligned}$$

Then $pr - \alpha^n(r)p$ is an element of $I \cap S$ with degree at most $n-1$, and so $pr - \alpha^n(r)p = 0$. In particular, it follows that $a_0r = \alpha^n(r)a_0$. Since this holds for any $r \in R$, we see that $a_0R \subseteq Ra_0$. On the other hand, taking $r = \alpha^{-n}(r')$ yields $r'a_0 = a_0\alpha^{-n}(r')$ for all $r' \in R$, whence $Ra_0 \subseteq a_0R$. Therefore $a_0R = Ra_0$.

Now $a_0R = Ra_0$ is a nonzero two-sided ideal of R, and it is an α-ideal because $\alpha(a_0) = a_0$. Since R is α-simple, we find that $a_0R = Ra_0 = R$, which tells us that a_0 is invertible in R. Consequently, the equations $a_0r = \alpha^n(r)a_0$ imply that α^n is an inner automorphism of R. Assumption (b) then forces $n = 0$. But now $p = 1$, and since $p \in I$, we conclude that $I = T$. Therefore T is a simple ring. $\square$

Corollary 1.18. *Let k be a field and $q \in k^\times$. Then $\mathcal{O}_q((k^\times)^2)$ is a simple ring if and only if q is not a root of unity.*

Proof. Set $T = \mathcal{O}_q((k^\times)^2)$. By Exercise 1O, $T = R[x^{\pm 1}; \alpha]$, where $R = k[y^{\pm 1}]$ and α is the k-algebra automorphism of R such that $\alpha(y) = qy$. Since R is commutative, the only inner automorphism of R is the identity.

If q is a root of unity, say $q^n = 1$ for some positive integer n, then α^n is the identity on R, and so Theorem 1.17 shows that T is not simple in this case.

Conversely, assume that q is not a root of unity. Then $\alpha^n(y) = q^n y \neq y$ for all $n > 0$, and so no positive power of α is inner. It remains to verify condition (a) of Theorem 1.17. Thus, let I be a nonzero α-ideal of R; we must show that $I = R$.

Observe that $I \cap k[y]$ is nonzero, and choose a monic polynomial $f \in I \cap k[y]$ of minimal degree, say $f = y^m + a_{m-1}y^{m-1} + \cdots + a_0$ for some $m \in \mathbb{Z}^+$ and $a_i \in k$. Since I is an α-ideal, we also have $\alpha(f) \in I \cap k[y]$. Now

$$\alpha(f) = q^m y^m + q^{m-1} a_{m-1} y^{m-1} + \cdots + a_0,$$

and so $\alpha(f) - q^m f$ is a polynomial in $I \cap k[y]$ with degree at most $m - 1$. By the minimality of m, we must have $\alpha(f) - q^m f = 0$, from which it follows that $q^i a_i = q^m a_i$ for all i, that is, $(q^{m-i} - 1)a_i = 0$. Since q is not a root of unity, we conclude that $a_i = 0$ for all $i \neq m$. Consequently, $f = y^m$, which is invertible in R. Therefore $I = R$, as desired. $\square$

Observe that the quantum tori $\mathcal{O}_q((k^\times)^2)$ are never division rings – for instance, $x + 1$ has no inverse in these algebras.

Exercise 1V. Let $R = k[y]$ be a polynomial ring over a field k of characteristic zero and α the k-algebra automorphism of R such that $\alpha(y) = y + 1$. Show that $R[x^{\pm 1}; \alpha]$ is simple. $\square$

• ADDITIONAL EXERCISES •

1W. Suppose that $\alpha_1, \dots, \alpha_n$ are commuting automorphisms of a ring R (that is, $\alpha_i \alpha_j = \alpha_j \alpha_i$ for all i, j).

(a) Construct an iterated skew polynomial ring from these data as follows. First, set $S_1 = R[x_1; \alpha_1]$. Next, show that α_2 extends uniquely to an automorphism $\hat{\alpha}_2$ of S_1 such that $\hat{\alpha}_2(x_1) = x_1$, and set $S_2 = S_1[x_2; \hat{\alpha}_2]$. Similarly, once S_i has been constructed for some $i < n$, construct $S_{i+1} = S_i[x_{i+1}; \hat{\alpha}_{i+1}]$, where $\hat{\alpha}_{i+1}$ is the unique automorphism of S_i such that $\hat{\alpha}_{i+1}|_R = \alpha_{i+1}$ and $\hat{\alpha}_{i+1}(x_j) = x_j$ for $j = 1, \dots, i$. Finally, let

$$S = S_n = R[x_1; \alpha_1][x_2; \hat{\alpha}_2] \cdots [x_n; \hat{\alpha}_n].$$

A standard notation for S, which indicates that the basic data are defined on R, is

$$S = R[x_1, \dots, x_n;\ \alpha_1, \dots, \alpha_n].$$

Note that $x_i x_j = x_j x_i$ for all i, j, and that $x_i r = \alpha_i(r)x_i$ for all i and all $r \in R$.

(b) State and prove a universal mapping property for S analogous to Lemma 1.11.

(c) Show that $S = R[x_{\pi(1)}, \dots, x_{\pi(n)};\ \alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}]$ for any permutation π of the index set $\{1, \dots, n\}$.

(d) Construct an analogous iterated skew-Laurent ring from R and the α_i. It may be denoted $R[x_1^{\pm 1}, \dots, x_n^{\pm 1};\ \alpha_1, \dots, \alpha_n]$. $\square$

1X. Let $S = R[x; \alpha]$ and $T = R[x^{\pm 1}; \alpha]$, where α is an automorphism of R. Suppose that α is inner; say there is a unit $u \in R$ with $\alpha(r) = u^{-1}ru$ for all $r \in R$. Show that $S = R[ux]$, an ordinary polynomial ring (that is, ux is a central indeterminate). Similarly, show that $T = R[(ux)^{\pm 1}]$, an ordinary Laurent polynomial ring. $\square$

1Y. Show that $R[x^{\pm 1}; \alpha]^{\mathrm{op}} = R^{\mathrm{op}}[x^{\pm 1}; \alpha^{-1}]$ if α is an automorphism of a ring R. (Cf. Exercise 1Q.) $\square$

1Z. The point of this exercise is to see what can be salvaged from Theorem 1.17 if the skew-Laurent ring $T = R[x^{\pm 1}; \alpha]$ is replaced by a skew polynomial ring $S = R[x; \alpha]$, where we continue to assume that α is an automorphism of R. We have already noted that S always has some nontrivial ideals, namely $Sx, Sx^2, \dots$.

(a) If I is any ideal of R, show that $Ix^n + Sx^{n+1}$ is an ideal of S for each $n \in \mathbb{N}$. Thus, if R is not simple, S has nonzero ideals not generated by powers of x.

(b) If R is simple and no positive power of α is inner, show that the only nonzero ideals of S are $S, Sx, Sx^2, \dots$.

(c) Now assume only that R is α-simple and no positive power of α is inner. Show that every nonzero ideal of S contains a power of x. [Hint: Given an ideal I of S, show that $\bigcup_{j=0}^{\infty} x^{-j} I x^{-j}$ is an ideal of T.] $\square$

1ZA. Let R be a ring and α an automorphism of R.

(a) Construct a *skew power series ring* $R[[x; \alpha]]$ whose elements are formal power series $\sum_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where $xr = \alpha(r)x$ for all $r \in R$.

(b) Construct a *skew Laurent series ring* $R((x; \alpha))$ consisting of formal Laurent series $\sum_{i=n}^{\infty} r_i x^i$ with $n \in \mathbb{Z}$ and coefficients $r_i \in R$, where $xr = \alpha(r)x$ for all $r \in R$.

(c) If R is right (left) noetherian, show that $R[[x; \alpha]]$ and $R((x; \alpha))$ are right (left) noetherian. $\square$

1ZB. Prove Proposition 1.2 using the condition "all submodules finitely generated" in place of the ACC on submodules. $\square$

• NOTES •

Noetherian Rings. The introduction of commutative rings with the ACC, and the source of their subsequent influence, was a paper of Noether [1921].

ACC Versus Finite Generation. Noether proved that a commutative ring has the ACC on ideals if and only if all ideals are finitely generated [1921, Satz I and comment following].

The Hilbert Basis Theorem. Hilbert's original basis theorems concerned homogeneous ideals in polynomial rings over fields and over $\mathbb{Z}$ [1890, Theorems I, II].

Skew Polynomial Rings. Skew polynomial rings in several variables with coefficients in a field K were introduced by Noether and Schmeidler [1920]; one of the cases they were particularly interested in was $K[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n]$, where K consists of (C^∞) functions in variables $y_1, \dots, y_n$ and each α_i is the automorphism of K sending y_i to $y_i + 1$ and fixing the other y_j.

Quantum Planes. The algebras $\mathcal{O}_q(k^2)$ were introduced (with a different notation) by Manin in [1987, §1.2].

Skew-Laurent Rings. The skew Laurent series ring $\mathbb{Q}(t)((x; \alpha))$, where α is the automorphism of $\mathbb{Q}(t)$ sending t to $2t$, was constructed by Hilbert to show the existence of a noncommutative ordered division ring [1903, Theorem 39].

Noetherian Skew Polynomial Rings. Finite generation of left ideals for skew polynomial rings in several variables over a field was proved by Noether and Schmeidler [1920, Satz III].

Noetherian Group Algebras. Theorem 1.16 was first proved by Hall in the case that the coefficient ring k is $\mathbb{Z}$ [1954, Theorem 1 and following remarks], but the proof for k a field is the same.

Simple Skew-Laurent Rings. Special cases were known well before a general criterion was formulated. For instance, Jacobson observed in [1956, p. 211] that if α is an automorphism of infinite order of a field K, then $K[x^{\pm 1}; \alpha]$ is simple, and Shamsuddin extended this result to the case where K is an α-simple commutative noetherian domain [1977, Theorem 3.1]. Other cases were subsumed in work on skew group rings. Theorem 1.17 as we have stated it was proved by Jordan in his dissertation [1975] but not published until [1984, Theorem 1].

2. Skew Polynomial Rings

We continue the study of skew polynomial rings in this chapter, first discussing the case where the multiplication is twisted by a derivation, and then developing the general case. Since our main motivation for looking at skew polynomial rings is to be able to construct and work with further important examples of noetherian rings, most of the later part of the book could be read independently of this chapter. Readers who are interested in immediately getting into the general theory of noetherian rings should feel free to skip to Chapter 3 and return to this chapter later.

• FORMAL DIFFERENTIAL OPERATOR RINGS •

Several of the examples discussed in the Prologue appear as polynomial rings in which multiplication by the indeterminate is twisted by a derivation rather than by an automorphism. This situation has several new features – in particular, the characteristic of the ring plays an important role – but it is still significantly simpler than the general case, in which both an automorphism and a derivation act. Thus, we begin the chapter by studying the derivation case. Since many of the results in this section are parallel to ones in Chapter 1 and will be special cases of later results for general skew polynomial rings, we leave most of the proofs as exercises.

Definition. A *derivation* on a ring R is any map $\delta : R \to R$ satisfying the usual sum and product rules for derivatives: $\delta(r+s) = \delta(r) + \delta(s)$ and $\delta(rs) = \delta(r)s + r\delta(s)$ for all $r, s \in R$. (If R is noncommutative, it is important to keep the order of the terms involving r and s consistent in the product rule.) Note that

$$\delta(1) = \delta(1{\cdot}1) = \delta(1)1 + 1\delta(1) = 2\delta(1),$$

whence we automatically have $\delta(1) = 0$.

For example, if R is the ring of all C^∞ real-valued functions on the real line, then the classical derivative d/dx is a derivation on R, and similarly if R is the ring of real polynomial functions on the line. The latter case can be mimicked for any polynomial ring:

Exercise 2A. Let $R = K[x]$ be a polynomial ring over an arbitrary ring K. Show that the rule

$$\delta\Big(\sum_i a_i x^i\Big) = \sum_i i a_i x^{i-1}$$

defines a derivation δ on R. It is convenient to label this derivation d/dx. Similarly, show that if $R = K[x_1, \dots, x_n]$ is a polynomial ring in n indeterminates, the partial derivative operations $\partial/\partial x_i$ give derivations on R. $\square$

Let R be a ring, δ a derivation on R, and x an indeterminate. Let S be the set of all formal expressions $a_0 + a_1 x + \cdots + a_n x^n$, where $n \in \mathbb{Z}^+$ and the $a_i \in R$, and define addition on S in the usual way. As suggested by examples from the Prologue, we would like to build a multiplication in S such that $xa = ax + \delta(a)$ for all $a \in R$. To fully describe this multiplication, we must iterate the above rule, which leads us to the formula $x^i a = \sum_{l=0}^{i} \binom{i}{l} \delta^{i-l}(a) x^l$ for $i \in \mathbb{Z}^+$ and $a \in R$. Thus, we define multiplication in S as follows:

$$\begin{aligned}\Big(\sum_i a_i x^i\Big)\Big(\sum_j b_j x^j\Big) &= \sum_{i,j} \sum_{l=0}^{i} \tbinom{i}{l} a_i \delta^{i-l}(b_j) x^{l+j} \\ &= \sum_k \Big(\sum_{l=0}^{k} \sum_{i \geq l} \tbinom{i}{l} a_i \delta^{i-l}(b_{k-l})\Big) x^k.\end{aligned}$$

Exercise 2B. Verify that the set S together with the operations discussed above is a ring, containing R as a subring. Give a formal description of S without using the symbol x, analogous to Exercise 1H. $\square$

Definition. Let R be a ring and δ a derivation on R. We write $S = R[x; \delta]$ to mean that

(a) S is a ring, containing R as a subring;
(b) x is an element of S;
(c) S is a free left R-module with basis $\{1, x, x^2, \dots\}$;
(d) $xr = rx + \delta(r)$ for all $r \in R$.

In this situation, we say that S is a *skew polynomial ring over* R or a *formal differential operator ring over* R. The latter terminology is prompted by the discussion in the Prologue and will be further justified below. It is also a helpful way to distinguish between $R[x; \delta]$ and the skew polynomial rings $R[x; \alpha]$ studied in the previous chapter.

Note that, given R and δ, Exercise 2B shows that there does exist a differential operator ring $R[x; \delta]$.

For example, let $R = k[y]$ be a polynomial ring over a field k, and let $\delta = y(d/dy)$, the unique k-linear derivation on R such that $\delta(y) = y$. In $R[x; \delta]$, we thus have the key relation $xy = yx + y$. We have met the algebra $R[x; \delta]$ before (although with x and y interchanged) – it is the enveloping algebra of the 2-dimensional Lie algebra over k with basis $\{x, y\}$ such that $[xy] = y$. The point of the following exercise is to verify this assertion.

Exercise 2C. Let S be the algebra over a field k presented by two generators x and y and one relation $xy - yx = y$. Show that $S = k[y][x; \delta]$, where $k[y]$ is a polynomial ring over k and $\delta = y(d/dy)$. $\square$

Now consider the polynomial ring $\mathbb{R}[y]$ over $\mathbb{R}$, and let $\delta = d/dy$ be the usual derivation on $\mathbb{R}[y]$. (In parallel with Exercise 2C, $\mathbb{R}[y][x; \delta]$ may be presented as the $\mathbb{R}$-algebra with generators x, y and relation $xy - yx = 1$. We shall discuss algebras with such presentations more generally in the following section.) Transformations of the form

$$a_0 + a_1\delta + \cdots + a_n\delta^n : \mathbb{R}[y] \to \mathbb{R}[y],$$

for $a_0, \ldots, a_n \in \mathbb{R}[y]$, are known as *linear differential operators* (with polynomial coefficients) on $\mathbb{R}[y]$. Since these operators have the same format as elements of the skew polynomial ring $\mathbb{R}[y][x; \delta]$, let us define a map θ from $\mathbb{R}[y][x; \delta]$ to $\mathrm{End}_{\mathbb{R}}(\mathbb{R}[y])$, the algebra of $\mathbb{R}$-linear transformations on the vector space $\mathbb{R}[y]$, by the rule

$$\theta\Big(\sum_i a_i x^i\Big) = \sum_i a_i \delta^i.$$

It is not hard to check that θ is an injective ring homomorphism (Exercise 2D), and so we can say that $\mathbb{R}[y][x; \delta]$ is isomorphic to the algebra of polynomial-coefficient linear differential operators on $\mathbb{R}[y]$.

Exercise 2D. Show that the map $\theta : \mathbb{R}[y][x; \delta] \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}[y])$ discussed above is an injective ring homomorphism. $\square$

Now consider a derivation δ on an arbitrary ring R. We can again define a map θ, this time from $R[x; \delta]$ to $\mathrm{End}_{\mathbb{Z}}(R)$, using the rule $\theta(\sum_i a_i x^i) = \sum_i a_i \delta^i$. In general, θ is still a ring homomorphism, but it need not be injective. For example, if $R = k[y]$, where k is a field of characteristic 2 and $\delta = d/dy$, then $\theta(x^2) = \delta^2 = 0$. Thus, the elements of $R[x; \delta]$ need not correspond precisely to the linear differential operators $\sum_i a_i \delta^i$, and to emphasize the distinction we refer to elements of $R[x; \delta]$ as *formal (linear) differential operators* on R. In spite of the example above, it is still possible to represent $R[x; \delta]$ in terms of actual differential operators, but on a ring larger than R, as follows.

Exercise 2E. Let δ be a derivation on a ring R. Form a polynomial ring $T = R[z_0, z_1, \ldots]$ in an infinite sequence of independent indeterminates over R, and show that there is a derivation ∂ on T such that $\partial(r) = \delta(r)$ for $r \in R$ and $\partial(z_j) = z_{j+1}$ for all j. (A convenient way to describe this derivation is to write $\partial = \delta + \sum_{j=0}^{\infty} z_{j+1}(\partial/\partial z_j)$.) Show that the rule $\xi(\sum_i a_i x^i) = \sum_i a_i \partial^i$ defines an injective ring homomorphism $\xi : R[x; \delta] \to \mathrm{End}_{\mathbb{Z}}(T)$. $\square$

As in the case of skew polynomial rings twisted by automorphisms, formal differential operator rings satisfy universal mapping properties and so are unique up to isomorphism. The proofs are parallel to Lemma 1.11 and Corollary 1.12.

Exercise 2F. Let δ be a derivation on a ring R and $S = R[x; \delta]$.

(a) Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi(r)y + \phi\delta(r)$ for all $r \in R$. Show that there is a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.

(b) Given $S' = R[x'; \delta]$, show that there is a unique ring isomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R$ is the identity on R. $\square$

In the proof of the Hilbert Basis Theorem that we gave in the last chapter (Theorem 1.9), the main items that we had to keep track of in computations were degrees and leading coefficients. The proof became more complicated for skew polynomial rings twisted by automorphisms (Theorem 1.14) mainly because in that case multiplying a polynomial by the indeterminate can change the leading coefficient. However, that cannot happen in a differential operator ring $R[x; \delta]$, because

$$x\big(a_n x^n + [\text{lower terms}]\big) = a_n x^{n+1} + [\text{lower terms}].$$

Consequently, the proof of Theorem 1.9 works without modifications in the differential operator case.

Exercise 2G. The point of this exercise is to check the details involved in recycling the proof of Theorem 1.9 in the case of a differential operator ring. Let δ be a derivation on a ring R. If R is right (left) noetherian, prove that $R[x; \delta]$ is right (left) noetherian. $\square$

Exercise 2G can be applied inductively to *iterated differential operator rings*

$$R[x_1; \delta_1][x_2; \delta_2] \cdots [x_n; \delta_n],$$

where δ_1 is a derivation on R, while δ_2 is a derivation on $R[x_1; \delta_1]$, etc. (Rings of this sort can arise as enveloping algebras of solvable Lie algebras.) A case of particular interest is the following analog of Exercise 1W.

Exercise 2H. Suppose that $\delta_1, \dots, \delta_n$ are commuting derivations on a ring R.

(a) Set $S_1 = R[x_1; \delta_1]$. Next, show that δ_2 extends uniquely to a derivation $\hat{\delta}_2$ on S_1 such that $\hat{\delta}_2(x_1) = 0$, and set $S_2 = S_1[x_2; \hat{\delta}_2]$. Similarly, once S_i has been constructed for some $i < n$, construct $S_{i+1} = S_i[x_{i+1}; \hat{\delta}_{i+1}]$, where $\hat{\delta}_{i+1}$ is the unique derivation on S_i such that $\hat{\delta}_{i+1}|_R = \delta_{i+1}$ and $\hat{\delta}_{i+1}(x_j) = 0$ for $j = 1, \dots, i$. Finally, let

$$S = S_n = R[x_1; \delta_1][x_2; \hat{\delta}_2] \cdots [x_n; \hat{\delta}_n].$$

The standard notation is $S = R[x_1, \dots, x_n;\ \delta_1, \dots, \delta_n]$.

(b) State and prove a universal mapping property for S analogous to Exercise 2F(a).

(c) Show that $S = R[x_{\pi(1)}, \dots, x_{\pi(n)};\ \delta_{\pi(1)}, \dots, \delta_{\pi(n)}]$ for any permutation π of the index set $\{1, \dots, n\}$. $\square$

• WEYL ALGEBRAS •

A fundamental class of skew polynomial rings is formed by taking differential operator rings over polynomial rings with respect to the standard (ordinary or partial) derivatives, as follows.

Definition. Let $K[y]$ be a polynomial ring over an arbitrary ring K, and let d/dy be the standard derivation on $K[y]$. The formal differential operator ring $K[y][x; d/dy]$ is called *the (first) Weyl algebra over* K and is denoted $A_1(K)$. (Unless K is commutative, the term "algebra" is out of place, but is commonly used.)

The Weyl algebra $A_1(K)$ is generated (as a ring) by the elements of K together with x and y, which commute with the elements of K and satisfy the equation $xy = yx + 1$. Rather than adapt the terminology of presentations to include "rings presented over subrings," let us phrase this description of $A_1(K)$ in terms of a universal property.

Exercise 2I. Let K and T be rings, $\phi : K \to T$ a ring homomorphism, and $u, v \in T$. Suppose that $uv - vu = 1$, and that u and v commute with all elements of $\phi(K)$. Show that ϕ extends uniquely to a ring homomorphism $\psi : A_1(K) \to T$ such that $\psi(x) = u$ and $\psi(y) = v$. In particular, if K is a field, this shows that $A_1(K)$ is the K-algebra presented by two generators x, y and one relation $xy - yx = 1$. $\square$

Definition. Let $K[y_1, \dots, y_n]$ be a polynomial ring in n independent indeterminates over a ring K. The formal partial derivatives $\partial/\partial y_1, \dots, \partial/\partial y_n$ are commuting derivations on $K[y_1, \dots, y_n]$, and so we may form the iterated differential operator ring

$$K[y_1, \dots, y_n][x_1, \dots, x_n;\ \partial/\partial y_1, \dots, \partial/\partial y_n].$$

This ring is called the n^{th} *Weyl algebra over* K and is denoted $A_n(K)$.

In view of Exercise 2G, if K is a right (left) noetherian ring, then all the Weyl algebras $A_n(K)$ are right (left) noetherian. Note also that it follows from the iterated differential operator construction that $A_n(K)$, for $n > 1$, can be identified with $A_1(A_{n-1}(K))$.

Exercise 2J. Formulate and prove a universal property for $A_n(K)$ analogous to Exercise 2I. $\square$

We now develop a simplicity criterion for differential operator rings, in order to show that certain Weyl algebras are simple rings. This is similar to but actually easier than Theorem 1.17, except that we have to restrict ourselves to characteristic zero to avoid problems with binomial coefficients, which occur because of the following formulas.

Exercise 2K. Let δ be a derivation on a ring R. Verify the *Leibniz Rule*:

$$\delta^n(rs) = \sum_{i=0}^{n} \tbinom{n}{i} \delta^{n-i}(r)\delta^i(s)$$

for $n \in \mathbb{N}$ and $r, s \in R$. Similarly, show that $x^n r = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(r) x^i$ in $R[x;\delta]$. Finally, if $\delta(r)$ commutes with r, show that $\delta(r^n) = nr^{n-1}\delta(r)$. □

Definition. Let R be a ring and $a \in R$. The rule $\delta_a(r) = ar - ra$ defines a derivation δ_a on R, called the *inner derivation induced by* a. Any derivation on R which is not an inner derivation is called an *outer derivation*.

Definition. Let δ be a derivation on a ring R. A δ*-ideal* of R is any ideal I of R such that $\delta(I) \subseteq I$. The ring R is called δ*-simple* if R is nonzero and the only δ-ideals of R are 0 and R.

Exercise 2L. Let δ be a derivation on a ring R, and set $S = R[x;\delta]$.

(a) If $\delta = \delta_a$ for some $a \in R$, show that $S = R[x - a]$, an ordinary polynomial ring, and conclude that $S(x - a)$ is a proper nonzero ideal of S.

(b) If I is a δ-ideal of R, show that $IS = SI$ and so IS is an ideal of S. If $I \neq R$ (respectively, $I \neq 0$), show that $IS \neq S$ (respectively, $IS \neq 0$). □

Proposition 2.1. *Let R be a $\mathbb{Q}$-algebra and δ a derivation on R. Then $R[x;\delta]$ is a simple ring if and only if R is δ-simple and δ is outer.*

Proof. Set $S = R[x;\delta]$. Exercise 2L shows that if either R is not δ-simple or δ is inner, then S is not simple. Conversely, assume that R is δ-simple and δ is outer.

Let I be a nonzero ideal of S, and let n be the minimum degree for nonzero elements of I. Let J be the subset of R consisting of 0 together with the leading coefficients of those elements of I which have degree n, and check that J is a nonzero ideal of R. We claim that J is a δ-ideal of R. Any nonzero $r \in J$ is the leading coefficient of some $p \in I$ with degree n, that is, $p = rx^n + r'x^{n-1} + [\text{lower terms}]$. Observe that $xp - px \in I$ and that

$$\begin{aligned} xp - px &= \big(rx^{n+1} + \delta(r)x^n + r'x^n + [\text{lower terms}]\big) \\ &\qquad - \big(rx^{n+1} + r'x^n + [\text{lower terms}]\big) \\ &= \delta(r)x^n + [\text{lower terms}]. \end{aligned}$$

Hence, $\delta(r) \in J$, and thus J is a δ-ideal, as claimed.

Since R is δ-simple, $J = R$. Hence, I contains an element q with degree n and leading coefficient 1. If $n = 0$, then $q = 1$ and $I = S$. We will show that the assumption that $n > 0$ leads to a contradiction.

Write $q = x^n + ax^{n-1} + [\text{lower terms}]$ for some $a \in R$. For any $r \in R$, observe that $rq - qr \in I$ and (using Exercise 2K) that

$$\begin{aligned} rq - qr &= \big(rx^n + rax^{n-1} + [\text{lower terms}]\big) \\ &\qquad - \big(rx^n + n\delta(r)x^{n-1} + arx^{n-1} + [\text{lower terms}]\big) \\ &= (ra - n\delta(r) - ar)x^{n-1} + [\text{lower terms}]. \end{aligned}$$

By the minimality of n, we must have $rq - qr = 0$, and hence $ra - n\delta(r) - ar = 0$. Since $n > 0$ and R is a $\mathbb{Q}$-algebra, we obtain $\delta(r) = (-a/n)r - r(-a/n)$ for all $r \in R$, contradicting the assumption that δ is outer.

Thus $n = 0$ and $I = S$. Therefore S is a simple ring. $\square$

Corollary 2.2. *If R is a simple $\mathbb{Q}$-algebra, then all the Weyl algebras $A_n(R)$ are simple rings.*

Proof. Recall that $A_n(R) \cong A_1(A_{n-1}(R))$ for all $n \geq 2$. Hence, it suffices to prove that A_1 of any simple $\mathbb{Q}$-algebra is simple.

Thus, consider $A_1(R) = R[y][x;\delta]$, where y is an indeterminate and $\delta = d/dy$. As $\delta(y) \neq 0$ and y is central in $R[y]$, we see that δ cannot be an inner derivation of $R[y]$. We next show that $R[y]$ is δ-simple.

Let I be any nonzero δ-ideal of $R[y]$, let n be the minimum degree for nonzero elements of I, and choose $p \in I$ with degree n. If p has leading coefficient r, then

$$\delta(p) = nry^{n-1} + [\text{lower terms}].$$

Since $\delta(p) \in I$, the minimality of n forces $\delta(p) = 0$, and so $nr = 0$. As $r \neq 0$ and R is a $\mathbb{Q}$-algebra, $n = 0$, whence p is a nonzero element of R. Now $RpR = R$ (because R is simple), and hence $I = R[y]$. Thus $R[y]$ is δ-simple.

By Proposition 2.1, $A_1(R)$ is a simple ring. $\square$

In characteristic $p > 0$, the Weyl algebras are not simple rings. For example, if k is a field of characteristic p, then $(d/dy)(y^p) = 0$ in the polynomial ring $k[y]$, whence y^p is in the center of the Weyl algebra $A_1(k)$. Consequently, $y^pA_1(k)$ is a nontrivial ideal of $A_1(k)$.

• GENERAL SKEW POLYNOMIAL RINGS •

As we have already pointed out in the Prologue, examples such as the enveloping algebra of $\mathfrak{sl}_2(k)$ and the quantized coordinate ring of $M_2(k)$ suggest that we should consider skew polynomial rings which have twists coming from automorphisms and derivations acting together. Also, for efficiency's sake, it is good to develop a context in which the two different types of skew polynomial rings we have already discussed can be treated simultaneously. Let us begin by asking how far we can go without losing the basic patterns we have seen.

Thus, let us analyze our requirements for a ring S to be a twisted polynomial ring in one indeterminate x over some ring R. If we continue to ask that polynomials be presented with left-hand coefficients, then elements of S should have the form $r_0 + r_1x + \cdots + r_nx^n$ for $n \in \mathbb{Z}^+$ and $r_i \in R$. In order for coefficients and degrees to be well defined, we should require that two polynomials cannot be equal unless their coefficients agree. This means that, so far, we are making the familiar demand that S be a free left R-module with basis $\{1, x, x^2, \dots\}$.

A further basic requirement that presents itself is that multiplication of polynomials should respect degrees as far as possible; in particular, the degree of a product of polynomials should not be larger than the sum of the degrees of the factors. To satisfy this requirement, it suffices to ensure that, for each $r \in R$, the product xr, when written with left-hand coefficients, has degree at most 1. In other words, $xr = r'x + r''$ for some $r', r'' \in R$. Since the coefficients of a given polynomial are supposed to be unique, r' and r'' should depend uniquely on r. Thus, there must be maps, say $\alpha, \delta : R \to R$, such that $xr = \alpha(r)x + \delta(r)$ for all $r \in R$. Now we may ask what properties are imposed on α and δ by the fact that we are working inside a ring.

The distributive law requires that $x(r+s) = xr + xs$ for all $r, s \in R$, and it follows that α and δ must be additive maps. From $x1 = x$ we obtain $\alpha(1) = 1$ and $\delta(1) = 0$. The associative law requires that $x(rs) = (xr)s$ for $r, s \in R$, whence

$$\alpha(rs)x + \delta(rs) = \big(\alpha(r)x + \delta(r)\big)s = \alpha(r)\big(\alpha(s)x + \delta(s)\big) + \delta(r)s.$$

Thus, we must have $\alpha(rs) = \alpha(r)\alpha(s)$ and $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for $r, s \in R$. In particular, α must be a ring endomorphism of R. The map δ is similar to a derivation except for the appearance of α, and it is called a *skew derivation* relative to α.

Definition. Let α be an endomorphism of a ring R. An *α-derivation* on R is any additive map $\delta : R \to R$ such that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. (Strictly speaking, we have defined a *left α-derivation*, but we shall not need the concept of a *right α-derivation*, which is any additive map $\delta : R \to R$ satisfying the rule $\delta(rs) = \delta(r)\alpha(s) + r\delta(s)$.) As in the case of an ordinary derivation, we do not need to include the condition $\delta(1) = 0$ in the definition of an α-derivation, since it follows from the skew product rule.

If α is the identity map on R, then α-derivations are just ordinary derivations. Here is a nontrivial example.

Exercise 2M. Let $k[y]$ be a polynomial ring over a field k, let $q \in k^\times$ with $q \neq 1$, and let α be the k-algebra automorphism of $k[y]$ such that $\alpha(y) = qy$. Show that the rule

$$\delta\big(f(y)\big) = \frac{f(qy) - f(y)}{qy - y} = \frac{\alpha(f) - f}{\alpha(y) - y}$$

defines an α-derivation on $k[y]$. This skew derivation is known as the *q-difference operator* or *Eulerian derivative*. $\square$

The discussion above leads us to try to construct a skew polynomial ring in which the multiplication is twisted by a ring endomorphism and associated skew derivation. Our goal may be defined in parallel with the earlier cases, as follows.

Definition. Let R be a ring, α a ring endomorphism of R, and δ an α-derivation on R. We shall write $S = R[x; \alpha, \delta]$ provided

(a) S is a ring, containing R as a subring;
(b) x is an element of S;
(c) S is a free left R-module with basis $\{1, x, x^2, \dots\}$;
(d) $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.

Such a ring S is called a *skew polynomial ring over* R, or an *Ore extension of* R (honoring O. Ore, who first systematically studied the general case).

The reader should be warned that some authors prefer their skew polynomial rings to have *right-hand* coefficients. To achieve this, one starts with a ring R, an endomorphism α of R, and a *right* α-derivation δ on R. The corresponding skew polynomial ring is a free *right* R-module with a basis $\{1, x, x^2, \dots\}$, where $rx = x\alpha(r) + \delta(r)$ for all $r \in R$.

So far, the definition just given is only a wish list – we have not proved that any ring of the form $R[x; \alpha, \delta]$ exists, nor that it would be unique up to isomorphism. These are our next tasks.

In order to proceed as we did in the cases $R[x; \alpha] = R[x; \alpha, 0]$ and $R[x; \delta] = R[x; \mathrm{id}_R, \delta]$, we would need to work out a general formula expressing $x^i r$, for any $i \in \mathbb{N}$ and $r \in R$, as a polynomial with left-hand coefficients. However, this soon gets rather involved – for instance,

$$x^3 r = \alpha^3(r)x^3 + \big[\delta\alpha^2(r) + \alpha\delta\alpha(r) + \alpha^2\delta(r)\big]x^2 \\ + \big[\delta^2\alpha(r) + \delta\alpha\delta(r) + \alpha\delta^2(r)\big]x + \delta^3(r).$$

Moreover, once an explicit multiplication rule for $R[x; \alpha, \delta]$ is written down, the process of checking the ring axioms is a very messy business. Thus, we change our tack – let us try to prove the existence of $R[x; \alpha, \delta]$ without explicitly defining the multiplication. After all, based on our experience so far, we will probably only need to know how to compute the leading (and occasionally next highest) coefficients of products in this ring.

Exercise 2E provides a clue as to how we might proceed. The point of that exercise was to show that any formal differential operator ring $R[x; \delta]$ is isomorphic to a ring of actual differential operators on a ring T. If we had not already constructed $R[x; \delta]$, we could proceed to define the ring T and the derivation ∂ as in Exercise 2E, identify R with its image in $\mathrm{End}_{\mathbb{Z}}(T)$ (as left multiplication operators), and then check that the subring of $\mathrm{End}_{\mathbb{Z}}(T)$

generated by $R \cup \{\partial\}$ is the required skew polynomial ring $R[x;\delta]$. Thus, let us try to build $R[x;\alpha,\delta]$ as a ring of operators (i.e., additive endomorphisms) of some abelian group. In particular, such a construction will give us the ring axioms for free.

Still anticipating the existence of $S = R[x;\alpha,\delta]$, we observe that S will embed in the additive endomorphism ring $\mathrm{End}_{\mathbb{Z}}(S)$ as left multiplication operators. To express elements of R in this fashion only requires us to know the R-module structure of S, and multiplication by x will be given by the rule

$$x\Big(\sum_i r_i x^i\Big) = \sum_i \big(\alpha(r_i)x + \delta(r_i)\big)x^i = \sum_i \big(\alpha(r_i)x^{i+1} + \delta(r_i)x^i\big).$$

In other words, we can readily express both x and elements of R as operators on the additive group of S, and can then construct S as the ring generated by these operators. To avoid confusion between the two roles S plays here – as abelian group and as skew polynomial ring – it is helpful to rewrite the abelian group $(S,+)$ as a polynomial ring in a new variable, say z.

Proposition 2.3. *Given a ring R, a ring endomorphism α of R, and an α-derivation δ on R, there exists a skew polynomial ring $R[x;\alpha,\delta]$.*

Proof. Let $E = \mathrm{End}_{\mathbb{Z}}(R[z])$, where $R[z]$ is an ordinary polynomial ring over R. Since $R[z]$ is a left R-module, there is a ring homomorphism $\lambda : R \to E$ sending elements of R to left multiplication operators, that is, $\lambda(r)(p) = rp$ for $r \in R$ and $p \in R[z]$. Clearly, λ is injective (e.g., because $\lambda(r)(1) = r$ for all $r \in R$). Thus, we can identify R with the subring $\lambda(R) \subset E$.

Next, define $x \in E$ according to the rule

$$x\Big(\sum_i r_i z^i\Big) = \sum_i \big(\alpha(r_i)z^{i+1} + \delta(r_i)z^i\big),$$

and let S be the subring of E generated by $R \cup \{x\}$. For any $r \in R$ and any polynomial $p = \sum_i r_i z^i$ in $R[z]$, we compute that

$$\begin{aligned}
(xr)(p) &= x\big(\sum_i r r_i z^i\big) = \sum_i \big(\alpha(r r_i)z^{i+1} + \delta(r r_i)z^i\big)\\
&= \sum_i \alpha(r)\alpha(r_i)z^{i+1} + \sum_i \big(\alpha(r)\delta(r_i) + \delta(r)r_i\big)z^i\\
&= \alpha(r)\sum_i \big(\alpha(r_i)z^{i+1} + \delta(r_i)z^i\big) + \delta(r)\sum_i r_i z^i = \big(\alpha(r)x + \delta(r)\big)(p).
\end{aligned}$$

Thus $xr = \alpha(r)x + \delta(r)$ for all $r \in R$. In particular, $xR \subseteq Rx + R$.

From the relation $xR \subseteq Rx + R$, it follows by induction that

$$x^i R \subseteq Rx^i + Rx^{i-1} + \cdots + Rx + R$$

for all $i \in \mathbb{Z}^+$, and consequently $(Rx^i)(Rx^j) \subseteq Rx^{i+j} + Rx^{i+j-1} + \cdots + Rx^j$ for all $i, j \in \mathbb{Z}^+$. Hence, the set $\sum_{i=0}^{\infty} Rx^i$ is a subring of E, and therefore $S = \sum_{i=0}^{\infty} Rx^i$. This shows that the set $\{1, x, x^2, \dots\}$ generates S as a left R-module. All that remains is to show that this set is left linearly independent over R, so that S will be a free left R-module with basis $\{1, x, x^2, \dots\}$.

Thus, consider an operator $r_0 + r_1 x + \cdots + r_n x^n$ in S for some $r_i \in R$. We shall apply this operator to the element $1 = z^0 \in R[z]$. Note that $x(z^j) = z^{j+1}$ for all $j \geq 0$, whence $x^i(1) = z^i$ for all i. Consequently,

$$(r_0 + r_1 x + \cdots + r_n x^n)(1) = r_0 + r_1 z + \cdots + r_n z^n,$$

and so the operator $r_0 + r_1 x + \cdots + r_n x^n$ can be the zero map only if the polynomial $r_0 + r_1 z + \cdots + r_n z^n$ is zero, and that happens only if all the $r_i = 0$. Therefore, the elements $1, x, x^2, \dots$ are indeed left linearly independent over R, as required. $\square$

For example, we can now proclaim the existence of a skew polynomial ring $k[y][x; \alpha, \delta]$ with $k[y]$, α, δ as in Exercise 2M. Note that $xy = qyx + 1$ in this ring, a relation similar to the defining relation for the first Weyl algebra.

Definition. Let k be a field and $q \in k^\times$. We write $A_1^q(k)$ to denote the k-algebra presented by two generators x and y and one relation $xy - qyx = 1$. This algebra is known as a *quantized Weyl algebra* over k. (Quantized Weyl algebras with more pairs of generators have been defined and extensively studied, but we shall not introduce them here.) Of course, $A_1^q(k) = A_1(k) = k[y][x; d/dy]$ when $q = 1$. When $q \neq 1$, we obtain the skew polynomial ring given above.

Exercise 2N. Let k be a field and $q \in k^\times$ with $q \neq 1$. Show that $A_1^q(k) = k[y][x; \alpha, \delta]$, where α is the k-algebra automorphism $f(y) \mapsto f(qy)$ and δ is the q-difference operator of Exercise 2M. [Hints: First construct an appropriate skew polynomial ring $A' = k[y'][x'; \alpha', \delta']$ with new variables x', y', and use the universal property of $A_1^q(k)$ to obtain a k-algebra homomorphism $\phi : A_1^q(k) \to A'$ such that $\phi(x) = x'$ and $\phi(y) = y'$. Then use the method of Exercise 1J(a) to show that ϕ is an isomorphism. The latter step can be simplified once Proposition 2.4 is proved, since then the proposition can be used to show the existence of a k-algebra homomorphism $\psi : A' \to A_1^q(k)$ such that $\psi(x') = x$ and $\psi(y') = y$.] $\square$

Definition. Let $R[x; \alpha, \delta]$ be a skew polynomial ring. Any nonzero element p in $R[x; \alpha, \delta]$ can be uniquely expressed in the form

$$p = r_n x^n + r_{n-1} x^{n-1} + \cdots + r_1 x + r_0$$

for some nonnegative integer n and some elements $r_i \in R$ with $r_n \neq 0$. The integer n is called the *degree* of p, abbreviated $\deg(p)$, and the element

r_n is called the *leading coefficient* of p. (In the differential operator ring case, namely $R[x;\delta]$, it is common to call n the *order* of p rather than the degree.) The zero element of $R[x;\alpha,\delta]$ is defined to have degree $-\infty$ and leading coefficient 0.

Strictly speaking, n and r_n should be called the *left degree* and the *left leading coefficient* of p, since if p can be written with right-hand coefficients, that is,

$$p = x^m r'_m + x^{m-1} r'_{m-1} + \cdots + x r'_1 + r'_0$$

for some $r'_i \in R$ with $r'_m \neq 0$ (which is not always possible), it can easily happen that $n \neq m$ or that $r_n \neq r'_m$. The following exercise implies that if $r'_m \in \ker(\alpha^m)$, then $n < m$, while if $r'_m \notin \ker(\alpha^m)$, then $n = m$ and $r_n = \alpha^n(r'_n)$.

Exercise 2O. Let $R[x;\alpha,\delta]$ be a skew polynomial ring. Show that if $r \in R$ and $n \in \mathbb{N}$, then

$$x^n r = \alpha^n(r)x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + \delta^n(r)$$

for some $a_{n-1}, \ldots, a_1 \in R$. Hence, if $r \neq 0$ and α is injective, $x^n r$ has degree n and leading coefficient $\alpha^n(r)$.

Now show that if R is a domain and α is injective, then

$$\deg(pq) = \deg(p) + \deg(q)$$

for all $p, q \in R[x;\alpha,\delta]$, and consequently $R[x;\alpha,\delta]$ is a domain. (Since we shall mostly be concerned with skew polynomial rings for which α is an automorphism, injectivity of α will usually be available.) $\square$

Now let us show the uniqueness of general skew polynomial rings $R[x;\alpha,\delta]$. This follows from a universal mapping property exactly parallel to Lemma 1.11 and Exercise 2F(a). However, the proof of the universal mapping property for $R[x;\alpha,\delta]$ is a bit different because we do not have an explicit formula for products in this ring.

Proposition 2.4. *Let $S = R[x;\alpha,\delta]$ be a skew polynomial ring. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi\alpha(r)y + \phi\delta(r)$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proof. There is a well-defined additive map $\psi : S \to T$ given by the rule

$$\psi\Big(\sum_i r_i x^i\Big) = \sum_i \phi(r_i) y^i,$$

with $\psi|_R = \phi$ and $\psi(x) = y$. It is clear that this is the only possibility for ψ, and so it is enough to show that ψ is a ring homomorphism.

First observe that if $t = \sum_j b_j x^j$ is an arbitrary element of S, then

$$\begin{aligned}\psi(xt) &= \psi\Big(\sum_j \alpha(b_j)x^{j+1} + \sum_j \delta(b_j)x^j\Big) = \sum_j \phi\alpha(b_j)y^{j+1} + \sum_j \phi\delta(b_j)y^j \\ &= \sum_j \big(\phi\alpha(b_j)y + \phi\delta(b_j)\big)y^j = \sum_j y\phi(b_j)y^j = y\psi(t).\end{aligned}$$

It follows by induction that $\psi(x^i t) = y^i\psi(t)$ for all $i \in \mathbb{Z}^+$ and $t \in S$. Moreover, if $a \in R$, then

$$\psi(at) = \sum_j \phi(ab_j)y^j = \sum_j \phi(a)\phi(b_j)y^j = \phi(a)\psi(t).$$

Consequently, given any $s = \sum_i a_i x^i$ in S, we have

$$\psi(st) = \sum_i \psi(a_i x^i t) = \sum_i \phi(a_i)\psi(x^i t) = \sum_i \phi(a_i)y^i\psi(t) = \psi(s)\psi(t).$$

Therefore ψ is a ring homomorphism. □

Corollary 2.5. *Let R be a ring, α a ring endomorphism of R, and δ an α-derivation on R. If $S = R[x;\alpha,\delta]$ and $S' = R[x';\alpha,\delta]$, there is a unique ring isomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R$ is the identity map on R.*

Proof. As Corollary 1.12. □

• A GENERAL SKEW HILBERT BASIS THEOREM •

We now turn to the question of whether (or when) $R[x;\alpha,\delta]$ is noetherian. In our treatment of the case $R[x;\alpha]$ (Theorem 1.14), we made several uses of the hypothesis that α was an automorphism. In fact, that theorem can fail when α is not an automorphism, as the following examples show. Consequently, we shall mainly restrict attention to skew polynomial rings $R[x;\alpha,\delta]$ when α is an automorphism.

Exercise 2P. (a) Let $R = k[t]$ be a polynomial ring over a field k, and let α be the k-algebra endomorphism of R given by the rule $\alpha\big(f(t)\big) = f(t^2)$. Show that $R[x;\alpha]$ is neither right nor left noetherian. [Hint: Look at the right ideal $\sum_{i=0}^{\infty} x^i txR[x;\alpha]$ and the left ideal $\sum_{i=0}^{\infty} R[x;\alpha]tx^i$.]

(b) Now let $R = k(t)$ be the quotient field of $k[t]$ and extend α to the k-algebra endomorphism of R given by the same rule $\alpha\big(f(t)\big) = f(t^2)$. Show that $R[x;\alpha]$ is not right noetherian. In this case, however, $R[x;\alpha]$ is left noetherian, as will follow from Theorem 2.8. □

Theorem 2.6. *Let $S = R[x; \alpha, \delta]$, where α is an automorphism of R. If R is right (left) noetherian, then so is S.*

Proof. In the right noetherian case, we can follow the same steps as in Theorem 1.14, with some help from Exercise 2O to keep track of leading coefficients. The set equation $R + Rx + \cdots + Rx^{n-1} = R + xR + \cdots + x^{n-1}R$ in Step 3 still holds, although a bit more work is needed to check it (Exercise 2O is helpful there too).

Now suppose that R is left noetherian. Then R^{op} is right noetherian, and Exercise 2R below shows that $R[x; \alpha, \delta]^{\mathrm{op}} = R^{\mathrm{op}}[x; \alpha^{-1}, -\delta\alpha^{-1}]$, where α^{-1} is viewed as an automorphism of R^{op}. By the case above, $R[x; \alpha, \delta]^{\mathrm{op}}$ is right noetherian, and therefore $R[x; \alpha, \delta]$ is left noetherian. $\square$

Exercise 2Q. Check the details of the proof of the right noetherian case of Theorem 2.6. $\square$

Exercise 2R. Let $R[x; \alpha, \delta]$ be a skew polynomial ring, and assume that α is an automorphism of R. Show that α^{-1} is an automorphism of the opposite ring R^{op}, that $-\delta\alpha^{-1}$ is an α^{-1}-derivation of R^{op}, and that $R[x; \alpha, \delta]^{\mathrm{op}} = R^{\mathrm{op}}[x; \alpha^{-1}, -\delta\alpha^{-1}]$. $\square$

For example, Theorem 2.6 shows that the quantized Weyl algebras $A_1^q(k)$ are noetherian domains. We shall discuss other examples in the following section.

Corollary 2.7. *Let $S = R[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$ be an iterated skew polynomial ring, where each α_i is an automorphism of the ring $R[x_1; \alpha_1, \delta_1] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$. If R is right (left) noetherian, then so is S.* $\square$

We conclude this section with a skew polynomial analog of the fact that a polynomial ring in one indeterminate over a field is a principal ideal domain. In the noncommutative case, we distinguish between the condition that all left ideals are principal and the right-hand analog.

Definition. A *principal right ideal domain* is a domain in which all right ideals are principal. *Principal left ideal domains* are defined analogously.

Theorem 2.8. *Let $S = R[x; \alpha, \delta]$, where R is a division ring. Then S is a principal left ideal domain. If α is an automorphism of R, then S is also a principal right ideal domain.*

Proof. Note that as R is a division ring, α must be injective, whence S is a domain.

Given a nonzero left ideal J of S, let m be the minimum degree for nonzero elements of J, and choose $p \in J$ with degree m. If r is the leading coefficient of p, then p may be replaced by $r^{-1}p$, and so there is no loss of generality in assuming that p has leading coefficient 1.

We claim that $J = Sp$. Obviously $Sp \subseteq J$, and we prove the reverse inclusion by induction on degree. The only element of J with degree less than m is 0, and certainly $0 \in Sp$. Now assume, for some integer $k \geq m$, that all elements of J with degree less than k lie in Sp. Let q be any element of J with degree k, and let a be the leading coefficient of q. Now $ax^{k-m}p$ has degree k and leading coefficient a, whence $q - ax^{k-m}p$ is an element of J with degree less than k. By the induction hypothesis, $q - ax^{k-m}p$ lies in Sp, and so $q \in Sp$. This completes the induction step, proving that $J = Sp$.

Therefore S is a principal left ideal domain.

The final statement of the theorem now follows with the help of Exercise 2R. $\square$

For example, Theorem 2.8 shows that the skew polynomial ring $k(t)[x;\alpha]$ discussed in Exercise 2P(b) is a principal left ideal domain. Since this ring is not right noetherian, it cannot be a principal right ideal domain. In fact, one can easily show that the right ideal generated by x and tx is not principal.

Another example is the differential operator ring $k(y)[x; d/dy]$, where $k(y)$ is a rational function field. (The derivation d/dy can be extended from $k[y]$ to $k(y)$ by the usual quotient rule; cf. Exercise 2ZD.) By Theorem 2.8, this ring is a principal right and left ideal domain.

• SOME EXAMPLES •

Now that the formalities of general skew polynomial rings have been set up, let us take a fresh look at the "mixed case" examples discussed in the Prologue – algebras that lead to skew polynomial rings $R[x;\alpha,\delta]$ in which α and δ are both nontrivial. Throughout this section, k will denote a base field for our algebras.

Recall that the standard basis for the Lie algebra $\mathfrak{sl}_2(k)$ is typically labelled $\{e, f, h\}$, where $[ef] = h$ and $[he] = 2e$ while $[hf] = -2f$. Consequently, the enveloping algebra $U(\mathfrak{sl}_2(k))$ is the k-algebra presented by three generators e, f, h and three relations

$$ef - fe = h \qquad\qquad he - eh = 2e \qquad\qquad hf - fh = -2f.$$

Let R be the subalgebra of $U(\mathfrak{sl}_2(k))$ generated by e and h. There are two ways to exhibit R as a skew polynomial ring, corresponding to Exercises 1L and 2C:

$$R = k[e][h;\delta_1] = k[h][e;\alpha_1],$$

where $k[e]$ and $k[h]$ are polynomial rings, δ_1 denotes the derivation $2e(d/de)$ on $k[e]$, and α_1 is the k-algebra automorphism of $k[h]$ such that $\alpha_1(h) = h-2$. (To make R exactly match the situations in Exercises 1L and 2C, at least when $\mathrm{char}(k) \neq 2$, replace h by $\frac{1}{2}h$.)

Since $fe = ef - h$ and $fh = (h+2)f$, it is easily checked that $fR \subseteq Rf + R$. We thus anticipate $U(\mathfrak{sl}_2(k))$ to be a skew polynomial ring $R[f;\alpha_2,\delta_2]$, where $\alpha_2(e) = e$ and $\alpha_2(h) = h + 2$, while $\delta_2(e) = -h$ and $\delta_2(h) = 0$. It is not hard to check that an automorphism α_2 and an α_2-derivation δ_2 on R with these properties do exist, so that we can form a skew polynomial ring $R[x; \alpha_2, \delta_2]$. Either of the methods given in Exercise 1J can be used to show that the identity map on R extends to a k-algebra isomorphism $U(\mathfrak{sl}_2(k)) \to R[x; \alpha_2, \delta_2]$ such that $f \mapsto x$, and therefore $U(\mathfrak{sl}_2(k)) = R[f; \alpha_2, \delta_2]$. To summarize:

Exercise 2S. Show that there are iterated skew polynomial presentations

$$U(\mathfrak{sl}_2(k)) = k[e][h; \delta_1][f; \alpha_2, \delta_2] = k[h][e; \alpha_1][f; \alpha_2, \delta_2],$$

where α_1, α_2, δ_1, δ_2 are the k-algebra automorphisms and (skew) derivations discussed above. Conclude, in particular, that $U(\mathfrak{sl}_2(k))$ is a noetherian domain. $\square$

In the quantized version of $U(\mathfrak{sl}_2(k))$, the generator h is replaced by an invertible generator K that is meant to model the exponential of λh for some scalar λ. If we operate formally with the expressions $K = \sum_{n=0}^{\infty} \lambda^n h^n/n!$ and $q = \sum_{n=0}^{\infty} \lambda^n/n!$, then, with the help of the relation $he = eh + 2e$ and the Leibniz Rule (Exercise 2K) for the derivation $2e(d/de)$, we are led to the equation $Ke = q^2eK$. These formal manipulations (which would require characteristic zero and other assumptions to be made precise) are only meant to motivate the choice of $Ke = q^2eK$ as a relation in the following example.

Definition. Now let $q \in k^\times$ (with k an arbitrary field) be any nonzero scalar such that $q \neq \pm 1$. The *quantized enveloping algebra of* $\mathfrak{sl}_2(k)$ corresponding to the choice of q is the k-algebra $U_q(\mathfrak{sl}_2(k))$ presented by four generators E, F, K, K^{-1} and five relations

$$KK^{-1} = K^{-1}K = 1 \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$
$$KE = q^2EK \qquad KF = q^{-2}FK.$$

Exercise 2T. Show that $U_q(\mathfrak{sl}_2(k))$ can be expressed as an iterated skew polynomial ring of the form $k[E][K^{\pm 1}; \alpha_1][F; \alpha_2, \delta_2]$. Conclude, in particular, that $U_q(\mathfrak{sl}_2(k))$ is a noetherian domain. $\square$

From the viewpoint of algebraic geometry, the set $M_2(k)$ of 2×2 matrices over k is just a 4-dimensional affine space. Its coordinate ring, $\mathcal{O}(M_2(k))$, is thus a polynomial ring in four indeterminates, usually written x_{11}, x_{12}, x_{21}, x_{22}, where x_{ij} corresponds to the function $M_2(k) \to k$ that records the (i, j)-entries of matrices. One thus expects a quantized coordinate ring of $M_2(k)$ to look like an iterated skew polynomial ring in four indeterminates. To motivate the choice of a particular iterated skew polynomial ring, one observes that the maps

$$M_2(k) \times k^2 \longrightarrow k^2 \qquad\qquad k^2 \times M_2(k) \longrightarrow k^2$$

given by matrix multiplication (where elements of k^2 are written as column vectors for the first map and as row vectors for the second) correspond to k-algebra homomorphisms

$$\begin{aligned}\lambda : \mathcal{O}(k^2) &\longrightarrow \mathcal{O}(M_2(k)) \otimes_k \mathcal{O}(k^2)\\ \rho : \mathcal{O}(k^2) &\longrightarrow \mathcal{O}(k^2) \otimes_k \mathcal{O}(M_2(k))\end{aligned}$$

such that

$$\lambda(x_i) = x_{i1} \otimes x_1 + x_{i2} \otimes x_2 \qquad\qquad \rho(x_j) = x_1 \otimes x_{1j} + x_2 \otimes x_{2j} \qquad (\dagger)$$

for all i, j. We have already discussed the quantized coordinate ring $\mathcal{O}_q(k^2)$, for $q \in k^\times$. Having chosen this algebra as our "quantization" of $\mathcal{O}(k^2)$, it is natural to seek a replacement $\mathcal{O}_q(M_2(k))$ for $\mathcal{O}(M_2(k))$ with four generators x_{ij} such that there exist k-algebra homomorphisms analogous to λ and ρ which satisfy the equations (†). These requirements force the x_{ij} to satisfy certain relations, which are then used to define $\mathcal{O}_q(M_2(k))$, as follows.

Definition. Given any $q \in k^\times$, the corresponding *quantized coordinate ring of* $M_2(k)$ is the k-algebra $\mathcal{O}_q(M_2(k))$ presented by four generators x_{11}, x_{12}, x_{21}, x_{22} and the six relations

$$\begin{aligned} x_{11}x_{12} &= qx_{12}x_{11} & x_{12}x_{22} &= qx_{22}x_{12}\\ x_{11}x_{21} &= qx_{21}x_{11} & x_{21}x_{22} &= qx_{22}x_{21}\\ x_{12}x_{21} &= x_{21}x_{12} & x_{11}x_{22} - x_{22}x_{11} &= (q - q^{-1})x_{12}x_{21}. \end{aligned}$$

This algebra is also called the *coordinate ring of quantum* 2×2 *matrices* over k, or the 2×2 *quantum matrix algebra* over k.

Exercise 2U. For any choice of $q \in k^\times$, show that there exist k-algebra homomorphisms

$$\begin{aligned}\lambda : \mathcal{O}_q(k^2) &\longrightarrow \mathcal{O}_q(M_2(k)) \otimes_k \mathcal{O}_q(k^2)\\ \rho : \mathcal{O}_q(k^2) &\longrightarrow \mathcal{O}_q(k^2) \otimes_k \mathcal{O}_q(M_2(k))\end{aligned}$$

such that the equations (†) hold for all i, j. Conversely, suppose that A is a k-algebra with generators x_{11}, x_{12}, x_{21}, x_{22} and that there exist k-algebra homomorphisms

$$\lambda : \mathcal{O}_q(k^2) \longrightarrow A \otimes_k \mathcal{O}_q(k^2) \qquad\qquad \rho : \mathcal{O}_q(k^2) \longrightarrow \mathcal{O}_q(k^2) \otimes_k A$$

such that (†) holds for all i, j. If $q^4 \neq 1$, show that the $x_{ij} \in A$ must satisfy the six relations in the definition of $\mathcal{O}_q(M_2(k))$. $\square$

The subalgebra of $\mathcal{O}_q(M_2(k))$ generated by x_{11}, x_{12}, x_{21} has the form of an iterated skew polynomial ring twisted by automorphisms, in fact a "quantum 3-space" $\mathcal{O}_{\mathbf{q}}(k^3)$ for suitable $\mathbf{q}$. It is only when the generator x_{22} is put into the picture that a twist by a skew derivation is needed.

Exercise 2V. Show that $\mathcal{O}_q(M_2(k))$ can be expressed as an iterated skew polynomial ring of the form $k[x_{11}][x_{12}; \alpha_{12}][x_{21}; \alpha_{21}][x_{22}; \alpha_{22}, \delta_{22}]$. $\square$

Since the group $SL_2(k)$ consists of those points of $M_2(k)$ where the determinant function $x_{11}x_{22} - x_{12}x_{21}$ is equal to 1, this group is also an algebraic variety, and its coordinate ring is the algebra

$$\mathcal{O}(SL_2(k)) = \mathcal{O}(M_2(k))/\langle x_{11}x_{22} - x_{12}x_{21} - 1\rangle.$$

In the noncommutative algebra $\mathcal{O}_q(M_2(k))$, the element $x_{11}x_{22} - x_{12}x_{21}$ is replaced by the *quantum determinant* $D_q := x_{11}x_{22} - qx_{12}x_{21}$ for reasons that we do not discuss here, except to point out the following property.

Exercise 2W. Show that the element D_q lies in the center of $\mathcal{O}_q(M_2(k))$. $\square$

Definition. The *quantized coordinate ring of* $SL_2(k)$, or *the (coordinate ring of) quantum* $SL_2(k)$, is the factor algebra $\mathcal{O}_q(SL_2(k)) = \mathcal{O}_q(M_2(k))/\langle D_q - 1\rangle$. This is a noetherian ring because $\mathcal{O}_q(M_2(k))$ is noetherian, and in fact it is a domain (Exercise 2X).

Exercise 2X. This exercise provides some sufficient conditions for the factor of a skew polynomial ring by a centrally generated principal ideal to be a domain, in order to show that $\mathcal{O}_q(SL_2(k))$ is a domain. Let $S = R[x; \alpha, \delta]$ be a skew polynomial ring where R is a domain and α is an automorphism; thus S is a domain. Assume that we have a central element $d \in S$ of the form $d = ax + b$, where $a, b \in R$ and $a \neq 0$. Note first that, since d is central, $\alpha(a) = a$ and $ra = \alpha(r)a$ for all $r \in R$; thus, $Ra = aR$ is an α-ideal of R. Assume further that R/Ra is a domain and that $b \notin Ra$.

(a) Show that if $s \in S$ and $as \in dS$, then $s \in dS$. [Hint: If $as = dt$, show that $bt \in aS$ and then that $t \in aS$.]

Now assume that $\bigcap_{n=0}^{\infty} a^n R = 0$.

(b) Show that if $r \in R$ and $s \in S$ with $rs \in dS$, then either $r = 0$ or $s \in dS$. [Hint: If the leading coefficient of s is in aR, try to write $s = dt + s'$ with $\deg(s') < \deg(s)$.]

(c) Show that if $r \in R$ and $s \in S$ with $sr \in dS$, then either $r = 0$ or $s \in dS$. [Hint: Exercise 2R.]

(d) Show that S/dS is a domain. [Hint: If not, choose $u, v \in S \setminus dS$ with $uv \in dS$ and $\deg(u) + \deg(v)$ as small as possible, and look first at the leading coefficient of uv.]

(e) Prove that $\mathcal{O}_q(SL_2(k))$ is a domain for any field k and any scalar $q \in k^\times$. $\square$

The points of the group $GL_2(k)$ form a subset of $M_2(k)$ determined by an "inequation," namely that the determinant be nonzero. This leads to a structure as an algebraic variety and to the coordinate ring $\mathcal{O}(GL_2(k)) = \mathcal{O}(M_2(k))[(x_{11}x_{22} - x_{12}x_{21})^{-1}]$ (this is just the subalgebra of the rational function field $k(x_{11}, x_{12}, x_{21}, x_{22})$ generated by the elements x_{11}, x_{12}, x_{21}, x_{22}, and $(x_{11}x_{22} - x_{12}x_{21})^{-1}$. Hence, the natural way to write down a quantized coordinate ring for $GL_2(k)$ would be to enlarge the algebra $\mathcal{O}_q(M_2(k))$ with an inverse for D_q. However, since $\mathcal{O}_q(M_2(k))$ is noncommutative, we require a division ring to take over the role of the quotient field; later, we will construct such a division ring, but that is not available to us at this point. Another way to proceed is to construct an algebra A presented by five generators x_{11}, x_{12}, x_{21}, x_{22}, y and eight relations – the six relations already used in defining $\mathcal{O}_q(M_2(k))$ as well as $(x_{11}x_{22} - qx_{12}x_{21})y = 1$ and $y(x_{11}x_{22} - qx_{12}x_{21}) = 1$. One then has to prove, for example, that the natural map $\mathcal{O}_q(M_2(k)) \to A$ is an embedding. A much more convenient way to adjoin an inverse for the quantum determinant is by construction of a "ring of fractions," which we shall study later. Therefore we defer introducing the algebra $\mathcal{O}_q(GL_2(k))$ until we have built up the basic theory of rings of fractions.

• ADDITIONAL EXERCISES •

2Y. Let $R[x;\alpha,\delta]$ be a skew polynomial ring. If there exists $d \in R$ such that $\delta(r) = dr - \alpha(r)d$ for all $r \in R$, then δ is called an *inner α-derivation* of R. In this case, show that $R[x;\alpha,\delta] = R[x-d;\alpha]$ (cf. Exercise 2L(a)). $\square$

2Z. Let $R[x;\alpha,\delta]$ be a skew polynomial ring. Assume that α is an inner automorphism of R; say there exists a unit $a \in R$ with $\alpha(r) = a^{-1}ra$ for all $r \in R$. Show that $a\delta$ is a derivation on R and that $R[x;\alpha,\delta] = R[ax;a\delta]$. $\square$

2ZA. Let $S = R[x;\alpha,\delta]$ be a skew polynomial ring and I an ideal of R such that $\alpha(I) \subseteq I$ and $\delta(I) \subseteq I$. Let $\hat{\alpha}$ and $\hat{\delta}$ denote the ring endomorphism and skew derivation on R/I induced by α and δ. Show that IS is a two-sided ideal of S such that $IS \cap R = I$, and that $IS = SI$ in case α is an automorphism and $\alpha(I) = I$. Then show that $S/IS \cong (R/I)[\hat{x};\hat{\alpha},\hat{\delta}]$. $\square$

2ZB. If $R[x;\alpha,\delta]$ is a skew polynomial ring, show that R can be made into a left $R[x;\alpha,\delta]$-module with a module multiplication $*$ such that $\left(\sum_i r_i x^i\right) * r = \sum_i r_i \delta^i(r)$ for all $\sum_i r_i x^i$ in $R[x;\alpha,\delta]$ and $r \in R$. $\square$

2ZC. Let α be an endomorphism of a ring R.

(a) Let δ be any map from R to R, and define a map $\phi : R \to M_2(R)$ by the rule $\phi(r) = \left(\begin{smallmatrix} \alpha(r) & \delta(r) \\ 0 & r \end{smallmatrix}\right)$. Show that δ is an α-derivation of R if and only if ϕ is a ring homomorphism.

(b) Now let $R[z]$ be a polynomial ring over a ring R and δ an α-derivation of R. Extend α to an endomorphism α' of $R[z]$ such that $\alpha'(z) = z$. Suppose that there is an element p of $R[z]$ satisfying $pr = \alpha(r)p$ for all $r \in R$. (E.g., if α is the identity map on R, this just requires that p be central in $R[z]$.) Show that δ extends uniquely to an α'-derivation δ' on $R[z]$ such that $\delta'(z) = p$. $\square$

2ZD. (a) Let R be a commutative domain and K its quotient field. Show that any derivation on R induces a derivation on K via the usual quotient rule for derivatives.

(b) If $F \subseteq K$ are fields of characteristic zero, show that any derivation δ on F extends to a derivation on K. [Hint: By Zorn's Lemma, we may assume that δ cannot be extended to any larger subfield of K. If $z \in K$ is transcendental over F, extend δ to $F[z]$ and then to $F(z)$, using Exercise 2ZC(b) and part(a). If $z \in K$ is algebraic over F, there is only one possible way to extend δ to $F(z)$.] $\square$

2ZE. Let R be a ring and δ a *locally nilpotent* derivation on R, meaning that, for each $r \in R$, there is some $n \in \mathbb{N}$ such that $\delta^n(r) = 0$.

(a) Construct a *skew power series ring* $R[[x;\delta]]$ consisting of formal power series $\sum_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where $xr = rx + \delta(r)$ for all $r \in R$.

(b) Suppose that $R = k[t]$ and $\delta = d/dt$, where k is a field of characteristic zero and t is an indeterminate. Show that in this case $R[[x;\delta]] \cong \mathrm{End}_k(R)$, and conclude that $R[[x;\delta]]$ is not noetherian on either side. $\square$

2ZF. Let R be a ring, α an automorphism of R, and δ an α-derivation of R.

(a) Construct a *skew inverse Laurent series ring* $R((x^{-1};\alpha,\delta))$ consisting of formal inverse Laurent series $\sum_{i=-\infty}^{n} r_i x^i$ with $n \in \mathbb{Z}$ and coefficients $r_i \in R$, where $xr = \alpha(r)x + \delta(r)$ for all $r \in R$. (In case α is the identity map, $R((x^{-1};\delta))$ is called a *formal pseudo-differential operator ring*.) Observe that the subset $R[[x^{-1};\alpha,\delta]]$ of $R((x^{-1};\alpha,\delta))$ consisting of inverse Laurent series of the form $\sum_{i=-\infty}^{0} r_i x^i$ is a subring of $R((x^{-1};\alpha,\delta))$.

(b) If R is right (left) noetherian, show that $R((x^{-1};\alpha,\delta))$ and $R[[x^{-1};\alpha,\delta]]$ are right (left) noetherian. $\square$

• NOTES •

Skew Polynomial Rings. We have already mentioned the work of Noether and Schmeidler [1920] in the notes for Chapter 1; here we point out that they were also interested in the case $K[x_1,\dots,x_n;\delta_1,\dots,\delta_n]$, where K consists of (C^∞) functions in variables $y_1,\dots,y_n$ and each $\delta_i = \partial/\partial y_i$. Later, Ore produced a systematic investigation of skew polynomial rings in one variable over a division ring [1933]; he in particular observed that, in the relation $xr = \alpha(r)x + \delta(r)$, the map α must be a ring endomorphism and the map δ must be an α-derivation.

Weyl Algebras. Manipulations with relations of the form $pq - qp = 1$ arising from quantum mechanics occurred in work of Dirac [1926] and Weyl [1928,

§§10, 18, 44]. The first investigation of an algebra with generators satisfying such a relation was carried out by Littlewood [1933], who proved that $A_1(\mathbb{R})$ and $A_1(\mathbb{C})$ are nonzero simple domains satisfying the left common multiple condition and possessing division algebras of left fractions [1933, Theorems VII, X, XII, XIX, XXI]. The appellation "Weyl algebra" was introduced into ring theory by Dixmier [1968, Introduction], following an analogous usage of "infinitesimal Weyl algebras" in differential geometry by Segal [1968, §2].

Simplicity of Weyl Algebras. The simplicity of $A_n(k)$ for k a field of characteristic zero was proved by Hirsch [1937, Theorem].

Skew Polynomial Principal Ideal Domains. The key ingredients in proving that a skew polynomial ring over a division ring is a PID, namely the left-hand division algorithm (in general) and the right-hand analog (in case α is an automorphism), are due to Ore [1933, pp. 483–484 and Theorem 6].

Quantum $\mathfrak{sl}_2(k)$. The now standard relations for the algebra $U_q(\mathfrak{sl}_2(k))$ evolved from equations developed by Kulish and Reshetikhin [1981].

Quantum Matrices and Quantum $SL_2(k)$. Relations for the algebras we have denoted $\mathcal{O}_q(M_2(\mathbb{C}))$ and $\mathcal{O}_q(SL_2(\mathbb{C}))$, with q viewed as a power series $e^{h/2}$ in an indeterminate h, were introduced by Drinfel'd [1987, §7, Eqns. 16–19; 1988, §7, Eqns. 16–19]. The general versions of $\mathcal{O}_q(M_2(k))$ and $\mathcal{O}_q(SL_2(k))$ were discussed by Manin in [1987, §§1.3, 1.7].

3. Prime Ideals

In trying to understand the ideal theory of a commutative ring, one quickly sees that it is important to first understand the prime ideals. We recall that a proper ideal P in a commutative ring R is prime if, whenever we have two elements a and b of R such that $ab \in P$, it follows that $a \in P$ or $b \in P$; equivalently, P is a prime ideal if and only if the factor ring R/P is a domain. (The terminology comes from algebraic number theory, where, for instance, one replaces the prime *numbers* in $\mathbb{Z}$ by the prime *ideals* in a Dedekind domain in order to preserve the unique factorization property.) The importance of prime ideals is perhaps clearest in the setting of algebraic geometry, for if R is the coordinate ring of an affine algebraic variety, the prime ideals of R correspond to irreducible subvarieties.

In the noncommutative setting, we define an integral domain just as we do in the commutative case (as a nonzero ring in which the product of any two nonzero elements is nonzero), but it turns out not to be a good idea to concentrate our attention on ideals P such that R/P is a domain. In fact, many noncommutative rings have no factor rings which are domains. (Consider a matrix ring over a field.) Thus a more relaxed definition for the concept of a prime ideal in the noncommutative case is desirable. The key is to change the commutative definition by replacing products of elements with products of ideals, which was first proposed in 1928 by Krull.

In the commutative case, there is a close connection between prime ideals and nilpotent elements. In particular, the intersection of all prime ideals equals the set of nilpotent elements. The noncommutative analog of this theory is presented in the opening sections of this chapter. We then see how prime ideals arise as annihilators, which is responsible for much of their significance. The most important class of prime ideals that arises in this way is the class of primitive ideals. We close the chapter by giving an analysis of the prime ideals in certain differential operator rings.

• PRIME IDEALS •

Definition. A *prime ideal* in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A *prime ring* is a ring in which 0 is a prime ideal.

Note that a prime ring must be nonzero.

That this definition coincides with the usual one in the commutative case follows from part (f) of the next proposition.

Proposition 3.1. *For a proper ideal P in a ring R, the following conditions are equivalent:*

(a) *P is a prime ideal.*

(b) *If I and J are any ideals of R properly containing P, then $IJ \not\subseteq P$.*

(c) *R/P is a prime ring.*

(d) *If I and J are any right ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.*

(e) *If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.*

(f) *If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.*

Proof. (a) $\Longrightarrow$ (b): This is clear.

(b) $\Longrightarrow$ (c): Given ideals I and J in R/P, there exist ideals $I' \supseteq P$ and $J' \supseteq P$ in R such that $I'/P = I$ and $J'/P = J$. If $IJ = 0$, then $I'J' \subseteq P$. By (b), either $I' = P$ or $J' = P$, and so either $I = 0$ or $J = 0$.

(c) $\Longrightarrow$ (a): If I and J are ideals of R satisfying $IJ \subseteq P$, then $(I+P)/P$ and $(J+P)/P$ are ideals of R/P whose product is zero. Then either $(I+P)/P = 0$ or $(J+P)/P = 0$, whence either $I \subseteq P$ or $J \subseteq P$.

(a) $\Longrightarrow$ (d): Since I is a right ideal, $(RI)(RJ) = RIJ \subseteq P$. Thus either $RI \subseteq P$ or $RJ \subseteq P$.

(d) $\Longrightarrow$ (f): Since $(xR)(yR) \subseteq P$, either $xR \subseteq P$ or $yR \subseteq P$.

(f) $\Longrightarrow$ (a): Given ideals $I \not\subseteq P$ and $J \not\subseteq P$, choose elements $x \in I \setminus P$ and $y \in J \setminus P$. Then $xRy \not\subseteq P$, whence $IJ \not\subseteq P$.

(a) $\Longleftrightarrow$ (e) by symmetry. $\square$

It follows immediately (by induction) from Proposition 3.1 that if P is a prime ideal in a ring R and $J_1, \ldots, J_n$ are right ideals of R such that $J_1 J_2 \cdots J_n \subseteq P$, then some $J_i \subseteq P$.

Recall that by a *maximal ideal* in a ring is meant a maximal *proper* ideal, i.e., an ideal which is a maximal element in the collection of proper ideals.

Proposition 3.2. *Every maximal ideal M of a ring R is a prime ideal.*

Proof. If I and J are ideals of R not contained in M, then $I + M = R$ and $J + M = R$. Now

$$R = (I + M)(J + M) = IJ + IM + MJ + M^2 \subseteq IJ + M,$$

and hence $IJ \not\subseteq M$. $\square$

Proposition 3.2 together with Zorn's Lemma guarantees that every nonzero ring has at least one prime ideal.

Exercise 3A. Let $R = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ be the ring of quaternions with integer coefficients. (In algebraic number theory, one meets a *ring of integer quaternions* which is slightly larger than the ring R.) Since R is contained in the division ring $\mathbb{H}$, it is a domain, and so 0 is a prime ideal of R.

(a) For any odd prime integer p, show that pR is a maximal ideal of R. [Hint: If I is a proper ideal containing pR, show that the cosets $1+I$, $i+I$, $j+I$, $k+I$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$.]

(b) For any odd prime integer p, show that R/pR is not a domain. [Hint: Find integers a, b in $\{0, 1, \ldots, (p-1)/2\}$ such that $a^2 + b^2 \equiv -1 \pmod{p}$, and look at $1 + ai + bj$ and $1 - ai - bj$.] Show that, in fact, $R/pR \cong M_2(\mathbb{Z}/p\mathbb{Z})$. [Hint: If $\alpha = a + p\mathbb{Z}$ and $\beta = b + p\mathbb{Z}$ for a, b as above, there is an isomorphism such that

$$i + pR \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad j + pR \mapsto \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \qquad k + pR \mapsto \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}.]$$

(c) Show that $2R$ is not a prime ideal of R, and that $2R+(1+i)R+(1+j)R$ is a maximal ideal of R.

(d) Show that the only prime ideals of R are 0 and $2R+(1+i)R+(1+j)R$ together with pR for all odd prime integers p. [Hint: If P is a prime ideal of R not containing any prime integer, show that R/P is a free abelian group with basis $\{1+P, i+P, j+P, k+P\}$.] $\square$

Definition. A *minimal prime ideal* in a ring R is any prime ideal of R that does not properly contain any other prime ideals.

For instance, if R is a prime ring, then 0 is a minimal prime ideal of R, and it is the only one.

Exercise 3B. Given an integer $n > 1$, show that the minimal prime ideals of $\mathbb{Z}/n\mathbb{Z}$ are exactly the prime ideals $p\mathbb{Z}/n\mathbb{Z}$, where p is any prime divisor of n. $\square$

Proposition 3.3. *Any prime ideal P in a ring R contains a minimal prime ideal.*

Proof. Let $\mathcal{X}$ be the set of those prime ideals of R which are contained in P. We may use Zorn's Lemma going downward in $\mathcal{X}$ provided we show that any nonempty chain $\mathcal{Y} \subseteq \mathcal{X}$ has a lower bound in $\mathcal{X}$.

The set $Q = \bigcap \mathcal{Y}$ is an ideal of R, and it is clear that $Q \subseteq P$. We claim that Q is a prime ideal.

Thus consider any $x, y \in R$ such that $xRy \subseteq Q$ but $x \notin Q$. Then $x \notin P'$ for some $P' \in \mathcal{Y}$. For any $P'' \in \mathcal{Y}$ such that $P'' \subseteq P'$, we have $x \notin P''$ and $xRy \subseteq Q \subseteq P''$, whence $y \in P''$. In particular, $y \in P'$. If $P'' \in \mathcal{Y}$ and $P'' \nsubseteq P'$, then $P' \subset P''$, and so $y \in P''$. Hence, $y \in P''$ for all elements P'' of $\mathcal{Y}$, and so $y \in Q$, which proves that Q is a prime ideal.

Now $Q \in \mathcal{X}$, and Q is a lower bound for $\mathcal{Y}$.

Thus, Zorn's Lemma applies, giving us a prime ideal $P^* \in \mathcal{X}$ that is minimal among the ideals in $\mathcal{X}$. Since any prime ideal contained in P^* is in $\mathcal{X}$, we conclude that P^* is a minimal prime ideal of R. $\square$

Given an ideal I in a ring R and a prime ideal P containing I, we may apply Proposition 3.3 in the ring R/I to see that the prime ideal P/I contains a minimal prime Q/I of R/I. Then Q is a prime ideal of R which contains I and is minimal among the primes containing I. By way of abbreviation, we say that Q is a *prime minimal over I*.

Theorem 3.4. *In a right or left noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) that equals zero.*

Proof. Note that the following proof does not require the full force of the right or left noetherian hypothesis, but only the ACC on two-sided ideals.

It suffices to prove that there exist prime ideals $P_1, \dots, P_n$ in R such that $P_1P_2 \cdots P_n = 0$. To see this, note that after replacing each P_i by a minimal prime ideal contained in it, we may assume that each P_i is minimal. Since any minimal prime P contains $P_1P_2 \cdots P_n$, it must contain some P_j, whence $P = P_j$ by minimality. Thus the minimal prime ideals of R are contained in the finite set $\{P_1, \dots, P_n\}$.

Suppose that no finite product of prime ideals in R is zero. Let $\mathcal{X}$ be the set of those ideals K in R that do not contain a finite product of prime ideals. Since $\mathcal{X}$ contains 0, it is nonempty. By the noetherian hypothesis (not Zorn's Lemma!), there exists a maximal element $K \in \mathcal{X}$.

As R/K is a counterexample to the theorem, we may replace R by R/K. Thus we may assume, without loss of generality, that no finite product of prime ideals in R is zero, while all nonzero ideals of R contain finite products of prime ideals.

In particular, 0 cannot be a prime ideal. Hence, there exist nonzero ideals I, J in R such that $IJ = 0$. Then there exist prime ideals $P_1, \dots, P_m, Q_1, \dots, Q_n$ in R with $P_1P_2 \cdots P_m \subseteq I$ and $Q_1Q_2 \cdots Q_n \subseteq J$. But then

$$P_1P_2 \cdots P_mQ_1Q_2 \cdots Q_n = 0,$$

contradicting our supposition.

Therefore some finite product of prime ideals in R is zero. $\square$

The use of the noetherian condition in the proof of Theorem 3.4 to pass from R to R/K is known as *noetherian induction*. Since R/K is as small as possible among factor rings of R violating the theorem, it is known as a *minimal criminal*. (For this terminology we are indebted to Reinhold Baer, who remarked that, as in the larger world, it is the minimal criminal who is apprehended.)

In general, a ring may have infinitely many minimal prime ideals, as the following example shows.

Exercise 3C. Let X be an infinite set, k a field, and R the ring of all functions from X to k. For $x \in X$, let P_x be the set of those functions in R which vanish at x. Show that each P_x is a minimal prime ideal of R, and also a maximal ideal. □

• SEMIPRIME IDEALS AND NILPOTENCE •

Definition. A *semiprime ideal* in a ring R is any ideal of R which is an intersection of prime ideals. (By convention, the intersection of the empty family of prime ideals of R is R, so that R is a semiprime ideal of itself.) A *semiprime ring* is any ring in which 0 is a semiprime ideal. Note that an ideal P in a ring R is semiprime if and only if R/P is a semiprime ring.

In $\mathbb{Z}$, the intersection of any infinite family of prime ideals is 0. The intersection of any finite list $p_1\mathbb{Z}, \dots, p_k\mathbb{Z}$ of prime ideals, where $p_1, \dots, p_k$ are distinct prime integers, is the ideal $p_1 p_2 \cdots p_k\mathbb{Z}$. Hence, the nonzero semiprime ideals of $\mathbb{Z}$ consist of the ideals $n\mathbb{Z}$ where n is any square-free positive integer (including $n = 1$).

The development of ideas in this section is made clearer if we first review the commutative case, even though it is well known.

Lemma 3.5. *Let R be a ring and X a subset of R such that $0 \notin X$ and such that X is closed under multiplication. Let P be an ideal of R chosen maximal with respect to the property that P and X are disjoint. Then P is a prime ideal.*

Proof. We verify condition (b) of Proposition 3.1: If I and J are ideals of R such that $I \supset P$ and $J \supset P$, then $IJ \not\subseteq P$. Since $I \supset P$, by the maximality of P there is an element $x \in X \cap I$, and similarly, since $J \supset P$, there is an element $y \in X \cap J$. Then $xy \in IJ$, and $xy \in X$ (since X is multiplicatively closed). Since P and X are disjoint, it follows that $IJ \not\subseteq P$, as desired. □

Proposition 3.6. *If R is a commutative ring, then:*

(a) *The intersection of all prime ideals of R is precisely the set of nilpotent elements of R.*

(b) *For every ideal I of R, the intersection of all of the prime ideals of R containing I is the set of elements $r \in R$ such that $r^n \in I$ for some positive integer n.*

(c) *The ring R is semiprime if and only if it contains no nonzero nilpotent elements.*

Proof. (a) If r is a nilpotent element of R, then r must be contained in every prime ideal, since if P is a prime ideal, then R/P has no nonzero nilpotent elements. Hence, all nilpotent elements are in the intersection of the prime ideals. Conversely, if r is not nilpotent, then, letting $X = \{r^n \mid n \in \mathbb{N}\}$, we can apply Lemma 3.5 to obtain a prime ideal P of R such that $r \notin P$, and so r is not in the intersection of the prime ideals.

Clearly, (b) follows from (a) by passing to the factor ring R/I, and (c) is a special case of (a). $\square$

It follows from Proposition 3.6 that an ideal I in a commutative ring R is semiprime if and only if, whenever $x \in R$ and $x^2 \in I$, it follows that $x \in I$. The example of a matrix ring over a field shows that this criterion fails in the noncommutative case. However, there is an analogous criterion, as we will see in the next theorem. One should think of it as parallel to condition (f) in Proposition 3.1 describing a prime ideal. The method of the proof is to generalize the idea of Lemma 3.5 to a set which is not quite multiplicatively closed. (This idea is further developed in Exercise 3Z.)

Theorem 3.7. [Levitzki, Nagata] *An ideal I in a ring R is semiprime if and only if*

(*) *Whenever $x \in R$ with $xRx \subseteq I$, then $x \in I$.*

Proof. First suppose that I equals the intersection of some family $\{P_j \mid j \in J\}$ of prime ideals. Given $x \in R$ with $xRx \subseteq I$, we have $xRx \subseteq P_j$ for each $j \in J$. Then x lies in each P_j, whence $x \in I$.

Conversely, assume that (*) holds. We shall prove that I equals the intersection of all those prime ideals of R which contain I. Hence, given any $x \in R \setminus I$, we need a prime ideal $P \supseteq I$ such that $x \notin P$.

Set $x_0 = x$. By (*), $x_0 R x_0 \nsubseteq I$, and so we may choose an element x_1 in $x_0 R x_0 \setminus I$. Applying (*) to x_1, we may choose an element x_2 in $x_1 R x_1 \setminus I$. Continuing in this manner, we obtain elements $x_0, x_1, x_2, \dots$ in $R \setminus I$ such that $x_{i+1} \in x_i R x_i$ for all i. Note (by induction) that if J is any ideal of R and some $x_j \in J$, then $x_n \in J$ for all $n \geq j$.

Now $x_i \notin I$ for all i. By Zorn's Lemma, there is an ideal $P \supseteq I$ maximal with respect to the property that $x_i \notin P$ for all i. In particular, $x = x_0 \notin P$, and P is a proper ideal of R. We claim that P is a prime ideal.

The argument is now the same as that in the proof of Lemma 3.5. Consider ideals J and K of R such that $J \supset P$ and $K \supset P$. By the maximality of P, some $x_j \in J$ and some $x_k \in K$. If m is the maximum of j and k, then $x_m \in J \cap K$, and so

$$x_{m+1} \in x_m R x_m \subseteq JK,$$

which proves that $JK \nsubseteq P$.

Thus, P is a prime ideal, as claimed. Therefore I does equal the intersection of all prime ideals of R containing I, whence I is semiprime. $\square$

The reader should be aware that many authors define semiprime ideals by the condition (*) in Theorem 3.7. From that viewpoint, the theorem then says that an ideal is semiprime if and only if it is an intersection of prime ideals.

Corollary 3.8. *For an ideal I in a ring R, the following conditions are equivalent:*

(a) *I is a semiprime ideal.*
(b) *If J is any ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*
(c) *If J is any ideal of R properly containing I, then $J^2 \nsubseteq I$.*
(d) *If J is any right ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*
(e) *If J is any left ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

Proof. (a) $\Longrightarrow$ (d): For any $x \in J$, we have $xRx \subseteq J^2 \subseteq I$, whence $x \in I$ by Theorem 3.7. Thus $J \subseteq I$.

(d) $\Longrightarrow$ (c): A priori.

(c) $\Longrightarrow$ (b): If $J \nsubseteq I$, then $I + J$ properly contains I. But since $(I + J)^2 = I^2 + IJ + JI + J^2 \subseteq I$, we have a contradiction to (c). Thus $J \subseteq I$.

(b) $\Longrightarrow$ (a): Given any $x \in R$ such that $xRx \subseteq I$, we have $(RxR)^2 = RxRxR \subseteq I$ and so $RxR \subseteq I$, whence $x \in I$. By Theorem 3.7, I is semiprime.

(a) $\Longleftrightarrow$ (e): By symmetry. $\square$

Corollary 3.9. *Let I be a semiprime ideal in a ring R. If J is a right or left ideal of R such that $J^n \subseteq I$ for some positive integer n, then $J \subseteq I$.*

Proof. In case $n = 1$, there is nothing to prove. Now let $n > 1$ and assume the corollary holds for lower powers. Since $n \geq 2$, we have $2n-2 \geq n$, whence

$$(J^{n-1})^2 = J^{2n-2} \subseteq J^n \subseteq I.$$

Then $J^{n-1} \subseteq I$ by Corollary 3.8, and so $J \subseteq I$ by the induction hypothesis. This completes the induction step. $\square$

Definition. A right or left ideal J in a ring R is *nilpotent* provided $J^n = 0$ for some positive integer n. More generally, J is *nil* provided every element of J is nilpotent.

Exercise 3G provides an example of a nil ideal that is not nilpotent. On the other hand, in noetherian rings all nil one-sided ideals are nilpotent (Theorem 6.21; Exercise 6M).

Corollary 3.9 shows that in a semiprime ring, the only nilpotent right or left ideal is 0. Conversely, if a ring R has no nonzero nilpotent ideals, then R is a semiprime ring by Corollary 3.8.

Definition. The *prime radical* of a ring R is the intersection of all the prime ideals of R.

If R is the zero ring, it has no prime ideals, and the prime radical equals R. If R is nonzero, it has at least one maximal ideal, which is prime by Proposition 3.2. Thus, the prime radical of a nonzero ring is a proper ideal.

Note that a ring R is semiprime if and only if its prime radical is zero. In any case, the prime radical of R is the smallest semiprime ideal of R, and because the prime radical is semiprime, it contains all nilpotent one-sided ideals of R (see Corollary 3.9).

Exercise 3D. Given an integer $n > 1$, find the prime radical of $\mathbb{Z}/n\mathbb{Z}$. □

Exercise 3E. Show that the prime radical of any ring is nil. □

Proposition 3.10. *In any ring R, the prime radical equals the intersection of the minimal prime ideals of R.*

Proof. This is immediate from the fact that every prime ideal of R contains a minimal prime (Proposition 3.3). □

The next theorem completes our development of a noncommutative analog to Proposition 3.6. (Note that we have only done this in the noetherian case.)

Theorem 3.11. *Let R be a right or left noetherian ring, and let N be the prime radical of R. Then N is a nilpotent ideal of R containing all the nilpotent right or left ideals of R.*

Proof. Since N is a semiprime ideal, it contains all the nilpotent one-sided ideals of R by Corollary 3.9. By Theorem 3.4, there exist (minimal) prime ideals $P_1, \dots, P_k$ in R such that $P_1P_2 \cdots P_k = 0$. Since N is contained in each P_i, we conclude that $N^k = 0$. □

Exercise 3F. Prove Theorem 3.11 without using Theorem 3.4, by considering a maximal nilpotent ideal of R. □

Exercise 3G. Show that the prime radical of the ring $R = \prod_{n=1}^{\infty} \mathbb{Z}/2^n\mathbb{Z}$ is nil but not nilpotent. [Hint: Show that the prime radical contains elements $x_1, x_2, \dots$ such that $x_n^n \neq 0$.] □

Exercise 3H. If R is a prime (semiprime) ring, show that each matrix ring $M_n(R)$ is prime (semiprime). □

• ANNIHILATORS AND ASSOCIATED PRIME IDEALS •

A basic principle of algebraic geometry is to study algebraic varieties via rings of functions on them. A key part of the theory is a correspondence between certain ideals and subvarieties that arises from "annihilation." (Such terminology is commonly used in reference to any process that results in a zero. For example, if $f(x) = 0$ or $cx = 0$, one says that x has been "annihilated" or even "killed" by f or c.) If R is a ring of functions on a set X, we can build a correspondence that takes a subset Y of X to an ideal

$$\{r \in R \mid r(y) = 0 \text{ for all } y \in Y\}$$

in R and takes an ideal I of R to a "zero-set"

$$\{x \in X \mid r(x) = 0 \text{ for all } r \in I\}$$

in X. In case X is an affine algebraic variety and R is its coordinate ring, then, under this correspondence, subvarieties of X correspond precisely to semiprime ideals of R and irreducible subvarieties of X to prime ideals of R.

In the theory of modules, there is a similar correspondence between certain ideals and certain submodules (using annihilation via multiplication in place of annihilation via functions), which we will study in this and the following sections. Here again it is important to study the submodules that correspond to prime ideals and to see how other modules are made up from these.

Definition. Let A be a right module over a ring R. Given any subset $X \subseteq A$, the *annihilator* of X is the set

$$\operatorname{ann}(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\},$$

which is a right ideal of R. In case the ring R must be made explicit, we write $\operatorname{ann}_R(X)$ and refer to the *annihilator of X in R*. Similarly, to emphasize that we are taking an annihilator on the right side of X (because A is a right module), we may write $\operatorname{r.ann}(X)$ for $\operatorname{ann}(X)$. When X consists of a single element x, we abbreviate $\operatorname{ann}(\{x\})$ to $\operatorname{ann}(x)$. We have already noted that $\operatorname{ann}(X)$ is a right ideal of R; moreover, if X is a submodule of A, then $\operatorname{ann}(X)$ is an ideal of R.

Annihilators of subsets of left R-modules are defined analogously, and are left ideals of R. In case $A = R$, we must specify whether an annihilator is taken with respect to the right module R_R or the left module ${}_RR$. Thus for $X \subseteq R$ we have a *right annihilator*

$$\operatorname{r.ann}(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$$

as well as a *left annihilator*

$$\operatorname{l.ann}(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}.$$

Finally, there are annihilators in modules to define. If A is a right R-module and Y is a subset of R, the *annihilator of Y in A* is the set

$$\operatorname{ann}_A(Y) = \{a \in A \mid ay = 0 \text{ for all } y \in Y\},$$

which is an additive subgroup of A. To emphasize that this annihilator is on the left side of Y, we may write $\operatorname{l.ann}_A(Y)$ for $\operatorname{ann}_A(Y)$. The annihilator of Y in a left R-module is defined analogously.

Note that the annihilator of a left ideal of R in a right R-module A is a submodule of A, and similarly the annihilator of a right ideal in a left module is a submodule.

Definition. A module A over a ring R is a *faithful* R-module if $\operatorname{ann}_R(A) = 0$.

Notice that a faithful module over a nonzero ring must be nonzero. Note also that the annihilator of an R-module A is an ideal of R, and that A is a faithful module over $R/\operatorname{ann}_R(A)$.

An important example to observe is that, in a prime ring, every nonzero right or left ideal is faithful.

Definition. A module A over a ring R is *fully faithful* provided A and all nonzero submodules of A are faithful R-modules. If A is a nonzero R-module which is fully faithful as a module over $R/\operatorname{ann}_R(A)$, then A is called a *prime module*; the next exercise justifies this terminology.

Exercise 3I. If A is a prime module over a ring R, show that $\operatorname{ann}_R(A)$ is a prime ideal of R. $\square$

The following proposition provides the key to finding fully faithful submodules over factor rings, at least in modules over noetherian rings.

Proposition 3.12. *Let A be a nonzero module over a ring R. Suppose that there exists an ideal P maximal among the annihilators of nonzero submodules of A. Then P is a prime ideal of R, and* $\operatorname{ann}_A(P)$ *is a fully faithful* (R/P)-*module.*

Proof. For specificity, suppose that A is a right R-module. There is a nonzero submodule B in A such that $P = \operatorname{ann}(B)$, and $P \neq R$ because $B \neq 0$. Suppose that I and J are ideals of R, properly containing P, such that $IJ \subseteq P$. Then $BI \neq 0$ and $\operatorname{ann}(BI) \supseteq J \supset P$, contradicting the maximality of P. Thus P is prime.

Now set $C = \operatorname{ann}_A(P)$ and note that C is a submodule of A with $P \subseteq \operatorname{ann}(C)$. Then $P = \operatorname{ann}(C)$, because $B \subseteq C$. Thus C is a faithful right (R/P)-module. Given any nonzero submodule $D \subseteq C$, we have $P = \operatorname{ann}(C) \subseteq \operatorname{ann}(D)$, whence $P = \operatorname{ann}(D)$ by maximality of P. Therefore C is fully faithful as a right (R/P)-module. $\square$

In general, given an R-module A, the family of annihilators of nonzero submodules of A need not have any maximal elements. However, if R is either right or left noetherian, the existence of maximal annihilators is automatic.

Definition. An *annihilator prime* for a module A over a ring R is any prime ideal P of R which equals the annihilator of some nonzero submodule of A. In this case, $\operatorname{ann}_A(P)$ is clearly nonzero and is a faithful (R/P)-module. An *associated prime* of A is any annihilator prime P which equals the annihilator of some prime submodule B of A; in other words, not only must P equal $\operatorname{ann}_R(B)$, it must also equal the annihilator of each nonzero submodule of B. Thus, B must be a *fully* faithful (R/P)-module. (We do not require, however, that $\operatorname{ann}_A(P)$ be a fully faithful (R/P)-module.) The set of all associated primes of A is denoted $\operatorname{Ass}(A)$.

Proposition 3.12 shows that any ideal maximal among the annihilators of nonzero submodules of a module A is an associated prime of A. In particular, it follows that every nonzero module over a right or left noetherian ring has at least one associated prime. Not every associated prime arises as a maximal annihilator, however. For instance, the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ has two associated primes, 0 and $2\mathbb{Z}$, and 0 is certainly not maximal among the annihilators of nonzero submodules of this module.

Over the ring $\mathbb{Z}$, the module A which is the direct sum of the cyclic modules $\mathbb{Z}/p\mathbb{Z}$ for all primes p is an example of a module whose annihilator is the prime ideal 0 but for which 0 is not an associated prime. However, this cannot occur for finitely generated modules over a commutative noetherian ring (cf. Exercise 5ZH, in which it is shown that annihilator primes and associated primes are the same for finitely generated modules over a commutative noetherian ring.) For finitely generated modules over a noncommutative noetherian ring, Execise 3ZD shows that an annihilator prime need not be associated, but these two classes of primes do coincide for important classes of rings; cf. Exercise 12K.

Exercise 3J. If B is a submodule of a module A, show that $\mathrm{Ass}(B) \subseteq \mathrm{Ass}(A)$ and that $\mathrm{Ass}(A) \subseteq \mathrm{Ass}(B) \cup \mathrm{Ass}(A/B)$. If every nonzero submodule of A has nonzero intersection with B, show that $\mathrm{Ass}(A) = \mathrm{Ass}(B)$. $\square$

Exercise 3K. Let $P_1, \dots, P_n$ be distinct associated primes of a module A, and for each i let B_i be a prime submodule of A such that $P_i = \mathrm{ann}_R(B_i)$. Show that $B_1, \dots, B_n$ are independent (that is, their sum is a direct sum). [Hint: Assume by induction that any $n-1$ of the B_i are independent. Re-index everything so that P_1 is minimal among the P_i and then show that $B_1 \cap (B_2 + \cdots + B_n) = 0$.] Conclude that a noetherian module has only finitely many associated primes. $\square$

Returning to Proposition 3.12, we note the following consequence: If A is a nonzero module over a noetherian ring, then A contains a prime submodule, say A_1. If $A \neq A_1$, then A/A_1 contains a prime submodule, say A_2/A_1. As long as A is finitely generated (and thus noetherian), this process must eventually terminate, resulting in a chain of submodules of A of the following form.

Definition. Let A be a nonzero module over a ring R. A *prime series* for A is a chain of submodules of the form

$$A_0 = 0 < A_1 < \cdots < A_n = A$$

such that, for each $i = 1, \dots, n$, the subfactor A_i/A_{i-1} is a prime module. Since each ideal $\mathrm{ann}_R(A_i/A_{i-1})$ is an associated prime of A/A_{i-1}, a prime series might also be called an *associated series* for A. The latter terminology can be misleading, however, since the primes $\mathrm{ann}_R(A_i/A_{i-1})$ need not be associated primes of A, as the following example shows.

Exercise 3L. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field, and let A be the right R-module $\begin{pmatrix} F & F \end{pmatrix}$. Show that A has exactly one prime series, which has the form $0 < A_1 < A$, and that the prime ideal $\mathrm{ann}_R(A/A_1)$ is not an associated prime of A. $\square$

Proposition 3.13. *Let A be a nonzero right module over a right noetherian ring R. If A is finitely generated, then A has a prime series. If $A_0 = 0 < A_1 < \cdots < A_n = A$ is a prime series for A and $P_i = \operatorname{ann}_R(A_i/A_{i-1})$ for $i = 1, \ldots, n$, then each P_i is a prime ideal of R and A_i/A_{i-1} is a fully faithful right (R/P_i)-module.*

Proof. The first conclusion follows from Proposition 3.12 and the fact that A is a noetherian module, as in the discussion above. The remainder is clear from Exercise 3I and the definition of a prime module. $\square$

The point of Proposition 3.13 is that it breaks up the module A into "slices" (namely, the subfactors A_i/A_{i-1}) which are fully faithful modules over prime factor rings of R. Thus, if we understand fully faithful modules over prime noetherian rings, we will be a good way towards understanding finitely generated modules over arbitrary noetherian rings. In particular, this is one indicator of the importance of prime noetherian rings within the general theory.

Proposition 3.14. *Let A be a module over a ring R, assume that A has a prime series $A_0 = 0 < A_1 < \cdots < A_n = A$, and set $P_i = \operatorname{ann}_R(A_i/A_{i-1})$ for $i = 1, \ldots, n$. If P is any prime minimal over $\operatorname{ann}_R(A)$, there is an index i such that $P = P_i$.*

Proof. Assume that A is a right R-module, say. Since $AP_nP_{n-1}\cdots P_1 = 0$, it follows that $P_nP_{n-1}\cdots P_1 \subseteq \operatorname{ann}_R(A) \subseteq P$. Hence, some $P_i \subseteq P$. On the other hand, it is clear that $\operatorname{ann}_R(A) \subseteq P_i$, and since P is minimal over $\operatorname{ann}_R(A)$, we conclude that $P = P_i$. $\square$

• SOME EXAMPLES FROM REPRESENTATION THEORY •

As indicated in the Prologue, one function of a group algebra $k[G]$ is to provide a ring whose modules coincide with the representations of G. The enveloping algebra $U(L)$ plays the analogous role for representations of a Lie algebra L, as follows. Let us assume that L is a Lie algebra over a field k, and let V be a vector space over k. A *representation of L on V* is a Lie algebra homomorphism $\phi : L \to \operatorname{End}_k(V)$, that is, a k-linear map such that $\phi([xy]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. (Recall that $[f, g]$ denotes the additive commutator $fg - gf$.) Any such representation ϕ extends uniquely to a k-algebra homomorphism $\phi' : U(L) \to \operatorname{End}_k(V)$; this universal property of enveloping algebras (which we do not prove here) is the reason for calling $U(L)$ the *universal* enveloping algebra of L. Once we have ϕ', we can make V into a left module over $U(L)$ by defining $av = \phi'(a)(v)$ for all $a \in U(L)$ and $v \in V$. Conversely, modules yield representations: If V is a left $U(L)$-module, then we can define a representation $\phi : L \to \operatorname{End}_k(V)$, where $\phi(x)(v) = xv$ for $x \in L$ and $v \in V$.

Let us look at a simple concrete case, where L is the 2-dimensional Lie algebra with a basis $\{x, y\}$ such that $[yx] = x$. To exhibit a representation of L, we just need a vector space V and linear transformations f and g on

V such that $[g, f] = f$; the corresponding representation of L is the unique linear map $\phi : L \to \mathrm{End}_k(V)$ such that $\phi(x) = f$ and $\phi(y) = g$. In view of the discussion above, we conclude that, given any $f, g \in \mathrm{End}_k(V)$ such that $[g, f] = f$, there is a unique way to make V into a left $U(L)$-module so that $xv = f(v)$ and $yv = g(v)$ for all $v \in V$.

Recall from the Prologue and Exercise 2C that $U(L) = k[x][y; x(d/dx)]$, a differential operator ring where $k[x]$ is a polynomial ring. In particular, $U(L)$ is a noetherian domain. The following exercises present some examples of $U(L)$-modules obtained from the representation-theoretic point of view and corresponding prime series for them.

Exercise 3M. Consider the 3×3 matrices $X = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$ and $Y = \begin{bmatrix} 3&0&0 \\ 0&2&0 \\ 0&0&1 \end{bmatrix}$, and check that $YX - XY = X$. Hence, the vector space $V = k^3$ becomes a left $U(L)$-module such that $xv = Xv$ and $yv = Yv$ for $v \in V$. (We must treat the vectors in V as column vectors, of course.) Show that V has a chain of submodules of the form

$$V_0 = 0 < V_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} < V_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} < V_3 = V,$$

and that these V_i are the only submodules of V. Show also that the above chain is the only prime series for V, and that $\mathrm{ann}_{U(L)}(V_i/V_{i-1}) = \langle x,\ y+i-4 \rangle$ for $i = 1, 2, 3$. $\square$

Exercise 3N. Let T be an infinite dimensional vector space over k with a countable basis $\{t_0, t_1, t_2, \dots\}$, define linear transformations f and g on T so that

$$f(t_i) = \begin{cases} t_{i+1} & (i \text{ even}) \\ 0 & (i \text{ odd}) \end{cases} \qquad g(t_i) = \begin{cases} t_{i+2} & (i \text{ even}) \\ t_i + t_{i+2} & (i \text{ odd}), \end{cases}$$

and check that $gf - fg = f$. Hence, T becomes a left $U(L)$-module. Show first that $\langle x \rangle^2 T = 0$, and conclude that T is not a prime module.

Next, observe that $T_1 = kt_1 + kt_3 + kt_5 + \cdots$ is a submodule of T, and show that $0 < T_1 < T$ is a prime series for T, with $\mathrm{ann}_{U(L)}(T_1) = \mathrm{ann}_{U(L)}(T/T_1) = \langle x \rangle$. Show also that $T_1 \cong T/T_1 \cong U(L)/\langle x \rangle$ as left modules.

The module T has infinitely many different prime series, although with more terms than the one above. For instance, if $T_i = kt_i + kt_{i+2} + kt_{i+4} + \cdots$, then $0 < T_{2n-1} < T_{2n-3} < \cdots < T_1 < T$ is a prime series, for any positive integer n. $\square$

Exercise 3O. Let W be an infinite dimensional vector space over k with a countable basis $\{w_0, w_1, w_2, \dots\}$, define linear transformations f and g on W so that $g(w_n) = w_{n+1}$ and $f(w_n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} w_i$ for all n, and check that $gf - fg = f$. Hence, W becomes a left $U(L)$-module.

(a) Show that $W \cong U(L)/U(L)(x-1)$.

Now assume that $\mathrm{char}(k) = 0$.

(b) Show that W is a simple $U(L)$-module. Thus, $0 < W$ is the only prime series for W.

(c) Given a nonzero element $a \in U(L)$, write $a = \sum_{i=0}^{n} y^i p_i$ for some n and some $p_i \in k[x]$, and let $(x-1)^m$ be the highest power of $x-1$ that divides all the p_i. Show that $aw_m \neq 0$, and conclude that W is a faithful $U(L)$-module. □

• PRIMITIVE AND SEMIPRIMITIVE IDEALS •

An almost trivial case of Proposition 3.12 is the case of a simple module A. Here A itself is the only nonzero submodule of A, and hence $\mathrm{ann}(A)$ is automatically maximal among annihilators of nonzero submodules of A. Such an annihilator is said to be "primitive," as follows.

Definition. An ideal P in a ring R is *right (left) primitive* provided $P = \mathrm{ann}_R(A)$ for some simple right (left) R-module A. A *right (left) primitive ring* is any ring in which 0 is a right (left) primitive ideal, i.e., any ring which has a faithful simple right (left) module.

Not all right primitive rings are left primitive, but the known examples are too involved to reproduce here. It is not known whether all right primitive noetherian rings are left primitive.

Proposition 3.15. *Every right or left primitive ideal in a ring R is a prime ideal. Every maximal ideal of R is a right and left primitive ideal.*

Proof. The primeness of primitive ideals follows from Proposition 3.12.

Given a maximal ideal M in R, choose a maximal right ideal K containing M. Then R/K is a simple right R-module and $\mathrm{ann}_R(R/K)$ is a right primitive ideal of R. Since $RM = M \subseteq K$, we have $(R/K)M = 0$, so that $M \subseteq \mathrm{ann}_R(R/K)$. Then $M = \mathrm{ann}_R(R/K)$ by the maximality of M, whence M is right primitive. By symmetry, M is also left primitive. □

Over a commutative ring R, any simple module is isomorphic to R/M for some maximal ideal M, and $\mathrm{ann}_R(R/M) = M$. Thus all primitive ideals of R are maximal; equivalently, all commutative primitive rings are simple rings. This does not hold among noncommutative rings, however. For example, if $U(L)$ is the enveloping algebra studied in Exercises 3M–O and the base field k has characteristic zero, then Exercise 3O shows that $U(L)/U(L)(x-1)$ is a faithful simple left $U(L)$-module; thus $U(L)$ is a left primitive ring. On the other hand, $U(L)$ has many proper nonzero ideals, such as $\langle x \rangle$ and $\langle x, y \rangle$, and so $U(L)$ is not a simple ring. The next exercise shows that $U(L)$ is also a right primitive ring.

Exercise 3P. Let L be the 2-dimensional Lie algebra with basis $\{x, y\}$ such that $[yx] = x$, over an arbitrary field k. Show that there is a k-algebra isomorphism $U(L) \to U(L)^{\mathrm{op}}$ sending $x \mapsto -x$ and $y \mapsto -y$. Conclude that if $\mathrm{char}(k) = 0$, then $U(L)$ is right primitive. $\square$

Proposition 3.16. *In any ring R, the following sets coincide:*

(a) *The intersection of all maximal right ideals of R.*
(b) *The intersection of all maximal left ideals of R.*
(c) *The intersection of all right primitive ideals of R.*
(d) *The intersection of all left primitive ideals of R.*

Proof. Let J_a, J_b, J_c, J_d denote the four intersections.

Given any maximal right ideal M of R, the annihilator of the simple right R-module R/M is a right primitive ideal P. Since $P \subseteq M$, we have $J_c \subseteq M$. Thus $J_c \subseteq J_a$.

We next show that J_a is an ideal of R. Consider any $x \in J_a$ and $r \in R$. Given any maximal right ideal M of R, either $r \in M$ or $M + rR = R$. If $r \in M$, then obviously $rx \in M$. If $r \notin M$, then left multiplication by r induces an isomorphism of R/L onto R/M, where $L = \{y \in R \mid ry \in M\}$. Hence, R/L is a simple right R-module and L is a maximal right ideal of R, from which we obtain $x \in J_a \subseteq L$ and so $rx \in M$. Now rx lies in all the maximal right ideals of R, whence $rx \in J_a$. Thus J_a is an ideal, as claimed.

Given any right primitive ideal P in R, there is a maximal right ideal M in R such that $\mathrm{ann}(R/M) = P$. Since J_a is an ideal, $RJ_a = J_a \subseteq M$ and so $(R/M)J_a = 0$, whence $J_a \subseteq P$. Thus $J_a \subseteq J_c$.

Therefore $J_a = J_c$. By symmetry, $J_b = J_d$.

We claim that whenever $x \in J_a$, then $1-x$ has a right inverse in R. If not, $(1-x)R \neq R$, and so $(1-x)R$ is contained in some maximal right ideal M. As $x \in J_a \subseteq M$, this is impossible. Thus $1-x$ must have a right inverse, as claimed.

Next, we claim that whenever $x \in J_a$, then $1-x$ is invertible in R. By the previous claim, $1-x$ has a right inverse $y \in R$. Then $(1-x)y = 1$ and so $y = 1 + xy$. Since $-xy \in J_a$, a second application of the previous claim shows that y has a right inverse $z \in R$. Then $z = (1-x)yz = 1-x$, whence $y(1-x) = 1$. Thus $1-x$ is invertible, as claimed.

We can now show that $J_a \subseteq J_b$. Consider any $x \in J_a$ and any maximal left ideal M of R. If $x \notin M$, then $rx + m = 1$ for some $r \in R$ and some $m \in M$. Since J_a is an ideal, $rx \in J_a$. By the previous claim, the element $m = 1 - rx$ is invertible in R, which is impossible. Hence, $x \in M$. Thus $J_a \subseteq J_b$.

By symmetry, $J_b \subseteq J_a$, and therefore $J_a = J_b$. $\square$

Definition. In any ring R, the ideal defined by the intersections given in Proposition 3.16 is called the *Jacobson radical* of R, denoted $J(R)$.

Since all right or left primitive ideals of R are prime (Proposition 3.15), the prime radical of R is contained in $J(R)$. For example, if $R = k[[x]]$ is a power series ring over a field k, then 0 is a prime ideal of R, and so the prime radical of R is 0. On the other hand, xR is the unique maximal ideal of R, whence $J(R) = xR$.

Definition. A ring R is *semiprimitive* (or *Jacobson semisimple*) provided $J(R) = 0$. A *semiprimitive ideal* (or *J-ideal*) in a ring R is any ideal I such that $J(R/I) = 0$.

By Proposition 3.16, an ideal I of a ring R is semiprimitive if and only if I is an intersection of right primitive ideals, if and only if I is an intersection of left primitive ideals. Hence, the semiprimitive ideals of R stand in the same relation to the primitive ideals as the semiprime ideals do to the prime ideals.

Note that all semiprimitive rings are semiprime. The example $k[[x]]$ of a power series ring over a field shows that not all semiprime (or even prime) rings are semiprimitive.

Exercise 3Q. Find the Jacobson radical of a formal triangular matrix ring $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$. □

Exercise 3R. Show that $J(M_n(R)) = M_n(J(R))$ for any ring R and any $n \in \mathbb{N}$. □

Exercise 3S. Show that a one-sided ideal I in a ring R is contained in $J(R)$ if and only if $1 - x$ is invertible for each $x \in I$. □

Exercise 3T. Show that the Jacobson radical of a ring R contains all nil one-sided ideals of R. □

Exercise 3U. If x is an element of a ring R and $x + J(R)$ is invertible in $R/J(R)$, show that x is invertible in R. □

Exercise 3V. If R is a commutative semiprime ring, show that the polynomial ring $R[x]$ is semiprimitive. [Hints: First reduce to the case that R is prime. If $p \in J(R[x])$, note that $1 + px$ is invertible in $R[x]$.] □

Theorem 3.17. [Jacobson, Azumaya] *If A is a finitely generated right module over a ring R and $AJ(R) = A$, then $A = 0$.*

Proof. If A is cyclic, say $A = aR$, then $A = AJ(R) = aJ(R)$, and so $a = ax$ for some $x \in J(R)$. Since $1 - x$ is invertible (Exercise 3S), we obtain $a = 0$ and thus $A = 0$.

Now suppose that $A = a_1R + \cdots + a_nR$, where $n > 1$, and that the theorem holds for modules with $n - 1$ generators. Since A/a_1R has $n - 1$ generators, and since

$$(A/a_1R)J(R) = A/a_1R,$$

it follows that $A/a_1R = 0$, that is, $A = a_1R$. Therefore, by the cyclic case, $A = 0$. □

Theorem 3.17 is often called *Nakayama's Lemma.* It implies in particular that if A is a finitely generated right R-module and $a_1, \ldots, a_n$ are elements of

A such that the cosets $a_i + AJ(R)$ generate $A/AJ(R)$, then $a_1, \dots, a_n$ generate A. (To see this, consider the module $A/(a_1R + \cdots + a_nR)$.)

• PRIME IDEALS IN DIFFERENTIAL OPERATOR RINGS •

In this section, we will in some sense write down all of the prime ideals of a special class of noetherian rings – the differential operator rings $R[x;\delta]$, where R is assumed to be a commutative noetherian $\mathbb{Q}$-algebra. This will illustrate some of the phenomena that occur in more general settings and will also give us more examples of primitive noetherian rings.

Lemma 3.18. *Let R be a ring, δ a derivation on R, and $S = R[x;\delta]$.*

(a) *If I is a right ideal of R, then IS is a right ideal of S and $IS \cap R = I$.*

(b) *If I is a δ-ideal of R, then IS is an ideal of S and $IS = SI$.*

(c) *If J is an ideal of S, then $J \cap R$ is a δ-ideal of R.*

Proof. Most of this is an easy computation. In (a), $IS \cap R = I$ because R is a direct summand of S as a left R-module. For (b), note that, if I is a δ-ideal and $a \in I$, then, because $xa = ax + \delta(a)$ and $\delta(a) \in I$, we have $xa \in IS$. For (c), note that if $a \in J \cap R$, then $\delta(a) = xa - ax \in J \cap R$. $\square$

Lemma 3.19. *Let R be a commutative integral domain of characteristic zero with a nonzero derivation δ, and let $S = R[x;\delta]$. If I is a nonzero ideal of S, then $I \cap R \neq 0$.*

Proof. Pick a nonzero element $s = s_nx^n + \cdots$ from I with degree n and leading coefficient s_n, and assume that $n \geq 1$. Choose $r \in R$ such that $\delta(r) \neq 0$, and look at the element $sr - rs$. An immediate calculation shows that

$$sr - rs = ns_n\delta(r)x^{n-1} + [\text{terms of degree less than } n-1].$$

Since under our hypotheses $ns_n\delta(r) \neq 0$, we see that I contains a nonzero element of degree $n-1$. Hence, iterating this argument, we conclude that I contains a nonzero element of degree 0. $\square$

Lemma 3.20. *If R is a ring, δ a derivation on R, and P a minimal prime ideal of R such that R/P has characteristic zero, then P is a δ-ideal.*

Proof. Let $Q = \{r \in R \mid \delta^n(r) \in P \text{ for all } n \geq 0\}$. Using Leibniz's Rule (Exercise 2K), it is clear that Q is an ideal of R and is contained in P. We show that Q is prime as follows. Consider any $a, b \in R \setminus Q$. Choose nonnegative integers r and s as small as possible so that $\delta^r(a)$ and $\delta^s(b)$ are not in P, and then choose $c \in R$ such that $\delta^r(a)c\delta^s(b) \notin P$. Now use Leibniz's Rule to expand $\delta^{r+s}(acb)$, as follows:

$$\begin{aligned}\delta^{r+s}(acb) &= \sum_{i=0}^{r+s} \tbinom{r+s}{i} \delta^{r+s-i}(a)\delta^i(cb) \\ &= \sum_{i=0}^{r+s}\sum_{j=0}^{i} \tbinom{r+s}{i}\tbinom{i}{j} \delta^{r+s-i}(a)\delta^{i-j}(c)\delta^j(b).\end{aligned}$$

Since $\delta^{r+s-i}(a) \in P$ whenever $i > s$ and $\delta^j(b) \in P$ whenever $j < s$, all of the terms in the last summation are in P except for $\binom{r+s}{s}\binom{s}{s}\delta^r(a)c\delta^s(b)$, which is not in P because $\delta^r(a)c\delta^s(b)$ is not and R/P has characteristic zero. Thus, $\delta^{r+s}(acb) \notin P$, and so $acb \notin Q$, which shows that Q is prime. Since P is a minimal prime, we must have $P = Q$, and then, since Q is clearly a δ-ideal, the result follows. $\square$

In the next two proofs, we shall make use of Exercise 2ZA. For the case of a differential operator ring, it may be phrased as follows. Let R be a ring, δ a derivation on R, and $S = R[x;\delta]$. If I is a δ-ideal of R and $\hat{\delta}$ the derivation on R/I induced by δ, then $S/IS \cong (R/I)[\hat{x};\hat{\delta}]$.

Lemma 3.21. *Let R be a noetherian $\mathbb{Q}$-algebra with a derivation δ. Let $S = R[x;\delta]$, and let P be a prime ideal of S. Then $P \cap R$ is a prime ideal of R.*

Proof. Since $P \cap R$ is a δ-ideal of R (Lemma 3.18), we can use Exercise 2ZA to reduce to a differential operator ring over $R/(P \cap R)$. Hence, we may assume that $P \cap R = 0$. If Q is any minimal prime of R, then R/Q has characteristic zero (since $R \supseteq \mathbb{Q}$), and so, by Lemma 3.20, Q is a δ-ideal. According to Theorem 3.4, there are minimal primes $Q_1, \dots, Q_m$ in R such that $Q_1Q_2\cdots Q_m = 0$. From Lemma 3.18, we infer that each Q_iS is an ideal of S, and that

$$(Q_1S)(Q_2S)\cdots(Q_mS) = Q_1Q_2\cdots Q_mS = 0.$$

Since P is prime, we have $Q_iS \subseteq P$ for some index i. Hence, $Q_i \subseteq P \cap R = 0$, and so $P \cap R = Q_i$ is a prime ideal, as claimed. $\square$

Theorem 3.22. *Let R be a commutative noetherian $\mathbb{Q}$-algebra and $S = R[x;\delta]$ a differential operator ring.*

(a) *If P is any prime ideal of S, then $P \cap R$ is a prime δ-ideal of R.*

(b) *If Q is a prime δ-ideal of R, then QS is a prime ideal of S such that $QS \cap R = Q$. Furthermore, if P is any prime ideal of S such that $P \cap R = Q$, then either $P = QS$ or $\delta(R) \subseteq Q$, and in the latter case S/QS and S/P are commutative rings.*

(c) *All prime factor rings of S are domains.*

Proof. (a) This is contained in Lemmas 3.18 and 3.21.

(b) By Lemma 3.18, QS is an ideal of S such that $QS \cap R = Q$. From Exercise 2ZA we have that $S/QS \cong (R/Q)[\hat{x};\hat{\delta}]$, where $\hat{\delta}$ is the derivation on R/Q induced by δ. Since R/Q is a domain, S/QS is a domain (Exercise 2O), and hence QS is a prime ideal of S.

If P is a prime ideal of S such that $P \cap R = Q$ but $P \neq QS$, then the image of P/QS in $(R/Q)[\hat{x};\hat{\delta}]$ is a nonzero ideal I such that $I \cap (R/Q) = 0$. It follows from Lemma 3.19 that $\hat{\delta} = 0$, whence $\delta(R) \subseteq Q$. Moreover, $(R/Q)[\hat{x};\hat{\delta}]$ is then an ordinary polynomial ring over the commutative ring R/Q. Thus in this case S/QS is commutative, as is S/P (since $P \supseteq QS$).

(c) In the notation of part (b), if $P = QS$, we have already seen that S/P is a domain. Otherwise, S/P is a commutative prime ring, and again it is a domain. $\square$

One way to summarize Theorem 3.22 is to say that the prime ideals of S are parametrized by the prime δ-ideals of R. If Q is a prime δ-ideal of R and $\delta(R) \not\subseteq Q$, there is a unique prime ideal of S that contracts to Q (that is, whose intersection with R equals Q), namely, QS. If Q is a prime δ-ideal of R and $\delta(R) \subseteq Q$, then S/QS is a commutative ring isomorphic to an ordinary polynomial ring $(R/Q)[\hat{x}]$. In this case, the primes of S that contract to Q correspond to the primes of $(R/Q)[\hat{x}]$ that contract to zero in R/Q; these in turn correspond precisely to the primes in $K[\hat{x}]$, where K is the quotient field of R/Q.

Exercise 3ZE below shows that Lemmas 3.20 and 3.21 and Theorem 3.22 are all false in characteristic p.

Exercise 3W. Let R be a polynomial ring $k[x]$ where k is an algebraically closed field of characteristic zero, and let $S = R[y; \delta]$ where $\delta = x\frac{d}{dx}$. Show that the only δ-ideals of R are 0 and the ideals $x^n R$ (for $n = 0, 1, \dots$). Show that the only prime ideals of S are 0 and xS together with $xS + (y - \alpha)S$ for all $\alpha \in k$. Then show (without the computations used in Exercises 3O,P) that S is right primitive. [Hint: If $\alpha \in k$ is nonzero, then $xS + (x - \alpha)S = S$. Hence, no proper right ideal containing $(x - \alpha)S$ can contain a nonzero prime ideal.] $\square$

Exercise 3X. Let R be a polynomial ring $k[x]$ where k is a field of characteristic zero, and let δ be any nonzero k-linear derivation on R. Show that there is a nonzero polynomial $g \in R$ such that $\delta = g\frac{d}{dx}$. If $S = R[y; \delta]$, show that S is right and left primitive. $\square$

We end the section with an example showing that Theorem 3.22 does not carry over to general skew polynomial rings $R[x; \alpha, \delta]$, even in characteristic zero.

Exercise 3Y. Let k be a field of characteristic 0, and let $U(\mathfrak{sl}_2(k)) = R[f; \alpha, \delta]$ as in Exercise 2S, where R is the k-subalgebra generated by e and h. Since $\mathfrak{sl}_2(k)$ is, by definition, a Lie subalgebra of $M_2(k)$, the vector space $V = k^2$ becomes a left $U(\mathfrak{sl}_2(k))$-module, such that the module multiplication of any element of $\mathfrak{sl}_2(k)$ with any column vector from V is given by matrix multiplication.

Show that V is a simple $U(\mathfrak{sl}_2(k))$-module, and conclude that its annihilator, call it P, is a left primitive ideal of $U(\mathfrak{sl}_2(k))$. (In fact, $U(\mathfrak{sl}_2(k))/P \cong M_2(k)$, and so P is a maximal ideal.) Now show that $P \cap R = \langle e^2, h^2 - 1 \rangle$, and conclude that $P \cap R$ is neither an α-ideal nor a δ-ideal of R. Finally, show that $P \cap R$ is not a prime (or even semiprime) ideal of R. $\square$

• ADDITIONAL EXERCISES •

3Z. (This puts the proof of Theorem 3.7 in a general setting, and is McCoy's noncommutative analog of Lemma 3.5.) An *m-system* in a ring R is a subset $X \subseteq R \setminus \{0\}$ such that for any $x, y \in X$ there exists $r \in R$ with $xry \in X$. Show that any ideal of R which is maximal with respect to being disjoint from X is a prime ideal. $\square$

3ZA. Let $R = \mathcal{O}_q(k^2)$, where k is a field and $q \in k^\times$ is not a root of unity. Show that $R/R(xy-1)$ is a faithful simple left R-module. [Hint: First show that the cosets of $1, x, y, x^2, y^2, \ldots$ form a basis for this module.] Similarly, show that $R/(xy-1)R$ is a faithful simple right R-module, and conclude that R is a left and right primitive ring. $\square$

3ZB. Let $S = A_1(k) = k[y][x; d/dy]$, where k is a field of characteristic zero, and set $R = k + xS$, which is a subring of S.

(a) Observe that xS is a proper nonzero ideal of R, so that R is not a simple ring. Show that xS is the only proper nonzero ideal of R. [Hint: If I is a nonzero ideal of R, then $SIxS = S$.]

(b) Show that xS is a maximal right ideal of S. [Hint: Given a right ideal $I \supseteq xS$, observe that $xp - px \in I$ for all $p \in I$.] Observe also that xS is a maximal right ideal of R.

(c) Show that R is a maximal right R-submodule of S. [Hint: Given $p \in S \setminus R$, observe that $px \notin xS$, whence $xS + pxS = S$.]

(d) Show that $(S/R)_R$ is a faithful simple right R-module. Thus R is a right primitive ring.

(e) Show that R is a right noetherian ring. [Hint: Given a right ideal J of R, show that there exist $a_1, \ldots, a_n \in J$ such that $JS = a_1S + \cdots + a_nS$. Then use the simplicity of $(S/R)_R$ to show that the right R-module $JS/(a_1R + \cdots + a_nR)$ is noetherian.]

The next exercise is to show that R is also left primitive and left noetherian. $\square$

3ZC. (a) In the situation of Exercise 3ZB, show that there is a k-algebra isomorphism $\phi : R \to k + Sx$ such that $x\phi(r) = rx$ for all $r \in R$.

(b) Show that there is a k-algebra isomorphism $\psi : S \to S^{\text{op}}$ such that ψ is the identity map on $k[y]$ while $\psi(x) = -x$. [Hint: Exercise 2R.] Observe that $\psi\phi$ gives a k-algebra isomorphism of R onto R^{op}. Hence, R is left primitive and left noetherian. $\square$

3ZD. In the situation of Exercise 3ZB, let $A = S/xS$, viewed as a right R-module.

(a) Show that A is finitely generated.

(b) Show that the only proper nonzero R-submodule of A is R/xS. [Hint: If B/xS is a proper R-submodule of A, observe that $BxS + xS$ is a proper right ideal of S.]

(c) Show that 0 is an annihilator prime of A but not an associated prime. $\square$

3ZE. Let k be a field of characteristic $p > 0$.

(a) Let z be an indeterminate, and let E be the ring of all $k[z^p]$-module endomorphisms of $k[z]$. (Note that since $k[z]$ is a free $k[z^p]$-module with basis $\{1, z, \dots, z^{p-1}\}$, there is an isomorphism of E onto $M_p(k[z^p])$.) Let $m_z \in E$ be multiplication by z. Show that E is generated as a k-algebra by m_z and d/dz. [Hint: Consider the endomorphisms

$$e_{ij} = \tfrac{1}{(p-1)!} m_z^i \left(\tfrac{d}{dz}\right)^{p-1} m_z^{p-1-j}$$

for $i, j = 0, \dots, p-1$.]

(b) Let $R = k[y]/y^p k[y]$ for an indeterminate y, and let u denote the coset $y + y^p k[y]$ in R. Show that there is a unique well-defined k-linear derivation δ on R such that $\delta(u) = 1$.

(c) Show that there is a k-algebra isomorphism of $R[x;\delta]$ onto E sending u to $-d/dz$ and x to m_z. Hence, $R[x;\delta] \cong M_p(k[z^p])$; in particular, $R[x;\delta]$ is a prime ring. $\square$

3ZF. Let $R = A_1(\mathbb{Z}) = \mathbb{Z}[y][x; d/dy]$, and let p be a prime integer.

(a) Show that pR is a prime ideal of R and that R/pR is a domain.

(b) Show that $pR + y^pR$ is a prime ideal of R and that $R/(pR + y^pR) \cong M_p((\mathbb{Z}/p\mathbb{Z})[t])$ for an indeterminate t.

(c) Finally, show that $pR + y^pR + x^pR$ is a maximal ideal of R and that $R/(pR + y^pR + x^pR) \cong M_p(\mathbb{Z}/p\mathbb{Z})$. $\square$

3ZG. If all prime ideals in a commutative ring R are finitely generated, show that R is noetherian. [Hint: If not, find an ideal maximal among non-finitely-generated ideals.] $\square$

3ZH. If an ideal I in a commutative ring R is contained in the union of finitely many prime ideals $P_1, \dots, P_n$, show that $I \subseteq P_j$ for some j. [Hint: Without loss of generality, $P_i \nsubseteq P_j$ for $i \neq j$. If $I \nsubseteq P_j$, choose $x_j \in I \cdot \prod_{i \neq j} P_i$ such that $x_j \notin P_j$.] $\square$

3ZI. Show that if R is a ring which satisfies the ascending chain condition on J-ideals, then every J-ideal is a finite intersection of prime J-ideals. [The most important non-noetherian rings satisfying this condition are the affine P.I. algebras – that is, algebras over a field which are finitely generated as algebras and which satisfy a polynomial identity.] $\square$

• NOTES •

Prime Ideals. The definition via products of ideals was introduced by Krull in both the commutative and noncommutative cases [1928a, p. 5; 1928b, Definition 3, p. 486].

Existence of Minimal Primes. That every prime contains a minimal prime was first proved in the commutative case by Krull [1929, Satz 5].

Finiteness of the Set of Minimal Primes. This was proved for rings with the ACC on semiprime ideals by Nagata [1951, Corollary to Proposition 34].

Semiprime Ideals. These were introduced in the commutative case by Krull [1929, p. 735] and in the noncommutative case by Nagata [1951, Definition 1].

Ideals Disjoint from Multiplicative Sets. That an ideal in a commutative ring maximal with respect to disjointness from a multiplicative set must be prime was observed by Krull in the proof of [1929, Lemma, p. 732].

Semiprime Commutative Rings. Krull proved that a commutative ring is semiprime if and only if it has no nonzero nilpotent elements [1929, Satz 4].

Criterion for Semiprime Ideals. That an ideal is semiprime precisely when the condition (*) of Theorem 3.7 holds is equivalent to a result proved independently by Levitzki [1951, Theorem 2] and Nagata [1951, Proposition 8 and Corollary], namely, that the "McCoy radical" of a ring R coincides with the "Baer lower radical." The McCoy radical is the set of those elements of R not contained in any m-system (see Exercise 3Z), and McCoy proved that this equals the intersection of the prime ideals of R [1949, Theorem 2]; the Baer lower radical of R is the smallest ideal N such that R/N has no nonzero nilpotent ideals.

Primitive Rings. These were introduced by Jacobson [1945a, Definition 3].

Jacobson Radical. Jacobson defined the radical of an arbitrary ring R to be the sum of those right ideals I of R such that $1+z$ is right invertible for all $z \in I$ [1945a, Definition 2], proved that this definition is left-right symmetric [1945a, Theorems 1, 2], and showed that the radical equals the intersection of the maximal right ideals of R [1945a, Corollary 2 to Theorem 18]. Earlier, Perlis had characterized the radical of a finite dimensional algebra R as the set of those $z \in R$ such that $u+z$ is a unit for all units u [1942, Theorem 1].

Nakayama's Lemma. This was first proved by Jacobson in the case that A is a right ideal contained in the radical [1945a, Theorem 10]; Azumaya then carried over Jacobson's proof to the module case [1951, Theorem 1] (see also Nagata [1950, Footnote 3, p. 67]). An alternate proof derived from a generalized result was presented by Nakayama [1951, (II)].

Invariance of Minimal Primes under Derivations. This was proved for associated primes as well as minimal primes in commutative noetherian $\mathbb{Q}$-algebras by Seidenberg [1967, Theorem 1]. For minimal completely prime ideals in noncommutative $\mathbb{Q}$-algebras it follows from a result of Dixmier [1966, Lemme 6.1], and his argument was extended to arbitrary minimal primes in $\mathbb{Q}$-algebras by Gabriel [1971, Lemme 3.4].

Contraction of Primes in Differential Operator Rings. Gabriel proved that in a differential operator ring $R[x;\delta]$ over a right noetherian $\mathbb{Q}$-algebra R, every prime of $R[x;\delta]$ contracts to a prime of R [1971, Proposition 3.3(b)].

4. Semisimple Modules, Artinian Modules, and Torsionfree Modules

In this chapter, we discuss several special types of modules that will make frequent appearances later. These include semisimple modules, artinian modules, and torsionfree modules, which have prototypes in vector spaces, finite abelian groups, and torsionfree abelian groups, respectively. That special types of modules have useful roles in the study of arbitrary modules may be seen already in the case of abelian groups (i.e., $\mathbb{Z}$-modules). In studying an arbitrary abelian group, an almost reflexive first step is to look at its torsion part (i.e., the torsion subgroup) and its torsionfree part (i.e., the factor group modulo the torsion subgroup), since entirely different techniques are available (and needed) for dealing with torsion groups and torsionfree groups. On the torsionfree side, vector spaces make an appearance due to the facts that the torsionfree divisible abelian groups are exactly the vector spaces over $\mathbb{Q}$ and that every torsionfree abelian group can be embedded in a divisible one. On the torsion side, many questions can be reduced to the case of finite abelian groups, since every torsion abelian group is a directed union of finite subgroups. In studying torsion abelian groups, one also reduces to the case of p-groups for various primes p. Vector spaces make another appearance here, since in an abelian p-group the set of elements of order p (together with 0) forms a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Viewed module-theoretically, vector spaces are distinguished by many nice decomposition properties. For instance, every vector space is a direct sum of 1-dimensional subspaces, and every subspace of a vector space is a direct summand. We view simple modules as analogs to 1-dimentional vector spaces, and the corresponding analogs to higher dimensional vector spaces are the *semisimple* modules: modules which are (direct) sums of simple submodules. Other decomposition properties follow; in particular, we shall find that a module is semisimple if and only if every submodule is a direct summand.

The finite abelian groups are perhaps more appropriately characterized as the finitely generated torsion abelian groups. Their "closest relatives" would then be the finitely generated torsion modules over a polynomial ring $k[x]$ with k a field; these are precisely the finite dimensional $k[x]$-modules. From this perspective, the essential features are the ascending and descending chain conditions on submodules. It has proved advantageous to study these conditions separately – in the form of *noetherian* modules and *artinian* modules – as well as together.

While it is clear enough what is meant by a torsionfree module over a commutative domain, it is not at all obvious what the appropriate notion is for modules over an arbitrary ring. In fact, there exists a large body of research on "torsion theories" which sidesteps the question of which notion to choose by axiomatizing the family of possible choices. We shall not discuss the most general notion, but it will be useful to develop concepts of "torsion" and "torsionfree" relative to various multiplicatively closed sets X in a ring R, where an R-module will be called "X-torsion" if each of its elements is annihilated by an element from X. For example, if $X = \mathbb{Z}\setminus\{0\}$, the X-torsion $\mathbb{Z}$-modules are the usual torsion abelian groups, i.e., those abelian groups in which every element has finite order. Moreover, if $Y = \{1, p, p^2, \dots\}$ for some prime p, then the Y-torsion $\mathbb{Z}$-modules are just the abelian p-groups.

• SEMISIMPLE MODULES •

Definition. The *socle* of a module A is the sum of all simple submodules of A and is denoted $\mathrm{soc}(A)$. (By convention, the sum of the empty family of submodules is the zero submodule. Hence, $\mathrm{soc}(A) = 0$ if and only if A has no simple submodules.) A *semisimple* (or *completely reducible*) module is any module A such that $\mathrm{soc}(A) = A$.

For example, any vector space A over a field k is a semisimple k-module, since all the 1-dimensional subspaces of A are simple k-submodules. For another example, the socle of an abelian group A consists of all elements of A of finite squarefree order (including order 1).

In any ring R, observe that $\mathrm{soc}(R_R)$ is an ideal of R. (If $r \in R$, recall that the function taking x to rx is a homomorphism of the right module R_R into itself. Hence, if A is a simple right ideal of R, then either $rA = 0$ or rA is simple, and so $rA \leq \mathrm{soc}(R_R)$ in either case.) Similarly, $\mathrm{soc}({}_RR)$ is an ideal of R, but these two socles need not coincide, as the following example shows.

Exercise 4A. If $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ for a field k, show that $\mathrm{soc}(R_R) \neq \mathrm{soc}({}_RR)$. □

Exercise 4B. Let R be a semiprime ring.

(a) Show that any simple right or left ideal of R is generated by an idempotent. [Hint: If I is a simple right ideal of R, show that there exists $x \in I$ for which $xI \neq 0$ and observe that $I \cap \mathrm{r.ann}(x) = 0$.]

(b) Given an idempotent $e \in R$, show that eR is a simple right ideal if and only if Re is a simple left ideal, if and only if eRe is a division ring.

(c) Show that $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$. □

Recall that a family $\{A_i \mid i \in I\}$ of submodules of a module A is *independent* if and only if the sum of the A_i is a direct sum – more precisely, if and only if the summation map $\bigoplus_{i\in I} A_i \to \sum_{i\in I} A_i$ is an isomorphism. In this case, we write $\sum_{i\in I} A_i = \bigoplus_{i\in I} A_i$, that is, we identify the internal and external direct sums of the A_i.

Proposition 4.1. *The socle of any module A is a direct sum of simple submodules of A.*

Proof. Let $\mathcal{B} = \{B_i \mid i \in I\}$ be a maximal independent family of simple submodules of A, and set $B = \sum_{i\in I} B_i = \bigoplus_{i\in I} B_i$. Then $B \le \operatorname{soc}(A)$, and if $B \ne \operatorname{soc}(A)$, there is a simple submodule $S \le A$ such that $S \not\le B$. Now $S \cap B \ne S$. As S is simple, $S \cap B = 0$. But then $\mathcal{B} \cup \{S\}$ is independent, contradicting the maximality of $\mathcal{B}$. Therefore $B = \operatorname{soc}(A)$. $\square$

Exercise 4C. Let B, C, D be submodules of a module A.

(a) Establish the *modular law*: If $B \le D$, then $(B+C)\cap D = B+(C\cap D)$.

(b) If $B \le D$ and $A = B \oplus C$, show that $D = B \oplus (C \cap D)$. $\square$

Proposition 4.2. *A module A is semisimple if and only if every submodule of A is a direct summand of A.*

Proof. Assume first that A is semisimple, and let B be a submodule of A. By Zorn's Lemma, there is a submodule C of A maximal with respect to the property $B \cap C = 0$. If $B \oplus C < A$, there is a simple submodule $S \le A$ such that $S \not\le B \oplus C$. As in the previous proof, $S \cap (B \oplus C) = 0$, and so $\{B, C, S\}$ is independent. But then $B \cap (C \oplus S) = 0$, contradicting the maximality of C. Therefore $B \oplus C = A$.

Conversely, assume that every submodule of A is a direct summand. In particular, $A = \operatorname{soc}(A) \oplus B$ for some submodule B. If $B \ne 0$, let C be a nonzero cyclic submodule of B. If c is a generator for C, then, by Zorn's Lemma, C has a submodule M which is maximal with respect to the property $c \notin M$. Then M is a maximal proper submodule of C. Now $A = M \oplus N$ for some submodule N, and $C = M \oplus (C \cap N)$ by Exercise 4C. Note that $C \cap N \cong C/M$, whence $C \cap N$ is a simple submodule of A, and so $C \cap N \le \operatorname{soc}(A)$. As $C \cap N \le B$, this is impossible. Thus $B = 0$, and therefore $A = \operatorname{soc}(A)$. $\square$

Corollary 4.3. *Any submodule of a semisimple module is semisimple.*

Proof. If C is a submodule of a semisimple module A, and B is a submodule of C, then B is a direct summand of A, and it follows from Exercise 4C that B is also a direct summand of C. $\square$

Exercise 4D. Let A be a module such that $A = \bigoplus_{i\in I} S_i$, where the S_i are simple submodules of A, and let B be a submodule of A. Show that there is a subset $J \subseteq I$ such that $A = B \oplus \left(\bigoplus_{j\in J} S_j\right)$, and conclude that $B \cong \bigoplus_{i\in I\setminus J} S_i$. $\square$

Exercise 4E. (a) A semisimple module A is said to be *homogeneous* if A is a direct sum of pairwise isomorphic simple submodules. Show that A is homogeneous if and only if all simple submodules of A are isomorphic to each other.

(b) Show that an arbitrary semisimple module A is a direct sum of homogeneous submodules A_i such that $\operatorname{Hom}(A_i, A_j) = 0$ when $i \neq j$. $\square$

Exercise 4F. Show that the endomorphism ring of any simple module is a division ring. (This result is known as *Schur's Lemma.*) If A is a nonzero finitely generated semisimple module, show that its endomorphism ring is isomorphic to a finite direct product of matrix rings over division rings. $\square$

• SEMISIMPLE RINGS •

Very few rings share the property with fields that all their modules are semisimple. The rings that do are those that appear in the Wedderburn-Artin Theorem. Since this is a standard part of many developments of the Wedderburn-Artin Theorem (which the reader is likely to have seen elsewhere), we will be somewhat sketchy in our treatment. We begin by recalling key properties of matrix rings over a division ring.

Exercise 4G. If $R = M_n(D)$ for some $n \in \mathbb{N}$ and some division ring D, show that R_R is a direct sum of n pairwise isomorphic simple right modules and that ${}_RR$ is a direct sum of n pairwise isomorphic simple left modules. [Hint: In case $n = 2$, look at the right ideals $I_1 = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$, and observe that $I_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} I_1$.] Show also that R is a simple ring. $\square$

Theorem 4.4. [Noether] *For any ring R, the following conditions are equivalent:*

(a) *All right R-modules are semisimple.*

(b) *All left R-modules are semisimple.*

(c) *R_R is semisimple.*

(d) *${}_RR$ is semisimple.*

(e) *Either R is the zero ring or $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for some positive integers n_i and some division rings D_i.*

Proof. Obviously (a) $\Longrightarrow$ (c). Conversely, if R_R is semisimple, then all cyclic right R-modules are semisimple, from which it is immediate that any right R-module is semisimple (since any module is the sum of its cyclic submodules). Thus (a) $\Longleftrightarrow$ (c), and similarly (b) $\Longleftrightarrow$ (d).

That (e) $\Longrightarrow$ (c) follows from Exercise 4G. On the other hand, if R_R is semisimple, then by Exercise 4F the endomorphism ring of R_R has the desired form, and since $R \cong \operatorname{End}_R(R_R)$, we have proved that (c) $\Longrightarrow$ (e). Finally, since (e) is clearly a symmetric condition, it follows that all of the given conditions are equivalent. $\square$

Definition. A ring satisfying the conditions of Theorem 4.4 is called a *semisimple ring*.

The reader should be warned that, in the older literature, "semisimple" is often used to mean "Jacobson semisimple," i.e., "semiprimitive." In that case, the rings in Theorem 4.4 are usually called "semisimple artinian" rings.

Exercise 4H. (This is for readers who are familiar with projective and injective modules.) Show that a ring R is semisimple if and only if all R-modules are projective, if and only if all R-modules are injective. □

• ARTINIAN MODULES •

Classically, the Jacobson and prime radicals were introduced in the context of artinian rings (where, as we will see, the notions coincide). To refresh the reader's memory, we recall some of the theory of artinian modules and rings.

Definition. A module A is *artinian* provided A satisfies the *descending chain condition* (or *DCC*) on submodules, i.e., there does not exist a properly descending infinite chain $A_1 > A_2 > \cdots$ of submodules of A. Equivalently, A is artinian if and only if every nonempty family of submodules of A has a minimal element. A ring R is *right (left) artinian* provided the right module R_R (left module ${}_RR$) is artinian. If both conditions hold, R is called an *artinian ring*.

For example, any finite dimensional algebra over a field is an artinian ring.

Our first few results concerning artinian modules are completely analogous to the corresponding results for noetherian modules (see Proposition 1.2 and Corollaries 1.3 and 1.4). The proofs of the artinian results may be obtained by imitating the proofs in the noetherian case, reversing inclusions when necessary.

Proposition 4.5. *Let B be a submodule of a module A. Then A is artinian if and only if B and A/B are both artinian.* □

Corollary 4.6. *Any finite direct sum of artinian modules is artinian.* □

Corollary 4.7. *If R is a right (left) artinian ring, then all finitely generated right (left) R-modules are artinian.* □

Definition. A *composition series* for a module A is a chain of submodules

$$A_0 = 0 < A_1 < \cdots < A_n = A$$

such that each of the factors A_i/A_{i-1} is a simple module. The number of gaps (namely n) is called the *length* of the composition series, and the factors A_i/A_{i-1} are called the *composition factors* of A corresponding to this composition series. By convention, the zero module is considered to have a composition series of length zero, with no composition factors. A *module of finite length* is any module which has a composition series.

Proposition 4.8. *A module A has finite length if and only if A is both noetherian and artinian.*

Proof. If A has finite length, then (since simple modules are clearly noetherian and artinian) it follows from Propositions 1.2 and 4.5 that A must be noetherian and artinian. Conversely, assume that A satisfies both chain conditions and set $A_0 = 0$. If $A \neq 0$, then, by the DCC, A contains a minimal nonzero submodule A_1, that is, A_1 is simple. Similarly, if $A_1 < A$, then A/A_1 contains a simple submodule A_2/A_1, and we continue in this manner. By the ACC, the chain $A_0 < A_1 < A_2 < \cdots$ must terminate at some stage n. Then $A_n = A$, and the chain $A_0 < A_1 < \cdots < A_n$ is a composition series for A. □

For instance, if R is an algebra over a field k, then any R-module which is finite dimensional over k has finite length.

Exercise 4I. Show that the $\mathbb{Z}$-modules $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ have finite length and that the first has just one composition series, while the second has two. Show that the $\mathbb{Q}$-module $\mathbb{Q} \oplus \mathbb{Q}$ has finite length, and that it has infinitely many composition series. □

If a module A has finite length, it may have many different composition series (as in the exercise above), and the obvious question to ask is how different composition series for A are related. It will turn out that all composition series for A have the same length and the same composition factors (up to a permutation of the list). This result can be obtained as a consequence of a general refinement theorem concerning arbitrary chains of submodules, and since we will need the refinement theorem later, we prove it next and then return to composition series.

Definition. A *submodule series* (or *normal series*) for a module A is any finite chain of submodules of the form

$$A_0 = 0 \leq A_1 \leq \cdots \leq A_n = A.$$

A *refinement* of this series is any submodule series that includes all the A_i, that is, a submodule series

$$B_0 = 0 \leq B_1 \leq \cdots \leq B_t = A$$

such that each A_i occurs in the list $B_0, \ldots, B_t$. Two submodule series

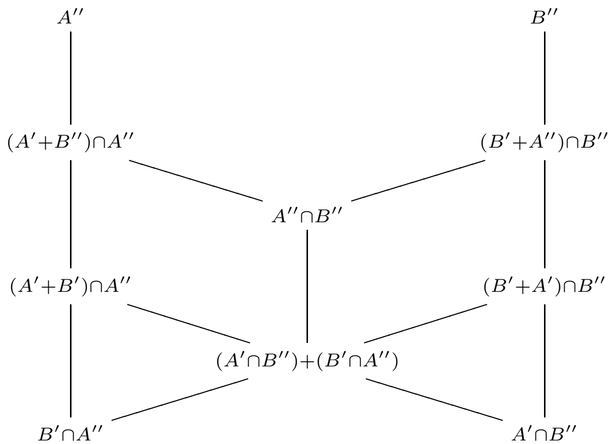
$$A_0 = 0 \leq A_1 \leq \cdots \leq A_n = A \qquad \text{and} \qquad B_0 = 0 \leq B_1 \leq \cdots \leq B_t = A$$

are said to be *isomorphic* (or *equivalent*) provided $n = t$ and there exists a permutation π of $\{1, 2, \ldots, n\}$ such that $A_i/A_{i-1} \cong B_{\pi(i)}/B_{\pi(i)-1}$ for all $i = 1, \ldots, n$.

Lemma 4.9. [Zassenhaus] *Let A', A'', B', B'' be submodules of a module A such that $A' \leq A''$ and $B' \leq B''$. Then*

$$\big[(A' + B'') \cap A''\big]/\big[(A' + B') \cap A''\big] \cong \big[(B' + A'') \cap B''\big]/\big[(B' + A') \cap B''\big].$$

The relevant submodules of A together with the inclusions among them may be arranged as in the following diagram:



Because of the shape of this diagram, the lemma is often called the *Butterfly Lemma*.

Proof. Observe that

$$\begin{aligned}
[(A' + B'') \cap A'']/[(A' + B') \cap A''] &= [A' + (B'' \cap A'')]/[A' + (B' \cap A'')] \\
&\cong (B'' \cap A'')/\big([B'' \cap A''] \cap [A' + (B' \cap A'')]\big) \\
&= (B'' \cap A'')/[(B'' \cap A') + (B' \cap A'')].
\end{aligned}$$

By symmetry, we also have

$$[(B' + A'') \cap B'']/[(B' + A') \cap B''] \cong (A'' \cap B'')/[(A'' \cap B') + (A' \cap B'')]. \qquad \square$$

Theorem 4.10. [Schreier] *Any two submodule series for a module A have isomorphic refinements.*

Proof. Given submodule series

$$A_0 = 0 \leq A_1 \leq \cdots \leq A_n = A \tag{s_1}$$
$$B_0 = 0 \leq B_1 \leq \cdots \leq B_t = A, \tag{s_2}$$

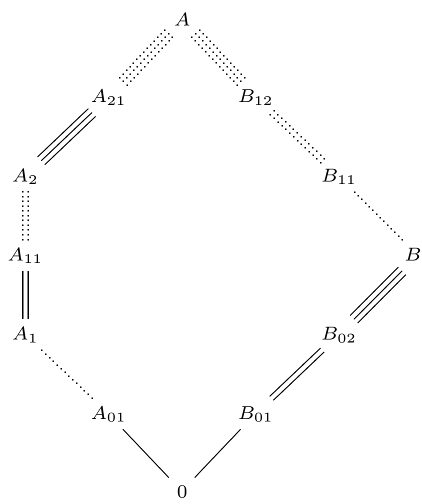
we must find a refinement of each series such that the two refinements are isomorphic. Set

$$A_{ij} = (A_i + B_j) \cap A_{i+1} \qquad (\text{for } i = 0, \ldots, n-1 \text{ and } j = 0, \ldots, t)$$
$$B_{ji} = (B_j + A_i) \cap B_{j+1} \qquad (\text{for } j = 0, \ldots, t-1 \text{ and } i = 0, \ldots, n).$$

Then we obtain refinements for (s_1) and (s_2) as follows:

$$\begin{aligned} A_{00} = 0 \le A_{01} \le \cdots \le A_{0t} = A_1 = A_{10} \le A_{11} \le \cdots \\ \le A_{1t} = A_2 = A_{20} \le \cdots \le A_{n-1,t} = A_n = A \end{aligned} \tag{r_1}$$
$$\begin{aligned} B_{00} = 0 \le B_{01} \le \cdots \le B_{0n} = B_1 = B_{10} \le B_{11} \le \cdots \\ \le B_{1n} = B_2 = B_{20} \le \cdots \le B_{t-1,n} = B_t = A. \end{aligned} \tag{r_2}$$

The diagram below illustrates this procedure in case $n = 3$ and $t = 2$; each pair of matching edges indicates a pair of isomorphic factors.



We leave to the reader the bookkeeping chore of relabelling (r_1) and (r_2) using single indices. Lemma 4.9 shows that $A_{i,j+1}/A_{ij} \cong B_{j,i+1}/B_{ji}$ for $i = 0, \ldots, n-1$ and $j = 0, \ldots, t-1$, from which we conclude that (r_1) and (r_2) are isomorphic. $\square$

Theorem 4.11. [Jordan, Hölder] *If a module A has finite length, then any two composition series for A are isomorphic. In particular, all composition series for A have the same length.*

Proof. Consider two composition series

$$A_0 = 0 < A_1 < \cdots < A_n = A \qquad \text{and} \qquad B_0 = 0 < B_1 < \cdots < B_t = A.$$

By Theorem 4.10, these two submodule series have isomorphic refinements, say

$$C_0 = 0 \leq C_1 \leq \cdots \leq C_m = A \qquad \text{and} \qquad D_0 = 0 \leq D_1 \leq \cdots \leq D_m = A.$$

There is a permutation σ of $\{1, 2, \ldots, m\}$ such that $C_k/C_{k-1} \cong D_{\sigma(k)}/D_{\sigma(k)-1}$ for all $k = 1, \ldots, m$.

Since each of the factors A_i/A_{i-1} is simple, there are no submodules lying strictly between A_{i-1} and A_i. Consequently, the refined series $C_0 \leq C_1 \leq \cdots \leq C_m$ consists of the submodules $A_0, A_1, \ldots, A_n$ in order but with possible repetitions. Hence, among the factors C_k/C_{k-1}, each factor A_i/A_{i-1} occurs exactly once, and the remaining factors are all zero. Similarly, among the factors D_k/D_{k-1}, each factor B_j/B_{j-1} occurs exactly once, and the remaining factors are all zero.

Since $C_k/C_{k-1} \neq 0$ if and only if $D_{\sigma(k)}/D_{\sigma(k)-1} \neq 0$, we conclude that $n = t$ and that there exists a permutation π of $\{1, 2, \ldots, n\}$ such that, whenever $C_k/C_{k-1} = A_i/A_{i-1}$, then $D_{\sigma(k)}/D_{\sigma(k)-1} = B_{\pi(i)}/B_{\pi(i)-1}$. Therefore $A_i/A_{i-1} \cong B_{\pi(i)}/B_{\pi(i)-1}$ for $i = 1, \ldots, n$, which proves that the two given composition series are isomorphic. $\square$

Definition. If A is a module of finite length, the common length of all composition series for A is called the *length* (or the *composition length*) of A, and we shall denote it by $\operatorname{length}(A)$.

For instance, the only module of length 0 is the zero module, and the modules of length 1 are precisely the simple modules. Note that a finitely generated semisimple module A has finite length, and if A is a direct sum of n simple submodules, then $\operatorname{length}(A) = n$.

Proposition 4.12. *Let A be a module of finite length. If B is any submodule of A, then*

$$\operatorname{length}(A) = \operatorname{length}(B) + \operatorname{length}(A/B).$$

Proof. Since this is clear if either $B = 0$ or $B = A$, we may assume that $0 < B < A$. In this case, choose composition series

$$B_0 = 0 < B_1 < \cdots < B_m = B$$
$$C_0/B = 0 < C_1/B < \cdots < C_n/B = A/B$$

for B and A/B. Since the chain

$$B_0 = 0 < B_1 < \cdots < B_m < C_1 < \cdots < C_n = A$$

is a composition series for A, the result follows. $\square$

In particular, if $A_1, \ldots, A_n$ are modules of finite length, then

$$\operatorname{length}(A_1 \oplus \cdots \oplus A_n) = \operatorname{length}(A_1) + \cdots + \operatorname{length}(A_n).$$

Exercise 4J. If A and B are submodules of a module C, and A and B both have finite length, show that $A+B$ has finite length and that

$$\mathrm{length}(A)+\mathrm{length}(B)=\mathrm{length}(A+B)+\mathrm{length}(A\cap B). \quad \square$$

• ARTINIAN RINGS •

Theorem 4.13. [Wedderburn, Artin] *For a ring R, the following conditions are equivalent:*

(a) *R is right artinian and $J(R)=0$.*
(b) *R is left artinian and $J(R)=0$.*
(c) *R is semisimple.*

Proof. (a) $\Longrightarrow$ (c): Let $\mathcal{B}$ be the set of those right ideals I of R such that R/I is a semisimple module and note that $\mathcal{B}$ is nonempty (e.g., $R\in\mathcal{B}$). Since R is right artinian, we may choose a right ideal K minimal in $\mathcal{B}$. If $K\neq 0$, then, since $J(R)=0$, there is a maximal right ideal M in R such that $K\not\leq M$. Since M is maximal, $K+M=R$, and hence

$$R/(K\cap M)\cong(R/K)\oplus(R/M).$$

But then $R/(K\cap M)$ is semisimple, and since $K\cap M<K$, this contradicts the minimality of K. Therefore $K=0$, and so R_R is semisimple.

(c) $\Longrightarrow$ (a): Write $R_R=S_1\oplus\cdots\oplus S_n$, where each S_i is a simple right R-module. (The direct sum must be finite because R_R is finitely generated.) Corollary 4.6 shows that R is right artinian (since each S_i is clearly artinian). Each of the annihilators $\mathrm{r.ann}_R(S_i)$ is a right primitive ideal of R and so contains $J(R)$. Thus $S_iJ(R)=0$ for each i, and consequently $J(R)=0$.

(b) $\Longleftrightarrow$ (c) by symmetry. $\square$

With a little study of socles over artinian rings, we can prove that every artinian ring is also noetherian, and also give a classical characterization of the Jacobson radical of an artinian ring.

Definition. The *socle series* of a module A is the ascending chain

$$\mathrm{soc}^0(A)\leq\mathrm{soc}^1(A)\leq\mathrm{soc}^2(A)\leq\cdots$$

of submodules of A defined inductively by setting $\mathrm{soc}^0(A)=0$ and

$$\mathrm{soc}^{n+1}(A)/\mathrm{soc}^n(A)=\mathrm{soc}\big(A/\mathrm{soc}^n(A)\big)$$

for all nonnegative integers n.

For example, if $A=\mathbb{Z}/p^k\mathbb{Z}$ for some prime integer p and some positive integer k, then $\mathrm{soc}^n(A)=p^{k-n}\mathbb{Z}/p^k\mathbb{Z}$ for $n=0,1,\ldots,k$ and $\mathrm{soc}^n(A)=A$ for all $n\geq k$.

Exercise 4K. Describe the socle series of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. $\square$

Proposition 4.14. *Let R be a ring such that $R/J(R)$ is semisimple, and let A be a right R-module. Then $\operatorname{soc}^n(A) = \operatorname{ann}_A(J(R)^n)$ for all nonnegative integers n.*

Proof. Set $J = J(R)$. Since $\operatorname{soc}^0(A) = 0$ and $J^0 = R$, the proposition is clear in case $n = 0$.

Now assume that the proposition holds for some nonnegative integer n and set $B = \operatorname{soc}^n(A)$. Note from the induction hypothesis that $BJ^n = 0$. Given any simple submodule $S \leq A/B$, note that $\operatorname{ann}_R(S)$ is a right primitive ideal of R. By definition, $J \subseteq \operatorname{ann}_R(S)$, and so $SJ = 0$. Thus $\operatorname{soc}(A/B)J = 0$, whence $\operatorname{soc}^{n+1}(A)J^{n+1} = 0$.

Set $C = \operatorname{ann}_A(J^{n+1})$. As $B = \operatorname{ann}_A(J^n)$ (by the induction hypothesis), we see that $B \leq C$ and $CJ \leq B$. Hence, $(C/B)J = 0$, so that C/B is a right module over R/J. By Theorem 4.4, C/B must be a semisimple module (over R/J, and hence also over R), whence $C/B \leq \operatorname{soc}(A/B)$. Thus $C \leq \operatorname{soc}^{n+1}(A)$. Since the reverse inclusion was shown in the previous paragraph, we therefore have $\operatorname{soc}^{n+1}(A) = \operatorname{ann}_A(J^{n+1})$, completing the induction step. $\square$

In particular, Proposition 4.14 applies to any right or left artinian ring R, since $R/J(R)$ is semisimple according to Theorem 4.13.

Theorem 4.15. [Hopkins, Levitzki] *If R is a right artinian ring, then R is also right noetherian, and $J(R)$ is nilpotent.*

Proof. Set $J = J(R)$. Since the powers of J form a descending chain of ideals, there must exist a positive integer n such that $J^{n+1} = J^n$. In view of Proposition 4.14, it follows that $\operatorname{soc}^{n+1}(R_R) = \operatorname{soc}^n(R_R)$. Hence, if $I = \operatorname{soc}^n(R_R)$, then $\operatorname{soc}((R/I)_R) = 0$.

If $I \neq R$, then R/I has a minimal nonzero right submodule M. But then M is a simple right submodule of R/I, contradicting the fact that $\operatorname{soc}((R/I)_R) = 0$. Thus $I = R$. Hence, by Proposition 4.14, $\operatorname{l.ann}_R(J^n) = \operatorname{soc}^n(R_R) = R$, and so $J^n = 0$. Therefore J is nilpotent.

Set $A_i = \operatorname{soc}^i(R_R)$ for $i = 0, 1, \ldots, n$. These A_i form a chain $A_0 = 0 \leq A_1 \leq \cdots \leq A_n = R$ of right ideals of R. Each of the factors A_i/A_{i-1} is a semisimple right R-module and so is a direct sum of simple modules, by Proposition 4.1.

Suppose that one of the factors A_i/A_{i-1} is a direct sum of an infinite family $\mathcal{B}$ of simple modules. Choose distinct $B_1, B_2, \ldots$ in $\mathcal{B}$, and for $k = 1, 2, \ldots$ let $C_k = \bigoplus_{j=k}^{\infty} B_j$. Then $C_1 > C_2 > \cdots$ is a strictly descending chain of submodules of A_i/A_{i-1}, whence A_i/A_{i-1} is not artinian. As R_R is artinian, this is impossible.

Thus A_i/A_{i-1} is a finite direct sum of simple right R-modules. As simple modules are noetherian, Corollary 1.3 shows that A_i/A_{i-1} is noetherian. Using Proposition 1.2, we conclude that each A_i is noetherian. Therefore, since $R_R = A_n$, the ring R is right noetherian. $\square$

Corollary 4.16. *For a right or left artinian ring, the Jacobson radical equals the prime radical.*

Proof. Since every primitive ideal is prime, the intersection of the primitive ideals contains the intersection of the prime ideals, so that the Jacobson radical contains the prime radical in any ring. Conversely, Theorem 4.15 shows that the Jacobson radical of a right or left artinian ring is nilpotent, and it is thus contained in the prime radical by Corollary 3.9. □

Corollary 4.17. [Noether] *For a ring R, the following conditions are equivalent:*

(a) *R is right artinian and semiprime.*
(b) *R is left artinian and semiprime.*
(c) *R is semisimple.*

Proof. Combine Theorem 4.13 and Corollary 4.16. □

Exercise 4L. Show that any right noetherian, left artinian ring is also right artinian. □

Corollary 4.18. *For a ring R, the following conditions are equivalent:*

(a) *R is prime and right artinian.*
(b) *R is prime and left artinian.*
(c) *R is simple and right artinian.*
(d) *R is simple and left artinian.*
(e) *R is simple and semisimple.*
(f) *$R \cong M_n(D)$ for some positive integer n and some division ring D.*

Proof. (a) $\Longrightarrow$ (f) by Corollary 4.17 and Theorem 4.4, (f) $\Longrightarrow$ (e) by Exercise 4G, (e) $\Longrightarrow$ (c) by Theorem 4.13, and (c) $\Longrightarrow$ (a) is clear. By symmetry, (b), (d), and (f) are also equivalent. □

Because of the symmetry in Corollary 4.18, the rings characterized there are referred to as *simple artinian rings*.

Proposition 4.19. *If R is a nonzero right or left artinian ring, then all prime ideals in R are maximal.*

Proof. If R contains a nonmaximal prime ideal P, then R/P is a prime right or left artinian ring which is not simple, contradicting Corollary 4.18. □

Proposition 4.20. *If R is a commutative noetherian ring, then R is artinian if and only if all prime ideals in R are maximal.*

Proof. Assume that all prime ideals in R are maximal. By Theorem 3.4, there are (minimal) prime ideals $P_1, \dots, P_n$ in R such that $P_1P_2 \cdots P_n = 0$. If $I_0 = R$ and $I_j = P_1P_2 \cdots P_j$ for $j = 1, \dots, n$, then each of the factors I_{j-1}/I_j is a finitely generated module over R/P_j. Moreover, since P_j is maximal, R/P_j is a field and hence artinian. It follows from Corollary 4.7 that each I_{j-1}/I_j is artinian, and we then conclude from Proposition 4.5 that R is artinian. □

• TORSION AND TORSIONFREE MODULES •

As mentioned at the beginning of the chapter, different kinds of "torsion" already make appearances in the theory of abelian groups. For instance, if A is an abelian group (written additively, as a $\mathbb{Z}$-module), then we have the *torsion subgroup*

$$T(A) = \{a \in A \mid ma = 0 \text{ for some nonzero } m \in \mathbb{Z}\}$$

and, for each prime p, the *p-torsion* (or *p-primary*) *subgroup*

$$T_p(A) = \{a \in A \mid p^n a = 0 \text{ for some } n \in \mathbb{N}\}.$$

The common factor in these definitions is that each of the above subgroups has the form $\{a \in A \mid xa = 0 \text{ for some } x \in X\}$, where X is a subset of $\mathbb{Z}$ which is closed under multiplication. Although some restrictions will be needed in transferring this idea to modules over a noncommutative ring, we can begin the discussion in complete generality.

Definition. A *multiplicative set* (more fully, a *unital multiplicatively closed set*) in a ring R is a subset $X \subseteq R$ such that $1 \in X$ and X is closed under multiplication. Now let A be an R-module. We say that A is *X-torsion* provided each element of A is annihilated by some element of X, and that A is *X-torsionfree* if the only element of A annihilated by any element of X is 0.

In particular, if $R = \mathbb{Z}$ and $X = \mathbb{Z} \setminus \{0\}$, then "$X$-torsion" and "$X$-torsionfree" coincide with the usual notions of "torsion" and "torsionfree" for abelian groups, while if we take $X = \{p^n \mid n \in \mathbb{N}\}$ for a prime p, we get the concepts of "p-torsion" and "p-torsionfree" abelian groups.

The reader is likely already wondering why we did not define a general notion of "X-torsion submodule" for a right R-module A, namely, as the set $t_X(A) = \{a \in A \mid ax = 0 \text{ for some } x \in X\}$. The reason is that this definition does not always produce submodules, as the following examples show.

Exercise 4M. Let k be a field.

(a) If $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and $X = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R \;\middle|\; z \neq 0 \right\}$, show that the set $t_X(R_R)$ is not an ideal of R.

(b) Let $R = \begin{pmatrix} k[t] & k[t] \\ 0 & k \end{pmatrix}$ and $X = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R \;\middle|\; x, z \neq 0 \right\}$, where $k[t]$ is a polynomial ring. If A denotes the right R-module $\begin{pmatrix} k[t] & k[t] \end{pmatrix} / \begin{pmatrix} tk[t] & tk[t] \end{pmatrix}$, show that the set $t_X(A)$ is not a submodule of A. □

It is not hard to find a necessary condition for $t_X(A)$ to be a submodule of any right R-module A, as follows. Let $x \in X$, and consider the cyclic module $A = R/xR$. Since the coset $1 + xR$ is annihilated by x, the set $t_X(A)$ can only be a submodule of A if it equals A. That requires that each coset $r + xR$

in A be annihilated by some element $y \in X$, and the equation $(r + xR)\cdot y = 0$ requires, in turn, that $ry = xs$ for some $s \in R$. This common right multiple property, first studied by Ore and Asano in the 1930s, turns out to be all that is needed to make the concept of X-torsion behave smoothly.

Definition. Let X be a multiplicative set in a ring R. Then X satisfies the *right Ore condition* provided that, for each $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$ such that $ry = xs$, that is, $rX \cap xR \neq \varnothing$. A multiplicative set satisfying the right Ore condition is called a *right Ore set* for short. The *left Ore condition* and *left Ore sets* are defined symmetrically. An *Ore set* is a multiplicative set which is both a right and a left Ore set.

For example, any multiplicative set in a commutative ring is an Ore set. Here are some noncommutative examples.

Exercise 4N. Let R be a right noetherian domain.

(a) Show that the intersection of any two nonzero right ideals of R is nonzero. [Hint: If not, then any nonzero right ideal of R contains a direct sum of two nonzero right ideals.]

(b) Show that $R \setminus \{0\}$ is a right Ore set in R. $\square$

Exercise 4O. If $S = R[x;\alpha]$, where α is an automorphism of R, show that $\{1, x, x^2, \dots\}$ is an Ore set in S. $\square$

Exercise 4P. Let $R = A_1(k) = k[y][x; d/dy]$, where k is an arbitrary field. Show that each of the sets

$$\{1, x, x^2, \dots\} \qquad \{1, y, y^2, \dots\} \qquad k[x] \setminus \{0\} \qquad k[y] \setminus \{0\}$$

is an Ore set in R. $\square$

Lemma 4.21. *Let X be a right Ore set in a ring R.*

(a) *Given any elements $x_1, \dots, x_n \in X$, there exist $s_1, \dots, s_n \in R$ such that $x_1 s_1 = \cdots = x_n s_n$ and $x_1 s_1 \in X$, that is, $x_1 R \cap \cdots \cap x_n R \cap X \neq \varnothing$.*

(b) *For any right R-module A, the set*

$$t_X(A) = \{a \in A \mid ax = 0 \text{ for some } x \in X\}$$

is a submodule of A.

Proof. (a) By induction, it is enough to establish the case $n = 2$. In this case, the right Ore condition gives $x_1 y = x_2 s$ for some $y \in X$ and $s \in R$, and $x_1 y \in X$ because X is multiplicatively closed.

(b) Given $a_1, a_2 \in t_X(A)$, there exist $x_1, x_2 \in X$ such that each $a_i x_i = 0$. By part (a), there is some $y \in x_1 R \cap x_2 R \cap X$, and $(a_1 \pm a_2)y = 0$, whence $a_1 \pm a_2 \in t_X(A)$. Given $r \in R$, the right Ore condition yields $rz = x_1 s$ for some $z \in X$ and $s \in R$. Then $a_1 rz = a_1 x_1 s = 0$, and hence $a_1 r \in t_X(A)$. $\square$

Definition. If X is a right Ore set in a ring R and A is a right R-module, the set $t_X(A)$ defined in Lemma 4.21(b) is called the *X-torsion submodule* of A. Note that $t_X(R_R)$ is an ideal of R, and that if $f : A \to B$ is a homomorphism of right R-modules, then $f(t_X(A)) \leq t_X(B)$. The definition of Y-torsion submodules of left R-modules, for a left Ore set Y in R, is symmetric.

Lemma 4.22. *Let X be a right Ore set in a ring R.*

(a) *If A is any right R-module, then $t_X(A)$ is an X-torsion module and $A/t_X(A)$ is an X-torsionfree module.*

(b) *All submodules, factor modules, and sums (direct or not) of X-torsion right R-modules are X-torsion.*

(c) *If $B \leq A$ are right R-modules with B and A/B both X-torsion, then A is X-torsion.*

(d) *All submodules and direct products of X-torsionfree right R-modules are X-torsionfree.*

(e) *Let $B \leq A$ be right R-modules such that B is X-torsionfree. If B has nonzero intersection with all nonzero submodules of A, then A is X-torsionfree.*

(f) *If $B \leq A$ are right R-modules with B and A/B both X-torsionfree, then A is X-torsionfree.*

Proof. Most of this is clear and does not require the Ore condition, which is only needed for $t_X(A)$ to be a submodule of A in part (a), for the last claim of part (b), and for part (e). For (b), note that if $\{A_i \mid i \in I\}$ is a collection of X-torsion submodules of a right R-module A, then each $A_i \leq t_X(A)$, whence $\sum_i A_i \leq t_X(A)$, and so $\sum_i A_i$ is X-torsion.

(e) Since $t_X(A)$ is a submodule of A and $B \cap t_X(A) = t_X(B) = 0$, we must have $t_X(A) = 0$. $\square$

In the terminology of the next chapter, part (e) of Lemma 4.22 says that all "essential extensions" of X-torsionfree right R-modules are X-torsionfree.

We conclude the chapter with a useful application of the X-torsion ideas introduced above.

Proposition 4.23. *Let X be a multiplicative set in a ring R and A an X-torsionfree, noetherian right R-module. If $f \in \operatorname{End}_R(A)$ and $A/f(A)$ is X-torsion, then* $\ker(f) = 0$.

Proof. Since $\ker(f) \leq \ker(f^2) \leq \cdots$ and A is noetherian, there exists a positive integer n such that $\ker(f^n) = \ker(f^{n+1})$. For any $i \in \mathbb{N}$, the endomorphism f^i induces an epimorphism from $A/f(A)$ onto $f^i(A)/f^{i+1}(A)$, whence $f^i(A)/f^{i+1}(A)$ is X-torsion. Consequently, $A/f^n(A)$ is X-torsion.

Now consider an arbitrary element $a \in \ker(f)$. Since $A/f^n(A)$ is X-torsion, there exist $x \in X$ and $b \in A$ such that $ax = f^n(b)$. Then $b \in \ker(f^{n+1}) = \ker(f^n)$, whence $ax = 0$. Therefore $a = 0$, because A is X-torsionfree. $\square$

Corollary 4.24. *Let X be a right Ore set in a right noetherian ring R. If $\mathrm{l.ann}_R(x) = 0$ for all $x \in X$, then all the elements of X are non-zero-divisors in R.*

Proof. Since $\mathrm{l.ann}_R(x) = 0$ for all $x \in X$, the right R-module R_R is X-torsionfree. Now fix some $x \in X$, let f be the endomorphism of R_R given by left multiplication by x, and observe that $R/f(R)$ is X-torsion (because X is a right Ore set). The proposition implies that $\ker(f) = 0$, that is, $\mathrm{r.ann}_R(x) = 0$. Therefore x is a non-zero-divisor in R. □

Corollary 4.25. *Let R be a right noetherian ring.*

(a) *Any surjective endomorphism of a finitely generated right R-module is an automorphism.*

(b) *If $x, y \in R$ and $xy = 1$, then $yx = 1$.*

Proof. (a) Let A be a finitely generated right R-module and f a surjective endomorphism of A, so that $A/f(A) = 0$. If $X = \{1\} \subseteq R$, then A is X-torsionfree and noetherian. By Proposition 4.23, $\ker(f) = 0$, and therefore f is an automorphism of A.

(b) Apply part (a) to the endomorphism of R_R given by left multiplication by x. □

• NOTES •

Socles. This concept was intruduced by Krull [1928c, Definition 5, p. 64]. The name was first used by Remak to label the product of all minimal normal subgroups of a group [1930, p. 4] and was later adopted by Dieudonné to define the right and left socles of a ring [1942, pp. 47, 51].

Structure of Semisimple Rings. Noether proved that a ring R is (right) semisimple if and only if all finitely generated (right) R-modules are semisimple, if and only if R is isomorphic to a finite direct product of matrix rings over division rings [1929, §§13, 14, 18].

Zassenhaus's Lemma. Lemma 4.9 was proved by Zassenhaus [1934, p. 107] in simplifying Schreier's original proof of the refinement theorem.

Schreier Refinement Theorem. This was first proved by Schreier for normal series of subgroups [1928, Satz 1], and he remarked in a footnote that his proof also works for groups with operators.

Jordan-Hölder Theorem. This was first developed for composition series of a finite group G. Jordan proved that, for any two composition series of G, the list of the orders of the composition factors in one series is a permutation of the corresponding list for the other series [1869a, p. 140; 1869b, §§19-21]. That any two composition series for G are isomorphic was proved by Hölder [1889, §10].

Wedderburn-Artin Theorem. Matrix representations were first considered for finite dimensional algebras. Molien proved (essentially) that any finite dimensional simple algebra over $\mathbb{C}$ must be isomorphic to a matrix algebra over $\mathbb{C}$ [1893, Satz 30]; later, Cartan explicitly proved this, along with the result that any finite dimensional simple algebra over $\mathbb{R}$ must be isomorphic to a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ [1898, §§71, 85]. Wedderburn proved that any finite dimensional semiprime algebra over a field F is isomorphic to a finite direct product of matrix algebras over finite dimensional division algebras over F [1908, Theorems 10, 17, 22, 23]. Artin developed the generalization to rings satisfying both the ACC and DCC for right ideals, but only gave the proofs up to the point where Wedderburn's arguments could be used [1927, cf. Satz 11]. The ACC hypothesis was not removed until Hopkins and Levitzki obtained Theorem 4.15.

ACC and Radical Nilpotence in Artinian Rings. The nilpotence of the radical in a right artinian ring was proved independently by Hopkins [1938, (3.1); 1939, (1.4)] and Levitzki [1939, Theorem 6]. Hopkins obtained the consequence that every right artinian ring is right noetherian [1939, (6.4)].

Semisimplicity of Semiprime Artinian Rings. That a ring is semisimple if and only if it is (right) artinian and semiprime was proved by Noether [1929, §13].

Ore Condition. The name honors Ore's use of this condition, relative to the set of nonzero elements in a domain R, to characterize when R can be embedded in a division ring whose elements are "fractions" rx^{-1} for $r, x \in R$ with $x \neq 0$ [1931, Theorems 1, II]. (See Theorem 6.8.) The condition was studied relative to more general multiplicative sets and rings of fractions by Asano [1939, 1949].

5. Injective Hulls

Injective modules may be regarded as modules that are "complete" in the following algebraic sense: Any "partial" homomorphism (from a submodule of a module B) into an injective module A can be "completed" to a "full" homomorphism (from all of B) into A. Other types of completeness often entail similar extension properties. For instance: (a) If X and Y are metric spaces with X complete, then any uniformly continuous map from a dense subspace of Y to X entends to a uniformly continuous map from Y to X; (b) if Y is a normed linear space, then any bounded linear map from a linear subspace of Y to $\mathbb{R}$ extends to a bounded linear map from Y to $\mathbb{R}$; and (c) if X and Y are boolean algebras with X complete, then any boolean homomorphism from a subalgebra of Y to X extends to a boolean homomorphism from Y to X.

In topological and order-theoretic contexts, incomplete objects can be investigated by enlarging them to their completions. Following this pattern, one way to study a module A is to "complete" it to an injective module, i.e., to embed A in an injective module E, called the "injective hull" of A, in some minimal fashion. The minimality is achieved by requiring E to be an "essential extension" of A, meaning that every nonzero submodule of E has nonzero intersection with A. (This is a very natural density condition, if one draws parallels between modules and topological spaces, and between nonzero submodules and nonempty open sets.) We begin the chapter with a discussion of injective modules and their basic properties, and then develop the concepts of essential extensions and injective hulls. After studying the general properties of injective hulls, we show how injective hulls may be used to develop a useful notion of "finite rank" for modules. Finally, we develop some relationships between injective hulls and prime ideals, setting out a theme that will recur often in later chapters.

Injective modules first appeared in the context of abelian groups ($\mathbb{Z}$-modules). Zippin observed in 1935 that an abelian group is divisible if and only if it is a direct summand of any larger group containing it as a subgroup, and that divisible abelian groups can be completely described. The general notion for modules was first investigated by Baer in 1940. It is interesting in retrospect that the theory of these modules was investigated long before the dual notion of projective modules was considered. The "injective" and "projective" terminology originated with Cartan and Eilenberg in 1956.

• INJECTIVE MODULES •

This introductory section covers the basic properties of injective modules. Since this is completely standard material, we merely sketch it; readers who have not met injective modules before should fill in the details for themselves or consult a text on homological algebra.

Definition. A right (left) module A over a ring R is *injective* provided that, for any right (left) R-module B and any submodule C of B, all homomorphisms $C \to A$ extend to homomorphisms $B \to A$. The following diagram is a mnemonic for this definition.

$$\begin{tikzcd} C \arrow[r, "\leq"] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ A & \end{tikzcd}$$

For example, over a semisimple ring all modules are injective (Exercise 4H). On the other hand, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module, since the homomorphism $f : 2\mathbb{Z} \to \mathbb{Z}$ given by the rule $f(2n) = n$ cannot be extended to a homomorphism $\mathbb{Z} \to \mathbb{Z}$.

Observe that all direct summands and direct products of injective modules are injective.

Proposition 5.1. [Baer's Criterion] *Let A be a right module over a ring R. Then A is injective if and only if, for every right ideal I of R and every $f \in \mathrm{Hom}_R(I, A)$, there exists $a \in A$ such that $f(r) = ar$ for all $r \in I$.*

Proof. If A is injective, then, given $I \leq R_R$, any $f \in \mathrm{Hom}_R(I, A)$ extends to some $f_1 \in \mathrm{Hom}_R(R, A)$, and $f(r) = f_1(r) = f_1(1)r$ for all $r \in I$. Conversely, assume that A satisfies the given condition, and consider right R-modules $C \leq B$ together with a homomorphism $f : C \to A$.

Let X be the set of all pairs (C_1, f_1), where C_1 is a submodule of B containing C and f_1 is a homomorphism from C_1 to A extending f. Define a relation $\leq$ on X by declaring that $(C_1, f_1) \leq (C_2, f_2)$ if and only if $C_1 \leq C_2$ and f_2 extends f_1. One then checks that this relation is a partial order on X and that every nonempty chain in X has an upper bound. By Zorn's Lemma, there is a maximal element (C^*, f^*) in X, and if $C^* = B$, we are done.

If not, choose $b \in B \setminus C^*$ and set $I = \{r \in R \mid br \in C^*\}$. The rule $r \mapsto f^*(br)$ defines a homomorphism $I \to A$, and hence, by assumption, there exists $a \in A$ such that $f^*(br) = ar$ for all $r \in I$. One checks that there is a well-defined homomorphism $f_1 : C^* + bR \to A$ such that $f_1(c + br) = f^*(c) + ar$ for all $c \in C^*$ and $r \in R$. But then $(C^* + bR,\, f_1) \in X$, which contradicts the maximality of (C^*, f^*). $\square$

Recall that a $\mathbb{Z}$-module A is *divisible* provided $nA = A$ for all nonzero $n \in \mathbb{Z}$.

Proposition 5.2. (a) *A $\mathbb{Z}$-module A is injective if and only if it is divisible.*

(b) *Every $\mathbb{Z}$-module is a submodule of a divisible module.*

Proof. (a) An element $a \in A$ is divisible by a nonzero integer n if and only if the homomorphism $n\mathbb{Z} \to A$ sending n to a extends to $\mathbb{Z}$.

(b) Any $\mathbb{Z}$-module is isomorphic to one of the form F/K, where F is free and $K \le F$. Now F is a direct sum of copies of $\mathbb{Z}$. If D is the corresponding direct sum of copies of $\mathbb{Q}$, then D and D/K are divisible and $F/K \le D/K$. $\square$

Recall that if R is a ring and D an abelian group, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ can be made into either a right or a left R-module. If $f \in \mathrm{Hom}_{\mathbb{Z}}(R, D)$ and $r \in R$, then fr is defined by the rule $(fr)(x) = f(rx)$ for $x \in R$, while rf is defined by the rule $(rf)(x) = f(xr)$.

Lemma 5.3. *If R is a ring and D a divisible $\mathbb{Z}$-module, then the group $H = \mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective right (or left) R-module.*

Proof. If $I \le R_R$ and $f \in \mathrm{Hom}_R(I, H)$, the rule $r \mapsto f(r)(1)$ defines a $\mathbb{Z}$-module homomorphism $I \to D$. Since D is an injective $\mathbb{Z}$-module, this extends to a $\mathbb{Z}$-module homomorphism $g : R \to D$. Now $g \in H$, and one checks that $f(r) = gr$ for all $r \in I$. $\square$

Theorem 5.4. [Baer] *Every module is a submodule of an injective module.*

Proof. Consider a right module A over a ring R. Viewed as a $\mathbb{Z}$-module, A is a submodule of a divisible $\mathbb{Z}$-module D. Then $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective right R-module, and there are right R-module embeddings

$$A \cong \mathrm{Hom}_R(R, A) \le \mathrm{Hom}_{\mathbb{Z}}(R, A) \le \mathrm{Hom}_{\mathbb{Z}}(R, D). \quad \square$$

Corollary 5.5. [Baer] *A module A is injective if and only if A is a direct summand of every module that contains it.*

Proof. If A is injective and $A \le B$, the identity map on A extends to a homomorphism $f : B \to A$, and then $B = A \oplus \ker(f)$. Conversely, if the direct summand condition holds, then A is a direct summand of an injective module. $\square$

• ESSENTIAL EXTENSIONS •

We now introduce the concepts of essential submodules and essential extensions. As remarked at the beginning of the chapter, it may be helpful to think of essential submodules as being dense in a suitable algebraic sense (analogous to density for subsets of topological spaces). The reader should note, however, that the term "dense submodule" is already used in the literature for a different property.

Definition. An *essential* (or *large*) submodule of a module B is any submodule A which has nonzero intersection with every nonzero submodule of B. We write $A \leq_e B$ to denote this situation, and we also say that B is an *essential extension* of A.

If A is a submodule of a right module B over a ring R, then $A \leq_e B$ if and only if for each nonzero element $b \in B$ there exists $r \in R$ such that $br \neq 0$ and $br \in A$.

Since any two nonzero $\mathbb{Z}$-submodules of $\mathbb{Q}$ have nonzero intersection, all nonzero $\mathbb{Z}$-submodules of $\mathbb{Q}$ are essential. For instance, $\mathbb{Z} \leq_e \mathbb{Q}$. Given a prime integer p and a positive integer n, all nonzero submodules of $\mathbb{Z}/p^n\mathbb{Z}$ are essential. At the other extreme, the only essential submodule (subspace) of a vector space V is V itself.

Exercise 5A. If I is a nonzero ideal in a prime ring R, show that I is both an essential right ideal and an essential left ideal of R. $\square$

Definition. Let A and B be modules. An *essential monomorphism* from A to B is any monomorphism $f : A \to B$ such that $f(A) \leq_e B$.

Note that a submodule A of a module B is essential if and only if the inclusion map $A \to B$ is an essential monomorphism.

Proposition 5.6. (a) *Let A, B, and C be modules with $A \leq B \leq C$. Then $A \leq_e C$ if and only if both $A \leq_e B$ and $B \leq_e C$.*

(b) *Let A_1, A_2, B_1, B_2 be submodules of a module C. If $A_1 \leq_e B_1$ and $A_2 \leq_e B_2$, then $A_1 \cap A_2 \leq_e B_1 \cap B_2$.*

(c) *Let A be a submodule of a module C and $f : B \to C$ a homomorphism. If $A \leq_e C$, then $f^{-1}(A) \leq_e B$. In particular, if $A \leq_e C$ are right modules over a ring R, then, for each $c \in C$, the right ideal $\{r \in R \mid cr \in A\}$ is essential in R_R.*

(d) *Let $\{B_i \mid i \in I\}$ be a collection of modules, and let $A_i \leq_e B_i$ for each $i \in I$. Then $\bigoplus_i A_i \leq_e \bigoplus_i B_i$.*

(e) *Let $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ be collections of submodules of a module C. If the A_i are independent and each $A_i \leq_e B_i$, then the B_i are independent and $\bigoplus_i A_i \leq_e \bigoplus_i B_i$.*

Proof. (a) If $A \leq_e C$, then obviously $A \leq_e B$. Since any nonzero submodule of C has nonzero intersection with A, it also has nonzero intersection with B. Thus $B \leq_e C$.

Conversely, assume that $A \leq_e B \leq_e C$. Given any nonzero submodule $M \leq C$, we have $B \cap M \neq 0$ because $B \leq_e C$. Then $B \cap M$ is a nonzero submodule of B, and so $A \cap B \cap M \neq 0$ because $A \leq_e B$. Thus $A \cap M \neq 0$, proving that $A \leq_e C$.

(b) Given any nonzero submodule $M \leq B_1 \cap B_2$, we have $A_2 \cap M \neq 0$ because $A_2 \leq_e B_2$. Then $A_2 \cap M$ is a nonzero submodule of B_1, whence $A_1 \cap A_2 \cap M \neq 0$ because $A_1 \leq_e B_1$. Thus $A_1 \cap A_2 \leq_e B_1 \cap B_2$.

(c) Let M be any nonzero submodule of B. If $f(M)=0$, then $M \le f^{-1}(A)$ and hence $f^{-1}(A) \cap M \neq 0$. If $f(M) \neq 0$, then $A \cap f(M) \neq 0$ because $A \le_e C$, whence $f^{-1}(A) \cap M \neq 0$. Thus $f^{-1}(A) \le_e B$. The final statement follows from what we have just proved because $\{r \in R \mid cr \in A\} = f^{-1}(A)$, where $f : R_R \to C$ is the homomorphism such that $f(1)=c$.

(d) If I consists of a single index, there is nothing to prove. Next, assume that $I = \{1, 2\}$. Applying (c) to the projection maps $B_1 \oplus B_2 \to B_i$, we find that $A_1 \oplus B_2$ and $B_1 \oplus A_2$ are essential in $B_1 \oplus B_2$. Then

$$A_1 \oplus A_2 = (A_1 \oplus B_2) \cap (B_1 \oplus A_2) \le_e B_1 \oplus B_2$$

by (b). Thus (d) holds for any 2-element index set. By induction, it follows that (d) holds for all finite index sets.

In the general case, given any nonzero submodule $M \le \bigoplus_i B_i$, there exists a finite subset $J \subseteq I$ such that $M \cap \big(\bigoplus_{j \in J} B_j\big) \neq 0$. Since $\bigoplus_{j \in J} A_j \le_e \bigoplus_{j \in J} B_j$, it follows that $M \cap \big(\bigoplus_{j \in J} A_j\big) \neq 0$. Therefore $\bigoplus_{i \in I} A_i \le_e \bigoplus_{i \in I} B_i$.

(e) Again, if I consists of a single index, there is nothing to prove. Next, assume that $I = \{1, 2\}$. Since $A_1 \cap A_2 = 0$, it follows from (b) that $0 \le_e B_1 \cap B_2$, whence $B_1 \cap B_2 = 0$. Thus B_1 and B_2 are independent. Now suppose that $I = \{1, \dots, n\}$ for some integer $n > 2$, and that $B_1, \dots, B_{n-1}$ are independent. In view of (d), we have $A_1 \oplus \cdots \oplus A_{n-1} \le_e B_1 \oplus \cdots \oplus B_{n-1}$. Since $(A_1 \oplus \cdots \oplus A_{n-1}) \cap A_n = 0$, it follows from what we have just proved that $(B_1 \oplus \cdots \oplus B_{n-1}) \cap B_n = 0$, whence $B_1, \dots, B_n$ are independent. Thus, by induction, all finite collections of the B_i are independent, and therefore the full collection is independent. The final statement of (e) now follows from (d). $\square$

Exercise 5B. Let $C = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ and let A, B_1, B_2 denote the cyclic subgroups of C generated by $(2, 0)$, $(1, 0)$, $(1,\, 1+2\mathbb{Z})$, respectively. Show that $A \le_e B_1$ and $A \le_e B_2$ but $A \not\le_e B_1 + B_2$. $\square$

Proposition 5.7. *Let A and B be submodules of a module C, and suppose that B is maximal with respect to the property $A \cap B = 0$. Then $A \oplus B \le_e C$ and $(A \oplus B)/B \le_e C/B$.*

Proof. If M is a submodule of C such that $(A \oplus B) \cap M = 0$, then A, B, M are independent, whence $A \cap (B \oplus M) = 0$. Then $B \oplus M = B$ by the maximality of B, and so $M = 0$. Thus $A \oplus B \le_e C$.

Any nonzero submodule of C/B has the form D/B for some submodule D of C which properly contains B. Then $A \cap D \neq 0$ by the maximality of B, whence $(A \oplus B) \cap D > B$, and so $[(A \oplus B)/B] \cap (D/B) \neq 0$. Thus $(A \oplus B)/B \le_e C/B$. $\square$

Corollary 5.8. *Any submodule of a module C is a direct summand of an essential submodule of C.*

Proof. Given a submodule A of C, Zorn's Lemma guarantees the existence of a submodule B of C maximal with respect to the property $A \cap B = 0$. $\square$

Corollary 5.8 guarantees a large supply of essential submodules in a module C, which can sometimes be used to reduce problems about arbitrary submodules to the essential case. For instance, to prove that C is noetherian, it suffices to show that all essential submodules of C are finitely generated.

A module C always has at least one essential submodule, namely C itself, but in the extreme case C may not have any other essential submodules. This occurs precisely when C is semisimple, as follows.

Corollary 5.9. *A module C is semisimple if and only if C has no proper essential submodules.*

Proof. Assume first that C is semisimple. If A is any proper submodule of C, then Proposition 4.2 implies that $C = A \oplus B$ for some nonzero submodule B. Since $B \cap A = 0$, we infer that $A \not\leq_e C$.

Conversely, if C has no proper essential submodules, then Corollary 5.8 shows that every submodule of C is a direct summand. Therefore, using Proposition 4.2 again, C is semisimple. $\square$

Exercise 5C. Show that, in any module C, the intersection of the essential submodules of C equals $\operatorname{soc}(C)$. More generally, show that any submodule of C containing $\operatorname{soc}(C)$ equals the intersection of some collection of essential submodules of C. $\square$

Exercise 5D. If A is a right module over a ring R, show that the set

$$Z(A) = \{x \in A \mid xI = 0 \text{ for some } I \leq_e R_R\} = \{x \in A \mid \operatorname{ann}_R(x) \leq_e R_R\}$$

is a submodule of A. It is called the *singular submodule* of A. $\square$

• INJECTIVE HULLS •

By an injective hull for a module A is meant, roughly, an injective module containing A which is as small as possible. Alternatively, an injective hull for A turns out to be an essential extension of A which is as large as possible. In order to construct injective hulls, and to exhibit this second description of them, we first look at the relationships between injectivity and essential extensions.

Definition. A *proper essential extension* of a module A is any module B such that $A \leq_e B$ while $B > A$. Note that A has a proper essential extension if and only if there exists an essential monomorphism $f : A \to C$ such that $f(A) < C$.

Proposition 5.10. [Eckmann-Schopf] *A module A is injective if and only if A has no proper essential extensions.*

Proof. First assume that A is injective, and consider an essential extension $A \leq_e B$. By Corollary 5.5, $B = A \oplus C$ for some submodule C. Since $A \cap C = 0$, we obtain $C = 0$ because $A \leq_e B$, whence $B = A$.

Conversely, if A is not injective, then by Corollary 5.5 there exists a module $C \geq A$ such that A is not a direct summand of C. Choose a submodule $B \leq C$ maximal with respect to the property $A \cap B = 0$, and note that $A \oplus B < C$. By Proposition 5.7, $(A \oplus B)/B \leq_e C/B$, whence the map $A \to C/B$ (given by composing the inclusion map $A \to C$ with the quotient map $C \to C/B$) is an essential monomorphism. Thus A has a proper essential extension. □

Definition. Let C be a module and A a submodule. We say that A is *essentially closed* in C provided A has no proper essential extensions within C, that is, the only submodule B of C for which $A \leq_e B$ is A itself. In short, A is essentially closed in C if and only if $A \leq_e B \leq C$ always implies $B = A$.

Exercise 5E. If a module A is a direct summand of a module C, show that A is essentially closed in C. □

Exercise 5F. Let $A \leq C$ be modules. Show that A is essentially closed in C if and only if there exists a submodule $B \leq C$ such that A is maximal with respect to the property $A \cap B = 0$, if and only if there exists a submodule $B \leq C$ such that $A \cap B = 0$ and $(A + B)/A \leq_e C/A$. □

Proposition 5.11. *Let A be a submodule of an injective module E. Then A is injective if and only if A is essentially closed in E.*

Proof. If A is injective, then, by Proposition 5.10, A is essentially closed in any module containing it.

Conversely, assume that A is essentially closed in E and consider any essential extension $A \leq_e B$. The inclusion map $A \to E$ extends to a homomorphism $f : B \to E$. Since $A \cap \ker(f) = 0$ and $A \leq_e B$, we obtain $\ker(f) = 0$, and so f provides an isomorphism of B onto $f(B)$. Then $A = f(A) \leq_e f(B) \leq E$, and so $f(B) = A$ (because A is essentially closed in E), whence $B = A$. Thus A has no proper essential extensions. By Proposition 5.10, A is injective. □

Definition. An *injective hull* (or *injective envelope*) for a module A is any injective module which is an essential extension of A.

For example, $\mathbb{Q}$ is an injective hull for $\mathbb{Z}$.

Exercise 5G. If R is any commutative domain, show that the quotient field of R is an injective hull for R_R. □

Theorem 5.12. [Baer, Eckmann-Schopf] *Let A be a module.*

(a) *Any injective module containing A contains an injective hull for A. In particular, there exist injective hulls for A.*

Now let E be any injective hull for A.

(b) *Whenever $A \leq_e B$, the identity map on A extends to a monomorphism $B \to E$.*

(c) *Whenever $A \leq E'$ with E' injective, the identity map on A extends to a monomorphism $E \to E'$.*

Proof. (a) Consider any injective module $F \geq A$. By Zorn's Lemma, there exists a submodule $E \leq F$ such that $A \leq E$ and E is maximal with respect to the property $A \leq_e E$. If E' is any submodule of F for which $E \leq_e E'$, then $A \leq_e E'$, and hence $E' = E$ by the maximality of E. Thus E is essentially closed in F, and so E is injective, by Proposition 5.11. Therefore E is an injective hull for A.

(b) Since E is injective, the inclusion map $A \to E$ extends to a homomorphism $g : B \to E$. Then $A \cap \ker(g) = 0$, whence $\ker(g) = 0$, because $A \leq_e B$. Thus g is a monomorphism.

(c) Since E' is injective, the inclusion map $A \to E'$ extends to a homomorphism $g : E \to E'$, and the argument used in (b) shows that g is a monomorphism. $\square$

Parts (b) and (c) of Theorem 5.12 may be summarized by saying that an injective hull for a module A is a "maximal essential extension" of A as well as a "minimal injective extension" of A. These properties may be rephrased in terms of homomorphisms as follows.

Exercise 5H. Let A be a module, E an injective hull for A, and $j : A \to E$ the inclusion map.

(a) Given an essential monomorphism $f : A \to B$, show that there exists a monomorphism $g : B \to E$ such that $gf = j$.

(b) Given a monomorphism $f : A \to E'$ with E' injective, show that there exists a monomorphism $g : E \to E'$ such that $gj = f$. $\square$

Injective hulls need not be unique as sets, even within a given injective module, as the following example shows. Let $R = \mathbb{Z}/4\mathbb{Z}$, and observe that R_R is injective. Set $F = R \oplus R$ and $A = (2,0)R \leq F$. Then $(1,0)R$ and $(1,2)R$ are submodules of F isomorphic to R, and so they are injective R-modules. Moreover, $A \leq_e (1,0)R$ and $A \leq_e (1,2)R$, so that $(1,0)R$ and $(1,2)R$ are both injective hulls for A.

However, injective hulls are unique up to isomorphism, as follows.

Proposition 5.13. *If E and E' are injective hulls for isomorphic modules A and A', then any isomorphism of A onto A' extends to an isomorphism of E onto E'. In particular, if E and E' are two injective hulls for a module A, the identity map on A extends to an isomorphism of E onto E'.*

Proof. Let $j : A \to E$ and $j' : A' \to E'$ be the inclusion maps and $f : A \to A'$ an isomorphism. Then $j'f : A \to E'$ is a monomorphism. By Exercise 5H, there exists a monomorphism $g : E \to E'$ such that $gj = j'f$.

$$\begin{tikzcd} E \arrow[r, "g"] & E' \\ A \arrow[u, "j"] \arrow[r, "f"] & A' \arrow[u, "j'"'] \end{tikzcd}$$

Then $A' = f(A) = g(A) \leq g(E)$ and so $g(E) \leq_e E'$, that is, g is an essential monomorphism. However, E – being injective – has no proper essential extensions (Proposition 5.10), and hence $g(E) = E'$. Thus g is an isomorphism. □

Definition. Given a module A, we use the notation $E(A)$ for an injective hull of A. As this is only unique up to isomorphism, general assertions about $E(A)$ must be valid for all injective hulls of A. The equation "$B = E(A)$" should only be used as an abbreviation for the statement "B is an injective hull for A."

For instance, given modules A_1 and A_2, we may state that

$$E(A_1) \oplus E(A_2) = E(A_1 \oplus A_2),$$

since the direct sum of any injective hull for A_1 with any injective hull for A_2 is an injective hull for $A_1 \oplus A_2$. Similarly, given modules $B \leq A$, we may write $E(B) \leq E(A)$, since any injective hull for A contains at least one injective hull for B.

Exercise 5I. Show that $E(A) \cong A \otimes_{\mathbb{Z}} \mathbb{Q}$ for any torsionfree $\mathbb{Z}$-module A. □

Exercise 5J. Let I be an ideal in a ring R, let A be a right (R/I)-module, and let E be an injective hull for A_R. Show that the (R/I)-module $\operatorname{ann}_E(I)$ is an injective hull for $A_{R/I}$. □

Exercise 5K. Given modules $B \leq A$, show that $E(A)$ is isomorphic to a submodule of $E(B) \oplus E(A/B)$. □

• MODULES OF FINITE RANK •

In abelian group theory, the "rank" of a torsionfree abelian group A is defined as the dimension of the vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. While this notion extends immediately to torsionfree modules over commutative domains, it is not readily apparent how to usefully adapt it to torsion modules or to modules over other rings. We proceed by relating the vector space viewpoint to injective hulls. Recall from Exercise 5I that the injective hull of a torsionfree abelian group A is isomorphic to $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Under this isomorphism, as is easily checked, 1-dimensional subspaces of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ correspond to nonzero indecomposable direct summands of $E(A)$. Hence, the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$, which may be calculated as the number of summands in a decomposition of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a direct sum of 1-dimensional subspaces, equals the number of nonzero summands in a decomposition of $E(A)$ as a direct sum of indecomposable submodules.

To adapt this idea to arbitrary modules, two obstacles must be overcome. First, not every injective module is a direct sum of indecomposable modules (Exercise 5L). Second, if an injective module is a finite direct sum of indecomposable submodules in two different ways, we need to know whether the number of nonzero summands is the same in both decompositions. The first obstacle we finesse by restricting attention to modules whose injective hulls are finite direct sums of indecomposable submodules, and within this context we prove that the second obstacle vanishes.

Exercise 5L. Let $R = \left(\prod_n F_n\right)/\left(\bigoplus_n F_n\right)$ for some fields $F_1, F_2, \dots$. Show that $E(R_R)$ has no nonzero indecomposable direct summands. [Hint: Show that no nonzero principal ideal of R is indecomposable.] $\square$

Definition. A module A has *finite rank* provided $E(A)$ is a finite direct sum of indecomposable submodules. (We shall consider a value for the rank of A later.) In the literature, a module of finite rank is sometimes called a *finite dimensional* module. Moreover, one or another of the equivalent conditions given in Proposition 5.15 and Theorem 5.17 below is often taken as the definition of finite rank.

As indicated in the discussion above, this notion of finite rank coincides with the traditional one when applied to torsionfree abelian groups. Thus, for instance, every subgroup of a finite dimensional vector space over $\mathbb{Q}$ is a $\mathbb{Z}$-module of finite rank. Moreover, we shall prove shortly that every noetherian module has finite rank (Corollary 5.18).

The obvious building blocks for finite rank modules should be those modules whose injective hulls are indecomposable. We approach this property from within the modules, as follows.

Definition. A *uniform module* is a nonzero module A such that the intersection of any two nonzero submodules of A is nonzero, or, equivalently, such that every nonzero submodule of A is essential in A.

Note that all nonzero submodules and all essential extensions of uniform modules are uniform. For an example, the quotient field of a commutative domain R is a uniform R-module.

Exercise 5M. Show that the finitely generated uniform $\mathbb{Z}$-modules are (up to isomorphism) exactly $\mathbb{Z}$ and $\mathbb{Z}/p^n\mathbb{Z}$ for prime integers p and positive integers n. $\square$

Lemma 5.14. *A nonzero module A is uniform if and only if $E(A)$ is indecomposable.*

Proof. First suppose that A is uniform and that $E(A) = B \oplus C$ for some submodules B and C. As $(B \cap A) \cap (C \cap A) = 0$, either $B \cap A = 0$ or $C \cap A = 0$ (because A is uniform), whence either $B = 0$ or $C = 0$ (because $A \leq_e E(A)$). Thus $E(A)$ is indecomposable.

Conversely, if A is not uniform, it has nonzero submodules B and C such that $B \cap C = 0$. Then $E(A)$ has a nonzero submodule E which is an injective hull for B, and $E \cap C = 0$ because $B \leq_e E$, whence $E \neq E(A)$. Thus E is a nontrivial direct summand of $E(A)$, and so $E(A)$ is not indecomposable. $\square$

In particular, Lemma 5.14 shows that an injective module is uniform if and only if it is nonzero and indecomposable. Thus the terms "uniform injective module" and "nonzero indecomposable injective module" are synonymous, and since the former is shorter, we shall generally use it.

We now show that the condition of finite rank can be characterized internally, using uniform submodules.

Proposition 5.15. *A module A has finite rank if and only if A has an essential submodule which is a finite direct sum of uniform submodules.*

Proof. First assume that A contains some independent uniform submodules $A_1, \dots, A_n$ such that $A_1 \oplus \cdots \oplus A_n \leq_e A$. Then $A_1 \oplus \cdots \oplus A_n \leq_e E(A)$, whence

$$E(A) = E(A_1 \oplus \cdots \oplus A_n) \cong E(A_1) \oplus \cdots \oplus E(A_n).$$

Since each $E(A_i)$ is indecomposable by Lemma 5.14, A has finite rank.

Conversely, if A has finite rank, then $E(A) = E_1 \oplus \cdots \oplus E_n$ for some indecomposable submodules E_i, and we may assume that each $E_i \neq 0$. Each E_i is uniform by Lemma 5.14. Each of the submodules $A_i = A \cap E_i$ is nonzero because $A \leq_e E(A)$, whence A_i is uniform. The A_i are clearly independent submodules of A, and each $A_i \leq_e E_i$ because E_i is uniform. Finally,

$$A_1 \oplus \cdots \oplus A_n \leq_e E_1 \oplus \cdots \oplus E_n = E(A),$$

and consequently $A_1 \oplus \cdots \oplus A_n \leq_e A$. □

Modules of finite rank need not be direct sums of uniform submodules, as the following examples show.

Exercise 5N. If $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \pmod 2\}$, a subring of $\mathbb{Z} \times \mathbb{Z}$, show that R_R has finite rank but is not a direct sum of uniform submodules. □

Exercise 5O. Let $S = k[x, y]$ be a polynomial ring over a field k, and let R be the ring $S/\langle x, y \rangle^2$. Show that R_R has finite rank but is not a direct sum of uniform submodules. □

Lemma 5.16. *If a module E is a finite direct sum of n uniform submodules, then E does not contain any direct sums of $n + 1$ nonzero submodules.*

Proof. If $n = 0$, then $E = 0$, while if $n = 1$, then E is uniform; in either case the conclusion is clear. Now let $n > 1$, and assume the lemma holds for direct sums of $n - 1$ uniform modules.

We are given that $E = E_1 \oplus \cdots \oplus E_n$ with each E_i uniform. Suppose that E contains a direct sum $A_1 \oplus \cdots \oplus A_{n+1}$ of $n + 1$ nonzero submodules. Set $A = A_1 \oplus \cdots \oplus A_n$. If $A \cap E_1 = 0$, then A embeds in $E_2 \oplus \cdots \oplus E_n$ (via the projection $E \to E_2 \oplus \cdots \oplus E_n$), whence $E_2 \oplus \cdots \oplus E_n$ contains a direct sum of n nonzero submodules, contradicting the induction hypothesis. Thus $A \cap E_1 \neq 0$, and similarly $A \cap E_i \neq 0$ for all i.

Since the E_i are uniform, $A \cap E_i \leq_e E_i$ for all i. Then

$$(A \cap E_1) \oplus \cdots \oplus (A \cap E_n) \leq_e E_1 \oplus \cdots \oplus E_n = E,$$

whence $A \leq_e E$. However, as $A \cap A_{n+1} = 0$, this is impossible. Therefore E does not contain a direct sum of $n+1$ nonzero submodules. $\square$

Theorem 5.17. [Goldie] *A module A has finite rank if and only if A contains no infinite direct sums of nonzero submodules.*

Proof. If A has finite rank, $E(A)$ is a direct sum of n uniform submodules for some $n \in \mathbb{Z}^+$. By Lemma 5.16, $E(A)$ cannot contain a direct sum of more than n nonzero submodules, and hence neither can A.

If A is not of finite rank, then $A \neq 0$ and $E(A)$ is not a finite direct sum of indecomposable submodules. Set $C_0 = E(A)$. Since C_0 is not indecomposable, $C_0 = M \oplus N$ for some nonzero submodules M and N; moreover, M and N cannot both be finite direct sums of indecomposable submodules. Hence, $C_0 = B_1 \oplus C_1$ for some nonzero submodules B_1 and C_1 such that C_1 is not a finite direct sum of indecomposable submodules.

Repeat this argument with respect to C_1 and continue inductively. We obtain submodules $B_1, C_1, B_2, C_2, \ldots$ of C_0 such that each $C_{n-1} = B_n \oplus C_n$ with B_n nonzero and C_n not a finite direct sum of indecomposable submodules. Since $B_k \leq C_n$ whenever $k > n$, we find that

$$B_n \cap \Big(\sum_{k=n+1}^{\infty} B_k \Big) \leq B_n \cap C_n = 0$$

for all $n \in \mathbb{N}$, from which it follows that $B_1, B_2, \ldots$ are independent submodules of $E(A)$.

Now $B_1 \cap A$, $B_2 \cap A$, $\ldots$ is an infinite sequence of independent submodules of A, and as $A \leq_e E(A)$, it follows that $B_n \cap A \neq 0$ for each $n \geq 1$. Therefore A contains an infinite direct sum of nonzero submodules. $\square$

Corollary 5.18. *Any noetherian module A has finite rank.*

Proof. If not, then, by Theorem 5.17, A contains an infinite direct sum of nonzero submodules, and hence A contains an infinite sequence $A_1, A_2, \ldots$ of independent nonzero submodules. But then

$$A_1 < A_1 \oplus A_2 < A_1 \oplus A_2 \oplus A_3 < \cdots$$

is a strictly ascending infinite chain of submodules of A, contradicting our noetherian hypothesis. (Alternatively, observe that the submodule $\bigoplus_{n=1}^{\infty} A_n$ of A cannot be finitely generated.) $\square$

Corollary 5.19. *Every nonzero noetherian module has a uniform submodule.* $\square$

Exercise 5P. Show that a module A has finite rank if and only if every submodule of A is an essential extension of a finitely generated submodule. $\square$

Exercise 5Q. Show that a module A has finite rank if and only if A satisfies the ACC on essentially closed submodules. $\square$

• UNIFORM RANK •

Definition. If A is a module of finite rank, there exists a nonnegative integer n such that $E(A)$ is a direct sum of n uniform submodules. Moreover, because of Lemma 5.16, any other decomposition of $E(A)$ into a direct sum of uniform submodules has exactly n summands. Thus n is uniquely determined by A. We shall call this integer the *uniform rank*, or just the *rank* of A, and denote it by $\operatorname{rank}(A)$. (In the literature, the rank of A is also called the *Goldie rank*, the *Goldie dimension*, the *uniform dimension*, or the *dimension* of A.)

Observe that, if $A_1, \dots, A_n$ are modules of finite rank, then $A_1 \oplus \cdots \oplus A_n$ has finite rank and

$$\operatorname{rank}(A_1 \oplus \cdots \oplus A_n) = \operatorname{rank}(A_1) + \cdots + \operatorname{rank}(A_n).$$

In other words, uniform rank is additive on direct sums.

Exercise 5R. Let R be a commutative domain with quotient field K. Show that $\operatorname{rank}(A) = \dim_K(A \otimes_R K)$ for all torsionfree R-modules A of finite rank. $\square$

Proposition 5.20. *Let A be a module and n a nonnegative integer. Then the following conditions are equivalent:*

(a) *A has finite rank n.*

(b) *A has an essential submodule which is a direct sum of n uniform submodules.*

(c) *A contains a direct sum of n nonzero submodules but no direct sum of $n+1$ nonzero submodules.*

Proof. (a) $\Longrightarrow$ (c): By assumption, $E(A) = E_1 \oplus \cdots \oplus E_n$ for some uniform submodules E_i. Lemma 5.16 then shows that $E(A)$ contains no direct sums of $n+1$ nonzero submodules, and hence A does not either. On the other hand, A contains the direct sum of the n nonzero submodules $E_1 \cap A, \dots, E_n \cap A$.

(c) $\Longrightarrow$ (b): Let $A_1, \dots, A_n$ be n independent nonzero submodules of A. There cannot be two nonzero submodules $B, C \leq A_1$ with $B \cap C = 0$, for then A would contain the direct sum $B \oplus C \oplus A_2 \oplus \cdots \oplus A_n$ of $n+1$ nonzero submodules. Thus A_1 is uniform, and similarly all the A_i are uniform. If $A_1 \oplus \cdots \oplus A_n$ is not essential in A, there is a nonzero submodule $A_{n+1} \leq A$ such that $(A_1 \oplus \cdots \oplus A_n) \cap A_{n+1} = 0$, but then A would contain the direct sum $A_1 \oplus \cdots \oplus A_{n+1}$ of $n+1$ nonzero submodules. Therefore $A_1 \oplus \cdots \oplus A_n \leq_e A$.

(b) $\Longrightarrow$ (a): See the proof of Proposition 5.15. $\square$

Corollary 5.21. *Let B be a submodule of a module A.*

(a) *Suppose that A has finite rank. Then B has finite rank, and* $\operatorname{rank}(B) \leq \operatorname{rank}(A)$. *Moreover,* $\operatorname{rank}(B) = \operatorname{rank}(A)$ *if and only if* $B \leq_e A$.

(b) *Now suppose that B and A/B have finite rank. Then A has finite rank, and* $\operatorname{rank}(A) \leq \operatorname{rank}(B) + \operatorname{rank}(A/B)$.

Proof. (a) By Proposition 5.20, B contains no direct sums of $\operatorname{rank}(A) + 1$ nonzero submodules. Hence, applying the proposition to B, we find that B has finite rank and that $\operatorname{rank}(B) \leq \operatorname{rank}(A)$.

Proposition 5.20 also says that B has an essential submodule C which is a direct sum of $\operatorname{rank}(B)$ uniform submodules. If $B \leq_e A$, then $C \leq_e A$, whence the proposition shows that $\operatorname{rank}(A) = \operatorname{rank}(B)$. Conversely, if $\operatorname{rank}(A) = \operatorname{rank}(B)$, the proposition says that A does not contain a direct sum of $\operatorname{rank}(B) + 1$ nonzero submodules, from which we conclude that $C \leq_e A$, and therefore $B \leq_e A$.

(b) Choose a submodule $C \leq A$ maximal with respect to the property $C \cap B = 0$. Then C is isomorphic to a submodule of A/B. By part (a), C has finite rank, and $\operatorname{rank}(C) \leq \operatorname{rank}(A/B)$. We also have $B \oplus C \leq_e A$ by Proposition 5.7. In view of Proposition 5.20, it follows that A has finite rank and that $\operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(C)$. $\square$

Corollary 5.22. *Let A be a module with finite rank. If $f : A \to A$ is a monomorphism, then $f(A) \leq_e A$.*

Proof. As $f(A)$ is isomorphic to A, it has the same rank as A. $\square$

Exercise 5S. Prove Corollary 5.22 using only the property that A contains no infinite direct sums of nonzero submodules. $\square$

Exercise 5T. If B is an essentially closed submodule in a module A of finite rank, show that A/B has finite rank and $\operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(A/B)$. (This additivity usually fails in case B is not essentially closed, as seen in the next exercise.) $\square$

Exercise 5U. Given $n \in \mathbb{N}$, show that $\mathbb{Z}$ contains a nonzero ideal B with $\operatorname{rank}(\mathbb{Z}/B) = n$. Note in particular that $\operatorname{rank}(\mathbb{Z}) < \operatorname{rank}(B) + \operatorname{rank}(\mathbb{Z}/B)$. $\square$

Exercise 5V. If $R = M_n(D)$ for some $n \in \mathbb{N}$ and some division ring D, show that $\operatorname{rank}(R_R) = \operatorname{rank}({}_R R) = n$. $\square$

• DIRECT SUMS OF INJECTIVE MODULES •

Theorem 5.23. [Papp, Bass] *A ring R is right noetherian if and only if every direct sum of injective right R-modules is injective.*

Proof. First assume that R is right noetherian and let $E = \bigoplus_{i \in I} E_i$ be a direct sum of injective right R-modules E_i. Let J be a right ideal of R and $f : J \to E$ a homomorphism.

Choose generators $x_1, \dots, x_n$ for J. Each $f(x_k)$ has only finitely many nonzero components in E, and so it lies in $\bigoplus_{i \in I_k} E_i$ for some finite subset $I_k \subseteq I$. If $I^* = I_1 \cup \cdots \cup I_n$ and $E^* = \bigoplus_{i \in I^*} E_i$, then each $f(x_k) \in E^*$, whence $f(J) \leq E^*$. Since I^* is finite, E^* is injective, and so f extends to a homomorphism $R_R \to E^* \leq E$. Therefore E is injective.

Conversely, assume that all direct sums of injective right R-modules are injective, and let $I_1 \leq I_2 \leq \cdots$ be an ascending chain of right ideals of R. Set

$$I = \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad E = \bigoplus_{n=1}^{\infty} E(R/I_n).$$

Then I is a right ideal of R, and E is an injective right R-module by hypothesis. Define a homomorphism

$$f : I \to \prod_{n=1}^{\infty} E(R/I_n)$$

so that $f(x)_n = x + I_n$ for all $x \in I$ and $n \in \mathbb{N}$. For any $x \in I$, we have $x \in I_k$ for some k, whence $x + I_n = 0$ for all $n \geq k$. Hence, $f(I) \leq E$.

As E is injective, there exists $z \in E$ such that $f(x) = zx$ for all $x \in I$. Then $z_k = 0$ for some k. For all $x \in I$, we have

$$x + I_k = f(x)_k = (zx)_k = z_k x = 0,$$

and so $x \in I_k$. Thus $I = I_k$, whence $I_n = I_k$ for all $n \geq k$.

Therefore R is right noetherian. $\square$

Corollary 5.24. [Matlis, Papp] *If R is a right noetherian ring, every injective right R-module is a direct sum of uniform injective modules.*

Proof. Let E be a nonzero injective right R-module. Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a maximal independent family of nonzero finitely generated submodules of E. If $\bigoplus_i A_i \not\leq_e E$, then E has a nonzero submodule B such that $\left(\bigoplus_i A_i\right) \cap B = 0$. Since B contains a nonzero finitely generated submodule, we may assume that B itself is finitely generated. But then $\mathcal{A} \cup \{B\}$ is independent, contradicting the maximality of $\mathcal{A}$. Thus $\bigoplus_i A_i \leq_e E$.

By Theorem 5.12, each A_i has an injective hull $E_i \leq E$. As $A_i \leq_e E_i$, Proposition 5.6 says that the E_i are independent. Hence, the sum of the E_i is a submodule $\bigoplus_i E_i$ of E, and it is essential in E because it contains $\bigoplus_i A_i$. On the other hand, $\bigoplus_i E_i$ is injective by Theorem 5.23, and so $E = \bigoplus_i E_i$.

Each A_i is a noetherian module and hence has finite rank, by Corollary 5.18. Thus each E_i is a (finite) direct sum of uniform submodules, all of which must be injective. Therefore E is a direct sum of uniform injective submodules. $\square$

Thus, over a right noetherian ring, finding all the injective right modules reduces to finding all the uniform ones. This is easy to do over a commutative noetherian ring, as follows.

Proposition 5.25. [Matlis] *Let R be a commutative noetherian ring and E an injective R-module. Then E is uniform if and only if $E \cong E((R/P)_R)$ for some prime ideal P of R.*

Proof. If $E \cong E((R/P)_R)$ for some prime ideal P, then, since R/P is a uniform R-module, so is E.

Conversely, assume that E is uniform. By Proposition 3.12, E has an associated prime P, and the submodule $A = \operatorname{ann}_E(P)$ is a fully faithful (R/P)-module. Choose a nonzero element $x \in A$, and note that $\operatorname{ann}(x) = \operatorname{ann}(xR) = P$, whence $xR \cong R/P$. Now E contains a submodule E' which is an injective hull for xR, and E' is a nonzero direct summand of E. As E is uniform, $E' = E$, and therefore $E = E(xR) \cong E((R/P)_R)$. $\square$

For example, Proposition 5.25 shows that every uniform injective $\mathbb{Z}$-module is isomorphic either to $E(\mathbb{Z})$ or to $E(\mathbb{Z}/p\mathbb{Z})$ for some prime integer p. On one hand, $E(\mathbb{Z}) = \mathbb{Q}$, while on the other, $E(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}(p^\infty)$ (which may be described as the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$). Thus every uniform injective $\mathbb{Z}$-module is isomorphic to one of $\mathbb{Q}$, $\mathbb{Z}(2^\infty)$, $\mathbb{Z}(3^\infty)$, $\mathbb{Z}(5^\infty)$, $\dots$. By Corollary 5.24, every injective $\mathbb{Z}$-module is isomorphic to a direct sum of some copies of these modules.

• ASSASSINATOR PRIMES •

We may view Proposition 5.25 as parametrizing the uniform injective modules over a commutative noetherian ring R via the prime ideals of R. An obvious question is whether this parametrization is a bijection, i.e., whether each prime ideal P is uniquely determined by the injective module $E((R/P)_R)$. The answer is positive, and to prove that we make use of the concept of "assassinator."

Lemma 5.26. *Let U be a uniform right module over a right noetherian ring R. Then there is a unique prime ideal P in R such that P equals the annihilator of some nonzero submodule of U and P contains the annihilators of all nonzero submodules of U. Moreover, P is the unique associated prime of U, and $\operatorname{ann}_U(P)$ is a fully faithful (R/P)-module.*

Proof. Choose a nonzero submodule A of U such that the ideal $P = \operatorname{ann}(A)$ is maximal among annihilators of nonzero submodules of U. Then P is an associated prime of U, and $\operatorname{ann}_U(P)$ is a fully faithful (R/P)-module (Proposition 3.12).

If B is a nonzero submodule of U, then $A \cap B \neq 0$ because U is uniform, and $P \subseteq \operatorname{ann}(A \cap B)$ because $AP = 0$. Then $\operatorname{ann}(A \cap B) = P$ by the maximality of P, whence $\operatorname{ann}(B) \subseteq P$. Thus P contains the annihilators of all nonzero submodules of U. The uniqueness of P in the first conclusion of the lemma is now clear.

Given an associated prime Q of U, there is a nonzero submodule C in U such that $\operatorname{ann}(C) = Q$ and C is a fully faithful (R/Q)-module. Since

$A \cap C \neq 0$, and since A is a fully faithful (R/P)-module, we conclude that $P = \operatorname{ann}(A \cap C) = Q$. Therefore P is the only associated prime of U. $\square$

Definition. If U is a uniform right module over a right noetherian ring, the unique associated prime of U is called the *assassinator* of U.

Exercise 5W. Identify the annihilator and the assassinator of each uniform submodule of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. $\square$

Exercise 5X. If $U \geq V$ are uniform right modules over a right noetherian ring, show that they have the same assassinator. $\square$

Exercise 5Y. If A is a right module over a right noetherian ring, show that $\operatorname{Ass}(A)$ equals the set of assassinators of uniform submodules of A. $\square$

Lemma 5.27. *Let P be a prime ideal in a right noetherian ring R, and let U be a uniform right ideal of R/P. Then $E(U_R)$ is a uniform injective right R-module, and its assassinator is P.*

Proof. Since U is uniform, so is $E(U_R)$. Observe that U is a fully faithful (R/P)-module, whence P is an associated prime of $E(U_R)$. Therefore P is the assassinator of $E(U_R)$. $\square$

Proposition 5.28. [Matlis] *If R is a commutative noetherian ring, then the rule $P \mapsto E((R/P)_R)$ provides a bijection between the set of prime ideals of R and the set of isomorphism classes of uniform injective R-modules. The inverse bijection is given by the rule $E \mapsto$ (assassinator of E).*

Proof. Proposition 5.25 and Lemma 5.27. $\square$

In the noncommutative case, a noetherian ring may have many more uniform injective modules than prime ideals. For instance, a simple noetherian ring R has only one prime ideal, namely 0, yet R may have more than one uniform injective module (up to isomorphism). This holds in particular for Weyl algebras over fields of characteristic zero, as follows.

Exercise 5Z. Let R be any simple noetherian domain which is not a division ring and A any simple right R-module. Show that $E(R_R)$ and $E(A)$ are uniform and not isomorphic. $\square$

• ADDITIONAL EXERCISES •

5ZA. If N is a nilpotent ideal in a ring R, show that $\operatorname{l.ann}_R(N)$ is an essential right ideal of R. $\square$

5ZB. If I is any ideal in a ring R, show that the ideal $I + \operatorname{l.ann}_R(I)$ is essential as a right ideal of R. $\square$

5ZC. Given a right module A over a ring R, show that A is injective if and only if every homomorphism from an essential right ideal of R to A extends to a homomorphism from R to A. $\square$

5ZD. Given modules A and B, show that $E(A) \cong E(B)$ if and only if there exist essential submodules $A' \leq_e A$ and $B' \leq_e B$ such that $A' \cong B'$. $\square$

5ZE. If R is a ring such that R_R has finite rank, show that any right or left invertible element in R is invertible. [Hint: If $x, y \in R$ with $yx = 1$, then $xR \leq_e R_R$ by Corollary 5.22. Check that $xR = xyR$ and xy is idempotent.] $\square$

5ZF. Let R be a commutative noetherian ring, J an ideal of R, and A a finitely generated right R-module. If A has an essential submodule B such that $BJ = 0$, show that there exists $n \in \mathbb{N}$ such that $AJ^n = 0$. [Hint: Find n such that $\operatorname{ann}_A(J^n) = \operatorname{ann}_A(J^{n+1})$. If $a \in A$ and $aJ^n \neq 0$, find $r \in R$ such that $0 \neq aJ^n r \subseteq B$.] $\square$

5ZG. If R is a commutative noetherian ring and $E = E((R/J)_R)$ for some ideal J of R, show that $E = \bigcup_{n=0}^{\infty} \operatorname{ann}_E(J^n)$. $\square$

5ZH. Let R be a commutative noetherian ring and A a finitely generated right R-module.

(a) Show that there exist $P_1, \dots, P_n \in \operatorname{Ass}(A)$ (with repetitions allowed) such that $AP_1P_2 \cdots P_n = 0$. [Hint: Exercises 3K and 5ZF.]

(b) If P is any annihilator prime for A, show that $P \in \operatorname{Ass}(A)$. [Hint: Apply (a) to the module $\operatorname{ann}_A(P)$.] Conclude that $\operatorname{Ass}(A)$ equals the set of annihilator primes for A. $\square$

5ZI. Show that any module A with finite rank n has submodules $A_1, \dots, A_n$ such that each A/A_i is uniform and $A_1 \cap \cdots \cap A_n = 0$. $\square$

• NOTES •

Existence of Injectives. Baer worked with what he called "complete" modules over a ring R, namely, modules A such that every homomorphism from a one-sided ideal of R to A extends to a homomorphism from R to A. He proved that every module is a submodule of a complete module [1940, Theorem 3], and that a module is complete if and only if it is a direct summand of every module that contains it [1940, Theorem 1]. The method we have given for embedding modules into injectives is due to Eckmann and Schopf [1953, §2].

Essential Submodules. The concept was introduced by Johnson [1951, p. 891], the name by Eckmann and Schopf [1953, (4.1)].

Injectivity Versus Inessentiality. That a module is injective if and only if it has no proper essential extensions is due to Eckmann and Schopf [1953, §4].

Injective Hulls. The concept was developed by Eckmann and Schopf [1953, §4]. The name "injective envelope" appeared in a paper of Matlis [1958, p. 512], the name "injective hull" in a paper of Rosenberg and Zelinsky [1959, p. 373].

Existence of Injective Hulls. Baer proved that any module A can be embedded in a complete module E with the property that every monomorphism from A to a complete module E' extends to a monomorphism from E to E' [1940, Theorem 4]. Eckmann and Schopf proved that any module has a maximal essential extension (in the sense of Theorem 5.12(b)), which is also a minimal injective extension (in the sense of Theorem 5.12(c)) [1953, §4].

Finite Rank. This was first developed from the standpoint of modules containing no infinite direct sums of nonzero submodules. The latter concept was introduced for rings by Goldie [1958b, p. 590] and later for modules [1960, p. 202], under the name "finite-dimensional modules."

Uniform Modules. Uniform right ideals were introduced by Goldie [1958b, p. 590] and later uniform modules [1960, p. 201].

Finite Rank Versus Infinite Direct Sums. Goldie proved that a module has finite rank if and only if it contains no infinite direct sums of nonzero submodules [1960, Theorem (1.1)].

Uniform Rank. This was introduced, under the name "dimension," by Goldie [1960, p. 202].

Injectivity of Direct Sums. That a ring R is right noetherian if and only if every direct sum of injective right R-modules is injective was proved independently by Papp [1959, Theorem 1] and Bass [1962, Theorem 1.1] (his result was actually donated to an earlier paper of Chase [1960, Proposition 4.1]).

Uniform Decompositions of Injectives. That every injective right module over a right noetherian ring is a direct sum of uniform injectives was proved independently by Matlis [1958, Theorem 2.5] and Papp [1959, Theorem 2].

Uniform Injectives over Commutative Noetherian Rings. Matlis developed the bijection between prime ideals of a commutative noetherian ring R and isomorphism classes of uniform injective R-modules [1958, Proposition 3.1].

6. Semisimple Rings of Fractions

One of the first constructions that an undergraduate student of algebra meets is the *quotient field* of a commutative integral domain, constructed as a set of fractions, that is, expressions a/b subject to an obvious equivalence relation. This leads to a very useful technique in commutative ring theory, namely, to pass from an arbitrary commutative ring R to a prime factor ring R/P and then to the quotient field of R/P. In the noncommutative case, we can ask whether it is possible to pass from a domain to a division ring built from fractions. While this is not always possible, it will turn out to be the case for any noetherian domain. However, since noncommutative noetherian rings need not have any factor rings that are domains, this is rather restrictive. Instead, recalling that prime rings are the most useful noncommutative analog of domains, we look for prime rings from which simple artinian rings can be built using fractions. The main result is *Goldie's Theorem*, which implies in particular that any prime noetherian ring has a simple artinian ring of fractions. It turns out to be little extra work to investigate rings from which semisimple rings of fractions can be built.

Our first task is to see how a ring of fractions can be constructed, given an appropriate set X of elements in a ring R to be used as denominators. Since the elements of X will become invertible in the ring of fractions, we cannot expect R to be embedded in the ring of fractions in case X contains any zero-divisors. While the presence of zero-divisors can be dealt with, it introduces some extra complications that we prefer to avoid for the present. Thus, in this chapter we restrict our discussion to non-zero-divisor denominators. The general theory of rings (and modules) of fractions will be developed in Chapter 10.

• RINGS OF FRACTIONS •

Definition. A *regular element* in a ring R is any non-zero-divisor, i.e., any element $x \in R$ such that $\mathrm{r.ann}_R(x) = 0$ and $\mathrm{l.ann}_R(x) = 0$.

Note that if $R \subseteq Q$ are rings and x is an element of R which is invertible in Q, then x must be a regular element of R.

Suppose that R is a ring and X a set of regular elements in R; we seek to build a ring whose elements are fractions with numerators from R and denominators from X. In the commutative case, the elementary notation r/x is convenient and familiar, but in the noncommutative case we must be more careful – since division by x amounts to multiplication by x^{-1}, we must decide whether to place the denominator on the right or the left of the numerator, i.e., whether we shall work with rx^{-1} or $x^{-1}r$. Thus, we already anticipate two possible rings of fractions – one with right-hand denominators and one with left-hand denominators. Note also that if we build a ring with denominators from X, the elements of X will become invertible in the new ring, as will all products of elements from X. Hence, we may as well include these products in X and start off with a multiplicatively closed set.

Definition. Let R be a ring and $X \subseteq R$ a multiplicative set of regular elements in R. A *right ring of fractions* (or *right quotient ring*) *for R with respect to X* is any overring $S \supseteq R$ such that:

(a) Every element of X is invertible in S.

(b) Every element of S can be expressed in the form ax^{-1} for some $a \in R$ and $x \in X$.

Left rings of fractions are defined analogously, using fractions of the form $x^{-1}a$. Of course, if a ring of fractions is commutative, the adjectives "right" and "left" are not needed.

This terminology obviously incorporates the classical case – the quotient field of a commutative domain R is a ring of fractions for R with respect to the multiplicative set $R \setminus \{0\}$.

For a noncommutative example, consider a skew polynomial ring $R = A[x;\alpha]$, where α is an automorphism of the ring A, and set $X = \{1, x, x^2, \dots\}$. The skew-Laurent ring $A[x^{\pm 1};\alpha]$ is both a right and a left ring of fractions for R with respect to X. Similarly, any quantum torus $\mathcal{O}_{\mathbf{q}}((k^{\times})^n)$ is a ring of fractions for the corresponding quantum affine space $\mathcal{O}_{\mathbf{q}}(k^n)$.

Based on experience with ordinary fractions, we expect computations in rings of fractions to rely heavily on common denominators, but in the noncommutative case we cannot always obtain common denominators just by multiplication. Let X be a multiplicative set of regular elements in a ring R, and suppose that there exists a right ring of fractions, say S, for R with respect to X. If we want to add fractions ax^{-1} and by^{-1} in S, we would like to write $ax^{-1} = a'z^{-1}$ and $by^{-1} = b'z^{-1}$ with a common denominator z, since then $ax^{-1} + by^{-1} = (a' + b')z^{-1}$. The problem is that, while $ax^{-1} = (ay)(xy)^{-1}$, we do not necessarily have $by^{-1} = (bx)(xy)^{-1}$ (unless x and y happen to commute). All we need, though, is an element $z \in X$ such that $z = xc = yd$ for some $c, d \in R$, since then $ax^{-1} = (ac)z^{-1}$ and $by^{-1} = (bd)z^{-1}$.

The key condition arises when we consider the product of fractions ax^{-1} and by^{-1}. We should not expect a simple formula like $(ax^{-1})(by^{-1}) = (ab)(yx)^{-1}$ unless x and b commute. On the other hand, since every element of S has (by assumption) the form of a fraction with right-hand denominator, the left-handed fraction $x^{-1}b$ must equal cz^{-1} for some $c \in R$ and $z \in X$, whence $(ax^{-1})(by^{-1}) = (ac)(yz)^{-1}$. Note that here we have a necessary condition for the existence of our ring of fractions: Given $b \in R$ and $x \in X$, there must exist $c \in R$ and $z \in X$ such that $x^{-1}b = cz^{-1}$, that is, $bz = xc$. This is precisely the right Ore condition that we met in Chapter 4. Moreover, this condition yields the common denominators needed for addition of fractions, by part (a) of Lemma 4.21.

Lemma 6.1. *Let R be a ring and X a multiplicative set of regular elements in R, and assume that there exists a right ring of fractions, say S, for R with respect to X.*

(a) *X is a right Ore set in R.*

(b) *Given any $s_1, \dots, s_n \in S$, there exist $a_1, \dots, a_n \in R$ and $x \in X$ such that each $s_i = a_i x^{-1}$.*

(c) *Let $a, b \in R$ and $x, y \in X$. Then $ax^{-1} = by^{-1}$ in S if and only if there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$.*

Proof. (a) This was noted in the discussion above.

(b) Each $s_i = b_i x_i^{-1}$ for some $b_i \in R$ and $x_i \in X$. By Lemma 4.21, there exist $x \in X$ and $c_1, \dots, c_n \in R$ such that $x = x_i c_i$ for all i. Since x and x_i are both invertible in S, so is c_i, and $x^{-1} = c_i^{-1} x_i^{-1}$. Thus $s_i = b_i c_i x^{-1}$ for all i.

(c) Suppose first that there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$. Then $ax^{-1} = ac(xc)^{-1} = bd(yd)^{-1} = by^{-1}$. Conversely, suppose that $ax^{-1} = by^{-1}$. By Lemma 4.21, there exist $c, d \in R$ such that $xc = yd \in X$. Hence,

$$ac(xc)^{-1} = ax^{-1} = by^{-1} = bd(yd)^{-1} = bd(xc)^{-1},$$

and therefore $ac = bd$. $\square$

The discussion above provides all the clues needed to construct rings of fractions. Thus, let X be a right Ore set of regular elements in a ring R. The construction can be summarized in the following five steps.

(1) Define a relation $\sim$ on $R \times X$ as follows: $(a, x) \sim (b, y)$ if and only if there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$. Then $\sim$ is an equivalence relation. Let $[a, x]$ denote the $\sim$-equivalence class of any pair (a, x) in $R \times X$, and let S denote the set of these equivalence classes.

(2) Given $[a, x]$ and $[b, y]$ in S, choose $c, d \in R$ such that $xc = yd \in X$, and set $[a, x] + [b, y] = [ac + bd,\ xc]$. Then $+$ is a well-defined operation on S.

(3) Given $[a, x]$ and $[b, y]$ in S, choose $c \in R$ and $z \in X$ such that $bz = xc$, and set $[a, x] \cdot [b, y] = [ac, yz]$. Then $\cdot$ is a well-defined operation on S.

(4) The system $(S, +, \cdot)$ is a ring.

(5) The rule $r \mapsto [r, 1]$ defines an isomorphism of R onto a subring of S, and when R is identified with this subring, S becomes a right ring of fractions for R with respect to X.

Exercise 6A. Check some of steps 1–5 above. □

The construction just outlined was developed by Ore and Asano in the 1930s and 1940s. It is completely straightforward but also tediously long. As we saw in the discussion of general skew polynomial rings $R[x;\alpha,\delta]$ in Chapter 2, it can be advantageous to build a ring as a ring of operators (i.e., endomorphisms) on some abelian group or module. Compared with a more abstract construction, this approach has the great benefit that addition and multiplication operations satisfying the ring axioms are already present. Let us see how this might work in the case of a ring of fractions.

Let X be a right Ore set of regular elements in a ring R, and suppose for the moment that there does exist a right ring of fractions for R with respect to X, say S. Then S embeds in the endomorphism ring $\operatorname{End}_R(S_R)$ as left multiplication operators, and we can try to mimic this embedding once we understand the right R-module structure of S. Note that any nonzero element $s \in S$ can be written as $s = ax^{-1}$ for some nonzero $a \in R$ and $x \in X$, whence $sx = a$ is a nonzero element of R. Consequently, S_R is an essential extension of R_R, and so we can think of S_R as a submodule of the injective hull $E(R_R)$. The observation above also shows that, for any $s \in S$, there exists $x \in X$ such that $sx \in R$. On the other hand, if $t \in E(R_R)$ and $y \in X$ with $ty \in R$, then $t = (ty)y^{-1} \in S$. Therefore we have determined S as a right R-module:

$$S_R = \{s \in E(R_R) \mid sx \in R \text{ for some } x \in X\}.$$

In order to recover S as an endomorphism ring of the above module, we first need to embed R into this endomorphism ring. That relies on the basic properties of injective modules – any element of R determines an endomorphism of R_R, which extends to an endomorphism of $E(R_R)$ by injectivity, and it is easily checked that the latter endomorphism maps the module S_R into itself. It must also be checked that the resulting endomorphism of S_R is uniquely determined by the initial element of R, but that is also easy.

The discussion above indicates how to describe a ring of fractions as an R-module and how to embed R in the endomorphism ring of this module. It is then routine to complete the construction, as follows.

Theorem 6.2. [Ore, Asano] *Let R be a ring and $X \subseteq R$ a multiplicative set of regular elements. Then there exists a right ring of fractions for R with respect to X if and only if X is a right Ore set.*

Proof. The necessity of the right Ore condition is given in Lemma 6.1. Now assume that X is a right Ore set.

Set $E = E(R_R)$ and $A = \{a \in E \mid ax \in R \text{ for some } x \in X\}$, and note that A is the inverse image (under the quotient map $E \to E/R$) of the X-torsion submodule $t_X(E/R)$. Thus A is a submodule of E, such that A/R is X-torsion while E/A is X-torsionfree. Since X consists of regular elements,

$t_X(E) \cap R = t_X(R) = 0$, and so $t_X(E) = 0$. Thus E and A are X-torsionfree. Set $S = \mathrm{End}_R(A)$.

Claim 1. Given any $a \in A$ and $x \in X$, there exists $b \in A$ such that $bx = a$.

Since x is regular, $xR \cong R_R$, and there is a homomorphism $f : xR \to A$ such that $f(x) = a$. By injectivity, f extends to a homomorphism $g : R \to E$. Setting $b = g(1)$, we obtain an element $b \in E$ such that $bx = g(x) = f(x) = a$. Since $bx \in A$ and E/A is X-torsionfree, we must have $b \in A$, and the claim is proved.

Claim 2. For each $r \in R$, there is a unique endomorphism $\phi_r \in S$ such that $\phi_r(b) = rb$ for all $b \in R$. In fact, ϕ_r is uniquely determined by the requirement $\phi_r(1) = r$.

Left multiplication by r defines an endomorphism of R_R, which extends to an endomorphism f of E such that $f(R) = rR$. For any $a \in A$, we have $ax \in R$ for some $x \in X$, whence $f(a)x \in f(R) \leq R$ and so $f(a) \in A$. Thus, f restricts to an endomorphism g of A, that is, $g \in S$ and $g(b) = f(b) = rb$ for all $b \in R$. Now consider any $g' \in S$ such that $g'(1) = r$. Then $R \leq \ker(g' - g)$, whence $(g' - g)(A)$ is a homomorphic image of A/R. Since A/R is X-torsion while E is X-torsionfree, we conclude that $(g' - g)(A) = 0$, and so $g' = g$. Therefore g is unique, establishing Claim 2.

Claim 3. The rule $r \mapsto \phi_r$ defines an isomorphism of R onto a subring $\widehat{R}$ of S.

Given any $r, s \in R$, we have $(\phi_r + \phi_s)(1) = r + s$ and $(\phi_r\phi_s)(1) = \phi_r(s) = rs$, whence $\phi_r + \phi_s = \phi_{r+s}$ and $\phi_r\phi_s = \phi_{rs}$ by uniqueness. Further, since $\mathrm{id}_A(1) = 1$, we see that $\mathrm{id}_A = \phi_1$, that is, $\phi_1 = 1$. Thus, the rule $r \mapsto \phi_r$ defines a ring homomorphism $R \to S$. Since $\phi_r(1) = r$, it is clear that $\phi_r = 0$ only when $r = 0$. Thus, the claim is proved.

Now set $\widehat{X} = \{\phi_x \mid x \in X\}$. By Claim 3, $\widehat{X}$ is a right Ore set of regular elements in $\widehat{R}$, and it will suffice to show that $\widehat{R}$ has a right ring of fractions with respect to $\widehat{X}$. We prove that S is the desired ring of fractions.

Given $x \in X$, note that $\ker(\phi_x) \cap R = \mathrm{r.ann}_R(x) = 0$ because x is regular, whence $\ker(\phi_x) = 0$. Since A/R and R/xR are both X-torsion, so is A/xR. Hence, for any $a \in A$ there exist $y \in X$ and $b \in R$ such that $ay = xb = \phi_x(b)$. By Claim 1, there exists $c \in A$ such that $cy = b$, whence $\phi_x(c)y = \phi_x(b) = ay$. Since A is X-torsionfree, $\phi_x(c) = a$, and so we have shown that $\phi_x(A) = A$. Thus ϕ_x is an automorphism of A, that is, an invertible element of S. Therefore all elements of $\widehat{X}$ are invertible in S.

Finally, consider an arbitrary endomorphism $s \in S$. Since $s(1) \in A$, there exist $x \in X$ and $r \in R$ such that $s(1)x = r$. Then $s\phi_x(1) = s(x) = r$, and hence $s\phi_x = \phi_r$ by the uniqueness of ϕ_r. Since ϕ_x is invertible in S, we conclude that $s = \phi_r\phi_x^{-1}$. Therefore S is a right ring of fractions for $\widehat{R}$ with respect to $\widehat{X}$, as desired. $\square$

Having constructed rings of fractions, the next question is uniqueness. As with many other concepts, uniqueness follows from a suitable universal mapping property.

Proposition 6.3. *Let R be a ring, $X \subseteq R$ a right Ore set of regular elements, and S a right ring of fractions for R with respect to X. Suppose that $\phi : R \to T$ is a ring homomorphism such that $\phi(x)$ is invertible in T for all $x \in X$. Then ϕ extends uniquely to a ring homomorphism $\psi : S \to T$.*

Proof. It is clear that there is at most one possibility for ψ: Given $a \in R$ and $x \in X$, we must have $\psi(ax^{-1}) = \phi(a)\phi(x)^{-1}$. The first task is to show that a well-defined map can be obtained in this way.

Suppose that $a, b \in R$ and $x, y \in X$ with $ax^{-1} = by^{-1}$. By Lemma 6.1, there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$. In particular, $\phi(x)$, $\phi(y)$, $\phi(xc)$, $\phi(yd)$ are all invertible in T, whence $\phi(c)$ and $\phi(d)$ are invertible, and so

$$\phi(a)\phi(x)^{-1} = \phi(ac)\phi(xc)^{-1} = \phi(bd)\phi(yd)^{-1} = \phi(b)\phi(y)^{-1}.$$

Therefore the rule $\psi(ax^{-1}) = \phi(a)\phi(x)^{-1}$ does indeed give a well-defined map $\psi : S \to T$. Note that $\psi(a) = \psi(a1^{-1}) = \phi(a)\phi(1)^{-1} = \phi(a)$ for all $a \in R$, so that ψ extends ϕ. In particular, $\psi(1) = 1$.

Let $a, b \in R$ and $x, y \in X$. There exist $c, d \in R$ such that $xc = yd \in X$, whence

$$\begin{aligned}\psi\big((ax^{-1}) + (by^{-1})\big) &= \psi\big((ac+bd)(xc)^{-1}\big) = \phi(ac+bd)\phi(xc)^{-1}\\ &= \phi(ac)\phi(xc)^{-1} + \phi(bd)\phi(yd)^{-1} = \psi(ax^{-1}) + \psi(by^{-1}).\end{aligned}$$

Further, there exist $e \in R$ and $z \in X$ such that $bz = xe$, whence

$$\psi\big((ax^{-1})(by^{-1})\big) = \psi\big((ae)(yz)^{-1}\big) = \phi(ae)\phi(yz)^{-1} = \phi(a)\phi(e)\phi(z)^{-1}\phi(y)^{-1}.$$

Since $\phi(b)\phi(z) = \phi(x)\phi(e)$, we have $\phi(e)\phi(z)^{-1} = \phi(x)^{-1}\phi(b)$, and thus

$$\psi\big((ax^{-1})(by^{-1})\big) = \phi(a)\phi(x)^{-1}\phi(b)\phi(y)^{-1} = \psi(ax^{-1})\psi(by^{-1}).$$

Therefore ψ is a ring homomorphism. □

Exercise 6B. Let R be a ring, $X \subseteq R$ a right Ore set of regular elements, and $T \supseteq R$ an overring such that every element of X is invertible in T. Show that the set of fractions $\{ax^{-1} \mid a \in R,\ x \in X\}$ is a subring of T, (a) with, and (b) without, using Proposition 6.3. □

Corollary 6.4. *Let R be a ring, $X \subseteq R$ a right Ore set of regular elements, and S, S' right rings of fractions for R with respect to X. Then the identity map on R extends uniquely to an isomorphism of S onto S'.*

Proof. Recall the pattern of proof for Corollary 1.12. □

Definition. If R is a ring and $X \subseteq R$ a right Ore set of regular elements, we shall write RX^{-1} to denote any right ring of fractions for R with respect to X. Similarly, we shall write $Y^{-1}R$ for a left ring of fractions.

Proposition 6.5. *Let R be a ring and $X \subseteq R$ a right and left Ore set of regular elements. Then $RX^{-1} = X^{-1}R$, that is, any right ring of fractions for R with respect to X is also a left ring of fractions for R with respect to X, and vice versa.*

Proof. Suppose that $S = RX^{-1}$; then, in particular, S is an overring of R and all elements of X are invertible in S. Given $s \in S$, we have $s = ax^{-1}$ for some $a \in R$ and $x \in X$. Since X is a left Ore set, there exist $b \in R$ and $y \in X$ such that $ya = bx$, whence $s = y^{-1}b$. Therefore $S = X^{-1}R$. $\square$

Exercise 6C. Let R be a ring and $X \subseteq R$ a right Ore set of regular elements. If R is right noetherian, show that RX^{-1} is right noetherian. [Hint: Review the proof of Corollary 1.15.] If R is a domain, show that RX^{-1} is a domain. $\square$

Recall that, for any field k and nonzero scalar $q \in k^{\times}$, the 2×2 quantum matrix algebra $\mathcal{O}_q(M_2(k))$ is an iterated skew polynomial ring over k (Exercise 2V). Hence, $\mathcal{O}_q(M_2(k))$ is a noetherian domain, and so the nonnegative powers of the quantum determinant, D_q, form a multiplicative set X consisting of regular elements. Since D_q is central in $\mathcal{O}_q(M_2(k))$ (Exercise 2W), X is an Ore set, and so there exists a (right and left) ring of fractions $\mathcal{O}_q(M_2(k))X^{-1}$. This algebra is generated by $\mathcal{O}_q(M_2(k))$ together with D_q^{-1}, and so it can be conveniently denoted $\mathcal{O}_q(M_2(k))[D_q^{-1}]$.

Definition. The *quantized coordinate ring of* $GL_2(k)$, or *the (coordinate ring of) quantum* $GL_2(k)$, is the ring of fractions $\mathcal{O}_q(GL_2(k)) = \mathcal{O}_q(M_2(k))[D_q^{-1}]$. (Recall the discussion at the end of Chapter 2.) This algebra is a noetherian domain by Exercise 6C.

• DIVISION RINGS OF FRACTIONS •

Definition. A *classical right quotient ring* for a ring R is a right ring of fractions for R with respect to the set of all regular elements in R; a *classical left quotient ring* is defined symmetrically. The situation when R has a classical right quotient ring Q is also denoted by saying that R *is a right order in* Q.

By Theorem 6.2, R has a classical right (left) quotient ring if and only if the set of regular elements in R is a right (left) Ore set. Moreover, Proposition 6.5 shows that if R has both a classical right and a classical left quotient ring, then these quotient rings coincide. In that case, we say that R has a *classical quotient ring*.

For example, every commutative ring has a classical quotient ring. In the case of a commutative domain, the classical quotient ring is its quotient field. A noncommutative domain need not have a classical quotient ring (see Exercise 6E), but if one exists, it will be a division ring, as we now show.

Definition. A *right Ore domain* is any domain R in which the nonzero elements form a right Ore set, i.e., for each nonzero $x, y \in R$ there exist $r, s \in R$ such that $xr = ys \neq 0$.

For example, every commutative domain is (right) Ore.

Lemma 6.6. *For a domain R, the following conditions are equivalent:*

(a) *R is a right Ore domain.*

(b) *R_R is uniform.*

(c) *R_R has finite rank.*

Proof. (a) $\iff$ (b): The right Ore condition is equivalent to saying that the intersection of any two nonzero principal right ideals of R is nonzero, which in turn is equivalent to the module R_R being uniform.

(b) $\Longrightarrow$ (c): This is clear.

(c) $\Longrightarrow$ (b): Suppose R_R is not uniform. Then there exist nonzero right ideals I_1 and J_1 in R such that $I_1 \cap J_1 = 0$. Choose a nonzero element $x_1 \in J_1$. Then $x_1 R \cong R$ because R is a domain, and so $x_1 R$ contains nonzero right ideals I_2 and J_2 such that $I_2 \cap J_2 = 0$. Continuing by induction, we obtain nonzero right ideals $I_1, J_1, I_2, J_2, \dots$ in R such that $I_n \cap J_n = 0$ and $J_n \geq I_{n+1} + J_{n+1}$ for all n. But then the right ideals $I_1, I_2, \dots$ are independent. Since R_R is assumed to have finite rank, this contradicts Theorem 5.17. Therefore R_R must be uniform. $\square$

Corollary 6.7. *Every right noetherian domain is right Ore.*

Proof. Corollary 5.18 and Lemma 6.6. $\square$

Exercise 6D. If R is a *right Bezout domain* (i.e., a domain in which every finitely generated right ideal is principal), show that R is right Ore. $\square$

Exercise 6E. Let $k(y)$ be a rational function field over a field k, let α be the k-algebra endomorphism of $k(y)$ given by the rule $\alpha(f) = f(y^2)$, and set $R = k(y)[x; \alpha]$. Show that R is left Ore but not right Ore. [Hint: Show that $xR \cap yxR = 0$.] Conclude that R does not have a classical right quotient ring. $\square$

Theorem 6.8. [Ore] *For a ring R, the following conditions are equivalent:*

(a) *There exists a right Ore set X of regular elements in R such that RX^{-1} is a division ring.*

(b) *R has a classical right quotient ring which is a division ring.*

(c) *R is a right Ore domain.*

Proof. (b) $\Longrightarrow$ (a): A priori.

(a) $\Longrightarrow$ (b): By assumption, RX^{-1} is an overring of R in which all elements have the form ax^{-1} for $a, x \in R$ with $x \neq 0$. Since all regular elements of

R are nonzero and hence invertible in RX^{-1}, we conclude that RX^{-1} is a classical right quotient ring of R.

(b) $\Longrightarrow$ (c): Under assumption (b), R must be a domain, and all nonzero elements of R are regular. By Theorem 6.2, $R \setminus \{0\}$ is a right Ore set, and thus R is a right Ore domain.

(c) $\Longrightarrow$ (b): Theorem 6.2 shows that R has a classical right quotient ring; call it D. Any nonzero element of D has the form ax^{-1} for some nonzero elements $a, x \in R$. Since a is a regular element of R, it is invertible in D, and so ax^{-1} is invertible in D. Therefore D is a division ring. $\square$

If R is a right Ore domain, its classical right quotient ring is usually called the *right Ore quotient (division) ring* of R. Of course, for a right and left Ore domain R, every right (left) Ore quotient ring is also a left (right) Ore quotient ring, because of Proposition 6.5, and so we refer just to *the Ore quotient ring* of R in this case.

In particular, Ore quotient rings provide a ready source of examples of transcendental division algebras, that is, division rings which contain elements transcendental over the center. In fact, there are division rings in which the only elements algebraic over the center are the elements of the center itself. The following exercises exhibit some important instances.

Exercise 6F. Let $R = S[x; \alpha, \delta]$, where S is a noetherian domain and α an automorphism of S. Then R is a right and left noetherian domain, and so R is right and left Ore. Let Q be the Ore quotient ring of R.

Given any $u \in Q$, write $u = ab^{-1}$ for some $a, b \in R$ with $b \neq 0$, and set $\deg(u) = \deg(a) - \deg(b)$. Show that $\deg(u)$ is well-defined. For all $u, v \in Q$, show that $\deg(uv) = \deg(u) + \deg(v)$ and that $\deg(u \pm v) \leq \max\{\deg(u), \deg(v)\}$. Given any nonzero element $u \in Q$ with degree $n \in \mathbb{Z}$, show that $u = (st^{-1})x^n + w$ for some $s, t \in S$ with $t \neq 0$ and some $w \in R$ with $\deg(w) < n$. The fraction st^{-1} is uniquely determined by u, and it is natural to call it the *leading coefficient* of u.

If F is a subfield of $S \cap Z(Q)$, show that any element of Q which is algebraic over F must have degree zero. $\square$

Exercise 6G. Let S be a division ring with center C, let δ be a derivation on S, and set $R = S[x; \delta]$. Let Q be the Ore quotient ring of R.

(a) Show that $\delta(C) \subseteq C$. Then show that the set $k = \{c \in C \mid \delta(c) = 0\}$ is a subfield of C. (It is called the *subfield of central constants* of S.) Show that k is contained in the center of Q.

(b) If $\operatorname{char}(S) = 0$ and δ is an outer derivation, show that k equals the center of Q. [Hint: Given $u \in Q \setminus S$, write $u = ax^n + bx^{n-1} + v$ for some $a, b \in S$ with $a \neq 0$, some $n \in \mathbb{Z}$, and some $v \in Q$ with $\deg(v) \leq n - 2$. Then show that $us \neq su$ for some $s \in S$. (The case $n = 0$ must be treated separately.)] $\square$

Exercise 6H. Let k be a field of characteristic zero and D the Ore quotient division ring of $A_1(k)$. Show that the center of D is k and that all elements of $D \setminus k$ are transcendental over k. [Hint: Write $A_1(k)$ as $k[y][x; d/dy]$ and show that D is also the Ore quotient ring of $k(y)[x; d/dy]$.] $\square$

Exercise 6I. Let S be a division ring with center C; let $R = S[x; \alpha]$, where α is an automorphism of S; and let Q be the Ore quotient ring of R.

(a) Show that the set $k = \{c \in C \mid \alpha(c) = c\}$ is a subfield of C and that k is contained in the center of Q.

(b) If no nonzero power of α is an inner automorphism, show that k equals the center of Q. [Hint: Exercise 6G.] $\square$

Exercise 6J. Let k be a field, $q \in k^\times$ a nonroot of unity, and D the Ore quotient ring of $\mathcal{O}_q(k^2)$. Show that the center of D is k, and that all elements of $D \setminus k$ are transcendental over k. [Hint: Write $\mathcal{O}_q(k^2)$ as $k[y][x; \alpha]$, where $\alpha(y) = qy$, and show that D is also the Ore quotient ring of $k(y)[x; \alpha]$.] $\square$

• GOLDIE'S THEOREM •

We now address the main goal of the chapter, Goldie's Theorem, which provides necessary and sufficient conditions for a ring to have a classical right quotient ring which is semisimple.

Definition. A *right (left) annihilator* in a ring R is any right (left) ideal of R which equals the right (left) annihilator of some subset of R.

Note that a right ideal I of R is a right annihilator if and only if $I = \mathrm{r.ann}(\mathrm{l.ann}(I))$. Namely, if $I = \mathrm{r.ann}(X)$ for some $X \subseteq R$, then $X \subseteq \mathrm{l.ann}(I)$, whence

$$I = \mathrm{r.ann}(X) \supseteq \mathrm{r.ann}(\mathrm{l.ann}(I)) \supseteq I.$$

Proposition 6.9. *Suppose that a ring R has a right noetherian classical right quotient ring Q. Then R_R has finite rank and R has the ACC on right annihhilators. Moreover, if Q is semisimple, then R must be semiprime.*

Proof. Let $I_1 \le I_2 \le \cdots$ be an ascending chain of right annihilators in R. Set $J_n = \mathrm{l.ann}(I_n)$ for all n. Then $J_1 \ge J_2 \ge \cdots$ is a descending chain of left annihilators in R and each $I_n = \mathrm{r.ann}(J_n)$. In Q, we have

$$\mathrm{r.ann}_Q(J_1) \le \mathrm{r.ann}_Q(J_2) \le \cdots,$$

an ascending chain of right ideals. Since Q is right noetherian, there is a positive integer m such that $\mathrm{r.ann}_Q(J_n) = \mathrm{r.ann}_Q(J_m)$ for all $n \ge m$. Consequently,

$$I_n = R \cap \mathrm{r.ann}_Q(J_n) = R \cap \mathrm{r.ann}_Q(J_m) = I_m$$

for all $n \ge m$. Therefore R has the ACC on right annihilators.

If R_R does not have finite rank, it contains an infinite sequence $A_1, A_2, \dots$ of independent nonzero right ideals (Theorem 5.17). Choose a nonzero element $a_i \in A_i$ for each i. Since Q is right noetherian, the right ideals $a_1Q, a_2Q, \dots$ cannot be independent, and so there exist $q_1, \dots, q_n \in Q$ such that $a_1q_1 + \cdots + a_nq_n = 0$ but $a_nq_n \neq 0$. There exist $b_1, \dots, b_n, x \in R$ with x regular such that each $q_i = b_ix^{-1}$. Then

$$a_1b_1 + \cdots + a_nb_n = (a_1q_1 + \cdots + a_nq_n)x = 0.$$

Since $A_1, \dots, A_n$ are independent, $a_nb_n = 0$. But then $a_nq_n = a_nb_nx^{-1} = 0$, a contradiction. Therefore R_R does have finite rank.

Finally, assume that Q is semisimple, and consider any ideal N of R such that $N^2 = 0$. Set $L = \mathrm{l.ann}_R(N)$ and note that L is an ideal of R. By Exercise 5ZA, L is essential as a right ideal in R. Next, we show that LQ is an essential right ideal of Q. Given any nonzero right ideal $B \leq Q_Q$, choose a nonzero element $b \in B$ and write $b = ax^{-1}$ for some $a, x \in R$ with $a \neq 0$ and x regular. Then $a = bx$ lies in $R \cap B$, whence $R \cap B$ is a nonzero right ideal of R. Consequently, $R \cap B \cap L \neq 0$, and hence $B \cap LQ \neq 0$. Thus $LQ \leq_e Q_Q$.

As Q_Q is semisimple, $LQ = Q$, by Corollary 5.9. Hence, there exist $y_1, \dots, y_n \in L$ and $q_1, \dots, q_n \in Q$ such that $y_1q_1 + \cdots + y_nq_n = 1$. There exist $a_1, \dots, a_n, x \in R$ with x regular such that each $q_i = a_ix^{-1}$. Then

$$x = (y_1q_1 + \cdots + y_nq_n)x = y_1a_1 + \cdots + y_na_n \in L,$$

whence $xN = 0$. Since x is regular, $N = 0$.

Therefore R is semiprime, by Corollary 3.8. $\square$

Definition. A *right Goldie ring* is any ring R such that R_R has finite rank and R has the ACC on right annihilators.

For example, every right noetherian ring is right Goldie.

Proposition 6.9 says that any ring which has a semisimple classical right quotient ring must be a semiprime right Goldie ring. The converse statement is the main content of Goldie's Theorem.

To prove Goldie's Theorem, we first have to show that the set of regular elements in a semiprime right Goldie ring R is a right Ore set. Thus, given $a, x \in R$ with x regular, we need to find $b, y \in R$ with y regular such that $ay = xb$, that is, we need a regular element y such that $ay \in xR$. Now since R_R has finite rank, it follows from Corollary 5.22 that $xR \leq_e R_R$, and so the right ideal $J = \{r \in R \mid ar \in xR\}$ is essential in R by Proposition 5.6(c). Hence, it will suffice to prove that every essential right ideal of R contains a regular element. This fact is the heart of Goldie's Theorem, and establishing it takes the majority of the work required to prove the theorem.

Proposition 6.10. *Let R be a ring and set*

$$J = \{a \in R \mid aI = 0 \text{ for some } I \leq_e R_R\} = \{a \in R \mid \mathrm{r.ann}(a) \leq_e R_R\}.$$

(a) *J is an ideal of R.*

(b) [Mewborn-Winton] *If R has the ACC on right annihilators, then J is nilpotent.*

(c) *If R is a semiprime right Goldie ring, then the left annihilator of every essential right ideal of R is zero.*

Proof. (a) By Exercise 5D, J is a right ideal of R. It is clear that J is also a left ideal.

(b) Note that $\text{r.ann}(J) \leq \text{r.ann}(J^2) \leq \cdots$. As R has the ACC on right annihilators, $\text{r.ann}(J^k) = \text{r.ann}(J^{k+1})$ for some $k \in \mathbb{N}$. We claim that $J^k = 0$.

If not, then we may choose an x in $R \setminus \text{r.ann}(J^k)$ such that $\text{r.ann}(x)$ is as large as possible. For any $a \in J$, we have $\text{r.ann}(a) \cap xR \neq 0$ since $\text{r.ann}(a) \leq_e R_R$. Hence, there is some $s \in R$ such that $axs = 0$ while $xs \neq 0$. Then $\text{r.ann}(x) < \text{r.ann}(ax)$, and so (by the maximality of $\text{r.ann}(x)$) we obtain $ax \in \text{r.ann}(J^k)$. Hence, $J^k ax = 0$; and since this holds for all $a \in J$, we infer that $x \in \text{r.ann}(J^{k+1})$. Since $\text{r.ann}(J^{k+1}) = \text{r.ann}(J^k)$, this contradicts our choice of x, and therefore $J^k = 0$.

(c) The right Goldie assumption says that R has the ACC on right annihilators, and so J is nilpotent by part (b). Semiprimeness then implies $J = 0$, and (c) follows. $\square$

Lemma 6.11. *Let R be a semiprime right Goldie ring. For any $x \in R$, the following conditions are equivalent:*

(a) *x is regular.*

(b) $\text{r.ann}_R(x) = 0$.

(c) $xR \leq_e R_R$.

Proof. Obviously (a) $\Longrightarrow$ (b), and (b) $\Longrightarrow$ (c) by Corollary 5.22.

(c) $\Longrightarrow$ (a): By Proposition 6.10, the left annihilator of every essential right ideal of R is zero. In particular, $\text{l.ann}(xR) = 0$, and thus $\text{l.ann}(x) = 0$.

Now set $I = \text{r.ann}(x)$ and choose a right ideal J of R maximal such that $I \cap J = 0$. Then $I + J \leq_e R_R$ by Proposition 5.7, and we claim that $xJ \leq_e xR$. Given a nonzero element $xr \in xR$, set $K = \{k \in R \mid rk \in I + J\}$ and note that $K \leq_e R_R$ by Proposition 5.6(c). Then $\text{l.ann}(K) = 0$, and hence $xrK \neq 0$. But $xrK \leq x(I+J) = xJ$, so we have shown that $xrR \cap xJ \neq 0$. Thus $xJ \leq_e xR$, as claimed.

By Corollary 5.21, $\text{rank}(xJ) = \text{rank}(xR) = \text{rank}(R_R)$. Since $J \cap \text{r.ann}(x) = 0$, we have $J \cong xJ$, and so $\text{rank}(J) = \text{rank}(R_R)$. A second application of Corollary 5.21 now shows that $J \leq_e R_R$, whence $I = 0$. Therefore x is regular. $\square$

Lemma 6.12. *If R is a semiprime right Goldie ring, then R has the DCC on right annihilators.*

Proof. By Proposition 6.10, the left annihilator of every essential right ideal of R is zero.

Let $I_1 \geq I_2 \geq \cdots$ be a descending chain of right annihilators in R and write each $I_j = \text{r.ann}(A_j)$ for some $A_j \subseteq R$. Then $\text{rank}(I_1) \geq \text{rank}(I_2) \geq \cdots$ is a descending chain of nonnegative integers, and so there exists an index m such that $\text{rank}(I_n) = \text{rank}(I_m)$ for all $n \geq m$. By Corollary 5.21, $I_n \leq_e I_m$ for all $n \geq m$, and we claim that $I_n = I_m$ for all $n \geq m$.

Fix $n \geq m$, consider $x \in I_m$, and set $J = \{r \in R \mid xr \in I_n\}$. Then $J \leq_e R_R$ and $A_n x J = 0$. Since $\text{l.ann}(J) = 0$, we obtain $A_n x = 0$, whence $x \in I_n$. Therefore $I_n = I_m$, as claimed. □

Proposition 6.13. [Goldie's Regular Element Lemma] *Let R be a semiprime right Goldie ring and I a right ideal of R. Then I is an essential right ideal if and only if I contains a regular element.*

Proof. If there is a regular element $x \in I$, then, as $xR \leq_e R_R$ by Lemma 6.11, we must have $I \leq_e R_R$.

Conversely, assume that $I \leq_e R_R$. We need to find an element of I whose right and left annihilators are zero. To get started, we look for an element of I whose right annihilator is as small as possible. By Lemma 6.12, R has the DCC on right annihilators. Hence, there is some $x \in I$ such that the right ideal $A = \text{r.ann}(x)$ is minimal among right annihilators of elements of I. We claim that $xR \leq_e I$.

Thus, consider any right ideal $B \leq I$ for which $B \cap xR = 0$. Given any $b \in B$, we have $bR \cap xR = 0$, from which it follows that

$$\text{r.ann}(b + x) = \text{r.ann}(b) \cap \text{r.ann}(x) \leq A.$$

As $b + x \in I$, the minimality of A forces

$$A = \text{r.ann}(b + x) = \text{r.ann}(b) \cap \text{r.ann}(x) \leq \text{r.ann}(b),$$

whence $bA = 0$. This shows that $BA = 0$. Now $(B \cap A)^2 \leq BA = 0$, and so $B \cap A = 0$ because R is semiprime. Similarly, $(RB \cap A)^2 \leq RBA = 0$, whence $RB \cap A = 0$.

The reason for introducing the ideal RB rather than working with the right ideal B is that $xRB \leq RB$ whereas $xB \cap B = 0$ (cf. Exercise 6K). Since $RB \cap \text{r.ann}(x) = 0$, left multiplication by x provides a monomorphism from $(RB)_R$ to itself. As $(RB)_R$ has finite rank, Corollary 5.22 shows that $xRB \leq_e RB$. However, we also have $B \cap xRB = 0$, and consequently $B = 0$.

Therefore $xR \leq_e I$, as claimed. Since $I \leq_e R_R$, it follows that $xR \leq_e R_R$. Therefore x is regular, by Lemma 6.11. □

Exercise 6K. The point of this exercise is to obtain $B = 0$ in the proof of Proposition 6.13 without introducing the ideal RB. First, show that $A \cap \sum_{i=0}^{\infty} x^i B = 0$. Then, show that the right ideals $B, xB, x^2B, \ldots$ are independent. □

There are many applications in which the Regular Element Lemma is used to show that certain two-sided ideals contain regular elements. The following corollary gives the application that will be of most use to us later.

Corollary 6.14. *If R is a prime right (or left) Goldie ring, then every nonzero ideal of R contains a regular element.*

Proof. By Exercise 5A, any nonzero ideal of R is both an essential right ideal and an essential left ideal. □

Theorem 6.15. [Goldie's Theorem] *A ring R has a semisimple classical right quotient ring if and only if R is a semiprime right Goldie ring.*

Proof. Necessity is given by Proposition 6.9. Conversely, assume that R is a semiprime right Goldie ring.

Given $a, x \in R$ with x regular, we have $xR \leq_e R_R$ by Lemma 6.11, whence the right ideal $J = \{r \in R \mid ar \in xR\}$ is essential in R. By Proposition 6.13, there exists a regular element $y \in J$, and $ay = xb$ for some $b \in R$. Thus, the set of regular elements in R satisfies the right Ore condition. Now, by Theorem 6.2, R has a classical right quotient ring; call it Q.

To prove that Q is semisimple, it suffices, by Corollary 5.9, to show that Q has no proper essential right ideals. Consider an essential right ideal I of Q; we claim that $I \cap R \leq_e R_R$. Given any nonzero element $b \in R$, there exists $q \in Q$ such that bq is a nonzero element of I. Write $q = ax^{-1}$ for some $a, x \in R$ with x regular. Then $ba = bqx$ is a nonzero element of $I \cap R$, proving the claim. Now, by Proposition 6.13, there exists a regular element $y \in I \cap R$; and since y is invertible in Q, we conclude that $I = Q$. Therefore Q has no proper essential right ideals, as desired. □

The most important part of Goldie's Theorem for us is the noetherian case:

Corollary 6.16. *Every semiprime right noetherian ring has a semisimple classical right quotient ring.* □

Lemma 6.17. *Let R be a ring with a semisimple classical right quotient ring Q. Then Q is a simple ring if and only if R is a prime ring.*

Proof. We first note that $R_R \leq_e Q_R$, since the product of any nonzero fraction in Q with its denominator is a nonzero element of R.

Suppose that R is prime. If I is any nonzero ideal of Q, then as $R_R \leq_e Q_R$ we see that $I \cap R$ is a nonzero ideal of R. Then $I \cap R \leq_e R_R$ (Exercise 5A), and so $I \cap R \leq_e Q_R$. As a result, $I_R \leq_e Q_R$, and hence $I_Q \leq_e Q_Q$. Since Q is semisimple, $I = Q$. Therefore Q is simple.

Conversely, assume that Q is simple and consider any ideals A and B of R such that $AB = 0$ but $A \neq 0$. Then QAQ is a nonzero ideal of Q, whence $QAQ = Q$, and so

$$p_1 a_1 q_1 + \cdots + p_n a_n q_n = 1$$

for some $p_i, q_i \in Q$ and some $a_i \in A$. There exist $b_1, \ldots, b_n, x \in R$ with x regular such that each $q_i = b_i x^{-1}$. Hence,

$$x = (p_1 a_1 q_1 + \cdots + p_n a_n q_n)x = p_1 a_1 b_1 + \cdots + p_n a_n b_n \in QA,$$

and so $xB \leq QAB = 0$. As x is regular, $B = 0$. Therefore R is prime. □

Theorem 6.18. [Goldie, Lesieur-Croisot] *A ring R has a simple artinian classical right quotient ring if and only if R is a prime right Goldie ring.*

Proof. Theorem 6.15 and Lemma 6.17. □

Corollary 6.19. *Every prime right noetherian ring has a simple artinian classical right quotient ring.* □

Definition. Let R be a semiprime right Goldie ring. Any classical right quotient ring of R is called a *right Goldie quotient ring* of R. Because of the uniqueness expressed in Corollary 6.4, we may speak of *the* right Goldie quotient ring of R.

Proposition 6.20. *If R is a semiprime right and left Goldie ring, then every right (left) Goldie quotient ring of R is also a left (right) Goldie quotient ring of R.*

Proof. Proposition 6.5. □

Because of Proposition 6.20, when R is a semiprime right and left Goldie ring we refer just to *the Goldie quotient ring* of R.

Exercise 6L. If a ring R has a classical right quotient ring Q, and R_R has finite rank, show that Q_Q has finite rank and that $\operatorname{rank}(Q_Q) = \operatorname{rank}(R_R)$. Conclude, in particular, that if R is a semiprime right Goldie ring with right Goldie quotient ring Q, then $\operatorname{length}(Q_Q) = \operatorname{rank}(R_R)$. □

• NIL SUBSETS •

Goldie's Theorem gives us the option of trying to study a noetherian ring R by working first within a semisimple ring (the Goldie quotient ring of R/N, where N is the prime radical of R), trying to pull information back from there to R/N, and finally trying to lift information from R/N to R. Later we will see this principle in action a number of times. Here, as a digression, we show how this method may be used to give an easy proof of a classical result – that nil subrings (nonunital subrings, of course!) of noetherian rings must be nilpotent. Actually, the details go more smoothly if we work more generally with nil multiplicatively closed subsets.

Recall that a subset S of a ring R is nil provided all elements of S are nilpotent. To say that S itself is nilpotent, on the other hand, means that $S^n = 0$ for some $n \in \mathbb{N}$. Now in standard ring-theoretic notation, S^n denotes the set of all sums of n-fold products of elements from S. Note, however, that $S^n = 0$ if and only if all n-fold products of elements from S equal zero.

Theorem 6.21. [Levitzki, Hopkins, Goldie] *If R is a right noetherian ring and S is a nil multiplicatively closed subset of R, then S is nilpotent.*

Proof. If N is the prime radical of R, then, since N is nilpotent, it suffices to show that the image of S in R/N is nilpotent. Moreover, by Goldie's

Theorem, R/N is a subring of an artinian ring, and so it is enough to show that nil multiplicatively closed subsets of artinian rings are nilpotent. Thus we may assume, without loss of generality, that R is artinian. We may also assume that $0 \in S$.

Let $\ell = \text{length}(R_R)$. We first show that every nilpotent multiplicatively closed subset N of R satisfies $N^\ell = 0$. Consider the descending chain

$$R \geq NR \geq N^2R \geq \cdots$$

of right ideals of R. Since R_R has length ℓ, there must be a nonnegative integer $m \leq \ell$ such that $N^mR = N^{m+1}R$. It follows (by induction) that $N^mR = N^kR$ for all $k > m$. Then, since N is nilpotent, $N^mR = 0$, and thus $N^\ell = 0$, as claimed.

Using Zorn's Lemma, there is a multiplicatively closed subset $N \subseteq S$ maximal with respect to the property $N^\ell = 0$. In view of the previous paragraph, N is maximal among nilpotent multiplicatively closed subsets of S. Note that $0 \in N$ (since otherwise $N \cup \{0\}$ would be a larger nilpotent multiplicatively closed subset of S).

If $S \neq N$, we claim that there is an element $s \in S \setminus N$ such that $sN \subseteq N$. Otherwise, there are elements $s_0 \in S \setminus N$ and $n_1, n_2, \cdots \in N$ such that $s_0n_1n_2 \cdots n_k \notin N$ for $k = 1, 2, \ldots$. But then $s_0n_1n_2 \cdots n_\ell \neq 0$, contradicting the fact that $N^\ell = 0$. Thus, there does exist $s \in S \setminus N$ such that $sN \subseteq N$, as claimed.

There is a positive integer m such that $s^m = 0$. Set

$$T = \{s^i \mid i = 1, 2, \ldots, m-1\} \cup \{ns^i \mid n \in N \text{ and } i = 0, 1, \ldots, m-1\}$$

and observe that T is a multiplicatively closed subset of S properly containing N. Any product of m elements of T either equals 0 or contains a factor of the form ns^i. Hence, all products of m elements from T belong to the set

$$U = \{ns^i \mid n \in N \text{ and } i = 0, 1, \ldots, m-1\}.$$

Now any product of ℓ elements from U has the form

$$n_1s^{i(1)}n_2s^{i(2)} \cdots n_\ell s^{i(\ell)}$$

for some $n_j \in N$ and $i(j) \in \{0, 1, \ldots, m-1\}$, and such a product is zero because

$$n_1,\ s^{i(1)}n_2,\ s^{i(2)}n_3,\ \ldots,\ s^{i(\ell-1)}n_\ell \in N.$$

Thus $U^\ell = 0$, whence $T^{m\ell} = 0$. However, this contradicts the maximality of N.

Therefore $S = N$, and S is nilpotent. $\square$

One is perhaps more likely to encounter nil one-sided ideals than nil multiplicatively closed subsets in a noetherian ring. The nilpotence of these, a result known as *Levitzki's Theorem*, may be obtained without using Goldie's Theorem, in the following manner.

Exercise 6M. Show (without using Goldie's Theorem) that any nil right or left ideal in a right noetherian ring R must be nilpotent. [Hints: First, reduce to the case that R is semiprime. Next, observe that if xR is a nil right ideal of R, then Rx is a nil left ideal. Finally, if I is a nonzero nil left ideal of R, look at a nonzero element of I whose right annihilator is as large as possible.] $\square$

• ADDITIONAL EXERCISES •

6N. Let R be a semiprime right Goldie ring. Show that R is left Goldie if and only if ${}_RR$ has finite rank. $\square$

6O. Let R be a semiprime right Goldie ring, and let $x, y \in R$. Show that xy is regular if and only if x and y are both regular. $\square$

6P. Let k be a field of characteristic zero, $T = A_1(k) = k[y][x; d/dy]$, and $R = \begin{pmatrix} k & T/xT \\ 0 & T \end{pmatrix}$. Observe that R is right noetherian, by Proposition 1.8. Show that R does not have the DCC on right annihilators. [Hint: Consider the elements $\begin{pmatrix} 0 & y^n + xT \\ 0 & 0 \end{pmatrix}$ for $n \in \mathbb{N}$.] Show that R is not isomorphic to a subring of a left noetherian ring. $\square$

6Q. Let R be a semiprime right noetherian ring with right Goldie quotient ring Q. If Q_R is finitely generated, show that $Q = R$. [Hint: Look at $x^{-n}R$ for $n \in \mathbb{N}$ and regular elements $x \in R$.] $\square$

6R. Let R be a prime ring, S a subring of the center of R, and $X = S \setminus \{0\}$. Then X is an Ore set of regular elements in R, and so RX^{-1} exists. If R is a finitely generated S-module, show that R is a Goldie ring and that RX^{-1} is its Goldie quotient ring. [Hint: Show that RX^{-1} is a prime finite dimensional algebra over the quotient field of S.] $\square$

6S. Let $R \subseteq Q$ be rings such that R is semiprime, Q is right Goldie, and $R_R \leq_e Q_R$.

(a) Show that R has the ACC on right annihilators.

(b) Show that every essential right ideal of R has zero left annihilator in Q. [Hint: First show that the left annihilators in R are zero and then use Exercise 5D.]

(c) Show that $J \leq_e (JQ)_R$ for all right ideals J of R.

(d) Show that R_R has finite rank; thus R is a right Goldie ring. $\square$

6T. Let $R \subseteq Q$ be rings such that R is semiprime, Q is semisimple, and $R_R \leq_e Q_R$. Then R is right Goldie by Exercise 6S. Show that Q is the right Goldie quotient ring of R. $\square$

• NOTES •

Existence of Rings of Fractions. Asano proved that a ring has a right ring of fractions with respect to its regular elements if and only if its regular elements satisfy the right Ore condition [1939, Satz 1] and later that a ring has a right ring of fractions with respect to a multiplicative set X of regular elements if and only if X satisfies the right Ore condition [1949, Satz I].

Quantum $GL_2(k)$. This algebra, which we are denoting $\mathcal{O}_q(GL_2(k))$, was introduced by Manin [1987, §1.7].

Division Rings of Fractions. Ore proved that a domain R has a classical right quotient ring which is a division ring if and only if R satisfies the right Ore condition [1931, Theorems 1, II].

Nilpotence of Singular Ideals. The ideal J described in Proposition 6.10 is called the *right singular ideal* of R; it was introduced by Johnson [1951, p. 894]. That J is nilpotent when R has the ACC on right annihilators comes from a proof of Mewborn and Winton [1969, Theorem 2.5].

Regular Elements in Essential Ideals. The existence of regular elements in essential right ideals was proved by Goldie first for prime right and left Goldie rings [1958a, Theorem 7; 1958b, Theorem 10] and later for semiprime right Goldie rings [1960, Theorem 3.9].

Goldie's Theorems. Initially, Goldie proved that a ring R has a simple artinian classical right and left quotient ring if and only if R is a prime right and left Goldie ring [1958a, Theorem A; 1958b, Theorem 13]. His methods were then modified by Lesieur and Croisot to prove a one-sided version of this result [1959a, Théorème 7; 1959b, Théorèmes 9, 10, 11]. After that, Goldie established the characterization of rings with semisimple classical right quotient rings [1960, Theorems 4.1, 4.4].

Nilpotence of Nil Subrings. This was first obtained by Levitzki for a ring with both the ACC and DCC on right ideals [1931, Hauptsatz, p. 625]. For right artinian rings, this was later obtained independently by Hopkins [1938, (3.3); 1939, (1.7)] and Levitzki [1939, Theorem 12]. Later still, Levitzki proved that nil subrings of a right and left noetherian ring are nilpotent [1945, Theorem 6], and that nil one-sided ideals in a right noetherian ring are nilpotent [1945, Theorem 5]. Finally, the nilpotence of nil subrings of right noetherian rings was obtained by Goldie [1960, Theorem 6.1].

7. Modules over Semiprime Goldie Rings

Many investigations of the structure of a right module A over a right noetherian ring R involve related modules over prime or semiprime factor rings of R. For instance, if A is finitely generated, then by using a prime series we may view A as built from a chain of subfactors each of which is a fully faithful module over a prime factor ring of R (see Proposition 3.13). Alternatively, we may relate the structure of A to the structure of the (R/N)-modules $A/AN, AN/AN^2, \dots,$ where N is the prime radical of R. Thus, we need a good grasp of the structure of modules over prime or semiprime noetherian rings. The fundamentals of such structure can be obtained with little extra effort for modules over prime or semiprime Goldie rings.

• MINIMAL PRIME IDEALS •

In working with the right Goldie quotient ring of a semiprime factor ring of a right noetherian ring R, say R/I, we shall often need to refer to the regular elements of R/I. It is then convenient to have a notation for the representatives of these cosets, as follows.

Definition. Let I be an ideal in a ring R. An element $x \in R$ is said to be *regular modulo* I provided the coset $x + I$ is a regular element of the ring R/I. The set of all such x is denoted by $\mathcal{C}(I)$, or by $\mathcal{C}_R(I)$ if the ring R needs emphasis. Thus, the set of all regular elements in R may be denoted $\mathcal{C}_R(0)$.

For example, if P is a prime ideal in a commutative ring R, then $\mathcal{C}_R(P)$ is just the complement $R \setminus P$.

Proposition 7.1. *Let R be a semiprime right Goldie ring and Q its right Goldie quotient ring.*

(a) *There are only finitely many distinct maximal ideals in the ring Q, say $M_1, \dots, M_n$, and $M_1 \cap \cdots \cap M_n = 0$.*

(b) *Each of the ideals $P_i = R \cap M_i$ is a minimal prime ideal of R, and $P_1, \dots, P_n$ are distinct. Each of the factor rings R/P_i is right Goldie, and Q/M_i is (via the natural embedding $R/P_i \to Q/M_i$) the right Goldie quotient ring of R/P_i.*

(c) *The only minimal prime ideals of R are $P_1, \dots, P_n$.*

(d) *Each $P_i = \mathrm{r.ann}_R\big(\bigcap_{j \neq i} P_j\big)$.*

Proof. (a) Since Q is semisimple, we may identify it with a finite direct product of simple artinian rings Q_i, say $Q = Q_1 \times \cdots \times Q_n$, so that the elements of Q may be written as n-tuples. The maximal ideals of Q are just the ideals $M_i = \{x \in Q \mid x_i = 0\}$, for $i = 1, \ldots, n$, and obviously $M_1 \cap \cdots \cap M_n = 0$.

(b) If $A_i = \bigcap_{j \neq i} M_j = \{x \in Q \mid x_j = 0 \text{ for all } j \neq i\}$, then $R \cap A_i = \bigcap_{j \neq i} P_j$. Since $R_R \leq_e Q_R$, we have $R \cap A_i \neq 0$. On the other hand,

$$R \cap A_i \cap P_i = P_1 \cap \cdots \cap P_n = 0,$$

and so $R \cap A_i \nsubseteq P_i$, whence $P_j \nsubseteq P_i$ for all $j \neq i$. In particular, $P_1, \ldots, P_n$ are distinct.

Note that P_i is the kernel of the natural map $\nu_i : R \to Q \to Q_i$ sending each element $r \in R$ to its i-th component r_i. Since $R_R \leq_e Q_R$ and R_R has finite rank, Q_R must have finite rank, whence $(Q_i)_R$ has finite rank. But the map ν_i induces a right R-module embedding of R/P_i into Q_i, so that $(R/P_i)_R$ has finite rank, and thus, finally, R/P_i has finite rank as a right module over itself.

We now claim that Q_i is a classical right quotient ring for $\nu_i(R)$. If $\nu_i(c)$ is a regular element of $\nu_i(R)$, then $c \in \mathcal{C}_R(P_i)$ and so $(cR+P_i)/P_i$ is an essential right ideal of R/P_i by Corollary 5.22, whence $cR + P_i \leq_e R_R$. Hence, $cR + P_i$ contains a regular element d (Proposition 6.13), and since d is invertible in Q it follows that $(cR + P_i)Q = Q$, whence $cQ + M_i = Q$. Consequently, $\nu_i(c)$ is right invertible in Q_i and hence invertible (see Corollary 4.25 or Exercise 5ZE). Thus, all regular elements of $\nu_i(R)$ are invertible in Q_i. Any element of Q_i has the form q_i for some $q \in Q$, and $q = ax^{-1}$ for some $a, x \in R$ with x regular. Then x_i is invertible in Q_i and is regular in $\nu_i(R)$. Since $qx = a$, we have $q_i x_i = a_i$ and hence $q_i = a_i x_i^{-1} = \nu_i(a)\nu_i(x)^{-1}$. Therefore Q_i is a classical right quotient ring for $\nu_i(R)$, as claimed.

By Theorem 6.18, $\nu_i(R)$ is a prime right Goldie ring, that is, P_i is a prime ideal of R and R/P_i is right Goldie. That Q_i is the right Goldie quotient ring of $\nu_i(R)$ is the same as to say that Q/M_i is the right Goldie quotient ring of R/P_i.

By Proposition 3.3, there is a minimal prime ideal $P_i^\circ \subseteq P_i$. Observe that

$$P_1 P_2 \cdots P_n \subseteq P_1 \cap \cdots \cap P_n = 0,$$

whence $P_1 P_2 \cdots P_n \subseteq P_i^\circ$. Then some $P_k \subseteq P_i^\circ \subseteq P_i$. As $P_j \nsubseteq P_i$ for all $j \neq i$, we must have $k = i$, and so $P_i = P_i^\circ$. Therefore P_i is a minimal prime.

(c) Given a minimal prime P in R, we observe (as above) that some $P_k \subseteq P$. Then $P = P_k$ by minimality.

(d) Set $B_i = \bigcap_{j \neq i} P_j$. Since $B_i P_i \subseteq P_1 \cap P_2 \cap \cdots \cap P_n = 0$, we see that $P_i \subseteq \text{r.ann}_R(B_i)$. On the other hand, as $P_1, \ldots, P_n$ are distinct minimal primes, $B_i \nsubseteq P_i$. Then, since $B_i \cdot \text{r.ann}_R(B_i) = 0 \subseteq P_i$, we conclude that $\text{r.ann}_R(B_i) \subseteq P_i$. $\square$

Lemma 7.2. *If I is an ideal in a semiprime ring R, then* $\mathrm{l.ann}(I) = \mathrm{r.ann}(I)$.

Proof. Note that $\mathrm{l.ann}(I)$ and $\mathrm{r.ann}(I)$ are ideals of R. Since

$$\big(I \cdot \mathrm{l.ann}(I)\big)^2 = I\big(\mathrm{l.ann}(I) \cdot I\big)\,\mathrm{l.ann}(I) = 0,$$

we obtain $I \cdot \mathrm{l.ann}(I) = 0$ and so $\mathrm{l.ann}(I) \subseteq \mathrm{r.ann}(I)$. The reverse inclusion is obtained symmetrically. $\square$

Proposition 7.3. *Let P be a prime ideal in a semiprime right Goldie ring R. Then the following conditions are equivalent:*

(a) *P is a minimal prime.*
(b) *P is a right annihilator.*
(c) *P is a left annihilator.*
(d) *P contains no regular elements.*

Proof. (a) $\Longrightarrow$ (b) by Proposition 7.1.

(b) $\Longrightarrow$ (c): If $L = \mathrm{l.ann}(P)$, then L is an ideal of R, and $P = \mathrm{r.ann}(L)$ because P is a right annihilator. By Lemma 7.2, $P = \mathrm{l.ann}(L)$ as well.

(c) $\Longrightarrow$ (d): By assumption, $P = \mathrm{l.ann}(X)$ for some $X \subseteq R$, and $X \not\subseteq \{0\}$ because P is a proper ideal. Thus P contains no regular elements.

(d) $\Longrightarrow$ (a): If P properly contains a prime ideal P_0, then P/P_0 is an essential right ideal of R/P_0 (Exercise 5A) and hence also an essential right R-submodule. But then P is an essential right ideal of R (Proposition 5.6(c)), and so P contains a regular element (Proposition 6.13), contradicting (d). $\square$

Exercise 7A. Show that in a semiprime right Goldie ring R, the minimal prime ideals are exactly those ideals P maximal with respect to the property that P contains no regular elements. $\square$

Lemma 7.4. *If R is a semiprime right Goldie ring and $P_1, \dots, P_n$ are its minimal primes, then the set of regular elements of R equals $\mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n)$.*

Proof. If $x \in \mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n)$, then $\mathrm{r.ann}_R(x) \subseteq P_1 \cap \cdots \cap P_n = 0$, and similarly $\mathrm{l.ann}_R(x) = 0$. Hence, x is regular.

Conversely, let x be a regular element of R. By Proposition 7.3 and Lemma 7.2, each

$$P_i = \mathrm{l.ann}_R(I_i) = \mathrm{r.ann}_R(I_i)$$

for some ideal I_i. If $r \in R$ and $xr \in P_i$, then $xrI_i = 0$, whence $rI_i = 0$ (because x is regular), and so $r \in P_i$. Similarly, $rx \in P_i$ implies $r \in P_i$, and therefore $x \in \mathcal{C}(P_i)$. $\square$

Exercise 7B. Let R be a semiprime right Goldie ring and P a minimal prime of R. Given $x \in \mathcal{C}(P)$, show that there exist $y, z \in \mathcal{C}(0)$ such that $xy \equiv z \pmod{P}$. [Hint: If $P_1, \dots, P_m$ are the other minimal primes of R, show that $\mathcal{C}(P) \cap \bigcap_i P_i$ and $P \cap \bigcap_i \mathcal{C}(P_i)$ are nonempty.] Thus, every regular element of R/P is a factor of a coset represented by a regular element of R. $\square$

Proposition 7.5. *Let N be a proper semiprime ideal in a right noetherian ring R, and let $P_1, \dots, P_n$ be the prime ideals of R minimal over N. Then $P_1, \dots, P_n$ are precisely the prime ideals of R that contain N but are disjoint from $\mathcal{C}(N)$, and*

$$\mathcal{C}(N) = \mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n).$$

Proof. After reducing to R/N, we may assume that $N = 0$. Then the first conclusion follows from Proposition 7.3 and the second from Lemma 7.4. □

Proposition 7.6. *If $P_1, \dots, P_n$ are the distinct minimal prime ideals of a semiprime right Goldie ring R, then the natural map*

$$f : R \to (R/P_1) \oplus \cdots \oplus (R/P_n)$$

is an essential monomorphism of right (or left) R-modules.

Proof. That f is a monomorphism is clear, since $P_1 \cap \cdots \cap P_n = 0$. For $j = 1, \dots, n$, set $I_j = \bigcap_{i \neq j} P_i$. Since $P_i \not\subseteq P_j$ for all $i \neq j$, we must have $I_j \not\subseteq P_j$, whence $(I_j + P_j)/P_j$ is a nonzero ideal of R/P_j. Then $(I_j + P_j)/P_j$ is essential as a right or left ideal of R/P_j (Exercise 5A) and thus also essential as a right or left R-submodule. Since

$$f(I_1 + \cdots + I_n) = \big((I_1 + P_1)/P_1\big) \oplus \cdots \oplus \big((I_n + P_n)/P_n\big),$$

we conclude that $f(\sum_j I_j)$ is essential as a right or left R-submodule of $\bigoplus_j (R/P_j)$, and therefore the same is true of $f(R)$. □

Corollary 7.7. *If N is a proper semiprime ideal in a right noetherian ring R and $P_1, \dots, P_n$ are the distinct prime ideals of R minimal over N, then*

$$E((R/N)_R) \cong E((R/P_1)_R) \oplus \cdots \oplus E((R/P_n)_R).$$

Proof. By Proposition 7.6, R/N is isomorphic to an essential submodule of the right R-module $(R/P_1) \oplus \cdots \oplus (R/P_n)$. □

Exercise 7C. Give alternative proofs (and interpretations) of Lemma 7.4 and Proposition 7.6 using the product decomposition of the Goldie quotient ring Q, as in the proof of Proposition 7.1. [Hint: For Lemma 7.4, note that $\mathcal{C}_R(P_i) = \{x \in R \mid x_i \text{ is invertible in } Q_i\}$.] □

• TORSION •

Consider a semiprime right Goldie ring R, and let $X = \mathcal{C}_R(0)$, the set of regular elements in R. As long as the context is clear, we shall omit the label X when referring to X-torsion submodules, X-torsionfree modules, etc. Thus, the *torsion submodule* of a right R-module A is the set

$$t(A) = \{a \in A \mid ax = 0 \text{ for some regular element } x \in R\}$$

(this is a submodule of A by Lemma 4.21), and we say that A is *torsion* (respectively, *torsionfree*) when $t(A) = A$ (respectively, $t(A) = 0$).

Let us recall the general properties of torsion and torsionfree modules from Chapter 4.

Proposition 7.8. *Let R be a semiprime right Goldie ring.*

(a) *If A is any right R-module, then $t(A)$ is a torsion module and $A/t(A)$ is a torsionfree module.*

(b) *All submodules, factor modules, sums (direct or not), and essential extensions of torsion right R-modules are torsion.*

(c) *Let B be a submodule of a right R-module A. If $B \leq_e A$, then A/B is torsion. If B and A/B are both torsion, then A is torsion.*

(d) *All submodules, direct products, and essential extensions of torsionfree right R-modules are torsionfree.*

(e) *Let B be a submodule of a right R-module A. If B and A/B are both torsionfree, then A is torsionfree.*

Proof. All of these properties, except for the last item of part (b) and the first of part (c), follow from Lemma 4.22.

Consider right R-modules $B \leq_e A$. For any element $a \in A$, the right ideal $I = \{r \in R \mid ar \in B\}$ is essential in R, and so by Goldie's Regular Element Lemma there exists a regular element $x \in I$, whence $ax \in B$. This shows that A/B is torsion. Now if B is torsion, then we conclude from part (c) that A is torsion. Therefore all essential extensions of torsion right R-modules are torsion. □

One should beware of using torsion terminology loosely when there is ambiguity about coefficient rings. In particular, if R is a semiprime right noetherian ring and N is a semiprime ideal of R, then R and R/N are both semiprime right Goldie rings, and the torsion submodule of a right (R/N)-module A may well differ from the torsion submodule of A viewed as an R-module. (We may distinguish these two submodules notationally by writing $t(A_{R/N})$ and $t(A_R)$.) For example, any vector space over $\mathbb{Z}/2\mathbb{Z}$ is torsionfree as a $(\mathbb{Z}/2\mathbb{Z})$-module but torsion as a $\mathbb{Z}$-module.

Proposition 7.9. *Let P be a minimal prime in a semiprime right Goldie ring R. Then R/P is torsionfree as both a right R-module and a right (R/P)-module, and $t(A_{R/P}) = t(A_R)$ for all right (R/P)-modules A.*

Proof. It is clear that R/P is torsionfree as a right module over itself. By Proposition 7.6, R is isomorphic to an essential submodule of the right R-module $(R/P_1) \oplus \cdots \oplus (R/P_n)$, where $P_1, \dots, P_n$ are the distinct minimal primes of R. Since R_R is torsionfree, so is $\big((R/P_1) \oplus \cdots \oplus (R/P_n)\big)_R$ (Proposition 7.8(d)), and thus $(R/P)_R$ is torsionfree.

Now let A be a right (R/P)-module. If $a \in t(A_R)$, then $ax = 0$ for some $x \in \mathcal{C}(0)$. Since $x \in \mathcal{C}(P)$ by Lemma 7.4, we have $a(x+P) = 0$ with $x + P$ regular in R/P, and so $a \in t(A_{R/P})$. Conversely, consider an element $b \in t(A_{R/P})$ and set $I = \{r \in R \mid br = 0\}$. In particular, I/P is a right ideal of R/P which contains some regular element of R/P. By Lemma 6.11, I/P is essential in R/P, whence I is essential in R. Thus I contains a regular element of R, whence $b \in t(A_R)$. Therefore $t(A_R) = t(A_{R/P})$. □

Proposition 7.10. *If U is a uniform right module over a semiprime right Goldie ring, then U is either torsion or torsionfree.*

Proof. If U is not torsionfree, then $t(U) \neq 0$, whence $t(U) \leq_e U$ (because U is uniform), and therefore U must be torsion by Proposition 7.8(b). $\square$

• TORSIONFREE INJECTIVE MODULES •

Definition. A right module A over a ring R is *divisible* provided $Ax = A$ for all regular elements $x \in R$.

For example, every injective module is divisible. Over $\mathbb{Z}$, or, more generally, over any principal right ideal domain, the divisible right modules are exactly the injective right modules. Over other domains, however, divisible modules need not be injective, as the following example shows.

Exercise 7D. Let $R = k[x, y]$ be a polynomial ring over a field k, and let $F = k(x, y)$ be the quotient field of R. Show that $F/(xR + yR)$ is a divisible R-module which is not injective. [Hint: Look at homomorphisms from $(xR + yR)/(xR + yR)^2$ to $F/(xR + yR)$.] $\square$

However, over a commutative domain all *torsionfree* divisible modules are injective, and the same result holds over semiprime Goldie rings, as follows.

Proposition 7.11. [Gentile, Levy] *Let A be a torsionfree right module over a semiprime right Goldie ring R. Then A is divisible if and only if it is injective.*

Proof. First assume that A is injective. Given $a \in A$ and a regular element $x \in R$, there is a well-defined homomorphism $f : xR \to A$ such that $f(xr) = ar$ for all $r \in R$. Then f extends to a homomorphism $g : R_R \to A$, and $g(1)x = f(x) = a$. Thus A is divisible.

Conversely, assume that A is divisible, let I be a right ideal of R, and let $f : I \to A$ be a homomorphism. By Corollary 5.8, I is a direct summand of an essential right ideal J, and f extends to a homomorphism $g : J \to A$. Now J contains a regular element x, and $xR \leq_e R_R$ by Lemma 6.11. In particular, J/xR is a torsion module.

Since A is divisible, $g(x) = ax$ for some $a \in A$. Define a homomorphism $h : R_R \to A$ so that $h(1) = a$, and observe that $h(x) = ax = g(x)$. Consequently, $(h - g)(xR) = 0$, whence $(h - g)(J)$ is a homomorphic image of the torsion module J/xR. As A is torsionfree, $(h - g)(J) = 0$. Thus h is an extension of g and hence an extension of f. Therefore A is injective. $\square$

Exercise 7E. If R is a semiprime right Goldie ring, show that all direct sums of torsionfree injective right R-modules are injective. $\square$

Corollary 7.12. *If R is a semiprime right Goldie ring with right Goldie quotient ring Q, then Q_R is an injective hull for R_R.*

Proof. We have observed in the previous chapter that $R_R \leq_e Q_R$ (cf. the proof of Lemma 6.17). Moreover, Q_R is a torsionfree divisible module, and so it is injective by Proposition 7.11. $\square$

Proposition 7.13. *Let R be a semiprime right Goldie ring with right Goldie quotient ring Q.*

(a) *All right Q-modules are torsionfree and injective as right R-modules, and their Q-submodules are exactly their divisible R-submodules.*

(b) *Every torsionfree divisible right R-module has a unique right Q-module structure compatible with its right R-module structure.*

(c) *A right Q-module is uniform as an R-module if and only if it is isomorphic to a minimal right ideal of Q.*

Proof. (a) Any right Q-module A is clearly torsionfree and divisible as an R-module. By Proposition 7.11, A_R is injective.

Obviously any Q-submodule of A is divisible as an R-module. Conversely, let B be a divisible R-submodule of A. Given $b \in B$ and $q \in Q$, write $q = ax^{-1}$ for some $a, x \in R$ with x regular. There exists $b' \in B$ such that $b'x = ba$, whence $(bq - b')x = 0$. As A is torsionfree, $bq - b' = 0$, and so $bq \in B$. Thus B is a Q-submodule of A.

(b) Let A be a torsionfree divisible right R-module and set $E = \operatorname{End}_{\mathbb{Z}}(A)$. Recall that left module structures on an abelian group correspond to ring homomorphisms to its endomorphism ring, while right module structures correspond to ring homomorphisms to the opposite of the endomorphism ring. In our case, there is a ring homomorphism $\phi : R \to E^{\text{op}}$ such that $\phi(r)(a) = ar$ for all $r \in R$ and $a \in A$. Since A is torsionfree and divisible, $\phi(x)$ is invertible in E^{op} for all $x \in \mathcal{C}(0)$. By the universal property of rings of fractions (Proposition 6.3), ϕ extends uniquely to a ring homomorphism $\psi : Q \to E^{\text{op}}$. Hence, we can make A into a right Q-module using the original addition together with a module multiplication $\cdot$ given by the rule $a{\cdot}q = \psi(q)(a)$. Since ψ extends ϕ, we have $a{\cdot}r = ar$ for all $a \in A$ and $r \in R$, and thus this Q-module structure on A is compatible with the original R-module structure.

The uniqueness of the compatible Q-module structure on A is just a restatement of the uniqueness of ψ.

(c) Let A be a right Q-module. If A is uniform as an R-module, it is indecomposable, and hence it is also indecomposable as a Q-module. Then, since Q is semisimple, A must be simple, whence A is isomorphic to a minimal right ideal of Q.

Conversely, assume that A is isomorphic to a minimal right ideal of Q, so that A is a simple right Q-module. If $A = B \oplus C$ for some R-submodules B and C, then B and C are divisible and so are Q-submodules of A, by (a). As A_Q is simple, either $B = 0$ or $C = 0$. Thus, A is indecomposable as an R-module and therefore uniform (because it is injective). $\square$

Corollary 7.14. *Let R be a semiprime right Goldie ring with right Goldie quotient ring Q. Up to isomorphism, the torsionfree uniform injective right R-modules are exactly the minimal right ideals of Q, and there are only finitely many isomorphism classes of them.* $\square$

Exercise 7F. Let R be a semiprime right Goldie ring with right Goldie quotient ring Q and A a right R-module. Let $f : A \to A \otimes_R Q$ be the natural map given by the rule $f(a) = a \otimes 1$, and $g : A \to A/t(A) \to E(A/t(A))$ the natural map. Show that there is an isomorphism (of right R-modules) $h : A \otimes_R Q \to E(A/t(A))$ such that $hf = g$. $\square$

Exercise 7G. If R is a semiprime right Goldie ring, show that its right Goldie quotient ring Q is a flat left R-module [i.e., given any monomorphism $t : A \to B$ of right R-modules, the induced map $t \otimes 1 : A \otimes_R Q \to B \otimes_R Q$ is also a monomorphism.] $\square$

Exercise 7H. Let R be a semiprime right Goldie ring with right Goldie quotient ring Q, let A be a right R-module, and let $f : A \to A \otimes_R Q$ be the natural map given by the rule $f(a) = a \otimes 1$. Note that f is a right R-module homomorphism.

(a) Show that $\ker(f) = t(A)$. [Hint: Exercise 7F.] Conclude that A is torsion if and only if $A \otimes_R Q = 0$.

(b) Show that $A/t(A)$ has finite rank if and only if $A \otimes_R Q$ has finite length (as a right Q-module) and that $\operatorname{rank}(A/t(A)) = \operatorname{length}(A \otimes_R Q)$ in this case. $\square$

Theorem 7.15. *If R is a semiprime right Goldie ring, there is a central idempotent $e \in R$ such that $eR = \operatorname{soc}(R_R)$. This yields a ring decomposition $R \cong S \times T$ (where $S = eR$ and $T = (1-e)R$) such that S is a semisimple ring and $\operatorname{soc}(T_T) = 0$.*

Proof. Set $J = \operatorname{soc}(R_R)$ and choose a right ideal K of R such that $J \oplus K \leq_e R_R$. Note that $KJ \leq K \cap J = 0$. There exists a regular element $x \in J \oplus K$. Since $xJ \leq J$ and left multiplication by x is a monomorphism, Corollary 5.22 shows that $xJ \leq_e J$. Thus $xJ = J$, because J_R is semisimple.

Write $x = y + z$ for some $y \in J$ and $z \in K$. As $xJ = J$, we must have $y = xe$ for some $e \in J$. For all $r \in J$, we have $zr \in KJ = 0$, whence $xr = yr + zr = yr = xer$ and so $r = er$, because x is regular. Thus, $e = e^2$ and $J = eR$.

Since J is an ideal, $(1-e)Re \subseteq (1-e)J = 0$. Then

$$\big(eR(1-e)R\big)^2 = eR\big[(1-e)Re\big]R(1-e)R = 0$$

and hence $eR(1-e)R = 0$, by semiprimeness. For all $s \in R$, we obtain

$$(1-e)se = es(1-e) = 0,$$

and so $se = ese = es$. Therefore e is central.

Now $S = eR$ and $T = (1-e)R$ are rings with unit and $R \cong S \times T$. As $\mathrm{soc}(R_R) = S$, we conclude that $\mathrm{soc}(S_S) = S$ while $\mathrm{soc}(T_T) = 0$. □

Corollary 7.16. *Let R be a prime right Goldie ring. If $\mathrm{soc}(R_R) \neq 0$, then R is a simple artinian ring.*

Proof. Since R is prime, the only central idempotents in R are 0 and 1. Hence, Theorem 7.15 implies that $\mathrm{soc}(R_R) = R$, that is, R is a semisimple ring. As R is prime, it must be simple artinian. □

Exercise 7I. If R is a semiprime right Goldie ring, show that all torsionfree semisimple right R-modules are injective. [Hint: In case $\mathrm{soc}(R_R) = 0$, show that every maximal right ideal of R is essential, and consequently all semisimple right R-modules are torsion.] □

• TORSIONFREE UNIFORM MODULES •

Lemma 7.17. *Let R be a semiprime right Goldie ring and A a right R-module. If A is not torsion, then A has a uniform submodule isomorphic to a right ideal of R.*

Proof. Choose an element $a \in A \setminus t(A)$. Then the right ideal $\mathrm{ann}_R(a)$ contains no regular elements of R, whence $\mathrm{ann}_R(a) \not\leq_e R_R$, and so there is a nonzero right ideal I in R such that $I \cap \mathrm{ann}_R(a) = 0$. Since R_R has finite rank, I must contain a uniform right ideal J, by Proposition 5.15. As $J \cap \mathrm{ann}_R(a) = 0$, we see that $J \cong aJ$, and therefore aJ is the desired uniform submodule of A. □

Lemma 7.17 guarantees that nonzero torsionfree right modules over a semiprime right Goldie ring R have uniform submodules isomorphic to right ideals of R. However, it does not say that all torsionfree uniform right R-modules are necessarily isomorphic to right ideals of R. For instance, if R is right noetherian, non-finitely-generated uniform right R-modules cannot be isomorphic to right ideals of R. (For example, $\mathbb{Q}$ is a uniform torsionfree $\mathbb{Z}$-module which is not isomorphic to an ideal of $\mathbb{Z}$.) More strictly, even finitely generated torsionfree uniform right R-modules need not be isomorphic to right ideals of R, as the following (left-handed) example shows. We shall see shortly that if R is both left and right Goldie, then finitely generated torsionfree uniform R-modules do always embed in R (Corollary 7.20).

Exercise 7J. Let $k(y)$ be a rational function field over a field k, let α be the k-algebra endomorphism of $k(y)$ given by the rule $\alpha(f) = f(y^2)$, and let $R = k(y)[x;\alpha]$. Then R is a principal left ideal domain (Theorem 2.8), whence R is a left Ore domain. Let Q be the left Ore quotient ring of R, and let A be the left R-submodule $R + Rx^{-1}yx \leq {}_RQ$. Show that A is a torsionfree uniform left R-module which is not isomorphic to any left ideal of R. [Hint: As mentioned in Exercise 6E, $xR \cap yxR = 0$.] □

Proposition 7.18. *If A is a torsionfree right module over a semiprime right Goldie ring R, then A has an essential submodule which is a direct sum of uniform submodules isomorphic to right ideals of R.*

Proof. Let $\mathcal{A}$ be the collection of those uniform submodules of A which are isomorphic to right ideals of R. Choose a maximal independent family $\mathcal{A}_0 \subseteq \mathcal{A}$ and set $A_0 = \bigoplus \mathcal{A}_0$. If B is a submodule of A satisfying $B \cap A_0 = 0$, then by the maximality of $\mathcal{A}_0$ we see that B has no uniform submodules isomorphic to right ideals of R. This forces B to be zero, by Lemma 7.17. Therefore $A_0 \leq_e A$. $\square$

Of course the direct sum of uniform submodules obtained in Proposition 7.18 is finite if A has finite rank.

Proposition 7.19. [Gentile, Levy] *If R is a semiprime right and left Goldie ring and A a finitely generated torsionfree right R-module, then A can be embedded in a finitely generated free right R-module.*

Proof. Let Q be the (right and left) Goldie quotient ring of R.

Since $E(A)$ is an essential extension of a torsionfree module, it is torsionfree. Hence, Proposition 7.13 shows that $E(A)$ can be made into a right Q-module. Now AQ is a finitely generated right Q-module, and so it is a finite direct sum of simple right Q-modules, each of which is isomorphic to a right ideal of Q. Consequently, A is isomorphic to a right R-submodule of Q^n for some $n \in \mathbb{N}$, and there is no loss of generality in assuming that $A \leq Q^n_R$.

Choose generators $a_1, \dots, a_t$ for A and write each $a_i = (q_{i1}, \dots, q_{in})$ for some $q_{ij} \in Q$. There exist $x, b_{ij} \in R$ with x regular such that each $q_{ij} = x^{-1}b_{ij}$. Then $xa_i \in R^n$ for $i = 1, \dots, t$, whence $xA \leq R^n_R$. As x is invertible in Q, we conclude that $A \cong xA$. $\square$

Corollary 7.20. *If R is a semiprime right and left Goldie ring and U a finitely generated torsionfree uniform right R-module, then U is isomorphic to a right ideal of R.*

Proof. In view of Proposition 7.19, there exist $f_1, \dots, f_n \in \mathrm{Hom}_R(U, R)$ such that

$$\ker(f_1) \cap \cdots \cap \ker(f_n) = 0.$$

Since U is uniform, $\ker(f_i)$ must be zero for some i. $\square$

Proposition 7.21. *Let R be a semiprime right Goldie ring and U a torsionfree uniform right R-module. Then $\mathrm{ann}(U)$ is a minimal prime ideal of R, and $E(U)$ is a fully faithful right $(R/\mathrm{ann}(U))$-module.*

Proof. Let Q be the right Goldie quotient ring of R. Since U is torsionfree, so is $E(U)$, and hence Corollary 7.14 shows that $E(U)$ is isomorphic to a minimal right ideal I of Q. Since Q is a semisimple ring, $\mathrm{r.ann}_Q(I)$ is a maximal ideal of Q. Hence, if

$$P = \mathrm{ann}_R(E(U)) = R \cap \mathrm{r.ann}_Q(I),$$

then P is a minimal prime ideal of R, by Proposition 7.1. Note that $E(U)$ is a faithful right (R/P)-module. By Proposition 7.9, $E(U)$ must be torsionfree as an (R/P)-module.

Given any nonzero R-submodule B in $E(U)$, note that $P \subseteq \operatorname{ann}_R(B)$. Since B is a torsionfree (R/P)-module, Lemma 7.17 implies that B has a nonzero submodule isomorphic to a right ideal J of R/P. Then $\operatorname{r.ann}_{R/P}(J) = 0$, whence $\operatorname{ann}_R(B) = P$. Therefore $E(U)$ is a fully faithful (R/P)-module. In particular, $\operatorname{ann}_R(U) = P$. $\square$

• TORSIONFREE MODULES OVER PRIME GOLDIE RINGS •

Restricting attention to a prime right Goldie ring R, we show that all torsionfree right R-modules are "essentially the same," and consequently that the rank of any torsionfree right R-module can be measured using copies of a single uniform right ideal of R.

Lemma 7.22. *Let A be a nonzero torsionfree right module over a prime right Goldie ring R. Then A is fully faithful, and every uniform right ideal of R is isomorphic to a submodule of A.*

Proof. If B is a nonzero submodule of A, then, by Lemma 7.17, B has a uniform submodule isomorphic to a right ideal V of R. Since R is prime, V is a faithful R-module, whence B is faithful. Thus A is fully faithful.

Let U be a uniform right ideal of R. Since A is faithful, there exists an element $a \in A$ such that $aU \neq 0$. Set $I = U \cap \operatorname{r.ann}(a)$ and note that $U/I \cong aU$. Since A is torsionfree, U/I must be torsionfree, and so $I \not\leq_e U$. Then $I = 0$ because U is uniform, and therefore $U \cong aU \leq A$. $\square$

Proposition 7.23. *If P is a prime ideal in a right noetherian ring R and U, V are torsionfree uniform right (R/P)-modules, then $E(U_R) \cong E(V_R)$.*

Proof. Let W be a uniform right ideal of R/P. Lemma 7.22 shows that W is isomorphic to a submodule of U, whence $E(W_R)$ is isomorphic to a nonzero direct summand of $E(U_R)$. Since U is uniform, $E(U_R)$ is indecomposable, and therefore $E(W_R) \cong E(U_R)$. Similarly, $E(W_R) \cong E(V_R)$. $\square$

Proposition 7.24. *Let R be a prime right Goldie ring, U a uniform right ideal of R, and A a torsionfree right R-module of finite rank. If $n = \operatorname{rank}(A)$, then A has an essential submodule isomorphic to $\oplus^n U$.*

Proof. Since A has rank n, it has an essential submodule $V_1 \oplus \cdots \oplus V_n$, where each V_i is uniform. By Lemma 7.22, each V_i has a submodule U_i isomorphic to U, and $U_i \leq_e V_i$ because V_i is uniform. Therefore $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of A, isomorphic to $\oplus^n U$. $\square$

Corollary 7.25. *Let R be a prime right Goldie ring, U a uniform right ideal of R, and $n = \operatorname{rank}(R_R)$. Then each essential right ideal of R contains an* *essential right ideal isomorphic to* $\oplus^n U$, *and* R_R *is isomorphic to an essential submodule of* $\oplus^n U$.

Proof. Since all essential right ideals of R have rank n, the first conclusion follows immediately from Proposition 7.24. In particular, R has at least one essential right ideal I which is isomorphic to $\oplus^n U$. There exists a regular element $x \in I$, and $xR \leq_e R_R$. Therefore $R_R \cong xR \leq_e I \cong \oplus^n U$. □

Corollary 7.26. *Let* R *be a prime right Goldie ring and* $n = \operatorname{rank}(R_R)$.

(a) *If* A *is any torsionfree right* R*-module, then* A^n *has an essential free submodule, which will be finitely generated in case* A *has finite rank.*

(b) *If* A *is an arbitrary right* R*-module, then* A^n *has a free submodule* F *such that* A^n/F *is torsion, and* F *will be finitely generated in case* $A/t(A)$ *has finite rank.*

Proof. (a) By Proposition 7.18, A has an essential submodule of the form $\bigoplus_i A_i$, where the A_i are uniform. Pick a uniform right ideal U in R; then Lemma 7.22 shows that U is isomorphic to an essential submodule of each A_i. Hence, some direct sum of copies of $\oplus^n U$ is isomorphic to an essential submodule of $\bigoplus_i A_i^n$, which is an essential submodule of A^n. By Corollary 7.25, R_R is isomorphic to an essential submodule of $\oplus^n U$, and thus any direct sum of copies of $\oplus^n U$ has an essential free submodule. This yields an essential free submodule F in A^n, and if A has finite rank, then F must also have finite rank and hence be finitely generated.

(b) Choose a submodule $B \leq A$ such that $B \oplus t(A) \leq_e A$. Then the quotient $A/(B \oplus t(A))$ is torsion, whence A/B is torsion. Since B is torsionfree, (a) provides us with an essential free submodule $F \leq B^n$. Then B^n/F is torsion, and so A^n/F is torsion. Since F embeds in $(A/t(A))^n$, it must be finitely generated in case $A/t(A)$ has finite rank. □

Exercise 7K. Let R be a prime right and left Goldie ring and A, B finitely generated torsionfree right R-modules.

(a) Show that A and B have finite rank. [Hint: Exercise 7H or Proposition 7.13.]

(b) Show that $\operatorname{rank}(A) \leq \operatorname{rank}(B)$ if and only if A is isomorphic to a submodule of B. [Hint: Inside $E(A)$, show that A is contained in a direct sum of $\operatorname{rank}(A)$ finitely generated uniform modules.]

(c) Show that $A^{\operatorname{rank}(B)}$ and $B^{\operatorname{rank}(A)}$ are isomorphic to essential submodules of each other. In particular, if $n = \operatorname{rank}(R_R)$, then A^n is isomorphic to an essential submodule of a free module. □

Exercise 7L. If U is a uniform right ideal in a prime right Goldie ring R, show that $\operatorname{End}_R(U)$ is a right Ore domain. [Hint: Let Q be the right Goldie quotient ring of R, and show that $\operatorname{End}_Q(UQ)$ is a classical right quotient ring for $\operatorname{End}_R(U)$.] □

Exercise 7M. Prove that a prime ring in which every right ideal is principal is isomorphic to a matrix ring over a right Ore domain. (This was proved by Goldie in 1962.) [Hint: Corollary 7.25 and Exercise 7L.] Generalize this to show that a semiprime ring in which every right ideal is principal is isomorphic to a direct product of such matrix rings. Generalize both results to semiprime right Goldie rings in which every finitely generated right ideal is principal. (This was done by Robson in 1967.) $\square$

• NOTES •

Much of this chapter consists of folklore, worked out by many hands in extending and applying Goldie's Theorems. Thus, we do not give many specific attributions for these results.

Divisibility Versus Injectivity. Gentile proved that, over a right Ore domain, a torsionfree right module is injective if and only if it is divisible [1960, Proposition 1.1]. That the same holds over a semiprime right Goldie ring follows from a result of Levy [1963, Theorem 3.3].

Embedding Torsionfree Modules in Free Modules. For a right Ore domain R, Gentile proved that all finitely generated torsionfree right R-modules can be embedded in free right R-modules if and only if R is also left Ore [1960, Proposition 4.1]. The analogous equivalence over a semiprime right Goldie ring follows from results of Levy [1963, Theorems 5.2, 5.3].

8. Bimodules and Affiliated Prime Ideals

Bimodules have become of increasing importance in the ideal theory of a noetherian ring R, particularly ideal factors I/J where $I \supseteq J$ are ideals of R, and overrings $S \supseteq R$, viewed as (S, R)-bimodules. To make the notation more convenient in both cases, we study bimodules in general. In this chapter, we investigate the structure of bimodules over noetherian rings, particularly bimodules which are noetherian or artinian on at least one side, and we illustrate the results by indicating a number of applications, particularly to the relationships between the prime ideals of a ring and the prime ideals of a subring. While the latter sections of the chapter are designed to show the reader several contexts in which bimodules have been successfully used, only the first four sections are strictly needed for later chapters of the book.

• NOETHERIAN BIMODULES •

By a "noetherian bimodule" is usually meant a bimodule ${}_RA_S$ which not only has the ACC on sub-bimodules, but also is noetherian as a left R-module and as a right S-module. (Some authors mean in addition that R and S are noetherian rings.) We state our results under somewhat more general hypotheses, while thinking mainly of applications to noetherian bimodules.

Our first result, although elementary, is fundamental to the entire development of noetherian bimodules. In particular, as the exercise following indicates, it gives severe restrictions on which modules can appear as one-sided submodules of noetherian bimodules.

Lemma 8.1. *Let ${}_RA_S$ be a bimodule and B a right S-submodule of A such that ${}_R(RB)$ is finitely generated.* (*The latter hypothesis is automatically satisfied if ${}_RA$ is noetherian.*) *If $I = \mathrm{r.ann}_S(B)$, there exists $n \in \mathbb{N}$ such that S/I is isomorphic to a* (*right*) *submodule of B^n.*

Proof. There is a finite set of generators for ${}_R(RB)$, each of which is a finite sum of products rb with $r \in R$ and $b \in B$. Hence, there exist $b_1, \dots, b_n \in B$ such that $RB = Rb_1 + \cdots + Rb_n$. Consequently,

$$I = \mathrm{r.ann}_S(B) = \mathrm{r.ann}_S(RB) = \mathrm{r.ann}_S(b_1) \cap \cdots \cap \mathrm{r.ann}_S(b_n).$$

Thus, the rule $s \mapsto (b_1 s, \dots, b_n s)$ defines a right S-module homomorphism $S \to B^n$ with kernel I. $\square$

Exercise 8A. Let R be a left noetherian ring and B a minimal right ideal of R (that is, B_R is simple). Show that $R/\operatorname{r.ann}_R(B)$ is a simple artinian ring. (In other words, the right primitive ideal of R that annihilates B must be "co-artinian.") $\square$

Proposition 8.2. *Let ${}_RA_S$ be a bimodule and B a right S-submodule of A such that ${}_R(RB)$ is finitely generated. (The latter hypothesis is automatically satisfied if ${}_RA$ is noetherian.) Set $I = \operatorname{r.ann}_S(B)$. If B_S is artinian (noetherian), then S/I is right artinian (right noetherian).*

Proof. Since any finite direct sum of copies of B is an artinian (noetherian) right (S/I)-module, this is immediate from Lemma 8.1. $\square$

Lemma 8.3. *Let ${}_RA_S$ be a bimodule such that ${}_RA$ is noetherian and S is a prime right Goldie ring. Let C be a sub-bimodule of A and B a right S-submodule containing C. If B/C is torsion as a right S-module, then there is a nonzero ideal I of S such that $BI \subseteq C$.*

Proof. We apply Lemma 8.1 to $B/C \subseteq A/C$. If $I = \operatorname{r.ann}_S(B/C)$, the lemma implies that S/I embeds in a finite direct sum of copies of $(B/C)_S$, and hence S/I is torsion as a right S-module. It follows that $I \neq 0$. $\square$

Proposition 8.4. *Let ${}_RA_S$ be a bimodule, where R is a prime left noetherian ring and S is a prime right noetherian ring, and suppose that the modules ${}_RA$ and A_S are both finitely generated and torsionfree. Let B and C be sub-bimodules of A with $B \supseteq C$. Then the following conditions are equivalent:*

(a) *B/C is torsion as a right S-module.*

(b) *B/C is torsion as a left R-module.*

(c) *There is a nonzero ideal I of S such that $BI \subseteq C$ (that is, B/C is an unfaithful right S-module).*

(d) *There is a nonzero ideal J of R such that $JB \subseteq C$ (that is, B/C is an unfaithful left R-module).*

(e) *C is essential as a right S-submodule of B.*

(f) *C is essential as a left R-submodule of B.*

Proof. (a) $\Longrightarrow$ (c) by Lemma 8.3.

(c) $\Longrightarrow$ (f): Since I is a nonzero ideal in the prime right Goldie ring S, it contains a regular element c. Then $Bc \subseteq C$, and we note that since B is torsionfree as a right S-module, right multiplication by c is an injective map of B into C. Right multiplication by c is also a homomorphism of left R-modules. By Corollary 5.22, $Bc \leq_e {}_RB$, whence ${}_RC \leq_e {}_RB$.

(f) $\Longrightarrow$ (b) by Proposition 7.8(c), and finally (b) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (a) by symmetry. $\square$

• AFFILIATED PRIME IDEALS •

If ${}_RA_S$ is a nonzero noetherian bimodule, then the right module A_S and the left module ${}_RA$ have prime series, by Proposition 3.13. We would like to have such a “right prime series” and a “left prime series” that coincide, but unless we choose them carefully, such series need not even consist of sub-bimodules of A. One clue for dealing with this problem can be found in Proposition 3.12: If P is an ideal of S maximal among annihilators of nonzero right S-submodules of A, then P is prime and $\operatorname{ann}_A(P)$ is a prime right S-submodule of A (by the proposition); moreover, $\operatorname{ann}_A(P)$ is a sub-bimodule of A. Continuing this procedure for the bimodule $A/\operatorname{ann}_A(P)$ and so on, we would eventually obtain a prime series for A_S which at least consists of sub-bimodules of A. Prime series chosen in this manner are also of interest for one-sided modules, and so we introduce corresponding terminology in the latter context first.

Definition. Let A be a nonzero module over a ring R. An *affiliated submodule* of A is any submodule of the form $\operatorname{ann}_A(P)$, where P is an ideal of R maximal among the annihilators of nonzero submodules of A. (We note that such an ideal P is an associated prime of A by Proposition 3.12 and that P equals the annihilator in R of the affiliated submodule $\operatorname{ann}_A(P)$.) An *affiliated series* for A is a series of submodules of the form

$$A_0 = 0 < A_1 < \cdots < A_n = A,$$

where for each $i = 1, \ldots, n$ the module A_i/A_{i-1} is an affiliated submodule of A/A_{i-1}. If $P_i = \operatorname{ann}_R(A_i/A_{i-1})$, then the list $P_1, \ldots, P_n$ is the list of *affiliated primes* of A corresponding to the given affiliated series. In general, an *affiliated prime* of A is a prime ideal of R which appears in the list of affiliated primes corresponding to some affiliated series of A.

A module A may have many affiliated series, and the corresponding sets of affiliated primes do not necessarily coincide, even disregarding the order in which each set of primes is listed (Exercise 8F).

Exercise 8B. Show that a noetherian module has only finitely many affiliated series and hence only finitely many affiliated primes. [Hint: If not, choose a minimal criminal and apply Exercise 3K.] $\square$

Proposition 8.5. *If A is a nonzero finitely generated right module over a right noetherian ring R, then A has an affiliated series. If $A_0 = 0 < A_1 < \cdots < A_n = A$ is such an affiliated series, and $P_1, \ldots, P_n$ are the corresponding affiliated primes, then each A_i/A_{i-1} is a fully faithful right (R/P_i)-module, and each $A_i = \operatorname{ann}_A(P_iP_{i-1}\cdots P_1)$. In particular, any affiliated series for A is a prime series.*

Proof. The existence of an affiliated series and the fact that each A_i/A_{i-1} is a fully faithful right (R/P_i)-module follow from Proposition 3.12 and the fact that A is noetherian. (Compare Proposition 3.13.) Now $A_1 = \operatorname{ann}_A(P_1)$ by definition of an affiliated submodule. Similarly, $A_2/A_1 = \operatorname{ann}_{A/A_1}(P_2)$, whence

$$A_2 = \{a \in A \mid aP_2 \subseteq \operatorname{ann}_A(P_1)\} = \operatorname{ann}_A(P_2P_1).$$

Continuing by induction, we conclude that $A_i = \operatorname{ann}_A(P_iP_{i-1}\cdots P_1)$ for all i. $\square$

Exercise 8C. Let A be a right module over a commutative ring R. Assume that A has an affiliated series $A_0 = 0 < A_1 < \cdots < A_n = A$, with corresponding affiliated primes $P_1, \dots, P_n$. Show that $P_i = \operatorname{ann}_R(A_iP_{i-1}P_{i-2}\cdots P_1)$ for each $i = 1, \dots, n$. Thus every affiliated prime of A is also an annihilator prime. $\square$

Exercise 8D. Let A be a module which has an affiliated series, with corresponding affiliated primes $P_1, \dots, P_n$.

(a) If B is a nonzero submodule of A, show that B has an affiliated series such that the corresponding affiliated primes are a subsequence of $P_1, \dots, P_n$.

(b) If P is any annihilator prime for A, show that P is one of $P_1, \dots, P_n$. Conclude that $\operatorname{Ass}(A) \subseteq \{P_1, \dots, P_n\}$. Thus annihilator primes and associated primes of A are affiliated in the strong sense that they appear among the affiliated primes for every affiliated series for A. $\square$

Proposition 8.6. *Let R be a right noetherian ring and A a nonzero finitely generated right R-module, and let $P_1, \dots, P_n$ be the affiliated primes corresponding to some affiliated series for A. Then, if P is a prime minimal over $\operatorname{ann}_R(A)$, there is an index i such that $P = P_i$.*

Proof. Since any affiliated series for A is also a prime series, this result follows from Proposition 3.14. $\square$

Exercise 8E. Let $T = k[x, y]$ be a polynomial ring over a field k and set $R = T/\langle x^2, xy\rangle$. Find an affiliated series for the module R_R and show that one of the corresponding affiliated primes is not a minimal prime ideal of R. $\square$

Exercise 8F. Set $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, view the row $(\,\mathbb{Z}\ \mathbb{Z}\,)$ as a right R-module, and set $A = (\,\mathbb{Z}\ \mathbb{Z}\,)/(\,0\ 2\mathbb{Z}\,)$. Find all affiliated series for the right R-module A (there are two). Show that the corresponding sets of affiliated primes are not the same, and that one of these affiliated primes is neither an associated prime nor an annihilator prime. $\square$

Associated primes are always annihilator primes (by definition), and all annihilator primes are affiliated primes (Exercise 8D). However, even for a finitely generated module, annihilator primes need not be associated (Exercise 3ZD) and affiliated primes need not be annihilators (Exercise 8F). On the other hand, Exercises 8C and 5ZH show that, for a finitely generated module over a commutative noetherian ring, the affiliated primes are precisely the associated primes. In fact, the affiliated primes were introduced partly in order to find a family of primes which would contain the same kind of information that is contained in the associated primes in the commutative case.

We next consider affiliated submodules and affiliated primes in the context of bimodules. If ${}_RA_S$ is a bimodule, an affiliated submodule for the module A_S is a nonzero submodule B such that the ideal $P = \text{r.ann}_S(B)$ is maximal among the right annihilators of nonzero S-submodules of A, and $B = \text{l.ann}_A(P)$. Observe that B is a sub-bimodule of A, and hence that P is maximal among the right annihilators of nonzero sub-bimodules of A. Conversely, if Q is an ideal of S maximal among the right annihilators of nonzero sub-bimodules of A, then Q is also maximal among the right annihilators of nonzero S-submodules of A (because $\text{r.ann}_S(C) = \text{r.ann}_S(RC)$ for any right S-submodule $C \leq A$), and so $\text{l.ann}_A(Q)$ is an affiliated submodule for A_S.

Thus, we may speak of "affiliated sub-bimodules" of a bimodule, as follows.

Definition. Let ${}_RA_S$ be a bimodule. A *right affiliated sub-bimodule* of A is any affiliated submodule for A_S, a *right affiliated series* for A is any affiliated series

$$A_0 = 0 < A_1 < A_2 < \cdots < A_m = A$$

for A_S, and the corresponding affiliated prime ideals $P_i = \text{r.ann}_S(A_i/A_{i-1})$ are called *right affiliated primes* of A. (In case of need, one can view a zero bimodule as having a right affiliated series with length zero and no affiliated primes.) *Left affiliated sub-bimodules*, *series*, and *primes* are defined analogously.

Proposition 8.7. *Let ${}_RA_S$ be a nonzero bimodule such that ${}_RA$ is noetherian and S is right noetherian. Then there exists a right affiliated series*

$$A_0 = 0 < A_1 < A_2 < \cdots < A_m = A$$

for A. If $P_1, \dots, P_m$ are the corresponding right affiliated primes, then each A_i/A_{i-1} is a torsionfree right (S/P_i)-module.

Proof. Since A has the ACC on sub-bimodules, the existence of a right affiliated series follows from Proposition 3.12. To prove the remaining statement of the proposition, it suffices to show that A_1 is a torsionfree right (S/P_1)-module.

Now A_1 is an $(R, S/P_1)$-bimodule. Let B be the torsion submodule of A_1 as a right (S/P_1)-module, and observe that B is an $(R, S/P_1)$-sub-bimodule of A. If $B \neq 0$, then Lemma 8.3 shows that $BI = 0$ for some nonzero ideal I of S/P_1, contradicting the maximality of P_1. Therefore $B = 0$, and so A_1 is torsionfree as a right (S/P_1)-module, as desired. $\square$

Corollary 8.8. *Let ${}_RA_S$ be a nonzero bimodule such that ${}_RA$ and A_S are finitely generated. If R is left noetherian and S is right noetherian, there exists a chain of sub-bimodules*

$$A_0 = 0 < A_1 < A_2 < \cdots < A_m = A \qquad (\alpha)$$

such that the ideals $Q_i = \text{l.ann}_R(A_i/A_{i-1})$ and $P_i = \text{r.ann}_S(A_i/A_{i-1})$ are prime, and A_i/A_{i-1} is torsionfree both as a left (R/Q_i)-module and as a right

(S/P_i)-module, for each $i = 1, \dots, m$. In particular, the chain (α) is both a prime series for A_S and a prime series for ${}_RA$.

Proof. It suffices to show that A contains a nonzero sub-bimodule B such that $\mathrm{l.ann}_R(B)$ and $\mathrm{r.ann}_S(B)$ are prime ideals and B is torsionfree on the left over $R/\mathrm{l.ann}_R(B)$ and on the right over $S/\mathrm{r.ann}_S(B)$.

Let C be a right affiliated sub-bimodule of A and B a left affiliated sub-bimodule of C; these exist by Proposition 8.7. Moreover, if $P = \mathrm{r.ann}_S(C)$ and $Q = \mathrm{l.ann}_R(B)$, then C is a torsionfree right (S/P)-module and B is a torsionfree left (R/Q)-module. In addition, $\mathrm{r.ann}_S(B) = P$ by the maximality of P, and since B is contained in C, it is a torsionfree right (S/P)-module. $\square$

The noetherian conditions are crucial to Proposition 8.7 and Corollary 8.8, as the following examples show.

Exercise 8G. Let $R = A_n(D)$ for some $n \in \mathbb{N}$ and some division ring D of characteristic zero. If A is any simple right R-module and $S = \mathrm{End}_R(A)$, show that the (S, R)-bimodule A contains no nonzero right submodule which is a torsionfree module over a prime factor ring of R. Similarly, if Q is the Ore quotient ring of R, show that the (R, R)-bimodule Q/R contains no nonzero right or left submodule which is a torsionfree module over a prime factor ring of R. $\square$

Comparing Proposition 8.7 with Proposition 8.5, we see that the advantage of working with a noetherian bimodule as opposed to a noetherian module is that in the former case we obtain submodules which are torsionfree, rather than just fully faithful, modules over prime factor rings.

In the situation of Corollary 8.8, it is often interesting to study what happens when we take first a right affiliated series for a bimodule and then refine it further by taking left affiliated series of each of the factors. The next result shows that the right annihilator primes of the resulting sequence of factors are just the original right affiliated primes from the right affiliated series.

Proposition 8.9. *Let ${}_RA_S$ be a nonzero bimodule, where R is a left noetherian ring and S is a prime right noetherian ring, and suppose that the modules ${}_RA$ and A_S are both finitely generated and that A_S is torsionfree. Let*

$$A_0 = 0 < A_1 < A_2 < \cdots < A_m = A$$

be a left affiliated series for A. Then each factor A/A_{i-1} (for $i = 1, \dots, m$) is a torsionfree right S-module. In particular, $\mathrm{r.ann}_S(A_i/A_{i-1}) = 0$.

Proof. It suffices to show that $(A/A_1)_S$ is torsionfree. Set $P_1 = \mathrm{l.ann}_R(A_1)$, so that $A_1 = \mathrm{r.ann}_A(P_1)$. If $a \in A$ and $ax \in A_1$ for some regular element $x \in S$, then $P_1ax = 0$. Since A_S is torsionfree, $P_1a = 0$, and so $a \in A_1$. Therefore A/A_1 is torsionfree as a right S-module. $\square$

• ARTINIAN BIMODULES •

We turn next to a basic symmetry result – that a noetherian bimodule which is artinian on one side is necessarily artinian on the other. (The corresponding result for a noetherian ring we have already met in Exercise 4L.)

Lemma 8.10. *Let ${}_RA_S$ be a bimodule such that S is a semiprime right Goldie ring and A_S is torsionfree. Let Q be the right Goldie quotient ring of S.*

(a) *If ${}_RA$ has finite length, then A_S is divisible.*

(b) *If A_S is divisible, its right S-module structure extends to a right Q-module structure and A becomes an (R,Q)-bimodule.*

Proof. (a) Let c be a regular element of S. Since A_S is torsionfree, right multiplication by c defines an injective map of A to itself. This map is also a left R-module endomorphism of A, whence Ac has the same length as A, and therefore $Ac = A$.

(b) By Proposition 7.13, A has a unique right Q-module structure compatible with its right S-module structure. Given $r \in R$, $a \in A$, $q \in Q$, write $q = sc^{-1}$ for some $s, c \in S$ with c regular. Then

$$[r(aq)]c = r(aqc) = r(as) = (ra)s,$$

whence $r(aq) = [(ra)s]c^{-1} = (ra)q$. Therefore A is an (R,Q)-bimodule. $\square$

Exercise 8H. Let R and S be semiprime noetherian rings, and let ${}_RA_S$ be a bimodule which is finitely generated and torsionfree on each side. Let $Q(R)$ and $Q(S)$ be the Goldie quotient rings of R and S, and set $B = Q(R) \otimes_R A$.

(a) Show that B has finite length as a left $Q(R)$-module.

(b) Show that B is torsionfree as a right S-module. [Hint: Every element of B has the form $c^{-1} \otimes a$, where $c \in R$, $a \in A$, and c is regular.] Now, by Lemma 8.10, B is a $(Q(R), Q(S))$-bimodule.

(c) Given $c \in R$, $a \in A$ with c regular, show that there exist $b \in A$, $d \in S$ with d regular such that $ad = cb$. [Hint: Corollary 5.22.]

(d) Show that B has finite length as a right $Q(S)$-module. [Hint: Map $A \otimes_S Q(S)$ onto B.] $\square$

Lemma 8.11. *Let ${}_RA_S$ be a bimodule such that ${}_RA$ and A_S are finitely generated. Assume that S is a semiprime right noetherian ring and that A is a torsionfree, faithful, divisible right S-module. Then S is a semisimple ring and A_S is a semisimple module with finite length.*

Proof. If Q is the right Goldie quotient ring of S, then in view of Lemma 8.10 we may view A as an (R,Q)-bimodule.

We next observe that A_Q is faithful. Namely, any nonzero $q \in Q$ can be written as sc^{-1} for some $s, c \in S$ with $s \neq 0$ and c regular, and $as \neq 0$ for some $a \in A$, whence $aq \neq 0$. Since ${}_RA$ is finitely generated, Lemma 8.1 shows that

Q_Q embeds in some finite direct sum of copies of A_Q. As A_S is noetherian, it follows that Q_S is noetherian.

Now, given any regular element $z \in S$, the chain $S \leq z^{-1}S \leq z^{-2}S \leq \cdots$ of right S-submodules of Q must terminate, i.e., there exists $m \in \mathbb{N}$ such that $z^{-m-1}S = z^{-m}S$. On multiplying by z^m, we obtain $z^{-1}S = S$, whence $z^{-1} \in S$. Thus all regular elements of S are invertible in S, whence $Q = S$.

Therefore S is a semisimple ring, and consequently A_S is a semisimple module. As A_S is finitely generated, it must have finite length. □

Theorem 8.12. [Lenagan] *Let ${}_RA_S$ be a bimodule such that ${}_RA$ has finite length and A_S is noetherian. Then A_S has finite length.*

Proof. We may assume that $A \neq 0$, and we may replace S by $S/\operatorname{r.ann}_S(A)$, in which case Proposition 8.2 shows that S is right noetherian. By Proposition 8.7, there exists a right affiliated series

$$A_0 = 0 < A_1 < A_2 < \cdots < A_m = A$$

for A, and for each $i = 1, \ldots, m$ the ideal $P_i = \operatorname{r.ann}_S(A_i/A_{i-1})$ is a prime ideal of S and A_i/A_{i-1} is a torsionfree right (S/P_i)-module. Each A_i/A_{i-1} has finite length as a left R-module, and it suffices to show that each A_i/A_{i-1} has finite length as a right (S/P_i)-module.

Thus we may assume, without loss of generality, that S is a prime ring and that A is a torsionfree faithful right S-module. By Lemma 8.11, it suffices to show that A_S is divisible, and this follows from Lemma 8.10. □

Corollary 8.13. *Let ${}_RA_S$ be a bimodule which is noetherian on each side. Then ${}_RA$ is artinian if and only if A_S is artinian.* □

Corollary 8.14. *Let I be an ideal in a noetherian ring R. Then ${}_RI$ is artinian if and only if I_R is artinian.* □

Corollary 8.15. *Let ${}_RA_S$ be a bimodule which is noetherian on each side. There exists a unique sub-bimodule $B \leq A$ such that B is artinian on each side and B contains all artinian right or left submodules of A.*

Proof. Let B be the sum of all the artinian right S-submodules of A, and let C be the sum of all the artinian left R-submodules of A. Since B_S is finitely generated, it must be the sum of finitely many artinian right S-submodules, whence B_S is artinian. Similarly, ${}_RC$ is artinian. For any $x \in R$, observe that $(xB)_S$ is an epimorphic image of B_S and so is artinian, whence $xB \leq B$. Thus B is a sub-bimodule of A, and similarly C is a sub-bimodule.

By Corollary 8.13, ${}_RB$ and C_S are artinian, and therefore $B = C$. In particular, ${}_RB$ is artinian and B contains all the artinian left R-submodules of A. The uniqueness of B is clear. □

Definition. The sub-bimodule B in Corollary 8.15 is called the *artinian radical* of A.

For example, if $R = \begin{pmatrix} k & k[x]/xk[x] \\ 0 & k[x] \end{pmatrix}$ for some field k and indeterminate x, the artinian radical of R (i.e., of the bimodule ${}_RR_R$) is the ideal $\begin{pmatrix} k & k[x]/xk[x] \\ 0 & 0 \end{pmatrix}$.

Exercise 8I. Let R be a subring of a ring S such that R is right noetherian and S_R is finitely generated, and let P be a prime ideal of S. Show that S/P is simple artinian if and only if $R/(P \cap R)$ is right artinian. $\square$

Here is a first application of Lenagan's Theorem, which will be used in the next chapter.

Theorem 8.16. [Ginn-Moss] *Let R be a noetherian ring. If* $\operatorname{soc}(R_R)$ *is essential as either a right or a left ideal of R, then R is an artinian ring.*

Proof. If $I = \operatorname{soc}(R_R)$, then, since I_R is finitely generated and semisimple it is artinian. By Corollary 8.14, ${}_RI$ is artinian as well. We now have symmetric hypotheses – namely, we have an ideal I of R which is artinian on both sides and essential on (at least) one side. Hence, it is enough to consider the case that $I_R \leq_e R_R$.

If $N = \operatorname{l.ann}_R(I)$, then, as ${}_RI$ is artinian, we see by Proposition 8.2 that R/N is left artinian. On the other hand, since $NI = 0$ and $I_R \leq_e R_R$, it follows from Proposition 6.10 that N is nilpotent. For $i = 0, 1, \dots,$ note that N^i/N^{i+1} is a finitely generated left (R/N)-module, whence ${}_R(N^i/N^{i+1})$ is artinian. As $N^k = 0$ for some $k \in \mathbb{N}$, we conclude that R is left artinian. By Corollary 8.14, R is right artinian as well. $\square$

Theorem 8.16 does not hold for one-sided noetherian rings, as the following example shows.

Exercise 8J. If $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, show that R is a right noetherian ring with $\operatorname{soc}(R_R) \leq_e R_R$ but that R is neither right nor left artinian. $\square$

Exercise 8K. Let I be a right ideal in a noetherian ring R. If I has an essential artinian submodule, show that I is artinian. [Hint: Look at $R/\operatorname{r.ann}(I)$.] $\square$

• PRIME IDEALS IN FINITE RING EXTENSIONS •

In this section, we begin the study of finite ring extensions, to which we will return in Chapters 11 and 14. Suppose that R is a noetherian ring and R is a subring of a ring S such that the modules ${}_RS$ and S_R are finitely generated; this immediately implies that S is noetherian. We can, of course, regard S itself as an (R, S)-bimodule or as an (S, R)-bimodule, but there are other bimodules arising from these which give even more information about the relation between the ideal theories of R and S. We will concentrate on properties of prime ideals of R and S. If P is a prime ideal of S, then (unlike the commutative case) $P \cap R$ need not be a prime (or even a semiprime) ideal of R – for instance, let $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ and $S = M_2(\mathbb{Q})$, while $P = 0$. In this section we will look at properties of P and see how they compare with properties of the prime ideals of R minimal over $P \cap R$.

Lemma 8.17. *Let R be a noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are finitely generated, and let P be a prime ideal of S. For any prime ideal Q of R which is minimal over $P \cap R$, there exist nonzero bimodules ${}_{R/Q}A_{S/P}$ and ${}_{S/P}B_{R/Q}$ which are finitely generated and torsionfree on each side. Similarly, if Q_1 and Q_2 are prime ideals of R which are both minimal over $P \cap R$, there exist nonzero bimodules ${}_{R/Q_1}C_{R/Q_2}$ and ${}_{R/Q_2}D_{R/Q_1}$ which are finitely generated and torsionfree on each side.*

Proof. Since we may factor P out of S and $P \cap R$ out of R, without loss of generality we may assume that $P = 0$, so that S is a prime ring.

Given a minimal prime Q of R, the first bimodule we need to find is an $(R/Q, S)$-bimodule which is finitely generated and torsionfree as a left (R/Q)-module and as a right S-module. We first regard S as an (R, S)-bimodule and choose a left affiliated series

$$S_0 = 0 < S_1 < \cdots < S_n = S$$

for this bimodule. Propositions 8.7 and 8.9 imply that each factor S_i/S_{i-1} is a torsionfree left (R/L_i)-module, where $L_i = \mathrm{l.ann}_R(S_i/S_{i-1})$, and a torsionfree right S-module. According to Proposition 8.6, since Q is minimal over $\mathrm{l.ann}_R(S)$, we must have $Q = L_j$ for some index j. Thus we obtain the desired bimodule A by choosing $A = S_j/S_{j-1}$. The bimodule B is obtained by a symmetric argument.

Now suppose that Q_1 and Q_2 are minimal primes of R. Let ${}_{R/Q_1}A_S$ be the bimodule obtained in the previous paragraph, and now regard A as an $(R/Q_1, R)$-bimodule. Since A is a faithful right S-module, it is also a faithful right R-module. Let

$$A_0 = 0 < A_1 < \cdots < A_m = A$$

be a right affiliated series for this bimodule, and let $P_j = \mathrm{r.ann}_R(A_j/A_{j-1})$ for $j = 1, \ldots, m$. Using Propositions 8.7 and 8.9 again, each factor A_j/A_{j-1} is torsionfree as a left (R/Q_1)-module and as a right (R/P_j)-module. As Q_2 is minimal over $\mathrm{r.ann}_R(A)$, we again apply Proposition 8.6 to see that $Q_2 = P_k$ for some index k. Therefore we may choose $C = A_k/A_{k-1}$. The corresponding argument (beginning with B as an $(R, R/Q_1)$-bimodule) completes the proof of the lemma. $\square$

Theorem 8.18. [Jategaonkar, Letzter] *Let R and S be prime noetherian rings and ${}_RB_S$ a nonzero bimodule which is finitely generated and torsionfree on each side.*

(a) *If R is semiprimitive, so is S.*

(b) *If R is right primitive, so is S.*

Proof. First observe that, since B is finitely generated and torsionfree as a left R-module, ${}_RB$ embeds in a free left R-module (Proposition 7.19). Hence, $\mathrm{Hom}_R({}_RB, {}_RR) \neq 0$. Next let

$$T = \sum \{f(B) \mid f \in \mathrm{Hom}_R({}_RB, {}_RR)\}$$

and observe that T is a nonzero ideal of R (called the *trace ideal of* ${}_RB$).

If I is a right ideal of R, we claim that $B/IB = 0$ only if $(R/I)T = 0$. Given any $f \in \mathrm{Hom}_R({}_RB, {}_RR)$, it follows from $B = IB$ that $f(B) = If(B) \leq I$. Hence, $T \leq I$, and so $(R/I)T = 0$, as claimed.

(a) Let $J = J(S)$. If $J \neq 0$, then, by Proposition 8.4 (applied to the sub-bimodules $B \supseteq BJ$), there is a nonzero ideal K of R such that $KB \subseteq BJ$. Since R is prime, $T \cap K \neq 0$, and then since R is semiprimitive we may choose a maximal right ideal I of R such that I does not contain $T \cap K$. It follows immediately that $K + I = R$, so that $KB + IB = B$. On the other hand, $(R/I)T \neq 0$. By the claim proved above, $B/IB \neq 0$, and so we can find a maximal right submodule C in B such that $IB \subseteq C$. Since B/C is a simple right S-module, $(B/C)J = 0$. Hence, $BJ \subseteq C$ and then $KB \subseteq C$. Thus $B = KB + IB \subseteq C$, which is a contradiction. Therefore $J = 0$, and S is semiprimitive.

(b) There exists a maximal right ideal I in R such that R/I is a faithful simple module. Then $(R/I)T \neq 0$, and the claim above shows that $B/IB \neq 0$. Let C be a maximal right submodule of B containing IB. We claim that the simple right S-module B/C is faithful. If not, then $BJ \subseteq C$ for some nonzero ideal J of S. Applying Proposition 8.4 again, there is a nonzero ideal K of R such that $KB \subseteq BJ$. Since I is a maximal right ideal not containing K (because R/I is faithful), we conclude that $K + I = R$, and so $KB + IB = B$. However, $KB \subseteq BJ \subseteq C$ and $IB \subseteq C$, so this contradicts the fact that $C \neq B$. Therefore B/C is a faithful simple right S-module and S is right primitive as required. $\square$

Corollary 8.19. *Let R be a noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are finitely generated, and assume that S is a prime ring. Then S is right* (*left*) *primitive if and only if at least one minimal prime ideal of R is right* (*left*) *primitive, if and only if every minimal prime ideal of R is right* (*left*) *primitive.*

Proof. Lemma 8.17 and Theorem 8.18. $\square$

Definition. A *Jacobson ring* (sometimes called a *Hilbert ring*) is a ring in which every prime ideal is semiprimitive, i.e., every prime factor ring has zero Jacobson radical.

Corollary 8.20. [Cortzen-Small] *Let R be a noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are finitely generated. If R is a Jacobson ring, then so is S.*

Proof. Lemma 8.17 and Theorem 8.18. □

Exercise 8L. Let R be a noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are finitely generated. If some minimal prime of R is right primitive, show that some minimal prime of S is right primitive. If all minimal primes of S are right primitive, show that all minimal primes of R are right primitive. □

• BIMODULE COMPOSITION SERIES •

If R is a subring of a ring S and P is a prime ideal of S, then it is natural (especially when R is noetherian) to study the primes in R minimal over $P \cap R$. However, it turns out that a slightly larger set of "affiliated primes" carries more information about the pair of rings. (When the extension is finite, as in the previous section, these primes can frequently be proved to be precisely the primes minimal over $P \cap R$, but this will not generally be true for infinite extensions.) The idea is to view the Goldie quotient ring C of S/P as a (C, R)-bimodule and to refine an affiliated series for this bimodule even further – to a bimodule composition series, which exists because C has finite length on the left. Thus we begin with another look at bimodules which have finite length on one side.

Most terminology relating to submodules of a module can be directly carried over to sub-bimodules of a bimodule. For instance, a bimodule ${}_TC_R$ is *simple* provided C is nonzero and its only sub-bimodules are 0 and C. A *bimodule composition series* for C is a chain

$$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$$

of sub-bimodules of C such that C_i/C_{i-1}, for $i = 1, \ldots, m$, is a simple bimodule. Recall from Exercise 1D that C can be made into a right $(T^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-module so that its $(T^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-submodules are precisely its (T, R)-sub-bimodules. Thus, a bimodule composition series for C is precisely a $(T^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-module composition series. One exists if and only if C is both artinian and noetherian as a $(T^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-module, i.e., if and only if C has both the DCC and the ACC on sub-bimodules. In particular, if C has finite length as either a left T-module or a right R-module, it will have a bimodule composition series.

Proposition 8.21. *Let ${}_TC_R$ be a bimodule such that ${}_TC$ has finite length, and let*

$$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$$

be a bimodule composition series for C. Let $Q_i = \mathrm{r.ann}_R(C_i/C_{i-1})$ for $i = 1, \dots, m$.

(a) *Each Q_i is a prime ideal of R, and if R/Q_i is right Goldie, then C_i/C_{i-1} is torsionfree as a right (R/Q_i)-module.*

(b) *If $D_0 = 0 < D_1 < D_2 < \cdots < D_n = C$ is another bimodule composition series for C, then $n = m$, and there exists a permutation π of $\{1, 2, \dots, m\}$ such that $\mathrm{r.ann}_R(D_i/D_{i-1}) = Q_{\pi(i)}$ for $i = 1, \dots, m$.*

Proof. (a) If I and J are ideals of R not contained in Q_i, then $(C_i/C_{i-1})I$ and $(C_i/C_{i-1})J$ are nonzero. As these are sub-bimodules of C_i/C_{i-1}, they must both equal C_i/C_{i-1}. Then $(C_i/C_{i-1})IJ = (C_i/C_{i-1})J \neq 0$, and so $IJ \not\subseteq Q_i$, proving that Q_i is prime.

Now assume that R/Q_i is right Goldie, let D be the torsion submodule of C_i/C_{i-1} as a right (R/Q_i)-module, and note that D is a sub-bimodule of C_i/C_{i-1}. Since C_i/C_{i-1} is a simple bimodule, either $D = 0$ or $D = C_i/C_{i-1}$. In the latter case, C_i/C_{i-1} would be torsion as a right (R/Q_i)-module, and since it is also finitely generated as a left T-module, there would be a regular element $c \in R/Q_i$ such that $(C_i/C_{i-1})c = 0$. However, that would contradict the fact that C_i/C_{i-1} is a faithful right (R/Q_i)-module. Therefore $D = 0$ and C_i/C_{i-1} is torsionfree over R/Q_i.

(b) This is immediate from the Jordan-Hölder Theorem (Theorem 4.11), which says that $n = m$ and there is a permutation π of $\{1, 2, \dots, m\}$ such that $D_i/D_{i-1} \cong C_{\pi(i)}/C_{\pi(i)-1}$ (as $(T^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-modules) for $i = 1, \dots, m$. $\square$

We intend to refer to the primes Q_i occurring in Proposition 8.21 as "right affiliated primes" of C, which a priori conflicts with our previous usage. As long as R is right noetherian, however, the next lemma shows that we may talk about right affiliated primes without ambiguity.

Lemma 8.22. *Let ${}_TC_R$ be a bimodule such that ${}_TC$ has finite length and R is right noetherian. Let*

$$C_0 = 0 < C_1 < \cdots < C_m = C$$

be a bimodule composition series for C and

$$B_0 = 0 < B_1 < \cdots < B_n = C$$

a right affiliated series for C. Let $Q_i = \mathrm{r.ann}_R(C_i/C_{i-1})$ for $i = 1, \dots, m$ and $P_j = \mathrm{r.ann}_R(B_j/B_{j-1})$ for $j = 1, \dots, n$. Then

$$\{Q_1, \dots, Q_m\} = \{P_1, \dots, P_n\}.$$

Proof. We may assume that $C \neq 0$. Since the set $\{Q_1, \dots, Q_m\}$ is independent of the choice of a bimodule composition series for C (by Proposition 8.21), we may assume that our bimodule composition series is a refinement of the given right affiliated series. Thus there exist integers

$$i(0) = 0 < i(1) < \cdots < i(n) = m$$

such that $C_{i(j)} = B_j$ for $j = 0, \dots, n$. To prove the lemma, it suffices to show that $Q_i = P_j$ for $j = 1, \dots, n$ and $i = i(j-1)+1, \dots, i(j)$, that is, we need only prove the lemma for each of the bimodules B_j/B_{j-1}.

Hence, we may assume that the right affiliated series for C is just

$$B_0 = 0 < B_1 = C,$$

and we must prove that all $Q_i = P_1$. Since $P_1 = \text{r.ann}_R(C) \subseteq Q_i$ for $i = 1, \dots, m$, we may replace R by R/P_1. Thus, we may assume that R is prime and C_R is faithful, and it remains to show that all $Q_i = 0$.

By Proposition 8.7, C is torsionfree as a right R-module. As each C_i is a bimodule which has finite length on the left, Lemma 8.10 says that C_i is divisible as a right R-module. Then $C_i x = C_i \nsubseteq C_{i-1}$ for all regular elements $x \in R$, whence the ideal Q_i contains no regular elements. Therefore $Q_i = 0$, as desired. $\square$

Definition. In the situation of Proposition 8.21, the prime ideals $Q_1, \dots, Q_m$ are called the *right affiliated primes of* C, and if a prime Q_j appears in the list $Q_1, \dots, Q_m$ exactly m_j times, we say that Q_j is *affiliated with multiplicity* m_j. It is clear from part (b) of Proposition 8.21 that the set $\{Q_1, \dots, Q_m\}$ and the multiplicities are independent of the choice of a bimodule composition series for C.

If instead of a bimodule composition series for C we take a right affiliated series, then, as long as R is right noetherian, the right affiliated primes will be the same (Lemma 8.22), but the multiplicities may differ. For instance, if $R = \mathbb{Q}$ and $T = C = M_2(\mathbb{Q})$, then a right affiliated series for ${}_TC_R$ is $0 < C$, whereas a bimodule composition series has length 2. The only right affiliated prime, namely 0, appears with multiplicity 1 with respect to the affiliated series but with multiplicity 2 with respect to the bimodule composition series.

Proposition 8.23. *Let ${}_TC_R$ be a bimodule such that ${}_TC$ has finite length, and let $A > B$ be sub-bimodules of C. Then any prime ideal of R minimal over* $\text{r.ann}_R(A/B)$ *is right affiliated to C. In particular, $R/\,\text{r.ann}_R(A/B)$ has only finitely many minimal prime ideals.*

Proof. By stringing together bimodule composition series for B, A/B, and C/A, we may choose a bimodule composition series for C that passes through B and A, from which it is clear that any prime of R right affiliated to A/B is also right affiliated to C. Hence, we may replace A/B by C.

If $Q_1, \ldots, Q_m$ is the list of right affiliated primes corresponding to a bimodule composition series

$$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C,$$

it is clear that $CQ_mQ_{m-1}\cdots Q_1 = 0$, whence $Q_mQ_{m-1}\cdots Q_1 \subseteq \text{r.ann}_R(C)$. On the other hand, $\text{r.ann}_R(C)$ annihilates each C_i/C_{i-1} and so is contained in each Q_i. Therefore any prime ideal Q of R containing $\text{r.ann}_R(C)$ must contain some Q_j, and if Q is minimal over $\text{r.ann}_R(C)$, then $Q = Q_j$. □

• ADDITIVITY PRINCIPLES •

Given a bimodule ${}_TC_R$ with ${}_TC$ having finite length, we wish to compare the length of ${}_TC$ with the ranks of R modulo the right affiliated primes. For this it suffices to assume that these prime factors of R are all right Goldie; however, to avoid excessive hypotheses, we shall assume that R is right noetherian. Also, for any prime ideal Q of R, we write $\text{rank}(R/Q)$ for the rank of R/Q as a right module over itself; note that this equals the rank of R/Q as a right R-module.

Lemma 8.24. *Let ${}_TC_R$ be a bimodule such that ${}_TC$ has finite length, R is prime right noetherian, and C_R is torsionfree. Then the rank of R divides the length of ${}_TC$.*

Proof. If Q is the right Goldie quotient ring of R, then by Lemma 8.10 we may view C as a (T, Q)-bimodule. As $\text{rank}(R) = \text{rank}(Q)$ (Exercise 6L), there is no loss of generality in replacing R by Q. Hence, we may assume that R is a simple artinian ring.

Now R is isomorphic to the $m \times m$ matrix ring over a division ring, where $m = \text{rank}(R)$, and so there exists a set of $m \times m$ matrix units $e_{ij} \in R$ (that is, $e_{ij}e_{kl} = 0$ for $j \neq k$ and $e_{ij}e_{jk} = e_{ik}$ while also $e_{11} + \cdots + e_{mm} = 1$). We now note that as a left T-module, C is the direct sum of the submodules Ce_{ii}, for $i = 1, \ldots, m$. Also, right multiplication by e_{ij} and by e_{ji} provide inverse isomorphisms between Ce_{ii} and Ce_{jj}. Hence, if Ce_{11} has length t, so does each Ce_{ii}, and therefore $\text{length}({}_TC) = mt$. □

Theorem 8.25. (The Additivity Principle) *Let ${}_TC_R$ be a nonzero bimodule such that ${}_TC$ has finite length and R is right noetherian, and let X be the set of primes in R right affiliated to C. Then there are positive integers z_Q, for $Q \in X$, such that*

$$\text{length}({}_TC) = \sum_{Q \in X} z_Q \, \text{rank}(R/Q).$$

Moreover, if $Q \in X$ is affiliated with multiplicity m_Q, then $z_Q \geq m_Q$.

Proof. Let $C_0 = 0 < C_1 < C_2 < \cdots < C_n = C$ be a bimodule composition series for C, and for $i = 1, \ldots, n$ set $B_i = C_i/C_{i-1}$ and $Q_i = \text{r.ann}_R(B_i)$. Then

$X = \{Q_1, \dots, Q_n\}$, each B_i is a torsionfree right (R/Q_i)-module (Proposition 8.21), and it follows from Lemma 8.24 that there are positive integers z_i such that $z_i \operatorname{rank}(R/Q_i) = \operatorname{length}({}_T B_i)$. We add up these equations to obtain

$$\operatorname{length}({}_T C) = \sum_{i=1}^{n} \operatorname{length}({}_T B_i) = \sum_{i=1}^{n} z_i \operatorname{rank}(R/Q_i),$$

from which the conclusions of the theorem are clear. $\square$

Corollary 8.26. *Let ${}_T C_R$ be a bimodule such that ${}_T C$ has finite length and R is right noetherian, and let Q be a prime ideal of R minimal over* $\text{r.ann}_R(C)$. *Then* $\operatorname{rank}(R/Q) \le \operatorname{length}({}_T C)$, *and if equality holds, then* $\text{r.ann}_R(C) = Q$.

Proof. If X is the set of primes in R right affiliated to C, then, by Proposition 8.23, $Q \in X$. As the coefficient z_Q in Theorem 8.25 is positive, it is immediate that $\operatorname{rank}(R/Q) \le \operatorname{length}({}_T C)$. Now if $\operatorname{rank}(R/Q) = \operatorname{length}({}_T C)$, we infer from Theorem 8.25 that $X = \{Q\}$ and that Q is affiliated to C with multiplicity 1. Consequently, C has a bimodule composition series of length 1 (that is, $0 < C$) and $Q = \text{r.ann}_R(C)$. $\square$

Note that if $\text{r.ann}_R(C) = Q$ in the situation of Corollary 8.26, we cannot necessarily conclude that $\operatorname{rank}(R/Q) = \operatorname{length}({}_T C)$ – for example, let $T = C = M_2(\mathbb{Q})$ while $R = \mathbb{Q}$ and $Q = 0$.

The Additivity Principle given in Theorem 8.25 is more general than the original, which applies to primes in ring extensions – see Theorem 8.27.

Definition. Let R be a subring of a right noetherian ring S and P a prime ideal of S. If C is the right Goldie quotient ring of S/P, we may view C as a (C, R)-bimodule, and of course ${}_C C$ has finite length. In this context, the right affiliated primes of C are called the *primes of R right affiliated to P*. If we had regarded C as an (R, C)-bimodule, we would have obtained primes of R *left affiliated to P*. That the primes left affiliated to P are the same as those right affiliated to P, with the same multiplicities, is immediate from the following exercise.

Exercise 8M. Let R be a subring of a simple artinian ring C, let

$$C_0 = 0 < C_1 < C_2 < \cdots < C_m = C$$

be a bimodule composition series for ${}_C C_R$, and set $Q_i = \text{r.ann}_R(C_i/C_{i-1})$ for $i = 1, \dots, m$. If $D_i = \text{r.ann}_C(C_i)$ for $i = 1, \dots, m$, show that $C_i = \text{l.ann}_C(D_i)$, that

$$D_m = 0 < D_{m-1} < \cdots < D_0 = C$$

is a bimodule composition series for ${}_R C_C$, and that $Q_i = \text{l.ann}_R(D_{i-1}/D_i)$ for $i = 1, \dots, m$. $\square$

If R is a subring of a right noetherian ring S and P a prime ideal of S, then by Proposition 8.23 all prime ideals of R minimal over $P \cap R$ are right affiliated to P. However, the primes affiliated to P need not all be minimal over $P \cap R$, as the following example shows.

Exercise 8N. Let k be a field, x an indeterminate, $S = M_2(k[x])$, and P the prime ideal 0 in S. Let R be the subring of S consisting of all matrices $\begin{pmatrix} f & 0 \\ 0 & f(0) \end{pmatrix}$ for $f \in k[x]$. Show that the primes in R right affiliated to P are 0 and $\begin{pmatrix} xk[x] & 0 \\ 0 & 0 \end{pmatrix}$. In particular, the latter is not minimal over $P \cap R$. $\square$

Theorem 8.27. [Joseph-Small, Borho, Warfield] *Let R be a right noetherian subring of a right noetherian ring S, let P be a prime ideal of S, and let X be the set of primes in R right affiliated to P. Then there are positive integers z_Q, for $Q \in X$, such that*

$$\operatorname{rank}(S/P) = \sum_{Q \in X} z_Q \operatorname{rank}(R/Q).$$

Moreover, if $Q \in X$ is affiliated with multiplicity m_Q, then $z_Q \geq m_Q$.

Proof. This is immediate from Theorem 8.25 by letting T and C equal the right Goldie quotient ring of S/P and noting that $\operatorname{length}({}_T C) = \operatorname{rank}(C) = \operatorname{rank}(S/P)$. $\square$

Corollary 8.28. *Let R be a right noetherian subring of a right noetherian ring S, let P be a prime ideal of S, and let Q be a prime ideal of R minimal over $P \cap R$. Then $\operatorname{rank}(R/Q) \leq \operatorname{rank}(S/P)$, and if equality holds, then $P \cap R = Q$.* $\square$

Exercise 8O. Let R be a right noetherian subring of a left artinian ring S, let N be the prime radical of R, and assume that every element of $\mathcal{C}_R(N)$ is invertible in S. Choose a bimodule composition series $S_0 = 0 < S_1 < \cdots < S_m = S$ for ${}_S S_R$.

(a) For $i = 1, \ldots, m$, show that S_i/S_{i-1} is a torsionfree divisible right (R/N)-module. [Hint: If this has been proved for $i = 1, \ldots, j$, show that $S_j c = S_j$ for all $c \in \mathcal{C}_R(N)$.]

(b) Show that the primes in R right affiliated to ${}_S S_R$ are precisely the minimal primes. $\square$

• NORMALIZING EXTENSIONS •

To conclude the chapter, we briefly consider the behavior of primes affiliated to a prime ideal in an extension ring generated by elements that "normalize" the subring in the following sense.

Definition. If R is a subring of a ring S, then an element $x \in S$ is said to *normalize* R if $xR = Rx$. (Note that we do not require that x commute with the elements of R, although that would certainly suffice.) We say that S is a *normalizing extension of* R if S is generated as a right (equivalently, left) R-module by a (possibly infinite) set of elements that normalize R. The reader should be warned that, in much of the literature, the term "normalizing extension" is reserved for a *finite* normalizing extension, i.e., a ring extension $S \supseteq R$ in which S is generated by a finite set of normalizing elements.

For instance, if α is an automorphism of a ring R, then $R[x;\alpha]$ and $R[x^{\pm 1};\alpha]$ are normalizing extensions of R. For another example, if k is a field and H a normal subgroup of a group G, the group algebra $k[G]$ is a normalizing extension of the group algebra $k[H]$. (Here the normalizing generators can be taken to be a set of coset representatives for H in G, and $k[G]$ is actually a free $k[H]$-module on this set of generators.)

Exercise 8P. Let R be a subring of a ring S, let x be an element of S that normalizes R, and let A be an R-submodule of a right S-module B. Observe that Ax is an R-submodule of B. Set $Q_1 = \operatorname{ann}_R(A)$ and $Q_2 = \operatorname{ann}_R(Ax)$. If $\operatorname{ann}_A(x) = 0$, show that $R/Q_1 \cong R/Q_2$. [Hint: Given $r \in R$, there exists $r' \in R$ such that $rx = xr'$; map $r + Q_1$ to $r' + Q_2$.] $\square$

Theorem 8.29. [Warfield] *Let R be a subring of a right noetherian ring S such that S is generated as an R-module by a (possibly infinite) set X of elements normalizing R, and let P be a prime ideal of S.*

(a) *$P \cap R$ is a semiprime ideal of R.*
(b) *R has only finitely many primes minimal over $P \cap R$, say $Q_1, \dots, Q_m$.*
(c) *The factor rings R/Q_i are all isomorphic.*
(d) *If X consists of units of S, then $\{Q_1, \dots, Q_m\} = \{xQ_1x^{-1} \mid x \in X\}$.*
(e) *If R is right noetherian, then* $\operatorname{rank}(R/Q_1)$ *is a divisor of* $\operatorname{rank}(S/P)$.

Proof. As before, we let C be the right Goldie quotient ring of S/P. First regard C as a (C,S)-bimodule. If C' is any (C,S)-sub-bimodule of C, then, since it has finite length on the left, by Lemma 8.10 it is divisible as a right (S/P)-module. Hence, C' is an ideal of C, and since C is a simple ring, either $C' = C$ or $C' = 0$. Thus C is a simple (C,S)-bimodule.

Now regard C as a (C,R)-bimodule and choose a simple sub-bimodule $A \leq C$. Note that if $x \in X$, then while right multiplication by x is not necessarily a bimodule map, it comes quite close: Because $Rx = xR$, the sets Ax and $\{a \in A \mid ax \in B\}$ (for any sub-bimodule $B \leq C$) are sub-bimodules of C. Hence, either $Ax = 0$ or Ax is a simple sub-bimodule of C.

We next look at the sub-bimodule $\sum_{x\in X} Ax = AS$. As this is a nonzero (C,S)-sub-bimodule of C, we obtain $C = \sum_{x\in X} Ax$, and thus C is a sum of simple (C,R)-sub-bimodules. Therefore C is a semisimple $(C^{\mathrm{op}} \otimes_{\mathbb{Z}} R)$-module, and so by Proposition 4.1 we can write it as a direct sum of simple sub-bimodules, in fact a finite direct sum because ${}_CC$ has finite length. Therefore $C = C_1 \oplus \cdots \oplus C_n$ for some simple sub-bimodules C_j, and each $C_j \cong Ax_j$ for some $x_j \in X$ (since $\sum_{x\in X} Ax$ is not contained in $\bigoplus_{i\neq j} C_i$). Observe that any simple sub-bimodule of C must be isomorphic to some C_j, and hence to Ax_j.

Since A was arbitrary, the same conclusion applies to any pair of simple sub-bimodules $B, B' \leq C$: There exists $x \in X$ such that $B' \cong Bx$. Note in this situation that $\mathrm{l.ann}_B(x)$ is a proper sub-bimodule of B, whence $\mathrm{l.ann}_B(x) = 0$.

(a) The ideals $Q'_j = \mathrm{r.ann}_R(C_j)$ (for $j = 1, \dots, n$) are the primes in R right affiliated to P, and clearly $P \cap R = \mathrm{r.ann}_R(C) = Q'_1 \cap \cdots \cap Q'_n$. Thus statement (a) is proved.

(b) By Proposition 8.23, the primes of R minimal over $P \cap R$ are all contained in the set $\{Q'_1, \dots, Q'_n\}$.

(c) If $i, j \in \{1, \dots, n\}$, then, as observed above, there exists $x_{ij} \in X$ such that $C_j \cong C_i x_{ij}$ and $\mathrm{l.ann}_{C_i}(x_{ij}) = 0$. Thus $R/Q'_i \cong R/Q'_j$ by Exercise 8P. (We will use this fact in the proof of part (e).) Statement (c) now follows using the proof of (b).

(d) As just observed, there are simple sub-bimodules $B_1, \dots, B_m \leq C$ such that each $Q_i = \mathrm{r.ann}_R(B_i)$, and each $B_i \cong B_1 y_i$ for some $y_i \in X$. Since $y_i R = R y_i$ and y_i is a unit of S, we have $y_i R y_i^{-1} = R$. For $r \in R$, observe that $r \in Q_i$ if and only if $B_1 y_i r y_i^{-1} = 0$, if and only if $y_i r y_i^{-1} \in Q_1$. Therefore $Q_i = y_i^{-1} Q_1 y_i$.

Conversely, if $x \in X$, then $x^{-1} R x = R$ and the rule $r \mapsto x^{-1} r x$ defines an automorphism of R. Hence, $x^{-1} Q_1 x$ must be a prime of R minimal over $x^{-1}(P \cap R)x$. However, $x^{-1} P x = P$ (because P is an ideal of S), whence $x^{-1}(P \cap R)x = P \cap R$, and thus $x^{-1} Q_1 x$ is minimal over $P \cap R$. Therefore $x^{-1} Q_1 x \in \{Q_1, \dots, Q_m\}$.

(e) As observed in the proof of (c), the rings R/Q'_j (for $j = 1, \dots, n$) are pairwise isomorphic, and so they all have the same rank. Statement (e) now follows from Theorem 8.27. $\square$

Corollary 8.30. *Let R be a noetherian ring and α an automorphism of R. If $S = R[x^{\pm 1}; \alpha]$ and P is a prime ideal of S, then there exist a prime ideal Q of R and a positive integer m such that $P \cap R = Q \cap \alpha(Q) \cap \cdots \cap \alpha^{m-1}(Q)$ and $\alpha^m(Q) = Q$.* $\square$

Exercise 8Q. Give a more direct proof of Corollary 8.30 by first observing that an ideal I of R is of the form $J \cap R$ for some ideal of S if and only if $\alpha(I) = I$. $\square$

Exercise 8R. Let k be a field and H a normal subgroup of a finite group G.

(a) If P is any maximal ideal of $k[G]$, show that $P \cap k[H] = M_1 \cap \cdots \cap M_n$ for some maximal ideals M_i of $k[H]$, and that each $M_i = g_i^{-1} M_1 g_i$ for some $g_i \in G$.

(b) If V is any simple $k[G]$-module, show that V is semisimple as a $k[H]$-module. (This is sometimes known as *Clifford's Theorem.*) $\square$

Exercise 8S. Let $T = \mathbb{Q}[x_1, x_2, \dots]$, where the x_i are independent commuting indeterminates, and let ϕ be the $\mathbb{Q}$-algebra endomorphism of T such that $\phi(x_1) = 0$ and $\phi(x_n) = x_{n-1}$ for all $n > 1$. Let $S = M_2(T)$, and let R be the subring of S consisting of all matrices of the form $\begin{pmatrix} t & 0 \\ 0 & \phi(t) \end{pmatrix}$, where $t \in T$.

Show that S is a (finite) normalizing extension of R but that one of the primes in R right affiliated to the prime 0 in S is not a minimal prime of R. □

• NOTES •

Affiliated Primes. Affiliated series and affiliated primes were first introduced for bimodules by Stafford [1979, pp. 265–266].

Lenagan's Theorem. Lenagan proved that if an ideal in a right noetherian ring has finite length on the left, then it also has finite length on the right [1975, Proposition].

Noetherian Rings with Essential Socles. That a noetherian ring with essential socle must be artinian is due to Ginn and Moss [1975, Theorem].

Transfer of (Semi-) Primitivity across Bimodules. The semiprimitive part of Theorem 8.18 is due to Jategaonkar [1979, Theorem D; 1981, Theorem 6.1] and the primitive part to Letzter [1989, Lemma 1.3].

Jacobson Condition in Finite Ring Extensions. That a finite ring extension of a noetherian Jacobson ring must be Jacobson is due to Cortzen and Small [1988, Theorem 1].

Additivity Principles. The additivity principle for ring extensions given in Theorem 8.27 was first proved by Joseph and Small for certain factor rings of enveloping algebras, where X turns out to be the set of primes of R minimal over $P \cap R$ [1978, Theorem 3.9]. Borho then developed a version for noetherian rings with suitable symmetric dimension functions, where X again turns out to be the set of primes of R minimal over $P \cap R$ [1982, Theorem 7.2]. The general theorem was proved by Warfield, assuming just that all prime factors of R are right Goldie and that S/P is Goldie on one side [1983, Theorem 1].

Primes in Infinite Normalizing Extensions. Theorem 8.29 is due to Warfield, assuming only suitable Goldie conditions on prime factors of R and S [1983, Theorem 3].

9. Fully Bounded Rings

One major obstacle to adapting commutative noetherian ring theory to the noncommutative case in general is the lack of ideals. For example, the Weyl algebras over division rings of characteristic zero are simple noetherian domains, yet their module structure is quite complicated. Thus, to derive much structure theory similar to the commutative theory, one should work in a context where a large supply of ideals is guaranteed. One such context is introduced and investigated in this chapter. The results obtained may serve to give a sample of what is known about noetherian rings satisfying a polynomial identity (P.I.), although the methods used are very different from those of P.I. theory.

• BOUNDEDNESS •

Definition. A ring R is *right bounded* if every essential right ideal of R contains an ideal which is essential as a right ideal.

For instance, every commutative ring is right bounded, as is every semisimple ring (since a semisimple ring has no proper essential right ideals). On the other hand, a simple ring cannot be right bounded unless it is artinian. Note that a prime ring R is right bounded if and only if every essential right ideal of R contains a nonzero ideal (recall Exercise 5A).

Definition. A ring R is *right fully bounded* provided every prime factor ring of R is right bounded.

A priori, a right fully bounded ring need not be right bounded. However, in a right fully bounded right noetherian ring, it can be shown that all factor rings are right bounded (Exercise 9G).

Definition. A *right* (*left*) *FBN ring* is any right (left) fully bounded right (left) noetherian ring. An *FBN ring* is any right and left FBN ring.

In order to emphasize what ingredients are going into our proofs, we state many results for right FBN rings or for left noetherian right FBN rings. However, as there are no known examples of noetherian rings which are fully bounded on one side but not on the other, the extra generality may be gratuitous.

The following proposition demonstrates two ways in which the ideals necessary for boundedness can be obtained from certain commutative subrings.

Proposition 9.1. *Let R be a module-finite algebra over a commutative ring S.*

(a) *All factor rings of R are right and left bounded, and in particular R is right and left fully bounded.*

(b) *If R is prime and S is noetherian, then every essential right or left ideal of R contains a nonzero central element and hence a nonzero centrally generated ideal.*

Proof. (a) Since all factor rings of R are module-finite S-algebras, it suffices to show that R is right and left bounded. Let $x_1, \dots, x_n$ be S-module generators for R.

Given $I \leq_e R_R$, set $I_j = \{r \in R \mid x_j r \in I\}$ for $j = 1, \dots, n$ and note that each $I_j \leq_e R_R$. Hence, the intersection $J = I_1 \cap \cdots \cap I_n$ is an essential right ideal of R. Then RJ is essential as a right ideal of R, and

$$RJ = (x_1S + \cdots + x_nS)J = (x_1J + \cdots + x_nJ)S \subseteq x_1I_1 + \cdots + x_nI_n \subseteq I.$$

Thus, R is right bounded and, by symmetry, left bounded.

(b) Note first that R is noetherian. If I is an essential right ideal of R, then by the Regular Element Lemma there exists a regular element $x \in I$. Since R is a noetherian S-module, the S-submodule $\sum_i x^i S$ is finitely generated, say by $1, x, x^2, \dots, x^{n-1}$. Then

$$x^n = 1s_0 + xs_1 + x^2s_2 + \cdots + x^{n-1}s_{n-1}$$

for some $s_i \in S$. Since x is regular, $x^n \neq 0$, and so not all s_i can be zero. If j is the least index such that $s_j \neq 0$, then

$$x^{n-j} = 1s_j + xs_{j+1} + \cdots + x^{n-j-1}s_{n-1}$$

(because x is regular). Thus $1s_j \in I$, and $1s_j$ is a nonzero central element of R. $\square$

Observe that Proposition 9.1(b) is equivalent to the following statement: If R is a prime module-finite algebra over a commutative noetherian ring, then the Goldie quotient ring of R equals the localization RX^{-1}, where X is the set of nonzero central elements of R. That such a statement also holds for prime P.I. rings is part of a fundamental result known as *Posner's Theorem*: If R is a prime P.I. ring with center Z and $X = Z \setminus \{0\}$, then RX^{-1} is a finite dimensional simple algebra over the quotient field of Z (see McConnell-Robson [2001, Theorem 13.6.5]). We conclude from Posner's Theorem that Proposition 9.1(b) also holds for prime P.I. rings.

In particular, Proposition 9.1 shows that any matrix ring over a commutative ring is fully bounded. It can be shown that any subring of a matrix ring over a commutative ring is fully bounded as well. As another example, Proposition 9.1 shows that the subring $\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ of the quaternions is an FBN ring.

Exercise 9A. Let (q_{ij}) be a multiplicatively antisymmetric matrix over a field k. If all the q_{ij} are roots of unity, show that $\mathcal{O}_q(k^n)$ and $\mathcal{O}_q((k^\times)^n)$ are FBN rings. [Hint: Certain powers of the generators x_i are central.] On the other hand, if $q \in k^\times$ is not a root of unity, it follows from Corollary 1.18 that $\mathcal{O}_q((k^\times)^2)$ is not fully bounded. $\square$

Exercise 9B. For any field k of characteristic $p > 0$, show that the Weyl algebras $A_n(k)$ are FBN rings. [Hint: The p-th powers of the standard generators are central.] $\square$

Exercise 9C. For any positive integer n, show that all proper factor rings of $A_n(\mathbb{Z})$ are FBN, but that $A_n(\mathbb{Z})$ itself is neither right nor left bounded. [Hint: Use the simplicity of $A_n(\mathbb{Q})$ to show that every nonzero ideal of $A_n(\mathbb{Z})$ contains a nonzero integer.] $\square$

Exercise 9D. Let k be a field of characteristic $p > 0$ and δ a derivation on k. If δ is a nilpotent map (that is, $\delta^n = 0$ for some $n \in \mathbb{N}$), show that $k[x;\delta]$ is an FBN ring. [Hint: Show that k is finite-dimensional over its subfield of constants, that is, $\ker(\delta)$.] $\square$

Lemma 9.2. *Let R be a prime right Goldie ring. Then R is right bounded if and only if R has no faithful finitely generated torsion right modules.*

Proof. If R is not right bounded, then R has an essential right ideal I which contains no nonzero ideals. Since $\operatorname{ann}_R(R/I)$ is an ideal contained in I, it must be zero. Thus R/I is a faithful finitely generated torsion right R-module.

Conversely, assume that R is right bounded, and let A be any finitely generated torsion right R-module. Choose generators $a_1, \dots, a_n$ for A. There exists a regular element $x \in R$ such that $a_j x = 0$ for all j, and xR is an essential right ideal of R. By assumption, there exists a nonzero ideal $J \subseteq xR$, and $a_j J = 0$ for each $j = 1, \dots, n$. Since J is an ideal, $AJ = 0$, and hence A is not faithful. $\square$

Corollary 9.3. *Let R be a right bounded prime right Goldie ring and A a finitely generated uniform right R-module. If A is faithful, then A is torsion-free.*

Proof. Proposition 7.10 and Lemma 9.2. $\square$

Exercise 9E. Let R be a right FBN ring and A a right R-module which has a prime series $A_0 = 0 < A_1 < A_2 < \cdots < A_n = A$, with corresponding primes $P_i = \operatorname{ann}_R(A_i/A_{i-1})$. Show that each A_i/A_{i-1} is a torsionfree (R/P_i)-module. $\square$

Proposition 9.4. *Let R be a right FBN ring. If P is a right primitive ideal of R (in particular, if P is a maximal ideal), then R/P is a simple artinian ring.*

Proof. Without loss of generality, $P = 0$. Then there exists a faithful simple right R-module, which we may write as R/M, for some maximal right ideal

M of R. As R/M is faithful, M does not contain a nonzero ideal of R, and hence $M \not\leq_e R_R$. Thus, there exists a nonzero right ideal $J \leq R_R$ for which $M \cap J = 0$. Then $J \cong R/M$, and so $J \leq \mathrm{soc}(R_R)$. As $\mathrm{soc}(R_R) \neq 0$, Corollary 7.16 shows that R is simple artinian. $\square$

It follows from Proposition 9.4 that every right primitive ideal in a right FBN ring is maximal.

Corollary 9.5. *Let R be a right FBN ring and A a simple right R-module. Then $\mathrm{ann}_R(A)$ is a maximal ideal of R and $R/\mathrm{ann}_R(A)$ is a simple artinian ring. Hence, A is isomorphic to a right ideal of $R/\mathrm{ann}_R(A)$ and $(R/\mathrm{ann}_R(A))_R$ is isomorphic to a finite direct sum of copies of A.* $\square$

Exercise 9F. Let $R = k[x][y; x(d/dx)]$, where k is a field of characteristic zero. Show that R is neither right nor left bounded. [Hint: Exercises 3O and 3P.] $\square$

• EMBEDDING MODULES INTO FACTOR RINGS •

Corollary 9.5 shows that any nonzero artinian right module over a right FBN ring R has nonzero submodules isomorphic to right ideals in prime factor rings of R. We now prove this for all right R-modules.

Theorem 9.6. *Let R be a right FBN ring and A a nonzero finitely generated right R-module. Then A has a prime series*

$$A_0 = 0 < A_1 < A_2 < \cdots < A_n = A$$

with corresponding prime ideals $P_i = \mathrm{ann}_R(A_i/A_{i-1})$, such that A_i/A_{i-1} is isomorphic to a uniform right ideal of R/P_i for each $i = 1, \dots, n$. In particular, each A_i/A_{i-1} is a torsionfree right (R/P_i)-module.

Proof. Since A is noetherian, it suffices to show that it has a nonzero submodule B such that $\mathrm{ann}_R(B)$ is a prime ideal of R and B is isomorphic to a uniform right ideal of $R/\mathrm{ann}_R(B)$.

By Corollary 5.19, A has a uniform submodule C. Let P be the assassinator of C, and set $D = \mathrm{ann}_C(P)$. Then P is a prime ideal of R, while D is a fully faithful right (R/P)-module and $P = \mathrm{ann}_R(D)$. Since D is finitely generated and uniform, Corollary 9.3 shows that D is a torsionfree right (R/P)-module. By Lemma 7.17, D has a submodule B isomorphic to a uniform right ideal of R/P. As D is fully faithful, $\mathrm{ann}_R(B) = P$. $\square$

Theorem 9.6 should be compared with Propositions 3.13 and 8.7. In particular, Theorem 9.6 does for right modules over a right FBN ring what Proposition 8.7 does only for bimodules over arbitrary noetherian rings. Of course boundedness is necessary in Theorem 9.6, as Lemma 9.2 shows.

Returning to Corollary 9.5, we observe that if R is a right FBN ring, M a maximal ideal of R, and A a finitely generated faithful right (R/M)-module, then R/M is a simple artinian ring and it embeds in some finite direct sum of copies of A. We shall prove a corresponding statement for faithful modules over arbitrary factor rings of R. As it is much easier for prime factor rings, we do that case first by way of illustration.

Proposition 9.7. *Let R be a prime right Goldie ring and A a finitely generated faithful right R-module. If R is right bounded, then R_R embeds in some finite direct sum of copies of A.*

Proof. By Lemma 9.2, A is not torsion. Then, by Lemma 7.17, A has a uniform submodule U which is isomorphic to a right ideal of R. Now, Corollary 7.25 shows that R_R embeds in U^n for some $n \in \mathbb{N}$, whence R_R embeds in A^n. $\square$

Lemma 9.8. *Let R be a prime right noetherian ring and A a finitely generated right R-module. If $A_0 > A_1 > \cdots > A_n$ is a chain of submodules of A such that each A_{i-1}/A_i is nonzero and torsionfree, then $n \leq \operatorname{rank}(A)$.*

Proof. Since A_{i-1}/A_i is not a torsion R-module, $A_i \not\leq_e A_{i-1}$, and so there is a nonzero submodule $B_i \leq A_{i-1}$ such that $B_i \cap A_i = 0$. Then $B_1 \oplus \cdots \oplus B_n \leq A$. $\square$

Proposition 9.9. [Cauchon] *Let A be a finitely generated right module over a right FBN ring R. Then R satisfies the DCC on right annihilators of subsets of A.*

Proof. If the proposition fails, we work with a minimal criminal A/B, i.e., we choose a submodule B of A maximal with respect to the property that the proposition fails for A/B. Thus, after replacing A by A/B, we may assume that, while R does not satisfy the DCC for right annihilators of subsets of A, it does satisfy the DCC for right annihilators of subsets of any fixed proper factor of A.

Choose subsets $X_1, X_2, \cdots \subseteq A$ such that $\operatorname{ann}_R(X_1) > \operatorname{ann}_R(X_2) > \cdots$. For each k, choose an element $r_k \in \operatorname{ann}_R(X_k) \setminus \operatorname{ann}_R(X_{k+1})$, and then choose an element $x_k \in X_{k+1}$ such that $x_k r_k \neq 0$. Now set $I_k = \operatorname{ann}_R(\{x_1, \ldots, x_k\})$. Since $r_k \in \operatorname{ann}_R(X_k) < \operatorname{ann}_R(X_{k-1}) < \cdots < \operatorname{ann}_R(X_1)$, we have $r_k \in I_{k-1} \backslash I_k$ for $k > 1$, and thus $I_1 > I_2 > \cdots$.

By Theorem 9.6, A contains a uniform submodule U which is isomorphic to a right ideal of R/P for some prime ideal P in R. Because A is a minimal criminal, R satisfies the DCC for right annihilators of subsets of A/U, and so there exists an index n such that

$$\operatorname{ann}_R(\{x_1 + U, \ldots, x_k + U\}) = \operatorname{ann}_R(\{x_1 + U, \ldots, x_n + U\})$$

for all $k \geq n$. In particular, it follows that $\operatorname{ann}_R(x_k + U) \geq I_n$ for all k, that is, $x_k I_n \leq U$ for all k. Note that $x_k I_n P \leq UP = 0$ for all k. Hence, if $I_\infty = \bigcap_{k=1}^{\infty} I_k$, then $I_n P \leq I_\infty$, so that I_n/I_∞ is a (finitely generated) right (R/P)-module.

Now, for any $k \geq n$, left multiplication by x_{k+1} defines a homomorphism $I_k \to U$ with kernel I_{k+1}, and so I_k/I_{k+1} is isomorphic to a right ideal of R/P. Thus, I_k/I_{k+1} is a torsionfree right (R/P)-module. However, since $I_n/I_\infty > I_{n+1}/I_\infty > \cdots$, this contradicts Lemma 9.8.

Therefore the proposition holds. $\square$

Theorem 9.10. [Cauchon] *Let A be a finitely generated right module over a right FBN ring R. Then there exists a finite subset $X \subseteq A$ such that* $\operatorname{ann}_R(X) = \operatorname{ann}_R(A)$. *Consequently, $R/\operatorname{ann}_R(A)$ is isomorphic (as a right R-module) to a submodule of some finite direct sum of copies of A.*

Proof. In view of Proposition 9.9, there exists a finite subset $X \subseteq A$ such that $\operatorname{ann}_R(X)$ is minimal among annihilators of finite subsets of A. Hence, $\operatorname{ann}_R(X) = \operatorname{ann}_R(X \cup \{x\})$ for all $x \in A$, and therefore $\operatorname{ann}_R(X) = \operatorname{ann}_R(A)$. Finally, if $X = \{x_1, \ldots, x_n\}$, then the map $r \mapsto (x_1, \ldots, x_n)r$ defines a homomorphism $R_R \to A^n$ with kernel $\operatorname{ann}_R(A)$. $\square$

Exercise 9G. If R is a right FBN ring, show that all factor rings of R are right bounded. [Hint: Exercise 5D.] $\square$

Exercise 9H. Let R be a right FBN ring, P a prime ideal of R, and A a finitely generated faithful right R-module. Show that A has a subfactor isomorphic to a uniform right ideal of R/P. [Hint: The Schreier Refinement Theorem.] $\square$

Corollary 9.11. *Let R be a right FBN ring and A a finitely generated right R-module. If A is artinian, then $R/\operatorname{ann}_R(A)$ is right artinian.* $\square$

Exercise 9I. Show that a right noetherian ring R is right fully bounded if and only if every finitely generated right R-module A has a finite subset X such that $\operatorname{ann}_R(X) = \operatorname{ann}_R(A)$. [This condition is sometimes called *Condition (H)* in the literature.] $\square$

• ARTINIAN MODULES •

Theorem 9.12. [Jategaonkar, Schelter] *Let R be a left noetherian right FBN ring and A a finitely generated right R-module. If A has an essential artinian submodule, then A is artinian.*

Proof. Without loss of generality, we may assume that A is faithful. Then, by Theorem 9.10, R_R embeds in A^n for some $n \in \mathbb{N}$. As A has an essential artinian submodule, so do A^n and R_R. As a result, every nonzero right ideal of R has a simple submodule, whence $\operatorname{soc}(R_R)$ is an essential right ideal of R. By Theorem 8.16, R is artinian, and therefore A is artinian. $\square$

Without the fully bounded hypothesis, Theorem 9.12 holds for right ideals of R (Exercise 8K) but not for arbitrary modules, as the following example shows.

Exercise 9J. Let $R = k[x][y; x(d/dx)]$, where k is a field of characteristic zero. Set $B = Ry/R(x-1)y$ and $A = R/R(x-1)y$, and observe from Exercise 3O that B is a simple left R-module. Show that $B \leq_e A$ but that A is not artinian. [Hints: (a) If $r \in R \setminus Ry$, then $r = sy + f$ for some $s \in R$ and some nonzero $f \in k[x]$. Set $\deg(f) = n$, and show that

$$(y-n)(y-n+1)\cdots(y-1)yr = \big((y-n)t + \alpha x^n\big)y$$

for some $t \in R$ and $\alpha \in k^\times$. (b) Observe that $R = (y-n)R \oplus k[x]$, and use this to see that $(y-n)t + \alpha x^n \notin R(x-1)$. (c) Observe that $R/Ry \cong k[x]$ as $k[x]$-modules, and then show that R/Ry is not an artinian R-module.] $\square$

Exercise 9K. Show that the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right FBN, and that R has a right ideal which has an essential simple submodule but is not artinian. $\square$

Exercise 9L. If R is a left noetherian right FBN ring and A an artinian right R-module, show that $E(A) = \bigcup_{n=1}^{\infty} \operatorname{soc}^n(E(A))$. $\square$

A special case of the Krull Intersection Theorem says that, in any commutative noetherian ring R, the intersection of the powers of the Jacobson radical is zero, that is, $\bigcap_{n=1}^{\infty} J(R)^n = 0$. That this should hold for any noetherian ring R is known as *Jacobson's Conjecture*, and we now verify it for FBN rings. Jacobson's Conjecture fails for one-sided noetherian rings, as Exercise 9M shows.

Theorem 9.13. [Cauchon, Jategaonkar, Schelter] *If R is a left noetherian right FBN ring, then*

$$\bigcap_{n=1}^{\infty} J(R)^n = 0.$$

Proof. Let $J = \bigcap_{n=1}^{\infty} J(R)^n$. If $J \neq 0$, then J_R has a maximal proper submodule, and so there is a simple right R-module A which is an epimorphic image of J_R. Since an epimorphism of J_R onto A extends to a homomorphism $R_R \to E(A)$, there exists $x \in E(A)$ such that $xJ = A$.

Note that $A \leq xR$. Thus, xR has an essential simple submodule, and so, by Theorem 9.12, xR is artinian. Consequently, xR has finite length, whence $(xR)J(R)^n = 0$ for some $n \in \mathbb{N}$ (because $J(R)$ annihilates each composition factor of xR). But then $xJ = 0$, contradicting the fact that $xJ = A$. Therefore $J = 0$. $\square$

Exercise 9M. Let $R = \begin{pmatrix} S & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, where S is the subring of $\mathbb{Q}$ consisting of all rational numbers with odd denominators. Show that R is a right FBN ring, $J(R) = \begin{pmatrix} 2S & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, and $\bigcap_{n=1}^{\infty} J(R)^n = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. $\square$

• UNIFORM INJECTIVE MODULES •

Consider trying to classify the uniform injective right modules over a right noetherian ring R by analogy with the commutative case (Proposition 5.28). Given a prime ideal P in R, the module $(R/P)_R$ need not be uniform (unless R/P is a domain), and so $E((R/P)_R)$ need not be uniform. To obtain a uniform injective module, we take a uniform right ideal $U \leq R/P$ and use $E(U_R)$, which by Lemma 5.27 is a uniform injective with assassinator P. In case R is right fully bounded, a prime ideal of R cannot appear as the assassinator of two nonisomorphic uniform injective right R-modules, as follows.

Proposition 9.14. *Let R be a right FBN ring. If E is a uniform injective right R-module and P its assassinator, then $E \cong E(U_R)$ for every uniform right ideal U of R/P. Consequently, every uniform injective right R-module with assassinator P is isomorphic to E.*

Proof. Choose a nonzero finitely generated submodule $A \leq \operatorname{ann}_E(P)$. Then A is a faithful uniform right (R/P)-module. By Corollary 9.3, A is torsionfree over R/P. If U is any uniform right ideal of R/P, then, by Lemma 7.22, U is isomorphic to a submodule of A, whence $E(U_R)$ is isomorphic to a direct summand of E. As E is uniform, $E(U_R) \cong E$. $\square$

Proposition 9.14 may fail if R has fewer prime ideals than uniform injective modules, as in Exercise 5Z. In fact, the only right noetherian rings for which Proposition 9.14 holds are the right fully bounded ones, as follows.

Theorem 9.15. [Gordon-Robson, Krause, Lambek-Michler] *Let R be a right noetherian ring. Then R is right fully bounded if and only if the rule*

$$E \longmapsto (\textit{assassinator of } E)$$

yields a bijection between the isomorphism classes of uniform injective right R-modules and the prime ideals of R.

Proof. If R is right fully bounded, Lemma 5.27 and Proposition 9.14 show that the given rule yields a bijection.

If R is not right fully bounded, there is a prime ideal P of R such that R/P is not right bounded. Then R has a right ideal $I > P$ such that I/P is an essential right ideal of R/P but I/P contains no nonzero ideals of R/P. By noetherian induction, we may assume that I is maximal with respect to the property that I/P contains no nonzero ideals of R/P. Since $\operatorname{ann}_{R/P}(R/I)$ is an ideal of R/P contained in I/P, it must be zero. Thus R/I is a faithful right (R/P)-module.

We claim that R/I is actually a fully faithful (R/P)-module, and that R/I is uniform.

Any nonzero submodule of R/I has the form J/I for some right ideal $J > I$. By the maximality of I, the factor J/P must contain a nonzero ideal of R/P, whence $\operatorname{ann}_R(R/J) > P$. As

$$\operatorname{ann}_R(R/J)\operatorname{ann}_R(J/I) \subseteq \operatorname{ann}_R(R/I) = P,$$

we obtain $\operatorname{ann}_R(J/I) = P$, and so J/I is a faithful (R/P)-module, as claimed.

If J and K are right ideals of R properly containing I, then, as above, $\operatorname{ann}_R(R/J) > P$ and $\operatorname{ann}_R(R/K) > P$. Then

$$P < \operatorname{ann}_R(R/J) \cap \operatorname{ann}_R(R/K) = \operatorname{ann}_R(R/(J \cap K)),$$

whence $J \cap K > I$. Thus $(J/I) \cap (K/I) \neq 0$, proving that R/I is uniform, as claimed.

Now set $E = E((R/I)_R)$, which is a uniform injective right R-module. Since R/I is a fully faithful (R/P)-module, the assassinator of E must be P.

Choose a uniform right ideal U in R/P and set $E' = E(U_R)$. By Lemma 5.27, E' is a uniform injective right R-module with assassinator P. Since R/I is torsion over R/P while U is torsionfree, R/I cannot be isomorphic to a submodule of E'. Therefore $E \not\cong E'$. $\square$

Exercise 9N. Let $R = A_n(k) = k[y_1, \dots, y_n][x_1, \dots, x_n; \partial/\partial y_1, \dots, \partial/\partial y_n]$ for some $n \in \mathbb{N}$ and some field k of characteristic zero. For $j = 0, 1, \dots, n$, set $B_j = R/(y_1R + \cdots + y_jR)$. Show that $E(B_0), \dots, E(B_n)$ are pairwise nonisomorphic uniform injective right R-modules with the same assassinator. [Hint: Show that B_j is torsionfree as a module over $k[y_{j+1}, \dots, y_n]$.] $\square$

Exercise 9O. Let $R = A_1(k) = k[y][x; d/dy]$ for some field k of characteristic zero. For each $\alpha \in k$, set $B_\alpha = R/(y - \alpha)R$. Show that the modules $E(B_\alpha)$ are pairwise nonisomorphic uniform injective right R-modules with the same assassinator. [Hint: Show that the B_α are pairwise nonisomorphic simple R-modules.] $\square$

• NOTES •

DCC on Annihilators of Subsets of Modules over FBN Rings. Proposition 9.9 was first proved by Cauchon [1976, Théorème II 7] for a right "T-ring," meaning a right noetherian ring over which nonisomorphic uniform injective right modules have distinct assassinators. (By Theorem 9.15, the right T-rings are exactly the right FBN rings.)

Finite Annihilation for Modules over FBN Rings. Theorem 9.10 was first proved by Cauchon for a right T-ring [1973, Proposition 2; 1976, Théorème II 8, p. 25].

Essential Extensions of Artinian Modules over FBN Rings. Theorem 9.12 was proved first by Jategaonkar for a right and left FBN ring [1973, Theorem 8; 1974b, Corollary 3.6] and then by Schelter for a left noetherian right FBN ring in which all left primitive factor rings are artinian [1975, Theorem].

Jacobson's Conjecture for FBN Rings. This was proved by Cauchon for a right and left T-ring [1974, Théorème 5; 1976, Théorème I 2, p. 36], by Jategaonkar for a right and left FBN ring [1973, Theorem 8; 1974b, Theorem 3.7], and by Schelter for a left noetherian right FBN ring in which all left primitive factor rings are artinian [1975, Corollary].

Assassinators of Uniform Injectives over FBN Rings. Theorem 9.15 was proved independently by Gordon and Robson [1973, Theorem 8.6 and Corollary 8.11] and Krause [1972, Theorem 3.5]. That assassinators yield a bijection between uniform injectives and primes over a right FBN ring was also proved by Lambek and Michler [1973, Corollary 3.12].

10. Rings and Modules of Fractions

In the study of commutative rings, one meets rings of fractions early on, first as quotient fields of integral domains but later as a general construction method. Given a commutative ring R and a subset X of R, we want to find a "larger" ring in which the elements of X become units. First of all, since all products of elements of X would necessarily become units in the new ring, we may enlarge X and assume that X is multiplicatively closed and that $1 \in X$. We then build a new ring RX^{-1} (often written R_X in the commutative literature, but this notation can cause confusion later) as a set of fractions r/x, where $r \in R$ and $x \in X$. There must be an equivalence relation on these fractions, and the situation is made slightly more complex by the fact that X may contain zero-divisors, in which case the map $R \to RX^{-1}$ taking r to $r/1$ is not injective. The correct equivalence relation turns out to be the following: We say that r/x and r'/x' define the same element of RX^{-1} if and only if $(rx' - r'x)y = 0$ for some $y \in X$. Some easy calculations show that we can define a ring RX^{-1} in this way. If we do, and if $\phi : R \to RX^{-1}$ is the ring homomorphism taking r to $r/1$, then we have the following conclusions:

(i) For each $x \in X$, the element $\phi(x)$ is a unit of RX^{-1}.
(ii) Each element of RX^{-1} has the form $a/x = \phi(a)\phi(x)^{-1}$ for some $a \in R$ and $x \in X$.
(iii) The kernel of ϕ is the ideal $\{r \in R \mid rx = 0 \text{ for some } x \in X\}$.
(iv) If $\psi : R \to T$ is any ring homomorphism (where T is a commutative ring) such that $\psi(x)$ is a unit of T for each $x \in X$, then ψ factors uniquely through ϕ, that is, there is a unique ring homomorphism $\eta : RX^{-1} \to T$ such that $\psi = \eta\phi$.

Extending this simple idea to the noncommutative case has turned out not to be so simple. The general question was first considered by Ore, who investigated whether a noncommutative integral domain R can be embedded in a division ring D such that every element of D is of the form rx^{-1} for some $r, x \in R$ with $x \neq 0$. We have seen his solution to this problem in Theorem 6.8 – a division ring D as described exists if and only if any two nonzero elements of R have a nonzero common right multiple. As we have also seen, a similar condition controls the existence of a ring of fractions with respect to a multiplicative set of regular elements (Theorem 6.2). One of our tasks in the present chapter is to see what conditions will allow us to construct a ring of fractions with respect to a multiplicative set that may contain zero-divisors. We close these introductory remarks by reminding the reader that a right noetherian domain always satisfies Ore's condition (Corollary 6.7). (This fact was not noticed for many years, presumably because the importance of the noetherian condition had not become clear.) That arbitrary domains (even principal left ideal domains!) need not satisfy Ore's condition (on the right) was observed in Exercise 6E.

• RINGS OF FRACTIONS •

Given any set X of elements in a ring R, one can look for a ring homomorphism $\phi : R \to S$ such that $\phi(x)$ is a unit of S for each $x \in X$, and such that any such ring homomorphism factors through ϕ. It is not hard to show that one can always do this, but we will not, because the resulting ring S is often quite unreasonable, as the following example shows.

Exercise 10A. Let V be an infinite-dimensional vector space over a field, R the ring of all linear transformations on V, and X the set of all surjective linear transformations in R. First note that the elements of X are already right invertible in R. If $\phi : R \to S$ is a ring homomorphism such that $\phi(x)$ is a unit of S for all $x \in X$, show that $S = 0$. $\square$

We are thus led to look for conditions on X that will lead to a useful construction of a ring in which the elements of X can be inverted. We start by specifying that we would like our end product to resemble the rings of fractions constructed in the commutative case. Thus, we extend the concept of a ring of fractions, as introduced in Chapter 6, to the case where the multiplicative set may contain zero-divisors. In this case, the resulting ring of fractions will only contain a factor ring of the original ring as a subring. To avoid cumbersome expressions involving fractions of cosets, we express the new concept in terms of a ring homomorphism.

Recall the convention (from Chapter 4) that a multiplicative set is always assumed to contain 1 (in addition to being multiplicatively closed).

Definition. Let R be a ring and $X \subseteq R$ a multiplicative set. A *right ring of fractions* (or *right Ore quotient ring*, or *right Ore localization*) *for* R *with respect to* X is a ring homomorphism $\phi : R \to S$ such that:

(a) $\phi(x)$ is a unit of S for all $x \in X$.

(b) Each element of S has the form $\phi(a)\phi(x)^{-1}$ for some $a \in R$ and $x \in X$.

(c) $\ker(\phi) = \{r \in R \mid rx = 0 \text{ for some } x \in X\}$.

When X is a right Ore set (as we will see shortly that it must), condition (c) just says that the kernel of ϕ equals the X-torsion submodule of R_R. By abuse of notation, we usually refer to S as the right ring of fractions, and later we will write elements of S in the form rx^{-1} for $r \in R$, $x \in X$. A *left ring of* *fractions for R with respect to X* is defined symmetrically; in particular, in a left ring of fractions the denominators are all written on the left.

Note that if $\phi : R \to S$ is a right ring of fractions for R with respect to X, then S must also be a right ring of fractions for the subring $\phi(R)$ with respect to the multiplicative set $\phi(X)$ in the sense of Chapter 6. Thus, by Lemma 6.1, $\phi(X)$ must be a right Ore set in $\phi(R)$. A similar argument shows that, in fact, X must be a right Ore set in R, as follows.

Lemma 10.1. *Let X be a multiplicative set in a ring R, and assume that there exists a right ring of fractions $\phi : R \to S$ with respect to X.*

(a) *X is a right Ore set in R.*

(b) *If $r \in R$ and $x \in X$ such that $xr = 0$, then there exists $x' \in X$ such that $rx' = 0$.*

(c) *Let $a, b \in R$ and $x, y \in X$. Then $\phi(a)\phi(x)^{-1} = \phi(b)\phi(y)^{-1}$ if and only if there exist $c \in R$ and $z \in X$ such that $az = bc$ and $xz = yc$.*

Proof. (a) Let $a \in R$ and $x \in X$. The element $\phi(x)^{-1}\phi(a)$ in S must have the form $\phi(b)\phi(y)^{-1}$ for some $b \in R$ and $y \in X$. Then $\phi(a)\phi(y) = \phi(x)\phi(b)$, and so $\phi(ay - xb) = 0$, whence $(ay - xb)z = 0$ for some $z \in X$. Thus $ayz = xbz$ with $yz \in X$.

(b) Since $\phi(x)\phi(r) = 0$ and $\phi(x)$ is invertible, $\phi(r) = 0$.

(c) If there exist $c \in R$ and $z \in X$ such that $az = bc$ and $xz = yc$, then $xz = yc \in X$, whence $\phi(a)\phi(x)^{-1} = \phi(az)\phi(xz)^{-1} = \phi(bc)\phi(yc)^{-1} = \phi(b)\phi(y)^{-1}$. Conversely, suppose that $\phi(a)\phi(x)^{-1} = \phi(b)\phi(y)^{-1}$. By the right Ore condition (part (a)), there exist $s \in R$ and $u \in X$ such that $xu = ys$. Then

$$\phi(au)\phi(xu)^{-1} = \phi(a)\phi(x)^{-1} = \phi(b)\phi(y)^{-1} = \phi(bs)\phi(ys)^{-1} = \phi(bs)\phi(xu)^{-1},$$

whence $\phi(au) = \phi(bs)$. Consequently, $(au - bs)v = 0$ for some $v \in X$. Therefore $auv = bsv$ and $xuv = ysv$ with $uv \in X$. $\square$

One can also prove a symmetric version of part (c) of the lemma, matching Lemma 6.1(c): $\phi(a)\phi(x)^{-1} = \phi(b)\phi(y)^{-1}$ if and only if there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$.

Definition. Let X be a multiplicative set in a ring R. Then X is *right reversible* if and only if X satisfies condition (b) of Lemma 10.1. A *right denominator set* is any right reversible right Ore set. Following the usual convention, a *denominator set* is any right and left denominator set. We shall prove that right rings of fractions exist precisely for right denominator sets (Theorem 10.3).

Exercise 10B. Show that the multiplicative set X in Exercise 10A is right Ore but not right reversible. $\square$

Exercise 10C. Show that in a ring with no nonzero nilpotent elements, every right Ore set is right reversible. □

We show later (Proposition 10.7) that, in a right noetherian ring, every right Ore set is right reversible.

Lemma 10.2. *Let X be a right Ore set in a ring R. Suppose that $\phi : R \to T$ is a ring homomorphism such that $\phi(x)$ is a unit of T for all $x \in X$.*

(a) *Given any $r_1, \dots, r_n \in R$ and $x_1, \dots, x_n \in X$, there exist $s_1, \dots, s_n \in R$ and $y \in X$ such that $\phi(r_i)\phi(x_i)^{-1} = \phi(s_i)\phi(y)^{-1}$ for all i (i.e., the "fractions" $\phi(r_i)\phi(x_i)^{-1}$ have a "common denominator" $\phi(y)$).*

(b) *The set $S = \{\phi(r)\phi(x)^{-1} \mid r \in R \text{ and } x \in X\}$ is a subring of T.*

Proof. (a) By Lemma 4.21, there exist $y \in X$ and $a_1, \dots, a_n \in R$ such that $y = x_i a_i$ for each i. Note from the equations $\phi(y) = \phi(x_i)\phi(a_i)$ that each $\phi(a_i)$ is a unit in T. Then, if $s_i = r_i a_i$ for each i, we obtain

$$\phi(s_i)\phi(y)^{-1} = \phi(r_i)\phi(a_i)\phi(a_i)^{-1}\phi(x_i)^{-1} = \phi(r_i)\phi(x_i)^{-1}.$$

(b) Obviously $1 = \phi(1)\phi(1)^{-1} \in S$, and it is clear from (a) that S is closed under addition and subtraction. Given $r_1, r_2 \in R$ and $x_1, x_2 \in X$, we may use the right Ore condition to obtain $r_2 y = x_1 s$ for some $y \in X$ and $s \in R$. Then $\phi(r_2)\phi(y) = \phi(x_1)\phi(s)$, and so $\phi(x_1)^{-1}\phi(r_2) = \phi(s)\phi(y)^{-1}$. Hence,

$$\big(\phi(r_1)\phi(x_1)^{-1}\big)\big(\phi(r_2)\phi(x_2)^{-1}\big) = \phi(r_1)\phi(s)\phi(y)^{-1}\phi(x_2)^{-1} = \phi(r_1 s)\phi(x_2 y)^{-1},$$

which lies in S. Therefore S is closed under multiplication. □

Let us now consider the construction of a right ring of fractions for a ring R with respect to a right denominator set X. We need to construct a ring homomorphism $\phi : R \to S$ such that $\ker(\phi) = t_X(R_R)$ and certain other properties hold. Because of the right Ore condition, $t_X(R_R)$ is an ideal of R, so a natural first step is to form the factor ring $\widehat{R} = R/t_X(R_R)$. The images of the elements of X form a multiplicative set $\widehat{X} \subseteq \widehat{R}$, and it is clear that $\widehat{X}$ satisfies the right Ore condition. Since $\widehat{R}$ is X-torsionfree as a right R-module, the elements of $\widehat{X}$ are right non-zero-divisors in $\widehat{R}$; that they are also left non-zero-divisors follows from the right reversibility of X. Thus $\widehat{X}$ consists of regular elements, and so we have a right ring of fractions $\widehat{R}\widehat{X}^{-1}$. All that remains is to note that the obvious map $R \to \widehat{R} \to \widehat{R}\widehat{X}^{-1}$ satisfies the desired properties for a ring of fractions.

Theorem 10.3. [Gabriel] *Let X be a multiplicative set in a ring R. Then there exists a right ring of fractions for R with respect to X if and only if X is a right denominator set.*

Proof. The necessity is given by Lemma 10.1. Conversely, assume that X is a right denominator set; we fill in the details of the construction outlined above.

Since X is a right Ore set, $t_X(R_R)$ is a right ideal of R (Lemma 4.21) and the factor $\widehat{R} = R/t_X(R_R)$ is an X-torsionfree right R-module (Lemma 4.22). It is clear that $t_X(R_R)$ is an ideal of R, so that $\widehat{R}$ is actually a ring. Let $\phi : R \to \widehat{R}$ denote the quotient map, set $\widehat{X} = \phi(X)$, and observe that $\widehat{X}$ is a right Ore set in $\widehat{R}$. If $r \in R$ and $x \in X$ such that $\phi(r)\phi(x) = 0$, then $\phi(r) = 0$ because $\widehat{R}_R$ is X-torsionfree. On the other hand, if $\phi(x)\phi(r) = 0$, then $xry = 0$ for some $y \in X$. In this case, by right reversibility there exists $x' \in X$ such that $ryx' = 0$, and again $\phi(r) = 0$. Therefore $\widehat{X}$ consists of regular elements of $\widehat{R}$.

By Theorem 6.2, there exists a right ring of fractions $S = \widehat{R}\widehat{X}^{-1}$. We now view ϕ as a ring homomorphism $R \to S$ and tick off the required properties: Each element of $\phi(X) = \widehat{X}$ is invertible in S; each element of S can be expressed in the form $\phi(a)\phi(x)^{-1}$ for some $\phi(a) \in \widehat{R}$ and $\phi(x) \in \widehat{X}$; and $\ker(\phi) = t_X(R_R)$. Therefore $\phi : R \to S$ is a right ring of fractions for R with respect to X. $\square$

Proposition 10.4. *Let $\phi : R \to S$ be a right ring of fractions for a ring R with respect to a right denominator set X. If $\psi : R \to T$ is any ring homomorphism such that $\psi(x)$ is invertible in T for each $x \in X$, then there exists a unique ring homomorphism $\eta : S \to T$ such that $\psi = \eta\phi$.*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[dr, "\psi"'] & S \arrow[d, dashed, "\eta"] \\ & T \end{tikzcd}$$

Proof. Clearly η must be unique if it exists, since it must satisfy

$$\eta\big(\phi(a)\phi(x)^{-1}\big) = \psi(a)\psi(x)^{-1}$$

for all $a \in R$ and $x \in X$.

If $r \in \ker(\phi)$, then $rx = 0$ for some $x \in X$, and so $\psi(r)\psi(x) = 0$. As $\psi(x)$ is invertible in T, it follows that $\psi(r) = 0$. Thus $\ker(\phi) \subseteq \ker(\psi)$, whence there exists a ring homomorphism $\eta_0 : \phi(R) \to T$ such that $\psi = \eta_0\phi$. As we have observed before, S must be a right ring of fractions for $\phi(R)$ with respect to $\phi(X)$ in the sense defined in Chapter 6. By Proposition 6.3, η_0 extends (uniquely) to a ring homomorphism $\eta : S \to T$, and the proof is complete. $\square$

Corollary 10.5. *Let X be a right denominator set in a ring R, and suppose that $\phi_1 : R \to S_1$ and $\phi_2 : R \to S_2$ are right rings of fractions for R with respect to X. Then there is a (unique) ring isomorphism $\eta : S_1 \to S_2$ such that $\eta\phi_1 = \phi_2$.*

Proof. This follows in the standard way from the universal property proved in Proposition 10.4. $\square$

Definition. Given a right denominator set X in a ring R, we now know that there exists a right ring of fractions $\phi : R \to S$ for R with respect to X. Because of the uniqueness given by Corollary 10.5, we shall denote S by RX^{-1}, and we shall refer to ϕ as *the natural map* from R to RX^{-1}. (When the ring RX^{-1} is called an *Ore localization* of R, the map ϕ is referred to as the *localization map*.) In the case of a left ring of fractions for R with respect to a left denominator set Y, we use the notation $Y^{-1}R$.

Given a right ring of fractions $\phi : R \to RX^{-1}$, it is cumbersome to use the precise notation $\phi(r)\phi(x)^{-1}$ for elements of RX^{-1}. Henceforth, we shall write elements of RX^{-1} in the form rx^{-1}, even when $t_X(R) \neq 0$, unless there is more than one ring of fractions under discussion. To avoid confusion in case $t_X(R) \neq 0$, the image in RX^{-1} of an element $r \in R$ should be denoted $r1^{-1}$, which emphasizes that it is being viewed as a fraction. (Remember that $r1^{-1} = 0$ if and only if $rx = 0$ for some $x \in X$.) There should be no confusion if for an element $x \in X$ the inverse of $x1^{-1}$ in RX^{-1} is denoted just x^{-1}, rather than $1x^{-1}$. Analogous notation ($x^{-1}r$, $1^{-1}r$, etc.) will of course be used in left rings of fractions.

Proposition 10.6. *If X is a right and left denominator set in a ring R, then $RX^{-1} = X^{-1}R$, that is, any right (left) ring of fractions for R with respect to X is also a left (right) ring of fractions for R with respect to X.*

Proof. Suppose that $\phi : R \to S$ is a right ring of fractions for R with respect to X. Then all elements of $\phi(X)$ are invertible in S, and since X is both right and left reversible, we see that $\ker(\phi) = t_X(R_R) = t_X({}_RR)$. That all elements of S have the form $\phi(y)^{-1}\phi(b)$ for $y \in X$ and $b \in R$ follows from the left Ore condition just as in Proposition 6.5. Therefore ϕ is a left ring of fractions for R with respect to X. $\square$

Because of Proposition 10.6, we may refer to *the* ring of fractions of a ring R with respect to a right and left denominator set. Of course, not all right denominator sets are also left denominator sets, as, for instance, the set of nonzero elements in a right Ore domain which is not left Ore.

Exercise 10D. Let $X = \{1, x, x^2, \dots\} \subseteq R[x;\alpha]$ for a ring R and an automorphism α of R. By Exercise 4O, X is an Ore set, and since it consists of regular elements, it must be a denominator set. Show that $R[x^{\pm1};\alpha] = R[x;\alpha]X^{-1}$. $\square$

We close this section by showing that in noetherian rings only the Ore condition needs to be checked, since then reversibility comes for free.

Proposition 10.7. *Let X be a right Ore set in a ring R. If R has the ACC on right annihilators of single elements, then X is right reversible.*

Proof. Consider $r \in R$ and $x \in X$ such that $xr = 0$. Using the ACC on right annihilators of elements, there exists a positive integer n such that $\mathrm{r.ann}(x^n) =$

r.ann(x^{n+1}). From the right Ore condition, there exist $b \in R$ and $y \in X$ such that $ry = x^n b$. As $x^{n+1}b = xry = 0$, we have

$$b \in \text{r.ann}(x^{n+1}) = \text{r.ann}(x^n),$$

and therefore $ry = x^n b = 0$. $\square$

Exercise 10E. Let X be a right Ore set in a right noetherian ring R and $I \supseteq J$ ideals of R. If $(I/J)_R$ is X-torsionfree, show that ${}_R(I/J)$ is X-torsionfree. (Recall that the concept of an X-torsionfree module was defined for arbitrary multiplicative sets X in Chapter 4.) $\square$

• MODULES OF FRACTIONS •

Having constructed the right ring of fractions RX^{-1} given a right denominator set X in a ring R, we turn to the construction of an analogous module of fractions AX^{-1} for each right R-module A. Of course, the elements of AX^{-1} should resemble fractions with numerators from A and denominators from X, the kernel of the map from A to AX^{-1} should be $t_X(A)$, and this map $A \to AX^{-1}$ should have a suitable universal property with respect to homomorphisms into RX^{-1}-modules. As with rings of fractions, we begin by specifying the minimal desired properties.

Definition. Let X be a right denominator set in a ring R and A a right R-module. A *module of fractions for A with respect to X* consists of a right RX^{-1}-module B together with an R-module homomorphism $\psi : A \to B$ such that:

(a) Each element of B has the form $\psi(a)x^{-1}$ for some $a \in A$ and $x \in X$.
(b) $\ker(\psi) = t_X(A)$.

With the usual abuse of notation, we often refer to B as the module of fractions.

The original construction of Ore and Asano for rings of fractions (outlined in Chapter 6) can be easily adapted to give a construction for a module of fractions for A, as a set of equivalence classes for a suitable equivalence relation on $A \times X$. The verifications involved are again straightforward but tedious. Let us instead adapt the method used in the proof of Theorem 6.2.

Recall that any module is isomorphic to a module of homomorphisms. In particular, for any RX^{-1}-module B, we have $\text{Hom}_{RX^{-1}}(RX^{-1}, B) \cong B$ via the map $f \mapsto f(1)$. Now suppose that B is to be a module of fractions for A. Since B must be X-torsionfree, it is easily checked that the RX^{-1}-module homomorphisms from RX^{-1} to B are the same as the R-module homomorphisms. This means that we can work with $\text{Hom}_R(RX^{-1}, B)$, and so we only need to anticipate the R-module structure of B. Because of requirement (b) in the definition above, we see that B should be an extension of $A/t_X(A)$, and since elements of B will eventually take the form of fractions with numerators from A, as an R-module B should be an essential extension of $A/t_X(A)$.

Thus, B should be a submodule of the injective hull of $A/t_X(A)$, and another moment's thought indicates that $B/\big(A/t_X(A)\big)$ should be the X-torsion submodule of $E\big(A/t_X(A)\big)/\big(A/t_X(A)\big)$. With these observations in hand, we are ready to proceed.

Theorem 10.8. *Let R be a ring, $X \subseteq R$ a right denominator set, and A a right R-module. Then there exists a module of fractions for A with respect to X.*

Proof. Set $S = RX^{-1}$ and let $\phi : R \to S$ be the natural map. Note that S can be viewed as a right (or left) R-module via ϕ, that it is X-torsionfree, and that ϕ is a right (or left) R-module homomorphism. Let $A' = A/t_X(A)$ and $E = E(A')$, and let $\pi : A \to A' \le E$ be the quotient map. Note that E is X-torsionfree. Set $B' = \{b \in E \mid bx \in A' \text{ for some } x \in X\}$, so that B' is a submodule of E containing A' and $B'/A' = t_X(E/A')$. Finally, set $B = \operatorname{Hom}_R(S, B')$. There is a standard right S-module structure on B: For $f \in B$ and $s \in S$, the product fs is the composition of f with left multiplication by s, that is, $(fs)(t) = f(st)$ for $t \in S$.

Claim 1. For each $a \in A$, there is a unique homomorphism $\psi_a \in B$ such that $\psi_a(1) = \pi(a)$.

There is a homomorphism $f : R \to A'$ such that $f(1) = \pi(a)$, and since A' is X-torsionfree, $\ker(\phi) = t_X(R_R) \le \ker(f)$. Hence, f induces a homomorphism $g : \phi(R) \to A'$ such that $g\phi = f$. Since E is injective, g extends to a homomorphism $h : S \to E$ such that $h(1) = g\phi(1) = \pi(a)$. For any $s \in S$, we have $sx = \phi(r)$ for some $x \in X$ and $r \in R$, whence $h(s)x = h\phi(r) = f(r) = \pi(a)r \in A'$, and so $h(s) \in B'$. Thus, h maps S to B', that is, $h \in B$. Now consider any $h' \in B$ such that $h'(1) = \pi(a)$. Then $\phi(R) \le \ker(h' - h)$, whence $(h' - h)(S)$ is X-torsion, and so $(h' - h)(S) = 0$. This shows that h is unique, establishing the claim.

Claim 2. The rule $a \mapsto \psi_a$ defines an R-module homomorphism $\psi : A \to B$ with kernel $t_X(A)$.

For any $a, c \in A$ and $r \in R$, we have $(\psi_a + \psi_c)(1) = \pi(a) + \pi(c) = \pi(a + c)$ and $(\psi_a r)(1) = \psi_a(r) = \psi_a(1)r = \pi(ar)$, whence $\psi_a + \psi_c = \psi_{a+c}$ and $\psi_a r = \psi_{ar}$ by uniqueness. Therefore ψ is an R-module homomorphism. If $a \in \ker(\psi)$, then $\pi(a) = \psi_a(1) = 0$ and $a \in t_X(A)$. On the other hand, if $a \in t_X(A)$, then, since $\pi(a) = 0$, we have $\psi_a = 0$ by uniqueness of ψ_a, whence $a \in \ker(\psi)$. Therefore $\ker(\psi) = t_X(A)$.

It remains to show that any homomorphism $f \in B$ has the form $\psi(a)x^{-1}$ for some $a \in A$ and $x \in X$. Since $f(1) \in B'$, there exist $x \in X$ and $a \in A$ such that $f(1)x = \pi(a)$. Then $(fx)(1) = \pi(a)$, and so $fx = \psi_a$ by uniqueness. Since $x1^{-1}$ is invertible in S, we conclude that $f = \psi_a x^{-1} = \psi(a)x^{-1}$. Therefore B is a module of fractions for A with respect to X, as claimed. □

It is possible to obtain the existence of modules of fractions just from the existence of rings of fractions. One way is via the trick of turning a right module into a right ideal of a triangular matrix ring, as in the following exercise. Another way is via tensor products, as we shall see later (Proposition 10.12).

Exercise 10F. Let X be a right denominator set in a ring R and A a right R-module. Set $T = \mathrm{End}_R(A)$, view A as a (T, R)-bimodule, and let $U = \left(\begin{smallmatrix} T & A \\ 0 & R \end{smallmatrix}\right)$. Show that the set $Y = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) \;\middle|\; x \in X \right\}$ is a right denominator set in U and that the set

$$B = \left\{ \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)^{-1} \;\middle|\; a \in A,\, x \in X \right\}$$

is a right ideal of UY^{-1}. Turn B into a right RX^{-1}-module and show that the map $a \mapsto \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$ is a module of fractions for A with respect to X. $\square$

Proposition 10.9. *Let R be a ring, $X \subseteq R$ a right denominator set, A a right R-module, and $\psi : A \to B$ a module of fractions for A with respect to X. If C is a right RX^{-1}-module and $f : A \to C$ an R-module homomorphism, there exists a unique RX^{-1}-module homomorphism $g : B \to C$ such that $f = g\psi$.*

Proof. Observe that C is X-torsionfree and consequently $\ker(f) \geq t_X(A) = \ker(\psi)$. Hence, there is a unique R-module homomorphism $g_0 : \psi(A) \to C$ such that $g_0\psi = f$. Then g_0 extends to an R-module homomorphism $g : B \to E(C_R)$, and of course $g\psi = g_0\psi = f$.

Any $b \in B$ has the form $\psi(a)x^{-1}$ for some $a \in A$ and $x \in X$. Then $bx = \psi(a)$, whence $g(b)x = g\psi(a) = f(a)$. As $f(a)$ lies in the RX^{-1}-module C, there exists $c \in C$ such that $cx = f(a)$, and since $E(C_R)$ is X-torsionfree (because C is), we conclude that $g(b) = c$. Therefore g actually maps B to C.

To show that g is an RX^{-1}-module homomorphism, it suffices to show that

$$g(bx^{-1}) = g(b)x^{-1}$$

for all $b \in B$ and $x \in X$. However, this is immediate from the observation that $g(bx^{-1})x = g(b)$. Finally, the uniqueness of g follows from the fact that it must satisfy $g(\psi(a)x^{-1}) = f(a)x^{-1}$ for all $a \in A$ and $x \in X$. $\square$

Corollary 10.10. *Let R be a ring, $X \subseteq R$ a right denominator set, and A a right R-module. Suppose that $\psi_1 : A \to B_1$ and $\psi_2 : A \to B_2$ are modules of fractions for A with respect to X. Then there is a (unique) RX^{-1}-module isomorphism $g : B_1 \to B_2$ such that $g\psi_1 = \psi_2$.* $\square$

Definition. Given a right denominator set X in a ring R and a right R-module A, we know that there exists a module of fractions $\psi : A \to B$ for A with respect to X. Because of the uniqueness given by Corollary 10.10, we shall denote B by AX^{-1}, and we shall refer to ψ as *the natural map* from A to

AX^{-1}, or the *localization map*. As with rings of fractions, we write elements of AX^{-1} in the form ax^{-1} and elements of $f(A)$ in the form $a1^{-1}$.

Now suppose that C is another right R-module, and let $f \in \mathrm{Hom}_R(A, C)$. In view of Proposition 10.9, there exists a unique right RX^{-1}-module homomorphism $g : AX^{-1} \to CX^{-1}$ such that $g(a1^{-1}) = f(a)1^{-1}$ for all $a \in A$. It follows that $g(ax^{-1}) = f(a)x^{-1}$ for $a \in A$ and $x \in X$. We refer to g as *the map induced by* f, and if a notation for g is needed, we shall write fX^{-1}.

Exercise 10G. Let X be a right denominator set in a ring R. Let A be a right R-module, $a, b \in A$, and $x, y \in X$. Show that $ax^{-1} = by^{-1}$ in AX^{-1} if and only if there exist $r \in R$ and $z \in X$ such that $az = br$ and $xz = yr$. $\square$

Definition. Let X be a multiplicative set in a ring R. We say that a right R-module A is *X-divisible* if and only if $Ax = A$ for all $x \in X$.

Proposition 10.11. *Let X be a right denominator set in a ring R.*

(a) *Every right or left RX^{-1}-module is X-torsionfree and X-divisible as an R-module.*

(b) *If A is an X-torsionfree, X-divisible right R-module, there exists a unique right RX^{-1}-module structure on A compatible with its right R-module structure.*

Proof. (a) This is clear from the fact that $X1^{-1}$ is contained in the set of units of RX^{-1}.

(b) Set $T = \mathrm{End}_{\mathbb{Z}}(A)^{\mathrm{op}}$, and let $\psi : R \to T$ be the ring homomorphism given by the right R-module structure on A, that is, $\psi(r)(a) = ar$ for all $r \in R$ and $a \in A$. A right RX^{-1}-module structure on A is just a ring homomorphism $\eta : RX^{-1} \to T$, and such a structure is compatible with the right R-module structure on A if and only if $\eta\phi = \psi$, where ϕ is the natural map from R to RX^{-1}. Thus, we just need to show that there is a unique ring homomorphism $\eta : RX^{-1} \to T$ such that $\eta\phi = \psi$. However, since A is X-torsionfree and X-divisible, $\psi(x)$ is a unit of T for each $x \in X$. Therefore the existence and uniqueness of η follow from Proposition 10.4. $\square$

Exercise 10H. Let X be a right denominator set in a ring R.

(a) If A is any X-torsionfree, X-divisible right R-module, show that the natural map $A \to AX^{-1}$ is an isomorphism.

(b) If A and B are right (left) RX^{-1}-modules, show that all R-module homomorphisms from A to B are also RX^{-1}-module homomorphisms. $\square$

The following proposition shows that any module of fractions AX^{-1} over a ring of fractions RX^{-1} is naturally isomorphic to $A \otimes_R RX^{-1}$. Hence, an alternative approach to modules of fractions would be to define them as tensor products.

Proposition 10.12. *Let X be a right denominator set in a ring R and A a right R-module. Then the "multiplication map" $A \times RX^{-1} \to AX^{-1}$,* *given by the rule* $(a,s) \mapsto (a1^{-1})s$*, induces an* RX^{-1}*-module isomorphism of* $A \otimes_R RX^{-1}$ *onto* AX^{-1}*.*

Proof. It is clear that the multiplication map induces a group homomorphism

$$g : A \otimes_R RX^{-1} \to AX^{-1}$$

such that $g(a \otimes s) = (a1^{-1})s$ for all $a \in A$ and $s \in RX^{-1}$, and that g is an RX^{-1}-module homomorphism. Any element of AX^{-1} has the form ax^{-1} for some $a \in A$ and $x \in X$, and $ax^{-1} = g(a \otimes x^{-1})$. Thus g is surjective.

Any element $m \in A \otimes_R RX^{-1}$ has the form $(a_1 \otimes s_1) + \cdots + (a_n \otimes s_n)$ for some $a_i \in A$ and $s_i \in RX^{-1}$. Using Lemma 10.2, we can find $r_1, \ldots, r_n \in R$ and $x \in X$ such that each $s_i = r_i x^{-1}$, whence

$$m = (a_1 r_1 + \cdots + a_n r_n) \otimes x^{-1}.$$

Thus, any element $m \in A \otimes_R RX^{-1}$ has the form $a \otimes x^{-1}$ for some $a \in A$ and $x \in X$, and $g(m) = ax^{-1}$. In case $m \in \ker(g)$, we must have $a1^{-1} = (ax^{-1})(x1^{-1}) = 0$, and hence $ay = 0$ for some $y \in X$. Consequently,

$$m = a \otimes x^{-1} = a \otimes (yy^{-1})x^{-1} = ay \otimes (xy)^{-1} = 0.$$

Therefore g is injective. $\square$

Corollary 10.13. *If* X *is a right denominator set in a ring* R*, then* RX^{-1} *is a flat left* R*-module. That is, if* $f : A \to B$ *is any monomorphism of right* R*-modules, then the induced map*

$$f \otimes 1 : A \otimes_R RX^{-1} \to B \otimes_R RX^{-1}$$

is a monomorphism.

Proof. There is a commutative diagram of RX^{-1}-modules and homomorphisms as follows:

$$\begin{tikzcd} A \otimes_R RX^{-1} \arrow[r, "f \otimes 1"] \arrow[d] & B \otimes_R RX^{-1} \arrow[d] \\ AX^{-1} \arrow[r, "fX^{-1}"] & BX^{-1} \end{tikzcd}$$

where the vertical maps are the isomorphisms induced from the multiplication maps as in Proposition 10.12. Hence, to show that $f \otimes 1$ is injective, it suffices to show that fX^{-1} is injective.

Consider an element $m \in \ker(fX^{-1})$. Then $m = ax^{-1}$ for some $a \in A$ and $x \in X$, and $f(a)x^{-1} = 0$ in BX^{-1}. It follows that $f(a)1^{-1} = 0$, and so $f(a)y = 0$ for some $y \in X$. Since f is injective, we obtain $ay = 0$, and thus $m = (ay)(xy)^{-1} = 0$. Therefore fX^{-1} is injective. $\square$

Corollary 10.14. *Let X be a right denominator set in a ring R.*

(a) *Every injective right RX^{-1}-module is X-torsionfree and injective as a right R-module.*

(b) *If A is an X-torsionfree injective right R-module, the natural map $A \to AX^{-1}$ is an R-module isomorphism and AX^{-1} is an injective right RX^{-1}-module.*

Proof. (a) The X-torsionfreeness is clear, and the injectivity follows from Corollary 10.13. (The latter implication is a standard fact: If $\phi : R \to S$ is a ring homomorphism under which S becomes a flat left R-module, then all injective right S-modules are also injective as R-modules. Readers who have not seen this should treat it as an exercise.)

(b) We first observe that A is X-divisible. For if $a \in A$ and $x \in X$, then

$$\text{r.ann}_R(x) \leq t_X(R_R) \leq \text{ann}_R(a)$$

(where the first inclusion holds because X is right reversible). Hence, there is a homomorphism from xR to A sending x to a; by injectivity we obtain $cx = a$ for some $c \in A$. Thus, the natural map $A \to AX^{-1}$ is an isomorphism, by Exercise 10H.

Now $B = AX^{-1}$ is a right RX^{-1}-module which is injective as an R-module. If $C \leq D$ are right RX^{-1}-modules and $g : C \to B$ is an RX^{-1}-module homomorphism, then g extends to an R-module homomorphism $h : D \to B$ and h is an RX^{-1}-module homomorphism by Exercise 10H. Therefore B is injective as an RX^{-1}-module. □

• SUBMODULES OF MODULES OF FRACTIONS •

As in the commutative case, the submodules of a module of fractions AX^{-1} are closely related to the submodules of A. In order to discuss this relationship without the notation obscuring the details, we use the following abbreviations.

Definition. Let X be a right denominator set in a ring R and A a right R-module. If B is a submodule of AX^{-1}, the set $\{a \in A \mid a1^{-1} \in B\}$ is called *the contraction of B to A* and is denoted B^{c}. If C is a submodule of A, the set $\{cx^{-1} \in AX^{-1} \mid c \in C,\ x \in X\}$ is called *the extension of C to AX^{-1}* and is denoted C^{e}.

A priori, we should be concerned about the potential ambiguity of using the same notation cx^{-1} for elements of C^{e} as for elements of CX^{-1}. However, a glance at Exercise 10G shows that two such fractions are equal in C^{e} if and only if they are equal in CX^{-1}, and so there is actually no ambiguity. Another way of presenting this conclusion is given in the following exercise.

Exercise 10I. Let X be a right denominator set in a ring R, and let $C \leq A$ be right R-modules. Show that the composition of the inclusion map $C \to A$ with the natural map $A \to AX^{-1}$ extends uniquely to an RX^{-1}-module isomorphism $CX^{-1} \to C^{\text{e}}$. □

Theorem 10.15. *Let X be a right denominator set in a ring R and A a right R-module.*

(a) *If B is any RX^{-1}-submodule of AX^{-1}, then B^{c} is an R-submodule of A, the factor A/B^{c} is X-torsionfree, and $B = B^{\mathrm{ce}}$.*

(b) *If C is any R-submodule of A, then C^{e} is an RX^{-1}-submodule of AX^{-1} and $C \le C^{\mathrm{ec}}$. Moreover, $C^{\mathrm{ec}}/C = t_X(A/C)$, and so $C = C^{\mathrm{ec}}$ if and only if A/C is X-torsionfree.*

(c) *Contraction and extension provide inverse lattice isomorphisms between the lattice of RX^{-1}-submodules of AX^{-1} and the lattice of those R-submodules C of A such that A/C is X-torsionfree.*

Proof. Let $\psi : A \to AX^{-1}$ be the natural map. Let us say that a submodule $C \le A$ is X-*closed* in A if and only if A/C is X-torsionfree. Observe that any intersection of X-closed submodules of A is X-closed, whence the X-closed submodules of A do form a lattice, with intersections for infima. (The supremum of a family $\{C_i\}$ of X-closed submodules in this lattice is the intersection of all X-closed submodules containing $\sum_i C_i$.)

(a) Obviously $B^{\mathrm{c}} = \psi^{-1}(B)$ is an R-submodule of A. Since ψ induces an R-module embedding of A/B^{c} into the RX^{-1}-module AX^{-1}/B, it is clear that A/B^{c} is X-torsionfree.

As $\psi(B^{\mathrm{c}}) \le B$, we certainly have $B^{\mathrm{ce}} \le B$. Given any $b \in B$, write $b = ax^{-1}$ for some $a \in A$ and $x \in X$. Then $\psi(a) = bx \in B$ and so $a \in B^{\mathrm{c}}$, whence $b \in B^{\mathrm{ce}}$. Thus $B^{\mathrm{ce}} = B$.

(b) Since any pair of elements of RX^{-1} has a common denominator, we see that C^{e} is the RX^{-1}-submodule of AX^{-1} generated by $\psi(C)$. Clearly $C \le C^{\mathrm{ec}}$. Given $a \in C^{\mathrm{ec}}$, we have $a1^{-1} = cx^{-1}$ for some $c \in C$ and $x \in X$. Then $ax1^{-1} = c1^{-1}$ and so $axy = cy$ for some $y \in X$, whence $axy \in C$. This shows that C^{ec}/C is X-torsion. On the other hand, A/C^{ec} is X-torsionfree by (a), and thus $C^{\mathrm{ec}}/C = t_X(A/C)$.

(c) From (a) and (b), we see that the rules $B \mapsto B^{\mathrm{c}}$ and $C \mapsto C^{\mathrm{e}}$ provide inverse bijections between the lattice $\mathcal{L}$ of RX^{-1}-submodules of AX^{-1} and the lattice $\mathcal{L}'$ of X-closed R-submodules of A. As these maps clearly preserve inclusions, they are order-isomorphisms (i.e., isomorphisms of partially ordered sets). Since infima and suprema are defined in terms of the order relations, we conclude that $(-)^{\mathrm{c}}$ and $(-)^{\mathrm{e}}$ are lattice isomorphisms. □

Corollary 10.16. *Let X be a right denominator set in a ring R and A a right R-module.*

(a) *If A is noetherian, or artinian, or of finite length, then so is AX^{-1} (as an RX^{-1}-module).*

(b) *If A is simple, then AX^{-1} is either zero or a simple RX^{-1}-module.*

Proof. (a) The lattice isomorphisms in Theorem 10.15 guarantee that any strictly ascending (descending) chain of RX^{-1}-submodules of AX^{-1} contracts to a strictly ascending (descending) chain of R-submodules of A.

(b) By Theorem 10.15, the only RX^{-1}-submodules of AX^{-1} are 0 and AX^{-1}. $\square$

Exercise 10J. Let X be a right denominator set in a ring R and $C \leq B \leq A$ right R-modules. In the notation of Theorem 10.15, show that $B^{\mathrm{e}}/C^{\mathrm{e}} \cong (B/C)X^{-1}$. [Hint: Exercise 10I.] $\square$

Exercise 10K. Let X be a right denominator set in a ring R and A an X-torsionfree right R-module. Show that the rank of AX^{-1} (as either an R-module or an RX^{-1}-module) is the same as the rank of A. $\square$

• IDEALS IN RINGS OF FRACTIONS •

The results of the previous section (especially Theorem 10.15) give us a good grasp of the submodule structure of a module of fractions. In particular, this applies to the right ideal structure of a right ring of fractions RX^{-1}. Thus, for instance, if R is right noetherian, then so is RX^{-1}. However, the connections between two-sided ideals of R and two-sided ideals of RX^{-1} are not quite so nice in general, as the following examples indicate.

Exercise 10L. Let $S = k[x_n \mid n \in \mathbb{Z}]$, where k is a field and the x_n are independent commuting indeterminates, α the k-algebra automorphism of S such that $\alpha(x_n) = x_{n+1}$ for all n, and $R = S[x;\alpha]$. By Exercise 10D, the multiplicative set $X = \{1, x, x, \dots\}$ is a denominator set in R. Show that $I = x_1R + x_2R + \dots$ is an ideal of R but that I^{e} is not an ideal of RX^{-1}. $\square$

Exercise 10M. Let k be a field, S the ring of all functions from $\mathbb{Z}$ to k, and α the automorphism of S such that $\alpha(f)(n) = f(n-1)$ for $f \in S$ and $n \in \mathbb{Z}$. Set $R = S[x;\alpha]$. By Exercise 10D, the multiplicative set $X = \{1, x, x^2, \dots\}$ is a denominator set in R. Show that RX^{-1} is a prime ring while R is not even semiprime. Thus, 0 is a prime ideal of RX^{-1} but 0^{c} is not a semiprime ideal of R. $\square$

Proposition 10.17. *Let X be a right denominator set in a ring R.*

(a) *If J is an ideal of RX^{-1}, then J^{c} is an ideal of R and $J = J^{\mathrm{ce}}$. Hence, all ideals of RX^{-1} are extended from ideals of R. However, if I is an ideal of R, then I^{e} need not be an ideal of RX^{-1}.*

(b) *If J is an ideal of RX^{-1} and J^{c} is prime* (*semiprime*), *then J is prime* (*semiprime*). *However, if J is prime, then J^{c} need not even be semiprime.*

Proof. (a) Let $\phi : R \to RX^{-1}$ be the natural map. Obviously $J^{\mathrm{c}} = \phi^{-1}(J)$ is an ideal of R, and $J = J^{\mathrm{ce}}$ by Theorem 10.15. That extensions of ideals need not be ideals is shown by Exercise 10L.

(b) First suppose that J^{c} is prime and note that, since J^{c} is proper, J is proper. If A and B are ideals of RX^{-1} such that $AB \subseteq J$, then $A^{\mathrm{c}}B^{\mathrm{c}} \subseteq (AB)^{\mathrm{c}} \subseteq J^{\mathrm{c}}$, and so either $A^{\mathrm{c}} \subseteq J^{\mathrm{c}}$ or $B^{\mathrm{c}} \subseteq J^{\mathrm{c}}$, whence either $A = A^{\mathrm{ce}} \subseteq J^{\mathrm{ce}} = J$ or $B \subseteq J$. Thus J is prime in this case. The proof for semiprimeness is analogous, and the final statement is shown by Exercise 10M. $\square$

In the noetherian case, the anomalies described in Proposition 10.17 do not occur. In fact, it suffices to assume that RX^{-1} is right noetherian, as follows.

Theorem 10.18. *Let X be a right denominator set in a ring R and assume that RX^{-1} is right noetherian. (For instance, R could be right noetherian.)*

(a) *If I is any ideal of R, then I^{e} is an ideal of RX^{-1}.*

(b) *Let I be an ideal of R such that $(R/I)_R$ is X-torsionfree. Then I is prime (semiprime) if and only if I^{e} is prime (semiprime).*

(c) *An ideal J of RX^{-1} is prime (semiprime) if and only if J^{c} is prime (semiprime).*

(d) *Let P be a prime (semiprime) ideal of R. Then $P = Q^{\mathrm{c}}$ for some prime (semiprime) ideal Q of RX^{-1} if and only if $X \subseteq \mathcal{C}_R(P)$.*

Proof. Let $\phi : R \to RX^{-1}$ be the natural map.

We first show that, for any ideal I of R, the image of X in R/I is right reversible. Thus, given $x \in X$ and $a \in R$ with $xa \in I$, we must show that $aw \in I$ for some $w \in X$.

Set $J_n = \{r \in R \mid x^n r \in I\}$ for $n = 0, 1, \dots$. As RX^{-1} is right noetherian, $J_n^{\mathrm{e}} = J_{n+1}^{\mathrm{e}}$ for some n. From the right Ore condition, there exist $y \in X$ and $b \in R$ such that $ay = x^n b$. Then $x^{n+1}b = xay \in I$, whence $b \in J_{n+1}$, and so $b1^{-1} \in J_{n+1}^{\mathrm{e}} = J_n^{\mathrm{e}}$. Thus $b \in J_n^{\mathrm{ec}}$. Since J_n^{ec}/J_n is X-torsion (Theorem 10.15), $bz \in J_n$ for some $z \in X$. Now $ayz = x^n bz \in I$, and as $yz \in X$, the claim is proved.

(a) To prove that I^{e} is an ideal of RX^{-1}, it suffices to show that $x^{-1}\phi(I) \subseteq I^{\mathrm{e}}$ for each $x \in X$. Given $a \in I$, we have $ay = xb$ for some $y \in X$ and $b \in R$, by the right Ore condition. Since then $xb \in I$, the reversibility proved above implies that $bz \in I$ for some $z \in X$. Therefore

$$x^{-1}\phi(a) = by^{-1} = (bz)(yz)^{-1} \in I^{\mathrm{e}},$$

as desired.

(b) By Theorem 10.15, $I^{\mathrm{ec}} = I$. Hence, if I is prime (semiprime), Proposition 10.17 shows that I^{e} is prime (semiprime). Conversely, assume that I^{e} is prime, note that I must be proper, and consider any ideals A and B in R such that $AB \subseteq I$. Since B^{e} is an ideal of RX^{-1}, we find that

$$\begin{aligned} A^{\mathrm{e}}B^{\mathrm{e}} &= \phi(A)RX^{-1}B^{\mathrm{e}} = \phi(A)B^{\mathrm{e}} = \phi(A)\phi(B)RX^{-1} \\ &= \phi(AB)RX^{-1} = (AB)^{\mathrm{e}} \subseteq I^{\mathrm{e}}. \end{aligned}$$

Thus either $A^{\mathrm{e}} \subseteq I^{\mathrm{e}}$ or $B^{\mathrm{e}} \subseteq I^{\mathrm{e}}$, and so either $A \subseteq A^{\mathrm{ec}} \subseteq I^{\mathrm{ec}} = I$ or $B \subseteq I$. Thus I is prime in this case, and the proof for semiprimeness is analogous.

(c) This follows immediately from (b), since $(R/J^{\mathrm{c}})_R$ is X-torsionfree and $J^{\mathrm{ce}} = J$, by Theorem 10.15.

(d) We prove only the prime case, since the semiprime case is analogous. If $X \subseteq \mathcal{C}_R(P)$, then clearly $(R/P)_R$ is X-torsionfree. In this case, P^{e} is a prime ideal of RX^{-1} by (b), and $P^{\mathrm{ec}} = P$ by Theorem 10.15.

Conversely, assume that $P = Q^{\mathrm{c}}$ for some (prime) ideal Q of RX^{-1}. As proved above, the image of X in R/P is right reversible. Since $(R/P)_R$ is X-torsionfree (Theorem 10.15), we conclude that, for any $x \in X$, the right and left annihilators of x in R/P are both zero. Therefore $X \subseteq \mathcal{C}_R(P)$. $\square$

Exercise 10N. Let X be a right denominator set in a ring R and I an ideal of R. If the image of X in R/I is right reversible, the proof of Theorem 10.18(a) shows that I^{e} is an ideal of RX^{-1}. Prove the converse. $\square$

Exercise 10O. Let X be a right denominator set in a ring R and assume that RX^{-1} is right noetherian. If Q is a semiprime ideal of RX^{-1}, show that R/Q^{c} is right Goldie, and that the right Goldie quotient rings of R/Q^{c} and RX^{-1}/Q are isomorphic. $\square$

Lemma 10.19. *Let X be a right Ore set in a right noetherian ring R and P a prime ideal of R such that $X \cap P = \varnothing$. Then $X \subseteq \mathcal{C}_R(P)$. In particular, $(R/P)_R$ is X-torsionfree.*

Proof. We may assume that $P = 0$. Recall from Proposition 10.7 that X must be right reversible. If the ideal $t_X(R)$ is nonzero, it must contain a regular element, say c. But then $cx = 0$ for some $x \in X$ and so $x = 0$, contradicting our hypotheses. Thus $t_X(R) = 0$. Now if $x \in X$ and $rx = 0$ for some $r \in R$, then $r \in t_X(R)$ and so $r = 0$. Since X is right reversible, if $xs = 0$ for some $s \in R$, then $sy = 0$ for some $y \in X$, whence $s = 0$. Therefore x is a regular element. $\square$

Theorem 10.20. *Let X be a right denominator set in a right noetherian ring R. Then contraction and extension provide inverse bijections between the set of prime ideals of RX^{-1} and the set of those prime ideals of R that are disjoint from X.*

Proof. If Q is a prime ideal of RX^{-1}, then Q^{c} is a prime ideal of R by Theorem 10.18, and $Q^{\mathrm{ce}} = Q$ by Theorem 10.15. Since Q is a proper ideal of RX^{-1}, it is clear that Q^{c} is disjoint from X.

Conversely, let P be a prime ideal of R disjoint from X. By Lemma 10.19, $(R/P)_R$ is X-torsionfree. Therefore P^{e} is a prime ideal of RX^{-1} by Theorem 10.18, and $P^{\mathrm{ec}} = P$ by Theorem 10.15. $\square$

Since a nonzero ring is simple precisely when it has no nonzero prime ideals, Theorem 10.20 leads to the following simplicity criterion for noetherian rings of fractions.

Corollary 10.21. *Let R be a prime right noetherian ring and X a right denominator set in R, and assume that $0 \notin X$. Then RX^{-1} is a simple ring if and only if all nonzero prime ideals of R have nonempty intersection with X.*

Proof. Since $0 \notin X$, Theorem 10.20 shows that RX^{-1} is a prime ring.

If RX^{-1} is simple, then 0 is its only prime ideal, and so, by Theorem 10.20, 0 is the only prime of R disjoint from X. If RX^{-1} is not simple, it has a nonzero maximal ideal, say M. By Theorem 10.20, $M = P^{\mathrm{e}}$ for some prime P of R disjoint from X, and $P \neq 0$ because $M \neq 0$. $\square$

For example, let $R = k[x][y; x(d/dx)]$, where k is an algebraically closed field of characteristic zero. By Exercise 3W, every nonzero prime ideal of R contains x. Now R is contained in the differential operator ring $S = k[x^{\pm 1}][y; x(d/dx)]$, and we observe that S is a (right and left) ring of fractions for R with respect to the multiplicative set $X = \{1, x, x^2, \dots\}$. In particular, X is a denominator set in R (cf. Exercise 10R), and it follows from Corollary 10.21 that S is a simple ring.

In the reverse direction, if we know that a prime noetherian ring R has a simple ring of fractions RX^{-1}, then Corollary 10.21 implies that all nonzero primes of R meet X. This can be helpful in determining the primes of R, as in the following exercise.

Exercise 10P. Let $R = \mathcal{O}_q(k^2)$, where k is a field and $q \in k^\times$ is not a root of unity.

(a) Show that the set $X = \{x^i y^j \mid i, j \in \mathbb{Z}^+\}$ is a denominator set in R, and that RX^{-1} is simple. [Hint: Corollary 1.18.]

(b) Show that every nonzero prime ideal of R contains x or y. [Hint: Show that $x^i y^j R = (xR)^i (yR)^j$ for all $i, j \geq 0$.]

(c) Conclude that if k is algebraically closed, the following is a complete list of the prime ideals of R (with one repetition): $\langle 0 \rangle$, $\langle x \rangle$, $\langle y \rangle$, $\langle x - \alpha,\, y \rangle$ $(\alpha \in k)$, $\langle x,\, y - \beta \rangle$ $(\beta \in k)$. $\square$

• PRIME IDEALS IN ITERATED DIFFERENTIAL OPERATOR RINGS •

In this section, we give an application of rings of fractions to the analysis of prime ideals in an iterated differential operator ring

$$S = R[x_1; \delta_1][x_2; \delta_2] \cdots [x_n; \delta_n].$$

Namely, we show that if R is a commutative noetherian algebra over the rational numbers, then every prime factor ring S/P is a domain. (We proved the case $n = 1$ of this in Theorem 3.22.) The most important example of an iterated diferential operator ring is the enveloping algebra of a solvable Lie algebra over the field of complex numbers, in which case this result includes the theorem of Lie that every finite dimensional irreducible representation is 1-dimensional. We will first need two more facts about differential operator rings.

Lemma 10.22. *Let D be a division ring of characteristic zero, δ a derivation of D, and $S = D[x; \delta]$. Then either S is a simple ring or S is a polynomial ring $D[y]$.*

Proof. It follows from Proposition 2.1 that S is simple unless δ is an inner derivation. If δ is inner, then there is an element $a \in D$ such that $\delta(d) = ad - da$ for all $d \in D$. In this case, S is a polynomial ring $D[x - a]$, by Exercise 2L(a). $\square$

Exercise 10Q. Let R be a subring of a ring S, let $W \subseteq S$, and assume that S is generated as a ring by $R \cup W$. Let X be a right Ore set in R and assume that, for all $w \in W$ and $x \in X$, there exist $s \in S$, $y \in X$, $r \in R$ such that $xs = wy + r$. Prove that X is a right Ore set in S. $\square$

Exercise 10R. Let X be a right denominator set in a ring R and δ a derivation on R.

(a) Show that X is a right denominator set in $R[x;\delta]$.

(b) If $\phi : R \to RX^{-1}$ is the natural map, show that there is a unique derivation ∂ on RX^{-1} such that $\partial\phi = \phi\delta$. [Hint: Exercise 2ZC(a).] Show that $R[x;\delta]X^{-1} \cong RX^{-1}[x;\partial]$. $\square$

Definition. A *completely prime ideal* in a ring R is any (prime) ideal P such that R/P is a domain.

Exercise 10S. Let X be a right denominator set in a ring R.

(a) If P is a completely prime ideal of R and P is disjoint from X, show that P^{e} is a completely prime ideal of RX^{-1}.

(b) If Q is a completely prime ideal of RX^{-1}, show that Q^{c} is a completely prime ideal of R. $\square$

Theorem 10.23. [Lie, Dixmier, Gabriel, Lorenz, Sigurdsson] *If R is a commutative noetherian $\mathbb{Q}$-algebra and $S = R[x_1;\delta_1][x_2;\delta_2]\cdots[x_n;\delta_n]$ is an iterated differential operator ring, then every prime ideal of S is completely prime.*

Proof. The proof is by induction on n, where the case $n = 0$ is clear. Now let $n > 0$, let $S = S_{n-1}[x_n;\delta_n]$ where $S_{n-1} = R[x_1;\delta_1][x_2;\delta_2]\cdots[x_{n-1};\delta_{n-1}]$, and note that S_{n-1} is noetherian. If P is a prime of S, then $P \cap S_{n-1}$ is a prime of S_{n-1} by Lemma 3.21. Thus, if $T = S_{n-1}/(P \cap S_{n-1})$, then by induction T is a noetherian domain. According to Lemma 3.18 and Exercise 2ZA, $P \cap S_{n-1}$ is a δ_n-ideal of S_{n-1}, and if we also use the name δ_n for the induced derivation on T, then $S/(P \cap S_{n-1})S \cong T[x_n;\delta_n]$. Under this isomorphism, $P/(P \cap S_{n-1})S$ corresponds to a prime Q of $T[x_n;\delta_n]$ such that $Q \cap T = 0$ and $T[x_n;\delta_n]/Q \cong S/P$. Thus, we need to show that Q is completely prime.

We now let X be the set of nonzero elements in T, and we let D be the Ore quotient ring of T. Then X is a denominator set in T and $TX^{-1} \cong D$. Using Exercise 10R, we find that X is also a denominator set in $T[x_n;\delta_n]$, that δ_n extends uniquely to a derivation ∂ on D, and that $T[x_n;\delta_n]X^{-1} \cong D[x;\partial]$. Since $Q \cap T = 0$, the prime Q is disjoint from X. According to Theorem 10.20, Q^{e} is a prime of $T[x_n;\delta_n]X^{-1}$ which contracts to Q. Hence, by Exercise 10S, it will suffice to show that Q^{e} is completely prime. Thus, it is enough to show that all primes of $D[x;\partial]$ are completely prime.

If $D[x;\partial]$ is simple, then our desired conclusion is immediate from the fact that $D[x;\partial]$ is a domain. Otherwise, according to Lemma 10.22, $D[x;\partial]$ is a polynomial ring $D[y]$. It follows that X is a denominator set in $T[y]$ and that $T[y]X^{-1} \cong D[x;\partial]$. Consequently, in view of Theorem 10.20 and Exercise 10S, we now see that it will suffice to show that all primes of $T[y]$ are completely prime. However, $T[y]$ is a factor ring of a polynomial ring $S_{n-1}[y]$, and we can finish the proof by showing that all primes of $S_{n-1}[y]$ are completely prime. To do this, we show that $S_{n-1}[y]$ may be viewed as an $(n-1)$-fold iterated differential operator ring over $R[y]$. This holds trivially if $n = 1$.

In case $n > 1$, we have $S_{n-1} = S_{n-2}[x_{n-1};\delta_{n-1}]$ for a suitable coefficient ring S_{n-2}. Observe that the inner derivation $\delta_{x_{n-1}}$ on $S_{n-1}[y]$ restricts to a derivation ∂_{n-1} on $S_{n-2}[y]$. Since $S_{n-1}[y]$ is clearly a free left $S_{n-2}[y]$-module in which the powers of x_{n-1} form a basis, we find that $S_{n-1}[y] = S_{n-2}[y][x_{n-1};\partial_{n-1}]$. Continuing in this fashion, we find that $S_{n-1}[y]$ is an iterated differential operator ring of the form $R[y][x_1;\partial_1]\cdots[x_{n-1};\partial_{n-1}]$. This iterated differential operator ring involves $n-1$ steps over the commutative noetherian $\mathbb{Q}$-algebra $R[y]$. Therefore, by induction, all primes of $S_{n-1}[y]$ are completely prime, and this completes the proof of the theorem. $\square$

• ADDITIONAL EXERCISES •

10T. If R is a semiprime right Goldie ring and P a minimal prime ideal of R, show that $\mathcal{C}(P)$ is a right denominator set in R and that $R\mathcal{C}(P)^{-1}$ is isomorphic to the right Goldie quotient ring of R/P. [Hint: Show that there is some $c \in \mathcal{C}(P)$ satisfying $Pc = 0$.] $\square$

10U. Let S be a prime right Goldie ring and ${}_SB_S$ a bimodule. Let R be the subring of $\begin{pmatrix} S & B \\ 0 & S \end{pmatrix}$ consisting of all $\begin{pmatrix} s & b \\ 0 & s \end{pmatrix}$, where $s \in S$ and $b \in B$, and let $P = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. Observe that P is a prime ideal of R.

(a) Show that $\mathcal{C}_R(P)$ is a right Ore set in R if and only if $(B/cB)_S$ is torsion for all regular elements $c \in S$.

(b) If S is noetherian, B_S is finitely generated, and ${}_SB$ is torsionfree, show that $\mathcal{C}_R(P)$ is a right denominator set in R. $\square$

10V. Let α be an endomorphism of a ring R and X a right denominator set in R such that $X \subseteq \alpha(X)$.

(a) Show that X is a right denominator set in $R[x;\alpha]$.

(b) If $\alpha(X) = X$ and if $\phi : R \to RX^{-1}$ is the natural map, show that there is a unique ring endomorphism β of RX^{-1} such that $\beta\phi = \phi\alpha$. Show that $R[x;\alpha]X^{-1} \cong RX^{-1}[x;\beta]$. $\square$

10W. Let $X \subseteq Y$ be multiplicative sets in a ring R, with X a right denominator set. Show that Y is a right denominator set in R if and only if the set $Y_1 = \{y1^{-1} \mid y \in Y\}$ is a right denominator set in RX^{-1}, in which case $RY^{-1} \cong (RX^{-1})Y_1^{-1}$. $\square$

10X. Let Y be the multiplicative set generated by a subset X of a ring R. Assume that for each $x \in X$ there exists $y \in Y$ such that $Ry \subseteq xR$. Show that Y is a right Ore set. $\square$

10Y. (a) Let X be a nonempty subset of a ring R, and let Y be the multiplicative set generated by X, that is, the set of all products of elements of X. (By convention, Y also contains the product of the empty set of elements from X, and this product is set equal to 1.) If for all $a \in R$ and $x \in X$ there exist $b \in R$ and $y \in X$ such that $ay = bx$, prove that Y is a right Ore set. Similarly, if for all $r \in R$ and $x \in X$ such that $xr = 0$ there exists $x' \in X$ such that $rx' = 0$, prove that Y is right reversible.

(b) Show that the multiplicative set generated by a nonempty family of right Ore (right reversible) sets is right Ore (right reversible).

(c) Conclude that, for any multiplicative set $Z \subseteq R$, there exists a unique largest right denominator set contained in Z. $\square$

10Z. Let $S = RX^{-1}$, where R is a prime right noetherian ring and $X \subseteq R$ a right Ore set of regular elements. Assume that S is not artinian, and that R/I is artinian for all essential right ideals I of R.

(a) Show that each simple right S-module contains a unique simple R-submodule.

(b) Show that the rule $A \mapsto \operatorname{soc}(A_R)$ induces a bijection from the set of isomorphism classes of simple right S-modules onto the set of isomorphism classes of X-torsionfree simple right R-modules.

These results apply, in particular, when $R = A_1(k) = k[y][x; d/dy]$ for a field k of characteristic zero and $X = k[y] \setminus \{0\}$, in which case $S = B_1(k) = k(y)[x; d/dy]$. (See Theorem 15.21 for the hypothesis on factors R/I.) Thus, up to isomorphism, the X-torsionfree simple right modules over $A_1(k)$ are paired bijectively with the simple right modules over the PID $B_1(k)$. The remaining simple right $A_1(k)$-modules can be determined as follows. This analysis is due to Block [1981].

(c) Show that the rule $M \mapsto R/MR$ induces a bijection from the set of maximal ideals of $k[y]$ onto the set of isomorphism classes of X-torsion right $A_1(k)$-modules. $\square$

• NOTES •

Much of the subject of this chapter developed as folklore, and so we do not give precise attributions for many of the results.

Existence of Rings of Fractions. As mentioned earlier, Asano proved that a ring has a right ring of fractions with respect to its regular elements if and only if its regular elements satisfy the right Ore condition [1939, Satz 1] and later that a ring has a right ring of fractions with respect to a multiplicative set X of regular elements if and only if X satisfies the right Ore condition [1949, Satz I]. For an arbitrary multiplicative set X in a ring R, Elizarov showed that a right ring of fractions exists if and only if the intersection of those ideals I such that $X \subseteq \mathcal{C}(I)$ and the image of X in R/I satisfies the right Ore condition yields an ideal with the same properties [1960, Theorem 2]. The general criterion using the Ore condition together with the reversibility condition was given by Gabriel [1962, Proposition 5, p. 415].

Two-Sided Rings of Fractions. Proposition 10.6 was first proved for multiplicative sets of regular elements by Asano [1949, Satz 6].

Reversibility of Ore Sets. The right reversibility of right Ore sets was first proved in the right noetherian case by Goldie [1964a, Proposition 2, p. 5-22].

Ideals in Rings of Fractions. Part (a) of Theorem 10.18 was first proved in a special case by Ludgate [1972, Theorem 5]. In the case that R is right noetherian and X consists of regular elements, parts (a) and (b) were proved by Jategaonkar [1974a, Proposition 1.4].

Complete Primeness in Iterated Differential Operator Rings. Lie's Theorem, one version of which appeared in [1893, §131, Satz 7], implies that for a solvable finite dimensional complex Lie algebra L, every finite dimensional irreducible representation is 1-dimensional. This is equivalent to saying that every finite dimensional prime factor ring of $U(L)$ is a domain. Dixmier proved that, for a solvable finite dimensional Lie algebra L over an algebraically closed field of characteristic zero, every prime of $U(L)$ is completely prime [1966, Théorème 1.3], and then Gabriel removed the algebraically closed hypothesis [1971, Théorème 3.2]. Lorenz proved that if R is an algebra over a field k of characteristic zero, and if for each field extension K of k the algebra $R \otimes_k K$ is right noetherian with all primes being completely prime, then the same holds for any differential operator ring $R[x;\delta]$, where δ is a k-linear derivation [1981, Theorem]. Sigurdsson proved that if R is a right noetherian $\mathbb{Q}$-algebra and if every prime in the polynomial ring $R[x]$ is completely prime, then the same holds for any iterated differential operator ring $R[x_1;\delta_1]\cdots[x_n;\delta_n]$ [1984, Corollary 2.6].

11. Artinian Quotient Rings

Goldie's Theorem gives a characterization of those rings which have a classical quotient ring that is semisimple and, in particular, artinian. This naturally gives rise to the question: Which rings have classical quotient rings that are artinian? While this question has a certain abstract interest of its own, its significance turns out to be much greater than one might initially suspect. Rings arising in a natural way frequently have artinian classical quotient rings, and this may be an important fact in their study. In particular, as we shall see in Chapter 14, if R is a subring of a ring S and P is a prime ideal in S, then, while $P \cap R$ need not be prime or semiprime, it is often possible to show that $R/(P \cap R)$ has an artinian classical quotient ring.

We first introduce a new notion of rank, known as "reduced rank" (different from the uniform rank introduced in Chapter 5), which is useful in many arguments involving noetherian rings, and we give two naive examples of its use. We then use reduced rank to derive necessary and sufficient conditions for a noetherian ring R to have an artinian classical quotient ring. This basic criterion is very satisfactory in some ways – for instance, it is phrased entirely in terms of properties of individual elements of R – but not in others. We then turn to criteria involving ideals, particularly affiliated prime ideals (parallel to the theorem in the commutative case that a commutative noetherian ring has an artinian classical quotient ring if and only if its associated prime ideals are all minimal).

• REDUCED RANK •

The notion of the rank of a module that was discussed in Chapter 5 goes back to the concept of the rank of a torsionfree abelian group. More generally, it has been found that a useful invariant of an arbitrary abelian group A is its "torsionfree rank," obtained by reducing A to the torsionfree group $A/t(A)$ and taking the rank of this group. This idea we shall carry over directly to modules over a semiprime Goldie ring, and then we shall extend it to modules over noetherian rings.

Definition. If A is a right module over a semiprime right Goldie ring R, the *reduced rank* (or *torsionfree rank*) of A is defined as

$$\rho_R(A) = \operatorname{rank}\big(A/t(A)\big),$$

which we may denote by $\rho(A)$ if the coefficient ring R is understood. (E.g., note that $\rho_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = 0$ whereas $\rho_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = 1$.) In case $A/t(A)$ does not have finite rank, we assign the value ∞ to $\rho_R(A)$. Other notations found in the literature for reduced rank include r.rank and red.rk.

Alternatively, if $Q(R)$ is the right Goldie quotient ring of R, we could define

$$\rho_R(A) = \text{length}\big(A \otimes_R Q(R)\big),$$

where $A \otimes_R Q(R)$ is viewed as a (semisimple) right $Q(R)$-module, because of Exercise 7H. This description of $\rho_R(A)$ will play a key role in some of our calculations.

Exercise 11A. If A is a finitely generated right module over a semiprime right Goldie ring R, show that $\rho_R(A)$ is finite. $\square$

If we wish to concoct a notion of reduced rank in more general circumstances, it is helpful to keep in mind, as an analog, the notion of length for artinian modules that are not semisimple. In particular, a module A of finite length has a chain of submodules

$$A_0 = 0 \le A_1 \le \cdots \le A_n = A$$

such that A_i/A_{i-1} is semisimple for $i = 1, \ldots, n$ and, of course,

$$\text{length}(A) = \sum_{i=1}^{n} \text{length}(A_i/A_{i-1}).$$

Thus, length could have been first defined for semisimple modules and then extended to other modules by the formula above.

We follow exactly this route in defining reduced rank for modules over a right noetherian ring R. Since the prime radical N of R is nilpotent (Theorem 3.11), say $N^n = 0$, a right R-module A has a chain of submodules

$$A \ge AN \ge AN^2 \ge \cdots \ge AN^n = 0$$

such that each of the factors AN^i/AN^{i+1} is a right (R/N)-module. Thus, each AN^i/AN^{i+1} has a reduced rank (as an (R/N)-module), and we could define a reduced rank for A to be

$$\sum_{i=0}^{n-1} \rho_{R/N}(AN^i/AN^{i+1}).$$

To make sure this behaves properly we need a lemma, and then we will give an alternate but slightly more useful definition.

Lemma 11.1. *Let R be a right noetherian ring, N its prime radical, and A a right R-module. Consider two chains of submodules of A,*

$$A_0 = A \geq A_1 \geq \cdots \geq A_m = 0 \qquad \text{and} \qquad B_0 = A \geq B_1 \geq \cdots \geq B_n = 0,$$

such that $(A_i/A_{i+1})N = 0$ and $(B_j/B_{j+1})N = 0$ for all indices i, j. Then

$$\sum_{i=0}^{m-1} \rho_{R/N}(A_i/A_{i+1}) = \sum_{j=0}^{n-1} \rho_{R/N}(B_j/B_{j+1}).$$

Proof. We first show this in case $AN = 0$. If X is the set of regular elements in R/N, then $(R/N)X^{-1}$ is the right Goldie quotient ring of R/N and, for any right (R/N)-module C, Proposition 10.12 shows that $C \otimes_R (R/N)X^{-1} \cong CX^{-1}$. Hence, $\rho_{R/N}(C) = \operatorname{length}(CX^{-1})$. Returning to our given module A and adopting the notation of Theorem 10.15, each A_i^{e} is an $(R/N)X^{-1}$ submodule of AX^{-1}, and $A_i^{\mathrm{e}}/A_{i+1}^{\mathrm{e}} \cong (A_i/A_{i+1})X^{-1}$ by Exercise 10J. Thus,

$$\sum_{i=0}^{m-1} \rho_{R/N}(A_i/A_{i+1}) = \sum_{i=0}^{m-1} \operatorname{length}(A_i^{\mathrm{e}}/A_{i+1}^{\mathrm{e}}) = \operatorname{length}(AX^{-1}),$$

and similarly for the second summation, which proves the lemma in case $AN = 0$.

It follows from this case that we may replace either of the given chains of submodules by a refinement without changing the value of the corresponding summation. (For instance, if the first chain is refined, the portion of the new summation dealing with the terms from A_i to A_{i+1} must add up to $\rho_{R/N}(A_i/A_{i+1})$.)

Finally, we cite the Schreier Refinement Theorem (Theorem 4.10) to say that the two chains have refinements in which the factors are isomorphic in pairs. That is, after replacing both chains by refinements, we may assume that $m = n$ and that there is a permutation π of $\{0, \ldots, m-1\}$ with $B_i/B_{i+1} \cong A_{\pi(i)}/A_{\pi(i)+1}$ for $i = 0, \ldots, m-1$. From this the desired result follows. $\square$

With this lemma in hand, we can immediately see that the following definition makes sense, i.e., that the value given for the reduced rank of A is independent of the chain of submodules chosen.

Definition. Let R be a right noetherian ring with prime radical N and A a right R-module. Choose a chain of submodules

$$A_0 = A \geq A_1 \geq \cdots \geq A_n = 0$$

such that $(A_i/A_{i+1})N = 0$ for $i = 0, \ldots, n-1$. (Reminder: Such chains exist because N is nilpotent.) The *reduced rank of* A, denoted by $\rho_R(A)$ or by $\rho(A)$ if the coefficient ring is understood, is defined to be

$$\rho_R(A) = \sum_{i=0}^{n-1} \rho_{R/N}(A_i/A_{i+1}).$$

Exercise 11B. If A is a finite abelian group and $I = \operatorname{ann}_{\mathbb{Z}}(A)$, show that $\rho_{\mathbb{Z}}(A) = 0$ whereas $\rho_{\mathbb{Z}/I}(A) = \operatorname{length}(A)$. $\square$

We record some easy but useful facts about reduced rank in the following lemmas.

Lemma 11.2. *If R is a right noetherian ring and A a finitely generated right R-module, then $\rho(A)$ is finite.* $\square$

Lemma 11.3. *If R is a right noetherian ring and $A' \leq A$ are right R-modules, then*

$$\rho(A) = \rho(A') + \rho(A/A').$$

Proof. This follows from the fact that the definition of reduced rank is independent of the chain of submodules chosen, so that the defining chain for $\rho(A)$ may be chosen as a refinement of the chain $A \geq A' \geq 0$. $\square$

The next lemma is a further special case.

Lemma 11.4. *Let R be a right noetherian ring and $A' \leq A$ right R-modules, such that $\rho(A)$ is finite. Then $\rho(A') = \rho(A)$ if and only if $\rho(A/A') = 0$.* $\square$

Lemma 11.4 will be useful because we can give a "torsion" interpretation of the condition "reduced rank zero," which is parallel to the usual condition when R is semiprime. Let N be the prime radical of R and $Q(R/N)$ the right Goldie quotient ring of R/N. If $AN = 0$, then $\rho(A) = 0$ if and only if $A \otimes_R Q(R/N) = 0$, if and only if A is a torsion (R/N)-module (recall Exercise 7H). In general, as the following lemma shows, R-modules with zero reduced rank can be thought of as $\mathcal{C}(N)$-torsion modules (recall that in Chapter 4 we have defined X-torsion and X-torsionfree modules for arbitrary multiplicative sets X).

Lemma 11.5. *Let R be a right noetherian ring, N its prime radical, and A a right R-module. Then $\rho(A) = 0$ if and only if A is $\mathcal{C}(N)$-torsion.*

Proof. Choose submodules $A_0 = A \geq A_1 \geq \cdots \geq A_n = 0$ such that $(A_i/A_{i+1})N = 0$ for $i = 0, \ldots, n-1$. Because of Lemma 11.3, $\rho(A) = 0$ if and only if $\rho(A_i/A_{i+1}) = 0$ for each i, and by our previous remarks this occurs if and only if each A_i/A_{i+1} is torsion as a right (R/N)-module. The result is now clear. $\square$

Exercise 11C. If Q is a prime ideal in a right noetherian ring R, show that $\rho_R(R/Q) > 0$ if and only if Q is a minimal prime. [Hint: Proposition 7.5.] $\square$

• APPLICATIONS OF REDUCED RANK TO FINITE RING EXTENSIONS •

Despite its innocent appearance, the notion of reduced rank has been useful in many contexts in recent work in ring theory. We give here two such applications. It should be emphasized that, although the results are simple, they were unknown until this method was used to prove them in 1986 and 1987.

If R is a subring of a ring S, one can ask many questions concerning the relations between primes of R and those of S. In the commutative case, one says that a prime P of S *lies over* a prime Q of R if $P \cap R = Q$. That is not a reasonable definition in the noncommutative setting since $P \cap R$ is not generally prime.

Definition. If R is a subring of a ring S, and P and Q are primes of S and R, respectively, then we say that P *lies over* Q if Q is minimal over $P \cap R$.

This definition has many advantages, but it has its dangers as well. For example, primes $P_1, P_2, P_3, \ldots$ could all lie over Q while the contractions $P_i \cap R$ could all be different, as in the following example.

Exercise 11D. Let k be a field and $S = M_2(k) \times \cdots \times M_n(k)$ for some integer $n > 2$. For $i = 2, \ldots, n$, let u_i be the matrix in $M_i(k)$ with $(j, j+1)$-entry equal to 1 for $j = 1, \ldots, i-1$ while all other entries are 0. If $u = (u_2, \ldots, u_n)$ and $R = k[u] \subseteq S$, show that all the minimal primes P_j of S lie over the prime uR, but that all the contractions $P_j \cap R$ are distinct. $\square$

Definition. Just as in the commutative case, we say that *LO* (*lying over*) holds in a ring extension $S \supseteq R$ if, for every prime Q of R, there is a prime P of S lying over Q. We say that *INC* (*incomparability*) holds if, for every prime Q of R, there do not exist primes P and P' of S lying over Q with $P > P'$.

Theorem 11.6. [Letzter] *If R is a right noetherian ring and R is a subring of a ring S such that S is finitely generated as a right R-module, then INC holds.*

Proof. Suppose P and P' are primes in S lying over the prime Q in R and $P > P'$. This means that Q is minimal over both $P \cap R$ and $P' \cap R$, and all of this will remain true if we factor P' out of S and $P' \cap R$ out of R. Therefore, without loss of generality, we may assume that $P' = 0$ and hence that S is prime. Since Q is minimal over $P' \cap R$, it follows that Q is a minimal prime of R.

We will now use reduced rank for right R-modules, noting that $\rho_R(S) < \infty$ (Lemma 11.2) and that $\rho_R(R/Q) > 0$ since Q is a minimal prime of R (Exercise 11C). Since $P \cap R \subseteq Q$, we conclude that $\rho_R(R/(P \cap R)) > 0$. Moreover, $R/(P \cap R)$ is isomorphic to a right R-submodule of S/P, and so $\rho_R(S/P) > 0$. Now P is a nonzero ideal of the prime right noetherian ring S and hence contains a regular element, say x. Since S and xS are isomorphic right R-modules, clearly $\rho_R(S) = \rho_R(xS)$; and since $\rho_R(xS) \leq \rho_R(P) \leq \rho_R(S)$, we conclude that $\rho_R(S) = \rho_R(P)$. From Lemma 11.4, it follows that $\rho_R(S/P) = 0$, a contradiction. $\square$

Theorem 11.7. [Letzter] *If R is a right FBN ring and R is a subring of a ring S such that S is finitely generated as a right R-module, then S is right FBN.*

Proof. Without loss of generality, we may assume that S is prime, and we need only prove that S is right bounded. Let I be an essential right ideal of S. To show that I contains a nonzero ideal of S, we again use reduced rank for S and I as right R-modules. Since I contains a regular element of S, it follows as in the previous proof that $\rho_R(S/I) = 0$. Lemma 11.5 then implies that S/I is $\mathcal{C}_R(N)$-torsion, where N is the prime radical of R.

According to Theorem 9.6, there is a chain

$$A_0 = 0 < A_1 < \cdots < A_n = S/I$$

of right R-submodules of S/I such that, for $i = 1, \ldots, n$, the factor A_i/A_{i-1} is isomorphic to a right ideal in R/P_i for some prime ideal P_i of R. Since S/I is $\mathcal{C}_R(N)$-torsion, it follows that none of the primes P_i can be minimal primes (see Proposition 7.9). Consequently, $P_n P_{n-1} \cdots P_1 \neq 0$, and hence S/I is not faithful as a right R-module. Since R is a subring of S, it follows that S/I is also not faithful as an S-module. Hence, I contains a nonzero ideal of S (the annihilator of S/I), as required. □

• SMALL'S THEOREM •

We now turn to Small's Theorem, which provides a criterion for a ring to have an artinian classical quotient ring. It is most often stated in the noetherian case, where the criterion is especially simple, and this is the case we shall do in the text. The original theorem did not have the noetherian condition but was very complicated to state. With more recently developed notions and methods, it is now possible to give a fairly simple statement (see the discussion at the end of the section), one which emphasizes the analogy with Goldie's Theorem.

Rather than asking for conditions under which a ring R has an artinian classical quotient ring, one might consider the apparently weaker condition that R embeds in a ring of fractions which is artinian. However, this is not really weaker, as the following exercise shows.

Exercise 11E. Let R be a ring and X a right denominator set in R. If RX^{-1} is right artinian and the natural map $R \to RX^{-1}$ is an embedding, show that RX^{-1} is a classical right quotient ring for R. □

Lemma 11.8. [Goldie] *Let R be a right noetherian ring and N its prime radical. Then $\mathcal{C}(0) \subseteq \mathcal{C}(N)$ and, given any $c \in \mathcal{C}(0)$ and $a \in R$, there exists $d \in \mathcal{C}(N)$ such that $ad \in cR$.*

Proof. Let $c \in \mathcal{C}(0)$. Since $\text{r.ann}_R(c) = 0$, we obtain $cR \cong R_R$, whence $\rho(cR) = \rho(R_R)$ and so $\rho(R/cR) = 0$. By Lemma 11.5, this implies that for any $a \in R$ there is some $d \in \mathcal{C}(N)$ with $ad \in cR$. This verifies the second conclusion of the lemma (sometimes called the *pseudo-Ore condition*).

In particular, from the case $a = 1$ we find that cR contains an element of $\mathcal{C}(N)$, whence $c(R/N)$ contains a regular element of R/N. Then $c + N$ itself must be a regular element of R/N, by Lemma 6.11. Thus $c \in \mathcal{C}(N)$. $\square$

Theorem 11.9. [Small, Talintyre] *A right noetherian ring R with prime radical N has a right artinian classical right quotient ring if and only if $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.*

Proof. Assuming $\mathcal{C}(0) = \mathcal{C}(N)$, Lemma 11.8 implies that $\mathcal{C}(0)$ is a right Ore set in R, and we can form the corresponding right ring of fractions $Q = R\,\mathcal{C}(0)^{-1}$. We identify R with its image in Q. Since $(R/N)_R$ is $\mathcal{C}(0)$-torsionfree (because $\mathcal{C}(0) = \mathcal{C}(N)$), we conclude from Theorems 10.18 and 10.15 that NQ is an ideal of Q satisfying $NQ \cap R = N$. Now the embedding $R/N \to Q/NQ$ makes Q/NQ the right Goldie quotient ring of R/N, whence Q/NQ is semisimple and, in particular, artinian.

As NQ is an ideal of Q, we have $QNQ = NQ$, and it follows that $(NQ)^i = N^iQ$ for all positive integers i. Thus NQ is nilpotent (because N is nilpotent). Each of the factors $N^iQ/N^{i+1}Q$ is a right module over Q/NQ and so is semisimple. Also, since Q is right noetherian (Corollary 10.16), $N^iQ/N^{i+1}Q$ is finitely generated. Hence, each $N^iQ/N^{i+1}Q$ is artinian as a right Q-module, and therefore Q is right artinian.

Conversely, if R has a right artinian classical right quotient ring Q, then $\mathcal{C}(0)$ is a right Ore set in R. Since $\mathcal{C}(0) \subseteq \mathcal{C}(N)$ (Lemma 11.8), $(R/N)_R$ is $\mathcal{C}(0)$-torsionfree, and so NQ is an ideal of Q satisfying $NQ \cap R = N$ (Theorems 10.18 and 10.15). As in the previous paragraph, NQ is nilpotent, and hence $NQ \subseteq J(Q)$.

Given $c \in \mathcal{C}_R(N)$ and $q \in Q$ with $cq \in NQ$, write $q = ad^{-1}$ with $a \in R$ and $d \in \mathcal{C}(0)$ and observe that $ca = cqd \in NQ \cap R = N$, whence $a \in N$ and $q \in NQ$. It follows that $c(Q/NQ) \cong Q/NQ$ and so $c(Q/NQ)$ has the same length as Q/NQ, yielding $c(Q/NQ) = Q/NQ$. Thus $c + NQ$ is right invertible in Q/NQ. By Corollary 4.25, $c + NQ$ is invertible in Q/NQ. As $NQ \subseteq J(Q)$, we conclude from Exercise 3U that c is invertible in Q. Consequently, $c \in \mathcal{C}_R(0)$, and therefore $\mathcal{C}_R(0) = \mathcal{C}_R(N)$. $\square$

For instance, consider the ring $R = k[x, y]/\langle y^2 \rangle$, where k is a field, viewed as a free $k[x]$-module with basis $\{1, y\}$ such that $y^2 = 0$. Then y generates the prime radical N of R, and $\mathcal{C}(N) = \{f + gy \mid f, g \in k[x],\ f \neq 0\}$. It is clear in this example that $\mathcal{C}(N) = \mathcal{C}(0)$, and so R has an artinian classical quotient ring. In fact, the latter ring is easy to identify – it is just $k(x)[y]/\langle y^2 \rangle$.

For comparison, let $R' = k[x, y]/\langle xy, y^2 \rangle$. The prime radical N' of R' is again generated by (the coset of) y, but this time $x \in \mathcal{C}(N') \setminus \mathcal{C}(0)$, since $xy = 0$ in R'. Thus, R' does not have an artinian classical quotient ring.

Since the condition $\mathcal{C}(0) = \mathcal{C}(N)$ is left-right symmetric, Theorem 11.9 shows that a noetherian ring R has a right artinian classical right quotient ring if and only if R has a left artinian classical left quotient ring. In that situation, the classical right and left quotient rings of R coincide, and we say that R *has an artinian classical quotient ring*.

Corollary 11.10. *A noetherian ring R with prime radical N has an artinian classical quotient ring if and only if $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.*

Proof. Theorem 11.9 and Proposition 6.5. □

The following exercise shows that the existence of artinian classical quotient rings passes to certain finite ring extensions. In particular, if R is a semiprime right noetherian ring and $S \supseteq R$ is a ring such that S_R is finitely generated and S is torsionfree as both a left and a right R-module, then S has a right artinian classical right quotient ring. For example, if R and T are semiprime noetherian rings and ${}_RB_T$ is a bimodule which is finitely generated and torsionfree on each side, then $\left(\begin{smallmatrix} R & B \\ 0 & T \end{smallmatrix}\right)$ has an artinian classical quotient ring.

Exercise 11F. Let $R \subseteq S$ be right noetherian rings such that S_R is finitely generated and $\mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$. Show that if R has a right artinian classical right quotient ring, then so does S. [Hint: First show that $\mathcal{C}_R(0)$ is a right Ore set in S, and that $S\mathcal{C}_R(0)^{-1}$ is right artinian.] □

The general (non-noetherian) version of Small's Theorem requires extending the notion of reduced rank to certain non-noetherian modules, as in the following exercises.

Exercise 11G. Let R be a ring with prime radical N such that R/N is right Goldie.

(a) Prove Lemma 11.1 under these hypotheses (assuming the module A has chains of submodules as described). Hence, define $\rho_R(A)$ for any right R-module A satisfying $AN^n = 0$ for some positive integer n.

(b) Prove Lemmas 11.3–11.5 in the present context, assuming $AN^n = 0$ for some n.

(c) If N is nilpotent and $\rho(R_R)$ is finite, prove Lemma 11.8 in the present context. □

Exercise 11H. Let R be a ring with prime radical N such that (i) R/N is right Goldie, (ii) N is nilpotent, (iii) $\rho(R_R)$ is finite, and (iv) $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

(a) Show that R has a classical right quotient ring Q.

(b) Show that NQ is a nilpotent two-sided ideal of Q.

(c) Show that Q/NQ is a semisimple ring.

(d) Show that $\rho(Q_R)$ is finite, and use this to see that $(N^iQ/N^{i+1}Q)_Q$ is artinian for all i.

(e) Conclude that Q is right artinian. □

Exercise 11I. Let R be a ring with prime radical N, and assume that R has a right artinian classical right quotient ring Q.

(a) If N' is the prime radical of Q, show that $N' \cap R$ is a nilpotent semiprime ideal of R, whence $N' \cap R = N$. In particular, N is nilpotent.

(b) Observe that $NQ = N'$. In particular, $NQ \cap R = N$.

(c) Show that $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

(d) Show that Q/NQ is a classical right quotient ring for R/N. Thus, R/N is right Goldie and Q/NQ is its right Goldie quotient ring.

(e) Finally, show that $\rho(R_R)$ is finite. [Hint: Show that $(N^i/N^{i+1}) \otimes_R Q \cong N^iQ/N^{i+1}Q$ for all $i = 0, 1, \ldots$.] □

To summarize the previous exercises: A ring R with prime radical N has a right artinian classical right quotient ring if and only if R/N is right Goldie, N is nilpotent, $\rho(R_R)$ is finite, and $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

• AFFILIATED PRIME IDEALS •

In this section, we develop ideal-theoretic criteria for a noetherian ring R to have an artinian classical quotient ring. In the commutative case, it is known that this occurs if and only if the associated primes of R are all minimal. As we will see, this fails in the noncommutative case (Exercise 11L). The appropriate modification is to look at affiliated primes rather than just associated primes or annihilator primes.

Lemma 11.11. *Let R be a right noetherian ring, X a right Ore set in R, and A a nonzero finitely generated X-torsionfree right R-module. If P is any annihilator prime for A, or any prime affiliated to A, then $(R/P)_R$ is X-torsionfree.*

Proof. If $P = \mathrm{ann}_R(Y)$ for some subset $Y \subseteq A$, then $(R/P)_R$ embeds in the direct product A^Y, whence $(R/P)_R$ is X-torsionfree.

Since X is right reversible (Proposition 10.7), the localizations $S = RX^{-1}$ and $B = AX^{-1}$ exist. Consider an affiliated series $A_0 = 0 < A_1 < \cdots < A_n = A$ with affiliated primes $P_1, \ldots, P_n$. If $I_j = P_jP_{j-1} \cdots P_1$ for $j = 1, \ldots, n$, then $A_j = \mathrm{ann}_A(I_j)$ by Proposition 8.5. Since A is X-torsionfree, we may identify it with a submodule of B. If $B_j = \mathrm{ann}_B(I_j)$, then $A \cap B_j = A_j$ and A/A_j embeds in B/B_j. On the other hand, the extension I_j^{e} of I_j to S is an ideal of S; and $B_j = \mathrm{ann}_B(I_j^{\mathrm{e}})$, whence B_j is an S-submodule of B. Thus B/B_j is X-torsionfree, and so is A/A_j.

Therefore $P_1, \ldots, P_n$ are annihilator primes of X-torsionfree R-modules, and hence $(R/P_j)_R$ is X-torsionfree for $j = 1, \ldots, n$, as shown in the first paragraph. □

Lemma 11.12. *If R is a right noetherian ring which has a right artinian classical right quotient ring, then every right annihilator prime for R and every right affiliated prime of R is a minimal prime.*

Proof. By assumption, $\mathcal{C}(0)$ is a right Ore set in R, and of course R_R is $\mathcal{C}(0)$-torsionfree. Hence, if P is an annihilator prime for R_R or an affiliated prime of R_R, Lemma 11.11 shows that $(R/P)_R$ is $\mathcal{C}(0)$-torsionfree. In particular, P is disjoint from $\mathcal{C}(0)$. If N is the prime radical of R, then $\mathcal{C}(0) = \mathcal{C}(N)$ by Small's Theorem, and so P is disjoint from $\mathcal{C}(N)$. By Proposition 7.5, P is therefore a minimal prime of R. $\square$

Theorem 11.13. [Stafford] *For a noetherian ring R the following conditions are equivalent:*

(a) *R has an artinian classical quotient ring.*

(b) *Every prime ideal of R which is either a right or a left affiliated prime of R is minimal.*

(c) *There exist a right affiliated series and a left affiliated series for R such that the affiliated primes for both series are all minimal.*

Proof. (a) $\Longrightarrow$ (b): Lemma 11.12.

(b) $\Longrightarrow$ (c) a priori.

(c) $\Longrightarrow$ (a): By Corollary 11.10 and Lemma 11.8, it suffices to show that $\mathcal{C}(0) \supseteq \mathcal{C}(N)$, where N is the prime radical of R.

There exists a right affiliated series $A_0 = 0 < A_1 < \cdots < A_n = R$ such that the corresponding right affiliated primes $P_1, \ldots, P_n$ are all minimal. Each of the factors A_i/A_{i-1} is torsionfree as a right (R/P_i)-module by Proposition 8.7 and hence also torsionfree as a right (R/N)-module, by Proposition 7.9.

Now, given $c \in \mathcal{C}(N)$, the left annihilator of c in A_i/A_{i-1} is zero for $i = 1, \ldots, n$, whence $A_i \cap \mathrm{l.ann}_R(c) \subseteq A_{i-1}$. We conclude by induction that $\mathrm{l.ann}_R(c) = 0$, and, since we have symmetric hypotheses, $\mathrm{r.ann}_R(c) = 0$. Therefore $c \in \mathcal{C}(0)$, completing the proof. $\square$

Exercise 11J. Show that in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$ there exists a right affiliated series for which the corresponding right affiliated primes are all minimal, as well as a right affiliated series for which one of the corresponding right affiliated primes is not minimal. $\square$

Exercise 11K. Show via Theorem 11.13 that a commutative noetherian ring R has an artinian classical quotient ring if and only if all annihilator primes of R are minimal. $\square$

Exercise 11L. Show that in the ring $R = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$ all right annihilator primes are minimal and all left annihilator primes are minimal, while nonetheless R fails to have a right artinian classical right quotient ring. $\square$

The following exercise isolates the role of the annihilator primes in the question of existence of an artinian classical quotient ring.

Exercise 11M. If R is a noetherian ring with prime radical N, show that R has a right artinian classical right quotient ring if and only if $\mathcal{C}(N)$ is a right Ore set and all right annihilator primes of R are minimal. $\square$

Exercise 11N. Let $T = A_1(k)$, where k is a field of characteristic zero, B a simple right T-module, $R = \begin{pmatrix} k & B \\ 0 & T \end{pmatrix}$, and N the prime radical of R. Show that $\mathcal{C}_R(N)$ is a right Ore set and that all prime ideals of R are minimal, but that R does not have a right artinian classical right quotient ring. $\square$

In using Stafford's criterion to check whether a noetherian ring R has an artinian classical quotient ring (Theorem 11.13), one first has to find a right affiliated series and a left affiliated series for R. Given a specific R, it is often easier to find a series of ideals in R such that the corresponding ideal factors are torsionfree bimodules over prime factor rings of R. The following exercise (together with Exercise 11M) shows that the primes arising in this fashion can be used in place of the affiliated primes.

Exercise 11O. Let R be a noetherian ring and N its prime radical. By Corollary 8.8, there exists a chain of ideals $A_0 = 0 < A_1 < \cdots < A_n = R$ such that, for $i = 1, \dots, n$, the ideals $P_i = \text{l.ann}_R(A_i/A_{i-1})$ and $Q_i = \text{r.ann}_R(A_i/A_{i-1})$ are prime and A_i/A_{i-1} is torsionfree as a left (R/P_i)-module and as a right (R/Q_i)-module.

(a) Show that $\mathcal{C}(N)$ is a right Ore set if and only if P_i is minimal whenever Q_i is minimal. [Hints: If $\mathcal{C}(N)$ is right Ore, use Exercise 10E. For the converse, show that if $c \in \mathcal{C}(N)$, then $A_i/(cA_i+A_{i-1})$ is torsion as a right (R/N)-module for each $i = n, n-1, \dots, 1$.]

(b) Show that the set $\{P_i \mid Q_i \text{ is minimal}\}$ is the same for any choice of the ideals A_i. [Hint: Schreier Refinement.] $\square$

• NOTES •

Reduced Rank. This was introduced by Goldie [1964b, p. 274].

INC and FBN in Finite Ring Extensions. Theorems 11.6 and 11.7 are due to Letzter [1990, Corollary 2.4 and Proposition 4.9].

Pseudo-Ore Condition. A slightly more general version of Lemma 11.8 (using one-sided regularity) was proved by Goldie [1972a, Theorem 2.5].

Small's Theorem. This is due to Small [1966a, Theorems 2.10, 2.11, 2.12]. The necessity of the condition $\mathcal{C}_R(0) = \mathcal{C}_R(N)$ was shown independently by Talintyre [1966, Theorem 2.1], who had earlier proved that, if $\mathcal{C}_R(0) = \bigcap_{r=1}^{\infty} \mathcal{C}_R(N^r)$, then R has a right artinian classical right quotient ring [1963, Theorems 2.2, 3.2]. Two different criteria for a non-noetherian ring to have a right artinian classical right quotient ring were obtained by Small [1966b, Theorem C] and Robson [1967, Theorem 2.10]; the criterion involving reduced rank discussed in Exercises 11G–I is due to Warfield [1979a, Theorem 3].

Affiliated Prime Criteria for Artinian Classical Quotient Rings. Theorem 11.13 is due to Stafford [1982a, Proposition 1.3].

12. Links Between Prime Ideals

The ways in which affiliated series may be put together to form modules over a noetherian ring leads us to study certain connections, called *links*, between the prime ideals of the ring. In this chapter, we will introduce these links and see how they can give insight into the structure of modules. This will naturally lead us to a class of rings for which this connection between the links and the module theory is particularly satisfactory.

• LINKS •

To investigate the structure of a finitely generated module M over a noetherian ring R, we may start by choosing an affiliated series

$$M_0 = 0 < M_1 < \cdots < M_n = M.$$

This breaks M up into "layers" M_i/M_{i-1}, each of which is a fully faithful module over a prime factor ring of R. The study of M may thus be separated into two topics: first, the study of fully faithful modules over prime noetherian rings, and, second, the ways in which the layers M_i/M_{i-1} can be put together to form the module M. The first topic, particularly torsionfree modules, we have studied to a certain extent in previous chapters. Here we shall begin to study certain aspects of the second topic.

Since uniform modules are ubiquitous (e.g., our module M contains an essential direct sum of uniform submodules, and M is in turn essential in a direct sum of uniform modules), it is reasonable to begin our discussion with uniform modules. In the commutative case, the situation is very nice, as the following exercise shows.

Exercise 12A. Let M be a finitely generated uniform module over a commutative noetherian ring R and P the assassinator of M. Show first that M is annihilated by a power of P. [Hint: Exercise 5ZF.] Moreover, show that M has a unique affiliated series

$$M_0 = 0 < M_1 < \cdots < M_n = M,$$

that the affiliated primes all equal P, and that each M_i/M_{i-1} is a torsionfree (R/P)-module. [Hint: Set $X = R \setminus P$ and consider the RX^{-1}-module MX^{-1}.] $\square$

Thus, in the commutative case, the prime affiliated to the first layer of a uniform module is also affiliated to the higher layers. In the noncommutative case, however, it is no longer true that the affiliated primes of a uniform module are all the same, as follows.

Exercise 12B. Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ for some field k and $M = \begin{pmatrix} k & k \end{pmatrix}$, viewed as a right R-module. Show that M is uniform, that M has a unique affiliated series, that M has two distinct affiliated primes, and that M is not annihilated by any power of its assassinator. $\square$

Thus the question arises, how are the affiliated primes of a uniform module over a noncommutative noetherian ring related to the assassinator of the module? More generally (since the assassinator of a uniform module is its unique associated prime), how are the affiliated primes of a finitely generated module M related to the associated primes? Since the associated primes of M are annihilators of submodules of M (rather than of subfactor modules), we may view them as being more "accessible" than the affiliated primes. Then, if we can find a way to relate the affiliated primes to the associated primes, we will have found a useful way to "approximate" the annihilator of M (because M is annihilated by a product of affiliated primes).

As a basic building block for relating prime ideals to each other, which in a large class of rings will provide a solution to the problems just discussed, we introduce a notion of "links" between prime ideals. We then study how links arise and try to relate affiliated primes to assassinators and associated primes via chains of links.

Links are probably most easily introduced in the context of simple modules with co-artinian annihilators, as follows. Suppose that M is a uniform right module over a noetherian ring R, with an affiliated series $0 < U < M$ such that U and M/U are simple modules. Let Q and P be the corresponding affiliated primes of M, and assume that R/P and R/Q are artinian. Then $R/(P \cap Q)$ is a semisimple ring. On the other hand, U is not a direct summand of M, and so M is not a semisimple module. Since $MPQ = 0$, the ring R/PQ cannot be semisimple, and therefore $PQ \neq P \cap Q$.

Conversely, if $PQ \neq P \cap Q$, there must exist a module M of the above form, as follows.

Exercise 12C. Let P and Q be maximal ideals in a noetherian ring R, such that R/P and R/Q are artinian. If $PQ \neq P \cap Q$, show that there exists a uniform right R-module M with an affiliated series $0 < U < M$ and corresponding affiliated primes Q, P such that U and M/U are simple modules. [Hint: Assume that $PQ = 0$ and $\mathrm{l.ann}_R(P \cap Q) = P$, and then show that P is essential as a right ideal of R.] $\square$

In the context just discussed above, we say there is a "link from P to Q" if $PQ \neq P \cap Q$. We will want to generalize this to arbitrary prime ideals P and Q and to generalize Exercise 12C as well. The general definition of a link is necessarily more complicated, since we do not just want $(P \cap Q)/PQ$ to be nonzero – we need it to be large in a suitable sense. The definition we present will be justified by the results that follow.

Definition. Let P and Q be primes in a noetherian ring R. We say there is a *link from P to Q*, written $P \rightsquigarrow Q$, if there is an ideal A of R such that $P \cap Q > A \geq PQ$ and $(P \cap Q)/A$ is nonzero and torsionfree both as a left (R/P)-module and as a right (R/Q)-module. In this case, the bimodule $(P \cap Q)/A$ is called a *linking bimodule* between Q and P.

A link as just defined is sometimes called a *second layer link*, to distinguish it from other types of links that are sometimes used. One such is an *ideal link* (or *internal bond*) from P to Q, meaning that R contains ideals $I > J$ with $PI \leq J$ and $IQ \leq J$ such that I/J is a nonzero torsionfree left (R/P)-module and a torsionfree right (R/Q)-module. More generally, there is a *bimodule link* (or *bond*) from P to Q if there exists a nonzero $(R/P, R/Q)$-bimodule which is finitely generated and torsionfree on each side.

The conditions for the existence of a link simplify considerably in FBN rings (and in a larger class of noetherian rings), as we will see in Chapter 14. Namely, if P and Q are prime ideals in an FBN ring, then $P \rightsquigarrow Q$ if and only if $(P \cap Q)/PQ$ is faithful both as a left (R/P)-module and as a right (R/Q)-module (Exercise 14G).

Although the definition of a link involves only two-sided ideals, it is nonetheless asymmetric in that P appears as the left annihilator of the linking bimodule $(P \cap Q)/A$ while Q appears as the right annihilator. Hence, given any theorem in which a link $P \rightsquigarrow Q$ is related to right R-modules, in the corresponding left module theorem the roles of P and Q must be interchanged. Perhaps the easiest way to keep track of this is by using the opposite ring, R^{op}. Note first that, viewed as subsets of R^{op}, the primes P and Q are also prime ideals of R^{op}. Then observe that $P \rightsquigarrow Q$ in R if and only if $Q \rightsquigarrow P$ in R^{op}.

One restriction on possible links can be noted immediately: In a prime noetherian ring, the prime ideal 0 is not linked to or from any prime. Note also that links lift from factor rings: If P and Q are primes in a noetherian ring R, and if there is an ideal $B \leq P \cap Q$ such that $P/B \rightsquigarrow Q/B$ in R/B, then $P \rightsquigarrow Q$.

In general, one takes the point of view that the existence of a link $P \rightsquigarrow Q$ in R forces the rings R/P and R/Q to be similar in some aspects. The following exercise is a first instance of this philosophy, which we shall pursue more extensively in Chapter 14.

Exercise 12D. Let $P \rightsquigarrow Q$ be a link in a noetherian ring R. Show that R/P is artinian if and only if R/Q is artinian. [Hint: Lenagan's Theorem.] $\square$

Definition. The set of prime ideals in a ring R is called the *prime spectrum* of R, denoted $\mathrm{Spec}(R)$. For the present, we just regard $\mathrm{Spec}(R)$ as a set, although in Chapter 16 we will turn it into a topological space.

Definition. If R is a noetherian ring, the *graph of links* of R is the directed graph whose vertices are the elements of $\mathrm{Spec}(R)$, with an arrow from P to Q whenever $P \rightsquigarrow Q$. The underlying undirected graph is a disjoint union of connected components, and the corresponding subsets of $\mathrm{Spec}(R)$ (i.e., the vertex sets of these components) are called *cliques*. If $P \in \mathrm{Spec}(R)$, then we denote by $\mathrm{Clq}(P)$ the (unique) clique containing P. To unwind this definition a bit, observe that a prime Q belongs to $\mathrm{Clq}(P)$ if and only if there exist primes $P_1 = P$, P_2, ..., $P_t = Q$ such that, for all $i = 1, \dots, t-1$, either $P_i \rightsquigarrow P_{i+1}$ or $P_{i+1} \rightsquigarrow P_i$.

The graph of links in a noetherian ring contains – by definition – no double arrows, i.e., for any vertices P and Q, there is at most one arrow from P to Q. In general, the only other restriction which is known is that, for any vertex P, there are at most countably many arrows leaving P and at most countably many arrows arriving at P (Theorem 16.23). As the following exercise shows, all finite directed graphs without double arrows appear as link graphs of noetherian rings, in fact, as link graphs of finite dimensional algebras.

Exercise 12E. Let k be a field, n a positive integer, and Γ a directed graph with n vertices and no double arrows. The graph Γ is determined by its *incidence matrix* (a_{ij}), which is an $n \times n$ matrix of 0's and 1's where a_{ij} gives the number of arrows from the i-th vertex of Γ to the j-th vertex. Let R be the k-algebra given by generators e_i and x_{ij} for $i, j = 1, \dots, n$ subject to the following relations:

$$\begin{aligned} e_i^2 &= e_i & e_ie_j &= 0 \qquad (i \neq j) \\ e_1 + \cdots + e_n &= 1 & x_{ij} &= a_{ij}x_{ij} \\ x_{ij} = e_ix_{ij} &= x_{ij}e_j & x_{ij}x_{lm} &= 0 \end{aligned}$$

for all i, j, l, m. Then R is a finite dimensional k-algebra, its radical is the ideal N generated by the x_{ij}, and its prime ideals are the ideals $P_i = \langle e_t \mid t \neq i \rangle + N$ for $i = 1, \dots, n$. Show that, for $i, j = 1, \dots, n$ (including the cases $i = j$), we have $P_i \rightsquigarrow P_j$ if and only if $a_{ij} = 1$. Therefore the link graph of R is isomorphic to Γ. $\square$

The following exercises exhibit the cliques in some specific rings. We will later find all the cliques in any artinian ring (Corollary 12.13) and in any ring which is module-finite over its center when the center is noetherian (Theorem 13.10).

Exercise 12F. In a commutative noetherian ring, show that all cliques are singletons. $\square$

Exercise 12G. Show that, for the ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is a field, the graph of links has the form $P \longrightarrow Q$. (Though this is the most trivial possible example, it is not hard to see that, in any noetherian ring R, if P and Q are primes and $P \rightsquigarrow Q$ by way of a linking bimodule $(P \cap Q)/A$, then the ring R/A tends to resemble the triangular matrix ring above.) $\square$

Exercise 12H. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, and set $P = \begin{pmatrix} 2\mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $Q = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$, which are maximal ideals of R. Show that the link graph of R has the form displayed below.

$$\begin{tikzcd}
\langle 0 \rangle & P \arrow[r, bend left] & Q \arrow[l, bend left] & \langle 3 \rangle \arrow[loop above] & \langle 5 \rangle \arrow[loop above] & \langle 7 \rangle \arrow[loop above] & \cdots
\end{tikzcd}$$

Hence, one clique of R contains two elements and the others are all singletons. $\square$

Exercise 12I. Consider the ring $R = k[x][y; x(d/dx)]$, where k is a field of characteristic zero. This ring has two nonmaximal primes, namely 0, which has no links of any kind, and xR, which is linked only to itself (cf. Exercise 12D). Now consider the maximal ideals $M_\alpha = xR + (y - \alpha)R$ for $\alpha \in k$. Show that each M_α is linked to itself, and that otherwise there is a link $M_\alpha \rightsquigarrow M_\beta$ if and only if $\beta = \alpha - 1$. If k is algebraically closed, the M_α are the only maximal ideals (Exercise 3W). Hence, in that case, two cliques in R are singletons, and the remaining cliques are all countably infinite, as displayed below. $\square$

$$\begin{tikzcd}
\cdots & \arrow[l] M_{\alpha-1} \arrow[loop above] & \arrow[l] M_\alpha \arrow[loop above] & \arrow[l] M_{\alpha+1} \arrow[loop above] & \arrow[l] M_{\alpha+2} \arrow[loop above] & \arrow[l] \cdots
\end{tikzcd}$$

• LINKS AND SHORT AFFILIATED SERIES •

We now turn to the central concern of this chapter, which is how links or chains of links arise naturally between affiliated primes of a module M. As will be shown in the following section, the question can be reduced to the case in which M has an affiliated series of length 2, say $0 < U < M$, where there are only two affiliated primes, say Q (the assassinator of M) and P (the annihilator of M/U). Although there need not be a link between P and Q, we will see that there is a dichotomy saying that either $P \rightsquigarrow Q$ in a natural way or else a very different situation occurs. This is the context of the following result, often called *Jategaonkar's Main Lemma*, which will prove very useful in our study of links.

Theorem 12.1. [Jategaonkar] *Let R be a noetherian ring and M a right R-module with an affiliated series $0 < U < M$ and corresponding affiliated prime ideals Q and P, such that $U \leq_e M$. Let M' be a submodule of M,* *properly containing U, such that the ideal $A = \mathrm{ann}_R(M')$ is maximal among annihilators of submodules of M properly containing U. Then exactly one of the following two alternatives occurs:*

(i) *$P < Q$ and $M'P = 0$. In this case, M' and M'/U are faithful torsion (R/P)-modules.*

(ii) *$P \rightsquigarrow Q$ and $(P \cap Q)/A$ is a linking bimodule between P and Q. In this case, if U is torsionfree as a right (R/Q)-module, then M'/U is torsionfree as a right (R/P)-module.*

Proof. The existence of a submodule M' as described follows from the ACC on ideals in R. Our hypotheses are retained if M is replaced by M'. Thus, we may assume that $M' = M$, that is, all submodules of M properly containing U have the same annihilator as M, namely A. Note that any submodule N of M that is not contained in U also has annihilator A. (Since $\mathrm{ann}_R(N) \leq \mathrm{ann}_R(N \cap U) = Q = \mathrm{ann}_R(U)$, we must have $\mathrm{ann}_R(N) = \mathrm{ann}_R(N + U) = A$.) Note also that alternatives (i) and (ii) are mutually exclusive, since in case (i), $A = P = P \cap Q$, while in case (ii), $A < P \cap Q$.

We first consider the case that $M(P \cap Q) = 0$. Since $MQ \neq 0$ (because $U = \mathrm{ann}_M(Q)$), it follows that $Q \neq P$. Also, $MQP = 0$ because $QP \leq P \cap Q$, and $MQ \cap U \neq 0$ because $U \leq_e M$. Hence, as U is a fully faithful (R/Q)-module, $P \leq Q$, and thus $P < Q$. In particular, $UP = 0$. Since $P < Q$ and $P = \mathrm{ann}_R(M/U)$, we have $(M/U)Q \neq 0$, and hence $MQ \nleq U$. Our added hypothesis on M implies that $\mathrm{ann}_R(MQ) = \mathrm{ann}_R(M)$, whence $MP = 0$. Therefore case (i) occurs.

Now, in case (i), M is an essential extension of U as an (R/P)-module, and so M/U is a torsion (R/P)-module. Moreover, U is torsion as an (R/P)-module (because $UQ = 0$), and hence M is a torsion (R/P)-module. Finally, M/U is a faithful (R/P)-module by hypothesis, and consequently M is a faithful (R/P)-module.

We have shown that, if $M(P \cap Q) = 0$, then statement (i) holds. We now assume that $M(P \cap Q) \neq 0$, and we will show from this that statement (ii) holds.

Clearly $P \cap Q > A \geq PQ$; we must show that $(P \cap Q)/A$ is torsionfree as a right (R/Q)-module and as a left (R/P)-module. If B/A is the right torsion submodule of $(P \cap Q)/A$, then, by Lemma 8.3, B/A must be annihilated on the right by an ideal I properly containing Q. Hence, $P \cap Q \geq B \geq A$ and $BI \leq A$. Then $MBI = 0$ and $(MB \cap U)I = 0$. Since U is a fully faithful (R/Q)-module, $MB \cap U = 0$, and thus $MB = 0$. This shows that $B = A$, and therefore $(P \cap Q)/A$ is torsionfree as a right (R/Q)-module. Similarly, if C/A is the left torsion submodule of $(P \cap Q)/A$, then $JC \leq A$ for some ideal J properly containing P, and $MJC = 0$. Since M/U is a faithful (R/P)-module, $(M/U)J \neq 0$, and so $MJ \nleq U$. By the remarks in the first paragraph, $\mathrm{ann}_R(MJ) = A$, whence $C = A$. Therefore $(P \cap Q)/A$ is torsionfree as a left (R/P)-module and $P \rightsquigarrow Q$ via $(P \cap Q)/A$. Thus, case (ii) occurs.

Finally, we must show that, in case (ii), if U is a torsionfree (R/Q)-module, then M/U is a torsionfree (R/P)-module. Let D/U be the torsion submodule of M/U and consider any $m \in D$. Then $mc \in U$ for some $c \in \mathcal{C}_R(P)$. Since left multiplication by c gives a monomorphism from $(P \cap Q)/A$ into itself, we see that $\big(c(P \cap Q) + A\big)/A$ is an essential right submodule of $(P \cap Q)/A$ (Corollary 5.22), whence $(P \cap Q)/\big(c(P \cap Q) + A\big)$ is a torsion right (R/Q)-module. As $mc \in U$ and $UQ = 0$, we have $m\big(c(P \cap Q) + A\big) = 0$, and hence $m(P \cap Q)$ is a homomorphic image of $(P \cap Q)\big(c(P \cap Q) + A\big)$. Consequently, $m(P \cap Q)$ is torsion as a right (R/Q)-module. However, $m(P \cap Q) \leq MP \leq U$, and U is assumed to be torsionfree as a right (R/Q)-module, so $m(P \cap Q) = 0$. Thus $D(P \cap Q) = 0$. Since $P \cap Q > A$, our hypothesis on annihilators of submodules of M implies that $D = U$. We have therefore shown that M/U is a torsionfree (R/P)-module, as required. $\square$

It is an open question whether, given a module M as in Theorem 12.1, there can exist two different choices of submodule M' such that for one choice alternative (i) occurs while for the other choice alternative (ii) occurs. This would, in particular, entail both $P < Q$ and $P \rightsquigarrow Q$, and no examples of prime ideals with this behavior are known.

Examples of alternative (ii) in Theorem 12.1 are very easily obtained. For instance, the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ has an affiliated series $0 < 2\mathbb{Z}/4\mathbb{Z} < \mathbb{Z}/4\mathbb{Z}$ with both affiliated primes being $2\mathbb{Z}$, and $2\mathbb{Z}$ is linked to itself via the linking bimodule $2\mathbb{Z}/4\mathbb{Z}$. The following exercise gives an instance when alternative (i) occurs.

Exercise 12J. Let $S = A_1(k) = k[y][x; d/dy]$, where k is a field of characteristic zero; and $R = k + xS$, a subring of S. By Exercises 3ZB and 3ZC, R is noetherian, R/xS and S/R are simple right R-modules, and $(S/R)_R$ is faithful. Show that $(S/xS)_R$ is uniform (see Exercise 3ZD), that $(S/xS)_R$ has an affiliated series $0 < R/xS < S/xS$ with affiliated primes xS and 0, and that 0 is not linked to xS. $\square$

We have shown that affiliated series of uniform modules sometimes give rise to links between primes. We will now show that all links between primes arise in this way. In fact, the existence of links is equivalent to the existence of uniform modules with affiliated series in which the corresponding factor modules are torsionfree.

For use in upcoming proofs, we make the following observation: If A is a uniform submodule of a direct sum $B_1 \oplus \cdots \oplus B_n$, then A embeds in B_j for some j. (To see this, note that the projections $p_j : A \to B_1 \oplus \cdots \oplus B_n \to B_j$ satisfy $\ker(p_1) \cap \cdots \cap \ker(p_n) = 0$. Since A is uniform, $\ker(p_j) = 0$ for some j.) In particular, if $A \leq B^n$, then A must embed in B.

Theorem 12.2. [Jategaonkar, Brown] *Let R be a noetherian ring and P and Q prime ideals of R. Then $P \rightsquigarrow Q$ if and only if there exists a finitely generated uniform right R-module M with an affiliated series $0 < U < M$*

such that U is isomorphic to a (uniform) right ideal of R/Q and M/U is isomorphic to a uniform right ideal of R/P.

Proof. If such a module M exists, all submodules of M/U are torsionfree as (R/P)-modules. Hence, case (i) of Theorem 12.1 cannot occur, and thus $P \rightsquigarrow Q$.

Conversely, assume that $P \rightsquigarrow Q$, and let $(P \cap Q)/A$ be a linking bimodule. Without loss of generality, we may assume that $A = 0$, so that $P \cap Q$ is a nonzero torsionfree right (R/Q)-module and a torsionfree left (R/P)-module. Since $\operatorname{l.ann}_R(P \cap Q) = P$, we conclude that $\operatorname{l.ann}_R(P) \leq P$ and $\operatorname{l.ann}_R(Q) = P$ (since $PQ = 0$ already). If I is a nonzero right ideal of R, then either $IP = 0$, and so $I \leq \operatorname{l.ann}_R(P) \leq P$, or $IP \neq 0$, whence $I \cap P \neq 0$. Hence, P is essential as a right ideal of R.

Now $P \cap Q$ is a torsionfree right (R/Q)-module; and $P/(P \cap Q)$ is isomorphic to an ideal of R/Q, and so it too is torsionfree as a right (R/Q)-module. Hence, P is a torsionfree right (R/Q)-module, and so P_R has an essential submodule isomorphic to V^n, where V is a uniform right ideal of R/Q and $n = \operatorname{rank}(P_R) = \operatorname{rank}(R_R)$ (Proposition 7.24). Since P_R is essential in R_R, we know that $E(P_R) = E(R_R)$, and thus $E(R_R) \cong E^n$, where $E = E(V_R)$. Note that E has an essential submodule which is a torsionfree (R/Q)-module, whence $\operatorname{ann}_E(Q)$ is torsionfree as an (R/Q)-module.

Since $\operatorname{l.ann}_R(Q) = P$, it follows that $P = R \cap \operatorname{ann}_{E(R_R)}(Q)$. Hence, $(R/P)_R$ embeds in $(E/\operatorname{ann}_E(Q))^n$, and it follows that any uniform right ideal of R/P embeds in $E/\operatorname{ann}_E(Q)$. Let K be a submodule of E such that $K > \operatorname{ann}_E(Q)$ and $K/\operatorname{ann}_E(Q)$ is isomorphic to a uniform right ideal of R/P. Choose an element $x \in K$ not annihilated by Q, and set $M = xR$ and $U = \operatorname{ann}_M(Q)$. Clearly M/U is isomorphic to a uniform right ideal of R/P and U is isomorphic to a uniform right ideal of R/Q by Corollary 7.20. Since E is uniform, so is M, and it is clear from the definition of U that $0 < U < M$ is an affiliated series for M. $\square$

• LINKS AND AFFILIATED PRIMES •

In the situation of Theorem 12.1, alternative (i) is regarded as the "bad" case, since little information is obtained linking or otherwise closely relating P to Q. Thus, we shall mostly study alternative (ii), and in rings where this alternative always occurs we can develop a satisfactory solution to the annihilator problem discussed at the beginning of the first section. The terminology we shall use to denote this desirable situation is based on a rather technical notion of first and second layers of certain modules, due to Jategaonkar. A general notion of the first layer and the second layer of a module turns out to be cumbersome at best. However, as at the beginning of this chapter, we can think of a module M with an affiliated series $M_0 = 0 < M_1 < \cdots < M_n = M$ as being equipped with a set of layers M_i/M_{i-1} (depending on the affiliated series). Then the alternatives (i) and (ii) of Theorem 12.1 are statements about the kinds of modules that can occur as second layers of a module M, and we view the ring R as "good" if only second layers of type (ii) occur.

Definition. A prime ideal Q in a noetherian ring R is said to satisfy the *right strong second layer condition* if, given the hypotheses of Theorem 12.1, conclusion (i) never occurs. Similarly, Q is said to satisfy the *right second layer condition* if, given the hypotheses of Theorem 12.1 and the additional hypothesis that U is torsionfree as an (R/Q)-module, conclusion (i) never occurs. We say that the ring R satisfies the *right strong second layer condition* or the *right second layer condition* if this holds for all prime ideals Q of R. The *left strong second layer condition* and the *left second layer condition* are defined similarly. Finally, we say that the ring itself satisfies the *strong second layer condition* (or the *second layer condition*) if it satisfies these conditions on both the left and the right.

For some other conditions equivalent to the plain and strong second layer conditions, including ideal-theoretic criteria, see Exercises 12K, 12R and Theorems 12.6, 12.9.

For example, any artinian ring R satisfies the strong second layer condition (since R contains no primes P and Q satisfying $P < Q$). A more important example is that any right fully bounded noetherian ring R satisfies the right strong second layer condition, since no prime factors of R possess fully faithful torsion modules (Lemma 9.2). Some non-fully-bounded examples appear in Theorems 12.18, 12.19, and 13.5. Exercise 12J provides an example of a ring that fails to satisfy the right second layer condition. No examples are known of noetherian rings satisfying the second layer condition but not the strong second layer condition.

As stated, the second layer condition and the strong second layer condition involve properties of modules which are not necessarily finitely generated. We show next that it suffices to check these conditions in finitely generated uniform modules.

Proposition 12.3. *Let Q be a prime ideal in a noetherian ring R.*

(a) *Q satisfies the right strong second layer condition if and only if there does not exist a finitely generated uniform right R-module M with an affiliated series $0 < U < M$ and corresponding affiliated prime ideals Q, P such that M/U is uniform, $P < Q$, and $MP = 0$.*

(b) *Q satisfies the right second layer condition if and only if there does not exist a finitely generated uniform right R-module M with an affiliated series $0 < U < M$ and corresponding affiliated prime ideals Q, P such that U is a torsionfree (R/Q)-module, M/U is uniform, $P < Q$, and $MP = 0$.*

Proof. (a) Clearly, if Q satisfies the right strong second layer condition, there cannot exist a module M as described. Conversely, if Q fails to satisfy the condition, there exists a right R-module M_1 with an affiliated series $0 < U_1 < M_1$ and corresponding affiliated primes Q, P such that $U_1 \leq_e M_1$ while $P < Q$

and $M_1P = 0$. (Here M_1 and U_1 have taken the roles of the modules M' and U in the statement of Theorem 12.1.) Next, choose an element $x \in M_1 \setminus U_1$, and set $M_2 = xR$ and $U_2 = M_2 \cap U_1$. Then M_2 has an affiliated series $0 < U_2 < M_2$ with affiliated primes Q, P, and $U_2 \leq_e M_2$ and $M_2P = 0$.

Since M_2 is finitely generated, $E(M_2) = E_1 \oplus \cdots \oplus E_n$ for some uniform injective modules $E_1, \ldots, E_n$. Now $U_2 \leq_e E(M_2)$, and so $U_2 \cap E_i \neq 0$ for $i = 1, \ldots, n$. As U_2 is a fully faithful (R/Q)-module, so is $U_2 \cap E_i$, and hence so is the affiliated submodule $A_i = \operatorname{ann}_{E_i}(Q)$. Set

$$U_3 = A_1 \oplus \cdots \oplus A_n = \operatorname{ann}_{E(M_2)}(Q)$$

and observe that U_3 is a fully faithful (R/Q)-module. If $M_3 = M_2 + U_3$, then

$$M_3/U_3 \cong M_2/(M_2 \cap U_3) = M_2/U_2.$$

Thus, M_3 has an affiliated series $0 < U_3 < M_3$ with affiliated primes Q, P, and $U_3 \leq_e M_3$. Since $P < Q$, we also have $U_3P = 0$, and so $M_3P = 0$. Therefore $M_3 \leq B_1 \oplus \cdots \oplus B_n$, where $B_i = \operatorname{ann}_{E_i}(P)$ for $i = 1, \ldots, n$.

Choose a uniform submodule $V \leq M_3/U_3$. Then V embeds in the direct sum $(B_1/A_1) \oplus \cdots \oplus (B_n/A_n)$, whence V embeds in B_j/A_j for some j. Choose a submodule $M_4 \leq B_j$ such that $M_4 > A_j$ and $M_4/A_j \cong V$. Then M_4 is a uniform module with an affiliated series $0 < A_j < M_4$ with affiliated primes Q, P while M_4/A_j is uniform and $M_4P = 0$.

Finally, choose an element $y \in M_4 \setminus A_j$, and set $M = yR$ and $U = M \cap A_j$. Then M is a finitely generated uniform right R-module with an affiliated series $0 < U < M$ and affiliated primes Q, P such that M/U is uniform and $MP = 0$. Since we already have $P < Q$, part (a) is proved.

(b) The proof is similar to that of (a) and is left to the reader. □

Observe from Proposition 12.3 (or directly from the definitions) that a noetherian ring R satisfies the right (strong) second layer condition if and only if all prime factor rings R/P satisfy the right (strong) second layer condition.

Definition. Let R be a noetherian ring. A subset X of $\operatorname{Spec}(R)$ is said to be *right link closed* if, whenever $Q \in X$ and $P \rightsquigarrow Q$, it follows that $P \in X$. The *right link closure* of a set X of primes is the smallest right link closed subset of $\operatorname{Spec}(R)$ containing X.

It may not be immediately clear why we call the above property "right link closed" rather than "left link closed," but the point is that right link closure is what arises in the study of right modules (as is already suggested by Theorems 12.1 and 12.2) and in the study of right Ore sets (see Chapter 14).

Theorem 12.4. [Jategaonkar] *Let R be a noetherian ring satisfying the right strong second layer condition and M a nonzero right R-module which has an* *affiliated series. Then all of the affiliated primes of M are in the right link closure of the set* $\mathrm{Ass}(M)$ *of associated primes. In particular, M is annihilated by some product of primes from the right link closure of* $\mathrm{Ass}(M)$.

Proof. We use induction on the length of an affiliated series for M, say

$$M_0 = 0 < M_1 < \cdots < M_n = M,$$

with corresponding affiliated primes $P_1, \ldots, P_n$. Since $P_1 \in \mathrm{Ass}(M)$, the result is trivial when $n = 1$. Now assume that $n > 1$ and that, by induction, the primes $P_1, \ldots, P_{n-1}$ are in the right link closure of $\mathrm{Ass}(M_{n-1})$. Since $\mathrm{Ass}(M_{n-1}) \subseteq \mathrm{Ass}(M)$, these primes are also in the right link closure of $\mathrm{Ass}(M)$.

If M_{n-1}/M_{n-2} is not essential in M/M_{n-2}, choose a submodule M' of M such that $M' > M_{n-2}$ and $M' \cap M_{n-1} = M_{n-2}$, and observe that M'/M_{n-2} is a fully faithful (R/P_n)-module. Hence, an affiliated series for M' is

$$M_0 = 0 < M_1 < \cdots < M_{n-2} < M',$$

with corresponding affiliated primes $P_1, \ldots, P_{n-2}, P_n$. This affiliated series has length $n-1$, and so P_n is in the right link closure of $\mathrm{Ass}(M')$ by induction. Thus P_n is in the right link closure of $\mathrm{Ass}(M)$.

If, on the other hand, M_{n-1}/M_{n-2} is essential in M/M_{n-2}, then we apply Theorem 12.1 to the module M/M_{n-2}. Because R satisfies the right strong second layer condition, we conclude that $P_n \rightsquigarrow P_{n-1}$. Since P_{n-1} is in the right link closure of $\mathrm{Ass}(M)$, so is P_n.

The final statement of the theorem is immediate, since $MP_nP_{n-1}\cdots P_1 = 0$. □

Suppose in particular, in the setup of Theorem 12.4, that M has an essential affiliated submodule U with corresponding affiliated prime Q. (For instance, this occurs if M is uniform.) Then $\mathrm{Ass}(M) = \{Q\}$, and consequently all affiliated primes of M are in the right link closure of $\{Q\}$.

We next use Theorem 12.4 to establish some alternative characterizations of the strong second layer condition.

Lemma 12.5. *Let $P < J$ be ideals in a ring R, with P prime. Then the following conditions are equivalent:*

(a) *For any right ideal $K \geq P$, there is an ideal $I > P$ such that $K \cap I \leq KJ + P$.*

(b) *For any finitely generated right (R/P)-module A and any submodule $B \leq A$, there is an ideal $I > P$ such that $B \cap AI \leq BJ$.*

(c) *There does not exist a faithful finitely generated right (R/P)-module containing an essential submodule annihilated by J.*

Proof. Since all of the statements in the lemma reduce to ideals and modules of R/P, there is no loss of generality in assuming that $P = 0$.

(a) $\Longrightarrow$ (c): Let A be a finitely generated right R-module with an essential submodule B such that $BJ = 0$; we must show that A is unfaithful. There is a finite set of generators for A, say $x_1, \dots, x_n$. Set $K_j = \{r \in R \mid x_j r \in B\}$ for each j and note that $x_j K_j J = 0$. By (a), there exists a nonzero ideal I_j such that $K_j \cap I_j \leq K_j J$, and so $x_j(K_j \cap I_j) = 0$. Now, if $r \in I_j$ and $x_j r \in B$, then $r \in I_j \cap K_j$ and $x_j r = 0$. Thus, $x_j I_j \cap B = 0$. Since $B \leq_e A$, we obtain $x_j I_j = 0$. Therefore $A(I_1 \cap \cdots \cap I_n) = 0$, and $I_1 \cap \cdots \cap I_n \neq 0$ because R is a prime ring.

(c) $\Longrightarrow$ (b): It suffices to find an ideal $I > 0$ such that $(B/BJ) \cap (A/BJ)I = 0$, and so there is no loss of generality in assuming that $BJ = 0$. Choose a submodule $C \leq A$ maximal such that $B \cap C = 0$. By Proposition 5.7, $(B + C)/C$ is an essential submodule of A/C. This submodule is annihilated by J, whence (c) implies that A/C is unfaithful. Thus, there exists a nonzero ideal I such that $AI \leq C$. Therefore $B \cap AI \leq B \cap C = 0$, establishing (b).

(b) $\Longrightarrow$ (a): Given K, take $A = R_R$ and $B = K$ and apply (b). $\square$

Theorem 12.6. *For a noetherian ring R, the following conditions are equivalent:*

(a) *R satisfies the right strong second layer condition.*

(b) *For each prime P of R, every finitely generated essential (R/P)-module extension of an unfaithful right (R/P)-module is unfaithful (as an (R/P)-module).*

(c) *For any primes $P < Q$ in R and any right ideal $K \geq P$, there exists an ideal $I > P$ such that $K \cap I \leq KQ + P$.*

Proof. (a) $\Longrightarrow$ (b): Let $P \in \operatorname{Spec}(R)$ and let $U \leq_e M$ be an essential extension of finitely generated right (R/P)-modules such that U is unfaithful; we must show that M is an unfaithful (R/P)-module. Since R/P satisfies the right strong second layer condition, there is no loss of generality in assuming that $P = 0$; hence, R is a prime ring.

Every associated prime of M is also an associated prime of U (because $U \leq_e M$) and so is nonzero (because U is unfaithful). Since the prime 0 cannot be linked to any nonzero prime, it follows that all primes in the right link closure of $\operatorname{Ass}(M)$ are nonzero. By Theorem 12.4, M is annihilated by a product of nonzero primes, and therefore M is unfaithful, as desired.

(b) $\Longrightarrow$ (a) is clear from Proposition 12.3, and (b) $\Longrightarrow$ (c) follows from Lemma 12.5 (taking $J = Q$).

(c) $\Longrightarrow$ (b): We must show that, for any ideal $J > P$, there does not exist a faithful finitely generated right (R/P)-module containing an essential submodule annihilated by J. By Lemma 12.5, it suffices to show that, for any right ideal $K \geq P$, there is an ideal $I > P$ such that $K \cap I \leq KJ + P$.

There exist primes $Q_1, Q_2, \dots, Q_n \geq J$ such that $Q_1 Q_2 \cdots Q_n \leq J$. Set $K_0 = K$ and $K_j = KQ_1Q_2 \cdots Q_j + P$ for $j = 1, \dots, n$. By (c), there exist ideals $I_j > P$ for $j = 1, \dots, n$ such that $K_{j-1} \cap I_j \leq K_{j-1} Q_j + P = K_j$.

Therefore

$$K \cap I_1 \cap \cdots \cap I_n \leq K_n = KQ_1Q_2 \cdots Q_n + P \leq KJ + P.$$

Since P is prime, $I_1 \cap \cdots \cap I_n > P$, and the proof is complete. $\square$

Exercise 12K. Show that a noetherian ring R satisfies the right strong second layer condition if and only if, for every finitely generated right R-module A, all annihilator primes of A are associated primes. $\square$

Theorem 12.7. [Jategaonkar] *Let R be a noetherian ring satisfying the right second layer condition, M a nonzero right R-module, and assume that M has an affiliated series*

$$M_0 = 0 < M_1 < \cdots < M_n = M,$$

with corresponding affiliated primes $P_1, \dots, P_n$. For all $P \in \operatorname{Ass}(M)$, assume that $\operatorname{ann}_M(P)$ is torsionfree as an (R/P)-module. Then, for $i = 1, \dots, n$, the prime P_i is in the right link closure of $\operatorname{Ass}(M)$ and M_i/M_{i-1} is torsionfree as a right (R/P_i)-module. In particular, M is annihilated by the product $P_nP_{n-1} \cdots P_1$ of primes from the right link closure of $\operatorname{Ass}(M)$.

Proof. The proof is analogous to that of Theorem 12.4, where at each stage we apply the following observation to conclude that the appropriate factors are torsionfree.

Claim: Let N be a right R-module with an affiliated series $0 < U < N$ and corresponding affiliated primes Q and P, such that $U \leq_e N$. If U is a torsionfree (R/Q)-module, then N/U is a torsionfree (R/P)-module.

If the claim fails, the torsion submodule of the (R/P)-module N/U equals V/U for some submodule V of N that properly contains U. Then V/U is a fully faithful (R/P)-module, and so $0 < U < V$ is an affiliated series for V, with corresponding affiliated primes Q and P. Choose a submodule V' of V, properly containing U, such that $\operatorname{ann}_R(V')$ is maximal among annihilators of submodules of V properly containing U. We now apply Theorem 12.1 to V. Since R has been assumed to have the right second layer condition, and since U is a torsionfree (R/Q)-module, case (ii) of the theorem must hold, and V'/U is a torsionfree (R/P)-module. But this is impossible, because V'/U is a nonzero submodule of the torsion module V/U. Therefore the claim holds. $\square$

Corollary 12.8. *Let R be a noetherian ring, E a uniform injective right R-module, and P the assassinator of E. Assume either that R satisfies the right strong second layer condition, or that R satisfies the right second layer condition and $\operatorname{ann}_E(P)$ is a torsionfree (R/P)-module. Then each finitely generated submodule of E is annihilated by a product of primes from the right link closure of $\{P\}$.*

Proof. Any nonzero finitely generated submodule $M \leq E$ has an affiliated series, and P is the unique associated prime of M. Consequently, either Theorem 12.4 or Theorem 12.7 shows that M is annihilated by a product of primes from the right link closure of $\{P\}$. $\square$

From Theorem 12.7, we next derive a version of Theorem 12.6 tailored to the plain second layer condition. This requires the following analog of Lemma 12.5.

Exercise 12L. Let $P < Q$ be prime ideals in a right noetherian ring R. Show that the following conditions are equivalent:

(a) For any right ideals $K \geq L \geq P$ such that K/L is a torsionfree (R/Q)-module, there is an ideal $I > P$ such that $K \cap I \leq L$.

(b) For any finitely generated right (R/P)-modules $A \geq B \geq C$ such that B/C is a torsionfree (R/Q)-module, there is an ideal $I > P$ such that $B \cap AI \leq C$.

(c) There does not exist a faithful finitely generated right (R/P)-module containing an essential submodule which is a torsionfree (R/Q)-module. $\square$

Theorem 12.9. *For a noetherian ring R, the following conditions are equivalent:*

(a) *R satisfies the right second layer condition.*

(b) *For all primes $P < Q$ in R, every finitely generated essential (R/P)-module extension of a torsionfree right (R/Q)-module is unfaithful (as an (R/P)-module).*

(c) *For any primes $P < Q$ in R and any right ideals $K \geq L \geq P$ such that K/L is a torsionfree (R/Q)-module, there exists an ideal $I > P$ such that $K \cap I \leq L$.*

Proof. This is proved in the same manner as Theorem 12.6, using Theorem 12.7 and Exercise 12L in place of Theorem 12.4 and Lemma 12.5. $\square$

To illustrate another consequence of the second layer condition, we prove the following result on the existence of links. Note that the proposition does not say that the primes P_i are distinct, nor that they are different from Q. In fact, if R is commutative, then all the P_i must equal Q.

Proposition 12.10. *Let R be a noetherian ring and Q a nonminimal prime in R. If R satisfies the right second layer condition, there exist nonminimal primes $P_1, P_2, \dots$ in R with links $\cdots \rightsquigarrow P_2 \rightsquigarrow P_1 \rightsquigarrow Q$. Similarly, if R satisfies the left second layer condition, there exist nonminimal primes $P_1', P_2', \dots$ in R with links $Q \rightsquigarrow P_1' \rightsquigarrow P_2' \rightsquigarrow \cdots$.*

Proof. Assume that R satisfies the right second layer condition, the case of the left second layer condition being symmetric.

Choose a prime $Q_0 < Q$. Since it suffices to find a sequence of primes linked to Q/Q_0 in R/Q_0, we may work in R/Q_0. Hence, there is no loss of generality in assuming that R is a prime ring and $Q \neq 0$. In particular, there exists a regular element $x \in Q$.

Set $E = E((R/Q)_R)$ and choose a nonzero element $a \in E$. As in Proposition 7.11, E is divisible, so there exists $b \in E$ such that $bx = a$. In particular, $bQ \neq 0$. Set $M = bR$ and let $M_0 = 0 < M_1 < \cdots < M_n = M$ be an affiliated series for M, with corresponding affiliated primes $P_1, \ldots, P_n$. Since $(R/Q)_R \leq_e E$, there is just one associated prime for E, namely Q, and $\operatorname{ann}_E(Q) \leq_e E$. Consequently, Q is also the unique associated prime of M, whence $P_1 = Q$ and $M_1 = \operatorname{ann}_M(Q) \leq_e M$. Note that since $(R/Q)_R$ is torsionfree as a right (R/Q)-module, so is M_1. Since $b \notin M_1$, we must have $n \geq 2$. Therefore, applying Theorem 12.1 to the module M_2 and invoking the second layer condition hypothesis, we conclude that $P_2 \rightsquigarrow Q$.

We have now proved that there exists a prime, which we relabel as P_1, such that $P_1 \rightsquigarrow Q$. Since 0 is not linked to any prime, we also have $P_1 \neq 0$. This result can now be iterated inductively, yielding a sequence of nonzero primes $P_1, P_2, \ldots$ such that $P_{i+1} \rightsquigarrow P_i$ for all i. $\square$

• ARTINIAN RINGS •

Any noetherian or artinian ring can be written in a unique way as a direct product of indecomposable rings. In this section, we show as an application of Theorem 12.4 that in an artinian ring (more generally, in a noetherian ring in which all primes are maximal), the cliques are in one-to-one correspondence with these indecomposable factors.

Definition. A ring R is *indecomposable* (as a ring) provided R is not isomorphic to a direct product of two nonzero rings. Note that R is indecomposable if and only if R cannot be expressed as a direct sum of two nonzero ideals.

If a ring R is isomorphic to a direct product $R_1 \times \cdots \times R_n$ of rings R_i, it is often convenient to identify R with $R_1 \times \cdots \times R_n$ and to identify each R_i with a nonunital subring of R, as follows. For $i = 1, \ldots, n$, let e_i denote the n-tuple in R with 1 in the i-th position and 0 elsewhere. These elements e_i form a *complete set of orthogonal central idempotents* in R, meaning that each e_i is an idempotent element of the center of R, that $e_i e_j = 0$ for $i \neq j$, and that $e_1 + \cdots + e_n = 1$. Each R_i is identified with the subset $e_i R$ of R, which is a ring with identity e_i and also an ideal of R. Observe that if I is any ideal of R, then

$$I = e_1 I + \cdots + e_n I = (I \cap R_1) + \cdots + (I \cap R_n).$$

In particular, if P is a prime ideal of R, then, since $R_i R_j = 0$ for $i \neq j$, we see that P must contain all but one of the R_i. Thus, if $P \supseteq R_i$ for all indices $i \neq k$, then $1 - e_k = \sum_{i \neq k} e_i \in P$ and $P = (1 - e_k)R + (P \cap R_k)$, where $P \cap R_k$ is a prime ideal of R_k. Hence, P is in an obvious sense associated with the factor R_k. Note also that P equals the inverse image of the prime

$P \cap R_k$ under the natural projection map $\pi_k : R \to R_k$. Thus, we can state the following conclusion:

$$\operatorname{Spec}(R) = \pi_1^{-1} \operatorname{Spec}(R_1) \sqcup \cdots \sqcup \pi_n^{-1} \operatorname{Spec}(R_n).$$

Our next observation, applied in the special case in which $c = 1 - e_k$, shows in particular that primes associated with different factors of a ring decomposition cannot be linked.

Lemma 12.11. *Let R be a noetherian ring, P a prime of R, and c a central element in R. If $c \in P$, then all primes in* $\operatorname{Clq}(P)$ *contain c.*

Proof. It suffices to show that if Q is a prime satisfying either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$, then $c \in Q$. We prove only the first case, the second being symmetric. Let $(P \cap Q)/A$ be a linking bimodule for a link $P \rightsquigarrow Q$. Then, since $PQ \subseteq A$, it follows that $c(P \cap Q) \subseteq A$. Since c is central, we have $(P \cap Q)c \subseteq A$, and therefore $c \in \operatorname{r.ann}_R\big((P \cap Q)/A\big) = Q$. $\square$

Proposition 12.12. *If R is an indecomposable noetherian ring in which all prime ideals are maximal, then* $\operatorname{Spec}(R)$ *consists of a single clique.*

Proof. Observe that R satisfies the strong second layer condition. Choose a clique X in $\operatorname{Spec}(R)$ and set $Y = \operatorname{Spec}(R) \setminus X$. Set

$$I = \{a \in R \mid aP_1P_2 \cdots P_m = 0 \text{ for some } P_1, \dots, P_m \in X\}$$
$$J = \{a \in R \mid aQ_1Q_2 \cdots Q_n = 0 \text{ for some } Q_1, \dots, Q_n \in Y\}$$

and observe that I and J are ideals of R. Since all primes of R are maximal, no maximal ideal of R can contain both a prime from X and a prime from Y. Consequently,

$$\big(P_1P_2 \cdots P_m\big) + \big(Q_1Q_2 \cdots Q_n\big) = R$$

for any $P_1, \dots, P_m \in X$ and $Q_1, \dots, Q_n \in Y$. It follows that $I \cap J = 0$.

If K/I is a nonzero right R-submodule of R/I, then K/I cannot be annihilated (on the right) by a prime in X, for then each element of K would be annihilated by a product of primes in X and so $K \subseteq I$. Thus, all the associated primes of $(R/I)_R$ are in Y. Since Y is right link closed, Theorem 12.4 shows that $(R/I)_R$ is annihilated by some product of primes from Y.

Now $R/(I + J)$ is annihilated (on the right) by some product of primes from Y. Similarly, it is also annihilated by some product of primes from X, and we conclude that $R/(I + J) = 0$. Consequently, $R = I \oplus J$. Hence, as R is indecomposable, either $I = R$ or $J = R$.

If $J = R$, some product of primes from Y equals zero, and so $\operatorname{Spec}(R) = Y$, which is impossible. Therefore $I = R$, and consequently $\operatorname{Spec}(R) = X$. $\square$

Corollary 12.13. *Let R be a noetherian ring in which all prime ideals are maximal, and let $R = R_1 \times \cdots \times R_n$, where each R_i is an indecomposable ring. Then each* $\operatorname{Spec}(R_i)$ *consists of a single clique. If $e_1, \ldots, e_n$ are the corresponding central idempotents in R (so that $e_1 + \cdots + e_n = 1$ and each $R_i = e_i R$), then the cliques in* $\operatorname{Spec}(R)$ *are the sets*

$$X_i = \{P \in \operatorname{Spec}(R) \mid 1 - e_i \in P\}$$

for $i = 1, \ldots, n$.

Proof. The existence of a ring decomposition $R = R_1 \times \cdots \times R_n$ with each R_i indecomposable follows from the chain conditions on R, and by Proposition 12.12 each R_i has only one clique of primes. It follows that each of the sets X_i is contained in a clique of $\operatorname{Spec}(R)$. Then, by Lemma 12.11, each X_i is a clique, and since any prime of R must contain one of the idempotents $1 - e_i$, there are no other cliques. $\square$

If R is an artinian ring, a decomposition $R = R_1 \times \cdots \times R_n$, where each R_i is an indecomposable ring, is called a *block decomposition* of R.

Corollary 12.14. *Let R be a noetherian ring and A a finitely generated right R-module such that $R/\operatorname{ann}_R(A)$ is artinian. There is a direct sum decomposition $A = A_1 \oplus \cdots \oplus A_n$ such that, for $i = 1, \ldots, n$, the annihilators of the composition factors of A_i all lie in the same clique of* $\operatorname{Spec}(R)$.

Proof. Set $I = \operatorname{ann}_R(A)$. By Corollary 12.13, there exist orthogonal central idempotents $e_1, \ldots, e_n$ in R/I such that $e_1 + \cdots + e_n = 1$ and each of the sets

$$X_i = \{P \in \operatorname{Spec}(R/I) \mid 1 - e_i \in P\}$$

is a clique of $\operatorname{Spec}(R/I)$. Then $A = A_1 \oplus \cdots \oplus A_n$, where each $A_i = Ae_i$.

If Y_i is the set of primes of R occurring as annihilators of composition factors of A_i, then $Q/I \in X_i$ for all $Q \in Y_i$, and so $\{Q/I \mid Q \in Y_i\}$ is contained in a clique of $\operatorname{Spec}(R/I)$. It follows that Y_i is contained in a clique of $\operatorname{Spec}(R)$. $\square$

• NORMAL ELEMENTS •

As we have just seen in Lemma 12.11, the presence of central elements in a prime P of a noetherian ring imposes certain restrictions on the primes in the clique of P. We now discuss a larger class of elements which impose similar restrictions but which occur more widely in noncommutative rings.

Definition. A *normal element* in a ring R is any element $x \in R$ such that $xR = Rx$. This definition should be compared with the concept of a normalizing element in a ring extension given in Chapter 8. A normal element in R also normalizes R, but it is helpful to be able to say "x is a normal element of R" to emphasize that $x \in R$.

Several immediate observations are in order. First, any product of normal elements is normal. Second, if x is a nonzero normal element in a prime ring R, then, since the ideal $xR = Rx$ has zero annihilator on each side, x is a regular element. Third, in any ring, the powers of a given normal element form an Ore set (Exercise 10X).

For example, if α is an automorphism of a ring S, then the indeterminate x is a normal element of both $S[x;\alpha]$ and $S[x^{\pm 1};\alpha]$. Similarly, the generators $x_1, \dots, x_n$ in any quantum affine space $\mathcal{O}_{\mathbf{q}}(k^n)$ or quantum torus $\mathcal{O}_{\mathbf{q}}((k^{\times})^n)$ are normal elements.

Lemma 12.15. *Let R be a noetherian ring, P a prime of R, and x a normal element in R. If $x \in P$, then all primes in $\operatorname{Clq}(P)$ contain x.*

Proof. By induction and symmetry, it suffices to prove that if $P \rightsquigarrow Q$, then $x \in Q$. Let $(P \cap Q)/A$ be a linking bimodule for this link. Since the same bimodule provides a link $P/A \rightsquigarrow Q/A$ in R/A, there is no loss of generality in assuming that $A = 0$. Now $PQ = 0$, and consequently $x(P \cap Q) = 0$. As noted above, the set $X = \{1, x, x^2, \dots\}$ is an Ore set in R, and then X is reversible by Proposition 10.7. Thus, $P \cap Q$ is X-torsion on the right, and hence $(P \cap Q)x^n = 0$ for some n. Since $P \cap Q$ is a nonzero torsionfree right (R/Q)-module, it is faithful, so we find that $x^n \in Q$. Normality then yields $(xR)^n = x^n R \le Q$, and therefore $x \in Q$. $\square$

For example, let $R = \mathcal{O}_q(k^2)$, where k is an algebraically closed field and $q \in k^{\times}$ is not a root of unity. Since the generators $x, y \in R$ are normal elements, Lemma 12.15 shows that any prime in the clique of xR (respectively, yR) must contain x (respectively, y). On the other hand, none of the co-artinian primes $\langle x - \alpha,\, y\rangle$ or $\langle x,\, y - \beta\rangle$ (for $\alpha, \beta \in k$) can be in the clique of xR or yR, because of Exercise 12D. Now, by Exercise 10P, these co-artinian primes together with xR, yR, and 0 are the only primes of R. Thus, the cliques of xR, yR, and 0 are all singletons. The remaining cliques in R can be calculated as follows.

Exercise 12M. Let $R = \mathcal{O}_q(k^2)$, where k is an algebraically closed field and $q \in k^{\times}$ is not a root of unity, and let $\alpha, \beta \in k$. Show that:

(a) $\langle x - \alpha,\, y\rangle \rightsquigarrow \langle x - \beta,\, y\rangle$ if and only if $\alpha = \beta$ or $\alpha = q\beta$.
(b) $\langle x,\, y - \alpha\rangle \rightsquigarrow \langle x,\, y - \beta\rangle$ if and only if $\alpha = \beta$ or $\beta = q\alpha$.
(c) $\langle x - \alpha,\, y\rangle \not\rightsquigarrow \langle x,\, y - \beta\rangle$ unless $\alpha = \beta = 0$.

Conclude that the cliques of these maximal ideals have the forms shown below, where $\lambda \in k^{\times}$. $\square$

$$\begin{tikzcd} \langle x, y\rangle \arrow[loop above] \end{tikzcd}$$

$$\begin{tikzcd}
\cdots & \langle x-q^{-1}\lambda,\, y\rangle \arrow[l] \arrow[loop above] & \langle x-\lambda,\, y\rangle \arrow[l] \arrow[loop above] & \langle x-q\lambda,\, y\rangle \arrow[l] \arrow[loop above] & \cdots \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \langle x,\, y-q^{-1}\lambda\rangle \arrow[r] \arrow[loop above] & \langle x,\, y-\lambda\rangle \arrow[r] \arrow[loop above] & \langle x,\, y-q\lambda\rangle \arrow[r] \arrow[loop above] & \cdots
\end{tikzcd}$$

Exercise 12N. Let $R = \mathcal{O}_q(k^n)$, where k is an algebraically closed field, $q \in k^\times$ is not a root of unity, and $n \geq 3$. Set $M_j(\alpha) = \langle x_j - \alpha\rangle + \langle x_i \mid i \neq j\rangle$ for $j = 1, \dots, n$ and $\alpha \in k$. For $\alpha, \beta \in k$ and $i, j = 1, \dots, n$, show that:

(a) $M_i(\alpha) \not\rightsquigarrow M_j(\beta)$ unless $i = j$ or $\alpha = \beta = 0$.
(b) $M_1(\alpha) \rightsquigarrow M_1(\beta)$ if and only if $\alpha \in \{\beta, q\beta\}$.
(c) $M_n(\alpha) \rightsquigarrow M_n(\beta)$ if and only if $\alpha \in \{\beta, q^{-1}\beta\}$.
(d) If $1 < j < n$, then $M_j(\alpha) \rightsquigarrow M_j(\beta)$ if and only if $\alpha \in \{\beta, q\beta, q^{-1}\beta\}$.

Conclude that the clique of $M_j(\alpha)$ consists of maximal ideals $M_j(\beta)$ satisfying the conditions above. In particular, for $1 < j < n$ and $\lambda \in k^\times$, the clique of $M_j(\lambda)$ has the form shown below. $\square$

$$\begin{tikzcd}
\cdots \arrow[r, shift left] & M_j(q^{-1}\lambda) \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & M_j(\lambda) \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & M_j(q\lambda) \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & \cdots \arrow[l, shift left]
\end{tikzcd}$$

Lemma 12.16. *Let R be a noetherian ring, $x \in R$ a normal element, and $B \leq_e A$ finitely generated right R-modules. If $Bx = 0$, then $Ax^n = 0$ for some $n \in \mathbb{N}$.*

Proof. Set $A_i = \operatorname{ann}_A(x^i)$ for $i = 0, 1, \dots$ and note that, since $x^i R = Rx^i$, the set A_i is a submodule of A. Moreover, $A_i \leq A_{i+1}$ for all i. Hence, there is a positive integer n such that $A_n = A_{n+1}$.

If $A_n \neq A$, choose an element $a \in A \setminus A_n$, so that $ax^n \neq 0$. Since $B \leq_e A$, there exists $r \in R$ such that $ax^n r$ is a nonzero element of B. In particular, $ax^n rx = 0$. By normality, $x^n r = sx^n$ for some $s \in R$. Then $asx^{n+1} = 0$, whence $as \in A_{n+1} = A_n$, and consequently $asx^n = 0$. But the last equation implies that $ax^n r = 0$, contradicting our choice of r. Therefore $A_n = A$, and the lemma is proved. $\square$

Exercise 12O. Show that Lemma 12.16 implies the following extension of Exercise 5ZF: If R is a noetherian ring, J an ideal of R generated by normal elements, and $B \leq_e A$ finitely generated right R-modules such that $BJ = 0$, then $AJ^n = 0$ for some $n \in \mathbb{N}$. $\square$

Lemma 12.16 will allow us to see that any noetherian ring with a suitably large supply of normal elements satisfies the strong second layer condition.

Definition. A ring R has *normal separation* if, for every comparable pair of prime ideals $P < Q$ in R, there exists an element $x \in Q \setminus P$ such that the coset $x + P$ is a normal element of the ring R/P.

For example, let $R = k[x][y; x(d/dx)]$, where k is an algebraically closed field of characteristic zero. Then x is a normal element in R and R/xR is commutative. As shown in Exercise 3W, every nonzero prime of R contains x. It follows immediately that R has normal separation.

Similarly, consider the algebra $R = \mathcal{O}_q(k^2)$, where k is an algebraically closed field and $q \in k^\times$ is not a root of unity. Here the standard generators x and y are normal elements, the factor rings R/xR and R/yR are commutative, and every nonzero prime of R contains x or y (Exercise 10P). Again, it follows immediately that R has normal separation.

Theorem 12.17. *Any noetherian ring R with normal separation satisfies the strong second layer condition.*

Proof. By Theorem 12.6, it suffices to show that, if P is any prime of R and $B \leq_e A$ any finitely generated (R/P)-modules such that B is unfaithful, then A is unfaithful. We may assume that $P = 0$, and that A and B are right modules. Let $I = \operatorname{ann}_R(B)$, a nonzero ideal of R. There exist primes $Q_1, Q_2, \dots, Q_t \geq I$ such that $Q_1 Q_2 \cdots Q_t \leq I$. By normal separation, each Q_i contains a nonzero normal element x_i. Since the x_i are regular, the product $x = x_1 x_2 \cdots x_t$ is a regular, normal element. Moreover, $x \in I$, whence $Bx = 0$. Lemma 12.16 now implies that $Ax^n = 0$ for some n, and therefore A is unfaithful, as desired. $\square$

Theorem 12.17 immediately implies that the algebras $k[x][y; x(d/dx)]$ and $\mathcal{O}_q(k^2)$ discussed above satisfy the strong second layer condition. Both of these are skew polynomial rings over commutative rings, and so one might say that they are only one step away from the commutative realm. In fact, the second layer condition holds in a large class of iterated differential operator rings over commutative noetherian rings, including all enveloping algebras of finite dimensional solvable Lie algebras. We shall address the "completely solvable" case (which includes all finite dimensional solvable Lie algebras over algebraically closed fields) in Theorem 12.19. The second layer condition also holds in a large class of quantized coordinate rings, which we now verify in the most accessible case.

Theorem 12.18. *Let $\mathbf{q} = (q_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over a field k. Then the algebras $\mathcal{O}_{\mathbf{q}}(k^n)$ and $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ both have normal separation and so satisfy the strong second layer condition.*

Proof. We first consider the algebra $R = \mathcal{O}_{\mathbf{q}}(k^n)$, with standard generators $x_1, \dots, x_n$. Note that R has a k-basis consisting of monomials $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, and that if x is such a monomial, then $x_i x = q_{i1}^{m_1} q_{i2}^{m_2} \cdots q_{in}^{m_n} x x_i$ for all i. Any element $r \in R$ is a linear combination of these monomials; let us refer to the number of monomials appearing with nonzero coefficients as the *length* of r.

Given primes $P < Q$ in R, choose an element $y \in Q \setminus P$ of minimal length; we claim that $y + P$ is normal in R/P. Write $y = \alpha_1 y_1 + \cdots + \alpha_\ell y_\ell$, where ℓ is the length of y, the y_j are distinct monomials, and the α_j are nonzero scalars. For $i = 1, \ldots, n$ and $j = 1, \ldots, \ell$, there is some $\lambda_{ij} \in k^\times$ such that $x_i y_j = \lambda_{ij} y_j x_i$. Hence,

$$x_i y - \lambda_{i1} y x_i = \alpha_2(\lambda_{i2} - \lambda_{i1}) y_2 x_i + \cdots + \alpha_\ell(\lambda_{i\ell} - \lambda_{i1}) y_\ell x_i,$$

which is an element of Q with length at most $\ell - 1$. By the minimality of ℓ, we must have $x_i y - \lambda_{i1} y x_i \in P$. Consequently, for any monomial x there exists $\lambda \in k^\times$ such that $xy \equiv \lambda yx \pmod{P}$. It follows that $y + P$ is a normal element of R/P, as claimed, which proves that R has normal separation.

Now let $S = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$, and let $P < Q$ be primes in S. Observe that S is a ring of fractions for R with respect to the multiplicative set generated by the x_i. Hence, P and Q contract to distinct primes $P \cap R < Q \cap R$ in R (Theorem 10.20). The proof above shows that there exist an element $y \in (Q \cap R) \setminus (P \cap R)$ and scalars $\lambda_i \in k^\times$ such that $x_i y - \lambda_i y x_i \in P \cap R$ for all $i = 1, \ldots, n$. Then $y \in Q \setminus P$ and $x_i^{\pm 1} y - \lambda_i^{\pm 1} y x_i^{\pm 1} \in P$ for all i, from which we see that $y + P$ is normal in S/P. Therefore S has normal separation. $\square$

An argument somewhat analogous to the proof of Theorem 12.18 can be used to show that the enveloping algebra of any finite dimensional completely solvable Lie algebra has normal separation. Rather than developing the necessary Lie theory here, we prove an equivalent theorem that can be stated in terms of iterated differential operator rings. We shall need the *reverse lexicographic order* on n-tuples of integers, denoted $\leq_{\text{rlex}}$. Recall that $(m_1, \ldots, m_n) \leq_{\text{rlex}} (m'_1, \ldots, m'_n)$ if and only if either $m_i = m'_i$ for all i, or there is an index s such that $m_i = m'_i$ for $i > s$ while $m_s < m'_s$.

Theorem 12.19. *Let $R = k[x_1][x_2; \delta_2] \cdots [x_n; \delta_n]$ be an iterated differential operator ring where k is a field, the derivations δ_i are k-linear, and $\delta_i(x_j) \in kx_1 + \cdots + kx_j$ for all $i > j$. Then R has normal separation, and so it satisfies the strong second layer condition.*

Proof. Note that R has a k-basis of monomials in the x_i. Let us write these monomials in the form $x^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ for n-tuples $\mathbf{m} = (m_1, \ldots, m_n)$ in $(\mathbb{Z}^+)^n$ and set $|\mathbf{m}| = m_1 + \cdots + m_n$. Define a relation $\leq$ on $(\mathbb{Z}^+)^n$ by the rule

$$\mathbf{m} \leq \mathbf{m}' \quad \iff \quad \begin{cases} |\mathbf{m}| < |\mathbf{m}'| & \text{or} \\ |\mathbf{m}| = |\mathbf{m}'| \text{ and } \mathbf{m} \leq_{\text{rlex}} \mathbf{m}'. \end{cases}$$

We leave it to the reader to check that this relation is a well-ordering on $(\mathbb{Z}^+)^n$. Note that $(1, 0, 0, \ldots, 0) < (0, 1, 0, \ldots, 0) < \cdots < (0, 0, \ldots, 0, 1)$ in the ordering just defined; this is the reason for using the reverse lexicographic rather than the lexicographic order.

In particular, any nonempty finite subset of $(\mathbb{Z}^+)^n$ has a (unique) maximum with respect to $\leq$. This allows us to define the *degree* of any nonzero $r \in R$ as the maximum of those $\mathbf{m} \in (\mathbb{Z}^+)^n$ such that $x^{\mathbf{m}}$ appears in r with a nonzero coefficient.

Next, for $i = 1, \dots, n$, let ∂_i denote the inner derivation induced by x_i on R. For $i > j$, our hypotheses give $\partial_i(x_j) = \delta_i(x_j) \in kx_1 + \cdots + kx_j$. For $i < j$, we have

$$\partial_i(x_j) = -\delta_j(x_i) \in kx_1 + \cdots + kx_i \subset kx_1 + \cdots + kx_j.$$

Since $\partial_i(x_i) = 0$, we see that there exist scalars $\lambda_{ij} \in k$ such that

$$\partial_i(x_j) \equiv \lambda_{ij} x_j \qquad (\operatorname{mod}\ kx_1 + \cdots + kx_{j-1})$$

for all $i, j = 1, \dots, n$. It follows (with some work, which we leave to the reader) that, for $i = 1, \dots, n$ and $\mathbf{m} \in (\mathbb{Z}^+)^n$, there exists $\lambda \in k$ such that

$$\partial_i(x^{\mathbf{m}}) \equiv \lambda x^{\mathbf{m}} \quad (\text{mod terms of degree } < \mathbf{m}).$$

In particular, whenever $r \in R$ and $\partial_i(r) \neq 0$, we have $\deg \partial_i(r) \leq \deg(r)$.

We are now ready to prove normal separation. In fact, we claim somewhat more: For any ideals $I > J$ in R, there exists an element $y \in I \setminus J$ such that $y + J$ is normal in R/J.

Let $\mathbf{m}$ be the minimum degree for elements of $I \setminus J$, and choose an element $y \in I \setminus J$ with degree $\mathbf{m}$. There exist scalars $\mu_i \in k$, for $i = 1, \dots, n$, such that $\partial_i(x^{\mathbf{m}}) \equiv \mu_i x^{\mathbf{m}}$ modulo terms of degree less than $\mathbf{m}$. Since the highest degree term of y is a scalar multiple of $x^{\mathbf{m}}$, and since ∂_i never raises degree, it follows that $\partial_i(y) - \mu_i y$ either vanishes or has degree less than $\mathbf{m}$. Note that $\partial_i(y) - \mu_i y = x_i y - y x_i - \mu_i y \in I$. By the minimality of $\mathbf{m}$, we must have $\partial_i(y) - \mu_i y \in J$, and thus $x_i y \equiv y(x_i + \mu_i) \pmod{J}$. Consequently, $y + J$ is a normal element of R/J, as desired. □

• ADDITIONAL EXERCISES •

12P. Show that if, in Theorem 12.1, we only assume that R is right noetherian, then the conclusions remain valid except that in (ii) we can only conclude that $(P \cap Q)/A$ is a fully faithful left (R/P)-module and a fully faithful right (R/Q)-module, and that if U is torsionfree as a right (R/Q)-module, then $(P \cap Q)/A$ is torsionfree as a right (R/Q)-module. (Here, in case (ii), when U is a torsionfree (R/Q)-module, it is an open question whether M'/U must be a torsionfree (R/P)-module.) □

12Q. If we want to work with a right noetherian ring R, then a natural definition of a link from P to Q would require that the linking bimodule $(P \cap Q)/A$ be torsionfree as a right (R/Q)-module and fully faithful as a left (R/P)-module (cf. Exercise 12P). Show that the proof of Theorem 12.2 carries over to give a result in this case as well, where, however, we cannot conclude that U is isomorphic to a right ideal of R/Q but only that it is uniform and torsionfree as an (R/Q)-module. □

12R. This is an analog of the equivalences (a) $\Longleftrightarrow$ (b) in Theorems 12.6 and 12.9. Let R be a noetherian ring, and for each prime Q of R consider the following condition:

(†) Whenever M is a finitely generated submodule of $E((R/Q)_R)$ containing R/Q such that $\operatorname{ann}_R(M)$ is a prime ideal, then $\operatorname{ann}_R(M) = Q$.

Show that if R satisfies the right second layer condition, then all primes of R satisfy (†). Conversely, show that any prime Q of R which satisfies (†) must satisfy the right second layer condition, and hence if all primes of R satisfy (†), then R satisfies the right second layer condition. □

12S. Occasionally in the literature, condition (†) of Exercise 12R is taken as the definition of the right second layer condition for a prime Q. However, the definition we have given is the most common, and it is strictly weaker than (†), as the following example shows. Let $R = k + xS$ as in Exercise 12J, and set $T = \begin{pmatrix} R & R \\ xS & R \end{pmatrix}$, $P = \begin{pmatrix} xS & R \\ xS & R \end{pmatrix}$, $Q = \begin{pmatrix} R & R \\ xS & xS \end{pmatrix}$, $A = \begin{pmatrix} R & R \\ S & S \end{pmatrix}$.

(a) Show that T is a prime noetherian ring, and that P and Q are prime ideals of T.

(b) Show that A is a right T-submodule of $M_2(S)$, and that $(T/Q)_T \leq_e (A/Q)_T$. Show also that $\operatorname{ann}_T(A/Q) = 0$, and conclude that the prime Q in T does not satisfy condition (†).

(c) Let $E = E((R/xS)_R)$ and make the row $E' = (\, E \;\; E \,)$ into a right $M_2(R)$-module in the obvious way. Show that E' is an injective right $M_2(R)$-module. [Hint: Given a right $M_2(R)$-module homomorphim $f : D \to E'$, consider the restriction of f to $D\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.] Observe that $M_2(R)$ is a projective left T-module [Hint: As a left T-module, $M_2(R)$ is a direct sum of two copies of a left ideal of T.] and conclude that E' is also injective as a right T-module.

(d) Observe that the row $B = (\, 0 \;\; R/xS \,)$ is a right T-submodule of E' and that $B \cong (T/Q)_T$. If M is any submodule of E' such that $M > B$, show that M contains the row $C = (\, R/xS \;\; R/xS \,)$. Conclude that if M has an affiliated series $M_0 = 0 < M_1 < \cdots < M_n = M$, the first two corresponding affiliated primes must be Q and P, and M_1 and M_2/M_1 must be torsionfree modules over T/Q and T/P, respectively.

(e) Show that Q satisfies the right strong second layer condition. [Hint: Proposition 12.3.] □

• NOTES •

Links. Ideal links (not named, but with the notation $\rightsquigarrow$) appeared in the work of Jategaonkar [1973, pp. 153–154]. Second layer links (called links, and with the notation $\rightsquigarrow$) between prime ideals in FBN rings appeared in the work of Müller [1976b, p. 235].

Jategaonkar's Main Lemma. The first version was proved for FBN rings [1974b, Lemma 2.4], where alternative (i) does not occur. A version with U

torsionfree (in the notation of Theorem 12.1) was given in [1981, Lemma 4.1] and the full version in [1982, Lemma 2.2]. Finally, a version for one-sided noetherian rings was proved in [1986, Lemma 6.1.3].

Characterization of Links via Uniform Modules. Theorem 12.2 was first proved by Jategaonkar under the hypothesis that Q is not properly contained in P [1986, Lemma 6.1.6]. Brown observed that this restriction is unnecessary [1985, Remark 2.3(ii)].

Second Layer Condition. The right second layer condition for a noetherian ring R (in the form given in Exercise 12R) was introduced by Jategaonkar in [1979, p. 167] under the name $(*)$, and in [1981, p. 386] he defined the condition $(*)_r$ for single primes. He presented and named the right second layer condition for a right noetherian ring R (in a form similar to the condition in Proposition 12.3(b)) in [1982, p. 47]. In the same paper, he introduced the right strong second layer condition for R (in the same form as Theorem 12.6(b)) under the name $\binom{*}{*}_r$ [1982, pp. 23, 24]. Finally, he presented and named the right strong second layer condition for single primes in [1986, p. 220]. For a survey of various forms of the second layer condition and some of their relations, see Kim-Krause [1998].

Links from Affiliated to Associated Primes. Jategaonkar first proved versions of Theorems 12.4 and 12.7 for noetherian bimodules (where the second layer condition hypotheses are not needed) in [1981, Theorem 4.2]. He then proved right module versions in [1982, Theorem 3.1] and [1986, Theorem 9.1.2].

Normal Separation Implies Strong Second Layer Condition. Theorem 12.17 was proved by Jategaonkar [1982, Proposition 4.2] as an application of a more general result that we shall meet in the next chapter (Theorem 13.4).

Normal Separation in Enveloping Algebras. The proof of Theorem 12.19 yields a stronger conclusion than normal separation – any ideal of an iterated differential operator ring R of the given form has a *normalizing sequence of generators* $x_1, \dots, x_t$, meaning that x_1 is a normal element of R and, for $i = 2, \dots, t$, the coset of x_i is a normal element of $R/\langle x_1, \dots, x_{i-1}\rangle$. McConnell proved that for any finite dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero, all ideals of $U(\mathfrak{g})$ have normalizing sequences of generators [1968, Theorem 3]. This enabled Jategaonkar to draw the conclusion that $U(\mathfrak{g})$ satisfies the strong second layer condition [1982, p. 61].

13. The Artin-Rees Property

The Artin-Rees property is a condition with a long history in the theory of commutative noetherian rings (where every ideal satisfies the condition). Versions of this property have also played important roles in many verifications of the second layer condition, and they place certain restrictions on the possible structure of cliques of prime ideals. We introduce a convenient form of this property and some of its uses in this chapter, which is a continuation of Chapter 12. The reader may also treat this chapter as an appendix if desired, since the Artin-Rees property will not appear later in the text aside from a few exercises in the following chapter.

• THE ARTIN-REES PROPERTY •

Definition. An ideal I in a ring R has the *right AR-property* if, for every right ideal K of R, there is a positive integer n such that $K \cap I^n \leq KI$. The *left AR-property* is defined symmetrically, and I has the *AR-property* if it has both the right and left AR-properties.

The reader should be warned that the definition just given is the weakest of several Artin-Rees properties discussed in the literature; in particular, in most of the commutative literature one finds a definition involving a stronger condition (see the proof of Lemma 13.2).

Exercise 13A. Show that of the two prime ideals in the ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is a field, one has the right AR-property but not the left AR-property, while the other has the left AR-property but not the right. $\square$

We should remark that no example is known of an ideal I in a *prime* noetherian ring such that I has the right AR-property but not the left.

Exercise 13B. Show that in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, the prime ideal $\begin{pmatrix} 2\mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ satisfies neither the right nor the left AR-property. $\square$

Lemma 13.1. *The right AR-property for an ideal I in a ring R is equivalent to each of the following properties:*

(a) *For every finitely generated right R-module A and every submodule $B \leq A$, there is a positive integer n such that $B \cap AI^n \leq BI$.*

(b) *For every finitely generated right R-module M with an essential submodule N satisfying $NI = 0$, there is a positive integer n such that $MI^n = 0$.*

Proof. The reader will notice that this proof is very similar to that of Lemma 12.5. We give the details since the present lemma is the original source of the arguments.

Suppose first that I has the right AR-property, and consider modules $N \leq_e M$ as in (b). Since M is finitely generated, to get $MI^n = 0$ we need only show that each element of a set of generators for M is annihilated by a power of I. Thus, let $x \in M$, and let $K = \{r \in R \mid xr \in N\}$. Note that $xKI \leq NI = 0$. By the right AR-property, there is a positive integer n such that $K \cap I^n \leq KI$, and we observe that $xI^n \cap N = x(K \cap I^n) \leq xKI = 0$. Since $N \leq_e M$, we conclude that $xI^n = 0$, as desired. This proves (b).

(b) $\Longrightarrow$ (a): Choose a submodule C of A maximal with respect to the property that $B \cap C = BI$. Then the embedding $B/BI \to A/C$ maps B/BI isomorphically onto an essential submodule of A/C (Proposition 5.7). Condition (b) now implies that $(A/C)I^n = 0$ for some positive integer n, or, in other words, $AI^n \leq C$. Thus $B \cap AI^n \leq B \cap C = BI$, which verifies (a).

Finally, it is clear that (a) implies the right AR-property (take $A = R$). □

It follows immediately from Lemmas 12.16 and 13.1 that, in a noetherian ring, any ideal generated by a normal element has the AR-property. In fact, taking Exercise 12O into account, the property holds for any ideal generated by a set of normal elements.

We now present an important method for verifying the AR-property, originally introduced by Rees.

Definition. If R is a ring and I an ideal of R, then the *Rees ring of I* is the subring $\mathcal{R}(I)$ of the polynomial ring $R[x]$ generated by $R + Ix$, that is,

$$\mathcal{R}(I) = R + Ix + I^2x^2 + \cdots + I^jx^j + \cdots .$$

Lemma 13.2. *If I is an ideal in a ring R and the Rees ring $\mathcal{R}(I)$ is right noetherian, then I satisfies the right AR-property.*

Proof. Let K be a right ideal of R, and let

$$\begin{aligned} K^* &= K[x] \cap \mathcal{R}(I) \\ &= K + (K \cap I)x + (K \cap I^2)x^2 + \cdots + (K \cap I^j)x^j + \cdots , \end{aligned}$$

a right ideal of $\mathcal{R}(I)$. If $\mathcal{R}(I)$ is right noetherian, then K^* must have a finite set of generators, say $\{k_1, \ldots, k_t\}$. Choose a positive integer n large enough so that the set $K + (K \cap I)x + (K \cap I^2)x^2 + \cdots + (K \cap I^{n-1})x^{n-1}$ contains $k_1, \ldots, k_t$. Since $(K \cap I^n)x^n \subseteq K^* = \left(\sum_{j=0}^{n-1}(K \cap I^j)x^j\right)\mathcal{R}(I)$, we see by comparing coefficients that

$$K \cap I^n = \sum_{j=0}^{n-1}(K \cap I^j)I^{n-j} \subseteq KI.$$

(In fact, since $(K \cap I^j)I^{n-j-1} \subseteq K \cap I^{n-1}$ for $j = 0, \dots, n-1$, we can conclude that $K \cap I^n = (K \cap I^{n-1})I$.) $\square$

Theorem 13.3. [Artin, Rees] *If R is a noetherian ring and I an ideal of R generated by central elements, then $\mathcal{R}(I)$ is noetherian, and hence I satisfies the AR-property.*

Proof. The noetherian assumption ensures that I can be generated by finitely many central elements, say $a_1, \dots, a_n$. Then $\mathcal{R}(I)$ is generated (as a ring) by R together with the central elements $a_1x, \dots, a_nx$. Consequently, $\mathcal{R}(I)$ is a homomorphic image of a polynomial ring $R[x_1, \dots, x_n]$, and hence it is noetherian by the Hilbert Basis Theorem. $\square$

The AR-property for ideals in commutative noetherian rings can also be obtained from Exercise 5ZF and Lemma 13.1. We have concentrated on the Rees ring approach because that provides the method we shall use to produce ideals with the AR-property in differential operator rings (Theorem 13.5).

Definition. A ring R is *right AR-separated* if, for every pair of prime ideals P and Q in R such that $P < Q$, there is an ideal I such that $P < I \leq Q$ and I/P satisfies the right AR-property in R/P. *Left AR-separated* is defined symmetrically, and R is *AR-separated* if it is both right and left AR-separated.

For example, any noetherian ring with normal separation is AR-separated. Thus, the following result generalizes Theorem 12.17.

Theorem 13.4. *If R is a noetherian ring which is right AR-separated, then R satisfies the right strong second layer condition.*

Proof. We verify condition (c) of Theorem 12.6. Thus, let $P < Q$ be primes in R and $K \geq P$ a right ideal. By hypothesis, there exists an ideal J such that $P < J \leq Q$ and J/P satisfies the right AR-property in R/P. Then there is a positive integer n such that $(K/P) \cap (J/P)^n \leq (K/P)(J/P)$, whence $K \cap J^n \leq KJ + P \leq KQ + P$. Since $J > P$ and $K \geq P$, it follows that $J^n + P > P$ and $K \cap (J^n + P) \leq KQ + P$, which establishes the desired condition. $\square$

Exercise 13C. Prove Theorem 13.4 by using the definition of the right strong second layer condition rather than Theorem 12.6. $\square$

Theorem 13.5. [Bell, Sigurdsson] *Let $S = R[x; \delta]$, where R is a commutative noetherian ring with a derivation δ. Then any ideal of S of the form IS, where I is a δ-ideal of R, has the AR-property. If, moreover, R is a $\mathbb{Q}$-algebra, then S is AR-separated and thus satisfies the strong second layer condition.*

Proof. First let I be a δ-ideal of R, and consider the Rees rings $\mathcal{R}(I)$ and $\mathcal{R}(IS)$. We need a new indeterminate, say y, to express these Rees rings. We can extend δ to a derivation δ^* on $R[y]$ such that $\delta^*(y) = 0$, and since I is a δ-ideal of R, we see that δ^* restricts to a derivation on $\mathcal{R}(I)$. As

$\mathcal{R}(I)$ is a noetherian ring (Theorem 13.3), we conclude that $\mathcal{R}(I)[x;\delta^*]$ is noetherian. On the other hand, a comparison of coefficients inside the ring $R[y][x;\delta^*] = R[x;\delta][y]$ shows that $\mathcal{R}(I)[x;\delta^*] = \mathcal{R}(IS)$. Therefore $\mathcal{R}(IS)$ is noetherian, and hence IS has the right and left AR-properties.

Now assume that R is a $\mathbb{Q}$-algebra, and let P and Q be primes of S with $P < Q$. We must find an ideal J such that $P < J \leq Q$ and J/P has the AR-property. If $P = (P \cap R)S$ and $Q \cap R > P \cap R$, the ideal $J = (Q \cap R)S$ satisfies $P < J \leq Q$. Since $Q \cap R$ is a δ-ideal of R (Lemma 3.18), J has the AR-property in S, and it follows that J/P has the AR-property in S/P.

Finally, if either $P > (P \cap R)S$ or $Q \cap R = P \cap R$, then there is a prime $K > (P \cap R)S$ such that $K \cap R = P \cap R$ (namely, either $K = P$ or $K = Q$). Then, according to Theorem 3.22, S/P is commutative, and we can take $J = Q$. Therefore S is AR-separated. $\square$

Theorem 13.5 remains true even if R is not necessarily a $\mathbb{Q}$-algebra, but we shall not develop the methods needed to prove this.

Extensions of Theorem 13.5, which we shall not prove here, were developed in order to show that the enveloping algebra of any finite dimensional solvable Lie algebra satisfies the second layer condition. We have seen a version of the completely solvable case in Theorem 12.19. The following exercise shows why that theorem is insufficient in general.

Exercise 13D. Let $S = R[z;\delta]$, where $R = \mathbb{R}[x,y]$, a polynomial ring in two indeterminates, and $\delta = y(\partial/\partial x) - x(\partial/\partial y)$. By Theorem 13.5, S is AR-separated and thus satisfies the strong second layer condition. Observe that $P_0 = (x^2 + y^2)R$ and $Q_0 = xR + yR$ are prime δ-ideals of R, and so $P_0S < Q_0S$ are primes of S. Show that Q_0S/P_0S contains no nonzero normal elements of S/P_0S, and conclude that S does not have normal separation. $\square$

Exercise 13E. Show that if R is a commutative noetherian ring and α an automorphism of R, then $R[x^{\pm 1};\alpha]$ is AR-separated and hence satisfies the strong second layer condition. [Hint: Use Theorem 1.17 and Proposition 9.1 in one step.] $\square$

• LINK-FINITENESS •

In this section, we show how the presence of ideals with the AR-property places restrictions on possible links between prime ideals. Our first observation is a generalization of Lemmas 12.11 and 12.15. We give a direct proof from the definitions; alternatively, Theorem 12.2 and Lemma 13.1 may be used.

Proposition 13.6. *Let I be an ideal in a noetherian ring R and P and Q primes of R with $P \rightsquigarrow Q$. If $I \leq Q$ and I satisfies the right AR-property, then $I \leq P$. Similarly, if $I \leq P$ and I satisfies the left AR-property, then $I \leq Q$.*

Proof. Assume first that $I \leq Q$ and I satisfies the right AR-property. Let $(P \cap Q)/A$ be a linking bimodule for the link $P \rightsquigarrow Q$. In particular, $PQ \leq A$

and $\mathrm{l.ann}_R\big((P\cap Q)/A\big) = P$. From the right AR-property, $P \cap Q \cap I^n \leq (P \cap Q)I$ for some positive integer n. Then

$$I^n(P\cap Q) \leq P \cap Q \cap I^n \leq (P\cap Q)I \leq PQ \leq A,$$

whence $I^n \leq \mathrm{l.ann}_R\big((P\cap Q)/A\big) = P$, and therefore $I \leq P$.

The final statement of the proposition is proved symmetrically. $\square$

Exercise 13F. Let $R = T[x;\delta]$, where T is a commutative noetherian ring with a derivation δ, and let P and Q be primes of R. If P and Q are in the same clique, show that $P \cap T = Q \cap T$. $\square$

Lemma 13.7. *If P is a minimal prime in a semiprime noetherian ring R, there are no primes of R linked to or from P.*

Proof. If $P \rightsquigarrow Q$ for some prime Q, then, by Theorem 12.2 there exists a uniform right R-module M with a nonzero submodule U such that M/U is isomorphic to a right ideal of R/P. In particular, M/U is a torsionfree (R/P)-module, and since P is a minimal prime, M/U must also be torsionfree as an R-module (Proposition 7.9). However, since $U \leq_e M$, this is impossible. Therefore P is not linked to any primes in R. By symmetry, no primes are linked to P either. $\square$

Theorem 13.8. [Müller, Brown] *Let P be a prime ideal in a noetherian ring R. If R is left AR-separated, P is linked to at most finitely many primes in R, while if R is right AR-separated, at most finitely many primes are linked to P.*

Proof. By symmetry, it suffices to prove the first statement. If the theorem fails, we may assume that R is a minimal criminal, i.e., that there is an infinite set $X \subseteq \mathrm{Spec}(R)$ such that P is linked to each prime in X, but that if I is a nonzero ideal contained in P, then P/I is linked to at most finitely many primes of R/I.

Choose an ideal T of R maximal with respect to the property that T equals the intersection of some infinite subset of X. Given ideals H_1, H_2 properly containing T, it follows from the maximality of T that each H_i is contained in at most finitely many primes from X. Hence, H_1H_2 is contained in at most finitely many primes from X, and so $H_1H_2 \not\leq T$. Thus, T is a prime ideal. After replacing X by $\{Q \in X \mid Q \geq T\}$, we may assume that $T = \bigcap X$.

For each $Q \in X$, there exists an ideal A_Q in R such that $P \cap Q > A_Q \geq PQ$ and $(P\cap Q)/A_Q$ is a linking bimodule for the link $P \rightsquigarrow Q$. Note that if I is an ideal of R and $I \leq A_Q$, then P/I and Q/I are primes of R/I satisfying $P/I \rightsquigarrow Q/I$. Thus, by our noetherian induction, any nonzero ideal of R can be contained in at most finitely many of the ideals A_Q. In other words, $\bigcap\{A_Q \mid Q \in Y\} = 0$ for any infinite subset $Y \subseteq X$.

In particular, it follows that

$$PT \leq \bigcap_{Q\in X} PQ \leq \bigcap_{Q\in X} A_Q = 0,$$

and hence T is a left (R/P)-module. Since $P \cap T \leq P \cap Q$ for all $Q \in X$, there is a natural left (R/P)-module homomorphism

$$P \cap T \longrightarrow \prod_{Q \in X} (P \cap Q)/A_Q,$$

and this map is injective because $\bigcap\{A_Q \mid Q \in X\} = 0$. As each $(P \cap Q)/A_Q$ is a torsionfree left (R/P)-module, it follows that $P \cap T$ must be torsionfree as a left (R/P)-module.

We claim that $P \cap T = 0$. If not, choose a nonzero ideal $B \leq P \cap T$ with $\operatorname{rank}({}_R B)$ as small as possible. Observe that if C is any nonzero ideal contained in B, then $\operatorname{rank}({}_R C) = \operatorname{rank}({}_R B)$ (by minimality of $\operatorname{rank}({}_R B)$), whence ${}_R C \leq_e {}_R B$ (Corollary 5.21), and consequently B/C is torsion as a left (R/P)-module.

Now, if $Q \in X$ and $B \cap A_Q \neq 0$, then $B/(B \cap A_Q)$ is torsion as a left (R/P)-module. However, $B/(B \cap A_Q)$ embeds in the torsionfree left (R/P)-module $(P \cap Q)/A_Q$, whence $B/(B \cap A_Q) = 0$ and so $B \leq A_Q$. On the other hand, if $Q \in X$ and $B \cap A_Q = 0$, then B embeds in the torsionfree right (R/Q)-module $(P \cap Q)/A_Q$, and hence $\operatorname{r.ann}_R(B) = Q$. This is obviously only possible for at most one prime $Q \in X$. Thus $B \leq A_Q$ for all but at most one $Q \in X$. However, as $B \neq 0$, this contradicts the fact that $\bigcap\{A_Q \mid Q \in Y\} = 0$ for all infinite subsets $Y \subseteq X$. Therefore $P \cap T = 0$, as claimed. In particular, R is a semiprime ring.

By Lemma 13.7, P cannot be a minimal prime. If P_0 is a minimal prime contained in P, then as $PT = 0 \leq P_0$ we must have $T \leq P_0 < P$, whence $T = P \cap T = 0$. Thus R is a prime ring.

Since R is left AR-separated, there is a nonzero ideal $J \leq P$ such that J satisfies the left AR-property. Proposition 13.6 then shows that $J \leq Q$ for all $Q \in X$. But now $J \leq \bigcap X = T = 0$, a contradiction.

Therefore the theorem holds. $\square$

Theorem 13.8 also holds for FBN rings, as we shall see later (Theorem 16.22). In general, a prime ideal in a noetherian ring can be linked to or from at most countably many primes (Theorem 16.23). Stafford has constructed an example of a noetherian ring containing a prime linked to infinitely many primes [1985, Theorem 4.4], but this example is too complicated to reproduce here.

• MODULE-FINITE ALGEBRAS •

Given a module-finite algebra R over a commutative noetherian ring S, we shall completely describe the cliques of prime ideals of R, in terms of their intersections with the center of R. Observe that the center of R is noetherian as an S-module and hence is a noetherian ring, and that R is a finitely generated module over its center. Hence, we may as well replace S by the center of R.

We proceed by using the AR-property and then by reducing the situation to an artinian ring, where we apply Proposition 12.12.

Lemma 13.9. *Let R be a ring module-finite over its center S, and assume that S is noetherian. If I is any ideal of S, there exists a positive integer n such that the images of S and the center of R/I^nR under the quotient maps $R \to R/IR$ and $R/I^nR \to R/IR$ coincide. In other words, the center of R/I^nR is contained in $(S + IR)/I^nR$.*

Proof. Choose elements $x_1, \dots, x_t$ that generate R as an S-module, and set $A = R^t$, viewed as a left S-module. Since R is a finitely generated S-module, so is A. We may define an S-module homomorphism $f : R \to A$ according to the rule

$$f(r) = (rx_1 - x_1r,\ rx_2 - x_2r,\ \dots,\ rx_t - x_tr),$$

and we observe that $\ker(f) = S$. Since I has the AR-property (Theorem 13.3), Lemma 13.1 shows that there exists a positive integer n such that $f(R) \cap I^nA \le If(R)$, that is, $f(R) \cap I^nA \le f(IR)$.

Since the quotient map $R \to R/I^nR$ sends S into the center Z_n of R/I^nR, it is clear that the image of S in R/IR is contained in the image of Z_n. Now consider a coset $r + I^nR$ in Z_n. Then $rx_i - x_ir \in I^nR$ for $i = 1, \dots, t$, from which we see that $f(r) \in I^nA$. As $f(R) \cap I^nA \le f(IR)$, we obtain $f(r) = f(x)$ for some $x \in IR$. Now $r - x \in \ker(f) = S$, and so we conclude that $r \in S + IR$. This shows that the image of $r + I^nR$ in R/IR (that is, the coset $r + IR$) is contained in the image of S. That $r \in S + IR$ also establishes the inclusion $Z_n \subseteq (S + IR)/I^nR$. $\square$

Lemma 13.9 cannot be strengthened to say that S maps onto the center of R/I^nR for some n, as the following example shows.

Exercise 13G. Let $R = U(\mathfrak{sl}_2(k))$, where k is a field of characteristic 2. By Exercise 2S, R can be expressed as an iterated skew polynomial ring in terms of the standard basis $\{e, f, h\}$ for $\mathfrak{sl}_2(k)$, and so R is a noetherian domain.

(a) Show that the center S of R is a polynomial ring $k[e^2, f^2, h]$.

(b) Set $I = e^2S + f^2S + hS$ and show that, for any positive integer n, the center of R/I^nR properly contains $(S + I^nR)/I^nR$. [Hint: Show that the cosets of e^{2n-1} and f^{2n-1} are central in R/I^nR.] $\square$

Exercise 13H. Let X be a right denominator set in a noetherian ring R, and P, Q primes of R disjoint from X. By Theorem 10.20, the extended ideals P^{e}, Q^{e} are primes of RX^{-1}. Show that $P^{\mathrm{e}} \rightsquigarrow Q^{\mathrm{e}}$ if and only if $P \rightsquigarrow Q$. $\square$

Theorem 13.10. [Müller] *Let R be a ring module-finite over its center S and assume that S is noetherian. Then, for any prime ideal P of R, the contraction $P \cap S$ is a prime ideal of S, and*

$$\mathrm{Clq}(P) = \{Q \in \mathrm{Spec}(R) \mid Q \cap S = P \cap S\}.$$

Proof. If $x, y \in S$ and $xy \in P \cap S$, then $xRy = xyR \subseteq P$ and so either $x \in P$ or $y \in P$, whence either $x \in P \cap S$ or $y \in P \cap S$. Thus $P \cap S$ is a prime of S.

We next remark, as an immediate consequence of Lemma 12.11, that any two linked primes of R have the same contraction in S. Consequently, $\mathrm{Clq}(P)$ is contained in the set

$$Y = \{Q \in \mathrm{Spec}(R) \mid Q \cap S = P \cap S\},$$

and it remains to show that $\mathrm{Clq}(P) \supseteq Y$.

Set $M = P \cap S$ and $X = S \setminus M$, and observe that X is a right (and left) denominator set in R. Also, (the natural image of) SX^{-1} equals the center of RX^{-1}. (Use the fact that any element of R which commutes with a set of generators for R_S will be in S.) By Theorem 10.20, each of the extended ideals Q^{e}, for $Q \in Y$, is a prime of RX^{-1}, and for such Q we note that

$$Q^{\mathrm{e}} \cap SX^{-1} = P^{\mathrm{e}} \cap SX^{-1} = M(SX^{-1}).$$

Moreover, if the primes in RX^{-1} contracting to $M(SX^{-1})$ all belong to the same clique in $\mathrm{Spec}(RX^{-1})$, it will follow from Exercise 13H that the primes in Y all belong to the same clique in $\mathrm{Spec}(R)$, and therefore $\mathrm{Clq}(P) = Y$.

Hence, without loss of generality, we may assume that M is a maximal ideal of S. Note that $MR \cap S \subseteq P \cap S$, whence $MR \cap S = M$.

According to Lemma 13.9, there exists a positive integer n such that the center of R/M^nR is contained in $(S + MR)/M^nR$. We use this property to show that R/M^nR is indecomposable (as a ring). Let e be a central idempotent in R/M^nR and write $e = a + M^nR$ for some $a \in R$. Then $a + MR = b + MR$ for some $b \in S$. Since e is idempotent, so is $a + MR$, whence $b - b^2 \in MR \cap S = M$. As S/M is a field, either $b \in M$ or $1 - b \in M$. Then either $a \in MR$ or $1-a \in MR$, whence either e or $1-e$ lies in MR/M^nR, and so either e or $1 - e$ is nilpotent. We conclude that either $e = 0$ or $e = 1$. Therefore R/M^nR is indecomposable, as claimed.

Finally, S/M^n is an artinian ring, and R/M^nR is finitely generated as an (S/M^n)-module, whence R/M^nR is an artinian ring. By Proposition 12.12, all primes of R/M^nR lie in the same clique of $\mathrm{Spec}(R/M^nR)$. It follows that all primes of R containing M^nR lie in the same clique of $\mathrm{Spec}(R)$, and therefore $Y \subseteq \mathrm{Clq}(P)$. $\square$

The two-element clique shown in Exercise 12H provides an illustration of the situation covered by Müller's Theorem. There, $S = \mathbb{Z}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and R has two primes that contract to the maximal ideal $\mathbb{Z}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in S. Another example follows.

Exercise 13I. Let k be an algebraically closed field, $q \in k^{\times}$ a primitive t-th root of unity for some $t \geq 2$, and $R = \mathcal{O}_q(k^2)$. (Recall that the existence of q

in k requires that $\operatorname{char}(k) \nmid t$.) Show that, for each $\alpha \in k^\times$, the clique of the maximal ideal $(x-\alpha)R + yR$ has t elements. [Hint: Show that the center of R is $k[x^t, y^t]$.] $\square$

We can think of Theorem 13.10 as parametrizing the cliques of R by the primes of S. In order to see that all primes of S actually appear in the parametrization, we need one standard fact from the localization theory of commutative rings, which we state as an exercise for those readers who have not seen it elsewhere.

Exercise 13J. Let S be a commutative ring, N a prime of S, and $X = S \setminus N$. Show that the ideal N^{e} in SX^{-1} equals the Jacobson radical of SX^{-1}. [Hint: Show that N^{e} is the unique maximal ideal of SX^{-1}, or use Exercise 3S.] $\square$

Corollary 13.11. *Let R be a ring module-finite over its center S, with S noetherian. For $N \in \operatorname{Spec}(S)$, set $X_N = S \setminus N$ and $R_N = RX_N^{-1}$, and let $\phi_N : R \to R_N \to R_N/NR_N$ be the composition of the localization and quotient maps. Then $\phi_N^{-1}\big(\operatorname{Spec}(R_N/NR_N)\big)$ is a clique in $\operatorname{Spec}(R)$, and the set map ϕ_N^{-1} provides a bijection from $\operatorname{Spec}(R_N/NR_N)$ onto this clique. Moreover, all cliques in $\operatorname{Spec}(R)$ have this form.*

Proof. Given $N \in \operatorname{Spec}(S)$, we first show that $R_N/NR_N \neq 0$. Set $S_N = SX_N^{-1}$ and note that R_N is a module-finite S_N-algebra. Since $0 \notin X_N$, we see that $R_N \neq 0$, and so it follows from Exercise 13J and Nakayama's Lemma that $NR_N \neq R_N$, as claimed. In particular, it follows that the ring R_N/NR_N has at least one prime ideal, and therefore $\phi_N^{-1}\big(\operatorname{Spec}(R_N/NR_N)\big)$ is nonempty.

Next, observe that since NS_N is a maximal ideal of S_N, the primes of R_N that contain NR_N are exactly those primes that contract to NS_N in S_N. In view of Theorem 10.20, we conclude that

$$\phi_N^{-1}\big(\operatorname{Spec}(R_N/NR_N)\big) = \{Q \in \operatorname{Spec}(R) \mid Q \cap S = N\},$$

and that ϕ_N^{-1} induces a bijection from $\operatorname{Spec}(R_N/NR_N)$ onto the above set. Since this set is nonempty, it is a clique by Müller's Theorem, and every clique in $\operatorname{Spec}(R)$ has this form. $\square$

Corollary 13.12. *If R is a ring module-finite over its center S, and if S is noetherian, then all cliques in $\operatorname{Spec}(R)$ are finite.*

Proof. By the previous corollary, any clique of $\operatorname{Spec}(R)$ is in bijection with $\operatorname{Spec}(R_N/NR_N)$ for some $N \in \operatorname{Spec}(S)$. Since R_N/NR_N is a finite dimensional algebra over the quotient field of S/N, it has only finitely many prime ideals. $\square$

Exercise 13K. Let R be a ring whose center S is noetherian and assume that R can be generated as an S-module by d elements, where $d < \infty$. Show that every clique in $\operatorname{Spec}(R)$ contains at most d primes. $\square$

Theorem 13.10 and its corollaries are specific to module-finite algebras – they do not generalize to FBN rings or even to noetherian rings satisfying polynomial identities, as the following example shows.

Exercise 13L. Let $R = k[x][y; x(d/dx)]$, where k is a field of characteristic zero. Show that R/x^2R is an FBN ring and that the clique of the maximal ideal $(xR + yR)/x^2R$ in R/x^2R is infinite. [Hint: Exercise 12I.] Observe also that, since R/xR is commutative, $(ab - ba)^2 = 0$ for all $a, b \in R/x^2R$, and thus R/x^2R is a P.I. ring. $\square$

As stated, Theorem 13.10 and Corollary 13.11 show how to match up the cliques in a module-finite algebra R over a noetherian center S with the primes in certain finite dimensional algebras, but these results do not indicate whether individual links are preserved. If P and Q are primes in R which both contract to the same prime N in S, and if $PR_N/NR_N \rightsquigarrow QR_N/NR_N$ in R_N/NR_N, then certainly $P \rightsquigarrow Q$ in R (recall Exercise 13H). Conversely, if $P \rightsquigarrow Q$, then $PR_N \rightsquigarrow QR_N$, but since the linking bimodule for the latter link is a factor of $(PR_N \cap QR_N)/PQR_N$, at first glance we might only conclude that there is a link $PR_N/N^2R_N \rightsquigarrow QR_N/N^2R_N$ in R_N/N^2R_N. In fact, there is almost always a link $PR_N/NR_N \rightsquigarrow QR_N/NR_N$ in R_N/NR_N – this occurs whenever $P \neq Q$, because of the following result.

Exercise 13M. Let R be a noetherian ring, $P \rightsquigarrow Q$ linked primes in R, and I a centrally generated ideal of R. If $P \neq Q$ and $I \leq P \cap Q$, show that $P/I \rightsquigarrow Q/I$. [Hint: Given a linking bimodule $(P \cap Q)/A$, show that $I \leq A$.] $\square$

• NOTES •

Artin-Rees Property. The AR-property for ideals in commutative noetherian rings – in a stronger form than we have given – was proved independently by Artin (apparently unpublished; see Nagata [1962, p. 212]) and Rees [1956, Lemma 1]. Rees's proof used a version of what is now called the Rees ring, as in Lemma 13.2.

AR-Separation Implies Strong Second Layer Condition. Jategaonkar first proved in [1979, Proposition 10] that AR-separated noetherian rings satisfy the second layer condition; he improved the conclusion to the strong second layer condition in [1982, Proposition 4.1]. He used this to show that enveloping algebras of solvable Lie algebras satisfy the strong second layer condition [1982, p. 61; 1986, Theorem A.3.9] and that group rings of polycyclic-by-finite groups with commutative noetherian coefficient rings satisfy it [1979, Proposition 10; 1982, Theorem 4.5].

Second Layer Condition in Differential Operator Rings. Bell proved that, in a large class of iterated differential operator rings with commutative noetherian coefficient rings, the second layer condition holds [1987, Theorem 7.3].

In the case $S = R[x;\delta]$, where R is a commutative noetherian $\mathbb{Q}$-algebra, Sigurdsson showed that S is AR-separated [1986, Proposition 2.4], from which it follows that S satisfies the strong second layer condition.

AR-Separation Implies Link-Finiteness. Müller proved that, if P is a prime ideal in a noetherian P.I. ring R, then at most finitely many primes of R can be linked to or from P [1979, Theorem 7; 1980, Theorem 7]; his proof used central elements where the proof we have given of Theorem 13.8 uses AR-ideals. Brown proved the corresponding result for primes in the group ring of a polycyclic-by-finite group with a suitable commutative noetherian coefficient ring [1981b, Theorem 6.4], and he remarked that his proof could be modified to obtain the result for AR-separated noetherian rings satisfying certain Krull dimension symmetry hypotheses [1981b, p. 279]. The general theorem appeared in Jategaonkar [1986, Proposition 8.1.8].

Cliques in Module-Finite Algebras. Müller defined a finite set X of incomparable prime ideals in a noetherian ring R to be *classical* provided $\mathcal{C}\big(\bigcap X\big)$ is an Ore set and the Jacobson radical of the corresponding localization has the AR-property, and he defined a *cycle* (later: *clan*) to be a classical set of primes such that no proper subset is classical. (In the presence of the second layer condition, it follows from Theorem 14.21 that cycles/clans are the same as finite cliques.) Müller proved that if R is a noetherian ring which is module-finite over its center S, then the cycles/clans in R are precisely the sets $\{P \in \operatorname{Spec}(R) \mid P \cap S = Q\}$ for $Q \in \operatorname{Spec}(S)$ [1974b, Theorem 6.1; 1976a, Theorem 7].

14. Rings Satisfying the Second Layer Condition

The main theme of this chapter is the exploration of the ideal theory of noetherian rings satisfying the second layer condition. This is a very large class of rings (as we began to see in the previous chapters), including many iterated differential operator rings, iterated skew-Laurent extensions, and quantized coordinate rings, as well as the group rings of polycyclic-by-finite groups and the enveloping algebras of finite dimensional solvable Lie algebras. It turns out that these rings have many properties that are not shared by other noetherian rings and that can be thought of as generalizations of well-known properties of commutative rings. We begin with a symmetry property of bimodules over these rings. This will give us immediate information about the graphs of links of these rings and will also give us the key tool to prove two intersection theorems – a strong form of Jacobson's Conjecture and an analogue of the Krull Intersection Theorem. Rings satisfying the second layer condition also behave well with respect to finite extensions. If R is a noetherian ring satisfying the second layer condition, and R is a subring of a ring S such that S is finitely generated as both a left and a right R-module, we prove that S also satisfies the second layer condition, and that "Lying Over" holds for the prime ideals in this setup. Finally, rings with the second layer condition turn out to be the natural class of rings in which to discuss localization at prime ideals and at families of prime ideals. The results in this direction are still not complete, but we will present what is probably going to remain the definitive result on localization at semiprime ideals and give some indications concerning the further development of the theory.

• CLASSICAL KRULL DIMENSION •

Given a link $P \rightsquigarrow Q$ in a noetherian ring R, we would like to be able to say that the factor rings R/P and R/Q are similar in some fashion, for instance, that there is some notion of "dimension" for rings which yields the same value for R/P as for R/Q. The dimension that has proved most useful in this regard, at least in the presence of the second layer condition, is "classical Krull dimension," which we introduce in this section. (A different "Krull dimension," better adapted to module theory, will be developed in the following chapter.) We should emphasize in advance that, except in the finite case, the actual value of this dimension for a ring has not had great significance. The importance of the dimension is that it has provided an invariant with certain good features and with the property that it distinguishes between a prime ring R and a proper prime factor ring R/P. In particular, classical Krull dimension provides a basis for proofs via transfinite induction.

In this and the following chapters, we will be using invariants with arbitrary ordinal values. In this way, we obtain invariants which apply to all noetherian rings. The reader who is uncomfortable with infinite ordinals may be relieved to know that in most of the important examples that arise in applications, the invariants are actually finite. The reader may assume, if desired, that all of the invariants are finite in these chapters, but then, of course, the results will not be as general as those stated here.

Classical Krull dimension was first defined for commutative noetherian rings, by counting lengths of chains of prime ideals. As with composition series, it is the *gaps* between the primes that are counted, so that a single prime is viewed as a chain of length 0 and a chain $P_0 > P_1 > \cdots > P_n$ has length n. The classical Krull dimension of a ring R was originally defined to be the supremum of the lengths of all chains of prime ideals in R. Then, in order to distinguish among rings with infinite classical Krull dimension, Krause introduced a refinement of the definition allowing infinite ordinal values.

The smallest classical Krull dimension that will occur for a nonzero ring is 0. In order to have a convenient value for the classical Krull dimension of the zero ring, and in order to conform with the standard usage for the "non-classical" Krull dimension to be introduced in the following chapter, we make the convention that -1 is to be considered an ordinal number.

Definition. Let R be a ring. We define, by transfinite induction, sets X_α of prime ideals of R for each ordinal α. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined for all ordinals $\beta < \alpha$, let X_α be the set of those prime ideals P in R such that all prime ideals properly containing P belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of maximal ideals of R.) If some X_γ contains all prime ideals of R, we say that Cl.K.dim(R) *exists*, and we set Cl.K.dim(R) – the *classical Krull dimension of* R – equal to the smallest such γ. We write "Cl.K.dim$(R) = \gamma$" as an abbreviation for the statement that Cl.K.dim(R) exists and equals γ.

For example, any simple ring has classical Krull dimension 0, as does any nonzero artinian ring. The ring $\mathbb{Z}$ has classical Krull dimension 1. Some other examples will be given in the exercises.

Proposition 14.1. *If R is a ring with the ACC on prime ideals, the classical Krull dimension of R exists.*

Proof. Define the sets X_α of prime ideals as in the definition above. Since there is a bound on the cardinalities of these sets (e.g., $2^{\text{card}(R)}$), the transfinite chain $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$ cannot be properly increasing forever. Hence, there exists an ordinal γ such that $X_\gamma = X_{\gamma+1}$.

If Cl.K.dim(R) does not exist, then X_γ does not contain all the prime ideals of R. Using the ACC on prime ideals, there is a prime P in R maximal with respect to the property $P \notin X_\gamma$. Hence, all primes properly containing P lie in X_γ. But then $P \in X_{\gamma+1} = X_\gamma$, a contradiction.

Therefore Cl.K.dim(R) exists. □

Exercise 14A. Let R be a ring for which Cl.K.dim(R) exists.

(a) Show that, for any ideal I of R, the classical Krull dimension of R/I exists and Cl.K.dim$(R/I) \leq$ Cl.K.dim(R).

(b) For any primes P and Q in R with $P < Q$, show that Cl.K.dim$(R/P) >$ Cl.K.dim(R/Q). Conclude that R satisfies the ACC on prime ideals.

(c) If R is right or left noetherian, P a prime of R, and I an ideal of R with $I > P$, show that Cl.K.dim$(R/I) <$ Cl.K.dim(R/P).

(d) If S is a polynomial ring in an infinite number of indeterminates over a nonzero ring, show that Cl.K.dim(S) does not exist. □

Exercise 14B. If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k in n independent indeterminates, show that Cl.K.dim$(R) = n$. [Hints: To see that Cl.K.dim$(R) \leq n$, it is enough to show that if $P_0 > P_1 > \cdots > P_n$ is a chain of primes in R of length n, then P_0 is a maximal ideal. By induction, $k(x_1)[x_2, \ldots, x_n]$ has classical Krull dimension $n-1$. Use this to show that $P_0 \cap k[x_1] \neq 0$, and similarly $P_0 \cap k[x_i] \neq 0$ for all i. Then show that R/P_0 is finite dimensional.] □

Lemma 14.2. *Let R be a ring with* Cl.K.dim$(R) = \gamma$. *If α is any nonnegative ordinal strictly less than γ, then there is a prime ideal P of R such that* Cl.K.dim$(R/P) = \alpha$. *If R is right or left noetherian, then there is a minimal prime P of R such that* Cl.K.dim$(R/P) = \gamma$.

Proof. Consulting the definition, we see for a prime P that Cl.K.dim$(R/P) = \alpha$ if and only if $P \in X_\alpha$ while $P \notin X_\beta$ for all $\beta < \alpha$. If there is no prime P such that Cl.K.dim$(R/P) = \alpha$, then we must have $X_\alpha = X_{\alpha+1}$, from which it would follow that $X_\beta = X_\alpha$ for all $\beta > \alpha$. We would then have Cl.K.dim$(R) \leq \alpha$, contrary to hypothesis. Therefore, there must be a prime P such that Cl.K.dim$(R/P) = \alpha$. This argument does not apply when $\alpha = \gamma$. However, it does show that Cl.K.dim(R) is the supremum of the ordinals Cl.K.dim(R/P) as P ranges over the set of prime ideals. Clearly, since every prime contains a minimal prime, we may restrict this set of primes to just the minimal primes. When R is right or left noetherian, there are only finitely many minimal primes, and therefore this supremum must actually be a maximum, so that for one of these minimal primes P we obtain Cl.K.dim$(R/P) = \gamma$. □

Exercise 14C. Let k be a field, $x_1, x_2, \dots$ independent indeterminates, and

$$R = k{\cdot}1 + \bigoplus_{n=1}^{\infty} k[x_1, \dots, x_n] \subseteq \prod_{n=1}^{\infty} k[x_1, \dots, x_n].$$

Show that $\mathrm{Cl.K.dim}(R) = \omega$ whereas $\mathrm{Cl.K.dim}(R/P)$ is finite for all primes P of R. $\square$

Exercise 14D. Let k be a field, $X = \{x_{ij} \mid i, j \in \mathbb{N} \text{ and } j \le i\}$ a collection of independent indeterminates, and $S = k[X]$. For $i \in \mathbb{N}$, set

$$P_i = x_{i1}S + x_{i2}S + \cdots + x_{ii}S.$$

Then set $U = \bigcup_{i \in \mathbb{N}} P_i$ and $R = S(S \setminus U)^{-1} \subseteq k(X)$.

(a) Show that any ideal J of S contained in U must be contained in some P_i. [Hint: Apply Exercise 3ZH to $J \cap k[x_{ij} \mid j \le i \le n]$ for all $n \in \mathbb{N}$ and then to J itself.]

(b) Show that no P_i contains an infinite ascending chain of prime ideals of S. [Hint: First show that $S(S \setminus P_i)^{-1}$ is noetherian.]

(c) Show that any prime ideal P of S contained in U is finitely generated. [Hint: Consider $\big(P \cap k[x_{ij} \mid i \le n]\big)S$ for each $n \in \mathbb{N}$.] Conclude that all prime ideals of R are finitely generated, and therefore, by Exercise 3ZG, that R is noetherian.

(d) Show that $\mathrm{Cl.K.dim}(R) = \omega$. $\square$

• BIMODULE SYMMETRY AND INTERSECTION THEOREMS •

Lemma 14.3. *Let S and R be noetherian rings, such that R satisfies the right second layer condition, and P a prime ideal of R. Let ${}_SB_R$ be a bimodule, finitely generated on each side, such that B_R is faithful. Then B has sub-bimodules $B' > B''$ such that B'/B'' is a torsionfree right (R/P)-module as well as a torsionfree left (S/Q)-module for some prime Q of S.*

Proof. By noetherian induction, we may assume that the lemma holds in any case where the right-hand ring is a proper factor of R.

Choose a right affiliated series $A_0 = 0 < A_1 < \cdots < A_m = B$ with corresponding right affiliated primes $P_1, \dots, P_m$. By Proposition 8.7, each factor A_i/A_{i-1} is a torsionfree right (R/P_i)-module, and hence also a faithful right (R/P_i)-module. Since B_R is faithful, $P_m P_{m-1} \cdots P_1 = 0$, and so $P_j \subseteq P$ for some index j. Hence, we may replace R, P, B by R/P_j, P/P_j, A_j/A_{j-1}. Thus, without loss of generality, we may assume that R is prime and that B is torsionfree as a right R-module.

If $P = 0$, just set $B'' = 0$ and let B' be any left affiliated sub-bimodule of B; by Proposition 8.7, B' is a torsionfree left module over $S/\,\mathrm{l.ann}_S(B')$.

Now assume that $P \neq 0$. Since B_R is faithful, R_R embeds in B_R^m for some positive integer m (Lemma 8.1), say $R_R \cong C \le B_R^m$. Then there is a submodule $C' < C$ such that $C/C' \cong (R/P)_R$. Now B_R^m has a submodule series $D_0 = 0 < D_1 < \cdots < D_m = B^m$ such that $D_i/D_{i-1} \cong B_R$ for $i = 1, \ldots, m$. By the Schreier Refinement Theorem, this series and the series $0 \leq C' < C \leq B^m$ have isomorphic refinements. Consequently, for some i there exist submodules $D_{i-1} \leq E' < E \leq D_i$ such that E/E' is isomorphic to a submodule of C/C'. Thus, B_R has submodules $B_1 > B_2$ such that B_1/B_2 is isomorphic to a right ideal of R/P.

In view of Theorem 12.9 and Exercise 12L (with $P < Q$ replaced by $0 < P$ and $A \geq B \geq C$ by $B \geq B_1 > B_2$), there exists a nonzero ideal I in R such that $B_1 \cap BI \leq B_2$. Since B_1/B_2 is a homomorphic image of $(B_1 + BI)/BI$, we have $I \leq \operatorname{ann}_R(B_1/B_2) = P$. Applying the induction hypothesis to the bimodule ${}_S(B/BI)_{R/I}$, we obtain sub-bimodules $B' > B'' \geq BI$ in B such that the desired properties hold. $\square$

Exercise 14E. Let R be a prime noetherian ring satisfying the right second layer condition. If the intersection of all nonzero ideals of R is nonzero, show that R is a simple ring. $\square$

Theorem 14.4. [Jategaonkar] *Let S and R be noetherian rings, such that R satisfies the right second layer condition, and let ${}_SB_R$ be a bimodule which is finitely generated on each side. If B_R is faithful, then* $\mathrm{Cl.K.dim}(R) \leq \mathrm{Cl.K.dim}(S)$.

Proof. We use induction on the ordinal $\alpha = \mathrm{Cl.K.dim}(R)$. Since the cases $\alpha = -1$ and $\alpha = 0$ are trivial, we may suppose that $\alpha > 0$.

By Lemma 14.2, $\mathrm{Cl.K.dim}(R/P) = \alpha$ for some minimal prime P of R. Then, by Lemma 14.3, there exist sub-bimodules $B' > B''$ of B such that B'/B'' is a torsionfree right (R/P)-module and a torsionfree left (S/Q)-module for some prime Q of S. Since it is enough to show that $\mathrm{Cl.K.dim}(S/Q) \geq \alpha$, we may reduce to the bimodule B'/B''. Thus, there is no loss of generality in assuming that R and S are prime rings and that B is torsionfree on both sides.

Given any ordinal $\beta < \alpha$, there exists a prime P of R (necessarily nonzero) such that $\mathrm{Cl.K.dim}(R/P) = \beta$ (Lemma 14.2). By Lemma 14.3, there exist sub-bimodules $C' > C''$ of B such that C'/C'' is a torsionfree right (R/P)-module. If $I = \mathrm{l.ann}_S(C'/C'')$, then $\mathrm{Cl.K.dim}(S/I) \geq \beta$ by the induction hypothesis. Since C'/C'' is unfaithful as a right R-module, Proposition 8.4 implies that it is also unfaithful as a left S-module, whence $I \neq 0$. As $\mathrm{Cl.K.dim}(S/I) \geq \beta$, we obtain $\mathrm{Cl.K.dim}(S) \geq \beta + 1$ (Exercise 14A). Therefore $\mathrm{Cl.K.dim}(S) \geq \alpha$. $\square$

Corollary 14.5. [Jategaonkar] *Let S and R be noetherian rings satisfying the second layer condition and ${}_SB_R$ a bimodule which is finitely generated and faithful on both sides. Then* $\mathrm{Cl.K.dim}(R) = \mathrm{Cl.K.dim}(S)$. $\square$

Corollary 14.6. [Jategaonkar] *If R is a noetherian ring satisfying the second layer condition, and if P and Q are prime ideals of R in the same clique, then* $\text{Cl.K.dim}(R/P) = \text{Cl.K.dim}(R/Q)$. *Hence, if P and Q are distinct primes in the same clique, then P and Q are incomparable.* □

Exercise 14F. Let $S = A_1(k)$ and $R = k + xS$ as in Exercises 3ZB and 3ZC. Show that, if we regard xS as an (R, S)-bimodule, then xS is finitely generated and faithful on both sides but the conclusion of Corollary 14.5 does not hold (that is, R and S have different classical Krull dimensions). □

Exercise 14G. Let R be a noetherian ring satisfying the second layer condition and P, Q primes of R. Show that $P \rightsquigarrow Q$ if and only if the bimodule $(P \cap Q)/PQ$ is faithful both as a left (R/P)-module and as a right (R/Q)-module. [Hint: Given the faithfulness assumption, choose an ideal A as large as possible such that $P \cap Q > A \geq PQ$ and $(P \cap Q)/A$ is faithful as a left (R/P)-module and as a right (R/Q)-module.] □

We next verify Jacobson's Conjecture for any noetherian ring R satisfying the second layer condition. In fact, we prove that $\bigcap J^n = 0$, where J is the intersection of all the maximal ideals of R. Since the Jacobson radical is the intersection of all the primitive ideals of R, the ideal J is a priori larger than $J(R)$. Our intersection result is therefore a strong form of Jacobson's Conjecture.

Lemma 14.7. *Let R be a noetherian ring. If I is a nonzero right ideal of R, there exists a maximal ideal M of R such that* $\text{Hom}_R(I, E((R/M)_R)) \neq 0$.

Proof. Set $J = \text{r.ann}_R(I)$. Since R is left noetherian, $(R/J)_R$ embeds in I^n for some positive integer n (Lemma 8.1). Choose a maximal ideal $M \geq J$ and observe that the quotient map $(R/J)_R \to (R/M)_R$ extends to a nonzero homomorphism $I^n \to E((R/M)_R)$. Therefore $\text{Hom}_R(I, E((R/M)_R)) \neq 0$. □

Theorem 14.8. [Jategaonkar] *Let R be a noetherian ring satisfying the second layer condition and J the intersection of all the maximal ideals of R. Then $\bigcap J^n = 0$. In particular, $\bigcap J(R)^n = 0$.*

Proof. Let E be the direct sum of $E((R/M)_R)$ as M ranges over all maximal ideals of R. By Lemma 14.7, the intersection of the kernels of all homomorphisms from R_R to E is zero, and so R_R embeds in a direct product of copies of E. Hence, it suffices to show that $\bigcap J^n$ annihilates E. Thus, it is enough to prove that any element x in any $E((R/M)_R)$ is killed by a power of J.

The right second layer condition implies that $xP_1P_2 \cdots P_n = 0$ for some primes P_i in the right link closure of $\{M\}$ (Theorem 12.7). From Corollary 14.6, we have

$$\text{Cl.K.dim}(R/P_i) = \text{Cl.K.dim}(R/M) = 0$$

for all i, and so each P_i is a maximal ideal. Therefore J is contained in each P_i, and hence $xJ^n = 0$, as desired. □

Another application of the Krull Intersection Theorem is that if R is a commutative noetherian domain and P a prime ideal in R, then $\bigcap P^n = 0$. This statement usually fails in the noncommutative case (Exercise 14H). There is, however, a version of this theorem for a prime noetherian ring with the second layer condition, as we now show. (Observe that Proposition 14.10 includes the commutative result just stated, since if R is commutative, $\{P\}$ is right link closed.)

Lemma 14.9. *Let R be a prime right noetherian ring and E a nonzero injective right R-module. If I is a nonzero right ideal of R, then* $\operatorname{Hom}_R(I, E) \neq 0$. *Consequently, E is a faithful module.*

Proof. Since the ideal RI is nonzero, it contains a regular element, and hence R_R embeds in I^n for some positive integer n. There exist nonzero homomorphisms $R_R \to E$, and they all extend to nonzero homomorphisms $I^n \to E$. Thus $\operatorname{Hom}_R(I, E) \neq 0$.

Now, given any nonzero element $x \in R$, there exists a nonzero homomorphism $f : xR \to E$. Then f extends to a homomorphism $g : R_R \to E$, whence $g(1)x = f(x) \neq 0$. Therefore E is faithful. $\square$

Proposition 14.10. *Let R be a prime noetherian ring satisfying the right second layer condition, P a prime ideal of R, and J the intersection of the right link closure of $\{P\}$. Then $\bigcap J^n = 0$.*

Proof. Set $E = E((R/P)_R)$. By Lemma 14.9, R_R embeds in a direct product of copies of E. The right second layer condition implies that any element of E is killed by a product of primes in the right link closure of $\{P\}$ (Theorem 12.7). Hence, $\bigcap J^n$ annihilates E, and therefore $\bigcap J^n = 0$. $\square$

Exercise 14H. Find an example of a prime fully bounded noetherian ring R with a prime ideal P such that $\bigcap P^n \neq 0$. $\square$

Exercise 14I. For a ring R, let $\beta(R)$ be the intersection of the maximal ideals of R. This ideal is sometimes called the *Brown-McCoy radical of R.* We showed in Theorem 14.8 that if R is a noetherian ring satisfying the second layer condition, then $\bigcap \beta(R)^n = 0$. Give an example to show that this may fail for noetherian rings not satisfying the second layer condition. $\square$

• FINITE RING EXTENSIONS •

In this section, we return to the study of prime ideals in finite ring extensions, which we considered earlier in Chapters 8 and 11. Here we concentrate on the situation in which the smaller ring satisfies the second layer condition. For some results – in particular, the first theorem – all that is needed from the second layer condition is the "bimodule symmetry" of classical Krull dimension obtained in Corollary 14.5. More precisely, the theorem can be proved for a noetherian ring with a suitable "symmetric dimension function," as indicated in Exercise 14J.

Lemma 14.11. *Let R be a noetherian ring satisfying the second layer condition, and suppose that R is a subring of a ring S such that ${}_RS$ and S_R are finitely generated. If S is a prime ring, then the minimal prime ideals of R are exactly those primes Q for which* $\text{Cl.K.dim}(R/Q) = \text{Cl.K.dim}(R)$*; moreover, the set of minimal primes of R is right and left link closed.*

Proof. By Lemma 14.2, R has at least one minimal prime Q' such that R/Q' has the same classical Krull dimension as R. If Q is any minimal prime of R, it follows from Lemma 8.17 and Corollary 14.5 that

$$\text{Cl.K.dim}(R/Q) = \text{Cl.K.dim}(R/Q') = \text{Cl.K.dim}(R).$$

On the other hand, if Q is a nonminimal prime of R, it properly contains a minimal prime Q'', and using Exercise 14A we conclude that

$$\text{Cl.K.dim}(R/Q) < \text{Cl.K.dim}(R/Q'') = \text{Cl.K.dim}(R).$$

The final conclusion of the lemma now follows from the above results and Corollary 14.6. $\square$

Theorem 14.12. [Joseph-Small, Borho, Warfield] *Let R be a noetherian ring satisfying the second layer condition, and suppose that R is a subring of a ring S such that ${}_RS$ and S_R are both finitely generated. If P is any prime ideal of S, the ring $R/(P \cap R)$ has an artinian classical quotient ring A. Moreover, if B is the Goldie quotient ring of S/P, the embedding $R/(P \cap R) \to S/P$ extends to an embedding $A \to B$. In other words, $\mathcal{C}_R(P \cap R) \subseteq \mathcal{C}_S(P)$.*

Proof. Without loss of generality, we may assume that $P = 0$.

To show that R has an artinian classical quotient ring, it will suffice, by Theorem 11.13, to show that R has a left affiliated series and a right affiliated series such that the affiliated primes for both series are all minimal. By symmetry, it is enough to find such an affiliated series on the left.

We first choose a left affiliated series $S_0 = 0 < S_1 < \cdots < S_n = S$ for the bimodule ${}_RS_S$ and show that the corresponding left affiliated primes $Q_1, \ldots, Q_n$ are all minimal primes of R. Then, by Exercise 8D, there will be a left affiliated series for R whose corresponding left affiliated primes are all from the set $\{Q_1, \ldots, Q_n\}$ and hence are minimal.

Let $T = S_i/S_{i-1}$ (for some $i \in \{1, \ldots, n\}$) and note that it follows from Propositions 8.7 and 8.9 that T is a (nonzero) torsionfree left (R/Q_i)-module as well as a torsionfree right S-module. In particular, T is faithful as a right S-module and hence also faithful as a right R-module. Next let $T_0 = 0 < T_1 < \cdots < T_m = T$ be a right affiliated series for the bimodule ${}_RT_R$. Any choice of a minimal prime Q of R must be a right affiliated prime corresponding to some factor in this series (Proposition 8.6), say T_j/T_{j-1}. Then T_j/T_{j-1} is a nonzero $(R/Q_i, R/Q)$-bimodule which is finitely generated and torsionfree on each side (Propositions 8.7 and 8.9). Using Corollary 14.5 and Lemma 14.11, we conclude that

$$\mathrm{Cl.K.dim}(R/Q_i) = \mathrm{Cl.K.dim}(R/Q) = \mathrm{Cl.K.dim}(R)$$

and hence that Q_i is a minimal prime of R, as desired.

Therefore, R does have an artinian classical quotient ring A. Now, by Small's Theorem, $\mathcal{C}_R(0) = \mathcal{C}_R(N)$, where N is the prime radical of R.

To obtain the embedding of A into B, we must show that every regular element of R is a unit in B (see Proposition 10.4); hence, it is enough to show that every regular element of R is also regular as an element of S. Using other terminology, we would like S to be $\mathcal{C}_R(0)$-torsionfree both as a left R-module and as a right R-module. Since the primes Q_i corresponding to our left affiliated series for ${}_RS_S$ are all minimal prime ideals of R, we conclude that $\mathcal{C}_R(0) = \mathcal{C}_R(N) \subseteq \mathcal{C}_R(Q_i)$ for $i = 1, \ldots, n$ (Proposition 7.5). As the factors S_i/S_{i-1} are already $\mathcal{C}_R(Q_i)$-torsionfree on the left, they must also be $\mathcal{C}_R(0)$-torsionfree. Therefore S is $\mathcal{C}_R(0)$-torsionfree on the left and, by symmetry, also on the right. □

Exercise 14J. Let R be a noetherian ring equipped with a *symmetric dimension function*, i.e., a function δ associating to each prime factor R/P of R an element of a partially ordered set (for example, an ordinal or a real number) such that the following two conditions are satisfied, for all prime ideals P and Q:

(a) If $P < Q$, then $\delta(R/P) > \delta(R/Q)$.

(b) If there exists a nonzero $(R/P, R/Q)$-bimodule which is finitely generated and torsionfree on each side, then $\delta(R/P) = \delta(R/Q)$.

(E.g., if R satisfies the second layer condition, δ can be taken to be the classical Krull dimension. There are other important examples, such as the *Gelfand-Kirillov dimension* for finitely generated algebras over a field, when this dimension is finite. For a discussion of this dimension, see Krause-Lenagan [2000] or McConnell-Robson [2001].) Show that the conclusions of Theorem 14.12 hold in this context, using δ in place of Cl.K.dim. □

The next two theorems relate the second layer condition and prime ideals in a noetherian ring R and a finite ring extension S. In both cases, we use the R-module structure of an S-module to produce a suitable ideal of S. We isolate the basic argument for this in a lemma (to avoid having to go through the argument three times).

Lemma 14.13. *Let R be a noetherian ring which is a subring of a prime ring S such that ${}_RS$ and S_R are finitely generated. Let M be a finitely generated right S-module, let $U > V$ be R-submodules of M, and assume that none of the associated primes of U/V is a minimal prime of R. Suppose either that R satisfies the strong second layer condition, or that R satisfies the second* *layer condition and, for all $T \in \mathrm{Ass}(U/V)$, the (R/T)-module $\mathrm{ann}_{U/V}(T)$ is torsionfree. Then there exists a nonzero ideal J in S such that $MJ \cap U \leq V$.*

Proof. In view of Theorems 14.12 and 11.9, $\mathcal{C}_R(N) = \mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$, where N is the prime radical of R.

Let W be an R-submodule of M such that $W \geq V$ and W is maximal with respect to the property $W \cap U = V$. Then U/V is isomorphic to an essential R-submodule of M/W. Now, if $T \in \mathrm{Ass}(M/W)$, then $T \in \mathrm{Ass}(U/V)$, and $\mathrm{ann}_{U/V}(T)$ is isomorphic to an essential submodule of $\mathrm{ann}_{M/W}(T)$. Hence, if $\mathrm{ann}_{U/V}(T)$ is torsionfree as an (R/T)-module, so is $\mathrm{ann}_{M/W}(T)$.

It now follows from either Theorem 12.4 or Theorem 12.7 that M/W is annihilated by a product of primes from the right link closure of $\mathrm{Ass}(U/V)$, say $MI_1I_2 \cdots I_n \leq W$ for suitable $I_1, \dots, I_n$ in the right link closure of $\mathrm{Ass}(U/V)$. Since, by assumption, there are no minimal primes in $\mathrm{Ass}(U/V)$, none of $I_1, \dots, I_n$ can be a minimal prime (Lemma 14.11).

By Proposition 7.5, each I_j contains an element of $\mathcal{C}_R(N)$. Since $\mathcal{C}_R(N) \subseteq \mathcal{C}_S(0)$, the ideal $I = I_1I_2 \cdots I_n$ therefore contains a regular element of S. Hence, S/SI is torsion as a left S-module, and so, by the left-hand version of Lemma 8.3, there is a nonzero ideal J of S such that $JS \leq SI$, that is, $J \leq SI$. Thus,

$$MJ \leq MSI = MI = MI_1I_2 \cdots I_n \leq W,$$

and therefore $MJ \cap U \leq W \cap U = V$. $\square$

Theorem 14.14. [Letzter] *Let R be a noetherian ring which is a subring of a ring S such that ${}_RS$ and S_R are both finitely generated. If R satisfies the second layer condition* (*or the strong second layer condition*)*, so does S.*

Proof. Suppose first that R satisfies the strong second layer condition but that S does not, say on the right. Then, by Proposition 12.3, there exists a finitely generated uniform right S-module M with an affiliated series $0 < U < M$ and corresponding affiliated primes Q, P such that $P < Q$ and $MP = 0$. Without loss of generality, we may assume that $P = 0$ and that M is faithful (because M/U is then faithful).

Let Y be the set of primes of R that contain $Q \cap R$. If Y contains a minimal prime T, then T is minimal over both $0 \cap R$ and $Q \cap R$, whence the primes 0 and Q in S both lie over T. However, this contradicts the property INC (Theorem 11.6). Therefore no element of Y is a minimal prime. Since $U(Q \cap R) = 0$, it follows that there are no minimal primes in $\mathrm{Ass}(U_R)$.

Applying Lemma 14.13 (with $V = 0$), we obtain a nonzero ideal J in S such that $MJ \cap U = 0$. Since MJ is an S-submodule of M and M_S is uniform, it follows that $MJ = 0$. But as M is a faithful S-module and $J \neq 0$, this is a contradiction.

Therefore S satisfies the strong second layer condition on the right and, by symmetry, on the left.

To prove that the second layer condition carries up from R to S, two changes are required in the above proof. First, we may assume that U is torsionfree as an (S/Q)-module, and second, in order to apply Lemma 14.13, we must show that, for any $T \in \mathrm{Ass}(U_R)$, the (R/T)-module $\mathrm{ann}_U(T)$ is torsionfree.

Since $\mathrm{ann}_U(T) \neq 0$ and U is torsionfree as an (S/Q)-module, T must be disjoint from $\mathcal{C}_S(Q)$. On the other hand, if $N/(Q \cap R)$ is the prime radical of $R/(Q \cap R)$, it follows from Theorems 14.12 and 11.9 that

$$\mathcal{C}_R(N) = \mathcal{C}_R(Q \cap R) \subseteq \mathcal{C}_S(Q).$$

Hence, T is disjoint from $\mathcal{C}_R(N)$. As $T \geq Q \cap R$, Proposition 7.5 now shows that T is minimal over N. Finally, $\mathrm{ann}_U(T)$ is torsionfree as an (R/N)-module (because $\mathcal{C}_R(N) \subseteq \mathcal{C}_S(Q)$), and therefore Proposition 7.9 shows that $\mathrm{ann}_U(T)$ is also torsionfree as an (R/T)-module, as desired. □

Corollary 14.15. *If R is a noetherian ring satisfying the second layer condition and R is a subring of a ring S such that ${}_RS$ and S_R are both finitely generated, then R and S have the same classical Krull dimension.*

Proof. Theorem 14.14 and Corollary 14.5 (applied to the bimodule ${}_RS_S$). □

Theorem 14.16. [Letzter] *If R is a ring satisfying the second layer condition and R is a subring of a ring S such that ${}_RS$ and S_R are both finitely generated, then Lying Over holds. That is, for every prime Q of R, there is a prime P of S such that Q is minimal over $P \cap R$.*

Proof. Let P be an ideal of S maximal with respect to the property $P \cap R \leq Q$. It is easy to see that P is prime: If A and B are ideals of S properly containing P, then $A \cap R \not\leq Q$ and $B \cap R \not\leq Q$, whence $(A \cap R)(B \cap R) \not\leq Q$, yielding $AB \cap R \not\leq Q$ and so $AB \not\leq P$. Without loss of generality, we may assume that $P = 0$, and, by way of contradiction, we assume that Q is not a minimal prime of R.

As a right R-module, R/Q has only one associated prime, namely Q, and of course R/Q is torsionfree as a right (R/Q)-module. We now apply Lemma 14.13, using $M = S_S$ while $U = R_R$ and $V = Q_R$. Thus, there is a nonzero ideal J in S such that $SJ \cap R \leq Q$, that is, $J \cap R \leq Q$. However, this contradicts the fact that 0 is the largest ideal of S whose contraction to R is contained in Q. Therefore Q must be a minimal prime of R. □

• LOCALIZATION AT A SEMIPRIME IDEAL •

If R is a commutative ring and P a prime ideal of R, then we can *localize* the ring R at the prime P by inverting the elements not in P. The resulting ring of fractions is denoted R_P, and it is a local ring with maximal ideal PR_P. The factor R_P/PR_P is naturally isomorphic to the quotient field of R/P. It is natural to try to imitate this in the noncommutative setting, but a naive attempt to do so does not work. If R is noetherian and P is a prime ideal, then we would like to invert the set $\mathcal{C}(P)$. If $\mathcal{C}(P)$ is a right Ore set, then we again write the corresponding ring of fractions as R_P, and it is easy to verify that PR_P is the unique maximal ideal of R_P, that it is also the only primitive ideal of R_P, and that R_P/PR_P can be naturally identified with the right Goldie quotient ring of R/P. The trouble with all of this is that $\mathcal{C}(P)$ is usually not a right Ore set. The first obstruction is indicated in the following lemma.

Lemma 14.17. *Let R be a noetherian ring and P and Q primes in R such that $Q \rightsquigarrow P$. Let $\mathcal{C}$ be a right Ore set in R and suppose that $\mathcal{C} \subseteq \mathcal{C}(P)$. Then $\mathcal{C} \subseteq \mathcal{C}(Q)$.*

Proof. If Q is linked to P via the linking bimodule $(Q \cap P)/A$, there is no loss of generality in assuming that $A = 0$. Now $Q \cap P$ is a torsionfree right (R/P)-module, and hence $Q \cap P$ is also $\mathcal{C}$-torsionfree on the right (because $\mathcal{C} \subseteq \mathcal{C}(P)$). By right reversibility (Proposition 10.7), $Q \cap P$ is also $\mathcal{C}$-torsionfree on the left.

As a left (R/Q)-module, $Q \cap P$ is nonzero and torsionfree, and hence faithful. It follows that R/Q embeds (as a left module) in some finite direct sum of copies of $Q \cap P$ (Lemma 8.1). Hence, R/Q is $\mathcal{C}$-torsionfree on the left; i.e., for each $c \in \mathcal{C}$ the coset $c + Q$ has zero right annihilator in R/Q. Therefore, by Lemma 6.11, these cosets are regular elements of R/Q, that is, $\mathcal{C} \subseteq \mathcal{C}(Q)$. $\square$

It is clear from Lemma 14.17 that if we are given a prime P in a noetherian ring R and we want to find a right Ore set contained in $\mathcal{C}(P)$, then the largest set we can consider as a possible right Ore set is the intersection of the $\mathcal{C}(Q)$ for all primes Q in the right link closure of $\{P\}$. This set can certainly be infinite, as examples in Chapter 12 show. We will concentrate at first on the special case in which such a set is finite. Our main considerations will be restricted to rings satisfying the second layer condition, and so the primes in the right link closure of $\{P\}$ will be incomparable (Corollary 14.6). This puts us in the following situation: We have a finite set $\{P_1, \ldots, P_n\}$ of incomparable primes, such that $\{P_1, \ldots, P_n\}$ is right link closed, and we want to know whether $\mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n)$ is a right Ore set.

In this situation, we will study the semiprime ideal $N = P_1 \cap \cdots \cap P_n$. Obviously, any prime minimal over N is one of the P_i; conversely, since the P_i are incomparable, they are all minimal over N. (Any P_i contains a prime Q minimal over N, and since $Q = P_j$ for some j, it follows from the incomparability of our set of primes that $P_i = P_j$, whence P_i is minimal over N.) Now, by Proposition 7.5,

$$\mathcal{C}(N) = \mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n).$$

Thus, to rephrase our problem: We have a semiprime ideal N such that the set of primes minimal over N is right link closed, and we want to know whether $\mathcal{C}(N)$ is a right Ore set. Assuming that R satisfies the right second layer condition, we shall prove that this is the case (Theorem 14.21). We first collect some elementary facts about the corresponding localization in case it exists.

Definition. A semiprime ideal N in a right noetherian ring R is called *right localizable* provided $\mathcal{C}(N)$ is a right Ore set. When this occurs, we denote the corresponding localization by R_N. The ideal N is *classically right localizable* if, in addition, the injective hull of the right R_N-module $R_N/J(R_N)$ is the union of its socle series. Of course, a semiprime ideal N in a noetherian ring R is called *(classically) localizable* if N is both right and left (classically) localizable.

If R is the right noetherian ring discussed in Exercise 9M, it is easily checked that $J(R)$ is right localizable but not classically right localizable. However, it is not known whether all right localizable semiprime ideals in a (right and left) noetherian ring are classically right localizable (cf. Exercise 14O).

Lemma 14.18. *Let N be a right localizable semiprime ideal of a right noetherian ring R. Then $NR_N = J(R_N)$, the ring R_N/NR_N is semisimple, and the natural map $R/N \to R_N/NR_N$ extends to a ring isomorphism from the Goldie quotient ring of R/N onto R_N/NR_N.*

Proof. Since R/N is $\mathcal{C}(N)$-torsionfree, Theorems 10.15 and 10.18 show that $NR_N = N^{\mathrm{e}}$ is an ideal of R_N that contracts to N. In particular, the natural map $R/N \to R_N/NR_N$ is a ring embedding. Now this map is a right ring of fractions for R/N with respect to the Ore set of regular elements of R/N, and therefore R_N/NR_N is the right Goldie quotient ring of the image of R/N.

In particular, R_N/NR_N is a semisimple ring, whence $J(R_N) \subseteq NR_N$. To prove the reverse inclusion, it suffices to show that any maximal right ideal I of R_N contains NR_N.

Now I^{c} is a proper right ideal of R such that $I^{\mathrm{c}}R_N = I$ (Theorem 10.15). It follows that I^{c} must be disjoint from $\mathcal{C}(N)$, and so $R/(I^{\mathrm{c}}+N)$ is not torsion as a right (R/N)-module. If $K/(I^{\mathrm{c}}+N)$ is the torsion submodule of $R/(I^{\mathrm{c}}+N)$, then K is a proper right ideal of R and R/K is $\mathcal{C}(N)$-torsionfree. By Theorem 10.15, $(KR_N)^{\mathrm{c}} = K$. Since KR_N is a proper right ideal of R_N containing $I^{\mathrm{c}}R_N$, which equals I, we must have $KR_N = I$. Therefore $NR_N \subseteq I$, as desired. $\square$

Exercise 14K. Let N be a right localizable semiprime ideal in a right noetherian ring R and $E = E((R/N)_R)$. Show that N is classically right localizable if and only if $E = \bigcup_{i=0}^{\infty} \operatorname{ann}_E(N^i)$. [Hint: Show that the injective hull of the right R_N-module $R_N/J(R_N)$ is isomorphic to E.] $\square$

If R is a noetherian ring and N its prime radical, we showed in Lemma 11.5 that an R-module A is $\mathcal{C}(N)$-torsion if and only if it has zero reduced rank. The following lemma can be thought of as a generalization of this result.

To see the connection, the reader may want to verify directly that a right R-module A has zero reduced rank if and only if $\operatorname{Hom}_R(A, E((R/N)_R)) = 0$.

Lemma 14.19. *Let R be a right noetherian ring, N a semiprime ideal of R, and $E = E((R/N)_R)$. Then a right R-module A is $\mathcal{C}(N)$-torsion if and only if* $\operatorname{Hom}_R(A, E) = 0$.

Proof. Suppose first that there exist $f \in \operatorname{Hom}_R(A, E)$ and $a \in A$ such that $f(a) \neq 0$. There is an element $r \in R$ such that $f(a)r \in R/N$ and $f(a)r \neq 0$. If A were $\mathcal{C}(N)$-torsion, then there would be an element $c \in \mathcal{C}(N)$ such that $arc = 0$, and hence $f(a)rc = 0$. Since R/N is $\mathcal{C}(N)$-torsionfree, this would imply that $f(a)r = 0$, which is false. Hence, $\operatorname{Hom}_R(A, E) \neq 0$ implies that A is not $\mathcal{C}(N)$-torsion.

Conversely, suppose that A is not $\mathcal{C}(N)$-torsion and choose $a \in A$ such that $ac \neq 0$ for all $c \in \mathcal{C}(N)$. We note first that $a + aN$ is not a $\mathcal{C}(N)$-torsion element of A/aN, since if $ac \in aN$ for some $c \in \mathcal{C}(N)$, then $ac = an$ for some $n \in N$, whence $a(c - n) = 0$, and clearly $c - n \in \mathcal{C}(N)$. Now aR/aN is an (R/N)-module which we have shown is not $\mathcal{C}(N)$-torsion, and so, by Lemma 7.17, aR/aN has a submodule U which is isomorphic to a uniform right ideal of R/N. The embedding of U into R/N extends to a nonzero homomorphism $A/aN \to E$. (Of course, if R were also left noetherian, we could conclude from Proposition 7.19 that $\operatorname{Hom}_R(aR/aN, R/N) \neq 0$.) Thus, we have shown that if A is not $\mathcal{C}(N)$-torsion, then $\operatorname{Hom}_R(A, E) \neq 0$. $\square$

Of course, if $\mathcal{C}(N)$ is a right Ore set, then Lemma 14.19 immediately implies that $E((R/N)_R)$ is $\mathcal{C}(N)$-torsionfree (since then the $\mathcal{C}(N)$-torsion elements of $E((R/N)_R)$ form a submodule). This turns out, in fact, to characterize the Ore condition for $\mathcal{C}(N)$.

Theorem 14.20. [Jategaonkar] *Let R be a right noetherian ring, N a semiprime ideal of R, and $E = E((R/N)_R)$. Then N is right localizable if and only if E is $\mathcal{C}(N)$-torsionfree.*

Proof. It is immediate from the previous lemma that if N is right localizable, then E is $\mathcal{C}(N)$-torsionfree. Conversely, suppose that E is $\mathcal{C}(N)$-torsionfree, and let $r \in R$ and $c \in \mathcal{C}(N)$. Since R/cR is generated by the $\mathcal{C}(N)$-torsion element $1 + cR$, and since E is $\mathcal{C}(N)$-torsionfree, we see that $\operatorname{Hom}_R(R/cR, E) = 0$, from which Lemma 14.19 allows us to conclude that R/cR is $\mathcal{C}(N)$-torsion. Hence, there is an element $c' \in \mathcal{C}(N)$ such that $(r + cR)c' = 0$, or, in other words, $rc' \in cR$. Therefore $\mathcal{C}(N)$ is a right Ore set. $\square$

Theorem 14.21. [Müller, Jategaonkar] *Let R be a noetherian ring satisfying the right second layer condition and N a semiprime ideal of R. Then N is classically right localizable if and only if the set of prime ideals minimal over N is right link closed in* $\operatorname{Spec}(R)$.

Proof. Let $E = E((R/N)_R)$.

Assume first that N is classically right localizable and consider a link $Q \rightsquigarrow P$, where P is a prime minimal over N. By Theorem 12.2, there exists a finitely generated uniform right R-module M with an affiliated series $0 < U < M$ such that U is isomorphic to a (uniform) right ideal of R/P and M/U is isomorphic to a uniform right ideal of R/Q. Since U is torsionfree as a right (R/P)-module, it is also torsionfree as a right (R/N)-module (Proposition 7.9), and so U is isomorphic to a right ideal of R/N (Corollary 7.20). Consequently, M can be embedded in E. Because of Exercise 14K, it follows that $MN^i = 0$ for some $i \in \mathbb{N}$. As M/U is a faithful right (R/Q)-module, we obtain $N^i \leq Q$, and hence $N \leq Q$.

We also have $\mathcal{C}(N) \subseteq \mathcal{C}(P)$ by Proposition 7.5, and then $\mathcal{C}(N) \subseteq \mathcal{C}(Q)$ by Lemma 14.17. Since Q contains N, we conclude from Proposition 7.5 that Q is minimal over N. Thus, the set of primes minimal over N is right link closed.

Conversely, assume that the set of primes minimal over N is right link closed. Observe using Corollary 7.7 that if P is any associated prime of E, then P is minimal over N and $\operatorname{l.ann}_E(P)$ is torsionfree as a right (R/P)-module. Hence, if M is a nonzero finitely generated submodule of E, and if $M_0 = 0 < M_1 < \cdots < M_k = M$ is an affiliated series for M with corresponding affiliated primes $P_1, \ldots, P_k$, then, according to Theorem 12.7, each P_i is in the right link closure of $\operatorname{Ass}(E)$ and each factor M_i/M_{i-1} is torsionfree as a right (R/P_i)-module. Our hypothesis on the primes minimal over N implies that each P_i contains N and is minimal over N. In particular, $MN^k = 0$. Since each P_i is minimal over N, each M_i/M_{i-1} is torsionfree as a right (R/N)-module (Proposition 7.9), that is, M_i/M_{i-1} is $\mathcal{C}(N)$-torsionfree. It follows that M is $\mathcal{C}(N)$-torsionfree.

We conclude from the previous paragraph that $E = \bigcup_{i=0}^{\infty} \operatorname{ann}_E(N^i)$ and that E is $\mathcal{C}(N)$-torsionfree. Therefore, by Theorem 14.20 and Exercise 14K, N is classically right localizable. $\square$

Corollary 14.22. *Let R be a noetherian ring satisfying the second layer condition and N its prime radical. If all factors of R by minimal prime ideals have the same classical Krull dimension, then N is classically localizable.*

Proof. Corollary 14.6 and Theorem 14.21. $\square$

Exercise 14L. Let R be a noetherian ring satisfying the right second layer condition and P a prime of R. Assume that the right link closure of $\{P\}$ is a finite set, say $\{P_1, \ldots, P_n\}$.

(a) Show that the set $\mathcal{C} = \mathcal{C}(P_1) \cap \cdots \cap \mathcal{C}(P_n)$ is the largest right Ore set contained in $\mathcal{C}(P)$, and set $S = R\mathcal{C}^{-1}$.

(b) For $i = 1, \ldots, n$, show that P_iS is a maximal ideal of S, that S/P_iS is a simple artinian ring isomorphic to the Goldie quotient ring of R/P_i, and that $E((S/P_iS)_S)$ is the union of its socle series.

(c) Show that $P_1S, \ldots, P_nS$ are the only right primitive ideals of S. $\square$

Exercise 14M. Let N be a semiprime ideal in a noetherian ring R and X the set of primes minimal over N. Show that N is classically right localizable if and only if X is right link closed and every prime in X satisfies the right second layer condition. (Cf. Exercise 11O. This is an improvement of Theorem 14.21 above. However, the only cases in which it has been applied in rings not necessarily satisfying the right second layer condition for *all* primes is when N is the prime radical, in which case one can recover some of the results of Chapter 11.) $\square$

The next two exercises address the question of whether the second layer condition is necessary for considerations of localization (as opposed to classical localization).

Exercise 14N. Let R be a prime noetherian ring which is not artinian but such that, for every essential right ideal I, the module R/I is artinian. (We say that R has *right Krull dimension* 1, according to the notion of Krull dimension to be discussed in the following chapter.)

(a) Show that if Q is a prime of R which fails to satisfy the right second layer condition, then there is a finitely generated right R-module M which has an essential submodule U isomorphic to $(R/Q)_R$, such that M/U is a faithful simple right R-module.

(b) Let N be a semiprime ideal of R such that the set X of primes minimal over N is right link closed. Show that N is right localizable if and only if every prime in X satisfies the right second layer condition. [Hint: If N is right localizable and $E = E((R/N)_R)$, use Theorem 14.20 to show that $E/\operatorname{ann}_E(N)$ is $\mathcal{C}(N)$-torsionfree.] $\square$

Exercise 14O. We give here three well-known open questions. The point of the exercise is not to prove them (although feel free to try), but to show that they are equivalent.

Conjecture (a): If R is a noetherian ring and N a localizable semiprime ideal in R, then N is classically localizable.

Conjecture (b): If R is a noetherian ring and N a localizable semiprime ideal in R, then each of the primes minimal over N must satisfy the second layer condition.

Conjecture (c): If R is a noetherian ring such that $R/J(R)$ is artinian, then $J(R)$ satisfies the AR-property. [Hint: Lenagan's Theorem.] $\square$

The next two exercises explore some of the distinctions between the AR-property and localizability.

Exercise 14P. Give an example of a noetherian ring containing a semiprime ideal which satisfies the right AR-property but is not right localizable. [Hint: The prime radical always satisfies the AR-property.] $\square$

Exercise 14Q. The point of this exercise is to construct a localizable semiprime ideal P in a noetherian ring R such that P does not satisfy the AR-property. In the example, P will be a minimal prime and R will be semiprime, and so P will be clearly localizable (Exercise 10T). An obvious way to have the AR-property fail is to have linked maximal ideals, $M \rightsquigarrow N$, such that $P \leq N$ while $P \not\leq M$ (Proposition 13.6). We construct such a ring as follows. Let T be the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and I the ideal $\begin{pmatrix} 2\mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. There is a ring homomorphism $\phi : T \to \mathbb{Z}/2\mathbb{Z}$ with kernel I. Let

$$R = \{(t, m) \in T \times \mathbb{Z} \mid \phi(t) = m + 2\mathbb{Z}\},$$

and verify that R is a ring of the desired type, with $P = I \times 0$ and $M = I \times 2\mathbb{Z}$. $\square$

• EMBEDDINGS INTO ARTINIAN RINGS •

As an application of the localization results obtained in the previous section, we show that any noetherian ring satisfying the second layer condition can be embedded in an artinian ring.

Lemma 14.23. *Let R be a noetherian ring satisfying the second layer condition and U a finitely generated uniform right R-module. Let P be the assassinator of U, and assume that $\operatorname{ann}_U(P)$ is torsionfree as a right (R/P)-module. Then the ring $R/\operatorname{ann}_R(U)$ has an artinian classical quotient ring.*

Proof. Without loss of generality, we may assume that U is a faithful R-module. By Lemma 11.8 and Corollary 11.10, it suffices to prove that $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, where N is the prime radical of R.

Choose an affiliated series $U_0 = 0 < U_1 < \cdots < U_n = U$ with corresponding affiliated primes $Q_1, \dots, Q_n$. Since P is the only associated prime of U, it follows from Theorem 12.7 that each of the primes Q_i is in the right link closure of $\{P\}$ and that each of the factors U_i/U_{i-1} is torsionfree as a right (R/Q_i)-module. Since U is faithful, the set $\{Q_1, \dots, Q_n\}$ contains all of the minimal primes of R (Proposition 8.6).

By Corollary 14.6, the factors R/Q_i all have the same classical Krull dimension, say α. Now all factors of R modulo minimal primes have classical Krull dimension α, and so we conclude that all of the primes Q_i are minimal. Hence, $\mathcal{C}(N) \subseteq \mathcal{C}(Q_i)$ for all i (Proposition 7.5). As a result, each of the factors U_i/U_{i-1} is $\mathcal{C}(N)$-torsionfree, and thus U is $\mathcal{C}(N)$-torsionfree. Since U is faithful, R_R embeds in a direct product of copies of U, and hence R_R is $\mathcal{C}(N)$-torsionfree. (This does not yet show that $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.)

Now, by Corollary 14.22, $\mathcal{C}(N)$ is an Ore set, and so we may use right reversibility (Proposition 10.7) to conclude from the $\mathcal{C}(N)$-torsionfreeness of R_R that $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, as desired. (The use of Small's Theorem in this proof can be avoided by a more careful analysis of the ring R_N. Once it is established that $\mathcal{C}(N) = \mathcal{C}(0)$, it just remains to show that R_N is artinian, as in the following exercise.) $\square$

Exercise 14R. Let R be a right noetherian ring, N a right localizable semiprime ideal of R, and $P_1, \dots, P_n$ the primes minimal over N. Show that R_N is right artinian if and only if all of the primes P_i are minimal primes of R. [Hint: Theorem 10.20 and Lemma 14.18.] $\square$

Theorem 14.24. [Jategaonkar] *Any noetherian ring R satisfying the second layer condition can be embedded in an artinian ring. More precisely, there exist ideals $A_1, \dots, A_n$ in R such that $A_1 \cap \cdots \cap A_n = 0$ and each of the factor rings R/A_i has an artinian classical quotient ring.*

Proof. We first claim that if U is any uniform right ideal of R, there exists an ideal A such that $U \cap A = 0$ and the ring R/A has an artinian classical quotient ring.

Let P be the assassinator of U, and recall from Proposition 7.10 that $\operatorname{ann}_U(P)$ is either torsion or torsionfree as a right (R/P)-module. It cannot be torsion since, according to Lemma 8.1, R/P can be embedded (as a right module) in a finite direct sum of copies of $\operatorname{ann}_U(P)$. Hence, $\operatorname{ann}_U(P)$ is a torsionfree right (R/P)-module. Now choose a right ideal V of R maximal with respect to the property that $U \cap V = 0$; then U is isomorphic to an essential submodule of R/V. It follows that R/V is uniform, that P is the assassinator of R/V, and that $\operatorname{ann}_{R/V}(P)$ is a torsionfree (R/P)-module. By Lemma 14.23, R/A has an artinian classical quotient ring, where $A = \operatorname{ann}_R(R/V)$. Since $A \leq V$, we have $U \cap A = 0$, and the claim is proved.

If R is nonzero, choose a uniform right ideal U_1. By the claim, there is an ideal A_1 such that $U_1 \cap A_1 = 0$ and R/A_1 has an artinian classical quotient ring. If $A_1 \neq 0$, choose a uniform right ideal $U_2 \leq A_1$. Then there is an ideal A_2 such that $U_2 \cap A_2 = 0$ and R/A_2 has an artinian classical quotient ring. Observe that $(U_1 \oplus U_2) \cap (A_1 \cap A_2) = 0$. Continue this process as long as possible (e.g., next with a uniform right ideal $U_3 \leq A_1 \cap A_2$). Since R_R has finite rank, the process must terminate at some point, and then $A_1 \cap \cdots \cap A_n = 0$ for some n. Since each of the rings R/A_i can be embedded in an artinian ring, so can R. $\square$

Exercise 14S. Show that if R is a noetherian ring satisfying the strong second layer condition and U a finitely generated uniform right R-module, then $R/\operatorname{ann}_R(U)$ has an artinian classical quotient ring. [Hint: Assume that U is faithful. Show that $\mathcal{C}(N)$ is an Ore set, where N is the prime radical of R, and then consider $UT \cap \operatorname{ann}_U(P)$, where T is the $\mathcal{C}(N)$-torsion ideal of R and P is the assassinator of U.] $\square$

• LOCALIZATION AT INFINITE SETS OF PRIME IDEALS •

The above results clearly do not exhaust the question of localizability in noetherian rings. What one would like is to be able to start with a prime ideal P in a noetherian ring R and to find a right Ore set (or a two-sided Ore set) in some canonical way such that in the corresponding ring of fractions R' it is true that R'/PR' can be identified with the Goldie quotient ring of R/P. It is clear from Lemma 14.17 that the best chance for such an Ore set is the intersection of the $\mathcal{C}(Q)$ for all Q in the right link closure of $\{P\}$ (or in the clique of P). This leads us naturally to the notion of a localization at an infinite set of primes. We will discuss this possibility in a series of exercises, since the results in this direction are not yet definitive.

Definition. Let X be a nonempty set of prime ideals in a right noetherian ring R. We define $\mathcal{C}(X) = \bigcap_{Q \in X} \mathcal{C}(Q)$, and in case $\mathcal{C}(X)$ is a right Ore set we denote the corresponding localization by R_X. If X is infinite, we do not have any analog of Lemma 14.18 available, and so we build the desired properties of R_X into our definition of localizability. Thus, we say that X is *right localizable* provided $\mathcal{C}(X)$ is a right Ore set, the rings R_X/QR_X (for $Q \in X$) are all simple artinian, and the ideals QR_X (for $Q \in X$) are the only right primitive ideals of R_X. Further, X is *classically right localizable* if, in addition, for each $Q \in X$ the injective hull of the right R_X-module R_X/QR_X is the union of its socle series. A nonempty set of primes in a noetherian ring is *(classically) localizable* provided it is both right and left (classically) localizable.

To see the necessity of the extra conditions in this definition of localizability, consider first a polynomial ring $R = \mathbb{C}[x, y]$ and let X be the set of all maximal ideals of R except $xR + yR$. Then X satisfies the first two conditions of the localizability definition but not the third. (Here $\mathcal{C}(X)$ is just the set of units of R.) For a second example, consider a polynomial ring $R = \mathbb{Q}[x, y]$ and let X be the set of all prime ideals of the form fR, where f is irreducible. Here again $\mathcal{C}(X)$ is the set of units of R, whence $\mathcal{C}(X)$ is an Ore set and $R_X = R$, yet none of the rings R/Q (for $Q \in X$) is artinian.

Exercise 14L shows that if R is a noetherian ring satisfying the second layer condition, X the right link closure of some prime of R, and X is finite, then X is classically right localizable.

Exercise 14T. Let $R = k[x][y; x(d/dx)]$, where k is an algebraically closed field of characteristic zero, and set $M_\alpha = xR + (y - \alpha)R$ for $\alpha \in k$. As shown in Exercise 12I, the set $X = \{M_\alpha \mid \alpha \in \mathbb{Z}\}$ is a clique in $\operatorname{Spec}(R)$.

(a) For each $\alpha \in \mathbb{Z}$, show that the module $E_\alpha = E((R/M_\alpha)_R)$ is the union of its socle series and that E_α is $\mathcal{C}(X)$-torsionfree.

(b) If I is a right ideal of R such that $\operatorname{Hom}_R(R/I, E_\alpha) = 0$ for all $\alpha \in \mathbb{Z}$, show that I contains an element of $\mathcal{C}(X)$. [Hint: Choose generators $t_1, \dots, t_n$ for I, write each $t_i = f_i + xr_i$ for some $f_i \in k[y]$ and $r_i \in R$, and consider the ideal in $k[y]$ generated by the f_i.]

(c) Show that X is classically localizable. [Hint: To see that R_X has no primitive ideals other than the $M_\alpha R_X$ for $\alpha \in \mathbb{Z}$, show that $J(R_X) = xR_X$.] $\square$

Exercise 14U. Let X be a right localizable set of primes in a right noetherian ring R. For $Q \in X$, show that the natural map $R/Q \to R_X/QR_X$ extends to a ring isomorphism from the right Goldie quotient ring of R/Q onto R_X/QR_X. $\square$

Definition. A nonempty set X of prime ideals in a right noetherian ring R satisfies the *right intersection condition* if each right ideal of R which contains an element of $\mathcal{C}(Q)$ for every $Q \in X$ also contains an element of $\mathcal{C}(X)$.

Exercise 14V. Show that a finite nonempty set X of incomparable primes in a right noetherian ring R satisfies the right intersection condition. [Hint: Corollary 7.7.] $\square$

Exercise 14W. Let X be a nonempty set of incomparable primes in a right noetherian ring R and assume that $\mathcal{C}(X)$ is a right Ore set.

(a) If X satisfies the right intersection condition, show that for each $Q \in X$ the ring R_X/QR_X is simple artinian. [Hint: If $c \in \mathcal{C}(Q)$, show that $cR + Q$ contains an element of $\mathcal{C}(X)$, and then use Corollary 4.25.]

(b) If X satisfies the right intersection condition, show that the only right primitive ideals in R_X are the ideals QR_X for $Q \in X$; hence, X is right localizable. [Hint: If I is a maximal right ideal of R_X, find $Q \in X$ such that $R/(I^{\mathrm{c}} + Q)$ is not torsion as a right (R/Q)-module and proceed as in Lemma 14.18.]

(c) If X is right localizable, show that X satisfies the right intersection condition. [Hint: If not, consider a right ideal I such that $I \cap \mathcal{C}(Q)$ is nonempty for each $Q \in X$ while I is maximal with respect to the property that $I \cap \mathcal{C}(X)$ is empty.] $\square$

Exercise 14X. Let R be a noetherian ring satisfying the right second layer condition and X a nonempty right link closed set of incomparable primes of R satisfying the right intersection condition.

(a) Modify the proof of Theorem 14.21 to show that if $Q \in X$, then $E((R/Q)_R)$ is $\mathcal{C}(X)$-torsionfree and each element of $E((R/Q)_R)$ is annihilated by a product of primes from X.

(b) Show that X is classically right localizable. [Hint: Modify the proof of Theorem 14.20 to show that $\mathcal{C}(X)$ is a right Ore set.] $\square$

The general theorem in Exercise 14X is due to Jategaonkar [1986, Theorem 7.1.5]. Obviously, it leaves open the question of what sets of primes satisfy the intersection condition. A more concrete result is that if R is a noetherian ring satisfying the second layer condition and R is an algebra over an uncountable field, and if X is a countable link-closed set of incomparable prime ideals such that there is a bound on the ranks of the factors R/Q for $Q \in X$, then X is classically localizable. (This was proved by Stafford [1987, Proposition 4.5] and Warfield [1986, Theorem 8].) As we will see in Chapter 16, every clique in a noetherian ring is countable, so this theorem is frequently applicable. (Also, by Corollary 14.6, in a noetherian ring satisfying the second layer condition, the primes in a clique are incomparable.) More concretely still, if R is an FBN ring which is an algebra over an uncountable field, then every clique in R is classically localizable (see Stafford [1987, Proposition 4.5]).

• NOTES •

Classical Krull Dimension. The relationship between chains of prime ideals and dimensions of algebraic varieties was explored by Noether. She showed that, first, if X is an irreducible algebraic variety over a field k, and if P is the prime ideal in the appropriate polynomial ring R over k consisting of those polynomials that vanish on X, then the dimension of X equals the transcendence degree over k of the quotient field of R/P [1923, Satz V], and second, (assuming that k is infinite) this dimension equals the maximum length of a chain of prime ideals ascending from P [1923, Satz VII]. Krull then developed this idea into a powerful tool for arbitrary commutative noetherian rings [1928a], and later writers gave the name *(classical) Krull dimension* to the supremum of the lengths of finite chains of prime ideals in a ring. The ordinal-valued definition of classical Krull dimension was introduced by Krause [1970, Definition 11].

Transfer of Classical Krull Dimension Across Noetherian Bimodules. Jategaonkar proved that if R and S are noetherian rings with the second layer condition, and if there exists a faithful noetherian (R, S)-bimodule, then R and S have the same classical Krull dimension [1979, Theorem H; 1982, Corollary 1.6 and Theorem 1.7]. He also gave the consequence that distinct primes in the same clique of a noetherian ring with the second layer condition are incomparable [1979, Theorem H; 1982, Theorem 1.8].

Second Layer Condition Implies Jacobson's Conjecture. This was proved by Jategaonkar [1979; Theorem H; 1982, Theorem 1.8]. The stronger intersection theorem given in Theorem 14.8 is due to Goodearl and Warfield [1989, Theorem 12.8].

Contraction of Prime Ideals in Finite Ring Extensions. Theorem 14.12 was first proved by Joseph and Small (with no second layer condition) for certain factor rings of enveloping algebras [1978, Corollary 3.7]. Borho then proved it for noetherian rings with suitable symmetric dimension functions, assuming just that ${}_R(RbR)$ is finitely generated for all $b \in S$ [1982, Theorem 7.2]. The version we have given was proved by Warfield, assuming the second layer condition and that the bimodules ${}_R(RbR)_R$ are finitely generated on both sides [1983, Corollary 2].

Transfer of Second Layer Condition in Finite Ring Extensions. Theorem 14.14 is due to Letzter [1990, Theorem 4.2].

Lying Over in Finite Ring Extensions. Theorem 14.16 was proved by Letzter [1990, Theorem 4.6].

Localizability of Semiprime Ideals. The localizability criterion given in

Theorem 14.20 is due to Jategaonkar [1974a, Theorem 3.2], who also gave an analogous criterion for classical localizability [1974a, Theorem 4.5]. The classical localizability criterion in terms of links (Theorem 14.21) was given for semiprime ideals in FBN rings by Müller [1976b, Theorem 5]. Jategaonkar proved that a semiprime ideal N in a noetherian ring is classically right localizable if and only if the set of primes minimal over N is right link closed and satisfies the right second layer condition [1986, Theorem 7.3.1].

Embeddability in Artinian Rings. Theorem 14.24 is due to Jategaonkar [1986, Theorem 8.3.9 and Proposition 8.3.5].

15. Krull Dimension

Krull dimension is a measurement of size of a ring that has an intrinsic importance of its own and is also a useful technical tool in the theory of noetherian rings. We have already discussed in the previous chapter the "classical" Krull dimension, which originated in commutative ring theory and is defined using the prime ideals of a ring. In the noncommutative theory, we need a notion of Krull dimension that does not depend on prime ideals, but which shares many of the important properties of the classical Krull dimension for commutative rings. For instance, we would like a notion of Krull dimension that gives some useful information even for simple rings. This is done by defining a dimension on modules rather than just on rings, which has the advantage that it in some sense replaces considerations involving two-sided ideals with considerations involving only one-sided ideals. The definition now used is due to Rentschler and Gabriel and will be defined in detail in the next section. While at first sight it appears completely unrelated to the classical definition, we shall see that it coincides with the classical Krull dimension on commutative noetherian rings and, in fact, on FBN rings.

• DEFINITIONS AND BASIC PROPERTIES •

Many of the results in this and subsequent sections follow fairly directly from the definitions. We have therefore left more of the details than usual to the reader. The Krull dimension to be defined and studied in this chapter is an ordinal-valued invariant defined for some modules but not all. It will, in particular, be defined for all noetherian modules, and in most of the "standard" examples it will be finite. As with classical Krull dimension, it is convenient to begin our list of ordinals with -1.

Definition. In order to define Krull dimension for modules over a ring R, we first define, by transfinite induction, classes $\mathcal{K}_\alpha$ of R-modules for all ordinals α. To start with, let $\mathcal{K}_{-1}$ be the class consisting precisely of the zero module. Next, consider an ordinal $\alpha \geq 0$; if $\mathcal{K}_\beta$ has been defined for all ordinals $\beta < \alpha$, let $\mathcal{K}_\alpha$ be the class of those R-modules M such that, for every (countable) descending chain

$$M_0 \geq M_1 \geq M_2 \geq \cdots$$

of submodules of M, we have $M_i/M_{i+1} \in \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for all but finitely many indices i. If an R-module M belongs to some $\mathcal{K}_\alpha$, then the least such α is the *Krull dimension of* M, denoted $\mathrm{K.dim}(M)$. On the other hand, if M does not belong to any $\mathcal{K}_\alpha$, we say that "$\mathrm{K.dim}(M)$ is not defined," or that "M does not have Krull dimension."

To avoid excessive statements about the existence of Krull dimension, we make the convention that a statement such as "$\mathrm{K.dim}(M) = \alpha$" is an abbreviation for "the Krull dimension of M exists and equals α." With this convention, the core of the definition above can be restated as follows, given a module M and an ordinal $\alpha \geq 0$. Namely, $\mathrm{K.dim}(M) \leq \alpha$ if and only if, for every descending chain $M_0 \geq M_1 \geq \cdots$ of submodules of M, we have $\mathrm{K.dim}(M_i/M_{i+1}) < \alpha$ for all but finitely many i.

For example, a module M has Krull dimension 0 if and only if $M \neq 0$ and M satisfies the DCC on submodules. In other words, the modules of Krull dimension 0 are precisely the nonzero artinian modules.

Exercise 15A. Show that $\mathbb{Z}$ and $k[x]$ (where k is a field and x an indeterminate), considered as modules over themselves, have Krull dimension 1. □

Exercise 15B. If M is a module with $\mathrm{K.dim}(M) \leq \alpha$ for some ordinal α and N is a submodule of M, show that $\mathrm{K.dim}(N) \leq \alpha$ and $\mathrm{K.dim}(M/N) \leq \alpha$. □

Exercise 15C. If a module M contains an infinite direct sum of copies of a nonzero module N, show that M does not have Krull dimension. [Hint: Find a chain of submodules $M_1 > M_2 > \cdots$ such that $M_i/M_{i+1} \cong M_1$ for all i.] □

Exercise 15D. Let M be a module with Krull dimension and $\alpha \geq 0$ an ordinal. Show that $\mathrm{K.dim}(M) > \alpha$ if and only if M contains a chain of submodules $M_0 \geq M_1 \geq \cdots$ such that $\mathrm{K.dim}(M_i/M_{i+1}) \geq \alpha$ for infinitely many i. □

Lemma 15.1. *Let M be a module and N a submodule. Then* $\mathrm{K.dim}(M)$ *is defined if and only if* $\mathrm{K.dim}(N)$ *and* $\mathrm{K.dim}(M/N)$ *are both defined, in which case*

$$\mathrm{K.dim}(M) = \max\{\mathrm{K.dim}(N), \mathrm{K.dim}(M/N)\}.$$

Proof. If M has Krull dimension, then, by Exercise 15B, N and M/N both have Krull dimension and

$$\mathrm{K.dim}(M) \geq \max\{\mathrm{K.dim}(N), \mathrm{K.dim}(M/N)\}.$$

Conversely, assume that N and M/N both have Krull dimension and set

$$\alpha = \max\{\mathrm{K.dim}(N), \mathrm{K.dim}(M/N)\}.$$

We proceed by induction on α to show that $\mathrm{K.dim}(M) \leq \alpha$, the case $\alpha = -1$ being clear. Thus, suppose that $\alpha \geq 0$.

Let $M_0 \geq M_1 \geq \cdots$ be a descending chain of submodules of M. Then the factors $(M_i + N)/N$ form a descending chain of submodules of M/N, and thus for all but finitely many indices i we must have

$$\text{K.dim}\big((M_i + N)/(M_{i+1} + N)\big) < \text{K.dim}(M/N) \leq \alpha.$$

Similarly, the intersections $M_i \cap N$ form a descending chain of submodules of N, and so for all but finitely many indices i we must have

$$\text{K.dim}\big((M_i \cap N)/(M_{i+1} \cap N)\big) < \text{K.dim}(N) \leq \alpha.$$

For any i, the kernel of the natural epimorphism

$$f_i : M_i/M_{i+1} \to (M_i + N)/M_{i+1} + N)$$

is isomorphic to $(M_i \cap N)/(M_{i+1} \cap N)$, and hence $\text{K.dim}(\ker(f_i)) < \alpha$ for all but finitely many i. By induction, we conclude that, for all but finitely many indices i,

$$\text{K.dim}(M_i/M_{i+1}) \leq \\ \max\big\{\text{K.dim}(\ker(f_i)),\ \text{K.dim}\big((M_i + N)/(M_{i+1} + N)\big)\big\} < \alpha,$$

which proves that $\text{K.dim}(M) \leq \alpha$. This completes the induction step, and the lemma follows. □

Just as with the ascending and descending chain conditions (Corollaries 1.3 and 4.6), Lemma 15.1 immediately applies to finite direct sums, as follows.

Corollary 15.2. *If $M_1, \ldots, M_k$ are modules with Krull dimension, then*

$$\text{K.dim}(M_1 \oplus \cdots \oplus M_k) = \max\{\text{K.dim}(M_1), \ldots, \text{K.dim}(M_k)\}. \qquad \square$$

So far, we have had almost no examples of modules which do have Krull dimension. We generously rectify this omission in the next lemma.

Lemma 15.3. *If M is a noetherian module, then* $\text{K.dim}(M)$ *is defined.*

Proof. Assume not. By noetherian induction, we may assume that all proper homomorphic images of M have Krull dimension. Let

$$\alpha = \sup\{\text{K.dim}(M/N) \mid N \text{ is a nonzero submodule of } M\}.$$

We will show that $\text{K.dim}(M) \leq \alpha + 1$ and thereby obtain a contradiction. Let $M_0 \geq M_1 \geq \cdots$ be submodules of M. We show that $\text{K.dim}(M_i/M_{i+1}) \leq \alpha$ for almost all i. If some $M_n = 0$, then $\text{K.dim}(M_i/M_{i+1}) = -1$ for all $i \geq n$, and so our condition is verified in this case. Otherwise, all the M_i are nonzero, in which case $\text{K.dim}(M_i/M_{i+1}) \leq \text{K.dim}(M/M_{i+1}) \leq \alpha$ for all i. Therefore $\text{K.dim}(M) \leq \alpha + 1$, yielding the desired contradiction. □

Definition. For any ring R, the Krull dimension of the right module R_R (if it exists) is called the *right Krull dimension of* R and is denoted $\text{r.K.dim}(R)$. Similarly, the *left Krull dimension of* R is the value $\text{l.K.dim}(R) = \text{K.dim}({}_RR)$.

It remains an unsolved problem whether the left and right Krull dimensions of a noetherian ring are the same, although it is easy to find examples of rings for which only one of the two Krull dimensions is defined. For instance, if $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{smallmatrix}\right)$, then R is right artinian and so $\text{r.K.dim}(R) = 0$, while ${}_RR$ contains an infinite direct sum of copies of a simple left module, and hence it follows from Exercise 15C that $\text{l.K.dim}(R)$ is not defined.

Proposition 15.4. *If M is a finitely generated right module over a right noetherian ring R, then* $\text{K.dim}(M) \leq \text{r.K.dim}(R)$.

Proof. Write $M \cong R^k/N$ for some $k \in \mathbb{N}$ and some submodule $N \leq R^k$. Apply, in turn, Lemma 15.3, Corollary 15.2, and Exercise 15B. $\square$

• PRIME NOETHERIAN RINGS •

Proposition 15.5. *If R is a nonzero right noetherian ring and N its prime radical, then*

$$\begin{aligned}\text{r.K.dim}(R) &= \text{r.K.dim}(R/N) \\ &= \max\{\text{r.K.dim}(R/P) \mid P \textit{ is a minimal prime ideal of } R\}.\end{aligned}$$

Proof. Clearly, $\text{r.K.dim}(R/N) = \text{K.dim}((R/N)_R)$, and so

$$\text{r.K.dim}(R/N) \leq \text{r.K.dim}(R).$$

Similarly, $\text{r.K.dim}(R/P) \leq \text{r.K.dim}(R/N)$ for any (minimal) prime P, because $P \supseteq N$.

There are minimal primes $P_1, \dots, P_n$ in R such that $P_1 P_2 \cdots P_n = 0$. For $i = 1, \dots, n$, we have

$$\begin{aligned}&\text{K.dim}\big((P_1P_2 \cdots P_{i-1}/P_1P_2 \cdots P_i)_R\big) = \\ &\qquad \text{K.dim}\big((P_1P_2 \cdots P_{i-1}/P_1P_2 \cdots P_i)_{R/P_i}\big) \leq \text{r.K.dim}(R/P_i)\end{aligned}$$

by Proposition 15.4, and so from Lemma 15.1 we conclude that

$$\text{r.K.dim}(R) \leq \max\{\text{r.K.dim}(R/P_1), \dots, \text{r.K.dim}(R/P_n)\}.$$

Therefore $\text{r.K.dim}(R) \leq \text{r.K.dim}(R/P)$ for some minimal prime P. $\square$

Lemma 15.6. *Let M be a nonzero module with Krull dimension and f an injective endomorphism of M. Then*

$$\text{K.dim}(M) \geq \text{K.dim}(M/f(M)) + 1.$$

Proof. Let $\alpha = \mathrm{K.dim}(M/f(M))$. Since M is nonzero, $\mathrm{K.dim}(M) \geq 0$, and so we are done if $\alpha = -1$. Now assume that $\alpha \geq 0$. In the chain of submodules $M \geq f(M) \geq f^2(M) \geq \cdots$, all the successive factors $f^i(M)/f^{i+1}(M)$ are isomorphic to $M/f(M)$ and hence have Krull dimension α. Therefore $\mathrm{K.dim}(M) > \alpha$. $\square$

Exercise 15E. Let R be a prime right noetherian ring. If I is any nonzero right ideal of R, show that $\mathrm{K.dim}(I) = \mathrm{r.K.dim}(R)$. [Hint: Corollary 7.25.] $\square$

Proposition 15.7. *Let R be a prime right noetherian ring and M a finitely generated right R-module. Then* $\mathrm{K.dim}(M) < \mathrm{r.K.dim}(R)$ *if and only if M is a torsion module.*

Proof. Set $\alpha = \mathrm{r.K.dim}(R)$. If x is any regular element in R, left multiplication by x defines a monomorphism $R_R \to R_R$, and consequently Lemma 15.6 implies that $\mathrm{K.dim}(R/xR) < \alpha$. It follows that if M is torsion, all cyclic submodules of M have Krull dimension less than α, and thus $\mathrm{K.dim}(M) < \alpha$. If M is not torsion, then, by Lemma 7.17, M has a uniform submodule U isomorphic to a right ideal of R, and $\mathrm{K.dim}(U) = \alpha$ by Exercise 15E. Therefore $\mathrm{K.dim}(M) = \alpha$ in this case. $\square$

Exercise 15F. Let R be a right noetherian ring. If P is a prime ideal of R and I an ideal with $I > P$, show that

$$\mathrm{r.K.dim}(R/I) < \mathrm{r.K.dim}(R/P).$$

Consequently, show that $\mathrm{Cl.K.dim}(R) \leq \mathrm{r.K.dim}(R)$. $\square$

It is easily possible to have $\mathrm{Cl.K.dim}(R) < \mathrm{r.K.dim}(R)$. For instance, if $R = A_1(k)$, where k is a field of characteristic zero, then, because R is a simple ring, $\mathrm{Cl.K.dim}(R) = 0$. On the other hand, R is not artinian, and so $\mathrm{r.K.dim}(R) > 0$. (We shall see later that $\mathrm{r.K.dim}(R) = 1$.)

• CRITICAL MODULES •

Definition. Let $\alpha \geq 0$ be an ordinal. A module M is *α-critical* provided $\mathrm{K.dim}(M) = \alpha$ while $\mathrm{K.dim}(M/N) < \alpha$ for all nonzero submodules N of M. A module is called *critical* (with respect to Krull dimension) if it is α-critical for some ordinal $\alpha \geq 0$. (We do not define "(-1)-critical.")

For example, the 0-critical modules are precisely the simple modules. Note that a module M (with Krull dimension) is 1-critical if and only if M is not artinian whereas all proper factors of M are artinian. For example, $\mathbb{Z}$ is a 1-critical module over itself, as is a polynomial ring in one variable over a field.

Exercise 15G. If M is a noetherian module with Krull dimension $\alpha \geq 0$, show that M has a proper submodule N such that M/N is α-critical. $\square$

Exercise 15H. If M is an α-critical module and N a nonzero submodule of M, show that N is α-critical. $\square$

Exercise 15I. If M is a critical module, show that M is uniform. $\square$

Exercise 15J. Let R be a semiprime right noetherian ring and I a right ideal of R. Show that I is critical if and only if I is uniform. $\square$

Exercise 15K. If M is a critical module and $f : M \to M$ a nonzero endomorphism, show that f is injective. $\square$

Exercise 15L. If R is a right noetherian ring, P a completely prime ideal of R, and $\alpha = \text{r.K.dim}(R/P)$, show that R/P is an α-critical right R-module. More generally, if P is any prime ideal of R and U a uniform right ideal of R/P, show that U is α-critical, where $\alpha = \text{r.K.dim}(R/P)$. $\square$

Lemma 15.8. *If M is a nonzero module with Krull dimension, then M has a critical submodule. However, M need not have a critical submodule with the same Krull dimension as M.*

Proof. Choose a nonzero submodule N_0 of M of minimal Krull dimension (which we may do, because ordinals satisfy the descending chain condition). Say $\text{K.dim}(N_0) = \alpha$. If N_0 is not critical, then N_0 has a nonzero submodule N_1 with $\text{K.dim}(N_0/N_1) = \alpha$. Necessarily, $\text{K.dim}(N_1) = \alpha$, since α is the smallest Krull dimension for nonzero submodules of M (and $\text{K.dim}(N_1) \leq \text{K.dim}(N_0)$ in any case). Now assume that we have constructed nonzero submodules $N_0 \geq N_1 \geq \cdots \geq N_k$ with

$$\text{K.dim}(N_i) = \text{K.dim}(N_{i-1}/N_i) = \alpha$$

for $i = 1, \ldots, k$. If N_k is not critical, there is a nonzero submodule $N_{k+1} \leq N_k$ such that

$$\text{K.dim}(N_{k+1}) = \text{K.dim}(N_k/N_{k+1}) = \alpha.$$

Since $\text{K.dim}(N_0) = \alpha$, this process cannot continue indefinitely, and therefore some N_k is critical.

The final statement of the lemma follows from Exercise 9J, in which there is an example of a noetherian ring R with a non-artinian cyclic module A possessing an essential simple submodule B. Since A is noetherian, it has Krull dimension, but since it is not artinian, $\text{K.dim}(A) > 0$. (In fact, it can be shown that $\text{K.dim}(A) = 1$.) If C is a critical submodule of A, then, since B is simple and essential, $B \leq C$. By Exercise 15H, it follows that $\text{K.dim}(C) = \text{K.dim}(B) = 0$. Now C is 0-critical and thus simple, whence $C = B$. Therefore B is the only critical submodule of A, and $\text{K.dim}(B) < \text{K.dim}(A)$. $\square$

Exercise 15M. Let M be a nonzero noetherian module and α the smallest ordinal which occurs as the Krull dimension of a nonzero submodule of M. Show that M has an α-critical submodule N such that M/N has no nonzero submodules of Krull dimension less than α. $\square$

Definition. A module M is *α-homogeneous* (for an ordinal α) provided M is nonzero and all its nonzero submodules have Krull dimension α. If M is α-homogeneous for some α, we just say that M is *Krull homogeneous* in case we do not wish to specify α.

For example, it is immediate from Exercise 15H that any α-critical module is α-homogeneous and from Exercise 15E that if R is a prime right noetherian ring, then R_R is Krull homogeneous. For another example, every nonzero artinian module is 0-homogeneous.

Exercise 15N. Show that the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ is a sum of two 1-homogeneous (in fact, 1-critical) submodules, yet M is not 1-homogeneous. $\square$

• CRITICAL COMPOSITION SERIES •

Definition. A *critical composition series* for a module M is a finite chain

$$M_0 = 0 < M_1 < \cdots < M_n = M$$

of submodules of M such that each of the factors M_i/M_{i-1} is critical and

$$\begin{aligned} \text{K.dim}(M_1) &\leq \text{K.dim}(M_2/M_1) \leq \cdots \\ &\leq \text{K.dim}(M_{n-1}/M_{n-2}) \leq \text{K.dim}(M/M_{n-1}). \end{aligned}$$

For example, any ordinary composition series for a nonzero module of finite length is also a critical composition series.

Exercise 15O. Let M be a module which has a critical composition series

$$M_0 = 0 < M_1 < \cdots < M_n = M.$$

For $i = 1, \ldots, n$, show that M/M_{i-1} has no nonzero submodules with smaller Krull dimension than M_i/M_{i-1}. In particular, if the factors M_i/M_{i-1} are all α-critical (for the same ordinal α), show that M is α-homogeneous. $\square$

Theorem 15.9. [Jategaonkar, Gordon] *If M is a nonzero noetherian module, then M has a critical composition series. Furthermore, any two critical composition series for M have the same length, and, given two such critical composition series, say*

$$M_0 = 0 < M_1 < \cdots < M_n = M \quad \textit{and} \quad N_0 = 0 < N_1 < \cdots < N_n = M,$$

there exists a permutation π of $\{1, 2, \ldots, n\}$ such that for each $i = 1, \ldots, n$ the factors M_i/M_{i-1} and $N_{\pi(i)}/N_{\pi(i)-1}$ contain nonzero isomorphic submodules.

Proof. Let α_1 be the smallest ordinal which occurs as the Krull dimension of a nonzero submodule of M. By Exercise 15M, M has an α_1-critical submodule

M_1 such that M/M_1 has no nonzero submodules of Krull dimension less than α_1. If $M_1 \neq M$ and α_2 is the smallest ordinal which occurs as the Krull dimension of a nonzero submodule of M/M_1, then $\alpha_2 \geq \alpha_1$, and M/M_1 has an α_2-critical submodule M_2/M_1 such that M/M_2 has no nonzero submodules of Krull dimension less than α_2. If $M_2 \neq M$, continue in the same fashion. Since M is noetherian, this process eventually terminates in a chain of submodules $M_0 = 0 < M_1 < \cdots < M_n = M$, where each M_i/M_{i-1} is α_i-critical and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. This chain is a critical composition series for M.

Now suppose that we are given two critical composition series,

$$M_0 = 0 < M_1 < \cdots < M_n = M \quad \text{and} \quad N_0 = 0 < N_1 < \cdots < N_k = M.$$

Consider first the case that all the factors M_i/M_{i-1} and N_j/N_{j-1} are α-critical (for the same α). By the Schreier Refinement Theorem, these two chains of submodules have isomorphic refinements. After removing any repeated terms, we may assume that both refinements are strictly ascending chains, say

$$\begin{aligned}
&A_0 = 0 = A_{p(0)} < A_1 < \cdots < A_{p(1)} = M_1 < \cdots < A_{p(2)} = M_2 < \cdots \\
&\qquad\qquad\qquad\qquad \cdots < A_{p(n-1)} = M_{n-1} < \cdots < A_t = M \\
&B_0 = 0 = B_{q(0)} < B_1 < \cdots < B_{q(1)} = N_1 < \cdots < B_{q(2)} = N_2 < \cdots \\
&\qquad\qquad\qquad\qquad \cdots < B_{q(k-1)} = N_{k-1} < \cdots < B_t = M.
\end{aligned}$$

Since each M_{i+1}/M_i is α-critical, each of the factors $A_{p(i)+1}/A_{p(i)}$ has Krull dimension α while all other factors A_m/A_{m-1} have Krull dimension less than α. Similarly, each of the factors $B_{q(j)+1}/B_{q(j)}$ has Krull dimension α while all other factors B_m/B_{m-1} have Krull dimension less than α.

Consequently, the pairing between the successive factors of these two refinements must match each $A_{p(i)+1}/A_{p(i)}$ with some $B_{q(j)+1}/B_{q(j)}$, and vice versa. Therefore $n = k$, and there is a permutation π of $\{1, 2, \ldots, n\}$ such that

$$A_{p(i-1)+1}/A_{p(i-1)} \cong B_{q(\pi(i)-1)+1}/B_{q(\pi(i)-1)}$$

for $i = 1, \ldots, n$. Thus, for $i = 1, \ldots, n$ the factors M_i/M_{i-1} and $N_{\pi(i)}/N_{\pi(i)-1}$ have nonzero isomorphic submodules. This concludes the proof in the case that the M_i/M_{i-1} and the N_j/N_{j-1} all have the same Krull dimension.

In the general case, set $\alpha_i = \text{K.dim}(M_i/M_{i-1})$ for $i = 1, \ldots, n$ and $\beta_j = \text{K.dim}(N_j/N_{j-1})$ for $j = 1, \ldots, k$. Then $\alpha_n = \text{K.dim}(M) = \beta_k$. Let p be the least index for which $\alpha_p = \alpha_n$ and q the least index for which $\beta_q = \alpha_n$. Now M/M_{p-1} has a critical composition series

$$M_{p-1}/M_{p-1} = 0 < M_p/M_{p-1} < \cdots < M_n/M_{p-1} = M/M_{p-1}$$

in which all the successive factors are α_n-critical. By Exercise 15O, M/M_{p-1} is α_n-homogeneous, and similarly M/N_{q-1} is α_n-homogeneous. The minimality of p implies that $\text{K.dim}(M_{p-1}) < \alpha_n$, whence the image of M_{p-1} in M/N_{q-1} has Krull dimension less than α_n. Since M/N_{q-1} is α_n-homogeneous, the image of M_{p-1} must be zero, that is, $M_{p-1} \leq N_{q-1}$. The reverse inclusion follows by symmetry, and thus $M_{p-1} = N_{q-1}$.

Applying the case proved above, $n - p + 1 = k - q + 1$ and the factors M_i/M_{i-1} (for $i = p, \ldots, n$) and N_j/N_{j-1} (for $j = q, \ldots, n$) can be paired in such a way that the members of each pair contain nonzero isomorphic submodules. Now M_{p-1} has a critical composition series of length less than n. By induction on the length of a critical composition series (or by induction on Krull dimension), we may conclude that $p - 1 = q - 1$ and that the factors M_i/M_{i-1} (for $i = 1, \ldots, p-1$) and N_j/N_{j-1} (for $j = 1, \ldots, q-1$) can be paired in such a way that the members of each pair contain nonzero isomorphic submodules. Therefore $n = k$ and all the factors M_i/M_{i-1} and N_j/N_{j-1} can be paired in the desired fashion. $\square$

Theorem 15.9 depends on the fact that, in the definition of a critical composition series, the sequence of Krull dimensions of successive factors is nondecreasing. If we consider arbitrary chains of submodules with successive factors critical, we lose uniqueness of lengths. For instance, given any nonnegative integer n, there is a chain of ideals

$$0 < 2^n\mathbb{Z} < 2^{n-1}\mathbb{Z} < \cdots < 2\mathbb{Z} < \mathbb{Z}$$

in $\mathbb{Z}$, of length $n+1$, such that all successive factors are critical $\mathbb{Z}$-modules.

Exercise 15P. If $M_0 = 0 < M_1 < \cdots < M_n = M$ and $N_0 = 0 < N_1 < \cdots < N_n = M$ are critical composition series for a module M, show that $\text{K.dim}(M_i/M_{i-1}) = \text{K.dim}(N_i/N_{i-1})$ for $i = 1, \ldots, n$. $\square$

Corollary 15.10. *Let $A_1, \ldots, A_n$ be critical modules and M a nonzero submodule of the direct sum $A_1 \oplus \cdots \oplus A_n$. Then M has a critical composition series of length at most n, and $M \leq_e A_1 \oplus \cdots \oplus A_n$ if and only if M has a critical composition series of length n.*

Proof. Set $\alpha_i = \text{K.dim}(A_i)$ for $i = 1, \ldots, n$ and re-order the A_i so that

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n.$$

Then set $M_0 = 0$ and $M_i = M \cap (A_1 \oplus \cdots \oplus A_i)$ for $i = 1, \ldots, n$. Since M_i/M_{i-1} embeds in A_i, it is either zero or α_i-critical. Hence, after discarding any possible repetitions in the series

$$M_0 = 0 \leq M_1 \leq \cdots \leq M_n = M,$$

we obtain a critical composition series for M, of length at most n. Moreover, if M is essential in $A_1 \oplus \cdots \oplus A_n$, then $M \cap A_i \neq 0$ for each i, whence

$$M_0 = 0 < M_1 < \cdots < M_n = M.$$

In this case, M has a critical composition series of length exactly n.

Conversely, assume that some (and hence all) critical composition series for M has length n. By the argument of the previous paragraph, M cannot be embedded in a direct sum of $n-1$ critical modules. Consequently, $M \cap A_i \neq 0$ for all i. Since the A_i are uniform (Exercise 15I), it follows that $M \cap A_i \leq_e A_i$ for all $i = 1, \ldots, n$. Thus

$$(M \cap A_1) \oplus \cdots \oplus (M \cap A_n) \leq_e A_1 \oplus \cdots \oplus A_n,$$

and therefore $M \leq_e A_1 \oplus \cdots \oplus A_n$. $\square$

Definition. Let M be a nonzero noetherian module, of Krull dimension α. Then the *Krull radical of* M, written $J_\alpha(M)$, is the intersection of the kernels of all homomorphisms from M to α-critical modules.

Proposition 15.11. *Let M be a nonzero noetherian module of Krull dimension α and $J_\alpha(M)$ its Krull radical. Then $M/J_\alpha(M)$ is α-homogeneous and $M/J_\alpha(M)$ is isomorphic to an essential submodule of a finite direct sum of α-critical modules.*

Proof. By Exercise 15G, $J_\alpha(M) < M$. In particular, $\text{K.dim}(M/J_\alpha(M)) = \alpha$. Now, without loss of generality, we may assume that $J_\alpha(M) = 0$. Choose a critical composition series

$$M_0 = 0 < M_1 < \cdots < M_n = M.$$

There exists a homomorphism f from M to an α-critical module C such that $f(M_1) \neq 0$, and so $\text{K.dim}(M_1) \geq \text{K.dim}(f(M_1)) = \alpha$. On the other hand,

$$\begin{aligned}\text{K.dim}(M_1) \leq \text{K.dim}(M_2/M_1) \leq \cdots \\ \leq \text{K.dim}(M_n/M_{n-1}) \leq \text{K.dim}(M) = \alpha.\end{aligned}$$

Thus $\text{K.dim}(M_i/M_{i-1}) = \alpha$ for all $i = 1, \ldots, n$, and hence, by Exercise 15O, M is α-homogeneous.

Choose a nonzero homomorphism $f_1 : M \to A_1$, where A_1 is α-critical, set $K_1 = \ker(f_1)$, and observe that M/K_1 is α-critical. If $K_1 \neq 0$, choose a homomorphism $f_2 : M \to A_2$, where A_2 is α-critical and $f_2(K_1) \neq 0$, set $K_2 = K_1 \cap \ker(f_2)$, and observe that K_1/K_2 is α-critical. Continue this process as long as possible.

If the process continues beyond n steps, we obtain a chain of submodules

$$K_0 = M > K_1 > \cdots > K_n > 0$$

such that K_{i-1}/K_i is α-critical for $i = 1, \ldots, n$. Since M is α-homogeneous, the successive factors in any critical composition series for K_n must all be α-critical. Hence, combining such a series with the chain above, we obtain a critical composition series for M, of length greater than n, which contradicts Theorem 15.9. Thus, our process must terminate, after at most n steps.

Therefore, we obtain a critical composition series $K_m = 0 < K_{m-1} < \cdots < K_1 < K_0 = M$, for some $m \leq n$. By Theorem 15.9, $m = n$. If f denotes the embedding

$$(f_1, \ldots, f_n) : M \to A_1 \oplus \cdots \oplus A_n,$$

we conclude from Corollary 15.10 that $f(M) \leq_e A_1 \oplus \cdots \oplus A_n$. $\square$

Corollary 15.12. *Let M be a nonzero noetherian module with Krull dimension α and Krull radical $J_\alpha(M)$, and let N be a submodule of M. Then* $\operatorname{K.dim}(M/N) < \alpha$ *if and only if* $(N + J_\alpha(M))/J_\alpha(M)$ *is essential in* $M/J_\alpha(M)$.

Proof. If $\operatorname{K.dim}(M/N) < \alpha$, then clearly $\operatorname{K.dim}\big(M/(N + J_\alpha(M))\big) < \alpha$. On the other hand, if $\operatorname{K.dim}(M/N) = \alpha$, then, by Exercise 15G, M has a submodule N' such that $N' \geq N$ and M/N' is α-critical. Since $J_\alpha(M) \leq N'$, it follows that $\operatorname{K.dim}\big(M/(N + J_\alpha(M))\big) = \alpha$ in this case. Thus, after replacing M and N by $M/J_\alpha(M)$ and $(N + J_\alpha(M))/J_\alpha(M)$, we may assume that $J_\alpha(M) = 0$. By Proposition 15.11, we may also assume that M is an essential submodule of a finite direct sum $A_1 \oplus \cdots \oplus A_n$ of α-critical modules.

Suppose first that $\operatorname{K.dim}(M/N) < \alpha$. Then none of the α-critical modules A_i can embed in M/N, whence $N \cap A_i \neq 0$ for $i = 1, \ldots, n$. As in the proof of Corollary 15.10, it follows that $N \leq_e A_1 \oplus \cdots \oplus A_n$, and thus $N \leq_e M$.

Conversely, if $N \leq_e M$, then $N \leq_e A_1 \oplus \cdots \oplus A_n$, and hence $N \cap A_i \neq 0$ for $i = 1, \ldots, n$. Since each A_i is α-critical, each factor $A_i/(N \cap A_i)$ has Krull dimension less than α. Therefore

$$\operatorname{K.dim}\big((A_1 \oplus \cdots \oplus A_n)/\big((N \cap A_1) \oplus \cdots \oplus (N \cap A_n)\big)\big) < \alpha,$$

whence $\operatorname{K.dim}\big((A_1 \oplus \cdots \oplus A_n)/N\big) < \alpha$, and thus $\operatorname{K.dim}(M/N) < \alpha$. $\square$

• FBN RINGS •

In this and the following sections, we obtain some specific results on Krull dimension that enable us to calculate the actual Krull dimension of several rings, including fully bounded rings, polynomial rings, Weyl algebras, and some quantized coordinate rings. Our first computation will show that, for many important rings, the new notion of Krull dimension agrees with the old (i.e., "classical") one, as defined in the previous chapter.

Theorem 15.13. [Gabriel, Gordon-Robson, Krause] *If R is a right FBN ring, then* $\operatorname{r.K.dim}(R) = \operatorname{Cl.K.dim}(R)$.

Proof. We already have $\operatorname{r.K.dim}(R) \geq \operatorname{Cl.K.dim}(R)$ by Exercise 15F. To prove the reverse inequality, we proceed by induction on the ordinal $\alpha = \operatorname{r.K.dim}(R)$, the cases in which $\alpha = -1, 0$ being trivial. Now assume that $\alpha > 0$ and that the result holds for right FBN rings with right Krull dimension less than α.

By Proposition 15.5, R has a minimal prime P such that $\text{r.K.dim}(R/P) = \alpha$, and it suffices to show that $\text{Cl.K.dim}(R/P) \geq \alpha$. Thus, without loss of generality, we may assume that R is prime.

Choose a uniform right ideal U of R and recall from Exercise 15E that $\text{K.dim}(U) = \alpha$. Hence, it suffices to show that $\text{K.dim}(U) \leq \text{Cl.K.dim}(R)$. Thus, consider a descending chain $U_0 \geq U_1 \geq \cdots$ of submodules of U; we must show that $\text{K.dim}(U_i/U_{i+1}) < \text{Cl.K.dim}(R)$ for all but finitely many indices i. As this is clear if some $U_i = 0$, we may assume that $U_i \neq 0$ for all i.

Now each factor U_i/U_{i+1} is torsion (since U is uniform) and hence unfaithful, by Lemma 9.2. If $A_i = \text{ann}_R(U_i/U_{i+1})$, then, by Exercise 15F, $\text{r.K.dim}(R/A_i) < \alpha$. It follows using our induction hypothesis and Exercise 14A(c) that

$$\text{K.dim}(U_i/U_{i+1}) \leq \text{r.K.dim}(R/A_i) = \text{Cl.K.dim}(R/A_i) < \text{Cl.K.dim}(R),$$

as desired. Therefore $\text{r.K.dim}(R) = \text{K.dim}(U) \leq \text{Cl.K.dim}(R)$, and the induction step is established. □

Corollary 15.14. *If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k in n independent indeterminates, then* $\text{r.K.dim}(R) = n$.

Proof. Exercise 14B and Theorem 15.13. □

We may also use Theorem 15.13 to see that the commutative noetherian domain constructed in Exercise 14D has Krull dimension ω.

It is immediate from Theorem 15.13 that any FBN ring has the same right and left Krull dimensions. More generally, Krull symmetry holds for noetherian bimodules over FBN rings, as follows.

Theorem 15.15. [Jategaonkar] *Let R and S be FBN rings and ${}_RB_S$ a bimodule which is finitely generated on both sides. Then* $\text{K.dim}({}_RB) = \text{K.dim}(B_S)$.

Proof. Without loss of generality, we may assume that ${}_RB$ and B_S are faithful. Since R and S satisfy the second layer condition, $\text{Cl.K.dim}(R) = \text{Cl.K.dim}(S)$ by Corollary 14.5. Applying Theorem 15.13, we find that $\text{l.K.dim}(R) = \text{r.K.dim}(S)$. By Lemma 8.1, there exists $n \in \mathbb{N}$ such that S_S can be embedded in B^n (as a right submodule). Consequently,

$$\text{r.K.dim}(S) = \text{K.dim}(B_S).$$

Similarly, $\text{l.K.dim}(R) = \text{K.dim}({}_RB)$, and the theorem is proved. □

• POLYNOMIAL AND SKEW POLYNOMIAL RINGS •

For precise calculations of Krull dimensions of specific rings, it is helpful to produce inequalities relating the Krull dimensions of different rings and modules, for instance, the Krull dimensions of a ring and a subring. In general, just because a ring R is a subring of a ring S, we cannot conclude that r.K.dim$(R) \le$ r.K.dim(S). (For example, let $R = \mathbb{Z}$ and $S = \mathbb{Q}$.) However, if S were a free left R-module, we could draw this conclusion, since the correspondence taking a right ideal I of R to the right ideal IS of S would be injective. (Note that if we write ${}_RS$ as a direct sum of copies of ${}_RR$, then IS is the set of all elements for which each coordinate is in I.)

The key idea here is to bound the right Krull dimension of R by that of S using a suitable map taking right ideals of R to right ideals of S. In the above example, we had an embedding (of ordered sets) from the lattice of right ideals of R into the lattice of right ideals of S, but we can get by with slightly weaker hypotheses, as follows.

Exercise 15Q. Let M and N be modules with Krull dimension (possibly over different rings), and let $\mathcal{L}(M)$ and $\mathcal{L}(N)$ be the corresponding lattices of submodules. Assume that there exists a map $g : \mathcal{L}(M) \to \mathcal{L}(N)$ which preserves strict inclusions (that is, whenever $A < B$ in $\mathcal{L}(M)$, then $g(A) < g(B)$ in $\mathcal{L}(N)$). Show that K.dim$(M) \le$ K.dim(N). [Hint: Prove for all ordinals α that whenever $A < B$ in $\mathcal{L}(M)$ with K.dim$(B/A) > \alpha$, then K.dim$(g(B)/g(A)) > \alpha$.] $\square$

Exercise 15R. If R is a right noetherian subring of a ring S such that S_R is finitely generated, and M is a finitely generated right S-module, show that K.dim$(M_S) \le$ K.dim(M_R). $\square$

Exercise 15S. If X is a right denominator set in a right noetherian ring R, show that r.K.dim$(RX^{-1}) \le$ r.K.dim(R). $\square$

Exercise 15Q applies in particular to the situation indicated at the beginning of the section, where $R \subseteq S$ are rings with right Krull dimension and S is a free left R-module. We will need a slightly more general form of this situation.

Definition. Let R be a subring of a ring S. We say that S is *left faithfully flat over* R if S is a flat left R-module and $M \otimes_R S \neq 0$ for every nonzero right R-module M.

Exercise 15T. Let R be a subring of a ring S. If S is free as a left R-module, show that S is left faithfully flat over R. If S is flat as a left R-module and $M \otimes_R S \neq 0$ for every *simple* right R-module M, show that S is left faithfully flat over R. $\square$

Exercise 15U. Let $R \subseteq S$ be right noetherian rings such that S is left faithfully flat over R. If M is any finitely generated right R-module, show that K.dim$(M) \le$ K.dim$(M \otimes_R S)$. In particular, r.K.dim$(R) \le$ r.K.dim(S). $\square$

Exercise 15V. Show that if k is a field, then r.K.dim$(A_n(k)) \ge n$ for all $n \in \mathbb{N}$. $\square$

In view of the preceding exercise, we now have examples of simple rings of large Krull dimension, though we have not yet computed the actual Krull dimension.

We now turn to some specific computations of Krull dimension. The rings we will look at are mainly skew polynomial rings. Among these, the ordinary polynomial ring $R[x]$ is particularly important, and it also turns out that knowing the Krull dimension of $R[x]$ gives us some control on the Krull dimensions of more general skew polynomial rings $R[y;\alpha,\delta]$. This becomes clear in the next lemma, which also shows that the Krull dimensions of certain modules over $R[x]$ are all that is needed to control the Krull dimension of $R[x]$ and of $R[y;\alpha,\delta]$.

We let $T = R[x]$, the polynomial ring over a right noetherian ring R. If M is a right R-module, then there is a corresponding right T-module, namely $M \otimes_R T$. It is convenient to write this module as $M[x]$, since every element of $M \otimes_R T$ can be written as a "polynomial"

$$f = (m_0 \otimes 1) + (m_1 \otimes x) + \cdots + (m_n \otimes x^n) \equiv m_0 + m_1 x + \cdots + m_n x^n$$

for some $m_i \in M$. If n is the index of the largest nonzero term in an expression for f, then n is the *degree* of f and m_n is the *leading coefficient*. Similarly, if $S = R[y;\alpha,\delta]$, we write the induced module $M \otimes_R S$ as $M[y]$, and we define degrees and leading coefficients for elements of $M[y]$ as in $M[x]$.

Lemma 15.16. *Let R be a right noetherian ring, $S = R[y;\alpha,\delta]$ a skew polynomial ring, and $T = R[x]$ a polynomial ring. Assume that α is an automorphism of R. If M is any finitely generated right R-module, then*

$$\text{K.dim}(M[y]) \leq \text{K.dim}(M[x]).$$

Moreover, if V is a nonzero S-submodule of $M[y]$, there exist a nonzero element $m \in M$ and a nonnegative integer n such that

$$\text{K.dim}(M[y]/V) \leq \text{K.dim}(M[x]/mx^nT).$$

Proof. The technique we use is essentially the method of "associated graded modules," which we do here only in a special case. We set up a map from the lattice of submodules of $M[y]$ to the lattice of submodules of $M[x]$ and then apply Exercise 15Q.

If A is a submodule of $M[y]$, then for $i = 0, 1, \ldots$ we let $g_i(A)$ be the subset of M consisting of 0 together with the leading coefficients of the nonzero elements of A of degree i. Since α is an automorphism, each $g_i(A)$ is a submodule of M. Note that $g_0(A) \leq g_1(A) \leq \cdots$. We then let

$$g(A) = g_0(A) + g_1(A)x + g_2(A)x^2 + \cdots .$$

Clearly, $g(A)$ is a submodule of $M[x]$, and if $A \leq B \leq M[y]$, then $g(A) \leq g(B)$.

We next observe that if A and B are submodules of $M[y]$ with $A < B$, then $g(A) < g(B)$. Suppose not; then $g(A) = g(B)$ and so $g_i(A) = g_i(B)$ for all i. Let b be an element of B not in A, and choose b to be of least possible degree, say degree j. Since $g_j(A) = g_j(B)$, there is an element $a \in A$ of the same degree and with the same leading coefficient as b. But then $b-a$ is an element of B of lower degree and not in A, a contradiction. Thus $g(A) < g(B)$, as claimed.

The inequality $\mathrm{K.dim}(M[y]) \leq \mathrm{K.dim}(M[x])$ is now immediate from Exercise 15Q. If V is a nonzero submodule of $M[y]$, choose a nonzero element of V, say with degree n and leading coefficient m, and observe that $mx^nT \leq g(V)$. The final conclusion of the lemma now follows from a second application of Exercise 15Q, using the map $A/V \mapsto g(A)/mx^nT$ from submodules of $M[y]/V$ to submodules of $M[x]/mx^nT$. $\square$

Theorem 15.17. [Rentschler-Gabriel] *Let R be a right noetherian ring, M a nonzero finitely generated right R-module, and x an indeterminate. Then*

$$\mathrm{K.dim}(M[x]) = \mathrm{K.dim}(M) + 1.$$

In particular, if R is nonzero, then $\mathrm{r.K.dim}(R[x]) = \mathrm{r.K.dim}(R) + 1$.

Proof. [Gordon-Robson] We let $T = R[x]$ and $U = M[x]$, and let $\beta = \mathrm{K.dim}(M)$. Now M can be made into a right T-module in a natural way, by letting x act trivially, and if we do this, then $\mathrm{K.dim}(M_R) = \mathrm{K.dim}(M_T)$. Next, note that $Ux^n/Ux^{n+1} \cong M$ (as right T-modules) and

$$\mathrm{K.dim}(Ux^n/Ux^{n+1}) = \beta$$

for all n, whence $\mathrm{K.dim}(U) > \beta$. (Here we use the fact that x is a central element of T.)

We now use a critical composition series for M to reduce to the case in which M is a β-critical module, and we may assume by induction that the theorem has been proved for ordinals smaller than β. We will show, in fact, that U is $(\beta+1)$-critical.

To show that U is $(\beta+1)$-critical, it will suffice to show, for every nonzero submodule V of U, that $\mathrm{K.dim}(U/V) \leq \beta$. Let us first assume that V has the special form $V = mx^nT$ (that is, V is generated by a monomial). Now we already know that $\mathrm{K.dim}(U/Ux^n) \leq \beta$, and so we only need to consider the factor Ux^n/mx^nT, which is isomorphic to U/mT. Since $U/mT \cong (M/mR)[x]$ and M is β-critical, it follows by induction that

$$\mathrm{K.dim}(Ux^n/mx^nT) = \mathrm{K.dim}(U/mT) = \mathrm{K.dim}(M/mR) + 1 \leq \beta,$$

and hence that $\mathrm{K.dim}(U/mx^nT) \leq \beta$.

To prove the theorem, we reduce the general case to this specific case. If we set $\alpha = 1$ and $\delta = 0$ in Lemma 15.16, then $M[y] = M[x] = U$, and the lemma provides a nonzero element $m \in M$ and a nonnegative integer n such that $\mathrm{K.dim}(U/V) \leq \mathrm{K.dim}(U/mx^nT)$. Then $\mathrm{K.dim}(U/V) \leq \beta$ by the previous paragraph, which completes the proof of the theorem. $\square$

For example, we may use Theorem 15.17 to extend Corollary 15.14 as follows: If $T = R[x_1, \dots, x_n]$ is a polynomial ring in n independent indeterminates over a right artinian ring R, then $\mathrm{r.K.dim}(T) = n$.

Corollary 15.18. *Let R be a right noetherian ring and $S = R[y; \alpha, \delta]$, where α is an automorphism of R. If M is any finitely generated right R-module, then*

$$\mathrm{K.dim}(M) \leq \mathrm{K.dim}(M \otimes_R S) \leq \mathrm{K.dim}(M) + 1.$$

In particular, $\mathrm{r.K.dim}(R) \leq \mathrm{r.K.dim}(S) \leq \mathrm{r.K.dim}(R) + 1.$

Proof. If $M = 0$, then M and $M \otimes_R S$ both have Krull dimension -1, and the desired inequalities are clear. Assuming $M \neq 0$, we have

$$\mathrm{K.dim}(M \otimes_R S) = \mathrm{K.dim}(M[y]) \leq \mathrm{K.dim}(M[x]) = \mathrm{K.dim}(M) + 1$$

by Lemma 15.16 and Theorem 15.17. On the other hand, since S is a free left R-module, it is left faithfully flat over R (Exercise 15T), and thus $\mathrm{K.dim}(M) \leq \mathrm{K.dim}(M \otimes_R S)$ by Exercise 15U. $\square$

Exercise 15W. If $S = k[y; \alpha, \delta]$ with k a field and α an automorphism, show that $\mathrm{r.K.dim}(S) = 1$. $\square$

In particular, inductive application of Corollary 15.18 shows that if

$$S = k[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$$

is an n-fold iterated skew polynomial ring over a field k, then $\mathrm{r.K.dim}(S) \leq n$. Equality holds for ordinary polynomial rings (Corollary 15.14) and, as we shall prove shortly, when all the skew derivations δ_i vanish. In general, however, the Krull dimension of S can be less than n, as we shall see in the case of Weyl algebras in characteristic zero (Theorem 15.21).

Theorem 15.19. *If $S = R[y; \alpha]$, where R is a nonzero right noetherian ring and α an automorphism, then* $\mathrm{r.K.dim}(S) = \mathrm{r.K.dim}(R) + 1$.

Proof. Let $\beta = \mathrm{r.K.dim}(R)$. Then $\mathrm{r.K.dim}(S) \leq \beta + 1$ by Corollary 15.18. Observe that $R \cong S/yS$ as right R-modules and that, under this isomorphism, the right ideals of R correspond to the right S-submodules of S/yS. Hence, $\mathrm{K.dim}((S/yS)_S) = \mathrm{K.dim}(R_R) = \beta$. Since left multiplication by y provides an injective endomorphism of S_S, it follows from Lemma 15.6 that $\mathrm{r.K.dim}(S) \geq \beta + 1$, and the theorem is proved. $\square$

Corollary 15.20. *If $\mathbf{q}$ is any multiplicatively antisymmetric $n \times n$ matrix over a field k, then* $\mathrm{r.K.dim}(\mathcal{O}_{\mathbf{q}}(k^n)) = n$. $\square$

Exercise 15X. Let k be a field, $q \in k^{\times}$, and $A = \mathcal{O}_q(M_2(k))$. Recall from Exercise 2V that A is an iterated skew polynomial ring of the form $R[x_{22}; \alpha_{22}, \delta_{22}]$, where $R = k[x_{11}][x_{12}; \alpha_{12}][x_{21}; \alpha_{21}]$. Hence, $\text{r.K.dim}(R) = 3$ by Theorem 15.19, and so it follows from Corollary 15.18 that $\text{r.K.dim}(A)$ is either 3 or 4. Show that $\text{Cl.K.dim}(A) = 4$ and conclude using Exercise 15F that $\text{r.K.dim}(A) = 4$.

Now show that $\text{r.K.dim}\big(\mathcal{O}_q(GL_2(k))\big) = 4$ and $\text{r.K.dim}\big(\mathcal{O}_q(SL_2(k))\big) = 3$. [Hint: Exercise 2X.] □

• WEYL ALGEBRAS •

We now turn to the problem of pinning down the Krull dimensions of the Weyl algebras $A_n(k)$, where k is a field. It is immediate from Corollary 15.18 that $\text{r.K.dim}(A_n(k)) \leq 2n$, and in fact $\text{r.K.dim}(A_n(k)) = 2n$ when k has positive characteristic (Exercise 15Z). However, when k has characteristic zero, the Krull dimension of $A_n(k)$ is just n, a result that we have to work somewhat harder to obtain.

Exercise 15Y. Let $R = A_n(k) = k[y_1, \dots, y_n][x_1, \dots, x_n; \partial/\partial y_1, \dots, \partial/\partial y_n]$ be a Weyl algebra over a field k.

(a) Show that there is a k-algebra automorphism ϕ of R such that $\phi(y_i) = x_i$ and $\phi(x_i) = -y_i$ for all $i = 1, \dots, n$. [Hint: Exercise 2F.]

(b) Set $X_i = k[x_i] \setminus \{0\}$ and $Y_i = k[y_i] \setminus \{0\}$ for $i = 1, \dots, n$. Show that X_i and Y_i are right and left denominator sets in R, and that $RX_i^{-1} \cong RY_i^{-1} \cong A_{n-1}(k(t))[x; \partial/\partial t]$ for some indeterminate t (where $A_0(k(t)) = k(t)$). [Hint: Exercise 10R.] □

Theorem 15.21. [Nouazé-Gabriel, Rentschler-Gabriel] *Let k be a field of characteristic zero. Then* $\text{r.K.dim}(A_n(k)) = n$*, for each positive integer n.*

Proof. Let $R = A_n(k)$, and recall from Exercise 15V that $\text{r.K.dim}(R) \geq n$. We proceed by induction on n, the case $n = 0$ (namely, $R = k$) being trivial. Now let $n > 0$ and assume the theorem holds for Weyl algebras of lower degree. We now embed R into $2n$ larger rings, as follows. Adopt the notation of Exercise 15Y and set $B_i = RX_i^{-1}$ and $C_i = RY_i^{-1}$ for $i = 1, \dots, n$. By induction together with Corollary 15.18, we conclude that $\text{r.K.dim}(B_i) \leq n$ and $\text{r.K.dim}(C_i) \leq n$.

Consider the "diagonal" embedding

$$R \longrightarrow S = \Big(\prod_{i=1}^{n} B_i\Big) \times \Big(\prod_{i=1}^{n} C_i\Big)$$

given by the rule $r \mapsto (r, r, \dots, r)$. It is clear that $\text{r.K.dim}(S) \leq n$ (e.g., by Proposition 15.5), and so the induction step will follow via Exercise 15U if we show that S is left faithfully flat over R.

The ring S is a flat left R-module because each B_i and C_i is flat (Corollary 10.13). Because of Exercise 15T, it now suffices to show that $(R/I) \otimes_R S \neq 0$ for each maximal right ideal I of R, that is, $S/IS \neq 0$. If not, $IS = S$, and hence $IB_i = B_i$ and $IC_i = C_i$ for all $i = 1, \ldots, n$. It follows that there exist monic polynomials $f_i \in k[x_i] \cap I$ and $g_i \in k[y_i] \cap I$, for each i. From this we conclude that R/I is finite dimensional over k. However, R^{op} embeds in $\text{End}_k(R/I)$ (because R is a simple algebra), and since R is infinite dimensional over k, we have a contradiction. Therefore $S/IS \neq 0$, as desired, and the induction step is complete. □

Exercise 15Z. Show that $\text{r.K.dim}(A_n(k)) = 2n$ for any field k of positive characteristic. [Hint: Show that $A_n(k)$ is faithfully flat over a polynomial ring in $2n$ indeterminates.] □

In particular, we conclude from Theorem 15.21 and Exercise 15Z that, for any field k, the Weyl algebras $A_1(k)$, $A_2(k)$, $\ldots$ are pairwise nonisomorphic.

Exercise 15ZA. If k is a field of characteristic zero and D_1 is the Ore quotient ring of $A_1(k)$, show that $\text{r.K.dim}(A_1(D_1)) = 2$. [Hint: Write $A_1(k) = k[y_1][x_1; d/dy_1]$ and $A_1(D_1) = D_1[y][x; \partial/\partial y]$, and show that left multiplication by $x - x_1$ induces an injective endomorphism of $A_1(D_1)/(y+y_1)A_1(D_1)$.] □

Exercise 15ZB. Let $A = \mathcal{O}_q((k^\times)^2) = k[y^{\pm 1}][x^{\pm 1}; \alpha]$, where k is a field and $q \in k^\times$.

(a) Show that $\text{r.K.dim}(A) = 2$ in case q is a root of unity. [Hint: If $q^m = 1$, show that A is faithfully flat over $k[y^{\pm m}, x^{\pm m}]$.]

(b) Show that $\text{r.K.dim}(A) = 1$ if q is not a root of unity. [Hint: As in the proof of Theorem 15.21, show that a ring of the form $k(y)[x^{\pm 1}; \alpha] \times k(x)[y^{\pm 1}; \beta]$ is faithfully flat over A.] □

• NOTES •

Krull Dimension. Classical Krull dimension is discussed in the notes for Chapter 14. Gabriel introduced an ordinal-valued dimension, which he named "Krull dimension," for objects in an abelian category [1962, p. 382], using a transfinite sequence of localizing subcategories. Rentschler and Gabriel presented the definition we have given (for finite ordinals only) in [1967, p. 712], commenting (without proof) that for a noetherian ring this agrees with Gabriel's definition [1967, Introduction]. The ordinal-valued version of the Rentschler-Gabriel definition was introduced by Krause [1970, Definition 9]. Gordon and Robson gave the name "Gabriel dimension" to Gabriel's original dimension after shifting the finite values by one [1973, p. 16; 1974, p. 461] and gave a proof that the Gabriel dimension of any noetherian module equals its Krull dimension plus one [1974, Proposition 2.3].

Critical Modules. These were introduced by Hart (under the name "restricted modules") [1971, p. 342] and Goldie (using the name "critical") [1972b, p. 162].

Critical Composition Series. Jategaonkar introduced critical composition series for a finitely generated module over an FBN ring (under the name "basic series") [1974b, p. 110] and proved that any two such series are equivalent in the sense of Theorem 15.9 [1974b, Theorem 3.1]. The general case was then developed by Gordon [1974, Corollary 2.8].

Krull Dimension of FBN Rings. For a right T-ring R (see the notes for Chapter 9), Gabriel showed that his version of Krull dimension can be computed in the following manner. If $E_\alpha = \{P \in \mathrm{Spec}(R) \mid \mathrm{K.dim}(R/P) \leq \alpha\}$ for each ordinal α, then E_{-1} is empty,

$$E_{\alpha+1} = \{P \in \mathrm{Spec}(R) \mid \text{all primes properly containing } P \text{ lie in } E_\alpha\}$$

for all α, and E_α is the union of the E_β for $\beta < \alpha$ whenever α is a limit ordinal [1962, Corollaire 2, p. 425]. That the later version of Krull dimension agrees with the classical Krull dimension for right FBN rings was obtained independently by Krause [1972, Theorem 2.4] and Gordon and Robson [1973, Theorem 8.12].

Krull Symmetry for Noetherian Bimodules over FBN Rings. Jategaonkar proved that if R and S are FBN rings and there exists a faithful noetherian (R, S)-bimodule, then R and S have the same (right) Krull dimension [1974b, Theorem 2.3].

Krull Dimension of Polynomial Rings. Rentschler and Gabriel showed that if R is a left noetherian ring with finite left Krull dimension, then $\mathrm{l.K.dim}(R[x]) = \mathrm{l.K.dim}(R) + 1$ [1967, Application 1°]. The general case of Theorem 15.17 was given by Gordon and Robson [1973, Theorem 9.2].

Krull Dimension of Weyl Algebras. Using Gabriel's original definition of Krull dimension, Nouazé and Gabriel showed that the Weyl algebra $A_n(k)$ over a field k of characteristic zero has Krull dimension at least n and at most $2n - 1$ [1967, Proposition, p. 83]. Using the new definition, Rentschler and Gabriel then showed that the Krull dimension of $A_n(k)$ is exactly n [1967, Application 3°].

16. Numbers of Generators of Modules

We turn in this chapter to a very "classical" problem – estimating the minimum number of generators needed for a finitely generated module A over a noetherian ring R. In case R is commutative, there is a theorem of Forster from 1964 giving an estimate for the number of generators of A in terms of "local data," namely, the values $g(A,P) + \text{K.dim}(R/P)$, where P is any prime ideal of R and $g(A,P)$ is the minimum number of generators of the localized module A_P over the local ring R_P. In the noncommutative case, we shall see that an appropriate analog of $g(A,P)$ is the minimum number of generators needed for the tensor product of A with the Goldie quotient ring of R/P. With this adjustment, we shall derive an analog of Forster's theorem for finitely generated modules over any FBN ring.

In order to handle data from all prime ideals at once, it is most convenient to work topologically. Thus, we first develop an appropriate topology for the prime spectrum of R, and then we develop a continuity theorem for a normalized version of $g(A,P)$ (considered as a function of P). In case R is FBN, this normalized function turns out to be locally constant, and the estimate for the number of generators of A can be obtained without too much further work. At the end of the chapter, we use the topological methods just developed to prove that all cliques of prime ideals in any noetherian ring are countable.

Some of our motivating discussion relies on the localization theory of commutative noetherian rings, which we have only briefly touched on in this book. The reader who is unfamiliar with this theory may simply ignore these discussions.

• TOPOLOGIES ON THE PRIME SPECTRUM •

Definition. For each ideal I in a ring R, define

$$V(I) = \{P \in \text{Spec}(R) \mid P \supseteq I\}$$
$$W(I) = \{P \in \text{Spec}(R) \mid P \not\supseteq I\}.$$

The family $\mathcal{W} = \{W(I) \mid I \text{ an ideal of } R\}$ is closed under finite intersections and arbitrary unions, since

$$W(I_1) \cap \cdots \cap W(I_n) = W(I_1 I_2 \cdots I_n) \quad \text{and} \quad \bigcup_{j \in J} W(I_j) = W\Big(\sum_{j \in J} I_j\Big)$$

for all ideals I_j in R. Moreover, $\operatorname{Spec}(R) = W(R)$ and the empty set equals $W(0)$. Therefore $\mathcal{W}$ is the family of open sets for a topology on $\operatorname{Spec}(R)$, known as the *Zariski topology* (or the *Stone topology*, or the *hull-kernel topology*). The closed subsets of $\operatorname{Spec}(R)$ in this topology are exactly the sets $V(I)$.

Exercise 16A. Find all the open sets for the Zariski topology on $\operatorname{Spec}(\mathbb{Z})$ and show that $\operatorname{Spec}(\mathbb{Z})$ with this topology is not Hausdorff. $\square$

Exercise 16B. Given a ring R, show that in the Zariski topology on $\operatorname{Spec}(R)$, all the closed sets are compact. Moreover, if R has the ACC on ideals, show that all subsets of $\operatorname{Spec}(R)$ are compact. $\square$

As we shall see in the following section, the Zariski topology is not suitable for some of our purposes, and so we introduce a second topology, with more open sets than the Zariski topology.

Definition. Let R be a ring, and let $\mathcal{V}$ be the family of all subsets of $\operatorname{Spec}(R)$ of the form $V(I) \cup W(J)$, where $V(I)$ is any Zariski-closed subset of $\operatorname{Spec}(R)$ and $W(J)$ is any Zariski-compact Zariski-open subset of $\operatorname{Spec}(R)$. Clearly $\mathcal{V}$ is closed under finite unions, and $\mathcal{V}$ contains $\operatorname{Spec}(R)$ and the empty set, since $\operatorname{Spec}(R)$ equals $V(0) \cup W(0)$ and the empty set equals $V(R) \cup W(0)$. Therefore $\mathcal{V}$ is a basis for the family of closed sets of a topology on $\operatorname{Spec}(R)$, known as the *patch topology* or the *constructible topology*. (Thus, the closed sets for the patch topology are precisely the intersections of sets from $\mathcal{V}$.)

We immediately restrict attention to the case that R has the ACC on ideals. Now all subsets of $\operatorname{Spec}(R)$ are Zariski-compact (Exercise 16B), and so

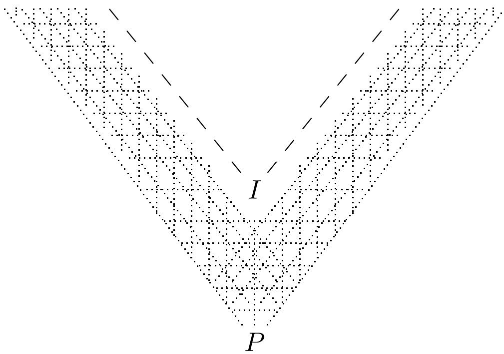
$$\mathcal{V} = \{V(I) \cup W(J) \mid I,\ J \text{ any ideals of } R\}.$$

Taking complements, we obtain the family

$$\mathcal{U} = \{V(J) \cap W(I) \mid I,\ J \text{ any ideals of } R\},$$

which is a basis for the *open* sets of the patch topology, i.e., the patch-open subsets of $\operatorname{Spec}(R)$ are precisely the unions of sets from $\mathcal{U}$.

Any patch-neighborhood of a point $P \in \operatorname{Spec}(R)$ must contain a neighborhood from $\mathcal{U}$, that is, a neighborhood of the form $V(J) \cap W(I)$, where $P \supseteq J$ and $P \not\supseteq I$. Since $P \in V(P)$ and $V(P) \subseteq V(J)$, we may replace $V(J)$ by $V(P)$. Then, since $V(P) \cap W(I) = V(P) \cap W(P+I)$, we may replace I by $P + I$. What we have shown is that every patch-neighborhood of P contains a neighborhood of the form $V(P) \cap W(I)$, where I is an ideal properly containing P, that is, the patch-neighborhoods $V(P) \cap W(I)$ form a base for the patch-neighborhoods of P.



Exercise 16C. Find all the open sets for the patch topology on $\operatorname{Spec}(\mathbb{Z})$, and show that $\operatorname{Spec}(\mathbb{Z})$ with this topology is homeomorphic to the one-point compactification of $\mathbb{N}$. $\square$

Proposition 16.1. *If R is a ring with the ACC on ideals, then* $\operatorname{Spec}(R)$ *with the patch topology is a compact, Hausdorff, totally disconnected space.*

Proof. Given distinct points $P, Q \in \operatorname{Spec}(R)$, either $P \nsubseteq Q$ or $Q \nsubseteq P$, say $P \nsubseteq Q$. Then $V(P) \cap W(R)$ is a patch-neighborhood of P and $V(Q) \cap W(P)$ is a patch-neighborhood of Q, and these neighborhoods are disjoint because $V(P)$ and $W(P)$ are disjoint. Therefore $\operatorname{Spec}(R)$ is Hausdorff in the patch topology.

For any ideal I of R, observe that the sets $V(I)$ and $W(I)$ are both patch-open, since $V(I) = V(I) \cap W(R)$ and $W(I) = V(0) \cap W(I)$. Since $V(I)$ and $W(I)$ are complements of each other, they are both patch-closed as well. Thus the basic open sets $V(I) \cap W(J)$, for ideals $I, J \subseteq R$, are all patch-closed. Therefore the patch topology on $\operatorname{Spec}(R)$ has a basis of open sets which are also closed, and hence $\operatorname{Spec}(R)$ is totally disconnected in this topology.

It remains to show that $\operatorname{Spec}(R)$ is patch-compact. Let X be a family of patch-open sets covering $\operatorname{Spec}(R)$ and suppose that no finite subfamily of X covers $\operatorname{Spec}(R)$. Since $\operatorname{Spec}(R) = V(0)$, we may use the ACC on ideals to choose an ideal Q maximal with respect to the property that no finite subfamily of X covers $V(Q)$. If A and B are ideals properly containing Q, there must be a finite subfamily Y of X that covers both $V(A)$ and $V(B)$. Then Y covers $V(AB)$, whence $V(AB) \nsupseteq V(Q)$ and so $AB \nsubseteq Q$. Thus, Q must be a prime ideal.

Choose $U \in X$ such that $Q \in U$. Then Q must have a patch-neighborhood $V(Q) \cap W(I)$, for some ideal $I > Q$, such that $V(Q) \cap W(I) \subseteq U$. Now $V(I)$

can be covered by some finite subfamily Y' of X. But

$$V(Q) \setminus V(I) = V(Q) \cap W(I) \subseteq U,$$

and so $V(Q)$ can be covered by $Y' \cup \{U\}$, contrary to our choice of Q. Thus, there must exist a finite subfamily of X which covers $\mathrm{Spec}(R)$.

Therefore $\mathrm{Spec}(R)$ is compact in the patch topology. $\square$

Exercise 16D. If R is a ring with the ACC on ideals and $X \subseteq \mathrm{Spec}(R)$, show that the patch-closure of X is the set of those primes Q which can be obtained as intersections of nonempty subsets of X. $\square$

• LOCAL NUMBERS OF GENERATORS •

Given a finitely generated module A over a commutative noetherian ring R, we may localize A with respect to the various prime ideals P of R and look at the number of generators required for each of these localized modules A_P. These "local numbers of generators" are of course bounded by the number of generators needed for A, and, conversely, we may try to use these local numbers of generators to estimate the number of generators for A. In order to find a suitable analog for the noncommutative case, we adjust the point of view a bit. First note that, by Nakayama's Lemma, the minimum number of generators needed for A_P as an R_P-module is the same as the minimum number of generators needed for A_P/PA_P as an (R_P/PR_P)-module, since $PR_P = J(R_P)$. (See Theorem 3.17 ff.) Now R_P/PR_P is isomorphic to the quotient field Q_P of R/P, and A_P/PA_P is correspondingly isomorphic to $Q_P \otimes_R A$. Thus, the number of generators needed for the localized module A_P is the same as the dimension of the vector space $Q_P \otimes_R A$. Since we have an appropriate noncommutative analog of the quotient field of a domain – namely, the Goldie quotient ring of a prime noetherian ring – we may define local numbers of generators in the following fashion.

Definition. Let R be a right noetherian ring, P a prime ideal of R, and A a finitely generated right R-module. Let Q_P be the right Goldie quotient ring of R/P. The *local number of generators of A at P*, denoted $g(A, P)$, is the minimum number of generators needed for $A \otimes_R Q_P$ as a right Q_P-module.

Note that the minimum number of generators needed for the zero module is 0, since the zero module is generated by the empty set.

Exercise 16E. Let R be a right noetherian ring, P a prime ideal of R, and A a finitely generated right R-module. Show that $g(A, P) = 0$ if and only if A/AP is torsion as a right (R/P)-module. $\square$

In the commutative case, fixing a module and computing the local numbers of generators at varying prime ideals results in a function with partial continuity properties, as follows.

Proposition 16.2. *If A is a finitely generated module over a commutative noetherian ring R, the function $g(A,-) : \mathrm{Spec}(R) \to \mathbb{Z}$ is upper semicontinuous with respect to the Zariski topology. Moreover, if A is projective, this function is Zariski-continuous.*

Proof. Upper semicontinuity means that, for each $m \in \mathbb{Z}$, the set

$$\{P \in \mathrm{Spec}(R) \mid g(A,P) \geq m\}$$

must be closed; equivalently, the set $\{P \in \mathrm{Spec}(R) \mid g(A,P) < m\}$ must be open. Thus, consider any $P \in \mathrm{Spec}(R)$ for which $g(A,P) < m$; we shall find a Zariski-neighborhood U of P such that $g(A,Q) < m$ for all $Q \in U$. Note that since $g(A,P) < m$, we must have $m > 0$.

Since $g(A,P) < m$, the R_P-module A_P can be generated by $m-1$ elements, say $a_1/r_1, \dots, a_{m-1}/r_{m-1}$, where the $a_i \in A$ and $r_i \in R \setminus P$. If B denotes the submodule of A generated by $a_1, \dots, a_{m-1}$, then $B_P = A_P$, whence $(A/B)_P = 0$. Since A is finitely generated, there exists $s \in R \setminus P$ such that $s(A/B) = 0$. Now let $U = W(sR)$, which is a Zariski-neighborhood of P in $\mathrm{Spec}(R)$. For $Q \in U$, we have $s \notin Q$ and $s(A/B) = 0$, whence $(A/B)_Q = 0$ and $A_Q = B_Q$, yielding $g(A,Q) \leq m-1$. This proves the upper semicontinuity of $g(A,-)$.

If A is projective, $A \oplus C \cong R^n$ for some finitely generated R-module C and some positive integer n. Then $g(A,-) + g(C,-)$ equals the constant function n, since

$$\begin{aligned} g(A,P) + g(C,P) &= \dim_{Q_P}(Q_P \otimes_R A) + \dim_{Q_P}(Q_P \otimes_R C) \\ &= \dim_{Q_P}\big(Q_P \otimes_R (A \oplus C)\big) = \dim_{Q_P}(Q_P^n) \end{aligned}$$

for $P \in \mathrm{Spec}(R)$. Now, since $g(C,-)$ is upper semicontinuous by our previous result, it follows that $g(A,-)$ is lower semicontinuous. Therefore $g(A,-)$ is continuous with respect to the Zariski topology in this case. $\square$

The function $g(A,-)$ in Proposition 16.2 need not be Zariski-continuous. For instance, if $R = \mathbb{Z}$ and $A = \mathbb{Z}/2\mathbb{Z}$, then $g(A, 2\mathbb{Z}) = 1$ and $g(A,0) = 0$, although $\{0\}$ is a Zariski-dense subset of $\mathrm{Spec}(\mathbb{Z})$. (Alternatively, $g(A, p\mathbb{Z}) = 0$ for all odd primes p, and the set $\{p\mathbb{Z} \mid p \text{ odd}\}$ is Zariski-dense in $\mathrm{Spec}(\mathbb{Z})$.) Note that in this example, $g(A,-)$ is continuous with respect to the patch topology, since the singleton $\{2\mathbb{Z}\}$ is a patch-open and patch-closed subset of $\mathrm{Spec}(\mathbb{Z})$.

In the noncommutative case, even the semicontinuity of Proposition 16.2 is lost. For instance, let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ and consider the right module $A = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$, which is finitely generated and projective (because it is isomorphic to the direct sum of two copies of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) R$). For each odd prime integer p, the ideal pR is a maximal ideal of R, and it is easily checked that A/Ap is a cyclic right (R/pR)-module, so that $g(A, pR) = 1$. There are two other maximal ideals of R, namely the ideals $M = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$ and $N = \begin{pmatrix} 2\mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, and $R/M \cong R/N \cong \mathbb{Z}/2\mathbb{Z}$. On one hand, $A = AM$ and so $g(A, M) = 0$, while on the other hand, $A/AN \cong (R/N)^2$ and so $g(A, N) = 2$. Since the set $\{pR \mid p \text{ odd}\}$ is Zariski-dense in $\operatorname{Spec}(R)$, we conclude that the function $g(A, -)$ is neither upper nor lower semicontinuous with respect to the Zariski topology.

As in the previous case, continuity could be saved by using the patch topology in this example. However, switching over to the patch topology is still not quite enough, as the following example shows.

Exercise 16F. Let $R = A_1(\mathbb{Z}) = \mathbb{Z}[y][x; d/dy]$ and $A = R/xR$. For all prime integers q, let $P_q = qR + x^qR + y^qR$, which is a maximal ideal of R (Exercise 3ZF). Show that $g(A, P_q) = 1$ for all q while $g(A, 0) = 0$. Show also that 0 is in the patch-closure of the set $\{P_q \mid q \text{ prime}\}$. Thus $g(A, -)$ is not patch-continuous. $\square$

Exercise 16G. Continuing the notation of Exercise 16F, show that each A/AP_q is a simple right (R/P_q)-module, whereas R/P_q is a simple artinian ring of length q (Exercise 3ZF). $\square$

In view of Exercises 16F and 16G, we might say that the lack of continuity of the function $g(A, -)$ in this example is due to the fact that $g(A, P_q)$ is too crude a measure of the size of A/AP_q. Instead, we might say that, relative to the ring R/P_q, the "size" of A/AP_q is just $1/q$. This "normalization" is all that remains to do to obtain a continuity theorem in general.

Definition. Let R be a right noetherian ring, $P \in \operatorname{Spec}(R)$, and Q_P the right Goldie quotient ring of R/P. For any finitely generated right R-module A, the *normalized rank of A at P* is the rational number

$$r_P(A) = \operatorname{length}(A \otimes_R Q_p)/\operatorname{length}(Q_p),$$

where "length" refers to the composition series length of right Q_P-modules.

Exercise 16H. Let R be a right noetherian ring, $P \in \operatorname{Spec}(R)$, and A a finitely gnerated right R-module.

(a) Show that $\operatorname{length}(A \otimes_R Q_P) = \rho_{R/P}(A/AP)$.

(b) Show that $g(A, P)$ is the smallest integer greater than or equal to $r_P(A)$.

(c) Show that $r_P(A) = 0$ if and only if A/AP is torsion as an (R/P)-module. $\square$

Proposition 16.3. *Let R be a right noetherian ring, $P \in \operatorname{Spec}(R)$, and $B \leq A$ finitely generated right R-modules. Then*

$$r_P(A) \leq r_P(B) + r_P(A/B).$$

In case either $AP = 0$ or B is a direct summand of A, equality holds: $r_P(A) = r_P(B) + r_P(A/B)$.

Proof. Let $f : B \to A$ and $g : A \to A/B$ be the inclusion and quotient maps. A basic property of tensor products is that the sequence of maps

$$B \otimes_R Q_P \xrightarrow{f\otimes 1} A \otimes_R Q_P \xrightarrow{g\otimes 1} (A/B) \otimes_R Q_P \to 0$$

is *exact*, meaning that the image of $f \otimes 1$ equals the kernel of $g \otimes 1$ and $g \otimes 1$ is surjective. Hence, if $K = (f \otimes 1)(B \otimes_R Q_P)$, then $(A \otimes_R Q_P)/K \cong (A/B) \otimes_R Q_P$, and so

$$\begin{aligned}\operatorname{length}(A \otimes_R Q_P) &= \operatorname{length}(K) + \operatorname{length}\big((A/B) \otimes_R Q_P\big)\\ &\le \operatorname{length}(B \otimes_R Q_P) + \operatorname{length}\big((A/B) \otimes_R Q_P\big).\end{aligned}$$

Dividing by $\operatorname{length}(Q_P)$, we obtain $r_P(A) \le r_P(B) + r_P(A/B)$.

If $AP = 0$, then, since Q_P is flat as a left (R/P)-module (Exercise 7G or Corollary 10.13), $f \otimes 1$ is injective and so $K \cong B \otimes_R Q_P$. On the other hand, if B is a direct summand of A, there exists a homomorphism $h : A \to B$ such that hf is the identity map on B. Then $(h \otimes 1)(f \otimes 1)$ is the identity map on $B \otimes_R Q_P$, whence again $f \otimes 1$ is injective and $K \cong B \otimes_R Q_P$. Therefore, in these cases

$$\operatorname{length}(A \otimes_R Q_P) = \operatorname{length}(B \otimes_R Q_P) + \operatorname{length}\big((A/B) \otimes_R Q_P\big),$$

and hence $r_P(A) = r_P(B) + r_P(A/B)$. $\square$

• PATCH-CONTINUITY OF NORMALIZED RANKS •

Our aim in this section is a noncommutative continuity theorem: Given a right noetherian ring R and a finitely generated right R-module A, we shall prove that the normalized rank $r_P(A)$ is a patch-continuous function of P.

Lemma 16.4. *Let R be a right noetherian ring and A a finitely generated, fully faithful right R-module. If* $\operatorname{K.dim}(A) < \operatorname{r.K.dim}(R)$, *then A has an essential cyclic submodule.*

Proof. Every nonzero submodule of A contains a critical submodule (Lemma 15.8), and so A has an essential submodule which is a direct sum of critical modules. Hence, after replacing A by this submodule, we may assume that $A = A_1 \oplus \cdots \oplus A_n$, where each A_i is α_i-critical for some α_i. We may also assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$.

We next construct elements $x_j \in A_j$ and right ideals $I_j \le R$, where $I_0 = R$ and $I_j = \operatorname{ann}_R(\{x_1, \dots, x_j\})$ for $j > 0$, such that each I_{j-1}/I_j is α_j-critical. First choose any nonzero element $x_1 \in A_1$ and observe that $I_0/I_1 \cong x_1R$, which is α_1-critical because A_1 is α_1-critical. Now suppose that $x_1, \dots, x_j$ have been chosen, for some $j < n$. Note that $R/I_j \cong (x_1 + \cdots + x_j)R \le A$. As

$$\operatorname{K.dim}(R/I_j) \le \operatorname{K.dim}(A) < \operatorname{r.K.dim}(R),$$

we find that $I_j \neq 0$. Since A is fully faithful, A_{j+1} is faithful, and so there must be an element $x_{j+1} \in A_{j+1}$ with $x_{j+1}I_j \neq 0$. It follows that $I_j/I_{j+1} \cong x_{j+1}I_j$, which is α_{j+1}-critical because A_{j+1} is. This completes our inductive construction of the elements x_j.

Now let $x = x_1 + \cdots + x_n$ and observe that $\operatorname{ann}_R(x) = I_n$. It follows that $xI_{j-1}/xI_j \cong I_{j-1}/I_j$ for each $j = 1, \ldots, n$, whence xI_{j-1}/xI_j is α_j-critical. Thus

$$xI_n = 0 < xI_{n-1} < \cdots < xI_1 < xI_0 = xR$$

is a critical composition series for xR, of length n. Therefore, by Corollary 15.10, xR is essential in A. $\square$

Exercise 16I. Let R be a right noetherian ring and A a simple right R-module. If $R/\operatorname{ann}_R(A)$ is not artinian, show that, for every positive integer n, the direct sum of n copies of A is cyclic. More generally, if B is a right R-module of finite length and $R/\operatorname{ann}_R(A)$ is not artinian for any composition factor A of B, show that B is cyclic. [Hint: Replace B by B modulo the intersection of its maximal submodules.] $\square$

Proposition 16.5. *Let R be a prime right noetherian ring, A a finitely generated torsion right R-module, and ϵ a positive real number. Then there exists a nonzero ideal I in R such that $r_P(A) < \epsilon$ for all primes $P \in W(I)$.*

Proof. We proceed by induction on the ordinal $\alpha = \text{K.dim}(A)$, the case $\alpha = -1$ being trivial. Now let $\alpha \geq 0$ and assume the result holds for finitely generated torsion modules of smaller Krull dimension. Choose a critical composition series

$$A_0 = 0 < A_1 < \cdots < A_m = A.$$

If, for each $j = 1, \ldots, m$, there exists a nonzero ideal I_j in R such that $r_P(A_j/A_{j-1}) < \epsilon/m$ for all $P \in W(I_j)$, it follows from Proposition 16.3 that $r_P(A) < \epsilon$ for all $P \in W(I_1I_2\cdots I_m)$. Hence, there is no loss of generality in assuming that A is α-critical.

Suppose first that A contains a nonzero unfaithful submodule B and set $I_1 = \operatorname{ann}_R(B)$. If $P \in W(I_1)$, then B/BP is an unfaithful (R/P)-module (because $(I_1 + P)/P \neq 0$) and so is torsion over R/P, whence $r_P(B) = 0$ (Exercise 16H). As $\text{K.dim}(A/B) < \alpha$ (because A is α-critical), we obtain from the induction hypothesis a nonzero ideal I_2 in R such that $r_P(A/B) < \epsilon$ for all $P \in W(I_2)$. Thus, $r_P(A) < \epsilon$ for all $P \in W(I_1I_2)$ (Proposition 16.3 again), completing the induction step in this case.

Finally, suppose that A is fully faithful and note that $\text{r.K.dim}(R) > \alpha$, because A is a torsion module (Proposition 15.7). Choose an integer $n > 2/\epsilon$. Then A^n is a fully faithful right R-module, and so by Lemma 16.4 it has an essential cyclic submodule C. Observe that A^n/C is a homomorphic image of a direct sum of proper factors of A; since A is α-critical, it follows that $\text{K.dim}(A^n/C) < \alpha$. Hence, by induction, R contains a nonzero ideal I such that $r_P(A^n/C) < 1$ for all $P \in W(I)$. For any such P, we have $r_P(C) \leq 1$ because C is cyclic, whence $r_P(A^n) < 2$ (Proposition 16.3). Therefore $r_P(A) < 2/n < \epsilon$ for all $P \in W(I)$, and the induction step is complete. $\square$

Theorem 16.6. [Stafford, Goodearl] *Let A be a finitely generated right module over a right noetherian ring R. Then the rule assigning to each prime ideal P the normalized rank $r_P(A)$ is a patch-continuous function from* $\mathrm{Spec}(R)$ *to* $\mathbb{Q}$.

Proof. It suffices to show that, given $Q \in \mathrm{Spec}(R)$ and a positive real number ϵ, there exists an ideal I properly containing Q such that

$$r_Q(A) - \epsilon < r_P(A) < r_Q(A) + \epsilon$$

for all primes $P \in V(Q) \cap W(I)$. As $r_P(A) = r_{P/Q}(A/AQ)$ for all $P \in V(Q)$, we may reduce to the case that $Q = 0$, with no loss of generality.

By Corollary 7.26, there is a positive integer n such that A^n has a free submodule F with A^n/F torsion. It follows that $r_0(A^n/F) = 0$. If $F \cong R^k$, then $F \otimes_R Q_0 \cong Q_0^k$, and so $r_0(F) = k$. Thus $r_0(A^n) = r_0(F) = k$ and $r_0(A) = k/n$, by Proposition 16.3. In view of Proposition 16.5, there is a nonzero ideal I_1 in R such that $r_P(A^n/F) < n\epsilon$ for all $P \in W(I_1)$. Using Proposition 16.3 once again, we obtain

$$r_P(A^n) \leq r_P(F) + r_P(A^n/F) < k + n\epsilon$$

for all $P \in W(I_1)$, and hence $r_P(A) < (k/n)+\epsilon = r_0(A)+\epsilon$ for all $P \in W(I_1)$.

To prove the remaining inequalities, note first that $A \cong R^m/K$ for some positive integer m and some submodule $K \leq R^m$. Then

$$m = r_0(R^m) = r_0(K) + r_0(A).$$

Applying the argument of the previous paragraph to K, there is a nonzero ideal I_2 in R such that $r_P(K) < r_0(K) + \epsilon$ for all $P \in W(I_2)$. Finally,

$$m = r_P(R^m) \leq r_P(K) + r_P(A)$$

for any prime P, and thus

$$r_P(A) \geq m - r_P(K) > m - r_0(K) - \epsilon = r_0(A) - \epsilon$$

for all $P \in W(I_2)$.

Therefore $|r_P(A) - r_0(A)| < \epsilon$ for all $P \in W(I_1 I_2)$. $\square$

Exercise 16J. Let R be a prime right noetherian ring and $n \in \mathbb{N}$. Show that there is a nonzero ideal I in R such that, for all primes $P \in W(I)$, either $\mathrm{rank}(R)$ divides $\mathrm{rank}(R/P)$ or $\mathrm{rank}(R/P) > n$. [Hint: If $r = \mathrm{rank}(R)$ and U is a uniform right ideal of R, then $r_0(U) = 1/r$. Apply Theorem 16.6 to the module U.] $\square$

• GENERATING MODULES OVER SIMPLE NOETHERIAN RINGS •

Our first application of the results of the previous section is to show that the number of generators for certain modules over simple noetherian rings can be estimated just from the Krull dimension of the module or the ring.

Theorem 16.7. [Stafford] *Let R be a right noetherian ring and A a finitely generated right R-module such that all nonzero factor modules A/B are fully faithful. If* $\mathrm{K.dim}(A) = m < \infty$ *and* $m < \mathrm{r.K.dim}(R)$, *then A can be generated by $m+1$ elements.*

Proof. The theorem is automatic in case $m = -1$. Now let $m \geq 0$ and assume the theorem holds for modules with Krull dimension less than m. If $J_m(A)$ is the Krull radical of A, then by hypothesis the factor module $A/J_m(A)$ is fully faithful and

$$\mathrm{K.dim}\big(A/J_m(A)\big) = \mathrm{K.dim}(A) < \mathrm{r.K.dim}(R).$$

Thus, the hypotheses of Lemma 16.4 are satisfied for $A/J_m(A)$, and hence there is an element $x \in A$ such that $(xR + J_m(A))/J_m(A)$ is essential in $A/J_m(A)$. It follows from Corollary 15.12 that $\mathrm{K.dim}(A/xR) < m$, and so by the induction hypothesis A/xR can be generated by m elements. Therefore A can be generated by $m+1$ elements. □

Corollary 16.8. [Stafford] *Let R be a simple right noetherian ring.*

(a) *If A is a finitely generated torsion right R-module with* $\mathrm{K.dim}(A) = m < \infty$, *then A can be generated by $m+1$ elements.*

(b) *If* $\mathrm{r.K.dim}(R) = n < \infty$, *then every right ideal of R can be generated by $n+1$ elements.*

Proof. (a) All nonzero right R-modules are fully faithful because R is simple, and $m < \mathrm{r.K.dim}(R)$ because A is torsion (Proposition 15.7). Hence, Theorem 16.7 applies.

(b) Any right ideal I of R is a direct summand of an essential right ideal J, and it suffices to show that J can be generated by $n+1$ elements. Hence, we may as well assume that $I \leq_e R_R$. Now there exists a regular element $x \in I$, and I/xR is a torsion module. If $m = \mathrm{K.dim}(I/xR)$, then $m < n$ by Proposition 15.7. By part (a), I/xR can be generated by $m+1$ elements, and hence by n elements. Therefore I can be generated by $n+1$ elements. □

The situation for a Weyl algebra $A_n(k)$ over a field k of characteristic zero is even better than that in Corollary 16.8(b). Although $A_n(k)$ has Krull dimension n (Theorem 15.21), Stafford has proved that every right ideal of $A_n(k)$ can be generated by two elements [1978, Corollary 3.2].

Exercise 16K. If R is a simple right noetherian ring with $\mathrm{r.K.dim}(R) = n < \infty$ and A is a finitely generated right R-module, show that A can be generated by $g(A,0)+n$ elements. □

Exercise 16L. If $R = k[x, y]$ is a polynomial ring over a field k, show that there is no finite bound on the numbers of generators needed for ideals of R. [Hint: If $M = xR + yR$, look at M^i/M^{i+1} for $i \in \mathbb{N}$.] $\square$

• GENERIC REGULARITY •

For certain rings, such as FBN rings, the patch-continuity theorem (Theorem 16.6) can be improved to say that, for any finitely generated right R-module A, the function $P \mapsto r_P(A)$ is actually *locally constant* with respect to the patch topology, that is, each point of the domain has an open neighborhood on which the function is constant. It would then follow that for all $r \in \mathbb{Q}$, the set $\{P \in \mathrm{Spec}(R) \mid r_P(A) = r\}$ is patch-open. If this is to happen, regularity modulo prime ideals must define patch-open sets, in the following manner. Consider a right noetherian ring R, a prime ideal Q of R, and an element $x \in \mathcal{C}(Q)$. Then $R/(xR + Q)$ is torsion as a right (R/Q)-module, whence $r_Q(R/xR) = 0$. If the map $P \mapsto r_P(R/xR)$ is locally constant in the patch topology, the set

$$U = \{P \in \mathrm{Spec}(R) \mid r_P(R/xR) = 0\}$$

must be patch-open. For all $P \in U$, observe that $R/(xR + P)$ is torsion as a right (R/P)-module, whence $(xR + P)/P$ is an essential right ideal of R/P, and so $x \in \mathcal{C}(P)$. Since x is regular modulo each of the prime ideals in the patch-open set U, we may say that x is "generically" regular (with respect to the patch topology).

For technical reasons, we define "generic regularity" for sets of prime ideals as well as for the full prime spectrum.

Definition. Let R be a ring. A subset X of $\mathrm{Spec}(R)$ satisfies the *generic regularity condition* (alternative terminology: X is *sparse*) provided that, for any prime ideal $Q \in \mathrm{Spec}(R)$ (not necessarily in X) and any element $x \in \mathcal{C}(Q)$, there exists a patch-neighborhood U of Q such that $x \in \mathcal{C}(P)$ for all $P \in X \cap U$.

Exercise 16M. If $R = A_1(\mathbb{Z})$, show that $\mathrm{Spec}(R)$ does not satisfy the generic regularity condition. [Hint: Exercise 16F.] $\square$

Exercise 16N. If R is a right noetherian ring and X is the set of completely prime ideals of R (that is, $X = \{P \in \mathrm{Spec}(R) \mid R/P \text{ is a domain}\}$), show that X satisfies the generic regularity condition. $\square$

Proposition 16.9. *If R is a right noetherian ring such that either R is right fully bounded or* $\mathrm{r.K.dim}(R) \le 1$, *then* $\mathrm{Spec}(R)$ *satisfies the generic regularity condition.*

Proof. Consider $Q \in \mathrm{Spec}(R)$ and $x \in \mathcal{C}(Q)$ and note that $(xR + Q)/Q$ is an essential right ideal of R/Q.

If R is right fully bounded, $(xR+Q)/Q$ contains a nonzero ideal I/Q. Given any P in $V(Q) \cap W(I)$, we have

$$xR + P \supseteq xR + Q \supseteq I$$

and so $(xR+P)/P$ contains the ideal $(I+P)/P$, which is nonzero because $P \in W(I)$. Thus $(xR+P)/P$ is an essential right ideal of R/P, whence $x \in \mathcal{C}(P)$. This verifies generic regularity in the fully bounded case.

Now assume that r.K.dim$(R) \leq 1$. If Q is a maximal ideal of R, then $\{Q\} = V(Q)$ and so $\{Q\}$ is a patch-neighborhood of Q, and we are done. Thus, we may assume that Q is not maximal. Because of Krull dimension 1, all proper factor rings of R/Q and all nonzero finitely generated torsion (R/Q)-modules have Krull dimension 0 (Proposition 15.7) and so are artinian. In particular, all prime ideals properly containing Q are maximal ideals, and $R/(xR+Q)$ must have finite length. Hence, there are at most finitely many maximal ideals $M_i > Q$ such that M_i annihilates a composition factor of $R/(xR+Q)$. Multiplying these M_i together and adding Q, we obtain an ideal $I > Q$ such that each $M_i \supseteq I$.

Let $P \in V(Q) \cap W(I)$. Since $x \in \mathcal{C}(Q)$ already, suppose that $P \neq Q$. Then P is a maximal ideal of R. If A is any composition factor of $R/(xR+Q)$, the annihilator of A is either Q or one of the M_i. In either case, $P \neq \operatorname{ann}_R(A)$ (because $P \neq Q$ and $P \not\supseteq I$), whence $P \not\subseteq \operatorname{ann}_R(A)$. Since P does not annihilate any composition factor of $R/(xR+Q)$, it follows that

$$\big[R/(xR+Q)\big]P = R/(xR+Q),$$

and hence $xR + P = R$. Therefore $x + P$ is a unit in R/P and so $x \in \mathcal{C}(P)$, verifying generic regularity in this case. $\square$

Proposition 16.10. *Let R be a right noetherian ring and $n \in \mathbb{N}$. Then the set*

$$X = \{P \in \operatorname{Spec}(R) \mid \operatorname{rank}(R/P) \leq n\}$$

satisfies the generic regularity condition.

Proof. For any $P \in X$, we have $\operatorname{length}(Q_P) = \operatorname{rank}(R/P) \leq n$ by Exercise 6L. Consequently, if A is any finitely generated right R-module, either $r_P(A) = 0$ or $r_P(A) \geq 1/n$.

Given $Q \in \operatorname{Spec}(R)$ and $x \in \mathcal{C}(Q)$, we have $r_Q(R/xR) = 0$. By Theorem 16.6, Q has a patch-neighborhood U such that $r_P(R/xR) < 1/n$ for all $P \in U$. Thus for $P \in X \cap U$ we have $r_P(R/xR) = 0$ and therefore $x \in \mathcal{C}(P)$. $\square$

Theorem 16.11. *Let R be a right noetherian ring, X a subset of* $\operatorname{Spec}(R)$ *satisfying the generic regularity condition, and A a finitely generated right R-module. Given any $Q \in$* $\operatorname{Spec}(R)$*, there is a patch-neighborhood U of Q such that $r_P(A) = r_Q(A)$ for all $P \in X \cap U$.*

Proof. Since $V(Q)$ is a patch-neighborhood of Q and $r_P(A) = r_{P/Q}(A/AQ)$ for all $P \in V(Q)$, we may replace R, A, and Q by R/Q, A/AQ, and 0. Thus, there is no loss of generality in assuming that $Q = 0$.

We first consider the case that $r_0(A) = 0$, whence A is torsion as a right R-module. Choose generators $a_1, \ldots, a_n$ for A. Then there exists $x \in \mathcal{C}(0)$ such that $a_i x = 0$ for $i = 1, \ldots, n$. By generic regularity, 0 has a patch-neighborhood U such that $x \in \mathcal{C}(P)$ for all $P \in X \cap U$. For any such P, it follows from the equations $a_i x = 0$ that A/AP is torsion as a right (R/P)-module, and thus $r_P(A) = 0$. This verifies the theorem in case $r_0(A) = 0$.

In the general case, we parallel the proof of Theorem 16.6. By Corollary 7.26, there exist integers $n > 0$ and $k \geq 0$ such that A^n has a submodule F with $F \cong R^k$ and A^n/F torsion. Then $r_0(F) = k$ and $r_0(A^n/F) = 0$, while $r_0(A) = k/n$. By the result of the previous paragraph, there is a patch-neighborhood U' of 0 such that $r_P(A^n/F) = 0$ for all $P \in X \cap U'$. Since $r_P(F) = k$ for all primes P, it follows that $r_P(A^n) \leq r_P(F) = k$ for all $P \in X \cap U'$, and therefore $r_P(A) \leq k/n = r_0(A)$ for all $P \in X \cap U'$.

Now $A \cong R^m/K$ for some positive integer m and some submodule $K \leq R^m$. Then $m = r_0(R^m) = r_0(K) + r_0(A)$. Applying the result of the last paragraph to the finitely generated module K, there exists a patch-neighborhood U'' of 0 such that $r_P(K) \leq r_0(K)$ for all $P \in X \cap U''$. For any prime P, we have $m = r_P(R^m) \leq r_P(K) + r_P(A)$. Consequently,

$$r_P(A) \geq m - r_P(K) \geq m - r_0(K) = r_0(A)$$

for all $P \in X \cap U''$.

Therefore $r_P(A) = r_0(A)$ for all $P \in X \cap U' \cap U''$. $\square$

Corollary 16.12. [Goodearl-Warfield] *If R is a right FBN ring and A a finitely generated right R-module, the rule $P \mapsto r_P(A)$ defines a function from* Spec(R) *to* $\mathbb{Q}$ *that is locally constant with respect to the patch topology.*

Proof. Proposition 16.9 and Theorem 16.11. $\square$

Exercise 16O. Let R be a prime right noetherian ring and X a subset of Spec(R) satisfying the generic regularity condition. Show that there is a nonzero ideal I in R such that rank(R) divides rank(R/P) for all $P \in X \cap W(I)$. [Cf. Exercise 16J.] $\square$

• GENERATING MODULES OVER FBN RINGS •

We have now developed most of the tools needed to derive estimates for the number of generators of a finitely generated module A over an FBN ring R, in terms of the "local data" $g(A, P)$ (for $P \in$ Spec(R)). We do not expect the maximum of the numbers $g(A, P)$ to bound the number of generators needed for A, but rather the maximum of numbers such as $g(A, P) +$ r.K.dim(R/P). Since R is FBN, r.K.dim$(R/P) =$ Cl.K.dim(R/P) (Theorem 15.13). In fact, we may replace Cl.K.dim(R/P) with a (smaller) number obtained from considering just chains of semiprimitive prime ideals, for which the following notation is used.

Definition. A *J-prime ideal* in a ring R is any prime J-ideal P, that is, any prime ideal P for which $J(R/P) = 0$. The *J-spectrum* of R, denoted J-Spec(R), is the collection of all J-prime ideals of R. The *J-dimension* of R, denoted J-dim(R), is the supremum of the lengths of all finite chains of J-prime ideals in R.

Obviously J-dim$(R) \leq$ Cl.K.dim(R) if Cl.K.dim(R) is finite. If R is local in any sense – for instance, assume only that $J(R)$ is a maximal ideal of R – then $J(R)$ is the only J-prime of R and so J-dim$(R) = 0$, regardless of the Krull dimension of R. Thus J-dim(R) may be considerably less than Cl.K.dim(R).

Lemma 16.13. *If R is a ring with the ACC on ideals, then* J-Spec(R) *is a patch-compact subset of* Spec(R). *Moreover,* J-Spec(R) *is the patch-closure of the set of right primitive ideals in* Spec(R).

Proof. The second statement is immediate from Exercise 16D. Then, since Spec(R) is patch-compact (Proposition 16.1), the first statement follows. □

Definition. Let R be a right noetherian ring and A a finitely generated right R-module. For $P \in$ Spec(R), recall that the local number of generators $g(A, P)$ is the minimum number of generators for the Q_P-module $A \otimes_R Q_P$, and that $g(A, P)$ is the smallest integer greater than or equal to $r_P(A)$ (Exercise 16H). Set

$$b(A, P) = \begin{cases} 0 & (\text{if } r_P(A) = 0) \\ g(A, P) + \text{J-dim}(R/P) & (\text{if } r_P(A) > 0). \end{cases}$$

We shall use the numbers $b(A, P)$ to estimate the number of generators of A, in case R is fully bounded.

Lemma 16.14. *Let R be a right noetherian ring and A a finitely generated right R-module, and set*

$$b = \sup\{b(A, P) \mid P \in \text{J-Spec}(R)\}.$$

If $0 < b < \infty$ and J-Spec(R) *satisfies the generic regularity condition, there are only finitely many J-primes P for which $b(A, P) = b$.*

Proof. Set $X =$ J-Spec(R). We claim that each $Q \in X$ has a patch-neighborhood $U(Q)$ such that $b(A, P) < b$ for all $P \neq Q$ in $X \cap U(Q)$.

If $r_Q(A) = 0$, then, by Theorem 16.11, Q has a patch-neighborhood $U(Q)$ such that $r_P(A) = 0$ for all $P \in X \cap U(Q)$. In this case, $b(A, P) = 0 < b$ for all $P \in X \cap U(Q)$.

Now assume that $r_Q(A) > 0$ and set $k = \text{J-dim}(R/Q)$. Note that

$$g(A, Q) + k = b(A, Q) \leq b < \infty,$$

whence $k < \infty$. By Theorem 16.11, Q has a patch-neighborhood $U(Q)$ such that $r_P(A) = r_Q(A)$ for all $P \in X \cap U(Q)$, and there is no loss of generality in assuming that $U(Q) \subseteq V(Q)$. Given any $P \in X \cap U(Q)$, note from $r_P(A) = r_Q(A)$ that $g(A, P) = g(A, Q)$. If $P \neq Q$, then $P > Q$ and so $\text{J-dim}(R/P) < k$ (since P and Q are both J-primes), whence

$$b(A, P) \leq g(A, P) + k - 1 = g(A, Q) + k - 1 = b(A, Q) - 1 < b.$$

This completes the proof of the claim.

Since X is patch-compact (Lemma 16.13), there exist $Q_1, \ldots, Q_n \in X$ such that $U(Q_1), \ldots, U(Q_n)$ cover X. Therefore any $P \in X$ for which $b(A, P) = b$ must be one of $Q_1, \ldots, Q_n$. $\square$

Lemma 16.14 is the key to an induction step, in which we must be able to reduce the values $b(A, P)$ at finitely many P simultaneously. This is accomplished by means of the next two lemmas.

Lemma 16.15. *Let R be a right noetherian ring, $P \in \text{Spec}(R)$, and A a finitely generated right R-module. Let $x \in A$ and $B \leq A$ such that $r_P(A/(xR + B)) = 0$. Then there exists $y \in B$ such that either*

$$g\big(A/(x+y)R,\, P\big) = 0 \qquad \text{or} \qquad g\big(A/(x+y)R,\, P\big) < g(A, P).$$

Proof. We may replace R by R/P and A by A/AP modulo its torsion submodule. Hence, there is no loss of generality in assuming that $P = 0$ and that A is torsionfree. Since $r_0(A/(xR + B)) = 0$, we know that $A/(xR + B)$ is a torsion module (Exercise 16H), and because A is torsionfree it follows that $xR + B \leq_e A$. Choose $y \in B$ such that $\text{rank}((x + y)R)$ is as large as possible and replace x by $x + y$. Thus we may assume that $\text{rank}(xR) \geq \text{rank}((x + z)R)$ for all $z \in B$, and we shall prove that the lemma holds with $y = 0$.

Choose $B' \leq B$ maximal with respect to the property $B' \cap xR = 0$. Then $B' \oplus (xR \cap B)$ is essential in B, whence $(xR + B)/(xR + B')$ is torsion, and so $A/(xR + B')$ is torsion. Thus we may replace B by B', that is, there is no loss of generality in assuming that $B \cap xR = 0$.

We next show that $B \cdot \text{ann}_R(x) = 0$. Consider any $z \in B$ and note that the projection $xR \oplus B \to xR$ maps $(x + z)R$ onto xR, the kernel of the latter map being $K = (x + z)R \cap B$. Since A is torsionfree, $(x + z)R/K$ is torsionfree, and hence it follows from Lemma 11.3 (or Proposition 16.3) that

$$\text{rank}\big((x + z)R\big) = \text{rank}(K) + \text{rank}(xR).$$

But $\operatorname{rank}(xR) \geq \operatorname{rank}((x+z)R)$ by assumption, whence $\operatorname{rank}(K) = 0$ and so $K = 0$. It follows that

$$z \cdot \operatorname{ann}_R(x) = (x+z) \cdot \operatorname{ann}_R(x) \subseteq K = 0.$$

Therefore $B \cdot \operatorname{ann}_R(x) = 0$.

If $\operatorname{ann}_R(x) \neq 0$, then $\operatorname{ann}_R(B) \neq 0$, and so $\operatorname{ann}_R(B)$ is a nonzero ideal in the prime ring R. In this case, $\operatorname{ann}_R(B)$ contains a regular element and B is torsion. Since A is torsionfree, we must have $B = 0$. In this case $xR \leq_e A$, whence A/xR is torsion, and therefore $g(A/xR, 0) = 0$.

On the other hand, if $\operatorname{ann}_R(x) = 0$, then $xR \cong R$ and $r_0(xR) = 1$. In this case,

$$r_0(A/xR) = r_0(A) - r_0(xR) = r_0(A) - 1,$$

and consequently $g(A/xR, 0) = g(A, 0) - 1$. □

Lemma 16.16. *Let R be a right noetherian ring, A a finitely generated right R-module, and X a finite subset of* $\operatorname{Spec}(R)$. *Then there exists an element $z \in A$ such that, for all $P \in X$, either*

$$g(A/zR,\, P) = 0 \qquad \textit{or} \qquad g(A/zR,\, P) < g(A, P).$$

Proof. If X contains only one prime, this follows from the case of Lemma 16.15 in which $x = 0$ and $B = A$. Now assume that X contains more than one prime and that the result holds for smaller sets of primes. Choose P minimal among the primes in X and set $Y = X \setminus \{P\}$. By the induction hypothesis, there exists $x \in A$ such that, for all $Q \in Y$, either $g(A/xR, Q) = 0$ or $g(A/xR, Q) < g(A, Q)$.

Let I be the product of the ideals in Y (in some order). Since P is minimal in X, it contains none of the ideals in Y, and so $P \not\supseteq I$. Then $(I+P)/P$ is a nonzero ideal of R/P, and so I contains an element of $\mathcal{C}(P)$. Since

$$(A/AI)/(A/AI)P \cong A/(AI + AP) \cong (A/AP)/(A/AP)I,$$

it follows that $(A/AI)/(A/AI)P$ is torsion as an (R/P)-module. Consequently, $r_P(A/AI) = 0$, and hence $r_P(A/(xR + AI)) = 0$.

By Lemma 16.15, there exists $y \in AI$ such that either

$$g\big(A/(x+y)R,\, P\big) = 0 \qquad \text{or} \qquad g\big(A/(x+y)R,\, P\big) < g(A, P).$$

For $Q \in Y$, we have $y \in AQ$ and so $g\big(A/(x+y)R,\, Q\big) = g(A/xR,\, Q)$. Therefore either $g(A/(x+y)R,\, Q) = 0$ or $g(A/(x+y)R,\, Q) < g(A, Q)$, and the induction step is established. □

Theorem 16.17. [Forster, Swan, Warfield] *Let R be a right FBN ring and A a finitely generated right R-module, and set*

$$b = \sup\{b(A,P) \mid P \in \text{J-Spec}(R)\}.$$

Then A can be generated by b elements.

Proof. We may obviously assume that $b < \infty$. Suppose first that $b = 0$, so that $g(A,P) = 0$ for all $P \in \text{J-Spec}(R)$. If $A \neq 0$, there exists an epimorphism of A onto a simple right R-module B. If $P = \operatorname{ann}_R(B)$, then P is a maximal ideal and R/P is a simple artinian ring (Corollary 9.5); in particular, $P \in \text{J-Spec}(R)$. Now R/P is its own Goldie quotient ring, and so $g(B,P) = 1$. But then $g(A,P) \geq 1$, contradicting the assumption that $g(A,P) = 0$. Thus $A = 0$, and A can be generated by b elements in this case.

Now let $b > 0$ and assume the theorem holds for modules with smaller values of b. By Proposition 16.9, J-Spec(R) satisfies the generic regularity condition, and hence Lemma 16.14 shows that the set

$$X = \{P \in \text{J-Spec}(R) \mid b(A,P) = b\}$$

is finite. Then, by Lemma 16.16, there exists $z \in A$ such that, for all $P \in X$, either $g(A/zR,P) = 0$ or $g(A/zR,P) < g(A,P)$. Since the first case implies $r_P(A/zR) = 0$ and so $b(A/zR,P) = 0$, we obtain $b(A/zR,P) < b$ for all $P \in X$. On the other hand, for all other $P \in \text{J-Spec}(R)$, we have

$$b(A/zR,P) \leq b(A,P) \leq b - 1.$$

Therefore $\sup\{b(A/zR,P) \mid P \in \text{J-Spec}(R)\} \leq b - 1$.

By the induction hypothesis, A/zR can be generated by $b - 1$ elements. Therefore A can be generated by b elements. $\square$

In the commutative case, the estimate given in Theorem 16.17 immediately yields an estimate coming from the local data at maximal ideals, since if P is a prime ideal and M a maximal ideal containing P, then A_P is a localization of A_M, whence $g(A,P) \leq g(A,M)$. However, such inequalities do not always hold in the noncommutative case, as the following example shows, and so some further work is needed to obtain an estimate from the numbers $g(A,M)$.

Exercise 16P. If R is the ring constructed in Exercise 3ZB and M is the maximal ideal xS, show that $g(M,0) = 1$ whereas $g(M,M) = 0$. $\square$

Lemma 16.18. *Let R be a right noetherian ring, $P \in \text{J-Spec}(R)$, and A a finitely generated right R-module. Then there exists a right primitive ideal M containing P such that $g(A,P) \leq g(A,M)$.*

Proof. Since $g(A,P)$ is the smallest integer greater than or equal to $r_P(A)$, we have $r_P(A) > g(A,P) - 1$. By Theorem 16.6, there is an ideal $I > P$ such that

$r_Q(A) > g(A,P) - 1$ for all $Q \in V(P) \cap W(I)$. As R/P is semiprimitive, P is an intersection of right primitive ideals, and so there exists a right primitive ideal $M \supseteq P$ such that $M \not\supseteq I$. Thus $M \in V(P) \cap W(I)$, and so

$$g(A,M) \geq r_M(A) > g(A,P) - 1.$$

Since $g(A,M)$ and $g(A,P)$ are integers, $g(A,M) \geq g(A,P)$. $\square$

Theorem 16.19. [Forster, Swan, Warfield] *Let R be a right FBN ring and A a finitely generated right R-module, and set*

$$m = \max\{g(A,M) \mid M \text{ a maximal ideal of } R\}.$$

Then A can be generated by $m + \text{J-dim}(R)$ elements.

Proof. It suffices to show that $b(A,P) \leq m + \text{J-dim}(R)$ for any $P \in \text{J-Spec}(R)$, because of Theorem 16.17. By Lemma 16.18, there exists a right primitive ideal $M \supseteq P$ such that $g(A,P) \leq g(A,M)$, and M is a maximal ideal by Proposition 9.4. Therefore

$$\begin{aligned} b(A,P) &\leq g(A,P) + \text{J-dim}(R/P) \\ &\leq g(A,M) + \text{J-dim}(R) \leq m + \text{J-dim}(R). \quad \square \end{aligned}$$

In the case of a finitely generated right module A over a right noetherian ring R which is not necessarily fully bounded, it is also possible to estimate the number of generators for A in terms of local data, but J-dimension must be replaced by Krull dimension. The proofs require a mixture of the methods used in the simple case (Theorem 16.7, Corollary 16.8) and the FBN case (Theorem 16.17), in order to deal with the appearance of fully faithful torsion modules over prime factor rings of R. Stafford has proved that the number of generators needed for A is bounded by each of the following:

$$\sup\{g(A,P) + \text{r.K.dim}(R/P) \mid P \in \text{J-Spec}(R)\};$$
$$\max\{g(A,P) \mid P \text{ a right primitive ideal of } R\} + \text{r.K.dim}\big(R/J(R)\big)$$

[1981, Corollaries 3.7 and 4.6]. (See also McConnell-Robson [2001, Corollary 11.7.10].)

• COUNTABILITY OF CLIQUES •

The patch-continuity and generic regularity results developed above have applications to parts of noetherian ring theory other than estimating numbers of generators of modules. Perhaps the most important application to date has been Stafford's theorem that all cliques in noetherian rings are countable. We conclude the chapter with a proof of this theorem. Note that while the proof may appear to rely entirely on generic regularity and local constantness (Theorem 16.11), the full patch-continuity theorem (Theorem 16.6) is needed to show that in any noetherian ring R, the sets $\{P \in \text{Spec}(R) \mid \text{rank}(R/P) \leq n\}$ satisfy generic regularity (Proposition 16.10).

Lemma 16.20. *Let P and Q be prime ideals in a noetherian ring R, such that $P \nsubseteq Q$. Then $r_Q(P) \geq 1$, and $r_Q(P) = 1$ if and only if $(P \cap Q)/PQ$ is torsion as a right (R/Q)-module. In particular, if $P \rightsquigarrow Q$, then $r_Q(P) > 1$. (In computing $r_Q(P)$ here, we view P as a* right *R-module.)*

Proof. Since P/PQ is a right (R/Q)-module, we have

$$r_Q(P) = r_Q(P/PQ) = r_Q\big(P/(P \cap Q)\big) + r_Q\big((P \cap Q)/PQ\big).$$

Note that $P/(P \cap Q) \cong (P+Q)/Q$, which is a nonzero ideal of R/Q (because $P \nsubseteq Q$). Hence, $(P+Q)/Q$ is an essential right ideal of R/Q, and so $(P+Q)/Q$ has the same rank as R/Q. Thus,

$$r_Q\big(P/(P \cap Q)\big) = r_Q\big((P+Q)/Q\big) = r_Q(R/Q) = 1.$$

Consequently, $r_Q(P) \geq 1$, and $r_Q(P) = 1$ if and only if $r_Q((P \cap Q)/PQ) = 0$, which occurs if and only if $(P \cap Q)/PQ$ is torsion as a right (R/Q)-module. If $P \rightsquigarrow Q$, the right (R/Q)-module $(P \cap Q)/PQ$ has a nonzero torsionfree factor, and therefore $r_Q(P) > 1$. □

Proposition 16.21. *Let P be a prime ideal in a noetherian ring R and X a subset of* $\operatorname{Spec}(R)$ *satisfying the generic regularity condition. There are at most finitely many primes $Q \in X$ such that either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$.*

Proof. By symmetry, it is enough to show that P is linked to at most finitely many primes in X. Assume that this conclusion fails. By noetherian induction, we may assume that the conclusion holds in all proper factor rings of R. Note that if I is any ideal of R, the set

$$\{Q/I \mid Q \in X \text{ and } Q \supseteq I\}$$

of primes of R/I satisfies the generic regularity condition.

By assumption, there is an infinite set $Y \subseteq X$ such that $P \rightsquigarrow Q$ for all $Q \in Y$, and we may replace X by Y. Following the argument of Theorem 13.8, we find that R must be a prime ring, that $\bigcap X = 0$, and that $P \neq 0$.

Lemma 16.20 now shows that $r_0(P) = 1$. By Theorem 16.11, there exists a patch-neighborhood U of 0 such that $r_Q(P) = 1$ for all $Q \in X \cap U$. There is no loss of generality in assuming that $U = W(I)$ for some nonzero ideal I.

Now $IP \neq 0$. Since $\bigcap X = 0$, there must exist a prime $Q \in X$ such that $Q \nsupseteq IP$. Then $Q \in X \cap U$, whence $r_Q(P) = 1$. However, as $P \nsubseteq Q$ and $P \rightsquigarrow Q$, this contradicts Lemma 16.20.

Therefore P cannot be linked to infinitely many primes from X. □

Theorem 16.22. [Stafford] *Let P be a prime ideal in a noetherian ring R. Assume that R is right fully bounded, or that* $\operatorname{r.K.dim}(R) \leq 1$, *or that there exists a positive integer n such that* $\operatorname{rank}(R/Q) \leq n$ *for all prime ideals Q in R. Then there are at most finitely many primes Q in R for which either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$.*

Proof. By either Proposition 16.9 or 16.10, $\operatorname{Spec}(R)$ satisfies the generic regularity condition. Hence, the theorem follows from Proposition 16.21. □

Theorem 16.23. [Stafford] *In any noetherian ring R, all cliques of prime ideals are countable.*

Proof. It suffices to show that, for any prime P in R, there are at most countably many primes linked to or from P. For $n \in \mathbb{N}$, let

$$X_n = \{Q \in \operatorname{Spec}(R) \mid \operatorname{rank}(R/Q) \leq n\},$$

and recall from Proposition 16.10 that X_n satisfies the generic regularity condition. Proposition 16.21 then shows that there are at most finitely many primes from each X_n linked to or from P. Therefore there are at most countably many primes linked to or from P, as desired. □

• NOTES •

Zariski Topology. Stone topologized the prime spectrum of a Boolean ring R by declaring the open sets to be those of the form $W(I)$ (in our notation) [1937, Theorem 1]. His idea was then used to topologize the set of maximal ideals of an arbitrary ring by Gelfand and Kolmogoroff [1939, p. 11] and to topologize the set of right primitive ideals of an arbitrary ring by Jacobson [1945b, p. 334]. In commutative algebra, this topology seems to have germinated from work of Zariski, who topologized a projective algebraic variety by declaring the closed subsets to be the algebraic subvarieties [1944, p. 684]; the affine analog of this is equivalent to topologizing the prime spectrum of a polynomial ring over a field by declaring the closed sets to be those of the form $V(I)$.

Patch Topology. This was introduced by Hochster for a "spectral space," meaning any topological space that is homeomorphic to the prime spectrum of a commutative ring with the Zariski topology [1969, p. 45]. He showed that the patch topology on any spectral space is compact, Hausdorff, and totally disconnected [1969, Theorem 1 and Proposition 4].

Patch-Continuity of Normalized Ranks. This was first proved for finitely generated projective right modules over right FBN rings and over right noetherian rings with right Krull dimension at most one by Goodearl and Warfield [1981, Propositions 4.4 and 4.10]. Theorem 16.6 was proved by Stafford over a two-sided noetherian ring [1981, Theorem 4.5, Lemma 6.1, and Corrigendum]. Goodearl then showed that the result also holds over a one-sided noetherian ring [1986, Theorem 1.4].

Generating Modules over Simple Noetherian Rings. Theorem 16.7 and Corollary 16.8 are due to Stafford [1976, Theorem 1.3 and Corollary 1.5].

Generic Regularity. This was introduced by Goodearl [1986, p. 89] (see also Warfield [1986, p. 180]).

Local Patch-Constantness of Normalized Ranks. A special case of Corollary 16.12, for a finitely generated torsionless right module over a right bounded prime right Goldie ring, is contained in a result of Warfield [1980, Theorem

4]. The corollary was then proved for finitely generated projective modules by Goodearl and Warfield, along with the corresponding result over a right noetherian ring with right Krull dimension at most one [1981, Propositions 4.4 and 4.10].

Generating Modules over FBN Rings. Theorems 16.17 and 16.19 were first proved by Forster for finitely generated modules over a commutative noetherian ring using Krull dimension in place of J-dimension [1964, Sätze 1, 2]. Swan then proved analogs for finitely generated modules over a module-finite algebra over a commutative J-noetherian ring R using J-dimension and data from localizations at J-primes of R [1967, Theorems 1, 2]. The theorems as we have stated them (with an extra left noetherian hypothesis in the case of Theorem 16.19) were proved by Warfield [1979b, Theorems A, B; 1980, Theorems A, C, 5].

Countability of Cliques. Theorems 16.22 and 16.23 are due to Stafford [1987, Corollaries 3.10, 3.13].

17. Transcendental Division Algebras

In this chapter we study another very “classical” topic, namely, transcendental division algebras (that is, division rings which are not algebraic over their centers). While at first glance it may not appear that the general theory of noetherian rings has anything to say about division rings, we shall see that much concrete information can be gained by applying noetherian methods to polynomial rings over division rings, in particular by applying what we have learned in previous chapters about injective modules, Ore localizations, and Krull dimension. We shall, for instance, derive analogs of the Hilbert Nullstellensatz for polynomial rings over division rings and over fully bounded rings. Information about a division ring D with center k will then be obtained by developing connections among the transcendence degree of D over k, the question of primitivity of a polynomial ring $D[x_1, \dots, x_n]$, and the Krull dimension of $D \otimes_k k(x_1, \dots, x_n)$, as well as connections between the noetherian condition on $D \otimes_k D$ and the question of finite generation of subfields of D. For technical reasons, and in order to be able to apply some of these results to Goldie quotient rings, we actually derive most of the results in this chapter for simple artinian rings rather than for division rings.

The transcendental division algebras most accessible to us are the so-called *Weyl division algebras*, which we label as follows. For any field k and any positive integer n, the Weyl algebra $A_n(k)$ is a noetherian domain (Exercise 2O and Corollary 2.7), and so it has an Ore quotient division ring, which we shall denote $D_n(k)$. In case $\operatorname{char}(k) = 0$, we have seen that the center of $D_1(k)$ is just k and that $D_1(k)$ is transcendental over k (Exercise 6H), and similar methods can be used to derive the same conclusions for any $D_n(k)$ (Exercise 17O). We shall also study a division algebra arising from quantum groups, as follows. Given a field k and a nonzero scalar $q \in k^\times$, recall that the quantum plane $\mathcal{O}_q(k^2)$ is a noetherian domain. Let us write $D_1^q(k)$ for the quotient division ring of $\mathcal{O}_q(k^2)$. The similarity with the notation $D_1(k)$ is deliberate, since $D_1^q(k)$ is isomorphic to the quotient division ring of the quantized Weyl algebra $A_1^q(k)$, as long as $q \neq 1$ (Exercise 17A). Consequently, $D_1^q(k)$ is sometimes called a *quantum Weyl division algebra*. In case q is not a root of unity, the center of $D_1^q(k)$ is k, and $D_1^q(k)$ is transcendental over its center (Exercise 6J).

Exercise 17A. If k is a field, $q \in k^\times$, and $q \neq 1$, show that $D_1^q(k)$ is isomorphic to the quotient division ring of $A_1^q(k)$. [Hint: Write $A_1^q(k) = k[y][x; \alpha, \delta]$ as in Exercise 2N and consider the element $x' = x + (q-1)^{-1}y^{-1} \in D_1^q(k)$.] $\square$

• POLYNOMIALS OVER DIVISION RINGS •

While polynomial rings over noncommutative rings are in some ways very similar to polynomial rings over commutative rings, there are important differences that should be noted. The most important difference is that one cannot simply evaluate a polynomial at a ring element. If $R[x]$ is a polynomial ring over a commutative ring R, then we can define a ring homomorphism from $R[x]$ to a commutative ring S by specifying a homomorphism from R to S and choosing an element $s \in S$ to be the image of x. The homomorphism $R[x] \to S$ is then defined by "evaluation," taking a polynomial in x to the corresponding polynomial in s. This is used particularly often when $S = R$. In the noncommutative case, the indeterminate x still commutes with the elements of R (and so x is in the center of $R[x]$). Hence, for any ring homomorphism $R[x] \to S$, the image of x must commute with the image of R. In particular, we can only "evaluate" polynomials from $R[x]$ at an element $r \in R$ if r is in the center of R. (An alternative viewpoint: The right (or left) ideal of $R[x]$ generated by $x - r$ is an ideal only if r is central.)

If k is a field, the chief importance of modules over a polynomial ring $k[x]$ is in the study of linear transformations, since if V is a vector space over k and f a linear transformation of V into itself, then V can be made into a $k[x]$-module by letting x act as the linear transformation f. (Conversely, every $k[x]$-module arises in this fashion.) In the noncommutative case we can proceed similarly, but care is necessary. Suppose then that we have a division ring D and a right vector space V over D. If f is a linear transformation on V, we let f act (as usual) on the left of V; this emphasizes the requirement that f commute with scalar multiplications by elements of D (which act on the right of V). Because f commutes with scalar multiplications, we can make V into a right module over $D[x]$ by letting x act (as before) as f, so that the module multiplication is given by

$$v \cdot (d_0 + d_1 x + \cdots + d_n x^n) = v d_0 + f(v) d_1 + \cdots + f^n(v) d_n$$

for vectors $v \in V$ and polynomials $d_0 + d_1 x + \cdots + d_n x^n \in D[x]$. The resulting $D[x]$-module is sometimes denoted (V, f). In particular, if V is finite dimensional, we may identify it with D^n for some n, in which case f will be given by an $n \times n$ matrix α. However, since we are viewing D^n as a *right* vector space, we should regard its elements as columns and f should act as *left* multiplication by the matrix α. Thus we obtain a right $D[x]$-module (D^n, α) in which D acts on the right via scalar multiplications while x acts on the left via α.

We do not want to develop the theory of linear transformations over a division ring here, although that point of view is occasionally useful. However, this point of view suggests a question that can be used as a motivation for our subsequent discussion: Does every linear transformation on a finite dimensional vector space satisfy a (minimal) polynomial? In module-theoretic terms, are the right $D[x]$-modules (D^n, α) discussed above necessarily unfaithful? It turns out that this is not the case if D is transcendental over its center, as we shall see shortly (Exercise 17B).

In order to use torsion terminology, we note that if S is a prime right noetherian ring, then any polynomial ring $S[x_1, \ldots, x_n]$ is also prime and right noetherian.

Proposition 17.1. *Let S be a simple artinian ring with center k and $S[x]$ a polynomial ring.*

(a) *Every finitely generated torsion $S[x]$-module has finite length as an S-module and hence also finite length as an $S[x]$-module.*

(b) *Every simple $S[x]$-module has finite length as an S-module.*

(c) *Every ideal of $S[x]$ is generated by a central element, that is, an element of $k[x]$.*

Proof. (a) Since it suffices to show this for all cyclic torsion $S[x]$-modules, it is enough to look at the case of a module $S[x]/J$, where J is an essential right ideal of $S[x]$.

Now $S_S = A_1 \oplus \cdots \oplus A_m$ for some simple right ideals A_i and

$$S[x] = A_1[x] \oplus \cdots \oplus A_m[x].$$

Each of the intersections $J \cap A_i[x]$ is nonzero, and since $S[x]/J$ is an epimorphic image of $\bigoplus_i A_i[x]/(J \cap A_i[x])$, it suffices to show that each of the factors $A_i[x]/(J \cap A_i[x])$ has finite length as an S-module. Relabelling, let A be a simple right ideal of S and K a nonzero right ideal of $S[x]$ contained in $A[x]$; we shall show that $A[x]/K$ has finite length as a right S-module. Since S is semisimple, $A = eS$ for some idempotent e.

Choose a nonzero polynomial $f \in K$, say with degree n and leading coefficient b. Since b is a nonzero element of A, there exists $s \in S$ such that $bs = e$, and we may replace f by fs. Hence, we may assume that the leading coefficient of f is e. Via the usual division algorithm, it follows that

$$A[x] = K + A + Ax + \cdots + Ax^{n-1},$$

and therefore $A[x]/K$ has finite length, as desired.

(b) We first observe that no right ideal H of $S[x]$ can be a simple $S[x]$-module, for if $H \neq 0$, then Hx is a proper nonzero submodule of H. Hence, all maximal right ideals of $S[x]$ are essential, and thus all simple right $S[x]$-modules are torsion.

(c) Let I be a nonzero ideal of $S[x]$, and let f be a nonzero polynomial of least degree in I, say with degree n and leading coefficient c. Since S is simple, there are elements $s_i, t_i \in S$, for $i = 1, \dots, m$, such that $\sum_{i=1}^{m} s_i c t_i = 1$. Hence, replacing f by $\sum_i s_i f t_i$, we may assume that $c = 1$. It now follows from the division algorithm that $I = fS[x] = S[x]f$. For any $s \in S$, we have $sf - fs \in I$, and since f is monic, it is clear that $sf - fs$ has lower degree than f. Hence, $sf - fs = 0$, and we conclude that $f \in k[x]$. $\square$

Exercise 17B. Let S be a simple artinian ring, x an indeterminate, $n \in \mathbb{N}$, and $\alpha \in M_n(S)$. Show that the right $S[x]$-module (S^n, α) is unfaithful if and only if α is algebraic over the center of S. $\square$

We now want to discuss the question of when a polynomial ring over a division ring D is primitive. As the previous exercise suggests, this requires a discussion of algebraic and transcendental elements, not only in D but also in matrix rings over D.

Definition. An algebra R over a field k is called *matrix-algebraic over* k if all the matrix algebras $M_n(R)$ are algebraic over k.

The reader should verify that a matrix ring $M_m(R)$ over a k-algebra R is matrix-algebraic over k if and only if R is matrix-algebraic over k. It is an unsolved problem (raised by Jacobson in 1945) whether a division ring algebraic over its center is necessarily matrix-algebraic. (For division rings with uncountable center, this was proved by Amitsur in [1956, Theorem 9].)

Theorem 17.2. [Jacobson] *Let S be a simple artinian ring with center k and x an indeterminate. Then the following conditions are equivalent:*

(a) *S is matrix-algebraic over k.*

(b) *$S \otimes_k k(x)$ is a simple artinian ring.*

(c) *$S[x]$ is right bounded.*

(d) *$S[x]$ is not right primitive.*

Proof. (a) $\Longrightarrow$ (b): We may assume that $S = M_m(D)$ for some $m \in \mathbb{N}$ and some division ring D, and that k is the center of D. Then we may identify $S \otimes_k k(x)$ with $M_m(D \otimes_k k(x))$. Since S is matrix-algebraic over k, so is D. We shall show that $D \otimes_k k(x)$ is a division ring, from which it is immediate that $S \otimes_k k(x)$ is simple artinian. Note that $D \otimes_k k(x)$ is (isomorphic to) an Ore localization $D[x]X^{-1}$, where $X = k[x] \setminus \{0\}$. Hence, to prove that $D \otimes_k k(x)$ is a division ring, it suffices to show that every nonzero element of $D[x]$ has a right inverse in $D[x]X^{-1}$.

Now let f be a nonzero polynomial in $D[x]$ and $n = \deg(f)$. The cyclic right module $M = D[x]/fD[x]$ is isomorphic, as a right vector space over D, to D^n, and under this isomorphism right multiplication by x on M corresponds to left multiplication by some matrix $\alpha \in M_n(D)$. (Actually, α is the usual "companion matrix" of ordinary linear algebra.) In other words, $M \cong (D^n, \alpha)$.

By assumption, α is algebraic over k, and so, by Exercise 17B, M is unfaithful. The nonzero ideal $\operatorname{ann}_{D[x]}(M)$ is generated by a nonzero central polynomial $g \in k[x]$, and since $Mg = 0$, we have $g = fh$ for some $h \in D[x]$. Thus hg^{-1} is a right inverse for f in $D[x]X^{-1}$, as desired.

(b) $\Longrightarrow$ (c): We note again that $S \otimes_k k(x) \cong S[x]X^{-1}$, where $X = k[x] \setminus \{0\}$. If I is any essential right ideal of $S[x]$, then we see by Theorem 10.15 that the extension I^{e} is an essential right ideal of $S[x]X^{-1}$. Since $S[x]X^{-1}$ is simple artinian, $I^{\mathrm{e}} = S[x]X^{-1}$, from which it follows that $I \cap X$ is nonempty. If $f \in I \cap X$, then $fS[x]$ is a nonzero ideal of $S[x]$ contained in I. As $S[x]$ is a prime ring, this proves that $S[x]$ is right bounded.

(c) $\Longrightarrow$ (d): As in Proposition 9.4, any right noetherian ring which is both right bounded and right primitive must be simple artinian. Therefore $S[x]$ cannot be right primitive.

(d) $\Longrightarrow$ (a): By assumption, all simple right $S[x]$-modules are unfaithful, and consequently all right $S[x]$-modules of finite length are unfaithful. Given a matrix $\alpha \in M_n(S)$, form the right $S[x]$-module (S^n, α) and observe that since (S^n, α) has finite length as a right S-module, it also has finite length as a right $S[x]$-module. Hence, (S^n, α) is unfaithful, and so, by Exercise 17B, α is algebraic over k. Therefore S is matrix-algebraic over k. $\square$

For example, if k is a field of characteristic zero, then $D_1(k)$ is transcendental over its center, and hence Theorem 17.2 shows that the polynomial ring $D_1(k)[x]$ is right (and left) primitive.

As an application of Theorem 17.2, we can give the following criterion for boundedness of more general polynomial rings. The question of primitivity for polynomial rings over bounded prime rings is partially answered in Theorem 17.10.

Theorem 17.3. *Let R be a right bounded prime right noetherian ring with right Goldie quotient ring Q. Then the polynomial ring $R[x]$ is right bounded if and only if Q is matrix-algebraic over its center.*

Proof. Because of Theorem 17.2, this amounts to showing that $R[x]$ is right bounded if and only if $Q[x]$ is right bounded. Note that $Q[x]$ may be identified with the right Ore localization $R[x]X^{-1}$, where X is the set of regular elements of R.

Assume first that $R[x]$ is right bounded. If I is an essential right ideal of $Q[x]$, then $I \cap R[x]$ is an essential right ideal of $R[x]$, and so it contains a nonzero ideal J. Obviously $JQ[x] \subseteq I$ and, by Theorem 10.18, $JQ[x]$ is an ideal of $Q[x]$. Thus $Q[x]$ is right bounded.

Conversely, suppose that $Q[x]$ is right bounded, and let I be an essential right ideal of $R[x]$. Then $IQ[x]$ is an essential right ideal of $Q[x]$, and so it contains a nonzero ideal J. By Proposition 17.1, J can be generated by some polynomial $f \in k[x]$, where k is the center of Q. Because $f \in J \subseteq IQ[x]$ and $Q[x] = R[x]X^{-1}$, there is an element $c \in X$ such that $fc \in I$. Now cR is an essential right ideal of R, and since R is right bounded, cR contains a nonzero ideal B. Note that $fB \subseteq fcR \subseteq I$, whence $fB[x] \subseteq I$. Since f is central in $Q[x]$, we conclude that $fB[x]$ is an ideal of $R[x]$. This proves that $R[x]$ is right bounded, as required. $\square$

Exercise 17C. Let R be a right noetherian ring such that, for each prime ideal P in R, the right Goldie quotient ring of R/P is matrix-algebraic over its center. If Q is a noninduced prime ideal in the polynomial ring $R[x]$ (that is, $Q > (Q \cap R)[x]$), show that the right Goldie quotient ring of $R[x]/Q$ is matrix-algebraic over its center. $\square$

Exercise 17D. Let R be a right noetherian ring with center S.

(a) If R is *integral over* S (that is, every element of R satisfies a monic polynomial with coefficients from S), show that R is right fully bounded.

(b) If R is prime and all matrix rings $M_t(R)$ are integral over S, show that the polynomial ring $R[x]$ is right bounded. [Hint: Consider the localization of R with respect to $S \setminus \{0\}$.] $\square$

• MORE VARIABLES •

If D is a division ring with center k, the ring $D \otimes_k k(x)$ appearing in Theorem 17.2 is sometimes called the ring of *rational functions* in one variable over D. It is not (usually) the quotient division ring of $D[x]$, since by the theorem $D \otimes_k k(x)$ is not a division ring unless D is matrix-algebraic over k. As noted in that proof, however, it can be thought of as a ring of fractions with numerators from $D[x]$ and denominators from $k[x]$; thus it is a subring of the quotient division ring of $D[x]$. Even when $D \otimes_k k(x)$ is not a division ring, it is nevertheless a simple noetherian ring, as is clear from the localization viewpoint just discussed together with Proposition 17.1. This gives, therefore, a new family of simple noetherian rings which we have not previously seen.

We can generalize this to polynomials in several variables, but it is convenient to first do the following lemma, which we state in greater generality than presently needed. In its proof the reader should notice a similarity to the proof of Proposition 17.1(c), which is, of course, a special case.

Lemma 17.4. *Let k be a field, A a simple k-algebra with center k, and B any k-algebra. Then the ideals of $A \otimes_k B$ are precisely the ideals $A \otimes_k I$, where I is an ideal of B. Moreover, the prime ideals of $A \otimes_k B$ are precisely the ideals $A \otimes_k P$, where P is a prime ideal of B.*

Proof. Let $C = A \otimes_k B$. It is clear that if I is an ideal of B, then $A \otimes_k I$ is an ideal of C, and that $C/(A \otimes_k I) \cong A \otimes_k (B/I)$. Observe that if elements $b_i \in B$ are linearly independent over k, then

$$A \otimes_k \Big(\sum_i kb_i\Big) = A \otimes_k \Big(\bigoplus_i kb_i\Big) = \bigoplus_i (A \otimes_k kb_i) = \bigoplus_i (A \otimes b_i),$$

and so the additive subgroups $A \otimes b_i \subseteq C$ are independent. In particular, if $a_i \in A$ are any elements such that $\sum_i a_i \otimes b_i = 0$, then all $a_i = 0$.

Let J be an ideal of C and set $I = \{b \in B \mid 1 \otimes b \in J\}$. Then I is an ideal of B and $A \otimes_k I = C(1 \otimes I) \subseteq J$. We claim that $J = A \otimes_k I$, and to show this we may work with the ideal $J/(A \otimes_k I)$ in the ring $C/(A \otimes_k I) \cong A \otimes_k (B/I)$. Thus, it suffices to assume that $I = 0$ and then show that $J = 0$.

Assuming that $J \neq 0$, choose a nonzero element $c \in J$ which can be written in the form

$$c = \sum_{i=1}^{m} a_i \otimes b_i$$

(where $a_i \in A$ and $b_i \in B$) with m as small as possible. It follows that the b_i must be linearly independent over k. (For instance, if $b_1 = \alpha_2 b_2 + \cdots + \alpha_m b_m$ for some $\alpha_i \in k$, then

$$c = \sum_{i=2}^{m} (a_i + \alpha_i a_1) \otimes b_i,$$

which contradicts the minimality of m.) For the same reason, $a_1 \neq 0$. Since A is simple, there are elements $r_j, s_j \in A$, for $j = 1, \ldots, n$, such that $\sum_{j=1}^{n} r_j a_1 s_j = 1$. Now observe that the element

$$c' = \sum_{j=1}^{n} (r_j \otimes 1) c (s_j \otimes 1) = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} r_j a_i s_j \Big) \otimes b_i$$

is a nonzero element of J. (That $c' \neq 0$ follows from the linear independence of the b_i.) Hence, replacing c by c', we may assume that $a_1 = 1$.

Now, for any $a \in A$, the element

$$(a \otimes 1)c - c(a \otimes 1) = \sum_{i=2}^{m} (a a_i - a_i a) \otimes b_i$$

lies in J, and so $(a \otimes 1)c - c(a \otimes 1) = 0$, by the minimality of m. Since the b_i are linearly independent, $a a_i - a_i a = 0$ for all i, and we conclude that all the $a_i \in k$. But then $b = \sum_{i=1}^{m} a_i b_i$ is a nonzero element of B such that $1 \otimes b = c \in J$, a contradiction.

This proves that all ideals of C have the desired form. The statement about prime ideals is a direct consequence. $\square$

Proposition 17.5. *Let S be a simple artinian ring, k the center of S, and $x_1, \ldots, x_n$ independent indeterminates.*

(a) *The center of the ring $S[x_1, \ldots, x_n]$ is exactly $k[x_1, \ldots, x_n]$.*

(b) *Every ideal of $S[x_1, \ldots, x_n]$ is generated by elements in the center.*

(c) *Extension and contraction provide inverse lattice isomorphisms between the lattice of ideals of $k[x_1, \ldots, x_n]$ and the lattice of ideals of $S[x_1, \ldots, x_n]$.*

These maps restrict to inverse bijections between $\operatorname{Spec}(k[x_1,\ldots,x_n])$ *and* $\operatorname{Spec}(S[x_1,\ldots,x_n])$.

(d) *The ring* $S \otimes_k k(x_1,\ldots,x_n)$ *is a simple noetherian ring.*

Proof. Note that $S[x_1,\ldots,x_n] \cong S \otimes_k k[x_1,\ldots,x_n]$. Statement (a) is clear, (b) and (c) are immediate from Lemma 17.4, and (d) follows from Corollary 10.16 and Proposition 10.17. □

Proposition 17.5 should be thought of as the n-variable analog of Proposition 17.1. Of course, the question of what kind of simple noetherian ring we obtain in (d) arises, as does the question of an analog for Theorem 17.2. It was conjectured at one time that a polynomial ring in several indeterminates over S would be primitive if and only if $S[x]$ were primitive, and it was even believed that this had been proved. The error was discovered by Resco, who conjectured the more subtle result that we will discuss below (Theorem 17.13). However, we must prove a noncommutative Nullstellensatz first.

• THE NULLSTELLENSATZ •

Hilbert's Nullstellensatz is traditionally stated in several different ways. One says that in a polynomial ring $R = k[x_1,\ldots,x_n]$ over a field k, every prime ideal is an intersection of maximal ideals. A second statement is that if M is a maximal ideal of R, then R/M is a finite dimensional extension of k. If we assume the statements for fields as known, then the first of these statements, for polynomials over a division ring, follows trivially from Proposition 17.5. There are two analogs of the second statement. Namely, if $R = D[x_1,\ldots,x_n]$ is a polynomial ring over a division ring D, we could ask whether all factors of R by maximal ideals are finite dimensional over D, or whether all factors of R by maximal one-sided ideals are finite dimensional over D, that is, whether all simple R-modules are finite dimensional. The first analog is again an easy consequence of Proposition 17.5; we give it as Exercise 17E in order to have the statement recorded. The second analog of the second form of the Nullstellensatz, however, is more difficult to prove, but it is also, as it turns out, far more useful. This result was first conjectured by Resco and then proved by Amitsur and Small.

Exercise 17E. If $R = S[x_1,\ldots,x_n]$ is a polynomial ring over a simple artinian ring and M a maximal ideal of R, show that R/M has finite length as a right or left S-module. □

We first make an observation that will allow us to reduce the case of simple artinian coefficients to that of division ring coefficients.

Exercise 17F. Let $T = M_t(R)$ for a ring R and some $t \in \mathbb{N}$, and identify R with the subring of scalar matrices in T. Show that any simple T-module A is finitely generated and semisimple when viewed as an R-module. [Hint: Multiply A by the standard matrix units.] □

If S is a simple artinian ring, we may assume that $S = M_t(D)$ for some division ring D and positive integer t, and then we may identify the polynomial ring $S[x_1, \dots, x_n]$ with $M_t(D[x_1, \dots, x_n])$. According to Exercise 17F, any simple module over $S[x_1, \dots, x_n]$ is finitely generated and semisimple over $D[x_1, \dots, x_n]$. Hence, if we know that all simple $D[x_1, \dots, x_n]$-modules are finite dimensional over D, then every simple $S[x_1, \dots, x_n]$-module is finite dimensional over D and hence has finite length over S.

Therefore, in proving the Nullstellensatz, we may reduce to the case that S is a division ring. This makes it easier to work with the rings $S[x_i]$, which, for instance, are principal right and left ideal domains (Theorem 2.8).

Exercise 17G. If $D[x]$ is a polynomial ring over a division ring D, show that there are infinitely many pairwise nonisomorphic unfaithful simple right $D[x]$-modules. [Hint: Let k be the center of D, and use either Euclid's method or Exercise 3V to find infinitely many pairwise nonassociate irreducible polynomials in $k[x]$.] □

Exercise 17H. If M is a right ideal of a polynomial ring $D[x_1, \dots, x_n]$ over a division ring D, and if $M \cap D[x_i] \neq 0$ for all $i = 1, \dots, n$, show that $D[x_1, \dots, x_n]/M$ is finite dimensional over D. □

Theorem 17.6. [Amitsur-Small] *If S is a simple artinian ring, then every simple module over the polynomial ring $S[x_1, \dots, x_n]$ has finite length as an S-module.*

Proof. As observed above, it is enough to deal with simple modules over a polynomial ring $T = D[x_1, \dots, x_n]$, where D is a division ring. The case $n = 1$ follows directly from the division algorithm (or Proposition 17.1). Now let $n > 1$, and assume that the theorem holds over polynomial rings in $n - 1$ indeterminates.

By symmetry, we need only show that any simple right T-module A has finite length (i.e., finite dimension) over D. We may assume that $A = T/M$, where M is a maximal right ideal of T. By Exercise 17H, it suffices to show that $M \cap D[x_i] \neq 0$ for all $i = 1, \dots, n$, and by symmetry it is enough to show that $M \cap D[x_1] \neq 0$. Set $R = D[x_1]$ and suppose that $M \cap R = 0$.

Now R embeds in A, and so the R-module A_R is not torsion. Since the torsion submodule of A_R is invariant under all endomorphisms of A_R, it must be invariant under multiplication by each x_i, and so it is a T-submodule. By simplicity, the torsion submodule of A_R must be zero, and hence A_R is torsionfree. Similarly, if k is the center of D and p is a nonzero polynomial in $k[x_1]$, then Ap is a nonzero T-submodule of A, whence $Ap = A$. Thus, A is divisible as a $k[x_1]$-module.

The set $X = R \setminus \{0\}$ is a right and left denominator set in both R and T, and TX^{-1} is isomorphic to $(RX^{-1})[x_2, \dots, x_n]$. Since A_R is torsionfree, AX^{-1} is nonzero, and so it is a simple module over TX^{-1}. As RX^{-1} is a division ring, it follows from our induction hypothesis that AX^{-1} is finite dimensional over RX^{-1}, and from this we conclude that A has finite rank as an R-module.

Now A has an essential R-submodule which is a finite direct sum of uniform submodules, and each of these, being torsionfree, contains a copy of R_R. Hence, A has an essential R-submodule B which is finitely generated and free.

Observe using Proposition 17.1 that any nonzero prime ideal P of R is a maximal ideal, that R/P is simple artinian, and that $P = pR$ for some nonzero $p \in k[x_1]$. Since A is divisible as a $k[x_1]$-module, it contains an R-submodule C such that $Cp = B$, and then, since A is torsionfree over R, we see that $C/B \cong B/Bp$, which is in turn isomorphic to a direct sum of copies of R/P. Therefore A/B contains at least one simple R-submodule annihilated by P. In view of Exercise 17G, it follows that A/B has infinitely many pairwise nonisomorphic (unfaithful) simple R-submodules.

Set $B_0 = B$, and then set $B_{i+1} = B_i + B_i x_1 + \cdots + B_i x_n$ for all $i = 0, 1, \ldots$. These B_i are finitely generated R-submodules of A, and their union is a T-submodule, whence $\bigcup_{i=0}^{\infty} B_i = A$. Since B_1/B_0 is a finitely generated torsion R-module, it has finite length, by Proposition 17.1. Let $\mathcal{S}$ be the set of isomorphism types of composition factors of B_1/B_0, and keep in mind that $\mathcal{S}$ is finite. Multiplication by any x_j induces a homomorphism $B_1/B_0 \to B_2/B_1$, and the sum of the images of these homomorphisms is all of B_2/B_1. In other words, B_2/B_1 is an epimorphic image of $(B_1/B_0)^n$, and consequently all composition factors of B_2/B_1 lie in $\mathcal{S}$. Likewise, all composition factors of each of the modules B_{i+1}/B_i lie in $\mathcal{S}$. Since A is the union of the B_i, it follows that the isomorphism types of all simple submodules of A/B are in $\mathcal{S}$. However, this contradicts the fact that A/B contains infinitely many pairwise nonisomorphic simple submodules.

Therefore $M \cap R \neq 0$, as desired. Consequently, A_D is finite dimensional, and the induction step is complete. $\square$

One form of the commutative Nullstellensatz is obviously contained in Theorem 17.6: If k is a field and M a maximal ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$, then R/M is finite dimensional over k. A second form may be obtained from the following exercise.

Exercise 17I. If k is a field and P a prime ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$, show that P is an intersection of maximal ideals. [Hint: If not, assume that P is maximal with respect to this property, observe that the intersection of the nonzero prime ideals of R/P is nonzero, and show that the quotient field of R/P is a homomorphic image of $R[x]$.] $\square$

Exercise 17J. This is an extended exercise that indicates how uncountability assumptions can simplify the proofs of both the commutative Nullstellensatz and the Amitsur-Small Nullstellensatz. Throughout, let k be an uncountable field and D a division ring with center k.

(a) Show that the field $k(x)$ of rational functions over k has uncountable dimension as a vector space over k. [Hint: The elements $1/(x-\alpha)$, for $\alpha \in k$, are linearly independent.]

(b) If M is a maximal ideal in the polynomial ring $T = k[x_1, \ldots, x_n]$, show that T/M is finite dimensional over k. [Hint: Use Exercise 17H.]

(c) Show that the ring $D \otimes_k k(x_1)$ has uncountable dimension over D. [Note that it is useless to show that the dimension over k is uncountable, since this will already be true of D itself if D is transcendental over k.]

(d) Show that $D \otimes_k k[x_1]$ is an essential $k[x_1]$-submodule of $D \otimes_k k(x_1)$.

(e) If M is a maximal right ideal in the polynomial ring $R = D[x_1, \ldots, x_n]$, show that R/M is finite dimensional over D. [Hint: If not, then by Exercise 17H we may assume that $M \cap D[x_1] = 0$. Make R/M into a $(k[x_1], D)$-bimodule and use the fact that the endomorphism ring of $(R/M)_R$ is a division ring to make R/M into a $(k(x_1), D)$-bimodule. Convert this bimodule structure into a right $D[x_1]$-module homomorphism $D \otimes_k k(x_1) \to R/M$ which is injective on $D \otimes_k k[x_1]$. Finally, use part (d) and part (c).] $\square$

• FULLY BOUNDED G-RINGS •

We now use the Amitsur-Small Nullstellensatz to study primitive ideals in polynomial rings. (We should point out that, although Proposition 17.5 seems to say a great deal about the ideals in a polynomial ring over a simple artinian ring, it does not say anything about which ideals are primitive.) The approach we shall take goes by way of the notion of a "G-ring," which is also important in some standard commutative treatments of the Nullstellensatz.

Definition. A *G-ring* is a prime ring in which the intersection of the nonzero prime ideals is nonzero. A *G-ideal* in a ring R is any prime ideal P such that R/P is a G-ring.

For instance, any simple ring is a G-ring. The local ring

$$\mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}$$

is a G-ring (since its only nonzero prime ideal is $2\mathbb{Z}_{(2)}$), while $\mathbb{Z}$ itself is not a G-ring. It can be proved that a commutative noetherian domain is a G-ring if and only if it has only finitely many prime ideals. In the noncommutative case, however, this no longer holds: For example, the noetherian domain S considered in Exercise 3W is a G-ring with infinitely many prime ideals.

Exercise 17K. Let P be a prime ideal in a ring R, and give $\text{Spec}(R)$ the Zariski topology. Show that P is a G-ideal if and only if P has an open neighborhood in which the singleton set $\{P\}$ is closed (in the relative topology). The latter condition says that P is a *locally closed point* in $\text{Spec}(R)$ (with respect to the Zariski topology), and so one often says that the G-ideals are the *locally closed prime ideals* of R. $\square$

A commutative domain R is a G-ring if and only if the quotient field of R can be obtained by inverting a single element b, that is, if every element of the quotient field can be written as a fraction with numerator from R and denominator a power of b. While the noncommutative analog of this equivalence is not generally true (Exercise 17L), it does hold for bounded prime Goldie rings, as Lemma 17.7 shows.

Exercise 17L. Let $R = A_1(k)$, where k is a field of characteristic zero. Show that we cannot have $D_1(k) = \{ab^{-n} \mid a \in R,\, n \in \mathbb{Z}^+\}$ for any nonzero element $b \in R$. [Hint: Suppose that such a b exists, and observe that R/bR has finite length. If R is expressed in the form $k[y][x; d/dy]$, then for $\alpha \in k$ show that $R/(y - \alpha)R$ is a simple module and that it is a composition factor of some R/b^nR.] $\square$

Lemma 17.7. *Let R be a right bounded prime right Goldie ring and Q its right Goldie quotient ring. Then the following conditions are equivalent:*

(a) *R is a G-ring.*

(b) *There is a regular element $b \in R$ such that every essential right ideal of R contains a power of b.*

(c) *There is a regular element $b \in R$ such that every element of Q can be expressed in the form ab^{-n} for some $a \in R$ and $n \in \mathbb{Z}^+$.*

(d) *There is a nonzero ideal I in R such that every element of Q/R can be annihilated on the right by a power of I.*

Proof. (a) $\Longrightarrow$ (b): The intersection of the nonzero prime ideals of R is a nonzero ideal I, and it must contain a regular element b.

If J is an essential right ideal of R, then, since R is right bounded, J contains a nonzero ideal K. Suppose that K is disjoint from the set $B = \{1, b, b^2, \dots\}$ and enlarge K to an ideal P maximal with respect to being disjoint from B. According to Lemma 3.5, P is prime. But then $b \in I \subseteq P$, a contradiction. Thus, there must be some power of b contained in K, and hence in J.

(b) $\Longrightarrow$ (c): Any element of Q has the form cd^{-1} for some $c, d \in R$ with d regular. Then dR is an essential right ideal of R, and so dR contains a power of b, say $b^n = de$, for some $n \in \mathbb{Z}^+$ and $e \in R$. Then $d^{-1} = eb^{-n}$ and thus $cd^{-1} = ceb^{-n}$.

(c) $\Longrightarrow$ (d): Since bR is an essential right ideal of R, by our right boundedness hypothesis there is a nonzero ideal I contained in bR. For all $n \in \mathbb{N}$, observe that

$$I^n \subseteq bRI^{n-1} = bI^{n-1} \subseteq b^2RI^{n-2} = b^2I^{n-2} \subseteq \cdots \subseteq b^nR.$$

Statement (d) follows from this and statement (c).

(d) $\Longrightarrow$ (a): Any nonzero prime ideal P of R is essential as a right ideal, and so P contains a regular element c. Then $c^{-1}I^n \subseteq R$ for some $n \in \mathbb{N}$, whence $I^n \subseteq cR \subseteq P$, and consequently $I \subseteq P$. Therefore the intersection of the nonzero prime ideals of R contains I. $\square$

Exercise 17M. If the hypotheses and the equivalent conditions of Lemma 17.7 hold, show that the set $X = \{1, b, b^2, \dots\}$ is a right denominator set in R and that RX^{-1} is the right Goldie quotient ring of R. $\square$

Exercise 17N. If P is a prime ideal in a polynomial ring $R[x_1, \dots, x_n]$, show that $P \cap R$ is a prime ideal of R. $\square$

Lemma 17.8. *If R is a G-ring which is either right or left Goldie, then, in the polynomial ring $T = R[x_1, \dots, x_n]$ (where $n > 0$), there is a right primitive ideal P such that $P \cap R = 0$.*

Proof. The intersection of all the nonzero primes of R is a nonzero ideal, which must contain a regular element b. Observe that $1 - bx_1$ is not right invertible in T. Let M be a maximal right ideal of T containing $(1 - bx_1)T$, and let P be the annihilator of the simple right T-module T/M. Then P is a right primitive ideal of T, and hence prime, and so, by Exercise 17N, $P \cap R$ is a prime ideal of R. Since $1 - bx_1 \in M$, we must have $b \notin M$, whence $b \notin P \cap R$. Therefore $P \cap R = 0$. $\square$

The converse of Lemma 17.8 is false in general, as we shall see below (Exercise 17P), but we shall prove it when R is bounded and Goldie on both sides.

Lemma 17.9. *Let R be a right and left bounded, prime, right and left Goldie ring, and A a simple right module over the polynomial ring $T = R[x_1, \dots, x_n]$. If A_R is faithful, then it is a torsionfree injective module of finite rank.*

Proof. If A_R is not torsionfree, choose a nonzero torsion element $a \in A$. There is an essential right ideal J in R such that $aJ = 0$, and since R is right bounded, there is a nonzero ideal I contained in J, whence $aI = 0$. Now $aT = A$, because A is a simple T-module. But then $AI = aTI = aIT = 0$, which contradicts the assumption that A_R is faithful. Thus A_R must be torsionfree.

Since A_R is now torsionfree, to prove that it is injective it is sufficient to prove that it is divisible (Proposition 7.11). Let c be a regular element of R. As Rc is an essential left ideal of R and R is left bounded, there is a nonzero ideal H contained in Rc, and $AH \neq 0$ because A_R is faithful. Observe that AH is a T-submodule of A. Since A_T is simple, we conclude that $A = AH \subseteq Ac$. Therefore A is divisible, and hence injective, as an R-module.

If Q is the Goldie quotient ring of R, then, by Proposition 7.13, A has a unique right Q-module structure compatible with its right R-module structure. For any regular element $b \in R$, since multiplication by b on A commutes with multiplication by each x_i, so does multiplication by b^{-1}. It follows that the Q-module structure on A extends to a $Q[x_1, \dots, x_n]$-module structure, compatible with the T-module structure. Obviously A is simple as a module over $Q[x_1, \dots, x_n]$. Applying Theorem 17.6, we conclude that A has finite length as a Q-module. Therefore A_R has finite rank. $\square$

A multivariable polynomial $f \in R[x_1, \ldots, x_n]$ has a number of possible degrees, such as the x_i-*degree*, which records the highest power of x_i that appears in any monomial in f. The *total degree* of f is obtained by giving all the x_i equal weight, as follows. The total degree of a monomial $x_1^{i(1)} x_2^{i(2)} \cdots x_n^{i(n)}$ is the sum of the exponents, namely $i(1) + i(2) + \cdots + i(n)$, and the total degree of f is the maximum of the total degrees of the monomials appearing in f.

Theorem 17.10. [Resco-Stafford-Warfield] *Let R be a right and left bounded, prime, right and left Goldie ring, and $T = R[x_1, \ldots, x_n]$ a polynomial ring, where $n > 0$. Then R is a G-ring if and only if there is a right primitive ideal P in T such that $P \cap R = 0$.*

Proof. One implication is given by Lemma 17.8. Conversely, assume that there is a right primitive ideal P in T such that $P \cap R = 0$. There is a simple right T-module A whose annihilator is P, and then A is faithful as an R-module. By Lemma 17.9, A_R is torsionfree injective of finite rank. Hence, A becomes a finitely generated semisimple module over the Goldie quotient ring Q of R. If A has length s and Q has length t, then $A^t \cong Q^s$, and so A^t has an essential R-submodule B such that $A^t/B \cong (Q/R)^s$. We may choose a finitely generated essential R-submodule $C \leq_e A$ such that $C^t \leq B$, and then $(Q/R)^s$ is an epimorphic image of $(A/C)^t$.

We shall find a nonzero ideal I in R such that every element of A/C is annihilated by a power of I. The same is then true in Q/R (on the right), and by Lemma 17.7 this is enough to show that R is a G-ring.

Since for each index $i = 1, \ldots, n$ the R-module $(C + Cx_i)/C$ is finitely generated and torsion, the right boundedness of R implies that there is a nonzero ideal I in R which annihilates all of these modules (see Lemma 9.2). In other words, $Cx_iI \subseteq C$ for all $i = 1, \ldots, n$. It follows that if f is any polynomial in T of total degree d, then $CfI^d \subseteq C$. Since A is simple as a T-module, $CT = A$. Therefore every element of A/C is annihilated by a power of I, as desired. □

Recall that a *Jacobson ring* is a ring in which all the prime ideals are semiprimitive. For instance, by the commutative Nullstellensatz, every finitely generated commutative algebra over a field is a Jacobson ring.

Corollary 17.11. [Resco-Stafford-Warfield] *Let R be a fully bounded noetherian Jacobson ring and A a simple right module over a polynomial ring $R[x_1, \ldots, x_n]$, where $n > 0$. Then there is a maximal ideal M of R such that $AM = 0$, and as an R-module A is finitely generated and semisimple.*

Proof. If $M = \operatorname{ann}_R(A)$, then A is a simple right module over the polynomial ring $T = (R/M)[x_1, \ldots, x_n]$ and A is faithful as an (R/M)-module. The annihilator of A in T is a right primitive ideal P such that $P \cap (R/M) = 0$. Since P is a prime ideal of T, it follows from Exercise 17N that 0 is a prime ideal of R/M. Theorem 17.10 then implies that R/M is a G-ring. Since R is a Jacobson ring, M is an intersection of right primitive ideals, and since R/M is a G-ring, this is only possible if M itself is right primitive. As R is fully bounded, M must be a maximal ideal and R/M must be simple artinian (Proposition 9.4). It follows that A is semisimple as an R-module. Lemma 17.9 implies that A_R has finite rank, and therefore it is finitely generated. □

• PRIMITIVITY AND TRANSCENDENCE DEGREE •

We return to the question of when a polynomial ring over a division ring D (or, more generally, over a simple artinian ring) can be primitive. The case of a single indeterminate has been treated in Theorem 17.2: $D[x]$ is primitive if and only if at least one of the matrix rings over D is transcendental over the center of D. In order to handle polynomial rings in many indeterminates, we need an appropriate notion of transcendence degree.

Definition. Let R be an algebra over a field k. The *transcendence degree of R over k* is the supremum of those nonnegative integers n for which there exists a k-algebra embedding of the polynomial ring $k[x_1, \dots, x_n]$ into R. (By convention, the polynomial ring in 0 indeterminates is just k. Hence, R has transcendence degree 0 over k precisely when R is algebraic over k.) The *matrix-transcendence degree of R over K* is the supremum of the transcendence degrees of all matrix rings $M_t(R)$.

The following lemma is frequently useful and shows that, for simple artinian rings, the notion of matrix-transcendence degree just defined agrees with another reasonable notion.

Lemma 17.12. *Let S be a simple artinian algebra over a field k and $n \in \mathbb{N}$. Then S has matrix-transcendence degree at least n over k if and only if there is a k-algebra embedding of the rational function field $k(x_1, \dots, x_n)$ (in n independent indeterminates) into some matrix ring $M_t(S)$.*

Proof. Obviously, if $k(x_1, \dots, x_n)$ embeds in $M_t(S)$, so does the polynomial ring $R = k[x_1, \dots, x_n]$, whence the matrix-transcendence degree of S over k is at least n.

Conversely, if this matrix-transcendence degree is at least n, there is a k-algebra embedding of R into some matrix ring $M_m(S)$. We use this to produce a bimodule as follows. We regard S^m as a left S-module and as a right module over the matrix ring $M_m(S)$. Since $M_m(S)$ is a simple ring, S^m is faithful as a right $M_m(S)$-module. Using the embedding of R into $M_m(S)$, we can turn S^m into an (S, R)-bimodule, finitely generated on the left and faithful on the right.

According to Lemma 8.1, R_R embeds in a finite direct sum of copies of S^m, and so $(S^m)_R$ is not torsion. Hence, if T is the torsion submodule of S^m as a right R-module, $S^m/T \neq 0$. Now T is a left S-submodule of S^m

and S^m/T is a finitely generated semisimple left S-module. If S has length s, then $(S^m/T)^s$ is a free left S-module, say of rank t, and so its endomorphism ring is isomorphic to $M_t(S)^{\mathrm{op}}$. Thus, there is a k-algebra embedding of the endomorphism ring of ${}_S(S^m/T)$ into $M_t(S)^{\mathrm{op}}$.

The bimodule ${}_S(S^m/T)_R$ is torsionfree on the right and has finite length on the left. By Lemma 8.10, its right R-module structure extends to a right $k(x_1, \dots, x_n)$-module structure, and S^m/T becomes an $(S,\, k(x_1, \dots, x_n))$-bimodule. Therefore $k(x_1, \dots, x_n)$ embeds in the opposite ring of the endomorphism ring of ${}_S(S^m/T)$ and hence in $M_t(S)$. $\square$

Theorem 17.13. [Resco, Amitsur-Small] *Let S be a simple artinian ring with center k and $n \in \mathbb{N}$. Then the polynomial ring $S[x_1, \dots, x_n]$ is right primitive if and only if the matrix-transcendence degree of S over k is at least n.*

Proof. If A is a faithful simple right $S[x_1, \dots, x_n]$-module, then, according to Theorem 17.6, A has finite length as an S-module. There are positive integers s, t such that $A^s \cong S^t$, whence $\mathrm{End}_S(A)$ embeds in $M_t(S)$ (as a k-algebra). Since $k[x_1, \dots, x_n]$ is the center of $S[x_1, \dots, x_n]$, there is an induced k-algebra map

$$k[x_1, \dots, x_n] \to \mathrm{End}_S(A) \to M_t(S),$$

and this map is injective because A is faithful over $S[x_1, \dots, x_n]$. Thus the matrix-transcendence degree of S over k is at least n.

Conversely, if the matrix-transcendence degree of S over k is at least n, then, by Lemma 17.12, some matrix ring $M_t(S)$ contains a k-subalgebra isomorphic to $k(x_1, \dots, x_n)$. This matrix ring is isomorphic to the endomorphism ring of the right S-module S^t, thus making S^t into a $(k(x_1, \dots, x_n),\, S)$-bimodule. Since $k(x_1, \dots, x_n)$ is commutative, this bimodule structure can be used to make S^t into a right module over the tensor product $k(x_1, \dots, x_n) \otimes_k S$ (as in Exercise 1D). Because $k(x_1, \dots, x_n) \otimes_k S$ is a simple ring (Proposition 17.5), the module we have just constructed is necessarily faithful. Thus, using the natural embedding of $S[x_1, \dots, x_n]$ into $k(x_1, \dots, x_n) \otimes_k S$, we have made S^t into a faithful right $S[x_1, \dots, x_n]$-module. It clearly has finite length, and so at least one of its composition factors must be faithful. Therefore $S[x_1, \dots, x_n]$ is right primitive. $\square$

Our next result is a remarkable theorem of Resco which connects the transcendence degree of an algebra with the Krull dimension of rational function rings over the algebra. It not only gives concrete information about the transcendence degree, but also leads to unexpected information about the primitivity of polynomial rings (e.g., Theorems 17.16 and 17.18).

Theorem 17.14. [Resco] *Let k be a field and R a right noetherian k-algebra which contains a subfield of transcendence degree n over k. If $x_1, \dots, x_n$ are independent indeterminates, then*

$$\mathrm{r.K.dim}\big(R \otimes_k k(x_1, \dots, x_n)\big) \geq n.$$

Proof. Note that $R\otimes_k k(x_1,\dots,x_n)$ is a right Ore localization of $R[x_1,\dots,x_n]$, whence it is right noetherian and so does have right Krull dimension.

By hypothesis, R contains a rational function field $K = k(t_1,\dots,t_n)$. Let $L = k(x_1,\dots,x_n)$, and identify $K\otimes_k L$ with the localization of $K[x_1,\dots,x_n]$ with respect to the multiplicative set $X = k[x_1,\dots,x_n]\setminus\{0\}$. Define ideals $P_m = \langle x_i - t_i \mid i = 1,\dots,m\rangle \subseteq K[x_1,\dots,x_n]$ for $m = 1,\dots,n$, and note that each P_m is a prime ideal of $K[x_1,\dots,x_n]$. The k-algebra map $k[x_1,\dots,x_n] \to K$ which sends each $x_i \mapsto t_i$ is injective because the t_i are algebraically independent over k. Since this map can be factored as

$$k[x_1,\dots,x_n] \xrightarrow{\subseteq} K[x_1,\dots,x_n] \xrightarrow{\text{quo}} K[x_1,\dots,x_n]/P_n \xrightarrow{\cong} K,$$

we conclude that P_n is disjoint from X. Hence, we have a chain $0 < P_1 < P_2 < \cdots < P_n$ of prime ideals in $k[x_1,\dots,x_n]$, all of which are disjoint from X. By Theorem 10.20, there is a corresponding chain of prime ideals of length n in $K\otimes_k L$, whence

$$\text{Cl.K.dim}(K\otimes_k L) \geq n.$$

Consequently, $\text{r.K.dim}(K\otimes_k L) \geq n$ (Exercise 15F).

Since R is free as a left K-module, we see that $R\otimes_k L$ is free as a left module over $K\otimes_k L$ and hence left faithfully flat. It follows from Exercise 15U that

$$\text{r.K.dim}(R\otimes_k L) \geq \text{r.K.dim}(K\otimes_k L) \geq n,$$

which proves the theorem. □

Corollary 17.15. [Resco] *Let S be a simple artinian algebra over a field k. If the matrix-transcendence degree of S over k is at least n, then*

$$\text{r.K.dim}\big(S\otimes_k k(x_1,\dots,x_n)\big) = n.$$

Proof. Set $L = k(x_1,\dots,x_n)$. We first observe that $S\otimes_k L$ is a right Ore localization of the polynomial ring $S[x_1,\dots,x_n]$, which has right Krull dimension n by Theorem 15.17. Hence, by Exercise 15S, $\text{r.K.dim}(S\otimes_k L) \leq n$. On the other hand, by Lemma 17.12, some matrix ring $M_t(S)$ contains a subfield of transcendence degree n over k, and hence the previous theorem shows that

$$\text{r.K.dim}\big(M_t(S)\otimes_k L\big) \geq n.$$

Since $M_t(S)\otimes_k L$ is finitely generated as a right module over $S\otimes_k L$, we conclude from Exercise 15R that the Krull dimension of $M_t(S)\otimes_k L$ as a right module over $S\otimes_k L$ is at least n, and therefore $\text{r.K.dim}\big(S\otimes_k L\big) \geq n$. □

• QUOTIENT DIVISION RINGS OF WEYL ALGEBRAS AND QUANTUM PLANES •

We now apply the results of the previous section to the Weyl division algebras $D_m(k)$ and the quantum Weyl division algebras $D_1^q(k)$. In particular, we determine the matrix-transcendence degrees of these division algebras in many cases (namely, when $\mathrm{char}(k) = 0$ in the first case, and when q is not a root of unity in the second). The calculations of matrix-transcendence degrees will allow us to see that $D_m(k) \not\cong D_n(k)$ when $m \neq n$; different methods allow us to determine when two division algebras of the form $D_1^q(k)$ are isomorphic.

Exercise 17O. If k is a field of characteristic zero and $m \in \mathbb{N}$, show that the center of $D_m(k)$ is k. [Hint: Exercises 6G and 6H.] $\square$

Theorem 17.16. [Resco, Amitsur-Small] *Let k be a field of characteristic zero, $m \in \mathbb{N}$, and $x_1, \dots, x_n$ independent indeterminates.*

(a) *The matrix-transcendence degree of $D_m(k)$ over k is exactly m.*

(b) $\mathrm{r.K.dim}\big(D_m(k) \otimes_k k(x_1, \dots, x_n)\big) = \min\{m, n\}$.

(c) $D_m(k)[x_1, \dots, x_n]$ *is right primitive if and only if $n \leq m$.*

Proof. By Exercise 17O, the center of $D_m(k)$ is k.

(b) Since $D_m(k)$ contains a polynomial ring in m indeterminates over k, Corollary 17.15 shows that $\mathrm{r.K.dim}\big(D_m(k) \otimes_k k(x_1, \dots, x_n)\big) = n$ when $n \leq m$.

Now suppose that $n > m$ and put $K = k(x_1, \dots, x_m)$. Then

$$\mathrm{r.K.dim}\big(D_m(k) \otimes_k K\big) = m.$$

Since $k(x_1, \dots, x_n)$ is a free K-module, $D_m(k) \otimes_k k(x_1, \dots, x_n)$ is free as a left module over $D_m(k) \otimes_k K$ and hence left faithfully flat. It follows from Exercise 15U that

$$\mathrm{r.K.dim}\big(D_m(k) \otimes_k k(x_1, \dots, x_n)\big) \geq \mathrm{r.K.dim}\big(D_m(k) \otimes_k K\big) = m.$$

On the other hand, since $D_m(k)$ is a right Ore localization of $A_m(k)$, we see that the algebra $D_m(k) \otimes_k k(x_1, \dots, x_n)$ is a right Ore localization of $A_m(k) \otimes_k k(x_1, \dots, x_n)$, which is isomorphic to $A_m\big(k(x_1, \dots, x_n)\big)$. We know from Theorem 15.21 that

$$\mathrm{r.K.dim}\Big(A_m\big(k(x_1, \dots, x_n)\big)\Big) = m,$$

and thus $\mathrm{r.K.dim}\big(D_m(k) \otimes_k k(x_1, \dots, x_n)\big) \leq m$, by Exercise 15S.

(a) and (c) now follow from Corollary 17.15 and Theorem 17.13. $\square$

Corollary 17.17. [Gelfand-Kirillov] *Let k, k' be fields of characteristic zero and m, n positive integers. Then $D_m(k) \cong D_n(k')$ if and only if $k \cong k'$ and $m = n$.* $\square$

Exercise 17P. Use Theorem 17.16 to give an example of a noetherian ring R such that the polynomial ring $R[x]$ is right primitive but R is not a G-ring. (Compare Theorem 17.10.) $\square$

Exercise 17Q. This exercise is designed to show that a division ring can be transcendental over its center and yet have maximal subfields which are algebraic. Let p be a prime integer, K the algebraic closure of the field $k = \mathbb{Z}/p\mathbb{Z}$, and α the Frobenius automorphism of K, given by the rule $\alpha(a) = a^p$. Let D be the Ore quotient ring of $K[x; \alpha]$.

(a) Show that the center of D is k. [Hint: Exercise 6I.]

(b) Show that K (which is algebraic over k) is a maximal subfield of D. Observe on the other hand that D is transcendental over k (e.g., x is transcendental). $\square$

Exercise 17R. Let $L \supseteq k$ be a finitely generated field extension of finite transcendence degree t. Let $T = L[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_m; \alpha_m, \delta_m]$ be an iterated skew polynomial ring, where the α_i are k-algebra automorphisms and the δ_i are k-linear α_i-derivations, and let D be the quotient division ring of T. Show that the matrix-transcendence degree of D over k is at most $t + m$. [Hints: Let $F = k(y_1, \ldots, y_n)$ be a rational function field with $n > t + m$ and study $D \otimes_k F$. Observe that $T \otimes_k F$ is an m-fold iterated skew polynomial ring over $L \otimes_k F$.] $\square$

We now develop analogs of Theorem 17.16 and Corollary 17.17 for the division algebras $D_1^q(k)$. Analysis of the quotient division algebras of general quantum affine spaces $\mathcal{O}_{\mathbf{q}}(k^n)$ requires further tools, which we do not develop here.

Theorem 17.18. *Let k be a field, $q \in k^\times$ a nonroot of unity, and $x_1, \ldots, x_n$ independent indeterminates, where $n > 0$.*

(a) *The matrix-transcendence degree of $D_1^q(k)$ over k is* 1.

(b) $\text{r.K.dim}\big(D_1^q(k) \otimes_k k(x_1, \ldots, x_n)\big) = 1$.

(c) *$D_1^q(k)[x_1, \ldots, x_n]$ is right primitive if and only if $n = 1$.*

Proof. By Exercise 6J, the center of $D_1^q(k)$ is k. Set $L = k(x_1, \ldots, x_n)$.

(b) Obviously $D_1^q(k)$ contains a polynomial ring $k[x]$, whence Corollary 17.15 implies that $\text{r.K.dim}\big(D_1^q(k) \otimes_k k(x_1)\big) = 1$. Since $D_1^q(k) \otimes_k L$ is faithfully flat as a left module over $D_1^q(k) \otimes_k k(x_1)$, it follows from Exercise 15U that

$$\text{r.K.dim}\big(D_1^q(k) \otimes_k L\big) \geq \text{r.K.dim}\big(D_1^q(k) \otimes_k k(x_1)\big) = 1.$$

On the other hand, since $D_1^q(k)$ is a right Ore localization of $\mathcal{O}_q((k^\times)^2)$, we see that $D_1^q(k) \otimes_k L$ is a right Ore localization of $\mathcal{O}_q((k^\times)^2) \otimes_k L$, which is isomorphic to $\mathcal{O}_q((L^\times)^2)$. By Exercises 15S and 15ZB,

$$\text{r.K.dim}\big(D_1^q(k) \otimes_k L\big) \leq \text{r.K.dim}\big(\mathcal{O}_q((L^\times)^2)\big) = 1.$$

Therefore $\text{r.K.dim}\big(D_1^q(k) \otimes_k L\big) = 1$.

For (a) and (c), apply Corollary 17.15 and Theorem 17.13. $\square$

Our next task is to study when two division algebras of the form $D_1^q(k)$ can be isomorphic. Note that if $r = q^{-1}$, then $\mathcal{O}_r(k^2) \cong \mathcal{O}_q(k^2)$ (just interchange the standard generators x and y), whence $D_1^r(k) \cong D_1^q(k)$. Thus, we ask whether the set $\{q, q^{-1}\}$ can be determined from $D_1^q(k)$, at least when q is not a root of unity. In that case, the cyclic subgroup $\langle q \rangle \subseteq k^\times$ is infinite, and its only generators are q and q^{-1}. Hence, it will suffice to determine the group $\langle q \rangle$ from $D_1^q(k)$. Now $\mathcal{O}_q(k^2)$ is generated by elements x and y satisfying $xy = qyx$, whence $q = xyx^{-1}y^{-1}$ in $D_1^q(k)$. This observation suggests that we approach our problem by studying the commutator subgroup of the multiplicative group $D_1^q(k)^\times$. Recall that, in any multiplicative group, the elements of the commutator subgroup are just products of commutators.

Lemma 17.19. *Let k be a field, $q \in k^\times$, and G the commutator subgroup of $D_1^q(k)^\times$. Then $G \cap k^\times = \langle q \rangle$.*

Proof. We have already observed that q is a commutator in $D_1^q(k)^\times$, and so $\langle q \rangle \subseteq G \cap k^\times$.

Write $\mathcal{O}_q(k^2) = k[y][x; \alpha]$, where α is the k-algebra automorphism of $k[y]$ such that $\alpha(y) = qy$. Note that α extends uniquely to an automorphism of $k(y)$ and that the subalgebra of $D_1^q(k)$ generated by $k(y)$ and x is a skew polynomial ring of the form $k(y)[x; \alpha]$. We may view $D_1^q(k)$ as the quotient division ring of $k(y)[x; \alpha]$. Let $\deg_x$ denote degree with respect to x in the skew polynomial ring $k(y)[x; \alpha]$ and extend $\deg_x$ to $D_1^q(k)$ as in Exercise 6F. Similarly, let $\deg_y$ denote degree with respect to y in the polynomial ring $k[y]$ and extend $\deg_y$ to $k(y)$.

Claim 1: If $u, v \in D_1^q(k)^\times$, then $uvu^{-1}v^{-1} = b\alpha^i(c)\alpha^{i+j}(b^{-1})\alpha^j(c^{-1}) + w$ for some $b, c \in k(y)^\times$, some $i, j \in \mathbb{Z}$, and some $w \in D_1^q(k)$ with $\deg_x(w) < 0$.

Let i, j and b, c be the degrees and leading coefficients of u, v, so that $u = bx^i + u_1$ and $v = cx^j + v_1$ for some $u_1, v_1 \in D_1^q(k)$ with $\deg_x(u_1) < i$ and $\deg_x(v_1) < j$. Since $\deg_x$ is additive on products, u^{-1} and v^{-1} have degrees $-i$ and $-j$. It is clear that their leading coefficients are b^{-1} and c^{-1}, whence $u^{-1} = b^{-1}x^{-i} + u_2$ and $v^{-1} = c^{-1}x^{-j} + v_2$ for some $u_2, v_2 \in D_1^q(k)$ with $\deg_x(u_2) < -i$ and $\deg_x(v_2) < -j$. Consequently, $uvu^{-1}v^{-1} = bx^icx^jb^{-1}x^{-i}c^{-1}x^{-j} + w$ for some $w \in D_1^q(k)$ with $\deg_x(w) < 0$, and

$$\begin{aligned} bx^icx^jb^{-1}x^{-i}c^{-1}x^{-j} &= b\alpha^i(c)x^{i+j}b^{-1}\alpha^{-i}(c^{-1})x^{-i-j} \\ &= b\alpha^i(c)\alpha^{i+j}(b^{-1})\alpha^j(c^{-1}). \end{aligned}$$

Claim 2: If $b, c \in k(y)^\times$ and $i, j \in \mathbb{Z}$, then $b\alpha^i(c)\alpha^{i+j}(b^{-1})\alpha^j(c^{-1}) = q^l + e$ for some $l \in \mathbb{Z}$ and $e \in k(y)$ with $\deg_y(e) < 0$.

Let m, n and β, γ be the degrees and leading coefficients of b, c, and note that b^{-1}, c^{-1} have degrees $-m$, $-n$ and leading coefficients β^{-1}, γ^{-1}. Hence,

$$\begin{aligned} b\alpha^i(c)\alpha^{i+j}(b^{-1})\alpha^j(c^{-1}) &= \beta y^m \alpha^i(\gamma y^n)\alpha^{i+j}(\beta^{-1}y^{-m})\alpha^j(\gamma^{-1}y^{-n}) + e \\ &= q^{in}q^{-im-jm}q^{-jn} + e \end{aligned}$$

for some $e \in k(y)$ with $\deg_y(e) < 0$.

Claim 3: Every element of G has the form $q^l + e + w$ for some $l \in \mathbb{Z}$, some $e \in k(y)$ with $\deg_y(e) < 0$, and some $w \in D_1^q(k)$ with $\deg_x(w) < 0$.

This statement is clear from Claims 1 and 2. It follows that $G \cap k^\times \subseteq \langle q \rangle$, and the lemma is proved. $\square$

Theorem 17.20. [Alev-Dumas] *Let k be a field and $q, r \in k^\times$ nonroots of unity. Then $D_1^q(k) \cong D_1^r(k)$ as k-algebras if and only if $q = r^{\pm 1}$.*

Proof. We have already noted that $D_1^q(k) \cong D_1^r(k)$ when $q = r^{\pm 1}$. Conversely, if $D_1^q(k)$ and $D_1^r(k)$ are isomorphic k-algebras, then $\langle q \rangle = \langle r \rangle$ by Lemma 17.19, and therefore $q = r^{\pm 1}$. $\square$

Corollary 17.21. *Let k, k' be fields and $q \in k^\times$ and $r \in (k')^\times$ nonroots of unity. Then $D_1^q(k) \cong D_1^r(k')$ if and only if there is an isomorphism $\phi : k \to k'$ such that $\phi(q) = r^{\pm 1}$.*

Proof. Exercise 6J and Theorem 17.20. $\square$

Exercise 17S. Let k be a field, $q \in k^\times$, and L the commutator subgroup of $(D_1^q(k), +)$ (that is, the set of all sums of additive commutators $uv - vu$). Show that $L \cap k = \{0\}$ and conclude that $D_1^q(k)$ is not isomorphic to any Weyl division algebra. $\square$

• FINITE GENERATION OF SUBFIELDS •

In this final section, we develop another direction in which "external" information about a division ring D can be used to yield "internal" information about subfields of D. We study the question of whether subfields of D containing the center k need to be finitely generated as field extensions of k. (Note that a field L containing k is a finitely generated field extension if and only if there exists a finite subset $E \subseteq L$ such that no proper sub*field* of L contains $k \cup E$. This occurs if and only if L is finite dimensional over a purely transcendental subfield of finite transcendence degree over k.) We begin with another characterization of finitely generated extension fields.

Theorem 17.22. [Vámos] *For fields $L \supseteq k$, the following conditions are equivalent:*

(a) *L is a finitely generated extension field of k.*

(b) *L satisfies the ACC on subfields containing k.*

(c) *$L \otimes_k L$ is a noetherian ring.*

Proof. (a) $\Longrightarrow$ (c): As L is finitely generated over k, it is finite dimensional over a rational function field $k(x_1,\dots,x_n)$. Since $L\otimes_k k(x_1,\dots,x_n)$ is a localization of the polynomial ring $L[x_1,\dots,x_n]$, it must be noetherian. Moreover, $L\otimes_k L$ is finitely generated as a module over $L\otimes_k k(x_1,\dots,x_n)$, and therefore $L\otimes_k L$ is noetherian.

(c) $\Longrightarrow$ (b): If not, there is a strictly ascending chain $L_1 \subset L_2 \subset \cdots$ of fields between k and L. For $n = 1, 2, \dots$, let I_n be the kernel of the natural ring homomorphism from $L\otimes_k L$ to $L\otimes_{L_n} L$; then $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in $L\otimes_k L$. If $\alpha \in L_{n+1} \setminus L_n$, then the element $\alpha\otimes 1 - 1\otimes\alpha$ in $L\otimes_k L$ lies in I_{n+1} but not in I_n (since 1 and α are linearly independent over L_n). Hence, $I_1 < I_2 < \cdots$ is a strictly ascending chain of ideals in $L\otimes_k L$, contradicting the noetherian assumption.

(b) $\Longrightarrow$ (a): This is clear. $\square$

Exercise 17T. Let $R \subseteq S$ be rings such that S is left faithfully flat over R. If S is right noetherian, show that R is right noetherian. $\square$

Proposition 17.23. *Let D be a division algebra over a field k and L a subfield of D containing k. Then $D\otimes_k L$ is right noetherian if and only if L is a finitely generated extension field of k.*

Proof. If L is finitely generated over k, we may proceed as in the proof of the implication (a) $\Longrightarrow$ (c) in the previous theorem to see that $D\otimes_k L$ is right (and left) noetherian. Conversely, assume that $D\otimes_k L$ is right noetherian. Since D is a free left L-module, $D\otimes_k L$ is a free left $(L\otimes_k L)$-module and hence left faithfully flat. Then $L\otimes_k L$ is (right) noetherian by Exercise 17T, and thus Theorem 17.22 shows that L is finitely generated over k. $\square$

Corollary 17.24. *Let D be a division algebra over a field k. If either $D\otimes_k D$ or $D\otimes_k D^{\mathrm{op}}$ is a right noetherian ring, then all subfields of D containing k are finitely generated extension fields of k.*

Proof. This follows from Proposition 17.23 and the observation that if L is a subfield of D containing k, then $D\otimes_k D$ and $D\otimes_k D^{\mathrm{op}}$ are left faithfully flat modules over $D\otimes_k L$. $\square$

Theorem 17.25. [Resco-Small-Wadsworth] *Let $L \supseteq k$ be fields,*

$$T = L[x_1;\alpha_1,\delta_1][x_2;\alpha_2,\delta_2]\cdots[x_n;\alpha_n,\delta_n]$$

an iterated skew polynomial ring where the α_i are k-algebra automorphisms and the δ_i are k-linear α_i-derivations, and D the Ore quotient division ring of T. Then the following conditions are equivalent:

(a) *$D\otimes_k D$ is right noetherian.*

(b) *$D\otimes_k D^{\mathrm{op}}$ is right noetherian.*

(c) *L is a finitely generated extension field of k.*

(d) *All subfields of D containing k are finitely generated extension fields of k.*

Proof. (a) and (b) imply (d) by Corollary 17.24, and (d) $\Longrightarrow$ (c) a priori.

(c) $\Longrightarrow$ (a): Observe that $D \otimes_k T$ is an iterated skew polynomial ring of the form

$$(D \otimes_k L)[1\otimes x_1; 1\otimes\alpha_1, 1\otimes\delta_1][1\otimes x_2; 1\otimes\alpha_2, 1\otimes\delta_2] \cdots [1\otimes x_n; 1\otimes\alpha_n, 1\otimes\delta_n],$$

where the maps $1 \otimes \alpha_i$ are automorphisms. Since $D \otimes_k L$ is right noetherian (Proposition 17.23), we know from Corollary 2.7 that $D \otimes_k T$ is right noetherian. Thus, since $D \otimes_k D$ is a right Ore localization of $D \otimes_k T$, it too must be right noetherian.

(c) $\Longrightarrow$ (b): The proof of (c) $\Longrightarrow$ (a) also shows, mutatis mutandis, that $D^{\mathrm{op}} \otimes_k D$ is right noetherian. But $D^{\mathrm{op}} \otimes_k D \cong D \otimes_k D^{\mathrm{op}}$. $\square$

Corollary 17.26. [Resco-Small-Wadsworth] *If k is a field and $n \in \mathbb{N}$, then all subfields of $D_n(k)$ containing k are finitely generated extension fields of k.*

Proof. View $D_n(k)$ as the quotient division ring of $A_n(k)$ and $A_n(k)$ as an iterated skew polynomial ring over k, and apply Theorem 17.25. $\square$

Corollary 17.27. *Let D be the Ore quotient division ring of $\mathcal{O}_{\mathbf{q}}(k^n)$ for some field k, some $n \in \mathbb{N}$, and some multiplicatively antisymmetric matrix $\mathbf{q} \in M_n(k^\times)$. Then all subfields of D containing k are finitely generated extension fields of k.* $\square$

Exercise 17U. (a) Let k be a field of characteristic zero and L a subfield of $D_1(k)$ that properly contains k. Show that any subfield of $D_1(k)$ that contains L must be finite dimensional over L.

(b) Let k be a field, $q \in k^\times$ a nonroot of unity, and L a subfield of $D_1^q(k)$ that properly contains k. Show that any subfield of $D_1^q(k)$ that contains L must be finite dimensional over L. $\square$

• NOTES •

Boundedness of Polynomial Rings over Division Rings. An argument given by Jacobson in [1956, p. 241] shows that if D is a division ring matrix-algebraic over its center, then every polynomial in $D[x]$ is a factor of a central polynomial (which shows that $D[x]$ is right and left bounded).

Hilbert's Nullstellensatz. In the original version, Hilbert proved that if $f_1, f_2, \ldots$ and $F_1, F_2, \ldots$ are homogeneous polynomials in n variables over an algebraically closed field k, and if the F_j vanish at all points of k^n where the f_i vanish, then there exists $r \in \mathbb{N}$ such that every product of r of the F_j lies in the homogeneous ideal generated by the f_i [1893, §3, pp. 320, 321]. The inhomogeneous analog of this amounts to saying that if J is the intersection of the maximal ideals containing an ideal I in $k[x_1, \ldots, x_n]$, then some power of J is contained in I; consequently, if I is semiprime, it must be an intersection of maximal ideals.

Amitsur-Small Nullstellensatz. This was proved for a polynomial ring over a division ring in [1978, Theorem 1].

Fully Bounded G-Rings. Theorem 17.10 and Corollary 17.11 were proved by Resco, Stafford, and Warfield for a ring R in which all prime factors are bounded and Goldie on both sides [1986, Theorem 4 and Corollary 4.1].

Primitivity of Polynomial Rings over Division Rings. Resco proved that if D is a division ring of transcendence degree at least n over its center, then $D[x_1, \dots, x_n]$ is primitive [1979, Theorem 3.13], and then Amitsur and Small proved that $D[x_1, \dots, x_n]$ is primitive if and only if the matrix-transcendence degree of D is at least n [1978, Theorem 2].

Krull Dimension of Tensor Products with Rational Function Fields. Theorem 17.14 and Corollary 17.15 are due to Resco [1979, Theorems 3.15, 3.16, Remark 3.17].

Transcendence Degree of Weyl Division Algebras. Parts (a) and (b) of Theorem 17.16 are due to Resco [1979, Theorem 4.2] and part (c) to Amitsur and Small [1978, Theorem 3].

Nonisomorphic Weyl Division Algebras. Gelfand and Kirillov proved that if k is a field of characteristic zero and m, n, p, q are positive integers, then the quotient division rings of $A_m(k[x_1, \dots, x_p])$ and $A_n(k[x_1, \dots, x_q])$ are isomorphic as k-algebras if and only if $m = n$ and $p = q$ [1966, Théorème 2].

Nonisomorphic Quantum Weyl Division Algebras. Lemma 17.19 and Theorem 17.20 were proved by Alev and Dumas [1994, Théorème 3.10(b), Corollaire 3.11(c)], along with the result that $D_1^q(k) \not\cong D_1(k)$ for any field k and any $q \neq 1$ [1994, Corollaire 3.11(a)].

Finite Generation of Field Extensions. Vámos proved Theorem 17.22 in [1978, Theorem 11].

Finite Generation of Subfields of Division Algebras. Theorem 17.25 was proved for the case where $T = L[x; \delta]$ by Resco, Small, and Wadsworth, along with Corollary 17.26 [1979, Theorems 5, 4].

Appendix.
Some Test Problems for Noetherian Rings

In this appendix, we briefly sketch some open questions in the theory of noetherian rings. These are not necessarily problems whose solutions would significantly advance the theory. Rather, we have concentrated on problems that seem to be good test questions, in the sense that a well-developed structure theory for a class of noetherian rings (e.g., FBN rings, or noetherian rings with the second layer condition) ought to be strong enough to answer some of these questions within that class. The questions to follow all have positive answers in the class of commutative noetherian rings. At the time the first edition of the book was written, all of these questions were unsolved in the class of (two-sided) noetherian rings, but many of them had been answered negatively for one-sided noetherian rings. In the meantime, only two questions (items 15 and 16) have been fully answered, but those solutions suggested revised questions. The reader who has not studied homological algebra may wish to skip over questions 5–8.

1. Jacobson's Conjecture.

Is the intersection of the powers of the Jacobson radical in a noetherian ring R equal to zero, i.e., is $\bigcap_{n=1}^{\infty} J(R)^n = 0$?

That this holds in a commutative noetherian ring R is a well-known consequence of the Krull Intersection Theorem (see, e.g., Kaplansky [1970, Theorem 79] or Matsumura [1980, (11.D), Corollary 2]). The noncommutative question was posed for one-sided noetherian rings R by Jacobson in [1956, p. 200]; he had earlier introduced transfinite powers of $J(R)$ (the intersection of the finite powers being $J(R)^{\omega}$) and had shown that some transfinite power of $J(R)$ must be zero [1945a, Theorem 11]. Counterexamples to the one-sided question were presented by Herstein [1965] and Jategaonkar [1968, Example 1], and Jategaonkar constructed counterexamples showing that arbitrarily high transfinite powers of $J(R)$ are needed [1969, Theorem 4.6].

The question in the two-sided noetherian case has been answered positively for FBN rings (see Theorem 9.13) by Cauchon [1974, Théorème 5; 1976, Théorème I 2, p. 36] and Jategaonkar [1973, Theorem 8; 1974b, Theorem 3.7] (see also Schelter [1975, Corollary]); for noetherian rings of Krull dimension one by Lenagan [1977, Theorem 4.4]; and for noetherian rings satisfying the second layer condition (see Theorem 14.8) by Jategaonkar [1979, Theorem H; 1982, Theorem 1.8].

2. The Artin-Rees Property for Jacobson Radicals.

When does the Jacobson radical of a noetherian ring R satisfy the Artin-Rees property? In particular, does this occur if either $R/J(R)$ is artinian or R is prime?

This of course occurs if R is commutative, since then all ideals of R have the AR-property (Theorem 13.3). In general, $J(R)$ need not satisfy the AR-property. For a one-sided noetherian example in which $R/J(R)$ is artinian, use the ring of Exercise 9M. A two-sided noetherian example is the ring $\left(\begin{smallmatrix} S/yS & S/yS \\ 0 & S \end{smallmatrix}\right)$ where $S = k[[x]][y]$ for some field k.

This problem is closely linked with the previous one, for if R is a (right) noetherian ring in which $J(R)$ satisfies the (right) AR-property, then Jacobson's Conjecture holds for R. (To see this, combine the AR-property with Nakayama's Lemma.) In the reverse direction, Jategaonkar has shown that if R is a noetherian ring such that $R/J(R)$ is artinian and all factor rings of R satisfy Jacobson's Conjecture, then $J(R)$ satisfies the AR-property [1981, Proposition 5.8]. Some other consequences of the AR-property for Jacobson radicals are discussed in Exercise 14O.

3. The Descending Chain Condition for Prime Ideals.

(a) *Does a noetherian ring R satisfy the descending chain condition on prime ideals?*

(b) *Does every prime ideal in R have finite height?*

(c) *Does every nonminimal prime ideal in R contain a prime ideal of height one?*

(The *height* of a prime ideal P is the supremum of the lengths of all finite chains $P > P_1 > \cdots > P_n$ of prime ideals descending from P.)

In case R is commutative, these are standard consequences of Krull's Generalized Principal Ideal Theorem (see, e.g., Atiyah-Macdonald [1969, Corollary 11.12], Kaplansky [1970, Theorem 152], or Matsumura [1980, (12.I), Theorem 18]). For one-sided noetherian rings, the general answers are negative, as shown by examples of Jategaonkar [1969, Theorem 4.6].

These questions do have obvious positive answers in case R has finite Krull dimension or just finite classical Krull dimension. There are also positive answers for noetherian P.I. rings (see Rowen [1980, Theorem 5.2.19] or McConnell-Robson [2001, Proposition 13.7.15]).

4. Countability of Chains of Ideals or Submodules.

(a) *In a noetherian ring R, are all chains* (*i.e., totally ordered collections*) *of ideals countable?*

(b) *In a finitely generated R-module, are all chains of submodules countable?*

Both properties were proved for commutative noetherian rings by Bass [1971, Theorem 1.1]. The examples of Jategaonkar mentioned above [1969, Theorem 4.6] show that they do not generally hold in one-sided noetherian rings. This problem appears to be closely linked with the previous one, for Brookfield and Goodearl [unpublished] have shown that if R is a left noetherian ring with the DCC on prime ideals, then all chains of subbimodules in any noetherian bimodule ${}_RB_S$ are countable, and that if R is a left FBN ring with the DCC on prime ideals, then all chains of submodules in any finitely generated left R-module are countable

5. Local Rings of Finite Global Dimension.

(a) *If R is a noetherian ring of finite global dimension which is local in the sense that $R/J(R)$ is simple artinian, is R a prime ring?*

(b) *If R is a noetherian ring of finite global dimension which is local in the stricter sense that $R/J(R)$ is a division ring, is R a domain?*

That (b) holds in the commutative case is part of the Auslander-Buchsbaum-Nagata-Serre Theorem that every commutative noetherian regular local ring is a unique factorization domain (see, e.g., Kaplansky [1970, Theorem 184], Matsumura [1980, (19.B), Theorem 48], or Rotman [1979, Theorem 9.64]).

Under one-sided assumptions as in (a) or (b), Ramras showed that the answers are positive in case R has an artinian classical quotient ring [1974, Theorem 4 and Corollary 5], and he raised the question of whether R must have an artinian classical quotient ring [1974, p. 586]. He also provided positive answers in case R has left and right global dimension 2 [1974, Proposition 7]. (An example of Stafford in Chatters-Hajarnavis [1980, Example 10.10] shows, however, that a right noetherian local ring with right global dimension 2 need not be semiprime.) Both questions were answered positively by Walker in case $J(R)$ has a regular normalizing set of generators, as well as question (b) in case R is nonsingular [1972, Theorems 2.7, 2.9]. Later, question (a) was answered positively in case R is either nonsingular or integral over its center, by Brown-Hajarnavis-MacEacharn [1982, Corollary 3.3; 1983, Theorem 6.7]. Snider has verified (b) in case $\mathrm{gl.dim}(R) = 3$ [1988, Theorem].

More recently, Stafford and Zhang proved that if R is a local noetherian P.I. ring of finite global dimension, then $R \cong M_n(D)$ for some n and some domain D [1994, p. 1016, bottom]; hence, both (a) and (b) have positive answers in the P.I. case. This result and the consequences were extended to FBN rings by Teo [1997, Corollary 3.10].

6. Krull Versus Global Dimension.

Is $\mathrm{r.K.dim}(R) \leq \mathrm{r.gl.dim}(R)$ *for every noetherian ring R of finite global dimension?*

In case R is commutative, more is true: $\mathrm{K.dim}(R) = \mathrm{gl.dim}(R)$ (see, e.g., Matsumura [1980, (18.G), proof of Theorem 45] or Northcott [1962, p. 208, Theorem 24 and Corollary] for the local case; the general result follows via localization). The inequality is not in general valid for one-sided noetherian rings, as seen by looking at Jategaonkar's examples [1969, Theorem 4.6] once again: These can be principal left ideal domains, in which case they have left global dimension 1, yet they have arbitrarily high Krull dimension.

Many cases are known in which the inequality holds. In case R is semiprime and $\mathrm{r.gl.dim}(R) \leq 1$, it follows from a result of Webber [1970, Theorem 4], and, with a modification of his argument, the semiprime hypothesis may be dropped (see Chatters-Hajarnavis [1980, Theorem 8.21]). Roos has proved the inequality in case R has a nonnegative filtration such that the associated graded ring is a commutative noetherian ring with finite global dimension. (This result was not published for many years but finally appeared in a paper of Björk [1985, §§1.4, 1.8].) In a sequence of results, the inequality was verified for semiprime noetherian P.I. rings by Resco-Small-Stafford [1982, Theorem 3.2]; for FBN rings which are algebras over uncountable fields by Brown-Warfield [1984, Corollary 12]; for arbitrary noetherian P.I. rings by Goodearl-Small [1984, Theorem D]; and for arbitrary FBN rings by Teo [1997, Corollary 3.3]. For an FBN ring which is an algebra over an uncountable field, Brown has shown that $\mathrm{r.K.dim}(R)$ is actually bounded by the injective dimension of the module R_R (if it is finite) [1990, Theorem B].

7. Global Dimension via Simple Modules.

Is the (*right*) *global dimension of a noetherian ring R equal to the supremum of the projective dimensions of the simple* (*right*) *R-modules?*

In the commutative case, this follows from the fact that, for any maximal ideal M of R, the global dimension of R_M equals the projective dimension of R/M (see Matsumura [1980, (18.B), Lemma 5 and Theorem 41], Northcott [1962, p. 195, Theorem 19], or Rotman [1979, Theorem 9.52 and Corollary 9.55]). A counterexample in the one-sided noetherian case was constructed by Fields [1970, p. 348].

The equality is known for rings which are finitely generated modules over their noetherian centers (see Bass [1968, Proposition III.6.7(a)]). It was established for noetherian rings of finite global dimension by Bhatwadekar [1976, Proposition 1.1] and Goodearl [1975, Theorem 16]. Thus, only the case of infinite global dimension remains open, and the problem may be stated this way: If there is a finite bound on the projective dimensions of the simple right R-modules, is the global dimension of R finite? This has been verified for prime noetherian rings of Krull dimension one by Stafford [1982b, Lemma 2.1]; for FBN rings which are algebras over uncountable fields by Warfield [1986, Corollary 14]; and then for arbitrary FBN rings as well as all noetherian rings of Krull dimension one by Rainwater [1987, Theorem 8 and Corollary 4].

8. Finite Projective Dimension for Finitely Generated Modules.

If all simple (right) modules over a noetherian ring R have finite projective dimension, do all finitely generated (right) R-modules have finite projective dimension?

In case R is commutative, this follows from a result of Bass and Murthy [1967, Lemma 4.5]. It can also be proved in case R is a module-finite algebra over a commutative noetherian ring S, the key step being found in Bass [1968, Corollary III.6.6], which reduces the problem to the case that S is local.

9. Krull Symmetry.

(a) *Do the right and left Krull dimensions of any noetherian ring coincide?*

(b) *Do the right and left Krull dimensions of any noetherian bimodule coincide?*

The zero-dimensional case was proved by Lenagan (Corollary 8.13): A noetherian bimodule has Krull dimension zero on one side if and only if it has Krull dimension zero on the other [1975, Proposition]. That (a) holds in case R is fully bounded follows immediately from the fact that the right and left Krull dimensions of R then equal the classical Krull dimension (see Theorem 15.13). Jategaonkar verified (b) for noetherian bimodules over FBN rings (Theorem 15.15) [1974b, Theorem 2.3; 1986, Theorem 8.2.17]. Krull symmetry for factor rings of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional solvable Lie algebra $\mathfrak{g}$ over a field k of characteristic zero has been proved by Heinicke [1981, Theorem 1], and Krull symmetry for noetherian $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodules has been established in case either $\mathfrak{g}$ is algebraic, by Brown and Smith [1985, Theorem 3.3], or k is algebraically closed, by Polo [1987, Théorème 2.7].

10. Transfer Across Noetherian Bimodules.

(a) *If R and S are noetherian rings and ${}_RB_S$ is a bimodule which is finitely generated and faithful on each side, what properties transfer from R (or ${}_RB$) to S (or B_S)?*

(b) *What if, in addition, R and S are prime and B is torsionfree on each side?*

Recall that bimodule properties are particularly important in regard to ring extensions – for instance, if $R \subseteq S$ are noetherian rings such that S is finitely generated as a left R-module, then S is a faithful noetherian (R, S)-bimodule.

A number of transferable properties are known in various cases. That ${}_RB$ is artinian if and only if B_S is artinian (Corollary 8.13) was proved by Lenagan [1975, Proposition]. In case R and S are fully bounded, Jategaonkar proved that they must have the same Krull dimension (Theorem 15.15) [1974b, Theorem 2.3; 1986, Theorem 8.2.17], while if R and S satisfy the second layer condition, he showed that they must have the same classical Krull dimension (Corollary 14.5) [1982, Theorems 1.5, 1.7; 1986, Theorem 8.2.8]. Under assumption (b), Jategaonkar proved that R is semiprimitive if and only if

S is semiprimitive [1981, Theorem 6.1; 1986, Theorem 5.2.15], while Letzter proved that if R is right primitive, then so is S [1989, Lemma 1.3] (see Theorem 8.18 for proofs of both results using common methods). Also under assumption (b), Warfield proved that R is a G-ring if and only if S is a G-ring [1991, Theorem 1]. For some other transferable properties, see Jategaonkar [1986, Section 5.2].

Warfield also showed that a number of properties transfer across B "up to trace ideals," where the *trace ideal* $\operatorname{Tr}({}_RB)$ is the sum of the images of all homomorphisms from ${}_RB$ to ${}_RR$, and $\operatorname{Tr}(B_S)$ is defined symmetriclly. These transfers work perfectly in case $\operatorname{Tr}({}_RB) = R$ or $\operatorname{Tr}(B_S) = S$ as appropriate; otherwise, assumptions on $R/\operatorname{Tr}({}_RB)$ or $S/\operatorname{Tr}(B_S)$ are needed. For example, if S and $R/\operatorname{Tr}({}_RB)$ are both right fully bounded (respectively, both Jacobson rings), then R is right fully bounded (respectively, a Jacobson ring) [1992, Propositions 3.1, 1.3]. If $\operatorname{Tr}(B_S) = S$, then $\text{Cl.K.dim}(S) \leq \text{Cl.K.dim}(R)$ [1992, Theorem 4.3], while if $R/\operatorname{Tr}({}_RB)$ is artinian, then $\text{r.K.dim}(S) \geq \text{r.K.dim}(R)$ [1992, Theorem 4.6].

11. Incomparability in Cliques.

(a) *Are distinct prime ideals in the same clique of a noetherian ring R always incomparable?*

(b) *Are distinct linked prime ideals in R always incomparable?*

Both statements of course hold if R is commutative, since then all cliques are singletons (Exercise 12F). They also hold in case R satisfies the second layer condition (Corollary 14.6), which was proved by Jategaonkar [1982, Theorem 1.8; 1986, Theorem 8.2.4]. A positive answer to question 9(b) would imply positive answers to these questions, since from the bimodule Krull symmetry it would follow that $\text{K.dim}(R/P) = \text{K.dim}(R/Q)$ for prime ideals P and Q in the same clique.

12. Localizability of Cliques.

Under what conditions is a clique X of prime ideals in a noetherian ring R (classically) localizable?

It is an open question whether localizable cliques are necessarily classically localizable. The main characterization of classical localizability is due to Jategaonkar: X is classically localizable if and only if X satisfies the second layer condition and the intersection condition [1986, Theorem 7.2.2]. Hence, the key problem is the intersection condition. In case R is an algebra over an uncountable field and X satisfies the generic regularity condition, the intersection condition was proved by Stafford [1987, Lemma 4.4] and Warfield [1986, Theorem 8]. In particular, if R is an algebra over an uncountable field, X satisfies the second layer condition, and either R is fully bounded or there is a finite bound on the ranks of R/P for $P \in X$, then X is classically localizable (see Jategaonkar [1986, Theorem 7.2.15] and Stafford [1987, Proposition 4.5]). Müller has proved that if R is a P.I. ring and a finitely generated algebra over a field (not necessarily uncountable), then all cliques in R are classically localizable [1985, Theorem 10].

13. Prime Middle Annihilators.

(a) *In a noetherian ring R, are there only finitely many prime middle annihilator ideals?*

(b) *Are there only finitely many maximal* (*proper*) *middle annihilator ideals?*

(A *middle annihilator ideal* of R is any ideal of the form $\{r \in R \mid ArB = 0\}$, where A and B are ideals of R.)

Positive answers have been obtained for (b) by Krause in case R satisfies certain Krull symmetry and primary decomposition conditions (which hold for FBN rings) [1980, Theorem 6], and for (a) by Small and Stafford in case R is fully bounded [1982, p. 417]; these cases were then subsumed by Dean's result that any ring embeddable in a left artinian ring has only finitely many middle annihilators [1988, Theorem]. In addition, Small and Stafford answered these questions positively in case $R \otimes_{Z(R)} R^{\mathrm{op}}$ has ACC on ideals (where $Z(R)$ is the center of R) [1982, Theorem 6.5], and Goldie and Krause gave positive answers in case the lattice of left annihilator ideals in R is modular [1987, Corollary 1.8].

14. Nilpotence Modulo One-Sided Ideals.

If I and J are right ideals in a noetherian ring R, and if each element of I has a power which lies in J, is some power of I contained in J?

In case J is a two-sided ideal, the answer is positive by Levitzki's Theorem (see Theorem 6.21 and Exercise 6M). The one-sided property was conjectured by Herstein, who proved it in case either R is a P.I. ring or there is a fixed positive integer n such that J contains the n-th power of each element of I [1966, Theorems 1, 2]. He also proved it in case R is either right or left artinian (not necessarily noetherian on the other side) [1986, Theorems 1.1, 1.5]. Some further cases were verified by Stafford [1990, Theorem], namely when $RI = R$, or R is right fully bounded, or $(I + J)/J$ is artinian. Using Stafford's results, Mushrub obtained positive answers in case some power of J is idempotent [1992a, Theorem 1], or R has a right associated prime P such that $(J + P)/P$ is artinian [1992a, Theorem 2], or R is a locally finite dimensional algebra over a field [1992b, Theorem 2].

15. Extension of a Base Field.

If R is a finitely generated noetherian algebra over a field k, is $R \otimes_k K$ noetherian for every extension field K of k?

This is easily checked when K is a finitely generated extension field. In case R is a P.I. algebra, the question has been answered positively by Small [1980, Proposition 53]. Small's result is actually more general: If R is finitely generated noetherian P.I. algebra and K is any noetherian algebra, then $R \otimes_k K$ is noetherian. Some other positive cases follow from a theorem of de Jong which appeared in a paper of Artin-Small-Zhang [1999, Theorem 5.1]. In particular, if $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded noetherian algebra over an algebraically closed field k, with $\dim_k(R_n) < \infty$ for all n, then $R \otimes_k K$ is noetherian for all fields $K \supseteq k$.

This problem was settled in the negative by Resco and Small [1993, Theorems 1,2], who constructed a simple right and left principal ideal domain R with center k such that R is generated (as a k-algebra) by two elements and such that $R \otimes_k K$ is not noetherian for some field extension $K \supset k$. They also showed that R is not finitely presented as a k-algebra [1993, Theorem 3], which leaves open the following restricted version of the problem:

15′. *If R is a finitely presented noetherian algebra over a field k, is $R \otimes_k K$ noetherian for every extension field K of k?*

16. Tensor Products of Noetherian Algebras.

If R and S are finitely generated noetherian algebras over a field k, is $R \otimes_k S$ noetherian?

This has been proved in case either R or S is P.I. by Small [1980, Proposition 53]. The problem was settled in the negative by Resco and Small [1993, Theorems 1,2], with the same algebra R mentioned above; in fact, they showed that $R \otimes_k R$ is not noetherian. The following restricted version of the question remains open:

16′. *If R and S are finitely presented noetherian algebras over a field k, is $R \otimes_k S$ noetherian?*

17. Classical Krull Dimension of Polynomial Rings.

If R is a nonzero noetherian ring, is $\mathrm{Cl.K.dim}(R[x]) = \mathrm{Cl.K.dim}(R) + 1$*?*

This is well known for commutative noetherian rings of finite Krull dimension (see, e.g., Matsumura [1980, (14.A), Theorem 22]). In the noncommutative case, certainly

$$\mathrm{Cl.K.dim}(R[x]) \geq \mathrm{Cl.K.dim}(R) + 1$$

(use Lemma 14.2). If R is fully bounded, the reverse inequality also holds, since then $\mathrm{Cl.K.dim}(R) = \mathrm{r.K.dim}(R)$ (Theorem 15.13), while

$$\mathrm{Cl.K.dim}(R[x]) \leq \mathrm{r.K.dim}(R[x]) = \mathrm{r.K.dim}(R) + 1$$

in any case (Theorem 15.17).

18. Symmetry of Primitivity.

Is every right primitive noetherian ring also left primitive?

This fails for non-noetherian rings, as shown by an example of Bergman [1964].

19. Generating Right Ideals in Simple Noetherian Rings.

Can every right ideal in a simple noetherian ring be generated by two elements?

Stafford has proved this for the Weyl algebras $A_n(k)$ over any field k of characteristic zero [1978, Corollary 3.2]. Recall also that, in a simple noetherian ring of right Krull dimension n, every right ideal can be generated by $n+1$ elements (Corollary 16.8).

20. Torsionfree Modules over Simple Noetherian Rings.

Over a simple noetherian ring R, is every finitely generated torsionfree right module isomorphic to a direct sum of right ideals of R?

Stafford has proved this for the Weyl algebras $A_n(k)$ over any field k of characteristic zero [1978, Theorem 3.3].

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