On the anti-Yetter-Drinfeld module-contramodule correspondence.

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Abstract

We study a functor from anti-Yetter Drinfeld modules to contramodules in the case of a Hopf algebra H. This functor is unpacked from the general machinery of [7]. Some byproducts of this investigation are the establishment of sufficient conditions for this functor to be an equivalence, verification that the center of the opposite category of H-comodules is equivalent to anti-Yetter Drinfeld modules in contrast to [5] where the question of H-modules was addressed, and the observation of two types of periodicities of the generalized Yetter-Drinfeld modules introduced in [4]. Finally, we give an example of a symmetric 2-contratrace on H-comodules that does not arise from an anti-Yetter Drinfeld module.

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1 Introduction.

This paper grew out of the author's attempts to better understand contramodules at least in some simple examples. The simplest case being the Hopf algebra kG where G is a discrete infinite group. Contramodules over a coalgebra were introduced by Eilenberg and Moore in 1965 and can be viewed either as algebraic structures allowing infinite combinations or a better behaved notion than that of modules over the dual algebra (see Remark 4.6). They do not strictly speaking generalize comodules, but do have a non-trivial intersection with them. In our investigations we found

[6] to be very helpful, in fact the phenomenon of this underived comodule-contramodule correspondence without the anti-Yetter-Drinfeld enhancement is investigated there as well.

The introduction of anti-Yetter-Drinfeld contramodule coefficients to the Hopf-cyclic cohomology theory in [2] that followed the definition of anti-Yetter-Drinfeld module coefficients in [3] can in retrospect be conceptually understood as being completely natural since they are seen to be exactly corresponding to the representable symmetric 2-contratraces, see [7] and [5]. The latter form a well behaved class of Hopf-cyclic coefficients explored in [4] and [7], that lead directly to Hopf-cyclic type cohomology theories.

Roughly speaking, the category of stable anti-Yetter-Drinfeld modules consists of H-modules and comodules such that the two structures are compatible in a way that ensures that they form the center of a certain bimodule category [4]. A similar statement with contramodule structure replacing the comodule one can be made about anti-Yetter-Drinfeld contramodules. In general understanding objects in these categories is not a simple task, however in the case of $H = kG$ the former category is known to consist of G-graded G-equivariant vector spaces, i.e., $\oplus_{g\in G} M_g$ with $x : M_g \to M_{xgx^{-1}}$. Stability, a condition that ensures cyclicity, translates to $x = Id_{M_x}$. We could find no similarly simple description of the anti-Yetter-Drinfeld contramodule category in the literature. It turns out, Corollary 4.10, that this category is also equivalent to G-graded G-equivariant vector spaces but the objects are now $\prod_{g\in G} M_g$. The Theorem 4.5 is a more general case of this correspondence.

The above anti-Yetter-Drinfeld module-contramodule correspondence was the motivation for the rest of the results in this paper. Namely, the Proposition 4.7 shows that the equivalence arises from a functor $M \mapsto \widehat{M}$ from comodules to contramodules. This functor can be found in [6] but arose independently from the considerations of [7] which furthermore demonstrate that it works on the anti-Yetter-Drinfeld versions as well. More precisely, for M a stable anti-Yetter-Drinfeld module we consider $\mathcal{F}(-) = \mathrm{Hom}(-, M)^H$ which is a symmetric 2-contratrace on H-comodules, i.e., a contravariant functor from $\mathcal{M}^H$ to Vec subject to a trace-like symmetry. Its pullback to the category ${}_H\mathcal{M}$ of H-modules is $\mathrm{Hom}_H(-, \widehat{M})$. The pullback construction reduces in this case to the observation that $H \in \mathcal{M}^H$ is an algebra and the category of H-bimodules in $\mathcal{M}^H$ is equivalent to ${}_H\mathcal{M}$. The pullback $\mathrm{Hom}_H(-, \widehat{M})$ is obtained as $\mathcal{F}_H$, i.e., the equalizer of the action maps $\mathcal{F}(V) \to \mathcal{F}(H \otimes V)$ and $\mathcal{F}(V) \to \mathcal{F}(V \otimes H)$ with the targets identified via the symmetry of $\mathcal{F}$. Though this can be used as the definition of $\widehat{M}$, we give an explicit construction of both the contramodule structure, essentially agreeing with [6], and the H-action on $\widehat{M} = \mathrm{Hom}(H, M)^H$.

It turns out that, not surprisingly, $M \mapsto \widehat{M}$ is not always an equivalence, but it does have a left adjoint, that we found in [6] and upgraded to the anti-Yetter-Drinfeld setting here. The key object when studying the question of equivalence is the ideal of left integrals for H as introduced in [8]. This object seems to be the first example of a generalized Yetter-Drinfeld module of charge other than 1 or -1, its charge is 2. These were introduced in [4] without any hope that anything other than ± 1 would be useful. In fact, the conditions for the comodule-contramodule correspondence are closely related to the presence of a 2-periodicity of the charges, see Remark 5.7. Furthermore, in studying the question of stability of anti-Yetter-Drinfeld modules/contramodules and the generalization of this concept to more general charges (in a way that was necessarily different from [4]) we observed a second kind of periodicity within a generalized Yetter-Drinfeld category of a fixed charge. The Remark 5.8 describes an action of $\mathbb{Z}/i\mathbb{Z}$ on Yetter-Drinfeld modules of charge $i-1$ and Yetter-Drinfeld contramodules of charge $i+1$. This action is compatible with the generalized $M \mapsto \widehat{M}$ that sends Yetter-Drinfeld modules of charge $i-1$ to Yetter-Drinfeld contramodules of charge $i+1$; the case of $i=0$ is the usual anti-Yetter-Drinfeld situation.

Identifying categories of interest with centers of bimodule categories such as was done in [4] and [5] is carried through this paper as well. We point to the summary Theorem 3.8 that is of the [4] flavor, and the Corollary 2.8 of the [5] flavor as examples. One of the natural questions that arose after [4] was if symmetric 2-contratraces give a more general class of coefficients for Hopf-cyclic type theories even in the case of Hopf algebras. It turns out [5] that for the case of ${}_H\mathcal{M}$ the representable symmetric 2-contratraces are equivalent to the anti-Yetter-Drinfeld contramodules, and similarly (Corollary 2.9) for the case of $\mathcal{M}^H$ the representable symmetric 2-contratraces are equivalent to the anti-Yetter-Drinfeld modules. Thus one needs only find a non-representable example of a contratrace in order to have a new coefficient in the H-comodule case. This is explained in Section 4.2.

The paper is arranged as follows: Section 2 is devoted to the establishment of the fact that $\mathcal{M}^H$ is biclosed, and thus it makes sense to consider the opposite bimodule category $\mathcal{M}^{Hop}$ with the adjoint action. The center is shown to consist of anti-Yetter-Drinfeld modules; this identifies symmetric representable 2-contratraces with stable anti-Yetter-Drinfeld modules.

Finally we introduce the functor $M \mapsto \widehat{M}$. In Section 3 we demonstrate that the adjoint pair of functors between comodules and contramodules: $N \mapsto N'$ and $M \mapsto \widehat{M}$ induce the same between the anti-Yetter-Drinfeld versions and also the stable anti-Yetter-Drinfeld versions. Section 4 deals with the question of equivalence established by $M \mapsto \widehat{M}$ and ends with an example of a new coefficient for the case of $H = kG$. In Section 5 we extend $M \mapsto \widehat{M}$ to the generalized Yetter-Drinfeld modules and discuss two types of periodicities and the compatibility of $M \mapsto \widehat{M}$ with them.

Some things to keep in mind: for a coalgebra C we use the following version of Sweedler notation $\Delta(c) = c^1 \otimes c^2$. For a right comodule N over C we use $\rho(n) = n_0 \otimes n_1$. All Hopf algebras have invertible antipodes S and are over a field k of characteristic 0. We denote by $\mathrm{Hom}(-,-)^H$ and $\mathrm{Hom}_H(-,-)$ the morphisms in $\mathcal{M}^H$ and ${}_H\mathcal{M}$ respectively, while $\mathrm{Hom}(-,-)$ stands for k-linear maps.

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2 The category of H-comodules.

This section is primarily dedicated to the establishment of the fact that the monoidal category $\mathcal{M}^H$ of H-comodules is biclosed, and the analysis of the center of the bimodule category over $\mathcal{M}^H$ resulting from considering $\mathcal{M}^{Hop}$. This establishes an analogue of a result in [5] describing the center as the category of aYD-modules for H.

2.1 Internal Homs in the category of H-comodules.

Motivated by the existence of internal Homs in ${}_H\mathcal{M}$, and thus the possibility of describing representable contratraces on ${}_H\mathcal{M}$ as central elements in the opposite category, we will now address the same question in $\mathcal{M}^H$, the monoidal category of H-comodules.

Since in the finite dimensional H case, we have that $\mathcal{M}^H \simeq {}_{H^*}\mathcal{M}$ so we have a suggestive way of obtaining the required formulas. We note that some modifications do need to be made to account for possible infinite dimensionality of H.

For $W, V \in \mathcal{M}^H$ consider $\rho : \mathrm{Hom}(W, V) \to \mathrm{Hom}(W, V \otimes H)$ given by

$$\rho f(w) = f(w_0)_0 \otimes f(w_0)_1 S(w_1). \tag{2.1}$$

Definition 2.1. *Define* $\mathrm{Hom}^l(W, V)$ *as the subspace of* $\mathrm{Hom}(W, V)$ *that consists of* f *such that* $\rho f \in \mathrm{Hom}(W, V) \otimes H$.

We can define two maps

$$\text{“}Id \otimes \Delta\text{”} = (Id \otimes \Delta) \circ -$$

and

$$\text{“}\rho \otimes Id\text{”} = (Id_V \otimes m \otimes Id_H) \circ (Id_{V \otimes H} \otimes \sigma_{H,H}) \circ (\rho_V \otimes Id_{H \otimes H}) \circ (f \otimes S) \circ \rho_W$$

from $\mathrm{Hom}(W, V \otimes H)$ to $\mathrm{Hom}(W, V \otimes H \otimes H)$. The latter can be written down more manageably as follows: let $f(w) = w^{(1)} \otimes w^{(2)}$ then

$$\text{“}\rho \otimes Id\text{”}(f)(w) = ((w_0)^{(1)})_0 \otimes ((w_0)^{(1)})_1 S(w_1) \otimes (w_0)^{(2)}.$$

A direct computation shows that

$$\text{“}Id \otimes \Delta\text{”} \circ \rho = \text{“}\rho \otimes Id\text{”} \circ \rho. \tag{2.2}$$

Note that when restricted to $\mathrm{Hom}(W, V) \otimes H$ the maps “$Id \otimes \Delta$” and “$\rho \otimes Id$” are actually $Id \otimes \Delta$ and $\rho \otimes Id$ respectively. The formula (2.2) has two important and immediate consequences: $\rho : \mathrm{Hom}^l(W, V) \to \mathrm{Hom}^l(W, V) \otimes H$, whereas before we only knew that it lands in $\mathrm{Hom}(W, V) \otimes H$, and ρ is a coaction.

It is not hard to see that $\mathrm{Hom}^l(W, V)$ is contravariant in W and covariant in V. More precisely, let $\phi \in \mathrm{Hom}(W', W)^H$ and $\theta \in \mathrm{Hom}(V, V')^H$, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(W, V) \arrow[r, "\theta \circ - \circ \phi"] \arrow[d, "\rho"] & \mathrm{Hom}(W', V') \arrow[d, "\rho"] \\
\mathrm{Hom}(W, V \otimes H) \arrow[r, "(\theta \otimes Id) \circ - \circ \phi"] & \mathrm{Hom}(W', V' \otimes H)
\end{tikzcd}$$

and so $\theta \circ - \circ \phi : \mathrm{Hom}^l(W, V) \to \mathrm{Hom}^l(W', V')$ and it is a map of H-comodules.

Lemma 2.2. *We have natural identifications*

$$\mathrm{Hom}(T \otimes W, V)^H \simeq \mathrm{Hom}(T, \mathrm{Hom}^l(W, V))^H,$$

i.e., $\mathrm{Hom}^l(-,-)$ *is the left internal Hom in* $\mathcal{M}^H$.

Proof. Note that $f \in \mathrm{Hom}(T \otimes W, V)^H$ if and only if

$$f(t \otimes w)_0 \otimes f(t \otimes w)_1 = f(t_0 \otimes w_0) \otimes t_1 w_1. \tag{2.3}$$

On the other hand $\phi \in \mathrm{Hom}(T, \mathrm{Hom}^l(W, V))^H$ if and only if

$$\rho\phi_t = \phi_{t_0} \otimes t_1 \tag{2.4}$$

where $\phi_t \in \mathrm{Hom}^l(W, V)$.

Let $f \in \mathrm{Hom}(T \otimes W, V)^H$ then if we define $f_t(w) = f(t \otimes w)$ we have

$$\begin{aligned}
\rho f_t(w) &= f(t \otimes w_0)_0 \otimes f(t \otimes w_0)_1 S(w_1) \\
&= f(t_0 \otimes w_{0,0}) \otimes t_1 w_{0,1} S(w_1) \\
&= f(t_0 \otimes w_0) \otimes t_1 w_1 S(w_2) \\
&= f(t_0 \otimes w) \otimes t_1 \\
&= f_{t_0} \otimes t_1.
\end{aligned}$$

So that $t \mapsto f_t \in \mathrm{Hom}(T, \mathrm{Hom}^l(W, V))^H$. Conversely, if $\phi \in \mathrm{Hom}(T, \mathrm{Hom}^l(W, V))^H$ then define $\phi(t \otimes w) = \phi_t(w)$ then $\phi(t_0 \otimes w) \otimes t_1 = \phi(t \otimes w_0)_0 \otimes \phi(t \otimes w_0)_1 S(w_1)$ so that

$$\begin{aligned}
\phi(t_0 \otimes w_0) \otimes t_1 w_1 &= \phi(t \otimes w_{0,0})_0 \otimes \phi(t \otimes w_{0,0})_1 S(w_{0,1}) w_1 \\
&= \phi(t \otimes w_0)_0 \otimes \phi(t \otimes w_0)_1 S(w_1) w_2 \\
&= \phi(t \otimes w)_0 \otimes \phi(t \otimes w)_1
\end{aligned}$$

and thus $t \otimes w \mapsto \phi_t(w) \in \mathrm{Hom}(T \otimes W, V)^H$.

So the usual bijection $f(t \otimes w) = f_t(w)$ establishes a natural identification between $\mathrm{Hom}(T \otimes W, V)^H$ and $\mathrm{Hom}(T, \mathrm{Hom}^l(W, V))^H$ as required. □

Remark 2.3. *From now on we will denote the coaction* ρ *of* (2.1) *by* ρ^l *since*

$$\rho^l : \mathrm{Hom}^l(W, V) \to \mathrm{Hom}^l(W, V) \otimes H.$$

Remark 2.4. *Note that we have a natural fully faithful embedding of* $\mathcal{M}^H$ *into* ${}_{H^*}\mathcal{M}$*. The right adjoint to it can be used to define* $\mathrm{Hom}^l(W,V)$*. Namely, the formula* (2.1) *defines a left* H^**-module structure on* $\mathrm{Hom}(W,V)$ *via* $\chi f = (Id_V \otimes \chi)(\rho f)$*. Then it is easy to see that*

$$\mathrm{Hom}(W,V)^{rat} = \mathrm{Hom}^l(W,V)$$

where $(-)^{rat}$ *is the right adjoint that features prominently in [8].*

Repeating the above considerations nearly verbatim, we define the right internal Hom for $\mathcal{M}^H$ as follows. Begin by defining

$$\rho^r f(w) = f(w_0)_0 \otimes S^{-1}(w_1) f(w_0)_1. \tag{2.5}$$

Definition 2.5. *Define* $\mathrm{Hom}^r(W,V)$ *as the subspace of* $\mathrm{Hom}(W,V)$ *that consists of* f *such that* $\rho^r f \in \mathrm{Hom}(W,V) \otimes H$*.*

We again obtain that $\rho^r : \mathrm{Hom}^r(W,V) \to \mathrm{Hom}^r(W,V) \otimes H$ and is a coaction. Furthermore, we have natural adjunctions:

$$\mathrm{Hom}(W \otimes T, V)^H \simeq \mathrm{Hom}(T, \mathrm{Hom}^r(W,V))^H.$$

As usual we now have the opposite category $\mathcal{M}^{Hop}$ with

$$V \triangleleft W = \mathrm{Hom}^r(W,V), \qquad \text{and} \qquad W \triangleright V = \mathrm{Hom}^l(W,V)$$

and we may examine its center $\mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{Hop})$.

Remark 2.6. *We observe that if* $V \in \mathcal{M}^H$ *is finite dimensional then*

$$\mathrm{Hom}^l(V,W) = W \otimes V^*$$

and

$$\mathrm{Hom}^r(V,W) = {}^*V \otimes W$$

where $V^* = \mathrm{Hom}^l(V,k)$ *and* ${}^*V = \mathrm{Hom}^r(V,k)$ *and both* V^* *and* ${}^*V$ *are* $\mathrm{Hom}_k(V,k)$ *as vector spaces. So that* $\mathcal{M}^H_{fd}$ *is rigid with*

$${}^{**}V = V^{S^{-2}} \quad \textit{and} \quad V^{**} = V^{S^2},$$

where $V^{S^{2i}}$ *denotes the* H*-comodule with the coaction modified by* S^{2i}*.*

2.2 The center of the opposite bimodule category.

We recall from [3] that a left-right anti-Yetter-Drinfeld module M over a Hopf algebra H is a left H-module and a right H-comodule satisfying

$$(hm)_0 \otimes (hm)_1 = h^2 m_0 \otimes h^3 m_1 S(h^1). \tag{2.6}$$

It is stable if $m_1 m_0 = m$ for all $m \in M$.

Recall, for example from [7], the notions of the opposite bimodule category and of the center of a bimodule category.

Proposition 2.7. *The category of aYD-modules for H is equivalent to $\mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{Hop})$.*

Proof. Let M be an aYD-module and define the central structure

$$\tau : \mathrm{Hom}(W, M) \to \mathrm{Hom}(W, M)$$

$$\tau f(w) = w_1 f(w_0).$$

Note that τ is invertible with $\tau^{-1} f(w) = S^{-1}(w_1) f(w_0)$. Define a map "$\tau \otimes Id$" : $\mathrm{Hom}(W, M \otimes H) \to \mathrm{Hom}(W, M \otimes H)$ by

$$\text{“}\tau \otimes Id\text{”} = (a \otimes Id) \circ \sigma_{M,H,H} \circ (f \otimes Id) \circ \rho_W$$

so that if $f(w) = w^{^{(1)}} \otimes w^{^{(2)}}$ then

$$\text{“}\tau \otimes Id\text{”}(f)(w) = w_1 (w_0)^{^{(1)}} \otimes (w_0)^{^{(2)}}.$$

A direct computation (using the aYD condition (2.6)) demonstrates that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(W, M) \arrow[r, "\tau"] \arrow[d, "\rho^l"] & \mathrm{Hom}(W, M) \arrow[d, "\rho^r"] \\ \mathrm{Hom}(W, M \otimes H) \arrow[r, "\text{“}\tau\otimes Id\text{”}"] & \mathrm{Hom}(W, M \otimes H) \end{tikzcd} \tag{2.7}$$

and since "$\tau \otimes Id$" restricted to $\mathrm{Hom}(W, M) \otimes H$ is actually $\tau \otimes Id$, so in fact $\tau : \mathrm{Hom}^l(W, M) \to \mathrm{Hom}^r(W, M)$ and it is an isomorphism in $\mathcal{M}^H$.

Observe that τ is natural in W, i.e., if $\phi : W' \to W$ is a morphism in $\mathcal{M}^H$ then $\phi(w_0) \otimes w_1 = \phi(w)_0 \otimes \phi(w)_1$ so that $w_1 f(\phi(w_0)) = \phi(w)_1 f(\phi(w)_0)$ and $\tau(f \circ \phi) = \tau f \circ \phi$.

If $\theta : M \to M'$ is a map of aYD-modules, then

$$\theta \circ \tau f(w) = \theta(w_1 f(w_0)) = w_1 \theta f(w_0) = \tau(\theta \circ f)(w)$$

so that a map of aYD-modules induces a map of central elements.

To check the commutativity of

$$\begin{tikzcd}
W \triangleright (V \triangleright M) \arrow[r, "Id \triangleright \tau"] & W \triangleright (M \triangleleft V) \arrow[r, "\tau \triangleleft Id"] & (M \triangleleft W) \triangleleft V \arrow[d] \\
(W \otimes V) \triangleright M \arrow[u] \arrow[rr, "\tau"] & & M \triangleleft (W \otimes V)
\end{tikzcd} \tag{2.8}$$

is to check that going along the bottom and obtaining $w \otimes v \mapsto (w_1 v_1) f(w_0 \otimes v_0)$ is the same as the long way around which gives $w \otimes v \mapsto w_1(v_1 f_{w_0}(v_0)) = w_1(v_1 f(w_0 \otimes v_0))$; and they are the same by the usual H-action axiom. Similarly, the unitality of the action implies that $k \triangleright M \to M \triangleleft k$ is the identity since $\tau : m \mapsto 1m$.

What has been shown so far is that if M is an aYD-module, then $(M, \tau) \in \mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{H op})$ and any $\theta : M \to M'$ a morphism of aYD-modules induces a morphism between the corresponding central elements.

Conversely, let $M \in \mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{H op})$ so that we have natural isomorphisms $\tau : \mathrm{Hom}^l(W, M) \to \mathrm{Hom}^r(W, M)$. Note that

$$\mathrm{Hom}(W, M) = \varprojlim_{\alpha} \mathrm{Hom}^l(W_\alpha, M)$$

where $W_\alpha \subset W$ is a finite dimensional sub-comodule since any $w \in W$ is contained in such an W_α and so

$$\mathrm{Hom}(W, M) = \mathrm{Hom}(\varinjlim_{\alpha} W_\alpha, M) = \varprojlim_{\alpha} \mathrm{Hom}(W_\alpha, M) = \varprojlim_{\alpha} \mathrm{Hom}^l(W_\alpha, M).$$

So we have a $\tau : \mathrm{Hom}(W, M) \to \mathrm{Hom}(W, M)$ that satisfies a version of all the properties that make the original τ so useful. Denote by r the composition

$$M \to \mathrm{Hom}(H, M) \to \mathrm{Hom}(H, M)$$

so that $m \mapsto \tau(h \mapsto \epsilon(h)m)$. Define

$$hm = r_m(h). \tag{2.9}$$

Note that we needed to use $\mathrm{Hom}(H, M)$ instead of $\mathrm{Hom}^l(H, M)$ since $h \mapsto \epsilon(h)m$ is not in $\mathrm{Hom}^l(H, M)$.

By the unitality of τ we have $ev_1 \circ \tau = ev_1$ so that

$$1m = r_m(1) = ev_1 \tau(h \mapsto \epsilon(h)m) = ev_1(h \mapsto \epsilon(h)m) = \epsilon(1)m = m.$$

Furthermore by the "associativity" of τ, i.e., the diagram (2.8) and its naturality, we have

$$\begin{aligned}
(xy)m &= r_m(xy) \\
&= \tau(h \mapsto \epsilon(h)m)(xy) \\
&= \tau(h \otimes h' \mapsto \epsilon(hh')m)(x \otimes y) \\
&= \tau(h \mapsto \tau(h' \mapsto \epsilon(hh')m)(y))(x) \\
&= \tau(h \mapsto \epsilon(h)\tau(h' \mapsto \epsilon(h')m)(y))(x) \\
&= \tau(h \mapsto \epsilon(h)r_m(y))(x) \\
&= r_{r_m(y)}(x) \\
&= x(ym).
\end{aligned}$$

Let $\theta : M \to M'$ be a map in the center, then we have

$$\begin{tikzcd}
M \arrow[r, "-\circ\epsilon"] \arrow[d, "\theta"] & \mathrm{Hom}(H, M) \arrow[r, "\tau"] \arrow[d, "\theta\circ-"] & \mathrm{Hom}(H, M) \arrow[d, "\theta\circ-"] \\
M' \arrow[r, "-\circ\epsilon"] & \mathrm{Hom}(H, M') \arrow[r, "\tau"] & \mathrm{Hom}(H, M')
\end{tikzcd}$$

where the left square commutes trivially and the right one commutes by definition, so that

$$\theta(hm) = \theta(r_m(h)) = r_{\theta(m)}(h) = h\theta(m).$$

Before proving that the H-action defined above satisfies the aYD-module condition (2.6) we will show that the definition of the action from τ and vice versa are mutually inverse. Let an H-action be given, then we set $\tau f(w) = w_1 f(w_0)$ so that the action becomes

$$r_m(h) = \tau(x \mapsto \epsilon(x)m)(h) = h^2\epsilon(h^1)m = hm,$$

i.e., the original action. On the other hand if $\tau : \mathrm{Hom}(W, M) \to \mathrm{Hom}(W, M)$ is given and we defined the action by $hm = r_m(h) = \tau(x \mapsto \epsilon(x)m)(h)$, then we obtain the following. Let $f \in \mathrm{Hom}(W, M)$, consider $f \otimes \epsilon \in \mathrm{Hom}(\underline{W} \otimes H, M)$ where $\underline{W}$ is a trivial H-comodule. Note that the co-action map ρ_W

is a morphism in $\mathcal{M}^H$ from W to $\underline{W} \otimes H$. So $\tau(f \otimes \epsilon \circ \rho_W) = \tau(f \otimes \epsilon) \circ \rho_W$ and the former is τf while the latter is

$$\begin{aligned} w \mapsto & \tau(w \otimes h \mapsto f(w)\epsilon(h))(w_0 \otimes w_1) \\ &= \tau(w \mapsto \tau(h \mapsto f(w)\epsilon(h))(w_1))(w_0) \end{aligned}$$

and since the coaction of H on $\underline{W}$ is trivial so

$$\begin{aligned} &= \tau(h \mapsto \epsilon(h) f(w_0))(w_1) \\ &= w_1 f(w_0). \end{aligned}$$

So that no matter if we start with a τ or an H-action, we always have

$$\tau f(w) = w_1 f(w_0). \tag{2.10}$$

Now recall the diagram (2.7), and note that it now commutes essentially by definition. Let $W = H$ and keep in mind the formula (2.10). We now get that for any $f \in \mathrm{Hom}(H, M)$ we have

$$h^3 f(h^1)_0 \otimes f(h^1)_1 S(h^2) = (h^2 f(h^1))_0 \otimes S^{-1}(h^3)(h^2 f(h^1))_1$$

and let us apply it to $f(h) = \epsilon(h)m$ to obtain

$$h^2 m_0 \otimes m_1 S(h^1) = (h^1 m)_0 \otimes S^{-1}(h^2)(h^1 m)_1$$

so that

$$\begin{aligned} h^2 m_0 \otimes h^3 m_1 S(h^1) &= (h^1 m)_0 \otimes h^3 S^{-1}(h^2)(h^1 m)_1 \\ &= (h^1 m)_0 \otimes \epsilon(h^2)(h^1 m)_1 \\ &= (hm)_0 \otimes (hm)_1 \end{aligned}$$

and M satisfies the aYD-module condition (2.6).

□

Recall that we denote by $\mathcal{Z}'_{\mathcal{M}^H}(\mathcal{M}^{H op})$ the full subcategory that consists of objects such that the identity map $Id \in \mathrm{Hom}(M, M)^H$ is mapped to itself via

$$\mathrm{Hom}(M, M)^H \simeq \mathrm{Hom}(1, M \triangleright M)^H \simeq \mathrm{Hom}(1, M \triangleleft M)^H \simeq \mathrm{Hom}(M, M)^H. \tag{2.11}$$

We have a straightforward corollary:

Corollary 2.8. *The category of $saYD$-modules for H is equivalent to $\mathcal{Z}'_{\mathcal{M}^H}(\mathcal{M}^{Hop})$.*

Proof. Recall that an aYD-module M is stable if $m_1 m_0 = m$ for all $m \in M$. On the other hand considering $\tau : \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, M)$ we see that according to (2.10) we have $\tau Id(m) = m_1 m_0$ and so $\tau Id = Id$ if and only if M is stable. $\square$

Thus we have established the following:

Corollary 2.9. *The category of $saYD$-modules for H is equivalent to the category of representable symmetric* 2*-contratraces on $\mathcal{M}^H$ via*

$$M \longleftrightarrow \mathrm{Hom}(-, M)^H.$$

Contrast that with the ${}_H\mathcal{M}$ case considered in [5] where the category of representable symmetric 2-contratraces is equivalent to the more unusual *saYD-contra*modules.

2.3 A functor from $(s)aYD$-modules to $(s)aYD$-contramodules

This section is motivated by the adjunction on cyclic cohomology of [7] that we explain below. Given an *saYD*-module M, i.e., a representable symmetric 2-contratrace $\mathrm{Hom}(-, M)^H$, as a special case of the theory developed in [7], we obtain an H-module $\widehat{M}$ such that $\mathrm{Hom}_H(-, \widehat{M})$ is a representable symmetric 2-contratrace.

We will need to recall from [2] that a right C-contramodule N, where C is a counital coassociative coalgebra, is equipped with the contraaction

$$\alpha : \mathrm{Hom}(C, N) \to N$$

satisfying

$$\alpha(x \mapsto \alpha(h \mapsto f(x \otimes h))) = \alpha(h \mapsto f(h^1 \otimes h^2)) \tag{2.12}$$

for any $f \in \mathrm{Hom}(C \otimes C, N)$ and

$$\alpha(h \mapsto \epsilon(h)n) = n \tag{2.13}$$

for any $n \in N$. Furthermore, a left-right aYD-contramodule N is a left H-module and a right H-contramodule such that for all $h \in H$ and any linear map $f \in \mathrm{Hom}(H, N)$ we have

$$h\alpha(f) = \alpha(h^2 f(S(h^3) - h^1)). \tag{2.14}$$

It is called stable, i.e., an $saYD$-contramodule, if for all $n \in N$ we have $\alpha(r_n) = n$ where $r_n(h) = hn$.

We will also recall the definitions from [7]: if M is an aYD-module then

$$\widehat{M} = \mathrm{Hom}(H, M)^H$$

has a left H-action via

$$h \cdot \phi(-) = h^2 \phi(S(h^3) - h^1) \tag{2.15}$$

and furthermore $\widehat{M}$ has a contraaction $\alpha : \mathrm{Hom}(H, \widehat{M}) \to \widehat{M}$ defined as follows. Let $\theta \in \mathrm{Hom}(H, \widehat{M})$ be viewed as $h \mapsto \theta_h(-)$ then

$$\alpha\theta(h) = \theta_{h^1}(h^2). \tag{2.16}$$

It is not hard to check all these statements directly (note that the aYD-module condition for M is only used to ensure that the action (2.15) preserves the H-comodule morphisms inside $\mathrm{Hom}(H, M)$), and most importantly we can also check that α is compatible with the action in the aYD-contramodule sense, i.e., the identity (2.14) holds.

The constructions above describe a functor

$$M \mapsto \widehat{M} \tag{2.17}$$

from $(s)aYD$-modules to $(s)aYD$-contramodules. Furthermore, the functor (2.17) is a special case of the pullback of contratraces [7] and so we have the following:

Proposition 2.10. *Given an H-module algebra A and a $saYD$-module M, we have an isomorphism of cyclic cohomologies:*

$$\widehat{HC}^n_H(A, \widehat{M}) \simeq HC^{n,H}(A \rtimes H, M)$$

where the theories considered are of the derived type.

We denote by $\widehat{HC}^n_H(A, \widehat{M})$ the cyclic cohomology obtained from an algebra A and a $saYD$-contramodule $\widehat{M}$ via the associated representable symmetric 2-contratrace $\mathrm{Hom}_H(-, \widehat{M})$ on ${}_H\mathcal{M}$, while $HC^{n,H}(A \rtimes H, M)$ denotes the Hopf-cyclic cohomology of an H-comodule algebra $A \rtimes H$ with coefficients in a $saYD$-module M obtained from the representable symmetric 2-contratrace $\mathrm{Hom}(-, M)^H$ on $\mathcal{M}^H$.

Remark 2.11. *In light of the Corollary 2.9 that shows the equivalence between saYD-modules and representable symmetric* 2*-contratraces on* $\mathcal{M}^H$ *and [5] where a similar result is demonstrated for saYD-contramodules and* ${}_H\mathcal{M}$*, the Proposition 2.10 is a concrete realization of the pullback of* representable *contratraces of [7].*

3 An adjoint pair of functors.

We will now analyze the functor $M \mapsto \widehat{M}$ with a view towards establishing some sufficient conditions for it being an equivalence. Consider the category $\mathcal{M}^H$ of right H-comodules and we are interested in comparing it to the category $\widehat{\mathcal{M}^H}$ of right H-contramodules. It turns out that the functor $M \mapsto \widehat{M}$, that appeared in [7] motivated by the pullback of contratraces has already appeared in the literature on comodule-contramodule correspondences [6], but considered without the extra H-module structure that we need. We will abuse notation somewhat and not usually distinguish between $\widehat{(-)} : \mathcal{M}^H \to \widehat{\mathcal{M}^H}$ of [6] and the upgraded version of [7] mentioned above (2.17). When we do want to emphasize the difference, the latter will be denoted by $\widehat{(-)}_H$.

Furthermore, $\widehat{(-)}$ has a left adjoint [6]

$$N \mapsto N' \tag{3.1}$$

where $N' = H \odot_H N$ is the cokernel of the difference between the maps $Id \otimes \alpha$ and $h \otimes f \mapsto h^2 \otimes f(h^1)$ between $H \otimes_k \mathrm{Hom}(H, N)$ and $H \otimes_k N$:

$$H \otimes_k \mathrm{Hom}(H, N) \to H \otimes_k N \to H \odot_H N \to 0.$$

The comodule structure on N' is given by

$$(h \otimes n)_0 \otimes (h \otimes n)_1 = (h^1 \otimes n) \otimes h^2. \tag{3.2}$$

When H is finite dimensional then $N' = H \otimes_{H^*} N$ so that the notation $\odot_H$ is a bit misleading.

The adjunctions are

$$H \odot_H \mathrm{Hom}(H, M)^H \to M$$

$$h \otimes f \mapsto f(h)$$

and

$$N \to \mathrm{Hom}(H, H \odot_H N)^H$$
$$n \mapsto \{h \mapsto h \otimes n\}.$$

Remark 3.1. *Just as the functor $M \mapsto \widehat{M}$ was upgraded from the functor between comodules and contramodules to a functor between aYD-modules and aYD-contramodules by converting an H-action on M to an H-action on $\widehat{M}$, we can do the same to its left adjoint directly. Namely, define an H-action on $H \otimes_k N$ via*

$$x \cdot (h \otimes n) = x^3 h S(x^1) \otimes x^2 n \tag{3.3}$$

then one can check that if N is an aYD-contramodule, then the action is well defined on the cokernel $H \odot_H N$ and gives N' the aYD-module structure.

We will now conceptually investigate if the adjoint pair of the functors above is compatible with the extra structure that we require. More precisely, $\mathcal{M}^H$ is a tensor category in the usual way with

$$\rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1$$

for $m \otimes n \in M \otimes N$ with $M, N \in \mathcal{M}^H$. Thus $\mathcal{M}^H$ is a bimodule category over itself.

On the other hand if $N \in \widehat{\mathcal{M}^H}$ and $T \in \mathcal{M}^H_{fd}$, i.e., T is a finite dimensional H-comodule, then we can define a natural contramodule structure on both $N \otimes T$ and $T \otimes N$. Namely, due to the finite dimensionality of T, we represent elements of $\mathrm{Hom}(H, N \otimes T)$ by $f \otimes t$ with $f \in \mathrm{Hom}(H, N)$ and $t \in T$, then

$$\alpha_{N \otimes T}(f \otimes t) = \alpha_N(f(-t_1)) \otimes t_0 \tag{3.4}$$

and similarly

$$\alpha_{T \otimes N}(t \otimes f) = t_0 \otimes \alpha_N(f(t_1 -)) \tag{3.5}$$

which makes $\widehat{\mathcal{M}^H}$ into a bimodule category over $\mathcal{M}^H_{fd}$.

The following is the key technical result of this section. It describes the exact nature of the compatibility of $M \mapsto \widehat{M}$ with the $\mathcal{M}^H_{fd}$-bimodule structure on both sides.

Proposition 3.2. *Let $W \in \mathcal{M}^H$ and $T, L \in \mathcal{M}^H_{fd}$ then we have:*

$$\mathrm{Hom}(H, T \otimes W \otimes L)^H \simeq T^{S^2} \otimes \mathrm{Hom}(H, W)^H \otimes L^{S^{-2}}$$
$$t \otimes f \otimes l \mapsto t_0 \otimes f(S^2(t_1) - S^{-2}(l_1)) \otimes l_0$$

a natural isomorphism in $\widehat{\mathcal{M}^H}$.

Proof. Recall that $\mathrm{Hom}^L(H,W)$ has a left H^*-action and a right H-contraaction and they commute. Namely,

$$(\chi \cdot f)(h) = f(h^1)_0\chi(f(h^1)_1 S(h^2))$$

and

$$\alpha(h \mapsto \theta_h(-)) = \{h \mapsto \theta_{h^1}(h^2)\}.$$

One quickly checks that the map

$$\begin{aligned} \mathrm{Hom}^L(H, T\otimes W) &\to \overline{T}\otimes \mathrm{Hom}^L(H,W) \\ t\otimes f &\mapsto t\otimes f \end{aligned} \tag{3.6}$$

is an isomorphism of both H^*-modules and H-contramodules, where $\overline{T}$ has the usual H^* structure, but is considered trivial for the purposes of defining the H-contraaction on the right hand side.

On the other hand

$$\begin{aligned} \mathrm{Hom}^L(H,W)\otimes \overline{T} &\to \underline{T}\otimes \mathrm{Hom}^L(H,W) \\ f\otimes t &\mapsto t_0\otimes f(t_1-) \end{aligned} \tag{3.7}$$

is also an isomorphism of both structures where $\underline{T}$ has trivial H^* structure but non-trivially modifies the contraaction on the right hand side.

So as H-contramodules we have:

$$\begin{aligned} \mathrm{Hom}(H, T\otimes W)^H &\simeq \mathrm{Hom}_{H^*}(k, \mathrm{Hom}^L(H, T\otimes W)) \\ &\simeq \mathrm{Hom}_{H^*}(k, \overline{T}\otimes \mathrm{Hom}^L(H,W)) \\ &\simeq \mathrm{Hom}_{H^*}(k, \mathrm{Hom}^L(H,W)\otimes \overline{T^{S^2}}) \\ &\simeq \mathrm{Hom}_{H^*}(k, \underline{T^{S^2}}\otimes \mathrm{Hom}^L(H,W)) \\ &\simeq T^{S^2}\otimes \mathrm{Hom}(H,W)^H \end{aligned}$$

where $\mathrm{Hom}_{H^*}(k, T\otimes V)\simeq \mathrm{Hom}_{H^*}(k, V\otimes T^{S^2})$ is due to the rigidity of $\mathcal{M}^H_{fd}$ and the isomorphism $T^{**}\simeq T^{S^2}$.

Analogously we have:

$$\begin{aligned} \mathrm{Hom}(H, W\otimes L)^H &\simeq \mathrm{Hom}_{H^*}(k, \mathrm{Hom}^R(H, W\otimes L)) \\ &\simeq \mathrm{Hom}_{H^*}(k, \mathrm{Hom}^R(H,W)\otimes \overline{L}) \\ &\simeq \mathrm{Hom}_{H^*}(k, \overline{L^{S^{-2}}}\otimes \mathrm{Hom}^R(H,W)) \\ &\simeq \mathrm{Hom}_{H^*}(k, \mathrm{Hom}^R(H,W)\otimes \underline{L^{S^{-2}}}) \\ &\simeq \mathrm{Hom}(H,W)^H\otimes L^{S^{-2}}. \end{aligned}$$

In the latter we have used the analogues of (3.6) and (3.7); namely the isomorphisms:

$$\mathrm{Hom}^R(H, W \otimes L) \to \mathrm{Hom}^R(H, W) \otimes \overline{L}$$
$$f \otimes l \mapsto f \otimes l$$

and

$$\overline{L} \otimes \mathrm{Hom}^R(H, W) \to \mathrm{Hom}^R(H, W) \otimes \underline{L}$$
$$l \otimes f \mapsto f(-l_1) \otimes l_0.$$

The result now follows after tracing through the isomorphisms. □

Denote by $\mathcal{M}^{H\#}$ the $\mathcal{M}^H_{fd}$ bimodule category with

$$T \cdot M \cdot L = T \otimes M \otimes L^{S^2}$$

and by ${}^{\#}\widehat{\mathcal{M}^H}$ the $\mathcal{M}^H_{fd}$ bimodule category with

$$T \cdot N \cdot L = T^{S^2} \otimes N \otimes L$$

then we immediately obtain the following as a Corollary of Proposition 3.2:

Corollary 3.3. *The functors*

$$\widehat{(-)} : \mathcal{M}^{H\#} \to {}^{\#}\widehat{\mathcal{M}^H}$$

and

$$(-)' : {}^{\#}\widehat{\mathcal{M}^H} \to \mathcal{M}^{H\#}$$

are bimodule functors over $\mathcal{M}^H_{fd}$ and so induce functors between the corresponding centers of bimodule categories.

Proof. The claim about $\widehat{(-)}$ is immediate from Proposition 3.2. Since $\mathcal{M}^H_{fd}$ is rigid, the statement about $(-)'$ follows from the one about $\widehat{(-)}$ through adjunction juggling, since they are adjoint functors.

□

Remark 3.4. *The adjunction manipulations mentioned in the proof of Corollary 3.3 can be traced through to obtain an explicit analogue of Proposition 3.2 for the functor $N \mapsto N'$. Namely, for $N \in \widehat{\mathcal{M}^H}$ and $T, L \in \mathcal{M}^H_{fd}$ we have a natural isomorphism in $\mathcal{M}^H$:*

$$H \odot_H (T \otimes N \otimes L) \simeq T^{S^{-2}} \otimes (H \odot_H N) \otimes L^{S^2}$$
$$h \otimes t \otimes n \otimes l \mapsto t_0 \otimes S^{-1}(t_1) h S(l_1) \otimes n \otimes l_0.$$

As in [4], we have central interpretations of aYD objects.

Lemma 3.5. *The center of $\mathcal{M}^{H\#}$ is equivalent to the category of anti-Yetter-Drinfeld modules, namely*

$$\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^{H\#}) \simeq aYD\text{-}mod.$$

Proof. The proof proceeds very much like that of Proposition 2.7 and so we provide only a sketch. Let $M \in {}_H\mathcal{M}^H$, i.e., it is both a left module and a right comodule, and let $T \in \mathcal{M}^H_{fd}$. Consider the map

$$\tau : T \otimes M \to M \otimes T^{S^2}$$

$$t \otimes m \mapsto t_1 m \otimes t_0.$$

It is an isomorphism with inverse $m \otimes t \mapsto t_0 \otimes S(t_1)m$. It is a map in $\mathcal{M}^H$ if and only if $M \in aYD$-mod. It is immediate that $(M, \tau) \in \mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^{H\#})$.

Conversely, let $M \in \mathcal{M}^H$ such that we have natural isomorphisms $\tau_T : T \otimes M \to M \otimes T^{S^2}$ for all $T \in \mathcal{M}^H_{fd}$. Now proceed in a by now familiar fashion. We need an action $H \otimes M \to M$ which we obtain via a limit over finite dimensional subcoalgebras $C \subset H$, i.e.,

$$\begin{aligned}
\operatorname{Hom}(H \otimes M, M) &= \operatorname{Hom}((\varinjlim C) \otimes M, M) \\
&= \operatorname{Hom}(\varinjlim (C \otimes M), M) \\
&= \varprojlim \operatorname{Hom}(C \otimes M, M)
\end{aligned}$$

and the latter contains $(Id \otimes \epsilon_C) \circ \tau_C$. $\square$

Remark 3.6. *Note that what these limit arguments demonstrate is that in contrast to the H-module case considered in [5], the H-comodule case is much easier as it reduces to the rigid category $\mathcal{M}^H_{fd}$. More precisely, Proposition 2.7 shows that $\mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{Hop}) \simeq aYD$-mod by essentially showing that $\mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{Hop}) \simeq \mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^{Hop})$, but the latter is clearly $\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^{H\#})$, which as we saw above is equivalent to $\mathcal{Z}_{\mathcal{M}^H}(\mathcal{M}^{H\#})$.*

Lemma 3.7. *The center of ${}^{\#}\widehat{\mathcal{M}^H}$ is equivalent to the category of anti-Yetter-Drinfeld contramodules, namely*

$$\mathcal{Z}_{\mathcal{M}^H_{fd}}({}^{\#}\widehat{\mathcal{M}^H}) \simeq aYD\text{-}ctrmd.$$

Proof. Repeat the proof of Lemma 3.5 verbatim with the exception that

$$\tau : T^{S^2} \otimes N \to N \otimes T$$

$$t \otimes n \mapsto t_1 n \otimes t_0$$

is a map in $\widehat{\mathcal{M}^H}$ if and only if $N \in aYD$-ctrmd. ◻

We summarize this section with the following Theorem.

Theorem 3.8. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^{H\#}) \arrow[r, shift left, "\mathcal{Z}_{\mathcal{M}^H_{fd}}(\widehat{(-)})"] & \mathcal{Z}_{\mathcal{M}^H_{fd}}(^{\#}\widehat{\mathcal{M}^H}) \arrow[l, shift left, "\mathcal{Z}_{\mathcal{M}^H_{fd}}((-)')"] \\
aYD\text{-}mod \arrow[u, "Lemma\ 3.5", "\simeq"'] \arrow[r, shift left, "\widehat{(-)}_H"] & aYD\text{-}ctrmd \arrow[u, "\simeq", "Lemma\ 3.7"'] \arrow[l, shift left, "(-)'_H"]
\end{tikzcd}$$

Recall that for $M \in aYD$-mod we equip $\widehat{M}$ with (2.16) *and* (2.15)*, whereas for $N \in aYD$-ctrmd we equip N' with* (3.2) *and* (3.3)*.*

Proof. For the $\widehat{(-)}$ case we have the map $T \otimes M \to M \otimes T^{S^2}$ with $t \otimes m \mapsto t_1 m \otimes t_0$ mapping to $\mathrm{Hom}(H, T \otimes M)^H \to \mathrm{Hom}(H, M \otimes T^{S^2})^H$ with $t \otimes f \mapsto t_1 f \otimes t_0$ which maps under the identification of Proposition 3.2 to $T^{S^2} \otimes \mathrm{Hom}(H, M)^H \to \mathrm{Hom}(H, M)^H \otimes T$ with $t_0 \otimes f(S^2(t_1)-) \mapsto t_2 f(-t_1) \otimes t_0$ and the latter coincides with $t \otimes g \mapsto t_1 \cdot g \otimes t_0$.

For the adjoint $(-)'$ we have $T^{S^2} \otimes N \to N \otimes T$ with $t \otimes n \mapsto t_1 n \otimes t_0$ mapping to $H \odot_H (T^{S^2} \otimes N) \to H \odot_H (N \otimes T)$ with $h \otimes t \otimes n \mapsto h \otimes t_1 n \otimes t_0$ which identifies with $T \otimes (H \odot_H N) \to (H \odot_H N) \otimes T^{S^2}$ with $t_0 \otimes S(t_1) h \otimes n \mapsto hS(t_1) \otimes t_2 n \otimes t_0$ under the isomorphisms of Remark 3.4 and the latter coincides with $t \otimes (x \otimes m) \mapsto t_1 \cdot (x \otimes m) \otimes t_0$. ◻

In the end we see that $((-)'_H, \widehat{(-)}_H)$ is an adjoint pair between aYD-mod and aYD-ctrmd extending the result of [6].

3.0.1 Stability.

Recall that in order to obtain cyclic cohomology we need to consider the coefficients in *stable* anti-Yetter-Drinfeld modules or contramodules. We now address the preservation of the stability conditions under the adjoint pair of functors of the previous section.

Recall the map

$$\sigma_M : M \to M$$

$$m \mapsto m_1 m_0$$

with the inverse $m \mapsto S^{-1}(m_1)m_0$; it defines an element $\sigma \in \mathrm{Aut}(Id_{aYD\text{-mod}})$. Similarly, there is a

$$\sigma_N : N \to N$$

$$n \mapsto \alpha(r_n)$$

with the inverse $n \mapsto \alpha(h \mapsto S^{-1}(h)n)$; it defines an element $\sigma \in \mathrm{Aut}(Id_{aYD\text{-ctrmd}})$.

It is an easy calculation to see that $\widehat{\sigma_M} : \widehat{M} \to \widehat{M}$ coincides with $\sigma_{\widehat{M}} : \widehat{M} \to \widehat{M}$ and also $(\sigma_N)' = \sigma_{N'}$. For example to prove the latter equality observe that the left hand side is $h \otimes n \mapsto h \otimes \alpha(r_n) = h^2 \otimes r_n(h^1) = h^2 \otimes h^1 n = (h \otimes n)_1 (h \otimes n)_0$ which is the right hand side.

Recall that $saYD$-mod is the full subcategory of aYD-mod that consists of M such that $\sigma_M = Id_M$. The definition of $saYD$-ctrmd is identical. We have proved the following Corollary to Theorem 3.8:

Corollary 3.9. *The functors $((-)'_H, \widehat{(-)}_H)$ is an adjoint pair between $saYD$-mod and $saYD$-ctrmd.*

4 A comodule-contramodule correspondence.

Here we will address the question of $\widehat{(-)}$ (equivalently $(-)'$) being an equivalence. Note that in light of the preceding discussion if $\widehat{(-)} : \mathcal{M}^H \to \widehat{\mathcal{M}^H}$ is an equivalence, then so is $\widehat{(-)}_H : aYD$-mod $\to aYD$-ctrmd and also $\widehat{(-)}_H : saYD$-mod $\to saYD$-ctrmd.

As usual, let us consider k as the trivial H-comodule, and let $J = \widehat{k}$ be its contramodule image under $\widehat{(-)}$. Note that this is nothing but the two-sided ideal in H^* consisting of right integrals [8]. Namely, $\chi \in J$ if and only if we have $\chi(h^1)h^2 = \chi(h)1$ for all $h \in H$. Strictly speaking it is left integrals that are considered in [8] but if χ is a left integral then $\chi(S(-))$ is right and vice versa. It is known [1] that $\mathrm{dim} J \leq 1$ and if $J \neq 0$ then S is invertible, which we have been assuming anyhow.

Remark 4.1. *Dually, we may consider k as the trivial contramodule, i.e., $\alpha : H^* \to k$ is evaluation at $1 \in H$. Let $K = k'$ and note that $K = H/I$ where I is generated by $\mu(h^1)h^2 - \mu(1)h$ for $\mu \in H^*$ and $h \in H$. Thus $K^* = I^\perp = \{\chi \in H^* | \mu(1)\chi(h) = \mu(h^1)\chi(h^2) \forall h\}$ and the latter is the ideal of left integrals.*

We are ready for the first negative result:

Lemma 4.2. *If $J = 0$ then $\widehat{(-)}$ is not an equivalence.*

Proof. Obviously we have that $\widehat{k} = 0$, but furthermore, by Proposition 3.2 we have that for $M \in \mathcal{M}^H_{fd}$, $\widehat{M} \simeq M^{S^2} \otimes J = 0$. ☐

On the other hand let us assume that $J \neq 0$. Let $\widehat{\mathcal{M}^H}_{rfd}$ denote the full subcategory of $\widehat{\mathcal{M}^H}$ consisting of finite dimensional, rational contramodules. By analogy with the H^*-module case, we say that a finite dimensional contramodule M is *rational* if the structure map α factors through $\mathrm{Hom}(C, M)$ for some C a finite dimensional subcoalgebra of H.

Lemma 4.3. *Let $J \neq 0$ then $\widehat{(-)} : \mathcal{M}^H_{fd} \simeq \widehat{\mathcal{M}^H}_{rfd}$.*

Proof. Again, for $M \in \mathcal{M}^H_{fd}$ we have that $\widehat{M} \simeq M^{S^2} \otimes J$. Note that by [8] the contramodule J is rational and thus so is $\widehat{M}$. On the other hand any rational finite dimensional contramodule is essentially a comodule (see Lemma 5.6) and so $(- \otimes {}^*J)^{S^{-2}}$ is the inverse of $\widehat{(-)}$. ☐

The above Lemma should be considered as in general a negative result. Namely, if exotic, i.e., non-rational contramodules are possible, then the equivalence fails. More precisely, let us consider the possibility of exotic contramodule structures on k. Let $\chi \in J$ and observe that

$$\alpha(x \mapsto \alpha(y \mapsto \chi(xS(y)))) = \alpha(h \mapsto \chi(h^1 S(h^2))) = \alpha(h \mapsto \epsilon(h))\chi(1) = \chi(1).$$

Since by [8], as x ranges over H, the functional $\chi(xS(-))$ ranges over H^{*rat} so if

$$\chi(1) \neq 0$$

then $\exists \mu \in H^{*rat}$ such that $\mu \cdot 1 = c \neq 0$. So that for any $\eta \in H^*$ we have $\eta \cdot 1 = \eta\mu\frac{1}{c} = \eta(\mu_1)\mu_0\frac{1}{c}$ and so the action of H^* on k factors through C^* and the structure on k is necessarily rational. On the other hand if $\chi(1) = 0$

then it is possible that the whole of H^{*rat} acts trivially without H^* doing the same, resulting in an exotic structure.

This suggests two possibilities for $\widehat{(-)}$ being an equivalence:

- $\exists \chi \in J$ with $\chi(1) \neq 0$.
- H is finite dimensional.

Note that the second case may appear trivial at first, but it isn't. It is true that there is no difference between H-comodules, H^*-modules and H-contramodules in the case when H is finite dimensional. However, we are not interested in the naive identification of the categories, rather the $\widehat{(-)}$ one. The latter functor is the one that translates the equivalence between comodules and contramodules to the equivalence between the $saYD$ versions that we need. Of course given all the work already done on this matter, the conclusion is easy to obtain, so we start with this case.

Proposition 4.4. *Let H be finite dimensional, then $\widehat{(-)}$ is an equivalence, and so is $\widehat{(-)}_H$.*

Proof. From [8] we know that $J \neq 0$. Furthermore, for $M \in \mathcal{M}^H$ we have $M = \varinjlim M_i$ with $M_i \in \mathcal{M}^H_{fd}$ so that $\widehat{M} = \mathrm{Hom}(H, M)^H = \mathrm{Hom}(H, \varinjlim M_i)^H$ which by the finite dimensionality of H is $\varinjlim \mathrm{Hom}(H, M_i)^H \simeq \varinjlim (M_i^{S^2} \otimes J) = M^{S^2} \otimes J$. Since there are no exotic contramodules here this proves the equivalence. $\square$

Moving on to the first case we get by [8] that the $\chi(1) \neq 0$ condition is actually very strict. Namely, we have that H is such that as a coalgebra $H = \bigoplus_i C_i$ where C_i are finite dimensional simple subcoalgebras. Let ϵ_i denote the counit of C_i with $\epsilon = \sum \epsilon_i$. For $x \in H$ let $x = \sum_i x_i$ denote its decomposition with respect to that of H.

Theorem 4.5. *The category of H-comodules and H-contramodules are equivalent. The former consists of $\bigoplus_i M_i$ and the latter of $\prod_i M_i$ where M_i are right C_i-comodules, i.e., $M_i \in \mathcal{M}^{C_i}$.*

Proof. The assertion about the comodules is immediate. Now let M be an H-contramodule, define $\alpha_i : M \to M$ via $\alpha_i(m) = \alpha(\epsilon_i(-)m)$. Note that

$$\alpha_i(\alpha_j(m)) = \alpha(x \mapsto \epsilon_i(x)\alpha(y \mapsto \epsilon_j(y)m)) = \alpha(h \mapsto \epsilon_i(h^1)\epsilon_j(h^2)m) = \delta_{ij}\alpha_i(m).$$

Let $M_i = \alpha_i(M)$ and consider $\beta : M \to \prod M_i$ such that

$$\beta(m) = (\alpha_i(m))_i$$

and $\iota : \prod M_i \to \mathrm{Hom}(H, M)$ via

$$\iota((m_i)_i)(x) = \sum \epsilon_i(x) m_i.$$

We have that

$$\begin{aligned}
\alpha\iota\beta(m) &= \alpha(x \mapsto \sum \epsilon_i(x)\alpha_i(m)) \\
&= \alpha(x \mapsto \sum \epsilon_i(x)\alpha(y \mapsto \epsilon_i(y)m)) \\
&= \alpha(h \mapsto \sum \epsilon_i(h^1)\epsilon_i(h^2)m) \\
&= \alpha(h \mapsto \epsilon(h)m) = m.
\end{aligned}$$

On the other hand we have that $\beta\alpha\iota((m_i)_i) = (\alpha_i(\alpha(x \mapsto \sum \epsilon_j(x)m_j)))_i$ and so we need to show that

$$m_i = \alpha_i(\alpha(x \mapsto \sum \epsilon_j(x)m_j)),$$

but the latter is

$$\begin{aligned}
\alpha(y \mapsto \epsilon_i(y)\alpha(x \mapsto \sum \epsilon_j(x)m_j)) &= \alpha(h \mapsto \sum_j \epsilon_i(h^1)\epsilon_j(h^2)m_j) \\
&= \alpha(h \mapsto \epsilon_i(h^1)\epsilon_i(h^2)m_i) \\
&= \alpha(\epsilon_i(-)m_i) = \alpha_i(m_i) = m_i.
\end{aligned}$$

Thus $\beta : M \simeq \prod_i M_i$ and using this identification we see that $\alpha : \mathrm{Hom}(H, M) \to M$ becomes

$$\begin{aligned}
\prod_i \mathrm{Hom}(H, M_i) &\to \prod_i M_i \\
(f_i)_i &\mapsto (\alpha(h \mapsto f_i(h_i)))_i
\end{aligned}$$

so that if we denote by $\alpha^i : \mathrm{Hom}(C_i, M_i) \to M_i$ the map $\alpha^i(f) = \alpha(h \mapsto f(h_i))$ then we see that the original α identifies with $\prod_i \alpha^i : \prod_i \mathrm{Hom}(C_i, M_i) \to \prod_i M_i$. It is immediate that α^i is a C_i-contramodule structure on M_i and since C_i is finite dimensional is the same as a C_i-comodule structure.

Conversely, given the data of $\rho_i : M_i \to M_i \otimes C_i$ we can define

$$\alpha^i : \mathrm{Hom}(C_i, M_i) = M_i \otimes C_i^* \to M_i$$

$$m \otimes \chi \mapsto \chi(m_1)m_0$$

and assemble the α^i into an $\alpha : \mathrm{Hom}(H, \prod_i M_i) \to \prod_i M_i$ that satisfies the contramodule axioms.

Now let $\phi : M \to N$ be a map of contramodules and let $m \in M$ with $m = (m_i)_i$ under the β identification, then

$$\phi(m)_i = \alpha_N(\epsilon_i(-)\phi(m)) = \phi(\alpha_M(\epsilon_i(-)m)) = \phi(m_i)$$

so that $\phi = \prod_i \phi_i$ with $\phi_i : M_i \to N_i$. It is immediate that $\phi_i \in \mathrm{Hom}(M_i, N_i)^{C_i}$ and that conversely, any such $(\phi_i)_i$ data can be reassembled into a $\phi : M \to N$ a map of contramodules.

$\square$

Remark 4.6. *The proof of Theorem 4.5 demonstrates a difference between H-contramodules and H^*-modules. While there is a forgetful functor from the former to the latter, the contramodule condition is better behaved than the H^*-module one in the case of the infinite dimensional H. Considering finite dimensional contramodules, that at first glance appear to be given an action indistinguishable from that of an H^*-module, it is the associativity that is strictly strengthened in the contramodule case. More precisely, there exist exotic 1-dimensional kG^*-modules (for example given by non-principal ultrafilters on G), yet any 1-dimensional kG-contramodule is supported at some $g \in G$, just as is the case for kG-comodules. The difference is due to the fact that in the contramodule case we have the freedom to work with the full $(H \otimes H)^*$ as opposed to only $H^* \otimes H^*$. Of course in the case when H is finite dimensional all three categories: H-contramodules, H-comodules and H^*-modules are equal.*

We need to connect the above to our adjoint pair of functors.

Proposition 4.7. *The correspondence*

$$\bigoplus_i M_i \leftrightarrow \prod_i M_i$$

of Theorem 4.5 is given, up to equivalence, by the adjoint functor pair $(\widehat{(-)}, (-)')$. Thus $\widehat{(-)}$ is an equivalence and so is $\widehat{(-)}_H$.

Proof. Observe that

$$\widehat{M} = \mathrm{Hom}(H, M)^H = \prod_i \mathrm{Hom}(C_i, M_i)^{H_i} = \prod_i \mathrm{Hom}_{A_i}(A_i^*, M_i)$$

where $A_i = C_i^*$ is a unital simple finite dimensional algebra. Let

$$\mu_i : A_i \to A_i^*$$

be given by $\mu_i(a)(b) = \mathrm{tr}_{A_i}(l_{ab})$, i.e., it is the trace of left multiplication by $ab \in A_i$. Note that μ_i is an A_i-bimodule map and $\mu_i(1)(1) = \mathrm{tr}_{A_i}(1) = \dim A_i \neq 0$ since $\mathrm{char} k = 0$, so that μ_i is an isomorphism by the simplicity of A_i. So $\widehat{M} \simeq \prod_i \mathrm{Hom}_{A_i}(A_i, M_i) \simeq \prod_i M_i$.

Similarly

$$N' = \bigoplus_i C_i \otimes_{C_i^*} N_i = \bigoplus_i A_i^* \otimes_{A_i} N_i \simeq \bigoplus_i N_i.$$

□

4.1 The case of $H = kG$.

Let G be an infinite discrete group. We ask that G be infinite as otherwise all of our considerations here become more or less trivial. Let M be a kG-contramodule, i.e., we view kG as a counital coalgebra with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. We have the following corollary of Theorem 4.5:

Corollary 4.8. *The category of kG-contramodules $\widehat{\mathcal{M}^{kG}}$ is equivalent to the category of G-graded vector spaces Vec_G. The equivalence is given by*

$$\Gamma(G,-) : Vec_G \to \widehat{\mathcal{M}^{kG}}.$$

Compare this with the well known equivalence

$$\Gamma_c(G,-) : Vec_G \to \mathcal{M}^{kG}$$

where Γ_c are global sections with compact support.

Proof. Note that $kG = \bigoplus_{g\in G} kg$ with $kg = k$ as coalgebras. □

It is well known that the category of anti Yetter-Drinfeld modules for kG (since $S^2 = Id$ it coincides with the category of Yetter-Drinfeld modules, and thus with the center of the monoidal category of kG-modules) is equivalent to the category $\mathrm{Vec}_{G/G}$ of G-equivariant G-graded vector spaces. More precisely, the kG-comodule part of the structure gives the G-grading, and the

kG-module part gives the G-action, while the Yetter-Drinfeld compatibility ensures that the action obeys

$$x : M_g \to M_{xgx^{-1}}.$$

We have an immediate Corollary to Proposition 4.7:

Corollary 4.9. *The category of aYD-contramodules for kG is equivalent to the category of G-graded G-equivariant vector spaces via*

$$\Gamma(G, -) : \mathit{Vec}_{G/G} \to aYD\text{-}\mathit{ctrmd}.$$

We now would like to address the question of stability. A stable aYD-module for kG is known to be G-graded G-equivariant vector space with the stability condition translating into

$$x \cdot m_x = m_x \tag{4.1}$$

for all $x \in G$ and all $m_x \in M_x$. Denote by $\mathrm{Vec}'_{G/G}$ the full subcategory of $\mathrm{Vec}_{G/G}$ consisting of objects for which (4.1) holds. We have another immediate Corollary to Proposition 4.7:

Corollary 4.10. *The functor*

$$\Gamma(G, -) : \mathit{Vec}'_{G/G} \to saYD\text{-}\mathit{ctrmd}$$

is an equivalence.

We can now restate the Proposition 2.10 more elegantly in the case of $H = kG$.

Proposition 4.11. *Let A be a G-equivariant algebra, and $\mathcal{M} \in \mathit{Vec}'_{G/G}$. Then*

$$\widehat{HC}^n_G(A, \Gamma(G, \mathcal{M})) \simeq HC^{n,G}(A \rtimes G, \Gamma_c(G, \mathcal{M})).$$

Remark 4.12. *While the right hand side of the above Proposition is definition invariant, the left hand side uses the definition of [7] and not the more classical one used in [4].*

4.2 A new "coefficient".

Since the introduction of coefficients in symmetric 2-contratraces in [4], there remained an obvious question: do these simply generalize the already well known coefficients in $saYD$-modules or contramodules to other settings, or do these traces furnish examples of coefficients that had not yet been considered even in the classical theories? In [7] we gave a derived version of the definition of cyclic cohomology with coefficients that restricted the possible symmetric 2-contratraces to the left exact ones. The results obtained in [5] immediately tell us that in the case of H-module algebras we need to look beyond the representable symmetric 2-contratraces if we are to obtain anything but the usual $saYD$-contramodule coefficients. In the present paper, Corollary 2.9 implies the same about H-comodule algebras, i.e., we need a non-representable contratrace to get away from the usual $saYD$-module coefficients. We will construct one below.

Let G have infinitely many conjugacy classes (such as when $G = \mathbb{Z}$ for example). Let $\mathcal{M}_{\langle g\rangle} \in \mathrm{Vec}'_{G/G}$ be supported on the conjugacy class $\langle g\rangle$, for example we can let

$$(\mathcal{M}_{\langle g\rangle})_x = \begin{cases} k, & x \in \langle g\rangle \\ 0, & else \end{cases}$$

with the trivial G-action. Then each $\mathcal{M}_{\langle g\rangle}$ yields a representable left exact symmetric 2-contratrace

$$\mathcal{F}_{\langle g\rangle}(V) = \mathrm{Hom}(V, \Gamma_c(G, \mathcal{M}_{\langle g\rangle}))^G,$$

yet

$$\bigoplus_{\langle g\rangle} \mathcal{F}_{\langle g\rangle} : V \mapsto \bigoplus_{\langle g\rangle} \mathcal{F}_{\langle g\rangle}(V)$$

is an example of a non-representable, left exact symmetric 2-contratrace on $\mathcal{M}^H$. Note that taking $\mathcal{M}$ to be the superposition of all $\mathcal{M}_{\langle g\rangle}$'s would result in $V \mapsto \prod_{\langle g\rangle} \mathcal{F}_{\langle g\rangle}(V)$.

5 Periodicities.

In this section we revisit the YD_i-modules from [4] and see that under the conditions that we have been looking at in this paper, there is nothing new that arises and we still only have the Yetter-Drinfeld and the anti-Yetter-Drinfeld modules and contramodules; this is the first observed periodicity.

In addition, we examine a natural symmetry on these objects and observe that it too is periodic; this we refer to as the second periodicity.

We recall the definition of YD_i-modules:

Definition 5.1. *Let M be a left module and a right comodule over H, and let $i \in \mathbb{Z}$. We say that M is a YD_i-module if*

$$(hm)_0 \otimes (hm)_1 = h^2 m_0 \otimes h^3 m_1 S^{-1-2i}(h^1), \tag{5.1}$$

for $h \in H$ and $m \in M$.

Remark 5.2. *Equivalently, we can define YD_i-modules by requiring that the comodule structure map $M \to M \otimes H$ is H-equivariant with respect to the H-structure on the right hand side given by $x \cdot (m \otimes h) = x^2 m \otimes x^3 h S^{-1-2i}(x^1)$.*

Note that YD_{-1}-modules are anti-Yetter-Drinfeld modules, while YD_0-modules are Yetter-Drinfeld modules.

We can rephrase the above a little more conceptually. Let $\mathbb{Z}$ act on $\mathcal{M}^H$ with $1 \cdot M = M^{S^2}$ so that we may consider the monoidal category $\mathcal{M}^H \rtimes \mathbb{Z}$. We get an immediate generalization of Lemma 3.5:

Lemma 5.3. *We have an equivalence of monoidal categories:*

$$\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H \rtimes \mathbb{Z}) \simeq \bigoplus_{i \in \mathbb{Z}} YD_{-i}\text{-}mod.$$

There are a few consequences of the above. First, if $M \in YD_i$-mod and $N \in YD_j$-mod then $M \otimes N \in YD_{i+j}$-mod with the usual comodule structure, but ${}_{S^{-2i}}N \otimes M$ as an H-module, i.e.,

$$(m \otimes n)_0 \otimes (m \otimes n)_1 = m_0 \otimes n_0 \otimes m_1 n_1$$

but

$$x \cdot (m \otimes n) = x^2 m \otimes S^{-2i}(x^1) n.$$

Second, if $M \in YD_i$-mod then so is $1 \cdot M = {}_{S^{-2}}M^{S^2} \in YD_i$-mod. Third, $\mathcal{M}^H$ has internal Homs, and so does $\mathcal{M}^H \rtimes \mathbb{Z}$, i.e.,

$$\mathrm{Hom}^l((M, j), (N, i)) = (\mathrm{Hom}^l((i-j)M, N), i-j)$$

and the same for right Homs. Consequently, $\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H \rtimes \mathbb{Z})$ has internal Homs as well. In particular $\mathcal{M}^H_{fd}$ is rigid, so is $\mathcal{M}^H_{fd} \rtimes \mathbb{Z}$ with $(V, i)^* =$

$((-i)V^*, -i)$ and ${}^*(V,i) = ((-i)^*V, -i)$ and so is $\mathcal{Z}_{\mathcal{M}^H_{fd}}(\mathcal{M}^H_{fd} \rtimes \mathbb{Z})$. Thus if $M \in YD^{fd}_i$-mod then we have elements $M^\star$ and ${}^\star M$ in YD^{fd}_{-i}-mod that are its right and left duals.

Just as we have generalized aYD-mod to YD_i-mod, we can do the same to aYD-ctrmd.

Definition 5.4. *Let M be a left H-module and a right H-contramodule, we say that M is a YD_i-contramodule if the contramodule structure α : $\mathrm{Hom}(H,M) \to M$ is H-equivariant with respect to the H-action on the left given by*

$$h \cdot f = h^2 f(S(h^3) - S^{2-2i}(h^1)), \tag{5.2}$$

where $h \in H$ and $f \in \mathrm{Hom}(H,M)$.

Note that YD_1-contramodules are aYD-contramodules.

5.1 The first periodicity.

We can easily generalize the content of Section 3 as follows. We have the Proposition/Definition below.

Proposition 5.5. *Let M be a YD_{i-1}-module, define $\widehat{M} = \mathrm{Hom}(H,M)^H$ and equip the latter with a left H-action via*

$$h \cdot f = h^2 f(S(h^3) - S^{-2i}(h^1))$$

and an H-contraaction as before (2.16). Let N be a YD_{i+1}-contramodule, define $N' = H \odot_H N$ and equip the latter with a left H-action via

$$h \cdot (x \otimes n) = S^{1-2i}(h^3) x S(h^1) \otimes h^2 n$$

and an H-coaction as before (3.2).

This defines an adjoint pair of functors $((-)'_H, \widehat{(-)}_H)$:

$$YD_{i-1}\text{-}mod \underset{(-)'_H}{\overset{\widehat{(-)}_H}{\rightleftarrows}} YD_{i+1}\text{-}ctrmd.$$

Let us again (see Remark 4.1 and the preceding discussion) consider the trivial YD_0-module k from which we obtain by the Proposition 5.5 the object $J = \widehat{k}$ which is now seen to be in YD_2^{fd}-ctrmd. Conversely, again considering the trivial YD_0-contramodule k, we obtain $K = k'$ which is now seen to be in YD_{-2}^{fd}-mod. If we denote by YD_i^{rfd}-ctrmd the full subcategory of YD_i-ctrmd that consists of objects that as contramodules are in $\widehat{\mathcal{M}^H}_{rfd}$ then $J \in YD_2^{rfd}$-ctrmd. Observe that we have an easy Lemma:

Lemma 5.6. *We have an equivalence (equality actually) of categories:*

$$\iota : YD_i^{fd}\text{-}mod \to YD_i^{rfd}\text{-}ctrmd$$

that does not change the underlying vector space M, nor the H-action, and sends the coaction to the contraaction:

$$\mathrm{Hom}(H, M) = H^* \otimes M \to M$$

$$\chi \otimes m \mapsto \chi(S^2(m_1))m_0.$$

As a consequence, we have $\iota^{-1}J \in YD_2^{fd}$-mod which is the dual of $K \in YD_{-2}^{fd}$-mod.

Remark 5.7. *If H is a Hopf algebra with $J \neq 0$ then both YD_i-mod and YD_i-ctrmd are 2-periodic, i.e.,*

$$J \otimes - : YD_i\text{-}mod \simeq YD_{i+2}\text{-}mod$$

and the same for contramodules.

For a finite dimensional H, the functor $\widehat{(-)}_H$ is essentially the periodicity above. Not so for the infinite dimensional case.

5.2 The second periodicity.

Recall our discussion of stability in Section 3.0.1. We observe that the

$$\sigma \in \mathrm{Aut}(Id_{aYD\text{-mod}})$$

that was used to define stability for aYD-modules (and its contramodule variant) can be generalized, with an interesting difference, to an arbitrary i for both modules and contramodules. More precisely,

$$\sigma \in \mathrm{Iso}(Id_{YD_{i-1}\text{-mod}}, (-i)\cdot),$$

i.e., for $M \in YD_{i-1}$-mod we have $\sigma_M : M \to {}_{S^{2i}}M^{S^{-2i}}$, with $m \mapsto S^{2i}(m_1)m_0$ and the inverse $m \mapsto S^{-1}(m_1)m_0$, is an identification in YD_{i-1}-mod.

Remark 5.8. *The above implies that if M is a Yetter-Drinfeld module, then it is canonically isomorphic to ${}_{S^{-2}}M^{S^2}$ as a Yetter-Drinfeld module. More generally, the action of $\mathbb{Z}$ on YD_{i-1}-mod factors through $\mathbb{Z}/i\mathbb{Z}$. We will see below that the same holds for YD_{i+1}-ctrmd.*

Note that the $M \mapsto {}_{S^{-2}}M^{S^2}$ symmetry of YD_i-modules also exists for YD_i-contramodules, i.e., ${}_{S^{-2}}N^{S^2}$ has its H-action modified by S^{-2} and $\alpha^{S^2}(f) = \alpha(f(S^2(-)))$. Then we have

$$\sigma \in \mathrm{Iso}(Id_{YD_{i+1}\text{-ctrmd}}, (-i)\cdot),$$

i.e., for $N \in YD_{i+1}$-ctrmd we have $\sigma_N : N \to {}_{S^{2i}}N^{S^{-2i}}$, with $n \mapsto \alpha(r_n)$ and the inverse $n \mapsto \alpha(h \mapsto S^{2i-1}(h)n)$, is an identification in YD_{i+1}-ctrmd.

Furthermore, the generalized functor $\widehat{(-)}_H$ of Proposition 5.5 is compatible with these symmetries, namely the diagram of isomorphisms

$$\begin{tikzcd} \widehat{M} \arrow[rr, "\widehat{\sigma_M}"] \arrow[dr, "\sigma_{\widehat{M}}"'] & & {}_{S^{2i}}\widehat{M^{S^{-2i}}} \\ & {}_{S^{2i}}\widehat{M}^{S^{-2i}} \arrow[ur, "f\mapsto f(S^{2i}(-))"'] & \end{tikzcd}$$

commutes in YD_{i+1}-ctrmd, where $M \in YD_{i-1}$-mod.

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