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Some Wonderful Rings in Algebraic Geometry

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Let $A = k \oplus A_1 \cdots$ be a graded ring over a field k. The ring is wonderful if $\mathrm{Tor}_i^A(k, k)$ has pure degree i for all i where $k = A/A_1$.

In a previous paper [4] it was discovered that the ring of regular functions on many cones over abelian varieties is wonderful. Similarly in [1] Backelin has shown that embeddings by high enough trists on ample sheaf on projective schemes give wonderful rings. We will estimate "high enough" and we will note that the coordinate ring of Grassmannians in Plücker coordinates gives wonderful rings. More exactly

THEOREM 1. *Any ring A with straightening law whose discrete algebra D is defined by quadratic monomials is wonderful.*

For the terminology see [2].

This paper will use a theorem of Fröberg [3] which says

THEOREM 2. *Any commutative graded ring generated by elements of degree one modulo quadratic monomials is wonderful.*

Thus in Theorem 1 we know that the discrete algebra D is wonderful and we want to conclude that A is wonderful. By [2] we know that A is a generalization of D by a flat deformation. Thus we need only prove

LEMMA 3. *Let $\tilde{A} = R \oplus \tilde{A}_1 \oplus \cdots$ be a finitely generated graded ring over a discrete valuation ring R. Let k be the residue field of R and K be its quotient. If $\tilde{A} \otimes k$ is wonderful then $\tilde{A} \otimes K$ is wonderful.*

Proof. Let $L_* \to k$ be a minimal projective $\tilde{A} \otimes k$-resolution where we know $L_i = \bigoplus \tilde{A} \otimes k[+i]$. It will suffice to show that we can lift this to a projective resolution $\tilde{L}_* \to R$. Assume that we have a partial lifting

$$\tilde{L}_j \xrightarrow{\tilde{\partial}_j} \cdots \longrightarrow R \longrightarrow 0.$$

* Partly supported by DFG.

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Then the kernel of d_j is the direct sum of flat R-modules which are finitely generated as $\tilde{A}$ is finitely generated. Thus the kernel of d_j is R-free. Now (kernel of $\tilde{d}_j$) $\otimes k$ = kernel of $\tilde{d}_j \otimes k$ by lifting. So (kernel of $\tilde{d}_j$) is generated by degree j elements which lift the (kernel of $\tilde{d}_j \otimes k$) by Nakayama's lemma. Thus we continue the lifting. Q.E.D.

We want to include another remark.
Let $P = k[A_1]$. Then A is a P-algebra and we may try to compute $\operatorname{Tor}_i^P(A, k)$.

LEMMA 4. *If A is wonderful then*

$$\operatorname{Tor}_i^P(A, k) \text{ is zero in degree } > 2i.$$

Proof. Consider the spectral sequence of change of rings

$$\operatorname{Tor}_p^A(\operatorname{Tor}_q^P(A, k), k) \Rightarrow \operatorname{Tor}_{p+q}^P(k, k).$$

This sequence contains $\operatorname{Tor}_q^P(A, k) = \operatorname{Tor}_0^A(\operatorname{Tor}_q^P(A, k), k)$ in the E_2-term but the abutment is zero if degree $> p+q$. So anything in $\operatorname{Tor}_q^P(A, k)$ of degree $> q$ dies in the spectral sequence. We prove the lemma by induction on q by noting the degree of elements that could kill $\operatorname{Tor}_q^P(A, k)$. These elements come from

$$\operatorname{Tor}_{i+1}^A(\operatorname{Tor}_{q-i}^P(A, k), k) = \operatorname{Tor}_{i+1}^A(k, k) \otimes \operatorname{Tor}_{q-i}^P(A, k)$$

but they have degree at most $2 + 2(q-1)$. Q.E.D.

Assume that $P \to A$ is surjective. Then there exists a positive number irr such that $\|\operatorname{Tor}_i^P(A, k)\| \leqslant \operatorname{irr} \cdot i$ for all the numbers i (finitely many) where $\| \ \|$ is the highest degree of a non-zero element of a graded module. In the geometric situation irr is estimated from the m-regularity of $\operatorname{Proj}(A)$.

LEMMA 5. $\|\operatorname{Tor}_{i+1}^A(k, k)\| \leqslant i \cdot irr$ *if* $i \geqslant 2$.

Proof. Consider the spectral sequence of Lemma 4 again. We need to see how an element of degree $> i \cdot \operatorname{irr}$ in $\operatorname{Tor}_{i+1}^A(k, k)$ can die in the spectral sequence. It must go to an element of the same degree found in $\operatorname{Tor}_{p+1}^A(k, k) \otimes \operatorname{Tor}_{i-1-p}^P(A, k)$. Thus this step is clear as $\operatorname{Tor}_1^A(k, k)$ has pure degree one $\leqslant$ irr. Q.E.D.

Next we may apply the details of the Backelin theorem to conclude

THEOREM 6. *If $j \geqslant irr$ then the projective coordinate ring of the j-Veronese embedding of X is wonderful.*

(This uses the notation of slope from [1].)

Proof. By Lemma 5 the slope of $A \leqslant \mathrm{irr}$. Thus the slope of $A^{(j)} \leqslant 1$ and hence $A^{(j)}$ is wonderful.

Remark. Lemma 5 gives a different method than [1] to complete the slope of the ring.

EXAMPLE. If X is a curve embedded by a complete linear system of degree $\geqslant 2$ genus $+2$, we have a projectively normal embedding and we know $\mathrm{Tor}_i^P(A, k)$ and $\mathrm{irr} \leqslant 2$. Then any double or better Veronese embedding of X has a wonderful ring.

Hopefully this result can be improved to other degrees.

Note added in proof. The reader should be informed that wonderful rings have been studied in algebra by many authors under the names, homogeneous preKoszul algebras, Koszul algebras, Fröberg algebras, Priddy rings, and formal rings. The following reference gives 18 different equivalent conditions for a ring to be wonderful: J. Backelin and R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolutions, *Rev. Roamaine Math. Pures Appl.* **30** (1985).

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