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The Grothendieck Festschrift
Volume III

A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck

P. Cartier

L. Illusie

N.M. Katz

G. Laumon

Yu.I. Manin

K.A. Ribet

Editors

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Pierre Cartier
Institut des Hautes Études Scientifiques
F-91440 Bures-sur-Yvette
France

Luc Illusie
Université de Paris-Sud
Département de Mathématiques
F-91405 Orsay
France

Nicholas M. Katz
Princeton University
Department of Mathematics
Princeton, NJ 08544
U.S.A.

Gérard Laumon
Université de Paris-Sud
Département de Mathématiques
F-91405 Orsay
France

Yuri I. Manin
Max-Planck Institut für Mathematik
D-53111 Bonn
Germany

Kenneth A. Ribet
University of California
Department of Mathematics
Berkeley, CA 94720
U.S.A.

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Institut des Hautes
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91440 Bures-sur-Yvette
France

Luc Illusie
Département de Mathématiques
Université de Paris-Sud
Centre d'Orsay
91405 Orsay Cedex
France

Nicholas M. Katz
Department of Mathematics
Princeton University
Princeton, NJ 08544
USA

Gérard Laumon
Département de Mathématiques
Université de Paris-Sud
Centre d'Orsay
91405 Orsay Cedex
France

Yuri Manin
Steklov Mathematical Institute
Academy of Sciences USSR
117966 Moscow GSP-1 USSR

Kenneth A. Ribet
Department of Mathematics
University of California
Berkeley, CA 94720
USA

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CONTENTS

VOLUME I

VOLUME II

Anneau de Grothendieck de la variété de drapeaux

ALAIN LASCOUX

dédié à A. Grothendieck

0. Introduction

Une variété de drapeaux relative se décompose en une suite de fibrations projectives; la combinatoire de cette variété s'obtient donc à partir de celle du projectif. Bien plus, ainsi que l'ont montré Bernstein-Gelfand-Gelfand et Demazure, on peut se réduire à des fibrations projectives en droites, i.e. l'objet géométrique de base est la variété $\mathbb{P}(V)$, où V est un fibré vectoriel de rang 2, avec pour groupe asocié le groupe symétrique $\mathfrak{S}(2)$.

La géométrie ayant ainsi fourni des morceaux élémentaires, c'est à la combinatoire qu'il faut faire appel pour procéder au recollement. Contrairement à Bernstein-Gelfand-Gelfand et Demazure, nous travaillons directement dans l'anneau des polynômes plutôt que ses quotients que sont l'anneau de Grothendieck (des classes de fibrés vectoriels) ou l'anneau de cohomologie. Ce faisant, on obtient des bases (polynômes de Schubert et polynômes de Grothendieck) stables par les plongements $\mathfrak{S}(n) \hookrightarrow \mathfrak{S}(n+1)$, compatibles aux variétés de drapeaux partiels, unifiant la cohomologie et l'anneau de Grothendieck.

Ces bases contiennent comme sous-famille les fonctions de Schur, i.e. les caractères irréductibles sur $\mathbb{C}$ du groupe linéaire (pour des caractères plus généraux, voir [K-P]). Ce sont d'ailleurs les fonctions de Schur qui décrivent l'anneau de cohomologie de la grassmannienne : elles sont alors interprétées comme *cycles de Schubert.*

Les fonctions de Schur peuvent s'obtenir, à la suite de Cauchy, par diagonalisation de la résultante $\prod(a_i - b_j)$ de deux ensembles d'indéterminées $\{a_i\}$ et $\{b_j\}$. Les polynômes de Grothendieck et Schubert s'obtiennent, quant à eux, à partir de $\mathbb{G}_\omega = \prod_{i+j\leq n}(1-b_j/a_i)$ ou $\mathbb{X}_\omega = \prod_{i+j\leq n}(a_i - b_j)$ à l'aide des opérateurs associés aux fibrations en droites projectives mentionnées ci-dessus (à peu de choses près, ces opérateurs ne sont autres que les *différences divisées de Newton*). La spécialisation $b_j \to 1$ de $\mathbb{G}_\omega$ (resp. $\mathbb{X}_\omega$) est la classe d'un point dans l'anneau de Grothendieck (resp.

de cohomologie). Il est remarquable qu'on puisse déduire de la classe d'un point celle de toutes les variétés de Schubert de la variété de drapeaux $\mathcal{F}$. On comprend mieux ce fait en partant de $\mathbb{G}_\omega$ et $\mathbb{X}_\omega$ plutôt que de leur spécialisation; $\mathbb{G}_\omega$ et $\mathbb{X}_\omega$ sont en effet les classes de $\mathcal{F}$ dans le plongement diagonal $\mathcal{F} \hookrightarrow \mathcal{F} \times \mathcal{F}$. C'est pourquoi, autre différence avec Bernstein-Gelfand-Gelfand et Demazure, nous utilisons deux ensembles d'indéterminées plutôt qu'un, quoiqu'étudiant les mêmes objets géométriques que ces auteurs. Pour ce qui est de la cohomologie, nous renvoyons à [B-G-G], [D3], [L2], [L-S2] .

Au paragraphe 1, nous donnons quelques propriétés du groupe symétrique agissant sur l'anneau des polynômes (opérateurs de symétrisation π_μ , $\mu \in \mathfrak{S}(\mathbb{A})$).

Aux paragraphes 2 et 3, nous définissons les objets fondamentaux (polynômes de Grothendieck $\mathbb{G}_\mu$) dont les classes sont les classes des faisceaux structuraux des variétés de Schubert. Nous résumons ensuite (th.2.8 et prop.3.4) toutes les propriétés combinatoires connues de l'anneau de Grothendieck de la variété de drapeaux pour le groupe linéaire. Les points essentiels sont l'existence d'un produit scalaire pour lequel les π_i sont auto-adjoints, et la remarque que tout permuté $(\mathbb{G}_\omega)^\mu$ de $\mathbb{G}_\omega$ s'annule pour la spécialisation $b_i \to a_i$, sauf lorsque μ est la permutation maximale de $\mathfrak{S}(\mathbb{A})$.

En 3.11, nous explicitons les permutations ("vexillaires") pour lesquelles le calcul est le même que dans le cas des grassmanniennes et conduit à des expressions déterminantales.

Le théorème de Riemann-Roch pour la variété de drapeaux est associé à l'opérateur de symétrisation maximal π_ω . Nous nous en inspirons pour donner en 4.4 une propriété des opérateurs plus généraux que sont les π_μ.

Le plongement de Plücker (associé au fibré inversible L^E) possède une symétrie (prop.5.2). Cette symétrie, jointe au décompte de certaines familles de tableaux de Young, permet de calculer facilement les dimensions ("postulation") de l'espace des sections des puissances de L^E au-dessus de toute variété de Schubert sans devoir recourir aux méthodes des paragraphes 2 et 3. La fonction génératrice des postulations est une fraction rationnelle $\mathcal{E}_\mu(z)/(1-z)^{\ell(\mu)+1}$ dont le numérateur est une "z-extension" du degré pour le plongement de Plücker (i.e. $\mathcal{E}_\mu(1)$ est le degré projectif de la variété de Schubert d'indice $\omega\mu$).

Les polynômes $\mathcal{E}_\mu(z)$ admettent pour sous-famille remarquable les polynômes d'Euler.

La structure multiplicative de l'anneau de cohomologie de la grassmannienne résulte, ainsi que l'a montré Giambelli [Gi], des formules de Pieri : l'intersection d'un cycle de Schubert avec un cycle spécial est une somme sans multiplicité de cycles de Schubert. Nous montrons que cette absence de multiplicité est encore vraie dans l'anneau de Grothendieck de la variété de drapeaux (th.6.4).

Au paragraphe 7, nous traduisons les résultats obtenus en termes d'anneau de Grothendieck.

Z! On note 1I, ..., nI les composantes d'un vecteur $I \in \mathbb{Z}^n$; une permutation $\mu \in \mathfrak{S}(n)$ est considérée comme un morphisme de $\{1, \ldots, n\}$ dans lui-même : $\{1, \ldots, n\} \longrightarrow \{1\mu, \ldots, n\mu\}$. Les opérateurs opèrent sur tout ce qui figure sur leur gauche . Ainsi, la proposition 4.4 que nous avons écrite

$$f \pi_{\mu^{-1}} \varepsilon = f\, a^E \, \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell(\omega\mu)} \nabla^{\ell(\mu)} \partial_\omega \, \varepsilon / \ell(\mu)\,!$$

peut aussi se lire, si l'on préfère,

$$\varepsilon\big(\pi_{\mu^{-1}}(f)\big) = \frac{1}{\ell(\mu)!}\, \varepsilon\left(\partial_\omega \left(\nabla^{\ell(\mu)}\big(\frac{1}{u^{\ell(\omega\mu)}}\, \mathfrak{g}_{\omega\mu} \cdot \underline{\mathrm{CH}}(a^E f)\big)\right)\right)$$

La majeure partie des résultats exposés est le fruit d'une collaboration soutenue avec M.P.Schützenberger.

1. Opérateurs de symétrisation

Soient V un fibré vectoriel de rang $n+1$ sur une base quelconque $\mathcal{M}$, $\mathcal{F}(V)$ la variété des drapeaux (complets) de V . La fibration $\mathcal{F}(V) \longrightarrow \mathcal{M}$ se décompose en une suite de fibrations en projectifs (cf. [Gr1]), ce qui permet de décrire aisément l'anneau de Grothendieck et l'anneau de cohomologie de $\mathcal{F}(V)$ à partir de ceux du projectif. Cette fibration donne $n+1$ fibrés inversibles $L_1, \ldots, L_{n+1}$ sur $\mathcal{F}(V)$, dits *tautologiques* .

On obtient ainsi que l'anneau de Grothendieck $\mathbf{K}\big(\mathcal{F}(V)\big)$ des modules cohérents sur $\mathcal{F}(V)$ est le quotient de l'anneau des polynômes (de *Laurent)* $\mathbf{K}[\mathbb{A}] = \mathbf{K}[a_1, 1/a_1, \ldots, a_{n+1}, 1/a_{n+1}]$ par l'idéal $\mathcal{J}$ engendré par la relation $V = a_1 + \cdots + a_{n+1}$ (en tant que λ-anneau) , où $\mathbf{K}$ est l'anneau de Grothendieck de $\mathcal{M}$. La classe de chaque L_i coïncide avec l'image de a_i dans $\mathbf{K}\big(\mathcal{F}(V)\big)$.

Plusieurs constructions donnent ce résultat, à partir de différentes interprétations de l'anneau $\mathbf{K}[\mathbb{A}]$ lui-même :

* Comme anneau de Grothendieck du classifiant de $U(n)$ (en transposant la méthode que Borel a donnée pour la cohomologie).
* Comme anneau des classes de représentations d'un sous-groupe de Borel dans le cas d'un groupe semi-simple déployé (*cf.* [D3]).
* Comme anneau de Grothendieck T-équivariant (*cf.* [K-K]).

Chacun de ces anneaux est muni d'une surjection canonique sur $\mathbf{K}\big(\mathcal{F}(V)\big)$ dont le noyau est l'idéal $\mathcal{J}$. Les constructions ici données sont compatibles avec le passage au quotient par $\mathcal{J}$, nous travaillerons désormais dans l'anneau des polynômes $\mathbf{K}[\mathbb{A}]$ sans choisir d'interprétation à cet anneau et nous renverrons au dernier paragraphe pour ce qui est de la géométrie.

La fibration $\mathcal{F}(V) \longrightarrow \mathcal{M}$ induit le morphisme suivant, noté π_ω , de $\mathbf{K}[\mathbb{A}]$ dans $\mathbf{K}$:

$$\mathbf{K}[\mathbb{A}] \ni f \longrightarrow \sum_{\mu} \Big(f / \prod_{i<j} (1 - a_j/a_i)\Big)^{\mu} \in \mathbf{K} \quad , \tag{1.1}$$

somme sur toutes les permutations $\mu \in \mathfrak{S}(\mathbb{A})$, c'est-à-dire les permutations de $\mathbb{A} = \{a_1, \ldots, a_{n+1}\}$.

En tant qu'opérateur sur l'anneau des polynômes, le morphisme précédent était utilisé par Jacobi pour définir les fonctions symétriques que l'on a appelé par la suite *fonctions de Schur* (*cf.* [Ma , I.3.1]). L'interprétation géométrique donne une factorisation de ce morphisme en un produit d'opérateurs élémentaires correspondant à des fibrations projectives de dimension 1, ansi que l'ont montré [B-G-G] et [D3].

Plus précisément, pour tout ensemble d'indéterminées totalement ordonné (dit *alphabet*) $\mathbb{A} = \{a_1 < \ldots < a_{n+1}\}$, pour tout p , $1 \leq p \leq n$, notant σ_p la transposition échangeant a_p et a_{p+1} , on définit le *symétriseur isobare élémentaire* $\pi_p = \pi_{\sigma_p}$ par :

$$\mathbf{K}[\mathbb{A}] \ni f \longrightarrow (a_p f)(\sigma_p - 1)\frac{1}{a_{p+1} - a_p} = f\,\pi_p \tag{1.2}$$

c'est-à-dire, plus explicitement,

$$f \longrightarrow \frac{a_{p+1} f(\ldots a_{p+1}, a_p \ldots) - a_p f(\ldots a_p, a_{p+1} \ldots)}{a_{p+1} - a_p} = f\,\pi_p$$

Les symétriseurs élémentaires vérifient les relations de Moore/Coxeter :

$$\begin{cases} \pi_p \pi_q = \pi_q \pi_p & \text{si } |p - q| \geq 2 \\ \pi_p \pi_{p+1} \pi_p = \pi_{p+1} \pi_p \pi_{p+1} & \end{cases} \tag{1.3}$$

Ces relations entrainent que les produits de symétriseurs élémentaires correspondent aux permutations, ainsi que l'indique le lemme suivant (th.1 de [D2] qui écrit L^0_μ au lieu de π_μ) :

Lemme 1.4. *Pour toute permutation $\mu \in \mathfrak{S}(\mathbb{A})$, toute décomposition réduite $\mu = \sigma\sigma'...\sigma''$, alors le produit $\pi_\sigma \pi_{\sigma'} \cdots \pi_{\sigma''}$ ne dépend que de μ ; il est noté π_μ.*

En itérant la définition 1.2, on voit que π_μ appartient à l'algèbre du groupe symétrique à coefficients les fonctions rationnelles en $\mathbb{A}$; plus précisément, pour tout μ , il existe des fonctions rationnelles $R_{\mu,\eta}$ telles que

$$\pi_\mu = \sum_{\eta \leq \mu} \eta \, R_{\mu,\eta} \, .$$

Dans le cas où la permutation μ est l'élément de longueur maximum de $\mathfrak{S}(\mathbb{A})$, noté ω , le symétriseur π_ω coïncide avec l'opérateur défini en 1.1 (cf. [L-S2], 1.9) .

Notons $\mathbb{A}_p$ le sous-alphabet $\mathbb{A}_p = \{a_1, \ldots, a_p\}$ et soient $S_j(\mathbb{A}_p)$ les *fonctions symétriques complètes* (ou *fonctions alephs* de Wronski) définies par la série génératrice

$$\sum_{-\infty}^{+\infty} z^j S_j(\mathbb{A}_p) = 1/(1-za_1)\cdots(1-za_p). \tag{1.5}$$

Ces fonctions sont préservées par les π_p :

Lemme 1.6. *1) Pour tout* $p : 1 \leq p \leq n$ *, toutes* f, g *dans* $\mathbf{K}[\mathbb{A}]$,

$$g = g\sigma_p \Rightarrow fg\pi_p = f\pi_p \, g$$

2) Pour tout $j \in \mathbb{Z}$ *,* $S_j(\mathbb{A}_p)\,\pi_p = S_j(\mathbb{A}_{p+1})$.

Preuve. Le morphisme 1.2 commute avec la multiplication par les fonctions symétriques en a_p et a_{p+1} . On a donc

$$\frac{1}{1-za_1}\cdots\frac{1}{1-za_p}\pi_p = \frac{1}{1-za_p}\pi_p \frac{1}{1-za_1}\cdots\frac{1}{1-za_{p-1}}$$
$$\frac{1}{(1-za_p)(1-za_{p+1})}\cdot\frac{1}{1-za_1}\cdots\frac{1}{1-za_{p-1}} \qquad \blacksquare$$

Plus généralement, étant donnés deux alphabets $\mathbb{A}$ et $\mathbb{B}$, les *fonctions complètes* $S_j(\mathbb{A}-\mathbb{B})$ sont définies par la série génératrice

$$\sum_{-\infty}^{+\infty} z^j S_j(\mathbb{A}-\mathbb{B}) = \prod_{b\in\mathbb{B}}(1-zb) / \prod_{a\in\mathbb{A}}(1-za) \tag{1.7}$$

Pour tout r , tout $I = (1I, 2I, ..., rI)$ dans $\mathbb{Z}^r$, tout r-uple de paires d'alphabets $(A^1, \mathbb{B}^1), \ldots, (\mathbb{A}^r, \mathbb{B}^r)$, la *fonction de Schur*

$S_I(\mathbb{A}^1 - \mathbb{B}^1, \ldots, \mathbb{A}^r - \mathbb{B}^r)$ est le déterminant

$$(1.8) \qquad S_I(\mathbb{A}^1 - \mathbb{B}^1, \ldots, \mathbb{A}^r - \mathbb{B}^r) = \left|S_{kI+k-h}(\mathbb{A}^k - \mathbb{B}^k)\right|_{1\le h,k\le n}$$

Plus généralement, dans un λ-anneau quelconque K , on pose, pour tout entier n et tout $x \in K$, $S_n(x) = (-1)^n\lambda^n(-x)$; on définit alors le *foncteur de Schur* S_I , $I \in \mathbb{Z}^r$, par :

$$K^r \ni (x_1, \ldots, x_r) \longrightarrow S_I(x_1, \ldots, x_r) = \left|S_{kI+k-h}(x_k)\right|_{1\le h,k\le n} \in K$$

On retrouve le cas précédent lorsque K est un anneau de polynômes muni de sa structure naturelle de λ-anneau (*cf.* [Gr2]), les “lettres” d'un alphabet étant des éléments de rang 1 de K , et un alphabet fini étant identifié à la somme de ses lettres.

Lorsque $0 \le 1I \le \cdots \le rI$ et que $\mathbb{A}^1 = \ldots = \mathbb{A}^r = \mathbb{A}$, $\mathbb{B}^1 = \ldots = \mathbb{B}^r = 0$, le déterminant $S_I(\mathbb{A}, \ldots, \mathbb{A}) = S_I(\mathbb{A})$ est la fonction de Schur classique sur l'alphabet $\mathbb{A}$, d'indice la partition I (cf. [Ma], p.25).

Un monôme a^I , pour tout I dans $\mathbb{N}^{n+1}$, peut se représenter comme une fonction de Schur :

$$(1.9) \qquad S_{I\omega}(\mathbb{A}_{n+1}, \ldots, \mathbb{A}_1) = a_{n+1}^{(n+1)I}\, a_n^{nI} \cdots a_1^{1I}$$

En effet, comme $S_j(\mathbb{A} \setminus \{a_1\}) = S_j(\mathbb{A}) - a_1 S_{j-1}(\mathbb{A})$, en soustrayant à chaque ligne (sauf la dernière) a_1 fois la suivante, on transforme le déterminant en le produit d'un déterminant d'ordre n du même type par $S_{(n+1)I}(a_1) = a_1^{(n+1)I}$ ∎

Par exemple, écrivant a, b, c pour a_1, a_2, a_3, on a
$c^2b^0a^4 = S_{204}(\mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1) =$

$$\begin{vmatrix} S_2(\mathbb{A}_3) & S_1(\mathbb{A}_2) & S_6(\mathbb{A}_1) \\ S_1(\mathbb{A}_3) & S_0(\mathbb{A}_2) & S_5(\mathbb{A}_1) \\ S_0(\mathbb{A}_3) & S_{-1}(\mathbb{A}_2) & S_4(\mathbb{A}_1) \end{vmatrix} = \begin{vmatrix} a^2+b^2+c^2+ab+ac+bc & a+b & a^6 \\ a+b+c & 1 & a^5 \\ 1 & 0 & a^4 \end{vmatrix}$$

Dans le cas où $\mathbb{A}^{(1)} = \mathbb{A}_p$, et où $\mathbb{A}^{(2)}, \ldots, \mathbb{A}^{(n+1)}, \mathbb{B}^{(1)}, \ldots, \mathbb{B}^{(n+1)}$ sont invariants par la transposition σ_p , alors pour tout $I \in \mathbb{N}^{n+1}$:

$$(1.10) \qquad S_I(\mathbb{A}_p - \mathbb{B}^{(1)}, \ldots, \mathbb{A}^{(n+1)} - \mathbb{B}^{(n+1)})\,\pi_p = S_I(\mathbb{A}_{p+1} - \mathbb{B}^{(1)}, \ldots, \mathbb{A}^{(n+1)} - \mathbb{B}^{(n+1)})$$

puisque les fonctions invariantes par σ_p sont des scalaires pour π_p et puisqu'en outre $S_j(\mathbb{A}_p)\pi_p = S_j(\mathbb{A}_{p+1})$ d'après 1.6 .

Il existe des opérateurs sur l'anneau des polynômes plus généraux que les π_p et vérifiant les relations de Moore/Coxeter; ils sont décrits dans [L-S5] . Comme cas particulier de ces opérateurs, on a les *différences divisées* ∂_p :

$$(1.11) \qquad f \longrightarrow f(\sigma_p - 1)/(a_{p+1} - a_p) = \frac{f(\ldots a_{p+1}, a_p \ldots) - f(\ldots a_p, a_{p+1} \ldots)}{a_{p+1} - a_p} = f\,\partial_p$$

notées A_{α_p} par [B-G-G], Δ_{α_p} par [D1], p.289 et D_{σ_p} par [D3], p.78.

On a aussi les opérateurs $\psi_p = \pi_p - 1$ (c'est-à-dire π_p - *identité*) et les opérateurs $\Xi_p = \pi_p - \partial_p$. En fait, Ξ_p n'est autre que l'image de π_p par le changement de variable $a_1 \to 1 - a_1, \ldots, a_{n+1} \to 1 - a_{n+1}$. Par produit, on dispose donc d'opérateurs ∂_μ, Ξ_μ et ψ_μ pour $\mu \in \mathfrak{S}(\mathbb{A})$.

Les permutations $\mu \in \mathfrak{S}(\mathbb{A}_n)$ sont considérées comme des morphismes $\mu : \{a_1, \ldots, a_n\} \longrightarrow \{1\mu, \ldots, n\mu\}$ ou comme des mots dans le monoïde sur $\mathbb{A}$: $\mu = (1\mu)(2\mu)\ldots(n\mu)$. Une permutation peut en outre se représenter bijectivement par son *code* $J = (1J, \ldots, nJ) \in \mathbb{N}^n$:

$$(1.12) \qquad \forall h : 1 \leq h \leq n, \quad hJ = card\{k : k > h \ \& \ k\mu < h\mu\} \quad .$$

La *longueur* $\ell(\mu)$ d'une permutation est la somme des composantes de son code : $\ell(\mu) = 1J + \ldots + nJ$; c'est le degré minimum d'une décomposition de μ comme produit de transpositions simples.

Ainsi $\mu : \{a_1, \ldots, a_6\} \longrightarrow \{a_4, a_1, a_3, a_6, a_2, a_5\}$ est notée comme le mot $a_4a_1a_3a_6a_2a_5$, a pour code 301200 et pour longueur 3+0+1+2+0+0 = 6 .

Si le code J de μ est décroissant (on dit *dominant* par référence à la théorie des *poids*) : $1J \geq \ldots \geq nJ$, μ est dite *dominante* .

Le groupe symétrique $\mathfrak{S}(\mathbb{A}_n)$ est canoniquement plongé dans $\mathfrak{S}(\mathbb{A}_{n+1})$ par adjonction du point fixe a_{n+1} , i.e.

$$\mathfrak{S}(\mathbb{A}_n) \ni (1\mu)\ldots(n\mu) \longrightarrow (1\mu)\ldots(n\mu)(a_{n+1}) \in \mathfrak{S}(\mathbb{A}_{n+1})$$

L'ordre d'Ehresmann/Bruhat sur le groupe symétrique est défini par comparaison des *facteurs gauches* des permutations : $\mu, \eta \in \mathfrak{S}(\mathbb{A}_n), \mu \leq \eta$ si et seulement si, pour tout p tel que $1 \leq p \leq n$, le vecteur réordonné en croissant de $\{1\mu, \ldots, p\mu\}$ est inférieur, composante à composante, au vecteur réordonné en croissant de $\{1\eta, \ldots, p\eta\}$ (*cf.* [E]). La comparaison des facteurs droits des permutations donnerait le même ordre. Un raffinement en est décrit dans la note 2.

Ainsi, les réordonnés des facteurs gauches succesifs des permutations 2314, 4132, 4231 sont respectivement

$$\begin{array}{llllll} \mu & = 2314 & \longrightarrow & 2, & 23, & 123, & 1234 \\ \nu & = 4132 & \longrightarrow & 4, & 14, & 134, & 1234 \\ \zeta & = 4231 & \longrightarrow & 4, & 24, & 234, & 1234 \end{array}$$

et l'on a donc $\mu < \zeta$, $\nu < \zeta$, mais non $\mu < \nu$ puisque 23 et 14 ne sont pas comparables.

Lemme 1.13. *Pour toute permutation* $\mu \in \mathfrak{S}(\mathbb{A})$,

$$\pi_\mu = \sum_{\eta \le \mu} \psi_\eta \quad \& \quad \psi_\mu = \sum_{\eta \le \mu} (-1)^{\ell(\mu)-\ell(\eta)} \pi_\eta$$

Preuve. Soient $\mu \in \mathfrak{S}(\mathbb{A}), \sigma$ transposition simple telle que $\ell(\mu\sigma) > \ell(\mu)$. On a donc $\pi_{\mu\sigma} = \pi_\mu \pi_\sigma$, et par l'hypothèse de récurence, $\pi_{\mu\sigma} = \left(\sum_{\eta \le \mu} \psi_\eta\right) \cdot (\pi_\sigma)$. D'après la propriété d'échange [B], IV.1.5), l'intervalle $[\textit{identité},\ \mu\sigma]$ se décompose en trois sous-ensembles : $\mathfrak{S}_1 = \{\nu : \nu \ \& \ \nu\sigma \le \mu\}$, $\mathfrak{S}_2 = \{\xi : \xi \le \mu \ \& \ \xi\sigma \not\le \mu\}$ et $\mathfrak{S}_3 = \{\zeta : \zeta \not\le \mu \ \& \ \zeta\sigma \le \mu\}$. L'expression de $\pi_{\mu\sigma}$ se décompose donc en deux parties :

$$\pi_{\mu\sigma} = \sum_{\nu \in \mathfrak{S}_1} \psi_\nu \, \pi_\sigma + \sum_{\xi \in \mathfrak{S}_2} \psi_\xi \, \pi_\sigma \quad .$$

La sous-somme $\sum \psi_\nu$ est invariante par π_σ puisqu'elle l'est par σ . Ecrivant $\pi_\sigma = 1 + \psi_\sigma$, on obtient pour la deuxième partie

$$\sum \psi_\xi \, \pi_\sigma = \sum \psi_\xi + \sum \psi_{\xi\sigma} = \sum \psi_\xi + \sum \psi_\zeta$$

ce qui démontre la première partie du lemme; la deuxième s'obtient par la même récurrence. ∎

Soient $E = (n, n-1, \ldots, 1, 0) \in \mathbb{N}^{n+1}$ et ♣ l'involution $\mathbb{A} \to \mathbb{A}^{-1}\omega$, i.e. $a_j \to 1/a_{n+2-j}$, $1 \le j \le n+1$. Se reportant à la définition 1.2, on voit que les opérateurs élémentaires vérifient

$$\omega \, a^{-E} \, \pi_j \, a^E \, \omega = -\, \psi_{\omega\sigma_j\omega} \quad , \tag{1.14}$$

$$♣ \, \pi_j \, ♣ = \pi_{\omega\sigma_j\omega} \quad . \tag{1.15}$$

Par produit, on a donc les deux involutions suivantes :

Lemme. — *Pour tout* $\mu \in \mathfrak{S}(\mathbb{A})$, *on a*

$$\omega \, a^{-E} \, \pi_\mu \, a^E \, \omega = (-1)^{\ell(\mu)} \, \psi_{\omega\mu\omega} \quad , \tag{1.16}$$

$$♣ \, \pi_\mu \, ♣ = \pi_{\omega\mu\omega} \quad . \tag{1.17}$$

2. Polynômes de Grothendieck

$\mathbf{K}[\mathbb{A}]$ est un module libre sur l'anneau des polynômes de Laurent symétriques en $\mathbb{A}$; une base de ce module consiste en les monômes $a_1^{1I} \cdots a_{n+1}^{n+1}$, avec $0 \leq 1I \leq n$, $0 \leq 2I \leq n-1$, $\ldots 0 \leq (n+1)I \leq 0$ (*cf.* [Gr1]). Une autre base (la matrice de changement de base est triangulaire de diagonale 1) est celle des *polynômes de Schubert* (cf. [L-S1]) dont les classes dans l'anneau de cohomologie de $\mathcal{F}(V)$ sont les *cycles de Schubert.* Nous définisssons dans ce paragraphe une base plus adaptée à l'anneau de Grothendieck.

Soient $\mathbb{B} = \{b_1, \ldots, b_{n+1}\}$ un deuxième alphabet de même cardinal que $\mathbb{A}$, $\mathbf{K}[\mathbb{A}, \mathbb{B}]$ l'anneau des polynômes de Laurent en $\mathbb{A}$ et $\mathbb{B}$. On définit les *polynômes (doubles) de Grothendieck* $\mathbb{G}_\mu$, $\mu \in \mathfrak{S}(\mathbb{A})$, par :

$$(2.1) \qquad \begin{cases} \mathbb{G}_\omega = \prod_{i+j \leq n+1} (1 - b_i/a_j) \\ \mathbb{G}_\mu = \mathbb{G}_\omega \, \pi_{\omega\mu} \end{cases}$$

où $\pi_{\omega\mu}$ est l'opérateur associé à la permutation $\omega\mu \in \mathfrak{S}(\mathbb{A})$; on aura à l'esprit que les opérateurs $\pi_{\omega\mu}$ sont $\mathbf{K}[b_1, 1/b_1, \ldots, 1/b_{n+1}]$-linéaires. Le *polynôme de Grothendieck (simple)* G_μ est l'image par la spécialisation $\varepsilon_{\mathbb{B}} : b_1 = \cdots = b_{n+1} = 1$ de $\mathbb{G}_\mu$, i.e. $G_\mu = \mathbb{G}_\mu \varepsilon_{\mathbb{B}}$.

Comme les fonctions symétriques en a_p et a_{p+1} sont invariantes par par π_p , d'après 1.6 *1)*, on a donc pour toute $\mu \in \mathfrak{S}(\mathbb{A})$, tout p tel que $1 \leq p \leq n$:

$$(2.2) \qquad p\mu < (p+1)\mu \ \Rightarrow \ \mathbb{G}_\mu = \mathbb{G}_\mu \, \sigma_p = \mathbb{G}_\mu \, \pi_p \quad .$$

Similairement, les *polynômes adjoints* $\mathbb{H}_\mu$, pour $\mu \in \mathfrak{S}(\mathbb{A})$, sont définis par

$$(2.3) \qquad \mathbb{H}_\omega = \prod_{i+j \leq n+1} (1 - a_{n+2-i}/b_j) \quad ,$$

$$(2.4) \qquad \mathbb{H}_\mu = \mathbb{H}_\omega \, \psi_{\omega\mu} \quad .$$

Sur $\mathbf{K}[\mathbb{A}, \mathbb{B}]$ on définit le produit scalaire $< \cdot \mid \cdot >$ par

$$(2.5) \qquad P, Q \in \mathbf{K}[\mathbb{A}, \mathbb{B}] \quad \Rightarrow \quad < P \mid Q >= PQ \, \pi_\omega$$

Aux notations près, c'est le produit scalaire utilisé dans [L-S1], [L-S2].

D'après 1.1), on peut calculer π_ω en sommant sur toutes les permutations μ dans $\mathfrak{S}(\mathbb{A})$:

$$(2.6) \qquad < P \mid Q >= \sum_\mu \big(PQ / \prod_{i<j} (1 - a_j/a_i) \big) \, \mu$$

Lemme 2.7. *Pour tout μ dans $\mathfrak{S}(\mathbb{A})$, les opérateurs π_μ et $\pi_{\mu^{-1}}$ sont adjoints.*

Preuve. Les opérateurs π_μ s'écrivent comme produit d'opérateurs élémentaires; il suffit donc de démontrer le lemme pour ceux-ci. On veut donc pour tout i, tous $P, Q \in \mathbf{K}[\mathbb{A}, \mathbb{B}]$, l'identité $< P\pi_i \,|\, Q >= < P \,|\, Q\pi_i >$. Or $< P\pi_i \,|\, Q >= (P\,\pi_i\, Q)\,\pi_\omega = (P\,\pi_i\, Q)\,\pi_i\,\pi_\omega$ puisque $\pi_i\,\pi_\omega = \pi_\omega$. Le polynôme $P\,\pi_i$, étant invariant par σ_i, commute avec π_i et donc $(P\,\pi_i)\,Q\,\pi_i = Q\,\pi_i(P\,\pi_i) = (Q\,\pi_i)(P\,\pi_i)$. Finalement, $< P\pi_i \,|\, Q > = < P\pi_i \,|\, Q\pi_i >$, d'où par symétrie, $< P\pi_i \,|\, Q >=< P \,|\, Q\pi_i >$ ∎

Comme les $\mathbb{G}_\mu$ sont définies comme images par les opérateurs π_μ de $\mathbb{G}_\omega$ et que les opérateurs π_i sont autoadjoints, il est clair que les produits scalaires $< P|\mathbb{G}_\mu >$, pour μ dans $\mathfrak{S}(\mathbb{A})$ et P dans $\mathbf{K}$[A,B], se calculent à l'aide des seuls $< P|\mathbb{G}_\omega >$, $P \in \mathbf{K}[\mathbb{A}, \mathbb{B}]$.

Le théorème suivant montre plus précisément que les $\mathbb{H}_\mu$ sont la base adjointe des polynômes de Grothendieck $\mathbb{G}_\zeta$ (on note θ la spécialisation $b_1 \to a_1, \ldots, b_{n+1} \to a_{n+1}$; $\omega\theta\omega$ est donc la spécialisation $b_1 \to a_{n+1}, \ldots, b_{n+1} \to a_1$).

Theoreme 2.8.

1) $\forall P \in \mathbf{K}[\mathbb{A}, \mathbb{B}] \quad , \quad P\,\mathbb{G}_\omega\,\pi_\omega\,\theta = P\,\omega\theta\omega$

2) $\forall P \in \mathbf{K}[\mathbb{A}, \mathbb{B}] \quad , \quad P\,\mathbb{H}_\omega\,\pi_\omega\,\theta = P\,\theta$

3) $\forall \mu, \zeta \in \mathfrak{S}(\mathbb{A})$, $< \mathbb{H}_{\omega\mu} \,\big|\, \mathbb{G}_\zeta >= 1$ ou 0 selon que $\mu = \zeta$ ou non

4) $\forall\, P \in \mathbf{K}[\mathbb{A}, \mathbb{B}]$, $P = \sum_{\mu \in S(\mathbb{A})} P\,\mathbb{H}_{\omega\mu}\,\pi_\omega\,\mathbb{G}_\mu = \sum_\mu P\,\psi_\mu\,\mathbb{H}_\omega\,\pi_\omega\,\mathbb{G}_{\mu^{-1}}$

Preuve. Pour 1) et 2), on remarque que lorsque $\mu \neq \omega$, au moins un des facteurs linéaires de $\mathbb{G}_\omega\,\mu$ s'annule par la spécialisation θ et donc $\mathbb{G}_\omega\,\mu\theta = 0$; il en est de même pour $\mathbb{H}_\omega\mu\theta$ lorsque $\mu \neq \omega\omega$. Calculant l'action de l'opérateur $\pi_\omega\theta$ à l'aide de 2.6, on voit que la sommation pour le polynôme $P\mathbb{G}_\omega$ (resp. $P\mathbb{H}_\omega$) se réduit à un seul terme. ∎

On démontre 3) par récurrence sur la longueur de μ. Il convient tout d'abord de remarquer que pour tout couple μ, ν le produit scalaire $< \mathbb{G}_\zeta|\mathbb{H}_{\omega\mu} >$ ne dépend pas de $\mathbb{A}$. En effet, $\mathbb{G}_\zeta\mathbb{H}_{\omega\mu}$ est une combinaison (à coefficients dans $\mathbb{B}$) de monômes $a_1^{1I} \cdots a_{n+1}^{(n+1)I}$, avec $-n \leq 1I \leq 0$,

$-n+1 \leq 2I \leq 1$, ... ,$0 \leq (n+1)I \leq n$. L'image d'un monôme a^I , pour I tel que $-n \leq 1I$, $-n+1 \leq 2I$, ... ,$0 \leq (n+1)I$ est la fonction de Schur $S_{I\omega}(\mathbb{A})$. Le déterminant 1.8 définissant cette fonction a deux colonnes identiques, sauf lorsque les entiers $1I+n, 2I+n-1, \ldots, (n+1)I+0$ sont tous différents (ces entiers sont les indices dans la première ligne du déterminant). Dans ce dernier cas, cet ensemble de $n+1$ entiers compris entre 0 et n est l'ensemble $\{0,1,\ldots,n\}$ à permutation près, ce qui implique qu'alors la fonction de Schur soit égale à $\pm S_{0\cdots 0}(\mathbb{A}) = \pm 1$. Ansi donc, l'assertion *3)* du théorème est équivalente à

$$3')\quad \forall \mu,\zeta \in \mathfrak{S}(\mathbb{A}) \,,\; < \mathbb{H}_{\omega\mu} \mid \mathbb{G}_\zeta > \theta = 1 \text{ ou } 0 \text{ selon que } \mu = \zeta \text{ ou non}$$

Montrons maintenant que $< \mathbb{G}_\zeta \,|\, \mathbb{H}_\omega > \theta = 0$ si $\zeta \neq$ *identité*; en effet $< \mathbb{G}_\zeta \,|\, \mathbb{H}_\omega > \theta = \mathbb{G}_\zeta\, \theta$ d'après *2)* . Or tout symétriseur π_μ s'exprime comme une combinaison linéaire (à coefficients rationnels en $\mathbb{A}$) de permutations $\eta : \eta \leq \mu$, et donc $\mathbb{G}_\zeta\, \theta$ est une combinaison linéaire des $\mathbb{G}_\omega\, \eta\, \theta$ avec $\eta \leq \omega\zeta$. La nullité de ces derniers lorsque $\zeta \neq$*identité* entraine celle de $< \mathbb{G}_\zeta \,|\, \mathbb{H}_\omega > \theta$. Lorsque $\zeta =$ *identité* , $\mathbb{G}_\zeta = 1$ et l'on vérifie directement que $< 1 \,|\, \mathbb{H}_\omega >= 1$.

Soit alors μ telle que pour tout $\zeta \neq \mu$, on ait $< \mathbb{G}_\zeta \mid \mathbb{H}_{\omega\mu} >= 0$ et telle que $< \mathbb{G}_\mu \mid \mathbb{H}_{\omega\mu} >= 1$. Soit de plus $\sigma = \sigma_i$ une transposition simple telle que $\ell(\mu\sigma) > \ell(\mu)$. Alors $\mathbb{H}_{\omega\mu\sigma} = \mathbb{H}_{\omega\mu}\psi_i$ et donc

$$< \mathbb{H}_{\omega\mu\sigma} | \mathbb{G}_\zeta > = < \mathbb{H}_{\omega\mu} | \mathbb{G}_\zeta \psi_i > = \begin{cases} < \mathbb{H}_{\omega\mu} | \mathbb{G}_{\zeta\sigma} - \mathbb{G}_\zeta > & \text{si } \ell(\zeta\sigma) < \ell(\zeta) \\ < \mathbb{H}_{\omega\mu} | 0 > & \text{sinon} \end{cases}$$

puisque $\mathbb{G}_\zeta \pi_i = \mathbb{G}_{\zeta\sigma}$ ou $\mathbb{G}_\zeta$ suivant que $\ell(\zeta\sigma) < \ell(\zeta)$ ou non. On ne peut donc espérer de non nullité que lorsque $\zeta\sigma$ ou ζ est égale à μ; il reste seulement à constater que $< \mathbb{H}_{\omega\mu\sigma} \mid \mathbb{G}_\mu >= 0$ puisque $\ell(\mu\sigma) > \ell(\mu)$ et que $< \mathbb{H}_{\omega\mu\sigma} \mid \mathbb{G}_{\mu\sigma} > = < \mathbb{H}_{\omega\mu} \mid \mathbb{G}_\mu - \mathbb{G}_{\mu\sigma} > = < \mathbb{H}_{\omega\mu} \mid \mathbb{G}_\mu >= 1$. ∎ ∎

Chacun des ensembles $\{\mathbb{G}_\mu\}$ et $\{\mathbb{H}_\mu\}$ étant de cardinal $n+1\,!$, la propriété 3) montre que ce sont deux bases adjointes de $\mathbf{K}[\mathbb{A},\mathbb{B}]$ en tant que module libre sur le sous-anneau des fonctions invariantes par $\mathfrak{S}(\mathbb{A})$. On a donc le développement

$$P = \sum P\,\mathbb{H}_{\omega\mu}\,\pi_\omega\,\mathbb{G}_\mu = \sum < P \mid \mathbb{H}_{\omega\mu} > \mathbb{G}_\mu$$

Comme $< P\,|\,\mathbb{H}_{\omega\mu} >=< P\,|\,\mathbb{H}_\omega\psi_\mu >=< P\psi_{-1}|\mathbb{H}_\omega >$, on peut aussi écrire

$$P = \sum_\mu P\,\psi_{\mu^{-1}}\,\mathbb{H}_\omega\,\pi_\omega\,\mathbb{G}_\mu \quad ,$$

ce qui termine la démonstration du théorème ∎ ∎ ∎

Les polynômes $\mathbb{H}_\omega$ et $\mathbb{G}_\omega$ sont échangés par l'involution ♣ : $a_i \to 1/a_{n+2-i}$, $b_j \to 1/b_j$, que nous continuons à noter comme sa restriction à $\mathbf{K}[\mathbb{A}]$ vue au paragraphe 1. Plus généralement :

Lemme *Pour toute $\mu \in \mathfrak{S}(\mathbb{A})$ on a*

1) $$\mathbb{H}_\omega\,\pi_\mu = \mathbb{G}_{\mu\omega}\,♣ \quad ,$$
2) $$(-1)^{\ell(\omega\mu)}\,\mathbb{H}_{\mu\omega}\;\omega = \mathbb{G}_{\omega\mu}\;a^E\,b^{-E} \quad .$$

Preuve. On vient de remarquer que $\mathbb{H}_\omega = \mathbb{G}_\omega$♣; il s'ensuit que $\mathbb{H}_\omega\,\pi_\mu = \mathbb{G}_\omega$♣$\pi_\mu$♣♣ $= \mathbb{G}_\omega\,\pi_{\omega\mu\omega}$♣ , d'après 1.7, ce qui donne *i)*. Par ailleurs, $(-1)^{\ell(\omega\mu)}\mathbb{H}_{\mu\omega}\,\omega = (-1)^{\ell(\omega\mu)}\mathbb{H}_\omega\,\psi_{\omega\mu\omega}\,\omega$, expression que l'on transforme grâce à 1.16 en $(-1)^{\ell(\omega)}\mathbb{H}_\omega\,\omega\,a^{-E}\,\pi_\mu\,a^E\,\omega\,\omega = \mathbb{G}_\omega\,b^{-E}\,\pi_\mu\,a^E$ $= \mathbb{G}_{\omega\mu}\,a^E\,b^{-E}$. ∎

Le polynôme $\mathbb{G}_\omega$ est un produit de facteurs linéaires et cette propriété est partagée par les polynômes $\mathbb{G}_\mu$ suivants :

Lemme 2.10. *Soit μ une permutation dominante de code J (i.e. telle que $1J \geq 2J \geq \cdots$). Alors*

$$\mathbb{G}_\mu = (1 - b_1/a_1)\cdots(1 - b_{1J}/a_1)(1 - b_1/a_2)\cdots(1 - b_{2J}/a_2)\cdots \quad .$$

Preuve. On peut trouver une chaîne, de ω à μ , de permutations dominantes. On est donc ramené par récurrence à calculer $\mathbb{G}_{\mu\sigma} = \mathbb{G}_\mu\,\pi_p$ pour μ dominante et p tel que $pJ = (p+1)J + 1$. Comme alors $\mathbb{G}_\mu/(1 - b_{pJ}/a_p)$ est symétrique en a_p et a_{p+1} , son image est donc égale à $(1 - b_{pJ}/a_p)\,\pi_p\,\mathbb{G}_\mu/(1 - b_{pJ}/a_p)$. ∎

Exemple. Pour $\mathfrak{S}(3)$, on a les polynômes de Grothendieck suivants, en écrivant $\mathbb{A} = \{a, b, c\}$ et $\mathbb{B} = \{x, y, z\}$. De fait, seule la permutation $\mu = 132$ n'est pas dominante.

$$\begin{tikzcd}
 & \mathbb{G}_{321} = (1-\frac{x}{a})(1-\frac{x}{b})(1-\frac{y}{a}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{G}_{231} = (1-\frac{x}{a})(1-\frac{x}{b}) \arrow[d, "\pi_2"'] & & (1-\frac{x}{a})(1-\frac{y}{a}) = \mathbb{G}_{312} \arrow[d, "\pi_1"] \\
\mathbb{G}_{213} = (1-\frac{x}{a}) \arrow[dr, "\pi_1"'] & & (1-\frac{xy}{ab}) = \mathbb{G}_{132} \arrow[dl, "\pi_2"] \\
 & \mathbb{G}_{123} = 1 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{H}_{321} = (1-\frac{c}{x})(1-\frac{b}{x})(1-\frac{c}{y}) \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\
\mathbb{H}_{231} = \frac{b}{x}(1-\frac{c}{x})(1-\frac{c}{y}) \arrow[d, "\psi_2"'] & & \frac{c}{y}(1-\frac{c}{x})(1-\frac{b}{x}) = \mathbb{H}_{312} \arrow[d, "\psi_1"] \\
\mathbb{H}_{213} = \frac{c}{x}(1-\frac{bc}{xy}) \arrow[dr, "\psi_1"'] & & \frac{bc}{xy}(1-\frac{c}{x}) = \mathbb{H}_{132} \arrow[dl, "\psi_2"] \\
 & \mathbb{H}_{123} = \frac{bcc}{xxy} &
\end{tikzcd}$$

On peut vérifier l'assertion 2.8 *2)* pour $P = a$, par exemple, écrivant $\pi_\omega = \pi_2\,\pi_1\,\pi_2$. En effet $a\mathbb{H}_\omega = a(1-c/x)(1-b/x)(1-c/y) \xrightarrow{\pi_2} a(1-c/x) \times (1-b/x) \xrightarrow{\pi_1} (1-c/x)(a+b-ab/x) \xrightarrow{\pi_2} a+b+c-(ab+ac)/x - bc/x + abc/xx \xrightarrow{\theta} a+b+c-(b+c)-bc/a+bc/a = a$.

De même, à titre d'exemple de 2.8 *3)*, et du fait qu'il n'est pas nécessaire de spécialiser le produit scalaire par θ , on a $< \mathbb{G}_{132} \mid \mathbb{H}_{312} >= 1$; en effet, dans le produit $(1-xy/ab)\frac{c}{y}(1-c/x-b/x+bc/xx)$, seul le monôme $(-xy/ab)(-cc/xy)$ a une image non nulle par π_ω (et égale à 1).

3. Postulation

Les polynômes de Grothendieck ont été obtenus comme images par les π_μ de l'élément $\mathbb{G}_\omega = \prod_{i+j\leq n+1}(1 - b_i/a_j)$. On peut se proposer de les exprimer en fonction des images des monômes en $\mathbb{A}$ par les π_μ .

Le théorème 2.8 montre qu'en fait chaque opérateur π_μ peut lui-même s'interpréter comme le produit scalaire avec le polynôme de Grothendieck $\mathbb{G}_{\omega\mu}$. En effet, d'après 2.8 *1)* , et puisque π_μ est adjoint de $\pi_{\mu^{-1}}$, on a la suite d' égalités : $a^I\pi_{\mu^{-1}} = a^I\pi_{\mu^{-1}}\omega\theta\omega =< \mathrm{a}^\mathrm{I}\pi_{\mu^{-1}} \mid \mathbb{G}_\omega > \theta =< \mathrm{a}^\mathrm{I} \mid \mathbb{G}_\omega\pi_\mu > \theta =< \mathrm{a}^\mathrm{I} \mid \mathbb{G}_{\omega\mu} > \theta$, c'est-à-dire qu'on a le lemme suivant :

Lemme 3.1. $\qquad a^I\,\pi_{\mu^{-1}} = a^I\,\mathbb{G}_{\omega\mu}\,\pi_\omega\,\theta$

La spécialisation $\varepsilon_{\mathbb{B}} : b_j \longrightarrow 1$ de ce lemme est due à [D3], th.2, p.87.

Lorsque I est *dominant* $\big($i.e. $1I \geq 2I \geq \cdots \geq (n+1)I \geq 0\big)$, le polynôme $a^I\,\pi_{\mu^{-1}}$ admet l'interprétation géométrique suivante : le fibré inversible $L^I = L_1^{1I} \otimes L_2^{2I} \otimes \cdots \otimes L_{n+1}^{(n+1)I}$ a sa cohomologie, au-dessus de toute variété de Schubert $Schub_{\omega\mu}$, concentrée en degré 0 $\big($cf. [D3], th1, p.84 et [Se], th1, p.362$\big)$; la classe de cet espace de cohomologie $\mathcal{H}^0(Schub_{\omega\mu}, \mathrm{L}^\mathrm{I})$ dans l'anneau de Grothendieck de la variété de drapeaux est donc $a^I\,\mathbb{G}_{\omega\mu}\,\pi_\omega\,\varepsilon_{\mathbb{B}} = a^I\,\pi_{\mu^{-1}}$.

Si $\nu \in \mathfrak{S}(\mathbb{A})$ stabilise a^I , alors $a^I\pi_{\nu\mu} = a^I\pi_\mu$. Pour tenir compte du stabilisateur de I , c'est-à-dire des répétitions des parts de I, on définit inductivement les $T_I \in \mathbb{Z}[\mathbb{A}], I \in \mathbb{N}^{n+1}$, comme suit :

$$(3.2) \qquad \begin{cases} I \quad \text{dominant} & \Rightarrow \quad T_I = a^I \\ \exists p : i_p \leq i_{p+1} & \Rightarrow \quad T_I = T_{I\sigma_p}\pi_p \end{cases}$$

Ainsi, pour $\mathfrak{S}(3)$, il n'y a que trois images différentes du monôme $a_1^5a_2^3a_3^3 = T_{533}$, à savoir $T_{533} = T_{533}\,\pi_2$; $T_{353} = T_{533}\,\pi_1 = T_{533}\,\pi_2\pi_1$; $T_{335} = T_{533}\,\pi_1\pi_2 = T_{533}\,\pi_1\pi_2\pi_3$.

Si I est une *partition* $\big($i.e. $0 \leq 1I \leq 2I \leq ... \leq (n+1)I\ \big)$, alors T_I est la fonction de Schur classique $S_I(\mathbb{A}_{n+1})$. La formule des caractères de Demazure [D2], prop.4, s'écrit, dans le cas du groupe linéaire :

Théorème 3.3. *Soient I dominant , $\sigma\sigma'\cdots\sigma''$ une décomposition (non nécessairement réduite) de l'élément maximal ω de $\mathfrak{S}(\mathbb{A})$. Alors a^I , $a^I\pi_\sigma$, $a^I\pi_\sigma\pi_{\sigma'},\ldots,a^I\pi_\sigma\pi_{\sigma'}\cdots\pi_{\sigma''} = S_{I\omega}(\mathbb{A})$ est une suite croissante de polynômes dans $\mathbb{Z}[\mathbb{A}]$.*

(On écrit $P \geq Q$ pour deux polynômes si les coefficients de $P - Q$ sont ≥ 0).

On a ainsi la suite croissante suivante : $a_1^2 a_2 = T_{210} \xrightarrow{\pi_1} T_{120} = a_1 a_2 (a_1 + a_2) \xrightarrow{\pi_2} T_{102} = a_1^2(a_2 + a_3) + a_1(a_2^2 + a_2 a_3 + a_3^2) \xrightarrow{\pi_1} T_{012} = \sum a_i^2 a_j + 2a_1 a_2 a_3$.

On a en fait une propriété de positivité plus forte que 3.3 :

Proposition 3.4. *Pour tout I dominant, toute permutation μ , le polynôme*

$$a^I \psi_\mu = \sum_{\eta \leq \mu} (-1)^{\ell(\mu)-\ell(\eta)} a^I \pi_\eta$$

a ses coefficients positifs ou nuls (la somme s'effectuant sur toutes les permutations dans l'intervalle $[id, \mu]$ pour l'ordre d'Ehresmann/Bruhat).

Ainsi, pour $I = 210, \mu = 321$, le polynôme $T_{210}\,\psi_{321} = T_{012} - T_{102} - T_{021} + T_{120} + T_{201} - T_{210}$ est positif (il est égal à $a_2 a_3^2$) .

Nous ne démontrerons pas la proposition 3.4 que l'on trouve dans [L-S3], th.5, comme corollaire de l'extension à l'algèbre libre (i.e. en variables non commutatives) des fonctions T_I (qui coïncident alors avec les “bases standard” de [L-M-S]; voir Note 1).

L'expression 1.13 de π_μ en fonction des ψ_η entraine le corollaire suivant :

Corollaire 3.5. *Pour tout I dominant, toute permutation μ ,*

$$a^I \pi_\mu = \sum_{\eta \leq \mu} a^I \psi_\eta \quad .$$

On retrouve bien ainsi le théorème 3.3 puisque dans le cas d'une permutation μ et d'une transposition simple σ telles que $\ell(\mu\sigma) = \ell(\mu) + 1$, on a

$$a^I \pi_{\mu\sigma} - a^I \pi_\mu = \sum_\eta a^I \psi_\eta$$

somme sur toutes les permutations $\eta \in [id, \mu\sigma] \setminus [id, \mu]$.

Dans la suite croissante 3.3, les deux termes extrêmes sont des fonctions de Schur (au sens 1.8), c'est-à-dire la suite interpole entre le monôme a^I et la fonction de Schur $S_{I\omega}(\mathbb{A})$. En fait, d'autres termes de cette suite sont aussi des fonctions de Schur, ceux correspondant aux vecteurs vexillaires (définis ci-dessous) :

Définition 3.6. $I \in \mathbb{N}^{n+1}$ est *vexillaire* ssi

1) pour tout $h : 1 \leq h \leq n$, l'inégalité $hI > (h+1)I$ implique $hI > kI$, pour tout $k > h$.

2) l'inégalité $hI \geq kI$ implique card $\{j, h < j < k \,\&\, jI < hI\} \leq hI - kI$.

A tout $I \in \mathbb{N}^{n+1}$ on associe un deuxième élément $J \in \mathbb{N}^{n+1}$ comme suit :

$$hJ = max\{j : h \leq j \leq n+1, hI \leq jI\} \tag{3.7}$$

Soit enfin $I\mathbf{D}$ l'élément de $\mathbb{N}^{n+1}$: $(1I\mathbf{D} \geq 2\mathrm{ID} \geq \cdots \geq (\mathrm{n}+1)\mathrm{ID})$ réordonné de J , i.e. tel que $\{1I\mathbf{D}, \ldots, (\mathrm{n}+1)\mathrm{ID}\} = \{1\mathrm{J}, \ldots, (\mathrm{n}+1)\mathrm{J}\}$. Le drapeau $\{\mathbb{A}_{1I\mathbf{D}}, \mathbb{A}_{2I\mathbf{D}}, \ldots, \mathbb{A}_{(n+1)I\mathbf{D}}\}$ est dit *drapeau associé* à I [L-S4].

Lemme 3.8. *Soient $I \in \mathbb{N}^{n+1}$ vexillaire, $I\mathbf{R}$ la partition obtenue en réordonnant I , $\{\mathbb{A}_{1I\mathbf{D}}, \ldots, \mathbb{A}_{(n+1)I\mathbf{D}}\}$ le drapeau associé à I . Alors*

$$T_I = S_{I\mathbf{R}}(\mathbb{A}_{1I\mathbf{D}}, \mathbb{A}_{2I\mathbf{D}}, \ldots, \mathbb{A}_{(n+1)I\mathbf{D}}) \quad .$$

La preuve consiste à exhiber une chaine : $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_k = I$ telle que I_1 soit dominant, que pour tout h, I_h soit vexillaire et qu'il existe une transposition simple σ^h : $T_{I_{h+1}} = T_{I_h} \pi_{\sigma^h}$. On peut alors appliquer répétitivement 1.10 et obtenir le déterminant cherché; cette preuve est exactement la même que pour les polynômes de Schubert vexillaires, au changement des π_i en les ∂_i près; nous renvoyons à [Wa], th.2.3, avec la réserve que le drapeau construit par cet auteur ne coïncide avec le drapeau associé au code de μ que dans le cas vexillaire . ∎

Par exemple, pour $n = 8$ et $I = 000\,353\,021$, on trouve $J = 999\,656\,989$ et $ID = 999\,998\,665$ ce qui montre d'après 3.8 que $T_{000353021} = S_{000012335}(\mathbb{A}_9, \mathbb{A}_9, \mathbb{A}_9, \mathbb{A}_9, \mathbb{A}_9, \mathbb{A}_8, \mathbb{A}_6, \mathbb{A}_6, \mathbb{A}_5) = S_{12335}(\mathbb{A}_9, \mathbb{A}_8, \mathbb{A}_6, \mathbb{A}_6, \mathbb{A}_5)$.

Pour atteindre tout $I \in \mathbb{N}^{n+1}$, on a besoin de l'image de fonctions de Schur dans un cas plus général que 1.10, c'est-à-dire dans le cas où il y a plus d'une colonne du déterminant 1.8 qui ne soit pas invariante par π_p . Par exemple, $T_{4602} = S_{246}(\mathbb{A}_4, \mathbb{A}_2, \mathbb{A}_2)$ est vexillaire; son image par π_2 , T_{4062} s'obtient en écrivant $\mathbb{A}_2 = \mathbb{A}_3 \setminus \{a_3\}$, d'où $T_{4602} = S_{246}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3) - a_3 S_{236}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3) - a_3 S_{245}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3) + a_3^2 S_{235}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3)$. Comme $a_3^0 \pi_2 = 1, a_3^1 \pi_2 = 0$ et $a_3^2 \pi_2 = -a_2 a_3$, on tire $T_{4062} = S_{246}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3) - a_2 a_3 S_{235}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_3)$.

La même méthode conduit à l'expression de tout T_I comme une somme de fonctions de Schur, somme qui ne nous semble présenter de propriétés remarquables qu'en liaison avec l'ensemble des décompositions réduites de la permutation dont I est le code (cf. [St], [E-G] et [L-S1]). Cela met en jeu une combinatoire différente de celle de cet article et pour cette raison, nous ne l'aborderons pas ici.

Comme d'après 3.1, $a^J \pi_{\mu^{-1}}$ est égal à la spécialisation par θ de $a^J \mathbb{G}_{\omega\mu}\pi_\omega$, il est plus intéressant de calculer, au lieu des T_I , les fonctions $a^J \mathbb{G}_{\omega\mu}\pi_\omega$. Tout d'abord, on a le lemme

Lemme 3.9. *Soient $p : 1 \leq p \leq n$ et μ une permutation dominante de code J ; soit $I \in \mathbb{N}^{n+1}, I \geq J\omega$. Alors*

$$a^I \, \omega \, \mathbb{G}_\mu = S_I\big(\mathbb{A}_{n+1} - \mathbb{B}_{(n+1)J}, \ldots, \mathbb{A}_1 - \mathbb{B}_{1J}\big) \quad .$$

En effet, $m \geq j \Rightarrow S_m(x - \mathbb{B}_j) = x^{m-j} S_j(x - \mathbb{B}_j) = x^{m-j}(x - b_1) \cdots \times (x - b_j)$. Soustrayant $\{a_1\} = \mathbb{A}_1$ dans toutes les lignes , sauf la dernière, du déterminant $S_I\big(\mathbb{A}_{n+1} - \mathbb{B}_{(n+1)J}, \ldots, \mathbb{A}_1 - \mathbb{B}_{1J}\big)$ on obtient comme dernière colonne $\big($posant $(n+1)I = i$, $1J = j\big)$ le vecteur

$$S_{i+n}(-\mathbb{B}_j), \ldots, S_{i+1}(-\mathbb{B}_j) \ , \ S_i(a_1 - \mathbb{B}_j),$$

i.e. $0, \ldots, 0$, $a_1^{i-j}(a_1 - b_1) \cdots (a_1 - b_j)$; on peut transformer de même le cofacteur de $S_i(a_1 - \mathbb{B}_j)$, et l'on obtient en fin de compte comme valeur du déterminant le produit de $a^I\omega$ par un polynôme qui n'est autre que $\mathbb{G}_\mu$ ∎

Le monôme $a_1 ... a_{n+1}$ étant un scalaire pour les opérateurs π_i , le lemme 3.9 a pour corollaire, pour toute permutation μ :

$$(3.10) \qquad (a_1 \cdots a_{n+1})^n \, \mathbb{G}_\mu = S_{n\,.\,n}\big(\mathbb{A}_{n+1} - \mathbb{B}_0, \ldots, \mathbb{A}_1 - \mathbb{B}_n\big)\, \pi_{\omega\mu}$$

Certains des polynômes $(a_1 \cdots a_{n+1})^n \, \mathbb{G}_\mu$ sont des fonctions de Schur, par exemple d'après 3.9, dans le cas où μ est une permutation dominante. La propriété 1.10 autorisant la présence d'un drapeau $\mathbb{B}$ invariant par les σ_j , la même méthode que pour 3.8 et que nous n'avons fait qu'esquisser, conduit à la proposition suivante plus générale qui montre que les polynômes de Grothendieck associés à des permutations vexillaires sont à un facteur $(a_1 \cdots a_{n+1})^n$ près des fonctions de Schur. C'est le cas en particulier de toutes les permutations correspondant aux sous-variétés de Schubert des grassmanniennes; il serait intéressant de caractériser géométriquement les autres permutations vexillaires.

Proposition 3.11. *Soient μ une permutation vexillaire, I son code, $\{\mathbb{A}_{1I\mathbf{D}},\ldots,\mathbb{A}_{(n+1)I\mathbf{D}}\}$ le drapeau d'alphabets associé à I . Soit η la plus petite permutation dominante $\eta\geq\mu$ telle que $\ell(\eta)-\ell(\eta^{-1}\mu)=\ell(\mu)$, H le code de η . Alors*

$$\begin{aligned}&(a_1\cdots a_{n+1})^n\,\mathbb{G}_\mu\\&=S_{n\ldots n}\big(\mathbb{A}_{1I\mathbf{D}}-\mathbb{B}_{(n+1)H},\mathbb{A}_{2I\mathbf{D}}-\mathbb{B}_{nH},\ldots,\mathbb{A}_{(n+1)I\mathbf{D}}-B_{1H}\big)\quad.\end{aligned}$$

Exemple. $\eta=3412$ est monomiale de code $2200=H$; on a donc $(a_1...a_4)^3\mathbb{G}_{3412}=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_2-\mathbb{B}_2,\mathbb{A}_1-\mathbb{B}_2)$; l'image par π_2 de cette identité est $(a_1...a_4)^3\mathbb{G}_{3142}=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_2,\mathbb{A}_1-\mathbb{B}_2)$, puis par π_1, $(a_1...a_4)^3\mathbb{G}_{1342}=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_2,\mathbb{A}_2-\mathbb{B}_2)=S_{3333}(\mathbb{A}_4,\mathbb{A}_4,\mathbb{A}_3-\mathbb{B}_2,\mathbb{A}_3-\mathbb{B}_2)$. Or $\mu=1342$ a pour code 0110 , ce qui par 3.7 donne la suite $J=4334$ (*cf.* 3.7) et le drapeau $\{\mathbb{A}_4,\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3\}$ ainsi que requis par 3.11 .

On trouve ainsi, pour $\mathfrak{S}(4)$ et en écrivant μ : $ijh|klm$ au lieu de $(a_1a_2a_3a_4)^3\mathbb{G}_\mu=S_{3333}(\mathbb{A}_4,\mathbb{A}_i-\mathbb{B}_k,\mathbb{A}_j-\mathbb{B}_l,\mathbb{A}_h-\mathbb{B}_m)$, les 23 polynômes vexilaires suivants :

4321 : 321|123; 3421 : 321|122, 4231 : 321|113, 4312 : 321|023;
2431 : 322|113, 3241 : 321|112, 3412 : 321|022, 4132 : 331|023 ,
4213 : 321|013; 1432 : 332|023, 2341 : 321|111, 2413 : 322|013 ,
3142 : 331|022; 4123 : 321|003, 3214 : 321|012; 1342 : 332|022 ,
1423 : 322|003, 2314 : 321|011, 3124 : 321|002; 1243 : 333|003 ,
1324 : 322|002, 2134 : 321|001, 1234 : 321|000 .

Seule $\mu=2143$ n'est pas vexillaire; partant de
$322|013=(a_1a_2a_3a_4)^3\mathbb{G}_{2413}=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_2-\mathbb{B}_1,\mathbb{A}_2-\mathbb{B}_3)$
$=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)-a_3S_{3323}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)$
$-a_3S_{3332}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)+a_3^2S_{3322}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)$,
on déduit par π_2 la valeur manquante

$$\begin{aligned}(a_1a_2a_3a_4)^3\,\mathbb{G}_{2143}&=S_{3333}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)\\&\quad-a_2a_3\,S_{3322}(\mathbb{A}_4,\mathbb{A}_3,\mathbb{A}_3-\mathbb{B}_1,\mathbb{A}_3-\mathbb{B}_3)\qquad,\end{aligned}$$

puisque $a_3\pi_2=0$ et $a_3^2\pi_2=-a_2a_3$. Explicitant, on trouve finalement l'identité $\mathbb{G}_{2143}=\mathbb{G}_{2134}\cdot\mathbb{G}_{1243}$ qui se déduit aussi du théorème 6.4.

4. Théorème de Riemann-Roch

Le théorème de Riemann-Roch exprime la compatibilité des projections de l'anneau de cohomologie et de l'anneau de Grothendieck de $\mathcal{F}(V)$ sur sa base. Plus précisément, d'après [Gr2] :

Théorème (Riemann-Roch). *Le pentagone suivant est commutatif :*

$$(4.1)\qquad \begin{tikzcd} \mathrm{K}(\mathcal{F}(V)) \arrow[r, "\underline{\mathbf{Ch}}"] \arrow[d] & \mathrm{H}(\mathcal{F}(V)) \arrow[r, "\underline{\mathbf{Todd}}"] & \mathrm{H}(\mathcal{F}(\mathrm{V})) \arrow[dl] \\ \mathrm{K}(Base) \arrow[r, "\underline{\mathbf{Ch}}"] & \mathrm{H}(Base) & \end{tikzcd}$$

Les flèches verticales commutant à la multiplication par les fonctions symétriques en $\mathbb{A}$, le pentagone de Riemann-Roch/Grothendieck est induit par l'heptagone commutatif suivant :

$$\begin{tikzcd} \mathrm{K}(\mathcal{F}(V)) \arrow[dr] & \mathrm{K}[\mathbb{A}] \arrow[l] \arrow[r, "\underline{\mathbf{Ch}}"] \arrow[d, "\pi_\omega"] & \mathrm{H}[[\Gamma]] \arrow[r, "\underline{\mathbf{Todd}}"] & \mathrm{H}[[\Gamma]] \arrow[r] \arrow[d, "\partial_\omega"'] & \mathrm{H}(\mathcal{F}(V)) \arrow[dl] \\ (4.2) & \mathrm{K}(Base) \arrow[rr, "\underline{\mathbf{Ch}}"] & & \mathrm{H}(Base) & \end{tikzcd}$$

où $\Gamma = \{\gamma_1, \ldots, \gamma_{n+1}\}$ est un alphabet supplémentaire, $\mathbf{H}[[\Gamma]]$ l'anneau des séries formelles en Γ , où $\underline{\mathbf{Ch}}$ est le morphisme de Chern $a_i \longrightarrow exp(\gamma_i)$, $1 \leq i \leq n+1$, $\underline{\mathbf{Todd}}$ le produit par l'élément $\Delta(\Gamma)/\prod_{i<j}\left(1-exp(\gamma_j-\gamma_i)\right)$, π_ω le morphisme $f \longrightarrow \sum\left[fa^E/\Delta(\mathbb{A})\right]^\mu$, ∂_ω le morphisme $f \longrightarrow \sum\left[f/\Delta(\Gamma)\right]^\mu$, la première somme s'effectuant sur toutes les permutations de $\mathfrak{S}(\mathbb{A})$, la deuxième sur celles de $\mathfrak{S}(\Gamma)$

Preuve. On a bien

$$\begin{aligned} f\,\pi_\omega\,\underline{\mathbf{Ch}} &= \sum\Big[\frac{f}{\prod_{i<j}(1-a_j/a_i)}\Big]^\mu \underline{\mathbf{Ch}} = \sum\Big[\frac{f\,\underline{\mathbf{Ch}}}{\prod_{i<j}(1-exp(\gamma_j-\gamma_i)}\Big]^\mu \\ &= \sum\left[f\,\underline{\mathbf{Ch}}\,\underline{\mathbf{Todd}}(1/\Delta(\Gamma))\right]^\mu = f\,\underline{\mathbf{Ch}}\,\underline{\mathbf{Todd}}\,\partial_\omega \quad , \end{aligned}$$

ce qui montre 4.2 et donc Riemann-Roch dans le cas d'une fibration en variétés de drapeaux (la partie difficile de ce théorème est le cas d'une immersion fermée, cf. [Gr2]).

On peut alors se poser la question : par quoi remplacer le diagramme 4.2 dans le cas d'un opérateur π_μ au lieu de π_ω ?

Pour apporter une réponse (dont nous n'avons pas d'interprétation géométrique), nous allons tout d'abord écrire différemment la compatibilité entre π_ω et ∂_ω. Introduisons une indéterminée supplémentaire u et définissons $\underline{\mathrm{CH}}$ comme le morphisme $\underline{\mathrm{CH}} : a_i \to exp(u\gamma_i)\,,\ 1 \leq i \leq n+1$. Soient ∇ la dérivée usuelle par rapport à u (opérant sur sa gauche!) , ε la spécialisation $a_1 \to 1, \ldots, a_{n+1} \to 1, u \to 0$. On remarquera que ∇ commute avec les différences divisées $\partial_i : f \to \big(f\sigma_i - f\big)/(\gamma_{i+1} - \gamma_i)$.

Lemme 4.3. *Pour tout $f \in \mathrm{K}[\mathbb{A}]$, on a*

$$f\,\pi_\omega\,\varepsilon = f\,a^E\,\underline{\mathrm{CH}}\,\nabla^{\ell(\omega)}\,\partial_\omega\,\varepsilon/\,\ell(\omega)\,! \quad .$$

Démonstration. Il suffit par linéarité de vérifier l'assertion pour un monôme $f = a^I$. Posons $I+E = (1I+n, 2I+n-1, \ldots, (n+1)I+0) = J$. Alors $f\pi_\omega$ est la fonction de Schur d'indice $I\omega$, dont la spécialisation (par ε) est $\Delta(J)/\Delta(E)$ (i.e. $\prod_{h<k}(j_h - j_k)/n!\cdots 0\,!$; cf. [Ma] Ex.4, p.28).

D'un autre côté , $\sum(-1)^{\ell(\mu)}\big[a^J\underline{\mathrm{CH}}\big]^\mu$ est le développement du déterminant $\big|exp(u\,j_k\,\gamma_h)\big|_{1\leq h,k\leq n+1}$, et est donc égal à

$$\frac{u^{\ell(\omega)}}{\Delta(E)}\sum(-1)^{\ell(\mu)}\Delta(J)\big[\gamma_1^n\cdots\gamma_{n+1}^0\big]^\mu + \begin{array}{l}\text{termes de degré}\\ \text{en } u \text{ supérieur à } \ell(\omega)\end{array} \quad .$$

On a donc

$$\begin{aligned} a^{I+E}\underline{\mathrm{CH}}\,\partial_\omega &= \sum(-1)^{\ell(\mu)}\big[a^J\underline{\mathrm{CH}}\big]^\mu/\Delta(\Gamma) \\ &= u^{\ell(\omega)}\Delta(J)/\Delta(E) + u^{\ell(\omega)+1}(\cdots) \quad . \end{aligned}$$

La fonction $a^{I+E}\,\underline{\mathrm{CH}}\,\partial_\omega\,\nabla^{\ell(\omega)}\,\varepsilon/\ell(w)\,!$ se réduit en fin de compte à $\big(u^{\ell(\omega)}\,\nabla^{\ell(\omega)}/\ell(\omega)\,!\big)\big(\Delta(J)/\Delta(E)\big)$ ∎

Le lemme précédent est à rapprocher de la "Formule des dimensions" de Weyl [We] .

D'après la formule de postulation 3.1 ,

$$f\pi_{\mu^{-1}}\varepsilon = fG_{\omega\mu}\pi_\omega\varepsilon \quad .$$

Posons $\mathfrak{g}_{\omega\mu} = G_{\omega\mu}\ \underline{\mathrm{CH}}\,\big|_{\ell(\omega)}$ (c'est-à-dire on limite à l'ordre $\ell(\omega)$ en u le développement de $G_{\omega\mu}\ \underline{\mathrm{CH}}$).

Par exemple, pour $\mathfrak{S}(3)$, avec $\Gamma = \{\alpha, \beta, \gamma\}$, on a $\mathfrak{g}_{321} = u^3\alpha^2\beta$; $\mathfrak{g}_{231} = u^2\alpha\beta - u^3\alpha\beta(\alpha+\beta)/2$; $\mathfrak{g}_{312} = u^2\alpha^2 - u^3\alpha^3$; $\mathfrak{g}_{213} = u\alpha - u^2\alpha^2/2 + u^3\alpha^3/6$; $\mathfrak{g}_{132} = u(\alpha+\beta) - u^2(\alpha+\beta)^2/2 + u^3(\alpha+\beta)^3/6$; $\mathfrak{g}_{123} = 1$.

Il est clair que $\mathfrak{g}_\mu u^{-\ell(\mu)}$ est un polynôme; combinant alors 4.3 et 3.1 , on obtient :

Proposition 4.4 . *Pour tout* $f \in \mathbf{K}[\mathbb{A}]$ *, on a*

$$f\, \pi_{\mu^{-1}}\, \varepsilon = f\, a^E\, \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell(\omega\mu)}\, \nabla^{\ell(\mu)}\, \partial_\omega\, \varepsilon / \ell(\mu)\,! \quad .$$

Ainsi, pour $\mathfrak{S}(3)$ et $f = a_1^7 a_2^4 a_3^1$, le membre de gauche de 4.4 s'écrit $a_1^7 a_2^4 a_3^1\, \pi_{231}\, \varepsilon = a_1^7 a_2^4 a_3^1\, \pi_2 \pi_3\, \varepsilon = S_{147}(\mathbb{A}_3, \mathbb{A}_3, \mathbb{A}_2)\varepsilon = 22$; celui de droite est $\exp\big(u(7\alpha + 4\beta + \gamma)\big) \cdot \exp\big(u(2\alpha+\beta)\big) \cdot \big(\alpha - u\alpha^2/2 + u^2\alpha^3/6\big)$ et a pour terme en u^2 : $\alpha(9\alpha + 5\beta + \gamma)^2/2 - \alpha^2(9\alpha + 5\beta + \gamma)/2 + \alpha^3/6$; son image par ∂_ω est $(-25 + 1 + 90 - 18)/2 - (5-1)/2 = 22$, puisque $\alpha^2\beta\partial_\omega = 1 = -\beta^2\alpha\partial_\omega = -\alpha^2\gamma\partial_\omega = \alpha\gamma^2\partial_\omega ; \alpha^3\partial_\omega = 0 = \alpha\beta\gamma\partial_\omega$.

Soient a^I un monôme, $\exp(u\hbar)$ son image par $\underline{\mathrm{CH}}$, μ une permutation , $\ell = \ell(\mu)$ sa longueur. La dérivée $a^E\big(1 - za^I\big)^{-1} \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell} \nabla^\ell / \ell\,!$ s'écrit $\big(1 - z\exp(u\hbar)\big)^{-\ell-1} \big[(z\hbar)^\ell \big(a^E \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell}\big) + \big(1 - z\exp(u\hbar)\big)(\ldots)\big]$. Par spécialisation ε , on a donc $a^E\big(1 - za^I\big)^{-1}\, \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell}\, \nabla^\ell\, \varepsilon / \ell\,! = (1-\mathrm{z})^{-\ell-1} \big[(\mathrm{z}\hbar)^\ell\, (\mathfrak{g}_{\omega\mu}\, \mathrm{u}^{-\ell}\, \varepsilon) + (1-\mathrm{z})(\ldots)\big]$.

Appliquant ∂_ω , on obtient que $a^E\big(1 - za^I\big)^{-1} \underline{\mathrm{CH}}\, \mathfrak{g}_{\omega\mu}\, u^{-\ell}\, \nabla^\ell\, \varepsilon\, \partial_\omega \times (1-z)^{\ell+1}/\ell\,!$ est un polynôme, noté $\mathcal{N}_{I,\mu}(z)$, de degré $\leq \ell$ en z .

La valeur en $z = 1$ de ce polynôme est $\hbar^\ell\, \mathfrak{g}_{\omega\mu}\, u^{-\ell}\, \varepsilon\, \partial_\omega$, c'est-à-dire d'après la note 2 , $\hbar^\ell\, \partial_{\mu^{-1}}$. Dans le cas où $1I > 2I > 3I > \cdots$, le fibré $L_1^{1I} \otimes L_2^{2I} \cdots$ définit un plongement projectif et le scalaire $\hbar^\ell \partial_{\mu^{-1}}$ est alors le degré de la variété $Schub_{\omega\mu}$ (cf. note 2) . Pour résumer, on a :

Proposition 4.5. *Soient* $I \in \mathbb{N}^{n+1}$ *,* $u\hbar$ *le logarithme de* a^I *,* μ *une permutation. Alors* $(1 - za^I)^{-1}\, \pi_{\mu^{-1}}\, \varepsilon\, (1-z)^{\ell(\mu)+1} = \mathcal{N}_{I,\mu}(z)$ *est un polynôme de degré* $\leq \ell(\mu)$ *dont la valeur en* $z = 1$ *est*

$$\mathcal{N}_{I,\mu}(1) = \hbar^{\ell(\mu)}\, \partial_{\mu^{-1}} \quad .$$

Remarque. L'image du polynôme G_μ par le changement de variable $1 - 1/a_1 \to \beta_1, \ldots, 1 - 1/a_{n+1} \to \beta_{n+1}$ a pour terme de degré minimum le polynôme de Schubert X_μ en les variables β (cf. [L-S2]). Les polynômes de Schubert sont une base de l'anneau de Chow, et l'on peut utiliser l'isomorphisme entre le gradué associé à $\mathbf{K}(\mathcal{F}(V))$ et l'anneau de Chow de $\mathcal{F}(V)$ (cf. [Gr2], p.678) pour transférer les résultats obtenus dans $\mathbf{K}[\mathbb{A}]$. Cependant, on obtient une information plus riche dans $\mathbf{K}[\mathbb{A}]$, d'où notre préférence pour ce dernier.

5. Plongement de Plücker

Le fibré $L = L^E = L_1^n \otimes L_2^{n-1} \otimes \cdots \otimes L_{n+1}^0$ (qui correspond au monôme a^E) est très ample et définit un plongement de $\mathcal{F}(V)$ dans un projectif de dimension relative $2^{\ell(\omega)}$. On se propose de calculer la *postulation* des puissances de L au dessus de chaque variété de Schubert , i.e. la dimension de l'espace des sections de tout $L^j, j \geq 0$, au dessus d'une variété quelconque $Schub_{\omega\mu}$. Cet espace de sections a pour classe $a^{jE} G_\mu \pi_\omega$ dans l'anneau de Grothendieck de la variété de drapeaux et nous l'avons calculé au paragraphe 3. Cependant, si l'on n'a en vue que les dimensions et non point les espaces eux-mêmes, les méthodes du paragraphe 3 sont loin d'être efficaces; nous en donnons dans ce paragraphe de plus adaptées au seul calcul des dimensions.

Nous voulons évaluer, pour tout $j \geq 0$, toute permutation μ, l'entier $a^{jE} G_\mu \pi_\omega \varepsilon$; d'après 3.1 , celui-ci est égal à $a^{jE} \pi_{\mu^{-1}} \varepsilon$. Il est commode de poser $F(z) = 1/(1 - za^E)$, où z est une indéterminée supplémentaire, et de calculer la série génératrice $F(z)\pi_{\mu^{-1}} \varepsilon$.

Comme par ailleurs $F(z)\omega = -F(1/za^{E+E\omega})/za^{E\omega}$ et $F(z)\clubsuit = F(z/a^{E+E\omega})$, on déduit de 1.16 et 1.17 :

Proposition 5.1. *Pour toute permutation* μ ,

$$(-1)^{\ell(\mu)+1} F(z)\, \psi_\mu\, a^E = \frac{1}{z} F\big(\frac{1}{z}\big)\, \pi_\mu \clubsuit \omega \quad .$$

Ainsi, pour $\mathfrak{S}(3)$, posant $\mathbb{A} = \{a, b, c\}$, on trouve

$$-F(z)\, \psi_1\, \psi_2\, a^E = \frac{(-zabc - zac^2 + z^2a^3bc^2 + z^3a^4b^2c^3)a^2b}{(1 - za^2b)(1 - zab^2)(1 - za^2c)(1 - zac^2)}$$

et cette fonction est bien l'image par l'involution $\omega \clubsuit$ de

$$\frac{1}{z} F\big(\frac{1}{z}\big)\, \pi_1\, \pi_2 = \frac{1}{z} \frac{(-a^3b^2cz^{-2} - a^3bc^2z^{-2} + abcz^{-1} + 1)}{(1 - a^2b/z)(1 - ab^2/z)(1 - a^2c/z)(1 - ac^2/z)} \quad .$$

On a vu en 4.5 que $F(z)\pi_{\mu^{-1}}\,\varepsilon\,(1-z)^{\ell(\mu)+1}$ est un polynôme $\mathcal{N}_{E,\mu}(z)$ que nous noterons $\mathcal{E}_\mu(z)$. De l'identité 1.13 : $\pi_\mu = \sum_{\nu\leq\mu}\psi_\nu$ et de la proposition 5.1 , on déduit la récurrence suivante pour les polynômes $\mathcal{E}_\mu(z)$:

Proposition 5.2. *Pour toute permutation* μ ,

$$\sum_{\nu\leq\mu}(1-z)^{\ell(\mu)-\ell(\nu)}z^{\ell(\nu)}\,\mathcal{E}_\nu(1/z) = \mathcal{E}_\mu(z) \quad .$$

Par exemple, l'ensemble des permutations en dessous de $\mu = 2413$ est

$$\begin{array}{ccc} & 2413 & \\ 1423 & 2143 & 2314 \\ 1243 & 1324 & 2134 \\ & 1234 & \end{array}$$

, les polynômes correspondants sont $\mathcal{E}_{1234} = 1 = \mathcal{E}_{1243} = \mathcal{E}_{1324} = \mathcal{E}_{2134}$, $\mathcal{E}_{1423} = \mathcal{E}_{2314} = 1+2z$, $\mathcal{E}_{2143} = 1+z$, $\mathcal{E}_{2413} = 1+9z+4z^2$ et l'on a bien l'identité
$(1-z)^3 + 3z(1-z)^2 + \big(2(2z+z^2)+(z+z^2)\big)(1-z) + \underline{(4z+9z^2+1z^3)}$
$= \underline{1+9z^2+4z^2}$.

Le polynôme $\mathcal{E}_\mu(z)$ est, d'après 4.5 de degré $\leq \ell(\mu)$, en fait de degré $\ell(\mu)-1$; on peut donc l'écrire $\mathcal{E}_\mu(z) = 1 + \ldots + c_\mu z^{\ell(\mu)-1}$; la proposition 5.2 entraine que $F(z)\psi_{\mu^{-1}}\varepsilon = (c_\mu z + \ldots + z^{\ell(\mu)})/(1-z)^{\ell(\mu)+1}$ et donc $c_\mu = a^E\psi_{\mu^{-1}}\varepsilon$.

La théorie des *bases standards* fournit une interprétation combinatoire explicite des coefficients de la série rationnelle $\big(1/(1-za^E\big)\psi_{\mu^{-1}}$, comme nombre de *tableaux de Young* satisfaisant certaines conditions (*cf.* Note 1 : tableaux associés à la permutation μ, de formes $E, E+E, E+E+E,\ldots$; on obtient les mêmes nombres en comptant les chaînes croissantes de tableaux de forme E). La donnée des $\big(\ell(\mu)+1\big)/2$ premiers de ces coefficients, ainsi que des polynômes $\mathcal{E}_\nu$, $\nu<\mu, \nu\neq\mu$, détermine $\mathcal{E}_\mu$ grâce à la symétrie 5.2. Un contrôle est fourni par la valeur $\mathcal{E}_\mu(1)$: celle-ci est égale (*cf.* Note 2), au degré projectif de la variété de Schubert $Schub_{\omega\mu}$ pour le plongement de Plücker et peut se calculer comme un nombre de chemins dans le groupe symétrique (pour l'ordre d'Ehresmann/Bruhat).

Pour $\mathfrak{S}(4)$, les polynômes sont :

$\mathcal{E}_{1234} = 1 = \mathcal{E}_{1243} = \mathcal{E}_{1324} = \mathcal{E}_{2134}$; $\mathcal{E}_{1342} = \mathcal{E}_{3124} = \mathcal{E}_{2314} = \mathcal{E}_{1423} = 1+2z$, $\mathcal{E}_{2143} = 1+z$; $\mathcal{E}_{2413} = 1+9z+4z^2$, $\mathcal{E}_{3142} = 1+8z+3z^2$, $\mathcal{E}_{2341} = \mathcal{E}_{4123} = 1+10z+5z^2$, $\mathcal{E}_{3214} = \mathcal{E}_{1432} = 1+4z+z^2$; $\mathcal{E}_{4132} = \mathcal{E}_{3241} = 1+18z+24z^2+3z^3$, $\mathcal{E}_{4213} = \mathcal{E}_{2431} = 1+19z+25z^2+3z^3$, $\mathcal{E}_{3412} = 1+25z+44z^2+8z^3$; $\mathcal{E}_{4312} = \mathcal{E}_{3421} = 1+38z+120z^2+58z^3+3z^4$, $\mathcal{E}_{4231} = 1+43z+150z^2+81z^3+5z^4$; $\mathcal{E}_{4321} = 1+57z+302z^2+302z^3+57z^4+z^5$.

Comme d'après la note 1, il y a 5 tableaux standards associés à 4231, le coefficient du terme dominant de $\mathcal{E}_{4231}$ doit être $c_{4231} = a^E\, \psi_{4231}\, \varepsilon = 5$, ce que donne bien la table; en outre $\mathcal{E}_{4231}(1) = 280$ est bien le degré de la variété $Schub_{1324}$.

Si μ est la permutation maximum d'un sous-groupe diagonal $\mathfrak{S}(p) \times \mathfrak{S}(q) \times \mathfrak{S}(r) \times \cdots$ (i.e. si $\mu = (p) \cdots (1)(p+q) \cdots (p+1)(p+q+r) \cdots (p+q+1) \cdots$), on peut alors montrer que le polynôme $\mathcal{E}_\mu$ est le *polynôme Eulérien* de degré $\ell(\mu) - 1$. Ainsi, $\mathcal{E}_{214365} = 1 + 4z + z^2 = \mathcal{E}_{32145}$; $\mathcal{E}_{32154} = 1 + 11z + 11z^2 + z^3$; $\mathcal{E}_{3215476} = 1 + 26z + 66z^2 + 26z^3 + z^4$; $\mathcal{E}_{321654} = 1 + 57z + 302z^2 + 302z^3 + 57z^4 + z^5 = \mathcal{E}_{43215}$.

Nous renvoyons à [F-S] pour la théorie de ces polynômes.

6. Structure multiplicative

Une des propriétés essentielles de l'anneau de cohomologie de la variété de drapeaux est que le produit d'un cycle de Schubert par un cycle de codimension 1 soit une somme sans multiplicité de cycles de Schubert, i.e. on a la formule de Monk [Mo] :

$$[Schub_\mu] \cdot [Schub_{\sigma_p}] = \sum [Schub_{\mu\tau_{ij}}]$$

somme sur toutes les transpositions τ_{ij} telles que $i \leq p$, $j > p$ et que $\ell(\mu\tau_{ij}) = \ell(\mu) + 1$, $[Schub_\mu]$ étant la classe de la variété de Schubert dans l'anneau de cohomologie.

Dans le cas de la grassmannienne, cette formule est due à Pieri, et pour des espaces homogènes plus généraux, à Chevalley. Les variétés de Schubert $Schub_\sigma$, σ transposition simple, sont des sections hyperplanes, et leurs classes sont les premières classes de Chern des fibrés inversibles L_1, $L_1 \otimes L_2, \ldots, L_1 \otimes L_2 \otimes \cdots \otimes L_n$ correspondant aux *poids fondamentaux.* Plus généralement, les classes de Chern des fibrés tautologiques de la variété de drapeaux (i.e. des fibrés images réciproques des fibrés tautologiques des grassmanniennes quotients de $\mathcal{F}(V)$) sont dites *cycles de Schubert spéciaux* et le produit d'un cycle de Schubert quelconque par un cycle spécial est une somme positive sans multiplicité de cyles de Schubert (dont la description est donnée en [L-S1]).

Il est remarquable que l'absence de multiplicité soit encore vérifiée dans l'anneau de Grothendieck et même, au niveau des polynômes doubles de Grothendieck. Nous n'examinerons ici que le produit par les polynômes correspondant aux sections hyperplanes, à partir du cas particulier suivant.

Proposition 6.1. *Soient p un entier, $1 \leq p \leq n$, μ une permutation dans $\mathfrak{S}(\mathbb{A})$ telle que $1\mu > \cdots > p\mu$ & $(p+1)\mu > \cdots > (n+1)\mu$ & $1\mu \neq n+1$. Alors*

$$\mathbb{G}_\mu \frac{b_{1\mu} \cdots b_{p\mu}}{a_1 \cdots a_p} = \mathbb{G}_\omega \, \psi_{\omega\mu} = \sum_{\nu \geq \mu} (-1)^{\ell(\nu)-\ell(\mu)} \mathbb{G}_\nu \quad .$$

Preuve. par récurrence sur p . Soit μ' : $1\mu' = 1\mu, \ldots, (p-1)\mu' = (p-1)\mu$ & $p\mu' > (p+1)\mu' > \cdots > (n+1)\mu'$; posons $p\mu = k$. Alors $\mu' . \sigma_{\mathrm{n+1-k}} \cdots \sigma_{\mathrm{p}} = \mu$ et $\ell(\mu') = \ell(\mu) + n + 1 - k - p$. On a donc :

$$\begin{aligned}
&\mathbb{G}_\mu \frac{b_{1\mu} \cdots b_{p\mu}}{a_1 \cdots a_p} \\
&= \mathbb{G}_{\mu'} \pi_{n+1-k} \cdots \pi_p (b_{1\mu} \cdots b_{(p-1)\mu}/a_1 \cdots a_{p-1})(b_k/a_p) \\
&\stackrel{1}{=} \mathbb{G}_{\mu'} (b_{1\mu'} \cdots b_{(p-1)\mu'}/a_1 \cdots a_{p-1}) \pi_{n+1-k} \cdots \pi_p \, b_k/a_p \\
&\stackrel{2}{=} \mathbb{G}_\omega \psi_{\omega\mu'} \pi_{n+1-k} \cdots \pi_p \, (b_k a_{p+1} a_{p+2} \cdots / a_p a_{p+1} a_{p+2} \cdots) \\
&\stackrel{3}{=} \mathbb{G}_\omega \psi_{\omega\mu'} \frac{a_p \cdots a_{n+1-k} a_{n+3-k} \cdots a_{n+1}}{a_p \cdots a_{n+1}} \partial_{n+1-k} a_{n+2-k} \cdots a_p \, \partial_p a_{p+1} b_k \\
&\stackrel{4}{=} \mathbb{G}_\omega \psi_{\omega\mu'} (b_k/a_{n+2-k})(\partial_{n+1-k} a_{n+2-k}) \cdots (\partial_p a_{p+1}) \\
&\stackrel{5}{=} \mathbb{G}_\omega \psi_{\omega\mu'} (b_k/a_{n+2-k}) \psi_{n+1-k} \cdots \psi_p \\
&\stackrel{6}{=} (\mathbb{G}_\omega/(1 - b_k/a_{n+1-k})) \psi_{\omega\mu'} (1 - b_k/a_{n+1-k})(b_k/a_{n+2-k}) \psi_{n+1-k} \cdots \psi_p \\
&\stackrel{7}{=} (\mathbb{G}_\omega/(1 - b_k/a_{n+1-k})) \psi_{\omega\mu'} (1 - b_k/a_{n+1-k}) \psi_{n+1-k} \cdots \psi_p \\
&\stackrel{8}{=} \mathbb{G}_\omega \psi_{\omega\mu'} \psi_{n+1-k} \cdots \psi_p = \mathbb{G}_\omega \psi_{\omega\mu} \\
&\stackrel{9}{=} \sum_{\nu \geq \mu} (-1)^{\ell(\nu)-\ell(\mu)} \mathbb{G}_\nu
\end{aligned}$$

($\stackrel{1}{=}$ puisque $a_1, \ldots, a_{p-1}$ commutent avec $\pi_p, \ldots, \pi_{n+1-k}$; $\stackrel{2}{=}$ par récurrence sur p ; $\stackrel{3}{=}$ et $\stackrel{4}{=}$ en utilisant que ∂_i commute avec tout a_j , pour $j \neq i, i+1$ et en explicitant $\pi_i = a_i \partial_i$; $\stackrel{5}{=}$ puisque $\psi_i = \partial_i a_{i+1}$; $\stackrel{6}{=}$ puisque $(1 - b_k/a_{n+1-k})$ est un scalaire pour $\pi_{\omega\mu'}$; $\stackrel{7}{=}$ car la symétrie de $(\mathbb{G}_\omega/(1 - b_k/a_{n+1-k}))\, \psi_{\omega\mu'}\, (1 - b_k/a_{n+1-k})(1 - b_k/a_{n+2-k})$ en a_{n+1-k} et a_{n+2-k} entraine que son image par ψ_{n+1-k} soit nulle ; $\stackrel{8}{=}$ en reportant à gauche le facteur $(1 - b_k/a_{n+k-1})$; $\stackrel{9}{=}$ d'après l'expression 1.13 de ψ en fonction de π) ∎

Dans le cas d'une permutation μ vérifiant les hypothèses 6.1, l'intervalle $[\omega, \mu]$ est très particulier et peut être décrit à l'aide des permutations inverses, ce qui revient à utiliser les opérateurs élémentaires ϕ_j définis par

$$(6.2) \qquad \mathfrak{G}_\mu \phi_j = \begin{cases} \mathfrak{G}_{\sigma_j\mu} - \mathfrak{G}_\mu & \text{si } \ell(\sigma_j\mu) < \ell(\mu) \\ 0 & \text{autrement} \end{cases} ,$$

c'est-à-dire $\mathfrak{G}_{..j+1\ j\ ..}\phi_j = \mathfrak{G}_{..j..j+1..} - \mathfrak{G}_{\ j+1..j..}$ et $\mathfrak{G}_{..j..j+1..}\phi_j = 0$.

Les produits d'opérateurs ϕ_j sont notés ϕ_ζ . Ces derniers commutent avec les opérateurs π_μ et ψ_μ . En outre, pour toute permutation μ , on a l'identité

$$(6.3) \qquad \mathfrak{G}_\omega \, \phi_{\omega\mu} = \mathfrak{G}_\omega \, \psi_{\omega\mu^{-1}} \quad ,$$

puisque $\omega\mu = \sigma_i\sigma_j\cdots\sigma_h \Rightarrow \omega\mu^{-1} = \omega\sigma_h\cdots\sigma_j\sigma_i\omega = (\omega\sigma_h\omega)\cdots(\omega\sigma_j\omega)\times(\omega\sigma_i\omega)$.

Ainsi donc, la proposition 6.1 prend la forme équivalente

$$\mathfrak{G}_\mu \, (b_{1\mu}...b_{p\mu}) \,/\, (a_1...a_p) = \mathfrak{G}_\omega \, \phi_{\omega\mu^{-1}} = \sum_{\nu\geq\mu} (-1)^{\ell(\nu)-\ell(\mu)} \, \mathfrak{G}_\nu$$

Par ailleurs, pour tout $j \neq p$, $\mathfrak{G}_\mu \frac{b_{1\mu}\cdots b_{p\mu}}{a_1\cdots a_p} \pi_j = \mathfrak{G}_\mu \, \pi_j \frac{b_{1\mu}\cdots b_{p\mu}}{a_1\cdot a_p}$. De plus, pour tout j, π_j commute avec les opérateurs ϕ_μ . Enfin, l'image par un opérateur π_j d'une somme $\sum_{\mathcal{S}}(-1)^{\ell(\nu)}\mathfrak{G}_\nu$, pour une famille quelconque $\mathcal{S}$, est une somme sans multiplicité puisque $\mathfrak{G}_\nu \, \pi_j = \mathfrak{G}_{\nu'} \, \pi_j$ & $\nu \neq \nu'$ implique que $\nu' = \nu\sigma_j$ & $(\mathfrak{G}_\nu - \mathfrak{G}_{\nu'})\,\pi_j = 0$. En résumé, sachant que $(b_1...b_p/a_1...a_p)$ n'est autre que $1 - \mathfrak{G}_{\sigma_p}$, on a le théorème suivant :

Théorème 6.4. *Soient η une permutation, p un entier : $1 \leq p \leq n$, μ la permutation de plus grande longueur dans la classe de η modulo $\mathfrak{S}(p) \times \mathfrak{S}(n+1-p)$. Alors*

$$\begin{aligned} \mathfrak{G}_\eta \cdot (1 - \mathfrak{G}_{\sigma_p}) \frac{b_{1\eta}...b_{p\eta}}{b_1...b_p} &= \mathfrak{G}_\omega \, \psi_{\omega\mu} \, \pi_{\mu^{-1}\eta} \\ &= \mathfrak{G}_\omega \, \phi_{\omega\mu^{-1}} \, \pi_{\mu^{-1}\eta} = \mathfrak{G}_{\omega\mu^{-1}\eta} \, \phi_{\omega\mu^{-1}} \end{aligned}$$

En outre, ce produit est une somme sans multiplicité de polynômes de Grothendieck.

Exemple. Soient $p = 2$, $\eta = 1342$ (et donc $\mu = 3142$). Alors $\mathfrak{G}_{1342}(1-\mathfrak{G}_{1324}) = \mathfrak{G}_{4321}\pi_1\phi_{3142} = \mathfrak{G}_{3421}\phi_{3142} = \mathfrak{G}_{3421}\phi_2\phi_1\phi_3 = (\mathfrak{G}_{2431} - \mathfrak{G}_{3421})\phi_1\phi_3 = (\mathfrak{G}_{1432} - \mathfrak{G}_{2431} - \mathfrak{G}_{3412} + \mathfrak{G}_{3421})\phi_3 = \mathfrak{G}_{342} - \mathfrak{G}_{1432} - \mathfrak{G}_{2341} + \mathfrak{G}_{2431} - 0 + 0$ et c'est aussi $(\mathfrak{G}_{4321}\phi_{3142})\pi_1 = (\mathfrak{G}_{3142} - \mathfrak{G}_{4132} - \mathfrak{G}_{3412} - \mathfrak{G}_{3241} + \mathfrak{G}_{4312} + \mathfrak{G}_{4231} + \mathfrak{G}_{3421} - \mathfrak{G}_{4321})\pi_1 = \mathfrak{G}_{1342} - \mathfrak{G}_{1432} - \mathfrak{G}_{3412} - \mathfrak{G}_{2341} + \mathfrak{G}_{3412} + \mathfrak{G}_{2431} + \mathfrak{G}_{3421} - \mathfrak{G}_{3421}$.

Nous attirons l'attention sur l'intérêt qu'il y aurait à expliciter les propriétés des ensembles de permutations apparaissant dans le produit $\mathfrak{G}_\eta(1-\mathfrak{G}_{\sigma_p})$. Ces ensembles ne sont en général pas des intervalles. De manière équivalente, on peut rechercher les relations entre les produits quelconques d'opérateurs ψ_μ et π_ζ. De tels calculs sont intimement liés aux propriétés des intervalles dans le groupe symétrique (pour l'ordre d'Ehresmann/Bruhat) et aux singularités des variétés de Schubert.

De la formule de Monk résulte aisément que si μ est un produit direct : $\mu = \mu' \times \mu''$ dans $\mathfrak{S}(k) \times \mathfrak{S}(n+1-k)$, alors dans l'anneau de cohomologie de la variété de drapeaux, on a :

$$[Schub_\mu] = [Schub_{\mu'}] \cdot [Schub_{\mu''}]$$

Il serait beaucoup plus compliqué de vérifier la propriété de factorisation analogue pour les polynômes de Grothendieck (simples) à partir de 6.4, puisque cette formule fait intervenir les permutations dans l'image d'un intervalle. Il est plus approprié d'utiliser les opérateurs $\Xi_p = \pi_p - \partial_p$ mentionnés au paragraphe 1.

Lemme 6.5. *Soit μ (resp $\nu : \nu \leq \mu$) une permutation dont le code J vérifie : pour tout p , $pJ - (p+1)J = 0$ ou 1 (resp. $pH - (p+1)H = 0$ ou 1). Alors les opérateurs $G_\mu\pi_{\mu^{-1}\nu}$ et $\Xi_{\mu^{-1}\nu}G_\nu$ sont identiques, i.e., pour toute f dans $\mathbf{K}[\mathbb{A}]$, on a*

$$f\, G_\mu\, \pi_{\mu^{-1}\nu} = f\, \Xi_{\mu^{-1}\nu}\, G_\nu$$

Preuve. Soit p tel que $pJ = (p+1)J + 1\, \&\, \mu\sigma \geq \nu$, avec $\sigma = \sigma_p$. Alors $G_\mu = (1-1/a_p)G_{\mu\sigma}$ et $G_\mu\, \pi_{\mu^{-1}\nu} = (1-1/a_p)G_{\mu\sigma}\, \pi_\sigma\, \pi_{\sigma\mu^{-1}\nu} \stackrel{1}{=} (1-1/a_p)$ $\pi_\sigma\, G_{\mu\sigma}\, \pi_{\sigma\mu^{-1}\nu} \stackrel{2}{=} (1-1/a_p)\, \pi_\sigma\, \Xi_{\sigma\mu^{-1}\nu}\, G_\nu \stackrel{3}{=} \Xi_\sigma\, \Xi_{\sigma\mu^{-1}\nu}\, G_\nu = \Xi_{\mu^{-1}\nu}\, G_\nu$ ($\stackrel{1}{=}$ puisque $G_{\mu\sigma}$ est symétrique en a_p et a_{p+1} ; $\stackrel{2}{=}$ par récurrence sur la longueur de $\mu^{-1}\nu$; $\stackrel{3}{=}$ puisque $\pi_\sigma = a_p\partial_\sigma$) ∎

Corollaire 6.6. *Soient k un entier , $\omega' = (k)(k-1)\cdots(1)(k+1)\cdots(n+1)$, $\omega'' = (1)\cdots(k)(n+1)(n)\cdots(k+1)$. Alors*

$$G_{\omega'\omega''} = G_{\omega'} \cdot G_{\omega''} \quad .$$

Preuve. Sur un exemple : Soit $\mu = 5671234$. On a $G_{3217654} = G_{7654321}\,\pi_\mu \overset{1}{=} G_{4321567} \cdot G_{4567321}\,\pi_\mu \overset{2}{=} G_{4321567}\,\Xi_\mu\, G_{3214567}$.

L'image de cette égalité par $\pi_1\,\pi_2\,\pi_1$ est

$$\begin{aligned} G_{1237654} &= {}_{4321567}\,\Xi_\mu\, G_{3214567}\pi_1\,\pi_2\,\pi_1 \overset{1}{=} G_{4321567}\Xi_\mu\,\Xi_{3214567} \\ &\overset{3}{=} G_{4321567}\,\Xi_{1234765}\,\Xi_\mu \overset{4}{=} G_{4321567}\,\Xi_\mu \end{aligned}$$

et l'on a bien

$$G_{3217654} = G_{3214567}\big(G_{4321567}\,\Xi_\mu\big) = G_{3214567} \cdot G_{1237654}$$

($\overset{1}{=}$ d'après l'expression de $G_\omega = (1-1/a_1)^6\cdots(1-1/a_7)^0$; $\overset{2}{=}$: lemme 6.5; $\overset{3}{=}$ puisque $\mu\sigma_1\sigma_2\sigma_1 = \sigma_5\sigma_6\sigma_5\mu$; $\overset{4}{=}$ car G_{4321} commute avec Ξ_5 et Ξ_6, et $1\,\Xi_5\,\Xi_6\,\Xi_5 = 1\,\pi_5\pi_6\pi_5 = 1$) ∎

Si $\mu = \mu' \times \mu''$ appartient au produit direct $\mathfrak{S}(k) \times \mathfrak{S}(n+1-k)$, alors $G_\mu = G_{\omega'\omega''}\,\pi_{\omega'\mu'}\,\pi_{\omega''\mu''}$; du corollaire 6.6 s'ensuit donc :

Proposition 6.7. *Soient k un entier, $\mu = \mu' \times \mu''$ une permutation appartenant au produit direct $\mathfrak{S}(k) \times \mathfrak{S}(n+1-k)$. Alors*

$$G_\mu = G_{\mu'} \cdot G_{\mu''} \quad .$$

7. Anneau de Grothendieck

Soient V un fibré vectoriel de rang $n+1$ sur une base quelconque $\mathcal{M}$, r un entier : $1 \leq r \leq n$, $D \in \mathbb{N}^r$ un vecteur croissant : $1D < 2D < \cdots < rD \leq n+1$, $\mathcal{F}(V)$ la variété des drapeaux complets dans V, $\mathcal{F}_D(V)$ la variété des drapeaux (quotients de V) de rangs respectifs $1D, 2D, \ldots, rD$. Soit de plus $\mathbb{A}$ un ensemble d'indéterminées de cardinal $n+1$.

L'anneau de Grothendieck $\mathbf{K}\big(\mathcal{F}(V)\big)$ de $\mathcal{F}(V)$ s'identifie au quotient de $\mathbf{K}(\mathcal{M})\big[a_1, 1/a_1, \ldots, a_{n+1}, 1/a_{n+1}\big] = \mathbf{K}(\mathcal{M})[\mathbb{A}]$ par l'idéal $\mathcal{J}$ engendré par la relation graduée :

$$(1+a_1)\cdots(1+a_{n+1}) = 1 + V + \Lambda^2 V + \cdots + \Lambda^{n+1} V$$

Les images des a_i sont les classes des fibrés inversibles tautologiques (l'image de a_1 est la classe de $\mathcal{O}(1)$, image réciproque du fibré inversible tautologique de $\mathbb{P}(V)$). L'anneau $\mathbf{K}(\mathcal{M})$ est le sous-anneau de $\mathbf{K}(\mathcal{F}(V))$ des invariants sous l'action de $\mathfrak{S}(\mathbb{A})$; $\mathbf{K}(\mathcal{F}(V))$ est un $\mathbf{K}(\mathcal{M})$-module libre de base les polynômes de Grothendieck.

Le morphisme $\pi_\omega : \mathbf{K}(\mathcal{M})[\mathbb{A}] \longrightarrow \mathbf{K}(\mathcal{M})$ factorise en la surjection $\mathbf{K}(\mathcal{M})[\mathbb{A}] \longrightarrow \mathbf{K}(\mathcal{M})[\mathbb{A}]/\mathcal{J} = \mathbf{K}(\mathcal{F}(V))$ et la projection canonique $\mathbf{K}(\mathcal{F}(V)) \longrightarrow \mathbf{K}(\mathcal{M})$ que nous noterons encore π_ω . L'anneau $\mathbf{K}(\mathcal{F}_D(V))$ s'identifie au sous-anneau de $\mathbf{K}(\mathcal{F}(V))$ engendré par les fonctions invariantes par le produit direct $\mathfrak{S}_D$ des groupes symétriques $\mathfrak{S}(1D) \times \mathfrak{S}(2D-1D) \times \cdots$, i.e. le sous-anneau des fonctions symétriques en $\{a_1, \ldots, a_{1D}\}$, symétriques en $\{a_{1D+1}, \ldots, a_{2D}\}, \ldots$. C'est en outre un $\mathbf{K}(\mathcal{M})$-module libre de base les polynômes de Grothendieck G_μ , pour les $\mu \in \mathfrak{S}(\mathbb{A})$ telles que $1\mu < \cdots < (1D)\mu$; $(1D+1)\mu < \cdots < (2D)\mu$; ... Soit ω_D la permutation $(1D)(1D-1)\cdots(1)(2D)\cdots(1D+1)\cdots((n+1)D)\cdots(rD+1)$, i.e. la permutation maximale dans la classe de l'identité modulo $\mathfrak{S}_D$. La projection $\mathbf{K}(\mathcal{F}(V)) \longrightarrow \mathbf{K}(\mathcal{F}_D(V))$ est induite par π_{ω_D} et la projection $\mathbf{K}(\mathcal{F}_D(V)) \longrightarrow \mathbf{K}(\mathcal{M})$ coïncide avec la restriction de π_ω .

De la construction de Demazure [D3] et Bernstein-Gelfand-Gelfand [B-G-G], il ressort que l'image de G_μ dans $\mathbf{K}(\mathcal{F}_D(V))$ est la classe du faisceau structural de la variété de Schubert d'indice μ .

Les grassmanniennes sont le cas des drapeaux de longueur 1 , i.e. le cas $r = 1$. Posons alors $1D = p$ et écrivons $\mathcal{G}_p(V)$ au lieu de $\mathcal{F}_D(V)$. Les polynômes de Grothendieck correspondant sont des fonctions (symétriques) de $a_1, \ldots, a_p$ seulement, puisque la croissance $(p+1)\mu < \cdots < (n+1)\mu$ implique qu'alors G_μ soit symétrique en $a_{p+1}, \ldots, a_{n+1}$, donc de degré 0 en chacune de ces variables (tout G_ν, $\nu \in \mathfrak{S}(\mathbb{A}_{n+1})$, est de degré 0 en a_{n+1}).

On a donc que $\mathbf{K}(\mathcal{G}_p(V))$ est le quotient de l'anneau des polynômes symétriques en $\mathbb{A}_p$ à coefficients les polynômes symétriques en $\mathbb{A}$: $\left(\mathbf{K}(\mathcal{M})[\mathbb{A}]^{\mathfrak{S}(\mathbb{A})}[\mathbb{A}_p]\right)^{\mathfrak{S}(\mathbb{A}_p)} = \mathbf{K}(\mathcal{M})[\mathbb{A}]^{\mathfrak{S}(\mathbb{A}_p)\times\mathfrak{S}(\mathbb{A}\setminus\mathbb{A}_p)}$ par les relations

$$\Lambda^{n-p+1}(\mathbb{A} - \mathbb{A}_p) = 0 = \ldots = \Lambda^{n+1}(\mathbb{A} - \mathbb{A}_p)$$

(on utilise ici la structure de λ-anneau de l'anneau des polynômes : les a_i sont des éléments de rang 1; $\mathbb{A}_p$ est identifié au polynôme : $a_1 + \cdots + a_p$ et son image est la classe du fibré quotient tautologique de $\mathcal{G}_p(V)$; l'image de $\mathbb{A} \setminus \mathbb{A}_p$ est la classe du fibré noyau tautologique. Les relations ci-dessus expriment que $\mathbb{A} \setminus \mathbb{A}_p$ est de rang $\leq n-p$).

Les permutations associées à la grassmanienne sont toutes vexillaires, et donc les polynômes G_μ pour les variétés de Schubert de la grassmannienne admettent l'expression 3.11. Dans [L1], ch.II, on trouve une expression plus générale pour $a^I\,G_\mu$ qui admet comme cas particulier la formule de postulation de Hodge et l'expression déterminantale d'un cycle de Schubert due à Giambelli [Gi] .

La projection $\mathbf{K}(\mathcal{G}_p(V)) \longrightarrow \mathbf{K}(\mathcal{M})$ est induite par π_ω , comme pour toute variété de drapeaux. Tenant compte des symétries, on peut en fait réduire la sommation 1.1 qui exprime l'opérateur π_ω à un sous-ensemble de représentants des classes de $\mathfrak{S}(\mathbb{A}_{n+1})$ modulo $\mathfrak{S}(\mathbb{A}_p) \times \mathfrak{S}(\mathbb{A} \setminus \mathbb{A}_p)$. On trouve dans les articles de Sylvester de nombreuses formules relatives à cette sommation (i.e. à la restriction de π_ω à l'anneau $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}(\mathbb{A}_p)\times\mathfrak{S}(\mathbb{A}\setminus\mathbb{A}_p)}$) qu'il avait obtenue comme généralisation de la formule d'interpolation de Lagrange.

Note 1. Bases standards

Un *tableau de Young* de forme une partition est un remplissage du diagramme de cette partition par des entiers positifs tel que l'on ait croissance stricte dans les colonnes de bas en haut et croissance large dans les lignes de gauche à droite. Soient $t_1, \ldots, t_r$ les colonnes successives de t , considérées comme des mots (croissants). L'*évaluation* $\mathbf{Ev}(t)$ d'un tableau est le monôme image du mot $t_1 \cdots t_r$ par le morphisme $i \longrightarrow a_i$. Chaque colonne t_i peut être lue, à la suite de Hodge, comme une *coordonnée de Plücker*, le tableau étant alors le produit des coordonnées de Plücker que sont ses colonnes (cf. [D-E-P1], [D-E-P2]).

Soient $\mu_0 = identité$, μ_i , $1 \le i \le r$, la plus petite permutation (pour l'ordre d'Ehresmann/Bruhat) qui ait pour facteur gauche t_i et telle que $\mu_i \ge \mu_{i-1}$. La dernière permutation μ_r est dite *clef droite* de t (*cf.* [L-S3]). L'ensemble des tableaux de clef droite une permutation donnée μ est dit *base standard associée à* μ par [L-M-S]. La théorie dévelopée par ces auteurs montre qu'une base de l'espace des sections de L^I, I dominant, au dessus de la variété de Schubert $Schub_{\omega\mu}$ est formée de l'ensemble des tableaux t de forme I de clef droite μ.

On préfère dans [L-S3] interpréter un tableau t comme le mot produit des lignes succesives de t dans l'algèbre libre sur $\mathbb{A}$; remontant à cette algèbre l'action du groupe symétrique et les opérateurs π_μ , on obtient, pour I dominant et μ quelconque :

$$(\cdots a_2^{2I}\, a_1^{1I})\, \psi_\mu = \sum t$$

somme sur tous les tableaux de forme I et de clef droite μ .

Par exemple, pour $I = 3210$, $\mu = 4231$, on a :

Tableaux	*Colonnes successives*	*Suite croissante de permutations*	*Mot dans l' algèbre libre*
4 23 124	$124, 23, 4$	$124\mathit{3} \leq 23\mathit{41} \leq 4\mathit{231}$	$a_4 \cdot a_2a_3 \cdot a_1a_2a_4$
4 33 124	$134, 23, 4$	$134\mathit{2} \leq 23\mathit{41} \leq 4\mathit{231}$	$a_4 \cdot a_3a_3 \cdot a_1a_2a_4$
4 34 124	$134, 24, 4$	$134\mathit{2} \leq 24\mathit{31} \leq 4\mathit{231}$	$a_4 \cdot a_3a_4 \cdot a_1a_2a_4$
4 33 224	$234, 23, 4$	$234\mathit{1} \leq 23\mathit{41} \leq 4\mathit{231}$	$a_4 \cdot a_3a_3 \cdot a_2a_2a_4$
4 34 224	$234, 24, 4$	$234\mathit{1} \leq 24\mathit{31} \leq 4\mathit{231}$	$a_4 \cdot a_3a_4 \cdot a_2a_2a_4$

L'assertion précédente est que, dans l'algèbre libre, $a_4^0 a_3^1 a_2^2 a_1^3\, \psi_{4231}$ est égale à la somme des mots figurant à droite, i.e. à $a_4 \cdot a_2a_3 \cdot a_1a_2a_4 + a_4 \cdot a_3a_3 \cdot a_1a_2a_4 + a_4 \cdot a_3a_4 \cdot a_1a_2a_4 + a_4 \cdot a_3a_3 \cdot a_2a_2a_4 + a_4 \cdot a_3a_4 \cdot a_2a_2a_4$.

On a donc, par image commutative dans $\mathbb{Z}[\mathbb{A}]$, $a_4^0 a_3^1 a_2^2 a_1^3\, \psi_{4231} = a_1a_2a_3a_4^2(a_2 + a_3 + a_4) + a_2^2a_3a_4^2(a_3 + a_4)$.

On peut définir symétriquement la *clef gauche* de tout tableau (*cf.* [L-S3]). On munit alors l'ensemble des tableaux de même forme de l'ordre : $t \leq t'$ si et seulement si la clef droite de t est inférieure à la clef gauche de t' (relativement à l'ordre d'Ehresmann). Cette construction permet de décrire les bases standards de forme multiple de I comme chaînes croissantes de tableaux de forme I et donne une description combinatoire des polynômes $\mathcal{E}_\mu$ vus au paragraphe 5.

Note 2. Degré d'une variété de Schubert pour un plongement projectif

L'ordre d'Ehresmann/Bruhat sur le groupe symétrique correspond à l'inclusion des cellules de Schubert dans la décomposition cellulaire de la variété de drapeaux (cf. [E]). Nous avons besoin d'enrichir cet ordre et de considérer le graphe orienté suivant (cf. [L-S2] pour plus de détails) :

- Les sommets sont les permutations dans $\mathfrak{S}(n+1)$
- Chaque arête a une *couleur*, i.e. il existe un morphisme de l'ensemble

• Il existe au moins une arête d'extrémités μ, ν si et seulement si $\ell(\nu) = \ell(\mu) + 1$ & $\nu\mu^{-1}$ est une transposition. Dans ce cas, $\nu\mu^{-1} = \tau_{i\,j+1}$ implique que μ et ν soient jointes par $j+1-i$ arêtes de couleurs respectives $c_i, c_{i+1}, \ldots, c_j$.

Par exemple, il y a trois arêtes c_2, c_3, c_4 joignant $\mu = 4\underline{2}61\underline{5}3$ à $\nu = 4\underline{5}61\underline{2}3$ puisque $\nu\mu^{-1} = \tau_{25}$ et que $\ell(\nu) = \ell(\mu) + 1$.

Deux permutations μ, η sont comparables pour l'ordre défini au paragraphe 1 si et seulement il existe au moins un chemin les joignant.

A tout chemin, on associe le mot de l'algèbre libre sur $\mathcal{C}$ (notée $\mathbf{Z}[\mathcal{C}^*]$) obtenu en lisant les arêtes dans l'ordre où elles se présentent sur ce chemin; à une permutation μ, on associe la somme $C_\mu \in \mathbf{Z}[\mathcal{C}^*]$ de tous les chemins de μ à ω .

Il est remarqué dans [L-S2] (en interprétant les chemins comme des produits par les cycles fondamentaux $[Schub_\sigma]$) que C_μ est invariant par l'action du groupe symétrique $\mathfrak{S}(\ell(\mu\omega))$, i.e. invariant par permutation, dans chaque mot, de l'ordre des facteurs.

Ainsi, pour $\mu = 3214$,

$$C_{3214} = (c_1 + c_2 + c_3)(c_2c_3 + c_3c_2 + c_3c_3) + \\ (c_2 + c_3)(c_1c_3 + c_3c_1) + c_3(c_1c_2 + c_2c_1 + c_2c_2)$$

L'élément C_μ est donc déterminé par le sous-ensemble des chemins croissants de μ à ω . Pour $\mu = 3214$, ce sous-ensemble est $\{c_1c_2c_3, c_1c_3c_3, c_2c_2c_3, c_2c_3c_3, c_3c_3c_3\}$.

Soit un plongement projectif de $\mathcal{F}(V)$ et $\hbar$ la classe (dans l'anneau de cohomologie) d'une section hyperplane. L'élément $[Schub_\mu] \cdot \hbar^{\ell(\omega\mu)}$ est égal, par définition, à $d_\mu[Schub_\omega]$, où d_μ est le *degré projectif* de la variété $Schub_\mu$, $[Schub_\omega]$ étant la classe de la variété de Schubert de codimension $\ell(\omega)$, c'est-à-dire la classe d'un point.

Cet entier d_μ est aussi, d'après [B-G-G], th.3.12*v)* , [D3], th.1 p.78 , égal à $\hbar^{\ell(\omega\mu)}\partial_{\mu^{-1}\omega} = d_\mu$. Soit $\hbar = h_1[Schub_{\sigma_1}] + h_2[Schub_{\sigma_2}] + \cdots$. Alors d_μ est l'image de C_μ par le morphisme $\mathbf{Z}[\mathcal{C}^*] \to \mathbf{Z} : c_i \to h_i$. Ainsi, d'après la valeur de C_{3214} calculée plus haut, on a $d_{3214} = 6h_1h_2h_3 + 3h_1h_3h_3 + 3h_2h_2h_3 + 3h_2h_3h_3 + h_3h_3h_3$.

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L.I.T.P., Université Paris 7,
2 Place Jussieu,
75251 Paris Cedex 05, France

New Results on Weight-Two Motivic Cohomology

S. LICHTENBAUM*

Dedicated to A. Grothendieck on his 60th birthday

Introduction

Among Grothendieck's manifold contributions to algebraic geometry is his emphasis on the search for a universal cohomology theory for algebraic varieties and a conjectured description of it in terms of motives [Ma]. Various authors have recently set out to describe the properties of and conjecturally define a cohomology theory for algebraic varieties, which has been baptized "motivic cohomology" by Beilinson, MacPherson, and Schechtman ([BMS],[Be],[Bl],[T],[L1],[L2]). It is hoped that this theory, when and if it is fully developed, will in some sense be universal and thus provide at least a partial response to Grothendieck's question.

Meanwhile, it should be pointed out that to a greater or lesser extent, all of the currently proposed definitions are *ad hoc* and so ultimately unsatisfactory. Presumably there will one day be a natural definition of motivic cohomology and the present attempts will fall into place as calculating devices which explicitly realize this definition (much as the "bar-construction" definition of group cohomology realizes the more natural derived-functor definition).

Let X be a regular noetherian scheme. Beilinson ([Be]) and the author ([L1]) have proposed the existence for each non-negative integer n of certain complexes of sheaves $\Gamma(n, X)$ on X satisfying certain properties, or "axioms". These complexes will be referred to as "motivic-cohomology

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complexes" and their hypercohomology as "motivic cohomology". Beilinson considered sheaves for the Zariski topology; the author used the étale site. We recall the étale site "axioms":

(0) $\Gamma(0,X)=\underline{\underline{Z}}$. $\Gamma(1,X)=G_m[-1]$.

(1) For $r\geq 1$, $\Gamma(r,X)$ is acyclic outside of $[1,r]$.

(2) Let α_* be the functor which assigns to every étale sheaf on X the associated Zariski sheaf. Then the Zariski sheaf $R^{q+1}\alpha_*\Gamma(q,X)=0$.

(3) Let n be a positive integer prime to all residue field characteristics of X. Then there exists a distinguished triangle in the derived category

$$\Gamma(r,X)\xrightarrow{n}\Gamma(r,X)\longrightarrow\mu_n^{\otimes r}\longrightarrow\Gamma(r,X)[1].$$

(4) There are product mappings $\Gamma(r,X)\overset{L}{\otimes}\Gamma(s,X)\to\Gamma(r+s,X)$, satisfying the usual properties.

(5) The cohomology sheaves $\mathfrak{H}^i(X,\Gamma(r,X))$ are isomorphic to the étale sheaves $\mathrm{gr}^r_\gamma\underline{\underline{K}}^{et}_{2r-i}(X)$ up to torsion involving primes $\leq(r-1)$.

(6) The Zariski sheaf $R^q\alpha_*\Gamma(q,X)$ is isomorphic to the sheaf of Milnor K_q-groups, $\underline{\underline{K}}^M_q(X)$.

In [L2] we constructed a candidate for $\Gamma(2,X)$. Namely, let A be a regular noetherian ring. Let $W=\operatorname{Spec}A[T]$, $Z=\operatorname{Spec}A[T]/T(T-1)$. Let $B=\{b_1,b_2,\dots,b_n\}$ be a finite sequence of "exceptional units" of A, i.e., b_i and $1-b_i$ are both units for all i. Let $Y_B=\operatorname{Spec}A[T]/\prod_{i=1}^{n}(T-b_i)$. Then there is an exact sequence

$$K_3(A)\longrightarrow K_2(W-Y_B,Z)\xrightarrow{\varphi_{A,B}}K_1'(Y_B)\longrightarrow K_2(A).$$

Let $C_{2,1}(A)=\varinjlim_B K_2(W-Y_B,Z)$, $C_{2,2}(A)=\varinjlim K_1'(Y_B)$ and $\varphi_A=\varinjlim\varphi_{A,B}$. Let $\widetilde{\Gamma}(2,A)$ be the two-term complex $(C_{2,1}(A)\xrightarrow{\varphi_A}C_{2,2}(A))$ with $C_{2,1}(A)$ in degree 1 and $C_{2,2}(A)$ in degree 2.

Now if $U=\operatorname{Spec}A$ is an open affine étale over X, the functors $U\to C_{2,i}(A)$ for $i=1,2$ evidently determine a two-term complex of presheaves on X for the étale topology and we define $\Gamma(2,X)$ to be the associated complex of sheaves.

Evidently $\Gamma(2,X)$ satisfies Axiom 1. In our paper [L2] Axiom 2 was proved up to 2-torsion and p-torsion if X was a scheme of finite type over a field of characteristic $p\geq 0$. In this paper we prove Axiom 2 up to 2-torsion. In [L2] Axioms 3 and 5 were proven for X finite type over a field and Axiom 4 for all X. Axiom 6 was proved only for $X=\operatorname{Spec}F$, a field, up to p-torsion. In this paper we prove Axiom 6 for fields, and prove it in general up to 2-torsion.

Also, we can now show that the two definitions we gave in [L2] of Beilinson's motivic-cohomology complexes in weight two on the Zariski site are equal (in the derived category, of course) up to 2-torsion. So these complexes satisfy Beilinson's "axioms" as described in [L2], except that $\mathfrak{H}^i(\Gamma(2))$ is only known to be $\mathrm{gr}^2_\gamma \underline{\underline{K}}_{4-i}$ for $i \geq 1$. For $i \leq 0$, $\mathfrak{H}^i(\Gamma(2)) = 0$, but even for fields, the vanishing of $\mathrm{gr}^2_\gamma K_{4-i}(F)$ is a special case of the Beilinson- Soulé conjecture.

The "motivic cycle" group $H^4(X, \Gamma(2))$ deserves serious investigation. By Theorem 2.13 it agrees up to torsion with the usual Chow group $CH^2(X)$ of codimension-2 cycles modulo rational equivalence, but it has in some ways better properties than $CH^2(X)$. $H^4(X, \Gamma(2))$ fits into a Kummer sequence and as a result, if X is projective and smooth over $\mathbb{C}$, every torsion cohomology class in $H^4(X, \mathbf{Z}_\ell(2))$ comes from a "motivic cycle" in $H^4(X, \Gamma(2))$. We should also mention that, at least up to 2-torsion, the groups $H^3(X, \Gamma(2))$ and $H^2(X, \Gamma(2))$ are the same as the groups $H^1_{\mathrm{zar}}(X, \underline{\underline{K}}_2)$ and $H^0_{\mathrm{zar}}(X, \underline{\underline{K}}_2)$ which have been the subject of study by Colliot-Thélène and Raskind [CTR].

We also prove a form of Gersten's conjecture for motivic cohomology on the étale site, and a weak purity theorem. As a corollary of the étale Gersten conjecture, we derive the existence of a cycle map into motivic cohomology which is compatible with the ℓ-adic and p-adic cycle maps, answering a question raised by Milne.

We close the introduction by recalling the three recently proved major results in algebraic K-theory which play an important role in this work:

Theorem 0.1. (Mercuriev-Suslin-Levine) ([MS];[Le]). *Let F be a field, and L a Galois extension of F with group G. Then the natural map from $K_3(F)_{\mathrm{ind}}$ to $(K_3(L)_{\mathrm{ind}})^G$ is an isomorphism.*

Theorem 0.2. (Mercuriev-Suslin) [MS]. *Let F be a field of characteristic p. Then $K_3(F)_{\mathrm{ind}}$ is uniquely p-divisible.*

Theorem 0.3. (Bloch-Kato) [BK]. *Let F be a field of characteristic p, and $\nu(2)$ the kernel of the map from Ω^2_F to $\Omega^2_F/d\Omega^1_F$ given by $1 - C^{-1}$, where C is the Cartier operator. Then $K_2(F)/pK_2(F) \simeq \nu(2)_F$.*

I would like to thank J.-L. Colliot-Thélène for several useful observations, including Remark 2.14.

Notation and Terminology

Cohomology groups and sheaves are understood to be relative to the small étale site, unless otherwise specified. X with the Zariski site is denoted by X_{zar}, but the category for the Zariski site will be the same as for the small étale site. Sometimes, if A is a ring, $H^i(A)$ is used to mean $H^i(\operatorname{Spec} A)$.

If C_* is a complex with maps $d_i : C_i \to C_{i+1}$, $t_{\leq n}C_*$ means the complex D_* such that $D_m = 0$ for $m > n$, $D_m = C_m$ for $m < n$, and $D_n = \operatorname{Ker} d_n$, with the evident maps. For a definition of $\mathrm{gr}^i_\gamma K_j(A)$, see [So]. If B is a group with Adams operators ψ^k acting on it, the subgroup $B^{(r)}$ of Adams weight r is $\{x \in B;\, \psi^k(x) = k^r x \text{ (for all } k)\}$. If F is a field, F_s denotes the separable closure of F.

All statements that are true "up to 2-torsion" are true after tensoring with $\mathbb{Z}[1/2]$. In particular, a map that is an isomorphism up to 2-torsion may not even exist until $\otimes\mathbb{Z}[1/2]$ has been applied.

Any unexplained notations can probably be found (with their explanations) in [L2].

Finally, the notation for "distinguished triangles" has been changed. What is now $A \to B \to C \to A[1]$ was denoted in [L2] by

$$\begin{tikzcd} & C \arrow[dl] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}\ .$$

1. Motivic cohomology and p-torsion

In this section we show that various theorems stated only modulo p-torsion in [L2] are still true even if p-torsion is considered.

Theorem 1.1. *Let F be a field of characteristic p. Then:*
(a) The natural map θ_F from $K_2(F)$ to $H^2(F,\Gamma(2))$ is an isomorphism
(b) $H^3(F,\Gamma(2)) = 0$.
(c) There is a triangle in the derived category:

$$\Gamma(2,F) \xrightarrow{p} \Gamma(2,F) \longrightarrow \nu(2)_F[-2] \to \Gamma(2,F)[1].$$

Proof. We begin with the proof of (a). In [L2] we showed that θ_F was injective, with p-torsion cokernel. Applying the obvious variant of Lemma 4.6 in [L2] it suffices now to show that $K_2(F)_p \to H^2(F,\Gamma(2,F))_p$ is surjective, and that $K_2(F)/pK_2(F)$ is isomorphic to $H^2(F,\Gamma(2))/pH^2(F,\Gamma(2))$.

We first show that $H^2(F,\Gamma(2))_p = 0$. Since the homology sheaves of $\Gamma(2,F)$ are given by $\mathcal{H}^1(F,\Gamma(2)) = K_3(F_s)_{\mathrm{ind}}$ and $\mathcal{H}^2(F,\Gamma(2)) = K_2(F_s)$, the others being zero, the second hypercohomology spectral sequence yields the short exact sequence

$$0 \to H^1(F, K_3(F_s)_{\mathrm{ind}}) \to H^2(F,\Gamma(2)) \to H^0(F, K_2(F_s)).$$

By Theorem 0.2, $H^1(F, K_3(F_s)_{\mathrm{ind}})$ has no p-torsion, and by a theorem of Suslin [Su], neither does $K_2(F_s)$, so $H^2(F, \Gamma(2))_p = 0$.

The same spectral sequence, together with Theorems 0.2, 0.3 and the above-mentioned theorem of Suslin, imply (c). Taking hypercohomology of the sequence in (c), and using Theorem 0.3 for F, we obtain the commutative diagram:

$$\begin{tikzcd}
& K_2(F)/pK_2(F) \arrow[r, "\sim"] \arrow[d, "\rho"] & \nu(2)_F \arrow[d, "\psi"] & \\
0 \arrow[r] & H^2(F,\Gamma(2))/pH^2(F,\Gamma(2)) \arrow[r] & (\nu(2)_{F_s})^{G_F} \arrow[r] & H^3(F,\Gamma(2))_p
\end{tikzcd}$$
$$\longrightarrow 0.$$

Since $\nu(2)$ is a sheaf, ψ is an isomorphism, hence ρ is an isomorphism and $H^3(F, \Gamma(2, F))_p = 0$, which completes the proof of (a) and proves (b).

We now observe that Theorem 0.1, Theorem 0.2, and Theorem 1.1 immediately combine to yield the following refinement (eliminating the p-torsion exception) of Theorem 10.1 of [L2]:

Theorem 1.2. *The following sequence is exact:*

$$0 \to \bigoplus_{\ell \neq p} (K_3(F)_{\mathrm{ind}} \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell) \to \bigoplus_{\ell \neq p} H^1(F, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) \to$$
$$K_2(F) \to K_2(F_s)^G \to \bigoplus_{\ell \neq p} H^2(F, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) \to 0.$$

2. The higher direct images of $\Gamma(2)$

Let α_* be the functor which associates with every sheaf for the étale site its restriction to the Zariski site. In this section, we will show that if X is a connected regular scheme of finite type over a field, then

(a) $R^1\alpha_*\Gamma(X,2) = \mathrm{gr}^2_\gamma \underline{\underline{K}}_3$, which is just the constant Zariski sheaf $\mathrm{gr}^2_\gamma K_3(F(X))$ on X (up to 2-torsion).

(b) $R^2\alpha_*\Gamma(X,2)$ is the Zariski sheaf $\underline{\underline{K}}_2$ on X.

(c) $R^3\alpha_*(X,2)$ is 2-torsion.

We begin by reviewing some basic facts about K_3.

Lemma 2.1. *Let A be a regular local ring with quotient field F. Assume that A is the localization of an algebra of finite type over an infinite field. Then, in the diagram of obvious maps:*

$$\begin{array}{ccccc} K_3(A)^{(2)} & \longrightarrow & K_3(A)_{\mathrm{ind}} & \longrightarrow & \widetilde{K}_3(A) \\ \downarrow & & \downarrow & & \downarrow \\ K_3(F)^{(2)} & \longrightarrow & K_3(F)_{\mathrm{ind}} & \longrightarrow & \widetilde{K}_3(F) \end{array}$$

all maps are isomorphisms up to 2-torsion.

Furthermore since $\mathrm{gr}^2_\gamma K_3(A)$ *may be identified with* $K_3(A)_{\mathrm{dec}}$, $K_3(A)_{\mathrm{ind}}$ *is naturally isomorphic to* $gr^2_\gamma K_3(A)$, *and of course similarly for* F.

Proof. That the horizontal arrows are isomorphisms up to 2-torsion, is contained in the discussion and proposition at the beginning of Section 8 of [L2], as is the last statement of the lemma. That the vertical maps are isomorphisms up to 2-torsion follows from Gersten's conjecture and Theorem 5 of [So].

Lemma 2.2. *Let* A *be a regular local ring which is the localization of an algebra of finite type over a field. Then the natural map from* $K_2(A)$ *to* $H^0(A, \underline{\underline{K}}_2)$ *has torsion kernel and cokernel.*

Proof. Let F be the quotient field of A, and let $j : \operatorname{Spec} F \to \operatorname{Spec} A$. Let $Y = \operatorname{Spec} A$. Let $\underline{\underline{K}}_2$ be the étale sheaf on Y associated with the étale presheaf $\underline{K}_2$ and let $\mathcal{M}$ be the sheaf on Y associated with the presheaf direct image of the presheaf $\underline{K}_{2,F}$ on $\operatorname{Spec} F$. If $y \in Y$ is of codimension one let i_y be the inclusion of y in Y. Then by Gersten's conjecture applied to étale extensions of A we have the exact sequence of sheaves

$$0 \to \underline{\underline{K}}_2 \to \mathcal{M} \to \coprod_y (i_y)_* G_m.$$

Now let $\underline{\underline{K}}_{2,F}$ be the sheaf on $\operatorname{Spec} F$ corresponding to the presheaf $\underline{K}_2$. Standard properties of the norm map on K_2 imply that the natural map from $\mathcal{M}$ to $j_* \underline{\underline{K}}_{2,F}$ is an isomorphism modulo torsion. This now gives rise to the complex of sheaves

$$0 \to \underline{\underline{K}}_2 \to j_* \underline{\underline{K}}_{2,F} \to \coprod_y (i_y)_* G_m$$

with torsion homology.

Taking sections, we obtain a complex of groups with torsion homology:

$$0 \to H^0(A, \underline{\underline{K}}_2) \to H^0(F, \underline{\underline{K}}_{2,F}) \to \coprod K_1(k(y)).$$

Comparing this with the sequence

$$0 \to K_2(A) \to K_2(F) \to \coprod K_1(y))$$

which is exact by Gersten's conjecture, we complete the proof of the lemma.

Now let $\underline{\underline{\widetilde{K}}}_3(A)$, as in [L2], denote the first homology sheaf of $\Gamma(2, A)$.

Lemma 2.3. *Let A be a regular semi-local ring containing a field. Then $H^1_{\text{ét}}(A, \underline{\underline{K}}_3)$ is torsion.*

Proof. Let B be any regular noetherian ring. Recall that $\widetilde{K}_3(B)$ is the kernel of the map from $C_{2,1}(B)$ to $C_{2,2}(B)$, and let $K_3^{(2)}(B)$ be the subgroup of $K_3(B)$ consisting of elements of Adams weight two. Then it follows from [L2] and [So] that there are natural maps

$$K_3(B) \xrightarrow{\psi} K_3(B)_{\text{ind}} \xrightarrow{\varphi} \widetilde{K}_3(B)$$

and Lemma 2.1 asserts that if B is regular and the localization of an algebra of finite type over an infinite field, then ψ and φ are isomorphisms up to 2-torsion.

Let F be the quotient field of B and j the natural map from $\operatorname{Spec} F$ to $\operatorname{Spec} B$. If $\widetilde{\theta}$ is the natural map from $\widetilde{K}_3(B)$ to $\widetilde{K}_3(F)$ it follows from Lemma 2.1 that $\widetilde{\theta}$ is an isomorphism up to 2-torsion. The existence of compatible norm maps on K_3 and K_3^M shows that $\underline{\underline{\widetilde{K}}}_3$ is isomorphic up to torsion to $j_*\underline{\underline{\widetilde{K}}}_3$. (We use here that strict Henselizations of local rings containing a field contain an infinite field). But since $H^i(B, j_*F)$ is torsion for $i > 0$ and any sheaf F, we are done.

For the rest of this section, let A be a regular local ring which is the localization of an algebra of finite type over a field K of characteristic $p \geq 0$. Let $\widetilde{K}_2(A)$ be the group $H^2_{\text{ét}}(A, \Gamma(2))$ and let ψ_A be the natural map from $K_2(A)$ to $\widetilde{K}_2(A)$. Our eventual goal is to show that ψ_A is an isomorphism.

Lemma 2.4. *ψ_A has torsion kernel and cokernel.*

Proof. We recall from [L2] that $\Gamma(2)$ is defined as a two-term complex of étale sheaves $\underline{\underline{C}}_{2,1} \to \underline{\underline{C}}_{2,2}$, corresponding to a map of presheaves $C_{2,1}(U) \xrightarrow{\varphi_U} C_{2,2}(U)$ such that if U is equal to $\operatorname{Spec} A$, $\operatorname{Coker} \varphi_U = K_2(A)$ and $\operatorname{Ker} \varphi_U = \widetilde{K}_3(A)$. It follows from the commutative diagram

$$\begin{array}{ccc}
C_{2,1}(A) & \xrightarrow{\varphi_U} & C_{2,2}(A) \\
\downarrow & & \downarrow \\
H^0(A, C_{2,1}) & \xrightarrow{\varphi} & H^0(A, C_{2,2})
\end{array}$$

that $K_2(A) = \operatorname{Coker} \varphi_U$ maps to $\operatorname{Coker} \varphi$, which in turn maps to $H^2(A, \Gamma(2))$, defining our map ψ_A. Now look at the diagram

$$\begin{tikzcd}
H^1(A,\underset{=}{\widetilde{K}}_3) \arrow[r] & H^2(A,\Gamma(2)) \arrow[r] & H^0(A,\underset{=}{K}_2) \\
& K_2(A) \arrow[u, "\psi_A"] \arrow[r, equal] & K_2(A) \arrow[u, "\theta"'] .
\end{tikzcd}$$

Since θ is an isomorphism up to torsion by Lemma 2.2 and $H^1(A,\underset{=}{\widetilde{K}}_3)$ is torsion by Lemma 2.3, ψ_A is an isomorphism up to torsion.

Lemma 2.5. *Let ℓ be a prime different from p. Then the map induced by ψ_A from $K_2(A)_\ell$ to $\widetilde{K}_2(A)_\ell$ is surjective.*

Proof. Let F be the quotient field of A. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_2(A)_\ell \arrow[r] & K_2(F)_\ell \arrow[r] & \prod_{x,\operatorname{cod}x=1} \mu_\ell(k(x)) \\
0 \arrow[r] & H^1_{\text{ét}}(A,\mu_\ell\otimes\mu_\ell) \arrow[r] \arrow[u, "\alpha"] & H^1_{\text{ét}}(F,\mu_\ell\otimes\mu_\ell) \arrow[r] \arrow[u, "\alpha_F"'] & \prod_{x,\operatorname{cod}x=1} H^0(F,\mu_\ell) \arrow[u, "\sim"']
\end{tikzcd}$$

where the top row comes from Gersten's conjecture for A and the bottom row is the Bloch-Ogus sequence [BO]. The map α_F comes from the Kummer sequence for $\Gamma(2,F) : H^1(F,\mu_\ell\otimes\mu_\ell) \to H^2(F,\Gamma(2))_\ell = K_2(F)_\ell$ and the commutativity is by direct computation if F contains a primitive ℓ-th root of unity and immediately reduces to that case, if not. This diagram then defines a unique map $\alpha : H^1_{\text{ét}}(A,\mu_\ell\otimes\mu_\ell) \to K_2(A)_\ell$ making the diagram commute.

In order to prove the surjectivity, it suffices to show that the diagram:

$$\begin{tikzcd}
H^1_{\text{ét}}(A,\mu_\ell\otimes\mu_\ell) \arrow[r, "\alpha"] \arrow[d, "1"'] & K_2(A)_\ell \arrow[d] \\
H^1_{\text{ét}}(A,\mu_\ell\otimes\mu_\ell) \arrow[r, "\beta"] & \widetilde{K}_2(A)_\ell
\end{tikzcd}$$

commutes, since the surjectivity of β follows from the Kummer sequence for $\Gamma(2)$ ([L2], Corollary 8.3). Essentially the same argument as in the proof of Theorem 4.5 in [L2] now shows the commutativity, and completes the proof.

Corollary 2.6. *The natural map ψ_A from $K_2(A)$ to $\widetilde{K}_2(A)$ is an isomorphism up to p-torsion.*

Proof. Theorem 9.1 of [L2] implies that $K_2(A)/\ell K_2(A)$ is isomorphic to $\widetilde{K}_2(A)/\ell\widetilde{K}_2(A)$. Now Lemma 4.6 of [L2] completes the proof, in view of Lemmas 2.4 and 2.5.

Let Y be a scheme over the prime field F_p. The differential symbol induces a map φ from $\underline{\underline{K}}_{2,Y}$ to $\Omega^2_{Y/\mathbb{Z}}$ of étale or Zariski sheaves on Y. Let $\nu_{2,Y}$ be the image of φ. In view of ([I] Thm. 02.4.2), if $Y = \operatorname{Spec} F$, Theorem 0.3 of the introduction implies that $\underline{\underline{K}}_{2,F}/p\underline{\underline{K}}_{2,F} \simeq \nu_{2,F}$ in either the étale or Zariski sites. Now assume that Y is regular and of finite type over a field.

Lemma 2.7. *There is a distinguished triangle (up to 2-torsion)*

$$\Gamma(2,Y) \xrightarrow{p} \Gamma(2,Y) \to \nu_Y(2)[-2] \to \Gamma(2,Y)[1].$$

Proof. Since the only non-vanishing homology sheaves of $\Gamma(2,Y)$ are $\mathcal{H}^1 = \mathrm{gr}_2\underline{\underline{K}}_3(Y)$ (up to 2-torsion) and $\mathcal{H}^2 = \underline{\underline{K}}_2(Y)$ ([L2], Corollary 8.4) it suffices to show that multiplication by p is an isomorphism on $\mathcal{H}^1$ and that there is a short exact sequence

$$0 \to \mathcal{H}^2 \xrightarrow{p} \mathcal{H}^2 \to \nu_Y(2) \to 0.$$

Let B be any strictly Hensel local ring on Y and let F be the quotient field of B. Lemma 2.1 and Theorem 0.2 show that multiplication by p is an isomorphism on $\mathcal{H}^1$.

The Gersten conjecture for A gives rise to the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K_2(A)/p & \longrightarrow & K_2(F)/p \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ & & \nu_A(2) & \longrightarrow & \nu_F(2)\ . \end{array}$$

Since β is an isomorphism by Theorem 0.3, and since α is surjective by definition, α is an isomorphism. Since $K_2(F)$ has no p-torsion by [Su] neither does $K_2(A)$, which completes the proof.

Lemma 2.8.

(i) *The natural map from $K_2(A)_p$ to $\widetilde{K}_2(A)_p$ is surjective.*
(ii) *The natural map from $K_2(A)/p$ to $\widetilde{K}_2(A)/p$ is bijective.*
(iii) $H^3(A,\Gamma(2))_p) = 0$.

Proof. Lemma 2.7 implies $\widetilde{K}_2(A)_p \stackrel{\text{def}}{=} H^2(A,\Gamma(2))_p = 0$, which proves (i). The homology sequence of the triangle in Lemma 2.7 implies also that

$$0 \to \widetilde{K}_2(A)/p \xrightarrow{\alpha} H^0(A,\nu_A(2)) \to H^3(X,\Gamma(2))_p \to 0$$

is exact. But now the commutative diagram

$$\begin{array}{ccccccc} & & K_2(A)/p & \xrightarrow{\sim} & \nu_A(2) & & \\ & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \widetilde{K}_2(A)/p & \xrightarrow{\alpha} & \nu_A(2) & \longrightarrow & H^3(X,\Gamma(2))_p \longrightarrow 0 \end{array}$$

shows that α is an isomorphism and $H^3(X,\Gamma(2))_p = 0$.

Proposition 2.9. *θ_A is an isomorphism.*

Proof. Since $K_2(F)$ has no p-torsion, Gersten's conjecture implies $K_2(A)_p = 0$, so Corollary 2.6 implies θ_A is injective. The obvious variant of Lemma 4.6 of [L2], Lemmas 2.7 and 2.8 and Corollary 2.6 imply that θ_A is surjective.

Still retaining the notations of [L2], if A is a regular ring, let $D(A) = \operatorname{Coker}(C_{2,1}(A) \to C_{2,2}(A))$, so Corollary 1.6 of [L2] yields a natural injection of $D(A)$ into $K_2(A)$. If all residue fields of A have at least three elements, then Lemma 1.7 of [L2] shows that the induced map on Zariski sheaves on $\operatorname{Spec} A$ from $\underline{\underline{D}}$ to $\underline{\underline{K}}_2$ is an isomorphism.

On the other hand, if φ is the induced map from $H^0_{\text{ét}}(A, \underline{\underline{C}}_{2,1})$ to $H^0_{\text{ét}}(A, \underline{\underline{C}}_{2,2})$, then $D(A)$ clearly maps to $\operatorname{Coker}\varphi$, which by the hypercohomology spectral sequence maps to $H^2_{\text{ét}}(A,\Gamma(2))$. Now applying Proposition 2.9, we have

Theorem 2.10. *Let X be a regular scheme of finite type over a field $\neq$ $\boldsymbol{F}_2$. Then the natural map from $K_{2,\text{zar}}$ to $R^2\alpha_*\Gamma(2,X)$ of Zariski sheaves is an isomorphism.*

Proposition 2.11. *Let X be a connected, regular scheme of finite type over a field. Then $R^1\alpha_*\Gamma(2,X) = \operatorname{gr}^2_\gamma \underline{\underline{K}}_3$, which is naturally isomorphic to the constant sheaf whose stalks are all $\operatorname{gr}^2_\gamma K_3(F(X))$, up to 2-torsion.*

Proof. By looking at stalks, it suffices to show that if A is local, $H^0(A, \operatorname{gr}^2_\gamma \underline{\underline{K}}_3) \xrightarrow{\delta} H^0(F, \operatorname{gr}^2_\gamma \underline{\underline{K}}_3)$ is bijective up to 2-torsion, where $F = F(X)$ is the quotient field of A. But we have the commutative diagram

$$\begin{array}{ccc} H^0(A, \operatorname{gr}^2_\gamma \underline{\underline{K}}_3) & \xrightarrow{\delta} & H^0(F, \operatorname{gr}^2_\gamma K_3) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \beta} \\ \operatorname{gr}^2_\gamma K_3(A) & \xrightarrow{\alpha} & \operatorname{gr}^2_\gamma K_3(F)\,. \end{array}$$

We know that α is an isomorphism up to 2-torsion by Lemma 2.1, β is an isomorphism by Theorem 0.1, and δ is injective up to 2-torsion again by Lemma 2.1 and Theorem 0.1.

Proposition 2.12. *$R^3\alpha_*\Gamma(2, X) = 0$ up to 2-torsion.*

Proof. This follows immediately from [L2], Corollary 9.7 and Lemma 2.8 (iii) of this paper.

Theorem 2.13. *Up to 2-torsion, we have:*

(a) $H^0_{\text{zar}}(X, \underline{\underline{K}}_2) \cong H^2_{\text{ét}}(X, \Gamma(2))$

(b) $H^1_{\text{zar}}(X, \underline{\underline{K}}_2) \cong H^3_{\text{ét}}(X, \Gamma(2))$

(c) *There is a natural injection from $CH^2(X) = H^2_{\text{zar}}(X, \underline{\underline{K}}_2)$ to $H^4_{\text{ét}}(X, \Gamma(2))$ with torsion cokernel.*

Proof. Consider the spectral sequence

$$H^p_{\text{zar}}(X, R^q\alpha_*\Gamma(2)) \Rightarrow H^{p+q}_{\text{ét}}(X, \Gamma(2)).$$

Ignoring 2-torsion, we have $R^q\alpha_*\Gamma(2) = 0$ for $q = 0$ and 3, and since $R^1\alpha_*\Gamma(2)$ is constant, $H^p_{\text{zar}}(X, R^1\alpha_*\Gamma(2)) = 0$ for $p > 0$. Also, $R^2\alpha_*\Gamma(2)$ is $\underline{\underline{K}}_{2,\text{zar}}$. Parts (a) and (b) of the theorem then follow immediately, and we obtain an exact sequence (up to 2-torsion) $0 \to H^2_{\text{zar}}(X, \underline{\underline{K}}_2) \to H^4_{\text{ét}}(X, \Gamma(2)) \to H^0_{\text{zar}}(X, R^4\alpha_*\Gamma(2)) \to H^3_{\text{zar}}(X, \underline{\underline{K}}_2)$. Since $R^4\alpha_*\Gamma(2)$ is easily seen to be torsion, part (c) follows.

Remark 2.14. A consideration of the Gersten resolution of $\underline{\underline{K}}_2$ shows that $H^p_{\text{zar}}(X, \underline{\underline{K}}_2) = 0$ for $p \geq 3$, hence the above map from $H^4_{\text{ét}}(X, \Gamma(2))$ to $H^0_{\text{zar}}(X, R^4\alpha_*\Gamma(2))$ is surjective.

Theorem 2.15. *Let X be a smooth projective variety over $\mathbb{C}$. The image of $H^4_{\text{ét}}(X, \Gamma(2))$ contains all torsion classes in $H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$.*

Proof. We start with the Kummer sequence

$$0 \to H^4_{\text{ét}}(X, \Gamma(2))/\ell^n \to H^4_{\text{ét}}(X, \mu_{\ell^n}^{\otimes 2}) \to H^5_{\text{ét}}(X, \Gamma(2))_{\ell^n} \to 0$$

and apply the inverse limit functor, which is exact since the groups are finite, to obtain

$$0 \to H^4_{\widehat{\text{ét}(X,\Gamma(2))}} \to H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)) \to T_\ell(H^5_{\text{ét}}(X, \Gamma(2)) \to 0.$$

Since T_ℓ is torsion-free, any torsion in $H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$ comes from $H^4_{\widehat{\text{ét}(X,\Gamma(2))}}$. Let $B(X)$ be the image of $H^4_{\text{ét}}(X, \Gamma(2))$ in $H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$, so $B(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_\ell$ maps onto the closure $\overline{B(X)}$ of $B(X)$ in $H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$, which contains all torsion classes.

We claim that the map from $B(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_\ell \to \overline{B(X)}$ is surjective with torsion kernel. The surjectivity is evident. Let $A(X)$ be the image of the codimension-2 cycles $CH^2(X)$ in $H^4_{\text{ét}}(X, \mathbf{Z}_\ell(2))$. Since we know by Theorem 2.13 that $CH^2(X)$ is a subgroup of $H^4_{\text{ét}}(X, \Gamma(2))$ with torsion quotient, it suffices to show that the map from $A(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_\ell \to \overline{A(X)}$ has torsion kernel.

Let z_i, $i = 1, \ldots, n$, represent a basis for codimension-two cycles modulo numerical equivalence. Since, by a theorem of Lieberman [Li], for codimension-two cycles numerical equivalence is the same as homological equivalence (mod torsion), any cycle in $A(X)$ has a multiple which is a linear combination of the z_i. So if $\beta \in \operatorname{Ker}(A(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_\ell \to \overline{A(X)})$, we may write $m\beta = \sum_{i=1}^n \alpha_i \otimes z_i$, $\alpha_i \in \mathbf{Z}_\ell$. Let w_j represent a basis for dimension-two cycles modulo numerical equivalence. By the compatibility of intersection product and cup product, $\sum_{i=1}^n \alpha_i(z_i, w_j) = 0$. Since $\det(z_i, w_j) \neq 0$, it follows that all the α_i are zero, i.e. $m\beta = 0$, so our claim is demonstrated.

So any torsion class comes from a torsion element in $B(X) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_\ell$, which in turn must come from a torsion element in $B(X)$, so be in the image of $H^4_{\text{ét}}(X, \Gamma(2))$.

Corollary 2.16. *There are projective smooth algebraic varieties over* $\mathbb{C}$ *where* $H^4_{\text{ét}}(X, \Gamma(2))$ *properly contains* $CH^2(X)$.

Proof. There are examples of Atiyah and Hirzebruch [A-H] of such varieties with codimension-two torsion classes in cohomology which do not come from algebraic cycles.

3. The construction of Beilinson's complex

Let X be a regular noetherian scheme. Let F be a sheaf on X for the big Zariski site and $\alpha^* F$ the associated sheaf for the étale site. The adjointness of the functors α^* and α_* gives rise to a map from F to $\alpha_*\alpha^* F$. This in turn induces a map in the derived category of Zariski sheaves $C \mapsto R\alpha_*\alpha^* C$. If we let $C = \Gamma_{\text{zar}}(2, X)$, observe that $\alpha^* C$ is $\Gamma_{\text{ét}}(2, X)$, and that since C is acyclic outside of [1,2] the map from C to $R\alpha_*\alpha^* C$ factors through $t_{\leq 2} R\alpha_*\alpha^* C$.

Proposition 3.1. *The natural map from* $\Gamma_{\text{zar}}(2, X)$ *to* $t_{\leq 2} R\alpha_*\Gamma_{\text{ét}}(2, X)$ *is an isomorphism up to 2-torsion.*

Proof. We must check that we have induced isomorphisms on the first and second Zariski homology sheaves $\mathcal{H}^i$ up to 2-torsion, and for this it suffices to look at stalks. Let A be a local ring on X. For $\mathcal{H}^1$ we must show that the natural map from $\widetilde{K}_3(A)$ to $H^0_{\text{ét}}(A, \underline{\underline{\widetilde{K}_3}})$ is an isomorphism up to 2-torsion. But up to 2-torsion this map is just the map γ in the proof of Proposition 2.11, and a corollary of that proof is that γ is an isomorphism up to 2-torsion. For $\mathcal{H}^2$ we are looking at the natural map from $K_2(A)$ to $H^2_{\text{ét}}(A, \Gamma(2))$, which is an isomorphism by Proposition 2.9.

We may conclude now (see [L2], Section 11) that $\Gamma_{\text{zar}}(2, X)$ satisfies Beilinson's conjectures as described in [L1], at least up to 2- torsion, and modulo the caveat in the introduction.

4. Gersten's conjecture and purity for motivic cohomology

Along with the many other suggested properties of motivic cohomology ([Be], [Bl], [L1]), there ought to be analogues, in the étale site, of Gersten's conjecture, and, closely related to it, the theorem of purity.

The general Gersten conjecture ought to be something like the following, although perhaps stronger in some sense:

Conjecture 4.1. *Let X be a regular scheme of finite type over a field k. Let $d = \dim X$. Let X_i denote the set of points of X of codimension i. If $x \in X$, let $j_x : \operatorname{Spec} k(x) \to X$. Then there exist objects U_i in the derived category, $i = 1, \dots, d$, together with the morphisms indicated below, such that there are distinguished triangles:*

$$\mathbf{Z}(X, r) \longrightarrow \bigoplus_{x \in X_0} t_{\le r+1} R(j_x)_* \mathbf{Z}(k(x), r) \longrightarrow U_1 \longrightarrow \mathbf{Z}(X, r)[1]$$

$$U_1 \longrightarrow \bigoplus_{x \in X_1} t_{\le r} R(j_x)_* \mathbf{Z}(k(x), (r-1))[-1] \longrightarrow U_2$$

$$\vdots$$

$$U_{d-1} \longrightarrow \bigoplus_{x \in X_{d-1}} t_{\le r+2-d} R(j_x)_* \mathbf{Z}(k(x), r-d+1)[1-d] \longrightarrow U_d$$

and an isomorphism

$$U_d \simeq \bigoplus_{x \in X_d} t_{\le r+1-d} R(j_x)_* \mathbf{Z}(k(x), r-d)[-d].$$

The referee suggests that this conjecture could be rephrased as follows:

Conjecture 4.1. *The complex $\mathbf{Z}(X, r)$ in the derived category is the image under the forgetful functor of an object $(\mathbf{Z}(X, r), F)$ in the filtered derived category (See [BBD]. Section 3 for basic definitions and terminology) such that the associated graded objects in the derived category F^{i-1}/F^i are equal to $\oplus_{X \in X_{i-1}} t_{\le r-i+2} R(j_*)_* \mathbf{Z}(k(x), r-i+1)[1-i]$. In this language the U_i are equal to $F^i/F^d[-i]$.*

The (weak) purity conjecture should take the following form:

Conjecture 4.2. *Let $i : Y \to X$ be a closed immersion of a regular scheme Y into a regular scheme X. Let c be the codimension of Y in X. Then*

$$t_{\leq r+c} Ri^{!}\mathbb{Z}(X,r) \simeq \mathbb{Z}(Y, r-c)[-2c].$$

Note that Conjecture 4.2 yields a composite map $\mathbb{Z}(Y,r-c)[-2c] \to t_{\leq r+c}Ri^{!}\mathbb{Z}(X,r) \to Ri^{!}\mathbb{Z}(X,r)$, which by adjointness gives rise to a map $Ri_*(\mathbb{Z}(Y,r-c)[-2c]) \to \mathbb{Z}(X,r)$, or from $Ri_*\mathbb{Z}(Y,r-c) \to \mathbb{Z}(X,r)[2c]$. In general, there should be a trace map, which for $f : Y \to X$ proper of relative dimension n, maps $Rf_*\mathbb{Z}(Y,n+r) \to \mathbb{Z}(X,r)[-2n]$, which could include the above ($f = i, n = -c$) as a special case.

This more general trace map could possibly follow from a more general purity conjecture with $Rf^{!}$ replacing $Ri^{!}$, but it is not clear to the author even how to define $Rf^{!}$. (One cannot expect a map $f^{!}$ to exist taking sheaves to sheaves, so $Rf^{!}$ would have to be defined directly on the level of complexes.)

In this section, we will show that the Gersten Conjecture (4.1) is valid for our proposed candidate $\Gamma(X,2)$ for $\mathbb{Z}(X,2)$ if X is a regular scheme of finite type over a field and we work modulo 2-torsion. Under the same conditions we prove the weak purity conjecture if $r = 2$ and $c = 1$. We also show that the Gersten conjecture leads to a natural construction of a cycle map.

Lemma 4.3. *Let $C_\bullet$ and $D_\bullet$ be complexes of objects in an abelian category which are acyclic outside of [1,2]. Let $\varphi : C_\bullet \to D_\bullet$ be a map in the derived category such that φ induces an isomorphism on H^1 and there exists an exact sequence*

$$0 \to H^2(C_\bullet) \xrightarrow{H^2(\varphi)} H^2(D_\bullet) \to W \to 0.$$

Then there exists a triangle in the derived category

$$C_\bullet \to D_\bullet \to W[-2] \to C_\bullet[1].$$

Proof. This is essentially the same as Exercise 3.1 of [L2], i.e., a straightforward exercise in derived categories.

Theorem 4.4. *Conjecture 4.1 is true up to 2-torsion for $\Gamma(X,2)$, i.e., there exists an object U in the derived category of étale sheaves on X and distinguished triangles (up to 2-torsion)*

$$\Gamma(X,2) \to \bigoplus_{x\in X_0} t_{\leq 3}R(j_x)_*\Gamma(k(x),2) \to U \to \Gamma(X,2)[1] \qquad (*)$$

and

$$U \to \bigoplus_{x\in X_1} t_{\le 2}R(j_x)_*\Gamma(k(x),1)[-1] \to \bigoplus_{x\in X_2} t_{\le 1}R(j_x)_*\Gamma(k(x),0)[-2] \to U[1]. \tag{**}$$

Proof. We clearly may suppose X is connected, hence irreducible. Let $\operatorname{Spec} F$ be the generic point of X, and let $j : \operatorname{Spec} F \to X$. We recall the definition of the map ψ from $\Gamma(X,2)$ to $t_{\le 3}Rj_*\Gamma(F,2)$. Since j_* and j^* are adjoint functors, there is a natural map from C to Rj_*Rj^*C for any object C in the derived category of sheaves on F. Since j^* is exact and $j^*\Gamma(X,2) = \Gamma(F,2)$ we get a map from $\Gamma(X,2)$ to $Rj_*\Gamma(F,2)$, which factors through $t_{\le 2}Rj_*\Gamma(F,2)$ since $\Gamma(X,2)$ is acyclic in degrees > 2, so certainly through $t_{\le 3}Rj_*\Gamma(F,2)$.

We now compute the effect of ψ on the homology sheaves $\mathcal{H}^i$ of $\Gamma(X,2)$. We first claim that ψ induces an isomorphism on $\mathcal{H}^1$. It suffices to check this on stalks, so we may assume that $X = \operatorname{Spec} A$, A strictly Hensel. In this case $\mathcal{H}^1(\Gamma(X,2)) = K_3^{(2)}(A)$, up to 2-torsion, by Proposition 8.1 of [L2] and $\mathcal{H}^1(Rj_*\Gamma(F,2)) = \mathcal{H}^1(\Gamma(F,2)) = K_3^{(2)}(F)$, up to 2-torsion by [L2], p.191 and Theorem 0.1. By Lemma 2.1, $K_3^{(2)}(A)$ is isomorphic to $K_3^{(2)}(F)$, up to 2-torsion.

Next we compute the effect of ψ on $\mathcal{H}^2$. As above if X is strictly Hensel, $X = \operatorname{Spec} A$, $\mathcal{H}^2(\Gamma(X,2)) = K_2(A)$, by Proposition 2.9. $\mathcal{H}^2(Rj_*\Gamma(F,2)) = \mathcal{H}^2(\Gamma(F,2)) = K_2(F)$ by [L2], p.191. By the Gersten conjecture for A, we have the exact sequence

$$0 \to K_2(A) \xrightarrow{\varphi} K_2(F) \xrightarrow{\psi} \coprod_{x\in X_1} K_1(k(x)) \to \coprod_{x\in X_0} K_0(k(x)) \to 0.$$

It follows that the cokernel of the map induced by ψ from $\mathcal{H}^2(\Gamma(X,2))$ to $\mathcal{H}^2(Rj_*\Gamma(F,2))$ is W, where

$$W = \operatorname{Ker}\left(\coprod_{x\in X_1} j_*G_m \to \coprod_{x\in X_0} j_*\mathbb{Z}\right).$$

Theorem 4.4 now follows from Lemma 4.3 and Hilbert Theorem 90 for $\Gamma(0)$, $\Gamma(1)$ and $\Gamma(2)$ (see Proposition 2.12) with of course $U = W[-2]$.

Theorem 4.5. *The weak purity conjecture (4.2) is true up to 2-torsion if $r = 2$ and $c = 1$, i.e.*

$$t_{\le 3}Ri^!\Gamma(X,2) \simeq \Gamma(Y,1)[-2].$$

Proof. We apply $Ri^!$ to the two distinguished triangles in the conclusion of Theorem 4.4. We may again assume X is connected, and let $j : \operatorname{Spec} F(X) \to X$. We claim first that

$$R^p i^!(t_{\leq 3} Rj_* \Gamma(F(X),2)) = 0 \quad \text{for } p \leq 3$$

or, equivalently, that

$$t_{\leq 3} Ri^!(t_{\leq 3} Rj_* \Gamma(F(X),2)) = 0.$$

But an easy argument shows that this last expression is equal to $t_{\leq 3} R(i^! j_*)\Gamma(F(X),2)$, which is zero since $i^! j_*$ is the zero functor. Hence the long exact homology sequence of $(*)$ shows that the map from U to $\Gamma(X,2)[1]$ induces an isomorphism of $t_{\leq 2} Ri^! U$ with $t_{\leq 2}(Ri^! \Gamma(X,2)[1]) = t_{\leq 3} Ri^! \Gamma(X,2)[1]$.

By definition of U and W and by Hilbert's Theorem 90, it follows from the triangle $(**)$ that we have the exact sequence of sheaves on Y :

$$0 \to i^! W \to \bigoplus_{x \in X_1} i^!(j_x)_* G_m \xrightarrow{\varphi} \bigoplus_{x \in X_2} i^!(j_x)_* \mathbf{Z} \to R^1 i^! W \to \bigoplus_{x \in X_1} R^1 i^!(j_x)_* G_m.$$

If $x \in X_1$ is not in Y, then $i^!(j_x)_* = 0$, and

$$0 = t_{\leq 1} R(i^!(j_x)_* G_m) = t_{\leq 1} Ri^! t_{\leq 1} R^1 j_* G_m = t_{\leq 1} Ri^! j_* G_m$$

by Hilbert 90, which implies $R^1 i^!(j_x)_* G_m = 0$ as well. If $x \in X_1 \cap Y$, then $i^!(j_x)_* = (h_x)_*$, where h_x is the inclusion of x in Y. Then $R^1(i^!(j_x)_*) = 0$ by Hilbert 90, and $R^1 i^!(j_x)_* G_m = 0$ by the composite-functor spectral sequence.

If $x \in X_2$ is not in Y, then $i^!(j_x)_* = 0$ and if $x \in X_2 \cap Y$, again $i^!(j_x)_* = (h_x)_*$. This all implies that we have an exact sequence

$$0 \to i^! W \to \bigoplus_{x \in Y_0} (h_x)_* G_m \xrightarrow{\varphi} \bigoplus_{x \in Y_1} (h_x)_* \mathbf{Z} \to R^1 i^! W \to 0.$$

Since $\operatorname{Ker} \varphi = G_{m,Y}$ and $\operatorname{Coker} \varphi = 0$, we have

$$i^! W = G_{m,Y}, \quad \text{hence}$$

$$t_{\leq 2} Ri^! U = t_{\leq 0} Ri^! W[-2] = i^! W[-2] = G_{m,Y}[-2] = \Gamma(Y,1)[-1], \text{so}$$

$$t_{\leq 3} Ri^! \Gamma(X,2) = \Gamma(Y,1)[-2].$$

5. Cycle maps

Let X be a regular scheme of finite type over a field K. We will give two different definitions of a map which associates an element of the group $H^4(X,\Gamma(2))$ with any codimension-2 cycle on X, and show that they are equivalent. We will go on to show that if X is smooth, this cycle map is compatible with the classical ℓ-adic and p-adic cycle maps. Throughout this section we work modulo 2-torsion. In particular, neither ℓ nor p is allowed to be 2.

Our first definition was given in Theorem 2.13 as a map θ from the group $H^2(X_{\text{zar}}, \underline{\underline{K}}_2)$ of codimension-2 cycles modulo rational equivalence to $H^4(X, \Gamma(2))$. Our second definition will be more refined, and will associate with any prime codimension-2 cycle Z a class $\psi(Z)$ (the fundamental class) in $H^4_Z(X, \Gamma(2))$ whose image in $H^4(X, \Gamma(2))$ will be $\theta(c\ell(Z))$. Let λ denote any of the natural maps : $H^4_Z(X, \Gamma(2)) \to H^4(X, \Gamma(2))$, $H^4_Z(X_{\text{zar}}, \Gamma(2)) \to H^4(X_{\text{zar}}, \Gamma(2))$, or $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2) \to H^2(X_{\text{zar}}, \underline{\underline{K}}_2)$.

Theorem 4.4 evidently gives us a map from $t_{\leq 1}(Rj_z)_*\mathbf{Z}$ to $\Gamma(X, 2)[4]$, and hence a map from $\mathbf{Z} = H^0_Z(X, t_{\leq 1}(Rj_z)_*\mathbf{Z})$ to $H^4_Z(X, \Gamma(2))$. We define the fundamental class $\psi(Z)$ to be the image of 1 in $H^4_Z(X, \Gamma(2))$.

Lemma 5.1. *The natural map $\beta : \Gamma(2) \longrightarrow \underline{\underline{K}}_2[-2]$ induces isomorphisms β_* from $H^4(X_{\text{zar}}, \Gamma(2))$ to $H^2(X_{\text{zar}}, \underline{\underline{K}}_2)$, and from $H^4_Z(X_{\text{zar}}, \Gamma(2))$ to $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$.*

Proof. Since the complex $\Gamma(2)$ has only two non-zero cohomology sheaves, $\mathcal{H}^1(\Gamma(2)) = \mathrm{gr}^2_\gamma \underline{\underline{K}}_3$ and $\mathcal{H}^2(\Gamma(2)) = \underline{\underline{K}}_2$, the hypercohomology spectral sequence degenerates into a long exact sequence

$$\to H^3(X_{\text{zar}}, \mathrm{gr}^2_\gamma \underline{\underline{K}}_3) \to H^4(X_{\text{zar}}, \Gamma(2)) \xrightarrow{\beta_*} H^2(X_{\text{zar}}, \underline{\underline{K}}_2) \to H^4(X_{\text{zar}}, \mathrm{gr}^2_\gamma \underline{\underline{K}}_3) \dots .$$

Since $\mathrm{gr}^2_\gamma(\underline{\underline{K}}_3)$ is a constant sheaf (Proposition 2.11), and X is irreducible, $H^i(X_{\text{zar}}, \mathrm{gr}^2_\gamma \underline{\underline{K}}_3)$ is zero for $i > 0$, which yields the conclusion of the lemma. The proof for cohomology with supports is similar.

Theorem 5.2. $\lambda(\psi(Z)) = \theta(c\ell(Z))$.

Proof. First we observe that Theorem 4.4 remains true in the Zariski site (the proof is similar, but simpler at key points). Let φ (resp. φ_Z) be the natural map from $H^4(X_{\text{zar}}, \Gamma(2))$ to $H^4(X, \Gamma(2))$ (resp. $H^4_Z(X_{\text{zar}}, \Gamma(2))$ to $H^4_Z(X, \Gamma(2))$. So we get an element $\psi_{\text{zar}}(Z)$ defined exactly as above in $H^4(X_{\text{zar}}, \Gamma(2))$ whose image via φ_Z in $H^4(X, \Gamma(2))$ is $\psi(Z)$.

Now look at the Gersten sequence for the sheaf $\underline{\underline{K}}_{2,\text{zar}}$:

$$0 \to \underline{\underline{K}}_2 \to \coprod_{x \in X_0} (j_x)_* \underline{\underline{K}}_2 \to \coprod_{x \in X_1} (j_x)_* \underline{\underline{K}}_1 \to \coprod_{x \in X_2} (j_x)_* \underline{\underline{K}}_0 \to 0.$$

Let z be the generic point of Z. The Gersten sequence induces a map ρ from $(j_z)_*\mathbf{Z}$ to $\underline{\underline{K}}_2[2]$ which takes 1_Z to the class of Z in $H^2(X_{\text{zar}}, \underline{\underline{K}}_2)$. It is easy to see that we have a commutative diagram

$$\begin{tikzcd} (j_z)_*\mathbf{Z} \arrow[r, "\psi_{\text{zar}}"] \arrow[d, "id"] & \Gamma(2)_{\text{zar}}[4] \arrow[d, "\beta"] \\ (j_z)_*\mathbf{Z} \arrow[r, "\rho"] & \underline{\underline{K}}_2[2] . \end{tikzcd}$$

So $\beta_*(\psi_{\text{zar}}(Z))$ is the class of Z in $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$, and $c\ell(Z) = \lambda(\beta_*(\psi_{\text{zar}}(Z))$.

Here we have written β_* for the map induced by β from $H^4_Z(X_{\text{zar}}, \Gamma(2)$ to $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$. We will also use β_* for the corresponding map without supports, and we remark that both these maps are isomorphisms by Lemma 5.1.

We now go back to the definition of θ. The cycle map θ is derived from the spectral sequence $H^p(X_{\text{zar}}, R^q\alpha_*\Gamma(2)) \Rightarrow H^{p+q}(X, \Gamma(2))$, which may be identified with the hypercohomology spectral sequence

$$H^p(X_{\text{zar}}, \mathcal{H}^q(R\alpha_*\Gamma(2))) \;\Rightarrow\; H^{p+q}(X, \Gamma(2)).$$

This evidently commutes with the spectral sequence

$$H^p(X_{\text{zar}}, \mathcal{H}^q(t_{\leq 2}R\alpha_*\Gamma(2))) \;\Rightarrow\; H^{p+q}(X_{\text{zar}}, t_{\leq 2}R\alpha_*\Gamma(2))).$$

Since we have seen that $t_{\leq 2}R\alpha_*\Gamma(2)$ is $\Gamma(2)_{\text{zar}}$, we easily derive the commutative diagram

$$\begin{tikzcd} H^2(X_{\text{zar}}, \underline{\underline{K}}_2) \arrow[r, "(\beta_*)^{-1}"] \arrow[d, "1"'] & H^4(X_{\text{zar}}, \Gamma(2)) \arrow[d, "\varphi"] \\ H^2(X_{\text{zar}}, \underline{\underline{K}}_2) \arrow[r, "\theta"] & H^4(X, \Gamma(2)) . \end{tikzcd}$$

So $\theta = \varphi \circ (\beta_*)^{-1}$. But now we have $\lambda(\psi(Z)) = \lambda(\varphi_Z(\psi_{\text{zar}}(Z)) = \lambda(\varphi_Z(\beta_*^{-1}(\beta_*\psi_{\text{zar}}(Z))) = \varphi\beta_*^{-1}\lambda(\beta_*\varphi_{\text{zar}}(Z)) = \varphi\beta_*^{-1}c\ell(Z) = \theta(c\ell(Z))$, which completes the proof of Theorem 5.2.

Lemma 5.3. *Our cycle class map is compatible with products. More precisely, if Z is an integral codimension 2 closed subscheme of X which is the scheme-theoretic intersection of two divisors Y_1 and Y_2 on X, then the fundamental class of Z in $H^4_Z(X, \Gamma(2))$ is the product of the fundamental classes of Y_1 and Y_2 in $H^2_{Y_1}(X, \Gamma(1))$ and $H^2_{Y_2}(X, \Gamma(1))$.*

Proof. Since products in the Zariski site are compatible with étale products, it suffices to prove this in the Zariski site. Since Remark 2.6 of

[L2] shows that the map from $\Gamma(2)$ to $\underline{\underline{K}}_2[-2]$ is compatible with products, and since we have seen that this map induces an isomorphism from $H^4_Z(X_{\text{zar}}, \Gamma(2))$ to $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$, it suffices to prove that fundamental classes are compatible with products into $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$. Recall that the fundamental class of Y_i is obtained by taking the Gersten resolution of G_m on X

$$G_m \to \coprod_{x \in X_0} (j_x)_* G_m \to \coprod_{x \in X_1} (j_x)_* \mathbf{Z}$$

which give us a map from $(j_{y_i})_*\mathbf{Z}$ to $G_m[1]$, where y_i is the generic point of Y_i. The class of 1 in $H^0_{Y_i}(X_{\text{zar}}, (j_{y_i})_*\mathbf{Z}) = \mathbf{Z}$ maps to the fundamental class in $H^1_{Y_i}(X_{\text{zar}}, G_m)$. Similarly, the fundamental class of Z is obtained by taking the Gersten resolution R_2 of $\underline{\underline{K}}_2$ and taking the image of 1 in $H^0_Z(X_{\text{zar}}, (j_z)_*\mathbf{Z}) = \mathbf{Z}$ in $H^2_Z(X_{\text{zar}}, \underline{\underline{K}}_2)$, where z is the generic point of Z.

To compute products, we replace the Gersten resolutions of G_m by the complexes C_i defined as follows : Let $U_i = X - Y_i$, $\gamma_i \; : \; U_i \to X$. Then C_i is the complex $(\gamma_i)_* G_m \to (j_{y_i})_*\mathbf{Z}$, which also represents G_m in the derived category. The fundamental class of Y_i can equally well be defined by mapping $(j_{y_i})_*\mathbf{Z}$ to C_i. But now we have an evident product map $C_1 \otimes C_2 \to R_2$, which is compatible with the fundamental class maps and the product map $(j_{y_1})_*\mathbf{Z} \otimes (j_{y_2})_*\mathbf{Z} \to (j_z)_*\mathbf{Z}$.

Proposition 5.4. *Let X be a smooth scheme over a field k. Let n be an odd integer prime to the characteristic of k. Let Z be a closed integral subscheme of X of codimension 2. Then the image of the cycle class $\psi(Z)$ corresponding to Z in $H^4_Z(X, \Gamma(2))$ via the Kummer sequence map to $H^4_Z(X, \mu_n^{\otimes 2})$ agrees with the classical cycle map defined by Deligne in [SGA4$\frac{1}{2}$].*

Proof. Let $i \; : \; Z \to X$. Since Deligne shows in [SGA4$\frac{1}{2}$] that $H^4_Z(X, \mu_n^{\otimes 2}) = H^0(Z, R^4 i^! \mu_n^{\otimes 2})$ may be computed locally on Z and is determined by its restriction to any non-empty open subset of Z, we may assume that Z is a complete intersection of prime divisors Y_1 and Y_2. Since our fundamental classes agree with Deligne's for divisors, and both are compatible with products ([SGA4$\frac{1}{2}$] and Lemma 5.2), as are the maps from $\Gamma(1)$ to μ_n and $\Gamma(2)$ to $\mu_n^{\otimes 2}$, the proposition follows.

Proposition 5.5. *The cycle map θ is compatible with the "classical" p-adic cycle map into $H^2(X, \nu_{2,n})$.* (See [M] for the precise definition).

Proof. Since the p-adic cycle map factors through $H^2(X, \underline{\underline{K}}_2)$ it suffices to show that there is a natural map σ from $H^4(X, \Gamma(2))$ to $H^2(X, \underline{\underline{K}}_2)$ such that the following diagram commutes :

$$\begin{array}{ccc} H^2(X_{\text{zar}}, \underline{\underline{K}}_2) & \xrightarrow{\theta} & H^4(X, \Gamma(2)) \\ \downarrow & & \downarrow{\scriptstyle\sigma} \\ H^2(X, \underline{\underline{K}}_2) & = & H^2(X, \underline{\underline{K}}_2) . \end{array}$$

But since the natural map from $\Gamma(2)$ to $\underline{\underline{K}}_2[-2]$ yields the commutative diagram

$$\begin{array}{ccc} H^4(X_{\text{zar}}, \Gamma(2)) & \xrightarrow{\beta_*} & H^2(Z_{\text{zar}}, \underline{\underline{K}}_2) \\ \downarrow{\scriptstyle\varphi} & & \downarrow \\ H^4(X, \Gamma(2)) & \longrightarrow & H^2(X, \underline{\underline{K}}_2) \end{array}$$

with β_* an isomorphism, and we have seen that $\theta = \varphi \circ \beta_*^{-1}$, we are done.

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Department of Mathematics
White Hall
Cornell University
Ithaca, New York 14853

Symmetric Spaces over a Finite Field

GEORGE LUSZTIG*

Dedicated to A. Grothendieck on his 60th birthday

Introduction

Let G be a (connected) reductive group defined over a finite field F_q (q odd) with a given involution $\theta: G \to G$ defined over F_q. The pair (G, θ) will be called a symmetric space (over F_q); we shall fix a closed subgroup K of the fixed point set G^θ such that K is defined over F_q and K contains the identity component $(G^\theta)^0$ of G^θ.

Let T be a maximal torus of G defined over F_q and let $\lambda: T(F_q) \to \overline{Q}_\ell^*$ be a character (ℓ is a fixed prime not dividing q); let R_T^λ be the virtual representation of $G(F_q)$ attached in [2] to (T, λ). This paper is concerned with the computation of the sum

$$(a) \qquad |K(F_q)|^{-1} \sum_{g \in K(F_q)} \mathrm{Tr}(g, R_T^\lambda);$$

we find for it an explicit formula. (See 3.3(a).)

There is a particular kind of symmetric spaces in $1-1$ correspondence with connected reductive groups G_1 defined over F_q (without involution); to G_1 corresponds the symmetric space $(G_1 \times G_1, \tau)$ where $\tau(g_1, g_1') = (g_1', g_1)$. It is well known that many objects attached to reductive groups can be redefined in terms of the associated symmetric space so that they make sense for an arbitrary symmetric space. For example:

(b) the Schubert varieties in the theory of reductive groups become a special case of the closures of G^θ-orbits on the flag manifold of G;

(c) the set of conjugacy class of the reductive group G_1 (or $G_1(F_q)$) becomes a special case of the set of double cosets $K\backslash G/K$ (or $K(F_q)\backslash G(F_q)/K(F_q)$);

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(d) the problem of classifying the irreducible representations of $G_1(F_q)$ becomes a special case of the problem of classifying the irreducible representations of $G(F_q)$ with a non-zero space of $K(F_q)$-invariant vectors;

(e) the character table of $G_1(F_q)$ becomes a special case of the character table (suitably normalized) of the Hecke algebra (algebra of double cosets) of $G(F_q)$ with respect to $K(F_q)$.

In the same spirit, our formula for (a) is a generalization of the orthogonality formula [2,6.8] which expresses the sum

$$|G_1(F_q)|^{-1} \sum_{g_1 \in G_1(F_q)} \mathrm{Tr}(g_1, R_{T_1}^{\lambda_1}) \mathrm{Tr}(g_1, R_{T_1'}^{\lambda_1'})$$

in terms of the Weyl group. (Here T_1, T_1' are maximal tori of G_1 defined over F_q and λ_1, λ_1' are characters of $T_1(F_q), T_1'(F_q)$.)

The proof of the orthogonality formula given in [2] is based in part on the disjointness theorem [2,6.2]. While that disjointness theorem extends to symmetric spaces (see 3.5), it is not strong enough to imply our formula for (a) in the general case. On the other hand, the argument given in [3, p.8-10] generalizes better.

In the case where λ is a character in general position of $T(F_q)$, our formula gives the dimension of the space of $K(F_q)$-invariant vectors in the irreducible representation $\pm R_T^\lambda$ of $G(F_q)$, hence it can be regarded as a solution of (d) as far as "generic" representations of $G(F_q)$ are concerned; this confirms in part a conjecture of Bannai-Kawanaka-Song [1,6.7].

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1. Recollections and notations

1.1. In this paper, G will always denote a connected reductive group defined over a finite field F_q with q elements (q odd), with a given involution $\theta: G \to G$ defined over F_q and a given closed subgroup K of the fixed point set G^θ such that K is defined over F_q and K contains the identity component $(G^\theta)^0$. We shall denote by $F: G \to G$ the Frobenius map corresponding to the F_q-structure. We denote $\mathbf{S} = \mathbf{G/K}$, a homogeneous variety for G.

1.2. If T is a maximal torus of G, we define $\Theta_T = \{f \in G | \theta(f^{-1}Tf) = f^{-1}Tf\}$. Then Θ_T is a union of $T - K$ double cosets and we denote $\mathbf{\Theta}_T$ the image of Θ_T under the canonical map $G \to \mathbf{S}$. Then $\mathbf{\Theta}_T$ is a T-stable subvariety of $\mathbf{S}$. The following analog of the Bruhat decomposition is well known.

Proposition 1.3. *Let T be a maximal torus of G and let B be a Borel subgroup of G containing T, with unipotent radical U.*

(a) *B has only finitely many orbits on $\mathbf{S}$.*

(b) *T has only finitely many orbits on $\mathbf{\Theta}_T$.*

(c) *We associate to any T-orbit $\mathcal{O}$ on $\mathbf{\Theta}_T$ the unique B-orbit $\mathcal{O}'$ on $\mathbf{S}$ such that $\mathcal{O} \subset \mathcal{O}'$. The resulting map {set of T-orbits on $\mathbf{\Theta}_T$} $\to$ {set of B-orbits on $\mathbf{S}$ } is a bijection.*

(d) *If $\mathcal{O} \subset \mathcal{O}'$ are as in* (c) *then the map $U \times \mathcal{O} \to \mathcal{O}'$ defined by the B-action is an affine space bundle.*

First note that if the proposition is true for a pair $(T \subset B)$, then it follows easily that it is true for any other pair $(xTx^{-1} \subset xBx^{-1}), (x \in G)$. Using [5,7.5], we are reduced to the case where both T and B are θ-stable. In this case, a proof can be found, for example, in [4, §4] at least under the additional assumption that $K = G^\theta$; however, essentially the same proof applies in the general case.

1.4. The variety of Borel subgroups of G will be denoted by $\mathcal{B}$. We shall denote by W the Weyl group of G and $\ell: W \to \mathbf{N}$ the usual length function. To two Borel subroups B, B' one can attach naturally an element $w \in W$ (relative position) as in [2,1.2]; we shall then write $B \overset{w}{\text{---}} B'$.

1.5. Let
$\Omega = \{B \in \mathcal{B} | B \text{ opposed to } \theta B\}$
$\mathcal{J}$ = variety of all θ-stable maximal tori T such that $T \subset B$ for some $B \in \Omega$
$\mathcal{P}$ = variety of all pairs $T \subset B$ with $T \in \mathcal{J}$, $B \in \Omega$.

(a) If Ω is non empty, then K^0 acts transitively on it, as well as on $\mathcal{J}$ and $\mathcal{P}$ (by conjugation).

1.6. If $f: V \to V$ is a morphism of a variety into itself we denote by V^f its fixed point set. For a closed subgroup G_1 of an algebraic group G_2 we denote by $Z(G_1)$ or $Z_{G_2}(G_1)$ the centralizer of G_1 in G_2 and by G_1^0 the identity component of G_1.

2. The homomorphism ϵ

2.1. We set $\sigma(G) = \begin{cases} 1, & \text{if } F_q\text{-rank } (G) \text{ is even;} \\ -1, & \text{otherwise} \end{cases}$

Let T be a maximal torus of G which is F-stable. It is well known that

$$(a) \qquad \sigma(T)\sigma(G) = (-1)^{\dim(U/U\cap FU)}$$

where B is any Borel subgroup of G containing T and U is its unipotent radical.

2.2. Assume now that T is a maximal torus of G which is both F-stable and θ-stable. Let $T_K = T \cap K$ and let T_K^0 be the identity component of T_K. It is well known that

$$(a) \qquad t \in T_K \Rightarrow t^2 \in T_K^0.$$

We denote by $Z = Z(T_K^0)$ the centralizer of T_K^0 in G; for each $t \in T_K^F$ we set $Z_t = Z \cap Z_G^0(t)$. Then Z, Z_t are connected reductive groups defined over F_q. We define

$$(b) \qquad \epsilon = \epsilon_T : T_K^F \to \pm 1$$

by $\epsilon(t) = \sigma(Z)\sigma(Z_t)$.

Proposition 2.3. *Let B be a Borel subgroup containing T, with unipotent radical U. For $t \in T_K^F$ let*

$$n(t) = \dim(U \cap Z/U \cap FU \cap Z) + \dim(U \cap Z_t/U \cap FU \cap Z_t).$$

Then

(a) $e(t) = (-1)^{n(t)}$

(b) $\epsilon(t) = 1$ *for all* $t \in (T_K^0)^F$

(c) ϵ *is a group homomorphism.*

Proof. Using 2.1(a) for Z and Z_t instead of G, we see that for $t \in T_K^F$ we have $(-1)^{n(t)} = \sigma(T)\sigma(Z) \cdot \sigma(T)\sigma(Z_t) = \epsilon(t)$, hence (a). If $t \in (T_K^0)^F$, we have $Z_t = Z$ hence $\epsilon(t) = \sigma(Z)^2 = 1$. To prove (c), we can clearly assume that $Z = G$, i.e. that T_K^0 is central in G. Let $\mathbf{\Phi}$ be the set of roots $\alpha : T \to \overline{F}_q^*$ such that the corresponding root subgroup is contained in U

but not in FU. For any $t \in T_K^F$ let $\mathbf{\Phi}_t$ be the set of roots $\alpha \in \mathbf{\Phi}$ such that $\alpha(t) = 1$. We have $n(t) = |\mathbf{\Phi}| + |\mathbf{\Phi}_t|$. Hence, to prove (c) it is enough to show, in view of (a), that given $t, t' \in T_K^F$, we have

$$(d) \qquad |\mathbf{\Phi}_t| + |\mathbf{\Phi}_{t'}| + |\mathbf{\Phi}_{tt'}| + |\mathbf{\Phi}| \equiv 0 (\text{mod } \mathbf{2}).$$

From (b) and 2.2(a) we see that $\alpha(t) \in \{\pm 1\}, \alpha(t') \in \{\pm 1\}$ for all roots α. Hence we can define a partition of $\mathbf{\Phi}$ into four pieces

$$\mathbf{\Phi}^{i,j} = \{\alpha \in \mathbf{\Phi} | \alpha(t) = i, a(t') = j\}, i, j \in \{\pm 1\}.$$

We have $|\mathbf{\Phi}_t| = |\mathbf{\Phi}^{1,1}| + |\mathbf{\Phi}^{1,-1}|, |\mathbf{\Phi}_{t'}| = |\mathbf{\Phi}^{1,1}| + |\mathbf{\Phi}^{-1,1}|, |\mathbf{\Phi}_{tt'}| = |\mathbf{\Phi}^{1,1}| + |\mathbf{\Phi}^{-1,-1}|, |\mathbf{\Phi}| = |\mathbf{\Phi}^{1,1}| + |\mathbf{\Phi}^{1,-1}| + |\mathbf{\Phi}^{-1,1}| + |\mathbf{\Phi}^{-1,-1}|$. Hence the left hand side of (d) is

$$4|\mathbf{\Phi}^{1,1}| + 2|\mathbf{\Phi}^{1,-1}| + 2|\mathbf{\Phi}^{-1,1}| + 2|\mathbf{\Phi}^{-1,-1}|$$

and (d) follows. The proposition is proved.

3. Statement of the main results

3.1. Let T be an F-stable maximal torus of G and let $\lambda: T^F \to \overline{Q}_\ell^*$ be a character. The variety Θ_T in 1.2 is F-stable and we denote

$$\Theta_{T,\lambda}^F = \left\{ f \in \Theta_T^F : {}^{f^{-1}}\lambda|_{(f^{-1}Tf)_K^F} = \epsilon_{f^{-1}Tf} \right\}.$$

Here ${}^{f^{-1}}\lambda: (f^{-1}Tf)^F \to \overline{Q}_\ell^*$ is the character defined by ${}^{f^{-1}}\lambda(t') = \lambda(ft'f^{-1})$, and $\epsilon_{f^{-1}Tf}$ is as in 2.2(b). It is clear that $\Theta_{T,\lambda}^F$ is a union of $T^F - K^F$ double cosets of G^F.

3.2. We recall the definition of the virtual representations R_T^λ of [2]. Let B be a Borel subgroup of G containing T, with unipotent radical U. Let

$$X = \{g \in G | g^{-1}F(g) \in FU\}.$$

Then G^F acts on X by $g_0: g \to g_0 g$ and T^F acts on X by $t: g \to gt^{-1}$, hence G^F, T^F acts on $H_c^i(X)$ (ℓ-adic cohomology with compact support; we generally omit writing the coefficients $\overline{Q}_\ell$).

If Y is any variety over $\overline{F}_q$ with T^F-action, we shall denote $H_c^i(Y)_{\lambda^{-1}}$ the subspace of $H_c^i(Y)$ on which T^F acts according to the character λ^{-1} and we define

$$(a) \qquad \chi(Y)_{\lambda^{-1}} = \sum_i (-1)^i \dim H_c^i(Y)_{\lambda^{-1}}$$

In particular $H^i_c(X)_{\lambda^{-1}}$ is well defined; it is naturally a G^F-module and, by definition [2,1.20]

$$R_T^\lambda = \sum_i (-1)^i H^i_c(X)_{\lambda^{-1}},$$

is a virtual representation of G^F of dimension $\chi(X)_{\lambda^{-1}}$.

Theorem 3.3.

(a) $$|K^F|^{-1} \sum_{g \in K^F} \mathrm{Tr}(g, R_T^\lambda) = \sum_f \sigma(T)\sigma(Z((f^{-1}Tf)^0_K)),$$

sum over a set of representatives f for the double cosets $T^F \backslash \Theta^F_{T,\lambda}/K^F$.

Theorem 3.4

(a) $$\sum_{\substack{g \in K^F \\ \text{unipotent}}} \mathrm{Tr}(g, R_T^1) = |T^F|^{-1} \sum_{f \in \Theta_T^F} \sigma(T)\sigma(Z((f^{-1}Tf)^0_K)).$$

Theorem 3.5. *Let ρ be an irreducible representation of G^F such that the space of K^F-invariants ρ^{K^F} is non-zero. Assume that ρ appears in the G^F-module $H^i_c(X)_{\lambda^{-1}}$ where T, λ, B, U, X are as in* 3.1, 3.2. *Then there exists $f \in \Theta_T$ and an integer $n \geq 1$ such that $F^n f = f$ and the character $\tilde{\lambda}: (f^{-1}Tf)^{F^n} \to \overline{Q}_\ell^*$ defined by $\tilde{\lambda}(t') = \lambda(N(ft'f^{-1}))$ is trivial on the subgroup $((f^{-1}Tf)^0_K)^{F^n}$. (Here, $N: T^{F^n} \to T^F$ is the norm map.) For that $f, n, \tilde{\lambda}$ we have $\tilde{\lambda}(\theta(t')) = \tilde{\lambda}(t')^{-1}$, for all $t' \in (f^{-1}Tf)^{F^n}$.*

3.6. *Remark.* Theorems 3.3, 3.4, 3.5 are generalizations of Theorems 6.8, 6.9, 6.2 of [2] respectively.

4. Proof of Theorem 3.5.

4.1. Let T, B, U be as in 3.1 and let $\mathcal{O}$ be a T-orbit on Θ_T. We define

(a) $$\mathfrak{S} = \mathfrak{S}_{\mathcal{O}} = \{(x, u, \phi) \in FU \times U \times \mathcal{O} | F(\phi) = xu\phi \text{ in.}\}$$

The variety $\mathfrak{S}$ has a T^F- action

(b) $$t: (x, u, \phi) \to (txt^{-1}, tut^{-1}, tf).$$

Actually, the same formula defines an action of a larger group $H_{\mathcal{O}}$ on $\mathfrak{S}$, where

(c) $$H_{\mathcal{O}} = \{t \in T | t^{-1}F(t) \in T_{F\mathcal{O}}\}$$

and, for any T-orbit $\mathcal{O}_1$ on Θ_T,

(d) $T_{\mathcal{O}_1}$ is the stabilizer in T of an element of $\mathcal{O}_1$ (this stabilizer is independent of the choice of an element of $\mathcal{O}_1$, since T is abelian). Clearly, $T^F \subset H_{\mathcal{O}}$.

We shall need the following generalization of [2,6.7].

Lemma 4.2. *Let $\psi: T \to T$ be an endomorphism of T which commutes with F^n for an integer $n \geq 1$, and let T_1 be the identity component of T^{ψ}. Let $H = \{t \in T | t^{-1}F(t) \in T^{\psi}\}$. Assume that $\lambda|_{H^0 \cap T^F}$ is trivial. Then $\lambda \circ N: T^{F^n} \to \overline{Q}_\ell^*$ is trivial on $T_1 \cap T^{F^n}$. (N is the norm map $T^{F^n} \to T^F$).*

Proof. Recall that N is the restriction to T^{F^n} of the homomorphism $\tilde{N}: T \to T, \tilde{N}(t) = tF(t) \cdots F^{n-1}(t)$. If $t \in T^{\psi}$, then $\tilde{N}(t) \in H$; indeed, $\tilde{N}(t)^{-1}F(\tilde{N}(t)) = t^{-1}F^n(t)$ and $\psi(t^{-1}F^n(t)) = t^{-1}F^n(t)$ since $\psi(t) = t$ and ψ commutes with F^n. Thus, $\tilde{N}(T^{\psi}) \subset H$, hence $\tilde{N}(T_1) \subset H^0$. It follows that $\lambda | N(T_1) \cap T^F$ is trivial, hence $\lambda \circ N | T_1 \cap T^{F^n}$ is trivial.

Proposition 4.3. *Assume that for some integer i, $H^i_c(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}}$ (defined in terms of the T^F-action, as in 3.2) is non-zero. Let $f' \in \Theta_T$ be such that $f'K \in \mathcal{O}$, let $n \geq 1$ be such that $F^n f' = f'$ and let $f = Ff'$. Then $\lambda \circ N: T^{F^n} \to T$ is trivial on the subgroup $f \cdot (f^{-1}Tf)^0_K \cdot f^{-1}$.*

Proof. We apply Lemma 4.2 to the automorphism $\psi: T \to T, \psi(t) = f\theta(f^{-1}tf)f^{-1}$. We have $T^{\psi} = f \cdot (f^{-1}Tf)_{G^\theta} \cdot f^{-1}$ hence $T_1 = f(f^{-1}Tf)^0_K f^{-1}$. We have $T_{F\mathcal{O}} = f(f^{-1}Tf)_K f^{-1}$. Since $T^{\psi}, T_{F\mathcal{O}}$ have the same identity component, H (of 4.2) and $H_{\mathcal{O}}$ (of 4.1) have the same identity component. Now $H_{\mathcal{O}}$ acts on $H^i_c(\mathfrak{S}_{\mathcal{O}})$ (see 4.1) and $H^0_{\mathcal{O}}$, being connected, must act trivially; hence $H^0_{\mathcal{O}} \cap T^F$ acts on $H^i_c(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}}$ both trivially and as the restriction of λ^{-1}. Since $H^i_c(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}} \neq 0$, it follows that λ is trivial on $H^0_{\mathcal{O}} \cap T^F$ hence on $H^0 \cap T^F$. Hence 4.2 is applicable and the proposition follows.

Proposition 4.4. *Assume that $K = (G^\theta)^0$ and consider the orbit space $K^F \backslash X$ of X in 3.2 with respect to the K^F-action $k_0: g \to k_0 g$. The map $g \mapsto (g^{-1}F(g), g^{-1}K)$ defines an isomorphism of $K^F \backslash X$ with*

$$Z = \{(x\phi) \in FU \times \mathfrak{s} | F\phi = x^{-1}\phi\}.$$

The proof is left to the reader; it uses Lang's theorem for G and K.

Now Z has a T^F-action $t\colon (x,\phi) \to (txt^{-1}, t\phi)$; this is compatible with the action of T^F on $K^F\backslash X$ induced by the action of T^F on X. Hence we have

$$(a) \qquad H_c^i(X)_{\lambda^{-1}}^{K^F} \cong H_c^i(K^F\backslash X)_{\lambda^{-1}} \cong H_c^i(Z)_{\lambda^{-1}}$$

for all i.

4.5. For each B-orbit $\mathcal{O}'$ on $\mathbf{S}$, we denote

$$Z_{\mathcal{O}'} = \{(x,\phi) \in Z | \phi \in \mathcal{O}'\}.$$

Then the $Z_{\mathcal{O}'}$ form a partition of Z into finitely many locally closed T^F-stable subvarieties; this partition has the property that the closure of a piece in the partition is a union of pieces in the partition. It follows that:
(a) If $H_c^i(Z)_{\lambda^{-1}} \neq 0$ for some i, then $H_c^i(Z_{\mathcal{O}'})_{\lambda^{-1}} \neq 0$ for some i and some $\mathcal{O}'$.
(b) $\chi(Z)_{\lambda^{-1}} = \sum_{\mathcal{O}'} \chi(Z_{\mathcal{O}'})_{\lambda^{-1}}$
where $\chi(Z)_{\lambda^{-1}}, \chi(Z_{\mathcal{O}'})_{\lambda^{-1}}$ are as in 3.2(a) and $\mathcal{O}'$ runs over all B-orbits in $\mathbf{S}$.

4.6. Let $\mathcal{O}$ be the unique T-orbit in Θ_T contained in the B-orbit $\mathcal{O}'$ in $\mathbf{S}$ (see 1.3(c)). Using 1.3(d) we see that the map $\mathfrak{S}_{\mathcal{O}} \to Z_{\mathcal{O}'}, (x,u,\phi) \to (x^{-1}F(u)^{-1}, u\phi)$, is an affine-space bundle, compatible with the T^F-actions. It follows that:
(a) If $H_c^i(Z_{\mathcal{O}'})_{\lambda^{-1}} \neq 0$ for some i, then $H_c^i(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}} \neq 0$ for some i
(b) $\chi(Z_{\mathcal{O}'})_{\lambda^{-1}} = \chi(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}}$

4.7. **Proof of Theorem 3.5.** It is sufficient to consider the case where $K = (G^\theta)^0$; the general case reduces immediately to this case. By assumption, we have $H_c^i(X)_{\lambda^{-1}}^{K^F} \neq 0$ for some i. Using 4.4(a), we deduce that $H_c^i(Z)_{\lambda^{-1}} \neq 0$ for some i. Using 4.5(a) we deduce that $H_c^i(Z_{\mathcal{O}'})_{\lambda^{-1}} \neq 0$ for some $\mathcal{O}'$ and some i and using 4.6(a) we deduce that $H_c^i(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}} \neq 0$ for some $\mathcal{O}$ and some i. We then apply 4.3, and we obtain the first assertion of 3.5. Let $f, n, \tilde{\lambda}$ be as in that assertion. The image of the homomorphism $t' \to t'\theta(t')$ of $f^{-1}Tf$ into itself is connected and contained in G^θ, hence it is contained in $(f^{-1}Tf)_K^0$. Hence if $t' \in (f^{-1}Tf)^{F^n}$, then $t'\theta(t') \in ((f^{-1}Tf)_K^0)^{F^n}$ and by the first assertion we have $\tilde{\lambda}(t'\theta(t')) = 1$. Hence the second assertion of 3.5 is proved.

5. Computations of some Euler characteristics

5.1. In this section we shall assume that G is simply connected and that there exists a maximal torus $T \subset G$ such that $\theta(t) = t^{-1}$ for all $t \in T$. We fix such a T and a Borel subgroup B containing T. In this case we must have $K = G^\theta = (G^\theta)^0$. (See [6,8.2]. Then $B \in \Omega$ (see 1.5) and the isotropy group of B for the transitive K-action on Ω (see 1.5(a)) is $T_K = \{t \in T | t^2 = 1\}$.

5.2. For each simple coroot $\check{\alpha}_j : \overline{F}_q^* \to T$ (with respect to B), let $t_j = \check{\alpha}_j(-1) \in T_K$. Then $t_1, t_2, \ldots, t_\ell$ ($\ell =$ rank G) form a basis of T_K as an F_2-vector space. Let $\chi_o : T_K \to \pm 1$ be the character such that $\chi_o(t_j) = -1$ for $j = 1, \ldots, \ell$. Let $\mathcal{L}$ be the 1-dimensional, K-equivariant $\overline{Q}_\ell$-local system on Ω such that the action of the isotropy group T_K on the stalk of $\mathcal{L}$ at B is given by χ_0.

5.3. Let D be the line in $\mathcal{B}$ consisting of all $B' \in \mathcal{B}$ which are contained in a fixed parabolic subgroup of semisimple rank 1. Then $D \cong \mathbf{P}^1, D - D \cap \Omega$ consists of two points and the restriction of $\mathcal{L}$ to $D \cap \Omega (\cong \mathbf{P}^1 - \{0, 1\})$ is isomorphic to the unique $\overline{Q}_\ell$-local system with monodromy of order exactly two. Hence
(a) $H_c^i(D \cap \Omega, \mathcal{L}) = 0$ for all i.

(We denote the restriction of $\mathcal{L}$ to various subvarieties of $\mathcal{B}$ again by $\mathcal{L}$.)

For each $w \in W$ let $Y_w = \{B' \in \Omega \,|\, B \stackrel{w}{\text{——}} B'\}$, a locally closed subvariety of Ω.

Proposition 5.4.

$$\dim H_c^i(Y_w, \mathcal{L}) = \begin{cases} 1, & \textit{if } i = \ell(w); \\ 0, & \textit{if } i \neq \ell(w). \end{cases}$$

Proof. When $w = e$, we have $Y_w = \{B\}$ and the result is clear. Assume now that $w \neq e$ and that the result is already known for w' of strictly smaller length than w. We can find $w', s \in W$ with $\ell(s) = 1, \ell(w') = \ell(w) - 1, w's = w$. We consider the following varieties

$$Y = \{(B', B'') \in \Omega \times \Omega \,|\, B \stackrel{w'}{\text{———}} B',\ B' \stackrel{s \text{ or } e}{\text{————}} B''\}$$

$$Y' = \{(B', B'') \in \Omega \times \Omega \,|\, B \stackrel{w'}{\text{———}} B',\ B' \stackrel{s}{\text{———}} B''\}$$

$$Y'' = \{(B', B'') \in \Omega \times \Omega \,|\, B \stackrel{w'}{\text{———}} B',\ B' = B''\}$$

$$Y''' = \{(B', B'') \in (\mathcal{B} - \Omega) \times \Omega \,|\, B \stackrel{w'}{\text{———}} B',\ B' \stackrel{s}{\text{———}} B''\}.$$

We can pull back $\mathcal{L}$ to each of these four varieties via the second projection; this pull back is denoted again as $\mathcal{L}$. We have

$$(a) \qquad H_c^i(Y,\mathcal{L}) = H_c^i(Y''',\, \mathcal{L}) = 0 \quad \text{for all i.}$$

Indeed, consider the first projection of Y or Y''' to $\mathcal{B}$. By the Leray spectral sequence it is enough to show that each fibre of this projection has vanishing $H_c^i(,\mathcal{L})$. In both cases, this follows from 5.3(a).

Now $Y = Y' \cup Y''$ with Y' open and Y'' closed in Y. Using (a) and the long exact sequence attached to this partition it follows that

$$(b) \qquad H_c^i(Y'',\, \mathcal{L}) \xrightarrow{\approx} H_c^{i+1}(Y',\, \mathcal{L}) \text{ for all i.}$$

We may identify Y' with an open subset of Y_w and Y''' with its complement in Y_w (via the map $(B',B'') \to B''$. Using (a) and the long exact sequence attached to this partition, it follows that

$$(c) \qquad H_c^i(Y',\, \mathcal{L}) \xrightarrow{\approx} H_c^i(Y_w,\mathcal{L}) \text{ for all i.}$$

The map $(B',\, B'') \to B''$ defines an isomorphism

$$(d) \qquad Y'' \cong Y_{w'}\ .$$

From (b), (c), (d) it follows that $H_c^i(Y_w,\mathcal{L}) \cong H_c^{i-1}(Y_{w'},\, \mathcal{L})$ and the proposition follows by induction.

Corollary 5.5. $\chi(Y_w) = (-1)^{\ell(w)}$.

(χ denotes Euler characteristic.) The proof will be given in 5.7. It is based on the following result.

Lemma 5.6[2,3.2]. *Let $\sigma\colon V \to V$ be an automorphism of finite order prime to q of an algebraic variety V over $\overline{F}_q$. Then* $\sum_i(-1)^i\mathrm{Tr}(\sigma^*, \mathrm{H}_c^i(\mathrm{V})) = \chi(\mathrm{V}^\sigma)$.

5.7. **Proof of Corollary 5.5.** We can regard K as a principal fibration over Ω with finite group T_K, via the map $\pi\colon K \to \Omega(g \to gBg^{-1})$ and the action of T_K on K by right translation. Let $\tilde{Y}_w = \pi^{-1}(Y_w)$. Then $H_c^i(Y_w,\mathcal{L})$ (resp. $H_c^i(Y_w)$) is isomorphic to the subspace of $H_c^i(\tilde{Y}_w)$ on which T_K acts according to χ (resp. trivially). Hence

$$(a) \quad \sum_i(-1)^i \dim H_c^i(Y_w,\mathcal{L}) = |T_K|^{-1} \sum_{t\in T_K}\sum_i(-1)^i\mathrm{Tr}(\mathrm{t}, \mathrm{H}_c^i(\tilde{\mathrm{Y}}_\mathrm{w}))\chi(\mathrm{t})$$

and

(b) $$\sum_i (-1)^i \dim H_c^i(Y_w) = |T_K|^{-1} \sum_{t \in T_K} \sum_i (-1)^i \mathrm{Tr}(t, \mathrm{H}_c^i(\tilde{\mathrm{Y}}_w)).$$

The right hand sides of (a),(b) can be evaluated using 5.6 (the fixed point set of t on $\tilde{Y}_w$ is empty for $t \neq e$); hence they are both equal to $|T_K|^{-1}\chi(\tilde{Y}_w)$. Thus, the left hand sides of (a) and (b) are equal. But the left hand side of (a) is equal to $(-1)^{\ell(w)}$ by 5.4; hence so is the left hand side of (b). The Corollary is proved.

6. Beginning of the study of $\mathfrak{S}$

6.1. In this section we shall assume that

(a) T, B, U, λ are as in 3.1., 3.2.

(b) $K = G^\theta$

(c) $\theta T = T, T_K^0 \subset \mathrm{centre(G)}$

(d) $\mathcal{O}$ is the T-orbit of the base point K in Θ_T.

6.2. Let $\mathfrak{S} = \mathfrak{S}_{\mathcal{O}}$ be the variety associated to these data as in 4.1(a). If we identify $\mathcal{O}$ with T/T_K, we have

$$\mathfrak{S} = \{(x,u,t) \in FU \times U \times T | F(t)^{-1}xut \in K\}$$

modulo the free action of $T_K, t_1: (x,u,t) \to (x,u,tt_1)$; the T^F action on $\mathfrak{S}$ is given by $t_0: (x,u,t) \to (t_0xt_0^{-1}, t_0ut_0^{-1}, t_0t)$.

We now state the main result of this section.

Proposition 6.3.

(a) $$\chi(\mathfrak{S}) = |T^F||(T_K^0)^F|^{-1}(-1)^{\dim(U/U\cap FU)}.$$

6.4. We make the change of variables $(x,u,t) \to (x',u',t)$ where $x' = t^{-1}xt, u' = t^{-1}ut$; then $\mathfrak{S}$ becomes

(a) $$\mathfrak{S} = \{(x,u,t) \in FU \times U \times T | F(t)^{-1}txu \in K\}$$

modulo the free action of $T_K, t_1: (x,u,t) \to (t_1^{-1}xt_1, t_1^{-1}ut_1, t_1t)$; the T^F action is $t_0: (x,u,t) \to (x,u,t_0t)$.

Let

(b) $\mathfrak{S}_1 = \{(x,u,\tau) \in FU \times U \times T | \tau xu \in K\}$.

(c) $\mathfrak{S}_2 = \mathfrak{S}_1$ modulo the free T_K^0 − action $\tilde{t}_1: (x,u,\tau) \to (x,u,\tilde{t}_1\tau)$.

On $\mathfrak{S}_1$ we have a T_K-action $t_1 : (x,u,\tau) \to (t_1^{-1}xt_1, t_1^{-1}ut_1, F(t_1)^{-1}t_1\tau)$; the same formula defines a T_K/T_K^0-action on $\mathfrak{S}_2$. Indeed, when $t_1 \in T_K^0$ we have $(t_1^{-1}xt_1, t_1^{-1}ut_1, F(t_1)^{-1}t_1\tau) = (x, u, F(t_1)^{-1}t_1\tau)$ (by 6.1(c)) and this is in the same orbit as (x,u,τ) for the T_K^0-action in (c). Moreover, using Lang's theorem in T_K^0 we see that the set of T_K-orbits on $\mathfrak{S}_1$ may be identified with the set of T_K/T_K^0-orbits on $\mathfrak{S}_2$ (which is an algebraic variety since T_K/T_K^0 is finite); using Lang's theorem in T, we see that the map $(x,u,t) \to (x,u,F(t)^{-1}t)$ defines an isomorphism $T^F\backslash\mathfrak{S} \cong T_K\backslash\mathfrak{S}_1 = (T_K/T_K^0)\backslash\mathfrak{S}_2$. It follows that

(*d*) $\chi(T^F\backslash\mathfrak{S}) = \chi((T_K/T_K^0)\backslash\mathfrak{S}_2)$.

6.5. Applying 1.5(a) to $(T \subset B) \in \mathcal{P}$ and $(T \subset FB) \in \mathcal{P}$ we see that there exists $n \in K$ normalizing T such that $FB = n^{-1}Bn$. We then also have $FU = n^{-1}Un$. Let

(*a*) $\mathfrak{S}_3 = (BnB) \cap K$ modulo the free T_K^0-action by left translation.

We define a map

(*b*) $\mathfrak{S}_2 \to \mathfrak{S}_3$

by $(x,u,\tau) \to n\tau xu$; this is well defined since for $(x,u,\tau) \in \mathfrak{S}_1$ we have $\tau xu \in K$ hence $n\tau xu \in K$ and $n\tau xu = (n\tau n^{-1}) \cdot (nxn^{-1})nu \in TUnU = BnB$; we also use the fact that T_K^0 is central. It is clear that the map (b) is an affine-space bundle. In particular, we have

(*c*) $\chi(\mathfrak{S}_2) = \chi(\mathfrak{S}_3)$.

6.6. On $\mathfrak{S}_3$ we have a free T_K/T_K^0 action by left translation. (This action is not compatible under (b) with the action of T_K/T_K^0 on $\mathfrak{S}_2$.) It is clear that the map $g \to g^{-1}Bg$ defines an isomorphism

(*a*) $$(T_K/T_K^0)\backslash\mathfrak{S}_3 \cong \{B' \in \mathcal{B} | B' \text{opposed to } \theta B', B \overset{w}{\text{———}} B'\}$$

where $w \in W$ is the relative position of $(B, n^{-1}Bn) = (B, FB)$. (We use the fact that $B \cap K = T_K$.)

The variety in the right side of (a) remains unchanged when G is replaced by the simply connected covering of its derived group (with the involution corresponding to θ) hence we can apply to it Corollary 5.5; using (a), we see that

(*b*) $\chi((T_K/T_K^0)\backslash\mathfrak{S}_3) = (-1)^{\ell(w)} = (-1)^{\dim(U/U\cap n^{-1}Un)} = (-1)^{\dim(U/U\cap FU)}$

6.7. When a finite group Γ of order prime to q acts on an algebraic variety V over $\overline{F}_q$, we have as in 5.7:

$$\begin{aligned}\chi(\Gamma\backslash V) &= |\Gamma|^{-1}\sum_{\gamma\in\Gamma}\sum_i(-1)^i\mathrm{Tr}(\gamma, H_c^i(V))\\ &= |\Gamma|^{-1}\sum_{\gamma\in\Gamma}\chi(V^\gamma)\end{aligned}$$

where V^γ is the fixed point set of γ on V. We apply this in three cases:

(a) $\chi((T_K/T_K^0)\backslash\mathfrak{S}_3) = |(T_K/T_K^0)|^{-1}\chi(\mathfrak{S}_3)$,

since the action of (T_K/T_K^0) on $\mathfrak{S}_3$ is free,

(b) $\chi(T^F\backslash\mathfrak{S}) = |T^F|^{-1}\sum_{t_0\in T^F}\chi(\mathfrak{S}^{t_0})$,

(c) $\chi((T_K/T_K^0)\backslash\mathfrak{S}_2) = |(T_K/T_K^0)|^{-1}\sum_{\bar{t}_0\in T_K/T_K^0}\chi(\mathfrak{S}_2^{\bar{t}_1})$.

From (a), 6.6(b) and 6.5(c), we see that

(d) $\chi(\mathfrak{S}_2) = |(T_K/T_K^0)|(-1)^{\dim(U/U\cap FU)}$.

6.8. Assume that $\bar{t}_1 \in T_K/T_K^0$ (representing $t_1 \in T_K$) has a fixed point (x, u, τ) on $\mathfrak{S}_2$.

Then $x \in FU\cap Z_G^0(t_1), u \in U\cap Z_G^0(t_1), F(t_1)^{-1}t_1 \in T_K^0$ hence $F(\bar{t}_1) = \bar{t}_1$. Thus $\mathfrak{S}_2^{\bar{t}_1}$ is empty unless $F\bar{t}_1 = \bar{t}_1$. If $F\bar{t}_1 = \bar{t}_1$, we may choose $t_1 \in T_K^F$ and we see that $\mathfrak{S}_2^{\bar{t}_1}$ may be identified with the variety defined exactly like $\mathfrak{S}_2$ but for $(G, U, T, \theta, \ldots)$ replaced by $(Z_G^0(t_1), U\cap Z_G^0(t_1), T, \theta, \ldots)$. To this variety we may apply formula 6.7(d). Hence 6.7(c) becomes:

(a) $\chi((T_K/T_K^0)\backslash\mathfrak{S}_2) = \sum_{t_1\in T_K^F}|(T_K^0)^F|^{-1}(-1)^{\dim(U^{t_1}/U^{t_1}FU^{t_1})}$,

where $U^{t_1} = U\cap Z_G^0(t_1)$.

Assume now that $t_0 \in T^F$ has a fixed point (x, u, t) on $\mathfrak{S}$ (in the description 6.4(a)). Then $x \in FU\cap Z_G^0(t), u \in U\cap Z_G^0(t), t_0 \in T_K^F$. Hence $\mathfrak{S}^{t_0}$ is empty unless $t_0 \in T_K^F$. If $t_0 \in T_K^F$, we see that $\mathfrak{S}^{t_0}$ may be identified with the variety defined axactly as in $\mathfrak{S}$ (in 6.4(a)) but for $(G, U, T, \theta, \ldots)$ replaced by $(Z_G^0(t_0), U\cap Z_G^0(t_0), T, \theta, \ldots)$. Thus (6.7b) becomes

(b) $\chi(T^F\backslash\mathfrak{S}) = \sum_{t_0\in T_K^F}|T^F|^{-1}\chi(\mathfrak{S}^{t_0})$.

Now 6.3(a) certainly holds when $G = \{e\}$.

We may assume that 6.3(a) holds for $(Z_G^0(t_0), U \cap Z_G^0(t_0), T, \theta, \ldots)$ instead of $(G, U, T, \theta, \ldots)$ whenever $t_0 \in T_K^F$ is not central in G (i.e. $Z_G^0(t_0)$ is a proper subgroup of G.) It follows that the terms corresponding to such t_0 in the sums in the right hand sides of (a), (b), match. Subtracting term by term the identities (a), (b) and using 6.4(d) we deduce

$$0 = \sum_{t_0 \in T_K^F \cap \mathrm{centre}(G)} (|T^F|^{-1}\chi(G) - |(T_K^0)^F|^{-1}(-1)^{\dim U/U\cap FU})$$

since $\mathfrak{S}^{t_0} = \mathfrak{S}$ for t_0 in the last sum. This implies that 6.3(a) holds.

Corollary 6.9.

$\chi(\mathfrak{S})_{\lambda^{-1}} = |(T_K^0)^F|^{-1} \sum_{t_0 \in T_K^F} (-1)^{\dim(U^{t_0}/U^{t_0} \cap FU^{t_0})} \lambda(t_0)$ where $U^{t_0} = U \cap Z_G^0(t_0)$.

Proof. We have

$$\begin{aligned}\chi(\mathfrak{S})_{\lambda^{-1}} &= |T^F|^{-1} \sum_{t_0 \in T^F} \sum_i (-1)^i \mathrm{Tr}(t_0, H_c^i(\mathfrak{S}))\lambda(t_0)\\ &= |T^F|^{-1} \sum_{t_0 \in T^F} \chi(\mathfrak{S}^{t_0})\lambda(t_0) \quad \text{(by 5.6)}\\ &= |T^F|^{-1} \sum_{t_0 \in T_K^F} \chi(\mathfrak{S}^{t_0})\lambda(t_0) \quad \text{(see 6.8)}\end{aligned}$$

and we apply 6.3 to $\mathfrak{S}^{t_0}$ (see 6.8).

Corollary 6.10.

$$\chi(\mathfrak{S}_{\lambda^{-1}}) = \begin{cases} |T_K^F| \cdot |(T_K^0)^F|^{-1}\sigma(T)\sigma(G), & \text{if } \lambda|_{T_K^F} = \epsilon_T \\ 0 & , \text{ otherwise.}\end{cases}$$

Proof. Using 6.9 and the definition of ϵ_T (2.2(b)) we have

$$\chi(\mathfrak{S}_{\lambda^{-1}}) = |(T_K^0)^F|^{-1}(-1)^{\dim(U/U\cap FU)} \sum_{t_0 \in T_K^F} \epsilon_T(t_0)\lambda(t_0).$$

Now ϵ_T is a character of T_K^F (see 2.3(c)) hence to the last sum we may apply the orthogonality formula for the characters of T_K^F; we also use 2.1(a). The desired result follows.

7. The study of $\mathfrak{S}$, continued

7.1. In this section we shall assume that

(a) T, B, U, λ are as in 3.1, 3.2.

(b) $\mathcal{O}$ is a T-orbit in Θ_T such that $F\mathcal{O} = \mathcal{O}$. (Hence $\mathcal{O}^F \neq \emptyset$.)

Then $FT_{\mathcal{O}} = T_{F\mathcal{O}} = T_{\mathcal{O}}$ (see 4.1). We set $\mathcal{G} = \mathcal{G}_{\mathcal{O}} = Z(T^0_{\mathcal{O}})$, a connected reductive group defined over F_q. Let $\tilde{\epsilon}_{\mathcal{O}}: T^F_{\mathcal{O}} \to \{\pm 1\}$ be defined by

(c) $\tilde{\epsilon}_{\mathcal{O}}(t) = \sigma(\mathcal{G})\sigma(Z^0_{\mathcal{G}}(t))$.

(d) Let $\tilde{\mathcal{O}}$ be the inverse image of $\mathcal{O}$ under $G \to G/K$.

7.2. Let $\phi \in \mathcal{O}^F$, and let $n \in G$ be defined by $n = f\theta(f)^{-1}$ where $f \in G$ satisfies $\phi = fK$. Then n depends only on ϕ (not on f), $Fn = n$ and $\theta n = n^{-1}$. Let $\theta_\phi: G \to G$ be the involution defined by $\theta_\phi(g) = n\theta(g)n^{-1}$. Then $\theta_\phi F = F\theta_\phi$ and $\theta_\phi T = T$ (since $f \in \Theta_T$); the fixed point set T^{θ_ϕ} is exactly $T_{\mathcal{O}}$. It follows that θ_ϕ leaves $\mathcal{G}$ stable; moreover, $(T^{\theta_\phi})^0 \subset$ centre $\mathcal{G}$. In particular $\tilde{\epsilon}_{\mathcal{O}}$ (see 7.1(c)) coincides with the homomorphism $\epsilon_T: (T^{\theta_\phi})^F \to \{\pm 1\}$ defined as in 2.2(b) for $(\mathcal{G}, T, \theta_\phi, F)$ instead of (G, T, θ, F).

7.3. Assume for a moment that $\tilde{\mathcal{O}}^F \neq \emptyset$. If $f \in \tilde{\mathcal{O}}^F$ then $f^{-1}Tf$ is F-stable and θ-stable and conjugation by f^{-1} carries $T^0_{\mathcal{O}}$ isomorphically onto $(f^{-1}Tf)^0_K$ and $\mathcal{G}$ isomorphically onto $\mathcal{G}_f = Z((f^{-1}Tf)^0_K)$. It follows that

(a) $$\sigma(\mathcal{G}) = \sigma(\mathcal{G}_f), \epsilon_{f^{-1}Tf}(f^{-1}tf) = \tilde{\epsilon}_{\mathcal{O}}(t) \text{ for all } t \in T^F_{\mathcal{O}}.$$

In particular, the following three conditions are equivalent:

(b) $$\tilde{\mathcal{O}}^F \cap \Theta^F_{T,\lambda} \neq \emptyset; \tilde{\mathcal{O}}^F \subset \Theta^F_{T,\lambda}; \lambda|_{T^F_{\mathcal{O}}} = \tilde{\epsilon}_{\mathcal{O}}.$$

Lemma 7.4. (a) *We have* $|T^F_{\mathcal{O}}| \cdot |T^{0F}_{\mathcal{O}}|^{-1} = |T^F \backslash \mathcal{O}^F|$
(b) *If* $K = (G^\theta)^0$, *then* $\tilde{\mathcal{O}}^F \neq \emptyset$ *and* $|T^F \backslash \mathcal{O}^F| = |T^F \backslash \tilde{\mathcal{O}}^F / K^F|$.

Proof. (a) Since T acts transitively on $\mathcal{O}$ and the stabilizer of any point (in particular a rational point) is $T_{\mathcal{O}}$, the number of T^F-orbits in $\mathcal{O}^F$ is equal to the cardinal of the cokernel of $F - 1: T_{\mathcal{O}}/T^0_{\mathcal{O}} \to T_{\mathcal{O}}/T^0_{\mathcal{O}}$, hence also to the number of fixed points of F on $T_{\mathcal{O}}/T^0_{\mathcal{O}}$. This equals $|T^F_{\mathcal{O}}| \cdot |T^{0F}_{\mathcal{O}}|^{-1}$.

(b) By Lang's theorem in K we have $\tilde{\mathcal{O}}^F/K^F \xrightarrow{\approx} \mathcal{O}^F$. The Lemma follows.

Proposition 7.5. *Let $\mathfrak{S} = \mathfrak{S}_{\mathcal{O}}$ be the variety associated to the data in* 7.1(a), (b) *as in* 4.1(a).

(a) *If $K = G^\theta$, then:*

$$\chi(\mathfrak{S})_{\lambda^{-1}} = \begin{cases} |T_{\mathcal{O}}^F| \cdot |(T_{\mathcal{O}}^0)^F|^{-1}\sigma(T)\sigma(\mathcal{G}), & \text{if } \lambda|_{T_{\mathcal{O}}^F} = \tilde{\epsilon}_{\mathcal{O}} \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If $K = G^\theta = (G^\theta)^0$, then:*

$$\chi(\mathfrak{S})_{\lambda^{-1}} = \sum_{f \in \tilde{\mathcal{O}}^F \cap \Theta_{T,\lambda}^F} |(f^{-1}Tf)_K^F| \cdot |T^F|^{-1}|K^F|^{-1}\sigma(T)\sigma(Z((f^{-1}Tf)_K^0)).$$

For the proof we shall need the following known result:

Lemma 7.6. *Let V be a variety over $\overline{F}_q$ with an action of T^F and with an action of a torus T_1 over $\overline{F}_q$, commuting with the T^F-action, with fixed point set V^{T_1}. Then $\chi(V)_{\lambda^{-1}} = \chi(V^{T_1})_{\lambda^{-1}}$.*

For a proof, based on [2,3.2], see [3,1.6].

7.7. For the proof of Proposition 7.5, we assume that $K = G^\theta$. We have $FT_{\mathcal{O}} = T_{F\mathcal{O}} = T_{\mathcal{O}}$ and $T_{\mathcal{O}} \subset H_{\mathcal{O}}$ (see 4.1). Since $T_{\mathcal{O}}, H_{\mathcal{O}}$ have the same dimension, it follows that $H_{\mathcal{O}}^0 = T_{\mathcal{O}}^0$. Now $H_{\mathcal{O}}$ acts on $\mathfrak{S}_{\mathcal{O}}$ (see 4.1).

Applying 7.3 to the restrictions of this action to T^F and to $H_{\mathcal{O}}^0 = T_{\mathcal{O}}^0$, we deduce

$$(a) \qquad \chi(\mathfrak{S})_{\lambda^{-1}} = \chi(\mathfrak{S}^{T_{\mathcal{O}}^0})_{\lambda^{-1}}$$

where

$$\mathfrak{S}^{T_{\mathcal{O}}^0} = \{(x, u, \phi) \in FU \times U \times \mathcal{O} | F\phi = xu\phi, x \in \mathcal{G}, u \in \mathcal{G}\},$$

hence

$$(b) \qquad \mathfrak{S}^{T_{\mathcal{O}}^0} = \{(x, u, \phi) \in FU' \times U' \times \mathcal{O} | F\phi = xu\phi\}, U' = U \cap \mathcal{G}.$$

Let $f, \phi, n, \theta_\phi: G \to G$ be as in 7.2.

We write θ' instead of θ_ϕ.

Let $\mathcal{O}_1$ be the T-orbit of the base point $\mathcal{G}^{\theta'}$ in $\mathcal{G}/\mathcal{G}^{\theta'}$. We have an imbedding $j: \mathcal{G}/\mathcal{G}^{\theta'} \hookrightarrow G/K, j(g\mathcal{G}^{\theta'}) = gfK$; this is well defined since $f^{-1}\mathcal{G}^{\theta'}f \subset K$. This imbedding is T-equivariant and it carries the T-orbit $\mathcal{O}_1$ isomorphically onto the T-orbit $\mathcal{O}$. From this and (b) we see that $(x, u, \phi') \to (x, u, j(\phi'))$ defines a T^F-equivariant isomorphism $\mathfrak{S}' \xrightarrow{\approx} \mathfrak{S}^{T_{\mathcal{O}}^0}$, where $\mathfrak{S}'$ is defined like $\mathfrak{S}$ in 6.1, in terms of $(\mathcal{G}, T, \theta', U \cap \mathcal{G}, \mathcal{O}_1, \ldots)$. This and (a) imply that $\chi(\mathfrak{S})_{\lambda^{-1}} = \chi(\mathfrak{S}')_{\lambda^{-1}}$. Now $\chi(\mathfrak{S}')_{\lambda^{-1}}$ is given by 6.10 for $(\mathcal{G}, T, \theta', \ldots)$; these satisfy the hypotheses in 6.1. Thus, 7.5(a) is proved. In the case where $K = G^\theta = (G^\theta)^0$, the formula in 7.5(a) can be rewritten in the form 7.5(b) using the remarks in 7.3 and Lemma 7.4. This completes the proof.

8. The study of $\mathfrak{S}$, concluded

8.1. In this section we shall assume that

(a) T, B, U, λ are as in 3.1, 3.2

(b) $K = G^\theta$

(c) $\mathcal{O}$ is a T-orbit in Θ_T such that $F\mathcal{O} \neq \mathcal{O}$.

Let $\mathfrak{S} = \mathfrak{S}_\mathcal{O}$ be the variety associated to the data above as in 4.1(a). The main result of this section is the following.

Proposition 8.2. $\chi(\mathfrak{S})_{\lambda^{-1}} = 0$.

The action 4.1 of $H_\mathcal{O}$ on $\mathfrak{S}$ restricts to the T^F-action and to an action of the torus $H^0_\mathcal{O}$. We shall write H, H^0 instead of $H_\mathcal{O}, H^0_\mathcal{O}$. Using 7.3, we have $\chi(\mathfrak{S})_{\lambda^{-1}} = \chi(\mathfrak{S}^{H^0})_{\lambda^{-1}}$. Hence the proposition is a consequence of the following result.

Proposition 8.3. *Under the hypothesis in* 8.1, *the fixed point set* $\mathfrak{S}^{H^0}$ *is empty.*

Proof. We assume that $(x, u, \phi) \in \mathfrak{S}^{H^0}$, and we shall prove that $\mathcal{O} \cap G^F \neq \emptyset$. Let $\mathcal{G} = Z(H^0)$ and let $\tilde{\mathcal{G}}$ be the derived group of $\mathcal{G}$. Our assumption implies that $x \in \mathcal{G}, u \in \mathcal{G}$; since x, u are unipotent, we have $x \in \tilde{\mathcal{G}}, u \in \tilde{\mathcal{G}}$ hence

$$xu \in \tilde{\mathcal{G}}. \tag{a}$$

Our assumption also implies that $t\phi = \phi$ for all $t \in H^0$, hence $H^0 \subset T_\mathcal{O}$ (see 4.1), hence $H^0 \subset T^0_\mathcal{O}$. Applying F to the equation $t\phi = \phi$ we have that $F(t)F(\phi) = F(\phi)$ for all $t \in H^0$. But for $t \in H$ we have $t^{-1}F(t)F(\phi) = F(\phi)$ (by the definition of H, see 4.1) hence $F(t)F(\phi) = tF(\phi)$. Thus $tF(\phi) = F(\phi)$ for all $t \in H^0$ hence $H^0 \subset T_{F\mathcal{O}}$ and $H^0 \subset T^0_{F\mathcal{O}}$. By the definition of H, we have $\dim H^0 = \dim T^0_{F\mathcal{O}}$ so that $H^0 = T^0_{F\mathcal{O}}$. We have $FT^0_\mathcal{O} = T^0_{F\mathcal{O}}$ hence $\dim T^0_\mathcal{O} = \dim T^0_{F\mathcal{O}} = \dim H^0$. Therefore the inclusion $H^0 \subset T_\mathcal{O}$ must be an equality. Thus

$$H^0 = T^0_\mathcal{O} = T^0_{F\mathcal{O}}. \tag{b}$$

Our assumption also implies that $F\phi = xu\phi$. Hence, if $f \in \Theta_T$ is a representative for ϕ, we have $Ff = g'fk$ where $g' = xu \in \tilde{\mathcal{G}}$ (see(a)) and $k \in K$. Hence $\theta(f^{-1}g'^{-1}F(f)) = f^{-1}g'^{-1}F(f)$. In terms of

$$n = f \cdot \theta(f)^{-1}, \tag{c}$$

the previous equality can be written

$$(d)\qquad F(n) = g'n\theta(g')^{-1}, (g' \in \tilde{\mathcal{G}}).$$

If $t \in H^0$, then $t \in T_{\mathcal{O}}$ and $t \in T_{F\mathcal{O}}$ (see (b)) hence $f^{-1}tf \in K$ and $F(f)^{-1}tF(f) \in K$, hence $\theta(f^{-1}tf) = f^{-1}tf$ and $\theta(F(f)^{-1}tF(f)) = F(f^{-1})tF(f)$, hence $\theta(t) = n^{-1}tn = F(n)^{-1}tF(n)$. It follows that $F(n)n^{-1}$ commutes with all elements of H^0, hence

$$(e)\qquad F(n)n^{-1} \in \mathcal{G}.$$

We now define $\theta': G \to G$ by $\theta'(g) = n\theta(g)n^{-1}$ and $\theta'': G \to G$ by $\theta''(g) = F(n)\theta(g)F(n)^{-1}$. Then θ', θ'' are involutions since $\theta(n) = n^{-1}$. It is clear that $\theta'T = T$, $\theta''T = T$ and

$$(f)\qquad T^{\theta'} = T_{\mathcal{O}}, T^{\theta''} = T_{F\mathcal{O}}.$$

In particular, θ' and θ'' acts as the identity on H^0 (see (b)), hence they map $\mathcal{G}$ into itself and $\tilde{\mathcal{G}}$ into itself. Let $\tilde{T} = T \cap \tilde{\mathcal{G}}$, a maximal torus of $\tilde{\mathcal{G}}$. Since T and $\tilde{\mathcal{G}}$ are both θ' and θ''-stable, it follows that $\tilde{T}$ is both θ' and θ''-stable. From (f) it follows that $\tilde{T}^{\theta'} \subset T_{\mathcal{O}}, \tilde{T}^{\theta''} \subset T_{F\mathcal{O}}$ hence, taking identity components, and using (b):

$$\tilde{T}^{\theta'} \subset H^0, \tilde{T}^{\theta''} \subset H^0.$$

Since $\tilde{T} \subset \tilde{\mathcal{G}}$, it follows that $\tilde{T}^{\theta'} \subset H^0 \cap \tilde{\mathcal{G}}, \tilde{T}^{\theta''} \subset H^0 \cap \tilde{\mathcal{G}}$. But $H^0 \cap \tilde{\mathcal{G}}$ is finite since H^0 is in the centre of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ is the derived group of $\mathcal{G}$. It follows that $\tilde{T}^{\theta'}, \tilde{T}^{\theta''}$ are finite. Hence the involutions θ', θ'' of $\tilde{T}$ must act on $\tilde{T}$ as $t \to t^{-1}$. We thus have $n\theta(t)n^{-1} = t^{-1}, F(n)\theta(t)F(n)^{-1} = t^{-1}$ for all $t \in T$. This implies $n^{-1}tn = F(n)^{-1}tF(n)$ for all $t \in \tilde{T}$ hence $F(n)n^{-1} \in Z_G(\tilde{T})$. Using now (e), we deduce that $F(n)n^{-1} \in Z_{\mathcal{G}}(\tilde{T})$. Clearly $Z_{\mathcal{G}}(\tilde{T}) = T$. Hence $F(n)n^{-1} \in T$ so that $F(n) = t_1 n$ for some $t_1 \in T$. Introducing this in (d) we have

$$(g)\qquad t_1 = g'n\theta(g')^{-1}n^{-1}, (t_1 \in T, g' \in \tilde{\mathcal{G}}).$$

We have $t_1 = g'\theta'(g')^{-1}$. Since $\theta'\tilde{\mathcal{G}} \subset \tilde{\mathcal{G}}$, we see that $t_1 \in \tilde{\mathcal{G}}$. Thus $t_1 \in \tilde{\mathcal{G}} \cap T$, hence $t_1 \in \tilde{T}$. We can find $t_2 \in \tilde{T}$ such that $t_2^2 = t_1$. Since $\theta'(t) = t^{-1}$ for all $t \in \tilde{T}$, we have $\theta'(t_2) = t_2^{-1}$ hence $t_2 n\theta(t_2)^{-1}n^{-1} = t_2\theta'(t_2)^{-1} = t_2 \cdot t_2 = t_1$. We have $F(n) = t_1 n = t_2 n\theta(t_2)^{-1}$. Using Lang's theorem for T, we write $t_2 = F(t_3)^{-1}t_3$ for some $t_3 \in T$. We have $F(n) = F(t_3)^{-1}t_3 n\theta(t_3)^{-1}\theta(F(t_3))$ hence $t_3 n\theta(t_3)^{-1} \in G^F$. Using (c), we deduce $t_3 f\theta(t_3 f)^{-1} \in G^F$. Let $f' = t_3 f$. We have $f'K \in \mathcal{O}$ and $f'\theta(f')^{-1} \in G^F$, hence $F(f'\theta(f')^{-1}) = f'\theta(f')^{-1}$, hence $f'^{-1}F(f') \in G^\theta = K$. Thus $F(f'K) = f'K$ so that $f'K$ is an F_q-rational point of $\mathcal{O}$. This contradicts our assumption $F\mathcal{O} \neq \mathcal{O}$ and proves the proposition.

9. Proof of Theorems 3.3 and 3.4

9.1. In this section, T, B, U, λ are as in 3.1, 3.2.

9.2. We first prove Theorem 3.3 under the assumption that G is simply connected. In this case we have necessarily $K = G^\theta = (G^\theta)^0$. (See [5,8.2].) Let A be the left hand side of 3.3(a).

Using 4.4(a), we see that $A = \chi(Z)_{\lambda^{-1}}$ (notation of 4.4); here we have used that $K = (G^\theta)^0$ (see the hypothesis of 4.4). Using 4.5(b) and 4.6(b) we deduce that

(a) $$A = \sum_{\mathcal{O}} \chi(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}}$$

where $\mathcal{O}$ runs over the T-orbits on Θ_T.

Using 8.2, we see that the last sum can be restricted to those $\mathcal{O}$ which satisfy $F\mathcal{O} = \mathcal{O}$. (Here we have used that $K = G^\theta$.) For each $\mathcal{O}$ such that $F\mathcal{O} = \mathcal{O}$, we can express $\chi(\mathfrak{S}_{\mathcal{O}})_{\lambda^{-1}}$ by 7.6. (Here we use that $K = G^\theta = (G^\theta)^0$.) The resulting expression for A is exactly the right hand side of 3.3(a).

9.3. Next we prove Theorem 3.4 under the assumption that G is simply connected. We can write the right hand side of 3.3(a) as

$$\sum_{f \in \Theta^F_{T,\lambda}} |(f^{-1}Tf)^F_K||T^F|^{-1}|K^F|\sigma(T)\sigma(Z((f^{-1}Tf)^0_K)).$$

We now sum the identities 3.3(a) over all characters λ of T^F. The sum of the left hand sides is

(a) $$|T^F||K^F|^{-1} \sum_{\substack{g \in K^F \\ unipotent}} \mathrm{Tr}(g, R^1_T)$$

since $\mathrm{Tr}(g, R^\lambda_T) = \mathrm{Tr}(g, R^1_T)$ for g unipotent and $\sum_\lambda \mathrm{Tr}(g, R^\lambda_T) = 0$ for g non-unipotent, (see [2,4.2]).

The sum of the right hand sides is

(b) $$|T^F|^{-1}|K^F|^{-1}\sigma(T) \sum_{f \in \Theta^F_T} A(f)\sigma(Z((f^{-1}Tf)^0_K))|(f^{-1}Tf)^F_K|$$

where

$$\begin{aligned}
A(f) &= \sharp\{\lambda | f \in \Theta^F_{T,\lambda}\} \\
&= \sharp\{\lambda | \lambda(ft'f^{-1}) = \epsilon_{f^{-1}Tf}(t'), \forall t' \in (f^{-1}Tf)^F_K\} \\
&= \text{number of characters of } (f^{-1}Tf)^F \text{whose restriction to} \\
&\qquad (f^{-1}Tf)^F_K \text{ is equal to } \epsilon_{f^{-1}Tf} \\
&= |(f^{-1}Tf)^F| \cdot |(f^{-1}Tf)^F_K|^{-1} \\
&= |T^F| \cdot |(f^{-1}Tf)^F_K|^{-1}.
\end{aligned}$$

Hence the expression (b) is equal to

$$|K^F|^{-1} \sum_{f \in \Theta^F_T} \sigma(T)\sigma(Z((f^{-1}Tf)^0_K)).$$

This must be equal to (a), and Theorem 3.4 is proved (for G simply connected).

9.4. We now prove Theorem 3.4 for general G. Let $\tilde{G} \to G$ be the simply connected covering of the derived group of G. Then $\theta, F, T, U, \ldots$ give rise to analogous objects $\tilde{\theta}, \tilde{F}, \tilde{T}, \tilde{U}, \ldots$ for $\tilde{G}$. We take $\tilde{K} = \tilde{G}^{\tilde{\theta}}$. The two sides of the identity 3.4(a) for G coincide with the respective sides of the identity 3.4(a) for $\tilde{G}$. (We use the fact that the unipotent elements of K are naturally in bijection with those of $\tilde{K}$, that the Green functions of $G^F, \tilde{G}^{\tilde{F}}$ are the same, and the fact that we have a natural bijection $\tilde{G}^F/\tilde{T}^F \xrightarrow{\sim} G^F/T^F$.) Since 3.4(a) is already known to hold for $\tilde{G}$, it follows that it also holds for G.

9.5. We now prove Theorem 3.3 for general G. We shall again make use of the character formula [2,4.2]. We have

$$(a) \qquad |K^F|^{-1} \sum_{g \in K^F} \mathrm{Tr}(g, R^\lambda_T)$$

$$= |K^F|^{-1} \sum_{\substack{s \in K^F \\ semis}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T}} |Z^0_G(s)^F|^{-1} \lambda(x^{-1}sx) \sum_{\substack{u \in K^F \cap Z^0_G(s) \\ unip}} \mathrm{Tr}(u, R^1_{xTx^{-1}, Z^0_G(s)})$$

$$= |K^F|^{-1} \sum_{\substack{s \in K^F \\ semis}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T}} |Z^0_G(s)^F|^{-1} \lambda(x^{-1}sx) |(xTx^{-1})^F| \times$$

$$\times \sum_{\substack{f \in Z^0_G(s)^F \\ \theta(f^{-1}xTx^{-1}f) \\ = f^{-1}xTx^{-1}f}} \sigma(T)\sigma(Z((f^{-1}xTx^{-1}f)^0_K) \cap Z^0_G(s))$$

(The last step used 3.4(a) for $Z_G^0(s), \theta, K\cap Z_G^0(s), \ldots$ instead of $G, \theta, K, \ldots$.) We now make a change of variables $(s,x,f) \to (t,x',f), x' = f^{-1}x, t = x^{-1}sx$, and use the definition of ϵ, (see 2.2(b)); the expression (a) becomes

(b)

$$|K^F|^{-1}|T^F|^{-1} \sum_{\substack{t\in T^F \\ x'\in G^F \\ x'tx'^{-1}\in K \\ \theta(x'Tx'^{-1})=x'Tx'^{-1}}} \lambda(t)\sigma(T)\sigma(Z((x'Tx'^{-1})_K^0))\epsilon_{x'Tx'^{-1}}(x'tx'^{-1})$$

We now make a change of variables $(t,x') \to (\bar{t},x'), \bar{t} = x'tx'^{-1}$, and the expression (b) becomes

$$(c) \quad |K^F|^{-1}|T^F|^{-1} \sum_{\substack{x'\in G^F \\ \theta(x'Tx'^{-1})=x'Tx'^{-1}}} \sigma(T)\sigma(Z((x'Tx'^{-1})_K^0)) \times$$

$$\times \sum_{\bar{t}\in(x'Tx'^{-1})_K^F} \lambda(x'^{-1}\bar{t}x')\epsilon_{x'Tx'^{-1}}(\bar{t}).$$

Since ϵ is a homomorphism (2.3(c)), the last sum is $|(x'Tx'^{-1})_K^F|$ if ${}^{x'}\lambda|_{(x'Tx'^{-1})_K^F} = \epsilon_{x'Tx'^{-1}}$ and is 0 otherwise; hence (c) is equal to the right hand side of 3.3(a). Thus, Theorem 3.3 is proved.

Using the remarks in 7.3 we can rewrite the identity 3.3(a) in the following form.

Corollary 9.6.

$$|K^F|^{-1} \sum_{g\in K^F} \mathrm{Tr}(g, R_T^\lambda) = \sum_{\mathcal{O}} |T^F\backslash\tilde{\mathcal{O}}^F/K^F|\sigma(T)\sigma(\mathcal{G}_{\mathcal{O}})$$

where $\mathcal{O}$ runs over all T-orbits in Θ_T such that $\tilde{\mathcal{O}}^F \neq \emptyset$ and $\lambda|_{T_{\mathcal{O}}^F} = \tilde{\epsilon}_{\mathcal{O}}$ (notations of 7.1).

In the case where $K = (G^\theta)^0$, we can replace $|T^F\backslash\tilde{\mathcal{O}}^F/K^F|$ by $|T_{\mathcal{O}}^F| \cdot |T_{\mathcal{O}}^{0F}|^{-1}$ (see 7.4), which is a power of 2, since $T_{\mathcal{O}}/T_{\mathcal{O}}^0$ is a 2-group.

10. Non singular characters

10.1. Let $\mathcal{J}$ be as in 1.5. It is known that for a θ-stable maximal torus T':

(a) $T' \in \mathcal{J}$ if and only if the involution induced by θ on the set of coroots (or roots) with respect to T' is fixed point free.

10.2. Let T be an F-stable maximal torus of G and let $\lambda: T^F \to \overline{Q}_\ell^*$ be a character. Following [2,5.15], we say that λ is non-singular if the following condition is satisfied:

(a) for any coroot $h: \overline{F}_q^* \to T$ and any (or equivalently, some) integer $n \geq 1$ such that $F^n h(x) = h(x^{q^n})$ for all $x \in F_{q^n}^*$, the restriction of $\lambda \cdot N$ to $h(F_{q^n}^*)(\subset T^{F^n})$ is non-trivial. (N is the norm $T^{F^n} \to T^F$). In this section we shall assume that λ is non-singular. It is known that this implies that $\sigma(T)\sigma(G)R_T^\lambda$ is an actual representation of G^F and that "most" irreducible representations of G^F are of this form. We state the main result of this section.

Theorem 10.3. *Recall that λ is non-singular. The space of K^F-invariant vectors of $\sigma(T)\sigma(G)R_T^\lambda$ is non-zero if and only if (T,λ) is G^F-conjugate to a pair (T',λ') such that $T' \in \mathcal{J}$ and $\lambda'|_{T_K'^F} = \epsilon_{T'}$.*

This result confirms in part a conjecture in [1,6.7].

For the proof, we need two lemmas.

Lemma 10.4. *If $f \in \Theta_{T,\lambda}^F$ then $T' = f^{-1}Tf \in \mathcal{J}$, (see 10.1).*

Proof. Let $\lambda': T'^F \to \overline{Q}_\ell^*$ be given by $\lambda' = {}^{f^{-1}}\lambda$ (see 3.1). Assume that there exists a coroot $h: \overline{F}_q^* \to T$ such that $h = \theta \circ h$. Let $n \geq 1$ be an integer such that $F^n h(x) = h(x^{q^n})$ for all $x \in \overline{F}_q^*$. Now λ' is non-singular, hence there exists $x \in F_{q^n}^*$ such that $\lambda'(N(h(x))) \neq 1$, where $N: T'^{F^n} \to T'^F$ is the norm map. By our assumption on f, we have $\lambda'|_{T_K'^F} = \epsilon_{T'}$. Using 2.3(b) it follows that $\lambda'|(T_K'^0)^F = 1$.

Let $\tilde{N}: T' \to T'$ be the homomorphism defined by $\tilde{N}(t) = tF(t)\cdots F^{n-1}(t)$; then $N = \tilde{N}|T'^{F^n}$. Clearly, $\tilde{N}(T_K'^0) \subset T_K'^0$ (since $T_K'^0$ is F-stable), hence $\tilde{N}((T_K'^0)^{F^n}) \subset T_K'^0$. We then have $N((T_K'^0)^{F^n}) = \tilde{N}((T_K'^0)^{F^n}) \subset T'^F \cap T_K'^0 = (T_K'^0)^F$. Since $\theta \circ h = h$, the image of h is a connected group contained in T'^θ hence in $T_K'^0$, hence $h(F_{q^n}^*) \subset (T_K'^0)^{F^n}$. We then have

$$N(h(F_{q^n}^*)) \subset N((T_K'^0)^{F^n}) \subset (T_K'^0)^F.$$

In particular, $N(h(x)) \in (T_K'^0)^F$. Since λ' is trivial on $(T_K'^0)^F$, we have $\lambda'(N(h(x))) = 1$. This contradiction proves the lemma (by the characterization 10.1(a) of $\mathcal{J}$).

Lemma 10.5. *Let $T' \in \mathcal{J}, FT' = T'$. Then $\sigma(Z(T_K'^0)) = \sigma(G)$.*

Proof. Choose a Borel subgroup B' containing T' such that B' is opposed to $\theta B'$. Let U' be the unipotent radical of B' and let $\overline{U}' = U' \cap Z(T_K'^0)$. Using 2.1(a) we see that it is enough to prove the congruence

$$(a) \qquad \dim(U'/U' \cap FU') \equiv \dim(\overline{U}'/\overline{U}' \cap F\overline{U}') \pmod 2.$$

Let $\mathcal{S}$ be the set of all roots with respect to T'; it has a natural action of θ. Let $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) be the set of roots corresponding to U' (resp. FU'). Since B' is opposed to $\theta B'$ (hence FB' is opposed to $\theta(FB')$), the involution $\tilde{\theta}: \mathcal{S} \to \mathcal{S}, \tilde{\theta}\alpha = \theta(\alpha)^{-1}$ leaves $\mathcal{S}_1$ and $\mathcal{S}_2$ stable. Let $\mathcal{S}'$ be the set of all roots α in $\mathcal{S}$ which are trivial on $T_K'^0$, i.e such that $\theta\alpha = \alpha^{-1}$; $\mathcal{S}'$ is the fixed point set of $\tilde{\theta}: \mathcal{S} \to \mathcal{S}$. Then $\mathcal{S}' \cap \mathcal{S}_1$ (resp. $\mathcal{S}' \cap \mathcal{S}_2$) are the roots corresponding to $\overline{U}'$ (resp. $F\overline{U}'$). Now (a) is equivalent to $|\mathcal{S}_1 \cap \theta\mathcal{S}_2| \equiv |\mathcal{S}' \cap \mathcal{S}_1 \cap \theta\mathcal{S}_2| \pmod 2$ and also to $|(\mathcal{S} - \mathcal{S}') \cap \mathcal{S}_1 \cap \theta\mathcal{S}_2| \equiv 0 \pmod 2$.

The involution $\tilde{\theta}$ leaves $(\mathcal{S} - \mathcal{S}') \cap \mathcal{S}_1 \cap \theta\mathcal{S}_2$ stable and is fixed point free on it; hence this set has an even number of elements. The lemma is proved.

10.6. We now prove Theorem 10.3. Using 10.4 and 10.5 we can rewrite 3.3(a) in our case as follows:

$$(a) \quad \dim((\sigma(T)\sigma(G)R_T^\lambda)^{K^F}) = \text{number of double cosets } T^F \backslash \Theta_{T,\lambda}^F / K^F.$$

This, together with 10.4, clearly imply 10.3.

10.7. Let $T' \in \mathcal{J}, FT' = T'$. Applying 10.5 to G and to $Z_G^0(t), (t \in T_K'^F)$ we see that

$$(a) \qquad \epsilon_{T'}(t) = \sigma(G)\sigma(Z_G^0(t)) \text{ for all } t \in T_K'^F.$$

11. An example

11.1. In this section we shall assume that G is simply connected, F_q-split and that there exists a maximal torus T of G such that $\theta t = t^{-1}$ for all $t \in T$. (The maximal tori with this property will then form the set $\mathcal{J}$ of 1.5.)

We fix $T \in \mathcal{J}$ with $FT = T$.

Theorem 11.2. *Let $\lambda: T^F \to \overline{Q}_\ell^*$ be a non-singular character. The dimension of $(\sigma(T)\sigma(G)R_T^\lambda)^{K^F}$ is 0 if $\lambda|T_K^F \not\equiv 1$ and is $|T_K^F|$ if $\lambda|T_K^F \equiv 1$.*

(Note that $T_K^F = \{t \in T^F | t^2 = 1\}$.)

The proof is based on the following result.

Lemma 11.3. $\epsilon_{T'} \equiv 1$ *for any* $T' \in \mathcal{J}^F$.

Proof. The definition of $\epsilon_{T'}$ shows that it is enough to prove that for any $g \in G^F$ such that $g = g^{-1}$ we have $\sigma(Z_G^0(g)) = \sigma(G)$. Let T_0 be an F_q-split maximal torus of G. We can find $t_0 \in T_0$ in the same G-conjugacy class as g. Not $t_0^2 = 1$ (since $g^2 = 1$) hence $t_0^q = t_0$ (since q is odd) hence $F(t_0) = t_0$ (since T_0 is F_q-split and $F(t_0) = t_0^q$). Thus $g, t_0 \in G^F$ are semisimple elements, conjugate under G. Since G is simply connected, $Z_G(g)$ is connected, hence g, t_0 are conjugate under G^F. Hence we can assume that $g \in T_0$. But then $T_0 \subset Z_G^0(g)$. Hence the F_q-rank of $Z_G^0(g)$ is the same as the F_q-rank of G. The Lemma is proved.

11.4. **Proof of Theorem 11.2.** Using 10.6(a), we are reduced to evaluating the number of double cosets $T^F \backslash \Theta_{T,\lambda}^F / K^F$. From 10.4 and 11.3 we see that $\Theta_{T,\lambda}^F = \{f \in \Theta_T^F | f^{-1}Tf \in \mathcal{J}, {}^{f^{-1}}\lambda | (f^{-1}Tf)_K^F \equiv 1\}$.

But for f in the last set,

$$\begin{aligned}(f^{-1}Tf)_K^F &= \{t' \in (f^{-1}Tf)^F | t'^2 = 1\} \\ &= f^{-1}\{t \in T^F | t^2 = 1\}f \\ &= f^{-1}T_K^F f\end{aligned}$$

hence the condition ${}^{f^{-1}}\lambda | (f^{-1}Tf)_K^F \equiv 1$ is equivalent to the condition $\lambda | T_K^F \equiv 1$. Thus $\Theta_{T,\lambda}^F$ is empty if $\lambda | T_K^F \not\equiv 1$. Assume now that $\lambda | T_K^F \equiv 1$. Then $\Theta_{T,\lambda}^F = R^F$ where $R = \{f \in G | f^{-1}Tf \in \mathcal{J}\}$. It remains to show that

(*a*) $\qquad T^F \backslash R^F / K^F$ has $|T_K^F|$ elements.

Since K acts transitively on $\mathcal{J}$, we have $R = N(T)K$ where $N(T)$ is the normalizer of T in G. Let B be a Borel subgroup containing T and let $n \in N(T)$. Applying 1.5(a) to $(T \subset B) \in \mathcal{P}, (T \subset nBn^{-1}) \in \mathcal{P}$, it follows that $n \in TK$. Thus $N(T) \subset TK$, hence $R = T \cdot K$. Now $T \times K$ acts transitively on R by $(t, k): r \to trk^{-1}$ and the isotropy group of 1 is T_K. Since $T \times K$ is connected, it follows that $T^F \backslash R^F / K^F$ is in bijection with the cokernel of $1 - F: T_K \to T_K$, hence has the same number of elements as T_K^F. The completes the proof of (a) and of the theorem.

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Le théorème de positivité de l'irrégularité pour les $\mathcal{D}_X$-modules

ZOGHMAN MEBKHOUT

dédié à Alexandre Grothendieck à l'occasion de son soixantième anniversaire.

Sommaire

0. Introduction

Soient X une variété algébrique complexe non singulière, Z une sous variété de X et $\mathcal{M}$, un coefficient de la catégorie $D^b_h(\mathcal{D}_X)$. Dans ce travail nous définissons le complexe d'irrégularité $\boldsymbol{IR}_Z(\mathcal{M})$ de $\mathcal{M}$ le long de Z qui est alors un complexe de faisceaux d'espaces vectoriels complexes algébriquement constructible et nous montrons que si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome et Z est une hypersurface de X le complexe $\boldsymbol{IR}_Z(\mathcal{M})$ est un *faisceau pervers* sur Z. Son cycle caractéristique qui est défini de façon purement algébrique *a priori* comme différence de deux cycles est alors un cycle lagrangien *positif* du fibré cotangent T^*X. Si X est une surface de Riemann la dimension de l'espace vectoriel complexe $\boldsymbol{IR}_Z(\mathcal{M})$ pour tout point Z de X est égale au nombre classique de Fuchs attaché à la singularité Z de $\mathcal{M}$ en vertu du théorème de Malgrange [M_1], [M_2]. Ainsi le faisceau $\mathbb{R}_Z(\mathcal{M})$, *objet d'une catégorie dérivée*, apparaît comme la généralisation vraiment naturelle du nombre de Fuchs en dimension supérieure et rend compte de la ramification de $\mathcal{M}$ en tout point de Z simultanément.

Le théorème de positivité est la base de toutes les propriétés de la ramification sur le corps des nombres complexes et en particulier de la ramification modérée. La chose la plus remarquable sans doute est qu'il se substitue avantageusement au théorème d'Hironaka [H] sur la résolution des singularités dans la démonstration du théorème de comparaison de Grothendieck [G_1] pour la cohomologie de de Rham d'une variété algébrique complexe non singulière et plus généralement dans la démonstration du théorème des coefficients de de Rham-Grothendieck [G_2] sur le corps des nombres complexes. D'ailleurs le faisceau $\mathbb{R}_Z(\mathcal{O}_X)$, *a priori* non nul comme nous le définissons ici, apparaît tout à fait explicitement dans la démonstration de Grothendieck [G_1] comme l'obstruction au théorème de comparaison entre cohomologies de de Rham d'une variété algébrique complexe. Cette idée de localisation d'un problème en apparence de nature globale s'est révélée comme le point crucial dans l'étude de la cohomologie des variétés algébriques et a été reprise à maints endroits. Cependant ceci va aussi à l'encontre de ce que pensait Grothendieck lui-même [G_3] à savoir que la résolution des singularités est vraiment au fond du problème dans l'étude de la cohomologie des variétés algébriques. Ceci amène à penser que le théorème de résolution des singularités n'est pas au fond du problème pour la cohomologie de de Rham aussi bien en caractéristique pure $p > 0$ qu'en inégale caractéristique et invite à chercher des méthodes directes de démonstration.

La théorie des $\mathcal{D}_X$-modules et ses succès sont avant tout une application de la théorie des catégories triangulées et des catégories dérivées. L'harmonie interne et la portée de cette dernière théorie étaient quelque peu masquées par la difficulté du théorème d'Hironaka. En effet le point crucial dans les démonstrations, une fois tout dévissage effectué, restait le théorème d'Hironaka. Ce faisant on perdait invariablement beaucoup d'informations dans l'application systématique de ce théorème qui a fini par dénaturer certaines questions concernant les singularités. Le théorème de positivité qui est une autre application de la théorie des catégories triangulées et des catégories dérivées montre qu'en ce qui concerne la théorie de de Rham-Grothendieck en caractéristique nulle le théorème de résolution des singularités n'est pas indispensable. A cette occasion particulière le lecteur nous permettra de rendre hommage à cette théorie et à son créateur.

La motivation du théorème de positivité était le théorème de semicontinuité de l'irrégularité d'une famille d'équations différentielles [Me_4]

analogue au théorème de semi-continuité de Deligne du conducteur de Swan [L_1].

Voici le contenu de ce travail. Dans le §1 nous rappelons quelques résultats généraux de la théorie des $\mathcal{D}_X$-modules utilisés dans la suite. Dans le §2 nous montrons le théorème de positivité et ses conséquences immédiates. Dans le §3 nous définissons le cycle d'irrégularité et nous montrons une formule de type Riemann-Roch exprimant la caractéristique d'Euler-Poincaré de la cohomologie de de Rham d'un fibré à connexion intégrable sur une variété ouverte non singulière sur un corps de caractéristique nulle à l'aide de sa ramification à l'infini. Dans le §4 nous précisons le théorème de semi continuité de l'irrégularité [Me_4] à partir du théorème de positivité. Dans le §5 nous indiquons comment le théorème de positivité remplace avantageusement le théorème d'Hironaka dans la démonstration du théorème des coefficients de de Rham-Grothendieck sur le corps des nombres complexes. Enfin dans le dernier § nous définissons une filtration du faisceau d'irrégularité par des sous-faisceaux indexés par des nombres réels ≥ 1 mais dont les sauts locaux sont des nombres rationnels : les pentes critiques locales du faisceau $\mathbb{R}_Z(\mathcal{M})$. La situation est toute pareille à celle du théorème de Hasse-Arf pour la filtration (numérotation supérieure) du groupe de Galois d'une extention galoisienne finie d'un corps local [Se_1]. Les résultats de ce travail ont été annoncés dans une Note aux Comptes rendus de l'Académie des Sciences : C. R. Acad. Sci. Paris, **303**, série I, 1986, 803-806.

Terminologie: Le lecteur trouvera peut-être paradoxal de qualifier le faisceau $\mathbb{R}_Z(\mathcal{M})$ (qui n'a rien de pathologique) "d'irrégulier". Il serait plus correct de l'appeler faisceau de ramification (qui n'a rien de sauvage) comme dans la théorie des corps locaux. Mais sur le corps des nombres complexes la terminologie "point singulier irrégulier" est profondément enracinée dans la théorie des équations différentielles depuis le siècle dernier qu'il paraît difficile de changer de terminologie. Nous utilisons dans cet article la terminologie de la théorie de Grothendieck, par exemple on appelera coefficient constructible un objet de la catégorie $D_c^b(C_X)$ pour mettre en valeur son caractère intrinsèque.

Une partie de ce travail a été réalisé pendant un séjour en 1986 à l'Institut for Advanced Study de Princeton. Nous remercions les membres de cet institut de leur hospitalité. Nous remercions B. Malgrange des discussions que nous avons eues à cette occasion. Il nous a fait savoir qu'il avait conjecturé de son côté le théorème de positivité. Nous remercions Y. Laurent des nombreuses discussions à propos de ses travaux. Nous remercions enfin J. J. Sansuc d'avoir relu le manuscrit.

1. Rappels de quelques résultats

Dans ce paragraphe nous allons rappeler quelques résultats généraux de la théorie des $\mathcal{D}_X$-modules utilisés dans cet article. Nous renvoyons le lecteur à [Me3], par exemple, pour leurs démonstrations et leurs références.

1.1. Soit $(X, \mathcal{O}_X)$ une variété algébrique non singulière sur un corps de caractéristique nulle ou une variété analytique complexe. On note $\mathcal{D}_X$ le faisceau des opérateurs différentiels d'ordre fini à coefficients dans $\mathcal{O}_X$ défini et noté $\mathcal{D}iff_{X/k}(\mathcal{O}_X)$ dans [EGA, IV, §16] pour tout espace annelé $(X, \mathcal{O}_X)$ en k-algèbres sur un corps k. Le faisceau $\mathcal{D}_X$ est un faisceau d'anneaux non commutatifs cohérent admettant une filtration $\mathcal{D}_l$, $l \in \boldsymbol{N}$ par les faisceaux des opérateurs différentiels d'ordre $\leq l$. Le spectre analytique du faisceau gradué de $\mathcal{O}_X$-algèbres commutatives $\operatorname{gr}(\mathcal{D}_X) := \otimes_l \mathcal{D}_{l+1}/\mathcal{D}_l$ est égal au fibré cotangent $\pi : T^*X \to X$.

Étant donné un $\mathcal{D}_X$-module (à gauche) $\mathcal{M}$, une filtration croissante $\mathcal{M}_k$ par des $\mathcal{O}_X$-modules cohérents telle que $\mathcal{M} = \cup_k \mathcal{M}_k$ est dite *bonne* si $\mathcal{D}_l \mathcal{M}_k \subset \mathcal{M}_{l+k}$ pour tout $l \in \boldsymbol{N}$ et tout $k \in \boldsymbol{N}$ avec égalité dès que k est assez grand. Un $\mathcal{D}_X$-module $\mathcal{M}$ est cohérent *si et seulement* s'il admet *localement* une bonne filtration. Le faisceau gradué $\operatorname{gr}(\mathcal{M}) := \otimes_l \mathcal{M}_{l+1}/\mathcal{M}_l$ associé à une bonne filtration de $\mathcal{M}$ est un $\operatorname{gr}(\mathcal{D}_X)$-module cohérent et le cycle du fibré cotangent T^*X associé au faisceau $\operatorname{gr}(\mathcal{M})$ ne dépend pas de la bonne filtration choisie. Nous noterons $\operatorname{CCh}(\mathcal{M})$ ce cycle que l'on appelle cycle caractéristique de $\mathcal{M}$. Si donc $\mathcal{M}$ est un $\mathcal{D}_X$-module cohérent, $\operatorname{CCh}(\mathcal{M})$ est un cycle homogène positif globalement défini. Son support noté $\operatorname{Ch}(\mathcal{M})$ est par définition la variété caractéristique de $\mathcal{M}$. Pour tout $\mathcal{D}_X$-module cohérent $\mathcal{M}$ non nul, on a l'inégalité de Bernstein $\dim(\operatorname{Ch}(\mathcal{M})) \geq \dim(X)$. En fait l'inégalité de Bernstein est aussi conséquence du théorème de l'involutivité des caractéristiques, cf. [G], mais elle est bien plus élémentaire, cf. ([M4] ou [Me3]). On déduit de l'inégalité de Bernstein que la dimension homologique des fibres du faisceau $\mathcal{D}_X$ est égale à la dimension de X.

Définition 1.1.1. On dit qu'un $\mathcal{D}_X$-module $\mathcal{M}$ est holonome s'il est cohérent et tel que $\dim(\operatorname{Ch}(\mathcal{M})) = \dim(X)$ ou s'il est nul.

La terminologie holonome est impropre, le lecteur préférera peut-être dire module de dimension minimale. On note $\operatorname{Mh}(\mathcal{D}_X)$ la catégorie des $\mathcal{D}_X$-modules holonomes. C'est une sous-catégorie pleine de la catégorie des $\mathcal{D}_X$-modules qui est abélienne et stable par extension. Si

$$(1.1.1) \qquad 0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$$

est une suite exacte de $\mathcal{D}_X$-modules holonomes on a l'égalité entre cycles caractéristiques

$$\text{(1.1.2)} \qquad \mathrm{CCh}(\mathcal{M}) = \mathrm{CCh}(\mathcal{M}_1) + \mathrm{CCh}(\mathcal{M}_2).$$

On note $D^b_h(\mathcal{D}_X)$ la sous-catégorie de $D(\mathcal{D}_X)$, catégorie dérivée de la catégorie des $\mathcal{D}_X$-modules à gauche, des complexes bornés à cohomologie holonome. La catégorie $D^b_h(\mathcal{D}_X)$ est une sous-catégorie pleine et triangulée de $D(\mathcal{D}_X)$.

1.2. Notons ω_X le faisceau sur X des formes différentielles de degré maximum. C'est de façon naturelle un $\mathcal{D}_X$-module à droite. Si $\mathcal{M}$ est un $\mathcal{D}_X$-module à gauche holonome on a $Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ si $i \neq \dim(X)$ et $Ext^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ est un $\mathcal{D}_X$-module à droite holonome où $n := \dim(X)$. Notons $\mathcal{M}^\nu$ le $\mathcal{D}_X$-module à droite $Ext^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ et $\mathcal{M}^*$ le $\mathcal{D}_X$-module à gauche $hom_{\mathcal{O}_X}(\omega_X, \mathcal{M}^\nu)$. Le foncteur de dualité $\mathcal{M} \to \mathcal{M}^*$ est une anti-involution de la catégorie des $\mathcal{D}_X$-modules à gauche holonomes dans elle-même.

1.3. Dans la situation de 1.1, si Z est une sous variété de X définie par un Idéal $\mathcal{I}_Z$, on définit deux foncteurs exacts à gauche de cohomologie locale algébrique de la catégorie des $\mathcal{D}_X$-modules à gauche dans elle-même en posant

$$\text{(1.3.1)} \qquad \mathcal{M}(*Z) := \varinjlim_k hom_{\mathcal{O}_X}(\mathcal{I}^k_Z, \mathcal{M})$$

$$\text{(1.3.2)} \qquad \mathrm{alg}\Gamma_Z(\mathcal{M}) := \varinjlim_k hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{M})$$

pour un $\mathcal{D}_X$-module $\mathcal{M}$ à gauche. Ces deux foncteurs se dérivent à droite pour donner naissance à des foncteurs da la catégorie $D^b(\mathcal{D}_X)$ dans elle-même $\mathcal{M} \to \boldsymbol{R}\mathcal{M}(*Z)$ et $\mathcal{M} \to \boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})$. De plus on a un triangle distingué de la catégorie $D^b(\mathcal{D}_X)$

$$\text{(1.3.3)} \qquad \boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}) \to \mathcal{M} \to \boldsymbol{R}\mathcal{M}(*Z)$$

muni d'un morphisme naturel dans le triangle de cohomologie locale topologique

$$\text{(1.3.4)} \qquad \boldsymbol{R}\Gamma_Z(\mathcal{M}) \to \mathcal{M} \to \boldsymbol{R}j_*j^{-1}\mathcal{M}.$$

Dans le cas algébrique, si $\mathcal{M}$ est à cohomologie $\mathcal{O}_X$ quasi-cohérente, le morphisme précédent entre (1.3.3) et (1.3.4) est un isomorphisme.

Théorème 1.3.5. *Si $\mathcal{M}$ est un complexe de la catégorie $D^b_h(\mathcal{D}_X)$ le triangle* (1.3.3) *de cohomologie locale algébrique est un triangle distingué de la catégorie $D^b_h(\mathcal{D}_X)$.*

Le théorème 1.3.5 résulte de l'équation fonctionnelle de Bernstein-Sato, cf. ([K3], [NM1]). On déduit du théorème 1.3.5 que si $f : X' \to X$ est morphisme de variétés non singulières l'image inverse

$$(1.3.6) \qquad \boldsymbol{L}f^*(\mathcal{M}) := \mathcal{O}_{X'} \overset{\boldsymbol{L}}{\underset{f^{-1}(\mathcal{O}_X)}{\otimes}} f^{-1}(\mathcal{M})$$

d'un complexe $\mathcal{M}$ de la catégorie $D^b_h(\mathcal{D}_X)$ est un complexe de la catégorie $D^b_h(\mathcal{D}_{X'})$. En particulier le produit tensoriel interne

$$(1.3.7) \qquad \mathcal{M}_1 \overset{\boldsymbol{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}_2$$

de deux coefficients $\mathcal{M}_1$ et $\mathcal{M}_2$ de la catégorie $D^b_h(\mathcal{D}_X)$ est encore un coefficient de la catégorie $D^b_h(\mathcal{D}_X)$.

1.3.8. Si Z est une sous-variété de X et si on pose $Z' := f^{-1}(Z)$ et $\mathcal{M}' := \boldsymbol{L}f^*(\mathcal{M})$, on a un morphisme de triangles de la catégorie $D^b_h(\mathcal{D}_{X'})$

$$\begin{array}{ccccc}
\boldsymbol{L}f^*(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})) & \longrightarrow & \boldsymbol{L}f^*(\mathcal{M}) & \longrightarrow & \boldsymbol{L}f^*(\boldsymbol{R}\mathcal{M}(*Z)) \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{R}\mathrm{alg}\Gamma_{Z'}(\mathcal{M}') & \longrightarrow & \mathcal{M}' & \longrightarrow & \boldsymbol{R}\mathcal{M}'(*Z')
\end{array}$$

qui est un isomorphisme.

1.4. Supposons que X est une variété complexe munie de la topologie transcendante. On note $D^b_c(\boldsymbol{C}_X)$ la catégorie triangulée des coefficients complexes algébriquement constructibles dans le cas algébrique et analytiquement constructibles dans le cas analytique. Le faisceau $\mathcal{O}_X$ est de façon naturelle un $\mathcal{D}_X$-module à gauche. On peut donc considérer les deux foncteurs exacts, l'un covariant, l'autre contravariant de la catégorie $D^b(\mathcal{D}_X)$ dans la catégorie $D^b(\boldsymbol{C}_X)$

$$(1.4.1) \qquad \boldsymbol{DR}(M) := \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$$

$$(1.4.2) \qquad \boldsymbol{S}(\mathcal{M}) := \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Le $\mathcal{D}_X$-module à gauche $\mathcal{O}_X$ est auto-dual et si $\mathcal{M}$ est un complexe de la catégorie $D^b(\mathcal{D}_X)$ dont la cohomologie est $\mathcal{D}_X$-cohérente, on a des isomorphismes canoniques

$$DR(\mathcal{M}) \cong S(\mathcal{M}^*) \tag{1.4.3}$$

$$S(\mathcal{M}) \cong DR(\mathcal{M}^*). \tag{1.4.4}$$

On a alors le théorème de constructibilité [K2] :

Théorème 1.4.5. *Si $\mathcal{M}$ est un complexe de $D^b_h(\mathcal{D}_X)$, les complexes $DR(\mathcal{M})$ et $S(\mathcal{M})$ sont constructibles.*

Les foncteurs DR et S envoient donc la catégorie $D^b_h(\mathcal{D}_X)$ dans la catégorie $D^b_c(C_X)$. Si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome, les faisceaux de cohomologie des complexes $DR(\mathcal{M})$ et $S(\mathcal{M})$ sont concentrés entre les degrés 0 et $\dim(X)$. De plus on a la proposition suivante [K2] :

Proposition 1.4.6. *Si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome, la dimension du support des faisceaux $Ext^i_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{M})$ et $Ext^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$ est inférieure ou égale à* $\dim(X) - i$ *pour tout i compris entre* 0 *et* $\dim(X)$.

Le lecteur trouvera dans [NM2] des démonstrations géométriques élémentaires du théorème de constructiblité 1.4.5 et de la proposition 1.4.6, indépendantes de tout théorème d'analyse.

1.5. On suppose toujours que X est une variété complexe. On a alors le théorème de dualité locale [Me1] :

Théorème 1.5.1. *Pour tout complexe $\mathcal{M}$ de la catégorie $D^b(\mathcal{D}_X)$ on a un morphisme* canonique de dualité

$$DR(\mathcal{M}) \to R hom_{C_X}(S(\mathcal{M}), C_X)$$

qui est un isomorphisme *si $\mathcal{M}$ est un complexe de la catégorie $D^b_h(\mathcal{D}_X)$.*

Autrement dit le foncteur $\mathcal{M} \to \mathcal{M}^*$ est compatible à la dualité locale des complexes constructibles. Ceci n'est pas vrai si on enlève l'hypothèse d'holonomie.

Définition 1.5.2. On dit qu'un coefficient constructible $\mathcal{F}$ a la propriété de *support* si $h^i(\mathcal{F})$ est nul pour $i \notin [0, \dim(X)]$ et si la dimension du support du faisceau $h^i(\mathcal{F})$ est inférieure ou égale à $\dim(X) - i$ pour tout $i \in [0, \dim(X)]$.

On note $h^i(\mathcal{F})$ le i-ème faisceau d'un complexe $\mathcal{F}$. Si donc $\mathcal{M}$ est $\mathcal{D}_X$-module holonome les complexes $DR(\mathcal{M})$ et $S(\mathcal{M})$ ont la propriété de support en vertu de 1.4.6.

Définition 1.5.3. On dit qu'un coefficient constructible $\mathcal{F}$ a la propriété de *co-support* si le complexe dual $\mathcal{F}^\nu := R hom_{C_X}(\mathcal{F}, C_X)$ a la propriété de support.

Si donc $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome les complexes $\boldsymbol{DR}(\mathcal{M})$ et $\boldsymbol{S}(\mathcal{M})$ ont la propriété de co-support en vertu du théorème de dualité locale 1.5.1. On pose alors cf. [BBD] où cette terminologie est proposée :

Définition 1.5.4. On dit qu'un coefficient constructible $\mathcal{F}$ est un faisceau *pervers* s'il a les conditions de support et de co-support.

Si donc $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome les complexes $\boldsymbol{DR}(\mathcal{M})$ et $\boldsymbol{S}(\mathcal{M})$ sont des faisceaux pervers. On note $\mathrm{Perv}(\boldsymbol{C}_X)$ la catégorie des faisceaux pervers. C'est une sous catégorie pleine de la catégorie $D^b_c(\boldsymbol{C}_X)$. Le couple $\mathrm{Perv}(\boldsymbol{C}_X)$, $D^b_c(\boldsymbol{C}_X)$ a les propriétés du couple $\mathrm{Mh}(\mathcal{D}_X)$, $D^b_h(\mathcal{D}_X)$. Par exemple on a alors le résultat suivant, cf. [BBD], :

Théorème 1.5.5. *La catégorie* $\mathrm{Perv}(\boldsymbol{C}_X)$ *est une sous catégorie* abélienne *de* $D^b_c(\boldsymbol{C}_X)$ *et un triangle distingué de* $D^b_c(\boldsymbol{C}_X)$ *donne naissance, à côté de la suite longue* de cohomologie ordinaire, *à une suite longue de* cohomologie perverse. *De plus la catégorie est un* champ *c'est-à-dire que ses objets et ses morphismes sont de nature locale, tout comme les objets et les morphismes de la catégorie* $\mathrm{Mh}(\mathcal{D}_X)$.

Notation 1.5.6. On note ${}^p h^i(\mathcal{F})$ le i-ème faisceau de cohomologie perverse d'un coefficient constructible $\mathcal{F}$.

Si $\mathcal{F}$ est un coefficient constructible sur une sous-variété Z de X on dit que c'est un faisceau pervers sur Z si le coefficient $\mathcal{F}[-\mathrm{codim}_X Z]$ vu comme complexe de la catégorie $D^b_c(\boldsymbol{C}_X)$ appartient à $\mathrm{Perv}(\boldsymbol{C}_X)$. On note $\mathrm{Perv}(\boldsymbol{C}_Z)$ la catégorie des faisceaux pervers sur Z. C'est une sous catégorie-abélienne pleine de la catégorie $D^b_c(\boldsymbol{C}_Z)$.

2. Le théorème de positivité de l'irrégularité

On utilise les notations du paragraphe 1.

2.1. *Enoncés.* Soit X une variété complexe (algébrique ou analytique) non singulière et Z une sous variété fermée de X de codimension p. Notons

$$j : U := X - Z \subset X \supset Z : i$$

les inclusions (transcendantes) canoniques. Soit $\mathcal{M}$ un complexe de la catégorie $D^b_h(\mathcal{D}_X)$. On pose

$$\boldsymbol{IR}_Z(\mathcal{M}) := \boldsymbol{R}\Gamma_Z(\boldsymbol{DR}(\boldsymbol{R}\mathcal{M}(*Z)))[1] \tag{2.1.1}$$

$$\boldsymbol{IR}^{\vee}_Z(\mathcal{M}) := i^{-1}\boldsymbol{S}(\boldsymbol{R}\mathcal{M}(*Z))[p]. \tag{2.1.2}$$

Les morphismes canoniques de complexes de $\mathcal{D}_X$-modules

$$\boldsymbol{R}\mathcal{M}(*Z) \to \boldsymbol{R}j_*j^{-1}\mathcal{M}$$

$$\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}) \to \mathcal{M}$$

donnent naissance aux morphismes de complexes constructibles

$$\boldsymbol{DR}(\boldsymbol{R}\mathcal{M}(*Z)) \to \boldsymbol{R}j_*j^{-1}\boldsymbol{DR}(\mathcal{M}) \tag{2.1.3}$$

$$i^{-1}\boldsymbol{S}(\mathcal{M}) \to \boldsymbol{S}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})) \tag{2.1.4}$$

de sorte que le complexe $\boldsymbol{IR}_Z(\mathcal{M})$, resp. $\boldsymbol{IR}_Z^{\vee}(\mathcal{M})$, est isomorphe au cône du morphisme (2.1.3) décalé de p vers la gauche, resp. du morphisme (2.1.4). On a ainsi défini de façon naturelle un foncteur covariant exact $\boldsymbol{IR}_Z$ et un foncteur contravariant exact $\boldsymbol{IR}_Z^{\vee}$ de catégories triangulées de la catégorie $D_h^b(\mathcal{D}_X)$ dans la catégorie $D_c^b(\boldsymbol{C}_Z)$.

Proposition 2.1.5. *Si $\mathcal{M}$ est un complexe de la catégorie $D_h^b(\mathcal{D}_X)$ les coefficients constructibles $\boldsymbol{IR}_Z(\mathcal{M})[-1]$ et $\boldsymbol{IR}_Z^{\vee}(\mathcal{M})[-p]$ de la catégorie $D_c^b(\boldsymbol{C}_X)$ sont en* dualité *:*

$$\boldsymbol{IR}_Z^{\vee}(\mathcal{M})[-p] \cong \boldsymbol{R}hom_{\boldsymbol{C}_X}(\boldsymbol{IR}_Z(\mathcal{M})[-1], \boldsymbol{C}_X)\,.$$

Preuve de 2.1.5. Cela résulte du théorème de dualité locale 1.5.1 parce que

$$\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})) \cong (\boldsymbol{S}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})))^{\vee}$$
$$\boldsymbol{R}\Gamma_Z(\boldsymbol{DR}(\mathcal{M})) \cong (i^{-1}\boldsymbol{S}(\mathcal{M}))^{\vee}\,.$$

Théorème 2.1.6. *Si Z est une hypersurface, c'est-à-dire défini localement par une équation, et si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome, les coefficients $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\vee}(\mathcal{M})$ sont des* faisceaux pervers *sur Z.*

Le preuve du théorème 2.1.6 sera donnée en 2.2. En vertu du théorème 2.1.6 les foncteurs $\boldsymbol{IR}_Z$ et $\boldsymbol{IR}_Z^{\vee}$ envoient la catégorie abélienne $\mathrm{Mh}(\mathcal{D}_X)$ dans la catégorie abélienne $\mathrm{Perv}(\boldsymbol{C}_Z)$ si Z est une hypersurface.

Corollaire 2.1.7. *Si Z est une hypersurface les foncteurs $\boldsymbol{IR}_Z$ et $\boldsymbol{IR}_Z^{\vee}$ sont des foncteurs* exacts *de catégories abéliennes entre* $\mathrm{Mh}(\mathcal{D}_X)$ *et* $\mathrm{Perv}(\boldsymbol{C}_Z)$.

Preuve de 2.1.7. En effet si $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ est une suite exacte de la catégorie $\mathrm{Mh}(\mathcal{D}_X)$ on a par construction les deux triangles distingués de la catégorie $D_c^b(\boldsymbol{C}_Z)$

$$0 \to \boldsymbol{IR}_Z(\mathcal{M}_1) \to \boldsymbol{IR}_Z(\mathcal{M}) \to \boldsymbol{IR}_Z(\mathcal{M}_2) \to 0$$

$$0 \to \mathbf{\mathit{IR}}^{\nu}_Z(\mathcal{M}_2) \to \mathbf{\mathit{IR}}^{\nu}_Z(\mathcal{M}) \to \mathbf{\mathit{IR}}^{\nu}_Z(\mathcal{M}_1) \to 0.$$

Prenons la suite longue de cohomologie perverse dans les triangles précédents; on trouve en vertu du théorème 2.1.6 qu'on a, en fait, affaire à des suites exactes courtes de la catégorie abélienne Perv($\mathbf{C}_Z$). En particulier le faisceau $\mathbf{\mathit{IR}}_Z(\mathcal{M})$ est nul si et seulement si les faisceaux $\mathbf{\mathit{IR}}_Z(\mathcal{M}_1)$ et $\mathbf{\mathit{IR}}_Z(\mathcal{M}_2)$ sont nuls. Le lecteur pourra mesurer le chemin parcouru s'il se rappelle qu'un résultat bien plus faible que ce dernier [Me$_2$] reposait de façon essentielle, avant qu'on dispose du théorème de positivité, sur le théorème d'Hironaka.

Corollaire 2.1.8. *Si $\mathcal{M}$ est un complexe de la catégorie $D^b_h(\mathcal{D}_X)$ et si Z est une hypersurface, on a des isomorphismes canoniques de faisceaux pervers, pour tout i :*

$$^p h^i(\mathbf{\mathit{IR}}_Z(\mathcal{M})) \cong \mathbf{\mathit{IR}}_Z\big(h^i(\mathcal{M})\big)$$
$$^p h^i(\mathbf{\mathit{IR}}^{\nu}_Z(\mathcal{M})) \cong \mathbf{\mathit{IR}}^{\nu}_Z\big(h^i(\mathcal{M})\big)\,.$$

Preuve de 2.1.8. On raisonne par récurrence sur l'amplitude de $\mathcal{M}$. Si $\mathcal{M}$ est concentré cohomologiquement en un seul degré c'est le théorème 2.1.6. Dans le cas général, soit $h^k(\mathcal{M})$ le dernier faisceau de cohomologie de $\mathcal{M}$. Le complexe $\mathcal{M}$ s'envoie sur son dernier faisceau de cohomologie et on a un triangle distingué de la catégorie $D^b_h(\mathcal{D}_X)$

$$\mathcal{M}' \to \mathcal{M} \to h^k(\mathcal{M})$$

où l'amplitude de $\mathcal{M}'$ est égale à celle de $\mathcal{M}$ diminuée d'une unité. D'où le triangle distingué de $D^b_c(\mathbf{C}_Z)$

$$\mathbf{\mathit{IR}}_Z(\mathcal{M}') \to \mathbf{\mathit{IR}}_Z(\mathcal{M}) \to \mathbf{\mathit{IR}}_Z\big(h^k(\mathcal{M})\big)\,.$$

Prenons la suite longue de cohomologie dans le premier triangle et appliquons le foncteur exact $\mathbf{\mathit{IR}}_Z$ puis la suite longue de cohomologie perverse dans le second triangle. En comparant les deux suites longues ainsi obtenues on obtient le corollaire 2.1.8 à partir de l'hypothèse de récurrence. Le deuxième isomorphisme se démontre de la même façon. En particulier pour un complexe $\mathcal{M}$ de la catégorie $D^b_h(\mathcal{D}_X)$ le complexe $\mathbf{\mathit{IR}}_Z(\mathcal{M})$ est nul si et seulement si les faisceaux $\mathbf{\mathit{IR}}_Z(h^i(\mathcal{M}))$ sont nuls pour tout i.

Soient $\mathcal{M}$ un complexe de la catégorie $D^b_h(\mathcal{D}_X)$ et

$$\mathcal{M}^* := hom_{\mathcal{O}_X}(\omega_X, \mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))\,[\dim(X)]$$

son complexe dual. Pour toute sous-variété Z les coefficients constructibles $\mathbf{\mathit{IR}}_Z(\mathcal{M})$ et $\mathbf{\mathit{IR}}_Z(\mathcal{M}^*)$ ne sont pas en dualité. Cependant notons $\chi(\mathcal{F})$ la fonction sur X d'Euler-Poincaré d'un coefficient constructible $\mathcal{F}$:

$$\chi(\mathcal{F})(x) := \sum_i (-1)^i \dim_{\boldsymbol{C}} h^i(\mathcal{F})_x .$$

Théorème 2.1.9. *Pour toute sous-variété Z de X et tout complexe de $\mathcal{M}$ de la catégorie $D_h^b(\mathcal{D}_X)$ on a l'égalité entre fonctions constructibles complexes sur X*

$$\chi\left(\boldsymbol{IR}_Z(\mathcal{M})\right) = \chi\left(\boldsymbol{IR}_Z(\mathcal{M}^*)\right).$$

La preuve de 2.1.9 sera donnée en 2.3.

Corollaire 2.1.10. *Si Z est une hypersurface et si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome, alors le faisceau $\boldsymbol{IR}_Z(\mathcal{M})$ est nul si et seulement si le faisceau $\boldsymbol{IR}_Z(\mathcal{M}^*)$ est nul.*

Preuve de 2.1.10. En effet un faisceau pervers est nul si et seulement si sa fonction d'Euler-Poincaré est nulle.

Définition 2.1.11. Si Z est une sous variété de X et si $\mathcal{M}$ est un complexe de la catégorie $D_h^b(\mathcal{D}_X)$, on appelle *complexe d'irrégularité de $\mathcal{M}$* le long de Z les coefficients constructibles $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^\nu(\mathcal{M})$.

Définition 2.1.12. Si Z est une hypersurface de X et si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome, on appelle *faisceau d'irrégularité de $\mathcal{M}$* le long de Z les faisceaux $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^\nu(\mathcal{M})$.

Exemple 2.1.13. Si X est une surface de Riemann, Z un point de X et $P := P(x, \frac{d}{dx})$ est une équation différentielle non nulle sur X qui a un point singulier en Z le nombre de Fuchs de P en Z qui est défini purement algébriquement est égal à la dimension de l'espace vectoriel $\boldsymbol{IR}_Z(\mathcal{M})$ pour $\mathcal{M} := \mathcal{D}_X/(P)$ où (P) est l'idéal à gauche de $\mathcal{D}_X$ engendré par P. C'est le théorème de comparaison de Malgrange ([M$_1$], [M$_2$]).

Exemple 2.1.14. Si X est une surface complexe non singulière, $\mathcal{M}$ un $\mathcal{D}_X$-module holonome qui est lisse en dehors d'une courbe Y sur X et Z une courbe sur X coupant Y en un seul point, alors le faisceau $\boldsymbol{IR}_Z(\mathcal{M})$ est un faisceau ponctuel placé en degré un. C'est le théorème 3.1 de [Me$_4$] qui est donc un cas particulier du théorème de positivité. Le lecteur trouvera des exemples de faisceaux d'irrégularité dans [Me$_4$].

Remarque 2.1.15. Si Z est une sous-variété de X de codimension plus grande que 1 le complexe $\boldsymbol{IR}_Z(\mathcal{M})$ n'est pas en général un faisceau pervers. Par exemple le lecteur trouvera des exemples dans [Me$_4$] où $\boldsymbol{IR}_Z(\mathcal{M})$ n'est pas concentré en un seul degré avec $\dim(X) = 2$ et Z est un point de X.

2.1.16. Soit $f : X \to C$ une fonction holomorphe non constante sur le plan complexe C et Z l'hypersurface $f^{-1}(0)$. Fixons les notations à l'aide du diagramme suivant

$$\begin{array}{ccccccc}
\widetilde{X}^* & \xrightarrow{p} X^* := X - Z & \overset{j}{\subset} & X & \overset{i}{\supset} & Z \\
\downarrow f^* & \downarrow f^* & & \downarrow f & & \downarrow f \\
\widetilde{C}^* & \longrightarrow C^* := C - 0 & \subset & C & \supset & 0,
\end{array}$$

où $\widetilde{C}^*$ un revêtement universel de C^*. Rappelons que le faisceau $\Psi_f(\mathcal{O}_X)$ sur Z de fonctions multiformes est défini dans [SGA 7, XIV] par

$$\Psi_f(\mathcal{O}_X) := i^{-1}j_*p_*p^{-1}j^{-1}\mathcal{O}_X.$$

C'est un $\mathcal{D}_X$-module à gauche muni d'une action de la monodromie notée T et qui commute à l'action de $\mathcal{D}_X$. Il contient comme sous-$\mathcal{D}_X$-module à gauche muni d'une action de la monodromie le faisceau des fonctions multiformes de détermination finie $\Psi_f^{df}(\mathcal{O}_X)$. C'est le faisceau des germes des fonctions multiformes dont l'action de T admet un polynôme minimal. On a alors le théorème suivant :

Théorème 2.1.17. *Pour tout polynôme $P \in \mathbf{C}[T]$ non nul l'action de $P(T)$ sur les $\mathcal{D}_X$-modules à gauche munis d'une action de la monodromie $\Psi_f(\mathcal{O}_X)$ et $\Psi_f^{df}(\mathcal{O}_X)$ est* surjective *et pour tout complexe $\mathcal{M}$ de la catégorie $D_h^b(\mathcal{D}_X)$ le morphisme canonique*

$$\mathbf{R}hom_{\mathcal{D}_X}\left(\mathcal{M}, \Psi_f^{df}(\mathcal{O}_X)\right) \to \mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \Psi_f(\mathcal{O}_X))$$

est un isomorphisme.

Preuve de 2.1.17. Voir [Me$_5$], §4.

2.1.18. Le faisceau $\Psi_f^{df}(\mathcal{O}_X)$ contient comme sous-$\mathcal{D}_X$-module à gauche muni d'une action de la monodromie le faisceau $\mathrm{Nils}_f(\mathcal{O}_X)$ des fonctions à croissance modérée, dites de classe de Nilson. Le cône du morphisme

$$\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathrm{Nils}_f(\mathcal{O}_X)) \to \mathbf{R}hom_{\mathcal{D}_X}\left(\mathcal{M}, \Psi_f^{df}(\mathcal{O}_X)\right)$$

pour un $\mathcal{D}_X$-module holonome $\mathcal{M}$ n'est pas un faisceau pervers. Cependant il apparait comme "limite inductive" de faisceaux pervers. De façon précise on a les isomorphismes $\mathrm{Nils}_f(\mathcal{O}_X) \cong \varinjlim_P \mathrm{Ker}(P, \mathrm{Nils}_f(\mathcal{O}_X))$ et $\Psi_f^{df}(\mathcal{O}_X) \cong \varinjlim_P \mathrm{Ker}(P, \Psi_f^{df}(\mathcal{O}_X))$ où la limite inductive est prise selon l'ensemble filtrant des polynômes P de $\mathbf{C}[T]$ ordonnés par la divisibilité. Mais pour chaque polynôme P fixé le $\mathcal{D}_C$-module $\mathrm{Ker}(P, \mathrm{Nils}_{\mathrm{id}}(\mathcal{O}_C))$ est somme directe de $\mathcal{D}_C$-modules qui sont extention successive de connexion méromorphes régulières de rang un [SGA 7, XIV]. C'est donc un $\mathcal{D}_C$-module holonome. Le $\mathcal{D}_X$-module $\mathrm{Ker}(P, \mathrm{Nils}_f(\mathcal{O}_X))$ est image inverse par la fonction f de $\mathrm{Ker}(P, \mathrm{Nils}_{\mathrm{id}}(\mathcal{O}_C))$ [SGA 7, XIV] c'est donc un $\mathcal{D}_X$-module holonome [K3] (voir aussi [NM2]). On peut aussi invoquer que $\mathrm{Ker}(P, \mathrm{Nils}_f(\mathcal{O}_X))$ est un $\mathcal{O}_X(*Z)$-module localement libre de type fini [SGA 7, XIV] muni d'une action de $\mathcal{D}_X$ et est donc holonome *loc. cit.* On a alors le théorème :

Théorème 2.1.19. *Pour tout polynôme P et tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ le cône du morphisme canonique*

$$\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}, Ker(P, \mathrm{Nils}_f(\mathcal{O}_X))) \to \boldsymbol{R}hom_{\mathcal{D}_X}\left(\mathcal{M}, Ker\left(P, \Psi_f^{df}(\mathcal{O}_X)\right)\right)$$

est un faisceau pervers sur Z.

Preuve de 2.1.19. En effet, par construction, le cône du morphisme du théorème (2.1.19) est isomorphe à $\boldsymbol{IR}_Z(\mathcal{M}^* \otimes_{\mathcal{O}_X} \mathrm{Ker}(P, \mathrm{Nils}_f(\mathcal{O}_X)))$ ce qui le fait apparaitre comme faisceau d'irrégularité d'un $\mathcal{D}_X$-module holonome le long d'une hypersurface Z. On réduit alors au théorème 2.1.6.

2.2. *Démonstration du théorème* 2.1.6. La question est locale pour la topologie transcendante. On peut donc supposer que X est une variété analytique complexe et que $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome analytique.

2.2.1. Préliminaires.

Lemme 2.2.1.1. *Pour tout $\mathcal{D}_X$-module holonome $\mathcal{N}$ il passe par tout point en dehors d'une partie analytique de X de dimension* nulle *une hypersurface de X lisse et non caractéristique pour $\mathcal{N}$.*

Preuve de 2.2.1.1. Rappelons qu'on dit qu'une sous-variété lisse X' de X est non caractéristique pour un $\mathcal{D}_X$-module cohérent $\mathcal{N}$ si la variété caractéristique $\mathrm{Ch}(\mathcal{N})$ de $\mathcal{N}$ ne coupe le conormal $T^*_{X'}X$ de X' dans X que le long de la section nulle du fibré cotangent T^*X. Si $\mathcal{N}$ est un $\mathcal{D}_X$-module holonome, sa variété caractéristique $\mathrm{Ch}(\mathcal{N})$ n'a localement qu'un nombre fini de composantes irréductibles qui en plus sont de dimension au plus égale à $\dim(X)$. Ces composantes qui sont homogènes se projettent sur X en des sous variétés de X. La partie analytique du lemme 2.2.1.1 est fournie par la réunion de ces projections qui sont de dimension nulle. En effet la fibre de $\mathrm{Ch}(\mathcal{N})$ au dessus de tout point en dehors de cette partie n'est pas égale à toute la fibre du fibré cotangent T^*X au dessus du même point pour des raisons de dimension. Prenons un vecteur cotangent non nul au dessus de ce point qui n'est pas dans $\mathrm{Ch}(\mathcal{N})$; alors toute hypersurface passant par ce point et conormal à ce vecteur est non caractéristique pour $\mathcal{N}$.

2.2.1.2. Soit Z une sous variété de X définie par un Idéal $\mathcal{I}_Z$ et

$$\mathcal{O}_{\widehat{X/Z}} := \varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k$$

le complété formel de X le long de Z qui est de façon naturelle un $\mathcal{D}_X$-module à gauche. On définit le faisceau $\mathcal{Q}_Z$ par la suite exacte de $\mathcal{D}_X$-modules à gauche

$$0 \to \mathcal{O}_{X/Z} \to \mathcal{O}_{\widehat{X/Z}} \to \mathcal{Q}_Z \to 0$$

où $\mathcal{O}_{X/Z} := i^{-1}\mathcal{O}_X$.

Lemme 2.2.1.3. *Pour tout complexe $\mathcal{N}$ de $\mathcal{D}_X$-modules à gauche à cohomologie bornée et cohérente le triangle de la catégorie $D^b(\boldsymbol{C}_X)$*

$$\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_{X/Z}) \to \boldsymbol{R}hom_{\mathcal{D}_X}\left(\mathcal{N}, \mathcal{O}_{\widehat{X/Z}}\right) \to \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{Q}_Z)$$

est isomorphe au triangle de la catégorie $D^b(\boldsymbol{C}_X)$

$$\begin{aligned} i^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) &\to \boldsymbol{R}hom_{\mathcal{D}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{N}), \mathcal{O}_X) \\ &\to i^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\boldsymbol{R}\mathcal{N}(*Z), \mathcal{O}_X)[+1]. \end{aligned}$$

Preuve de 2.2.1.3. Il suffit de définir un isomorphisme

$$\boldsymbol{R}hom_{\mathcal{D}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{N}), \mathcal{O}_X) \cong \boldsymbol{R}hom_{\mathcal{D}_X}\left(\mathcal{N}, \mathcal{O}_{\widehat{X/Z}}\right).$$

Mais on a un isomorphisme naturel d'adjonction

$$\begin{aligned} &\boldsymbol{R}hom_{\mathcal{D}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{N}), \mathcal{O}_X) \\ &\qquad \cong \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \boldsymbol{R}hom_{\mathcal{O}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{O}_X), \mathcal{O}_X)) \end{aligned}$$

et un isomorphisme de complexes de $\mathcal{D}_X$-modules qui permet de passer du point de vue des pro-modules au point de vue des ind-modules :

$$\mathcal{O}_{\widehat{X/Z}} \cong \boldsymbol{R}hom_{\mathcal{O}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{O}_X), \mathcal{O}_X).$$

Rappelons sa démonstration :

(a) Si $\mathcal{M}$ et $\mathcal{N}$ sont des $\mathcal{D}_X$-modules à gauche $hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ est un $\mathcal{D}_X$-module à gauche par $\partial\phi : m \to \partial\phi(m) - \phi(\partial m)$ où ∂ est une dérivation, ϕ une section locale de $hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ et m est une section locale de $\mathcal{M}$. Si $\mathcal{N}$ est un complexe de $\mathcal{D}_X$-modules à gauche le complexe $hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ est un complexe de $\mathcal{D}_X$-modules à gauche et si $\mathcal{M}$ est un complexe de $\mathcal{D}_X$-modules à gauche le complexe $hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ est un complexe de $\mathcal{D}_X$-modules à gauche. Soit $\mathcal{O}_X \to \mathcal{I}$ une résolution

$\mathcal{D}_X$-injective de $\mathcal{O}_X$ qui est alors $\mathcal{O}_X$-injective. Le complexe $\boldsymbol{R}hom_{\mathcal{O}_X}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{O}_X), \mathcal{O}_X)$ est représenté par $hom_{\mathcal{O}_X}(\mathrm{alg}\Gamma_Z(\mathcal{I}), \mathcal{I})$ qui est donc un complexe de $\mathcal{D}_X$-modules à gauche.

(b) Pour tout k on a un morphisme de bidualité qui est un isomorphisme

$$\mathcal{O}_X/\mathcal{I}_Z^k \to hom_{\mathcal{O}_X}\left(hom_{\mathcal{O}_X}\left(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}\right), \mathcal{I}\right).$$

D'où un morphisme

$$(*)\quad \varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k \to \varprojlim_k hom_{\mathcal{O}_X}\left(hom_{\mathcal{O}_X}\left(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}\right), \mathcal{I}\right)$$
$$\cong hom_{\mathcal{O}_X}\left(\mathrm{alg}\Gamma_Z(\mathcal{I}), \mathcal{I}\right).$$

Voyons que ce morphisme est $\mathcal{D}_X$-linéaire. Il suffit de voir que le diagramme suivant est commutatif pour tout k :

$$\begin{array}{ccc} \mathcal{O}_X/\mathcal{I}_Z^{k+1} & \longrightarrow & hom_{\mathcal{O}_X}\left(hom_{\mathcal{O}_X}\left(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{I}^p\right), \mathcal{I}^p\right) \\ {\scriptstyle\partial}\downarrow & & {\scriptstyle\partial}\downarrow \\ \mathcal{O}_X/\mathcal{I}_Z^k & \longrightarrow & hom_{\mathcal{O}_X}\left(hom_{\mathcal{O}_X}\left(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}^p\right), \mathcal{I}^p\right). \end{array}$$

Notons f_k une section locale de $\mathcal{O}_X/\mathcal{I}_Z^k$, g_k une section locale de $hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}_Z^p)$ et ϕ_k une section locale de $hom_{\mathcal{O}_X}(hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}^p), \mathcal{I}^p)$. Alors $\partial\phi_{k+1}$ est l'application qui à g_k associe $\partial\phi_{k+1}(g_k) - \phi_{k+1}(\partial g_k)$ où on regarde g_k comme section locale de $hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{I}^p)$ par l'injection naturelle de $hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}^p)$ dans $hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{I}^p)$. Maintenant ∂g_k est l'application qui à f_{k+1} associe $\partial g_k(f_{k+1}) - g_k(\partial f_{k+1})$. En tenant compte des ces actions des dérivations on trouve que diagramme est commutatif.

(c) Voyons que le morphisme $(*)$ est un isomorphisme. Notons $\mathcal{J} := \varprojlim_k \mathcal{J}_k$ le complexe $\varprojlim_k hom_{\mathcal{O}_X}(hom_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{I}), \mathcal{I})$. Les composantes du complexe $\mathcal{J}_k$ sont flasques et pour un ouvert V les morphismes de transition dans le système projectif $\varprojlim_k \Gamma(V; \mathcal{J}_k)$ sont surjectifs. Il résulte alors de ([EGA 0 III], (13.3.1), Publ. IHES **11**, (1961), pp. 1-82) que les composantes de $\mathcal{J}$ sont acycliques pour le foncteur section globale. Pour un ouvert V de Stein les morphismes de transition entre cohomologies de degré zéro de $\Gamma(V; \mathcal{J}_k)$ sont surjectifs en vertu du théorème B de Cartan et donc la cohomologie commute à la limite projective *loc. cit.* On trouve alors pour un ouvert V de Stein assez petit que la cohomologie de $\Gamma(V; \mathcal{J})$ est nulle en degré positif et isomorphe à $\Gamma(V; \varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k)$ en degré nul. Le complexe $\mathcal{J}$ est acyclique en degrés positifs et son faisceau de cohomologie de degré nul est isomorphe à $\varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k$. D'où l'isomorphisme cherché. On peut remplacer le faisceau $\mathcal{O}_X$ par n'importe quel fibré à connexion intégrable. Mais par contre cet isomorphisme est en défaut pour un $\mathcal{D}_X$-module singulier.

En particulier si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome et Z est une hypersurface de X, on a donc les isomorphismes

$$I\!R^{\nu}_{Z}(\mathcal{M}) \cong i^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}(*Z), \mathcal{O}_X)[+1] \cong R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z).$$

Lemme 2.2.1.4. *Soient $\mathcal{F}$ un coefficient constructible sur X, $X = \cup_\alpha X_\alpha$ une stratification de Whitney de X telle que la restriction à chaque strate X_α de chaque faisceau de cohomologie de $\mathcal{F}$ est lisse et x un point de X. Alors si X' est une hypersurface lisse passant par x et transverse à la stratification $\cup_\alpha X_\alpha$ au voisinage de x, le morphisme canonique*

$$f^{-1}R\mathcal{H}om_{C_X}(\mathcal{F}, C_X) \to R\mathcal{H}om_{C_{X'}}(f^{-1}\mathcal{F}, C_{X'})$$

est un isomorphisme au voisinage de x où f est l'inclusion canonique de X' dans X.

Preuve de 2.2.1.4. La question est locale. On peut supposer par dévissage à la Grothendieck que $\mathcal{F}$ est un faisceau lisse sur une strate X_α et nul en dehors de cette strate. Fixons les notations à l'aide du diagramme suivant

$$\begin{array}{ccc} j : X_\alpha \cap X' & \subset & \overline{X}_\alpha \cap X' \\ \downarrow f & & \downarrow f \\ j : X_\alpha & \subset & \overline{X}_\alpha . \end{array}$$

Il suffit de voir que l'on a l'isomorphisme entre complexes d'espaces vectoriel complexes

$$\left(f^{-1}Rj_*\mathcal{F}^{\nu}\right)_x \cong \left(Rj_*f^{-1}\mathcal{F}^{\nu}\right)_x$$

pour tout point x de $\overline{X}_\alpha - X_\alpha$. Mais si B est un voisinage assez petit de x dans X les types d'homotopie de $B \cap X_\alpha \cap X'$ et de $B \cap X_\alpha$ sont les mêmes en vertu du premier théorème d'isotopie de Thom-Mather, cf. ([LT$_2$], Thm. 1.2.8) et le lemme en résulte.

Corollaire 2.2.1.5. *Soient $\mathcal{F}$ un coefficient constructible, $X = \cup_\alpha X_\alpha$ une stratification de Whitney de X, un point x de X et X' une hypersurface comme dans le lemme 2.2.1.4. Alors au voisinage de x on a les isomorphismes pour tout i de faisceaux pervers*

$$f^{-1\,p}h^i(\mathcal{F}) \cong {}^p h^i(f^{-1}\mathcal{F}).$$

Preuve de 2.2.1.5. Sous les conditions de 2.2.1.5, si $\mathcal{F}$ est un coefficient constructible qui a la condition de support, le coefficient $f^{-1}\mathcal{F}$ a la condition de support. En vertu de 2.2.1.4, si $\mathcal{F}$ a la condition de co-support, $f^{-1}\mathcal{F}$ a la condition de co-support. Donc si $\mathcal{F}$ est un faisceau pervers sur X, $f^{-1}\mathcal{F}$ est un faisceau pervers sur X' et le corollaire 2.2.1.5 est vrai dans ce cas là. Dans le cas général soit $[a,b]$ l'amplitude perverse d'un coefficient $\mathcal{F}$; on a alors un triangle distingué de $D^b_c(\boldsymbol{C}_X)$

$$\mathcal{F}' \to \mathcal{F} \to {}^p h^b(\mathcal{F})$$

où l'amplitude de $\mathcal{F}'$ est égale à $[a, b-1]$. On a un triangle distingué de $D^b_c(\boldsymbol{C}_{X'})$

$$f^{-1}\mathcal{F}' \to f^{-1}\mathcal{F} \to f^{-1}\,{}^p h^b(\mathcal{F}).$$

Appliquons le foncteur i^{-1} à la suite longue de cohomologie perverse associée au premier triangle puis prenons la suite longue de cohomologie perverse du second triangle; on déduit le corollaire 2.2.1.5 en raisonnant par récurrence sur la longueur $b-a$.

2.2.1.6. Soient $P(x, \partial_X)$ un opérateur différentiel d'ordre m défini au voisinage d'un point x_0 de X et X' une hypersurface lisse définie au voisinage de x_0 et non caractéristique pour $P(x, \partial_X)$. Posons $\mathcal{N} := \mathcal{D}_X/\mathcal{D}_X P$. Alors on a, au voisinage de x_0,

$$\boldsymbol{L}f^*\mathcal{N} \cong \mathcal{D}^m_{X'}$$

où f désigne l'inclusion canonique de X' dans X. On a en vertu du théorème de Cauchy-Kovalewski un isomorphisme canonique

$$f^{-1}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \cong \mathcal{O}^m_{X'}$$

et la nullité du faisceau $Ext^1_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X)$. En résumé, on a alors un isomorphisme canonique au voisinage de x_0

$$f^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \cong \boldsymbol{R}hom_{\mathcal{D}_{X'}}(f^*\mathcal{N}, \mathcal{O}_{X'}).$$

On en déduit par dévissage [K1] que si $\mathcal{N}$ est un $\mathcal{D}_X$-module cohérent défini au voisinage de x_0 tel que X' est non caractéristique pour $\mathcal{N}$, alors

$$\boldsymbol{L}f^*\mathcal{N} \cong f^*\mathcal{N},$$

$f^*\mathcal{N}$ est un $\mathcal{D}'_X$-module cohérent et on a un isomorphisme canonique au voisinage de x_0

$$f^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \cong \boldsymbol{R}hom_{\mathcal{D}_{X'}}(f^*\mathcal{N}, \mathcal{O}_{X'}).$$

2.2.2. **Preuve du théorème** 2.1.6. Nous allons raisonner par récurrence sur $\dim(X)$. Soit donc un triplet $(X, Z, \mathcal{M})$ où X est une variété analytique complexe, Z une hypersurface de X éventuellement singulière et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome. Il s'agit de montrer que les coefficients constructibles $\boldsymbol{I\!R}_Z(\mathcal{M})$ et $\boldsymbol{I\!R}^\nu_Z(\mathcal{M})$ sont des faisceaux pervers sur Z. Il suffit de montrer, en vertu du théorème de dualité locale 1.5.1, que le coefficient

$$\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M})[-1] \cong i^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}(*Z), \mathcal{O}_X)$$

est un faisceau pervers sur X, où i désigne l'inclusion canonique de Z dans X. Remarquons qu'en vertu de (2.2.1.3) on a l'isomorphisme

$$i^{-1}hom_{\mathcal{D}_X}(\mathcal{M}(*Z), \mathcal{O}_X) \cong Ext^{-1}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z) = 0.$$

En particulier si $\dim(X) = 1$, le complexe $\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M})[-1]$ est un faisceau pervers sur X puisque le faisceau $Ext^{1}_{\mathcal{D}_X}(\mathcal{M}(*Z), \mathcal{O}_X)$ est à support de dimension nulle. D'où le théorème 2.1.6 en dimension un. Supposons démontré le théorème 2.1.6 pour toutes les variétés analytiques complexes de dimension égale à $\dim(X)-1$ et soit $\cup_\alpha X_\alpha$ une stratification de Whitney de X telle que Z est réunion de strates et que les restrictions aux strates X_α des faisceaux de cohomologie (ordinaire) de $\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M})[-1]$ sont lisses. En vertu du lemme 2.2.1.1, puisque $\mathcal{M}(*Z)$ est un $\mathcal{D}_X$-module holonome, il passe par tout point de Z, en dehors d'une partie analytique de Z de dimension nulle, une hypersurface lisse non caractéristique pour $\mathcal{M}(*Z)$ au voisinage de ce point. Il passe par tout point de Z, en dehors d'une partie analytique de Z de dimension nulle, une hypersurface lisse qui est transverse à la stratification $\cup_\alpha X_\alpha$. Donc il passe par tout point de Z, en dehors d'une partie analytique de Z de dimension nulle, une hypersurface lisse non caractéristique pour $\mathcal{M}(*Z)$ et transverse à la stratification $\cup_\alpha X_\alpha$. Soient X' une telle hypersurface et f l'inclusion canonique de X' dans X. Posons $Z' := Z \cap X'$ et $\mathcal{M}' := f^*(\mathcal{M})$ qui est un $\mathcal{D}_{X'}$-module holonome en vertu du 1.3.6. On a donc un triplet $(X', Z', \mathcal{M}')$ analogue au triplet $(X, Z, \mathcal{M})$ avec $\dim(X') = \dim(X)-1$. En vertu de l'hypothèse de récurrence le coefficient constructible $\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M}')[-1]$ est un faisceau pervers sur X'. En vertu de 1.3.8 et de 2.2.1.5, on a les isomorphismes

$$\boldsymbol{L}f^*(\mathcal{M}(*Z)) \cong \boldsymbol{L}f^*(\mathcal{M}(*Z')) \cong \mathcal{M}'(*Z').$$

En vertu du théorème de Cauchy-Kovalewski on a un isomorphisme canonique

$$f^{-1}\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}(*Z), \mathcal{O}_X) \cong \boldsymbol{R}hom_{\mathcal{D}_{X'}}(\mathcal{M}'(*Z'), \mathcal{O}_{X'})$$

et donc

$$f^{-1}\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M})[-1] \cong \boldsymbol{IR}^{\nu}_{Z'}(\mathcal{M}')[-1].$$

En vertu du corollaire 2.2.1.4 et de l'hypothèse de récurrence les faisceaux de cohomologie perverse de degré non nul du coefficient $\boldsymbol{IR}^{\nu}_{Z}(\mathcal{M})[-1]$ sont nuls en dehors d'une partie analytique de Z de dimension nulle. Le théorème 2.1.6 est donc vrai en dehors d'une partie analytique de Z de dimension nulle. Reste à exorciser ce qui se passe sur cette partie de dimension nulle. Remarquons que le coefficient $I\!R_Z^{\nu}(\mathcal{M})[-1]$ qui est concentré cohomologiquement en degrés $[1, \dim(X)]$ a certainement la condition de support. Voyons qu'il a alors la condition de co-support ou que son complexe dual qui n'est autre que $I\!R_Z(\mathcal{M})[-1]$ a la propriété de support. Notons j l'inclusion canonique $U := X - Z$ dans X. Par construction, on a l'isomorphisme :

$$I\!R_Z(\mathcal{M})[-1] \cong I\!Rhom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z)[-1]$$

où $\mathcal{L}_Z$ est le $\mathcal{D}_X$-module à gauche défini par la suite exacte de $\mathcal{D}_X$-modules à gauche

$$0 \to \mathcal{O}_X(*Z) \to j_*j^{-1}\mathcal{O}_X \to \mathcal{L}_Z \to 0.$$

Donc le complexe $I\!R_Z(\mathcal{M})[-1]$ qui est *a priori* concentré cohomologiquement en degrés $[1, \dim(X)+1]$ a la propriété de support en dehors d'une partie analytique de Z de dimension nulle. Ce qui l'empêche d'avoir la propriété de support est son dernier faisceau de cohomologie $Ext^n_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z)$ où $n := \dim(X)$. La démonstration du théorème 2.1.6 est achevée par le lemme suivant :

Lemme 2.2.2.1. *Pour tout $\mathcal{D}_X$-module à gauche holonome $\mathcal{N}$, le faisceau $Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{L}_Z)$ est* nul.

Preuve de 2.2.2.1. Les arguments sont calqués sur ([M$_2$], théorème 2.3). On a une suite exacte de faisceaux de support de dimension nulle

$$Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X(*Z)) \to Ext^n_{\mathcal{D}_X}(\mathcal{N}, j_*j^{-1}\mathcal{O}_X) \to Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{L}_Z) \to 0.$$

Nous allons voir que le premier morphisme est surjectif, ce qui démontrera le lemme 2.2.2.1. Posons $\mathcal{N}^{\nu} := Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{D}_X)$ qui est un $\mathcal{D}_X$-module à droite cohérent. Pour tout $\mathcal{D}_X$-module à gauche $\mathcal{G}$ on a l'isomorphisme

$$Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{G}) \cong \mathcal{N}^{\nu} \underset{\mathcal{D}_X}{\otimes} \mathcal{G}.$$

Soit $\mathcal{D}_X^q \xrightarrow{\phi} \mathcal{D}_X^p \longrightarrow \mathcal{N}^{\nu} \to 0$ une présentation locale de $\mathcal{N}^{\nu}$. On a donc localement l'isomorphisme :

$$\mathcal{N}^{\nu} \underset{\mathcal{D}_X}{\otimes} \mathcal{G} \cong \mathcal{G}^p / \phi(\mathcal{G}^q).$$

Le morphisme

$$Ext^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X(*Z)) \to Ext^n_{\mathcal{D}_X}(\mathcal{N}, j_*j^{-1}\mathcal{O}_X)$$

se représente par le morphisme naturel de faisceaux d'espaces vectoriels complexes de dimension finie

$$\mathcal{O}_X^p(*Z)/\phi(\mathcal{O}_X^q(*Z)) \to (j_*j^{-1}\mathcal{O}_X^p)/\phi(j_*j^{-1}\mathcal{O}_X^q).$$

Soit B un voisinage de Stein d'un point o de X. Pour B assez petit on a deux suites exactes:

$$\begin{array}{ccccccccc} 0 \to & \Gamma(B, \phi(\mathcal{O}_X^q(*Z))) & \to & \Gamma(B, \mathcal{O}_X^p(*Z)) & \to & (\mathcal{O}_X^p(*Z)/\phi(\mathcal{O}_X^q(*Z)))_0 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma(B, \phi(j_*j^{-1}\mathcal{O}_X^q)) & \to & \Gamma(B, j_*j^{-1}\mathcal{O}_X^p) & \to & (j_*j^{-1}\mathcal{O}_X^p/\phi(j_*j^{-1}\mathcal{O}_X^q))_0 & \to 0. \end{array}$$

Lemme 2.2.2.2. *Soient F un espace vectoriel complexe topologique de Fréchet et E un sous-espace vectoriel de F de codimension finie image d'un espace de Fréchet par une application linéaire continue; alors E est fermé dans F.*

Preuve de 2.2.2.2. C'est une conséquence directe du théorème de Banach: une application linéaire continue surjective entre espaces de Fréchet est un homomorphisme, cf. [Se$_2$], ([G$_8$], exercice 4, p. 57).

Appliquons ceci à $\Gamma(B, j_*j^{-1}\mathcal{O}_X^p)$ qui est un espace de Fréchet. Pour B assez petit, $\Gamma(B, \phi(j_*j^{-1}\mathcal{O}_X^q))$, qui est isomorphe à $\phi(\Gamma(B, j_*j^{-1}\mathcal{O}_X^q))$, ceci se voit en supposons que le présention ϕ de $\mathcal{N}^\nu$ est le début d'une résolution locale finie par des modules libres de type fini et en utilisant le théorème B de Cartan. Donc $\Gamma(B, \phi(j_*j^{-1}\mathcal{O}_X^q))$ est fermé dans $\Gamma(B, j_*j^{-1}\mathcal{O}_X^p)$ en vertu du lemme 2.2.2.2. L'espace $\Gamma(B, j_*j^{-1}\mathcal{O}_X^p)/\Gamma(B, \phi(j_*j^{-1}\mathcal{O}_X^q))$ muni de la topologie quotient est *séparé*. Mais, pour B assez petit, Z est définie par une équation et l'espace $\Gamma(B, \mathcal{O}_X^p(*Z))$ est dense dans $\Gamma(B, j_*j^{-1}\mathcal{O}_X^p)$. Donc l'image du morphisme $\Gamma(B, \mathcal{O}_X^p(*Z))/\Gamma(B, \phi(\mathcal{O}_X^q(*Z))) \to \Gamma(B, j_*j^{-1}\mathcal{O}_X^p)/\Gamma(B, \phi(j_*j^{-1}\mathcal{O}_X^q))$ est dense. C'est donc un morphisme surjectif. Le morphisme

$$(\mathcal{O}_X^p(*Z)/\phi(\mathcal{O}_X^q(*Z)))_0 \to ((j_*j^{-1}\mathcal{O}_X^p)/\phi(j_*j^{-1}\mathcal{O}_X^q))_0$$

est donc surjectif. D'où le lemme 2.2.2.1.

Remarque 2.2.2.3. Le lemme 2.2.2.1 qui est de nature transcendante est pour l'instant le principal obstacle à une démonstration purement algébrique de théorème de positivité, cf. §3.

2.3. *Démonstration du théorème* 2.1.9. Il s'agit de montrer que si $\mathcal{M}$ est un complexe de la catégorie $D_h^b(\mathcal{D}_X)$ les fonctions constructibles complexes, sur X, $\chi((\mathbf{R}_Z(\mathcal{M}))$ et $\chi((\mathbf{R}_Z(\mathcal{M}^*))$ sont égales pour toute sous-variété Z de X. On peut supposer que $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome. La question est locale. Si $Z = Z_1 \cap Z_2$, on a par construction le triangle de Mayer-Vietoris de la catégorie $D_c^b(\mathbf{C}_X)$:

$$\boldsymbol{IR}_Z(\mathcal{M}) \to \boldsymbol{IR}_{Z_1}(\mathcal{M}) \oplus \boldsymbol{IR}_{Z_2}(\mathcal{M}) \to \boldsymbol{IR}_{Z_1 \cup Z_2}(\mathcal{M}).$$

En raisonnant par récurrence sur le nombre d'équations définissant Z on est ramené à supposer que Z est définie par une équation. D'autre part on a, par définition du faisceau d'irrégularité, l'égalité :

$$\chi\left(\boldsymbol{IR}_Z(\mathcal{M})\right) = \chi\left(\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}))\right) - \chi\left(\boldsymbol{DR}(\boldsymbol{R}\Gamma_Z(\mathcal{M}))\right).$$

Mais en vertu du théorème de dualité locale 1.5.1, on a l'égalité

$$\chi\left(\boldsymbol{DR}(\boldsymbol{R}\Gamma_Z(\mathcal{M}))\right) = \chi\left(\boldsymbol{DR}(\boldsymbol{R}\Gamma_Z(\mathcal{M}^*))\right).$$

Donc, pour démontrer le théorème 2.1.9, il suffit de montrer que l'on a l'égalité :

$$\chi\left(\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}))\right) = \chi\left(\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}^*))\right)$$

pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ et toute hypersurface Z. En prenant une équation de Z et en considérant son graphe on est réduit à supposer que Z est lisse parce que tous les foncteurs en $\mathcal{M}$ qui interviennent dans l'égalité précédente commutent avec une immersion fermée, cf. [Me$_3$]. Considérant alors la V-filtration ([M$_3$], [K$_4$]) de $\mathcal{D}_X$ indexée par $\boldsymbol{Z}$ et définie par :

$$V_k(\mathcal{D}_X) := \left\{P \in \mathcal{D}_X, P(\mathcal{I}_Z^p) \subset \mathcal{I}_Z^{p-k}, p \in \boldsymbol{Z}\right\}$$

où $\mathcal{I}_Z$ est l'Idéal de Z avec $\mathcal{I}_Z^p = \mathcal{O}_X$ si p est négatif ou nul. Si t est une équation locale de Z elle est d'ordre -1 pour la V-filtration. Le gradué d'ordre 0, $\mathrm{gr}_0^V(\mathcal{D}_X) := V_0(\mathcal{D}_X)/V_{-1}(\mathcal{D}_X)$, contient comme section distinguée le champ d'Euler Eu. Il résulte alors de l'équation fonctionnelle de Bernstein-Sato, cf. ([K$_3$], [NM$_1$]), que tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ admet une unique filtration croissante de Malgrange-Kashiwara, indexée par $\boldsymbol{Z}$, par des $V_0(\mathcal{D}_X)$-modules cohérents $V_k(\mathcal{M})$, telle que l'action du champ d'Euler Eu sur le gradué d'ordre k, $\mathrm{gr}_k^V(\mathcal{M}) := V_k(\mathcal{M})/V_{k-1}(\mathcal{M})$, admet localement un polynôme minimal dont les zéros sont dans la bande $-k-1 \leq \mathrm{Re}\alpha < -k$ du plan complexe pour tout $k \in \boldsymbol{Z}$, ([M$_3$], [K$_4$]). Il en résulte alors que les $\mathcal{D}_Z$- modules $\mathrm{gr}_k^V(\mathcal{M})$ sont cohérents. En fait on a le résultat suivant, cf. ([Sa$_1$], [SM]) :

Théorème 2.3.1. *Pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ les $\mathcal{D}_Z$-modules $\mathrm{gr}_k^V(\mathcal{M})$ sont holonomes pour tout $k \in \boldsymbol{Z}$. De plus les $\mathcal{D}_Z$-modules $\mathrm{gr}_k^V(\mathcal{M})$ et $\mathrm{gr}_k^V(\mathcal{M}^*)$ sont en dualité pour tout $k \in \boldsymbol{Z}$:*

$$\mathrm{gr}_k^V(\mathcal{M}))^* \cong \mathrm{gr}_k^V(\mathcal{M}^*).$$

Pour la preuve du théorème 2.3.1, voir [SM]. On définit avec ([M$_3$], [K$_4$]):

Définition 2.3.2. Pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ et toute hypersurface lisse Z, on pose $\Psi_Z^m(\mathcal{M}) := \mathrm{gr}_Z^V(\mathcal{M})$ et $\Phi_Z^m(\mathcal{M}) := \mathrm{gr}_0^V(\mathcal{M})$.

Les foncteurs Ψ_Z^m et Φ_Z^m sont exacts de la catégorie $\mathrm{Mh}(\mathcal{D}_X)$ dans la catégorie $\mathrm{Mh}(\mathcal{D}_Z)$. Si t est une équation locale de Z le complexe $0 \to \Psi_Z^m(\mathcal{M}) \xrightarrow{t} \Phi_Z^m(\mathcal{M}) \to 0$ représente le complexe $\boldsymbol{Li}^*\mathcal{M}$ de la catégorie $D_h^b(\mathcal{D}_Z)$ pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$ où i est l'inclusion canonique de Z dans X. Mais le complexe $\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M})[1]$ est l'image directe par l'immersion i au sens des $\mathcal{D}_Z$-modules du complexe $\boldsymbol{Li}^*\mathcal{M}$ [Me2]. Il résulte alors du théorème 2.3.1 et du théorème de dualité 1.5.1 que l'on a l'égalité entre fonctions constructibles complexes :

$$\chi(\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}))) = \chi(\boldsymbol{DR}(\boldsymbol{R}\mathrm{alg}\Gamma_Z(\mathcal{M}^*))).$$

Ceci termine la démonstration du théorème 2.1.9.

Si $f : X \to C$ est une fonction holomorphe non constante et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome, notons $\mathcal{M}_f$ l'image directe de $\mathcal{M}$ par le morphisme graphe de f. Si $Z := X \times 0$ les $\mathcal{D}_Z$-modules $\Psi_Z^m(\mathcal{M}_f)$ et $\Phi_Z^m(\mathcal{M}_f)$ sont à support dans $f^{-1}(0)$ et donc ils proviennent de $\mathcal{D}_X$-modules holonomes, que nous noterons encore $\Psi_Z^m(\mathcal{M}_f)$ et $\Phi_Z^m(\mathcal{M}_f)$.

Définition 2.3.3. On pose $\Psi_f^m(\mathcal{M}) := \Psi_Z^m(\mathcal{M}_f)$ et $\Phi_f^m(\mathcal{M}) := \Phi_Z^m(\mathcal{M}_f)$.

Les foncteurs Ψ_f^m et Φ_f^m sont exacts de la catégorie des $\mathcal{D}_X$-modules holonomes dans la catégorie des $\mathcal{D}_X$-modules holonomes à support dans $f^{-1}(0)$.

3. Le cycle d'irrégularité

Dans le §2 nous avons défini le faisceau d'irrégularité par voie transcendante. Mais comme tout faisceau pervers le faisceau d'irrégularité admet un cycle caractéristique positif du fibré cotangent défini à l'aide de la théorie des cycles évanescents topologiques de SGA 7, cf. [LM]. Le cycle caractéristique de faisceau d'irrégularité admet une description purement algébrique qui garde un sens sur un corps k de caractéristique nulle. Cependant le passage du faisceau d'irrégularité à son cycle fait perdre beaucoup d'information sur la ramification le long de l'hypersurface, ne serait-ce que parce que tout cycle lagrangien positif n'est pas en général le cycle caractéristique d'un faisceau pervers. Aussi le théorème de positivité suggère de construire à partir d'un $\mathcal{D}_X$-module holonome $\mathcal{M}$ et d'une hypersurface Z d'une variété non singulière X sur k un $\mathcal{D}_X$-module holonome à support dans Z tel que lorsque k est le corps des nombres complexes son complexe de de Rham (transcendant) soit isomorphe au moins virtuellement au faisceau $\boldsymbol{IR}_Z(\mathcal{M})$. Cette construction n'est pas encore au point; aussi nous nous limitons dans ce § à définir le cycle d'irrégularité. Nous supposerons toujours pour simplifier que les corps de base qui interviennent sont algébriquement clos.

3.1 *Cas des fibrés à connexion intégrable.* Soient une variété algébrique $(X, \mathcal{O}_X)$ non singulière sur un corps k de caractéristique nulle, Z une hypersurface de X et $\mathcal{M}$ un $\mathcal{D}_U$-module holonome lisse de rang r sur $U := X - Z$. Si j désigne l'inclusion canonique de U dans X on a $j_*\mathcal{O}_U \cong \mathcal{O}_X(*Z)$ et on pose

$$\mathcal{M}(*Z) := j_*\mathcal{M}$$

qui est donc un $\mathcal{D}_X$-module holonome.

Définition 3.1.1. On définit le cycle d'irrégularité de $\mathcal{M}$, le long de Z, $\mathrm{CCh}(\mathcal{M})$ en posant :

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) := \mathrm{CCh}(\mathcal{M}(*Z)) - r\,\mathrm{CCh}(\mathcal{O}_X(*Z)).$$

En vertu du théorème de l'involutivité des caractéristiques [G], $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ est un cycle lagrangien du fibré cotangent T^*X : ses composantes irréductibles sont les adhérences des fibrés conormaux de la partie lisse de leur projection sur X.

Théorème 3.1.2. *Sous les conditions précédentes le cycle d'irrégularité $CCh(\boldsymbol{IR}_Z(\mathcal{M}))$ est* positif.

Preuve du théorème 3.1.2. Par le principe de Lefschetz on peut supposer que $k = \boldsymbol{C}$. Notons alors par "an" en exposant le passage d'un objet défini sur k à l'objet transcendant associé. En vertu de [LM] on a les égalités de cycles lagrangiens :

$$\begin{aligned} \mathrm{CCh}(\mathcal{M}(*Z)) &= \mathrm{CCh}(\boldsymbol{DR}(\mathcal{M}^{\mathrm{an}}(*Z^{\mathrm{an}})) \\ r\mathrm{CCh}(\mathcal{O}_X(*Z)) &= \mathrm{CCh}(\boldsymbol{R}j_*^{\mathrm{an}}\boldsymbol{DR}(\mathcal{M}^{\mathrm{an}})). \end{aligned}$$

Donc par définition le cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ apparaît comme égal au cycle caractéristique du faisceau $\boldsymbol{IR}_Z(\mathcal{M}^{\mathrm{an}}(*Z^{\mathrm{an}}))[-1]$ qui est donc positif.

Si l'on a une suite exacte de fibrés sur U à connexion intégrable

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_2 \longrightarrow 0$$

on a l'égalité

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) = \mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}_1)) + \mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}_2)).$$

D'où un morphisme

$$\boldsymbol{IR}_Z : K(\mathcal{O}_U) \rightarrow Z_n(T^*X)$$

entre le groupe de Grothendieck des fibrés sur U à connexion intégrable et le groupe des cycles lagrangiens de T^*X.

Exemple 3.1.3. Si $\dim(X) = 1$ et si Z est un point, la multiplicité de $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ est égale au nombre de Fuchs. Ainsi le cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ apparaît comme la généralisation naturelle en dimension supérieure de l'irrégularité de Fuchs.

Plus généralement si Z est une hypersurface d'une sous-variété Y de X telle que $Y - Z$ est lisse et $\mathcal{M}$ est un fibré sur $Y - Z$ de rang r à connexion intégrable on définit son cycle d'irrégularité le long de Z en posant

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) := \mathrm{CCh}(\overline{\mathcal{M}}(*Z)) - r\mathrm{CCh}(H^p_Y(\mathcal{O}_X(*Z)))$$

où $\overline{\mathcal{M}}$ désigne l'image directe au sens des $\mathcal{D}_X$-modules de $\mathcal{M}$ par l'immersion de $Y - Z$ dans X et $p := \mathrm{codim}_Y(X)$. Le cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ est encore un cycle lagrangien positif.

Théorème 3.1.4. *Soit un fibré vectoriel de rang r à connexion intégrable $\mathcal{M}$ sur un ouvert U, complémentaire d'une hypersurface Z d'une variété algébrique X propre et non singulière sur un corps de caractéristique nulle. On a alors la formule de type* Riemann-Roch *:*

$$\chi(U, \boldsymbol{DR}(\mathcal{M})) = r\chi(U, \boldsymbol{DR}(\mathcal{O}_U)) + (-1)^{\dim(U)} CCh(\boldsymbol{IR}_Z(\mathcal{M})).T^*_X X.$$

Preuve du théorème 3.1.4. On a bien sûr noté $\chi(U, \boldsymbol{DR}(\mathcal{M}))$ la caractéristique d'Euler-Poincaré du complexe de de Rham global $\boldsymbol{R}\mathrm{Hom}_{\mathcal{D}_U}(\mathcal{O}_U, \mathcal{M})$ d'un $\mathcal{D}_U$-module $\mathcal{M}$ et $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})).T^*_X X$ le nombre d'intersection [F] du cycle d'irrégularité et de la section nulle du fibré cotangent T^*X. Si j désigne l'inclusion canonique de U dans X on a :

$$\chi(U, \boldsymbol{DR}(\mathcal{M})) = \chi(X, Rj_*\boldsymbol{DR}(\mathcal{M})) = \chi(X, \boldsymbol{DR}(\mathcal{M}(*Z))).$$

Mais si $\mathcal{N}$ est un $\mathcal{D}_X$-module holonome on a la formule globale [L3] :

$$\chi(X, \boldsymbol{DR}(\mathcal{N})) = (-1)^{\dim(X)} \mathrm{CCh}(\mathcal{N}).T^*_X X.$$

Appliquons cette formule à $\mathcal{N} = \mathcal{M}(*Z)$ et à $\mathcal{N} = \mathcal{O}_X(*Z)$: on trouve la formule du théorème 3.1.4. Bien sûr le point important dans la formule du théorème 3.1.4 est la positivité du cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$. Autrement dit à cette situation on attache canoniquement un nombre fini de couples (m_α, Z_α) où les m_α sont des entiers naturels *positifs* ou nuls (les irrégularités génériques) et les Z_α sont des sous-variétés irréductibles de Z si bien qu'on a la formule plus familière [D] :

$$\chi(U, \boldsymbol{DR}(\mathcal{M})) = r\chi(U, \boldsymbol{DR}(\mathcal{O}_U)) + (-1)^{\dim(U)} \sum_\alpha m_\alpha T^*_{Z_\alpha} X . T^*_X X.$$

On a noté comme d'habitude $T^*_{Z_\alpha} X$ l'adhérence du conormal de la partie lisse de Z_α. Pour les formules analogues en théorie l-adique voir [L2], [Ka].

Plus généralement si $\mathcal{M}$ est un fibré vectoriel à connexion intégrable sur un ouvert lisse U, complémentaire d'une hypersurface Z d'une sous-variété éventuellement singulière Y d'une variété algébrique X propre et non singulière sur un corps de caractéristique nulle, on a une formule de type Riemann-Roch analoque à celle du théorème 3.1.4. C'est cette dernière situation qui se produit le plus souvent dans les applications.

3.2. *Cas général.* Pour définir le cycle d'irrégularité d'un $\mathcal{D}_X$-module holonome le long d'une hypersurface nous avons besoin de l'isomorphisme d'Euler [Mc] entre le groupe $\mathbf{Z}(X)$ des cycles de X et le groupe des fonction constructibles $\mathsf{F}(X)$ à valeurs dans $\boldsymbol{Z}$. Rappelons que si T est une sous-variété réduite de X supposée définie sur un corps de caractéristique nulle on définit de façon purement algébrique l'obstruction d'Euler $\mathrm{Eu}_x(T)$ en un point x de T qui est un entier, cf. [Gs], [LT]. Elle coïncide avec l'obstruction d'Euler de MacPherson [Mc] définie par voie transcendante quand le corps de base est $\boldsymbol{C}$. Ceci entraîne que la fonction $x \rightarrow \mathrm{Eu}_x(T)$ est constructible et que le morphisme de groupes :

$$\mathrm{Eu} : \mathbf{Z}(X) \rightarrow \mathsf{F}(X)$$

est un isomorphisme. D'autre part l'application qui à une sous variété irréductible de X associe l'adhérence du conormal de sa partie lisse se prolonge de façon naturelle en un isomorphisme entre $\mathbf{Z}(X)$ et le groupe $\mathbf{Z}_n(T^*X)$ des cycles lagrangiens homogènes du fibré T^*X. En particulier à un $\mathcal{D}_U$-module holonome $\mathcal{M}_U$ sur le complémentaire U d'une hypersurface Z de X correspond la fonction constructible $\mathrm{Eu}(\mathrm{CCh}(\mathcal{M}_U))$ sur U et au prolongement par zéro $j_!\mathrm{Eu}(\mathrm{CCh}(\mathcal{M}_U))$ à X de cette fonction correspond un cycle lagrangien homogène noté $j_!(\mathrm{CCh}(\mathcal{M}_U))$. Si $\mathcal{M}_U$ est lisse de rang r, alors $j_!(\mathrm{CCh}(\mathcal{M}_U)) = r\,\mathrm{CCh}(\mathcal{O}_X(*Z))$. La description de ce dernier cycle se fait à l'aide de la modification de Nash de Z [LM]. Ceci montre que la description directe de $j_!(\mathrm{CCh}(\mathcal{M}_U))$ à partir de $\mathrm{CCh}(\mathcal{M}_U))$ n'est pas aisée dans le cas général cf. [Gi].

Définition 3.2.1. Pour tout triplet $(X, Z, \mathcal{M})$ comme ci-dessus où $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome on définit le cycle d'irrégularité de $\mathcal{M}$, le long de Z, $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ en posant

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) := \mathrm{CCh}(\mathcal{M}(*Z)) - j_!\left(\mathrm{CCh}(j^{-1}\mathcal{M})\right).$$

Le cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ est un cycle lagrangien homogène. Si on pose $j_!j^{-1}\mathcal{M} := (\mathcal{M}^*(*Z))^*$ on a en vertu du théorème 2.1.9 l'égalité entre cycles caractéristiques $\mathrm{CCh}(\mathcal{M}(*Z)) = \mathrm{CCh}(j_!j^{-1}\mathcal{M})$ et le cycle d'irrégularité prend la forme plus symétrique

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) = \mathrm{CCh}(j_!j^{-1}\mathcal{M}) - j_!\left(\mathrm{CCh}(j^{-1}\mathcal{M})\right).$$

Théorème 3.2.2. *Pour tout triplet* $(X, Z, \mathcal{M})$ *comme ci-dessus, le cycle* $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ *est* positif.

Preuve du théorème 3.2.2. On peut supposer que le corps de base est $\boldsymbol{C}$. Dans ce cas le cycle $\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M}))$ est le cycle caractéristique du faisceau $\boldsymbol{IR}_Z(\mathcal{M})[-1]$ qui est donc positif. On obtient donc un morphisme de groupes :

$$\boldsymbol{IR}_Z : K(\mathrm{Mh}(\mathcal{D}_X)) \to \mathbf{Z}_n(T^*X).$$

Définition 3.2.3. Si $\mathcal{M}$ est un complexe de la catégorie $D_h^b(\mathcal{D}_X)$ on définit son cycle d'irrégularité en posant :

$$\mathrm{CCh}(\boldsymbol{IR}_Z(\mathcal{M})) := \sum_1 \mathrm{CCh}\left(\boldsymbol{IR}_Z\left(h^i(\mathcal{M})\right)\right).$$

Cette définition est imposée par le corollaire 2.1.8. C'est donc un cycle positif qui est nul si et seulement si les faisceaux de cohomologie de $\mathcal{M}$ sont réguliers cf. §5, 5.1.5.

4. Le théorème de semi-continuité de l'irrégularité

4.1. Nous allons montrer dans ce § que le théorème de positivité précise le théorème de semi-continuité de l'irrégularité dont il est issu [Me4]. Soit donc $f : X \to S$ un morphisme lisse de dimension relative égale à un entre variétés algébriques non singulières sur un corps de caractéristique nulle, ou entre variétés analytiques complexes, et Z une hypersurface de X telle que la restriction de f à Z soit finie. Soit $\mathcal{M}$ un $\mathcal{D}_X$-module holonome dont la restriction à $U := X - Z$ est lisse de rang r. Posons $X_s := f^{-1}(s)$, $Z_s := Z \cap X_s$ et $\mathcal{M}_s := j_s^*(\mathcal{M})$ où j_s désigne l'inclusion canonique de X_s dans X, $\mathcal{M}_s$ est donc un $\mathcal{D}_{X_s}$-module holonome lisse sur $U_s := X_s - Z_s$ de rang r et on note $\mathrm{ir}_x(\mathcal{M}_s)$ la multiplicité de $T_x^*X_s$ dans le cycle d'irrégularité $\boldsymbol{IR}_x(\mathcal{M}_s)$ pour tout point x de X_s. On définit les fonctions Ψ et Φ sur S et à valeurs dans $\boldsymbol{N}$ en posant

$$\Psi(s) := \sum_{x \to s} \mathrm{ir}_x(\mathcal{M}_s) \quad \text{et} \quad \Phi(s) := \Psi(s) + r\#Z_s,$$

où $\#Z_s$ est égal au nombre de points de Z_s. On a alors les résultats suivants [Me4] : (i) les fonctions Ψ et Φ sont constructibles; (ii) la fonction Φ est semi-continue inférieurement; (iii) si la fonction Φ est localement constante, la variété caractéristique de $\mathcal{M}(*Z)$ est égale à $T_X^*X \cup T_Z^*X$ (on suppose que r est non nul). En fait on a le résultat plus précis le suivant :

Théorème 4.1.1. *Sous les conditions précédentes, si la fonction* Φ *est localement constante, le morphisme* $Z \to S$ *est* étale.

Preuve du théorème 4.1.1. On peut supposer que le corps de base est le corps des nombres complexes, puis que f est un morphisme entre variétés analytiques complexes. La question est locale sur S. On peut supposer par changement de base que $\dim(S) = 1$. Dans cette situation, si la fonction Φ est localement constante, toutes les fibres X_s sont non caractéristiques pour $\mathcal{M}(*Z)$ [Me4]. En particulier

$$j_s^{-1} \boldsymbol{IR}_Z^{\nu}(\mathcal{M}) \cong \boldsymbol{IR}_{Z_s}^{\nu}(\mathcal{M}_s).$$

Ainsi si s_0 est un point spécial et s_t un point général voisin, le saut de la fonction Ψ est égal à

$$\sum_{x \to s_t} \dim_{\boldsymbol{C}} \boldsymbol{IR}_Z^{\nu}(\mathcal{M})_x - \sum_{x \to s_0} \dim_{\boldsymbol{C}} \boldsymbol{IR}_Z^{\nu}(\mathcal{M})_x.$$

Considérons le triangle distingué de la catégorie $D_c^b(\boldsymbol{C}_{Z_0})$ des cycles évanescents [SGA 7, XIV]:

$$j_{s_0}^{-1} \boldsymbol{IR}_Z^{\nu}(\mathcal{M}) \to \boldsymbol{R}\Psi_f\,(\boldsymbol{IR}_Z^{\nu}(\mathcal{M})) \to \boldsymbol{R}\Phi_f\,(\boldsymbol{IR}_Z^{\nu}(\mathcal{M}))\,.$$

Mais alors puisque $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$ est un faisceau pervers sur Z, le complexe $\boldsymbol{R}\Phi_f(\boldsymbol{IR}_Z^{\nu}(\mathcal{M}))$ est un faisceau pervers sur Z_0, il est donc concentré en degré zéro puisque $\dim(Z_0) = 0$ et en particulier le morphisme:

$$j_{s_0}^{-1} \boldsymbol{IR}_Z^{\nu}(\mathcal{M}) \to j_{s_t}^{-1} \boldsymbol{IR}_Z^{\nu}(\mathcal{M})$$

est injectif et le saut de la fonction Ψ, égal à $\sum_{x \to s_0} \dim_{\boldsymbol{C}} \boldsymbol{R}\Phi_f(\boldsymbol{IR}_Z^{\nu}(\mathcal{M}))_x$, est positif ou nul. Mais le saut de la fonction Φ apparaît alors comme la somme de deux entiers *positifs* non nuls:

$$\Phi(s_t) - \Phi(s_0) = \Psi(s_t) - \Psi(s_0) + r\left(\#Z_{s_t} - \#Z_{s_0}\right).$$

Comme ce saut est nul par hypothèse, il en résulte que le saut de la fonction Ψ est nul et que $\#Z_{s_t} = \#Z_{s_0}$. Le morphisme $Z \to S$ est *étale*. C'est cette propriété de positivité qui est à la base du théorème de positivité.

Remarque 4.1.2. Dans la situation précédente le saut de la fonction Ψ est positif ou nul chaque fois que $\boldsymbol{IR}_{X_{s_0}}^{\nu}(\mathcal{M}(*Z))$ est nul. C'est par exemple le cas si le support du faisceau $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$ n'est pas égal à Z tout entier, cf. §5. En fait c'est là une conjecture générale.

4.2. Dans la situation précédente supposons que $\dim(S) = 1$ et notons $\mathcal{D}_{X/S}$ le faisceau des opérateurs différentiels sur X relativement à S [EGA, IV, §16]. Alors si $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome lisse sur U on lui associe de façon naturelle un $\mathcal{D}_{X/S}$-module et les fonctions Ψ et Φ ne dépendent que de ce $\mathcal{D}_{X/S}$-module. Si on part d'un $\mathcal{O}_X(*Z)$-module cohérent muni d'une structure de $\mathcal{D}_{X/S}$-module à gauche, les fonctions Ψ et Φ sont encore définies de façon évidente et il est facile de fabriquer des exemples où la fonction Ψ n'est pas semi-continue inférieurement à cause des phénomènes bien connus de confluence. Mais la fonction Φ est toujours semi-continue inférieurement comme l'a montré Deligne dans une lettre à N. Katz datée du 1.12.1976.

Conjecture 4.2.1. (Malgrange). La fonction Ψ associée à un $\mathcal{D}_X$-module holonome lisse sur U est *semi-continue inférieurement.*

Autrement dit, dans le cas d'une famille d'équations différentielles provenant d'un $\mathcal{D}_X$-module holonome lisse sur U il ne se produit pas de confluences. Le théorème 4.1.1. fournit un support substantiel à cette conjecture. Cependant les méthodes de cet article ne semblent pas adaptées pour démontrer cette conjecture et qu'il faille plutôt connaître la structure des connexions méromorphes irrégulières formelles le long d'une courbe sur une surface complexe non singulière.

5. Le théorème des coefficients de de Rham-Grothendieck en caractéristique nulle

Nous allons montrer dans ce § comment le théorème de positivité remplace avantageusement le théorème de résolution des singularités d'Hironaka [H] dans la démonstration du théorème de comparaison entre cohomologies de de Rham d'une variété algébrique complexe [G_1] et plus généralement dans la démonstration du théorème des coefficients de de Rham-Grothendieck [G_2]. C'est là un exemple très important où la résolution des singularités n'est pas au fond du problème, contrairement à une opinion généralement bien admise jusqu'alors [G_3]. Pour les démonstrations des résultats de ce § nous renvoyons le lecteur à l'article [Me_5] consacré systématiquement à la situation de singularités régulières et au théorème d'existence de Riemann du point de vue du théorème de positivité. Pour alléger l'exposé dans ce § qui est surtout destiné à informer le lecteur on se limite au cas algébrique, mais il est clair que les énoncés valent aussi bien dans le cas analytique.

5.1. Soient X une variété algébrique complexe non singulière, Z une hypersurface de X et $\mathcal{M}$ de $\mathcal{D}_X$-module holonome. Notons Y le support de $\mathcal{M}$ et supposons que la trace de Z sur Y est une hypersurface.

Théorème 5.1.1. *Etant donné un triplet* $(X, Z, \mathcal{M})$ *comme ci-dessus tel que* $Y - Z$ *soit lisse et que* $\mathcal{M}$ *soit lisse au dessus de* $Y - Z$, *alors le faisceau* $\mathbb{R}_Z(\mathcal{M})$ *est* nul, *si et seulement si la codimension dans* $Z \cap Y$ *de son support est au moins égale à un.*

Preuve de 5.1.1. Voir [Me_5] §3.

Corollaire 5.1.2. (Grothendieck) [G_1]. *Si* X *est une variété algébrique* *complexe non-singulière, les morphismes canoniques*

$$(*) \qquad H^{\cdot}_{DR}(X) \to H^{\cdot}_{DR}(X^{\mathrm{an}})$$

entre cohomologie de de Rham $H^{\cdot}_{DR}(X) := \boldsymbol{H}^{\cdot}(X; \boldsymbol{DR}(\mathcal{O}_X))$ *de* X *et cohomologie de de Rham* $H^{\cdot}_{DR}(X^{\mathrm{an}}) := \boldsymbol{H}^{\cdot}(X^{\mathrm{an}}; \boldsymbol{DR}(\mathcal{O}_{X^{\mathrm{an}}})) \cong H^{\cdot}(X^{\mathrm{an}}; \boldsymbol{C}_{X^{\mathrm{an}}})$ *de la variété transcendante* X^{an} *associée à* X *sont des isomorphismes.*

Preuve de 5.1.2. La suite spectrale de de Rham-Cech nous réduit à supposer que X est affine [G₁]. Si $\overline{X}$ désigne l'adhérence de X dans un espace projectif P^m l'obstruction aux isomorphismes $(*)$ est égale, en vertu du théorème GAGA de Serre, à la cohomologie de $Z := \overline{X} - X$ à valeurs dans le faisceau $\boldsymbol{IR}_Z(H^p_{\overline{X}}(\mathcal{O}_{P^m}))$ où $p = m - \dim(X)$ [G₁]. Mais en vertu du théorème 5.1.1 le faisceau $\boldsymbol{IR}_Z((\mathcal{O}_{P^m}))$ est nul pour toute hypersurface Z de P^m. Ceci entraîne par un argument combinatoire de Mayer-Vietoris que le complexe $\boldsymbol{IR}_Z(\boldsymbol{R}\Gamma_Y(\mathcal{O}_{P^m}))$ est nul pour toute sous-variétés Y et Z de P^m. D'où le corollaire 5.1.2.

Corollaire 5.1.3. (Deligne) [D]. *Si* $\mathcal{E}$ *est un fibré à connexion intégrable sur une variété complexe non singulière* X *tel que les nombres* $\dim_{\boldsymbol{C}} \boldsymbol{IR}_\infty(\pi^*\mathcal{E})$ *soient nuls pour tout point à l'infini* ∞ *de toute courbe non singulière* $\pi : C \to X$ *au-dessus de* X*, alors les morphismes canoniques*

$$(**) \qquad H^{\cdot}_{DR}(X, \mathcal{E}) \to H^{\cdot}_{DR}(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$$

entre cohomologie de de Rham $H^{\cdot}_{DR}(X, \mathcal{E}) := \boldsymbol{H}^{\cdot}(X; \boldsymbol{DR}(\mathcal{E}))$ *de* X *à valeurs dans* $\mathcal{E}$ *et cohomologie de de Rham* $H^{\cdot}_{DR}(X^{\mathrm{an}}; \mathcal{E}^{\mathrm{an}}) := \boldsymbol{H}^{\cdot}(X^{\mathrm{an}}; \boldsymbol{DR}(\mathcal{E}^{\mathrm{an}}))$ *de la variété transcendante* X^{an} *associée à* X *à valeurs dans* $\mathcal{E}^{\mathrm{an}}$ *sont des* isomorphismes.

Preuve de 5.1.3. Comme dans le théorème 5.1.2 on peut supposer que X est affine. Soit $\overline{X}$ l'adhérence de X dans un espace projectif P^m. Quitte à prendre la normalisation projective, bien que cela ne soit pas indispensable [Me₅], on peut supposer que $\overline{X}$ est normale. Soit $\overline{\mathcal{E}}$ l'image directe au sens des $\mathcal{D}_X$-modules de $\mathcal{E}$ par l'immersion canonique de X dans P^m. C'est alors un $\mathcal{D}_{P^m}$-module holonome à support dans $\overline{X}$. Alors en vertu du théorème GAGA de Serre, l'obstruction aux isomorphismes $(**)$ est égale à la cohomologie de $Z := \overline{X} - X$ à valeurs dans $\boldsymbol{IR}_Z(\overline{\mathcal{E}})$. En faisant passer une courbe par un point assez général de Z on voit que ce faisceau est nul en vertu du théorème 5.1.1. D'où le corollaire 5.1.3.

Théorème 5.1.4. *Soit* $\mathcal{M}$ *un* $\mathcal{D}_X$*-module holonome sur une variété affine complexe non singulière* X*. Alors son faisceau d'irrégularité le long* *de tout diviseur, y compris ceux de l'infini, est nul si et seulement si son image inverse sur toute courbe non singulière au-dessus de X n'a que des singularités régulières, y compris à l'infini.*

Preuve de 5.1.4. Voir [Me_5] §3.

Ceci amène à poser la definition :

Définition 5.1.5. On dit qu'un $\mathcal{D}_X$-module holonome sur une variété algébrique non singulière sur un corps de caractéristique nulle est *régulier* si son image inverse sur toute courbe non singulière au-dessus de X n'a que des singularités régulières, resp. son cycle d'irrégularité le long de tout diviseur d'un ouvert affine de X est nul.

On note $\mathrm{Mh}r(\mathcal{D}_X)$ la sous catégorie de $\mathrm{Mh}(\mathcal{D}_X)$ des $\mathcal{D}_X$-modules réguliers et $D^b_{hr}(\mathcal{D}_X)$ la sous catégorie triangulée $D^b_h(\mathcal{D}_X)$ des complexes dont la cohomologie est dans $\mathrm{Mh}r(\mathcal{D}_X)$.

Corollaire 5.1.6. *Soient $\mathcal{M}_i$ $(i = 1, 2)$ des coefficients de la catégorie $D^b_{hr}(\mathcal{D}_X)$ pour une variété algébrique complexe non singulière X. Alors le morphisme canonique*

$$(***)\qquad \boldsymbol{R}\mathrm{Hom}_{\mathcal{D}_X}(X;\mathcal{M}_1,\mathcal{M}_2) \to \boldsymbol{R}\mathrm{Hom}_{\mathcal{D}_{X^{\mathrm{an}}}}(X^{\mathrm{an}};\mathcal{M}_1^{\mathrm{an}},\mathcal{M}_2^{\mathrm{an}})$$

est un isomorphisme.

Preuve de 5.1.6. Voir [Me_5] §3.

Corollaire 5.1.7. *Si Y est une sous-variété d'une variété algébrique complexe non singulière X on a les* isomorphismes *canoniques*

$$\begin{aligned} \boldsymbol{H}^{\cdot}\left(X;\Omega^{\cdot}_{\widehat{X/Y}}\right) &\cong \boldsymbol{E}\mathrm{xt}^{\cdot}_{\mathcal{D}_X}(X;\boldsymbol{R}\Gamma_Y(\mathcal{O}_X),\mathcal{O}_X) \cong \boldsymbol{H}^{\cdot}\left(X^{\mathrm{an}};\Omega^{\cdot}_{\widehat{X^{\mathrm{an}}/Y^{\mathrm{an}}}}\right) \\ &\cong \boldsymbol{E}\mathrm{xt}^{\cdot}_{\mathcal{D}_{X^{\mathrm{an}}}}(X^{\mathrm{an}};\boldsymbol{R}\mathrm{alg}\Gamma_{Y^{\mathrm{an}}}(\mathcal{O}_{X^{\mathrm{an}}}),\mathcal{O}_{X^{\mathrm{an}}}) \cong \boldsymbol{H}^{\cdot}(Y^{\mathrm{an}};\boldsymbol{C}_{Y^{\mathrm{an}}}). \end{aligned}$$

Preuve du corollaire 5.1.7. On a noté bien sûr $\Omega^{\cdot}_{\widehat{X/Y}}$ le complété formel du complexe de de Rham de X le long de Y. Ce corollaire résulte du fait que $\boldsymbol{R}\Gamma_Y(\mathcal{O}_X)$ appartient à $D^b_{hr}(\mathcal{D}_X)$ par le théorème 5.1.1 et du théorème 5.1.6. Ce résultat a été obtenu par plusieurs auteurs (Deligne (1969), Herrera-Libermann,...) à l'aide du théorème d'Hironaka. Remarquons cependant que le corollaire 5.1.7 apparaît dans la théorie des $\mathcal{D}_X$-modules comme équivalent au théorème de comparaison de Grothendieck [G_1] par le théorème de dualité locale pour les $\mathcal{D}_X$-modules holonomes [Me_1]. On obtient donc, puisque la cohomologie du complété formel du complexe de de Rham de X le long de Y est isomorphe à la cohomologie cristalline de Y, une démonstration, *sans faire appel au théorème* *de la résolution des singularités*, de l'isomorphisme entre la cohomologie cristalline d'une variété algébrique complexe et la cohomologie de l'espace localement compact sous-ajacent.

Si f est un morphisme entre variétés algébriques non singulières sur un corps de caractéristique nulle, notons f^* et $f_{c^\bullet}$ les foncteurs image inverse et image directe cristalline (cf. par exemple [Me3]) entre catégories dérivées de $\mathcal{D}_X$-modules. Posons pour tout complexe $\mathcal{M}$ de $\mathcal{D}_X$-modules $f^!(\mathcal{M}) := (f^*(\mathcal{M})^*)^*$ et $f_{c!}(\mathcal{M}) := (f_{c^\bullet}(\mathcal{M})^*)^*$ où $(\ )^*$ désigne le foncteur de dualité de $\mathcal{D}_X$-modules. Il résulte du théorème de dualité pour les complexes de $\mathcal{D}_X$- modules à cohomologie bornée et cohérente que le foncteur $f_{c!}$ est isomorphe au foncteur $f_{c^\bullet}$ si f est propre [Me3]. Les catégories $D^b_h(\mathcal{D}_X)$ sont stables par les opérations f^*, $f^!$, $f_{c^\bullet}$, $f_{c!}$, $\otimes$, $(\ )^*$ et les foncteurs de cycles évanescents Ψ^m_g et Φ^m_g. On déduit [Me5] du théorème 5.1.1 le théorème suivant :

Théorème 5.1.8 (des coefficients de de Rham-Grothendieck). *Les catégories $D^b_{hr}(\mathcal{D}_X)$ sont stables par les opérations cohomologiques f^*, $f^!$, $f_{c^\bullet}$, $f_{c!}$, $\otimes$, $(\ )^*$, Ψ^m_g et Φ^m_g. De plus si le corps de base est le corps des nombres complexes, la restriction* $\boldsymbol{DR}_r$ *du foncteur de de Rham transcendant aux catégories $D^b_{hr}(\mathcal{D}_X)$ est pleinement fidèle et respecte les opérations analogues pour les catégories de coefficients constructibles $D^b_c(\boldsymbol{C}_X)$.*

Prendre seulement garde que le foncteur $\boldsymbol{DR}_r$ transforme bien $f_{c^\bullet}$ en f_*, mais transforme f^* en $f^!$. On n'a pas à proprement parler un formalisme des six opérations de Grothendieck pour les catégories $D^b_{hr}(\mathcal{D}_X)$ analogue à celui des catégories $D^b_c(\boldsymbol{C}_X)$. Ceci a amené Grothendieck à introduire la notion de co-cristal analogue à la notion de $\mathcal{D}_X$-module à droite. Mais on a un formalisme et des compatibilités entre les opérations tout à fait similaires, cf. [Me3].

Le théorème 5.1.8 contient comme cas particuliers plusieurs résultats dont les démonstrations reposaient de façon essentielle sur le théorème d'Hironaka. Ainsi, par exemple, le fait que les catégories $D^b_{hr}(\mathcal{D}_X)$ sont stables par image directe $f_{c^\bullet}$ contient comme cas particulier le théorème de régularité de la connexion de Gauss-Manin [Kat]. Ou encore le fait que le foncteur $\boldsymbol{DR}_r$ commute avec le foncteur Ψ^m_g a pour conséquence que le théorème de monodromie locale d'un germe de fonction holomorphe (cf. [Le]) est rigoureusement équivalent au théorème de rationalité des zéros du polynôme de Bernstein-Sato associé à ce germe [M3] réalisant ainsi un peu mieux l'idée originale de Malgrange. On obtient donc une démonstration du théorème de rationalité des zéros du polynôme de Bernstein-Sato d'un germe de fonction holomorphe sans faire appel au théorème d'Hironaka comme souhaité par Lê [Le]. Il n'est pas fortuit de constater que c'est le même homme qui est à l'origine aussi bien du théorème de comparaison [G1] que du théorème de monodromie ([SGA 7, I] ou l'Appendice de l'article de J. P. Serre, J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492-517). Le théorème 5.1.8 a visiblement été prédit par Grothendieck [G3] page 312, ligne 7 du bas. Nous regardons son énoncé et sa démonstration comme un des succès les plus éclatants de la théorie de Grothendieck, l'une des plus admirables s'il en est, par son unité, sa généralité et son originalité. Une fois aquis le théorème de comparaison on a alors une meilleure approche, beaucoup plus précise, du théorème d'existence de Riemann, proche de celle de Grauert-Rammert pour le théorème analogue pour les revêtements finis, et sans faire appel au théorème d'Hironaka [Me3] §5. Il devient alors intéressant de chercher une démonstration du théorème des coefficients de Hodge-Deligne sans faire appel au théorème d'Hironaka.

A partir du théorème 5.1.8 on déduit [Me2] le théorème analogue pour les coefficients holonomes d'ordre infini, voir ([Me3], Chap. II, thm, 9.5.7) pour l'énoncé complet. Nous rappelons cela parce que ce sont les coefficients holonomes complexes d'ordre *infini* qui doivent servir de modèle aux coefficients p-adiques en caractéristique $p > 0$. En effet il y a une analogie naturelle entre une algèbre de Dwork-Monsky-Washnitzer, complétée faible d'une algèbre localisée le long d'une hypersurface, dans le cas p-adique et une algèbre de fonctions holomorphes ayant des singularités essentielles le long d'une hypersurface dans le cas complexe, cf. [NM2]. Pour cette raison la théorie des modules d'ordre infini a commencée à être développée par P. Berthelot, cf. [B] et, indépendament, par L. Narvaez et l'auteur, cf. ([NM2], [NM3]). Bien que l'on n'a pas encore démontré tout ce que suggère la théorie complexe on commence à avoir de nombreuses indications qui montrent que c'est la bonne théorie de de Rham-Grothendieck en caractéristique $p > 0$. Cette théorie doit fournir les coefficients p-adiques analogues aux coefficients l-adiques ($l \neq p$) et aux coefficients de de Rham-Grothendieck en caractéristique nulle. Soient $W := W(k)$ l'anneau des vecteurs de Witt d'un corps parfait k de caractéristique $p > 0$, $X \to W$ un schéma formel lisse topologiquement de type fini sur W, resp. un schéma formal faible lisse topologiquement de type fini sur W, alors le faisceau $\mathcal{D}^{\dagger}_{X/W}$ des opérateurs différentiels p-adiques d'ordre infini est défini [B], resp. ([NM2], [NM3]), comme le complété de Dwork-Monsky-Washnitzer du faisceau des opérateurs différentiels $\mathcal{D}\mathrm{iff}_{X/W}(\mathcal{O}_X)$ de ([EGA IV], §16), resp. du faisceau défini dans ([NM2], 4.4.3). Le faisceau $\mathcal{D}^{\dagger}_{X/W}$ est l'analogue p-adique du faisceau $\mathcal{D}^{\infty}_{X}$ des opérateurs différentiels d'ordre infini dans le cas complexe. Mais comme toujours la situation est bien meilleure en géométrie algébrique qu'en géométrie analytique. En effet P. Berthelot [B] a démontré que le faisceau $\mathcal{D}^{\dagger}_{X/W}$ est limite inductive de ses sous-faisceaux des opérateurs qu'il apelle d'échelon $j (j \in \boldsymbol{N})$ qui sont noethériens et cohérents. Il en déduit que le faisceau $\mathcal{D}^\dagger_{X/W} \otimes \boldsymbol{Q}$ est un faisceau *cohérent* d'anneaux [B]. A partir de là on peut définir, cf. [NM3], la catégorie des $\mathcal{D}^\dagger_{X/W} \otimes \boldsymbol{Q}$-modules holonomes qui doit être la catégorie de base pour une théorie des coefficients p-adiques réalisant ainsi l'idée de Grothendieck ([G3], 7.3, p.355) du topos de Monsky-Washnitzer. Cependant pour l'instant on n'a pas encore démontré le point clef de cette théorie à savoir qu'une sous-catégorie convenable de $\mathcal{D}^\dagger_{X/W} \otimes \boldsymbol{Q}$-modules holonomes est stable par image directe par une *immersion ouverte* d'où doit résulter que ces catégories des coefficients holonomes p-adiques sont stables par les six opérations de Grothendieck, en particulier le théorème de finitude de la cohomologie de Dwork-Monsky-Washnitzer et de la cohomologie rigide de Berthelot. L'analogue complexe pour les coefficients holonomes d'ordre infini [Me2] repose dans l'état actuel de nos connaissances sur le théorème d'existence de Riemann. Mais là encore on peut penser que la situation dans la cas p-adique est plus favorable et doit être intermédiaire entre le cas algébrique de caractéristique nulle et le cas analytique complexe bien que la théorie du polynôme de Bernstein-Sato pour les algébres de Dwork-Monsky-Washnitzer est insuffisante [NM2]. Mais il y a lieu d'être optimiste d'autant plus que l'hypothèque du théorème de résolution des singularités est levée en caractéristique nulle. Le lecteur trouvera de plus amples renseignements sur cette théorie dans ([B], [NM3]).

6. Filtration du faisceau d'irrégularité

Soient X une variété algébrique complexe non singulière, Z une hypersurface de X et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome. Nous allons définir des filtrations des faisceaux $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^\vee(\mathcal{M})$ par des sous-faisceaux indexés par les nombres réels r compris entre 1 et ∞, mais dont les sauts locaux sont des nombres rationnels. La situation est toute analogue à celle du théorème de Hasse-Arf pour la filtration (numérotation supérieure) du groupe de Galois d'une extension galoisienne finie d'un corps local [Se1]. De plus ces filtrations sont purement algébriques au niveau du cycle caractéristique et gardent un sens sur tout corps de caractéristique nulle. Là encore comme dans le §4 la situation doit être considérée comme provisoire. Il nous faut pour cela généraliser le théorème de positivité. On utilise la théorie de Y. Laurent ([La1], [La2], [La3], [La4]). Les ingrédients sont le théorème de constructibilité et le théorème de dualité locale. Pour les résultats sur les espaces vectoriels topologiques et en particulier sur les espaces nucléaires nous renvoyons le lecteur aux articles de Grothendieck ([G4], [G5], [G6], [G7], [G8]). Il y trouvera toutes les démonstrations qui peuvent lui manquer.

6.1. Soient X une variété analytique complexe et Z une sous-variété lisse de X de codimension p. On a alors les deux suites exactes de $\mathcal{D}_X$-modules à gauche

$$0 \to \mathcal{O}_{X/Z} \to \mathcal{O}_{\widehat{X/Z}} \to \mathcal{Q}_Z \to 0$$

$$0 \to \mathrm{alg}H^p_Z(\mathcal{O}_X) \to H^p_Z(\mathcal{O}_X) \to \mathcal{L}_Z \to 0.$$

Si $(x,z) = (x_1, \ldots, x_p, z_1, \ldots, z_{n-p})$ est un système de coordonnées locales tel que Z soit définie par $x_1 = \ldots x_p = 0$, un germe au point $x = z = 0$ du faisceau $\mathcal{O}_{\widehat{X/Z}}$, resp. du faisceau $H^p_Z(\mathcal{O}_X)$, est représenté par une série $\sum_{\alpha \in N^p} a_\alpha x^\alpha$, resp. $\sum_{\beta \in N^p} b_\beta x^{-\beta-1}$, où a_α, resp. b_β, est une suite de fonctions holomorphes définies sur le même ouvert U de Z, resp. telle que pour tout compact K de U, pour tout ϵ positif, il existe une constante positive ou nulle K_ϵ telle que $\sup_K |b_\beta| \leq K_\epsilon(\beta!)^{-1}\epsilon^{|\beta|}$. On a noté $\beta+1$ pour $\beta+(1,\ldots,1)$ appartenant à $\boldsymbol{N}^p$, $\beta!$ pour $\beta_1!\ldots\beta_p!$ et $|\beta|$ pour $\beta_1 + \ldots \beta_r$. Pour tout réel r supérieur ou égal à 1, on définit le sous-faisceau des germes de fonctions, resp. des ultradistributions, de classe de Gevrey d'ordre r, $\mathcal{O}_{X/Z}\{r\}$, resp $\mathcal{B}_{Z/X}\{r\}$, du faisceau $\mathcal{O}_{\widehat{X/Z}}$, resp. du faisceau $H^p_Z(\mathcal{O}_X)$, comme le faisceau des germes des séries $\sum_{\alpha \in N^p} a_\alpha x^\alpha$, resp. $\sum_{\beta \in N^p} b_\beta x^{-\beta-1}$, telle que la série $\sum_{\alpha \in N^p} a_\alpha(\alpha!)^{1-r}x^\alpha$ converge, resp. pour tout compact K de U et tout ϵ positif il existe un constante K_ϵ telle que $\sup_K |b_\beta| \leq K_\epsilon(\beta!)^{-r}\epsilon^{|\beta|}$. On a alors $\mathcal{O}_{X/Z}\{1\} = \mathcal{O}_{X/Z}$ et $\mathcal{B}_{Z/X}\{1\} = H^p_Z(\mathcal{O}_X)$ et on pose $\mathcal{O}_{X/Z}\{\infty\} := \mathcal{O}_{\widehat{X/Z}}$ et $\mathcal{B}_{Z/X}\{\infty\} := \mathrm{alg}H^p_Z(\mathcal{O}_X)$. La filtration $\mathcal{O}_{X/Z}\{r\}$, resp. $\mathcal{B}_{Z/X}\{r\}$, est une filtration croissante, resp. décroissante, de $\mathcal{D}_X$-modules à gauche de $\mathcal{O}_{\widehat{X/Z}}$, resp. de $H^p_Z(\mathcal{O}_X)$, cf. ([La$_1$], [La$_2$], [La$_3$], [La$_4$]).

Pour tout r, $1 \leq r < \infty$, on a les isomorphismes de faisceaux *d'espaces* vectoriels complexes

$$\mathcal{O}_{X/Z}\{r\} \to \mathcal{O}_{X/Z}\{1\}$$
$$\mathcal{B}_{Z/X}\{1\} \to \mathcal{B}_{Z/X}\{r\}$$

qui à la série $\sum_{\alpha \in N^p} a_\alpha x^\alpha$, resp. $\sum_{\beta \in N^p} b_\beta x^{-\beta-1}$, associe la série convergente $\sum_{\alpha \in N^p} a_\alpha(\alpha!)^{1-r}x^\alpha$, resp. la série $\sum_{\beta \in N^p} b_\beta(\beta!)^{1-r}x^{-\beta-1}$.

Pour tout polycylindre K, resp. tout ouvert de Stein V, assez petit de Z l'espace $\Gamma(K;\mathcal{O}_{X/Z}\{1\})$ resp. $\Gamma(V;\mathcal{B}_{Z/X}\{1\})$, est un espace vectoriel topologique de type ***DFN*** (dual de Fréchet-nucléaire), resp. ***FN*** (Fréchet-nucléaire) cf. ([G$_4$], [G$_5$], [G$_6$], [G$_7$], [G$_8$]). Choississons des équations locales de Z on a alors les isomorphsimes d'espaces vectoriels topologiques qui *commutent* à l'action de $\Gamma(K;\mathcal{D}_{X/Z})$, resp. $\Gamma(V;\mathcal{D}_{X/Z})$,

$$\Gamma(K;\mathcal{O}_Z)\widehat{\otimes}\mathcal{O}_{C^p/0}\{1\} \to \Gamma\left(K;\mathcal{O}_{X/Z}\{1\}\right)$$
$$\Gamma(V;\mathcal{O}_Z)\widehat{\otimes}\mathcal{B}_{0/C^p}\{1\} \to \Gamma\left(V;\mathcal{B}_{Z/X}\{1\}\right).$$

Le produit tensoriel topologique complété est défini sans ambiguité puisque on a affaire à des espaces nucléaires ([G4], [G5]). On en déduit, pour tout r, $1 \leq r < \infty$, un diagramme commutatif d'isomorphismes où les isomorphismes verticaux ne sont pas compatibles à l'action de $\Gamma(K;\mathcal{D}_{X/Z})$, resp. $\Gamma(V;\mathcal{D}_{X/Z})$, mais où les isomorphismes horizontaux sont compatibles à l'action de $\Gamma(K;\mathcal{D}_{X/Z})$, resp. de $\Gamma(V;\mathcal{D}_{X/Z})$,

$$\begin{array}{ccc} \Gamma(K;\mathcal{O}_Z)\widehat{\otimes}\mathcal{O}_{C^p/0}\{r\} & \longrightarrow & \Gamma(K;\mathcal{O}_{X/Z}\{r\}) \\ \downarrow & & \downarrow \\ \Gamma(K;\mathcal{O}_Z)\widehat{\otimes}\mathcal{O}_{C^p/0}\{1\} & \longrightarrow & \Gamma(K;\mathcal{O}_{X/Z}\{1\}) \end{array}$$

resp.

$$\begin{array}{ccc} \Gamma(V;\mathcal{O}_Z)\widehat{\otimes}\mathcal{B}_{0/C^p}\{1\} & \longrightarrow & \Gamma(V;\mathcal{B}_{Z/X}\{1\}) \\ \downarrow & & \downarrow \\ \Gamma(V;\mathcal{O}_Z)\widehat{\otimes}\mathcal{B}_{0/C^p}\{r\} & \longrightarrow & \Gamma(V;\mathcal{B}_{Z/X}\{r\}) . \end{array}$$

D'où une topologie ***DFN***, resp. ***FN***, sur $\Gamma(K;\mathcal{O}_{X/Z}\{r\})$, resp. $\Gamma(V;\mathcal{B}_{Z/X}\{r\})$, pour tout r, $1 \leq r < \infty$. De plus l'espace $\Gamma(K;\mathcal{O}_{X/Z}\{r\})$, resp. $\Gamma(V;\mathcal{B}_{Z/X}\{r\})$ s'interprète comme l'espace des fonctions holomorphes au voisinage de K, resp. sur V, à valeurs dans $\mathcal{O}_{C^p/0}\{r\}$, resp. dans $\mathcal{B}_{0/C^p}\{r\}$ [G6].

Considérons le complexe de Dolbeaut $\mathcal{D}'^{0,\cdot}(\mathcal{O}_{C^p/0}\{r\})$, resp. $\mathcal{E}^{0,\cdot}(\mathcal{B}_{0/C^p}\{r\})$, des courants, resp. des formes différentielles indéfiniment différentiables, sur Z à valeurs dans $\mathcal{O}_{C^p/0}\{r\}$, resp. dans $\mathcal{B}_{0/C^p}\{r\}$ ([Sc1], I). Pour tout compact K, resp. tout ouvert V, de Z on a les isomorphismes d'espaces vectoriels topologiques ([Sc1], I) compatibles à l'action des opérateurs différentiels holomorphes

$$\Gamma\left(K;\mathcal{D}'^{0,\cdot}\right)\widehat{\otimes}\mathcal{O}_{C^p/0}\{r\} \to \Gamma\left(K;\mathcal{D}'^{0,\cdot}\left(\mathcal{O}_{C^p/0}\{r\}\right)\right)$$

resp.

$$\Gamma\left(V;\mathcal{E}^{0,\cdot}\right)\widehat{\otimes}\mathcal{B}_{0/C^p}\{r\} \to \Gamma\left(V;\mathcal{E}^{0,\cdot}\left(\mathcal{B}_{0/C^p}\{r\}\right)\right) .$$

Rappelons que si $F^\cdot$ est un complexe acyclique d'espaces de type ***F*** dont les différentielles sont continues (donc des homomorphismes) et F est un espace de type ***FN*** alors le complexe $F^\cdot\widehat{\otimes}F$ est acyclique. Ceci est un cas particulier de la formule de Künneth topologique [Sc2], voir aussi [Do]. On en déduit que si $E^\cdot$ est un complexe acyclique d'espaces de type ***DF*** réfléxifs dont les différentielles sont continues et E un espace de type ***DFN*** réfléxif le complexe $E^\cdot\widehat{\otimes}E$ est acyclique. En effet le complexe dual fort $E^{\cdot\prime}$ est un complexe d'espaces de type ***F*** dont les différentielles sont donc des homomorphismes [G8]. Par dualité ce complexe est acyclique [Se2]. Donc le complexe $E^{\cdot\prime}\hat{\otimes}E'$ est acyclique et son complexe dual qui est $E^{\cdot}\hat{\otimes}E$ ([G4], [G5]) est acyclique. Appliquons ceci, toutes les hypothèses précédentes sont requises, on trouve que si K, resp. V, est un polycylindre, resp. un ouvert de Stein, assez petit de Z le complexe $\Gamma(K;D'^{0,\cdot}(\mathcal{O}_{C^p/0}\{r\}))$, resp. $\Gamma(V;\mathcal{E}^{0,\cdot}(\mathcal{B}_{0/C^p}\{r\}))$, est une résolution de l'espace $\Gamma(K;\mathcal{O}_{X/Z}\{r\})$, resp $\Gamma(V;\mathcal{B}_{Z/X}\{r\})$ en vertu du théorème B de Cartan. Par conséquent au dessus d'un ouvert assez petit de Z le complexe $\mathcal{D}'^{0,\cdot}(\mathcal{O}_{C^p/0}\{r\})$, resp. $\mathcal{E}^{0,\cdot}(\mathcal{B}_{0/C^p}\{r\})$, est une résolution $\mathcal{D}_{X/Z}$-linéaire du faisceau $\mathcal{O}_{X/Z}\{r\}$, resp $\mathcal{B}_{Z/X}\{r\}$, pour un choix d'équations locales de Z. Nous avons besoin de la généralisation suivante du théorème de dualité de Serre [Se2] :

Théorème 6.1.1. (i) *Pour tout r, $1 \leq r < \infty$, et tout ouvert de Stein V assez petit de X l'espace $H^i_c(V;\mathcal{O}_{X/Z}\{r\})$ est nul si i est différent de* $\dim(Z)$*; l'espace $H^i(V;\mathcal{B}^n_{Z/X}\{r\})$ est nul si i est différent de* 0.

(ii) *Il existe un unique (à isomorphisme prés) couple de toplogie* ***FN−DFN*** *sur $\Gamma(V;\mathcal{B}^n_{Z/X}\{r\}) - H^{n-p}_c(V;\mathcal{O}_{X/Z}\{r\})$ est un accouplement compatible à l'action de $\Gamma(V;\mathcal{D}_X)$ qui est une dualité parfaite entres ces espaces c'est à dire qu'il identifie un espace au dual fort de l'autre.*

Preuve de (i). On a posé $\mathcal{B}^n_{Z/X}\{r\} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{B}_{Z/X}\{r\}$. Les composantes du complexe $\mathcal{E}^{0,\cdot}(\mathcal{B}_{0/C^n}\{r\}))$ sont acycliques pour le foncteur sections globales. En choisissant des équations locales de Z on trouve en vertu de ce qui précède et du théorème B de Cartan que $H^i(V;\mathcal{B}^n_{Z/X}\{r\})$ est nul si i est non nul et l'isomorphisme de $\Gamma(V;\mathcal{D}_X)$-modules à droite

$$\Gamma(V\cap Z;\omega_Z)\hat{\otimes}\mathcal{B}^p_{0/C^p}\{r\} \to \Gamma\left(V\cap Z;\mathcal{B}^n_{Z/X}\{r\}\right).$$

Les composantes du complexes $\mathcal{D}'^{0,\cdot}(\mathcal{O}_{C^p/0}\{r\})$ sont acycliques pour le foncteur sections à support compact. Mais l'espace $\mathcal{O}_{C^p/0}\{r\}$ étant précisément de type ***DF*** on les isomorphismes ([Sc1], Chap. I §3 p. 62)

$$\Gamma_c\left(V\cap Z;\mathcal{D}'^{0,\cdot}\right)\hat{\otimes}\mathcal{O}_{C^p/0}\{r\} \to \Gamma_c\left(V\cap Z;\mathcal{D}'^{0,\cdot}\left(\mathcal{O}_{C^p/0}\{r\}\right)\right).$$

On trouve en vertu de ce qui précède et du théorème de dualité de Serre [Se2] que l'espace $H^i_c(V;\mathcal{O}_{X/Z}\{r\})$ est nul si i est différent de $n-p$ et l'isomorphisme de $\Gamma(V;\mathcal{D}_X)$-modules à gauche

$$H^{n-p}_c(V\cap Z;\mathcal{O}_Z)\hat{\otimes}\mathcal{O}_{C^p/0}\{r\} \to H^{n-p}_c\left(V;\mathcal{O}_{X/Z}\{r\}\right).$$

Preuve de (ii). Pour un choix d'équations locales de Z le couple $\Gamma(V;\mathcal{B}^n_{Z/X}\{r\}) - H^{n-p}_c(V;\mathcal{O}_{X/Z}\{r\})$ est isomorphe au couple d'espaces ***FN−DFN***, $\Gamma(V\cap Z;\omega_Z)\hat{\otimes}\mathcal{B}^p_{0/C^p}\{r\} - H^{n-p}_c(V\cap Z;\mathcal{O}_Z)\hat{\otimes}\mathcal{O}_{C^p/0}\{r\}$. Le couple d'espaces ***FN − DFN*** $\Gamma(V\cap Z;\omega_Z) - H_c^{n-p}(V\cap Z;\mathcal{O}_Z)$ est muni d'un accouplement qui est une dualité parfaite [Se2]. Le couple d'espaces ***FN–DFN*** $\mathcal{B}^p_{0/C^p}\{r\}-\mathcal{O}_{\mathbf{C}^p/0}\{r\}$ est muni d'un accouplement residu qui est compatible avec l'accouplement residu du couple isomorphe $\mathcal{B}^p_{0/C^p}\{1\} - \mathcal{O}_{\mathbf{C}^p/0}\{1\}$. Supposons que $p \geq 2$ (le cas $p = 1$ se traite de la même façon) alors le couple $\mathcal{B}^p_{0/C^p}\{1\} - \mathcal{O}_{C^p/0}\{1\}$ est isomorphe au couple $H^{p-1}(C^p - 0;\omega_{C^p}) - H^1_c(C^p - 0;\mathcal{O}_Z)$. Il en résulte en vertu du théorème de dualité de Serre que l'accouplement residu qui est compatible à l'action des opérateurs différentiels est une dualité parfaite entre le couple ***FN − DFN*** ,$\mathcal{B}^p_{0/C^p}\{r\} - \mathcal{O}_{C^p/0}\{r\}$. En vertu de ([G4], [G5]) le couple d'espaces ***FN − DFN*** ,$\Gamma(V\cap Z;\omega_Z)\widehat{\otimes}\mathcal{B}^p_{0/C^p}\{r\} - H_c^{n-p}(V\cap Z;\mathcal{O}_Z)\widehat{\otimes}\mathcal{O}_{C^p/0}\{r\}$ est en dualité parfaite. Il en résulte des topologies ***FN − DFN*** sur le couple $\Gamma(V;\mathcal{B}^n_{Z/X}\{r\} - H_c^{n-p}(V;\mathcal{O}_{X/Z}\{r\})$ et une dualité parfaite compatible à l'action de $\Gamma(V;\mathcal{D}_X)$. La seule chose qui reste à voir est que cet accouplement ne dépend pas des équations locales de Z. Mais l'on a un morphisme naturel

$$\mathcal{O}_{X/Z}\{r\} \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^n_{Z/X}\{r\} \to \mathcal{B}^n_{Z/X}\{r\}$$

d'où un morphisme par cup-produit

$$H_c^{n-p}\left(V;\mathcal{O}_{X/Z}\{r\}\right)\otimes\Gamma\left(V;\mathcal{B}^n_{Z/X}\{r\}\right) \to H_c^{n-p}\left(V;\mathcal{B}^n_{Z/X}\{r\}\right).$$

Ce morphisme suivi du morphisme résidu $H_c^{n-p}(V;\mathcal{B}^n_{Z/X}\{r\}) \to H^n_c(V;\omega_X) \to \boldsymbol{C}$ coincide avec l'accouplement précédent une fois restreint au couple $\Gamma(V\cap Z;\omega_Z)\otimes B^p_{0/C^p}\{r\} - H_c^{n-p}(V\cap Z;\mathcal{O}_Z)\otimes\mathcal{O}_{C^p/0}\{r\}$. Il suffit donc de démontrer que ce morphisme est *continu*. Mais on a un diagramme commutatif pour V assez petit

$$\begin{tikzcd}
\mathcal{O}_{V/Z}\{r\} \underset{\mathcal{O}_V}{\otimes} \mathcal{B}^n_{Z/X}\{r\} \arrow[r] \arrow[d] & \mathcal{B}^n_{Z/X}\{r\} \arrow[d] \\
\mathcal{D}'^{0,\cdot}\left(\mathcal{O}_{C^p/0}\{r\}\right) \underset{\mathcal{O}_V}{\otimes} \mathcal{E}^{n-p,\cdot}\left(\mathcal{B}^p_{0/C^p}\{r\}\right) \arrow[r] & \mathcal{D}'^{n-p,\cdot}\left(\mathcal{B}^p_{0/C^p}\{r\}\right)
\end{tikzcd}$$

où le deuxième morphisme horizontal est le produit *croisé* ([Sc1], Chap. II §2) du morphisme $\mathcal{D}'^{0,\cdot}\otimes_{\mathcal{O}_{V'}}\mathcal{E}^{n-p,\cdot} \to \mathcal{D}'^{n-p,\cdot}$ au dessus d'un petit voisinage V' de l'origine dans C^{n-p} et de l'application bilinéaire continue $\mathcal{O}_{C^p/0}\{r\}\times\mathcal{B}^p_{0/C^p}\{r\} \to \mathcal{B}^p_{0/C^p}\{r\}$. En vertu de ([Go], Chap. II, thm. 6.6.1) le cup produit

$$H_c^{n-p}\left(V;\mathcal{O}_{X/Z}\{r\}\right)\otimes\Gamma\left(V;\mathcal{B}^n_{Z/X}\{r\}\right) \to H_c^{n-p}\left(V;\mathcal{B}^n_{Z/X}\{r\}\right)$$

est représenté par le morphisme de complexe

$$\Gamma_c\left(V \cap Z; \mathcal{D}'^{0,\cdot}\left(\mathcal{O}_{C^p/0}\{r\}\right)\right) \otimes \Gamma\left(V \cap Z; \mathcal{E}^{n-p,\cdot}\left(\mathcal{B}^p_{0/C^p}\{r\}\right)\right)$$
$$\rightarrow \Gamma_c\left(V \cap Z; \mathcal{D}'^{n-p,\cdot}\left(\mathcal{B}^p_{0/C^p}\{r\}\right)\right).$$

En vertu de ([Sc$_1$], Chap. II, §2, cor. de la prop. 3 p. 38) le morphisme entre ces deux complexes est formé d'applications bilinéaires *continues*. Par ailleurs le morphisme, pour un petit voisinage de l'origine V'' dans C^p, $(\mathcal{B}^p_{0/C^p}\{r\}) \rightarrow H^p_c(V''; \omega_{C^p})$ induit un morphisme continu $\Gamma_c(V \cap Z; \mathcal{D}'^{n-p,\cdot}(\mathcal{B}^p_{0/C^p}\{r\}) \rightarrow \Gamma_c(V \cap Z; \mathcal{D}'^{n-p,\cdot}(H^p_c(V''; \omega_{C^p}))$. Comme $H^p_c(V''; \omega_{C^p})$ est un espace ***DFN*** la cohomologie du dernier complexe est isomorphe à $H^{n-p}_c(V'; \omega^{n-p}) \widehat{\otimes} H^p_c(V'; \omega_{C^p}) \cong H^n_c(V; \omega_X)$. D'où un morphisme continu

$$H^{n-p}_c\left(V; \mathcal{O}_{X/Z}\{r\} \otimes \Gamma\left(V; \mathcal{B}^n_{Z/X}\{r\}\right)\right) \rightarrow H^n_c\left(V; \omega_X\right),$$

qui suivi du morphisme résidu qui est continu est l'accouplement obtenu par cup produit entre le couple $H^{n-p}_c(V; \mathcal{O}_{X/Z}\{r\}) - \Gamma(V; \mathcal{B}^n_{Z/X}\{r\})$ qui est donc continu.

Remarque 6.1.2. Pour tout r, $1 \leq r < \infty$ l'accouplement entre le couple $H^{n-p}_c(V; \mathcal{O}_{X/Z}\{r\}) - \Gamma(V; \mathcal{B}^n_{Z/X}\{r\})$ est *compatible* avec l'accouplement du couple qui lui est isomorphe $H^{n-p}_c(V; \mathcal{O}_{X/Z}\{1\}) - \Gamma(V; \mathcal{B}^n_{Z/X}\{1\})$.

Remarque 6.1.3. En prenant un recouvrement de X par des ouverts de Stein assez petit on déduit de 6.1.1 le théorème analogue global.

6.2. Supposons que Z est une sous-variété lisse de codimension p d'une variété analytique complexe X. Le théorème suivant est dû à Ramis [R] en dimension un et à Laurent ([La$_1$], [La$_2$]) en dimension supérieure.

Théorème 6.2.1. ([La$_1$], [La$_2$]). *Pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$, le complexe $\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Z/X}\{r\})$ est constructible pour tout r. De plus pour tout ϵ positif assez petit le morphisme canonique*

$$\mathbf{R}hom_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{B}_{Z/X}\{r+\epsilon\}\right) \rightarrow \mathbf{R}hom_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{B}_{Z/X}\{r\}\right)$$

est localement un isomorphisme *pour tout r, sauf pour un nombre* fini *d'entre eux qui sont les pentes critiques locales de $\mathcal{M}$, le long de Z et qui sont des nombres* rationnels.

Théorème 6.2.2. *Pour tout complexe $\mathcal{M}$ de la catégorie $D^b_h(\mathcal{D}_X)$, et tout nombre réel r, $1 \leq r \leq \infty$, on a un morphisme canonique de la catégorie $D^b(\boldsymbol{C}_X)$*

$$\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r\}) \rightarrow \mathbf{R}hom_{\boldsymbol{C}_X}(\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{r\}), \boldsymbol{C}_X)\,[+p]$$

qui est un isomorphisme.

Si $r = 1, \infty$, le théorème 6.2.2 se réduit au théorème de dualité locale 1.5.1 qui correspond au cas $p = 0$. Pour $1 < r < \infty$ nour allons démontrer le théorème 6.2.2 en reprenant les arguments de la démonstration du théorème de dualité locale [Me₁].

Lemme 6.2.3. (i) *Pour tout* $1 \leq r < \infty$ *le faisceau* $\mathcal{E}xt^i_{\boldsymbol{C}}(\mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X)$ *est nul si* $i \neq \dim(X) + p$.

(ii) *On a un morphisme canonique de* $\mathcal{D}_X$*-modules à droite*

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_{Z/X}\{r\} \to \mathcal{E}xt^{n+p}_{\boldsymbol{C}}\left(\mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X\right) \quad \text{où} \quad n := \dim(X).$$

Preuve de 6.2.3 (i). Pour tout r, $1 \leq r < \infty$, le faisceau $\mathcal{E}xt^i_{\boldsymbol{C}}$ $(\mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X)$ est associé au préfaisceau $V \to \mathrm{Ext}^i_{\boldsymbol{C}}(V; \mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X)$. Il suffit donc de montrer que pour tout ouvert de Stein V assez petit l'espace vectoriel complexe $\mathrm{Ext}^i_{\boldsymbol{C}}(V; \mathcal{O}_{X/Z}\{1\}, \boldsymbol{C}_X)$ est nul si i est différent de $n+p$. Mais en vertu du théorème de dualité de Poincaré pour les coefficients généraux sur les espaces localement compacts [V] on a les isomorphismes de dualité

$$\mathrm{Hom}_{\boldsymbol{C}}\left(H^i_c X, \mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}\right) \approx \mathrm{Ext}^{2n-i}_{\boldsymbol{C}}\left(V; \mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X\right).$$

Mais en vertu de 6.1.1 (i) l'espace vectoriel $H^i_c(V, \mathcal{O}_{V/Z}\{r\})$ est nul si i est différent de $n - p$. D'où 6.2.3.(i).

Preuve de 6.2.3 (ii). Il résulte de 6.1.1 (ii) que si V est de Stein assez petit l'espace $H^{n-p}_c(V - Z, \mathcal{O}_{V/Z}\{r\})$ est de type ***DFN*** dont le dual topologique est l'espace de type ***FN*** $\Gamma(V, \mathcal{B}^n_{Z/X}\{r\})$. D'où un morphisme canonique

$$\Gamma\left(V, \mathcal{B}^n_{Z/X}\{r\}\right) \to \mathrm{Ext}^{n+p}_{\boldsymbol{C}}\left(V; \mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X\right)$$

qui est $\Gamma(V, \mathcal{D}_V)$-linéaire. En passant aux faisceaux associés on trouve un morphisme de $\mathcal{D}_X$-modules à droite

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_{Z/X}\{r\} \to \mathcal{E}xt^{n+p}_{\boldsymbol{C}}\left(\mathcal{O}_{X/Z}\{r\}, \boldsymbol{C}_X\right).$$

D'où 6.2.3.(ii).

Preuve du théorème 6.2.2. Pour tout complexe $\mathcal{M}$ de $\mathcal{D}_X$-modules à cohomologie bornée et cohérente on a un isomorphisme canonique

$$\boldsymbol{R}hom_{\mathcal{D}_X}\left(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r\}\right) \approx \mathcal{M} \overset{\boldsymbol{L}}{\underset{\mathcal{D}_X}{\otimes}} \left(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_{Z/X}\{r\}\right)[-n].$$

D'où en vertu du lemme 6.2.3 un morphisme

$$\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r\}) \to \mathcal{M} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathbf{R}hom_{\mathbf{C}_X}(\mathcal{O}_{X/Z}\{r\}, \mathbf{C}_X)[p].$$

Mais en vertu du lemme du "way out" foncteur on a un isomorphisme canonique pour tout complexe $\mathcal{M}$ de $\mathcal{D}_X$-modules à cohomologie bornée et cohérente :

$$\mathcal{M} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathbf{R}hom_{\mathbf{C}_X}(\mathcal{O}_{X/Z}\{r\}, \mathbf{C}_X)[p] \approx \mathbf{R}hom_{\mathbf{C}_X}(\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{r\}), \mathbf{C}_X)[p].$$

D'où le morphisme du théorème 6.2.1. Pour montrer que c'est un isomorphisme si $\mathcal{M}$ est un complexe de la catégorie $D^b_h(\mathcal{D}_X)$, on peut supposer que $\mathcal{M}$ est un $\mathcal{D}_X$-module holonome puis qu'il admet une résolution libre, puisque la question est locale sur X, par des $\mathcal{D}_X$-modules libres de type fini

$$0 \to \mathcal{D}_X^{r_N} \to \dots \qquad \dots \to \mathcal{D}_X^{r_0} \to 0.$$

D'où un représentant local du morphisme du théorème 6.2.2

$$\begin{array}{ccccccc}
0 \to & \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^{r_N}_{Z/X}\{r\} & \to \dots & \dots \to & \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^{r_0}_{Z/X}\{r\} & \to 0 \\
 & \downarrow & & & \downarrow & \\
0 \to & Ext^{n+p}_C\left(\mathcal{O}^{r_N}_{X/Z}\{r\}, \mathbf{C}_X\right) & \to \dots & \dots \to & Ext^{n+p}_C\left(\mathcal{O}^{r_0}_{X/Z}\{r\}, \mathbf{C}_X\right) & \to 0.
\end{array}$$

Pour montrer que c'est un isomorphisme il suffit de montrer que l'analogue global

$$\begin{array}{ccccccc}
0 \to & \Gamma\left(V, \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^{r_N}_{Z/X}\{r\}\right) & \to \dots & \dots \to & \Gamma\left(V, \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^{r_0}_{Z/X}\{r\}\right) & \to 0 \\
 & \downarrow & & & \downarrow & \\
0 \to & \mathrm{Ext}^{n+p}_C\left(V, \mathcal{O}^{r_N}_{X/Z}\{r\}, \mathbf{C}_X\right) & \to \dots & \dots \to & \mathrm{Ext}^{n+p}_C\left(V, \mathcal{O}^{r_0}_{X/Z}\{r\}, \mathbf{C}_X\right) & \to 0.
\end{array}$$

pour V de Stein assez petit est un isomorphisme. Mais en vertu du théorème de constructibilité, pour V assez petit, la cohomologie du premier complexe d'espaces de type ***FN*** est de dimension finie et donc ses différentielles sont des homomorphismes [Se$_2$]. En vertu du théorème 6.1.1 c'est le dual fort topologique du complexe

$$0 \to H_c^{n-p}\left(V, \mathcal{O}_{X/Z}^{r_0}\{r\}\right) \dots \qquad \dots \to H_c^{n-p}\left(V, \mathcal{O}_{X/Z}^{r_N}\{r\}\right) \to 0.$$

Ce dernier complexe d'espaces de type ***DFN*** est à différentielles d'images fermées puisque dual d'un complexe d'espaces de type ***F*** dont les différentielles sont des homomorphismes [G8]. Il en résulte que ce complexe est à cohomologie de dimension finie et séparée. En vertu du théorème de dualité de Poincaré pour les coefficients généraux sur les espaces localement compacts [V] le dual algébrique de ce dernier complexe est précisément le deuxième complexe du diagramme précédent. Les morphismes verticaux induisent des isomorphismes en cohomologie puisque le dual topologique et le dual algébrique d'un espace topologique séparé de dimension finie coïncident. D'où le théorème 6.2.2.

Corollaire 6.2.4. *Pour toute sous-variété lisse Z de X, tout complexe $\mathcal{M}$ de la catégorie $D_h^b(\mathcal{D}_X)$ et tout réel r, $1 \leq r \leq \infty$, le complexe $\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{r\})$ est* constructible.

Preuve du corollaire 6.2.4. Cela résulte du théorème de constructibilité 6.2.1, du théorème de dualité locale 6.2.2 et du théorème de bidualité locale pour les coefficients constructibles qui implique qu'un complexe borné de faisceaux d'espaces vectoriels complexes est constructible si et seulement si son complexe dual est constructible.

6.3. Supposons que Z est une sous-variété lisse de codimension p d'une variété analytique complexe X. Pour tous réels r', r, $1 \leq r' \leq r \leq \infty$, définissons les deux $\mathcal{D}_X$-modules $\mathcal{Q}_Z\{r, r'\}$ et $\mathcal{L}_Z\{r, r'\}$ par les suites exactes de $\mathcal{D}_X$-modules

$$0 \to \mathcal{O}_{X/Z}\{r'\} \to \mathcal{O}_{X/Z}\{r\} \to \mathcal{Q}_Z\{r, r'\} \to 0$$

$$0 \to \mathcal{B}_{Z/X}\{r\} \to \mathcal{B}_{Z/X}\{r'\} \to \mathcal{L}_Z\{r, r'\} \to 0.$$

Définition 6.3.1. Pour tout complexe $\mathcal{M}$ de la catégorie $D_h^b(\mathcal{D}_X)$, tous réels r', r, $1 \leq r' \leq r \leq \infty$, et toute sous-variété lisse Z de codimension p posons

$$\begin{aligned} \boldsymbol{IR}_Z^{\nu}\{r, r'\}(\mathcal{M}) &:= \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z\{r, r'\}) \\ \boldsymbol{IR}_Z\{r, r'\}(\mathcal{M}) &:= \boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z\{r, r'\}). \end{aligned}$$

On a ainsi défini deux foncteurs exacts $\boldsymbol{IR}_Z^{\nu}\{r, r'\}, \boldsymbol{IR}_Z\{r, r'\}$ de la catégorie triangulée $D_h^b(\mathcal{D}_X)$ dans la catégorie triangulée $D_c^b(\boldsymbol{C}_Z)$. Par définition donc

$$\mathbb{R}_Z\{\infty,1\}(\mathcal{M}) = \mathbb{R}_Z(\mathcal{M}) \quad \text{et} \quad \mathbb{R}^\vee_Z\{\infty,1\}(\mathcal{M}) = \mathbb{R}^\vee_Z(\mathcal{M}).$$

Théorème 6.3.2. *Pour tout complexe $\mathcal{M}$ de la catégorie $D^b_h(\mathcal{D}_X)$, tous réels $r', r, 1 \leq r' \leq r \leq \infty$, et toute sous variété lisse Z de codimension p les complexes $\mathbb{R}^\vee_Z\{r,r'\}(\mathcal{M})$ et $\mathbb{R}_Z\{r,r'\}(\mathcal{M})$ sont en* dualité:

$$\mathbb{R}^\vee_Z\{r,r'\}(\mathcal{M})[-p] \approx \mathbf{R}hom_{\mathbf{C}_X}(\mathbb{R}_Z\{r,r'\}[-1], \mathbf{C}_X).$$

Preuve de 6.3.2. Cela résulte du théorème de dualité locale 6.2.3.

Théorème 6.3.3. *Pour tout $\mathcal{D}_X$-module holonome $\mathcal{M}$, tous réels r', r, $l \leq r' \leq r \leq \infty$, et toute hypersurface lisse Z, les complexes $\mathbb{R}^\vee_Z\{r,r'\}(\mathcal{M})$ et $\mathbb{R}_Z\{r,r'\}(\mathcal{M})$ sont des* faisceaux pervers sur Z.

Remarquons que $\mathbb{R}_Z\{r,r'\}(\mathbf{R}\mathrm{alg}\Gamma_Z(\mathcal{M}))$ est nul pour tous les r et r' puisqu'on a les isomorphismes

$$\mathrm{alg}H^1_Z(\mathcal{O}_X) \cong \mathbf{R}\mathrm{alg}\Gamma_Z(\mathcal{B}_{Z/X}\{r'\}) \cong \mathbf{R}\mathrm{alg}\Gamma_Z(\mathcal{B}_{Z/X}\{r\}).$$

On a donc les isomorphismes $\mathbb{R}_Z\{r,r'\}(\mathcal{M}) \cong \mathbb{R}_Z\{r,r'\}(\mathcal{M}(*Z))$. On peut donc supposer que $\mathcal{M}$ est isomorphe à son localisé $\mathcal{M}(*Z)$. Nous allons procéder en deux étapes pour démontrer le théorème 6.3.3. On peut supposer en vertu du théorème 6.2.1 que r et r' sont des nombres rationnels.

Preuve du théorème 6.3.3. Cas (i); $r = \infty$. Si $r = \infty$ et $r' = 1$ le théorème 6.3.3 se réduit au théorème de positivité 2.1.6. Si $r = \infty$ et $r' \neq 1$ il suffit de démontrer que le complexe $\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{r\})[1]$ est un faisceau pervers sur Z puisque que $\mathbf{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{\infty\})$ est nul si $\mathcal{M}$ est isomorphe à son localisé $\mathcal{M}(*Z)$.

Considérons le faisceau $\mathcal{D}_{X/Z}\{r\}$ des opérateurs micro-différentiels de type r qui opère sur le faisceau $\mathcal{O}_{X/Z}\{r\}$ [La3]. Rappelons [La3] que le faisceau $\mathcal{D}_{X/Z}\{r\}$ est la restriction à Z du faisceau $\mathcal{D}^2_\Lambda\{r\}$ sur $\Lambda := T^*_Z X$ lui même restriction du faisceau des opérateurs 2-microdifférentiels, de type r, $\mathcal{E}^2_\Lambda\{r,r\}$ sur $T^*\Lambda$. Le faisceau $\mathcal{D}_{X/Z}\{r\}$ est un faisceau cohérent d'anneaux muni d'une filtration croissante indexée par $\mathbf{Z}$ dont la dimension homologique est égale à $n = \dim(X)$. En coordonnées locales (x,t) où t est une équation locale de Z une section locale P d'ordre $m (m \in \mathbf{Z})$ de $\mathcal{D}_{X/Z}\{r\}$ s'écrit

$$P(x,t,\partial_x,\partial_t) = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma}(x) t^\alpha \partial_t^\beta \partial_x^\gamma$$

où $a_{\alpha,\beta,\gamma}(x)$ est une suite de fonctions holomorphes, α, $\beta \in \mathbf{N}$, $\gamma \in \mathbf{N}^{n-1}$ tels qu'il existe une constant C positive avec les deux conditions

$p(\beta-\alpha)+q(\alpha+|\gamma|)\leq m$ et $|a_{\alpha,\beta,\gamma}(x)|\leq C^{\alpha+\beta+|\gamma|}((\alpha-\beta)!)^r/(\alpha+|\gamma|)!$ sur tout compact où $r=p/q$.

Le faisceau $\mathcal{D}_{X/Z}\{1\}$ se réduit simplement à la restriction à Z du faisceau $\mathcal{D}_X$. L'extension $\mathcal{D}_{X/Z}\{1\}\to\mathcal{D}_{X/Z}\{r\}$ est plate mais non fidèlement plate [La3]. Il nous suffit donc de démontrer que le complexe $\mathbf{R}hom_{\mathcal{D}_{X/Z}\{r\}}(\mathcal{D}_{X/Z}\{r\}\otimes_{\mathcal{D}_X}\mathcal{M},\mathcal{O}_{X/Z}\{r\})[1]$ est un faisceau pervers sur Z. Le $\mathcal{D}_{X/Z}\{r\}$-module $\mathcal{D}_{X/Z}\{r\}\otimes_{\mathcal{D}_X}\mathcal{M}$ est holonome [La4]. Mais on a de façon évidente $\mathcal{O}(*Z)\otimes_{\mathcal{O}_X}\mathcal{D}_{X/Z}\{r\}\cong\mathcal{D}_{X/Z}\{r\}\otimes_{\mathcal{O}_X}\mathcal{O}(*Z)$ et la localisation le long de Z commute au changement de base $\mathcal{D}_{X/Z}\{1\}\to\mathcal{D}_{X/Z}\{r\}$. Il suffit de démontrer que le complexe $\mathbf{R}hom_{\mathcal{D}_{X/Z}\{r\}}(\mathcal{M},\mathcal{O}_{X/Z}\{r\})[1]$ est un faisceau pervers sur Z pour tout $\mathcal{D}_{X/Z}\{r\}$-module holonome $\mathcal{M}$ isomorphe à son localisé le long de Z.

Théorème 6.3.4. *Pour tout r, $1\leq r<\infty$, et pour tout triplet $(X,Z,\mathcal{M})$ comme ci-dessus où $\mathcal{M}$ est un $\mathcal{D}_{X/Z}\{r\}$-module holonome $\mathcal{M}$ isomorphe à son localisé le long de Z le complexe $\mathbf{R}hom_{\mathcal{D}_{X/Z}\{r\}}(\mathcal{M},\mathcal{O}_{X/Z}\{r\})[1]$ est un faisceau* pervers sur Z.

Preuve du théorème 6.3.4. Les arguments de la démonstration du théorème 2.1.6 se transposent puisque on a les mêmes ingrédients. On raisonne par récurrence sur $\dim(X)$. Si $\dim(X)=1$ le faisceau $hom_{\mathcal{D}_{X/Z}\{r\}}(\mathcal{M},\mathcal{O}_{X/Z}\{r\})$ est nul parce que si t est équation locale de Z l'opérateur t qui est dans $\mathcal{D}_{X/Z}\{r\}$ est inversible dans $\mathcal{M}$. D'où le théorème dans ce cas là. En dimension supérieure on a

(1) le théorème de constructibilité de Laurent 6.2.1;

(2) le théorème de Cauchy-Kowalewski de type $\{r\}$ à valeur dans le faisceau $\mathcal{O}_{X/Z}\{r\}$ ([La3], 3.2.2): si X' est une hypersurface non caractéristique de type $\{r\}$ pour $\mathcal{M}$ le système induit $\mathcal{M}':=\mathcal{O}_{X'}\otimes_{\mathcal{O}_X}\mathcal{M}$ est un $\mathcal{D}_{X'/Z'}\{r\}$-module holonome isomorphe à son localisé le long de $Z':=Z\cap X'$ et on a l'isomorphisme $\mathbf{R}hom_{\mathcal{D}_{X/Z}}\{r\}(\mathcal{M},\mathcal{O}_{X/Z}\{r\})_{|X'}\cong\mathbf{R}hom_{\mathcal{D}_{X'/Z'}\{r\}}(\mathcal{M}',\mathcal{O}_{X'/Z'}\{r\})$;

(3) en dehors d'un ensemble de dimension nulle de Z il passe une hypersurface X' transverse à Z non micro-caractéristique de type $\{r\}$, $\mathrm{Ch}(\mathcal{M})\{r\}\cap T^*_{\Lambda'}\Lambda$ où $\Lambda':=T^*_{Z'}Z$ puisque la variété caractéristique $\mathrm{Ch}(\mathcal{M})\{r\}$ [La3] de type $\{r\}$ d'un module holonome est lagrangienne [La4];

(4) le théorème de dualité locale pour les $\mathcal{D}_{X/Z}\{r\}$-modules holonomes qui s'énonce et se démontre exactement comme le théorème 6.2.2 à partir du 6.1.1 parce que les opérateurs de $\mathcal{D}_{X/Z}\{r\}$ opérent *continument* sur les espace $\Gamma(V,\mathcal{B}_{Z/X}\{r\})$ et par transposition sur les espaces $H^{n-p}_c(V,\mathcal{O}_{X/Z}\{r\})$ pour V assez petit;

(5) et enfin l'hypothése de récurrence sur $\dim(X)$ entraine que le faisceau $Ext^n_{\mathcal{D}_{X/Z}\{r\}}(\mathcal{M},\mathcal{L}_Z\{\infty,r\})$ est à support de dimension nulle puis qu'il est

nul par les mêmes arguments qui démontrent le lemme 2.2.2.1.

Preuve du théorème 6.3.3 Cas (ii); $r \neq \infty$. En vertu du cas (i) les complexes $\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X/Z}\{r\})[1]$ sont des faisceaux pervers sur Z si $\mathcal{M}$ est isomorphe à son localisé le long de Z et donc par dualité les complexes $\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r\})$ sont des faisceaux pervers sur Z. Du triangle distingué pour $r' \leq r$

$$\begin{aligned}\boldsymbol{R}hom_{\mathcal{D}_X}\,(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r\}) \to \boldsymbol{R}hom_{\mathcal{D}_X}\,&(\mathcal{M}^*, \mathcal{B}_{Z/X}\{r'\}) \\ &\to \boldsymbol{R}hom_{\mathcal{D}_X}\,(\mathcal{M}^*, \mathcal{L}_Z\{r,r'\})\end{aligned}$$

on déduit que le complexe $\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z\{r,r'\})$ a la condition de *support*. Reste à voir qu'il a la condition de *co-support*. Par dualité il suffit de montrer que pour toute variété lisse S de Z le complexe $\boldsymbol{R}\Gamma_S(\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z\{r,r'\}))[\mathrm{codim}_Z(S)]$ est concentré en degrés positifs. Mais $\boldsymbol{R}\Gamma_S(\mathcal{B}_{Z/X}\{r\})[\mathrm{codim}_Z(S)] \cong H^q_S(\mathcal{B}_{Z/X}\{r\})$ pour tout r où $q = \mathrm{codim}_Z(S)$. De plus si $r' \leq r$ le morphisme $H^q_S(\mathcal{B}_{Z/X}\{r\}) \to H^q_S(\mathcal{B}_{Z/X}\{r'\}$ est injectif. D'où la suite exacte de $\mathcal{D}_X$-modules

$$0 \to H^q_S\,(\mathcal{B}_{Z/X}\{r\}) \to H^q_S\,(\mathcal{B}_{Z/X}\{r'\}) \to H^q_S\,(\mathcal{L}_Z\{r,r'\}) \to 0$$

qui montre bien que le complexe $\boldsymbol{R}\Gamma_S(\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{L}_Z\{r,r'\}))$ $[\mathrm{codim}_Z(S)]$ qui est isomorphe au complexe $\boldsymbol{R}hom_{\mathcal{D}_X}(\mathcal{M}^*, H^q_S(\mathcal{L}_Z\{r,r'\}))$ est concentré en degrés positifs.

Par conséquent les complexes $\boldsymbol{IR}^\nu_Z\{r,r'\}(\mathcal{M})$ et $\boldsymbol{IR}_Z\{r,r'\}(\mathcal{M})$ sont des faisceaux pervers sur Z pour tous r et r'. D'où le théorème 6.3.3.

Corollaire 6.3.5. *Pour toute hypersurface lisse Z de X, $\boldsymbol{IR}^\nu_Z\{r,r'\}$, $\boldsymbol{IR}_Z\{r,r'\}$ sont des foncteurs* exacts *de catégories abéliennes entre la catégorie* $\mathrm{Mh}(\mathcal{D}_X)$ *et la catégorie* $\mathrm{Perv}(\boldsymbol{C}_Z)$.

Preuve du corollaire 6.3.5. Cela résulte du théorème 6.3.3 et de la suite longue de cohomologie perverse.

Corollaire 6.3.6. *Pour toute hypersurface lisse Z de X, pour r, r', r'' tels que $1 \leq r'' \leq r' \leq r \leq \infty$ le faisceau $\boldsymbol{IR}_Z\{r,r'\}(\mathcal{M})$ est un* sous-faisceau *du faisceau $\boldsymbol{IR}_Z\{r,r''\}(\mathcal{M})$ et le faisceau $\boldsymbol{IR}^\nu_Z\{r',r''\}(\mathcal{M})$ est une* sous-faisceau *du faisceau $\boldsymbol{IR}^\nu_Z\{r,r''\}(\mathcal{M})$.*

Preuve du corollaire 6.3.6. On a alors deux suites exactes de $\mathcal{D}_X$-modules

$$0 \to \mathcal{Q}_Z\{r',r''\} \to \mathcal{Q}_Z\{r,r''\} \to \mathcal{Q}_Z\{r,r'\} \to 0$$

$$0 \to \mathcal{L}_Z\{r,r'\} \to \mathcal{L}_Z\{r,r''\} \to \mathcal{L}_Z\{r',r''\} \to 0.$$

Et le corollaire 6.3.6 résulte du théorème 6.3.3 et de la suite longue de cohomologie perverse.

Définition 6.3.7. Toujours pour une hypersurface lisse Z et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome, posons pour tout réel r, $1 \leq r \leq \infty$

$$\boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M}) := \boldsymbol{IR}_Z^{\nu}\{r,1\}(\mathcal{M}) \quad \text{et} \quad \boldsymbol{IR}_Z\{r\}(\mathcal{M}) := \boldsymbol{IR}_Z\{\infty, r\}(\mathcal{M}).$$

On obtient une filtration décroissante avec r du faisceau $\boldsymbol{IR}_Z(\mathcal{M}) := \boldsymbol{IR}_Z\{1\}(\mathcal{M}) \supset \boldsymbol{IR}_Z\{r\}(\mathcal{M}) \supset \ldots \supset \boldsymbol{IR}_Z\{\infty\}(\mathcal{M}) = 0$ et une filtration croissante avec r du faisceau $\boldsymbol{IR}_Z^{\nu}(\mathcal{M}) := \boldsymbol{IR}_Z^{\nu}\{\infty\}(\mathcal{M}) \supset \boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M}) \supset \ldots \supset \boldsymbol{IR}_Z^{\nu}\{1\}(\mathcal{M}) = 0$. Ces filtrations, qui se déterminent l'une l'autre par dualité, sont localement finies, et leurs sauts locaux sont des nombres *rationnels*, qui sont les pentes topologiques critiques de $\mathcal{M}$ le long de Z. En particulier si $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$ est nul ou si $\boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M})$ est nul, les faisceaux $\boldsymbol{IR}_Z\{r\}(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M})$ sont nuls pour tout r.

6.4. Si Z est une hypersurface de X qui est singulière nous allons encore définir des filtrations des faisceaux $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$ pour un $\mathcal{D}_X$-module holonome $\mathcal{M}$ par des sous-faisceaux indexés par les nombres réels r, $1 \leq r \leq \infty$, localement finie dont les sauts locaux sont encore des nombres rationnels qui seront les pentes topologiques critiques de $\mathcal{M}$ le long de Z.

Si f est une équation locale réduite de Z au dessus d'un ouvert U de X, notons δ_f le morphisme graphe $U \to U' := U \times C$, Z' l'hypersurface non singulière $U \times 0$ de U' et $\mathcal{M}_f$ le $\mathcal{D}_{U'}$-module holonome image directe au sens des $\mathcal{D}_X$-modules où C désigne le plan complexe. On a des isomorphismes de faisceaux pervers sur Z'

$$\boldsymbol{IR}_{Z'}(\mathcal{M}_f) \cong \boldsymbol{IR}_Z(\mathcal{M})[-1] \quad \text{et} \quad \boldsymbol{IR}_{Z'}^{\nu}(\mathcal{M}_f) \cong \boldsymbol{IR}_Z^{\nu}(\mathcal{M})[-1].$$

Les filtrations $\boldsymbol{IR}_{Z'}\{r\}(\mathcal{M}_f)$ et $\boldsymbol{IR}_{Z'}^{\nu}\{r\}(\mathcal{M}_f)$ se transposent au-dessus de U' aux faisceaux $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$. Il faut voir maintenant que ces filtrations ne dépendent pas de l'équation f de Z. Ceci résulte de la proposition suivante :

Proposition 6.4.1. *Si δ est une immersion d'un couple lisse (X, Z) dans un autre couple lisse (X', Z'), on a un morphisme canonique de projection*

$$\mathcal{N}_\delta \overset{L}{\underset{\mathcal{D}_{X'}}{\otimes}} \left(\omega_{X'} \underset{\mathcal{O}_{X'}}{\otimes} \mathcal{B}_{Z'/X'}\{r\}\right) \to \delta_* \left(\mathcal{N} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \left(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_{Z/X}\{r\}\right)\right)$$

qui est un isomorphisme *pour tout* $\mathcal{D}_X$*-module à droite cohérent* $\mathcal{N}$ *et tout réel* r, $1 \leq r \leq \infty$.

Preuve de 6.4.1. Le morphisme est précisément le morphisme de projection. Pour vérifier que c'est un isomorphisme, le lemme du "way out" foncteur nous ramène à supposer que $\mathcal{N} = \mathcal{D}_X$, auquel cas c'est immédiat.

Pour un triplet $(X, Z, \mathcal{M})$ les faisceaux $\boldsymbol{IR}_Z(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\nu}(\mathcal{M})$ se trouvent *localement* munis de filtrations $\boldsymbol{IR}_Z\{r\}(\mathcal{M})$ et $\boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M})$ par des faisceaux (pervers). Mais comme la nature fait bien les choses, ces filtrations locales se recollent pour donner naissance à des filtrations globales cf. [BBD]. Ainsi si Z est un hypersurface on a défini des foncteurs exacts $\boldsymbol{IR}_Z^{\nu}\{r\}$ et $\boldsymbol{IR}_Z^{\nu}\{r\}$ entre catégories abéliennes $\mathrm{Mh}(\mathcal{D}_X)$ et $\mathrm{Perv}(\boldsymbol{C}_X)$ qui ont toutes les propriétés des foncteurs analogues quand Z est lisse.

6.5. Pour tout r les cycles caractéristiques $\mathrm{CCh}(\boldsymbol{IR}_Z\{r\}(\mathcal{M}))$, $\mathrm{CCh}(\boldsymbol{IR}_Z^{\nu}\{r\}(\mathcal{M}))$ sont des cycles lagrangiens positifs du fibré cotangent T^*X pour tout triplet $(X, Z, \mathcal{M})$ où Z est une hypersurface et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome. Ces cycles sont différences de deux cycles lagrangiens qui sont définis de façon purement algébrique. En effet si Z est lisse le complexe $\boldsymbol{Rhom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Z/X}\{r\})$ apparaît dans la théorie de Y. Laurent ([La$_1$], [La$_2$]) comme la restriction à Z vue comme section nulle du fibré normal T_Z^*X d'un complexe constructible sur T_Z^*X dont le cycle caractéristique est défini par voie algébrique. Par l'intermédiaire de l'isomorphisme d'Euler le cycle caractéristique du complexe $\boldsymbol{Rhom}_{\mathcal{D}_X}$ $(\mathcal{M}, \mathcal{B}_{Z/X}\{r\})$ est défini par voie purement algébrique. Là encore la situation doit être considérée comme provisoire. On doit pouvoir définir de façon purement algébrique une filtration par des modules holonomes du module "d'irrégularité", qui doit contenir toute l'information précédente.

6.6. Pour un triplet $(X, Z, \mathcal{M})$ comme ci-dessus où Z est une hypersurface lisse le faisceau $\boldsymbol{IR}_Z(\mathcal{M})$ est donc muni d'une filtration par des sous-faisceaux dont les sauts sont localement finis. Ce sont des nombres rationnels que l'on appellera les *pentes topologiques* de $\mathcal{M}$ le long de Z. Ces pentes varient de façon constructibles sur Z. Elles sont contenues en vertu du théorème de Laurent 6.2.1 dans les *pentes algébriques* définies pour tout module cohérent et toute sous-variété lisse [La$_4$]. Les pentes algébriques sont aussi localement finies et varient de façon constructible sur Z ([La$_4$], [Sa$_2$]). Nous démontrerons dans un travail en colaboration avec Y. Laurent que pour un $\mathcal{D}_X$-module holonome les pentes topologiques *coïncident* avec les pentes algébriques le long d'une hypersurface. Nous définirons l'analogue du polygône de Newton pour les $\mathcal{D}_X$-modules holonomes le long d'une hypersurface et nous préciserons les résultats de ce numéro en donnant un analogue complexe du théorème de Hasse-Arf [Se$_1$] en dimension supérieure.

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UA 212, Mathématiques
Université de Paris VII
2 Place Jussieu
75251 Paris Cedex 05, France

The Convergent Topos in Characteristic p

ARTHUR OGUS [1]

Dedicated to A. Grothendieck on his 60th birthday

The purpose of this note is to investigate some of the foundational questions concerning convergent cohomology as introduced in [?] and [?], using the language and techniques of Grothendieck topologies. In particular, if X is a scheme of finite type over a perfect field k of characteristic p and with Witt ring W, we define the "convergent topos $(X/W)_{conv}$," and we study the cohomology of its structure sheaf $\mathcal{O}_{X/W}$ and of $\mathcal{K}_{X/W} := \mathbf{Q} \otimes \mathcal{O}_{X/W}$. Since the topos $(X/W)_{conv}$ is not noetherian, formation of cohomology does not commute with tensor products, and these are potentially quite different. In fact we find the following, when k is algebraically closed:

Theorem 0.0.1. *If X/k is smooth and proper, there are natural isomorphisms:*

$$H^i_{conv}(X/W, \mathcal{O}_{X/W}) \cong H^i(X_{\acute{e}t}, \mathbf{Z}_p) \otimes W$$

$$H^i_{conv}(X/W, \mathcal{K}_{X/W}) \cong \mathbf{Q} \otimes H^i_{cris}(X/W, \mathcal{O}_{X/W}) \cong H^i_{rig}(X).$$

Actually many of the results we prove are more general, dealing with cohomology with coefficients in a crystal and with schemes over ramified DVR's and which are not necessarily smooth. When X/k is smooth, k is perfect, and V is the Witt ring of k, we also verify that the category of convergent isocrystals on X/V is a full subcategory of the category of crystals (in the usual divided power sense) up to isogeny.

I decided to look carefully at these questions in response to Grothendieck's remark that most mathematicians don't have the patience to think carefully about foundational matters. In addition to motivation, nearly all the fundamental ideas and techniques—indeed the very concept of geometry—in this paper are due to Grothendieck, with influences from Dwork

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and Berthelot as well. It is an honor to be able to dedicate this work to him.

1. Widenings and Enlargements

We begin by recalling the definition of enlargements made in [7], incorporating some slight modifications (one of which was introduced by Faltings in [5]). We will also find it convenient to define a more general object which we call a "widening," after Coleman's notion of a "wide open set" in rigid geometry. Let V be a complete DVR of mixed characteristic p and let π be a nonzero element of the maximal ideal of V; k is the residue field of V and K is its fraction field. Let V_1 denote $V/\pi V$ and let X be a V-scheme of finite type. By a "formal V-scheme" we mean a morphism of formal schemes $S \to SpfV$, which we tacitly assume is essentially of finite type. (We do *not* assume that S has the p-adic topology.) For any such S we denote by S_1 the closed formal subscheme defined by $\pi\mathcal{O}_S$, and by S_0 the largest reduced formal subscheme of S_1.

Definition 0.1.1. *A "widening of (X/V)" is a triple (S, Z, z), where S is a flat formal V-scheme, $Z \subseteq S$ is a subscheme of definition, and $z: Z \to X$ is a V-morphism. We say a widening (S, Z, z) is an "enlargement of (X/V)" iff $S_0 \subseteq Z$. If $f: X \to X'$ is a morphism of V-schemes and (S', Z', z') is a widening of (X'/V), then a "morphism of widenings $g: (S, Z, z) \to (S', Z', z')$ over f" is a morphism $g_S: S \to S'$ inducing a commutative diagram:*

$$\begin{array}{ccccc} X & \xleftarrow{z} & Z & \longrightarrow & S \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g_Z} & & \downarrow{\scriptstyle g_S} \\ X' & \xleftarrow{z'} & Z' & \longrightarrow & S' \end{array}$$

In particular, we let $Wide(X/V)$ denote the category whose objects are the widenings of X/V and whose morphisms are the morphisms of widenings over id_X; $Enl(X/V)$ stands for the analogous category of enlargements. We say a widening (S, Z, z) is "affine" iff the morphism z is an affine morphism, and we say (S, Z, z) is "absolutely affine" iff Z is an affine scheme. Notice that the category $Wide(X/V)$ has products and fiber products. In fact, if $T' := (S', Z', z')$ and $T'' := (S'', Z'', z'')$, we let $S' \hat{\times}_V S''$ be the formal completion of $S' \times S''$ along $Z' \times_X Z''$, and it is clear that $(S' \hat{\times}_V S'', Z' \times_X Z'', pr_X)$ is the product of T' and T'' in the category $Wide(X/V)$. Similarly, if $T := (S, Z, z)$ is another widening and $g': T' \to T$ and $g'': T'' \to T$ are morphisms, then $(S' \hat{\times}_S S'', Z' \times_Z Z'', pr_X)$ is the fiber product $T' \times_T T''$ in $Wide(X/V)$. If T' and T'' are enlargements of X/V it need not be the case that $T' \times T''$ is an enlargement, because $(Z' \times_X Z'')_{red} \neq (S' \hat{\times}_V S'')_0$ in general. On the other hand, if T, T', and T'' are all enlargements of X/V then so is the fiber product $T' \times_T T''$. Thus, $Enl(X/V)$ has fiber products, but not products, in general.

We endow the category $Enl(X/V)$ with the Zariski topology and denote the topos of sheaves on $Enl(X/V)$ by $(X/V)_{conv}$. For example, define $\mathcal{O}_{X/V}(T) := \mathcal{O}_S(S)$, $\mathcal{O}_X(T) := \mathcal{O}_Z(Z)$, $I_{X/V}(T) := I_Z(S)$, and $\mathcal{K}_{X/V} := K \otimes \mathcal{O}_S(S)$ for any enlargement $T = (S, Z, z)$ of X/V. One sees easily that these are all sheaves on $Enl(X/V)$.

Remark 0.1.2. The definition we have given above works even if X is purely in characteristic zero, and gives something quite close to the infinitesimal site of X. In fact, if we took $\pi = 0$, we would have $S_0 = S_{red}$, and then a widening is an enlargement iff S is an infinitesimal thickening of Z. In particular, if X is a K-scheme and $\pi = 0$, then the category $Enl(X/V)$ is just the category of infinitesimal thickenings of schemes over X, and $(X/V)_{conv}$ is the "big" infinitesimal topos of X/K as defined by Grothendieck. However, due to lack of time and space, we shall in this paper always assume that π annihilates X, *i.e.* that X is in fact a V_1-scheme. Then it follows that Z is also a V_1-scheme, and hence $Z \subseteq S_1$ automatically and the condition that Z contain S_0 just says that S has the π-adic topology.

We define crystals following Grothendieck's model [3].

Definition 0.1.3. *A "crystal of $\mathcal{O}_{X/V}$-modules" is a sheaf E of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$ such that each morphism $g: S' \to S$ in $Enl(X/V)$ gives rise to an isomorphism $\rho_g: g_S^*(E_S) \to E_{S'}$.*

Similarly a crystal of $\mathcal{K}_{X/V}$-modules is a crystal of $\mathcal{O}_{X/V}$-modules such that each E_S is a sheaf of $\mathcal{K}_S$-modules. We have to refer to [7] for the basic facts about sheaves of $\mathcal{K}_S$-modules on formal schemes. It should be clear that a crystal of $\mathcal{K}_{X/V}$-modules is the same notion as that of a "convergent isocrystal" as discussed there.

For any enlargement T of X/V, widening T' of X'/V, and morphism $f: X \to X'$, we let $f^*(T')(T)$ denote the set of morphisms $T \to T'$ over f. If $f = id_X$ we denote this just by $h_{T'}(T)$. One sees immediately that $f^*(T')$ is a sheaf on $Enl(X/V)$, *i.e.* an object of $(X/V)_{conv}$. We will see later that if T' is a widening of X/V, then the sheaf $h_{T'}$ determines T' uniquely and will simply write T' in place of $h_{T'}$. By abuse of language, we will say that an object of $(X/V)_{conv}$ is a "widening" (resp., an "affine widening") if it is isomorphic to h_T for some (affine) widening T of X/V. In particular, the product of two (affine) widenings in $(X/V)_{conv}$ is again an (affine) widening. Similarly if $i: X \to X'$ is a closed immersion and (S', Z', z') is an (affine) enlargement of X', then $i^*(S', Z', z')$ is an (affine) widening of X. Indeed, $Z := X \times_{X'} Z'$ is a closed subscheme of Z' and hence of S', and we let S be the formal completion of S' along Z and $z := pr_X: Z \to X$; note that z is affine if z' is. Then (S, Z, z) is a widening of X/V such that $i^*(S', Z', z') \cong h_{(S',Z',z')}$.

If E is an object of $(X/V)_{conv}$ and $T = (S, Z, z)$ is a an object of $Enl(X/V)$, we can construct a sheaf E_T on the Zariski topology of S (which is the same as the Zariski topology of Z) in the obvious way: for any open set $U \subseteq S$, $E_T(U) := E(U, Z \cap U, z|_{Z\cap U})$. For any morphism $g: T' \to T$ in $Enl(X/V)$ there is an evident transition map $\rho_g: g_T^* E_T \to E_{T'}$, and it is clear that E is determined by the collection of Zariski sheaves E_T and transition maps ρ_g. Conversely, one can easily check that any such collection will define an an object of $(X/V)_{conv}$ if the transition maps satisfy the obvious compatibility conditions. For a complete discussion in the crystalline case, *c.f.* [2].

The verification of the following lemma is completely straightforward and involves nothing special about our definitions.

Lemma 0.1.4. *Suppose T is an enlargement of X/V and $E.$ is a finite inverse system of sheaves in $(X/V)_{conv}$. Then there is a natural isomorphism*

$$(\varprojlim E.)_T \cong \varprojlim (E_T.).$$

A sequence Σ of abelian objects of $(X/V)_{conv}$ is exact iff each Σ_T is.

The functoriality of our construction of the convergent topos is an immediate consequence of its definition.

Proposition 0.1.5. *A morphism of V_1-schemes $f: X \to X'$ induces a canonical morphism of topoi:*

$$f_{conv}: (X/V)_{conv} \to (X'/V)_{conv}$$

Proof: We define a functor: $\phi: Enl(X/V) \to Enl(X'/V)$ by $(S, Z, z) \mapsto (S, Z, f \circ z)$. Then according to [4] we have by abstract nonsense a constellation of functors on presheaves:

$$\phi_!, \phi_*: Enl(X/V)^\frown \to Enl(X'/V)^\frown$$

$$\phi^*: Enl(X'/V)^\frown \to Enl(X/V)^\frown$$

Here ϕ_* is right adjoint to ϕ^* and $\phi_!$ is left adjoint to ϕ^* and extends ϕ. The functors ϕ_* and ϕ^* make up a morphism of topoi provided that ϕ is "cocontinuous." This means that for any object (S, Z, z) of $Enl(X/V)$, and any covering sieve R of $\phi(S, Z, z)$, the set of arrows $g: (\tilde{S}, \tilde{Z}, \tilde{z}) \to (S, Z, z)$ such that $\phi(g)$ factors through R generates a covering sieve of (S, Z, z)—a triviality in this case, because S and its Zariski topology do not change when we apply ϕ. It implies (in fact is equivalent to) the fact that the functor ϕ_* above takes sheaves to sheaves. We should also note that if (S, Z, z) is an object of $Enl(X/V)$ and (T', Z', z') is an object of $Enl(X'/V)$ it follows trivially from our definitions that

$$Mor(\phi(S, Z, z), (S', Z', z',)) \cong Mor((S, Z, z), \phi^*(S', Z', z'))$$

is the set of morphisms of enlargements $(S, Z, z) \to (S', Z', z')$ over f. Furthermore, one can check easily that ϕ^* takes sheaves to sheaves, so that we have $f^* \cong \phi^*$. Q.E.D.

2. Universal Enlargements

A crucial tool for the study of the convergent topos is the construction of "universal enlargements." These are the analogs of Berthelot's "tubes" in rigid geometry. We have to modify slightly the construction given in [7].

Proposition 0.2.1. *Suppose $T = (S, Z, z)$ is a widening of X/V. Then the sheaf h_T in $(X/V)_{conv}$ is canonically a direct limit of enlargements: $h_T \cong \varinjlim T_{X,n}(S, Z, z)$. Moreover, each $T_{X,n}(S, Z, z) := (T_n(S), Z_n, z_n)$ is (absolutely) affine if (S, Z, z) is (absolutely) affine.*

Proof: Note that $\pi \in I_Z$. Denote by $B_Z(S)$ the formal blow up of S by the ideal I_Z, and let $T_Z(S)$ denote the affine piece of $B_Z(S)$ defined by π, with natural map $\lambda: T_Z(S) \to S$. Then $\lambda^{-1}Z \subset T_Z(S)$ is defined by the ideal π, and $\lambda: T_Z(S) \to S$ is universal with this property. If $n > 0$, let $\lambda_n: T_{Z,n}(S) \to S$ denote this construction with Z replaced by the intersection Y_n of its $(n-1)$st infinitesimal neighborhood in S with S_1, *i.e.* the subscheme define by $I_Z^n + \pi\mathcal{O}_S$. Let us note that λ_n maps $T_{Z,n}(S)_1$ to Y_n; we denote $\lambda_n^{-1}(Z)$ by Z_n. We find a commutative diagram:

$$\begin{tikzcd}
Z_n \arrow[r] \arrow[d] & T_{Z,n}(S)_1 \arrow[r] \arrow[d] & T_{Z,n}(S) \arrow[d] \\
Z \arrow[r] \arrow[d] & Y_n \arrow[r] & S \\
X & &
\end{tikzcd}$$

This diagram defines a map $z_n: Z_n \to X$, and then

$$T_{X,n}(S,Z,z) := (T_{Z,n}(S), Z_n, z_n)$$

is an enlargement of X endowed with an element of $h_T(T_{X,n}(S,Z,z))$. Thus we have morphisms of functors $\{T_{X,n}(S,Z,z) \to h_{(S,Z,z)} : n \in \mathbf{N}\}$ which patch together to give a morphism: $\varinjlim\{T_{X,n}(S,Z,z)\} \to h_T$.

To see that this morphism is surjective, let $(\tilde{S},\tilde{Z},\tilde{z})$ be an enlargement of X/V and $g: (\tilde{S},\tilde{Z},\tilde{z}) \to (S,Z,z)$ a morphism. Then $g|_{\tilde{Z}}$ factors through Z and so $g^* I_Z \mathcal{O}_{\tilde{S}} \subset I_{\tilde{Z}}$. Since $S_0 \subseteq \tilde{Z}$, $I_{\tilde{Z}}^n \subseteq \pi\mathcal{O}_{\tilde{S}}$ for some n and hence $g^* I_Z^n \mathcal{O}_{\tilde{S}} \subset \pi\mathcal{O}_{\tilde{S}}$. It follows that $g^*(I_Z^n + \pi\mathcal{O}_S) = \pi\mathcal{O}_{\tilde{S}}$, and by the universal property of $\lambda_n: T_{Z,n}(S) \to S$, g factors through λ_n. One verifies immediately that the restriction of g to $\tilde{Z}$ factors through Z_n, so that we have a morphism of enlargements of $X/V: (\tilde{S},\tilde{Z},\tilde{z}) \to T_{X,n}(S,Z,z)$.

The injectivity of our morphism will follow from the following stronger result, which we will need again later.

Lemma 0.2.2. *Suppose $T = (S,Z,z)$ is a widening of X/V, with $T \cong \varinjlim T_n$ as above. Then the morphisms: $h_{T_n} \to h_{T_{n+1}}$ and $h_{T_n} \to h_T$ are monomorphisms.*

Proof: It suffices to prove that each $h_{T_n} \to h_T$ is a monomorphism, *i.e.* that for every enlargement T' of X/V, the map $h_{T_n}(T') \to h_T(T')$ is injective. We may suppose without loss of generality that $T' \cong (S', Z', z')$, with $S' \cong SpfC$, and that $S \cong SpfB$, with Z defined by an ideal I; then T_n is affine, say $SpfB_n$. We claim that two homomorphisms $B_n \to C$ with the same composition $B \to B_n \to C$ must be equal. Since C is flat over V, it suffices to check this after tensoring with the fraction field K of V. Then our claim follows from the fact that C is separated in the π-adic topology and that $B \otimes K$ has dense image in $B_n \otimes K$. Let us record this fact and its proof explicitly.

Lemma 0.2.3. *The image of $K \otimes B \to K \otimes B_n$ is dense in the π-adic topology.*

Proof: Let C_n denote the ring of the affine piece of the blow up of $I^n + \pi B$ cut out by π, so that B_n is the π-adic completion of C_n. Evidently $K \otimes B \to K \otimes C_n$ is an isomorphism. Suppose $x \in K \otimes B_n$ and $m \geq 0$. Then there is a positive integer j such that $\pi^j x \in B_n$. Now we can write $\pi^j x = y + \pi^{m+j} z$, with $y \in C_n$ and $z \in B_n$. Then $x = \pi^{-j} y + \pi^m z$, where now $\pi^{-j} y \in K \otimes C_n = K \otimes B$. Q.E.D.

We have already remarked that if $i: X \to X'$ is a closed immersion and (S', Z', z') is an (affine) enlargement of X', then $i^*(S', Z', z')$ is a widening (resp, an affine widening) of X. In particular, it is canonically a direct limit of enlargements $\varinjlim T_{X,n}(S,Z,z)$. Similarly we see that the product of two widenings is canonically a direct limit of enlargements.

The following lemma follows immediately from the universal property of the construction of $T_Z(S)$.

Lemma 0.2.4. *Suppose $g: (S', Z', z') \to (S, Z, z)$ is a morphism of widenings of X/V, and $Z' = g^{-1}(Z)$. Then for any $n > 0$, there is a natural isomorphism: $T_{Z',n}(S') \to T_{Z,n}(S) \times_S S'$.*

The next lemma gives us a bound on the influence of the choice of Z on the construction of the direct system $\{T_{X,n}(S, Z, z)\}$.

Lemma 0.2.5. *Let (S, Z, z) and (S, Z', z') be widenings of X/V with $Z \subseteq Z'$ and $z'|_Z = z$, and suppose there exists a sheaf of ideals J in $\mathcal{O}_S$ such that $J^{m+1} \subseteq \pi\mathcal{O}_S$ and $I_Z \subseteq I_{Z'} + J$. Then there are maps*

$$h_n: T_{Z',n}(S, Z', z') \to T_{Z,n+m}(S, Z, z)$$

which when composed with the natural maps

$$T_{Z,n+m}(S, Z, z) \to T_{Z',n+m}(S, Z', z')$$

in either direction become the obvious transition morphisms. In particular, the natural maps:

$$T_{Z,n}(S, Z, z) \to T_{Z',n}(S, Z', z')$$

induce an isomorphism of Artin-Rees ind-objects.

Proof: It follows from our hypothesis that $I_Z^{n+m} \subseteq I_{Z'}^n + J^{m+1} \subseteq I_{Z'}^n + \pi\mathcal{O}_S$ for any $n > 0$. Then $I_Z^{n+m} + \pi\mathcal{O}_S \subseteq I_{Z'}^n + \pi\mathcal{O}_S$ for any $n > 0$, and it follows from the construction that we get an induced map $T_{Z',n}(S) \to T_{Z,n+m}(S)$. It is apparent that this map has the desired properties. Q.E.D.

Since the map $T_{Z',n}(S) \to T_{Z,n+m}(S)$ constructed above does not map Z'_n to Z_{n+m} in general, we cannot assert the existence of a morphism of enlargements $T_{X,n}(S, Z', z') \to T_{X,n+m}(S, Z, z)$. However, if we let $\tilde{Z}_n$ denote the inverse image of Z in $T_{Z',n}(S, Z', z')$ we do get a commutative diagram of morphisms of enlargements:

$$\begin{tikzcd}
\tilde{T}_n := (T_{Z'_n}(S), \tilde{Z}_n, \tilde{z}_n) \arrow[r, "\tilde{h}_n"] \arrow[d, "h_n"] & (T_{Z'_n}(S), Z'_n, z'_n) := T'_n \\
T_{n+m} := (T_{Z,n+m}(S), Z_{n+m}, z_{n+m})
\end{tikzcd}$$

Now if E is a sheaf of $\mathcal{O}_{X/V}$-modules, we have a natural map

$$\rho_{\tilde{h}_n}: \tilde{h}_n^* E_{T'_n} \to E_{\tilde{T}_n},$$

and since the morphism of formal schemes underlying $\tilde{h}_n$ is the identity map, we can identify $\tilde{h}_n^* E_{T'_n}$ with $E_{T'_n}$. If in addition E is a crystal, $\rho_{\tilde{h}_n}$ is an isomorphism, and we can therefore identify $E_{T'_n}$ with $E_{\tilde{T}_n}$. Finally, we have an isomorphism

$$\rho_{h_n}: h_n^* E_{T_{n+m}} \to E_{\tilde{T}_n}.$$

Putting these all together, we get the following result:

Proposition 0.2.6. *Let $(S, Z, z) \to (S, Z', z')$ be as in the previous lemma, and let E be a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$. Then there are natural isomorphisms:*

$$h_n^* E_{(T_{Z,n+m}(S), Z_{n+m}, z_{n+m})} \to E_{(T_{Z',n}(S), Z'_n z'_n)}.$$

For example, if (S', Z', z') and (S', Z'', z'') are enlargements of X', with $Z' \subset Z''$ and $z''|_{Z'} = z'$, and if $i: X \to X'$ is a closed immersion, the above lemma applies to the morphism of widenings $i^*(S', Z', z') \to i^*(S', Z'', z'')$. Indeed, let $Y := X \times_{X'} Z$ and $Y' := X \times_{X'} Z'' = Y \cap Z''$. Then $I_{Y'} = I_Y + I_{Z''}$, and since (S', Z'', z'') is an enlargement, $I_{Z''}^{m+1} \subset \pi$ for some integer m.

3. Sheaves on Widenings

We shall have to deal with widenings as systematically as enlargements. To this end we generalize the construction of the Zariski sheaf E_T associated to a sheaf E in $(X/V)_{conv}$ and an enlargement T to allow T to be a widening. If $U \subseteq S$ is a Zariski open subset of S, then $T|_U := (U, Z \cap U, z|_{Z \cap U})$ is again a widening of X/V, and so if E is a sheaf in $(X/V)_{conv}$, $E(T|_U) =: Mor(T|_U, E)$ is defined. Define a presheaf E_T on the topological space $S = Z$ by $E_T(U) := E(T|_U)$; one checks immediately that in fact E_T is a sheaf. Now by Proposition 0.2.1, we have an isomorphism: $h_T \cong \varinjlim\{h_{T_n}\}$, where each T_n is an enlargement, and hence $E(T) \cong Mor(h_T, E) \cong \varprojlim\{Mor(h_{T_n}, E)\} \cong \varprojlim E(T_n)$. Restricting to open subsets, we find that $E_T \cong \varprojlim\{\lambda_{n*} E_{T_n}\}$.

We will find it convenient to construct a sort of direct limit topos $\vec{T}$ as follows. We form the category whose objects are the open subsets of some T_n and where, if U_n is open in T_n and U_m is open in T_m with $n \leq m$, $Mor(U_n, U_m)$ is the set of maps $U_n \to U_m$ covering the given transition morphism $\phi_{m,n}: T_n \to T_m$. (Note that this set has at most one element, so we can write $U \subseteq U'$ if U and U' are objects and there exists a morphism $U \to U'$.) The coverings of U_n are just the ordinary coverings of $U_n \subseteq T_n$ in the Zariski topology. One verifies immediately that we have thus constructed a pretopology and that to give a sheaf on this pretopology is equivalent to giving a family of sheaves F_n on each T_n and transition morphisms $\psi_n: F_{n+1} \to \phi_n(F_n)$, where $\phi_n := \phi_{n+1,n}$. We denote the topos consisting of this category of sheaves by $\vec{T}$. Similarly, we can construct a topos $\vec{S}$ by setting $S_n := S$ for every n with the identity maps as transition maps. This topos is equivalent to the category of inverse system of sheaves on S, and we have an obvious morphism: $\gamma: \vec{S} \to S$ for which γ_* is just the inverse limit functor. If E is a sheaf in $(X/V)_{conv}$, then for each n we have a sheaf E_{T_n} on T_n and a transition map $\psi_n: E_{n+1} \to \phi_{n*}(E_n)$; these data define an object of $\vec{T}$ which we will denote simply by $E_{\vec{T}}$. In particular, we have obvious sheaves of rings $\mathcal{O}_{\vec{T}} := \mathcal{O}_{X/V,\vec{T}}$ and $\mathcal{K}_{\vec{T}} := \mathcal{K}_{X/V,\vec{T}} \cong K \otimes \mathcal{O}_{\vec{T}}$ on $\vec{T}$. There is also a commutative diagram of topoi:

$$\begin{tikzcd} T_n \arrow[r] \arrow[d, "\lambda_n"] & \vec{T} \arrow[d, "\lambda_{\vec{T}}"] \arrow[dr, "\gamma_T"] & \\ S \arrow[r] & \vec{S} \arrow[r, "\gamma"] & S \end{tikzcd}$$

It follows from the definitions and the remarks at the beginning of the previous paragraph that $\gamma_{T*}E_{\vec{T}} \cong E_T$ for any sheaf E in $(X/V)_{conv}$.

We shall call a sheaf E of $\mathcal{O}_{\vec{T}}$-modules "coherent" iff each E_n is coherent as an $\mathcal{O}_{T_n}$-module. We refer to [7] for the precise definition of a coherent sheaf of K_T-modules on a V-adic formal scheme T; a coherent sheaf of $\mathcal{K}_{\vec{T}}$-modules is a similarly compatible collection of coherent sheaves of $\mathcal{K}_{T_n}$-modules and transition maps. We say that a sheaf E of $\mathcal{O}_{\vec{T}}$ is "crystalline" iff the morphisms $\tilde{\psi}_n: E_{n+1} \otimes_{\mathcal{O}_{T_{n+1}}} \mathcal{O}_{T_n} \to E_n$ are isomorphisms; we say that E is "admissible" iff these morphisms are surjections. For example, the sheaves $\mathcal{K}_{X/V,\vec{T}}$, $\mathcal{O}_{X/V,\vec{T}}$, and$\mathcal{O}_{X,\vec{T}}$ are crystalline, and $I_{X/V,\vec{T}}$ is admissible.

We are now ready for the following key result.

Proposition 0.3.1. *Let $T := (S, Z, z)$ be a widening of X/V. Then*

$$\mathcal{O}_{X/V}(T) \cong \mathcal{O}_S(S) \quad \text{and} \quad \mathcal{O}_{X/V,T} \cong \mathcal{O}_S.$$

Before giving the proof, let us note the following immediate consequence.

Corollary 0.3.2. *The functor $h: Wide(X/V) \to (X/V)_{conv}$ is fully faithful.*

Proof of Proposition 0.3.1: It suffices to show that $\Gamma(\vec{T}, \mathcal{O}_{\vec{T}}) \cong \Gamma(S, \mathcal{O}_S)$, or that $\gamma_{T*}\mathcal{O}_{\vec{T}} \cong \mathcal{O}_S$. In fact, we will establish a more general result.

Proposition 0.3.3. *For any coherent sheaf of $\mathcal{O}_S$-modules F on S, the natural map $F \to \gamma_{T*}\gamma_T^* F$ is an isomorphism. For any coherent crystalline sheaf of $\mathcal{O}_{\vec{T}}$-modules E in $\vec{T}$, the natural map $\gamma_T^*\gamma_{T*}E \to E$ is an isomorphism. In fact, γ_T induces an equivalence between the categories of coherent sheaves of $\mathcal{O}_S$-modules on S and of coherent crystalline $\mathcal{O}_{\vec{T}}$-modules in $\vec{T}$.*

Proof: We may assume without loss of generality that T is absolutely affine; say $S = SpfB$, with Z the subscheme of definition defined by the ideal I. For each positive integer n, let J_n be the ideal $I^n + \pi B$, and let C_n be the affine piece of the blowup of J_n in B defined by π. (That is, C_n is the affine ring of $D^+(\pi) \subset ProjG^n$, where G^n is the graded ring $(B \oplus J_n \oplus J_n^2 \oplus \cdots)$.) Finally, we let B_n denote the π-adic completion of C_n, so that $T_n = SpfB_n$. Let $\mu_n: B \to C_n$ and $\lambda_n: B \to B_n$ be the natural maps.

Lemma 0.3.4. *For any two positive integers n and m, the image of the map $B_{n+mn} \to B_n$ is contained in $Im(\lambda_n) + \pi^m B_n$.*

Proof: It suffices to prove this before passing to completions. If $x \in J_{m+mn}$ we denote by $[x]$ the element y regarded as an element of degree 1 in G^{m+mn}. Then C_{m+nm} is the degree zero part of the localization of G^{m+mn} by $[\pi]$, which as a B-algebra is generated by elements y of the form $\{[x]/[\pi] : x \in I^{n+mn}\}$. But the map $C_{n+mn} \to C_n$ takes $[I^{n+mn}]$ to $I^{mn}[I^n]$, and any such element y will map to an element of $I^{mn}C_n \subseteq \pi^m C_n$. It follows immediately that $Im(C_{n+mn} \to C_n) \subseteq Im(\mu_n) + \pi^m C_n$. Q.E.D.

Lemma 0.3.5. *For any positive integers m and n, let*

$$J_{m,n} := Ker(B \to B_n/\pi^m B_n).$$

Then $I^{mn} \subseteq J_{m,n}$. Furthermore, there exists an integer N such that $J_{m,n} \subseteq I^m$ for all $n \geq N$. In particular, for $n \geq N$, the topology on B induced by the π-adic topology on B_n is the I-adic topology, and $\lambda_n: B \to B_n$ is injective.

Proof: It suffices to prove this statement with C_n instead of B_n. The inclusion $I^{nm} \subseteq J_n^m$ holds for any n and m, and hence $\mu_n(I^{nm}) \subseteq \mu_n(J_n^m) \subseteq \pi^m B_n$. For the reverse inclusion, note that by the Artin-Rees lemma there exists an integer N such that $I^{\nu+N} \cap \pi B \subseteq I^\nu \pi$ for all $\nu > 0$. I claim that for $n \geq N$, $\mu_n^{-1}(\pi^m B_n) \subseteq I^m$. Suppose that $b \in \mu_n^{-1}(\pi^m B_n)$. If $\mu_n(b) = \pi^m g$ with $g \in B_n$, we can write $g = x/[\pi]^d$, where $x \in G_d^n$, say.

Since B is π-torsion-free, so is G^n, and this means that $[\pi]^d b = \pi^m x$ in $G^n_d = J^d_n$. We can express this in the original ring B by simply saying that there is an element x of J^d_n such that $\pi^d b = \pi^m x$. Since $\pi \in I$, we conclude that $\pi^d b \in I^m J^d_n$. If $d = 0$ we see that $b \in I^m$, as claimed. If $d > 0$, note that $J^d_n = (I^n + \pi)^d = I^{nd} + I^{n(d-1)}\pi + \cdots \pi^d = I^{nd} + \pi J^{d-1}_n$, and so $\pi^d b \in I^{m+nd} + I^m \pi J^{d-1}_n$. Write $\pi^d b = y + \pi z$, where $y \in I^{m+nd}$ and $z \in I^m J^{d-1}_n$. Then since $d > 0$ we have $y \in I^{m+nd} \cap \pi B$, and since $n \geq N$, $y \in \pi I^{m+n(d-1)}$. We can now write $y = \pi y'$ and $\pi^{d-1} b = y' + z \in I^{m+n(d-1)} + I^m J^{d-1}_n = I^m J^{d-1}_n$. Continuing by induction, we find that in fact $b \in I^m$ as desired. Q.E.D.

Lemma 0.3.4 implies that the image of the map $B_{n+mn} \to B_n/\pi^m B_n$ is just $B/J_{m,n}$. Thus we have maps:

$$B \to C_{n+mn} \to B/J_{m,n} \to C_n/\pi^m C_n.$$

The composition of either pair of maps is just the natural transition map. Tensoring with any finitely generated B-module F, passing to the limit in n and m, and applying Lemma 0.3.5, we find that the natural map $F \to \varprojlim F \otimes B_n$ is an isomorphism. This proves half of Proposition 0.3.3.

For the other half, suppose $E = \{E_n\}$ is a compatible collection of finitely generated B_n-modules, with $E_{n+1} \otimes_{B_{n+1}} B_n \cong E_n$. By Lemmas 0.3.4 and 0.3.5, we have a natural map $B_{n+mn} \to B/J_{m,n} \to B/I^m$, and we set $F_m := E_{n+mn} \otimes_{B_{n+mn}} B/I^m$ for any $n \geq N$. Then there are natural isomorphisms: $F_{m+1} \otimes_{B/I^{m+1}} B/I^m \to F_m$, and hence $F := \varprojlim F_m$ is a finitely generated B-module such that $F \otimes_B B/I^m \cong F_m$ for each m. Moreover,

$$F \otimes_B B_n \cong F \otimes_B \varprojlim B_n/\pi^m B_n \cong \varprojlim F \otimes_B B_n/\pi^m B_n \cong$$

$$\varprojlim F \otimes_B \otimes B/I^{mn} \otimes_{B/I^{mn}} B_n/\pi^m B_n \cong \varprojlim F_{mn} \otimes_{B/I^{mn}} B_n/\pi^m B_n$$

$$\cong \varprojlim E_{n+mn^2} \otimes_B B_n/\pi^m B_n \cong \varprojlim E_n/\pi^m E \cong E_n.$$

Q.E.D.

Remark 0.3.6. If $T = (S, Z, z)$ is a widening of X/V, $\mathcal{K}_{X/V}(T)$ is the set of rigid analytic functions on the open tubular neighborhood $]S[_Z$ of radius one of Z in S as defined by Berthelot [1].

We shall also have to consider the higher derived functors of the functor γ_{T*}.

Proposition 0.3.7. *Let T be a widening of X/V. If $E_{\vec{T}}$ is an admissible sheaf of coherent $\mathcal{O}_{\vec{T}}$ or $\mathcal{K}_{\vec{T}}$-modules, then $R^q\gamma_{T*}E_{\vec{T}} = 0$ for $q > 0$. If T is absolutely affine, $H^q(\vec{T}, E_{\vec{T}}) = 0$ for $q > 0$.*

Proof: Clearly the second statement implies the first, so we may as well suppose that $T = (S, Z, z)$ with $S = Spf B$ and Z defined by an ideal I. Let **N** denote the topos consisting of inverse systems of sheaves indexed by the natural numbers **N**. We have an obvious morphism $\vec{\gamma}: \vec{T} \to \mathbf{N}$ for which $\vec{\gamma}_*$ takes a sheaf F of $\vec{T}$ to the inverse system of abelian groups $\{\Gamma(T_n, F_n)\}$; we are going to use the Leray spectral sequence for this morphism. Evidently $(R^q\vec{\gamma}_*F)_n \cong H^q(T_n, F_n)$. Since T_n is an affine formal scheme and E_n is coherent, each $H^q(T_n, E_n) = 0$ for $q > 0$. (*Cf.* [7] for the case of $\mathcal{K}_{T_n}$-modules.) Thus, $R^q\vec{\gamma}_*E_{\vec{T}} = 0$ for $q > 0$, and it suffices to show that $H^p(\mathbf{N}, \vec{\gamma}_*E_{\vec{T}}) = 0$ for $p > 0$. Now each $\Gamma(T_n, E_n)$ is a finitely generated B_n (or K_n)-module, which we denote again by E_n. Since $H^p(\mathbf{N}, \gamma_*E) \cong R^p \varprojlim\{E_n : n \in \mathbf{N}\}$, our proposition will follow from the following result in commutative algebra.

Proposition 0.3.8. *Let $\{E_n, \psi_n : n \in \mathbf{N}\}$ be a collection of finitely generated B_n or $K \otimes B_n$ modules and transitions maps inducing surjections:*

$$\tilde{\psi}_n: E_{n+1} \otimes_{B_{n+1}} B_n \to E_n.$$

Then $R^p \varprojlim\{E_n\} = 0$ for $p > 0$.

Proof: For $p > 1$ this is trivial, and for $p = 1$ it amounts to the following statement: any sequence $e. \in \Pi E_n$ is a coboundary, *i.e.* there exists a sequence $f \in \Pi E_n$ such that $\psi_{n+1}(f_{n+1}) - f_n = e_n$ for all n.

Claim 0.3.9. *For any positive integers m and n, let*

$$F^m E_n := Im(E_{n+mn} \to E_n).$$

Then for any $m' \geq m$ and $n' \geq n$,

$$F^m E_n \subseteq Im(F^{m'} E_{n'} \to E_n) + \pi^m E_n.$$

Proof: It suffices to show that for $j \geq n + mn$, the image of the map $E_j \to E_n/\pi^m E_n$ is independent of j, *i.e.*, that it contains the image of E_{n+mn}. To see this, note that since the map $: E_j \otimes_{B_j} B_{n+mn} \to E_{n+mn}$ is surjective, the image of E_{n+mn} in $E/\pi^m E_n$ is contained in the sub-B_{n+mn}-module generated by the image of E_j. But Lemma 0.3.4 above implies that B_{n+mn} and B_j have the same image in $B_n/\pi^m B_n$, and hence this image already is a B_{n+mn}-submodule. Q.E.D.

Now let

$$\begin{aligned} f_n &:= e_n + e_{n+1} + \cdots e_{n^2+n} \\ e'_n &:= e_{n^2+n+1} + \cdots e_{(n+1)^2+n+1}. \end{aligned}$$

(In the above sums, the elements e_{n+j} are first mapped into E_n and the addition is performed there.) Then we find that

$$\psi_n(f_{n+1}) - f_n = e'_n - e_n$$

Hence it suffices to prove that $e.'$ is a coboundary. Note that $e'_n \in F^n E_n$ for all n. We now proceed to choose sequences $g.$ and $e.''$ with $g_n \in F^n E_n$ and $e''_n \in \pi^n E_n$ such that

$$e'_n + g_n = \psi_n(g_{n+1}) + e''_n.$$

We do this inductively, starting with $g_1 = 0$, using Claim 0.3.9 above. Then $e.'$ (and hence also $e.$) will be a coboundary if $e.''$ is. But now as m tends to infinity, the image of e''_{n+m} in E_n tends to zero, and hence the infinite sum $h_n := e''_n + e''_{n+1} + \cdots$ (where again all the elements e''_{n+m} are mapped to E_n) converges. Since $h_n - \psi_n(h_{n+1}) = -e''_n$, our result is proved.

The case of $K \otimes \mathcal{O}_{\mathcal{T}}$-modules was treated by Kiehl [6] with slightly different hypotheses; we include it here for the sake of completeness. Suppose E_n is a finitely generated $K \otimes B_n$-module for each n and we have maps $\psi_n : E_{n+1} \to E_n$ inducing surjections $\tilde{\psi}_n$ as above. If L_n is a lattice in E_n and L_{n+1} is a lattice in E_{n+1}, the intersection of L_{n+1} with the inverse image of L_n is a lattice in E_{n+1}. Thus if we replace L_{n+1} by this intersection we may assume that the transition maps ψ_n are compatible with the lattice structure. Moreover, it follows from Lemma 0.2.3 that the image of ψ_n is dense in E_n. Now if we are given a sequence $e. \in \Pi E_n$, we can inductively find elements $f_n \in E_n$ and $e'_n \in \pi^n L_n$ such that $e_n + f_n = \psi_n(f_{n+1}) + e'_n$. Then the sequence $e.$ is cohomologous to $e'.$, and since the latter converges to zero, we can construct a coboundary for it as in the last step above.

Q.E.D.

4. Cohomology of Widenings

We are now ready to begin the study of the cohomology of sheaves in the convergent topos. By and large we can follow the techniques used in [2] to study crystalline cohomology. Of course, these techniques are variations on themes of Grothendieck and developments of Berthelot. First of all we can construct a morphism of topoi: $u_{X/V} : (X/V)_{conv} \to X_{zar}$ just as in the crystalline case: if $j : U \to X$ is a Zariski open immersion and E is a sheaf in $(X/V)_{conv}$, $u_{X/V*}(E)(U) := j^*_{conv}(E)(U/V_{conv})$, and if F is a sheaf on X_{zar} and (S,Z,z) is an enlargement of X/V, $u^*_{X/V}(F)_T := z^*F$. The reader can easily check that these are adjoint, and the exactness of $u^*_{X/V}$ follows from the exactness of z^* and Lemma 0.1.4.

If T is an object of $(X/V)_{conv}$ we let $H^q_{conv}(T, \ )$ denote the qth derived functor of the functor $\Gamma(T, \ )$. To compare this with the cohomology of the entire topos $(X/V)_{conv}$, it is useful to introduce the restricted topos $(X/V)_{conv}|_T$, *i.e.* the category of morphisms g in $(X/V)_{conv}$ with target T. There is a morphism $j_T\colon (X/V)_{conv}|_T \to (X/V)_{conv}$; recall that $j_T^*(G) = pr_T\colon G \times T \to T$, that $j_{T*}(g\colon F \to T)$ is the sheaf of cross-sections of g, and that j_T^* has an exact left adjoint $j_{T!}$ which takes a morphism $g\colon F \to T$ to its source F. All this is standard; *c.f.* [2] or [4]. It follows easily that, for any abelian object E of $(X/V)_{conv}$, there is a natural isomorphism:

$$H^q_{conv}(T,E) \to H^q((X/V)_{conv}|_T, j_T^*E).$$

Let $u_T\colon (X/V)_{conv}|_T \to X_{zar}$ denote the composite $u_{X/V} \circ j_T$. If T is an enlargement (S,Z,z) of X/V we can construct a commutative diagram of topoi which will allow us to compare various topologies, just as in the crystalline case [2]. Although the Zariski topologies of S and Z are the same, as ringed topoi they are different, and we emphasize this by the choice of notation in the diagram below.

$$\begin{tikzcd}
(X/V)_{conv}|_T \arrow[r, "\phi_T"] \arrow[d, "j_T"'] & S_{zar} \arrow[d, "z_S"] & Z_{zar} \arrow[l] \arrow[dl, "z"] \\
(X/V)_{conv} \arrow[r, "u_{X/V}"] & X_{zar} &
\end{tikzcd}$$

To construct the morphism ϕ_T, let F be a sheaf on Z_{zar}. It suffices to specify $\phi_T^*(F)$ on objects $g\colon T' \to T$ of $(X/V)_{conv}|_T$ whose source is an enlargement $T' := (S',Z',z')$ of X/V. Then g induces a morphism $g_Z\colon Z'_{zar} \to Z_{zar}$, and we define $\phi_T^*(F)(g)$ to be $g_Z^*(F)(Z')$. If G is a sheaf on $(X/V)_{conv}|_T$ and $U \subseteq Z$ is a Zariski open subset, we define $\phi_{T*}(G)(U)$ to be $G(g_U)$, where g_U is the inclusion $(S|_U, U, z|_U) \to (S,Z,z)$. The necessary adjointness and exactness properties of these constructions, as well as the commutativity of the above, are immediate. Note that $\phi_{T*}j_T^*F$ is just the Zariski sheaf F_T associated to G and $T = (S,Z,z)$.

If T is a widening of X/V we have to modify these techniques a little. The techniques above define a morphism of topoi $\phi_T\colon (X/V)_{conv}|_T \to S$, and a functor $\phi_{\vec{T}*}(X/V)_{conv}|_T \to \vec{T}$, but it does not seem to be possible to construct an adjoint $\phi^*_{\vec{T}}$ in any natural way. We still have a commutative diagram:

$$\begin{array}{ccccc} (X/V)_{conv}|_T & \xrightarrow{\phi_{\vec{T}*}} & \vec{T} & \xrightarrow{\gamma_{T*}} & S_{zar} \\ {\scriptstyle j_{T*}}\downarrow & & \downarrow{\scriptstyle z_{\vec{T}*}} & & \updownarrow \\ (X/V)_{conv} & \xrightarrow{u_{X/V*}} & X_{zar} & \xleftarrow{z*} & Z_{zar} \end{array}$$

It is useful to remark that we *can* define $\phi^*_{\vec{T}}$ on the category of "crystalline" sheaves. For example, if F is a crystalline sheaf of $\mathcal{O}_{\vec{T}}$-modules and $g: T' \to T$ is an object of $(X/V)_{conv}|_T$, with $T' = (S', Z', z')$ an enlargement of X/V, we have seen that for n large enough g factors uniquely through a map $g_n: T' \to T_n$. The crystalline nature of F implies that the sheaf $g_n^* F_n$ is independent of the choice of n, so we can define $\phi^*_{\vec{T}}(F)$ to be $g_n^*(F_n)$. It is clear that this construction is compatible with morphisms in $(X/V)_{conv}|_T$, and thus $\phi^*_{\vec{T}}(F)$ is a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}|_T$. In particular, if F is a sheaf of $\mathcal{O}_S$-modules, then $\gamma^*_{\vec{T}}(F)$ is a crystalline sheaf of $\mathcal{O}_{\vec{T}}$-modules and $\phi^*_{\vec{T}}\gamma^*_{\vec{T}}(F) \cong \phi^*_T(F)$ is a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}|_T$. Conversely, it is apparent from the definition that if E is any crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}|_T$, $\phi_{\vec{T}*}(F)$ is a crystalline sheaf of $\mathcal{O}_{\vec{T}}$-modules, and the natural map $\phi^*_{\vec{T}}\phi_{\vec{T}*}(E) \to E$ is an isomorphism.

If $g: T' \to T$ is an object of $(X/V)_{conv}|_T$ with T' a widening of X/V and if E is any object of $(X/V)_{conv}|_T$, we can define in the obvious way a sheaf $E_{\vec{g}}$ in the topos $\vec{T}'$; this sheaf will be a sheaf of $\mathcal{O}_{\vec{T}}$-modules if E is a sheaf of $\mathcal{O}_{X/V}$-modules. We will say that E is "admissible" iff each $E_{\vec{g}}$ is coherent and admissible, with a similar definition for sheaves of $\mathcal{K}_{X/V}$-modules.

Because of the unfortunate fact that $\phi_{\vec{T}*}$ is not part of a morphism of topoi it is not *a priori* clear that it will be compatible with Leray spectral sequences. We shall have to verify that $\phi_{\vec{T}*}$ takes injectives to acyclics "by hand." We shall say that a sheaf F in $\vec{T}$ is "flasque" iff whenever $U \subseteq U'$, the transition map $F(U') \to F(U)$ is surjective. It is quite standard to verify that an abelian flasque sheaf is acyclic for the global section functor $\Gamma(\vec{T},\ )$ and for the functor γ_T.

Lemma 0.4.1. *If T is a widening of X/V, the functor $\phi_{\vec{T}*}: (X/V)_{conv}|_T \to \vec{T}$ takes injectives sheaves to flasque sheaves.*

Proof: We have a morphism of topoi $j_n: (X/V)_{conv}|_{T_n} \to (X/V)_{conv}|_{T_{n+1}}$. The functor j_n^* has an exact adjoint $j_{n!}$. Its abelian incarnation (for which we abusively use the same notation) is also exact. Specifically, if G is an abelian sheaf in $(X/V)_{conv}|_{T_n}$, then $j_{n!}$ is the sheaf associated to the presheaf P:

$$P(h: S \to T_{n+1}) = \oplus\{G(g) : \phi_n \circ g = h\}.$$

Since the morphism $\phi_n: T_n \to T_{n+1}$ is a monomorphism by Lemma 0.2.2, we see that $P(h)$ is zero unless h factors through ϕ_n, and if it does factor, the factorization $\phi_n \circ g = h$ is unique and $P(h) = G(g)$. Thus, $j_{n!}$ deserves to be called "extension by zero," as one would hope. If H is an abelian sheaf in $(X/V)_{conv}|_{T_{n+1}}$, $j_n^*(H)(g) = H(\phi_n \circ g)$, and it is clear that the adjunction morphism $j_{n!}j_n^*H \to H$ is injective. In fact, the same argument applies whenever $j: U \subseteq U'$ is a morphism in the site defining $\vec{T}$. Now if I is an injective object of $(X/V)_{conv}|T$, its restriction to $(X/V)_{conv}|_{U'}$ is still injective. Since the map $j_!j^*\mathbf{Z}_{U'} \to \mathbf{Z}_{U'}$ is injective, the map $I(U') \to I(U)$ is surjective. Q.E.D.

Proposition 0.4.2. *Let $T := (S, Z, z)$ be a widening of X/V. Then the functor $\phi_{\vec{T}*}: (X/V)_{conv}|_T \to \vec{T}$ is exact. If E is an abelian sheaf in $(X/V)_{conv}|_T$, there is a natural isomorphism: $H^q(\vec{T}, E_{\vec{T}}) \cong H^q((X/V)_{conv}|_T, E)$. If T is absolutely affine and E is an admissible sheaf of $\mathcal{O}_{X/V}$ or $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}|_T$ then these groups vanish for $q > 0$.*

Proof: The first statement follows immediately from the definitions and the analogue of Lemma 0.1.4 for the localized topos $(X/V)_{conv}|_T$. We have seen that $\Gamma((X/V)_{conv}|_T, \ )$ can be written as a composite of the two functors $\Gamma(\vec{T}, \ )$ and $\phi_{\vec{T}*}$. Lemma 0.4.1 above tells us that the spectral sequence for the composite of these functors exists, and the exactness of $\phi_{\vec{T}*}$ implies that the spectral sequence degenerates. Since $\phi_{\vec{T}*}E \cong E_{\vec{T}}$, the second statement is proved. Finally, Proposition 0.3.7 implies that $H^q(\vec{T}, E_{\vec{T}}) = 0$ if E is a coherent sheaf of $\mathcal{O}_{X/V}$ or $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}|_T$ such that $E_{\vec{T}}$ is admissible. Q.E.D.

Proposition 0.4.3. *If $T = (S, Z, z)$ is an affine widening of X/V and E is an admissible sheaf of $\mathcal{O}_{X/V}$ or $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}|_T$ then $R^q u_{T*}E$ and $R^q j_{T*}E$ vanish for $q > 0$, while $u_{T*}E \cong E_T$.*

Proof: We may assume without loss of generality that X is affine. Then so is Z, and the previous proposition then tells us that $H^q((X/V)_{conv}|_T, E) = 0$ for $q > 0$. Since $R^q u_{T*}E$ is the sheaf associated to the presheaf $U \mapsto H^q((U/V)_{conv}|_T, E)$, in fact $R^q u_{T*}E = 0$ for $q > 0$. To see that $R^q j_{T*}(E) = 0$, it is enought to show that $R^q j_{T*}(E)_{T'} = 0$ for each enlargement T'. We have a diagram:

$$\begin{tikzcd}
S'\hat{\times}S \arrow[dr, "pr_{S'}"] & T'\vec{\times}T \arrow[l, "\gamma_{T'\times T}"'] \arrow[d, "pr_{\vec{T}'}"] & (X/V)_{conv}|_{T'\times T} \arrow[l, "\phi_{T'\vec{\times}T*}"'] \arrow[r, "j_{T'|_T}"] \arrow[d, "j_{T|_{T'}}"] & (X/V)_{conv}|_T \arrow[d, "j_T"] \\
 & S' & (X/V)_{conv}|_{T'} \arrow[l, "\phi_{T'}"'] \arrow[r, "j_{T'}"] & (X/V)_{conv}
\end{tikzcd}$$

One verifies immediately that $j_{T'}^* \circ j_{T*} \cong j_{T|_{T'}*} \circ j_{T'|_T}^*$. Moreover, because the left adjoint $j_{T'!}$ of $j_{T'}^*$ is exact, the functor $j_{T'}^*$ takes injectives to injectives. This argument applies as well to the functor $j_{T'|_T}^*$. Using these facts, Lemma 0.4.1, and the exactness of $j_{T'|_T}^*$, $\phi_{T'*}$, $\phi_{T'\vec{\times}T*}$ and $j_{T'}^*$, we see that

$$\begin{aligned} (R^q j_{T*}E)_{T'} &\cong \phi_{T'*} j_{T'}^* R^q j_{T*} E \\ &\cong \phi_{T'*} R^q j_{T|_{T'}*} j_{T'|_T}^* E \\ &\cong R^q pr_{T'*} \phi_{T'\vec{\times}T*} j_{T'|_T}^* E \end{aligned}$$

All this holds for any abelian sheaf E on $(X/V)_{conv}|_T$. Assume that $T' = (S', Z', z')$ is an absolutely affine enlargement of X/V. Then $T' \times T$ is also absolutely affine, and if E is admissible, Proposition 0.4.2 implies the vanishing of $H^q(T'\vec{\times}T, \phi_{T'\vec{\times}T*}E)$ for $q > 0$. Sheafifying on S we find that $R^q pr_{T'*}E = 0$ for $q > 0$, and the proposition is proved. Q.E.D.

The following corollary is an immediate consequence of the two previous propositions.

Corollary 0.4.4. *If T is an affine widening of X/V and if E is an admissible sheaf of $\mathcal{O}_{T/V}$ or $\mathcal{K}_{T/V}$ modules on $(X/V)_{conv}|_T$, $j_{T*}(E)$ is acyclic for $u_{X/V*}$.*

5. Crystals and Connections

We begin with a description of the behaviour of crystals under pushing forward via closed immersions. Although this result will not be used explicitly, it does provide a prototype for dealing with some of the technicalities that arise later.

Proposition 0.5.1. *Let E be a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$ and let $i: X \to X'$ be a closed immersion. Then i_*E is an Artin-Rees pro-crystal in $(X'/V)_{conv}$ in the following sense: there is a family $\{E_n : n \in \mathbf{N}\}$ of $\mathcal{O}_{X'/V}$-modules in $(X'/V)_{conv}$ such that $i_*E \cong \varprojlim E_n$ and such that for every morphism $g: T' \to T$ in $Enl(X'/V)$, the natural maps*

$$\rho_{g_{E_n}}: g^* E_{n,T} \to E_{n,T'}$$

induce an isomorphism of Artin-Rees pro-objects.

Proof: Let $T := (S, Z, z)$ be an enlargement of X'/V. Then $i_* E(T) \cong E(i^* T)$ and $i^* T \cong \varinjlim\{T_n : n \in \mathbf{N}\}$. Hence $i_* E(T) \cong \varprojlim E(T_n)$. If we define E_n to be $\lambda_{n*}(E_{T_n})$, where $\lambda_n : T_n \to S$ is the natural map, it is clear that $i_* E \cong \varprojlim\{E_n : n \in \mathbf{N}\}$. Suppose that $T' \to T$ is a morphism of enlargements of X'/V, where $T' := (S', Z', z')$ and $T := (S, Z, z)$. Let $Z'' := g_{S'}^{-1}(Z) \subseteq S'$, let $z'' := Z'' \to Z \to X'$, and let $T'' := (S', Z'', z'')$. Then we can factor the morphism g as $g = g'' \circ g'$, where $g' : T' \to T''$ and $g'' : T'' \to T$.

We observe first that the maps $g''^* E_{n,T} \to E_{n,T''}$ are isomorphisms. Indeed, we have by Lemma 0.2.4 a cartesian diagram:

$$\begin{array}{ccc} T_{Z',n}(S') & \xrightarrow{\tilde{g}} & T_{Z,n}(S) \\ \downarrow{\scriptstyle \lambda''_n} & & \downarrow{\scriptstyle \lambda'_n} \\ S' & \xrightarrow{g_{S'}} & S. \end{array}$$

Since the maps λ''_n and λ'_n are affine, our diagram will commute with push-foward and pullback. We find:

$$\begin{aligned} g_{S'}^* E_{n,T} &\cong g_{S'}^* \lambda'_{n*} E_{T_{X,n}(S,Z,z)} \\ &\cong \lambda''_{*n} \tilde{g}^* E_{T_{X,n}(S,Z,z)} \\ &\cong \lambda''_{*n} E_{T_{X,n}(S',Z'',z'')} \\ &\cong E_{n,T''} \end{aligned}$$

Next we consider the map $g' : T' \to T''$. Here we are in the situation of lemmas 0.2.5 and 0.2.6, and we obtain maps as follows:

$$\begin{aligned} E_{(T_{Z',n+m}(S'), Z'_{n+m}, z'_{n+m})} &\to h_{n*} E_{(T_{Z'',n}(S'), Z''_n, z''_n)} \\ \lambda_{n+m*} E_{(T_{Z',n+m}(S'), Z'_{n+m}, z'_{n+m})} &\to \lambda_{n*} E_{(T_{Z'',n}(S'), Z''_n, z''_n)} \\ E_{n+m,T'} &\to E_{n,T''} \\ E_{n+m,T'} &\to g'^* E_{n,T''}. \end{aligned}$$

It should be clear that these maps define the requisite inverse of the associated Artin-Rees pro-object. Q.E.D.

Theorem 0.5.2. *There is a natural equivalence of categories between the categories of coherent convergent and coherent infinitesimal crystals of $\mathcal{O}_{X/V}$-modules.*

Proof: This is an immediate consequence of Proposition 0.3.3 above.

Q.E.D.

Note that if (S, Z, z) is an enlargement of X/V, then any nilpotent immersion $i: S \to S'$ defines a new enlargement (S', Z, z) of X/V if S' is flat over V. In particular, if E is a crystal of $\mathcal{O}_{X/V}$-modules on $(X/V)_{conv}$, the value $E_{(S',Z,z)}$ of E on (S', Z, z) provides a canonical extension of $E_{(S,Z,z)}$ to S'. This construction almost defines a crystal of $\mathcal{O}_S$-modules on the nilpotent site of S/V—the only trouble is the flatness requirement on S'. If E is a crystal of $\mathcal{K}_{X/V}$-modules this flatness becomes irrelevant, and we obtain for each (S, Z, z) a crystal of $K \otimes \mathcal{O}_S$-modules as explained in [7].

We need to give a similar construction for the Artin-Rees objects associated to widenings. Let $T := (S, Z, z)$ be a widening of X/V, let E be a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$, and let $E_T := \varprojlim\{E_n\}$ be the associated Zariski sheaf on S. We shall see that E_T has an integrable connection as an $\mathcal{O}_S$-module, in a suitable sense. Let S' denote the mth infinitesimal neighborhood of the diagonal of $S \times S$, with its two projections $pr_i: S' \to S$. Note that S' will be flat over V if, for example, S/V is smooth, as we shall assume. Let $Z_i' := pr_i^{-1}(Z)$, let $z_i': Z_i' \to X := z \circ pr_i$ and let $Z \to S'$ via the diagonal. Then we have morphisms of widenings:

$$\begin{array}{ccc} (S', Z, z) & \longrightarrow & (S', Z_i', z_i') \\ & \searrow & \downarrow{\scriptstyle pr_i} \\ & & (S, Z, z) \end{array}$$

Let J be the ideal of the diagonal in S', so that $J^{m+1} = 0$. Since $I_Z = I_{Z_i'} + J$, we can construct as in Proposition 0.2.6 maps

$$E_{T_{Z,n+m}(S',Z,z)} \to E_{T_{Z,n}(S',Z_i',z_i')}.$$

It follows from the base change lemma 0.2.4 that

$$T_n(S', Z_i', z_i') \cong T_n(S, Z, z) \times_{S,pr_i} S',$$

so that there are canonical isomorphisms: $pr_i^* E_n \cong E_{T_n(S',Z_i',z_i')}$. We deduce the existence of maps:

$$pr_2^* E_{n+m} \cong E_{T_{n+m}(S',Z_i',z_i')} \to E_{T_{n+m}(S',Z,z)} \to E_{T_{Z,n}(S',Z_i',z_i')} \cong pr_1^* E_n.$$

These maps fit together to form a stratification on the Artin-Rees pro-object associated to $\{E_n\}$. Let us summarize this as follows:

Proposition 0.5.3. *Let $T := (S, Z, z)$ be a widening of X/V, with S/V formally smooth, let E be a crystal of $\mathcal{O}_X$-modules in $(X/V)_{conv}$, and let $\{E_n\}$ be the associated inverse system of sheaves of $\mathcal{O}_S$-modules. Then for each n there is a map*

$$\nabla_{E_S}: E_{n+1} \to \Omega_{S/V} \otimes E_n$$

inducing an integrable connection on the Artin-Rees pro-$\mathcal{O}_S$-module associated to $\gamma_{T*}E_{\tilde{T}}$.

We now must develop the Poincaré lemma in the context of convergent cohomology. This can be done just as in the crystalline case. We choose a method of exposition that avoids explicit mention of the linearization of differential operators, but in fact that is what is happening.

We suppose again that X can be embedded as a closed subscheme of a formal scheme Y which is formally smooth over V. Then $T := (Y, X, id_X)$ is a widening of X/V, and for any coherent sheaf F of $\mathcal{O}_Y$-modules, $\phi_T^* F$ is a crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}|_T$. For any crystal E of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$, we set $\Omega^i_T(E) := j_{T*}(\phi_T^*\Omega^i_{Y/V} \otimes j_T^* E)$, a sheaf in $(X/V)_{conv}$. We shall construct maps of sheaves:

$$d^i : \Omega^i_T(E) \to \Omega^{i+1}_T(E)$$

which form a complex $\Omega^{\cdot}_T(E)$.

Proposition 0.5.4. *Let E be a coherent crystal of $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$ and let $T = (Y, X, id_X)$, with Y/V formally smooth. Then there exist a natural complex $\Omega^{\cdot}_T(E)$ and a quasi-isomorphism $E \to \Omega^{\cdot}_T(E)$ of complexes of sheaves in $(X/V)_{conv}$.*

Proof: Let $T' = (S', Z', z')$ be an enlargement of X/V. To simplify the notation, we assume that T and T' are absolutely affine; the general case follows by gluing. Now $\Omega^0_T(E) \cong j_{T*}j_T^*E$, so $\Omega^0_T(E)(T') = E(T' \times T)$. Note that this group has the structure of an $\mathcal{O}_Y(Y)$-module, we shall see that in fact it has an integrable connection as $\mathcal{O}_Y(Y)$-module; we then obtain the complex $\Omega_T(E)^{\cdot}$ as the De Rham complex of this connection. The construction will be essentially the same as the construction in Proposition 0.5.3.

Let $E' := j_T^*(E)$. Then $(j_{T'*}E')(T') \cong \varprojlim\{E'_n\}$, where $E'_n := E'_{T_{X,n}(T' \times T)}$. Each of these modules E'_n can be viewed as an $\mathcal{O}_Y(Y)$-modules, and we will see that the corresponding Artin-Rees pro-object has a stratification. Let Y' denote the mth infinitesimal neighborhood of the diagonal of $Y \times Y$, let $X'_i := pr_i^{-1}(X)$, and let $z'_i : X'_i \to X$ be the obvious projection. We can also embed X in Y' via the diagonal. Then (Y', X, id_X) and (Y', X'_i, z'_i) are widenings of X/V, with $T' \times (Y', X, id_X) = (S' \hat{\times} Y', Z', z')$ and $T' \times (Y', X'_i, z'_i) = (S' \hat{\times} Y', Z'_i, z'_i)$, where $Z'_i := Z \times_X X'_i$. There is a commutative diagram:

$$\begin{array}{ccc} (S'\hat{\times}Y', Z', z') & \longrightarrow & (S'\hat{\times}Y', Z'_i, z'_i) \\ & \searrow & \Big\downarrow{\scriptstyle pr_i} \\ & & (S'\hat{\times}Y, Z', z') \end{array}$$

Let J be the ideal of the diagonal in Y' and let J' be its pullback to $S'\hat{\times}Y'$. Then the ideal of Z' in $S'\hat{\times}Y'$ is the sum of J' and $I_{Z'_i}$. so again we can construct maps

$$E_{T_{X,n+m}(S'\hat{\times}Y',Z',z')} \to E_{T_{X,n}(S'\hat{\times}Y',Z'_i,z'_i)}.$$

As $Z'_i = pr_i^{-1}(Z')$, the base change lemma basechange tells us that

$$T_{X,n}(S'\hat{\times}Y, Z', z') \times_{S'\hat{\times}Y, pr_i} Y' \cong T_{X,n}(S'\hat{\times}Y', Z'_i, z'_i).$$

We conclude as before the existence of maps: $pr_2^* E'_{n+m} \to pr_1^* E'_n$, which fit together to form a stratification as desired.

The previous paragraph applies to any crystal E' of $\mathcal{K}_{X/V}$-modules on $(X/V)_{conv}|_T$. If $E' = j_T^* E$, we get a natural map $E \to j_{T*} j_T^* E$. We shall see that this map determines a quasi-isomorphism $E \to \Omega^{\cdot}_T(E)$. This verification is a local question; *i.e.* it suffices to check that for each absolutely affine enlargement T' of X/V, the map $E_{T'} \to \Omega^{\cdot}_T(E)_{T'}$ is a quasi-isomorphism. Now

$$(j_{T*}\phi_T^*\Omega^i_{Y/V} \otimes j_T^* E)_{T'} \cong pr_{T'*}(\phi_T^*\Omega^i_{Y/V} \otimes j_T^* E)_{T'\times T},$$

and $T'\times T \cong (S'\hat{\times}Y, Z, z)$. It is clear from 0.3.6 that the value of $\phi_T^*(\Omega^i_{Y/V}) \otimes j_T^* E)$ on $T' \times T$ can be identified with the global sections of the sheaf $\Omega_{S'\times Y/Y} \otimes E_T$ on the rigid analytic tube $(S' \times Y)_{]Z[}$, and we can argue as does Berthelot in his proof of the rigid analytic Poincaré lemma [1] to conclude that our map is a quasi-isomorphism. Q.E.D.

6. Cohomology of the Convergent Topos

Let us first note that convergent cohomology is invariant under infinitesimal thickenings.

Proposition 0.6.1. *If $i: X \to X'$ is a nilpotent immersion, the functor*

$$i_*: (X/V)_{conv} \to (X'/V)_{conv}$$

is exact. Moreover, $i_*\mathcal{O}_{X/V} \cong \mathcal{O}_{X'/V}$ and $i_*\mathcal{K}_{X/V} \cong \mathcal{K}_{X'/V}$, so that we have natural isomorphisms:

$$H^q_{conv}(X'/V, \mathcal{O}_{X'/V}) \to H^q_{conv}(X/V, \mathcal{O}_{X/V})$$

$$H^q_{conv}(X'/V, \mathcal{K}_{X'/V}) \to H^q_{conv}(X/V, \mathcal{K}_{X/V}).$$

Proof: If (S', Z', z') is an enlargement of X'/V, $Z := X \times_{X'} Z' \to Z'$ is a nilpotent immersion, and hence Z is a subscheme of definition of S'. Then $z := z'|_Z$ defines the structure of an enlargement of X; and (S', Z, z) represents the functor $i^*(S', Z', z')$. It follows that $i_*(F)_{(S',Z',z')} \cong F_{(S',Z,z)}$, and this makes everything clear. Q.E.D.

Now suppose that $i: X \to Y$ is a closed immersion from X to a formal scheme Y which is formally smooth over V. It is immediate that the object i^*Y of $(X/V)_{conv}$ covers the final object, which we denote by X/V. In particular, a sequence Σ of abelian sheaves in $(X/V)_{conv}$ is exact iff $j^*_{i^*(Y)}\Sigma$ is. We can use this remark to construct resolutions in the usual way. Let $Y(n) := i^*(Y \hat{\times} Y \hat{\times} \dots Y)$ (where we have taken $n+1$ copies of Y in the product). If E is an abelian sheaf on $(X/V)_{conv}$ let $C^n_Y(E) := j_{Y(n)*}j^*_{Y(n)}(E)$. Then the obvious maps $C^n_Y(E) \to C^{n+1}_Y(E)$ make this collection into a complex, and it follows that this complex is a resolution of E. If the sheaf E is admissible as $\mathcal{O}_{X/V}$ or $\mathcal{K}_{X/V}$-module, our resolution is even acyclic for $u_{X/V*}$, by 0.4.4. Moreover, $u_{X/V*}C^n_Y(E) \cong E_{Y(n)}$; we denote the corresponding complex of Zariski sheaves on X by $CA^{\cdot}_Y(E)$.

Theorem 0.6.2. *There is a canonical isomorphism in the derived category:*

$$Ru_{X/V*}E \cong CA^{\cdot}_Y(E).$$

For example, if E is a coherent sheaf of $\mathcal{O}_X$-modules, define a sheaf $i_{X/V*}(E)$ on $(X/V)_{conv}$ by $i_{X/V*}(E)(S, Z, z) := z^*(E)(Z)$ for any enlargement (S, Z, z) of X/V. This obviously defines an admissible sheaf of $\mathcal{O}_{X/V}$-modules, so the sheaves $C^n_Y(E)$ are $u_{X/V*}$-acyclic. It follows from Proposition 0.3.3 that $E^n_Y \cong E$ for all n. Since the boundary maps are zero in even degrees and the identity in odd degrees, we can conclude:

Corollary 0.6.3. *If E is a coherent sheaf of $\mathcal{O}_X$-modules, there are canonical isomorphisms:*

$$H^q_{conv}(X/V, i_{X/V*}(E)) \cong H^q_{zar}(X, E).$$

Theorem 0.6.4. *Let E be a coherent crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{conv}$ and let $\tilde{E}$ be the corresponding crystal of $\mathcal{O}_{X/V}$-modules in $(X/V)_{inf}$. Then there is a canonical isomorphism: $H^q_{inf}(X/V, \tilde{E}) \cong H^q_{conv}(X/V, E)$ for all q.*

Proof: We prove this under the assumption that X can be embedded in a smooth V-scheme Y, leaving the to the reader the task of applying cohomological descent to deduce the general case. Our language expresses

Grothendieck's calculation of the infinitesimal cohomology of $\tilde{E}$ as the existence of a canonical isomorphism: $Ru_{X/V*}\tilde{E} \cong CA^{\cdot}(E)_Y$, where $\tilde{E}^{\cdot}_Y$ is constructed in the infinitesimal topos just as $E^{\cdot}_Y$ was in the convergent topos. But Proposition 0.3.3 tells us that there is a canonical isomorphism $\tilde{C}A^n(E)_Y \cong E^n_Y$ for each n. Since these isomorphisms are evidently compatible with the boundary maps, the theorem follows. Q.E.D.

Combining this with the main result of [8], we find:

Corollary 0.6.5. *If X/V is proper and k is algebraically closed, there are canonical isomorphisms:*

$$H^q_{conv}(X/V, \mathcal{O}_{X/V}) \cong H^q_{inf}(X/V, \mathcal{O}_{X/V}) \cong H^q_{et}(X, \mathbf{Z}_p) \otimes V.$$

Theorem 0.6.6. *If E is a coherent sheaf of $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$ and $X \to Y$ is an embedding of X in a formally smooth formal V-scheme Y, then there is a natural isomorphism in the derived category:*

$$Ru_{X/V*}E \cong u_{T*}(\Omega^{\cdot}_Y \otimes E_Y),$$

where $(\Omega^{\cdot}_{Y/V} \otimes E_Y)$ is the De Rham complex of the integrable connection on E_Y constructed in Proposition 0.5.3.

Proof: We have just seen that there is a quasi-isomorphism:
$E \to \Omega^{\cdot}_T(E)$ in $(X/V)_{conv}$. Moreover, it follows from Proposition 0.4.4 that each sheaf of the complex $\Omega^{\cdot}_T(E)$ is acyclic for $u_{X/V*}$. Finally, we have only to note that $u_{X/V*}\Omega^{\cdot}_T(E) \cong (\Omega^{\cdot}_{Y/V} \otimes E_Y)$. Q.E.D.

Theorem 0.6.7. *If X/k is proper, there is a canonical isomorphism*

$$H^i_{conv}(X/V, \mathcal{K}_{X/V}) \cong H^i_{rig}(X/K)$$

Proof: In fact, this follows directly from the above result and Berthelot's definition of rigid cohomology. Q.E.D.

Corollary 0.6.8. *If Y/V is a smooth formal lifting of X/k, then there is a canonical isomorphism:*

$$H^i_{conv}(X/V, \mathcal{K}_{X/V}) \cong K \otimes H^i_{DR}(Y/V).$$

Proof: This is an immediate consequence of 0.6.6. Q.E.D.

7. Crystals and isocrystals

For most of this section we suppose that V is the Witt ring of k, $\pi = p$, and X is a smooth k scheme of finite type. Our goal is to justify our habit of referring to coherent crystals of $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$ as "convergent isocrystals." Let us recall the definition of the category of isocrystals.

Definition 0.7.1. *The category of isocrystals in $CrisX/W$ is the category whose objects are the crystals E of finite type $\mathcal{O}_{X/V}$-modules and with morphisms given by*

$$Mor_{iso}(E, F) := K \otimes Mor_{cris}(E, F)$$

for any two crystals E and F.

Thus, we have an essentially surjective functor from the category of crystals of finite type $\mathcal{O}_{X/V}$-modules in $CrisX/W$ to the category of isocrystals. It is natural to denote this functor by $E \mapsto K \otimes E$.

We are now ready to state the main result of this section.

Theorem 0.7.2. *Suppose that $V = W$ and X/k is smooth. Then the category of coherent crystals of $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$ is equivalent to a full subcategory of the category of isocrystals in $CrisX/W$.*

Before beginning the proof, we have to investigate the category of isocrystals in some detail. If L is a crystal of finite type $\mathcal{O}_{X/V}$-modules in $CrisX/W$ and T is an object of $CrisX/W$ we can set $(K \otimes L)(T) := K \otimes L(T)$. This defines a functor from the category of isocrystals to the category of abelian sheaves in $CrisX/W$. Now if $X \subseteq Y$ is a closed immersion of X into a formally smooth p-adic formal V-scheme Y, let $D = D_X(Y)$ be the (p-adically complete) divided power envelope of X in Y, and let $D(1) = D_X(Y \times Y)$ and $D(2) = D_X(Y \times Y \times Y)$, with the usual maps p_i, p_{ij}. We obtain a sheaf E_D of finite type $K \otimes \mathcal{O}_{D_X(Y)}$-modules together with an element

$$\epsilon \in Hom_{D(1)}(p_2^* E_D, p_1^* E_D)$$

satisfying the cocycle coditions:

1. $\Delta^*(\epsilon) = id \in End_D(E_D)$

2. $p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon) \in Hom_{D(2)}(p_3^* E_D, p_1^* E_D)$

We shall refer to such an ϵ as an "HPD iso-stratification" on E_D. Evidently we have a functor which associates to any isocrystal E a sheaf of finite type $K \otimes \mathcal{O}_D$-modules E_D with HPD iso-stratification.

Proposition 0.7.3. *The category of isocrystals in $Cris X/W$ is equivalent to the category of finite type $K \otimes \mathcal{O}_D$-modules with HPD iso-stratification.*

Proof: It is easy to see that our functor is fully faithful. The main point is to show that it is essentially surjective. We begin by discussing the liftable case. The next lemma will serve as our technical key; evidently it proves the proposition in the liftable case.

Lemma 0.7.4. *Suppose Y/V is a formally smooth p-adic formal scheme and L is a coherent sheaf of $\mathcal{O}_Y$-modules with HPD iso-stratification ϵ on $K \otimes L$. Let $U \subseteq Y$ be an open subset of Y, possibly empty, and let L' be a coherent sheaf of $\mathcal{O}_Y$- modules on U with an isomorphism $\alpha: K \otimes L' \to K \otimes L|_U$ such that L' is stable under the HPD iso-stratification $\epsilon|_U$. Then there exist an ϵ-stable coherent extension L'' of L' to Y and an isomorphism $\alpha': K \otimes L'' \to K \otimes L$ extending α and compatible with ϵ.*

Proof: Replacing α by $p^n\alpha$ for a suitable n, we may assume that α comes from an actual morphism $L' \to L|_U$. Replacing L' by its image in $L|_U$, we may assume that $L' \subseteq L|_U$. Choose m so that p^m annihilates $L|_U/L'$, and let L_m be the restriction of L to the subscheme of definition Y_m defined by p^m and $L'_m \subseteq L_m$ the image of L' in L_m. Choose a coherent extension $M_m \subseteq L_m$ of L'_m to Y_m and let $M \subseteq L$ be the inverse image of M_m. Then $K \otimes M \cong K \otimes L$ and $M|_U \cong L'$. Thus, without loss of generality, we may assume in our proof of the lemma that $L|_U \cong L'$. Let E be the sheaf $K \otimes L$ on Y.

The rest of the argument is based on the proof of rigid analytic faithfully flat descent due to O. Gabber *c.f.* (1.9) of [7].

Let $D_Y(1)$ be the p-adically complete divided power envelope of the diagonal of $Y \times Y$ and let $D_Y(2)$ be the divided power envelope of the diagonal of $Y \times Y \times Y$. We have maps

$$p_i: D_Y(1) \to Y, \qquad p_{ij}: D_Y(2) \to D_Y(1).$$

Let $\theta: E \to p_1^* E$ be the map given by $x \mapsto \epsilon(p_2^* x)$.

Recall from (3.32) of [2] that the two projections maps $p_i: D_Y(1) \to Y$ are flat; it follows that $p_1^* L \subseteq p_1^* E$. Let $\phi := \theta|_L$ and $L'' := \theta^{-1} p_1^* L$; we shall show that L'' is coherent and compatible with θ. Because $L|_U = L'$ is already stable under θ, it is clear from the construction that $L''|_U = L|_U$. Note that $K \otimes L'' \cong E$: because tensoring with K is exact, $K \otimes \theta^{-1} p_1^* L \cong \theta^{-1} p_1^* K \otimes L \cong E$.

First let us check that $L'' \subseteq L$. In fact, the first cocyle condition implies that if we let

$$\Delta_E : p_1^* E \to E \quad \text{by} \quad x \otimes \gamma \mapsto x\Delta^*(\gamma) \qquad \text{for } x \in E_D, \gamma \in \mathcal{O}_{D(1)},$$

then $\Delta_E \circ \theta = id$. Since L is an $\mathcal{O}_Y$-submodule of E, Δ_E maps $p_1^* L$ to L. If x is a local section of L'', $\theta(x) \in p_1^* L$ by definition, and hence $x = \Delta_E(\theta(x)) \in L$ also. Since Y is noetherian, we can also conclude that L'' is coherent.

The flatness of p_1 implies that $p_1^* L'' \subseteq p_1^* L$; our goal is to prove that the map $\phi' := \theta|_{L''} : L'' \to p_1^* L$ factors through $p_1^* L''$. This can be checked locally, so we may assume that Y is affine, say $Y = Spec\, A$. Let $A' := A \otimes A$ and $A'' := A \otimes A \otimes A$. Then $D_Y(1)$ and $D_Y(2)$ are also affine, say $D_Y(1) = Spf\, B'$,and $D_Y(2) = Spf\, B''$, and write E for $E(S)$, *etc.* As usual we denote by $B' \otimes_B B'$ the tensor product of B' with itself over B, using p_2^* for the B' which appears on the left and p_1^* for the B' which appears on the right. Note that here we really do mean the ordinary tensor product—not its π-adic completion, to which we add a "^". Recall that there is a natural isomorphism:

$$A' \otimes_A A' \to A'' \quad \text{given by} \quad a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_1 \otimes a_2 a_3 \otimes a_4 .$$

This isomorphism identifies the ideal $I' \otimes A' + A' \otimes I'$ with the ideal I'' of the multidiagonal of A''. By (3.7) of [2], the ideal $I'B' \hat{\otimes}_B B' + B' \hat{\otimes}_B B' I'$ has divided powers, and so this isomorphism induces a map $\beta^* : B'' \to B' \hat{\otimes}_B B'$.

We denote by δ^* the map $B' \to B' \hat{\otimes}_B B'$ in the bottom row of the following commutative diagram:

$$\begin{array}{ccccc} A' & \xrightarrow{p_{13}^*} & A'' & \longrightarrow & A' \otimes_A A' \\ \downarrow & & \downarrow & & \downarrow \\ B' & \xrightarrow{p_{13}^*} & B'' & \xrightarrow{\beta^*} & B' \hat{\otimes}_B B' \end{array}$$

Then the cocycle condition (2) above can be expressed as the commutativity of the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E \otimes_B B' \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id \otimes \delta^*} \\ E \otimes_B B' & \xrightarrow{\theta \hat{\otimes} id} & E \otimes_B B' \hat{\otimes}_B B' \end{array}$$

We can now prove that L'' is compatible with θ. Consider the following diagram:

$$\begin{array}{ccccc}
 & & L'' \otimes_B B' & \xrightarrow{\phi'\otimes id} & L \otimes_B B' \otimes_B B' \\
 & & \downarrow & & \downarrow \\
L'' & \xrightarrow{\phi'} & L \otimes_B B' & \xrightarrow{\phi\otimes id} & E \otimes_B B' \otimes_B \otimes B' \\
\downarrow & & \downarrow & & \| \\
E & \xrightarrow{\theta} & E \otimes_B B' & \xrightarrow{\theta\otimes id} & E \otimes_B B' \otimes_B B'
\end{array}$$

By definition, the square on the bottom left is Cartesian. Since p_1 is flat, it remains Cartesian when tensored on the right by B', and so the large rectangle on the right is also Cartesian. Thus it suffices to prove that the composite arrow:

$$L'' \to E \xrightarrow{\theta} E \otimes_B B' \xrightarrow{\theta\otimes id} E \otimes_B B' \otimes_B B' \xrightarrow{pr\otimes id} E/L \otimes_B B' \otimes_B B'$$

is the zero map. Since E/L is π-torsion, the natural map

$$E/L \otimes_B B' \otimes_B B' \to E/L \otimes_B B' \hat{\otimes}_B B'$$

is an isomorphism, so it suffices to show that our map becomes zero after we follow it with this isomorphism. This follows easily from the cocycle condition: if $x \in L''$, $\theta(x) \in L \otimes_B B'$ and hence $(\theta \hat{\otimes} id)\theta(x) = id \otimes \delta^*(\theta(x)) \in L \otimes_B B' \hat{\otimes}_B B'$. Q.E.D.

It is now clear that Proposition 0.7.3 will follow from the following lemma.

Lemma 0.7.5. *The category of isocrystals on X/k is of a local nature.*

Proof: This means that if we are given an open cover $\{X_i\}$ of X, a finite type crystal E_i on each X_i, and compatible isomorphisms of isocrystals $\alpha_{ij}: K \otimes E_i \cong K \otimes E_j$ on each $X_i \cap X_j$, then there exists a finite type crystal E on X and isomorphisms $K \otimes E|_{X_i} \cong E_i$ giving back the isomorphisms α_{ij}. To prove this we may as well assume that each X_i has a lifting Y_i, and then we argue by induction on the number n of elements in the covering (which we may assume is finite because X is quasicompact). Let $X_1' := \cup\{X_i : i < n\}$; then by the induction assumption there exists a finite type crystal E_1' on X_1' with the desired properties, and hence inducing an isomorphism $K \otimes E'|_{X_1' \cap X_n'} \cong K \otimes E_n|_{X_1' \cap X_n'}$. Thus, we may assume without loss of generality that $n = 2$ and that X_2 has a smooth lifting Y. Let $U := X_1 \cap X_2$. The crystal E_2 defines a coherent $\mathcal{O}_Y$-module L with HPD-stratification ϵ_1, and the crystal $E_1|_U$ defines a coherent $\mathcal{O}_Y$-module

L' with HPD-stratification ϵ_2. Moreover, we are given an isomorphism $\alpha: K \otimes L \to K \otimes L'$ compatible with the stratifications. Lemma 0.7.4 tells us that we may replace L by an isogenous L'' which is still stable under its stratification and which prolongs L'. This L' defines a crystal on X_2, and since the category of crystals is of a local nature, the proof of the lemma is complete. Q.E.D.

Before beginning the proof of Theorem 0.7.2 it will be helpful to discuss briefly the following way of passing between the crystalline and convergent topoi.

Remark 0.7.6. Note that if $X \subseteq Y$ is a closed immersion of X into any smooth formal V-scheme Y, the divided power envelope $D_X(Y)$ of X in Y (p-adically completed) is flat over V. Since the ideal J of X in $D_X(Y)$ has divided powers, the pth power of any of its elements is divisible by p. If Y is sufficiently small and affine, the ideal I of X in Y has $r := \operatorname{codim}(X, Y)$ generators, and I^{rp} becomes divisible by p in $D_X(Y)$. It follows that the natural map $D_X(Y) \to Y$ factors through $T_{X,rp}(Y)$.

Proof of Theorem 0.7.2: By Lemma 0.7.5 we can prove this locally, and hence may assume X lifts to a smooth S/V. Suppose that E is a coherent crystal of $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$, choose n large and let S' and S'' be the formal schemes underlying $T_{X,n}(S \times S)$ and $T_{X,n}(S \times S \times S)$. For n large enough we will have maps

$$D(1) := D_X(S \times S) \to S', \text{and } D(2) := D_X(S \times S \times S) \to S''.$$

By Lemma (1.5) of [7], we can find a coherent sheaf L_S of $\mathcal{O}_S$-modules such that $K \otimes L \cong E_S$. We obtain a sheaf L_S of finite type S-modules together with an element

$$\epsilon \in K \otimes Hom_{D(1)}(p_2^* L_S, p_1^* L_S)$$

satisfying the cocycle coditions.

Thus any crystal of finite type $\mathcal{K}_{X/V}$-modules in $(X/V)_{conv}$ gives rise to a module with HPD iso-stratification (E, ϵ) and in fact this gives us a functor from the category of isocrystals in $(X/V)_{conv}$ to the category of finite type $\mathcal{O}_S$-modules with HPD iso-stratification. By Lemma 0.7.4 (E, ϵ) "is" an isocrystal. Also, theorem (2.15) of [7] allows us to identify $\Gamma_{conv}(X/V, E)$ with the set of horizontal sections of E_Y with respect to its connection ∇_Y, which is the same as $K \otimes \Gamma_{cris}(X/V, L)$. This shows us that our functor is fully faithful and completes the proof. Q.E.D.

Let us recall the following result of Berthelot:

Theorem 0.7.7. *Suppose X/k is smooth and proper, V is the Witt ring of k, and L is a coherent crystal of $\mathcal{O}_{X/V}$-modules in $Cris X/W$ which is convergent, i.e. which corresponds via Theorem 0.7.2 to a coherent crystal of $\mathcal{K}_{X/V}$-modules E in $(X/V)_{conv}$. Then there is a canonical isomorphism:*

$$K \otimes H^q_{cris}(X/V, L) \cong H^q_{conv}(X/V, E)$$

Proof: Let us sketch the idea of the proof. Embed X in a smooth Y/V as usual, and note that for any $n > rp$ we have a map $D_X(Y) \to T_{X,n}(Y)$, where $r = \operatorname{codim}(X, Y)$. Passing to the limit as $n \to \infty$, we obtain a map of complexes of abelian sheaves on X:

$$\Omega^{\cdot}_Y \otimes E_Y \to \Omega^{\cdot}_D \otimes E_D$$

We can check that it is a quasi-isomorphism locally on X, and hence we may assume that there is a lifting Y' of X to V. These complexes both map quasi-isomorphically to their analogues with Y' in place of Y, so the theorem is reduced to the case in which Y is itself a smooth lifting of X. In that case, $D_X(Y) = Y$ and $T_{X,n}(Y) = Y$ for every n, and our morphism is just the identity map. Q.E.D.

Corollary 0.7.8. *If X/k is proper, there is a canonical isomorphism*

$$H^i_{conv}(X/V, \mathcal{K}_{X/V}) \cong K \otimes H^i_{cris}(X/W).$$

We can deduce from this comparison theorem and 0.7.2 a new proof of the following finiteness result, first proved by Berthelot by rather different methods.

Theorem 0.7.9. *If E is a convergent isocrystal on a smooth proper X/k, then the cohomology groups $H^q_{conv}(X/V, E)$ are finite dimensional K-vector spaces.*

Proof: If $V = W$, this is immediate. If there is a convergent isocrystal E' on X/W such that $E \cong V \otimes E$, it is a consequence of the base change formula: $H^q_{conv}(X/V, V \otimes E) \cong V \otimes H^q_{conv}(X/V, E)$. In general, we may find a finite extension V' of V which is Galois over W, and it will suffice to prove that $V' \otimes E$ is a direct summmand of some E'' which descends to W. This is clear: take for E'' the sum of the conjugates of E under $Aut(V''/W)$. Q.E.D.

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Department of Mathematics
University of California
Berkeley, California 94720

Finiteness Theorems and Hyperbolic Manifolds

A. N. PARSHIN

Dedicated to A. Grothendieck

Introduction

Let $f : X \to S$ be a proper smooth holomorphic family of projective algebraic varieties. If we fix a base point $s_0 \in S$, then we have monodromy action of the fundamental group

$$\rho : \pi_1(S, s_0) \to \text{ Aut } H^p(X_0, \mathbb{Z})$$

where X_0 is a fiber over s_0. The following results were proved using a hyperbolic metric which was introduced by S. Kobayashi [10], [11].

Borel's Theorem. *Let $S = \Delta^*$ be a punctured disk and T be an image of a generator of $\pi_1(S, 0)$ under ρ. Then there exist non-negative integers M and N such that*

$$(T^M - \mathrm{Id})^N = 0.$$

Deligne's Theorem. *Let S be an algebraic variety. Fix a Hodge type of the fibers of f. Then the set of characters of representations ρ over $\mathbb{R}$ arising from all families with given type is finite.*

For a more complete description of the applications of hyperbolic analysis to monodromy problems and related problems (Griffiths' theory), we refer to [5]. A proof of Deligne's theorem can be found in [8] (and an outline in [24]). In this paper we apply these kind of ideas to a study of *sections* of the families $f : X \to S$. Motivated by diophantine geometry over functional fields we are interested here in the finiteness theorems for the set of sections.

The paper is organized as follows. Section 1 contains a short survey of hyperbolic manifolds. In Section 2, we give a new proof of Mordell's conjecture for algebraic curves over functional fields using a hyperbolic argument. At last in Section 3, these ideas are applied to a study of rational points on the projective subvarieties of abelian varieties (a new proof of Raynaud's theorem [17]) and of integer points on the affine open subsets of abelian varieties. In the last case we prove Lang's finiteness conjecture for integer points [12] (under a mild restriction on the hyperplane section). These results can be considered as some arguments in favour of the general Lang conjecture that the Mordell property is valid for the hyperbolic varieties (see [13]). Some hints about the possible application of these ideas to the number case are collected in an appendix.

This paper is indebted very much to the works of Borel, Griffiths, and especially Deligne, mentioned earlier. Recently I was familiarized with Grothendieck's letter [7] in which he has outlined new conjectures about the relation between the maps of algebraic varieties defined over number fields and the corresponding maps of their fundamental groups. Going back probably to his older paper [6] these conjectures should be valid for the "anabelschen" manifolds (in his terminology) which have something in common with hyperbolic manifolds (and coincide with them in dimension 1). The problems which Grothendieck considers in [7] can be also formulated for the holomorphic maps of algebraic varieties. Under the strong conditions of the "negativeness" type a positive solution of the problems is contained in the theorems of Borel-Narasimhan [1] and Siu [20] (the last result is the rigidity theorem for Kahler manifolds - a holomorphic variant of Mostow's rigidity theorem for locally symmetric spaces). All these results and conjectures show the deep link between the hyperbolicity and the arithmetical properties of the variety.

This work was finished during my stay at IHES in September-October 1986 and I would like to express my gratitude to Professors M. Berger and N. Kuiper for their warm hospitality.

1. Hyperbolic Manifolds

We consider here complex manifolds X not necessarily compact. General references about the hyperbolic manifolds are [10], [11], and [14].

Let $x, y \in X$. Choose the following structure σ : the points $x_0, x_1, \ldots, x_n$ with $x_0 = x$, $x_n = y$, holomorphic maps $f_i : \Delta \to X$ (where $\Delta = \{Z \in \mathbb{C} : |Z| < 1\}$) and points $a_i, b_i \in \Delta$, $i = 0, 1, \ldots, n-1$ such that $f_i(a_i) = x_i, f_i(b_i) = x_{i+1}$.

Definition 1. Let $d_X(x,y) = \inf_\sigma \sum_i \rho(a_i, b_i)$ where ρ is the Poincaré metric in Δ.

On every complex manifold, d_X satisfies symmetry and triangle conditions but it can happen that $d_x(x,y) = 0$ for different points x, y.

Definition 2. A complex manifold X is *hyperbolic* iff $d_X(x,y) > 0$ for all points $x \neq y$.

In this case, d_x is a hyperbolic metric on X. It has the following properties:

1. For any holomorphic map $f : X \to Y$ and $x, y \in X$

$$d_X(x,y) \geq d_Y(f(x), f(y))$$

2. $$d_\Delta = \rho.$$

3. $$d_{X\times Y}(x_1,y_1),(x_2,y_2)) \geq \max(d_X(x_1,x_2), d_X(y_1,y_2))$$

4. If $\pi : X \to Y$ is an unramified covering $x, y \in X$, then

$$d_Y(\pi(x), \pi(y)) = \inf_{y' \in \pi^{-1}(\pi(y))} d_X(x, y')$$

It means that the product of hyperbolic manifolds is hyperbolic, that is, if $f : X \to Y$ is a covering then X is hyperbolic iff Y is.

By definition $d_{\mathbf{C}} = 0$ and we get that a compact Riemann surface is hyperbolic iff its genus $g > 1$. Also, if we remove from some compact Riemann surface (of arbitrary genus) a finite number of disjoint closed disks (with sufficiently smooth boundaries), then we get a hyperbolic surface with a complete metric.

5. Let $f : X \to S$ be a proper holomorphic map. Then the set of $s \in S'$ such that fiber X_s is hyperbolic is open in the classical topology [2].

6. Let Y be a relatively compact complex subspace of a complex manifold X. Denote by $\overline{Y}$ its closure in X and assume that there is no non-constant holomorphic maps $\mathbf{C} \to \overline{Y}$. Then X is hyperbolic [4], [13].

7. Let W be a relatively compact open subset of a complex manifold X, D is a closed complex subspace in W and $\overline{W}, \overline{D}$ are their closures in X.

Assume that there are no holomorphic maps $g : \mathbf{C} \to \overline{D}$ and $g : \mathbf{C} \to \overline{W} - \overline{D}$. Then $W - D$ is hyperbolic and there exists an Hermitian metric ρ on X such that

$$d_{W-D} \geq \rho|_{W-D}.$$

This result is actually proved by Green [4, proof of Theorem 3] but not stated as above. His proof can be applied to this more general situation without any change (see [13, p.175]).

To the examples of hyperbolic manifolds already mentioned, we can add bounded domains in $\mathbf{C}^N$ (because they are contained in Δ^N) and also

8. If A is a complex torus and $D \subset A$ is a closed complex subspace, then D is hyperbolic iff D does not contain a translation of some complex subtorus [4].

This result is quite obvious in one direction because $d_{\mathbf{C}^N} = 0$, and consequently $d_A = 0$ for any complex torus.

2. Algebraic Curves

The main purpose here is to prove the following:

Theorem 1. *Let X be an algebraic curve defined over the field $K = \mathbb{C}(B)$ of algebraic functions of one variable. Assume that X is projective, smooth and of genus $g > 1$, and cannot be defined over $\mathbb{C}$ even after some finite extension of K. Then, the set $X(K)$ of rational points is finite.*

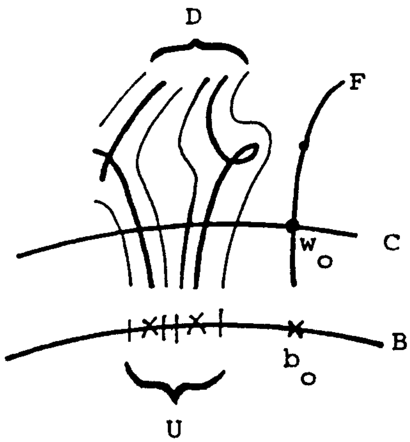
Here B is a projective algebraic curve (= compact Riemann surface) over $\mathbb{C}$. This result is a geometrical analogy of Mordell's conjecture and was proved by Manin. The other proofs were given later by Grauert and the author (see [12]).* The theorem can be reformulated in purely geometrical terms.

Theorem 2. *Let V be a projective smooth surface, and B be a projective smooth curve over $\mathbb{C}$. Consider a fibration $f : V \to B$ having as fibers projective smooth curves of genus $g > 1$, possibly with a finite number of exceptions. Assume that for any finite covering $B' \to B$ the fibration $V \times_B B' \to B'$ is not trivial.*

Then the set of sections of the map f is finite.

First of all we introduce the following notations. Let S be a finite subset in B, corresponding to non-smooth fibers of f. Denote by U a disjoint union of small closed disks around the points from S. Let

$$D = f^{-1}(U)$$
$$W = V - D$$
$$b_0 \in B - U = B_0$$
$$w_0 \in f^{-1}(b_0)$$



Fix a point w_0 and consider sections C of the map f which are going through the point w_0 of the fiber F.

*A very nice effective proof (our proof here is not effective) was found by H. Esnault and E. Viehweg. It gives the best known bounds for the height of rational points.

The fibration $f : W \to B_0$ is smooth and that gives an exact sequence of fundamental groups

$$1 \to \pi_1(F, w_0) \to \pi_1(W, w_0) \overset{i_c}{\underset{\longrightarrow}{\dashleftarrow}} \pi_1(B_0, b_0) \to 1 \tag{1}$$

All the manifolds are topologically $K(\pi, 1)$-spaces.

If the section C goes through w_0 it defines a splitting i_c of a sequence (1). If not, take the point w_1 of intersection C and F and connect w_1 and w_0 by some path γ on F. As above, C defines a splitting of a sequence like (1) but with the point w_1 instead of w_0. Then the path γ gives an isomorphism of the new exact sequence with (1). It means that we can associate with C a conjugacy class of splittings i_c.

Theorem 2 will follow from the two statements.

Proposition 1. *A class of homomorphisms i_c defines the corresponding section C up to a finite number of possibilities.*

Proposition 2. *The set $\{i_c\}$ of conjugacy classes of the splittings coming from the sections C is finite.*

Proof of Proposition 1. We have a commutative diagram with exact rows

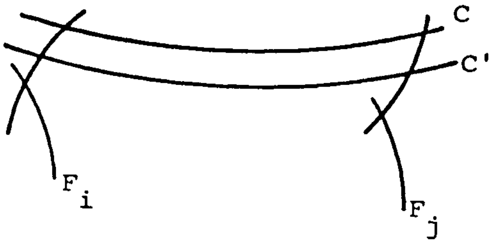
$$\begin{array}{rcccc} & H_2(V, D) & \leftarrow H_2 V) & \leftarrow & H_2(D) \\ & \uparrow & \uparrow & & \uparrow \\ I = \text{ Image of } & \mathrm{NS}(V) \leftarrow & \mathrm{NS}(V) & \leftarrow & \{F_i\} \end{array} \tag{2}$$

where $\mathrm{NS}(V)$ is the Neron-Severi group of the surface V and $\{F_i\}$ is a subgroup generated by the components of non-smooth fibers. A curve C gives us elements (denoted also by C) in $\mathrm{NS}(V)$ and by (2) in $H_2(V)$ and $H_2(V, D)$.

The conjugacy class i_c defines C as an element of $H_2(V, D)$ because all the spaces involved here are $K(\pi, 1)$-spaces and their maps are conjugacy classes of the homomorphisms of fundamental groups. If we know C in $H_2(V, D)$ then we know it in the group $H_2(V)$ by

Lemma 1. *Let C, C' be the sections and $C = C' + \Sigma n_i F_i$ where F_i are the components of fibers. Then $C = C'$ up to a finite number of possibilities.*

Proof. We say that two sections C, C' are equivalent iff they intersect the same component of each non-smooth fiber.



So we can assume that C is equivalent to C'. I claim that $C = C'$. If not, remember that on a non-isotrivial fibration we have $C.C < 0$ and $C'.C' < 0$ (Arakelov's Theorem). By assumption $C.C' \geq 0$ and we see that

$$C.C = C.C' + \Sigma n_i(F_i.C) \leq 0$$
$$C'.C = C'.C' + \Sigma n_i(F_i.C') \geq 0.$$

The definition of equivalency implies that $\Sigma n_i(F_i.C) = \Sigma n_i(F_i.C')$ and we get a contradiction.

It remains to show that the knowledge of C in $H_2(V)$ defines the curve C up to a finite number of possibilities. If we imbed our surface V to $\mathbf{P}_N$, then the degree of C in this imbedding is defined by its intersection with a hyperplane section as with the classes in $H_2(V)$. The above mentioned inequality $C.C < 0$ means that the sections are impossible to deform as the curves on V. This gives the finiteness statement.

Proposition 3. *The surface W is hyperbolic.*

This follows immediately from Proposition 6, Section 1 because both B_0 and the fibers F are hyperbolic.

Remark. It is possible to give two other proofs of this fact. The first way is to remove some smooth fibers also and then apply Griffiths' Theorem [5] that the surface W can be uniformized by a bounded domain in $\mathbf{C}^2$. The other way is to construct over W a smooth non-trivial family of algebraic curves of some genus (using an author's construction of ramified coverings, see [24]). This gives a finite map to the product of moduli varieties (= Teichmüller space) by the base curve B_0 which is hyperbolic.

Proposition 4. *Any section C considered as a submanifold of the surface W with an hyperbolic metric d_X is a totally geodesic submanifold.*

It means that for any points $x, y \in C$

$$d_{\mathbf{C}}(x,y) = d_W(x,y). \tag{3}$$

Proof. Applying Proposition 1, Section 1 to the maps $C \hookrightarrow W \xrightarrow{f} B \xrightarrow{\sim} C$ we get

$$d_{\mathbf{C}}(x,y) \geq d_W(x,y) \geq d_B(f(x),f(y)) \geq d_{\mathbf{C}}(x,y)$$

and that implies (3).

Proof of Proposition 2. Choose the loops $\gamma_1,\dots,\gamma_n$ on (B_0,b_0) which are represented by the generators of $\pi_1(B_0,b_0)$. Also choose a system of paths connecting all pairs w_0, w_1 of points on F. This gives some representative $i : \pi_1(B,b_0) \to \pi_1(W,w_0)$ in any conjugacy class i_c such that the images $i(\gamma_1),\dots,i(\gamma_n)$ have the following expression in $\pi_1(W,w_0)$

$$i(\gamma_k) = \gamma\tilde{\gamma}_k\gamma^{-1}, \quad k = 1,\dots,n. \tag{4}$$

Here $\tilde{\gamma}_k$ are the loops γ_k themselves but considered on the section $C \xrightarrow{\sim} B$ and γ is a path connecting w_0 with $w_1 = C \cap F$. By Proposition 4 the length of $\tilde{\gamma}_k$ in the hyperbolic metric of W is bounded and it is true for $i(\gamma_k)$ by the compactness of F.

We can enlarge slightly the set U to U' without touching the point b_0 and the loops $\gamma_1,\dots,\gamma_n$. Let $W' = f^{-1}(\overline{U}')$, $\overline{U}'$ be a closure of U'. Then $\pi_1(W',w_0) = \pi_1(W,w_0)$ and *all* loops $i(\gamma),\dots,i(\gamma_n)$ belong to $W'.W'$ is a compact manifold with a boundary and we have

Lemma 2. *The set of elements of $\pi_1(W',w_0)$ represented by the loops of bounded length is finite.*

The proof can be done along different lines. Take, for example, a universal covering $\widetilde{W}' \to W'$. Its Galois group Γ is $\pi_1(W')$ and the loops of bounded length on W' correspond to the paths on $\widetilde{W}'$ contained in some ball. The intersection of any (compact) ball with an orbit of Γ is finite and we are done. To get the statement of Proposition 2, we need to mention only that our representative i is completely defined by its values (4) which constitute a finite set by the lemma and by the bound for the length in the hyperbolic metric.

Remark. If the hyperbolic metric would be a Riemannian metric of non-negative curvature, then the classical argument on the uniqueness of geodesical loops in each homotopy class states that the class i_c defines the corresponding section C in a unique way.

3. Subvarieties of Abelian Varieties

To get finiteness theorems on the abelian varieties, it is necessary to avoid some trivial exceptions. Let A be an abelian variety defined over a field $K = \mathbb{C}(B)$ of algebraic functions on some projective curve B.

Definition 1. Let A_0 be a maximal abelian subvariety of A which is defined over $\mathbb{C}(K/\mathbb{C}$-trace, it exists and is unique [12]). Then we say that a set $M \subset A(K)$ is a *finite modulo trace* iff the M intersects with only finitely many cosets of $A_0(\mathbb{C})$.

Theorem 1. [17]. *Let $X \subset A$ be a projective subvariety of an abelian variety, defined over a functional field $K = \mathbb{C}(B)$. If X does not contain a translation of an abelian subvariety, then the set $X(K)$ of rational points is finite modulo trace.*

Denote by S_A the subset of B where A has a bad reduction. Then there exists a group scheme $f : \mathcal{A} \to B$ (Néron's minimal model) with a general fiber A and which is smooth and projective over the base outside S_A. Rational points from $A(K)$ define then the sections $C \hookrightarrow \mathcal{A}$ of the map f.

Definition 2. Fix some finite subset S of $B - S_A$ and some projective imbedding of A. Let H be a hyperplane section of $\mathcal{A}$ and let D be its closure in $\mathcal{A}$. A rational point $P \in A(K)$ is an *S-integer point* iff

$$f(C \cap D) \subset S_A \cup S$$

for the corresponding section C.

Theorem 2. *Let A be an abelian variety over a functional field $K = \mathbb{C}(B)$ and let H be a hyperplane section in some projective imbedding of A.*

Assume that H does not contain a translation of an abelian subvariety. (*)

Then any S set of S-integer points on A is finite modulo trace.

Without the assumption (*) it is Lang's conjecture [12].

I do not know whether the assumption (*) is really necessary or not. There are examples of abelian varieties with hyperplane sections containing an abelian subvariety [13, p.174].

Corollary 1. *If A is a simple abelian variety, then the set of S-integer points is always finite modulo trace.*

Corollary 2. *Let H be a general hyperplane section on an abelian variety A. Then the set of S-integer points is finite modulo trace.*

Proof of the theorems will be done simultaneously along the same lines as in Section 2.

We remove from B a finite number of disjoint closed disks and let B_0 be a remaining part of B_0 and $W = f^{-1}(B_0)$, D is a closure in W of the subvariety X (Theorem 1) or a closure of the hyperplane section H (Theorem 2). We assume that at least $S_A \not\subset B_0$ but the exact description of removed disks will be done later.

There is an exact sequence of fundamental groups (we fix the base points as in Section 2):

$$1 \to H_1(A_{b_0}) \to \pi_1(W, w_0) \xrightarrow{i_c} \pi_1(B_0, b_0) \to 1. \tag{1}$$

Any rational point $P \in A(K)$ defines a section C of the fibration $f : W \to B_0$ and a splitting i_c (or conjugacy class of splittings) of the sequence (1).

Proposition 1. *The set of rational points $P \in A(K)$ with a given class i_c is finite modulo trace.*

Proof. Let $G = \pi_1(B_0, b_0)$. The manifold W is an abelian group scheme over B_0. It has a zero section which defines some splitting i_0 of (1). The difference $i_c - i_0$ is a map from G to $H_1(A_{b_0})$ and actually is a 1-cocycle on G. It gives a homomorphism $\alpha : A(K) \to H^1(G, \Gamma)$, $\Gamma = H_1(A_{b_0}, \mathbb{Z})$.

On the other side, if $S \supset S_A$ are the centers of the removed disks then $\widehat{G} = \mathrm{Gal}(K_s/K)$ where K_s is a maximal Galois extension of K unramified outside S. An exact sequence

$$0 \to A_{1^n} \to A(K_s) \xrightarrow{1^n} A(K_s) \to 0$$

induces the following commutative diagram

$$\begin{array}{ccc} A(K)/1^n A(K) & \hookrightarrow & H^1(\widehat{G}, A_{1^n}) \\ \uparrow & & \downarrow \\ A(K) \xrightarrow{\alpha} H^1(G,\Gamma) & \xrightarrow{\beta} & H^1(G, A_{1^n}) \end{array}$$

where the map β is coming from the natural maps $\Gamma \to H^1(A_{B_0}, \mathbb{Z}/1^n\mathbb{Z}) = A_{1^n}$. Now an equality $\alpha(P) = 0$ implies that P is infinitely divisible in $A(K)$. Because the group $A(K)$ is finitely generated modulo trace (Mordell-Weil theorem) the point P is contained in $A_0(\mathbb{C})$. The proposition is proved.

Proposition 2. *There exists a finite number of disjoint closed disks $U \subset B$ such that the subvarieties D and $W-D$ lying over $B_0 = B-U$ are hyperbolic. There exists an Hermitian metric ρ on W such that $d_{W-D} \geq \rho|_{W-D}$.*

Proof. Consider the family $D_b, b \in B$. By assumption (from both theorems) a general fiber of the family does not contain a translation of an abelian subvariety. From the general theorems about Hilbert schemes of subvarieties, it is easy to deduce that there exists a countable subset $R \subset B$ such that outside of R any fiber D_b does not contain a translation of an abelian variety. By Proposition 8, Section 1, it is hyperbolic. From Proposition 5, Section 1, we know that the set $\{b \in B : D_b \text{ is hyperbolic}\}$ is open in complex topology. Consequently, the set $\{b \in B : D_b \text{ is not hyperbolic}\}$ is closed and no more than countable.

Lemma. *Let R be a closed countable subset of a compact Riemann surface B. Then R is contained in a finite number of disjoint closed disks.*

Proof. For any point $x \in R$ there exists a closed disk U_x such that $x \in U_x$ and $\partial U_x \cap R = \emptyset$. Taking these disks sufficiently small, we can assume that they do not intersect each other. By compactness, there are only finitely many of them.

Now we apply the construction of the manifolds W and D introduced above to the set of disks which are slightly more than the disks from the lemma and which contain the points $S \subset B$ in the case of Theorem 2.

If $g : \mathbb{C} \to \overline{D}$ or $g : \mathbb{C} \to \overline{W} - \overline{D}$ are holomorphic maps then $f \circ g$ are trivial maps by the hyperbolicity of B_0. But the fibers D_b and $(W-D)_b$, are hyperbolic by Propositions 8 and 7. So our maps g are trivial. Applying again Proposition 7 (for Theorem 2) and Proposition 6 (for Theorem 1) we get that both D and $W-D$ are hyperbolic, as well as an existence of metric ρ.

The theorems follow now from

Proposition 3. *The set of classes of splittings $\{i_c\}$ arising from the sections belonging to D (Theorem 1), or from the sections corresponding to S-integer points in W, are finite.*

Proof. First, we consider the case of Theorem 1. Then we proceed exactly as in the proof of Proposition 2, Section 2, using Proposition 2 instead of Proposition 3, Section 2. This gives a finite number of (classes) splittings of the exact sequence

$$(3) \qquad 1 \to \pi_1(D_{b_0}, w_0) \to \pi_1(D, w_0) \overset{\dashleftarrow}{\longrightarrow} \pi_1(B_0, b_0) \to 1.$$

There is a map of the sequence (3) to (1) and thus our splittings completely define the maps i_c from (1). An application of Proposition 1 gives us the theorem.

In the case of Theorem 2, we are working with the fibrations $f : W - D \to B_0$ which are not proper. Using the exact sequence for $\pi_1(W - D), w_0)$ we get the bounds for the lengths of $i'_c(\gamma_x)$ in the hyperbolic metric d_{W-D}. Here $i'_c : \pi_1(B_0, b_0) \to \pi_1(W - D, w_0)$ is a splitting corresponding to the section C which goes through w_0. We do the same for any section C with $C \cap (W - D)_{b_0} = w_1$. The bounds for the images $i'_c(\gamma_x)$ in $\pi_1(W - D, w_1)$ will be the same. Using the second part of Proposition 2 we get the upper bounds in some metric ρ on W, and because W_{b_0} is compact, the distances $\rho(w_0, w_1)$ are bounded for all w_1. Returning as in Section 2 to conjugacy classes in $\pi_1(W - D, w_0)$ and applying the natural map from $\pi_1(W - D, w_0)$ to $\pi_1(W, w_0)$, we get the bound for the images of i_c in the sequence (1). Now we can use Lemma 2 for the metric ρ on W, and then apply Proposition 1.

Remark 1. These results are related to the following general problem. Let $f : X \to S$ be a proper algebraic family of hyperbolic manifolds over a Riemannian surface S (S can either be an algebraic curve, or a compact Riemann surface without a finite number of disks). Is the set of algebraic sections considered modulo deformations finite? See the discussions in [13] and [18]. A positive answer is known for families with strong additional restrictions (triviality of f; the fibers should have a negative tangent bundle in Grauert's sense. See a review in [18]).

The most general result along these lines is Noguchi's finiteness theorem [15] for the families of hyperbolic manifolds. He uses the following condition: if $f : V \to B$ is the family and $U \subset V$ is the set of all smooth fibers then U should be hyperbolically imbedded into V in some sense. The general theorem is applied in [15, Section 5] to prove the Mordell conjecture in the functional case by using hyperbolic metric. It is done there by checking that U is hyperbolically imbedded for some model V of the general fiber. M. G. Seidenberg has developed a different approach. He could prove that the desired property can be satisfied after some base change (see [19]).

Remark 2. Theorem 1, Section 2 is a particular case of Theorem 1. If we imbed a curve X in its Jacobian A, then the intersection $X \cap J_0(\mathbb{C})$ is finite for non-isotrivial X. Also Proposition 1, Section 2 can be replaced by the argument used in the proof of Proposition 1 above.

Appendix

The loops, primes and Frobenius

Let K be a number field, Λ_K its integers, and X an algebraic curve of genus > 1 over K. Is it possible to apply the approach developed above to this situation? (Same question for problems from Section 3). First of all, we can consider Neron's minimal model over Spec Λ_K instead of the fibration $f : V \to B$. A definition of fundamental groups was done by Grothendieck (see SGA 1 and [9]). We have an analogue of the exact sequence (1) and the finiteness problem can be also divided in two parts. It can be shown that Proposition 1 remains valid in the new situation, and the main difficulty is to find some substitute for the loops and their lengths from the proof of Proposition 2. Not pretending to answer this question, we give here a remarkable analogy between loops on complex manifolds and closed points (= primes) on algebraic varieties defined over a finite field $\mathbb{F}_q$.

1. a complex manifold X over $\mathbb{C}$ with a base point x_0	an algebraic variety X over $\mathbb{F}_q$
2. a loop γ on X	a closed point v on X

x_0  γ

3. length $\ell_X(\gamma)$ of the loop γ in hyperbolic metric	norm Nv of the point v
4. The set of homotopy classes of loops γ with $\ell_X(\gamma) \leq C$ is finite (X is completely hyperbolic)	the set of closed points v with $Nv \leq C$ is finite

5. $f : Y \to X$ is a holomorphic map preserving base points, or an algebraic morphism defined over $\mathbf{F}_q$

If γ is a loop on Y, then a loop $f_X(\gamma)$ is defined on X and	If v is a closed point on $Y, F_*(v)$ is a closed point on X and
$\ell_Y(\gamma) \geq \ell_X(f_*\gamma)$	$Nv \geq N(f_* v)$

6. $\pi : Y \to X$ is a covering

Let $y \in \pi^{-1}(x_0)$ and $\sigma = \sigma(\gamma, y) \in \mathrm{Gal}(Y/X)$ is such that γ is covered with a path	Let $y \in \pi^{-1}(v)$ and $F = \mathrm{Fr}_v$ be the Frobenius automorphism of $k(y)$ over $k(v)$

$y' = \gamma y$

$\sigma' = \gamma^{-1}\sigma\gamma$	$F' = \gamma^{-1}F\gamma$

7. $f : Y \to X$ gives a map

$f_*(\sigma(\gamma, y))$	$f_* : \pi_1(Y) \to \pi_1(X)$	$f_*\,(\mathrm{Fr}_v^y)$
$= \sigma(f_*(\gamma), \tilde{f}(y))$	and a map $\tilde{f} : \widetilde{Y} \to \widetilde{X}$ of universal covering	$= \mathrm{Fr}_{f(v)}^{\tilde{f}(y)}$

8. $\pi : Y \to X$ is a finite covering

Fix y on Y $\forall\ \sigma \in \mathrm{Gal}(Y/X)$ $\exists\ \gamma\sigma = \sigma(\gamma, y)$	Chebotarëv's theorem: every $\sigma \in \mathrm{Gal}(Y/X)$ is represented by a Frobenius automorphism
9. a loop of minimal length in the class σ	Frobenius automorphism of smallest norm in the class σ
10. Ruelle zeta function (for the conjugacy classes of geodesic loops)	Weil zeta function (for the rational points over $\mathbf{F}_q$)

This type of analogue was used already at least twice:

(1) If we consider a compact Riemann surface of genus ≥ 2 and the distribution of lengths of closed geodesics on it, then one has an asymptotical law which is exactly the same as for primes (from Λ_K, or for closed points on curves defined over $\mathbf{F}_q$). This is also true for Riemannian manifolds with negative sectional curvature (see [16], [21], [22]).

(2) Faltings has proved an analogy of Deligne's theorem from the introduction for the number case [3]. His use of the Riemann-Weil hypothesis to bound the norms of Frobenius automorphisms has great similarity with Deligne's bounds for the trace of monodromy operators.

Remark 1. Still, there is a great difference between the objects which we have compared above. We can associate with the loops the *elements* of the fundamental group. In the number case, the Frobenius automorphisms are defined only *up to conjugacy.* They look much more like the free loops (without a base point). It is exactly the difficulty which appears in the straightforward attempts to construct non-abelian class field theory.

Remark 2. In [23] P. Vojta has introduced a new kind of analogy between number and functional cases (see, in particular, the dictionary on p.34). It would be interesting to find a relation of his analogy with the previous one.

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Steklov Mathematical Institute
Academy of Sciences of USSR
Ul. Vavilova 42
117966 Moscow GSP-1, USSR

p-groupes et réduction semi-stable des courbes

MICHEL RAYNAUD(*)

dédié à A. Grothendieck

Soit R un anneau de valuation discrète complet, de corps des fractions K, de corps résiduel k de caractéristique $p > 0$. Considérons une R-courbe propre et lisse X, à fibres géométriques connexes et un revêtement fini étale galoisien de groupe $G : Y_K \to X_K$ de la fibre générique X_K de X. Lorsque l'ordre de G est premier à p, il résulte de la théorie du groupe fondamental de Grothendieck [9], qu'après extension modérée éventuelle de R, Y_K s'étend en un revêtement fini étale $Y \to X$. Nous nous proposons d'examiner ici le cas où G est un *p-groupe*.

Théorème 1. *Supposons que G soit un p-groupe. Alors:*

(i) *La jacobienne $J(Y_K)$ de Y_K a potentiellement bonne réduction.*

(ii) *Après extension finie éventuelle de R, la courbe Y_K admet une réduction semi-stable $\tilde{Y}$ sur R et les composantes connexes du graphe de la fibre spéciale $\tilde{Y}_k$ de $\tilde{Y}$ sont des arbres. De plus, les composantes irréductibles éclatées de $\tilde{Y}_k$ (i.e., celles qui se projettent sur des points de X_k), ont une cohomologie cristalline dont les pentes se situent strictement entre* 0 *et* 1.

Le théorème 1 se déduit d'une version locale (théorème 2) et repose sur des propriétés élémentaires des actions de groupes nilpotents sur les schémas réguliers (proposition 1). Nous examinons ensuite quelques exemples de revêtements cycliques d'ordre p.

1. Action d'un groupe nilpotent

Rappelons qu'un schéma X est *géométriquement unibranche* (confer [8] 0_{IV} 23.2.1 ou [13] p. 100) si, pour tout point x de X, l'hensélisé strict de $O_{X,x}$ a un spectre irréductible. Si X est intègre, il revient au même de

(*) Unité associée au CNRS N° 752.

dire que $\tilde{X} \to X$ est un homéomorphisme, où $\tilde{X}$ est le normalisé de X. Si X est géométriquement unibranche, il en est de même de tout schéma X' étale sur X.

Proposition 1. *Soient S un schéma régulier localement noethérien intègre, M son corps des fractions, N une M-algèbre finie étale galoisienne de groupe de galois G nilpotent, T le normalisé de S dans N, $f : T \to S$ le morphisme canonique. Soit $\overline{S}$ le fermé de S, purement de codimension 1, au-dessus duquel f est ramifié. On suppose que $\overline{S}$ est géométriquement unibranche. Alors:*

(i) $\overline{T} = f^{-1}(\overline{S})$ *est géométriquement unibranche.*

(ii) *Soient $\overline{T}_i$ une composante irréductible de $\overline{T}$, $\overline{S}_i$ son image dans $\overline{S}$. Alors le sous-groupe d'inertie I de G est constant en tous les points de $\overline{T}_i$. Soit T' le quotient de T par I et $\overline{T}'_i$ l'image de $\overline{T}_i$ dans T'. Alors T' est étale sur S au voisinage de $\overline{T}'_i$; $\overline{T}_i \to \overline{T}'_i$ est un homéomorphisme et $\overline{T}'_i \to \overline{S}_i$ est un revêtement étale galoisien de groupe de Galois D/I, où D est le sous-groupe de décomposition de $\overline{T}_i$.*

Comme $\overline{S}$ est géométriquement unibranche et localement noethérien, $\overline{S}$ est localement irréductible. Quite à restreindre S à un ouvert, on peut donc supposer que $\overline{S} = \overline{S}_i$ est irréductible. Soit s le point générique de $\overline{S}$ et soit t au-dessus de s le point générique de $\overline{T}_i$. Le groupe de décomposition D est donc le stabilisateur de t. Notons I le sous-groupe d'inertie de G en t.

Supposons d'abord $D \neq G$. Comme G est nilpotent, on peut trouver un sous-groupe invariant H de G, contenant D et distinct de G ([4] Section 6.3 Proposition 8). Notons $T'' = T/H$ le quotient de T par H. T'' est un revêtement galoisien de S, de groupe $G'' = G/H$. Notons t'' l'image de t dans T''. T'' est évidemment étale sur S au-dessus de $S - \overline{S}$. D'autre part, vu le choix de H, T'' est aussi étale sur S au-dessus de s ([13] p. 103), donc est étale sur S en codimension ≤ 1. D'après le théorème de pureté de Zariski, ([9] Chapter X 3.1), T'' est alors étale sur S. Comme H contient D, T'' est décomposé au-dessus de s. $\overline{S}$ étant géométriquement unibranche, T'' est alors décomposé au-dessus de $\overline{S}$. Quitte alors à remplacer S par T'', qui est encore régulier, on est ramené au cas où $G = H$. De proche en proche, on peut donc supposer que $G = D$.

Maintenant $T' = T/I$ est un revêtement galoisien de S, de groupe D/I, et $\overline{T} = \overline{T}_i$ est irréductible. Comme plus haut, on voit que T' est étale sur S. L'image réciproque $\overline{T}'$ de $\overline{S}$ dans T' est irréductible et est un revêtement étale de $\overline{S}$ de groupe de Galois D/I, en particulier $\overline{T}'$ est géométriquement unibranche. Finalement $\overline{T} \to \overline{T}'$ est un morphisme entier, radiciel au point générique, donc est un homéomorphisme puisque $\overline{T}'$ est géométriquement unibranche. Comme T' est étale sur S en tout point de T' le groupe d'inertie I en t est aussi le groupe d'inertie en tout point de $\overline{T}$, d'où la proposition.

Désormais, on considère un anneau de valuation discrète R complet, de corps des fractions K, de corps résiduel k; on note π une uniformisante de R.

Corollaire 1. *Soit X un R-schéma régulier et plat, $Y_K \to X_K$ un revêtement fini étale galoisien, de groupe G nilpotent et soit Y le normalisé de X dans Y_K. Supposons que X_k soit géométriquement unibranche, alors il en est de même de Y_k.*

Exemple. On prend pour X une R-courbe lisse. Alors X_k est géométriquement unibranche et le corollaire 1 s'applique; il équivaut au fait que les seules singularités $(Y_k)_{\mathrm{red}}$ sont des points de rebroussement.

L'exemple précédent peut être généralisé en acceptant un diviseur relatif à croisements normaux, le long duquel le revêtement est modérément ramifié. De façon précise, on a le résultat suivant :

Corollaire 2. *Soient X un R-schéma lisse, D un diviseur relatif à croisements normaux de X, U l'ouvert $X-D$, $V_K \to U_K$ un revêtement fini étale galoisien de groupe de Galois G nilpotent, modérément ramifié le long de D_K. Soit Y le normalisé de X dans V_k. Alors Y_k est géométriquement unibranche.*

Démonstration. Plaçons nous en un point x de X_k. Après localisation stricte de X en x, on peut supposer Y local et on doit montrer que Y_k est irréductible. Le diviseur D a pour équation $t_i \cdots t_r = 0$, où les t_i font partie d'un système de coordonnées relatives de X/R en x. De plus, il existe des entiers $n_i \geq 1$, premiers à la caractéristique de K, tels que l'indice de ramification de V_K le long de la composante de D d'équation $t_i = 0$, soit n_i. Notons X^1 le schéma fini sur X obtenu en extrayant pour tout i une racine n_i-ème de t_i. Alors X^1 est l'hensélisé strict d'un schéma *lisse sur* R et donc X^1_k est intègre. Notons Y^1 le normalisé de $Y \times_X X^1$. Commes Y_K est modérément ramifié le long de D_k, il résulte du lemme d'Abhyankar ([9], Chapter XIII Proposition 5.2) que $Y^1_K \to X^1_K$ est étale. On peut appliquer la proposition 1 et conclure que Y^1_k est géométriquement unibranche. Soit alors un composant local de Y^1. Il domine par un morphisme fini surjectif le schéma local normal Y, donc le schéma irréductible

Y_k^1 domine Y_k et celui-ci est aussi irréductible.

2. Démonstration du théorème 1

On suppose désormais que R est de caractéristique résiduelle $p > 0$.

Soit (X, x) un germe de courbe relative lisse sur R. Donc $X = \mathrm{Spec}\,(\mathcal{O})$, où $\mathcal{O}$ est un anneau local régulier de dimension 2, localisé d'une R courbe lisse. Soit D un diviseur à croisements normaux relatifs sur X (c'est à dire un sous-schéma fermé de X, étale sur R) et $U = X - D$.

Soit G un p-groupe fini et considérons $V_K \to U_K$ un revêtement étale galoisien de groupe G, étale en dehors de D_K, et modérément ramifié le long de D_K (lorsque K est également de caractéristique p et $G \neq \{1\}$, D est nécessairement vide). Soit Y le normalisé de X dans V_K.

On suppose que Y admet un R-modèle semi-stable, c'est à dire qu'il existe un éclatement $\tilde{Y} \to Y$ de Y tel que :

(i) La fibre fermée $\tilde{Y}_k$ est géométriquement réduite.

(ii) Les seules singularités de $\tilde{Y}_k$ sont des points doubles ordinaires.

(iii) Toute droite projective de $\tilde{Y}_k$ rencontre les autres composantes en au moins trois points géométriques distincts.

Il résulte du théorème de réduction semi-stable pour les courbes ([2], [6], [12]) que, quitte à faire une extension finie de R, Y admet un R-modèle semi-stable. De plus, celui-ci est essentiellement unique et s'obtient à partir d'une désingularisation minimale en contractant les chaînes de droites projectives de self-intersection (-2). En particulier l'action de G sur Y s'étend en une action de G sur $\tilde{Y}$.

Notons que toute composante irréductible de Y_k est en correspondance birationnelle avec sa transformée stricte dans $\tilde{Y}$; en particulier Y_k est réduit. D'autre part, il résulte du corollaire 2 à la proposition 1 que Y_k est géométriquement unibranche.

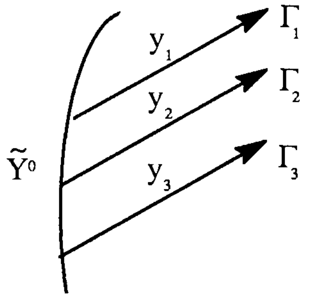
Théorème 2. *Les composantes connexes de $\tilde{Y}_k$ ont des graphes qui sont des arbres. De plus, les composantes propres de $\tilde{Y}_k$ ont une cohomologies cristalline dont les pentes appartiennent à* $]0, 1[$.

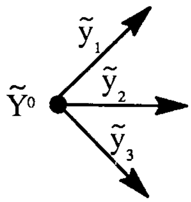
On peut supposer k algébriquement clos. Commençons par établir la première assertion.

Puisque Y_k est géométriquement unibranche et est réduit, ses seules singularités sont des points de rebroussement (cusps). En particulier, Y_k est somme disjointe de ses composantes irréductibles. Quitte à remplacer G par le groupe de décomposition D d'une composante irréductible de Y_k, et X par un localisé du quotient Y/D, on est donc ramené au cas où Y_k est irréductible. Notons y_i, $i \in I$, la famille des points fermés du schéma semi-local Y.

Soit $\tilde{Y}^0$ la composante irréductible de $\tilde{Y}_k$, transformée stricte de Y_k. Comme $\tilde{Y}_k$ a pour seules singularités des points doubles ordinaires et que Y_k est géométriquement unibranche, $\tilde{Y}^0$ n'est autre que la normalisée de Y_k et $\tilde{Y}^0 \to Y_k$ est un homéomorphisme. On note $\tilde{y}_i$ *l'unique* point fermé de $\tilde{Y}^0$ au-dessus du point fermé y_i de Y_k.

Prenons $\tilde{Y}^0$ comme composante origine dans le graphe de $\tilde{Y}_k$ et soit $\tilde{g} : \tilde{Y} \to Y$ la projection. Les fibres fermées de g sont les $g^{-1}(y_i)$ et $g^{-1}(y_i)$ est liée à $\tilde{Y}^0$ par le seul point $\tilde{y}_i$. Pour voir que le graphe de $\tilde{Y}_k$ est un arbre Γ, il suffit de montrer que le graphe de $g^{-1}(y_i)$ est un arbre Γ_i : On passe en effet des Γ_i à Γ en branchant les Γ_i sur la composante origine $\tilde{Y}^0$, via les ponts $\tilde{y}_i$.





Graphe dual

Soit I le sous-groupe d'inertie de G commun à tout les points de Y_k (Proposition 1). Alors I est invariant dans G et $Z = Y/I$ est étale sur X, donc essentiellement lisse et $Y_k \to Z_k$ est un homéomorphisme. Le groupe I agit sur $\tilde{Y}$ en respectant chacune des fibres $g^{-1}(y_i)$. Fixons l'un des points y_i et soit z_i son image dans Z. Quitte à remplacer X par Z localisé en z_i et Y par Y localisé en y_i, on est ramené au cas où $G = I$ et Y possède un seul point fermé noté y. Soit $\tilde{y}$ son unique relèvement dans $\tilde{Y}^0$.

Nous allons procéder par récurrence sur la longueur h du graphe de $\tilde{Y}_k$.

Si $h = 0$, $\tilde{Y} = Y$ et Γ est bien un arbre.

Supposons $h > 0$. Alors $\tilde{y}$ est un point double de $\tilde{Y}_k$ et il existe une *unique composante* $\tilde{Y}^1$ de $\tilde{Y}_k$ qui passe par $\tilde{y}$ et est distincte de $\tilde{Y}^0$. Elle est nécessairement stable par G.

Considérons le quotient $\tilde{X}$ de $\tilde{Y}$ par G. C'est un éclatement de X en x, dont la fibre spéciale est réduite avec pour seules singularités des points doubles ordinaires (confer appendice). On note $\tilde{f} : \tilde{X} \longrightarrow X$ le morphisme canonique.

Comme X est *régulier*, la fibre spéciale de $\tilde{X}$ au-dessus de x est un *arbre de droites projectives.* Le transformé strict de X_k par $\tilde{f}$ est une composante $\tilde{X}^0$. Il rencontre $f^{-1}(x)$ en un point double $\tilde{x}$. Notons $\tilde{X}^1$ la droite projective de $\tilde{X}_k$ qui passe par $\tilde{x}$. Le morphisme fini $\tilde{Y} \longrightarrow \tilde{X}$ envoie respectivement $\tilde{Y}^0, \tilde{Y}^1, \tilde{y}$, sur $\tilde{X}^0, \tilde{X}^1, \tilde{x}$.

Introduisons le schéma Y' déduit de $\tilde{Y}$ en contractant les composantes irréductibles de $\tilde{Y}_k$ autres que $\tilde{Y}^0$ et $\tilde{Y}^1$ ([1] ou [3] chapitre 6.6). Le morphisme $\tilde{g} : \tilde{Y} \longrightarrow Y$ se factorise en les morphismes propres : $\tilde{Y} \xrightarrow{\tilde{g}'} Y' \xrightarrow{g} Y$. De plus, $\tilde{g}'$ est un isomorphisme au voisinage de $\tilde{y}$. Notons $y' = \tilde{g}'(y)$. Y'_k a deux composantes irréductibles, images respectives par $\tilde{g}'$ de $\tilde{Y}^0$ et $\tilde{Y}^1$ et notées Y'^0 et Y'^1. Ces deux composantes se coupent au point double y'.

Soit de même X' le schéma déduit de $\tilde{X}$ en contractant les composantes autres que $\tilde{X}^0$ et $\tilde{X}^1$. On obtient une factorisation de $\tilde{f}$:

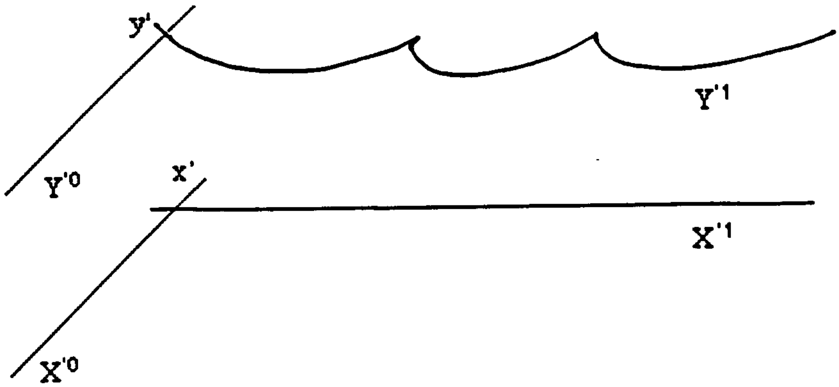
$$\tilde{X} \xrightarrow{\quad \tilde{f}' \quad} X' \xrightarrow{\quad f \quad} X.$$

Alors X'_k a deux composantes X'^0 et X'^1 qui se coupent transversalement en x'. Comme la fibre X'_k est réduite et que X est *lisse* sur R, X'^1 est nécessairement une droite projective et donc X' est lisse sur R aux points de X'^1 autres que x'.

Finalement on obtient un diagramme commutatif dont les flèches verticales sont des morphismes finis de passage au quotient sous l'action de G :

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "\tilde{g}'"] \arrow[d] & Y' \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
\tilde{X} \arrow[r, "\tilde{f}'"] & X' \arrow[r, "f"] & X
\end{tikzcd}$$

Considérons le morphisme fini $Y' \longrightarrow X'$. Il induit un homéomorphisme $Y'^0 \longrightarrow X'^0$; en particulier y' est l'unique point au-dessus de x'. Soit D' le transformé strict dans X' du diviseur relatif D de X. Aux points de X'^1 autres que x', on a vu que X' était lisse sur R et en ces points D' est un diviseur relatif à croisements normaux. Il résulte alors du corollaire 2 à la proposition 1 que, en dehors de y', Y'^1 présente pour seules singularités des points de rebroussement.



Bien sûr $\tilde{g}' : \tilde{Y} \to Y'$ est une réduction semi-stable de Y'. Si alors on choisit un point fermé x'^1 de X'^1, autre que x' et si on localise le morphisme $Y' \to X'$ au-dessus de x'^1, on obtient une situation analogue à la situation de départ, avec une réduction semi-stable de la singularité éventuelle au dessus de x'^1 dont le graphe est clairement de longueur $< h$. Par hypothèse de récurrence, les composantes connexes de ce graphe sont des arbres. Chacun de ces arbres se branche en un point de Y^1, d'où le fait que le graphe de Y_k est un arbre, ce qui prouve la première assertion du théorème 2.

Avant d'établir la dernière assertion, revenons sur la démonstration précédente et sur le morphisme $Y' \to X'$. Il y a deux cas possibles :

(a) Le groupe G opère trivialement sur $\tilde{Y}^1$, donc aussi sur Y'^1. $\tilde{Y}^1$ est alors radiciel sur $\tilde{X}^1$, donc est une droite projective. Vu le caractère minimal du modèle semi-stable $\tilde{Y}$, $\tilde{Y}^1$ rencontre au moins trois autres composantes de $\tilde{Y}_k$, donc au moins deux composantes autres que $\tilde{Y}^0$. C'est dire que Y'^1 présente au moins deux points de rebroussement.

(b) Le groupe G n'opère pas trivialement sur $\tilde{Y}^1$. Soit alors H le sous-groupe d'inertie au point générique de $\tilde{Y}^1$. H est invariant dans G et distinct de G. Considérons le morphisme $Y' \to X'$ que l'on factorise à travers $Z' = Y'/H$. Z' est alors étale au-dessus du point générique de X'^1. D'après le théorème de pureté de Zariski, $Z' \to X'$ est donc étale au-dessus des points de $X'^1 - \{x'\}$, c'est à dire au-dessus des points d'une droite affine sur k. Par suite, l'inertie en chacun des points de Y'^1, autres que y' est égale à H. De plus,

— ou bien $\tilde{Y}^1$ est une droite projective, auquel cas Y'^1 présente au moins deux points de rebroussement et G/H est un p-sous-groupe du groupe additif de k donc est ismorphe à $(\mathbb{Z}/p\mathbb{Z})^d$.

— ou bien $\tilde{Y}^1$ est de genre > 0.

D'où le complément suivant au théorème 2 :

Proposition 2. *Sous les conditions du théorème 2, supposons de plus k algébriquement clos et Y_k connexe (donc irréductible d'après le corollaire 1 à la proposition 1). Alors :*

(i) *le graphe Γ de $\tilde{Y}_k$ est un arbre d'origine $\tilde{Y}^0$ le transformé strict de Y_k.*

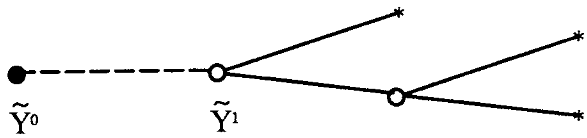
(ii) *Soit Z une composante irréductible propre de $\tilde{Y}_k$, c'est à dire distincte de $\tilde{Y}^0$. Notons y le point de Z qui lie Z aux autres composantes en direction de $\tilde{Y}^0$, z^j, $j \in J$ les points de Z qui lient Z à d'autres composantes Z^j quand on s'éloigne de $\tilde{Y}^0$. Soient D et I les groupes de décomposition et d'inertie au point générique de Z. Alors : Le quotient de Z par $H = D/I$ est une droite projective P sur k. Si $H \neq \{1\}$, le revêtement galoisien $Z \to P$ est ramifié au seul point y; il est même totalement ramifié en ce point. Le groupe de décomposition D^j d'une composante Z^j est égal à I; le groupe H opère librement sur les points z^j et sur les composantes Z^j. Les groupes de décomposition et groupes d'inertie des diverses composantes de $\tilde{Y}_k$ vont en décroissant quand on se déplace sur Γ en s'éloignant de $\tilde{Y}^0$.*

(iii) *Les bouts de Γ, autres que $\tilde{Y}^0$, correspondent à des courbes de genre > 0.*

La dernière assertion traduit le caractère minimal de la réduction semi-stable $\tilde{Y}$: un bout de Γ qui est une droite projective peut être contracté sans introduire de singularité.

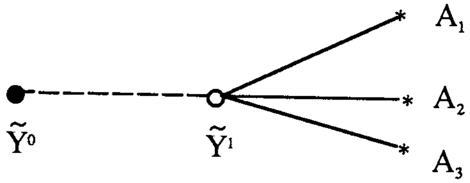
L'assertion (ii) a pratiquement été démontrée. En effet, par la récurrence déjà utilisée, il suffit de traiter le cas où Z est la composante contiguë à $\tilde{Y}^0$ et notée $\tilde{Y}^1$. Avec les notations ci-dessus, l'assertion résulte alors du fait que $Y'/I \to X'$ est un revêtement étale au-dessus de la droite affine $X'^1 - \{x'\}$ et est totalement ramifié au-dessus de x'. Le stabilisateur de tout point de Y'^1, autre que y' est donc égal à I, d'où le fait que les groupes de décomposition D^j soient égaux à I.

Exemples : (1) $G = \mathbb{Z}/p\mathbb{Z}$. Les sommets de Γ autres que les bouts correspondent à des droites projectives sur lesquelles G opère trivialement. Les bouts de Γ distincts de $\tilde{Y}^0$ sont des courbes de genre > 0, revêtements d'Artin-Schreier de la droite projective, ramifiés en un seul point.



o : Droite projective.
$*$: Courbe de genre > 0.

(2) $G = \mathbb{Z}/9\mathbb{Z}$.



$\mathbb{Z}/9\mathbb{Z}$ est le stabilisateur de $\tilde{Y}^0$.
$\mathbb{Z}/3\mathbb{Z}$ est le stabilisateur de $\tilde{Y}^1$.
Le quotient $\mathbb{Z}/3\mathbb{Z}$ de $\mathbb{Z}/9\mathbb{Z}$ agit transitivement sur $\{A_1, A_2, A_3\}$.

Prouvons maintenant l'assertion du théorème 2 qui concerne les pentes.

Si C est une courbe propre et lisse sur k algébriquement clos de caractéristique $p > 0$, on note

$$p_C = \dim_{\mathbb{Q}_p} H^1(C, \mathbb{Q}_p),$$

le p-rang de C.

Soit alors Z une composante propre de $\tilde{Y}_k$. Dire que les pentes de la cohomologie cristalline de Z sont dans $]0,1[$ équivaut à $p_Z = 0$. D'après la proposition 2 (ii), Z est un revêtement galoisien, de groupe H un p-groupe, de la droite projective P, ramifié en un seul point ∞. D'après la formule de Crew ([5] corollaire 1.8), on a :

$$1 - p_Z = (\text{Card } H)(1 - p_P) - (e_\infty - 1)$$

où e_∞ est l'indice de ramification en ∞, donc est égal ici à Card(H).

Comme $p_P = 0$, on trouve bien que $p_Z = 0$.

Ceci achève la démonstration du théorème 2. QED

Démontrons maintenant le théorème 1, ou plutôt la variante suivante où l'on accepte une ramification modérée horizontale.

Théorème 1′. *Soient X une R-courbe propre et lisse, D un diviseur relatif étale sur R, $U = X - D$, $Y_K \to X_K$ un revêtement fini galoisien de groupe G nilpotent, étale au-dessus de U_K et modérément ramifié le long de D_K. Alors :*

Après extension finie de R, le normalisé Y de X dans Y_K admet un éclatement $\tilde{Y}$ qui est une réduction semi-stable de Y_K. Les composantes connexes du graphe de $\tilde{Y}_k$ sont des arbres. Les composantes contractées en des points dans le morphisme $\tilde{Y} \to Y$ ont une cohomolgie cristalline dont les pentes sont dans $]0,1[$. La jacobienne de Y_K a potentiellement bonne réduction sur R.

La première assertion résulte aisément du theéorème de réduction semi-stable pour Y_K; notons toutefois que lorsque Y_k contient des composantes rationnelles, les transformées strictes de ces composantes dans $\tilde{Y}$ sont des droites rationnelles qui ne rencontrent pas nécessairement les autres composantes en au moins trois points.

Pour établir les autres assertions, on se ramène au cas oú Y_K est géométriquement connexe; il en est alors de même de Y_k qui est donc géométriquement irréductible d'après le corollaire 1 à la proposition 1. Soit p la caractéristique résiduelle de R. Le groupe nilpotent G est produit de ses l sous-groupes de Sylow pour tous les nombres premiers l ([4] Section 7 théorème 4). Si G est d'ordre premier à p, il résulte de la théorie du groupe fondamental modéré([9] Chapitre XIII)) que Y est lisse sur R. On peut donc se limiter au cas où G est un p-groupe. Soit alors x un point de X au-dessus duquel Y n'est pas lisse sur R et appliquons le théorème 2 après localisation en x. On conclut que le graphe de $\tilde{Y}_k$ est un arbre d'origine la transformée stricte de Y_k dans $\tilde{Y}$, avec un branchement pour chaque point y où Y n'est pas lisse sur R.

Enfin il est classique (confer. [3] Chapitre 9) que la jacobienne de Y_K a potentiellement bonne réduction si et seulement si la réduction semi-stable de Y_K a un graphe dont les composantes connexes sont simplement connexes.

Donnons maintenant un énoncé analogue au théorème 1′, mais dans lequel on ne considère plus la réduction semi-stable de Y_K.

Corollaire 1′. *Sous les hypothèses du théorème 1′, considérons la normalisée Y de X dans Y_K et une désingularisation $\hat{Y}$ de Y. Alors $Pic(\hat{Y}_k/k)$ ne contient pas de sous-tore non nul ou, de façon équivalente, les composantes irréductibles réduites de $\hat{Y}_k$ sont homéomorphes à leur normalisée et les composantes connexes du graphe de $\hat{Y}_k$ sont des arbres.*

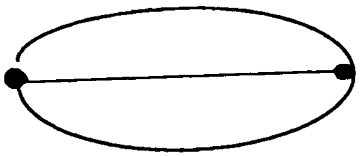
Soit J_K la jacobienne de Y_K et J son modèle de Néron sur R. D'après le théorème 1′, J_k a potentiellement bonne réduction. On en déduit facilement que J_k ne contient pas de sous-tore non nul. Lorsque le *pgcd* des multiplicités géométriques des composantes irréductibles de Y_k est égal à 1, J_k est la composante neutre de Pic $\hat{Y}_k/k$ ([14] Théorème 8.2.1). D'où le résultat dans ce cas. Dans le cas général, on a une suite exacte *fppf* de faisceaux :

$$0 \to E \to \mathrm{Pic}^0(\hat{Y}_k/k) \to J^0 \to 0,$$

où E est l'adhérence schématique de la section unité, dans $\mathrm{Pic}^0(\hat{Y}/R)$ ([14] Section 4). De plus, la fibre fermée $E \times_R k$ de E est un k-groupe algébrique dont la composante neutre ne contient pas de sous-tore non nul (loc. cit. 6.3.8), d'où la conclusion.

Enfin il est classique (confer [3] Chapitre 9) que Pic $(\hat{Y}_k/k)$ ne contient pas de sous-tore non nul si et seulement si les composantes irréductibles de $\hat{Y}_k$ sont géométriquement unibranches et si le graphe de $\hat{Y}_k$ est sans circuit.

Remarque. Si dans les théorèmes 1 ou 1′ on ne suppose plus X lisse mais simplement que X_K est lisse et que X_k est semi-stable, les conclusions ne sont plus valables, même lorsque le graphe de X_k est un arbre. Par exemple, prenons pour X une courbe de genre 2, à fibre générique lisse dégénérant en deux courbes elliptiques ordinaires ayant un point commun. Après extension éventuelle de R, et pour tout l premier, on peut trouver un revêtement étale $Y \to X$, galoisien de groupe $\mathbb{Z}/l\mathbb{Z}$ qui induit sur chacune des courbes elliptiques un revêtement connexe. Alors le graphe de Y_k n'est plus un arbre.



graphe de Y_k pour $\ell = 3$.

graphe de X_k.

3. Exemples

On suppose désormais que la caractéristique résiduelle de R est $p > 0$.

Partons d'une R-courbe propre et lisse X, à fibres géométriquement connexes de genre g et soit J la jacobienne de X.

Pour tout R schéma en groupes commutatif fini et plat G, de dual de Cartier G', on a une bijection canonique :

$$H^1(X,G) \xrightarrow{\sim} \mathrm{Hom}\,(G',J),$$

compatible avec les changements de base sur R et où la cohomologie est calculée pour la topologie *fppf* ([14] 6.2.3).

Prenons pour G' un sous-schéma en groupes de J, annulé par une puissance de p à fibre générique de type multiplicatif. A l'immersion $G' \to J$ correspond donc un G-torseur $Y \to X$, avec G_K étale. Après extension éventuelle de K qui déploie G_K, on obtient ainsi tous le revêtements étales $Y_K \to X_K$, géométriquement connexes, galoisiens de groupe un p-groupe commutatif.

On suppose désormais que G est d'ordre p. Comme G'_k est un sous-groupe non trivial de J_k, $Y_k \to X_k$ est un torseur non trivial sous G_k ; en particulier Y_k est irréductible, géométriquement réduit, ayant pour seules singularités des points de rebroussement; Y est normal.

Il y a essentiellement trois cas suivant le type de G :

— G' est de type multiplicatif. Alors G est étale, donc $Y \to X$ est étale.

— G' est étale. Comme G'_K est supposé de type multiplicatif, ce cas ne peut se produire que lorque K est de caractéristique 0. Après extension étale de R on se ramène au cas où $G' = \mathbb{Z}/p\mathbb{Z}$. Le revêtement $Y \to X$ est alors un torseur de groupe μ_p décrit par un faisceau inversible L sur X d'ordre exactement p sur X_K et sur X_k. Si $h : Y \to X$ est le morphisme structural, $h_*(O_Y) = \oplus_i L^i$, $i = 0, 1, \ldots, p-1$ et la multiplication est donnée par la trivialisation de L^p.

— $G'_k = G_k = \alpha_p$. Alors G dégénère d'un groupe étale sur K en un groupe α_p sur k.

Lorque G_k est radiciel, le genre topologique de Y_k est égal à celui g de X/R, tandis que le genre arithmétique est celui de la fibre générique Y_K :

$$p(g-1)+1-g = (p-1)(g-1).$$

Mis à part le cas $g = 1$, la courbe Y_k présente donc effectivement des points de rebroussement.

Supposons k algébriquement clos et étudions localement les singularités de Y_k lorsque $G_k = \mu_p$ ou α_p. Soient x un point fermé de X_k, t un paramètre local de X_k centré en x, y le point de Y_k au-dessus de x. Au voisinage de x le torseur $Y_k \to X_k$ admet une équation de la forme $v^p = a$, où a est une section de $O_{X,x}$. Le torseur n'est pas génériquement trivial, donc a n'est pas une puissance p-ème. Localement pour la topologie étale, l'équation du torseur se ramène à $v^p = t^n$, pour un certain entier n avec $(n,p) = 1$. Y_k est singulier en y si et seulement si $n > 1$.

Calculons la contribution au genre arithmétique de Y_k de la singularité local en y, c'est à dire la longueur de $\tilde{\mathcal{O}}/\mathcal{O}$ où $\mathcal{O}$ est l'anneau local de Y en y et $\tilde{\mathcal{O}}$ l'anneau local de la normalisée. $\tilde{\mathcal{O}}$ admet comme paramètre centré une racine p-ème τ de t. Modulo une unité de $\tilde{\mathcal{O}}$, on peut écrire :

$$v = \tau^n \ , \ dv = \tau^{n-1} d\tau.$$

Le module dualisant Ω de la courbe d'équation $v^p = t^n$, admet pour base dv/t^{n-1}, celui de la normalisée $\tilde{\Omega}$ est engendré par $d\tau$. On a $d\tau = (t/\tau)^{n-1}$ dv/t^{n-1}. Donc $\Omega/\tilde{\Omega}$ est de longeur $(n-1)(p-1)$. Il résulte alors de ([16] Chapitre IV Section 3 N° 11) que la contribution de la singularité au genre arithmétique est $(n-1)(p-1)/2$.

Considérons maintenant un revêtement d'Artin Schreier non constant Z de la droite affine de paramètre t. Il admet une équation de la forme $U^p - U = g(t)$, où $g(t)$ est un polynôme de degré $m > 0$, que l'on peut supposer sans monôme de degré multiple de p. Etudions le genre de la courbe normale complétée $\overline{Z}$ de Z. Après localisation étale à l'infini et changement de t en $1/t$, on se ramène à l'équation :

$$U^P - U = 1/t^m.$$

Soit ∞ l'unique point à l'infini de Z et soit τ un paramètre de Z centré en ∞. La fonction U admet en ∞ un pôle d'ordre m. D'où, à des unités près, les relations :

$$d\tau/\tau^{m+1} = du = dt/t^{m+1}.$$
$$dt = (t/\tau)^{m+1} d\tau.$$

La valuation de la différente au point ∞ est donc $(p-1)(m+1)$. Si alors h est le genre de Z, on a :

$$2h - 2 = p(-2) + (p-1)(m+1),$$

d'où

$$h = (m-1)(p-1)/2.$$

Revenons à notre R-torseur Y sous G.

Proposition 3. *Supposons que Y_k ait au point y une singularité minimale donc, localement pour la topologie étale, admette une équation de la forme $v^p = t^2$ si $p > 2$ et $v^2 = t^3$ pour $p = 2$. Alors, la contribution de la singularité en y, à la réduction semi-stable Y de Y_K est une seule* *composante qui est une courbe de genre* $(p-1)/2$ *pour* $p > 2$ *et une courbe de genre* 1 *pour* $p = 2$.

En effet, on sait (proposition 2) que les bouts de l'arbre relatif à cette singularité sont des revêtements d'Artin-Schreier de la droite projective, ramifiés en un seul point et de genre > 0. La somme de ces genres est égale à la contribution de la singularité au genre arithmétique à savoir $(p-1)/2$ si $p > 2$ et 1 si $p = 2$. La formule précédente donnant le genre d'un revêtement d'Artin-Schreier ne laisse alors qu'une possiblité : il y a un seul bout, dont le genre est $(p-1)/2$ pour $p > 2$ et 1 pour $p = 2$.

Lorsque la singularité de Y_k, n'est plus minimale, il est possible que la réduction semi-stable de Y_K dépende non seulement de la singularité de Y_k, mais aussi de l'équation du torseur $Y \to X$ sur R.

Exemple : torseur sous μ_p sur une courbe générique

Partons d'une courbe X propre et lisse de genre g définie sur un corps k de caractéristique $p > 0$ et d'un torseur non trivial Y sous μ_p, associé à un faisceau inversible L sur X, d'ordre p.

Si L est défini par le 1-cocycle g_{ij}, avec $g_{ij}^p = f_i f_j^{-1}$, le torseur Y a pour équation locale $v^p = f_i$. Au faisceau L on associe la forme différentielle globale $\omega = df_i/f_i$ qui est une forme fixe sous l'opération de Cartier. Lorsque k est parfait, l'application $L \mapsto \omega$ établit une bijection entre faisceaux inversibles d'ordre divisant p et formes fixes sous l'opération de Cartier ([17] Proposition 10). Lorsque k n'est pas parfait, à une forme sur X fixe par l'opération de Cartier correspond un faisceau inversible L d'ordre p qui est défini seulement sur la courbe X^1, image réciproque de X par le Frobenius du corps k.

Les singularités de Y correspondent aux zéros de la forme ω. Plus précisément, supposons k algébriquement clos. En un point fermé x de X, choisissons une coordonnée centrée t. Quitte à modifier le 1-cocycle qui définit L, on peut choisir f_i en x, de la forme $f_i = 1 + t^n$, mod t^{n+1}, avec $(n,p) = 1$. Y est singulier au-dessus de x si et seulement si $n > 1$ et ω présente alors en x un zéro d'ordre $n-1$.

Exemples. Si $p = 2$, en un point singulier de Y, on a $n \geq 3$. Si $n = 3$, ω présente en x un zéro double; la courbe Y admet, localement pour la topologie étale, une équation de la forme $v^2 = t^3$; sa normalisée a pour équation $w^2 = t$ avec $w = v/t$. La contribution de la singularité au genre arithmétique de Y est 1.

Si $p > 2$, en un point singulier on a $n \geq 3$. Si $n = 2$, ω présente un zéro simple, Y a pour équation localement pour la topologie étale, $v^p = t^2$.

Posons $p = 2s - 1$, $w = v^s/t$. Alors la normalisée de Y a pour équation $w^p = t$. Dans l'anneau local $\tilde{\mathcal{O}}$ de la normalisée de Y, w a pour valuation 1, v pour valuation 2, $\tilde{\mathcal{O}}/\mathcal{O}$ est de longueur $s = (p-1)/2$, engendré par les images de $w, wv, \ldots, wv^{s-1}$. La contribution de la singularité au genre arithmétique est $(p-1)/2$, ce qui est en accord avec la formule générale donnée plus haut.

Proposition 4. *Soit X une courbe algébrique* générique *de genre $g > 1$ sur un corps k algébriquement clos de caractéristique $p > 0$. Alors :*

X est ordinaire et possède $p^g - 1$ formes différentielles ω, non nulles, fixes par l'opération de Cartier.

Pour $p = 2$, chacune de ces formes possède $g - 1$ zéros doubles. Le torseur sous μ_p associé à ω possède $g - 1$ points de rebroussement; chacun d'eux contribue au genre arithmétique pour 1.

Pour $p > 2$, chacune des formes ω a $2g - 2$ zéros simples; le torseur sous μ_p associé possède $2g - 2$ points de rebroussement; chacun d'eux contribue au genre arithmétique pour $(p-1)/2$.

Montrons d'abord qu'il existe une forme fixe par Cartier, non nulle, dont les zéros ont les propriétés mentionnées ci-dessus. Pour cela, considérons une courbe elliptique ordinaire E sur k et une forme ω sur E, non nulle, fixe par Cartier.

Si $p \geq 3$, on construit une courbe X, de genre g comme revêtement double de E en extrayant une racine carrée d'un faisceau inversible sur E, de degré $g-1$ assez général. L'image réciproque de ω sur X a les propriétés requises. Si $p = 2$, on procède de même à partir d'un revêtement cyclique triple X de E obtenu en extrayant une racine troisième d'un faisceau inversible de degré $g - 1$ assez général.

Soit alors η le point générique en caractéristique p, de la variété modulaire M_g des courbes de genre g et soit X_η une courbe générique de genre g au-dessus de η (X_η est unique dès que g est ≥ 3). Les considérations qui précèdent et un argument de spécialisation montrent que, sur une clôture algébrique de $k(\eta)$, il existe une forme invariante par l'opération de Cartier sur X_η, dont les zéros ont les multiplicités souhaitées.

Par ailleurs, on sait que X_η est ordinaire [11]. Soit J_η la jacobienne de X_η. Le noyau ${}_pJ_\eta$ de la multiplication par p dans J_η est extension d'une partie étale ${}_pJ_{et}$ de rang g par une partie de type multiplicatif. D'après Ekedahl [7], la représentation de monodromie a pour image $Gl_g(\mathbb{F}_p)$. En particulier, elle agit transitivement sur les points d'ordre p, à valeurs dans une clôture algébrique de $k(\eta)$. L'action de Galois est donc aussi transitive sur les formes différentielles non nulles invariantes par l'opération de Cartier et toutes ces formes ont des zéros de la multiplicité souhaitée.

Corollaire 4. *Soit R un anneau de valuation discrète complet d'inégale caractéristique, de caractéristique résiduelle $p > 0$. Soit X une courbe propre et lisse sur R, à fibres géométriques connexes de genre $g > 1$. On suppose que X_k est* générique. *Soit $Y \to X$ un torseur sous μ_p, non trivial en réduction sur X_k. Alors le graphe de la fibre spéciale de la réduction semi-stable de Y_K est un arbre qui admet :*

$2g - 2$ bouts, chacun d'eux étant une courbe de genre $(p-1)/2$, pour $p > 2$,

$g - 1$ bouts, chacun d'eux étant une courbe de genre 1 si $p = 2$.

Appendice : quotient d'une courbe semi-stable.

Faute de référence nous allons établir le résultat suivant utilisé dans la démonstration du théorème 1′ :

Proposition 5. *Soient R un anneau de valuation discrète complet, X une R-courbe de type fini, normale, intègre de corps des fractions L, M une extension finie de K, Y la normalisée de X dans M. On suppose que Y_k est géométriquement réduite et a pour seules singularités des points doubles ordinaires. Alors il en est de même de X_k.*

Corollaire. *Soit Y une R-courbe plate, normale, séparée, sur laquelle opère un groupe fini G et soit $X = Y/G$. On suppose que Y_k est géométriquement réduite avec pour seules singularités des points doubles ordinaires. Alors il en est de même de X_k.*

(Noter que R étant complet, Y est quasi-projective et le schéma quotient Y/G existe).

Démonstration de la proposition. Soit η un point générique d'une composante de X_k et τ un point de Y_k au-dessus de η. L'anneau local $\mathcal{O}_{Y,\tau}$ est un anneau de valuation discrète et il en est donc de même de $\mathcal{O}_{X,\eta}$. Par ailleurs, Y_k étant réduit, une uniformisante π de R est aussi une uniformisante de $\mathcal{O}_{Y,\tau}$, donc aussi de $\mathcal{O}_{X,\eta}$. Il en résulte que X_k est réduit en η. Par ailleurs X est normal donc S_2; par suite X_k est S_1, donc réduit. Ces propriétés sont conservées après extension plate de l'anneau de valuation discrète R. On en déduit que X_k est géométriquement réduit.

Pour établir la proposition, on peut donc gonfler R et supposer k algébriquement clos.

Soit x un point fermé de X_k et y un point fermé de Y_k au-dessus de x. Pour établir la proposition, on peut faire une localisation étale de X en x et se ramener au cas où y est l'unique point de Y au-dessus de x.

Supposons tout d'abord que Y soit lisse en y et montrons que X est lisse sur R en x. Soit $\bar{t}$ un paramètre de Y_k centré en y et soit t un relèvement de $\overline{T}$ dans $\mathcal{O}_{Y,y}$. Considérons $s = \text{Norme}_{Y/X}(t)$. Notons que X_k est intègre et soit $\tilde{X}_k$ son normalisé et x le point de $\tilde{X}_k$ au-dessous de y. La réduction $\overline{S}$ de s mod π est une section de $\mathcal{O}_{X,x}$ qui induit une uniformisante de l'anneau de valuation discrète $\mathcal{O}_{\tilde{X}_k,x}$. On en déduit immédiatement que l'anneau intègre $\mathcal{O}_{\tilde{X}_k,x}$ est un anneau de valuation discrète, puis que X est lisse sur R en x.

Supposons que Y_k ait un point double en y. Après hensélisation, on peut supposer que $\mathcal{O}_{Y,y}$ est l'hensélisé de $R[t,t']/tt' - \pi^e$. Distinguons deux cas :

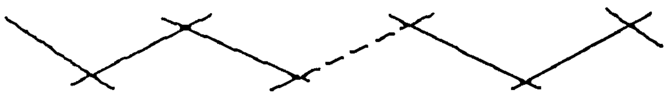
Premier cas. X_k possède deux branches en x. Le morphisme de schémas intègres :

$$\mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,x})$$

est fini et plat en dehors de x, d'un certain rang n. Notons s et s' les normes respectives de t et t'. On a $ss' = \pi^{ne}$. D'où un morphisme : $R[s,s']/ss' - \pi^{ne} \longrightarrow \mathcal{O}_{X,x}$, dont on vérifie qu'il est formellement étale (c'est à dire induit un isomorphisme sur les complétés). En particulier, x est un point double ordinaire de X_k.

Deuxième cas. X possède une seule branche en x. On doit montrer que X est alors lisse en x.

Notons que Y reste normale après extension finie, normale et génériquement étale de l'anneau de valuation discrète R; par suite la formation de X commute à un telle extension. Quitte à faire une telle extension de R, on peut supposer que l'épaisseur e de la singularité de Y en y est paire. Soit $\tilde{Y} \longrightarrow Y$ la désingularisation minimale de Y en y. La fibre spéciale de $\tilde{Y}$ est encore géométriquement réduite, à croisements normaux et consiste en une chaîne de $e-1$ droites projectives joignant les transformées strictes des deux composantes de Y_k en y.



Considérons $\tilde{Y} \times_X \tilde{Y}$ et ses deux projections sur $\tilde{Y}$. Soit Z_K la fibre générique de $\tilde{Y} \times_X \tilde{Y}$ et $\tilde{Z}$ son adhérence schématique dans $\tilde{Y} \times_X \tilde{Y}$. Alors, les projections de $\tilde{Z}$ sur Y sont finies et plates en codimension ≤ 1. Comme $\tilde{Y}$ est régulier, le saturé S_2 de $\tilde{Z}$, soit $\tilde{\tilde{Z}}$ est alors fini et plat sur $\tilde{Y}$. De plus $\tilde{Z} \to \tilde{Y}$ définit une relation d'équivalence en codimension ≤ 1, donc $\tilde{\tilde{Z}} \to \tilde{Y}$ définit un groupoïde fini et plat sur $\tilde{Y}$. Soit $\tilde{X}$ le quotient de $\tilde{Y}$ par ce groupoïde ([10] Exposé V). Par hypothèse le groupoïde fini $Y \to X$ échange les deux composantes de Y_k, donc le groupoïde $\tilde{Y} \to \tilde{Z}$ échange les deux bouts du graphe de $\tilde{Y}_k$. Par suite, ce groupoïde échange les composantes symétriques des extrémités du graphe de $\tilde{Y}_k$ et en particulier fixe la composante médiane.

D'après les cas déjà traités, on voit alors que $\tilde{X}_k$ est réduit avec pour seules singularités des points doubles ordinaires. De plus $\tilde{X}_k$ est une chaîne de $e/2$ droites projectives liée à un bout lisse : le transformé strict de X_k. On obtient X par contraction de droites projectives et donc X est lisse sur R.

Remarque. Sous les hypothèses de la proposition 5, supposons de plus que X soit propre. Alors on peut voir que X_k a pour seules singularités des points doubles ordinaires par voie globale, de la façon suivante :

Soit $J_{Y/R}$ la composante neutre de $\mathrm{Pic}_{Y/R}$. C'est un schéma en groupes dont la fibre spécial est de rang unipotent nul (confer. [3] chapitre 9). Il en est donc de même de la fibre générique et en fait Y_K étant normale est lisse. Soit de même la composante neutre $J_{X/R}$ de $\mathrm{Pic}_{X/R}$ qui est un schéma en groupes lisse (*loc. cit.*). L'application de norme relative au morhpisme fini $Y \to X$, fournit un morphisme $u : J_{Y/R} \to J_{X/R}$ qui est surjectif sur la fibre générique. Alors u se factorise à travers $v : G \to J_{X/R}$, avec G semi-abélien et v_K un isomorphisme. L'étude des points de l^n-torsion, avec l premier inversible, montre alors que $J_{X/R}$ est semi-abélien. La fibre spéciale X_k étant géométriquement réduite, on en déduit que localement pour la topologie étale, les singularités de X_k sont "comme des axes de coordonnées" (*loc. cit.*). Mais X_k est dominé par Y_k, donc les singularités de X_k sont au plus des points doubles ordinaires.

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Université de Paris-Sud
Centre d'Orsay
Mathématique - Bât. 425
91405 Orsay Cédex, France

Drawing Curves Over Number Fields

G.B. SHABAT and V.A. VOEVODSKY

"Lucky we know the Forest so well, or we might get lost"— said Rabbit, and he gave the careless laugh you give when you know the Forest so well that you can't get lost.

A.A. Milne,

The world of Winnie-the-Pooh.

Introduction

0.0. This paper develops some of the ideas outlined by Alexander Grothendieck in his unpublished *Esquisse d'un programme* [0] in 1984.

We draw our curves by means of what Grothendieck called "dessins d'enfant" on the topological Riemann surfaces. In the sequel we shall call them simply "dessins." By definition, a desssin D on a compact oriented connected surface X is a pair

$$D = (K(D), [\iota]),$$

where

$K(D)$ is a connected 1-complex ;

$[\iota]$ is an isotopical class of inclusions $\iota : K(D) \hookrightarrow X$.

We denote by $K_0(D)$ the set of vertices of $K(D)$.

It is supposed that

(a) the complement of $\iota(K(D))$ in X is a disjoint union of open cells ;

(b) the complement of $K_0(D)$ in $K(D)$ is a disjoint union of open segments.

The main construction we work with is based on the theorem of Gennady Belyi [1]. To a pair $D = (K, [\iota])$ it assigns a smooth algebraic curve together with some non-constant rational function on it over some number field. Throughout the paper we denote this curve by X_D and this function by β_D. We called them the *Belyi pair* associated to the dessin D.

According to [0], the realization of the possibility of such an assignment was one of the most striking events in Grothendieck's mathematical life. The only one he could compare it with was the following: "... vers l'âge de douze ans, j'étais interné au camp de concentration de Riecros (près de Mende). C'est là que j'ai appris, par une détenue, Maria, qui me donnait des leçons particulières bénévoles, la définition du cercle. Celle-ci m'avait impressionné par sa simplicité et son évidence, alors que la propriété de "rotondité parfaite" du cercle m'apparaissait auparavant comme une réalité mystérieuse au delà des mots".

The correspondence between the curves over number fields and dessins indeed seems to be very fundamental; in the end of this introduction we outline this construction in both directions.

We are interested in the constructive aspects of this correspondence. In the spirit of D. Mumford's monograph [2] we consider 5 ways of defining complex algebraic curves :

(1) Writing an equation.

(2) Defining generators of the uniformizing fuchsian groups.

(3) Specifying a point in the moduli space.

(4) Introducing a metric.

(5) Defining jacobian.

Our general approach is: given D, can we say anything about X_D and β_D?

Overview Of The Main Results

0.1. We use the following terminology. Let D be a dessin on a surface X. When $K(D)$ has no loops and each edge of $K(D)$ lies in the closure of exactly 2 components of $X\backslash K(D)$, we call

(a) a *valency of a vertex* $V \in K_0(D)$ the number of edges from $K(D)\backslash K_0(D)$, whose closures contain V.

(b) a *valency of a component* W of $X\backslash K(D)$ the number of edges from $K(D)\backslash K_0(D)$ that lie in the closure of W.

For the general dessins, see 1.4 below.

For a dessin D on a surface X we call its 0-*valency* the least common multiple of the valencies of all the vertices from $K_0(D)$ and its *2-valency* the least common multiple of the valencies of all the components of $X\backslash K(D)$. Sometimes we denote them $v_0(D)$ and $v_2(D)$, respectively.

Call the dessin D *trigonal* if the valencies of all the components of $X\backslash K(D)$ are 3. The trigonal dessins with some regularity assumptions define the triangulations of X.

We call the dessin D on X *balanced* if all the valencies of the vertices of $K_0(D)$ are equal (to $v_0(D)$) and all the valencies of all the components of $X\backslash K(D)$ are equal (to $v_2(D)$).

Similar objects were explored in the classical topology from the combinatorial point of view (see e.g., [3], [15]). We do not pretend any terminological compatibility with this line of research because the technique we use is completely different. (Threlfall, e.g., called our balanced dessins the "regelmässig Zellsystem").

0.1.1. *Equations.* Here we have no general theory and only give a number of examples. We also consider the opposite question: can we actually draw a given curve? The answer is yes for some famous ones: Fermat curves, Klein quartic, and some modular curves are among them.

The completeness of our results decrease rapidly with growing genus; we are able to give some complete lists (of non-trivial experimental material) for genus 0, but for genera exceeding 3 we are able to give only some general remarks.

0.1.2. *Uniformization.* Let $p = v_0(D)$, $q = v_2(D)$. Consider the subgroup $\Gamma_{p,q}$ of $PSL_2(\mathbb{R})$ consisting of those transformations of the Poincaré upper half-plane $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ that respect the tesselation obtained by the reflections of the regular p-gon with the angles $2\pi/q$ (see Coxeter [3]). We show that there exists a subgroup Γ of finite index in $\Gamma_{p,q}$ such that the quotient $\mathcal{H}/\Gamma$ is isomorphic to X_D. We describe this Γ explicitly by the combinatorics of D. For the balanced dessins D this construction describes the universal covering of X_D.

0.1.3. *Moduli.* Here we consider only the trigonal dessins D and use the results from R.C. Penner's preprint [4], where one finds several constructions equivalent to ours (without reference to the Grothendieck program).

Penner introduced the extended Teichmuller spaces $\tilde{T}_{g,n}$, consisting of marked Riemann surfaces together with the horocycles about each puncture. Using dessins, Penner coordinates $\tilde{T}_{g,n}$'s, essentially by considering the metric on $X\backslash K_0(D)$ of constant curvature -1 and by assigning to it some functions of the lengths of parts of edges of $K(D)$ lying between the horocycles. By this construction $\tilde{T}_{g,n}$ turns out to be homeomorphic to $\mathbb{R}_{>0}^{6g-6+3n}$. We deduce from Penner's results, that under this coordinatisation X_D corresponds to $(\sqrt{2}, \dots, \sqrt{2})$.

Penner builds the cell decomposition

$$\tilde{T}_{g,n} = \bigcup_{\substack{\text{dessins} \\ D \text{ on curves} \\ \text{of genus } g \\ \text{with } n \text{ vertices}}} C_D$$

where the trigonal dessins D correspond to the open cells. For $n = 1$ it gives the cell decomposition of the space $T_{g,1}$ itself. We deduce from Penner that in some sense $X_D \in C_D$.

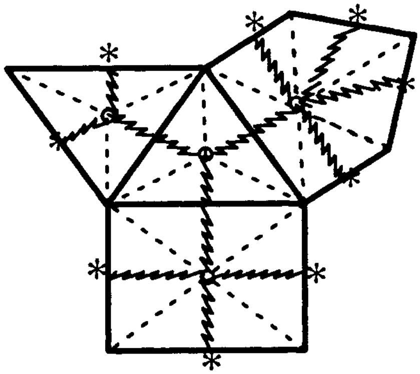
0.1.4. *Metric.* Here we consider only triangulations of X_D. Instead of riemannian metrics we work with "piecewise-euclidean" ones; they also allow us to define a complex structure on X_D. We show that X_D corresponds to the equilateral metrics.

0.1.5. *Jacobians.* Here we also work only with such dessins D that triangulate X. We define "approximate" jacobians $J_D(X)$, about which we think, that their limit, when D becomes finer, is the usual jacobian $J(X)$.

0.2. In the rest of the introduction we explain the essence of the assignment of Belyi pairs to dessins. In both directions we use :

Theorem [1] (Belyi). *A complex non-singular complete complex curve can be defined over some number field if and only if there exists a meromorphic function on it with only three critical values.*

0.2.1. *From dessins to curves.* We are supposed to be given a dessin D on a surface X. Choose a point in each 2-cell, connect the points of the neighbouring 2-cells by an edge of a different type, and connect all these points with the vertices of the original graph inside the 2-cells when it is possible.



Now we have three types of vertices and three types of edges. Since we are going to map the whole picture onto $\mathbf{P}^1(\mathbb{C})$, we mark them :

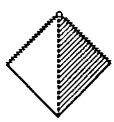
$$\begin{array}{lll} \circ & \text{over} & \text{“}0\text{”}, \\ \cdot & \text{over} & \text{“}\infty\text{”}, \\ * & \text{over} & \text{“}1\text{”}; \end{array}$$

we find the edges

$$\begin{array}{lll} \text{———} & \text{over} & (0,1), \\ & \text{over} & (1,\infty), \\ ---- & \text{over} & (\infty,0). \end{array}$$

Note that the whole surface has become divided into triangles, each with vertices of all three types. The orientability of the surface results in the possibility of painting it in two colours in such a way that if we move around the black and the white triangles counter-clockwise, the order of vertices is “0”-“1”-“∞” and “0”-“∞”-“1”, respectively.

Now look at the “butterflies”—the pairs of the adjacent triangles of different colours and think of the surface as the union of the butterflies.



If all the butterflies put their wings together to become S^2-like and are identified, we get the desired map

$$\beta_D : X_D \longrightarrow \mathbf{P}^1(\mathbb{C}),$$

ramified only over $\{0, 1, \infty\}$.

After we restore the unique algebraic structure of X_D (by the Riemann existence theorem) in which β_D is rational, we use the easier part of Belyi ("if" in the above formulation) theorem to conclude that X_D and β_D are defined over $\overline{\mathbb{Q}}$. The Belyi pair (X_D, β_D) thus obtained depends only on the original dessin D.

The Belyi function β_D obtained through this construction has all the ramifications over "1" or order 2. We shall call such Belyi functions clean.

The image of $K(D)$ in X can be reconstructed as the β_D-preimage of the segment $[1, \infty]$.

0.2.2. *From curves to dessins.* In this direction we outline the proof of the more striking half of the Belyi theorem ("only if" in the above formulation), closely following [1]. Suppose we are given a curve X over some number field L ; take a non-constant element f of the function field $L(X)$ and consider it as a ramified covering

$$f : X(\mathbb{C}) \longrightarrow \mathbf{P}^1(\mathbb{C});$$

its ramification points belong to $\mathbf{P}^1(\overline{\mathbb{Q}})$.

We are going to transform f to the desired covering by a sequence of replacements of f by $P \circ f$, where P is a polynomial with rational coefficients, considered as a map $\mathbf{P}^1(\mathbb{C}) \longrightarrow \mathbf{P}^1(\mathbb{C})$. After such replacements, infinity will always go to infinity; denote for $F : X(\mathbb{C}) \longrightarrow \mathbf{P}^1(\mathbb{C})$ by W_F the set of finite critical values of F.

By the first step of the construction, we reduce the situation to the case $W_F \subset \mathbb{Q}$. We proceed by induction on $\#(W_F \backslash \mathbb{Q})$. At each inductive step we take for P a generator of the annihilator in $\mathbb{Q}[T]$ of W_F. It is clear that $\# W_{P \circ F} = \# W_F - 1$. So we suppose $W_F \subset \mathbb{Q}$ and proceed to the second step.

Replacing f by $Af + B$ with suitable A, B we can assume $\{0, 1, \infty\} \subset W_F$. Proceed by the induction in $\#(W_F \backslash \{0, 1, \infty\})$. Assuming that there exists $r \in W_F \backslash \{0, 1, \infty\}$ with $0 < r < 1$ (which can again be achieved by a change of f to $Af + B$, denote $r = \frac{m}{m+n}$ with natural n, m. Now use

$$P(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

and check that $P(0) = P(1) = 0$, $P(r) = 1$ and that if $P^1(z) = 0$, then $z \in \{0, 1, r\}$; so replacing f by $P \circ f$ reduces $\#(W_F \backslash \{0, 1, \infty\})$ by 1, which completes the inductive argument.

We have obtained the Belyi function f; to get a clean one put

$$\beta = 4f(1-f).$$

The desired dessin D is defined by setting $K(D) = \beta^{-1}([1,\infty])$ with $K_0(D) = \beta^{-1}(0)$.

0.3. The style of our exposition is far from the standards of modern mathematics; we hope that this flaw is partially compensated for by the explicitness of the results. Besides, we do not prove some of our assertions. This is not only the result of space and time constraints, but rather of the feeling that the proper language for the mathematics of the Grothendieck Program has not yet been found.

The main reasons to publish our results in the present state is our eagerness to invite our colleagues into the world of the divine beauty and simplicity we have been living in since we have been guided by the Esquisse [0]. We emphasize that in the present text we use only a very small part of the ideas one can find in the epoch-making paper.

We are indebted to Yu. I. Manin for the useful discussions and to A.A. Migdal and his colleagues for their interest. We are grateful to I. Gabitov, without whose assistance the preparation of the present text would have been impossible.

Part 1. Generalities

Let D be a dessin on a surface X. We choose a representative of the isotopical class of the inclusions $K(D) \overset{\iota}{\hookrightarrow} X$ and in what follows consider $K(D)$ as a subset of X; all our constructions are independent of this choice.

1.1.1 The *flag set* $\mathsf{F}(D)$ is, by one of the definitions, a set of triples (U, E, V) such that U is a component of $X \backslash K(D)$, E a component of $K(D) \backslash K_0(D)$, V a vertice from $K_0(D)$ and

(i) E lies in the closure of U;

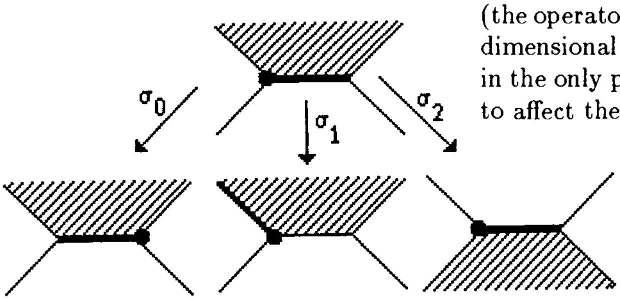
(ii) V lies in the closure of E.

This definition is suitable only for the fine enough dessins; later, we will give a universal definition.

1.1.2. Consider a *"cartographical"* group

$$\mathcal{C}_2 = \langle \sigma_0, \sigma_1, \sigma_2 | \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1 \rangle$$

It acts on the flags in the following way :



(the operator σ_q changes the q-dimensional element of the flag in the only possible way as not to affect the other elements).

Because of the connectedness of X, the flag space $\mathsf{F}(D)$ is $\mathcal{C}_2$-homogeneous.

In analogy with the linear case, we call a (non-oriented) *Borel* subgroup of the flag $F \in \mathsf{F}(D)$ the stationary group of this action:

$$B_{D,F} = \{c \in \mathcal{C}_2 | c \cdot F = F\}$$

1.1.3. The *orientability* of X (in the spirit of the Introduction) results in the possibility of defining the map

$$o : \mathsf{F}(D) \longrightarrow \{\pm 1\}$$

satisfying

$$o(\sigma_q \cdot F) = -o(F) \text{ for all } F \in \mathsf{F}(D), q \in \{0, 1, 2\}.$$

We consider the set of *positively oriented flags*

$$\mathsf{F}^+(D) = o^{-1}(1).$$

The *oriented cartographical* group $\mathcal{C}_2^+$ is the one that respects all the sets

$$\mathsf{F}^+(D) \subset \mathsf{F}(D).$$

It is the subgroup of index 2 of $\mathcal{C}_2$ generated by the words in $\sigma_0, \sigma_1, \sigma_2$ of even length. We take the generators

$$\rho_0 = \sigma_2 \sigma_1$$
$$\rho_1 = \sigma_0 \sigma_2$$
$$\rho_2 = \sigma_1 \sigma_0$$

satisfying

$$\rho_2 \rho_1 \rho_0 = 1$$

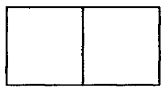
Since all the Borel subgroups corresponding to one dessin are conjugate, in cases where we need only the conjugacy class of $B_{D,F}$, we shall omit the reference to F.

The dessin is called a *Galois* one if the group B_D is normal in $\mathcal{C}_2^+$.

We call an automorphism of the dessin a compatible triple consisting of a permutation $K_0(D) \longrightarrow K_0(D)$ and isotopical classes of homeomorphisms $K(D) \longrightarrow K(D)$, $X \longrightarrow X$.

It is clear, that in the case of a Galois dessin D with Borel subgroup B the factor $\mathcal{C}_2^+/B$ acts on D, transitively on the vertices, on the edges and on the components of $X \backslash K(D)$. Therefore, the Galois dessins are balanced.

The converse is wrong: the dessin

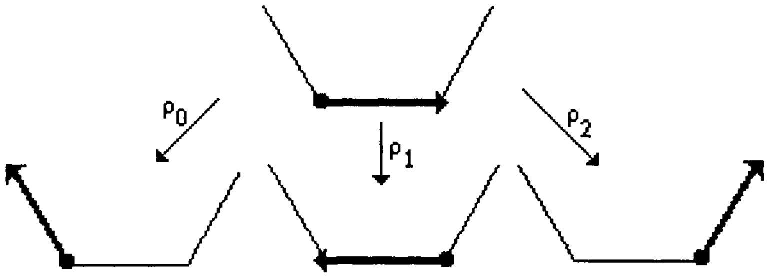


with identified opposite sides gives an example of a balanced non-Galois dessin on the torus.

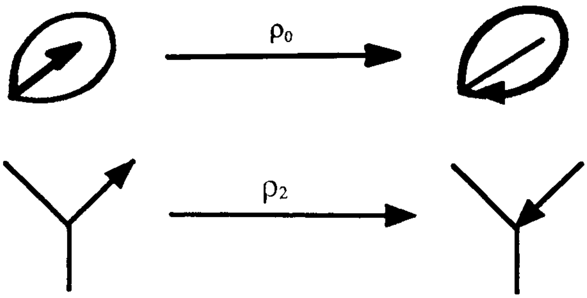
All the automorphisms of the dessin D on a surface X are realizable as conformal automorphisms of X.

1.1.4. The positively oriented flags are in canonical one-to-one correspondence with the oriented edges of $K(D)$. We use the following convention:

Under this convention the oriented cartographical group acts on the oriented edges in the following way:



In this way the definition of the action of the oriented cartographical group on the oriented flags is naturally extended to all the dessins, with no regularity assumptions. In the degenerate cases this action may look, for instance, like this:



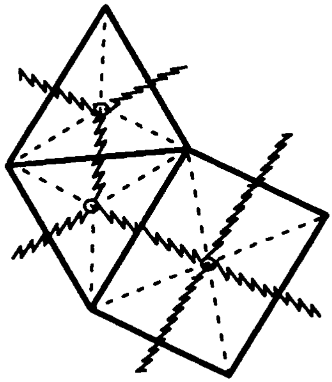
The orders of $F \in \mathbb{F}^+(D)$ with respect to ρ_0, ρ_2 are related with the general valencies (see 0.1) as follows :

$$\#\{\langle\rho_0\rangle\}F = v_0(\text{0-component of } F)$$
$$\#\{\langle\rho_2\rangle\}F = v_2(\text{2-component of } F).$$

In particular, if $v_0(D) = p$, $v_2(D) = q$, then for any $F \in \mathbb{F}^+(D)$

$$\langle\rho_0^p, \rho_2^q\rangle \subset B_{D,F}$$

1.1.5. For a dessin D on a surface X the dual dessin D^* is defined as follows. The set of vertices $K_0(D^*)$ is in canonical one-to-one correspondence with the components of $X \backslash K(D)$, and the set of edges of D—with that of D. The pair of dual dessins looks like



1.1.6. It is very important [0] that the correspondence between the dessins and the Belyi pairs defines the action of the Galois group Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$ on the dessins. In particular, the

FIELD OF DEFINITION OF A DESSIN

makes sense.

Part 2. Results

2.1. *Equations.* We start with some of the most simple dessins and try to determine which Belyi pairs over what number fields they define. Grothendieck [0] doubts that the problem can be solved by a uniform method. We can confirm from our experience that if such a method exists, it is quite involved.

We adopt the following drawing conventions:

(a) for genus 0 we put the dessins in the euclidian plane; if we have a vertex at infinity, we put arrow marks on the edges going there;

(b) for genus 1 we draw everything in the period parallelogram;

(c) for genus 2 and higher we realize the surfaces as polygons with identified boundary edges and draw our dessins on these polygons; the pairs of identified edges are directed controversially (which means that one edge goes clockwise while another one goes counterclockwise).

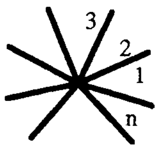
2.1.0. *Genus* 0. Since the curves of genus 0 have no moduli, the only thing to calculate is the Belyi function, which in this case is just a rational map $\mathbf{P}^1(\mathbb{C})\longrightarrow\mathbf{P}^1(\mathbb{C})$ with three critical values. It is defined up to composition with $PSL_2(\mathbb{C})$-transformations from the left and from the right. The proper choice of the representatives seems to be connected with delicate arithmetical questions which we do not discuss here; in the examples below we try to choose them in the shortest form. As a result, the critical values of the Belyi functions vary.

We consider first the simplest case : suppose that $X \backslash K(D)$ is connected. Then $K(D)$ is just a tree inside an oriented 2-sphere S^2, and we shall draw it in the plane (where the orientation is essential!).

Using our right to choose the $PSL_2(\mathbb{C})$-representatives as we like, we normalize the Belyi maps in such a way that the "centre" of the only 2-cell lies at ∞ and goes to ∞ and the intersections of $K(D)$ and $K(D^*)$ will go to 0. Then the Belyi map is represented by a polynomial, all the zeroes of which are double; therefore it is a square of a polynomial, which we denote h_D or simply h. We are going to present a number of h for some simple dessins D.

We present three infinite series of tree-like dessins for which h can be specified:

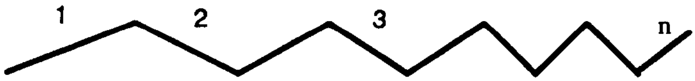
1.



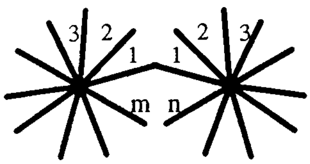
$h(z) = z^n$

2.

$h(z)$ is the n-th Chebyshev polynomial: $\cos(nt) = h(\cos t)$



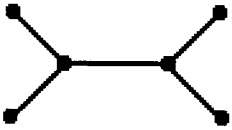
3.



$h(z) = z^m(1-z)^n$
(up to multiplication by constant the Belyi function from the introduction)

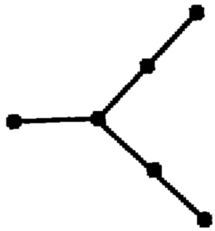
To cover all the trees with the number of edges not exceeding 6, we should add eight individual dessins (three of them equivalent under the Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$-action).

1.

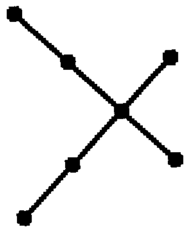


$h(z) = (z+1)^3(3z^2 - 9z + 8)$

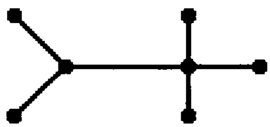
2. $h(z) = z^3(9z^2 - 15z + 40)$



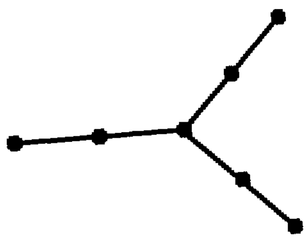
3. $h(z) = z^4(36z^2 + 36z + 25)$



4. $h(z) = z^3(6z^2 + 96z + 25)$



5. $h(z) = z^3(z^3 + 1)$



Note that all these dessins, as well as the infinite series above, are defined over the rationals $\mathbb{Q}$.

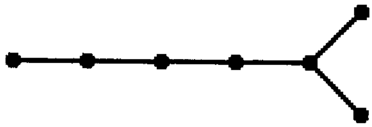
The remaining three dessins are defined over three cubic fields, permutable by the Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$. They lie in the field of decomposition of the polynomial

$$25t^3 - 12t^2 - 24t - 16 = 25(t-a)(t-a_+)(t-a_-)$$

(we agree that $a \in \mathbb{R}$, Im $(a_+) > 0$, Im $(a_-) < 0$).

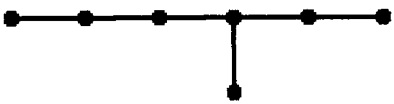
The dessin

D_a:



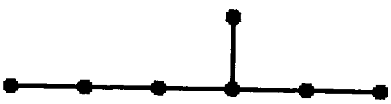
is defined over $\mathbb{Q}(a)$,

D_{a_+}:



over $\mathbb{Q}(a_+)$,

D_{a_-}:



over $\mathbb{Q}(a_-)$.

For each $b \in \{a, a_\pm\}$ we have

$$h_{D_c}(z) = z^3(z+1)^2(z+b).$$

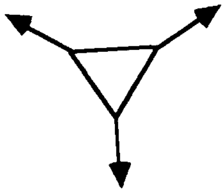
Now we turn our attention to the Galois dessins. They constitute two families: corresponding to the plane polygons and to the Platonic solids. As for the plane n-gons, in one of the normalizations they are described by the rational function

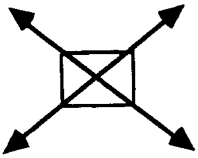
$$z^n + (1/z)^n$$

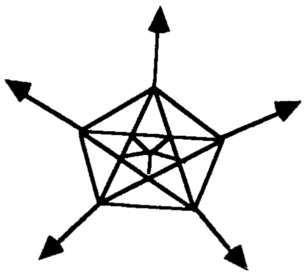
To discuss the platonic solids, introduce the notation

$$\dot{X}_D = X_D \backslash K_0(D)$$

and denote by Y_n the affine modular curve $\mathcal{H}/\Gamma(n)$, where $\Gamma(n)$ is a principal congruence subgroup of $PSL_2(\mathbb{Z})$ (the kernel of the natural homomorphism $PSL_2(\mathbb{Z} \longrightarrow PSL_2(\mathbb{Z}/n\mathbb{Z}))$. We claim that the platonic solids with the triangle 2-cells correspond to the modular curves in the following way:

—for tetrahedral $D =$  $\dot{X}_D = Y_3$

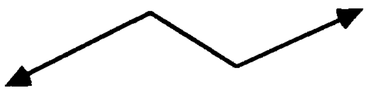
—for octahedral $D =$  $\dot{X}_D = Y_4$

—for icosahedral $D =$  $\dot{X}_D = Y_5$.

For all of them, the Belyi map corresponds to the canonical projection

$$Y_n = \mathcal{H}/\Gamma(n) \longrightarrow \mathbb{P}^1(\mathbb{C}) \simeq \mathcal{H}/PSL_2(\mathbb{Z}).$$

The curve Y_2 did not enter this list because the corresponding

$D =$ 

is not a platonic solid. But this dessin is a remarkable one: the corresponding rational function in one of the normalizations is

$$\frac{27(z^3 - z + 1)}{4z^2(z-1)},$$

which is exactly the expression for the J-invariant of the elliptic curve

$$y^2 = x(x-1)(x-2).$$

Left composition of this function with any canonically normalized Belyi function corresponds to the barycentric subdivision of the corresponding dessin.

2.1.1. *Genus* 1. On the curves of genus 1 there exist balanced dessins of valencies $v_0 = v_2 = 4$, $v_0 = 6$, $v_2 = 3$ and $v_0 = 3$, $v_2 = 6$.

The corresponding curves are isogenic to the elliptic curve

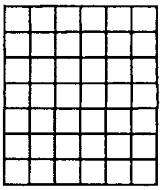
$$y^2 = x^3 - x$$

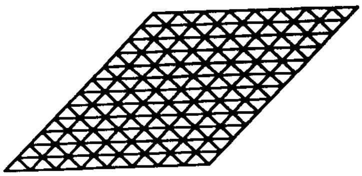
in the first case and to the curve

$$y^2 = x^3 - 1$$

in the second and in the third cases.

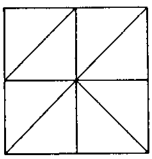
The Galois dessins represent exactly these curves. (See [3] for the proofs of the classification results.) In the period parallelogram these dessins look like



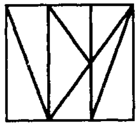


$\{v_0, v_2\} = \{4, 4\}$ $\{v_0, v_2\} = \{6, 3\}$

The problem of determining the J-invariant of the curves drawn by a dessin on the torus seems to be rather hard. For instance, consider the dessin on the torus which in the fundamental parallelogram looks like

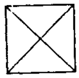


The corresponding curve is defined over $\mathbb{Q}(\sqrt{7})$. The $(\sqrt{7} \to -\sqrt{7})$-conjugated dessin looks like



The J-invariants are $-\frac{1}{16\,27}(8 \mp 3\sqrt{7})^2(2 \pm \sqrt{7})^6(10 \pm 3\sqrt{7})^3$. This calculation answers the question of A. A. Migdal, 1986.

Sometimes the curve can be determined by the additional symmetries of the dessin. For instance, for

D= 

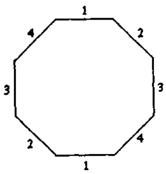
X_D is determined by the equation

$$y^2 = x^3 - x$$

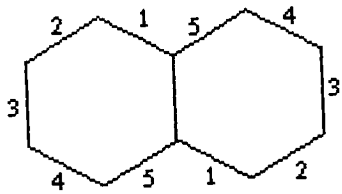
with Belyi function $-27x^4/(x^2-4)^3$ because of the symmetry of 4th order.

2.1.2. *Genus* 2. We are able to give some results only for Galois dessins. They are listed in [3]; there are 10 of them, but we choose only one from each pair of the dual ones.

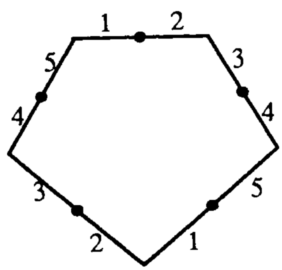
DESSIN — BELYI PAIR

1. 

$y^2 = x^5 - x$
$\beta = x^4$

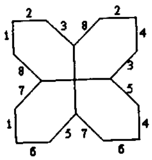
2. 

$y^2 = x^5 - 1$
$\beta = x^5$

3. 

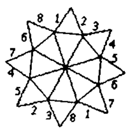
$y^2 = x^6 - 1$
$\beta = x^6$

4.



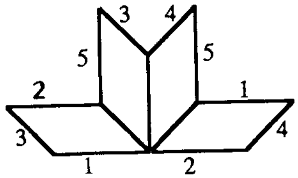
$$y^2 = x^6 - 1$$
$$\beta = (1 + x^6)^2/4x^6$$

5.



$$y^2 = x^5 - x$$
$$\beta = 31^3 \frac{4x^2(1 + x^4)^4}{[27(1 + x^4)^2 - 8x^4]^3}$$

6.



$$y^2 = x^5 - x$$
$$\beta = 4x^4/(1 + x^4)^2$$

2.1.3. *Genus* 3. Here we discuss only three curves. The Klein quartic is defined in the homogeneous coordinates by the equation

$$x_0 x_1^3 + x_1 x_2^3 + x_2 x_0^3 = 0$$

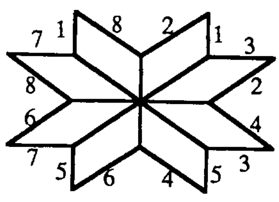
Its full automorphism group has 168 elements; the quotient by this group is isomorphic to $\mathbf{P}^1(\mathbf{C})$, and the natural projection defines a clean Belyi map.

We do not attempt to draw the corresponding dessin and refer the reader to Klein's paper [5], where one finds a very beautiful triangular tesselation of the fundamental domain of the Klein quartic on the universal covering (this was the figure from which the uniformization started). After suitable identifications, we get some triangular dessin D on a curve of genus 3.

Thus the Klein quartic itself has been drawn; indeed, the automorphism group of D and consequently of the curve X has 168 elements, and Klein quartic is the only curve of genus 3 with this (highest possible) number of automorphisms [6].

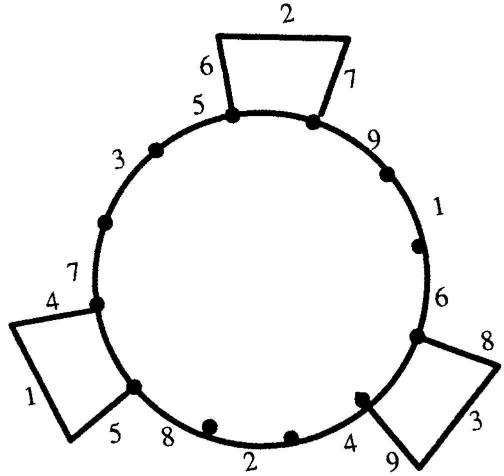
Here is the dessin for the Fermat quartic

$$x^4 + y^4 = 1 :$$



with the Belyi function $\beta = (x^4+1)^2/(4x^4)$.

Here is the dessin for the Picard curve [7] $y^3 = x^4 - 1$



with the Belyi function of the same form $\beta = (x^4+1)^2/(4x^4)$.

2.1.4. *Higher Genera.* We propose two infinite families of curves for which something can be done.

One is the family of the generalized Fermat curves

$$x^m + y^n = 1$$

It is easy to check that the functions x^m and y^n are the Belyi ones. Taking $n = 2$, we get a drawable curve of any given genus. The other series is the family of modular curves $X_0(n)$ (the factors $\mathcal{H}/\Gamma_0(n)$, where $\Gamma_0(n)$ is the preimage of the upper triangular matrices under the canonical projection $PSL_2(\mathbb{Z}) \longrightarrow PSL_2(\mathbb{Z}/n\mathbb{Z})$). The dessins on these curves are the projections of the $PSL_2(\mathbb{Z})$-orbits of the arc from $\exp\,(2\pi i/3)$ to $\exp\,(\pi i/3)$ on the boundary of the modular figure. For the discussion of an explicit description of modular curves see [8], [9].

2.2 Uniformization

Let D be an arbitrary dessin on a surface X. Let $p = v_0(D)$, $q = v_2(D)$. We canonically associate to D a conjugacy class of discrete subgroups

$$\Gamma_D \subset PSL_2(\mathbf{R})$$

and then show that

$$X_D \simeq \mathcal{H}/\Gamma_D$$

2.2.0. *Construction.* The reference for the material below is [3].

Consider in the Poincaré upper half-plane $\mathcal{H}$ the regular q-gon $\Pi_{p,q}$ with angles $2\pi/p$. The half-plane is tesselated by its reflections through its sides.

Consider the group $\Gamma_{p,q} \subset PSL_2(\mathbf{R})$ that respects this tesselation. It is generated by two elliptic elements

$$\begin{aligned} \beta = \quad & \text{the rotation of } \Pi_{p,q} \text{ about the centre of} \\ & \text{angle } 2\pi/q. \\ \gamma = \quad & \text{the rotation of } \mathcal{H} \text{ about one of the vertices} \\ & \text{of } \Pi_{p,q} \text{ of angle } 2\pi/p. \end{aligned}$$

It is geometrically obvious that these elements satisfy the relations

$$\begin{aligned} \beta^q &= \gamma^p = 1, \\ (\beta\gamma)^2 &= 1 \end{aligned}$$

(the transformation $\beta\gamma$ being a symmetry around the centre of the side of $\Pi_{p,q}$).

These relations allow us to define the homomorphism

$$\begin{aligned} \mathcal{H}_{p,q} : C_2^+ &\longrightarrow \Gamma_{p,q} \subset PSL_2(\mathbf{R}) \\ \rho_0 &\longmapsto \gamma \\ \rho_2 &\longmapsto \beta \end{aligned}$$

Fix some $F \in \mathbf{F}(D)$. The desired group is

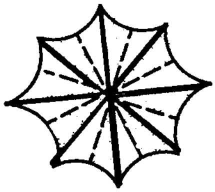
$$\Gamma_{D,F} = \mathcal{H}_{p,q}(B_{D,F})$$

Its conjugacy class is independent of all the choices involved. Denote it by Γ_D.

2.2.1. Theorem.

$$\mathcal{H}/\Gamma_D \simeq X_D$$

Sketch of the proof. If we conect the centre of $\Pi_{p,q}$ by geodesics with all the vertices and all the centres of the sides and then paint (as Klein did in [5]) these triangles in black and white this way



we realize that for every subgroup $\Gamma \subset \Gamma_{p,q}$ of finite index these triangles are in one-to-one correspondence with the flags of the dessin on $\mathcal{H}/\Gamma$ obtained by the projection of the above infinite dessin on $\mathcal{H}$. The orientation of the flags corresponds to the colour of the triangles.

It remains to realize that this dessin is isomorphic to the original one and that the natural projection

$$\mathcal{H}/\Gamma_D \longrightarrow \mathcal{H}/\Gamma_{p,q} \simeq \mathbf{P}^1(\mathbb{C})$$

is the same as the one defined by this dessin by our construction (see Introduction).

2.2.2. Theorem. *If D is balanced, then the natural map*

$$\mathcal{H} \longrightarrow \mathcal{H}/\Gamma_D$$

is isomorphic to the universal covering of X.

Indeed, for general D the ramification index of the map $\mathcal{H} \longrightarrow \mathcal{H}/\Gamma$ equals $p/v_0(V)$ over $V \in K_0(D)$ and $q/v_2(S)$ over the "centre" of the component S of $X \backslash K(D)$.

Therefore, this map is unramified for the balanced D.

In this way we can effectively describe the universal coverings of all the curves that were drawn by the balanced dessins in the previous section. For the Klein quartic it was done by Klein [5].

2.2.3. Now we turn to the universal coverings of the curves $\dot{X}_D$ with trigonal D.

Proposition. *The group C_2^+ with the additional relation $\rho_2^3 = 1$ is isomorphic to $PSL_2(\mathbb{Z})$.*

Using [3], *define this isomorphism by the assignment*

$$\rho_2 \longmapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_1 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_0 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Theorem. *For trigonal dessins D the curve $\dot{X}_D$ is isomorphic to $\mathcal{H}/\Gamma$, where $\Gamma \subset PSL_2(\mathbb{Z})$ corresponds to the Borel subgroup B_D under the above isomorphism.*

2.3 Moduli

2.3.0. The results of this part are based on Penner's paper [4]. Now we work with the above curve $\dot{X}_D$.

Introduce the notation $[D] = \#(K_0(D))$. The curves $\dot{X}_D$ after a suitable marking turn into the points of the Teichmuller space $T_{g,n}$ (see, e.g., [16]). We always assume $2g - 2 + [D] > 0$ and interpret the points of $T_{g,n}$ as metrics with constant curvature -1 on $\dot{X}_D$.

We start with a review of Penner's approach to the Teichmuller spaces $T_{g,n}$ with $n > 0$. Penner introduces the augmented Teichmuller spaces $\tilde{T}_{g,n}$, whose points correspond to the points of the usual Teichmuller space T together with the horocycles about each puncture (a closed curve, orthogonal to all the geodesics, going to the cusp).

Denote for any dessin $D = (K, [\iota])$ by $K_1(D)$ the set of (nonoriented!) edges of K, i.e., the set of the connected components of $K(D)\backslash K_0(D)$.

Also denote by $K_2(D)$ the set of connected components of $X\backslash K(D)$.

In this section we consider only trigonal dessins D. In what follows, $n = [D]$.

Lemma. $\dim_{\mathbb{R}}\tilde{T}_{g,n} = \#K_1(D) = 6g - 6 + 3[D]$.

Proof. For the Euler characteristic we have

$$\#K_0(D) - \#K_1(D) + \#K_2(D) = 2 - 2g,$$

and the number of "1, 2-flags" in the pair $(X, K(D))$ equals

$$2\#K_2(D) = 3\#K_1(D).$$

From these two equalities we have

$$\#K_1(D) = \#K_0(D) + \#K_2(D) + 2g - 2 = \#K_0(D) + \frac{2}{3}\#K_1(D) + 2g - 2,$$

and

$$\begin{aligned}\#K_1(D) = 3([D] + 2g - 2) &= [D] + 2(3g - 3 + [D]) \\ &= [D] + \dim_{\mathbf{R}} T_{g,n} = \dim_{\mathbf{R}} \tilde{T}_{g,n}\end{aligned}$$

So it is natural to try to coordinatize $\tilde{T}_{g,n}$ by the functions on the set of edges $K(D)$. On the part of $\tilde{T}_{g,n}$, on which the horocycles are so small that they do not intersect, there exists a natural function: think of every edge from $K(D)$ as an (infinite) line from puncture to puncture in $X(D)$, deform it to the geodesic and take the lengths of the part between the horocycles. It turns out that the analytic continuation of this construction gives the global coordinatization of $\tilde{T}_{g,n}$!

To establish it, Penner uses the Minkowski space $\mathbf{M}^3$ with the coordinates $x = (x_0, x_1, x_2)$ and with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2$$

induced from the scalar product $\langle\ ,\ \rangle$ of signature $(- + +)$. It follows from the local isomorphity of $SL\,(\mathbf{R})$ and $SO\,(1,2)$ and some easy Lie group considerations, that the Poincaré upper half-plane is isometric to a connected component of the hyperboloid

$$\langle x, x\rangle = -1,$$

and the space of horocycles on it to the future light cone

$$\langle x, x\rangle = 0, x > 0.$$

Denote by ℓ the Poincaré length of the above part of geodesic; let the intersections of this geodesic with the horocycles be represented by the Minkowski space points u, v. Then (see Lemma 2.1 from [4])

$$-\langle u, v\rangle = 2\exp(\ell),$$

and this formula allows the length coordinatization to be continued to the whole $\tilde{T}_{g,n}$.

Denote the length map thus described by

$$\mathrm{Pen} : \tilde{T}_{g,n} \longrightarrow \mathbf{R}_{>0}^{6g-6+3n}$$

Theorem 3.1 from [4] says, that this map is a real-analytic homeomorphism.

Now we can state our results.

2.3.1. Theorem. *There exists a set of horocycles on $\dot{X}_D$, endowed with the metric of the constant curvature, such that the above map Pen sends the corresponding point of $\tilde{T}_{g,n}$ to*

$$(\sqrt{2}, \ldots, \sqrt{2}) \in \mathbb{R}_{>0}^{6g-6+3n}.$$

The proof follows from Penner's proposition 6.5 from [4], which states that the point

$$\mathrm{Pen}^{-1}(\sqrt{2}, \ldots, \sqrt{2})$$

is uniformizable by a subgroup of $PSL_2(\mathbb{Z})$, and from our Theorem 2.3 from the previous section.

2.3.2. Next, we describe Penner's universal cell decomposition of the space $\tilde{T}_{g,n}$. Its points are interpreted as the conjugacy classes of the fuchsian groups $\Gamma \subset SO^+(2,1)$ together with the Γ-invariant set B on the light cone (the only point of B on each light ray corresponds to the choice of the horcycle). To such a pair (Γ, B) the convex hull of the discrete set $\Gamma \cdot B$ is associated. The cells of $\tilde{T}_{g,n}$'s correspond to the pairs (Γ, B) with the fixed combinatorics of the boundary of this convex hull. Projecting the edges of this boundary to the hyperboloid $\langle x, x \rangle = -1$, we get a Γ-invariant tesselation of it. Dividing then by Γ, we get the dessins (with fixed number $n = [D]$ of vertices) that parametrize the cells of the decomposition we are describing.

From now on, we suppose $[D] = 1$; then the above construction defines a decomposition of the Teichmuller space $T_{g,1}$ itself. Since this decomposition is invariant under the Teichmuller modular group, it induces a finite cell decomposition of the moduli space $M_{g,1}$. For a dessin D, denote by C_D the cell corresponding to it. The cells C_D of maximal dimension correspond to the trigonal dessins D.

2.3.3. Theorem. *For $[D] = 1$ the point of $M_{g,1}$ corresponding to X_D lies inside C_D.*

The proof follows from Penner's reasoning on the non-emptiness of C_D (Corollary 6.3 of [4]), combined with Proposition 6.5.

2.3.4. Without proof we state one more result concerning the position of the curves X_D with the Galois D's in the moduli spaces. Since they have many automorphisms, they correspond to some strong singularities of the moduli spaces.

Theorem. *The curve X over $\overline{\mathbb{Q}}$ can be realized as X_D with a Galois dessin D if and only if it is a projection of an isolated fixed point of some finite subgroup of the Teichmuller modular group.*

2.4 Metrics

In this section, we consider only piecewise-euclidean metrics—the ones that are formed by putting together compatible flat polygons; the resulting metric is flat away from the isolated points where the vertices meet and where the discrete curvature occurs (as a difference of 2π and the sum of the flat angles).

The piecewise-euclidean metrics define complex structures as well as the riemannian ones; any complex structure on a Riemann surface can be obtained in this way. A nice proof of this fact can be obtained using Strebel differentials; for a modern exposition of this theory see Douady-Hubbard [11] (though they do not formulate explicitly the result we need).

2.4.0. We are going to work only with the piecewise-euclidean metrics, in which all the polygons are the equilateral triangles; we call them the equilateral metrics.

The equilateral metrics have no continuous moduli and depend only on the combinatorics of the triangles; so the trigonal dessins D appear.

Denote the corresponding curves Y_D.

They define the countable set of points in the moduli spaces.

2.4.1. Theorem. *For any trigonal dessin D the curve Y_D is isomorphic to X_D.*

2.4.2. Theorem. *The set of curves Y_D for all the dessins D on the surfaces of genus g "is" exactly the set $M_g(\overline{\mathbb{Q}})$ of curves over all the number fields.*

For the proof, see our paper [10].

2.4.3. This result can be interpreted in terms of string physics (see, for instance, [12]; in fact, we were influenced by the authors of this paper). It shows that integration over all the metrics on Riemann surfaces using the lattice-like method of approximation of Riemann metrics uncovers the arithmetical nature of the subject.

In the spirit of fashionable ideas of modern theoretical physics, it is natural to suggest that the non-archimedian components would also be taken into account in this approach. For the discussion of these ideas see Manin [13].

We suppose, that the dessins on the Riemann surfaces may turn to be quite fundamental in quantum physics, being the natural analogues of Feynman graphs in the pre-string theories.

2.5 Jacobians

In this section we work with the trigonal dessins D on the Riemann surfaces of positive genus g.

2.5.0. The dessin D defines on X the structure of a cell complex, which will be used in the realisation of the cohomology classes. Denote by $\mathcal{O}$ the structure sheaf, Ω the sheaf of germs of holomorphic differentials on the curve X_D.

Using the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & H^1(X_D,\mathbf{Z}) \arrow[r, phantom, "="] \arrow[d, hook] & H^1(X_D,\mathbf{Z}) \arrow[d] & \\
0 \arrow[r] & H^0(X_D,\Omega) \arrow[r, "I"] & H^1(X_D,\mathbf{C}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[r] \arrow[d] & 0 \\
 & & & J(X_D) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

where the horizontal exact sequence comes from the exact sheaf sequence

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathcal{O} \xrightarrow{\quad d \quad} \Omega \longrightarrow 0,$$

we realize the jacobian $J(X)$ as the double coset space

$$J(X_D) = I(H^D(X,\Omega))\backslash H^1(X_D,\mathbf{C})/H^1(X_D,\mathbf{Z}).$$

2.5.1. The steps of our construction are:

(i) Construct a piecewise-linear analogue of the abelian differentials on X and as a result obtain a g-dimensional space

$$L_D \subset H^1(X_D, \mathbb{C}),$$

where the RHS is interpreted as the space of the cell cohomologies of X_D.

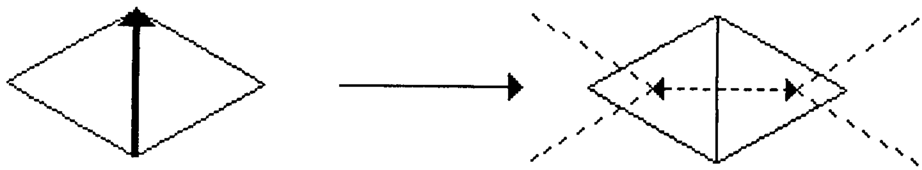
(ii) Iterate the refinements

and use the canonical isomorphisms of X_D and $X_{\alpha D}$, which results in the canonical isomorphism

$$H^1(X_D, \mathbb{C}) = H^1(X_{\alpha D}, \mathbb{C}).$$

Under this isomorphism the spaces $L_{\alpha^n D}$ form a sequence of the g-dimensional subspaces of $H^1(X_D, \mathbb{C})$, whose limit is (we hope)* the I-image of the space of abelian differentials in $H^1(X_D, \mathbb{C})$.

2.5.2. The space L_D is constructed in the following way: Denote by $C^1(D)$ the space of the cell cochains of X_D with complex coefficients. To a 1-cochain on D we associate the 1-cochain on D as it is shown below:



Thus we get an operator

$$\star_D : C^1(D) \longrightarrow C^1(D^*)$$

which, we suppose, is the proper analogue of the harmonic Hodge operator (see also [14]). It enjoys the following properties:

(a) $$\star_D \star_{D^*} = -1.$$

* (Added in proof). It is really so. The demonstration will be published in a forthcoming paper.

(b) $$\text{If } \mathcal{H}^1(D) = Z^1(D) \cap (\star_D^{-1}(Z^1(D^*)))$$

(Z^1 denoting the cocycles), then the projection

$$\mathcal{H}^1(D) \longrightarrow H^1(X_D, \mathbb{C})$$

is an isomorphism.

Thus we have obtained the diagram of isomorphisms

$$\begin{tikzcd} \mathcal{H}^1(D) \arrow[r, shift left, "\star_D"] \arrow[d] & \mathcal{H}^1(D^*) \arrow[l, shift left, "\star_{D^*}"] \arrow[d] \\ H^1(X_D, \mathbb{C}) \arrow[r, phantom, "="] & H^1(X_{D^*}, \mathbb{C}) \end{tikzcd}$$

which defines the operators on $H^1(X_D, \mathbb{C})$ for which we use the notations $\underline{\star}_D$ and $\underline{\star}_{D^*}$; they also satisfy

$$\underline{\star}_D \underline{\star}_{D^*} = -1.$$

Denote by $\wedge$ the cup-product on $H^1(X_D, \mathbb{C})$.

For x, y from $H^1(X_D, \mathbb{C})$, define

$$(x, y) = \star_D(x) \wedge y,$$

Proposition. (,) *is symmetric and positively defined. Denote by* $\{\ell_\alpha | \alpha = 1, \ldots, 2g\}$ *the set of eigenvalues of* $\underline{\star}_D$ *and* $\{L_\alpha\}$ *the corresponding eigenspaces.*

Proposition.

(a) $\forall \alpha, \ell_\alpha \notin \mathbb{R}$.

(b) *For any* $\ell \in \{\ell_\alpha\}$ *also* $-\ell \in \{\ell_\alpha\}$; *the corresponding eigenspaces are equidimensional.*

Set $L_D = \oplus_{\mathrm{Im} L_\alpha < 0} L_\alpha$. It follows from the last proposition that $\dim(L_D) = g$. Thus the approximate Jacobians can be defined as

$$L_D \backslash H^1(X_D, \mathbb{C}) / H^1(X_D, \mathbb{Z}).$$

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G.B. Shabat
Profsoyuznaya ul., 115-1-253,
Moscow 117321, USSR

V.A. Voevodsky
Moscow State University,
Mech.-Math faculty,
Moscow, USSR

Sur les propriétés numériques du dualisant relatif d'une surface arithméthique

L. SZPIRO

pour A. Grothendieck

Une courbe lisse, géométriquement connexe et de genre non nul sur un corps de nombres, possède un modèle régulier (Abhyankar), minimal (Shafarevich), unique. C'est cet objet qu'on appelle surface arithmétique. Arakelov, ayant introduit en 1972 une théorie des intersections pour les diviseurs "compactifiés" sur une telle surface, il est très tentant d'essayer de développer une "géométrie italienne" dans le cadre diophantien. J'ai indiqué ailleurs [Sz 4] les difficultés d'une telle tâche. On a cependant un corpus de base dû à Arakelov [A] et Faltings [F 1]. Nous appliquons ici ces fondements pour essayer de "mesurer" le dualisant relatif. Ce dernier faisceau inversible est central dans bien des questions. Nous avons, par exemple, dans [Sz 1], exploité au maximum la situation, dans le cas géométrique, pour obtenir un "Mordell effectif" en toute caractéristique par la construction de "petits points". En genre un, nous montrons plus bas, qu'une borne supérieure, polynomiale en le conducteur, pour le "degré d'Arakelov" du dualisant relatif d'une courbe elliptique donnerait Fermat par la construction de Frey.

Il est remarquable que pour ce genre de questions, une borne supérieure pour la self-intersection du dualisant relatif soit suffisante. Cependant l'expérience montre qu'on n'obtient celle-ci qu'après avoir obtenu des résultats de positivité (i.e., une borne inférieure pour les invariants numériques de ce faisceau). Si les bornes supérieures en genre un, comme en genre au moins deux, sont toujours conjecturales, (1-4 et 2-3) nous apportons ici un modeste éclairage sur les bornes inférieures. Le plan de cet article est suivant :

On se reportera à l'introduction de chacun des chapitres pour une description plus détaillée de leur contenu. Indiquons que dans les deux cas (genre un et genre au moins deux) nous réussissons à donner une interprétation diophantienne d'invariants obtenus de manière transcendante par la théorie d'Arakelov (Théorème 2, Proposition 1 et son corollaire; Théorème 3, Section 2.3).

La lecture de ce travail requiert une certaine familiarité avec la théorie d'Arakelov. On peut acquérir celle-ci en consultant [A], [F 1], [Sz 4] et surtout [Sz 2].

L'objet mathématique le plus utilisé dans ce travail est le "dualisant relatif". C'est A. Grothendieck ([G]) qui a dégagé le premier cette notion. D'autre part il est clair que la théorie des schémas a "magnifié" le traitement identique des corps de nombres et des corps de fonctions après l'intuition qu'en avaient Dedekind, Chevalley et Weil. On voit ainsi que, non seulement ce lignes sont respectueusement dédiées à A. Grothendieck pour son soixantième anniversaire, mais que comme bien d'autres, je n'aurais pas pu "penser" à ces problèmes sans les travaux du maître de la géométrie algébrique de cette deuxième moitié du siècle.

1. Courbes elliptiques

Dans ce chapitre nous étudions les isogénies de courbes elliptiques sur un corps de nombres du point de vue de la théorie d'Arakelov. Le premier résultat notable est le théorème 2 qui dit que le degré d'Arakelov d'une courbe elliptique semi-stable est un douzième du logarithme de la norme du discriminant minimal. On voit ainsi que ce degré n'est jamais négatif et est différent de la "hauteur de Faltings", résultat qui semble avoir échappé à certains auteurs. Dans la section 3 nous calculons explicitement les valeurs des fonctions de Green d'Arakelov aux points de deux torsions des courbes de Frey. Nous rappelons enfin à la section 4 la conjecture du discriminant et ses variantes. Toute reproduction et démonstration fausse de cette conjecture, peut être librement traduite ou adaptée même sans indication d'origine.

1.1. *Isogénies et fonctions de Green.* Arakelov a déjà noté le manque de "commutation raisonnable" de sa théorie des intersections pour les morphismes finis, même étales, entre courbes de genre au moins deux. Le lemme 1 qui suit exprime que pour une isogénie de courbes elliptiques un modeste miracle se produit.

Lemme 1. *Soient $\pi : E \longrightarrow E'$ une isogénie de courbes elliptiques sur un corps de nombres, L_1 et L_2 deux faisceaux inversibles sur E' munis de métriques permises. Alors pour chaque $i = 1, 2$, $\pi^* L_i$ est muni par image réciproque d'une métrique permise et on a :*

$$(\pi^* L_1 \cdot \pi^* L_2) = (\deg \pi)(L_1 \cdot L_2).$$

Corollaire 1. *Soit $\pi : E \longrightarrow E'$ une isogénie de courbes elliptiques sur un corps de nombres K. Supposons que son* noyau *H soit* fini *et* déployé *sur $\mathcal{O}_K$. Notons $H = \sum_{i=1}^{h} E_i$ où E_i est une section de E sur O_K et h le degré de π. Alors on a :*

$$\sum_{i \neq j} (E_i \cdot E_j) = h(d - d')$$

où d et d' sont les degrés d'Arakelov de ω_E et $\omega_{E'}$ respectivement.

Démonstration du corollaire. Si $0'$ est la section nulle de E' on a $h\big(\mathcal{O}_{E'}(0') \cdot \mathcal{O}_{E'}(0')\big) = -hd' = (\Sigma E_i \cdot \Sigma E_i) = -hd + \sum_{i \neq j} (E_i \cdot E_j)$ c.q.f.d.

Démonstration du lemme 1. Rappelons que si $d\mu$ désigne la métrique plate, invariante par translation, de volume total égal à 1, sur une courbe elliptique E sur $\mathbf{C}$, une métrique $|\ |$, sur un faisceau inversible L sur E est dite permise si "l'équation de courbure" suivante est satisfaite :

$$-\frac{1}{2i\pi} d\bar{d} \log |\, s \,| = (\deg L) d\mu - \delta_D$$

où s est une section méromorphe de L, D le diviseur lui correspondant et δ_D la fonction de Dirac de support D.

Dans la situation du lemme 1, munissons $\pi^* L$ de la métrique image réciproque de celle donnée sur L. On a pour tout point P de E

$$|(\pi^* s)(P)| = |s(\pi(P))|$$

donc $-\frac{1}{2i\pi} d\bar{d} \log |\pi^* s| = \pi^*(-\frac{1}{2i\pi} d\bar{d} \log |s|) = \pi^*\big((\deg L) d\mu_{E'} - \delta_D\big)$ $= \frac{(\deg \pi^* L)}{\deg \pi} d\mu_E (\deg \pi) - \pi^* \delta_D = (\deg \pi^* L) d\mu - \delta_{\pi^* D}$.

Nous avons utilisé dans les deux dernières égalités les trois faits suivants (faciles à établir):

$$\begin{aligned} \deg \pi^* L &= (\deg \pi) \deg L \\ \pi^* d\mu_{E'} &= (\deg \pi) d\mu_E \\ \pi^* \delta_D &= \delta_{\pi^* D}. \end{aligned}$$

Nous avons donc établi la première partie du lemme 1.

Pour la deuxième partie, considérant que L_1 et L_2 sont associés à des diviseurs D_1 et D_2, en exprimant l'intersection d'Arakelov comme somme de termes locaux on voit :

(a) que la partie à distance finie ne pose pas de problème,

(b) qu'à l'infini, il suffit de vérifier $\prod_{\pi(P_i)=P} |\pi^*(s)(P_i)| = |s(P)|^h$ ce qui est clair.

Corollaire 2. *Soit E une courbe elliptique sur un corps de nombre K et n un entier au moins égal à 2. Supposons que tous les points de n torsion de E soient rationnels sur K et notons E_i, $i = 1, 2, \dots, n^2$, les sections du modèle minimal géométrique de E sur $\mathcal{O}_K$ correspondantes. Alors on a la relation suivante :*

$$\sum_{i \neq j} (E_i \cdot E_j) = 0.$$

Le corollaire 2 se déduit du corollaire 1 ci-dessus en prenant $\pi = [n]$ et $E' = E$ (donc $d = d'$ et $h = n^2$).

Note. Le corollaire 2 donne un exemple agréable d'un des obstacles au développement de la théorie d'Arakelov : deux diviseurs effectifs, irréductibles, horizontaux d'une surface arithmétique peuvent très bien avoir une intersection négative. (On verra plus bas dans l'exemple des courbes de Frey que les $(E_i \cdot E_j)$ ne sont pas tous nuls). Dans le cas géométrique si n est premier à la caractéristique chacun des $(E_i \cdot E_j)$ est nul. Par contre quand $n = p$ est la caractéristique du corps de base tous les $(E_i \cdot E_j)$ sont positifs ou nuls et certains sont non nuls.

Théorème 1 (Energie du noyau d'une isogénie). *Soit $\pi : E \to E'$ une isogénie de courbes elliptiques semi-stables définies sur un corps de nombre K. Soit H le noyau de cette isogénie et h son degré. Pour chaque place σ de K notons P_i^σ, $i = 1, \ldots, h$, les points de H sur $\mathbb{C}$. Alors on a*

$$\sum_\sigma \sum_{i \neq j} g(P_i^\sigma, P_j^\sigma) = \frac{K : \mathbb{Q}}{2} h \log h + \frac{h}{12} \sum_\sigma \log \left| \frac{\Delta(\tau'_\sigma)(\mathrm{Im}\ \tau'_\sigma)^6}{\Delta(\tau_\sigma)(\mathrm{Im}\ \tau_\sigma)^6} \right|.$$

Remarquons de suite qu'on peut se demander si une telle formule (où l'on retire les $\sum_\sigma$; les σ, et $K : \mathbb{Q}$) est valable pour une isogénie de courbes elliptiques sur $\mathbb{C}$. C'est en tout cas un fait quand $K = \mathbb{Q}$ et que l'hypothèse sur H est satisfaite !! Notons dans ce sens le manuscrit de N. Elkies [E].

Démonstration. Par [F 2] ou [S i] on a

$$\begin{aligned} h_F &= \frac{1}{12}(\log \Delta - \sum_\sigma \log |\Delta(\tau)(\mathrm{Im}\ \tau)^6|) \\ h'_F &= \frac{1}{12}(\log \Delta' - \sum_\sigma \log |\Delta(\tau')(\mathrm{Im}\ \tau')^6|). \end{aligned}$$

On a d'autre part $[F\,2]$ (formule du changement de la hauteur modulaire par isogénie)

$$[K : \mathbb{Q}](h_F - h'_F) = \log \bar{\delta}_{H/\mathcal{O}_K} - \frac{K : \mathbb{Q}}{2} \log h$$

où $\bar{\delta}_{H/\mathcal{O}_K}$ est l'image directe du dualisant relatif du schéma en groupes H sur $\mathcal{O}_K$ (supposé fini et plat) (cf. [R]). On a par adjonction [Sz 4]

$$(\omega_E \cdot H) + (H \circ H) = \log \delta_{H/\mathcal{O}_K} - \sum_\sigma \sum_{i \neq j} g(P_i^\sigma, P_j^\sigma).$$

Si d est le degré d'Arakelov de ω_E, on a donc :

$$hd - hd + \sum_{i \neq j} (E_i \cdot E_j) = h \log \bar{\delta}_{H/\mathcal{O}_K} - \sum_\sigma \sum_{i \neq j} g(P_i^\sigma, P_j^\sigma).$$

Par le corollaire 1 du lemme 1 on obtient donc

$$h(d - d') = h \log \bar{\delta}_{H/\mathcal{O}_K} - \sum_\sigma \sum_{i \neq j} g(P_i^\sigma, P_j^\sigma).$$

Par le théorème 2 et la formule du changement de hauteur modulaire par isogénie [F 2] on obtient le résultat voulu.

1.2. *Points de torsion et degré d'Arakelov du dualisant relatif.* Nous commençons par établir une formule, surprenante à première vue, pour le degré d'Arakelov du dualisant relatif d'une courbe elliptique semi-stable.

Théorème 2. *Soit E une courbe elliptique semi-stable sur un corps de nombre K. Soit $\Delta_{\min}$ son discriminant minimal et $|\Delta_{\min}|$ sa norme. Alors le degré d'Arakelov de l'image directe du dualisant relatif de E sur $\mathcal{O}_K$ satisfait :* $12 \deg_{\mathrm{Ar}} \omega_E = \log|\Delta_{\min}|$.

Remarque 1. Certains auteurs ont fortement conjecturé que $\deg_{\mathrm{Ar}} \omega_E = [K:\mathbb{Q}]h_F(E)$. Le théorème 1 leur donne tort. Le lecteur curieux de ces choses s'amusera à trouver dans la bibliographie de cet article, au moins un auteur réputé qui s'est laissé prendre.

Remarque 2. La formule donnée par le théorème 2 peut se traduire par :

$$\prod_{\sigma} |\Delta|_{\mathrm{Ar},\sigma} = 1$$

où Δ est considéré comme une section de $\omega_E^{\otimes/2}$ et $|\Delta|_{\mathrm{Ar},\sigma}$ est sa norme d'Arakelov à la place σ. Il est naturel de se demander si un tel énoncé est valable pour une courbe elliptique sur $\mathbb{C}$ (i.e., à chaque place à l'infini).

Avant de passer à la démonstration du théorème 1 nous insérons le calcul des diviseurs verticaux $\emptyset$ correspondants à deux sections en une place où la réduction est un "cycle".

Lemme 2. *Soit V un anneau de valuation discrète $f : E \to V$ une courbe elliptique dont la fibre générique est lisse et la fibre spéciale est réduite et est un cycle de $\mathbb{P}^1$ de self-intersection -2. Soient $F_0 \cdots F_{n-1}$ les composantes de cette fibre spéciale. Si E_1 et E_2 sont deux sections de f telles que : $(E_1 \cdot F_0) \neq \emptyset$ et $(E_2 \cdot F_k) \neq \emptyset$, le diviseur à coefficients rationnels vertical $\emptyset$ défini à un multiple de la fibre spéciale près, par l'équation $\big((E_1 - E_2 + \emptyset) \cdot F_i\big) = 0$ pour tout i satisfait la formule suivante*

$$-\emptyset^2 = \frac{k(n-k)}{n} \log N(v).$$

Notons que si V est d'inégale caractéristique à corps résiduel fini (situation arithmétique) $N(v) =$ cardinal du corps résiduel, et si V est d'égale caractéristique on prendra $\log N(v) = 1$.

Démonstration du lemme 2. Ecrivons $\emptyset = \Sigma x_i F_i$ où les x_i sont dans $\mathbf{Q}$.

On doit avoir $\big((\Sigma x_i F_i + E_1 - E_2)\cdot F_j\big) = 0$ pour tout j. Notons d'abord que $(E_1 - E_2 + \emptyset \cdot \emptyset) = 0$, donc

$$-\emptyset^2 = (x_1 - x_k)\log Nv,$$

un calcul facile permet alors de conclure.

Passons maintenant à la **preuve du théorème 2** : E étant semi-stable la formule à montrer est invariante par changement de base. On peut donc supposer que les points de torsion sont rationnels sur K. Soient $E_1, \dots, E_{n^2}$ les sections du modèle minimal géométrique de E sur $\mathcal{O}_K$ correspondantes. Pour tout $i \neq j$ on définit comme plus haut $\emptyset_{i,j}$ diviseur vertical à coefficients rationnels par

$$(E_i - E_j + \emptyset_{ij} \cdot V) = 0$$

pour tout diviseur vertical V. On a donc :

$$\big((E_i - E_j + \emptyset_{i,j}) \cdot \emptyset_{ij}\big) = 0$$

et

$$(E_i - E_j + \emptyset_{i,j})^2 = 0.$$

Cette dernière formule étant due au fait que $E_i - E_j$ est de torsion et à la formule de comparaison entre l'intersection d'Arakelov et la hauteur de Néron-Tate ([F 1], [H], [SPA] p. 83).

Ecrivant $\sum_{i \neq j}(E_i - E_j + \emptyset_{i,j})^2 = 0$ et utilisant la première formule ci-dessus on obtient :

$$(n^4 - n^2)\deg_{\mathrm{Ar}} \omega_E = -\sum_{i \neq j} \emptyset_{i,j}^2.$$

En effet $-E_i^2 = \deg_{\mathrm{Ar}} \omega_E$ pour tout i.

Comme on sait que les points de n-torsion sur une fibre singulière qui est un cycle sont distribués en n paquets de n points équidistribués, le lemme 2 donne une formule pour $-\sum_{i \neq j} \emptyset_{i,j}^2$. Appliquant ceci à $n = 2$, on obtient $12 \deg_{\mathrm{Ar}} \omega_E = \sum_{v|N} n_v \log N(v)$ où N est le conducteur de E, et n_v le nombre de composantes de la fibre de v. Maintenant E étant semi-stable on sait que $v(\Delta_{\min}) = n_v$ pour tout $v|N$. D'où le résultat.

Au cours de la démonstration précédente nous avons établi le lemme suivant qui nous sera utile :

Lemme 3. *Soit E une courbe elliptique sur un corps de nombre K. Soient n un point positif et P un point rationnel de E sur K d'ordre exactement n. Soient 0 la section nulle du modèle géométrique minimal de E sur $\mathcal{O}_K$ et E_p la section de ce modèle correspondant à P. Soit $\emptyset_P$ un diviseur vertical à coefficients rationnels tel que $(0 - E_p + \emptyset_p \cdot V) = 0$ pour tout diviseur vertical V. Alors on a*

$$(0 \cdot E_p) = -\deg_{\mathrm{Ar}} \omega_E - \emptyset_P^2/2.$$

1.3. *Points d'ordre deux et courbes de Frey.* Les formules établies précédemment permettent de "pousser" les calculs assez loin pour les courbes elliptiques définies sur $\mathbb{Q}$ dont les points d'ordre deux sont rationnels. (Ces courbes méritent désormais le nom de "courbes de Frey" [Fr 1], [Fr 2], [Fr 3]. Nous suivons la présentation de G. Frey.

Soient a, b, c des entiers sans facteur commun tels que $a + b = c$. Considérons la courbe elliptique $E_{a,b,c}$ définie par $y^2 = x(x+a)(x-b)$ (*). Frey a établi les faits suivants :

(1) Notant a celui de ces trois nombres qui est pair, si 2^4 divise a et si on a choisi b tel que $b \equiv -1(4)$, alors $E_{a,b,c}$ est semi-stable, son équation minimale est donnée par :

$$y^2 + xy = x^3 + \frac{a-b-1}{4}x^2 - \frac{ab\,x}{16}$$

(2) Son discriminant minimal vaut alors :

$$\Delta_{\min} = 2^{-8}a^2b^2c^2.$$

Les trois points d'ordre 2 significatifs correspondent, dans l'équation initiale (*) respectivement à $x = 0$, $x = -a$, $x = b$. Nous noterons ces points P_0, P_a, P_b, les sections du modèle minimal géométrique E_0, E_a, E_b, le discriminant minimal des courbes elliptiques quotientes de E par, respectivement, chacun de ces points, Δ_0, Δ_a, Δ_b.

Proposition 1. *Avec les notations précédentes on a :*

$$g(0, P_0) = \frac{1}{12}\log\ |c^2/2^{-4}ab|\,;\ |\Delta_0| = |2^{-4}abc^4|$$
$$g(0, P_a) = \frac{1}{12}\log\ |2^{-8}a^2/bc|\,;\ |\Delta_a| = |2^{-16}a^4bc|$$
$$g(0, P_b) = \frac{1}{12}\log\ |2^8b^2/ac|\ \ ;\ |\Delta_b| = |2^{-4}ab^4c|.$$

Par le lemme 3 et lemme 1 appliqués aux isogénies à noyau cyclique d'ordre 2 engendrés par respectivement chaque P_i, $i = 0, a, b$, on a

$$(0 \cdot E_i) = -d - \frac{\emptyset_i^2}{2} = d - d_i$$

où $d_i = \frac{1}{12}\log|\Delta_i|$.

(i) *Calcul des* $-\emptyset_i^2$. Considérons la contribution en un nombre premier p divisant Δ est nulle si E_i ne passe pas par le point singulier de la fibre en p, dans le modèle minimal arithmétique. Ceci a lieu :

$$\begin{aligned} &\text{pour } E_0 \text{ si } p \mid c \\ &\text{pour } E_a \text{ si } p \mid 2^{-4}a \\ &\text{pour } E_b \text{ si } p \mid b. \end{aligned}$$

Par contre, dans le cas contraire, par le lemme 2, la contribution est de $-\frac{v_p(\Delta)}{4}\log$ p. Ceci a lieu :

$$\begin{aligned} &\text{pour } E_0 \text{ si } p \mid 2^{-4}ab \\ &\text{pour } E_a \text{ si } p \mid bc \\ &\text{pour } E_b \text{ si } p \mid 2^{-4}ac. \end{aligned}$$

On obtient donc $-\emptyset_0^2 = \frac{1}{2}\log 2^{-4}ab$, $-\emptyset_\infty^2 = \frac{1}{2}\log bc$, $-\emptyset_b^2 = \frac{1}{2}\log 2^{-4}ac$.

(ii) *Calcul des* $(0 \cdot E_i)_f$ (contribution locale à distance finie de l'intersection). Pour ceci notons que dans l'équation minimale homogénéisée :

$$y^{2t} + xyt = x^3 + \frac{a-b-1}{4}tx^2 - \frac{abxt^2}{16}$$

les quatres points de 2 torsions sont donnés par :

$$\begin{array}{llll} E_0 & : t = 1 & , x = 0 & , y = 0 \\ E_q & : t = 1 & , x = -\frac{a}{4} & , y = \frac{a}{8} \\ E_b & : t = 8 & , x = 2b & , y = -b \\ 0 & : t = 0 & , x = 0 & , y = 1. \end{array}$$

On voit donc que $(0 \cdot E_b)_f$ est le seul des $(0 \cdot E_i)_f$ qui est non nul. Le nombre réel $(0 \cdot E_b)_f$ est donc un multiple entier α de log 2. Nous calculerons α plus bas. On obtient ainsi

(i) $d_0 = \frac{1}{12}\log|\Delta_0| = \frac{1}{12}\log|2^{-4}abc^4|$

(ii) $d_a = \frac{1}{12}\log|\Delta_a| = \frac{1}{12}\log|2^{-16}a^4bc|$

(iii) $d_b = \frac{1}{12}\log|\Delta_b| = \frac{1}{12}\log|2^{-4}ab^4c|$.

Il nous reste à calculer les valeurs des fonctions de Green.

Par le théorème 1 on trouve

$$\sum_{i=0,a,b} g(0,P_i) = \log 2$$

(on a utilisé $g(P_i,P_j) = g(0,P_i - P_j)$).

Par le (ii) ci-dessus $-g(0,P_0) = \frac{1}{12}\log\left|\frac{2^{-8}a^2b^2c^2}{2^{-4}abc^4}\right|$ soit $g(0,P_0) = \frac{1}{12}\log\left|\frac{c^2}{2^{-4}ab}\right|$ de même $g(0,P_a) = \frac{1}{12}\log\left|\frac{2^{-8}a^2}{bc}\right|$ et $g(0,P_b) = \frac{1}{12}\log\left|\frac{b^2}{2^{-4}ac}2^\alpha\right|$ d'où $\alpha = 12$. On en déduit la proposition.

Corollaire. *Si E est une courbe de Frey correspondant à $a+b=c$ avec $a>0$, $2^4|a$, et $b \equiv -1(4)$, alors le maximum de la fonction de Green sur $E(\mathbb{C})$ est*

$$\frac{1}{12}\log\frac{2^4c^2}{ab} \quad \text{si} \quad b \quad \textit{est positif}$$

$$\frac{1}{12}\log\frac{2^8b^2}{ac} \quad \text{si} \quad b \quad \textit{est négatif.}$$

Par ce que nous venons de montrer ces valeurs sont bien le maximum des $g(0,P_i)$, $i = 0,a,b$. Il est facile de voir que les points de deux torsions sont des extrêmes pour $g(0,P)$.

Pour conclure nous avons besoin du résultat plus délicat suivant :

Théorème (Showu-Zang). *Soit E une courbe elliptique sur $\mathbb{C}$, dont le τ dans le domaine fondamental du demi-plan de Poincaré pour l'action de $SL(2,\mathbb{Z})$ est* purement imaginaire. *Alors le maximum des $g(0,z)$ est obtenu pour $z = \frac{1+\tau}{2}$.*

La démonstration sera publiée prochainement dans la thèse de Showu-Zang [Z].

1.4. *Problèmes et conjectures.* Il est facile par la méthode de la classe de Kodaira-Spencer introduite dans [Sz 1] exposé 3, de montrer que le degré de discriminant minimal d'une courbe lisse de genre g, sur un corps k est majoré par $p^e6(2g-2+s)$ où, e est le degré d'inséparabilité de l'application $j : C \to \mathbf{P}^1$ correspondant à cette courbe elliptique, et s le nombre de points géométriques de C dont la fibre n'est pas lisse. Il est donc naturel de se poser la question suivante :

C.D. *Conjecture du discriminant.* Soit K un corps de nombres, il existe une constante k ne dépendant que de K telle que

$$\operatorname{Norme}(\Delta_E) \leq N_E^k \text{ pour toute courbe elliptique } E \text{ sur } K$$

où Δ_E est le discriminant minimal de E sur K et N_E son conducteur.

Si on veut se rapprocher de l'énoncé géométrique signalé plus haut on peut faire une conjecture plus audacieuse :

C.D.F. *Conjecture du discriminant* (forme forte). Soit K un corps de nombres, alors pour tout $\epsilon > 0$, il existe une constante $C(K, \epsilon)$ telle que

$$\operatorname{Norme}(\Delta_E) \leq C(K, \epsilon) N_E^{6+\epsilon} \text{ pour toute courbe elliptique } E \text{ sur } K.$$

Notons d'abord qu'on ne peut prendre $\epsilon = 0$, ni même $N_E^6(\log N_E)^\alpha$ pour tout α (ce dernier point a été montré par Masser [M]).

Appliquant ceci aux courbes de Frey du paragraphe précédent on obtient l'énoncé conjectural suivant : Il existe une constante k telle que : si a, b, c sont des entiers non nuls sans facteurs communs tels que $a + b = c$, alors

$$|a\,b\,c| \leq \left(\prod_{p \mid a\,b\,c} p \right)^k$$

(sous la forme forte : pour tout $\epsilon > 0$ il existe une constante $c(\epsilon)$ telle que $|a\,b\,c| \leq c(\epsilon) \left(\prod_{p \mid a\,b\,c} p \right)^{3+\epsilon}$. On sait l'étonnant tour de passe-passe que de tels énoncés autorisent avec une solution non triviale de l'équation de Fermat $x^p + y^p = z^p$.

Signalons l'exposé de Oesterlé [O], le livre [Ba] et le séminaire à paraître [Sz 7] où d'autres implications et des raffinements de ces conjectures sont étudiés (cf. aussi Vojta [V]). On notera, dans un genre totalement différent, que Ribet a réduit récemment la conjecture de Fermat à montrer que les courbes de Frey sont modulaires. Par Weil il suffit donc de montrer que la fonction L d'une telle courbe à une équation fonctionnelle raisonnable. Ce travail de Ribet donne la première évidence sérieuse de la conjecture de Fermat.

2. Courbes de genre au moins deux

G. Faltings a montré dans [F 1] que la self-intersection du dualisant relatif d'une courbe de genre au moins deux n'est jamais strictement négative. Nous montrons ici (au moins pour les courbes ayant bonne réduction partout) que la non nullité de cette self-intersetion est essentiellement équivalente à une conjecture de Bogomolov qui dit que les points arithmétiques d'une courbe plongée dans sa jacobienne sont discrets pour la "topologie de Néron-Tate".

Nous établissons cette conjecture pour les points dont l'image canonique dans la jacobienne n'est pas de torsion (Théorème 3). L'ensemble fini de points restants (M. Raynaud), donne du fil à retordre quand justement la self-intersection du dualisant relatif est nulle. Nous ne savons pas si une telle situation existe. Nous l'analysons cependant au Section 2. On notera qu'un théorème d'évanouissement de la partie fixe (conjectural) rendrait bien des services.

Au Section 3 nous indiquons une nouvelle "conjecture des petits points". Cette fois c'est la hauteur de Néron-Tate qui doit être "petite". Nous déduisons de cette conjecture quelque corollaire médiatique.

2.1. La conjecture de Bogomolov. Soit X une courbe de genre $g \geq 2$ sur un corps de nombres K. Nous établissons dans ce chapitre une généralisation du théorème de Raynaud (conjecture de Lang) sur la finitude de l'ensemble des points de $X(\bar{K})$ qui sont de torsion après plongement dans la jacobienne. Cette généralisation remplace "hauteur de Néron-Tate nulle" par "hauteur de Néron-Tate" inférieure à $\epsilon > 0$. (Un résultat conjecturé par Bogomolov).

Malheureusement, par manque d'un théorème de disparition asymptotique des parties fixes, nous n'établissons pas le résultat, pour les points P de $X(\bar{K})$ qui sont proportionnels à ω_X dans $\mathrm{Pic}(X)(\bar{K})$ (nous appelons ces points : points qui divisent ω) lorsque $(\omega_{x/\mathcal{O}_K} \cdot \omega_{x/\mathcal{O}_K}) = 0$. Notons que nous ne connaissons pas d'exemple où cette égalité est satisfaite.

Rappelons la formule établie dans [Sz 3] qui relie la hauteur de Néron-Tate $\langle x_p \rangle^2$, de l'image x_P (dite ici canonique) d'un point P de $X(K)$ dans sa jacobienne par l'application

$$P \to \text{classe de } \Omega^1_{X/K} \otimes \mathcal{O}_X((2-2g)P),$$

à l'intersection d'Arakelov notée $(.\,,.)$.

$$\langle x_{P_1} \cdot x_{P_2} \rangle = \frac{\langle x_{P_1} \rangle^2 \cdot \langle x_{P_2} \rangle^2}{2g} + \frac{1}{K:\mathbf{Q}}(\frac{1}{g} - 1)(\omega_{x/\mathcal{O}_K} \cdot \omega_{x/\mathcal{O}_K})$$
$$-(2g-2)^2(E_{P_1} \cdot E_{P_2}) - \frac{\emptyset^2_{P_1} + \emptyset^2_{P_2}}{2g} + (\emptyset_{P_1} \cdot \emptyset_{P_2})(^{**}).$$

Comme à l'habitude dans cette formule $\omega_{X/\mathcal{O}_K}$ est le dualisant relatif du modèle minimal régulier de X sur $\mathcal{O}_K$, E_{P_i} la section de X sur $\mathcal{O}_K$ correspondant à P_i et $\emptyset_{P_i}$ un diviseur vertical à coefficients rationnels défini par

$$\left(\omega_{X/\mathcal{O}_K} \otimes \mathcal{O}_X\left((2-2g)E_{P_i}\right) \otimes \mathcal{O}_X(\emptyset_{P_i}) \cdot \mathcal{O}_X(V)\right) = 0$$

pour tout diviseur vertical V. En particulier on a : ([SPA] exposé XI)

$$2g(2g-2)(-E_{P_i}^2) - \omega_{X/\mathcal{O}_K}^2 = [K:P]\langle x_{P_i}\rangle^2 - \emptyset_{P_i}^2. \qquad (***)$$

Théorème 3. (Conjecture de Bogomolov pour les courbes lisses). *Soit X une courbe lisse, géométriquement irréductible, de genre $g \geq 2$, sur un corps de nombre K. Soit P un point rationnel sur la cloture algébrique $\bar{K}$ de K, dont l'image canonique x_P dans la jacobienne est de hauteur de Néron-Tate $\langle x_P\rangle^2$ non nulle. Supposons que X ait bonne réduction partout. Alors il existe un nombre réel $\epsilon_p > 0$, calculable, tels que : l'ensemble des points Q de $X(\bar{K})$, tels que la hauteur de Néron-Tate de $P - Q$ soit inférieure à ϵ_p, sont en nombre fini.*

De plus s'il existe une suite infinie de points distincts dans $X(\bar{K})$ tels $\langle x_p\rangle^2$ tend vers 0, alors $\omega_{X/\mathcal{O}_K^2} = 0$.

Puisque nous avons supposé que X avait une bonne réduction, les termes "$\emptyset$" dans les formules (**) et (***) disparaissent. Par un théorème classique, déjà utilisé par G. Faltings dans [F 1] il existe, pour tout $\epsilon > 0$ un entier N tel que si les $P_1, \ldots, P_n$, $n \geq N$ sont n points distincts de $X(\bar{K})$ et L un corps de nombres où les P_i sont rationnels et si σ est une place à l'infini d'un tel corps, on a : $\sum_{i\neq j} g(P_i^\sigma, P_j^\sigma) < \epsilon n^2$ (où g est la fonction de Green d'Arakelov).

Pour un point P tel que $\langle X_P\rangle^2 \neq 0$ choisissons ϵ_P de telle façon que si $\langle P-Q\rangle^2 < \epsilon_P$ on ait :

$$\langle x_P \cdot x_Q\rangle = \left(\langle x_P\rangle^2 \langle x_Q\rangle^2\right)^{1/2} \cos\theta_{P,Q}$$

avec $(\cos\theta_{P,Q} - 1)$ assez petit (donc $|\theta_{P,Q}|$ petit). S'il existait un ensemble infini de points proches de P pour la hauteur de Néron-Tate, leurs corps de rationalité aurait un degré tendant vers l'infini. On aurait donc pour tout $\epsilon > 0$ un point P_1, possédant N conjugués distincts sur K, $P_1, \ldots, P_N$, soit en calculant sur un corps L de rationalité de ces N points, et en additionant les égalités (**) :

$$\frac{(N^2-N)}{L:\mathbb{Q}}\left(\omega_{X/\mathcal{O}_L}^2\left(1-\frac{1}{g}\right)\right) \leq \langle x_P\rangle^2 \sum_{i\neq j}\left(-\cos\theta_{ij} + \frac{1}{g}\right)$$
$$+\frac{(2g-2)^2}{L:\mathbb{Q}}\sum_\sigma \sum_{i\neq j} g(P_i^\sigma, P_j^\sigma)$$

où $\theta_{i,j}$ est l'angle pour l'accouplement de Néron-Tate entre x_{P_i} et x_{P_j}. Comme $g \geq 2$, on s'est assuré, que $\frac{1}{g} - \cos\theta_{i,j} \leq -\frac{1}{3}$ (θ_{ij} petit). On obtient donc, puisque $\omega^2_{X/\mathcal{O}_L} = \omega^2_{X/\mathcal{O}_K}[L:K]$, $\omega^2_{X/\mathcal{O}_K}(1-\frac{1}{g}) \leq -\langle x_p\rangle^2 \cdot \frac{1}{3} + (2g-2)^2\epsilon\frac{N^2}{N^2-N}$ pour tout $\epsilon > 0$ et N assez grand, ce qui force $\omega^2_{X/\mathcal{O}_K} < 0$, contrairement au théorème de Faltings [F 2] affirmant $\omega^2_{X/\mathcal{O}_K} \geq 0$. Ceci termine la démonstration de la première partie du théorème. Pour la deuxième partie, reprenant l'argument de le démonstration précédente et choisissant $\epsilon > 0$ et N points distincts dans cette suite tels que $\langle x_{P_i}\rangle^2 \leq \epsilon$, N grand, on obtient :

$$\frac{1}{L:\mathbb{Q}}(N^2-N)\omega^2_{X/\mathcal{O}_L}(1-\frac{1}{g}) \leq \epsilon(\frac{1}{g}+1) + \frac{1}{L:\shortparallel}(L:\mathbb{Q}) \in N^2$$

d'où $\omega^2_{X/\mathcal{O}_K} \leq 3\epsilon$ pour tout $\epsilon > 0$.

Remarque. Le résultat s'étend par la même démonstration à une courbe (qu'on peut supposer semi-stable) n'ayant pas nécessairement bonne réduction partout si on remplace $\langle x_P\rangle^2 > 0$ dans l'énoncé par : il existe une constante $\alpha(X)$ ne dépendant que du nombre de composantes des mauvaises fibres de X sur $\mathcal{O}_K$ telles que, si $\langle x_P\rangle^2 > \alpha(X)$ alors... En effet les $\frac{\text{“}(\emptyset_i : \emptyset_j)\text{”}}{K:\mathbb{Q}}$ ne dépendent que du nombre de composantes des fibres.

2.2. *La question de la nullité éventuelle de* $\omega^2_{X/\mathcal{O}_K}$. Pour une courbe X ayant bonne réduction partout sur K, si $\omega^2_{X/\mathcal{O}_K} > 0$ l'argument précédent donne aussi la conjecture de Bogomolov pour les points tels que $\langle x_P\rangle^2 = 0$. On prendra alors $\epsilon_P = \omega^2_{X/\mathcal{O}_K}/10$ par exemple. La question de savoir si $\omega^2_{X/\mathcal{O}_K}$ est strictement positif reste ouverte même pour une courbe lisse. La remarque qui suit renforce la liaison de cette question à la conjecture de Bogomolov.

Supposons que X ait bonne réduction partout et supposons que $\omega^2_{X/\mathcal{O}_K} = 0$. Nous allons essayer, sous ces hypothèses d'établir la réciproque de théorème précédent.

Pour tout $\omega > 0$, $\omega_{X/\mathcal{O}_K} \otimes \mathcal{O}_X(\epsilon F)$ où F est une fibre à l'infini, est de self-intersection positive. Par le théorème d'existence de Faltings ([F 2]), il existe $N(\epsilon)$ tel que pour $n \geq N(\omega)$ $(\omega_{X/\mathcal{O}_K} \otimes \mathcal{O}_X(\epsilon F))^{\otimes n}$ ait une section positive au sens d'Arakelov. Ecrivons le diviseur positif D après changement de base, à un corps $L \supset K$ assez grand : $D = \sum_1^{n(2g-2)} E_i + V$

où les E_i sont des sections de X sur $\mathcal{O}_L$ et V est vertical et positif. On a :

$$(\omega_{X/\mathcal{O}_L} \cdot \mathcal{O}_X(D)) = \sum_{1}^{n(2g-2)} (-E_\imath)^2 + (\omega_{X/\mathcal{O}_L} \cdot \mathcal{O}_X(V)) = n\epsilon(2g-2)[L:K].$$

Comme $(\omega_{X/\mathcal{O}_L} \cdot \mathcal{O}_X(V)) \geq 0$, il existe un i tel que $-\frac{E_i^2}{L:\mathbf{Q}} = \frac{\langle X_i \rangle^2}{2g(2g-2)} \leq \epsilon$. On pourrait croire qu'on a la réciproque cherchée. Malheureusement on ne sait pas qu'un tel point correspondant à E_i satisfait à $-E_i^2 \neq 0$. Par M. Raynaud il n'y a qu'un nombre fini de tels points sur $\bar{K}$. Il manque donc, pour conclure, de savoir si un faisceau du type $\omega_{X/\mathcal{O}_K} \otimes \mathcal{O}_X(\epsilon F)$ peut avoir une partie fixe "horizontale" asymptotiquement !

Exemple. Prenons une courbe X de genre deux sur un corps de nombre K qui ait une bonne réduction partout. Soient $E_1, \ldots, E_6$ les sections de X sur $\mathcal{O}_K$ correspondantes aux points de Weierstrass supposés rationnels sur K. Alors pour chaque $i = 1, \ldots, 6$ on a :

$$\begin{aligned} &\omega_{X/\mathcal{O}_K} \text{ numériquement équivalent à} \\ &\mathcal{O}_X(2E_i + \alpha_i F) \quad (F \text{ fibre à l'infini}) \end{aligned}$$

d'où l'on tire : $\alpha_i = 3E_i^2$ et $\omega^2_{X/\mathcal{O}_K} = -4E_i^2$.

Comme la courbe a bonne réduction sur $\mathcal{O}_K$ pour tout $i \neq j$, on a $(E_i - E_j)^2 = 0$, d'où $4E_i^2 = 4(E_i \cdot E_j) = -\omega^2_{X/\mathcal{O}_K}$. On voit ainsi qu'ou bien $\omega^2_{X/\mathcal{O}_K} = 0$ ou bien l'intersection d'Arakelov $(E_i \cdot E_j)$, $i \neq j$ de deux points de Weierstrass est négative. On trouvera dans [M-B,M,B] des exemples où $\omega^2_{X/\mathcal{O}_K} > 0$.

2.3. *Problèmes et conjectures.* Aux paragraphes précédents nous nous sommes heurtés au problème de borner inférieurement $\omega^2_{X/\mathcal{O}_K}$. La question d'une borne supérieure, si elle a une réponse heureuse, donne, on le sait par la construction de Kodaïra-Parshin, un "Mordell effectif" (cf. [Sz 1]).

Nous avons à ce propos proposé ailleurs [Sz 6] une conjecture dite des *petits points.* En fait par le théorème de l'index de Hodge en théorie d'Arakelov ([F 1], [H]) on a ([Sz 1]) : $2g(2g-2)(-E^2) \geq \omega^2_{X/\mathcal{O}_K}$ donc toute borne supérieure pour un point donne une borne supérieure pour $\omega^2_{X/\mathcal{O}_K}$. Cette conjecture s'énonce en termes d'intersections d'Arakelov, nous en proposons une plus bas en termes uniquement de hauteur de Néron-Tate.

Conjecture des deux petits points pour Néron-Tate (C.P.P.N.T.). Soit g un entier et K un corps de nombres, alors il existe deux constantes $a(g)$ et $b(g)$ ne dépendant que de g, telles que : pour toute courbe X de genre g sur K, il existe deux points distincts P_1 et P_2 dans $X(\bar{K})$ dont la hauteur de Néron-Tate de l'image canonique dans la jacobienne $\langle x_{P_i}\rangle^2$ satisfait :

$$\langle x_{P_i}\rangle^2 \leq a(g)A(X)\log |D_{K/\mathbb{Q}}| + b(g)B(X)$$

où $A(X)$ et $B(X)$ ne dépendent que de la géométrie de la mauvaise réduction de X, et $D_{K/\mathbb{Q}}$ est le discriminant de K.

Nous allons montrer que (C.P.P.N.T.) implique

$$\omega^2_{X_Q/\mathcal{O}_K} \leq a'(g,X)\log D_{K/\mathbb{Q}} + b'(g,X)$$

pour X_Q la fibre de la famille de Kodaïra-Parshin correspondant au point Q rationnel sur K d'une *courbe fixe* X. Les constantes $a'(g,X)$ et $b'(g,X)$ *ne dépendent que* du genre et de la géométrie de la mauvaise réduction de la courbe fixe X.

On sait, par Parshin [P], qu'une telle conjecture implique la conjecture du discriminant $\Delta \leq N^k$ du Chapitre 1, Section 3. Est-il utile de rappeler que la construction de G. Frey lui a permis de remarquer que cette dernière conjecture donne des renseignements étonnants sur les équations diophantiennes ?

Passons à la démonstration de l'implication. Soit L un corps où les P_i sont rationnels (rappelons qu'on peut supposer que X et les X_Q sont semi-stables si nécessaire), on a :

$$\langle x_{P_1}\cdot x_{P_2}\rangle \leq \frac{\langle x_{P_1}\rangle^2+\langle x_{P_2}\rangle^2}{2g} + \frac{1}{L:\mathbb{Q}}\left(\omega^2_{X_Q/\mathcal{O}_L}(\frac{1}{g}-1)\right) + (2g-2)^2c(X)+\alpha(X)$$

où $c(X)$ est une borne supérieure pour les fonctions de Green sur toute les X_Q, $Q\in X(\bar{K})$ (la famille de Kodaïra-Parshin est à base compacte !!), et $\alpha(x)$ est une borne pour les termes en "$\emptyset$" dans (**) qui donc ne dépend que de g et de la matrice d'intersection des mauvaises fibres de X_Q et donc, que de la géométrie des mauvaises fibres de X (On peut lire les détails sur la construction de Kodaïra-Parshin dans [D]). L'implication s'en déduit facilement.

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C.N.R.S.
Institut Henri Poincaré
11 rue Pierre et Marie Curie
75005 Paris, France

Higher Algebraic K-Theory of Schemes and of Derived Categories

R. W. THOMASON* and THOMAS TROBAUGH

to Alexander Grothendieck on his 60th birthday

In this paper we prove a localization theorem for the K-theory of commutative rings and of schemes, Theorem 7.4, relating the K-groups of a scheme, of an open subscheme, and of the category of those perfect complexes on the scheme which are acyclic on the open subscheme. The localization theorem of Quillen [Q1] for K'- or G-theory is the main support of his many results on the G-theory of noetherian schemes. The previous lack of an adequate localization theorem for K-theory has obstructed development of this theory for the fifteen years since 1973. Hence our theorem unleashes a pack of new basic results hitherto known only under very restrictive hypotheses like regularity. These new results include the "Bass fundamental theorem" 6.6, the Zariski (Nisnevich) cohomological descent spectral sequence that reduces problems to the case of local (hensel local) rings 10.3 and 19.8, the Mayer-Vietoris theorem for open covers 8.1, invariance mod ℓ under polynomial extensions 9.5, Vorst-van der Kallen theory for NK 9.12, Goodwillie and Ogle-Weibel theorems relating K-theory to cyclic cohomology 9.10, mod ℓ Mayer-Vietoris for closed covers 9.8, and mod ℓ comparison between algebraic and topological K-theory 11.5 and 11.9. Indeed most known results in K-theory can be improved by the methods of this paper, by removing now unnecessary regularity, affineness, and other hypotheses.

We also develop the higher K-theory of derived categories, which is an essential tool in the above results. Our techniques here rest on the brilliant work of Waldhausen [W], who has extended and deepened the foundation of K-theory beyond that laid down by Quillen, allowing it to bear a heavier load.

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The key ideas that make all our results possible go back to the theory of K_0 of the derived category, which was conceived by Grothendieck, and was developed by him with Illusie and Berthelot in [SGA 6]. We especially need his concept of a perfect complex, a sheaf of chain complexes that is locally quasi-isomorphic to a bounded complex of algebraic vector bundles. These ideas have remained dormant for some time, especially because Quillen [Q1] discovered the higher K-theory of exact categories in a form which did not immediately extend to define a higher K-theory of derived categories. Thus one worked with the exact category of algebraic vector bundles, and not with the derived category of perfect complexes. Waldhausen's work [W] first made it clear how to define such a K-theory of a derived category, or more precisely, of a category of chain complexes provided with a notion of "weak equivalence" like quasi-isomorphism. Several people, among them Brinkmann [Bri], Gabber, Gillet [Gi2], [Gi4], Hinich and Shekhtman [HS], Landsburg, Waldhausen, and ourselves then became aware of this possibility of returning to the ideas of [SGA 6]. The intrinsic appeal of those ideas did not instantly overcome public inertia, and they did not at once appear strictly necessary to further progress. However, they turn out to be essential to the very statement of our localization theorem, if not to all its consequences. Moreover, the key geometric fact behind the theorem is the fact that the only obstruction to extending up to quasi-isomorphism a perfect complex on the open subscheme to the full scheme is its class in K_0. The naive analogous statement for algebraic vector bundles is false, as shown long ago by Serre ([Se], Section 5, a)). Furthermore, the proof of this extension fact depends essentially on the very Grothendieckian idea that perfect complexes are finitely presented objects in the derived category.

Of course, Grothendieck's ideas completely pervade modern mathematics, and it would be a hopeless task to isolate and acknowledge all intellectual debts to him. But we hope our case illustrates that despite their widespread influence, and nearly two decades after Grothendieck's withdrawal from public mathematical life, many of Grothendieck's ideas are still full of unexhausted potential, and will amply repay further development. Remarkably, his by now classic works can still surprise and instruct the serious reader. We dedicate this paper to him with profound admiration.

The reader may find a brief sketch of the contents of this long paper useful. Section 1 recalls for the convenience of the reader the results of Waldhausen [W]. An expert might skip this section, but should glance at biWaldhausen categories 1.2.4 and 1.2.11 to allow dualization of arguments, the inductive construction of chain complexes 1.9.5, the fact that K-theory is invariant under functors inducing equivalences of derived categories 1.9.8, and the cofinality theorem 1.10.1. Section 2 recalls the theory of perfect complexes on schemes from [SGA 6]. The expert might skip this, but should look at the characterization of perfect complexes as finitely presented objects 2.4, the fact that a complex with quasi-coherent cohomology on a nice scheme is a direct colimit of perfect complexes 2.3, and the basis for the excision theorem laid down in 2.6. Section 3 contains the definitions and basic functorialities of K-theory. Section 4 contains the projective space bundle theorem. Section 5 proves the key extension result for perfect complexes, and contains the first crude form of the localization theorem. Section 6 proves the Bass fundamental theorem, and defines K-groups also in negative degrees. Section 7 extends the previous results into negative degrees, putting them in their final form. In particular, Section 7 contains the good form of the localization theorem. This section gives the best quick summary of the fundamental results. The other basic results occur as consequences in Sections 8 - 11. The appendices contain various results needed in the text, but which are not limited to K-theory. In particular, Appendix B merely summarizes from all points of EGA and SGA the relations between the categories of $\mathcal{O}_X$-modules and of quasi-coherent $\mathcal{O}_X$-modules, as a help to the conscientious reader when he becomes as confused about this as we were.

The paper should be comprehensible to anyone with a good first year graduate knowledge in algebraic geometry, and with a bit of algebraic topology. We must formulate our results in the language of spectra in the sense of topology but this may be picked up easily by skimming through [A] III Sections 1-6 (ignore any pointless examples involving baroque curiosities like "MU," "MSO," "MSpin," or the "Steenrod algebra"), and a glance at [Th1] Section 5 and A. This formulation in terms of spectra is much more powerful than the naive formulation in terms of disembodied abelian groups, and is not subject to certain unstable pathologies like the formulation in terms of spaces as in [Q1] and [W]. Indeed, the spectral formulation works just like Grothendieck's formulation of homological algebra in terms of the derived category, as explained in [Th1] Sections 5. To see the proofs of the results quoted in Section 1, the reader must see [W], although the sufficiently trusting need not. [Q1] is still good reading, although not necessary for this paper, except for the homotopy theory of categories of [Q1] Section 1.

The first author must state that his coauthor and close friend, Tom Trobaugh, quite intelligent, singularly original, and inordinately generous, killed himself consequent to endogenous depression. Ninety-four days later, in my dream, Tom's simulacrum remarked, "The direct limit characterization of perfect complexes shows that they extend, just as one extends a coherent sheaf." Awaking with a start, I knew this idea had to be wrong, since some perfect complexes have a non-vanishing K_0 obstruction to extension. I had worked on this problem for 3 years, and saw this approach to be hopeless. But Tom's simulacrum had been so insistent, I knew he wouldn't let me sleep undisturbed until I had worked out the argument and could point to the gap. This work quickly led to the key results of this paper. To Tom, I could have explained why he must be listed as a coauthor. During his lifetime, Tom also pointed out the interesting comparison of the careers of Grothendieck and Newton.

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1. Waldhausen K-theory and K-theory of derived categories

1.0. In this section we review some definitions and results of Waldhausen's framework for K-theory, [W]. Our only claims to some originality in Section 1 are the general cofinality theorem 1.10.1 which is slightly different from previous results, and the results 1.9.5 and 1.9.8 which make it easier to apply Waldhausen's approximation theorem.

1.1.1. Let $\mathcal{A}$ be an abelian category. Consider chain complexes $C^\cdot$ in $\mathcal{A}$. We use the algebraic geometer's indexing, so differentials increase degree: $\partial : C^n \to C^{n+1}$.

Recall the standard notation $Z^k C^\cdot = \ker \partial : C^k \to C^{k+1}$, and $B^k C^\cdot = \operatorname{im} \partial : C^{k-1} \to C^k$.

A complex $C^\cdot$ is (strictly) bounded above if there is an integer N such that $C^n = 0$ for all $n \geq N$. The category of bounded above complexes is denoted $\mathcal{C}^-(\mathcal{A})$. A complex $C^\cdot$ is cohomologically bounded above if there is an N such that $H^n(C^\cdot) = 0$ for all $n \geq N$. Dually for bounded below, $\mathcal{C}^+(\mathcal{A})$, and cohomologically bounded below. A complex is bounded if it is bounded both above and below. The category of strict bounded complexes is $\mathcal{C}^b(\mathcal{A})$.

A chain map $f : C^\cdot \to D^\cdot$ is a chain homotopy equivalence if there is a chain map $g : D^\cdot \to C^\cdot$ and chain homotopies $fg \simeq 1_D$, $gf \simeq 1_C$. More generally, a chain map $f : C^\cdot \to D^\cdot$ is a quasi-isomorphism if it induces an isomorphism on all cohomology groups $H^*(f) : H^*(C^\cdot) \cong H^*(D^\cdot)$. For any integer m, a chain map f is an m-quasi-isomorphism if $H^k(f)$ is an isomorphism for $k > m$ and an epimorphism for $k = m$.

The derived category $D(\mathcal{A})$ (cf. [H], [V]) is formed from the category of all chain complexes in $\mathcal{A}$ by localizing this category of complexes so that precisely its quasi-isomorphisms become isomorphisms in $D(\mathcal{A})$. The variant subcategories $D^-(\mathcal{A})$, $D^+(\mathcal{A})$, $D^b(\mathcal{A})$ are formed similarly from the categories of cohomologically bounded above, cohomologically bounded below, and cohomologically bounded complexes, respectively. $D(\mathcal{A})$ admits a 2-sided calculus of fractions as a localization of the chain homotopy category that results from the category of complexes by identifying chain homotopic maps, as in 1.9.6 below, or in [V] I Section 2, [H] I Section 3. The additional structure of $D(\mathcal{A})$ as a triangulated category results from the construction of homotopy pushouts and pullbacks, which we review next.

1.1.2. Let $f : A^{\cdot} \to F^{\cdot}$ and $g : A^{\cdot} \to G^{\cdot}$ be chain maps of complexes. The canonical homotopy pushout

$$F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}$$

is the complex given by

$$\left(F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}\right)^{n} = F^n \oplus A^{n+1} \oplus G^n \tag{1.1.2.1}$$

with differential,

$$\partial(x, a, y) = (\partial_F x + fa,\ -\partial_A a,\ \partial_G y - ga)\,. \tag{1.1.2.2}$$

(We describe ∂ as if objects of $\mathcal{A}$ had "elements," by the standard abuse.)

Chain maps from this canonical homotopy pushout to a complex $C^{\cdot}$ correspond bijectively to data (h, k, H) where $h : F^{\cdot} \to C^{\cdot}$ and $k : G^{\cdot} \to C^{\cdot}$ are chain maps and H is a chain homotopy $hf \simeq kg : A^{\cdot} \to C^{\cdot}$. Thus H consists of maps $A^n \to C^{n-1}$ for all n such that $\partial H + H\partial = hf - kg$. To (h, k, H) corresponds the map from the homotopy pushout to $C^{\cdot}$ sending (x, a, y) to $hx + Ha + ky$.

Given another $f' : A^{\cdot\prime} \to F^{\cdot\prime}$ $g' : A^{\cdot\prime} \to G^{\cdot\prime}$, suppose there are maps $a : A^{\cdot} \to A^{\cdot\prime}$, $b : F^{\cdot} \to F^{\cdot\prime}$, $c : G^{\cdot} \to G^{\cdot\prime}$ and chain homotopies $f'a \simeq bf$, $g'a \simeq cg$. The maps a, b, c, and choice of chain homotopies then determine a map of canonical homotopy pushouts $F \overset{h}{\underset{A^{\cdot}}{\cup}} G \to F' \overset{h}{\underset{A'}{\cup}} G'$, as one sees by the universal mapping property of the preceding paragraph. This map will be a quasi-isomorphism if each of the maps a, b, c is a quasi-isomorphism. This last fact follows from the 5-lemma and the long exact sequence of cohomology groups

(1.1.2.3)
$$\ldots \xrightarrow{\partial} H^n(A) \to H^n(F^{\cdot}) \oplus H^n(G^{\cdot}) \to H^n(F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot}) \xrightarrow{\partial} H^{n+1}(A^{\cdot}) \to \ldots$$

which results from the short exact sequence of complexes

$$0 \to F^{\cdot} \oplus G^{\cdot} \to F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot} \to A^{\cdot}[1] \to 0. \tag{1.1.2.4}$$

Here $A^{\cdot}[k]$ is the complex $A^{\cdot}$ shifted in degree, so $A^{\cdot}[k]^n = A^{k+n}$, and $H^n(A[k]) = H^{n+k}(A^{\cdot})$.

Several special cases of the homotopy pushout construction are particularly important. When $f : A^{\cdot} = F^{\cdot}$ is the identity map, the canonical homotopy pushout is the mapping cylinder of $g : A^{\cdot} \to G^{\cdot}$, considered in 1.3.4. When $f : A^{\cdot} \to F^{\cdot} = 0$ is the map to 0, the canonical homotopy pushout is the mapping cone of $g : A^{\cdot} \to G^{\cdot}$.

There is a canonical map from the homotopy pushout to the (strict) pushout, induced by $(x, a, y) \longmapsto (x, y) \bmod A^{\cdot}$.

$$F^{\cdot} \overset{h}{\underset{A^{\cdot}}{\cup}} G^{\cdot} \to F^{\cdot} \underset{A^{\cdot}}{\cup} G^{\cdot}.$$

This map is a quasi-isomorphism whenever $A^n \to F^n \oplus G^n$ is a monomorphism for all n, as is seen by the 5-lemma applied to the map of the long exact sequence 1.1.2.3 to its analog resulting from the short exact sequence of complexes $0 \to A^{\cdot} \to F^{\cdot} \oplus G^{\cdot} \to F^{\cdot} \underset{A^{\cdot}}{\cup} G^{\cdot} \to 0$.

Dually, given $f : F^{\cdot} \to A^{\cdot}$ and $g : G^{\cdot} \to A^{\cdot}$ one has a canonical homotopy pullback

$$(1.1.2.5) \qquad \begin{aligned} \left(F^{\cdot} \overset{h}{\underset{A}{\times}} G^{\cdot}\right) &= F^n \oplus A^{n-1} \oplus G^n \\ \partial(x, a, y) &= (\partial_F x, \ -\partial_A a + fx - gy, \ \partial_G y) . \end{aligned}$$

This indeed corresponds to the homotopy pushout in the dual category of complexes in $\mathcal{A}^{\mathrm{op}}$, and so has all the dual properties. As special cases, dual to the mapping cylinder is the mapping path space, and dual to the mapping cone is the homotopy fibre.

1.1.3. Several standard truncation functors are useful. Let $C^{\cdot}$ be a complex. There is brutal truncation

$$\sigma^k C^{\cdot} = \sigma^{\geq k} C^{\cdot} = \cdots \to 0 \to 0 \to C^k \to C^{k+1} \to C^{k+2} \to \dots .$$

This is a subcomplex of $C^{\cdot}$. The quotient $C^{\cdot}/\sigma^k C^{\cdot}$ is another brutal truncation, denoted $\sigma^{\leq k-1} C^{\cdot}$.

There is also the good truncation

$$C^{\cdot}\langle -k\rangle = \tau^k C^{\cdot} = \tau^{\geq k} C^{\cdot} = \cdots \to 0 \to \ \operatorname{im} \partial C^{k-1} \to C^k \to C^{k+1} \to \dots .$$

There is a quotient map $C^{\cdot} \twoheadrightarrow \tau^k C^{\cdot}$ which induces an isomorphism on cohomology H^n for all $n \geq k$. For $n \leq k-1$, $H^n(\tau C^{\cdot}) = 0$. The kernel of

$C^\cdot \to \tau^k C^\cdot$ is denoted $\tau^{\leq k-1}C^\cdot$. For $n \leq k-1$, $H^n(\tau^{\leq k-1}C^\cdot) = H^n(C^\cdot)$, while $H^n(\tau^{\leq k-1}C^\cdot) = 0$ for $n \geq k$.

1.2.1. *Definition* ([W]). A category with cofibrations $\boldsymbol{A}$ is a category with a zero object 0, together with a chosen subcategory $\mathrm{co}(\boldsymbol{A})$ satisfying the three axioms:

1.2.1.1. Any isomorphism in $\boldsymbol{A}$ is a morphism in $\mathrm{co}(\boldsymbol{A})$.

1.2.1.2. For every object A in $\boldsymbol{A}$, the unique map $0 \to A$ is in $\mathrm{co}(\boldsymbol{A})$.

1.2.1.3. If $A \to B$ is a map in $\mathrm{co}(\boldsymbol{A})$, and $A \to C$ is a map in $\boldsymbol{A}$, then the pushout $B \underset{A}{\cup} C$ exists in $\boldsymbol{A}$, and the canonical map $C \to B \underset{A}{\cup} C$ is in $\mathrm{co}(\boldsymbol{A})$. In particular, $\boldsymbol{A}$ has finite coproducts.

1.2.2. One calls the morphisms in $\mathrm{co}(\boldsymbol{A})$ cofibrations. To distinguish them in diagrams, one usually denotes them by a feathered arrow "$\rightarrowtail$." Given $A \rightarrowtail B$, let the quotient B/A be the pushout $B \underset{A}{\cup} 0$ along $A \to 0$. One says that $A \rightarrowtail B \to C$ is a cofibration sequence if $B \to C$ is the canonical map $B \to B/A$ up to an isomorphism $C \cong B/A$. One then says $B \twoheadrightarrow C$ is a quotient map and denotes it by a double- headed arrow. Cofibration sequences are also called exact sequences.

In general, the set of quotient maps need not be closed under composition. Suppose however for a given category with cofibrations $\boldsymbol{A}$ that the set of quotient maps do form a subcategory $\mathrm{quot}(\boldsymbol{A})$, and moreover that the opposite category $\boldsymbol{A}^{\mathrm{op}}$ is a category with cofibration $\mathrm{co}(\boldsymbol{A}^{\mathrm{op}}) = \mathrm{quot}(\boldsymbol{A})^{\mathrm{op}}$. Suppose further that the canonical map $A \cup B \to A \times B$ is always an isomorphism, where $A \times B$ is the product in $\boldsymbol{A}$, and the coproduct in $\boldsymbol{A}^{\mathrm{op}}$. Suppose also that $A \to B \to C$ is a cofibration sequence in $\boldsymbol{A}$ iff the dual sequence $A \leftarrow B \leftarrow C$ is a cofibration sequence in $\boldsymbol{A}^{\mathrm{op}}$. Under these conditions, one says $\boldsymbol{A}$ is a category with bifibrations. This concept is self dual, so $\boldsymbol{A}^{\mathrm{op}}$ is then a category with bifibrations.

Note that in a category with bifibrations, given a cofibration sequence $A \rightarrowtail B \twoheadrightarrow C$, that A is the quotient of B by C in $\boldsymbol{A}^{\mathrm{op}}$, so dually A must be the kernel of $B \twoheadrightarrow C$ in $\boldsymbol{A}$.

1.2.3. *Definition.* A Waldhausen category (in [W], "a category with cofibrations and weak equivalences") is a category with cofibrations $\boldsymbol{A}$, $\mathrm{co}(\boldsymbol{A})$, together with a subcategory $\boldsymbol{w}(\boldsymbol{A})$ of $\boldsymbol{A}$ satisfying the two axioms:

1.2.3.1. Any isomorphism in $\boldsymbol{A}$ is a morphism in $\boldsymbol{w}(\boldsymbol{A})$.

1.2.3.2. ("gluing lemma") Given a commutative diagram in $\boldsymbol{A}$

$$\begin{array}{ccccc} B & \leftarrowtail & A & \to & C \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ B' & \leftarrowtail & A' & \to & C' \end{array}$$

with the two maps $A \rightarrowtail B$, $A' \rightarrowtail B'$ being cofibrations, and with the three maps $A \xrightarrow{\sim} A'$, $B \xrightarrow{\sim} B'$, and $C \xrightarrow{\sim} C'$ being in $\boldsymbol{w}(\boldsymbol{A})$, then the induced map $B \cup_A C \to B' \cup_{A'} C'$ is also in $\boldsymbol{w}(\boldsymbol{A})$.

The Waldhausen category consists of the triple data $\boldsymbol{A}$, $\mathrm{co}(\boldsymbol{A})$, $\boldsymbol{w}(\boldsymbol{A})$, but one usually abbreviates it as $\boldsymbol{A}$, or as $\boldsymbol{wA}$ when the choice of $\boldsymbol{w}(\boldsymbol{A})$ is particularly important.

One says the maps in $\boldsymbol{w}(\boldsymbol{A})$ are "weak equivalences," and denotes them by arrows with tildes"$\xrightarrow{\sim}$".

1.2.4. *Definition.* A biWaldhausen category is a category with bifibrations $\boldsymbol{A}$, $\mathrm{co}(\boldsymbol{A})$, $\mathrm{quot}(\boldsymbol{A})$, together with a subcategory $\boldsymbol{w}(\boldsymbol{A})$ such that both ($\boldsymbol{A}$, $\mathrm{co}(\boldsymbol{A})$, $\boldsymbol{w}(\boldsymbol{A})$) and the dual ($\boldsymbol{A}^{\mathrm{op}}$ $\mathrm{quot}(\boldsymbol{A})^{\mathrm{op}}$, $\boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}}$) are Waldhausen categories. That is, 1.2.3.1, 1.2.3.2, and the dual of 1.2.3.2 concerning pullbacks with $A \twoheadleftarrow B$ and $A' \twoheadleftarrow B'$ being in $\mathrm{quot}(\boldsymbol{A})$ all hold in the category with bifibrations $\boldsymbol{A}$.

This concept is self-dual, in that if $\boldsymbol{A}$ is a biWaldhausen category, so is $\boldsymbol{A}^{\mathrm{op}}$.

1.2.5. *Definition.* A saturated Waldhausen or biWaldhausen category is one where $\boldsymbol{w}(\boldsymbol{A})$ satisfies the saturation axiom: Given $A \xrightarrow{a} B \xrightarrow{b} C$ composible morphisms in $\boldsymbol{A}$, if any two of a, b, ba are in $\boldsymbol{w}(\boldsymbol{A})$, then so is the third.

1.2.6. *Definition.* An extensional Waldhausen or biWaldhausen category is one that satisfies the extension axiom: Given a commutative diagram whose rows are cofibration sequences

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

if both a and c are in $\boldsymbol{w}(\boldsymbol{A})$, then so is b.

1.2.7. *Definition.* A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between two Waldhausen categories is exact if $F(\mathrm{co}(\boldsymbol{A})) \subseteq \mathrm{co}(\boldsymbol{B})$, if $F(\boldsymbol{w}(\boldsymbol{A})) \subseteq \boldsymbol{w}(\boldsymbol{B})$, and if F preserves pushouts along a cofibration. The last condition means that the canonical map $FC \cup_{FA} FB \to F(C \cup_A B)$ is an isomorphism whenever $A \rightarrowtail B$ is in $\mathrm{co}(\boldsymbol{A})$.

A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between two biWaldhausen categories is exact if both $F : (\boldsymbol{A}\ \mathrm{co}(\boldsymbol{A}),\ \boldsymbol{w}(\boldsymbol{A})) \to (\boldsymbol{B},\ \mathrm{co}(\boldsymbol{B}), \boldsymbol{w}(\boldsymbol{B}))$ and the dual $F^{\mathrm{op}} : (\boldsymbol{A}^{\mathrm{op}},\ \mathrm{quot}(\boldsymbol{A})^{\mathrm{op}}, \boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}}) \to (\boldsymbol{B}^{\mathrm{op}},\ \mathrm{quot}(\boldsymbol{B})^{\mathrm{op}}, \boldsymbol{w}(\boldsymbol{B})^{\mathrm{op}})$ are exact functors between Waldhausen categories. This is equivalent to saying that $F : (\boldsymbol{A},\ \mathrm{co}(\boldsymbol{A}), \boldsymbol{w}(\boldsymbol{A})) \to (\boldsymbol{B},\ \mathrm{co}(\boldsymbol{B}), \boldsymbol{w}(\boldsymbol{B}))$ is exact and F preserves pullbacks where one of the maps is a quotient map.

1.2.8. Any category with cofibrations $\boldsymbol{A}$ becomes a Waldhausen category by taking $\boldsymbol{w}(\boldsymbol{A})$ to have as morphisms all the isomorphisms in $\boldsymbol{A}$. Henceforth, we identify all categories with cofibrations to Waldhausen categories in this way.

1.2.9. *Example.* Let $\boldsymbol{A}$ be an abelian category (or more generally an exact category in the sense of Quillen [Q1]). Let $\mathrm{co}(\boldsymbol{A})$ consist of all monomorphisms in $\boldsymbol{A}$ (in the exact category case, let $\mathrm{co}(\boldsymbol{A})$ consist of all admissible monomorphisms [Q1]). Let $\boldsymbol{w}(A)$ consist of all isomorphisms in $\boldsymbol{A}$. Then $\boldsymbol{A}$ is a Waldhausen category, and in fact a biWaldhausen category.

1.2.10. *Example* (optional). Let $\boldsymbol{A}$ be the category of simplicial sets. Let $\mathrm{co}(\boldsymbol{A})$ consist of all monomorphisms. Let $\boldsymbol{w}(A)$ consist of all "weak equivalences," i.e., all maps that induce homotopy equivalences between geometric realizations. Then $\boldsymbol{A}$ is a Waldhausen category.

1.2.11. *Definition.* A complicial biWaldhausen category is a saturated extensional biWaldhausen category $\boldsymbol{A}$ formed from a category of chain complexes as follows: One takes an abelian category $\mathcal{A}$, whose choice is part of the structure. $\boldsymbol{A}$ is to be a full additive subcategory of the category $\mathcal{C}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. $\mathrm{co}(\boldsymbol{A})$ is to contain at least all maps of complexes in $\boldsymbol{A}$ that are degree-wise split monomorphisms such that the quotient chain complex lies in $\boldsymbol{A}$. That is, $\mathrm{co}(\boldsymbol{A})$ contains all maps $C^{\cdot} \to D^{\cdot}$ in $\boldsymbol{A}$ such that for all integers n the map $C^n \to D^n$ is a split monomorphism in $\mathcal{A}$ and such that moreover the quotient chain complex $D^{\cdot}/C^{\cdot}$ in $\mathcal{C}(\mathcal{A})$ is also isomorphic to a complex in $\boldsymbol{A}$. $\mathrm{co}(\boldsymbol{A})$ may possibly contain other maps. One does require that if $C^{\cdot} \to D^{\cdot}$ is in $\mathrm{co}(\boldsymbol{A})$, then for all n, $C^n \to D^n$ is a monomorphism in $\mathcal{A}$, but not necessarily split.

Of course $\mathrm{co}(\boldsymbol{A})$ and the corresponding $\mathrm{quot}(A)$ must satisfy axiom 1.2.1.3 and its corresponding dual. We also demand that the pushouts and pullbacks required in $\boldsymbol{A}$ by these axioms are also the pushouts and pullbacks in the category of $\mathcal{C}(\mathcal{A})$; i.e., that $\boldsymbol{A}$ is a subcategory closed under the required pushouts and pullbacks.

$\boldsymbol{w}(A)$ is to contain all maps in $\boldsymbol{A}$ which are quasi-isomorphisms in the full category of complexes in $\mathcal{A}$. $\boldsymbol{w}(A)$ may contain other morphisms. Of course $\boldsymbol{w}(\boldsymbol{A})$ must satisfy the usual axioms 1.2.3.2 and its dual 1.2.4, and also the saturation and extension axioms 1.2.5 and 1.2.6.

In specifying a complicial biWaldhausen category, we make the default convention that unless explicitly specified otherwise, $\boldsymbol{w}(\boldsymbol{A})$ is to consist of exactly the quasi-isomorphisms, and that $\mathrm{co}(\boldsymbol{A})$ is to consist of exactly those degree-wise split monomorphisms whose cokernel is in $\boldsymbol{A}$.

1.2.12. *Example.* Let $\mathcal{E}$ be an exact category [Q1]. Then there is an abelian category $\mathcal{A}$ and a fully-faithful Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$, reflecting exactness and such that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, (cf Appendix A).

Let $\boldsymbol{A}$ be the full subcategory of complexes $C^\cdot$ in $\mathcal{A}$ with each C^k in $\mathcal{E}$, and with $C^k = 0$ unless $k = 0$. Take the cofibrations to be admissible monomorphisms, and the weak equivalences to be the quasi-isomorphisms. As $H^0(C^\cdot) = C^0$ for $C^\cdot$ in $\boldsymbol{A} \cong \mathcal{E}$, these weak equivalences are just the isomorphisms in $\mathcal{E}$.

This $\boldsymbol{A}$ is a complicial biWaldhausen category, which in fact is the biWaldhausen category of 1.2.9.

A more interesting example would be take $\boldsymbol{A}$ the category of complexes in $\mathcal{A}$ which are degree-wise in $\mathcal{E}$. See 1.11.6 below.

1.2.13. *Example.* Let $\mathcal{A}$ be an abelian category, and let $\boldsymbol{A}$ be the category of all chain complexes in $\mathcal{A}$. Then with the default conventions for $\mathrm{co}(\boldsymbol{A})$ and $\boldsymbol{w}(\boldsymbol{A})$, $\boldsymbol{A}$ is a complicial biWaldhausen category.

There are variants where $\boldsymbol{A}$ consists of the bounded complexes, the bounded above complexes, the cohomologically bounded complexes, etc..

1.2.14. *Example.* Let $\mathcal{A}$ be an abelian category, and let $\boldsymbol{A}$ be the category of all bounded below complexes $C^\cdot$ in $\mathcal{A}$ such that each C^k is an injective object of $\mathcal{A}$. Then $\boldsymbol{A}$ is complicial biWaldhausen.

1.2.15. *Example.* Let $\mathcal{A}$ be an abelian category with a thick abelian subcategory $\mathcal{B}$. Let $\boldsymbol{A}$ be the category of complexes $C^\cdot$ in $\mathcal{A}$ such that $C^\cdot$ is cohomologically bounded and with all cohomology groups $H^k(C^\cdot)$ in $\mathcal{B}$. Then $\boldsymbol{A}$ is complicial biWaldhausen.

1.2.16. *Definition.* Let $\boldsymbol{A}$, $\boldsymbol{B}$ be complicial biWaldhausen categories. A complicial exact functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is an exact functor of biWaldhausen categories in the sense of 1.2.7, with the additional property that the functor F of complexes is induced by degree-wise application of some additive functor $f : \mathcal{A} \to \mathcal{B}$ between the abelian categories chosen as part of the complicial structure.

Hence $F(C^\cdot) = \cdots \to f(C^k) \to f(C^{k+1}) \to \ldots .$

1.3.1. *Definition* ([W] 1.6). Let $\boldsymbol{A}$ be a Waldhausen category. A cylinder functor T on $\boldsymbol{A}$ is a functor $T : \ \mathrm{Cat}(\mathbf{1}, \boldsymbol{A}) \to \boldsymbol{A}$ to $\boldsymbol{A}$ from the category of morphisms in $\boldsymbol{A}$, together with three natural transformations p, j_1, j_2, satisfying the conditions below.

Thus to each morphism $f : A \to B$ in $\boldsymbol{A}$, T assigns an object Tf of $\boldsymbol{A}$. To each commutative square (1.3.1.1) in $\boldsymbol{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{1.3.1.1}$$

T assigns functorially a morphism $T(a,b) : Tf \to Tf'$.

The natural transformations are maps $j_1 : A \to Tf$, $j_2 : B \to Tf$, and $p : Tf \to B$ such that $pj_1 = f : A \to B$, $pj_2 = 1 : B \to B$, and such that (1.3.1.2) commutes.

$$\begin{array}{ccccc} A \cup B & \xrightarrow{j_1 \cup j_2} & Tf & \xrightarrow{p} & B \\ {\scriptstyle a \cup b}\downarrow & & \downarrow{\scriptstyle T(a,b)} & & \downarrow{\scriptstyle b} \\ A' \cup B' & \xrightarrow[j_1 \cup j_2]{} & Tf' & \xrightarrow[p]{} & B' \end{array} \tag{1.3.1.2}$$

We also require conditions 1.3.1.3 - 1.3.1.6 to hold.

1.3.1.3. $j_1 \cup j_2 : A \cup B \rightarrowtail Tf$ is in $\mathrm{co}(\boldsymbol{A})$.

1.3.1.4. If a and b are in $\boldsymbol{w}(\boldsymbol{A})$, then $T(a,b)$ is in $\boldsymbol{w}(\boldsymbol{A})$.

1.3.1.5. If a and b are in $\mathrm{co}(\boldsymbol{A})$, then not only is $T(a,b)$ in $\mathrm{co}(\boldsymbol{A})$, but also the map $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ induced by the left square of (1.3.1.2) is in $\mathrm{co}(\boldsymbol{A})$.

1.3.1.6. $T(0 \to A) = A$, with p and j_2 the identity map.

To define a "cylinder functor satisfying the cylinder axiom," one imposes the extra cylinder axiom:

1.3.1.7. For all f, $p : Tf \to B$ is in $\boldsymbol{w}(\boldsymbol{A})$.

(Note our 1.3.1.3 - 1.3.1.5 are equivalent to [W] 1.6 "Cyl 1").

1.3.2. *Definition.* If $\boldsymbol{A}$ is a biWaldhausen category, a cocylinder functor or mapping path space functor is a functor $M : \ \mathrm{Cat}(\mathbf{1}, \boldsymbol{A}) \to \boldsymbol{A}$ and natural transformations

$$\begin{array}{ccccc} & & M(f : A \to B) & & \\ & k_2 \swarrow & & \searrow k_1 & & \nwarrow q \\ A & & B & & A \end{array} \tag{1.3.2.1}$$

such that M is a cylinder functor on the dual ($\boldsymbol{A}^{\mathrm{op}}$, quot$(\boldsymbol{A})^{\mathrm{op}}$, $\boldsymbol{w}(\boldsymbol{A})^{\mathrm{op}}$). M satisfies the cocylinder axiom if the dual of 1.3.1.7 holds, i.e., if $q : A \to M(A \to B)$ is always a weak equivalence.

1.3.3. *Example* (optional). The usual mapping cylinder of algebraic topology is a cylinder functor satisfying the cylinder axiom in the Waldhausen category of simplicial sets 1.2.10.

1.3.4. *Example.* Let $\mathcal{A}$ be an abelian category, and $\boldsymbol{A}$ the biWaldhausen category of all chain complexes in $\mathcal{A}$. Then $\boldsymbol{A}$ has well-known cylinder and cocylinder functors satisfying the cylinder and cocylinder axioms respectively. For given $f : A \to B$, let Tf be the canonical homotopy pushout $A \overset{h}{\underset{A}{\cup}} B$ of $1 : A \to A$ and $f : A \to B$ constructed as in 1.1.2. The maps $j_1 : A \to A \overset{h}{\underset{A}{\cup}} B$ and $j_2 : B \to A \overset{h}{\underset{A}{\cup}} B$ are the canonical inclusions $j_1(a) = (a, 0, 0)$, $j_2(b) = (0, 0, b)$. The map $p : A \overset{h}{\underset{A}{\cup}} B \to B$ is the morphism induced by $f : A \to B$, $1 : B \to B$, and the trivial homotopy $f1 \simeq 1f$ so that $p(a, a', b) = fa + b$. Dually, the canonical homotopy pullback $A \overset{h}{\underset{B}{\times}} B$ provides a cocylinder.

1.3.5. *Example.* Let $\boldsymbol{A}$ be a complicial biWaldhausen category with associated chosen abelian category $\mathcal{A}$. Suppose that those canonical homotopy pullbacks and canonical homotopy pushouts, formed in the category of complexes in $\mathcal{A}$ starting from diagrams in $\boldsymbol{A}$, are in fact objects of $\boldsymbol{A}$. Then we claim that the mapping cylinder and cocylinder functors of 1.3.4 induce mapping cylinder and cocylinder functors on the subcategory $\boldsymbol{A}$, provided only that the cofibration axiom 1.3.1.5 and its dual hold. Moreover, the cylinder axiom 1.3.1.7 and its dual cocylinder axiom hold automatically.

Note 1.3.1.3 holds automatically, as $j_1 \cup j_2$ is a degree-wise split monomorphism whose cokernel is the homotopy pushout of $A \to 0$ along $A \to 0$, and hence in $\boldsymbol{A}$. Thus $j_1 \cup j_2$ is in co$(\boldsymbol{A})$. As p is a quasi-isomorphism, it is in $\boldsymbol{w}(\boldsymbol{A})$ and so 1.3.1.7 holds. Axiom 1.3.1.4 now follows from saturation 1.2.5, and axiom 1.3.1.6 is trivial. This leaves only 1.3.1.5 in doubt, proving the claim.

To verify 1.3.1.5 it suffices to show $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ is in co$(\boldsymbol{A})$, for the canonical map of Tf into $Tf \underset{A \cup B}{\cup} A' \cup B'$ is a cofibration by 1.2.1.3. Hence if the first map is a cofibration so is the composite $Tf \to Tf'$.

The map $Tf \underset{A \cup B}{\cup} A' \cup B' \to Tf'$ is given degree-wise as a sum of $A^{n+1} \to A'^{n+1}$ with identity maps

$$A'^n \oplus A^{n+1} \oplus B'^n \to A'^n \oplus A'^{n+1} \oplus B'^n. \tag{1.3.5.1}$$

If co($\boldsymbol{A}$) consists exactly of the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{A}$, or else consists of all monomorphisms whose quotients lie in $\boldsymbol{A}$ then 1.3.1.5 and its dual also hold automatically. This also works if co($\boldsymbol{A}$) is defined to be all maps that are degree-wise admissible monomorphisms whose quotients lie in $\boldsymbol{A}$ for some exact subcategory $\mathcal{E} \subseteq \mathcal{A}$ that contains A^n for all $A^\cdot$ in $\boldsymbol{A}$ and all degrees n. Thus in all the usual examples, and in fact in all cases arising in Sections 2-11, axiom 1.3.1.5 also holds automatically.

1.3.6. *Example.* Let $\boldsymbol{A}$ be a complicial biWaldhausen category with associated abelian category $\mathcal{A}$. Suppose co($\boldsymbol{A}$) is one of the cases listed in 1.3.5 that make 1.3.1.5 automatic, e.g., all degree-wise admissible monomorphisms with quotients lying in $\boldsymbol{A}$. Suppose $\boldsymbol{A}$ is closed under finite degree shifts so that if $A^\cdot$ is in $\boldsymbol{A}$, so is $A^\cdot[k]$. Suppose $\boldsymbol{A}$ is closed under extensions, i.e., that if $0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$ is an exact sequence of complexes in $\mathcal{A}$ with $A^\cdot$ and $C^\cdot$ in $\boldsymbol{A}$ then $B^\cdot$ is isomorphic to a complex in $\boldsymbol{A}$. Then $\boldsymbol{A}$ has a mapping cylinder and cocylinder satisfying the cylinder and cocylinder axioms. This will follow from 1.3.5 once we see $\boldsymbol{A}$ is closed under formation of canonical homotopy pushouts and canonical homotopy pullbacks. But this follows from the hypotheses that $\boldsymbol{A}$ is closed under extensions and degree shifts and the exact sequence (1.1.2.4) and is dual.

1.4. We henceforth consider only small Waldhausen categories, those with a set, as opposed to a class of morphisms. Hence, when we speak of a Waldhausen category of all chain complexes of abelian groups or of $\mathcal{O}_X$-modules on a scheme, it is implicit that we are looking at the category of such complexes in a Grothendieck universe so that it is small with respect to a larger universe [SGA 4] I Appendice. (As in [SGA 4], the K-theory spectrum of the biWaldhausen categories of Section 2 and Section 3 will be independent of the choice of universe at least up to homotopy, as these biWaldhausen categories in the various universes will have equivalent derived categories, so 1.9.8 applies. See Appendix F).

1.5.1. *Definition* [W]. For $\boldsymbol{A}$ a Waldhausen category with weak equivalences $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{A})$ define $\boldsymbol{w}S_\cdot\boldsymbol{A}$ to be the following simplicial category.

The objects of the category in degree n, $\boldsymbol{w}S_n\boldsymbol{A}$ are the functors A, meeting the conditions below, to $\boldsymbol{A}$ from the partially ordered set of pairs of integers (i,j), with $0 \le i \le j \le n$. The partial order is defined by $(i,j) \le (i',j')$ iff both $i \le i'$ and $j \le j'$. The functors A must meet the conditions that for all j, $A(j,j) = 0$, and that for all (i,j,k) with $i \le j \le k$, the maps $A(i,j) \rightarrowtail A(i,k) \twoheadrightarrow A(j,k)$ form a cofibration sequence.

The morphisms of $\boldsymbol{w}S_n(\boldsymbol{A})$ are the natural transformations $A \to A'$

such that for all (i,j), $A(i,j) \to A'(i,j)$ is in $\boldsymbol{w}(\boldsymbol{A})$. By the gluing Lemma 1.2.3.2 and the cofibration sequence condition on A for $0 \leq i \leq j$, it suffices that all $A(0,k) \to A'(0,k)$ are in $\boldsymbol{w}(\boldsymbol{A})$.

Given $\varphi : n \to k$ in Δ^{op} corresponding to a monotone map $\varphi : \{0,1,\ldots,k\} \to \{0,1,\ldots,n\}$, the simplicial operator φ on $\boldsymbol{w}S.\boldsymbol{A}$ is the functor $\varphi : \boldsymbol{w}S_n\boldsymbol{A} \to \boldsymbol{w}S_k\boldsymbol{A}$ that sends the object $(i,j) \mapsto A(i,j)$ to the object $(r,s \mapsto A(\varphi(r),\varphi(s))$.

1.5.1.2. The category $\boldsymbol{w}S_n\boldsymbol{A}$ is equivalent via the forgetful functor to the category of subdiagrams $A_1 \rightarrowtail \cdots \rightarrowtail A_n = A(0,1) \rightarrowtail A(0,2) \rightarrowtail \cdots \rightarrowtail A(0,n)$. Indeed $A(i,j)$ is $A(0,j)/A(0,i) = A_j/A_i$ up to canonical natural isomorphism, so specifying the $A(i,j)$ for $i \neq 0$ just adds choices of objects determined up to isomorphism. However, it is necessary to specify these choices to make the simplicial identities hold strictly in $\boldsymbol{w}S.\boldsymbol{A}$, instead of just up to natural isomorphism.

1.5.2. For $\boldsymbol{A}$ a small Waldhausen category, taking the nerve in each degree of the simplicial category $\boldsymbol{w}S.\boldsymbol{A}$ yields a bisimplicial set $N.\boldsymbol{w}S.\boldsymbol{A}$. Waldhausen defines the K-theory space of $\boldsymbol{A}$ to be the loops on the geometric realization of this bisimplicial set, $K(\boldsymbol{A}) = \Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$. One also denotes this $K(\boldsymbol{w}\boldsymbol{A})$ when it is important to distinguish among several possible choices of weak equivalences. The K-groups of $\boldsymbol{A}$ are the homotopy groups of $K(\boldsymbol{A})$.

This space $K(\boldsymbol{A})$ is in fact an "infinite loop space" by [W] 1.3.3 and 1.5.3, as Waldhausen shows that it is the zero-th space of a spectrum, i.e., of a sequence of spaces each of which is homotopy equivalent by a given map to the loops on the next space in the sequence. It is in fact better to work with this spectrum than with the space. The proofs of [W] (and of [Q1], [Grl], etc.) immediately generalize to give "infinite loop space" versions of its results which are then valid for $K(\boldsymbol{A})$ as a spectrum.

1.5.3. *Definition.* For $\boldsymbol{A}$ a small Waldhausen category with weak equivalences $\boldsymbol{w}$, define $K(\boldsymbol{A}) = K(\boldsymbol{w}\boldsymbol{A})$ to be the spectrum constructed from $\boldsymbol{A}$ by the process of [W] 1.3.3 and remark, 1.5.3, and whose 0th space is $\Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$. Define the Waldhausen K-groups $K_n(\boldsymbol{w}\boldsymbol{A})$ to be the homotopy groups of the spectrum, $\pi_n K(\boldsymbol{w}\boldsymbol{A})$. Note these groups are 0 if $n \leq -1$, and are isomorphic to the homotopy groups of the space $\Omega|N.\boldsymbol{w}S.\boldsymbol{A}|$ for $n \geq 0$.

1.5.4. An exact functor $F : \boldsymbol{A} \to \boldsymbol{B}$ induces a simplicial functor $\boldsymbol{w}S.\boldsymbol{A} \to \boldsymbol{w}S.\boldsymbol{B}$, and a map of spectra $KF : K(\boldsymbol{A}) \to K(\boldsymbol{B})$. This makes K a functor.

If $\eta : F \to G$ is a natural transformation of exact functors $\boldsymbol{A} \to \boldsymbol{B}$, and if for all objects A in $\boldsymbol{A}$, $\eta A : FA \xrightarrow{\sim} GA$ is in $\boldsymbol{w}(\boldsymbol{B})$, then η induces a homotopy $\boldsymbol{w}S.F \simeq \boldsymbol{w}S.G$, and in fact a homotopy of maps of spectra

$KF \simeq KG$.

See [W] for details, and [Th3] A for homotopies of maps of spectra.

1.5.5. If $\boldsymbol{A}$ is a biWaldhausen category, we define $K(\boldsymbol{A})$ using the underlying Waldhausen category $(\boldsymbol{A}, \ \mathrm{co}(\boldsymbol{A}), \ w(\boldsymbol{A}))$. But since $A \to B \to C$ is a cofibration sequence in $\boldsymbol{A}$ iff $C \to B \to A$ is a cofibration sequence in $\boldsymbol{A}^{\mathrm{op}}$, there is a canonical isomorphism of simplicial categories $(\boldsymbol{w}S_.\boldsymbol{A})^{\mathrm{op}} \cong \boldsymbol{w}S_.(\boldsymbol{A}^{\mathrm{op}})$. From the canonical isomorphism between the classifying space $|N_.(\ )|$ of a category and of its dual ([Q1] Section 1(3)), we deduce a canonical duality isomorphism of spectra $K(\boldsymbol{A}) \cong K(\boldsymbol{A}^{\mathrm{op}})$. This allows us to dualize all theorems of [W] when applied to biWaldhausen categories.

1.5.6. It is easy to derive the following formula for $K_0(\boldsymbol{wA})$ from the edge-path group presentation of $K_0(\boldsymbol{wA}) = \pi_0\Omega|N_.\boldsymbol{w}S_.\boldsymbol{A}| = \pi_1|N_.\boldsymbol{w}S_.\boldsymbol{A}|$.

$K_0(\boldsymbol{wA})$ is the free group (or the free abelian groups) on generators $[A]$ as A runs over the objects of $\boldsymbol{A}$, modulo the two relations

1.5.6.1. $[A] = [B]$ if there is a map $A \xrightarrow{\sim} B$ in $\boldsymbol{w}(\boldsymbol{A})$.

1.5.6.2. $[B] = [A][B/A]$ for all cofibration sequences $A \rightarrowtail B \twoheadrightarrow B/A$.

Note that relation 1.5.6.2 applied to $A \rightarrowtail A \cup B \twoheadrightarrow B$ and $B \rightarrowtail A \cup B \twoheadrightarrow A$ forces $[A][B] = [A \cup B] = [B][A]$. Thus $K_0(\boldsymbol{wA})$ is abelian and we usually write the relation additively: $[B] = [A] + [B/A]$. (Also 1.5.6.2 forces $[0] = 0$.)

1.5.7. If $\boldsymbol{A}$ has a mapping cylinder satisfying the cylinder axiom, let ΣA be the cone of $A \to 0$. Thus $A \rightarrowtail T(A \to 0) \twoheadrightarrow \Sigma A$ is a cofibration sequence. As $T(A \to 0) \simeq 0$, it follows that $-[A] = [\Sigma A]$ in $K_0(\boldsymbol{A})$. Hence $[B] - [A] = [B \vee \Sigma A]$. Thus every element of $K_0(\boldsymbol{A})$ is the class $[C]$ of some C in $\boldsymbol{A}$.

1.6. We now turn to the basic theorems of Waldhausen K- theory: the additivity, localization, approximation, and cofinality theorems. We will not expose Waldhausen's cell filtration theorem [W] 1.7, but it will be cited later in an optional exercise 5.7.

1.7.1. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be small Waldhausen categories, with exact functors $\boldsymbol{A} \to \boldsymbol{C}$ and $\boldsymbol{B} \to \boldsymbol{C}$ which are inclusions of the underlying categories.

Let the category of "exact sequences" $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ be the category whose objects are those cofibration in $\boldsymbol{C}$, $A \rightarrowtail C \twoheadrightarrow B$, which have A in $\boldsymbol{A}$ and B in $\boldsymbol{B}$. The morphism of $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ are commutative diagrams in $\boldsymbol{C}$

$$\begin{array}{ccccc} A & \rightarrowtail & C & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \rightarrowtail & C' & \twoheadrightarrow & B' \end{array}$$

with $A \to A'$ a morphism in $\boldsymbol{A}$ and $B \to B'$ a morphism in $\boldsymbol{B}$. Such a morphism in $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ is a cofibration if $A \to A'$, $B \to B'$, and $A' \cup_A C \to C'$ are cofibrations in $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ respectively. Note then that $C \to C'$ is a cofibration in $\boldsymbol{C}$, as it is the composite $A \cup_A C \rightarrowtail A' \cup_A C \rightarrowtail C'$.

A morphism in $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ is a weak equivalence if $A \to A'$, $B \to B'$, and $C \to C'$ are weak equivalences in $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ respectively.

This gives $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ the structure of a Waldhausen category. There are exact functors s, t, q from $E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ to $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{B}$ respectively, sending $A \rightarrowtail C \twoheadrightarrow B$ to A, C, and B respectively. There is also an exact functor $\cup : \boldsymbol{A} \times \boldsymbol{B} \to E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$ sending (A,B) to $A \rightarrowtail A \cup B \twoheadrightarrow B$. This functor splits $(s,q) : E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B}) \to \boldsymbol{A} \times \boldsymbol{B}$.

1.7.2. **Additivity Theorem**([W] 1.3.2, 1.4.2). *Take the notation and make the hypotheses of* 1.7.1. *Then the exact functors* (s,q) *induce a homotopy equivalence of* K*- theory spectra*

$$K(s,q) : K(E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})) \xrightarrow{\sim} K(\boldsymbol{A}) \times K(\boldsymbol{B}).$$

A natural homotopy inverse to this map is $K(\cup)$ *induced by* $\cup : \boldsymbol{A} \times \boldsymbol{B} \to E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$.

Proof. See [W] 1.3.2, Section 1.4.

1.7.3. **Corollary** ([W] 1.3.2(4)). *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be small Waldhausen categories, and let* F, F', F'' : $\boldsymbol{A} \to \boldsymbol{B}$ *be three exact functors. Suppose there are natural transformations* $F' \to F$ *and* $F \to F''$ *such that the following two conditions hold:*

1.7.3.1. *For all* A *in* $\boldsymbol{A}$, $F'A \rightarrowtail FA \twoheadrightarrow F''A$ *is a cofibration sequence.*

1.7.3.2. *For any cofibration* $A' \rightarrowtail A$ *in* $\boldsymbol{A}$, *the induced map* $F'A \cup_{F'A'} FA' \to FA$ *is a cofibration.*

Then there is a homotopy of maps of spectra $KF \simeq KF' + KF''$: $K(\boldsymbol{A}) \to K(\boldsymbol{B})$.

Proof. The natural cofibration sequence $F' \rightarrowtail F \twoheadrightarrow F''$ induces an exact functor $\boldsymbol{A} \to E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B})$. The additivity theorem 1.7.2 implies a homotopy $Kt \simeq Ks + Kq$ of maps $K(E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B})) \to K(\boldsymbol{B})$, since these maps become equal after composing with the homotopy equivalence $K(\cup)$. Composing the homotopy $Kt \simeq Ks + Kq$ with the map $K(\boldsymbol{A}) \to K(E(\boldsymbol{B},\boldsymbol{B},\boldsymbol{B}))$ yields a homotopy $KF \simeq KF' + KF''$.

1.7.4. When $\boldsymbol{B}$ is a complicial biWaldhausen category, hypothesis 1.7.3.2 is superfluous as it follows automatically from 1.7.3.1 and exactness of F', F, and F''. For one notes that

$$F'A \cup_{F'A'} FA'/F'A \cong 0 \cup_{F'A'} FA' \cong FA'/F'A' \cong F''A'.$$

Thus the diagram below has cofibration sequences as rows

$$\begin{tikzcd} F'A \arrow[r, tail] \arrow[d, equal] & F'A \underset{F'A'}{\cup} FA' \arrow[r, two heads] \arrow[d] & F''A' \arrow[d, tail] \\ F'A \arrow[r, tail] & FA \arrow[r, two heads] & F''A \arrow[d, two heads] \\ & & F''(A/A') \end{tikzcd}$$

As $\boldsymbol{B}$ is biWaldhausen, the composite map $FA \twoheadrightarrow F''(A/A')$ is in quot($\boldsymbol{B}$), and its kernel is a cofibration into FA. But as $\boldsymbol{B}$ is complicial, this kernel is the same as the kernel taken in the category of chain complexes in the associated abelian category $\mathcal{B}$. Applying the snake lemma to the above diagram in the category of chain complexes, we see that the kernel is $F'A \underset{F'A'}{\cup} FA' \to FA$, and so this map is a cofibration as required by 1.7.3.2.

1.8.1. Let $\boldsymbol{A}$ be a small category with cofibrations. Suppose $\boldsymbol{A}$ has two subcategories $\boldsymbol{v}(\boldsymbol{A})$ and $\boldsymbol{w}(\boldsymbol{A})$, each of which is the category of weak equivalences for a Waldhausen category structure on $\boldsymbol{A}$, $\boldsymbol{vA}$ and $\boldsymbol{wA}$. Suppose $\boldsymbol{v}(\boldsymbol{A}) \subseteq \boldsymbol{w}(\boldsymbol{A})$, and that $\boldsymbol{wA}$ satisfies the extension and saturation axioms.

Let $\boldsymbol{A}^w$ be the full subcategory of $\boldsymbol{A}$ whose objects are the A such that $0 \to A$ is in $\boldsymbol{w}(\boldsymbol{A})$, i.e., which are $\boldsymbol{w}$-acyclic. This $\boldsymbol{A}^w$ becomes a Waldhausen category $\boldsymbol{vA}^w$ with $\mathrm{co}(\boldsymbol{A}^w) = \mathrm{co}(\boldsymbol{A}) \cap \boldsymbol{A}^w$ and $\boldsymbol{v}(\boldsymbol{A}^w) = \boldsymbol{v}(\boldsymbol{A}) \cap \boldsymbol{A}^w$. If $\boldsymbol{vA}$ and $\boldsymbol{wA}$ are biWaldhausen, so is $\boldsymbol{vA}^w$. If A has a functor T which is a cylinder functor both for $\boldsymbol{vA}$ and for $\boldsymbol{wA}$, T induces a cylinder functor on $\boldsymbol{A}^w$.

1.8.2. **Localization Theorem** ([W] 1.6.4 Fibration Theorem). *With the notation and hypotheses of* 1.8.1, *suppose also that* $\boldsymbol{A}$ *has a functor* T *which is a cylinder functor both for* $\boldsymbol{vA}$ *and for* $\boldsymbol{wA}$, *and that* T *satisfies the cylinder axiom* 1.3.1.7 *for* $\boldsymbol{wA}$.

Then the exact inclusion functors $\boldsymbol{vA}^w \to \boldsymbol{vA}$, $\boldsymbol{vA} \to \boldsymbol{wA}$, *induce a homotopy fibre sequence of spectra*

$$K(\boldsymbol{vA}^w) \to K(\boldsymbol{vA}) \to K(\boldsymbol{wA}).$$

(The requisite chosen nullhomotopy of $K(\boldsymbol{vA}^w) \to K(\boldsymbol{wA})$ *is induced by the natural weak equivalence* $0 \xrightarrow{\sim} A$ *in* $\boldsymbol{wA}$ *for* A *in* $\boldsymbol{vA}^w$.)

Proof. [W] 1.6.4.

1.9.1. **Approximation Theorem** ([W] 1.6.7). *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be small saturated Waldhausen categories. Suppose* $\boldsymbol{A}$ *has a cylinder functor satisfying the cylinder axiom 1.3.1.7. Let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be an exact functor satisfying the two conditions:*

1.9.1.1. A morphism f *of* $\boldsymbol{A}$ *is in* $\boldsymbol{w}(\boldsymbol{A})$ *if and only if* Ff *is in* $\boldsymbol{w}(\boldsymbol{B})$.

1.9.1.2. Given any A in $\boldsymbol{A}$ and any $x : FA \to B$ in $\boldsymbol{B}$, there is an A' in $\boldsymbol{A}$, a map $a : A \to A'$ in $\boldsymbol{A}$, and a weak equivalence $x' : FA' \xrightarrow{\sim} B$ in $\boldsymbol{w}(\boldsymbol{B})$ such that $x = x' \circ Fa$.

Then under these conditions, F induces a homotopy equivalence $KF : K(\boldsymbol{A}) \xrightarrow{\sim} K(\boldsymbol{B})$.

Proof. This results from Waldhausen's version [W] 1.6.7. Condition 1.9.1.2 appears to be weaker than the corresponding condition "App 2" of [W] in that 1.9.1.2 does not require the map $a : A \to A'$ to be a cofibration. But given $x = x' \circ Fa$ as in 1.9.1.2, one applies the cylinder functor to $a : A \to A'$ to factor $a = a'' \circ a'$, with a' the cofibration $A \rightarrowtail A'' = T(a)$, and a'' the weak equivalence $A'' = T(a) \xrightarrow{\sim} A'$. Then $x'' = x' \circ Fa'' : FA'' \xrightarrow{\sim} B$ is a weak equivalence, $a' : A \rightarrowtail A''$ is a cofibration, and $x = x'' \circ Fa'$. Hence 1.9.1.2 implies "App 2" of [W] in the presence of the other hypotheses.

1.9.2.**Theorem.** *Let $\boldsymbol{A}$ be a small complicial biWaldhausen category. Let $\boldsymbol{A}'$ be the new complicial biWaldhausen structure on $\boldsymbol{A}$ where $\boldsymbol{w}(\boldsymbol{A}') = \boldsymbol{w}(\boldsymbol{A})$, but where $co(\boldsymbol{A}')$ consists exactly of those degree-wise split monomorphisms whose quotient lies in $\boldsymbol{A}$. Suppose that $\boldsymbol{A}'$ has a cylinder functor satisfying the cylinder axiom, as often occurs* (cf. 1.3.5, 1.3.6).

Then the exact inclusion functor $\boldsymbol{A}' \to \boldsymbol{A}$ induces a homotopy equivalence $K(\boldsymbol{A}') \xrightarrow{\sim} K(\boldsymbol{A})$.

Proof. Apply 1.9.1 to the inclusion $\boldsymbol{A}' \to \boldsymbol{A}$. Condition 1.9.1.1 is obvious, and 1.9.1.2 holds trivially with $a = x$, $x' = 1$.

Something much like 1.9.2 was proved in [HS] by Hinich and Shekhtman before the unveiling of Waldhausen's approximation theorem.

1.9.2.1 *Remark.* Theorem 1.9.2 shows it is usually harmless in K-theory to impose the condition that co($\boldsymbol{A}$) consists of precisely the degree-wise split monomorphisms with quotients lying in $\boldsymbol{A}$, at least in the presence of cylinders. This is convenient, since then any complicial functor $F : \boldsymbol{A} \to \boldsymbol{B}$ induced by an additive $f : \mathcal{A} \to \mathcal{B}$ automatically will preserve cofibrations and pushouts along cofibrations.

1.9.3. In applications of the approximation theorem, most of the effort involved is expended in verifying condition 1.9.1.2. The following lemmas 1.9.4 and 1.9.5 are useful tools for this, and also for some other purposes. Lemma 1.9.5 serves to build bounded above complexes quasi-isomorphic to a given complex, and lemma 1.9.4 serves to truncate them to strict bounded complexes.

1.9.4. **Lemma.** *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Suppose that $\mathcal{B}$ is closed under extensions in $\mathcal{A}$, and is closed under kernels of epimorphisms. (More precisely, $\mathcal{B}$ is to be closed under taking kernels of maps in $\mathcal{B}$ that are epimorphisms in $\mathcal{A}$.) Let $C^\cdot$ be a strictly bounded above complex in $\mathcal{B} \subseteq \mathcal{A}$. Then:*

1.9.4(a). *If for an integer n, one has $H^k(C^\cdot) = 0$ for $k \neq n$ and $C^k = 0$ for $k < n$, then $H^n(C^\cdot) = Z^n C^\cdot$ is an object of $\mathcal{B}$.*

1.9.4(b). *If for an integer n, one has $H^k(C^\cdot) = 0$ ıor $k > n$, then $Z^n C^\cdot$ is an object of $\mathcal{B}$, and the complex $C^\cdot$ is quasi-isomorphic to the subcomplex $\tau^{\leq n} C^\cdot$, which is the complex in $\mathcal{B}$*

$$\cdots \to C^{n-2} \to C^{n-1} \to Z^n C \to 0 \to 0 \to \cdots .$$

1.9.4(c). *If for an integer n, one has $H^k(C^\cdot) = 0$ for $k \neq n$, then $H^n(C^\cdot)$ has a resolution by objects of $\mathcal{B}$*

$$\cdots \to C^{n-2} \to C^{n-1} \to Z^n C \to H^n(C^\cdot) \to 0.$$

If $C^\cdot$ is also strictly bounded below, this resolution has finite length.

1.9.4(d). *If $C^\cdot$ is an acyclic complex, so $H^k(C^\cdot) = 0$ for all k, then all $Z^k C^\cdot = B^k C^\cdot$ are objects of $\mathcal{B}$, and $C^\cdot$ has a natural filtration by acyclic complexes in $\mathcal{B}$, $F_n C^\cdot = \tau^{\leq n} C^\cdot$, so that $F_{n+1} C^\cdot / F_n C^\cdot$ is isomorphic to the acyclic complex*

$$\cdots \to 0 \to 0 \to B^{n+1} C^\cdot \cong B^{n+1} C^\cdot \to 0 \to 0 \to \cdots .$$

Proof. As $C^\cdot$ is strict bounded above, there is an integer N such that $C^p = 0$ for $p > N$. If $N \leq n$, $Z^n C^\cdot = C^n$ and 1.9.4(b) is obvious. One proceeds to prove 1.9.4(b) by induction on $N - n$. If it is already known to be true for smaller values of $N - n$, and $N - n \geq 1$, consider $C^\cdot = \cdots \to C^{N-1} \to C^N \to 0 \to \cdots$. As $N > n$, $H^N(C^\cdot) = 0$ and $C^{N-1} \to C^N$ is an epimorphism in $\mathcal{A}$ of objects in $\mathcal{B}$. AS $\mathcal{B}$ is closed under taking kernels of such epimorphisms, the kernel $Z^{N-1} C^\cdot$ is in $\mathcal{B}$. Then $C^\cdot$ is quasi-isomorphic to the shorter subcomplex

$$C^{\cdot\prime} = \tau^{\leq N-1} C^\cdot = \cdots \to C^{N-2} \to Z^{N-1} C^\cdot \to 0 \to 0 \to \cdots$$

which is also a complex in $\mathcal{B}$. Clearly $Z^k C^\cdot = Z^k C^{\cdot\prime}$ for all $k \leq N - 1$, and as $C^{\cdot\prime}$ is shorter we get $Z^k C^{\cdot\prime}$ is in $\mathcal{B}$ for all $k \geq n$ by the induction hypotheses. Thus $Z^k C^\cdot$ is in $\mathcal{B}$ for $n \leq k \leq N - 1$. We have that $Z^N C^\cdot = C^N$ is in $\mathcal{B}$, and clearly $Z^k C^\cdot = 0$ is in $\mathcal{B}$ for $k > N$. Thus $Z^k C^\cdot$ is in $\mathcal{B}$ for all $k \geq n$, completing the induction step, and hence the proof of 1.9.4(b). Now 1.9.4(a) and 1.9.4(c) are immediate corollaries and 1.9.4(d) is a porism (i.e., follows from the proof of 1.9.4(b)).

1.9.5. **Lemma** (cf. [SGA 6] I 1.4). *Let $\mathcal{A}$ be an abelian category, let $\mathcal{D}$ be an additive category, and let $F : \mathcal{D} \to \mathcal{A}$ be an additive functor. Let $\mathcal{C}$ be a full subcategory of the category $C(\mathcal{A})$ of chain complexes in $\mathcal{A}$, such that any complex quasi-isomorphic to a complex in $\mathcal{C}$ is also in $\mathcal{C}$. Suppose that every complex in $\mathcal{C}$ is cohomologically bounded above. Suppose that if $D^{\cdot}$ is any strict bounded complex in $\mathcal{D}$, the $F(D^{\cdot})$ is in $\mathcal{C}$, and that $\mathcal{C}$ contains the mapping cone of any map of complexes $F(D^{\cdot}) \to C^{\cdot}$ with $C^{\cdot}$ in $\mathcal{C}$ and $D^{\cdot}$ strict bounded in $\mathcal{D}$.*

Suppose the key condition 1.9.5.1 holds, so "$\mathcal{D}$ has enough objects to resolve":

1.9.5.1. For any integer n, any $C^{\cdot}$ in $\mathcal{C}$ such that $H^i(C^{\cdot}) = 0$ for $i \geq n$, and any epimorphism in $\mathcal{A}$, $A \twoheadrightarrow H^{n-1}(C^{\cdot})$, then there exists a D in $\mathcal{D}$ and a map $FD \to A$ such that the composite $FD \twoheadrightarrow H^{n-1}(C^{\cdot})$ is an epimorphism in $\mathcal{A}$.

Suppose finally that the functor $F : \mathcal{D} \to \mathcal{A}$ satisfies the following two conditions (which trivially hold in the usual case where F is a fully faithful inclusion):

1.9.5.2. Given a morphism $f : FD_2 \to FD_1$ in $\mathcal{A}$, there is a map $d : D_3 \to D_2$ in $\mathcal{D}$ such that $Fd : FD_3 \to FD_2$ is an epimorphism in $\mathcal{A}$ and such that $f \circ Fd = Ff'$ for some map $f' : D_3 \to D_1$ in $\mathcal{D}$.

1.9.5.3. Given $h : D_2 \to D_1$ in $\mathcal{D}$ with $Fh = 0$, there is a map $d : D_3 \to D_2$ in $\mathcal{D}$ with $Fd : FD_3 \to FD_2$ an epimorphism in $\mathcal{A}$ and $hd = 0$ in $\mathcal{D}$.

Then: For any $D^{\cdot}$ in $C^-(\mathcal{D})$ with $F(D^{\cdot})$ in $\mathcal{C}$, any $C^{\cdot}$ in $\mathcal{C}$, and any map $x : FD^{\cdot} \to C^{\cdot}$, there exists a D' in $C^-(\mathcal{D})$ with $F(D'^{\cdot})$ in $\mathcal{C}$, a degree-wise split monomorphism $a : D^{\cdot} \to D'^{\cdot}$, and a quasi- isomorphism $x' : FD'^{\cdot} \xrightarrow{\sim} C^{\cdot}$ such that $x = x' \circ Fa$.

Moreover, if $x : FD^{\cdot} \to C^{\cdot}$ is an n-quasi-isomorphism for some integer n (i.e., $H^i(x)$ is iso for $i > n$ and epi for $i = n$), then one may choose $D'^{\cdot}$ above so that $a : D^k \to D'^k$ is an isomorphism for $k \geq n$.

Proof. We construct $D'^{\cdot}$ by induction, constructing what will become the brutal truncation $\sigma^n D'^{\cdot}$ given $\sigma^{n+1} D'^{\cdot}$ (recall σ from 1.1.3). To start, take n large enough so that $H^i(x)$ is an isomorphism for $i > n$ and an epimorphism for $i = n$. This is possible because $FD^{\cdot}$ and $C^{\cdot}$ are cohomologically bounded above, so $H^i(FD^{\cdot}) = 0 = H^i(C^{\cdot})$ for $i >> 0$. For this large n, we set $\sigma^n D'^{\cdot} = \sigma^n D^{\cdot}$ to begin the induction.

Now assume as induction hypothesis that for some n we have $\sigma^n D'^{\cdot}$ and maps $\sigma^n a$, $\sigma^n x'$ so that

1.9.5.4.1. $\sigma^n D'^{\cdot}$ is bounded complex in $\mathcal{D}$, and $\sigma^n(a) : \sigma^n D^{\cdot} \to \sigma^n D'^{\cdot}$ is a degree-wise split monomorphism. (Moreover, we need to assume that $(\sigma^n D'^{\cdot})^k = 0$ for $k < n$, so that $\sigma^n D'^{\cdot}$ could be the truncation σ^n of some complex.)

1.9.5.4.2. $\sigma^n x = \sigma^n x' \cdot F(\sigma^n a)$.

1.9.5.4.3. $\sigma^n x'$ is an n-quasi-isomorphism.

The induction step consists of defining a complex $\sigma^{n-1}D'^{\cdot}$ in $\mathcal{D}$ whose σ^n is indeed the given $\sigma^n D'^{\cdot}$, and maps $\sigma^{n-1}x'$, $\sigma^{n-1}a$ which restrict to the given $\sigma^n x'$, $\sigma^n a$, and which satisfy 1.9.5.4.1 - 1.9.5.4.3 with n replaced by $n-1$.

Let $M^{\cdot}$ be the mapping cone of $\sigma^n x' : F(\sigma^n D'^{\cdot}) \to C^{\cdot}$

$$(1.9.5.6) \quad M^{\cdot} = \cdots \to C^{n-3} \to C^{n-2} \to FD'^n \oplus C^{n-1} \to FD'^{n+1} \oplus C^n \to \ldots .$$

Then $M^{\cdot}$ is in $\mathcal{C}$. The long exact cohomology sequence for the mapping cone and the fact that $\sigma^n x'$ is an n-quasi-isomorphism yield that $H^i(M^{\cdot}) = 0$ for $i \geq n$. By the key hypothesis 1.9.5.1, there is a $D^{\wedge}$ in $\mathcal{D}$ and a map $FD^{\wedge} \to Z^{n-1}M^{\cdot} \twoheadrightarrow H^{n-1}(M^{\cdot})$ so that $FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ is an epimorphism. As $M^{n-1} = FD'^n \oplus C^{n-1}$, and (1.1.2.2) shows that the differential is given by $\partial(d,c) = (-\partial d,\ \partial c - (\sigma^n x')(d))$ in $M^n = FD'^{n+1} \oplus C^n$, one easily checks that $Z^{n-1}M^{\cdot} = \ker \partial$ is the fibre product of $\partial : C^{n-1} \to Z^n C$ and $\sigma^n x' : Z^n F \sigma^n D'^{\cdot} \to Z^n C^{\cdot}$

$$(1.9.5.7) \qquad Z^{n-1}M^{\cdot} = Z^n F\sigma^n D'^{\cdot} \underset{Z^n C^{\cdot}}{\times} C^{n-1} \subseteq FD'^n \oplus C^{n-1}.$$

To simplify notation we write $Z^n FD'^{\cdot}$ for $Z^n F\sigma^n D'^{\cdot}$, as if the rest of $D'^{\cdot}$ already existed. Consider now the composite map induced by the canonical projection and inclusion maps

$$FD^{\wedge} \to Z^{n-1}M^{\cdot} \cong Z^n FD'^{\cdot} \underset{Z^n C^{\cdot}}{\times} C^{n-1} \to Z^n FD'^{\cdot} \to FD'^n.$$

By 1.9.5.2, there is a map $D^{\sim} \to D^{\wedge}$ with $FD^{\sim} \twoheadrightarrow FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ an epimorphism and with the map $FD^{\sim} \to FD^{\wedge} \to FD'^n$ being F of a map $D^{\sim} \to D'^n$. Replacing the old $D^{\wedge}$ by $D^{\sim}$, we may assume that $FD^{\wedge} \to FD'^n$ is F of a map $D^{\wedge} \to D'^n$. As $FD^{\wedge} \to FD'^n$ factors through $Z^n FD'^{\cdot}$, the composite $FD^{\wedge} \to FD'^n \to FD'^{n+1}$ is 0. By 1.9.5.3 there is a $D^{\sim} \to D^{\wedge}$ in $\mathcal{D}$ so that $FD^{\sim} \twoheadrightarrow FD^{\wedge} \twoheadrightarrow H^{n-1}(M^{\cdot})$ is still an epimorphism, and such that the composite $D^{\sim} \to D^{\wedge} \to D'^n \to D'^{n+1}$ is 0. Replacing the old $D^{\wedge}$ with this $D^{\sim}$, we may assume that $D^{\wedge} \to D'^n$ factors through $Z^n D'^{\cdot}$ $(= Z^n \sigma^n D'^{\cdot})$.

Now set D'^{n-1} to be $D^{n-1} \oplus D^{\wedge}$. Define $\partial : D'^{n-1} \to D'^n$ to be $a \circ \partial : D^{n-1} \to D^n \to D'^n$ on the summand D^{n-1}, and to be $D^{\wedge} \to D'^n$ on the summand $D^{\wedge}$. As $D^{\wedge} \to D'^n$ factors through $Z^n D'^{\cdot}$, and as $\partial a \partial = a\partial^2 = 0$ on D^{n-1}, we see that the composite $\partial^2 : D'^{n-1} \to D'^n \to$

D'^{n+1} is 0. Now let $\sigma^{n-1}D'^{\cdot}$ be the chain complex formed from $\sigma^n D'^{\cdot}$ by replacing the old 0 in degree $n-1$ by D'^{n-1}, and with $\partial : D'^{n-1} \to D'^n$ as the new boundary operator here. Let the map $\sigma^{n-1}(a)$ agree with $\sigma^n(a)$ in degrees above $n-1$, and to be the inclusion of the summand $D^{n-1} \to D^{n-1} \oplus D^{\wedge} = D'^{n-1}$ in degree $n-1$. Let the map $\sigma^{n-1}(x')$ agree with $\sigma^n(x')$ in degrees above $n-1$, and to be given in degree $n-1$ on $FD'^{n-1} \cong FD^{n-1} \oplus FD^{\wedge}$ as $x^{n-1} : FD^{n-1} \to C^{n-1}$ on the summand FD^{n-1} and on the summand $FD^{\wedge}$ as the composite of the map $FD^{\wedge} \to Z^{n-1}M^{\cdot}$ and the projection $Z^{n-1}M^{\cdot} \to C^{n-1}$ determined by the isomorphism 1.9.5.7 of $Z^{n-1}M^{\cdot}$ with a fibre product. It is easy to verify that $\sigma^{n-1}(x')$ and $\sigma^{n-1}(a)$ are chain maps and that 1.9.5.4.1 and 1.9.5.4.2 hold with $n-1$ in place of n.

To verify condition 1.9.5.4.3 that $\sigma^{n-1}(x')$ is an $(n-1)$-quasi-isomorphism, consider its mapping cone $\widetilde{M}^{\cdot}$. This is the mapping cone $M^{\cdot}$ of $\sigma^n(x')$ with an additional term $FD^{n-1} \oplus FD^{\wedge} = F(D'^{n-1})$ added:

(1.9.5.8)
$$\widetilde{M}^{\cdot} = \cdots \to C^{n-3} \to FD^{n-1} \oplus FD^{\wedge} \oplus C^{n-2} \to FD'^n \oplus C^{n-1} \to \cdots$$

By construction the boundary map from the summand $FD^{\wedge}$ in $\widetilde{M}^{n-2}$ maps onto $H^{n-1}(M^{\cdot}) = Z^{n-1}(M^{\cdot})/B^{n-1}(M^{\cdot})$. Hence $FD^{\wedge} \oplus B^{n-1}(M^{\cdot})$ and *a fortiori* $FD^{\wedge} \oplus M^{n-2} = FD^{\wedge} \oplus C^{n-2}$ and $FD^{n-1} \oplus FD^{\wedge} \oplus C^{n-2}$ map onto $Z^{n-1}M^{\cdot} = Z^{n-1}\widetilde{M}^{\cdot}$. Then $H^{n-1}(\widetilde{M}^{\cdot}) = 0$, as well as $H^i(\widetilde{M}^{\cdot}) = H^i(M^{\cdot}) = 0$ for $i \geq n$. Now the long exact cohomology sequence for the mapping cone $\widetilde{M}^{\cdot}$ of $\sigma^{n-1}(x')$ shows that $\sigma^{n-1}(x')$ is an $(n-1)$-quasi-isomorphism as required. This completes the induction step.

Now given the inductively constructed $\sigma^n D'^{\cdot}$ for all n, we set $D'^{\cdot} = \varinjlim \sigma^n D'^{\cdot}$ as $n \to -\infty$. As $\sigma^n D'^{\cdot}$ and $\sigma^{n-1}D'^{\cdot}$ agree in degrees above $n-1$, we have $D'^k = (\sigma^n D')^k$ for any $n \leq k$. We define a, x' similarly. It is then clear from 1.9.5.4 that $D'^{\cdot}$, a, and x' meet the requirements, completing the proof of Lemma 1.9.5.

1.9.5.9. *Porism.* If 1.9.5.1 holds only for those $n \geq N+1$ for some fixed N, the proof still constructs a $\sigma^N D'^{\cdot}$ in $C^b(\mathcal{D})$, and N-quasi-isomorphism $\sigma^N(x') : F(\sigma^N D'^{\cdot}) \to C^{\cdot}$, and a degree-wise split monomorphism $\sigma^N(a) : \sigma^N D^{\cdot} \to \sigma^N(D'^{\cdot})$ such that $\sigma^N(x) = \sigma^N(x') \circ \sigma^N(a)$.

If $\mathcal{A}$ has two additive full subcategories $\mathcal{D}_1 \subseteq \mathcal{D}_2$ so that the inclusion $\mathcal{D}_1 \to \mathcal{A}$ satisfies 1.9.5.1 for $n \geq N+1$ and $\mathcal{D}_2 \to \mathcal{A}$ satisfies 1.9.5.1 for all n, then the proof constructs a quasi-isomorphism $x' : D'^{\cdot} \to C^{\cdot}$ with D'^k in $\mathcal{D}_1$ for $k \geq N$ and D'^k in $\mathcal{D}_2$ for all k. Moreover, if $x : D^{\cdot} \to C^{\cdot}$ is given with $D^{\cdot} \in C^-(\mathcal{D}_2)$ and with D^k in $\mathcal{D}_1$ for $k \geq N$, then we may construct the quasi-isomorphism $x' : D'^{\cdot} \xrightarrow{\sim} C^{\cdot}$ so that there is a degree-wise split monomorphism $a : D^{\cdot} \to D'^{\cdot}$ with $x' = x \circ a$.

These variants of Lemma 1.9.5 will come to seem less bizarre in Section 2.2 below.

1.9.6. Let $\boldsymbol{A}$ be a complicial biWaldhausen category with associated abelian category $\mathcal{A}$. So $\boldsymbol{A}$ is a full subcategory of the category of chain complexes $\mathcal{C}(\mathcal{A})$. Suppose $\boldsymbol{A}$ is closed under the formation of canonical homotopy pushouts and canonical homotopy pullbacks in $\mathcal{C}(\mathcal{A})$ 1.1.2, as is often the case 1.3.6. Then the "derived" or homotopy category of $\boldsymbol{A}$ will be a "triangulated category," and there will be a "calculus of fractions." We recall parts of this theory of Grothendieck and Verdier in our context (cf. [V] or [H] I).

Let $\boldsymbol{A}/\simeq$ be the quotient category where two maps of $\boldsymbol{A}$ are identified to each other if they are chain homotopic as maps of complexes. The category $\boldsymbol{A}/\simeq$ is a full subcategory of the chain homotopy category $\mathcal{K}(\mathcal{A}) = \mathcal{C}(\mathcal{A})/\simeq$. If one defines the distinguished triangles in $\boldsymbol{A}/\simeq$ to be those chain homotopy equivalent to those coming in the usual way from mapping cone sequences in $\boldsymbol{A}$, then $\boldsymbol{A}/\simeq$ satisfies the axioms for a triangulated category ([V] I Section 1, nos. 1-2, or [H] I Section 1), and is a subtriangulated category of $\mathcal{K}(\mathcal{A})$ with its usual mapping cone sequence triangulated structure ([V] I Section 1 no. 2, or [H] I Section 2).

Let $\boldsymbol{w}$ denote the image of $\boldsymbol{w}(A)$ in $\boldsymbol{A}/\simeq$. As the complicial $\boldsymbol{A}$ is saturated and extensional, it is easy to see that $\boldsymbol{w}$ is a saturated multiplicative system in $\boldsymbol{A}/\simeq$ in the sense of ([V] I Section 2, nos. 1-2, or [H] I Section 3). The corresponding thick triangulated subcategory of $\boldsymbol{A}/\simeq$ is $\boldsymbol{A}^w/\simeq$, the full subcategory of objects A such that the unique map $0 \to A$ is in $\boldsymbol{w}$.

Let $\mathrm{Ho}(\boldsymbol{A}) = \boldsymbol{w}^{-1}\boldsymbol{A}$ be the "derived" or homotopy category formed from $\boldsymbol{A}$ by localizing the category $\boldsymbol{A}$ to make the maps in $\boldsymbol{w}(\boldsymbol{A})$ isomorphisms in $\boldsymbol{w}^{-1}\boldsymbol{A}$. As chain homotopy equivalences are quasi-isomorphisms, hence are in $\boldsymbol{w}(\boldsymbol{A})$, they become isomorphisms in $\boldsymbol{w}^{-1}\boldsymbol{A}$. In particular, in $\boldsymbol{w}^{-1}\boldsymbol{A}$ a complex A becomes isomorphic to the complex "$A \times I$" $= A \overset{h}{\underset{A}{\cup}} A$ that parameterizes chain homotopies of maps out of A. Thus $\boldsymbol{w}^{-1}\boldsymbol{A}$ is also the localization $\boldsymbol{w}^{-1}\boldsymbol{A}/\simeq$ of $\boldsymbol{A}/\simeq$ at $\boldsymbol{w}$. The work in ([V] I Section 2 or [H] I Sections 3-4) shows that $\boldsymbol{w}$ is exactly the set of all morphisms in $\boldsymbol{A}/\simeq$ (or in $\boldsymbol{A}$) that become isomorphisms in $\boldsymbol{w}^{-1}\boldsymbol{A} = \mathrm{Ho}(\boldsymbol{A})$. Also $\boldsymbol{w}^{-1}\boldsymbol{A}$ has an induced triangulated structure. (Not only mapping cone sequences, but also general cofibration sequences turn out to yield distinguished triangles in $\boldsymbol{w}^{-1}\boldsymbol{A}$, as in ([V] II Section 1, nos. 1-5, or [H] I 6.1).)

Furthermore, the passage from $\boldsymbol{A}/\simeq$ to $\boldsymbol{w}^{-1}\boldsymbol{A} = \boldsymbol{w}^{-1}\boldsymbol{A}/\simeq$ admits a "calculus of fractions"([H] I Section 3 or [V] I Section 2 no. 3). In particular, the morphisms in $\boldsymbol{w}^{-1}\boldsymbol{A}$ from A to A' correspond to the equivalence classes of data (1.9.6.1) in $\boldsymbol{A}/\simeq$,

$$A \to A'' \xleftarrow{\sim} A' \tag{1.9.6.1}$$

where $A \to A''$ is a morphism in $\boldsymbol{A}/\simeq$, and $A' \xrightarrow{\sim} A''$ is a map in $\boldsymbol{w}$ in $\boldsymbol{A}/\simeq$. Two such data $A \to A_1'' \xleftarrow{\sim} A'$ and $A \to A_2'' \xleftarrow{\sim} A'$ are equivalent if there exists a commutative diagram (1.9.6.2) in $\boldsymbol{A}/\simeq$

$$\begin{tikzcd}
 & A_1'' \arrow[d, "\sim"] & \\
A \arrow[ur] \arrow[r] \arrow[dr] & A_3'' & A' \arrow[ul, "\sim"'] \arrow[l, "\sim"'] \arrow[dl, "\sim"] \\
 & A_2'' \arrow[u, "\sim"'] &
\end{tikzcd} \tag{1.9.6.2}$$

The calculus of fractions insures that this is an equivalence relation, and yields the composition of morphisms represented as data by constructing the C'' and the bottom arrows in (1.9.6.3). Here C'' is the canonical homotopy pushout in $\boldsymbol{A}$ of a choice of maps in $\boldsymbol{A}$ to represent the chain homotopy classes of maps $B \xrightarrow{\sim} A''$ and $B \to B''$ in $\boldsymbol{A}/\simeq$, and the bottom arrows are the canonical maps into the homotopy pushout. (It is easy to check that $B'' \xrightarrow{\sim} C''$ is a weak equivalence, cf. the construction of 1.9.8.4 below.)

$$\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl, "\sim"'] \arrow[dr] & & C \arrow[dl, "\sim"'] \\
 & A'' \arrow[dr] & & B'' \arrow[dl, "\sim"'] & \\
 & & C'' & &
\end{tikzcd} \tag{1.9.6.3}$$

Dually, morphisms in $\boldsymbol{w}^{-1}\boldsymbol{A}$ from A to A' may be represented by equivalence classes of data $A \xleftarrow{\sim} A'' \to A'$, with equivalence relation

$$\begin{tikzcd}
 & A_1'' \arrow[dl, "\sim"'] \arrow[dr] & \\
A & A_3'' \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] \arrow[d, "\sim"] & A' \\
 & A_2'' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd} \tag{1.9.6.4}$$

and with composition coming from homotopy pullbacks.

When there is danger of confusion as to whether a map $A \to A'$ is to be a map in $\boldsymbol{A}$ or in $\mathrm{Ho}(\boldsymbol{A}) = \boldsymbol{w}^{-1}\boldsymbol{A}$, we refer to morphisms in $\boldsymbol{A}$ as strict maps.

1.9.7. Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a complicial exact functor between biWaldhausen categories (1.2.16). As F is induced by degree-wise application to complexes of an additive functor $f : \mathcal{A} \to \mathcal{B}$ of associated abelian categories, the functor F preserves canonical homotopy pushouts and canonical homotopy pullbacks. In particular, F preserves mapping cones and mapping cylinders. Thus if $\boldsymbol{A}$ and $\boldsymbol{B}$ are closed under the formation of homotopy pushouts and homotopy pullbacks, $F : \boldsymbol{A}/\simeq \to \boldsymbol{B}/\simeq$ and $w^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ are triangulated functors.

From the calculus of fractions we see that $\boldsymbol{w}^{-1}F : w^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ is an equivalence of homotopy categories if $F : \boldsymbol{A} \to \boldsymbol{B}$ satisfies the following four conditions:

1.9.7.0. For any map a in $\boldsymbol{A}$, a is in $\boldsymbol{w}(\boldsymbol{A})$ if and only if Fa is in $\boldsymbol{w}(\boldsymbol{B})$.

1.9.7.1. For any B in $\boldsymbol{B}$, there is an A in $\boldsymbol{A}$ and a map $FA \xrightarrow{\sim} B$ in $\boldsymbol{w}(\boldsymbol{B})$.

1.9.7.2. For any map $b : FA' \to FA''$ in $\boldsymbol{B}$, there are maps $a' : A \xrightarrow{\sim} A'$ in $\boldsymbol{w}(\boldsymbol{A})$ and $a'' : A \to A''$ in $\boldsymbol{A}$, such that there is a chain homotopy $b \circ Fa' \simeq Fa''$ in $\boldsymbol{B}$.

1.9.7.3. For any map $a' : A' \to A''$ in $\boldsymbol{A}$ such that $Fa' \simeq 0$ is chain nullhomotopic in $\boldsymbol{B}$, there is a map $a : A \xrightarrow{\sim} A'$ in $\boldsymbol{w}(\boldsymbol{A})$ such that $a' \cdot a \simeq 0$ is chain nullhomotopic in $\boldsymbol{A}$.

The last two conditions (1.9.7.2 and 1.9.7.3) trivially hold whenever $F : \boldsymbol{A} \to \boldsymbol{B}$ is fully faithful. Note then that F is also full and faithful for chain homotopies between maps, as a chain homotopy between maps $C \to D$ corresponds to a chain map "$C \times I$" $= C \overset{h}{\underset{C}{\cup}} C \to D$.

1.9.8. **Theorem**. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two complicial biWaldhausen categories, each of which is closed under the formation of canonical homotopy pushouts and canonical homotopy pullbacks* (1.9.6, 1.2.11). *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a complicial exact functor* (1.2.16). *Suppose that F induces an equivalence of the derived homotopy categories $\boldsymbol{w}^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$. Then F induces a homotopy equivalence of K-theory spectra*

$$K(F) : K(\boldsymbol{A}) \to K(\boldsymbol{B}).$$

Proof. By 1.9.2, we reduce to the case where cofibrations in $\boldsymbol{A}$ and $\boldsymbol{B}$ are precisely the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ have cylinder and cocylinder functors satisfying the cylinder and cocylinder axioms, thanks to 1.3.5.

Let $\boldsymbol{C}$ be the category whose objects are data $(A,\ FA \xrightarrow{\sim} B)$ where A is an object of $\boldsymbol{A}$, B is an object of $\boldsymbol{B}$, and $FA \xrightarrow{\sim} B$ is a map in $\boldsymbol{w}(\boldsymbol{B})$.

A map in $\boldsymbol{C}$ from $(A,\ FA \xrightarrow{\sim} B)$ to $(A',\ FA' \xrightarrow{\sim} B')$ consists of a map $A \to A'$ in $\boldsymbol{A}$ and a map $B \to B'$ in $\boldsymbol{B}$ such that

$$\begin{tikzcd} FA \arrow[r, "\sim"] \arrow[d] & B \arrow[d] \\ FA' \arrow[r, "\sim"] & B' \end{tikzcd}$$

commutes. Call a map in $\boldsymbol{C}$ a cofibration (or respectively, weak equivalence) if both maps $A \to A'$ in $\boldsymbol{A}$ and $B \to B'$ in $\boldsymbol{B}$ are cofibrations (resp. weak equivalences). This makes $\boldsymbol{C}$ a biWaldhausen category. (Indeed, it is even a complicial biWaldhausen category, with associated abelian category of data $(A,\ fA \to B)$ with A in $\mathcal{A}$, B in $\mathcal{B}$, and $fA \to B$ a map in $\mathcal{B}$.) As in 1.9.7, $F : \boldsymbol{A} \to \boldsymbol{B}$ preserves the mapping cylinders and cocylinders. Thus $\boldsymbol{C}$ has a cylinder functor induced by the cylinder functors of $\boldsymbol{A}$ and $\boldsymbol{B}$. Dually $\boldsymbol{C}$ has a cocylinder functor. These satisfy the cylinder and cocylinder axiom 1.3.1.7.

The functor $F : \boldsymbol{A} \to \boldsymbol{B}$ factors as the composite of exact functors $\boldsymbol{A} \to \boldsymbol{C}$ and $\boldsymbol{C} \to \boldsymbol{B}$. Here $\boldsymbol{A} \to \boldsymbol{C}$ sends A to $(A,\ FA = FA)$, and $\boldsymbol{C} \to \boldsymbol{B}$ sends $(A,\ FA \xrightarrow{\sim} B)$ to B. We will show both these functors induce homotopy equivalences on K-theory spectra.

The functor $\boldsymbol{A} \to \boldsymbol{C}$ is split by an exact functor $\boldsymbol{C} \to \boldsymbol{A}$ sending $(A,\ FA \xrightarrow{\sim} B)$ to A. So $\boldsymbol{A} \to \boldsymbol{C} \to \boldsymbol{A}$ is the identity functor. The composite $\boldsymbol{C} \to \boldsymbol{A} \to \boldsymbol{C}$ is naturally weak equivalent to the identity by the natural transformation $(A,\ FA = FA) \to (A,\ FA \xrightarrow{\sim} B)$ induced by $A = A$ and $FA \xrightarrow{\sim} B$. Thus $K(\boldsymbol{A}) \to K(\boldsymbol{C})$ and $K(\boldsymbol{C}) \to K(\boldsymbol{A})$ are inverse homotopy equivalences by 1.5.4.

Consider now the exact functor $\boldsymbol{C} \to \boldsymbol{B}$ of biWaldhausen categories. We claim it induces a homotopy equivalence $K(\boldsymbol{C}) \xrightarrow{\sim} K(\boldsymbol{B})$ by the dual to the approximation theorem 1.9.1.

First we note that the dual hypothesis to 1.9.1.1 holds. For suppose $(a,b) : (A,\ FA \xrightarrow{\sim} B) \to (A',\ FA' \xrightarrow{\sim} B')$ is a map in $\boldsymbol{C}$, whose image $b : B \xrightarrow{\sim} B'$ in $\boldsymbol{C}$ is a weak equivalence. Then by saturation, $Fa : FA \xrightarrow{\sim} FA'$ is also a weak equivalence. Hence Fa becomes an isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{B}$. As by hypothesis, $\boldsymbol{w}^{-1}F : \boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ is an equivalence of categories, the map a becomes an isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{A}$. Hence as in 1.9.6, the map a is in $\boldsymbol{w}(\boldsymbol{A})$. Thus both maps a and b are weak equivalences, so (a,b) is a weak equivalence in $\boldsymbol{C}$, as required by 1.9.1.1.

To verify the dual hypothesis to 1.9.1.2, we must show that given a diagram (1.9.8.1) corresponding to a map $x : B' \to (B \text{ of } (FA \to B))$, this can be completed to a commutative diagram (1.9.8.2) corresponding to the factorization required by 1.9.1.2

$$\begin{tikzcd}
FA \arrow[r, "\sim"] & B \\
 & B' \arrow[u]
\end{tikzcd} \tag{1.9.8.1}$$

$$\begin{tikzcd}
FA \arrow[r, "\sim"] & B \\
FA'' \arrow[u, "Fa''"] \arrow[r, "\sim"] & B'' \arrow[u] \\
 & B' \arrow[u] \arrow[uu, bend right=60, "b'"']
\end{tikzcd} \tag{1.9.8.2}$$

As a first approximation, we construct a diagram like (1.9.8.2) in $\boldsymbol{B}/\simeq$, i.e., a version that chain homotopy commutes. We begin by noting that as $\boldsymbol{w}^{-1}F$ is an equivalence of homotopy categories, B' is isomorphic in $\boldsymbol{w}^{-1}\boldsymbol{B}$ to some FA_1. By calculus of fractions, this isomorphism corresponds to a datum in $\boldsymbol{B}$: $FA_1 \xrightarrow{\sim} B_1 \xleftarrow{\sim} B'$. Composing this isomorphism in $\boldsymbol{w}^{-1}\boldsymbol{B}$ with the map $b : B' \to B$ and with the inverse isomorphism to $FA \xrightarrow{\sim} B$ yields a map in $\boldsymbol{w}^{-1}\boldsymbol{B}$ from FA_1 to FA. As $\boldsymbol{w}^{-1}F$ is an equivalence of categories, this map is $\boldsymbol{w}^{-1}F$ of some map from A_1 to A in $\boldsymbol{w}^{-1}\boldsymbol{A}$. This map from A_1 to A is represented by a datum $A_1 \to A_2 \xleftarrow{\sim} A$ in $\boldsymbol{A}$. Applying the formulae of the calculus of fractions for composition and equivalence of data representing maps in $\boldsymbol{w}^{-1}\boldsymbol{B}$, as given in 1.9.6, we deduce that there exists in $\boldsymbol{B}/\simeq$ a commutative diagram, which after removal of intermediate constructions becomes:

$$\begin{tikzcd}
FA \arrow[r, "\sim"] \arrow[d, "F(\ )"', "\sim"] & B \arrow[d, "\sim"] \\
FA_2 \arrow[r, "\sim"] & B_2 \\
FA_1 \arrow[u, "F(\ )"] \arrow[r, "\sim"] & B_1 \arrow[u] \\
 & & B' \arrow[ul, "\sim"'] \arrow[uuul, bend right=40, "b'"']
\end{tikzcd} \tag{1.9.8.3}$$

We choose representatives of these maps in $\boldsymbol{A}$ and $\boldsymbol{B}$, so that (1.9.8.3) becomes a chain homotopy commutative diagram in $\boldsymbol{B}$, where the indicated maps are F of maps in $\boldsymbol{A}$.

Now let A_3 be the canonical homotopy pullback of $A_1 \to A_2$ and $A \xrightarrow{\sim} A_2$, and let B_3 be the canonical homotopy pullback of $B_1 \to B_2$ and $B \xrightarrow{\sim} B_2$, as in 1.1.2. The FA_3 is the canonical homotopy pullback of $FA_1 \to FA_2$ and $FA \to FA_2$, as in 1.9.7.

By 1.1.2, the chain homotopy commutative diagram (1.9.8.3), after a choice of chain homotopies, induces a map of homotopy pullbacks $FA_3 \to B_3$, which is an edge of a homotopy commutative cube with vertices FA_3, FA_2, FA_1, FA, B_3, B_2, B_1, and B. The map $FA_3 \to B_3$ is a weak equivalence. To see this, we first note that the projection map $B_3 \to B$ from the homotopy pullback is the pullback along $B \to B_2$ of the canonical map

$$k_1 : B_2 \overset{h}{\underset{B_2}{\times}} B_1 = \text{ cocylinder } (B_1 \to B_2) \twoheadrightarrow B_2.$$

As k_1 is a degree-wise split epimorphism, and even a map in quot($\boldsymbol{B}$), it follows that the pullback $B_3 \twoheadrightarrow B$ is in quot($\boldsymbol{B}$). Similarly $FA_3 \twoheadrightarrow FA$ is in quot($\boldsymbol{B}$). Now as $FA_3 \to B_3$ is induced by the horizontal arrows in (1.9.8.3), each of which is a weak equivalence, it follows from the dual of the gluing lemma axiom 1.2.3.2 that $FA_3 \xrightarrow{\sim} B_3$ is a weak equivalence.

Also, as $B \to B_2$ is a weak equivalence, the extension axiom 1.2.6 shows that the induced map

$$B_3 \xrightarrow{\sim} B_2 \overset{h}{\underset{B_2}{\times}} B_1 = \text{ cocylinder } (B_1 \to B_1)$$

is a weak equivalence. As the projection of the cocylinder onto B_1 is even a quasi-isomorphism, it follows that the canonical map $B_3 \xrightarrow{\sim} B_1$ is a weak equivalence.

The homotopy commutative right half of (1.9.8.3), together with a choice of homotopy, determines a map $B' \to B_3$ by the universal mapping property of homotopy pullbacks, dual to the mapping property of homotopy pushouts explained in 1.1.2. As $B_3 \to B_1$ and $b' : B' \to B_1$ are weak equivalences, the saturation axiom implies that $B' \xrightarrow{\sim} B_3$ is a weak equivalence. Thus we have constructed a homotopy commutative diagram (1.9.8.4), the desired first approximation to (1.9.8.2).

(1.9.8.4)
$$\begin{tikzcd}
FA \arrow[r, "\sim"] & B \\
FA_3 \arrow[u, "F(\ )"] \arrow[r, "\sim"] & B_3 \arrow[u] \\
& & B' \arrow[ul, hook'] \arrow[uul, bend right, "b'"']
\end{tikzcd}$$

To finish, it remains to replace (1.9.8.4) by a strictly commutative diagram in $\boldsymbol{B}$, as opposed to $\boldsymbol{B}/\simeq$. Let B'' be the homotopy pullback of $B \to B$ and $B_3 \to B$. Then the projection $B'' \xrightarrow{\sim} B_3$ is a quasi-isomorphism, hence a weak equivalence. Choices of homotopies in the homotopy commutative diagram (1.9.8.4) determine maps $FA_3 \to B''$ and $B' \to B''$. As $B'' \xrightarrow{\sim} B_3$ and $FA_3 \xrightarrow{\sim} B_3$ are weak equivalences, the saturation axiom shows that $FA_3 \xrightarrow{\sim} B''$ is a weak equivalence. Similarly, $B' \xrightarrow{\sim} B''$ is a weak equivalence. Now consider the map $B'' \to B$ which is the canonical projection of the homotopy fibre product onto B. By construction, the composite $FA_3 \to B'' \to B$ is $FA_3 \to FA \to B$, and $B' \to B'' \to B$ is $b' : B' \to B$. Thus we have a strictly commutative (1.9.8.2) on taking $FA'' = FA_3$ and $B'' = B''$. This completes the verification of the dual of hypothesis 1.9.1.2. Now 1.9.1 applies to complete the proof of the theorem.

1.9.9. The theorem 1.9.8 is very useful in providing K-theoretic equivalences directly from off-the-shelf data, as found in [SGA 6] for example. Morally, it says that $K(\boldsymbol{A})$ essentially depends only on the derived category $\boldsymbol{w}^{-1}\boldsymbol{A}$, and thus that Waldhausen K-theory gives essentially a K-theory of the derived category. However, it is true that to so define a K-theory of a derived category, one must find some underlying model $\boldsymbol{A}$ which is complicial biWaldhausen. Also, we know independence of the choice of model only when the models are related by some additive functor exact in the sense of 1.2.16. These caveats are annoying, but do not seem to cause serious problems in practice.

We also note an equivalence $\boldsymbol{w}^{-1}\boldsymbol{A} \to \boldsymbol{w}^{-1}\boldsymbol{B}$ often induces equivalences of homotopy categories $\boldsymbol{w}^{-1}\boldsymbol{A}' \to \boldsymbol{w}^{-1}\boldsymbol{B}'$ for various naturally defined complicial BiWaldhausen subcategories $\boldsymbol{A}'$, $\boldsymbol{B}'$, of $\boldsymbol{A}$, $\boldsymbol{B}$. Then 1.9.8 shows that $K(\boldsymbol{A}') \to K(\boldsymbol{B}')$ will also be a homotopy equivalence of K-theory spectra. Thus the equivalences of 1.9.8 have a nice tendency towards inheritance by natural subcategories.

1.10.1. **Cofinality Theorem.** *Let $\boldsymbol{vA}$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom. Let G be an abelian group, and $\pi : K_0(\boldsymbol{vA}) \to G$ an epimorphism. Let $\boldsymbol{A}^w$ be the full subcategory of those A in $\boldsymbol{A}$ for which the class $[A]$ in $K_0(\boldsymbol{vA})$ has $\pi[A] = 0$ in G. Make $\boldsymbol{A}^w$ a Waldhausen category with $\boldsymbol{v}(\boldsymbol{A}^w) = \boldsymbol{A}^w \cap \boldsymbol{v}(\boldsymbol{A})$, $\mathrm{co}(\boldsymbol{A}_w) = \boldsymbol{A}^w \cap \mathrm{co}(\boldsymbol{A})$. Let "$G$" denote G considered as a Eilenberg-MacLane spectrum whose only non-zero homotopy group is a G in dimension 0.*

Then there is a homotopy fibre sequence

$$K(\boldsymbol{vA}^w) \to K(\boldsymbol{vA}) \to \text{“}G\text{”} \tag{1.10.1.1}$$

In particular,

$$(1.10.1.2)\qquad \begin{aligned} K_i(\boldsymbol{v}\boldsymbol{A}^w) &= K_i(\boldsymbol{v}\boldsymbol{A}) \qquad \textit{for } i > 0 \\ K_0(\boldsymbol{v}\boldsymbol{A}^w) &= \; \mathit{Ker}\ \pi : K_0(\boldsymbol{v}\boldsymbol{A}) \twoheadrightarrow G \end{aligned}$$

Proof. Define $\boldsymbol{w}(\boldsymbol{A})$ to be the set of maps in $\boldsymbol{A}$ whose mapping cones have their K_0 class in the kernel of $\pi : K_0(\boldsymbol{v}\boldsymbol{A}) \to G$. It is easy to check using 1.5.6 that $\boldsymbol{w}\boldsymbol{A}$ is a Waldhausen category and $\boldsymbol{v}(\boldsymbol{A}) \subseteq \boldsymbol{w}(\boldsymbol{A})$. Clearly $\boldsymbol{w}\boldsymbol{A}$ satisfies the extension and saturation axioms. Appealing then to the homotopy fibre sequence given by the localization theorem 1.8.2, it suffices to show that $K(\boldsymbol{w}\boldsymbol{A})$ is homotopy equivalent to "G".

For each non-negative integer n, consider $\overset{n}{\Pi} G$ as a category whose objects are n-tuples of elements of G, $(g_1, g_2, \ldots, g_n)$. The category $\overset{n}{\Pi} G$ has only identity morphisms. There is a functor $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi} G$ induced in terms of the description 1.5.1.2 by sending the object $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ of $\boldsymbol{w}S_n\boldsymbol{A}$ to the n-tuple $(\pi[A_1], \pi[A_2] - \pi[A_1], \pi[A_3] - \pi[A_2], \ldots, \pi[A_n] - \pi[A_{n-1}])$.

(In the more precise description 1.5.1, this functor sends $A(\ ,\ )$ to $(\pi[A(1,0)],\ \pi[A(2,0)] - \pi[A(1,0)],\ \ldots,\ \pi[A(n,0)] - \pi[A(n-1,0)]) = (\pi[A(1,0)],\ \pi[A(2,1)],\ \pi[A(3,2)],\ \ldots,\ \pi[A(n,n-1)]).)$

We claim that for each n, $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi} G$ induces a homotopy equivalence of nerves of categories. As $\overset{n}{\Pi} G$ is a discrete category, i.e., has only identity morphisms, it suffices to show for all $(g_1, g_2, \ldots, g_n)$ that the category $\pi^{-1}(g_1, \ldots g_n)$ has contractible nerve. The fibre $\pi^{-1}(0, 0, \ldots, 0)$ has initial object $0 \rightarrowtail 0 \rightarrowtail \cdots \rightarrowtail 0$, and so is contractible. We plead that all fibres $\pi^{-1}(g_1, \ldots, g_n)$ are homotopy equivalent to $\pi^{-1}(0, 0, \ldots, 0)$, and hence are contractible. First note by 1.5.7 and the hypothesis that $\pi : K_0(\boldsymbol{v}\boldsymbol{A}) \to G$ is onto every element g in G is $\pi[C]$ for some C in $\boldsymbol{A}$. Given $(g_1, \ldots, g_n)$ then choose C_i in $\boldsymbol{A}$ so $\pi[C_i] = g_i$. Consider the objects $C_. = C_1 \rightarrowtail C_1 \cup C_2 \rightarrowtail C_1 \cup C_2 \cup C_3 \rightarrowtail \cdots \rightarrowtail C_1 \cup C_2 \cup \cdots \cup C_n$, and $\Sigma C_. = \Sigma C_1 \rightarrowtail \Sigma C_1 \cup \Sigma C_2 \rightarrowtail \cdots \rightarrowtail \Sigma C_1 \cup \Sigma C_2 \cup \cdots \cup \Sigma C_n$ in $\boldsymbol{w}S_n\boldsymbol{A}$. Clearly $\pi(C_.) = (g_1, g_2, \ldots, g_n)$. Also $\pi(\Sigma C_.) = (-g_1, -g_2, \ldots, -g_n)$ as $[\Sigma C] = -[C]$ by 1.5.7. Then the functor $\cup C_. : \boldsymbol{w}S_n\boldsymbol{A} \to \boldsymbol{w}S_n\boldsymbol{A}$ sending $(A_1 \rightarrowtail \cdots \rightarrowtail A_n)$ to $(A_1 \cup C_1 \rightarrowtail A_2 \cup C_1 \cup C_2 \rightarrowtail \cdots \rightarrowtail A_n \cup C_1 \cup \cdots \cup C_n)$ restricts to a functor $\cup C_. : \pi^{-1}(0, 0, \ldots, 0) \to \pi^{-1}(g_1, g_2, \ldots, g_n)$. Similarly $\cup\ \Sigma C_.$ gives a functor $\pi^{-1}(g_1, g_2, \ldots, g_n) \to \pi^{-1}(0, 0, \ldots 0)$. The maps $0 \to C_i \cup \Sigma C_i$, $C_i \cup \Sigma C_i \to 0$ are in $\boldsymbol{w}\boldsymbol{A}$ as $[C \cup \Sigma C] = [C] - [C] = 0$, and they induce natural transformations between the identity functors on $\pi^{-1}(0, 0, \ldots, 0)$ and $\pi^{-1}(g_1, g_2, \ldots, g_n)$ and the composites of the functor $\cup\ C_.$ and $\cup\ \Sigma C_.$. Thus these functors induce homotopy equivalences between $\pi^{-1}(0, 0, \ldots, 0)$ and $\pi^{-1}(g_1, g_2, \ldots, g_n)$, as was to be shown. This completes the proof of the claim that $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi}G$ is a homotopy equivalence.

The map $\pi : \boldsymbol{w}S_n\boldsymbol{A} \to \overset{n}{\Pi}G$ for various n are compatible with the simplicial operators, and so induce a functor between simplicial categories which is a homotopy equivalence of classifying spaces in each degree. Here the simplicial operators on $\boldsymbol{w}S_n\boldsymbol{A}$ are as in 1.5.1 - 1.5.2, and on $\overset{\cdot}{\Pi}G$ are defined so that $d_0(g_1,\ldots,g_n) = (g_2,\ldots,g_n)$, $d_i(g_1,\ldots,g_n) = (g_1,g_2,\ldots,g_{i-1},g_ig_{i+1},g_{i+2},\ldots,g_n)$ for $1 \leq i < n$, $d_n(g_1,\ldots,g_n) = (g_1,\ldots,g_{n-1})$, and $s_i(g_1,\ldots,g_n) = (g_1,\ldots,g_i,0,g_{i+1},\ldots,g_n)$. With this structure, the simplicial category $\overset{\cdot}{\Pi}G$ is actually the simplicial set $BG = NG$, the bar construction on G. The nerve of this degree-wise discrete simplicial category thus collapses to NG. The degree-wise homotopy equivalence π induces a homotopy equivalence of spaces

$$|N\boldsymbol{w}S_.(\boldsymbol{A})| \to |NG| \simeq BG.$$

In fact this homotopy equivalence is a map of infinite loop spaces. For one checks easily that the iterated $S_.$ construction on $\boldsymbol{wA}$ that defines the Waldhausen spectrum structure ([W] 1.3.5 Remark) corresponds under π to the iterated bar construction on G that defines the Eilenberg- MacLane spectrum "G". (Or one notes that π is a simplicial symmetric monoidal functor, and feeds it to an infinite loop space machine [Th3] A). Thus π induces a homotopy equivalence of spectra $K(\boldsymbol{wA}) \xrightarrow{\sim}$ "G", as required.

(We found this proof in 1985; it has since become folklore.)

1.10.2. *Exercise* (optional). Theorem 1.10.1 is all the cofinality that we need for Sections 2 - 11. Other well-known cofinality results often have a different flavor, (see [Gr2] 6.1 and [Sta] 2.1 for some latest versions). In particular, the Waldhausen strict cofinality theorem [W] 1.5.9 at first seems quite different in purpose from our 1.10.1. Combining Waldhausen strict cofinality, our 1.10.1 and Grayson's cofinality trick [Gr3] Section 1, prove the following cofinality result:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Waldhausen categories. Suppose $\boldsymbol{A}$ is a full subcategory of $\boldsymbol{B}$ closed under extensions, that $\boldsymbol{w}(\boldsymbol{A}) = \boldsymbol{A} \cap \boldsymbol{w}(\boldsymbol{B})$, and that a map in $\boldsymbol{A}$ is a cofibration in $\boldsymbol{A}$ iff it is a cofibration in $\boldsymbol{B}$ with quotient isomorphic to an object of $\boldsymbol{A}$. Suppose that $\boldsymbol{B}$ has mapping cylinders satisfying the cylinder axiom, and that $\boldsymbol{A}$ is closed under them. Suppose finally that $\boldsymbol{A}$ is cofinal in $\boldsymbol{B}$ in that for all B in $\boldsymbol{B}$ there is a B' in $\boldsymbol{B}$ such that $B \cup B'$ is isomorphic to an object of $\boldsymbol{A}$.

Then $K(\boldsymbol{A}) \to K(\boldsymbol{B}) \to$ "$K_0(\boldsymbol{B})/K_0(\boldsymbol{A})$" is a homotopy fibre sequence.

1.11.1. To close Section 1 we compare the Quillen K-theory [Q1] of an exact category $\mathcal{E}$ to the Waldhausen K-theory of $\mathcal{E}$, and to the Waldhausen K-theory of a category of bounded complexes in $\mathcal{E}$. The latter category has cylinders and cocylinders and is complicial biWaldhausen. This allows one to apply the results 1.10.1, 1.10.2, 1.9.8, 1.9.1, 1.8.2 to Quillen K-theory of exact categories. See [Gr2] and [Sta] for rederivations of all Quillen's basic results on K-theory of exact categories in the Waldhausen framework. We pause to mention two open problems. First, find a general result for Waldhausen categories that specializes to Quillen's devissage theorem when applied to the category of bounded complexes in an abelian category. Second, show under the conditions 1.2.15, that $K(\boldsymbol{A}) \simeq K(\mathcal{B})$. This would make Quillen's localization theorem for abelian categories an immediate consequence of 1.8.2.

Logically, one should now read Appendix A, and then return to 1.11.2.

1.11.2. **Theorem** (Waldhausen). *Let $\mathcal{E}$ be an exact category in the sense of Quillen* [Q1]. *Consider $\mathcal{E}$ as a biWaldhausen category as in* 1.2.9. *Then the Quillen and the Waldhausen K-theory spectra of $\mathcal{E}$ are naturally homotopy equivalent.*

Proof. [W] 1.9, or [Gi2] 9.3.

1.11.3. Let $\mathcal{A}$ be an abelian category, and let $i : \mathcal{E} \to \mathcal{A}$ be an exact functor which is full and faithful. Assume that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, and that if a sequence in $\mathcal{E}$ is exact in $\mathcal{A}$, then it must be exact in $\mathcal{E}$. We also make the following stronger assumption (which will be harmless by 1.11.10):

1.11.3.1. If f is a map in $\mathcal{E}$ such that $i(f)$ is an epimorphism in $\mathcal{A}$, then f is an admissible epimorphism in $\mathcal{E}$.

1.11.4. *Example.* Let X be a scheme, and let $\mathcal{E}$ be the exact category of algebraic vector bundles on X. Let $\mathcal{A}$ be either the abelian category of all $\mathcal{O}_X$-modules, or else the abelian subcategory of all quasi-coherent $\mathcal{O}_X$-modules. Let $i : \mathcal{E} \to \mathcal{A}$ be the canonical inclusion. Then this inclusion satisfies all the conditions of 1.11.3.

1.11.5. *Example.* Let $\mathcal{E}$ be an exact category satisfying the condition that all weakly split epimorphisms of $\mathcal{E}$ are admissible epimorphisms. That is, suppose that for any $r : E \to E''$ in $\mathcal{E}$ such that there is an $s : E'' \to E$ with $rs = 1$, then $r : E \twoheadrightarrow E''$ is an admissible epimorphism in $\mathcal{E}$. Let $i : \mathcal{E} \to \mathcal{A}$ be the Gabriel-Quillen embedding (cf. A 7.1), i.e., let $\mathcal{A}$ be the abelian category of left exact additive functors $\mathcal{E}^{\mathrm{op}} \to \mathbf{Z}$-modules, with i the Yoneda embedding $i(E) = \hom_{\mathcal{E}}(\ ,E)$. Then $i : \mathcal{E} \to \mathcal{A}$ satisfies the hypotheses of 1.11.3, including 1.11.3.1. For a proof, see Appendix A.7.1 and A.7.16.

1.11.6. Given $i : \mathcal{E} \to \mathcal{A}$ as in 1.11.3, consider the category $\boldsymbol{E}^{\sim}$ of bounded chain complexes in $\mathcal{E}$ as a full subcategory of the category of chain complexes $C(\mathcal{A})$. Define $\mathrm{co}(\boldsymbol{E}^{\sim})$ to be the degree-wise admissible monomorphisms. Define $\boldsymbol{w}(\boldsymbol{E}^{\sim})$ to be those maps in $\boldsymbol{E}^{\sim}$ which are quasi-isomorphisms in $\mathcal{A}$. (This appears to depend on the choice of $\mathcal{A}$, but in fact does not given 1.11.3.1, see 1.11.8.)

Then $\boldsymbol{E}^{\sim}$ is a complicial biWaldhausen category. There is a canonical complicial exact functor $\mathcal{E} \to \boldsymbol{E}^{\sim}$ sending E in $\mathcal{E}$ to the complex which is E in degree 0 and 0 in other degrees.

Let $\boldsymbol{E}$ be $\boldsymbol{E}^{\sim}$, but now with $\mathrm{co}(\boldsymbol{E})$ being the degree-wise split monomorphisms whose quotients lie in $\boldsymbol{E}$. Then $\boldsymbol{E}$ is a complicial biWaldhausen category, and the inclusion functor $\boldsymbol{E} \to \boldsymbol{E}^{\sim}$ is complicial exact.

By 1.3.6, both $\boldsymbol{E}$ and $\boldsymbol{E}^{\sim}$ have cylinder and cocylinder functors, satisfying the cylinder and cocylinder axioms 1.3.1.7.

1.11.7. **Theorem** (Gillet-Waldhausen). *Under the hypotheses of 1.11.3 and with the notation of 1.11.6. the canonical exact inclusions induce homotopy equivalences of K-theory spectra*

$$K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim}) \xleftarrow{\sim} K(\boldsymbol{E})$$

Proof. The map $K(\boldsymbol{E}) \to K(\boldsymbol{E}^{\sim})$ is a homotopy equivalence by 1.9.2. The homotopy equivalence $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim})$ is due to Gillet, who patterned his argument [Gi2] 6.2 on a proof for special cases due to Waldhausen. Gillet's statement [Gi2] 6.2 does not make the extra hypothesis 1.11.3.1 on the embedding $\mathcal{E} \to \mathcal{A}$, but the proof given there needs 1.11.3.1 in order to work. (Gillet attempts to evade 1.11.3.1 by appealing to Quillen's resolution theorem, but in fact 1.11.3.1 is needed to verify one of the hypotheses of the resolution theorem, [Q1] Section 4 Thm 3i).) We will give the complete proof that $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim})$ is a homotopy equivalence.

For integers $a \leq b$, let $\boldsymbol{E}_a^{\sim b}$ be the full subcategory of those complexes $E^{\cdot}$ in $\boldsymbol{E}^{\sim}$ such that $E^i = 0$ for $i \leq a-1$ and for $i \geq b+1$. Hence $\mathcal{E} = \boldsymbol{E}_0^{\sim 0}$, and $\boldsymbol{E}^{\sim}$ is the direct colimit of the $\boldsymbol{E}_a^{\sim b}$ as b goes to $+\infty$ and a goes to $-\infty$. Let $\boldsymbol{w}(\boldsymbol{E}^{\sim})$ be the quasi-isomorphisms of complexes as in 1.11.6, and let $\boldsymbol{i}(\boldsymbol{E}^{\sim})$ be isomorphisms of complexes. Set $\boldsymbol{w}(\boldsymbol{E}_a^{\sim b}) = \boldsymbol{w}(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$ $\boldsymbol{i}(\boldsymbol{E}_a^{\sim b}) = \boldsymbol{i}(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$, and $\mathrm{co}(\boldsymbol{E}_a^{\sim b}) = \mathrm{co}(\boldsymbol{E}^{\sim}) \cap \boldsymbol{E}_a^{\sim b}$. Then $\boldsymbol{w}\boldsymbol{E}_a^{\sim b}$ and $\boldsymbol{i}\boldsymbol{E}_a^{\sim b}$ are Waldhausen categories. Let $\boldsymbol{E}_a^{\sim bw}$ be the full subcategory of $\boldsymbol{E}_a^{\sim b}$ of those complexes quasi-isomorphic to 0, with $\mathrm{co}(\boldsymbol{E}_a^{\sim bw}) = \boldsymbol{E}_a^{\sim bw} \cap \mathrm{co}(\boldsymbol{E}^{\sim})$. Then $\boldsymbol{i}\boldsymbol{E}_a^{\sim bw}$ is a Waldhausen category.

Consider the exact functor

$$\boldsymbol{i}\boldsymbol{E}_a^{\sim b} \to \prod^{b-a+1} \mathcal{E} \tag{1.11.7.1}$$

sending a complex $E^{\cdot}$ to $(E^a, E^{a+1}, \ldots, E^b)$. We claim that this functor induces a homotopy equivalence on Waldhausen K-theory. For $b = a$, this is clear, as the exact functor is then an isomorphism. The proof of the claim now proceeds by induction on $b-a$, and consists of showing that a functor

$$(1.11.7.2) \qquad \boldsymbol{iE}_a^{\sim b} \to \boldsymbol{iE}_{a+1}^{\sim b} \times \mathcal{E}$$

induces a homotopy equivalence on K-theory. This functor sends a complex $(E^a \to \cdots \to E^b)$ to the pair consisting of the subcomplex $(0 \to E^{a+1} \to \cdots \to E^b)$ and the quotient $E^a = (E^a \to 0 \to \cdots \to 0)$. This functor does induce a homotopy equivalence on K-theory by the Additivity Theorem 1.7.2 with $\boldsymbol{A} = \boldsymbol{iE}_{a+1}^{\sim b}$, $\boldsymbol{B} = \mathcal{E} = \boldsymbol{iE}_a^{\sim a}$ and $\boldsymbol{C} = \boldsymbol{iE}_a^{\sim b}$, as the canonical filtration defining our functor induces an equivalence of categories $\boldsymbol{iE}_a^{\sim b} \simeq E(\boldsymbol{A},\boldsymbol{C},\boldsymbol{B})$. This proves of claim.

On the other hand, we also claim that similarly, $K(\boldsymbol{iE}_a^{\sim bw})$ is homotopy equivalent to $\prod^{b-a} K(\mathcal{E})$. If $b = a$, this holds trivially as $\boldsymbol{E}_a^{\sim aw} = \mathcal{E}^w$ is equivalent to the 0 category. It also holds if $a = b-1$, as the category $\boldsymbol{iE}_{b-1}^{\sim bw}$ is the category of complexes $\partial : E^{b-1} \to E^b$ with ∂ an isomorphism, and this category is equivalent to $\mathcal{E}$. The proof now proceeds by induction on $b-a$ and consists of producing a homotopy equivalence

$$(1.11.7.4) \qquad K\left(\boldsymbol{iE}_a^{\sim bw}\right) \xrightarrow{\sim} K(\boldsymbol{iE}_a^{\sim(b-1)w}) \times K\left(\boldsymbol{iE}_{b-1}^{\sim bw} = \mathcal{E}\right)$$

This homotopy equivalence results by applying the additivity theorem 1.7.2 to an equivalence of categories (cf. 1.9.4(d))

$$(1.11.7.5) \qquad \boldsymbol{iE}_a^{\sim bw} \cong E(\boldsymbol{iE}_a^{\sim(b-1)w},\ \boldsymbol{iE}_a^{\sim bw},\ \boldsymbol{iE}_{b-1}^{\sim bw})$$

To see the equivalence, we must associate an extension to a complex $E^\cdot$ in $\boldsymbol{iE}_a^{\sim bw}$. As $E^\cdot$ is acyclic and $Z^bE = E^b$ as $E^{b+1} = 0$, the map $E^{b-1} \twoheadrightarrow E^b$ is an epimorphism in $\mathcal{A}$. Hence $E^{b-1} \twoheadrightarrow E^b$ is an admissible epimorphism in $\mathcal{E}$ by 1.11.3.1. Thus its kernel $Z^{b-1}E^\cdot$ is in $\mathcal{E}$ and $Z^{b-1}E \rightarrowtail E^{b-1} \twoheadrightarrow E^b$ is an exact sequence in $\mathcal{E}$. The complex $\tau^{\leq b-1}E^\cdot = (E^a \to E^{a+1} \to \cdots \to E^{b-2} \to Z^{b-1})$ is thus in $\boldsymbol{iE}_a^{\sim(b-1)w}$. The complex $\tau^bE^\cdot = (B^b \xrightarrow{\cong} E^b)$ is in $\boldsymbol{iE}_{b-1}^{\sim(b-1)w}$, and $E^\cdot$ fits into a canonical cofibration sequence $\tau^{\leq b-1}E^\cdot \rightarrowtail E^\cdot \twoheadrightarrow \tau^bE^\cdot$ which defines an object of $E(\boldsymbol{iE}_a^{\sim(b-1)w},\ \boldsymbol{iE}_a^{\sim bw},\ \boldsymbol{iE}_{b-1}^{\sim bw})$

(1.11.7.6)

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
0 \arrow[d] & 0. \arrow[d] & \arrow[d] \\
E^a \arrow[r, equal] \arrow[d] & E^a \arrow[r] \arrow[d] & 0 \arrow[d] \\
E^{a+1} \arrow[r, equal] \arrow[d] & E^{a+1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^{b-2} \arrow[r, equal] \arrow[d] & E^{b-2} \arrow[r] \arrow[d] & 0 \arrow[d] \\
Z^{b-1} \arrow[r, tail] \arrow[d] & E^{b-1} \arrow[r, two heads] \arrow[d] & B^b \arrow[d, equal, "\wr"'] \\
0 \arrow[r] \arrow[d] & E^b \arrow[r, equal] \arrow[d] & E^b \arrow[d] \\
\vdots & 0 \arrow[d] & 0 \arrow[d] \\
 & \vdots & \vdots
\end{tikzcd}
$$

The inverse equivalence of the categories takes the total complex $E^{\cdot}$ and forgets the extensions. It is easy to check that both the equivalence of categories and its inverse are exact functors. This completes the proof of our second claim that $K(\boldsymbol{iE}_a^{\sim bw})$ is homotopy equivalent to $\prod^{b-a} K(\mathcal{E})$. In fact our proof shows that for $E^{\cdot}$ in $\boldsymbol{E}_a^{\sim bw}$, the $Z^k E^{\cdot} = B^k E^{\cdot}$ are objects of $\mathcal{E}$, that $E^{\cdot} \mapsto B^k(E^{\cdot})$ is an exact functor, and that our claimed homotopy equivalence is induced by the exact functor $\boldsymbol{iE}_a^{\sim bw} \to \overset{b-a}{\Pi}\mathcal{E}$ given by sending $E^\cdot$ to $(B^{a+1}E^\cdot,\ B^{a+2}E^\cdot,\ \ldots,\ B^bE)$.

Now consider the exact inclusion $\boldsymbol{iE}_a^{\sim bw} \to \boldsymbol{iE}_a^{\sim b}$. This induces a map on K-theory spectra, which by the two claims above is homotopy equivalent to a map $\overset{b-a}{\Pi} K(\mathcal{E}) \to \overset{b-a+1}{\Pi} K(\mathcal{E})$

$$\begin{array}{ccc} K\left(\boldsymbol{iE}_a^{\sim bw}\right) & \longrightarrow & K\left(\boldsymbol{iE}_a^{\sim b}\right) \\ \big\downarrow\wr & & \big\downarrow\wr \\ \overset{b-a}{\Pi} K(\mathcal{E}) & \longrightarrow & \overset{b-a+1}{\Pi} K(\mathcal{E}) \end{array} \tag{1.11.7.7}$$

For $E^\cdot$ in $\boldsymbol{iE}_a^{\sim bw}$, the corresponding term in $\overset{b-a+1}{\Pi}\mathcal{E}$ is $(E^a,\ E^{a+1},\ \ldots,\ E^b)$ while the corresponding term in $\overset{b-a}{\Pi}\mathcal{E}$ is $(B^{a+1},\ \ldots,\ B^b)$. From the exact sequence $Z^k \rightarrowtail E^k \twoheadrightarrow B^{k+1}$ for $E^\cdot$ and the fact that $Z^k = B^k$ by acyclicity, the Additivity Theorem 1.7.2 shows the map on K-theory spectra induced by sending $E^\cdot$ to E^k is homotopic to the sum of the maps induced on K-theory induced by sending $E^\cdot$ to B^k and to B^{k+1}. Considering also that $B^a = \ \mathrm{im}E^{a-1} = 0$, we see that our map $\overset{b-a}{\Pi} K(\mathcal{E}) \to \overset{b-a+1}{\Pi} K(\mathcal{E})$ in (1.11.7.7) is that induced by the exact functor:

(1.11.7.8)
$$(B^{a+1},\ \ldots,\ B^b) \longmapsto (B^{a+1},\ B^{a+1}\oplus B^{a+2},\ \ldots,\ B^{b-1}\oplus B^b)\,.$$

The homotopy cofibre of this map is $K(\mathcal{E})$, with $\overset{b-a+1}{\Pi} K(\mathcal{E}) \to K(\mathcal{E})$ induced by $(x_a, \ldots, x_b) \mapsto \Sigma(-1)^k x_k$. Taking the direct colimit as a goes to $-\infty$ and b goes to $+\infty$, we get a homotopy cofibre sequence

$$K\left(\boldsymbol{iE}^{\sim w}\right) \to K(\boldsymbol{iE}^{\sim}) \to K(\mathcal{E}) \tag{1.11.7.9}$$

where $K(\boldsymbol{iE}^{\sim}) \to K(\mathcal{E})$ sends $E^\cdot$ to its Euler characteristic $\Sigma(-1)^k E^k$.

But by the localization theorem 1.8.2, the homotopy cofibre spectrum of $K(\boldsymbol{iE}^{\sim w}) \to K(\boldsymbol{iE}^{\sim})$ is $K(\boldsymbol{wE}^{\sim}) = K(\boldsymbol{E}^{\sim})$. Thus there is a homotopy equivalence $K(\mathcal{E}) \xrightarrow{\sim} K(\boldsymbol{E}^{\sim})$, which in fact is the map induced by the exact functor $\mathcal{E} \to \boldsymbol{E}^{\sim}$. This proves the theorem.

1.11.8. *Porism.* Under the hypotheses of 1.11.3 and with the notation of 1.11.6, a complex $E^\cdot$ in $\mathcal{E}$ is acyclic in $\mathcal{C}(\mathcal{A})$ if and only if all cycle and boundary objects $Z^kE^\cdot$ and $B^kE^\cdot$ are in $\mathcal{E} \subseteq \mathcal{A}$, $B^kE^\cdot = Z^kE^\cdot$ in $\mathcal{E}$, and the sequences $Z^k \rightarrowtail E^k \twoheadrightarrow B^{k+1}$ are exact in $\mathcal{E}$. In particular, this is independent of the choice of $\mathcal{A}$, provided only it satisfies 1.11.3 and especially 1.11.3.1.

Also, the set $\boldsymbol{w}(\boldsymbol{E})$ of quasi-isomorphisms of complexes in $\mathcal{E}$ is independent of the choice of $\mathcal{A}$.

For the first paragraph is a porism of the proof of 1.11.7. To verify the second paragraph, we note that a map in $\boldsymbol{E}$ is a quasi-isomorphism (with respect to $\mathcal{A}$) iff its mapping cone is acyclic. The formation of mapping cones uses only the additive structure of $\mathcal{E}$, and acyclicity is independent of the choice of $\mathcal{A}$ by the first paragraph.

1.11.9. *Remark.* An exact functor $f : \mathcal{E} \to \mathcal{E}'$ induces a compatible additive functor of the associated Gabriel-Quillen abelian categories $f^* : \mathcal{A} \to \mathcal{A}'$, by A.8.2. Although $f^* : \mathcal{A} \to \mathcal{A}'$ need not preserve exact sequences in $\mathcal{A}$, it does preserve exact sequences in $\mathcal{E}$, A.8.5. Hence it preserves quasi-isomorphisms of complexes in $\mathcal{E}$ by 1.11.8, and induces complicial exact functors $f^* : \boldsymbol{E} \to \boldsymbol{E}'$, $\boldsymbol{E}^\sim \to \boldsymbol{E}'^\sim$. Thus 1.11.7 becomes natural in $\mathcal{E}$ with $\mathcal{E} \to \mathcal{A}$ chosen as the Gabriel- Quillen embedding 1.11.5.

1.11.10. *Remark.* If $\mathcal{E}$ is an exact category not satisfying the hypothesis of 1.11.5 that weakly split epimorphisms are admissible epimorphisms, we let $\mathcal{E}'$ be its Karoubianization A.9.1. Then 1.11.7 applies to the Gabriel-Quillen embedding of $\mathcal{E}'$, as $\mathcal{E}'$ satisfies the hypothesis of 1.11.5. As $K(\mathcal{E})$ is a cover of $K(\mathcal{E}')$ so $K_i(\mathcal{E}) = K_i(\mathcal{E}')$ for $i > 0$, by classical cofinality A.9.1, this shows the extra hypothesis of 1.11.3.1 is essentially harmless. Indeed by 1.11.1 we get in general that $K(\mathcal{E})$ is homotopy equivalent to the K-theory of the category of those bounded chain complexes in $\mathcal{E}'$ whose Euler characteristic lies in $K_0(\mathcal{E}) \subseteq K_0(\mathcal{E}')$.

2. Perfect complexes on schemes

2.0. In this section, we review and extend the theory of perfect complexes on a scheme. This theory was discovered and developed by Grothendieck and his school (especially Illusie) in [SGA 6] as a more flexible replacement for the naive theory of algebraic vector bundles on a scheme in K-theory. The somewhat new results of this section are the characterizations of pseudo-coherence and perfection in 2.4, and some of the functoriality statements in 2.6 and 2.7. Aside from a few minor improvements, the rest of this material is already in [SGA 6].

2.1.1. *Definition.* A scheme with an ample family of line bundles is a scheme X, which is quasi-compact and quasi-separated, and which has a family of line bundles $\{\mathcal{L}_\alpha\}$ which satisfy any one of the following equivalent conditions ([SGA 6] II 2.2.3 and proof thereof).

(a) Let n run over all positive integers $n \geq 1$ and let $\mathcal{L}_\alpha$ run over the family of line bundles. Let $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ run over all the global sections of all the $\mathcal{L}_\alpha^{\otimes n}$. Then the resulting family of open set $X_f = \{x \in X | f(x) \neq 0\}$ is a basis for the Zariski topology of X.

(b) One may choose a set of positive integers $n \geq 1$, a set of line bundles $\mathcal{L}_\alpha$ in the family, and a set of global sections $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ such that the set $\{X_f\}$ is a basis for the Zariski topology of X and all these X_f are affine schemes.

(c) One may choose a set of positive integers $n \geq 1$, line bundles $\mathcal{L}_\alpha$ in the family, and global sections $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$ such that $\{X_f\}$ is a cover of X by affine schemes.

(d) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the evaluation map

$$\bigoplus_{\alpha, n \geq 1} \Gamma\left(X, \mathcal{F} \otimes \mathcal{L}_\alpha^{\otimes n}\right) \otimes \mathcal{L}_\alpha^{-\otimes n} \twoheadrightarrow \mathcal{F}$$

is an epimorphism.

2.1.2. *Examples.* (a) Any scheme with an ample line bundle in the sense of [EGA] II 4.5.3 and IV 1.7 has an ample family (consisting of one line bundle) in the present sense. As special cases we have (b) and (c):

(b) Any affine scheme has an ample family of line bundles.

(c) Any scheme quasi-projective over an affine scheme has an ample family of line bundles. In particular, any quasi-affine scheme has an ample family of line bundles.

(d) Any separated regular noetherian scheme has an ample family of line bundles ([SGA 6] II 2.2.7.1).

(e) If Y has an ample family of line bundles, and $U \subseteq Y$ is a quasi-compact open, then U has an ample family of line bundles, given as the restriction of the family on Y. This is clear by 2.1.1(a).

(f) Let Y be a scheme with an ample family of line bundles $\{\mathcal{L}_\alpha\}$. Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes. Suppose X has an f-ample family of line bundles $\{\mathcal{K}_\beta\}$. That is, suppose Y is covered by (affine) opens $U \subseteq Y$ such that $\{\mathcal{K}_\beta | f^{-1}(U)\}$ is an ample family on each $f^{-1}(U)$. Then $\{f^*\mathcal{L}_\alpha^{\otimes k} \otimes \mathcal{K}_\beta^{\otimes n} | k \geq 1,\ n \geq 1\}$ is an ample family on X, as clearly follows from criterion 2.1.1(a) and [EGA] I 6.8.1. As special cases we have (g) and (h):

(g) If Y has an ample family of line bundles and $f : X \to Y$ is an affine map of schemes, then X has an ample family of line bundles.

(h) If Y has an ample family of line bundles and $f : X \to Y$ is a quasi-projective map of schemes, then X has an ample family of line bundles.

2.1.3. **Lemma.** *Let X have an ample family of line bundles. Then*

(a) *For any quasi-coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$*, there is a locally free* $\mathcal{O}_X$*-module* $\mathcal{E}$ *and an epimorphism* $\mathcal{E} \twoheadrightarrow \mathcal{F}$*.*

(b) *For any quasi-coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ *of finite type, there is an algebraic vector bundle (i.e., a locally free* $\mathcal{O}_X$*-module of finite type)* $\mathcal{E}$ *on* X*, and an epimorphism* $\mathcal{E} \twoheadrightarrow \mathcal{F}$*.*

(c) *For any epimorphism* $\mathcal{G} \to \mathcal{F}$ *of quasi-coherent* $\mathcal{O}_X$*-modules with* $\mathcal{F}$ *of finite type, there is an algebraic vector bundle* $\mathcal{E}$ *and a map* $\mathcal{E} \to \mathcal{G}$ *such that the composite map* $\mathcal{E} \to \mathcal{G} \to \mathcal{F}$ *is an epimorphism onto* $\mathcal{F}$*.*

In all cases (a), (b), (c), $\mathcal{E}$ *may be taken to be a direct sum of tensor powers of line bundles of the family.*

Proof. First note by quasi-compactness of X and 2.1.1(c) that X has an ample finite subfamily of line bundles, $\mathcal{L}_{\alpha_1}, \ldots, \mathcal{L}_{\alpha_n}$. As each $\mathcal{L}_{\alpha_i}$ is locally free, X is covered by open sets where all the $\mathcal{L}_{\alpha_i}$ are simultaneously free. Thus any direct sum of $\mathcal{L}_{\alpha_i}^{-\otimes n}$ is locally free on X, even if it is an infinite sum.

To prove (a) we appeal to 2.1.1(d), and let $\mathcal{E}$ be the sum with one factor $\mathcal{L}_{\alpha_i}^{-\otimes n}$ for each global section in $\Gamma(X,\ \mathcal{F} \otimes \mathcal{L}_{\alpha_i}^{\otimes n})$.

Case (b) follows from case (c) on setting $\mathcal{G} = \mathcal{F}$.

To prove case (c), we consider the epimorphism $\oplus \mathcal{L}_{\alpha_i}^{-\otimes n} \twoheadrightarrow \mathcal{G}$ constructed in the proof of case (a). The composite of this epimorphism with $\mathcal{G} \twoheadrightarrow \mathcal{F}$ is also an epimorphism. As $\mathcal{F}$ has finite type and X is quasi-compact, some finite subsum of the factors $\mathcal{L}_{\alpha_i}^{-\otimes n}$ maps epimorphically to $\mathcal{F}$. We let $\mathcal{E}$ be this finite subsum, and take $\mathcal{E} \subseteq \oplus \mathcal{L}_{\alpha_i}^{-\otimes n} \twoheadrightarrow \mathcal{G}$ the induced map.

2.2. Logically, the reader should now examine Appendix B before returning to 2.2.1. We note the convention that the word "$\mathcal{O}_X$-module" means "a sheaf on the scheme X which is a sheaf of modules over the sheaf of rings $\mathcal{O}_X$," and does not apply to a general presheaf of modules over the sheaf of rings $\mathcal{O}_X$. That is, an "$\mathcal{O}_X$-module" is a $\mathcal{O}_X$-module in the Zariski topos of X.

2.1.1. *Definition* ([SGA 6] I 2.1). For any integer m, a chain complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be strictly m-pseudo-coherent if E^i is an algebraic vector bundle on X for all $i \geq m$ and $E^i = 0$ for all i sufficiently large. A complex $E^{\cdot}$ is strictly pseudo-coherent if it is strictly m-pseudo-coherent for all m, i.e., if it is a bounded above complex of algebraic vector bundles.

2.2.2. *Definition* ([SGA 6] I 2.1). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules is strictly perfect if it is strictly pseudo-coherent and strictly bounded below. That is, a strict perfect complex is a strict bounded complex of algebraic vector bundles.

2.2.3. **Lemma** ([SGA 6] I 2.10b). *Let $A^{\cdot}$ be a complex of $\mathcal{O}_X$-modules with $H^i(A^{\cdot}) = 0$ for $i \geq m+1$. Then $H^m(A^{\cdot})$ is an $\mathcal{O}_X$-module of finite type iff for each $x \in X$ there is an open neighborhood U of x and an isomorphism in the derived category $D(\mathcal{O}_U\text{-Mod})$ between the restriction $A^{\cdot}|U$ and a strictly m-pseudo-coherent complex on U.*

Proof. Suppose $H^m(A^{\cdot})$ is of finite type, and let $x \in X$. Then for some positive integer k there is an epimorphism of stalks $\oplus^k \mathcal{O}_{X,x} \twoheadrightarrow H^m(A)_x$. As $\oplus^k \mathcal{O}_{X,x}$ is a free module over $\mathcal{O}_{X,x}$, this map lifts to the group of cycles $Z^m(A^{\cdot})_x \twoheadrightarrow H^m(A^{\cdot})_x$. The lifted map extends over some open *nbd* U of x to a map $\oplus^k \mathcal{O}_U \to Z^m(A^{\cdot})|U \subseteq A^m|U$. Shrinking U, we may assume that $\oplus^k \mathcal{O}_U \to H^m(A^{\cdot})|U$ is epimorphic. Then the complex consisting only of $\oplus^k \mathcal{O}_U$ in degree m maps to the complex $A^{\cdot}|U$ by a map inducing an isomorphism on the cohomology $H^p = 0$ for $p > m$, and an epimorphism $\oplus^k \mathcal{O}_U \twoheadrightarrow H^m(A^{\cdot})|U$ on H^m. We now apply Lemma 1.9.5 with $\mathcal{A} = \mathcal{D} =$ category of $\mathcal{O}_U$-modules and $\mathcal{C} = \mathcal{C}^-(\mathcal{A})$. This inductive construction lemma produces a complex $D^{\cdot}$ on U and a quasi-isomorphism $D^{\cdot} \xrightarrow{\sim} A^{\cdot}|U$. Moreover, $D^{\cdot}$ satisfies $D^m = \oplus^k \mathcal{O}_U$, and $D^p = 0$ for $p > m$. Thus $D^{\cdot}$ is strict m-pseudo-coherent on U, as required.

Conversely, suppose $A^{\cdot}$ is locally quasi-isomorphic to a strict m-pseudo-coherent complex. As $H^m(A^{\cdot})$ is a quasi-isomorphism invariant, and as being of finite type is a local question, passing to $U \subseteq X$ we may assume that $A^{\cdot}$ is strict m-pseudo-coherent. Recall that $H^i(A^{\cdot}) = 0$ for $i \geq m+1$. Applying 1.9.4(a), with $\mathcal{A} = \mathcal{O}_U$-Mod and $\mathcal{B} =$ the category of algebraic vector bundles on U, to the truncated complex of vector bundles $\sigma^m A^{\cdot}$, we see that $Z^m(A^{\cdot}) = Z^m(\sigma^m A^{\cdot})$ is an algebraic vector bundle on U, and hence is of finite type. As $H^m(A^{\cdot})$ is a quotient of $Z^m A^{\cdot}$, it is also of finite type, as required.

2.2.4. **Lemma** ([SGA 6]). *Let U be a scheme, and x a point of U. Consider the solid arrow diagram of complexes of $\mathcal{O}_U$-modules, $F^{\cdot}$, $G^{\cdot}$, $E^{\cdot}$:*

(2.2.4.1)
$$\begin{tikzcd} & E'^{\cdot} \arrow[d, dashed, "c"] \\ F^{\cdot} \arrow[ur, dashed, "d"] \arrow[r, "a"] & G^{\cdot} \arrow[d, "b"] \\ & E^{\cdot} \end{tikzcd}$$

Then under any of the following sets of conditions, there exists a smaller nbd V of x, a complex $E'^{\cdot}$ of $\mathcal{O}_V$-modules on V, and maps $c : E'^{\cdot} \to$

$G^{\cdot}|V$, $d : F^{\cdot}|V \to E'^{\cdot}$ such that $cd = a|V$, and which satisfy the extra conclusions attached to the corresponding set of conditions:

(a) If $E^{\cdot}$ and $F^{\cdot}$ are strict m-pseudo-coherent, and the truncation $\tau^m(b) : \tau^m G^{\cdot} \xrightarrow{\sim} \tau^m E^{\cdot}$ is a quasi-isomorphism, then we may take $E'^{\cdot}$ to be strict m-pseudo-coherent and c to be a quasi-isomorphism.

(b) If $E^{\cdot}$ is strict m-pseudo-coherent, and $F^{\cdot}$ is strict perfect, with $\tau^m(b) : \tau^m G^{\cdot} \xrightarrow{\sim} \tau^m E^{\cdot}$ a quasi-isomorphism, then we may take $E'^{\cdot}$ to be strict perfect and c to be an m-quasi-isomorphism.

(c) If $E^{\cdot}$ and $F^{\cdot}$ are strict perfect and $b : G^{\cdot} \xrightarrow{\sim} E^{\cdot}$ is a quasi-isomorphism, then we may take $E'^{\cdot}$ to be strict perfect and c to be a quasi- isomorphism.

Proof. In all cases $E^{\cdot}$ and $F^{\cdot}$ are strictly bounded above. Hence $G^{\cdot}$ is cohomologically bounded above, and replacing $G^{\cdot}$ by the subcomplex $\tau^{\leq k} G^{\cdot}$ for some sufficiently large k, we may assume that $G^{\cdot}$ is strictly bounded above.

Consider the germs of complexes $E^{\cdot}_x$, $F^{\cdot}_x$, $G^{\cdot}_x$ at the point $x \in U$. We apply to $F^{\cdot}_x \to G^{\cdot}_x$ the inductive construction Lemma 1.9.5 with $\mathcal{A}$ the category of modules over $\mathcal{O}_{X,x}$, $\mathcal{D}$ the category of free modules of finite type over $\mathcal{O}_{X,x}$, and $\mathcal{C}$ the category of complexes in $\mathcal{A}$ with a map b to a strict m-pseudo-coherent complex such that $\tau^m(b)$ is a quasi- isomorphism. The quasi-isomorphism $\tau^m(b)$ and Lemma 2.2.3 show that hypothesis 1.9.5.1 is met for $n-1 \geq m$. For if $C^{\cdot}$ is in $\mathcal{C}$ with $H^i(C^{\cdot}) = 0$ for $i \geq n \geq m+1$, $H^{n-1}(C^{\cdot})$ is isomorphic to an $\mathcal{O}_{X,x}$ module of finite type via $\tau^m(b)$ and 2.2.3. So there is an epimorphism $\oplus^k \mathcal{O}_{X,x} \to H^{n-1}(C^{\cdot})$. As $\oplus^k \mathcal{O}_{X,x}$ is a free module over the local ring $\mathcal{O}_{X,x}$, this epimorphism lifts along any epimorphism of modules over $\mathcal{O}_{X,x}$, $A_x \twoheadrightarrow H^{n-1}(C^{\cdot})$. This verifies 1.9.5.1 as long as $n-1 \geq m$.

Now the variant 1.9.5.9 of 1.9.5 provides a strict perfect complex $\sigma^m E'^{\cdot}_x$, a map $\sigma^m d : \sigma^m F^{\cdot}_x \to \sigma^m E'^{\cdot}_x$, and an m-quasi-isomorphism $\sigma^m E'^{\cdot}_x \to \sigma^m G^{\cdot}_x \to G^{\cdot}_x$, which is the germ at x of a truncated version of (2.2.4.1).

As $\sigma^m E'^{\cdot}_x$, $\sigma^m F^{\cdot}$, and $\sigma^m E^{\cdot}$ are all bounded complexes of free modules of finite type, there is a small open *nbd* V of x in U over which $\sigma^m E'^{\cdot}_x$ extends to a strict perfect complex $\sigma^m E'^{\cdot}$, over which the maps d and c extend, and over which the map $bc : E'^{\cdot} \to E^{\cdot}$ is an m-quasi-isomorphism (i.e., its mapping cone is acyclic in degrees $\geq m$). Then as $\tau^m(b)$ is a quasi-isomorphism, it follows that $\sigma^m(c) : \sigma^m E'^{\cdot} \to G^{\cdot}$ is an m-quasi-isomorphism on V.

Now we apply 1.9.5 again, now with $\mathcal{A} = \mathcal{O}_V$-Mod, $\mathcal{D} = \mathcal{O}_V$-Mod to construct the rest of $E'^{\cdot}$ and d, c, leaving $\sigma^m E'^{\cdot}$ unchanged. Then E' is strict m-pseudo-coherent, as $\sigma^m E'^{\cdot}$ is strict perfect. This proves 2.2.4(a).

To prove (b), let $E'^{\cdot}$ be the pushout of $\sigma^m F^{\cdot} \to F^{\cdot}$ and the map $\sigma^m F^{\cdot} \to \sigma^m E'^{\cdot}$ constructed above. Let $c : E'^{\cdot} \to G^{\cdot}$ be the map induced by $a : F^{\cdot} \to G^{\cdot}$ and $\sigma^m(c) : \sigma^m E'^{\cdot} \to G^{\cdot}$. Then $E'^{\cdot} \to G^{\cdot}$ is an m-quasi-isomorphism as both $\sigma^m(c) : \sigma^m E'^{\cdot} \to G^{\cdot}$ and $\sigma^m E'^{\cdot} \to E'^{\cdot}$ are. As the pushout of strict perfect complexes along a degree-wise split monomorphism, $E'^{\cdot}$ is strict perfect.

To prove (c), we take $m \ll 0$ so that $E^i = 0 = F^i$ for $i \leq m+1$. We apply the proof of (b), which gives a strict perfect $E'^{\cdot}$ such that $\sigma^m E'^{\cdot} = E'^{\cdot}$ (as $\sigma^m F^{\cdot} = F^{\cdot}$), and a map $bc : E'^{\cdot} \to E^{\cdot}$ which is an m-quasi-isomorphism on V. The mapping cone $M^{\cdot}$ of bc is strict perfect, is 0 below degree $m-1$ and is acyclic except for $H^{m-1}(M^{\cdot}) = \ker(H^m(E'^{\cdot}) \twoheadrightarrow H^m(E^{\cdot}))$. By 1.9.4(a) $H^{m-1}(M^{\cdot}) = Z^{m-1}M^{\cdot}$ is an algebraic vector bundle. As $E'^{m-1} = 0 = E^m$, we note that $Z^{m-1}M^{\cdot} = \ker(Z^m E'^{\cdot} \to Z^m E^{\cdot}) = Z^m E'^{\cdot}$. As the map $b : G^{\cdot} \to E^{\cdot}$ is a quasi-isomorphism, $H^{m-1}(M^{\cdot})$ goes to 0 in $H^m(G^{\cdot}) \cong H^m(E^{\cdot})$. Then $Z^m E'^{\cdot}_x \to Z^m G^{\cdot}_x$ lifts along $\partial G^{m-1}_x \to Z^m G_x$ as a map of germs of $\mathcal{O}_X$-modules over the local ring $\mathcal{O}_{X,x}$. As $Z^m E'^{\cdot} = Z^{m-1}M^{\cdot}$ is locally free of finite type, this lift extends to a map of $\mathcal{O}_V$-modules $Z^m E'^{\cdot} \to G^{m-1}$ lifting $Z^m E'^{\cdot} \to Z^m G^{\cdot}$ on some smaller open *nbd* V of x. We now extend $\sigma^m E'^{\cdot}$ to a new $E'^{\cdot}$ by $E'^{m-1} = Z^m E'^{\cdot}$ with boundary $\partial : E'^{m-1} \to E'^m$ given by the inclusion $Z^m E'^{\cdot} \subseteq E'^m$. As $Z^m E'^{\cdot} = Z^{m-1}M^{\cdot}$ is a vector bundle, the new $E'^{\cdot}$ is still strict perfect. The map $\sigma^m c : \sigma^m E'^{\cdot} \to G^{\cdot}$ is extended to the new $E'^{\cdot}$ by using $Z^m E'^{\cdot} \to G^{m-1}$ in degree $m-1$. Now clearly $c : E'^{\cdot} \to G^{\cdot}$ is a quasi- isomorphism, and the other conditions of (c) are met.

2.2.5. **Lemma** ([SGA 6] I 2.2). *On a scheme X, the following conditions are equivalent for any complex $E^{\cdot}$ of $\mathcal{O}_X$-modules:*

2.2.5.1. For every point $x \in X$, there is a nbd U of x, a strict n-pseudo-coherent complex $F^{\cdot}$, and a quasi-isomorphism $F^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.5.2. For every point $x \in X$, there is a nbd U of x, a strict perfect complex $F^{\cdot}$, and an n-quasi-isomorphism $F^{\cdot} \to E^{\cdot}|U$.

2.2.5.3. For every point $x \in X$, there is a nbd U of x, a strict n-pseudo-coherent complex $F^{\cdot}$, and an isomorphism between $F^{\cdot}$ and $E^{\cdot}|U$ in the derived category $D(\mathcal{O}_U$-Mod$)$.

2.2.5.4. For every point $x \in X$, there is a nbd U of x, a strict perfect complex $F^{\cdot}$, and an n-quasi-isomorphism $F^{\cdot} \to E^{\cdot}|U$ in the derived category $D(\mathcal{O}_U$-Mod$)$ (that is, there is a map in the derived category inducing an epimorphism on H^n and an isomorphism on H^k for $k \geq n+1$).

Proof. We see that (1) $\Rightarrow$ (2) by replacing $F^{\cdot}$ in (1) by the strict perfect $\sigma^n F^{\cdot}$. Clearly (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3). It suffices then to show that (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1).

To see that (3) $\Rightarrow$ (1), we consider an isomorphism $F^{\cdot} \to E^{\cdot}|U$ in $D(\mathcal{O}_U$-Mod$)$. By the calculus of fractions, this is represented by a datum of strict maps which are quasi-isomorphisms $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$. By 2.2.4(a), after shrinking the *nbd* U, there is a strict n-pseudo-coherent

$F'^{\cdot}$ and a quasi-isomorphism $F'^{\cdot} \xrightarrow{\sim} G^{\cdot}$. Then the composite $F'^{\cdot} \xrightarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$ is the strict quasi-isomorphism required by (1).

To see that (4) $\Rightarrow$ (1), we represent the n-quasi-isomorphism in $D(\mathcal{O}_U\text{-Mod})$ by a datum of strict maps $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \to E^{\cdot}|U$ where $G^{\cdot} \to E^{\cdot}|U$ is an n-quasi- isomorphism. After shrinking U, by 2.2.4(c) there is a strict perfect $F'^{\cdot}$ and a quasi-isomorphism $F'^{\cdot} \to G^{\cdot}$. Then $F'^{\cdot} \to E^{\cdot}|U$ is an n-quasi-isomorphism. Applying the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \mathcal{D} = \mathcal{O}_U$-Mod and $\mathcal{C} = \mathcal{C}^-(\mathcal{A})$, we obtain a complex F'' and a quasi-isomorphism $F'' \xrightarrow{\sim} E^{\cdot}|U$, such that $\sigma^n F'' = \sigma^n F'$. Thus $\sigma^n F''$ is strict perfect, so F'' is strict n-pseudo-coherent as required.

2.2.6. *Definition* ([SGA 6] I 2.3). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be n-pseudo-coherent if any of the equivalent conditions 2.2.5.1 - 2.2.5.4 hold. The complex $E^{\cdot}$ is said to be pseudo-coherent if it is n- pseudo-coherent for all integers n.

2.2.7. Clearly pseudo-coherence of $E^{\cdot}$ depends only on the quasi-isomorphism class of $E^{\cdot}$, and is a local property on X. The cohomology sheaves $H^*(E^{\cdot})$ of a pseudo-coherent complex are all quasi-coherent $\mathcal{O}_X$-modules. If X is quasi-compact, and $E^{\cdot}$ is pseudo-coherent, there is an N such that $H^k(E^{\cdot}) = 0$ for all $k > N$, as this is true locally on X and since any open cover of X has a finite subcover.

A strict pseudo-coherent complex is pseudo-coherent. It will follow from 2.3.1(b) below that any pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules is locally quasi-isomorphic to a strict pseudo-coherent complex. In fact, it will be quasi-isomorphic to a strict pseudo-coherent complex on any affine open subscheme. For a pseudo-coherent complex of general $\mathcal{O}_X$-modules, there will locally be n-quasi-isomorphisms with a strict pseudo-coherent complex, but the local *nbds* where the n-quasi-isomorphisms are defined may shrink as n goes to $-\infty$, and so may fail to exist in the limit. So there may not be a local quasi-isomorphism with a strict pseudo-coherent complex. This phenomenon renders the auxiliary concept of n-pseudo-coherent necessary to our work.

2.2.8. *Example* ([SGA 6] I Section 3). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a noetherian scheme X is pseudo-coherent iff $E^{\cdot}$ is cohomologically bounded above and all the $H^k(E^{\cdot})$ are coherent $\mathcal{O}_X$-modules.

Proof. If E is pseudo-coherent, then $E^{\cdot}$ is locally $(k-1)$-quasi-isomorphic to a complex of coherent locally free sheaves. Computing $H^k(E^{\cdot})$ as H^k of the latter complex, we see that $H^k(E^{\cdot})$ is coherent.

Conversely, suppose $E^{\cdot}$ is cohomologically bounded above with coherent cohomology. Let m be an integer and x a point of X. We apply the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \mathcal{O}_{X,x}$-modules, $\mathcal{D} =$ free $\mathcal{O}_{X,x}$-

modules of finite type, and $\mathcal{C}$ = cohomologically bounded above complexes with coherent cohomology to produce a strict pseudo-coherent complex of $\mathcal{O}_{X,x}$-modules $F_x^{\cdot}$ and a quasi-isomorphism $F_x^{\cdot} \xrightarrow{\sim} E_x^{\cdot}$. Then $\sigma^m F_x^{\cdot}$ is strict perfect, and $\sigma^m F_x^{\cdot} \to E_x^{\cdot}$ is an m-quasi-isomorphic. The strict perfect complex $\sigma^m F_x^{\cdot}$ and the map $\sigma^m F_x^{\cdot} \to E_x^{\cdot}$ extend to a strict perfect complex $\sigma^m F^{\cdot}$ and map $\sigma^m F^{\cdot} \to E^{\cdot}|V$ on some small *nbd* V of x, as this requires only finitely many extensions of germs of sections of various sheaves at x. Choosing V smaller and using coherence of cohomology, we can arrange that $\sigma^m F^{\cdot} \to E^{\cdot}$ is an m-quasi-isomorphism on V. This shows criterion 2.2.5.2 holds for $E^{\cdot}$.

2.2.9. **Lemma** ([SGA] I Section 4). *The following conditions on a complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme are equivalent:*

2.2.9.1. For each point $x \in X$, there is an nbd U of x, a strict perfect complex $F^{\cdot}$ on U, and a quasi-isomorphism $F^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.9.2. For each point $x \in X$, there is an nbd U of x, a strict perfect complex $F^{\cdot}$ on U, and an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $E^{\cdot}|U$ and F.

Proof. Clearly (1) $\Rightarrow$ (2). To show (2) $\Rightarrow$ (1), we represent the isomorphism in the derived category via calculus of fractions as a datum of strict quasi-isomorphisms $F^{\cdot} \xleftarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$. Then we apply 2.2.4 (c) to $G^{\cdot} \xrightarrow{\sim} F^{\cdot}$ to produce after shrinking U a strict perfect $F'^{\cdot}$, and a strict quasi-isomorphism $F'^{\cdot} \xrightarrow{\sim} G^{\cdot} \xrightarrow{\sim} E^{\cdot}|U$.

2.2.10. *Definition* ([SGA 6] I 4.2). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme is perfect if it is locally quasi-isomorphic to a strict perfect complex, i.e., if 2.2.9.1 or 2.2.9.2 hold for $E^{\cdot}$.

2.2.11. *Definition* ([SGA 6] I 5.1). A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules has Tor-amplitude contained in $[a, b]$ for integers $a \leq b$ if for all $\mathcal{O}_X$-modules $\mathcal{F}$, $H^k(E^{\cdot} \otimes^L_{\mathcal{O}_X} \mathcal{F}) = 0$ unless $a \leq k \leq b$. If such an a and b exist, one says $E^{\cdot}$ has (globally) finite Tor-amplitude. If X is covered by opens U such that a and b exist on each U for $E^{\cdot}|U$, one says that $E^{\cdot}$ has locally finite Tor-amplitude.

2.2.12. **Proposition** ([SGA 6] I 5.8.1). *A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules is perfect iff $E^{\cdot}$ is pseudo-coherent and has locally finite Tor-amplitude.*

Proof. A strict perfect complex E' is flat and strictly bounded, so that $E'^{\cdot} \otimes^L \mathcal{F} = E'^{\cdot} \otimes \mathcal{F}$ is cohomologically bounded. So $E'^{\cdot}$ is pseudo-coherent and of finite Tor-amplitude. As a perfect complex $E^{\cdot}$ is locally quasi-isomorphic to a strict perfect $E'^{\cdot}$, it is pseudo-coherent and of locally finite Tor-amplitude.

Conversely, suppose $E^\cdot$ is pseudo-coherent of locally finite Tor-amplitude. Take any point $x \in X$, and pick a *nbd* U of x on which $E^\cdot$ has Tor-amplitude in $[a,b]$. By 2.2.5.3, after shrinking U, we may replace $E^\cdot$ up to quasi-isomorphism and assume that $E^\cdot$ is strict $(a-2)$-pseudo-coherent. Then as $\sigma^a E^\cdot$ is a strict perfect complex strictly bounded below by a, it has Tor-amplitude bounded below by a. Applying 2.2.11 with $\mathcal{F} = \mathcal{O}_X$, we see that $E^\cdot$ is cohomologically bounded between a and b. Thus $\sigma^a E^\cdot \to E^\cdot$ induces an isomorphism on cohomology H^k for $k \neq a$, and on H^a induces an epimorphism $H^a(\sigma^a E^\cdot) = Z^a E^\cdot \twoheadrightarrow H^a(E^\cdot)$. Let B be the kernel of this epimorphism, and $B[a]$ the complex consisting of B in degree a. Then $B[a] \to \sigma^a E^\cdot \to E^\cdot$ is a homotopy fibre sequence, i.e., 2 sides of a distinguished triangle in the derived category. Considering the induced long exact sequence for $H^*(\mathcal{F} \otimes^L (\ ))$, we get that $\mathrm{Tor}^k(\mathcal{F}, B) = H^{a-k}(\mathcal{F} \otimes^L B[a])$ is 0 for $k \geq 1$. Thus B is a flat $\mathcal{O}_U$-module. B is also the only non-vanishing cohomology group H^{a-1} of the mapping cone $M^\cdot$ of $\sigma^a E^\cdot \to E^\cdot$. As $\sigma^a E^\cdot$ is strictly perfect and $E^\cdot$ is strict $(a-2)$-pseudo-coherent, the mapping cone $M^\cdot$ is strict $(a-2)$-pseudo-coherent. By 1.9.4(b), $Z^{a-1}(M^\cdot)$ is a vector bundle, as is M^{a-2}. The exact sequence $M^{a-2} \to Z^{a-1}(M^\cdot) \to H^{a-1}(M^\cdot) \to 0$ shows that $B = H^{a-1}(M^\cdot)$ is a finitely presented $\mathcal{O}_U$-module. But as B is also flat, it is then a locally free $\mathcal{O}_U$-module of finite type, i.e., a vector bundle (e.g., [SGA 6] I 5.8.3, or Bourbaki). Now consider the truncated complex $\tau^a E^\cdot$. This differs from the strict perfect $\sigma^a E$ only in degree $a-1$, where $\tau^a E^\cdot$ has $B^a(E^\cdot) = \ker(Z^a E^\cdot \to H^a(E^\cdot)) = B$. As B is a vector bundle, $\tau^a E^\cdot$ is strict perfect. But as $E^\cdot$ is cohomologically bounded below by a, $E^\cdot \xrightarrow{\sim} \tau^a E^\cdot$ is a quasi-isomorphism on U. Thus $E^\cdot$ is perfect locally, and hence perfect, as required.

2.2.13. **Proposition** ([SGA 6] I). *Let x be a scheme.*

(a) *Suppose $A^\cdot \to B^\cdot \to C^\cdot$ is a homotopy fibre sequence in $D(\mathcal{O}_X\text{-}Mod)$, i.e., forms two sides of a distinguished triangle. Then:*

If $A^\cdot$ is $(n+1)$-pseudo-coherent and $B^\cdot$ is n-pseudo-coherent, then $C^\cdot$ is n-pseudo-coherent.

If $A^\cdot$ and $C^\cdot$ are n-pseudo-coherent, then $B^\cdot$ is n-pseudo-coherent.

If $B^\cdot$ is n-pseudo-coherent, and $C^\cdot$ is $(n-1)$-pseudo-coherent, then $A^\cdot$ is n-pseudo-coherent.

(b) *If $A^\cdot$, $B^\cdot$, $C^\cdot$ are the three vertices of a distinguished triangle in $D(\mathcal{O}_X\text{-}Mod)$, and 2 of these 3 vertices are pseudo-coherent (resp. perfect), then the third vertex is also pseudo-coherent (resp. perfect).*

(c) *The complex $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) iff both summands $F^\cdot$ and $G^\cdot$ are n-pseudo-coherent (resp. pseudo-coherent, resp. perfect).*

Proof. To prove (a), we first note by "rotating the triangle" ([V]

TR2 I Section 1 no. 1-1, or [H] I Section 1, TR2) that $A^\cdot \to B^\cdot \to C^\cdot$ is a homotopy fibre sequence iff $C^\cdot[-1] \xrightarrow{\partial} A^\cdot \to B^\cdot$ and $B^\cdot[-1] \to C^\cdot[-1] \to A^\cdot$ are. Also, it is clear that a shifted complex $F^\cdot[k]$ is $(n+k)$-pseudo-coherent iff $F^\cdot$ is n-pseudo-coherent. So it suffices to prove the first statement of (a). So assume that $A^\cdot$ is $(n+1)$-pseudo-coherent and that $B^\cdot$ is n-pseudo-coherent. We need to show that $C^\cdot$ is n-pseudo-coherent. This question is local, so it suffices to show it in an arbitrarily small open *nbd* U of each point $x \in X$. By definition, for U small, we may choose representatives of the quasi-isomorphism classes of $A^\cdot$ and $B^\cdot$ so that $A^\cdot$ is strict $(n+1)-$pseudo-coherent and $B^\cdot$ is strict n-pseudo-coherent. The map $A^\cdot \to B^\cdot$ in the derived category $D(\mathcal{O}_U\text{-Mod})$ is represented by a datum of strict maps $A^\cdot \xleftarrow{\sim} G^\cdot \to B^\cdot$. Applying 2.2.4(a) and shrinking U, there is a strict $(n+1)$-pseudo-coherent $A'^\cdot$, and a quasi- isomorphism $A'^\cdot \xrightarrow{\sim} G^\cdot \xrightarrow{\sim} A^\cdot$. Replacing $A^\cdot$ by $A'^\cdot$, we may assume we have a strict map $A^\cdot \to B^\cdot$. Now $C^\cdot$ is quasi-isomorphic to the mapping cone of $A^\cdot \to B^\cdot$. In degree k this cone is $A^{k+1} \oplus B^k$, and so is a vector bundle for $k \geq n$. So the cone is strict n-pseudo-coherent. So $C^\cdot$ is n-pseudo-coherent as required, proving (a).

Clearly (a) implies (b) for pseudo-coherence. To prove (b) for perfection, we reduce by rotating the triangle to show that if $A^\cdot$ and $B^\cdot$ are perfect, then the mapping cone $C^\cdot$ is perfect. We now argue as in the proof of (a), locally taking strict perfect representatives for $A^\cdot$ and $B^\cdot$, using 2.2.4(c) to reduce to the case where there is a strict map of complexes $A^\cdot \to B^\cdot$. Then the mapping cone is strict perfect, and is quasi-isomorphic to $C^\cdot$, which is hence perfect. This proves (b).

The non-trivial part of (c) is to show that if $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) then both factors $F^\cdot$ and $G^\cdot$ have the same property.

Suppose $F^\cdot \oplus G^\cdot$ is n-pseudo-coherent. We must show that for a small *nbd* U of $x \in X$ that $F^\cdot|U$ is n-pseudo-coherent. As the question is local, we may assume that $F^\cdot \oplus G^\cdot$ is quasi-isomorphic to a strict n-pseudo-coherent complex. Then there is an integer $N \gg 0$ such that $H^k(F^\cdot) \oplus H^k(G) = H^k(F \oplus G) = 0$ for $k \geq N$. Then $F^\cdot$ and $G^\cdot$ are trivially N-pseudo-coherent, as $0 \to F$ is an N-quasi-isomorphism. By descending induction on p for $n \leq p \leq N$ we show that $F^\cdot$ and $G^\cdot$ are p-pseudo-coherent. To do the induction step, suppose we already know that $F^\cdot$ and $G^\cdot$ are $(p+1)$-pseudo-coherent. Shrinking U, we may assume that $F^\cdot$ and $G^\cdot$ are strict $(p+1)$-pseudo-coherent, on replacing their representatives up to quasi-isomorphism. Consider the canonical homotopy fibre sequence, noting that $\sigma^{\leq p-1}A = A/\sigma^p A$

$$\sigma^{p+1}F^\cdot \oplus \sigma^{p+1}G^\cdot \to F^\cdot \oplus G^\cdot \to \sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot. \tag{2.2.13.1}$$

As $F^\cdot$ and $G^\cdot$ are strict $(p+1)$-pseudo-coherent, $\sigma^{p+1}F^\cdot \oplus \sigma^{p+1}G^\cdot$ is strict perfect. As $p \geq n$, $F^\cdot \oplus G^\cdot$ is p-pseudo-coherent, by 2.2.13(a) proved above, $\sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot$ is p-pseudo-coherent. By 2.2.3, $H^p(\sigma^{\leq p}F^\cdot) \oplus H^p(\sigma^{\leq p}G^\cdot) = H^p(\sigma^{\leq p}F^\cdot \oplus \sigma^{\leq p}G^\cdot)$ is an $\mathcal{O}_U$-module of locally finite type. Thus the summand $H^p(\sigma^{\leq p}F^\cdot)$ is locally of finite type. By 2.2.3 again, $\sigma^{\leq p}F^\cdot$ is p-pseudo-coherent. As $\sigma^{p+1}F^\cdot$ was even strict perfect, 2.2.13(a) now shows that $F^\cdot$ is p-pseudo-coherent. Similarly, $G^\cdot$ is p-pseudo-coherent. This completes the induction step. When the induction stops at $n = p$, it has proved (c) for the n-pseudo-coherent case. The pseudo-coherent case follows immediately.

It remains to do the perfect case. But if $F^\cdot \oplus G^\cdot$ is perfect, it is pseudo-coherent and of locally finite Tor-amplitude by 2.2.12. It is clear from definition 2.2.11 that the summands $F^\cdot$ and $G^\cdot$ have locally finite Tor-amplitude. Both are pseudo-coherent by the above. Thus by 2.2.12, $F^\cdot$ and $G^\cdot$ are perfect, as required.

2.3. In the presence of an ample family of line bundles, we have "global resolution" results:

2.3.1. **Proposition** (cf. [SGA 6] II). . *Let X be a quasi-compact and quasi-separated scheme with an ample family of line bundles. Then*

(a) *If $E^\cdot$ is a strict perfect complex, $F^\cdot$ any perfect strictly bounded below complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ is any strict map of complexes, then there exists a strict perfect complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x = x' \cdot a$.*

(b) *If $E^\cdot$ is any strict pseudo-coherent complex, $F^\cdot$ any pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ any strict map, then there exists a strict pseudo-coherent complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x' \cdot a = x$.*

(c) *If $E^\cdot$ is any strict n-pseudo-coherent complex, $F^\cdot$ any n-pseudo-coherent complex of quasi-coherent $\mathcal{O}_X$-modules, and $x : E^\cdot \to F^\cdot$ is any map, then there is a strict n-pseudo-coherent complex $F'^\cdot$ on X, a map $a : E^\cdot \to F'^\cdot$, and a quasi-isomorphism $x' : F'^\cdot \xrightarrow{\sim} F^\cdot$ such that $x' \cdot a = x$.*

(d) *If $F^\cdot$ is any perfect complex of $\mathcal{O}_X$-modules (perhaps not quasi-coherent), then there is a strict perfect complex $E^\cdot$ and an isomorphism in the derived category $D(\mathcal{O}_X\text{-Mod})$ between $E^\cdot$ and $F^\cdot$.*

(e) *Let $F^\cdot$ be any pseudo-coherent complex of $\mathcal{O}_X$-modules (perhaps not quasi-coherent). Suppose either that $F^\cdot \in D^b(\mathcal{O}_X\text{-Mod})$ is cohomologically bounded, or else that X is noetherian of finite Krull dimension. Then there is a strict pseudo-coherent complex $E^\cdot$ on X and an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $E^\cdot$ and $F^\cdot$.*

Proof. To prove (b), we apply the Inductive Construction Lemma 1.9.5 with $\mathcal{A} =$ Qcoh(X), $\mathcal{D} =$ the category of algebraic vector bundles, and $\mathcal{C} =$ the category of pseudo-coherent complexes in Qcoh(X). The hypothesis 1.9.5.1 holds by 2.2.3 and the ample family 2.1.3(c).

The proof of (c) is similar, using 1.9.5.9.

To prove (a), we note that (b) yields $E \to F''^{\cdot}$ and $F''^{\cdot} \xrightarrow{\sim} F^{\cdot}$ with F'' strict pseudo-coherent. $F''^{\cdot}$ is perfect as $F^{\cdot}$ is. We take an integer $n \ll 0$ so that $E^k = F^k = 0 = H^k(F''^{\cdot})$ for all $k \leq n$. As in the proof of 2.2.12, $B^n = \ker Z^n(F'') \to H^n(F'')$ is perfect, and hence a module of locally finite Tor-dimension, as $B^n[n]$ is quasi-isomorphic to the homotopy fibre of the map of perfect complexes $\sigma^n F''^{\cdot} \to F''^{\cdot}$. As X is quasi-compact, $B^n = B^n(F'')$ has globally finite Tor-dimension, say N. As $H^k(F'') = 0$ for $k \leq n$, $Z^k F'' = B^k F''$ for $k \leq n$, and $0 \to B^k F'' \to F''^k \to B^{k+1} F'' \to 0$ is exact for $k \leq n$. Considering the induced long exact sequence for $\mathrm{Tor}^*_{\mathcal{O}_x}(\ ,\mathcal{F})$ and the fact that F''^k is a vector bundle and hence is flat, we see that $B^{n-1}(F''^{\cdot})$ has Tor-dimension $N-1$, etc. Thus $B^{n-N}(F''^{\cdot})$ is flat. The exact sequence $F''^{n-N-2} \to F''^{n-N-1} \to B^{n-N}(F''^{\cdot}) \to 0$ shows that $B^{n-N}(F''^{\cdot})$ is also finitely presented. Thus $B^{n-N}(F''^{\cdot})$ is a vector bundle ([SGA 6] I 5.6.3, or Bourbaki). Thus the good truncation $\tau^{n-N}(F''^{\cdot})$ is strict perfect. By choice of n, $F''^{\cdot} \to F^{\cdot}$ factors into quasi-isomorphisms $F''^{\cdot} \xrightarrow{\sim} \tau^{n-N}(F''^{\cdot}) \xrightarrow{\sim} F^{\cdot}$. Setting $F'^{\cdot} = \tau^{n-N}(F''^{\cdot})$ then proves (a).

To prove (d), we note that the perfect $F^{\cdot}$ is cohomologically bounded. The coherator B.16 yields an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $F^{\cdot}$ and a perfect complex of quasi-coherent modules $RQ(F^{\cdot})$. (Note X satisfies the semiseparation hypothesis of B.16 because of the ample family of line bundles, cf. B.7.) For $n << 0, RQ(F^{\cdot})$ is quasi-isomorphic to $\tau^n RQ(F^{\cdot})$, which is still quasi-coherent. Now we apply (a) to $0 \to \tau^n RQ(F^{\cdot})$ to produce a strict perfect complex $E^{\cdot}$ quasi-isomorphic to $RQ(F^{\cdot})$ and $F^{\cdot}$.

The proof of (e) is similar to that of (d), using (b) instead of (a) at the last step.

2.3.2. **Proposition.** *Let X be a scheme with an ample family of line bundles. Let $E^{\cdot}$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a direct system of strict perfect complexes $\{F^{\cdot}_\alpha\}$, and a quasi-isomorphism*

$$\varinjlim F^{\cdot}_\alpha \xrightarrow{\sim} E^{\cdot}. \tag{2.3.2.1}$$

Proof. Let $E^{\cdot}(n)$ be the subcomplex $\tau^{\leq n}(E^{\cdot})$ of $E^{\cdot}$:

$$E^{\cdot}(n) = \ldots \to E^{n-2} \to E^{n-1} \to Z^n E^{\cdot} \to 0 \to 0 \to \ldots . \tag{2.3.2.2}$$

Then $E^\cdot$ is the direct colimit $E^\cdot = \varinjlim E^\cdot(n)$ as n goes to ∞.

By increasing induction on n, starting with $n = 0$, we construct a complex $E'^\cdot\{n\}$, which in each degree is an infinite direct sum of line bundles, and which is 0 in degrees above n. We construct this so that there is a quasi-isomorphism $E'^\cdot\{n\} \xrightarrow{\sim} E^\cdot(n)$, and a degree-wise split monomorphism $E'^\cdot\{n-1\} \rightarrowtail E'^\cdot\{n\}$ such that (2.3.2.3) commutes.

$$\text{(2.3.2.3)} \qquad \begin{array}{ccc} E'^\cdot\{n-1\} & \xrightarrow{\sim} & E^\cdot(n-1) \\ \downarrow & & \downarrow \\ E'^\cdot\{n\} & \xrightarrow{\sim} & E^\cdot(n) \end{array}$$

This construction is possible by the Inductive Construction Lemma 1.9.5 with $\mathcal{A} = \text{Qcoh}(X)$ and $\mathcal{D}$ = the category of sums of line bundles. Hypothesis 1.9.5.1 holds because of the ample family 2.1.3(a).

Now we consider the directed system whose objects consist of an integer $n \geq 0$ and a strict bounded subcomplex $F_\alpha^\cdot$ of $E'^\cdot\{n\}$ such that in each degree $F_\alpha^\cdot$ is a finite subsum of the given direct sum of line bundles which is $E'^\cdot\{n\}$ in that degree. The morphisms in the direct system are the obvious increases in n with inclusions of subcomplexes of $\varinjlim E'^\cdot\{n\}$. Given any finite subsum in $E'^\cdot\{n\}$, ∂ of it is of finite type, so is contained in a finite subsum with all degrees shifted one higher. Continuing in this way until we hit the bounding degree n, we see that any finite subsum in $E'^\cdot\{n\}$ is contained in a complex $F_\alpha^\cdot$ in the directed system. Thus for the subsystem with n fixed, $\varinjlim F_\alpha^\cdot = E'^\cdot\{n\}$. Thus over the full directed system $\varinjlim F_\alpha^\cdot = \varinjlim E'^\cdot\{n\}$, which is quasi-isomorphic to $\varinjlim E^\cdot(n) = E^\cdot$, as required.

2.3.2.4. *Porism.* If $E^\cdot$ in 2.3.2 also has $H^k(E^\cdot) = 0$ for $k > N$, we can choose the $F_\alpha^\cdot$ to be strictly 0 in degrees $k > N$. Indeed, then $E'^\cdot\{N\} \simeq E^\cdot(n) \simeq E^\cdot$ are quasi-isomorphisms, and we take the subsystem of $F_\alpha^\cdot$ in $E'^\cdot\{N\}$.

2.3.2.5. *Remark.* A general scheme is locally affine, and hence locally has an ample family of line bundles. Thus 2.3.2 holds locally on a general scheme. In Deligne's terms ([SGA 4] V 8.2) a quasi-coherent complex is a local inductive limit of strict perfect complexes.

2.3.3. **Corollary**. *Let X have an ample family of line bundles. Let $E^\cdot$ be a complex of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Suppose either $E^\cdot \in D^+(\mathcal{O}_X\text{-Mod})$ is cohomologically bounded below, or else that X is noetherian of finite Krull dimension.*

Then there is a direct system of strict perfect complexes $F_\alpha^\cdot$ in $C(Qcoh(X))$, and an isomorphism in $D(\mathcal{O}_X\text{-Mod})$ between $\varinjlim F_\alpha^\cdot$ and

$E^\cdot$. If $E^\cdot$ is also cohomologically bounded above (by N), then all the $F^\cdot_\alpha$ may be chosen to be strictly bounded above (by N).

Proof. Recall that X is semi-separated because of the ample family, B.7. Then by B.16, B.17, the coherator gives a quasi-isomorphism of $E^\cdot$ to a complex of quasi-coherent modules $RQ(E^\cdot)$. We conclude by applying 2.3.2 to this $RQ(E^\cdot)$.

2.4.1. **Theorem**. *Let X be a scheme, and $E^\cdot$ a perfect complex of $\mathcal{O}_X$-modules. Then:*

(a) *The derived functor $R\mathcal{H}om(E^\cdot, \ ) : D(\mathcal{O}_X\text{-Mod}) \to D(\mathcal{O}_X\text{-Mod})$ is defined on all $D(\mathcal{O}_X\text{-Mod})$, not just on $D^+(\mathcal{O}_X\text{-Mod})$.*

(b) *$R\mathcal{H}om(E^\cdot, \ )$ is locally of finite cohomological dimension. That is, for each point $x \in X$, there is an open nbd U of x and integers $a \leq b$ such that if $F^\cdot$ is any complex of $\mathcal{O}_X$-modules on U with $H^k(F^\cdot) = 0$ unless $c \leq k \leq d$ (resp. unless $c \leq k$; resp. unless $k \leq d$), then $H^k(R\mathrm{Hom}(E^\cdot, F^\cdot)) = 0$ unless $c - b \leq k \leq d - a$ (resp. unless $c - b \leq k$, resp. unless $k \leq d - a$).*

(c) *If $F^\cdot$ is a complex with quasi-coherent cohomology, the $R\mathcal{H}om(E^\cdot, F^\cdot)$ has quasi-coherent cohomology.*

(d) *For any U open in X, for any direct system $F^\cdot_\alpha$ of complexes of $\mathcal{O}_U$-modules, and for any integer k, the canonical map (2.4.1.1) is an isomorphism of sheaves of $\mathcal{O}_U$-modules:*

$$(2.4.1.1)\quad \varinjlim_\alpha H^k\left(R\mathrm{Hom}\left(E^\cdot|U, F^\cdot_\alpha\right)\right) \xrightarrow{\cong} H^k\left(R\mathrm{Hom}\left(E^\cdot|U, \varinjlim F^\cdot_\alpha\right)\right).$$

(e) *If X has noetherian underlying space of finite Krull dimension, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules, then the canonical map (2.4.1.2) is an isomorphism*

(2.4.1.2)
$$\varinjlim_\alpha \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, F^\cdot_\alpha\right) \xrightarrow{\cong} \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^\cdot, \varinjlim_\alpha F^\cdot_\alpha\right).$$

This says roughly that $\mathrm{Mor}_D(E^\cdot, \ )$ preserves direct colimits, with the qualification that the direct system and its colimit are taken in the strict category of complexes $C(\mathcal{O}_X-\mathrm{Mod})$, not in $D(\mathcal{O}_X-\mathrm{Mod})$.

(f) *If X is quasi-compact and quasi-separated, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology, then (2.4.1.2) is an isomorphism.*

(g) *If X is quasi-compact and quasi-separated, and $F^\cdot_\alpha$ is any direct system of complexes of $\mathcal{O}_X$-modules which is uniformly cohomologically* *bounded below (i.e.,* $\exists n \forall \alpha \forall k < n\ H^k(F^\cdot_\alpha) = 0$*), then* (2.4.1.2) *is an isomorphism.*

Proof. Recall the construction of the mapping complex $\mathrm{Hom}(E^\cdot, F^\cdot) = \prod \mathrm{Hom}\ (E^p, F^q)$. (See [H] II Section 3 for the usual details). The statement (a) on extension from D^+ to D follows from the finite cohomological dimension given by (b) in the usual way, (cf. [H] I 5.3 γ).

To prove (b), (c), and (d), we note these are local questions. Given a point $x \in X$, we take V to be a small *nbd* so that $E^\cdot|V$ is quasi-isomorphic to a strict perfect complex, and so reduce to the case where $E^\cdot$ is a bounded complex of vector bundles

$$(2.4.1.3) \qquad E^\cdot = \ldots \to 0 \to E^a \to E^{a+1} \to \ldots \to E^b \to 0 \to \ldots .$$

Shrinking V further, we may assume all the E^i are free of finite ranks k_i. Then $\mathrm{Hom}(E^i, F^\cdot)$ is $\oplus^{k_i} F^\cdot$, and $\mathrm{Hom}(E^\cdot, F^\cdot)$ is the total complex of the bicomplex (2.4.1.4) consisting of finitely many finite sums of shifted copies of $F^\cdot$:

(2.4.1.4)
$$\mathrm{Hom}(E^\cdot, F^\cdot)|V = \oplus^{k_a} F^\cdot[-a] \leftarrow \oplus^{k_{a+1}} F^\cdot[-a-1] \leftarrow \ldots \leftarrow \oplus^{k_b} F^\cdot[-b].$$

Clearly this $\mathrm{Hom}(E^\cdot, F^\cdot)|V$ is exact in $F^\cdot$, and so represents $R\mathrm{Hom}(E^\cdot, F^\cdot)|V$. Now it is clear that (b) holds with a and b the given strict bounds on $E^\cdot$. Also (c) is clear, and (d) is clear as (2.4.1.4) commutes with direct colimits.

To prove (g) we recall ([H] I 4.7, [V] II Section 1 no. 2-3 4, or our 1.9.5 dualized) that any $F^\cdot$ in $D^+(\mathcal{O}_X\text{-Mod})$ is quasi-isomorphic to a complex of injective $\mathcal{O}_X$-modules $I^\cdot$. Then $\mathrm{Hom}(E^\cdot, I^\cdot)$ is a complex of flasque sheaves ([SGA 4] V 4.10), and thus is deployed for computing $R\Gamma(X,\ )$. Hence $R\Gamma(X,\ R\mathrm{Hom}(E^\cdot, F^\cdot))$ is represented by $\Gamma(X,\ \mathrm{Hom}(E^\cdot, I^\cdot)) = \mathrm{Hom}_X(E^\cdot, I^\cdot)$, which also represents $R\mathrm{Hom}_X(E^\cdot, F^\cdot)$. The cohomology in degree 0, H^0, of the complex $\mathrm{Hom}_X(E^\cdot, I^\cdot)$ is the group of chain homotopy classes of maps $E^\cdot \to I^\cdot$, and as $I^\cdot$ is injective this is exactly $\mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(E^\cdot, F^\cdot)$.

Consider the Grothendieck spectral sequence

$$(2.4.1.5)\quad E_2^{p,q} = H^p\left(X; H^q R\mathrm{Hom}\left(E^\cdot, F^\cdot_\alpha\right)\right) \Rightarrow H^{p+q} R\mathrm{Hom}_X\left(E^\cdot, F^\cdot_\alpha\right).$$

By (d), $H^q R\mathrm{Hom}(E^\cdot,\ )$ preserves direct colimits, and by B.6, $H^p(X;\ )$ preserves direct colimits, so the $E_2^{p,q}$ term of the spectral sequence for $\varinjlim F^\cdot_\alpha$ is the direct colimit of the $E_2^{p,q}$ terms for the F_α. By the hypothesis of (g) that $\exists n \forall \alpha \forall k < n\ H^k(F^\cdot_\alpha) = 0$, the convergence of the spectral sequences is uniform in α, and it follows that $H^* R\mathrm{Hom}_X(E^\cdot,\ )$

preserves direct colimits of such systems $F^{\cdot}_{\alpha}$ (cf. [Th1] 5.50, 1.40). For $H^0(R\mathrm{Hom}_X(E^{\cdot},\ ))$, this yields the desired conclusion of (g).

It remains to prove (e) and (f). First we do this under the extra hypothesis that $E^{\cdot}$ is globally quasi-isomorphic to a strict perfect complex. Then on replacing $E^{\cdot}$ by a strict perfect complex, we get that $R\mathrm{Hom}(E^{\cdot},\ )$ is represented by $\mathrm{Hom}(E^{\cdot},\ )$, which is a well-defined functor into the category of complexes, not just into $D(\mathcal{O}_X\text{-Mod})$. We consider the Grothendieck spectral sequence. As above, the $E_2^{p,q}$ term, $H^p(X; H^q R\mathrm{Hom}(E^{\cdot},\ )$ preserves direct colimits. Also, there is an integer N such that $H^p(X; H^q R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha}) = 0$ for $p > N$. In case (e), this is because X of finite Krull dimension has finite Zariski cohomological dimension by [Gro] 3.6.5. In case (f), this is because all $H^q R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha})$ are quasi-coherent $\mathcal{O}_X$-modules by (c), and because X has finite cohomological dimension for quasi-coherent modules by B.11 applied to $X \to \mathrm{Spec}(\mathbb{Z})$. This N gives uniform convergence of the Grothendieck spectral sequence to $H^{p+q}(R\Gamma(X; R\mathrm{Hom}(E^{\cdot}, F^{\cdot}_{\alpha}))$. From this uniform convergence and the fact that the $E_2^{p,q}$ terms preserve direct colimits, it follows that the $H^*(R\Gamma(X; R\mathrm{Hom}(E^{\cdot},\ ))$ preserve the appropriate direct colimits (cf. [Th1] 1.40).

It remains in cases (e) and (f) to identify $R\Gamma(X, R\mathrm{Hom}(E^{\cdot}, F^{\cdot}))$ to $R\mathrm{Hom}_X(E^{\cdot}, F^{\cdot})$, or more precisely to show that the canonical augmentation map (2.4.1.6) is an isomorphism

$$\text{(2.4.1.6)} \qquad \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(E^{\cdot}, F^{\cdot}) \to H^0(X, R\mathrm{Hom}(E^{\cdot}, F^{\cdot})).$$

As we are considering the case where $E^{\cdot}$ is strict perfect, the obvious devissage shows it suffices to prove (2.4.1.6) is an isomorphism when $E^{\cdot}$ is a single vector bundle E^i. But as $H^0(X;\ )$ is clearly isomorphic to $\mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}(\mathcal{O}_X\ )$, this reduces to the obvious adjointness of the functors $\otimes E^i$ and $\mathrm{Hom}(E^i,\ )$. This proves (e) and (f) when $E^{\cdot}$ satisfies the extra hypothesis that it is globally quasi-isomorphic to a strict perfect complex.

This extra hypothesis is always met if X has an ample family of line bundles 2.3.1(d). We prove (e) and (f) that the map (2.4.1.2) is an isomorphism without the extra hypothesis by induction on the number of open quasi-affine schemes needed to cover X. The induction starts because a quasi-affine scheme has an ample line bundle $\mathcal{O}_X$. The induction step follows immediately by comparing the exact sequence of 2.4.1.7 below for $\varinjlim F^{\cdot}_{\alpha}$ and the direct colimit of these exact sequences for the $F^{\cdot}_{\alpha}$. Thus the proof of Lemma 2.4.1.7 below will complete the proof of Theorem 2.4.1.

2.4.1.7. **Lemma.** *Let $U \cup V$ be a scheme, covered by open subschemes U and V. Denote the open immersions by $j : U \to U \cup V$, k :*

$V \to U \cup V$, and $\ell : U \cap V \to U \cup V$. Then for any complexes $E^{\cdot}$ and $F^{\cdot}$ of $\mathcal{O}_{U \cup V}$-modules, there is a long exact sequence Mayer-Vietoris of groups of morphisms in the derived categories $D(X) = D(\mathcal{O}_X$-modules) for $X = U \cup V$, U, V, and $U \cap V$;

(2.4.1.8)

$$
\begin{array}{ccc}
 & \cdots & \\
 & \downarrow & \\
\mathrm{Mor}_{D(U)}(j^*E^{\cdot}[1], j^*F^{\cdot}) & \oplus & \mathrm{Mor}_{D(V)}(k^*E^{\cdot}[1], k^*F^{\cdot}) \\
 & \downarrow & \\
 & \mathrm{Mor}_{D(U\cap V)}(\ell^*E^{\cdot}[1], \ell^*F^{\cdot}) & \\
 & \downarrow{\scriptstyle \partial} & \\
 & \mathrm{Mor}_{D(U\cup V)}(E^{\cdot}, F^{\cdot}) & \\
 & \downarrow & \\
\mathrm{Mor}_{D(U)}(j^*E^{\cdot}, j^*F^{\cdot}) & \oplus & \mathrm{Mor}_{D(V)}(k^*E^{\cdot}, k^*F^{\cdot}) \\
 & \downarrow & \\
 & \mathrm{Mor}_{D(U\cap V)}(\ell^*E^{\cdot}, \ell^*F^{\cdot}) & \\
 & \downarrow{\scriptstyle \partial} & \\
 & \mathrm{Mor}_{D(U\cup V)}(E^{\cdot}[-1], F^{\cdot}) & \\
 & \downarrow & \\
 & \cdots &
\end{array}
$$

Proof. Recall that j is flat, so $j^* = Lj^*$ is exact. Let $j! : \mathcal{O}_U$-Mod $\to \mathcal{O}_{U \cup V}$-Mod extension by 0, the functor left adjoint to j^*, ([SGA 4] IV 11.3.1). Then $j!$ is also exact. As both functors are exact and adjoint, they induce adjoint functors on the derived category. So there is a canonical isomorphism for all complexes $G^{\cdot}$ in $D(U)$ and $H^{\cdot}$ in $D(U \cup V)$

$$\mathrm{Mor}_{D(U\cup V)}\left(j!G^{\cdot}, H^{\cdot}\right) \cong \mathrm{Mor}_{D(U)}\left(G^{\cdot}, j^*H^{\cdot}\right) \tag{2.4.1.9}$$

There is a canonical exact sequence of complexes on $U \cup V$

$$0 \to \ell!\ell^*E^{\cdot} \to j!j^*E^{\cdot} \oplus k!k^*E^{\cdot} \to E^{\cdot} \to 0 \tag{2.4.1.10}$$

The maps in (2.4.1.10) are induced by the adjunction maps. Locally, and in fact after restriction to U and to V, this sequence is split exact, hence exact globally (cf. 3.20.4 below). This exact sequence induces a long exact sequence of $\mathrm{Ext}^*_{D(U\cup V)}(\ \ , F^{\cdot}) = \mathrm{Mor}_{D(U\cup V)}((\ \ )[-*], F^{\cdot})$ in the usual way ([V] I Section 1 1-2, or [H] I 6.1). Interpreting this long exact sequence by means of the isomorphisms (2.4.1.9) for j, k, and ℓ yields the long exact sequence (2.4.1.8).

2.4.2. **Theorem**. *Let X have an ample family of line bundles. Let $E^{\cdot}$ be a complex of $\mathcal{O}_X$-modules, with quasi-coherent cohomology. Then the following are equivalent:*

(a) *$E^{\cdot}$ is pseudo-coherent.*

(b) *For all integers n and k, and all direct systems $\{F_\alpha^{\cdot}\}$ of complexes of $\mathcal{O}_X$-modules, the canonical map (2.4.2.1) is an isomorphism of $\mathcal{O}_X$-modules*

(2.4.2.1)

$$\varinjlim_\alpha H^k\left(R\mathrm{Hom}\left(E^{\cdot}, \tau^n F_\alpha^{\cdot}\right)\right) \xrightarrow{\cong} H^k\left(R\mathrm{Hom}\left(E^{\cdot}, \varinjlim_\alpha \tau^n F_\alpha^{\cdot}\right)\right).$$

(c) *For all n, k, and all direct systems of strict perfect complexes $F_\alpha^{\cdot}$, the map (2.4.2.1) is an isomorphism.*

(d) *$E^{\cdot}$ is in $D^-(\mathcal{O}_X\text{-Mod})$, and for all n, k and all direct systems of strict perfect complexes $F_\alpha^{\cdot}$ which are uniformly bounded above ($\exists\, m\, \forall\, \alpha$ $\forall\, i > m$ $F_\alpha^i = 0$), the map (2.4.2.1) is an isomorphism.*

(e) *For all integers n, and all direct systems $\{F_\alpha^{\cdot}\}$ of complexes of $\mathcal{O}_X$-modules, the canonical map (2.4.2.2) is an isomorphism*

(2.4.2.2)

$$\varinjlim_\alpha \ \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^{\cdot}, \tau^n F_\alpha^{\cdot}\right) \xrightarrow{\cong} \ \mathrm{Mor}_{D(\mathcal{O}_X-\mathrm{Mod})}\left(E^{\cdot}, \varinjlim_\alpha \tau^n F_\alpha^{\cdot}\right)$$

That is, roughly speaking, $\mathrm{Mor}_D(E^{\cdot},\ )$ preserves direct colimits of uniformly cohomologically bounded below systems.

(f) *For all integers n, and all direct systems $\{F_\alpha^{\cdot}\}$ of strict perfect complexes, the map (2.4.2.2) is an isomorphism.*

(g) *$E^{\cdot}$ in $D^-(\mathcal{O}_X\text{-Mod})$, and for all n and all direct systems of strict perfect complexes $F_\alpha^{\cdot}$ which are uniformly bounded above, the map (2.4.2.2) is an isomorphism.*

Proof. We fix an integer n, and note that the good truncation τ^n preserves direct colimits, so $\tau^n(\varinjlim F_\alpha^{\cdot}) = \varinjlim \tau^n F_\alpha^{\cdot}$. Also, any map $f : E^{\cdot} \to \tau^n F^{\cdot}$ factors uniquely through $E^{\cdot} \to \tau^n E^{\cdot}$ as the composite with $\tau^n f : \tau^n E^{\cdot} \to \tau^n \tau^n F^{\cdot} = \tau^n F^{\cdot}$. Similarly, $R\mathrm{Hom}(E^{\cdot}, \tau^n F^{\cdot})$ is quasi-isomorphic to $R\mathrm{Hom}(\tau^{n-k-1}E^{\cdot}, \tau^n F^{\cdot})$ in degrees less than or equal to k. So all the conclusions of statements (b), (c), $\ldots$, (g) depend only on $\tau^m E^{\cdot}$ for some m, where m possibly depends on n and k.

Now as $E^{\cdot}$ has quasi-coherent cohomology, $\tau^{m-1}E^{\cdot}$ is quasi-isomorphic to a complex of quasi-coherent modules by the coherator B.16.

We show (a) $\Rightarrow$ (b). As $E^{\cdot}$ is pseudo-coherent, $\tau^{m-1}E^{\cdot}$ is $(m-1)$-pseudo-coherent. By 2.3.1(c) applied to a complex of quasi-coherent modules quasi-isomorphic to $\tau^{m-1}E^{\cdot}$, $\tau^{m-1}E^{\cdot}$ is quasi-isomorphic to some strict $(m-1)$-pseudo-coherent complex $E'^{\cdot}$. Then $\sigma^{m-1}E'^{\cdot}$ is strict perfect, and $\tau^m\sigma^{m-1}E'^{\cdot}$ is quasi-isomorphic to $\tau^m E^{\cdot}$. So to prove (b) for given n and k, we take an appropriate m, and replace $E^{\cdot}$ with a strict perfect complex which has truncation τ^m quasi-isomorphic to $\tau^m E^{\cdot}$. Then it suffices to prove (b) for $E^{\cdot}$ perfect, which follows from 2.4.1(d).

Clearly (b) $\Rightarrow$ (c).

To show (c) $\Rightarrow$ (d), the problem is to show that $E^{\cdot}$ is cohomologically bounded above. It suffices to show $\tau^0 E^{\cdot}$ is so. As $\tau^0 E^{\cdot}$ is in $D^+(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology, it is quasi-isomorphic to the direct colimit of a direct system of strict perfect complexes $F^{\cdot}_\alpha$ by 2.3.3. Consider the induced quasi- isomorphism in $D^+(\mathcal{O}_X\text{-Mod})$, $\tau^0 E \to \tau^0(\varinjlim F^{\cdot}_\alpha) = \varinjlim \tau^0 F^{\cdot}_\alpha$. This gives a section of the sheaf $H^0(R\mathrm{Hom}(\tau^0 E^{\cdot}, \tau^0(\varinjlim F^{\cdot}_\alpha)))$, and hence via the map induced by $E^{\cdot} \to \tau^0 E^{\cdot}$, it gives a section of $H^0(R\mathrm{Hom}(E^{\cdot}, \tau^0(\varinjlim F^{\cdot}_\alpha)))$. By hypothesis (c), at every point $x \in X$, the germ of this section comes from some section of a $H^0(R\mathrm{Hom}(E^{\cdot}, \tau^0(F^{\cdot}_\alpha)))_x$. Replacing $\tau^0 F^{\cdot}_\alpha$ by a quasi-isomorphic complex of injectives $I^{\cdot}$ to compute $R\mathrm{Hom}$, we get a germ in $H^0(\mathrm{Hom}(E^{\cdot}, I^{\cdot}))_x$. This means there is a *nbd* U of x, and a chain map $E^{\cdot} \to I^{\cdot}$ on U, representing the class of a map $E \to \tau^0 F^{\cdot}_\alpha$ in $D^+(\mathcal{O}_U\text{-Mod})$, and such that $E^{\cdot} \to \tau^0 F^{\cdot}_\alpha \to \tau^0(\varinjlim F^{\cdot}_\alpha) \simeq \tau^0(E^{\cdot})$ is the canonical map $E^{\cdot} \to \tau^0 E^{\cdot}$ in $D^+(\mathcal{O}_U\text{-Mod})$. Thus $\tau^0 E^{\cdot}$ splits off $\tau^0 F^{\cdot}_\alpha$ in $D^+(\mathcal{O}_U\text{- Mod})$. As $F^{\cdot}_\alpha$ is strict perfect, it is bounded above. Hence on U, $\tau^0 F^{\cdot}_\alpha$ and so $\tau^0 E^{\cdot}$ are cohomologically bounded above. Thus $E^{\cdot}$ is locally cohomologically bounded above. As X is quasi-compact (2.1.1), it follows that $E^{\cdot}$ is globally cohomologically bounded above, as required.

To see that (b) $\Rightarrow$ (e), (c) $\Rightarrow$ (f) and (d) $\Rightarrow$ (g), we note that the hypotheses imply that $E^{\cdot}$ is in $D^-(\mathcal{O}_X\text{-Mod})$ as above. Also, the $\tau^n F^{\cdot}_\alpha$ are uniformly cohomologically bounded below. Combining these facts, we see that $R\mathrm{Hom}(E, \tau^n F^{\cdot}_\alpha)$ is uniformly cohomologically bounded below. This shows that the Grothendieck spectral sequence (2.4.1.5) computing $H^0(R\mathrm{Hom}_X(E^{\cdot}, \tau^n F^{\cdot})) = \mathrm{Mor}_{D(X)}(E^{\cdot}, \tau^n F^{\cdot})$ converges uniformly in α. Now (b) $\Rightarrow$ (e), (c) $\Rightarrow$ (f), and (d) $\Rightarrow$ (*g*) follow as in the proof of 2.4.1(g).

Clearly, (e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) as (f) implies that $E^{\cdot}$ is cohomologically bounded above similarly to the proof of (d) above.

Finally (g) $\Rightarrow$ (a). For it suffices to show for all n that $\tau^n E^{\cdot}$ is n-pseudo-coherent, since then $E^{\cdot}$ is $(n-1)$-quasi-isomorphic to the n-pseudo-coherent $\tau^n E^{\cdot}$, and thus locally n-quasi-isomorphic to a strict perfect complex, 2.2.5.4. As $\tau^n E^{\cdot}$ is cohomologically bounded, 2.3.3 shows that $\tau^n E^{\cdot}$ is quasi-isomorphic with a direct colimit of a direct system of strict perfect complexes $F^{\cdot}_\alpha$ which are uniformly bounded above. Then there are quasi-isomorphisms $\tau^n E^{\cdot} \simeq \varinjlim F^{\cdot}_\alpha \simeq \varinjlim \tau^n F^{\cdot}_\alpha$. By hypothesis (g), the map $E^{\cdot} \to \tau^n E^{\cdot}_\alpha \simeq \varinjlim \tau^n F^{\cdot}_\alpha$ factors $E^{\cdot} \to \tau^n F^{\cdot}_\alpha \to \varinjlim \tau^n F^{\cdot}_\alpha$ for some α.

Then $\tau^n E^\cdot$ is a retract of $\tau^n F_\alpha^\cdot$. As $F_\alpha^\cdot$ is strict pseudo-coherent, $\tau^n F_\alpha^\cdot$ is n-pseudo-coherent, and hence $\tau^n E^\cdot$ is n-pseudo-coherent by 2.2.13(c). This completes the proof.

2.4.2.3. As pseudo-coherence is a local property, and any scheme is locally affine, hence locally has an ample family of line bundles, 2.4.2 is a useful criterion to apply locally on a general scheme.

2.4.3. **Theorem.** *Let X be a scheme with an ample family of line bundles. Let $E^\cdot$ be a complex of $\mathcal{O}_X$-modules. Suppose that $E^\cdot$ has quasi-coherent cohomology. Then the following are equivalent:*

(a) *$E^\cdot$ is perfect.*

(b) *$E^\cdot$ is quasi-isomorphic with a strict perfect complex.*

(c) *$E^\cdot$ is cohomologically bounded below, and for any direct system of complexes $F_\alpha^\cdot$ with quasi-coherent cohomology, $\mathrm{Mor}_{D(X)}(E^\cdot, \ )$ preserves the direct colimit in that the map (2.4.1.2) is an isomorphism.*

(d) *$E^\cdot$ is cohomologically bounded, and the map (2.4.1.2) is an isomorphism for any direct system $F_\alpha^\cdot$ of strict perfect complexes which are uniformly cohomologically bounded above.*

Proof. (a) $\Rightarrow$ (b) by 2.3.1(d). We have (b) $\Rightarrow$ (c) by 2.4.1(f). To show (c) $\Rightarrow$ (d), the main point is to show that $E^\cdot$ is also cohomologically bounded above. But this follows from the proof of 2.4.2(c) $\Rightarrow$ (d).

Now to show (d) $\Rightarrow$ (a), we note by 2.3.3 that $E^\cdot$ is quasi-isomorphic to the direct colimit of a direct system of strict perfect complexes which are uniformly bounded above. By (d), the quasi-isomorphism in $D(\mathcal{O}_X\text{-Mod})$, $E^\cdot \to \varinjlim F_\alpha^\cdot$, factors through some $F_\alpha^\cdot$. Thus in $D(\mathcal{O}_X\text{-Mod})$, $E^\cdot$ is a summand of the perfect $F_\alpha^\cdot$. Hence by 2.2.13(c), $E^\cdot$ is perfect as required.

2.4.3.1. As perfection is a local property, the theorem may be applied to the open affines on a general scheme to test perfection there.

2.4.4. To summarize, 2.4.3 roughly characterizes perfect complexes on schemes with ample families of line bundles as the finitely presented objects (in the sense of Grothendieck [EGA] IV 8.14 that Mor out of them preserves direct colimits) in the derived category $D(\mathcal{O}_X\text{-Mod})_{qc}$ of complexes with quasi-coherent cohomology. On a general scheme, the prefect complexes are the locally finitely presented objects in the "homotopy-stack" of derived categories. (We must say "roughly characterizes" as we always take our direct systems in the category $C(\mathcal{O}_X\text{-Mod})$ of chain complexes, and have not examined the question of lifting a direct system if $D(\mathcal{O}_X\text{-Mod})$ to $C(\mathcal{O}_X\text{-Mod})$ up to cofinality.)

This characterization, with Lemma 2.3.3, will be the basis for the key extension Lemma 5.5.1. A more immediate application will be to the functoriality statement 2.6.3 below.

2.5.1. ([SGA 6] I.2). Let $f : X \to Y$ be a map of schemes. For $E^{\cdot}$ a strict perfect complex on Y, $f^*E^{\cdot}$ is clearly a strict perfect complex on X. This complex represents $Lf^*E^{\cdot}$, as the vector bundles E^i are flat over $\mathcal{O}_Y$ and hence deployed for Lf^*.

In general, $Lf^* : D^-(\mathcal{O}_Y\text{-Mod}) \to D^-(\mathcal{O}_X\text{- Mod})$ sends perfect complexes to perfect complexes. For the question is local on Y, hence reduces to the case where Y is affine and where any perfect complex is quasi-isomorphic to a strict perfect one.

Similarly, $Lf^* = f^*$ preserves strict pseudo-coherent complexes, and it follows that $Lf^* : D^-(\mathcal{O}_Y\text{-Mod}) \to D^-(\mathcal{O}_X\text{Mod})$ preserves pseudo-coherence. If the map f also has finite Tor-dimension, and so induces a functor $Lf^* : D^b(\mathcal{O}_Y\text{-Mod}) \to D^b(\mathcal{O}_X\text{-Mod})$, Lf^* preserves cohomologically bounded pseudo-coherent complexes. (Recall [SGA 6] III 3.1 that f is said to have finite Tor-dimension if $\mathcal{O}_X$ is of finite Tor-dimension as a sheaf of modules over the sheaf of rings $f^{-1}(\mathcal{O}_Y)$ on X.)

If $E^{\cdot}$ and $F^{\cdot}$ are strict pseudo-coherent, $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot} = E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is also strict pseudo-coherent. If $E^{\cdot}$ and $F^{\cdot}$ are strict n-pseudo-coherent and strict m-pseudo-coherent respectively, $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ may be taken to be isomorphic to the strict $(m+n)$-pseudo-coherent $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot}$ in degrees greater than or equal to $m+n$. (We apply 1.9.5 with $\mathcal{D} =$ flat $\mathcal{O}_X$-modules to replace $E^{\cdot}$ by a quasi-isomorphic flat complex without changing $E^{\cdot}$ in degree $\geq m$, etc.) It follows that if $E^{\cdot}$ and $F^{\cdot}$ are pseudo-coherent, then $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is pseudo-coherent. If $E^{\cdot}$ is perfect, hence of finite Tor-amplitude, and $F^{\cdot}$ is cohomologically bounded and pseudo-coherent, then $E^{\cdot} \otimes^L_{\mathcal{O}_X} F^{\cdot}$ is also cohomologically bounded and pseudo-coherent.

2.5.2. *Definition* ([SGA 6] III). Let $f : X \to Y$ be a map locally of finite type between schemes.

The map f is n-pseudo-coherent at $x \in X$ if there is a *nbd* U of x and an open $V \subseteq Y$ with $f : U \to V$ factoring as $f = gi$, where $i : U \to Z$ is a closed immersion with $i_*\mathcal{O}_U$ n-pseudo-coherent as a complex on Z, and where $g : Z \to V$ is smooth. (The property that $i_*\mathcal{O}_U$ is n-pseudo-coherent is independent of the choice of Z meeting the other conditions by [SGA 6] III 1.1.4. Hence this property depends only on f.)

The map f is n-pseudo-coherent if f is n-pseudo-coherent at x for all points $x \in X$.

The map f is pseudo-coherent if it is n-pseudo-coherent for all integers n.

The map f is perfect if f is pseudo-coherent and of locally finite Tor-dimension.

2.5.3. *Examples* ([SGA 6]).

(a) For Y noetherian, any $f : X \to Y$ locally of finite type is pseudo-coherent.

(b) Any smooth map $f : X \to Y$ is perfect.

(c) Any regular closed immersion ([SGA 6] VII 1.4) $f : X \to Y$ is perfect ([SGA 6] III 1.1.2).

(d) Any locally complete intersection morphism ([SGA 6] VIII 1.1) is perfect.

(e) For Y not noetherian, a closed immersion need not be pseudo-coherent, for $\mathcal{O}_X$ need not be even finitely presented over $\mathcal{O}_Y$.

2.5.4. **Theorem** ([SGA6] III 2.5, 4.8.1). *Let $f : X \to Y$ be a proper map of schemes. Suppose either that f is projective, or that Y is locally noetherian. Suppose that f is a pseudo-coherent (respectively, a perfect) map. Then if $E^{\cdot}$ is a pseudo-coherent (resp. perfect) complex on X, $Rf_*(E^{\cdot})$ is pseudo-coherent (resp. perfect) on Y.*

Proof. The case f projective will follow from the slightly more general results 2.7 below on taking Z there to be a projective space bundle over Y locally.

The case Y locally neotherian follows from the Grothendieck Finiteness Theorem ([EGA] III 3.2) that $R^p f_*$ preserves coherence, the finite cohomological dimension of Rf_* (B.11), the strongly converging spectral sequence $R^p f_*(H^q(E^{\cdot})) \Rightarrow H^{p+q}(Rf_*(E^{\cdot}))$ and criterion 2.2.8 that a complex on a noetherian scheme is pseudo-coherent iff it is cohomologically bounded above with coherent cohomology. This shows $Rf_*(E^{\cdot})$ is pseudo-coherent when $E^{\cdot}$ is. When f and $E^{\cdot}$ are also perfect, we show that $Rf_*(E^{\cdot})$ has locally finite-Tor-amplitude using base-change 2.5.5 locally. Then criterion 2.2.12 shows that $Rf_*(E^{\cdot})$ is perfect.

2.5.5. **Theorem** ([SGA 6] III 3.7). *Let Y be a quasi-compact scheme, and let $f : X \to Y$ be a quasi-compact and quasi-separated map. Let $E^{\cdot}$ be a cohomologically bounded complex of $\mathcal{O}_X$-modules with quasi-coherent cohomology, and let $F^{\cdot}$ be a cohomologically bounded complex of $\mathcal{O}_Y$-modules with quasi-coherent cohomology. Assume either that $F^{\cdot}$ has finite Tor-amplitude over $\mathcal{O}_Y$, or else that $E^{\cdot}$ has finite Tor-amplitude over $\mathcal{O}_X$. Then the canonical map is a quasi-isomorphism in* $D(\mathcal{O}_Y-\mathrm{Mod})$

$$Rf_*(E^{\cdot}) \otimes^L_{\mathcal{O}_Y} F^{\cdot} \xrightarrow{\sim} Rf_*\left(E^{\cdot} \otimes^L_{\mathcal{O}_X} Lf^* F^{\cdot}\right). \tag{2.5.5.1}$$

Proof. See [SGA 6] III 3.7. We sketch the argument for $F^{\cdot}$ perfect, the main case used in this paper. The question is local, so we may assume Y is affine. Then $F^{\cdot}$ is quasi-isomorphic to a strict perfect complex 2.3.1(d). For $F^{\cdot}$ strict perfect, filtering $F^{\cdot}$ by $\sigma^n F$ and comparing the long exact sequences of cohomology of the two sides of (2.5.5.1) induced by $0 \to \sigma^{n+1}F \to \sigma^n F \to F^n \to 0$, we reduce to the case where $F^{\cdot}$ is a single vector bundle F^n. Shrinking Y further, we may assume that $F^{\cdot}$ is free, so is a $\oplus^k \mathcal{O}_Y$. Then (2.5.5.1) reduces to the sum of k copies of the identity map of $Rf_*(E^{\cdot})$, and so is clearly an isomorphism.

2.5.6. **Theorem** ([SGA 6] IV 3.1). *Let (2.5.6.1) be a pull-back square of schemes*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "g"] \end{tikzcd} \tag{2.5.6.1}$$

Suppose Y is quasi-compact, and that f is a quasi-compact and quasi-separated map. Suppose that f and g are Tor-independent over Y so that given $x \in X$, $y' \in Y'$ with $f(x) = y = g(y')$, then for all integers $p \geq 1$ we have

$$\mathrm{Tor}^p_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x},\ \mathcal{O}_{Y',y'}) = 0. \tag{2.5.6.2}$$

Let $E^{\cdot}$ be a cohomologically bounded complex on X, with quasi-coherent cohomology. Suppose either that $E^{\cdot}$ has finite Tor-amplitude over the sheaf of rings $f^{-1}(\mathcal{O}_Y)$ on the space X, or else that the map g has finite Tor-dimension. Then there is a canonical base-change quasi-isomorphism

$$Lg^* Rf_* E^{\cdot} \xrightarrow{\sim} Rf'_* Lg'^* E^{\cdot}. \tag{2.5.6.3}$$

Proof. [SGA 4] XVII 4.2.12 defines the map, and [SGA 6] IV 3.1 shows it is an isomorphism in the derived category (cf. 3.18. below).

2.5.6.4. An examination of the proof shows that the Tor-independence hypothesis (2.5.6.2) may be weakened to be required only for those x in a closed subspace $Z \subseteq X$ such that $E^{\cdot}$ is acyclic on $X - Z$.

2.5.7. **Proposition** (Trivial Duality). *Let $f : X \to Y$ be a map of schemes. Then for $E^{\cdot}$ in $D^-(\mathcal{O}_Y\text{-Mod})$ and $G^{\cdot}$ in $D^+(\mathcal{O}_X\text{-Mod})$ there is a canonical isomorphism derived from adjointness between f^* and f_**

$$\mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Lf^*E^{\cdot},\ G^{\cdot}\right) \cong \mathrm{Mor}_{D(\mathcal{O}_Y\text{-Mod})}\left(E^{\cdot},\ Rf_*G^{\cdot}\right). \tag{2.5.7.1}$$

Proof. This is [SGA 4] XVII 2.3.7. See also [SGA 6] IV 3, [H] II 5.1, [V] II, Section 3 no. 3. Also see [SGA $4\frac{1}{2}$], Erratum to [SGA 4], to correct an argument preceding all our cited base-change results.

2.6.1. **Lemma**. *Let X be a quasi-compact scheme, $|Y|$ a closed subspace, and $U = X - |Y|$ the complementary open subscheme. Then*

(a) If there is a finitely presented closed immersion $i : Y \to X$ with underlying space $|Y|$, then U is quasi-compact and $j : U \to X$ is a quasi-compact map.

(b) If $Y = \mathrm{Spec}(\mathcal{O}_X/\mathcal{J})$ and $Y' = \mathrm{Spec}(\mathcal{O}_X/\mathcal{I})$ are two finitely presented closed subschemes of X, both with underlying space $|Y|$, then there exists an integer $n \geq 1$ such that $\mathcal{J}^n \subseteq \mathcal{I}$ and $\mathcal{I}^n \subseteq \mathcal{J}$.

(c) If X is also quasi-separated, and if U is quasi-compact, then there exists a finitely presented closed immersion $i : Y \to X$ with underlying space $|Y|$.

Proof. (a) It suffices to show that j is a quasi-compact map. Let $\mathrm{Spec}(A) \subseteq X$ be an affine open, and let $Y \cap \mathrm{Spec}(A) = \mathrm{Spec}(A/J)$. As Y is finitely presented, $J \subseteq A$ is a finitely generated ideal, say $J = (a_1, \ldots, a_n)$. Then $U \cap \mathrm{Spec}(A)$ is quasi-compact as required, for it is covered by the finitely many affine opens $\mathrm{Spec}(A[1/a_i])$ for $i = 1, \ldots, n$.

(b) As the ideals $\mathcal{J}$ and $\mathcal{I}$ are of finite type, and as the support of both $\mathcal{O}/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}$ are $|Y|$, there is a positive integer n so that $\mathcal{J}^n(\mathcal{O}/\mathcal{I}) = 0 = \mathcal{I}^n(\mathcal{O}/\mathcal{J})$ by [EGA] I 6.8.4. Then $\mathcal{J}^n \subseteq \mathcal{I}$ and $\mathcal{J}^n \subseteq \mathcal{J}$.

(c) $|Y|$ is the underlying space of a reduced closed subscheme $Y_{\mathrm{red}} = \mathrm{Spec}(\mathcal{O}_X/\mathcal{K})$. The ideal $\mathcal{K}$ is the direct colimit of its finitely generated subideals $\mathcal{J}_\alpha$, so $\mathcal{K} = \varinjlim \mathcal{J}_\alpha$, by [EGA] I 6.9.9. Each $Y_\alpha = \mathrm{Spec}(\mathcal{O}/\mathcal{J}_\alpha)$ is a finitely presented closed subscheme of X. As $\mathcal{O}/\mathcal{K} = \varinjlim \mathcal{O}/\mathcal{J}_\alpha$, $Y_{\mathrm{red}} = \varprojlim Y_\alpha$, and $U = X - Y_{\mathrm{red}}$ is the direct colimit of the open subschemes $X - Y_\alpha$ of X, $U = \varinjlim(X - Y_\alpha)$. As U is quasi-compact, there is an α such that $U = X - Y_\alpha$. Then Y_α has underlying space $|Y|$ as required.

2.6.2.1. *Definition*. Let $f : X' \to X$ be a map of schemes, with X quasi-compact. Let $|Y| \subseteq X$ be a closed subspace, which is the underlying space of some finitely presented closed immersion $i : Y \to X$ (see 2.6.1(a,c)).

We say f is an isomorphism infinitely near Y if the following two conditions hold:

(a) f is flat over the points of Y; that is, for all $x' \in X'$ with $y = f(x')$ in $|Y| \subseteq X$, $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{X,y}$.

(b) f induces an isomorphism of schemes $Y' = Y \underset{X}{\times} X' \to Y$.

2.6.2.2. **Lemma-Definition**. *In the presence of hypothesis 2.6.2.1(a), the condition 2.6.2.1(b) does not depend on the choice of finitely presented closed subscheme Y with underlying space $|Y|$. In particular, if (b) holds for $Y = \mathrm{Spec}(\mathcal{O}_X/\mathcal{J})$, it also holds for any of the* *infinitesimal thickenings* $Y(n) = \operatorname{Spec}(\mathcal{O}_X/\mathcal{J}^n)$. *We may then say* f *is an isomorphism infinitely near the subspace* $|Y|$.

Proof. For $Y = \operatorname{Spec}(\mathcal{O}_X/\mathcal{J})$, Y' is $\operatorname{Spec}(\mathcal{O}_{X'}/\mathcal{J}') \subseteq X'$ with $\mathcal{J}'$ the ideal generated by $f^{-1}(\mathcal{J})$. As f is flat over the points of $|Y|$, and these are the only points x where $\mathcal{J}_x \neq \mathcal{O}_{X,x}$, we see that $f^*\mathcal{J} \to f^*\mathcal{O}_X = \mathcal{O}_{X'}$, is a monomorphism. So, in fact, $\mathcal{J}' = f^*\mathcal{J}$. Similarly $f^*\mathcal{J}^n \to \mathcal{O}_{X'}$ is a monomorphism, so $\mathcal{J}'^n = f^*\mathcal{J}^n$. It follows that $\mathcal{J}'^k/\mathcal{J}'^{k+n} = f^*(\mathcal{J}^k/\mathcal{J}^{k+n})$, as f^* preserves cokernels. Consider the exact sequence (2.6.2.2.1) of sheaves supported on the space $|Y|$:

$$(2.6.2.2.1) \qquad 0 \to \mathcal{J}^{k+1}/\mathcal{J}^{k+n} \to \mathcal{J}^k/\mathcal{J}^{k+n} \to \mathcal{J}^k/\mathcal{J}^{k+1} \to 0.$$

As f is flat over the points of $|Y|$, f^* of this sequence is also exact, and is in fact the exact sequence on $|Y'|$ where $\mathcal{J}^m$ is replaced by $\mathcal{J}'^m$ everywhere. The map f induces a map between these two exact sequences. As $f : Y' \to Y$ is an isomorphism of schemes,the induced map $\mathcal{O}/\mathcal{J} \to \mathcal{O}/\mathcal{J}'$, and even $\mathcal{J}^k/\mathcal{J}^{k+1} = \mathcal{J}^k \otimes \mathcal{O}/\mathcal{J} \to \mathcal{J}^k \otimes \mathcal{O}/\mathcal{J}' = \mathcal{J}'^k/\mathcal{J}'^{k+1}$ are isomorphisms of sheaves on the space $|Y| = |Y'|$. By induction on n, using the 5-lemma on the map between the two exact sequences, we get that f induces an isomorphism for all k and $n \geq 1$

$$(2.6.2.2.2) \qquad \mathcal{J}^k/\mathcal{J}^{k+n} \cong \mathcal{J}'^k/\mathcal{J}'^{k+n}.$$

In particular, for $k = 0$ we get $\mathcal{O}_X/\mathcal{J}^n \cong \mathcal{O}_{X'}/\mathcal{J}'^n$, so that $Y'^{(n)} \to Y^{(n)}$ is an isomorphism of schemes for all $n \geq 1$.

Now for $Z = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ a finitely presented closed subscheme of X with underlying space $|Y|$, there exists an n such that $\mathcal{J}^n \subseteq \mathcal{I}$ by 2.6.1(b). Then the closed immersion $Z \to X$ factors as $Z \to Y^{(n)} \to X$. Thus $Z' = Z \underset{X}{\times} X' \to Z$ is the pullback of the isomorphism $Y'^{(n)} = Y^{(n)} \underset{X}{\times} X' \to Y^{(n)}$, and so $Z' \to Z$ is an isomorphism as required.

2.6.3. **Theorem**. *Let* $f : X' \to X$ *be a quasi-separated map of quasi-compact schemes. Let* $i : Y \to X$ *be a finitely presented closed immersion. Suppose that* f *is an isomorphism infinitely near* Y (2.6.2.1). *Set* $Y' = f^{-1}(Y) = Y \underset{X}{\times} X'$. *Then*

(a) *For* $E^\cdot$ *in* $D^-(\mathcal{O}_X\text{-Mod})$ *with quasi-coherent cohomology and which is acyclic on* $X - Y$, *the canonical map* $E^\cdot \to Rf_*Lf^*E^\cdot$ *is an isomorphism in* $D^-(\mathcal{O}_X\text{-Mod})$.

(b) *For* $E'^\cdot$ *in* $D^-(\mathcal{O}_{X'}\text{-Mod})$ *with quasi-coherent cohomology and which is acyclic on* $X' - Y'$, *the canonical map* $Lf^*Rf_*E'^\cdot \to E'^\cdot$ *is an isomorphism in* $D^-(\mathcal{O}_{X'}\text{-Mod})$.

(c) *If $E'^{\cdot}$ is pseudo-coherent on X' and acyclic on $X' - Y'$, then $Rf_* E'^{\cdot}$ is pseudo-coherent on X.*

(d) *If $E'^{\cdot}$ is perfect on X' and acyclic on $X' - Y'$, then $Rf_* E'^{\cdot}$ is perfect on X.*

Proof. First we note that f is a quasi-compact map ([EGA] I 6.1.10 iii). Thus Rf_* has bounded cohomological dimension on complexes with quasi-coherent cohomology (B.6, B.11), and so Rf_* is defined on $D(\mathcal{O}_{X'}\text{-Mod})$ and $D^-(\mathcal{O}_{X'}\text{-Mod})$, as well as the usual $D^+(\mathcal{O}_{X'}\text{-Mod})$. Using the functorial Godement resolution T to compute sheaf cohomology as in Deligne's treatment in [SGA 4] XVII 4.2, we realize Rf_* as an exact functor defined on the level of complexes, as $Rf_* = f_* \circ T$ (cf. [Th1] Section 1). As $R^p f_*$ preserves direct colimits of modules by B.6, the usual uniformly converging spectral sequence argument à la 2.4.1 shows that Rf_* preserves up to quasi-isomorphism the colimits of directed systems of complexes with quasi-coherent cohomology.

Let $j : U \to X$ be the open complement to Y. Then j is a quasi-compact map by 2.6.1(a), and is also quasi-separated. Hence the discussion of the preceding paragraph applies also to Rj_*. We define local cohomology $R\Gamma_Y$ as the canonical homotopy fibre of the map of complexes $1 \to Rj_* j^*$, as justified by [SGA 4] V 6.5. (More precisely we use the map $1 \to j_* j^* \to j_* T j^* = Rj_* j^*$ induced by the augmentation $1 \to T$ into the Godement resolution.) Then $R\Gamma_Y$ preserves up to quasi-isomorphism the colimits of directed systems of complexes with quasi-coherent cohomology, as this is true of 1 and $Rj_* j^*$. Similarly, $R\Gamma_Y$ has finite cohomological dimension on complexes with quasi-coherent cohomology, and preserves quasi-coherence of cohomology.

Now to prove (a). Using the finite cohomological dimension of Rf_*, for any k, $H^k(Rf_* Lf^* E^{\cdot})$ equals $H^k(Rf_* Lf^* \tau^n E^{\cdot})$ for all n sufficiently small. Thus we reduce to the case where $E^{\cdot}$ is cohomologically bounded below as well as above. Now the usual devissage reduces us to the case where $E^{\cdot}$ is a single quasi-coherent module considered as a complex concentrated in one degree. For we induct on n using the homotopy fibre sequence $H^n(E^{\cdot}) \to \tau^n E^{\cdot} \to \tau^{n+1} E^{\cdot}$ and the 5-lemma to reduce to proving the theorem for the $H^n(E^{\cdot})$ as complexes. Now as Rf_* and Lf^* preserve direct colimits, writing the module E as the direct colimit of its submodules of finite type ([EGA] I 6.9.9), we reduce to the case where E is a quasi-coherent module of finite type, and which vanishes off Y. As the defining ideal $\mathcal{J}$ of Y is of finite type, there is a positive integer n such that $\mathcal{J}^n E = 0$ ([EGA] I 6.8.4). By devissage, it suffices to prove that the canonical map $1 \to Rf_* Lf^*$ is a quasi-isomorphism for $\mathcal{J}^k E / \mathcal{J}^{k+1} E$ with $k = 0, 1, \ldots, n-1$. Thus we reduce to the case where E is an $\mathcal{O}_X/\mathcal{J} = \mathcal{O}_Y$ module, $E = i_* \widetilde{E} = Ri_* \widetilde{E}$. Then using

2.5.6, which applies as f is flat over the points of Y (recall 2.5.6.4), we have $Rf_*Lf^*E \simeq Rf_*Lf^*i_*E^\sim \simeq Rf_*i'_*Lf'^*E^\sim \simeq i_*Rf'_*Lf'^*E^\sim$. But as $f' : Y' \to Y$ is an isomorphism, we continue the chain of quasi-isomorphisms: $i_*Rf'_*Lf'^*E^\sim \simeq i_*E^\sim \simeq E$. This proves (a). The proof of (b) requires only a change of notation in this argument (e.g., Lf^*Rf_* in place of Rf_*Lf^*, X' in place of X, etc.).

To prepare to prove (c) and (d) we first note that if $E^\cdot$ is acyclic on $X - Y$, and if $F^\cdot$ is any complex on X with quasi-coherent cohomology, then $R\Gamma_Y F^\cdot \to F^\cdot$ induces an isomorphism:

$$\text{(2.6.3.1)} \qquad \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(E^\cdot,\ R\Gamma_Y F^\cdot\right) \overset{\cong}{\to} \mathrm{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(E^\cdot,\ F^\cdot\right).$$

For the homotopy fibre sequence $R\Gamma_Y F^\cdot \to F^\cdot \to Rj_*j^*F$ induces a long exact sequence:

$$\text{(2.6.3.2)} \qquad \begin{array}{ccccc}
\downarrow & & & & \\
\mathrm{Mor}_D(E[1], Rj_*j^*F) & \cong & \mathrm{Mor}_{D(U)}(j^*E[1], j^*F) & = & 0 \\
\partial\downarrow & & & & \\
\mathrm{Mor}_D(E, R\Gamma_Y F) & & & & \\
\downarrow & & & & \\
\mathrm{Mor}_D(E, F) & & & & \\
\downarrow & & & & \\
\mathrm{Mor}_D(E, Rj_*j^*F) & \cong & \mathrm{Mor}_{D(U)}(j^*E, j^*F) & = & 0 \\
\downarrow & & & &
\end{array}$$

The horizontal isomorphisms result from trivial duality 2.5.7. for $j : U \to X$, and the fact that $j^*E^\cdot \simeq 0$ as $E^\cdot$ is acyclic on U. The exactness of (2.6.3.2) yields (2.6.3.1), as required. In particular, (2.6.3.1) holds for $E^\cdot = Rf_*E'^\cdot$ in (c) and (d).

As final preparation, we note that $R\Gamma_Y$ is also represented on the level of complexes as Γ_Y of the Godement resolution on X. This yields $R\Gamma_Y$ as a complex of modules over the localization of $\mathcal{O}_X$ along Y. As f is flat over the points of Y, $f^*R\Gamma_Y$ would then represent $Lf^*R\Gamma_Y$ on the level of complexes. We now switch over to this representative of $R\Gamma_Y$. As a statement in the derived category, (2.6.3.1) remains true.

Now we prove (c) that $Rf_*E'^\cdot$ is pseudo-coherent. As the question is local on X, we may assume X is affine and hence has an ample family of line bundles. We appeal to criterion 2.4.2(g). Let $\{F_\alpha^\cdot\}$ be a direct system of strict perfect complexes on X. Then we have a sequence of isomorphisms:

(2.6.3.3)

$$\begin{aligned}
&\varinjlim_\alpha \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ \tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_\alpha \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_\alpha \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ Rf_*Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_\alpha \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(Lf^*Rf_*E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \varinjlim_\alpha \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ \varinjlim_\alpha Lf^*R\Gamma_Y\tau^n F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_{X'}\text{-Mod})}\left(Lf^*Rf_*E'^{\cdot},\ Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ Rf_*Lf^*R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ R\Gamma_Y\tau^n \varinjlim F_\alpha^{\cdot}\right)\\
&\cong \operatorname{Mor}_{D(\mathcal{O}_X\text{-Mod})}\left(Rf_*E'^{\cdot},\ \tau^n \varinjlim F_\alpha^{\cdot}\right).
\end{aligned}$$

Here the isomorphisms are successively justified by (2.6.3.1), (a), trivial duality 2.5.7, (b), pseudo-coherence of $E'^{\cdot}$ with the finite Tor-dimension of $Lf^*R\Gamma_Y = f^*R\Gamma_Y$ and 2.4.2(e), the fact that Lf^* and $R\Gamma_Y$ preserve direct colimits of complexes with quasi-coherent cohomology, (b), trivial duality 2.5.7, (a), and (2.6.3.1). By 2.4.2(g), the isomorphism between the first and last terms of (2.6.3.3) shows that $Rf_*E'^{\cdot}$ is pseudo-coherent, proving (c).

The proof of (d) that $Rf_*E'^{\cdot}$ is perfect requires only removal of the truncation τ^n in the proof of (c), and the use of 2.4.3(d) on X and 2.4.1(f) on X' in place of 2.4.2(g) and 2.4.2(e).

2.7. **Proposition** ([SGA 6]). . *Let $f : X \to Y$ be a proper map of schemes. Suppose that locally on Y, the map factors as $f = h \circ i$, with $h : Z \to Y$ a flat and finitely presented map, and $i : X \to Z$ a closed immersion. Then*

(a) *If $i_*\mathcal{O}_X$ is perfect on Z, and $E^{\cdot}$ is a perfect complex on X, then $Rf_*E^{\cdot}$ is perfect on Y.*

(b) *If* $i_*\mathcal{O}_X$ *is pseudo-coherent on* Z, *and if* $h : Z \to Y$ *has an* h*-ample family of line bundles* (2.1.2(f)), *and if* $E^\cdot$ *is a pseudo-coherent complex on* X, *then* $Rf_*E^\cdot$ *is pseudo-coherent on* Y.

Proof. The question is local on Y, so we may assume that $Y = \mathrm{Spec}(A)$ is affine, and that the factorization $f = h \circ i$ exists globally over Y. In case (b), Z will then have an ample family of line bundles 2.1.2(f).

We write $A = \varinjlim A_\alpha$ as a direct colimit of its noetherian subrings of finite type over $\mathbb{Z}$. On passing to a cofinal system of α, there will be flat and finitely presented maps $h_\alpha : Z_\alpha \to Y_\alpha = \mathrm{Spec}(A_\alpha)$ and closed immersions $i_\alpha : X_\alpha \to Z_\alpha$, such that $f_\alpha = h_\alpha \circ i_\alpha$ is proper, $Z_\beta = Z_\alpha \underset{(A_\alpha)}{\times} (A_\beta)$, $Z = \varinjlim Z_\beta$, etc. This all follows by [EGA] IV 8.9.1, 8.10.5, 11.2.6. Let $g_\alpha : Y \to Y_\alpha$, $g'_\alpha : Z \to Z_\alpha$ be the canonical maps.

We use this noetherian approximation to do case (a). The complex $i_*E^\cdot$ is pseudo-coherent on Z by [SGA 6] III 1.1.1. As $E^\cdot$ has finite Tor-amplitude over X and $i_*\mathcal{O}_X$ has finite Tor-dimension over Z, the complex $i_*E^\cdot$ has finite Tor-amplitude over Z ([SGA 6] III 3.7.2). Thus $i_*E^\cdot$ is perfect on Z by 2.2.12. By 3.20 below, whose proof does not depend on this 2.7 (or by [SGA 6] IV 3.2), there is an α in the direct system and a perfect complex $E^\cdot_\alpha$ on Z_α such that $i_*E^\cdot$ is quasi-isomorphic to $Lg'^*_\alpha E^\cdot_\alpha$. As $i_*E^\cdot$ is acyclic on $Z - X$, by taking α larger we may assume that $E^\cdot_\alpha$ is acyclic on $Z_\alpha - X_\alpha$, again by 3.20. (Danger: We do not know that the maps i_α and g'_α are Tor-independent, and so cannot appeal to 2.5.6. Thus we do not claim that we can take $E^\cdot_\alpha$ to be $i_{\alpha *}$ of a perfect complex on X_α.)

As Z_α is noetherian, $E^\cdot_\alpha$ has coherent cohomology sheaves by 2.2.8, which are modules over $\mathcal{O}_{X_\alpha(n)}$ for some infinitesimal thickening $X_\alpha(n)$ of X_α in Z_α ([EGA] I 6.8.4). The maps $f_\alpha(n) = h_\alpha| : X_\alpha(n) \to Y_\alpha$ are proper as $f_{\alpha\mathrm{red}} = f_\alpha(n)_{\mathrm{red}}$ is proper ([EGA] II 5.4.6, I 5.3.1). Thus $Rh_{\alpha *}(E^\cdot_\alpha)$ has coherent cohomology, and in fact is pseudo-coherent by the noetherian case of 2.5.4 proved above.

As $h_\alpha : Z_\alpha \to Y_\alpha$ is flat and $E^\cdot_\alpha$ has finite Tor-amplitude over $\mathcal{O}_{Z_\alpha}$, it follows that $E^\cdot_\alpha$ has finite Tor-amplitude over $f^{-1}(\mathcal{O}_{Y_\alpha})$. Also, $Rh_{\alpha *}$ has finite cohomological dimension by B.11. Then the projection formula 2.5.5 applies to show that $Rh_{\alpha *}(E^\cdot_\alpha)$ has finite Tor-amplitude. Thus $Rh_{\alpha *}(E^\cdot_\alpha)$ is perfect on Y_α by 2.2.12. Then $Lg^*_\alpha Rh_{\alpha *}(E^\cdot_\alpha)$ is perfect on Y. But as h_α is flat, the Base-change Theorem 2.5.6 yields a quasi-isomorphism $Lg^*_\alpha Rh_{\alpha *}(E^\cdot_\alpha) \simeq Rh_*(Lg^*_\alpha E^\cdot_\alpha)$. But this complex is quasi-isomorphic to $Rh_*(i_*E) \simeq Rf_*(E^\cdot)$. Thus $Rf_*(E^\cdot)$ is perfect, proving (a).

To prove (b), it suffices to show for all n that the truncation $\tau^n Rf_*E^\cdot = \tau^n Rh_*(i_*E)$ is n-pseudo-coherent. By B.11, Rh_* has finite cohomological dimension, say, k. Then $\tau^n Rh_*(i_*E)$ is quasi-isomorphic to $\tau^n Rh_*(\tau^{n-k}$

$i_* E^\cdot$). By the hypothesis of pseudo-coherence of $i_* \mathcal{O}_X$ on Z and the devissage of [SGA 6] III 1.1.1, the complex $i_* E$ is pseudo-coherent on Z. The complex $\tau^{n-k} i_* E$ is then $(n-k-1)$-pseudo-coherent by 2.2.6 and 2.2.5, as $i_* E \to \tau^{n-k} i_* E$ is an $(n-k-1)$-quasi-isomorphism. By use of the coherator B.16, we see that the cohomologically bounded $\tau^{n-k} i_* E^\cdot$ is quasi-isomorphic to a complex of quasi-coherent $\mathcal{O}_Z$-modules. (Note Z meets the semi-separated hypothesis of B.16, as Z has an ample family of line bundles.) Then using 2.3.1(c), we see that $\tau^{n-k} i_* E^\cdot$ is quasi-isomorphic to a strict $(n-k-1)$-pseudo-coherent complex $E''^\cdot$. Then $\sigma^{n-k-1} E''$ is strict perfect, and $\tau^{n-k} i_* E$ is quasi-isomorphic to $\tau^{n-k} \sigma^{n-k-1} E''$. So $\tau^n Rf_* E^\cdot$ is quasi-isomorphic to $\tau^n Rh_*(\tau^{n-k} i_* E)$, hence to $\tau^n Rh_*(\tau^{n-k} \sigma^{n-k-1} E'')$, hence to $\tau^n Rh_*(\sigma^{n-k-1} E'')$. But this is n-pseudo-coherent since $Rh_*(\sigma^{n-k-1} E'')$ is perfect by part (a). This concludes the proof of (b).

2.7.1. *Porism.* Let $h : Z \to Y$ be a finitely presented map, $U \subseteq Z$ a quasi-compact open. Suppose $\mathcal{O}_{Z,z}$ is flat over $\mathcal{O}_{Y,y}$ with $y = h(z)$, for all z in $Z - U$. Suppose for some locally finitely presented closed subscheme $X \to Z$ with $Z - X = U$ (2.6.1) that $h|X \to Z$ is proper.

Then for any perfect complex $E^\cdot$ on Z which is acyclic on U, $Rh_*(E^\cdot)$ is perfect on Y.

If there is an h-ample family of line bundles on Z and if $E^\cdot$ is a pseudo-coherent complex on Z which is acyclic on U, then $Rh_* E$ is pseudo-coherent on Y.

Proof. This is the main point of the argument proving 2.7, with attention now paid to the fact that the appeals to 2.5.6 and 2.5.5 really need only the flatness hypothesis in 2.7.1.

2.7.2. *Remark.* If $f = h \circ i$ with $h : Z \to Y$ a smooth map and $i : X \to Z$ a closed immersion, then $i_* \mathcal{O}_X$ is pseudo-coherent on Z iff f is a pseudo-coherent morphism by ([SGA 6] III 1.1.4, 1.2, 1.1). We have that $i_* \mathcal{O}_X$ is perfect on Z iff f is a perfect morphism by ([SGA 6] III 1.1, 3.6, 4.1, or III 4.4).

3. K-theory of schemes: definition, models, functorialities, excision, limits

3.1. *Definition.* For X a scheme $K(X)$ is the K-theory spectrum of the complicial biWaldhausen category (1.2.11) of perfect complexes of globally finite Tor-amplitude (2.2.11), in the abelian category of all $\mathcal{O}_X$-modules. (By the default conventions of 1.2.11, the cofibrations in this biWaldhausen category are the degree-wise split monomorphisms, and the weak equivalences are the quasi-isomorphisms.)

For Y a closed subspace of X, $K(X \text{ on } Y)$ is the K-theory spectrum of the complicial biWaldhausen subcategory of those perfect complexes on X which are acyclic on $X - Y$.

3.1.1. It is clear from the description of K_0 in 1.5.6. that $K_0(X)$ is indeed the Grothendieck group "$K^\cdot(X)$" of [SGA 6] IV 2.2.

3.1.2. For X quasi-compact, any complex of locally finite Tor-amplitude, and in particular any perfect complex, automatically has globally finite Tor-amplitude.

3.2. *Definition.* For X a scheme, $K^{\text{naive}}(X)$ is the K-theory spectrum of the complicial biWaldhausen category of strict perfect complexes in the category of $\mathcal{O}_X$-modules. Similarly for $K^{\text{naive}}(X \text{ on } Y)$.

3.2.1. Indeed, $K_0^{\text{naive}}(X)$ is the naive Grothendieck group "$K^\cdot(X)_{\text{naif}}$" of [SGA 6] IV 2.2. Soon we will see that $K^{\text{naive}}(X)$ is the Quillen K-theory of X in general, and that $K^{\text{naive}}(X)$ is homotopy equivalent to $K(X)$ when X has an ample family of line bundles. Thus locally K^{naive} and K agree, but it will be K that has good local-to-global properties (cf. Sections 8, 10, esp. 8.5, 8.6).

3.3. *Definition.* For X a scheme, $G(X)$ is the K-theory spectrum of the complicial biWaldhausen category of all pseudo-coherent complexes with globally bounded cohomology in the abelian category of all $\mathcal{O}_X$-modules. For Y a closed subspace of X, $G(X \text{ on } Y)$ is the K-theory spectrum of the subcategory of those complexes acyclic on $X - Y$.

3.3.1. Indeed $G_0(X)$ is the Grothendieck group "$K_\cdot(X)$" of [SGA 6] IV 2.2. Quillen [Q1] defined higher K'- or G-theory only for noetherian schemes, and for these his $K'(X)$ is homotopy equivalent to $G(X)$ as we will see.

3.4. The construction of the K-theory spectrum proceeds not from the derived category, but from the underlying complicial biWaldhausen category, and is known to be functorial only in the biWaldhausen category. By 1.9.8, the choice of underlying biWaldhausen category is not critical in that all related choices give the same K-theory. But to exhibit all the required functorialities in K-theory, many different underlying model categories must be employed. Hence we proceed to compile lists of the most useful models.

3.5. **Lemma**. *Let X be a quasi-compact scheme. Consider the following list of complicial biWaldhausen categories* (cf. 1.2.11) *in the abelian category of all $\mathcal{O}_X$-modules (or in cases* 6, 7, 8 *in the abelian* *category of diagrams $A \to B$ of $\mathcal{O}_X$-modules):*

3.5.1. *perfect complexes,*

3.5.2. *perfect strict bounded complexes,*

3.5.3. *perfect bounded above complexes of flat $\mathcal{O}_X$-modules,*

3.5.4. *perfect bounded below complexes of injective $\mathcal{O}_X$-modules,*

3.5.5. *perfect bounded below complexes of flasque $\mathcal{O}_X$-modules,*

3.5.6. *diagrams $E^\cdot \xrightarrow{\sim} F^\cdot$ consisting of a quasi-isomorphism between perfect complexes,*

3.5.7. *diagrams as in 3.5.6, but with $E^\cdot$ degree-wise flat bounded above and $F^\cdot$ degree-wise injective bounded below, perfect complexes,*

3.5.8. *diagrams as in 3.5.6, but with $E^\cdot$ degree-wise flat bounded above and $F^\cdot$ degree-wise flasque bounded below perfect complexes.*

Then the obvious inclusion functors induce homotopy equivalences of all their K-theory spectra, so all are homotopy equivalent to $K(X)$. Similarly there are homotopy equivalences to $K(X \text{ on } Y)$ from the K-theory spectra of the various subcategories of complexes which are also acyclic on $X - Y$.

Moreover, the results will hold for non-quasi-compact X if we add everywhere the extra condition that the perfect complexes have globally finite Tor-amplitude.

Proof. More precisely, we need to show that the inclusion functors of categories 3.5.2 through 3.5.5 into 3.5.1 induce homotopy equivalences on K-theory spectra. We also show that the inclusions of 3.5.8 and 3.5.7 into 3.5.6 induce homotopy equivalences. The two functors from 3.5.6 to 3.5.1 sending $E^\cdot \xrightarrow{\sim} F^\cdot$ to $E^\cdot$ and to $F^\cdot$ respectively both induce homotopic homotopy equivalences, inverse to the homotopy equivalence induced by the functor from 3.5.1 to 3.5.6 sending $E^\cdot$ to the diagram $1 : E^\cdot = E^\cdot$.

The last statement follows immediately from 1.5.4. The inclusion functors will induce homotopy equivalences in K-theory because they will induce equivalences on the derived categories of the complicial biWaldhausen categories, allowing appeal to 1.9.8. (One could also appeal directly to Waldhausen approximation 1.9.1 given 1.9.5 and the facts below.)

By 1.9.7 (or its dual), to prove the equivalence of derived categories it suffices to show for all B in the target category that there is an A in the source category and a quasi-isomorphism $A \xrightarrow{\sim} B$ (resp., $B \xrightarrow{\sim} A$). As the category of $\mathcal{O}_X$-modules has enough flat and enough injective objects (e.g., [H] II Section 1), and as all injectives are flasque, 1.9.5 yields the well-known fact that for any cohomologically bounded complex $B^\cdot$, one has quasi-isomorphisms $A^\cdot \xrightarrow{\sim} B^\cdot \xrightarrow{\sim} C^\cdot$ where $A^\cdot$ is bounded above and degree-wise flat, and $C^\cdot$ is bounded below and degree-wise injective, *a fortiori* degree-wise flasque. This immediately proves the equivalence of derived categories for the inclusions of 3.5.3, 3.5.4, and 3.5.5 into 3.5.1. If one considers an intermediate biWaldhausen category whose objects are diagrams of quasi-isomorphisms $E^{\cdot} \xrightarrow{\sim} F^{\cdot}$ of perfect complexes with $E^{\cdot}$ degree-wise flat and bounded above, one sees similarly that the inclusion of this category into 3.5.6 and the inclusions of 3.5.7 and 3.5.8 into this intermediate category induce equivalences of derived categories.

It remains only to show that the inclusion of 3.5.2 into 3.5.1 induces an equivalence of derived categories. But a perfect complex of globally finite Tor-amplitude is globally cohomologically bounded. Thus such a perfect $E^{\cdot}$ is quasi-isomorphic to the truncations for suitable $n \ll 0$ and $m \gg 0 : E^{\cdot} \xrightarrow{\sim} \tau^n E^{\cdot} \xleftarrow{\sim} \tau^{\leq m}(\tau^n E^{\cdot})$. As $\tau^n E^{\cdot}$ is strict bounded below and $\tau^{\leq m}(\tau^n E^{\cdot})$ is strict bounded, these quasi-isomorphisms show that the inclusion of 3.5.2 into the intermediate biWaldhausen category of perfect strict bounded below complexes and the further inclusion of this intermediate category into 3.5.1 induce equivalences of derived categories.

Hence all the inclusions induce equivalences of derived categories, and 1.9.8 yields the result. Clearly the proof works if we impose everywhere the extra condition that the complexes are to be acyclic on $X - Y$, yielding the result for $K(X \text{ on } Y)$.

3.6. **Lemma**. *For X either quasi-compact and semi-separated* (B.7), *or else noetherian, Lemma 3.5 remains true if the following categories are added to the list in 3.5. So all have K-theory spectra homotopy equivalent to $K(X)$;*

3.6.1. *perfect complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.6.2. *perfect complexes of injective objects in* Qcoh(X).

Proof. The inclusion of 3.6.2 into 3.6.1 induces an equivalence of derived categories as Qcoh(X) has enough injectives (B.3). The inclusion of 3.6.1 into 3.5.1 has an inverse equivalence on the derived category, given by the coherator (B.16). Now we apply 1.9.8.

3.7. **Lemma**. *For X noetherian, Lemmas* 3.5 *and* 3.6 *remain true if the following categories are added to the lists. Hence all have K-theory spectra homotopy equivalent to $K(X)$:*

3.7.1. *perfect complexes of coherent $\mathcal{O}_X$-modules,*

3.7.2. *perfect strict bounded complexes of coherent $\mathcal{O}_X$-modules.*

Similarly for $K(X \text{ on } Y)$ if we add the extra condition that the complexes are acyclic on $X - Y$.

Proof. The inclusion of 3.7.1 into 3.6.1 induces an equivalence of derived categories, as we prove by the Inductive Construction Lemma 1.9.5, whose hypothesis 1.9.5.1 is met by 2.2.8 and [EGA] I 6.9.9. We conclude by the usual 1.9.7 and 1.9.8.

The inclusion of 3.7.2 into 3.7.1 induces an equivalence of derived categories, as we see using truncations just as for the inclusion of 3.5.2 into 3.5.1.

3.8. **Lemma**. *For X with an ample family of line bundles (2.1.1), Lemma 3.5 remains true if the following categories are added to the list in 3.5 and 3.6. In particular, all their K-theory spectra are homotopy equivalent to $K(X)$:*

3.8.1. *perfect complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.8.2. *perfect bounded above complexes of flat quasi-coherent $\mathcal{O}_X$-modules,*

3.8.3. *strict perfect complexes.*

Similarly for $K(X \text{ on } Y)$ if we add everywhere the extra condition that the complexes are acyclic on $X - Y$.

Proof. We note by 2.1.1 and B.7 that X is quasi-compact and semi-separated, so that 3.6 applies. We note 3.8.1 = 3.6.1. Finally the inclusion of 3.8.3 into 3.8.2 and into 3.8.1 induces an equivalence of derived categories by 2.3.1(d).

3.9. **Corollary**. *For X a scheme with an ample family of line bundles, there is a natural homotopy equivalence $K^{\text{naive}}(X) \xrightarrow{\sim} K(X)$.*

Proof. Lemma 3.8.3 and Definition 3.2. (cf. [SGA 6] IV 2.9).

3.10. **Proposition**. *For X any scheme, $K^{\text{naive}}(X)$ is naturally homotopy equivalent to Quillen's K-theory spectrum of X ([Q1]).*

For X with an ample family of line bundles, $K(X)$ is naturally homotopy equivalent to the Quillen K-theory spectrum of X.

Proof. The first statement follows from 1.11.7, as strict perfect complexes are precisely bounded complexes in the exact category of algebraic vector bundles on X. The second statement follows by 3.9.

3.11. **Lemma**. *The obvious inclusions of the complicial biWaldhausen categories in the following lists induce homotopy equivalences on K-theory under the conditions preceding each list. In particular, the K-theory spectra are all homotopy equivalent to $G(X)$.*

For X a scheme:

3.11.1. *cohomologically bounded pseudo-coherent complexes of $\mathcal{O}_X$-modules,*

3.11.2. *pseudo-coherent strict bounded complexes of $\mathcal{O}_X$-modules,*

3.11.3. *cohomologically bounded pseudo-coherent complexes of flat $\mathcal{O}_X$-modules,* 3.11.4. *cohomologically bounded pseudo-coherent complexes of injective $\mathcal{O}_X$-modules,*

3.11.5. *cohomologically bounded pseudo-coherent complexes of flasque $\mathcal{O}_X$-modules,*

3.11.6. *diagrams of quasi-isomorphisms $E^{\cdot} \xrightarrow{\sim} F^{\cdot}$ of cohomologically bounded pseudo-coherent complexes, with $E^{\cdot}$ degree-wise flat and $F^{\cdot}$ degree-wise flasque,*

3.11.7. *as in 3.11.6., except $F^{\cdot}$ is degree-wise injective instead of merely flasque.*

For X either a quasi-compact and semi-separated scheme, or else noetherian, one may add to the list:

3.11.8. *cohomologically bounded pseudo-coherent complexes of quasi-coherent $\mathcal{O}_X$-modules,*

3.11.9. *cohomologically bounded pseudo-coherent complexes of injectives in* Qcoh(X),

For X with an ample family of line bundles, one may add to the list:

3.11.10. *cohomologically bounded strict pseudo-coherent complexes.*

Proof. Note "cohomologically bounded" means "globally cohomologically bounded," and recall the default conventions of 1.2.11 for cofibrations and weak equivalences.

The proof of 3.11 exactly parallels that of 3.5-3.8, and hence we leave it to the reader.

3.12. **Lemma**. *For X a noetherian scheme, the inclusion of the following biWaldhausen categories into* 3.11.8 *induce homotopy equivalences of their K-theory spectra to $G(X)$.*

3.12.1. *cohomologically bounded complexes of coherent $\mathcal{O}_X$-modules,*

3.12.2. *strict bounded complexes of coherent $\mathcal{O}_X$-modules.*

Proof. One argues as in 3.7, recalling also that coherent complexes are pseudo-coherent by 2.2.8.

3.13. **Corollary**. *For X a noetherian scheme, $G(X)$ is naturally homotopy equivalent to the Quillen G- or K'-spectrum of X defined in* [Q1].

Proof. This follows from 3.12 and 1.11.7.

3.14. $K(X)$ and $K^{\text{naive}}(X)$ are contravariant functors in the scheme X, as a map of schemes $f : X \to X'$ induces a complicial exact (1.2.16) functor $Lf^* = f^*$ between the biWaldhausen categories of perfect bounded above complexes of flat modules (3.5.3), or those of strict perfect complexes. Similarly f induces a map $f^* : K(X' \text{ on } Y') \to K(X \text{ on } f^{-1}(Y'))$.

3.14.1. If $f : X \to X'$ is a map of globally finite Tor-dimension, then $Lf^*E^\cdot$ is cohomologically bounded if $E^\cdot$ is. It follows then f^* is a complicial exact functor between categories of cohomologically bounded pseudo-coherent complexes of flat modules (3.11.3), and so induces a map $f^* : G(X') \to G(X)$. This makes $G(\ )$ a contravariant functor for maps of finite Tor-dimension (cf. [SGA 6] IV 2.12 for G_0, [Q1] Section 7 2.5).

3.15. The functor $(E^\cdot, F^\cdot) \mapsto E^\cdot \underset{\mathcal{O}_X}{\otimes} F^\cdot$ preserves degree-wise split monomorphisms in either variable. This functor represents $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ and preserves quasi-isomorphisms if either $E^\cdot$ or $F^\cdot$ runs over a category of bounded above complexes of flat $\mathcal{O}_X$-modules. Thus it is biexact, i.e., exact in each variable, on a pair of biWaldhausen categories if either category consists of flat complexes. It is clear that $E^\cdot \underset{\mathcal{O}_X}{\otimes} F^\cdot$ is strict perfect of both $E^\cdot$ and $F^\cdot$ are strict perfect. Applied locally on X, this shows $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ is perfect if both $E^\cdot$ and $F^\cdot$ are. It is easy to see that if $E^\cdot$ has finite Tor-amplitude, then $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ has finite Tor-amplitude (respectively, is cohomologically bounded) if $F^\cdot$ has finite Tor-amplitude (resp. is cohomologically bounded). (See [SGA 6] I 5.6, 5.3 if you get stuck.) We have already seen that $E^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} F^\cdot$ is pseudo-coherent if both $E^\cdot$ is perfect (even pseudo-coherent) and $F^\cdot$ is pseudo-coherent, back in 2.5.1. Thus $\otimes$ induces biexact functors between various biWaldhausen categories 3.5.3, 3.8.3, 3.11.3. As biexact functors induce pairings between K-theory spectra ([W] just after 1.5.3), we get various pairings (cf. [SGA 6] IV 2.7, 2.10):

$$K(X) \wedge K(X) \to K(X) \tag{3.15.1}$$

$$K^{\text{naive}}(X) \wedge K^{\text{naive}}(X) \to K^{\text{naive}}(X) \tag{3.15.2}$$

$$K(X) \wedge G(X) \to G(X) \tag{3.15.3}$$

and even:

(3.15.4)
$$K(X \text{ on } Y) \wedge K(X \text{ on } Z) \to K(X \text{ on } Y \cap Z)$$

(3.15.5)
$$K(X \text{ on } Y) \wedge G(X \text{ on } Z) \to G(X \text{ on } Y \cap Z).$$

As the tensor product $\otimes$ is associative and commutative up to "coherent natural isomorphism," $K(X)$ and $K^{\text{naive}}(X)$ are in fact "homotopy-everything" ring spectra, and the canonical map $K^{\text{naive}}(X) \to K(X)$ and also $f^* : K(X') \to K(X)$ and $f^* : K^{\text{naive}}(X') \to K^{\text{naive}}(X)$ are maps of such ring spectra. The spectrum $G(X)$ is a module spectrum over $K(X)$, and when it exists, $f^* : G(X') \to G(X)$ is a map of module spectra over $K(X')$ (cf. e.g., [Ma2]).

Moreover $K(X \text{ on } Y)$ has a commutative and associative multiplication up to "coherent homotopy," but fails to have a unit when $X \neq Y$.

There are also external pairings induced by $(E^{\cdot}, F^{\cdot}) \to E^{\cdot} \underset{\mathcal{O}_S}{\otimes} F^{\cdot}$ for X flat over S and Z over S.

$$K(X) \wedge K(Z) \to K(X \underset{S}{\times} Z) \tag{3.15.6}$$

$$K(X) \wedge G(Z) \to G(X \underset{S}{\times} Z) \tag{3.15.7}$$

See [SGA 6] IV 3.3 for how to go further.

3.16. Let $f : X \to Y$ be a map of schemes. Then f_* is a complicial exact functor in the sense of 1.2.16 between the complicial biWaldhausen categories of all bounded below complexes of flasque $\mathcal{O}$-modules on X and on Y. For flasque modules are deployed for f_*, so f_* represents Rf_* and preserves quasi-isomorphisms on such complexes; and f_* also preserves flasqueness ([SGA 4] V 4.9, and [H] I 5.3 β or [V] II Section 2 no. 2). To conclude that f_* induces maps $f_* : K(X) \to K(Y)$ or $f_* : G(X) \to G(Y)$, we need only find conditions that make Rf_* to preserve the required perfection, pseudo-coherence, and the global bounds on cohomological dimension or Tor-amplitude in the definitions of the biWaldhausen categories 3.5.5 and 3.11.5. Considering B.11, 2.5.4 (= [SGA 6] III 2.5, 4.8.1) and 2.7, we get variously (cf. [SGA 6] IV 2.11, [Th4] 1.13, [Q1] Section 7 2.7):

3.16.1. $G(\ )$ is a covariant functor on the category of noetherian schemes and proper maps.

(Note this improves upon [Q1] 7.2.7, which only made $G(\ )$ a functor up to homotopy and for finite or projective maps. Our method avoids the fuss of Gillet's Chow envelope method [Gil] of proving 3.16.1. Also after the usual rectification to make $f_* g_* = (fg)_*$ on $\mathcal{O}$-modules strictly, instead of up to natural isomorphism, our method yields a strictly functorial $G(\ )$, instead of a functor up to homotopy.)

3.16.2. $G(\ )$ is a covariant functor on the category of quasi-compact schemes and flat proper maps with a relatively ample family of line bundles.

3.16.3. $G(\ )$ is a covariant functor on the category of quasi-compact schemes and pseudo-coherent projective morphisms.

3.16.4. $K(\ )$ is a covariant functor on the category of noetherian schemes, with proper maps of finite Tor-dimension (i.e., perfect proper maps).

3.16.5. $K(\ )$ is a covariant functor on the category of quasi-compact schemes and perfect projective morphisms.

3.16.6. $K(\ )$ is a covariant functor on the category of quasi-compact schemes and flat proper morphisms.

3.16.7. There are analogs of 3.16.2 - 3.16.6 for $G(X$ on $Y)$ and $K(X$ on $Y)$, using 2.6.3 and 2.7.1. We also note that if $Z \subseteq Y \subseteq X$ with Z and Y closed subspaces in X, the exact functor forgetting part of the acyclicity requirement yields a canonical map $K(X$ on $Z) \to K(X$ on $Y)$.

3.17. **Proposition**. Projection Formula (cf. [SGA 6] IV 2.12, [Q1] Section 7 2.10). *Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes with Y quasi-compact. Suppose that f is such that Rf_* preserves pseudo-coherence (respectively, preserves perfection), hence induces a map $f_* : G(X) \to G(Y)$, (resp. $f_* : K(X) \to K(Y)$). For example, let f be as in* 3.16.1 - 3.16.3 *(resp.* 3.16.4 - 3.16.6*). Then f_* is a map of module spectra over the ring spectra $K(Y)$. That is, the diagram (*3.17.1*) (resp. the similar diagram where all $G(\ )$'s are replaced by $K(\ )$'s) commutes up to canonically chosen homotopy:*

(3.17.1)
$$\begin{tikzcd}
 & K(X) \wedge G(X) \arrow[r, "\otimes"] & G(X) \arrow[dd, "f_*"] \\
K(Y) \wedge G(X) \arrow[ur, "f^* \wedge 1"] \arrow[dr, "1 \wedge f_*"'] & & \\
 & K(Y) \wedge G(Y) \arrow[r, "\otimes"] & G(Y)
\end{tikzcd}$$

Proof. It is convenient to represent Rf_* on the chain level as $f_* \circ T$ where T is the total complex (totaled via sums, and not products) of the functorial flasque Godement resolution indexed by all the points of X, as in [SGA 4] XVII 4.2. This $Rf_* = f_* \circ T$ is exact on all complexes.

Now consider $F^\cdot$ a bounded above perfect complex of flat $\mathcal{O}_Y$-modules, and $E^\cdot$ a pseudo-coherent (resp. perfect) complex of $\mathcal{O}_X$-modules. By deployment, we have $F^\cdot \overset{L}{\underset{\mathcal{O}_Y}{\otimes}} (\ ) = F^\cdot \underset{\mathcal{O}_Y}{\otimes} (\ )$, etc. Then 2.5.5 shows that the canonical map (3.17.2) is a natural quasi-isomorphism:

(3.17.2)

$$F^{\cdot} \underset{\mathcal{O}_Y}{\overset{L}{\otimes}} Rf_*E^{\cdot} = F^{\cdot} \underset{\mathcal{O}_Y}{\otimes} f_*TE^{\cdot} \to f_*\left(f^*F^{\cdot} \underset{\mathcal{O}_Y}{\otimes} TE^{\cdot}\right) \to f_*T\left(f^*F^{\cdot} \underset{\mathcal{O}_X}{\otimes} E^{\cdot}\right) \,\Vert\, Rf_*\left(f^*F^{\cdot} \underset{\mathcal{O}_X}{\overset{L}{\otimes}} E^{\cdot}\right).$$

As in 1.5.4, this quasi-isomorphism of two functors biexact in $(E^{\cdot}, F^{\cdot})$ yields the desired homotopy between two maps $K(Y) \wedge G(X) \to G(Y)$ (resp., $K(Y) \wedge K(X) \to K(Y)$).

3.18. **Proposition** (cf. [SGA 6] IV 3.1.1, [Q1] Section 7 2.11). *Let (3.18.1) be a pullback diagram of quasi-coherent schemes, with f a quasi-separated map.*

(3.18.1)
$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "g"] \end{tikzcd}$$

Suppose f and g are Tor-independent over Y (2.5.6.2). Suppose that g has finite Tor-dimension (resp., that f has finite Tor-dimension) and that f and f' are such that Rf_* and Rf'_* preserve pseudo-coherence (resp., that Rf_* and Rf'_* preserve perfection), and so define maps $f_* : G(X) \to G(Y)$ and $f'_* : G(X') \to G(Y')$, (resp., $f_* : K(X) \to K(Y)$ and $f'_* : K(X') \to K(Y')$). For example, we could suppose that f and f' are as in 3.16.1 - 3.16.3 (resp., 3.16.4 - 3.16.6).

Then there is a canonical homotopy between $g^*f_* \simeq f'_*g'^* : G(X) \to G(Y')$ (resp., $g^*f_* \simeq f'_*g'^* : K(X) \to K(Y')$).

Proof. The idea is to use the base change Theorem 2.5.6. We follow Deligne's proof of 2.5.6 in [SGA 4] XVII 4.2, and build yet another model of $G(X)$. Consider $E^{\cdot}$ in the category of bounded above pseudo-coherent complexes of flat $\mathcal{O}_X$-modules. Let $T(E^{\cdot})$ be the total complex, totaled by sums, of the Godement resolution as in [SGA 4] XVII Section 4. Then each stalk $T(E^{\cdot}_x)$ is chain homotopic to $E^{\cdot}_x$ so $g'^*T(E^{\cdot})_x \simeq g'^*E^{\cdot}_x = Lg'^*E^{\cdot}_x$, and $T(E^{\cdot})$ is deployed for Lg'^*. As $T(E^{\cdot})$ is flasque, it is deployed for Rf_* (using B.11 to allow $T^{\cdot}(E)$ to be unbounded below). Using this functor T, the augmentation quasi-isomorphism $E^{\cdot} \simeq T(E^{\cdot})$, 3.11, and the methods of 3.5, we see the complicial biWaldhausen category of cohomologically bounded pseudo-coherent complexes deployed for both Lg'^* and Rf_*, that of cohomologically bounded above pseudo-coherent complexes of flat $\mathcal{O}_X$-modules, and that of cohomologically bounded pseudo-coherent complexes, all have equivalent derived categories. Indeed, after identifying the last two derived categories by their known equivalence, the inverse equivalences to the bideployed derived category are induced by T and the inclusion (cf. [SGA 4] XVII 4.2.10). So by 1.9.8, the bideployed biWaldhausen category has K-theory spectrum homotopy equivalent to $G(X)$.

The techniques of proof of 3.11 and 3.5 show that the derived category of the bideployed biWaldhausen category is equivalent to the derived category of the complicial biWaldhausen category whose objects are data consisting of a cohomologically bounded pseudo-coherent $E^\cdot$ on X which is deployed for both f_* and g'^*, a bounded above degree-wise flat $F^\cdot$ on Y and a quasi-isomorphism $F^\cdot \xrightarrow{\sim} f_*E^\cdot$, and a bounded below degree-wise flasque $G^\cdot$ on X' and a quasi-isomorphism $g'^*E^\cdot \xrightarrow{\sim} G^\cdot$ on X'. The associated abelian category is that of all diagrams $(A \to f_*B,\ g'^*B \to C)$ with A an $\mathcal{O}_Y$-module, B an $\mathcal{O}_X$-module , and C an $\mathcal{O}_{X'}$-module. (Compare [SGA 4] XVII 4.2.12.) The K-theory spectrum of this biWaldhausen category is thus also homotopy equivalent to $G(X)$.

In this model of $G(X)$, g^*f_* is represented by the exact functor sending $(F^\cdot \xrightarrow{\sim} f_*E^\cdot,\ g'^*E^\cdot \xrightarrow{\sim} G^\cdot)$ to $g^*F^\cdot$ (recall, $F^\cdot$ is flat). The map $f'_*g'^*$ is represented by the exact functor sending the object $(F^\cdot \xrightarrow{\sim} f_*E^\cdot,\ g'^*E^\cdot \xrightarrow{\sim} G^\cdot)$ to $f'_*G^\cdot$. There is a natural transformation

$$g^*F^\cdot \to g^*f_*E^\cdot \to g^*f_*g'_*g'^*E^\cdot = g^*g_*f'_*g'^*E^\cdot \to f'_*g'^*E^\cdot \to f'_*G^\cdot.$$

This is the canonical base change map of Deligne ([SGA 4] XVII 4.2.12), and is a quasi-isomorphism by 2.5.6 = [SGA 6] IV 3.1. This natural quasi-isomorphism then induces the desired homotopy $g^*f_* \simeq f'_*g'^*$ of maps on $G(\ )$ by 1.5.4. The proof for $K(\ )$ is essentially the same. Clearly, there are analogs for $K(X \text{ on } Z)$, etc.

3.19. **Proposition** (Excision). *Let $f : X' \to X$ be a map of quasi-compact and quasi-separated schemes. Let $Y \subseteq X$ be a closed subspace such that $X - Y$ is quasi-compact. Set $Y' = f^{-1}(Y) \subseteq X$. Suppose that f is an isomorphism infinitely near Y in the sense of* 2.6.2.2 , 2.6.1. *Then f^* induces homotopy equivalences*

$$\begin{aligned} f^* &: K(X \text{ on } Y) \xrightarrow{\sim} K(X' \text{ on } Y') \\ f^* &: G(X \text{ on } Y) \xrightarrow{\sim} G(X' \text{ on } Y'). \end{aligned} \tag{3.19.1}$$

Proof. We note that the map f is quasi-separated ([EGA] I 6.1.10). By definition 2.6.2.1, we may choose a scheme structure on Y so that

$i : Y \to X$ is a finitely presented closed immersion with f inducing an isomorphism $f^{-1}(Y) = Y \underset{X}{\times} X' \to Y$. The proposition then results from 2.6.3 which shows Rf_* and Lf^* are inverse up to natural quasi-isomorphism, once we realize Rf_* and Lf^* as exact functors on appropriate model bi-Waldhausen categories. We represent Rf_* by the exact $f_* \circ T$, for T the Godement resolution, and note that Rf_* preserves perfection and pseudo-coherence of complexes acyclic on $X' - Y'$ by 2.6.3, and preserves cohomological boundness by the finite cohomological dimension of Rf_*, B.11. The most appropriate models for $K(X \text{ on } Y)$ and $K(X' \text{ on } Y')$ (resp., $G(X \text{ on } Y)\ldots$) are the complicial biWaldhausen category of perfect (resp., cohomologically bounded pseudo-coherent) complexes of $\mathcal{O}_X$-modules which are strictly 0 on $X - Y$. The inclusion functor of this new model into the category of perfect complexes of $\mathcal{O}_X$-modules which are acyclic on $X - Y$ is exact, and $\Gamma_Y \circ T$ provides an exact functor which is inverse to the inclusion up to natural quasi-isomorphism. Thus the new model category indeed has K-theory spectrum homotopy equivalent to $K(X \text{ on } Y)$, (resp. $\ldots$).

As f is flat over the points of Y (2.6.2.1), f^* is exact on the category of complexes strictly 0 on $X - Y$. Thus f^* and $f_* \circ T$ induce exact functors on the new models, which are inverse up to natural quasi-isomorphism by 2.6.3. By 1.5.4, this yields the result.

3.19.2. *Examples.* Let A be a noetherian ring, and $I \subseteq A$ an ideal. Then the map to the completion $A \to A\hat{_I}$ induces a homotopy equivalence:

$$K(\mathrm{Spec}(A) \text{ on } \mathrm{Spec}(A/I)) \to K(\mathrm{Spec}(A\hat{_I}) \text{ on } \mathrm{Spec}(A\hat{_I}/IA\hat{_I}))$$

For $A\hat{_I}/IA\hat{_I} = A/I$, and for A noetherian $A \to A\hat{_I}$ is flat over the points of $\mathrm{Spec}(A/I)$, as $A_{(I)}$ is a Zariski ring at (I).

For a general commutative ring A, and $I \subseteq A$ a finitely generated ideal, let A^h_I be the henselization of A along $\mathrm{Spec}(A/I)$. Then the map $A \to A^h_I$ induces a homotopy equivalence $K((A) \text{ on } (A/I)) \xrightarrow{\sim} K((A^h_I) \text{ on } (A^h_I/IA^h_I))$.

If $j : V \to X$ is an open immersion with $Y \subseteq V$, and V is quasi-compact, then $K(X \text{ on } Y) \xrightarrow{\sim} K(V \text{ on } Y)$ is a homotopy equivalence. For this example, it is in fact trivial to prove 2.6.3 directly.

3.20. **Proposition** (cf. [SGA 6] IV 3.2, [Q1] Section 7 2.2). *Let $X = \varprojlim X_\alpha$ be the limit of an inverse system of schemes, where the "bonding" maps $f_{\alpha\beta} : X_\alpha \to X_\beta$ in the system are affine. Suppose that all the X_α are quasi-compact and quasi-separated. Let $Y_\alpha \subseteq X_\alpha$ be a system of closed subspaces with $f_{\alpha\beta}^{-1}(Y_\beta) = Y_\alpha$, and all $X_\alpha - Y_\alpha$ quasi-compact. Then:*

3.20.1. *The derived category of strict perfect (resp., perfect) complexes on X is the direct colimit of the derived categories of strict perfect (resp., perfect) complexes on the X_α, where the maps in the direct system are the $Lf^*_{\alpha\beta}$. Similarly, there is an equivalence of derived categories where one imposes the conditions of acyclicity on $X - Y$ and $X_\alpha - Y_\alpha$ on the complexes in the derived categories.*

3.20.2. *The canonical maps induced by the Lf^*_α are homotopy equivalences:*

$$\varinjlim_\alpha K(X_\alpha) \xrightarrow{\sim} K(X)$$

$$\varinjlim_\alpha K(X_\alpha \text{ on } Y_\alpha) \xrightarrow{\sim} K(X \text{ on } Y).$$

Proof. The construction of $K(\ )$, see 1.5.2 -1.5.3, clearly preserves direct colimits of biWaldhausen categories, and also converts complicial exact functors inducing an equivalence of derived categories into homotopy equivalences of K-theory spectra, by 1.9.8. Hence 3.20.2 follows from 3.20.1.

To prove 3.20.1, we first consider the biWaldhausen category of strict perfect complexes (possibly imposing the condition of acyclicity on $X_\alpha - Y_\alpha$, $X - Y$). As a strict perfect complex is a finite complex of finitely presented modules, it follows from [EGA] IV Section 8 as quoted in C.4 that the biWaldhausen category on X is equivalent to the direct colimit of the biWaldhausen categories on the X_α, and *a fortiori* that the derived categories are equivalent. In more detail, we see that [EGA] IV 8.5.2(ii) applied to the modules in each degree and [EGA] IV 8.5.2(i) applied to the differentials, each applied finitely any times, shows that each strict perfect $E^\cdot$ on X is isomorphic to $f^*_\alpha E^\cdot_\alpha$ for a strict perfect $E^\cdot_\alpha$ on some X_α. Given a morphism $e : E^\cdot \to E'^\cdot$ between strict perfect complexes on X, we apply [EGA] IV 8.5.2(i), first to get maps $E^i_\alpha \to E'^i_\alpha$ defined on some X_α, and then to make them satisfy the identities $\partial e^i = e^{i+1}\partial$, and so to obtain a chain map $e_\alpha : E^\cdot_\alpha \to E'^\cdot_\alpha$ on X_α for α sufficiently large, such that $e = f^*_\alpha(e_\alpha)$. Now if V_α is an affine open in X_α, and $E^\cdot$ is acyclic on $V = V_\alpha \underset{X_\alpha}{\times} X$, then as a complex of projective modules it is chain homotopic to 0 on the affine V. As above, [EGA] IV 8.5.2 shows that this chain nulhomotopy is defined on $E^\cdot_\beta | V_\beta$ for all sufficiently large β, so $E^\cdot_\beta | V_\beta$ is acyclic. Applied to affine covers, this shows that if the strict perfect complex $E^\cdot$ is acyclic on X, or on $X - Y$, then for all sufficiently large β the complex $E^\cdot_\beta$ is acyclic on X_β or $X_\beta - Y_\beta$. Finally e is a quasi-isomorphism iff its mapping cone $\text{cone}(e) = f^*_\alpha \text{cone}(e_\alpha)$ is acyclic on X. By the above, this occurs iff for some β sufficiently large cone (e_β)

is acyclic on X_β, i.e., iff for β sufficiently large e_β is a quasi-isomorphism on X_β. This completes the proof of 3.20.1 for categories of strict perfect complexes.

To prove 3.20.1 for perfect complexes, we work in the biWaldhausen category of perfect strict bounded above complexes of flat modules. This has the correct derived category by 3.5, and all the f^* and f_α^* are exact on this category. By replacing the system of X_α by the cofinal system of all $\alpha \geq \alpha_0$, we may assume that there is a terminal X_0. Then X and all X_α are affine over X_0.

If X_0, and hence X_α and X, has an ample family of line bundles, the derived category of perfect complexes is equivalent to the derived category of strict perfect complexes as in 3.8, and 3.20.1 reduces to the strict case proved above.

We now prove in 3.20.3 - 3.20.6 the result 3.20.1 by induction on the number n of affine open subschemes needed to cover X_0. If $n = 1$, X_0 is affine, hence has an ample family of line bundles, and 3.20.1 is known. To do the induction step, let $n > 1$, and suppose the result is known for schemes covered by fewer affines. Then we write $X_0 = U_0 \cup V_0$ with U_0 open affine, and V_0 open and covered by $n - 1$ open affines. Let $U_\alpha = X_\alpha \underset{X_0}{\times} U_0$, etc. Then 3.20.1 is known for $U = \varprojlim U_\alpha$, $V = \varprojlim V_\alpha$, and $U \cap V = \varprojlim U_\alpha \cap V_\alpha$. We note that $U_0 \cap V_0$ is quasi-affine, and so has an ample family of line bundles, so the above indeed yields 3.20.1 for it.

3.20.3. If $E_\alpha^{\cdot}$ is a bounded above flat perfect complex on X_α such that $f_\alpha^* E_\alpha^{\cdot}$ is acyclic on X (or on $X - Y$), then by induction hypothesis for all β sufficiently large $f_{\beta\alpha}^* E_\alpha^{\cdot} = E_\beta^{\cdot}$ has $E_\beta^{\cdot}|U_\beta$ acyclic on U_β (or on $U_\beta - Y_\beta$), and $E_\beta^{\cdot}|V_\beta$ acyclic on V_β (or on $V_\beta - Y_\beta$). As acyclicity is a local question, $E_\beta^{\cdot}$ is then acyclic on X_β (resp., on $X_\beta - Y_\beta$) for all sufficiently large β. We apply this to the mapping cone of a map $e_\alpha : E_\alpha^{\cdot} \to E_\alpha'^{\cdot}$ to show that if $f_\alpha^* e_\alpha$ is a quasi-isomorphism on X, then $e_\beta = f_{\beta\alpha}^* e_\alpha$ is a quasi-isomorphism on X_β for all sufficiently large β.

The next step is to show that any bounded above flat perfect complex on X is quasi-isomorphic to f_α^* of a bounded above flat perfect complex on X_α, for some α. For this, we make use of the patching construction of 3.20.4.

3.20.4. Let U and V form an open cover of a scheme X. Denote the various open immersions as in:

$$\begin{tikzcd} U \arrow[r, "j"] & X \\ U \cap V \arrow[u, "k'"] \arrow[ur, "\ell"] \arrow[r, "j'"'] & V \arrow[u, "k"'] \end{tikzcd} \qquad (3.20.4.1)$$

Suppose $(E^{\cdot}_U, F^{\cdot}_V, G^{\cdot}_{U\cap V}, \varphi, \psi)$ is a datum consisting of flat perfect complexes $E^{\cdot}_U$, $F^{\cdot}_V$, $G^{\cdot}_{U\cup V}$ on U, V, $U \cup V$ respectively, and of quasi-isomorphisms $\varphi : G^{\cdot}_{U\cap V} \xrightarrow{\sim} k'^* E^{\cdot}_U$, $\psi : G^{\cdot}_{U\cap V} \xrightarrow{\sim} j'^* F^{\cdot}_V$ on $U \cap V$. Let $j! : \mathcal{O}_U\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ be the left adjoint to j^*. This $j!$ is the extension by 0 functor, is exact, and preserves flatness (e.g., [SGA 4] IV 11.3.3, V 1.3.1). The quasi-isomorphism φ induces a map on X:

$$!\varphi : \ell! G^{\cdot}_{U\cap V} \to \ell! k'^* E^{\cdot}_U = j! k'! k'^* E^{\cdot}_U \to j! E^{\cdot}_U.$$

The restriction to U, $j^*(!\varphi)$ is isomorphic to the map $k'! G_{U\cap V} \to E^{\cdot}_U$ adjoint to φ. On V, $k^*(!\varphi)$ is the quasi-isomorphism

$$j'!\varphi : j'! G^{\cdot}_{U\cap V} \xrightarrow{\sim} j'! k'^* E^{\cdot}_U = k^* j! E^{\cdot}_U.$$

Let $C^{\cdot}(E^{\cdot}_U, F^{\cdot}_V, G^{\cdot}_{U\cap V}, \varphi, \psi)$ be the homotopy pushout (1.1.2) of $!\varphi : \ell! G^{\cdot}_{U\cap V} \to j! E^{\cdot}_U$ and the similar map $!\psi : \ell! G^{\cdot}_{U\cap V} \to k! F^{\cdot}_V$.

$$C^{\cdot}\left(E^{\cdot}_U, F^{\cdot}_V, G^{\cdot}_{U\cap V}, \varphi, \psi\right) = j! E^{\cdot}_U \overset{h}{\underset{\ell! G^{\cdot}_{U\cap V}}{\cup}} k! F^{\cdot}_V \equiv !E \overset{h}{\underset{!G}{\cup}} !F. \tag{3.20.4.2}$$

For any complex $J^{\cdot}$ on X, a choice of maps $E^{\cdot}_U \to j^* J^{\cdot}$ and $F^{\cdot}_V \to k^* J^{\cdot}$, together with a choice of homotopy between the two restrictions of this maps via φ and ψ to maps $G^{\cdot}_{U\cap V} \to \ell^* J^{\cdot}$, determine maps $j! E^{\cdot}_U \to J^{\cdot}$, $k! F^{\cdot}_V \to J^{\cdot}$, and a homotopy of maps $\ell! G^{\cdot}_{U\cap V} \to J^{\cdot}$, and hence (1.1.2) determine a map $C^{\cdot}(E, F, G, \varphi, \psi) \to J$.

The complex $C^{\cdot}(E, F, G, \varphi, \psi)$ is flat, as the $j!$, $k!$, $\ell!$ preserve flatness, as does the construction of the homotopy pushout (1.1.2.1).

As $j'!(\varphi) = k^*(!\varphi) : k^*!G \to k^*(!E)$ is a quasi-isomorphism on V, the corresponding canonical map into the homotopy pushout

$$F^{\cdot}_V \xrightarrow{\sim} F^{\cdot}_V \overset{h}{\underset{k^*(!G^{\cdot})}{\cup}} k^*(!E^{\cdot}) = k^* C^{\cdot}(E, F, G, \varphi, \psi) = k^* C^{\cdot}$$

is a quasi-isomorphism on V. Hence $k^* C^{\cdot} = C^{\cdot}|V$ is perfect. Similarly $C^{\cdot}|U = j^* C^{\cdot}$ is quasi-isomorphic to $E^{\cdot}_U$, and so perfect. Hence $C^{\cdot}(E, F, G, \varphi, \psi)$ is locally perfect, hence perfect.

Suppose $J^{\cdot}$ is a perfect complex on X, from which we obtain a datum $(E^{\cdot}_U, F^{\cdot}_V, G^{\cdot}_{U\cap V}, \varphi, \psi)$ by $E^{\cdot}_U = j^* J^{\cdot} = J^{\cdot}|U$, $F^{\cdot}_V = k^* J^{\cdot} = J^{\cdot}|V$, $G^{\cdot}_{U\cap V} = \ell^* J^{\cdot} = J^{\cdot}|U \cup V$, and $\varphi = 1$, $\psi = 1$. Then the adjunction maps $j! E^{\cdot}_U = j! j^* J^{\cdot} \to J^{\cdot}$, $k! F^{\cdot}_V = k! k^* J^{\cdot} \to J^{\cdot}$, and the 0 homotopy between identical maps $\ell! G^{\cdot}_{U\cap V} = \ell! \ell^* J^{\cdot} \to J^{\cdot}$ induce a map $C^{\cdot}(j^* J, k^* J, \ell^* J, 1, 1,) \to J^{\cdot}$ which is a quasi-isomorphism, as one checks locally on U and V. (Indeed we have already seen this, as $C^{\cdot}$ here is the mapping cone of the left map in the exact sequence (2.4.1.10) with $E^{\cdot} = J^{\cdot}$.)

3.20.5. Now let $J^{\cdot}$ be a bounded above flat perfect complex on X. We want to show that it is quasi-isomorphic to f_α^* of a bounded above flat perfect complex on some X_α. By induction hypothesis on U and V, for α sufficiently large, there are flat bounded above perfect complexes $E^{\cdot}_{U_\alpha}$ on U_α, $F^{\cdot}_{V_\alpha}$ on V_α, and isomorphisms in the derived category $f_\alpha^* E^{\cdot}_{U_\alpha} \sim J^{\cdot}|U$, $f_\alpha^* F^{\cdot}_{V_\alpha} \sim J^{\cdot}|V$. By the calculus of fractions we represent these by strict quasi-isomorphisms of flat bounded above perfect complexes $f_\alpha^* E^{\cdot}_{U_\alpha} \xleftarrow{\simeq} A^{\cdot} \xrightarrow{\simeq} J^{\cdot}|U$, $f_\alpha^* F^{\cdot}_{V\alpha} \xleftarrow{\simeq} B^{\cdot} \xrightarrow{\simeq} J^{\cdot}|V$. In the derived category on $U \cap V$, we have a composite isomorphism $f_\alpha^* E^{\cdot}_{U_\alpha}|U \cap V \sim J^{\cdot} \sim f_\alpha^* F^{\cdot}_{V\alpha}|U \cap V$. By 3.20.1 for the quasi-affine $U_\alpha \cap V_\alpha$, we see that for α sufficiently large that this isomorphism is f_α^* of an isomorphism on $U_\alpha \cap V_\alpha$, represented in the calculus of fractions by strict quasi-isomorphisms $E^{\cdot}_{U_\alpha}|U_\alpha \cap V_\alpha \xleftarrow{\simeq} G^{\cdot}_{U_\alpha \cap V_\alpha} \xrightarrow{\simeq} F^{\cdot}_{V_\alpha}|U_\alpha \cap V_\alpha$. On $U \cap V$, the criterion of the calculus of fractions for equivalence of representations of maps in the derived category gives a chain homotopy commutative diagram:

$$
\begin{array}{lccccc}
 & J^{\cdot}|U \cap V & \xleftarrow{\ 1\ } & J^{\cdot}|U \cap V & \xrightarrow{\ 1\ } & J^{\cdot}|U \cap V \\
 & {\scriptstyle \sim|U\cap V}\uparrow & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim|U\cap V} \\
(3.20.5.1) & A^{\cdot}|U \cap V & \xleftarrow{\ \sim\ } & D^{\cdot} & \xrightarrow{\ \sim\ } & B^{\cdot}|U \cap V \\
 & {\scriptstyle \sim|U\cap V}\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim|U\cap V} \\
 & f_\alpha^* E^{\cdot}_\alpha|U_\alpha \cap V_\alpha & \underset{f_\alpha^*\varphi}{\xleftarrow{\ \sim\ }} & f_\alpha^* G^{\cdot}_{U_\alpha \cap V_\alpha} & \underset{f_\alpha^*\psi}{\xrightarrow{\ \sim\ }} & f_\alpha^* F^{\cdot}_{V_\alpha}|U_\alpha \cap V_\alpha
\end{array}
$$

We make choices of homotopies in this diagram, and appeal to the mapping properties of the construction of 3.20.4. This yields quasi-isomorphisms of perfect complexes

$$J^{\cdot} \xleftarrow{\simeq} !J^{\cdot}|U \overset{h}{\underset{!J^{\cdot}U\cap V}{\cup}} !J^{\cdot}|V \xleftarrow{\simeq} !A^{\cdot} \overset{h}{\underset{!D^{\cdot}}{\cup}} !B^{\cdot} \xrightarrow{\simeq} f_\alpha^*\left(!E^{\cdot}_\alpha \underset{!G^{\cdot}_\alpha}{\cup} !F^{\cdot}_\alpha \equiv C^{\cdot}_\alpha \right).$$

Thus $J^{\cdot}$ is quasi-isomorphic to $f_\alpha^* C^{\cdot}_\alpha$ for a bounded above flat perfect complex $C^{\cdot}_\alpha$ on some X_α, as required. If $J^{\cdot}|X - Y$ is to be acyclic, $C^{\cdot}_\beta = f_{\beta\alpha}^* C^{\cdot}_\alpha$ will be acyclic on $X_\beta - Y_\beta$ for some β sufficiently large.

3.20.6. To complete the induction step, it remains only to show that if $E^{\cdot}_\alpha$ and $E'^{\cdot}_\alpha$ are bounded above flat perfect complexes on X_α, then with $E^{\cdot}_\beta = f_{\beta\alpha}^* E^{\cdot}_\alpha$ and $E^{\cdot} = f_\alpha^* E^{\cdot}_\alpha$, etc., that the canonical map is an isomorphism:

$$\text{(3.20.6.1)} \qquad \varinjlim_{\beta} \ \mathrm{Mor}_{D(X_\beta)}\left(E^{\cdot}_\beta,\ E'^{\cdot}_\beta\right) \xrightarrow{\cong} \mathrm{Mor}_{D(X)}\left(E^{\cdot}, E'^{\cdot}\right).$$

By the induction hypothesis, the corresponding maps to (3.20.6.1) for U, V and for the quasi-affine $U \cap V$ are isomorphisms. The result follows for X by the 5-lemma applied to the map (3.20.6.1) between the Mayer-Vietoris exact sequences (2.4.1.8).

This completes the proof of the induction step, hence of 3.20.1 and of the proposition.

3.21. **Theorem** (Poincaré duality) (cf. [SGA 6] IV 2.5, [Q1] Section 7-1). *Let X be a quasi-compact scheme. Suppose for every local ring $\mathcal{O}_{X,x}$ of X, that every finitely presented $\mathcal{O}_{X,x}$-module has finite Tor-dimension over $\mathcal{O}_{X,x}$. (In fact, it suffices to suppose that every pseudo-coherent $\mathcal{O}_{X,x}$-module has finite Tor-dimension over $\mathcal{O}_{X,x}$.) (Note that any regular noetherian scheme meets these hypotheses.) Then the canonical map is a homotopy equivalence:*

$$K(X) \xrightarrow{\sim} G(X).$$

Proof. This follows from the definitions once we show that any cohomologically bounded pseudo-coherent complex $E^{\cdot}$ on X is perfect. As X is quasi-compact, there is no need to worry about global bounds, and the question is local. So we take a point $x \in X$, and restrict to a small affine *nbd* U of x. By 2.3.1(e), $E^{\cdot}|U$ is quasi-isomorphic to a strict pseudo-coherent complex, so we may assume that $E^{\cdot}$ is strict pseudo-coherent. As $E^{\cdot}$ is cohomologically bounded, there is an integer k such that $H^n(E^{\cdot}) = 0$ for $n \leq k$. Then $Z^nE^{\cdot} = B^nE^{\cdot}$ for $n \leq k$. For $n \leq k$, $E^{n-2} \to E^{n-1} \to Z^nE \to 0$ is then exact, so that Z^nE is finitely presented. (In fact $Z^nE^{\cdot}$ is resolved by the exact complex of algebraic vector bundles $\sigma^{\leq n-1}E^{\cdot}$, and so is a pseudo-coherent module.) Then the stalk $Z^kE^{\cdot}_x$ has finite Tor-dimension, say p. By descending induction, using the facts that E^n_x is free and that $0 \to Z^{n-1}E^{\cdot}_x \to E^{n-1}_x \to Z^nE^{\cdot}_x \to 0$ is exact for $n \leq k$, we get that $Z^{k-p}E^{\cdot}_x$ is flat and finitely presented over $\mathcal{O}_{X,x}$, and hence free. Then the finitely presented $Z^{k-p}E^{\cdot}$ is free over some smaller open *nbd* U of x. Thus $\tau^{k-p}E^{\cdot}$ is strict perfect on U, and is also quasi-isomorphic to $E^{\cdot}|U$. So $E^{\cdot}$ is perfect, as required.

3.22. To prepare the key Localization Theorem 5.1, 7.4, we must consistently use K-theory with supports, $K(X \text{ on } Y)$. We remark that the absolute case $K(X)$ is the special case $K(X) = K(X \text{ on } X)$. Also $K(X \text{ on } \phi) = 0$ for ϕ empty.

4. Projective space bundle theorem

4.1. **Theorem** (Projective space bundle Theorem). *Let X be a quasi-compact and quasi-separated scheme. Let Y be a closed subspace such that $X - Y$ is quasi-compact. Let $\mathcal{E}$ be an algebraic vector bundle of rank r over X, and let $\pi : \mathbf{P}\mathcal{E}_X \to X$ be the associated projective space bundle. Then there are natural homotopy equivalences*

$$\prod^{r} K(X) \xrightarrow{\sim} K(\mathbf{P}\mathcal{E}_X) \tag{4.1.1}$$

$$\prod^{r} K(X \text{ on } Y) \xrightarrow{\sim} K(\mathbf{P}\mathcal{E}_X \text{ on } \mathbf{P}\mathcal{E}_Y). \tag{4.1.2}$$

These equivalences are given by the formula

$$(x_0, x_1, \ldots, x_{r-1}) \mapsto \sum_{i=0}^{r-1} \pi^*(x_i) \otimes [\mathcal{O}_{\mathbf{P}\mathcal{E}}(-i)]. \tag{4.1.3}$$

Proof. 4.3 - 4.12 below.

4.2 Theorem 4.1 for K_0 is proved in Berthelot's exposé, [SGA 6] VI, and for K_0^{naive} goes back to Grothendieck's early work on Riemann-Roch and Chern classes. Modifying these arguments to make them functorial on the Q-category, Quillen ([Q1] Section 8) proved (4.1.1) for K^{naive}. Below, we will modify Quillen's argument to make it work for K.

Logically, the reader should now proceed to Appendix C before returning to 4.3.

4.3. We first reduce to the case where X is noetherian. By C.9, X is the inverse limit $\varprojlim X_\alpha$ of an inverse system of schemes in which all the bounding maps $X_\beta \to X_\alpha$ are affine, and in which all the X_α are finitely presented over $\mathrm{Spec}(\mathbb{Z})$, and hence noetherian.

As $U = X - Y$ is quasi-compact, and X is quasi-separated, the open immersion $j : U \to X$ is finitely presented ([EGA] I 6.1.10(iii), 6.3.8(i)). Of course $\mathcal{E}$ is a finitely presented $\mathcal{O}_X$-module. Then by restricting to a cofinal system of α, we may assume there are quasi-compact opens $U_\alpha \subseteq X_\alpha$ and vector bundles $\mathcal{E}_\alpha$ on X_α such that $f_{\beta\alpha}^{-1}(U_\alpha) = U_\beta$, $f_{\beta\alpha}^*(\mathcal{E}_\alpha) = \mathcal{E}_\beta$, $U = \varprojlim U_\alpha$, $\mathbf{P}\mathcal{E} = \varprojlim \mathbf{P}\mathcal{E}_\alpha$ is the pullback of $\pi_\alpha : \mathbf{P}\mathcal{E}_\alpha \to X_\alpha$ along $X \to X_\alpha$, etc., ([EGA] IV Section 8, as quoted in C.3, C.4).

We set $Y_\alpha = X_\alpha - U_\alpha$. Then by 3.20 we obtain the diagram (4.3.1) in which the indicated maps are homotopy equivalences:

$$
\begin{tikzcd}
\varinjlim_\alpha \prod^r K(X_\alpha \text{ on } Y_\alpha) \arrow[r, "\sim"] \arrow[d] & \prod^r K(X \text{ on } Y) \arrow[d] \\
\varinjlim_\alpha K(\mathbf{P}\mathcal{E}_{X_\alpha} \text{ on } \mathbf{P}\mathcal{E}_{Y_\alpha}) \arrow[r, "\sim"] & K(\mathbf{P}\mathcal{E}_X \text{ on } \mathbf{P}\mathcal{E}_Y)
\end{tikzcd}
\tag{4.3.1}
$$

Thus it suffices to prove the theorem for the noetherian schemes X_α.

4.4. We assume X is noetherian for the rest of Section 4. We need to recall some standard facts about the cohomology of coherent sheaves on $\mathbf{P}\mathcal{E}$, due to Serre and Grothendieck, and to recall Mumford's notion of a "regular" coherent sheaf, to set up Quillen's argument.

4.5. *Recollection.* (a) For all integers q, $R^q\pi_*$ preserves quasi-coherence and coherence.

(b) For $q \geq r = \operatorname{rank} \mathcal{E}$, and $\mathcal{F}$ any quasi-coherent sheaf on $\mathbf{P}\mathcal{E}$, $R^q\pi_*\mathcal{F} = 0$.

(c) For $\mathcal{F}$ coherent on $\mathbf{P}\mathcal{E}$, there is an integer $n_0(\mathcal{F}) = n_0$ such that for all $n \geq n_0$ and all $q \geq 1$, $R^q\pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) = 0$.

(d) For $\mathcal{F}$ quasi-coherent on $\mathbf{P}\mathcal{E}$ and for $\mathcal{M}$ flat and quasi-coherent on X there is a canonical isomorphism

$$R^q\pi_*(\mathcal{F} \otimes \pi^*\mathcal{M}) \cong R^q\pi_*(\mathcal{F}) \otimes \mathcal{M}.$$

(e) For all integers n, there are natural isomorphisms

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) = \begin{cases} 0 & q \neq 0,\ r-1 \\ S^n\mathcal{E} & q = 0 \\ (S^{-r-n}\mathcal{E})^\vee \otimes (\Lambda^r\mathcal{E})^\vee & q = r-1 \end{cases}$$

where $S^k\mathcal{E}$ is the k-th symmetric power of $\mathcal{E}$, considered to be 0 for $k \leq -1$, $\Lambda^r\mathcal{E}$ is the maximal exterior power of $\mathcal{E}$, and $(\ )^\vee$ sends a vector bundle to its dual, $(\ )^\vee = \operatorname{Hom}(\ ,\mathcal{O}_X)$.

(f) For all quasi-coherent sheaves $\mathcal{M}$ on X, there is a natural isomorphism

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n) \otimes \pi^*\mathcal{M}) \cong R^q\pi_*(\mathcal{O}_{\mathbf{P}\mathcal{E}}(n)) \otimes \mathcal{M}.$$

Proof. Of course, (a), (b), and (c) are very well-known. ([EGA] III 1.4.10, 2.2.2, 2.2.1). The formula (e) results from a standard Čech cohomology computation [EGA] III 2.1.15, 2.1.16 or [Q1] Section 8.1.1(c).

(Note both these citations give the formula (e) with different typographical errors!) We see from (e) that $R^q\pi_*\mathcal{O}(n)$ is a flat $\mathcal{O}_X$-module. Now (d) and (f) are recovered from the quasi-isomorphism (2.5.5.1), as this flatness of the cohomology groups forces the Tor-spectral sequences of Künneth to degenerate (cf. [EGA] III Section 6, Section 7).

4.6. On $\mathbf{P}\mathcal{E}$, the canonical map

$$(\pi^*\mathcal{E})\otimes\mathcal{O}(-1) = (\pi^*\pi_*\mathcal{O}(1))\otimes\mathcal{O}(-1) \twoheadrightarrow \mathcal{O}(1)\otimes\mathcal{O}(-1) = \mathcal{O}_{\mathbf{P}\mathcal{E}}$$

is an epimorphism by [EGA] II 4.1.6. Locally on $\mathbf{P}\mathcal{E}$, the vector bundle $\pi^*\mathcal{E}\otimes\mathcal{O}(-1)$ is free, hence locally is a sum of line bundles $\oplus^i\mathcal{L}_i$. As then $\oplus^i\mathcal{L}_i \twoheadrightarrow \mathcal{O}$ is epimorphic, for each point p some $\mathcal{L}_i$ has image not contained in the maximal ideal of the local ring $\mathcal{O}_p$. For this $\mathcal{L}_i$, $\mathcal{L}_{ip}\to\mathcal{O}_p$ is an epimorphism of rank 1 free modules over the local ring $\mathcal{O}_p$, and so is an isomorphism.

The Koszul complex of $\pi^*\mathcal{E}\otimes\mathcal{O}(-1)\to\mathcal{O}_{\mathbf{P}\mathcal{E}}$ is thus locally isomorphic to a tensor product of complexes $\otimes^r(\mathcal{L}_i\to\mathcal{O})$, where at each point p one of the complexes $\mathcal{L}_i\to\mathcal{O}$ is acyclic. Thus the Kozsul complex of $\pi^*\mathcal{E}\otimes\mathcal{O}(-1)\to\mathcal{O}$ is acyclic (cf. [EGA] III 1.1, [SGA 6] VII 1). Expanding out the Koszul complex yields the well-known long exact sequence of algebraic vector bundles on $\mathbf{P}\mathcal{E}$ ([SGA 6] VI 1.11, [Q1] Section 8):

$$\begin{aligned}(4.6.1)\quad 0\to\pi^*\left(\overset{r}{\wedge}\mathcal{E}\right)\otimes\mathcal{O}(-r)\to\pi^*\left(\overset{r-1}{\wedge}\mathcal{E}\right)\otimes\mathcal{O}(1-r)\to\dots\\ \dots\to\pi^*\left(\overset{2}{\wedge}\mathcal{E}\right)\otimes\mathcal{O}(-2)\to\pi^*\mathcal{E}\otimes\mathcal{O}(-1)\to\mathcal{O}\to 0.\end{aligned}$$

Dually, there is the exact sequence:

$$\begin{aligned}(4.6.2)\quad 0\to\mathcal{O}\to(\pi^*\mathcal{E}^\vee)\otimes\mathcal{O}(1)\to\pi^*\left(\overset{2}{\wedge}\mathcal{E}^\vee\right)\otimes\mathcal{O}(2)\to\dots\\ \dots\to\pi^*\left(\overset{r}{\wedge}(\mathcal{E}^\vee)\right)\otimes\mathcal{O}(r)\to 0.\end{aligned}$$

4.7.0 Let m be an integer. A quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}\mathcal{E}$ is said to be m-regular in the sense of Mumford if $R^q\pi_*(\mathcal{F}(m-q))=0$ for all $q\geq 1$ (cf. [Q1] Section 8, [SGA 6] XIII Section 1, [Mum] 14). We note that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(n)$ is $(m-n)$-regular.

If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}\mathcal{E}$, there exists an integer m_0 such that $\mathcal{F}$ is n-regular for all $n\geq m_0$. This follows from 4.5(b) and (c) on taking $m_0=n_0+r-1$.

4.7.1. **Lemma** [Q1], [SGA 6], [Mum]). *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of quasi-coherent sheaves on $\mathbf{P}\mathcal{E}$. Then*

(a) If $\mathcal{F}$ and $\mathcal{H}$ are n-regular, $\mathcal{G}$ is n-regular.

(b) If $\mathcal{G}$ is n-regular and $\mathcal{F}$ is $(n+1)$-regular, then $\mathcal{H}$ is n-regular.

(c) If $\mathcal{G}$ is $(n+1)$-regular and $\mathcal{H}$ is n-regular and if $\pi_*\mathcal{G}(n) \to \pi_*\mathcal{H}(n)$ is an epimorphism, then $\mathcal{F}$ is $(n+1)$-regular.

Proof. All these results follow from the long exact sequence in cohomology:

$$\cdots \to R^{q-1}\pi_*\mathcal{G}(n-q) \to R^{q-1}\pi_*\mathcal{H}(n-q) \xrightarrow{\partial} R^q\pi_*\mathcal{F}(n-q) \to R^q\pi_*\mathcal{G}(n-q)$$
$$\to R^q\pi_*\mathcal{H}(n-q) \xrightarrow{\partial} R^{q+1}\pi_*\mathcal{F}(n-q) \to \dots.$$

4.7.2. **Lemma** ([SGA 6] XIII 1.3, [Q1] 8.1.3, 8.1.7). *If $\mathcal{F}$ is m-regular on $\mathbf{P}\mathcal{E}$, then for all $k \geq m$ we have that:*

(a) *$\mathcal{F}$ is k-regular.*

(b) *The product map $\pi_*(\mathcal{F}(k)) \otimes \mathcal{E} = \pi_*(\mathcal{F}(k)) \otimes \pi_*\mathcal{O}(1) \to \pi_*\mathcal{F}(k+1)$ is an epimorphism on X.*

(c) *$\pi^*\pi_*\mathcal{F}(k) \to \mathcal{F}(k)$ is an epimorphism on $\mathbf{P}\mathcal{E}$.*

Proof. To prove (a), we induct on $k-m$. We must show that if $\mathcal{F}$ is k-regular, then it is $(k+1)$-regular. We tensor the Koszul exact sequence 4.6.1 with $\mathcal{F}(k)$, to obtain an exact sequence

(4.7.2.1)
$$0 \to \left(\pi^* \overset{r}{\wedge} \mathcal{E}\right) \otimes \mathcal{F}(k-r) \to \cdots \to (\pi^*\mathcal{E}) \otimes \mathcal{F}(k-1) \to \mathcal{F}(k) \to 0.$$

This sequence breaks up into short exact sequences (4.7.2.2), where the Z_p are the kernels of the maps in (4.7.2.1).

$$(4.7.2.2) \qquad 0 \to Z_p \to \left(\pi^* \overset{p}{\wedge} \mathcal{E}\right) \otimes \mathcal{F}(k-p) \to Z_{p-1} \to 0.$$

The sheaf $(\pi^* \overset{p}{\wedge} \mathcal{E}) \otimes \mathcal{F}$ is k-regular by 4.5. Hence $(\pi^* \wedge^p \mathcal{E}) \otimes \mathcal{F}(k-p)$ is p-regular. By descending induction on p, starting from the $(r+1)$-regular $0 = Z_r$, and applying 4.7.1(b) to (4.7.2.2), we see that Z_{p-1} is p-regular. In particular, $Z_0 = \mathcal{F}(k)$ is 1-regular; i.e., $\mathcal{F}$ is $(k+1)$-regular, as required to prove (a).

We also have obtained that Z_1 is 2-regular,, so $R^1\pi_*(Z_1(1)) = 0$. The long exact cohomology sequence induced by (4.7.2.2) for $p = 1$ then shows that $\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}(k-1+1)) \to \pi_*(Z_0(1) = \mathcal{F}(k+1))$ is an epimorphism for all $k \geq m$. This proves (b).

From (b), we get that (4.7.2.3) is an epimorphism of graded quasi-coherent $\mathcal{O}_X$-modules for $k \geq m$:

$$S^{\cdot}\mathcal{E} \underset{\mathcal{O}_X}{\otimes} \pi_*\mathcal{F}(k) \to \coprod_{n\geq 0} \pi_*\mathcal{F}(n+k). \tag{4.7.2.3}$$

The result (c) now follows on applying the functor Proj to (4.7.2.3) to show that the corresponding canonical map $\pi^*\pi_*\mathcal{F}(k) \to \mathcal{F}(k)$ is an epimorphism on $\mathbf{P}\mathcal{E}$ ([EGA] II 3.2, 3.3, 3.4.4).

4.8. Next we recall Quillen's functorial resolution for quasi-coherent 0-regular sheaves on $\mathbf{P}\mathcal{E}$, [Q1] 8.1.11.

For any quasi-coherent $\mathcal{F}$ on $\mathbf{P}\mathcal{E}$, we inductively define quasi-coherent sheaves $T_n\mathcal{F}$ on X and $Z_n\mathcal{F}$ on $\mathbf{P}\mathcal{E}$. We start with $Z_{-1}\mathcal{F} = \mathcal{F}$. Let $T_n\mathcal{F} = \pi_*((Z_{n-1}\mathcal{F})(n))$, and let $Z_n\mathcal{F}$ be the kernel of the product map $\mathcal{O}(-n) \otimes \pi^*T_n\mathcal{F} \to Z_{n-1}\mathcal{F}$, inductively defining these for $n \geq 0$.

Clearly Z_n and T_n are additive functors, and preserve coherence.

If $\mathcal{F}$ is 0-regular, then by induction on n we see that $Z_{n-1}(\mathcal{F})(n)$ is 0-regular. This is clear for $n = 0$ and $Z_{-1}\mathcal{F} = \mathcal{F}$. The induction step results from 4.7.1(c) applied to the exact sequence

$$0 \to (Z_n\mathcal{F})(n) \to \pi^*T_n\mathcal{F} \to (Z_{n-1}\mathcal{F})(n) \to 0. \tag{4.8.1}$$

Note that $\pi^*T_n\mathcal{F} = \pi^*\pi_*((Z_{n-1}\mathcal{F})(n)) \to (Z_{n-1}\mathcal{F})(n)$ is an epimorphism by 4.7.2(c), and (4.8.1) is exact in the other places by definition of $Z_n\mathcal{F}$. We also note that $\pi_*\pi^*T_n\mathcal{F} \to \pi_*((Z_{n-1}\mathcal{F})(n))$ is not only epimorphic as required by 4.7.1(c), but is actually an isomorphism as we see from the definition of $T_n\mathcal{F}$ and the fact that $\pi_*\pi^* = 1$ for $\pi : \mathbf{P}\mathcal{E} \to X$. From this remark and (4.8.1), we also see that $\pi_*(Z_n(\mathcal{F})(n)) = 0$.

As $R^q\pi_* = 0$ for $q \geq 1$ on the category of 0-regular sheaves, the functor π_* is exact on this exact subcategory of quasi-coherent sheaves. By induction on n, we then see that $Z_{n-1}(\mathcal{F})$ and $T_n\mathcal{F} = \pi_*((Z_{n-1}\mathcal{F})(n))$ are exact functors on the exact category of 0-regular sheaves.

Next we note that $Z_{r-1}\mathcal{F} = 0$ for $r = \operatorname{rank} \mathcal{E}$ and $\mathcal{F}$ 0-regular. For (4.8.1) yields a long exact sequence in cohomology for any $n \geq 0$:

$$\begin{array}{ll}(4.8.2) \quad \ldots \to R^{q-1}\pi_*\left(\left(Z_{n+q-1}\mathcal{F}\right)(n)\right) \xrightarrow{\partial} & R^q\pi_*\left(\left(Z_{n+q}\mathcal{F}\right)(n)\right) \\ & \downarrow \\ & R^q\pi_*\left(\mathcal{O}(-q) \otimes \pi^*T_{n+q}\mathcal{F}\right) \to \ldots.\end{array}$$

Using the facts that $\pi_*((Z_n\mathcal{F})(n)) = 0$ and the formula of 4.5, we obtain from (4.8.2) by ascending induction on q that $R^q\pi_*((Z_{n+q}\mathcal{F})(n)) = 0$. As

$R^q\pi_* = 0$ for $q \geq r$, this shows that $(Z_{r-1}\mathcal{F})(r-1)$ is 0-regular. Then $\pi_*(Z_{r-1}\mathcal{F})(r-1) = 0$ combined with 4.7.2(c) shows that $Z_{r-1}(\mathcal{F})(r-1) = 0$ as required.

Now we tensor (4.8.1) with $\mathcal{O}(-n)$, and splice together into a long exact sequence, to obtain Quillen's functorial resolution of a 0-regular sheaf.

4.8.4. **Lemma**. *On the exact category of 0-regular coherent sheaves on* $\mathbf{P}\mathcal{E}$, *there are exact functors* T_i, $i = 0, 1, 2, \dots, r-1$, *to the category of coherent sheaves on* X, *and a functorial exact sequence on* $\mathbf{P}\mathcal{E}$:

$$0 \to \mathcal{O}(-r+1) \otimes \pi^* T_{r-1}(\mathcal{F}) \to \cdots \to \mathcal{O} \otimes \pi^* T_0(\mathcal{F}) \to \mathcal{F} \to 0.$$

4.8.4. These T_i extend to functors on the category of strict bounded complexes of 0-regular coherent sheaves on $\mathbf{P}\mathcal{E}$, by $T_i(E^\cdot)^k = T_i E^k$. These extended T_i preserve mapping cones of complexes. Suppose $F^\cdot$ is an acyclic complex, which is a bounded complex of 0-regular coherent sheaves. By increasing induction on n starting from $n \ll 0$ where $F^n = 0$, using the exact sequence $0 \to Z^n F \to F^n \to B^{n+1} F \to 0$, the isomorphisms $Z^{n+1}F^\cdot \cong B^{n+1}F^\cdot$ and Lemma 4.7.1(b) and 4.7.2(a), we see that $Z^{n+1}F^\cdot = B^{n+1}F^\cdot$ is 0-regular and coherent for all n. It follows that $0 \to T_i(Z^n F^\cdot) \to T_i F^n \to T_i(B^{n+1}F^\cdot) \to 0$ is exact, so $T_i(B^n F^\cdot) = B^n(T_i F^\cdot)$, $T_i(Z^n F^\cdot) = Z^n(T_i F^\cdot)$, and so $Z^n(T_i F^\cdot) = B^n(T_i F^\cdot)$ for all n. Hence $T_i F^\cdot$ is acyclic. Applying this to mapping cones $F^\cdot$, we see that each functor T_i preserves quasi-isomorphisms between strict bounded complexes of 0-regular coherent sheaves. Thus the T_i will be exact functors between the biWaldhausen categories that we will soon introduce in 4.9.

Also, if $F^\cdot$ is acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y = \mathbf{P}\mathcal{E}_X - \pi^{-1}(Y)$, the above argument shows that $T_i F^\cdot$ is acyclic on $X - Y$.

4.8.5. Tensoring the exact sequence of 4.8.3 with $\mathcal{O}(k)$ for $0 \leq k \leq r-1$, applying π_* and considering 4.5 and 4.7, we obtain for any strict bounded complex of 0-regular coherent sheaves $F^\cdot$ on $\mathbf{P}\mathcal{E}$, an exact sequence of complexes on X

$$\begin{aligned} \ldots \to 0 \to T_k(F^\cdot) \to \mathcal{E} \otimes T_{k-1}(F^\cdot) \to S^2\mathcal{E} \otimes T_{k-2}(F^\cdot) \to \ldots \\ \to \cdots \to S^k\mathcal{E} \otimes T_0(F^\cdot) \to \pi_* F^\cdot \to 0. \end{aligned} \tag{4.8.5.1}$$

Also, $\pi_* F^\cdot$ represents $R\pi_* F^\cdot$ as $R^q \pi_* F^i = 0$ for all $q \geq 1$. Suppose now that $F^\cdot$ is perfect. Then $\pi_* F^\cdot = T_0(F^\cdot)$ is perfect by 2.7 for the proper flat $\pi : \mathbf{P}\mathcal{E} \to X$. By ascending induction on k, using (4.8.5.1) and 2.2.13(b), it follows that $T_k(F^\cdot)$ is perfect.

4.9. Consider the following complicial biWaldhausen categories (1.2.11) (with associated abelian categories the categories of all quasi-coherent sheaves).

$\boldsymbol{A}$: perfect strict bounded complexes of 0-regular coherent sheaves on $\mathbf{P}\mathcal{E}_X$ (resp., such as are also acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y$).

$\boldsymbol{B}$: perfect strict bounded complexes of coherent sheaves on $\mathbf{P}\mathcal{E}_X$ (resp., such as are also acyclic on $\mathbf{P}\mathcal{E}_X - \mathbf{P}\mathcal{E}_Y$).

$\boldsymbol{C}$: perfect strict bounded complexes of coherent sheaves on X (resp., such as are also acyclic on $X - Y$).

From 4.8.3, 4.8.4, and 4.8.5, we get:

There is an obvious exact inclusion $I : \boldsymbol{A} \to \boldsymbol{B}$.

There are exact functors $T_k : \boldsymbol{A} \to \boldsymbol{C}$ for $k = 0, 1, \ldots, r-1$.

There are exact functors $\mathcal{O}(-k) \otimes \pi^*(\ ) : \boldsymbol{C} \to \boldsymbol{B}$ for all k.

There is a natural quasi-isomorphism (4.9.1) in $\boldsymbol{B}$ for $A^\cdot$ in $\boldsymbol{A}$

(4.9.1)
$$I(A^\cdot) \xleftarrow{\sim} \text{Total complex}\left[\mathcal{O}(-r+1) \otimes \pi^* T_{r-1} A^\cdot \to \cdots \to \mathcal{O} \otimes \pi^* T_0 A^\cdot\right].$$

As X and $\mathbf{P}\mathcal{E}_X$ are noetherian, we see from 3.7 that $K(\boldsymbol{B})$ is $K(\mathbf{P}\mathcal{E}_X)$ (resp., $K(\mathbf{P}\mathcal{E}_X \text{ on } \mathbf{P}\mathcal{E}_Y)$) and that $K(\boldsymbol{C})$ is $K(X)$ (resp., $K(X \text{ on } Y)$). Thus the proof of Theorem 4.1 is reduced to showing that the exact functor

$$\bigoplus_{k=0}^{r-1} \mathcal{O}(-k) \otimes \pi^*(\ ) : \prod^{r} \boldsymbol{C} \to \boldsymbol{B} \tag{4.9.2}$$

induces a homotopy equivalence on K-theory spectra. We show the map on K-theory is both a split epimorphism and a split monomorphism in the homotopy category of spectra.

4.10. First we show that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is split mono up to homotopy. The formulae of 4.5 show that $R\pi_*(\mathcal{O}(n-k) \otimes \pi^* E^\cdot) = 0$ for $0 \le n < k \le r-1$, and that $R\pi_*(\mathcal{O} \otimes \pi^* E^\cdot) = R\pi_* \pi^* E^\cdot = E^\cdot$, for $E^\cdot$ any complex of quasi-coherent sheaves on X.

Consider the map $K(\boldsymbol{B}) \to \prod^r K(\boldsymbol{C})$ induced by $F^\cdot \mapsto (R\pi_* F^\cdot, R\pi_*(F^\cdot(1)), \ldots, R\pi_*(F^\cdot(r-1)))$. Composing this map with the $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ induced by (4.9.2), we get an endomorphism of $\prod^r K(\boldsymbol{C})$. This endomorphism is represented in the homotopy category of spectra by an $r \times r$ matrix of maps $K(\boldsymbol{C}) \to K(\boldsymbol{C})$. The calculation of the preceding paragraph shows that this matrix has 0's above the diagonal and has 1's along the diagonal. Thus the matrix is invertible, and the composite endomorphism of $\prod^r K(\boldsymbol{C})$ is a homotopy equivalence. This shows that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is a split mono, as required.

4.11. To show that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is a split epi, we use the T_i. By 4.9.1, the inclusion $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is homotopic to the map induced by the total complex of the $\mathcal{O}(-k) \otimes \pi^* T_k$. Filtering this total complex so that the $\mathcal{O}(-k) \otimes \pi^* T_k$ are the filtration quotients, and appealing to the Additivity Theorem (1.7.3, 1.7.4), we see that the map induced by this total complex is homotopic to $\Sigma(-1)^k \mathcal{O}(-k) \otimes \pi^* T_k$, where the sum is over $k = 0, 1, \ldots, r-1$. Thus up to homotopy the map $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is $\Sigma(-1)^k \mathcal{O}(-k) \otimes \pi^* T_k$ and thus factors through the map (4.9.2) $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ via the map $K(\boldsymbol{A}) \to \prod^r K(\boldsymbol{C})$ given by $(K(T_0), -K(T_1), \ldots, (-1)^{r-1} K(T_{r-1}))$.

Thus it suffices to show that $I : K(\boldsymbol{A}) \to K(\boldsymbol{B})$ is a homotopy equivalence. This will follow from the approximation theorem in the form 1.9.8, set up by the dual of 1.9.7, once we show that for every $B^\cdot$ in $\boldsymbol{B}$ there is an $A^\cdot$ in $\boldsymbol{A}$ and a quasi-isomorphism $B^\cdot \xrightarrow{\sim} A^\cdot$. So let $B^\cdot$ be a bounded complex of coherent sheaves on $\mathbf{P}\mathcal{E}_X$. By 4.7.0, there is an n such that every B^i is n-regular. If $n \leq 0$, $B^\cdot$ is in $\boldsymbol{A}$, as every B^i is 0-regular by 4.7.2(a). We now proceed by descending induction on n, for $n > 0$. To do the induction step, suppose the result is known for complexes of $(n-1)$-regular sheaves. For $k \geq 1$ all $B^i(k)$ are $(n-1)$-regular by 4.7.0 and 4.7.2. Tensoring the (locally split) exact Koszul sequence (4.6.2) with $B^\cdot$ yields an exact sequence of complexes. We reinterpret this as a quasi-isomorphism of $B^\cdot = B^\cdot \otimes \mathcal{O}$ into the total complex of the rest of the sequence

$$B^\cdot \to \text{ Total complex } \left[\pi^* \mathcal{E}^\vee \otimes B^\cdot(1) \to \cdots \to \pi^* \left(\overset{r}{\wedge} \mathcal{E}^\vee\right) \otimes B^\cdot(r)\right] \equiv B'^\cdot.$$

The total complex $B'^\cdot$ consists of $(n-1)$-regular sheaves. By induction hypothesis, there is then a quasi-isomorphism $B'^\cdot \xrightarrow{\sim} A^\cdot$ and so $B \xrightarrow{\sim} B'^\cdot \xrightarrow{\sim} A^\cdot$ with $A^\cdot$ in $\boldsymbol{A}$, as required.

This completes the proof that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is split epi up to homotopy.

4.12. We have shown that $\prod^r K(\boldsymbol{C}) \to K(\boldsymbol{B})$ is both split mono and split epi in the homotopy category of spectra. It follows that this map is a homotopy equivalence, with homotopy inverse given by either of the splitting maps.

This completes the proof of 4.1.

5. Extension of perfect complexes, and the proto-localization theorem

5.1. **Theorem** (Proto-localization, cf. 7.4). *Let X be a quasi-compact and quasi-separated scheme. Let $U \subseteq X$ be a quasi-compact open subscheme, and set $Y = X - U$, a closed subspace of X. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then aside from possible failure of surjectivity for $K_0(X) \to K_0(U)$ and $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$, the usual maps give homotopy fibre sequences*

$$K(X \text{ on } Y) \to K(X) \to K(U) \tag{5.1.1}$$

$$K(X \text{ on } Y \cap Z) \to K(X \text{ on } Z) \to K(U \text{ on } U \cap Z). \tag{5.1.2}$$

That is, (5.1.1) *becomes a homotopy fibre sequence of spectra after $K(U)$ is replaced by the covering spectrum $K(U)^{\sim}$ with $\pi_i K(U)^{\sim} = K_i(U)$ for $i > 0$, and $\pi_0 K(U)^{\sim} =$ image $K_0(X) \to K_0(U)$. Similarly for* (5.1.2).

Proof. The proof will occupy 5.2 - 5.6. First, we make some remarks.

5.1.3. Later in Section 6, we will use 5.1 and 4.1 to define non-connective deloopings $K^B(X \text{ on } Y)$, etc., with $\pi_n K^B(X \text{ on } Y) = K_n(X \text{ on } Y)$ for $n \geq 0$, but possibly $\neq 0$ for $n < 0$. In 7.4, we will show that K^B analogs of (5.1.1) and (5.1.2) are homotopy fibre sequences without any covering spectrum fudge. This will be the mature localization theorem.

5.1.4. The fibre terms $K(X \text{ on } Y)$, $K(X \text{ on } Y \cap Z)$ have explicit descriptions in terms of complexes, and satisfy excision 3.19. This makes Theorem 5.1 very useful. Exercise 5.7 will give an alternate description of the fibre term comparable to the traditional kind of fibre terms in the very special cases where some form of localization theorem has been previously established.

5.1.5. Unlike our results in Sections 1 - 4, which have been at most minor improvements on the work of Grothendieck, Illusie, Berthelot, Quillen, and Waldhausen, this result is a revolutionary advance. Quillen proved a localization theorem for the G-theory of noetherian schemes [Q1], which is the most important tool in that subject. For K-theory Quillen proved [Gr1] a localization homotopy fibre sequence similar to (5.1.1) only in the case where U is affine, and where Y is a divisor defined by a section s of a line bundle $\mathcal{L}$ which is a monomorphism $s : \mathcal{O} \rightarrowtail \mathcal{L}$, i.e., such that the local equation $t = 0$ of Y has t a non-zero-divisor in $\mathcal{O}_X$. These restrictions greatly hinder applications of the result even to the K-theory of rings, and have obstructed the development of K-theory to any level approaching that of G-theory. For noteworthy previous attempts to break out of these restrictions, see the work of Gersten [Ge] Sections 5, 6, 7; Levine [L1]; and Weibel [We4], [We5].

5.2. We begin the proof of 5.1. Let $K(X \text{ on } Z \text{ for } U)$ be the K-theory spectrum of the complicial biWaldhausen category of those perfect complexes on X which are acyclic on $X - Z$, but where now the weak equivalences are the maps of complexes on X which are quasi-isomorphisms when restricted to U. The open immersion $j : U \to X$ induces an exact functor j^*, and so a map $j^* : K(X \text{ on } Z \text{ for } U) \to K(U \text{ on } Z \cap U)$.

The Waldhausen Localization Theorem 1.8.2 immediately gives a homotopy fibre sequence (5.2.1), after we note that a complex acyclic on $X - Z$ and quasi-isomorphic to 0 on $U = X - Y$ is acyclic on $(X-Z) \cup (X-Y) = X - Z \cap Y$.

(5.2.1) $$K(X \text{ on } Z \cap Y) \to K(X \text{ on } Z) \to K(X \text{ on } Z \text{ for } U).$$

This reduces the proof of (5.1.2) to showing that $j^* : K(X \text{ on } Z \text{ for } U) \to K(U \text{ on } U \cap Z)$ induces an isomorphism on homotopy groups π_i for $i > 0$, and induces a monomorphism on π_0. Cofinality 1.10.1 reduces this to showing that j^* is a homotopy equivalence of $K(X \text{ on } Z \text{ for } U)$ to the K-theory spectrum of the biWaldhausen category of those perfect complexes on U which are acyclic on $U - U \cap Z$, and which have Euler characteristic in the image of $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$. By the Approximation Theorem in form 1.9.8, this in turn reduces to showing that j^* induces an equivalence of the derived categories of the two complicial biWaldhausen categories. This equivalence follows from the results 5.2.2., 5.2.3, and 5.2.4, below.

5.2.2. **Key Proposition.** *Adopt the hypotheses and notations of 5.1. Then:*

(a) A perfect complex $F^\cdot$ on U is isomorphic in the derived category $D(\mathcal{O}_U\text{-Mod})$ to the restriction $j^*E^\cdot$ of some perfect complex $E^\cdot$ on X, if and only if the class $[F^\cdot]$ in $K_0(U)$ is in the image of $K_0(X)$.

(b) More generally, for a perfect complex $F^\cdot$ on U which is acyclic on $U - U \cap Z$, there exists a perfect complex $E^\cdot$ on X which is acyclic on $X - Z$ and an isomorphism between $F^\cdot$ and $j^*E^\cdot$ in the derived category $D(\mathcal{O}_U\text{-Mod})$ if and only if the class $[F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$ is in the image of $K_0(X \text{ on } Z)$.

5.2.3. **Proposition.** *Adopt the hypotheses and notations of* 5.1. *Then:*

(a) For any two perfect complexes $E^\cdot$ and $E'^\cdot$ on X, and for any map $b : j^*E^\cdot \to j^*E'^\cdot$ in the derived category on U, $D(\mathcal{O}_U\text{-Mod})$, there is a

perfect complex $E''^{\cdot}$ *on* X *and maps* $a : E''^{\cdot} \to E^{\cdot}$, $a' : E''^{\cdot} \to E'^{\cdot}$ *in the derived category on* X, $D(\mathcal{O}_X\text{-}Mod)$, *such that* j^*a *is an isomorphism in* $D(\mathcal{O}_U\text{-}Mod)$ *and* $b \cdot j^*a = j^*a'$.

(b) Moreover, if in (a) *both* $E^{\cdot}$ *and* $E'^{\cdot}$ *are acyclic on* $X - Z$, *then* $E''^{\cdot}$ *may also be taken to be acyclic on* $X - Z$.

5.2.4. **Proposition.** *Adopt the hypotheses and notations of* 5.1. *Then:*

(a) Let $E^{\cdot}$ *and* $E'^{\cdot}$ *be two perfect complexes on* X. *Suppose that* a, $b : E^{\cdot} \to E'^{\cdot}$ *are two maps in the derived category on* X, $D(\mathcal{O}_X\text{-}Mod)$, *such that* $j^*a = j^*b$ *in* $D(\mathcal{O}_U\text{-}Mod)$ *on* U. *Then there is a perfect complex* $E''^{\cdot}$ *on* X, *and a map* $c : E''^{\cdot} \to E^{\cdot}$ *in* $D(\mathcal{O}_X\text{-}Mod)$, *such that* $ac = bc$, *and such that* $j^*(c)$ *is an isomorphism in* $D(\mathcal{O}_U\text{-}Mod)$ *on* U.

(b) Moreover, if in (a) $E^{\cdot}$ *and* $E'^{\cdot}$ *are acyclic on* $X - Z$, *then* $E''^{\cdot}$ *may also be taken to be acyclic on* $X - Z$.

5.2.5. These three propositions will be proved in 5.2.6 - 5.6. This will complete the proof of 5.1.

5.2.6. We begin by showing that 5.2.4 in fact follows from 5.2.3. First we note that to prove 5.2.4 it suffices to show that if $j^*(a-b) = 0$ then there is a $c : E''^{\cdot} \to E^{\cdot}$ as in 5.2.4 with $(a-b)c = 0$. Thus we reduce 5.2.4 to the special case where $b = 0$.

In this case, let $F^{\cdot}$ be the homotopy fibre of $a : E^{\cdot} \to E'^{\cdot}$ with $f : F^{\cdot} \to E^{\cdot}$ the canonical map. Then $j^*F^{\cdot} \to j^*E^{\cdot} \to j^*E'^{\cdot}$ is a homotopy fibre sequence on U. As $j^*a = 0$ by hypothesis, the long exact sequence of $\mathrm{Mor}(j^*E[^*],\ )$ resulting from this fibre sequence shows that there is a map $g : j^*E^{\cdot} \to j^*F^{\cdot}$ in $D(\mathcal{O}_U\text{-Mod})$ such that $j^*f \cdot g = 1$. Then granting 5.2.3, there is a perfect $E''^{\cdot}$ on X and maps $d : E''^{\cdot} \to F^{\cdot}$, $d' : E''^{\cdot} \to E^{\cdot}$ such that $j^*(d')$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$ and $g \cdot j^*(d') = j^*(d)$. If $E^{\cdot}$ and $E'^{\cdot}$, hence also $F^{\cdot}$, are acyclic on $X - Z$, then $E''^{\cdot}$ can be taken to be acyclic on $X - Z$ by 5.2.3(b). Now $f \cdot d : E''^{\cdot} \to E^{\cdot}$ has $a \cdot f \cdot d = 0$ as $a \cdot f = 0$. Also $j^*(f \cdot d) = j^*f \cdot g \cdot j^*(d') = j^*(d')$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$. Thus $E''^{\cdot}$ and $f \cdot d$ satisfy the conclusion of 5.2.4.

This proves that 5.2.4(a) follows from 5.2.3(a) and 5.2.4(b), from 5.2.3(b).

5.3. To prove Proposition 5.2.2 and 5.2.3, we begin by reducing them to the case where X is noetherian of finite Krull dimension. (This will be convenient for studying extensions of morphisms in the derived categories as the coherator and injectives in $\mathrm{Qcoh}(X)$ work well for such X, (cf. Appendix B)).

For X quasi-compact and quasi-separated as in 5.1, by C.9 with $\Lambda = \mathbb{Z}$ we have $X = \varprojlim X_\alpha$ for $\{X_\alpha\}$ an inverse system of schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$, in which the bonding maps $X_\beta \to X_\alpha$ are affine. Then the

X_α are noetherian schemes of finite Krull dimension. As $X - Y = U$ and $X - Z$ are quasi-compact open, by C.2 we may pass to a cofinal subsystem to get opens $X_\alpha - Y_\alpha = U_\alpha$, $X_\alpha - Z_\alpha$, with $\varprojlim U_\alpha = U$, etc. Now 3.20.1 shows that the various derived categories of perfect complexes on X, on U, on X and acyclic on U, etc., are the direct colimits of the corresponding systems of derived categories of perfect complexes on the X_α, on the U_α, on the X_α and acyclic on U_α. Also $K_0(X) = \varinjlim K_0(X_\alpha)$, by 3.20.2. So it will suffice to prove 5.2.2 and 5.2.3 for each of the noetherian X_α. If X had an ample of line bundles, we may assume the X_α do, by C.9.

5.4. We next turn to the case where X has an ample family of line bundles, and study the extension of morphisms, i.e., 5.2.3.

5.4.1. **Lemma.** *Let X be quasi-compact and quasi-separated Let $\mathcal{L}$ be a line bundle on X, $s \in \Gamma(X, \mathcal{L})$ a global section, and let $U = X_s$ be the non-vanishing locus, with $j : U \to X$ the open immersion.*

Let $E^\cdot$ be a strict perfect complex on X, and $F^\cdot$ a complex of quasi-coherent $\mathcal{O}_X$-modules. Then

(a) For any strict map of the restrictions of the complexes to U, $f : j^*E^\cdot \to j^*F^\cdot$, there exists an integer $k > 0$ and a strict map of complexes on X, $\tilde{f} : E^\cdot \otimes \mathcal{L}^{-k} \to F^\cdot$, such that $j^*\tilde{f} = f \cdot s^k$.

(b) Given any two strict maps of the complexes on X, $\tilde{f}_1$, $\tilde{f}_2 : E^\cdot \to F^\cdot$, such that $j^*\tilde{f}_1 = j^*\tilde{f}_2$ on U, there is an $n > 0$ such that $s^n\tilde{f}_1 = s^n\tilde{f}_2 : E^\cdot \otimes \mathcal{L}^{-n} \to F^\cdot$.

(c) Given any two strict maps $\tilde{f}_1$, $\tilde{f}_2 : E^\cdot \to F^\cdot$ on X, such that $j^*\tilde{f}_1 \simeq j^*\tilde{f}_2$ are chain homotopic on U, there is an $m > 0$ such that $s^m\tilde{f}_1$ and $s^m\tilde{f}_2$ are chain homotopic as maps $E^\cdot \otimes \mathcal{L}^{-m} \to F^\cdot$ on X.

(Note in (a) and (b), equality of maps means strict equality in the category of chain complexes, not in the derived category.)

Proof. Under the adjointness of j^* and j_* on categories of complexes, a map $f : j^*E^\cdot \to j^*F^\cdot$ corresponds to a map $E^\cdot \to j_*j^*F^\cdot$. But $j_*j^*F^\cdot$ is $F^\cdot[1/s]$, i.e., is the direct colimit

$$(5.4.1.1)\quad F^\cdot[1/s] = \varinjlim \left(F^\cdot \xrightarrow{s} F^\cdot \otimes \mathcal{L} \xrightarrow{s} F^\cdot \otimes \mathcal{L}^2 \xrightarrow{s} F^\cdot \otimes \mathcal{L}^3 \to \dots\right).$$

Indeed, there is an obvious map of this colimit into $j_*j^*F^\cdot$, which is easily seen to be an isomorphism by looking at open affines in X.

The complex $E^\cdot$ is finitely presented, as it is even a finite complex of vector bundles. Hence the mapping complex $\mathrm{Hom}^\cdot(E^\cdot, \ )$ preserves direct colimits (cf. (2.4.1.4)). Thus we have isomorphisms of mapping complexes:

(5.4.1.2)

$$\begin{aligned}\mathrm{Hom}^{\cdot}(\, j^*E^{\cdot}, j^*F^{\cdot}) &\cong \mathrm{Hom}^{\cdot}\left(E^{\cdot}, j_*j^*F^{\cdot}\right) \cong \mathrm{Hom}^{\cdot}\left(E^{\cdot}, \varinjlim F^{\cdot}\otimes\mathcal{L}^k\right)\\ &\cong \varinjlim_k \mathrm{Hom}^{\cdot}\left(E^{\cdot}, F^{\cdot}\otimes\mathcal{L}^k\right) \cong \varinjlim_k \mathrm{Hom}^{\cdot}\left(E^{\cdot}\otimes\mathcal{L}^{-k}, F^{\cdot}\right).\end{aligned}$$

As the cycle group $Z^0\mathrm{Hom}^{\cdot}$ is the group of chain maps of complexes and the cohomology group $H^0\mathrm{Hom}^{\cdot}$ is the group of chain homotopy classes of maps, applying these functors to (5.4.1.2) yields 5.4.1(a), (b), and (c). Compare [EGA] I 6.8.

5.4.2. **Proposition.** *Let X be noetherian, and have an ample family of line bundles. Let $j : U \to X$ be an open immersion.*

Let $E^{\cdot}$ be a perfect complex on X, and $F^{\cdot}$ a complex on X with quasi-coherent cohomology and which is cohomologically bounded below (i.e., $F^{\cdot} \in D^+(\mathcal{O}_X\text{-Mod})_{qc}$).

Let $a : j^*E^{\cdot} \to j^*F^{\cdot}$ be a map in the derived category of U, $D(\mathcal{O}_U\text{-Mod})$.

Then there is a perfect complex $E'^{\cdot}$ on X, a map $b : E'^{\cdot} \to F^{\cdot}$ in the derived category of X, and a map $c : E'^{\cdot} \to E^{\cdot}$ in the derived category of X such that $j^*(c)$ is an isomorphism in the derived category of U and such that $a \cdot j^*(c) = j^*(b)$ there. Moreover, if $E^{\cdot}$ is acyclic off a closed subspace $Z \subseteq X$, $E'^{\cdot}$ may be chosen to be acyclic there.

Proof. We note that the open U is quasi-compact, as X is noetherian. As X has an ample family of line bundles, there is a finite set of line bundles $\mathcal{L}_i$, $i = 1, 2, \ldots n$, and sections $s_i \in \Gamma(X, \mathcal{L}_i)$ such that $X_{s_i} \subseteq U$ and $U = \cup_{i=1}^n X_{s_i}$. This follows from 2.1.1(b), letting the $\mathcal{L}_i$ be tensor powers of line bundles in the ample family. Note $X_{s_i} = U_{s_i}$.

As X has an ample family, we may choose a strict perfect representative $E^{\cdot}$ of the quasi-isomorphism class of the original $E^{\cdot}$, by 2.3.1(d). As X is noetherian, the coherator B.16 allows us to choose a representative $F^{\cdot}$ of its quasi-isomorphism class which is a complex of quasi-coherent $\mathcal{O}_X$-modules. As $F^{\cdot}$ is cohomologically bounded below, we may then replace it by a quasi-isomorphic complex of injective objects in the category of quasi-coherent $\mathcal{O}_X$-modules (B.3). As X is noetherian these injectives in $\mathrm{Qcoh}(X)$ are still injective in the category of all $\mathcal{O}_X$-modules, and j^*F is a complex of injectives in $\mathrm{Qcoh}(U)$, as we see by B.4 and B.5. Henceforth, we use these representatives for $E^{\cdot}$ and $F^{\cdot}$.

As $j^*F^{\cdot}$ is a complex of injectives, the map in the derived category from $j^*E^{\cdot}$ to $j^*F^{\cdot}$ is represented by a strict map of complexes $a : j^*E^{\cdot} \to j^*F^{\cdot}$.

By 5.4.1(a), there is a positive integer k such that for $i = 1, 2, \ldots, n$, the map $s_i^k a | U_{s_i} = X_{s_i}$ extends to a strict map of complexes on X, $b_i : E^{\cdot} \otimes \mathcal{L}_i^{-k} \to F^{\cdot}$. On $X_{s_i} \cap X_{s_j} = X_{s_i s_j}$, we have $s_j^k b_i = s_j^k s_i^k a = s_i^k b_j$.

Hence by 5.4.1(b) there is an $m > 0$ such that $s_i^m s_j^m s_j^k b_i = s_i^m s_j^m s_i^k b_j$ on X. Taking m large enough to work for all pairs (i, j), we get $s_j^{m+k}(s_i^m b_i) = s_i^{m+k}(s_j^m b_j)$ on X for all (i, j). Then replacing b_i by $s_i^m b_i$ and k by $m+k$, we may assume that $s_j^k b_i = s_i^k b_j$ on X. We have then $s_i^k a = b_i|U$.

Consider now the Koszul complex of $s_1^k, \dots, s_n^k$, that is the tensor product of the complexes $\mathcal{L}_i^{-k} \to \mathcal{O}_X$

(5.4.2.1)

$$K(s_1^k, \dots, s_n^k) = \bigotimes_{i=1}^{n} (\mathcal{L}_i^{-k} \xrightarrow{s_i^k} \mathcal{O}_X)$$

$$\|$$

$$\bigwedge^{n}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \bigwedge^{n-1}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \dots \to \bigwedge^{1}\left(\bigoplus_{i=1}^{n} \mathcal{L}_i^{-k}\right) \to \mathcal{O}_X$$

Let K^+ be the part of the complex outside of $\mathcal{O}_X$, that is, the part which consists of the $\wedge^p(\oplus\mathcal{L}_i^{-k})$ for $p \geq 1$. Let $K^+ \to \mathcal{O}_X$ be the obvious map, so that $K(s_1^k, \dots, s_n^k)$ is the mapping cone of $K^+ \to \mathcal{O}_X$. As s_i^k is an isomorphism on X_{s_i}, the complex $K(s_1^k, \dots, s_n^k)$ is acyclic there because $\mathcal{L}_i^{-k} \to \mathcal{O}_X$ is acyclic there, and so $K^+ \to \mathcal{O}_X$ is a quasi-isomorphism there. Thus $K^+ \to \mathcal{O}_X$ is a quasi-isomorphism and $K(s_1^k, \dots, s_n^k)$ is acyclic on $U = \cup X_{s_i}$.

Let $E'^{\cdot}$ be $K^+ \otimes E^{\cdot}$, and let $c : E'^{\cdot} \to E^{\cdot}$ be the map induced by tensoring $K^+ \to \mathcal{O}_X$ with $E^{\cdot}$. Then as $E^{\cdot}$ is strict perfect, hence flat, $j^*(c)$ is a quasi-isomorphism on U. As K^+ is strict perfect, hence flat, $K^+ \otimes E^{\cdot}$ is acyclic on $X - Z$ if $E^{\cdot}$ is acyclic there. Also $K^+ \otimes E^{\cdot}$ is strict perfect.

The map $b : E'^{\cdot} = K^+ \otimes E^{\cdot} \to F^{\cdot}$ will be the map induced on total complexes by the map of bicomplexes (5.4.2.2) induced by the $b_i : E^{\cdot} \otimes \mathcal{L}_i^{-k} \to F^{\cdot}$

(5.4.2.2)

$$\begin{array}{ccccccc}
\bigwedge^{n}\left(\bigoplus^{n}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} & \longrightarrow \dots \longrightarrow & \bigwedge^{2}\left(\bigoplus^{n}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} & \xrightarrow{\delta} & \left(\bigoplus^{n}\mathcal{L}_i^{-k}\right) \otimes E^{\cdot} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma b_i} \\
0 & \longrightarrow \dots \longrightarrow & 0 & \longrightarrow & F^{\cdot}
\end{array}$$

To check this is a map of bicomplexes, we need to see that $\Sigma b_i \circ \delta = 0$ on $\wedge^2(\oplus\mathcal{L}_i^{-k}) \otimes E^{\cdot} = \oplus_{i<j} \mathcal{L}_i^{-k} \otimes \mathcal{L}_j^{-k} \otimes E^{\cdot}$. But on the factor $\mathcal{L}_i^{-k} \otimes \mathcal{L}_j^{-k} \otimes E^{\cdot}$, $(\Sigma b_i) \circ \delta = s_i^k b_j - s_j^k b_i = 0$.

We check that $j^* b = a \cdot j^* c$ by restricting to the summands $\mathcal{L}_i^{-k} \otimes E^{\cdot}$ of $(\overset{n}{\oplus}\mathcal{L}_i^{-k}) \otimes E^{\cdot}$ where the equation reduces to the valid $j^* b_i = a \cdot j^* s_i^k$.

Thus taking this $E'^{\cdot}$, (b) and (c) we have proved 5.4.2.

5.4.3. Then specialized to the case where $F^{\cdot}$ is perfect, 5.4.2 yields 5.2.3 and hence 5.2.4 for X noetherian with an ample family of line bundles.

5.4.4. By the reduction of 5.3, we conclude that 5.2.3 and 5.2.4 hold whenever X is a scheme with an ample family of line bundles (hence quasi-compact and quasi-separated (2.1.1)).

5.5. We now study the extension of perfect complexes when X has an ample family of line bundles. We first consider the case $X = Z$ of unrestrained support of 5.2.2(a).

5.5.1. **Lemma.** *Let X be a scheme with an ample family of line bundles, a fortiori a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Then for every perfect complex $F^{\cdot}$ on U, there exists a perfect complex $E^{\cdot}$ on X such that $F^{\cdot}$ is isomorphic to a summand of $j^*E^{\cdot}$ in the derived category $D(\mathcal{O}_U\text{-}Mod)$.*

Proof. Consider $Rj_*F^{\cdot}$ on X. This complex is cohomologically bounded below with quasi-coherent cohomology (B.6), and so by 2.3.3 is quasi-isomorphic to a colimit of a directed system of strict perfect complexes $E^{\cdot}_\alpha$,

$$\varinjlim_\alpha E^{\cdot}_\alpha \simeq Rj_*F^{\cdot}. \tag{5.5.1.1}$$

We consider the induced isomorphism in $D^+(\mathcal{O}_U\text{-Mod})$

$$\varinjlim_\alpha j^*E^{\cdot}_\alpha = j^*\left(\varinjlim E^{\cdot}_\alpha\right) \simeq j^*Rj_*(F^{\cdot}) \simeq F^{\cdot}. \tag{5.5.1.2}$$

By 2.4.1(f), the map (5.5.1.3) is an isomorphism

$$\varinjlim_\alpha \ \mathrm{Mor}_{D(U)}\left(F^{\cdot},\ j^*E^{\cdot}_\alpha\right) \cong \ \mathrm{Mor}_{D(U)}\left(F^{\cdot},\ \varinjlim j^*E^{\cdot}_\alpha\right). \tag{5.5.1.3}$$

Thus in $D(\mathcal{O}_U\text{-Mod})$ the inverse isomorphism to (5.5.1.2) must factor through some $j^*E^{\cdot}_\alpha$. Thus $F^{\cdot}$ is a summand of $j^*E^{\cdot}_\alpha$ in $D(\mathcal{O}_U\text{-Mod})$, proving the lemma.

5.5.2. The idea of 5.5.1 is that perfect complexes are finitely presented objects in the derived category 2.4.4, and so we may adapt Grothendieck's method of extending finitely presented sheaves ([EGA] I 6.9.1), as suggested by the Trobaugh simulacrum. While this adaptation does not allow us to extend all perfect complexes, it does lead quickly to the determination of which perfect complexes do extend.

Despite the flagrant triviality of the proof of 5.5.1, this result is the key point in the paper.

5.5.3. **Lemma.** *Let X have an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact. Suppose $F_1^{\cdot} \to F_2^{\cdot} \to F_3^{\cdot}$ is a homotopy fibre sequence in $D(\mathcal{O}_U\text{-Mod})$, i.e., two sides of a distinguished triangle. Suppose the three $F_i^{\cdot}$ are perfect complexes on U, and that two of the three are isomorphic in $D(\mathcal{O}_U\text{-Mod})$ to the restrictions of perfect complexes on X. Then the third is also isomorphic in $D(\mathcal{O}_U\text{-Mod})$ to the restriction of a perfect complex X.*

Proof. By "rotating the triangle," we see that $F_2^{\cdot} \to F_3^{\cdot} \to F_1^{\cdot}[1]$ and $F_3^{\cdot} \to F_1^{\cdot}[1] \to F_2^{\cdot}[1]$ are also homotopy fibre sequences of perfect complexes. Thus we reduce to the case where $F_1^{\cdot}$ and $F_2^{\cdot}$ are quasi-isomorphic to j^* of perfect complexes $E_1^{\cdot}$ and $E_2^{\cdot}$ on X. By 5.4.4, after replacing E_1 with a new perfect complex whose $j^*E_1^{\cdot}$ is quasi-isomorphic to the old $j^*E_1^{\cdot} \simeq F_1^{\cdot}$, we may assume that $F_1 \to F_2$ is $j^*(e)$ of a map $E_1 \to E_2$ in $D(\mathcal{O}_X\text{-Mod})$. Then the mapping cone cone(e) is perfect on X, and there are isomorphisms in $D(\mathcal{O}_U\text{-Mod})$, $j^*(\text{cone}(e)) \simeq \text{cone}(j^*e) \simeq \text{cone}(F_1 \to F_2) \simeq F_3$, as required.

5.5.4. **Proposition.** *Let X be a scheme with an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact.*

Then a perfect complex $F^{\cdot}$ on U is quasi-isomorphic to the restriction to U of some perfect complex on X if and only if the class $[F^{\cdot}]$ in $K_0(U)$ is in the image of $j^* : K_0(X) \to K_0(U)$.

Proof. This will follow from 5.5.1 and 5.5.3 by a cofinality trick of Grayson, (cf. [Gr3] Section 1).

Let π be presented as the free abelian monoid generated by the quasi-isomorphism classes $\langle F \rangle$ of perfect complexes on U, modulo the relations

(5.5.4.1) $$\langle F_1 \rangle + \langle F_2 \rangle = \langle F_1 \oplus F_2 \rangle$$

(5.5.4.2) $$\langle F \rangle = 0 \text{ if } F \simeq j^*E^{\cdot} \text{ for some } E^{\cdot} \text{ perfect on } X.$$

By 5.5.1, for each F there is an F' such that $F \oplus F'$ is quasi-isomorphic to the restriction of a perfect complex on X. Then $\langle F \rangle + \langle F' \rangle = \langle F \oplus F' \rangle = 0$ and π is a group.

Suppose $\langle G \rangle = 0$ in π. This means that there is an F such that $G \oplus F$ is quasi-isomorphic to $H \oplus F$ for some H quasi-isomorphic to the restriction of a perfect complex on X. Let F' be an "inverse to F" as above. Then $G \oplus F \oplus F' \simeq H \oplus F \oplus F'$, and both H and $F \oplus F'$, hence $G \oplus F \oplus F' \simeq H \oplus (F \oplus F')$ extend to perfect complexes on X. Then by 5.5.3 applied to $F \oplus F' \to G \oplus F \oplus F' \to G$, we see that G extends to a perfect complex on X. Thus $\langle G \rangle = 0$ in π iff G is quasi-isomorphic to $j^*E^{\cdot}$ for some perfect complex $E^{\cdot}$ on X.

Hence it remains only to show that π is isomorphic to $K_0(U)/\text{ im } K_0(X)$. Comparing the presentation of π by (5.5.4.1) and (5.5.4.2) with the presentation of $K_0(U)/\text{ im } K_0(X)$ resulting from 1.5.6, we see that it suffices to show that if $F_1 \to F_2 \to F_3$ is a homotopy fibre sequence of perfect complexes on U, then $\langle F_2 \rangle = \langle F_1 \rangle + \langle F_3 \rangle$ in π.

Let F_1', F_3' be such that $F_1 \oplus F_1'$ and $F_3 \oplus F_3'$ extend to perfect complexes on X. Thus $\langle F_1' \rangle = -\langle F_1 \rangle$ and $\langle F_3' \rangle = -\langle F_3 \rangle$ in π. There is a homotopy fibre sequence $F_1 \oplus F_1' \to F_2 \oplus F_1' \oplus F_3' \to F_3 \oplus F_3'$, obtained by adding $F_1' \to F_1' \to 0$ and $0 \to F_3' \to F_3'$ to the given $F_1 \to F_2 \to F_3$. By 5.5.3, we see that $F_2 \oplus F_1' \oplus F_3'$ extends to a perfect complex on X, as $F_1 \oplus F_1'$ and $F_3 \oplus F_3'$ do. Hence in π, $0 = \langle F_2 \oplus F_1' \oplus F_3'' \rangle = \langle F_2 \rangle + \langle F_1' \rangle + \langle F_3' \rangle = \langle F_2 \rangle - \langle F_1 \rangle - \langle F_3 \rangle$, as required.

5.5.5. **Proposition.** *Let X have an ample family of line bundles. Let $j : U \to X$ be an open immersion with U quasi-compact. Let $Z \subseteq X$ be a closed subspace with $X - Z$ quasi-compact.*

Then for a perfect complex $F^{\cdot}$ on U which is acyclic on $U - U \cap Z$, there exists a perfect complex $E^{\cdot}$ on X which is acyclic on $X - Z$ and is such that j^*E is isomorphic to $F^{\cdot}$ in $D(\mathcal{O}_U\text{-Mod})$, if and only if the class $[F^{\cdot}]$ in $K_0(U \text{ on } U \cap Z)$ is in the image of $K_0(X \text{ on } Z)$.

Proof. The "only if" part is trivial.

To prove the "if" direction, we suppose that $[F^{\cdot}]$ is the image of a class in $K_0(X \text{ on } Z)$. Let $F'^{\cdot} = k!F^{\cdot}$ be the extension of $F^{\cdot}$ by 0 along the open immersion $k : U \to U \cup X - Z$. Recall that the functor $k!$ is exact and is left adjoint to the exact k^* ([SGA 4] IV 11.3.1). As $F^{\cdot}$ is acyclic on $U - U \cap Z$, $F'^{\cdot}$ is acyclic, hence perfect on $X - Z$. The restriction $k^* F'^{\cdot}$ of $F'^{\cdot}$ to U is isomorphic to $F^{\cdot}$, hence $F'^{\cdot}$ is perfect on U. Thus $F'^{\cdot}$ is perfect on $U \cup X - Z$.

Now we consider the commutative diagram of K_0's.

(5.5.5.1)
$$\begin{tikzcd}
K_0(X) \arrow[r] & K_0(U \cup (X - Z)) \\
K_0(X \text{ on } Z) \arrow[u] \arrow[r] \arrow[dr] & K_0(U \cup (X-Z) \text{ on } U \cap Z) \arrow[u] \arrow[d, "k^*\cong"] \\
& K_0(U \text{ on } U \cap Z)
\end{tikzcd}$$

The map k^* is an isomorphism $K_0(U \cup (X - Z) \text{ on } U \cap Z) \cong K_0(U \text{ on } U \cap Z)$ by excision 3.19, as U is an open *nbd* of $U \cap Z$ in $U \cup (X - Z)$. The class $[F'^{\cdot}]$ in $K_0(U \cup (X - Z) \text{ on } Z)$ goes to the class $[F^{\cdot}]$ in $K_0(U \text{ on } U \cap Z)$ under k^*. Then the hypothesis implies that $[F'^{\cdot}]$ is the image of a class in $K_0(X \text{ on } Z)$. It follows that the class $[F'^{\cdot}]$ in $K_0(U \cup X - Z)$ is the image of a class in $K_0(X)$.

Then by 5.5.4 there is a perfect complex $E^\cdot$ on X such that $E^\cdot|U \cup (X - Z)$ is quasi-isomorphic to $F'^\cdot$. Then the restriction to U, $j^*E^\cdot$ is quasi-isomorphic to $F'^\cdot|U \simeq F^\cdot$, and $E^\cdot|X - Z \simeq F'^\cdot|X - Z$ is acyclic. This proves the proposition.

5.5.6. **Corollary.** *If X has an ample family of line bundles (so a fortiori, is quasi-compact and quasi-separated), then Proposition 5.2.2, 5.2.3, 5.2.4, and Theorem 5.1 are true for X.*

Proof. 5.5.5, 5.5.4, 5.4.4, 5.2.

5.6. We now proceed to remove the hypothesis of an ample family of line bundles, using the techniques of 3.20.4-6.

5.6.1. **Lemma.** *Let X be a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Let Z be a closed subspace of X with $X - Z$ quasi-compact.*

Let V be an open subscheme of X, such that V has an ample family of line bundles. Suppose $X = U \cup V$. Then

(a) *Suppose $F^\cdot$ is a perfect complex on U, and that $F^\cdot$ is acyclic on $U - U \cap Z$. Suppose the class $[F^\cdot|U \cap V]$ in $K_0(U \cap V$ on $Z \cap U \cap V)$ is in the image of $K_0(V$ on $V \cap Z)$. Then there is a perfect complex $E^\cdot$ on X, such that $E^\cdot$ is acyclic on $X - Z$ and $j^*E^\cdot$ is quasi-isomorphic to $F^\cdot$ on U.*

(b) *Suppose $E_1^\cdot$ and $E_2^\cdot$ are perfect complexes on X which are acyclic on $X - Z$. Suppose $a : j^*E_1^\cdot \to j^*E_2^\cdot$ is a map in the derived category on U, $D(\mathcal{O}_U$-Mod$)$. Then there is a perfect complex $E'^\cdot$ on X which is acyclic on $X - Z$, and maps $c : E'^\cdot \to E_1^\cdot$, $b : E'^\cdot \to E_2^\cdot$ in $D(\mathcal{O}_X$-Mod$)$ such that $j^*(c)$ is an isomorphism in $D(\mathcal{O}_U$-Mod$)$ and $a \cdot j^*(c) = b$ there.*

(c) *Moreover, in (b) we may choose $E'^\cdot$ so that $[E'^\cdot] = [E_1^\cdot]$ in $K_0(X$ on $Z)$.*

(d) *The conclusion (5.1.2) of Theorem 5.1 is valid for this X, U, and Z. In particular, there is an induced exact sequence of homotopy groups for $Y = X - U$:*

$$(5.6.1.1) \quad \cdots \to K_0(X \text{ on } Z \cap Y) \to K_0(X \text{ on } Z) \to K_0(U \text{ on } Z \cap U).$$

Proof. First we note that both V and its open subscheme $U \cap V$ are quasi-compact, quasi-separated, and have an ample family of line bundles.

By 5.5.5 applied to the open immersion $j' : U \cap V \to V$, there is a perfect complex $F_V^\cdot$ on V, acyclic on $V - V \cap Z$, and a quasi-isomorphism of $F_V^\cdot|U \cap V$ with $F^\cdot|U \cap V$. By the calculus of fractions, this quasi-isomorphism is represented by data consisting of a complex $G^\cdot$ on $U \cap V$ and strict quasi-isomorphisms

(5.6.1.2) $$F_V^{\cdot}|U \cap V \xleftarrow{\sim} G^{\cdot} \xrightarrow{\sim} F^{\cdot}|U \cap V.$$

In the notation of 3.20.4, let $E^{\cdot}$ be the complex on X given by

(5.6.1.3) $$E^{\cdot} = !F_V^{\cdot} \overset{h}{\underset{!G^{\cdot}}{\cup}} !F^{\cdot} = k!F_V^{\cdot} \overset{h}{\underset{\ell!G^{\cdot}}{\cup}} j!F^{\cdot}.$$

Then, as in 3.20.4, $E^{\cdot}|U = j^*E^{\cdot}$ is quasi-isomorphic to $F^{\cdot}$, and $E^{\cdot}|V = k^*E^{\cdot}$ is quasi-isomorphic to $F_V^{\cdot}$. As $X = U \cup V$, this shows $E^{\cdot}$ is perfect on X and acyclic on $X - Z$. This proves (a).

To prove (b), we consider $a|U \cap V : E_1^{\cdot}|U \cap V \to E_2^{\cdot}|U \cap V$ in $D(\mathcal{O}_{U\cap V}$-Mod). By 5.4.4 applied to $U \cap V \to V$, there is a perfect complex $E_V'^{\cdot}$ on V, which is acyclic on $V - V \cap Z$, and there are maps in $D(\mathcal{O}_V$-Mod) $b_V : E_V'^{\cdot} \to E_2|V$, $c_V : E_V'^{\cdot} \to E_1|V$ such that $c_V|U \cap V$ is an isomorphism in $D(\mathcal{O}_{U\cap V}$-Mod) and $a \cdot c_V|U \cap V = b_V|U \cap V$ there. We choose representatives of the quasi-isomorphism classes $E_1^{\cdot}$ and $E_2^{\cdot}$ among complexes of injective $\mathcal{O}_X$-modules. Then b_V, c_V, and a are represented by strict maps of complexes, and $a \cdot c_V|U \cap V$ is chain homotopic to $b_V|U \cap V$.

Now in the notation of 3.20.4 we set

(5.6.1.4)
$$E'^{\cdot} = !E_V'^{\cdot} \underset{!E_V'^{\cdot}|U\cap V}{\cup} !E_1^{\cdot}|U = C^{\cdot}\left(E_V'^{\cdot},\ E_V'^{\cdot}|U \cap V,\ E_1^{\cdot}|U,\ 1,\ c_V|U \cap V\right).$$

Then $E'^{\cdot}|U$ is quasi-isomorphic to $E_1^{\cdot}|U$ and $E'^{\cdot}|V$ is quasi-isomorphic to $E_V'^{\cdot}$. Thus $E'^{\cdot}$ is perfect and acyclic on $X - Z$.

Let $c : E'^{\cdot} = C^{\cdot}(E_V'^{\cdot}, E_V'^{\cdot}|U \cap V, E_1^{\cdot}|U, 1, c_V|U \cap V) \to E_1^{\cdot}$ be the map induced by $1 : E_1^{\cdot}U \to E_1^{\cdot}|U$ and $c_V : E_V'^{\cdot} \to E_1'^{\cdot}|V$ according to the mapping property of 3.20.4. Then $j^*(c) = c|U$ is a quasi-isomorphism, and in fact is inverse to the canonical quasi-isomorphism $E_1^{\cdot}|U \to E'^{*}|U$.

Let $b : E'^{\cdot} \to E_2^{\cdot}$ be the map induced by $b_V : E_V'^{\cdot} \to E_2^{\cdot}|V$, $a : E_1^{\cdot}|U \to E_2^{\cdot}|U$, and a choice of chain homotopy between $b_V|U \cap V$ and $a \cdot c_V|U \cap V$. Then it is easy to see that $a \cdot c|U = b|U$ in $D(\mathcal{O}_U$-Mod), and are even equal up to chain homotopy of strict maps of complexes, as $E'^{\cdot}|U$ deformation retracts to the summand $E_1^{\cdot}|U$ on which $a \cdot c = b$ reduces to $a \cdot 1 = a$.

This completes the proof of (b).

To prove (c), it suffices to find a new perfect complex on X, which is acyclic on U, and whose class in $K_0(X \text{ on } Z)$ is $[E_1^{\cdot}] - [E'^{\cdot}]$. For then we may replace the old complex $E'^{\cdot}$ by its direct sum with this new perfect complex, and extend the maps b and c to be 0 on this new summand. But clearly the mapping cone of $c : E'^{\cdot} \to E_1$ meets the requirements to be the new summand.

Now (d) follows as 5.6.1(a), (b) proves 5.2.2 and 5.2.3 for this X, U, and Z; and then 5.2 shows that this implies 5.1 for this X, U, and Z.

5.6.2. **Lemma.** *Let X be quasi-compact and quasi-separated. Let $j : U \to X$ be an open immersion with U quasi-compact. Set $Y = X - U$. Let Z be a closed subspace of X such that $X - Z$ is quasi-compact.*

Then 5.2.2, 5.2.3 and 5.1 are true for X, U, Z. That is

(a) *If $F^\cdot$ is a perfect complex on U, acyclic on $U - U \cap Z$, and with its class $[F^\cdot]$ in the image of $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$, then there is a perfect complex $E^\cdot$ on X which is acyclic on $X - Z$ and with $j^* E^\cdot$ quasi-isomorphic to $F^\cdot$ on U.*

(b) *If $E_1^\cdot$ and $E_2^\cdot$ are perfect complexes on X which are acyclic on $X - Z$, and if $a : j^* E_1^\cdot \to j^* E_2^\cdot$ is a map in $D(\mathcal{O}_U\text{-Mod})$, then there is a perfect complex $E'^\cdot$ on X, which is acyclic on $X - Z$, and has $[E'^\cdot] = [E_1^\cdot]$ in $K_0(X \text{ on } Z)$, and there exist maps $b : E'^\cdot \to E_2^\cdot$ and $c : E'^\cdot \to E_1^\cdot$ in $D(\mathcal{O}_X\text{-Mod})$ such that $j^*(c)$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$ and $a \cdot j^*(c) = j^*(b)$ there.*

(c) *There is a homotopy fibre sequence as in 5.1. In particular, there is an induced exact sequence*

$$(5.6.2.1) \quad \ldots \to K_0(X \text{ on } Z \cap Y) \to K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z).$$

Proof. There exist a finite set $\{V_1, \ldots, V_n\}$ of open affine subschemes of X such that $X = U \cup V_1 \cup V_2 \cdots \cup V_n$. We prove the results by induction on the number n of affines in the set.

For $n = 1$, the result follows from 5.6.1, as the the affine V_1 has an ample family of line bundles. (Note the hypothesis of 5.6.1(a) is indeed weaker than that of 5.6.2(a).)

To do the induction step, we suppose the results are known for schemes with a set of less than n affine V's. Set $W = U \cap V_1 \cup \cdots \cup V_{n-1}$. Then $X = W \cup V_n$, and the results hold for W.

To do the induction step for (a), we note by induction hypothesis that there is a perfect complex $F^{\sim\cdot}$ on W, acyclic on $W - Z \cap W$, and such that $F^{\sim\cdot}|U$ is quasi-isomorphic to $F^\cdot$. By hypothesis and 1.5.7, there is a perfect complex $H^\cdot$ on X, acyclic on $X - Z$, and such that $[H^\cdot|U] = [F^\cdot]$ in $K_0(U \text{ on } U \cap Z)$. Then $[H^\cdot|W] - [F^{\sim\cdot}]$ in $K_0(W \text{ on } W \cap Z)$ goes to 0 in $K_0(U \text{ on } U \cap Z)$. By 5.6.2(c) for $U \to W$, known by the induction hypothesis, and 1.5.7, there is a $[G^\cdot]$ in $K_0(W \text{ on } W \cap Y \cap Z)$ such that $[F^{\sim\cdot} \oplus G^\cdot] = [H^\cdot|W]$ in $K_0(W \text{ on } W \cap Z)$. This $[G^\cdot]$ is the class of a perfect complex $G^\cdot$ on W which is acyclic on $U \cup (W - Z)$. Then $F^{\sim\cdot} \oplus G^\cdot|U$ is quasi-isomorphic to $F^{\sim\cdot} \oplus 0|U \simeq F^\cdot$. Replacing the old $F^{\sim\cdot}$ by $F^{\sim\cdot} \oplus G^\cdot$, we may assume that $[F^{\sim\cdot}]$ is in the image of $K_0(X \text{ on } Z) \to K_0(W \text{ on } W \cap Z)$. But then $[F^{\sim\cdot}|V_n \cap W]$ is in the image of $K_0(V_n \text{ on } Z \cap V_n) \to K_0(W \cap V_n \text{ on } Z \cap W \cap V_n)$. We now appeal to 5.6.1(a) with $V = V_n$, $U = W$ to get a perfect $E^\cdot$ on X, acyclic on $X - Z$, and a quasi-isomorphism $E^{\cdot}|W \simeq F^{\sim\cdot}$. Thus there is a quasi-isomorphism $E^{\cdot}|U \simeq F^{\sim\cdot}|U \simeq F^{\cdot}$. This proves (a).

To prove (b), we note by the induction hypothesis that there is a perfect complex $E^{\sim\cdot}$ on W, acyclic on $W - W \cap Z$, and maps $b^{\sim} : E^{\sim\cdot} \to E_2^{\cdot}|W$, $c^{\sim} : E^{\sim\cdot} \to E_1^{\cdot}|W$ in $D(\mathcal{O}_W\text{-Mod})$ such that $c^{\sim}|U$ is an isomorphism in $D(\mathcal{O}_U\text{-Mod})$ and $a \cdot c^{\sim}|U = b^{\sim}|U$ there. Also we can arrange that $[E^{\sim\cdot}] = [E_1^{\cdot}|W]$ in $K_0(W \text{ on } W \cap Z)$. Because of the last condition, 5.6.1(a) shows that $E^{\sim\cdot}$ on W extends to a perfect complex on X which is acyclic on $X - Z$. Henceforth, we denote this perfect complex by $E^{\sim\cdot}$, and the old $E^{\sim\cdot}$ on W by $E^{\sim\cdot}|W$. Now by 5.6.1(d), we may arrange that $[E^{\sim\cdot}] = [E_1^{\cdot}]$ in $K_0(X \text{ on } Z)$, adding to $E^{\sim\cdot}$ a perfect complex acyclic on $W \cup (X - Z)$ if necessary.

Applying 5.6.1(b) twice on $X = W \cup V_n$, we get perfect complexes $G^{\cdot}$ and $H^{\cdot}$ on X, acyclic on $X - Z$, and maps $G^{\cdot} \to E^{\sim\cdot}$, $H^{\cdot} \to E^{\sim\cdot}$ in $D(\mathcal{O}_X\text{-Mod})$ that are quasi-isomorphisms on W, and also maps $G^{\cdot} \to E_1^{\cdot}$, $H^{\cdot} \to E_2^{\cdot}$ in $D(\mathcal{O}_X\text{-Mod})$, forming diagrams (5.6.2.2) in $D(\mathcal{O}_X\text{-Mod})$ and (5.6.2.3) in $D(\mathcal{O}_W\text{-Mod})$

(5.6.2.2)
$$\begin{tikzcd}
E'^{\cdot} \arrow[r, dashed] \arrow[d] & H^{\cdot} \arrow[r] \arrow[d] & E_2^{\cdot} \\
G^{\cdot} \arrow[r] \arrow[d] & E^{\sim\cdot} & \\
E_1^{\cdot} & &
\end{tikzcd}$$

(5.6.2.3)
$$\begin{tikzcd}
E'|W \arrow[r, dashed, "\sim"] \arrow[d, dashed, "\sim"'] & H|W \arrow[r] \arrow[d, "\sim"] & E_2|W \\
G|W \arrow[r, "\sim"] \arrow[d] & E^{\sim}|W \arrow[ur, "b^{\sim}"'] \arrow[dl, "c^{\sim}"] & \\
E_1|W & &
\end{tikzcd}$$

By 5.6.1(c) we may assume that $[G^{\cdot}] = [H^{\cdot}] = [E^{\sim\cdot}] = [E_1^{\cdot}]$ in $K_0(X$ on $Z)$. We choose representatives of the quasi-isomorphism classes of $E_1^{\cdot}$, $E^{\sim\cdot}$, $E_2^{\cdot}$ among complexes of injective $\mathcal{O}_X$- modules. Then the diagrams (5.6.2.2) and (5.6.2.3) exist as chain homotopy commutative diagrams of strict maps of chain complexes.

Let $E'^{\cdot}$ be the canonical homotopy pullback of $G^{\cdot}$ and $H^{\cdot}$ over $E^{\sim\cdot}$ (1.1.2), so $E'^{\cdot} \to G^{\cdot} \oplus H^{\cdot} \to E^{\sim\cdot}$ is a homotopy fibre sequence. Then $E'^{\cdot}$ is perfect on X and acyclic on $X - Z$. In $K_0(X \text{ on } Z)$, $[E'^{\cdot}] = -[E^{\sim\cdot}] + [G^{\cdot}] + [H^{\cdot}] = -[E^{\sim\cdot}] + [E^{\sim\cdot}] + [E^{\sim\cdot}] = [E^{\sim\cdot}] = [E_1^{\cdot}]$.

Let $c : E'^{\cdot} \to G^{\cdot} \to E_1^{\cdot}$ and $b : E'^{\cdot} \to H^{\cdot} \to E_2^{\cdot}$ be the compositions of the canonical projections of $E'^{\cdot}$ onto $G^{\cdot}$ and $H^{\cdot}$ with the maps $G^{\cdot} \to E_1^{\cdot}$

and $H^\cdot \to E_2^\cdot$. On $U \subseteq W$, the restriction of diagram (5.6.2.3) shows that $c|U :\xrightarrow{\sim} E'^\cdot|U \xrightarrow{\sim} G^\cdot|U \to E_1^\cdot \xrightarrow{\sim} |U$ is a quasi-isomorphism, and that $a \cdot c|U : E'^\cdot|U \to E_1^\cdot|U \to E_2^\cdot|U$ is chain homotopic to $b : E'^\cdot|U \to E_2^\cdot|U$ as required. This proves (b).

Statement (c) for X, U, Z follows from (a) and (b) by 5.2. This completes the induction step and proves 5.6.2.

5.6.3. With 5.6.2, the proofs of 5.2.2, 5.2.3, 5.2.4, and Theorem 5.1 are complete.

5.7. *Exercise* (Optional). Let X be a scheme with an ample family of line bundles. Let $i : Y \to X$ be a regular closed immersion ([SGA 6] VII Section 1) defined by ideal $\mathcal{J}$. Suppose Y has codimension k in X.

Then show that $K(X \text{ on } Y)$ is homotopy equivalent to the Quillen K-theory of the exact category of pseudo-coherent $\mathcal{O}_X$-modules supported on the subspace Y and of Tor-dimension $\leq k$ on X.

(a) Begin by noting that $\mathcal{O}_X/\mathcal{J}$ is pseudo-coherent and of Tor-dimension $\leq k$ by the Koszul resolution. As $\mathcal{J}^n/\mathcal{J}^{n+1}$ is locally a sum of copies of $\mathcal{O}_X/\mathcal{J}$, $\mathcal{J}^n/\mathcal{J}^{n+1}$ is pseudo-coherent of Tor-dimension $\leq k$ ([SGA 6] VII 1.3 iii). By induction, using the exact sequence $0 \to \mathcal{J}^{n+1}/\mathcal{J}^{n+p} \to \mathcal{J}^n/\mathcal{J}^{n+p} \to \mathcal{J}^n/\mathcal{J}^{n+1} \to 0$, show that all $\mathcal{J}^n/\mathcal{J}^{n+p}$ and in particular, all $\mathcal{O}_X/\mathcal{J}^p$ are pseudo-coherent of Tor-dimension $\leq k$ (cf. 2.2.13).

(b) Using the functor $R(\text{Qcoh})\, \underline{\Gamma}_Y = \varinjlim \underline{\text{Ext}}^\cdot(\mathcal{O}_X/\mathcal{J}^p, \;)$, calculated using injective resolutions in $\text{Qcoh}(X)$, construct a map between appropriate models of $K(X \text{ on } Y)$ (3.6.2, 3.6.1) and show $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of quasi-coherent modules that vanish on $X - Y$.

(c) Let $\mathcal{A}$ be the abelian category of quasi-coherent modules that vanish on $X - Y$. Note every submodule of finite type of an object of $\mathcal{A}$ is annihilated by all $\mathcal{J}^p$ for p sufficiently large. Let $\mathcal{D}$ be the additive category generated by all $\mathcal{L}_\alpha^m \otimes \mathcal{O}_X/\mathcal{J}^p$ with $m \in \mathbb{Z}$, $p \geq 1$, and $\mathcal{L}_\alpha$ a line bundle in the ample family. All objects of $\mathcal{D}$ are pseudo-coherent of Tor-dimension $\leq k$ over $\mathcal{O}_X$. The inclusion $\mathcal{D} \to \mathcal{A}$ satisfies the hypotheses of 1.9.5. Hence $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of pseudo-coherent modules of Tor-dimension $\leq k$ supported on the subspace Y.

(d) Now appeal to 1.11.7, (or to [W] 1.7.1) to conclude that $K(X \text{ on } Y)$ is homotopy equivalent to the K-theory spectrum of the exact category as claimed.

(e) An $\mathcal{O}_X$-module of Tor-dim $\leq k$ is pseudo-coherent iff it has a resolution by vector bundles of length $\leq k$ iff it is k-pseudo-coherent. Thus for $k = 1$ with $Y \to X$ a regularly immersed divisor, the exact category is that of finitely presented $\mathcal{O}_X$-modules supported on Y and of Tor-dim ≤ 1. For X with an ample family of line bundles, prove the conjecture of Gersten [Ge1] Section 7, and recover the localization theorem of [Gr1]. Recover the results of [L1].

(f) If U is affine, and open in X, any strict perfect complex on U is trivially a summand of a strict perfect complex of free $\mathcal{O}_U$-modules. If also $U = X_s$ for $s : \mathcal{O} \to \mathcal{L}$, use [EGA] I 6.8.1 to show that any strict bounded complex of free modules on U extends to a strict perfect complex on X. This yields trivially for such U the strict perfect analog of 5.5.1. Combining this with 5.4.1, prove the analog of 5.1 for K^{naive} of such U. Now assuming also that $s : \mathcal{O}_X \rightarrowtail \mathcal{L}$ is a monomorphism, use the ideas of (a) - (d) to recover Quillen's localization theorem of [Gr1] in general, without assuming that X has an ample family of line bundles.

(g) Contemplate Deligne's counterexample in [Ge1] Section 7 to an attempt to generalize the identification of the fibre of $K(X) \to K(X-Y)$ to $K(\ )$ of an exact category as above when $Y \to X$ is not a regular immersion.

6. Bass fundamental theorem and negative K-groups, K^B

6.0. To control the failure of surjectivity of $K_0(X) \to K_0(U)$ and $K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z)$ in the proto-localization Theorem 5.1, one wants to find a non-connective spectrum K^B with K as its -1-connective cover, so that $K^B(X \text{ on } Y) \to K^B(X) \to K^B(U)$ is a homotopy fibre sequence in 5.1 without fudging, and so that the resulting long exact sequence of homotopy groups extends through the K_n^B for $n < 0$. This is done by combining Sections 4 and 5 with ideas of Bass [B] (cf. also Carter [Ca].)

For once it is notationally easier to work first on the level of abelian-group valued functors, and then to produce a spectrum level version.

6.1. **Theorem** (Bass fundamental proto-theorem) (cf. [B] XII 7; [Gr1]; 6.6 below). *Let X be quasi-compact and quasi-separated. Set $X[T] = X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[T]$. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then*

(a) *For $n \geq 1$, there is an exact sequence*

(6.1.1)

$$0 \to K_n(X \text{ on } Z) \xrightarrow{(p_1^*, -p_2^*)} K_n(X[T] \text{ on } Z[T]) \oplus K_n\left(X[T^{-1}] \text{ on } Z[T^{-1}]\right)$$
$$\xrightarrow{(j_1^*, j_2^*)} K_n\left(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]\right) \xrightarrow{\partial_T} K_{n-1}(X \text{ on } Z) \to 0.$$

Here p_1^*, p_2^* are induced by the projections $(X[T]) \to X$, etc. and j_1^*, j_2^* are induced by the obvious open immersions $(X[T,T^{-1}]) \to (X[T])$, etc.

The sum of these exact sequences for $n = 1, 2, 3, \ldots$ is an exact sequence of graded $K_*(X)$-modules.

(b) For $n \geq 0$, $\partial_T : K_{n+1}(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \to K_n(X \text{ on } Z)$ is naturally split by a map h_T of $K_*(X)$ modules. Indeed, cup product with $T \in K_1(\mathbb{Z}[T,T^{-1}])$ splits ∂_T up to a natural automorphism of $K_n(X \text{ on } Z)$.

(c) There is an exact sequence for $n = 0$

$$\begin{tikzcd} 0 \arrow[r] & K_0(X \text{ on } Z) \arrow[r, "(p_1^*, p_2^*)"] & K_0(X[T] \text{ on } Z[T]) \oplus K_0\left(X[T^{-1}] \text{ on } Z[T^{-1}]\right) \arrow[d, "(j_1^*, j_2^*)"] \\ & & K_0\left(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]\right) \end{tikzcd}$$

Proof. Consider $\mathbb{P}^1_X$. By 4.1, there is an isomorphism $K_*(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \cong K_*(X \text{ on } Z) \oplus K_*(X \text{ on } Z)$, where the two summands are $K_*(X \text{ on } Z)[\mathcal{O}]$ and $K_*(X \text{ on } Z)[\mathcal{O}(-1)]$ with respect to the external product $K(X \text{ on } Z) \wedge K(\mathbb{P}^1_{\mathbb{Z}}) \to K(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$ and with $[\mathcal{O}]$, $[\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^1_{\mathbb{Z}})$. We prefer now to shift to a direct sum decomposition of $K_*(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z)$ with basis $\{[\mathcal{O}],\ [\mathcal{O}] - [\mathcal{O}(-1)]\}$.

We consider the cover of $\mathbb{P}^1_X$ by opens $X[T]$ and $X[T^{-1}]$, intersecting in $X[T,T^{-1}]$. The proto-localization Theorem 5.1 shows that the columns in (6.1.2) are homotopy fibre sequences. Here the $K(\ )^{\sim}$ are covering spectra of the $K(\ )$ to change π_0 suitably, as in 5.1.

$$\begin{tikzcd} K(\mathbb{P}^1_X \text{ on } (T=0) \cap \mathbb{P}^1_Z) \arrow[r, "\simeq"] \arrow[d] & K(X[T] \text{ on } (T=0) \cap Z[T]) \arrow[d] \\ K(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \arrow[r, "k_1^*"] \arrow[d, "k_2^*"] & K(X[T] \text{ on } Z[T]) \arrow[d, "j_2^*"] \\ K(X[T^{-1}] \text{ on } Z[T^{-1}])^{\sim} \arrow[r, "j_1^*"] & K(X[T,T^{-1}] \text{ on } Z[T,T^{-1}])^{\sim} \end{tikzcd} \tag{6.1.2}$$

As $X[T]$ is an open *nbd* of the locus $(T = 0)$ in $\mathbb{P}^1_X$, excision 3.19 shows that the top horizontal map in (6.1.2) is a homotopy equivalence. Thus the bottom square of (6.1.2) is homotopy cartesian. Thus it yields a long exact Mayer-Vietoris sequence on homotopy groups. Recalling that $\pi_i K(\ )^{\sim} = \pi_i K(\ )$ for $i > 0$, we see this long exact sequence is:

$$(6.1.3)\qquad \begin{array}{c} \downarrow \\ K_{n+1}(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow{\scriptstyle \partial_T} \\ K_n(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \\ \downarrow{\scriptstyle (k_1^*,-k_2^*)} \\ K_n(X[T] \text{ on } Z[T]) \oplus K_n(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \downarrow{\scriptstyle (j_1^*,j_2^*)} \\ K_n(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow{\scriptstyle \partial_T} \\ \cdots \\ \downarrow{\scriptstyle \partial_T} \\ K_0(\mathbb{P}^1_X \text{ on } \mathbb{P}^1_Z) \\ \downarrow{\scriptstyle (k_1^*,-k_2^*)} \\ K_0(X[T] \text{ on } Z[T]) \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}])^{\sim} \end{array}$$

Here $K_0(X[T^{-1}]$ on $Z[T^{-1}])^{\sim}$ is some subgroup of $K_0(X[T^{-1}]$ on $Z[T^{-1}])$, namely the image under k_2^* of $K_0(\mathbb{P}^1_X$ on $\mathbb{P}^1_Z)$.

Now for $k = k_1$ and k_2, $k^*(\mathcal{O}_{\mathbb{P}}(i)) = \mathcal{O}$. Hence $k^*([\mathcal{O}]) = [\mathcal{O}] = 1$, and $k^*([\mathcal{O}]-[\mathcal{O}(-1)]) = [\mathcal{O}]-[\mathcal{O}] = 0$ in $K_0(\mathbb{Z}[T])$ or $K_0(\mathbb{Z}[T^{-1}])$. Thus on the summand $K_n(X$ on $Z)[\mathcal{O}]$ of $K_n(\mathbb{P}^1_X$ on $\mathbb{P}^1_Z)$, k_1^* is the map p_1^* induced by the canonical $p_1 : X[T] \to X$. The map p_1 has a section $T = 0$, so p_1^* is a split monomorphism. Similarly, on this summand k_2^* is p_2^*, which is a split monomorphism. On the summand $K_n(X$ on $Z)([\mathcal{O}] - [\mathcal{O}(-1)])$, k_1^* and k_2^* are 0. Hence in (6.1.3) the boundary map ∂_T is onto this summand. Thus the long exact sequence (6.1.3) breaks up into short exact sequences, yielding 6.1(a).

To prove (b), it suffices to show that the natural map $\partial_T \cdot (T \cup p^*(\ ))$: $K_n(X$ on $Z) \to K_{n+1}(X[T,T^{-1}]$ on $Z[T,T^{-1}]) \to K_n(X$ on $Z)$ is an automorphism of K_n for $n \geq 0$. For we then define h_T to be $(T \cup p^*(\ ))$ composed with the inverse of this automorphism. By the 5-lemma applied to diagram (6.1.4) where the rows are the exact localization sequences 5.1 for X and $X - Z$ and for $X[T,T^{-1}]$ and $(X - Z)[T,T^{-1}]$, it suffices to prove the map is an automorphism in the absolute case $K_*(S)$ for $S = X$ and for $S = X - Z$. (In diagram (6.1.4) we abbreviate $X[T,T^{-1}]$ as $X[T^{\pm}]$, etc.)

(6.1.4)

$$\begin{tikzcd}
\dots \arrow[r] & K_{n+1}(X) \arrow[r] \arrow[d, "\cup T"] & K_{n+1}(X-Z) \arrow[r] \arrow[d, "\cup T"] & {} \\
\dots \arrow[r] & K_{n+2}(X[T^{\pm}]) \arrow[r] \arrow[d, "\partial_T"] & K_{n+2}((X-Z)[X[T^{\pm}]) \arrow[r] \arrow[d, "\partial_T"] & {} \\
\dots \arrow[r] & K_{n+1}(X) \arrow[r] & K_{n+1}(X-Z) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
K_n(X \text{ on } Z) \arrow[r] \arrow[d, "\cup T"] & K_n(X) \arrow[r] \arrow[d, "\cup T"] & \dots \\
K_{n+1}(X[X[T^{\pm}] \text{ on } Z[T^{\pm}]) \arrow[r] \arrow[d, "\partial_T"] & K_{n+1}(X[T^{\pm}]) \arrow[r] \arrow[d, "\partial_T"] & \dots \\
K_n(X \text{ on } Z) \arrow[r] & K_n(X) \arrow[r] & \dots
\end{tikzcd}$$

(We see that the squares of (6.1.4) involving ∂_T commute, as they are derived in a canonical way from a commutative 3×3 diagram of homotopy fibre sequences of spectra.)

To prove that our map is an automorphism in the absolute case $K_*(S)$, it suffices to prove that $\partial_T T = \pm 1 = \pm[\mathcal{O}]$ in $K_0(S)$, as this is a generator of $K_*(S)$ as a free $K_*(S)$ module, and $\partial_T \cdot (T \cup \ ) = \partial_T T \cup (\ )$ is a map of $K_*(S)$ modules. By naturality in S, it suffices to prove this for $S = \operatorname{Spec}(\mathbb{Z})$, i.e., that $\partial_T : K_1(\mathbb{Z}[T, T^{-1}]) \to K_0(\mathbb{Z})$ sends T to ± 1. This is known classically (cf. [B], [Gr1]). (Briefly one has $K_0(\mathbb{Z}) = \mathbb{Z}$, $K_1(\mathbb{Z}[T, T^{-1}])$ = units in $\mathbb{Z}[T, T^{-1}] \cong \mathbb{Z} \oplus \mathbb{Z}/2$ generated by T and -1, and that ∂_T is onto (examine (6.1.1) and note that $K(\mathbb{Z}) \simeq K(\mathbb{Z}[T]) \simeq K(\mathbb{Z}[T^{-1}])$ as $\mathbb{Z}$ is regular noetherian). So the torsion element $\partial_T(-1)$ of $\mathbb{Z}$ must be 0, and $\partial_T T$ must be a generator ± 1.)

(A careful calculation of $\partial_T(T)$ by building categorical models of everything via [Th3], and considering our choice of signs in forming a Mayer-Vietoris sequence from a homotopy cartesian square yields that in fact $\partial_T T = 1$. In fact, $\partial \cdot (T \cup \ ) = 1$ on $K(X \text{ on } Z)$.) This proves (b).

To prove (c), we consider the diagram:

(6.1.5)

$$\begin{array}{ccccccc}
0 & \to & K_0(X \text{ on } Z) & \to & K_0(X[T] \text{ on } Z[T] \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}]) & \to & K_0(X[T^{\pm}] \text{ on } Z[T^{\pm}]) \\
 & & \downarrow h_S & & \downarrow h_S \oplus h_S & & \downarrow h_S \\
0 & \to & K_1(X[S^{\pm}] \text{ on } Z[S^{\pm}] & \to & \begin{array}{c} K_1(X[S^{\pm}][T] \text{ on } Z[S^{\pm}][T]) \\ \oplus \\ K_1(X[S^{\pm}][T^{-1}] \text{ on } Z[S^{\pm}][T^{-1}]) \end{array} & \to & K_1(X[S^{\pm}][T^{\pm}] \text{ on } Z[S^{\pm}][T^{\pm}]) \\
 & & \downarrow \partial_S & & \downarrow \partial_S \oplus \partial_S & & \downarrow \partial_S \\
0 & \to & K_0(X \text{ on } Z) & \to & K_0(X[T] \text{ on } Z[T] \oplus K_0(X[T^{-1}] \text{ on } Z[T^{-1}]) & \to & K_0(X[T^{\pm}] \text{ on } Z[T^{\pm}])
\end{array}$$

The middle row is exact by (a). By (b), the composite $\partial_S \cdot h_S$ is 1, so the bottom row is a retract of the middle row. Hence the bottom row is also exact, as required.

6.2. **Lemma** (Bass construction) (cf. [B] XII 7). *By descending induction on $k = 1, 0, -1, -2, -3, \ldots$ one may define contravariant abelian group valued functors B_k on the category of quasi-compact and quasi-separated schemes X with a chosen closed subspace Z such that $X - Z$ is quasi-compact. One may also define natural transformations*

$$\begin{aligned}
d_{kT} &: B_{k+1}\left(X\left[T,T^{-1}\right] \text{ on } Z\left[T,T^{-1}\right]\right) \to B_k(X \text{ on } Z) \\
h_{kT} &: B_k(X \text{ on } Z) \to B_{k+1}\left(X\left[T,T^{-1}\right] \text{ on } Z\left[T,T^{-1}\right]\right).
\end{aligned}$$

These functors and natural transformations are uniquely characterized by the following properties that they satisfy:

(a) *$B_1(X$ on $Z) = K_1(X$ on $Z)$, $B_0(X$ on $Z) = K_0(X$ on $Z)$, and $d_1 = \partial_1$, h_1 are the maps of* 6.1.

(b) *The sequence induced by the natural maps as in* 6.1(a) *is exact:*

$$\begin{aligned}
0 \to B_{k+1}&(X \text{ on } Z) \\
&\to B_{k+1}(X[T] \text{ on } Z[T]) \oplus B_{k+1}\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right) \\
&\to B_{k+1}\left(X\left[T,T^{-1}\right] \text{ on } Z\left[T,T^{-1}\right]\right) \xrightarrow{d_{kT}} B_k(X \text{ on } Z) \to 0.
\end{aligned}$$

(c) $d_{kT} \cdot d_{(k+1)S} = d_{kS} \cdot d_{(k+1)T} : B_{k+2}(X[T,T^{-1}][S,S^{-1}]$ *on* $Z[T,T^{-1}][S,S^{-1}]) \to B_k(X$ *on* $Z)$ *provided that* $k \leq -1$.

(d) $d_k \cdot h_k = 1$ *on* $B_k(X$ *on* $Z)$.

(e) *The sequence induced by the natural maps as in* 6.1(a) *is exact*

$$\begin{aligned}
0 \to B_k(X \text{ on } Z) \to B_k(X[T] \text{ on } Z[T]) &\oplus B_k\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right) \\
&\to B_k\left(X\left[T,T^{-1}\right] \text{ on } Z\left[T,T^{-1}\right]\right).
\end{aligned}$$

Proof. We start by abbreviating $X[T,T^{-1}]$ as $X[T^{\pm}]$. We will give maps and diagrams for the absolute case $B_k(X)$. To obtain the correct diagrams for the case with supports, the reader should replace every $B_k(X[T^{-1}][S^{\pm}])$ by $B_k(X[T^{-1}][S^{\pm}]$ on $Z[T^{-1}][S^{\pm}])$.

We prove statements (b) - (e) by descending induction on k, starting with $k=0$. The statements (b) - (e) hold for $k=0$ by 6.1, on taking (a) as a definition. Note (c) says nothing in this case.

To do the induction step, we assume that $k \leq 0$, and that the B_k have been constructed and satisfy (a) - (e). Then for $k-1$ we define $B_{k-1}(X)$ to be the cokernel of the map $B_k(X[T]) \oplus B_k(X[T^{-1}]) \to B_k(X[T^{\pm}])$, and $d_{(k-1)T} : B_k(X[T^{\pm}]) \to B_{k-1}(X)$ to be the cokernel map. Now (b) for $k-1$ results from this definition and (e) for k.

Consider the diagram (6.2.1)

(6.2.1)

$$\begin{array}{ccccc}
\begin{array}{c} B_{k+1}(X[T][S^{-1}]) \oplus B_{k+1}(X[T^{-1}][S^{-1}]) \\ \oplus B_{k+1}(X[T][S]) \oplus B_{k+1}(X[T^{-1}][S]) \end{array} & \to & \begin{array}{c} B_{k+1}(X[T^{\pm}][S^{-1}]) \\ \oplus\, B_{k+1}(X[T^{\pm}][S]) \end{array} & \to & \begin{array}{c} B_k(X[S^{-1}]) \\ B_k(X[S]) \end{array} \\
\downarrow & & \downarrow & & \downarrow \\
B_{k+1}(X[T][S^{\pm}]) \oplus B_{k+1}(X[T^{-1}][S^{\pm}]) & \to & B_{k+1}(X[T^{\pm}][S^{\pm}]) & \xrightarrow{d_T} & B_k(X[S^{\pm}]) \\
\downarrow{\scriptstyle d_S \oplus d_S} & & \downarrow{\scriptstyle d_S} & & \downarrow{\scriptstyle d_S} \\
B_k(X[T]) \oplus B_k(X[T^{-1}]) & \to & B_k(X[T^{\pm}]) & \xrightarrow{d_T} & B_{k-1}(X)
\end{array}$$

All the small squares of (6.2.1) except possibly the lower right one commute by naturality. All rows and columns are cokernel sequences. Since colimits commute and the d_T, d_S into $B_{k-1}(X)$ are defined to be the canonical cokernel map, this shows that the lower right square commutes. This proves (c) for $k-1$. (As the reader expects, we will identify $B_k(X)$ with $K_k^B(X) = \pi_k K^B(X)$ for $k<0$, and d_T will become a boundary map in a long exact sequence of homotopy subgroups coming from a homotopy fibre sequence. So the reader might have expected to see $\partial_T \partial_S = -\partial_S \partial_T$ instead of $d_T d_S = d_S d_T$. But the natural identification will be $B_k(X) = \pi_0 \Sigma^k K^B(X)$, and the degree k shift changes the sign conventions.)

Now to construct $h_{(k-1)T}$ and prove (d) for $k-1$, we consider diagram (6.2.2) which commutes by naturality and (c).

(6.2.2)

$$\begin{array}{ccccccc}
B_k(X[S]) \oplus B_k(X[S^{-1}]) & \to & B_k(X[S^{\pm}]) & \xrightarrow{d_S} & B_{k-1}(X) & \to & 0 \\
\downarrow{\scriptstyle h_{kT} \oplus h_{kT}} & & \downarrow{\scriptstyle h_{kT}} & & \vdots & & \\
B_{k+1}(X[S][T^{\pm}] \oplus B_{k+1}(X[S^{-1}][T^{\pm}]) & \to & B_{k+1}(X[S^{\pm}][T^{\pm}]) & \xrightarrow{d_S} & B_k(X[T^{\pm}]) & \to & 0 \\
\downarrow{\scriptstyle d_{kT} \oplus d_{kT}} & & \downarrow{\scriptstyle d_{kT}} & & \downarrow{\scriptstyle d_{(k-1)T}} & & \\
B_k(X[S]) \oplus B_k(X[S^{-1}]) & \to & B_k(X[S^{\pm}]) & \xrightarrow{d_S} & B_{k-1}(X) & \to & 0
\end{array}$$

The rows are exact by definition of B_{k-1}. We define $h_{(k-1)T} : B_{k-1}(X) \to B_k(X[T^{\pm}])$ to be the map induced on the cokernels by h_{kT} in (6.2.2). As $d_{kT}h_{kT} = 1$ by (d) for k, it follows that $d_{(k-1)T} \cdot h_{(k-1)T} = 1$, proving (d) for $k-1$. Note h_T is "$T\cup$" composed with an automorphism on $\oplus_{n\geq k} B_n(X)$.

It remains to prove (e) for $k-1$. This follows from (d) $d_S h_S = 1$ and the diagram (6.2.3)

(6.2.3)

$$\begin{array}{ccccccc}
0 & \to & B_{k-1}(X) & \to & B_{k-1}(X[T]) \oplus B_{k-1}(X[T^{-1}]) & \to & B_{k-1}(X[T^{\pm}]) \\
 & & \downarrow{\scriptstyle h_S} & & \downarrow{\scriptstyle h_S \oplus h_S} & & \downarrow{\scriptstyle h_S} \\
0 & \to & B_k(X[S^{\pm}]) & \to & B_k(X[S^{\pm}][T]) \oplus B_k(X[S^{\pm}][T^{-1}]) & \to & B_k(X[S^{\pm}][T^{\pm}]) \\
 & & \downarrow{\scriptstyle d_S} & & \downarrow{\scriptstyle d_S \oplus d_S} & & \downarrow{\scriptstyle d_S} \\
0 & \to & B_{k-1}(X) & \to & B_{k-1}(X[T]) \oplus B_{k-1}(X[T^{-1}]) & \to & B_{k-1}(X[T^{\pm}])
\end{array}$$

This diagram exhibits (e) for $B_{k-1}(X)$ as a retract of the exact sequence (e) for $B_k(X[S^{\pm}])$. This proves (e) for $k-1$.

This completes the induction step, and hence the proof of the theorem.

6.3. **Lemma** (Bass Spectral Lemma). *There exist for $k = 0, -1, -2, -3, \cdots$ contravariant functors F^k from the category of (X, Z) as in 6.2 to the category of spectra, such that:*

(a) *$F^0(X$ on $Z) = K(X$ on $Z)$.*

(b) *There is a natural homotopy fibre sequence*

$$
\begin{array}{c}
F^k(X[T] \text{ on } Z[T]) \underset{F^k(X \text{ on } Z)}{\overset{h}{\cup}} F^k\left(X\left[T^{-1}\right] \text{ on } Z\left[T^{-1}\right]\right) \\
\downarrow b \\
F^k\left(X\left[T^{\pm}\right] \text{ on } Z\left[T^{\pm}\right]\right) \\
\downarrow d \\
\Sigma F^{k-1}(X \text{ on } Z).
\end{array}
$$

(In particular, we have a natural nullhomotopy of $d \cdot b$.) The map b from the homotopy pushout is induced by the obvious open immersions $(X[T^{\pm}]) \to (X[T])$, $(X[T^{-1}])$, and the trivial constant homotopy of the two equal maps $F^k(X \text{ on } Z) \to F^k(X[T^{\pm}] \text{ on } Z[T^{\pm}])$.

(c) *The fibre sequence in* (b) *is a fibre sequence of $K(X)$ modules.*

(d) *The homotopy groups of F^k are given by*

$$
\pi_n F^k(X \text{ on } Z) = \begin{cases} K_n(X \text{ on } Z) & n \geq 0 \\ B_n(X \text{ on } Z) & 0 \geq n \geq k \\ 0 & k-1 \geq n. \end{cases}
$$

The map $\pi_n(b)$ in (b) *is a monomorphism for all n.*

(e) *The map $f : F^k \to F^{k-1}$, given by*

$$
\begin{array}{c}
F^k(X \text{ on } Z) \\
\downarrow T\cup \\
\Omega F^k\left(X\left[T^{\pm}\right] \text{ on } Z[T^{\pm}]\right) \\
\downarrow \Omega d \\
\Omega\Sigma F^{k-1}(X \text{ on } Z) \simeq F^{k-1}(X \text{ on } Z)
\end{array}
$$

induces an isomorphism on homotopy π_n for $n \geq k$.

Proof. The proof is by descending induction on k. $F^0 = K$ is defined by (a). Inductively define $F^{k-1}(X \text{ on } Z)$ as $\Omega = \Sigma^{-1}$ of the canonical mapping cone of the map b of F^k in (b). As the map b is strictly natural in X, and is a strict $K(X)$ module map, its mapping cone is functorial in X and is a $K(X)$ module. Hence so is F^{k-1}. This constructs $F^{k-1}(X \text{ on } Z)$ and proves statements (b) and (c) by induction.

As $F^k(X \text{ on } Z) \to F^k(X[T] \text{ on } Z[T])$ is canonically split by the map induced by the 0-section map of schemes $X \to X[T] = X \times A^1$, the Mayer-Vietoris sequence for π_* of the homotopy pushout in (b) breaks up into split short exact sequences, of which we will print only the absolute case, by the convention of the proof of 6.2

$$(6.3.1) \quad 0 \to \pi_n F^k(X) \to \pi_n F^k(X[T]) \oplus \pi_n F^k\left(X\left[T^{-1}\right]\right) \\ \to \pi_n\left(F^k(X[T]) \underset{F^k(X)}{\overset{h}{\cup}} F^k\left(X\left[T^{-1}\right]\right)\right) \to 0.$$

We substitute (6.3.1) into the long exact sequence of π_* induced by the homotopy fibre sequence of (b). This breaks up this long exact sequence into short exact sequences like the top two rows of (6.3.2), which we print only in the absolute case:

(6.3.2)

$$\begin{tikzcd}
0 \arrow[r] & \pi_n\left(F^k(X[T]) \underset{F^k(X)}{\overset{h}{\cup}} F^k\left(X\left[T^{-1}\right]\right)\right) \arrow[r] \arrow[d, equal, "\wr"'] & \pi_n F^k\left(X\left[T^{\pm}\right]\right) \arrow[r] \arrow[d, equal] & \pi_n \Sigma F^{k-1}(X) \arrow[r] \arrow[d, equal, "\wr"'] & 0 \\
0 \arrow[r] & \operatorname{coker}\left(\pi_n F^k(X) \to \pi_n F^k(X[T]) \oplus \pi_n F^k(X[T^{-1}])\right) \arrow[r] \arrow[d, equal, "\wr"'] & \pi_n F^k\left(X\left[T^{\pm}\right]\right) \arrow[r] \arrow[d, equal, "\wr"'] & \pi_{n-1} F^{k-1}(X) \arrow[r] \arrow[d, equal, "\wr"'] & 0 \\
0 \arrow[r] & \operatorname{coker}\left(B_n(X) \to B_n(X[T]) \oplus B_n(X[T^{-1}])\right) \arrow[r] & B_n\left(X\left[T^{\pm}\right]\right) \arrow[r, "d_T"] & B_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

By descending induction on k, with an inner descending induction on n for $0 \geq n \geq k$, we compare these short exact sequences for $n \geq k$ to the exact sequence of B_n in the bottom of (6.3.2) that results from the long exact sequence 6.2(b). The induction starts at $k = 0$ by 6.2(a) and 6.3(a), and proves (d) and also that the exact sequences in (6.3.2) for $\pi_n F^k$ and for B_n correspond under the isomorphisms of (d) for $n \geq k$.

To prove (e), we first note that "$T\cup$" : $F^k(X \text{ on } Z) \to \Omega F^k(X[T^{\pm}] \text{ on } Z[T^{\pm}])$ factors as the composite of the map p^* induced by the projection $(X[T^{\pm}]) \to X$ and the cup product on $\pi_* F^k(X[T^{\pm}] \text{ on } Z[T^{\pm}])$ with $T \in K_1(X[T^{\pm}])$. Thus we see that the homotopy fibre sequence of (b) is natural with respect to this map. Now comparing the map induced by "$T \cup$" in the top of (6.3.2) with a corresponding map "$T \cup$" on the B_n, and with the inductive construction of the h_T in (6.2.2) and 6.1(b), we see that "$T \cup$" is identified by the isomorphisms of (d) to h_T composed with some natural automorphism of $B_n(X \text{ on } Z)$. Also the "d" on $\pi_*(F)$ is identified to the "d" on B up to sign. Then $\pi_n f = \pi_n(d \cdot (T\cup\ ))$ is identified to $d_T \cdot h_T$ composed with a natural automorphism of $B_n(X \text{ on } Z)$. Thus (e) follows from 6.2(d). This completes the proof.

6.4. *Definition.* Let $K^B(X \text{ on } Z)$ be the homotopy colimit of the solid arrow diagram (6.4.1), where the F^k are the $F^k(X \text{ on } Z)$ of 6.3, and the maps $F^k \to \Omega\Sigma F^{k-1}$ are the $\Omega d \cdot (T\cup\ )$ of 6.3(e), corresponding to the maps f of 6.3 under the homotopy equivalence $F^{k-1} \xrightarrow{\sim} \Omega\Sigma F^{k-1}$.

(6.4.1)
$$\begin{tikzcd}
F^0 \arrow[r] & \Omega\Sigma F^{-1} \arrow[r, dashed] & F'^{-2} \arrow[r, dashed] & F'^{-3} \arrow[r, dashed] & \cdots \\
& F^{-1} \arrow[u, "\sim"] \arrow[r] & \Omega\Sigma F^{-2} \arrow[u, dashed] & & \\
& & F^{-2} \arrow[u, "\sim"] \arrow[r] & \Omega\Sigma F-3 \arrow[uu, dashed] & \\
& & & F^{-3} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Equivalently, let $F'^{-2}, F'^{-3}, \ldots$ be the canonical homotopy pushouts of the indicated squares. Let $\Omega\Sigma F^{-1} = F'^{-1}$. Then let $K^B(X \text{ on } Z)$ be the colimit of the direct system $F^0 \to F'^{-1} \to F'^{-2} \to F'^{-3} \to \ldots$.

As the maps $F^k \to F'^k$ are homotopy equivalences, $K^B(X \text{ on } Z)$ is homotopy equivalent to the direct colimit of $F^0 \to F^{-1} \to F^{-2} \to \ldots$ where the bounding maps are the maps f of 6.3(e). Thus we have $\pi_n K^B(X \text{ on } Z) = K_n(X \text{ on } Z)$ for $n \geq 0$, and $= B_n(X \text{ on } Z)$ for $n \leq 0$.

6.5. So defined, $K^B(X \text{ on } Z)$ is a contravariant functor in X, just like $K(X \text{ on } Z)$. It has the same covariant functoriality with respect to flat proper, perfect projective, and (for noetherian schemes) perfect proper maps as does $K(X \text{ on } Z)$, (cf. 3.16). It is also covariant with respect to enlarging Z, as in 3.16.7. The canonical homotopy $g^* f_* \simeq f'_* g'^*$ of 3.18 extends from K to K^B.

There is a canonical map $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$, natural with respect to both contravariant and covariant functorialities, and which induces isomorphisms $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$ for $n \geq 0$.

$K^B(X \text{ on } Z)$ is a module spectrum over $K(X)$. The projection formula 3.17 holds in that $f_* : K^B(X) \to K^B(Y)$ will be a map of $K(Y)$ modules under the hypotheses of 3.17.

All these remarks follow by passing the relevant properties of K through the inductive construction of 6.3 and the direct colimit of 6.4. (If the reader wishes to verify exactly the compatibilities of various systems of homotopies involved, he should build symmetric monoidal category models of everything starting from the simplicial symmetric monoidal $\boldsymbol{wS.A}$ and using [Th3], realize the involved homotopies as symmetric monoidal natural transformations, and calculate compatibilities using the calculus of 2-categories as in [Th3]).

6.6. **Theorem** (Bass Fundamental Theorem) (cf. [B] XII Section 7; [Gr1]). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subscheme with $X - Z$ quasi-compact. Then*

(a) *The natural map $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ induces isomorphism on π_n for $n \geq 0$: $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$, $K_n(X) \cong K_n^B(X)$.*

(b) *For all integers $n \in \mathbb{Z}$, there are natural exact sequences:*

$$\begin{array}{l} 0 \to K_n^B(X \text{ on } Z) \to K_n^B(X[T] \text{ on } Z[T]) \oplus K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \qquad\qquad\qquad\qquad \downarrow \partial \\ \qquad\qquad\qquad K_{n-1}^B(X \text{ on } Z) \to 0 \end{array}$$

$$\begin{array}{r} 0 \to K_n^B(X) \to K_n^B(X[T]) \oplus K_n^B(X[T^{-1}]) \to K_n^B(X[T,T^{-1}]) \\ \downarrow \partial \qquad \\ K_{n-1}^B(X) \to 0. \end{array}$$

(c) *There is a homotopy fibre sequence*

$$\begin{array}{c} K^B(X[T] \text{ on } Z[T]) \underset{K^B(X \text{ on } Z)}{\overset{h}{\cup}} K^B(X[T^{-1}] \text{ on } Z[T^{k-1}]) \\ \downarrow b \\ K^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \xrightarrow{d} \Sigma K^B(X \text{ on } Z). \end{array}$$

(d) *For all integers $n \in \mathbb{Z}$, and all positive integers $k \geq 1$, the composition $\partial_k \cdot \ldots \cdot \partial_1(T_1 \cup \cdots \cup T_k \cup)$ is an isomorphism*

$$\begin{array}{c} K_n^B(X \ \ on \ \ Z) \\ \big\downarrow T_1 \cup \cdots \cup T_k \cup \\ K_{n+k}^B(X[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}] \ \ on \ \ Z[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}]) \\ \big\downarrow \partial_{T_k} \cdot \partial_{T_{k-1}} \cdots \partial_{T_1} \\ K_n^B(X \ \ on \ \ Z) \end{array}$$

In particular, this holds in the absolute case of $K^B(X) = K^B(X \ \ on \ \ X)$.

(e) *For all positive integers* $k \geq 1$, *the composition* $d_k \cdot \ldots \cdot d_1(T_1 \cup \cdots \cup T_k \cup)$ *is a homotopy equivalence*

$$\begin{array}{c} \Sigma^k K^B(X \ \ on \ \ Z) \\ \big\downarrow T_1 \cup \cdots \cup T_k \cup \\ K^B(X[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}] \ \ on \ \ Z[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}]) \\ \big\downarrow d_{T_k} \cdots d_{T_1} \\ \Sigma^k K^B(X \ \ on \ \ Z) \end{array}$$

In particular, this holds in the absolute case of $K^B(X) = K^B(X \ \ on \ \ X)$.

Proof. Parts (a), (b), (c) follow from 6.4, 6.3, 6.2, 6.1, and in fact just combine pieces of these.

Parts (d) and (e) follow by induction on k from 6.3(e), 6.1(b), and the definition 6.4.

6.7. By 6.6(d), for $n > 0$, $K_{-n}^B(X)$ is a natural retract of $K_0(X[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}])$ and of $K_1(X[T_1, T_1^{-1}, \ldots, T_{n+1}, T_{n+1}^{-1}])$. Thus a statement that certain natural classes of maps (invariant under Laurent extensions to $X[T^{\pm}]$) induce isomorphisms or exact sequences on the K-groups K_n for $n > 0$ quickly extend to the K_n^B for all n. For a retract of an isomorphism is an isomorphism, and the retract of an exact sequence is exact. Section 7 will be devoted to results for K^B derived by this method. Similarly 6.6(e) allows one to deduce spectra level versions.

6.8 **Proposition.** *Let X be a regular noetherian scheme. Then*

(a) $p^* : K(X) \xrightarrow{\sim} K(X[T])$ *is a homotopy equivalence.*

(b) $K(X) \xrightarrow{\sim} K^B(X)$ *is a homotopy equivalence, so* $K_n^B(X) = 0$ *for* $n < 0$.

Proof. Statement (a) results from Poincaré duality 3.21 and the corresponding result for G-theory, [Q1] Section 7, 4.1. Similarly, Quillen's localization theorem for G-theory [Q1] Section 7, 3.2 applied as in (6.1.2) gives $G(X[T,T^{-1}]) \simeq G(X) \times \Sigma G(X)$ using $G(X[T]) \simeq G(X)$. By Poincaré duality, we have $K(X[T,T^{-1}]) \simeq K(X) \times \Sigma K(X)$. In particular, the map $K_0(X) \to K_0(X[T,T^{-1}])$ is onto, so $K^B_{-1}(X) = 0$ by 6.2(b). Now by descending induction on n, using 6.6(b), $K^B_n(X) = 0$ for $n < 0$.

6.9. *Exercise* (Optional). Show $K^B(X)$ is a homotopy ring spectrum. Let $K^B(X) > k <$ be the coPostnikov truncation killing off π_n for $n < k$. Thus $K^B(X) > 0 <\simeq K(X)$. Consider for k, $p \geq 0$ the map

$$
\begin{array}{c}
\Sigma^k \left(K^B(X) > -k <\right) \wedge \Sigma^p \left(K^B(X) > -p <\right) \\
\downarrow T_1 \cup \ldots \cup T_k \wedge S_1 \cup \ldots \cup S_p \\
\left(K^B \left(X \left[T_1, T_1^{-1}, \ldots, T_k T_k^{-1}\right]\right) > 0 <\right) \wedge \left(K^B \left(X \left[S_1, S_1^{-1}, \ldots, S_p, S_p^{-1}\right]\right) > 0 <\right) \\
\wr | \\
K \left(X \left[T, T^{-1}, \ldots, T_k, T_k^{-1}\right]\right) \wedge K \left(X \left[S_1, S_1^{-1}, \ldots, S_p, S_p^{-1}\right]\right) \\
\downarrow \otimes^L_{\mathcal{O}_X} \\
K \left(X \left[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}, S_1 . S_1^{-1}, \ldots, S_p, S_p^{-1}\right]\right) \\
\wr | \\
K^B \left(X \left[T_1, T_1^{-1}, \ldots, S_p . S_p^{-1}\right]\right) > 0 < \\
\downarrow \partial_{T_k} \quad \cdots \partial_{S_1} \\
\Sigma^{k+p} \left(K^B(X) > -p-k <\right)
\end{array}
$$

This defines $(K^B(X) > -k <) \wedge (K^B(X) > -p <) \to K^B(X) > -p-k <$. Now take the colimit as $k \to \infty$ and $p \to \infty$.

7. Basic theorems for K^B, including the Localization Theorem

7.0. We recall that $K^B(X \text{ on } Z)$ is defined for X a quasi-compact and quasi-separated scheme X with a closed subspace Z such that $X - Z$ is quasi-compact. K^B has the same functorialities as K. There is a natural transformation $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ inducing an isomorphism on homotopy groups π_n for $n \geq 0$. However $K^B_n(X \text{ on } Z)$ could be non-zero for $n < 0$.

7.1. **Theorem** (Excision). *Let $f : X' \to X$ be a map of quasi-compact and quasi-separated schemes. Let Y be a closed subspace of X with $X - Y$ quasi-compact. Set $Y' = f^{-1}(Y)$.*

Suppose f is an isomorphism infinitely near Y, in the sense 2.6.2.2.

Then $f^* : K^B(X \text{ on } Y) \xrightarrow{\sim} K^B(X' \text{ on } Y')$ is a homotopy equivalence.

Proof. It suffices to show that $\pi_n f^*$ is an isomorphism on homotopy groups for all n. We use the trick of 6.7. As $\pi_{-k} f^*$ for X is a retract of $\pi_0 f^*$ for $X[T_1, T_1^{-1}, \ldots, T_k, T_k^{-1}]$ when $k > 0$ by 6.6(d), it suffices to show that $\pi_n f^*$ is an isomorphism for $n \geq 0$ on $K_n^B = K_n$. But this holds by excision 3.19.

7.2. **Theorem** (Continuity). *Let $X = \varprojlim X_\alpha$ be the limit of an inverse system of schemes X_α in which the bonding maps $f_{\alpha\beta} : X_\alpha \to X_\beta$ are affine. Suppose all the X_α are quasi-compact and quasi-separated. Let $Y_\alpha \subseteq X_\alpha$ be a system of closed subspaces with $f_{\alpha\beta}^{-1}(Y_\beta) = Y_\alpha$ and with $X_\alpha - Y_\alpha$ quasi-compact. Then the canonical maps are homotopy equivalences*

$$\varinjlim_{\alpha} K^B(X_\alpha) \xrightarrow{\sim} K^B(X)$$

$$\varinjlim_{\alpha} K^B(X_\alpha \text{ on } Y_\alpha) \xrightarrow{\sim} K^B(X \text{ on } Y).$$

Proof. We use the trick of 6.7. It suffices to show that the maps induce isomorphisms on homotopy groups π_n. By 6.6.(d), it suffices to do so for $n \geq 0$. Then the result follows from continuity 3.20 as $K_n^B = K_n$ for $n \geq 0$.

7.3. **Theorem** (Projective space bundle theorem). *Let X be a quasi-compact and quasi-separated scheme, and let $Y \subseteq X$ be a closed subspace with $X - Y$ quasi-compact. Let $\mathcal{E}$ be a vector bundle of rank r on X. Then the maps sending $(x_0, x_1, \ldots, x_{r-1})$ to $\Sigma\pi^*(x_i) \otimes [\mathcal{O}(-i)]$ (using the module structure of $K^B(\mathbb{P}\mathcal{E}_X \text{ on } \mathbb{P}\mathcal{E}_Y)$ over $K(\mathbb{P}\mathcal{E}_X)$) induce homotopy equivalences:*

$$\prod_1^r K^B(X) \xrightarrow{\sim} K^B(\mathbb{P}\mathcal{E}_X)$$

$$\prod_1^r K^B(X \text{ on } Y) \xrightarrow{\sim} K^B(\mathbb{P}\mathcal{E}_Y \text{ on } \mathbb{P}\mathcal{E}_Y).$$

Proof. Again, we use the trick of 6.7 to reduce this to 4.1.

7.4. **Theorem** (Localization Theorem). *Let X be a quasi-compact and quasi-separated scheme. Let $j : U \to X$ be an open immersion with U quasi-compact. Set $Y = X - U$. Let Z be a closed subspace of X with $X - Z$ quasi-compact. Then there are homotopy fibre sequences, induced by the obvious maps and nullhomotopies:*

$$\begin{aligned} & K^B(X \text{ on } Y) \to K^B(X) \to K^B(U) \\ (7.4.1) \qquad & K^B(X \text{ on } Y \cap Z) \to K^B(X \text{ on } Z) \to K^B(U \text{ on } U \cap Z). \end{aligned}$$

There are resulting long exact sequences of homotopy groups:

(7.4.2)

$$\cdots \xrightarrow{\partial} K_n^B(X \text{ on } Y) \to K_n^B(X) \to K_n^B(U) \xrightarrow{\partial} K_{n-1}^B(X \text{ on } Y) \to \dots$$

$$\begin{aligned} & \cdots \xrightarrow{\partial} K_n^B(X \text{ on } Y \cap Z) \to K_n^B(X \text{ on } Z) \to K_n^B(U \text{ on } U \cap Z) \\ (7.4.3) \qquad & \qquad \to K_{n-1}^B(X \text{ on } Y \cap Z) \to \cdots . \end{aligned}$$

Proof. It suffices to prove that (7.4.1) are homotopy fibre sequences. For this, we must first specify a natural nullhomotopy of the composed map $K^B(X \text{ on } Y \cap Z) \to K^B(U \text{ on } U \cap Z)$. The map $K(X \text{ on } Y \cap Z) \to K(U \text{ on } U \cap Z)$ is canonically nullhomotopic, as any complex on X acyclic on $X - (Y \cap Z) \supseteq X - Y = U$ is naturally quasi-isomorphic to 0 on U. Thus 1.5.4 provides the nullhomotopy. This nullhomotopy is strictly natural in X, and in particular is natural for the maps $(X[T, T^{-1}]) \to X$. Thus the nullhomotopy is natural with respect to the map b in 6.3(b), and by inductive construction, as in 6.3, it induces a natural nullhomotopy of $F^k(X \text{ on } Y \cap Z) \to F^k(U \text{ on } U \cap Z)$ for $k = 0, -1, -2, \dots$. By 6.4, on taking the colimit as k goes to $-\infty$, we get a natural nullhomotopy on K^B.

This specified natural nullhomotopy determines a natural map from $K^B(X \text{ on } Y \cap Z)$ to the canonical homotopy fibre of the map $K^B(X \text{ on } Z) \to K^B(U \text{ on } U \cap Z)$. It remains to show this map to the canonical homotopy fibre is a homotopy equivalence. It suffices to show it induces an isomorphism on homotopy groups π_n. By the trick of 6.7, it suffices to show that it induces an isomorphism on $\pi_n = K_n$ for $n \geq 0$. But this is true by the Proto-localization Theorem 5.1. This proves the result for $K^B(X \text{ on } Z)$, and so for $K^B(X) = K^B(X \text{ on } X)$.

7.5. **Theorem** (Bass Fundamental Theorem). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subscheme with $X - Z$ quasi-compact. Then*

(a) *The natural map $K(X \text{ on } Z) \to K^B(X \text{ on } Z)$ induces isomorphisms on π_n for $n \geq 0$: $K_n(X \text{ on } Z) \cong K_n^B(X \text{ on } Z)$, $K_n(X) \cong K_n^B(X)$.*

(b) *For all integers $n \in \mathbb{Z}$, there are natural exact sequences:*

$$\begin{array}{c} 0 \\ K_n^B(X \text{ on } Z) \\ \downarrow \\ K_n^B(X[T] \text{ on } Z[T]) \oplus K_n^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \downarrow \\ K_n^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow \partial \\ K_{n-1}^B(X \text{ on } Z) \\ \downarrow \\ 0 \end{array}$$

$$\begin{array}{r} 0 \to K_n^B(X) \to K_n^B(X[T]) \oplus K_n^B(X[T^{-1}]) \to K_n^B(X[T,T^{-1}]) \\ \downarrow \partial \quad \\ K_{n-1}^B(X) \to 0 \end{array}$$

(c) There is a homotopy fibre sequence:

$$\begin{array}{c} K^B(X[T] \text{ on } Z[T]) \overset{h}{\underset{K^B(X \text{ on } Z)}{\cup}} K^B(X[T^{-1}] \text{ on } Z[T^{-1}]) \\ \downarrow b \\ K^B(X[T,T^{-1}] \text{ on } Z[T,T^{-1}]) \\ \downarrow d \\ \Sigma K^B(X \text{ on } Z). \end{array}$$

Proof. This is part of 6.6.

7.6. **Theorem.** *If X has an ample family of line bundles, then $K^{\text{naive}}(X) \to K(X) \to K^B(X)$ induces a natural isomorphism for $n \geq 0$: $K_n^{\text{naive}}(X) \cong K_n^B(X)$, where $K_n^{\text{naive}}(X)$ is Quillen's K_n of X as in [Q1].*

Proof. Combine 7.5(a) with 3.8. and 3.10.

8. Mayer-Vietoris theorems

8.1. **Theorem**. *Let X be a quasi-separated scheme. Let U and V be quasi-compact open subschemes of X. Let $Z \subseteq U \cup V$ be a closed subspace, with $U \cup V - Z$ quasi-compact. Then the squares (8.1.1) are homotopy cartesian:*

(8.1.1)
$$\begin{array}{ccc} K^B(U \cup V) & \longrightarrow & K^B(U) \\ \downarrow & \square & \downarrow \\ K^B(V) & \longrightarrow & K^B(U \cap V) \end{array}$$

$$\begin{array}{ccc} K^B(U \cup V \text{ on } Z) & \longrightarrow & K^B(U \text{ on } U \cap Z) \\ \downarrow & \square & \downarrow \\ K^B(V \text{ on } V \cap Z) & \longrightarrow & K^B(U \cap V \text{ on } U \cap V \cap Z) \end{array}$$

Thus, there are long exact Mayer-Vietoris sequences:

(8.1.2)
$$\cdots \xrightarrow{\partial} K_n^B(U \cup V) \to K_n^B(U) \oplus K_n^B(V) \to K_n^B(U \cap V) \xrightarrow{\partial} K_{n-1}^B(U \cup V) \to \ldots$$

There is a similar sequence for $K_n^B($ on $Z)$.

Proof. Consider (8.1.3)

$$\begin{array}{ccc} K^B(U \cup V \text{ on } ((U \cup V) - V) \cap Z) & \xrightarrow{\simeq} & K^B(U \text{ on} (U - V) \cap Z) \\ \downarrow & & \downarrow \\ K^B(U \cup V \text{ on } Z) & \longrightarrow & K^B(U \text{ on } U \cap Z) \\ \downarrow & & \downarrow \\ K^B(V \text{ on } V \cap Z) & \longrightarrow & K^B(U \cap V \text{ on} U \cap V \cap Z) \end{array} \qquad (8.1.3)$$

The columns of (8.1.3) are homotopy fibre sequences by the Localization Theorem 7.4. The induced map on the fibres is a homotopy equivalence by excision 7.1. Hence by Quetzalcoatl, the square on the bottom of (8.1.3) is homotopy cartesian, as required (or more naively, apply the 5-lemma to show the map of $K^B(U \cup V$ on $Z)$ into the homotopy pullback induces an isomorphism on homotopy groups).

8.1.4. **Corollary.** *Let X be a quasi-compact and quasi-separated scheme. Let Y_1 and Y_2 be two closed subspaces with both $X - Y_1$ and $X - Y_2$ quasi-compact. Then* (8.1.5) *is a homotopy cartesian square*

$$\text{(8.1.5)} \qquad \begin{tikzcd} K^B(X \text{ on } Y_1 \cap Y_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & K^B(X \text{ on } Y_1) \arrow[d] \\ K^B(X \text{ on } Y_2) \arrow[r] & K^B(X \text{ on } Y_1 \cup Y_2) \end{tikzcd}$$

Proof. Consider diagram (8.1.6)

(8.1.6)

$$\begin{tikzcd}[column sep=small] K^B(X \text{ on } Y_1 \cap Y_2) \arrow[rr] \arrow[dr] \arrow[dd] & & K^B(X) \arrow[rr] \arrow[dr] \arrow[dd] & & K^B(X - Y_1 \cup X - Y_2) \arrow[dr] \arrow[dd] & \\ & K^B(X \text{ on } Y_1) \arrow[rr] \arrow[dd] & & K^B(X) \arrow[rr] \arrow[dd] & & K^B(X - Y_1) \arrow[dd] \\ K^B(X \text{ on } Y_2) \arrow[rr] \arrow[dr] & & K^B(X) \arrow[rr] \arrow[dr] & & K^B(X - Y_2) \arrow[dr] & \\ & K^B(X \text{ on } Y_1 \cup Y_2) \arrow[rr] & & K^B(X) \arrow[rr] & & K^B((X - Y_1) \cap (X - Y_2)) \end{tikzcd}$$

The rows are homotopy fibre sequences by Localization 7.4. The right vertical plane is homotopy cartesian by 8.1, as is the middle plane of $K^B(X)$'s. Hence by Quetzalcoatl, the left hand plane is homotopy cartesian, as required.

8.2. To formulate Mayer-Vietoris theorems for covers by more than two open sets, we recall the Čech hypercohomology spectrum of a cover, $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$, from [Th1] Section 1. (The reader unfamiliar with this may skip ahead to Section 9 and ignore the rest of Section 8).

8.2.1. Let $\mathcal{E}^{\sim}$ be a Grothendieck topos with terminal object X, and with a site $\mathcal{E}$ containing X and closed under pullbacks ([SGA 4]). For us, $\mathcal{E}$ will usually be the category of Zariski open subsets of a scheme X, and $\mathcal{E}^{\sim}$ the category of sheaves of sets on X.

Let $F : \mathcal{E}^{\mathrm{op}} \to \text{Spectra}$ be a presheaf of spectra defined on the site $\mathcal{E}$. (Upon replacing F by a homotopy equivalent presheaf, we may assume it satisfies the technical topological conditions to be a "presheaf of fibrant spectra," (see [Th1] 5.2.)).

Let $\mathcal{U} = \{U_i \to X \mid i \in I\}$ be a cover of X in the site $\mathcal{E}$.

8.2.2. Recall ([Th1] 1.9) that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is defined as the homotopy limit of the Čech cosimplicial spectrum of F and $\mathcal{U}$

$$\begin{array}{c} \check{\mathbb{H}}^{\cdot}(\mathcal{U};F) \\ \wr\| \\ \operatorname{holim}_{\Delta}\left(\prod_{i_0\in I}F(U_{i_0}) \rightrightarrows \prod_{(i_0,i_1)\in I^2} F\left(U_{i_0}\underset{X}{\times}U_{i_1}\right) \substack{\leftarrow\\\rightrightarrows\\\rightarrow}\cdots\right) \end{array} \tag{8.2.3}$$

There is a natural augmentation $F(X)\to\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$.

$\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ is a covariant functor in F, and preserves homotopy fibre sequences and homotopy equivalences of presheaves F ([Th1] 1.15). $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ is a covariant functor with respect to maps of covers, where a map $\mathcal{U}\to\mathcal{V}$ of covers consists of a function $\varphi: J\to I$ from the indexing set of $\mathcal{V}$ to that of $\mathcal{U}$, and a family of maps over X, $V_j\to U_{\varphi(j)}$, one for each $j\in J$. Up to homotopy, the induced map $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)\to\check{\mathbb{H}}^{\cdot}(\mathcal{V};F)$ is independent of the choice of φ or the particular maps $V_j\to V_{\varphi(j)}$, and exists whenever $\mathcal{V}$ is a "refinement" of the cover $\mathcal{U}$ ([Th1] 1.20). Hence if $\mathcal{U}$ and $\mathcal{V}$ refine each other, their $\check{\mathbb{H}}^{\cdot}(\ ;F)$ are homotopy equivalent ([Th1] 1.21]).

Let $f:\mathcal{F}\to\mathcal{E}$ be a map of sites ([SGA 4] III 1, IV 4.9). Then the cover $\mathcal{U}$ on $\mathcal{E}$ induces a cover $f^{-1}(\mathcal{U})=\{f^{-1}(U_i)|i\in I\}$ in $\mathcal{F}$. There is a canonical isomorphism $\check{\mathbb{H}}^{\cdot}(\mathcal{U};f_{\#}G=G\cdot f^{-1})\cong\check{\mathbb{H}}^{\cdot}(f^{-1}(\mathcal{U});G)$. This makes $\check{\mathbb{H}}^{\cdot}$ a contravariant functor with respect to the site $\mathcal{E}$.

There is a spectral sequence relating Čech hypercohomology $\check{\mathbb{H}}^{\cdot}(\mathcal{U};F)$ to the usual Čech cohomology of presheaves of abelian groups

$$E_2^{p,q}=\check{H}^p(\mathcal{U};\pi_qF)\Longrightarrow\pi_{q-p}\check{\mathbb{H}}^{\cdot}(\mathcal{U};F). \tag{8.2.4}$$

This spectral sequence converges strongly if either there exists an integer N so that $\pi_qF=0$ for all $q\geq N$, or else, if there exists an integer M such that $\check{H}^p(\mathcal{U};\pi_qF)=0$, for all $p\geq M$ and all q. See [Th1] 1.16, and note we follow [Th1] in using the Bousfield-Kan indexing of spectral sequences, so $d_r: E_r^{p,q}\to E_r^{p+r,q+r-1}$.

8.2.5. **Lemma**. *Assume 8.2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X, and suppose there is a map of covers $\mathcal{U}\to\mathcal{V}$, so $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

Suppose that for every finite set I of $U_\iota\to X$ drawn from $\mathcal{U}$, and for the fibre product U_I over X of the elements of I and for the induced cover $\mathcal{V}\underset{X}{\times}U_I$ of

$$U_I=U_{\iota_0}\underset{X}{\times}U_{i_1}\underset{X}{\times}\cdots\underset{X}{\times}U_{\iota_n} \tag{8.2.6}$$

that the augmentation map (8.2.7) is a homotopy equivalence

$$(8.2.7) \qquad F(U_I) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{V} \underset{X}{\times} U_I; F\right).$$

In particular, for $I = \phi$ we suppose that $F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{V}; F)$ is a homotopy equivalence.

Then the augmentation map (8.2.8) for $\mathcal{U}$ is also a homotopy equivalence

$$(8.2.8) \qquad F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F).$$

Proof. We consider the diagram of maps induced by the various augmentation maps ϵ

$$(8.2.9) \qquad \begin{array}{ccc} F(X) & \xrightarrow{\epsilon} & \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \\ \epsilon \downarrow \sim & & \sim \downarrow \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \epsilon) \\ \check{\mathbb{H}}^{\cdot}(\mathcal{V}; F) & \xrightarrow{\epsilon} & \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \check{\mathbb{H}}^{\cdot}(\mathcal{V} \underset{X}{\times}\ ; F)) \\ \check{\mathbb{H}}^{\cdot}(\mathcal{V}; \epsilon) & \searrow & \wr\| \\ & & \check{\mathbb{H}}^{\cdot}(\mathcal{V}; \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times}\ ; F)) \end{array}$$

By the hypothesis the left vertical map of (8.2.9) is a homotopy equivalence. By hypothesis (8.2.7) and the fact that holim preserves homotopy equivalences, inspection of formula (8.2.3) shows that the right vertical map $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; \epsilon)$ is also a homotopy equivalence (cf. [Th1] 1.15).

The isomorphism at the bottom right of (8.2.9) is deduced from the fact that holims commute (e.g., [Th1] 5.7), so that there is an isomorphism:

(8.2.10)

$$\begin{aligned}
&\check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; \check{\mathbb{H}}^{\cdot}\left(\mathcal{V} \underset{X}{\times}\ ; F\right)\right) \\
&= \underset{\Delta}{\operatorname{holim}}\left(p \mapsto \prod_{(i_0, i_1, \ \ , i_p)} \right. \\
&\qquad \left.\left. \operatorname{holim}\left(q \mapsto \prod_{(j_0, j_1, \ \cdot, j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} U_{i_p} \underset{X}{\times} V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right)\right) \\
&\cong \underset{\Delta}{\operatorname{holim}}\left(p \mapsto \left(\underset{\Delta}{\operatorname{holim}}\left(q \mapsto \prod_{(i_0, i_1, \ \ , i_p)} \prod_{(j_0, j_1, \ \ , j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right)\right) \\
&\cong \underset{\Delta}{\operatorname{holim}}\left(q \mapsto \underset{\Delta}{\operatorname{holim}}\left(p \mapsto \prod_{(i_0, i_1, \ \ , i_p)} \prod_{(j_0, j_1, \ \ , j_q)} F\left(U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}\right)\right)\right) \\
&\cong \check{\mathbb{H}}^{\cdot}\left(\mathcal{V}; \check{\mathbb{H}}^{\cdot}\left(\mathcal{U} \underset{X}{\times}\ ; F\right)\right).
\end{aligned}$$

It is easy to see that this isomorphism carries the augmentation ϵ to $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ as claimed in (8.2.9).

We claim that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ is a homotopy equivalence since $\mathcal{V}$ is a refinement of $\mathcal{U}$. It will suffice to show for each $V_J = V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q}$ that $\epsilon : F(V_J) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times} V_J;\ F)$ is a homotopy equivalence, for then we use the argument above that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\ )$ preserves homotopy equivalences. But as $\mathcal{V}$ is a refinement of $\mathcal{U}$, $V_{j_0} \to X$ factors as $V_{J_0} \to U_{\varphi(j_0)} \to X$ for some $U_{\varphi(j_0)} \to X$ in $\mathcal{U}$. But then the identity map $V_J \to V_J$ factors through an element of $\mathcal{U} \underset{X}{\times} V_J$:

(8.2.11)
$$\begin{tikzcd}
V_J = V_{j_0} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q} \arrow[r, "\Delta\times 1\times\cdots\times 1"] \arrow[dd, "1"] & V_{j_0} \underset{X}{\times} V_{j_0} \underset{X}{\times} V_{j_1} \underset{X}{\times} \cdots \underset{X}{\times} V_{j_q} \arrow[d] \\
& U_{\varphi(j_0)} \underset{X}{\times} V_J \arrow[d] \\
V_J \arrow[r, "1"] & V_J = X \underset{X}{\times} V_J
\end{tikzcd}$$

This gives a map of covers from $\mathcal{U} \underset{X}{\times} V_J$ to the trivial cover $\{V_J = V_J\}$ of V_J, so $\mathcal{U} \underset{X}{\times} V_J$ is refined by the trivial cover. There is a canonical map of covers from the trivial cover to any cover. So $\mathcal{U} \underset{X}{\times} V_J$ and the trivial cover of V_J refine each other. It follows that these maps of covers induce homotopy equivalences of Čech hypercohomologies for these covers ([Th1] 1.21). But the Čech hypercohomology for the trivial cover of V_J and with coefficients F is homotopy equivalent via the augmentation map to $F(V_J)$. (E.g., this is well-known for abelian group presheaves of coefficients, and the general case then follows by collapse of the spectral sequence (8.2.4); or else use (8.2.3) for the constant Čech cosimplicial spectrum coming from the trivial cover, and the dual of [Th1] 5.21.) It follows that $\epsilon : F(V_J) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U} \underset{X}{\times} V_J; F)$ is a homotopy equivalence, and hence that $\check{\mathbb{H}}^{\cdot}(\mathcal{V};\epsilon)$ is a homotopy equivalence, as claimed.

Now we have shown that three sides of the square in (8.2.9) are homotopy equivalences. It follows that the fourth side $\epsilon : F(X) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is also a homotopy equivalence, proving the lemma.

8.3. **Proposition** (cf. 8.4). *Let X be a quasi-compact and quasi-separated scheme, and let $Z \subseteq X$ be a closed subspace with $X - Z$ quasi-compact. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a cover of X by finitely many Zariski* *open subschemes, each of which is quasi-compact. Then the augmentation maps are homotopy equivalences*

$$\begin{aligned} K^B(X) &\xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B\right) \\ K^B(X \text{ on } Z) &\xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}\left(\mathcal{U}; K^B((\ ) \text{ on } (\ ) \cap Z)\right). \end{aligned} \tag{8.3.1}$$

There are strongly converging Mayer-Vietoris spectral sequences (with Bousfield-Kan indexing, so $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$):

$$\begin{aligned} E_2^{p,q} &= \check{\mathbb{H}}^p\left(\mathcal{U}; K_q^B\right) \Longrightarrow K_{q-p}^B(X) \\ E_2^{p,q} &= \check{\mathbb{H}}^p\left(\mathcal{U}; K_q^B((\ ) \text{ on } (\ ) \cap Z)\right) \Longrightarrow K_{p-q}^B(X \text{ on } Z). \end{aligned} \tag{8.3.2}$$

Proof. The last statement (8.3.2) follows from (8.3.1) by taking the canonical spectral sequence (8.2.4), and noting that $H^p(\mathcal{U};\ ) = 0$ for $p > n$ as $\mathcal{U}$ has at most n distinct elements, so the alternating Čech cochain complex for $\mathcal{U}$ vanishes in degree $p > n$ (cf. [EGA] 0_{III} 11.8 and [Th1] 1.49). Thus it suffices to prove (8.3.1). To simplify notation, we give the argument in the absolute case $K^B(X)$,which in fact implies the case with supports as a corollary, using 7.4, the fact $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ preserves homotopy fibre sequences, and the 5-lemma.

We prove (8.3.1) by induction on n, the number of open sets in the cover $\mathcal{U}$. For $n = 1$, $\mathcal{U}$ is the trivial cover and (8.3.1) is trivially true (cf. proof 8.2.5).

To do the induction step, we suppose the result is known for covers of schemes by $\leq n-1$ opens. We set $V = U_1 \cup \ldots U_{n-1}$, and set $\mathcal{V}$ to be the cover $\{U_1, \ldots, U_{n-1}\}$ of V.

By the Mayer-Vietoris Theorem 8.1, for any quasi-compact open W in X, there is a homotopy cartesian square, natural in W

$$\begin{array}{ccc} K^B(W) & \longrightarrow & K^B(W \cap U_n) \\ \downarrow & & \downarrow \\ K^B(W \cap V) & \longrightarrow & K^B(W \cap V \cap U_n) \end{array} \tag{8.3.3}$$

As W varies, this is a homotopy cartesian square of presheaves on X. Applying $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ which preserves homotopy fibre sequences and hence preserves homotopy cartesian squares ([Th1] 1.15), we get a cube

(8.3.4)

$$\begin{tikzcd}
K^B(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \check{H}^{\cdot}(\mathcal{U};K^B(\ )) \arrow[dr] \arrow[dd] & \\
& K^B(U_n) \arrow[rr] \arrow[dd] & & \check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap U_n)) \arrow[dd] \\
K^B(V) \arrow[rr] \arrow[dr] & & \check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap V)) \arrow[dr] & \\
& K^B(U_n \cap V) \arrow[rr] & & \check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap U_n \cap V))
\end{tikzcd}$$

The left-to-right arrows in the cube are all augmentation maps. The left and right sides of this cube are homotopy cartesian.

For W open in X, $\check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap W))$ is naturally isomorphic to $\check{H}^{\cdot}(\mathcal{U}\cap W;K^B(\ ))$, as we see by inspection of (8.2.3) (cf. [Th1] 1.25).

In particular, $\check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap U_n))$ is $\check{H}^{\cdot}(\mathcal{U}\cap U_n;K^B)$. As $\mathcal{U}\cap U_n$ contains $U_n \to U_n$, there is a map of covers from $\mathcal{U}\cap U_n$ to the trivial cover $\{U_n \to U_n\}$, in addition to the canonical map of covers going the other way. As before in the proof of 8.2.5, this shows that $\check{H}^{\cdot}(\mathcal{U}\cap U_n;K^B)$ is homotopy equivalent to $K^B(U_n)$, and in fact that the augmentation map $K^B(U_n) \to \check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap U_n))$ is a homotopy equivalence.

Also, we get that $\check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap V))$ is $\check{H}^{\cdot}(\mathcal{U}\cap V;K^B)$. As $\mathcal{V} = \{U_1,\dots,U_{n-1}\}$ is a subset of $\{U_1,\dots,U_{n-1},U_n\cup V\} = \mathcal{U}\cap V$, there is a map of covers $\mathcal{U}\cap V \to \mathcal{V}$. By the induction hypothesis, for any $W \subseteq V$, $K^B(W) \to \check{H}^{\cdot}(W\cap\mathcal{V};K^B)$ is a homotopy equivalence. Then as $\mathcal{V}$ refines $\mathcal{U}\cap V$, Lemma 8.2.5 shows that $K^B(V) \to \check{H}^{\cdot}(\mathcal{U}\cap V;K^B)$ is also a homotopy equivalence. Thus $K^B(V) \to \check{H}^{\cdot}(\mathcal{U};K^B((\ )\cap V))$ is a homotopy equivalence. The argument of the preceding paragraph applies also to $V\cap U_n$, to show that $K^B(V\cap U_n) \to \check{H}^{\cdot}(\mathcal{U}\cap V\cap U_n;K^B)$ is a homotopy equivalence.

Thus we have seen that three of the four left-to-right arrows in (8.3.4) are homotopy equivalences. As the left and right sides of (8.3.4) are homotopy cartesian, and as taking homotopy pullbacks preserves homotopy equivalences of diagrams, it follows that the fourth left-to-right arrow $K^B(X) \to \check{H}^{\cdot}(\mathcal{U};K^B)$ is also a homotopy equivalence. This completes the proof of the induction step, and hence of the theorem.

8.4. **Theorem.** *Let X be a quasi-compact and quasi-separated scheme. Let Z be a closed subspace of X with $X-Z$ quasi-compact. Let $\mathcal{U} = \{U_i | i\in I\}$ be any open cover of X by quasi-compact open U_i.*

Then the augmentation maps are homotopy equivalences:

$$K^B(X) \xrightarrow{\sim} \check{H}^{\cdot}(\mathcal{U};K^B)$$
$$K^B(X \text{ on } Z) \to \check{H}^{\cdot}(\mathcal{U};K^B((\ ) \text{ on } (\ )\cap Z)).$$

Proof. If $\mathcal{U}$ is a finite cover, the augmentation maps are homotopy equivalences by 8.3. If $\mathcal{U} = \{U_i | i \in I\}$ is not finite, it has a finite subcover $\mathcal{V} = \{U_{\varphi(k)} | k = 1, \ldots, n\}$ as X is quasi-compact. There is an obvious map of covers $\mathcal{U} \to \mathcal{V}$. It now follows from Lemma 8.2.5, where hypothesis (8.2.7) is met by 8.3 for the finite cover $\mathcal{V}$, that the augmentation map for $\mathcal{U}$ is a homotopy equivalence.

8.5. *Exercise* (Optional). (a) The homotopy cofibre of $K \to K^B$ has non-zero homotopy groups π_q only for $q < 0$. Conclude from the spectral sequence (8.2.4) that $\pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; \mathrm{cofibre}(K \to K^B)) = 0$ for $n \geq 0$. Deduce that $K_n(X) \to \pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K)$ is an isomorphism for $n \geq 0$.

(b) Let X be quasi-compact and quasi-separated. Take an open cover $\mathcal{U}$ of X by affines. Then the intersections $U_I = U_{i_0} \cap \ldots \cap U_{i_n}$ of the U_i in $\mathcal{U}$ are quasi-affine, and hence the U_I have an ample family of line bundles. Thus $K^{\mathrm{naive}}(U_I) \simeq K(U_I)$. Conclude that for $n \geq 0$, $K_n(X)$ is isomorphic to $\pi_n \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^{\mathrm{naive}})$. Thus it is necessary to use K and not Quillen's K^{naive} to make the Mayer-Vietoris theorem work for any scheme X where these theories are not equivalent. (Recall that such a bad X cannot have an ample family of line bundles.)

8.6. *Exercise* (Optional). Let k be a field, and let n be an integer, $n \geq 2$. Let X be affine n-space with the origin doubled, the union of two copies of affine n-space $\mathbb{A}^n$ glued together on the open $\mathbb{A}^n - \{0\}$. This X is noetherian.

Using Poincaré duality 3.21, and Quillen's localization sequences for G-theory, $G(k) \to G(\mathbb{A}^n) \to G(\mathbb{A}^n - 0)$, $G(k) \to G(X) \to G(\mathbb{A}^n)$, and the homotopy equivalence $G(\mathbb{A}^n) \simeq G(k)$, show there are homotopy equivalences

$$K(X) \simeq G(X) \simeq G(\mathbb{A}^n) \times G(k) \simeq G(k) \times G(k) \simeq K(k) \times K(k).$$

On the other hand, show that the open immersion $\mathbb{A}^n \to X$ induces homotopy equivalences

$$K^{\mathrm{naive}}(X) \xrightarrow{\sim} K^{\mathrm{naive}}(\mathbb{A}^n) \simeq K(\mathbb{A}^n) \simeq K(k).$$

(Hint: $j : \mathbb{A}^n \to X$ induces an isomorphism of the categories of algebraic vector bundles. For j^* is fully faithful as $X - \mathbb{A}^n$ is a codimension ≥ 2 in X. Also any vector bundle on X consists of two vector bundles on the two copies of $\mathbb{A}^n$ together with a patching isomorphism on $\mathbb{A}^n - 0$. But as 0 has codimension ≥ 2 in $\mathbb{A}^n$, this isomorphism extends over $\mathbb{A}^n$ and the vector bundle on X is a pullback of a vector bundle on $\mathbb{A}^n$ via the canonical map identifying the two origins $X \to \mathbb{A}^n$. See [EGA] IV 5.10, 5.9).

Conclude that $K^{\mathrm{naive}}(X) \neq K(X)$. Also note that $G(X) = K(X)$, but $G(X) \neq K^{\mathrm{naive}}(X)$.

9. Reduction to the affine case, and the homotopy, closed Mayer-Vietoris, and invariance-under-infinitesimal -thickenings properties of K-theory with coefficients.

9.1. Suppose $F(X)$ and $F'(X)$ are homotopy limits of diagrams of $K^B(Z)$'s for a diagram of schemes Z over X natural with respect to base change. Suppose there is a natural map $F(X) \to F'(X)$ and one wishes to prove it is a homotopy equivalence. Then if this result is known for X affine, it will follow for X quasi-compact and quasi-separated. First one proves it for X quasi-compact and separated. Such an X has a finite open cover $\{U_i\}$ with all U_i, and hence all finite intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ being affine schemes. Then $F(U_{i_0} \cap \cdots \cap U_{i_n}) \to F'(U_{i_0} \cap \cdots \cap U_{i_n})$ is a homotopy equivalence, and it follows that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ is a homotopy equivalence. But as homotopy limits commute, $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is a homotopy limit of the $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B((\ )\underset{X}{\times} Z))$ for $K^B(Z)$'s in the diagram for $F(X)$. Hence by the Mayer-Vietoris Theorem 8.3, it follows that $F(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$ is a homotopy equivalence. Similarly $F'(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ is a homotopy equivalence, and it follows that $F(X) \to F'(X)$ is a homotopy equivalence for X is quasi-compact and and separated. In particular this is true for X quasi-affine. Now if X is quasi-compact and quasi-separated, it has a finite open cover $\{U_i\}$ by affines, and all the finite intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ are quasi-affine. Now arguing as above, we conclude that $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; F')$ and $F(X) \to F'(X)$ are homotopy equivalences as required. We note we could also use 8.1 and induction on the cardinality of $\mathcal{U}$ in place of using 8.3 above.

9.2. To apply this method of reduction to the affine case, we need to find some K-theory results known in the affine case. There are not too many, since lack of a localization theorem like 7.4 for X affine but U quasi-affine has hindered the development of K-theory of commutative rings. For example, for X the Spec of an integral domain A, and $\mathcal{U} = \{U_1, \ldots, U_n\}$ a cover by affine opens of the form $U_i = \mathrm{Spec}(A[1/a_i])$, the Mayer-Vietoris Theorem 8.3 was hitherto known for $n = 2$, but not for $n \geq 3$, as the proof of the latter requires consideration of the typically non-affine scheme $U_1 \cup \ldots \cup U_{n-1}$.

However, a few results are known in the affine case. These are: the Bass Fundamental Theorem, which we have already proved in general; Swan's theorem on quadric hypersurfaces over an affine [Sw]; Gabber's rigidity theorem for henselian pairs of rings [Gab]; and the work of Stienstra, Vorst, van der Kallen, Goodwillie, Ogle, and especially Weibel on the failure of $K(A) \to K(A[T])$ to be a homotopy equivalence for a general commutative ring A. This last failure is closely related to the failure of $K(A) \to K(A/I)$ to be a homotopy equivalence when I is a nil ideal, and to the failure of K to send fibre squares of rings to homotopy fibre squares of K-spectra. These failures can be remedied by passing to K-theory with appropriate coefficients. In this section we will extend these results to schemes.

9.3. We recall K-theory with coefficients first considered by Karoubi and Browder.

For $n \geq 2$ an integer, let $K^B/n(X)$ be the mod n reduction of the spectrum $K^B(X)$, that is, its smash product with a mod n Moore spectrum Σ^∞/n. It fits in a homotopy fibre sequence

$$K^B(X) \xrightarrow{n} K^B(X) \to K^B/n(X). \tag{9.3.1}$$

The long sequence of homotopy groups of (9.3.1) induces short exact universal coefficient sequences, (which are split if $n \geq 3$)

$$0 \to K_k^B(X) \otimes \mathbb{Z}/n \to K_k^B(X) \to \mathrm{Tor}^1_{\mathbb{Z}}\left(K_{k-1}^B(X); \mathbb{Z}/n\right) \to 0. \tag{9.3.2}$$

$K^B/n(X)$ is a product of the $K^B/\ell^\nu(X)$ for the prime powers ℓ^ν dividing n, so usually we consider only the $K^B/\ell^\nu(X)$. As reduction mod ℓ^ν preserves homotopy equivalences and homotopy fibre sequences, and commutes with $\check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ (all this is clear from the fibre sequence (9.3.1)), all results of Sections 7 and 8 immediately adapt to $K^B/\ell^\nu(X)$.

Similarly, for any multiplicative subset S of $\mathbb{Z}$, we form a spectrum $K^B(X) \otimes \mathbb{Z}_{(S)}$ by taking the colimit along the direct system of multiplication maps $n : K^B(X) \to K^B(X)$ for $n \in S$. Then clearly we have

$$\pi_k\left(K^B(X) \otimes \mathbb{Z}_{(S)}\right) \cong K_k^B(X) \otimes \mathbb{Z}_{(S)}. \tag{9.3.3}$$

9.4. For X a quasi-compact and quasi-separated scheme, we define the group $NK_n^B(X)$ as the kernel of the map induced by the 0-section $X \to (X[T])$ embedding X as $(T = 0)$ in $(X[T])$

$$NK_n^B(X) = \ker\left(K_n^B(X[T]) \to K_n^B(X)\right).$$

As the 0-section splits the projection $p : (X[T]) \to X$, the map $K^B_*(X[T]) \to K^B_*(X)$ is a naturally split epimorphism, and $NK^B_n(X)$ is naturally isomorphic to the cokernel of p^*.

For $X = \mathrm{Spec}\,(A)$, Stienstra, following work of Almkvist and Grayson, showed that $NK^B_n(A)$ was a module over the ring of Witt vectors of A ([We6] or [We2] for $n \geq 0$, hence for $n < 0$ by 6.7). As Weibel noted, it follows that if $1/\ell \in A$, then $NK^B_n(A)$ is a $\mathbb{Z}[1/\ell]$-module, and if $\ell^m = 0$ in A, $NK^B_n(A)$ consists of ℓ-torsion elements (as it is a "continuous" module over the ring of Witt vectors). Then considering the universal coefficient sequence (9.3.2) and (9.3.3) leads to the affine case of the following results.

9.5. **Theorem** (cf. Weibel, [We2], [We3], [We6]). *Let X be a quasi-compact and quasi-separated scheme. Let ℓ be a prime integer, and ℓ^ν a prime power. Suppose $1/\ell \in \mathcal{O}_X$. Then*

(a) *The projection $p : (X[T]) \to X$ induces a homotopy equivalence*

$$K^B/\ell^\nu(X) \xrightarrow{\sim} K^B\ell^\nu(X[T]).$$

(b) *More generally, for $\mathcal{S}_. = \bigoplus_{n\geq 0} \mathcal{S}_n$ any sheaf of positively graded commutative quasi-coherent $\mathcal{O}_X$-algebras, with $\mathcal{S}_0 = \mathcal{O}_X$, the projection induces a homotopy equivalence*

$$K^B\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu\,(\mathrm{Spec}_X\,(\mathcal{S}_.))\,.$$

(c) *If $p : W \to X$ is a torsor under a vector bundle, then p induces a homotopy equivalence*

$$p^* : K^B/\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu(W).$$

Proof. The method of reduction to the affine case 9.1 shows that it suffices to prove the maps are homotopy equivalences when X is affine. For X affine, (a) is [We3] 1.1, generalized from K/ℓ^ν by the trick of 6.7. Similarly (b) follows from [We6]. We may also deduce it from (a). We show the zero section $X \to \mathrm{Spec}(\mathcal{S}_.)$ induced by $\mathcal{S}_. \twoheadrightarrow \mathcal{S}_0 = \mathcal{O}_X$ induces on K^B/ℓ^ν a homotopy inverse to the map of (b). Consider the map $\mathcal{S}_. \to \mathcal{S}_.[T]$ of algebras sending an element $s \in \mathcal{S}_n$ to sT^n, and the induced map $\mathrm{Spec}(\mathcal{S}_.[T]) \to \mathrm{Spec}(\mathcal{S}_.)$. When composed with the section at $T = 1$, $\mathrm{Spec}(\mathcal{S}_.) \to \mathrm{Spec}(\mathcal{S}_.[T])$, this map yields the identity map of $\mathrm{Spec}(\mathcal{S}_.)$. When composed with the section at $T = 0$, this yields the composite $\mathrm{Spec}\,(\mathcal{S}_.) \to X \to \mathrm{Spec}(\mathcal{S}_.)$. But both sections yield homotopic maps $K^B/\ell^\nu(\mathrm{Spec}(\mathcal{S}_.[T])) \to K^B/\ell^\nu(\mathrm{Spec}(\mathcal{S}_.))$, as they are both inverse to the homotopy equivalence of (a) for $\mathrm{Spec}(\mathcal{S}_.)$ in place of X. Thus the composite map $K^B/\ell^\nu(\mathrm{Spec}(\mathcal{S}_.)) \to K^B/\ell^\nu(X) \to K^B/\ell^\nu(\mathrm{Spec}(\mathcal{S}_.))$ is homotopic to the identity. As the composite $K^B/\ell^\nu(X) \to K^B/\ell^\nu(\mathrm{Spec}(\mathcal{S}_.)) \to K^B/\ell^\nu(X)$ is also the identity, this proves (b).

To prove (c), we reduce to the case X is affine. Torsors under a vector bundle space $\mathbf{V}(\mathcal{E})$ on X are classified by the cohomology groups $H^1(X;\mathcal{E}^\vee)$, which is 0 as X is affine. Thus the torsor is trivial on X, and W is isomorphic to $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(S^\cdot(\mathcal{E}))$. Thus the affine case of (c) reduces to (b). (See [Gir], [Jo], [We1] for torsors.)

9.6. **Theorem** (cf. Weibel, [We2], [We3], [We6]). *Let X be a quasi-compact and quasi-separated scheme. Let ℓ be a prime integer, and ℓ^ν a prime power. Suppose ℓ is nilpotent in $\mathcal{O}_X$. Then*

(a) *The projection $p : (X[T]) \to X$ induces a homotopy equivalence*

$$K^B(X)\otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(X[T])\otimes \mathbb{Z}[1/\ell].$$

(b) *For $\mathcal{S}_.$ a sheaf of positively graded commutative quasi-coherent $\mathcal{O}_X$-algebras with $\mathcal{S}_0 = \mathcal{O}_X$, the projection induces a homotopy equivalence*

$$K^B(X)\otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(\mathrm{Spec}(\mathcal{S}_.))\otimes \mathbb{Z}[1/\ell].$$

(c) *If $p : W \to X$ is a torsor under a vector bundle, then p induces a homotopy equivalence*

$$p^* : K^B(X)\otimes \mathbb{Z}[1/\ell] \xrightarrow{\sim} K^B(W)\otimes \mathbb{Z}[1/\ell].$$

Proof. First we observe that for a finite open cover $\mathcal{U}$, $\check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ preserves direct colimits up to homotopy, as this is trivially true for $\check{\mathbb{H}}^p(\mathcal{U};\pi_p(\ ))$, and we have the strongly converging spectral sequence (8.2.4) with $\check{\mathbb{H}}^p(\mathcal{U};\ ) = 0$ for $p > N =$ number of open sets in the cover $\mathcal{U}$. In particular, $\check{\mathbb{H}}^\cdot(\mathcal{U};\ )$ commutes with formation of $\otimes\mathbb{Z}[1/\ell]$. Now the method of 9.1 goes through to reduce the problem to the case where X is affine. Then (a) follows from [We2] 5.2. As in 9.5 (a) implies (b) and (b) implies (c).

9.7. **Theorem** (cf. Weibel, [We2], [We3], [We1]). *Let X be a quasi-compact and quasi-separated scheme. Let $i : X' \to X$ be a closed immersion defined by a nil ideal of $\mathcal{O}_X$. Let ℓ be a prime integer, and ℓ^ν a prime power. Then*

(a) *If $1/\ell \in \mathcal{O}_X$, $i^* : K^B/\ell^\nu(X) \xrightarrow{\sim} K^B/\ell^\nu(X')$ is a homotopy equivalence.*

(b) *If ℓ is nilpotent in $\mathcal{O}_X$, $i^* : K^B(X)\otimes\mathbb{Z}[1/\ell] \to K^B(X')\otimes\mathbb{Z}[1/\ell]$ is a homotopy equivalence.*

Proof. The method of 9.1 reduces this to the case X, and hence X', is affine. The obvious direct colimit argument reduces us to the case where the nil ideal is finitely generated, and hence nilpotent. Then (a) follows by [We3] 1.4 and its proof. Similarly (b) follows by [We2] 5.4.

9.8. **Theorem** (Weibel, [We2], [We3], [We1]). *Let X be a quasi-compact and quasi-separated scheme. Let Y and Z be closed subschemes of X, such that $X = Y \cup Z$ as spaces. Let $Y \cap Z$ be the fibre product scheme of Y and Z over X. Let ℓ be a prime integer and ℓ^ν a prime power. Then*

(a) *If $1/\ell \in \mathcal{O}_X$, the square (9.8.1) is homotopy cartesian*

$$
\begin{tikzcd}
K^B/\ell^\nu(X) \arrow[r] \arrow[d] & K^B/\ell^\nu(Y) \arrow[d] \\
K^B/\ell^\nu(Z) \arrow[r] & K^B/\ell^\nu(Y \cap Z)
\end{tikzcd}
\tag{9.8.1}
$$

(b) *If ℓ is nilpotent in $\mathcal{O}_X$, the square (9.8.2) is homotopy cartesian*

$$
\begin{tikzcd}
K^B(X) \otimes \mathbb{Z}[1/\ell] \arrow[r] \arrow[d] & K^B(Y) \otimes \mathbb{Z}[1/\ell] \arrow[d] \\
K^B(Z) \otimes \mathbb{Z}[1/\ell] \arrow[r] & K^B(Y \cap Z) \otimes \mathbb{Z}[1/\ell]
\end{tikzcd}
\tag{9.8.2}
$$

Proof. The method of 9.1 reduces this to the case where $X = \operatorname{Spec}(A)$ is affine. Then Y and Z are affine, corresponding to ideals I and J of A. The hypothesis that $X = Y \cup Z$ says that any prime ideal of A either contains I or else contains J. Thus any prime of A contains $I \cap J$, so the ideal $I \cap J$ is nil. By 9.7, we may replace A by $A/I \cap J$ without changing the relevant K-theory spectrum, and so may assume that $I \cap J = 0$. Then $0 \to A \to A/I \oplus A/J \to A/I + J \to 0$ is exact, so A is the pullback of A/I and A/J over $A/I + J$. The result then follows by [We3] 1.3 and [We2] 5.5.

9.9. We remark that the integral K^B analogs of 9.5, 9.6, 9.7, 9.8 are false even for affine X unless one adds additional hypotheses. Indeed, $NK_*(A)$ may be non-zero (e.g., [We2] 4.4, 4.5), $K_1(\mathbb{Q}[x]/x^2) \to K_1(\mathbb{Q})$ is not an isomorphism as the unit $1 + x$ is in the kernel, and 9.8 fails as first shown by Swan's famous "counter-example to excision" (cf. [We3] 1.6).

9.10. *Exercise* (Optional). For X affine over $\mathbb{Q}$ and $i : X' \to X$ a closed immersion defined by a nil ideal, the fibre of $i^* : K^B(X') \to K^B(X)$ is homotopy equivalent to the fibre of i^* on cyclic cohomology HC^- by a theorem of Goodwillie [Goo]. As cyclic cohomology is calculable, this gives some control on the fibre of i^*.

Similarly, Ogle and Weibel have proved ([OW], after reducing to $I \cap J = 0$) that the double relative K^B measuring failure of 9.8 is homotopy equivalent to double relative HC^- for X affine over $\mathbb{Q}$.

Using the result of Brylinski, Loday, J. Bloch, *et. al.* that HC^- satisfies the Mayer-Vietoris property for open covers ([Lo] 3.4), generalize the results of Goodwillie, Ogle,and Weibel to quasi-compact and quasi-separated schemes over $\mathbb{Q}$. (This generalization is due to Weibel, who suggested this exercise.)

9.11. *Exercise* (Optional). For X a quasi-compact and quasi-separated scheme, define $K^B H(X)$ as the homotopy colimit of K^B applied to the product of X with the Gersten resolution of $\mathbb{Z}$ by polynomial rings

$$K^B H(X) = \underset{\Delta^{\mathrm{op}}}{\mathrm{hocolim}} \left(p \mapsto K^B \left(X\left[T_0, T_1, \dots, T_p\right]/T_0 + \cdots + T_p = 1\right)\right).$$

See [Wel] for details in the affine case.

(a) Show $K^B H(X) \xrightarrow{\sim} K^B H(X[T])$ is a homotopy equivalence, essentially by construction.

(b) Show for a finite cover $\mathcal{U}$ by open subsets, that $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$ commutes with homotopy colimits, not only with homotopy limits. For homotopy pullback squares of spectra are also homotopy pushout squares, and so commute with homotopy colimits. Now argue by induction on the number of open sets in the cover $\mathcal{U}$, as in the proof of 8.3.

(c) From (b), and 8.3, conclude that $K^B H(X) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{U}; K^B H)$ is a homotopy equivalence for $\mathcal{U}$ a finite open cover of X.

(d) Use the method of 9.1 to generalize the results of [Wel] for $K^B H$ of affines to quasi-compact and quasi-separated schemes. In particular show that:

(e) If $i : X' \to X$ is a closed immersion defined by a nil ideal, then $K^B H(X) \to K^B H(X')$ is a homotopy equivalence.

(f) If $X = Y \cup Z$ as in 9.8, then the following square is homotopy cartesian

$$\begin{array}{ccc} K^B H(X) & \longrightarrow & K^B H(Y) \\ \downarrow & & \downarrow \\ K^B H(Z) & \longrightarrow & K^B H(Y \cap Z) \end{array}$$

(g) If $1/\ell \in \mathcal{O}_X$, then $K^B/\ell^\nu(X) \to K^B H/\ell^\nu(X)$ is a homotopy equivalence.

(h) If ℓ is nilpotent in $\mathcal{O}_X$, then $K^B(X) \otimes \mathbb{Z}[1/\ell] \to K^B H(X) \otimes \mathbb{Z}[1/\ell]$ is a homotopy equivalence.

9.12. *Exercise* (Optional). The Čech cohomological descent results of [Vo] 1.7 and [vdK] 1.3 for NK of commutative rings are cluttered with stupid hypotheses that the rings are reduced, or that every non-zero-divisor is contained in a minimal prime. These hypotheses are required to justify appeal to [Vo] 1.4, which needs the hypothesis that f is a non-zero-divisor (or more generally, that $\exists g$, $fg = 0$, $f + g$ a non-zero-divisor) to be able to appeal to Quillen's Localization Theorem for projective modules [Gr1]. Use the Excision Theorem 7.1 and Localization Theorem 7.4 to remove this hypothesis on f in [Vo] 1.4, and hence to remove the stupid hypotheses from [Vo] and [vdK].

(a) Note $NK_*(A) \cong \operatorname{coker}(K_*(A) \to K_*(A[T]))$.

(b) Argue as in [Vo] 1.4, but using the coker formulation of NK_* in place of Vorst's $\ker(K_*(A[T]) \to K_*(A))$ formulation to show that the critical Vorst isomorphism $NK_*(A_f) \simeq NK_*(A)_{[f]}$ results from showing the canonical map is an isomorphism (9.12.1):

(9.12.1)
$$\operatorname{coker}(K_n(A) \to K_n(A + XA_f[X])) \xrightarrow{\cong} \operatorname{coker}(K_n(A_f) \to K_n(A_f[X])) .$$

(c) Deduce that (9.12.1) is an isomorphism for any $f \in A$ by applying 7.1 and 7.4 to the diagram

(9.12.2)
$$\begin{array}{ccc}
K^B(A \text{ on } (f=0)) & \xrightarrow{\sim} & K^B(A + XA_f[X] \text{ on } (f=0)) \\
\downarrow & & \downarrow \\
K^B(A) & \longrightarrow & K^B(A + XA_f[X]) \\
\downarrow & & \downarrow \\
K^B(A_f) & \longrightarrow & K^B(A_f[X])
\end{array}$$

9.13. *Exercise* (Optional). Let X be quasi-compact and quasi-separated.

(a) Let $\mathcal{E}$ be a vector bundle on X. The 0-section $i : X \to \mathbf{V}_X\mathcal{E}$ is a regular immersion. Using ideas of 2.7 as in 3.16.5, show there is a map $i_* : K^B(X) \to K^B(\mathbf{V}\mathcal{E} \text{ on } i(X))$.

(b) For $\mathcal{E} = \mathcal{O}_X$, $\mathbf{V}\mathcal{E} = X[T]$, consider the diagram (9.13.1)

(9.13.1)

$$\begin{tikzcd}
K^B\left(\mathbb{P}^1_X \text{ on } (T=0)\right) \arrow[r] \arrow[d, "\simeq"] & K^B\left(\mathbb{P}^1_X\right) \arrow[r] & K^B\left(X\left[T^{-1}\right]\right) \\
K^B(X[T] \text{ on } (T=0)) & & \\
K^B(X) \arrow[r] \arrow[u, "i_*"] & K^B(X) \times K^B(X) \arrow[r] \arrow[uu, "\simeq"] & K^B(X) \arrow[uu, "\pi^*"]
\end{tikzcd}$$

Note that the rows are homotopy fibre sequences, and conclude that there is a homotopy equivalence

$$\text{fibre}\left(\pi^* : K^B(X) \to K^B\left(X\left[T^{-1}\right]\right)\right) \simeq \text{cofibre}\left(i_* : K^B(X) \to K^B(X[T] \text{ on } (T=0))\right).$$

(c) Now suppose X has an ample family of line bundles. Use 5.7 and the ideas of [Gr1] to show $K(X[T] \text{ on } (T=0))$ is the K-theory of the exact category of vector bundles on X together with a nilpotent endomorphism. Thus the cofibre of i_* is a sort of Nil$K$, and in fact is the usual Nil$K$ in the case X is affine.

(d) Generalize the "Nil$K$" form of the Bass Fundamental Theorem from the affine case of [Gr1] to the case X has an ample family of line bundles.

9.14. It need not be true that $K^B/\ell^\nu_n(X) = 0$ for $n < 0$ if $1/\ell \in \mathcal{O}_X$. See [We4] 0.3 for $\ell = 2$, and X affine of finite type over $\mathbb{Q}$.

10. Brown-Gersten spectral sequences and descent

10.1. Let X be a scheme with underlying space a noetherian space of finite Krull dimension. For example, X should be a finite dimensional noetherian scheme.

10.2. A theorem of Grothendieck [Gro] 3.6.5 reveals that the Zariski cohomological dimension of such an X is at most its Krull dimension dim X, so $H^p_{\text{Zar}}(X;\ ) = 0$ for $p > \dim X$.

10.3. **Theorem.** *Let X be an in 10.1, and $Y \subsetneq X$ a closed subspace. Then the augmentation maps into Zariski hypercohomology spectra are homotopy equivalences (10.3.1):*

$$\begin{aligned} &K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; K^B(\ )\right) \\ &K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; K^B((\ ) \text{ on } (\ )\cap Y)\right). \end{aligned} \tag{10.3.1}$$

Thus there are stronlgy converging spectal sequences

$$\begin{aligned} &E_2^{p,q} = H^p_{\mathrm{Zar}}\left(X; \tilde{K}^B_q\right) \Longrightarrow K^B_{q-p}(X) \\ &E_2^{p,q} = H^p_{\mathrm{Zar}}\left(X; \tilde{K}^B_q((\ ) \text{ on } (\ )\cap Y)\right) \Longrightarrow K^B_{q-p}(X \text{ on } Y). \end{aligned} \tag{10.3.2}$$

(In (10.3.2), $\tilde{K}^B_q$ is the sheafification of the presheaf $\pi_q K^B(\ )$, and the spectral sequences have the Bousfield-Kan indexing with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.)

Proof. For Zariski hypercohomology $\mathbb{H}^{\cdot}_{\mathrm{Zar}}$, see [Th1] Section 1, and references there to earlier work of Brown.

The theorem follows from the Mayer-Vietoris property 8.1 by a result of Brown and Gersten ([BG] Thm 4). For the reader's convenience and to prepare for the generalization to the Nisnevich topology, we will give a complete proof following the sketch in [Th1] Exercise 2.5. For notational simplicity, we will give the proof for the absolute case $X = Y$. To prove the case with supports in Y, one just replaces all $K^B((\ ) \text{ on } (\ ) \cap Z)$ below with $K^B((\ ) \text{ on } (\ ) \cap Z \cap Y)$.

The spectral sequences (10.3.2) follow from (10.3.1) and 10.2 by [Th1] 1.36. It remains to prove (10.3.1).

Let $Z \subseteq X$ be a locally closed subspace, the intersection of a closed and open subspace of X. Then $Z = \bar{Z} \cap U$ for some open U and $\bar{Z}$ the closure of Z. We define $K^B(X \text{ on } Z)$ to be the direct colimit of $K^B(U \text{ on } Z)$ over the inverse system of such opens U with Z closed in U. Note by excision 7.1 that all $K^B(U \text{ on } Z)$ are homotopy equivalent, and hence all are homotopy equivalent to $K^B(X \text{ on } Z)$. (We need to take the colimit to avoid an arbitrary choice of U, so that $K^B(X \text{ on } Z)$ will be strictly functorial.)

If Z' and Z are locally closed in X with Z' closed in Z, the Localization Theorem 7.4 for $Z' \subseteq Z \subseteq U$, Z closed in U, shows that (10.3.3) is a homotopy fibre sequence

$$\begin{array}{ccccc} K^B(X \text{ on } Z') & \longrightarrow & K^B(X \text{ on } Z) & \longrightarrow & K^B(X \text{ on } Z - Z') \\ \wr\| & & \wr\| & & \wr\| \\ K^B(U \text{ on } Z') & \longrightarrow & K^B(U \text{ on } Z) & \longrightarrow & K^B(U - Z' \text{ on } Z - Z') \end{array} \tag{10.3.3}$$

This fibre sequence is natural for open immersions $V \to X$, $Z \cap V \to Z$, $Z' \cap V \to Z'$, and hence yields a homotopy fibre sequence of presheaves on X:

(10.3.4)
$$K^B((\ )\text{ on } Z'\cap(\ )) \to K^B((\ )\text{ on } Z\cap(\ )) \to K^B((\ )\text{ on } (Z-Z')\cap(\ )).$$

Now for $p \geq 0$ a non-negative integer, we define

$$S^pK^B((\ )\text{ on } Y\cap(\ )) = \varinjlim_{\bar{Z}'} K^B((\ )\text{ on } Y\cap\bar{Z}'\cap(\ )) \tag{10.3.5}$$

as the direct colimit over all closed $\bar{Z}'$ of codimension $\geq p$ in X. Note $S^pK^B = 0$ if $p > \dim X$.

We claim that the obvious map $S^{p+1}K^B \to S^pK^B$ induced by inclusion of direct systems is part of a homotopy fibre sequence (10.3.6), where the last term is a wedge over the points x of codimension p in X of skyscraper sheaves $i_*F(x)$ supported at x:

$$S^{p+1}K^B(\ ) \to S^pK^B(\ ) \to \bigvee_{\substack{i\in X \\ \text{codim } X = p}} i_xF(x). \tag{10.3.6}$$

In fact, the $F(x)$ are given by

$$F(x) = K^B(\operatorname{Spec}(\mathcal{O}_{X,x})\text{ on } x) \tag{10.3.7}$$

and $i_xF(x)(V) = F(x)$ if $x \in V$, and $= 0$ if $x \notin V$. (In the case with supports Y, $F(x) = K^B(\operatorname{Spec}(\mathcal{O}_{X,x})$ on $x \cap Y)$, so $F(x) = 0$ if $x \notin Y$.)

To begin the proof of the claim, we fix a codimension p closed subspace Z in X. We look at the fibre sequence of (10.3.4) for all $Z' \subseteq Z$ closed and of codimension $\geq p+1$ in X.

$$\begin{aligned}(10.3.8)\quad & K^B(V\text{ on } Z'\cap V) \to K^B(V\text{ on } Z\cap V) \\ & \to K^B(V\text{ on } (Z-Z')\cap V) \simeq K^B(V-Z'\text{ on } (Z-Z')\cap V).\end{aligned}$$

As the space of X is noetherian, Z contains finitely many points $z_1, z_2, \ldots, z_k$ of codimension p in X. The union of the $\bar{z}_i \cap \bar{z}_j$ over all distinct pairs $z_i \neq z_j$ is a closed subspace of codimension $\geq p+1$ in X, and we consider only the cofinal system of Z' that contains this union. Then $Z - Z'$ is a disjoint union of k components $\bar{z}_i - Z'$. Thus $Z - Z'$ has an open *nbd* $V_1 \cup V_2 \cup \cdots \cup V_k$ so that $V_i \cap (Z - Z') = (\bar{z}_i - Z')$. Then $V_I \cap (Z-Z') = \phi$ for $V_I = V_{i_0} \cap \ldots V_{i_n}$ if $I = \{i_0, \ldots, i_n\}$ contains two distinct indices i.

Adding $X - Z$ to each V_i, we may assume that $\{V_i\} = \mathcal{V}$ is a cover of $X - Z'$. As $K^B(V_I \text{ on } V_I \cap (Z - Z')) = K^B(V_I \text{ on } \phi) \simeq 0$ if I contains two distinct indices,the Mayer-Vietoris spectral sequence (8.3.2) collapses (even at the E^1 term if this is calculated by alternating cochains) yielding homotopy equivalences

$$\text{(10.3.9)}\quad K^B(V - Z' \text{ on } Z - Z') \simeq \bigvee_{i=1}^{k} K^B(V \cap V_i \text{ on } \bar{z}_i \cap V_i - Z') \simeq \bigvee_{i=1}^{k} K^B(V - Z' \text{ on } \bar{z}_i - Z').$$

(One may also prove this from 8.1 by induction on k). Substituting (10.3.9) into (10.3.8) and taking the direct colimit over Z' contained in our fixed Z and of codimension $\geq p+1$ in X, we obtain a homotopy fibre sequence

$$\text{(10.3.10)}\quad \begin{array}{c} S^{p+1}K^B(V \text{ on } Z \cap V) \to K^B(V \text{ on } Z) \\ \downarrow \\ \bigvee_{1}^{k} \varinjlim_{Z'} K^B(V - Z', \bar{z}_i \cap V - Z'). \end{array}$$

By excision 7.1, the right hand term of (10.3.10) is not changed up to homotopy if $V - Z'$ is replaced by an open $W \subseteq V - Z'$ such that $\bar{z}_i \cap W = \bar{z}_i \cap V - Z'$. As Z' runs over the codimension $\geq p+1$ subspaces of Z, the inverse limit of $\bar{z}_i - Z'$ is the point z_i, and the inverse limit of the various W's for the various Z''s is the spectrum of the local ring $\operatorname{Spec}(\mathcal{O}_{X,z_i})$. The subsystem of such W's which are affine is cofinal in the full system. Applying continuity 7.2 to this subsystem, we get a homotopy equivalence

$$\text{(10.3.11)}\quad \varinjlim_{Z'} K^B(V - Z' \text{ on } \bar{z}_i \cap V - Z') \simeq K^B(V \cap \operatorname{Spec}(\mathcal{O}_{X,z_i}) \text{ on } z_i \cap V) = i_{z_i}F(z_i)(V).$$

Now substituting (10.3.11) into (10.3.10), and then taking the direct colimit over all Z closed and of codimension $\geq p$ in X yields a homotopy fibre sequence which is (10.3.6), proving our claim. We now prove the theorem.

As $\vee i_x F(x)$ is a skyscraper sheaf, it is Zariski cohomologically trivial, and the augmentation map (10.3.12) is a homotopy equivalence

$$\bigvee_x F(x) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; \vee i_x F(x)). \tag{10.3.12}$$

(In more detail, $\pi_n({}_x\vee i_x F(x)) = \oplus i_x \pi_n F(x)$ is for all n a skyscraper sheaf of abelian groups, hence flasque, hence Zariski cohomologically acyclic. Then the hypercohomology spectral sequence [Th1] 1.36 collapses, yielding (10.3.12).)

Now we prove by descending induction on p that the augmentation is a homotopy equivalence

$$S^p K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^p K^B(\ )). \tag{10.3.13}$$

For $p > \dim X$, this is trivial as $S^p K^B \simeq 0$. To do the induction step, suppose the augmentation is known to be a homotopy equivalence for $S^{p+1}K^B$. We consider the diagram

$$\begin{tikzcd} S^{p+1}K^B(X) \arrow[r, "\sim"] \arrow[d] & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^{p+1}K^B(\ )) \arrow[d] \\ S^p K^B(X) \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^p K^B(\ )) \arrow[d] \\ \bigvee_x F(X) \arrow[r, "\sim"] & \mathbb{H}^{\cdot}_{\mathrm{Zar}}\left(X; \bigvee_x i_x F(x)\right) \end{tikzcd} \tag{10.3.14}$$

The columns are homotopy fibre sequences by (10.3.6) and the fact that hypercohomology $\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X;\ )$ preserves homotopy fibre sequences ([Th1] 1.35). The top horizontal arrow is a homotopy equivalence by induction hypothesis, and the bottom arrow is such by (10.3.12). Hence the middle arrow is a homotopy equivalence by the 5-lemma, completing the proof of the induction step. Hence (10.3.13) is a homotopy equivalence. For $p = 0$, $S^0 K^B = K^B$ so this yields the theorem.

10.4. *Remark.* The sheafification $\tilde{K}_q^B$ in the Zariski topology of the presheaf K_q^B has as stalk at a point $x \in X$ the K_q^B of the local ring $\mathcal{O}_{X,x}$ in X

$$\left(\tilde{K}_q^B\right)_x \cong K_q^B(\mathcal{O}_{X,x}). \tag{10.4.1}$$

This is immediate from continuity 7.2 (cf. [Th1] 1.44). Thus the spectral sequence (10.3.2) reduces problems in K-theory to the question of what happens at a local ring.

10.5. **Corollary.** *Let X have underlying space noetherian of finite Krull dimension, and let $Y \subseteq Z$ be a closed subspace. Then there is a natural homotopy equivalence*

$$K^B(X \text{ on } Y) \to \mathbb{H}_Y^{\cdot}(X; K^B(\ ))_{\mathrm{Zar}}. \tag{10.5.1}$$

Thus there is a strongly converging spectral sequence of cohomology with supports in Y

$$E_2^{p,q} = H_Y^p\left(X; \tilde{K}_q^B(\ )\right)_{\mathrm{Zar}} \Longrightarrow K_{q-p}^B(X \text{ on } Y). \tag{10.5.2}$$

Proof. (10.5.1) follows from the 5-lemma applied to diagram (10.5.3) where the indicated maps are homotopy equivalences by 10.3, and the rows are homotopy fibre sequences by localization 7.4, and the definition of $\mathbb{H}_Y^{\cdot}(X;\ )$ as the homotopy fibre of $\mathbb{H}^{\cdot}(X;\ ) \to \mathbb{H}^{\cdot}(X-Y;\ )$, (see Appendix D).

(10.5.3)
$$\begin{array}{ccccc}
K^B(X \text{ on } Y) & \longrightarrow & K^B(X) & \longrightarrow & K^B(X-Y) \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq \\
\mathbb{H}_Y^{\cdot}(X; K^B)_{\mathrm{Zar}} & \longrightarrow & \mathbb{H}_{\mathrm{Zar}}^{\cdot}(X; K^B) & \longrightarrow & \mathbb{H}_{\mathrm{Zar}}^{\cdot}(X-Y; K^B)
\end{array}$$

(The canonical choice of nullhomotopy of $K^B(X \text{ on } Y) \to K^B(X-Y)$ and the strictly commutative right half of (10.5.3) determine a strictly natural map of fibres $K^B(X \text{ on } Y) \to \mathbb{H}_Y^{\cdot}(X; K^B)$.)

The spectral sequence (10.5.2) follows from (10.5.1) and D.4.

10.6. *Remark.* The analog of 10.3 for G-theory of noetherian schemes, and hence for K-theory of regular noetherian schemes is due to Brown and Gersten [BG] using Quillen's Localization Theorem for G-theory. Quillen constructed another version of the G-theory spectral sequence [Q1]. Much work has been done to perturb Quillen's ideas to produce 10.3 in special cases where X has dimension ≤ 1, or where X has isolated or otherwise very mild singularities. See [Co], [PW], [L1], [L2], [We4], [We5], [Gi3].

Note that the result 10.3 is new even for X affine of dimension ≥ 2. One was not able to prove this affine case using Quillen's projective module Localization Theorem [Gr1], as the proof involves considering all open sets $U \subseteq X$, which will not in general be affine.

10.7. By modifying the above argument, we can prove $K^B(X \text{ on } Y)$ has cohomological descent for the Nisnevich topology on X. This topology is close to the Zariski topology, but has as its local rings the henselian local rings. Thus the descent theorem for this topology allows us to reduce problems in K-theory to the case of henselian local rings, bringing them within range of Gabber's form of the Rigidity Theorem, 11.6. This step will be essential in the comparison of algebraic and topological K-theory in Section 11.

The Nisnevich topology also plays an essential role in the work of Kato and Saito on higher-dimensional global class field theory [KS], where it is called the henselian topology, and originated in Nisnevich's work on algebraic group schemes [N1], where it is called the completely decomposed topology.

The basic references for the Nisnevich topology are [N2], [N3] Section 1, and [KS] 1.1-1.2.5. We collect the basic facts in Appendix E, which the logical reader will turn to before resuming 10.8.

10.8. **Theorem.** *Let X be a noetherian scheme of finite Krull dimension, and let $Y \subseteq X$ be a closed subscheme. Then the augmentation maps into Nisnevich hypercohomology are homotopy equivalences*

$$\begin{aligned} K^B(X) &\xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; K^B(\ )\right) \\ K^B(X \text{ on } Y) &\xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; K^B((\ ) \text{ on } (\ ) \cap Y)\right). \end{aligned} \tag{10.8.1}$$

Thus there are strongly converging spectral sequences

$$\begin{aligned} E_2^{p,q} = H^p_{\mathrm{Nis}}\left(X; \widetilde{K}_q^B\right) &\Longrightarrow K^B_{q-p}(X) \\ E_2^{p,q} = H^p_{\mathrm{Nis}}\left(X; \widetilde{K}_q^B((\ ) \text{ on } (\ ) \cap Y)\right) &\Longrightarrow K^B_{q-p}(X \text{ on } Y). \end{aligned} \tag{10.8.2}$$

Proof. The spectral sequences result from (10.8.1) via the standard hypercohomology spectral sequence [Th1] 1.36. The strong convergence holds as X has finite Nisnevich cohomological dimension by E.6(c). So it suffices to prove (10.8.1).

As pullback along flat, *a fortiori* along étale, maps preserves local codimension at each point ([EGA] 0_{IV} 14.2.3, 14.2.4, IV 6.1.4, IV 2.4.6), the constructions $S^pK^B((\ ) \text{ on } Y \times_X (\ ))$ of (10.3.5) extend to presheaves on the Nisnevich site. The localization fibre sequence (10.3.6) is natural for flat, hence for étale maps, and so induces a fibre sequence of presheaves on the Nisnevich site, whose value at $U \to X$ is the sequence

$$\begin{aligned} S^{p+1}K^B\left(U \text{ on } U \times_X Y\right) &\to S^pK^B\left(U \text{ on } U \times_X Y\right) \\ &\to \bigvee_{\substack{z \in U \\ \operatorname{codim} z = p}} K^B\left(\operatorname{Spec}(\mathcal{O}_{U,z}) \text{ on } z \times_X Y\right). \end{aligned} \tag{10.8.3}$$

The henselization $\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \to \mathrm{Spec}\,(\mathcal{O}_{U,z})$ is pro-étale, hence flat, and induces an isomorphism of residue fields $k(z) = k(z)$. Hence it is an isomorphism infinitely near z (2.6.2.2), and so by excision 7.1 it induces a homotopy equivalence

(10.8.4)
$$K^B\left(\mathrm{Spec}\,(\mathcal{O}_{U,z}) \text{ on } z \underset{X}{\times} Y\right) \xrightarrow{\sim} K^B\left(\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \text{ on } z \underset{X}{\times} Y\right).$$

For $x \in X$ of codimension p, let $F(x)$ be the presheaf of spectra on the Nisnevich site of the residue field $k(x)$, which associates to each étale $\mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k(x))$, the spectrum $K^B(\mathrm{Spec}\,(\mathcal{O}^h_{U,z}) \text{ on } z \underset{X}{\times} Y)$ where $\mathcal{O}^h_{U,z}$ is the henselization of the local ring $\mathcal{O}_{X,x}$ at the residue extension $k(x) \subseteq k(z) = k'$, that is, $\mathcal{O}^h_{U,z}$ is the stalk of the structure sheaf $\mathcal{O}_X$ at the point of the Nisnevich topos corresponding to $\mathrm{Spec}(k(x)) \to \mathrm{Spec}\,(k(x)) \to X$ (see E.4, E.5). For the closed immersion of x into X, $i:\ \mathrm{Spec}\,(k(x)) \to X$, let $i_\# F(x)$ be the induced presheaf of spectra on X_{Nis}, so $(i_\# F)(V) = F(i^{-1}(V))$. Then using E.5, we get isomorphisms

$$\text{(10.8.5)}\quad (i_\# F(x))\,(U) \cong F(x)\left(i^{-1}(U)\right) \cong \bigvee_{z \in U \underset{X}{\times} k(x)} F(x)(k(z))$$
$$\cong \bigvee_{z \in U \underset{X}{\times} k(x)} K^B\left(\mathcal{O}^h_{U,z} \text{ on } z \underset{X}{\times} Y\right).$$

Thus the fibration sequence (10.8.3) becomes a homotopy fibre sequence

$$\text{(10.8.6)}\quad S^{p+1}K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right) \to S^p K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right)$$
$$\to \bigvee_{\substack{x \in X \\ \mathrm{codim}\ x = p}} i_\# F(x).$$

Now by descending induction on p, we show that the augmentation (10.8.7) is a homotopy equivalence for $S^p K^B$

$$\text{(10.8.7)}\quad S^p K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{Nis}}\left(X; S^p K^B\left((\ ) \text{ on } (\ ) \underset{X}{\times} Y\right)\right).$$

For $p >$ Krull $\dim X$, both sides are 0, which starts the induction. To do the induction step, as in (10.3.14), by using the localization sequence (10.8.6) and the 5-lemma, we reduce to proving that the augmentation is a homotopy equivalence for $\bigvee i_\# F(x)$:

$$(10.8.8) \qquad \bigvee_{\substack{x \\ \operatorname{codim} x = p}} F(x)(k(x)) \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; \bigvee_{\substack{x \\ \operatorname{codim} x = p}} i_{\#}F(x)\right).$$

To show this, we consider the sheafification of the presheaf of homotopy groups $\tilde{\pi}_n(\vee i_{\#}F(x)) \cong \oplus\tilde{\pi}_n i_{\#}F(x) \cong \oplus i_*\tilde{\pi}_n F(x)$. In the Nisnevich topos of $k(x)$, the only covering sieve of a field k' is the trivial sieve of all objects over $\operatorname{Spec}(k')$, by E.4. Thus every sheaf in the Nisnevich site of $k(x)$ is acyclic for the topos in the sense of [SGA 4] VI 4.1, because the $\operatorname{Spec}(k')$ have the cohomology of points, and every object in the site is a disjoint union of such $\operatorname{Spec}(k')$. It follows that $i_*\tilde{\pi}_n F(x) = \tilde{\pi}_n i_{\#}F(x)$ for $i: \operatorname{Spec}(k(x)) \to X$ is acyclic for the Nisnevich site of X ([SGA 4] V 4.9). As $H^*_{\mathrm{Nis}}(X; \ )$ preserves direct colimits (E.6) and finite sums, it also preserves infinite sums. Hence we get $\oplus i_*\tilde{\pi}_n F(x)$ is acyclic, so

$$(10.8.9) \qquad \begin{aligned} H^0_{\mathrm{Nis}}\left(X; \tilde{\pi}_n\left(\bigvee_x i_{\#}F(x)\right)\right) &= \oplus\pi_n F(x)(k(x)) \\ H^p_{\mathrm{Nis}}\left(X; \tilde{\pi}_n\left(\bigvee_x i_{\#}F(x)\right)\right) &= 0 \quad \text{for } p > 0. \end{aligned}$$

Plugging this into the hypercohomology spectral sequence of [Th1] 1.36 for the right hand side of (10.8.8) yields that (10.8.8) is a homotopy equivalence. This completes the induction step to prove (10.8.7). For $p = 0$, (10.8.7) yields the theorem.

10.9. *Remark.* By continuity 7.2, and the description of stalks in the Nisnevich topology E.5, the stalks of the sheaves $\widetilde{K}^B_q((\ ) \text{ on } (\ ) \underset{X}{\times} Y)$ at the point corresponding to $\operatorname{Spec}(k') \to \operatorname{Spec}(k(x)) \to X$ are the $K^B_q(\mathcal{O}^h_{U,z} \text{ on } z \underset{X}{\times} Y)$, where $\mathcal{O}^h_{U,z}$ is the henselization of the local ring $\mathcal{O}_{X,x}$ at an étale residue field extension $k(x) \to k' = k(z)$, or equivalently the usual henselization $\mathcal{O}^h_{U,z}$ of an appropriate U étale over X at a point z over x with $k(z) = k'$ over $k(x)$.

Thus the spectral sequence (10.8.2) reduces problems in the K-theory of X to the case of hensel local rings of schemes étale over X.

10.10. **Corollary.** *Under the hypotheses of 10.9, there is a natural homotopy equivalence*

$$(10.10.1) \qquad K^B(X \text{ on } Y) \xrightarrow{\sim} \mathbf{H}^{\cdot}_Y\left(X; K^B(\ )\right)_{\mathrm{Nis}}.$$

Thus, there is a strongly converging spectral sequence

(10.10.2) $$E_2^{p,q} = H^p_Y\left(X; \tilde{K}^B_q(\ )\right)_{\mathrm{Nis}} \Longrightarrow K^B_{q-p}(X \text{ on } Y).$$

Proof. This follows from 10.8 as 10.5 follows from 10.3.

10.11. *Remark.* The analog of 10.8 for G-theory of noetherian schemes, and hence for K-theory of regular neotherian schemes is due to Nisnevich, circa 1983, although the paper has just now appeared as a preprint [N3]. This paper also has a proof that under certain hypotheses on F, that $F(X) \simeq H^{\cdot}_{\mathrm{Nis}}(X; F)$, which hypotheses are met by K^B thanks to 7.1 and 7.4.

11. Étale cohomological descent and comparison with topological K-theory

11.1. *Hypotheses.*

11.1.0. Let X be a noetherian scheme of finite Krull dimension. Let ℓ be a prime integer, and ℓ^ν a prime power.

Suppose $1/\ell \in \mathcal{O}_X$. If $\ell = 2$, also suppose $\sqrt{-1} \in \mathcal{O}_X$.

11.1.1. Suppose there is a uniform bound on the étale cohomological dimension with respect to ℓ-torsion coefficient sheaves, of all the residue fields $k(x)$ of X.

11.1.2. Suppose further that the extension of each residue field $k(x)$ to its separable closure has a Tate-Tsen filtration by subextensions of cohomological dimension 1, [Th1] 2.112.

11.2. *Remark.* If k is a field with $1/\ell \in k$ (and $\sqrt{-1} \in k$ if $\ell = 2$), and k is of finite transcendence degree over a separably closed field, or over a global field (e.g., over $\mathbb{Q}$), or over a local field (e.g., over $\mathbb{F}_p$, $\mathbb{F}_q((t))$, $\mathbb{Q}_p^\wedge$); then k has finite étale cohomological dimension for ℓ-torsion sheaves (in fact, it is $\leq 2 +$ the transcendence degree of k), and has a Tate-Tsen filtration. This follows from [SGA 4] X, (cf. [Th1] 2.44).

11.3. *Remark.* If X is a scheme with $1/\ell \in \mathcal{O}_X$, (and with $\sqrt{-1} \in \mathcal{O}_X$ if $\ell = 2$), and if X is of finite type over any of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{F}_q((t))$, $\mathbb{Q}_p^\wedge$, $\mathbb{Z}_p^\wedge$, or over a separably or algebraically closed field, then X satisfies all the hypotheses of 11.1. This results from [SGA 4] X and various obvious facts.

In particular, 11.1.1 and 11.1.2 follows from 11.1.0 in the cases of interest to number theory or classical algebraic geometry.

11.4. Let $K/\ell^\nu(X)[\beta^{-1}]$ be the localization of the ring spectrum $K/\ell^\nu\,(X)$ by inverting the Bott element as in [Th1] Appendix A. The discussion there extends immediately to our $K(X)$ from Quillen's $K^{\text{naive}}(X)$. (We note that for $\ell = 2$ or 3, there are technical complications in putting a ring spectrum structure on mod ℓ reductions, see [Th1] A for details of what to do in this case.)

As in [Th1] A, $K/\ell^\nu(X)[\beta^{-1}]$ is homotopy equivalent to the mod ℓ^ν reduction of the Bousfield K-localization of $K(X)$, $K/\ell^\nu(X)_K$.

We have by [Th1] A that for a suitable integer N depending on X and ℓ^ν ($N = 1$ for X over a separably closed field), there is a Bott element β^N in $K/\ell^\nu_{2N}(X)$, and so an isomorphism

(11.4.1)

$$K/\ell^\nu_n(X)[\beta^{-1}] \cong \varinjlim_k \left(\cdots \to K/\ell^\nu_{n+k(2N)}(X) \xrightarrow{\cup\beta^N} K/\ell^\nu_{n+(k+1)(2N)}(X) \to \cdots \right).$$

Similarly for $K^B/\ell^\nu(X)[\beta^{-1}]$, the localization of the module spectrum over $K/\ell^\nu(X)$. As $n + k(2N)$ becomes positive as k increases, and as $K/\ell^\nu_q(X) \cong K^B/\ell^\nu_q(X)$ for $q \geq 1$ (by 6.6(a) and the universal coefficient sequences like (9.3.2)), the canonical map $K(X) \to K^B(X)$ induces a homotopy equivalence

$$\begin{array}{ccc} K/\ell^\nu(X)\,[\beta^{-1}] & \xrightarrow{\sim} & K^B/\ell^\nu(X)\,[\beta^{-1}] \\ \wr| & & \wr| \\ K/\ell^\nu(X)_K & & K^B/\ell^\nu(X)_K \end{array} \tag{11.4.2}$$

We also note that localization $(\ )[\beta^{-1}]$ and $(\ )_K = (\ ) \wedge \Sigma_K$ of spectra preserve homotopy equivalences, direct colimits, and homotopy fibre sequences.

11.5. **Theorem.** *Under the hypotheses of 11.1, the augmentation is a homotopy equivalence into étale hypercohomology*

$$K/\ell^\nu(X)\,[\beta^{-1}] \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\text{et}}\left(X; K/\ell^\nu(\ )\,[\beta^{-1}]\right). \tag{11.5.1}$$

Moreover, the sheaf of homotopy groups in the étale topology is given by

$$\tilde{\pi}_q K/\ell^\nu(\ )\,[\beta^{-1}] \cong \begin{cases} \mathbb{Z}/\ell^\nu(i) \cong \mu_{\ell^\nu}^{\otimes i} & q = 2i,\ i \in \mathbb{Z} \\ 0 & q \text{ odd} \end{cases} \tag{11.5.2}$$

where $\mathbb{Z}/\ell^\nu(i)$ is generated locally by β^i. Hence there is a spectral sequence which converges strongly to $K/\ell^\nu(X)[\beta^{-1}]$ from étale cohomology

(11.5.3) $$E_2^{p,q} = \begin{Bmatrix} H^p_{\mathrm{et}}(X; \mathbb{Z}/\ell^\nu(i)) & q = 2i \\ 0 & q \text{ odd} \end{Bmatrix} \Longrightarrow K/\ell^\nu_{q-p}(X)\,[\beta^{-1}]\,.$$

(The spectral sequence has Bousfield-Kan indexing, so $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.)

Proof. As the hypotheses 11.1 imply that X has finite étale cohomological dimension for ℓ-torsion sheaves ([SGA 4] X), the strongly converging spectral sequence (11.5.3) results from (11.5.2) and (11.5.1) via the canonical hypercohomology spectral sequence [Th1] 1.36.

The proof of (11.5.1) and (11.5.2) will spread over 11.6 - 11.8. Both depend on Gabber's form of the Gabber-Gillet-Thomason Rigidity Theorem.

11.6. **Theorem** (Gabber [Gab]). *Let A be a commutative ring with $1/\ell \in A$, and let I be an ideal of A such that $(Spec(A),\ Spec(A/I))$ is a henselian pair ([EGA] IV 18.5.5). Then the map $A \to A/I$ induces a homotopy equivalence*

(11.6.1) $$K/\ell^\nu(A) \xrightarrow{\sim} K/\ell^\nu(A/I).$$

In particular, if $\mathcal{O}^h_{U,z}$ is a hensel local ring with residue field $k(z)$, and if $1/\ell \in \mathcal{O}^h_{U,z}$, then $\mathcal{O}^h_{U,z} \to k(z)$ induces a homotopy equivalence

(11.6.2) $$K/\ell^\nu\left(\mathcal{O}^h_{U,z}\right) \xrightarrow{\sim} K/\ell^\nu(k(z)).$$

Proof. This is Theorem 1 of [Gab]. Gabber's proof extends ideas of Suslin who proved the special case of A an algebra over a field in [Su2] 2.1 from an even more special case due to Gabber and to Gillet and Thomason [GT], which in turn was inspired by a theorem of Suslin [Su1].

11.7. To prove (11.5.2), we note that the inclusion of the subgroup generated by β^i induces a map of sheaves $\mathbb{Z}/\ell^\nu(i) \to \tilde{K}/\ell^\nu_{2i}(\ )[\beta^{-1}]$. It suffices to show that this map is an isomorphism and that $0 \to \tilde{K}/\ell^\nu_q(\ )[\beta^{-1}]$ is an isomorphism for q odd. For this, it suffices to show the maps are isomorphisms on the stalks of sheaves for each point in the étale topology. The stalk of $\tilde{K}/\ell^\nu_*(\ )[\beta^{-1}]$ at a point x is $K/\ell^\nu_*(\mathcal{O}^{\mathrm{sh}}_{X,x})$ for $\mathcal{O}^{\mathrm{sh}}_{X,x}$ the strict local henselization of X at x (e.g., [Th1] 1.29, 1.43). This $\mathcal{O}^{\mathrm{sh}}_{X,z}$ is a hensel local ring whose residue field $\overline{k(x)}$ is separably closed. So by Gabber rigidity (11.6), $K/\ell^\nu(\mathcal{O}^{\mathrm{sh}}_{X,z})[\beta^{-1}] \xrightarrow{\sim} K/\ell^\nu(\overline{k(x)})[\beta^{-1}]$ is a homotopy equivalence. Thus it suffices to show that (11.5.2) gives the values of $K/\ell^\nu_q(\overline{k(x)})[\beta^{-1}]$ for $\overline{k(x)}$ a separably closed field. But this is true by [Th1] 3.1 or by a trivial extension of the results of Suslin [Su1], [Su2] 3.13 from the algebraically closed to the separably closed case. This proves (11.5.2).

11.7.1. *Remark.* Suslin's result shows that without inverting β, $\tilde{K}/\ell^\nu_q$ is 0 for $q \leq 0$ or for q odd, and is $\mathbf{Z}/\ell^\nu(i)$ for $q = 2i \geq 0$. Thus inverting β does not change $\pi_n \mathbf{H}^{\cdot}_{\mathrm{et}}(X; K/\ell^\nu(\ ))$ for $n \geq 0$. The inversion of β is necessary to make this isomorphic to $K/\ell^\nu_n(X)[\beta^{-1}]$, and inverting β makes the minimum possible change to $K/\ell^\nu_n(X)$ to create this isomorphism. However, this minimum change is not zero in general, as one sees by the examples $K/\ell^\nu_0(X) \neq K/\ell^\nu_0(X)[\beta^{-1}]$ for X a $K3$ surface, or for $X = \operatorname{Spec}(R[1/\ell])$ for R any ring of integers which has at least two distinct primes over ℓ, ([Th1] 4.5).

11.8. It remains to prove (11.5.1) is a homotopy equivalence. The augmentation map (11.5.1) is strictly natural in X, and hence induces a map of presheaves on the Nisnevich site of X. Hence we have a commutative diagram

(11.8.1)
$$\begin{tikzcd}
K/\ell^\nu(X)\,[\beta^{-1}] \arrow[r] \arrow[d, "\simeq"'] & \mathbf{H}^{\cdot}_{\mathrm{et}}\left(X; K/\ell^\nu(\ )\,[\beta^{-1}]\right) \arrow[d, "\simeq"] \\
\mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; K/\ell^\nu(\ )\,[\beta^{-1}]\right) \arrow[r] & \mathbf{H}^{\cdot}_{\mathrm{Nis}}\left(X; \mathbf{H}^{\cdot}_{\mathrm{et}}\left((\ ); K/\ell^\nu(\ )\,[\beta^{-1}]\right)\right)
\end{tikzcd}$$

The left vertical map of (11.8.1) is a homotopy equivalence by Nisnevich cohomological descent 10.8 (using also the facts that $\mathbf{H}^{\cdot}_{\mathrm{Nis}}(X;\ )$ preserves homotopy fibre sequences as in reduction mod ℓ^ν, and direct colimits as in forming the localization $(\ )[\beta^{-1}]$, E.6(d)). The right hand vertical map is a homotopy equivalence because the étale topology is finer than the Nisnevich topology, and hence étale cohomology cohomologically descends for the Nisnevich topology. (In more detail: Apply the Cartan-Leray Theorem [Th1] 1.56 for the map of sites $f : X_{\mathrm{et}} \to X_{\mathrm{Nis}}$, considering [Th1] 1.55. The cohomological dimension hypotheses of [Th1] 1.56 hold by the proof of [Th1] 1.48.)

As the two vertical maps in (11.8.1) are homotopy equivalences, the top horizontal map will be a homotopy equivalence iff the bottom horizontal map is so. To show the latter is, it suffices by the strongly converging hypercohomology spectral sequence [Th1] 1.36 to show that $\tilde{K}/\ell^\nu_q(\ )[\beta^{-1}] \to \tilde{\pi}_n \mathbf{H}^{\cdot}_{\mathrm{et}}(\ ; K/\ell^\nu(\ )[\beta^{-1}])$ is an isomorphism of sheaves in the Nisnevich topos of X. For this, it suffices that it is an isomorphism on stalks. From the description of stalks in E.5, and the continuity of K-theory and étale cohomology (7.1, [Th1] 1.43, 1.45), this in turn reduces to showing that the augmentation map (11.5.1) is a homotopy equivalence whenever X is replaced by $\mathrm{Spec}(\mathcal{O}^h_{U,z})$, the henselization at a point z of a scheme U étale over X.

To prove this reduced statement, we consider diagram (11.8.2)

$$\begin{array}{ccc} K/\ell^\nu\,(\mathcal{O}^h_{U,z})\,[[\beta^{-1}] & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{et}}\left(\mathcal{O}^h_{U,z}; K/\ell^\nu(\ )\,[\beta^{-1}]\right) \\ \simeq\Big\downarrow & & \simeq\Big\downarrow \\ K/\ell^\nu\,(k(z))\,[\beta^{-1}] & \longrightarrow & \mathbf{H}^{\cdot}_{\mathrm{et}}\left(k(z); K/\ell^\nu(\ )\,[\beta^{-1}]\right) \end{array} \tag{11.8.2}$$

The vertical maps are induced by the map of a hensel local ring $\mathcal{O}^h_{U,z}$ to its residue field $k(z)$. The left vertical map is a homotopy equivalence by Gabber 11.6. The right vertical map we see to be a homotopy equivalence by combining the formula (11.5.2) for the coefficients, the strongly converging hypercohomology spectral sequence [Th1] 1.36, and the isomorphism $H^*_{\mathrm{et}}(\mathcal{O}^h_{U,z}; \mathbb{Z}/\ell^\nu(i)) \cong H^*_{\mathrm{et}}(k(z); \mathbb{Z}/\ell^\nu(i))$ for the hensel local ring as provided by [SGA 4] VIII 8.6.

By diagram (11.8.2) we further reduce to proving that the augmentation is a homotopy equivalence in the special case of a field, $k(z)$. This is hard, but was done in [Th1] 2.43. This quote completes the proof of the theorem.

11.9. **Corollary**. *Under the hypotheses of* 11.1, *the Dwyer-Friedlander map induces a homotopy equivalence from* $K/\ell^\nu(X)[\beta^{-1}]$ *to the étale topological* K*-theory of* X *([DF])*

$$\rho : K/\ell^\nu(X)\,[\beta^{-1}] \xrightarrow{\sim} K/\ell^{\nu\,\mathrm{top}}(X). \tag{11.9.1}$$

Proof. This follows from 11.5 by the method of [Th1] 4.11, 4.12. The naive idea is that the spectral sequence (11.5.3) has the same E_2 term as the Atiyah-Hirzebruch spectral sequence of [DF].

11.10. **Proposition**. *For* X *a noetherian scheme of finite Krull dimension, the augmentation map is a homotopy equivalence:*

$$K^B(X) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{et}}\left(X; K^B(\ ) \otimes \mathbb{Q}\right). \tag{11.10.1}$$

Proof. Note that $K^B(X) \otimes \mathbb{Q}$ is a direct colimit of $K^B(X)$ along a system of multiplication by intergers, so E.6(d) shows that $\mathbf{H}^{\cdot}_{\mathrm{Nis}}(X;\ )$ and $(\ ) \otimes \mathbb{Q}$ commute. We also note that all schemes étale over X have bounded étale cohomological dimension for $\mathbb{Q}$-sheaves, by the methods of [SGA 4] X (cf. [Th1], proof of 1.4.8).

Then by Nisnevich cohomological descent for the two sides of (11.10.1), as in (11.8.1), it suffices to prove that the map is a homotopy equivalence in the special case where $X = \mathrm{Spec}(R_x)$ is a hensel local ring with residue field $k(x)$.

Let k_α run over the finite Galois extensions of $k(x)$, and let G_α be the Galois group $\mathrm{Gal}(k_\alpha/k)$. As X is hensel local, to each k_α there is a corresponding finite étale covering of hensel local rings, $f_\alpha : X_\alpha \to X$, inducing $\mathrm{Spec}(k_\alpha) \to \mathrm{Spec}(k)$ over the closed point of X ([EGA] IV 18.5.15). As f_α is flat and finite, it induces a transfer map $f_* : K^B(X_\alpha) \to K^B(X)$, e.g., by 3.16.6 and 6.5. By 3.17 and 6.5, the composite $f_*f^* : K^B(X) \to K^B(X)$ is multiplication by $[f_*f^*\mathcal{O}_X] = [f_*\mathcal{O}_{X_\alpha}]$ in $K_0(X)$. As X is local, $f_*\mathcal{O}_{X_\alpha}$ is free of rank equal to the degree $[k_\alpha : k]$ of the extension. So f_*f^* is multiplication by the integer $[k_\alpha : k]$ and $f_*f^* \otimes \mathbb{Q}$ is a homotopy equivalence. By 3.18 and 6.5, we see that the other composite $f^*f_* : K^B(X_\alpha) \to K^B(X_\alpha)$ is induced by the functor

$$\mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} (\ ).$$

Galois theory gives an isomorphism

$$\kappa : \mathcal{O}_{X_\alpha} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \xrightarrow{\cong} \prod_{G_\alpha} \mathcal{O}_{X_\alpha} \tag{11.10.2}$$

where $\kappa(x \otimes y)$ has component $x \cdot gy$ in the factor indexed by $g \in G_\alpha$. Indeed, κ corresponds to the action map $G_\alpha \times X_\alpha \to X_\alpha \underset{X}{\times} X_\alpha$ sending $(g, x) \mapsto (x, gx)$, which is an isomorphism as $X_\alpha \to X$ is a Galois covering, hence a torsor under $G_\alpha = \mathrm{Gal}(X_\alpha/X) = \mathrm{Gal}(k_\alpha/k)$ ([EGA] IV 18.5.15, [SGA 1] V). This isomorphism κ shows that $f^*f_* = \Sigma g^*$ equals the sum of the g^* for $g : X_\alpha \to X_\alpha$ in the Galois group (cf. [Th1] 1.50, 2.12 - 2.13). It follows by a standard transfer argument from $f^*f_* = \Sigma g^*$ and $f_*f^* = [k_\alpha : k] =$ order of G_α that the augmentation induces an isomorphism (cf. [Th1] 2.14)

(11.10.3)
$$K_n^B(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^0(G_\alpha;\ K_n^B(X_\alpha) \otimes \mathbb{Q}) \cong \varinjlim_\alpha H^0(G_\alpha; K_n^B(X_\alpha) \otimes \mathbb{Q})\,.$$

As G_α is a finite group, its rational cohomology is trivial by a standard transfer argument, so

$$0 \cong \varinjlim_\alpha H^p(G_\alpha;\ K_*^B(X_\alpha) \otimes \mathbb{Q}) \quad \text{for } p \geq 1. \tag{11.10.4}$$

But by ([SGA 4] VIII 8.6, 2.3), for any sheaf of abelian groups $\tilde{K}_*(\ )$ on the étale site of the hensel local ring X, there is a canonical isomorphism

$$\varinjlim_{\alpha} H^p\left(G_\alpha;\ \tilde{K}_*(X_\alpha)\right) \cong H^p_{\text{et}}\left(X;\ \tilde{K}_*\right). \tag{11.10.5}$$

Combining (11.10.5), (11.10.4), and (11.10.3), we see that the hypercohomology spectral sequence [Th1] 1.36 collapses to yield an isomorphism for X hensel local and all integers n

$$K^B_n(X) \otimes \mathbb{Q} \cong \pi_n \mathbb{H}^{\cdot}_{\text{et}}\left(X;\ K^B(\ ) \otimes \mathbb{Q}\right). \tag{11.10.6}$$

This proves that (11.10.1) is a homotopy equivalence for X hensel local. But by our previous reductions, this proves the theorem.

11.11. **Theorem.** *Let X be a noetherian scheme of finite Krull dimension. Let S be a set of prime integers such that the hypotheses of 11.1 hold for every $\ell \in S$. Let $K^B(\ )_K$ denote the Bousfield K-localization of K^B,* [Th1] A*), and let $K^B(\ )_K \otimes \mathbb{Z}_{(S)}$ be the further localization by inverting all primes not in S.*

Then the augmentation is a homotopy equivalence

$$K^B(X)_K \otimes \mathbb{Z}_{(S)} \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\text{et}}\left(X;\ K^B(\ )_K \otimes \mathbb{Z}_{(S)}\right). \tag{11.11.1}$$

Proof. This follows from 11.10 and 11.5 for the various ℓ in S. Indeed the homotopy fibre of $K^B(X)_K \to \mathbb{H}^{\cdot}_{\text{et}}(X;\ K^B(\ )_K)$ becomes trivial upon forming $\otimes\mathbb{Q}$ by 11.10, and so its homotopy groups are torsion. By 11.5, the mod ℓ^ν reductions of the fibre are trivial for ℓ in S, and hence the homotopy groups are uniquely ℓ-divisible torsion groups for ℓ in S (9.3.2). Thus the torsion groups have no ℓ-torsion for ℓ in S, and $\otimes\mathbb{Z}_{(S)}$ of them are zero. Thus $\otimes\mathbb{Z}_{(S)}$ of the homotopy fibre is homotopy trivial, so (11.11.1) is a homotopy equivalence.

11.12. The analogs of 11.5, 11.10, 11.11 for G-theory appeared in [Th1].

Appendix A

Exact categories and the Gabriel-Quillen embedding

A1. We recall Quillen's definition of an exact category [Q1] Section 2. An exact category $\mathcal{E}$ is an additive category together with a choice of a class of sequences $\{E_1 \rightarrowtail E_2 \twoheadrightarrow E_3\}$ said to be exact. This determines two classes of morphisms: the admissible epimorphisms $E_2 \twoheadrightarrow E_3$ and the admissible monommorphisms $E_1 \rightarrowtail E_2$. The exact category is to satisfy the following axioms: The class of admissible monomorphisms is closed under composition and is closed under cobase change by pushout along an arbitrary map $E_1 \to E_1'$ (cf. 1.2.1.3). Dually, the class of admissible epimorphisms is closed under composition and under base change by pullback along an arbitrary map $E_3' \to E_3$. Any sequence isomorphic to an exact sequence is exact, and any "split" sequence

$$E \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} E \oplus F \xrightarrow{[0,1]} F$$

is to be exact. In any exact sequence $E_1 \rightarrowtail E_2 \twoheadrightarrow E_3$, the map $E_1 \rightarrowtail E_2$ is a kernel for $E_2 \twoheadrightarrow E_3$, and $E_2 \twoheadrightarrow E_3$ is a cokernel for $E_1 \rightarrowtail E_2$. Finally, there is the obscure axiom:

A.1.1. Let $i : E \to F$ be a map in $\mathcal{E}$ which has a cokernel in $\mathcal{E}$. If there exists a map $k : F \to G$ such that $ki : E \rightarrowtail G$ is an admissible monomorphism, then $i : E \rightarrowtail F$ is itself an admissible monomorphism.

Dually if $i : F \to E$ has a kernel in $\mathcal{E}$, and if there exists a $k : G \to F$ such that $ik : G \twoheadrightarrow \mathrm{E}$ is an admissible epimorphism, then $i : F \twoheadrightarrow E$ is an admissible epimorphism.

A.2. The concept of exact category is self-dual, so $\mathcal{E}$ is exact iff the opposite category $\mathcal{E}^{\mathrm{op}}$ is exact, where $E_1 \to E_2 \to E_3$ is exact in $\mathcal{E}$ iff $E_1 \leftarrow E_2 \leftarrow E_3$ is exact in $\mathcal{E}^{\mathrm{op}}$.

A.3. An exact functor $f : \mathcal{E}_1 \to \mathcal{E}_2$ is one that sends exact sequences in $\mathcal{E}_1$ to exact sequence in $\mathcal{E}_2$. An exact functor *reflects* exactness if whenever f of a sequence in $\mathcal{E}_1$ is exact in $\mathcal{E}_2$, the original sequence is exact in $\mathcal{E}_1$.

If $\boldsymbol{C}$ is a full subcategory of an exact category $\mathcal{E}$, we say $\boldsymbol{C}$ is closed under extensions in $\mathcal{E}$ if whenever $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathcal{E}$ with A and C in $\boldsymbol{C}$, then B is isomorphic to an object of $\boldsymbol{C}$.

A.4. Let $\mathcal{E}$ be an additive full subcategory of an abelian category $\mathcal{A}$. Suppose $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. Declare a sequence $E_1 \to$

$E_2 \to E_3$ in $\mathcal{E}$ to be exact iff it is short exact in $\mathcal{A}$; i.e., iff $E_1 \to E_2$ is the kernel of $E_2 \to E_3$, and $E_2 \to E_3$ is the cokernel of $E_1 \to E_2$ in $\mathcal{A}$. Then $\mathcal{E}$ is an exact category.

A.5. Many exact categories satisfy a stronger version of A.1.1, namely:

A.5.1. Axiom: If $f : E \to F$ is a map in $\mathcal{E}$, and there is a map $s : F \to E$ which splits f so $f \cdot s = 1_F$, then f is an admissible epimorphism $E \twoheadrightarrow F$.

A.5.2. Assuming that A.5.1 holds for $\mathcal{E}$, suppose $g : F \to E$ has a $t : E \to F$ with $t \cdot g = 1_F$. Then by A.5.1, t is an admissible epimorphism and so has a kernel in $\mathcal{E}$. Then there is an isomorphism $E \cong F \oplus \ker t$, under which g corresponds to the canonical inclusion of the summand F. Thus g is an admissible monomorphism. Hence A.5.1 implies its dual in the presence of the axioms A.1.1.

A.6.1. *Definition.* An additive category $\mathcal{E}$ is Karoubian (in Karoubi's terminology [K] 1.2.1, "pseudo-abelienne") if whenever $p : E \to E$ is an idempotent endomorphism in $\mathcal{E}$ (i.e., $p^2 = p$), then there is an isomorphism in $\mathcal{E}$, $E \cong E' \oplus E''$ under which p corresponds to the endomorphism $1 \oplus 0$. Note then E' is an image for p, and E'' is a kernel for p.

A.6.2. **Lemma.** *If an exact category $\mathcal{E}$ is Karoubian, it satisfies the extra axiom* A.5.1.

Proof. Given f, s as in A.5.1, $sf : E \to E$ is idempotent as $sfsf = s(1)f = sf$. Hence $E \cong \mathrm{im}(sf) \oplus \ker(sf)$. Clearly $s : F \to E$ induces an isomorphism of F onto $\mathrm{im}(sf)$, and $f : E \to F$ corresponds to the projection $F \oplus \ker(sf) \to F$. Thus f is an admissible epi, as required by A.5.1.

A.7.1. **Theorem** (Gabriel-Quillen Embedding Theorem) (cf. [Ga] II Section 2, [Q1] Section 2). *Let $\mathcal{E}$ be a small exact category. Then there is an abelian category $\mathcal{A}$, and a fully faithful exact functor $i : \mathcal{E} \to \mathcal{A}$ that reflects exactness. Moreover $\mathcal{E}$ is closed under extensions in $\mathcal{A}$.*

$\mathcal{A}$ may be canonically chosen to be the category of left exact functors $\mathcal{E}^{\mathrm{op}} \to \mathbb{Z}$-modules, and $i : \mathcal{E} \to \mathcal{A}$ to be the Yoneda embedding $i(E) = \mathrm{Hom}_{\mathcal{E}}(\ ,E)$.

A.7.2. The proof of A.7.1 and its elaborations will occupy all of Section A.7, and is derived from the Grothendieck-Verdier theory of sheafification in [SGA 4] II.

Let $\mathcal{B}$ be the abelian category of additive functors $F : \mathcal{E}^{\mathrm{op}} \to \mathbb{Z}$-mod where $\mathbb{Z}$-mod is the category of abelian groups. Limits and colimits exist in $\mathcal{B}$, and are formed pointwise, so $(\varprojlim F_\alpha)(E) = \varprojlim(F_\alpha(E))$, etc. Then it is clear that direct colimits in $\mathcal{B}$ are exact, i.e., Grothendieck's axiom AB5 holds. Also, $\mathcal{B}$ has a set of generators consisting of the functors $yE = \mathrm{Hom}(\ ,E)$ for E in $\mathcal{E}$. The Yoneda embedding $y : \mathcal{E} \to \mathcal{B}$ is fully faithful by the Yoneda lemma. Thus $\mathcal{B}$ is a Grothendieck abelian category, as is well-known.

A.7.3. *Definition.* Let $G : \mathcal{E}^{\mathrm{op}} \to \mathbb{Z}\text{-mod}$ be an object of $\mathcal{B}$. One says G is *separated* if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, the induced map $G(F) \to G(E)$ is a monomorphism. One says that G is "*left exact*" if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, then (A.7.4) is a difference kernel, where the maps d are induced by the two projections $p : E \underset{F}{\times} E \to E$:

$$G(F) \to G(E) \underset{d^1=G(p_1)}{\overset{d^0=G(p_0)}{\rightrightarrows}} G\left(E \underset{F}{\times} E\right). \tag{A.7.4}$$

Thus $G(F)$ is the kernel of $d^0 - d^1 : G(E) \to G(E \underset{F}{\times} E)$.

A.7.5. Let $\mathcal{A}$ be the full subcategory of $\mathcal{B}$ consisting of the "left exact" functors $\mathcal{E}^{\mathrm{op}} \to \mathbb{Z}\text{-mod}$. Let $j_* : \mathcal{A} \to \mathcal{B}$ be the inclusion. Later, we will show that j_* has a left adjoint j^* so that $j^*j_* = 1_{\mathcal{A}}$. Then $\mathcal{A}$ will be a Grothendieck abelian category such that j^* is an exact functor, and j_* is left exact (in the covariant abelian sense that j_* preserves kernels).

A.7.6. The Yoneda embedding $y : \mathcal{E} \to \mathcal{B}$ factors through $\mathcal{A}$, so $y = j_* \cdot i$ for a functor $i : \mathcal{E} \to \mathcal{A}$. To show this, it suffices to show that $yG = \mathrm{Hom}(\ ,G)$ is "left exact" for all G in $\mathcal{E}$. But for any admissible epimorphism $E \twoheadrightarrow F$, $E \underset{F}{\times} E \rightarrowtail E \oplus E \twoheadrightarrow F$ is exact in $\mathcal{E}$; so $E \oplus E \twoheadrightarrow F$ is the cokernel of $E \underset{F}{\times} E \to E \oplus E$. Thus

(A.7.7)
$$0 \to \mathrm{Hom}(F,G) \to \mathrm{Hom}(E,G) \oplus \mathrm{Hom}(E,G) \to \mathrm{Hom}\left(E \underset{F}{\times} E, G\right)$$

is exact, so $\mathrm{Hom}(F,G)$ is the kernel of $d^0 - d^1 : \mathrm{Hom}(E,G) \to \mathrm{Hom}(E \underset{F}{\times} E, G)$. Thus $\mathrm{Hom}(\ ,G)$ is "left exact" as required.

A.7.8. (cf. [SGA 4] II 3.0.5, 2.4). For E in $\mathcal{E}$, let $\mathcal{C}_E$ be the following directed category. The objects of $\mathcal{C}_E$ are admissible epimorphisms $e' : E' \twoheadrightarrow E$. There is at most one map between any two objects of $\mathcal{C}_E$. There exists a map $(e' : E' \twoheadrightarrow E) \to (e'' : E'' \twoheadrightarrow E)$ in $\mathcal{C}_E$ iff there is a map (backwards!) $a : E'' \to E'$ in $\mathcal{E}$ such that $e'a = e''$. It is easy to check that $\mathcal{C}_E$ is directed.

Note that any two choices of a_1, $a_2 : E'' \to E'$ with $e' \cdot a_i = e''$ induce the same map on the kernels of $d^0 - d^1$ in (A.7.9)

$$\text{(A.7.9)}\qquad \begin{array}{ccc} \ker\left(G(E')\right. & \xrightarrow{d^0-d^1} & \left.G(E' \underset{E}{\times} E')\right) \\ \big\downarrow{\scriptstyle G(a_i)} & & \big\downarrow{\scriptstyle G(a_i\times a_i)} \\ \ker\left(G(E'')\right. & \xrightarrow{d^0-d^1} & \left.G(E'' \underset{E}{\times} E'')\right) \end{array}$$

For $a_1 \perp a_2 : E'' \to E' \underset{E}{\times} E'$ induces a "homotopy" $h : G(E' \underset{F}{\times} E') \to G(E'')$ such that $h \cdot (d^0 - d^1) = G(a_1) - G(a_2)$ on $G(E')$, showing that $G(a_1) - G(a_2) = 0$ on $\ker(d^0 - d^1)$.

Thus sending E' to $\ker(G(E') \to G(E' \underset{F}{\times} E'))$ is a functor from the directed category $\mathcal{C}_E$ to the category of abelian groups. Define $LG(E)$ to be the direct colimit

$$\text{(A.7.10)}\qquad LG(E) = \varinjlim_{(E' \twoheadrightarrow E) \text{ in } \mathcal{C}_E} \ker\left(G(E') \overset{d^0-d^1}{\longrightarrow} G\left(E' \underset{E}{\times} E'\right)\right).$$

We note that LG is a covariant functor in E, and that LG is an additive functor as G is additive and as $\mathcal{C}_{E_1 \oplus E_2} = \mathcal{C}_{E_1} \times \mathcal{C}_{E_2}$, since $E' \twoheadrightarrow E_1 \oplus E_2$ canonically decomposes as $E' = E' \underset{E}{\times} E_1 \oplus E' \underset{E}{\times} E_2$ for $E' \underset{E}{\times} E_i \twoheadrightarrow E_i$ in $\mathcal{C}_{E_i}$. There is a natural transformation $\eta : G \to LG$ induced by the obvious map $G(E) \to \ker(G(E') \to G(E' \underset{E}{\times} E'))$. As kernels and directed colimits commute with finite limits in $\mathbb{Z}$-mod, and as limits in $\mathcal{B}$ are formed by pointwise taking limits in $\mathbb{Z}$-mod, the functor $L : \mathcal{B} \to \mathcal{B}$ preserves finite limits.

A.7.11. **Lemma.** (a) *For any $x \in LG(E)$, there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ in $\mathcal{E}$, and a $y \in G(E')$, such that $\eta(y) = LG(e)(x)$ in $LG(E')$.*

(b) *For any $x \in G(E)$, then $\eta(x) = 0$ in $LG(E)$ iff there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ such that $G(e)(x) = 0$ in $G(E')$.*

(c) *$LG = 0$ iff for all E in $\mathcal{E}$ and all $x \in G(E)$, there exists an admissible epimorphism $e : E' \twoheadrightarrow E$ such that $G(e)(x) = 0$ in $G(E')$.*

(d) *If G is separated, then for all E in $\mathcal{E}$ the map $\eta(E) : G(E) \to LG(E)$ is a monomorphism.*

(e) *If G is left exact, then for all E in $\mathcal{E}$, $\eta(E) : G(E) \to LG(E)$ is an isomorphism.*

Proof. Statements (a), (b), (d), and (e) are clear from the definitions. Statement (c) follows immediately from (a) and (b).

A.7.12. **Lemma** (cf. [SGA 4] II 3.2).

(a) *For all G in $\mathcal{B}$, LG is separated.*

(b) *For all separated G in $\mathcal{B}$, LG is "left-exact."*

Proof. (a) Suppose $x \in LG(A)$, and that $b : B \twoheadrightarrow A$ is an admissible epimorphism for which $LG(b)(x) = 0$. We need to show then that $x = 0$. By construction of $LG(A)$, x is represented by $y \in \ker(G(C) \rightrightarrows G(C \underset{A}{\times} C))$ for some $C \twoheadrightarrow A$ in $\mathcal{C}_A$. As x goes to 0 in $LG(B)$, the image of y in $\ker(G(C \underset{A}{\times} B) \rightrightarrows G((C \underset{A}{\times} B) \underset{B}{\times} (C \underset{A}{\times} B)))$ is equivalent to 0 in the direct colimit over $\mathcal{C}_B$ that defines $LG(B)$. Hence there is a map $D \to C \underset{A}{\times} B$ in $\mathcal{E}$ such that the composite with the projection to B is an admissible epimorphism $D \twoheadrightarrow B$, and such that y goes to 0 in $G(D)$. But then $D \twoheadrightarrow B \twoheadrightarrow A$ is in $\mathcal{C}_A$, and y is equivalent to 0 in the direct colimit over $\mathcal{C}_A$ that defines $LG(A)$. Hence $x = 0$ in $LG(A)$, as required.

(b) Suppose G is separated, we must show for any admissible epi $B \twoheadrightarrow A$ in $\mathcal{E}$, that $LG(A) \to LG(B) \rightrightarrows LG(B \underset{A}{\times} B)$ is a difference kernel. As G is separated, $LG(A) \to LG(B)$ is a monomorphism. It remains to show that if $x \in LG(B)$ has $d^0x = d^1x$, i.e., $LG(p_1)(x) = LG(p_2)(x)$ in $LG(B \underset{A}{\times} B)$, then x is in the image of $LG(A)$. But by A.7.11, there is a $c : C \twoheadrightarrow B$ and a $y \in G(C)$ such that $\eta(y) = LG(c)(x)$. Then $\eta G(p_1)(y) = \eta G(p_2)(y)$ in $LG(C \underset{A}{\times} C) \supseteq LG(B \underset{A}{\times} B)$. As G is separated, $\eta : G \to LG$ is a monomorphism, so $G(p_1)(y) = G(p_2)(y)$ in $G(C \underset{A}{\times} C)$. Hence $y \in \ker(G(C) \rightrightarrows G(C \underset{A}{\times} C))$ is a class in $LG(A)$ which represents x. This shows x is in $LG(A)$, as required.

A.7.13. **Proposition.** *Let $j^* : \mathcal{B} \to \mathcal{A}$ be $j^* = L \cdot L$. Then j^* is left adjoint to $j_* : \mathcal{A} \to \mathcal{B}$, and the adjunction map $j^*j_* \to 1_{\mathcal{A}}$ is an isomorphism. Hence $\mathcal{A}$ is a reflexive subcategory of $\mathcal{B}$, and j_* is fully faithful.*

$\mathcal{A}$ is an abelian category, and j^* is an exact functor. The functor j_* is left exact, i.e., it preserves kernels.

$\mathcal{A}$ has all limits and colimits, and is a Grothendieck abelian category.

Proof. The adjointness of j^* to j_* and the isomorphism $j^*j_* \cong 1_{\mathcal{A}}$ follow immediately from the fact $\eta : G \to LG$ is an isomorphism for G in the subcategory $\mathcal{A}$ of "left exact" functors.

The cokernel of a map in $\mathcal{A}$ is simply j^* of the cokernel taken in $\mathcal{B}$. As $j^* = L \cdot L$ preserves finite limits, it preserves kernels. It is then clear that $\mathcal{A}$ is abelian since $\mathcal{B}$ is, and that $j^* : \mathcal{B} \to \mathcal{A}$ is exact. Then the right adjoint j_* must be left exact.

$\mathcal{A}$ has all limits and colimits, and is a Grothendieck abelian category, since it is a retract of $\mathcal{B}$ which has these properties.

A.7.14. **Proposition.** *The Yoneda functor $i : \mathcal{E} \to \mathcal{A}$ of A.7.6 is fully faithful and exact.*

Proof. As $y : \mathcal{E} \to \mathcal{B}$ is fully faithful by the Yoneda lemma, and as $j_* : \mathcal{A} \to \mathcal{B}$ is fully faithful, and as $y = j_* \cdot i$, it follows that $i : \mathcal{E} \to \mathcal{A}$ is fully faithful.

As $y : \mathcal{E} \to \mathcal{B}$ is clearly left exact, and $i = 1 \cdot i = j^* \cdot j_* \cdot i = j^* \cdot y$, the functor $i : \mathcal{E} \to \mathcal{A}$ is left exact. It remains to show that the functor i is right exact.

Let $A \rightarrowtail B \twoheadrightarrow C$ be an exact sequence in $\mathcal{E}$. We have already shown that $0 \to iA \to iB \to iC$, i.e., $0 \to \mathrm{Hom}(\ ,A) \to \mathrm{Hom}(\ ,B) \to \mathrm{Hom}(\ ,C)$ is left exact in $\mathcal{A}$. Let H be the cokernel of $\mathrm{Hom}(\ ,B) \to \mathrm{Hom}(\ ,C)$ in $\mathcal{B}$. For $0 \to iA \to iB \to iC \to 0$ to be exact in $\mathcal{A}$, it suffices that $j^*H = 0$. For this, it suffices to show that $LH = 0$. We show $LH = 0$ by applying criterion A.7.11(c). Take any $\bar{x} \in H(E)$. We must show $\bar{x}$ goes to 0 in $H(E')$ for some $E' \twoheadrightarrow E$. As $\bar{x} \in H(E) = \mathrm{Hom}(E,C)/\mathrm{Hom}(E,B)$, $\bar{x}$ is represented by a map $x : E \to C$. We consider the pullback along x of $B \twoheadrightarrow C$, $B \underset{C}{\times} E \twoheadrightarrow E$. Then $\bar{x}$ goes to 0 in $H(B \underset{C}{\times} E) = \mathrm{Hom}(B \underset{C}{\times} E, C)/\mathrm{Hom}(B \underset{C}{\times} E, B)$ as the map $x' : B \underset{C}{\times} E \twoheadrightarrow E \to C$ factors through B as $B \underset{C}{\times} E \to B \twoheadrightarrow C$. Thus $B \underset{C}{\times} E$ is the required E'.

A.7.15. **Lemma.** *Let $e : E \to F$ be a map in $\mathcal{E}$. Then $i(e)$ is an epimorphism in $\mathcal{A}$ iff there is a $k : E' \to E$ in $\mathcal{E}$ such that $ek : E' \twoheadrightarrow F$ is an admissible epimorphism.*

More generally, for any A in $\mathcal{A}$ and F in $\mathcal{E}$, a map $e : A \to i(F)$ in $\mathcal{A}$ is an epimorphism in $\mathcal{A}$ iff there is a $k : i(E') \to A$ (i.e., a $k \in A(E')$) such that $ek : E' \twoheadrightarrow F$ is an admissible epimorphism in $\mathcal{E}$.

Proof. If $ek : E' \twoheadrightarrow F$ is an admissible epimorphism in $\mathcal{E}$, the exact $i : \mathcal{E} \to \mathcal{A}$ preserves the exact sequence $\ker(ek) \rightarrowtail E' \twoheadrightarrow F$, so $ek : i(E') \to i(F)$ is an epimorphism in $\mathcal{A}$. Hence $e : A \to i(F)$ is an epimorphism in $\mathcal{A}$.

Conversely suppose that $e : A \to i(F)$ is epi in $\mathcal{A}$. We let H be the cokernel of e in $\mathcal{B}$. Then $j^*H = \mathrm{coker}\ e = 0$ in $\mathcal{A}$. As $0 = j^*H = LLH$, and LH is separated, it follows that $LH = 0$. Consider $\bar{x} \in H(F) = \mathrm{Hom}(F,F)/\mathrm{Hom}(F,A)$ corresponding to 1 in $\mathrm{Hom}(F,F)$. As $\eta(\bar{x}) = 0$ in $LH(F)$, by A.7.11(b) there is an admissible epimorphism $E' \twoheadrightarrow F$ in $\mathcal{E}$ such that $\bar{x}$ goes to 0 in $H(E') = \mathrm{Hom}(E',F)/\mathrm{Hom}(E',A)$. That is to say, that $iE' \twoheadrightarrow iF$ factors as the composite ke of a $k : iE' \to A$ and $e : A \to iF$, as required.

A.7.16. **Proposition.** (a) *The embedding $i : \mathcal{E} \to \mathcal{A}$ reflects exactness.*

(b) *If $\mathcal{E}$ satisfies the extra axiom* A.5.1, *and if e is a map in $\mathcal{E}$ such that $i(e)$ is an epimorphism in $\mathcal{A}$, then e is an admissible epimorphism in $\mathcal{E}$.*

Proof. Let $A \to B \to C$ be a sequence in $\mathcal{E}$ such that $0 \to iA \to iB \to iC \to 0$ is a short exact in A. Then $iA \to iB$ is the kernel of $iB \to iC$ in $\mathcal{A}$. As $i : \mathcal{E} \to \mathcal{A}$ is fully faithful, $A \to B$ is the kernel of $B \to C$ in $\mathcal{E}$.

By A.7.15, as $iB \to iC$ is epi in $\mathcal{A}$, there is a $B' \to B$ such that the composite $B' \to B \to C$ is an admissible epimorphism $B' \twoheadrightarrow C$ in $\mathcal{E}$. As $B \to C$ has a kernel in $\mathcal{E}$, this implies that $B \twoheadrightarrow C$ is an admissible epimorphism in $\mathcal{E}$, by the hitherto obscure axiom A.1.1. Then the kernel $A \to B$ is an admissible monomorphism and $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathcal{E}$. This proves (a).

Suppose now that $\mathcal{E}$ satisfies A.5.1, and that $e : B \to C$ is a map in $\mathcal{E}$ such that $i(e)$ is an epimorphism in $\mathcal{A}$. By A.7.15, there is a $B' \to B$ in $\mathcal{E}$ such that the composite $B' \to B \to C$ is an admissible epimorphism in $\mathcal{E}$. If we knew that $B \to C$ had a kernel in $\mathcal{E}$, we would conclude that $e : B \twoheadrightarrow C$ is an admissible epimorphism by A.1.1. Hence it suffices to show that $e : B \to C$ has a kernel. We consider the pullback square in $\mathcal{E}$:

$$\text{(A.7.17)} \qquad \begin{tikzcd} B \underset{C}{\times} B' \arrow[r] \arrow[d, twoheadrightarrow] \arrow[dr, phantom, "\square"] & B' \arrow[d, twoheadrightarrow] \\ B \arrow[r] & C \end{tikzcd}$$

The map $1 : B' \to B'$ and $B' \to B$ induce a map $B' \to B \underset{C}{\times} B'$ that splits the map $B \underset{C}{\times} B' \to B'$. By axiom A.5.1, we conclude that $B \underset{C}{\times} B' \to B'$ is an admissible epimorphism in $\mathcal{E}$, and so has a kernel in $\mathcal{E}$. But as (A.7.17) is a pullback, the kernel of $B \underset{C}{\times} B' \to B'$ is also a kernel of $B \to C$ in $\mathcal{E}$, as sought.

A.7.18. **Lemma.** *$\mathcal{E}$ is closed under extension in $\mathcal{A}$.*

Proof. Let $0 \to A \to G \to B \to 0$ be a short exact sequence in $\mathcal{A}$ with A and B in $\mathcal{E}$. By A.7.15, the condition that $G \to B$ is epi in $\mathcal{A}$ implies that there is an admissible epimorphism $C \twoheadrightarrow B$ in $\mathcal{E}$ that factors as $C \to G \to B$. We consider the pullback diagram in $\mathcal{A}$:

(A.7.19)
$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d] & A \arrow[d] \\ & C \times_B G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ K \arrow[r, tail] & C \arrow[r, two heads] \arrow[d] & B \arrow[d] \\ & 0 & 0 \end{tikzcd}$$

The maps $1 : C \to C$ and $C \to G$ induce a map $C \to C \times_B G$ that splits $C \times_B G \to C$. Hence in $\mathcal{A}$, $C \times_B G$ is isomorphic to $A \oplus C$, an object of $\mathcal{E}$. Let $i : K \rightarrowtail C$ be the kernel of the admissible epi $C \twoheadrightarrow B$. As (A.7.19) is a pullback, $K \to C \times_B G$ is the kernel of the epimorphism in $\mathcal{A}$, $C \times_B G \to G$. We write $K \to C \times_B G \cong A \oplus C$ as $\begin{bmatrix} a \\ i \end{bmatrix}$, $a : K \to A$, $i : K \rightarrowtail C$. Then the exact sequence $K \to A \oplus C \to G \to 0$ in $\mathcal{A}$ shows that G is isomorphic to the pushout

(A.7.20)
$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, tail, "i"'] & A \arrow[d] \\ C \arrow[r] & G \end{tikzcd}$$

But as K, A, C are in $\mathcal{E}$, and as i is an admissible monomorphism in $\mathcal{E}$, this square also has a pushout G' in $\mathcal{E}$. Then by Lemma A.8.1 below applied to the exact functor $i : \mathcal{E} \to \mathcal{A}$, we have an isomorphism $G \cong G'$ of G to an object of $\mathcal{E}$, as required.

A.7.21. Modulo A.8.1, this completes the proof of A.7.1.

A.8.1. **Lemma.** *Let $f : \mathcal{E} \to \mathcal{E}'$ be an exact functor between exact categories. The f preserves pushouts along an admissible mono, and f preserves pullbacks along an admissible epi.*

Proof. Consider $A \rightarrowtail B$ and $A \to C$ in $\mathcal{E}$, and the pushout $C \cup_A B$. We have an exact sequence $C \rightarrowtail C \cup_A B \twoheadrightarrow B/A$. Taking f of this sequence yields an exact sequence in $\mathcal{E}'$. Also, $fA \rightarrowtail fB$ is an admissible mono in $\mathcal{E}'$, so $\mathcal{E}'$ has a pushout $fC \cup_{fA} fB$ and an exact sequence $fC \rightarrowtail fC \cup_{fA} fB \twoheadrightarrow fB/fA$. Consider the diagram in $\mathcal{E}'$:

$$\begin{array}{ccccc} fC & \rightarrowtail & fC \underset{fA}{\cup} fB & \twoheadrightarrow & fB/fA \\ \| & & \downarrow & & \downarrow \cong \\ fC & \rightarrowtail & f(C \underset{A}{\cup} B) & \twoheadrightarrow & f(B/A) \end{array}$$

Note $f(B/A) \cong fB/fA$ by exactness. We consider the fully faithful exact embedding $\mathcal{E}' \to \mathcal{A}'$ of A.7.14. By the 5-lemma in the abelian category $\mathcal{A}'$ applied to the diagram, we see that $fC \underset{fA}{\cup} fB \to f(C \underset{A}{\cup} B)$ is an isomorphism in $\mathcal{A}'$, hence in $\mathcal{E}'$, as required. Dually, f preserves pullbacks along admissible epis.

A.8.2. **Proposition.** *Let $f : \mathcal{E} \to \mathcal{E}'$ be an exact functor between exact categories. Let $i : \mathcal{E} \to \mathcal{A}$ and $i' : \mathcal{E}' \to \mathcal{A}'$ be the Gabriel-Quillen embeddings into the categories of "left exact" functors.*

Then there is a right exact additive functor $f^* : \mathcal{A} \to \mathcal{A}'$ extending f in that $f^* \cdot i \cong i' \cdot f$. This f^* has an additive left exact right adjoint functor $f_* : \mathcal{A}' \to \mathcal{A}$.

Proof. We follow the analogy with [SGA 4] III. Consider $f_\# : \mathcal{B}' \to \mathcal{B}$ given by sending the additive functor $G : \mathcal{E}'^{\mathrm{op}} \to \mathbb{Z}\text{-mod}$ to $f_\# G = G \cdot f$ with $(f_\# G)(E) = G(f(E))$ for E in $\mathcal{E}$. We claim that if G is 'left exact," so is $f_\# G$, so that $f_\#$ restricts to a functor $f_* : \mathcal{A}' \to \mathcal{A}$. For let G be "left exact" on $\mathcal{E}'$, and let $E \twoheadrightarrow F$ be an admissible epi in $\mathcal{E}$. Then $fE \twoheadrightarrow fF$ is an admissible epi in $\mathcal{E}'$, and $f(E \underset{F}{\times} E) \cong fE \underset{fF}{\times} fE$ by A.8.1. Then (A.8.3) is a difference kernel, as required (A.7.3)

$$G(fF) \to G(fE) \rightrightarrows G\left(f\left(E \underset{F}{\times} E\right)\right). \tag{A.8.3}$$

Clearly $f_\# : \mathcal{B}' \to \mathcal{B}$ preserves all limits. As the inclusions $\mathcal{A}' \to \mathcal{B}'$, $\mathcal{A} \to \mathcal{B}$ preserve and reflect all limits, it follows that the induced $f_* :$ $\mathcal{A}' \to \mathcal{A}$ preserves all limits. In particular, it preserves finite products and kernels, so is additive and left exact.

As $\mathcal{A}'$ has limits and has a set of generators, the special adjoint functor theorem shows that f_* has a left adjoint $f^* : \mathcal{A} \to \mathcal{A}'$. This f^* must preserve all colimits, in particular direct sums and cokernels. So f^* is additive and right exact. As $\mathrm{Hom}(fE, G) \cong G(fE) \cong (f_*G)(E) \cong \mathrm{Hom}(E, f_*G)$ for E in $\mathcal{E}$ and G in $\mathcal{A}'$, it is clear that this adjoint f^* is isomorphic to f when restricted to $\mathcal{E}$.

A.8.4. In general $f^* : \mathcal{A} \to \mathcal{A}'$ need not be an exact functor of abelian categories. If R is a ring, and $\mathcal{E}$ is the exact category of finitely generated projective R-modules, $\mathcal{A}$ is the abelian category of all R-modules. For $R \to S$ a ring map, $f = S\underset{R}{\otimes}$ extends to $f^* = S\underset{R}{\otimes} : R\text{-mod} \to S\text{-mod}$, which need not be exact. Compare [SGA 4] IV 4.9.1.

However if $\mathcal{E}$ and $\mathcal{E}'$ have all pushouts and if $f : \mathcal{E} \to \mathcal{E}'$ preserves all pushouts, that is if $\mathcal{E}$ and $\mathcal{E}'$ are abelian categories with some exactness structure (possibly not the canonical one) and if $f : \mathcal{E} \to \mathcal{E}'$ is exact with respect to both the chosen and the canonical exactness structures, then $f^* : \mathcal{A} \to \mathcal{A}'$ is exact. We will not need this, but the interested reader may prove it as an exercise, guided by [SGA 4] IV 4.9.2, III 1.3.5, ...

A.8.5. Although $f^* : \mathcal{A} \to \mathcal{A}'$ may not be exact, it does preserve the exact sequences in $\mathcal{E}$, as $\mathcal{E} \to \mathcal{E}' \to \mathcal{A}'$ does so.

A.9.1. **Theorem.** *Let $\mathcal{E}$ be an exact category. Then*

(a) *There is a Karoubian* (A.6.1) *additive category $\mathcal{E}'$ and a fully faithful additive functor $f : \mathcal{E} \to \mathcal{E}'$ such that any additive functor from $\mathcal{E}$ to a Karoubian additive category factors uniquely-up-to-natural-isomorphism through $\mathcal{E} \to \mathcal{E}'$.*

(b) *Every object in $\mathcal{E}'$ is a direct summand in $\mathcal{E}'$ of an object in $\mathcal{E}$. We say a sequence in $\mathcal{E}'$ is exact iff it is a direct summand of an exact sequence in $\mathcal{E}$. This makes $\mathcal{E}'$ an exact category. The inclusion functor $f : \mathcal{E} \to \mathcal{E}'$ is exact and reflects exactness, and $\mathcal{E}$ is closed under extensions in $\mathcal{E}'$.*

(c) *$K(\mathcal{E})$ is a covering spectrum of $K(\mathcal{E}')$, in fact f induces an isomorphism of Quillen K-groups $K_n(\mathcal{E}) \xrightarrow{\cong} K_n(\mathcal{E}')$ for $n \geq 1$, and a monomorphism $K_0(\mathcal{E}) \subseteq K_0(\mathcal{E}')$.*

Proof. (Compare Karoubi [K] 1.2.2.) Let $\mathcal{E}'$ be the category whose objects are pairs (E, p), with E an object of $\mathcal{E}$ and $p = p^2$ an idempotent endomorphism of E. A map $e : (E,p) \to (E',p')$ in $\mathcal{E}'$ is a map $e : E \to E'$ such that $p'e = ep$. (The identity map of (E,p) is p.)

The functor $f : \mathcal{E} \to \mathcal{E}'$ sending E to $(E, 1)$ is fully faithful.

The category $\mathcal{E}'$ is additive with $(E,p)\oplus(E',q) = (E\oplus E', p\oplus q)$. (E,p) is a summand of E, as there are obvious isomorphisms $(E,p)\oplus(E,1-p) \cong (E \oplus E, p \oplus 1 - p) \cong (E \oplus E, 1 \oplus 0) \cong (E,1) = E$. It is easy to check that $\mathcal{E}'$ is Karoubian, and has the universal mapping property claimed for $\mathcal{E} \to \mathcal{E}'$.

To show that $\mathcal{E}'$ is an exact category, we consider the Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$. This induces a fully faithful functor between Karoubianizations, $\mathcal{E}' \to \mathcal{A}'$. By definition of exact sequence in the Karoubianization, and the fact $\mathcal{E} \to \mathcal{A}$ preserves and reflects exactness, the induced functor $\mathcal{E}' \to \mathcal{A}'$ preserves and reflects exact sequences. But as $\mathcal{A}$ already has images of idempotents, $\mathcal{A}'$ is equivalent to the abelian category $\mathcal{A}$. We claim that $\mathcal{E}'$ is closed under extensions in $\mathcal{A}$. For let

$0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$ with A and C in $\mathcal{E}'$. Then there are A', C' in $\mathcal{E}'$ so that $A \oplus A'$ and $C \oplus C'$ are isomorphic to objects of $\mathcal{E}$. The sequence $0 \to A \oplus A' \to C' \oplus B \oplus A' \to C' \oplus C \to 0$ is exact in $\mathcal{A}$, and shows that $C' \oplus B \oplus A'$ is isomorphic to an object of $\mathcal{E}$, since $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. Thus B is a summand of an object of $\mathcal{E}$, hence is isomorphic to the image of an idempotent in $\mathcal{E}$, and hence is isomorphic to an object of $\mathcal{E}'$. This proves the claim. Now $\mathcal{E}'$ is an exact category by A.4. As the functors $\mathcal{E} \to \mathcal{A}$ and $\mathcal{E}' \to \mathcal{A}$ preserve and reflect exactness, so does the functor $\mathcal{E} \to \mathcal{E}'$.

It remains to prove part (c). But this follows from (b) and the classical cofinality theorems in Quillen K-theory, e.g., [Gr2] 6.1 or [Sta] 2.1.

A.9.2. The point is, that because of A.9.1(c), it is no real loss of generality in K-theory to consider only Karoubian exact categories $\mathcal{E}$. For these exact categories, A.6.2 and A.7.16(b) show that the Gabriel-Quillen embedding $\mathcal{E} \to \mathcal{A}$ satisfies hypothesis 1.11.3.1. So it is harmless to make this hypothesis in K-theory.

Appendix B

Modules vs. Quasi-coherent modules

B.0. This appendix reviews the relations between the categories of quasi-coherent $\mathcal{O}_X$-modules and of all $\mathcal{O}_X$-modules in the Zariski topos of a scheme X. Most of these facts are well-known in outline, although many people exhibit some confusion and fuzziness on the details when pressed. The theory of the "coherator" is more esoteric, but essential for this paper.

The results in this appendix are all due to Grothendieck and his school, and are collected from scattered parts of [Gro], [EGA], [SGA 4], and [SGA 6], with some slight sharpening due to the new concept of "semi-separated."

B.1. For X a scheme, let $\mathcal{O}_X$-Mod be the abelian category of all sheaves of $\mathcal{O}_X$-modules (for the Zariski topology on X), and let $D(X) = D(\mathcal{O}_X\text{-Mod})$ be the derived category of $\mathcal{O}_X$-Mod.

The category $\mathcal{O}_X$-Mod has all limits and colimits, and has a set of generators. Direct colimits are exact. Hence $\mathcal{O}_X$-Mod is a Grothendieck abelian category and has enough injectives. Also it has an internal hom sheaf, Hom(,) and a tensor product $\otimes_{\mathcal{O}_X}$ ([Gro], [SGA 4] IV).

B.2. Let Qcoh(X) be the full subcategory of $\mathcal{O}_X$-Mod consisting of the quasi-coherent $\mathcal{O}_X$-modules, i.e., those which locally on X have a presentation by free $\mathcal{O}_X$-modules. This category Qcoh(X) includes all $\mathcal{O}_X$-modules of finite presentation.

Let φ : Qcoh$(X) \to \mathcal{O}_X$-Mod be the inclusion. Then Qcoh(X) is an abelian category, closed under extensions and tensor products in $\mathcal{O}_X$-Mod. The functor φ is exact and reflects exactness. In particular, φ preserves all finite limits and colimits. It also preserves and reflects infinite direct sums, and hence all colimits. Thus Qcoh(X) has all colimits, and satisfies Grothendieck's axiom AB5 that direct colimits are exact. For $\mathcal{F}$ a finitely presented $\mathcal{O}_X$-module, and $\mathcal{G}$ a quasi-coherent $\mathcal{O}_X$-module, Hom$(\mathcal{F}, \mathcal{G})$ is quasi-coherent. ([EGA] I 2.2).

It seems to be unknown whether, for general schemes X, Qcoh(X) has a set of generators, enough injectives, or even all limits.

When X is affine, say $X = \operatorname{Spec}(A)$, the category Qcoh$(X)$ is equivalent to the category of A-modules. Of course, in this case Qcoh(X) has all limits, a set of generators, and enough injectives.

In general, let $D(\text{Qcoh}(X))$ be the derived category of Qcoh(X).

B.3. If X is a quasi-compact and quasi-separated scheme, every sheaf in Qcoh(X) is a direct colimit of its sub-$\mathcal{O}_X$-modules of finite type. Also, every sheaf in Qcoh(X) is a filtering colimit of finitely presented $\mathcal{O}_X$-modules. ([EGA] I 6.9.9, 6.9.12.) In this case, the set of finitely presented $\mathcal{O}_X$-modules forms a set of generators for Qcoh(X), which is then a Grothendieck abelian category and has enough injectives (cf. B.12.).

B.4. DANGER: For a general quasi-compact and quasi-separated X, φ : Qcoh$(X) \to \mathcal{O}_X$-Mod does *not* preserve injectives, nor need it send injectives in Qcoh(X) to flasque sheaves in $\mathcal{O}_X$-Mod. For Verdier's counterexample in which X is even affine with noetherian underlying space, see [SGA 6] II App. I 0.1.

On the other hand, if X is a noetherian scheme, then φ does preserve injectives. For let $\mathcal{F}$ be injective in Qcoh(X). By [H] II 7.18, there is a $\mathcal{G}$ in Qcoh(X) with $\varphi(\mathcal{G})$ injective in $\mathcal{O}_X$-Mod, and a monomorphism $\mathcal{F} \rightarrowtail \mathcal{G}$. In Qcoh($X$) this splits as $\mathcal{F}$ is injective. Then $\varphi(\mathcal{F})$ is a direct summand of the injective $\varphi(\mathcal{G})$, and so is injective in $\mathcal{O}_X$-Mod, as required.

The fact that φ does preserve injectives in the noetherian case can lure one to a false sense of security. In general, when one computes by injective resolutions various derived functors evaluated on a quasi-coherent sheaf $\mathcal{F}$, one must distinguish between the possibly different derived functors taken in $\mathcal{O}_X$-Mod and those taken in Qcoh(X) (see [SGA 6] II App. I 0.2). We may add "$\mathcal{O}_X$-Mod" or "Qcoh(X)" to the name of the derived functor to indicate the distinction, so that we have $R^n(\mathcal{O}_X\text{-Mod})f_*(\mathcal{F})$ vs. $R^n(\text{Qcoh}(X))f_*(\mathcal{F})$ for a quasi-coherent $\mathcal{F}$. When these are shown to be equivalent in some cases, the notation reverts back to $R^n f_*(\mathcal{F})$ (e.g., B.8).

B.5. If $j : U \to X$ is an open immersion of schemes, and $\mathcal{F}$ is an injective in $\mathcal{O}_X$-Mod, then $j^*\mathcal{F}$ is an injective in $\mathcal{O}_X$-Mod. For j^* has an exact left adjoint functor $j!$, extension by 0 off U ([SGA 4] V 4.11, IV 11.3.1). As $j!$ does not preserve quasi-coherence, this argument does not apply to injectives in Qcoh(X), and in fact j^* need not send them to injectives in Qcoh(U) ([SGA 6] II App. I).

However, if X is noetherian and $\mathcal{F}$ is injective in Qcoh(X) then $j^*\mathcal{F}$ is injective in Qcoh(U). For by B.4, $\varphi\mathcal{F}$ is injective in $\mathcal{O}_X$-Mod, so $j^*\varphi\mathcal{F} = \varphi j^*\mathcal{F}$ is injective in $\mathcal{O}_U$-Mod. But as φ is exact and fully faithful, this implies that $j^*\mathcal{F}$ is injective in Qcoh(U). (For another proof that $j^*\mathcal{F}$ is injective in the noetherian case, use the pro-existing left adjoint denoted $j!$ in Deligne's letter in [H] p.411.)

B.6. **Lemma.** *For X a quasi-compact and quasi-separated scheme, the cohomology functors $H^k(X;\ )$: $\mathcal{O}_X$- Mod $\to$ $\mathbb{Z}$-Mod preserve direct colimits.*

For $f : X \to Y$ a quasi-compact and quasi-separated map of schemes,

$R^k f_* = R^k(\mathcal{O}_X\text{-Mod})f_* : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}$ preserves direct colimits. Also, if $\mathcal{F}$ is a quasi-coherent sheaf, $f_*\mathcal{F}$ and indeed the $R^k(\mathcal{O}_X\text{-Mod})f_*(\mathcal{F})$ are quasi-coherent for such an f.

Proof. To prove the first statement, we note by [SGA 4] VI 1.22 that the Zariski topos is coherent in the sense of [SGA 4] VI, and then we appeal to [SGA 4] VI 5.2. Similarly, $R^k f_*$ preserves direct colimits by appeal to [SGA 4] VI 5.1, or by the obvious reduction to the first statement.

The last statement is [EGA] III 1.4.10, IV 1.7.21, and is based on using Čech complexes of finite affine hypercovers ([SGA 4] V Section 7) to compute cohomology.

B.7. We say a scheme is *semi-separated* if there is a set $\mathcal{A} = \{U_\alpha\}$ of open subschemes of X which is a basis for the topology of X such that each U_α in $\mathcal{A}$ is affine, and which is such that the intersection $U_\alpha \cap U_\beta$ of any two members of $\mathcal{A}$ is also in $\mathcal{A}$. This $\mathcal{A}$ is then said to be a *semi-separating affine basis.* Note that any open or closed subscheme of a semi-separated scheme is itself semi-separated.

We say an open cover $\mathcal{B} = \{V_\alpha\}$ of a scheme X is a *semi-separating cover* if all the V_α, and also all the pairwise intersections $V_\alpha \cap V_\beta$ are affine schemes. Then the open immersions $V_\alpha \to X$ are affine maps, and it follows all finite intersections of V_α are affine. If X has a semi-separating cover $\mathcal{B}$, then X is semi-separated, for we take a semi-separating affine basis $\mathcal{A}$ to consist of all open affine subschemes U of X for which there is some V_β in $\mathcal{B}$ with $U \subseteq V_\beta$. Similarly, X is semi-separated if it has an open cover $\{V_\alpha\}$ with each V_α semi-separated and each open immersion $V_\alpha \cap V_\beta \to V_\beta$ an affine morphism.

We say a map $f : X \to Y$ of schemes is semi-separated if for every affine scheme Z and map $Z \to Y$, then the fibre product $Z \underset{Y}{\times} X$ is a semi-separated scheme. The class of semi-separated maps is closed under composition and base-change. If $f : X \to Y$ is a semi-separated map and Y is a semi-separated scheme, then X is a semi-separated scheme. (Consider $\mathcal{B} = \{f^{-1}(U_\alpha)\}$ on X for $\{U_\alpha\}$ a semi-separating basis for Y. Note each $f^{-1}(U_\alpha)$ is semi-separated, and that each $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to f^{-1}(U_\beta)$ is an affine map as it is the base-change of the affine map $U_\alpha \cap U_\beta \to U_\beta$.)

Given a morphism $f : X \to Y$, suppose Y has an open cover by affines $\{V_\alpha\}$ with all $V_\alpha \cap V_\beta \to V_\alpha$ being affine morphisms and all $f^{-1}(V_\alpha)$ being semi-separated schemes. Then $f : X \to Y$ is a semi-separated morphism. In particular, any map between semi-separated schemes is a semi-separated morphism.

A semi-separated scheme or morphism is quasi-separated ([EGA] I

6.1.12). A separated scheme is semi-separated, with semi-separating basis $\mathcal{A}$ consisting of all affine open subschemes of X. A separated map is semi-separated.

A scheme with an ample family of line bundles (2.1.1, or [SGA 6] II 2.2.4) is semi-separated. For let $\mathcal{A}$ be the set of all affine opens of the form $X_f = \{x | f(x) \neq 0\}$ as f runs over the set of all those global sections of tensor powers of line bundles in the family for which X_f is indeed affine. Then $\mathcal{A}$ is a basis for the topology by ampleness. Also $X_f \cap X_g = X_{fg}$, and this is affine if either X_f or X_g is by [EGA] II 5.5.8.

B.8. **Proposition.** *Let X be either noetherian, or else quasi-compact and semi-separated* (B.7). *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the canonical map is an isomorphism for all integers $k \geq 0$:*

$$R^k(Qcoh)\Gamma(X;\mathcal{F}) \xrightarrow{\cong} R^k(\mathcal{O}_X\text{-}Mod)\Gamma(X;\mathcal{F}) = H^k(X;\mathcal{F}). \tag{B.8.1}$$

Moreover, if $\mathcal{V} = \{V_\alpha\}$ is a semi-separating open cover of X (so the V_α and all their finite intersections are affine), then there is a canonical isomorphism to the Čech cohomology of $\mathcal{V}$

$$H^k(X;\mathcal{F}) \cong \check{H}^k(\mathcal{V};\mathcal{F}). \tag{B.8.2}$$

Proof. B.8.2 follows from the collapse of the Cartan-Leray spectral sequence in the usual way, [God] II 5.4.1, just as in [EGA] III 1.4.1. The key point is that since the intersections $V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$ are affine, $H^q(V_{\alpha_1} \cap \ldots V_{\alpha_n}; \varphi\mathcal{F}) = 0$ for $q > 0$ by Serre's Theorem [EGA] III 1.3.1.

For X noetherian, B.8.1 holds as $\varphi :$ Qcoh$(X) \to \mathcal{O}_X$-Mod preserves injective resolutions by B.4.

To prove B.8.1 for X quasi-compact and semi-separated, we take a semi-separating cover $\mathcal{V} = \{\mathcal{V}_\alpha\}$. We may assume $\mathcal{V}$ is a finite cover, on passing to a subcover on the quasi-compact X. For all finite sequences of indices $I = (\alpha_1, \ldots, \alpha_n)$, let $V_I = V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$, and let $j_I = V_I \to X$ be the open immersion. As $\mathcal{V}$ is semi- separating, each V_I is an affine scheme and each j_I is an affine map. In particular j_{I*} preserves quasi-coherence.

We consider the Čech complex of quasi-coherent sheaves

(B.8.3)
$$0 \to \mathcal{F} \to \bigoplus_{\alpha} j_{\alpha*} j_\alpha^* \mathcal{F} \to \bigoplus_{I=(\alpha_0,\alpha_1)} j_{I*} j_I^* \mathcal{F} \to \bigoplus_{I=(\alpha_0,\alpha_1,\alpha_2)} j_{I*} j_I^* \mathcal{F} \to \ldots .$$

This is an exact sequence of sheaves, and in fact the complex has a canonical chain contraction when restricted to any V_α.

As V_I is affine, $\Gamma(V_I;\ )$ is exact on Qcoh(V_I), so $R^k(\text{Qcoh})\Gamma(V_I;\mathcal{F}) = 0$ for $k > 0$. Similarly, as j_I is an affine map, $R^k(\text{Qcoh})j_{I*} = 0$ for

$k > 0$. So in the derived category $R^{\cdot}(\text{Qcoh})\Gamma(V_I;\ ) = \Gamma(V_I;\ )$, and $R^{\cdot}(\text{Qcoh})j_{I*} = j_{I*}$.

We consider the Grothendieck spectral sequence (B.8.4) for the derived functors of the composite $\Gamma(V_I;\ ) = \Gamma(X;\ ) \cdot j_{I*}$

(B.8.4)
$$E_2^{p,q} = R^p(\text{Qcoh})\Gamma(X; R^q(\text{Qcoh})j_{I*}(\ )) \Longrightarrow R^{p+q}(\text{Qcoh})\Gamma(V_I;\ ).$$

By the above, it collapses to yield isomorphisms for $k > 0$, $R^k(\text{Qcoh})\Gamma(X; j_{I*}(\ )) \cong R^k(\text{Qcoh})\Gamma(V_I;\ ) \cong 0$. So the sheaves $j_{I*}j_I^*\mathcal{F}$ are acyclic for $R^{\cdot}(\text{Qcoh})\Gamma(X;\ )$. Now the usual hypercohomology spectral sequence that results from applying $R^{\cdot}(\text{Qcoh})\Gamma(X;\ )$ to the Čech resolution B.8.3 of $\mathcal{F}$ collapses, yielding isomorphisms for $k \geq 0$

(B.8.5) $R^k(\text{Qcoh})\Gamma(X;\mathcal{F})$
$$\cong H^k\left(\Gamma\left(X; \bigoplus_{\alpha} j_{\alpha*}j_\alpha^*\mathcal{F}\right) \to \dots \Gamma\left(X; \bigoplus_{I} j_{I*}j_I^*\mathcal{F}\right) \to \dots\right) \cong \check{H}^k(\mathcal{V};\mathcal{F}).$$

Comparing this with (B.8.2) yields (B.8.1).

B.9. **Corollary.** *Let $f : X \to Y$ be either a quasi-compact and semi-separated map of schemes, or else a quasi-compact and quasi-separated map of schemes with X locally noetherian. Then for an quasi-coherent sheaf $\mathcal{F}$ on X, the canonical map is an isomorphism for all integers k*

$$\varphi : R^k(\mathit{Qcoh})f_*\mathcal{F} \xrightarrow{\cong} R^k(\mathcal{O}_X\text{-}\mathit{Mod})f_*\mathcal{F}.$$

Proof. This follows from B.8, as R^kf_* is the sheafification of $V \mapsto R^k\Gamma(f^{-1}(V);\ )$. We apply B.8 to $f^{-1}(V)$ for V affine open in Y.

(Note $\text{Qcoh}(X)$ might not have enough injectives under our hypothesis, but that the $\text{Qcoh}(f^{-1}(V))$ will for V affine in Y, and this suffices to define $R^*(\text{Qcoh})f_*$.)

B.10. The conclusion of B.9 is that there is a natural isomorphism

$$R^{\cdot}(\mathcal{O}_X\text{-Mod})f_* \cdot \varphi \cong \varphi \cdot R(\text{Qcoh})f_* : D^+(\text{Qcoh}(X)) \to D^+(Y).$$

B.11. **Proposition.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated map with Y quasi-compact. Then there exists an integer N such that for all $k \geq N$ and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, one has $R^kf_*(\mathcal{F}) = 0$ (i.e., $R^k(\mathcal{O}_X\text{-Mod})f_*(\mathcal{F}) = 0$ for $k \geq N$).*

Moreover N can be chosen to be universal in that same N works for any pullback $f' : X' \to Y'$ of f by any map $Y' \to Y$.

Proof. ([EGA] III 1.4.12, IV 1.7.21). The question is local on Y for Y quasi-compact, so we reduce to the case where Y is affine, and hence where $X = f^{-1}(Y)$ is quasi-compact and quasi-separated. We first consider the case where X is also semi-separated. Then let $\mathcal{W}$ be a finite semi-separating cover, and let N be the number of opens in $\mathcal{W}$. By B.8, $H^k(X, \mathcal{F}) \cong \check{H}^k(\mathcal{W}; \mathcal{F})$. But computing $\check{H}^k(\mathcal{W}; \mathcal{F})$ with the Čech complex of alternating cochains shows that it is 0 for $k > N$. Moreover, given any $Y' \to Y$, let V' be any affine open in Y'. Then $V' \underset{Y}{\times} \mathcal{W}$ is a semi-separating cover of $V \underset{Y}{\times} X$, so $H^k(V' \underset{Y}{\times} X; \mathcal{F}') = 0$ for $k > N$. Sheafifying this yields $R^k f'_*(\mathcal{F}') = 0$ for $k > N$ and $\mathcal{F}'$ quasi-coherent on $X' = Y' \underset{Y}{\times} X$.

Now we do the general case without assuming X is semi-separated. Let $\mathcal{W}$ be a finite affine cover of X. As X is quasi-separated, the $W_I = W_{i_0} \cap \ldots \cap W_{I_n}$ are quasi-compact open in the affine W_{i_0}, so all the W_I are quasi-affine, hence semi-separated. (For V' affine in Y', $V' \to Y$ is affine as Y is affine, so $V' \underset{Y}{\times} \mathcal{W}$ is an affine cover of $V' \underset{Y}{\times} X$.) By the semi-separated case, there is an integer N_1 such that for all the finitely many W_I, $H^k(W_I; \mathcal{F}) = 0$ for $k > N_1$. (Moreover, N_1 is universal in that $H^k(V' \underset{Y}{\times} W_I; \mathcal{F}') = 0$ for $k > N_1$ for the $V' \underset{Y}{\times} W_I$ which are affine over W_I, and any quasi-coherent sheaf $\mathcal{F}'$.) We now consider the Cartan-Leray spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{W}; H^q(W_J; \mathcal{F})) \Longrightarrow H^{p+q}(X; \mathcal{F}). \tag{B.11.1}$$

If $\mathcal{W}$ has N_2 open sets, $\check{H}^p(\mathcal{W};\ ) = 0$ for $p > N_2$, and it follows that $H^k(X; \mathcal{F}) = 0$ for $k > N_1 + N_2$ and $\mathcal{F}$ quasi-coherent. (This holds also for $V' \underset{Y}{\times} X$.)

B.12. **Lemma** ([SGA 6] II 3.2). *Let X be a quasi-compact and quasi-separated scheme. Then the exact inclusion functor $\varphi : Qcoh(X) \to \mathcal{O}_X$-Mod has a right adjoint, the coherator $Q : \mathcal{O}_X$-Mod $\to Qcoh(X)$.*

The adjunction map $1 \to Q \cdot \varphi$ is an isomorphism, so $Qcoh(X)$ is a reflexive subcategory of $\mathcal{O}_X$-Mod. In particular, $Qcoh(X)$ has all limits.

Proof. By B.3 and B.2, Qcoh(X) has a set of generators and all colimits. As φ preserves colimits, the special adjoint functor theorem insures that φ has a right adjoint Q. As φ is fully faithful, the adjunction map $1 \to Q\varphi$ is an isomorphism. (For, under the adjunction isomorphism Mor$(\varphi\mathcal{F}, (\ )) \cong$ Mor$(\mathcal{F}, Q(\ ))$, the map induced by $1 \to$

$Q\varphi$, $\mathrm{Mor}(\mathcal{F},\mathcal{G}) \to \mathrm{Mor}(\mathcal{F}, Q\varphi\mathcal{G})$ is the isomorphism $\varphi : \mathrm{Mor}(\mathcal{F},\mathcal{G}) \cong \mathrm{Mor}(\varphi\mathcal{F}, \varphi\mathcal{G})$.)

As the category $\mathcal{O}_X$-Mod has all limits, so does its reflexive subcategory Qcoh(X), as Q sends limits taken in $\mathcal{O}_X$-Mod to limits in Qcoh(X).

B.13. The coherator $Q : \mathcal{O}_X\text{-Mod} \to \mathrm{Qcoh}(X)$ preserves all limits as it is the right adjoint of the functor φ. As φ is exact, hence preserves monomorphisms, Q sends injectives in $\mathcal{O}_X$-Mod to injectives in Qcoh(X).

Let $f : X \to Y$ be a quasi-compact and quasi-separated map of schemes. As f^* preserves quasi-coherence, $\varphi_X \cdot f^* = f^* \cdot \varphi_Y$. As Q is right adjoint to φ and f_* is right adjoint to f^*, it follows that $Q_Y \cdot f_* = f_* \cdot Q_X$. (We note that indeed f_* restricts to a functor $f_* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ by B.6.)

B.14. For $X = \mathrm{Spec}(A)$ an affine scheme, Q_X is clearly the functor sending an $\mathcal{O}_X$-module $\mathcal{F}$ to the quasi-coherent sheaf associated to the A-module $\Gamma(X;\mathcal{F})$. For this functor is the adjoint to φ.

To deduce a formula for Q_X on a general quasi-compact and quasi-separated scheme X, we let $\{U_i\}$ be a finite cover of X by open affines. Each $U_i \cap U_j$ is quasi-compact, so we can choose a finite cover $\{U_{ijk}\}$ of $U_i \cap U_j$ by open affines. We denote all the various open immersions $U_{ijk} \to X$ as $j : U_{ijk} \to X$.

For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the sheaf axioms show that $\mathcal{F}$ is the difference kernel of the start of a Čech hypercover complex, where the two right maps are induced by $U_{ijk} \to U_i \cap U_j \to U_i$ and $U_{ijk} \to U_i \cap U_j \to U_j$

$$0 \to \mathcal{F} \to \bigoplus_i j_*(\mathcal{F}|U_i) \rightrightarrows \bigoplus_{(i,j,k)} j_*(\mathcal{F}|U_{ijk}). \tag{B.14.1}$$

Hence by B.13, we have

$$Q_X\mathcal{F} = \ker\left(\bigoplus_i j_*(Q_{U_i}(\mathcal{F}|U_i)) \rightrightarrows \bigoplus_{(ijk)} j_*\left(Q_{U_{ijk}}(\mathcal{F}|U_{ijk})\right)\right). \tag{B.14.2}$$

Here, Q_{U_i} and $Q_{U_{ijk}}$ are given by the first paragraph, as the U_i and U_{ijk} are affine.

B.15. **Lemma.** *For X a quasi-compact and quasi-separated scheme, $Q : \mathcal{O}_X$-Mod $\to$ Qcoh(X) preserves direct colimits.*

Proof. The global section functor $\Gamma(X;\ )$ preserves direct colimits by B.6. For A a commutative ring, the "associated sheaf" equivalence A-Mod $\to$ Qcoh(Spec(A)) preserves direct colimits. This proves the lemma for $X = \mathrm{Spec}(A)$ affine, by B.14. The case for a general X now follows from (B.14.2), the fact that difference kernels commute with the exact direct colimits of a Grothendieck abelian category, the result for Q_U in the case U is affine, and B.6 for the maps j_*.

B.16. **Proposition** (cf. [SGA 6] II 3.5). *Let X be either quasi-compact and semi-separated, or else noetherian. Then for any positive integer $q > 0$ and any quasi-coherent sheaf $\mathcal{F}$ on X, one has $(R^q Q)(\varphi \mathcal{F}) = 0$. We recall that $R^0 Q \varphi \mathcal{F} = Q \varphi \mathcal{F} = \mathcal{F}$.*

For any complex $E^\cdot$ in $D^+(Qcoh(X))$, the canonical map $E^\cdot \to (RQ)(\varphi E^\cdot)$ is a quasi-isomorphism.

For any complex $F^\cdot$ in $D^+(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology, the canonical map $\varphi(RQ(F^\cdot)) \to F^\cdot$ is a quasi-isomorphism.

Proof. The last two statements result from the first by the collapse of the usual hypercohomology spectral sequence

$$R^p Q\left(H^q\left(G^\cdot\right)\right) \Longrightarrow H^{p+q}\left(R^\cdot Q(G^\cdot)\right). \tag{B.16.1}$$

To prove the first statement, we first consider the case where X is affine. Then by B.14, $R^q Q(\varphi \mathcal{F}) = H^q(X; \varphi \mathcal{F})$. But this is 0 for $q > 0$ and X affine, by Serre's Theorem [EGA] III 1.3.1.

For the general case of X quasi-compact and quasi-separated, we consider the exact Čech complex of sheaves (B.8.3) of a finite semi-separating cover $\mathcal{V}$. Applying $RQ \cdot \varphi$ and considering the resulting bicomplex, we get a canonical spectral sequence

$$H^p\left(\ldots \to \bigoplus_I R^q\left(Q j_{I*}\right)\left(j_I^* \varphi \mathcal{F}\right) \to \ldots\right) \Longrightarrow \left(R^{p+q} Q\right)(\varphi \mathcal{F}). \tag{B.16.2}$$

As $j_I : V_I \to X$ is an affine map, $R^k j_{I*}(\mathcal{G}) = 0$ for $k > 0$ and $\mathcal{G}$ quasi-coherent. As V_I is affine $R^k Q(\mathcal{G}) = 0$ for $k > 0$ and $\mathcal{G}$ an $\mathcal{O}_{V_I}$-module by the affine case done above. As $Q_X \cdot j_* = j_* \cdot Q_{V_I}$, the resulting collapse of the Grothendieck spectral sequence $R^a j_*(R^b Q_{V_I}(\mathcal{G})) \Longrightarrow R^{a+b}(Q_X j_{I*})(\mathcal{G})$ yields that $R^q(Q j_{I*}) = 0$ for $q > 0$. Then the spectral sequence (B.16.2) also collapses. This yields that $R^k Q(\varphi \mathcal{F})$ is just H^k of the complex formed by applying Q to the complex (B.8.3) for $\varphi \mathcal{F}$. But commuting φ past the j_* and j^*, and using $Q\varphi \cong 1$, this is just the complex (B.8.3) for $\mathcal{F}$, which is exact. Thus $R^k Q(\varphi \mathcal{F}) = 0$ for $k > 0$.

To prove the first statement in the noetherian case, we take an injective resolution of $\mathcal{F}$ in Qcoh(X). By B.4, for X noetherian, φ of this resolution is an injective resolution of $\varphi \mathcal{F}$ in $\mathcal{O}_X$-Mod. Taking Q of this yields the original resolution as $Q\varphi \cong 1$, so using this resolution to compute $R^* Q$ yields that $R^k Q(\varphi \mathcal{F}) = 0$ for $k > 0$.

B.17. The right exact functor Q induces a derived functor RQ : $D^+(\mathcal{O}_X\text{-Mod}) \to D^+(\mathrm{Qcoh}(X))$. But this derived functor does not extend to unbounded complexes without further assumptions. Suppose however that RQ has finite cohomological dimension, i.e., that there is an integer N such that for all $q > N$ and all $\mathcal{O}_X$-modules $\mathcal{G}$, $R^q Q(\mathcal{G}) = 0$. Then RQ extends to derived functors $RQ : D(\mathcal{O}_X\text{-Mod}) \to D(\mathrm{Qcoh}(X))$ and $RQ : D^-(\mathcal{O}_X\text{-Mod}) \to D^-(\mathrm{Qcoh}(X))$, by [H] I Section 7, or [V] II Section 2 no. 2 Corollary 2-2 to Theorem 2.2 (one learns to appreciate the elegant and complete Wittgenstein-Grothendieck multi-decimals).

If X is quasi-compact and semi-separated, and if there is a bound on the $\mathcal{O}_U$-Mod cohomological dimension of the functor $H^*(U;\ )$ for all finite intersections $U = V_I = V_{i_0} \cap \ldots \cap V_{i_n}$ of the opens in some finite semi-separating cover $\mathcal{V} = \{V_i\}$ of X, then RQ has finite cohomological dimension, as we see by examining the proof of B.16. Indeed, the argument of [SGA 6] II 3.7 to prove this in the separated case immediately generalizes to the semi-separated case.

If X is neotherian, and if there is a uniform bound on the $\mathcal{O}_U$-Mod cohomological dimension of $H^*(U;\ )$ for all open U in X, then RQ has finite cohomological dimension by [SGA 6] II 3.7.

We recall that if X has a noetherian underlying space of finite Krull dimension, then for any open U in X, the $\mathcal{O}_U$-Mod cohomological dimension of $H^*(U;\ )$ is at most the Krull dimension of X by [Gro] 3.6.5.

In any case where RQ has finite cohomological dimension and extends to a derived functor on $D(\mathcal{O}_X\text{-Mod})$, the canonical maps $E^\cdot \xrightarrow{\sim} RQ(\varphi E^\cdot)$ and $\varphi RQ(F^\cdot) \xrightarrow{\sim} F^\cdot$ will be quasi-isomorphisms for $E^\cdot$ in $D(\mathrm{Qcoh}(X))$ and for $F^\cdot$ in $D(\mathcal{O}_X\text{-Mod})$ with quasi-coherent cohomology. This follows as in B.16 by collapse of the spectral sequence B.16.1, which converges strongly even for unbounded complexes thanks to the finite cohomological dimension of RQ.

In particular, B.16 remains true if we delete the hypotheses that $E^\cdot$ and $F^\cdot$ are cohomologically bounded below, and at the same time add the hypothesis that either X is noetherian of finite Krull dimension, or else is semi-separated and has underlying space a noetherian space of finite Krull dimension.

Appendix C:
Absolute noetherian approximation

In this appendix we review part of Grothendieck's theory of inverse limits of schemes from [EGA] IV Section 8. We then extend his theory of noetherian approximation to the case of general quasi-compact and quasi-separated schemes which are not necessarily finitely presented over an affine. Presumably, this would have been in [EGA] V or VI.

C.1. ([EGA] IV 8.2) Consider an inverse system of quasi-compact and quasi-separated schemes X_α, where the maps of the system (the "bonding maps") $f_{\alpha\beta} : X_\alpha \to X_\beta$ are all affine morphisms. Then an inverse limit scheme $X = \varprojlim X_\alpha$ exists, and the canonical maps $f_\alpha : X \to X_\alpha$ are all affine morphisms. Indeed, over an affine open $\mathrm{Spec}(A_\beta) \subseteq X_\beta$, $f^{-1}(\mathrm{Spec}(A_\beta))$ is $\mathrm{Spec}(A)$ for $A = \varinjlim A_\alpha$, where A_α runs over the direct system of rings A_α such that $\mathrm{Spec}(A_\alpha) = f_{\alpha\beta}^{-1}(\mathrm{Spec}(A_\beta)) \subseteq X_\alpha$ for $\alpha \geq \beta$.

C.2. ([EGA] IV 8.3.11, 8.6.3). For a system as in C.1, given any quasi-compact open $U \subseteq X = \varprojlim X_\alpha$, there is an α and a quasi-compact open $U_\alpha \subseteq X_\alpha$ such that $U = f_\alpha^{-1}(U_\alpha)$. If we set $U_\beta = X_\beta \times_{X_\alpha} U_\alpha$, then $U = \varprojlim U_\beta$, with the limit taken over the cofinal system $\beta \geq \alpha$. Also the closed subspace $X - U$ will be $f_\beta^{-1}(X_\beta - U_\beta)$ for any $\beta \geq \alpha$.

C.3. ([EGA] IV 8.8, 8.10.5). Given a system X_α as in C.1, let $g : Y \to X$ be a scheme finitely presented over X. Then there is an α and a finitely presented $g_\alpha : Y_\alpha \to X_\alpha$ such that

$$g = g_\alpha \times_{X_\alpha} X : Y = Y_\alpha \times_{X_\alpha} X \to X.$$

Then $Y = \varprojlim Y_\beta$ for $Y_\beta = Y_\alpha \times_{X_\alpha} X_\beta$ over the cofinal system of $\beta \geq \alpha$.

If $h : Z \to X$ is also finitely presented, and $k : Z \to Y = \varprojlim Y_\beta$ is any map over X, it follows that $k : Z \to Y$ is finitely presented. Hence k is $Y \times_{Y\alpha} k_\alpha$ for some α and some $k_\alpha : Z_\alpha \to Y_\alpha$. The finitely presented map $k : Z \to Y$ is respectively an immersion, a closed immersion, and open immersion, separated, surjective, affine, quasi-affine, finite, quasi-finite, proper, projective, or quasi-projective, iff k_β has the same property for all $\beta \geq \alpha$ for some α, iff k_α has the same property for some α.

C.4. ([EGA] IV 8.5). Suppose $X = \varprojlim X_\alpha$ for a system as in C.1. Then for any finitely presented, hence quasi-coherent, $\mathcal{O}_X$-module $\mathcal{F}$, there exists an α, a finitely presented $\mathcal{O}_X$-module $\mathcal{F}_\alpha$ on X_α, and an isomorphism $\mathcal{F} \cong f_\alpha^* \mathcal{F}_\alpha$. Let $\mathcal{F}_\beta = f_{\beta\alpha}^* \mathcal{F}_\alpha$ on X_β for all $\beta \geq \alpha$. Then $\mathcal{F} \cong \varinjlim \mathcal{F}_\beta$ as a module over $\mathcal{O}_X = \varinjlim \mathcal{O}_{X_\beta}$ as a sheaf on X_α.

The sheaf $\mathcal{F}$ is a vector bundle on X iff there is an α such that $\mathcal{F}_\alpha$ is a vector bundle on X_α. When the later condition is satisfied, $\mathcal{F}_\beta$ will be a vector bundle on X_β for all $\beta \geq \alpha$.

For any map $k : \mathcal{F} \to \mathcal{G}$ between finitely presented $\mathcal{O}_X$-modules, there will be an α and a map $k_\alpha : \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ such that k corresponds to $f_\alpha^*(k_\alpha)$ under the isomorphisms $\mathcal{F} \cong \mathcal{F}_\alpha$, $\mathcal{G} \cong f_\alpha^* \mathcal{G}_\alpha$. The map k is an isomorphism on X iff there is an α such that k_α is an isomorphism on X_α, and hence such that k_β is an isomorphism on X_β for all $\beta \geq \alpha$. A sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of finitely presented modules is right exact on X iff there is an α such that $\mathcal{F}_\alpha \to \mathcal{G}_\alpha \to \mathcal{H}_\alpha \to 0$ is right exact on X_α, and hence such that $\mathcal{F}_\beta \to \mathcal{G}_\beta \to \mathcal{H}_\beta \to 0$ is right exact on X_β for all $\beta \geq \alpha$.

A sequence of finitely presented modules on X is locally split short exact iff there is an α such that the corresponding sequence on X_α is locally split short exact. In particular, a sequence of algebraic vector bundles on X is short exact iff there is an α such that the corresponding sequence on X_α is short exact, iff there is an α such that the corresponding sequence on X_β is short exact for all $\beta \geq \alpha$.

C.5. ([EGA] IV 8.14.1). Let $f : T \to S$ be a map of schemes. Then f is locally finitely presented iff for all inverse systems of schemes over S, $\{X_\alpha\}$, satisfying the conditions of C.1, the canonical map (C.5.1) is an isomorphism

$$\text{(C.5.1)} \qquad \varinjlim_\alpha \mathrm{Mor}_S(X_\alpha, T) \xrightarrow{\cong} \mathrm{Mor}_S\left(\varprojlim X_\alpha, T\right).$$

C.6. **Proposition.** *Let Λ be a commutative ring. Let $\{X_\alpha\}$ be an inverse system of schemes as in* C.1, *and with all X_α finitely presented over* Spec(Λ) *and all bonding maps $f_{\beta\alpha} : X_\beta \to X_\alpha$ being maps over* Spec(Λ). *Then if $X = \varprojlim X_\alpha$ is an affine scheme, there exists an α such that X_β is affine for all $\beta \geq \alpha$.*

Proof. First note that we cannot quote C.3 or [EGA] IV 8.10.5, since we do not assume that X is finitely presented over Spec(Λ).

As the $f_{\beta\alpha}$ are all affine maps, it suffices to show some X_α is affine.

Let $A = \Gamma(X, \mathcal{O}_X)$, so $X = \mathrm{Spec}(A)$. We write $A = \varinjlim A_\gamma$ as a direct colimit of algebras finitely presented over Λ. Applying C.5 to $\varprojlim$ $\mathrm{Spec}(A_\gamma) = X \to X_\beta$, we see that there is a γ so that $X \to X_\beta$ factors as $X = \mathrm{Spec}(A) \to \mathrm{Spec}(A_\gamma) \to X_\beta$. We now apply C.5 to $\varprojlim X_\alpha = X \to$ $\mathrm{Spec}(A_\gamma)$ to see that there is an α such that $X \to \mathrm{Spec}(A_\gamma)$ factors as

$X \to X_\alpha \to \mathrm{Spec}(A_\gamma)$. By C.5, we may choose α sufficiently large so that the composite $X_\alpha \to \mathrm{Spec}(A_\gamma) \to X_\beta$ is $f_{\alpha\beta}$. As $\mathrm{Spec}(A_\gamma)$ is affine, hence a separated scheme, the map $\mathrm{Spec}(A_\gamma) \to X_\beta$ is a separated map. Hence the graph of the map $X_\alpha \to \mathrm{Spec}(A_\gamma)$ gives a closed immersion of X_α into the fibre product of X_α and $\mathrm{Spec}(A_\gamma)$ over X_β, as in (C.6.1)

(C.6.1)
$$\begin{tikzcd} X_\alpha \arrow[r] & X_\alpha \underset{X_\beta}{\times} \mathrm{Spec}(A_\gamma) \arrow[r] \arrow[d] & \mathrm{Spec}(A_\gamma) \arrow[d] \\ & X_\alpha \arrow[r, "f_{\alpha\beta}"'] & X_\beta \end{tikzcd}$$

As $f_{\alpha\beta}$ is an affine map, the right top arrow of (C.6.1) is affine. As closed immersions are affine maps, the composite map $X_\alpha \to \mathrm{Spec}(A_\gamma)$ is then an affine map, and X_α is an affine scheme as required (cf. [EGA] I 9.1, esp. 9.1.16(v) and 9.1.11.).

C.7. **Proposition.** *Let Λ be a commutative ring, and let $\{X_\alpha\}$ be an inverse system over* $\mathrm{Spec}(\Lambda)$ *of schemes finitely presented over* $\mathrm{Spec}(\Lambda)$, *satisfying the conditions of* C.1. *Then if $X = \varprojlim X_\alpha$ is a separated scheme, there is an α such that for all $\beta \geq \alpha$, X_β is separated.*

Proof. As the maps $X_\beta \to X_\alpha$ are affine, hence separated, it suffices to show some X_α is separated.

Let $\{U_i\}$ for $i = 1, \ldots, n$ be a finite open cover of X by affines. As X is separated, the maps $U_i \cap U_j \to U_i \times U_j$ are closed immersions, so the $U_i \cap U_j$ are affine and the maps of rings $\Gamma(U_i, \mathcal{O}) \otimes_{\mathbb{Z}} \Gamma(U_j, \mathcal{O}) \to \Gamma(U_i \cap U_j, \mathcal{O})$ are onto. Passing to a cofinal system of α by C.2, we may assume that $U_i = \varprojlim U_{i\alpha}$, $U_i \cap U_j = \varprojlim U_{i\alpha} \cap U_{j\alpha}$. By C.6, on passing to a cofinal subsystem, we may assume that all $U_{i\alpha}$ and all $U_{i\alpha} \cap U_{j\alpha}$ are affine. We denote $\Gamma(U_{i\alpha}, \mathcal{O}) = A_{i\alpha}$, $\Gamma(U_{i\alpha} \cap U_{j\alpha}, \mathcal{O}) = A_{ij\alpha}$. Then if for all (i,j) the map $A_{i\alpha} \otimes A_{j\alpha} \to A_{ij\alpha}$ is onto, $U_{i\alpha} \cap U_{j\alpha} \to U_{i\alpha} \times U_{j\alpha}$ will be a closed immersion for all (i,j), and X_α will be separated ([EGA] I 5.3.6).

As $U_{i\beta} \cap U_{j\beta}$ is the pullback of $U_{i\alpha} \cap U_{j\alpha}$ under $X_\beta \to X_\alpha$ or $U_{i\beta} \to U_{i\alpha}$, there is an isomorphism of coordinate rings

(C.7.1) $$A_{ij\beta} \cong A_{ij\alpha} \otimes_{A_{i\alpha}} A_{i\beta}.$$

As X is separated, $A_i \otimes A_j = \varinjlim (A_{i\alpha} \otimes A_{j\alpha}) \to \varinjlim (A_{ij\alpha}) = A_{ij}$ is onto. Fix a γ in the direct system of rings. As $A_{ij\gamma}$ is finitely generated as an algebra over Λ, there is an $\alpha \geq \gamma$ in the direct system of rings such that for all $\beta \geq \alpha$ and all the finitely many pairs (i,j), the image of $A_{i\alpha} \otimes A_{j\alpha}$ in $A_{ij\beta}$ contains all the generators of, and hence the image of, $A_{ij\gamma}$. Thus the image of $A_{i\alpha} \otimes A_{j\alpha}$ in $A_{ij\alpha}$ contains the image of $A_{i\alpha} \otimes A_{ij\gamma}$, and hence is all of $A_{ij\alpha}$ by (C.7.1). Thus $A_{i\alpha} \otimes A_{j\alpha} \to A_{ij\alpha}$ is onto for all (i,j), and X_α is separated, as required.

C.8. **Proposition.** *Let Λ be a commutative ring. Suppose $X = \varprojlim X_\alpha$ is an inverse limit of an inverse system over* Spec(Λ), *satisfying* C.1, *and with the schemes X_α finitely presented over* Spec(Λ). *Then if X has an ample family of line bundles* (2.1.1, or [SGA6] II 2.2), *there is an α such that for all $\beta \geq \alpha$, X_β has an ample family of line bundles.*

Proof. As all bonding maps $X_\beta \to X_\alpha$ are affine, it suffices to show some X_α has an ample family (2.1.2(g)).

If X has an ample family, there is a finite set of line bundles, $\mathcal{L}_i^{k_i}$, $i = 1, \ldots, n$, which are tensor powers of line bundles in the family, and sections $s_i \in \Gamma(X, \mathcal{L}_i^{k_i})$ such that each X_{s_i} is affine, and $X = \bigcup_{i=1}^{n} X_{s_i}$. By C.4, by taking α sufficiently large we may assume that the $\mathcal{L}_i$ are $f_\alpha^* \mathcal{L}_{i\alpha}$ for line bundles $\mathcal{L}_{i\alpha}$ on X_α, and that the sections s_i on X are induced by sections $s_{i\alpha} \in \Gamma(X_\alpha, \mathcal{L}_{i\alpha}^{\otimes k_i})$ on X_α. Then $X_{s_i} = f^{-1}(X_{\alpha s_{i\alpha}})$. As each X_{s_i} is affine, by C.6, on taking α sufficiently large, we may assume the $X_{\alpha s_{i\alpha}}$ are affine. As the X_{s_i} cover X, taking α larger still, we may assume that the $X_{\alpha s_{i\alpha}}$ cover X_α (apply C.3 to the finitely presented surjection $\coprod X_{s_i} \to X$). But then $\{\mathcal{L}_{i\alpha}\}$ is an ample family of line bundles for X_α by criterion 2.1.1(c).

C.8.1. In the situation of C.8, if X has a single line bundle $\mathcal{L}$ which is ample, there is an α such that for all $\beta \geq \alpha$, X_β has an ample line bundle $\mathcal{L}_\beta$ with $f_\beta^* \mathcal{L}_\beta = \mathcal{L}$, as the proof of C.8 shows.

C.9. **Theorem.** *Let Λ be a commutative noetherian ring (e.g., $\mathbb{Z}$). Let X be a quasi-compact and quasi-separated scheme over* Spec(Λ). *Then X is the limit $\varprojlim X_\alpha$ of an inverse system over* Spec(Λ) *of schemes X_α finitely presented over Spec(Λ). The bonding maps of the system $X_\alpha \to X_\beta$ are all affine maps, and are schematically dominant, so $\mathcal{O}_{X_\beta} \to f_{\alpha\beta}^* \mathcal{O}_{X_\alpha}$ is a monomorphism ([EGA] I 5.4).*

All the X_α are noetherian. If Λ has finite Krull dimension, all the X_α will have finite Krull dimension.

If X has an ample family of line bundles, we may arrange that all the X_α do. If X is semi-separated, we may arrange that all the X_α are semi-separated. If X is separated, we may arrange that all the X_α are separated.

Proof. As X is quasi-compact, it has a finite cover $\{U_1, \ldots, U_n\}$ by affine open subschemes. The construction of the inverse system proceeds by induction on the number n in such a cover.

If $n = 1$, X is affine. Say $X = \mathrm{Spec}(A)$. We write $A = \varinjlim A_\alpha$ as the direct colimit of those subrings A_α which are finitely generated over Λ. As Λ is noetherian, each A_α is then finitely presented over Λ, and $X = \varprojlim X_\alpha$ for $X_\alpha = \mathrm{Spec}(A_\alpha)$.

To do the induction step, we suppose the result is known for such X as are covered by $n-1$ affines, and in particular for $V = U_2 \cup U_3 \cup \ldots \cup U_n$. So we write $V = \varprojlim V_\alpha$. Set $U = U_1$. As X is quasi-separated, $U \cap V$ is quasi-compact, and so the open immersion $W = U \cap V \to V$ is finitely presented. By C.3, on passing to a cofinal system of α, we may assume that $W = \varprojlim W_\alpha$ for a system of finitely presented open immersions $W_\alpha \to V_\alpha$. Then each W_α is finitely presented over $\mathrm{Spec}(\Lambda)$.

As $W \subseteq U$ is quasi-affine, $\mathcal{O}_W$ is an ample line bundle for W. By C.8.1 and C.4, on passing to a cofinal system of α, we may assume that $\mathcal{O}_{W_\alpha}$ is an ample line bundle for the quasi-compact W_α. Then each W_α is quasi-affine. In fact, let $A = \Gamma(U, \mathcal{O}_U)$, so $U = \mathrm{Spec}(A)$. As W is quasi-affine in U, there are elements $g_i \in A$ for $i = 1, \ldots, n$ such that the $W_{g_i} = \mathrm{Spec}(A[1/g_i])$ are affine and cover W. For α sufficiently large, the $g_i \in \Gamma(W, \mathcal{O}_W) \supseteq \Gamma(U, \mathcal{O}_U)$ are in $\Gamma(W_\alpha, \mathcal{O}_{W_\alpha})$. By C.6, on taking α large, we may assume the $W_{\alpha g_i}$ are affine. Then $W_{\alpha g_i} = \mathrm{Spec}(\Gamma(W_\alpha, \mathcal{O}_{W_\alpha})[1/g_i])$, and it is of finite type over Λ since W_α is.

Consider now the pullback diagram of rings, where the indicated maps are monomorphisms by the schematic dominance of the maps $V_\alpha \to V_\beta$, and of the cover $\cup W_{\alpha g_i} \to W_\alpha$

(C.9.1)

$$\begin{tikzcd}
B_\alpha \arrow[rrr, tail] \arrow[d] \arrow[drrr, phantom, "\square"] & & & A \arrow[d] \\
\Gamma(W_\alpha, \mathcal{O}_{W_\alpha}) \arrow[r, tail] & \prod_1^n \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha}) \arrow[r, tail] & \prod_{i=1}^n \Gamma(W_{g_i}, \mathcal{O}_X) \arrow[r, equal] & \prod_1^n A[1/g_i]
\end{tikzcd}$$

As localization and direct colimits commute with pullbacks and finite products, we see that $B_\alpha[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$, and that $A = \varinjlim B_\alpha$.

We consider the direct system whose objects (α, A') consist of an α in the system of V_α, and a subring $A' \subseteq B_\alpha$, such that A' is of finite type over Λ, contains the g_i for $i = 1, \ldots, n$, and satisfies $A'[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$ for all i. As $W_{\alpha g_i}$ is affine and of finite type of Λ, and as $B_\alpha[1/g_i] = \Gamma(W_{\alpha g_i}, \mathcal{O}_{W_\alpha})$, such A' exist for each α. A morphism $(\alpha_1, A'_1) \to (\alpha_2, A'_2)$ is an $\alpha_1 \leq \alpha_2$ in the system of V_α (corresponding to a map $V_{\alpha_2} \to V_{\alpha_1}$), and an inclusion of rings $A'_1 \subseteq A'_2$ induced by the monomorphism $B_{\alpha_1} \subseteq B_{\alpha_2}$.

For each $\gamma = (\alpha, A')$ in this system, the map $W_\alpha \to \mathrm{Spec}(A')$ induced by $A' \to B_\alpha \to \Gamma(W_\alpha, \mathcal{O}_{W_\alpha})$ is an open immersion. Let X_γ be the scheme obtained by patching V_α and $\mathrm{Spec}(A')$ along the open W_α. Then the X_γ form an inverse system of schemes, with affine and schematically dominant bonding maps. The inverse system $X_\gamma \cap V_\alpha$ is clearly cofinal with the original system of V_α, so $\varprojlim X_\gamma \cap V_\alpha \cong \varprojlim V_\alpha = V \subseteq X$. For each fixed

α, $\varinjlim A' = B_\alpha$. As $\varinjlim B_\alpha = A$, taking the colimit of A' for all $\gamma = (\alpha, A')$ yields $\varinjlim A' = A$. Hence $\varprojlim X_\gamma \cap \operatorname{Spec}(A') = \operatorname{Spec}(A) = U \subseteq X$. Thus $\varprojlim X_\gamma = X$, as required. This completes the induction step, proving the first paragraph of the statement of the theorem.

We note the X_γ are of finite type over the noetherian Λ, hence are noetherian, and have finite Krull dimension if Λ does.

If X has an ample family of line bundles, C.8 shows that all the X_α will after passing to a cofinal subsystem. If X is semi-separated, applying C.6 to a semi-separating cover of X by affines shows that all the X_α in a cofinal subsystem will be semi-separated (B.7). By C.7, if X is separated, a cofinal subsystem of X_α will be separated.

Appendix D

Hypercohomology with supports

D.1. Let X be a topos with enough points. Let $Y \subseteq X$ be a closed subtopos, and $X - Y$ its open complement. Recall that $X - Y$ and Y have enough points ([SGA 4] IV 6, 9).

Let G be a presheaf of spectra on a site for X. (As in [Th1], one should assume G is a presheaf of "fibrant spectra." We can always attain this by replacing G by a homotopy equivalent presheaf.)

D.2. *Definition.* Let $\mathbf{H}^{\cdot}_Y(X;G)$ be the canonical homotopy fibre of the restriction map on the hypercohomology spectra of [Th1], $\mathbf{H}^{\cdot}(X;G) \to \mathbf{H}^{\cdot}(X-Y;G)$

$$\mathbf{H}^{\cdot}_Y(X;G) \to \mathbf{H}^{\cdot}(X;G) \to \mathbf{H}^{\cdot}(X-Y;G). \tag{D.2.1}$$

D.3. **Lemma.** *$\mathbf{H}^{\cdot}_Y(X;\ )$ preserves homotopy equivalences and homotopy fibre sequences of coefficient spectra. If both $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$ preserve direct limits up to homotopy, so does $\mathbf{H}^{\cdot}_Y(X;\ )$.*

Proof. This is clear from the homotopy fibre sequence (D.2.1) using the 5-lemma, the Quetzalcoatl lemma, the fact direct colimits preserve homotopy fibre sequences, and the corresponding properties of $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$, [Th1] 1.35, 1.39.

D.4. **Theorem.** *There is a hypercohomology spectral sequence (with Bousfield-Kan indexing) of homotopy groups*

$$E_2^{p,q} = H^p_Y(X;\tilde{\pi}_q G) \Longrightarrow \pi_{q-p} H^{\cdot}_Y(X;G). \tag{D.4.1}$$

Here the E_2 term is cohomology with supports ([SGA 4] V Section 6) *with the coefficients in the sheafification $\tilde{\pi}_q G$ of the presheaf $\pi_q G$ on X.*

The spectral sequence converges strongly if either there exists an integer M such that $\tilde{\pi}_q G = 0$ for all $q \geq M$, or else if $H^*_Y(X;\ )$ has finite cohomological dimension for the $\tilde{\pi}_* G$ so there exists an integer N such that $H^p_Y(X,\tilde{\pi}_q G) = 0$ for all q and for all $p \geq N$. (We note that $H^*_Y(X;\ )$ has finite cohomological dimension for $\tilde{\pi}_* G$ if both $H^*(X;\ )$ and $H^*(X-Y;\ )$ do, thanks to the long exact sequence of [SGA 4] V (6.5.4).)

Proof. Let $\{G\langle n\rangle\}$ for $n \in \mathbb{Z}$ be the Postnikov tower of G as in [Th1] 5.51, so $\pi_q G\langle n\rangle = \pi_q G$ for $q \leq n$, and $\pi_q G\langle n\rangle = 0$ for $q > n$. The map to the homotopy inverse limit of $H_Y^{\cdot}(X;\ )$ of this tower is a homotopy equivalence

$$\text{(D.4.2)} \qquad \mathbf{H}_Y^{\cdot}(X;G) \xrightarrow{\sim} \underset{\overleftarrow{n}}{\mathrm{holim}}\, \mathbf{H}_Y^{\cdot}(X;G\langle n\rangle).$$

This homotopy equivalence follows from the 5-lemma and (D.2.1), and the corresponding equivalences for $\mathbf{H}^{\cdot}(X;\ )$ and $\mathbf{H}^{\cdot}(X-Y;\ )$ given by [Th1] 1.37. The spectral sequence (D.4.1) will be the canonical spectral sequence of this $\underleftarrow{\mathrm{holim}}$, or of the $\underleftarrow{\lim}$ of a homotopy equivalent tower of fibrations, as in [Th1] 5.54, 5.43. Aside from the identification of the E_2 term, all results follow from [Th1] 5.43. The E_2 term of the canonical spectral sequence is given as $E_2^{p,q} = \pi_{q-p}\mathbf{H}_Y^{\cdot}(X;K(\pi_qG,q))$, where $K(\pi_qG,q)$ is the homotopy fibre of $G\langle q\rangle \to G\langle q-1\rangle$, and thus is equivalent to the presheaf of Eilenberg-Maclane spectra associated to the presheaf of abelian groups π_qG shifted q degrees [Th1] 5.52. It remains to identify this E_2 term with the cohomology with supports as in (D.4.1).

Shifting degrees q times by looping, we reduce to showing that if A is a presheaf of abelian groups, with sheafification $\tilde{A}$, and if $K(A,0)$ is a presheaf of spectra with $\pi_qK(A,0) = 0$ for $q \neq 0$, $\pi_0K(A,0) = A$, then there is a natural isomorphism for all p

$$\text{(D.4.3)} \qquad \pi_{-p}\mathbf{H}_Y^{\cdot}(X;K(A,0)) \cong H_Y^p(X;\tilde{A}).$$

We know from [Th1] 1.36 that we do have natural isomorphisms

$$\text{(D.4.4)} \qquad \begin{aligned} \pi_{-p}\mathbf{H}^{\cdot}(X;K(A,0)) &\cong H^p(X;\tilde{A}) \\ \pi_{-p}\mathbf{H}^{\cdot}(X-Y;K(A,0)) &\cong H^p(X-Y;\tilde{A}). \end{aligned}$$

As $K(A,0) \to K(\tilde{A},0)$ induces isomorphisms on $\pi_*\mathbf{H}^{\cdot}(X;\ )$ and $\pi_*\mathbf{H}^{\cdot}(X-Y;\ )$ by D.4.4, the 5-lemma and the long exact sequence of homotopy groups resulting from the defining fibration sequence D.2.1 shows that the map $\pi_*\mathbf{H}_Y^{\cdot}(X;K(A,0)) \to \pi_*\mathbf{H}_Y^{\cdot}(X;K(\tilde{A},0))$ is also an isomorphism. Thus we may assume that $A = \tilde{A}$ is a sheaf.

The isomorphisms (D.4.4) and the long exact sequence homotopy groups of (D.2.1), together with the obvious fact $H^p(X;\ ) = H^p(X-Y;\ ) = H_Y^p(X;\ ) = 0$ for $p < 0$, show that (D.4.3) trivially holds for $p < 0$, as both sides are 0. This argument also shows that $\pi_0\mathbf{H}_Y^{\cdot}(X;K(\tilde{A},0))$ is the kernel of $H^0(X;\tilde{A}) \to H^0(X-Y;\tilde{A})$, which is $H_Y^0(X;\tilde{A})$ by definition ([SGA 4] V 6). This proves (D.4.3) for $p = 0$.

If $\tilde{A}$ is an injective sheaf on X, it is also injective on U, so $H^p(X;\tilde{A}) = 0 = H^p(X-Y;\tilde{A})$ for $p > 0$. Also for $\tilde{A}$ injective, the long exact sequence [SGA 4] V 6.5.4 collapses into the short exact sequence

$$(\text{D.4.5}) \qquad 0 \to H^0_Y(X;\tilde{A}) \to H^0(X;\tilde{A}) \to H^0(X-Y;\tilde{A}) \to 0.$$

Comparing this with the long exact sequence of homotopy groups induced by (D.2.1), using the isomorphisms already established, we see that $\pi_{-p}\mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) = 0$ for $p > 0$ when $\tilde{A}$ is injective, so (D.4.3) is an isomorphism for all p when $\tilde{A}$ is injective.

Now suppose $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$ is some short exact sequence of sheaves. Then $\tilde{A} \to \tilde{B}$ is still a monomorphism in the category of presheaves. Let C be the cokernel presheaf, so $0 \to \tilde{A} \to \tilde{B} \to C \to 0$ is exact in the category of presheaves. (Note that $\tilde{C}$ is indeed the sheafification of C.) From the exact sequence, it follows that $K(\tilde{A},0) \to K(\tilde{B},0) \to K(C,0)$ is a homotopy fibre sequence of presheaves of spectra. Then by D.3, we have a homotopy fibre sequence of spectra

$$(\text{D.4.6}) \qquad \begin{array}{r} \mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) \to \mathbb{H}^{\cdot}_Y(X;K(\tilde{B},0)) \to \mathbb{H}^{\cdot}_Y(X;K(C,0)) \\ \wr\| \\ \mathbb{H}^{\cdot}_Y(X;K(\tilde{C},0)). \end{array}$$

This induces a long exact sequence of homotopy groups $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ \ ,0))$ for $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. Thus $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ \ ,0))$ is a cohomological ∂-functor on the category of sheaves. We recall $\pi_0\mathbb{H}^{\cdot}_Y(X;K(\ \ ,0)) \cong H^0_Y(X;\ \ )$ and that $\pi_{-p}\mathbb{H}^{\cdot}_Y(X;K(\tilde{A},0)) = 0$ for $\tilde{A}$ injective and $p \neq 0$. It follows that the $\pi_*\mathbb{H}^{\cdot}_Y(X;K(\ \ ,0))$ are the derived functors of $H^0_Y(X;\ \ )$, i.e., they are the $H^*_Y(X;\ \ )$.

This proves D.4.3 for all $\tilde{A}$ and all p, as required.

D.5. (Optional). Since [Th1] was written, André Joyal has shown that the category of simplicial objects in any topos has the structure of a Quillen closed model category [Q2]. Jardine has extended the result to provide a closed model structure on the category of presheaves of spectra on a site, and on that of sheaves of spectra in the topos [Ja].

This allows a more flexible and simple construction of $\mathbb{H}^{\cdot}_Y(X;\ \ )$ and $\mathbb{H}^{\cdot}(X;\ \ )$ than the canonical Godement resolution construction of [Th1]. Now to form $\mathbb{H}^{\cdot}(X;G)$ of a presheaf of spectra G, one takes any homotopy equivalent $G^{\sim}$ that is fibrant for the model structure, and takes global sections $\Gamma(X;G^{\sim}) \cong \mathbb{H}^{\cdot}(X;G)$. To make a version of $\mathbb{H}^{\cdot}(X;G)$ that is strictly functorial in the topos X, one works in the model category of the huge topos (cf. [SGA 4] IV 2.5, 4.10) that contains every topos in the universe.

Appendix E

The Nisnevich topology

E.1. *Definition.* The Nisnevich site of a scheme X is the category of schemes étale (hence finitely presented) over X, $U \to X$, with the following Grothendieck pretopology: A family $\{V_\alpha \to U\}$ in the site is cover if for all points $x \in U$, there is an α and a point $y_\alpha \in V_\alpha$ such that $V_\alpha \to U$ sends y_α to x and induces an isomorphism of residue fields $k(x) \xrightarrow{\cong} k(y_\alpha)$.

E.2. Clearly a map $f : X \to X'$ of schemes induces a preimage functor f^{-1} that determines a map of Nisnevich sites and topoi $f : X_{\mathrm{Nis}} \to X'_{\mathrm{Nis}}$ ([SGA 4] III 1.6, IV 4.9).

There are obvious natural morphisms of sites and topoi $X_{\mathrm{et}} \to X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$, as the Nisnevich topology is coarser than the étale topology, but finer than the Zariski topology.

E.3. An integral (en français "entier," pas "intègre") radicial surjective map $X' \to X$, and in particular the closed immersion $X_{\mathrm{red}} \to X$, induces an equivalence of Nisnevich sites and topoi $X'_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$.

Proof. By [SGA 4] VIII 1.1, the map induces an equivalence of étale sites. Under this equivalence, corresponding objects $U' \to U$ of the sites have isomorphic residue fields at corresponding points. Hence Nisnevich covers correspond under the equivalence, and the result follows.

E.4. *Example.* Let k be a field. The Nisnevich site of k consists of all finite products of fields étale over k, $\mathrm{Spec}(\Pi k_i') = \cup\ \mathrm{Spec}(k_i) \to\ \mathrm{Spec}(k)$. A family of fields covers a field k' exactly when a member of the family is isomorphic to k' by the given map. Thus the Nisnevich topos of a field k consists of copies of the trivial Zariski topoi of all fields k' étale over k, but with the copies related by a map of topoi for every map $\mathrm{Spec}(k') \to \mathrm{Spec}(k'')$ over $\mathrm{Spec}(k)$. Thus the Nisnevich topos of the field k is sort of a bigger Zariski topos of k, cf. [SGA 4] IV 4.10. Indeed there is a map of topoi $i : (k')_{\mathrm{Zar}} \to (k)_{\mathrm{Nis}}$ for k' étale over k, with i^* given be restriction of a sheaf to the Zariski topos of k'.

E.5. **Lemma** (Nisnevich [N3]). *Let X be a scheme. Then*

(a) *The Nisnevich topos X_{Nis} has enough points. In fact for every field k' étale over a residue field $k(x)$ of X, consider the map of topoi Sets $= (\mathrm{Spec}(k'))_{\mathrm{Zar}} \to (\mathrm{Spec}(k(x)))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$, where $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}}$ is as in E.4, and $(k(x))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ is the map induced by $\mathrm{Spec}(k(x)) \to X$. Then this family of morphisms of topoi Sets $\to X_{\mathrm{Nis}}$ is a conservative* *family of points.*

(b) *The filtering system of neighborhoods of such a point* $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ *is the system of all diagrams (E.5.1) with* $U \to X$ *étale, and* $(k') \to (k(x)) \to X$ *the given maps.*

(E.5.1)
$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "y"] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k(x)) \arrow[r] & X \end{tikzcd}$$

A cofinal subsystem is the category of such diagrams were $y : \mathrm{Spec}(k') \to U$ *induces an isomorphism of* k' *to the residue field* $k(y)$ *of* U *at the point* $y \in U$.

(c) *The stalk of the structure sheaf* $\mathcal{O}_X$ *in* X_{Nis} *at the point* $(k')_{\mathrm{Zar}} \to (k(x))_{\mathrm{Nis}} \to X$ *is the henselization of* $\mathcal{O}_{X,x}$ *at the residue field extension* $k(x) \to k'$ *([EGA] IV 18.8), or equivalently, the henselization* $\mathcal{O}^h_{U,y}$ *of the local ring* $\mathcal{O}_{U,y}$ *of any* U *in the cofinal system of (b) for which* $k' \cong k(y)$.

Proof. Part (a) follows from the criterion of [SGA 4] IV 6.5, that a set of points that distinguishes covering families from non-covering families in the site is then a conservative set of points for the topos. Part (b) follows from the definition, [SGA 4] IV 6.3. The cofinality statement results from first applying [EGA] IV 18.1 to extend $\mathrm{Spec}(k') \to \mathrm{Spec}(k(x))$ to an étale cover of the local ring $\mathrm{Spec}(\mathcal{O}_{X,x})$ by a local ring, and then extending this cover to an étale map $U \to X$ by a limit argument [EGA] IV 17.7.8. Part (c) follows from the definitions [SGA 4] IV 6.3, [EGA] IV 18.5, 18.6.

Note the analogy of this with the proofs of the corresponding results for the étale topology [SGA 4] VIII 3.5, 3.9 Section 4. See also [N3], but beware of its funny definition of point, which is not equivalent to that of [SGA 4], but is rather an acyclic category of classic points.

E.6. **Lemma** (Nisnevich, Kato-Saito). *Let* X *be a noetherian scheme. Then*

(a) The Nisnevich topos X_{Nis} *is a coherent topos, and even a noetherian topos in the sense of [SGA 4] VI 2.3, 2.11.*

(b) If $f : Y \to X$ *is a finite map, then* $f_* : Y_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$ *is exact and* $R^q f_* = 0$ *for* $q > 0$.

If $Y \to X$ *is a closed immersion,* Y_{Nis} *is a closed subtopos of* X_{Nis}*, with open complement* $(X - Y)_{\mathrm{Nis}}$.

(c) If X *has finite Krull dimension* N*, the cohomological dimension of* X_{Nis} *is at most* N.

(d) If X *has finite Krull dimension,* $\mathbf{H}^{\cdot}_{\mathrm{Nis}}(X;\ )$ *preserves up to homotopy direct colimits of presheaves of spectra.*

(e) Suppose X has finite Krull dimension. Let $\{U_\alpha\}$ be an inverse system with affine bonding maps of schemes étale over X, and with inverse limit $U = \varprojlim U_\alpha$. Let F be a presheaf of spectra on the category of quasi-compact and quasi-separated schemes, which is continuous in the sense that the canonical map $\varinjlim F(X_\alpha) \xrightarrow{\sim} F(\varprojlim X_\alpha)$ is a homotopy equivalence for any inverse system with affine bonding maps of quasi-compact and quasi-separated schemes X_α (cf. [Th1] 1.42). Then the canonical map

$$\varinjlim \mathbf{H}^{\cdot}_{\mathrm{Nis}}(U_\alpha; F) \xrightarrow{\sim} \mathbf{H}^{\cdot}_{\mathrm{Nis}}(U; F)$$

is a homotopy equivalence.

Proof. To prove (a), it suffices to show that for any U in the Nisnevich site of X that any Nisnevich cover $\{V_\alpha \to U\}$ of U has a finite subcover. As U is finitely presented over X, it is noetherian. So it suffices to show any Nisnevich cover of a noetherian scheme U has a finite subcover. We proceed by noetherian induction. Suppose the result is known for all closed subschemes $Y \neq U$. Let η be a generic point of U. By definition of Nisnevich cover E.1, there is a V_1 in the cover and a point $\eta' \in V_1$ such that $V_1 \to U$ induces an isomorphism $k(\eta') \to k(\eta)$. We claim that $V_1 \to U$ induces an isomorphism of an open *nbd* of η' onto an open *nbd* of η. If U and hence the étale V_1 are reduced, $k(\eta)$ and $k(\eta')$ are the local rings of U and V_1 at these generic points. The inverse isomorphism $k(\eta) \to k(\eta')$ then extends to an inverse isomorphism of some open *nbd*s by the finite presentation of $V_1 \to U$. In the general case, we apply the equivalence of sites E.3 of U and U_{red}, to reduce to the case where U is reduced. This proves our claim. So $V_1 \to U$ is a Nisnevich cover when restricted to the open *nbd* $W \subseteq U$ over which V_1 has an *nbd* isomorphic to W. As $U - W$ is a closed subspace and is not all U, the induction hypothesis shows that there is a finite set of V_β such that the induced $V_\beta \times_U (U - W) \to (U - W)$ cover $U - W$. Then $V_1 \to U$ and these $V_\beta \to U$ form a finite subcover of U, as required. This proves (a) (cf. [KS] 1.2.1).

Statement (b) follows by an argument parallel to the proof of the corresponding statements for the étale topology in [SGA 4] VII 6.3, 6.1, 5.5, replacing the descriptions of the stalks in the étale topology everywhere by E.5(b) and E.5(c). Whenever the étale case appeals to [SGA 4] VII 5.4 and 4.6, we instead use the fact that a finite extension of a hensel local ring is a hensel ring ([EGA] IV 18.5.10).

To prove (c), one proves the stronger statement that $H^q_{\mathrm{Nis}}(X; \mathcal{F}) = 0$ for all $q \geq p$ if $\mathcal{F}_y = 0$ for all $k(y)$ étale over $k(x)$ for those $x \in X$ with closure $\bar{x}$ of Krull dimension $\geq p$. This proof proceeds by induction on p and $\dim X$, using (b) and the method of [SGA 4] X 4.1. One starts by noting that if k is a field, $H^q_{\mathrm{Nis}}(k; \mathcal{F}) = 0$ for $q > 0$, as the global section functor $H^0_{\mathrm{Nis}}(k;\ )$ is isomorphic to taking the stalk at the point $Sets \to (k)_{\mathrm{Zar}} \to (k)_{\mathrm{Nis}}$ and so is exact. In general if $\dim k = 0$, k is an Artin ring, so k_{red} is a product of fields, and the result follows by E.3. This starts the induction, and one proceeds as in [SGA 4] X 4.1. Where [SGA 4] X makes an appeal to the theory of constructible sheaves, we note that these are just the coherent objects in the topos, so ([SGA 4] VI 2.14, 2.9) gives an adequate theory of constructible sheaves in any coherent topos like X_{Nis}.

For an even less detailed, hence more psychologically convincing, proof of (b) and (c), see [KS] 1.2.5. The final version of [N3] should also contain proofs.

Statements (d) and (e) follow from (a) and (c) using [Th1] 1.39, 1.41.

Appendix F

Invariance under change of universe

Let X be a scheme, and $\mathcal{U}_1 \subseteq \mathcal{U}_2$ two Grothendieck universes containing X ([SGA 4] I Appendix). We see successively for each of the categories in the following list that the change of universe functor is an equivalence of categories:

(a) category of all finitely presented $\mathcal{O}_X$-modules in the universe;

(b) category of all algebraic vector bundles on X in the universe;

(c) complicial biWaldlhausen category of all the strict perfect complexes on X in the universe;

(d) complicial biWaldhausen category of all the strict pseudo-coherent complexes on X in the universe.

For X quasi-compact and quasi-separated, we may add to this list

(e) the homotopy category of the biWaldhausen category of all perfect complexes on X in the universe;

(f) the homotopy category of the biWaldhausen category of all cohomologically bounded pseudo-coherent complexes on X in the universe.

For locally on affines of X, (e) and (f) hold by 2.3.1(d) and 2.3.1(e) and the equivalence of the homotopy categories of (c) and (d). The case of a general quasi-compact and quasi-separated X follows by the methods of 3.20.4 - 3.20.6.

Now 1.9.8 applies to show $G(X)$, $K(X)$, and hence $K^B(X)$ are invariant up to homotopy under change of universe. Similarly for $K(X \text{ on } Y)$.

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Department of Mathematics
The Johns Hopkins University

Université de Paris-Sud
Mathématiques, Bât. 425
91405 Orsay Cedex, France

Solitons elliptiques

A. TREIBICH et J.-L. VERDIER*
avec un appendice de J. Oesterlé

dédié à A. Grothendieck

1. Introduction

1.1. Dans ce mémoire nous étudions les solutions de l'équation de Korteweg-de Vries (KdV) :

$$u_t = 6uu_x - u_{xxx}$$

où $u(x,t)$ est une fonction à valeurs complexes, méromorphe dans $\mathbb{C} \times U$ où U est un voisinage ouvert connexe de zéro, et doublement périodique par rapport à un réseau $\Lambda \subset \mathbb{C}$ en la variable $x \in \mathbb{C}$. Cette étude a été commencée dans l'article [1] de H. Airault, H.P. McKean et J. Moser dont nous utilisons les résultats. Notons X la courbe elliptique $\mathbb{C}/\Lambda$, $\partial/\partial x$ l'image sur X du champ de vecteurs tangents canonique sur $\mathbb{C}$, et interprétons les fonctions périodiques par rapport à Λ comme des fonctions sur X. Les solutions de KdV du type décrit ci-dessus sont appelées dans ce travail des *solitons elliptiques de base X de l'équation KdV* ou simplement, lorsqu'aucune confusion n'en résulte, des *solitons elliptiques.* Une fonction méromorphe v sur X est appelée un *potentiel hyperelliptique de l'équation KdV* ou plus simplement potentiel hyperelliptique s'il existe un soliton elliptique $(x,t) \mapsto u(x,t)$ tel que $u(x,0) = v(x)$.

1.2. D'après *loc. cit.*, on sait que pour tout soliton elliptique u de base X, il existe un entier $n \geq 0$, une constante $c \in \mathbb{C}$ et une application analytique $t \mapsto (x_i(t)_{1 \leq i \leq n})$ de U dans le produit symétrique n-ième de X tels que

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** C'est avec beaucoup de tristesse que nous avons appris le décès de Jean-Louis Verdier, survenu accidentellement le 25 Août 1989. Le Comité éditorial.

$$u(x,t) = 2\sum_{1}^{n} \wp(x - x_i(t)) + c,$$

où $\wp$ est la fonction de Weierstrass de $(X, \partial/\partial x)$. L'entier n, l'application $t \mapsto (x_i(t)_{1\leq i\leq n})$ et la constante c sont uniquement déterminés par le soliton u. L'entier n s'appelle *le degré du soliton*. De plus, les $x_i(t)$ sont deux à deux distincts pour au moins une valeur de t.

1.3. On sait qu'à toute donnée de Krichever [9 et 10] hyperelliptique: (Γ, p, U, W, F), où Γ est une courbe algébrique hyperelliptique en $p \in \Gamma$, de genre arithmétique g, U, W sont des vecteurs tangents à l'origine de la variété jacobienne de Γ associés à une coordonnée locale antisymétrique de Γ en p, F un faisceau sur Γ localement libre de rang 1, de degré $g-1$ et en position générale, on associe [10] une solution $u(x,t)$ de l'équation de KdV, méromorphe en (x,t). Le faisceau F est appelé la donnée initiale de la solution $u(x,t)$. Nous allons montrer que les solitons elliptiques de base X de l'équation de KdV proviennent d'une donnée de Krichever (2.7). Donc à tout soliton elliptique est associé une donnée de Krichever qui lui donne naissance et qui est déterminée à isomorphisme unique près. En particulier le genre arithmétique de la courbe Γ est appelé le *genre* du soliton elliptique.

1.4. Soit u une fonction méromorphe d'une variable complexe $x \in \mathbb{C}$. Pour tout entier $l \geq 0$, notons $K_{2l+1}(u)$ le l-ième flot de la hierarchie KdV [3]. C'est un polynôme universel en u et ses dérivés, et les flots $\partial u/\partial t_l = K_{2l+1}(u)$ commutent deux à deux. On a en particulier $K_3(u) = 6uu_x - u_{xxx}$ de sorte que le premier flot n'est autre que l'équation KdV. Pour $(t_0, \ldots, t_l) \in \mathbb{C}^{l+1}$, notons $e(t_0K_0 + \ldots + t_lK_l)(u)$ l'intégrale de ces flots. On dit que deux fonctions méromorphes u_1 et u_2 sur $\mathbb{C}$ sont équivalentes s'il existe un entier $l \geq 0$, une constante c et $(t_0, \ldots, t_l) \in \mathbb{C}^{l+1}$ tels que $u_2(x) = e(t_0K_0 + \ldots + t_lK_l)(u_1)(x) + c$. On démontre dans [1] que toute fonction équivalente à un potentiel hyperelliptique sur X est encore un potentiel hyperelliptique sur X [cf. aussi (2.7)]. Deux potentiels hyperelliptiques équivalents ont même degré, même courbe Γ et par suite même genre.

1.5. Nos résultats portent sur les classes d'équivalence de potentiels hyperelliptiques. Nous démontrons que pour tout entier $n \geq 0$ et toute courbe elliptique X, il n'existe qu'un nombre fini de classe de potentiels hyperelliptiques sur X de degré n (4.9) et que tout potentiel hyperelliptique de degré n est de genre $g \leq \gamma(n) = \sup\{p \in \mathbb{N} | p(p+1)/2 \leq n\}$ (*loc. cit.*). On dit qu'un potentiel hyperelliptique u sur X est *primitif* s'il n'existe pas d'isogénie $\iota : X \longrightarrow X'$ de degré > 1 et de fonction u' sur X' tel que $u = u' \circ \iota$. Nous démontrons que pour tout entier $n \geq 0$, il existe des potentiels hyperelliptiques sur X de degré n et de genre $\gamma(n)$ (6.2 et 6.3), et que tout potentiel hyperelliptique de degré n et de genre $\gamma(n)$ est primitif si $n \neq 2$ et $n \neq 0$ (4.13). Enfin nous démontrons que le nombre de classes de potentiels hyperelliptiques de degré n et de genre $\gamma(n)$ ne peut être borné indépendamment de n.

1.6. Ces résultats sont à comparer avec ceux de [1] où sont étudiés les potentiels *rationnels*. Ce cas peut être considéré comme un cas limite du cas elliptique obtenu en faisant tendre vers l'infini les deux périodes génératrices du réseau Λ. Les classes de potentiels rationnels n'existent que si n est un nombre triangulaire (i.e. $n = \gamma(n)(\gamma(n)+1)/2$). Lorsque n est triangulaire, il n'existe qu'une seule classe: la classe du potentiel $u(x) = 2n/x^2$. Cette classe est de genre $\gamma(n)$. Ce cas dégénéré ne rend pas compte de la complexité du problème dans le cas général.

1.7. Ces résultats, de nature énumérative, sont obtenus en étudiant la courbe hyperelliptique Γ associée à une classe de potentiels hyperelliptiques de degré $n \geq 1$. Cette courbe est de manière naturelle, un revêtement tangentiel de X (2.2). On introduit la surface S, complétion du module des fibrés vectoriels de rang 1 à connexion sur X (3.1). On montre que tout revêtement tangentiel minimal de degré n de X est un élément d'un système linéaire de diviseurs sur S que l'on décrit (3.10). Comme on s'intéresse aux revêtements tangentiels hyperelliptiques, on est amené à étudier les diviseurs symétriques sur S par rapport à l'involution naturelle τ. Ces diviseurs proviennent de la surface quotient S/τ. Comme cette surface est singulière, on introduit la surface $S^\sim$ désingularisée minimale de S/τ (4.1). Cette surface est une surface rationnelle (4.2) [7, p. 513] et les revêtements tangentiels hyperelliptiques minimaux de X correspondent aux courbes rationnelles de certains systèmes linéaires sur $S^\sim$ que nous décrivons (4.2). Certaines de ces courbes sont des diviseurs exceptionnels de $S^\sim$ et correspondent à des classes de potentiels hyperelliptiques que nous appellons *exceptionnelles* (6.2). Ce sont ces classes (6.3) et quelques autres (6.6) que l'on peut énumérer. Le présent travail démontre les résultats annoncés ou conjecturés dans [20].

1.8. Pour tout $n \in \mathbf{N}$ et pour toute courbe elliptique réelle $(X, \partial/\partial x)$ dont l'ensemble des points réels $X_{\mathbf{R}}$ possède deux composantes connexes, posons $u(x) = 2n\wp(x+\omega)$ où ω est une demi-periode qui n'est pas dans la composante connexe de l'origine de $X_{\mathbf{R}}$. On sait depuis les travaux de Ince [8] (voir aussi [1]) que l'opérateur $-\partial^2/\partial x^2 + u(x)$ ne possède qu'un nombre fini de zones d'instabilité si et seulement si n est triangulaire. De plus Novikov [14] et McKean-Van Moerbeke [12] ont montré que les opérateurs périodiques réels $-\partial^2/\partial x + u(x)$ qui n'ont qu'un nombre fini de zones d'instabilité, possèdent des opérateurs différentiels de degré impair commutants dont les coéfficients sont des polynômes en u et ses dérivées. On en conclut par (2.7) (voir aussi [1] p. 136) que pour tout $p \in \mathbf{N}$ et pour toute courbe elliptique X comme ci dessus $p(p+1)\wp(x)$ est un potentiel hyperelliptique de degré $p(p+1)/2$ et il résulte de [8] que le genre de ce potentiel est p. Il résulte alors de (6.4) que le revêtement tangentiel hyperelliptique correpondant à ce potentiel est l'unique revêtement tangentiel hyperelliptique exceptionnel de X de degré $p(p+1)/2$ et de genre p. Notons que pour ces courbes elliptiques réelles, ce revêtement tangentiel est lisse, ainsi qu'il résulte de la théorie des solutions périodiques de KdV [12]. Mais cela n'est pas vrai pour toutes les courbes elliptiques comme on le constate aisement pour la courbe elliptique possédant un automorphisme d'ordre 3 et le revêtement tangentiel hyperelliptique de degré 3 et de genre arithmétique 2.

Pour les courbes elliptiques quelconques, Dubrovin et Novikov [4] (voir aussi [5]) ont étudié tous les solitons elliptiques de la forme $u(x,t) = 2(\wp(x-x_1(t)) + \wp(x-x_2(t)) + \wp(x-x_3(t)))$ et ont en particulier montré que $6\wp(x)$ est un potentiel hyperelliptique. Nous démontrons, en utilisant un résultat de Fay [6], que pour toute courbe elliptique et pour tout entier $n > 0$, le potentiel $2n\wp(x)$ est hyperelliptique si et seulement si n est triangulaire et que le revêtement tangentiel associé est dans ce cas l'unique revêtement tangentiel hyperelliptique de degré n et de genre arithmétique $\gamma(n)$ (6.5).

1.9. Pour décrire des solutions elliptiques de l'équation KP, Kritchever dans [11] introduit des courbes spectrales revêtements d'une courbe elliptique dont il donne les équations sous forme de polynômes caractéristiques de matrices à coéfficients fonctions elliptiques qui généralisent les matrices de Calogero [2] et Moser [13]. On peut montrer que des désingularisées convenables des courbes introduites par Krichever sont des revêtements tangentiels minimaux [18]. Nous reviendrons dans une publication ultérieure sur les matrices de Calogero-Moser-Krichever et préciserons les résultats de Krichever en nous débarrassant des hypothéses de généricité implicites dans [11].

1.10 *Sommaire.*

2. Potentiels hyperelliptiques et revêtements tangentiels.

2.1. Soit $\mathrm{Sym}^n(X)$ le produit symétrique n-ième et notons Δ le diviseur des diagonales partielles de $\mathrm{Sym}^n(X)$. Dans $\mathrm{Sym}^n(X) - \Delta$, les n-équations indéxées par $i, 1 \leq i \leq n$:

$$\sum_{j=1, j\neq i}^{n} \wp'(x_j - x_i) = 0,$$

définissent une sous-variété algébrique dont on note $I^n(X)$ l'adhérence dans $\mathrm{Sym}^n(X)$.

2.2. Soient Γ une courbe, i.e. une variété algébrique complexe (irréductible et réduite) complète de dimension 1, $p \in \Gamma$ un point lisse et $\pi : \Gamma \to X$ un morphisme fini de degré n, tel que $\pi(p) = q$ où q est l'élément neutre de X. Identifions X à sa jacobienne. L'application d'image inverse des fibrés vectoriels de rang 1 et de degré 0, définit un morphisme fini de groupes algébriques $\pi^* : X \to \mathrm{Jac}\,\Gamma$, de X dans la variété jacobienne de Γ. On dispose par ailleurs de l'application d'Abel-Jacobi, Ab: $\Gamma^\circ \to \mathrm{Jac}\,\Gamma$, définie sur l'ouvert de lissité Γ° de Γ, qui envoit p sur l'élément neutre de $\mathrm{Jac}\,\Gamma$. On dit que π (ou Γ) est un *revêtement tangentiel* de X, si la courbe elliptique $\pi^*(X)$ est tangente à la courbe $\mathrm{Ab}(\Gamma^\circ)$ en l'élément neutre de $\mathrm{Jac}\,\Gamma$.

2.3. Un revêtement tangentiel est étale en p. En effet l'application d'Albanese $F \mapsto \mathrm{Alb}(F) = \det(\pi_* F) \otimes \det(\pi_* O_\Gamma)^{-1}$ de $\mathrm{Jac}\,\Gamma$ dans X est telle que Alb $\circ\pi^*$: $X \to X$ est la multiplication par n. Elle induit donc un morphisme étale sur $\pi^*(X)$ et par suite un morphisme étale à l'origine sur $\mathrm{Ab}(\Gamma^o)$. Donc $\pi = \mathrm{Alb}\circ\mathrm{Ab}$ est étale en p. Si z est la coordonnée locale canonique sur X au voisinage de q (telle que $\partial z/\partial x = 1$), alors $z\circ\pi$ est une coordonnée locale sur Γ au voisinage de p. On pose $U = n.\partial \mathrm{Ab}/\partial(z\circ\pi)|_{z=0}$, $V = n^2.\partial^2\mathrm{Ab}/\partial(z\circ\pi)^2|_{z=0}$, $W = n^3.\partial^3\mathrm{Ab}/\partial(z\circ\pi)^3|_{z=0}$. On définit ainsi trois vecteurs tangents à Jac Γ en l'élément neutre.

2.4. Un revêtement tangentiel $\pi : \Gamma \to X$ est dit *symétrique* s'il existe une involution τ de Γ qui fixe le point p et qui est au dessus de l'involution canonique τ de X. Une telle involution est alors uniquement déterminée. Supposons que Γ soit hyperelliptique en p i.e. qu'il existe une involution σ de Γ qui fixe p et telle que $\Gamma/\sigma \simeq \mathbf{P}^1$, alors le revêtement tangentiel $\pi : \Gamma \to X$, dit hyperelliptique, est symétrique et la symétrie τ est l'involution hyperelliptique σ. La coordonnée locale $z\circ\pi$ sur Γ (2.3) est alors antisymétrique et par suite le vecteur V correspondant est nul.

2.5. Soient g le genre arithmétique de Γ revêtement tangentiel de X, $\mathrm{Jac}_{g-1}\Gamma$ la variété des fibrés vectoriels de rang 1 et de degré $g-1$ sur Γ, $\Theta \subset \mathrm{Jac}_{g-1}\Gamma$ le diviseur canonique. Nous dirons qu'un fibré $F \in \mathrm{Jac}_{g-1}\Gamma$ est en position générale par rapport à X si $\pi^*(X).F \not\subset \Theta$. En fait, tous les $F \in \mathrm{Jac}_{g-1}\Gamma$ sont en position générale par rapport à X. Cela est classique lorsque Γ est lisse et on peut montrer qu'il en est de même dans le cas général en utilisant les résultats de G. Segal et G. Wilson [16], mais nous n'utiliserons pas ce fait.

2.6. Soient $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique de degré n et de genre g, α un nombre complexe et $F \in \mathrm{Jac}_{g-1}\Gamma$ en position générale par rapport à X. Alors $(\Gamma, p, U, W + \alpha U, F)$ est une donnée de Krichever que nous appellons *donnée de Krichever tangentielle à X*. Par la construction de Krichever [9], on en déduit une fonction méromorphe $(x,t) \mapsto u(x,t)$ solution de l'équation KdV. Comme le morphisme π est tangentiel, la trajectoire de tout $G \in Jac_{g-1}$ sous le flot linéaire défini par U est $\pi^*(X).G$. Par suite, pour tout t voisin de zéro, $x \mapsto u(x,t)$ est une fonction méromorphe sur $\mathbf{C}$, périodique de période Λ. C'est donc un soliton elliptique et en particulier $u(x,0)$ est un potentiel hyperelliptique sur X (1.1) appellée le potentiel hyperelliptique associé à la donnée de Krichever tangentielle à X.

Théorème 2.7. *Soient u une fonction méromorphe sur X et n un entier > 0. Les propriétés suivantes sont équivalentes:*

(i) *La fonction u est un potentiel hyperelliptique de degré n (1.1).*

(ii) *Il existe un opérateur différentiel* $M = \partial^k/\partial x^k + a_1\partial^{k-1}/\partial x^{k-1} + \ldots$ *de degré impair et à coéfficients méromorphes sur* X *qui commute à* $L = -\partial^2/\partial x^2 + u$.

(iii) *Il existe un revêtement tangentiel hyperelliptique* $\pi : \Gamma \to X$, *de degré* n *et de genre* g, *un* $F \in \mathrm{Jac}_{g-1}\Gamma$ *en position générale par rapport à* X, *un nombre complexe* α *tel que* u *soit le potentiel hyperelliptique associé à la donnée de Krichever* $(\Gamma, p, U, W + \alpha U, F)$ *tangentielle à* X *(2.5).*

2.8. Démonstration. Démontrons (i) $\Longrightarrow$ (ii). Pour $(y_i) \in I^n(X)$ et $c \in \mathbf{C}$ posons $u_n(x, (y_i), c) = 2\sum_1^n \wp(x - y_i) + c$. On démontre dans [1] qu'il existe $(x_i) \in I^n(X)$ (2.1) et un nombre complexe c tel que $u(x) = u_n(x, (x_i), c)$. De même dans *loc. cit.* sont construits des champs de vecteurs algébriques X_{2l+1} tangents à $\mathrm{Sym}^n(X)$ et définis sur $Sym^n(X)\backslash\Delta$. De plus les champs de vecteurs X_{2l+1} commutent entre eux et sont tangents à $I^n(X)\backslash\Delta$ et on a $K_{2l+1}u_n = X_{2l+1}u_n$. Soient $t \mapsto x_i(t)$ des germes d'application tels que $u(x,t) = 2\sum_i^n \wp(x - x_i(t)) + c$ soit un soliton elliptique (1.2). Pour $t \neq 0$ et t voisin de zéro, on a $(x_i(t)) \in I^n(X)\backslash\Delta$. Soit Z le plus petit sous-ensemble algébrique fermé dans $I^n(X)$ contenant les $(x_i(t))$ et dont l'intersection avec $I^n(X)\backslash\Delta$ soit stable par les flots X_{2l+1}, $l \in \mathbf{N}$. Alors Z est irréductible. Il existe un ouvert de Zariski, dense et lisse, de Z, un plus petit entier d, des fonctions algébriques $\alpha_i, 0 \leq i \leq d$, dont une est une constante non nulles sur cette ouvert, tels que $\Sigma\alpha_i X_{2i+1} = 0$. Comme les champs de vecteurs X_{2l+1} commutent entre eux, on a $X_{2l+1}(\alpha_i) = 0$ pour tout l, en vertu de la minimalité de d. Donc les fonctions α_i sont constantes en vertu de la minimalité de Z et on a $\Sigma\alpha_i X_{2i+1} = 0$ sur $Z\backslash\Delta$. On a donc $(\Sigma\alpha_i K_{2i+1})u_n = 0$ pour tout $(y_i) \in Z\backslash\Delta$ et par suite on a $(\Sigma\alpha_i K_{2i+1})u = 0$. Par construction [3], on a $K_{2i+1}u = [(L^i(\sqrt{L}))_+, L]$ où $(L^i(\sqrt{L}))_+$ est la partie entière de l'opérateur pseudodifférentielle $L^i(\sqrt{L})$. Par suite, on a $[(\Sigma\alpha_i L^i(\sqrt{L}))_+, L] = 0$ et on a ainsi construit un opérateur différentiel de degré impair qui commute à L. On a vu en (2.6) que (iii) $\Longrightarrow$ (i). Il reste à démontrer (ii) $\Longrightarrow$ (iii). Soit A l'algèbre des opérateurs différentiels qui commutent à L. On sait [19] que A est l'algèbre affine d'une courbe hyperelliptique complète Γ privée d'un point de Weierstrass lisse p. Choisissons un vecteur tangent t_p à Γ en p, tel que $t_p^{\otimes 2} \in O_p(2p)$ soit la partie polaire en p de la fonction $L \in A = H^0(\Gamma - p, O_\Gamma)$. Notons $D \subset X$ la réunion des supports des diviseurs polaires des coéfficients des éléments de L. C'est une partie finie de X ; posons $X' = X\backslash D$. Par une construction de Krichever globale sur X' [19], on obtient un module cohérent M sur $\Gamma \times X'$ muni d'une section σ_p sur $\{p\} \times X'$ et d'une $O_{X'}$-dérivation $\nabla : M \to M(\{p\} \times X')$, au dessus de $\partial/\partial x$ qui possède les propriétés suivantes:

(1) M est localement libre de rang 1[1].

(2) Pour tout $x \in X'$, on a $\chi(M_x) = 0, h^0(M_x) = 0$.

(3) La restriction de M à $\{p\} \otimes X'$ est libre de rang 1, trivialisée par σ_p.

(4) La restriction de ∇ à $\{p\} \otimes X'$ est O'_X-linéaire et on a $\nabla(\sigma_p) = \sigma_p \otimes t_p$.

Il résulte de (1), (2) et (3), que M est décrit par un morphisme $\mu : X' \to \mathrm{Jac}_{g-1}\Gamma$ où g est le genre arithmétique de Γ. Il résulte de (4) qu'en chaque point $x \in X'$, $d\mu(\partial/\partial x)$ est parallèle au vecteur $U = \partial\mathrm{Ab}(t_p)$. On en déduit que le sous-groupe analytique à un paramètre Y engendré par U est un sous groupe algébrique de dimension 1 de $\mathrm{Jac}\,\Gamma$ muni d'un morphisme $X' \to Y$ qui transforme le champ de vecteurs $\partial/\partial x$ en un champ de vecteurs invariant par translation sur Y. Il en résulte que Y est une courbe elliptique et qu'il existe une isogénie $\iota : X \to Y$ et un $F \in \mathrm{Jac}_{g-1}\Gamma$ tel que $\mu(x) = \iota(x).F$ pour tout $x \in X'$. En particulier le morphisme μ s'étend en un morphisme encore noté μ de X dans $\mathrm{Jac}_{g-1}\Gamma$ et donc le module M s'étend en un module localement libre $\underline{M}$ sur $\Gamma \times X$ dont la restriction à $\{p\} \times X$ est triviale. Le module $\underline{M}$ définit un morphisme $\pi : \Gamma \to X$ obtenu en identifiant X à sa jacobienne. Le morphisme $\pi^* : X \to \mathrm{Jac}\,\Gamma$ n'est autre que le composé de ι avec l'injection canonique $Y \to \mathrm{Jac}\,\Gamma$. Il résulte donc de ce qui précède que π est un revêtement tangentiel (2.2) et de (2) que F est en position générale. Par construction, le potentiel u est associé à la donnée de Krichever $(\Gamma, p, \partial\mathrm{Ab}/\partial\lambda, \partial^3\mathrm{Ab}/\partial\lambda^3, F)$ où λ est la coordonnée locale analytique en le point p telle que $d\lambda(t_p) = 1$ et $\lambda^2 L = 1$. L'endomorphisme $\mathrm{Alb}\circ\pi^*$ de X est la multiplication par n et, par construction, on a $d\pi^*(\partial/\partial x) = d\mathrm{Ab}(t_p)$. On a donc $n\partial/\partial x = d\mathrm{Alb}\circ d\mathrm{Ab}(t_p)$ et comme on a $\pi = \mathrm{Alb}\circ\mathrm{Ab}$, on en déduit $d\pi(t_p) = n\partial/\partial x$. Par suite la fonction $\lambda - n^{-1}z\circ\pi$ est antisymétrique et à une dérivée nulle à l'origine. Il existe donc $\alpha \in \mathbb{C}$ tel que la fonction $\lambda - n^{-1}z\circ\pi + (\alpha/6)n^{-3}(z\circ\pi)^3$ soit d'ordre ≥ 5 en p. On en déduit aussitôt $\partial\mathrm{Ab}/\partial\lambda = U, \partial^3\mathrm{Ab}/\partial\lambda^3 = W + \alpha U$ (2.3). c.q.f.d.

2.9. Il résulte de la théorie de l'équation KdV que faire agir les flots de la hierarchie KdV sur un potentiel hyperelliptique u revient à changer le fibré initial F [16] de sa donnée de Krichever tangentielle. Il est élémentaire de vérifier que si $u(x,t)$ est une solution de l'équation KdV, alors pour

[1] Un module minimal [maximal dans la terminologie de 16] sur une courbe à singularités planaires (comme les courbes hyperelliptiques) est localement libre: Soit F non localement libre, génériquement de rang 1, sans torsion sur Γ. Pour des raisons de dimension [15], il existe un L localement libre; non trivial sur Γ; tel que $F \otimes L = F$. On a donc End $F \otimes L =$ End F et par suite End $F \neq O_\Gamma$.

toute constante β, $v(x,t) = u(x + 6\beta t, t) + \beta$ est encore une solution de l'équation KdV. Par suite changer la constante c (1.2) revient à changer la constante α de la donnée de Krichever tangentielle $(\Gamma, p, U, W + \alpha U, F)$. Le théorème (2.7) établit donc une correspondance biunivoque entre les classes d'équivalence de potentiels hyperelliptiques de degré n sur X et les classes d'isomorphismes de revêtements tangentiels hyperelliptiques de degré n de base X. Dans cette correspondance, les classes des potentiels hyperelliptiques primitifs (1.5) correspondent aux *revêtements tangentiels primitifs*, c'est à dire les revêtements tangentiels hyperelliptiques $\Gamma \to X$ qui ne se factorisent pas à travers une isogénie non triviale $X' \to X$.

2.10. Soient $\Gamma_i \to X$, $i = 1, 2$, deux revêtements tangentiels de degré n de X. Une *domination* de Γ_1 sur Γ_2 est un X-morphisme de Γ_1 sur Γ_2 qui préserve les points bases. Une domination de Γ_1 sur Γ_2 est un isomorphisme birationnel. Réciproquement si $\Gamma \to X$ est un revêtement tangentiel et si $\Gamma' \to \Gamma$ est un morphisme qui est un isomorphisme birationnel, alors le revêtement composé $\Gamma' \to X$ est tangentiel. Un revêtement tangentiel (resp. tangentiel hyperelliptique) $\Gamma \to X$ est dit minimal (resp.hyperelliptique minimal) si toute domination de Γ sur un revêtement tangentiel (resp. tangentiel hyperelliptique) $\Gamma' \to X$ est un isomorphisme. On prendra garde au fait qu'un revêtement tangentiel hyperelliptique minimal n'est pas nécéssairement un revêtement tangentiel minimal.

3. La surface S et les revêtements tangentiels

3.1. Dans ce paragraphe, X désigne une courbe elliptique complexe, $\partial/\partial x$, un champ de vecteurs tangents non nuls, invariants par translation sur X et $q \in X$ est l'élément neutre de X. On identifie X à sa jacobienne. Soit Δ l'espace des classes d'isomorphismes des fibrés vectoriels F de rang 1 sur X muni d'une dérivation ∇ au dessus de $\partial/\partial x$. Alors Δ est un groupe algébrique commutatif pour le produit tensoriel et $\Delta(\mathbb{C})$ est analytiquement isomorphe à $\mathrm{Hom}(\Pi_1(X), \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^2$, l'isomorphisme étant donné par l'application qui à (F, ∇) associe le système local $\ker\nabla_{\mathrm{an}}$. L'homomorphisme d'oubli de la dérivation $\pi : \Delta \to \mathrm{Jac}\, X = X$ est un homorphisme surjectif et le noyau en est le groupe des dérivations ∇ de O_X isomorphe au groupe additif G_a par l'homomorphisme $\nabla \mapsto \nabla(1)$. On a donc une suite exacte de groupes algébriques commutatifs considérée par J.P. Serre [17] :

$$0 \to G_a \to \Delta \to X \to 0$$

Il s'ensuit que Δ est un fibré de base X principal sous G_a. Notons que ce fibré est analytiquement non trivial: L'espace de Stein $\Delta(\mathbb{C})$ n'est pas analytiquement isomorphe à $\mathbb{C} \times X$ qui n'est pas de Stein. Considérons Δ comme un espace linéaire affine au dessus de X. Soient $\Delta \to W$ le plongement linéaire affine universel dans un fibré vectoriel W sur X de rang 2 , $S = \mathbf{P}(W)$ la complétion projective canonique de Δ et notons encore $\pi : S \to X$ la projection canonique. Le complémentaire de Δ dans S est un diviseur C_0 image d'une section de π. Notons S_q la fibre de π au dessus de q. Posons $p = C_0 \cap S_q$. Comme le groupe structural de Δ est le sous-groupe G_a du groupe affine, le fibré $\Lambda^2 W$ est trivial et le sous-fibré de W correspondant à la section C_0 est trivial. On a donc une suite exacte $0 \to O_X \to W \to O_X \to 0$. Comme Δ est un fibré non trivial, cette suite exacte n'est pas scindable.

Par transport de structures, la symétrie canonique τ de X se relève en une symétrie de S encore notée τ qui induit sur Δ la symétrie canonique. De même le groupe additif G_a agit sur le fibré principal Δ de base X et cette opération se prolonge par transport de structures en une opération de G_a sur S qui préserve π. On peut montrer que l'homomorphisme ainsi défini de G_a dans le groupe $\mathrm{Aut}(S/X)$ des automorphismes de S qui préserve π, est un isomorphisme.

L'application $L \mapsto \pi^* L$ de $\mathrm{Jac}\, X = X$ dans le groupe de Picard $\mathrm{Pic}(S)$ est un isomorphisme de X sur la composante connexe de l'élément neutre $\mathrm{Pic}^0(S)$. Le groupe quotient $NS(S) = \mathrm{Pic}\,(S)/\mathrm{Pic}^0(S)$ est un groupe libre de rang 2 engendré par les classes des diviseurs C_0 et S_q. On a $(C_0)^2 = (S_q)^2 = 0$ et $C_0.S_q = 1$. Le quotient canonique de $\pi^*(W)$ est $O_S(C_0)$ et le diviseur canonique de S est $O_S(-2C_0)$.

Nous aurons besoin ultérieurement des lemmes suivants :

Lemme 3.2. *Posons, pour tout* $(i,n,m) \in \mathbb{Z}^3$, $h^i(S,n,m) = \dim H^i(S, O_S(nC_0 + mS_q))$.

(1) *Pour* $n \geq 0$, *on a* $h^i(S,n,0) = 1$ *si* $i = 0$ *ou 1 et* $h^2(S,n,0) = 0$.

(2) *Pour* $n = -1$ *et tout* m, *et pour* $n \geq 0$ *et* $m > 0$ *on a* $h^0(S,n,m) = (n+1)m$ *et* $h^i(S,n,m) = 0$ *si* $i = 1$ *ou 2*.

(3) *Pour tout* $(i,n,m) \in \mathbb{Z}^3$, *on a* $h^i(S,n,m) = h^{2-i}(S,-2-n,-m)$.

(4) *Pour* $n < -1$ *et tout* m, *on a* $h^i(S,n,m) = h^{1-i}(S,-2-n,m)$.

(5) *Pour tout diviseur* δ *sur* X *de degré* $m > 0$, *on a* $h^0(S, O_S\ (nC_0 + \pi^*(d))) = (n+1)m$

Démonstration. Résulte immédiatement de la suite spectrale de Leray du morphisme $\pi : S \to X$, des dualités globales et relatives et de l'isomorphisme de W avec son dual.

Lemme 3.3. *Soient C une courbe, C° l'ouvert de lissité, p un point de C°, $A_C : C^\circ \to$ Jac C l'application d'Abel correspondante, m un point de C°, T_m l'espace tangent en m. Identifions par translation le fibré tangent à* Pic C *au fibré trivial de fibre $H^1(C, O_C)$. Soit*

$$0 \to O_C \to O_C(m) \to O_m(m) \to 0 \tag{1}$$

la suite exacte canonique et $\partial : O_m(m) \to H^1(O_C)$ le cobord de la longue suite exacte de cohomologie associée. On a $\partial(O_m(m)) = dA_C(T_m)$.

3.4 Démonstration. Ce résultat est classique. Faute de référence dans le cas singulier nous proposons la démonstration suivante valable en toute caractéristique. Soient V une variété lisse, $\underline{L}$ un faisceau inversible sur $V \times C$. Il existe alors un morphisme $u : V \to$ Pic C uniquement déterminé et, localement sur V, un isomorphisme de $\underline{L}$ avec l'image inverse par u du faisceau de Poincaré sur (Pic C) $\times$ C. Soit $\underline{J}(\underline{L})$ le fibré des jets d'ordre 1 horizontaux de $\underline{L}$. On a une suite exacte $0 \to pr^*\Omega^1 \otimes \underline{L} \to \underline{J}(\underline{L}) \to \underline{L} \to 0$; d'où , en prenant le dual à valeurs dans $\underline{L}$, une suite exacte $0 \to O_{V\times C} \to \mathrm{Hom}(\underline{J}(\underline{L}), \underline{L}) \to pr^*T_V \to 0$. En prenant les images directes par la projection $V \times C \to V$, on obtient un morphisme cobord appelé morphisme de Kodaira-Spencer et noté $KS : T_V \to H^1(C, O_C) \otimes_{\mathbf{C}} O_V$ qui, compte tenu des identifications, s'interprète comme un morphisme $KS : T_V \to u^*T_{\mathrm{Pic}\, C}$. Il résulte de la théorie classique des déformations que KS est la différentielle de u.

Appliquons ce qui précède au cas où $V = C^\circ$ et $\underline{L} = O_{C^\circ \times C}(\Delta)$ où Δ est la diagonale de $C^\circ \times C$. Le morphisme u qu'on obtient est $m \mapsto O_C(m)$. C'est un translaté de A_C et par conséquent le morphisme de Kodaira-Spencer correspondant est la différentielle de A_C. Considérons la suite exacte canonique : $0 \to O_{C^\circ \times C} \to O_{C^\circ \times C}(\Delta) \to T_\Delta \to 0$. En prenant les jets d'ordre 1 on obtient un diagramme commutatif de suites exactes :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & pr^*\Omega^1 \arrow[r] \arrow[d] & \underline{J}(O_{C^\circ \times C}) \arrow[r] \arrow[d] & O_{C^\circ \times C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & pr^*\Omega^1(\Delta) \arrow[r] \arrow[d] & \underline{J}(O_{C^\circ \times C}(\Delta)) \arrow[r] \arrow[d] & O_{C^\circ \times C}(\Delta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O_\Delta \arrow[r] \arrow[d] & E_\Delta \arrow[r] \arrow[d] & T_\Delta \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 ,&
\end{tikzcd}$$

et en prenant les duaux à valeurs dans $O_{C^\circ\times C}(\Delta)$, on obtient :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & O_{C^\circ\times C} & \longrightarrow & \underline{J}(O_{C^\circ\times C}(\Delta))^*(\Delta) & \longrightarrow & pr^*T_{C^\circ} & \longrightarrow 0 \\
 & \downarrow \alpha & & \downarrow & & \downarrow & \\
0 \to & O_{C^\circ\times C}(\Delta) & \longrightarrow & \underline{J}(O_{C^\circ\times C})^*(\Delta) & \longrightarrow & pr^*T_{C^\circ}(\Delta) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & T_\Delta & \longrightarrow & E_\Delta{}^*\otimes T_\Delta^2 & \longrightarrow & T_\Delta^2 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Des deux premières suites horizontales et de la première suite verticale, on tire, en prenant les images directes, un diagramme commutatif :

$$\begin{array}{ccccccc}
 & & T_{C^\circ} & = & T_{C^\circ} & & \\
 & & \downarrow dA_C & & \downarrow \partial_h & & \\
0 \to & T_{C^\circ} & \xrightarrow{\partial_v} & H^1(C, O_C)\otimes_{\mathbf{C}} O_{C^\circ} & \xrightarrow{\alpha} & R^1\, pr_* O_{C^\circ\times C}(\Delta) & \to 0,
\end{array}$$

où ∂_h est le cobord de la deuxième suite horizontale et ∂_v le cobord de la première suite verticale. On sait que A_C est une submersion et par suite dA_C est une injection localement facteur directe. La deuxième suite horizontale est le produit tensoriel par $O_{C^\circ\times C}(\Delta)$ d'une suite exacte provenant de la base. Il s'en suit, par la formule de projection, que le cobord ∂_h est nul. On a donc $dA_C(T_{C^\circ}) = \partial_v(T_{C^\circ})$. Le lemme résulte alors de ce qu'en chaque point $m \in C^\circ$, $\partial_v(m)$ est le cobord de la suite (1). C.Q.F.D.

3.5. Dans la suite lorsque le genre arithmétique de C est ≥ 1, nous identifierons l'espace tangent T_m à la courbe C en $m \in C^\circ$ avec son image $dA_C(T_m) = \partial(O_m(m))$ dans $H^1(C, O_C)$.

Lemme 3.6. *Soient C une courbe, $\alpha \in H^1(O_C)$, $0 \to O_C \xrightarrow{a} W \xrightarrow{b} O_C \to 0$ une suite exacte dans la classe α, $c : W \to L$ un morphisme surjectif sur un faisceau inversible. Alors on a*

(1) deg $L \geq 0$,

(2) $\alpha \neq 0$ *et* deg $L = 0 \Longrightarrow L \simeq O_C$ *et c est proportionnel à b,*

(3) *si* deg $L > 0$, *$c\circ a$ est non nul et α est dans l'image de $H^0(L/c\circ a(O_C))$ par le cobord,*

(4) *en particulier si deg $L = 1$, il existe $p \in C^\circ$ tel que $L \simeq O_C(p)$. De plus si le genre de C est $\neq 0$, on a $0 \neq \alpha \in T_p \subset H^1(O_C)$ (3.5) et le point $p \in C^\circ$ tel que $L \simeq O_C(p)$ est uniquement déterminé par L.*

Réciproquement soient $p \in C^{\circ}$, $\alpha \in H^1(O_C)$ un élément $\neq 0$ de T_p, $0 \to O_C \to W \to O_C \to 0$ une suite exacte dans la classe α. Alors

(5) *Il existe un morphisme surjectif $c : W \to O_C(p)$.*

Soient s une section $\neq 0$ de $O_C(p)$, $c_i, i = 1, 2$, deux morphismes surjectifs de W sur $O_C(p)$. Alors

(6) *il existe $\lambda \in \mathbf{C}^*$ et $\mu \in \mathbf{C}$, uniques, tels que $c_1 = \lambda c_2 + \mu s \circ b$,*

(7) *pour tout $\lambda \in \mathbf{C}^*$ et tout $\mu \in \mathbf{C}$, le morphisme $\lambda c_2 + \mu s \circ b$ est surjectif.*

Démonstration. Démontrons (1) et (2). Si $c \circ a = 0$, alors c se factorise par b suivi d'un morphisme surjectif $O_C \to L$ et par suite L est isomorphe à O_C et si $c \circ a \neq 0$, L possède une section. Donc dans tous les cas on a $\deg L \geq 0$, d'où (1). Si $\deg L = 0$, on a nécéssairement $c \circ a = 0$ car si non, $c \circ a$ serait un isomorphisme et a possèderait une section ce qui contredit $\alpha \neq 0$, d'où (2). Démontrons (3). Comme $\deg L > 0$, on a $c \circ a \neq 0$ car sinon on a $L \simeq O_C$. On a un diagramme commutatif de suites exactes :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & O_C \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & O_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & O_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L\ /c \circ a(O_C) \arrow[r] \arrow[d] & W'/W \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

où W' est la somme amalgamée de W et L au dessus de O_C. L'existence de c implique que la suite exacte horizontale du milieu est scindée. Par suite $c \circ a$ induit un morphisme $H^1(O_C) \to H^1(L)$ qui annule α. Donc α est dans l'image du cobord associé à la suite exacte verticale. Démontrons (4). Le morphisme $c \circ a : O_C \to L$ est non nul car sinon, on aurait un isomorphisme $O_C \simeq L$ et L serait de degré 0. Comme L est de degré 1, $L/c \circ a(O_C)$ est de longueur 1 donc isomorphe au corps résiduel d'un point $p \in C$ et ce corps résiduel admet une résolution localement libre finie. Par suite $L \simeq O_C(p)$ et $p \in C^{\circ}$. Supposons que le genre de C soit non nul et supposons $\alpha = 0$. Alors $W \simeq (O_C)^2$; donc le point p est mobile dans sa classe d'équivalence rationnelle et par suite $C \simeq \mathbf{P}^1$ contrairement à l'hypothèse. On a donc $\alpha \neq 0$. L'image de $L/c \circ a(O_C)$ dans $H^1(O_C)$ par le cobord est T_p (2.9). Cette image contient α d'après (3). Enfin le point p est uniquement déterminé dans ce cas car le genre de C est > 0. Démontrons la réciproque. Construisons le diagramme commutatif de suites exactes :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & O_C \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & O_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O_C(p) \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & O_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \chi_p \arrow[r] \arrow[d] & \chi_p \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

où W' est la somme amalgamée. Par hypothèse la classe α est dans l'image de χ_p par le cobord, donc son image par s dans $H^1(O_C(p))$ est nulle. Par suite la deuxième suite exacte horizontale est scindable, d'où, par composition, un morphisme $c : W \to O_C(p)$ tel que $c \circ a = s$. Ce morphisme est surjectif car sinon son image serait $s(O_C)$ et la première suite exacte horizontale serait elle aussi scindée ce qui contredit $\alpha \neq 0$, d'où (5). Démontrons (6). Comme $h^0(O_C(p)) = 1$ et $c_i \circ a \neq 0$ pour $i = 1, 2$, il existe $\lambda \in \mathbb{C}^*$ tel que $c_1 \circ a = \lambda c_2 \circ a$. Donc $c_1 - \lambda c_2$ se factorise par b d'où (6). Démontrons (7). Le morphisme $(\lambda c_1 + \mu s \circ b) \circ a = \lambda c \circ a$ de O_C dans $O_C(p)$ est non nul donc injectif. L'image de $\lambda c_1 + \mu s \circ b$ ne peut être égale à $s(O_C)$ car $\alpha \neq 0$. Elle est donc égale à $O_C(p)$ et le morphisme $\lambda c_1 + \mu s \circ b$ est surjectif. C.Q.F.D.

Proposition 3.7 (Critère cohomologique de tangence). *Soient Γ une courbe, $\pi : \Gamma \to X$ un revêtement, $p \in \Gamma^\circ$ un point lisse tel que $\pi(p) = q$, $0 \to O_X \to W \to O_X \to 0$ la suite exacte canonique* (3.1). *Les conditions suivantes sont équivalentes :*

(i) *π est un revêtement tangentiel en p.*

(ii) *Il existe un morphisme surjectif $\pi^*(W) \to O_\Gamma(p)$.*

Démonstration. Notons $\alpha \in H^1(X, O_X)$ la classe de la suite exacte canonique $0 \to O_X \to W \to O_X \to 0$. On sait que $\alpha \neq 0$ (3.1). La classe $\pi^*(\alpha) \in H^1(O_\Gamma)$ est la classe de la suite exacte déduite de $0 \to O_X \to W \to O_X \to 0$ par π^*. Comme X est une courbe elliptique, on a $h^1(X, O_X) = 1$ et par suite $T_q = H^1(X, O_X)$ (3.5). De plus le morphisme $\pi^* : H^1(X, O_X) \to H^1(\Gamma, O_\Gamma)$ est injectif (cf. 2.3). Dire que π est un revêtement tangentiel (2.2) équivaut à dire que $\pi^*(\alpha) \in T_p \subset H^1(O_\Gamma)$. Supposons cette dernière propriété satisfaite. D'après l'assertion (5) de (3.6), il existe un morphisme surjectif $\pi^*(W) \to O_\Gamma(p)$, d'où (ii). Supposons (ii) satisfaite. Comme Γ revêt une courbe elliptique, le genre de Γ est $\neq 0$. Alors, d'après l'assertion (4) de (3.6), on a $\pi^*(\alpha) \in T_p \subset H^1(O_\Gamma)$. C.Q.F.D.

Théorème 3.8. (Critère géométrique de tangence) *Soit $\pi : \Gamma \to X$ un revêtement. Les conditions suivantes sont équivalentes :*

(i) *π est un revêtement tangentiel en p.*

(ii) *Il existe un morphisme de degré 1, au dessus de $X, \iota : \Gamma \to S$ qui fait coincider les points p.*

Supposons π tangentiel en p et soient ι_i, $i = 1, 2$, deux morphismes comme dans (ii). Il existe un automorphisme μ de S au dessus de X tel que $\mu \circ \iota_1 = \iota_2$. Cet automorphisme est unique.

Démonstration. Supposons π tangentiel en p. D'après (3.7) il existe un morphisme surjectif $\pi^*(W) \to O_\Gamma(p)$. Un tel morphisme correspond à un morphisme $\iota : \Gamma \to S$, au dessus de X, tel que $\iota^* O_S(1) \simeq O_\Gamma(p)$. Par suite ι est de degré 1. Comme $O_S(1) = O_S(C_0)$, on a $\iota(p) \in C_0$ et comme $\iota(p) \in S_q$, ι fait coincider les points p, d'où (ii). Réciproquement supposons (ii) satisfait et soit ι un morphisme comme dans (ii). Comme ι est de degré 1 et fait coincider les points p, on a $\iota^* O_S(1) \simeq \iota^* O_S(C_0) \simeq O_\Gamma(p)$. Il existe donc un morphisme surjectif $\pi^* W \to O_\Gamma(p)$ et par suite π est tangentiel en p (3.7). La dernière assertion résulte de la description des automorphismes de S (3.1) et des assertions (6) et (7) de (3.6). C.Q.F.D.

Lemme 3.9.

(1) *Soit D un diviseur effectif de la surface S. Il existe un diviseur ∂ effectif ou nul de X et un entier $m \geq 0$ tel que $D \in |mC_0 + \pi^*\partial|$.*

(2) *Soient n un entier ≥ 1, $D \in |nC_0 + S_q|$ un diviseur réductible de la surface S. Alors $D = D_1 + C_0$ où $D_1 \in |(n-1)C_0 + S_q|$.*

(3) *Les conditions (i) à (iv) suivantes sur un diviseur effectif D sont équivalentes :*

(i) *D est irréductible et on a $D \in |nC_0 + S_q|$ pour un $n \geq 0$.*

(ii) *le support de D est irréductible et on a $D \in |nC_0 + S_q|$ pour un $n \geq 0$.*

(iii) *p est un point lisse de D, $D \cap C_0 = \{p\}$, D et C_0 se coupent transversalement en p.*

(iv) *On a* $p \in D \cap C_0$, *dim*$(D \cap C_0) = 0$ *et* $D.K_S = -2$ *où* K_S *est le diviseur canonique de* S.

(4) *Le système* $|nC_0 + S_q|$ *n'a pas de composantes fixes. Le point* p *est l'unique point fixe de ce système.*

Démonstration. Démontrons (1). Soit m le degré de D sur X. On a $m \geq 0$ et $D \in |mC_0 + \pi^*\partial|$ où ∂ est un diviseur de X. On a $\pi_* O_S(D) = S^m W(\partial)$ avec $h^0(S^m W(\partial)) > 0$. Montrons par récurrence sur m que cette dernière inégalité entraine que ∂ est effectif ou nul. Cela est clair si $m = 0$. Supposons $m > 0$. On a une suite exacte $0 \to S^{m-1}W(\partial) \to S^m W(\partial) \to O_S(\partial) \to 0$. Si on a $h^0(S^{m-1}W(\partial)) > 0$, alors ∂ est effectif ou nul par hypothèse de récurrence. Sinon, on a $h^0(O_S(\partial)) > 0$ et par suite ∂ est effectif ou nul. Démontrons (2). Le diviseur D possède une composante D_2 distinct de D et d'un multiple de C_0. On a $D = D_2 + D_3$ et $D_3 \in |mC_0 + \pi^*\Delta|$, $m \geq 0$, et deg $\Delta \geq 0$. On a donc $D_3.C_0 \geq 0$, $D_2.C_0 > 0$ et $1 = D.C_0 = D_3.C_0 + D_2.C_0$. Donc $D_3.C_0 = 0$, par suite $\Delta = 0$ et $D_3 \in |mC_0|$. On a donc $D_3 = mC_0$ car mC_0 est fixe. Donc on a $D = D_1 + C_0$ où $D_1 \in |(n-1)C_0 + S_q|$.

Démontrons (3). L'implication (i) $\Longrightarrow$ (ii) est tautologique et (ii) $\Longrightarrow$ (i) résulte de (2).

(i) $\Longrightarrow$ (iii) : Soit D comme dans (i). Le cycle $D \circ C_0$ est rationnellement équivalent à $(nC_0 + S_q) \circ C_0 = p$. Comme D est irréductible, C_0 n'est pas une composante de D et par suite $D \cap C_0$ est un ensemble fini, support d'un cycle rationnellement équivalent à p. Donc $D \cap C_0$ est réduit à un point rationnellement équivalent à p et par suite contenu dans S_q. Donc $D \cap C_0$ est réduit à p ; p est un point lisse de D et D et C_0 se coupent transversalement en p.

(iii) $\Longrightarrow$ (iv) : Supposons que D satisfasse (iii). On a donc $p \in D \cap C_0$, $\dim(D \cap C_0) = 0$ et $D.C_0 = 1$. Comme K_S est rationnellement équivalent à $-2C_0$ (3.1), on a $D.K_S = -2$.

(iv) $\Longrightarrow$ (i) : Supposons (iv) satisfait. D'après (1), on a

$$D \in |nC_0 + \pi^*(\delta)|$$

pour un $n \geq 0$ et un δ effectif ou nul. On a

$$\begin{aligned} D.K_S &= -2D.C_0 \\ &= -2\pi^*(\delta).C_0 \\ &= -2. \end{aligned}$$

Donc δ est effectif de degré 1 et par suite δ est un point de X. On a $D.C_0 = 1$ et comme C_0 n'est pas une composante de D, $D \cap C_0$ est réduit à un point et comme $p \in D \cap C_0$, on a $D \cap C_0 = \{p\}$ et le cycle $D o C_0$ est rationnellement équivalent à p. Le cycle $D \circ C_0$ est par ailleurs rationnellement équivalent à $C_0 \cap S_\delta$; donc p est rationnellement équivalent à $C_0 \cap S_\delta$ et par suite on a $\delta = q$. Donc on a $D \in |nC_0 + S_q|$ et comme C_0 n'est pas une composante de D, le diviseur D est irréductible d'après (2).

(4) Il résulte de (2) que s'il existe une composante fixe, alors C_0 est une composante fixe. Mais si C_0 est une composante fixe de $|nC_0 + S_q|$, alors l'application $D \mapsto D + C_0$ de $|(n-1)C_0 + S_q|$ dans $|nC_0 + S_q|$ est une bijection ce qui est absurde pour des raisons de dimensions (3.2). Comme les points fixes de $|nC_0 + S_q|$ sont stables sous-l'action de $\mathrm{Aut}(S/X)$, ce sont des points de C_0 et il résulte alors de (3) que le seul point fixe est p. C.Q.F.D

Corollaire 3.10. *Soit n un entier ≥ 1.*

(1) *Soient $\pi : \Gamma \longrightarrow X$ un revêtement tangentiel en p de degré n, $\iota : \Gamma \longrightarrow S$ un morphisme de degré 1 au dessus de X qui fait coincider les points p (3.8). Alors $\iota(\Gamma)$ appartient au système linéaire $|nC_0 + S_q|$ et π est un revêtement tangentiel minimal si et seulement si ι est un plongement.*

(2) *Tout revêtement tangentiel domine un revêtement tangentiel minimal unique à isomorphisme unique près.*

(3) *Tout diviseur irréductible Γ du système linéaire $|nC_0 + S_q|$ contient p et le morphisme $\pi : \Gamma \longrightarrow X$ induit par la projection $\pi : S \longrightarrow X$ est un revêtement tangentiel en p minimal et de degré n.*

(4) *Soit $\pi : \Gamma \longrightarrow X$ un revêtement tangentiel en p de degré n. Alors le genre arithmétique de Γ est $\leq n$. Il est égal à n si et seulement si π est un revêtement minimal.*

Démonstration.

(1) et (2) Comme $\pi : \Gamma \longrightarrow X$ est étale en p (2.3), ι est une immersion en p et comme on a $\iota^*(O_S(C_0)) = O_\Gamma(p)$, ι induit un isomorphisme birationnel de Γ sur $\iota(\Gamma)$ et p est un point lisse de $\iota(\Gamma)$. De plus $\iota(\Gamma) \cap C_0 = \{p\}$ et l'intersection est transversal en p. Donc la projection $\iota(\Gamma) \longrightarrow X$ est un revêtement tangentiel d'après (3.8), dominé par Γ et par suite de degré n. Il résulte alors de (3.8) que $\Gamma \longrightarrow X$ est un revêtement tangentiel minimal si et seulement si ι est un plongement et que tout revêtement tangentiel domine un revêtement tangentiel minimal unique à isomorphisme unique près. Enfin il résulte de (3.9) qu'on a $\iota(\Gamma) \in |nC_0 + S_q|$.

(3) D'après (3.9) $\Gamma \cap C_0$ est réduit à p ; p est un point lisse de Γ et Γ et C_0 se coupent transversalement en p. Donc l'injection canonique $\iota : \Gamma \longrightarrow S$ est un morphisme de degré 1 qui fait coincider les points p et par suite $\pi : \Gamma \longrightarrow X$ est un revêtement tangentiel (3.8) minimal et de degré n d'après (1).

(4) D'après (1), tout revêtement tangentiel de degré n domine une courbe $\Gamma \in |nC_0 + S_q|$. Donc le genre arithmétique de ce revêtement est inférieur au genre de Γ et il lui est égal si et seulement si ce revêtement tangentiel est minimal. Soit Γ une courbe de $|nC_0 + S_q|$. Son genre arithmétique est donné par la formule d'adjonction [7, p. 471] : $g_a(\Gamma) = 1 + (\Gamma.\Gamma + \Gamma.K_S)/2$. On a $\Gamma.\Gamma = 2n$ (3.1) et $\Gamma.K_S = -2$ (3.9) ; d'où $g_a(\Gamma) = n$. C.Q.F.D.

Théorème 3.11. *Soit n un entier > 0.*

(1) *Le système linéaire $|nC_0 + S_q|$ est un espace projectif de dimension n.*

(2) *La variété $V(n,X)$ des diviseurs irréductibles de $|nC_0 + S_q|$ est le complémentaire de l'hyperplan image de $|(n-1)C_0 + S_q|$ par le morphisme $D \mapsto D + C_0$. En particulier $V(n,X)$ est un espace linéaire affine de dimension n.*

(3) *La variété $V(n,X)/\mathrm{Aut}(S/X)$ est un espace linéaire affine de dimension $n-1$ et ses points sont en correspondance biunivoque avec l'ensemble des classes d'isomorphismes de revêtements tangentiels minimaux de degré n.*

(4) *L'ensemble des $\Gamma \in V(n,X)$ qui sont des courbes lisses est un ouvert de Zariski dense dans $V(n,X)$.*

Démonstration

(1) D'après (3.2), on a $h^0(S,n,1) = n+1$, d'où l'assertion.

(2) Résulte de (3.9,2)).

(3) Notons a la section canonique de $O_S(C_0)$. Pour tout p tel que $0 \leq p \leq n$, le produit tensoriel par $a^{\otimes n-p}$ induit une injection $H^0(S,pC_0 + S_q) \longrightarrow H^0(S,nC_0 + S_q)$ permettant d'identifier $H^0(S,pC_0 + S_q)$ à son image. On a donc une filtration

$$H^0(S,S_q) \subset H^0(S,C_0 + S_q) \subset \ldots \subset H^0(S,nC_0 + S_q).$$

Le groupe $\mathrm{Aut}(S/X) = G_a$ opère de manière unipotente sur $H^0(S,nC_0 + S_q)$, respecte cette filtration et a pour sous-espace fixe $H^0(S,S_q)$ qui est de dimension 1. D'après (2), $V(n,X)$ est le complémentaire de $\mathbf{P}(H^0(S,(n-1)C_0 + S_q))$ dans $\mathbf{P}(H^0(S,nC_0 + S_q))$. Il résulte alors de la description de l'action de G_a, qu'il existe une fonction linéaire affine $l : V(n,X) \to \mathbf{C}$, unique à une constante additive près, telle que $l(\lambda\Gamma) = l(\Gamma) + \lambda$ pour tout $\Gamma \in V(n,X)$ et tout $\lambda \in G_a = \mathbf{C}$. Posons $H = l^{-1}(0)$. Le morphisme cano nique $H \to V(n,X)/Aut(S/X)$ est un isomorphisme qui définit, par transport de structure, une structure d'espace linéaire affine de dimension $n-1$ sur $V(n,X)/\mathrm{Aut}(S/X)$. On vérifie que cette structure ne change pas lorsqu'on ajoute à l une constante. Elle est donc canonique. On remarquera cependant que la projection canonique $V(n,X) \to V(n,X)/\mathrm{Aut}(S/X)$ n'est pas linéaire affine. Notons M l'ensemble des classes d'isomorphismes de revêtements tangentiels minimaux de degré n. L'assertion (3.10,3)) définit une application de $V(n,X)$ dans M et il résulte de (3.8) que cette application se factorise en le morphisme canonique $V(n,X) \to V(n,X)/\mathrm{Aut}(S/X)$ suivie d'une injection $V(n,X)/\mathrm{Aut}(S/X) \to M$. D'après (3.10,1)), cette application est surjective.

(4) Résulte du théorème de Bertini [7, p. 137], compte tenu du fait que le seul point fixe p du système linéaire $|nC_0 + S_q|$ (3.9,4)) est un point lisse des $\Gamma \in V(n,X)$ (3.9,3)). C.Q.F.D.

3.12. Notons τ l'involution canonique de S et de X (3.1) et pour tout $n > 0$, soit $SV(n,X)$ l'espace $V(n,X)^\tau$ des points fixes de $V(n,X)$ sous τ. Comme τ est une involution linéaire affine de $V(n,X)$, $SV(n,X)$ est un sous-espace linéaire affine de $V(n,X)$. On peut montrer (*cf.* (5.3)) que $SV(n,X)$ est de dimension $[n/2]$. Tout $\Gamma \in SV(n,X)$ est globalement fixe sous l'involution τ et par suite τ induit sur Γ une involution encore notée τ.

3.13. Posons $\omega_0 = q$ et pour $i = 1,2,3$, notons ω_i les trois autres points d'ordre 2 de X. Les points ω_i, $0 \leq i \leq 3$, sont les points fixes de X sous l'involution τ. De même, notons r_0 l'origine du groupe Δ (3.1) et pour $i = 1,2,3$, notons r_i les trois autres points d'ordre 2 de Δ. On a $\pi(r_i) = \omega_i$ pour $0 \leq i \leq 3$ et les points r_i sont les quatre points de Δ fixes sous l'involution τ. Enfin, pour $0 \leq i \leq 3$, posons $s_i = C_0 \cap S_{\omega_i}$, de sorte qu'on a $p = s_0$. Les huits points r_i et s_i, $0 \leq i \leq 3$, sont les points fixes de S sous l'involution τ.

Pour tout revêtement tangentiel symétrique $\pi : \Gamma \to X$, on note $\mu_i(\Gamma)$ la multiplicité de $\iota(\Gamma)$ en r_i où $\iota : \Gamma \to S$ est un X-morphisme de degré 1 qui fait coincider les points p (3.8) et qui commute aux involutions ι. On note $\mu(\Gamma)$ le quadruplet $(\mu_i(\Gamma)_{0\leq i\leq 3}) \in \mathbf{N}^4$. Le quadruplet $\mu(\Gamma)$ est appelé *le type du revêtement* Γ. On pose $\varepsilon_0 = 1$ et pour $i = 1,2,3$, $\varepsilon_i = 0$. Nous dirons qu'un quadruplet $\mu = ((\mu_i)_{0\leq i\leq 3}) \in \mathbf{N}^4$ est *adapté à* n si on a $\mu_i + \varepsilon_i \equiv n$, mod.2 pour $0 \leq i \leq 3$.

Proposition 3.14.

(1) *Soient $\pi : \Gamma \longrightarrow X$ un revêtement tangentiel en p, symétrique de degré $n > 0$, τ l'involution canonique. Il existe un unique morphisme $\iota : \Gamma \longrightarrow S$, au-dessus de X, τ-équivariant, de degré 1 qui fait coincider les points p. On a $\iota(\Gamma) \in SV(n, X)$. Réciproquement, soient $\Gamma' \in SV(n, X)$ et $\Gamma \longrightarrow \Gamma'$ un morphisme qui est un isomorphisme birationnel et tel que l'involution τ sur Γ' se prolonge à Γ. Alors le morphisme composé $\Gamma \longrightarrow X$ est un revêtement tangentiel en p, symétrique de degré n.*

(2) *Soit $\pi : \Gamma \longrightarrow X$ un revêtement tangentiel symmétrique de degré n. Alors $\mu(\Gamma)$ est adapté à n.*

Démonstration

(1) Il existe, d'après (3.8), un X-morphisme $\rho : \Gamma \longrightarrow S$ de degré 1 qui fait coincider les points p. Le morphisme $\tau \circ \rho \circ \tau$ possède les mêmes propriétés et par suite (*loc. cit.*) il existe un unique $g_0 \in \mathrm{Aut}\,(S/X)$, tel que $\tau \circ \rho \circ \tau = \rho \circ g_0$. On cherche $g \in \mathrm{Aut}\,(S/X)$ tel que $\rho \circ g$ commute à τ. On doit donc avoir $\tau \circ \rho \circ g \circ \tau = \rho \circ g$, c'est à dire $\rho \circ g_0 \circ \tau \circ g \circ \tau = \rho \circ g$, ce qui équivaut (3.8) à $g_0 \circ \tau \circ g \circ \tau = g$. Pour tout $g \in \mathrm{Aut}\,(S/X)$, on a $\tau \circ g \circ \tau = g^{-1}$, et par suite g doit satisfaire $g^2 = g_0$, équation qui a une seule solution car $\mathrm{Aut}(S/X)$ est isomorphe à $\mathbb{C}$; d'où l'existence et l'unicité de $\iota = \tau \circ g$. La réciproque résulte de (3.10).

(2) D'après (1), on peut, quitte à remplacer Γ par $\iota(\Gamma)$ supposer que $\Gamma \in SV(n, X)$. Démontrons d'abord un lemme :

Lemme 3.15. *Soient $n > 0$ et $\Gamma \in SV(n,X)$. Il existe une fonction rationnelle f_Γ sur S, $(-1)^n$-τ-équivariante, dont le diviseur des zéros est Γ et dont le diviseur des pôles est $nC_0 + S_q$.*

Démonstration. Notons a (resp. α) la section canonique de $O_S(C_0)$ (resp. $O_S(S_q)$) interprété comme le faisceau des fonctions méromorphes sur S dont les pôles sont majorés par C_0 (resp S_q). Les sections a et α correspondent à la fonction constante et égale à 1 et par suite sont invariantes sous τ. Il s'ensuit que τ agit par l'identité sur les espaces vectoriels (de dimension 1) $H^0(S, O_S(C_0))$ et $H^0(S, O_S(S_q))$. On a une suite exacte τ-équivariante $0 \to O_S(S_q - C_0) \to O_S(S_q) \to O_{C_0}(S_q) \to 0$ qui donne naissance à une longue suite exacte de cohomologie τ-équivariante. Comme $h^i(S, O_S(S_q - C_0)) = 0$ pour tout i, on en déduit que τ agit par l'identité sur $H^0(S, O_{C_0}(S_q))$. Par ailleurs τ agit par multiplication par -1 sur $H^1(X, O_X)$ et comme l'homomorphisme d'image inverse $H^1(X, O_X) \to H^1(S, O_S)$ est un isomorphisme τ-équivariant, τ agit par -1 sur $H^1(S, O_S)$. On a une suite exacte τ-équivariante $0 \to O_S \to O_S(C_0) \to O_{C_0}(C_0) \to 0$ qui donne naissance à une longue suite exacte de cohomologie τ-équivariante et en particulier à un isomorphisme $H^0(S, O_{C_0}(C_0)) \to H^1(S, O_S)$. Donc τ agit par -1 sur l'espace vectoriel (de dimension 1) $H^1(S, O_{C_0}(C_0))$. Pour tout $n \geq 0$, on a un isomorphisme τ-équivariant $H^0(S, O_{C_0}(S_q)) \otimes (H^0(S, O_{C_0}(C_0))^{\otimes n}) \to H^0(S, O_{C_0}(nC_0 + S_q))$. Donc τ agit par $(-1)^n$ sur ce dernier espace. Considérons la suite exacte $0 \to O_S((n-1)C_0 + S_q) \to O_S(nC_0 + S_q) \to O_{C_0}(nC_0 + S_q) \to 0$ qui donne naissance à la suite exacte τ-équivariante $0 \to H^0(S, O_S((n-1)C_0 + S_q)) \to H^0(S, O_S(nC_0 + S_q)) \to H^0(S, O_{C_0}(nC_0 + S_q) \to 0$. La courbe Γ est le diviseur des zéros d'une fonction rationnelle $f_\Gamma \in H^0(S, O_S(nC_0 + S_q))$ et comme Γ est invariant par τ, la fonction f_Γ est invariante ou anti-invariante par τ. Comme Γ est irréductible, C_0 n'est pas une composante de Γ et par suite l'image de f_Γ dans $H^0(S, O_{C_0}(nC_0 + S_q))$ est non nulle. Comme τ opère par $(-1)^n$ sur ce dernier espace, f_Γ est $(-1)^n$-τ-équivariante. C.Q.F.D.

3.16. Achevons maintenant la démonstration de 3.14. Soit f_Γ une fonction rationnelle comme dans 3.15. Pour tout i, $1 \leq i \leq 3$, $f_\Gamma = 0$ est une équation locale de Γ au voisinage de r_i. Comme f_Γ est $(-1)^n$-τ-équivariante, la multiplicité $\mu_i(\Gamma)$ en le point fixe r_i est congrue à n mod.2. Soit z une uniformisante locale sur X en q, anti-invariante. Alors $(z \circ \pi) f_\Gamma$ est une équation locale de Γ au voisinage de r_0. Comme cette équation est $(-1)^{n-1}$-τ-équivariante, la multiplicité $\mu_0(\Gamma)$ en le point fixe r_0 est congrue à $n-1$ mod.2.

3.17. Montrons que $SV(1, X)$ ne comporte qu'un seul élément que nous noterons C_1. Soit C et C' deux éléments distincts de $SV(1, X)$. D'après (3.14), les trois points $r_i, 1 \leq i \leq 3$, sont des points de C et de C'. On a donc $C.C' \geq 3$. Mais on a $(C_0 + S_q).(C_0 + S_q) = 2$; contradiction ! Enfin l'application identique de X dans X est un revêtement tangentiel symétrique de degré 1. Donc on a $i(X) \in SV(1, X)$ (3.14). Comme C_1 est de degré 1 sur X, la projection $\pi : C_1 \to X$ est un isomorphisme.

4. Revêtements tangentiels hyperelliptiques

4.1. Les courbes symétriques tracées sur S sont des images inverses de courbes tracées sur la surface quotient S/τ. Cette surface possède huit points singuliers quadratiques, images des r_i et des s_i pour $0 \leq i \leq 3$. Considérons alors la surface $S^\sim$ désingularisée minimale de S/τ. On sait qu'elle est isomorphe au quotient $S^\perp/\tau$ où $S^\perp$ se déduit de S en éclatant les huits points r_i et s_i pour $0 \leq i \leq 3$, et où τ est l'extension canonique à $S^\perp$ de l'involution de S. On a un diagramme cartésien:

$$\begin{array}{ccc} S^\perp & \xrightarrow{\phi} & S^\sim \\ \downarrow{\scriptstyle e} & & \downarrow \\ S & \longrightarrow & S/\tau, \end{array} \tag{4.1.1}$$

où ϕ est un morphisme plat.

Dans la suite, on utilise le signe $=$ pour l'égalité de cycles et $\equiv$ pour l'équivalence linéaire de cycles. On note S_i les fibres de S au dessus des ω_i, $0 \leq i \leq 3$ (3.13), de sorte qu'on a $S_0 = S_q$. On note $C_0^\perp$, $C_1^\perp$, $S_i^\perp$ les transformés stricts dans $S^\perp$, des diviseurs correspondants de S et $r_i^\perp$, $s_i^\perp$, les diviseurs exceptionnels de l'éclatement $S^\perp \to S$. On note $C_0^\sim$, $C_1^\sim$, $S_i^\sim$, $r_i^\sim$, $s_i^\sim$ les images réduites dans $S^\sim$ des diviseurs correspondants de $S^\perp$. Pour tout $(\mu_i) \in \mathbf{N}^4$, on pose $\mu^{(1)} = \Sigma\mu_i$, $\mu^{(2)} = \Sigma\mu_i^2$. Enfin on note $K, K^\perp, K^\sim$ les diviseurs canoniques des surfaces $S, S^\perp, S^\sim$ respectivement.

Lemme 4.2.

(1) *La surface $S^\sim$ est rationnelle [7, p. 513].*

(2) *Pour tout $n \geq 0$ et tout μ adapté à n (3.13), il existe un unique $\lambda(n,\mu) \in \operatorname{Pic}(S^\sim)$ tel que $\phi^*(\lambda(n,\mu)) \equiv e^*(nC_0 + S_0) - s_0^\perp - \Sigma\mu_i r_i^\perp$.*

(3) *Pour tout $n \geq 0$ et tout μ adapté à n, on a $(\lambda(n,\mu))^2 = (2n-1-\mu^{(2)})/2$ et $\lambda(n,\mu).K^\sim = -1$.*

(4) *On a $(K^\sim)^2 = 0$. Le groupe $\operatorname{Pic}(S^\sim)$ est libre de rang 10.*

Démonstration

(1) Comme ϕ est un revêtement, les morphismes canoniques $H^0(S^\sim, \Omega^1_{S^\sim}) \to H^0(S^\perp, \Omega^1_{S^\perp})^\tau$ et $H^0(S^\sim, (\Omega^2_{S^\sim})^2) \to H^0(S^\perp, (\Omega^2_{S^\perp})^2)$ sont injectifs. Comme $S^\perp$ est birationnellement isomorphe à S donc birationnellement isomorphe à $X \times \mathbf{P}^1$ on a

$$H^0(S^\perp, \Omega^1_{S^\perp})^\tau \simeq H^0(X \times \mathbf{P}^1, \Omega^1_{X\times\mathbf{P}^1})^\tau.$$

Mais on a $H^0(X\times\mathbf{P}^1, \Omega^1_{X\times\mathbf{P}^1}) = H^0(X, \Omega^1_X)$ et τ agit par -1 sur ce dernier groupe, on a donc $H^0(S^\perp, \Omega^1_{S^\perp})^\tau = 0$ et par suite $H^0(S^\sim, \Omega^1_{S^\sim}) = 0$. De même, on a $h^0(S^\perp, (\Omega^2_{S^\perp})^2) = h^0(X, (\Omega^1_X)^2) = h^0(\mathbf{P}^1, (\Omega^1_{\mathbf{P}^1})^2) = 0$ et par suite on a $H^0(S^\sim, (\Omega^2_{S^\sim})^2) = 0$. On a donc, en utilisant les notations traditionnelles, $q(S^\sim) = P_2(S^\sim) = 0$ et par suite $S^\sim$ est une surface rationnelle d'après le critère de Castelnuovo-Enriques [7, p. 536].

(2) Comme $S^\sim$ est rationnelle, le groupe $\mathrm{Pic}(S^\sim)$ est un groupe libre de type fini et en particulier, il est sans torsion. On dispose de l'homorphisme de norme $\boldsymbol{N}$: Pic $(S^\perp) \to$ Pic $(S^\sim)$ et le morphisme composé $\boldsymbol{N} \circ \phi^*$ est la multiplication par 2 sur $\mathrm{Pic}(S^\sim)$, donc injectif. Par suite ϕ^* : Pic $(S^\sim) \to$ Pic $(S^\perp)$ est injectif, d'où l'unicité de $\lambda(n,\mu)$. Supposons n pair. Posons $\lambda(n,\mu) = nC_0^\sim + S_0^\sim + (n/2)\Sigma s_i^\sim - \Sigma((\mu_i - \varepsilon_i)/2)r_i^\sim$. Pour vérifier qu'on a la relation cherchée, on utilise les relations $\phi^*(C_0^\sim) = C_0^\perp$, $\phi^*(S_i^\sim) = S_i^\perp$, $\phi^*(s_i^\sim) = 2s_i^\perp$, $\phi^*(r_i^\sim) = 2r_i^\perp$, $e^*(C_0) = C_0^\perp + \Sigma s_i^\perp$, $e^*(S_0) = S_0^\perp + r_0^\perp + s_0^\perp$, et la vérification est immédiate. On vérifie de même que $C_1^\sim \equiv \lambda(1,(0,1,1,1))$. On en déduit aussitôt que pour n impair, on a $\lambda(n,\mu) \equiv C_1^\sim + (n-1)C_0^\sim + ((n-1)/2)\Sigma s_i^\sim - \Sigma((\mu_i - 1 + \varepsilon_i)/2)r_i^\sim$.

(3) Pour tout $\lambda \in$ Pic $(S^\sim)$, on a $\phi_*\phi^*(\lambda) = 2\lambda$. De plus, comme $\Sigma s_i^\perp + r_i^\perp$ est le diviseur de ramification de ϕ et le diviseur de contraction de e, on a $\phi^* K^\sim = K^\perp - \Sigma s_i^\perp + r_i^\perp = e^* K$. Enfin e^*, e_* et ϕ^*, ϕ_*, sont adjoints pour la forme d'intersection. On a donc

$$\begin{aligned}
2\lambda(n,\mu).K^\sim &= \phi_*\phi^*(\lambda(n,\mu)).K^\sim \\
&= \phi^*(\lambda(n,\mu)).\phi^*(K^\sim) \\
&= (e^*(nC_0 + S_0) - s_0^\perp - \Sigma\mu_i r_i^\perp).e^*(K) \\
&= (nC_0 + S_0).K = -2,
\end{aligned}$$

d'où la deuxième égalité. De même, on a

$$\begin{aligned}
2\lambda(n,\mu).\lambda(n,\mu) &= \phi_*\phi^*(\lambda(n,\mu)).\lambda(n,\mu) \\
&= (\phi^*(\lambda(n,\mu)))^2 \\
&= (e^*(nC_0 + S_0) - s_0^\perp - \Sigma\mu_i r_i^\perp)^2 \\
&= (e^*(nC_0 + S_0))^2 - 1 - \mu^{(2)} = 2n - 1 - \mu^{(2)}.
\end{aligned}$$

(4) On a vu dans la démonstration de (3), que $\phi^* K^\sim = e^* K$. On a donc

$$\begin{aligned}
2K^\sim.K^\sim &= \phi_*\phi^* K^\sim.K^\sim \\
&= \phi^* K^\sim.\phi^* K^\sim \\
&= e^* K.e^* K \\
&= e_* e^* K.K \\
&= K.K = 0
\end{aligned}$$

On sait que $S^{\sim}$ étant rationnelle, on a rang (Pic $(S^{\sim})$) $= 10 - K^{\sim}.K^{\sim}$, d'où rang (Pic $(S^{\sim})$) $= 10$. C.Q.F.D.

Lemme 4.3. *Soient $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique, $\iota : \Gamma \to S$, le morphisme au dessus de X, τ-équivariant, de degré 1, qui fait coincider les points p (3.14). Il existe un unique morphisme $\iota^{\perp} : \Gamma \to S^{\perp}$ qui relève ι. Le morphisme $\iota^{\perp}$ est τ-équivariant.*

Démonstration. Le morphisme ι qui commute à τ, fournit par passage au quotient un morphisme $\iota/\tau : \Gamma/\tau \to S/\tau$. Comme Γ est hyperelliptique, on a $\Gamma/\tau \cong \mathbf{P}^1$ et en particulier Γ/τ est lisse. Donc le morphisme ι/τ se relève en un morphisme $(\iota/\tau)^{\sim} : \Gamma/\tau \to S^{\sim}$. Comme le diagramme (4.1.1) est cartésien, le morphisme ι se relève en un morphisme $\iota^{\perp} : \Gamma \to S^{\perp}$. L'unicité de $\iota^{\perp}$ résulte de ce que $\iota^{\perp}$ coincide avec ι sur un ouvert dense. La commutation à τ résulte de l'unicité. C.Q.F.D.

4.4. Soient n un entier > 0, μ un quadruplet adapté à n (3.13). Notons $\mathrm{RTH}(n,\mu)$ l'ensemble des classes d'isomorphismes de revêtements tangentiels hyperelliptiques (2.10) de degré n et de type μ (3.13) et $|\lambda(n,\mu)|$ l'espace projectif des diviseurs effectifs de $S^{\sim}$ appartenant au système linéaire $\lambda(n,\mu)$. Soient $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique de degré n et de type μ, $\iota^{\perp} : \Gamma \to S^{\perp}$ le morphisme décrit en (4.3). On note $\rho(\Gamma)$ l'image réduite de Γ dans $S^{\sim}$ par le morphisme $\phi \circ \iota^{\perp}$. On a $\rho(\Gamma) \in |\lambda(n,\mu)|$: en effet $\iota^{\perp}(\Gamma)$ est le transformé strict de $\iota(\Gamma)$. On a $\iota(\Gamma) \in |nC_0 + S_0|$ (3.10). Par suite $\iota^{\perp}(\Gamma) \in |e^*(nC_0 + S_0) - s_0^{\perp} - \Sigma\mu_i r_i^{\perp}|$ et comme $\iota^{\perp}(\Gamma)$ est symétrique et n'est pas contenu dans le lieu de ramification, on a $\iota^{\perp}(\Gamma) = \phi^*(\rho(\Gamma))$. On en déduit $\rho(\Gamma) \in |\lambda(n,\mu)|$ par unicité des $\lambda(n,\mu)$ (4.2).

Théorème 4.5.

(1) *L'application $\Gamma \mapsto \rho(\Gamma)$ établit une bijection entre les éléments hyperelliptiques minimaux (2.10) de $RTH(n,\mu)$ et l'ensemble des éléments de $|\lambda(n,\mu)|$ qui sont des courbes rationnelles (i.e. de genre géométrique nul). On a $\Gamma = \widehat{\rho(\Gamma)} \times_{S^{\sim}} S^{\perp}$ où $\widehat{\rho(\Gamma)}$ est la normalisée de $\rho(\Gamma)$.*

(2) *Tout revêtement tangentiel hyperelliptique domine un revêtement tangentiel hyperelliptique minimal déterminé à isomorphisme unique près.*

(3) *Soient $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique et $\Gamma' \to \Gamma$ un morphisme qui est un isomorphisme birationnel. Alors le morphisme composé $\Gamma' \to X$ est un revêtement tangentiel hyperelliptique.*

Démonstration. Soit $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique de degré n et de type μ. Il est clair tout d'abord que $\rho(\Gamma)$ est une courbe rationnelle car le morphisme $\phi \circ \iota^{\perp}$ se factorise en le morphisme canonique de passage au quotient $\Gamma \to \Gamma/\tau \cong \mathbf{P}^1$ suivi du morphisme $(\iota/\tau)^{\sim}$. Cette factorisation nous fournit de plus une factorisation de $\iota^{\perp}$ en un morphisme $\Gamma \to \Gamma/\tau \times_{S^{\sim}} S^{\perp}$ qui est un morphisme birationnel τ-équivariant de Γ sur une courbe irreductible et réduite, suivi de la deuxième projection $\Gamma/\tau \times_{S^{\sim}} S^{\perp} \to S^{\perp}$. Posons $\Gamma' = \Gamma/\tau \times_{S^{\sim}} S^{\perp}$. Comme l'image de Γ' dans S est aussi l'image de Γ dans S, le morphisme $\Gamma' \to X$ est un revêtement tangentiel (3.8) dominé par Γ. De plus le morphisme $\Gamma' \to \Gamma/\tau$ défini par la première projection induit un isomorphisme de Γ'/τ sur Γ/τ et, en particulier, Γ' est une courbe hyperelliptique. Donc $\Gamma' \to X$ est un revêtement tangentiel hyperelliptique dominé par $\Gamma \to X$ et par suite $\Gamma' = \Gamma$ lorsque Γ est minimal. La courbe Γ' est uniquement déterminé par $\rho(\Gamma)$ car Γ/τ n'est autre que la normalisée de $\rho(\Gamma)$ et par suite Γ' est minimal. On a donc démontré l'assertion (2), la dernière assertion de (1) et l'injectivité de l'application ρ. La surjectivité de ρ résulte du lemme suivant :

Lemme 4.6. *Soient C une courbe rationnelle appartenant à $|\lambda(n,\mu)|$, $\widehat{C}$ la courbe normalisée de C. Alors $\Gamma = \widehat{C} \times_{S^{\sim}} S^{\perp}$ est une courbe et le morphisme $\Gamma \to X$ obtenu en composant la deuxième projection $\Gamma \to S^{\perp}$ et le morphisme $\pi \circ e : S^{\perp} \to X$, est un revêtement tangentiel hyperelliptique minimal de degré n et de type μ.*

Démonstration. Remarquons d'abord que C n'est pas égal à une des composantes $r_i^{\sim}$., ou $s_i^{\sim}$ car cela entrainerait que, pour un indice i, $2r_i^{\perp}$ ou $2s_i^{\perp}$ est linéairement équivalent à $e^*(nC_0 + S_0) - s_0^{\perp} - \Sigma\mu_i r_i^{\perp}$ ce qui n'est pas possible : les diviseurs $C_0^{\perp}$, $S_0^{\perp}$, $s_\iota^{\perp}$, $r_i^{\perp}$ sont linéairement indépendants. Par suite le diviseur ϕ^*C est réduit, symétrique et possède une ou deux composantes toutes distinctes de $s_0^{\perp}$. Mais on a $(\phi^*C).s_0^{\perp} = (e^*(nC_0 + S_0) - s_0^{\perp} - \Sigma\mu_i r_i^{\perp}).s_0^{\perp} = 1$. Par suite ϕ^*C est irréductible et réduit. Donc Γ est une courbe hyperelliptique, et $\Gamma \to X$ est un revêtement tangentiel hyperelliptique (3.8), minimal d'après ce qui précède. C.Q.F.D.

4.7. Achevons maintenant la démonstration de (4.5) et démontrons (3). Il est clair que $\Gamma' \to X$ est un revêtement tangentiel (3.8). Montrons que Γ' est hyperelliptique. On a sur Γ/τ deux injections $O_\Gamma \to O_{\Gamma'} \to O_{\widehat{\Gamma}}$ où $\widehat{\Gamma}$ est la normalisée de Γ. L'involution hyperelliptique opère sur O_Γ et sur $O_{\widehat{\Gamma}}$ et opère par -1 sur $O_\Gamma/O_{\widehat{\Gamma}}$. Donc elle opère aussi sur $O_{\Gamma'}$ et par suite Γ' est hyperelliptique. C.Q.F.D.

Corollaire 4.8. *Soit $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique de degré n et de type μ (3.13).*

(1) *On a $2n + 1 - \mu^{(2)} \geq 0$.*

(2) *Le genre arithmétique $g(\Gamma)$ de la courbe Γ est $\leq (\mu^{(1)} - 1)/2$. On a $g(\Gamma) = (\mu^{(1)} - 1)/2$ si et seulement si π est un revêtement tangentiel hyperelliptique minimal.*

Démonstration

(1) On sait que $\rho(\Gamma)$ est une courbe du système linéaire $|\lambda(n,\mu)|$. Son genre arithmétique est donné par la formule d'adjonction [7, p. 471] : $g(\rho(\Gamma)) = 1 + (\rho(\Gamma).\rho(\Gamma) + K^{\sim}.\rho(\Gamma))/2$. On obtient

$$g(\rho(\Gamma)) = (2n + 1 - \mu^{(2)})/4$$

(4.2). L'inégalité annoncée résulte de $g(\rho(\Gamma)) \geq 0$.

(2) Posons $\Gamma^{\perp} = \iota^{\perp}(\Gamma) = \phi^*(\rho(\Gamma))$. On sait qu'on a

$$\Gamma^{\perp} \in |e^*(nC_0 + S_0) - s_0^{\perp} - \Sigma\mu_i r_i^{\perp}|$$

(3.10). On a donc $\Gamma^{\perp}.\Gamma^{\perp} = 2n - 1 - \mu^{(2)}$ et $\Gamma^{\perp}.K^{\perp} = \mu^{(1)} - 1$, et par suite $g(\Gamma^{\perp}) = n - (\mu^{(2)} - \mu^{(1)})/2$. Le morphisme canonique $\Gamma/\tau \to \rho(\Gamma)$ est un isomorphisme birationnel donc $O_{\Gamma/\tau}/O_{\rho(\Gamma)}$ est un faisceau de longueur finie sur $\rho(\Gamma)$. Comme Γ/τ est isomorphe à $\mathbf{P}^1$, on a $\mathrm{long}(O_{\Gamma/\tau}/O_{\rho(\Gamma)}) = g(\rho(\Gamma)) = (2n + 1 - \mu^{(2)})/4$. Posons $\Gamma' = \Gamma/\tau \times_{\rho(\Gamma)} \Gamma^{\perp}$. On sait que le morphisme canonique $\Gamma \to \Gamma'$ est un isomorphisme (4.6). Le morphisme $\Gamma' \to \Gamma^{\perp}$ est un isomorphisme birationnel et par suite $O_{\Gamma'}/O_{\Gamma^{\perp}}$ est un faisceau de longueur finie sur $\Gamma^{\perp}$. Comme le morphisme $\Gamma^{\perp} \to \rho(\Gamma)$ est fini et plat de degré 2, on a $\mathrm{long}(O_{\Gamma'}/O_{\Gamma^{\perp}}) = 2\,\mathrm{long}(O_{\Gamma/\tau}/O_{\rho(\Gamma)}) = (2n + 1 - \mu^{(2)})/2$. D'où

$$\begin{aligned} g(\Gamma') &= g(\Gamma^{\perp}) - \mathrm{long}(O_{\Gamma'}/O_{\Gamma^{\perp}}) \\ &= (2n - (\mu^{(2)} - \mu^{(1)}))/2 - (2n + 1 - \mu^{(2)})/2 \\ &= (\mu^{(1)} - 1)/2. \end{aligned}$$

Le morphisme canonique $\Gamma \to \Gamma'$ est un isomorphisme birationnel et par suite on a $g(\Gamma) \leq g(\Gamma')$ et on a $g(\Gamma) = g(\Gamma')$ si et seulement si Γ est un revêtement tangentiel hyperelliptique minimal. C.Q.F.D.

Théorème 4.9. *Soit X une courbe elliptique. Pour tout $n > 0$, il n'existe qu'un nombre fini de classes d'isomorphismes de revêtements* *tangentiels hyperelliptiques de degré n de X. Le genre arithmétique de ces revêtements est inférieur ou égal à* $\gamma(n) = \sup\{p \in \mathbf{N} | (p(p+1))/2 \leq n\}$ (1.5).

Démonstration Le type μ d'un revêtement tangentiel hyperelliptique de degré n satisfait l'inégalité $2n + 1 - \mu^{(2)} \geq 0$ (4.8). Il n'y a donc qu'un nombre fini de types possible pour un degré n donné et il suffit donc de démontrer la finitude de l'ensemble des classes d'isomorphismes de revêtements tangentiels hyperelliptiques de degré n et de type μ. Démontrons d'abord le lemme suivant :

Lemme 4.10. *Soient Σ une surface analytique complexe, K_Σ la classe canonique, $f : \mathbf{P}^1 \to \Sigma$, un morphisme. On suppose qu'on a $f(\mathbf{P}^1).K_\Sigma = -1$. Alors pour tout germe de déformation analytique $t \mapsto f_t$ de $f = f_0$, on a $f_t(\mathbf{P}^1) = f_0(\mathbf{P}^1)$ pour tout t dans un voisinage de* 0.

Démonstration. Notons T_Σ et $T_{\mathbf{P}^1}$ les fibrés tangents et pour tout $x \in \mathbf{P}^1$, posons $f^\cdot(x,t) = \partial f/\partial t(x)$. On obtient ainsi une famille de sections analytiques du fibré f^*T_Σ. La différentielle (en x) de f_t fournit une suite exacte $0 \to T_{\mathbf{P}^1} \to f^*T_\Sigma \to N_t \to 0$ où N_t est un faisceau cohérent sur $\mathbf{P}^1$. L'hypothèse implique que le degré de f^*T_Σ est égal à 1. Comme le degré de $T_{\mathbf{P}^1}$ est égal à 2, pour tout t dans un voisinage de zéro, N_t est somme d'un faisceau localement libre de rang 1 et de degré strictement négatif, et d'un faisceau de torsion. Par suite l'image dans N_t de la section $f^\cdot(x,t)$ est nulle sauf au plus en un nombre fini de points. On en déduit alors le lemme par une suite d'arguments classiques. C.Q.F.D.

4.11. L'ensemble des éléments de $|\lambda(n,\mu)|$ qui sont des courbes rationnelles est un ensemble algébrique constructible. Cet ensemble constructible est de dimension 0 d'après (4.10) qui s'applique en vertu de (4.2,3)). Cet ensemble est donc fini. Pour achever la démonstration de la finitude, on est ramené, d'après (4.5), à démontrer que l'ensemble des classes d'isomorphismes de domination birationnelles d'une courbe hyperelliptique est fini. Mais cette finitude est vraie plus généralement pour toute les courbes dont les singularités sont analytiquement isomorphes à des points doubles de courbes planes, comme le sont, ainsi qu'on le vérifie facilement, les singularités des courbes hyperelliptiques. Enfin la dernière assertion de (4.9), résulte de (4.8) et du lemme ci-après.

Lemme 4.12.

(1) *Soient $n \in \mathbf{N}$, $n > 0$, et $\mu \in \mathbf{N}^4$, adapté à n (3.13) tel que*

$$2n + 1 - \mu^{(2)} \geq 0.$$

Il existe un entier m, $0 \leq m \leq n$, de même parité que n tel que $2m+1 = \mu^{(2)}$. De plus, μ est adapté à m. On a $2n+1-\mu^{(2)} \equiv 0$, mod.4.

(2) *Sous les hypothèses de (1), on a $(\mu^{(1)}-1)/2 \leq \gamma(n)$. On a $(\mu^{(1)}-1)/2 = \gamma(n)$ si et seulement si on a $\sum_{i<j}(\mu_i-\mu_j)^2 \leq 4\mu^{(1)}-1$.*

Démonstration.

(1) S'obtient immédiatement en considérant les deux cas n pair et n impair.

(2) Posons $d = m-(\mu^{(1)}-1)(\mu^{(1)}+1)/8$ et montrons d'abord qu'on a $(\mu^{(1)}-1)/2 \leq \gamma(m)$, c'est à dire $d \geq 0$. Or on a $8d = 4\mu^{(2)} - \mu^{(1)2} - 3 = (\Sigma_{i<j}(\mu_i-\mu_j)^2) - 3$ qui est positif car $\mu_0 - \mu_i$ est non nul pour $1 \leq i \leq 3$. D'après ce qui précède, l'égalité $(\mu^{(1)}-1)/2 = \gamma(n)$ équivaut à l'inégalité $n < (\mu^{(1)}+1)(\mu^{(1)}+3)/8$ où encore à $2d < \mu^{(1)}+1$. Comme $\mu^{(1)}$ est impair, cela équivaut à $2d \leq \mu^{(1)}-1$ ou encore, d'après ce qui précède, à $(\Sigma_{i<j}(\mu_i-\mu_j)^2) \leq 4\mu^{(1)}-1$. C.Q.F.D.

Proposition 4.13. *Soit $\pi : \Gamma \rightarrow X$ un revêtement tangentiel hyperelliptique de degré n tel que le genre arithmétique de Γ soit $\gamma(n)$. Alors π est un revêtement tangentiel hyperelliptique minimal. Si $n = 2$, Γ est une courbe elliptique et π est une isogénie de degré 2. Si $n \neq 2$, π est un revêtement primitif* (2.9).

Démonstration. La première assertion résulte de (4.9) et de (4.8,2)). Supposons $n > 1$. Soit $\Gamma \rightarrow X'$ et $X' \rightarrow X$ une factorisation de π tel que $\Gamma \rightarrow X'$ soit un revêtement de degré ν et $X' \rightarrow X$ une isogénie de degré $\delta > 1$. On a $n = \nu\delta$. D'après (4.9), on a $g_a(\Gamma) \leq \gamma(\nu)$, car $\Gamma \rightarrow X'$ est un revêtement tangentiel hyperelliptique. On a donc $\gamma(\nu) = \gamma(n)$, d'où $n < (\gamma(\nu)+1)(\gamma(\nu)+2)/2$ ce qui implique $n \leq \nu + \gamma(\nu)$, c'est à dire $\nu(\delta-1) \leq \gamma(\nu)$, ce qui implique encore $\nu \leq \gamma(\nu)$ car on a $\delta > 1$. L'inégalité $\nu \leq \gamma(\nu)$ implique $\nu = 1$, d'où $g_a(\Gamma) = 1 = \gamma(n)$, ce qui implique $n = 2$. Réciproquement, $n = 2$ implique $\gamma(n) = 1$. Comme Γ revêt la courbe elliptique X, son genre géométrique est ≥ 1. Donc Γ est une courbe elliptique et π est une isogénie de degré 2. C.Q.F.D.

5. Diviseurs remarquables sur $S^{\sim}$

5.1. Choisissons une fibre l de $\pi : S \rightarrow X$ au dessus d'un point de X qui n'est pas un point de 2-division. Notons $l^{\perp}$ le transformé strict de l, et

$l^\sim$ l'image réduite de $l^\perp$ dans $S^\sim$.

Lemme 5.2.

(1) *Le groupe* $Pic(S^\sim)$ *est libre de rang* 10. *Les* 10 *diviseurs* $C_0^\sim, l^\sim, S_i^\sim, r_i^\sim$ $(0 \leq i \leq 3)$ *en forment une base sur* $\mathbf{Z}$.

(2) *Pour* $0 \leq i \leq 3$, *on a la relation* $l^\sim \equiv 2S_i^\sim + r_i^\sim + s_i^\sim$. *On a* $C_1^\sim \equiv C_0^\sim + 2l^\sim - \Sigma_1^3(S_i^\sim + r_i^\sim)$.

(3) *Pour tout* $n \geq 0$ *et tout* $\mu \in \mathbf{N}^4$, *adapté à n (3.13), on a (4.2)* $\lambda(n,\mu) \equiv n(C_0^\sim + 2l^\sim) - \Sigma\{(n-\varepsilon_i)S_i^\sim + ((n+\mu_i-\varepsilon_i)/2)r_i^\sim\}$.

(4) *Soient* $n \geq 0$, $\mu \in \mathbf{N}^4$, *adapté à* n *et* $x \in |\lambda(n,\mu)|$. *Il existe un entier* $m \geq 0$, *un quadruplet* ν *adapté à* m *et une courbe irréductible* $C^\sim \equiv \lambda(n,\mu)$ *tels que :*

$.\mu_i \leq \nu_i, \quad$ pour $0 \leq i \leq 3$,

$.n \equiv m,$ mod.2,

$.\nu^{(2)} \leq 2m+1 \leq 2n+1,$

$.x = C^\sim + (n-m)C_0^\sim + \Sigma\{((n-m)/2)s_i^\sim + ((\nu_i-\mu_i)/2)r_i^\sim\}.$

Démonstration

(1) Notons NS($S^\perp$) le groupe de Néron-Severi de $S^\perp$. Comme $S^\perp$ se déduit de S par éclatement de 8 points, ce groupe est libre de rang 10 . On sait que Pic($S^\sim$) est libre de rang 10 (4.2) et les morphismes ϕ^* et ϕ_* induisent des morphismes encore notés ϕ^* et ϕ_* entre NS($S^\perp$) et Pic($S^\sim$). On a $\phi_*\phi^* = 2\mathrm{Id}$. Par suite ϕ^* et ϕ_* sont injectifs et des isomorphismes modulo torsion. De plus on a $\#\mathrm{coker}\phi^*.\#coker\phi_* = 2^{10}$, où $\#$ désigne le cardinal d'un ensemble. Comme ϕ_* et ϕ^* sont adjoints pour le produit d'intersection, on a $\#\mathrm{coker}\phi^* = \#\mathrm{coker}\phi_* = 2^5\#$. Les classes des diviseurs $C_0^\perp, l^\perp, S_i^\perp, r_i^\perp, 0 \leq i \leq 3$, forment une base de NS($S^\perp$). Par suite les classes linéaires de diviseurs $2C_0^\sim = \phi_*C_0^\perp$, $l^\sim = \phi_*l^\perp$, $2S_i^\sim = \phi_*S_i^\perp$, $r_i^\sim = \phi_*r_i^\perp$ forment une $\mathbf{Q}$-base de Pic($S^\sim$). Donc les diviseurs $C_0^\sim, l^\sim, S_i^\sim, r_i^\sim$ forment une $\mathbf{Q}$-base de Pic($S^\sim$) de déterminant 2^{-5} par rapport à la première. Ils forment donc une base sur $\mathbf{Z}$ de Pic($S^\sim$).

(2) Pour tout i, $l^\perp$ est algébriquement équivalent à $S_i^\perp + r_i^\perp + s_i^\perp$. Les premières relations s'en déduisent en appliquant ϕ_* et en remarquant que l'équivalence algébrique sur $S^\sim$ entraine l'équivalence linéaire. Posons $D = C_0^\sim + 2l^\sim - \Sigma_1^3(S_i^\sim + r_i^\sim)$. Compte tenu des relations déja démontrées, on a $2D \equiv (2C_0^\sim + \Sigma s_i^\sim) + (2S_0^\sim + r_0^\sim + s_0^\sim) - s_0^\sim - \Sigma_1^3 r_i{}^\sim$. On a donc $2D \equiv \phi_*(e^*(C_0+S_0) - s_0^\perp - \Sigma_1^3 r_i{}^\perp)$, d'où $2D \equiv \phi_*(\phi^*(\lambda(1,(0,1,1,1))))$ (4.2) et par suite $2D \equiv 2C_1^\sim$. Comme Pic($S^\sim$) est sans torsion , on a $D \equiv C_1^\sim$.

(3) Les égalités résultent immédiatement de (2) et de la description des $\lambda(n,\mu)$ donnée dans la démonstration de (4.2).

(4) Il existe un diviseur effectif y dont les composantes sont distinctes des $r_i^\sim$ et des $s_i^\sim$ et des entiers α_i et β_i positifs tels que $x = y + \Sigma(\alpha_i s_i^\sim + \beta_i r_i^\sim)$. Le diviseur symétrique $\phi^*(y)$ est le transformé strict de $z = e_*\phi^*(y)$ et on a $\phi^*(y) \equiv e^*(nC_0 + S_0) - \Sigma(\alpha_i - \varepsilon_i)s_i{}^\perp - \Sigma(\beta_i + \mu_i)r_i{}^\perp$. On a donc $z \in |nC_0 + S_0|$ et par suite (3.9) on a $z = (n-m)C_0 + C$ où C est irréductible et symétrique, $C \in |mC_0 + S_0|$ et $0 \le m \le n$. Soient ν le type de C (3.13) et $C^\perp$ le transformé strict de C. On a $C^\perp \equiv e^*(mC_0 + S_0) - s_0{}^\perp - \Sigma\nu_i r_i{}^\perp \equiv \phi^*(\lambda(m,\nu))$ (4.2). Comme $C^\perp$ est irréductible, symétrique et distinct des $r_i^\sim$ et des $s_i^\sim$, on a $\phi_* C^\perp = 2C^\sim$ où $C^\sim$ est irréductible. On en déduit $2C^\sim = \phi_* C^\perp \equiv \phi_*\phi^*(\lambda(m,\nu)) = 2\lambda(m,\nu)$, d'où $C^\sim \equiv \lambda(m,\nu)$ car $\mathrm{Pic}(S^\sim)$ est sans torsion. Comme $\phi^*(y)$ est le transformé strict de z, on a $\phi^*(y) = (n-m)C_0{}^\perp + C^\perp$, d'où $y = (n-m)C_0^\sim + C^\sim$. On en déduit $x \equiv (n-m)C_0^\sim + C^\sim + \Sigma(\alpha_i s_i^\sim + \beta_i r_i^\sim)$. En exprimant le second membre de cette dernière égalité dans la base donnée en (1), on obtient compte tenu de (2) et de (3)

$$x \equiv nC_0^\sim + (2m + \Sigma\alpha_i)l^\sim - \Sigma\{(m - \varepsilon_i + 2\alpha_i)S_i^\sim + ((m + \nu_i - \varepsilon_i)/2 - \beta_i + \alpha_i)r_i^\sim\}.$$

En comparant avec l'expression de $\lambda(n,\mu)$ donnée dans (3), on obtient $\alpha_i = (n-m)/2$ et $\beta_i = (\nu_i - \mu_i)/2$ pour $0 \le i \le 3$. Comme les α_i sont entiers on a $n \equiv m$, mod.2, et comme les β_i sont ≥ 0, on a $\mu_i \le \nu_i$. Comme $C^\sim \in |\lambda(m,\nu)|$ est irréductible, on a, par la formule d'adjonction [7, p. 471] $0 \le g_a(C^\sim) = 1 + (C^\sim.C^\sim + C^\sim.K^\sim)/2 = (2m + 1 - \nu^{(2)})/4$ d'après (4.2). C.Q.F.D.

Proposition 5.3. *Soient $n \ge 0$ et μ un quadruplet adapté à n* (3.13). *Alors l'espace $|\lambda(n,\mu)|$ est vide si $2n + 1 - \mu^{(2)} < 0$ et on a $\dim|\lambda(n,\mu)| = (2n + 1 - \mu^{(2)})/4$ si $2n + 1 - \mu^{(2)} \ge 0$.*

Démonstration. Démontrons d'abord qu'on a $h^2(S^\sim, O_{S^\sim}(\lambda(n,\mu)) = 0$ pour $n \ge 0$. L'injection canonique $O_{S^\sim} \to \phi_* O_{S^\perp}$ est scindable : un faisceau supplémentaire est le sous-faisceau de $\phi_* O_{S^\perp}$ des sections anti-invariantes par τ. Par suite, pour tout faisceau L de $O_{S^\sim}$-modules, l'injection canonique $L \to \phi_*\phi^* L$ est scindable et par conséquent, les applications canoniques $H^j(S^\sim, L) \to H^j(S^\perp, \phi^* L)$ sont injectives. Il suffit donc de montrer que $h^2(S^\perp, O_{S^\perp}(\phi^*\lambda(n,\mu))) = 0$. D'après (4.2), on a une suite exacte de faisceaux sur $S^\perp$, $0 \to O_{S^\perp}(\phi^*\lambda(n,\mu)) \to e^*(O_S(nC_0 + S_0)) \to O_{s_0^\perp + \Sigma\mu_i r_i^\perp}(e^*(nC_0 + S_0)) \to 0$. On a $e^*(nC_0 + S_0).r_i{}^\perp = e^*(nC_0 + S_0).s_i{}^\perp = 0$ pour $0 \le i \le 3$ et les $r_i{}^\perp$ et $s_i{}^\perp$ sont deux à deux disjoints. Par suite le faisceau quotient de cette suite exacte est isomorphe à

$\bigoplus_0^3 O_{\mu_i r_i^\perp} \oplus O_{s_0^\perp}$ et ce dernier faisceau est acyclique en degré > 0. On a donc

$$\begin{aligned} h^2(S^\perp, O_{S^\perp}(\phi^* \lambda(n,\mu))) &= h^2(S^\perp, e^*(O_S(nC_0 + S_0))) \\ &= h^2(S, O_S(nC_0 + S_0))) \end{aligned}$$

et ce dernier entier est nul d'après (3.2).

Supposons tout d'abord $2n + 1 < \mu^{(2)}$ et supposons qu'il existe $x \in |\lambda(n,\mu)|$. Il résulte de (5.2,4)), qu'il existe un entier m et un quadruplet ν tels que $\mu^{(2)} \leq \nu^{(2)} \leq 2m + 1 \leq 2n + 1$, contradiction ! Donc $|\lambda(n,\mu)|$ est vide dans ce cas.

Supposons maintenant que $2n + 1 = \mu^{(2)}$. D'après le théorème de Riemann-Roch [7, p. 472], on a

$$\begin{aligned} \chi(O_{S^\sim}(\lambda(n,\mu))) &= \chi(O_{S^\sim}) + (\lambda(n,\mu).\lambda(n,\mu) - \lambda(n,\mu).K^\sim)/2 \\ &= 1 + ((2n - 1 - \mu^{(2)})/2 + 1)/2 = 1 \end{aligned}$$

(4.2), et comme on a $h^2(S^\sim, O_{S^\sim}(\lambda(n,\mu))) = 0$ on en déduit

$$h^0(S^\sim, O_{S^\sim}(\lambda(n,\mu))) > 0.$$

Par suite il existe $x \in |\lambda(n,\mu)|$ et on a donc $\dim|\lambda(n,\mu)| \geq 0$. D'après (5.2,4)), tout $x \in |\lambda(n,\mu)|$ est irréductible et d'après (4.2), on a $x.x = -1$. Donc $|\lambda(n,\mu)|$ ne comporte qu'un seul élément et on a $\dim|\lambda(n,\mu)| = 0$ dans ce cas.

Il reste à montrer que $\dim|\lambda(n,\mu)| = (2n + 1 - \mu^{(2)})/4$ lorsque $2n + 1 - \mu^{(2)} > 0$. Comme on a $h^2(S^\sim, O_{S^\sim}(\lambda(n,\mu))) = 0$ et

$$\chi(O_{S^\sim}(\lambda(n,\mu))) = 1 + (2n + 1 - \mu^{(2)})/4$$

d'après le théorème de Riemann-Roch [7, p. 472] et (4.2), l'égalité à démontrer équivaut à $h^1(S^\sim, O_{S^\sim}(\lambda(n,\mu))) = 0$ dans ce cas. Remarquons que ce qui précède implique $h^1(S^\sim, O_{S^\sim}(\lambda(n,\mu))) = 0$ lorsque $2n + 1 = \mu^{(2)}$.

Il résulte de (4.2,2) et (3)) qu'on a $\lambda(n + 2,\mu) \equiv \lambda(n,\mu) + C^\sim$ où $C^\sim \equiv 2C_0^\sim + \Sigma s_i^\sim$. On a donc une suite exacte de faisceaux sur $S^\sim$:

(5.3.1) $0 \to O_{S^\sim}(\lambda(n,\mu)) \to O_{S^\sim}(\lambda(n + 2,\mu)) \to O_{C^\sim}(\lambda(n + 2,\mu)) \to 0,$

et, par récurrence croissante sur n, la nullité de $h^1(S^\sim, O_{S^\sim}(\lambda(n,\mu)))$ est impliqué par la nullité de $h^1(S^\sim, O_{C^\sim}(\lambda(n,\mu)))$. Il résulte de (4.2,3)) qu'on a $C_0^\sim.\lambda(n,\mu) = 0$ et $s_i^\sim.\lambda(n,\mu) = \varepsilon_i$ pour $0 \leq i \leq 3$. De plus, une classe d'équivalence rationnelle de diviseurs sur $C^\sim$ est caractérisée par ses degrés sur les différentes composantes. Donc $O_{C^\sim}(\lambda(n,\mu))$ est isomorphe à $O_{C^\sim}(\lambda(2,\underline{\mu}))$ où $\underline{\mu} = (1,2,0,0)$ et il suffit de démontrer qu'on a $h^1(S^\sim, O_{C^\sim}(\lambda(2,\underline{\mu}))) = 0$.

En reprenant la suite exacte (5.3.1), on obtient dans ce cas

$$0 \to O_{S^\sim}(\lambda(0,\underline{\mu})) \to O_{S^\sim}(\lambda(2,\underline{\mu})) \to O_{C^\sim}(\lambda(2,\underline{\mu})) \to 0.$$

On a $|\lambda(0,\underline{\mu})| = \emptyset$ d'après ce qui précède. On a donc $h^0(S^\sim, O_{S^\sim}(\lambda(0,\underline{\mu}))) = 0$ et rappellons qu'on a montré qu'on a $h^2(S^\sim, O_{S^\sim}(\lambda(0,\underline{\mu}))) = 0$. De plus la formule de Riemann-Roch [7, p. 472] fournit $\chi(O_{S^\sim}(\lambda(0,\underline{\mu}))) = 0$ (4.2). On a donc aussi $h^1(S^\sim, O_{S^\sim}(\lambda(0,\underline{\mu}))) = 0$. Il en résulte qu'on a $h^1(S^\sim, O_{S^\sim}(\lambda(2,\underline{\mu}))) = h^1(S^\sim, O_{C^\sim}(\lambda(2,\underline{\mu})))$. Mais comme on a $\underline{\mu}^{(2)} = 5$ et $2n+1 = 5$ lorsque $n = 2$, on a $h^1(S^\sim, O_{S^\sim}(\lambda(2,\underline{\mu}))) = 0$ d'après ce qui précède, et par suite on a $h^1(S^\sim, O_{C^\sim}(\lambda(2,\underline{\mu}))) = 0$. C.Q.F.D.

5.4. Rappelons qu'on appelle *diviseur exceptionnel* d'une surface algébrique complète lisse Σ, les diviseurs de Σ isomorphes à $\mathbf{P}^1$, d'auto-intersection -1. On sait qu'un diviseur $D \subset \Sigma$, irréductible, est exceptionnel si et seulement si D est le diviseur d'une contraction de Σ sur une surface lisse, ou bien si et seulement si on a $D.D = -1$ et $D.K_\Sigma = -1$ où K_Σ est le diviseur canonique de Σ.

Notons ${}_2X$ (resp. ${}_2\Delta$) le groupe (d'ordre 4) des points de 2-division de X (resp. Δ). La projection $\Delta \to X$ induit un isomorphisme de ${}_2\Delta$ sur ${}_2X$ permettant de les identifier. Le groupe ${}_2X$ opére donc par translation sur Δ et par transport de structure sur S. Cette opération commute à l'involution τ. Elle passe au quotient et fournit une opération de ${}_2X$ sur S/τ et donc sur $S^\sim$. Le groupe ${}_2X$ opére librement sur $S^\sim$.

Proposition 5.5.

(1) *Le groupe ${}_2X$ opère librement sur l'ensemble des diviseurs de $S^\sim$ tels que $D.K^\sim = -1$. Pour tout diviseur D tel que $D.K^\sim = -1$, il existe un unique $\alpha \in {}_2X$ tel que αD coupe $s_0^\sim$.*

(2) *Soient n un entier ≥ 0, μ un quadruplet adapté à n, $D \in |\lambda(n,\mu)|$ un diviseur irréductible. Alors on a $D.K^\sim = -1$ et D coupe $s_0^\sim$.*

(3) *Réciproquement soit $D \subset S^\sim$ un diviseur irréductible qui coupe $s_0^\sim$ et tel que $D.K^\sim = -1$. Posons $\mu_i = D.r_i^\sim$ pour $0 \leq i \leq 3$, $\mu = (\mu_i)$, $n = D.l^\sim$. Alors n est un entier ≥ 0, μ est un quadruplet adapté à n et on a $D \in |\lambda(n,\mu)|$.*

Démonstration.

(1) Notons que l'on a $K^\sim \equiv -2C_0^\sim - \Sigma s_i^\sim$ et que D n'est pas un des diviseurs $C_0^\sim$, $s_i^\sim$, $r_i^\sim$, $0 \leq i \leq 3$, car pour ces derniers on a $K^\sim.C_0^\sim = K^\sim.s_i^\sim = K^\sim.r_i^\sim = 0$. On a $1 = -D.K^\sim = D.(2C_0^\sim + \Sigma s_i^\sim)$. Comme l'intersection de deux diviseurs irréductibles distincts est positive et strictement positive si les diviseurs se coupent on a $D.C_0^\sim = 0$, $D.s_i^\sim = 1$ pour un seul indice i et 0 pour les autres indices. Comme ${}_2X$ opère librement sur l'ensemble $\{s_i^\sim | 0 \leq i \leq 3\}$, il existe un unique $\alpha \in {}_2X$ tel que $\alpha D.s_0^\sim = 1$, $\alpha D.s_i^\sim = 0$ pour $1 \leq i \leq 0$.

(2) On a $D.K^\sim = -1$ d'après (4.2). En utilisant (5.2,3), on vérifie que $D.s_0^\sim = 1$.

(3) Supposons démontré que $D \in |\lambda(n,\mu)|$ pour un entier $n \geq 0$ et un quadruplet μ adapté à n. Il résulte alors de (5.2,3), que l'on a $\mu_i = D.r_i^\sim$, pour $0 \leq i \leq 3$ et $D.l^\sim = n$. Le diviseur $D^\perp = \phi^* D$ est symétrique. Il est irréductible et réduit, car il possède au plus deux composantes irréductibles permutées par l'involution τ et qu'on a, d'après la démonstration de (1), $D^\perp.s_0^\perp = D.\phi_* s_0^\perp = D.s_0^\sim = 1$. Comme D est distinct des $r_i^\sim$ et des $s_i^\sim$, $D^\perp$ est le transformé strict de $\delta = e_* D^\perp$. On a $\delta.C_0 = D^\perp.e^*(C_0) = D^\perp.(C_0^\perp + \Sigma s_i^\perp) = D.(2C_0^\sim + \Sigma s_i^\sim) = 1$ et comme on a de plus $D^\perp.s_0^\perp = 1$, δ passe par le point s_0. Il s'ensuit que δ est un élément symétrique du système linéaire $|nC_0 + S_0|$ où $n = \delta.S_0$. Le type μ de δ est un quadruplet adapté à n (3.14). On a $D^\perp \equiv e^*(nC_0 + S_0) - {s_0}^\perp - \Sigma \mu_i {r_i}^\perp$. Par suite (4.2), on a $D \in |\lambda(n,\mu)|$. C.Q.F.D.

Corollaire 5.6. *Il existe une correspondance biunivoque entre les classes d'isomorphismes de revêtements tangentiels hyperelliptiques minimaux et les courbes rationnelles C tracées sur $S^\sim$ qui coupent $s_0^\sim$ et $l^\sim$, telles que $C.K^\sim = -1$.*

Démonstration. Une telle courbe rationnelle C appartient à $|\lambda(n,\mu)|$ où n est strictement positif (5.5), et le corollaire résulte de (4.5). C.Q.F.D.

Corollaire 5.7. *Soient n un entier > 0 et m un quadruplet adapté à n tels que $2n + 1 = \mu^{(2)}$. Notons $D(n,\mu)$ l'unique élément de $|\lambda(n,\mu)|$ (5.3). Alors $D(n,\mu)$ est un diviseur exceptionnel de $S^\sim$ qui coupe $s_0^\sim$ et $l^\sim$. Réciproquement, pour tout diviseur exceptionnel C de $S^\sim$ qui coupe $s_0^\sim$ et $l^\sim$, il existe un unique couple (n,μ), où n est un entier > 0 et μ un quadruplet adapté à n tel que $2n + 1 = \mu^{(2)}$ et $C = D(n,\mu)$.*

Démonstration. On sait qu'on a $D(n,\mu).D(n,\mu) = -1$ et $D(n,\mu).K^\sim = -1$ (4.2) et il résulte de (5.2,4)) que $D(n,\mu)$ est irréductible.

Donc $D(n,\mu)$ est un diviseur exceptionnel. On sait d'après (5.5) que ce diviseur coupe $s_0^{\sim}$ et $l^{\sim}$. La réciproque résulte de (5.5) et du fait que pour tout diviseur exceptionnel C on a $C.K^{\sim} = -1$. C.Q.F.D.

6. Les revêtements tangentiels hyperelliptiques exceptionnels et les autres

Proposition 6.1. *Soit $\pi : \Gamma \to X$ un revêtement tangentiel hyperelliptique minimal. Les propriétés suivantes sont équivalentes :*

(i) *La courbe $\rho(\Gamma)$ (4.4) est un diviseur exceptionnel de $S^{\sim}$.*

(ii) *Le morphisme $\iota^{\perp} : \Gamma \to S^{\perp}$ (4.3) est un plongement.*

Démonstration. On a un morphisme birationnel (4.6) $\Gamma \longrightarrow \iota^{\perp}(\Gamma) \times_{\rho(\Gamma)} \rho(\Gamma)^{\sim}$ où $\rho(\Gamma)^{\sim}$ est la courbe normalisée de $\rho(\Gamma)$. Supposons (i) satisfait. Alors Γ est isomorphe à $\iota^{\perp}(\Gamma)$, d'où (ii). Supposons (ii) satisfait. Alors la première projection $\iota^{\perp}(\Gamma) \times_{\rho(\Gamma)} \rho(\Gamma)^{\sim} \to \iota^{\perp}(\Gamma)$ est un isomorphisme. Comme $\iota^{\perp}(\Gamma) \to \rho(\Gamma)$ est fidèlement plat, on en déduit que $\rho(\Gamma)^{\sim} \to \rho(\Gamma)$ est un isomorphisme et par suite que $\rho(\Gamma)$ est isomorphe à $\mathbf{P}^1$. Comme d'après (4.5), $\rho(\Gamma)$ appartient à un $|\lambda(n,\mu)|$, on a $\rho(\Gamma).K^{\sim} = -1$ (4.2), on en déduit par la formule de Noether que $\rho(\Gamma).\rho(\Gamma) = -1$. Donc $\rho(\Gamma)$ est un diviseur exceptionnel. C.Q.F.D.

6.2. Nous appelons *revêtement tangentiel hyperelliptique exceptionnel*, les revêtement tangentiels hyperelliptiques qui possèdent les propriétés équivalentes de la proposition (6.1). Il résulte de (5.7) que l'application qui à un revêtement tangentiel hyperelliptique associe son degré n et son type μ établit une correspondance biunivoque entre les classes d'isomorphismes de revêtements tangentiels hyperelliptiques exceptionnels et les couples (n,μ) où n est un entier > 0 et μ un quadruplet adapté à n tel que $2n+1 = \mu^{(2)}$. Il résulte de (4.8) que le genre d'un revêtement tangentiel hyperelliptique minimal de type μ est $(\mu^{(1)} - 1)/2$. Par suite (4.8 et 4.9), le genre g de tout revêtement tangentiel hyperelliptique exceptionnel de degré n est soumis aux inégalités $((2n+1)^{1/2} - 1)/2 \leq g \leq \gamma(n)$.

6.3. Pour tout $n > 0$, notons $\chi(n)$ (resp. $\psi(n)$) le cardinal de l'ensemble des revêtements tangentiels hyperelliptiques exceptionnels de degré n (resp. de degré n et de genre $\gamma(n)$). On déduit des résultats de J. Oesterlé et de (6.2) les faits suivants (*cf* appendice pour des estimations plus précises):

(1) La fonction $n \mapsto \chi(n)$ est croissante en O(nloglogn).

(2) La fonction $n \mapsto \sup\{\psi(p) | 0 < p \leq n\}$ tend vers l'infini quand n tend vers l'infini. Cette fonction est $o(n^{1/4} \log^2 n)$.

(3) On a $\psi(n) = 1$ si et seulement si $n = \gamma(n)(\gamma(n) + 1)/2$ i.e. si et seulement si n est un nombre triangulaire. On en déduit qu'il n'existe qu'une seule classe d'isomorphismes de revêtements tangentiels hyperelliptiques exceptionnels de degré n et de genre $\gamma(n)$, si et seulement si n est un nombre triangulaire (4.12). Si $n = g(g+1)/2$, les revêtements tangentiels hyperelliptiques de degré n et de genre g sont de type μ où $\mu_0 = 3[g/2] - g + 1$ et $\mu_i = g - [g/2]$ pour $i > 0$.

Proposition 6.4. *Soient $n = \gamma(\gamma + 1)/2$ un nombre triangulaire et $\pi : \Gamma \rightarrow X$ un revêtement tangentiel hyperelliptique de degré n et de genre γ. Alors $\Gamma \rightarrow X$ est l'unique revêtement tangentiel hyperelliptique exceptionnel de degré n et de genre γ de X et le type de $\Gamma \rightarrow X$ est $\mu_0 = 3[\gamma/2] - \gamma + 1$ et $\mu_i = \gamma - [\gamma/2]$ pour $i > 0$. Ce revêtement est lisse sauf pour un nombre fini de classes d'isomorphismes de courbes elliptiques X (dependant a priori de l'entier n).*

Démonstration Notons que $\gamma = \gamma(n)$ et par suite $\Gamma \rightarrow X$ est un revêtement tangentiel minimal (4.13). On sait qu'on a $\gamma = (\mu^{(1)} - 1)/2$ (4.8). On en déduit l'inégalité $\mu^{(2)} \geq \gamma^2 + \gamma + 1/4$ et comme par ailleurs on a $\mu^{(2)} \leq 2n + 1 = \gamma^2 + \gamma + 1$ (4.8), on a $\mu^{(2)} = 2n + 1$ et par suite $\Gamma \rightarrow X$ est exceptionnel (6.2). Mais lorsque n est triangulaire, on a $\psi(n) = 1$ et par suite $\Gamma \rightarrow X$ est l'unique revêtement tangentiel exceptionnel de degré n. On vérifie immédiatement que le type de $\Gamma \rightarrow X$ et celui annoncé. Lorsque X est une courbe elliptique qui possède une structure réelle telle que l'ensemble des points réels ne soit pas connexe, il résulte des travaux de Ince [8] et de la théorie des solutions périodiques de KdV [12], que la courbe Γ correspondante est lisse. Comme cette courbe Γ varie dans une famille plate dépendant de l'invariant $j(X)$, on en déduit que Γ est lisse sauf au plus pour un nombre fini de valeurs de $j(X)$. C.Q.F.D.

Théorème 6.5.

(1) *Pour toute courbe hyperelliptique $(X, \partial/\partial x)$, la fonction $2n\mathfrak{P}(x)$ est un potentiel hyperelliptique si et seulement si n est un entier triangulaire.*

(2) Supposons que n soit triangulaire. Le revêtement tangentiel hyperelliptique associé au potentiel $2n\mathfrak{P}(x)$ (2.7) est l'unique revêtement tangentiel hyperelliptique Γ de degré n de X, de genre $g = \gamma(n)$. La donnée initiale correspondante est $O_\Gamma((g-1)p)$.

Démonstration Supposons que $2n\mathfrak{P}(x)$ soit un potentiel hyperelliptique. Il existe une fonction $u(x,t)$, méromorphe en x et t, telle que $u(x,0) = 2n\mathfrak{P}(x)$, qui soit solution de l'équation KdV. En reportant les developpements de Laurent en x de u dans l'équation KdV, on constate immédiatement que n est un nombre triangulaire. La condition sur n est donc nécéssaire. Supposons maintenant que n soit triangulaire; posons $g = \gamma(n)$ et démontrons la réciproque ainsi que (2). Soient $\pi : \Gamma \to X$ l'unique revêtement tangentiel hyperelliptique de degré n et de genre g (6.4) et $v(x,t)$ la solution de KdV associée à la donnée de Kritchever tangentielle $(\Gamma, p, U, W+\alpha U, \mathcal{O}_\Gamma(e))$ où $e = (g-1)p$ (2.6). D'après un résultat de Fay [6], on sait que le nombre d'intersection en e du diviseur Θ avec le flot défini par U est $g(g+1)/2$ lorsque la courbe Γ est lisse. Par un argument facile de spécialisation, on en déduit que ce nombre d'intersection est $\geq g(g+1)/2$ dans le cas général (on peut en fait montrer qu'il est encore égal à $g(g+1)/2$). Comme $v(x,t)$ est un soliton elliptique de degré $n = g(g+1)/2$, il s'ensuit que $v(x,0) = g(g+1)\mathfrak{P}(x) + cst$. On peut choisir la constante α de façon que $v(x,0) = g(g+1)\mathfrak{P}(x)$. C.Q.F.D.

6.6. Etudier les revêtements tangentiels hyperelliptiques non exceptionnels revient à étudier les courbes rationnelles des systèmes linéaires $|\lambda(n,\mu)|$ (4.5) lorsque $d = (2n+1-\mu^{(2)})/4 = \dim|\lambda(n,\mu)|$ est strictement positif (5.3). Nous n'avons pas de résultats généraux sur ces diviseurs et en particulier nous n'avons pas d'estimation sur le nombre des classes d'isomorphismes de revêtements tangentiels hyperelliptiques de degré n (resp. de revêtements tangentiels primitifs de degré n, resp. de revêtements tangentiels de degré n et de genre $\gamma(n)$) de X. Cependant pour $d = 1$ ou 2 nous avons des résultats partiels sur le nombre $R(n,\mu,X)$ de telles courbes qui peuvent appuyer quelques conjectures. Ces résultats sont obtenus en étudiant le morphisme rationnel de $S^{\sim}$ dans $|\lambda(n,\mu)|^{\vee}$ définit par $\lambda(n,\mu)$. Nous les énonçons sans démonstration dans l'alinéa suivant.

6.7. Supposons tout d'abord que $d = 1$. Ils se présentent alors 4 cas :

(A) $n = 2(k^2+k+1)$ et $\mu = (2k+1,0,0,0)$, pour $k \in \mathbf{N}$. Alors on a $R(n,\mu,X) = 0$.

(B) Il existe deux indices $i > 0$ tels que $\mu_i = 0$, ce qui implique n pair. On a alors $0 \leq R(n,\mu,X) \leq 2$ (En fait pour $n = 4$ et $\mu = (1,2,0,0)$ on a $R(n,\mu,X) = 2$).

(C) Il existe un seul indice i, tel que $\mu_i = 0$ (par exemple $n = 3$, $\mu = (0,1,1,1)$, ou $n = 6$ et $\mu = (1,0,2,2)$). On a alors $2 \leq R(n,\mu,X) \leq 4$ (En fait pour $n = 3$ et $\mu = (0,1,1,1)$, on a $R(n,\mu,X) = 4$).

(D) Pour tout i, on a $\mu_i > 0$ (par exemple $n = 5$, $\mu = (2,1,1,1)$ ou bien

$n = 8$, $\mu = (1, 2, 2, 2)$). On a alors $2 \leq R(n, \mu, X) \leq 6$ (On a en fait $R(n, \mu, X) = 6$ dans les deux exemples cités, pour X générique).

On déduit de cette analyse, qu'il existe des entiers n (non triangulaires cf. (6.4)) et des revêtements tangentiels hyperelliptiques de degré n, et de genre $\gamma(n)$ (donc primitif et minimal (4.13)) qui ne sont pas exceptionnels. Par exemple, il existe 6 revêtements tangentiels hyperelliptiques de degré 5 et de genre $2 = \gamma(5)$ qui sont de type $(2, 1, 1, 1)$ et par suite non exceptionnels. Il existe aussi trois revêtements tangentiels hyperelliptiques de degré 5, de genre 2 et de type $(0, 3, 1, 1)$ (à permutation près des μ_i), donc exceptionnels.

Supposons maintenant que $d = 2$ et que $\mu_i \geq 2$ pour tout i. On a alors $9 \leq R(n, \mu, X) \leq 27$ et nous conjecturons que, pour X générique, ce nombre est égal à 27.

7. Appendice

Estimations des fonctions ψ et χ (par J. Oesterlé).

Le but de cet appendice est de donner des formules explicites permettant de calculer les expressions $\chi(n)$ et $\psi(n)$ introduites en (6.3), et dont les définitions sont rappelées dans les propositions 1 et 2 ci-après.

Notons $r_k(m)$ le nombre de k-uplets $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ tels que $\sum_{i=1}^{k} x_i^2 = m$. Les nombres $\chi(n)$ et $\psi(n)$ s'expriment simplement à l'aide des nombres $r_k(m)$, avec $k \leq 4$. En effet :

Proposition 1. *Pour tout entier $n \geq 0$, le nombre $\chi(n)$ de quadruplets $(\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbb{N}^4$ tels que*

$$\begin{cases} 2n + 1 = \sum_{i=0}^{3} \mu_i^2 \\ \mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 \equiv n \qquad (\text{mod.} 2) \end{cases}$$

est donné par la formule

$$\chi(n) = \frac{1}{64}(r_4(m) + 4r_3(m) + 6r_2(m) + 4r_1(m)), \qquad \text{où} \;\; m = 2n + 1.$$

Soient $(\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbb{N}^4$ et $n \in \mathbb{N}$ tels que $2n + 1 = \sum_{i=0}^{3} \mu_i^2$. En réduisant cette égalité modulo 4, on constate que trois exactement des μ_i ont même parité que n. Il en résulte aussitôt que $4\chi(n)$ est le nombre de quadruplets $(\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbb{N}^4$ tels que $2n+1 = \sum_{i=0}^{3} \mu_i^2$ (sans restriction de parité sur μ_i) . On a alors, en posant $m = 2n+1$ et en notant $s_k(m)$ le nombre de k-uplets $(x_1, \ldots, x_k) \in (\mathbf{N} - \{0\})^k$ tels que $m = \sum_{i=1}^k x_i^2$

(1) $$4\chi(n) = s_4(m) + 4s_3(m) + 6s_2(m) + 4s_1(m).$$

Par ailleurs, il est clair que l'on a

(2) $$\begin{cases} r_4(m) = 16s_4(m) + 32s_3(m) + 24s_2(m) + 8s_1(m) \\ r_3(m) = 8s_3(m) + 12s_2(m) + 6s_1(m) \\ r_2(m) = 4s_2(m) + 4s_1(m) \\ r_1(m) = 2s_1(m). \end{cases}$$

La proposition 1 résulte de (1) et (2).

Proposition 2. *Soit $n \in \mathbf{N}$. Comme en* (1.5) *et* (6.3), *notons $\gamma(n)$ le plus grand entier $p \geq 0$ tel que $p(p+1)/2 \leq n$. Le nombre $\psi(n)$ de quadruplets $(\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbf{N}^4$ tels que*

$$\begin{cases} 2n+1 = \sum_{i=0}^{3} \mu_i^2 \\ \mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 \equiv n \qquad (\text{mod.} 2) \\ \sum_{i=0}^{3} \mu_i = 2\gamma(n) + 1 \end{cases}$$

est donné par la formule $\psi(n) = \frac{1}{8} r_3(8n + 4 - a^2)$, où a est le plus grand entier impair tel que $a^2 \leq 8n+1$.

Il est clair que l'entier a est égal à $2\gamma(n) + 1$. Comme dans la démonstration de la proposition 1, on voit que $4\psi(n)$ est le cardinal de l'ensemble $A(n)$ des quadruplets $(\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbf{N}^4$ vérifiant

(3) $$2n + 1 = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2$$

(4) $$\sum_{i=0}^{3} \mu_i = a$$

(et pas de relation de parité sur les μ_i).

Introduisons l'ensemble $B(n)$ des quadruplets $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$ vérifiant

$$(5) \qquad 4(2n+1) = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

$$(6) \qquad a_0 = a$$

$$(7) \qquad a_0 + a_1 + a_2 + a_3 \equiv 0 \qquad (\text{mod.}4)$$

Si $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_2, \mu_3)$ appartient à $A(n)$, montrons que le quadruplet $\lambda(\boldsymbol{\mu}) = (a_0, a_1, a_2, a_3)$ défini par

$$(8) \qquad \begin{cases} a_0 = \mu_0 + \mu_1 + \mu_2 + \mu_3 \\ a_1 = \mu_0 + \mu_1 - \mu_2 - \mu_3 \\ a_2 = \mu_0 - \mu_1 + \mu_2 - \mu_3 \\ a_3 = \mu_0 - \mu_1 - \mu_2 + \mu_3 \end{cases}$$

appartient à $B(n)$: la relation (5) résulte en effet de (3), la relation (6) de (4) , et la relation (7) de l'égalité $a_0 + a_1 + a_2 + a_3 = 4\mu_0$.

Nous avons ainsi défini une application λ de $A(n)$ dans $B(n)$. Cette application est clairement injective, car le déterminant du système d'équations linéaires (8) est non nul. Montrons que λ est bijective : il s'agit de prouver que si (a_0, a_1, a_2, a_3) appartient à $B(n)$, le quadruplet $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_2, \mu_3)$ défini par

$$(9) \qquad \begin{cases} \mu_0 = (a_0 + a_1 + a_2 + a_3)/4 \\ \mu_1 = (a_0 + a_1 - a_2 - a_3)/4 \\ \mu_2 = (a_0 - a_1 + a_2 - a_3)/4 \\ \mu_3 = (a_0 - a_1 - a_2 + a_3)/4 \end{cases}$$

appartient à $A(n)$. Parce que a est impair, les relations (5) et (6) impliquent que les entiers a_i sont impairs ; le fait que les μ_i soient des entiers relatifs résulte alors de (7); la relation (3) résulte de (5) et la relation (4) de (6). Il nous reste à vérifier que les entiers μ_i sont positifs. Par définition de a, on a $(a+2)^2 \geq 8n+9$, d'où , en vertu de (5) et (6) ,

$$a_1^2 + a_2^2 + a_3^2 = 8n + 4 - a^2 \leq 4a - 1.$$

On en déduit, par l'inégalité de Cauchy-Schwarz,

$$|a_1| + |a_2| + |a_3| \leq \sqrt{3(a_1^2 + a_2^2 + s_3^2)} \leq \sqrt{12a - 3} \leq a + 3 = a_0 + 3$$

d'où pour $0 \leq i \leq 4$

$$\mu_i \geq \frac{a_0 - (|a_1| + |a_2| + |a_3|)}{4} \geq -\frac{3}{4}$$

et donc $\mu_i \geq 0$ puisque μ_i est entier.

En conclusion, nous avons

$$(10). \qquad 4\psi(n) = \mathrm{Card}(A(n)) = \mathrm{Card}(B(n))$$

Soit $C(n)$ l'ensemble des quadruplets $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$ vérifiant (5) et (6) (mais pas nécessairement (7)). On a

$$(11) \qquad \mathrm{Card}(C(n)) = 2\mathrm{Card}(B(n))$$

car $C(n)$ est réunion disjointe de $B(n)$ et de l'image de $B(n)$ par l'application $(a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_2, -a_3)$.

Soit $D(n)$ l'ensemble des triplets $(b_1, b_2, b_3) \in \mathbb{Z}^3$ tels que $b_1^2 + b_2^2 + b_3^2 = 8n + 4 - a^2$. L'application $(a_0, a_1, a_2, a_3) \mapsto (a_1, a_2, a_3)$ est une bijection de $C(n)$ sur $D(n)$. De (10) et (11) on déduit donc

$$8\psi(n) = 2\mathrm{Card}\ (B(n)) = \mathrm{Card}\ (C(n)) = \mathrm{Card}\ (D(n)) = r_3(8n + 4 - a^2),$$

ce qui démontre la proposition 2.

Formules explicites pour $r_k(m)$. On peut obtenir des expresssions explicites de $r_k(m)$ en se servant du fait que

$$\sum_{m=0}^{\infty} r_k(m) q^m = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^k$$

est le développement de Taylor à l'infini d'une forme modulaire de poids $k/2$ pour le groupe de congruence $\Gamma_0(4)$ et que l'on sait trouver une base de l'espace de ces formes modulaires. Les calculs sont exposés en détail dans la thèse d'Henri Cohen (chapitre I, n° 1.7). Ils permettent de retrouver les résultats classiques suivants, où nous supposons $m \geq 1$:

a) $k = 1$

$$r_1(m) = \begin{cases} 0 & \text{si } m \text{ n'est pas un carré} \\ 27 & \text{si } m \text{ est un carré} \end{cases}$$

b) $k = 2$

$$r_2(m) = 4 \sum_{d \mid m} \left(\frac{-4}{d} \right)$$

où $\left(\frac{-4}{d}\right)$ vaut 0 si d est pair, 1 si $d \equiv 1(\text{mod.}4)$ et -1 si $d \equiv 3(\text{mod.}4)$. Pour tout nombre premier p, notons $v_p(m)$ l'exposant de p dans m. On peut alors écrire

$$r_2(m) = 4 \prod_{\substack{p \mid m \\ p \text{ premier}}} a_p(m)$$

avec $a_2(m) = 1$, $a_p(m) = 1 + v_p(m)$ si $p \equiv 1(\text{mod.}4)$, $a_p(m) = 0$ si $p \equiv 3(\text{mod.}4)$ et $v_p(m)$ est impair, $a_p(m) = 1$ si $p \equiv 3(\text{mod.}4)$ et $v_p(m)$ est pair. On a $r_2(m) = o(m^\varepsilon)$ pour tout $\varepsilon > 0$.

c) $k = 3$ Supposons que l'entier m soit sans facteurs carrés, et notons h le nombre de classes de l'anneau des entiers de $\mathbb{Q}(\sqrt{-m})$. On a

$$r_3(m) \begin{cases} 6 \text{ si } m = 1 \\ 12h \text{ si } m \equiv 1,2,5,6 \ (\text{mod.}8), \quad m \neq 1 \\ 8 \text{ si } m = 3 \\ 24h \text{ si } m \equiv 3(\text{mod.}8), m \neq 3 \\ 0 \text{ si } m \equiv 7(\text{mod.}8). \end{cases}$$

Supposons maintenant que m s'écrive $m = m'f^2$, avec m' sans facteurs carrés. On a alors :

$$r_3(m) = r_3(m') \prod_{\substack{p \mid f \\ p \text{ premier} \\ p \neq 2}} \left[\frac{p^{v_p(f)+1} - 1}{p-1} - \left(\frac{-m'}{p}\right) \frac{p^{v_p(f)} - 1}{p-1} \right]$$

Compte tenu de majorations élémentaires des nombres de classes, on en déduit que l'on a

$$r_3(m) = o\left(\sqrt{m} \log^2 m\right).$$

d) $k = 4$ Si m est impair, on a

$$r_4(m) = 8 \sum_{d \mid m} d.$$

Si m est pair, $m = 2^a m'$ avec m' impair, on a

$$r_4(m) = 3 r_4(m').$$

Applications à $\chi(n)$ et $\psi(n)$

(a) Lorsque n tend vers l'infini, $\chi(n)$ est équivalent à $\frac{1}{8}\sum_{d\,|\,m} d$, où $m = 2n+1$. Plus précisément, on a

$$\chi(n) - \frac{1}{8}\sum_{d\,|\,m} d = o(\sqrt{m}\log^2 m)$$

d'après la proposition 1 et les résultats ci-dessus. On en déduit que l'on a $\chi(n) = O(n \log\log n)$. Par ailleurs la proposition 1 et l'inégalité $r_4(m) \geq 8m$ impliquent que $\chi(n)$ est minoré par $n/4$.

(b) On a $\psi(n) > 0$ pour tout n : en effet d'après la proposition 2, $\psi(n)$ est égal à $r_3(8n+4-a^2)/8$, où a est le plus grand entier impair tel que $a^2 \leq 8n+4$. On a $8n+4-a^2 \equiv 3(\text{mod}.8)$, donc $r_3(8n+4-a^2) \neq 0$.

(c) Le comportement asymptotique de $\psi(n)$ est chaotique. Par exemple ψ prend une infinité de fois la valeur 1 : on a $\psi(n) = 1$ si et seulement si n est de la forme $\frac{u(u+1)}{2}$; d'autre part, $\psi(n)$ prend des valeurs arbitrairement grandes. On déduit de la proposition 2 et des résultats concernant r_3 que l'on a $\psi(n) = o\left(n^{1/4}\log^2 n\right)$.

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Index des notations: X (1.1), $\gamma(n)$ (1.5), S (1.7), Γ, p, q, π (2.2), τ (2.4), g (2.5), π, Δ, S, C_0, S_q (3.1), $V(n,X)$ (3.11), τ, $\mathrm{SV}(n,X)$ (3.12), i (3.8), $\omega_i^\perp$, r_i, s_i, μ_i, ε_i, μ (3.13), ι (3.14), C_1 (3.17), $S^\perp$, $S^\sim$, ϕ, e, $C_0^\perp$, $C_1^\perp$, $S_i^\perp$, $s_i^\perp$, $r_i^\perp$, $C_0^\sim$, $C_1^\sim$, $S_i^\sim$, $s_i^\sim$, $r_i^\sim$, S_i, K, $K^\perp$, $K^\sim$, $\mu^{(1)}$, $\mu^{(2)}$ (4.1), $\lambda(n,\mu)$ (4.2), $\iota^\perp$ (4.3), $\mathrm{RTH}(n,\mu)$, ρ (4.4), l, $l^\perp$, $l^\sim$ (5.1), $R(n,\mu,X)$ (6.6).

A. Treibich
IRMAR
Université de Rennes I
Campus de Beaulieu
35042 Rennes, France

J.-L. Verdier

J. Oesterlé
U.F.R. de Mathématiques
Université de Paris VI
Paris, France

Linear Simple Lie Algebras and Ranks of Operators

YU. G. ZARHIN

Abstract: We discuss spectra of operators in irreducible finite-dimensional representations of simple Lie algebras. We give lower bounds for the rank of non-zero operators in the representations.

1. Definitions, statements and reductions

Let V be a non-zero finite-dimensional vector space over an algebraically closed field C of characteristic zero. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be an irreducible linear Lie algebra; irreducibility means that the natural representation of $\mathfrak{g}$ on V is irreducible. Let $f : V \to V$ be a non-zero operator, lying in $\mathfrak{g}$. Let $k = \dim(fV)$ be the rank of f.

Let us assume that $k = 1$. If f is semisimple then a theorem of Kostant [4] asserts that $\mathfrak{g} = \mathrm{End}(V)$. If one drops the semisimplicity assumption, then one of the following conditions holds ([2], Ch. 8, sect. 13, ex. 15; Guillemin, Quillen, Sternberg [3]): (1) $\mathfrak{g} = \mathrm{End}(V)$; (2) $\mathfrak{g} = sl(V)$ is the Lie algebra of all operators on V with zero trace; (3) there exists a non-degenerate skew-symmetric bilinear form on V such that either $\mathfrak{g} = sp(V)$ is the Lie algebra of the corresponding symplectic group of V or $\mathfrak{g} = sp(V) \oplus C\,id$, where $id : V \to V$ is the identity map.

Let us assume that $k = 2$ and $\mathfrak{g}$ is semisimple. If f is semisimple then either $\mathfrak{g} = sl(V)$ or $\mathfrak{g} = so(V)$ is the Lie algebra of "the" orthogonal group of V or $\mathfrak{g} = sp(V)$ is the Lie algebra of "the" symplectic group of V. For V given with a non-degenerate $\mathfrak{g}$-bilinear form this result is due to Kostant[4]; the general case is an easy corollary of results of ([5], sect. 11; [6], Appendix).

The following two theorems are the main results of this paper.

Theorem 1. *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible simple Lie algebra of rank r, $f : V \to V$ a non-zero semisimple operator of rank k, lying in $\mathfrak{g}$. Then:*

(1) $k \geq (2/3)\dim(V)/r$.

(2) *if V is a symplectic representation of $\mathfrak{g}$, i.e., if there exists a non-degenerate skew-symmetric $\mathfrak{g}$-invariant bilinear form on V, then $k \geq \dim(V)/r$.*

(3) *let us choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, containing f and let V_0 be the weight zero subspace of V with respect to the action of $\mathfrak{h}$, i.e.,*

$$V_0 = \{v \in V \mid xv = 0 \quad \text{for all} \quad x \in \mathfrak{h} \subset \mathfrak{g}\}.$$

Then

$$\dim(V) - \dim(V_0) \geq k \geq (\dim(V) - \dim(V_0))/r.$$

Examples. If $r = 1$, i.e., $\mathfrak{g}$ is isomorphic to $sl(2, C) \cong so(3, C)$ then $k = \dim(V) - \dim(V_0)$ for any semisimple f in $\mathfrak{g}$. This means that $k = \dim(V)$ if $\dim(V)$ is even, i.e., if V is symplectic, and $k = \dim(V) - 1$ if $\dim(V)$ is odd. If $\dim(V) = 3$ then $\mathfrak{g} = so(V)$ and $k = 2 = (2/3)\dim(V)/r$.

If either $\mathfrak{g} = so(V)$ or $\mathfrak{g} = sp(V)$ then $(\dim(V) - \dim(V_0))/r = 2$. If $\mathfrak{g} = sl(V)$ then $V_0 = \{0\}$ and $(\dim(V) - \dim(V_0))/r = (r+1)/r \leq 2$. Clearly in all these cases there exists a semisimple f in $\mathfrak{g}$ of rank 2.

Theorem 2. *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be linear irreducible simple Lie algebra of rank r, $f : V \to V$, a non-zero semisimple operator, lying in $\mathfrak{g}$. Then for each non-zero scalar $a \in C$*

$$\dim((f - a\,id)V) \geq \dim(V)/(r+1).$$

Remarks on the proof of Theorem 1. (a) The inequality $\dim(V) - \dim(V_0) \geq k$ is an obvious corollary of the inclusions $f \in \mathfrak{h}$, $V_0 \subset \mathrm{Ker}(f)$. (b) In order to prove Theorem 1, it suffices to check its third assertion. Indeed, the second assertion is an immediate corollary of the third one because $V_0 = \{0\}$ for symplectic representations ([2], Ch.8, sect.7, ex.3). The first assertion is an immediate corollary of the third one and the following easy Lemma, whose proof will be given at the end of this section.

Lemma 1. $\dim(V_0) \leq \dim(V)/3$.

Proof of Theorems 1 and 2. Let us choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, containing f. Let

$$\mathfrak{h}^* = \mathrm{Hom}_C(\mathfrak{h}, C)$$

be the C-vector space, dual to $\mathfrak{h}$. The dimensions of $\mathfrak{h}$ and $\mathfrak{h}^*$ coincide and equal to the rank r of $\mathfrak{g}$. The element f of $\mathfrak{h}$ defines a non-zero linear functional

$$q : \mathfrak{h}^* \to C, \ \ \mu \mid \to \mu(f).$$

Let $\mathrm{Supp}(V) \subset \mathfrak{h}^*$ be the collection of weights of $\mathfrak{h}$ in V, i.e., the collection of all $\mu \in \mathfrak{h}^*$, such that the weight subspace

$$V_\mu : = \{v \subset V \mid xv = \mu(x)v \quad \text{for all} \quad x \in \mathfrak{h}\} \subset V$$

is not the zero subspace $\{0\}$. Its dimension $\dim(V_\mu)$ is called multiplicity of μ and denoted by $\mathrm{mult}(\mu)$. It is well known ([2]) that

$$V = \oplus V_\mu, \ \ (\mu \in \mathrm{Supp}(V)).$$

The image $q(\mathrm{Supp}(V)) \subset C$ coincides with the set $\mathrm{spec}(f)$ of eigenvalues of f on V. In addition, the multiplicity of each eigenvalue c of f is equal to

$$\sum \mathrm{mult}(\mu), \ \ (\mu \in \mathrm{Supp}(V), \ \ q(\mu) = c\}.$$

This implies that the rank of $f - a\,id$, i.e., $\dim((f - a\,id)V)$ is equal to

$$\sum \mathrm{mult}(\mu), \ \ (\mu \in \mathrm{Supp}(V), \ \ q(\mu) \neq a\} \quad \text{for} \quad a \in C.$$

In particular,

$$k = \dim(fV) = \sum \mathrm{mult}(\mu), \ \ (\mu \in \mathrm{Supp}(V), \ \ q(\mu) \neq 0\}.$$

Clearly,

$$k = \dim(fV) = \sum \mathrm{mult}(\mu), \ \ (\mu \in \mathrm{Supp}(V)\backslash\{0\}, \ \ q(\mu) \neq 0\}$$

and for each non-zero $a \in C$

$$\dim((f - a\,id)V) = \dim(V_0) + \sum \mathrm{mult}(\mu), \ \ (\mu \in \mathrm{Supp}(V)\backslash\{0\}, \ \ q(\mu) \neq a\}.$$

Let $R \subset \mathfrak{h}^*$ be the root system attached to $(\mathfrak{g}, \mathfrak{h})$, i.e., the set of all non-zero $\alpha \in \mathfrak{h}^*$, such that there exists a non-zero element $X_\alpha \in \mathfrak{g}$ with

$$\mathrm{ad}(x)X_\alpha = [x, X_\alpha] = \alpha(x)X_\alpha \quad \text{for all} \quad x \in \mathfrak{h}.$$

Such an element X_α is defined uniquely up to multiplication by a non-zero scalar in C ([2]). One has the inclusions ([2])

$$X_\alpha V_\mu \subset V_{\mu+\alpha} \quad \text{for all} \quad \alpha \in R, \quad \mu \in \mathrm{Supp}(V).$$

Let $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ be the Weyl group of the root system R [1]. It is a finite group, acting linearly and ***irreducibly*** on $\mathfrak{h}^*$, because $\mathfrak{g}$ is simple [2]. The collection $\mathrm{Supp}(V)$ is W-invariant: it is the union of a finite number of W-orbits in $\mathfrak{h}^*$. In addition, if S is a W-orbit in $\mathfrak{h}^*$ then all weights in S have the same multiplicity in V.

In the following we will write $\mathrm{card}(A)$ for the number of elements of a finite set A.

Lemma 2. *Let S be a non-zero W-orbit in $\mathfrak{h}^*$. Then:*

(0) $\mathrm{card}(S) \geq r+1$;

(1) $\mathrm{card}\{x \in S \subset \mathfrak{h}^* \mid q(x) \neq 0\} \geq \mathrm{card}(S)/r$;

(2) *for each non-zero $a \in C$*

$$\mathrm{card}\{x \in S \subset \mathfrak{h}^* \mid q(x) \neq a\} \geq \mathrm{card}(S)/(r+1).$$

This Lemma will be proven in the next section.

End of the proof of Theorems 1 and 2 (modulo Lemmas 1 and 2). Notice that $\mathrm{Supp}(V)\backslash\{0\}$ is the union of finite number of W-orbits and

$$\mathrm{card}\{\mu \in \mathrm{Supp}(V) \text{ (with multiplicities)}\} = \dim(V),$$

$$\mathrm{card}\{\mu \in \mathrm{Supp}(V)\backslash\{0\} \text{ (with multiplicities)}\} = \dim(V) - \dim(V_0).$$

Let $S_1, \ldots, S_t$ be the all W-orbits in $\mathrm{Supp}(V)\backslash\{0\}$. Clearly, different W-orbits do not meet each other and therefore

$$\mathrm{Supp}(V)\backslash\{0\} = S_1 \cup \cdots \cup S_t$$

and

$$\begin{aligned}\dim(V) - \dim(V_0) &= \mathrm{card}\{\mu \in \mathrm{Supp}(V)\backslash\{0\} \text{ (with multiplicities)}\} \\ &= \sum \mathrm{mult}(S_i)\mathrm{card}(S_i), \quad (1 \leq i \leq t),\end{aligned}$$

where

$$\mathrm{mult}(S_i) := \mathrm{mult}(\mu) \quad \text{for each} \quad \mu \in S_i.$$

Applying Lemma 2 to the W-orbit S_i, we obtain the following inequalities.

(1) $\mathrm{card}\{x \in S_i \mid q(x) \neq 0\} \geq \mathrm{card}(S_i)/r$;

(2) for each non-zero $a \in C$

$$\mathrm{card}\{x \in S_i \mid q(x) \neq a\} \geq \mathrm{card}(S_i)/(r+1).$$

Multiplying by $\mathrm{mult}(S_i)$ and summing up over i we obtain:

(1)

$$\begin{aligned} k = \dim(fV) &= \sum_{\substack{\mu \in \mathrm{Supp}(V)\setminus\{0\} \\ q(\mu)\neq 0}} \mathrm{mult}(\mu) \\ &= \sum_i \mathrm{mult}(S_i)\,\mathrm{card}\{\mu \in S_i,\ q(\mu) \neq 0\} \geq \sum_i \mathrm{mult}(S_i)\,\mathrm{card}\{S_i)/r \\ &= \big(\sum_i \mathrm{mult}(S_i)\,\mathrm{card}\{S_i)\big)/r \\ &= (\dim(V) - \dim(V_0))/r; \end{aligned}$$

(2) for each non-zero $a \in C$

$$\begin{aligned} \dim((f - a\,id)V) = \dim(V_0) &+ \sum_{\substack{\mu \in \mathrm{Supp}(V)\setminus\{0\} \\ q(\mu)\neq a}} \mathrm{mult}(\mu) \\ &= \dim(V_0) + \sum_i \mathrm{mult}(S_i)\,\mathrm{card}\,\{\mu \in S_i,\ q(\mu) \neq a\} \\ &\geq \dim(V_0) + \sum_i \mathrm{mult}(S_i)\,\mathrm{card}\,\{S_i)/(r+1) \\ &= \dim(V_0) + \big(\sum_i \mathrm{mult}(S_i)\,\mathrm{card}\,\{S_i)\big)/(r+1) \\ &= \dim(V_0) + (\dim(V) - \dim(V_0))/(r+1) \\ &\geq \dim(V)/(r+1). \end{aligned}$$

QED

Proof of Lemma 1. Let us choose a basis B of the root system R and let $R_+ \subset R$ be the positive root system attached to B. We have

$$B \subset R_+ \subset R = (R_+) \cup (-R_+) \subset \mathfrak{h}^*.$$

The theory of highest weight modules [2] implies the existence of a non-zero highest weight $\lambda \in \mathrm{Supp}(V)$ with the following properties:

(a) the weight subspace V_λ is one-dimensional and

$$X_\alpha V_\lambda = \{0\} \quad \text{for all positive roots} \quad \alpha;$$

(b) V is spanned as a vector space by V_λ and all the subspaces

$$X_{-\alpha_1} \cdots X_{-\alpha_m} V_\lambda \subset V_{\lambda - \alpha_1 - \cdots - \alpha_m}$$

where $m \geq 1$ and $\alpha_1, \dots, \alpha_m$ is an arbitrary sequence of (not necessarily distinct) positive roots. In particular, for each $\mu \in \mathrm{h}^*$, $\mu \neq \lambda$,

$$V_\mu = \sum X_{-\alpha} V_{\mu+\alpha} \qquad (\alpha \in R_+).$$

This implies that

$$\dim(V_\mu) \leq \sum \dim(V_{\mu+\alpha}) \qquad (\alpha \in R_+).$$

Putting $\mu = 0$ we obtain

$$\dim(V_0) \leq \sum \dim(V_\alpha) \qquad (\alpha \in R_+).$$

Each positive root α is conjugate to the negative root $-\alpha$ with respect to the action of the Weyl group [1]; in particular,

$$\dim(V_\alpha) = \dim(V_{-\alpha}) \quad \text{for all} \quad \alpha \in R_+.$$

This implies that

$$\begin{aligned} \dim(V_0) \leq \sum_{\alpha \in R_+} \dim(V_\alpha) &= (1/2)(\sum_{\alpha \in R} \dim(V_\alpha)) \\ &\leq (\dim(V) - \dim(V_0))/2, \end{aligned}$$

i.e.,

$$\dim(V_0) \leq (\dim(V) - \dim(V_0))/2.$$

Hence, $\dim(V_0) \leq \dim(V)/3$. This proves Lemma 1.

Remark. Since each finite-dimensional representation of $\mathfrak{g}$ is isomorphic to the direct sum of irreducible representations, the third assertion of Theorem 1 immediately implies the following corollary.

Corollary to Theorem 1. *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear, not necessarily irreducible, simple Lie algebra of rank r, f a non-zero semisimple element of $\mathfrak{g}$. Then $\dim(fV) \geq (\dim(V) - \dim(V_0))/r$.*

2. Hyperplanes and orbits of finite linear groups

Our Lemma 2 is a special case of a general theorem about orbits of finite linear groups and hyperplanes. This section contains the statement and the proof of this theorem.

Let L be a vector space of finite dimension r over an arbitrary field K, $G \subset \mathrm{Aut}(L)$ a finite group of linear automorphisms of L. For each non-zero linear functional $q : L \to K$ and for each $a \in K$ we denote by

$$H = H(q, a) = \{x \in L \mid q(x) = a\}$$

the corresponding (affine) hyperplane; the hyperplane $H(q, a)$ is a *vector subspace* of L if and only if $a = 0$.

Theorem 3. *Let $S \subset L$ be a G-orbit which is not contained in H, i.e., $S \backslash H := \{x \in S \subset L \mid q(x) \neq a\}$ is non-empty. Then:*

(1) *if H is a vector subspace, i.e., if $a = 0$, then*

$$\operatorname{card}(S \backslash H) \geq \operatorname{card}(S)/r;$$

(2) *if H is not a vector subspace, i.e., if $a \neq 0$, then*

$$\operatorname{card}(S \backslash H) \geq \operatorname{card}(S)/(r+1).$$

Remark. Replacing, if necessary, L by the subspace of L spanned by S, we may and will assume that the linear hull of S coincides with L. In particular, S contains basis $e_1, \ldots, e_r$ of L. Notice that no basis of L is contained in any proper vector subspace of L.

We begin our proof of Theorem 3 with the following elementary remarks. For each $x, y \in S$ let us put

$$G(x, y) := \{g \in G \mid gy = x\}.$$

In particular, $G(x, x) = G(x)$ is the stabilizer of x. Clearly

$$\operatorname{card}(G(x, y)) = \operatorname{card}(G(x)) = \operatorname{card}(G(y)) = \operatorname{card}(G)/\operatorname{card}(S).$$

Lemma 3. *Let X, Y be two non-empty finite subsets of S. Let us assume that X meets gY for all $g \in G$ where $gY = \{gy \mid y \in Y\}$. Then* $\operatorname{card}(X)\operatorname{card}(Y) \geq \operatorname{card}(S)$, i.e.,

$$\operatorname{card}(X) \geq \operatorname{card}(S)/\operatorname{card}(Y).$$

Proof of Lemma 3. Let us assume that $\operatorname{card}(X)\operatorname{card}(Y) < \operatorname{card}(S)$. Then

$$\begin{aligned}\operatorname{card}(G) &= \operatorname{card}(S)(\operatorname{card}(G)/\operatorname{card}(S)) \\ &\quad > \operatorname{card}(X)\operatorname{card}(Y)(\operatorname{card}(G)/\operatorname{card}(S)) \\ &= \sum_{\substack{x \in X \\ y \in Y}} \operatorname{card}(G(x, y)) \geq \operatorname{card}\Big(\bigcup_{\substack{x \in X \\ y \in Y}} G(x, y)\Big).\end{aligned}$$

This implies that there exists $g \in G$ such that $g \notin G(x,y)$ for all $x \in X$, $y \in Y$. But this means that X does not meet gY (by the definition of the sets $G(x,y)$). Contradiction.

We will prove Theorem 3 by applying Lemma 3 to $X = S \backslash H$ and to certain Y, with $\operatorname{card}(Y) = r$ if H is a vector subspace and with $\operatorname{card}(Y) \leq r+1$ if H is not a vector subspace.

Proof of Theorem 3. (1) If H is a vector subspace, choose a basis $e_1, \ldots, e_r$ of L with all e_i lying in S, and put $Y = \{e_1, \ldots, e_r\} \subset S$. Clearly, $\operatorname{card}(Y) = r$ and for all $g \in G$ the set $gY = \{ge_1, \ldots, ge_r\}$ is a basis of L, lying in S. Since gY is a basis of L, it is not a subset of the proper vector subspace H. This means that the set $gY \backslash H$ is non-empty, i.e., $X = S \backslash H$ meets gY. So we may apply Lemma 3 to our X and Y and obtain the inequality

$$\operatorname{card}(S \backslash H) = \operatorname{card}(X) \geq \operatorname{card}(S)/\operatorname{card}(Y) = \operatorname{card}(S)/r.$$

(2) Assume that H is not a vector subspace, i.e., $a \neq 0$. If S is a subset of a hyperplane $H(q,b)$ parallel to H with $b \neq a$ then $S \backslash H = S$ and there is nothing to prove. So we may and will exclude this trivial case and assume that the restriction of q to S is non-constant, i.e., there exist $x, y \in S$ with $q(x) \neq q(y)$. Let L' be the vector subspace of L generated by all vectors $x - y$ where $x, y \in S$. Put $m := \dim(L')$. Clearly L' is G-invariant, $m \leq r$ and the restriction of q to L' is not identically zero. For each $e \in S$ the set $\{e - x \mid x \in S\}$ spans L' as a vector subspace, because $x - y = (e-y) - (e-x)$ for all $x, y \in S$. Hence there is a set $F := \{f_1, \ldots, f_m\}$ in S consisting of m elements, and such that $e - F := \{e - f_1, \ldots, e - f_m\}$ is a basis of L'. Let Y be the union of one-point set $\{e\}$ and the set F, i.e., $Y = \{e, f_1, \ldots, f_m\}$. Clearly, $\operatorname{card}(Y) = m + 1 \leq r + 1$. Thus for $g \in G$, $gY = \{ge, gf_1, \ldots, gf_m\}$, and the set

$$g(e-F) := \{g(e - f_i) \mid 1 \leq i \leq m\} :=$$
$$\{g(e-f_1) = ge - gf_1, \ldots, g(e-f_i) = ge - gf_i, \ldots, g(e-f_m) = ge - gf_m\}$$

is a basis of L', because L' is G-invariant. Since q does not vanish identically on L', there exists an element $ge - gf_i$ of this basis with $q(ge - gf_1) \neq 0$ for some i with $1 \leq i \leq m$. This means that $q(ge) \neq q(gf_i)$. Therefore, either $q(ge) \neq a$ or $q(gf_i) \neq a$, i.e., either ge or gf_i is an element of $S \backslash H = X$. However, since ge and gf_i are elements of gY, the set X meets gY. Applying Lemma 3 to $X = S \backslash H$ and Y we obtain the inequalities

$$\begin{aligned}\operatorname{card}(S \backslash H) &= \operatorname{card}(X) \geq \operatorname{card}(S)/\operatorname{card}(Y) \\ &= \operatorname{card}(S)/(m+1) \geq \operatorname{card}(S)/(r+1).\end{aligned} \qquad \text{QED}$$

Corollary of Theorem 3. *Let us assume that* $\mathrm{card}(G) > 1$ *and that the action of* G *on* L *is irreducible. Let* S *be a non-zero* G*-orbit in* L *and let* $H = H(q, a)$ *be an arbitrary hyperplane in* L*. Then:*

(1) *if* H *is a vector subspace then* $\mathrm{card}(S \backslash H) \geq \mathrm{card}(S)/r$,

(2) *if* H *is not a vector subspace then* $\mathrm{card}(S \backslash H) \geq \mathrm{card}(S)/(r+1)$.

(3) $\mathrm{card}(S) \geq r + 1$.

Proof. For parts (1) and (2), one has only to check that the set $S \backslash H$ is non-empty and to apply Theorem 3. But in order to prove that S is not contained in *any* hyperplane, it suffices to check that the restriction of q to S is non-constant. Once we know that S is not contained in any hyperplane in L, then $\mathrm{card}(S) \geq r + 1$ just because $\dim(L) = r$.

Since L is an irreducible representation of G, the G-orbit S spans L as a vector space. Hence, $\mathrm{card}(S) \geq \dim(L) = r$. If $r = 1$ then, obviously, $\mathrm{card}(S) = \mathrm{card}(G) > 1$ and if $r > 1$ then $\mathrm{card}(S) \geq r > 1$. Thus $\mathrm{card}(S) \geq 2$ in all cases, and therefore, as L is G-irreducible, it is spanned by all vectors $x - y$ where $x, y \in S$. But this implies that the restriction of q to S is non-constant: otherwise, we would have $q(x - y) = 0$ for all $x, y \in S$ and therefore that $q = 0$ on L. So, S is not a subset of any hyperplane in the r-dimensional vector space L. QED

Proof of Lemma 2. One has only to apply this Corollary to $K = C$, $L = \mathfrak{h}^*$ and $G = W$.

3. Non-semisimple operators and asymptotic results

Theorem 4. *Let* $\mathfrak{g} \subset \mathrm{End}(V)$ *be a linear irreducible simple Lie algebra of rank* r*,* $f : V \to V$ *a non-zero, not necessarily semisimple operator of rank* k*, lying in* $\mathfrak{g}$*. Then*

(1) $k := \dim(fV) \geq \dim(V)/3r$;

(2) *for each non-zero* $a \in C$

$$\dim((f - a\,id)V) \geq \dim(V)/(r+1).$$

Proof. There is a Jordan decomposition

$$f = f_S + f_n; \quad f_S, f_n \in \mathfrak{g}, \ [f_S, f_n] = 0$$

where f_S is semisimple and f_n is a nilpotent. Clearly,

$$\dim(f_S V) \leq \dim(fV), \ \dim((f_S - a\,id)V) \leq \dim((f - a\,id)V).$$

Hence, if $f_S \neq 0$ then one has only to apply Theorems 1 and 2 to the non-zero semisimple element $f_S \in \mathfrak{g}$.

Assume now that $f_S = 0$, i.e., $f = f_n$ is a non-zero nilpotent element of $\mathfrak{g}$. According to a theorem of Jacobson-Morozov ([2], Ch.8, sect.11) there exists a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $\mathfrak{g}'$ contains x and is isomorphic to $sl(2, C)$. Let us slightly change notation and put

$$x := f = f_n \in \mathfrak{g}' \subset \mathfrak{g}.$$

Since $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$, the a vector space V becomes a $\mathfrak{g}'$-module. Let us choose a non-zero semisimple element h of $\mathfrak{g}'$. The well-known representation theory of sl(2) ([2]) implies that

$$\dim(xV) \geq \dim(hV)/2.$$

Applying Theorem 1 to the non-zero semisimple $h \in \mathfrak{g}$ we have

$$\dim(xV) \geq \dim(hV)/2 \geq \dim(V)/3r.$$

Now, if a is any non-zero element of C then $f - a\,id$ is an automorphism of V, because f is nilpotent. Hence, $(f - a\,id)V = V$ and

$$\dim((f - a\,id)V) = \dim(V) > \dim(V)/(r+1). \qquad \text{QED}$$

Corollary of Theorem 4. *Notations and assumptions as in Theorem 4, if either*

$$24\dim(fV) < \dim(V),$$

or if for some non-zero $a \in C$ we have

$$9\dim((f - a\,id)V) < \dim(V),$$

then $\mathfrak{g}$ is a classical simple Lie algebra, i.e., $\mathfrak{g}$ is a Lie algebra of type A_r, B_r, C_r or D_r.

Proof. Indeed, Theorem 4 implies that $r > 8$. QED

The following statement will be proven at the end of this section.

Lemma 4. *There exists a positive real number D with the following property. For any algebraically closed field C of characteristic zero, any non-zero finite-dimensional C-vector space V, any linear irreducible simple Lie algebra $\mathfrak{g} \subset \mathrm{End}(V)$ such that $\mathfrak{g}$ is neither $sl(V)$ nor $so(V)$ nor $sp(V)$, putting $r := \mathrm{rank}(\mathfrak{g})$ we have $\dim(V) > Dr^2$.*

Theorem 5. *There exists a positive real number D_1, with the following property. Let C be an algebraically closed field of characteristic* *zero, V a finite-dimensional C-vector space. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible simple Lie algebra of rank r. Let f be a non-zero element of $\mathfrak{g}$ such that either*

$$D_1 r > \dim(fV)$$

or, for some $a \in C$,

$$D_1 r > \dim((f - a\,id)V).$$

Then either $\mathfrak{g} = sl(V)$ or $\mathfrak{g} = so(V)$ or $\mathfrak{g} = sp(V)$.

Proof of Theorem 5. Let D be as in Lemma 4. One has only to put $D_1 = D/3$ and then apply Lemma 4 and Theorem 4.

Theorem 6. *There exists a positive real number D_2 with the following property. Let C be an algebraically closed field of characteristic zero, V a finite-dimensional C-vector space. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible simple Lie algebra of rank r. Let f be a non-zero element of $\mathfrak{g}$ such that either*

$$D_2(\dim(fV))^2 < \dim(V)$$

or, for some $a \in C$,

$$D_2(\dim((f - a\,id)V))^2 < \dim(V).$$

Then either $\mathfrak{g} = sl(V)$ or $\mathfrak{g} = so(V)$ or $\mathfrak{g} = sp(V)$.

Proof of Theorem 6. Let D be as in Lemma 4. One has only to put $D_2 = 9/D$ and then apply Lemma 4 and Theorem 4.

Remark. In fact, this paper arose from a question of N. L. Gordeev (Leningrad), who conjectured that for each positive integer k there exists a positive integer N, depending only on k with the following property. Let C be an algebraically closed field of characteristic zero, V a finite-dimensional C-vector space. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible classical simple Lie algebra. Let f be a non-zero element of $\mathfrak{g}$ such that $k = \dim(fV)$. If $\dim(V) > N$ then either $\mathfrak{g} = sl(V)$ or $\mathfrak{g} = so(V)$ or $\mathfrak{g} = sp(V)$. Clearly Theorem 6 implies the conjecture of Gordeev.

Proof of Lemma 4. Recall that λ is the highest weight of the simple $\mathfrak{g}$-module V, and W is the Weyl group. Clearly,

$$\dim(V) \geq \mathrm{card}(W\lambda) \geq r + 1.$$

Let us put $D = 1/8$. If $r \leq 8$ then

$$\dim(V) \geq r+1 = 8D(r+1) > Dr^2.$$

So, we may assume that $r > 8$; in particular, $\mathfrak{g}$ is a classical simple Lie algebra.

First, let us assume that V is a fundamental $\mathfrak{g}$-module, i.e.,λ is a fundamental weight. Then Bourbaki's Tables [2] show us that if $\mathfrak{g}$ is neither $sl(V)$ nor $sp(V)$ nor $so(V)$ then $\dim(V) > (1/8)r^2 = Dr^2$. Notice that:

type **(A)** $\mathfrak{g} = sl(V)$ iff $\mathfrak{g}$ is of type A_r and either $\lambda = \overline{\omega}_1$ or $\lambda = \overline{\omega}_r$;

types **(B, D)** $\mathfrak{g} = so(V)$ iff $\lambda = \overline{\omega}_1$ and either $\mathfrak{g}$ is of type B_r or $\mathfrak{g}$ is of type D_r;

type **(C)** $\mathfrak{g} = sp(V)$ iff $\mathfrak{g}$ is of type C_r and $\lambda = \overline{\omega}_1$.

Here we use the notations from Bourbaki's Tables [1]; in particular, the basis $B = \{\alpha_1, \dots, \alpha_r\}$ is the set of simple roots and $\overline{\omega}_i$ is the fundamental weight attached to the simple root α_i $(1 \leq i \leq r)$.

In general, λ is a linear combination of fundamental weights with non-negative integral coefficients, i.e.,

$$\lambda = \sum c_i \overline{\omega}_i \quad (1 \leq i \leq r)$$

where all c_i are non-negative integers. The Weyl formula for the dimension of V ([2]) immediately implies that $\dim(V)$ is a strictly increasing function in all variables c_i. So, in order to prove that

$$\dim(V) > Dr^2 = (1/8)r^2, \quad (r > 8)$$

it suffices to check this inequality in the following cases:

(A) $\mathfrak{g}$ is of type A_r and either $\lambda = 2\overline{\omega}_1$ or $\lambda = 2\overline{\omega}_r$ or $\lambda = \overline{\omega}_1 + \overline{\omega}_r$.
(B) $\mathfrak{g}$ is of type B_r and $\lambda = 2\overline{\omega}_1$;
(C) $\mathfrak{g}$ is of type C_r and $\lambda = 2\overline{\omega}_1$;
(D) $\mathfrak{g}$ is of type D_r and $\lambda = 2\overline{\omega}_1$.

We leave this easy check to the reader as an exercise. QED

4. Ranks of operators and the Coxeter number

The goal of this section is to prove the following refinement of the first assertion of Theorem 1.

Theorem 7. *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible simple Lie algebra of rank r, $f : V \to V$ a non-zero semisimple operator of rank k, lying in $\mathfrak{g}$. Then*

$$k \geq (h/(h+1))\dim(V)/r = h\dim(V)/\dim(\mathfrak{g})$$

where h is the Coxeter number of the root system R attached to $\mathfrak{g}$.

Example. If V is the adjoint representation, then $\dim(V) = \dim(\mathfrak{g})$ and, therefore, always $k \geq h$.

Proof of Theorem 7. It suffices to apply the third assertion of Theorem 1 to the following refinement of Lemma 1.

Lemma 5. $\dim(V_0) \leq \dim(V)/(h+1)$ *where h is the Coxeter number of the root system R.*

Proof of Lemma 5. We begin with the following remark to the proof of Lemma 1. Recall that the elements of the basis B are called simple roots and $\operatorname{card}(B) = r$. Notice that V is still spanned as a vector space by V_λ and all the subspaces

$$X_{-\alpha_1}\dots X_{-\alpha_m}V_\lambda \subset V_{\lambda-\alpha_1-\dots-\alpha_m}$$

where $m \geq 1$ and $\alpha_1, \dots, \alpha_m$ is an arbitrary sequence of (not necessarily distinct) *simple* roots, because the Lie subalgebra of $\mathfrak{g}$ spanned by all $X_{-\alpha}\,(\alpha \in B)$ coincide with the vector subspace of $\mathfrak{g}$ spanned by all $X_{-\alpha}\,(\alpha \in R_+)$ ([2], Ch.8, sect.3, prop.9). As in the proof of Lemma 1 this implies that

$$V_\mu = \sum X_{-\alpha}V_{\mu+\alpha} \quad (\alpha \in B),$$

$$\dim(V_\mu) \leq \sum \dim(V_{\mu+\alpha}) \quad (\alpha \in B),$$

and

$$\dim(V_0) \leq \sum \dim(V_\alpha) \quad (\alpha \in B).$$

Recall that $R = WB$,

$$\operatorname{card}(R) = h \operatorname{card}(B) = hr$$

[1] and the inequality

$$\dim(V_0) \leq \dim(V) - \sum \dim(V_\alpha) \quad (\alpha \in R)$$

holds. So, in order to prove Lemma 5 it suffices to check that

$$\sum_{\alpha\in R} \dim(V_\alpha) = h \sum_{\alpha\in B} \dim(V_\alpha).$$

This equality is obvious if all roots in R are conjugate to each other under the action of the Weyl group W, because in this case $\dim(V_\alpha)$ does not depend on the choice of a root α. If this is not the case, then R is the disjoint union of two W-orbits, say R_1 and R_2. Let B_1 be the intersection of R_1 and B and B_2 be the intersection of R_2 and B. Then B is the disjoint union of B_1 and B_2, $R_1 = WB_1, R_2 = WB_2$ and

$$\mathrm{card}(R_1) = h\,\mathrm{card}(B_1), \quad \mathrm{card}(B_2) = h\,\mathrm{card}(B_2)$$

([1], Ch.6, sect.1, prop.33 and ex.20). This implies that

$$\sum_{\alpha\in R_1} \dim(V_\alpha) = h \sum_{\alpha\in B_1} \dim(V_\alpha),$$
$$\sum_{\alpha\in R_2} \dim(V_\alpha) = h \sum_{\alpha\in B_2} \dim(V_\alpha).$$

In order to obtain the desired equality one has only to notice that

$$\sum_{\alpha\in R} \dim(V_\alpha) = \sum_{\alpha\in R_1} \dim(V_\alpha) + \sum_{\alpha\in R_2} \dim(V_\alpha)$$
$$\sum_{\alpha\in B} \dim(V_\alpha) = \sum_{\alpha\in B_1} \dim(V_\alpha) + \sum_{\alpha\in B_2} \dim(V_\alpha).$$

Remark. Notice that always $h \geq r+1$ (see ([1], Ch.6, sect.1, remark to prop.25 and prop.31). Combining this inequality with Theorem 7 we obtain that always

$$k \geq (\,(r+1)/(r(r+2))\,)\dim(V) > \dim(V)/(r+1).$$

Combining the latter inequality with Theorem 2 we obtain the following assertion.

Theorem 8. *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a linear irreducible simple Lie algebra of rank r, $f : V \to V$ a non-zero semisimple operator, lying in $\mathfrak{g}$. Then for all $a \in C$*

$$\dim((f - a\,id)V) \geq \dim(V)/(r+1).$$

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Research Computing Center USSR
Academy of Sciences, Moscow