Local cohomology for non-commutative graded algebras

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LOCAL COHOMOLOGY FOR NON-COMMUTATIVE GRADED ALGEBRAS

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ABSTRACT. We generalize the theory of local cohomology and local duality to a large class of non-commutative $\mathbb{N}$-graded noetherian algebras; specifically, to any algebra, B, that can be obtained as graded quotient of some noetherian AS-Gorenstein algebra, A.

As an application, we generalize three "classical" commutative results. For any graded module M over B we have the Bass-numbers $\mu^i(M) = \dim_k \operatorname{Ext}^i_B(k, M)$, and we can then prove that for M finitely generated, we have

- $\operatorname{id}(M) = \sup\{i \mid \mu^i(M) \neq 0\}$;
- the Bass-theorem: if $\operatorname{id}(M) < \infty$, then $\operatorname{id}(M) = \operatorname{depth}(B)$;
- the "No Holes"-theorem: if $\operatorname{depth}(M) \leq i \leq \operatorname{id}(M)$, then $\mu^i(M) \neq 0$,

where $\operatorname{id}(M)$ is M's injective dimension as an object in the category of graded modules, while $\operatorname{depth}(M)$ is the smallest i such that $\operatorname{Ext}^i_B(k, M) \neq 0$.

As a further application, we also generalize a non-vanishing result for local cohomology. It states that if M is a finitely generated graded B-module, then

$$\sup\{i \mid H^i_{\mathfrak{n}}(M) \neq 0\} = \operatorname{GKdim}(M).$$

Here $H^i_{\mathfrak{n}}(M) = \varinjlim \operatorname{Ext}^i_B(B/B_{\geq n}, M)$ is the i'th local cohomology-module of M. To prove this result, we need the AS-Gorenstein algebra, A, of which B is a quotient, to satisfy the so-called Similar Submodule Condition, SSC, defined in [11] (for instance, A could be PI).

0. INTRODUCTION

It is a recent discovery that in many respects, in particular the homological ones, non-commutative N-graded noetherian algebras behave in ways very similar to commutative local noetherian rings. For instance, in the non-commutative case, the notions of Gorenstein-ness and Cohen-Macaulay-ness can be defined, and in good cases, graded Gorenstein-algebras are Cohen-Macaulay, see [11]. Another example is the fact that the Auslander-Buchsbaum-theorem can be generalized to the non-commutative graded situation, see [6].

It is therefore very natural also to try generalizing to the non-commutative graded case the theory of local cohomology and local duality, which is really one of the high-points of the application of homological methods to commutative local rings; see for instance [5, chp. V] or [4, sec. 2]. Such a generalization was carried out in [9].

The present paper presents another generalization, which follows the same overall lines as the commutative theory, and which is related to the generalization in [9], but which bypasses some of its problems. In particular, [9] only makes the machinery of local cohomology work for some rather special algebras, whereas the theory of the present paper deals with a large class of algebras, namely, all graded quotients of noetherian AS-Gorenstein algebras.

The larger generality of the theory presented has the advantage that a number of applications from commutative algebra, which form the real motivation for studying local cohomology in that case, can also be generalized to the large class of non-commutative algebras studied. The results in question here are several "classical" statements about Bass-numbers: the fact that a finitely generated module's injective dimension is determined by non-vanishing of its Bass-numbers, Bass' theorem, which determines the possible values for the injective dimension of a finitely generated module, and the "no-holes"-theorem for Bass-numbers. For the commutative cases of these theorems, see [4, p. 19], [7, thm. 18.9], and [2, thm. 1.1] respectively [8, thm. 2].

This paper should be seen as a direct continuation of [6]. Firstly, Bass' theorem forms the natural "dual" to the Auslander-Buchsbaum-theorem, and it was a problem in connection with [6] that it could only prove Auslander-Buchsbaum, and not Bass (or rather, that it could only prove Bass in a special case). It is therefore very satisfying that the theory in the present paper gives Bass' theorem for a large class of algebras. Secondly, the reader familiar with the commutative theory of local cohomology will know that it can only be developed in a setting of derived categories. This also turns out to apply to the non-commutative graded case, and so we shall continue the hyperhomological line laid out in [6]. So the theory described in the following pages is thoroughly hyperhomological, always dealing with complexes of modules rather than single modules. The expected results for modules, as described in the abstract, can be obtained by specializing to complexes concentrated in degree zero.

As mentioned, the overall lines of the theory to be presented are the same as in the commutative case; on the other hand, there are some complications

introduced by the lack of commutativity. Among other things, it is necessary to use both left- and right-modules, and to use dualizing complexes consisting of bi-modules. The article is therefore organized as follows: section 1 contains some background from the theory of complexes of modules. In particular, it introduces the local cohomology-functor

$$R\Gamma_{\mathfrak{m}} : D(\mathrm{GrMod}(A)) \longrightarrow D(\mathrm{GrMod}(A))$$

defined on the derived category of the category of graded modules. Section 2 proves the local duality-theorem,

$$R\Gamma_{\mathfrak{m}}(X)' \cong R\underline{\mathrm{Hom}}_A(X, R\Gamma_{\mathfrak{m}}(A)')$$

for a right-bounded complex $X \in D(\mathrm{GrMod}(A))$ ($\underline{\mathrm{Hom}}$ is graded Hom of graded modules, see the definition in section 1). Section 3 uses this to prove an important fact: the local cohomology-functors induce an equivalence

$$(1) \qquad D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(A)) \xrightleftharpoons[{R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(-)'}]{{R\Gamma_{\mathfrak{m}}(-)'}} D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(A^{\mathrm{opp}}))^{\mathrm{opp}}.$$

of categories. As in the commutative theory, this is a central fact and key to many of the applications of local cohomology. Here A^{opp} is the opposite algebra to A; a left-module over A^{opp} is the same thing as a right-module over A. It is a crucial property of the theory developed here that the above duality connects a category of left-modules to one of right-modules, cf. the lack of commutativity. Section 4 uses the equivalence to prove the theorems for Bass-numbers alluded to above. Most importantly, it proves the Bass-theorem, which for a finitely generated graded module M states that

$$\mathrm{id}_A(M) < \infty \implies \mathrm{id}_A(M) = \mathrm{depth}_A(A).$$

Finally, section 5 gives an application of the theory which is unrelated to Bass-numbers. It proves, if we restrict to modules, that the largest i for which the local cohomology $H_{\mathfrak{m}}^i(M)$ is non-zero, is $i = \mathrm{GKdim}(M)$.

Note that each of these results requires the algebra A to satisfy some conditions. As mentioned above, many will only be proved for graded quotients of noetherian AS-Gorenstein algebras, but some require more, some less — see the detailed statements below.

1. HYPERHOMOLOGICAL BACKGROUND

This section provides a brief introduction to hyperhomology over non-commutative graded algebras. For more material, we refer to the literature: the hyperhomological theory of graded modules over an $\mathbb{N}$-graded k-algebra is presented in [9, secs. 1 and 2] and in [6, sec. 1].

We let k denote a fixed field. Let A be an $\mathbb{N}$-graded k-algebra. We have the category $\mathrm{GrMod}(A)$ of graded A-left-modules and homomorphisms of degree zero. Shifting of modules is denoted by ( ), and we define the graded Hom-

functor by

$$\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{GrMod}(A)}(M, N(i)).$$

The words "module", "homomorphism", "injective", "projective", and "flat" are to be understood in the graded sense, as are the notions of injective, projective and flat dimension, id, pd and fd (cf. [6, sect. 1]). Also, A^{opp}-left-modules are identified with A-right-modules, and modules over $A^e = A \otimes_k A^{\mathrm{opp}}$ are identified with A-bi-modules.

We write $\mathfrak{m} = A_{\geq 1}$. The algebra A is called connected if $A_0 = k$. In this case, we equip k as an A-bi-module concentrated in graded degree zero by $k = A/\mathfrak{m}$. The $\mathbb{N}$-graded connected algebra A is called AS-Gorenstein if $\mathrm{id}_A(A) = \mathrm{id}_{A^{\mathrm{opp}}}(A) = d < \infty$ and

$$\underline{\mathrm{Ext}}_A^i(k, A) = \underline{\mathrm{Ext}}_{A^{\mathrm{opp}}}^i(k, A) = \begin{cases} 0 & \text{for } i \neq d, \\ k(\ell) & \text{for } i = d. \end{cases}$$

The category $\mathrm{GrMod}(A)$ has a derived category, $D(\mathrm{GrMod}(A))$, and there are various full subcategories of left- resp. right- resp. bounded complexes, denoted by D^- resp. D^+ resp. D^b. There are also full subcategories consisting of complexes of finitely generated resp. locally finite modules, denoted by D_{fg} resp. D_{lf}. Super- and subscripts are combined freely. Since A^{opp}-modules are identified with A-right-modules, we think of $D(\mathrm{GrMod}(A^{\mathrm{opp}}))$ as the derived category of the category of graded A-right-modules and homomorphisms of degree zero. Twisting of complexes is denoted by [ ]. The i'th cohomology-functor defined on a derived category is denoted h^i (capital H's are reserved for the local cohomology-functors, see below), and if a chain-map of complexes $X \longrightarrow Y$ is a quasi-isomorphism, we will decorate it with a "$\simeq$". We will use the notation

$$\sup(X) = \sup\{i \mid h^i(X) \neq 0\}$$

and similarly for $\inf(X)$.

The graded functors $\underline{\mathrm{Hom}}$ and $\odot$ have derived functors, $R\underline{\mathrm{Hom}}$ and $^L\odot$, defined on derived categories. The usual boundedness-conditions have to be imposed, the usual recipes for computations (with injective and projective resolutions) can be used, and the usual isomorphisms between $R\underline{\mathrm{Hom}}$'s and $^L\odot$'s computed via resolutions in different variables can be obtained. An extra complication is the fact that if X is a complex of A-bi-modules, the functor $\underline{\mathrm{Hom}}_A(X, -)$ defined on complexes of A-left-modules takes values in complexes of A-left-modules; its derived functor therefore takes values in $D(\mathrm{GrMod}(A))$. To compute $R\underline{\mathrm{Hom}}_A(X, Y)$ in this case, we may use either a bi-module-resolution of X into a complex of modules that are projective as left-modules or a left-module-resolution of Y into a complex of injective modules. Similar remarks apply to the second variable in $\underline{\mathrm{Hom}}$ and to both variables in $\odot$, cf. [9, thms. 2.2 and 2.5].

Similar remarks also apply to the functor $\Gamma_{\mathfrak{m}}$, defined by

$$\Gamma_{\mathfrak{m}}(M) = \{m \in M \mid \text{ For } n \gg 0 \text{ we have } A_{\geq n}m = 0\}$$

and restriction of homomorphisms (the subscript "$_{\mathfrak{m}}$" refers to the ideal $\mathfrak{m} = A_{\geq 1}$ of A; in situations where this ideal has a different name, say $\mathfrak{n}$, the notation for the functor will be changed correspondingly). The elements in $\Gamma_{\mathfrak{m}}(M)$ will be called the torsion-elements of M, and we shall be working with the derived functor $R\Gamma_{\mathfrak{m}}$. We use the notation $H_{\mathfrak{m}}^i(X) = R^i\Gamma_{\mathfrak{m}}(X)$: this is called the i'th local cohomology of X. Note that

$$H_{\mathfrak{m}}^i(X) = \varinjlim \operatorname{Ext}_A^i(A/A_{\geq n}, X).$$

The cohomological dimension of $\Gamma_{\mathfrak{m}}$ is called the local cohomological dimension of A: it is denoted $\operatorname{lcd}(A)$. The functor $\Gamma_{\mathfrak{m}}$ is usually considered to be a functor from $\operatorname{GrMod}(A)$ to itself. However, $\Gamma_{\mathfrak{m}}$ can also be considered as defined on bi-modules, since the left-submodule of torsion-elements in a bi-module, K, is in fact also a right-submodule. If we want to construct the right-derived of $\Gamma_{\mathfrak{m}} : \operatorname{GrMod}(A^e) \longrightarrow \operatorname{GrMod}(A^e)$, we use a resolution consisting of $\Gamma_{\mathfrak{m}}$-acyclic objects of $\operatorname{GrMod}(A^e)$. However, since the forgetful functor $\operatorname{GrMod}(A^e) \longrightarrow \operatorname{GrMod}(A)$ preserves injectivity, a bi-module is $\Gamma_{\mathfrak{m}}$-acyclic precisely when it is $\Gamma_{\mathfrak{m}}$-acyclic as a left-module. Therefore, if Res_A is the canonical functor $D(\operatorname{GrMod}(A^e)) \longrightarrow D(\operatorname{GrMod}(A))$, we have

$$\operatorname{Res}_A \circ R\Gamma_{\mathfrak{m}}(X) \cong R\Gamma_{\mathfrak{m}} \circ \operatorname{Res}_A(X)$$

for any $X \in D(\operatorname{GrMod}(A^e))$ where the two sides are defined. Note that as a consequence, if $\operatorname{lcd}(A) < \infty$, the functor $\Gamma_{\mathfrak{m}}$ on bi-modules also has finite cohomological dimension.

If $\alpha : A \longrightarrow A$ is an automorphism of A as a graded k-algebra and M is an A-left-module, we define a new A-left-module by

$$a \cdot m = \alpha(a)m.$$

The new module is denoted $_\alpha M$. Note that as bi-modules, $_\alpha A \cong A_{\alpha^{-1}}$. Clearly $M \longmapsto {}_\alpha M$ determines an auto-equivalence of $\operatorname{GrMod}(A)$, and since, moreover,

$$\operatorname{Hom}_A({}_\alpha M, N) \cong \operatorname{Hom}_A(M, {}_{\alpha^{-1}}N) \quad \text{and} \quad M_\alpha \otimes_A N \cong M \otimes_A {}_{\alpha^{-1}}N.$$

the equivalence preserves injectivity, projectivity and flatness. The equivalence lifts to an auto-equivalence of $D(\operatorname{GrMod}(A))$, for which we use the same notation as for the equivalence on modules.

Let α be an automorphism of A as a graded k-algebra. If X is a complex of bi-modules and Y a complex of left-modules, we can easily see that

$$\operatorname{Hom}_A(X_\alpha, Y) \cong {}_\alpha\operatorname{Hom}_A(X, Y).$$

Consequently,

$$R\operatorname{Hom}_A(X_\alpha, Y) \cong {}_\alpha R\operatorname{Hom}_A(X, Y).$$

Similar remarks can be made if α is applied in the second argument and, indeed, for $\overset{L}{\otimes}_A$ and $R\Gamma_{\mathfrak{m}}$.

We can also, for any graded module M, define the graded module M' by

$$(M')_i = (M_{-i})'$$

where the prime denotes dualization with respect to k. The functor $M \longmapsto M'$ is exact and lifts to $D(\mathrm{GrMod}(A))$, and if M is locally finite, $M'' \cong M$. Therefore, $M \longmapsto M'$ is an auto-equivalence of the category of locally finite modules, lifting to an auto-equivalence of $D_{\mathrm{lf}}(\mathrm{GrMod}(A))$. Note that $(_{\alpha}X)' \cong (X')_{\alpha}$ and that as bi-modules $_{\alpha}(A') \cong (A')_{\alpha^{-1}}$.

If A is connected, we introduce the Bass-numbers

$$\mu^i(X) = \dim_k \underline{\mathrm{Ext}}_A^i(k, X)$$

for each $X \in D(\mathrm{GrMod}(A))$. We will use the depth of X,

$$\mathrm{depth}_A(X) = \inf R \underline{\mathrm{Hom}}_A(k, X) = \inf\{i \mid \mu^i(X) \neq 0\},$$

and the k-injective dimension of X,

$$\mathrm{k.id}_A(X) = \sup R \underline{\mathrm{Hom}}_A(k, X) = \sup\{i \mid \mu^i(X) \neq 0\}.$$

We say that A satisfies the condition χ_i° if $\mu^j(M) < \infty$ for every finitely generated module $M \in \mathrm{grmod}(A)$ and every $j \leq i$. If A satisfies the condition χ_i° for any i, we say that A satisfies χ°.

Note the concept of minimal free and injective complexes, cf. [6, defs. 1.1 and 1.2]. If the algebra A is connected, and if L resp. I is a minimal free resp. a minimal injective complex, the differentials in $\underline{\mathrm{Hom}}_A(L, k)$ resp. $\underline{\mathrm{Hom}}_A(k, I)$ are zero. According to [6, thm. 1.4], any right-complex has a minimal injective resolution and if A is connected and left-noetherian, any left-complex consisting of finitely generated modules has a minimal free resolution.

Consider the torsion-injectives, i.e. the injective modules consisting of torsion-elements. If A is connected and left-noetherian, by [1, p. 275] they all have the form

$$\coprod_i A'(\ell_i).$$

For this reason, when A is connected and left-noetherian, the condition χ_i° is equivalent to requiring that, for each finitely generated module $M \in \mathrm{grmod}(A)$, the number of summands isomorphic to some $A'(\ell)$ in the first i terms of M's minimal resolution is finite, cf. [1, prop. 7.7(2)]. Also, if A satisfies χ° and $X \in D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(A))$ has the minimal injective resolution $X \xrightarrow{\simeq} I$, then we can easily prove by induction on the number of modules in X that every module I^i contains only finitely many summands isomorphic to $A'(\ell)$.

Finally, let us quote two results from the literature:

Theorem 1.1. *Let A be a noetherian AS-Gorenstein k-algebra, write $d = \mathrm{id}_A(A) = \mathrm{id}_{A^{\mathrm{opp}}}(A)$.*

Then $\mathrm{lcd}(A) = \mathrm{lcd}(A^{\mathrm{opp}}) = d < \infty$, and A and A^{opp} satisfy the condition χ°.

Proof. This can be found as [10, cor. 4.3]. $\square$

Theorem 1.2. *Let A be a noetherian AS-Gorenstein k-algebra, write $d = \mathrm{id}_A(A) = \mathrm{id}_{A^{\mathrm{opp}}}(A)$ and $\underline{\mathrm{Ext}}_A^d(k, A) = \underline{\mathrm{Ext}}_{A^{\mathrm{opp}}}(k, A) = k(\ell)$.*

Then there exists an automorphism $\alpha : A \longrightarrow A$ of A as a graded k-algebra, such that there are isomorphisms in $D(\mathrm{GrMod}(A^e))$

$$R\Gamma_{\mathfrak{m}}(A)' \cong A_\alpha(-\ell)[d] \cong R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(A)'.$$

Proof. This is a direct consequence of the results of [9]: A is itself a dualizing complex in the sense of [9, def. 3.3]. According to [9, thm. 3.9], any other complex which is a dualizing complex in this sense will have the form $A_\alpha(m)[n]$ for a suitable degree zero-automorphism $\alpha : A \longrightarrow A$ and numbers $m, n \in \mathbb{Z}$.

According to [9, cor. 4.14], A has a balanced dualizing complex, cf. [9, p. 42, def.]. Thus, we can find α and m and n as above such that

$$R\Gamma_{\mathfrak{m}}(A_\alpha(m)[n])' \cong A \cong R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(A_\alpha(m)[n])'$$

in $D(\mathrm{GrMod}(A^e))$. This means

$$R\Gamma_{\mathfrak{m}}(A)' \cong A_\alpha(m)[n] \cong R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(A)',$$

where, necessarily, $m = -\ell$ and $n = d$. $\square$

2. The local duality theorem

This section gives two hyperhomological isomorphisms and applies the first to the proof of theorem 2.3, which is a version of the local duality theorem. For a different version, see [9, thm. 4.18].

Proposition 2.1. *Let A be an $\mathbb{N}$-graded left-noetherian k-algebra which has $\mathrm{lcd}(A) < \infty$.*

For $X \in D^b(\mathrm{GrMod}(A^e))$ and $Y \in D^-(\mathrm{GrMod}(A))$, we have a functorial isomorphism

$$R\Gamma_{\mathfrak{m}}(X \overset{L}{\otimes}_A Y) \cong R\Gamma_{\mathfrak{m}}(X) \overset{L}{\otimes}_A Y.$$

Proof. Note that $R\Gamma_{\mathfrak{m}}$ is defined on all of $D(\mathrm{GrMod}(A))$, and can be computed via resolutions into complexes of $\Gamma_{\mathfrak{m}}$-acyclic objects, cf. [5, proof of cor. 5.3(γ)]. Let $X \overset{\simeq}{\longrightarrow} I$ be an injective resolution over A^e; we choose the complex I left-bounded. Since all objects in I are injective left-modules and $\Gamma_{\mathfrak{m}} : \mathrm{GrMod}(A) \longrightarrow \mathrm{GrMod}(A)$ has finite cohomological dimension, we may truncate I to get a bounded X-resolution consisting of bi-modules that are $\Gamma_{\mathfrak{m}}$-acyclic when considered as left-modules. Call the truncation Q. As noted above, the objects in Q are also acyclic for $\Gamma_{\mathfrak{m}} : \mathrm{GrMod}(A^e) \longrightarrow \mathrm{GrMod}(A^e)$.

Choose also a free resolution $L \overset{\simeq}{\longrightarrow} Y$ with L right-bounded. We have

$$X \overset{L}{\otimes}_A Y = X \otimes_A L \simeq Q \otimes_A L,$$

and the modules in $Q \otimes_A L$ are direct sums of objects from Q, so they are acyclic for $\Gamma_{\mathfrak{m}}$. But then

$$R\Gamma_{\mathfrak{m}}(X \overset{L}{\otimes}_A Y) = \Gamma_{\mathfrak{m}}(Q \otimes_A L) \cong \Gamma_{\mathfrak{m}}(Q) \otimes_A L = R\Gamma_{\mathfrak{m}}(X) \overset{L}{\otimes}_A Y.$$

since the "$\cong$" is easy to establish.   □

Proposition 2.2. *Let A be an $\mathbb{N}$-graded left-noetherian k-algebra which has* $\mathrm{lcd}(A^{\mathrm{opp}}) < \infty$.

For $X \in D^-_{\mathrm{fg}}(\mathrm{GrMod}(A))$ and $Y \in D^b(\mathrm{GrMod}(A^e))$, we have a functorial isomorphism

$$R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(R\underline{\mathrm{Hom}}_A(X,Y)) \cong R\underline{\mathrm{Hom}}_A(X, R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}Y).$$

Proof. Again we let $Y \overset{\simeq}{\longrightarrow} I$ be an injective resolution over A^e where I is left-bounded. The functor $\Gamma_{\mathfrak{m}^{\mathrm{opp}}} : \mathrm{GrMod}(A^{\mathrm{opp}}) \longrightarrow \mathrm{GrMod}(A^{\mathrm{opp}})$ has finite cohomological dimension, so we truncate I to get a bounded Y-resolution consisting of bi-modules that are $\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$-acyclic when considered as right-modules over A. As in the last proof, we call the truncation Q and note that Q consists of objects that are acyclic for $\Gamma_{\mathfrak{m}^{\mathrm{opp}}} : \mathrm{GrMod}(A^e) \longrightarrow \mathrm{GrMod}(A^e)$.

Choose also a free resolution $L \overset{\simeq}{\longrightarrow} X$, where L is a left-complex consisting of finitely generated free modules.

We have

$$R\underline{\mathrm{Hom}}_A(X,Y) = \underline{\mathrm{Hom}}_A(L,Y) \simeq \underline{\mathrm{Hom}}_A(L,Q)$$

and the modules in $\underline{\mathrm{Hom}}_A(L,Q)$ are finite direct sums of objects from Q, so they are acyclic for $\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$. But then

$$
\begin{aligned}
R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(R\underline{\mathrm{Hom}}_A(X,Y)) &= \Gamma_{\mathfrak{m}^{\mathrm{opp}}}(\underline{\mathrm{Hom}}_A(L,Q)) \\
&\cong \underline{\mathrm{Hom}}_A(L, \Gamma_{\mathfrak{m}^{\mathrm{opp}}}Q) \\
&= R\underline{\mathrm{Hom}}_A(X, R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}Y),
\end{aligned}
$$

since it is again easy to prove the "$\cong$". For this, we use the fact that the free modules in L are finitely generated and the fact that Q is bounded.   □

Theorem 2.3. *(The local duality theorem) Let A be an $\mathbb{N}$-graded left-noetherian k-algebra which has* $\mathrm{lcd}(A) < \infty$.

Then there is an isomorphism

$$R\Gamma_{\mathfrak{m}}(X)' \cong R\underline{\mathrm{Hom}}_A(X, R\Gamma_{\mathfrak{m}}(A)'),$$

functorial in $X \in D^-(\mathrm{GrMod}(A))$.

Proof. We may compute

$$R\Gamma_{\mathfrak{m}}(X)' = R\Gamma_{\mathfrak{m}}(A \overset{L}{\otimes}_A X)' \cong (R\Gamma_{\mathfrak{m}}(A) \overset{L}{\otimes}_A X)' \cong R\underline{\mathrm{Hom}}_A(X, R\Gamma_{\mathfrak{m}}(A)').$$

where the first "$\cong$" is obtained from proposition 2.1, while the second "$\cong$" follows from [6, prop. 2.2].   □

3. THE FUNCTOR $R\Gamma_{\mathfrak{m}}(-)'$ AS EQUIVALENCE OF CATEGORIES

This section proves the property stated in theorem 3.3 of local cohomology over quotients of noetherian AS-Gorenstein algebras: the functors $R\Gamma_{\mathfrak{n}}(-)'$ and

$R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(-)'$ are inverse to each other on the categories of bounded complexes of finitely generated modules. Hence, they induce the equivalence of categories in equation (1). As mentioned in the introduction, this is a central property of local cohomology, and lies at the heart of many applications of the theory.

However, we begin by proving the result contained in proposition 3.2, which allows us to compute local cohomology over quotients of an algebra, A, by means of local cohomology over A itself; this makes possible theorem 3.3, since local cohomology over noetherian AS-Gorenstein algebras is so well-behaved.

Lemma 3.1. *Let A be an $\mathbb{N}$-graded connected noetherian k-algebra with a graded ideal $\mathfrak{a}$, and suppose that $A/\mathfrak{a}$ satisfies the condition χ°. Write $B = A/\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{a}$.*

Then if $M \in \mathrm{GrMod}(B)$ is acyclic for $\Gamma_{\mathfrak{n}}$, it is also, when viewed as an A-module, acyclic for $\Gamma_{\mathfrak{m}}$.

Proof. Given any integers p, n and d, there is a canonical homomorphism

$$\underline{\mathrm{Ext}}^p_B(B/B_{\geq n}, M)_{\geq d} \longrightarrow \underline{\mathrm{Ext}}^p_A(A/A_{\geq n}, M)_{\geq d},$$

cf. [1, formula 8.1.5]. Because of [1, lem. 8.2(2) and (3)] and the χ°-condition on B, if we fix p and d and choose n large enough, the homomorphism becomes an isomorphism. But that means that there is an isomorphism

$$R^p\Gamma_{\mathfrak{n}}(M)_{\geq d} \overset{\cong}{\longrightarrow} R^p\Gamma_{\mathfrak{m}}(M)_{\geq d}$$

for any p and d. This implies the lemma's statement. □

Proposition 3.2. *Let A be an $\mathbb{N}$-graded connected noetherian k-algebra with a graded ideal $\mathfrak{a}$. Suppose that $\mathrm{lcd}(A) < \infty$ and that $A/\mathfrak{a}$ satisfies the condition χ°. Write $B = A/\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{a}$.*

Then

- *The local cohomological dimension satisfies $\mathrm{lcd}(B) \leq \mathrm{lcd}(A) < \infty$;*
- *The functors $R\Gamma_{\mathfrak{m}}$ respectively $R\Gamma_{\mathfrak{m}/\mathfrak{a}}$ are defined on the full derived categories $D(\mathrm{GrMod}(A))$ respectively $D(\mathrm{GrMod}(A/\mathfrak{a}))$, and if*

$$I : D(\mathrm{GrMod}(A/\mathfrak{a})) \longrightarrow D(\mathrm{GrMod}(A))$$

is the canonical functor, then

$$R\Gamma_{\mathfrak{m}} \circ I = I \circ R\Gamma_{\mathfrak{m}/\mathfrak{a}}.$$

Proof. Consider the first statement. If $M \in \mathrm{GrMod}(B)$ has the injective resolution

$$0 \to M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

over B, then according to lemma 3.1, the complex I will be a $\Gamma_{\mathfrak{m}}$-acyclic resolution of M over A. And it is easy to see that for any $N \in \mathrm{GrMod}(B)$, we have $\Gamma_{\mathfrak{n}}(N) = \Gamma_{\mathfrak{m}}(N)$. That is,

$$R^i\Gamma_\mathfrak{n}(M) = h^i\Gamma_\mathfrak{n}(I) = h^i\Gamma_\mathfrak{m}(I) = R^i\Gamma_\mathfrak{m}(M).$$

and this implies the first claim in the proposition.

Now look at the second statement. The fact that $\Gamma_\mathfrak{m}$ and $\Gamma_\mathfrak{n}$ are defined everywhere in the relevant derived categories follows for instance from [5, cor. 5.3(γ)].

Also, according to the proof of [5, cor. 5.3(γ)], we may, given any complex X over B, choose a $\Gamma_\mathfrak{n}$-acyclic resolution $X \xrightarrow{\simeq} J$. According to lemma 3.1, J also consists of $\Gamma_\mathfrak{m}$-acyclic modules, so

$$R\Gamma_\mathfrak{m} \circ I(X) = \Gamma_\mathfrak{m}(J) = \Gamma_\mathfrak{n}(J) = I \circ R\Gamma_\mathfrak{n}(X).$$

$\square$

Theorem 3.3. *Let A be a noetherian AS-Gorenstein k-algebra with a graded ideal $\mathfrak{a}$, and put $B = A/\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{a}$.*

For each $X \in D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(B))$, there is an isomorphism in $D(\mathrm{GrMod}(B))$,

$$R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_\mathfrak{n}(X)')' \cong X,$$

and this is functorial with respect to X.

Proof. Note that by theorem 1.1, the algebras A and A^{opp} satisfy χ° and have finite local cohomological dimension. By [1, cor. 8.4(2)] and the first part of proposition 3.2, this is also the case for B and B^{opp}.

We let

$$I : D(\mathrm{GrMod}(B)) \longrightarrow D(\mathrm{GrMod}(A)) \qquad \text{and}$$
$$I^{\mathrm{opp}} : D(\mathrm{GrMod}(B^{\mathrm{opp}})) \longrightarrow D(\mathrm{GrMod}(A^{\mathrm{opp}}))$$

be the canonical functors, and we let $X \in D^{\mathrm{b}}(\mathrm{GrMod}(B))$. Then by proposition 3.2, in the derived category $D(\mathrm{GrMod}(A))$,

$$I(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_\mathfrak{n}(X)')') \cong R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(I^{\mathrm{opp}}(R\Gamma_\mathfrak{n}(X)'))' \cong R\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(R\Gamma_\mathfrak{m}(IX)')' = (*),$$

and by theorem 2.3 and theorem 1.2, this is

$$
\begin{aligned}
(*) \quad &\cong \quad R\operatorname{\underline{Hom}}_{A^{\mathrm{opp}}}(R\operatorname{\underline{Hom}}_A(IX, A_\mathfrak{a}(-\ell)[d]), A_\mathfrak{a}(-\ell)[d]) \\
&\cong \quad R\operatorname{\underline{Hom}}_{A^{\mathrm{opp}}}(R\operatorname{\underline{Hom}}_A(IX, A), A) \\
&\cong \quad IX,
\end{aligned}
$$

the last "$\cong$" valid when X consists of finitely generated modules. The "$\cong$"'s are all functorial with respect to X.

Now I commutes with taking cohomology, so if we take a finitely generated module $M \in \mathrm{grmod}(B)$ and consider $R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_\mathfrak{n}(M)')'$ as a complex in $D(\mathrm{GrMod}(B))$, we see that its zeroth cohomology is M and all its other cohomology-groups vanish, so there is an isomorphism in $D(\mathrm{GrMod}(B))$,

$$(2) \qquad\qquad\qquad R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_\mathfrak{n}(M)')' \cong M,$$

functorial with respect to $M \in \mathrm{grmod}(B) \subset D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(B))$.

In particular, $R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(B)')'$ is concentrated in degree zero when considered as an element of $D(\mathrm{GrMod}(B))$, and consequently also so concentrated when considered as an element of $D(\mathrm{GrMod}(B^e))$. We therefore identify it with a B-bi-module, K, and note that K is locally finite.

Now for $X \in D^b_{\mathrm{fg}}(\mathrm{GrMod}(B))$, we perform the computation

$$
\begin{aligned}
(3) \qquad R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(X)')' &= R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(B \overset{L}{\otimes}_B X)')' \\
&\cong R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}((R\Gamma_{\mathfrak{n}}(B) \overset{L}{\otimes}_B X)')' \\
&\cong R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\underline{\mathrm{Hom}}_B(X, R\Gamma_{\mathfrak{n}}(B)'))' \\
&\cong R\underline{\mathrm{Hom}}_B(X, R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(B)'))' \\
&\cong (R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(B)')' \overset{L}{\otimes}_B X)'' \\
&= (K \overset{L}{\otimes}_B X)'' \\
&\cong K \overset{L}{\otimes}_B X
\end{aligned}
$$

which is functorial in X. The first "$\cong$" used proposition 2.1, the second and the fourth "$\cong$" used [6, prop. 2.2], and the third "$\cong$" used proposition 2.2. The final "$\cong$" came about since $K \overset{L}{\otimes}_B X$ is locally finite due to the local finiteness of K.

However, equations (2) and (3) in conjunction tell us that if $M \in \mathrm{grmod}(B)$, then

$$M \cong K \overset{L}{\otimes}_B M$$

functorially in M, and this implies that K is flat from the right, and that $K \otimes_B M \cong M$ functorially for any finitely generated module $M \in \mathrm{grmod}(B)$. But then

$$X \cong K \overset{L}{\otimes}_B X$$

functorially in $X \in D^b_{\mathrm{fg}}(\mathrm{GrMod}(B))$. Along with equation (3) this shows

$$R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(X)')' \cong X.$$

$\square$

Note that since $B = A/\mathfrak{a}$ satisfies χ°, we know that if $X \in D^b_{\mathrm{fg}}(\mathrm{GrMod}(B))$, each module in the minimal injective resolution I of X only contains a finite number of direct summands isomorphic to B'. Hence $R\Gamma_{\mathfrak{n}}(X)'$ may be taken to consist of finitely generated modules. Also, since $\mathrm{lcd}(B) < \infty$, we can take $R\Gamma_{\mathfrak{n}}(X)'$ to be bounded. And replacing A with A^{opp} (and accordingly B with B^{opp}), we get $R\Gamma_{\mathfrak{n}}(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(Y)')' \cong Y$ for $Y \in D^b_{\mathrm{fg}}(\mathrm{GrMod}(B^{\mathrm{opp}}))$. Thus the local cohomology-functors induce the equivalence of categories from equation (1) of the introduction,

$$
D^b_{\mathrm{fg}}(\mathrm{GrMod}(B)) \xrightleftharpoons[R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(-)']{R\Gamma_{\mathfrak{n}}(-)'} D^b_{\mathrm{fg}}(\mathrm{GrMod}(B^{\mathrm{opp}}))^{\mathrm{opp}}.
$$

4. APPLICATIONS TO BASS-NUMBERS

This section will prove the theorems about Bass-numbers quoted in the introduction. Throughout this section, we will suppose that A is a noetherian

AS-Gorenstein algebra. that $\mathfrak{a}$ is a graded ideal in A, and that $B = A/\mathfrak{a}$. Note that by theorem 1.1. this entails that A satisfies $\chi°$ and that $\mathrm{lcd}(A)$ is finite: the same holds for B by [1, cor. 8.4(2)] and the first part of proposition 3.2. We will write $\mathfrak{m} = A_{\geq 1}$ and $\mathfrak{n} = B_{\geq 1} = \mathfrak{m}/\mathfrak{a}$.

The section is organized as follows. The three principal results are theorems 4.5, 4.6, and 4.8. They state, respectively, that over an algebra B as described. the injective dimension of a finitely generated module is determined by non-vanishing of its Bass-numbers, that Bass' theorem holds, and that the sequence of Bass-numbers of a finitely generated module has no holes. The rest of the results are included primarily to assist in the proofs of theorems 4.5, 4.6, and 4.8.

Note that the proofs of lemma 4.7 and theorem 4.8 have been taken over almost verbatim from the commutative case, cf. [3, prop. 10.24 and cor. 10.25]: these are in turn inspired by [8, thm. 2].

Proposition 4.1. *If $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(B))$ is non-trivial, then*

$$\mathrm{depth}_B(X) < \infty.$$

Proof. If $\mathrm{depth}_B(X)$ is infinite. there is no torsion in the minimal injective resolution of X. and thus $R\Gamma_{\mathfrak{n}}(X) = 0$. But when X is bounded and consists of finitely generated modules, this entails $X = 0$ in the derived category by theorem 3.3. i.e. X is a trivial complex.   □

Lemma 4.2. *If $X, Y \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(B))$. then for each i we have*

$$\underline{\mathrm{Ext}}^i_B(X, Y) \cong \underline{\mathrm{Ext}}^i_{B^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(Y)', R\Gamma_{\mathfrak{n}}(X)').$$

Proof. Due to the fact that $R\Gamma_{\mathfrak{n}}(-)'$ induces an equivalence of full subcategories of $D(\mathrm{GrMod}(B))$ and $D(\mathrm{GrMod}(B^{\mathrm{opp}}))^{\mathrm{opp}}$, as described at the end of section 3. we know that

$$\mathrm{Hom}_{D(\mathrm{GrMod}(B))}(X, Y) \cong \mathrm{Hom}_{D(\mathrm{GrMod}(B^{\mathrm{opp}}))}(R\Gamma_{\mathfrak{n}}(Y)', R\Gamma_{\mathfrak{n}}(X)').$$

But twisting Y by i and taking sums of shifts produces the isomorphism of the lemma. since

$$\mathrm{Ext}^i_{\mathrm{GrMod}(B)}(X, Y) = \mathrm{Hom}_{D(\mathrm{GrMod}(B))}(X, Y[i]).$$

□

Proposition 4.3. *Let C be an $\mathbb{N}$-graded connected left-noetherian k-algebra satisfying $\chi°$.*
For $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(C))$ we have

$$\mathrm{depth}_C(X) = -\sup(R\Gamma_{\mathfrak{m}}(X)').$$
$$\mathrm{k.id}_C(X) = \mathrm{pd}_{C^{\mathrm{opp}}}(R\Gamma_{\mathfrak{m}}(X)').$$

Proof. We will write $\mathfrak{o} = C_{\geq 1}$. Consider a minimal injective resolution $X \xrightarrow{\simeq} I$. Because of the $\chi°$-condition. each I^i only contains a finite number of direct summands isomorphic to C', so $R\Gamma_{\mathfrak{o}}(X)' = \Gamma_{\mathfrak{o}}(I)' = F$ is a complex

of finitely generated free modules. Moreover, the minimality of I implies the minimality of F. The two equations can now be read off. $\qquad\square$

Proposition 4.4. *Let* $X, Y \in D_{fg}^b(\mathrm{GrMod}(B))$. *If* $\mathrm{k.id}_B(Y) < \infty$, *we have*

$$\sup R\underline{\mathrm{Hom}}_B(X, Y) = \mathrm{k.id}_B(Y) - \mathrm{depth}_B(X).$$

Proof. If either complex is trivial, the statement follows immediately, so we assume that both X and Y are non-trivial.

According to lemma 4.2, the left-hand side of the proposition's equation is equal to $\sup R\underline{\mathrm{Hom}}_{B^{opp}}(R\Gamma_{\mathfrak{n}}(Y)', R\Gamma_{\mathfrak{n}}(X)')$, and using [6, prop. 2.2], we see that this is equal to

$$\sup((R\Gamma_{\mathfrak{n}}(X) \overset{L}{\otimes}_{B^{opp}} R\Gamma_{\mathfrak{n}}(Y)')') = -\inf(R\Gamma_{\mathfrak{n}}(X) \overset{L}{\otimes}_{B^{opp}} R\Gamma_{\mathfrak{n}}(Y)') = (*).$$

Let us consider the two complexes being tensored here.

By the first equation of proposition 4.3 we have $\mathrm{depth}_B(X) = \inf R\Gamma_{\mathfrak{n}}(X)$, and this number is finite by proposition 4.1, so by truncation below degree $\mathrm{depth}_B(X)$, we may assume that the lowest module in $R\Gamma_{\mathfrak{n}}(X)$ has number $\mathrm{depth}_B(X)$. Since all modules in $R\Gamma_{\mathfrak{n}}(X)$ are right-bounded due to the fact that B satisfies χ°, and since cohomology number $\mathrm{depth}_B(X)$ of $R\Gamma_{\mathfrak{n}}(X)$ is non-zero, we can suppose that the kernel of differential number $\mathrm{depth}_B(X)$ contains an element, t, annihilated by $\mathfrak{n}$.

Also, by the second equation of proposition 4.3 we have

$$\mathrm{pd}_{B^{opp}}(R\Gamma_{\mathfrak{n}}(Y)') = \mathrm{k.id}_B(Y) < \infty,$$

whence we can replace $R\Gamma_{\mathfrak{n}}(Y)'$ with a minimal free complex, L, with lowest term in degree $-\mathrm{pd}_{B^{opp}}(R\Gamma_{\mathfrak{n}}(Y)')$.

Now, due to the existence of t and the minimality of L, it is easy to see that

$$\begin{aligned} \inf(R\Gamma_{\mathfrak{n}}(X) \overset{L}{\otimes}_{B^{opp}} R\Gamma_{\mathfrak{n}}(Y)') &= -\mathrm{pd}_{B^{opp}}(R\Gamma_{\mathfrak{n}}(Y)') + \mathrm{depth}_B(X) \\ &= -\mathrm{k.id}_B(Y) + \mathrm{depth}_B(X). \end{aligned}$$

and thus

$$(*) = \mathrm{k.id}_B(Y) - \mathrm{depth}_B(X).$$

$\qquad\square$

Theorem 4.5. *If* $X \in D_{fg}^b(\mathrm{GrMod}(B))$ *then*

$$\mathrm{id}_B(X) = \mathrm{k.id}_B(X).$$

Proof. If X is trivial or $\mathrm{k.id}_B(X) = \infty$, the result is clear. So assume X non-trivial and $\mathrm{k.id}_B(X) < \infty$. Then if $M \in \mathrm{grmod}(B)$, proposition 4.4 tells us that

$$\sup R\underline{\mathrm{Hom}}_B(M, X) = \mathrm{k.id}_B(X) - \mathrm{depth}_B(M) \leq \mathrm{k.id}_B(X).$$

But then by [6, prop. 1.7], we see $\mathrm{id}_B(X) \leq \mathrm{k.id}_B(X)$, and the opposite inequality is of course also satisfied. $\qquad\square$

Theorem 4.6. *(Bass) Let* $X \in D_{fg}^b(\mathrm{GrMod}(B))$ *be non-trivial with* $\mathrm{id}_B(X) < \infty$.

Then we have

$$\mathrm{id}_B(X) = \mathrm{depth}_B(B) + \sup(X).$$

Proof. When $\mathrm{id}_B(X)$ is finite, $\mathrm{k.id}_B(X)$ is of course also finite. We therefore use proposition 4.4 with B in the first argument of $R \underline{\mathrm{Hom}}$, and X in the second. Then, using theorem 4.5, we get

$$\sup(X) = \sup R \underline{\mathrm{Hom}}_B(B, X) = \mathrm{k.id}_B(X) - \mathrm{depth}_B(B) = \mathrm{id}_B(X) - \mathrm{depth}_B(B).$$

whence the conclusion of the theorem follows.   □

Lemma 4.7. *Suppose that the complex* $X \in D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(B))$ *has* $\mu^i(X) = 0$.
Then there exist complexes $Y, Z \in D_{\mathrm{fg}}^{\mathrm{b}}(\mathrm{GrMod}(B))$ *such that*

$$\mathrm{id}_B(Y) < i < \mathrm{depth}_B(Z)$$

and $X \cong Y \oplus Z$ *in* $D(\mathrm{GrMod}(B))$.

Proof. If $X \xrightarrow{\simeq} I$ is the minimal injective resolution of X, the assumption means that there is no torsion in the module I^i. Accordingly, $F = R\Gamma_{\mathfrak{n}}(X)'$ looks like

$$\cdots \to F^{-i-2} \to F^{-i-1} \to 0 \to F^{-i+1} \to \cdots \to F^{-\inf(X)} \to 0 \to \cdots,$$

and if we put H equal to the left-hand part and G equal to the right-hand part of this complex, F is isomorphic to $G \oplus H$ and clearly $\mathrm{pd}_{B^{\mathrm{opp}}}(G) < i$ and $\sup(H) < -i$.

Now set $Y = R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(G)'$ and $Z = R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(H)'$. Then by theorem 4.5, proposition 4.3 and theorem 3.3,

$$\begin{aligned}
\mathrm{id}_B(Y) &= \mathrm{id}_B(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(G)') = \mathrm{pd}_{B^{\mathrm{opp}}}(R\Gamma_{\mathfrak{n}}(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(G)')') \\
&= \mathrm{pd}_{B^{\mathrm{opp}}}(G) < i
\end{aligned}$$

and by proposition 4.3 and theorem 3.3,

$$\begin{aligned}
\mathrm{depth}_B(Z) &= \mathrm{depth}_B(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(H)') = -\sup(R\Gamma_{\mathfrak{n}}(R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(H)')') \\
&= -\sup(H) > i,
\end{aligned}$$

and

$$Y \oplus Z = R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(G)' \oplus R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(H)' \cong R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(G \oplus H)' = R\Gamma_{\mathfrak{n}^{\mathrm{opp}}}(R\Gamma(X)')' = X.$$

□

Theorem 4.8. *If* $M \in \mathrm{grmod}(B)$ *is a finitely generated module, then*

$$\mu^i(M) > 0 \quad \text{whenever} \quad \mathrm{depth}_B(M) \leq i \leq \mathrm{id}_B(M).$$

Proof. We may suppose $M \neq 0$. Let $\mathrm{depth}_B(M) \leq i \leq \mathrm{id}_B(M)$. If $\mu^i(M) = 0$, we use lemma 4.7; the fact that $M \cong Y \oplus Z$ means that Y and Z must be concentrated in degree zero, so we get modules $M_1 = h^0(Y)$ and $M_2 = h^0(Z)$ with $M \cong M_1 \oplus M_2$ and $\mathrm{id}_B(M_1) < i < \mathrm{depth}_B(M_2)$. But proposition 4.4 and theorem 4.5 imply that

$$\sup R\underline{\mathrm{Hom}}_B(M_2, M_1) = \mathrm{id}_B(M_1) - \mathrm{depth}_B(M_2) < 0,$$

and this is possible only if M_1 or M_2 is zero.

However. either possibility produces a contradiction — if $M_1 = 0$. we get

$$\mathrm{depth}_B(M) = \mathrm{depth}_B(M_2) > i \geq \mathrm{depth}_B(M),$$

and if $M_2 = 0$.

$$\mathrm{id}_B(M) = \mathrm{id}_B(M_1) < i \leq \mathrm{id}_B(M).$$

$\square$

5. NON-VANISHING OF LOCAL COHOMOLOGY

In this section we will prove a generalization to quotients of certain good noetherian AS-Gorenstein algebras of a "classical" non-vanishing result for local cohomology. The commutative version of the result can be found as [4. lem. 2.1(1)]. The AS-Gorenstein algebras in question are the ones satisfying the SSC-condition of [11. sec. 2]. Note that there are many such algebras: by [11. sec. 2]. all PI-algebras and all noetherian AS-regular algebras of dimension 3 are included.

Definition 5.1. *Let* $X \in D(\mathrm{GrMod}(A))$. *Then we set*

$$\mathrm{GKdim}(X) = \sup\{\ell + \mathrm{GKdim}(h^\ell X) \mid h^\ell(X) \neq 0\}.$$

Theorem 5.2. *Let A be a noetherian AS-Gorenstein k-algebra which satisfies the Similar Submodule Condition. SSC, cf. [11. sec. 2]. Let $\mathfrak{a}$ be a graded ideal in A. and put $B = A/\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{a}$.*

Then for any $X \in D_{\mathrm{fg}}^-(\mathrm{GrMod}(B))$, we have

$$\sup R\Gamma_\mathfrak{n}(X) = \mathrm{GKdim}(X).$$

Proof. We may clearly suppose that X has non-zero cohomology. since the theorem is trivially true for the zero-complex.

Think of X as a complex over A. Choose a free resolution $L \xrightarrow{\simeq} X$. consisting of finitely generated modules. Choose also an injective resolution of A as an A-left-module.

$$0 \to A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots.$$

Consider the double-complex given by

$$M^{pq} = \underline{\mathrm{Hom}}_A(L_p, I^q).$$

It gives rise to two convergent spectral sequences. The first one yields

$$
\begin{aligned}
h^i \mathrm{Tot}(M) &= \underline{\mathrm{Ext}}_A^i(X, A) \\
&= h^i(R\underline{\mathrm{Hom}}_A(X, A[d])[-d]) \\
&= h^{i-d} R\underline{\mathrm{Hom}}_A(X, A[d]) \\
&\cong h^{i-d}(R\Gamma_\mathfrak{m}(X)'(-\ell))
\end{aligned}
$$

$$= (R^{d-i}\Gamma_{\mathfrak{m}}(X))'(-\ell)$$
$$\cong (R^{d-i}\Gamma_{\mathfrak{n}}(X))'(-\ell);$$

here the first "$\cong$" is a consequence of theorem 1.2 and theorem 2.3, while the second "$\cong$" follows from proposition 3.2. This means that

(4) $\inf \mathrm{Tot}(M) = d - \sup R\Gamma_{\mathfrak{n}}(X)$.

The second spectral sequence has

$$E_2^{pq} = \underline{\mathrm{Ext}}_A^p(h_q L, A) = \underline{\mathrm{Ext}}_A^p(h^{-q} X, A).$$

To learn something from this, we recall some facts from the proof of [11, thm. 3.1]. Given any finitely generated module $N \in \mathrm{grmod}(A)$, it tells us that

(i): $\mathrm{grade}(N) = \inf\{i \mid \underline{\mathrm{Ext}}_A^i(N, A) \neq 0\} = d - \mathrm{GKdim}(N)$;

(ii): $\mathrm{GKdim}(\underline{\mathrm{Ext}}_A^j(N, A)) = \mathrm{GKdim}(N)$ when $j = \mathrm{grade}(N)$;

(iii): For $\mathrm{grade}(N) \leq i \leq d$, we have $\mathrm{GKdim}(\underline{\mathrm{Ext}}_A^i(N, A)) \leq d - i$.

Now, first, let us find the lowest-lying line of the form $p + q = T$ on which some E_2^{pq} is different from zero. Due to (i) above, it must have

(5) $T = \min\{d - \mathrm{GKdim}(h^{-q}X) + q \mid h^{-q}(X) \neq 0\} = d - \mathrm{GKdim}(X)$.

Next, let us consider what will happen to a contribution E_2^{rs} on this line. Because of (i) again, it has $r = d - \mathrm{GKdim}(h^{-s}X)$, and by (ii), it has $\mathrm{GKdim}(E_2^{rs}) = \mathrm{GKdim}(h^{-s}X)$. Now, the E_*^{rs}-terms will, of course, not be hit by anything different from zero when we pass through successive terms of the spectral sequence. On the other hand, they will map to modules E_*^{pq} having p-values larger than r. However, by (iii) above, this ensures that they map to modules having GKdim strictly smaller than $\mathrm{GKdim}(E_2^{rs})$: moreover, after finitely many steps, they will only map to zero. Therefore E_2^{rs} survives to infinity, so

(6) $\inf \mathrm{Tot}(M) = T$.

Combining equations (4), (6) and (5), we get

$$d - \sup R\Gamma_{\mathfrak{n}}(X) = d - \mathrm{GKdim}(X),$$

that is, $\sup R\Gamma_{\mathfrak{m}}(X) = \mathrm{GKdim}(X)$.   □

As noted in the introduction, if we specialize this result to a complex concentrated in degree zero, i.e. to a finitely generated module $M \in \mathrm{grmod}(B)$, it just states that

$$\sup\{i \mid R^i\Gamma_{\mathfrak{n}}(M) \neq 0\} = \mathrm{GKdim}(M).$$

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It should be noted that the remarks from [6] about Hans-Bjørn Foxby's work on the hyperhomology of commutative rings, as described in [3], apply very much to this paper as well. which could not have been written without inspiration from the commutative case. The key idea of the previous pages is simply to generalize the commutative hyperhomological theory directly, and the general framework of the theory presented here is the same as that of the commutative theory. In fact. even the proofs of lemma 4.7 and theorem 4.8 were taken over from [3] (but see also [8, proof of thm. 2]).

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