STRING BASES FOR QUANTUM GROUPS OF TYPE A_r

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To I.M.Gelfand on his 80th birthday

0. Introduction

For every positive integer r let $\mathcal{A}_r$ denote the associative algebra with unit over the field of rational functions $\mathbf{Q}(q)$ generated by the elements $x_1, \ldots, x_r$ subject to the relations:

$$x_i x_j = x_j x_i \text{ for } |i-j| > 1, \tag{0.1}$$

$$x_i^2 x_j - (q + q^{-1}) x_i x_j x_i + x_j x_i^2 = 0 \text{ for } |i-j| = 1. \tag{0.2}$$

This is the *quantum deformation* (or $q-$deformation) of the algebra of polynomial functions on the group N_{r+1} of upper unitriangular $(r+1) \times (r+1)$ matrices. In this paper we introduce and study a class of bases in $\mathcal{A}_r$ which we call *string bases*. The main example of a string basis is given as follows. Let $U_+ = U_{+,r}$ be the quantized universal enveloping algebra of the Lie algebra $\underline{n}_{r+1}$ of N_{r+1} (see e.g., [10]). Then $\mathcal{A}_r$ is seen to be the graded dual of U_+, and the basis in $\mathcal{A}_r$ dual to the Lusztig's canonical basis in U_+ is a string basis. The string bases are defined by means of so called *string axioms* which we find easier to work with than the axioms imposed by Lusztig or those by Kashiwara. The string axioms seem to be rather strong, and it is even conceivable that the string basis is unique but we do not know this in general. We prove the uniqueness of a string basis for $\mathcal{A}_2$ and $\mathcal{A}_3$.

The main advantage of string bases is that they seem to have nicer multiplicative properties than the canonical basis. We say that $x, y \in \mathcal{A}_r$ *quasicommute* if $xy = q^n yx$ for some integer n. We conjecture that every string basis B has the following property: two elements $b, b' \in B$ quasicommute if and only if $q^N bb' \in B$ for some integer N. We prove this for $\mathcal{A}_2$ and $\mathcal{A}_3$, and provide some supporting evidence for general $\mathcal{A}_r$.

Before giving precise formulations of the results we would like to put this work into historic context. Let $\underline{g}$ be a semisimple complex Lie algebra of rank r with fixed Cartan decomposition $\underline{g} = \underline{n}_- \oplus \underline{h} \oplus \underline{n}_+$. Our main motivation was to study "good bases" in irreducible $\underline{g}-$modules. Good bases were introduced independently in [5] and [1]. Let $P \subset \underline{h}^*$ denote the weight lattice of $\underline{g}$, and $P_+ \subset P$ denote the semigroup of dominant integral weights, i.e., weights of the form $n_1\omega_1 + \ldots + n_r\omega_r$, where $\omega_1, \ldots, \omega_r$ are fundamental weights of $\underline{g}$, and $n_1, \ldots, n_r$ are nonnegative integers. For $\lambda \in P_+$ let V_λ denote the

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irreducible (finite-dimensional) $\underline{g}-$module with the highest weight λ. For $\beta \in P$ we denote by $V_\lambda(\beta)$ the weight subspace of weight β in V_λ. For $\nu = \sum_i n_i\omega_i \in P_+$ we set

$$V_\lambda(\beta;\nu) = \{x \in V_\lambda(\beta) : e_i^{n_i+1}x = 0 \text{ for } i = 1, \ldots, r\},$$

where $e_1, \ldots, e_r$ are standard generators of $\underline{n}_+$ whose weights are simple roots $\alpha_1, \ldots, \alpha_r$ corresponding to fundamental weights $\omega_1, \ldots, \omega_r$. A basis B of V_λ is said to be *good* if every subspace of the form $V_\lambda(\beta;\nu)$ is spanned by a part of B. This definition was motivated by the classical result that for every three irreducible finite-dimensional $\underline{g}-$modules V_λ, V_μ, V_ν there is a natural isomorphism

$$\mathrm{Hom}_{\underline{g}}(V_\mu, V_\lambda \otimes V_\nu) \simeq V_\lambda(\mu - \nu; \nu).$$

It follows that a good basis "resolves the multiplicities" in the tensor product $V_\lambda \otimes V_\nu$, i.e., provides its decomposition into irreducible components.

It was conjectured in [5] that good bases always exist (the proof given in [1] turned out to be inadequate). For $\underline{g} = sl_{r+1}$ the existence of good bases follows from [3]: as shown in [5], the special basis constructed in [3] is good. For arbitrary $\underline{g}$ the existence of good bases was proven by Mathieu [14].

Each V_λ has two important realizations dual to each other. First, there is a canonical epimorphism $U(\underline{n}_+) \to V_\lambda$ sending each $u \in U(\underline{n}_+)$ to $u\xi_\lambda$, where ξ_λ is the lowest vector in V_λ. Hence, V_λ can be realized as a quotient of $U(\underline{n}_+)$. For the dual realization we notice that the dual space $U(\underline{n}_+)^*$ can be identified with the algebra $\mathbf{C}[N_+]$ of polynomial functions on the unipotent group N_+ whose Lie algebra is $\underline{n}_+$. Hence, each V_λ can be canonically realized as a subspace of $\mathbf{C}[N_+]$.

It was suggested in [5] and further pursued in [15] that there should exist a basis B in $\mathbf{C}[N_+]$ such that for every $V_\lambda \subset \mathbf{C}[N_+]$ the set $B \cap V_\lambda$ is a good basis for V_λ. Such a basis was constructed for $\underline{g} = sl_3$ in [5] and for $\underline{g} = sp_4$ in [15]. In both cases B consists of some monomials in a finite number of generators. It was conjectured in [15] that there exist some natural conditions which fix B to be *unique* but the problem of finding these conditions remained open.

Two remarkable solutions of this problem were given by Lusztig [10] and Kashiwara [6]. Both solutions provide a system of axioms that determines a basis in $U(\underline{n}_+)$ uniquely. The fundamental idea beyond these axioms is that in order to determine the basis uniquely one has to pass from $U(\underline{n}_+)$ to its $q-$deformation $U_q(\underline{n}_+)$. Thus, both authors construct a basis in $U_q(\underline{n}_+)$, called *canonical* by Lusztig and *(lower) global crystal* by Kashiwara (it was later proven by Lusztig [11] that these bases coincide). One recovers the basis in $U(\underline{n}_+)$ by specializing $q = 1$. As shown in [11, Theorem 4.4 (c)], the dual basis B of $\mathbf{C}[N_+]$ has the property that $B \cap V_\lambda$ is a good basis for V_λ for all $V_\lambda \subset \mathbf{C}[N_+]$. This gives another and much more constructive proof of the existence of good bases.

The papers [6], [7], [8], [10], [11], [12] reveal many important properties and applications of canonical (or crystal) bases. But these exciting developments essentially leave aside the structure of the dual basis.

The algebra $\mathcal{A}_r$ introduced above is the $q-$deformation of $\mathbf{C}[N_+]$ for $\underline{g} = sl_{r+1}$. Many of the results and arguments below make sense for arbitrary semisimple Lie algebras or even for arbitrary Kac-Moody algebras but for the sake of simplicity we shall treat only this case here.

The material is organized as follows. Main results of the paper are collected in sections 1 and 2, which can be considered as an expanded introduction. In §1 we introduce string axioms and describe the string bases of $\mathcal{A}_r$ for $r = 2, 3$. Our main conjecture on the multiplicative property of string bases is also given here (Conjecture 1.7).

In §2 we introduce our main tool for the study of string bases, the notion of the *string* of an element $x \in \mathcal{A}_r$ in a given direction. These strings are certain finite sequences of nonnegative integers, used as combinatorial labels for the elements of a string basis. Precise definitions and our main results on strings are collected in §2.

Sections 3 to 10 are devoted to the proofs of all theorems of §§1,2. More detailed directions to the proof of each theorem can be found after its formulation in Sections 1 and 2.

The article is concluded with the appendix where we discuss basic properties of $\mathcal{A}_r$ and the duality between $\mathcal{A}_r$ and $U_{+,r}$. The results we need (Propositions 1.1 to 1.3) seem to be well-known to experts in the field, but we were unable to locate exact references. For the sake of convenience of the reader we sketch the proofs in the appendix.

1. String bases and quasicommutative monomials

Let $U_+ = U_{+,r}$ be the quantized universal enveloping algebra of the maximal nilpotent subalgebra $\underline{n}_+$ of sl_{r+1}. This is an algebra with unit over the field of rational functions $\mathbf{Q}(q)$ generated by the elements $E_1, \ldots, E_r$ subject to the relations:

$$E_i E_j = E_j E_i \text{ for } |i-j| > 1, \tag{1.1}$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \text{ for } |i-j| = 1. \tag{1.2}$$

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots of sl_{r+1} in the standard numeration, and Q_+ the semigroup generated by $\alpha_1, \ldots, \alpha_r$. The algebra U_+ is Q_+-graded via $\deg(E_i) = \alpha_i$. For $\gamma \in Q_+$ let $U_+(\gamma)$ denote the homogeneous component of degree γ in U_+. Since U_+ is a deformation of $U(\underline{n}_+)$, we have

$$\dim(U_+(\gamma)) = p(\gamma), \tag{1.3}$$

where $p(\gamma)$ is the number of partitions of γ into the sum of positive roots (the Kostant partition function [9]).

Comparing (1.1), (1.2) with (0.1), (0.2) we see that the correspondence $E_i \mapsto x_i$ extends to an algebra isomorphism $U_+ \to \mathcal{A}_r$. We transfer the Q_+-grading from U_+ to $\mathcal{A}_r$ via this isomorphism, so we have $\deg(x_i) = \alpha_i$, and $\dim(\mathcal{A}_r(\gamma)) = p(\gamma)$ for $\gamma \in Q_+$. But it is important for us to keep distinguishing U_+ and $\mathcal{A}_r$. In fact, we wish to identify each graded component $\mathcal{A}_r(\gamma)$ with the dual space $U_+(\gamma)^*$. To do this we introduce an action of U_+ on $\mathcal{A}_r$, which will play the crucial part in the sequel.

Proposition 1.1. *There exists a unique action $(E, x) \mapsto E(x)$ of the algebra U_+ on $\mathcal{A}_r$ satisfying the following properties:*

(a) (Homogeneity) If $E \in U_+(\alpha), x \in \mathcal{A}_r(\gamma)$ then $E(x) \in \mathcal{A}_r(\gamma - \alpha)$.

(b) (Leibnitz formula)

$$E_i(xy) = E_i(x)y + q^{-(\gamma,\alpha_i)} x E_i(y) \text{ for } x \in \mathcal{A}_r(\gamma), y \in \mathcal{A}_r \tag{1.4}$$

(here and in the sequel (γ, α) is the usual scalar product on Q_+ defined by means of the Cartan matrix).

(c) (Normalization) $E_i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, r$.

Proposition 1.2.

(a) If $\gamma \in Q_+ \setminus \{0\}$, and x is a non-zero element of $\mathcal{A}_r(\gamma)$ then $E_i(x) \neq 0$ for some $i = 1, \ldots, r$.

(b) For every $\gamma \in Q_+$ the mapping $(E, x) \mapsto E(x)$ defines a non-degenerate pairing

$$U_+(\gamma) \times \mathcal{A}_r(\gamma) \to \mathcal{A}_r(0) = \mathbf{Q}(q).$$

Both propositions will be proven in the appendix.

Now we are in a position to define the string bases in $\mathcal{A}_r$, the main object of study in this paper. We shall use the notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [1][2]\cdots[n],$$

and define the *divided powers* to be

$$E_i^{(n)} = \frac{E_i^n}{[n]!}.$$

For a subset $B \subset \mathcal{A}_r$ we denote by $[B]^+ \subset \mathcal{A}_r$ the set of all linear combinations of elements of B with coefficients from $\mathbf{Z}_+[q, q^{-1}]$.

Let B be a basis in $\mathcal{A}_r$. We say that B is a string basis if it satisfies the following *string axioms*:

(S0) B consists of homogeneous elements and contains 1.

(S1) For every $b, b' \in B$ the product bb' belongs to $[B]^+$.

(S2) $E_i(b) \in [B]^+$ for $b \in B,\ i = 1, \ldots, r$.

(S3) If $b \in B$, and l is the maximal integer such that $E_i^l(b) \neq 0$ then $E_i^{(l)}(b) \in B$.

Proposition 1.3. *The basis in $\mathcal{A}_r$ dual to the Lusztig's canonical basis in U_+ is a string basis.*

This will be proven also in the appendix. Note that the axiom (S3) is analogous to [7, Lemma 5.1.1].

Now we construct some elements belonging to every string basis of $\mathcal{A}_r$. We need the $q-$analogs of the natural coordinates (matrix entries) on the group N_{r+1}. For every two elements $x \in \mathcal{A}_r(\gamma), y \in \mathcal{A}_r(\gamma')$ we define their $q-$*commutator* as follows:

$$[x, y] = \frac{xy - q^{(\gamma,\gamma')}yx}{q - q^{-1}}. \tag{1.5}$$

Then $[x, y] \in \mathcal{A}_r(\gamma + \gamma')$. Now let $T = T_r = (t_{ij})$ be the $(r + 1) \times (r + 1)$ matrix with entries in $\mathcal{A}_r$ defined as follows. We set $t_{ij} = 0$ for $i > j$, $t_{ii} = 1$ for all i, $t_{i,i+1} = x_i$ for $i = 1, \ldots, r$, and finally for $j > i + 1$ we define t_{ij} inductively by $t_{ij} = [t_{i,j-1}, t_{j-1,j}]$. Clearly, t_{ij} is homogeneous of degree $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$. In particular,

$$T_2 = \begin{pmatrix} 1 & x_1 & \frac{x_1x_2 - q^{-1}x_2x_1}{q-q^{-1}} \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

For every two increasing sequences of indices $I = (i_1 < i_2 < \ldots < i_s)$, $J = (j_1 < j_2 < \ldots < j_s)$ from $\{1, 2, \ldots, r+1\}$ we denote by $\Delta(I; J)$ the *quantum minor* of T with rows I and columns J:

$$\Delta(I; J) = \sum_\sigma (-q)^{l(\sigma)} t_{i_{\sigma(1)},j_1} \cdots t_{i_{\sigma(s)},j_s}, \tag{1.6}$$

where the summation is over the symmetric group S_s, and $l(\sigma)$ is the length of a permutation σ.

Theorem 1.4. *Every string basis of $\mathcal{A}_r$ contains all non-zero minors $\Delta(I; J)$.*

Theorem 1.4 will be proven in §7.

Now we are able to describe the string bases for $\mathcal{A}_2$ and $\mathcal{A}_3$. A minor $\Delta(I; J)$ will be called *primitive* if $i_1 < j_1$ and $i_k \leq j_{k-1}$ for $k = 2, \ldots, s$. Let $\mathcal{P} = \mathcal{P}_r$ denote the set of all primitive minors of T_r. We recall that two elements $x, y \in \mathcal{A}_r$ *quasicommute* if $xy = q^n yx$ for some integer n. We call a product $b_1 b_2 \cdots b_k$ *quasicommutative* if every b_i and b_j quasicommute (some of the elements b_i may coincide).

Theorem 1.5. *Let $r = 2$ or $r = 3$. Then there is only one string basis B in $\mathcal{A}_r$. Every $b \in B$ has the form $b = q^N b_1 b_2 \cdots b_k$, where $N \in \mathbf{Z}$, and $b_1 \cdots b_k$ is a quasicommutative product of elements of $\mathcal{P}_r$. Conversely, for every quasicommutative product $b_1 \cdots b_k$ of elements of $\mathcal{P}_r$ there is an integer N such that $q^N b_1 b_2 \cdots b_k \in B$.*

The case $r = 2$ will be treated in §5, and the case $r = 3$ in §§9, 10.

Consider the simplicial complex on $\mathcal{P}_r$ whose simplices are all mutually quasicommutative subsets. Theorem 1.5 says that for $r = 2$ or $r = 3$ the structure of the string basis B in $\mathcal{A}_r$ is controlled by this simplicial complex. The set $\mathcal{P}_2$ consists of 4 primitive minors:

$$\Delta(1;2) = t_{12} = x_1, \quad \Delta(2;3) = t_{23} = x_2,$$

$$\Delta(1;3) = t_{13} = [x_1, x_2], \ \Delta(1,2;2,3) = t_{12}t_{23} - qt_{13} = [x_2, x_1]. \tag{1.7}$$

In this case Theorem 1.5 can be refined as follows (for the proof see §5).

Theorem 1.6. *The simplicial complex $\mathcal{P}_2$ has two maximal simplices: $\mathcal{P}_2 \setminus \{x_1\}$ and $\mathcal{P}_2 \setminus \{x_2\}$ (i.e., every two elements of $\mathcal{P}_2$ quasicommute with the only exception of x_1 and x_2). The string basis B of $\mathcal{A}_2$ is given by*

$$B = \{q^{\binom{m_1+m_{12}}{2}+\binom{m_2+m_{21}}{2}} x_1^{m_1} x_2^{m_2} [x_1, x_2]^{m_{21}} [x_2, x_1]^{m_{12}}\},$$

where $(m_1, m_2, m_{12}, m_{21})$ runs over all 4-tuples of nonnegative integers such that $\min(m_1, m_2) = 0$.

The set $\mathcal{P}_3$ consists of 12 primitive minors: $\Delta(1;2), \Delta(2;3), \Delta(3;4), \Delta(1;3), \Delta(2;4)$, $\Delta(1;4), \Delta(1,2;2,3), \Delta(1,2;2,4), \Delta(1,2;3,4), \Delta(1,3;3,4), \Delta(2,3;3,4), \Delta(1,2,3;2,3,4)$. The simplicial complex $\mathcal{P}_3$ has 14 maximal simplices; it will be described in §9.

Now we return to arbitrary $\mathcal{A}_r$, and state our main conjecture.

Conjecture 1.7. *Let B be a string basis in $\mathcal{A}_r$. Two elements $b, b' \in B$ quasicommute if and only if $q^N bb' \in B$ for some integer N.*

Conjecture 1.7 would imply that elements of a string basis B are quasicommutative monomials in some set of generators $\tilde{\mathcal{P}}_r$. Here $\tilde{\mathcal{P}}_r$ consists of all elements $b \in B$ that cannot be decomposed into a quasicommutative product of two elements of smaller degree. It follows easily from Theorem 1.5 that Conjecture 1.7 is true for $r = 2, 3$, and in these cases we have $\tilde{\mathcal{P}}_r = \mathcal{P}_r$. In general, it is not even clear whether $\tilde{\mathcal{P}}_r$ is finite.

2. Strings: main results

Here we introduce the main tool for our study of string bases. Let x be a non-zero homogeneous element of $\mathcal{A}_r$. For each $i = 1, \ldots, r$ we set

$$l_i(x) = \max\{l \in \mathbf{Z}_+ : E_i^l(x) \neq 0\}. \tag{2.1}$$

We shall use the following notation:

$$E_i^{(top)}(x) := E_i^{(l_i(x))}(x). \tag{2.2}$$

Now let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a sequence of indices from $\{1, 2, \ldots, r\}$ such that no two consecutive indices are equal to each other. We call such a sequence *admissible*. We associate to x and $\mathbf{i}$ a nonnegative integer vector $a(\mathbf{i}; x) = (a_1, \ldots, a_m)$ defined by

$$a_k = l_{i_k}(E_{i_{k-1}}^{(top)} E_{i_{k-2}}^{(top)} \cdots E_{i_1}^{(top)}(x)). \tag{2.3}$$

We call $a(\mathbf{i}; x)$ the *string* of x in direction $\mathbf{i}$. We abbreviate

$$E_{\mathbf{i}}^{(top)}(x) = E_{i_m}^{(top)} E_{i_{m-1}}^{(top)} \cdots E_{i_1}^{(top)}(x). \tag{2.4}$$

Note that $E_{\mathbf{i}}^{(top)}(x)$ is a non-zero homogeneous element of $\mathcal{A}_r$ of degree $\deg(x) - \sum_k a_k \alpha_{i_k}$.

¿From now on we fix a string basis B in $\mathcal{A}_r$. By the string axiom (S3), if $x \in B$ then $E_{\mathbf{i}}^{(top)}(x) \in B$ for all $\mathbf{i}$.

Theorem 2.1. *For every admissible sequence* $\mathbf{i}$ *an element* $b \in B$ *is uniquely determined by the string* $a(\mathbf{i}; b)$ *and the element* $E_{\mathbf{i}}^{(top)}(b)$.

Theorem 2.1 will be proven in §4.

Let $W = S_{r+1}$ be the Weyl group of type A_r. For each $w \in W$ we denote by $R(w)$ the set of all reduced decompositions of w, i.e., the set of sequences $\mathbf{i} = (i_1, i_2, \ldots, i_l)$ such that $l = l(w)$, and w is equal to the product of simple reflections $s_{i_1} s_{i_2} \cdots s_{i_l}$.

Theorem 2.2. *Let* $w \in W$, *and* $\mathbf{i}, \mathbf{i}' \in R(w)$. *Then* $E_{\mathbf{i}}^{(top)}(b) = E_{\mathbf{i}'}^{(top)}(b)$ *for* $b \in B$. *Furthermore, there is a piecewise-linear automorphism* ${}_{\mathbf{i}'}T_{\mathbf{i}} : \mathbf{R}^l \to \mathbf{R}^l$ *preserving the lattice* $\mathbf{Z}^l$ *and such that* $a(\mathbf{i}'; b) = {}_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b))$ *for* $b \in B$.

According to Theorem 2.2, if $b \in B, \mathbf{i} \in R(w)$ then the element $E_{\mathbf{i}}^{(top)}(b) \in B$ depends only on b and w, so we shall denote it simply by $E_w^{(top)}(b)$. The most important for us will be the case when $w = w_0$, the element of maximal length in W.

Theorem 2.3. *We have* $E_{w_0}^{(top)}(b) = 1$ *for all* $b \in B$.

Theorem 2.4. *For every* $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in R(w_0)$ *the correspondence* $b \mapsto a(\mathbf{i}; b)$ *is a bijection between* B *and the semigroup* $C_{\mathbf{Z}}(\mathbf{i})$ *of all integral points of some polyhedral convex cone* $C(\mathbf{i}) \subset \mathbf{R}_+^m$.

One easily checks that the number $m = l(w_0)$ is equal to $\binom{r+1}{2}$, and that the sequence $\mathbf{i}(1) = (1,\, 2, 1,\, 3, 2, 1, \ldots, r, r-1, \ldots, 1)$ belongs to $R(w_0)$.

Theorem 2.5. *The cone* $C(\mathbf{i}(1))$ *is a simplicial cone in* $\mathbf{R}_+^m$ *defined by the inequalities* $a_k \leq a_{k-1}$ *whenever* $\mathbf{i}(1)_k < \mathbf{i}(1)_{k-1}$.

The description of the cones $C(\mathbf{i})$ for other reduced decompositions of w_0 is more complicated. They can be computed by means of the maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ from Theorem 2.2. These maps are closely related to another family of piecewise-linear maps $R_{\mathbf{i}}^{\mathbf{i}'}$ introduced by Lusztig [10, 2.1, 2.6]. An explicit description of the maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ is given as follows. First, they are *local* in the following sense.

Proposition 2.6. *Let* $w \in W$, *and* $\mathbf{i} = (i_1, \ldots, i_l), \mathbf{i}' = (i'_1, \ldots, i'_l) \in R(w)$. *Suppose there are two indices* $s < t$ *such that* $i_k = i'_k$ *for* $k < s$ *or* $k > t$, *and let* $\mathbf{i}_0 = (i_s, \ldots, i_t), \mathbf{i}'_0 = (i'_s, \ldots, i'_t)$. *Then the map* ${}_{\mathbf{i}'}T_{\mathbf{i}}$ *leaves the components* a_k *with* $k < s$ *or* $k > t$ *unchanged, and transforms the vector* $(a_s, \ldots, a_t)$ *according to* ${}_{\mathbf{i}'_0}T_{\mathbf{i}_0}$.

It is well-known (cf. [10, §2]) that any two reduced decompositions of $w \in W$ can be transformed into each other by a sequence of elementary transformations of two kinds:

$$(\mathbf{i_1}, i, j, \mathbf{i_2}) \mapsto (\mathbf{i_1}, j, i, \mathbf{i_2}) \text{ for } |i-j| > 1, \tag{2.5}$$

$$(\mathbf{i_1}, i, j, i, \mathbf{i_2}) \mapsto (\mathbf{i_1}, j, i, j, \mathbf{i_2}) \text{ for } |i-j| = 1, \tag{2.6}$$

Taking into account Proposition 2.6, this allows us to reduce the computation of ${}_{\mathbf{i}'}T_{\mathbf{i}}$ to the following two special cases.

Theorem 2.7.
(a) If $\mathbf{i} = (i,j)$, $\mathbf{i}' = (j,i)$ *with* $|i-j| > 1$ *then*

$${}_{\mathbf{i}'}T_{\mathbf{i}}(a_1, a_2) = (a_2, a_1). \tag{2.7}$$

(b) If $\mathbf{i} = (i,j,i)$, $\mathbf{i}' = (j,i,j)$ *with* $|i-j| = 1$ *then*

$${}_{\mathbf{i}'}T_{\mathbf{i}}(a_1, a_2, a_3) = (\max(a_3, a_2 - a_1), a_1 + a_3, \min(a_1, a_2 - a_3)). \tag{2.8}$$

The proofs of Theorems 2.2 to 2.7 will be completed in §6.

Multiplicative properties of string bases are closely related to geometric properties of the maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$. Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. By an $\mathbf{i}-$*wall* we mean a hyperplane in $\mathbf{R}^m$ given by the equation $a_k - a_{k+1} + a_{k+2} = 0$ for some index k such that $i_k = i_{k+2} = i_{k+1} \pm 1$. Let $C(\mathbf{i})^0$ be the interior of the cone $C(\mathbf{i})$ from Theorem 2.4. We say that a point $a \in C(\mathbf{i})^0$ is $\mathbf{i}-$*regular* if for every $\mathbf{i}' \in R(w_0)$ the point ${}_{\mathbf{i}'}T_{\mathbf{i}}(a)$ does not lie on any $\mathbf{i}'-$wall. We call $\mathbf{i}-$*linearity domains* the closures of connected components of the set of $\mathbf{i}-$regular points. This term is justified by the following.

Proposition 2.8. *Every $\mathbf{i}-$linearity domain is a polyhedral convex cone in $C(\mathbf{i})$. Two points $a, a' \in C(\mathbf{i})$ lie in the same $\mathbf{i}-$linearity domain if and only if*

$$ {}_{\mathbf{i}'}T_{\mathbf{i}}(a + a') = {}_{\mathbf{i}'}T_{\mathbf{i}}(a) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a') $$

for each $\mathbf{i}' \in R(w_0)$.

The following theorem adds some support to Conjecture 1.7.

Theorem 2.9. *Suppose two elements $b, b' \in B$ satisfy at least one of the following two conditions:*

(1) b and b' quasicommute;

(2) $q^N bb' \in B$ for some integer N.

Then for every $\mathbf{i} \in R(w_0)$ the strings $a(\mathbf{i}; b)$ and $a(\mathbf{i}; b')$ belong to the same $\mathbf{i}-$linearity domain.

As a consequence of Theorem 2.9, we obtain the following "first approximation" to Conjecture 1.7.

Theorem 2.10. *Suppose $b, b' \in B$ quasicommute. Then there is exactly one element $b'' \in B$ occurring in the B-decomposition of bb' such that $a(\mathbf{i}; bb') = a(\mathbf{i}; b'')$ for every $\mathbf{i} \in R(w_0)$.*

Proposition 2.8 and Theorems 2.9, 2.10 will be proven in §8.

We describe the $\mathbf{i}(1)-$linearity domains for $r = 2$ in §5, and for $r = 3$ in §9. Using this description we can recover the simplicial complex $\mathcal{P}_r$ for $r = 2, 3$ (see §1) in the following way.

Theorem 2.11. *For $r = 2, 3$ the edges of $\mathbf{i}(1)-$linearity domains are exactly the rays $\mathbf{R}_+ a(\mathbf{i}(1); b)$ for $b \in \mathcal{P}_r$.*

Theorem 2.12. *For $r = 2, 3$ there is a decomposition $C(\mathbf{i}(1)) = \cup_{C \in \mathcal{D}} C$ into the union of simplicial cones, satisfying the following properties:*

(a) Every $\mathbf{i}(1)-$linearity domain is a union of some $C \in \mathcal{D}$.

(b) Each edge of a cone $C \in \mathcal{D}$ is an edge of some $\mathbf{i}(1)-$linearity domain i.e., has the form $\mathbf{R}_+ a(\mathbf{i}(1); b)$ for $b \in \mathcal{P}_r$.

(c) *Two elements $b, b' \in \mathcal{P}_r$ quasicommute if and only if the rays $\mathbf{R}_+ a(\mathbf{i}(1); b)$ and $\mathbf{R}_+\ a(\mathbf{i}(1); b')$ are edges of the same $C \in \mathcal{D}$.*

We expect some analogs of Theorems 2.11, 2.12 to hold for general r.

3. Bases of PBW type

In this section we collect together some properties of the algebra $\mathcal{A}_r$ which will be used later for the study of string bases. We use freely the notation and terminology introduced above.

We start with a generalization of the Leibnitz formula (1.4). Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence of indices from $\{1, 2, \ldots, r\}$, and $a = (a_1, \ldots, a_m)$ be a nonnegative integral vector of the same length. We define an element $E_{\mathbf{i}}^{(a)} \in U_+$ by the following formula:

$$E_{\mathbf{i}}^{(a)} = E_{i_m}^{(a_m)} E_{i_{m-1}}^{(a_{m-1})} \cdots E_{i_1}^{(a_1)}. \tag{3.1}$$

The element $E_{\mathbf{i}}^{(a)}$ is homogeneous of degree $\sum_{k=1}^m a_k \alpha_{i_k}$.

Proposition 3.1. *Let $x \in \mathcal{A}_r(\gamma), y \in \mathcal{A}_r$. Then*

$$E_{\mathbf{i}}^{(a)}(xy) = \sum_{a', a''} q^{\Phi(a', a'')} E_{\mathbf{i}}^{(a')}(x) E_{\mathbf{i}}^{(a'')}(y), \tag{3.2}$$

where the summation is over all $a', a'' \in \mathbf{Z}_+^m$ with $a' + a'' = a$, and

$$\Phi(a', a'') = \Phi_{\mathbf{i},\gamma}(a', a'') = \sum_{1 \le k \le m} a'_k a''_k + \sum_{1 \le k < l \le m} a'_k a''_l (\alpha_{i_k}, \alpha_{i_l}) - (\gamma, \sum_{1 \le k \le m} a''_k \alpha_{i_k}). \tag{3.3}$$

In particular, for $m = 1$ i.e., $a \in \mathbf{Z}_+$ we have

$$E_i^{(a)}(xy) = \sum_{a'=0}^{a} q^{(a-a')(a'-(\gamma,\alpha_i))} E_i^{(a')}(x) E_i^{(a-a')}(y). \tag{3.4}$$

Proof. We first deduce (3.4) from (1.4) using induction on a and representing $E_i^{(a)}$ as $\frac{1}{[a]} E_i^{(a-1)} E_i$. The formula (3.2) follows from (3.4) by induction on m. ◁

Corollary 3.2. *The algebra $\mathcal{A}_r$ has no zero-divisors.*

Proof. It is enough to show that $xy \neq 0$ for every non-zero homogeneous $x, y \in \mathcal{A}_r$. We prove this by induction on the degree of x (here and in the sequel we use the partial order on the grading semigroup Q_+ given by $\gamma \ge \gamma'$ if $\gamma - \gamma' \in Q_+$). There is nothing to prove if $\deg(x) = 0$. Suppose $\deg(x) > 0$. By Proposition 1.2 (a), $E_i(x) \neq 0$ for some i. Now we apply (3.4) for $a = l_i(x) + l_i(y)$. Then all the summands on the right hand side of (3.4) vanish except the one with $a' = l_i(x)$. By inductive assumption, this remaining summand is non-zero, and we are done. ◁

The same argument as in the proof of Corollary 3.2 implies the following.

Corollary 3.3. *Let x, y be non-zero homogeneous elements of $\mathcal{A}_r$, and* $\deg(x) = \gamma$. *Then for every admissible sequence $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ we have*

$$a(\mathbf{i}; xy) = a(\mathbf{i}; x) + a(\mathbf{i}; y), \tag{3.5}$$

$$E_{\mathbf{i}}^{(top)}(xy) = q^{\Phi_{\mathbf{i},\gamma}(a(\mathbf{i};x), a(\mathbf{i};y))} E_{\mathbf{i}}^{(top)}(x) E_{\mathbf{i}}^{(top)}(y). \tag{3.6}$$

Our next task is to construct a number of bases in $\mathcal{A}_r$ of the Poincaré-Birkhoff-Witt type. Such a construction was developed by Lusztig (cf. [10, Sec.2]). For the convenience of the reader we present an independent and simplified construction in the special cases needed for our purposes.

Fix $m = \binom{r+1}{2}$, and let $\mathcal{R}$ denote the set of all pairs of integers (i, j) such that $1 \leq i < j \leq r+1$. Let $\mathbf{Z}^{\mathcal{R}}$ denote the integer lattice of rank m with coordinates d_{ij}, $(i, j) \in \mathcal{R}$. We choose the following linear order on $\mathcal{R}$: $(i, j) \prec (i', j')$ if $j < j'$ or $j = j', i > i'$. Let $\varphi : \mathcal{R} \to \{1, 2, \ldots, \binom{r+1}{2}\}$ denote the order-preserving bijection; one checks easily that $\varphi(i, j) = \binom{j}{2} - i + 1$. To illustrate the use of φ we notice that the sequence $\mathbf{i}(1)$ from Theorem 2.5 can be defined by $\mathbf{i}(1)_{\varphi(i,j)} = i$. We define the semigroup Γ by

$$\Gamma = \{(a_1, \ldots, a_m) \in \mathbf{Z}_+^m : a_{\varphi(i,j)} \leq a_{\varphi(i',j)} \text{ for } i < i' < j\}. \tag{3.7}$$

(In the notation of §2, $\Gamma = C_{\mathbf{Z}}(\mathbf{i}(1))$ is the semigroup of integral points in the cone $C(\mathbf{i}(1))$.)

In §1 we associated to each $(i, j) \in \mathcal{R}$ a homogeneous element $t_{ij} \in \mathcal{A}_r$ of degree $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$. For every $d = (d_{ij}) \in \mathbf{Z}_+^{\mathcal{R}}$ let t^d denote the monomial $t^d = \prod_{(i,j)\in\mathcal{R}} t_{ij}^{d_{ij}}$, the product taken in the linear order just introduced. We set $\psi(d) := a(\mathbf{i}(1); t^d)$.

Proposition 3.4.

(a) The correspondence $d \mapsto \psi(d)$ is a semigroup isomorphism between $\mathbf{Z}_+^{\mathcal{R}}$ and Γ.

(b) For every $d \in \mathbf{Z}_+^{\mathcal{R}}$ the element $E_{\mathbf{i}(1)}^{(top)}(t^d)$ has the form $q^N \in \mathcal{A}_r(0) = \mathbf{Q}(q)$ for some $N \in \mathbf{Z}$.

Proof. (a). According to (3.5), ψ is a linear map. Therefore, to compute it we have only to compute the string $a(\mathbf{i}(1); t_{ij})$ for every $(i, j) \in \mathcal{R}$.

Lemma 3.5. *Let $(i, j) \in \mathcal{R}$, and $k = 1, \ldots, r$. Then $E_k(t_{ij}) = 0$ unless $k = j - 1$, and $E_{j-1}(t_{ij}) = t_{i,j-1}$.*

Proof of Lemma 3.5. We proceed by induction on $j - i$. If $j - i = 1$ then $t_{ij} = x_{j-1}$, and our statement follows from Proposition 1.1 (c). So we can assume that $j - i \geq 2$, and that our statement is valid for $t_{i,j-1}$. By definition,

$$t_{ij} = [t_{i,j-1}, x_{j-1}] = \frac{t_{i,j-1}x_{j-1} - q^{-1}x_{j-1}t_{i,j-1}}{q - q^{-1}}.$$

Applying E_k and using the inductive assumption and (1.4), we see that $E_k(t_{ij}) = 0$ unless $k = j-1$ or $k = j-2$, and we have

$$E_{j-1}(t_{ij}) = t_{i,j-1},\ E_{j-2}(t_{ij}) = [t_{i,j-2}, x_{j-1}] = \frac{t_{i,j-2}x_{j-1} - x_{j-1}t_{i,j-2}}{q - q^{-1}}.$$

It remains to show that $[t_{i,j-2}, x_{j-1}] = 0$. Since $t_{i,j-2}$ is a polynomial in $x_i, \ldots, x_{j-3}$, it commutes with x_{j-1} in view of (0.1). $\triangleleft$

Using Lemma 3.5 and the definition (3.3), we conclude that

$$a(\mathbf{i}(1); t_{ij}) = \sum_{i'=i}^{j-1} e_{\varphi(i',j)}, \tag{3.8}$$

where e_k stands for the vector $(a_1, \ldots, a_m)$ with $a_i = \delta_{ki}$. Clearly, the vectors of the form (3.8) for all $(i,j) \in \mathcal{R}$ form a $\mathbf{Z}-$basis of the semigroup Γ. This completes the proof of Proposition 3.4 (a).

(b) According to (3.6), it is enough to prove our statement for $t^d = t_{ij}$. But in this case it follows from (3.8) and Lemma (3.5). $\triangleleft$

For every $\gamma \in Q_+$ we set

$$\mathbf{Z}_+^{\mathcal{R}}(\gamma) = \{(d_{ij}) \in \mathbf{Z}_+^{\mathcal{R}} : \sum_{(i,j)\in\mathcal{R}} d_{ij}(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}) = \gamma\}, \tag{3.9}$$

$$\Gamma(\gamma) = \{(a_1, \ldots, a_m) \in \Gamma : \sum_{(i,j)\in\mathcal{R}} a_{\varphi(i,j)}\alpha_i = \gamma\}. \tag{3.10}$$

Corollary 3.6. *Let $\gamma \in Q_+$. Then*
(a) The elements t^d for $d \in \mathbf{Z}_+^{\mathcal{R}}(\gamma)$ form a basis of $\mathcal{A}_r(\gamma)$.
(b) The elements $E_{\mathbf{i}(1)}^{(a)}$ for $a \in \Gamma(\gamma)$ form a basis of $U_+(\gamma)$.

Proof. We prove both statements at the same time. Clearly, $t^d \in \mathcal{A}_r(\gamma)$ for $d \in \mathbf{Z}_+^{\mathcal{R}}(\gamma)$, and $E_{\mathbf{i}(1)}^{(a)} \in U_+(\gamma)$ for $a \in \Gamma(\gamma)$. By definition,

$$\mathrm{card}\,(\mathbf{Z}_+^{\mathcal{R}}(\gamma)) = p(\gamma),$$

the Kostant partition function. By Proposition 3.4 (a) and (3.8), ψ is a bijection between $\mathbf{Z}_+^{\mathcal{R}}(\gamma)$ and $\Gamma(\gamma)$. It follows that

$$\mathrm{card}\,(\Gamma(\gamma)) = p(\gamma).$$

Taking into account (1.3), it remains to show that each of the families $\{t^d : d \in \mathbf{Z}_+^{\mathcal{R}}(\gamma)\}$ and $\{E_{\mathbf{i}(1)}^{(a)} : a \in \Gamma(\gamma)\}$ is linearly independent. It is enough to show that the pairing matrix $(c_{ad} = E_{\mathbf{i}(1)}^{(a)}(t^d))$ is non-degenerate.

Consider the *lexicographic linear ordering* of $\mathbf{Z}^m$ defined as follows: $(a'_1, \ldots, a'_m) \prec (a_1, \ldots, a_m)$ if the first non-zero difference among $a_1 - a'_1, \ldots, a'_m - a_m$ is positive. By definition of the string, we have

$$E^{(a)}_{\mathbf{i}(1)}(t^d) = 0 \text{ for } \psi(d) \prec a. \tag{3.11}$$

Furthermore, if $a = \psi(d)$ then $E^{(a)}_{\mathbf{i}(1)}(t^d) \neq 0$ in view of Proposition 3.4 (b). It follows that if we identify $\mathbf{Z}^{\mathcal{R}}_+(\gamma)$ and $\Gamma(\gamma)$ by means of ψ, and order both sets lexicographically then the pairing matrix (c_{ad}) becomes triangular with non-zero diagonal entries, hence non-degenerate. $\triangleleft$

Remark. The basis in Corollary 3.6 (b) is analogous to the Verma bases constructed in [13].

Now we fix a number $s \in \{1, 2, \ldots, r\}$, and consider the sequence

$$\mathbf{i}(s) = (s, \overline{s-1, s}, \overline{s-2, s}, \ldots, \overline{1, s}, \overline{s+1, 1}, \overline{s+2, 1}, \ldots, \overline{r, 1}), \tag{3.12}$$

where for $i < j$ the symbol $\overline{i, j}$ stands for the sequence $i, i+1, \ldots, j$, and $\overline{j, i}$ stands for the sequence $j, j-1, \ldots, i$. The sequence $\mathbf{i}(1)$ has been introduced earlier. It is easy to see that $\mathbf{i}(s) \in R(w_0)$ for all s. We shall extend Proposition 3.4 and Corollary 3.6 to $\mathbf{i}(s)$.

Let $x \mapsto x^*$ be the $\mathbf{Q}(q)-$linear antiautomorphism of $\mathcal{A}_r$ such that $x_i^* = x_i$ for $i = 1, \ldots, r$ (it is well-defined in view of the defining relations (0.1), (0.2)). Clearly, $x \mapsto x^*$ is a degree-preserving involution of $\mathcal{A}_r$. In particular, for every $(i, j) \in \mathcal{R}$ the element t^*_{ij} is homogeneous of the same degree $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$ as t_{ij}.

Now we define $t(s)_{ij}$ to be t_{ij} for $j > s+1$, and to be t^*_{ij} for $j \leq s+1$. For every $d = (d_{ij}) \in \mathbf{Z}^{\mathcal{R}}_+$ let $t(s)^d$ denote the monomial $t(s)^d = \prod_{(i,j) \in \mathcal{R}} t(s)_{ij}^{d_{ij}}$, the product in the same linear order as above. Finally, let σ_s denote the automorphism of the semigroup $\mathbf{Z}^{\mathcal{R}}_+$ given by $\sigma_s(d)_{ij} = d_{ij}$ for $j > s+1$, and $\sigma_s(d)_{ij} = d_{s+2-j, s+2-i}$ for $j \leq s+1$.

Proposition 3.7.

(a) *For every $d \in \mathbf{Z}^{\mathcal{R}}_+$ we have $a(\mathbf{i}(s); t(s)^d) = \psi(\sigma_s(d))$. Hence the correspondence $d \mapsto a(\mathbf{i}(s); t(s)^d)$ is a semigroup isomorphism between $\mathbf{Z}^{\mathcal{R}}_+$ and Γ.*

(b) *For every $d \in \mathbf{Z}^{\mathcal{R}}_+$ the element $E^{(top)}_{\mathbf{i}(s)}(t(s)^d)$ has the form $q^N \in \mathcal{A}_r(0) = \mathbf{Q}(q)$ for some $N \in \mathbf{Z}$.*

The proof of Proposition 3.7 is totally analogous to that of Proposition 3.4. We have only to replace Lemma 3.5 by the following lemma.

Lemma 3.8. *Let $(i, j) \in \mathcal{R}$, and $k = 1, \ldots, r$. Then $E_k(t^*_{ij}) = 0$ unless $k = i$, and $E_i(t^*_{ij}) = t^*_{i+1,j}$.*

Corollary 3.9. *Let $\gamma \in Q_+$. Then*

(a) *The elements* $t(s)^d$ *for* $d \in \mathbf{Z}_+^{\mathcal{R}}(\gamma)$ *form a basis of* $\mathcal{A}_r(\gamma)$.
(b) *The elements* $E_{\mathbf{i}(s)}^{(a)}$ *for* $a \in \psi\sigma_s\psi^{-1}(\Gamma(\gamma))$ *form a basis of* $U_+(\gamma)$.

This follows from Proposition 3.7 in exactly the same way as Corollary 3.6 from Proposition 3.4.

Let $E \mapsto E^*$ be the $\mathbf{Q}(q)-$linear antiautomorphism of $U_{+,r}$ such that $E_i^* = E_i$ for $i = 1, \ldots, r$ (it is well-defined in view of the defining relations (1.1), (1.2)). This is a degree-preserving involution of $U_{+,r}$.

Proposition 3.10. *Suppose* $E \in U_{+,r}$ *and* $z \in \mathcal{A}_r$ *are homogeneous elements of the same degree. Then* $E(z) = E^*(z^*)$.

Proof. It is enough to consider the case when E is the monomial $E_{\mathbf{i}}^{(a)}$ from (3.1), and $z = xy$ is a product of two elements of smaller degree. By definition, $(E_{\mathbf{i}}^{(a)})^* = E_{\mathbf{i}_{opp}}^{(a_{opp})}$, where $\mathbf{i}_{opp}$ and a_{opp} stand for the sequences $(i_m, \ldots, i_1)$ and $(a_m, \ldots, a_1)$ respectively. Now we expand $E(z) = E_{\mathbf{i}}^{(a)}(xy)$ with the help of (3.2). Since $\deg(E) = \deg(xy)$, for each non-zero summand in (3.2) we have

$$\sum_{1 \le k \le m} a'_k \alpha_{i_k} = \deg(x), \quad \sum_{1 \le k \le m} a''_k \alpha_{i_k} = \deg(y). \tag{3.13}$$

Using induction on degree of E we can assume that

$$E_{\mathbf{i}}^{(a')}(x) = E_{\mathbf{i}_{opp}}^{(a'_{opp})}(x^*), \; E_{\mathbf{i}}^{(a'')}(y) = E_{\mathbf{i}_{opp}}^{(a''_{opp})}(y^*). \tag{3.14}$$

Using (3.3) and (3.13), we check directly that

$$\Phi_{\mathbf{i},\deg(x)}(a', a'') = \Phi_{\mathbf{i}_{opp},\deg(y)}(a''_{opp}, a'_{opp}). \tag{3.15}$$

Substituting the expressions (3.14), (3.15) into (3.2) we conclude that $E_{\mathbf{i}}^{(a)}(xy) = E_{\mathbf{i}_{opp}}^{(a_{opp})}(y^* x^*)$, which proves our statement. $\triangleleft$

We conclude this section with some commutation relations for the elements t_{ij}.

Proposition 3.11.
(a) *If* $i \le j < i' \le j'$ *then* $[t_{ij}, t_{i',j'}] = 0$.
(b) *If* $i < i' \le j' \le j$ *then* $[t_{ij}, t_{i',j'}] = 0$.
(c) *If* $i < i' \le j < j'$ *then* $[t_{ij}, t_{i',j'}] = t_{i,j'} t_{i',j}$.

Proof. Part (a) is clear since t_{ij} is a (non-commutative) polynomial in $x_i, x_{i+1}, \ldots, x_{j-1}$, and $t_{i',j'}$ is a (non-commutative) polynomial in $x_{i'}, x_{i'+1}, \ldots, x_{j'-1}$, and each of $x_i, x_{i+1}, \ldots, x_{j-1}$ commutes with each of $x_{i'}, x_{i'+1}, \ldots, x_{j'-1}$. We prove (b) and (c) simultaneously by induction on $(j - i) + (j' - i')$. Part (b) is evident if $i' = j'$, so we can assume that $i' < j'$.

First consider the case $j' < j$ in (b). By Proposition 1.2 (a), it is enough to show that $E_k([t_{ij}, t_{i',j'}]) = 0$ for all k. According to (1.4) and Lemma 3.5, we have only to consider $k = j' - 1, j - 1$, and in these cases we have

$$E_{j'-1}([t_{ij}, t_{i',j'}]) = [t_{ij}, t_{i',j'-1}],\ E_{j-1}([t_{ij}, t_{i',j'}]) = [t_{i,j-1}, t_{i',j'}],$$

both expressions being 0 by the inductive assumption.

In the remaining case $i < i' < j' = j$ in (b) we proceed in the same way. Now the only non-trivial thing is to show that $E_{j-1}([t_{ij}, t_{i',j}]) = 0$. By a straightforward calculation using (1.4), Lemma 3.5, and the inductive assumption that $t_{ij}t_{i',j-1} = t_{i',j-1}t_{ij}$, we obtain

$$E_{j-1}([t_{ij}, t_{i',j}]) = [t_{i,j-1}, t_{i',j}] - t_{ij}t_{i',j-1}.$$

But the last expression is 0 by the inductive assumption in part (c).

To prove (c) we start with the case $i < i' = j < j'$, when it becomes the identity

$$[t_{ij}, t_{j,j'}] = t_{i,j'} \text{ for } i < j < j'. \tag{3.16}$$

If $j = j' - 1$ then (3.16) is the definition of $t_{i,j'}$, so we can assume that $j < j' - 1$. Again we apply all E_k to both sides of (3.16). It suffices to consider $k = j' - 1, j - 1$. We have

$$E_{j'-1}([t_{ij}, t_{j,j'}]) = [t_{ij}, t_{j,j'-1}] = t_{i,j'-1} = E_{j'-1}(t_{i,j'}),$$

$$E_{j-1}([t_{ij}, t_{j,j'}]) = [t_{i,j-1}, t_{j,j'}] = 0 = E_{j-1}(t_{i,j'})$$

(in the first case we use the inductive assumption, and in the second the part (a) which is already proven).

It remains to consider the case $i < i' < j < j'$. As before, it is enough to check that both parts in (c) give the same result under the action of E_{j-1} and $E_{j'-1}$. Applying E_{j-1} we get

$$E_{j-1}([t_{ij}, t_{i',j'}]) = [t_{i,j-1}, t_{i',j'}] = t_{i,j'}t_{i',j-1} = E_{j-1}(t_{i,j'}t_{i',j}),$$

as required. Applying $E_{j'-1}$ we have to distinguish two cases: $j < j' - 1$, and $j = j' - 1$. If $j < j' - 1$ then we get

$$E_{j'-1}([t_{ij}, t_{i',j'}]) = [t_{i,j}, t_{i',j'-1}] = t_{i,j'-1}t_{i',j} = E_{j'-1}(t_{i,j'}t_{i',j}),$$

as required. Finally, for $j = j' - 1$ we use the following identity (the proof is straightforward):

$$E_j([t_{ij}, t_{i',j+1}]) = t_{ij}t_{i',j} + q^{-1}[t_{ij}, t_{i',j}]. \tag{3.17}$$

By the inductive assumption in (b), the second summand in (3.17) is 0, and we are done. $\triangleleft$

4. String parametrizations of string bases

From now on we fix a string basis B in $\mathcal{A}_r$. In this section we show that for every $s = 1, \ldots, r$ the strings in direction $\mathbf{i}(s)$ provide a parametrization of B. This allows us to prove Theorem 2.1 and Theorem 2.4 for $\mathbf{i} = \mathbf{i}(s)$.

We shall use the following notation: for $x \in \mathcal{A}_r, b \in B$ let $[x : b]$ denote the coefficient of b in the $B-$expansion of x. We say that b is a *constituent* of x if $[x : b] \neq 0$. Recall that $[B]^+$ stands for the set of all $x \in \mathcal{A}_r$ such that $[x : b] \in \mathbf{Z}_+[q, q^{-1}]$ for all $b \in B$. The following statement is an immediate consequence of the definition of strings and string axioms (S2) and (S3).

Proposition 4.1. *Suppose $x \in [B]^+$ is non-zero, and $\mathbf{i}$ is an admissible sequence. Then for every constituent b of x the string $a(\mathbf{i}; b)$ either is equal to $a(\mathbf{i}; x)$ or precedes $a(\mathbf{i}; x)$ in the lexicographic order. There is at least one constituent b of x such that $a(\mathbf{i}; b) = a(\mathbf{i}; x)$; for every such b the element $E_{\mathbf{i}}^{(top)}(b)$ is a constituent of $E_{\mathbf{i}}^{(top)}(x)$.*

Recall from §3 that to each pair of indices $(i, j) \in \mathcal{R}$ (i.e., such that $1 \leq i < j \leq r + 1$) there are associated two homogeneous elements $t_{ij}, t_{ij}^* \in \mathcal{A}_r$ of the same degree $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$.

Proposition 4.2. *The elements t_{ij}, t_{ij}^* belong to B for all $(i, j) \in \mathcal{R}$.*

Proof. We deal only with the elements t_{ij}, the case of t_{ij}^* being totally similar. We proceed by induction on $j - i$. In view of (S0) and (S3), $t_{i,i+1} = x_i \in B$ because $E_i^2(x_i) = 0$ and $E_i(x_i) = 1 \in B$. Now let $j \geq i + 1$ and assume that $t_{i,j'} \in B$ for all $j' = i + 1, \ldots, j$. We have to show that $t_{i,j+1} \in B$. Consider the product $x_j t_{ij}$. By (S1) and the inductive assumption, $x_j t_{ij} \in [B]^+$. Using (1.4) and Lemma 3.5, we see that $E_k(x_j t_{ij}) = 0$ unless $k = j - 1, j$, and

$$E_{j-1}(x_j t_{ij}) = q x_j t_{i,j-1},\ E_j(x_j t_{ij}) = t_{ij}.$$

Furthermore, $E_j^2(x_j t_{ij}) = 0$, hence $E_j^{(top)}(x_j t_{ij}) = t_{ij}$. By Proposition 4.1, $x_j t_{ij}$ has a constituent $b \in B$ such that $E_j(b) = t_{ij}$. Clearly, $[x_j t_{ij} : b] = 1$. It remains to show that $b = t_{i,j+1}$.

By Proposition 1.2 (a), it is enough to prove that $E_k(b) = E_k(t_{i,j+1})$ for all k. By Lemma 3.5 and our choice of b, we have $E_j(b) = E_j(t_{i,j+1}) = t_{ij}$, and $E_k(b) = E_k(t_{i,j+1}) = 0$ for $k \neq j-1, j$. It remains to show that $E_{j-1}(b) = 0$. Suppose this is not so i.e., $E_{j-1}(b) = b' \neq 0$. Since $E_{j-1}^2(x_j t_{ij}) = 0$, it follows that $E_{j-1}^2(b) = 0$ hence $b' = E_{j-1}^{(top)}(b) \in B$. Therefore, b' is a constituent of $E_{j-1}(x_j t_{ij}) = q x_j t_{i,j-1}$, and $[q x_j t_{i,j-1} : b'] = 1$. We get a contradiction by showing that $x_j t_{i,j-1} \in B$.

Let $\gamma = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-2} + \alpha_j$, so that $x_j t_{i,j-1} \in \mathcal{A}_r(\gamma)$. Since x_j commutes with each of $x_i, \ldots, x_{j-2}$ it follows that every element of $\mathcal{A}_r(\gamma)$ has the form $x_i y$ for some $y \in \mathcal{A}_r(\gamma - \alpha_j)$. By (S1) and the inductive assumption, $x_j t_{i,j-1} \in [B]^+$. Let $x_j y \in B$ be a constituent of $x_j t_{i,j-1}$. We have $E_j(x_j y) = y, E_j^2(x_j y) = 0$, hence $y \in B$ in view of (S3). On the other hand, by Proposition 4.1, y is a constituent of $t_{i,j-1}$. Hence, $y = t_{i,j-1}$, and we are done. $\triangleleft$

For $s = 1, \ldots, r$ let $\mathbf{i}(s)$ be the reduced decomposition of w_0 defined by (3.12). Our next result is a special case of Theorems 2.3 and 2.4.

Proposition 4.3.
(a) *The correspondence $b \mapsto a(\mathbf{i}(s); b)$ is a bijection between B and Γ.*
(b) *We have $E_{\mathbf{i}(s)}^{(top)}(b) = 1$ for all $b \in B$.*

Proof. Consider the basis in $\mathcal{A}_r$ formed by monomials $t(s)^d$ for $d \in \mathbf{Z}_+^{\mathcal{R}}$ (see Corollary 3.9 (a)). By Proposition 4.2 and the string axiom (S1), all these monomials belong to $[B]^+$. By Proposition 4.1, every monomial $t(s)^d$ has a constituent $b = b(s,d) \in B$ such that $a(\mathbf{i}(s); b) = a(\mathbf{i}; t(s)^d)$. In view of Proposition 3.7 (a), all elements $b(s,d)$ are distinct, hence they exhaust B. Now both statements of Proposition 4.3 follow from the corresponding statements of Proposition 3.7. $\triangleleft$

Proof of Theorem 2.1. Using induction on the length of an admissible sequence $\mathbf{i}$ we can assume that $\mathbf{i}$ has length 1 i.e., consists of one index s. Suppose $b, b' \in B$ are such that $l_s(b) = l_s(b')$, and $E_s^{(top)}(b) = E_s^{(top)}(b')$. Since the sequence $\mathbf{i}(s)$ has the first term s, it follows that $a(\mathbf{i}(s); b) = a(\mathbf{i}(s); b')$. Hence by Proposition 4.3 (a), $b = b'$, as required. $\triangleleft$

The above proof actually establishes the following.

Proposition 4.4. *If $b, b' \in B$ and $E_s^n(b) = E_s^n(b') \neq 0$ for some $s = 1, \ldots, r,\ n \geq 0$ then $b = b'$.*

We conclude this section with two helpful corollaries of Theorem 2.1.

Proposition 4.5. *Let x be a non-zero homogeneous element of $\mathcal{A}_r$. Then $l_s(b) \leq l_s(x)$ for each constituent b of x and each $s = 1, \ldots, r$.*

Proof. Fix s and let l be the maximal possible value of $l_s(b)$ for a constituent b of x. Let $b_1, \ldots, b_k$ be all the constituents of x with $l_s(b) = l$. By Theorem 2.1, $E_s^{(l)}(b_1), \ldots, E_s^{(l)}(b_k)$ are distinct elements of B, hence are linearly independent. But $E_s^{(l)}(x)$ is their non-trivial linear combination, hence $E_s^{(l)}(x) \neq 0$, and so $l \leq l_s(x)$, as required. $\triangleleft$

For every $\gamma \in Q_+$, $\nu = (n_1, \ldots, n_r) \in \mathbf{Z}_+^r$ we denote

$$\mathcal{A}_r(\gamma; \nu) = \{x \in \mathcal{A}_r(\gamma) : l_s(x) \leq n_s \text{ for } s = 1, \ldots, r\}. \tag{4.1.}$$

Proposition 4.5 can be reformulated as follows.

Proposition 4.6. *Every subspace of the type $\mathcal{A}_r(\gamma;\nu)$ is spanned by a subset of B.*

Remarks. (a) Using Proposition 4.5, one can show that Proposition 4.1 remains true even without the assumption that $x \in [B]^+$.

(b) Let $\mathcal{A}_r^\nu = \oplus_\gamma \mathcal{A}_r(\gamma;\nu)$. One can show that for every ν the subspace $\mathcal{A}_r^\nu \subset \mathcal{A}_r$ is invariant under the action of all E_i. Moreover, the action of U_+ on $\mathcal{A}_r^\nu$ extends to an action of the whole algebra $U_q(sl_{r+1})$ so that $\mathcal{A}_r^\nu$ becomes an irreducible $U_q(sl_{r+1})-$module with lowest weight $-\nu$ (cf. [10, Sec. 8]). We see that each string basis B gives rise to a basis in every irreducible finite-dimensional $U_q(sl_{r+1})-$module. Specializing $q=1$ we obtain a basis in every irreducible finite-dimensional $sl_{r+1}-$module. If B is dual to the Lusztig's canonical basis then the bases in irreducible finite-dimensional $sl_{r+1}-$modules constructed in this way are *good* (see Introduction).

5. The string basis for sl_3

In this section we prove Theorems 1.5, 1.6, and all theorems in §2 in the case $r=2$, i.e., for the algebra $\mathcal{A}_2$ having only two generators x_1 and x_2.

Proof of Theorems 1.5, 1.6. The general definition of the elements t_{ij}, t^*_{ij} given above takes the form

$$t_{13} = [x_1, x_2],\ \ t^*_{13} = [x_2, x_1] \tag{5.1}$$

(recall that [x,y] stands for the $q-$commutator (1.5)). Comparing (5.1) with (1.7), we see that $\mathcal{P}_2 = \{x_1, x_2,\ t_{13}, t^*_{13}\}$. One easily checks that the defining relations (0.2) can be rewritten as

$$[x_1, t_{13}] = [t_{13}, x_2] = 0, \tag{5.2}$$

or, equivalently as

$$[x_2, t^*_{13}] = [t^*_{13}, x_1] = 0. \tag{5.3}$$

This implies the first statement in Theorem 1.6: every two elements of $\mathcal{P}_2$ quasicommute with the only exception of x_1 and x_2.

As in §4, assume that we are given a string basis B in $\mathcal{A}_2$. Let M denote the set of all 4-tuples $m = (m_1, m_2, m_{12}, m_{21})$ of nonnegative integers such that $\min(m_1, m_2) = 0$. For every $m \in M$ we set

$$b(m) = q^{\binom{m_1+m_{12}}{2} + \binom{m_2+m_{21}}{2}} x_1^{m_1} x_2^{m_2} t_{13}^{m_{21}} (t^*_{13})^{m_{12}}, \tag{5.4}$$

and temporarily denote by B_0 the set of all monomials $b(m)$ for $m \in M$. To complete the proofs of Theorems 1.5 and 1.6 we have to show that $B = B_0$.

First of all, $\mathcal{P}_2 \subset B$ by Proposition 4.2. By (S1), we have $B_0 \subset [B]^+$.

Now let $\mathbf{i} = (1,2,1)$, $\mathbf{i}' = (2,1,2)$ be two reduced decompositions of $w_0 \in W = S_3$ (in the notation of previous sections, $\mathbf{i} = \mathbf{i}(1), \mathbf{i}' = \mathbf{i}(2)$). The following formulas are proven by a straightforward computation using Proposition 3.1, (3.5), and Lemmas 3.5 and 3.8. For $m = (m_1, m_2, m_{21}, m_{12}) \in M$ we have

$$a(\mathbf{i}; b(m)) = (m_1 + m_{12}, m_2 + m_{12} + m_{21}, m_{21}),$$

$$a(\mathbf{i}'; b(m)) = (m_2 + m_{21}, m_1 + m_{12} + m_{21}, m_{12}), \tag{5.5}$$

$$E_{\mathbf{i}}^{(top)}(b(m)) = E_{\mathbf{i}'}^{(top)}(b(m)) = 1. \tag{5.6}$$

In particular,

$$l_1(b(m)) = m_1 + m_{12},\ l_2(b(m)) = m_2 + m_{21}. \tag{5.7}$$

We abbreviate $\eta(m) = a(\mathbf{i}; b(m)), \eta'(m) = a(\mathbf{i}'; b(m))$. Recall that $\Gamma = \{(a_1, a_2, a_3) \in \mathbf{Z}_+^3 : a_2 \geq a_3\}$.

Proposition 5.1. *Each of the maps η and η' is a bijection between M and Γ.*

The proof follows directly from (5.5): the inverse bijections are given by

$$\eta^{-1}(a_1, a_2, a_3) = (\max(0, a_1 + a_3 - a_2), \max(0, a_2 - a_1 - a_3), \min(a_1, a_2 - a_3), a_3),$$

$$\eta'^{-1}(a_1, a_2, a_3) = (\max(0, a_2 - a_1 - a_3), \max(0, a_1 + a_3 - a_2), a_3, \min(a_1, a_2 - a_3)). \tag{5.8}$$

For $a \in \Gamma$ we denote by $b(1; a)$ and $b(2; a)$ the elements from B such that $a(\mathbf{i}; b(1; a)) = a(\mathbf{i}'; b(2; a)) = a$ (see Proposition 4.3 (a)). Taking into account Proposition 4.1, we obtain the following.

Corollary 5.2. *The monomials $b(m)$ for $m \in M$ form a basis in $\mathcal{A}_2$, and we have*

$$[b(m) : b(1; \eta(m))] = [b(m) : b(2; \eta'(m))] = 1 \tag{5.9}$$

for all $m \in M$.

To show that $B_0 = B$ it remains to prove that all monomials $b(m)$ belong to B. In view of Proposition 4.6 and (5.6), this is a consequence of the following.

Proposition 5.3. *Let $m \in M$, and let $\gamma \in Q_+$ be the degree of $b(m)$, and $\nu = (l_1(b(m)), l_2(b(m)))$. Then $\mathcal{A}_2(\gamma; \nu)$ is a one-dimensional space spanned by $b(m)$.*

Proof. Fix $\gamma = g_1\alpha_1 + g_2\alpha_2$. Without loss of generality we can assume that $g_1 \leq g_2$. Then the condition $b(m) \in \mathcal{A}_2(\gamma)$ means that $m_1 = 0$ and

$$m_{12} + m_{21} = g_1,\ m_2 + m_{12} + m_{21} = g_2. \tag{5.10}$$

It follows that the elements m such that $b(m) \in \mathcal{A}_2(\gamma)$ are exactly those of the form $m = (0, g_2 - g_1, n, g_1 - n)$ for $n = 0, 1, \ldots, g_1$; we abbreviate $b(0, g_2 - g_1, n, g_1 - n)$ as b_n. By (5.7),

$$l_1(b_n) = n,\ l_2(b_n) = g_2 - n. \tag{5.11}$$

Now choose $\nu = (n, g_2 - n)$ and suppose that $x \in \mathcal{A}_2(\gamma; \nu)$. Then the condition $l_1(x) \leq n$ implies that x is a linear combination of elements $b_k \in \mathcal{A}_2(\gamma)$ with $k \leq n$. Likewise, the condition $l_2(x) \leq g_2 - n$ implies that x is a linear combination of elements $b_l \in \mathcal{A}_2(\gamma)$ with $l \geq n$. It follows that x is proportional to b_n, and we are done. $\triangleleft$

Theorems 1.5 and 1.6 are proven.

Proposition 5.4. *Theorems 2.2 to 2.12 are valid for $\mathcal{A}_2$.*

Proof. The first statement in Theorem 2.2, and Theorem 2.3 follow from (5.6). Theorems 2.4 and 2.5 follow from Proposition 5.1. To establish existence of the map ${}_{\mathbf{i}'}T_{\mathbf{i}} : \mathbf{R}^3 \to \mathbf{R}^3$ in Theorem 2.2, we notice that according to (5.9), $b(m) = b(1; \eta(m)) = b(2; \eta'(m))$ for all $m \in M$. It follows that the map ${}_{\mathbf{i}'}T_{\mathbf{i}}$ exists and is equal to $\eta'\eta^{-1}$. Computing this composition with the help of (5.5) and (5.8), we obtain (2.8). As for Proposition 2.6 and (2.7), there is nothing to prove in our case.

In our situation the cone $C = C(\mathbf{i}(1))$ has the form

$$C = \{(a_1, a_2, a_3) \in \mathbf{R}^3 : a_1 \geq 0, a_2 \geq a_3 \geq 0\}. \tag{5.12}$$

By definition (see §2), there are two $\mathbf{i}(1)-$linearity domains given by

$$C_+ = \{(a_1, a_2, a_3) \in C : a_2 \geq a_1 + a_3\},\ C_- = \{(a_1, a_2, a_3) \in C : a_2 \leq a_1 + a_3\}. \tag{5.13}$$

Since the only non-trivial transformation ${}_{\mathbf{i}'}T_{\mathbf{i}}$ is given by (2.8), Proposition 2.8 becomes clear (we have only to check that if $a = (a_1, a_2, a_3) \in C_+^0, a' = (a_1', a_2', a_3') \in C_-^0$ then ${}_{\mathbf{i}'}T_{\mathbf{i}}(a + a') \neq {}_{\mathbf{i}'}T_{\mathbf{i}}(a) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a')$.)

Each of the cones C, C_+, C_- is simplicial: C is generated by $e_1, e_2, e_2 + e_3$, C_+ is generated by $e_2, e_1 + e_2, e_2 + e_3$, and C_- is generated by $e_1, e_1 + e_2, e_2 + e_3$, where e_1, e_2, e_3 is the standard basis in $\mathbf{R}^3$. On the other hand, by (5.5) we have

$$a(\mathbf{i}; x_1) = e_1,\ a(\mathbf{i}; x_2) = e_2,\ a(\mathbf{i}; t_{13}) = e_2 + e_3,\ a(\mathbf{i}; t_{13}^*) = e_1 + e_2.$$

Theorems 2.9, 2.11 and 2.12 follow by inspection (the decomposition in Theorem 2.12 is simply $C = C_+ \cup C_-$). Finally, we do not have to bother about Theorem 2.10, because the above results establish the stronger Conjecture 1.7 (for $r = 2$). $\triangleleft$

We conclude this section with some corollaries on the structure of the dual algebra U_+. We retain the notation of the proof of Proposition 5.3, i.e., fix $\gamma = g_1\alpha_1 + g_2\alpha_2$ with $g_1 \leq g_2$, and abbreviate $b_n = b(0, g_2 - g_1, n, g_1 - n)$. For $n = 0, \ldots, g_1$ we set

$$u_n = E_1^{(g_1-n)} E_2^{(g_2)} E_1^{(n)},\ u'_n = E_2^{(n)} E_1^{(g_1)} E_2^{(g_2-n)}. \tag{5.14}$$

Proposition 5.5.

(a) The elements u_n form a basis in $U_+(\gamma)$ dual to the basis $\{b_n : 0 \leq n \leq g_1\}$ i.e., $u_n(b_k) = \delta_{kn}$.

(b) The transition matrix between (u_n) and (u'_n) is unitriangular i.e.,

$$u_n = u'_n + \text{ linear combination of } u'_0, \ldots, u'_{n-1},$$

$$u'_n = u_n + \text{ linear combination of } u_0, \ldots, u_{n-1}.$$

Proof. Part (a) is proven by a straightforward computation. In view of (a), to prove (b) it is enough to show that $u'_n(b_k) = 0$ for $k > n$, and $u'_n(b_n) = 1$, which is also quite straightforward (we can actually spare some of the calculations by observing that ${}_{\mathbf{i}'}T_{\mathbf{i}}(n, g_2, g_1 - n) = (g_2 - n, g_1, n)$). $\triangleleft$

Recalling Proposition 1.3, we see that Proposition 5.5 (a) recovers the Lusztig's result that the canonical basis in $U_+(\gamma)$ consists of the elements u_n (cf. [10, 3.4]).

6. Proofs of Theorems 2.2 to 2.7: general case

Proofs of Theorem 2.2, Proposition 2.6, and Theorem 2.7. We prove all these statements simultaneously. The argument is divided into several steps.

Step 1. Proof of Theorem 2.2 in the case of Theorem 2.7 (a). Let $\mathbf{i} = (i, j)$, $\mathbf{i}' = (j, i)$ with $|i - j| > 1$. Let $b \in B$, and suppose $a(\mathbf{i}; b) = (a_1, a_2)$, $E_{\mathbf{i}}^{(top)}(b) = b_0$. We have to show that $a(\mathbf{i}'; b) = (a_2, a_1)$, $E_{\mathbf{i}'}^{(top)}(b) = b_0$. Since E_i and E_j commute, it suffices to show that

$$a_1 = l_i(b),\ a_2 = l_j(b). \tag{6.1}$$

The equality $a_1 = l_i(b)$ is just the definition (2.3), and the inequality $a_2 \leq l_j(b)$ is clear since

$$b_0 = E_j^{(a_2)} E_i^{(a_1)}(b) = E_i^{(a_1)} E_j^{(a_2)}(b) \neq 0.$$

To prove the reverse inequality $a_2 \geq l_j(b)$ we first show that

$$E_i(b_0) = E_j(b_0) = 0. \tag{6.2}$$

Indeed, $E_j E_j^{(a_2)} E_i^{(a_1)}$ is proportional to $E_j^{(a_2+1)} E_i^{(a_1)}$, and $E_i E_j^{(a_2)} E_i^{(a_1)}$ is proportional to $E_j^{(a_2)} E_i^{(a_1+1)}$ since E_i and E_j commute. Hence (6.2) follows from the definition (2.4). Now consider the element $x = x_i^{a_1} x_j^{a_2} b_0$. By (S1) and (S3), $x \in [B]^+$. Using (1.4) and (6.2), we see that $a(\mathbf{i}; x) = (a_1, a_2)$, and $E_{\mathbf{i}}^{(top)}(x) = b_0$. Using Proposition 4.1 and Theorem 2.1, we conclude that b is a constituent of x. Hence $l_j(b) \leq l_j(x) = a_2$, which completes the proof of (6.1). Theorem 2.2 for $w = s_i s_j$, and the formula (2.7) are proven.

Step 2. Proof of Theorem 2.2 in the case of Theorem 2.7 (b). Let $\mathbf{i} = (i, j, i)$, $\mathbf{i}' = (j, i, j)$ with $|i - j| = 1$. Let $b \in B$, and suppose $a(\mathbf{i}; b) = (a_1, a_2, a_3)$. We have to show that $a(\mathbf{i}'; b)$ is given by (2.8), and $E_{\mathbf{i}'}^{(top)}(b) = E_{\mathbf{i}}^{(top)}(b)$.

Without loss of generality we can assume that $i = s, j = s - 1$ i.e., $\mathbf{i}$ is the initial segment of the sequence $\mathbf{i}(s)$ (see (3.12)). It follows that the string $a := a(\mathbf{i}(s); b)$ begins with a_1, a_2, a_3. In particular, by Proposition 4.3, $a_2 \geq a_3$. Consider the string $a^{(0)}$ obtained from a by replacing the initial segment (a_1, a_2, a_3) with $(0, 0, 0)$. Clearly, $a^{(0)} \in \Gamma$, hence by Proposition 4.3, there is a unique $b_0 \in B$ such that $a(\mathbf{i}(s); b_0) = a^{(0)}$. It follows that $E_{s-1}(b_0) = E_s(b_0) = 0$. We claim that

$$E_{\mathbf{i}}^{(top)}(b) = b_0. \tag{6.3}$$

Let $b_1 \in B$ be the element with $a(\mathbf{i}(s); b_1) = a - a^{(0)} = (a_1, a_2, a_3, 0, \ldots, 0)$. By (S3), $b_1 b_0 \in [B]^+$. Using (3.5) and (3.6), we see that $a(\mathbf{i}(s); b_1 b_0) = a$, and $E_{\mathbf{i}}^{(top)}(b_1 b_0) = b_0$. By Propositions 4.1 and 4.3, b is a constituent of $b_1 b_0$. Hence $E_{\mathbf{i}}^{(top)}(b)$ is a constituent of $E_{\mathbf{i}}^{(top)}(b_1 b_0) = b_0$. But $E_{\mathbf{i}}^{(top)}(b) \in B$ according to (S3). This proves (6.3).

The degree of b_1 is equal to $\gamma = a_2 \alpha_{s-1} + (a_1 + a_3)\alpha_s$. Hence b_1 belongs to the subalgebra of $\mathcal{A}_r$ generated by x_{s-1}, x_s, and we can apply to it the results of §5. In particular, we see that $a(\mathbf{i}'; b_1) = (a'_1, a'_2, a'_3)$ is given by (2.8), and $E_{\mathbf{i}'}^{(top)}(b_1) = 1$. Since $E_{s-1}(b_0) = E_s(b_0) = 0$, it follows that $a(\mathbf{i}'; b_1 b_0) = (a'_1, a'_2, a'_3)$ and $E_{\mathbf{i}'}^{(top)}(b_1 b_0) = b_0$. Since b is a constituent of $b_1 b_0$, we have $l_{s-1}(b) \leq l_{s-1}(b_1 b_0) = a'_1$. By Proposition 4.1, the string $a(\mathbf{i}'; b)$ is either equal to (a'_1, a'_2, a'_3) (and in this case $E_{\mathbf{i}'}^{(top)}(b) = b_0$) or precedes (a'_1, a'_2, a'_3) in the lexicographic order. It remains only to exclude the latter possibility.

By Proposition 5.5, the element $E_{\mathbf{i}}^{(a_1, a_2, a_3)} \in U_+(\gamma)$ has the form $E_{\mathbf{i}'}^{(a'_1, a'_2, a'_3)} +$ (linear combination of other elements $E_{\mathbf{i}'}^{(a''_1, a''_2, a''_3)}$ of the same degree). If $a_2 \geq a_1 + a_3$ then by Proposition 5.5 (b), all the terms $E_{\mathbf{i}'}^{(a''_1, a''_2, a''_3)}$ appearing in this decomposition have $a''_1 > a'_1$, hence vanish at b. If $a_2 \leq a_1 + a_3$ then by Proposition 5.5 (a), all the terms $E_{\mathbf{i}'}^{(a''_1, a''_2, a''_3)}$ vanish at b_1, hence vanish at $b_1 b_0$, hence vanish at b too. We conclude that in both cases

$$E_{\mathbf{i}}^{(a_1, a_2, a_3)}(b) = E_{\mathbf{i}'}^{(a'_1, a'_2, a'_3)}(b).$$

Theorem 2.2 for $w = s_i s_j s_i$, and the formula (2.8) are proven.

Step 3. End of the proof. The results of two previous steps imply Theorem 2.2 and Proposition 2.6 in the case when the transformation $\mathbf{i} \mapsto \mathbf{i}'$ is given by (2.5) or (2.6). Since any two reduced decompositions of the same element $w \in W$ can be transformed into each other by a sequence of these elementary transformations, we conclude that Theorem 2.2 is true in general. Once we know that the maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ exist, Proposition 2.6 becomes an immediate consequence of the definitions (2.3), (2.4), and the first statement in Theorem 2.2. $\triangleleft$

Proof of Theorem 2.3. Taking into account Proposition 1.2 (a), we see that Theorem 2.3 is a consequence of the following.

Proposition 6.1. *Let $w \in W, b \in B$. Then $E_i(E_w^{(top)}(b)) = 0$ for all $i = 1, \ldots, r$ such that $l(ws_i) < l(w)$.*

Proof. Clearly, the condition $l(ws_i) < l(w)$ means that there is a reduced decomposition $\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$ with $i_l = i$. Hence $E_i(E_w^{(top)}(b)) = E_i(E_{\mathbf{i}}^{(top)}(b))$ vanishes by definition of $E_{\mathbf{i}}^{(top)}(b)$. $\triangleleft$

Proofs of Theorems 2.4 and 2.5. Theorem 2.4 for $\mathbf{i} = \mathbf{i}(1)$ and Theorem 2.5 are established by Proposition 4.3 (a) because the semigroup Γ coincides with $C_{\mathbf{Z}}(\mathbf{i}(1))$. Now let $\mathbf{i}$ be an arbitrary reduced decomposition of w_0. By Theorem 2.2, the correspondence $b \mapsto a(\mathbf{i}; b)$ is a bijection between B and ${}_{\mathbf{i}}T_{\mathbf{i}(1)}(\Gamma)$. Let $C =_{\mathbf{i}} T_{\mathbf{i}(1)}(C(\mathbf{i}(1)))$. Since ${}_{\mathbf{i}}T_{\mathbf{i}(1)}$ is a piecewise-linear automorphism preserving $\mathbf{Z}^m$, it follows that C is a polyhedral cone, and ${}_{\mathbf{i}}T_{\mathbf{i}(1)}(\Gamma)$ coincides with the set $C_{\mathbf{Z}} = C \cap \mathbf{Z}^m$ of integral points in C. It remains to show that C is convex. Clearly, $C_{\mathbf{Q}} = C \cap \mathbf{Q}^m$ is dense in C. So it is enough to show that $C_{\mathbf{Q}}$ is closed under taking rational convex combinations. This in turn follows from the fact that $C_{\mathbf{Z}}$ is a semigroup. But the last statement follows from (3.5) and Proposition 4.1: for every $b', b'' \in B$ there is a constituent b of $b'b''$ such that $a(\mathbf{i}; b) = a(\mathbf{i}; b') + a(\mathbf{i}; b'')$. $\triangleleft$

7. Proof of Theorem 1.4

We recall that the quantum minors $\Delta(I;J)$ are defined by (1.6). Here $I = (i_1 < i_2 < \ldots < i_s)$, $J = (j_1 < j_2 < \ldots < j_s)$ are two increasing sequences of indices from $\{1, 2, \ldots, r+1\}$.

Proposition 7.1. *We have $E_j(\Delta(I;J)) = 0$ unless $j \notin J, (j+1) \in J$. If $j \notin J, (j+1) \in J$ then*

$$E_j(\Delta(I;J)) = \Delta(I; J \cup \{j\} \setminus \{j+1\}). \tag{7.1}$$

Proof. Applying E_j term by term to the expression (1.6), and using (1.4) and Lemma 3.5, we obtain (7.1) and the fact that $E_j(\Delta(I;J)) = 0$ whenever $(j+1) \notin J$. It remains to check that $E_j(\Delta(I;J)) = 0$ if $j, j+1 \in J$. Concentrating on the contribution to $\Delta(I;J)$ of the jth and $(j+1)$th column, we see that it suffices to show that $E_j(\Delta(i, i'; j, j+1)) = 0$ for all $i < i'$. The only non-trivial case is $i < i' < j$, when $E_j(\Delta(i, i'; j, j+1))$ is easily seen to be proportional to $[t_{ij}, t_{i',j}]$ (see (1.5)). But $[t_{ij}, t_{i',j}] = 0$ by Proposition 3.11 (b). $\triangleleft$

We say that $\Delta(I;J)$ is *non-trivial* if $i_k \leq j_k$ for $k = 1, \ldots, s$.

Corollary 7.2. *We have $\Delta(I;J) \neq 0$ if and only if $\Delta(I;J)$ is non-trivial.*

Proof. Since the matrix $T = (t_{ij})$ is unitriangular, $\Delta(I;J) = 0$ unless $\Delta(I;J)$ is non-trivial. Conversely, suppose $\Delta(I;J)$ is non-trivial. If $I = J$ then $\Delta(I;J)$ is unitriangular, and hence equal to 1. So we can assume that $I \neq J$. Let k be the minimal index such that $i_k < j_k$. Then $j_{k-1} = i_{k-1} < i_k \leq j_k - 1$, hence $(j_k - 1) \notin J$. Applying (7.1) for $j = j_k - 1$ and using induction on $(j_1 - i_1) + \ldots + (j_s - i_s)$, we conclude that $E_j(\Delta(I;J)) \neq 0$ hence $\Delta(I;J) \neq 0$. $\triangleleft$

According to Corollary 7.2, we can state Theorem 1.4 as follows.

Theorem 7.3. *Every string basis B in $\mathcal{A}_r$ contains all non-trivial minors $\Delta(I;J)$.*

Proof. As in the proof of Corollary 7.2, we use induction on $(j_1 - i_1) + \ldots + (j_s - i_s)$. If $I = J$ then $\Delta(I;J) = 1$ belongs to B in view of the string axiom (S0). Hence we assume that $I \neq J$. As above, choose $j = j_k - 1$, where k is the minimal index such that $i_k < j_k$. Let $J' = J \cup \{j\} \setminus \{j+1\}$. By Proposition 7.1, $E_j(\Delta(I;J)) = \Delta(I;J')$, and $E_i(\Delta(I;J)) = 0$ for $i < j$.

Let $x = x_j \Delta(I;J')$. By induction, we can assume that $\Delta(I;J') \in B$, hence by (S1), $x \in [B]^+$. Clearly, $E_j(x) = \Delta(I;J')$, and $E_j^2(x) = 0$. By Proposition 4.1, there is a constituent b of x such that $E_j(b) = \Delta(I;J')$. It remains to show that $\Delta(I;J) = b$. By Proposition 1.2 (a), it suffices to prove that $E_i(\Delta(I;J)) = E_i(b)$ for all i.

If $i < j$ then $E_i(\Delta(I;J)) = E_i(x) = 0$ by Proposition 7.1 and our choice of j. By (S2), $E_i(b) = 0$, as required. If $i = j$ we have nothing to prove. So it remains to treat $i > j$.

First consider the case $i > j+1$. Consider two admissible sequences $\mathbf{i} = (j,i), \mathbf{i}' = (i,j)$. Clearly,

$$a(\mathbf{i}; b) = a(\mathbf{i}; \Delta(I;J)),\ E_{\mathbf{i}}^{(top)}(b) = E_{\mathbf{i}}^{(top)}(\Delta(I;J)).$$

Moreover, using (7.1) we see that there are only two possibilities for the string $a(\mathbf{i}; \Delta(I;J))$: it can be equal either $(1,0)$ or $(1,1)$. Furthermore, (7.1) implies that if $a(\mathbf{i}; \Delta(I;J)) = (1,0)$ (resp. $(1,1)$) then $a(\mathbf{i}'; \Delta(I;J)) = (0,1)$ (resp. $(1,1)$); we have also $E_{\mathbf{i}'}^{(top)}(\Delta(I;J)) = E_{\mathbf{i}}^{(top)}(\Delta(I;J))$. On the other hand, by Theorem 2.7 (a) the element b satisfies the same properties. Hence $a(\mathbf{i}'; b) = a(\mathbf{i}'; \Delta(I;J))$ and $E_{\mathbf{i}'}^{(top)}(b) = E_{\mathbf{i}'}^{(top)}(\Delta(I;J))$. If $a(\mathbf{i}'; b) = a(\mathbf{i}'; \Delta(I;J)) = (0,1)$ then $E_i(\Delta(I;J)) = E_i(b)$, as required. If $a(\mathbf{i}'; b) = a(\mathbf{i}'; \Delta(I;J)) = (1,1)$ then $E_i(b) \in B$ by (S3), and $E_i(\Delta(I;J)) \in B$ by the inductive assumption. Since two basis elements $E_i(b)$ and $E_i(\Delta(I;J))$ have the same string in direction (j), and $E_j(b) = E_j(\Delta(I;J))$ they must coincide in view of Proposition 4.4.

It remains to treat the case $i = j+1$. The proof in this case is parallel to the previous one. Consider two admissible sequences $\mathbf{i} = (j,i,j), \mathbf{i}' = (i,j,i)$. Clearly,

$$a(\mathbf{i}; b) = a(\mathbf{i}; \Delta(I;J)),\ E_{\mathbf{i}}^{(top)}(b) = E_{\mathbf{i}}^{(top)}(\Delta(I;J)).$$

By (7.1), there are only two possibilities for the string $a(\mathbf{i}; \Delta(I;J))$: it can be equal either $(1,0,0)$ or $(1,1,0)$. Furthermore, $E_i(\Delta(I;J)) = 0$. On the other hand, the string $a(\mathbf{i}'; b)$ can be computed by (2.8). Applying (2.8) to each of $(1,0,0)$ and $(1,1,0)$ we obtain the string with $a_1 = 0$. Hence, $E_i(b) = 0$, which completes the proof of Theorem 7.3. $\triangleleft$

Remarks. (a) It follows easily from Proposition 7.1 that for every non-trivial minor $\Delta(I;J)$ and every admissible sequence $\mathbf{i}$ the string $a(\mathbf{i}; \Delta(I;J))$ consists of zeros and ones. The arguments similar to those in the proof of Theorem 7.3 imply the following converse statement: if an element $b \in B$ has the property that the string $a(\mathbf{i}; b)$ in every admissible direction consists of zeros and ones then b coincides with some non-trivial minor $\Delta(I;J)$.

(b) One can show that the set of all non-trivial minors $\Delta(I;J)$ is invariant under the antiautomorphism $x \mapsto x^*$ (see §3). In particular, we have $t_{ij}^* = \Delta(i, i+1, \ldots, j-1; i+1, i+2, \ldots, j)$.

8. Linearity domains: proofs of Proposition 2.8 and Theorems 2.9, 2.10

Proof of Proposition 2.8. Fix $\mathbf{i} \in R(w_0)$. We call $\mathbf{i}-$*chambers* the connected components of the set of $\mathbf{i}-$regular points (so $\mathbf{i}-$linearity domains are the closures of $\mathbf{i}-$chambers). Clearly, two $\mathbf{i}-$regular points a, a' lie in the same $\mathbf{i}-$chamber if and only if for every $\mathbf{i}' \in R(w_0)$ the points ${}_{\mathbf{i}'}T_{\mathbf{i}}(a)$ and ${}_{\mathbf{i}'}T_{\mathbf{i}}(a')$ lie on the same side of each $\mathbf{i}'-$wall. Since all maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ are homogeneous of degree 1, it follows that every $\mathbf{i}-$chamber is a cone. To complete the proof of Proposition 2.8 it remains only to show the following.

Lemma 8.1. *Let $a, a' \in C(\mathbf{i})$.*

(a) If a, a' belong to the same $\mathbf{i}-$chamber C^0 then $a + a'$ also belongs to C^0, and

$$ {}_{\mathbf{i}'}T_{\mathbf{i}}(a + a') = {}_{\mathbf{i}'}T_{\mathbf{i}}(a) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a') \tag{8.1} $$

for each $\mathbf{i}' \in R(w_0)$.

(b) If a, a' do not belong to the same $\mathbf{i}-$linearity domain then ${}_{\mathbf{i}'}T_{\mathbf{i}}(a + a') \neq {}_{\mathbf{i}'}T_{\mathbf{i}}(a) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a')$ for some $\mathbf{i}' \in R(w_0)$.

Proof of Lemma 8.1. We know that each $\mathbf{i}' \in R(w_0)$ can be reached by $\mathbf{i}$ by a number of moves of type (2.5) and (2.6). Let $d(\mathbf{i}, \mathbf{i}')$ denote the minimal number of moves of type (2.6) needed for the transition from $\mathbf{i}$ to $\mathbf{i}'$. We prove the equality (8.1) in (a) by induction on $d(\mathbf{i}, \mathbf{i}')$. If $d(\mathbf{i}, \mathbf{i}') = 0$ then by Proposition 2.6 and Theorem 2.7 (a), ${}_{\mathbf{i}'}T_{\mathbf{i}}$ is some permutation of coordinates $(a_1, \ldots, a_m)$ hence is a linear map, and we are done. If $d(\mathbf{i}, \mathbf{i}') = d > 0$ then we can decompose ${}_{\mathbf{i}'}T_{\mathbf{i}}$ into a product ${}_{\mathbf{i}'}T_{\mathbf{i}_2} \circ {}_{\mathbf{i}_2}T_{\mathbf{i}_1} \circ {}_{\mathbf{i}_1}T_{\mathbf{i}}$, where $d(\mathbf{i}, \mathbf{i}_1) = 0$, $d(\mathbf{i}_2, \mathbf{i}') = d - 1$, and $\mathbf{i}_2$ is obtained from $\mathbf{i}_1$ by a move of type (2.6). Clearly, the points $a_1 = {}_{\mathbf{i}_1}T_{\mathbf{i}}(a)$ and $a'_1 = {}_{\mathbf{i}_1}T_{\mathbf{i}}(a')$ lie in the same $\mathbf{i}_1-$chamber, in particular, are on the same side of each $\mathbf{i}_1-$wall. It follows from Proposition 2.6 and Theorem 2.7 (b) that

$$ {}_{\mathbf{i}_2}T_{\mathbf{i}_1}(a_1 + a'_1) = {}_{\mathbf{i}_2}T_{\mathbf{i}_1}(a_1) + {}_{\mathbf{i}_2}T_{\mathbf{i}_1}(a'_1). \tag{8.2} $$

Applying ${}_{\mathbf{i}'}T_{\mathbf{i}_2}$ to both sides of (8.2) and using induction, we obtain (8.1).

To complete the proof of (a) we assume that $a, a' \in C^0$ but $(a + a') \notin C^0$. By definition, this means that there exist $\mathbf{i}' \in R(w_0)$ and an $\mathbf{i}'-$wall U such that ${}_{\mathbf{i}'}T_{\mathbf{i}}(a)$ and ${}_{\mathbf{i}'}T_{\mathbf{i}}(a')$ are on one side of U, but ${}_{\mathbf{i}'}T_{\mathbf{i}}(a + a')$ is on the other side. But this contradicts (8.1).

To prove (b) suppose that a, a' do not belong to the same $\mathbf{i}-$linearity domain. This means that there exist $\mathbf{i}' \in R(w_0)$ and an $\mathbf{i}'-$wall U separating ${}_{\mathbf{i}'}T_{\mathbf{i}}(a)$ from ${}_{\mathbf{i}'}T_{\mathbf{i}}(a')$. Let $\mathbf{i}' \mapsto \mathbf{i}''$ be the move of type (2.6) corresponding to the wall U. It follows easily from (2.8) that

$$ {}_{\mathbf{i}''}T_{\mathbf{i}'}({}_{\mathbf{i}'}T_{\mathbf{i}}(a) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a')) \neq {}_{\mathbf{i}''}T_{\mathbf{i}'}({}_{\mathbf{i}'}T_{\mathbf{i}}(a)) + {}_{\mathbf{i}''}T_{\mathbf{i}'}({}_{\mathbf{i}'}T_{\mathbf{i}}(a')). $$

This implies (b). Lemma 8.1 and hence Proposition 2.8 are proven. $\triangleleft$

Proof of Theorem 2.9. Let $b, b' \in B$. First suppose that b, b' satisfy (2), i.e $b'' = q^N bb' \in B$ for some $N \in \mathbf{Z}$. In view of (3.5), $a(\mathbf{i}; b'') = a(\mathbf{i}; b) + a(\mathbf{i}; b')$ for all $\mathbf{i} \in R(w_0)$. It follows that

$$_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b) + a(\mathbf{i}; b')) = {}_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b)) + {}_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b'))$$

for each $\mathbf{i}' \in R(w_0)$. By Proposition (2.8), $a(\mathbf{i}; b)$ and $a(\mathbf{i}; b')$ belong to the same $\mathbf{i}-$linearity domain, as claimed.

Now suppose that b and b' quasicommute. To show that the strings $a(\mathbf{i}; b)$ and $a(\mathbf{i}; b')$ belong to the same $\mathbf{i}-$linearity domain, it is enough to show that they are not separated by any $\mathbf{i}-$wall. Suppose an $\mathbf{i}-$wall U corresponds to the move

$$\mathbf{i} = (\mathbf{i_1}, \mathbf{i}_0, \mathbf{i_2}) \mapsto (\mathbf{i_1}, \mathbf{i}'_0, \mathbf{i_2}) = \mathbf{i}'$$

of type (2.6), where $\mathbf{i}_0 = (i, j, i), \mathbf{i}'_0 = (j, i, j)$, $|i - j| = 1$. By (3.6), the elements $E^{(top)}_{\mathbf{i}_1}(b)$ and $E^{(top)}_{\mathbf{i}_1}(b')$ quasicommute. Replacing if needed b and b' by $E^{(top)}_{\mathbf{i}_1}(b)$ and $E^{(top)}_{\mathbf{i}_1}(b')$ we can assume that $\mathbf{i} = (\mathbf{i}_0, \mathbf{i_2}), \mathbf{i}' = (\mathbf{i}'_0, \mathbf{i_2})$. Let $a(\mathbf{i}_0; b) = a = (a_1, a_2, a_3)$, $a(\mathbf{i}'_0; b) = a' = (a'_1, a'_2, a'_3)$. We have to show that the numbers $a_2 - a_1 - a_3$ and $a'_2 - a'_1 - a'_3$ are of the same sign.

Let $b_0 = E^{(top)}_{\mathbf{i}_0}(b), b'_0 = E^{(top)}_{\mathbf{i}_0}(b')$. Then we have $b'b = q^n bb', b'_0 b_0 = q^{n_0} b_0 b'_0$ for some integers n, n_0. Applying (3.6), we obtain

$$E^{(top)}_{\mathbf{i}_0}(bb') = q^{\Phi_{\mathbf{i}_0,\gamma}(a,a')} b_0 b'_0, E^{(top)}_{\mathbf{i}_0}(b'b) = q^{\Phi_{\mathbf{i}_0,\gamma'}(a',a)} b'_0 b_0,$$

where $\gamma = \deg(b), \gamma' = \deg(b')$. It follows that

$$\Phi_{\mathbf{i}_0,\gamma}(a, a') - \Phi_{\mathbf{i}_0,\gamma'}(a', a) = n_0 - n. \tag{8.3}$$

By Theorem 2.2, we have $b_0 = E^{(top)}_{\mathbf{i}'_0}(b), b'_0 = E^{(top)}_{\mathbf{i}'_0}(b')$. Using the same argument and applying Theorem 2.7 (b), we obtain

$$\Phi_{\mathbf{i}'_0,\gamma}(T(a), T(a')) - \Phi_{\mathbf{i}'_0,\gamma'}(T(a'), T(a)) = n_0 - n, \tag{8.4}$$

where T is the operator given by (2.8).

Combining (8.3) and (8.4), we see that

$$\Phi_{\mathbf{i}_0,\gamma}(a, a') - \Phi_{\mathbf{i}_0,\gamma'}(a', a) = \Phi_{\mathbf{i}'_0,\gamma}(T(a), T(a')) - \Phi_{\mathbf{i}'_0,\gamma'}(T(a'), T(a)). \tag{8.5}$$

Using (3.3), we can rewrite (8.5) in the form

$$\Psi(a, a') = \Psi(T(a), T(a')), \tag{8.6}$$

where $\Psi(a, a')$ is a skew-symmetric bilinear form on $\mathbf{R}^3$ given by

$$\Psi(a, a') = 2(a_1 a_3' - a_3 a_1') + a_2(a_1' - a_3') - (a_1 - a_3)a_2'. \tag{8.7}$$

To complete the proof it remains to show that (8.6) implies that $a_2 - a_1 - a_3$ and $a_2' - a_1' - a_3'$ are of the same sign. Suppose this is not so, i.e., say $a_2 - a_1 - a_3 > 0, a_2' - a_1' - a_3' < 0$. Then $T(a) = (a_2 - a_1, a_1 + a_3, a_1), T(a') = (a_3', a_1' + a_3', a_2' - a_3')$. Substituting these vectors into (8.7), we obtain after a straightforward calculation that

$$\Psi(T(a), T(a')) - \Psi(a, a') = 2(a_2 - a_1 - a_3)(a_2' - a_1' - a_3') < 0,$$

which contradicts (8.6). Theorem 2.9 is proven. $\triangleleft$

The argument in the proof of Theorem 2.9 implies the following.

Proposition 8.2. *Let $w \in W$, and $\mathbf{i}, \mathbf{i}' \in R(w)$. If $b, b' \in B$ quasicommute then*

$$_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b) + a(\mathbf{i}; b')) = {_{\mathbf{i}'}T_{\mathbf{i}}}(a(\mathbf{i}; b)) + {_{\mathbf{i}'}T_{\mathbf{i}}}(a(\mathbf{i}; b')).$$

Proof of Theorem 2.10. Choose an arbitrary decomposition from $R(w_0)$ say $\mathbf{i}(1)$. By Proposition 4.1 and (3.5), there is exactly one constituent b'' of bb' such that $a(\mathbf{i}(1); b'') = a(\mathbf{i}(1); b) + a(\mathbf{i}(1); b')$. By Proposition 8.2, $a(\mathbf{i}; b'') = a(\mathbf{i}; b) + a(\mathbf{i}; b')$ for every $\mathbf{i} \in R(w_0)$, and we are done. $\triangleleft$

9. The string basis for sl_4

In this section we prove Theorems 1.5, 2.11, and 2.12 in the case $r = 3$.

Recall that the reduced expression $\mathbf{i}(1)$ of $w_0 \in W$ is given by $\mathbf{i}(1) = (1,2,1,3,2,1)$, and the cone $C = C(\mathbf{i}(1))$ is given by

$$C = \{a \in \mathbf{R}^6 : a_1 \geq 0, a_2 \geq a_3 \geq 0, a_4 \geq a_5 \geq a_6 \geq 0\}.$$

It will be convenient for us to rename the elements $b \in \mathcal{P}_3$ according to the first column of Table 1 below.

Table 1. Generators of the string basis in $\mathcal{A}_3$

$b \in \mathcal{P}_3$	deg (b)	$a(\mathbf{i}(1); b)$	l_1	l_2	l_3	n_1	n_2	n_3
$x_1 = t_{12}$	α_1	(1,0,0,0,0,0)	1	0	0	1	0	0
$x_2 = t_{23}$	α_2	(0,1,0,0,0,0)	0	1	0	0	1	0
$x_{21} = t_{13}$	$\alpha_1 + \alpha_2$	(0,1,1,0,0,0)	0	1	0	1	0	0
$x_{12} = t^*_{13}$ $= \Delta(1,2;2,3)$	$\alpha_1 + \alpha_2$	(1,1,0,0,0,0)	1	0	0	0	1	0
$x_3 = t_{34}$	α_3	(0,0,0,1,0,0)	0	0	1	0	0	1
$x_{32} = t_{24}$	$\alpha_2 + \alpha_3$	(0,0,0,1,1,0)	0	0	1	0	1	0
$x_{23} = t^*_{24}$ $= \Delta(2,3;3,4)$	$\alpha_2 + \alpha_3$	(0,1,0,1,0,0)	0	1	0	0	0	1
x_{132} $= \Delta(1,2;2,4)$	$\alpha_1 + \alpha_2 + \alpha_3$	(1,0,0,1,1,0)	1	0	1	0	1	0
x_{213} $= \Delta(1,3;3,4)$	$\alpha_1 + \alpha_2 + \alpha_3$	(0,1,1,1,0,0)	0	1	0	1	0	1
$x_{321} = t_{14}$	$\alpha_1 + \alpha_2 + \alpha_3$	(0,0,0,1,1,1)	0	0	1	1	0	0
$x_{123} = t^*_{14}$ $= \Delta(1,2,3;2,3,4)$	$\alpha_1 + \alpha_2 + \alpha_3$	(1,1,0,1,0,0)	1	0	0	0	0	1
x_{2132} $= \Delta(1,2;3,4)$	$\alpha_1 + 2\alpha_2 + \alpha_3$	(0,1,1,1,1,0)	0	1	0	0	1	0

The strings $a(\mathbf{i}(1); b)$ and the values of $l_1(b), l_2(b), l_3(b)$ displayed in Table 1 are obtained by using repeatedly Proposition 7.1 and Corollary 7.2. The functions $n_1(b), n_2(b)$, and $n_3(b)$ will be defined and used in §10.

Let $M = M(3)$ denote the set of all 12-tuples $m = (m_1, m_2, m_{21}, m_{12}, m_3, m_{32}, m_{23}, m_{132}, m_{213}, m_{321}, m_{123}, m_{2132})$ of nonnegative integers satisfying at least one of the following 14 conditions:

$$(C1) \qquad m_1 = m_2 = m_{21} = m_{12} = m_{132} = m_{213} = 0,$$

$$(C2) \qquad m_1 = m_{21} = m_{12} = m_3 = m_{132} = m_{213} = 0,$$

$$(C3) \qquad m_1 = m_2 = m_{21} = m_{12} = m_{32} = m_{132} = 0,$$

$$(C4) \qquad m_1 = m_{21} = m_3 = m_{23} = m_{132} = m_{213} = 0,$$

$$(C5) \qquad m_1 = m_3 = m_{32} = m_{23} = m_{132} = m_{213} = 0,$$

$$(C6) \qquad m_1 = m_2 = m_{21} = m_{12} = m_{23} = m_{213} = 0,$$

$$(C7) \qquad m_2 = m_{21} = m_{12} = m_{32} = m_{23} = m_{213} = 0,$$

$$(C8) \qquad m_2 = m_{21} = m_{12} = m_{32} = m_{23} = m_{132} = 0,$$

$$(C9) \qquad m_2 = m_{12} = m_3 = m_{32} = m_{23} = m_{132} = 0,$$

$$(C10) \qquad m_1 = m_2 = m_{21} = m_3 = m_{23} = m_{213} = 0,$$

$$(C11) \qquad m_2 = m_{21} = m_3 = m_{32} = m_{23} = m_{213} = 0,$$

$$(C12) \qquad m_2 = m_3 = m_{32} = m_{23} = m_{132} = m_{213} = 0,$$

$$(C13) \qquad m_1 = m_{12} = m_3 = m_{32} = m_{132} = m_{213} = 0,$$

$$(C14) \qquad m_1 = m_2 = m_{12} = m_3 = m_{32} = m_{132} = 0.$$

For each $j = 1, \ldots, 14$ let $M_j \subset M$ denote the semigroup of points satisfying (Cj). Let C_j denote the cone generated by strings $a(\mathbf{i}(1); x_\rho)$ for all indices $\rho = 1, 2, \ldots, 2132$ not appearing in the condition (Cj).

For each $m \in M$ we abbreviate $x^m = x_1^{m_1} x_2^{m_2} \cdots x_{2132}^{m_{2132}}$. Iterating (3.6), we see that $E_{\mathbf{i}(1)}^{(top)}(x^m)$ is an integer power of q. We set

$$b(m) = (E_{\mathbf{i}(1)}^{(top)}(x^m))^{-1} x^m, \tag{9.1}$$

so that $E_{\mathbf{i}(1)}^{(top)}(b(m)) = 1$. The following theorem refines Theorems 1.5 and 2.12.

Theorem 9.1.

(a) *There is a unique string basis in $\mathcal{A}_3$. It is formed by the elements $b(m)$ for $m \in M$.*

(b) *Two elements $b(m)$ and $b(m')$ for $m, m' \in M$ quasicommute if and only if m and m' belong to the same semigroup M_j for some $j = 1, \dots, 14$.*

(c) *The collection $\mathcal{D}$ of cones $C_1, \dots, C_{14}$ satisfies all conditions of Theorem 2.12.*

Proof. We start with the description of the $\mathbf{i}(1)-$linearity domains.

Proposition 9.2. *There are thirteen $\mathbf{i}(1)-$linearity domains in C: the simplicial cones $C_1, \dots, C_{12}$, and the cone $\tilde{C}_{13} = C_{13} \cup C_{14}$.*

Proof of Proposition 9.2. The set $R(w_0)$ is displayed on Figure 1. Here dotted lines correspond to moves of type (2.5), and solid lines correspond to moves of type (2.6).

$(1,2,3,1,2,1)$ $(1,2,1,3,2,1)$

$(1,2,3,2,1,2)$ $(2,1,2,3,2,1)$

$(1,3,2,3,1,2)$ $(2,1,3,2,3,1)$

$(1,3,2,1,3,2)$ $(3,1,2,3,1,2)$ $(2,3,1,2,3,1)$ $(2,1,3,2,1,3)$

$(3,1,2,1,3,2)$ $(2,3,1,2,1,3)$

$(3,2,1,2,3,2)$ $(2,3,2,1,2,3)$

$(3,2,1,3,2,3)$ $(3,2,3,1,2,3)$

Fig.1. Reduced decompositions of w_0 for S_4.

Now we can find the $\mathbf{i}(1)-$linearity domains in C by a tedious but completely straightforward calculation using Proposition 2.6 and Theorem 2.7. We get thirteen domains defined by linear constraints. This is again a straightforward task to identify these domains with the cones $C_1, \dots, C_{12}$ and $\tilde{C}_{13}$. The description in terms of linear constraints is as follows:

$$C_1 = \{a_2 \geq a_1 + a_3, a_5 \geq a_3 + a_6, a_3 + a_4 \geq a_2 + a_5, a_1 \geq 0, a_3 \geq 0, a_6 \geq 0\},$$

$$C_2 = \{a_4 \geq a_1 + a_5, a_5 \geq a_3 + a_6, a_2 + a_5 \geq a_3 + a_4, a_1 \geq 0, a_3 \geq 0, a_6 \geq 0\},$$

$$C_3 = \{a_2 \geq a_1 + a_3, a_3 + a_6 \geq a_5 \geq a_6, a_4 \geq a_2 + a_6, a_1 \geq 0, a_6 \geq 0\},$$

$$C_4 = \{a_2 \geq a_1 + a_3, a_1 + a_5 \geq a_4 \geq a_5 \geq a_3 + a_6, a_3 \geq 0, a_6 \geq 0\},$$

$$C_5 = \{a_2 \geq a_1 + a_3, a_1 + a_5 \geq a_4 \geq a_5 \geq a_6 \geq 0, a_3 + a_6 \geq a_5\},$$

$$C_6 = \{a_1 + a_3 \geq a_2 \geq a_3, a_3 + a_4 \geq a_2 + a_5 \geq a_1 + 2a_3 + a_6, a_3 \geq 0, a_6 \geq 0\},$$

$$C_7 = \{a_2 \geq a_3, a_3 + a_4 \geq a_2 + a_5, a_5 \geq a_3 + a_6, a_1 + 2a_3 + a_6 \geq a_2 + a_5, a_3 \geq 0, a_6 \geq 0\},$$

$$C_8 = \{a_1 + a_3 \geq a_2 \geq a_3, a_3 + a_6 \geq a_5 \geq a_6 \geq 0, a_4 \geq a_2 + a_6\},$$

$$C_9 = \{a_1 + a_3 \geq a_2 \geq a_3, a_3 + a_4 \geq a_2 + a_5, a_5 \geq a_6 \geq 0, a_2 + a_6 \geq a_4\},$$

$$C_{10} = \{a_1 + a_3 \geq a_2, a_2 + a_5 \geq a_3 + a_4, a_4 \geq a_5, a_2 + a_5 \geq a_1 + 2a_3 + a_6, a_3 \geq 0, a_6 \geq 0\},$$

$$C_{11} = \{a_1 + 2a_3 + a_6 \geq a_2 + a_5 \geq a_3 + a_4, a_4 \geq a_5 \geq a_3 + a_6, a_3 \geq 0, a_6 \geq 0\},$$

$$C_{12} = \{a_1 + a_3 \geq a_2, a_2 + a_5 \geq a_3 + a_4, a_4 \geq a_5, a_3 + a_6 \geq a_5 \geq a_6 \geq 0\},$$

$$\tilde{C}_{13} = \{a_2 \geq a_1 + a_3, a_4 \geq a_1 + a_5, a_3 + a_6 \geq a_5 \geq a_6 \geq 0, a_2 + a_6 \geq a_4, a_1 \geq 0\}.$$

It remains to show that $\tilde{C}_{13} = C_{13} \cup C_{14}$. This follows easily once we represent C_{13} and C_{14} in terms of linear constraints:

$$C_{13} = \{a_4 \geq a_1 + a_5, a_3 + a_6 \geq a_5 \geq a_6 \geq 0, a_2 + a_5 \geq a_3 + a_4, a_1 \geq 0\},$$

$$C_{14} = \{a_2 \geq a_1 + a_3, a_3 + a_4 \geq a_2 + a_5, a_2 + a_6 \geq a_4, a_5 \geq a_6 \geq 0, a_1 \geq 0\}$$

(the wall between C_{13} and C_{14} in $\tilde{C}_{13}$ is given by the equation $a_2 + a_5 = a_3 + a_4$.) $\triangleleft$

In particular, looking at the edges of all cones $C_1, \ldots, C_{12}$ and $\tilde{C}_{13}$, we obtain Theorem 2.11.

We define $\eta : M \to \mathbf{Z}_+^6$ by $\eta(m) = a(\mathbf{i}(1); b(m))$. In view of (3.5), η is a restriction to M of a linear map $\mathbf{R}^{12} \to \mathbf{R}^6$. Therefore, it is totally determined by the column $a(\mathbf{i}(1); b)$ of Table 1. By definition, for each $j = 1, \ldots, 14$ the map η is a semigroup isomorphism between M_j and the semigroup of integral points in C_j. Since $C = \cup_j C_j$, we obtain the following.

Proposition 9.3. *The map η is a bijection between M and the semigroup Γ of integral points in C.*

As before, we fix a string basis B of $\mathcal{A}_3$. Let B_0 denote the set of all elements $b(m)$ for $m \in M$. By Theorem 1.4 and the string axiom (S3), all $b(m)$ belong to $[B]^+$. Hence, Propositions 9.3, 4.1 and 4.3 imply that B_0 is a basis of $\mathcal{A}_3$.

The following Table 2 exhibits the expansion of the product of any two elements of $\mathcal{P}_3$ in the basis B_0. Here the entry in the row x_ρ and the column $x_{\rho'}$ is equal to the product $x_\rho x_{\rho'}$. All the entries can be checked by induction on degrees with the help of Proposition 1.2 (a). We exclude from the table the last three elements of $\mathcal{P}_3$, namely x_{321}, x_{123}, and x_{2132}, because it is easy to check that each of them quasicommutes with all elemens of $\mathcal{P}_3$.

Table 2. Products of pairs of elements from $\mathcal{P}_3$

	x_1	x_2	x_{21}	x_{12}	x_3
x_1	x_1^2	qx_{21} $+x_{12}$	x_1x_{21}	x_1x_{12}	x_1x_3
x_2	qx_{12} $+x_{21}$	x_2^2	x_2x_{21}	x_2x_{12}	qx_{32} $+x_{23}$
x_{21}	$q^{-1}x_1x_{21}$	qx_2x_{21}	x_{21}^2	$x_{21}x_{12}$	qx_{321} $+x_{213}$
x_{12}	qx_1x_{12}	$q^{-1}x_2x_{12}$	$x_{21}x_{12}$	x_{12}^2	qx_{132} $+x_{123}$
x_3	x_1x_3	qx_{23} $+x_{32}$	qx_{213} $+x_{321}$	qx_{123} $+x_{132}$	x_3^2
x_{32}	qx_{132} $+x_{321}$	$q^{-1}x_2x_{32}$	$q^{-1}x_2x_{321}$ $+x_{2132}$	$x_{12}x_{32}$	qx_3x_{32}
x_{23}	qx_{123} $+x_{213}$	qx_2x_{23}	$x_{21}x_{23}$	qx_2x_{123} $+x_{2132}$	$q^{-1}x_3x_{23}$
x_{132}	qx_1x_{132}	$q^{-1}x_{12}x_{32}$ $+x_{2132}$	$q^{-1}x_{12}x_{321}$ $+x_1x_{2132}$	$q^{-1}x_{12}x_{132}$	qx_3x_{132}
x_{213}	$q^{-1}x_1x_{213}$	$qx_{21}x_{23}$ $+x_{2132}$	$qx_{21}x_{213}$	$qx_{21}x_{123}$ $+q^{-1}x_1x_{2132}$	$q^{-1}x_3x_{213}$

	x_{32}	x_{23}	x_{132}	x_{213}
x_1	qx_{321} $+x_{132}$	qx_{213} $+x_{123}$	x_1x_{132}	x_1x_{213}
x_2	x_2x_{32}	x_2x_{23}	$qx_{12}x_{32}$ $+x_{2132}$	$q^{-1}x_{21}x_{23}$ $+x_{2132}$
x_{21}	qx_2x_{321} $+x_{2132}$	$x_{21}x_{23}$	$q^{-1}x_1x_{2132}$ $+x_{12}x_{321}$	$x_{21}x_{213}$
x_{12}	$x_{12}x_{32}$	$q^{-1}x_2x_{123}$ $+x_{2132}$	$x_{12}x_{132}$	$x_{21}x_{123}$ $+x_1x_{2132}$
x_3	x_3x_{32}	x_3x_{23}	x_3x_{132}	x_3x_{213}
x_{32}	x_{32}^2	$x_{32}x_{23}$	$x_{32}x_{132}$	$x_{23}x_{321}$ $+x_3x_{2132}$
x_{23}	$x_{32}x_{23}$	x_{23}^2	$q^{-1}x_3x_{2132}$ $+x_{32}x_{123}$	$x_{23}x_{213}$
x_{132}	$q^{-1}x_{32}x_{132}$	$q^{-1}x_{32}x_{123}$ $+x_3x_{2132}$	x_{132}^2	$x_1x_3x_{2132}$ $+x_{321}x_{123}$
x_{213}	$qx_{23}x_{321}$ $+q^{-1}x_3x_{2132}$	$qx_{23}x_{213}$	$q^{-2}x_1x_3x_{2132}$ $+x_{321}x_{123}$	x_{213}^2

Table 2 allows us to check by inspection that if $m, m' \in M$ belong to the same M_j then $b(m)$ and $b(m')$ quasicommute. This proves a half of Theorem 9.1 (b). To prove the converse statement we notice that according to Theorem 2.9, if $b(m)$ and $b(m')$ quasicommute then $\eta(m)$ and $\eta(m')$ belong to the same $\mathbf{i}(1)-$linearity domain in C. By Proposition 9.2, we have only to check that if $m \in C_{13} \setminus C_{14}, m' \in C_{14} \setminus C_{13}$ then $b(m)$ and $b(m')$ do not quasicommute. This also follows from Table 2. Theorem 9.1 (b) is proven. Combining Theorem 9.1 (b) and Proposition 9.2, we obtain Theorem 9.1 (c).

To complete the proof of Theorem 9.1 it remains to show the uniqueness of a string basis in $\mathcal{A}_3$. It is enough to show that $B_0 \subset B$. We deduce it from the following result whose proof will be given in §10.

Proposition 9.4. *Let $m, m' \in M$ be two distinct elements such that $b(m)$ and $b(m')$ have the same degree. Then there exists a product E of divided powers $E_1^{(n_1)}, E_2^{(n_2)}, E_3^{(n_3)}$ such that $E(b(m)) = 0, E(b(m')) \neq 0$.*

Using Proposition 9.4, we complete the proof of Theorem 9.1 as follows. Suppose $b(m) \notin B$ for some $m \in M$. Choose $m \in M$ so that $b(m) \notin B$, and the string $\eta(m)$ is lexicographically smallest possible. We have mentioned already that $b(m) \in [B]^+$. By our choice, $b(m)$ has a constituent $b' \in B$ which has the form $b' = b(m')$ for some $m' \in M$, $m' \neq m$. But then applying to $b(m)$ and $b(m')$ a monomial E from Proposition

9.4, we obtain a contradiction with the string axiom (S2). $\triangleleft$

Remarks. (a) Table 2 provides us with a straightening type algorithm for expanding every monomial in the elements of $\mathcal{P}_3$ in the basis B_0. More precisely, we assign weights to the elements of $\mathcal{P}_3$ as follows:

$$\omega(x_1) = \omega(x_{21}) = \omega(x_{12}) = \omega(x_3) = \omega(x_{32}) = \omega(x_{23}) = 3,$$

$$\omega(x_2) = \omega(x_{132}) = \omega(x_{213}) = 4,\ \omega(x_{123}) = \omega(x_{321}) = \omega(x_{2132}) = 0.$$

(we use the following rule: the weight of a generator x_ρ is defined as the number of other generators $x_{\rho'}$ such that x_ρ and $x_{\rho'}$ do not quasicommute.)

Inspecting Table 2, one observes the following: for each product of the generators $x = x_\rho x_{\rho'}$ that is not proportional to an element from B_0 the weight $\omega(x)$ is greater than the weight of every monomial occurring in the decomposition of x given by Table 2. Now the standard argument shows that every monomial in $\mathcal{P}_3$ can be transformed into a linear combination of elements of B_0 by a sequence of operations consisting of rearrangements of terms and replacements of each monomial $x = x_\rho x_{\rho'}$ as above by its expression from Table 2. Using this algorithm, one can verify in a straightforward way that B_0 is a string basis.

(b) Consider $\mathcal{P}_3$ as a simplicial complex whose simplices are subsets of mutually quasicommuting elements (see §1). By Theorem 9.1 (b), there are fourteen maximal simplices: to each of the conditions (Cj) ($j = 1, \ldots, 14$) is associated a simplex $\{x_{\rho_1}, \ldots, x_{\rho_6}\}$, where $m_{\rho_1}, \ldots, m_{\rho_6}$ are the exponents not occurring in (Cj). More transparent description of the simplices can be given as follows. First, each of the elements x_{321}, x_{123}, and x_{2132} has the property that adding it to each simplex of $\mathcal{P}_3$ gives us again a simplex. Hence, it is enough to describe simplices in $\mathcal{P}_3 \setminus \{x_{321}, x_{123}, x_{2132}\}$. One checks readily that these simplices correspond to complete subgraphs of the graph displayed in Figure 2 below:

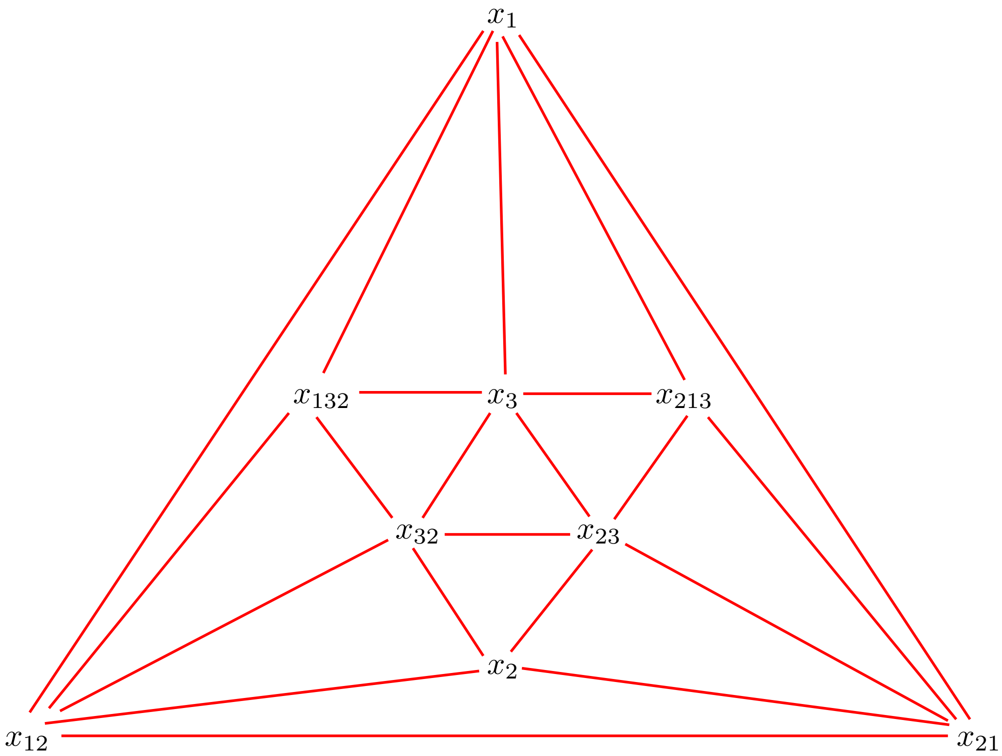

Fig.2. The simplicial complex on $\mathcal{P}_3 \setminus \{x_{321}, x_{123}, x_{2132}\}$.

10. Invariants $n_i(b)$ and the proof of Proposition 9.4

To prove Proposition 9.4 we need some notation which makes sense for general $\mathcal{A}_r$. For $i = 1, \ldots, r$ and $x \in \mathcal{A}_r$ we define $n_i(x) = l_i(x^*)$, where the antiautomorphism $x \mapsto x^*$ was defined in §3. It follows readily from Proposition 3.10 that

$$n_i(x) = \max\{n \in \mathbf{Z}_+ : E_i^{(n)}(E(x)) \text{ is a non} - \text{zero scalar for some monomial } E\}. \quad (10.1)$$

Here by a monomial E we mean any product of divided powers $E_j^{(n_j)}$. In view of (3.5) we have also

$$n_i(xy) = n_i(x) + n_i(y) \quad (10.2)$$

for $i = 1, \ldots, r$.

Now we return to the case $r = 3$. We retain the notation of §9. A direct calculation shows that the map $x \mapsto x^*$ leaves $\mathcal{P}_3$ invariant (cf. Remark (b) in the end of §7). More precisely, we have $x_\rho^* = x_{\rho_{opp}}$, where the index sequence ρ_{opp} is obtained from ρ by reversing the order of indices (we use the convention that two consecutive indices 1 and 3 can be interchanged without changing an element x_ρ, so $x_{132}^* = x_{213}, x_{213}^* = x_{132}, x_{2132}^* = x_{2132}$.) The values of $n_1(b), n_2(b), n_3(b)$ for $b \in \mathcal{P}_3$ are given in the last three columns of Table 1.

By (3.5) and (10.2), each of the functions $m \mapsto l_i(b(m))$ and $m \mapsto n_i(b(m))$ for $i = 1, 2, 3$ is a restriction to M of some linear form on $\mathbf{R}^{12}$. By slight abuse of notation, we denote these linear forms by the same symbols l_i or n_i. They can be read off the last six columns of Table 1, e.g., we have

$$l_1(m) = m_1 + m_{12} + m_{132} + m_{123}.$$

We also define the linear forms $g_1(m), g_2(m), g_3(m)$ as the coefficients in the formula

$$\deg (b(m)) = g_1(m)\alpha_1 + g_2(m)\alpha_2 + g_3(m)\alpha_3. \tag{10.3}$$

These forms can be read off from the second column of Table 1, e.g., we have

$$g_1(m) = m_1 + m_{21} + m_{12} + m_{132} + m_{213} + m_{321} + m_{123} + m_{2132}.$$

We shall deduce Proposition 9.4 from the following lemma.

Lemma 10.1.
(a) Every $m \in M$ satisfies the following linear constraints:

$$l_1 + l_2 + l_3 \geq g_1, \tag{10.4}$$

$$l_1 + l_2 + l_3 \geq g_3, \tag{10.5}$$

$$l_1 + l_2 + l_3 + n_2 \geq g_2, \tag{10.6}$$

$$l_1 + l_2 + n_1 \geq g_1, \tag{10.7}$$

$$l_1 + n_1 \geq 2g_1 - g_2, \tag{10.8}$$

$$l_2 + n_2 \geq 2g_2 - g_1 - g_3, \tag{10.9}$$

$$l_3 + n_3 \geq 2g_3 - g_2. \tag{10.10}$$

(b) The inequality (10.4) becomes an equality on $M_9 \cup M_{12}$, (10.5) becomes an equality on $M_1 \cup M_3$, (10.6) becomes an equality on M_{14}, (10.7) becomes an equality on M_{10}, (10.8) becomes an equality on $M_7 \cup M_8 \cup M_{11}$, (10.9) becomes an equality on $M_2 \cup M_4 \cup M_5 \cup M_{13}$, and (10.10) becomes an equality on M_6.

Proof. All the inequalities (10.4) to (10.10) are treated in the same way. For instance, consider (10.9). It follows from Table 1 that

$$l_2 + n_2 - (2g_2 - g_1 - g_3) = m_1 + m_3 + m_{132} + m_{213}. \tag{10.11}$$

This implies (10.9). Inspecting conditions (C1) to (C14), we see that the right hand side of (10.11) vanishes on $M_2 \cup M_4 \cup M_5 \cup M_{13}$, which proves the equality condition in (b). ◁

Proof of Proposition 9.4. Suppose the statement is not true, and let $m, m' \in M$ provide a counter-example of minimal degree. Using (2.1) and (10.1), we see that $l_i(m') \leq l_i(m), n_i(m') \leq n_i(m)$ for $i = 1, 2, 3$. Furthermore, suppose $l_i(m') = l_i(m) = l$ for some i. By Proposition 9.3, we have $E_i^{(l)}(b(m)) = b(m_0), E_i^{(l)}(b(m')) = b(m_0')$ for some distinct $m_0, m_0' \in M$. Then m_0 and m_0' also provide a counter-example to our statement. Since we have chosen m, m' as a counter-example of minimal degree, we conclude that $l = 0$. Summarizing, we have reduced the proof of Proposition 9.4 to the following statement.

Lemma 10.2. *Suppose $m, m' \in M$ satisfy the following properties:*
(1) $g_i(m) = g_i(m')$ for $i = 1, 2, 3$.
(2) $l_i(m') \leq l_i(m), n_i(m') \leq n_i(m)$ for $i = 1, 2, 3$.
(3) If $l_i(m') = l_i(m) = l$ for some i then $l = 0$.
Then $m' = m = 0$.

Proof of Lemma 10.2. We shall repeatedly use the following statement: if one of the inequalities (10.4) to (10.10) say (10.?), becomes an equality at m then we have $l_i(m) = 0$ for all the forms l_i appearing in the left hand side of (10.?). This follows at once by combining (10.?) with the conditions (1) to (3). Now we proceed in several steps.

Step 1. Suppose $m \in M_1 \cup M_3 \cup M_9 \cup M_{12} \cup M_{14}$. By Lemma 10.1 (b), at least one of (10.4), (10.5) and (10.6) becomes an equality at m, hence $l_1(m) = l_2(m) = l_3(m) = 0$. This obviously implies that $m' = m = 0$.

Step 2. Suppose $m \in M_{10}$. By Lemma 10.1 (b), (10.7) becomes an equality at m, hence $l_1(m) = l_2(m) = 0$. Combining this condition with (C10), we conclude that all the components of m are equal to 0 except maybe m_{32} and m_{321}. But then $m \in M_1$, and we find ourselves in the situation of Step 1.

The same argument works for m in each of the classes M_7, M_8 or M_{11}, which make (10.8) an equality. Now we obtain $l_1(m) = 0$. Combining this with (C8), we conclude that all the components of m are equal to 0 except maybe m_3, m_{213}, m_{321} and m_{2132}. But then $m \in M_3$, which again brings us to the situation of Step 1.

For $m \in M_6$ we use (10.10), and the same routine implies that m must belong to M_1, so we are again in the situation of Step 1.

Step 3. Finally, for $m \in M_2 \cup M_4 \cup M_5 \cup M_{13}$ we use (10.9) and conclude that all the components of m are equal to 0 except maybe m_{12}, m_{32}, m_{321} and m_{123}. But then $m \in M_{10}$, which brings us to the situation of Step 2.

Since we have covered all fourteen classes M_j, the proof of Lemma 10.2, and hence that of Proposition 9.4 is completed. $\triangleleft$

Remarks. (a) It is well-known that the canonical basis in $U_{+,r}$ is unvariant under the antiautomorhism $E \mapsto E^*$ (this was shown for arbitrary symmetrizable Kac-Moody algebras in [8, §2]). Using Proposition 3.10, we conclude that the dual basis in $\mathcal{A}_r$ is invariant under $x \mapsto x^*$. We do not know whether this is true for arbitrary string bases.

(b) The results in §§9,10 allow us to construct explicitly good bases of irreducible finite-dimensional sl_4-modules. This construction implies all the conjectures on good bases made in [2] for the case of sl_4. These questions will be treated in detail in a separate publication.

Appendix : Proofs of Propositions 1.1 to 1.3

Proofs of Propositions 1.1, 1.2. We recall that Q_+ is the semigroup generated by simple roots of the root system of type A_r, and that $U_+ = U_{+,r}$ is the Q_+-graded algebra over $\mathbf{Q}(q)$ generated by the elements $E_1, \ldots, E_r$ subject to the relations (1.1), (1.2). For $\gamma \in Q_+$ let $U_+^*(\gamma) = \mathrm{Hom}_{\mathbf{Q}(q)}(U_+(\gamma), \mathbf{Q}(q))$ be the dual of the homogeneous component in U_+, and let $U_+^* = \oplus_{\gamma \in Q_+} U_+^*(\gamma)$. To prove Propositions 1.1 and 1.2 we turn things around: we make U_+^* an associative algebra and check that it satisfies all the desired properties, and then show that it can be identified with our algebra $\mathcal{A}_r$.

Let $\mathcal{K} = \mathbf{Q}(q)[K_1, K_1^{-1}, \ldots, K_r, K_r^{-1}]$ be the (commutative) algebra of Laurent polynomials in variables $K_1, \ldots, K_r$. Let $\mathcal{K}U_+$ be an associative algebra over $\mathbf{Q}(q)$ generated by $\mathcal{K}$ and U_+ subject to the relations

$$K_i E_j = E_j K_i \text{ for } |i-j| > 1, \tag{A.1}$$

$$K_i E_j = q^{-1} E_j K_i \text{ for } |i-j| = 1. \tag{A.2}$$

The algebra $\mathcal{K}U_+$ is a subalgebra of $U_q(sl_{r+1})$ (see e.g., [10, Sec.1]). The following proposition is well-known (cf. [11, 4.1]).

Proposition A.1. *The algebra $\mathcal{K}U_+$ is a Hopf algebra with the comultiplication $\Delta : \mathcal{K}U_+ \to \mathcal{K}U_+ \otimes \mathcal{K}U_+$ given by*

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \ \Delta(K_i) = K_i \otimes K_i. \tag{A.3}$$

We extend the Q_+-grading from U_+ to $\mathcal{K}U_+$ by setting $(\mathcal{K}U_+)(\gamma) = \mathcal{K}(U_+(\gamma))$. We define

$$(\mathcal{K}U_+)^* = \oplus_{\gamma \in Q_+} (\mathcal{K}U_+)^*(\gamma),$$

where

$$(\mathcal{K}U_+)^*(\gamma) = \mathrm{Hom}_{\mathbf{Q}(q)}((\mathcal{K}U_+)(\gamma), \mathbf{Q}(q)).$$

The multiplication map $(\mathcal{K}U_+)^* \otimes (\mathcal{K}U_+)^* \to (\mathcal{K}U_+)^*$ adjoint to the comultiplication $\Delta : \mathcal{K}U_+ \to \mathcal{K}U_+ \otimes \mathcal{K}U_+$ makes $(\mathcal{K}U_+)^*$ a Q_+-graded associative algebra over $\mathbf{Q}(q)$.

Let $\varepsilon : \mathcal{K} \to \mathbf{Q}(q)$ be an algebra homomorphism specializing all K_i to 1. For $x \in U_+^*$ let $\tilde{x} \in (\mathcal{K}U_+)^*$ denote the extension of a linear form x given by $\tilde{x}(KE) = \varepsilon(K)x(E)$

for $K \in \mathcal{K}, E \in U_+$. Clearly, the correspondence $x \mapsto \tilde{x}$ is an embedding of graded spaces $U_+^* \to (\mathcal{K}U_+)^*$, and its image $\tilde{U}_+^*$ consists of all forms $y \in (\mathcal{K}U_+)^*$ such that $y(KH) = \varepsilon(K)y(H)$ for $K \in \mathcal{K}, H \in \mathcal{K}U_+$.

Proposition A.2. *The image $\tilde{U}_+^*$ of the map $x \mapsto \tilde{x}$ is a subalgebra of $(\mathcal{K}U_+)^*$.*

Proof. Let $y, z \in \tilde{U}_+^*$. It is enough to show that $(yz)(K_iE) = (yz)(E)$ for $E \in U_+$. Using definitions and (A.3), we see that

$$(yz)(K_iE) = (y \otimes z)(\Delta(K_iE)) = (y \otimes z)(\Delta(K_i)\Delta(E)) = (y \otimes z)((K_i \otimes K_i)\Delta(E))$$

$$= (y \otimes z)(\Delta(E)) = (yz)(E),$$

as desired. ◃

Using Proposition A.2, we transfer the multiplicative structure from $\tilde{U}_+^*$ to U_+^* via the bijection $x \mapsto \tilde{x}$. Thus, U_+^* becomes a graded associative algebra.

Let $(E, x) \mapsto E(x)$ be the action of U_+ on U_+^* adjoint to the action of U_+ on itself by right multiplication, i.e., given by $(E(x), E') = (x, E'E)$. For $i = 1, \ldots, r$ we define an element $x_i \in U_+^*(\alpha_i)$ by the normalization condition $(x_i, E_i) = 1$. Then all the statements in Propositions 1.1, 1.2 are valid if we replace $\mathcal{A}_r$ by U_+^*. They follow directly from definitions (for Proposition 1.2 (a) we use also the fact that U_+ is generated by $E_1, \ldots, E_r$). It remains to show that U_+^* can be identified with $\mathcal{A}_r$. More precisely, it suffices to prove the following.

Proposition A.3. *The elements $x_1, \ldots, x_r \in U_+^*$ generate the algebra U_+^*. They satisfy relations (0.1) and (0.2) from the introduction, and these relations form a defining set of relations for U_+^*.*

Proof. First we show that relations (0.1) and (0.2) are formal consequences of Propositions 1.1, 1.2, hence must hold in U_+^*. By Proposition 1.2 (a), it is enough to show that (0.1) and (0.2) remain true after applying every E_i to both sides. If $|i - j| > 1$ then by Proposition 1.1 (b), (c) we have $E_k(x_ix_j) = E_k(x_jx_i) = 0$ for $k \neq i, j$, $E_i(x_ix_j) = E_i(x_jx_i) = x_j$, and $E_j(x_ix_j) = E_j(x_jx_i) = x_i$. This implies (0.1). If $|i - j| = 1$, and x is the left hand side of (0.2) then using again Proposition 1.1 (b), (c) we obtain $E_k(x) = 0$ for $k \neq i, j$, and

$$E_i(x) = (1 + q^{-2})x_ix_j - (q + q^{-1})(x_jx_i + q^{-1}x_ix_j) + (q + q^{-1})x_jx_i = 0,$$

$$E_j(x) = q^2x_i^2 - (q + q^{-1})qx_i^2 + x_i^2 = 0.$$

This implies (0.2).

It remains to prove two statements:

(1) $x_1, \ldots, x_r$ generate U_+^*.

(2) The relations (0.1) and (0.2) form a system of defining relations for U_+^*.

Looking through the arguments in §3, we observe that both statements actually were not used there. We needed only the relations (0.1) and (0.2), Propositions 1.1 and 1.2, and the fact that $\mathcal{A}_r$ and U_+ have the same dimensions of homogeneous components. If we replace $\mathcal{A}_r$ by U_+^* the latter statement becomes obvious since $U_+^*(\gamma) = U_+(\gamma)^*$. We conclude that all the results in §3 remain true for U_+^*. In particular, the analog of Corollary 3.6 (b) implies the statement (1) above. The statement (2) follows from the fact that $\dim(\mathcal{A}_r(\gamma)) = \dim(U_+^*(\gamma))$ for all γ. This completes the proof of Proposition A.3, and hence the proofs of Propositions 1.1, 1.2. $\triangleleft$

Remark. We see that the $q-$deformations of the (commutative) algebra of functions $\mathbf{C}[N_+]$ and of the universal enveloping algebra $U(\underline{n}_+)$ are isomorphic to each other. This remarkable property was first observed by Drinfeld [4].

Proof of Proposition 1.3. Let B^* be the Lusztig's canonical basis in U_+, and B be the basis in $\mathcal{A}_r$ dual to B^*. In the course of the above proof of Propositions 1.1, 1.2 we have interpreted the multiplication in $\mathcal{A}_r$ and the action of U_+ on $\mathcal{A}_r$ in terms of the Hopf algebra structure on $\mathcal{K}U_+$. This allows us to reformulate each of the properties (S0) to (S3) in terms of B^*. Now the property (S0) is clear, (S1) is dual to [12, Theorem 11.5 (b)], (S2) is dual to [11, Proposition 7.2(a)], and (S3) is dual to [11, Theorem 7.5]. $\triangleleft$

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