UNITS OF INTEGRAL GROUP RINGS - A SURVEY

SUDARSHAN K. SEHGAL
Department of Mathematics, University of Alberta,
Edmonton, Alberta, Canada T6G 2G1

Introduction. Let $\mathbb{Z}G$ be the integral group ring of a finite group G and $U = U(\mathbb{Z}G)$ its group of units. Thus, for example, $\pm g, g \in G$, are units of $\mathbb{Z}G$. Further, if $A = \langle a \rangle$ is cyclic of order 10, then

$$u = -372099 + 114985a + 301035a^2 - 301035a^3 - 114985a^4 + 372100a^5$$
$$- 114985a^6 - 301035a^7 + 301035a^8 + 114985a^9$$

has an inverse

$$u^{-1} = -372099 - 301035a - 114985a^2 + 114985a^3 + 301035a^4 + 372100a^5$$
$$+ 301035a^6 + 114985a^7 - 114985a^8 - 301035a^9.$$

For a noncommutative example, let $G = S_3 = \langle a^3 = 1 = b^2,\ a^b = a^{-1} \rangle$ be the symmetric group on three letters. Then writing $\eta = (1-b)a(1+b) = a - a^2 + ab - a^2b$ we have $\eta^2 = 0$ and $1 + \eta = 1 + a - a^2 + ab - a^2b$ is a unit of infinite order with inverse $1 - \eta$. Moreover, $(1+\eta)a(1-\eta)$ is a nontrivial unit of order 3. We see that units can be complicated and fascinating. In this talk we shall give a survey of questions regarding their constructibility and related results. Incidently, there are only a few groups G where $U(\mathbb{Z}G)$ has been described in some way:

Abelian groups and K_8 by Higman [11], S_3 by Hughes-Pearson [17], D_4 by Polcino Milies [21], A_4 by Allen-Hobby [1], D_p by Passman-Smith [20], $G = C_p \rtimes C_q$ where q is a prime dividing $p-1$ by Galovitch-Reiner-Ullom [10], $|G| = p^3$ by Ritter-Sehgal [24], $G = D_n$ where n is an odd number which is a product of distinct primes by Kleinert [18], S_4 by Allen-Hobby [2] and D_n by Fernandes [9].

§2. Notations and preliminaries.

Let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $\varepsilon(\sum a_g g) = \sum_g a_g$ be the augmentation map and $\Delta = \Delta(G)$ the kernel of ε. More generally, for a normal subgroup N of G, $\Delta(G, N)$ is the kernel of the natural map $\mathbb{Z}G \to \mathbb{Z}(G/N)$. It is the ideal of $\mathbb{Z}G$ generated by $\Delta(N)$. Let $U_1(\mathbb{Z}G)$ be the units of $\mathbb{Z}G$ having augmentation 1 then $U(\mathbb{Z}G) = \pm U_1(\mathbb{Z}G)$.

2.1. Lemma. *Let $u = \sum u(g)g \in U_1(\mathbb{Z}G)$ be of finite multiplicative order with $u(1) \neq 0$. Then $u = u(1) = 1$.*

Proof: Consider $T(u)$ the trace of the regular representation of G. Then $T(u) = \sum u(g)T(g) = u(1)T(1) = u(1)|G|$. But $T(u)$ is the sum of eigenvalues ε_i of a diagonalizable matrix of finite order. Thus $u(1)|G| = \sum_1^{|G|} \varepsilon_i$, with ε_i being complex roots of unity. It follows that all ε_i are equal to 1 and $u = u(1) = 1$.

2.2. Corollary. *If A is a finite abelian group then the torsion units of $\mathbb{Z}A$ are all trivial.*

We need to know what the remaining units of $\mathbb{Z}A$ are. Let C be a cyclic group of order n. Then for the rational group algebra we have

$$\mathbb{Q}C \simeq \sum_{d|n}^{\oplus} \mathbb{Q}(\varsigma^d)$$

where ς is a primitive nth root of unity. The isomorphism is given by $a \to \sum^{\oplus} \varsigma^d$. Then $\mathbb{Z}C$ is mapped into the maximal order, $\mathbb{Z}C \to \sum^{\oplus} \mathbb{Z}[\varsigma^d] = M$. This means that units of $\mathbb{Z}M$ have the same torsion free rank as $U(\mathbb{Z}C)$. By applying Dirichlet's unit theorem we have

2.3. THEOREM OF HIGMAN. *If A is a finite abelian group then $U(\mathbb{Z}A) = \pm A \times F$ where F is free of rank $\rho(F) = \frac{1}{2}(|A| + n_2 - 2c + 1)$. Here, n_2 is the number of elements of order 2 in G and c is the number of cyclic subgroups of G.*

PROOF: See ([3], [11], [32]).

It is possible sometimes that $\rho(F) = 0$. This happens if every $\mathbb{Z}[\varsigma^d]$ above has no units of infinite order, i.e. if $n = 1,2,3,4$ or 6. Moreover, $U(\mathbb{Z}K_8) = \pm K_8$ if K_8 is the quaternion group of order 8. We have

2.4. THEOREM (HIGMAN). *$U(\mathbb{Z}G) = \pm G$ if and only G is abelian of exponent 2,3,4 or 6 or $G = K_8 \times E$ with $E^2 = 1$.*

PROOF: See [32].

We have a complete description (2.3) of U if G is an abelian group. But this does not give us all the units explicitly. It is futile to expect this as one can not write down the fundamental units of $\mathbb{Z}[\varsigma]$. However, it is possible to construct U upto finite index in this case. We shall do so now.

Suppose $a \in G$, $0(a) = n$, $|G| = m$. Then $\varphi(n) \mid \varphi(m)$. Let ς be an nth root of unity and $(i,n) = 1$. Then $\varsigma^i - 1/\varsigma - 1 = 1 + \varsigma + \cdots + \varsigma^{i-1}$ is a unit of $\mathbb{Z}[\varsigma]$ with the inverse $\frac{\varsigma-1}{\varsigma^i-1} = \frac{\varsigma^{ik}-1}{\varsigma^i-1} = 1 + \varsigma^i + \cdots + \varsigma^{i(k-1)}$ where $ik \equiv 1(\text{mod } n)$. This is a circular unit. Correspondingly, if we write $v = 1 + a + \cdots + a^{i-1} \in \mathbb{Z}\langle a\rangle$ then the projection of v in every component of

$$\mathbb{Z}\langle a\rangle \to \sum_{d|n}^{\oplus} \mathbb{Z}[\varsigma^d] = M$$

is a unit except when $d = n$ and then the projection is i. Since $(i,n) = 1$ and $\varphi(n) \mid \varphi(m), i^{\varphi(m)} = 1 + \ell n, \ell \in \mathbb{Z}$. So we modify v to

$$u = (1 + a + \cdots + a^{i-1})^{\varphi(m)} - \ell\hat{a}, \quad \hat{a} = 1 + a + \cdots + a^{n-1}.$$

Then $\hat{a}$ maps to 0 in every component but one, where it maps to n. We see that u is a unit of M and hence of $\mathbb{Z}\langle a\rangle$.

2.5. DEFINITION. *The units above as a varies over G and i runs over integers less than $0(a)$ and relatively prime to $0(a)$ are called the Bass cyclic units of $\mathbb{Z}G$.*

We have

2.6. THEOREM (BASS). *If G is abelian then the Bass cyclic units of $\mathbb{Z}G$ generate a subgroup of finite index in $U(\mathbb{Z}G)$.*

Bass [5] actually proved more; we shall need this version.

2.7. THEOREM (BASS). *Let G be a finite group and let $j : U(\mathbb{Z}G) \to K_1(\mathbb{Z}G)$ be the natural map. Then the images of the Bass cyclic units of $\mathbb{Z}G$ generate a subgroup of finite index in $K_1(\mathbb{Z}G)$.*

Let A be abelian. We have seen that $U(\mathbb{Z}C)$ as C runs over cyclic subgroups of A generate a subgroup of finite index in $U(\mathbb{Z}A)$. We shall call this index the Bass-Milnor index of A. Is there an estimate on this index?

2.8. THEOREM. *(i) Let A be an elementary abelian p-group of order p^{n+1}. Then the Bass-Milnor index of $A \leq p^N$ where*

$$N = n/2 \cdot (p-3)/2 \cdot (1 + p + \cdots + p^n).$$

(ii) Let A be an abelian p-group where p is a regular prime then the Bass-Milnor index of A is 1.

(iii) If A is an abelian p-group where p is not regular then the Bass-Milnor index of A is > 1.

Part (i) is contained in [16]. Parts (ii) and (iii) are results of Hoechsmann (For (ii) see [13], (iii) is to be published yet.)

§3. Construction in the Noncommutative Case.

In this section G is not necessarily commutative. Let B_1 be the group generated by the Bass Cyclic units of $\mathbb{Z}G$. Let H be the group generated by all $U(\mathbb{Z}C)$ where C runs through cyclic subgroups of G. Then H is not in general of finite index in $U(\mathbb{Z}G)$, to wit, for $G = S_3$ it follows from (2.4) that $H = \pm S_3$ which is not of finite index as $1+\eta$ of the introduction is of infinite order. But there is the

3.1. Theorem (Kleinert). *But for a few exceptions, H^U, the normal closure of H in U is of finite index in U.*

Proof: See [19]. The exceptions concern the 2×2 matrix rings, if any, appearing as Wedderburn Components of $\mathbb{Q}G$.

In order to generate U upto finite index we need more units. For $a, b \in G$, let $\eta = (a-1)b\hat{a}$, $\hat{a} = 1 + a + \cdots + a^{0(a)-1}$. Then $\eta^2 = 0$, $(1+\eta)(1-\eta) = 1$. We have a unit

$$u_{a,b} = 1 + (a-1)b\hat{a}.$$

3.2. Definition. *We call the units $u_{a,b}$ as $a, b \in G$, the bicyclic units of $\mathbb{Z}G$. We denote by B_2 the group generated by them.*

Let Z be the centre of U. Then we have

3.3. Theorem (Ritter-Sehgal). *But for a few exceptions, $\langle B_2, Z\rangle^U$ is of finite index in U if and only if G has no noncommutative homomorphic image which is fixed point free.*

Proof: See [26].

A group F is said to be fixed point free if it has a complex irreducible representation T such that for every nonidentity element f of F, $T(f)$ has no eigenvalue equal to 1. These groups are well known and were characterized by Zassenhaus in a classical paper [39].

The last two theorems do not really construct the units. However, fortunately, we have

3.4. THEOREM (RITTER-SEHGAL). *Let G be a nilpotent group. Suppose that $QG = \Sigma(K_i)_{n_i \times n_i}$, K_i fields. Suppose further that if $n_i = 2$ then $K_i \neq Q$ or $Q(\sqrt{-1})$. Then $\langle B_1, B_2\rangle$ is of finite index in $U(\mathbb{Z}G)$.*

The proof (see [27]) is dependent on the congruence subgroup theorems of Bass-Milnor-Serre [3] and Serre [34]. We make a few comments regarding the proof. Suppose that we have the Wedderburn decomposition:

$$QG = \sum^{\oplus}(K_i)_{n_i \times n_i}, \quad 0_i = \quad \text{the ring of integers in the algebraic number field } K_i.$$

Let π_i be the ith projection. We see first by using (2.7) that

(*) it is enough to prove: $B_2 \geq \prod_i F_i$, F_i of finite index in $SL(n_i, 0_i)$ if $n_i > 1$ and $F_i = 1$ elsewhere.

In order to prove (*) we prove

(3.5) $$(SL(n_i, 0_i) : \pi_i(B_2)) < \infty \quad \text{for all} \quad n_i > 1; \quad \text{and,}$$

(3.6) For $i \neq j$, $n_i, n_j > 1$, there exists $b \in B_2$ such that $\pi_i(b) = 1$ and $\pi_j(b)$ noncentral.

Theorem 3.4 is not true for all groups. We give below a group G of order 16 for which B is of infinite index in $U(\mathbb{Z}G)$.

Example 3.7. Let $G = \langle a, b \mid a^4 = 1 = b^4,\ a^b = a^{-1}\rangle$. Then $G \to \bar{G} = G/\langle b^2\rangle = D_8$, the dihedral group of order 8. We have

$$\mathbb{Q}\bar{G} = \mathbb{Q}(\bar{G}/\bar{G}') \oplus (\mathbb{Q})_{2\times 2}\,.$$

Let T be the representation of G corresponding to the faithful representation $\bar{T}$ of $\bar{G}$, namely,

$$T(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad T(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\bar{T}(B_2(D_8)) = \left\langle \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 4 \\ 0 & 1, \end{pmatrix},\ \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix},\ \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \right\rangle.$$

Notice that $\hat{\bar{x}} = r\,\hat{\bar{x}}$ with $r = 0(x)/0(\bar{x})$. Thus

$$T(u_{x,y}) = \bar{T}(\bar{u}_{x,y}) = \bar{T}(1 + r(\bar{x}-1)\bar{y}\hat{\bar{x}}) = \bar{T}(u_{\bar{x},\bar{y}})^r.$$

In our case, $0(\bar{x}) \neq 2 \Rightarrow u_{\bar{x},\bar{y}} = 1$, $0(\bar{x}) = 2 \Rightarrow 2 \mid r$. Thus

$$T(B_2(G)) \leq \left\langle \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^2,\ \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}^2 \right\rangle.$$

This group is of infinite index in $(\mathbb{Z})_{2\times 2}$.

§4. Torsion units.

(a) Zassenhaus Conjectures

Let us denote by $TU(\mathbb{Z}G)$ the set of torsion units of $\mathbb{Z}G$ and by $TU(\mathbb{Z}G)$ those of augmentation one. Then for $g \in G, \alpha \in U(\mathbb{Z}G)$, $\alpha^{-1}g\alpha = g^\alpha \in TU(\mathbb{Z}G)$. But these are not all the torsion units as is seen in S_3 already [17]. Keeping this in mind, Zassenhaus made the following conjectures:

(ZC1) $u \in TU_1(\mathbb{Z}G) \Rightarrow \exists\, g \in G,\ \alpha \in U(\mathbb{Q}G)$ such that $u = g^\alpha$.

(ZC2) $\mathbb{Z}G = \mathbb{Z}H \Rightarrow \exists\, \alpha \in U(\mathbb{Q}G)$ such that $H^\alpha = G$.

Both $ZC1$ and $ZC2$ are special case of a third conjecture:

(ZC3) H a finite subgroup of $U_1(\mathbb{Z}G) \Rightarrow \exists\ \alpha \in U(\mathbb{Q}G)$ such that $H^\alpha \subset G^\alpha$.

Recently $(ZC2)$ was proved for nilpotent groups by Roggenkamp and Scott [30] whereas $(ZC3)$ was proved by Weiss [37] for p-groups. The results proved are even stronger in that the conjugating element α is chosen from the p-adic group ring $\mathbb{Z}_pG$ by Weiss and in the appropriate semilocalization by Roggenkamp-Scott. Roggenkamp and Scott, at a conference in Oberwolfach (June 88), announced a counterexample to $(ZC2)$. Besides the p-group case, $(ZC1)$ is known to be true for the following cases:

(1) Split metacyclic groups $C_m \rtimes C_n$ with $(m,n) = 1$, due to Policino Milies - Ritter - Sehgal [22].

(2) Split metabelian $A \rtimes B$, B acting faithfully irreducibly on A, due to Sehgal - Weiss [33].

<u>(b) orders of torsion elements</u>

In order for $(ZC1)$ to be true we must know at least

$$u \in TU_1(\mathbb{Z}G), \quad 0(u) = m \Rightarrow \exists\ g \in G, \quad 0(g) = m. \tag{4.3}$$

In this regard, we can say

4.4. LEMMA. $u \in TU_1(\mathbb{Z}G), \quad 0(u) = m \Rightarrow m \mid |G|$.

PROOF: Consider $e = (1 + u + \cdots + u^{m-1})/m$. Then $e^2 = e = \sum e(g)g$ is an idempotent. We observe that due to (2.1), $e(1) = 1/m$. Moreover, by the regular representation argument in (2.1) we have $e(1) = s/|G|$ where s is the rank of the matrix of e. Thus $mS = |G|$ as required.

Of course, (4.3) is true for p-groups by the theorem of Weiss. But there is an easy argument of Zassenhaus for this.

4.5. THEOREM. *Let G be a finite p-group and let $u \in TU(\mathbb{Z}G)$ have order p^n. Then there is a $g \in G$ of order p^n.*

PROOF: Write $u = \sum u(g)g$. Then by [32, p. 4],

$$u^{p^{n-1}} \equiv \Sigma u(g) g^{p^{n-1}} + \lambda (\text{mod } p) \quad \text{for some} \quad \lambda \in [\mathbb{Z}G, \mathbb{Z}G].$$

Since $\lambda(1) = 0$ for $\lambda \in [\mathbb{Z}G, \mathbb{Z}G]$ we have

$$u^{p^{n-1}}(1) = \sum_{g^{p^{n-1}}=1} u(g) (\text{mod } p).$$

By (2.1) above we have $0 = u^{p^{n-1}}(1) = \sum_{g^{p^{n-1}}=1} u(g)(\text{mod } p)$. Now, we have

$$\begin{aligned} 1 = u^{p^n}(1) &= \sum_{g^{p^n}=1} u(g)(\text{mod } p) \\ &= \sum_{g^{p^{n-1}}=1} u(g) + \sum_{0(g)=p^n} u(g)(\text{mod } p). \end{aligned}$$

It follows that the second sum cannot be zero. Thus there is a $g \in G$ of order p^n with $u(g) \neq 0$.

4.6. COROLLARY. *Let G be nilpotent and $u \in TU_1(\mathbb{Z}G)$. Then there is a $g \in G$ with $0(g) = 0(u)$.*

Now let G be metabelian with a normal abelian subgroup A such that G/A is abelian. Let $u \in TU_1(\mathbb{Z}G)$. In the homomorphic image $G/A = \bar{G}$, we see that by (2.2), $\bar{u} = \bar{g}_0, \quad g_0 \in G$. Thus

$$u = g_0 + \delta, \quad \delta \in \Delta(G, A) = \langle (1-a) \mid a \in A \rangle_{\mathbb{Z}G}.$$

We have,

$$u = g_0 + \Sigma\alpha(a-1), \quad \alpha \in \mathbb{Z}G, \quad a \in A. \qquad \text{Then}$$

$$u \equiv g_0 + \Sigma z(a-1) \quad \text{mod } (\Delta G \Delta A), \quad z = \epsilon(\alpha).$$

Using the identity $(x-1)+(y-1)=(xy-1)-(x-1)(y-1)$ we conclude that

$$u \equiv g_0 + (a_0 - 1) \bmod (\Delta G \Delta A), \quad \text{for some } a_0 \in A$$

$$\equiv 1 + (g_0 - 1) + (a_0 - 1) \bmod (\Delta G \Delta A).$$

Again, by the same argument

$$u \equiv g \bmod (\Delta G \Delta A), \quad g = g_0 a_0.$$

It is known [6] that in this case $U(1+\Delta G \Delta A)$ is torsion free form which is follows that $0(u) = 0(g)$.

(c) Normal Complements

In some cases, $U(1+\Delta G \Delta A)$ is a normal torsion free complement of G in $U_1(\mathbb{Z}G)$. This would be an extension of (2.2) to the noncommuative case, as was proposed by K. Dennis [7]. Namely,

(4.6) Is it true that $\exists\, N \triangleleft U_1(\mathbb{Z}G)$, N torsion free so that $U_1(\mathbb{Z}G) = N \rtimes G$?

This was proved to be true for groups G having an abelian subgroup of index 2 by Passman Smith [20] and for circle groups of nilpotent rings by Sandling [30] and Passman-Smith [20]. Further, Cliff - Sehgal - Weiss [6] proved the following

4.7. THEOREM. *Let G be a finite group having an abelian normal subgroup A such that either*

a) *G/A is abelian of exponent 2,3,4,6 or*

b) *G/A is abelian of odd order.*

Then G has a normal torsion free complement in $U_1(\mathbb{Z}G)$.

Roggenkamp and Scott [28] have given examples to show that this theorem can not be extended to arbitrary metabelian groups. The counterexamples are $G = C_{241} \rtimes C_{10}$

and $G = C_{73} \rtimes C_8$. However, the theorem is true for nilpotent class two groups. One may consult [28, p. 39] for one of the proofs of this. It remains to decide:

(4.8) Is (4.6) true for nilpotent groups?

The normal complement of (4.7) is a generalization of $1+\Delta G\Delta A$. It is quite hard to prove that it is, in fact, torsion free [6]. We ask

(4.9) Let G be a finite group and A a normal abelian subgroup. Is $U(1+\Delta G\Delta A)$ torsion free?

(4.9) is easily seen to be true if G is nilpotent. Use induction and factor by a central subgroup $\langle z\rangle$. Then if $u \in TU(1+\Delta G\Delta A)$ we conclude that $u \equiv 1 \mod \Delta(G,\langle z\rangle)$. It follows that there is an element z_1 of $\langle z\rangle$ in the support of u. Thus $uz_1^{-1} = \sum \alpha(g)g$ with $\alpha(1) \neq 0$ is a torsion unit. It follows by (2.1) that $u = z_1$. We have $z_1 \equiv 1(\Delta G\Delta A)$, $z_1 \in G$. Now, an easy argument gives that $z_1 = 1$.

References

[1] P.J. Allen and C. Hobby, *A characterization of the units in $\mathbb{Z}[A_4]$*, J. Algebra 66 (1980), 534–543.

[2] P.J. Allen and C. Hobby, *A characterization of units in $\mathbb{Z}S_4$*, Comm. Algebra 16 (7) 1988, 1479–1505.

[3] R.G. Ayoub and C. Ayoub, *On the group ring of a finite abelian group*, Bull. Austral. Math. Soc. **1** (1969), 245–261.

[4] H. Bass, *The Dirichlet unit theorem and Whitehead groups of finite groups*, Topology **4** (1966), 391–410.

[5] H. Bass, J. Milnor and J.P. Serre, *Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n} (n \geq 2)$*; Publ. Math. I.H.E.S. **33** (1967), 59–137.

[6] G.H. Cliff, S.K. Sehgal and A.R. Weiss, *Units of integral group rings of metabelian groups*, J. Algebra **73** (1981), 167–185.

[7] R.K. Dennis, The structure of the unit group of group rings, In "Ring Theory II", Proc. Second Cong., Univ. Oklahoma, Norman, Okla, 1975, pp. 103–130, Dekker, New York, 1977.

[8] N. Fernandes, Zassenhaus Conjecture for S_4, Bol. Soc. Brasil. Mat. (to appear).

[9] N. Fernandes, A characterization of units in $\mathbb{Z}D_n$, preprint.

[10] S. Galovitch, I. Reiner and S. Ullom, *Class groups for integral representations of metacyclic groups*, Mathematika **19** (1972), 105–111.

[11] G. Higman, *The units of group rings*, Proc. London Math. Soc. (2) **46** (1940), 231–248.

[12] K. Hoechsmann, *Units and class-groups in integral elementary abelian group rings*, J. Pure and Applied Algebra (to appear).

[13] K. Hoechsmann, *On the Bass-Milnor index of abelian p-groups*, Contemporary Mathematics (to appear).

[14] K. Hoechsmann and S.K. Sehgal, *Units in regular elementary abelian group rings*, Arch. Math., **47** (1986), 413–417.

[15] K. Hoechsmann and S.K. Sehgal, *Units in regular abelian p-group rings*, J. Number Theory, to appear.

[16] K. Hoechsmann, S.K. Sehgal and A. Weiss, *Cyclotomic units and the unit group of*

an elementary abelian group ring, Arch. Math. **45** (1985), 5–7.

[17] I. Hughes and K.R. Pearson, *The group of units of the integral group ring $\mathbb{Z}S_3$*, Can. Math. Bull. **15** (1972), 529–534.

[18] E. Kleinert, *Einheiten in $\mathbb{Z}[D_{2m}]$*, J. Number Theory **13** (1981), 541–561.

[19] E. Kleinert, *A theorem on units of integral group rings*, J. Pure and Applied Algebra, **49** (1987), 161–171.

[20] D.S. Passman and P.F. Smith, *Units in integral group rings*, J. Algebra **69** (1981), 213–239.

[21] C. Polcino Milies, *The units of the integral group ring $\mathbb{Z}D_4$*, Bol. Soc. Brasil. Mat. **4** (1972), 85–92.

[22] C. Polcino Milies, J. Ritter and S.K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings II*, Proc. Amer. Math. Soc. **97** (1986), 201–206.

[23] C. Polcino Milies and S.K. Sehgal, *Torsion units in integral group rings of metacyclic groups*, J. Number Theory **19** (1984), 103–114.

[24] J. Ritter and S.K. Sehgal, *Integral group rings of some p-groups*, Can. J. Math. **26** (1982), 233–246.

[25] J. Ritter and S.K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings*, Math. Ann. **264** (1983), 257–270.

[26] J. Ritter and S.K. Sehgal, *Generators of subgroups of $U(\mathbb{Z}G)$*, Contemporary Math., to appear.

[27] J. Ritter and S.K. Sehgal, *Construction of units in integral group rings of finite nilpotent groups*, preprint.

[28] J. Ritter and S.K. Sehgal, *Isomorphism of group rings*, Arch. Math. **40** (1983), 32–39.

[29] K.W. Roggenkamp and L.L. Scott, "Units in metabelian group rings II, Non-splitting examples for normalized units", Quart. J. Math. Oxford Series (2) 1981.

[30] K. Roggenkamp and L. Scott, *Isomorphisms of p-adic group rings*, Ann. of Math. **126** (1987), 593–647.

[31] R.S. Sandling, *Group rings of circle and unit groups*, Math. Z. **140** (1974), 195–202.

[32] S.K. Sehgal, Topics in Group Rings, Dekkar, New York, 1978.

[33] S.K. Sehgal and A. Weiss, *Torsion units in integral group rings of some metabelian groups*, J. Algebra.

[34] J.P. Serre, *Le problème des groupes de congruence pour SL_2*, Ann. of Math. **92** (1970), 489–527.

[35] L.N. Vaserstein, *On the group SL_2 over Dedekind rings of arithmetic type*, Math. USSR Sbornik **18** (1973), 321–332.

[36] L.N. Vaserstein, *The structure of classic arithmetic groups of rank greater than one*, Math. USSR Sbornik **20** (1973), 465–492.

[37] A. Weiss, *Rigidity of p-adic p-torsion*, Ann. of Math. **127** (1988), 317–332.

[38] A. Whitcomb, The group rings problem, Ph.D. Thesis, University of Chicago (1968).

[39] H. Zassenhaus, *über endliche Fastkörper*, Hamburg Abb. **11** (1935), 187–220.