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Изв. Акад. Наук СССР
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ON CERTAIN COMMUTATIVE SUBALGEBRAS OF A UNIVERSAL ENVELOPING ALGEBRA

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È. B. VINBERG

ABSTRACT. The question is posed of lifting, to the universal enveloping algebra of a semisimple Lie algebra, the commutative subalgebras constructed by A. S. Mishchenko and A. T. Fomenko in the corresponding Poisson-Lie-Berezin algebra by the translation-of-invariants method. It is shown that such a lifting is in any case possible up to degree 2. A characterization is given of the commutative subspaces of elements of degree 2 so obtained in the universal enveloping algebra, as well as a description of the subspaces obtained from them by a limit passage. In the case of the algebra of all matrices with zero trace, a connection is exhibited between one of these limit subspaces and the construction of a Gel'fand-Tsetlin basis.

Bibliography: 7 titles.

0.1. The universal enveloping algebra $\mathscr{U}(\mathscr{G})$ of a Lie algebra $\mathfrak{g}$ has an increasing filtration

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{k=0}^{\infty} \mathscr{U}^k(\mathfrak{g}), \tag{1}$$

where $\mathscr{U}^k(\mathfrak{g})$ is the subspace of elements representable as (non-commutative) polynomials of degree $\leq k$ in the elements of $\mathfrak{g}$. From the defining relations of the algebra $\mathscr{U}(\mathfrak{g})$ it follows that

$$[\mathscr{U}^k(\mathfrak{g}), \mathscr{U}^l(\mathfrak{g})] \subset \mathscr{U}^{k+l-1}(\mathfrak{g}). \tag{2}$$

The associated graded algebra

$$\operatorname{gr}\mathscr{U}(\mathfrak{g}) = \mathscr{P}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathscr{P}_k(\mathfrak{g}), \qquad \mathscr{P}_k(\mathfrak{g}) = \mathscr{U}^k(\mathfrak{g})/\mathscr{U}^{k-1}(\mathfrak{g}), \tag{3}$$

is canonically isomorphic, by the Poincaré-Birkhoff-Witt theorem, to the symmetric algebra of the space $\mathfrak{g}$. But besides the commutative-associative operation of multiplication, there is defined in it, thanks to relation (2), a natural Lie operation $\{\ ,\ \}$—the "Poisson-Lie-Berezin bracket" (we shall call it simply the *Poisson bracket*), for which

$$\{\mathscr{P}_k(\mathfrak{g}), \mathscr{P}_l(\mathfrak{g})\} \subset \mathscr{P}_{k+l-1}(\mathfrak{g}). \tag{4}$$

Namely, for $u \in \mathscr{U}^k(\mathfrak{g})$ and $v \in \mathscr{U}^l(\mathfrak{g})$ we take

$$\{u + \mathscr{U}^{k-1}(\mathfrak{g}), v + \mathscr{U}^{l-1}(\mathfrak{g})\} = [u, v] + \mathscr{U}^{k+l-2}(\mathfrak{g}).$$

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(Because of (2), this is well defined.) For other, equivalent definitions, see [5].

The Poisson bracket is connected with multiplication by the *Leibniz identity*

$$\{x, yz\} = \{x, y\}z + y\{x, z\}. \tag{5}$$

It is uniquely determined by this property and the "initial conditions" $\{x, y\} = [x, y]$ for $x, y \in \mathfrak{g}$.

Two elements $x, y \in \mathscr{P}(\mathfrak{g})$ will be said to commute if $\{x, y\} = 0$. It is in accordance with this definition that we shall understand such terms as "commutative subspace" (in particular, "commutative subalgebra") and "centralizer" (of an arbitrary subset) in the algebra $\mathscr{P}(\mathfrak{g})$. From (5) it follows that the subalgebra generated by any pairwise commuting elements of $\mathscr{P}(\mathfrak{g})$ is commutative.

0.2. Let G be a connected Lie group, of which $\mathfrak{g}$ is the tangent algebra. If we interpret the elements of the algebra $\mathscr{P}(\mathfrak{g})$ as being polynomials on the space $\mathfrak{g}^*$ conjugate to $\mathfrak{g}$, then the Poisson bracket in $\mathscr{P}(\mathfrak{g})$ coincides with the usual Poisson bracket of functions on the orbits of the coadjoint representation of the group G with respect to the canonical symplectic structure on these orbits [5]. It follows that the transcendence degree of any commutative subalgebra of $\mathscr{P}(\mathfrak{g})$ does not exceed $d(\mathfrak{g}) = (\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$, where $\operatorname{ind} \mathfrak{g}$ (the index of $\mathfrak{g}$) is the codimension of the orbit in general position of the coadjoint representation of G. (For $\mathfrak{g}$ semisimple, the index is equal to the rank.) In [4] (see also [5]), for any semisimple Lie algebra $\mathfrak{g}$ commutative subalgebras of $\mathscr{P}(\mathfrak{g})$ are constructed, by a translation-of-invariants method, whose transcendence degree is equal to $d(\mathfrak{g})$.

If A is a commutative subalgebra of $\mathscr{U}(\mathfrak{g})$, then $\operatorname{gr} A = L$ is a commutative subalgebra of $\mathscr{P}(\mathfrak{g})$. We shall say that A is a subalgebra of finite type if L is finitely generated. It is easily shown (see Proposition 1.1) that in this case the transcendence degrees of the algebras A and L coincide. Thus, the transcendence degree of any commutative subalgebra of $\mathscr{U}(\mathfrak{g})$ of finite type cannot exceed $d(\mathfrak{g})$. Do there always exist in $\mathscr{U}(\mathfrak{g})$ commutative subalgebras of finite type with transcendence degree *equal* to $d(\mathfrak{g})$? There *are* examples. Specifically, the construction of a *Gel′fand-Tsetlin basis* [3] in the space of an irreducible representation of the Lie algebra $\mathfrak{Sl}_n$ or $\mathfrak{So}_n$ is connected with such a subalgebra of, respectively, $\mathscr{U}(\mathfrak{Sl}_n)$ or $\mathscr{U}(\mathfrak{So}_n)$.

The present paper attempts, for an arbitrary semisimple Lie algebra $\mathfrak{g}$, to "quantize"—i.e., lift into $\mathscr{U}(\mathfrak{g})$—the commutative subalgebras of $\mathscr{P}(\mathfrak{g})$ that are obtained by the translation-of-invariants method. Unfortunately, the author has so far succeeded in doing this only up to degree 2. An indirect argument in favor of the possibility of such a lifting is the fact that the image in $\mathscr{P}(\mathfrak{Sl}_n)$ of the above-mentioned commutative subalgebra of $\mathscr{U}(\mathfrak{Sl}_n)$ connected with the Gel′fand-Tsetlin basis, although it is not actually obtained by the translation-of-invariants method, is at least a limit of subalgebras so obtained (see §6 and the example in §0.5).

Our discussion also provides a new approach to the definition of these same Mishchenko-Fomenko subalgebras, revealing a greater naturalness than could have been supposed on the basis of their original definition.

§**0.3.** We come now to a precise statement of the results. First, some notation.

Let $\mathfrak{g}$, to begin with, be an arbitrary Lie algebra. Then $\mathscr{U}(\mathfrak{g})$ and $\mathscr{P}(\mathfrak{g})$ can both be regarded as Lie algebras relative to the operations of commutation and Poisson bracket, respectively; and the algebra $\mathfrak{g}$ is canonically imbedded in each of them. The image of an element $x \in \mathfrak{g}$ in $\mathscr{P}(\mathfrak{g})$ will be identified with x (and denoted by the same letter), while the image of x in $\mathscr{U}(\mathfrak{g})$ will be denoted by $\hat{x}$. The imbedding

$\mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ extends to a vector space isomorphism $\mathscr{P}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$, for which

$$x_1 \cdots x_k \to \widehat{x_1 \cdots x_k} \doteqdot \frac{1}{k!} \sum_{\sigma \in S_k} \hat{x}_{\sigma 1} \cdots \hat{x}_{\sigma k} \qquad (x_1, \dots, x_k \in \mathfrak{g}). \tag{6}$$

The imbeddings of $\mathfrak{g}$ into $\mathscr{U}(\mathfrak{g})$ and $\mathscr{P}(\mathfrak{g})$ define on each of these spaces a $\mathfrak{g}$-module structure. The mapping (6) is a $\mathfrak{g}$-module isomorphism. Put $\mathscr{U}_k(\mathfrak{g}) = \widehat{\mathscr{P}_k}(\mathfrak{g})$. Then

$$\mathscr{U}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathscr{U}_k(\mathfrak{g}) \tag{7}$$

is a decomposition into a direct sum of $\mathfrak{g}$-submodules (but not, in general, a grading of the algebra!).

Now let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $(\ , \ )$ be any invariant inner product in $\mathfrak{g}$. In each root subspace $\mathfrak{g}_\alpha$ choose an element e_α so that $(e_\alpha, e_{-\alpha}) = 1$.

We shall be examining the commutative subalgebras of $\mathscr{P}(\mathfrak{g})$ (resp. $\mathscr{U}(\mathfrak{g})$) that contain $\mathfrak{h}$ (resp. $\hat{\mathfrak{h}}$).

The centralizer of $\mathfrak{h}$ in $\mathscr{P}_1(\mathfrak{g}) = \mathfrak{g}$ coincides with $\mathfrak{h}$; the centralizer of $\mathfrak{h}$ in $\mathscr{P}_2(\mathfrak{g})$ is the direct sum of $\mathscr{P}_2(\mathfrak{h})$ and the linear span of products of the form $e_\alpha e_{-\alpha}$ $(\alpha \in \Delta)$, which we denote by Q. Correspondingly, the centralizer of $\hat{\mathfrak{h}}$ in $\mathscr{U}_2(\mathfrak{g})$ is $\mathscr{U}_2(\mathfrak{h}) \oplus \widehat{Q}$.

Every element $q \in Q$ has a unique representation in the symmetric form

$$q = \sum_{\alpha \in \Delta} q_\alpha e_\alpha e_{-\alpha} \qquad (q_\alpha = q_{-\alpha} \in \mathbf{C}). \tag{8}$$

Then also

$$\hat{q} = \sum_{\alpha \in \Delta} q_\alpha \hat{e}_\alpha \hat{e}_{-\alpha}. \tag{9}$$

In particular, if $\{h_1, \dots, h_l\}$ is an orthonormalized basis for $\mathfrak{h}$, then the element

$$c(\mathfrak{g}) = \sum_{i=1}^{l} h_i^2 + \sum_{\alpha \in \Delta} e_\alpha e_{-\alpha} \in \mathscr{P}_2(\mathfrak{h}) \oplus Q$$

lies in the center of the whole algebra $\mathscr{P}(\mathfrak{g})$. What corresponds to it in $\mathscr{U}(\mathfrak{g})$ is the *Casimir element*

$$\hat{c}(\mathfrak{g}) = \sum_{i=1}^{l} \hat{h}_i^2 + \sum_{\alpha \in \Delta} \hat{e}_\alpha \hat{e}_{-\alpha} \in \mathscr{U}_2(\mathfrak{h}) \oplus \widehat{Q},$$

lying in the center of $\mathscr{U}(\mathfrak{g})$. The element $c(\mathfrak{g})$ will also be called a Casimir element.

0.4. The next step is to ascertain what subspaces in Q (resp. in $\widehat{Q}$) are commutative. Consider an arbitrary set $(q_1, \dots, q_k) \subset Q$, where

$$q_i = \sum_{\alpha \in \Delta} q_{i,\alpha} e_\alpha e_{-\alpha}, \qquad (q_{i,\alpha} = q_{i,-\alpha} \in \mathbf{C}). \tag{10}$$

To each root $\alpha \subset \Delta$ assign the point

$$p_\alpha = (q_{1,\alpha}, \dots, q_{k,\alpha}) \in \mathbf{C}^k \tag{11}$$

(so that $p_\alpha = p_{-\alpha}$).

THEOREM 1. *The following conditions are equivalent:*

1) $\{q_i, q_j\} = 0$ *for all* i *and* j.
2) $[\hat{q}_i, \hat{q}_j] = 0$ *for all* i *and* j.
3) *For any roots* $\alpha, \beta, \gamma \in \Delta$ *such that* $\alpha + \beta + \gamma = 0$, *the points* $p_\alpha, p_\beta, p_\gamma$ *are collinear.*

A set $(q_1, \dots, q_k)$ that satisfies these equivalent conditions will be called *commutative.* The next theorem gives an explicit description of commutative sets, under certain restrictions.

We say that $(q_1, \dots, q_k)$ is a *separating set* if $p_\alpha \neq p_\beta$ for $\alpha \neq \pm\beta$.

THEOREM 2. *For any regular element* $h_0 \in \mathfrak{h}$ *and any elements* $h_1, \dots, h_k \in \mathfrak{h}$, *the set* $(q_1, \dots, q_k) \subset Q$ *defined by*

$$q_i = \sum_{\alpha \in \Delta} \frac{\alpha(h_i)}{\alpha(h_0)} e_\alpha e_{-\alpha} \qquad (i = 1, \dots, k) \tag{12}$$

is commutative. Conversely, if $\mathfrak{g}$ *is simple, then any separating commutative set* $(q_1, \dots, q_k) \subset Q$ *of rank* ≥ 3 *is obtained in this way.*

The rank of the set $(q_1, \dots, q_k)$ defined by (12) is obviously equal to that of the set $(h_1, \dots, h_k)$, and therefore does not exceed $l = \operatorname{rk} \mathfrak{g}$.

COROLLARY. *For any semisimple Lie algebra* $\mathfrak{g}$, *the space* Q *has no separating commutative set in it of rank* $> l = \operatorname{rk} \mathfrak{g}$.

The separating condition in the statement of the theorem is essential, as shown by the following example. Suppose $\mathfrak{g}$ contains a proper semisimple regular subalgebra $\mathfrak{g}_1$ of the same rank l. Let $\Delta_1 \subset \Delta$ be the root system for $\mathfrak{g}_1$, and Q_1 the linear span of products $e_\alpha e_{-\alpha}$, where $\alpha \in \Delta_1$. Using (12) applied to $\mathfrak{g}_1$, we construct a linearly independent commutative set of l elements in Q_1. Then adjoining to it the element $\sum_{\alpha \in \Delta} e_\alpha e_{-\alpha}$ (the projection on Q of the Casimir element of $\mathscr{P}(\mathfrak{g})$), we obtain a linearly independent commutative set of $l + 1$ elements of Q.

The condition on the rank is also essential, since the projection on Q of the Casimir element forms together with any element $q \in Q$ a commutative set, but in general it is hardly the case that an arbitrary element $q \in Q$ can be represented in the form

$$\sum_{\alpha \in \Delta} \frac{\alpha(h)}{\alpha(h_0)} e_\alpha e_{-\alpha}.$$

According to the first part of the theorem, to every regular element $h_0 \in \mathfrak{h}$ one can associate an l-dimensional commutative subspace

$$Q(h_0) = \left\{ \sum_{\alpha \in \Delta} \frac{\alpha(h)}{\alpha(h_0)} e_\alpha e_{-\alpha} : h \in \mathfrak{h} \right\} \subset Q. \tag{13}$$

On the other hand, the translation-of-invariants method constructs for each regular element $h_0 \in \mathfrak{h}$ a homogeneous commutative subalgebra $L(h_0)$ of $\mathscr{P}(\mathfrak{g})$ with transcendence degree $d(\mathfrak{g})$ [5]. It is known [6] that

$$L(h_0) \cap \mathfrak{g} = \mathfrak{h}, \qquad L(h_0) \cap \mathscr{P}_2(\mathfrak{g}) = \mathscr{P}_2(\mathfrak{h}) \oplus Q(h_0). \tag{14}$$

By Theorem 1, the subspace $Q(h_0) \subset Q$ lifts to a commutative subspace $\widehat{Q}(h_0) \subset \widehat{Q}$, whose elements, furthermore, commute with those of $\hat{\mathfrak{h}}$. Thus, the generators of degree one and two of the algebra $L(h_0)$ lift to pairwise commuting elements of the algebra $\mathscr{U}(\mathfrak{g})$.

0.5. By passing to limits, we can obtain from the subspaces of the form $Q(h_0)$ other l-dimensional commutative subspaces of the space Q. All the subspaces so obtained (including the subspaces $Q(h_0)$ themselves) we shall call the *principal commutative subspaces*. To describe them explicitly, we now introduce certain additional notions.

We shall be considering convergent power series with coefficients in $\mathfrak{h}$, in a variable t taking values in $\mathbf{C}$. To distinguish series as such from their values, we denote the series by boldface letters and their values by the corresponding ordinary letters (with the value of the variable indicated). We call a series

$$\boldsymbol{h}_0 = h_0^{(0)} + h_0^{(1)}t + h_0^{(2)}t^2 + \cdots \qquad (h_0^{(k)} \in \mathfrak{h}) \tag{15}$$

regular if it satisfies the following equivalent conditions:

1) $\mathbf{h}_0$ is a regular (semisimple) element of the algebra $\mathfrak{g} \otimes \mathbf{C}((t))$ over the field $\mathbf{C}((t))$ of Laurent series.

2) The element $h_0(t) \in \mathfrak{h}$ is regular for all sufficiently small $t \neq 0$.

3) No root $\alpha \in \Delta$ vanishes on every element $h_0^{(0)}$, $h_0^{(1)}$, $h_0^{(2)}$, ...

For any series

$$\boldsymbol{h} = h^{(0)} + h^{(1)}t + h^{(2)}t^2 + \cdots \qquad (h^{(k)} \in \mathfrak{h}) \tag{16}$$

we denote by $\Delta_k(\boldsymbol{h})$ the subsystem of roots that vanish on $h^{(0)}$, $h^{(1)}, \ldots, h^{(k-1)}$. Clearly,

$$\Delta = \Delta_0(\boldsymbol{h}) \supset \Delta_1(\boldsymbol{h}) \supset \Delta_2(\boldsymbol{h}) \supset \cdots \tag{17}$$

The series $\boldsymbol{h}_0$ is regular if and only if $\Delta_m(\boldsymbol{h}_0) = \varnothing$ for some m. We say that a series $\boldsymbol{h}$ is *subordinate* to a series $\boldsymbol{h}_0$ if

$$\Delta_k(\boldsymbol{h}) \supset \Delta_k(\boldsymbol{h}_0) \quad \text{for all } k. \tag{18}$$

In this case, for $\alpha \in \Delta_k(\boldsymbol{h}_0) \backslash \Delta_{k+1}(\boldsymbol{h}_0)$ there exists the limit

$$\lim_{t\to 0} \frac{\alpha(h(t))}{\alpha(h_0(t))} = \frac{\alpha(h^{(k)})}{\alpha(h_0^{(k)})}, \tag{19}$$

and therefore, if $\boldsymbol{h}_0$ is regular, the limit

$$\lim_{t\to 0} \sum_{\alpha\in\Delta} \frac{\alpha(h(t))}{\alpha(h_0(t))} e_\alpha e_{-\alpha} \in Q. \tag{20}$$

For fixed $\boldsymbol{h}_0$ the elements of the form (20) form a subspace, which we denote by $Q(\boldsymbol{h}_0)$.

THEOREM 3. *For every regular series* $\boldsymbol{h}_0$,

$$Q(\boldsymbol{h}_0) = \lim_{t\to 0} Q(h_0(t)),$$

and therefore $Q(\boldsymbol{h}_0)$ *is a principal commutative subspace of the space* Q. *Conversely, every principal commutative subspace is obtained in this way.*

Different regular series can, of course, determine the same commutative subspace. Taking this into account, we can give a more effective description of the principal commutative subspaces.

We use the invariant inner product to identify $\mathfrak{h}^*$ with $\mathfrak{h}$; the roots can then be regarded as elements of $\mathfrak{h}$. We call a regular series $\boldsymbol{h}_0$ *canonical* if for every k the following conditions hold:

(C1) $h_0^{(k)} \in \langle \Delta_k(\boldsymbol{h}) \rangle$.

(C2) The projection of $h_0^{(k)}$ on the linear span of every indecomposable component of the system $\Delta_k(\boldsymbol{h}_0)$ is different from zero.

Condition (C2) implies that if $\Delta_k(\boldsymbol{h}_0) \neq \varnothing$, then $\Delta_{k+1}(\boldsymbol{h}_0 \neq \Delta_k(\boldsymbol{h}_0)$. Hence a canonical series is automatically finite.

Two canonical series $\boldsymbol{h}_0$ and $\tilde{\boldsymbol{h}}_0$ are called *equivalent* if for any k the following conditions hold:

(E1) $\Delta_k(\boldsymbol{h}_0) = \Delta(\tilde{\boldsymbol{h}}_0)$.

(E2) The projections of the elements $h_0^{(k)}$ and $\tilde{h}_0^{(k)}$ on the linear span of every indecomposable component of the system $\Delta_k(\boldsymbol{h}_0)$ are proportional.

(Note that if (E1) and (E2) are satisfied for any one k, then automatically $\Delta_{k+1}(\boldsymbol{h}_0) = \Delta_{k+1}(\tilde{\boldsymbol{h}}_0)$.)

THEOREM 4. *Every principal commutative subspace of the space Q is of the form $Q(\boldsymbol{h}_0)$, where $\boldsymbol{h}_0$ is a canonical regular series; and $\boldsymbol{h}_0$ is determined uniquely up to equivalence.*

To this can be added the observation that if $\boldsymbol{h}_0$ is a canonical regular series, then in the definition above of the subspace $Q(\boldsymbol{h}_0)$ we can restrict ourselves to those (automatically finite) series $\boldsymbol{h}$ in which $h^{(k)} \in \langle \Delta_k(\boldsymbol{h}_0) \rangle$ for all k.

EXAMPLE. Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{h}$ the subalgebra of diagonal matrices, and $\boldsymbol{h}_0 = h_0^{(0)} + h_0^{(1)} t + \cdots + h_0^{(n-2)} t^{n-2}$ where

$$h_0^{(k)} = \operatorname{diag}(\underbrace{1, \ldots, 1}_{n-k-1}, -(n-k-1), 0, \ldots, 0).$$

If we denote by ε_i the linear function on $\mathfrak{h}$ equal to the ith diagonal element, then $\Delta_k(\boldsymbol{h}_0)$ is the subsystem of roots of the form $\varepsilon_i - \varepsilon_j$, where $i, j \leq n-k$, i.e., the root system of the algebra $\mathfrak{sl}_{n-k}$ naturally imbedded (as the "upper left corner") in $\mathfrak{sl}_n$. In particular, $\Delta_{n-1}(\boldsymbol{h}_0) = \varnothing$, so that the series $\boldsymbol{h}_0$ is regular (and canonical). The series

$$\boldsymbol{h}_k = h_0^{(0)} + h_0^{(1)} t + \cdots + h_0^{(k-1)} t^{k-1} \qquad (k = 1, \ldots, n-1),$$

is obviously subordinate to the series $\boldsymbol{h}_0$, and

$$\lim_{t \to 0} \sum_{\alpha \in \Delta} \frac{\alpha(h_k(t))}{\alpha(h_0(t))} e_\alpha e_{-\alpha} = \sum_{\alpha \in \Delta \setminus \Delta_k(\boldsymbol{h}_0)} e_\alpha e_{-\alpha}.$$

Consequently, the subspace $Q(\boldsymbol{h}_0)$ coincides with the linear span of the elements

$$\sum_{\alpha \in \Delta_k(\boldsymbol{h}_0)} e_\alpha e_{-\alpha} \qquad (k = 0, 1, \ldots, n-2),$$

that are the projections on Q of the Casimir elements of the subalgebras $\mathfrak{sl}_{n-k}$.

The subspace $Q(\boldsymbol{h}_0)$ in this example is part of the commutative subalgebra L of $\mathscr{P}(\mathfrak{sl}_n)$ generated by the subalgebra $\mathfrak{h}$ and the centers of the subalgebras

$\mathscr{P}(\mathfrak{sl}_{n-k})$, $k = 0, 1, \dots, n-2$. This subalgebra L lifts to the commutative subalgebra A of $\mathscr{U}(\mathfrak{sl}_n)$ generated by the subalgebra $\hat{\mathfrak{h}}$ and the centers of the subalgebras $\mathscr{U}(\mathfrak{sl}_{n-k})$. The weight subspaces of the algebra A in the space V of any irreducible linear representation of $\mathfrak{sl}_n$ are one-dimensional; and their basis vectors constitute a Gel′fand-Tsetlin basis for V.

§1. Some properties of the universal enveloping algebra

1.1. It is known [7] that the transcendence degree of a commutative-associative algebra A coincides with its Gel′fand-Kirillov dimension

$$\operatorname{Dim} A = \sup_{\alpha} \overline{\lim_{N\to\infty}} \frac{\ln d(\alpha, N)}{\ln N},$$

where $\alpha = (a_1, \dots, a_m)$ is an arbitrary finite set of elements of A and $d(\alpha, n)$ is the dimension of the linear span of the words of length $\leq N$ in $a_1, \dots, a_m$. If $A = \bigcup_1^\infty A^k$ is an increasing filtration of A, then for any set $\alpha \subset A^p$ we have $d(\alpha, N) \leq \dim A^{pN}$, and therefore

$$\operatorname{Dim} A \leq \overline{\lim_{N\to\infty}} \frac{\ln \dim A^{pN}}{\ln N} \leq \overline{\lim_{N\to\infty}} \frac{\ln \dim A^N}{\ln N}. \tag{21}$$

On the other hand, if $L = \bigoplus_0^\infty L_k$ is a finitely generated graded algebra, then taking for α a system of homogeneous generators, we find that $d(\alpha, N) \geq \dim \sum_0^N L_k$, and so

$$\operatorname{Dim} L \geq \overline{\lim_{N\to\infty}} \frac{\ln \dim \sum_0^N L_k}{\ln N}. \tag{22}$$

A filtered algebra $A = \bigcup_0^\infty A^k$ will be called an *algebra of finite type* if the associated graded algebra $\operatorname{gr} A = L$ is finitely generated. In this case, since $\dim A^N = \dim \sum_0^N L_k$, it follows from (21) and (22) that $\operatorname{Dim} A \leq \operatorname{Dim} L$. On the other hand, it is obvious that the lifts into A of any algebraically independent homogeneous elements of L are algebraically independent, so that $\operatorname{tr.deg.} A \geq \operatorname{tr.deg.} L$. From this follows

PROPOSITION 1.1. *The transcendence degree of a filtered algebra of finite type is equal to that of the associated graded algebra.*

COROLLARY. *The transcendence degree of any commutative subalgebra of $\mathscr{U}(\mathfrak{g})$ of finite type (where $\mathfrak{g}$ is any Lie algebra) cannot exceed $d(\mathfrak{g}) = (\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$.*

(See §0.2.)

1.2. Let $\mathfrak{g}$ be an arbitrary Lie algebra. From the form of the defining relations of the algebra $\mathscr{U}(\mathfrak{g})$ it follows that there exists a (unique) involutive antiautomorphism σ of $\mathscr{U}(\mathfrak{g})$ such that $\sigma(\hat{x}) = -\hat{x}$ for $x \in \mathfrak{g}$. Clearly,

$$\sigma(\hat{x}_1 \cdots \hat{x}_k) = (-1)^k \hat{x}_k \cdots \hat{x}_1 \quad \text{for } x_1, \dots, x_k \in \mathfrak{g}, \tag{23}$$

and therefore σ acts on $\mathscr{U}_k(\mathfrak{g})$ (see §0.3) as multiplication by $(-1)^k$.

The mapping σ is also an antiautomorphism of $\mathscr{U}(\mathfrak{g})$ as a Lie algebra relative to the operation of commutation. Consequently, $-\sigma$ is an automorphism of this Lie algebra. This means that *the decomposition*

$$\mathscr{U}(\mathfrak{g}) = \left(\bigoplus_{k=0}^{\infty} \mathscr{U}_{2k+1}(\mathfrak{g})\right) \oplus \left(\bigoplus_{k=0}^{\infty} \mathscr{U}_{2k}(\mathfrak{g})\right) \tag{24}$$

is a $\mathbf{Z}_2$*-grading of the Lie algebra* $\mathscr{U}(\mathfrak{g})$ (where the first summand is the grading subspace of degree zero).

1.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists an involutive automorphism θ of $\mathfrak{g}$ (the Weyl automorphism) that acts on the Cartan subalgebra $\mathfrak{h}$ as multiplication by -1 ([2], Chapter VIII, §4, Proposition 5). Choosing the elements e_α (see §0.3) appropriately, we can arrange that

$$\theta(e_\alpha) = -e_{-\alpha}. \tag{25}$$

The automorphism θ, like any automorphism of $\mathfrak{g}$, induces in a canonical fashion an automorphism of $\mathscr{U}(\mathfrak{g})$, which we shall denote by the same letter θ. Since $\theta(\hat{e}_\alpha \hat{e}_{-\alpha}) = \hat{e}_{-\alpha}\hat{e}_\alpha$, *the automorphism* θ *is the identity on* $\widehat{Q}$ (see the notation in §0.3).

§2. Proof of Theorem 1

2.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Define the constants $N_{\alpha\beta}(\alpha, \beta, \alpha+\beta \in \Delta)$ from the condition

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}. \tag{26}$$

Clearly,

$$N_{\beta\alpha} = -N_{\alpha\beta}. \tag{27}$$

Furthermore, if $\alpha, \beta, \gamma \in \Delta$ are such that $\alpha+\beta+\gamma = 0$, then from the equalities $([e_\alpha, e_\beta], e_\gamma) = ([e_\beta, e_\gamma], e_\alpha) = ([e_\gamma, e_\alpha], e_\beta)$ it follows that

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}. \tag{28}$$

If $\alpha, \beta \in \Delta$ but $\alpha+\beta \notin \Delta$, we put $N_{\alpha\beta} = 0$.

2.2. A straightforward calculation in the algebra $\mathscr{P}(\mathfrak{g})$, using the Leibniz formula (5), shows that

$$\begin{aligned} \{e_\alpha e_{-\alpha}, e_\beta e_{-\beta}\} = {} & N_{\alpha\beta} e_{-\alpha} e_{-\beta} e_{\alpha+\beta} + N_{-\alpha, \beta} e_\alpha e_{-\beta} e_{-\alpha+\beta} \\ & + N_{\alpha, -\beta} e_{-\alpha} e_\beta e_{\alpha-\beta} + N_{-\alpha, -\beta} e_\alpha e_\beta e_{-\alpha-\beta}. \end{aligned} \tag{29}$$

Now suppose elements $q_1, \ldots, q_k \in Q$ are given by (10). Using (29), we find that

$$\begin{aligned} \{q_i, q_j\} &= 4 \sum_{\alpha, \beta \in \Delta} q_{i,\alpha} q_{j,\beta} N_{-\alpha,-\beta} e_\alpha e_\beta e_{-\alpha-\beta} \\ &= 4 \sum_{\substack{\alpha, \beta, \gamma \in \Delta \\ \alpha+\beta+\gamma=0}} N_{-\alpha,-\beta} q_{i,\alpha} q_{j,\beta} e_\alpha e_\beta e_\gamma. \end{aligned}$$

Combining like terms by means of (27) and (28), we obtain for the coefficient of $e_\alpha e_\beta e_\gamma$ $(\alpha+\beta+\gamma = 0)$ the quantity

$$\begin{aligned} & 4N_{-\alpha,-\beta}(q_{i,\alpha}q_{j,\beta} - q_{i,\beta}q_{j,\alpha} + q_{i,\beta}q_{j,\gamma} - q_{i,\gamma}q_{j,\beta} + q_{i,\gamma}q_{j,\alpha} - q_{i,\alpha}q_{j,\gamma} \\ & = 4N_{-\alpha,-\beta} \begin{vmatrix} 1 & 1 & 1 \\ q_{i,\alpha} & q_{i,\beta} & q_{i,\gamma} \\ q_{j,\alpha} & q_{j,\beta} & q_{j,\gamma} \end{vmatrix}. \end{aligned}$$

This implies the equivalence of conditions 1) and 3) of Theorem 1.

2.3. Suppose $q_1, q_2 \in Q$. If $[\hat{q}_1, \hat{q}_2] = 0$, then certainly, by definition of the Poisson bracket, $\{q_1, q_2\} = 0$.

Conversely, suppose $\{q_1, q_2\} = 0$. Then

$$[\hat{q}_1, \hat{q}_2] \in \mathscr{U}^2(\mathfrak{g}) = \mathscr{U}_0(\mathfrak{g}) \oplus \mathscr{U}_1(\mathfrak{g}) \oplus \mathscr{U}_2(\mathfrak{g});$$

but in view of the grading (24) of $\mathscr{U}(\mathfrak{g})$ we must in fact have $[\hat{q}_1, \hat{q}_2] \in \mathscr{U}_1(\mathfrak{g}) = \hat{\mathfrak{g}}$. Furthermore, since $\hat{q}_1$ and $\hat{q}_2$ both belong to the centralizer of $\hat{\mathfrak{h}}$, so also must $[\hat{q}_1, \hat{q}_2]$. This means that $[\hat{q}_1, \hat{q}_2] \in \hat{\mathfrak{h}}$.

Consider now the automorphism θ of $\mathscr{U}(\mathfrak{g})$ defined in §1.3. Since it is the identity on $\hat{q}_1$ and $\hat{q}_2$, it must also be the identity on $[\hat{q}_1, \hat{q}_2]$. On the other hand, it acts on $\hat{\mathfrak{h}}$ as multiplication by -1. Consequently, $[\hat{q}_1, \hat{q}_2] = 0$. This proves the equivalence of conditions 1) and 2) of Theorem 1.

§3. Some properties of root systems

3.1. Let $\mathbf{E}$ be a Euclidean space, $\Delta \subset \mathbf{E}$ a given root system, W its Weyl group, Π a fixed system of simple roots, and C the corresponding Weyl chamber.

For any vector $\lambda \in C$ we denote by $P(\lambda)$ the convex hull of the orbit $W\lambda$. The vector λ is called *admissible* if its projection on the linear span of every indecomposable component of the system Δ is different from zero. In this (and only this) case, $\langle W\lambda \rangle = \mathbf{E}$, so that $P(\lambda)$ is a convex polyhedron in the space $\mathbf{E}$. (It is being assumed that $\langle \delta \rangle = \mathbf{E}$.) We describe now the combinatorial structure of this polyhedron.

By construction, the polyhedron $P(\lambda)$ is invariant with respect to the group W. Consider any (closed) face F. For a transformation $w \in W$ it is obvious that the following conditions are equivalent:

1) $wF = F$.
2) w leaves fixed the center of mass of F.
3) w leaves fixed some interior point of F.

In particular, if the mirror of a reflection $r \in W$ passes through an interior point of F, then $rF = F$. This can happen in two ways: if the mirror of the reflection r entirely contains the plane of F, or if it is perpendicular to this plane and passes through the center of mass of F. We denote by W_F the group generated by the mappings of the second of these two types.

The mirrors of the reflections in W_F separate the face F into pieces that are transitively permuted by the group W_F and constitute the intersections of F with the chambers wC, $w \in W$, for which $\dim(F \cap wC) = \dim F$. It follows that if two points of F are equivalent with respect to W, then they are already equivalent with respect to W_F. In particular, the group W_F acts transitively on the set of vertices of F.

The center of mass of this face F must be the only fixed point of the group W_F in the plane of F. Otherwise, the W_F-orbit of any point of this plane would lie in a plane of lower dimension; whereas one such orbit is, by the observation just made, the set of vertices of F, and this cannot lie in a plane of lower dimension.

Let $\mathbf{E}_F$ be the directional subspace of the plane of F. Then the group W_F is simply the Weyl group of the root system $\Delta_F = \Delta \cap \mathbf{E}_F$, and its action in the plane of F is isomorphic to its action in the space $\mathbf{E}_F$. Consequently, the group W_F has a unique fixed point in the space $\mathbf{E}_F$, which means that $\langle \Delta_F \rangle = \mathbf{E}_F$.

We say that the face F is *dominant* if $\dim(F \cap C) = \dim F$. Obviously, every face is W-equivalent to a unique dominant one. Furthermore, every face contained in a dominant face F is W_F-equivalent to a dominant face. In particular, every dominant face contains a unique dominant vertex λ.

Let F be a dominant face. Then $F \cap C$ is a fundamental domain for the action of the group W_F on F, and therefore $\Pi_F = \Delta_F \cap \Pi$ is a system of simple roots for the root system Δ_F. Clearly, $|\Pi_F| = \dim F$. The face F can be reconstructed from the subset $\Pi_F \subset \Pi$ as the convex hull of the orbit $W_F\lambda$.

Conversely, let $\Pi' \subset \Pi$ be an arbitrary subset, $\Delta' \subset \Delta$ the root subsystem generated by it, $W' \subset W$ the Weyl group of this subsystem, and $F = \operatorname{conv} W'\lambda$ the convex hull of the orbit $W'\lambda$. In order that F be a convex polyhedron in the plane $\lambda + \langle\Pi'\rangle$ (i.e., that $\dim F = |\Pi'|$), it is necessary and sufficient that the projection of λ on the linear span of every indecomposable component of the system Π' be different from zero; in other words, that there exist in every indecomposable component of Π' a root that is not orthogonal to λ. A subset $\Pi' \subset \Pi$ satisfying this condition will be called *λ-admissible.*

We prove now that for any λ-admissible subset $\Pi' \subset \Pi$ the set $F = \operatorname{conv} W'\lambda$ is a (dominant) face of the polyhedron $p(\lambda)$. (And then, obviously, $\mathbf{E}_F = \langle\Pi'\rangle$, $\Delta_F = \Delta'$, $W_F = W'$, and $\Pi_F = \Pi'$). This will be a consequence of the following lemma.

LEMMA 3.1. $P(\lambda) \subset \lambda - \operatorname{con}\Pi$, *where* $\operatorname{con}\Pi$ *means the convex cone spanned by* Π.

PROOF. It suffices to show that every vertex $w\lambda$ $(w \in W)$ of the polyhedron $P(\lambda)$ lies in $\lambda - \operatorname{con}\Pi$. We denote by r_α the reflection corresponding to the root α. Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ $(\alpha_1, \dots, \alpha_k \in \Pi)$ be the shortest representation of the element w as a product of reflections corresponding to simple roots. Then for every $i = 1, \dots, k$ the chambers C and $r_{\alpha_{i+1}} \cdots r_{\alpha_k} C$ lie on the same side of the hyperplane orthogonal to α_i ([1], Chapter V, §3, Theorem 1), so that $(\alpha_i, r_{\alpha_{i+1}} \cdots r_{\alpha_k}\lambda) \geq 0$. Consequently,

$$r_{\alpha_i} r_{\alpha_{i+1}} \cdots r_{\alpha_k}\lambda = r_{\alpha_{i+1}} \cdots r_{\alpha_k}\lambda - p_i\alpha_i, \quad \text{where } p_i \geq 0,$$

and by reverse induction on i it follows that $w\lambda \in \lambda - \operatorname{con}\Pi$.

Now let $\Pi' \subset \Pi$ be an arbitrary λ-admissible subset. We have $F = \operatorname{conv} W'\lambda \subset P(\lambda)$. Consider a linear function f on $\mathbf{E}$ that equals 0 on Π' and 1 on $\Pi \backslash \Pi'$. It follows from the lemma that $f(\mu) \leq f(\lambda)$ for all $\mu \in P(\lambda)$, with equality if and only if $\mu \in \lambda + \langle\Pi'\rangle$, and in particular for $\mu \in F$. Since Π' is λ-admissible, we have $\dim F = |\Pi'|$. Therefore $\lambda + \langle\Pi'\rangle$ is the plane of some face $\widetilde{F}$ of $P(\lambda)$ that contains F and has the same dimension. As shown above, the vertices of the face $\widetilde{F}$ constitute a single $W_{\widetilde{F}}$-orbit. But obviously $W_{\widetilde{F}} = W'$. Hence $\widetilde{F}$ has the same vertices as F, and so $\widetilde{F} = F$.

We sum up the main results so far:

PROPOSITION 3.2. *Let* $\lambda \in C$ *be an arbitrary admissible vector. Every face of the polyhedron* $P(\lambda)$ *is* W*-equivalent to a unique dominant face. There exists a one-to-one correspondence between the set of all dominant faces and the set of all* λ*-admissible subsets* $\Pi' \subset \Pi$, *assigning to each subset* Π' *the face* $F = \operatorname{conv} W'\lambda$, *of dimension* $|\Pi'|$.

§**3.2.** By specializing the choice of the vector $\lambda \in C$, we shall now associate with every indecomposable root system Δ of rank ≥ 3, except for a system of type D_4, a convex polyhedron P that will play an important role in what follows.

Let α_1 be a simple root with the following properties:

1) There exists a unique simple root α_3 different from α_1 and not orthogonal to α_1.

2) $|\alpha_1| = |\alpha_2|$.

3) There exists a unique simple root α_2 different from α_1 and α_2 and not orthogonal to α_2.

Such a simple root α_1 exists for any indecomposable root system of rank ≥ 3 different from D_4. Furthermore, if the system Δ is different from B_3 C_3 and F_4, the following additional condition is satisfied:

4) $|\alpha_2| = |\alpha_3|$.

Now we choose a nonzero vector $\lambda \in C$ orthogonal to every simple root except α_1, and put $P = P(\lambda)$. Let us establish some properties of the polyhedron P.

First of all, it is obvious that with this choice of λ the unique λ-admissible subset consisting of one root (resp. two roots, three roots) is the subset $\Pi_1 = \{\alpha_1\}$ (resp. $\Pi_2 = \{\alpha_1, \alpha_2\}, \Pi_3 = \{\alpha_1, \alpha_2, \alpha_3\}$). This means that the polyhedron P has a unique dominant edge e_0, lying on the line $\lambda + \langle\alpha_1\rangle$, a unique dominant 2-face f_0, lying in the plane $\lambda + \langle\alpha_1, \alpha_2\rangle$, and a unique dominant 3-face c_0, lying in the plane $\lambda + \langle\alpha_1, \alpha_2, \alpha_3\rangle$.

Since $|\alpha_1| = |\alpha_2|$, the subset Π_2 generates a root subsystem Δ_2 of type A_2. Its Weyl group W_2 is isomorphic to S_3. The vector λ remains fixed under the reflection r_{α_2}. Therefore the orbit $W_2\lambda$ consists of three points, and the face $f_0 = \operatorname{conv} W_2\lambda$ is a triangle. This means that *all* 2-faces of P are triangles.

If the system Δ is different from B_3, C_3, and F_4, then the subset Π_3 generates a root subsystem Δ_3 of type A_3. Its Weyl group W_3 is isomorphic to S_4. The vector λ remains fixed under the reflections r_{α_2} and r_{α_3}, which generate a subgroup isomorphic to S_3. Therefore the orbit $W_3\lambda$ consists of four points, and the face $c_0 = \operatorname{conv} W_3\lambda$ is a tetrahedron. This means that *all* 3-faces of P are tetrahedra.

When Δ is a system of type F_4, there are two variants possible for the choice of the vertex α_1. We shall suppose that for α_1 we have chosen a short root. Then Δ_3 is a system of type C_3, and the face c_0 is an octahedron. This means that all the 3-faces of P in this case are octahedra. For a suitable normalization of the vector λ the vertices of P are short roots of the system Δ. Opposite vertices of each 3-face are orthogonal short roots, generating a subsystem of type B_2. In particular, the vertex of c_0 opposite λ is of the form

$$\mu = r_\beta\lambda = \lambda - \beta, \tag{30}$$

where $\beta = 2\alpha_1 + 2\alpha_2 + \alpha_3$.

To each face F of P we assign the subsystem $\Delta^F \subset \Delta$ (not to be confused with the subsystem Δ_F of §3.1!) consisting of those roots that are orthogonal to $\langle F\rangle$. Obviously, if $v_1, \dots, v_s$ are all the vertices of F, then

$$\Delta^F = \Delta^{v_1} \cap \cdots \cap \Delta^{v_s}. \tag{31}$$

Denote by l the rank of the system Δ (i.e., the dimension of the space E). From the choice of the root α_1 and the description of the faces of P, it follows that:

1) For any vertex v of P the subsystem Δ^v is W-equivalent to the subsystem Δ^λ generated by the subset $\Pi\backslash\Pi_1$, and is therefore an indecomposable root system of rank $l-1$.

2) For any edge e of P the subsystem Δ^e is W-equivalent to the subsystem Δ^{e_0} generated by the subset $\Pi\backslash\Pi_2$, and is therefore an indecomposable root system of rank $l-2$.

3) For any 2-face f of P the subsystem Δ^f is W-equivalent to the subsystem Δ^{f_0} generated by the subset $\Pi\backslash\Pi_3$, and is therefore of rank $l-3$.

3.3. We come now to the statement and proof of the main result of this section.

Let V be a complex vector space, PV the associated projective space, and π the canonical mapping $V \to PV$ (undefined at zero).

PROPOSITION 3.3. *Let $\Delta \subset \mathbf{E}$ be an indecomposable root system, and $p\colon \Delta \to PV$ a mapping with the following properties:*

1) *$p(\alpha) = p(\beta)$ if and only if $\alpha = \pm\beta$.*
2) *If α, β, $\gamma \in \Delta$ are such that $\alpha+\beta+\gamma = 0$, then the points $p(\alpha)$, $p(\beta)$, $p(\gamma) \in PV$ are collinear.*
3) *The set $p(\Delta)$ is not collinear.*

Then there exists a linear mapping $\varphi\colon \mathbf{E}\otimes\mathbf{C} \to V$ such that $\operatorname{Ker}\varphi \cap \Delta = \varnothing$ *and $p = \pi\varphi|_\Delta$. The mapping φ is uniquely determined up to a scalar factor.*

We first prove uniqueness. It is a consequence of the following slightly more general lemma.

LEMMA 3.4. *Let $\Delta \subset \mathbf{E}$ be an indecomposable root system, and φ_1, $\varphi_2\colon \mathbf{E}\otimes\mathbf{C} \to V$ two linear mappings such that* $\operatorname{Ker}\varphi_1 \cap \Delta = \varnothing$, $\operatorname{Ker}\varphi_2 \cap \Delta = \varnothing$ *and $\pi\varphi_1|_\Delta = \pi\varphi_2|_\Delta$, with $\pi\varphi_1(\alpha) \neq \pi\varphi_1(\beta)$ for α, $\beta \in \Delta$, $\alpha \neq \pm\beta$. Then $\varphi_1 = \lambda\varphi_2$, where $\lambda \in \mathbf{C}^*$.*

PROOF. For any $\alpha \in \Delta$ we have $\varphi_1(\alpha) = \lambda_\alpha\varphi_2(\alpha)$, where $\lambda_\alpha \in \mathbf{C}^*$. We must prove that λ_α is in fact independent of α. Let α, $\beta \in \Delta$ be two roots such that $\alpha \neq \pm\beta$ and $(\alpha, \beta) \neq 0$. For definiteness, suppose $(\alpha, \beta) < 0$. Then $\alpha+\beta \in \Delta$. From the equality

$$\varphi_1(\alpha+\beta) = \varphi_1(\alpha) + \varphi_1(\beta)$$

we obtain

$$\lambda_{\alpha+\beta}(\varphi_2(\alpha) + \varphi_2(\beta)) = \lambda_\alpha\varphi_2(\alpha) + \lambda_\beta\varphi_2(\beta),$$

and therefore, since by assumption $\varphi_2(\alpha)$ and $\varphi_2(\beta)$ are linearly independent, $\lambda_{\alpha+\beta} = \lambda_\alpha = \lambda_\beta$. Since the system Δ is indecomposable, this implies that $\lambda_\alpha = \lambda_\beta$ for any α, $\beta \in \Delta$.

PROOF OF PROPOSITION 3.3. Observe first of all that from conditions 1) and 2) it follows that the image of any subsystem of rank 2 is collinear. Therefore the complete inverse image of any line is a subsystem that together with any two non-proportional roots belonging to a subsystem of rank 2 contains the whole subsystem. This comes down to the fact that together with any two roots the subsystem contains their half-sum whenever the latter is a root. Subsystems with this property will be called *saturated*.

We carry out the proof by induction on $l = \operatorname{rk}\Delta$. Observe that condition 3) implies that $l \geq 3$.

Suppose first that $l = 3$. Then Δ is a system of type A_3, B_3, or C_3. It is easily seen that none of these systems contains proper saturated subsystems of rank 3. It follows that the complete inverse image of any line is a subsystem of rank ≤ 2. In particular, the images of different indecomposable subsystems of rank 2 lie on different lines.

Suppose Δ is a system of type A_3, and let $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ be its extended system of simple roots. Then $\Delta = \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\beta, \pm\gamma\}$, where (with appropriate numbering of the roots α_i) $\beta = \alpha_0+\alpha_1 = -\alpha_2-\alpha_3$ and $\gamma = \alpha_1+\alpha_2 = -\alpha_3-\alpha_0$. There exists a linear mapping $\varphi\colon \mathbf{E}\otimes\mathbf{C} \to V$ such that $\pi\varphi(\alpha_i) = p(\alpha_i)$ for

$i = 0, 1, 2, 3$. The point $\pi\varphi(\beta)$ lies on the intersection of the line passing through $\pi\varphi(\alpha_0) = p(\alpha_0)$ and $\pi\varphi(\alpha_1) = p(\alpha_1)$, and the line passing through $\pi\varphi(\alpha_2) = p(\alpha_2)$ and $\pi\varphi(\alpha_3) = p(\alpha_3)$. The same is true of the point $p(\beta)$. Therefore $\pi\varphi(\beta) = p(\beta)$. Similarly, $\pi\varphi(\gamma) = p(\gamma)$. Thus, $p(\alpha) = \pi\varphi(\alpha)$ for all $\alpha \in \Delta$.

Suppose Δ is of type B_3 or C_3. The subset $\Delta' \subset \Delta$ consisting of all long roots in the case B_3, and all short in the case C_3, is a root system of type A_3 (in the case C_3 it is not a subsystem of the system Δ, but this is of no consequence). By what has just been proved, there exist a linear mapping $\varphi\colon \mathbf{E} \otimes \mathbf{C} \to V$ such that $p(\alpha) = \pi\varphi(\alpha)$ for $\alpha \in \Delta'$. An argument similar to that in the case of A_3 then allows us to show that $p(\alpha) = \pi\varphi(\alpha)$ for all $\alpha \in \Delta$.

Now suppose $l \geq 4$ and Δ is not a system of type D_4. Let $P \subset \mathbf{E}$ be the convex polyhedron described in §3.2. A vertex v of P will be called *good* if the set $p(\Delta^v)$ is not collinear. By the induction assumption, there exist in this case a linear mapping $\varphi^v\colon \langle\Delta^v\rangle \otimes \mathbf{C} \to V$ such that $kp(\alpha) = \pi\varphi^v(\alpha)$ for all $\alpha \in \Delta^v$. The mapping φ^v is determined up to a scalar factor.

If v_1 and v_2 are two adjacent vertices, then $\Delta^{v_1} \cap \Delta^{v_2}$ is an indecomposable subsystem of rank $l - 2 \geq 2$ (see §3.2).

We assert that two adjacent vertices v_1 and v_2 cannot both be bad. Indeed, assume the contrary. Suppose $p(\Delta^{v_1})$ lies on a line a_1, and $p(\Delta^{v_2})$ on a line a_2. Then $p(\Delta^{v_1} \cap \Delta^{v_2})$ lies on the intersection of these lines. Since the system $\Delta^{v_1} \cap \Delta^{v_2}$ contains nonproportional roots, the set $p(\Delta^{v_1} \cap \Delta^{v_2})$ consists of more than one point. Therefore $a_1 = a_2$; i.e., the set $p(\Delta^{v_1} \cup \Delta^{v_2})$ is collinear. But it is easily seen that the set $\Delta^{v_1} \cup \Delta^{v_2}$ generates the entire system Δ (it suffices to verify this for $v_1 = \lambda$ and $v_2 = r_{\alpha_1}\lambda$; whatever the case, $\Delta^{v_1} \supset \Pi\backslash\{\alpha_1\}$, while $\Delta^{v_2} = r_{\alpha_1}\Delta^{v_1} \ni r_{\alpha_1}\alpha_2 = \alpha_1 + \alpha_2$). Consequently, all of $p(\Delta)$ is collinear, contradicting the hypothesis.

A similar proof shows that when Δ is of type F_4, two opposite vertices v and w of a 3-face cannot both be bad. For this it suffices to verify that $\Delta^v \cap \Delta^w$ is a subsystem of rank 2 and that there is no proper saturated subsystem of Δ that contains $\Delta^v \cup \Delta^w$. In turn, it suffices to do this for $v = \lambda$ and $w = \mu$ (see (30)); whatever the case, Δ^v is generated by the subset $\{\alpha_2, \alpha_3, \alpha_4\}$, and $\Delta^w = r_\beta\Delta^v$ by the subset $\{\alpha_2, \alpha_3, \alpha_4 + \beta\}$; and all is evident.

Thus, in any case, every 3-face of P contains at most one bad vertex.

If v_1 and v_2 are two adjacent good vertices, then, by Lemma 3.4 applied to the system $\Delta^{v_1} \cap \Delta^{v_2}$, the restrictions of the mappings φ^{v_1} and φ^{v_2} to $\langle\Delta^{v_1} \cap \Delta^{v_2}\rangle$ differ only by a scalar factor. Multiplying by a suitable scalar factor, we can therefore combine them into a single linear mapping $\mathbf{E} \otimes \mathbf{C} \to V$, which we denote by φ^e, where e is the edge joining v_1 and v_2. This mapping, defined up to a scalar factor, has the property that $p(\alpha) = \pi\varphi^e(\alpha)$ for all $\alpha \in \Delta^{v_1} \cup \Delta^{v_2}$.

An edge of which both endpoints are good vertices will itself be called *good*. We prove now that the linear mappings φ^{e_1}, $\varphi^{e_2}\colon \mathbf{E} \otimes \mathbf{C} \to V$ associated with any two good edges e_1 and e_2 must coincide up to a scalar factor.

Two edges that belong to the same k-face will be called *k-adjacent*. For any $k \geq 3$ the following holds:

(R_k) Any two k-adjacent good edges e_1 and e_2 can be included in a sequence of good edges in which any two successive terms are $(k-1)$-adjacent edges.

Indeed, suppose e_1, and e_2 belong to a k-face F; and let F_1 and F_2 be $(k-1)$-faces containing them that belong to the boundary ∂F of F. Since ∂F is homeomorphic to a sphere, the faces F_1 and F_2 can be included in a sequence of $(k-1)$-faces belonging to ∂F in which any two successive terms intersect in a $(k-2)$-face. Since any 2-face contains a good edge, this implies (R_k) for $k \geq 4$.

The same argument also goes through for $k = 3$, provided all the vertices of F are good. If $k = 3$ and F has a bad vertex (in which case, as we have seen, it can have only one), then instead of ∂F we must take the union $\partial' F$ of those faces belonging to ∂F that do not contain v. The faces F_1 and F_2 can be chosen to belong to $\partial' F$. Since $\partial' F$ is homeomorphic to a cell, the argument also goes through in this case.

In view of (R_k), to prove our assertion concerning proportionality of the linear mappings φ^{e_1} and φ^{e_2} it suffices to consider just 2-adjacent edges e_1 and e_2. We call such edges simply *adjacent*.

Let e_1 and e_2 be two adjacent good edges belonging to a 2-face f, v the common vertex of these two edges, and v_1 and v_2 the other vertices of f belonging to e_1 and e_2, respectively. We can suppose that the mappings φ^{e_1} and φ^{e_2} coincide on $\langle \Delta^v \rangle$. To prove that they coincide in general, it suffices to find two nonproportional vectors no belonging to $\langle \Delta^v \rangle$ such that the images of each under the mappings φ^{e_1} and φ^{e_2} are proportional. For these vectors we can take any two nonproportional roots in the set

$$(\Delta^{v_1} \cap \Delta^{v_2}) \backslash \Delta^v = (\Delta^{v_1} \cap \Delta^{v_2}) \backslash \Delta^f .$$

Such roots always exist, since $\Delta^{v_1} \cap \Delta^{v_2}$ is an indecomposable root system of rank $l - 2 \geq 2$ and has Δ^f as a subsystem of rank $l - 3$ (see §3.2).

Thus, there exists a linear mapping $\varphi \colon \mathbf{E} \otimes \mathbf{C} \to V$ that coincides up to a scalar factor with the mappings φ^e for every good edge e, and therefore has the property that $p(\alpha) = \pi\varphi(\alpha)$ for all $\alpha \in M = \bigcup_v \Delta^v$, where the union is taken over all good vertices v. We prove now that $M = \Delta$.

Since every edge contains at least one good vertex, the set M contains the W-invariant subset $M' = \bigcup_e \Delta^e$, where the union is taken over *all* edges e. If all the roots of the system Δ have the same length, then W acts transitively on Δ; hence $M' = \Delta$ and so, certainly, $M = \Delta$. If Δ contains roots of different lengths—"short" and "long"—then W has two orbits in Δ, consisting of the short roots and the long roots, respectively. It is easily seen that if Δ is different from F_4, then Δ^{e_0} contains both short and long roots (the subsystem Δ^{e_0} is generated by the subset $\Pi \backslash \{\alpha_1, \alpha_2\}$), and so again we obtain $M' = \Delta$.

If Δ is of type F_4, then instead of the subsystems of the form Δ^e we consider those of the form $\Delta^v \cap \Delta^w$, where v and w are opposite vertices of a 3-face. Denote the union of these by M''. This subset is W-invariant and contained in M. It contains the subsystem $\Delta^\lambda \cap \mathbb{D}^\mu$ (see (30)), which, as is easily seen, is generated by the subset $\{\alpha_2, \alpha_3\}$ and in particular contains roots both short and long. Then as above it follows that $M'' = \Delta$ and so, certainly, $M = \Delta$.

This completes the induction step in the case that the system Δ is different from D_4. It remains to prove the proposition for a system of type D_4.

Thus, consider the case that Δ is of type D_4. Encase the system Δ in a system $\widehat{\Delta}$ of type F_4 as the subsystem consisting of all the long roots, and consider the convex polyhedron $\widehat{P}$ constructed for $\widehat{\Delta}$, as in §3.2. We carry out the proof of the proposition for the system Δ in the same way as above, taking for the basic polyhedron the polyhedron $\widehat{P}$.

For any face F of $\widehat{P}$ we have $\Delta^F = \widehat{\Delta}^F \cap \Delta$; i.e., Δ^F is the subsystem consisting of all the long roots of the subsystem $\widehat{\Delta}^F$. This makes it easy to obtain the necessary information about the subsystems Δ^F of Δ from the corresponding information about the subsystems of $\widehat{\Delta}$.

In particular, we can prove the following:

1) For any vertex v of the polyhedron $\widehat{P}$, the subsystem Δ^v is of type A_3.

2) If v_1 and v_2 are adjacent vertices, then $\Delta^{v_1} \cap \Delta^{v_2}$ is a subsystem of type A_2, and $\Delta^{v_1} \cup \Delta^{v_2}$ generates the system Δ.

3) If v and w are opposite vertices of a 3-face, then $\Delta^v \cap \Delta^w$ is a subsystem of type $A_1 + A_2$, and $\Delta^v \cup \Delta^w$ generates the system Δ.

4) For any 2-face f, the subsystem Δ^f is of type A_1.

These facts now allow us to carry out the induction step in this final case.

§4. Proof of Theorem 2

4.1. Let $(q_1, \dots, q_k) \subset Q$ be a set defined by (12). Then the corresponding points $p_\alpha \in \mathbf{C}^k$ $(\alpha \in \Delta)$ are of the form

$$p_\alpha = (\alpha(h_1)/\alpha(h_0), \dots, \alpha(h_k)/\alpha(h_0)).$$

Let $\alpha, \beta, \gamma \in \Delta$ be roots such that $\alpha + \beta + \gamma = 0$. Then for any i and j we have

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{\alpha(h_i)}{\alpha(h_0)} & \frac{\beta(h_i)}{\beta(h_0)} & \frac{\gamma(h_i)}{\gamma(h_0)} \\ \frac{\alpha(h_j)}{\alpha(h_0)} & \frac{\beta(h_j)}{\beta(h_0)} & \frac{\gamma(h_j)}{\gamma(h_0)} \end{vmatrix} = \frac{1}{\alpha(h_0)\beta(h_0)\gamma(h_0)} \begin{vmatrix} \alpha(h_0) & \beta(h_0) & \gamma(h_0) \\ \alpha(h_i) & \beta(h_i) & \gamma(h_i) \\ \alpha(h_j) & \beta(h_j) & \gamma(h_j) \end{vmatrix} = 0,$$

since the sum of the columns of the second determinant is zero. This means that the points p_α, p_β, p_γ are collinear. By Theorem 1, it follows that the set $(q_1, \dots, q_k)$ is commutative. This proves the first part of Theorem 2.

4.2. Conversely, suppose the algebra $\mathfrak{g}$ is simple and $(q_1, \dots, q_k) \subset Q$ is a separating commutative set of rank ≥ 3.

Let $\mathbf{C}^{k+1}$ be the vector space of sequences $(x_0, x_1, \dots, x_k)$, and $P\mathbf{C}^{k+1}$ the associated projective space. We regard the space $\mathbf{C}^k$ as the affine chart of the space $P\mathbf{C}^{k+1}$ given by $x_0 \neq 0$.

Let $p_\alpha \in \mathbf{C}^k$ $(\alpha \in \Delta)$ be the points corresponding to the set $(q_1, \dots, q_k)$. Define the mapping $p\colon \Delta \to P\mathbf{C}^{k+1}$ by the formula $p(\alpha) = p_\alpha$. Then the conditions of Proposition 3.3 are satisfied. Specifically, condition 1) follows from the separating property of the set $(q_1, \dots, q_k)$; condition 2), from its commutativity; and condition 3), from the fact that $\operatorname{rk}(q_1, \dots, q_k) \geq 3$.

Consequently, there exists a linear mapping $\varphi\colon \mathbf{E} \otimes \mathbf{C} \to \mathbf{C}^{k+1}$ such that $p_\alpha = \pi\varphi(\alpha)$ for all $\alpha \in \Delta$. This mapping φ has the form

$$\varphi(\lambda) = (\lambda(h_0), \lambda(h_1), \dots, \lambda(h_k)),$$

where $h_0, h_1, \dots, h_k \in \mathfrak{h}$. Since $\varphi(\alpha)$, for $\alpha \in \Delta$, lies in the affine chart $x_0 \neq 0$, the element h_0 is regular. We have then, for $\alpha \in \Delta$,

$$p_\alpha = \pi\varphi(\alpha) = (\alpha(h_1)/\alpha(h_0), \dots, \alpha(h_k)/\alpha(h_0)),$$

which means precisely that the set $(q_1, \dots, q_k)$ is given by (12). This proves the second part of Theorem 2.

§5. Proof of Theorems 3 and 4

5.1. Let $\boldsymbol{h}_0$ be a regular series. Since the Plücker coordinates of the subspace $Q(h_0(t))$ are represented as Laurent series in t, it is a priori clear that $\lim_{t\to 0} Q(h_0(t))$

exists. From (20) it follows that $Q(\boldsymbol{h}_0) \subset \lim_{t\to 0} Q(h_0(t))$. We prove now that $\dim Q(\boldsymbol{h}_0) = l$; this will imply that

$$Q(\boldsymbol{h}_0) = \lim_{t\to 0} Q(h_0(t)).$$

Since for fixed $\boldsymbol{h}_0$ the limit (20) depends only on the numbers $\alpha(h^{(k)})$, where $k = 0, 1, \ldots$ and $\alpha \in \Delta_k(h_0)\backslash\Delta_{k+1}(\boldsymbol{h}_0)$, we can restrict ourselves in the definition of the subspace $Q(\boldsymbol{h}_0)$ to those series $\boldsymbol{h}$ in which $h^{(k)} \in \langle\Delta_k(\boldsymbol{h}_0)\rangle$. The subordination condition means that $h^{(k)}$ must be orthogonal to $\langle\Delta_{k+1}(\boldsymbol{h}_0)\rangle$. Otherwise the choice of $h^{(k)}$ is arbitrary. Denote by $h^{(k)}$ the orthogonal complement to $\langle\Delta_{k+1}(\boldsymbol{h}_0)\rangle$ in $\langle\Delta_k(\boldsymbol{h}_0)\rangle$, and consider the subspace H of (automatically finite) series (16) in which $h^{(k)} \in \mathfrak{h}^{(k)}$. The mapping that assigns to every series $\boldsymbol{h} \in H$ the limit (20) is easily seen to be an isomorphism of H onto the space $Q(\boldsymbol{h}_0)$. Since $\mathfrak{h} = \mathfrak{h}^{(0)} \oplus \mathfrak{h}^{(1)} \oplus \cdots$, we have

$$\dim Q(\boldsymbol{h}_0) = \dim H = \dim \mathfrak{h}^{(0)} + \dim \mathfrak{h}^{(1)} + \cdots = l,$$

proving what was required.

5.2. Suppose the algebra $\mathfrak{g}$ is simple. Let $\mathfrak{h}^{\mathrm{reg}}$ be the (open) subset of regular elements of the Cartan subalgebra $\mathfrak{h}$, and $\mathrm{Gr}_l(Q)$ the Grassmann manifold of all l-dimensional subspaces of Q. We examine from the analytic point of view the mapping $\sigma\colon \mathfrak{h}^{\mathrm{reg}} \to \mathrm{Gr}_l(Q)$ that assigns to each element $h_0 \in \mathfrak{h}^{\mathrm{reg}}$ the subspace $Q(h_0) \subset Q$.

Observe first of all that $Q(ch_0) = Q(h_0)$ for $c \in \mathbf{C}^*$, so that the mapping σ reduces to the mapping

$$\hat{\sigma}\colon P\mathfrak{h}^{\mathrm{reg}} \to \mathrm{Gr}_l(Q), \tag{32}$$

where $P\mathfrak{h}^{\mathrm{reg}}$ is the (open) subset of the projective space $P\mathfrak{h}$ that corresponds to the subset $\mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h}$.

PROPOSITION 5.1. *The mapping $\hat{\sigma}$ is injective, and has at all points the maximum possible rank, namely, $l - 1 = \dim P\mathfrak{h}^{\mathrm{reg}}$.*

PROOF. Orient in any way the edges of the Dynkin diagram of the algebra $\mathfrak{g}$ (this has nothing to do with the orientation given to the multiple edges in the definition of the Dynkin diagram), and to the oriented edge joining the ith vertex to the jth associate the function α_{ij} on $P\mathfrak{h}^{\mathrm{reg}}$ given by

$$\alpha_{ij}(\pi h_0) = \alpha_j(h_0)/\alpha_i(h_0) \quad (h_0 \in \mathfrak{h}^{\mathrm{reg}}),$$

where π is the canonical mapping $\mathfrak{h} \to P\mathfrak{h}$. It is easily seen that the functions α_{ij} so defined constitute a global coordinate system on $P\mathfrak{h}^{\mathrm{reg}}$.

We now define a coordinate system in a certain domain of the manifold $\mathrm{Gr}_l(Q)$ that contains the image of the mapping $\hat{\sigma}$.

Denote by f_α $(\alpha \in \Delta)$ the linear function on Q given by $f_\alpha(q) = q_\alpha$, where q_α is the coefficient of $e_\alpha e_{-\alpha}$ in the expression (8) for the element $q \in Q$. The functions f_α corresponding to the positive roots α form a basis of the conjugate space Q^*. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is a system of simple roots, then on every subspace $Q(h_0)$, $h_0 \in \mathfrak{h}^{\mathrm{reg}}$ the functions $f_{\alpha_1}, \ldots, f_{\alpha_l}$ are linearly independent, and all the functions f_α $(\alpha \in \Delta)$ can be expressed linearly in terms of them. More precisely, if

$\alpha = \sum_1^l k_i\alpha_i$ $(k_i \in \mathbf{Z})$, then on $Q(h_0)$ we have

$$f_\alpha = \sum_{i=1}^{l} k_i \frac{\alpha_i(h_0)}{\alpha(h_0)} f_{\alpha_i}. \tag{33}$$

Consider the open subset U in $\mathrm{Gr}_l(Q)$ consisting of all the subspaces on which the functions $f_{\alpha_1}, \dots, f_{\alpha_l}$ are linearly independent. On each subspace $P \in U$ we have equalities of the form

$$f_\alpha = \sum_{i=1}^{l} g_{\alpha,i}(P) f_{\alpha_i} \qquad (\alpha \in \Delta). \tag{34}$$

The functions $g_{\alpha,i}$ corresponding to nonsimple positive roots α form a coordinate system on U. From (33) it follows that

$$g_{\alpha,i}(\hat{\sigma}(\pi h_0)) = g_{\alpha,i}(Q(h_0)) = k_i\alpha_i(h_0)/\alpha(h_0) \tag{35}$$

for $\alpha = \sum_1^l k_i\alpha_i$.

In particular, if the ith and jth vertices of the Dynkin diagram have an edge joining them, then $\alpha_i + \alpha_j \in \Delta$, and we can take $\alpha = \alpha_i + \alpha_j$. In this case

$$g_{\alpha,i}(\hat{\sigma}(\pi h_0)) = \frac{\alpha_i(h_0)}{\alpha_i(h_0) + \alpha_j(h_0)} = \frac{1}{1 + \alpha_{ij}(\pi h_0)}.$$

Thus, among the coordinate functions defining the mapping $\hat{\sigma}$ are the $l-1$ functions $1/(1+\alpha_{ij})$ corresponding to all the oriented edges of the Dynkin diagram. From this, in an obvious manner, the proposition follows.

COROLLARY. *The set of all subspaces of the form* $Q(h_0)$, $h_0 \in \mathfrak{h}^{\mathrm{reg}}$, *is a nonsingular* $(l-1)$*-dimensional irreducible algebraic variety.*

5.3. We now prove the second part of Theorem 3. We note first of all that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then all the constructions to be considered can be carried out independently in $\mathfrak{g}_1$ and $\mathfrak{g}_2$. It suffices, therefore, to prove the theorem for the case that $\mathfrak{g}$ is simple; and we assume this in what follows.

Let X be the set of all principal commutative subspaces of the space Q. It is an $(l-1)$-dimensional irreducible algebraic subvariety of $\mathrm{Gr}_l(Q)$. The set X^0 of subspaces of the form $Q(h_0)$, $h_0 \in \mathfrak{h}^{\mathrm{reg}}$, is an open subset of X in the Zariski topology.

Consider an arbitrary principal commutative subspace $P \in X$. There exists a homeomorphism F of the unit disk $\mathbf{D} = \{t \in \mathbf{C} : |t| < 1\}$ into $\mathrm{Gr}_l(Q)$ with the following properties:

1) $F(\mathbf{D}) \subset X$.
2) $F(0) = P$.
3) $F(t) \in X^0$ for all $t \neq 0$.

Thus, the punctured disk $\mathbf{D}\backslash\{0\}$ is mapped into X^0. This mapping is of the form $F(t) = \hat{\sigma}(\pi h_0(t))$, where the coordinates α_{ij} of the point $\pi h_0(t) \in P\mathfrak{h}^{\mathrm{reg}}$, as defined in the proof of Proposition 5.1, are Laurent series in t. By suitably normalizing the element $h_0(t) \in \mathfrak{h}$ we can arrange for them to be power series in t. Thus $F(t) = Q(h_0(t))$ for $t \neq 0$, where $h_0(t)$ is the value of some (automatically regular) series $\boldsymbol{h}_0$. Then

$$P = F(0) = \lim_{t\to 0} F(t) = \lim_{t\to 0} Q(h_0(t)) = Q(\boldsymbol{h}_0).$$

5.4. Next we prove the first part of Theorem 4. The argument in §5.1 shows that for an arbitrary regular series $\boldsymbol{h}_0$,

$$Q(\boldsymbol{h}_0) = Q_0(\boldsymbol{h}_0) \oplus Q_1(\boldsymbol{h}_0) \oplus Q_2(\boldsymbol{h}_0) \oplus \cdots , \tag{36}$$

where

$$Q_k(\boldsymbol{h}_0) = \left\{ \sum_{\alpha\in\Delta_k(\boldsymbol{h}_0)\setminus\Delta_{k+1}(\boldsymbol{h}_0)} \frac{\alpha(h)}{\alpha(h_0^{(k)})} e_\alpha e_{-\alpha} : h \in \mathfrak{h}^{(k)} \right\}. \tag{37}$$

For the definition of the subspace $Q(\boldsymbol{h}_0)$, therefore, all that is relevant is the family of subsystems $\Delta_k(\boldsymbol{h}_0)$ $(k = 0, 1, 2, \ldots)$ of the root system Δ and, for each k, the orthogonal projection of $h_0^{(k)}$ on $\langle\Delta_k(\boldsymbol{h}_0)\rangle$. Consequently, we can suppose that $h_0^{(k)} \in \langle\Delta_k(\boldsymbol{h}_0)\rangle$.

Furthermore, it is clear that if $h_0^{(k)} = 0$ for some k, so that $\Delta_{k+1}(\boldsymbol{h}_0) = \Delta_k(\boldsymbol{h}_0)$, then $Q_k(\boldsymbol{h}_0) = 0$ and, without changing the subspace $Q(\boldsymbol{h}_0)$, we can "shrink" $\boldsymbol{h}_0$, replacing it by the series

$$h_0^{(0)} + h_0^{(1)}t + \cdots + h_0^{(k-1)}t^{k-1} + h_0^{(k+1)}t^k + h_0^{(k+2)}t^{k+1} + \cdots .$$

Consequently, we can assume that $h_0^{(k)} \neq 0$ for any k such that $\Delta_k(\boldsymbol{h}_0) \neq \varnothing$.

More generally, suppose the system $\Delta_k(\boldsymbol{h}_0)$ is the union of two orthogonal subsystems Δ'_k and Δ''_k. Then for any $m \geq k$ we have $\Delta_m(\boldsymbol{h}_0) = \Delta'_m \cup \Delta''_m$, where $\Delta'_m \subset \Delta'_k$ and $\Delta''_m \subset \Delta''_k$, and $Q_m(\boldsymbol{h}_0) = Q'_m \oplus Q''_m$, where Q'_m (resp. Q''_m is contained in $\langle e_\alpha e_{-\alpha} : \alpha \in \Delta'_m\rangle$ (resp. in $\langle e_\alpha e_{-\alpha} : \alpha \in \Delta''_m\rangle$). The constructions of the subspaces Q'_k, Q'_{k+1}, ... on the one hand, and of the subspaces Q''_k, Q''_{k+1}, ... on the other, go through independently. For the definition of the first, all that is relevant is the projections $h_0^{(m)'}$ of the elements $h_0^{(m)}$ $(m = k, k+1, \ldots)$ on $\langle\Delta'_m\rangle$; for the definition of the second, the projections $h_0^{(m)''}$ of these elements on $\langle\Delta''_m\rangle$. Therefore, if $h_0^{(k)''} = 0$, then $Q''_k = 0$, and without changing the subspace $Q(\boldsymbol{h}_0)$ we can replace $\boldsymbol{h}_0$ by the series

$$h_0^{(0)} + h_0^{(1)}t + \cdots + h_0^{(k-1)}t^{k-1} + (h_0^{(k)'} + h_0^{(k+1)''})t^k + (h_0^{(k+1)'} + h_0^{(k+2)''})t^{k+1} + \cdots .$$

Consequently, we can assume that for any k the projection of $h_0^{(k)}$ on the linear span of each indecomposable component of the system $\Delta_k(\boldsymbol{h}_0)$ is different from zero.

5.5. We come now to the proof of the second part of Theorem 4. Let $\boldsymbol{h}_0$ be a canonical regular series. We show how, starting from the subspace $Q(\boldsymbol{h}_0)$, to reconstruct the subsystems $\Delta_k(\boldsymbol{h}_0)$ $(k = 0, 1, 2, \ldots)$ each in turn and, up to proportionality, the projections of the elements $h_0^{(k)}$ on the linear spans of the indecomposable components of $\Delta_k(\boldsymbol{h}_0)$.

First we prove a lemma of a general nature.

LEMMA 5.2. *Let Δ be an indecomposable root system, and Δ_1 a proper subsystem. Then for any root $\delta \in \Delta_1$ there exist roots β, $\gamma \in \Delta\setminus\Delta_1$ such that $\gamma - \beta = \delta$.*

PROOF. Let W be the Weyl group of the system Δ_1, naturally imbedded in the Weyl group of Δ, and let Δ'_1 be the indecomposable component of Δ_1 containing δ. Since Δ is indecomposable, there exists a root $\beta \in \Delta\setminus\Delta'_1$ that is not orthogonal to Δ'_1. Clearly, $\beta \notin \Delta_1$. Furthermore, since $\langle W_1\delta\rangle = \langle\Delta'_1\rangle$, there exists an element

$w_1 \in W_1$ such that $(\beta, w_1\delta) \neq 0$. Replacing β by $\pm w_1^{-1}\beta$, we can arrange that $(\beta, \delta) < 0$. Then $\gamma = \beta + \delta \in \Delta$, and the roots β and γ satisfy the requirements of the lemma.

Suppose now that the subsystem $\Delta_k(\boldsymbol{h}_0)$ is already known. Then the subsystem $\Delta_{k+1}(\boldsymbol{h}_0)$ can be reconstructed from $Q(\boldsymbol{h}_0)$ by using the following lemma.

LEMMA 5.3. *Let Δ'_k be any indecomposable component of the system $\Delta_k(\boldsymbol{h}_0)$. Then $\Delta_{k+1}(\boldsymbol{h}_0) \cap \Delta'_k$ is the largest proper subsystem Δ'_{k+1} of Δ'_k with the following property:*

(D) *The intersection of $Q(\boldsymbol{h}_0)$ with $\langle e_\alpha, e_{-\alpha} : \alpha \in \Delta'_k\rangle$ is the (direct) sum of the intersections of $Q(\boldsymbol{h}_0)$ with $\langle e_\alpha e_{-\alpha} : \alpha \in \Delta'_k \backslash \Delta'_{k+1}\rangle$ and with $\langle e_\alpha e_{-\alpha} : \alpha \in \Delta'_{k+1}\rangle$.*

PROOF. From (36) and (37) it follows that the subsystem $\Delta_{k+1}(\boldsymbol{h}_0) \cap \Delta'_k$ does have property (D). Suppose now that some proper subsystem Δ'_{k+1} of Δ'_k not contained in $\Delta_{k+1}(\boldsymbol{h}_0)$ also has this property.

Take $d \in \Delta'_{k+1} \backslash \Delta_{k+1}(\boldsymbol{h}_0)$. Since

$$\langle \Delta_{k+1}(\boldsymbol{h}_0)\rangle \cap \Delta = \Delta_{k+1}(\boldsymbol{h}_0),$$

we have $\delta \notin \langle \Delta_{k+1}(\boldsymbol{h}_0)\rangle$. There exists therefore an element $h \in \langle \Delta'_k\rangle$ orthogonal to $\Delta_{k+1}(\boldsymbol{h}_0)$ but not to δ. Consider the element

$$q = \sum_{\alpha \in \Delta'_k \backslash \Delta_{k+1}(\boldsymbol{h}_0)} \frac{\alpha(h)}{\alpha(h_0^{(k)})} e_\alpha e_{-\alpha} \in Q(\boldsymbol{h}_0).$$

By (D), we have

$$\overline{q} = \sum_{\alpha \in \Delta'_{k+1} \backslash \Delta_{k+1}(\boldsymbol{h}_0)} \frac{\alpha(h)}{\alpha(h_0^{(k)})} e_\alpha e_{-\alpha} \in Q(\boldsymbol{h}_0). \tag{38}$$

Since $\overline{q} \in \langle e_\alpha e_{-\alpha} : \alpha \in \Delta'_k \backslash \Delta_{k+1}(\boldsymbol{h}_0)\rangle$, it must be that

$$\overline{q} = \sum_{\alpha \in \Delta'_k \backslash \Delta_{k+1}(\boldsymbol{h}_0)} \frac{\alpha(\overline{h})}{\alpha(h_0^{(k)})} e_\alpha e_{-\alpha} \tag{39}$$

for some element $\overline{h} \in \langle \Delta'_k\rangle$ orthogonal to $\Delta_{k+1}(\boldsymbol{h}_0)$.

By Lemma 5.2, there exist roots β, $\gamma \in \Delta'_k \backslash \Delta'_{k+1}$ such that $\gamma - \beta = \delta$. Comparing coefficients of $e_\beta e_{-\beta}$ and $e_\gamma e_{-\gamma}$ in (38) and (39), we see that $\beta(\overline{h}) = \gamma(\overline{h}) = 0$, so that $\delta(\overline{h}) = 0$. But comparing coefficients for $e_\delta e_{-\delta}$, we find that $\delta(\overline{h}) \neq 0$. The contradiction proves the lemma.

5.6. If we know the indecomposable component Δ'_k of the system $\Delta_k(\boldsymbol{h}_0)$ and its subsystem $\Delta'_{k+1} = \Delta_{k+1}(\boldsymbol{h}_0) \cap \Delta'_k$, then we know the subspace

$$\begin{aligned} Q'_k &= Q(\boldsymbol{h}_0) \cap \langle e_\alpha e_{-\alpha} : \alpha \in \Delta'_k \backslash \Delta'_{k+1}\rangle \\ &= \left\{ \sum_{\alpha \in \Delta'_k \backslash \Delta'_{k+1}} \frac{\alpha(h)}{\alpha(h_0^{(k)})} e_\alpha e_{-\alpha} : h \in \mathfrak{h}^{(k)} \cap \langle \Delta'_k\rangle \right\}. \end{aligned}$$

To reconstruct up to proportionality the projection of the element $h_0^{(k)}$ on $\langle \Delta'_k\rangle$, it suffices to find the ratios $\beta(h_0^{(k)})/\gamma(h_0^{(k)})$ for all β, $\gamma \in \Delta'_k \backslash \Delta'_{k+1}$.

Since $h_0^{(k)} \in \mathfrak{h}^{(k)}$, it suffices to restrict ourselves to those cases where the projections of β and γ on $\mathfrak{h}^{(k)}$ are not proportional; if they are, the desired ratio $\beta(h_0^{(k)})/\gamma(h_0^{(k)})$ is equal simply to the coefficient of proportionality between the two projections.

Consider the case that $\beta + \gamma \in \Delta$. In this case we use the argument that worked for Proposition 5.1. Namely, we observe that the functions f_β, f_γ and $f_{\beta+\gamma}$ are connected on Q'_k by the linear dependence

$$f_{\beta+\gamma} = \frac{\beta(h_0^{(k)})f_\beta + \gamma(h_0^{(k)})f_\gamma}{\beta(h_0^{(k)}) + \gamma(h_0^{(k)})}. \tag{40}$$

If the projections of β and γ on $\mathfrak{h}^{(k)}$ are not proportional, then f_β and f_γ are linearly independent on Q'_k, and the desired ratio $\beta(h_0^{(k)})/\gamma(h_0^{(k)})$ can be found from (40).

A similar argument applies to the case $\beta - \gamma \in \Delta$.

Consequently, we can find the ratio $\beta(h_0^{(k)})/\gamma(h_0^{(k)})$ whenever the roots β and γ can be included (as the extreme terms) in a sequence of roots from $\Delta'_k \backslash \Delta'_{k+1}$ for which either the sum or the difference of any two successive terms is also a root.

But this is sufficient, in view of the following general lemma.

LEMMA 5.4. *Let Δ be an indecomposable root system, and Δ_1 a proper subsystem. Then any two roots β, $\gamma \in \Delta\backslash\Delta_1$ can be included in a sequence of roots from $\Delta\backslash\Delta_1$ for which either the sum or the difference of any two successive terms is also a root.*

PROOF. Assume the contrary. Then $\Delta\backslash\Delta_1$ splits into two nonempty subsets Γ' and Γ'' such that $\beta \pm \gamma \notin \Delta$ for any $\beta \in \Gamma'$ and $\gamma \in \Gamma''$. In particular, these subsets are orthogonal. By Lemma 5.2, for any root $\delta \in \Delta_1$ there exist roots β, $\gamma \in \Delta\backslash\Delta_1$ such that $\gamma - \beta = \delta$. Then both these roots must belong to one of the subsets Γ' or Γ''. Therefore, if Δ' (resp. Δ'') is the subsystem generated by the subset Γ' (resp. Γ''), then $\Delta = \Delta' \cup \Delta''$, with Δ' and Δ'' orthogonal. Since Δ was assumed indecomposable, we have a contradiction.

This completes the proof of Theorem 4.

§6. Commutative subalgebras of the algebra $\mathscr{P}(\mathfrak{sl}_n)$

6.1. In this section we prove that *in the algebra $\mathscr{P}(\mathfrak{sl}_n)$, the commutative subalgebra generated by the subalgebra $\mathfrak{h}$ and the centers of the subalgebras $\mathscr{P}(\mathfrak{sl}_{n-k})$, $k = 0, 1, \dots, n-2$ (see the example in* §0.5) *is the limit of commutative subalgebras obtained by the translation-of-invariants method.*

It will be technically more convenient to deal, not with the algebra $\mathfrak{sl}_n$, but with the algebra $\mathfrak{gl}_n$, even though the latter is not semisimple. We have the projection homomorphism

$$p\colon \mathfrak{gl}_n \to \mathfrak{sl}_n, \qquad X \mapsto X - \tfrac{1}{n}(\operatorname{tr} X)E. \tag{41}$$

It induces a projection homomorphism $\mathscr{P}(\mathfrak{gl}_n) \to \mathscr{P}(\mathfrak{sl}_n)$, which we denote by the same letter p. This homomorphism allows us to derive from any result that we obtain for $\mathfrak{gl}_n$ a corresponding result for $\mathfrak{sl}_n$.

6.2. The center Z_n of the algebra $\mathscr{P}(\mathfrak{gl}_n)$ consists, as is known, of the invariants of the natural action of $\mathfrak{gl}_n$ on $\mathscr{P}(\mathfrak{gl}_n)$. We describe its generators.

Let e_{ij} $(1 \le i, j \le n)$ be the matrix units constituting a basis for the algebra $\mathfrak{gl}_n$, and define the "principal minors of order r" of $\mathfrak{gl}_n$ as the elements

$$M_{i_1 \cdots i_r} = \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma) e_{i_1 i_{\sigma 1}} \cdots e_{i_r i_{\sigma r}} \qquad (i_1 < \cdots < i_r) \tag{42}$$

(where multiplication is understood not as matrix multiplication, but as an operation on $\mathscr{P}(\mathfrak{gl}_n)$). Then the sums

$$F_r = \sum_{i_1 < \cdots < i_r} M_{i_1 \cdots i_r} \qquad (r = 1, \ldots, n) \tag{43}$$

are algebraically independent and generate the center Z_n.

Denote by $\partial_{ij} \in \operatorname{Der} \mathscr{P}(\mathfrak{gl}_n)$ the derivation with respect to e_{ij}. For any matrix $A = (a_{ij})$ put

$$\partial_A u = \sum_{i,j} a_{ij} \partial_{ij} u \qquad (u \in \mathscr{P}(\mathfrak{gl}_n)). \tag{44}$$

In particular, let $A = \operatorname{diag}(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are all different. Then by a general result of Mishchenko and Fomenko [4], the elements

$$F_{r,k}(a_1, \ldots, a_n) = \frac{1}{k!} \partial_A^k F_r \qquad (r = 1, \ldots, n;\ k = 0, 1, \ldots, r-1) \tag{45}$$

of $\mathscr{P}(\mathfrak{gl}_n)$ are pairwise commutative and algebraically independent. The commutative subalgebra generated by them we denote by $L(a_1, \ldots, a_n)$. It has the maximum possible transcendence degree, equal to $d(\mathfrak{gl}_n) = n(n+1)/2$.

We note that the element $F_{r,k}(a_1, \ldots, a_n)$ is the sum of all possible terms obtained from those of the expression for the element F_r in terms of matrix units by replacing any k diagonal matrix units by the corresponding diagonal elements of the matrix A.

6.3. We prove now that the algebra

$$L = \lim_{t \to 0} L(t^{n-1}, \ldots, t, 1) \subset \mathscr{P}(\mathfrak{gl}_n) \tag{46}$$

is generated by the centers Z_{n-k} of the subalgebras $\mathscr{P}(\mathfrak{gl}_{n-k})$, $k = 0, 1, \ldots, n-1$.

In the expression for the element $F_{r,k}(t^{n-1}, \ldots, t, 1)$, the terms in t of lowest degree, equal to $k(k-1)/2$, are obtained by replacing the factors $e_{n-k+1,n-k+1}, \ldots, e_{nn}$ in the appropriate terms of the expression for F_r by the corresponding powers of t. For the coefficient of $t^{k(k-1)/2}$ we obtain the sum of the principal minors of order $r-k$ of the algebra $\mathfrak{gl}_{n-k}$ (imbedded in $\mathfrak{gl}_n$ as the upper left corner). Consequently, this sum belongs to L. Our assertion follows from this.

6.4. To pass to the algebra $\mathfrak{sl}_n$, we observe that the subalgebra $\mathscr{P}(\mathfrak{sl}_n) \subset \mathscr{P}(\mathfrak{gl}_n)$ is annihilated by the operator ∂_E, so that

$$\partial_A u = \partial_{pA} u \in \mathscr{P}(\mathfrak{sl}_n) \quad \text{for } u \in \mathscr{P}(\mathfrak{sl}_n) \text{ and } A \in \mathfrak{gl}_n. \tag{47}$$

Therefore, applying the translation-of-invariants method to the algebra $\mathfrak{sl}_n$, without loss of generality we can take for the initial diagonal matrix A a matrix of arbitrary trace.

The center of the algebra $\mathscr{P}(\mathfrak{sl}_n)$ is equal to pZ_n, and is generated by the elements pF_r, $r = 2, \ldots, n$. Since these together with the element $F_1 = E$ can be taken as generators of Z_n, the commutative subalgebra of $\mathscr{P}(\mathfrak{sl}_n)$ obtained by the translation-of-invariants method starting from the matrix $A = \operatorname{diag}(a_1, \ldots, a_n) \in \mathfrak{gl}_n$ coincides with the subalgebra $pL(a_1, \ldots, a_n)$.

The subalgebra

$$pL = \lim_{t \to 0} pL(t^{n-1}, \dots, t, 1)$$

is generated by the projections on $\mathscr{P}(\mathfrak{sl})_n$ of the centers Z_{n-k} of the subalgebras $\mathscr{P}(\mathfrak{gl}_{n-k})$, $k = 0, 1, \dots, n-1$. Choose for the generators of Z_{n-k} the projections on $\mathscr{P}(\mathfrak{sl}_{n-k})$ of the sums of the principal minors of orders ≥ 2 of the algebra $\mathfrak{sl}_{n-k}$, together with the identity matrix $E_{n-k} \in \mathfrak{gl}_{n-k}$ (whose projection on $\mathscr{P}(\mathfrak{sl}_{n-k})$ is equal to zero). Then we find that the subalgebra pL is generated by the centers of the subalgebras $\mathscr{P}(\mathfrak{sl}_{n-k})$ and the projections on $\mathscr{P}(\mathfrak{sl}_n)$ of the matrices E_{n-k}, $k = 1, \dots, n-1$, which constitute, as is easily seen, a basis for the space of all diagonal matrices with zero trace.

Thus, in the algebra $\mathscr{P}(\mathfrak{sl}_n)$, the commutative subalgebra generated by the subalgebra of diagonal matrices (with zero trace) and the centers of the subalgebras $\mathscr{P}(\mathfrak{sl}_{n-k})$ is the limit of subalgebras obtained by the translation-of-invariants method.

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