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EXOTIC ELLIPTIC ALGEBRAS

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ABSTRACT. The 4-dimensional Sklyanin algebras, over $\mathbb{C}$, $A(E,\tau)$, are constructed from an elliptic curve E and a translation automorphism τ of E. The Klein vierergruppe Γ acts as graded algebra automorphisms of $A(E,\tau)$. There is also an action of Γ as automorphisms of the matrix algebra $M_2(\mathbb{C})$ making it isomorphic to the regular representation. The main object of study is the algebra $\widetilde{A} := \big(A(E,\tau)\otimes M_2(\mathbb{C})\big)^\Gamma$. Like $A(E,\tau)$, $\widetilde{A}$ is noetherian, generated by 4 elements modulo six quadratic relations, Koszul, Artin-Schelter regular of global dimension 4, has the same Hilbert series as the polynomial ring on 4 variables, satisfies the χ condition, and so on. These results are special cases of general results proved for a triple (A,T,H) consisting of a (often co-semisimple) Hopf algebra H, a (often graded) H-comodule algebra A, and an H-torsor T. Those general results involve transferring properties between A, $A\otimes T$, and $(A\otimes T)^{\mathrm{coH}}$. We then investigate $\widetilde{A}$ from the point of view of non-commutative projective geometry. We examine its point modules, line modules, and a certain quotient $\widetilde{B} := \widetilde{A}/(\Theta,\Theta')$ where Θ and Θ' are homogeneous central elements of degree two. In doing this we show that $\widetilde{A}$ differs from A in interesting ways. For example,the point modules for A are parametrized by E and 4 more points but $\widetilde{A}$ has exactly 20 point modules. Although $\widetilde{B}$ is not a twisted homogeneous coordinate ring in the sense of Artin and Van den Bergh a certain quotient of the category of graded $\widetilde{B}$-modules is equivalent to the category of quasi-coherent sheaves on the curve $E/E[2]$ where $E[2]$ is the 2-torsion subgroup.

1. Introduction

1.1. The 3- and 4-dimensional Sklyanin algebras are among the most interesting algebras that have appeared in non-commutative algebraic geometry. Such an algebra determines and is determined by an elliptic curve, E, a translation automorphism, τ, of E, and an invertible $\mathcal{O}_E$-module $\mathcal{L}$ of degree 3, and 4, respectively. The representation theory of the Sklyanin algebra $A(E,\tau,\mathcal{L})$ and, what is almost the same thing, the geometric features of the non-commutative projective space $\mathrm{Proj}_{nc}\big(A(E,\tau,\mathcal{L})\big)$, is governed by the geometry of E and τ when E is embedded as a cubic or quartic curve in $\mathbb{P}\big(H^0(E,\mathcal{L})^*\big)$. We refer the reader to [2] and [31] for overviews of the 3- and 4-dimensional Sklyanin algebras. The n in "n-dimensional" refers to the Gelfand-Kirillov dimension of $A(E,\tau)$, or its global dimension, or the dimension of $A(E,\tau,\mathcal{L})_1$ which is equal to $H^0(E,\mathcal{L})$.

Odesskii and Feigin have defined generalizations of the 4-dimensional Sklyanin algebras in [22], [23], and [11]. The algebras they construct depend on a pair (E,τ), as before, but now a higher degree line bundle is used to construct $A(E,\tau,\mathcal{L})$. In particular, when $\deg(\mathcal{L})=n^2$, $n\geq 2$, Odesskii and Feigin construct an algebra that they denote by $Q_{n^2}(E,\tau)$.

Following an idea of Odesskii in [21] we construct for every such pair (E,τ) and integer $n\geq 2$ a connected graded algebra $\widetilde{Q}=\widetilde{Q}_{n^2}(E,\tau)$ by a kind of Galois descent procedure applied to $Q_{n^2}(E,\tau)$. We show that the algebras obtained in this manner inherit many of the good properties enjoyed by $Q_{n^2}(E,\tau)$. For example, they are Artin-Schelter regular.

It is our hope that the algebras $\widetilde{Q}$ are new examples with interesting properties.

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1.2. This paper examines the case $n = 2$ and shows that the $\widetilde{Q}$ do indeed exhibit novel features. They are still governed very strongly by the geometry of E and τ. For this reason we call them "elliptic algebras", the name Odesskii and Feigin adopted for their algebras, and we append the adjective "exotic" to indicate that they are somewhat novel when compared to the familiar 4-dimensional Sklyanin algebras and other 4-dimensional Artin-Schelter regular algebras.

1.3. The procedure behind the construction of the exotic elliptic algebras is quite general. Let H be a finite dimensional Hopf algebra over a field k and A an H-comodule algebra. One might ask in addition that A be a graded algebra and that every homogeneous component be a subcomodule. Let T be an H-torsor (see §3.1) and define the H-comodule algebra $A' := A \otimes T$. If A is graded one places T in degree zero to make A' a graded algebra. Let $\widetilde{A}$ denote the subalgebra of A' consisting of the H-coinvariant elements. In §3 and §4 we show how various properties pass back and forth between A, A', and $\widetilde{A}$. For example, we consider the noetherian property, that of being finite as a module over its center, and numerous homological properties that play an important role in non-commutative algebraic geometry.

In §4 we assume that $\dim_k(H) < \infty$, that H is semisimple and co-semisimple, and (usually) A is a connected graded H-comodule algebra. We show that T must be semisimple. We observe that A is Koszul (m-Koszul) if and only if $\widetilde{A}$ is. We show A is Artin-Schelter regular of dimension d if and only if $\widetilde{A}$ is. We show that A is Artin-Schelter regular of dimension d if and only is $\widetilde{A}$ is. We show that $\widetilde{A}$ satisfies the χ condition, introduced in [7], if A does.

1.4. We expect the construction $A \rightsquigarrow \widetilde{A}$, and our results about properties shared by A and $\widetilde{A}$, to be of use in many other situations. For example, it would be sensible to examine the effect of this construction on 2- and 3-dimensional Artin-Schelter regular algebras now that J.J. Zhang and his co-authors have determined (many/all?) the finite dimensional Hopf algebras for which such algebras can be comodule algebras. Even the case when A is a polynomial ring, or an enveloping algebra, deserves investigation.

1.5. Let $Q = A(E, \tau, \mathcal{L})$ be a 4-dimensional Sklyanin algebra. It was shown in [33] that $\Gamma = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ acts as graded algebra automorphisms of Q when the base field is $\mathbb{C}$. The action there is induced by the translation action of the 2-torsion subgroup, $E[2]$, on E. Here, working over an arbitrary algebraically closed field k of characteristic $\neq 2$, we define an action of Γ as graded k-algebra automorphisms of Q and show that this "corresponds" to the translation action of $E[2]$ on E.

In the language of §1.3, we take H to be the Hopf algebra of k-valued functions on Γ and T to be $M_2(k)$, the ring of 2×2 matrices, with an appropriate H-comodule algebra structure. We then have $\widetilde{Q} = (Q \otimes T)^{\mathrm{coH}} = (Q \otimes T)^{\Gamma}$. Generally speaking, the results in §3 and §4 show that $\widetilde{Q}$ has all the good properties Q has.

It is a noetherian domain, has global dimension 4, has the same Hilbert series as the polynomial ring on 4 indeterminates, is Artin-Schelter regular, satisfies the χ condition, and so on.

1.6. Among the most important results about Sklyanin algebras are classifications of their point and line modules. The point modules of a 3-dimensional Sklyanin algebra are naturally parametrized by E or, more informatively, by a natural copy of E as a smooth cubic curve in $\mathbb{P}^2 = \mathbb{P}(Q_1^*)$. The point modules for a 4-dimensional Sklyanin are parametrized by a natural copy of E as a smooth quartic curve in $\mathbb{P}^3 = \mathbb{P}(Q_1^*)$ and 4-additional points, those being the vertices of the 4 singular quadrics that contain the copy of E. The line modules are, in both cases, parametrized by the secant lines to E, the lines in $\mathbb{P}(Q_1^*)$ that meet E with multiplicity ≥ 2.

The situation changes dramatically when Q is replaced by $\widetilde{Q}$. For example, $\widetilde{Q}$ has only 20 point modules. In a privately circulated note in 1988 [10], Van den Bergh showed that a generic 4-dimensional AS-regular algebra (with some other properties) has exactly 20 point modules. Since then, there have been a number of examples showing that particular algebras, rather than the ephemeral "generic algebras", have exactly 20 point modules. We believe that ours are the first such examples that turn up "in vivo", so to speak.

1.7. Van den Bergh and Tate [38] showed that the Odesskii and Feigin algebras Q_{n^2} are noetherian, Koszul, Artin-Schelter regular algebras of dimension n^2 with Hilbert series $(1-t)^{-n^2}$. It follows from the relations for Q_{n^2} that $\Gamma = (\mathbb{Z}/n) \times (\mathbb{Z}/n)$, realized as the n-torsion subgroup $E[n] \subset E$, acts as graded algebra automorphisms of Q_{n^2}. It is an easy matter to see that the ring of $n \times n$ matrices $M_n(\mathbb{C})$ is an H-torsor where H is the Hopf algebra of k-valued functions on Γ. In §5 we show that for all $n \geq 2$, $\widetilde{Q_{n^2}} = \big(Q_{n^2} \otimes M_n(k)\big)^\Gamma$ has "the same" properties as Q_{n^2}.

1.8. In §6 we begin a detailed examination of the algebra $\widetilde{Q}$ described in §1.5. We give explicit generators and relations for $\widetilde{Q}$. It has 4 generators and 6 quadratic relations (Proposition 6.1). Since $\Gamma = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ acts on Q_1 it acts as autoomorphisms of $\mathbb{P}(Q_1)^* = \mathbb{P}^3$. This $\mathbb{P}^3$ has a natural copy of E embedded as a quartic curve. It is easy to see that Γ restricts to an action as automorphisms of E.

In §7 we show that this action is the same as the translation action of the 2-torsion subgroup $E[2]$. Each $\gamma \in \Gamma$ acts as an auto-equivalence $M \rightsquigarrow \gamma^* M$of the graded-module category $\mathsf{Gr}(Q)$. Because Γ acts as $E[2]$ does, if M_p, $p \in E$, is the point module corresponding to $p \in E$, then $\gamma^* M_p \cong M_{p+\omega}$ for a suitable $\omega \in E[2]$. There is a similar result for line modules: $\gamma^* M_{p,q} \cong M_{p+\omega, q+\omega}$.

1.9. By [32], there is a regular sequence in Q consisting of two homogeneous central elements of degree 2, Ω and Ω' say, such that $Q/(\Omega, \Omega')$ is a twisted homogeneous coordinate ring, $B(E, \tau, \mathcal{L})$, in the sense of Artin and Van den Bergh [6]. The main result in [6] tells us that the quotient category $\mathsf{QGr}(B(E, \tau, \mathcal{L}))$ is equivalent to $\mathsf{Qcoh}(E)$, the category of quasi-coherent sheaves on E.

We observe that $\widetilde{Q}$ also has a regular sequence consisting of two homogeneous central elements of degree 2, Θ and Θ' say. However, $\widetilde{B} := \widetilde{Q}/(\Theta, \Theta')$ is not a twisted homogeneous coordinate ring in the sense of [6]. Nevertheless, Theorem 8.1 proves that $\mathsf{QGr}(\widetilde{B})$ is equivalent to $\mathsf{Qcoh}(E/E[2])$. Incidentally, because k is algebraically closed of characteristic not 2, $E/E[2] \cong E$ as algebraic groups. Despite this, $\widetilde{B}$ has no point modules. The points on $E/E[2]$ correspond to fat point modules of multiplicity 2 over $\widetilde{B}$.

1.10. In §9 we prove that $\widetilde{Q}$ has exactly 20 point modules. These modules correspond to 20 points in $\mathbb{P}^3 = \mathbb{P}(\widetilde{Q}_1^*)$ that we determine explicitly. The "meaning" of these 20 points eludes us. Let $\mathfrak{P}$ denote that set of 20 points. The degree shift functor $M \rightsquigarrow M(1)$ induces a permutation $\theta : \mathfrak{P} \to \mathfrak{P}$ of order 2. A beautiful result of Shelton and Vancliff shows that the data $(\mathfrak{P}, \theta)$ determines $\widetilde{Q}$ in the sense that the subspace $R \subseteq Q_1 \otimes Q_1$ of bihomogeneous forms vanishing on the graph of θ has the property that $\widetilde{Q}$ is isomorphic to $T(Q_1)/(R)$, the tensor algebra on Q_1 modulo the ideal generated by R. In §10, we exhibit six families of line modules for $\widetilde{Q}$, each parametrized by a copy of $E/E[2]$. We do not know if these are all the line modules for $\widetilde{Q}$.

1.11. In late January 2015, after proving most of the results in this paper, one of us found an announcement on the web of a seminar talk given at the University of Manchester in January 2014 by Andrew Davies that appeared to contain many of the results we prove here. A few days later (on 1/20/2015) we found a copy of Andrew Davies Ph.D. dissertation [9] that does indeed have substantial overlap with this paper. Davies also proves many things we don't. For example, he describes $\widetilde{B}$ (when τ has infinite order) in the manner of Artin and Stafford [3]. Nevertheless, some of what we do is more general, and some of our arguments differ from his so there seems to be some value in presenting our results to a wider audience. In particular, when dealing with the Sklyanin algebras we make no assumption on the order of τ and we do not restrict our base field to the complex numbers. Also, several of the results in §3 and §4 for arbitrary H and T are proved by Davies only in the case H is the ring of k-valued functions on a finite abelian group.

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2. Preliminaries

In §3.1 we work over an arbitrary field k. Once we begin discussing the 4-dimensional Sklyanin algebras k will be an algebraically closed field of characteristic $\neq 2$.

2.1. **Notation and Terminology.** We will use what is now standard terminology and notation for graded rings and non-commutative projective algebraic geometry. There are several sources for unexplained terminology: The papers by Artin-Tate-Van den Bergh that started the subject of non-commutative projective algebraic geometry, [4] and [5]; The survey by Stafford and Van den Bergh [34]; The papers by Stafford and Smith [32] and Levasseur and Smith [17] on 4-dimensional Sklyanin algebras; The survey [31] on 4-dimensional Sklyanin algebras; The paper by Artin and Van den Bergh on twisted homogeneous coordinate rings [6]; The paper by Artin and Zhang on non-commutative projective schemes [7].

Suppose A is an $\mathbb{N}$-graded k-algebra such that $\dim_k(A_i) < \infty$ for all i. The category of $\mathbb{Z}$-graded left A-modules with degree-preserving A-module homomorphisms is denoted by $\mathsf{Gr}(A)$. The full subcategory of $\mathsf{Gr}(A)$ consisting of those modules that are the sum of their finite dimensional submodules is denoted by $\mathsf{Fdim}(A)$. This is a Serre subcategory so we can form the quotient category

$$\mathsf{QGr}(A) \;:=\; \frac{\mathsf{Gr}(A)}{\mathsf{Fdim}(A)}.$$

In fact, $\mathsf{Fdim}(A)$ is a localizing subcategory so the quotient functor $\pi^* : \mathsf{Gr}(A) \to \mathsf{QGr}(A)$ has a right adjoint π_*. The functor π^* is exact. By definition, $\mathsf{QGr}(A)$ has the same objects as $\mathsf{Gr}(A)$. Since $\pi_*\pi^*$ is isomorphic to the identity functor we may view objects in $\mathsf{QGr}(A)$ as objects in $\mathsf{Gr}(A)$.

2.2. Throughout H is a co-semisimple Hopf algebra over k; i.e., its category of right comodules, $\mathcal{M}^H$, is semisimple. Further, A is a right H-comodule-algebra, i.e., an algebra object in $\mathcal{M}^H$, T is a right H-torsor (see §3.1), $A' := A \otimes T$, and $\widetilde{A} := (A')^{\mathrm{coH}}$, the subalgebra of H-coinvariants.

Let Υ be an abelian group. We call A an Υ-graded H-comodule algebra or an Υ-graded algebra in $\mathcal{M}^H$ if it is an H-comodule algebra such that each homogeneous component, A_i, is an H-subcomodule. For example, if V is a right H-comodule and $R \subseteq V \otimes V$ an H-subcomodule, then the tensor algebra, TV, and its quotient $TV/(R)$, are $\mathbb{Z}$-graded algebras in $\mathcal{M}^H$.

We write $\mathsf{Mod}(R)$ for the the category of left modules over a ring R. We write ${}_{A'}\mathcal{M}^H$ for the category of A'-modules internal to the category of H-comodules, i.e., vector spaces V equipped with an A'-module structure and an H-comodule structure such that $A' \otimes V \to V$ is an H-comodule map. If A is an Υ-graded algebra in $\mathcal{M}^H$ we write ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ for the category of Υ-graded A-modules internal to $\mathcal{M}^H$, i.e., each homogeneous component M_i is an H-comodule.

3. Torsors, twisting, and descent

In this section we prove some general results on the inheritance of various properties for certain rings of (co)invariants so relating various good properties of A or A' to those of $\widetilde{A}$.

3.1. Torsors.

An H-torsor is a comodule-algebra $T \in \mathcal{M}^H$ such that the ring T^{coH} of coinvariants is k and the linear map

$$T \otimes T \xrightarrow{\mathrm{id} \otimes \rho} T \otimes T \otimes H \xrightarrow{m \otimes \mathrm{id}} T \otimes H \tag{1}$$

is bijective, where $\rho : T \to T \otimes H$ is the comodule structure and $m : T \otimes T \to T$ is multiplication, and $T \cong H$ in $\mathcal{M}^H$.

Remark 3.1. A comodule algebra for which the composition in (1) is an isomorphism is sometimes called a right H-Galois object (see e.g. [8, Defn. 1.1]). Loc. cit. and the references therein are good sources for background on torsors. The additional condition that $T \cong H$ in $\mathcal{M}^H$ makes the torsor *cleft*; this condition is automatic when H is finite-dimensional, which is the case we are really interested in here. This is (part of) [8, Thm. 1.9], which cites [16] for a proof. ♦

3.2. Descent.

The following descent result is the key to much that follows.

Proposition 3.2. *The categories ${}_{A'}\mathcal{M}^H$ and $\mathsf{Mod}(\widetilde{A})$ are equivalent via the mutually quasi-inverse functors*

$$\begin{tikzcd} \mathsf{Mod}(\widetilde{A}) \arrow[r, bend left, "A' \otimes_{\widetilde{A}} \bullet"] & {}_{A'}\mathcal{M}^H \arrow[l, bend left, "\bullet^{\mathrm{coH}}"] \end{tikzcd} \tag{2}$$

Furthermore, the extension $\widetilde{A} \to A'$ is faithfully flat on the right and on the left.

Proof. By [26, Thm. I], both assertions follow if A' is injective as an H-comodule and the map

$$A' \otimes A' \xrightarrow{\mathrm{id} \otimes \rho} A' \otimes A' \otimes H \xrightarrow{m \otimes \mathrm{id}} A' \otimes H \tag{3}$$

analogous to (1) is onto. We are assuming H is cosemisimple, so *every* H-comodule is injective. Since (1) is an isomorphism so is the composition

$$A' \otimes T = A \otimes T \otimes T \to A' \otimes T \otimes H = A \otimes T \otimes T \otimes H \to A \otimes T \otimes H = A' \otimes H,$$

i.e., the restriction of (3) to $A' \otimes T \subseteq A' \otimes A'$ already surjects onto $A' \otimes H$. ■

3.3. **Finite generation, the noetherian property, and GK-dimension.**

Proposition 3.3. *If A an Υ-graded H-comodule algebra, then* $\dim_k(A_i) = \dim_k(\widetilde{A}_i)$ *for all* $i \in \Upsilon$.

Proof. We are assuming $T \cong H$ in $\mathcal{M}^H$ so $W \otimes T \cong W \otimes H$ in $\mathcal{M}^H$ for all $W \in \mathcal{M}^H$. The map

$$W \otimes H \ni w \otimes h \mapsto w_0 \otimes w_1 h \in W \otimes H$$

is an isomorphism between $W \otimes H$ with the diagonal H-coaction and $W \otimes H$ with the regular H-coaction on the right-hand tensorand. As a consequence, there is a vector space isomorphism $W \cong (W \otimes T)^{\mathrm{coH}}$. Now apply this fact with W equal to each homogeneous component of A. ■

Lemma 3.4. [15, Lem. 6.1] *Let A be an $\mathbb{N}$-graded k-algebra such that* $\dim_k(A_i) < \infty$ *for all i, and M a finitely generated graded A-module. Then*

$$\mathrm{GKdim}(M) = 1 + \limsup \log_n(\dim_k(M_n)).$$

Proposition 3.5. *If A is a $\mathbb{Z}$-graded comodule algebra such that* $\dim_k(A_i) < \infty$ *for all i, then A and $\widetilde{A}$ have the same Gelfand-Kirillov dimension.* ■

Lemma 3.6. *The functor* FORGET : ${}_{\mathsf{Gr}(A)}\mathcal{M}^H \to \mathsf{Gr}(A)$ *preserves projectivity, as does the analogous functor for ungraded modules.*

Proof. This follows from the fact that FORGET is left adjoint to an exact functor, namely $\bullet \otimes H$: $\mathsf{Gr}(A) \to {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. The same proof works in the ungraded case. ■

Proposition 3.7. *Suppose H is finite-dimensional. If A is noetherian so is $\widetilde{A}$.*

Proof. For any ring noetherianness is equivalent to the fact that the lattice of subobjects of a projective generator of its module category satisfies the ascending chain condition. If $\mathsf{Mod}(A)$ has this property then so does $\mathsf{Mod}(A')$ because H is finite-dimensional. From this and the fact that the forgetful functor ${}_{A'}\mathcal{M}^H \to \mathsf{Mod}(A')$ preserves projectivity it follows that ${}_{A'}\mathcal{M}^H$ has the same property. Finally, ${}_{A'}\mathcal{M}^H$ is equivalent to $\mathsf{Mod}(\widetilde{A})$ by Proposition 3.2. ■

Proposition 3.8. *If H is finite-dimensional and A is noetherian and module-finite over its center Z, then $\widetilde{A}$ is module-finite over its center $\widetilde{Z}$.*

Proof. Since T is finite-dimensional, $A' = A \otimes T$ is module-finite over Z. By [30, Thm. 6.2 (iii)], Z is module-finite over Z^{coH}. Since the action of Z^{coH} preserves the decomposition of A' into H-isotypic components, the coinvariants $\widetilde{A} = (A')^{\mathrm{coH}}$ are module-finite over $Z^{\mathrm{coH}} \leq \widetilde{Z}$. ■

The Artin-Tate Lemma as stated in [1] says the following: If $R \subseteq S \subseteq T$ are commutative rings such that R is noetherian, and T a finitely generated R-algebra and a finitely generated S-module, then S is a finitely generated R-algebra. It was later observed that the proof did not require T to be commutative, just R and S.

Proposition 3.9. *Suppose H is finite-dimensional. If A is a finitely generated k-algebra and module-finite over its center, then $\widetilde{A}$ has the same properties. Furthermore, under these hypotheses, the centers of A and $\widetilde{A}$ are finitely generated k-algebras.*

Proof. Let Z denote the center of A. By the Artin-Tate Lemma, Z is a finitely generated k-algebra, hence noetherian. By Proposition 3.8, $\widetilde{A}$ is module-finite over its center. Hence $\widetilde{A}$ is a finitely generated k-algebra if Z^{coH} is. However, Z is module-finite over Z^{coH} so the Artin-Tate Lemma implies that Z^{coH} is also a finitely generated k-algebra. ■

Lemma 3.10. *If A is a $\mathbb{Z}$-graded algebra in $\mathcal{M}^H$ and $N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ is finitely generated over A, then N^{coH} is finitely generated over A^{coH}.*

Remark 3.11. There is a completely analogous ungraded version. ♦

Proof. Being finitely generated can be characterized in category-theoretic terms as follows.

Let I be a *filtered* small category in the sense of [18, Section IX.1]: Every two objects i, i' fit inside a diagram

$$\begin{tikzcd} i \arrow[dr, dashed, bend left] & \\ & k \\ i' \arrow[ur, dashed, bend right] & \end{tikzcd}$$

and every solid left hand wedge as in the picture below can be completed to a commutative diagram by a dotted right hand wedge

$$\begin{tikzcd} & i \arrow[dr, dashed, bend left] & \\ j \arrow[ur, bend left] \arrow[dr, bend right] & & k \\ & i' \arrow[ur, dashed, bend right] & \end{tikzcd}$$

For any functor $F : I \to \mathsf{Gr}(A)$ we have a canonical map

$$\varinjlim_{i\in I} \mathrm{Hom}_{\mathsf{Gr}(A)}(N, F(i)) \to \mathrm{Hom}_{\mathsf{Gr}(A)}(N, \varinjlim_{i} F(i)). \tag{4}$$

We leave it to the reader to check that N is finitely generated if and only if for every filtered I and every functor F such that every arrow $F(i \to i')$ is an embedding the map (4) is an isomorphism. Also, the hom spaces on the two sides of the arrow are H-comodules, and the isomorphism respects these comodule structures.

By the graded version of [26, Thm. I], the functor $(\bullet)^{\mathrm{coH}}$ gives an equivalence $\mathsf{Gr}(A^{\mathrm{coH}}) \equiv {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. We identify the two categories. Let $F : I \to {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ be a functor from a filtered small category such that all $F(i \to i')$ are monomorphisms. Since the equivalence $\mathsf{Gr}(A^{\mathrm{coH}}) \equiv {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ is effected by the functor $(\bullet)^{\mathrm{coH}}$ which preserves filtered colimits $\varinjlim_{i\in I}$ as in (4), the analogue of (4) over A^{coH} is obtained by applying $(\bullet)^{\mathrm{coH}}$ to (4). Since the latter is an isomorphism, so is its restriction to the H-coinvariants. ■

Proposition 3.12. *Let A be a $\mathbb{Z}$-graded algebra in $\mathcal{M}^H$. If $M \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ and is a finitely generated A-module, then $(A' \otimes_A M)^{\mathrm{coH}}$ is a finitely generated $\widetilde{A}$-module.*

Proof. Apply Lemma 3.10, substituting A' for A, $\widetilde{A}$ for A^{coH}, and $A' \otimes_A M$ for N. ■

Proposition 3.13. *Suppose H is finite-dimensional. Let A be a $\mathbb{Z}$-graded H-comodule algebra and $M \in {}_{\mathsf{Gr}(A')}\mathcal{M}^H$. If M is a finitely generated A'-module, then M and the $\widetilde{A}$-module M^{coH} have the same GK-dimension.*

Proof. By the graded version of Proposition 3.2, $M \cong A' \otimes_{\widetilde{A}} M^{\mathrm{coH}}$.

The map $H \to A'$ that sends H to $1 \otimes T$ via the isomorphism $H \cong T$ in $\mathcal{M}^H$ is a morphism in $\mathcal{M}^H$ and is convolution invertible because the isomorphism $H \to T$ is convolution invertible. Since $\widetilde{A} = (A')^{\mathrm{coH}}$, it follows from [19, Defn. 7.2.1 and Thm. 7.2.2] that A' is isomorphic to a "twisted smash product" $\widetilde{A}\sharp_\sigma H$, where $\sigma : H \otimes H \to \widetilde{A}$ is a convolution-invertible map satisfying a cocycle-type condition. We do not spell out what this means in detail (see [19, Defn. 7.1.1 and Lem. 7.1.2] for this) but it implies that as a left $\widetilde{A}$-module, A' is isomorphic to $\widetilde{A} \otimes H$ and this is an isomorphism of graded $\widetilde{A}$-modules when H concentrated in degree zero.

Everything in the previous paragraph applies to the left coaction of H on A' obtained by twisting by the antipode, so A' is also isomorphic to $H \otimes \widetilde{A}$ as a right graded $\widetilde{A}$-module. Therefore

$$M \cong A' \otimes_{\widetilde{A}} M^{\mathrm{coH}} \cong (H \otimes \widetilde{A}) \otimes_{\widetilde{A}} M^{\mathrm{coH}} \cong H \otimes M^{\mathrm{coH}}.$$

Hence $\dim_k(M_i) = \dim(H) \times \dim_k\left((M^{\mathrm{coH}})_i\right)$. The conclusion follows from Lemma 3.4. ■

Remark 3.14. For future use, we record the observation in the previous proof that $A' \cong \widetilde{A} \otimes H$ as graded left $\widetilde{A}$-modules and $A' \cong H \otimes \widetilde{A}$ as graded right $\widetilde{A}$-modules. ♦

Corollary 3.15. *Under the hypotheses of Proposition 3.13 every proper quotient of M^{coH} has GK-dimension $< \mathrm{GKdim}\big(M^{\mathrm{coH}}\big)$ if and only if every proper quotient of M in ${}_{A'}\mathcal{M}^{\mathrm{coH}}$ has GK-dimension $< \mathrm{GKdim}(M)$.* ■

4. Homological properties under twisting

We keep the notation and conventions from the previous section, under the assumption that H is finite dimensional, semisimple and co-semisimple.

4.1. We will prove that semisimplicity of H implies that every H-torsor T is also semisimple. The first step towards this is to show that semisimplicity of H implies that every H-torsor T is separable. Recall what this means ([14, Definition 2]):

Definition 4.1. An algebra T is separable if there is an element $\sum r_i \otimes s_i \in T \otimes T$ such that

(1) $\sum r_i s_i = 1$ and
(2) $\sum t r_i \otimes s_i = \sum r_i \otimes s_i t$ for all $t \in T$.

♦

Remark 4.2. The separability condition has many alternate characterizations. For example, it is equivalent to requiring that the multiplication $T \otimes T \to T$ split as a map of T-bimodules. To pass from 4.1 to the latter condition note that

$$T \ni t \mapsto \sum t r_i \otimes s_i \in T \otimes T$$

is left inverse to the multiplication.

Moreover, separability is well known to imply semisimplicity; see e.g. [14, Proposition 2.6]. ♦

Proposition 4.3. *If H is finite-dimensional and semisimple then every torsor T over H is separable.*

Proof. The properties of T ensure that $(\bullet \otimes T)^{\mathrm{coH}}$ is an exact monoidal functor from $\mathcal{M}^H$ to $\mathrm{V{\scriptsize ECT}}$, and T is the image through this functor of the regular right H-comodule H.

A suitable modification of definition 4.1 makes sense in any monoidal category $\mathcal{C}$: If T is an algebra object in $\mathcal{C}$, then the definition simply requires the existence of a map η from the monoidal unit $1_{\mathcal{C}}$ to $T \otimes T$ (analogous to $k \ni 1 \mapsto \sum r_i \otimes s_i \in T \otimes T$) such that

(1) the composition $1_{\mathcal{C}} \xrightarrow{\eta} T \otimes T \xrightarrow{\text{mult}} T$ coincides with the unit of T and

(2) the two parallel arrows

$$\begin{tikzcd} & T\otimes T\otimes T \arrow[dr, "\text{mult.}\otimes\mathrm{id}"] & \\ T \arrow[ur, "\mathrm{id}\otimes\eta"] \arrow[dr, "\eta\otimes\mathrm{id}"'] & & T\otimes T \\ & T\otimes T\otimes T \arrow[ur, "\mathrm{id}\otimes\text{mult.}"'] & \end{tikzcd}$$

are equal.

Moreover, monoidal functors preserve separability in this generalized sense, i.e., a monoidal functor between monoidal categories turns separable algebras in the domain category into separable algebras in the codomain.

The upshot of all of this is that in order to prove the separability of T in VECT it suffices to prove that the H-comodule algebra H (with the regular right comodule structure) is separable in $\mathcal{M}^H$.

By the general version of Maschke's theorem, the Hopf algebra H is semisimple if and only if there is an *integral* $\xi \in H$ such that $\varepsilon(\xi) = 1$. Here, being an integral means that $h\xi = \xi h = \varepsilon(h)\xi$ for all $h \in H$ and $\varepsilon : H \to k$ is the counit of H; this is [24, Thm. 10.3.2].

It is now routine to check that the map $k \to H \otimes H$ defined by $1 \mapsto \xi_1 \otimes S\xi_2$ makes H into a separable algebra in $\mathcal{M}^H$. ■

Since, as mentioned before, semisimplicity is weaker than separability, we finally have

Corollary 4.4. *Under the hypotheses of Proposition 4.3, T is semisimple.* ■

4.2. Let A be a (usually connected) graded k-algebra. For $M, N \in \mathsf{Gr}(A)$ we denote the graded vector space

$$\underline{\mathrm{Hom}}(M, N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(M, N(d)),$$

where $N(d)$ is the degree shift of N by d and Hom here is understood from context to be the space of degree-preserving A-module maps. Just like ordinary Hom, $\underline{\mathrm{Hom}}$ has derived functors $\underline{\mathrm{Ext}}^i$ taking values in graded vector spaces, which will feature heavily below. We denote the degree-j component of $\underline{\mathrm{Ext}}^i(M, N)$ by $\underline{\mathrm{Ext}}^i(M, N)_j$, as usual.

Remark 4.5. If A is noetherian and M is finitely generated then $\underline{\mathrm{Ext}}(M, -)$ and $\mathrm{Ext}(M, -)$ agree (or rather the latter is the vector space obtained by forgetting the grading of the former). This is not the case in general though. ♦

Let A be a connected graded k-algebra in $\mathcal{M}^H$. If we make $A\sharp H^*$ a $\mathbb{Z}$-graded k-algebra by placing H^* in degree 0, then ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ is equivalent to $\mathsf{Gr}(A\sharp H^*)$. Therefore every $M \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$ has a resolution by projective objects in ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$. Let $(P_\bullet, d)$ be such a projective resolution; it is also a projective resolution in $\mathsf{Gr}(A)$ by Lemma 3.6. If $N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$, then the homology of $\underline{\mathrm{Hom}}_A(P_\bullet, N)$ is in $\mathcal{M}^H$. Thus, if $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$, then every $\underline{\mathrm{Ext}}^i_A(M, N)_j$ is in $\mathcal{M}^H$:

Lemma 4.6. *Let A be a connected graded H-comodule algebra and $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. Then the components $\underline{\mathrm{Ext}}^i_A(M, N)_j$ acquire H-comodule structures natural in $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$.* ■

Remark 4.7. Similarly, if $M, N \in {}_A\mathcal{M}^H$, then $\mathrm{Ext}^i_A(M, N) \in \mathcal{M}^H$, naturally in M and N. ♦

The properties of T ensure that $\widetilde{\bullet} = (\bullet \otimes T)^{\mathrm{coH}} : \mathcal{M}^H \to$ VECT is an exact monoidal functor. It lifts to a functor from the category of $\mathbb{Z}$-graded H-comodules to $\mathbb{Z}$-graded vector spaces, and since it turns A into $\widetilde{A}$ it induces a functor $(\bullet \otimes T)^{\mathrm{coH}} : {}_{\mathsf{Gr}(A)}\mathcal{M}^H \to \mathsf{Gr}(\widetilde{A})$. We denote all of these functors along with their ungraded and/or right module analogues by $M \mapsto \widetilde{M}$, relying on context to differentiate between them.

Theorem 4.8. *Let A be a connected graded H-comodule algebra and $M, N \in {}_{\mathsf{Gr}(A)}\mathcal{M}^H$. There is an isomorphism of bigraded vector spaces*

$$\underline{\operatorname{Ext}}^*_A(M,N)_\bullet \;\cong\; \underline{\operatorname{Ext}}^*_{\widetilde{A}}\big(\widetilde{M},\widetilde{N}\big)_\bullet.$$

Proof. Let $(P_\bullet, d)$ be a projective resolution of ${}_AM$ in ${}_{\mathsf{Gr}(A)}\mathcal{M}^H$ (and hence also in $\mathsf{Gr}(A)$ by Lemma 3.6). Then $(P_\bullet \otimes T, d \otimes \mathrm{id}_T)$ is a projective resolution of $M \otimes T$ as an H-equivariant graded A'-module (if we place T in degree zero).

Now, $\underline{\operatorname{Hom}}_{\mathsf{Gr}(A')}(P_\bullet \otimes T, N \otimes T)$ is a complex in $\mathcal{M}^H$, and hence its homology groups are H-comodules. By Proposition 3.2, the functor $(\bullet)^{\mathrm{coH}}$ is an equivalence ${}_{A'}\mathcal{M}^H \to \mathsf{Gr}(\widetilde{A})$ so $(\bullet)^{\mathrm{coH}}$ applied to the homology of the complex $\underline{\operatorname{Hom}}_{\mathsf{Gr}(A')}(P_\bullet \otimes T, N \otimes T)$ produces $\underline{\operatorname{Ext}}^i_{\widetilde{A}}\big(\widetilde{M},\widetilde{N}\big)_j$.

On the other hand, the finite-dimensionality of T implies

$$\underline{\operatorname{Hom}}_{\mathsf{Gr}(A')}(P_\bullet \otimes T, N \otimes T) \cong \underline{\operatorname{Hom}}_{\mathsf{Gr}(A)}(P_\bullet, N \otimes T) \cong \underline{\operatorname{Hom}}_A(P_\bullet, N) \otimes T. \tag{5}$$

These homology groups are the same as those of $\underline{\operatorname{Hom}}_{\mathsf{Gr}(A)}(P_\bullet, N) \otimes T$, namely $\underline{\operatorname{Ext}}^*_A(M,N)_\bullet \otimes T$. The argument used in Proposition 3.3 shows that the dimension of the H co-invariants of $\underline{\operatorname{Ext}}^i_A(M,N)_j \otimes T$ is the same as the dimension of $\underline{\operatorname{Ext}}^i_A(M,N)_j$, and by the paragraph preceding (5) we get the desired vector space isomorphism

$$\underline{\operatorname{Ext}}^*_A(M,N)_\bullet \;\cong\; \underline{\operatorname{Ext}}^*_{\widetilde{A}}\big(\widetilde{M},\widetilde{N}\big)_\bullet.$$

The naturality is easily seen to be built into the construction. ■

Remark 4.9. The result goes through with only the obvious modifications for ungraded modules $M, N \in {}_A\mathcal{M}^H$; see also Remark 4.7. ♦

Corollary 4.10. *Let A be a connected graded H-comodule algebra. If $A \cong TV/(R)$, then $A \cong TV/(\widetilde{R})$ where $\widetilde{R}$ and R are isomorphic as graded vector spaces.*

Proof. This follows by applying Theorem 4.8 to $M = N = k$ from the fact that there are isomorphisms $\operatorname{Ext}^1_A(k,k) \cong V^*$ and $\operatorname{Ext}^2_A(k,k) \cong R^*$ of bigraded vector spaces. ■

4.3. The Koszul property.

Definition 4.11. Let m be an integer ≥ 2. A connected graded algebra A is m-Koszul if $A \cong TV/(R)$ with $\deg(V) = 1$, $R \subseteq V^{\otimes m}$, and $\operatorname{Ext}^i_A(k,k)$ is concentrated in just one degree for all i. ♦

Corollary 4.12. *Let m be an integer ≥ 2. A connected graded H-comodule algebra A is m-Koszul if and only if $\widetilde{A}$ is.*

Proof. This follows immediately from Corollary 4.10 and Theorem 4.8 applied to $M = N = k$. ■

4.4. Artin-Schelter regularity.

We begin by recalling the relevant notions.

Definition 4.13. A connected graded algebra A is Artin-Schelter Gorenstein (AS-Gorenstein for short) of dimension d if the left and right injective dimensions of A as a graded A-module equal d and

$$\underline{\operatorname{Ext}}^i_A(k,A) = \underline{\operatorname{Ext}}^i_{A^\circ}(k,A) \cong \delta_{id}\, k(\ell). \tag{6}$$

for some integer ℓ.

If A is AS-Gorenstein we say it is Artin-Schelter regular (AS-regular for short) of dimension d if in addition $\operatorname{gldim}(A) = d < \infty$. ♦

Artin and Schelter's original definition of regularity included a restriction on the growth of $\dim_k(A_i)$ but in some situations it is sensible to avoid that restriction. We will show that if A is AS-regular of dimension d then so is $\widetilde{A}$. Since $\dim_k(A_i) = \dim_k(\widetilde{A}_i)$ for all i (Proposition 3.5), if A is AS-regular with the growth restriction so is $\widetilde{A}$.

Proposition 4.14. *For all algebras $A \in \mathcal{M}^H$, $\operatorname{gldim}(\widetilde{A}) \le \operatorname{gldim}(A)$.*

Proof. Since T is semisimple, $\operatorname{gldim}(A) = \operatorname{gldim}(A' = A \otimes T)$. It therefore suffices to show that the functor $F = A' \otimes_{\widetilde{A}} \bullet$ from $\mathsf{Mod}(\widetilde{A})$ to $\mathsf{Mod}(A')$ does not increase projective dimension.

Because A' is flat over $\widetilde{A}$ (Proposition 3.2), $\operatorname{Ext}^*_{A'}(F(\bullet), -) \cong \operatorname{Ext}^*_{\widetilde{A}}(\bullet, -)$ for $\bullet$ ranging over $\mathsf{Mod}(\widetilde{A})$ and $-$ ranging over $\mathsf{Mod}(A')$. It now suffices to show that every $M \in \mathsf{Mod}(\widetilde{A})$ is an $\widetilde{A}$-module direct summand of $F(M)$. Because H is co-semisimple, $\widetilde{A} \to A'$ splits as an $\widetilde{A}$-bimodule map. Therefore the inclusion $M \to F(M) = A' \otimes_{\widetilde{A}} M$, $m \mapsto 1 \otimes m$, presents M as an $\widetilde{A}$ summand of $F(M)$. ■

Theorem 4.15. *If a connected graded algebra $A \in \mathcal{M}^H$ is AS-regular of dimension d so is $\widetilde{A}$.*

Proof. By Proposition 4.14, $\operatorname{gldim}(\widetilde{A}) \le d$. Theorem 4.8 and its right handed version applied to $M = k$ and $N = A$ show that (6) holds (or does not hold) simultaneously for A and $\widetilde{A}$. ■

Corollary 4.16. *If A is a twisted Calabi-Yau algebra, so is $\widetilde{A}$.*

Proof. By [25, Lem. 1.2], an algebra is twisted Calabi-Yau if and only if it is AS-regular. ■

4.5. **Condition χ.** In this subsection we prove that the finiteness condition χ introduced in [7] is preserved under twisting. Throughout, A will be an $\mathbb{N}$-graded algebra.

Definition 4.17. [7, Defn. 3.7] We say that A has property χ if for every finitely generated $M \in \mathsf{Gr}(A)$ and all positive integers i, d there is an integer n_0 such that $\underline{\operatorname{Ext}}^i_A(A/A_{\ge n}, M)_{\ge d}$ is finitely generated over A for all $n \ge n_0$. ♦

Remark 4.18. The left A-module structure on $\underline{\operatorname{Ext}}$ comes from the right A-action on $A/A_{\ge n}$. ♦

Although technical, the χ condition is crucial in proving Serre-type results on finiteness of cohomology for non-commutative projective schemes (see e.g. [7, Thm. 7.4]).

Theorem 4.19. *If the $\mathbb{N}$-graded algebra $A \in \mathcal{M}^H$ has property χ then so does $\widetilde{A}$.*

Proof. Let $M \in \mathsf{Gr}(\widetilde{A})$ and fix integers i, d as in 4.17. Because A has property χ, there is some n_0 for which the finiteness condition in 4.17 holds for the graded A-module $M' = A' \otimes_{\widetilde{A}} M$ (the A-module structure is obtained by restricting scalars from $A' = A \otimes T$ to A). We will show that n_0 satisfies the requirements of 4.17 for M.

Once more, apply the graded analogue of Proposition 3.2 to identify $\mathsf{Gr}(\widetilde{A})$ with ${}_{\mathsf{Gr}(A')}\mathcal{M}^H$. Arguing as in the proof of Theorem 4.8 we see that the $\widetilde{A}$-module $\underline{\operatorname{Ext}}^i_{\widetilde{A}}(\widetilde{A}/\widetilde{A}_{\ge n}, M)_{\ge d}$ that we are interested in is precisely the space of H-coinvariants in

$$\underline{\operatorname{Ext}}^i_{A'}(A'/A'_{\ge n}, A' \otimes_{\widetilde{A}} M)_{\ge d} \cong \underline{\operatorname{Ext}}^i_A(A/A_{\ge n}, A' \otimes_{\widetilde{A}} M)_{\ge d}. \tag{7}$$

To conclude, apply Lemma 3.10 substituting A' for A, $\widetilde{A}$ for A^{coH} and (7) for N respectively. ■

5. "Exotic" elliptic algebras

We are now ready to apply the above results to Sklyanin algebras.

We fix a positive integer $n \geq 2$ and assume that k is an algebraically closed field whose characteristic does not divide n. We fix a primitive n^{th} root of unity $\zeta \in k$.

5.1. We will take A to be the Sklyanin algebra $Q = Q_{n^2,1}(E,\tau)$ from [21] and H to be the algebra of functions on the group $(\mathbb{Z}/n)^2$ whose two generators act on the degree-one piece $V = Q_1$ by $x_i \mapsto x_{i+n}$ (indices modulo n) and $x_i \mapsto \zeta^i x_i$ for some primitive n^{th} root of unity ζ.

In this section, T will be the $n \times n$ matrix algebra $M_n(k)$ with the action of $(\mathbb{Z}/n)^2$ on $M_n(k)$ implemented by conjugation by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By duality, the action $(\mathbb{Z}/n)^2$ as automorphisms of $M_n(k)$ gives $M_n(k)$ the structure of an H-comodule algebra. It is easily seen to be a torsor in the sense of §3.1. To see this, note first that $M_n(k)^{\mathrm{coH}}$ is indeed k. In fact, every $(\mathbb{Z}/n)^2$-character has multiplicity one in $M_n(k)$.

The second condition required for being a torsor is that M_n be H-Galois (Remark 3.1). For this observe that H is the group algebra of the Pontryagin dual of $(\mathbb{Z}/n)^2$, which is again isomorphic to $(\mathbb{Z}/n)^2$. In other words, an action by $(\mathbb{Z}/n)^2$ is the same thing as a $(\mathbb{Z}/n)^2$-grading. A result of Ulbrich ([19, Thm. 8.1.7]) characterizes for any group Υ those Υ-graded algebras T that are Galois as comodules over $k\Upsilon$: They are exactly the strongly graded ones, i.e. those for which $T_\gamma T_\delta = T_{\gamma\delta}$ for $\gamma, \delta \in \Upsilon$ (where A_γ is the γ-homogeneous component of the grading). This is clearly the case here, since every homogeneous component is the k-span of an invertible element.

Now set $\widetilde{Q} = (Q \otimes M_n(k))^{\mathrm{coH}}$. All the hypotheses of Propositions 3.3 and 3.7, Corollary 4.12, and Theorem 4.15 are satisfied by [38, Thm. 1.1, Cor. 1.3] so we have the following result.

Proposition 5.1. *The algebra $\widetilde{Q}$ is AS-regular of dimension n^2, Koszul and noetherian and has Hilbert series $(1-t)^{-n^2}$.* ■

6. Generators and relations for $\widetilde{Q_4}$

Let k be an algebraically closed field whose characteristic is not 2.

We now specialize the discussion from Section 5 to $n = 2$, considering the action of the group $\Gamma = \mathbb{Z}/2 \times \mathbb{Z}/2$ on $Q = Q_{n^2} = Q_4$.

6.1. Let $\alpha_1, \alpha_2, \alpha_3 \in k$ be such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$, $\{\pm 1\} \not\subseteq \{\alpha_1, \alpha_2, \alpha_3\}$, and $\{\alpha_1, \alpha_2, \alpha_3\} \cap \{0, \pm 1\} = \varnothing$. Often we write $\alpha = \alpha_1$, $\beta = \alpha_2$, and $\gamma = \alpha_3$.

We fix $a, b, c, i \in k$ such that $a^2 = \alpha$, $b^2 = \beta$, $c^2 = \gamma$, and $i^2 = -1$.

6.2. Let $Q = k[x_0, x_1, x_2, x_3]$ be the quotient of the free algebra $k\langle x_0, x_1, x_2, x_3\rangle$ by the six relations

$$x_0x_i - x_ix_0 = \alpha_i(x_jx_k + x_kx_j), \qquad x_0x_i + x_ix_0 = x_jx_k - x_kx_j, \tag{8}$$

where (i, j, k) runs over the cyclic permutations of $(1, 2, 3)$.

6.3. The co-semisimple Hopf algebra to which the previous results will be applied is the Hopf algebra H of k-valued functions on

$$\Gamma \;=\; \langle g_1, g_2\rangle \;=\; \{1, g_1, g_2, g_3 = g_1g_2\} \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The action of Γ as k-algebra automorphisms of Q is given by

$$\begin{cases} g_1(x_0, x_1, x_2, x_3) \;=\; (x_0, x_1, -x_2, -x_3), \\ g_2(x_0, x_1, x_2, x_3) \;=\; (x_0, -x_1, x_2, -x_3), \\ g_3(x_0, x_1, x_2, x_3) \;=\; (x_0, -x_1, -x_2, x_3). \end{cases} \tag{9}$$

The irreducible characters for Γ are $\chi_0, \chi_1, \chi_2, \chi_3$ where

$$\begin{aligned} \chi_0 &= \text{the trivial representation,} \\ \chi_1(g_1) &= \chi_2(g_2) = 1, \\ \chi_1(g_2) &= \chi_2(g_1) = -1, \text{ and} \\ \chi_3 &= \chi_1\chi_2. \end{aligned}$$

Thus $g(x_j) = \chi_j(g)x_j$ for all $g \in \Gamma$ and $j = 0, 1, 2, 3$.

6.4. **A quaternionic basis for $M_2(k)$ and the conjugation action of Γ on $M_2(k)$.** Define

$$q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad q_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad q_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

Then $q_1^2 = q_2^2 = q_3^2 = -1$ and if (i, j, k) is a cyclic permutation of $(1, 2, 3)$, then $q_iq_j = q_k$ and $q_iq_j + q_jq_i = 0$.

Define an action of Γ as automorphisms of $M_2(k)$ by $g_j(a) := q_jaq_j^{-1}$, i.e., $g(q_j) = \chi_j(g)q_j$.

As before, $\widetilde{Q} = (Q \otimes M_2(k))^\Gamma$. If $g \in \Gamma$, then $g(x_iq_j) = \chi_i(g)\chi_j(g)x_iq_j$ so

$$y_0 := x_0, \quad y_1 := x_1q_1, \quad y_2 := x_2q_2, \quad y_3 := x_3q_3,$$

are Γ-invariant elements of $Q \otimes M_2(k)$.

Proposition 6.1. *The algebra $\widetilde{Q}$ is generated by y_0, y_1, y_2, y_3 modulo the relations*

$$y_0y_i - y_iy_0 \;=\; \alpha_i(y_jy_k - y_ky_j) \qquad \text{and} \qquad y_0y_i + y_iy_0 \;=\; y_jy_k + y_ky_j, \tag{11}$$

were (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The function $y_j \mapsto -y_j$, $j = 0, 1, 2, 3$, extends to an algebra anti-automorphism of $\widetilde{Q}$.

Proof. Because $\widetilde{Q}$ is Koszul with Hilbert series $(1 - t)^{-4}$, it is generated by 4 degree-one elements subject to 6 degree-two relations. Since y_0, y_1, y_2, y_3 are Γ-invariant elements of degree one, they generate $\widetilde{Q}$. It follows from the quadratic relations for Q_4 that $(x_0x_i - x_ix_0)q_i = \alpha_i(x_jx_k + x_kx_j)q_jq_k$ and $(x_0x_i + x_ix_0)q_i = (x_jx_k - x_kx_j)q_jq_k$. Rewriting these relations in terms of y_0, y_1, y_2, y_3 gives the relations in (11). ∎

Since $\widetilde{Q}$ is a regular noetherian algebra of global dimension and GK-dimension 4, it is a domain by [5, Thm.3.9].

Proposition 6.2. *There is an action of Γ as graded k-algebra automorphisms of $\widetilde{Q}$ given by*

$$\begin{aligned} g_1(y_0,y_1,y_2,y_3) &= (y_0,y_1,-y_2,-y_3), \\ g_2(y_0,y_1,y_2,y_3) &= (y_0,-y_1,y_2,-y_3), \\ g_3(y_0,y_1,y_2,y_3) &= (y_0,-y_1,-y_2,y_3). \end{aligned}$$

Using the conjugation action of Γ as automorphisms of $M_2(k)$, this gives an action of Γ as automorphisms of $\widetilde{Q}\otimes M_2(k)$. The invariant subalgebra $(\widetilde{Q}\otimes M_2(k))^\Gamma$ is generated by

$$z_0 := y_0, \quad z_1 := y_1q_1, \quad z_2 := y_2q_2, \quad z_3 := y_3q_3$$

and is isomorphic to Q via $z_j \mapsto x_j$.

Proof. A calculation shows that the action of Γ respects the relations (11). Because $(\widetilde{Q}\otimes M_2(k))^\Gamma$ is Koszul with Hilbert series $(1-t)^{-4}$, it is generated by 4 degree-one elements subject to 6 degree-two relations. The elements z_0,z_1,z_2,z_3 are Γ-invariant so generate $(\widetilde{Q}\otimes M_2(k))^\Gamma$. It follows from the quadratic relations for $\widetilde{Q}$ that $(y_0y_i-y_iy_0)q_i = \alpha_i(y_jy_k-y_ky_j)q_jq_k$ and $(y_0y_i+y_iy_0)q_i = (y_jy_k+y_ky_j)q_jq_k$. Rewriting these relations in terms of z_0,z_1,z_2,z_3 gives the relations $z_0z_i-z_iz_0 = \alpha_i(z_jz_k+z_kz_j)$ and $z_0z_i+z_iz_0 = z_jz_k-z_kz_j$. ■

6.5. **Central elements in $\widetilde{Q}$.** In [29, Thm.2], Sklyanin proved that

$$\Omega := -x_0^2+x_1^2+x_2^2+x_3^2 \qquad \text{and} \qquad \Omega' := x_1^2+\left(\frac{1+\alpha_1}{1-\alpha_2}\right)x_2^2+\left(\frac{1-\alpha_1}{1+\alpha_3}\right)x_3^2 \tag{12}$$

belong to the center of Q when $k=\mathbb{C}$. By the Principle of Permanence of Algebraic Identities, Ω and Ω' are central for all k.

The elements x_0^2,x_1^2,x_2^2,x_3^2 are fixed by the action of Γ. Since $y_j^2=-x_j^2$ for $j=1,2,3$, the elements

$$\Theta := y_0^2+y_1^2+y_2^2+y_3^2 \qquad \text{and} \qquad \Theta' := y_1^2+\left(\frac{1+\alpha_1}{1-\alpha_2}\right)y_2^2+\left(\frac{1-\alpha_1}{1+\alpha_3}\right)y_3^2$$

belong to the center of $\widetilde{Q}$. We note that $\Theta=-\Omega$ and $\Theta'=-\Omega'$.

Proposition 6.3. *The following elements belong to the center of $\widetilde{Q}$:*

$$\begin{aligned} \Theta_0 &:= (1+\gamma)y_1^2+(1+\alpha\gamma)y_2^2+(1-\alpha)y_3^2, \\ \Theta_1 &:= (1+\gamma)y_0^2+(\gamma-\alpha\gamma)y_2^2+(\alpha+\gamma)y_3^2, \\ \Theta_2 &:= (1+\alpha\gamma)y_0^2+(\alpha\gamma-\gamma)y_1^2+(\alpha+\alpha\gamma)y_3^2, \\ \Theta_3 &:= (1-\alpha)y_0^2-(\alpha+\gamma)y_1^2-(\alpha+\alpha\gamma)y_2^2, \end{aligned}$$

and have the property that

$$\begin{aligned} \alpha(\Theta_0+\Theta_1) &= \Theta_2-\Theta_3 \\ \beta(\Theta_0+\Theta_2) &= \Theta_3-\Theta_1, \\ \gamma(\Theta_0+\Theta_3) &= \Theta_1-\Theta_2. \end{aligned}$$

Proof. The elements Θ_i come from those at [17, p.39]. Simple calculations give

$$\begin{aligned}
\Theta_0+\Theta_1 &= (1+\gamma)(y_0^2+y_1^2+y_2^2+y_3^2),\\
\Theta_0+\Theta_2 &= (1+\alpha\gamma)(y_0^2+y_1^2+y_2^2+y_3^2),\\
\Theta_0+\Theta_3 &= (1-\alpha)(y_0^2+y_1^2+y_2^2+y_3^2),\\
\Theta_1-\Theta_2 &= \gamma(1-\alpha)(y_0^2+y_1^2+y_2^2+y_3^2),\\
\Theta_2-\Theta_3 &= \alpha(1+\gamma)(y_0^2+y_1^2+y_2^2+y_3^2),\\
\Theta_3-\Theta_1 &= -(\alpha+\gamma)(y_0^2+y_1^2+y_2^2+y_3^2),\\
&= \beta(1+\alpha\gamma)(y_0^2+y_1^2+y_2^2+y_3^2).
\end{aligned}$$

The result follows immediately. ■

7. Γ acts on E as translation by 2-torsion points

7.1. The action of Γ on Q_1 induces an action of Γ as automorphisms of Q_1^*. If we use x_0,x_1,x_2,x_3 as an ordered set of coordinate functions on Q_1^* then the action of Γ on a point $(w,x,y,z)\in Q_1^*$ is given by the formulas

$$\begin{cases} g_1(w,x,y,z)=(w,x,-y,-z)\\ g_2(w,x,y,z)=(w,-x,y,-z)\\ g_3(w,x,y,z)=(w,-x,-y,z). \end{cases} \tag{13}$$

We will write $\mathbb{P}^3$ for $\mathbb{P}(Q_1^*)$, the projective space of lines in Q_1^*. The action of Γ on Q_1^* induces an action of Γ as automorphisms of $\mathbb{P}^3$ given by the formulas in (13).

The relations for Q, which are elements of $Q_1\otimes Q_1$, are bi-homogeneous forms on $\mathbb{P}^3\times\mathbb{P}^3$. We write $R=\ker(Q_1\otimes Q_1\xrightarrow{\text{mult}}Q_2)$ and define the subscheme

$$V \;:=\; \{(\mathbf{u},\mathbf{v})\mid r(\mathbf{u},\mathbf{v})=0 \text{ for all } r\in R\}\;\subseteq\;\mathbb{P}^3\times\mathbb{P}^3.$$

Let $\mathrm{pr}_i:\mathbb{P}^3\times\mathbb{P}^3\to\mathbb{P}^3$, $i=1,2$, be the projections of V onto the left and right copies of $\mathbb{P}^3$.

Proposition 7.1. [32, Props. 2.4, 2.5] *With the above notation,*

$$\mathrm{pr}_1(V)\;=\;\mathrm{pr}_2(V)\;=\;E\cup\big\{(1,0,0,0),\,(1,0,0,0),\,(1,0,0,0),\,(1,0,0,0)\big\}$$

where E is the intersection of the quadrics

$$\begin{aligned}
x_0^2+x_1^2+x_2^2+x_3^2 &= 0,\\
(1-\gamma)x_1^2+(1+\alpha\gamma)x_2^2+(1+\gamma)x_3^2 &= 0.
\end{aligned}$$

Furthermore, E is an elliptic curve.

The reader will notice that we use the same notation for elements in Q as for elements in the symmetric algebra $S(Q_1)$. Thus, in Proposition 7.1, $x_0^2+x_1^2+x_2^2+x_3^2$ is an element in $S(Q_1)$, i.e., a degree-two form on $\mathbb{P}^3$, whereas in (12), $-x_0^2+x_1^2+x_2^2+x_3^2$ denotes an element in Q.

It is clear that Γ fixes the points in $\{(1,0,0,0),\,(1,0,0,0),\,(1,0,0,0),\,(1,0,0,0)\}$. It is also clear that E is stable under the action of Γ (indeed, that must be so because R is Γ-stable). The map $\Gamma\to\mathrm{Aut}(E)$ is injective so we will identity Γ with a subgroup of E. Once we have fixed a group law $\oplus$ on E we will identify E with the subgroup of $\mathrm{Aut}(E)$ consisting of the translation automorphisms, i.e., $E\to\mathrm{Aut}(E)$ sends a point $\mathbf{v}\in E$ to the automorphism $\mathbf{u}\mapsto\mathbf{u}\oplus\mathbf{v}$.

Once we have defined the group $(E,\oplus)$ we will write o for its identity element and

$$E[2] \;:=\; \{\mathbf{v}\in E \mid \mathbf{v}\oplus\mathbf{v}=o\}.$$

The next main result, Theorem 7.6, shows we can define $\oplus$ such that $\Gamma = E[2]$ as subgroups of $\operatorname{Aut}(E)$. We will then identify Γ with $E[2]$. In anticipation of that result we define an involution $\ominus : E \to E$ and a distinguished point $o \in E$ by

$$\ominus(w,x,y,z) \;:=\; (-w,x,y,z)$$

and

$$o \;:=\; \big(0,\sqrt{\nu-1},\,\sqrt{1-\mu},\,\sqrt{\mu-\nu}\,\big)$$

where

$$\mu := \frac{1-\gamma}{1+\alpha} \qquad \text{and} \qquad \nu := \frac{1+\gamma}{1-\beta}$$

and $\sqrt{\nu-1}$, $\sqrt{1-\mu}$, and $\sqrt{\mu-\nu}$ are some fixed square roots.[1] The restrictions on the values of α, β, γ, imply that $|\{1,\mu,\nu\}|=3$. We use this fact in the proof of Lemma 7.5.

Lemma 7.2.

$$\begin{aligned} E\cap\{x_0=0\} \;&=\; \{p\in E \mid p=\ominus p\} \\ &=\; \Big\{\big(0,\pm\sqrt{\nu-1},\pm\sqrt{1-\mu},\pm\sqrt{\mu-\nu}\,\big)\Big\}. \end{aligned}$$

Proof. It follows from the definition of $\ominus$ that $E\cap\{x_0=0\}=\{p\in E \mid p=\ominus p\}$. Computing $E\cap\{x_0=0\}$ reduces to computing the intersection of the plane conics $x_1^2+x_2^2+x_3^2=0$ and $\mu x_1^2+\nu x_2^2+x_3^2=0$. The conics meet at four points, namely

$$\big(\pm\sqrt{\nu-1},\,\pm\sqrt{1-\mu},\,\pm\sqrt{\mu-\nu}\big)\in\mathbb{P}^2.$$

The result follows. ■

Lemma 7.3. *There is a degree-two morphism $\pi : E \to \mathbb{P}^1$ such that $\pi(p)=\pi(\ominus p)$ for all $p\in E$, i.e., the fibers of π are the sets $\{p,\ominus p\}$, $p\in E$. In particular, the ramification locus of π is $\{p\in E \mid p=\ominus p\}=\{o,\omega_1,\omega_2,\omega_3\}$ where*

$$\begin{aligned} o \;&:=\; \big(0,\sqrt{\nu-1},\,\sqrt{1-\mu},\,\sqrt{\mu-\nu}\,\big) \\ \omega_1 \;&:=\; g_1(o) \;=\; \big(0,\,\sqrt{\nu-1},\,-\sqrt{1-\mu},\,\sqrt{\mu-\nu}\,\big) \\ \omega_2 \;&:=\; g_2(o) \;=\; \big(0,\,-\sqrt{\nu-1},\,\sqrt{1-\mu},\,\sqrt{\mu-\nu}\,\big) \\ \omega_3 \;&:=\; g_3(o) \;=\; \big(0,\,-\sqrt{\nu-1},\,-\sqrt{1-\mu},\,\sqrt{\mu-\nu}\,\big). \end{aligned}$$

Proof. The conic C, given by $\mu x_1^2+\nu x_2^2+x_3^2=0$, is smooth so isomorphic to $\mathbb{P}^1$. Define $\pi : E\to C$ by $\pi(w,x,y,z)=(x,y,z)$. The result is now obvious. ■

Proposition 7.4. *Let $E'\subseteq\mathbb{P}^2$ be the curve $y^2z=x(x-z)(x-\lambda z)$ where*

$$\lambda \;:=\; \frac{\nu-\mu\nu}{\nu-\mu} \;=\; \frac{1}{\gamma}\left(\frac{1+\gamma}{1+\alpha}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right),$$

and consider the group $(E',\oplus)$ in which $(0,1,0)$ is the identity and three points of E' sum to zero if and only if they are collinear.

[1] The choice of square root doesn't matter—as one takes the different square roots one obtains 4 different candidates for o. But, as we will see, with the choice of $\oplus$ we eventually make, those 4 points are the points in $E[2]$. The situation is analogous to that of a smooth plane cubic: there are nine inflection points and if one chooses the group law so that one of those points is the identity, then the inflection points are the points in $E[3]$, the 3-torsion subgroup.

(1) *There is an isomorphism of varieties $g : E \to E'$ such that*
$$g(o) = \infty = (0,1,0), \quad g(\omega_1) = (0,0,1), \quad g(\omega_2) = (1,0,1), \quad g(\omega_3) = (\lambda,0,1).$$

(2) *If $(E, \oplus)$ is the unique group law such that $g : (E,\oplus) \to (E',\oplus)$ is an isomorphism of groups, then $E[2] = \{p \mid p = \ominus p\} = \{o, \omega_1, \omega_2, \omega_3\}$, and*

(3) *$p \oplus (\ominus p) = o$ for all $p \in E$, and*

(4) *4 points on E are coplanar if and only if their sum is zero.*

Proof. (1) Let $\pi : E \to C = \{\mu x_1^2 + \nu x_2^2 + x_3^2 = 0\}$ be the morphism $\pi(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$ in Lemma 7.3 and $f : C \to \mathbb{P}^1$ the isomorphism
$$f(x_1, x_2, x_3) = (\sqrt{-\nu}x_2 + \sqrt{\mu}x_1,\ x_3) = (x_3,\ \sqrt{-\nu}x_2 - \sqrt{\mu}x_1)$$
with inverse
$$f^{-1}(s,t) = \Big(\tfrac{1}{\sqrt{\mu}}(s^2 - t^2), \tfrac{1}{\sqrt{-\nu}}(s^2+t^2), 2st\Big).$$
Let $h = f \circ \pi : E \to \mathbb{P}^1$. The ramification locus of π, and hence of h, is obviously $\{p \in E \mid p = \ominus p\}$. Let E' be the plane cubic $y^2 z = x(x-z)(x-\lambda z)$ and $h' : E' \to \mathbb{P}^1$ the morphism $h'(x,y,z) = (x,z)$.

Consider the following diagram:

$$\begin{tikzcd} & E' \arrow[r, bend left, "h'"] & \mathbb{P}^1 \\ E \arrow[ur, dashed, "g"] \arrow[r, bend right, "\pi"'] & C \arrow[ur, bend right, "f"'] & \end{tikzcd} \tag{14}$$

The following result is implicit in [13, Ch.4, §4]: If E and E' are elliptic curves and $h : E \to \mathbb{P}^1$ and $h' : E' \to \mathbb{P}^1$ are degree 2 morphisms having the same branch points, then there is an isomorphism of varieties $g : E \to E'$ such that $h'g = h$.

The four branch points for h are
$$\Big(\pm\sqrt{\mu\nu - \nu} \pm \sqrt{\mu\nu - \mu},\ \sqrt{\mu - \nu}\Big) = \Big(\sqrt{\mu-\nu},\ \pm\sqrt{\mu\nu - \nu} \mp \sqrt{\mu\nu-\mu}\Big).$$
The cross-ratios of these four points are $\big\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\big\}$ where
$$\lambda \ :=\ \frac{\nu - \mu\nu}{\nu - \mu} \ =\ \frac{1}{\gamma}\left(\frac{1+\gamma}{1+\alpha}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right).$$

The four branch points for $h' : E' \to \mathbb{P}^1$ have the same cross-ratios so $E \cong E'$. In particular, there is an isomorphism of varieties $g : E \to E'$ such that
$$g(o) = \infty = (0,1,0), \quad g(\omega_1) = (0,0,1), \quad g(\omega_2) = (1,0,1), \quad g(\omega_3) = (\lambda,0,1).$$

(2) Let $\oplus$ be the unique group law on E such that $g(p \oplus p') = g(p) \oplus g(p')$ for all $p, p' \in E$. Then g is an isomorphism of algebraic groups. Since $E'[2] = \{0,1,0),(0,0,1),(1,0,1),(\lambda,0,1)\}$, $E[2] = \{o, \omega_1, \omega_2, \omega_3\} = \{p \in E \mid p = \ominus p\}$.

(3) Since $g : E \to E'$ is a group isomorphism it suffices to show that $g(p) \oplus g(\ominus p) = o$. The fibers of h consist of points that sum to zero so it suffices to show that $h(g(p)) = h(g(\ominus p))$. However, $hg = f\pi$ and $\pi(p) = \pi(\ominus p)$ so $hg(p) = hg(\ominus p)$.

(4) Let $\Phi : \mathrm{Div}(E) \to E$ be the map $\Phi\big((q_1) + \ldots + (q_m) - (r_1) \ldots - (r_n)\big) := q_1 \oplus \ldots \oplus q_m \ominus r_1 \ominus r_n$. It is easy to show that if D and D' are divisors of the same degree, then $D \sim D'$ if and only if $\Phi(D) = \Phi(D')$. The points $\{o, \omega_1, \omega_2, \omega_3\}$ are coplanar. Four points $q_0, \ldots, q_3 \in E$ are coplanar

if and only if $(o)+(\omega_1)+(\omega_2)+(\omega_3)\sim(q_0)+(q_1)+(q_2)+(q_3)$. Since $o\oplus\omega_1\oplus\omega_2\oplus\omega_3=o$, $q_0,\ldots,q_3\in E$ are coplanar if and only if $q_0\oplus q_1\oplus q_2\oplus q_3=o$. ■

Lemma 7.5. *There are exactly four singular quadrics that contain E, namely*

$$\begin{aligned} Q_0 &= \{\mu x_1^2+\nu x_2^2+x_3^2=0\},\\ Q_1 &= \{\mu x_0^2+(\mu-\nu)x_2^2+(\mu-1)x_3^2=0\},\\ Q_2 &= \{\nu x_0^2+(\nu-\mu)x_1^2+(\nu-1)x_3^2=0\},\\ Q_3 &= \{x_0^2+(1-\mu)x_1^2+(1-\nu)x_2^2=0\}.\end{aligned}$$

Let $p\in E$. For each i, the line through $\ominus p$ and $g_i(p)$ lies on Q_i.

Proof. Since the equation defining each Q_i is a linear combination of the equations in Proposition 7.1, Q_i contains E. Each Q_i has a unique singular point, namely e_i where

$$e_0:=(1,0,0,0),\quad e_1:=(0,1,0,0),\quad e_2:=(0,0,1,0),\quad e_3:=(0,0,0,1).$$

Thus Q_i is a union of lines and every line on Q_i passes through e_i.

Let f_1,f_2 be quadratic forms such that $E=\{f_1=f_2=0\}$. A quadric contains E if and only if it is the zero locus of $\lambda_1f_1+\lambda_2f_2$ for some $(\lambda_1,\lambda_2)\in\mathbb{P}^1$; conversely, for all $(\lambda_1,\lambda_2)\in\mathbb{P}^1$ the zero locus of $\lambda_1f_1+\lambda_2f_2$ is a quadric that contains E. Since $|\{1,\mu,\nu\}|=3$, there are exactly 4 singular quadrics in the pencil of quadrics that contain E; these are the quadrics Q_i (see [17, Prop. 3.4]).

Let $p=(w,x,y,z)\in E$. Let L be line through $\ominus p$ and e_0. Thus $L=\{(t-sw,sx,sy,sz)\mid(s,t)\in\mathbb{P}^1\}$. The line L lies on Q_0 and meets E when

$$(t-sw)^2+(sx)^2+(sy)^2+(sz)^2 \;=\; \mu(sx)^2+\nu(sy)^2+(sz)^2 \;=\; 0.$$

The second expression is zero for all s. The first expression is zero if and only if $t^2-2stw=0$; one solution to this is $t=0$ and it corresponds to the point $\ominus p\in L\cap E$. The other solution occurs when $t-2sw=0$ and corresponds to the point $(w,x,y,z)=p$. Another way of saying this is that if $w\neq0$, then the line through p and $\ominus p$ lies on Q_0.

The line through $\ominus p$ and e_1 is $\{(-sw,sx+t,sy,sz)\mid(s,t)\in\mathbb{P}^1\}$. It lies on Q_1 and meets E when

$$(-sw)^2+(sx+t)^2+(sy)^2+(sz)^2 \;=\; \mu(-sw)^2+\nu(sy)^2+(sz)^2 \;=\; 0.$$

The second expression is zero for all s and the first is zero if and only if $t^2+2stx=0$. The solution $t=0$ to this equation corresponds to the point $\ominus p\in L\cap E$. The other solution occurs when $t+2sx=0$ and gives the point $(-w,-x,y,z)=g_1(p)$. Another way of saying this is that if $x\neq0$, then the line through $(-w,x,y,z)$ and $(w,x,-y,-z)$ lies on Q_1; i.e., the line through $\ominus p$ and $g_1(p)$ lies on Q_1.

Similar calculations show that the line through $\ominus p$ and $g_i(p)$ lies on Q_i for $i=2,3$. ■

The statement of Lemma 7.5 doesn't quite make sense if $\ominus p=g_i(p)$. It should be changed to say that the line through e_i and $\ominus(p)$ meets E again at $g_i(p)$, i.e., the line is tangent to E.

Theorem 7.6. *There is a group law $\oplus$ on E such that each element in Γ acts as translation by a point in $E[2]$.*

Proof. Let g_i be the automorphism in (9) and let ω_i be point in Lemma 7.3. We will show that g_i is translation by ω_i, i.e., $\omega_i=g_i(o)$.

Let p and q be arbitrary points of E. The line through $\ominus p$ and $g_i(p)$ lies on Q_i. So does the line through $\ominus q$ and $g_i(q)$. Because these lines are on Q_i they meet at e_i. The lines therefore span a plane, i.e., $\ominus p$, $g_i(p)$, $\ominus q$, and $g_i(q)$, are coplanar. Therefore $(\ominus p)\oplus g_i(p)\oplus(\ominus q)\oplus g_i(q)=o$. Taking $q=o$ and rearranging the equation gives $p=g_i(p)\oplus g_i(o)$ or, $g_i(p)=p\ominus g_i(o)=p\oplus g_i(o)$. ■

7.2. **Twisting a Q-module by g_i.** Let $\gamma\in\Gamma$ and M a graded left Q-module. We define γ^*M to be the graded Q-module which is equal to M as a graded vector space and has the new Q-action

$$r\bullet_\gamma m \;:=\; \gamma(r)m$$

for $r\in Q$ and $m\in\gamma^*M=M$.

Proposition 7.7. *Let $p,q\in E$ and let M_p and $M_{p,q}$ be the associated point and line modules. Then $g_i^*M_p\cong M_{p+\omega_i}$ and $g_i^*M_{p,q}\cong M_{p+\omega_i,q+\omega_i}$.*

Proof. Let $r\in Q_1$ and $p\in\mathbb{P}^3=\mathbb{P}(Q_1^*)$. The action of g_i on Q_1 and Q_1^* is such that $g_i(r)(p)=r(g_i^{-1}(p))=r(g_i(p))$. Thus, $r(p)=0$ if and only if $g_i(r)$ vanishes at $g_i(p)$. Since $M_p=Q/Qp^\perp$ where $p^\perp$ is the subspace of Q_1 vanishing at p, $g_i^*M_p=Q/Q(p+\omega_i)^\perp$. A similar argument works for line modules. ■

8. Properties of $\widetilde{B}$

By [32, §3.9], $Q/(\Omega,\Omega')$ is isomorphic to the twisted homogeneous coordinate ring $B(E,\tau,\mathcal{L})$. Since Ω and Ω' are fixed by Γ, there is an induced action of Γ on $Q/(\Omega,\Omega')$. The quotient $\widetilde{Q}/(\Omega,\Omega')$ is isomorphic to $\widetilde{B}:=(B(E,\tau,\mathcal{L})\otimes M_2(k))^\Gamma$.

8.1. **The category $\mathsf{QGr}(\widetilde{B})$.** Let $B=B(E,\tau,\mathcal{L})$, $B'=B\otimes M_2(k)$, $\widetilde{B}=(B')^\Gamma$, and $\mathcal{B}=M_2(\mathcal{O}_E)$. The main result in this subsection is

Theorem 8.1. *There is an equivalence of categories $\mathsf{QGr}(\widetilde{B})\equiv\mathsf{Qcoh}(E/E[2])$.*

Corollary 8.2. *The set of isomorphism classes of simple $\mathsf{QGr}(\widetilde{B})$-objects is in natural bijection with $E/E[2]$.* ■

The plan is to work our way through the chain of equivalences

$$\mathsf{QGr}(\widetilde{B})\equiv\mathsf{QGr}(B')^\Gamma\equiv\mathsf{Qcoh}(\mathcal{B})^\Gamma\equiv\mathsf{Qcoh}(\mathcal{B}^\Gamma)\equiv\mathsf{Qcoh}(E/E[2]). \tag{15}$$

The notation needs some unpacking.

First, Γ acts on the categories $\mathsf{QGr}(B')$ and $\mathsf{Qcoh}(\mathcal{B})$. Such an action comprises an auto-equivalence γ^* of the respective category for each $\gamma\in\Gamma$ together with natural isomorphisms $t_{\delta,\gamma}:\delta^*\circ\gamma^*\cong(\gamma\delta)^*$ for $\gamma,\delta\in\Gamma$ such that

$$\begin{tikzcd} \varepsilon^*\circ\delta^*\circ\gamma^* \arrow[r] \arrow[d] & (\delta\varepsilon)^*\circ\gamma^* \arrow[d] \\ \varepsilon^*\circ(\gamma\delta)^* \arrow[r] & (\gamma\delta\varepsilon)^* \end{tikzcd} \tag{16}$$

commutes for all $\gamma,\delta,\varepsilon\in\Gamma$.

The action of Γ as automorphisms of B' induces an action of Γ on $\mathsf{Gr}(B')$ as described in §7.2. Since the subcategory $\mathsf{Fdim}(B')$ is stable under each γ^*, the Γ-action passes to the quotient category $\mathsf{QGr}(B')$. The action on $\mathsf{Qcoh}(\mathcal{B})$ comes from translation on E by $E[2]$ together with twisting via the Γ-action on the $M_2(k)$ tensorand in $\mathcal{B}=\mathcal{O}_E\otimes M_2(k)$.

If Γ acts on a category $\mathcal{C}$ we can then form the category of Γ-equivariant objects $\mathcal{C}^\Gamma$. The objects of $\mathcal{C}^\Gamma$ are objects $c \in \mathcal{C}$ equipped with isomorphisms $\gamma : c \to \gamma^* c$ for $\gamma \in \Gamma$ such that

$$\begin{tikzcd} & \delta^* c \arrow[r, "\delta^*(\gamma)"] & \delta^*(\gamma^* c) \arrow[d, "t_{\delta,\gamma}"] \\ c \arrow[ur, bend left, "\delta"] \arrow[rr, bend right, "\gamma\delta"'] & & (\gamma\delta)^* c \end{tikzcd} \tag{17}$$

commutes and the morphisms are those in $\mathcal{C}$ that preserve all the structure. This elucidates the notation $\mathcal{C}^\Gamma$ in (15) for $\mathcal{C} = \mathsf{QGr}(B')$ or $\mathsf{Qcoh}(\mathcal{B})$.

Finally, $\mathcal{B}^\Gamma$ denotes the sheaf of algebras on $E/E[2]$ obtained by descent from $\mathcal{B}$. To make sense of this, let $\rho : E \to E/E[2]$ be the étale quotient morphism. Now recall

Proposition 8.3. [20, Prop. 2, p.70] *The functors*

$$\mathcal{G} \rightsquigarrow \rho^* \mathcal{G} \qquad \textit{and} \qquad \mathcal{F} \rightsquigarrow (\rho_* \mathcal{F})^\Gamma$$

are mutually inverse equivalences between $\mathsf{Qcoh}(E/E[2])$ *and* $\mathsf{Qcoh}(E)^\Gamma$. ■

The equivalences in Proposition 8.3 are monoidal, because ρ^* is, so they identify Γ-equivariant sheaves of algebras on E with sheaves of algebras on $E/E[2]$. Keeping this in mind, $\mathcal{B}^\Gamma$ is simply shorthand for the sheaf of algebras on $E/E[2]$ corresponding to $\mathcal{B} \in \mathsf{Qcoh}(E)^\Gamma$, i.e. $(\rho_* \mathcal{B})^\Gamma$.

Proof of Theorem 8.1. We go through the equivalences in (15) one by one, moving rightward.

First equivalence. The graded version of Proposition 3.2 (applied to B' coacted upon by the function algebra of Γ) provides the equivalence $\mathsf{Gr}(\widetilde{B})$ and $\mathsf{Gr}(B)^\Gamma$. The equivalence restricts to the subcategories $\mathsf{Fdim}(\widetilde{B})$ and $\mathsf{Fdim}(B')^\Gamma$ so descends to the quotient categories QGr.

Second equivalence. By [6, Thm. 3.12], $\mathsf{QGr}(B) \equiv \mathsf{Qcoh}(\mathcal{O}_E)$. Since $\mathcal{B} = \mathcal{O}_E \otimes M_2$, Morita equivalence lifts this to

$$\mathsf{QGr}(B') \equiv \mathsf{Qcoh}(\mathcal{B}). \tag{18}$$

Now note that Γ acts on the geometric data $(E, \tau, \mathcal{L})$ that gives rise to $B = B(E, \tau, \mathcal{L})$ in the sense that it acts on E, commutes with τ, and there is an Γ-equivariant structure on $\mathcal{L}$. Moreover, it acts in the same way on the M_2 tensorands in $B' = B \otimes M_2$ and $\mathcal{B} = \mathcal{O}_E \otimes M_2$. This observation together with the precise description of the equivalence $\mathsf{QGr}(B) \equiv \mathsf{Qcoh}(E)$ from [6, Thm. 3.12] shows that (18) intertwines the Γ-actions on the two categories. This implies the desired result that it lifts to an equivalence between the respective categories of Γ-equivariant objects.

Third equivalence. This is also follows from Proposition 8.3. As observed before that equivalence is monoidal, and it identifies $\mathcal{B} \in \mathsf{Qcoh}(E)^\Gamma$ with $\mathcal{B}^\Gamma \in \mathsf{Qcoh}(E/E[2])$. The monoidality then ensures that it implements an equivalence between the categories of modules over $\mathcal{B}$ and $\mathcal{B}^\Gamma$ internal to $\mathsf{Qcoh}(E)^\Gamma$ and $\mathsf{Qcoh}(E/E[2])$ respectively.

Fourth equivalence. Because $\rho : E \to E/E[2]$ is étale and $\rho^*(\mathcal{B}^\Gamma) \cong M_2(\mathcal{O}_E)$, $\mathcal{B}^\Gamma$ is a sheaf of Azumaya algebras on $E/E[2]$. The fourth equivalence now follows from Morita equivalence and the fact that $\mathcal{B}^\Gamma$ is Azumaya and hence (because we are working over an algebraically closed field) of the form $\mathcal{E}nd(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on $E/E[2]$. ■

We can actually find an explicit vector bundle $\mathcal{V}$ on $E/E[2]$ such that $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{V})$.

Proposition 8.4. *Let $\mathcal{F}$ be the unique non-split extension* $0 \longrightarrow \mathcal{O}_{E/E[2]} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{E/E[2]} \longrightarrow 0$. *There is an isomorphism of $\mathcal{O}_{E/E[2]}$-algebras $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{F})$.*

Proof. We already know that $\mathcal{B}^\Gamma$ is trivial Azumaya, hence $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{V})$ for some rank 2 vector bundle $\mathcal{V}$. By Atiyah's classification of vector bundles on elliptic curves, either $\mathcal{V}$ is decomposable, or isomorphic to $\mathcal{F} \otimes \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic}(E/E[2])$. If $\mathcal{V}$ is decomposable, the $\mathcal{O}_{E/E[2]}$-module $\mathcal{B}^\Gamma$ contains two copies of $\mathcal{O}_{E/E[2]}$ as direct summands, whence $\dim H^0(\mathcal{B}^\Gamma) \geqslant 2$. Since $\dim H^0(\mathcal{B}^\Gamma) = \dim H^0(\mathcal{B})^\Gamma = 1$, we must have $\mathcal{B}^\Gamma \cong \mathcal{E}nd(\mathcal{F} \otimes \mathcal{L}) \cong \mathcal{E}nd(\mathcal{F})$. ■

8.2. **Fat point modules for $\widetilde{Q}$.** Let $p \in E$. Let $p^\perp \subset Q_1$ be the subspace of Q_1 vanishing at p. We call $M_p := Q/Qp^\perp$ the point module associated to p. We view k^2 as a left $M_2(k)$-module in the natural way. Let $p \in E$. Then $M_p \otimes k^2$ is a left $Q \otimes M_2(k)$-module, and hence a left $\widetilde{Q}$-module.

Since (Ω, Ω') annihilates M_p, $M_p \otimes k^2$ is a $\widetilde{B}$-module.

Lemma 8.5. *If $p \in E$, then at most one of $\{x_0, x_1, x_2, x_3\}$ vanishes at p.*

Proof. Suppose $x_r(p) = x_s(p) = 0$ and $r \neq s$. Let $t \in \{0,1,2,3\} - \{r,s\}$. There are non-zero scalars λ, μ, ν such that $\lambda x_r^2 + \mu x_s^2 + \nu x_t^2$ vanishes on E so $x_t(p) = 0$ also. But $x_0^2 + x_1^2 + x_2^2 + x_3^2$ vanishes on E so it would follow that $x_j(p) = 0$ for all j. That is absurd. ■

Proposition 8.6. *Let $p \in E$. If $m \otimes v$ is a non-zero element in $(M_p \otimes k^2)_n$, then $\widetilde{Q}(m \otimes v) \supseteq (M_p \otimes k^2)_{\geq n+1}$. In particular, every quotient of $M_p \otimes k^2$ by a non-zero graded $\widetilde{Q}$-submodule has finite dimension; i.e., $M_p \otimes k^2$ is 1-critical.*

Proof. Let N be a non-zero graded $\widetilde{Q}$-submodule of $M_p \otimes k^2$. Let $e_n \otimes v$ be a non-zero element in N where $\{e_n\}$ is a basis for the degree-n component of M_p and $v \in k^2 - \{0\}$.

Every non-zero element in $(kq_0 + kq_2) \cup (kq_0 + kq_3) \cup (kq_1 + kq_2) \cup (kq_1 + kq_3)$ has rank 2 so

$$(kq_0 + kq_2)v \;=\; (kq_0 + kq_3)v \;=\; (kq_1 + kq_2)v \;=\; (kq_1 + kq_3)v \;=\; k^2.$$

If $p + n\tau = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ with respect to the coordinates $x_0, \ldots, x_3$, then there is a basis $\{e_{n+1}\}$ for the degree-$(n+1)$ component of M_p such that $x_j e_n = \lambda_j e_{n+1}$ for $j = 0, \ldots, 3$.

By Lemma 8.5, at least one element in $\{x_0, x_1\}$ and at least one element in $\{x_2, x_3\}$ does not vanish at $p + n\tau$. Suppose, for the sake of argument, that $x_1(p + n\tau) \neq 0$ and $x_2(p + n\tau) \neq 0$. Then $x_1 e_n$ and $x_2 e_n$ are non-zero. It follows that $(kx_1 \otimes q_1 + kx_2 \otimes q_2) \cdot (e_n \otimes v) = e_{n+1} \otimes k^2$. Thus, $\widetilde{Q}_1(e_n \otimes v) = e_{n+1} \otimes k^2$. The same sort of argument can be used in the other cases (for example, if $x_0(p + n\tau)$ and $x_2(p + n\tau)$ are non-zero) to show that $\widetilde{Q}_1(e_n \otimes v)$ is always equal to $e_{n+1} \otimes k^2$.

It now follows by induction on n that $\widetilde{Q}(e_n \otimes v) \supseteq (M_p)_{\geq n+1} \otimes k^2$. The result follows. ■

Corollary 8.7. *If $p \in E$, then the image of $M_p \otimes k^2$ in $\mathsf{QGr}(\widetilde{Q})$ is a simple object. By Corollary 8.2, the $\pi^*(M_p \otimes k^2)$ exhaust the simple objects in $\mathsf{QGr}(\widetilde{B})$ of GK-dimension one.*

The previous result is the reason that $M_p \otimes k^2$ is called a *fat point module* for $\widetilde{Q}$: "point" because in algebraic geometry simple objects in $\mathsf{Qcoh}(X)$ correspond to closed points, "fat" because $\operatorname{Hom}_{\mathsf{QGr}(\widetilde{Q})}(\widetilde{Q}, \pi^*(M_p \otimes k^2)) = 2$, not 1.

Proposition 8.8. *If $\omega \in E[2]$ and $p \in E$, then there is an isomorphism of $\widetilde{Q}$-modules*

$$M_p \otimes k^2 \;\cong\; M_{p+\omega} \otimes k^2.$$

Proof. Write $E[2]=\{o,\omega_1,\omega_2,\omega_3\}$. If $\omega=o$ the identity map is an isomorphism. Fix $i\in\{1,2,3\}$.

Let $\{e_n \mid n\ge 0\}$ be a homogeneous basis for M_p with $\deg(e_n)=n$. For each n, let $\xi_{nj}\in k$, $j=0,1,2,3$, be the unique scalars such that

$$x_je_n=\xi_{nj}e_{n+1}.$$

Thus, $(\xi_{n0},\xi_{n1},\xi_{n2},\xi_{n3})=p+n\tau$. Let $\xi'_{n0}=\xi_{n0}$, $\xi'_{ni}=\xi_{ni}$, and $\xi'_{nj}=-\xi_{nj}$ when $j\in\{1,2,3\}-\{i\}$. Therefore $p+n\tau+\omega_i=(\xi'_{n0},\xi'_{n1},\xi'_{n2},\xi'_{n3})$. Let $\{f_n \mid n\ge 0\}$ be the unique homogeneous basis for $M_{p+\omega_i}$ with $\deg(f_n)=n$ such that $x_jf_n=\xi'_{nj}f_{n+1}$ for $j=0,1,2,3$.

Define $\varphi_i: M_p\otimes k^2 \longrightarrow M_{p+\omega_i}\otimes k^2$ by $\varphi_i(e_n\otimes v):=f_n\otimes q_iv$.

It follows that

$$\varphi_i\big(y_j\cdot(e_n\otimes v)\big)=\varphi_i(x_je_n\otimes q_jv)=\varphi_i(\xi_je_{n+1}\otimes q_jv)=\xi_jf_{n+1}\otimes q_iq_jv$$

and

$$y_j\cdot\varphi_i(e_n\otimes v)=y_j\cdot(f_n\otimes q_iv)=x_jf_{n+1}\otimes q_jq_iv)=\xi'_jf_{n+1}\otimes q_jq_iv.$$

For all j, $\xi_jf_{n+1}\otimes q_iq_jv=\xi'_jf_{n+1}\otimes q_jq_iv$ because

- if $j\in\{0,i\}$, then $\xi_j=\xi'_j$ and $q_iq_j=q_jq_i$;
- if $j\in\{1,2,3\}-\{i\}$, then $\xi_j=-\xi'_j$ and $q_iq_j=-q_jq_i$.

Therefore $\varphi_i\big(y_j\cdot(e_n\otimes v)\big)=y_j\cdot\varphi_i(e_n\otimes v)$ for $j=0,1,2,3$. This proves that φ_i is a homomorphism of graded $\widetilde{Q}$-modules. It is obviously bijective so the proof is complete. ■

Proposition 8.9. *$\widetilde{B}$ is not a domain. In particular, in $\widetilde{B}$*

$$\begin{aligned}0=y_0^2+y_1^2+y_2^2+y_3^2 &= (y_0-y_1-y_2-y_3)^2\\ &= (y_0-y_1+y_2+y_3)^2\\ &= (y_0+y_1-y_2+y_3)^2\\ &= (y_0+y_1+y_2-y_3)^2.\end{aligned}$$

Proof. This is a straightforward calculation: $(y_0-y_1-y_2-y_3)^2$ equals

$$y_0^2+y_1^2+y_2^2+y_3^2-\sum_{i=1}^{3}(y_0y_i+y_iy_0-y_jy_k-y_ky_j)$$

where (i,j,k) is a cyclic permutation of $1,2,3$. But $y_0y_i+y_iy_0=y_jy_k+y_ky_j$ when (i,j,k) is a cyclic permutation of $1,2,3$ and $y_0^2+y_1^2+y_2^2+y_3^2=-\Omega$ which is zero in $\widetilde{B}$. Similar calculations show that the squares of the other 3 elements are zero in $\widetilde{B}$. Or, one can use the fact that Γ acts as automorphisms of $\widetilde{B}$ and these four elements in $\widetilde{B}_1$ form an Γ-orbit. ■

9. Point modules for $\widetilde{Q}$

A point module for over a connected graded algebra A is a graded left A-module M such that $M=AM_0$ and $\dim_k(M_i)=1$ for all $i\ge 1$. The importance of point modules is that they are simple objects in $\mathsf{QGr}(A)$.

9.1. Suppose M is a point module for $\widetilde{Q}$. Its degree-zero component, M_0, is annihilated by a 3-dimensional subspace of $\widetilde{Q}_1$. That 3-dimensional subspace determines and is determined by a point in $\mathbb{P}^3$, its vanishing locus. We will show that the only points in $\mathbb{P}^3$ that arise in this way are those in Table 1 where the coordinates are written with respect to the coordinate system (y_0, y_1, y_2, y_3). We write $\mathfrak{P}$ for this set of points.

Recall that a, b, c, i are fixed square roots of $\alpha, \beta, \gamma, -1$.

$\mathfrak{P}_\infty$	$\mathfrak{P}_0$	$\mathfrak{P}_1$	$\mathfrak{P}_2$	$\mathfrak{P}_3$	Γ
$(1,0,0,0)$	$(1,1,1,1)$	$(bc,-i,-ib,-c)$	$(ac,-a,-i,-ic)$	$(ab,-ia,-b,-i)$	
$(0,1,0,0)$	$(1,1,-1,-1)$	$(bc,-i,ib,c)$	$(ac,-a,i,ic)$	$(ab,-ia,b,i)$	g_1
$(0,0,1,0)$	$(1,-1,1,-1)$	$(bc,i,-ib,c)$	$(ac,a,-i,ic)$	$(ab,ia,-b,i)$	g_2
$(0,0,0,1)$	$(1,-1,-1,1)$	$(bc,i,ib,-c)$	$(ac,a,i,-ic)$	$(ab,ia,b,-i)$	g_3

TABLE 1. The points in $\mathfrak{P}$.

The points in $\mathfrak{P}_\infty$ are fixed by Γ and every other $\mathfrak{P}_i$ is a Γ-orbit. If $\mathbf{u}$ is the topmost point in one of the columns $\mathfrak{P}_i$, $i = 0, 1, 2, 3$, the other points in that column are $g_1(\mathbf{u})$, $g_2(\mathbf{u})$, and $g_3(\mathbf{u})$, in that order.

We define a permutation θ of $\mathfrak{P}$ with the property $\theta^2 = \mathrm{id}_{\mathfrak{P}}$ by

$$\theta(\mathbf{u}) := \begin{cases} \mathbf{u} & \text{if } \mathbf{u} \in \mathfrak{P}_\infty \cup \mathfrak{P}_0 \\ g_i(\mathbf{u}) & \text{if } \mathbf{u} \in \mathfrak{P}_i, \ i = 1, 2, 3. \end{cases} \tag{19}$$

9.2. **Point modules and Γ.** Let V denote the linear span of y_0, y_1, y_2, y_3. The defining relations for $\widetilde{Q}$ belong to $V^{\otimes 2}$. Non-zero elements in $V^{\otimes 2}$ are forms of bi-degree $(1,1)$ on $\mathbb{P}(V^*) \times \mathbb{P}(V^*) = \mathbb{P}^3 \times \mathbb{P}^3$. Let

$$\Gamma := \text{ the subscheme of } \mathbb{P}^3 \times \mathbb{P}^3 \text{ where the quadratic relations for } \widetilde{Q} \text{ vanish.}$$

We will show that Γ is a reduced scheme consisting of 20 points.

Lemma 9.1. *If $(\mathbf{u}, \mathbf{v}) \in \Gamma$, then $(\mathbf{v}, \mathbf{u}) \in \Gamma$.*

Proof. As remarked in Proposition 6.1, there is an anti-automorphism of $\widetilde{Q}$ given by $y_i \mapsto -y_i$ for $i = 0, 1, 2, 3$. Thus, if $r = \sum \mu_{ij} y_i \otimes y_j$ is a quadratic relation for $\widetilde{Q}$ so is $r' = \sum \mu_{ij} y_j \otimes y_i$. Obviously, r vanishes at $(\mathbf{u}, \mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3$ if and only if r' vanishes at $(\mathbf{v}, \mathbf{u})$. The lemma now follows from the fact that Γ is the zero locus of the set of quadratic relations for $\widetilde{Q}$. ■

Suppose M is a point module for $\widetilde{Q}$. Let $e_0, e_1, \ldots$ be a basis for M with $\deg(e_n) = n$. Define $\lambda_{nj} \in k$ by the requirement that $y_j e_n = \lambda_{nj} e_{n+1}$. Because M is a point module, for each n, some λ_{nj} is non-zero. The point $p_n := (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3}) \in \mathbb{P}^3$ does not depend on the basis $\{e_n\}_{n \geq 0}$. Since $y_j(p_n) = \lambda_{nj}$, the p_n's belong to $\mathbb{P}(V^*)$.

Because M is a $\widetilde{Q}$-module, each quadratic relation $r \in V^{\otimes 2}$ has the property that $r \cdot e_n = 0$ for all n. Thus, r viewed as a (1,1) form on $\mathbb{P}^3 \times \mathbb{P}^3$ vanishes at (p_{n+1}, p_n). Hence $(p_{n+1}, p_n) \in \Gamma$.

Proposition 9.2. *Let $\mathbf{u} \in \mathfrak{P}$. Let θ be the function defined at (19) and for each $n \geq 0$ write $\theta^n(\mathbf{u}) = (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3})$ where the coordinates are written with respect to (y_0, y_1, y_2, y_3). There is a point module, $M_{\mathbf{u}}$, with homogeneous basis $e_0, e_1, \ldots$, $\deg(e_n) = n$, and action*

$$y_j e_n := \lambda_{nj} e_{n+1}. \tag{20}$$

These 20 point modules are pair-wise non-isomorphic.

Proof. It is clear that $M_{\mathbf{u}}$ is generated by e_0 so it suffices to show that (20) really does define a left $\widetilde{Q}$-module. To do this we must show that every relation for $\widetilde{Q}$ annihilates every e_n. In other words, we must show that every quadratic relation for $\widetilde{Q}$, when viewed as a form of bi-degree $(1,1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$, vanishes at $\big((\lambda_{n+1,0}, \lambda_{n+1,1}, \lambda_{n+1,2}, \lambda_{n+1,3}), (\lambda_{n0}, \lambda_{n1}, \lambda_{n2}, \lambda_{n3})\big) \in \mathbb{P}^3 \times \mathbb{P}^3$ for all $n \geq 0$.; i.e., it suffices to show that these forms vanish at $(\theta(\mathbf{v}), \mathbf{v})$ for all $\mathbf{v} \in \mathfrak{P}$. Since $\theta^2 = 1$, this is equivalent to showing they vanish at $(\mathbf{v}, \theta(\mathbf{v}))$ for all $\mathbf{v} \in \mathfrak{P}$.

The relations for $\widetilde{Q}$ are the entries in the matrix $\mathsf{M}_1\mathsf{x}$ where

$$\mathsf{M}_1 = \begin{pmatrix} -y_1 & y_0 & \alpha y_3 & -\alpha y_2 \\ -y_2 & -\beta y_3 & y_0 & \beta y_1 \\ -y_3 & \gamma y_2 & -\gamma y_1 & y_0 \\ y_1 & y_0 & -y_3 & -y_2 \\ y_2 & -y_3 & y_0 & -y_1 \\ y_3 & -y_2 & -y_1 & y_0 \end{pmatrix} \quad \text{and} \quad \mathsf{x} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We must therefore show that $\mathsf{M}_1(\mathbf{v})\theta(\mathbf{v})^{\mathsf{T}} = 0$ for all $\mathbf{v} \in \mathfrak{P}$. This is a routine calculation. We give one example to illustrate the process.

Let $\mathbf{v} = (\nu_0, \nu_1, \nu_2, \nu_3) \in \mathfrak{P}_1$. Then $\theta(\mathbf{v}) = g_1(\mathbf{v}) = (\nu_0, \nu_1, -\nu_2, -\nu_3)$ so

$$\mathsf{M}_1(\mathbf{v})\theta(\mathbf{v})^{\mathsf{T}} = \begin{pmatrix} -\nu_1 & \nu_0 & \alpha\nu_3 & -\alpha\nu_2 \\ -\nu_2 & -\beta\nu_3 & \nu_0 & \beta\nu_1 \\ -\nu_3 & \gamma\nu_2 & -\gamma\nu_1 & \nu_0 \\ \nu_1 & \nu_0 & -\nu_3 & -\nu_2 \\ \nu_2 & -\nu_3 & \nu_0 & -\nu_1 \\ \nu_3 & -\nu_2 & -\nu_1 & \nu_0 \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \\ -\nu_2 \\ -\nu_3 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ -\nu_0\nu_2 - \beta\nu_3\nu_1 \\ -\nu_0\nu_3 + \gamma\nu_1\nu_2 \\ \nu_0\nu_1 + \nu_2\nu_3 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to check that this 6×1 matrix is 0 for all $\mathbf{v} \in \mathfrak{P}_1$.

The annihilator of e_0 in $\widetilde{Q}_1$ is the subspace that vanishes at $\mathbf{u}$. Hence if $\mathbf{u}$ and $\mathbf{v}$ are different points of $\mathfrak{P}$, $M_{\mathbf{u}} \not\cong M_{\mathbf{v}}$. ■

Theorem 9.3. *The point modules for $\widetilde{Q}$ are exactly the 20 point modules $M_{\mathbf{u}}$, $\mathbf{u} \in \mathfrak{P}$, constructed in Proposition 9.2.*

Proof. Let M be a point module for $\widetilde{Q}$. Let $\{e_n \mid n \geq 0\}$ be a homogeneous basis for M with $\deg(e_n) = n$. Let p_n, $n \geq 0$, be the points in $\mathbb{P}^3$ determined by the procedure described in §9.2. Then $(p_{n+1}, p_n) \in \Gamma$ for all $n \geq 0$. By Lemma 9.1, $(p_n, p_{n+1}) \in \Gamma$. Thus, to prove the Theorem it suffices to show that $\Gamma = \big\{\big(\mathbf{u}, \theta(\mathbf{u})\big) \bigm| \mathbf{u} \in \mathfrak{P}\big\}$. This is what we do in Theorem 9.4 below. ■

Theorem 9.4. *Let $\Gamma \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ be the subscheme defined in §9.2. Then*

$$\Gamma \;=\; \{(\mathbf{u}, \mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid \mathsf{M}_1(\mathbf{u})\mathbf{v} = 0\} \;=\; \big\{\big(\mathbf{u}, \theta(\mathbf{u})\big) \bigm| \mathbf{u} \in \mathfrak{P}\big\}.$$

In particular, Γ is the graph of the automorphism θ of $\mathfrak{P}$.

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2 : \Gamma \to \mathbb{P}^3$ denote the projections onto the first and second factors of $\mathbb{P}^3 \times \mathbb{P}^3$. We will show that $\mathrm{pr}_1(\Gamma) = \mathfrak{P}$. Let $\mathbf{u} \in \mathrm{pr}_1(\Gamma)$. There is a point $\mathbf{v} \in \mathbb{P}^3$ such that $(\mathbf{u}, \mathbf{v}) \in \Gamma$, i.e., such that $\mathsf{M}_1(\mathbf{u})\mathbf{v} = 0$. This implies that $\mathrm{rank}(\mathsf{M}_1(\mathbf{u})) \leq 3$. Thus the 4×4 minors of M_1 vanish at $\mathbf{u}$.

SAGE [35] computed the 4×4 minors of M_1. After removing a common factor of 2, they are

$$
\begin{aligned}
&-b\gamma y_0y_1^3 - \alpha\gamma y_0y_1y_2^2 + \beta\gamma y_1^2y_2y_3 + a\gamma y_2^3y_3 - \alpha\beta y_0y_1y_3^2 + \alpha\beta y_2y_3^3 - y_0^3y_1 + y_0^2y_2y_3\\
&\qquad = (y_2y_3 - y_0y_1)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&-\beta\gamma y_0y_1^2y_2 - \alpha\gamma y_0y_2^3 + \beta\gamma y_1^3y_3 + \alpha\gamma y_1y_2^2y_3 - \alpha\beta y_0y_2y_3^2 + \alpha\beta y_1y_3^3 - y_0^3y_2 + y_0^2y_1y_3\\
&\qquad = (y_1y_3 - y_0y_2)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&\beta\gamma y_1^3y_2 + \alpha\gamma y_1y_2^3 - \beta\gamma y_0y_1^2y_3 - \alpha\gamma y_0y_2^2y_3 + \alpha\beta y_1y_2y_3^2 - \alpha\beta y_0y_3^3 + y_0^2y_1y_2 - y_0^3y_3\\
&\qquad = (y_1y_2 - y_0y_3)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2),\\
&-\alpha\beta y_1^2y_3^2 + \alpha\beta y_2^2y_3^2 - \beta y_0^2y_1^2 - \alpha y_0^2y_2^2 + \beta y_1^2y_3^2 + \alpha y_2^2y_3^2 - y_0^2y_1^2 + y_0^2y_2^2,\\
&-\alpha\beta y_1^2y_2y_3 + \alpha\beta y_2y_3^3 + \beta y_0y_1^3 - \alpha y_0^2y_2y_3 + \alpha y_2^3y_3 - \beta y_0y_1y_3^2 + y_0^3y_1 - y_0y_1y_2^2\\
&\qquad = (y_0y_1 - \alpha y_2y_3)(y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2),\\
&-\alpha\beta y_1y_2^2y_3 + \alpha\beta y_1y_3^3 - \alpha y_0y_2^3 + \beta y_0^2y_1y_3 - \beta y_1^3y_3 + \alpha y_0y_2y_3^2 + y_0^3y_2 - y_0y_1^2y_2\\
&\qquad = (y_0y_2 + \beta y_1y_3)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2),\\
&\alpha\gamma y_1^2y_2y_3 - \alpha\gamma y_2^3y_3 + \gamma y_0y_1^3 - \gamma y_0y_1y_2^2 - \alpha y_0^2y_2y_3 + \alpha y_2y_3^3 - y_0^3y_1 + y_0y_1y_3^2,\\
&\qquad = (y_0y_1 + \alpha y_2y_3)(-y_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2),\\
&\alpha\gamma y_1^2y_2^2 - \alpha\gamma y_2^2y_3^2 - \gamma y_0^2y_1^2 + \gamma y_1^2y_2^2 - \alpha y_0^2y_3^2 + \alpha y_2^2y_3^2 + y_0^2y_1^2 - y_0^2y_3^2,\\
&\alpha\gamma y_1y_2^3 - \alpha\gamma y_1y_2y_3^2 - \gamma y_0^2y_1y_2 + \gamma y_1^3y_2 - \alpha y_0y_2^2y_3 + \alpha y_0y_3^3 + y_0^3y_3 - y_0y_1^2y_3\\
&\qquad = (y_0y_3 - \gamma y_1y_2)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2),\\
&\alpha y_0y_1y_2^2 + \alpha y_2^3y_3 - \alpha y_0y_1y_3^2 - \alpha y_2y_3^3 - y_0^3y_1 + y_0y_1^3 - y_0^2y_2y_3 + y_1^2y_2y_3\\
&\qquad = (y_0y_1 + y_2y_3)(-y_0^2 + y_1^2 + \alpha y_2^2 - \alpha y_3^2),\\
&-\beta\gamma y_1^3y_3 + \beta\gamma y_1y_2^2y_3 + \gamma y_0y_1^2y_2 - \gamma y_0y_2^3 + \beta y_0^2y_1y_3 - \beta y_1y_3^3 - y_0^3y_2 + y_0y_2y_3^2\\
&\qquad = (y_0y_2 - \beta y_1y_3)(-y_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2),\\
&-\beta\gamma y_1^3y_2 + \beta\gamma y_1y_2y_3^2 - \gamma y_0^2y_1y_2 + \gamma y_1y_2^3 - \beta y_0y_1^2y_3 + \beta y_0y_3^3 - y_0^3y_3 + y_0y_2^2y_3\\
&\qquad = (y_0y_3 + \gamma y_1y_2)(-y_0^2 - \beta y_1^2 + y_2^2 + \beta y_3^2),\\
&-\beta\gamma y_1^2y_2^2 + \beta\gamma y_1^2y_3^2 - \gamma y_0^2y_2^2 + \gamma y_1^2y_2^2 - \beta y_0^2y_3^2 + \beta y_1^2y_3^2 - y_0^2y_2^2 + y_0^2y_3^2,\\
&-\beta y_0y_1^2y_{-}\beta y_1^3y_3 + \beta y_0y_2y_3^2 + \beta y_1y_3^3 - y_0^3y_2 + y_0y_2^3 - y_0^2y_1y_3 + y_1y_2^2y_3\\
&\qquad = (y_0y_2 + y_1y_3)(-y_0^2 - \beta y_1^2 + y_2^2 + \beta y_3^2),\\
&\gamma y_1^3y_2 - \gamma y_1y_2^3 + \gamma y_0y_1^2y_3 - \gamma y_0y_2^2y_3 - y_0^2y_1y_2 - y_0^3y_3 + y_1y_2y_3^2 + y_0y_3^3\\
&\qquad = (y_0y_3 + y_1y_2)(-x_0^2 + \gamma y_1^2 - \gamma y_2^2 + y_3^2).
\end{aligned}
$$

Some reorganization and changes of sign show that the linear span of the above 15 polynomials is the same as the linear span of the following 15 polynomials:

$$
\begin{gathered}
(y_2y_3 - y_0y_1)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2)\\
(y_1y_3 - y_0y_2)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2)\\
(y_1y_2 - y_0y_3)(y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2)\\
(y_0y_1 + y_2y_3)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2)\\
(y_0y_2 + \beta y_1y_3)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2)
\end{gathered}
$$

$$
\begin{aligned}
&(y_0y_3 - \gamma y_1y_2)(y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2)\\
&(y_0y_1 - \alpha y_2y_3)(y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2)\\
&(y_0y_2 + y_1y_3)(y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2)\\
&(y_0y_3 + \gamma y_1y_2)(y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2)\\
&(y_0y_1 + \alpha y_2y_3)(y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2)\\
&(y_0y_2 - \beta y_1y_3)(y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2)\\
&(y_0y_3 + y_1y_2)(y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2)\\
&\alpha\beta y_1^2y_3^2 - \alpha\beta y_2^2y_3^2 + \beta y_0^2y_1^2 - \beta y_1^2y_3^2 + \alpha y_0^2y_2^2 - \alpha y_2^2y_3^2 + y_0^2y_1^2 - y_0^2y_2^2,\\
&\beta\gamma y_1^2y_2^2 - \beta\gamma y_1^2y_3^2 + \gamma y_0^2y_2^2 - \gamma y_1^2y_2^2 + \beta y_0^2y_3^2 - \beta y_1^2y_3^2 + y_0^2y_2^2 - y_0^2y_3^2,\\
&\alpha\gamma y_1^2y_2^2 - \alpha\gamma y_2^2y_3^2 + \alpha y_2^2y_3^2 - \gamma y_0^2y_1^2 + \gamma y_1^2y_2^2 - \alpha y_0^2y_3^2 + y_0^2y_1^2 - y_0^2y_3^2.
\end{aligned}
$$

The proof of Proposition 9.2 showed that $\mathsf{M}_1(\mathbf{u})\theta(\mathbf{u})^\mathsf{T} = 0$ for all $\mathbf{u} \in \mathfrak{P}$ so these 15 polynomials vanish at the points in $\mathfrak{P}$. One can also check this directly by evaluating these quartic polynomials at $\mathbf{u} \in \mathfrak{P}$. For example, it is obvious that y_iy_j vanishes on $\mathfrak{P}_\infty$ if $i \neq j$ from which it immediately follows that all 15 polynomials vanish on $\mathfrak{P}_\infty$. As another example, $y_2y_3 - y_0y_1$, $y_1y_3 - y_0y_2$, and $y_1y_2 - y_0y_3$, vanish on $\mathfrak{P}_0$, whence the first 3 of the 15 polynomials vanish on $\mathfrak{P}_0$; the other twelve polynomials belong to the ideal $(y_0^2 - y_1^2, y_0^2 - y_2^2, y_0 - y_3^2)$ so they too vanish on $\mathfrak{P}_0$. As a final example, consider $\mathfrak{P}_2$. The first three quartics vanish on $\mathfrak{P}_2$ because $y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2$ does. The second three quartics vanish on $\mathfrak{P}_2$ because $y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2$ does. The third three quartics vanish on $\mathfrak{P}_2$ because the ideal $(y_0y_1 - \alpha y_2y_3, y_0y_2 + y_1y_3, y_0y_3 - +\gamma y_1y_2)$ does. The fourth three quartics vanish on $\mathfrak{P}_2$ because $y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2$ does. A calculation shows the last three quartics vanish on $\mathfrak{P}_2$.

It remains to show that if these 15 quartics vanish at $\mathbf{u}$, then $\mathbf{u} \in \mathfrak{P}$.

This determinant

$$
\det \begin{pmatrix} 1 & \beta\gamma & \alpha\gamma & \alpha\beta \\ 1 & -1 & -\alpha & \alpha \\ 1 & \beta & -1 & -\beta \\ 1 & -\gamma & \gamma & -1 \end{pmatrix} \;=\; -(1 + \alpha\beta + \beta\gamma + \gamma\alpha)^2
$$

is non-zero because the hypothesis that $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ implies $1 + \alpha\beta + \beta\gamma + \gamma\alpha = (1 + \alpha)(1 + \beta)(1 + \gamma)$ which is non-zero because we are assuming that $\{\alpha, \beta, \gamma\} \cap \{0, \pm 1\} = \varnothing$. Because the determinant is non-zero the polynomials

$$
\left\{ \begin{array}{l} y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2, \\ y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2, \\ y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2, \\ y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2, \end{array} \right. \tag{21}
$$

are linearly independent. Their linear span is therefore the same as that of $\{y_0^2, y_1^2, y_2^2, y_3^2\}$. Hence the common zero locus of the polynomials in (21) is empty and at most three of them vanish at $\mathbf{u}$.

Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2 + \beta\gamma y_1^2 + \alpha\gamma y_2^2 + \alpha\beta y_3^2$. Then

$$
y_0y_1 - y_2y_3 = y_0y_2 - y_1y_3 = y_0y_3 - y_1y_2 = 0 \tag{22}
$$

at $\mathbf{u}$. If one of the coordinate functions y_0, y_1, y_2, y_3 vanishes at $\mathbf{u}$, then three of do so

$$
\mathbf{u} \;\in\; \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\} \;=\; \mathfrak{P}_\infty. \tag{23}
$$

If none of y_0, y_1, y_2, y_3 vanishes at $\mathbf{u}$, then it follows from (22) that

$$\mathbf{u} \;\in\; \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\} \;=\; \mathfrak{P}_0.$$

Suppose $\mathbf{u}$ is not in the zero locus of $y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2$. Then

$$y_0y_1 + y_2y_3 = y_0y_2 + \beta y_1y_3 = y_0y_3 - \gamma y_1y_2 = 0 \tag{24}$$

at $\mathbf{u}$. If one of y_0, y_1, y_2, y_3 vanishes at $\mathbf{u}$, then three of them do so (23) applies. Suppose none of y_0, y_1, y_2, y_3 vanishes at $\mathbf{u}$. There is no loss of generality in assuming that the y_0-coordinate of $\mathbf{u}$ is bc. We make that assumption; i.e., $\mathbf{u} = (bc, y_1, y_2, y_3)$. It follows from (24) that $y_0^3(y_1y_2y_3) = \beta\gamma(y_1y_2y_3)^2$. Therefore $bc = y_1y_2y_3$. It also follows from (24) that $\beta\gamma y_1^2 = \gamma y_2^2 = -\beta y_3^2$. Some case-by-case analysis shows that

$$\mathbf{u} \;\in\; \{(bc,-i,ib,c),(bc,-i,-ib,-c),(bc,i,ib,-c),(bc,i,-ib,c)\} \;\subseteq\; \mathfrak{P}.$$

Similarly, if $\mathbf{u}$ is not in the zero locus of $y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2$, then either (23) holds or

$$\mathbf{u} \;\in\; \{(ac,a,-i,ic),(ac,a,i,-ic),(ac,-a,-i,-ic),(ac,-a,i,ic) \;\subseteq\; \mathfrak{P}.$$

Finally, if $\mathbf{u}$ is not in the zero locus of $y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2$, then either (23) holds or

$$\mathbf{u} \;\in\; \{(ab,ia,b,-i),(ab,ia,-b,i),(ab,-ia,b,i),(ab,-ia,-b,-i)\} \;\subseteq\; \mathfrak{P}.$$

This completes the proof that $\mathrm{pr}_1(\Gamma) \subset \mathfrak{P}$. Thus $\mathrm{pr}_2(\Gamma) = \mathfrak{P}$.

By Lemma 9.1, $\mathrm{pr}_2(\Gamma) = \mathfrak{P}$ also. Since $\mathrm{pr}_2(\Gamma)$ does not contain a line, the rank of $\mathsf{M}_1(\mathbf{u})$ is 3 for all $\mathbf{u} \in \mathrm{pr}_1(\Gamma)$. Let $\mathbf{u} \in \mathfrak{P}$. Since $\mathsf{M}_1(\mathbf{u})\theta(\mathbf{u})^{\mathsf{T}} = 0$, $\theta(\mathbf{u})^{\mathsf{T}}$ is the only $\mathbf{v} \in \mathbb{P}^3$ such that $\mathsf{M}_1(\mathbf{u})\mathbf{v}^{\mathsf{T}} = 0$. Hence $(\mathbf{u}, \theta(\mathbf{u}))$ is the only point in $\mathrm{pr}_1^{-1}(\mathbf{u})$. It follows that $\Gamma = \{(\mathbf{u}, \theta(\mathbf{u})) \mid \mathbf{u} \in \mathfrak{P}\}$. ■

Proposition 9.5. *The central element* $\Theta = y_0^2 + y_1^2 + y_2^2 + y_3^2$ *does not annihilate any point modules for* $\widetilde{Q}$. *Consequently,* $\widetilde{B}$ *has no point modules.*

Proof. Let $\mathbf{u} \in \mathfrak{P}$. To describe the action of Θ on $M_{\mathbf{u}}$ we must fix a basis for $M_{\mathbf{u}}$.

We pick a basis for $M_{\mathbf{u}}$ that is compatible with the entries in Table 1. To do this it is helpful, for a moment, to think of the entries in Table 1 as points in k^4. Suppose $\mathbf{u} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is one of the entries in Table 1. Let e_0 be any non-zero element in $(M_{\mathbf{u}})_0$. There is a unique element $e_1 \in (M_{\mathbf{u}})_0$ such that $y_ie_0 = \lambda_ie_1$ for $i = 0,1,2,3$. Likewise, if $(\lambda_0', \lambda_1', \lambda_2', \lambda_3')$ is the entry in the table for $\theta(\mathbf{u})$ there is a unique element $e_2 \in (M_{\mathbf{u}})_2$ such that $y_ie_1 = \lambda_i'e_2$ for $i = 0,1,2,3$.

If $\mathbf{u} \in \mathfrak{P}_\infty$, then $\Theta e_0 = e_2$. If $\mathbf{u} \in \mathfrak{P}_0$, then $\Theta e_0 = 4e_2$.

Let $\mathbf{u} = (bc, -i, -ib, -c) \in \mathfrak{P}_1$. Then $\theta(\mathbf{u}) = (bc, -i, ib, c)$. Therefore

$$\begin{aligned}
\Theta e_0 &= (y_0^2 + y_1^2 + y_2^2 + y_3^2)e_0 \\
&= (bcy_0 - iy_1 - iby_2 - cy_3)e_1 \\
&= \big((bc)^2 - 1 + b^2 - c^2\big)e_2 \\
&= (\beta - 1)(\gamma + 1)e_2
\end{aligned}$$

Likewise, if $\mathbf{u} = (bc, i, -ib, c) \in \mathfrak{P}_1$, then $\theta(\mathbf{u}) = (bc, i, ib, -c)$ and a similar calculation shows that $\Theta e_0 = (\beta - 1)(\gamma + 1)e_2$. Thus, $\Theta e_0 = (\beta - 1)(\gamma + 1)e_2$ for all $\mathbf{u} \in \mathfrak{P}_1$.

Similar calculations show that $\Theta e_0 = (\alpha + 1)(\gamma - 1)e_2$ for all $\mathbf{u} \in \mathfrak{P}_2$. Finally, if $\mathbf{u} \in \mathfrak{P}_3$, then $\Theta e_0 = (\alpha - 1)(\beta + 1)e_2$. ■

The scalar multiples of e_2 that appear in the proof of Proposition 9.5 are,

$$1, \quad 4, \quad (\beta-1)(\gamma+1), \quad (\gamma-1)(\alpha+1), \quad (\alpha-1)(\beta+1)$$

for $\mathfrak{P}_\infty$, $\mathfrak{P}_0$, $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$.

9.3. Not only do the relations for $\widetilde{Q}$ determine Γ, but Γ determines the defining relations for $\widetilde{Q}$: the quadratic relations for $\widetilde{Q}$ are precisely the elements of $V^{\otimes 2}$ that vanish at $\mathfrak{P}$. This is a consequence of the following remarkable result.

Theorem 9.6 (Shelton-Vancliff)**.** [28] *Let V be a 4-dimensional vector space and $R \subseteq V^{\otimes 2}$ a 6-dimensional subspace. Let TV denote the tensor algebra on V and let $\Gamma \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ be the scheme-theoretic zero locus of R. If $\dim(\Gamma) = 0$, then*

$$R = \{f \in V^{\otimes 2} \mid f|_\Gamma = 0\}.$$

9.4. There has been some interest in Artin-Schelter regular algebras with Hilbert series $(1-t)^{-4}$ that have only finitely many point modules [39], [27], [36], [37]. The interest arises because this phenomenon does not occur for Artin-Schelter regular algebras with Hilbert series $(1-t)^{-3}$; the point modules for the latter algebras are parametrized either by a cubic divisor in $\mathbb{P}^2$ or by $\mathbb{P}^2$. In 1988, M. Van den Bergh circulated a short note showing that a generic 4-dimensional Artin-Schelter regular algebra with Hilbert series $(1-t)^{-4}$ has exactly 20 point modules [10]. Van den Bergh's example is a generic Clifford algebra. In particular, it is a finite module over its center.

Davies [9, §5.1] shows, when the translation automorphism has infinite order, that $\widetilde{Q}$ is not isomorphic to any of the previously found examples of 4-dimensional regular algebras having 20 point modules.

Proposition 9.7. *The point modules $M_{\mathbf{u}}$ for $\mathbf{u} \in \mathfrak{P}_\infty \cup \mathfrak{P}_0$ are quotient rings of $\widetilde{Q}$. If $\mathbf{u} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathfrak{P}_\infty \cup \mathfrak{P}_0$, then*

$$M_{\mathbf{u}} \;\cong\; \frac{\widetilde{Q}}{(\lambda_j y_i - \lambda_i y_j \mid 0 \le i, j \le 3)} \;\cong\; k[t].$$

9.5. **The action of the center on $M_{\mathbf{u}}$.**

Lemma 9.8. *If $\mathbf{u} \in \mathfrak{P}_2$, then $(\gamma\Omega_0 - \Omega_1)M_{\mathbf{u}} = 0$.*

Proof. Let $\mathbf{u} \in \mathfrak{P}_2$. The points in $\mathfrak{P}_2$ are the points in the Γ-orbit of $(ac, -a, -i, -ic)$. If e_0 is a basis for $(M_{\mathbf{u}})_0$, then there is a basis e_2 for $(M_{\mathbf{u}})_2$ such that

$$y_0^2 e_0 = \alpha\gamma e_2, \quad y_1^2 e_0 = \alpha e_2, \quad y_2^2 e_0 = e_2, \quad y_3^2 e_0 = \gamma e_2.$$

Therefore $\Omega_0 e_0 = \big((1+\gamma)\alpha + (1+\alpha\gamma) + (1-\alpha)\gamma\big)e_2$ and $\Omega_1 e_0 = \big((1+\gamma)\alpha\gamma + (\gamma - \alpha\gamma) + (\alpha+\gamma)\gamma\big)e_2$. Therefore $(\gamma\Omega_0 - \Omega_1)e_0$. Since $\gamma\Omega_0 - \Omega_1$ is central and e_0 generates $M_{\mathbf{u}}$, $(\gamma\Omega_0 - \Omega_1)M_{\mathbf{u}} = 0$. ■

Lemma 9.9. *Let $\mathbf{u} \in \mathfrak{P}_i$. If $i \in \{\infty, 0\}$, then $M_{\mathbf{u}}$ has a family of 1-dimensional quotients parametrized by $\mathbb{A}^1$. If $i \in \{0,1,2,3\}$, then $M_{\mathbf{u}}$ has a family of 2-dimensional simple quotient modules parametrized by $\mathbb{A}^1 - \{0\}$.*

Proof. If $i \in \{\infty, 0\}$, then $M_{\mathbf{u}}$ is a quotient *ring* of $\widetilde{Q}$ isomorphic to the polynomial ring $k[t]$ so the quotients $k[t]/(t-\lambda)$, $\lambda \in k$, give a family of 1-dimensional $\widetilde{Q}$-modules.

Suppose $i \in \{0,1,2,3\}$. Then $(M_{\mathbf{u}})_{\ge n}(-n) \cong M_{\theta^n(\mathbf{u})}$. Let $\varphi : M \to M$ be a homomorphism with image $M_{\ge 2}$. Then $M/(\mathrm{id} - \lambda\varphi)M$ is a 2-dimensional simple module for all $\lambda \in \mathbb{A}^1 - \{0\}$. ■

Proposition 9.10. *The scheme-theoretic zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the relations for $\widetilde{Q}$ is a reduced scheme with 20 points.*

Proof. (Van den Bergh [10].) We have already seen that the relations for $\widetilde{Q}$ vanish at 20 points in $\mathbb{P}^3 \times \mathbb{P}^3$. Let X denote the image of the Segre embedding $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^{15}$. If we view $\mathbb{P}^{15}$ as the space of 4×4 matrices, then X is the space of rank-one matrices. By [12, §18.15], for example, the degree of X is $\binom{6}{3} = 20$. The 6 defining relations for $\widetilde{Q}$ are linear combinations of terms $x_i x_j$ which, under the Segre embedding, become linear combinations of the coordinate functions x_{ij}. Hence the vanishing locus of the relations in $\mathbb{P}^{15}$ is the vanishing locus of 6 linear forms, hence a linear subspace, L say, of dimension 9. Hence, by Bézout's Theorem, if the scheme-theoretic intersection $L \cap X$ is finite it has degree 20. But, $L \cap X$ consists of 20 different points so it is reduced. ■

10. Line Modules

In this section we find six families of line modules for $\widetilde{Q}$, but we do not know whether we have found all line modules. Each family is parametrized by the elliptic curve $E/E[2]$.

By [17, Thm. 4.5], the line modules of Q are parametrized by lines in $\mathbb{P}^3$ secant to E, i.e., lines whose intersection with $E \subset \mathbb{P}^3$ has degree ≥ 2 as a divisor on E. Since E has no trisecants, these lines can be identified with the unordered pairs $p, q \in E$ of points where they intersect the elliptic curve; we denote the corresponding module by $M_{p,q}$. By Proposition 7.7, $g_i^* M_{p,q} \cong M_{p+\omega_i, q+\omega_i}$.

By Proposition 3.2 we immediately get

Proposition 10.1. *The isomorphism classes of line $\widetilde{Q}$-modules are in natural bijection with the isomorphism classes of Γ-equivariant modules over $Q' = Q \otimes M_2(k)$ with Hilbert series $\frac{4}{(1-t)^2}$.* ■

By Morita equivalence, an equivariant module as in the statement above must be of the form $M \otimes k^2$ for some Q-module M with Hilbert series $\frac{2}{(1-t)^2}$ (a "fat line" of multiplicity two over Q). Moreover, the equivariance ensures that the isomorphism class of M is invariant under translation by the 2-torsion of E.

In conclusion, fat Q-lines will be the main ingredient in constructing $\widetilde{Q}$-lines. The easy examples are those of the form $M_{p,q} \oplus M_{r,s}$, and the invariance condition demands that $p, q, r, s \in E$ form an orbit under translation by the 2-torsion of E. The main result of this section now provides the examples announced above.

Proposition 10.2. *If the points $p, q, r, s \in E$ comprise an $E[2]$-orbit, then the Q'-module $M = (M_{p,q} \oplus M_{r,s}) \otimes k^2$ admits precisely two Γ-equivariant structures up to isomorphism.*

Proof. Note first that M is by construction *invariant* under the twisting action by Γ on $\mathsf{Gr}(Q')$, in the sense that $M \cong \gamma^* M$ for all $\gamma \in \Gamma$. This allows us to define a right action, $(a, \gamma) \mapsto a \triangleleft \gamma$, of Γ on $\operatorname{Aut}_{Q'}(M) \cong (k^\times)^2$ by declaring that $a \triangleleft \gamma$ is the unique element in $\operatorname{Aut}_{Q'}(M)$ such that

$$\begin{tikzcd} M \arrow[r, "\varphi_\gamma"] \arrow[d, "a\triangleleft\gamma"'] & \gamma^* M \arrow[d, "\gamma^*(a)"] \\ M \arrow[r, "\varphi_\gamma"'] & \gamma^* M \end{tikzcd}$$

commutes where $a \in \mathrm{Aut}_{Q'}(M)$ and $\varphi_\gamma : M \to \gamma^* M$ are arbitrary isomorphisms. A different choice of φ's does not change $a \triangleleft \gamma$ since any other $\varphi'_\gamma : M \to \gamma^* M$ would be a multiple of φ_γ by an element in $\mathrm{Aut}_{Q'}(\gamma^* M)$ which is an abelian group.

The action $\triangleleft$ mimics the permutation action of $\Gamma \cong E[2]$ on p, q, r, s. If $\delta \in G$ interchanges p and q and η interchanges p and r then $\triangleleft\delta$ acts trivially on $(k^\times)^2$ (because the two $k^\times$ are the automorphism groups of $M_{p,q} \otimes k^2$ and $M_{r,s} \otimes k^2$ respectively and these are invariant under δ) while η interchanges the two copies of $k^\times$.

Step 1: Existence of an equivariant structure. The φ_γ were chosen arbitrarily, so there is no reason why they should satisfy the compatibility condition (17) required in order that they form an equivariant structure. Their failure to do so is measured by

$$a_{\delta,\gamma} = \varphi_{\gamma\delta}^{-1} \circ t_{\delta,\gamma} \circ \delta^*(\varphi_\gamma) \circ \varphi_\delta, \quad \delta, \gamma \in \Gamma, \tag{25}$$

where $t_{\delta,\gamma}$ is as in (17) and the displayed expression is simply the clockwise composition of all the automorphisms in that picture (with φ_γ substituted for γ).

This is easily seen to be a 2-cocycle of Γ valued in the Γ-module $\mathrm{Aut}_{Q'}(M) \cong (k^\times)^2$ defined above, and the φ_γ can be corrected into an equivariant structure if and only if this cocycle is cohomologous to zero. We skip the routine proof, but observe for instance that if $(a_{\delta,\gamma})$ is the coboundary of $\Gamma \ni \gamma \mapsto a_\gamma \in \mathrm{Aut}_{Q'}(M)$ then $\varphi_\gamma \circ a_\gamma$ will be an equivariant structure.

Using the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a(\mathbb{Z}/2, H^b(\mathbb{Z}/2, (k^\times)^2)) \Rightarrow H^{a+b}(\Gamma, (k^\times)^2)$$

and the cohomology of cyclic groups it can be shown that $H^2(\Gamma, (k^\times)^2)$ is trivial, and hence the obstruction cocycle $a_{\delta,\gamma}$ from (25) is indeed cohomologous to zero.

Step 2: Classification of equivariant structures. We know from Step 1 that there is at least one equvariant structure, consisting of say the maps $t_\gamma : M \to \gamma^* M$ for $\gamma \in \Gamma$.

Now let $(s_\gamma)_{\gamma \in \Gamma}$ be another equivariant structure. Running through the compatibility conditions comprising equivariance, The maps $a_\gamma = (t_\gamma)^{-1} \circ s_\gamma$ can be seen to form a 1-cocycle of Γ valued in the Γ-module $\mathrm{Aut}_{Q'}(M) \cong (k^\times)^2$. We similarly leave it to the reader to check that cocycles (a_γ) and (a'_γ) give rise to isomorphic equivariant structures

$$s_\gamma = t_\gamma a_\gamma \text{ and } s'_\gamma = t_\gamma a'_\gamma$$

if and only if they are cohomologous. In other words, the set of equivariant structures is acted upon simply and transitively by $H^1(\Gamma, (k^\times)^2)$. Using the Hochschild-Serre spectral sequence once more we get $H^1(\Gamma, (k^\times)^2) \cong \mathbb{Z}/2$. ■

Remark 10.3. The proof of Proposition 10.2 illustrates a familiar pattern in obstruction theory: The class of structures we are interested in (equivariant structures in this case) is a *pseudotorsor* over a cohomology group: Whether or not it is empty is controlled by an obstruction living in a cohomology group (H^2 for us, as in Step 1 of the proof), but if this obstruction vanishes then the cohomology group one degree lower (H^1) acts on the class of structures simply transitively. ♦

We now have a family of line modules for $\widetilde{Q}$ parametrized by six copies of $E/E[2]$: Each $E[2]$-orbit $p, q, r, s \in E$ can be partitioned into two halves in three ways, and for each of these three choices (say p, q and r, s) there are, according to Proposition 10.2, two equivariant structures on the corresponding Q'-module $(M_{p,q} \oplus M_{r,s}) \otimes k^2$.

Under quite general conditions, which $\widetilde{Q}$ satisfies, Shelton and Vancliff prove that every irreducible component of the scheme parametrizing the line modules has dimension ≥ 1 [28, Cor.2.6] and that every point module is a quotient of a line module [28, Prop.3.1].

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