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Rational and polynomial representations of Yangians

Sergey Khoroshkin [a,b], Maxim Nazarov [c], Alexander Shapiro [a,d]

[a] *Institute for Theoretical and Experimental Physics, Moscow 117259, Russia*
[b] *Department of Mathematics, Higher School of Economics, Moscow 117312, Russia*
[c] *Department of Mathematics, University of York, York YO10 5DD, United Kingdom*
[d] *Department of Mathematics, University of California, Berkeley, CA 94720, United States*

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ABSTRACT

We define natural classes of rational and polynomial representations of the Yangian of the general linear Lie algebra. We also present the classification and explicit realizations of all irreducible rational representations of the Yangian.

0. Introduction

In a recent series of works the first two authors studied certain functors between categories of modules of the complex general linear Lie algebra $\mathfrak{gl}_m$ and of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$; see [10,11] and references therein. Using the Howe duality [8,9] between the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$, these functors arise from the centralizer construction of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ due to Olshanski [18]. They can also be defined as direct sums over $N = 1, 2, \ldots$ of the compositions of two well known functors. The first functor in the composition, due to Cherednik [5], is between the categories of modules of $\mathfrak{gl}_m$ and of the degenerate affine Hecke algebra of $\mathfrak{gl}_N$. The second functor in the composition, due to Drinfeld [6], is between the latter category and the category of $\mathrm{Y}(\mathfrak{gl}_n)$-modules.

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In the above mentioned series of works the Zhelobenko operators [25] were used to study intertwining operators between certain $\mathrm{Y}(\mathfrak{gl}_n)$-modules. These modules correspond to pairs of weights $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of the Lie algebra $\mathfrak{gl}_m$ subject to a condition that each difference $\nu_a = \lambda_a - \mu_a$ is a non-negative integer not exceeding n. In the present article we denote by Φ_μ^λ the corresponding $\mathrm{Y}(\mathfrak{gl}_n)$-module, see (1.9). By definition, this module is a tensor product over $a = 1, \ldots, m$ of certain pullbacks of the fundamental modules $\Lambda^{\nu_a}(\mathbb{C}^n)$ of $\mathfrak{gl}_n$. Each pullback also depends on a complex number μ_a, while the tensor product is taken by using the comultiplication (1.5) on $\mathrm{Y}(\mathfrak{gl}_n)$. In particular, when $\nu_a = 1$ this pullback is called a vector $\mathrm{Y}(\mathfrak{gl}_n)$-module. Note that in the present article we use notation different from that of [10,11]. Most significantly, here the weights λ and μ of the Φ_μ^λ correspond to the weights $\lambda + \rho$ and $\mu + \rho$ in [10,11] where $\rho = (0, -1, \ldots, 1 - m)$.

Regard the symmetric group $\mathfrak{S}_m$ as the Weyl group of $\mathfrak{gl}_m$ and let $\sigma_0 \in \mathfrak{S}_m$ be the longest element. One of the principal results of [11] was a new proof of irreducibility of the image of the canonical intertwining operator

$$\Phi_\mu^\lambda \to \Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)} \tag{0.1}$$

under the condition of *dominance* of the weight λ of $\mathfrak{gl}_m$. This condition on λ means that

$$\lambda_a - \lambda_b \neq -1, -2, \ldots \quad \text{whenever } a < b. \tag{0.2}$$

The study of these operators has been started by Cherednik [4]. The proof of irreducibility in [11] was based on the results of [13]. Other proofs follow from the results of Akasaka and Kashiwara [1] and of Nazarov and Tarasov [17]. Note that a connection between the intertwining operators for tensor products of $\mathrm{Y}(\mathfrak{gl}_n)$-modules and the Zhelobenko operators has been first discovered by Tarasov and Varchenko [23].

It was also demonstrated in [11] that up to an automorphism of the form (1.1) of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$, any irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module arises as the image of some intertwining operator (0.1). Furthermore, for that particular purpose it suffices to consider only the operators corresponding to dominant weights λ while μ satisfy the extra conditions

$$\mu_a - \mu_b \leqslant 0 \quad \text{whenever } \lambda_a = \lambda_b \text{ and } a < b. \tag{0.3}$$

For any dominant λ the canonical intertwining operator (0.1) is defined only up to a scalar multiplier. These multipliers can be chosen in such a way that for any fixed $\nu = (\nu_1, \ldots, \nu_m)$ the operator (0.1) depends rationally on the weight λ. One such choice was made in [11].

In the normalization used in [11] the operator (0.1) vanishes if any of the extra conditions (0.3) is not satisfied. However, other arguments [1,11,19,24] indicate that in this case the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_μ^λ should still have a unique irreducible quotient. Another normalization of (0.1) was then provided in [12]. In that normalization for any dominant λ the operator (0.1) does not vanish, and moreover is constructed explicitly by using the *fusion procedure* from [4]. As a $\mathrm{Y}(\mathfrak{gl}_n)$-module the image of (0.1) is then isomorphic to that of the operator obtained by replacing in (0.1) the weight μ by any weight $\sigma(\mu)$ where $\sigma \in \mathfrak{S}_m$ fixes λ.

The construction in [12] is combinatorial and rather involved. The first goal of the present article is to provide another explicit expression (2.3) for the canonical intertwining operator (0.1) which is easier to use. In this form the operator (0.1) has the same normalization as in [12]. Yet the existence of this new form of (0.1) is far from obvious. It is obtained by also using another kind of Zhelobenko operators. The two kinds of these operators were studied in [14] and are closely related. They are proportional for the generic values of their parameters and admit analytic continuations to the domains of non-singularity of each other. By [10] our intertwining operator (0.1) corresponds to a product of Zhelobenko operators of both kinds.

A more general goal of the present article is to eliminate the use of the automorphisms (1.1) in [11,12]. The first steps in this direction have been made in [16,20]. In particular, the Howe duality between the Lie algebras $\mathfrak{u}_{p,q}$ and $\mathfrak{gl}_n$ has been used in [20] to construct $\mathrm{Y}(\mathfrak{gl}_n)$-modules which do not arise as subquotients of tensor products of the vector modules. Thus we come to the notions of polynomial and rational modules naturally generalizing the corresponding notions for the general linear group GL_n: the polynomial modules are subquotients of tensor products of vector modules, the rational modules are subquotients of tensor products of vector and covector modules. These covector modules can be described as the pullbacks of the vector modules relative to the automorphism (1.3) of $\mathrm{Y}(\mathfrak{gl}_n)$.

Up to twisting by the automorphisms of $\mathrm{Y}(\mathfrak{gl}_n)$ of the form (1.1), the isomorphism classes of irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules are labelled by sequences of $n-1$ monic polynomials with complex coefficients, called the Drinfeld polynomials [7]. The polynomial and the rational irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-modules have parametrizations very similar to that of [7]. In addition to the $n-1$ Drinfeld polynomials we have one more monic polynomial for labelling the polynomial irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules, or one more rational function for labelling the rational irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules. This result is stated as Theorem 1.2 below. It comes from the analysis of the action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ on the tensor products of vector and covector modules. Moreover, we demonstrate that all the polynomial and the rational irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-modules arise as images of the intertwining operators studied in [10,11] and [20] respectively.

Our article is organized as follows. Section 1 recalls the basic definitions and results from the representation theory of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. Sections 2 and 3 give realizations of the polynomial and the rational $\mathrm{Y}(\mathfrak{gl}_n)$-modules. All proofs are gathered in Section 4.

1. Basic results

1.1. The *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$ is a complex unital associative algebra with a family of generators $T_{ij}^{(1)}, T_{ij}^{(2)}, \dots$ with $i, j = 1, \dots, n$. These generators are customarily gathered into the generating series

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \dots$$

where u is a formal parameter. The defining relations of $\mathrm{Y}(\mathfrak{gl}_n)$ can be then written as

$$(u-v)\big[T_{ij}(u), T_{kl}(v)\big] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)$$

where v is another formal parameter. These relations imply that for any formal power series $f(u)$ in u^{-1} with the leading term 1 and all the coefficients in $\mathbb{C}$ the assignments

$$T_{ij}(u) \mapsto f(u)T_{ij}(u) \tag{1.1}$$

define an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Further, for any $z \in \mathbb{C}$ the assignments

$$\tau_z : T_{ij}(u) \mapsto T_{ij}(u-z) \tag{1.2}$$

define an automorphism τ_z of $\mathrm{Y}(\mathfrak{gl}_n)$. Here each of the formal series $T_{ij}(u-z)$ in $(u-z)^{-1}$ should be re-expanded in u^{-1} so that the assignment (1.2) becomes a correspondence between the respective coefficients of the series in u^{-1}. In this article we also employ the involutive automorphism ω of $\mathrm{Y}(\mathfrak{gl}_n)$ defined by the assignments

$$\omega : T_{ij}(u) \mapsto T_{ji}(-u). \tag{1.3}$$

The quotient of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ by the relations $T_{ij}^{(2)} = T_{ij}^{(3)} = \dots = 0$ for all indices $i, j = 1, \dots, n$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The defining relations of $\mathrm{Y}(\mathfrak{gl}_n)$ contain in particular the commutation relations

$$\big[T_{ij}^{(1)}, T_{kl}^{(1)}\big] = \delta_{kj} T_{il}^{(1)} - \delta_{il} T_{kj}^{(1)}.$$

Hence the images of the generators $T_{ij}^{(1)}$ in the quotient algebra can be identified with the standard matrix units in $\mathfrak{gl}_n$ having the same indices i, j. Denote by π the corresponding quotient map $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$. We will also use the composite homomorphisms $\pi_z, \pi'_z :$ $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ defined for any parameter $z \in \mathbb{C}$ by

$$\pi_z = \pi \circ \tau_z, \qquad \pi'_z = \pi \circ \omega \circ \tau_z. \tag{1.4}$$

The associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ contains $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra. Again, here we identify the generators $T_{ij}^{(1)}$ with the corresponding matrix units in $\mathfrak{gl}_n$. Hence the homomorphism π is identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$. The same is true for π_z with any $z \in \mathbb{C}$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a Hopf algebra. The counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ and the antipodal antihomomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$ are defined by mappings $T_{ij}(u) \mapsto \delta_{ij}$ and $T(u) \mapsto T^{-1}(u)$ respectively, while the comultiplication $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$ is defined by

$$T_{ij}(u) \mapsto \sum_{k=1}^{n} T_{ik}(u) \otimes T_{kj}(u). \tag{1.5}$$

1.2. For any non-negative integer d consider the exterior power $\Lambda^d(\mathbb{C}^n)$ of the defining $\mathfrak{gl}_n$-module $\mathbb{C}^n$. Let us denote by Φ_z^d and Φ_z^{-d} the $\mathrm{Y}(\mathfrak{gl}_n)$-modules obtained by pulling the $\mathfrak{gl}_n$-module $\Lambda^d(\mathbb{C}^n)$ back through the homomorphisms π_z and π'_{z-1} respectively, see (1.4). Clearly, these $\mathrm{Y}(\mathfrak{gl}_n)$-modules are non-zero if and only if $d \leqslant n$.

For any $z \in \mathbb{C}$ the $\mathrm{Y}(\mathfrak{gl}_n)$-modules Φ_z^1 and Φ_z^{-1} are called *vector* and *covector* modules respectively. Their underlying space is $\mathbb{C}^n$. Let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbb{C}^n$. The action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the basis vectors is given by

$$T_{ij}(u)e_k = \delta_{ij}e_k + \frac{\delta_{jk}e_i}{u-z} \quad \text{in } \Phi_z^1, \tag{1.6}$$

$$T_{ij}(u)e_k = \delta_{ij}e_k - \frac{\delta_{ik}e_j}{u-z+1} \quad \text{in } \Phi_z^{-1}. \tag{1.7}$$

Further, for any $z \in \mathbb{C}$ put

$$\Delta_z = \Phi_z^n \quad \text{and} \quad \Delta'_z = \Phi_z^{-n}. \tag{1.8}$$

We call Δ_z and Δ'_z the *determinantal* $\mathrm{Y}(\mathfrak{gl}_n)$-modules. By definition, a $\mathrm{Y}(\mathfrak{gl}_n)$-module is *polynomial* if it is isomorphic to a subquotient of a tensor product of the vector modules. More generally, a $\mathrm{Y}(\mathfrak{gl}_n)$-module is *rational* if it is isomorphic to a subquotient of a tensor product of vector and covector modules. In Subsection 4.1 we prove the following

Proposition 1.1. *If $d \in \{0, 1, \dots, n\}$ and $z \in \mathbb{C}$ then:*

(i) the module Φ_z^d is polynomial;
(ii) the module Φ_z^{-d} is rational;
(iii) the one-dimensional module Φ_z^0 is trivial;
(iv) the one-dimensional modules Δ_z and Δ'_z are cocentral;
(v) the module Φ_z^{-d} is isomorphic to $\Phi_z^{n-d} \otimes \Delta'_z$.

Remark. By the last two parts of Proposition 1.1 any rational $\mathrm{Y}(\mathfrak{gl}_n)$-module is isomorphic to a tensor product of a polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-module and a number of determinantal modules of the form Δ'_z. Further, every polynomial or rational $\mathrm{Y}(\mathfrak{gl}_n)$-module admits an action of the complex general linear group GL_n compatible with the embedding $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$ as described above. This action of the group GL_n is polynomial or rational respectively. $\square$

Now let $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_m)$ be two elements of $\mathbb{C}^m$ such that each difference $\nu_a = \lambda_a - \mu_a$ is an integer. The corresponding *standard rational* $\mathrm{Y}(\mathfrak{gl}_n)$-module is the tensor product

$$\Phi_\mu^\lambda = \Phi_{\mu_1}^{\nu_1} \otimes \dots \otimes \Phi_{\mu_m}^{\nu_m}. \tag{1.9}$$

We will be occasionally calling λ and μ the weights of Φ_μ^λ. If $\nu_1, \dots, \nu_m \geqslant 0$ then Φ_μ^λ is the *standard polynomial* $\mathrm{Y}(\mathfrak{gl}_n)$-module as employed in (0.1). Note that by Proposition 1.1 the standard polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-modules are indeed polynomial in our terminology, and the standard rational $\mathrm{Y}(\mathfrak{gl}_n)$-modules are indeed rational.

1.3. Let us recall some basic facts from the representation theory of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. They were first obtained by Tarasov [21,22] in the case $n = 2$ and then generalized by Drinfeld [7] to any n. For a detailed exposition of the proofs of these basic facts see [15].

Let Ψ be any irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module. There exists a non-zero vector $v \in \Psi$ called *highest*, such that $T_{ij}(u)v = 0$ for all $i < j$. This vector is unique up to a multiplier from $\mathbb{C}$. For every index i there exists a formal power series $A_i(u)$ in u^{-1} with coefficients in $\mathbb{C}$ and the leading term 1 such that $T_{ii}(u)v = A_i(u)v$. For any $i < n$ we have

$$\frac{A_i(u)}{A_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)} \tag{1.10}$$

where $P_i(u)$ are monic polynomials in u. The polynomials $P_1(u), \dots, P_{n-1}(u)$ are called the *Drinfeld polynomials* of Ψ. The series $A_1(u), \dots, A_n(u)$ determine the $\mathrm{Y}(\mathfrak{gl}_n)$-module Ψ uniquely up to isomorphism. Any sequence of formal power series $A_1(u), \dots, A_n(u)$ in u^{-1} with coefficients in $\mathbb{C}$ and the leading terms 1 satisfying (1.10) for some monic polynomials $P_1(u), \dots, P_{n-1}(u)$ occurs in this way.

The sequence of series $A_1(u), \dots, A_n(u)$ satisfying (1.10) can be recovered from any one of them, say from the $A_n(u)$, and from the sequence of monic polynomials $P_1(u), \dots, P_{n-1}(u)$. Here the series $A_n(u)$ itself can be chosen arbitrary, provided its leading term is 1. Our first (and rather elementary) result is the next theorem. We prove it in Subsections 4.2 and 4.3.

Theorem 1.2. *The irreducible finite-dimensional* $\mathrm{Y}(\mathfrak{gl}_n)$*-module determined by* $A_n(u)$ *and by* $P_1(u), \dots, P_{n-1}(u)$ *is polynomial or rational respectively if and only if*

$$A_n(u) = \frac{Q_n(u+1)}{Q_n(u)} \tag{1.11}$$

for some polynomial or rational function $Q_n(u)$ *in* u.

Clearly, in the case of a polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-module the polynomial function $Q_n(u)$ in (1.11) can be chosen monic. More generally, the rational function $Q_n(u)$ can be chosen as a ratio of two monic polynomials. This choice defines the function $Q_n(u)$ in (1.11) uniquely.

Note that equalities (1.10) and (1.11) imply that for each $i = 1, \ldots, n$

$$A_i(u) = \frac{Q_i(u+1)}{Q_i(u)} \quad \text{where } Q_i(u) = P_i(u) \ldots P_{n-1}(u) Q_n(u). \tag{1.12}$$

The functions $Q_1(u), \ldots, Q_n(u)$ are polynomial or rational if the corresponding irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module is respectively polynomial or rational. Here the converse (the only if) statement is untrue, because by definition the functions $Q_1(u), \ldots, Q_n(u)$ are related by polynomials $P_1(u), \ldots, P_{n-1}(u)$.

2. Polynomial modules

2.1. For any positive integer m take the general linear Lie algebra $\mathfrak{gl}_m$. For $a, b = 1, \ldots, m$ let $E_{ab} \in \mathfrak{gl}_m$ be the standard matrix units. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}_m$ with the basis $\{E_{11}, \ldots, E_{mm}\}$. Identify the dual vector space $\mathfrak{h}^*$ with $\mathbb{C}^m$ by using this basis of $\mathfrak{h}$. Hence any element of $\lambda \in \mathfrak{h}^*$ will be written as $(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$. The Weyl group of $\mathfrak{gl}_m$ is isomorphic to the symmetric group $\mathfrak{S}_m$. Thus there is a natural action of the group $\mathfrak{S}_m$ on $\mathfrak{h}^*$ such that $\sigma(\lambda)_a = \lambda_{\sigma^{-1}(a)}$ for $\sigma \in \mathfrak{S}_m$. We denote by σ_0 the longest element of $\mathfrak{S}_m$ so that we have $\sigma_0(a) = m + 1 - a$ for each $a = 1, \ldots, m$.

Choose the triangular decomposition $\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'$ where $\mathfrak{n}$ and $\mathfrak{n}'$ are the opposite nilpotent subalgebras of $\mathfrak{gl}_m$ spanned by the vectors E_{ab} where $a > b$ and $a < b$ respectively. Then the positive roots of $\mathfrak{gl}_m$ are the weights $\lambda \in \mathfrak{h}^*$ with the only non-zero coordinates $\lambda_a = 1$ and $\lambda_b = -1$ for some pair of indices $a < b$.

Denote by $\mathcal{G}_{mn}$ the Grassmann algebra on the mn anticommuting variables x_{ai} where $a = 1, \ldots, m$ and $i = 1, \ldots, n$. Let ∂_{ai} be the operator of left derivation in $\mathcal{G}_{mn}$ relative to x_{ai}. Then for any $a, b = 1, \ldots, m$ consider the first order differential operator on $\mathcal{G}_{mn}$

$$D_{ab} = \sum_{k=1}^{n} x_{ak} \partial_{bk}.$$

The assignment $E_{ab} \mapsto D_{ab}$ defines an action of the Lie algebra $\mathfrak{gl}_m$ on $\mathcal{G}_{mn}$. We will use the weight decomposition of $\mathcal{G}_{mn}$ under the action of $\mathfrak{h} \subset \mathfrak{gl}_m$. We will also use the action of the symmetric group $\mathfrak{S}_m$ on $\mathcal{G}_{mn}$ by permutations of the first indices of the variables:

$$\sigma(x_{ai}) = x_{\sigma(a)i} \quad \text{for any } \sigma \in \mathfrak{S}_m.$$

Now take any two indices a and b such that $a < b$. Let $\lambda \in \mathfrak{h}^*$ be any weight such that $\lambda_a - \lambda_b \neq -1, -2, \ldots$. Define the linear operators X_{ab}^{λ} and Y_{ab}^{λ} on the vector space $\mathcal{G}_{mn}$ by

$$X_{ab}^{\lambda}=1+\sum_{r=1}^{\infty}\frac{(-1)^r D_{ba}^r D_{ab}^r}{r!(\lambda_a-\lambda_b+1)_r}\quad\text{and}\quad Y_{ab}^{\lambda}=1+\sum_{r=1}^{\infty}\frac{(-1)^r D_{ab}^r D_{ba}^r}{r!(\lambda_a-\lambda_b+1)_r}$$

where $(u)_r=u(u+1)\dots(u+r-1)$ is the Pochhammer function. The operators X_{ab}^{λ} and Y_{ab}^{λ} on $\mathcal{G}_{mn}$ are well defined and preserve the subspace $\mathcal{G}_{mn}^{\nu}\subset\mathcal{G}_{mn}$ of every weight $\nu\in\mathfrak{h}^*$.

2.2. For any two weights $\lambda,\mu\in\mathfrak{h}^*$ let $\nu=\lambda-\mu$ be their difference. Suppose that each coordinate ν_a of ν is a non-negative integer not exceeding n. Let us consider the standard polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_{μ}^{λ}. By the definition (1.9) its underlying vector space is

$$\Lambda^{\nu_1}(\mathbb{C}^n)\otimes\dots\otimes\Lambda^{\nu_m}(\mathbb{C}^n).\tag{2.1}$$

Using the basis vectors $e_1,\dots,e_n\in\mathbb{C}^n$ we can now define an isomorphism of vector spaces

$$G_{\nu}:\Phi_{\mu}^{\lambda}\to\mathcal{G}_{mn}^{\nu}$$

as follows. For an element of the vector space (2.1)

$$w=(e_{i_1}\wedge\dots\wedge e_{i_{\nu_1}})\otimes\dots\otimes(e_{j_1}\wedge\dots\wedge e_{j_{\nu_m}})$$

we set

$$G_{\nu}(w)=(x_{1i_1}\dots x_{1i_{\nu_1}})\dots(x_{mj_1}\dots x_{mj_{\nu_m}}).\tag{2.2}$$

Now suppose that $\lambda\in\mathfrak{h}^*$ satisfies the dominance condition (0.2). Define a linear map

$$I_{\mu}^{\lambda}:\Phi_{\mu}^{\lambda}\to\Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)}$$

as the composition

$$I_{\mu}^{\lambda}=(-1)^N G_{\sigma_0(\nu)}^{-1}\sigma_0 Z_{\mu}^{\lambda}G_{\nu}\tag{2.3}$$

where

$$N=\sum_{1\leqslant a<b\leqslant m}\nu_a\nu_b,\tag{2.4}$$

$$Z_{\mu}^{\lambda}=\prod_{1\leqslant a<b\leqslant m}^{\longrightarrow}\begin{cases}X_{ab}^{\lambda} & \text{if } \nu_a\geqslant\nu_b\\ Y_{ab}^{\mu} & \text{if } \nu_a<\nu_b\end{cases}\tag{2.5}$$

and the ordering of the factors in the product over the pairs of indices $a < b$ corresponds to any normal ordering[1] of the positive roots of the Lie algebra $\mathfrak{gl}_m$. It means that the factor corresponding to the pair $a < c$ stands between those corresponding to $a < b$ and to $b < c$. The right hand side of (2.3) is defined due to the dominance of λ. In Subsections 4.4 to 4.7 we will derive from the results of the works [10–12] the following proposition.

Proposition 2.1. *For any weights $\lambda, \mu \in \mathfrak{h}^*$ such that λ is dominant and all coordinates $\nu_a = \lambda_a - \mu_a$ are non-negative integers not exceeding n, the operator I^λ_μ defined by (2.3):*

(i) is not zero;
(ii) does not depend on the choice of the normal ordering;
(iii) commutes with the action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$;
(iv) has the image irreducible under the latter action.

2.3. Let us describe the highest vector and the series $A_1(u), \ldots, A_n(u)$ for the irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module

$$\Psi^\lambda_\mu = \operatorname{Im} I^\lambda_\mu \subset \Phi^{\sigma_0(\lambda)}_{\sigma_0(\mu)}. \tag{2.6}$$

Let

$$v_d = e_1 \wedge \ldots \wedge e_d \tag{2.7}$$

be the highest vector of the $\mathfrak{gl}_n$-module $\Lambda^d(\mathbb{C}^n)$. Here we use the triangular decomposition of the Lie algebra $\mathfrak{gl}_n$ similar to that of $\mathfrak{gl}_m$, see Subsection 2.1. For the vector

$$v^\lambda_\mu = v_{\nu_1} \otimes \ldots \otimes v_{\nu_m} \in \Phi^\lambda_\mu \tag{2.8}$$

by using (1.5) we get

$$T_{ij}(u) v^\lambda_\mu = 0 \quad \text{if } i < j. \tag{2.9}$$

We also get

$$T_{ii}(u) v^\lambda_\mu = A_i(u) v^\lambda_\mu \tag{2.10}$$

where for each index i

[1] A total ordering $\prec$ of the system of positive roots of a reductive Lie algebra is called *normal* if for any positive roots α, β such that $\alpha \prec \beta$ and $\alpha + \beta$ is also a positive root, the relation $\alpha \prec (\alpha + \beta) \prec \beta$ holds; see [2]. There is a natural bijection between the set of normal orderings of the system of positive roots of a reductive Lie algebra and the set of reduced decompositions of the longest element of the Weyl group [25].

$$A_i(u) = \prod_{a:\nu_a \geqslant i} \frac{u - \mu_a + 1}{u - \mu_a}. \tag{2.11}$$

In the course of proving our Proposition 2.1 we will establish the equality

$$I_\mu^\lambda\left(v_\mu^\lambda\right) = v_{\sigma_0(\mu)}^{\sigma_0(\lambda)} \tag{2.12}$$

where according to (2.8)

$$v_{\sigma_0(\mu)}^{\sigma_0(\lambda)} = v_{\nu_m} \otimes \ldots \otimes v_{\nu_1}. \tag{2.13}$$

Hence the irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module Ψ_μ^λ has the highest vector $v_{\sigma_0(\mu)}^{\sigma_0(\lambda)}$ and the corresponding series $A_1(u), \ldots, A_n(u)$ are given by (2.11). By (1.10), (1.11), (1.12) it now follows that for Ψ_μ^λ

$$P_i(u) = \prod_{a:\nu_a = i} (u - \mu_a) \quad \text{where } i = 1, \ldots, n-1;$$
$$Q_i(u) = \prod_{a:\nu_a \geqslant i} (u - \mu_a) \quad \text{where } i = 1, \ldots, n.$$

While proving Theorem 1.2 we will also obtain the following corollary to Proposition 2.1.

Corollary 2.2. *The Ψ_μ^λ exhaust all the non-trivial irreducible polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-modules.*

For any $\sigma \in \mathfrak{S}_m$ such that both λ and $\sigma(\lambda)$ are dominant, the proof of [12, Proposition 2.9] yields an isomorphism

$$\Psi_\mu^\lambda \to \Psi_{\sigma(\mu)}^{\sigma(\lambda)}$$

of $\mathrm{Y}(\mathfrak{gl}_n)$-modules. Further suppose that all coordinates of the weight ν are positive, see Proposition 1.1(iii). Then take another irreducible polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-module $\Psi_{\mu'}^{\lambda'}$ where λ' is dominant and all coordinates of the weight $\lambda' - \mu'$ are positive integers not exceeding n. The $\mathrm{Y}(\mathfrak{gl}_n)$-modules Ψ_μ^λ and $\Psi_{\mu'}^{\lambda'}$ are isomorphic if and only if λ, μ and λ', μ' are weights of the same $\mathfrak{gl}_m$ and there exists $\sigma \in \mathfrak{S}_m$ such that $\lambda' = \sigma(\lambda)$ and $\mu' = \sigma(\mu)$.

3. Rational modules

3.1. Take a sequence of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ where $\varepsilon_a \in \{-1, +1\}$ for each $a = 1, \ldots, m$. For any given $a, b = 1, \ldots, m$ let D_{ab}^ε be a differential operator on $\mathcal{G}_{mn}$ of order at most two,

$$D_{ab}^\varepsilon = \sum_{i=1}^n q_{ai} p_{bi}$$

where

$$q_{ai} = \begin{cases} x_{ai} & \text{if } \varepsilon_a = +1 \\ \partial_{ai} & \text{if } \varepsilon_a = -1 \end{cases} \quad \text{and} \quad p_{bi} = \begin{cases} \partial_{bi} & \text{if } \varepsilon_b = +1 \\ x_{bi} & \text{if } \varepsilon_b = -1. \end{cases} \tag{3.1}$$

Now take any two indices a and b such that $a < b$. Let $\lambda \in \mathfrak{h}^*$ be any weight such that $\lambda_a - \lambda_b \neq -1, -2, \ldots$. Define the linear operators $X_{ab}^{\lambda\varepsilon}$ and $Y_{ab}^{\lambda\varepsilon}$ on the vector space $\mathcal{G}_{mn}$ by

$$X_{ab}^{\lambda\varepsilon} = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r (D_{ba}^{\varepsilon})^r (D_{ab}^{\varepsilon})^r}{r!(\lambda_a - \lambda_b + 1)_r} \quad \text{and} \quad Y_{ab}^{\lambda\varepsilon} = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r (D_{ab}^{\varepsilon})^r (D_{ba}^{\varepsilon})^r}{r!(\lambda_a - \lambda_b + 1)_r}$$

where we again use the Pochhammer function $(u)_r = u(u+1)\ldots(u+r-1)$.

Take any two weights $\lambda, \mu \in \mathfrak{h}^*$ such that each coordinate ν_a of the difference $\nu = \lambda - \mu$ is an integer with the absolute value $|\nu_a| \leqslant n$. Consider the corresponding standard rational $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_μ^λ. By the definition (1.9) its underlying vector space is

$$\Lambda^{|\nu_1|}\big(\mathbb{C}^n\big) \otimes \ldots \otimes \Lambda^{|\nu_m|}\big(\mathbb{C}^n\big). \tag{3.2}$$

Denote by $|\nu|$ the weight $(|\nu_1|, \ldots, |\nu_m|)$ of $\mathfrak{gl}_m$. We will use the linear map $G_{|\nu|}$ to identify the vector space (3.2) with the weight subspace $\mathcal{G}_{mn}^{|\nu|} \subset \mathcal{G}_{mn}$, see the definition (2.2).

Choose the sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ so that ε_a is $+1$ or -1 depending on whether $\nu_a \geqslant 0$ or $\nu_a < 0$. Define the weights $\bar\lambda$ and $\bar\nu$ of $\mathfrak{gl}_m$ by setting $\bar\lambda = \mu + \bar\nu$ where

$$\bar\nu_a = \begin{cases} \nu_a & \text{if } \nu_a \geqslant 0 \\ n + \nu_a & \text{if } \nu_a < 0. \end{cases}$$

Then denote

$$\bar N = \sum_{1 \leqslant a < b \leqslant m} \bar\nu_a \bar\nu_b. \tag{3.3}$$

Suppose that $\bar\lambda$ is dominant. Note that here we do not require the dominance of λ itself. Define a linear map

$$I_\mu^\lambda : \Phi_\mu^\lambda \to \Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)}$$

as the composition

$$I_\mu^\lambda = (-1)^{\bar N} G_{\sigma_0|\nu|}^{-1} \sigma_0 Z_\mu^\lambda G_{|\nu|} \tag{3.4}$$

where now

$$Z_\mu^\lambda = \prod_{1\leqslant a<b\leqslant m}^{\longrightarrow} \begin{cases} X_{ab}^{\bar\lambda\varepsilon} & \text{if } \bar\nu_a \geqslant \bar\nu_b \\ Y_{ab}^{\mu\varepsilon} & \text{if } \bar\nu_a < \bar\nu_b \end{cases}$$

and the ordering of the factors in the product over the pairs of indices $a < b$ corresponds to any normal ordering of the positive roots of the Lie algebra $\mathfrak{gl}_m$. The next theorem extends our Proposition 2.1 to rational $\mathrm{Y}(\mathfrak{gl}_n)$-modules. The proof will be given in Subsection 4.8.

Theorem 3.1. *For any $\lambda, \mu \in \mathfrak{h}^*$ such that $\bar\lambda$ is dominant and all coordinates of $\lambda - \mu$ are integers with absolute values not exceeding n, the operator I_μ^λ defined by (3.4):*

(i) is not zero;
(ii) does not depend on the choice of the normal ordering;
(iii) commutes with the action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$;
(iv) has the image irreducible under the latter action.

3.2. Let us describe the highest vector and the series $A_1(u), \dots, A_n(u)$ for the irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module defined as the image of the operator (3.4). We will still denote this module by Ψ_μ^λ thus generalizing the notation (2.6).

For any positive integer $d \leqslant n$ consider the $\mathfrak{gl}_n$-module obtained by pulling $\Lambda^d(\mathbb{C}^n)$ back through the restriction of the automorphism (1.3) to the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$. Let

$$v_{-d} = e_{n-d+1} \wedge \ldots \wedge e_n$$

be the highest vector of the resulting $\mathfrak{gl}_n$-module. Recall that the highest vector of $\Lambda^d(\mathbb{C}^n)$ itself is (2.7). Define the vector $v_\mu^\lambda \in \Phi_\mu^\lambda$ as in (2.8) where every coordinate ν_a can also be negative now. Then (2.9) and (2.10) still hold, however now for each $i = 1, \dots, n$

$$A_i(u) = \prod_{a:\nu_a<i-n} \frac{u-\mu_a}{u-\mu_a+1} \cdot \prod_{a:\nu_a\geqslant i} \frac{u-\mu_a+1}{u-\mu_a}. \tag{3.5}$$

In the course of proving our Theorem 2.1 we will show that the equalities (2.12), (2.13) also hold now. Hence $v_{\sigma_0(\mu)}^{\sigma_0(\lambda)}$ is the highest vector of Ψ_μ^λ and the corresponding series $A_1(u), \dots, A_n(u)$ are given by (3.5). It now follows that for Ψ_μ^λ

$$P_i(u) = \prod_{a:\nu_a=i,i-n} (u-\mu_a) \quad \text{where } i = 1, \dots, n-1;$$

$$Q_i(u) = \prod_{a:\nu_a<i-n} \frac{1}{u-\mu_a} \cdot \prod_{a:\nu_a\geqslant i} (u-\mu_a) \quad \text{where } i = 1, \dots, n.$$

While proving Theorem 1.2 we will also obtain the following corollary to Theorem 3.1.

Corollary 3.2. *The Ψ_μ^λ exhaust all the non-trivial irreducible rational* $\mathrm{Y}(\mathfrak{gl}_n)$*-modules.*

There is no uniqueness in realizing a given non-trivial irreducible rational $\mathrm{Y}(\mathfrak{gl}_n)$-module as some Ψ_μ^λ where $\bar\lambda$ is dominant and all coordinates of the weight $\lambda-\mu$ are non-zero integers with absolute values not exceeding n. Take Proposition 1.1(v) with $0<d<n$ as an example:

$$\begin{aligned} \Phi_z^{-d} &= \Psi_\mu^\lambda \quad \text{where } m=1,\ \lambda_1 = z-d \text{ and } \mu_1 = z; \\ \Phi_z^{n-d}\otimes\Delta'_z &= \Psi_\mu^\lambda \quad \text{where } m=2,\ (\lambda_1,\lambda_2)=(z+n-d,z-n) \text{ and } (\mu_1,\mu_2)=(z,z). \end{aligned}$$

All choices of λ and μ with minimal number of coordinates m are described in Subsection 4.3.

4. The proofs

4.1. Here we prove Proposition 1.1. Part (i) is well known, see for instance [12, Lemma 2.1]. Part (ii) immediately follows from (i) because $\omega\circ\tau_z=\tau_{-z}\circ\omega$. By definition, the trivial $\mathrm{Y}(\mathfrak{gl}_n)$-module is one-dimensional and is defined by the counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n)\to\mathbb{C}$. Composition of the homomorphism $\pi_z:\mathrm{Y}(\mathfrak{gl}_n)\to\mathrm{U}(\mathfrak{gl}_n)$ with the representation of $\mathrm{U}(\mathfrak{gl}_n)$ in $\Lambda^0(\mathbb{C}^n)$ coincides with the counit. Therefore we get (iii). Further, the cocentrality in (iv) means that for any $\mathrm{Y}(\mathfrak{gl}_n)$-module M the flip of two tensor factors yields isomorphisms

$$\mathrm{M}\otimes\Delta_z\to\Delta_z\otimes\mathrm{M} \quad \text{and} \quad \mathrm{M}\otimes\Delta'_z\to\Delta'_z\otimes\mathrm{M}.$$

Thus (iv) follows from the definition (1.5) of the comultiplication and an observation that the defining homomorphisms of one-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-modules (1.8) are given respectively by

$$T_{ij}(u)\mapsto\delta_{ij}\frac{u-z+1}{u-z} \quad \text{and} \quad T_{ij}(u)\mapsto\delta_{ij}\frac{u-z}{u-z+1}.$$

Using the latter description of the $\mathrm{Y}(\mathfrak{gl}_n)$-module Δ'_z, (v) follows from [12, Lemma 2.3]. However, we will still include another proof of (v) here. It will be then used in Subsection 4.8.

Let $\mathcal{G}_n$ be the Grassmann algebra on n variables $x_1,\dots,x_n$. For each $d=0,1,\dots,n$ let $\mathcal{G}_n^d\subset\mathcal{G}_n$ be the subspace of degree d. Define a bijective linear map $G_d:\Lambda^d(\mathbb{C}^n)\to\mathcal{G}_n^d$ by

$$G_d(e_{i_1}\wedge\ldots\wedge e_{i_d})=x_{i_1}\ldots x_{i_d}.$$

Let us carry the structures of $\mathrm{Y}(\mathfrak{gl}_n)$-modules from Φ_z^{-d} and $\Phi_z^{n-d}\otimes\Delta'_z$ to the vector spaces $\mathcal{G}_n^d$ and $\mathcal{G}_n^{n-d}\otimes\mathcal{G}_n^n$ via the maps G_d and $G_{n-d}\otimes G_n$ respectively. The action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ on the resulting two modules can be described by the assignments

$$T_{ij}(u) \mapsto \delta_{ij} - \frac{x_j \partial_i}{u - z + 1} \quad \text{and} \quad T_{ij}(u) \mapsto \left(\delta_{ij} + \frac{x_i \partial_j}{u - z} \right) \otimes \frac{u - z}{u - z + 1}$$

respectively, where ∂_i is the operator of left derivation in $\mathcal{G}_n$ relative to x_i.

Let $\mathcal{D}_n$ be the ring of left differential operators on the Grassmann algebra $\mathcal{G}_n$. Denote by S the involutive automorphism of $\mathcal{D}_n$ defined by setting $S(x_i) = \partial_i$ for $i = 1, \ldots, n$. Put

$$R(x) = S(x) \cdot x_1 \ldots x_n \quad \text{for } x \in \mathcal{G}_n.$$

Note that the operator R^2 is $(-1)^{n(n-1)/2}$ times the identity map. Then we have the relation

$$R\big(Y(x)\big) = S(Y)\big(R(x)\big) \quad \text{for } Y \in \mathcal{D}_n. \tag{4.1}$$

Indeed, if Y is the operator of multiplication by any $y \in \mathcal{G}_n$ then we get (4.1) because S is a homomorphism:

$$R(yx) = S(yx) \cdot x_1 \ldots x_n = S(y)\big(R(x)\big).$$

By substituting here $R^{-1}(x)$ for x, switching the left and the right hand sides of the resulting equality, and then applying R to both sides we get

$$R\big(S(y)(x)\big) = yR(x) = S\big(S(y)\big)\big(R(x)\big).$$

Thus we also get (4.1) for $Y = S(y)$. Since the operators $Y = y$ and $Y = S(y)$ generate the ring $\mathcal{D}_n$ we obtain the relation (4.1) in general.

An isomorphism of $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\mathcal{G}_n^d \to \mathcal{G}_n^{n-d} \otimes \mathcal{G}_n^n$ can now be defined by mapping any element $x \in \mathcal{G}_n^d$ to the element $R(x) \otimes (x_1 \ldots x_n)$. Due to the relation (4.1) the intertwining property of this map follows from the operator equality

$$\delta_{ij} - \frac{x_j \partial_i}{u - z + 1} = \frac{u - z}{u - z + 1} \left(\delta_{ij} + \frac{\partial_i x_j}{u - z} \right).$$

Hence the $\mathrm{Y}(\mathfrak{gl}_n)$-modules $\mathcal{G}_n^d$ and $G_n^{n-d} \otimes \mathcal{G}_n^n$ are isomorphic. By the definition of these two modules, we now get part (v) of Proposition 1.1.

4.2. In this subsection we prove the *only if* part of Theorem 1.2. This part follows from a more general property of $\mathrm{Y}(\mathfrak{gl}_n)$-modules stated as the lemma below. Let

$$\Phi = \Phi_{z_1}^{\varepsilon_1} \otimes \ldots \otimes \Phi_{z_m}^{\varepsilon_m}$$

be the tensor product of any number m of the vector and covector $\mathrm{Y}(\mathfrak{gl}_n)$-modules. Here we have $\varepsilon_a \in \{-1, 1\}$ and $z_a \in \mathbb{C}$ for each index $a = 1, \ldots, m$.

Lemma 4.1. *For any index i all eigenvalues of $T_{ii}(u)$ on the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ have the form*

$$\prod_{a\in I}\frac{u-z_a+1}{u-z_a}\cdot\prod_{a\in J}\frac{u-z_a}{u-z_a+1}$$

where I and J are subsets of the sets of indices a such that $\varepsilon_a=1$ and $\varepsilon_a=-1$ respectively.

Proof. We will prove the lemma by induction on m. In the base case $m=0$ the statement of the lemma is trivial. Suppose that $m>0$ and the statement is true for $m-1$ instead of m. We will consider the cases $\varepsilon_m=1$ and $\varepsilon_m=-1$ separately. In both cases we will denote

$$\Phi'=\Phi_{z_1}^{\varepsilon_1}\otimes\ldots\otimes\Phi_{z_{m-1}}^{\varepsilon_{m-1}}.$$

First suppose $\varepsilon_m=1$. For any index $i=1,\ldots,n$ let $W_i\subset\mathbb{C}^n$ be the span of the vectors $e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n$. Then by (1.5), (1.6) for any vector $w\in\Phi'$ and for any index $k\neq i$

$$T_{ii}(u)(w\otimes e_k)=\big(T_{ii}(u)w\big)\otimes e_k,$$
$$T_{ii}(u)(w\otimes e_i)=\frac{u-z_m+1}{u-z_m}\big(T_{ii}(u)w\big)\otimes e_i\quad\mathrm{mod}\ \Phi'\otimes W_i\big[\big[u^{-1}\big]\big].$$

In particular, the action of the coefficients of the series $T_{ii}(u)$ on Φ preserves the subspace $\Phi'\otimes W_i$. Hence in this case any eigenvalue of $T_{ii}(u)$ on Φ is equal to an eigenvalue of $T_{ii}(u)$ on Φ' multiplied either by 1 or by

$$\frac{u-z_m+1}{u-z_m}.$$

Next suppose that $\varepsilon_m=-1$. Then by (1.5), (1.7) for any $w\in\Phi'$ and for any $k\neq i$

$$T_{ii}(u)(w\otimes e_k)=\big(T_{ii}(u)w\big)\otimes e_k\quad\mathrm{mod}\ \Phi'\otimes e_i\big[\big[u^{-1}\big]\big],$$
$$T_{ii}(u)(w\otimes e_i)=\frac{u-z_m}{u-z_m+1}\big(T_{ii}(u)w\big)\otimes e_i.$$

In particular, the action of the coefficients of the series $T_{ii}(u)$ on Φ preserves the subspace $\Phi'\otimes e_i$. Hence in this case any eigenvalue of $T_{ii}(u)$ on Φ is equal to an eigenvalue of $T_{ii}(u)$ on Φ' multiplied either by 1 or by

$$\frac{u-z_m}{u-z_m+1}.\qquad\square$$

4.3. In this subsection we will prove the *if* part of Theorem 1.2. Then we will derive our Corollaries 2.2 and 3.2. Let $P_1(u), \dots, P_n(u)$ and $P_{-n}(u)$ be any monic polynomials in u with complex coefficients. Let $Q_n(u) = P_n(u)/P_{-n}(u)$. Assume that the polynomials $P_n(u)$ and $P_{-n}(u)$ have no common zeroes. Determine the series $A_1(u), \dots, A_n(u)$ by (1.12).

We need to prove that the irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_n)$-module corresponding to $A_1(u), \dots, A_n(u)$ is rational, and is polynomial if $P_{-n}(u) = 1$. Write

$$P_i(u) = \prod_{s=1}^{m_i} (u - z_{is}) \quad \text{for } i = 1, \dots, n, -n.$$

Set $m = m_1 + \ldots + m_n + m_{-n}$. Denote by $\mathcal{P}$ the set of m pairs (i, z_{is}) where $s = 1, \dots, m_i$ and $i = 1, \dots, n, -n$. Let λ and μ be any two weights of $\mathfrak{gl}_m$ such that for $\nu = \lambda - \mu$ the set

$$\{(\nu_a, \mu_a) \mid a = 1, \dots, m\} = \mathcal{P}.$$

Define the weight $\bar{\lambda}$ as in Subsection 3.1. Note that if $\nu_a < 0$ for any a here, then $\nu_a = -n$. In particular, if $P_{-n}(u) = 1$ then all the coordinates ν_a are positive integers not exceeding n.

Consider the corresponding $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_μ^λ and its vector v_μ^λ. Note that the definition of this vector does not require the dominance of the weight $\bar{\lambda}$. Moreover, the equalities (2.9) and (2.10) hold for any $\bar{\lambda}$, not necessarily dominant. It now follows that the $\mathrm{Y}(\mathfrak{gl}_n)$-module Φ_μ^λ has an irreducible subquotient such that its highest vector is the image of v_μ^λ, see for instance the proof of [3, Theorem 2.16]. The series $A_1(u), \dots, A_n(u)$ corresponding to this irreducible subquotient are given by (1.12). Hence we get the *if* part of Theorem 1.2.

So far the weights λ and μ have been determined up to any simultaneous permutation of their coordinates. We can now choose λ and μ so that the weight $\bar{\lambda}$ is dominant. Then due to (2.12) the irreducible subquotient of Φ_μ^λ considered above becomes the quotient relative to the kernel of the operator I_μ^λ. This quotient is isomorphic to Ψ_μ^λ. Corollaries 2.2 and 3.2 now follow from Proposition 2.1 and Theorem 3.1 respectively, by the *only if* part of Theorem 1.2.

Note that if $P_{-n}(u) \neq 1$, then the irreducible module Ψ_μ^λ considered above does not necessarily have the minimal number m possible for the given polynomials $P_1(u), \dots, P_{n-1}(u)$ and the rational function $Q_n(u)$. Suppose the set $\mathcal{P}$ contains the two pairs (d, z) and $(-n, z)$ for some $z \in \mathbb{C}$ and $d > 0$. Then $d < n$ because the polynomials $P_n(u)$ and $P_{-n}(u)$ have no common zeroes. Replace the two pairs (d, z) and $(-n, z)$ in $\mathcal{P}$ by the single pair $(d - n, z)$. Let $\mathcal{P}'$ be any set obtained by repeating this replacement step until possible. This $\mathcal{P}'$ may be not unique. However, all resulting sets $\mathcal{P}'$ have the same size which will denote by m'. Let λ' and μ' be any weights of $\mathfrak{gl}_{m'}$ such that for $\nu' = \lambda' - \mu'$

$$\{(\nu'_a, \mu'_a) \mid a = 1, \dots, m'\} = \mathcal{P}'.$$

Assuming that $\overline{\lambda'}$ is dominant, the irreducible $\mathrm{Y}(\mathfrak{gl}_n)$-module $\Psi^{\lambda'}_{\mu'}$ has the same polynomials $P_1(u), \dots, P_{n-1}(u)$ and rational function $Q_n(u)$ as Ψ^{λ}_{μ}. In particular, it is isomorphic to Ψ^{λ}_{μ}.

4.4. This and the next three subsections are devoted to the proof of Proposition 2.1. First we recall the construction of Zhelobenko operators for the Lie algebra $\mathfrak{gl}_m$ from [14]; see also [10,13]. Let A be any associative algebra containing $\mathrm{U}(\mathfrak{gl}_m)$ as a subalgebra. Then A can be regarded as a $\mathrm{U}(\mathfrak{gl}_m)$-bimodule. Suppose that the following two conditions are satisfied:

(i) the corresponding adjoint action of the Lie algebra $\mathfrak{gl}_m$ on A is locally finite and can be lifted to an algebraic action of the group GL_m by automorphisms of the algebra A;
(ii) the algebra A contains a vector subspace V invariant under the action of GL_m, such that multiplication in A gives an isomorphism of GL_m-modules $\mathrm{U}(\mathfrak{gl}_m) \otimes \mathrm{V} \to \mathrm{A}$.

Consider $\mathfrak{S}_m$ as the subgroup of GL_m consisting of the permutation matrices. Note that under the above conditions the group $\mathfrak{S}_m$ acts by automorphisms on the algebra A. In the beginning of Subsection 2.1 we fixed a triangular decomposition $\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'$. Now let $\bar{\mathrm{U}}(\mathfrak{h})$ and $\bar{\mathrm{A}}$ be the rings of fractions of $\mathrm{U}(\mathfrak{h})$ and of A relative to the set of denominators, multiplicatively generated by all the elements

$$E_{aa} - E_{bb} + z \in \mathrm{U}(\mathfrak{h}) \quad \text{where } a < b \text{ and } z \in \mathbb{Z}.$$

For every index $a = 1, \dots, m-1$ consider a standard $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_m$:

$$E_a = E_{a,a+1}, \qquad F_a = E_{a+1,a} \quad \text{and} \quad H_a = E_{aa} - E_{a+1,a+1}. \tag{4.2}$$

Then one can define a linear map $\xi_a : \mathrm{A} \to \bar{\mathrm{A}}/\mathfrak{n}\bar{\mathrm{A}}$ by setting for each $x \in \mathrm{A}$

$$\xi_a(x) = \sigma_a(x) + \sum_{r=1}^{\infty} \big(r! H_a(H_a - 1) \dots (H_a - r + 1)\big)^{-1} E_a^r \, \mathrm{ad}_{F_a}^r \big(\sigma_a(x)\big) \mod \mathfrak{n}\bar{\mathrm{A}}$$

where $\sigma_a \in \mathfrak{S}_m$ is the transposition of a and $a+1$, regarded as an element of $\mathrm{Aut}(\mathrm{A})$.

The so defined map ξ_a can be canonically extended to the *Zhelobenko operators*

$$\xi_a : \bar{\mathrm{A}}/\mathfrak{n}\bar{\mathrm{A}} \to \bar{\mathrm{A}}/\mathfrak{n}\bar{\mathrm{A}} \quad \text{and} \quad \xi_a : \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}'\big) \to \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}'\big)$$

which we denote by the same symbol ξ_a with a slight abuse of notation. By [25] the operators $\xi_1, \dots, \xi_{m-1}$ on $\bar{\mathrm{A}}/\mathfrak{n}\bar{\mathrm{A}}$ and on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ satisfy the braid group relations

$$\xi_a\xi_{a+1}\xi_a = \xi_{a+1}\xi_a\xi_{a+1} \quad \text{for } a < m-1, \tag{4.3}$$

$$\xi_a\xi_b = \xi_b\xi_a \quad \text{for } |a-b| > 1. \tag{4.4}$$

Moreover, for any $H \in \mathfrak{h}$ and $x \in \bar{\mathrm{A}}$ we have

$$\xi_a(Hx) = (\sigma_a \circ H)\xi_a(x) \quad \text{and} \quad \xi_a(xH) = \xi_a(x)(\sigma_a \circ H)$$

where

$$\sigma_a \circ E_{bb} = E_{\sigma_a(b),\sigma_a(b)} - \delta_{ab} + \delta_{a+1,b} \tag{4.5}$$

defines the *shifted* action of the Weyl group $\mathfrak{S}_m$ on $\mathrm{U}(\mathfrak{h})$.

Another family of Zhelobenko operators on the double coset space $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ was also studied in [14]. For any $a = 1, \dots, m-1$ we can define a linear map $\eta_a : \mathrm{A} \to \bar{\mathrm{A}}/\bar{\mathrm{A}}\mathfrak{n}'$ by

$$\eta_a(x) = \sigma_a(x) + \sum_{r=1}^{\infty}(-1)^r \operatorname{ad}^r_{E_a}\big(\sigma_a(x)\big)F_a^r\big(r!H_a(H_a-1)\dots(H_a-r+1)\big)^{-1} \mod \bar{\mathrm{A}}\mathfrak{n}'.$$

It can be then extended to the operators

$$\eta_a : \bar{\mathrm{A}}/\bar{\mathrm{A}}\mathfrak{n}' \to \bar{\mathrm{A}}/\bar{\mathrm{A}}\mathfrak{n}' \quad \text{and} \quad \eta_a : \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}'\big) \to \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}'\big)$$

which we denote by the same symbol η_a by an abuse of notation. The operators $\eta_1, \dots, \eta_{m-1}$ satisfy the same braid group relations as $\xi_1, \dots, \xi_{m-1}$. Moreover, for any $H \in \mathfrak{h}$ and $x \in \bar{\mathrm{A}}$

$$\eta_a(Hx) = (\sigma_a \circ H)\eta_a(x) \quad \text{and} \quad \eta_a(xH) = \eta_a(x)(\sigma_a \circ H).$$

For any $a = 1, \dots, m-1$ we will now give a more explicit description of the operators ξ_a and η_a on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}}+\bar{\mathrm{A}}\mathfrak{n}')$. Let $\mathfrak{g}_a \subset \mathfrak{gl}_m$ be Lie subalgebra spanned by the three elements (4.2). Choose $j \in \{0, \frac{1}{2}, 1, \dots\}$ and take any $x \in \mathrm{V}$ from an irreducible $\mathfrak{g}_a$-submodule of V of dimension $2j+1$. Here we use the restriction of the adjoint action of $\mathfrak{gl}_m$ to the subspace $\mathrm{V} \subset \mathrm{A}$. Suppose that x is of the weight $2h$ relative to the action H_a:

$$[H_a, x] = 2hx \quad \text{where } h \in \{-j, -j+1, \dots, j-1, j\}.$$

Then by [11, Section 1.4] the double coset $\xi_a(x) \in \bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}}+\bar{\mathrm{A}}\mathfrak{n}')$ is that of the element of $\bar{\mathrm{A}}$

$$\begin{aligned}&\prod_{i=1}^{j-h}(H_a-i+1)^{-1} \cdot \sigma_a(x) \cdot \prod_{i=1}^{j-h}(H_a+i+1)\\ &\quad= \prod_{i=0}^{j+h}(H_a+i+1) \cdot \sigma_a(x) \cdot \prod_{i=0}^{j+h}(H_a-i+1)^{-1}\end{aligned}$$

while the double coset $\eta_a(x) \in \bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ is that of the element of $\bar{\mathrm{A}}$

$$\prod_{i=0}^{j-h}(H_a - i + 1)^{-1} \cdot \sigma_a(x) \cdot \prod_{i=0}^{j-h}(H_a + i + 1)$$
$$= \prod_{i=1}^{j+h}(H_a + i + 1) \cdot \sigma_a(x) \cdot \prod_{i=1}^{j+h}(H_a - i + 1)^{-1}.$$

As a corollary we get the relations in $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$

$$\xi_a(x)(H_a + 1) = (H_a + 1)\eta_a(x), \tag{4.6}$$

$$(\xi_a\eta_a)(x) = (\eta_a\xi_a)(x) = x \mod \big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}'\big). \tag{4.7}$$

4.5. Let $\mathcal{D}_{mn}$ be the ring of left differential operators on the Grassmann algebra $\mathcal{G}_{mn}$, see Subsection 2.1 for further notation. Choose the algebra A from the previous subsection as

$$\mathrm{A} = \mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{D}_{mn}.$$

The assignment $E_{ab} \mapsto D_{ab}$ defines a homomorphism $\varphi : \mathrm{U}(\mathfrak{gl}_m) \to \mathcal{D}_{mn}$. The algebra A contains a diagonally embedded subalgebra $\mathrm{U}(\mathfrak{gl}_m)$ generated by elements $X \otimes 1 + 1 \otimes \varphi(X)$ where $X \in \mathfrak{gl}_m$. The subspace $1 \otimes \mathcal{D}_{mn} \subset \mathrm{A}$ is invariant under the adjoint action of $\mathfrak{gl}_m$ and can be chosen as the subspace V from the previous subsection. In particular, the symmetric group $\mathfrak{S}_m \subset \mathrm{GL}_m(\mathbb{C})$ acts on $\mathcal{D}_{mn}$ by permuting the first indices of the anticommuting variables x_{ai} and the corresponding left derivations ∂_{ai}. For any $\lambda \in \mathfrak{h}^*$ let J_λ and $\bar{\mathrm{J}}_\lambda$ be the right ideals of respectively A and $\bar{\mathrm{A}}$ generated by all elements $E_{aa} - \lambda_a$ where $a = 1, \ldots, m$. Let I_μ and $\bar{\mathrm{I}}_\mu$ be the left ideals of respectively A and $\bar{\mathrm{A}}$ generated by all elements $\partial_{ai}, E_{aa} - \mu_a$ where $a = 1, \ldots, m$ and $i = 1, \ldots, n$.

Take any pair of weights $\lambda, \mu \in \mathfrak{h}^*$ such that all coordinates ν_a of the difference $\nu = \lambda - \mu$ are non-negative integers not exceeding n. Consider the quotient vector space

$$\mathrm{A}_\mu^\lambda = \mathrm{A}/\big(\mathfrak{n}\mathrm{A} + \mathrm{A}\mathfrak{n}' + \mathrm{J}_\lambda + \mathrm{I}_\mu\big).$$

One can define an isomorphism of vector spaces

$$A_\nu : \mathcal{G}_{mn}^\nu \to \mathrm{A}_\mu^\lambda$$

by mapping any $x \in \mathcal{G}_{mn}^\nu$ to the coset of $1 \otimes x \in \mathrm{A}$. The vector space A_μ^λ comes equipped with a natural structure of a $\mathrm{Y}(\mathfrak{gl}_n)$-module, see [10, Section 3]. Further, let $\rho \in \mathfrak{h}^*$ be the weight with coordinates $\rho_a = 1 - a$. Then by [10, Corollary 2.4] the composite map

$$A_\nu G_\nu : \Phi^{\lambda-\rho}_{\mu-\rho} \to \mathrm{A}^\lambda_\mu \tag{4.8}$$

is an isomorphism of $\mathrm{Y}(\mathfrak{gl}_n)$-modules, see also the beginning of Subsection 2.2 above.

In what follows we will also consider the shifted action of the Weyl group $\mathfrak{S}_m$ on $\mathfrak{h}^*$: using the weight ρ determined above, for any $\sigma \in \mathfrak{S}_m$ we have

$$\sigma \circ \lambda = \sigma(\lambda + \rho) - \rho.$$

Note that by regarding the elements of $\mathrm{U}(\mathfrak{h})$ as polynomial functions on $\mathfrak{h}^*$ we then recover the action of $\mathfrak{S}_m$ on $\mathrm{U}(\mathfrak{h})$ defined by (4.5).

If λ is *generic*, that is if $\lambda_a - \lambda_b \notin \mathbb{Z}$ for all $a \neq b$, then the weight μ is generic as well. Then the quotient vector space A^λ_μ can be identified with another quotient vector space

$$\bar{\mathrm{A}}^\lambda_\mu = \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}' + \bar{\mathrm{J}}_\lambda + \bar{\mathrm{I}}_\mu\big).$$

Namely, for each $x \in \mathcal{G}^\nu_{mn}$ one can then identify the cosets of $1 \otimes x$ in A^λ_μ and $\bar{\mathrm{A}}^\lambda_\mu$.

Proposition 4.2. *Suppose that the weight $\lambda \in \mathfrak{h}^*$ is generic. Then for any $a = 1, \ldots, m-1$:*

(i) the Zhelobenko operator ξ_a on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ induces a linear map

$$U_a : \mathrm{A}^\lambda_\mu \to \mathrm{A}^{\sigma_a \circ \lambda}_{\sigma_a \circ \mu}; \tag{4.9}$$

(ii) the latter map is $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining;
(iii) we have

$$A^{-1}_{\sigma_a(\nu)} U_a A_\nu = \sigma_a \cdot \sum_{r=0}^{\infty} \frac{(-1)^r D^r_{a+1,a} D^r_{a,a+1}}{r!(\lambda_a - \lambda_{a+1} + 2)_r}.$$

Proof. Since the weight λ is generic, we can identify the quotient vector spaces A^λ_μ and $\mathrm{A}^{\sigma_a\circ\lambda}_{\sigma_a\circ\mu}$ respectively with $\bar{\mathrm{A}}^\lambda_\mu$ and $\bar{\mathrm{A}}^{\sigma_a\circ\lambda}_{\sigma_a\circ\mu}$ as above. Using this identification, parts (i) and (ii) of the proposition have been proved in [10, Section 3]. Further, part (iii) is a particular case of [11, Proposition 3.5 and Corollary 3.6]. Namely see [11, Equation 3.12]. □

Let us now fix $\nu = \lambda - \mu$ and consider the map U_a as a function of the parameter $\lambda \in \mathfrak{h}^*$. It immediately follows from part (iii) of Proposition 4.2 that this function admits an analytical continuation from the generic λ to all weights λ such that

$$\lambda_a - \lambda_{a+1} + 1 \neq -1, -2, \ldots.$$

Using the isomorphisms (4.8) and then changing notation from $\lambda + \rho$ and $\mu + \rho$ to λ and μ respectively, we now obtain an intertwining operator of standard polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$I_a : \Phi^\lambda_\mu \to \Phi^{\sigma_a(\lambda)}_{\sigma_a(\mu)}$$

for any pair of weights $\lambda, \mu \in \mathfrak{h}^*$ such that all coordinates ν_a of the difference $\nu = \lambda - \mu$ are non-negative integers not exceeding n while

$$\lambda_a - \lambda_{a+1} \neq -1, -2, \ldots. \tag{4.10}$$

Moreover, by part (iii) of Proposition 4.2 we have an explicit formula for this operator:

$$I_a = G^{-1}_{\sigma_a(\nu)} \left(\sigma_a \cdot \sum_{r=0}^{\infty} \frac{(-1)^r D^r_{a+1,a} D^r_{a,a+1}}{r!(\lambda_a - \lambda_{a+1} + 1)_r} \right) G_\nu. \tag{4.11}$$

4.6. Proposition 4.2 and its implications as described above have their counterparts for the Zhelobenko operator η_a instead of ξ_a. We give them here. Take again any weights $\lambda, \mu \in \mathfrak{h}^*$ such that all coordinates ν_a of $\nu = \lambda - \mu$ are non-negative integers not exceeding n.

Proposition 4.3. *Suppose that the weight $\lambda \in \mathfrak{h}^*$ is generic. Then for any $a = 1, \ldots, m-1$:*

(i) the Zhelobenko operator η_a on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ induces a linear map

$$V_a : \mathrm{A}^\lambda_\mu \to \mathrm{A}^{\sigma_a \circ \lambda}_{\sigma_a \circ \mu};$$

(ii) the latter map is $\mathrm{Y}(\mathfrak{gl}_n)$-intertwining;
(iii) we have

$$A^{-1}_{\sigma_a(\nu)} V_a A_\nu = \sigma_a \cdot \sum_{r=0}^{\infty} \frac{(-1)^r D^r_{a,a+1} D^r_{a+1,a}}{r!(\mu_a - \mu_{a+1} + 2)_r}.$$

Proof. Firstly we make a general observation. In the setting of Subsection 4.4 suppose that the algebra A admits an involutive anti-automorphism such that its restriction to $\mathfrak{gl}_m \subset \mathrm{A}$ coincides with the matrix transposition $E_{ab} \mapsto E_{ba}$ for $a, b = 1, \ldots, m$. Then the Zhelobenko operators ξ_a and η_a on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ are conjugate to each other by the involutive linear operator on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ induced by this anti-automorphism.

Now choose $\mathrm{A} = \mathrm{U}(\mathfrak{gl}_m) \otimes \mathcal{D}_{mn}$ as in Subsection 4.5. Then the matrix transposition on $\mathfrak{gl}_m$ extends to an involutive anti-automorphism of A which maps $x_{ai} \mapsto \partial_{ai}$ for $a = 1, \ldots, m$ and $i = 1, \ldots, n$. Using our observation with this choice of the involutive anti-automorphism of A, part (i) follows from another property of ξ_a stated below. Let $\bar{\mathrm{J}}'_\lambda$ be the left ideal of $\bar{\mathrm{A}}$ generated by all elements $E_{aa} - \lambda_a$ where $a = 1, \ldots, m$. Let $\bar{\mathrm{I}}'_\mu$ be the right ideal of $\bar{\mathrm{A}}$ generated by all elements $x_{ai}, E_{aa} - \mu_a$ where $a = 1, \ldots, m$ and $i = 1, \ldots, n$. Consider the quotient vector space

$$\mathrm{B}^\lambda_\mu = \bar{\mathrm{A}}/\big(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}' + \bar{\mathrm{J}}'_\lambda + \bar{\mathrm{I}}'_\mu\big).$$

Then the Zhelobenko operator ξ_a on $\bar{\mathrm{A}}/(\mathfrak{n}\bar{\mathrm{A}} + \bar{\mathrm{A}}\mathfrak{n}')$ induces a linear map $\mathrm{B}_\mu^\lambda \to \mathrm{B}_{\sigma_a \circ \mu}^{\sigma_a \circ \lambda}$. This property of ξ_a can be proved in the same way as part (i) of Proposition 4.2.

Further, since the weight λ is generic we can use Proposition 4.2 and define the operator (4.9). It then readily follows from the relation (4.6) that

$$(\mu_a - \mu_{a+1} + 1)U_a = (\lambda_a - \lambda_{a+1} + 1)V_a. \tag{4.12}$$

In particular, part (ii) of Proposition 4.3 follows from part (ii) of Proposition 4.2. Part (iii) of Proposition 4.3 can be derived from the respective part of Proposition 4.2 by using the above chosen involutive anti-automorphism of the algebra A. □

Let us again fix $\nu = \lambda - \mu$ and consider the map V_a as a function of the parameter λ. It immediately follows from part (iii) of Proposition 4.3 that this function admits an analytical continuation from the generic λ to all weights λ such that

$$\mu_a - \mu_{a+1} + 1 \neq -1, -2, \ldots.$$

Using the isomorphisms (4.8) and then changing notation from $\lambda + \rho$ and $\mu + \rho$ to λ and μ respectively, the V_a yields an intertwining operator of standard polynomial $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$J_a : \Phi_\mu^\lambda \to \Phi_{\sigma_a(\mu)}^{\sigma_a(\lambda)}$$

for any pair of weights $\lambda, \mu \in \mathfrak{h}^*$ such that all coordinates ν_a of the difference $\nu = \lambda - \mu$ are non-negative integers not exceeding n while

$$\mu_a - \mu_{a+1} \neq -1, -2, \ldots. \tag{4.13}$$

Moreover, by part (iii) of Proposition 4.3 we have an explicit formula for this map:

$$J_a = G_{\sigma_a(\nu)}^{-1} \left(\sigma_a \cdot \sum_{r=0}^{\infty} \frac{(-1)^r D_{a,a+1}^r D_{a+1,a}^r}{r!(\mu_a - \mu_{a+1} + 1)_r} \right) G_\nu.$$

Let us compare J_a with the intertwining operator I_a defined from the previous subsection. Suppose that the weights λ and μ satisfy the conditions (4.10) and (4.13) respectively, so that both operators I_a and J_a are defined. It then immediately follows from (4.12) that

$$(\mu_a - \mu_{a+1})I_a = (\lambda_a - \lambda_{a+1})J_a. \tag{4.14}$$

Here we have taken into account the change from $\lambda + \rho$ and $\mu + \rho$ to λ and μ. The relation (4.14) shows that the operator I_a vanishes on the hyperplane $\lambda_a = \lambda_{a+1}$ in $\mathfrak{h}^*$ while the operator J_a vanishes on the hyperplane $\mu_a = \mu_{a+1}$. Thus the operator J_a can be regarded as a renormalization of the operator I_a that is also regular on the hyperplane $\lambda_a = \lambda_{a+1}$.

Now suppose that the weight λ still satisfies the condition (4.10) while the weight μ obeys

$$\mu_a - \mu_{a+1} \neq 1, 2, \ldots \tag{4.15}$$

instead of (4.13). Then by using Proposition 4.3 and the subsequent argument we can define instead of J_a an intertwining operator $\Phi^{\sigma_a(\lambda)}_{\sigma_a(\mu)} \to \Phi^{\lambda}_{\mu}$. Denote it by J'_a. The relation (4.7) now implies the equalities

$$J'_a I_a = \mathrm{Id} \quad \text{and} \quad I_a J'_a = \mathrm{Id} \tag{4.16}$$

on Φ^{λ}_{μ} and $\Phi^{\sigma_a(\lambda)}_{\sigma_a(\mu)}$ respectively. In particular, under (4.15) the operator I_a is invertible.

4.7. In this subsection we will complete the proof of Proposition 2.1. Let again $\lambda, \mu \in \mathfrak{h}^*$ be any weights satisfying all conditions of that proposition. Choose any reduced decomposition $\sigma_0 = \sigma_{a_1} \ldots \sigma_{a_\ell}$ of the longest element of $\mathfrak{S}_m$. Here $\ell = m(m-1)/2$. Using Proposition 4.2 and the subsequent argument we can define for each $s = 1, \ldots, \ell$ an intertwining operator

$$I_{a_s} : \Phi^{\sigma(\lambda)}_{\sigma(\mu)} \to \Phi^{\sigma_{a_s}\sigma(\lambda)}_{\sigma_{a_s}\sigma(\mu)} \quad \text{where } \sigma = \sigma_{a_{s+1}} \ldots \sigma_{a_\ell}. \tag{4.17}$$

Let $I = I_{a_1} \ldots I_{a_\ell}$ be the composition of these operators. We get an intertwining operator

$$I : \Phi^{\lambda}_{\mu} \to \Phi^{\sigma_0(\lambda)}_{\sigma_0(\mu)}$$

which does not depend on the choice of reduced decomposition of σ_0 due to the braid group relations (4.3), (4.4). Moreover by using (4.11) repeatedly, we get an explicit formula for the latter operator: in the notation of Subsections 2.1 and 2.2

$$I = G^{-1}_{\sigma_0(\nu)} \sigma_0 Z G_\nu$$

where

$$Z = \prod_{1 \leqslant a < b \leqslant m}^{\longrightarrow} X^{\lambda}_{ab} \tag{4.18}$$

and the factors X^{λ}_{ab} are ordered so that for every $s = 1, \ldots, \ell$ the sth factor from the left has the indices $a = \sigma^{-1}(a_s)$ and $b = \sigma^{-1}(a_s + 1)$. Here we use the permutation σ from (4.17).

Now take the vector (2.8). Due to [10, Proposition 3.7] the vector

$$I\left(v_\mu^\lambda\right) \in \Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)}$$

is equal to the vector (2.13) multiplied by $(-1)^N$ in the notation (2.4), and by the product

$$\prod_{\substack{1 \leqslant a<b \leqslant m \\ \nu_a<\nu_b}} \frac{\lambda_a-\lambda_b}{\mu_a-\mu_b}.$$

Let us now replace every factor X_{ab}^λ with $\nu_a<\nu_b$ in (4.18) by the corresponding factor Y_{ab}^μ. Here $\mu_a-\mu_b>\lambda_a-\lambda_b$ so that $\mu_a-\mu_b \neq 0,-1,-2, \ldots$ and the operator Y_{ab}^μ is defined.

These replacements change the product (4.18) to (2.5). On the other hand, due to (4.14) for each such replacement

$$(\mu_a-\mu_b) X_{ab}^\lambda=(\lambda_a-\lambda_b) Y_{ab}^\mu .$$

It now follows that the operator I_μ^λ defined by (2.3) satisfies (2.12). This argument completes the proof of parts (i)–(iii) of Proposition 2.1. Further, due to (2.12) our operator I_μ^λ coincides with the intertwining operator $I(\mu-\rho)$ in the notation of [12]. Therefore the last part (iv) of our Proposition 2.1 follows directly from [12, Theorem 1.1 and Proposition 2.9].

Remark. Due to (4.14) any factor X_{ab}^λ with $\nu_a=\nu_b$ equals Y_{ab}^μ. Hence the strict inequality in the second line of the definition (2.5) can be replaced by a non-strict one. □

Remark. The proof of [12, Proposition 2.9] has been based on the equalities (4.16) which hold under the conditions (4.10) and (4.15). In the present paper we gave an independent proof of these two equalities, by using the properties of Zhelobenko operators. □

4.8. In this subsection we will prove Theorem 3.1. Suppose the weights λ and $\nu=\lambda-\mu$ satisfy all conditions of the theorem. Determine the sequence of signs ε together with the weights $\bar{\lambda}$ and $\bar{\nu}$ as in Subsection 3.1. Then we can define an isomorphism of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$J_\mu^\lambda: \Phi_\mu^\lambda \rightarrow \Phi_\mu^{\bar{\lambda}} \otimes\Big(\bigotimes_{a: \nu_a<0} \Delta_{\mu_a}^{\prime}\Big) \tag{4.19}$$

as follows. Consider the bijective linear maps

$$G_{|\nu|}: \Phi_\mu^\lambda \rightarrow \mathcal{G}_{mn}^{|\nu|} \quad \text{and} \quad G_{\bar{\nu}}: \Phi_\mu^{\bar{\lambda}} \rightarrow \mathcal{G}_{mn}^{\bar{\nu}} .$$

Using the notation (3.1) let S_ε be the involutive automorphism of the algebra $\mathcal{D}_{mn}$ such that

$$S_\varepsilon(x_{ai}) = q_{ai} \quad \text{and} \quad S_\varepsilon(\partial_{ai}) = p_{ai} \quad \text{for } a = 1, \dots, m \text{ and } i = 1, \dots, n.$$

For every $x \in \mathcal{G}_{mn}$ put

$$R_\varepsilon(x) = S_\varepsilon(x) \prod_{a:\nu_a<0}^{\longrightarrow} (x_{a1} \dots x_{an}) \tag{4.20}$$

where the factors corresponding to the indices a with $\nu_a < 0$ are ordered from left to right as the indices increase. Then

$$R_\varepsilon : \mathcal{G}_{mn}^{|\nu|} \to \mathcal{G}_{mn}^{\bar\nu}$$

and moreover by (4.1) for any operator $Y \in \mathcal{D}_{mn}$ we have the relation

$$R_\varepsilon\big(Y(x)\big) = S_\varepsilon(Y)\big(R_\varepsilon(x)\big). \tag{4.21}$$

Subsection 4.1 shows that the isomorphism (4.19) can be defined by mapping any $w \in \Phi_\mu^\lambda$ to

$$G_{\bar\nu}^{-1}\big(R_\varepsilon\big(G_{|\nu|}(w)\big)\big) \otimes \Big(\bigotimes_{a:\nu_a<0} v_n\Big)$$

where according to (2.7)

$$v_n = e_1 \wedge \dots \wedge e_n \in \Delta'_{\mu_a}.$$

Similarly, we can define an isomorphism of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$J_{\sigma_0(\mu)}^{\sigma_0(\lambda)} : \Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)} \to \Phi_{\sigma_0(\mu)}^{\sigma_0(\bar\lambda)} \otimes \Big(\bigotimes_{a:\nu_a<0} \Delta'_{\mu_a}\Big).$$

Note that in the latter definition the order of the tensor factors Δ'_{μ_a} is chosen to be the same as in (4.19) by using Proposition 1.1(iv). But the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\Phi_\mu^{\bar\lambda}$ is polynomial while the weights $\bar\lambda$ and $\bar\nu$ satisfy the conditions of Proposition 2.1. We get an intertwining operator

$$I_\mu^{\bar\lambda} : \Phi_\mu^{\bar\lambda} \to \Phi_{\sigma_0(\mu)}^{\sigma_0(\bar\lambda)}.$$

Take the composition

$$\big(J_\mu^{\bar\lambda}\big)^{-1}\big(I_\mu^{\bar\lambda} \otimes \mathrm{Id}\big) J_\mu^\lambda : \Phi_\mu^\lambda \to \Phi_{\sigma_0(\mu)}^{\sigma_0(\lambda)}. \tag{4.22}$$

By Proposition 2.1 the composite operator has the properties (i)–(iv) from Theorem 3.1. Put

$$K = \sum_{a<b:\nu_a<0} \nu_a \nu_b \quad \text{and} \quad L = \sum_{a>b:\nu_a<0} \nu_a \nu_b .$$

Then by the definitions (2.4), (3.3) and (4.20) we have the operator relation

$$R_{\sigma_0(\varepsilon)} \sigma_0 = (-1)^{K+L+N+\bar{N}} \sigma_0 R_\varepsilon .$$

By using the definitions of J^λ_μ and $J^{\bar{\lambda}}_\mu$ along this relation, the operator (4.22) equals

$$\begin{aligned}
&(-1)^N \big(G^{-1}_{\sigma_0(\bar{\nu})} R_{\sigma_0(\varepsilon)} G_{\sigma_0|\nu|}\big)^{-1} \big(G^{-1}_{\sigma_0(\bar{\nu})} \sigma_0 Z^{\bar{\lambda}}_\mu G_{\bar{\nu}}\big) \big(G^{-1}_{\bar{\nu}} R_\varepsilon G_{|\nu|}\big) \\
&\quad = (-1)^{K+L+\bar{N}} G^{-1}_{\sigma_0|\nu|} \sigma_0 R^{-1}_\varepsilon Z^{\bar{\lambda}}_\mu R_\varepsilon G_{|\nu|} \\
&\quad = (-1)^{K+L+\bar{N}} G^{-1}_{\sigma_0|\nu|} \sigma_0 S_\varepsilon \big(Z^{\bar{\lambda}}_\mu\big) G_{|\nu|} = (-1)^{K+L} I^\lambda_\mu ;
\end{aligned}$$

see also the definitions (2.3), (3.4) and the relation (4.21). Thus we have proved Theorem 3.1.

Furthermore, put

$$M = \sum_{a:\nu_a<0} \nu_a(\nu_a - 1)/2 .$$

Then by the definition (4.20) we have the equality

$$J^\lambda_\mu \big(v^\lambda_\mu\big) = (-1)^{K+M} v^{\bar{\lambda}}_\mu \otimes \Big(\bigotimes_{a:\nu_a<0} v_n \Big).$$

Similarly, we have

$$J^{\sigma_0(\lambda)}_{\sigma_0(\mu)} \big(v^{\sigma_0(\lambda)}_{\sigma_0(\mu)}\big) = (-1)^{L+M} v^{\sigma_0(\bar{\lambda})}_{\sigma_0(\mu)} \otimes \Big(\bigotimes_{a:\nu_a<0} v_n \Big).$$

We also have

$$I^{\bar{\lambda}}_\mu \big(v^{\bar{\lambda}}_\mu\big) = v^{\sigma_0(\bar{\lambda})}_{\sigma_0(\mu)} .$$

The last three displayed equalities imply that the operator I^λ_μ also has the property (2.12).

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