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Koszulity for Nonquadratic Algebras

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It is known that a Koszul algebra is defined as being a quadratic algebra with a "pure" resolution of the ground field. In this paper, we extend Koszulity to algebras whose relations are homogeneous of degree $s > 2$. A cubic Artin–Schelter regular algebra has motivated our work. Generalized Koszulity is connected to lattice distributivity and to confluence. A generalized symmetric algebra is proved to be generalized Koszul, and the bimodule version of the generalized Koszul resolution is used for investigating its Hochschild homology.

1. INTRODUCTION

A natural first step for making non-commutative algebraic geometry is to define non-commutative analogs of polynomial algebras. Such analogs are the (Artin–Schelter) regular algebras [2]. Those algebras are defined as graded algebras of polynomial growth with some good homological properties. Let $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded k-algebra which is regular of global dimension 3 and generated in degree 1. It is shown in [2] that the minimal projective resolution of the trivial left A-module k in the category of graded left A-modules has the form

$$0 \to A(-s-1) \xrightarrow{X^t} A(-s)^r \xrightarrow{M} A(-1)^r \xrightarrow{X} A(0) \to k \to 0, \quad (1.1)$$

where $(r, s) = (3, 2)$ or $(2, 3)$. Moreover, A has r generators of degree 1 and r relations of degree s. The notation $A(l)$ means that the gradation of A is shifted by the integer l, i.e., $A(l)_n = A_{n+l}$. For example, $A(0)$ is the module A (generated in degree 0), $A(-1)^r$ is the module A^r generated in

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degree 1, $A(-s)^r$ is A^r generated in degree s, and $A(-s-1)$ is A generated in degree $s+1$. Then we can read in the projective resolution (1.1) the two following characteristic properties of generalized Koszulity:

(i) Each module is free and generated in only one degree ("purity").

(ii) The successive degrees are $0, 1, s, s+1, 2s, 2s+1, 3s, 3s+1, \ldots$

In the quadratic case $(r, s) = (3, 2)$, we find in (ii) the sequence of the natural numbers as expected for usual Koszulity [6]. An explicit example in the cubic case $(r, s) = (2, 3)$ is the regular algebra (so-called of type A) whose two relations in the generators x, y are

$$\begin{aligned} ay^2x + byxy + axy^2 + cx^3 &= 0 \\ cy^3 + ayx^2 + bxyx + ax^2y &= 0. \end{aligned} \tag{1.2}$$

The parameters a, b, c are elements of $k \supset \mathbf{Q}$ which are algebraically independent over $\mathbf{Q}$. For this example, (1.1) is defined by the matrices

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad M = \begin{pmatrix} ay^2 + cx^2 & byx + axy \\ ayx + bxy & cy^2 + ax^2 \end{pmatrix}.$$

The definition of generalized Koszulity for algebras A with relations homogeneous of degree $s \geq 2$ is introduced in Section 2. The sequence of degrees in (ii) is obtained inductively by imposing condition (i) in lowest degree at each step. But instead of searching minimal resolutions in matrix form as (1.1), we prefer to construct a "canonical" minimal projective resolution of k (still in the graded category). In fact, for any A, one has a minimal projective resolution of k which starts as (tensor products are over k)

$$A \otimes R \to A \otimes V \to A \to k \to 0, \tag{1.3}$$

where $V = A_1$ and $R \subseteq V^{\otimes s}$ is the vector space of relations of A. The graded modules $A \otimes V$ and $A \otimes R$ are generated in degrees 1 and s, respectively. The differential is defined by natural inclusions $V \to A$ and $R \to A_{s-1} \otimes V$. The algebra A is said to be (generalized) Koszul if (1.3) has a "natural" prolongation satisfying (i) and (ii). The modules K^i of the resolution thus obtained are of the form $K^i = A \otimes E$, where E is concentrated in an adequate degree, and the differential is defined by inclusion of E in K^{i-1}.

J. Backelin has shown that (usual) Koszulity is equivalent to distributivity of lattices $\mathcal{T}_n$, $n \geq 4$ [4] (see also [5]). Here a lattice is a partially ordered set in which each pair has an infimum and a supremum. In fact, the lattice $\mathcal{T}_n$ defined by a quadratic algebra A is a set of vector subspaces (ordered by inclusion) of the tensor power $V^{\otimes n}$ and hence is modular (distributivity is stronger than modularity). We show in Section 2 that generalized Koszulity is equivalent to a sequence of distributivity relations plus an extra condition if $s > 2$. The extra condition is not reducible to a distributivity relation. For example, if $s = 3$, this condition is

$$(V^{\otimes 2} \otimes R) \cap (R \otimes V^{\otimes 2} + V \otimes R \otimes V) = (V^{\otimes 2} \otimes R) \cap (V \otimes R \otimes V). \tag{1.4}$$

It is an easy exercise to show that (1.4) is equivalent to the distributivity of the triple $(V^{\otimes 2} \otimes R, V \otimes R \otimes V, R \otimes V^{\otimes 2})$ and to the inclusion

$$(V^{\otimes 2} \otimes R) \cap (R \otimes V^{\otimes 2}) \subseteq V \otimes R \otimes V. \tag{1.5}$$

If A is the cubic algebra (1.2), one can check that $(V^{\otimes 2} \otimes R) \cap (R \otimes V^{\otimes 2}) = 0$; hence (1.5) is trivially satisfied in this case. It is easy to find non-trivial inclusions (1.5). For example, if R is generated by monomials, (1.5) is a rather large combinatorial condition, which is providing many generalized Koszul algebras (all distributivity relations hold if R is generated by monomials).

Since there are infinitely many lattices, no algorithm is available for deciding distributivity from the data of generators and relations. However, as in the usual situation, $s = 2$ (see [7]), there exists an algorithm for deciding confluence of the algebra A. Confluence for associative algebras was first defined by G. M. Bergman [9]. In our framework, confluence implies distributivity and includes the monomial case. Some interesting non-monomial algebras are proved to be generalized Koszul by use of confluence. It is the case (in characteristic zero) of the algebra whose relations are the antisymmetrizers of degree s (if $s = 2$, this is the symmetric algebra). Confluence, along with the latter example, is discussed in Section 3. We also investigate in Section 3 the problem of expressing for $s > 2$ the distributivity relations in a nice way, as in Backelin's theorem. Only a partial answer is given (Theorem 3.6).

It is well known that the Koszul resolution allows us to get the Hochschild homology of a polynomial algebra [20]. This procedure has been extended to the symmetric algebra associated to a solution of the Yang–Baxter equation by M. Wambst [19] and to any (usual) Koszul algebra by M. Van den Bergh [18]. For any $s \geq 2$, a generalized Koszul complex of bimodules is introduced in Section 5. If A is generalized Koszul, the generalized bimodule Koszul resolution of A provides a complex computing the Hochschild homology of A in the same spirit as in [18]. A part of the computation is carried out (in characteristic zero) for the algebra whose relations are the antisymmetrizers of degree s.

2. PURE RESOLUTIONS OF THE GROUND FIELD

Throughout this paper, k is a field and V is a finite-dimensional vector space over k. The tensor power $V^{\otimes n}$, $n \geq 0$, will be more conveniently denoted by $V^{(n)}$. We fix a natural number $s \geq 2$ and a subspace R of $V^{(s)}$. The two-sided ideal $I = I(R)$ generated by R in the tensor algebra $\mathrm{Tens}(V)$ is graded by the subspaces I_n given by $I_n = 0$, $0 \leq n \leq s-1$, and

$$I_n = \sum_{i+j+s=n} V^{(i)} \otimes R \otimes V^{(j)}, \qquad n \geq s. \tag{2.1}$$

The algebra $A = \mathrm{Tens}(V)/I$ is called an s-homogeneous algebra on V, with R as space of relations. The algebra A is graded by the subspaces $A_n = V^{(n)}/I_n$ and generated by V (hence in degree 1). Clearly, $A_n = V^{(n)}$ for $0 \leq n \leq s-1$.

We begin to collect some known facts dealing with the graded category which can be found in [11], or more abstractly in [12]. As noted in [3], the results of [11] concerning positively graded modules can obviously be extended to left-bounded graded modules. In this section and the two following, a module M is a graded left A-module which is supposed left bounded; i.e., for some n_0, we have $M_n = 0$, $n < n_0$. The modules M form an abelian category called the (left-bounded) graded category of A. The morphisms of this category are the degree-preserving homomorphisms. Following the terminology of Eilenberg [12], the graded category of A is perfect and satisfies Axiom 6. The shift $M(l)$ of M is the module M graded by $M(l)_n = M_{n+l}$. A module M is said to be graded-free if M has a basis formed of homogeneous elements or, equivalently, if M is isomorphic to a direct sum of shifts $A(-l_i)$ of A, where the various degrees l_i (with possible repetitions) are bounded below. The first fact is

PROPOSITION 2.1. *A module M is projective in the graded category if and only if M is graded-free.*

A surjective morphism $f\colon N \to M$ is called essential if, for any morphism $g\colon X \to N$ with $f \circ g$ surjective, g is surjective. An essential surjective morphism $f\colon N \to M$ with N projective is called a projective cover of M in the graded category. The second fact is

PROPOSITION 2.2. *Any module M has a projective cover in the graded category, unique up to a non-unique isomorphism.*

An immediate consequence of the latter result is that any M has a projective resolution (in the graded category)

$$\cdots \to P^i \xrightarrow{d_i} P^{i-1} \to \cdots \to P^1 \xrightarrow{d_1} P^0 \xrightarrow{d_0} M \to 0, \tag{2.2}$$

which is minimal, that is, each surjective morphism $P^i \to \mathrm{im}(d_i)$ induced by d_i is essential. Any minimal projective resolution of M is isomorphic to (2.2) (but the isomorphism is non-unique), and any projective resolution of M contains a minimal one as direct summand.

In the following definition, the pure modules are more general than those of [6].

DEFINITION 2.3. A module M such that there exists l satisfying $M = M_l$ (respectively, $M = A.M_l$) is said to be *concentrated* (respectively, *pure*) in degree l.

In both cases, l is uniquely determined if $M \neq 0$. Concentrated in degree l implies pure in degree l. Any module concentrated in degree l is isomorphic to a direct sum of shifts $k(-l)$. Any projective module M pure in degree l is isomorphic to a direct sum of shifts $A(-l)$ and is isomorphic to $A \otimes_k M_l$, where M_l is considered as a module concentrated in degree l. Note that the objects which are simple (respectively, indecomposable projective) in the graded category are exactly the modules $k(-l)$ (respectively, $A(-l)$), $l \in \mathbf{Z}$. The next result is a criterion of essentiality in the pure situation.

PROPOSITION 2.4. *Let f: $M \to M'$ be a surjective morphism in the graded category. Assume that M is pure in degree l. Then M' is pure in degree l. Moreover, f is essential if and only if the linear map f_l: $M_l \to M'_l$ induced by f is bijective.*

Proof. The first assertion is clear. Let us prove the second assertion. The Nakayama lemma in the graded category asserts that f is essential if and only if the linear map $\bar{f}$: $k \otimes_A M \to k \otimes_A M'$ naturally defined by f is bijective (see Proposition 7 in [11]). But purity implies that $k \otimes_A M$ and $k \otimes_A M'$ are canonically identified to M_l and M'_l respectively, so that $\bar{f}$ becomes f_l in this identification. ∎

Now we want to know when it is possible to construct a pure resolution of k. The natural projection ϵ: $A \to k$ is a projective cover of k, and its kernel is pure in degree 1 with $(\ker \epsilon)_1 = V$. A projective cover of $\ker \epsilon$ is the morphism $A \otimes V \to \ker \epsilon$ induced by the inclusion of V in $\ker \epsilon$. Including $\ker \epsilon$ in A, we get δ_1: $A \otimes V \to A$ defined by $\bar{a} \otimes v \mapsto \overline{av}$, where $\bar{a}$ denotes the class in A of an element a of $\mathrm{Tens}(V)$ and $v \in V$. Clearly, $\ker \delta_1$ vanishes in degree $< s$ and is R in degree s. For $n \geq s$,

$$(\ker \delta_1)_n = \frac{I_{n-1} \otimes V + V^{(n-s)} \otimes R}{I_{n-1} \otimes V} \tag{2.3}$$

shows that $\ker \delta_1$ is pure in degree s. A projective cover of $\ker \delta_1$ is $A \otimes R \to \ker \delta_1$ induced by the inclusion of R in $\ker \delta_1$. Including $\ker \delta_1$

in $A \otimes V$, we get δ_2: $A \otimes R \to A \otimes V$ defined by the restriction of the linear map $\bar{a} \otimes v \otimes w \mapsto \overline{av} \otimes w$, where $v \in V^{(s-1)}$, $w \in V$.

Thus there is no obstruction to beginning a pure resolution as

$$A \otimes J_s \xrightarrow{\delta_2} A \otimes J_1 \xrightarrow{\delta_1} A \otimes J_0 \xrightarrow{\epsilon} k \to 0, \tag{2.4}$$

where $J_0 = k$, $J_1 = V$, $J_s = R$ are modules concentrated in respective degrees $0, 1, s$. For any $n \geq s$, define the subspace J_n (concentrated in degree n) of $V^{(n)}$ as being

$$J_n = \bigcap_{i+j+s=n} V^{(i)} \otimes R \otimes V^{(j)}. \tag{2.5}$$

In the next steps described below, some obstructions to the construction of a pure resolution of k occur. If the obstructions are removed, the successive concentrated modules will be $J_{s+1} = (V \otimes R) \cap (R \otimes V)$, followed by $J_{2s}, J_{2s+1}, J_{3s}, J_{3s+1}$, and so on. The spaces J_n are directly linked to the dual algebra of A (Section 4). Thus the generalized Koszul resolution will be again an emanation of the dual algebra of A, as for $s = 2$.

Clearly, $\ker \delta_2$ vanishes in degree $\leq s$ and is J_{s+1} in degree $s+1$. So the projective cover of $\ker \delta_2$ contains $A \otimes J_{s+1}$ but may also contain pure modules in degree $> s+1$ as direct summands. For $n = s + m$, $2 \leq m \leq s-1$, we have

$$(\ker \delta_2)_n = (V^{(m)} \otimes R) \cap (R \otimes V^{(m)} + \cdots + V^{(m-1)} \otimes R \otimes V), \tag{2.6}$$

which obviously contains $V^{(m-1)} \otimes J_{s+1}$. Thus $(\ker \delta_2)_{s+m} = A_{m-1}.J_{s+1}$ is equivalent to the relation

$$(V^{(m)} \otimes R) \cap (R \otimes V^{(m)} + \cdots + V^{(m-1)} \otimes R \otimes V) = V^{(m-1)} \otimes J_{s+1}. \tag{2.7}$$

Note that (2.7) for $m = s-1$ implies

$$\begin{aligned}(V^{(s-1)} \otimes R) \cap (V^{(s-1-m)} \otimes R \otimes V^{(m)} + \cdots + V^{(s-2)} \otimes R \otimes V)\\ \subseteq V^{(s-2)} \otimes J_{s+1},\end{aligned} \tag{2.8}$$

for $2 \leq m \leq s-2$, and hence implies the other relations (2.7) after simplifying by $V^{(s-1-m)}$ on the left.

Relation (2.7) for $m = s-1$ is called the extra condition (void for $s = 2$) and is denoted by (ec). Clearly, (ec) is equivalent to

$$\begin{aligned}(V^{(s-1)} \otimes R) \cap (R \otimes V^{(s-1)} + \cdots + V^{(s-2)} \otimes R \otimes V)\\ \subseteq V^{(s-2)} \otimes R \otimes V.\end{aligned} \tag{2.9}$$

Recall that a triple (E, F, G) of subspaces of a vector space is said to be distributive if

$$E \cap (F + G) = (E \cap F) + (E \cap G). \tag{2.10}$$

In fact, the inclusion $\subseteq$ suffices in (2.10). Three distinct one-dimensional subspaces in a two-dimensional space do not satisfy (2.10), but (2.10) holds for any subspaces such that E contains F (the equality (2.10) is then called the modular equality).

PROPOSITION 2.5. *(ec) holds if and only if for $m = 2, \dots, s-1$, the triple*

$$(V^{(m)} \otimes R, R \otimes V^{(m)}, V \otimes R \otimes V^{(m-1)} + \cdots + V^{(m-1)} \otimes R \otimes V) \tag{2.11}$$

is distributive and we have the inclusion

$$(V^{(m)} \otimes R) \cap (R \otimes V^{(m)}) \subseteq V^{(m-1)} \otimes R \otimes V. \tag{2.12}$$

Proof. Assume (ec) holds. Relation (2.7) for $m = 2, \dots, s-1$ shows the distributivity of (2.11) and the inclusion (2.12). Conversely, assuming the distributivity of (2.11) and the inclusion (2.12) for $m = 2, \dots, s-1$, we see that $(V^{(s-1)} \otimes R) \cap (R \otimes V^{(s-1)} + \cdots + V^{(s-2)} \otimes R \otimes V)$ is successively contained in the subspaces

$$\begin{aligned}
&V^{(s-2)} \otimes R \otimes V \\
&\quad + V \otimes \big((V^{(s-2)} \otimes R) \cap (R \otimes V^{(s-2)} + \cdots + V^{(s-3)} \otimes R \otimes V)\big), \\
&\quad\; V^{(s-2)} \otimes R \otimes V \\
&\quad + V^{(2)} \otimes \big((V^{(s-3)} \otimes R) \cap (R \otimes V^{(s-3)} + \cdots + V^{(s-4)} \otimes R \otimes V)\big), \\
&\quad\; \dots, V^{(s-2)} \otimes R \otimes V \\
&\quad + V^{(s-3)} \otimes \big((V^{(2)} \otimes R) \cap (R \otimes V^{(2)} + V \otimes R \otimes V)\big),
\end{aligned}$$

and the latter is contained in $V^{(s-2)} \otimes R \otimes V$. ∎

It will be useful to express the inclusions (2.12) in a more symmetric way, as in the following lemma (the proof of which is left to the reader).

LEMMA 2.6. *For any $m = 2, \dots, s-1$, $(V^{(m)} \otimes R) \cap (R \otimes V^{(m)})$ is contained in $V^{(m-1)} \otimes R \otimes V$ if and only if for any $m = 2, \dots, s-1$, we have*

$$(V^{(m)} \otimes R) \cap (R \otimes V^{(m)}) = J_{s+m}.$$

PROPOSITION 2.7. *For $\ker \delta_2$ to be pure in degree $s+1$, it is necessary and sufficient that (ec) holds and that for any $n \geq 2s$, the triple (E, F, G)*

where

$$\begin{aligned} &E = V^{(n-s)} \otimes R, \qquad F = R \otimes V^{(n-s)} + \cdots + V^{(n-2s)} \otimes R \otimes V^{(s)}, \\ &G = V^{(n-2s+1)} \otimes R \otimes V^{(s-1)} + \cdots + V^{(n-s-1)} \otimes R \otimes V \end{aligned} \tag{2.13}$$

is distributive.

Proof. We have already proved that $(\ker \delta_2)_n = A_{n-s-1}.J_{s+1}$ for $s+1 \leq n \leq 2s-1$ if and only if (ec) holds. In particular, for proving the remainder, we can assume that (ec) is true. Fix $n \geq 2s$. The subspace $(\ker \delta_2)_n$ of $A_{n-s} \otimes R$ can be viewed as a quotient whose denominator is $I_{n-s} \otimes R$ and numerator N is

$$N = E \cap (F + G). \tag{2.14}$$

If (E, F, G) is distributive and since (ec) is true, one can write down

$$N = I_{n-s} \otimes R + V^{(n-s-1)} \otimes J_{s+1}, \tag{2.15}$$

so that $(\ker \delta_2)_n = A_{n-s-1}.J_{s+1}$. Conversely, assuming the latter equality, we deduce (2.15). Therefore, N is contained in $E \cap F + V^{(n-s-1)} \otimes J_{s+1}$. But using (ec), we can set $E \cap G$ equal to $V^{(n-s-1)} \otimes J_{s+1}$. Thus (E, F, G) is distributive. ∎

From now on, we assume that $\ker \delta_2$ is pure in degree $s+1$. A projective cover of $\ker \delta_2$ is $A \otimes J_{s+1} \to \ker \delta_2$ induced by the inclusion of J_{s+1} in $\ker \delta_2$. Including $\ker \delta_2$ in $A \otimes R$, we get $\delta_3 \colon A \otimes J_{s+1} \to A \otimes R$ defined by restriction of the linear map $\bar{a} \otimes v \otimes w \to \overline{av} \otimes w$, where $v \in V$, $w \in V^{(s)}$. Let us examine $\ker \delta_3$. It is obvious that $\ker \delta_3$ vanishes in degree $< 2s$ and that

$$(\ker \delta_3)_{2s} = \left(V^{(s-1)} \otimes J_{s+1}\right) \cap \left(R \otimes V^{(s)}\right). \tag{2.16}$$

But Lemma (2.6) implies that $(V^{(s-1)} \otimes R) \cap (R \otimes V^{(s-1)}) = J_{2s-1}$, so that $(\ker \delta_3)_{2s} = J_{2s}$.

PROPOSITION 2.8. *Assume* $\ker \delta_2$ *pure in degree* $s+1$. *For* $\ker \delta_3$ *to be pure in degree* $2s$, *it is necessary and sufficient that for any* $n \geq 2s+1$, *the triple* (E, F, G) *where*

$$\begin{gathered} E = V^{(n-s-1)} \otimes J_{s+1}, \qquad F = I_{n-s-1} \otimes V^{(s+1)}, \\ G = V^{(n-2s)} \otimes R \otimes V^{(s)} \end{gathered} \tag{2.17}$$

is distributive.

Proof. For each $n \geq 2s+1$, we can consider $(\ker \delta_3)_n$ as a quotient whose denominator is $I_{n-s-1} \otimes J_{s+1}$ and numerator is

$$N = E \cap \left(I_{n-s-1} \otimes V \otimes R + V^{(n-2s)} \otimes R \otimes R\right). \tag{2.18}$$

In fact, N coincides with $E \cap (F+G)$ since E is included in $V^{(n-s)} \otimes R$. Thus (E, F, G) is distributive if and only if

$$N = I_{n-s-1} \otimes J_{s+1} + V^{(n-2s)} \otimes J_{2s}, \tag{2.19}$$

which is equivalent to $(\ker \delta_3)_n = A_{n-2s}.J_{2s}$. ∎

We detail again the next step to convince the reader that no extra condition other than (ec) is necessary. Assume $\ker \delta_3$ pure in degree $2s$. From the projective cover of $\ker \delta_3$ and the inclusion of $\ker \delta_3$ in $A \otimes J_{s+1}$, we get δ_4: $A \otimes J_{2s} \to A \otimes J_{s+1}$ defined by restriction of the linear map $\bar{a} \otimes v \otimes w \mapsto \overline{av} \otimes w$, where $v \in V^{(s-1)}$, $w \in V^{(s+1)}$. Clearly, $\ker \delta_4$ vanishes in degree $\leq 2s$ and is J_{2s+1} in degree $2s+1$. For $n = 2s+m$, $2 \leq m \leq s-1$, we have

$$\begin{aligned}(\ker \delta_4)_n &= \left(V^{(m)} \otimes J_{2s}\right) \cap \left((R \otimes V^{(m-1)} + \cdots + V^{(m-1)} \otimes R) \otimes J_{s+1}\right)\\ &= \left((V^{(m)} \otimes R) \cap (R \otimes V^{(m)} + \cdots + V^{(m-1)} \otimes R \otimes V) \otimes V^{(s)}\right)\\ &\quad \cap \left(V^{(m)} \otimes J_{2s}\right).\end{aligned} \tag{2.20}$$

Applying (2.7) in (2.20), we arrive at $(\ker \delta_4)_n = A_{m-1}.J_{2s+1}$. Then it is easy to mimic the proof of Proposition 2.7 and get the following.

PROPOSITION 2.9. *Assume* $\ker \delta_2$ *pure in degree* $s+1$ *and* $\ker \delta_3$ *pure in degree* $2s$. *For* $\ker \delta_4$ *to be pure in degree* $2s+1$, *it is necessary and sufficient that for any* $n \geq 3s$, *the triple* (E, F, G) *where*

$$\begin{aligned}&E = V^{(n-2s)} \otimes J_{2s}, \qquad F = R \otimes V^{(n-s)} + \cdots + V^{(n-3s)} \otimes R \otimes V^{(2s)},\\ &G = V^{(n-3s+1)} \otimes R \otimes V^{(2s-1)} + \cdots + V^{(n-2s-1)} \otimes R \otimes V^{(s+1)}\end{aligned} \tag{2.21}$$

is distributive.

Setting $n(2j) = js$, $n(2j+1) = js+1$ for $j \geq 2$, we introduce the notation

$$K^i = A \otimes J_{n(i)}, \qquad i \geq 0. \tag{2.22}$$

In (iii) of Theorem 2.11 below, we state the distributivity relations removing, besides (ec), all the obstructions to continuing the process. The proof presents no difficulty and is left to the reader. On the other hand, the fact that no obstruction occurs can be formulated in terms of functor Tor. As a matter of fact, assume that A is any s-homogeneous algebra. It is easily seen by induction that the graded vector space $\mathrm{Tor}_i^A(k,k)$ lives in the degrees $\geq n(i)$ for any $i \geq 3$. Then no obstruction occurs if and only if $\mathrm{Tor}_i^A(k,k)$ is pure in degree $n(i)$ for any $i \geq 3$.

DEFINITION 2.10. An s-homogeneous algebra A is said to be (*generalized*) *Koszul* if the graded vector space $\mathrm{Tor}_i^A(k,k)$ is pure in degree $n(i)$ for any $i \geq 3$.

THEOREM 2.11. *Let $A = \mathrm{Tens}(V)/I$ be an s-homogeneous algebra on V, with R as space of relations. The following are equivalent.*

(i) *A is Koszul.*

(ii) *There is no obstruction to construct* (*in the category of the left-bounded graded left A-modules*) *a pure projective resolution*

$$\cdots \to K^i \xrightarrow{\delta_i} K^{i-1} \to \cdots \to K^1 \xrightarrow{\delta_1} K^0 \xrightarrow{\epsilon} k \to 0, \tag{2.23}$$

of k following the process started above.

(iii) *The extra condition* (*ec*) *holds, and for any $j \geq 1$, one has the distributivity of the triples (E, F, G) for $n \geq (j+1)s$ and the distributivity of the triples (E', F', G') for $n \geq (j+1)s+1$, where*

$$\begin{aligned} E &= V^{(n-js)} \otimes J_{js}, \qquad F = I_{n-js} \otimes V^{(js)}, \\ G &= V^{(n-(j+1)s+1)} \otimes I_{2s-2} \otimes V^{((j-1)s+1)}, \\ E' &= V^{(n-js-1)} \otimes J_{js+1}, \qquad F' = I_{n-js-1} \otimes V^{(js+1)}, \\ G' &= V^{(n-(j+1)s)} \otimes R \otimes V^{(js)}. \end{aligned} \tag{2.24}$$

Definition 2.10 shows that it is equivalent to working with right modules in (ii), so that there is a dual formulation of (iii) that is easy to write down. If A is Koszul, (2.23) is called the (left) Koszul resolution of k. It is a minimal projective resolution since the construction of (2.23) allows us to apply Proposition 2.4 at each step. Note also that it is possible to generalize the Koszul complex as defined by Priddy in the quadratic case [14, 17]. In fact, for any s-homogeneous algebra A, one can define the following complex (called the (left) Koszul complex of A)

$$\cdots \to K^i \xrightarrow{\delta_i} K^{i-1} \to \cdots \to K^1 \xrightarrow{\delta_1} K^0 \to 0, \tag{2.25}$$

in which δ_i is again defined by inclusion of J_{js} or J_{js+1} in K^{i-1} according to $i = 2j$ or $i = 2j + 1$. Clearly, δ_i can alternatively be defined by restriction to K^i of the A-linear map

$$f_i\colon A \otimes \left(V^{(js)} \text{ or } V^{(js+1)}\right) \to A \otimes \left(V^{((j-1)s+1)} \text{ or } V^{(js)}\right) \qquad (2.26)$$

given by $\bar{a} \otimes v \otimes w \mapsto \overline{av} \otimes w$, where v belongs to $V^{(s-1)}$ or V, and w belongs to $V^{((j-1)s+1)}$ or $V^{(js)}$, according to $i = 2j$ or $i = 2j + 1$. But J_{js} or J_{js+1} is contained in $R \otimes V^{(j-1)s}$ or $R \otimes V^{((j-1)s+1)}$, respectively. Therefore the composition $f_{i-1}f_i$ vanishes on J_{js} or J_{js+1}, since the image of R in A is zero. Thus (2.25) is really a complex. On the other hand, in the graded category, (2.25) is naturally a complex whose modules are projective and pure.

PROPOSITION 2.12. *For an s-homogeneous algebra A to be Koszul, it is necessary and sufficient that its left (or right) Koszul complex is exact in degree > 0.*

Proof. The condition is necessary by Theorem 2.11. It is sufficient by Definition 2.10. ∎

Let A be a regular algebra of global dimension 3, generated in degree 1, in the cubic case $s = 3$ (see Introduction). Then (1.1) is a pure projective resolution of k with the same sequence of degrees as the Koszul complex. Hence A is Koszul. The comparison between (1.1) and the Koszul resolution comes from the following general result (see [6] for the quadratic case).

PROPOSITION 2.13. *Let $A = \mathrm{Tens}(V)/I$ be an s-homogeneous algebra on V, with R as a space of relations. Assume that k has a projective resolution (in the graded category)*

$$\cdots \to P^i \xrightarrow{d_i} P^{i-1} \to \cdots \to P^0 \xrightarrow{d_0} k \to 0, \qquad (2.27)$$

such that for any $i \geq 0$, P^i is pure in degree $n(i)$. Definition 2.10 says that A is Koszul. Then there exists a unique morphism (in the graded category) from the resolution (2.27) to the Koszul relation (2.23), and this morphism is an isomorphism.

Proof. We proceed by induction. First, it is clear that there is a unique morphism $f^0\colon P^0 \to K^0$ satisfying $\epsilon f^0 = d_0$. Furthermore, f^0 is bijective. Next, fix $i \geq 1$. Homological algebra in the graded category tells us that there is a resolution morphism f from (2.27) to (2.23). We put in the induction hypothesis the fact that the morphisms $f^0\colon P^0 \to K^0, \ldots, f^{i-1}\colon P^{i-1} \to K^{i-1}$ are unique and bijective. Write down the commutative diagram

$$\begin{array}{ccccc} P^{i+1} & \xrightarrow{d_{i+1}} & P^{i} & \xrightarrow{d_i} & P^{i-1} \\ \big\downarrow{\scriptstyle f^{i+1}} & & \big\downarrow{\scriptstyle f^{i}} & & \big\downarrow{\scriptstyle f^{i-1}} \\ K^{i+1} & \xrightarrow{\delta_{i+1}} & K^{i} & \xrightarrow{\delta_i} & K^{i-1}. \end{array} \tag{2.28}$$

Let us prove the uniqueness of f^i. Let g be another resolution morphism from (2.27) to (2.23), coinciding with f up to f^{i-1}. Homological algebra in the graded category tells us again that there exists a chain homotopy (s_n) from f to g. In particular, $f^i - g^i = s_{i-1}d_i + \delta_{i+1}s_i$. Since P^i is generated in degree $n(i)$ and K^{i+1} lives in degree $> n(i)$, we get $s_i = 0$. In the same manner, s_{i-1} vanishes; hence f^i is unique. By an analogous homotopical argument, we see that f^i has a left and right inverse. So f^i is an isomorphism. ∎

There is an alternative definition of generalized Koszulity in terms of functor Ext. We shall closely follow the set-up of the quadratic case [6]. Functors Hom and Ext in the graded category will be denoted by $\hom_A$ and ext_A. The following characterization has the same proof as Proposition 2.1.3 of [6] and is left to the reader.

PROPOSITION 2.14. *Let $A = \mathrm{Tens}(V)/I$ be an s-homogeneous algebra on V, with R as space of relations. The following are equivalent.*

(i) *A is Koszul.*

(ii) *For any modules M and N which are concentrated respectively in degrees m and n, we have $\mathrm{ext}_A^i(M, N) = 0$ whenever $n \neq m + n(i)$.*

(iii) $\mathrm{ext}_A^i(k, k(-n)) = 0$ *whenever $n \neq n(i)$.*

3. LATTICE DISTRIBUTIVITY AND CONFLUENCE

In this section, some sufficient conditions for Koszulity are presented. For that, we have to examine more closely the infinitely many distributivity relations introduced in the previous section as obstructions to Koszulity. For the convenience of the reader, we begin to recall some basic facts on lattices formed of vector subspaces. Let $\mathscr{V}$ be any vector space on k. Denote by $\mathscr{L}(\mathscr{V})$ the set of the subspaces of $\mathscr{V}$. Ordered by inclusion, this set is a lattice. The infimum (supremum) of two subspaces is their intersection (sum). This lattice is modular. It means that $E \cap (F + G) = F + (E \cap G)$ for any subspaces E, F, G of $\mathscr{V}$ such that E contains F. A sublattice is a subset of $\mathscr{L}(\mathscr{V})$ which is stable by intersection and sum. Such a sublattice $\mathscr{T}$ is obviously modular, but we are interested in knowing if it is distributive. Let $E_1, \dots, E_n$ be subspaces of $\mathscr{V}$. The sublattice $\mathscr{T}$

generated by $E_1, \dots, E_n$ is the smaller sublattice containing them. Here is a first criterion of distributivity [4, 5].

PROPOSITION 3.1. *$\mathscr{T}$ is distributive if and only if there is a basis B of $\mathscr{V}$ distributing $E_1, \dots, E_n$; i.e., for $i = 1, \dots, n$, $B \cap E_i$ is a basis of E_i.*

The condition is easily seen as sufficient. In fact, if the condition holds, $\mathscr{T}$ is contained in the sublattice of all subspaces generated by a part of B. But the latter is isomorphic to the lattice (ordered by inclusion) of all subsets of the set B and hence is distributive.

To state the second criterion needs some preliminaries. The sublattice $\mathscr{T}$ generated by $E_1, \dots, E_n$ is said to be predistributive if for each $i = 1, \dots, n$, the sublattice generated by the E_j, $j \neq i$, is distributive. On the other hand, for $2 \leq i \leq n-1$, we let $E_i^{\cap} = E_1 \cap \cdots \cap E_{i-1}$ and $E_i^{+} = E_{i+1} + \cdots + E_n$. The following criterion due to R. Musti and E. Buttafuoco [15] is very useful as far as Koszulity is concerned. For example, the proof given in [5] of the hard part of the Backelin theorem is based on this criterion.

PROPOSITION 3.2. *$\mathscr{T}$ is distributive if and only if $\mathscr{T}$ is predistributive and the triples $(E_i^{\cap}, E_i, E_i^{+})$, $2 \leq i \leq n-1$, are distributive.*

The first non-trivial situation is $n = 3$ and says that E_1, E_2, E_3 generate a distributive lattice if and only if the triple (E_1, E_2, E_3) is distributive (this result goes back to Öre [16]). It is worth noticing that, in Proposition 3.2, we can eliminate predistributivity by induction. Let us call a characteristic triple of the list $E_1, \dots, E_n$ any triple (M, N, P) of $\mathscr{T}$ such that $N = E_i$, $2 \leq i \leq n-1$, M is the intersection of some of the subspaces $E_1, \dots, E_{i-1}$, and P is the sum of some of the subspaces $E_{i+1}, \dots, E_n$. The following is clear from Proposition 3.2.

PROPOSITION 3.3. *$\mathscr{T}$ is distributive if and only if all the characteristic triples of the list $E_1, \dots, E_n$ are distributive.*

The elimination of predistributivity in Proposition 3.2 and a direct counting show that the number of characteristic triples is

$$\begin{aligned} \sum_{k=0}^{n-3} (n-2-k)\binom{n}{n-k} &= \sum_{i=2}^{n-1} (2^{i-1}-1)(2^{n-i}-1) \\ &= (n-4)2^{n-1} + n + 2. \end{aligned}$$

For the remainder of this section, $A = \mathrm{Tens}(V)/I$ is an s-homogeneous algebra on V, with R as space of relations. For any $n \geq s$, we denote by $\mathscr{T}_n$ the sublattice of $\mathscr{L}(V^{(n)})$ generated by the subspaces $V^{(i)} \otimes R \otimes V^{(j)}$, $i + j + s = n$. We say that the algebra A is distributive if all lattices $\mathscr{T}_n$ are distributive. Clearly, it suffices to let $n \geq s + 2$. In Theorem 2.11, each triple (E,F,G) or (E',F',G') is contained in some $\mathcal{T}_n$, so we immediately get the following.

PROPOSITION 3.4. *Assume that A is distributive. Then A is Koszul if and only if (ec) holds. Moreover, according to Proposition* 2.5, (ec) *is equivalent to the inclusions*

$$(V^{(m)} \otimes R) \otimes (R \otimes V^{(m)}) \subseteq V^{(m-1)} \otimes R \otimes V, \qquad 2 \leq m \leq s-1. \tag{3.1}$$

In particular, for $s = 2$, if A is distributive, A is Koszul. Thus we have obtained a part (the easy part!) of the Backelin theorem:

THEOREM 3.5 [4, 5]. *In the quadratic case $s = 2$, A is Koszul if and only if A is distributive.*

When $s > 2$, the distributivity of the lattices $\mathcal{T}_n$ seems too strong. In fact, we are going to show that, besides (ec), the distributivity of well-chosen sublattices $\mathcal{T}'_n$ will be sufficient. It is a bit more involved, but the idea is to exploit the fact that R and $I_{2s-2} = V^{(s-2)} \otimes R + \cdots + R \otimes V^{(s-2)}$ occur alternately while moving to the left (see Theorem 2.11(iii)). If $s = 2$, $I_{2s-2} = R$ and $\mathcal{T}_n$ will be recovered. For any n of the form $n = is$ or $n = is + s - 1$ where $i \geq 1$, let us introduce the sublattice $\mathcal{T}'_n$ of $\mathcal{T}_n$ generated by the subspaces

$$\begin{gathered} V^{(n-s)} \otimes R, V^{(n-2s+1)} \otimes I_{2s-2} \otimes V, V^{(n-2s)} \otimes R \otimes V^{(s)}, \\ V^{(n-3s+1)} \otimes I_{2s-2} \otimes V^{(s+1)}, \dots \end{gathered} \tag{3.2}$$

up to $R \otimes V^{(n-s)}$ or $I_{2s-2} \otimes V^{(n-2s+2)}$, according to $n = is$ or $n = is + s - 1$. In other words, in the list

$$V^{(n-s)} \otimes R, V^{(n-s-1)} \otimes R \otimes V, V^{(n-s-2)} \otimes R \otimes V^{(2)}, \dots, R \otimes V^{(n-s)}$$

generating $\mathcal{T}_n$, we alternately keep the subspace or add the $s-1$ following ones. For $n = is + s - 2$ or $n = is + s - 1$, we also introduce the sublattice $\mathcal{T}''_n$ of $\mathcal{L}(V^{(n)})$ generated by the subspaces

$$V^{(n-2s+2)} \otimes I_{2s-2}, V^{(n-2s+1)} \otimes R \otimes V^{(s-1)}, V^{(n-3s+2)} \otimes I_{2s-2} \otimes V^{(s)}, \dots \tag{3.3}$$

Clearly, $\mathcal{T}''_n \otimes V$ is contained in $\mathcal{T}'_{n+1}$. If $s = 2$, $\mathcal{T}'_n = \mathcal{T}''_n = \mathcal{T}_n$ for any $n \geq 2$.

THEOREM 3.6. *Let $A = \mathrm{Tens}(V)/I$ be an s-homogeneous algebra on V, with R as space of relations. Assume that (ec) holds and the lattices $\mathcal{T}'_n$, $n \geq s$, are distributive. Then A is Koszul.*

Proof. Fix $j \geq 1$ and $n \geq (j+1)s$ (resp. $n \geq (j+1)s+1$). Consider the characteristic triple (M, N, P) of the list (3.2) where

$$N = V^{(n-(j+1)s+1)} \otimes I_{2s-2} \otimes V^{((j-1)s+1)}$$
$$(\text{resp. } V^{(n-(j+1)s)} \otimes R \otimes V^{(js)}),$$

M is the intersection of all subspaces before N in the list, and P is the sum of all subspaces after N in the list. But (ec) means that

$$(V^{(s-1)} \otimes R) \cap (I_{2s-2} \otimes V) = V^{(s-2)} \otimes J_{s+1}, \tag{3.4}$$

and Lemma 2.6 implies that $(V^{(s-1)} \otimes R) \cap (R \otimes V^{(s-1)}) = J_{2s-1}$. Therefore, we get $M = V^{(n-js)} \otimes J_{js}$ (resp. $V^{(n-js-1)} \otimes J_{js+1}$). On the other hand, $P = I_{n-js} \otimes V^{(js)}$ (resp. $I_{n-js-1} \otimes V^{(js+1)}$). Since (M, N, P) is distributive, Theorem 2.11 allows us to conclude that A is Koszul. ∎

We have not succeeded in proving the converse of Theorem 3.6. There exists some evidence for the validity of this converse. Let us try to follow the same method as in the proof of the Backelin theorem given in [5]. (I am indebted to Wolfgang Soergel for his (unpublished) work on the proof of [5].) Assume that A is Koszul. Try to proceed by induction on $i \geq 1$. In an obvious manner, $\mathcal{T}'_s, \mathcal{T}''_{2s-2}, \mathcal{T}'_{2s-1}, \mathcal{T}''_{2s-1}$ are distributive. Now fix $i \geq 2$ and let $n = is$ (resp. $n = is + s - 1$). Suppose that the lattices $\mathcal{T}'_{n-1}$ (resp. $\mathcal{T}'_{n-s+1}$) and $\mathcal{T}''_{n-1}$ are distributive. We would like to prove that $\mathcal{T}'_n$ is distributive. Consider any characteristic triple (M, N, P) of the list (3.2). Using Proposition 3.3, it would suffice to prove that (M, N, P) is distributive. Let us call support of (M, N, P) the sublist of (3.2) formed of the subspaces occurring in the definition of M, N, P. If this support is maximal, then (M, N, P) is distributive since A is Koszul. Thus we can assume that the support of (M, N, P) is not maximal. We can likewise assume that the first (the last) subspace of the support of (M, N, P) is the first (the last) subspace of the list (3.2) (otherwise, the induction hypothesis is applied). On the other hand, a subspace of the list (3.2) alternately contains R or I_{2s-2} as a factor, and it will be respectively called of R-type or of I-type.

Claim 3.7. *(M, N, P) is distributive if at least one I-type subspace of the list* (3.2) *does not belong to the support of* (M, N, P).

Proof. Denote by E a I-type subspace not belonging to the support of (M, N, P). The idea is that the R's of the subspaces of the list (3.2) located before E do not overlap the R's of the subspaces of the list (3.2) located after E, so that induction and Proposition 3.1 can be used. More precisely, we can decompose $V^{(n)} = V^{(l)} \otimes V^{(k)}$, $k \geq s$, $l \geq s$, such that the subspaces before E are of the form $V^{(l)} \otimes F_u$, and the subspaces after E are of the form $G_v \otimes V^{(k)}$, with u and v running over suitable indexing sets. Moreover, the F_u generate the distributive lattice $\mathscr{T}_k'$, and the G_v generate the distributive lattice $\mathscr{T}_l'$. Proposition 3.1 shows that there exist a basis (e_α) of $V^{(k)}$ distributing the subspaces F_u, and a basis (f_β) of $V^{(l)}$ distributing the subspaces G_v. Then $(f_\beta \otimes e_\alpha)$ is a basis of $V^{(n)}$ distributing the subspaces $V^{(l)} \otimes F_u$ and $G_v \otimes V^{(k)}$. By Proposition 3.1, those subspaces generate a distributive lattice, and the latter contains (M, N, P). ∎

When $s = 2$, the previous claim suffices to get the hard part of Backelin's theorem. When $s > 2$, it would remain to examine the case where all I-type subspaces are in the support of (M, N, P), but we shall not go further in this discussion.

R. Fröberg has shown that any quadratic algebra for which R is generated by monomials is Koszul [13]. The generalization of this result is the following.

PROPOSITION 3.8. *Let A be an s-homogeneous algebra on V, with R as space of relations. Let $X = (x_1, \dots, x_r)$ be a basis of V. Assume that A is X-monomial; i.e., R is generated by any collection C of words $x_{i_1} \dots x_{i_s}$ of length s. Then A is Koszul if and only if C has the following overlap property*:

Any word $x_{i_1} \dots x_{i_n}$, $s + 2 \leq n \leq 2s - 1$, such that the first subword $x_{i_1} \dots x_{i_s}$ of length s and the last subword $x_{i_{n-s+1}} \dots x_{i_n}$ of the length s belong to C, has all its subwords of length s in C (for us, a subword is always a connected *part of the word, so the latter are of the form $x_{i_j} \dots x_{i_{j+s-1}}$, $1 \leq j \leq n - s + 1$).*

Proof. Fix $n \geq s$. The words of length n form a basis of $V^{(n)}$ distributing the subspaces $V^{(i)} \otimes R \otimes V^{(j)}$, $i + j + s = n$. Therefore $\mathscr{T}_n$ is distributive by Proposition 3.1. Thus A is distributive, and Proposition 3.4 implies that A is Koszul if and only if we have the inclusions

$$(V^{(m)} \otimes R) \cap (R \otimes V^{(m)}) \subseteq V^{(m-1)} \otimes R \otimes V, \qquad 2 \leq m \leq s - 1. \tag{3.5}$$

Translating these inclusions in terms of C, we get the overlap property. ∎

The overlap property is a rather large combinatorial condition. For example, if C has the overlap property, any C^σ has the overlap property. Here σ is a permutation of the indices $1, \dots, r$, and C^σ denotes the collection of words $x_{i_1} \dots x_{i_s}$ such that $x_{\sigma(i_1)} \dots x_{\sigma(i_s)}$ belongs to C. On the other hand, let C and C' be two collections having the overlap property.

Assume that there is no word $x_{i_1} \dots x_{i_n}$, $s + 2 \le n \le 2s - 1$, such that the first subword of length s belongs to one of the two collections and the last subword of length s belongs to the other collection. Then the union of C and C' has the overlap property. In particular, any collection of powers x_i^s, $1 \le i \le r$, has the overlap property. Let us give another example. Assume $s = 3$ and C is reduced to a single word a. Then C has the overlap property if and only if a is not of the form $x_i x_j x_i$ with $j \ne i$. Even for $s = 3$, it seems difficult to find all collections which satisfy the overlap property.

Assume again that A is X-monomial. Fix $n \ge s$. It is obvious that the classes in A of the words $x_{i_1} \dots x_{i_n}$ such that no subword of length s belongs to C form a basis of A_n. This situation can be generalized as follows. Let A be any s-homogeneous algebra on V, with R as space of relations. Let $X = (x_1, \dots, x_r)$ be a basis of V. The set X is supposed to be ordered by $x_1 < \cdots < x_r$. The words of length n are naturally lexicographically ordered. For example, if $n = 2$, we have $x_1x_1 < x_1x_2 < \cdots$ $x_1x_r < x_2x_1 < \cdots < x_rx_r$. Introduce the X-reduction operator S on $V^{(s)}$ such that $\ker(S) = R$ [7]. An X-reduction operator S on $V^{(s)}$ is an idempotent endomorphism of $V^{(s)}$ such that, for any word a of length s, either $S(a) = a$ or $S(a) < a$ (the latter inequality means that either $S(a) = 0$ or the highest word appearing in the linear decomposition of $S(a)$ is $< a$). A word a of length s is called S-reduced if $S(a) = a$, and S-nonreduced if $S(a) < a$. For instance, if A is X-monomial, the S-nonreduced words are exactly the elements of the collection C (the endomorphism S is diagonal in the basis $(x_{i_1} \dots x_{i_s})$ if and only if A is X-monomial). Fix $n \ge s$. A word $x_{i_1} \dots x_{i_n}$ is said to be S-reduced if any subword of length s is S-reduced. It is clear that the classes in A of the S-reduced words of length n generate the vector space A_n. In general, those classes are not linearly independent. We are going to describe a situation (containing the monomial case) for which the linear independence is ensured. First a definition is needed (see [7] for more details).

DEFINITION 3.9. Let T and U be two X-reduction operators of any tensor power $V^{(n)}$. The pair (T, U) is said to be confluent if

$$\dots TUT = \dots UTU. \tag{3.6}$$

In the "braided" product $\dots TUT$, the (finite) number of factors is not specified because it can be shown that such a product stabilizes if the number of factors is large enough (twice the dimension of $V^{(n)}$ suffices). Note that any commuting pair is confluent. We also need a notation. Fix $n \ge s$. For $1 \le i \le n - s + 1$, we set

$$S_i^{(n)} = 1_{V^{(n-s-i+1)}} \otimes S \otimes 1_{V^{(i-1)}}, \tag{3.7}$$

where $1_{V^{(j)}}$ denotes the identity endomorphism of $V^{(j)}$. Clearly, $S_i^{(n)}$ is an X-reduction operator of $V^{(n)}$. Let us recall that the set of X-reduction operators of $V^{(n)}$ is a lattice anti-isomorphic to the lattice $\mathscr{L}(V^{(n)})$ by the bijection $T \mapsto \ker(T)$. Since $\ker(S_i^{(n)}) = V^{(n-s-i+1)} \otimes R \otimes V^{(i-1)}$, one has

$$I_n = \ker\left(S_1^{(n)} \wedge S_2^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}\right). \tag{3.8}$$

Therefore, the projection $S_1^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}$ induces a linear isomorphism from A_n onto $\operatorname{im}(S_1^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)})$. In other words, the classes of the $S_1^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}$-reduced words form a basis of A_n. Note that we always have the inclusion $\operatorname{im}(T \wedge U) \subseteq \operatorname{im}(T) \cap \operatorname{im}(U)$, the equality meaning exactly that the pair (T, U) is confluent. Accordingly, the $S_1^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}$-reduced words are S-reduced words, but the converse does not hold in general.

Definition 3.10. Let $A = \operatorname{Tens}(V)/I$ be an s-homogeneous algebra on V, with R as space of relations. Let X be a totally ordered basis of V. Let S be the X-reduction operator of $V^{(s)}$ such that $\ker(S) = R$. If the X-reduction operators $S_1^{(2s-1)}, S_2^{(2s-1)}, \ldots, S_s^{(2s-1)}$ are pairwise confluent, A is said to be X-confluent. (If A is X-monomial, A is X-confluent since the operators $S_1^{(2s-1)}, S_2^{(2s-1)}, \ldots, S_s^{(2s-1)}$ are pairwise commuting.)

Assume that A is X-confluent. Show by induction on $n \geq s$ that $S_1^{(n)}, \ldots, S_{n-s+1}^{(n)}$ are pairwise confluent. The X-confluence of A implies that the property holds up to $n = 2s - 1$. Suppose that $n \geq 2s$ and that the property holds up to $n - 1$. It remains to prove that $S_{n-s+1}^{(n)}$ is confluent with each of the previous ones. But the X-confluence of A also implies that $S_{n-s+1}^{(n)}$ is confluent with $S_{n-2s+2}^{(n)}, \ldots, S_{n-s}^{(n)}$. On the other hand, $S_{n-s+1}^{(n)}$ commutes (hence is confluent) with $S_1^{(n)}, \ldots, S_{n-2s+1}^{(n)}$. Thus the property holds for n. Since $S_1^{(n)}, \ldots, S_{n-s+1}^{(n)}$ are pairwise confluent, $S_1^{(n)}$ is confluent with $S_2^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}$, and hence we have

$$\operatorname{im}\left(S_1^{(n)} \wedge S_2^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}\right) = \operatorname{im}\left(S_1^{(n)}\right) \cap \operatorname{im}\left(S_2^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}\right). \tag{3.9}$$

It follows by induction that

$$\begin{aligned}\operatorname{im}&\left(S_1^{(n)} \wedge S_2^{(n)} \wedge \cdots \wedge S_{n-s+1}^{(n)}\right)\\ &= \operatorname{im}\left(S_1^{(n)}\right) \cap \operatorname{im}\left(S_2^{(n)}\right) \cap \cdots \cap \operatorname{im}\left(S_{n-s+1}^{(n)}\right),\end{aligned} \tag{3.10}$$

so that the classes in A of the S-reduced words of length n form a basis of A_n.

Theorem 3.11. *Let A be an s-homogeneous algebra on V. Assume that A is X-confluent for a totally ordered basis X of V. Then A is distributive.*

Proof. Fix $n \geq s$. We know that $S_1^{(n)}, \ldots, S_{n-s+1}^{(n)}$ are pairwise confluent. By a general result [7], the sublattice generated by these X-reduction operators is distributive. Using the bijection $T \mapsto \ker(T)$, the latter is anti-isomorphic to $\mathscr{T}_n$. Thus $\mathscr{T}_n$ is distributive. ∎

As a corollary, we get Priddy's theorem [17]: if $s = 2$ and A is X-confluent, then A is Koszul (see also [1, 7]). The reader is invited to refer to [8] for various examples and counterexamples in the quadratic case. Our purpose is now to apply Theorem 3.11 to the algebra whose relations are the antisymmetrizers of degree s. The characteristic of the ground field k is supposed to be zero. Fix $s \geq 2$. For any integer $n \geq 1$, $\mathbf{S}_n$ denotes the permutation group of $1, \ldots, n$. For σ in $\mathbf{S}_n$ and $a_1, \ldots, a_n$ elements of V, we set

$$\sigma . a_1 \otimes \cdots \otimes a_n = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}.$$

By linearity, that defines an action $(\sigma, a) \mapsto \sigma . a$ of the group $\mathbf{S}_n$ on $V^{(n)}$. Define the subspace R of $V^{(s)}$ as being

$$R = \left\{ a \in V^{(s)};\ \sigma . a = \operatorname{sgn}(\sigma) a \text{ for any } \sigma \in \mathbf{S}_s \right\}, \tag{3.11}$$

where $\operatorname{sgn}(\sigma)$ is the sign of σ. Since $\operatorname{car}(k) = 0$, one has

$$R = \left\{ \sum_{\sigma \in \mathbf{S}_s} \operatorname{sgn}(\sigma) \sigma . a;\ a \in V^{(s)} \right\}. \tag{3.12}$$

Moreover, R is naturally isomorphic to the homogeneous part of degree s of the exterior (Grassmann) algebra on V. Evidently $R = 0$ if $r < s$. Choose now a totally ordered basis X: $x_1 < \cdots < x_r$ of V. A basis of R is formed by the elements

$$\sum_{\sigma \in \mathbf{S}_s} \operatorname{sgn}(\sigma) \sigma . (x_{i_1} \ldots x_{i_s}), \qquad i_1 > \cdots > i_s.$$

Let S be the X-reduction operator such that $\ker(S) = R$. As $x_{i_1} \ldots x_{i_s}$ is the highest word in the previous sum, we see that the S-nonreduced words of length s are exactly the words $x_{i_1} \ldots x_{i_s}$ with $i_1 > \cdots > i_s$. If $a = x_{i_1} \ldots x_{i_s}$, $i_1 > \cdots > i_s$, then $\sigma . a$ belongs to $\operatorname{im}(S)$ whenever σ is not the identity permutation, and

$$(1_{V^{(s)}} - S)(a) = \sum_{\sigma \in \mathbf{S}_s} \operatorname{sgn}(\sigma) \sigma . a. \tag{3.13}$$

LEMMA 3.12. *Assume* $1 \leq m \leq s - 1$ *and* a *is an element of* $V^{(s+m)}$. *Then* a *belongs to* $(V^{(m)} \otimes R) \cap (R \otimes V^{(m)})$ *if and only if for any* σ *in* $\mathbf{S}_{s+m}$ *one has*

$$\sigma . a = \operatorname{sgn}(\sigma) a. \tag{3.14}$$

Proof. Assume a in $(V^{(m)} \otimes R) \cap (R \otimes V^{(m)})$. It suffices to prove (3.14) when σ is any transposition $(i, i+1)$, $1 \leq i \leq s+m-1$. Since $s \geq m+1$, two cases can be considered: the case $1 \leq i \leq s-1$ and the case $m+1 \leq i \leq s+m-1$. In the former (resp. the latter) case, we write down $a = \sum_j u_j \otimes v_j$ (resp. $a = \sum_j v_j \otimes u_j$) with u_j in R and v_j in $V^{(m)}$. Using the definition of R, we get (3.14) in both cases.

Conversely, assume that a satisfies (3.14) for any σ in S_{s+m}. Let (v_j) be a basis of $V^{(m)}$. Writing down $a = \sum_j u_j \otimes v_j$ with u_j in $V^{(s)}$, we draw from (3.14)

$$\sum_j (\sigma . u_j) \otimes v_j = \operatorname{sgn}(\sigma) \sum_j u_j \otimes v_j \tag{3.15}$$

for any σ in $\mathbf{S}_s$. Therefore $\sigma . u_j = \operatorname{sgn}(\sigma) u_j$, and u_j belongs to R. Thus a belongs to $R \otimes V^{(m)}$. In the same manner, a belongs to $V^{(m)} \otimes R$. ∎

Let A be the s-homogeneous algebra on V whose space of relations is R as defined by (3.11). Assume $m = 1, \ldots, s-1$. Abbreviating the notation (3.7), let $S_1 = S_1^{(s+m)}$ and $S_{m+1} = S_{m+1}^{(s+m)}$. Consider the pair $P = (S_1, S_{m+1})$ of X-reduction operators on $V^{(s+m)}$. To prove that A is X-confluent, it suffices to prove that P is confluent. A word of length $s+m$ which is both S_1-nonreduced and S_{m+1}-nonreduced is said to be P-ambiguous. Since the S's in S_1 and in S_{m+1} overlap, the P-ambiguous words of length $s+m$ are exactly the words $x_{i_1} \ldots x_{i_{s+m}}$ where $i_1 > \cdots > i_{s+m}$. Denote by $\operatorname{Amb}(P)$ the subspace generated by those words. Lemma 3.12 implies that the dimensions of $(V^{(m)} \otimes R) \cap (R \otimes V^{(m)})$ and of $\operatorname{Amb}(P)$ are the same. In other words, we have

$$\dim(\ker(S_1 \vee S_{m+1})) = \dim(\operatorname{Amb}(P)). \tag{3.16}$$

As the non-ambiguous words generate $\operatorname{im}(S_1) + \operatorname{im}(S_{m+1})$, we deduce

$$\dim(\operatorname{im}(S_1 \vee S_{m+1})) = \dim(\operatorname{im}(S_1) + \operatorname{im}(S_{m+1})). \tag{3.17}$$

But for any pair (T, U), the inclusion $\operatorname{im}(T) + \operatorname{im}(U) \subseteq \operatorname{im}(T \vee U)$ always holds, and the equality means that the pair is confluent. Thus P is confluent.

Theorem 3.11 allows us to assert that A is distributive. On the other hand, (ec) is satisfied since the inclusions

$$(V^{(m)} \otimes R) \cap (R \otimes V^{(m)}) \subseteq V^{(m-1)} \otimes R \otimes V, \qquad 2 \leq m \leq s-1. \tag{3.18}$$

are an immediate consequence of Lemma 3.12. We have

THEOREM 3.13. *Assume* $\operatorname{car}(k) = 0$ *and* $s \geq 2$. *Let* A *be the* s*-homogeneous algebra on* V *whose space of relations is* R *as defined by* (3.11). *Then, for any totally ordered basis* X *of* V, A *is* X*-confluent. Furthermore, the extra condition* (*ec*) *holds for* A. *Accordingly,* A *is Koszul.*

As an application, let us compute the global dimension of A. It is known that the left and right global dimensions of A are the same, and they coincide with the length of a minimal resolution of the trivial left module k [3]. Hence $\operatorname{gl.dim}(A)$ is the highest i such that $K^i \neq 0$. The proof of Lemma 3.12 shows that J_n is always the space of the antisymmetrizers of degree n, so that J_n vanishes if and only if $n > r$. But n is of the form js or $js + 1$. Define the non-negative integers q and q' by $r = qs + q'$, $q' < s$. Thus $\operatorname{gl.dim}(A)$ is $2q$ or $2q + 1$ according to whether q' vanishes or not. If $s = 2$, A is the symmetric algebra on V, and we recover that $\operatorname{gl.dim}(A) = r$.

Remark 3.14. Let us go back to the confluent pair $P = (S_1, S_{m+1})$. Following the general formalism of the reduction operators [7], an idempotent endomorphism of $V^{(s+m)}$, denoted by Λ^P, is defined as being

$$\Lambda^P = \ldots (1_{V^{(s+m)}} - S_1)(1_{V^{(s+m)}} - S_{m+1})(1_{V^{(s+m)}} - S_1). \qquad (3.19)$$

Let a be a P-ambiguous word. It is clear from (3.19) that $\Lambda^P(a)$ belongs to $(V^{(m)} \otimes R) \cap (R \otimes V^{(m)})$, and that a is the single ambiguous word appearing in $\Lambda^P(a)$. Consequently, Lemma 3.12 allows us to write

$$\Lambda^P(a) = \sum_{\sigma \in \mathbf{S}_{s+m}} \operatorname{sgn}(\sigma)\sigma.a. \qquad (3.20)$$

4. OPPOSITE ALGEBRA AND DUAL ALGEBRA

Throughout this section, $A = \operatorname{Tens}(V)/I(R)$ is an s-homogeneous algebra on V, with R as space of relations. Introduce the endomorphism τ of $\operatorname{Tens}(V)$ by $\tau(1) = 1$ and for any $n \geq 1$,

$$\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_n \otimes \cdots \otimes v_2 \otimes v_1, \qquad v_1 \in V, \ldots, v_n \in V. \qquad (4.1)$$

Clearly, τ is an algebra anti-isomorphism which naturally induces an algebra anti-isomorphism

$$\bar{\tau} \colon A \to \operatorname{Tens}(V)/I(\tau(R)). \qquad (4.2)$$

Denoting by A° the s-homogeneous algebra on V having $\tau(R)$ as space of relations, we see that $\bar{\tau}$ is an algebra isomorphism from the opposite algebra of A onto A°. We shall say that A° is the opposite s-homogeneous algebra of A.

PROPOSITION 4.1. *A° is Koszul if and only if A is Koszul.*

Proof. Using $\bar{\tau}$, there is an obvious isomorphism from the category of the left-bounded graded right A-modules to the category of the left-bounded graded left A°-modules. In particular, A is right Koszul if and only if A° is left Koszul. But Definition 2.10 implies that left and right Koszulities are equivalent. ∎

DEFINITION 4.2. The s-homogeneous algebra A is said to be τ-commutative if $\tau(R) = R$, which is equivalent to $A^\circ = A$.

If $s = 2$, any A which is commutative in the usual sense (or even skew-commutative for a given parity) is τ-commutative. The algebra A whose relations are the antisymmetrizers of degree s (end of Section 3) is τ-commutative. Another example is provided by the cubic algebra (1.2) of the Introduction.

Let V^* be the dual vector space of V. Denote by $R^\perp$ the subspace of the elements of $V^{*(s)}$ which are orthogonal for the natural pairing to any element of R. Define the dual s-homogeneous algebra of A as being the s-homogeneous algebra $A^!$ on V^* whose space of relations is $R^\perp$. We have $I(R^\perp)_n = J(R)_n^\perp$, where $J(R)_n = V^{(n)}$, $0 \leq n \leq s-1$, and

$$J(R)_n = \bigcap_{i+j+s=n} V^{(i)} \otimes R \otimes V^{(j)}, \qquad n \geq s. \tag{4.3}$$

PROPOSITION 4.3. *Let A be an s-homogeneous algebra on V. Assume that A is distributive. Then A° and $A^!$ are distributive.*

Proof. Fix $n \geq s$. The linear isomorphism τ from $V^{(n)}$ onto itself induces naturally a lattice automorphism of $\mathscr{L}(V^{(n)})$. This lattice automorphism transforms $\mathscr{T}_n$ into the sublattice generated by the subspaces $V^{(i)} \otimes \tau(R) \otimes V^{(j)}$, $i+j+s=n$. Hence the latter is distributive. Thus A° is distributive. On the other hand, define the map ψ from $\mathscr{L}(V^{(n)})$ to $\mathscr{L}(V^{*(n)})$ by $\psi(E) = E^\perp$ for any subspace E of $V^{(n)}$. Clearly, ψ is a lattice anti-isomorphism of $\mathscr{L}(V^{(n)})$ to $\mathscr{L}(V^{*(n)})$. Moreover, ψ transforms $\mathscr{T}_n$ into the sublattice generated by the subspaces $V^{*(i)} \otimes R^\perp \otimes V^{*(j)}$, $i+j+s=n$. So we can conclude that $A^!$ is distributive. ∎

The Backelin theorem shows that if $s = 2$ and A is Koszul, then A° and $A^!$ are Koszul. When $s > 2$, according to Proposition 4.1, we do not have to examine the extra condition for the opposite algebra. But we can also show directly

PROPOSITION 4.4. *(ec) for A° is equivalent to (ec) for A.*

Proof. Assume that (ec) holds for A. We wish to prove (ec) for A° by using Proposition 2.5. Applying τ to the triple (2.11), we obtain a distributive triple. Next, τ is applied to the inclusions (2.12) expressed in a symmetric way as in Lemma 2.6, and we can conclude similarly. ∎

On the other hand, using ψ, we see that (ec) for $A^{!}$ is equivalent to

$$V^{(s-2)} \otimes R \otimes V \subseteq (V^{(s-1)} \otimes R) + (R \otimes V^{(s-1)} \cap \cdots \cap V^{(s-2)} \otimes R \otimes V). \quad (4.4)$$

Proposition 3.4 immediately provides the following result.

PROPOSITION 4.5. *Let A be a distributive s-homogeneous algebra on V. Then $A^{!}$ is Koszul if and only if we have the inclusions*

$$V^{(m-1)} \otimes R \otimes V \subseteq (V^{(m)} \otimes R) + (R \otimes V^{(m)}), \qquad 2 \leq m \leq s-1. \quad (4.5)$$

Assume that A is X-monomial, defined by a collection C of words of length s. Clearly, A° is X-monomial, and $A^{!}$ is X^{*}-monomial (X^{*} is the dual basis of X). Moreover, $A^{!}$ is Koszul if and only if the complement $\overline{C}$ of C has the overlap property. It is easy to find for any $s > 2$ a Koszul A such that $A^{!}$ is not Koszul. For example, the singleton C reduced to x_1^s has the overlap property. But $\overline{C}$ does not have the overlap property (consider the word $x_2 x_1^s x_2$).

5. BIMODULE RESOLUTIONS OF THE ALGEBRA

In this section, A denotes an s-homogeneous algebra on V. We are concerned in resolutions of A by bimodules. Denote by $\mathscr{C}$ the abelian category of the left-bounded graded A-A-bimodules, the morphisms being the degree-preserving homomorphisms. Define as usual the algebra $A^e = A \otimes_k A^{op}$. We can naturally view $\mathscr{C}$ as the category of the left-bounded graded left A^e-modules. The description of $\mathscr{C}$ comes from this identification. In the sequel, a bimodule (a morphism) will always be an object (a morphism) of the category $\mathscr{C}$. The identity maps of k and A are denoted by 1_k and 1_A, respectively. The shift $M(l)$ of a bimodule M is the bimodule M graded by $M(l)_n = M_{n+l}$. A bimodule M is concentrated (respectively, pure) in degree l if there exists l satisfying $M = M_l$ (respectively, $M = A.M_l.A$). A bimodule is concentrated in degree l if and only if it is isomorphic to a direct sum of shifts $k(-l)$. The shift $A^e(-l)$ is pure in degree l. A bimodule M is graded-free if and only if M is isomorphic to a direct sum of $A^e(-l)$, where l can vary while being bounded below. Any bimodule is projective (as an object of $\mathscr{C}$) if and only if it is graded-free, and any bimodule has a projective cover which is unique up to a non-unique isomorphism. Furthermore, any bimodule M has a minimal projective resolution which is unique up to a non-unique isomorphism, and any projective resolution of M contains a minimal one as direct summand. Note also that any projective bimodule M pure in degree l is isomorphic to $A \otimes_k M_l \otimes_k A$, where M_l is considered as a bimodule concentrated in degree l. For a later use, we recall below some elementary facts.

LEMMA 5.1. *Let $\mathscr{I} = A_1 \oplus A_2 \oplus \cdots$ be the augmentation ideal of A. Let M be a bimodule (i.e., an object of $\mathscr{C}$).*

(i) *The map $M \to M \otimes_A k$, $m \mapsto m \otimes_A 1$, defines an isomorphism from $M/M.\mathscr{I}$ to $M \otimes_A k$.*

(ii) *The map $M \to k \otimes_A M$, $m \mapsto 1 \otimes_A m$, defines an isomorphism from $M/\mathscr{I}.M$ to $k \otimes_A M$.*

(iii) *The map $M \to k \otimes_A M \otimes_A k$, $m \mapsto 1 \otimes_A m \otimes_A 1$, defines an isomorphism from $M/M.\mathscr{I} + \mathscr{I}.M$ to $k \otimes_A M \otimes_A k$.*

(iv) *Assume M pure in degree l. The map $M_l \to k \otimes_A M \otimes_A k$, $m \mapsto 1 \otimes_A m \otimes_A 1$, is a linear isomorphism.*

Proof. The map in (i) defines clearly a morphism $f: \overline{m} \mapsto m \otimes_A 1$ from $M/M.\mathscr{I}$ to $M \otimes_A k$. On the other hand, the map $(m, 1) \mapsto \overline{m}$ defines a morphism g from $M \otimes_A k$ to $M/M.\mathscr{I}$. Moreover, f and g are the inverse of each other, proving (i). The proof of (ii) is analogous, and (iii) comes from (i) and (ii). Finally, if M is pure in degree l, then $M.\mathscr{I} + \mathscr{I}.M = \oplus_{n>l} M_n$, and we deduce (iv) from (iii). ∎

LEMMA 5.2 (Graded Nakayama lemma). *Let M be a bimodule. If $M \otimes_A k$ (respectively, $k \otimes_A M$, $k \otimes_A M \otimes_A k$) vanishes, then M vanishes.*

Proof. Assume $M \otimes_A k = 0$. Lemma 5.1 implies $M = M.\mathscr{I}$. Assume $M \neq 0$. The left-bounded assumption in the definition of $\mathscr{C}$ allows us to consider the lowest n such that $M_n \neq 0$. The inclusion of M_n in $M.\mathscr{I}$ leads to a contradiction. The other cases are treated in the same manner. ∎

LEMMA 5.3. *Let $f: M \to M'$ be a morphism in $\mathscr{C}$. Then f is surjective if and only if $f \otimes_A 1_k$ (resp. $1_k \otimes_A f$, $1_k \otimes_A f \otimes_A 1_k$) is surjective.*

Proof. It is an immediate consequence of the previous lemma. ∎

LEMMA 5.4. *Let $f: M \to M'$ be a morphism in $\mathscr{C}$.*

(i) *f is surjective and essential (in the category $\mathscr{C}$) if and only if $1_k \otimes_A f \otimes_A 1_k$ is bijective.*

(ii) *Assume that f is surjective and M is pure in degree l. Then f is essential if and only if $f_l: M_l \to M'_l$ is bijective.*

Proof. (i) It suffices to adapt the proof of Proposition 7 of [11].

(ii) Since M' is also pure in degree l, we can use Lemma 5.1(iv). ∎

LEMMA 5.5. *The complex (in $\mathscr{C}$)*

$$M'' \xrightarrow{g} M' \xrightarrow{f} M \tag{5.1}$$

is exact if the following is exact:

$$M'' \otimes_A k \xrightarrow{g \otimes_A 1_k} M' \otimes_A k \xrightarrow{f \otimes_A 1_k} M \otimes_A k. \tag{5.2}$$

Proof. Assume that (5.2) is exact; i.e., the morphism

$$M'' \otimes_A k \xrightarrow{g \otimes_A 1_k} \ker(f \otimes_A 1_k) \tag{5.3}$$

is surjective. It is obvious that $\mathrm{im}(g \otimes_A 1_k) = \mathrm{im}(g) \otimes_A k$ and that $\ker(f) \otimes_A k \subseteq \ker(f \otimes_A 1_k)$. Therefore, (5.3) is the composition

$$M'' \otimes_A k \xrightarrow{g \otimes_A 1_k} \mathrm{im}(g) \otimes_A k \xrightarrow{i \otimes_A 1_k} \ker(f) \otimes_A k \hookrightarrow \ker(f \otimes_A 1_k), \tag{5.4}$$

where i is the natural injection of $\mathrm{im}(g)$ in $\ker(f)$. Hence the morphism

$$M'' \otimes_A k \xrightarrow{g \otimes_A 1_k} \ker(f) \otimes_A k \tag{5.5}$$

is surjective. Lemma 5.3 allows us to conclude that $g: M'' \to \ker(f)$ is surjective. ∎

The left Koszul complex (2.25) will be denoted by K_{L} with differential d_{L}, while its right counterpart will be denoted by K_{R} with differential d_{R}. Then $K_{\mathrm{L-R}} = K_{\mathrm{L}} \otimes_k A = A \otimes_k K_{\mathrm{R}}$ is a bimodule complex for the differentials $d'_{\mathrm{L}} = d_{\mathrm{L}} \otimes_k 1_A$ and $d'_{\mathrm{R}} = 1_A \otimes_k d_{\mathrm{R}}$. The latter commute, so that $K_{\mathrm{L-R}}$ is a bimodule complex for the differential d' defined in homological degree i by $d'_{\mathrm{L}} + (-1)^i d'_{\mathrm{R}}$. Since $K^i_{\mathrm{L-R}}$ is $A \otimes_k J_{js} \otimes_k A$ or $A \otimes_k J_{js+1} \otimes_k A$ according to $i = 2j$ or $i = 2j + 1$, $(K_{\mathrm{L-R}}, d')$ is naturally a complex in the category $\mathscr{C}$, which is projective and pure. It is called the bimodule Koszul complex of A. Using Lemma 5.5, we see that this complex, with the multiplication μ: $K^0_{\mathrm{L-R}} = A \otimes_k A \to A$ as augmentation, begins by the exact sequence

$$K^2_{\mathrm{L-R}} \xrightarrow{d'_2} K^1_{\mathrm{L-R}} \xrightarrow{d'_1} K^0_{\mathrm{L-R}} \xrightarrow{\mu} A \to 0. \tag{5.6}$$

THEOREM 5.6. *Let A be an s-homogeneous algebra on V. The augmented bimodule Koszul complex*

$$\cdots \xrightarrow{d'_3} K^2_{\mathrm{L-R}} \xrightarrow{d'_2} K^1_{\mathrm{L-R}} \xrightarrow{d'_1} K^0_{\mathrm{L-R}} \xrightarrow{\mu} A \to 0 \tag{5.7}$$

is exact if and only if A is Koszul.

Proof. Assume that A is Koszul. Applying the functor $- \otimes_A k$ to (5.7), we get the exact complex K_{L} with augmentation ϵ. Thus (5.7) is exact by Lemma 5.5. For the converse, we follow the arguments used by Butler and King in the quadratic case [10]. Assume that (5.7) is exact. Denote by $\mathscr{C}_{\mathrm{R}}$ the abelian category of the left-bounded graded right A-modules. Then (5.7) is a projective resolution of A in $\mathscr{C}_R$. But

$$\cdots \to 0 \to A \xrightarrow{1_A} A \to 0 \tag{5.8}$$

is also a projective resolution of A in $\mathscr{C}_{\mathrm{R}}$. Homological algebra in the category $\mathscr{C}_{\mathrm{R}}$ tells us that μ: $K^0_{\mathrm{L-R}} \to A$ extends to a resolution morphism f from (5.7) to (5.8) and that i: $A \to K^0_{\mathrm{L-R}}$, $a \mapsto 1 \otimes_k a$, extends to a resolution morphism g from (5.8) to (5.7). Therefore, the identity morphism on (5.7) is homotopic (still in $\mathscr{C}_{\mathrm{R}}$) to $g \circ f$ and hence is null homotopic in degree > 0. Applying the functor $- \otimes_A k$, we conclude that the left Koszul complex is exact in degree > 0. ∎

From now on, assume that A is Koszul. Then (5.7) is called the bimodule Koszul resolution of A. It is a minimal projective resolution in the category $\mathscr{C}$ (use Lemma 5.4(ii)). Let us show how this resolution allows us to compute the Hochschild homology of A. Recall that the Hochschild homology $HH_*(A)$ is isomorphic to $\mathrm{Tor}^{A^e}_*(A, A)$. Therefore, considering $K_{\mathrm{L-R}}$ as a complex of left A^e-modules which is a resolution of the left A^e-module A, we get

$$HH_*(A) \cong H_*(A \otimes_{A^e} K_{\mathrm{L-R}}, 1_A \otimes_{A^e} d'). \tag{5.9}$$

Fix $i \geq 0$. If i is even (resp. odd), set $i = 2j$ (resp. $2j+1$) and $n(i) = js$ (resp. $js+1$). Recall that $K^i_{\mathrm{L-R}} = A \otimes_k J_{n(i)} \otimes_k A$. Then the linear maps f and g defined by

$$A \otimes_{A^e} \left(A \otimes_k J_{n(i)} \otimes_k A\right) \xrightarrow{f} A \otimes_k J_{n(i)},$$

$$\bar{a} \otimes_{A^e} \left(\bar{b} \otimes_k m \otimes_k \bar{c}\right) \mapsto \overline{cab} \otimes_k m$$

$$A \otimes_k J_{n(i)} \xrightarrow{g} A \otimes_{A^e} \left(A \otimes_k J_{n(i)} \otimes_k A\right),$$

$$\bar{a} \otimes_k m \mapsto \bar{a} \otimes_{A^e} (1 \otimes_k m \otimes_k 1)$$

are inverses of each other. Thus, $A \otimes_{A^e} K_{\mathrm{L-R}}$ is identified with the vector space K_{L}. The differential $1_A \otimes_{A^e} d'$ is denoted by $\tilde{d}$ on K_{L}; therefore we obtain the complex of vector spaces over k,

$$\cdots \to A \otimes J_{2s} \xrightarrow{\tilde{d}_4} A \otimes J_{s+1} \xrightarrow{\tilde{d}_3} A \otimes J_s \xrightarrow{\tilde{d}_2} A \otimes J_1 \xrightarrow{\tilde{d}_1} A \otimes J_0 \to 0, \quad (5.10)$$

whose homology is $HH_*(A)$. Using the identifications, we get

$$\tilde{d}_1(\bar{a} \otimes v) = \overline{av} - \overline{va}, \qquad v \in V, \quad (5.11)$$

$$\tilde{d}_2(\bar{a} \otimes v_1 v_2 v_3) = \overline{a v_1 v_2} \otimes v_3 + \overline{v_2 v_3 a} \otimes v_1,$$

$$v_1 \in V, v_2 \in V^{(s-2)}, v_3 \in V, \quad (5.12)$$

$$\tilde{d}_i(\bar{a} \otimes v w v') = \overline{av} \otimes w v' + (-1)^i \overline{v'a} \otimes v w, \qquad i \geq 3. \quad (5.13)$$

In (5.13), v and v' both belong to $V^{(s-1)}$ (resp. V), and w belongs to $V^{(js-2s+2)}$ (resp. $V^{(js-1)}$) if i is even (resp. odd). Note that in the formulas (5.12) and (5.13), we have written down the natural extensions of $\tilde{d}_i$ to $A \otimes V^{(n(i))}$. Our next purpose is to investigate the complex (5.10) when A is the algebra whose relations are the antisymmetrizers of degree s.

From now on, $\operatorname{car}(k) = 0$, $r \geq s$, and A is the s-homogeneous algebra whose space R of relations is defined by (3.11). We have seen that A is Koszul. Let $X = (x_1, x_2, \ldots, x_r)$ be a basis of the vector space V. Some notations are needed. For any subset P of the index set $\{1, 2, \ldots, r\}$, the number of elements of P is denoted by $|P|$. Suppose $|P| = n$ with $1 \leq n \leq r$. Writing down the elements of P in the natural order as $u_1 < \cdots < u_n$, we set

$$[x_{u_1}, \ldots, x_{u_n}] = \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma) x_{u_{\sigma(1)}} \ldots x_{u_{\sigma(n)}}. \quad (5.14)$$

It will often be convenient to abbreviate $[x_{u_1}, \ldots, x_{u_n}]$ by $[P]$. Recall that the family of the $[P]$ such that $|P| = n$ is a basis of J_n. We also need the shuffles of P. Let Q be a subset of P with $0 < |Q| = m < n$. The elements of Q are $u_{i_1}, \dots, u_{i_m}$ with $i_1 < \cdots < i_m$. The elements of the complement Q^c are $u_{j_1}, \dots, u_{j_{n-m}}$ with $j_1 < \dots < j_{n-m}$. The Q-shuffle of P is the permutation σ_Q^P in $\mathbf{S}_n$ defined by

$$\sigma_Q^P(1) = i_1, \dots, \sigma_Q^P(m) = i_m, \qquad \sigma_Q^P(m+1) = j_1, \dots, \sigma_Q^P(n) = j_{n-m}.$$

Denoting by q the integer part of the quotient of r by s, recall that gl.dim$(A) = 2q$ or $2q+1$ according to whether s divides r or not. Clearly, $HH_i(A) = 0$ if $i >$ gl.dim(A). Fix i such that $2 \le i \le$ gl.dim(A). If i is even (resp. odd), set again $i = 2j$ (resp. $2j+1$) and $n(i) = js$ (resp. $js+1$). Assume that a belongs to Tens(V) and P is a subset of $\{1, 2, \dots, r\}$ with $|P| = n(i)$. Then for i even ≥ 2, one has

$$\tilde{d}_i(\bar{a} \otimes [P]) = \sum_{Q \subset P, |Q| = s-1} \operatorname{sgn}\left(\sigma_Q^P\right)\left(\overline{a[Q]} - (-1)^s \overline{[Q]a}\right) \otimes [Q^c], \tag{5.15}$$

and for i odd ≥ 3, one has

$$\tilde{d}_i(\bar{a} \otimes [P]) = \sum_{Q \subset P, |Q| = 1} \operatorname{sgn}\left(\sigma_Q^P\right)\left(\overline{a[Q]} - (-1)^{js} \overline{[Q]a}\right) \otimes [Q^c]. \tag{5.16}$$

These formulas are easily drawn from (5.12) and (5.13). For example, let us prove (5.16). The elements of P are denoted by $u_1 < \cdots < u_{js+1}$ following the natural order. Formula (5.13) shows that $\tilde{d}_i(\bar{a} \otimes [P])$ is equal to

$$\sum_\sigma \operatorname{sgn}(\sigma)\overline{ax_{u_{\sigma(1)}}} \otimes x_{u_{\sigma(2)}} \dots x_{u_{\sigma(js+1)}} - \sum_\sigma \operatorname{sgn}(\sigma)\overline{x_{u_{\sigma(js+1)}}a} \otimes x_{u_{\sigma(1)}} \dots x_{u_{\sigma(js)}}, \tag{5.17}$$

where σ runs over $\mathbf{S}_{js+1}$. For $l = 1, \dots, js+1$, introduce the singleton $Q_l = \{u_l\}$ of P. Any σ in $\mathbf{S}_{js+1}$ such that $\sigma(1) = l$ is uniquely written as $\sigma = \tau \circ \sigma_{Q_l}^P$, where τ is a permutation on $\{1, \dots, l-1, l+1, \dots, js+1\}$ considered in $\mathbf{S}_{js+1}$ by letting $\tau(l) = l$. Thus the first sum in (5.17) is equal to

$$\sum_{l=1}^{js+1} \sum_\tau (-1)^{l+1} \operatorname{sgn}(\tau)\overline{ax_{u_l}} \otimes x_{u_{\tau(1)}} \dots \check{x}_{u_{\tau(l)}} \dots x_{u_{\tau(js+1)}}. \tag{5.18}$$

Here, $\check{x}_{u_{\tau(l)}}$ means that the factor $x_{u_{\tau(l)}}$ is removed. Clearly (5.18) becomes

$$\sum_{l=1}^{js+1} (-1)^{l+1} \overline{ax_{u_l}} \otimes \left[x_{u_1}, \dots, \check{x}_{u_l}, \dots, x_{u_{js+1}}\right]. \tag{5.19}$$

Analogously, any σ in $\mathbf{S}_{js+1}$ such that $\sigma(js+1) = l$ is uniquely written as $\sigma = \tau \circ \sigma^{P}_{Q_l^c}$, where τ has the same meaning. Thus the second sum in (5.17) is equal to

$$\sum_{l=1}^{js+1} \sum_{\tau} (-1)^{js+l+1} \operatorname{sgn}(\tau) \overline{x_{u_l} a} \otimes x_{u_{\tau(1)}} \dots \check{x}_{u_{\tau(l)}} \dots x_{u_{\tau(js+1)}}, \tag{5.20}$$

which is easily transformed into

$$\sum_{l=1}^{js+1} (-1)^{js+l+1} \overline{x_{u_l} a} \otimes \left[x_{u_1}, \dots, \check{x}_{u_l}, \dots, x_{u_{js+1}} \right]. \tag{5.21}$$

Joining (5.19) and (5.21) together, we get (5.16). The proof of (5.15) is similar.

Let us examine some consequences of the formulas (5.11), (5.15), (5.16). First, assume $s = 2$. Then A is commutative, so $\tilde{d}$ vanishes in this case. We recover the well-known fact that the Hochschild homology of the symmetric algebra A is such that $HH_i(A) \cong \bigwedge^i_A(A^r)$ for $i = 0, \dots, r$ [20].

Assume now $s \geq 3$. Denote by n the degree of the graded complex (5.10). Then $HH_i(A)$ is naturally graded by subspaces $HH_i^{(n)}(A)$. It is obvious that $HH_0^{(0)}(A) = k$ and $HH_i^{(0)}(A) = 0$ for $i > 0$. Assume $0 < n < s$. The complex (5.10) in degree n is

$$0 \to V^{(n)} \xrightarrow{\tilde{d}_1} V^{(n)} \to 0, \tag{5.22}$$

where $\tilde{d}_1$ is given by (5.11). Accordingly, $HH_*^{(n)}(A)$ is the same as for the tensor algebra $\operatorname{Tens}(V)$ [20]. So $HH_i^{(n)}(A)$ vanishes when $i \geq 2$, and $HH_0^{(1)}(A) = HH_1^{(1)}(A) = V$. Moreover, if $1 < n < s$, denote by $t\colon V^{(n)} \to V^{(n)}$ the cyclic permutation $t(v_1 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$. Then $HH_0^{(n)}(A)$ is the space of the coinvariants $(V^{(n)})_t$, while $HH_1^{(n)}(A)$ is the space of the invariants $(V^{(n)})^t$ for the obvious group action defined by t.

On the other hand, fix i with $2 \leq i \leq \operatorname{gl.dim}(A)$. Clearly $HH_i^{(n)}(A) = 0$ whenever $n < n(i)$. Let us compute $HH_i^{(n(i))}(A)$. Suppose that i is *even*. Then (5.15) shows that

$$\tilde{d}_i([P]) = \begin{cases} 0 & \text{if } s \text{ is even} \\ 2[P] & \text{if } s \text{ is odd.} \end{cases} \tag{5.23}$$

Thus we have

$$HH_i^{(n(i))}(A) = \begin{cases} J_{n(i)} & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd.} \end{cases} \tag{5.24}$$

Supposing that i is odd, (5.16) gives, similarly,

$$HH_i^{(n(i))}(A) = \begin{cases} J_{n(i)} & \text{if } js \text{ is even} \\ 0 & \text{if } js \text{ is odd.} \end{cases} \tag{5.25}$$

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