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Symmetric Functions, Schubert Polynomials and Degeneracy Loci

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Fonctions Symétriques, Polynômes de Schubert et Lieux de Dégénérescence
(Symmetric Functions, Schubert Polynomials and Degeneracy Loci)

by Laurent Manivel

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Introduction

THE QUESTIONS ASKED IN THIS COURSE. This text is derived from a third-year graduate course given at l'Institut Fourier (Université Joseph Fourier, Grenoble 1) during the 1995–1996 academic year. On one hand, it serves as an introduction to the combinatorics of symmetric functions, more precisely to those of Schur polynomials, and hence to the combinatorics of Schubert polynomials. On the other hand, we study the geometry of Grassmannians, flag varieties, and especially their Schubert varieties. The profound connections which, despite our artificial classifications, unite these two subjects—subjects which *a priori* have so little in common—have for quite some time curiously remained unnoticed.

Schur polynomials were made explicit by Jacobi as far back as 1841 [**43**]. Their importance, however, derives above all from the role they play in the theory of group representations. Indeed, it was their eponym, Isaiah Schur, student of Frobenius, who in a celebrated memoir in 1901 recognized these functions as the irreducible characters of the complex linear group $GL(n, \mathbf{C})$ [**82**]. Furthermore, Schur polynomials also describe the representations of permutation groups, studied by A. Young with the aid of the tableaux to which he left his name [**98**].

Schubert varieties, on the other hand, are subvarieties of Grassmannians, spaces which parametrize the linear subspaces of given dimension of a vector space, in this case complex. They were introduced in the last century because of the needs of enumerative geometry [**81**]. How many lines in projective space intersect a given family of linear subspaces? How many lines does a hypersurface of given degree contain in the same projective space? How many conics of the plane are tangent to five given lines? Schubert varieties are some of the tools forged by German and Italian geometers in order to answer these questions. Indeed, one may recast these questions in terms of the intersections of certain varieties parametrizing the objects under consideration. In particular, it happens that the problem reduces to counting the number of points of an intersection of Schubert varieties.

Now, in an absolutely formal way, calculating products of Schur polynomials and calculating intersections of Schubert varieties (in homology at the very least, or in the Chow ring) are one and the same! But if the general answer to the first of these problems was given by Littlewood and Richardson (without, however, an entirely rigorous proof) as early as 1934 [**64**], it was only in 1947 that Lesieur made the analogy clear [**62**]. The knot then noticed between, on one hand, the combinatorics of partitions and permutations, and, on the other, the geometry of Grassmannians and flag varieties, has not ceased tightening. All this is the subject of this course, and here, more precisely, are the contents.

WHAT THE COURSE CONTAINS... I have divided the course into three chapters. The first is devoted to symmetric functions and especially to Schur polynomials.

These are the polynomials with positive integer coefficients in which each of the monomials corresponds to a Young tableau with the property of being "semistandard."

The combinatorial properties of these tableaux were successively explored by Robinson ([**77**], 1938), Schensted ([**80**], 1961), Knuth ([**48**], 1970), and then by Schützenberger and his school [**83, 84, 96**]. They remain today, more than ever, a subject of great interest. Through the introduction of the plactic monoid, it is these combinatorial properties, together with other properties of symmetric functions, which allow us to prove the celebrated Littlewood-Richardson rule which governs the multiplication of Schur polynomials.

I adopted this approach for at least two reasons. First, because the approach offers a different point of view from Macdonald's treatment [**66**], now already classical, and second, because it gave me the opportunity to make more accessible a theory which, it seems to me, is relatively unknown and yet for several years has only grown in importance.

I explain how Schur polynomials allow us to describe the irreducible characters of symmetric groups $\mathcal{S}_n$. I establish the following dictionary between symmetric functions and representations of groups of permutations:

Representations of $\mathcal{S}_n$	Symmetric functions
Irreducible representations (1.6.6)	Partitions of weight n (§1.1.1)
Degrees (1.6.8)	Number of standard tableaux (1.4.12)
Young's Rule (1.6.14)	Pieri's Formula (1.2.5)
Induction (§1.6.2)	Littlewood-Richardson Rule (1.5.23)
Tensor product	??????????

This last row signals that the problem, however fundamental, of the decomposition of a tensor product of irreducible representations of a symmetric group—a decomposition about which we know little—remains one of the obscure points of the theory.

Once the preceding dictionary is established, we consider the *Kostka-Foulkes polynomials*. These polynomials are the "q-generalizations" of the classical Kostka numbers (1882), the interest in which derives from the fact that they appear simultaneously in the theory of characters of linear groups over finite fields $\mathrm{GL}(n, \mathbb{F}_q)$ (Green 1955), in the description of the cohomology of nilpotent orbits of $\mathrm{GL}(n, \mathbb{C})$ (Kraft 1981, DeConcini and Procesi 1981, Garsia and Procesi 1992), as Kazhdan-Lusztig polynomials (Lusztig 1983), and as describing the charge statistic on Young tableaux (Lascoux and Schützenberger 1978 [**57**]). We take up their study from this last vantage point in order to establish some of the most remarkable properties.

The second chapter is devoted to Schubert polynomials. These polynomials were discovered, or invented, in 1982 by A. Lascoux and M.-P. Schützenberger [**58**], who deeply probed their combinatorial properties. We see, for example, that they support the subtle connections between problems of enumeration of reduced decompositions of permutations and the Littlewood-Richardson rule, a particularly efficacious version of which may be derived from these connections.

We are likewise indebted to Macdonald for a nice account of a not inconsiderable portion of this theory, now considered as a reference [**67**]. Nevertheless, I have repeatedly decided to stray from it. In particular, I have preferred to follow the elegant approach to Schubert polynomials which Fomin and Kirillov proposed, making connections with the Yang-Baxter equation and the Hecke algebras of the symmetric groups [**15**]. I have also chosen to take advantage of the lattice path method of Gessel and Viennot, a combinatorial method which has continued to prove its relevance since its appearance around 1986 [**28, 29**].

We note that Schubert polynomials are far from having revealed all of their secrets. We know almost nothing, for example, about their multiplication, and about a rule of Littlewood-Richardson type which must govern them.

If the first two thirds of this course are essentially of a combinatorial nature, the third and last chapter is essentially geometric. Indeed, it is devoted to Schubert varieties, subvarieties of Grassmannians, and flag varieties, defined by certain incidence conditions with fixed subspaces. These varieties were first introduced for the needs of enumerative geometry.

Since the first steps of algebraic topology, important efforts have been undertaken to make rigorous the work of Schubert and enumerative geometers [**94**], work which solved Hilbert's 15th problem. The problem is as follows [**38**]:

> Rigorous determination of numbers of enumerative geometry, and this by fixing as precisely as possible the limits on their validity, and, in particular, numbers Schubert found by relying on the principle of his enumerative calculus, called "special position" or "conservation of number."

So far, this program has been only incompletely carried out [**47**]. The cohomology ring of Grassmannians, however, was one of the first to be correctly understood [**94**], and uncovers an astonishing formal analogy with the ring of symmetric functions. One of my first objectives has been to establish a dictionary which permits the translation of problems of intersection of Schubert varieties into problems in terms of symmetric functions. This dictionary is established as follows:

Grassmannians	Symmetric functions
Schubert varieties (§3.2.1)	Partitions (§1.1.1)
Incidence (3.2.3)	Inclusion of partitions (§1.1.1)
Fundamental classes (3.2.9)	Schur polynomials (§1.2.1)
Degrees (§3.2.2)	Kostka numbers (§1.2.3)
Pieri's Formula (3.2.8)	Pieri's Formula (1.2.5)
Giambelli's Formula (3.2.10)	Jacobi-Trudi Formula (1.2.13)
Intersection (3.2.11)	Littlewood-Richardson Rule (1.5.23)
Postulation (3.3.5)	Plane partitions (§1.4.4)

The cohomology ring of flag varieties, moreover, has given rise to a number of important results. We cite the work of Ehresmann ([**13**], 1934), of Borel ([**4**], 1953) describing the ring as a quotient of a polynomial ring, of Chevalley ([**9**], circa 1958), and later that of Bernstein, Gelfand, and Gelfand ([**2**], 1973) and Demazure ([**10**], 1974). The last authors used in an essential way Newton's divided differences, on which rests even the definition of Schubert polynomials.

These polynomials appear also as representatives, in Borel's presentation, of the Poincaré duals of fundamental classes of Schubert varieties, representatives whose properties are so remarkable, we must note, that the definition of their analogues for simple complex Lie groups other than $\mathrm{SL}(n, \mathbb{C})$ continues to pose difficulties.

In the case of complete flag varieties, we may write down a part of the following dictionary:

Flag varieties	Polynomials
Schubert varieties (§3.6.2)	Permutations (§2.1.1)
Incidence (3.6.5)	Bruhat order (§2.1.2)
Fundamental classes (3.6.13)	Schubert polynomials (2.3.4)
Monk's Formula (3.6.12)	Monk's Formula (2.7.1)
Degrees	Number of chains of permutations
Postulation	Standard monomials
Intersection	???????????

Grassmannians may be realized as projective varieties thanks to the Plücker embeddings. After having shown how to obtain the equations of the Schubert varieties, we will be interested in their singular loci, which we describe explicitly. In the case of flag varieties, we consider a simple smoothness criterion, and we construct some desingularizations of Schubert varieties.

We then make explicit the connection that exists between Schubert varieties and the characteristic classes that Chern associates to a complex vector bundle on a differentiable variety. One may interpret these classes as representing, for example, the subvarieties defined by the annihilation of a suitable section of the bundle. More generally, the homology classes of the degeneracy loci of morphisms between fiber bundles, that is, of loci defined by certain conditions on rank, may be expressed in terms of characteristic classes. Without a doubt, the most famous example is the formula of Thom and Porteous, which dates from 1971 [**72**].

In 1992, W. Fulton obtained a vast generalization of this formula—and of its relatives which had been understood up to then (Kempf and Laksov 1974, Pragacz 1988)—by proving that the degeneracy loci of morphisms between vector bundles endowed with flags of subbundles may be described by means of certain Schubert polynomials [**21**]: the Schubert varieties in this situation play the role of universal degeneracy loci. In particular, for a very particular class of permutations, called vexillary by Lascoux and Schützenberger, one obtains determinantal formulas, certain examples of which go back to Giambelli. The course concludes with a proof of this result, which I hope will be appreciated as much for the remarkable generality of its statement as for the simplicity of its proof.

We are further indebted to Fulton for a recent book on Young tableaux and their use in geometry, a work of which this course has taken advantage for part of its substance [**22**]. The interested reader will find details there on the combinatorics of tableaux, a subject on which we preferred not to linger too long. The geometric part of Fulton's book, on the contrary, is very succinct. It was in part from a desire to extend it that this book was born.

...AND WHAT IT HAS NOT BEEN POSSIBLE TO COVER. This text has not, it must be said, the least pretension to completeness. Its only ambition is to lay some stones of an edifice which could be much more vast. I would therefore like to note in passing some themes which could complete it, if we had world enough and time.

I have not considered the theory of projective representations of symmetric groups (Schur 1911), and I have kept myself from mentioning the problems posed by positive characteristic, both in extending the theory of complex representations of symmetric groups to representations over finite fields, as well as in comprehending—a problem which is in some sense an inverse of the other—the theory of complex representations of the linear groups with coefficients in a finite field $\mathrm{GL}(n, \mathbb{F}_q)$ [**30, 66**].

Similarly, I have taken a detour around the classical description of Schur functors, which define representations of the complex linear group. I have already mentioned the fact that Schur polynomials describe their irreducible characters; still further, it is possible to establish a fairly rich dictionary between symmetric functions and representations of $\mathrm{GL}(n, \mathbb{C})$ [**97, 66, 49**].

Some connections which are as subtle as they are remarkable have also been discovered between, on one hand, certain aspects of the combinatorics of tableaux and plane partitions and, on the other, the representations of the other classical Lie groups and Lie algebras—in particular, via the Weyl character formula [**97**].

On the geometric side, the problems that we have grappled with represent only certain aspects of the study of Schubert varieties in generalized flag varieties. These are defined as quotients $X = G/P$ of a semisimple complex Lie group G by certain types of algebraic subgroups P called parabolic. These are projective varieties, the Schubert subvarieties of which are indexed by a certain quotient W^P of the Weyl group W of G: in the case of ordinary flags, for which $G = \mathrm{SL}(n, \mathbb{C})$, the group $W = W^P$ is none other than the group of permutations; while in the case of Grassmannians, W^P is the set of Grassmannian permutations with given descent. Here one may still obtain a cellular decomposition of X, different descriptions of its integral cohomology ring, divided difference operators, and subtle ties to the combinatorics of the Bruhat order on the Weyl group [**4, 9, 2, 10**].

Similarly, the understanding of ideals of Schubert varieties requires the definition of standard monomials, a theory which goes back to Hodge and which we have outlined in the case of Grassmannians. This theory could carry us fairly far into the study of the representations of complex Lie groups [**52**].

A great deal of attention has similarly been devoted to the singularities of generalized Schubert varieties, but they still remain poorly understood. The study of these singularities is nevertheless connected to those of the Kazhdan-Lusztig polynomials (Kazhdan and Lusztig 1979, Lascoux and Schützenberger 1981, Lascoux 1995). These polynomials, on which we have been able to spend some time, themselves lead us to the study of Hecke algebras, which we have only touched on.

In a related direction, we have also taken some forays into the "quantum" world. We have done so first because certain "q-deformations" of the classical symmetric functions—beginning with the Kostka-Foulkes polynomials—are connected to the representations of quantum groups (Lascoux, Leclerc and Thibon 1995), and also because of the recent definition of "quantum cohomology" rings, which are deformations of the cohomology rings of certain varieties. In particular, for Grassmannians and flag varieties, it is possible to develop a quantum Schubert calculus, the combinatorics of which is even now the object of intense study (Bertram 1994, Givental and Kim 1995, Ciocan-Fontanine 1995, Fomin, Gelfand, and Postnikov 1996, Bertram, Ciocan-Fontanine, and Fulton 1997).

Finally, it has been interesting to study the degeneracy loci of morphisms between vector bundles when some symmetry conditions are imposed. These have led to the introduction of some combinatorial tools which are as remarkable as the Schur functions, and moreover nearly homonymic: the Schur Q-polynomials, which are directly connected to the modular representations of the symmetric groups. A detailed treatment of this theme is found in the monograph [**25**].

SOME REMARKS BY WAY OF CONCLUSION. We end this introduction with some remarks on method. I have already said that this course has no encyclopedic ambition. At most, I have attempted to make accessible a certain number of results, which are not as many as one might wish. Perhaps it will also serve as a stepping stone from which one may scale more ambitious heights.

Neither does it pretend to originality: it contains neither a new result nor a new proof. I have picked and chosen from several sources in order to extract what seemed to me to constitute the substantive core, and at the same time I have attempted to make it a coherent whole, largely open to its possible extensions.

Moreover, I have tried, as much as possible, to remain elementary. This was easy for the first two chapters, which require no prior knowledge, and was somewhat less so for the third, where we use some rudimentary notions of topology and algebraic geometry. In order to help the reader who might find useful a brief "digest" of singular homology, or in any case of some notions that we have borrowed, I have made use of and adapted the best appendix, contained in [**22**], which concerns the topology of algebraic varieties.

On the other hand, due to the elementary character of many of the results of this course, if I may say so, I have permitted myself most of the time to provide proofs without unnecessary details—but I hope always complete, with some rare (and intentional!) exceptions. Some effort will therefore be necessary on the part of the reader, effort which in any case is indispensable, for instance, to understand the combinatorics of tableaux. Who could pretend, after all, to understand the Knuth correspondence without having sufficiently practiced it? In order to assist in gaining this experience, I have sprinkled the text with a certain number of exercises, rarely difficult, often accompanied by succinct directions.

May all this sustain the willing reader in the discovery of a theory which boasts many jewels!

CHAPTER 1

The Ring of Symmetric Functions

The basic object of study in this first chapter is the ring Λ_n of symmetric polynomials with integral coefficients in n independent variables. We are particularly interested in the family of Schur polynomials, the monomials of which we show correspond to semistandard tableaux of integers.

The different combinatorial operations on these tableaux which may be defined using Knuth's insertion algorithm permit us to establish some of the most remarkable properties of this family of symmetric functions. With the aid of these functions we establish the celebrated Littlewood-Richardson rule which describes the decomposition of the product of two Schur functions in terms of these same functions.

We then consider how Schur polynomials permit the description of the characters of irreducible representations of permutation groups, and we devote some time to the study of these representations, namely Specht modules.

Finally, we conclude this chapter with a study of Kostka-Foulkes polynomials, which are the "q-generalizations" of the numbers of semistandard tableaux of given shape and weight. In particular, we establish a conjecture of Foulkes, proved by Lascoux and Schützenberger, according to which these polynomials are described by a certain statistic on the tableaux, called the charge. We also explore the connections with the action of the symmetric group on semistandard tableaux.

1.1. Ordinary Functions

1.1.1. Elementary Symmetric Functions. We begin with some notation concerning *partitions*, which are finite decreasing sequences λ of natural numbers $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$. The integers $\lambda_1, \ldots, \lambda_l$ are the *parts*. The *length* $l(\lambda)$ designates the number of nonzero parts, and the *weight* $|\lambda|$ designates the sum of the parts. We will be concerned very little with these zero parts, and, in particular, we allow the addition or deletion of zero parts when the need arises.

The *Ferrers diagram* of λ is obtained by lining up, from top to bottom, rows of lengths given by the parts of λ and sharing the same leftmost column. By diagonal symmetry, we obtain the Ferrers diagram of the *conjugate partition*, denoted λ^*.

EXAMPLE 1.1.1. Figure 1 is the diagram of the partition $\lambda = (5,3,3,2)$, of length 4 and weight 13, and of its conjugate partition $\lambda^* = (4,4,3,1,1)$, of length 5 and the same weight. If there is no ambiguity, we will denote such partitions as $\lambda = 5332$ and $\lambda^* = 44311$.

We may endow the set of partitions with a natural partial order, defined by the relation of inclusion for Ferrers diagrams. It is more useful, however, to use the partial order given by *dominance*, according to which $\lambda \geq \mu$ if the two partitions

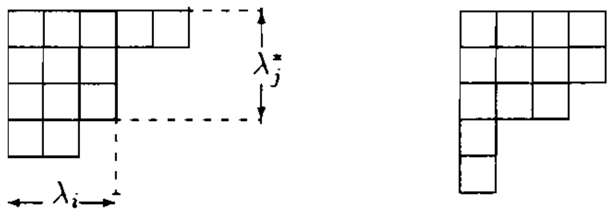


FIGURE 1. Ferrers diagram of a partition and its conjugate

have the same weight and, for all integers i, the inequality

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

is satisfied.

EXERCISE 1.1.2. Show that the set of partitions, endowed with either partial order, forms a lattice: given two partitions, there exists a unique minimal element among those which dominate (or are dominated by) each of them. Verify that for the dominance order, the partitions of given weight form a symmetric lattice, in the sense that $\lambda \geq \mu$ if and only if $\mu^* \geq \lambda^*$.

The most natural way to construct a symmetric polynomial in n variables is, undoubtedly, to begin with a monomial $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and to symmetrize it. We then obtain the *monomial symmetric functions*

$$m_\lambda = \sum_{\alpha \in \mathcal{S}_n(\lambda)} x^\alpha .$$

Only monomials which are pairwise distinct appear in this expression, for λ is a partition with n parts $\lambda_1, \ldots, \lambda_n$; $\mathcal{S}_n$ denotes the group of permutations of these integers between 1 and n; and $\mathcal{S}_n(\lambda)$ denotes the set of n-tuples α of integers such that $\alpha_i = \lambda_{w(i)}$ for a (not necessarily unique) permutation $w \in \mathcal{S}_n$.

The monomial symmetric functions form a base for the ring Λ_n of symmetric functions in the n variables $x_1, \ldots, x_n$ with integral coefficients. For partitions such that the nonzero parts are all equal to one, these monomial functions give in particular the *elementary symmetric functions*

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k},$$

for $1 \leq k \leq n$. For each partition λ, we let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$.

The following result is sometimes called the fundamental theorem of symmetric functions (N. Bourbaki, *Algèbre*, chapitre 4):

FUNDAMENTAL THEOREM 1.1.3. *The polynomials e_λ, where λ runs through the set of partitions of arbitrary length with parts less than or equal to n, form a base of Λ_n. In other words,*

$$\Lambda_n = \mathbb{Z}[e_1, \ldots, e_n].$$

1.1.2. Complete Symmetric Functions. Closely related to the elementary symmetric functions are the *complete symmetric functions*

$$h_k = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k},$$

sums of all monomials of degree $k \geq 1$. Their products will be denoted $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_n}$, and we set $h_0 = e_0 = 1$.

EXAMPLE 1.1.4. If $n = 3$, we obtain, among others, the following polynomials:

$$\begin{aligned}
e_1 = h_1 = m_1 &= x_1 + x_2 + x_3, \\
e_2 = m_{11} &= x_1x_2 + x_1x_3 + x_2x_3, \\
h_2 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3, \\
m_2 &= x_1x_2x_3, \\
e_3 = m_{111} &= x_1^3 + x_2^3 + x_3^3 + x_1x_2x_3 + \\
&\quad x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.
\end{aligned}$$

PROPOSITION 1.1.5. *The ring homomorphism $\omega\colon \Lambda_n \to \Lambda_n$, defined by $\omega(e_k) = h_k$, is an involution. Consequently,*

$$\Lambda_n = \mathbb{Z}[h_1, \ldots, h_n].$$

PROOF. Consider the formal generating series

$$e(t) = \sum_{k \geq 0} e_k t^k = \prod_{i=1}^{n} (1 + tx_i),$$

$$h(t) = \sum_{k \geq 0} h_k t^k = \prod_{i=1}^{n} (1 - tx_i)^{-1}.$$

These series satisfy the relation $e(t)h(-t) = 1$, hence $\sum_{i+j=k} (-1)^i e_i h_j = 0$ for all integers $k > 0$. Because these last relations are symmetric with respect to e and h, we deduce by induction that if $h_k = P_k(e_1, \ldots, e_n)$ for a polynomial P_k, then we must necessarily also have $e_k = P_k(h_1, \ldots, h_n) = \omega(h_k)$. ☐

Symmetric functions, both elementary and complete, may be expressed in terms of monomial functions as follows.

PROPOSITION 1.1.6. *Let $a_{\lambda\mu}$ (respectively $b_{\lambda\mu}$) be the number of matrices with entries zero or one (respectively, with integer entries) for which the successive row and column sums are given by the parts of λ and μ. Then*

$$e_\lambda = \sum_\mu a_{\lambda\mu} m_\mu \qquad \textit{and} \qquad h_\lambda = \sum_\mu b_{\lambda\mu} m_\mu.$$

EXERCISE 1.1.7. Verify the proposition, and show that if $a_{\lambda\mu} > 0$, then $\mu \leq \lambda^*$.

1.2. Schur Functions

There are different ways to define Schur functions. We choose the oldest approach, that of Jacobi, which has the inconvenience of making it difficult to write down the polynomials explicitly. It is necessary first to develop a number of preliminary results in this section, and then we begin to explore the connection between symmetric functions and tableaux. It is only in the next section that the theorem of Littlewood solves the thorny problem of making the functions explicit.

1.2.1. Jacobi's Definition. Multiplication by the Vandermonde determinant $\det(x_i^{n-j})_{1\le i,j\le n} = \prod_{1\le i<j\le n}(x_i - x_j)$ gives an isomorphism, shifting the degree, between symmetric and antisymmetric polynomials. Just as monomial functions give the most natural bases of symmetric functions, we obtain bases of antisymmetric functions by antisymmetrizing the monomials. Hence we set, for μ an n-tuple of natural numbers,

$$a_\mu = \sum_{w\in S_n} \varepsilon(w) x^{w(\mu)},$$

where $\varepsilon(w)$ is the signature of the permutation w. This polynomial is zero if μ has two equal components and remains unchanged, up to sign, if one permutes the components of μ. We may then restrict ourselves to strictly decreasing partitions. Such partitions take the form $\mu = \lambda + \delta$, where λ is a partition and $\delta = (n-1, n-2, \ldots, 1, 0)$ is the smallest strictly decreasing partition.

In the other direction, by dividing these polynomials by the Vandermonde, which is none other than a_δ, we obtain some symmetric functions introduced for the first time by Jacobi [**43**], but commonly called *Schur functions:*[1]

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} = \frac{\det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}}{\det(x_i^{n-j})_{1\le i,j\le n}}.$$

The fact that this procedure induces an isomorphism between symmetric and antisymmetric functions with integer coefficients translates into the following statement:

PROPOSITION 1.2.1. *As λ runs through the set of partitions of length at most n, the Schur functions s_λ form a base of Λ_n.*

REMARK 1.2.2. If α is an n-tuple of natural numbers, not necessarily decreasing, we again set $s_\alpha = a_{\alpha+\delta}/a_\delta$. Two cases may then result. If $\alpha+\delta$ has two equal components, then $s_\alpha = 0$. If there exist a partition λ and a permutation w such that $\alpha + \delta = w(\lambda+\delta)$, then $s_\alpha = \varepsilon(w)s_\lambda$.

EXAMPLE 1.2.3. For a partition having only one nonzero part, equal to k, we obtain the complete symmetric function h_k. For a partition composed of k nonzero parts, all equal to 1, we obtain the elementary symmetric function e_k.

EXERCISE 1.2.4. Verify that $s_\delta = \prod_{1\le i<j\le n}(x_i + x_j)$.

The definition of Schur functions that we have given, as the quotient of two determinants, suffers from a major fault: it does not permit us to write the functions explicitly. It is the theorem of Littlewood 1.4.1 which mitigates this inconvenience.

For the moment, we are content to remark that the elementary and complete symmetric functions are particular Schur functions. They are associated, respectively, to partitions with nonzero parts equal to one, and to their conjugates, for which only one part is nonzero:

$$h_k = s_k \qquad \text{and} \qquad e_k = s_{1^k},$$

where 1^k denotes the partition of weight k for which each nonzero part is 1.

[1]They were in fact rediscovered by Schur who, studying the complex linear representations of $GL(n,\mathbb{C})$, interpreted them as the characters of irreducible representations of this group [**82**].

1.2.2. Pieri's Formulas. We will see that the Schur functions can, more generally, be expressed in the form of determinants of elementary or complete symmetric functions. But we first establish a formula for the product of a Schur function and an elementary or complete symmetric function.

Let λ be a partition and k an integer. We denote $\lambda \otimes k$ (respectively, $\lambda \otimes 1^k$) the set of partitions obtained by adding k boxes, or cells, to λ, at most one per column (respectively, at most one per row).

PIERI'S FORMULAS 1.2.5. *With the preceding notation,*

$$s_\lambda e_k = \sum_{\mu \in \lambda \otimes 1^k} s_\mu \qquad \textit{and} \qquad s_\lambda h_k = \sum_{\mu \in \lambda \otimes k} s_\mu .$$

PROOF. The first of these formulas is deduced from the expansion

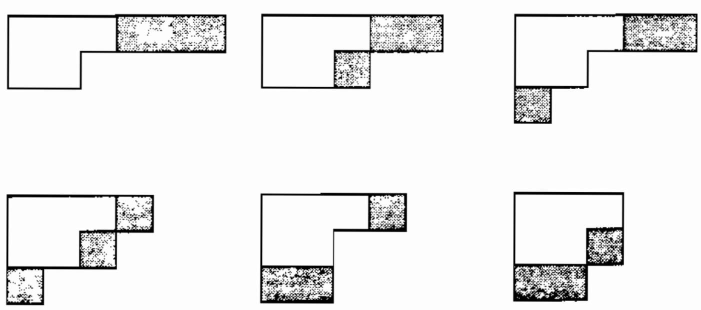
$$\begin{aligned} a_{\lambda+\delta} e_k &= \sum_{w \in S_n} \sum_{i_1 < \cdots < i_k} \varepsilon(w) x^{w(\lambda+\delta)} x_{w(i_1)} \cdots x_{w(i_k)} \\ &= \sum_{\alpha \in \{0,1\}^n} a_{\lambda+\alpha+\delta}, \end{aligned}$$

taking into account the fact that $a_{\lambda+\alpha+\delta}$ is nonzero if and only if $\lambda+\alpha$ is a partition. For the second formula, we obtain the same identity

$$a_{\lambda+\delta} h_k = \sum_{|\alpha|=k} a_{\lambda+\alpha+\delta}.$$

Suppose that there exists an integer i such that $\alpha_{i+1} > \lambda_i - \lambda_{i+1}$. Define a sequence β by $\beta_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$, $\beta_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$, and $\beta_j = \alpha_j$ for $j \neq i, i+1$. Then $a_{\lambda+\alpha+\delta} = -a_{\lambda+\beta+\delta}$, and the result follows after cancellations. $\square$

EXAMPLE 1.2.6. We have $s_{32} h_3 = s_{62} + s_{53} + s_{521} + s_{431} + s_{422} + s_{332}$, where the different terms of the sum correspond to the following Ferrers diagrams, where the cells added are in gray:



EXERCISE 1.2.7. We say that a partition is *even* when all of its parts are even. Deduce from Pieri's formulas the formal identity

$$\left(\sum_{\mu \text{ even}} s_\mu \right) \times \left(\sum_{k=0}^{n} e_k \right) = \sum_{\lambda} s_\lambda .$$

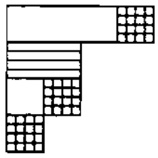


FIGURE 2. Decomposition into bands

1	1	1	4
2	2		
3	4		
4			

FIGURE 3. Semistandard tableau

1	2	3	9
4	5		
6	8		
7			

FIGURE 4. Standard tableau

1.2.3. Tableaux. Pieri's formulas may be conveniently interpreted in terms of Ferrers diagrams: to calculate the products $s_\lambda e_k$ or $s_\lambda h_k$, one adds to λ some vertical or horizontal bands, of total length k. In particular, if one part of a diagram is empty, taking products with complete symmetric functions amounts to constructing diagrams by adjoining horizontal bands. Filling in these different bands with successive integers, we obtain a diagram numbered in an increasing fashion along the rows (from left to right), and strictly increasing along the columns (from top to bottom).

We call such a numbered diagram T a *semistandard tableau*. While one usually calls these Young tableaux, we omit this reference to Alfred Young solely to simplify our terminology. The *shape* $\lambda(T)$ of the tableau T is the partition consisting of its support. Its *weight* $\mu(T)$ is the sequence consisting of the numbers of occurrences in T of successive integers; $\mu(T)_i$ is the number of entries of T equal to i. A semistandard tableau numbered with successive integers, each appearing only once and therefore of weight $(1, \dots, 1)$, is called *standard*.[2]

EXAMPLE 1.2.8. The semistandard tableau T of figure 3 has for its shape the partition $\lambda(T) = 4221$ and for its weight $\mu(T) = 3213$.

COROLLARY 1.2.9. *Let $K_{\lambda\mu}$ be the number of semistandard tableaux with shape λ and weight μ. (These integers are called* Kostka numbers.*) Then*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda \quad \textit{and} \quad e_\mu = \sum_\lambda K_{\lambda\mu} s_{\lambda^*}.$$

[2]These tableaux were introduced by the Reverend Alfred Young in his work on the representations of the permutation group [**98**].

In particular, if K_λ is the number of standard tableaux of shape λ, then

$$e_1^n = \sum_{|\lambda|=n} K_\lambda s_\lambda.$$

COROLLARY 1.2.10. *If $|\lambda| \leq n$, then $\omega(s_\lambda) = s_{\lambda^*}$.*

EXERCISE 1.2.11. Show that $K_{\lambda\mu} \neq 0$ if and only if $\lambda \geq \mu$.

EXERCISE 1.2.12. Prove the identities

$$K_\lambda = \sum_{\lambda\in\mu\otimes 1} K_\mu, \quad (1+|\lambda|)K_\lambda = \sum_{\mu\in\lambda\otimes 1} K_\mu, \quad \text{and} \quad \sum_{|\lambda|=l} K_\lambda^2 = l!,$$

the second by induction on the weight of λ and the third by induction on l from the second. We will see in section 1.6 that this last formula, which may be derived immediately from the Robinson correspondence, may be interpreted as giving a particular case, that of the symmetric group $\mathcal{S}_l$, of the following general fact: the sum of the squares of the dimensions of the irreducible representations of a finite group is equal to the order of the group.

1.2.4. The Jacobi-Trudi Formulas. We have come to see how the elementary or complete symmetric functions may be decomposed into Schur functions. Inversely, the Schur functions may be expressed in terms of elementary or complete symmetric functions, according to the following formulas:

JACOBI-TRUDI FORMULAS 1.2.13. *If a partition λ has length at most n, then*

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1\leq i,j\leq n} \quad \text{and} \quad s_{\lambda^*} = \det(e_{\lambda_i - i+j})_{1\leq i,j\leq n}.$$

PROOF. Observe first of all that the formulas remain unchanged if we restrict the order of these determinants to the length l of λ. This observation permits an induction on l. For the first formula, for example, we obtain by expanding the determinant along the last column the alternating sum

$$\sum_{i=1}^{l} (-1)^{l-i} s_{\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1}-1,\ldots,\lambda_l-1} \times h_{\lambda_i+l-i}.$$

Using Pieri's formulas, we may write the ith term of this sum in the form

$$\sum_{\mu\in J_i} s_\mu + \sum_{\mu\in J_{i+1}} s_\mu,$$

where J_i is the set of partitions μ of the same weight as λ, such that $\lambda_j \leq \mu_j \leq \lambda_{j-1}$ for $j < i$, and $\lambda_{j+1} - 1 \leq \mu_j \leq \lambda_j - 1$ for $j \geq i$. After cancellations, we have the first stated formula. The second may be proved in the same fashion. □

EXERCISE 1.2.14. Show that the first of these formulas remains valid for $l(\lambda) > n$, if we set $s_\lambda = 0$. With this convention, the second formula is similarly correct in complete generality.

1.2.5. Giambelli's Formula. In the same spirit as the Jacobi-Trudi formulas, we establish a determinantal formula attributed to Giambelli. The role of the elementary or complete symmetric functions is played in this situation by the Schur functions associated to partitions for which the diagram is a *hook*, that is, the diagram has shape $(j+1, 1, \ldots, 1)$. We denote by $(j|k)$ this partition if it has length

FIGURE 5. Decomposition into hooks

$k+1$. Its weight is then $j+k+1$, and its conjugate partition is none other than $(k|j)$. Pieri's formula gives $h_j e_k = s_{(j-1|k)} + s_{(j|k-1)}$, whence the identity

$$s_{(j|k)} = \sum_{l=0}^{k} (-1)^l h_{j+1+l} e_{k-l}.$$

EXERCISE 1.2.15. By expressing the products $h_j e_k$ in two different ways, derive the formula

$$\sum_{\lambda} q^{l(\lambda)-1} m_\lambda = \sum_{j,k} (q-1)^k s_{(j|k)}.$$

Each partition λ may be decomposed into hooks as in figure 5.

Let l be the length of the diagonal of the diagram of λ. We then write

$$\lambda = (\alpha_1, \dots, \alpha_l | \beta_1, \dots, \beta_l),$$

where the integers $\alpha_i = \lambda_i - i$ are the lengths of the horizontal bands lying on the right of the diagonal of λ, and the $\beta_j = \lambda_j^* - j$ are the lengths of the vertical bands lying below the same diagonal. This is the *Frobenius notation*.

GIAMBELLI'S FORMULA 1.2.16. *If* $\lambda = (\alpha_1, \dots, \alpha_l | \beta_1, \dots, \beta_l)$, *then*

$$s_\lambda = \det(s_{(\alpha_i|\beta_j)})_{1 \le i,j \le l}.$$

PROOF. If j or k is negative, we again define a symmetric function $s_{(j|k)}$ by the identity

$$s_{(j|k)} = \sum_{l=0}^{k} (-1)^l h_{j+1+l} e_{k-l}.$$

An elementary calculation shows that $s_{(j|k)} = 0$, except if $j+k=-1$, in which case we have that $s_{(j|k)} = (-1)^k$. Furthermore, the product of the matrices with entries $h_{\lambda_i - i + j}$ and $(-1)^{j-1} e_{n+1-j-k}$ is equal to the matrix with entries $s_{(\lambda_i - i|n-k)}$. Passing to determinants, and taking into account the Jacobi-Trudi formulas, we deduce the identity

$$s_\lambda = \det(s_{(\lambda_i - i|n-k)})_{1 \le i,k \le n}.$$

For $i > l$, however, the ith row of this last matrix has only one nonzero entry, located in the column of index k such that $n-k = i - \lambda_i - 1$. Then it is easy to verify that the complement in $\{0, \dots, n-1\}$ of the positive integers $i - \lambda_i - 1$ (so that $i > l$) is precisely the set of positive integers $\beta_j = \lambda_j^* - j$ (so that $j \le l$). In fact, we have arranged all n integers between 0 and $n-1$, and we never have $(\lambda_j^* - j) - (i - \lambda_i - 1) = (\lambda_j^* - i) + (\lambda_i - j) + 1 = 0$, since $\lambda_j^* - i$ and $\lambda_i - j$

are simultaneously positive or strictly negative, according to whether the Ferrers diagram of λ contains the cell (i, j) or not.

Expanding the preceding determinant with respect to the last $n - l$ rows, we obtain, after a verification of the sign which we leave to the goodwill of the reader, Giambelli's formula. □

EXAMPLE 1.2.17. For the partition 3211, we have the following determinantal expressions of the associated Schur function:

$$s_{3211} = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_4 & e_5 & e_6 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix} = \begin{vmatrix} s_{(2|3)} & s_{(2|0)} \\ s_{(0|3)} & s_{(0|0)} \end{vmatrix}.$$

1.2.6. The Ring of Symmetric Functions. We conclude this section with several formal remarks. If Λ_n^k denotes the set of symmetric polynomials in n variables with integer coefficients which are homogeneous of degree k, we have restriction maps

$$r_{n+1}^k : \Lambda_{n+1}^k \to \Lambda_n^k,$$

defined by setting $x_{n+1} = 0$. For example, if λ is a partition of weight k, $r_{n+1}^k(s_\lambda) = s_\lambda$ if $l(\lambda) \leq n$, and $r_{n+1}^k(s_\lambda) = 0$ otherwise. Consequently, r_{n+1}^k is an isomorphism when $n \geq k$.

The projective limit $\Lambda^k = \lim_{\leftarrow n} \Lambda_n^k$ therefore admits a base consisting of the family of elements that we will still denote s_λ, indexed by the finite set of partitions λ of weight k. These elements specialize in Λ_n^k to the Schur functions of n variables.

DEFINITION 1.2.18. *We call the direct sum*

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

the ring of symmetric functions. *This ring is endowed with the structure of a graded ring; it is a ring of polynomials over a countable family of indeterminates:*

$$\Lambda = \mathbb{Z}[e_k,\ k > 0] = \mathbb{Z}[h_k,\ k > 0],$$

e_k *and* h_k *being of degree* k.

All of the formulas established thus far remain valid in Λ, without the restrictions of the type appearing in the statement of Corollary 1.2.10; ω extends to an involutory automorphism of Λ, and for all partitions λ, we have $\omega(s_\lambda) = s_{\lambda^*}$. Hence, in general we work in Λ directly and specialize to a given number of variables when necessary. This specialization simply annihilates the Schur functions associated to partitions of length greater than this number.

1.3. The Knuth Correspondence

In the last section, we saw that tableaux and matrices with integer entries appear naturally. These objects share some subtle combinatorial connections, of which the Knuth correspondence [48] is among the most important. The point of departure is the following problem: how to add, in a canonical manner, a given integer to a tableau?

1.3.1. Insertion. Let T be a semistandard tableau and n an integer. Knuth insertion is a procedure which adds to T a cell numbered with the integer n, in a manner which makes sense combinatorially. We will have many occasions in the following pages to appreciate the virtues of this algorithm, which we now describe without further justification.

Consider first the top row of T. If it contains only integers less than or equal to n, we place the new cell on the right of this row and we are done. If not, we consider the cell of this first row of T which is located the furthest to the left among those which are numbered with integers strictly larger than n. We replace the integer p in this cell by n, and then move to the following row with this integer p. We continue until the procedure stops. We are finally left with a new semistandard tableau S, numbered with the same integers as the tableau T together with the inserted integer n.

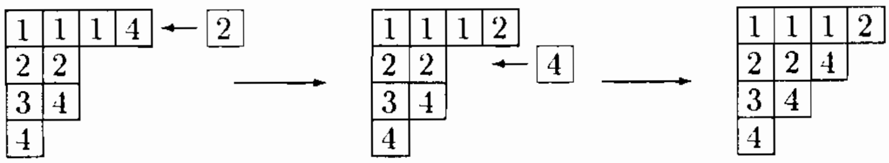


FIGURE 6. Knuth insertion

An important property of this procedure is that it is *reversible*; if we know which cell has been added to a semistandard tableau S, we may recover the tableau T and the integer n which was inserted to obtain S. It suffices to consider the integer p which lies in the added cell. We remove this cell from its row and place it on the preceding row where we find the integer the furthest to the right among those which are strictly smaller than p. We begin again with the cell which we replaced, and continue until we have taken out a integer n from the top row, recovering the original tableau T.

EXERCISE 1.3.1. We insert in T an integer n, hence a new cell x, and then an integer n', hence a new cell x'. Verify that if $n \leq n'$, x is strictly to the left of x', while if $n > n'$, x is strictly below x'.

We may also define, in a symmetric manner, a column insertion. Given an integer n and a tableau T, we place n at the end of the first column of T if n is the greatest among all the numbers contained in the column. If not, we place it in the cell occupied by the smallest integer of this column which n is greater than or equal to, and we move to the following column, until the procedure ends.

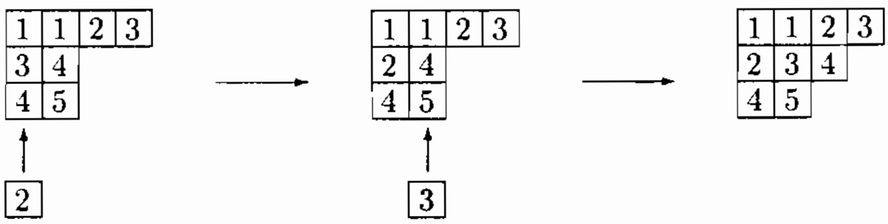


FIGURE 7. Insertion by columns

1.3.2. Correspondences. If we are given a finite sequence m of strictly positive integers, which is called a *word* of the alphabet $\mathbb{N}^*$, it will be of interest to describe the semistandard tableau T obtained from the insertion procedure. Simply stated, we successively insert the letters of the word, from left to right, into the empty tableau. We will say that the tableau T obtained is the *rectification* of the word m.

From the preceding remark, it is possible to reconstruct the word m from the tableau T if one knows the order of appearance of its cells, this order being given by a standard tableau S of the same shape as T. This may be stated in the following fashion:

SCHENSTED CORRESPONDENCE 1.3.2. *There exists a bijective correspondence between words m of the alphabet $\{1, \dots, n\}$ and pairs (T, S) of tableaux of the same shape, numbered with integers between 1 and n, the first of which is semistandard and the second of which is standard.*

We remark that T is standard if and only if the word m to which it is associated is of the form $w(1)w(2)\dots w(n)$ for a certain permutation $w \in S_n$. Hence we have a predecessor of the preceding correspondence:

ROBINSON CORRESPONDENCE 1.3.3. *There exists a bijective correspondence between permutations $w \in S_n$ and pairs (T, S) of standard tableaux of size n of the same shape.*

$$w = 279385146 \quad \longrightarrow \quad T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} 9 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad , \quad S = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 9 \\ \hline 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

FIGURE 8. Robinson correspondence

The Schensted correspondence may be extended to pairs (T, S) of semistandard tableaux with the following convention. If these tableaux are of size l, let s_l be the largest entry of S. We choose, among the cells of S numbered with this integer, the one furthest to the right. We remove it, and bump out the corresponding cell of T by the inverse process of insertion, which removes from this tableau an integer t_l.

Repeating this procedure, we obtain two sequences of integers $s = (s_1, \dots, s_l)$ and $t = (t_1, \dots, t_l)$, with $s_1 \leq \dots \leq s_l$ and $t_k \leq t_{k+1}$ if $s_k = s_{k+1}$. Indeed, if we bump out two cells lying at the corners of a tableau, beginning with the rightmost, this first integer is necessarily greater than or equal to the second integer bumped out. We denote by s and t these sequences of entries of the tableaux S and T. The first is ordered in an increasing fashion, and we say that the second is ordered in a fashion compatible with the first.

Inversely, given sequences s and t satisfying the preceding conditions of monotonicity, we reconstruct the original pair of tableaux by successively inserting $t_1, \dots, t_l$ to construct T, while S is obtained by placing s_k where we inserted t_k in constructing T. The increasing properties of the double sequence guarantees the semistandard character of S (see exercise 1.3.1 above).

We remark that being given s and t is equivalent to being given pairs (s_1, t_1), $\dots$, (s_l, t_l), in any order. These are then also equivalent to being given a matrix

$A = \sum_i E_{s_i, t_i}$, where $E_{p,q}$ is the matrix consisting of all zeros except for a one in row p, column q. In conclusion:

KNUTH CORRESPONDENCE 1.3.4. *There exists a bijective correspondence between matrices A with positive integer entries, almost all zero, and pairs (T, S) of semistandard tableaux of the same shape. Under this correspondence, the column sums (respectively, the row sums) of A are given by the weight of T (respectively, of S).*

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \quad T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}, \quad S = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

FIGURE 9. Knuth correspondence

If we begin with a permutation, apply to it the Robinson correspondence, and then apply the Knuth correspondence to the pair of standard tableaux of the result, we simply obtain the matrix of the permutation.

1.4. Some Applications to Symmetric Functions

The interest in these manipulations of tableaux which we have begun to study lies in the fact that they permit us to obtain, without difficulty, a whole series of nontrivial properties of symmetric functions.

1.4.1. Littlewood's Theorem. Littlewood's Theorem gives the most convenient way to calculate explicitly a Schur function. Given a partition, each semistandard tableau with that partition as its shape corresponds to a monomial of the associated Schur function.

THEOREM 1.4.1. *For all partitions λ we have the identity*

$$s_\lambda = \sum_{\lambda(T)=\lambda} x^{\mu(T)},$$

where the sum runs over the set of semistandard tableaux with shape λ.

PROOF. Denote by t_λ the right-hand side of the identity above. It is not at all evident *a priori* that this is a symmetric function! Nevertheless, we expand the product

$$h_\mu = h_{\mu_1} \cdots h_{\mu_m} = \sum_A \prod_{j=1}^{n} x_j^{a_{1j}+\cdots+a_{nj}}.$$

This sum runs over the set of $n \times n$ matrices $A = (a_{ij})$ with integer entries such that $\sum_k a_{lk} = \mu_l$ for each l. Via the Knuth correspondence, we obtain the identity

$$h_\mu = \sum_{\substack{\lambda(T)=\lambda(S), \\ \mu(T)=\mu}} x^{\mu(S)} = \sum_\lambda K_{\lambda\mu} t_\lambda.$$

In other words, the functions h_μ have the same expansion into Schur functions s_λ as into the functions t_λ. Recognizing that this expansion is invertible, we have that it corresponds to a change of base inside the ring of symmetric functions. We deduce therefore that $s_\lambda = t_\lambda$ for all partitions λ. □

REMARK 1.4.2. This result suggests that there exists an action, preserving weight, from the symmetric group to the semistandard tableaux. This action does in fact exist, but is very delicate to define. We will describe it in section 1.8.

1.4.2. Cauchy Formulas. We similarly demonstrate two remarkable identities due, in a slightly different form, to Cauchy. Consider two sets of independent variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.

THE FIRST CAUCHY FORMULA 1.4.3.

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

PROOF. We brutally expand

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_A \prod_{i,j}(x_i y_j)^{a_{ij}},$$

this formal sum running over the set of matrices with integer entries, almost all zero. The Knuth correspondence then gives us

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda(T)=\lambda(S)} x^{\mu(T)} y^{\mu(S)} = \sum_\lambda s_\lambda(x) s_\lambda(y),$$

which we were to prove. □

We define on the ring Λ_n an integer-valued scalar product by requiring that the family of Schur functions be orthonormal. This product extends to Λ and makes the involution ω an isometry.

EXERCISE 1.4.4. Deduce from the Cauchy formula the following result: if a_λ and b_λ are families of symmetric functions, homogeueous of degree $|\lambda|$ and indexed by partitions, such that

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda a_\lambda(x) b_\lambda(y),$$

then these form bases for Λ_n, which are dual with respect to the scalar product considered.

PROPOSITION 1.4.5. *The two bases of Λ_n consisting on one hand of complete symmetric functions h_λ and on the other of monomial symmetric functions m_λ are dual.*

PROOF. The statement follows from the identity

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \prod_i \sum_j x_i^j h_j(y) = \sum_\lambda m_\lambda(x) h_\lambda(y)$$

and the preceding exercise. □

We may then apply the involution ω, on the variables $y_1, \ldots, y_n$, to the identity

$$\sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

We obtain the following variant of formula 1.4.3.

THE SECOND CAUCHY FORMULA 1.4.6.

$$\prod_{i,j}(1+x_iy_j)=\sum_\lambda s_\lambda(x)s_{\lambda^*}(y).$$

EXERCISE 1.4.7. Let $z_1,\ldots,z_{m+n}$ be auxiliary variables. If J is a subset of size n of $\{1,\ldots,m+n\}$, we set [8]

$$t_J(x,z)=\prod_{\substack{1\le i\le n\\ k\notin J}}(x_i-z_k)\Big/\prod_{\substack{j\in J\\ k\notin J}}(z_j-z_k).$$

If I is also such a subset, we denote by z_I the corresponding subset of variables. Taking note of the fact that $t_J(z_I,z)=\delta_{I,J}$, show that these functions form a base of those symmetric polynomials in x for which the degree with respect to each variable x_i does not exceed m. Verify moreover that such a polynomial f may be expanded in the form

$$f(x)=\sum_{\#J=n}f(z_J)t_J(x,z).$$

By applying this interpolation formula to the function $f=1$ after having replaced each variable x_i by x_i^{-1}, establish the identity

$$(z_1\cdots z_{m+n})^m\prod_{i,j}(1-x_iz_j)^{-1}=\sum_{\#J=n}\prod_{\substack{j\in J\\ 1\le i\le n}}(1-x_iz_j)^{-1}\prod_{\substack{k\in J\\ l\notin J}}(z_k-z_l)^{-1}.$$

Thanks to the first formula of Cauchy, deduce then that for each partition λ of length at most $n+m$,

$$s_{\lambda_1-m,\ldots,\lambda_{m+n}-m}(z)=\sum_{\#J=n}s_\lambda(z_J)\Big/\prod_{\substack{j\in J\\ k\notin J}}(z_j-z_k).$$

1.4.3. The Number of Standard Tableaux. We may similarly use the Cauchy formula to determine the number K_λ of standard tableaux with shape λ, as well as the number $K_\lambda(n)$ of semistandard tableaux with shape λ, numbered with integers at most equal to n. By Littlewood's Theorem,

$$K_\lambda(n)=s_\lambda(\underbrace{1,\ldots,1}_{n}).$$

We will see in corollary 1.4.11 that this integer is a polynomial function of n. For the moment, set $y_1=\cdots=y_n=1/n$ in the first Cauchy formula. We obtain the identity

$$\prod_1^n\left(1-\frac{x_i}{n}\right)^{-n}=\sum_\lambda\frac{K_\lambda(n)}{n^{|\lambda|}}s_\lambda(x).$$

As n tends toward infinity, the term on the left-hand side converges to $\exp(e_1)$. On the other hand, if the partition λ is of size l, the coefficient of s_λ in e_1^l is equal, by corollary 1.2.9, to K_λ. Therefore

$$K_\lambda/|\lambda|!=\lim_{n\to+\infty}n^{-|\lambda|}K_\lambda(n).$$

This last statement signifies none other than the following: if n is large, almost all semistandard tableaux with shape λ, numbered with integers at most n, are numbered with pairwise distinct integers.

DEFINITION 1.4.8. *To each cell x with coordinates (i, j), of the Ferrers diagram of the partition λ, we associate on one hand its* content $c(x) = j - i$, *and on the other hand its* hook length $h(x) = \lambda_i + \lambda_j^* - i - j + 1$, *which is the length of the hook traced over λ to the right and below x. We furthermore set $n(\lambda) = \sum_i (i-1)\lambda_i$.*

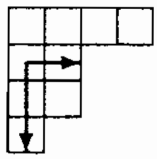


FIGURE 10. Hooks

7	5	2	1
4	2		
3	1		
1			

FIGURE 11. Hook lengths

0	1	2	3
-1	0		
-2	-1		
-3			

FIGURE 12. Contents

EXERCISE 1.4.9. Verify that the hook lengths of the ith row of λ constitute the sequence of integers between 1 and $\lambda_i + n - i$ excepting the integers $\lambda_i - \lambda_j - i + j$ for $j > i$.

We now determine what is called the principal specialization of Schur functions, which is derived by evaluating such a function on a geometric series [**65**].

PROPOSITION 1.4.10. *Let q be an indeterminate. Then*

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}}.$$

PROOF. Begin with the determinantal expression of antisymmetric polynomials $a_{\lambda+\delta}$. If we set $x_i = q^{i-1}$, we obtain a Vandermonde determinant, which leads easily to the expression

$$a_{\lambda+\delta}(1, q, \dots, q^{n-1}) = q^{n(\lambda)+n(n-1)(n-2)/6} \prod_{i<j} (1 - q^{\lambda_i - \lambda_j - i + j}).$$

The preceding exercise, however, implies the identity

$$\prod_{x \in \lambda} (1 - q^{h(x)}) \prod_{i<j} (1 - q^{\lambda_i - \lambda_j - i + j}) = \prod_{i=1}^{n} \prod_{k=1}^{\lambda_i + n - i} (1 - q^k).$$

Comparing this expression with that obtained for $a_{\lambda+\delta}(1, q, \ldots, q^{n-1})$, divided by the same expression for $\lambda = 0$, we end with the stated formula. □

For $q = 1$, taking into account Littlewood's Theorem 1.4.1, we obtain the following formula for $K_\lambda(n)$:[3]

COROLLARY 1.4.11. *The number of semistandard tableaux with shape λ, numbered with integers less than or equal to n, is equal to*

$$K_\lambda(n) = \prod_{x \in \lambda} \frac{n + c(x)}{h(x)}.$$

It remains to make n tend towards infinity to obtain a remarkable formula for the integer K_λ [**18**]:

FRAME-ROBINSON-THRALL FORMULA 1.4.12. *Let λ be a partition, and denote by $h(\lambda)$ the product of its hook lengths. Then the number of standard tableaux with shape λ is the integer*

$$K_\lambda = |\lambda|!/h(\lambda).$$

REMARK 1.4.13. This result has a classical probabilistic interpretation, in that it answers the following problem, called the *ballot problem*. Imagine an election with n candidates who garner respectively, in decreasing order, $\lambda_1, \ldots, \lambda_n$ votes. What is the probability that at each instant of voting, the ranking of the candidates is the same as the final ranking (or is at least compatible with that ranking, in the case of equalities)?

We represent the votes for the different candidates, numbered according to their ranking in the polls, in a tableau. The possible allocations of votes may then be identified with the tableaux with shape λ, increasing along the rows, and their number is equal to $|\lambda|!/\lambda_1! \cdots \lambda_n!$. Among them, the votes which are at each instant compatible with the final ranking correspond to the standard tableaux. The desired probability is therefore

$$p(\lambda) = \lambda_1! \ldots \lambda_n!/h(\lambda).$$

EXERCISE 1.4.14. Let J_λ be the right-hand side of the Frame-Robinson-Thrall formula. In order to show that $J_\lambda = K_\lambda$, verify that is suffices to establish the recurrence relation

$$J_\lambda = \sum_i J_{\lambda - \varepsilon_i},$$

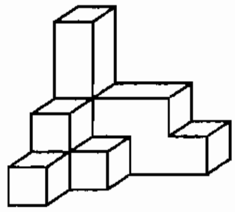
this sum running over the integers i such that (i, λ_i) is a corner of the diagram of λ. (The cell of the tableau with the greatest number necessarily occupies a corner.) We imagine then the following random process. We choose by chance a cell (a_1, b_1) of λ and then, on the hook for which it is the vertex, another cell (a_2, b_2), continuing until we reach a corner. Show that the probability of ending with a corner (i, λ_i) is precisely $J_{\lambda-\varepsilon_i}/J_\lambda$. Conclude.[4]

[3]We point out that the integer $K_\lambda(n)$ is also the dimension of the irreducible $\mathrm{GL}(n, \mathbb{C})$-module, called the Schur module, associated to the partition λ, and that the formula which follows is often more compact and convenient than the dimension formula of Weyl.

[4]This proof of the Frame-Robinson-Thrall formula is due to Greene, Nijenhuis, and Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. Math. **31** (1979), 104–109.

1.4.4. Enumeration of Plane Partitions. Schur functions have some remarkable applications to problems of enumeration of *plane partitions*. These objects are three-dimensional generalizations of the usual partitions, defined as finite sets P of triplets of nonzero natural numbers such that if $(i,j,k) \in P$ and $i' \leq i$, $j' \leq j$, $k' \leq k$, then $(i',j',k') \in P$. We denote by $\#P$ the cardinality of P.

A plane partition may be coded by projecting it on the horizontal plane, numbering each cell of the projection with the number of boxes situated above it. We obtain a partition $\lambda(P)$, the *support* of P, numbered in a decreasing fashion along its rows and columns.



4	2	2	1
2	1		
1			

FIGURE 13. Plane partitions

A plane partition is said to be *strict* if the associated partition is numbered in a strictly decreasing fashion along its columns. If we impose in addition that it be numbered with integers at most equal to n, we may derive a tableau by replacing each integer i with which it is numbered with the integer $n+1-i$. Proposition 1.4.10 then implies the following result.

PROPOSITION 1.4.15. *The generating function of the strict plane partitions with a given partition λ as support is*

$$\sum_{\lambda(P)=\lambda} q^{\#P} = q^{n(\lambda)+|\lambda|} \prod_{x\in\lambda} (1-q^{h(x)})^{-1}.$$

In particular, we consider the plane partitions P which are inscribed in a parallelopiped $P(l,m,n)$ with sides of length l,m,n. We add to the numbered partition $\lambda(P)$, over each $l \times m$ rectangle which contains it, the integer $l+1-i$ on each of the m cells of the ith row, and this for $1 \leq i \leq l$. We then obtain a strict plane partition, the support of which is an $l \times m$ rectangle, and the height of which is bounded above by $l+n$. Moreover, each of the plane partitions possessing these properties may be obtained in this fashion.

We then apply the preceding formula to the case in which λ is an $l \times m$ rectangle, and let the integers l,m,n tend towards infinity. We obtain the following result:

PROPOSITION 1.4.16. *The generating function of the plane partitions is*

$$\sum_P q^{\#P} = \prod_{n=1}^{+\infty} (1-q^n)^{-n}.$$

One may of course compare this with the generating function of Euler for ordinary partitions:

$$\sum_\lambda q^{|\lambda|} = \prod_{n=1}^{+\infty} (1-q^n)^{-1}.$$

Moreover, one has analogous results for plane partitions which are symmetric with respect to a diagonal, or cyclically symmetric. They are however more difficult to prove, and we will restrict ourselves to stating them.

We denote $ht(x) = i + j + k - 2$ if $x = (i, j, k) \in C = (\mathbb{N}^*)^3$. Consider the subgroups G_0, G_1, and G_2 of $\mathcal{S}_3$ (which acts on C by permuting the coordinates), generated respectively by the identity, the permutation of the two first coordinates, and the cyclic permutation of the three coordinates. Then,[5] for $0 \le i \le 2$,

$$\sum_{G_i(P)=P} q^{\#P} = \prod_{\overline{x} \in C/G_i} \frac{1 - q^{\mathrm{ht}(\overline{x}) + \#\overline{x}}}{1 - q^{\mathrm{ht}(\overline{x})}},$$

where $\bar{x}$ runs over the orbits of G_i in C, $\#\overline{x}$ is the cardinality of the orbit $\overline{x}$, and

$$\mathrm{ht}(\overline{x}) = \sum_{y \in \overline{x}} \mathrm{ht}(y).$$

1.5. The Littlewood-Richardson Rule

This important combinatorial rule permits the decomposition of a product of Schur functions into sums of functions of the same type. The proof that we give rests on the insertion procedure and the Knuth correspondence.

1.5.1. The Symmetry Theorem. It is possible to give a presentation of the Knuth correspondence which is independent of the insertion algorithm and which has the advantage of clarifying certain properties. This presentation is due to Viennot [**96**] for the particular case of the Robinson correspondence, and in the general case to Fulton [**22**], whose approach we present in terms of b-matrices.

Let us take a matrix A with integer entries, positive or zero, and place in the cell with coordinates (i, j) a chain of a_{ij} balls, oriented NW-SE (northwest-southeast). Given a ball b on this cell, we will say that a ball b' of (i', j') is NW of b if $i' < i$ and $j' < j$ and, when $i' = i$ and $j' = j$, if b' is NW of b in the chain containing these two balls.

We number them by requiring that b be numbered with the integer k if all of the balls situated NW of b have already been numbered, the maximum of their numbers being $k - 1$. We denote by $\overline{A}$ the object thus obtained, which we call the b-matrix associated to A. For the matrix given in the example of the Knuth correspondence (figure 9), we obtain the b-matrix of figure 14.

Consider the balls of $\overline{A}$ numbered with a given integer k. By construction, they form a SW-NE chain—in other words, they have coordinates $(i_1, j_1), \ldots, (i_l, j_l)$ with $i_1 > \cdots > i_l$ and $j_1 < \cdots < j_l$. Replace this chain by the chain of balls with coordinates $(i_1, j_2), \ldots, (i_{l-1}, j_l)$.

If we do the same for each integer k, we obtain a matrix of balls that one may number according to the same rule as above. We then obtain the b-matrix $\partial\overline{A}$, called the derived b-matrix of $\overline{A}$. This b-matrix is itself associated to an ordinary

[5] For $i = 0$, this is none other than a rewriting of the preceding identity. The case $i = 1$ appears in [**66**], page 50, and the case $i = 2$ is proved in Mills, Robbins, and Rumsey, *Proof of the Macdonald conjecture*, Invent. Math. **66** (1982), 73–87. The interested reader might equally well consult R. Stanley, *A baker's dozen of conjectures concerning plane partitions*, pp. 285–283 of Combinatoire énumérative, Lecture Notes in Mathematics **1234**, Springer-Verlag, 1986, and J. Stembridge, *The enumeration of totally symmetric plane partitions*, Adv. Math. **111** (1995), 227–243.

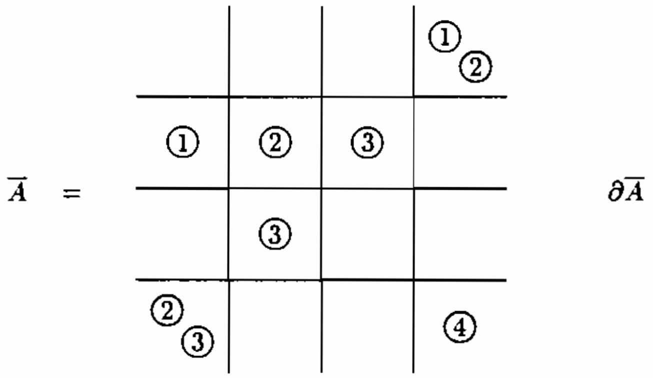


FIGURE 14. b-matrix $\overline{A}$ associated to the matrix A

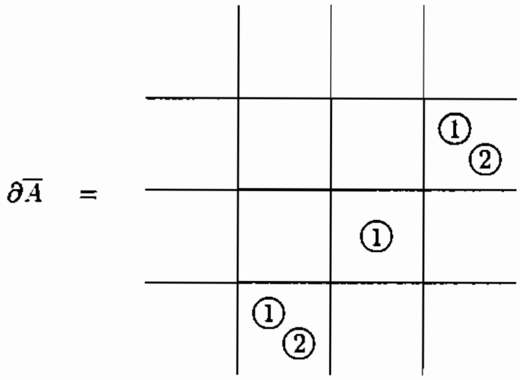


FIGURE 15. Derived b-matrix

matrix that we denote by ∂A. For the preceding example, the derived b-matrix is given in figure 15.

We then iterate this procedure as many times as is possible. Then let p_{ij} (respectively q_{ij}) be the smallest index of the columns (respectively of the rows) of $\partial^{i-1}\overline{A}$ where the integer j appears. Let $P(A)$ (respectively $Q(A)$) be the tableau in which the cell (i, j) is numbered with the integer p_{ij} (respectively q_{ij}). By construction, these two tableaux are semistandard and of the same shape.

PROPOSITION 1.5.1. *The map which associates to the matrix A the pair $(P(A), Q(A))$ of semistandard tableaux coincides with the Knuth correspondence.*

PROOF. We work by induction on the sum of the entries of A. Let us name the *extremal ball* of $\overline{A}$ the ball which appears furthest to the right on the last occupied line, and let (q, p) be its coordinates. Let k be the integer with which it is numbered, and denote by $(q_1, p_1), \ldots, (q_l, p_l)$ the chain of balls numbered with this same integer, with $(q, p) = (q_1, p_1)$. If we remove the extremal ball, we obtain a b-matrix $\overline{B}$ associated to a matrix B to which we may apply the inductive hypothesis. It remains, therefore, to show that $P(A)$ results from inserting p to $P(B)$, and $Q(A)$ from adding the integer q to $Q(B)$ in the cell thus created.

By construction, the first row of $P(A)$ has a p, and the first row of $Q(A)$ a q, in the kth position. If $l = 1$, $\overline{A}$ does not contain a ball numbered $k + 1$, and therefore the first row of $P(A)$ is of length k, and the first row of $P(B)$ is obtained by removing the last cell. Similarly for $Q(A)$ and $Q(B)$. Moreover, $\partial A = \partial B$, and hence the rest of the rows of these tableaux are identical. As a result, $(P(A), Q(A))$

is derived from $(P(B), Q(B))$ by Knuth insertion, which consists here simply by adding a cell to the right of the first row.

If $l \geq 2$, the first row of $P(B)$ has p_2 in the kth position, and the first row of $P(A)$ has p_1 in the same position—and the other entries are identical. In the $(k-1)$st position appears therefore an integer less than or equal to p_1, which implies that the first row of $P(A)$ is obtained by inserting p_1 in the first row of $P(B)$, which bumps out p_2. Because the extremal ball of ∂A has coordinates (q_1, p_2), it suffices to iterate this reasoning to conclude, since the last insertion step corresponds to the preceding case. □

The maps $A \to \overline{A}$ and ∂ commute with the transpose, and as a result, $P(A^t) = Q(A)$ and $Q(A^t) = P(A)$. The interest in this version of the Knuth correspondence is that it makes the following result of M.-P. Schützenberger clear.

THEOREM 1.5.2. *If the Knuth correspondence associates to a matrix A the pair (P, Q) of semistandard tableaux, then it also associates to the transposed matrix A^t the pair of tableaux (Q, P).*

In particular, to a symmetric matrix A is associated a pair (P, P) of identical tableaux. Hence, if we recall that the Robinson correspondence for permutations is a special case of the Knuth correspondence, the double corollary follows:

COROLLARY 1.5.3. *The Knuth correspondence restricts to a one-to-one correspondence between, on one hand, symmetric matrices and semistandard tableaux, and, on the other, involutory permutations and standard tableaux.*

EXERCISE 1.5.4. Deduce, in the same fashion that we deduced the Cauchy formula of the Knuth correspondence, the identity

$$\sum_{\lambda} s_\lambda = \prod_i (1 - x_i)^{-1} \prod_{j<k} (1 - x_j x_k)^{-1}.$$

Using the exercise which followed the statement of Pieri's formulas 1.2.5, show then that

$$\sum_{\mu \text{ even}} s_\mu = \prod_{j \leq k} (1 - x_j x_k)^{-1}, \quad \text{and} \quad \sum_{\mu \text{ even}} s_{\mu^*} = \prod_{j<k} (1 - x_j x_k)^{-1}.$$

More generally, if $o(\lambda)$ is the number of columns of odd length in the diagram of the partition λ, then

$$\sum_{\lambda} t^{o(\lambda)} s_\lambda = \prod_i (1 - t x_i)^{-1} \prod_{j<k} (1 - x_j x_k)^{-1}.$$

EXERCISE 1.5.5. The algorithm described above associates to a matrix A a matrix ∂A and two increasing rows of the same length. Describe the inverse algorithm. Deduce a direct description of the inverse Knuth correspondence $(P, Q) \rightsquigarrow A$.

EXERCISE 1.5.6. Let i_l be the number of standard tableaux of size l. Establish the relation $i_{l+1} = i_l + l i_{l-1}$. Deduce the identity

$$\sum_{l \geq 0} i_l q^l = \exp(q + \frac{q^2}{2}),$$

where we set $i_0 = 1$.

REMARK 1.5.7. In the case of permutations, the Fulton construction simplifies greatly. One begins with the graph of a permutation $w \in S_n$, that is, the set of points $(i, w(i))$ for $1 \leq i \leq n$. The numbering procedure that we described for the b-matrices may be interpreted in the following fashion. We consider that each point of the graph casts a shadow in the region SE of it. The points of the graph which are not the shadow of any point are then connected by a dashed line of which these points are the NW corners. We continue then in the same fashion for the remaining points, which gives us a sequence of *shadow lines*.

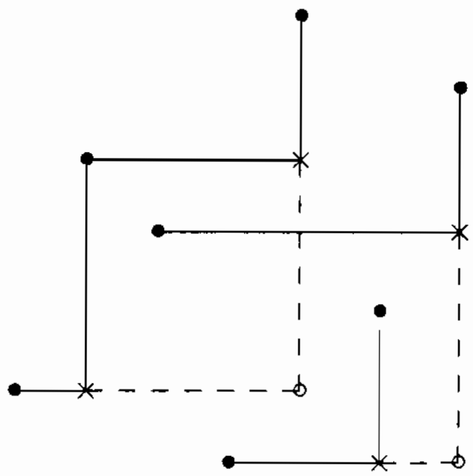


FIGURE 16. Shadow lines

The passage to the derived graph corresponds exactly to that for the b-matrices, when one considers only the SE corners of each line. The standard tableaux associated in the Robinson correspondence are obtained by placing in the jth row the x-values of the leftmost points (respectively, the y-values of the rightmost points) of the jth shadow lines of the successive diagrams [**96**].

1.5.2. The Plactic Ring. We now encode the combinatorics of Knuth insertion into an algebraic structure, namely the plactic ring, which is an intermediate object between the ring of polynomials in a finite family of variables and its noncommutative analogue.

For this we associate to a tableau T the word $m(T)$ obtained by reading its entries from bottom to top and from left to right. If we consider how the word is modified when one inserts the integer n, we realize without difficulty that the word associated to the tableau S obtained after insertion results from the word $m(T)n$ by applying the following operations to groupings of three consecutive letters:

$$xzy \sim zxy \text{ if } x \leq y < z, \quad yzx \sim yxz \text{ if } x < y \leq z.$$

Here the symbol $\sim$ indicates that one may pass from one to the other of the two equivalent forms. These elementary operations are called *Knuth relations* [**48**]. Two words are said to be equivalent in the sense of Knuth if one may pass from one to the other by a succession of elementary operations.

REMARK 1.5.8. One may memorize these elementary Knuth relations by distinguishing between the strict and the nonstrict inequalities. In the case of the first, the elementary relations signify that one may commute two letters when a letter between the two is placed immediately on their left, or on their right. In the case

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,, \qquad m(T) = 6452351224.$$

FIGURE 17. Word $m(T)$ of a tableau T

of the second, they signify that if $x < y$, yx commutes at one and the same time with x and y.

One may similarly remark that for the first relation, zxy is the word associated to the tableau obtained by rectifying xzy, just as for the second, yxz is the word associated to the tableau obtained by rectifying yzx. These three-letter words suffice therefore to define the rectification operation.

EXERCISE 1.5.9. Verify that column insertion is compatible with Knuth equivalence.

EXERCISE 1.5.10. Show that if two words are equivalent, they remain so when one removes their leftmost smallest letters, or their rightmost largest letters. In the same vein: show that if we keep only the letters belonging to a given interval, we obtain again two equivalent words. These simple remarks are useful for proofs by induction.

PROPOSITION 1.5.11. *Each Knuth equivalence class contains the word of a unique tableau.*

PROOF. The existence is a consequence of the fact that one may transform a word into a tableau by the successive insertion of its letters, a procedure which respects Knuth equivalence. The uniqueness is more delicate to establish. We need the following definition.

DEFINITION 1.5.12. *If m is a word, let $l_k(m)$ be the maximum of the sum of the lengths of k disjoint, weakly decreasing sequences taken from m.*

LEMMA 1.5.13. *These integers depend only on the Knuth equivalence class of m.*

PROOF OF THE LEMMA. It suffices to verify that the integers are invariant under an elementary Knuth transformation. Suppose for example that $m = axzyb$ and $n = azxyb$, a and b being words and x, y, and z some letters such that $x \leq y < z$. If we have k increasing sequences for n, they remain so for m, hence $l_k(m) \geq l_k(n)$. The inverse remains true as well, unless the sequence is an increasing sequence of m of the form $a'xzb'$, a' being extracted from a and b' from b. But it suffices then to replace it by $a'xyb'$, substituting y by z if it appears in another sequence. Therefore $l_k(n) \geq l_k(m)$. □

The following statement is due to C. Greene [**32**]:

LEMMA 1.5.14. *If m is the word of a tableau T with shape λ, then*

$$l_k(m) = \lambda_1 + \cdots + \lambda_k.$$

PROOF OF THE LEMMA. A weakly increasing sequence extracted from m corresponds to a sequence of cells of T in which the column indices are strictly increasing. For k sequences we therefore take at most k cells per column, which implies that $l_k(m) \leq \lambda_1 + \cdots + \lambda_k$. But the inequality is reached by the subsequences given by the first k rows. □

END OF PROOF OF THE PROPOSITION. Consider therefore a word m which is equivalent to the word of a tableau T. By the two preceding lemmas, the shape of T is uniquely determined by m. We conclude by induction: let m' be the word obtained by removing from m its greatest letter, say l, situated furthest to the right. Let T' be the tableau resulting from T by the same procedure.

Then m', by the preceding exercise, is equivalent to the word of T', which by the inductive hypothesis is therefore uniquely determined. But we know the shape of T', and the cell that it is necessary to add to obtain the shape of T: we must necessarily place the integer l there, and T is therefore uniquely determined. □

COROLLARY 1.5.15. *Let w be a permutation to which the Robinson correspondence associates the standard tableaux P and Q. Then the lengths of the successive rows of these tableaux are the integers $l_i(w)$. Similarly, the lengths of the successive columns are the integers $l_j(ww_0)$, where w_0 is the permutation of maximal length in the symmetric group to which w belongs, that is, the maximum of the sums of the lengths of j decreasing sequences extracted from the word of w.*

PROOF. For the statement concerning the rows, this is an immediate consequence of the preceding proposition. For the statement concerning the columns, we remark that for the permutations, the insertion algorithm for columns is simply the transpose of the insertion algorithm for rows. The rectification of the word ww_0, which is obtained by reading from right to left the word of w, is therefore simply the transposed tableau of the rectification of w. □

DEFINITION 1.5.16. *Denote by $\mathbb{Z}\{x_1, \ldots, x_n\}$ the ring of polynomials in n noncommutative variables. The* plactic ring[6] *$\mathcal{P}_n$ is the quotient of this ring by the two-sided ideal generated by the elementary Knuth relations (one identifies each variable with its index). Because these relations are compatible with commutative evaluation, we have ring morphisms*

$$\mathbb{Z}\{x_1, \ldots, x_n\} \to \mathcal{P}_n \to \mathbb{Z}[x_1, \ldots, x_n].$$

REMARK 1.5.17. More generally, one may replace the interval $1, \ldots, n$ with any *alphabet*, if one means by that a totally ordered finite set, and then consider as a *word* a noncommutative monomial.

We have thus proved that the plactic ring has a set of tableaux for a base over the integers. Moreover, the product of two tableaux may be obtained by rectifying the word obtained by writing the words to which they are associated, one next to the other. An amusing algorithm for determining such a product is provided by the *jeu de taquin*.

[6]From the Greek $\pi\lambda\alpha\xi$, flat place, stone plate, tablet. This terminology is due to Lascoux and Schützenberger. These authors also define the nilplactic ring, which is none other than the algebra of divided differences, or the nilCoxeter algebra, which we encounter in the second chapter. We note that P. Littelmann has recently associated a plactic algebra to each complex semisimple Lie algebra: *A plactic algebra for semisimple Lie algebras*, Adv. Math. **124** (1996), 312–331.

1.5.3. Jeu de Taquin. Let λ and μ be two partitions such that λ contains μ. We denote by λ/μ the complement of the diagram of μ in that of λ; it is a *skew partition*. We call any numbering of a diagram with shape λ/μ, obeying the same constraints as for semistandard tableaux—increasing along rows, strictly increasing along columns—a *skew tableau*.

The *jeu de taquin* permits the transformation of a skew tableau into an ordinary tableau. It is defined in the following fashion: choose a corner of μ, and slide into its place that cell of λ immediately below or to the right which is numbered with the smallest integer. If there is any ambiguity, choose the one below. We will have created a "hole" to which one applies the same procedure, repeated until it is no longer possible.

$$\begin{array}{|c|c|}\hline & y \\ \hline x & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xrightarrow{x\le y} \begin{array}{|c|c|}\hline x & y \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline & y \\ \hline x & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xrightarrow{x>y} \begin{array}{|c|c|}\hline y & \\ \hline x & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

FIGURE 18. Jeu de taquin

We then obtain a skew tableau of the shape λ'/μ', where λ' and μ' are derived from λ and μ by removing one cell. By reiterating this procedure, we finish by obtaining a semistandard tableau. We will show that this tableau does not depend on the choice of corners made at each step.

PROPOSITION 1.5.18. *The jeu de taquin is compatible with Knuth equivalence.*

PROOF. In other words, if we associate a word to a skew tableau in the same fashion as to a standard tableau, its equivalence class is not altered by the jeu de taquin. This is clear when one slides a cell to the right, because the word does not change. If we slide a cell toward the top, we may restrict ourselves to the two rows of the tableau which are involved in this operation. It is then elementary to verify that the tableau obtained by rectifying the word associated to the two original rows is identical to the rectification of the word associated to the two new rows. $\square$

COROLLARY 1.5.19. *The jeu de taquin permits rectifying skew tableaux. In particular, the ordinary tableau that results from a skew tableau does not depend on the interior corners chosen at each step.*

$$\begin{array}{ccc} & \bullet & 1 \\ & 2 & 2 \\ 3 & & \end{array} \quad \begin{array}{ccc} & 1 & \bullet \\ & 2 & 2 \\ 3 & & \end{array} \quad \begin{array}{ccc} & 1 & 2 \\ \bullet & 2 & \\ 3 & & \end{array} \quad \begin{array}{ccc} \bullet & 1 & 2 \\ 2 & & \\ 3 & & \end{array} \quad \begin{array}{ccc} 1 & \bullet & 2 \\ 2 & & \\ 3 & & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 2 & \\ 3 & \end{array}$$

FIGURE 19. Rectification of a skew tableau

As was mentioned at the end of the preceding section, the jeu de taquin permits the calculation of the product of two tableaux. For this procedure, it suffices to begin with a skew tableau constructed by joining the rightmost point of the first tableau to the point of the second tableau the least to the right. The tableau obtained by the jeu de taquin is the desired product.

1.5.4. Littlewood-Richardson Coefficients. The multiplication rule for Schur functions is obtained by means of skew tableaux. The essential proposition is the following, in the proof of which the symmetry theorem 1.5.2 of Schützenberger plays an important role.

PROPOSITION 1.5.20. *Let U_0 be a tableau with shape μ, V_0 a tableau with shape ν, both semistandard. Then there exists a one-to-one correspondence between pairs (T, S) of tableaux with shapes (λ, μ) and product V_0 on one hand, and skew tableaux U with shape μ/λ and rectification U_0 on the other.*

PROOF. Let T and S be tableaux with shapes λ and μ and product V_0. Let s and u_0 be the sequences associated by the Knuth correspondence to the pair (S, U_0) of tableaux, which are of the same shape: the second sequence is increasing, and the first is ordered in a compatible fashion. If we insert s into the tableau T, and if we place the letters corresponding to u_0 in the cells successively created, we obtain a skew tableau U with shape ν/λ. We show that the rectification of U is U_0.

To do so, consider an auxiliary tableau T_0 with shape λ but numbered with *negative* integers. If the Knuth correspondence associates to the pair (T, T_0) the sequences t and t_0, then the sequences t_0u_0 and ts are suitably ordered, and the inverse Knuth correspondence associates to them the pair of tableaux $(T.S = V_0, V)$. This last tableau, with shape ν, is the superposition of the negative tableau T_0 and the skew tableau U.

By Schützenberger's symmetry theorem, the Knuth correspondence similarly gives the associations

$$(U_0, S) \sim (x, y) \quad \text{and} \quad (V, TU) \sim (x', y')$$

where x is obtained by ordering t in increasing fashion and y in ordering t_0 in a compatible fashion, while x' is obtained by ordering ts in increasing fashion and y' in ordering t_0u_0 in a compatible fashion. But then, since t_0 consists of negative integers, the parts of u_0 are ordered in the same fashion in y and in y'. As a consequence, y is the word obtained by removing from y' its $|\lambda|$ smallest letters. Because y' is equivalent to $m(V)$, y is therefore equivalent to the word obtained by removing from $m(V)$ its $|\lambda|$ smallest letters, which is to say $m(U)$. But we know that y is equivalent to $m(U_0)$, which is therefore equivalent to $m(U)$.

Conversely, given U and T_0, their superposition is a tableau V with shape ν. The Knuth correspondence associates to the pair (V_0, V) sequences that we will write (ts, t_0u_0), t and t_0 being of length $|\lambda|$, and s and u_0 of length $|\mu|$. But then, to the sequences (t, t_0) and (s, u_0) are associated the pairs of tableaux (T, T_0) and (U, U_0), with $TS = V_0$. The correspondence thus defined is the inverse of the one above. □

An essential consequence of this result is that the cardinality of the sets it concerns depends only on the partitions λ, μ, and ν, not the chosen tableaux; we denote by $c^{\nu}_{\lambda\mu}$ this cardinality. We may translate this cardinality in the following fashion. Denote by S_λ the sum, in the plactic ring, of the tableaux with shape λ, and then call this sum a Schur function. By the Littlewood's Theorem 1.4.1, the commutative evaluation of S_λ is the ordinary Schur function s_λ.

COROLLARY 1.5.21. *In the plactic ring, a product of Schur functions decomposes as a sum of Schur functions.*

As a consequence, the sums S_λ generate a commutative subring of the plactic ring, which lifts the ring of symmetric functions. We note in passing that if $c^\nu_{\lambda\mu}$ is nonzero, then $|\nu| = |\lambda| + |\mu|$ and ν contains λ and μ. Moreover,[7] we have some symmetry relations

$$c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda} = c^{\nu^*}_{\lambda^*\mu^*} = c^{\nu^*}_{\mu^*\lambda^*}.$$

Furthermore, the preceding proposition permits the determination of the Littlewood-Richardson coefficients by choosing the tableau U_0 to be as simple as possible. Given a partition λ, we call the *canonical tableau* with shape λ the tableau in which the cells are numbered with the number of their row. We further say that a word $x_1 \dots x_r$ is *Yamanouchi* if, for all s and for all i, the word $x_s \dots x_r$ contains at least as many i's as $i+1$'s.

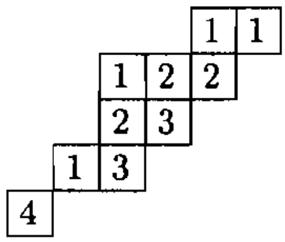


FIGURE 20. A skew tableau with Yamanouchi word

PROPOSITION 1.5.22. *The Yamanouchi words are those with canonical rectification.*

PROOF. It suffices to verify that the property of being Yamanouchi is preserved by Knuth equivalence, which is elementary, and that a tableau T for which the word $w(T)$ is Yamanouchi is canonical, which is more elementary still. □

We finally come to the celebrated rule of Littlewood and Richardson[8], in its original form:

LITTLEWOOD-RICHARDSON RULE 1.5.23. *The coefficient $c^\nu_{\lambda\mu}$ of s_ν in the product $s_\lambda s_\mu$ is equal to the number of skew tableaux with shape ν/μ and weight λ, for which the associated word is Yamanouchi.*

EXAMPLE 1.5.24. We remark that this rule is completely asymmetric in λ and μ. In the simple example which follows, we determined the product $s_{22}s_{21}$ by applying the Littlewood-Richardson rule in two possible ways.

[7] There are some other symmetry relations which are not made apparent by the Littlewood-Richardson rule that we are going to state. These are, however, made explicit by Berenstein and Zelevinsky, *Triple multiplicities for $sl(r+1)$ and the spectrum of the exterior algebra of the adjoint representation*, J. Algebraic Combin. **1** (1992), 7–22.

[8] This rule was stated for the first time by Littlewood and Richardson in 1934, but only fairly recently has it been possible to give a complete proof. The proof that we have developed is, up to some details, that of M.-P. Schützenberger [**83**]. See also G. P. Thomas, *On Schensted's construction and the multiplication of Schur functions*, Adv. in Math. **30** (1978), 8–32, as well as the book by Macdonald [**66**] for different proofs of the same rule. Since then several variants have appeared. We mention the polytopes of Berenstein and Zelevinsky, *op. cit.*, and the path model of P. Littelmann, the formalism of which has the immense advantage of extending to the set of complex semisimple Lie algebras, and beyond: *The Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Inventiones Math. **116** (1994), 329–346. We similarly find in section 2.7.4 of the second chapter of this text a version of the Littlewood-Richardson rule in terms of trees of permutations.

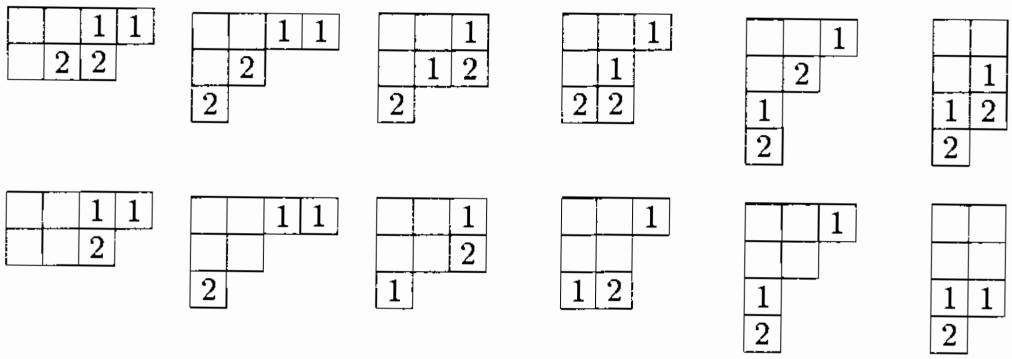


FIGURE 21. $s_{22}s_{21} = s_{43} + s_{421} + s_{331} + s_{322} + s_{3211} + s_{2221}$

1.5.5. Skew Schur Functions. The preceding discussion prompts us to introduce the *skew Schur functions*

$$s_{\nu/\mu} = \sum_{\lambda} c^{\nu}_{\lambda\mu} s_{\lambda}.$$

We denote by (,) the scalar product that we defined in paragraph 1.4.2 by the requirement that the family of Schur functions be orthonormal. Then

$$(s_\lambda, s_{\nu/\mu}) = (s_\lambda s_\mu, s_\nu).$$

Moreover, theorem 1.4.1 and proposition 1.5.20 imply the following corollary.

PROPOSITION 1.5.25. *The skew tableaux of a given shape lift to skew Schur functions in the plactic ring. In other words,*

$$s_{\lambda/\mu} = \sum_{S} x^{\mu(S)},$$

the sum running over the set of skew tableaux with shape λ/μ.

The Jacobi-Trudi formulas extend similarly to skew Schur functions. We begin with a summation formula.

PROPOSITION 1.5.26. *Let x and y be two sets of indeterminates. Then*

$$s_{\lambda/\mu}(x,y) = \sum_{\nu} s_{\lambda/\nu}(x) s_{\nu/\mu}(y).$$

PROOF. Let x, y, z, and w be sets of indeterminates. The definition of skew Schur functions and the Cauchy formula successively permit writing

$$\begin{aligned}\sum_{\lambda,\mu} s_{\lambda/\mu}(x,y)s_\mu(z)s_\lambda(w) &= \sum_{\mu,\nu} s_\nu(x,y)s_\mu(z)s_\mu(w)s_\nu(w)\\ &= \prod(1-x_iw_j)^{-1}\prod(1-y_kw_l)^{-1}\prod(1-z_mw_n)^{-1}.\end{aligned}$$

By symmetry, we have, for all partitions λ,

$$\sum_{\mu} s_{\lambda/\mu}(x,y)s_\mu(z) = s_\lambda(x,y,z) = \sum_{\nu} s_{\lambda/\nu}(x)s_\nu(y,z).$$

If x is empty, we obtain in particular the statement for $\mu = 0$. It remains only to insert it in the identity above to deduce the general expression. $\square$

PROPOSITION 1.5.27. *If λ and μ are of length at most n,*

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{1\leq i,j\leq n}, \quad \text{and} \quad s_{\lambda^*/\mu^*} = \det(e_{\lambda_i-\mu_j-i+j})_{1\leq i,j\leq n}.$$

PROOF. By using the first Cauchy formula above, we obtain

$$\sum_\lambda s_{\lambda/\mu}(x)s_\lambda(y) = s_\mu(y)\sum_\nu h_\nu(x)m_\nu(y).$$

Then, if y is a family of n indeterminates,

$$\sum_\lambda s_{\lambda/\mu}(x)a_{\lambda+\delta}(y) = \sum_\alpha h_\alpha(x)\sum_{w\in\mathcal{S}_n}\varepsilon(w)y^{\alpha+w(\mu+\delta)},$$

where α is an n-tuple and not necessarily a partition. As a result, $s_{\lambda/\mu}(x)$ is the coefficient of $y^{\lambda+\delta}$ on the right-hand side, and the first formula follows easily. The second follows similarly via the second Cauchy formula. □

1.6. The Characters of the Symmetric Group

Having up to now described the ring of symmetric polynomials by itself, we examine in this section its connections with the representations of permutation groups. Sums of powers, or Newton symmetric functions, first permit us to associate to each of these representations a symmetric polynomial. We then show that the irreducible representations correspond thus to Schur polynomials. From this correspondence we deduce different ways to determine the character table of a permutation group. Finally, after having pointed out the Hopf algebra structure of the ring of symmetric polynomials, we give some concrete descriptions of these irreducible representations: these are the Specht modules.

1.6.1. Sums of Powers. Among the simplest symmetric functions appear the sums of powers, or Newton sums,

$$p_k = x_1^k + \cdots + x_n^k,$$

where k is a strictly positive integer. We denote their products by $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_m}$, where m is a partition. The generating function

$$p(t) = \sum_{k\geq 0} p_{k+1}t^k$$

satisfies the two following formal identities:

$$p(t) = h'(t)/h(t), \quad p(-t) = e'(t)/e(t).$$

These identities are equivalent to the *Newton relations*

$$kp_k = \sum_{i+j=k} p_ih_j = \sum_{i+j=k}(-1)^{j-1}p_ie_j.$$

By writing these relations in the form of Cramer systems, it is easy to express the Newton sums in terms of elementary or complete symmetric functions, and vice versa. Thus,

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 & 0 \\ p_2 & p_1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & n-1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 \end{vmatrix}.$$

These relations imply, in particular, that as λ runs through the partitions of length at most n, the functions p_λ form a base of the algebra $\Lambda_n \otimes \mathbb{Q}$ of symmetric functions in n variables with coefficients not integers, but rationals. Moreover,

$$\omega(p_\lambda) = (-1)^{|\lambda|+l(\lambda)} p_\lambda.$$

We similarly note that the preceding formulas may be reversed. For example, the relation $p(t) = h'(t)/h(t)$ integrates to $h(t) = \exp(\sum_k p_k t^k/k)$. If the integer i appears m_i times in λ, then set

$$z_\lambda = \prod_i i^{m_i} m_i!.$$

Further denote by $m(\lambda)$ the sequence of the integers m_i. Then we have

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda, \quad \text{and} \quad e_n = \sum_{|\lambda|=n} (-1)^{|\lambda|+l(\lambda)} z_\lambda^{-1} p_\lambda.$$

The scalar product that we defined in section 1.4.2 clearly extends to the space of symmetric functions with rational coefficients. The Newton sums constitute an orthogonal basis by virtue of the exercise following the statement of Cauchy's first formula 1.4.3 and the following expansion.

PROPOSITION 1.6.1. $\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y)$.

EXERCISE 1.6.2. Verify the proposition. If f is a symmetric function, denote by D_f the adjoint operator of multiplication by f, always with respect to the same scalar product on the symmetric functions. Show that $D_{s_\mu}(s_\lambda) = s_{\lambda/\mu}$. Under the interpretation of $\Lambda \otimes \mathbb{Q}$ as a ring of polynomials in the Newton functions, show that $D_{p_k} = k \frac{\partial}{\partial p_k}$.

1.6.2. Representations of Finite Groups. We recall briefly some general properties of representations of finite groups, which we soon illustrate with the example of permutation groups.[9]

Let G be a finite group and $\#G$ its order. Each of its complex representations of finite dimension $\rho \colon G \to \mathrm{GL}(V)$ is completely reducible, or, in other words, decomposes into a sum of irreducible representations. Such a representation is moreover determined by its character $\chi_\rho(g) = \operatorname{trace} \rho(g)$, which is a class function. The characters of the irreducible representations form a basis of the space of class functions on G, which is orthonormal with respect to the hermitian product

$$(\chi, \chi') = \frac{1}{\#G} \sum_{g \in G} \chi(g) \overline{\chi'(g)}.$$

If H is a subgroup of G, we may restrict the representation ρ of G to a representation of H, that we denote $\mathrm{Res}_H^G \rho$. Inversely, we may define, beginning with a representation σ of H on a space W an *induced representation* of G, that we denote $\mathrm{Ind}_H^G \sigma$. Recall that W is a module over the algebra $\mathbb{C}[H]$ of the group H. The induced representation $\mathrm{Ind}_H^G \sigma$ is then simply associated to the $\mathbb{C}[G]$-module

[9]This theory essentially goes back to the work of Frobenius. To the reader who seeks details, we suggest the treatment of J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, 1967, from which the first part and the beginning of the second should amply meet our needs.

$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. If χ_σ is the character of σ, the character $\operatorname{Ind}_H^G \chi_\sigma$ of the induced representation is given by the following formula:

$$\operatorname{Ind}_H^G \chi_\sigma(g) = \frac{1}{\# H} \sum_{\substack{s \in G, \\ s^{-1} g s \in H}} \chi_\sigma(s^{-1} g s).$$

In particular, the dimension (the degree) of $\operatorname{Ind}_H^G \sigma$ is equal to the degree of σ multiplied by the index of H in G.

At the level of characters, and with respect to the hermitian product defined on class functions, these two operations of restriction and induction are mutually adjoint: this is *Frobenius reciprocity*.

For future use we cite *Mackey's Theorem*: if H and K are two subgroups of G, let S be a system of representatives of the set $K\backslash G/H$ of double cosets of G modulo (K, H), which means that G is the disjoint union of the KsH as s runs through S. For all $s \in S$, the intersection $H_s = sHs^{-1} \cap K$ is then a subgroup of K. If ρ is a representation of H, we obtain a representation ρ_s of H_s by setting $\rho_s(h) = \rho(s^{-1} h s)$, and the representation $\operatorname{Res}_K^G(\operatorname{Ind}_H^G \rho)$ then decomposes into a sum of induced representations:

$$\operatorname{Res}_K^G(\operatorname{Ind}_H^G \rho) = \bigoplus_{s \in S} \operatorname{Ind}_{H_s}^K \rho_s.$$

1.6.3. The Ring of Characters. Let R^n be the free $\mathbb{Z}$-module generated by the irreducible characters of the symmetric group $\mathcal{S}_n$, and set $R^0 = \mathbb{Z}$. We may endow the direct sum

$$R = \bigoplus_{n \geq 0} R^n$$

with the structure of an associative and commutative graded ring in the following fashion: if ϕ and ψ are the characters of the representations Φ and Ψ of $\mathcal{S}_n$ and $\mathcal{S}_m$, their product is defined to be the character of the representation of $\mathcal{S}_{m+n}$ induced by the product representation $\Phi \times \Psi$ of the subgroup $\mathcal{S}_n \times \mathcal{S}_m$:

$$\phi \cdot \psi = \operatorname{Ind}_{\mathcal{S}_n \times \mathcal{S}_m}^{\mathcal{S}_{m+n}} (\phi \times \psi).$$

The ring R is similarly endowed with a scalar product induced by the hermitian product on the characters of each of the permutation groups.

Recall that the conjugacy class of a permutation $w \in \mathcal{S}_n$ is determined by its cycle decomposition, hence by a partition $\rho(w)$ of size n. We therefore define the *characteristic map* $\operatorname{ch}\colon R \to \Lambda \otimes \mathbb{C}$ by setting, for $\phi \in R^n$,

$$\operatorname{ch}(\phi) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \phi(w) p_{\rho(w)} = \sum_{|\rho| = n} z_\rho^{-1} \phi_\rho p_\rho,$$

where ϕ_ρ denotes the value taken by ϕ on the conjugacy class of $\mathcal{S}_n$ defined by ρ. The integer z_ρ appears here as the order of the centralizer of an element of the conjugacy class of $\mathcal{S}_n$ associated to the partition ρ (that, for the action of $\mathcal{S}_n$ on itself by conjugation).

PROPOSITION 1.6.3. *The characteristic map defines a graded ring isomorphism, as well as an isometry, from the ring R of the characters of the symmetric group, to the ring Λ of symmetric functions.*

PROOF. Frobenius reciprocity implies that the characteristic map is a morphism of rings. The compatibility with scalar products follows from the fact that $(p_\lambda, p_\mu) = z_\lambda \delta_{\lambda,\mu}$. Finally, if 1_n is the trivial character of $\mathcal{S}_n$, we have $\mathrm{ch}(1_n) = h_n$. As a consequence, the image of the characteristic map contains Λ, and is therefore equal to Λ because ch is an isometry. □

EXERCISE 1.6.4. Let $\mathcal{P}_k$ be the space of polynomials of degree k in n variables with complex coefficients. The symmetric group $\mathcal{S}_n$ acts naturally on this space by permutation of variables. Let π_k be the corresponding character. Noting that the monomials of degree k fixed by a permutation w may be identified with n-tuples of integers with sum k, constant on each cycle of w, establish the formal identity

$$\sum_k q^k \,\mathrm{ch}(\pi_k) = \sum_\mu z_\mu^{-1}(1-q^{\mu_1})^{-1}\cdots(1-q^{\mu_n})^{-1}p_\mu.$$

The right-hand side is none other than the expression of complete symmetric functions in terms of Newton functions, modulo the substitution of $(1-q^m)^{-1}p_m$ for p_m. Deduce that

$$\sum_k q^k \,\mathrm{ch}(\pi_k) = \prod_{i,j}(1-q^j x_i)^{-1} = \sum_\lambda s_\lambda(1,q,\ldots,q^n)s_\lambda.$$

EXERCISE 1.6.5. Prove the identities

$$\frac{1}{n!}\sum_{w\in\mathcal{S}_n} p_{\rho(w)} = h_n, \qquad \frac{1}{n!}\sum_{w\in\mathcal{S}_n} p_{\rho(w^2)} = \sum_{|\lambda|=n} s_\lambda.$$

1.6.4. The Frobenius Character Formula. The preceding results permit us to determine the irreducible characters of $\mathcal{S}_n$. Recall that these characters form an orthonormal basis of R^n, and that a base of a $\mathbb{Z}$-module, when it exists, is unique up to order and up to signs. Up to signs, the desired characters are therefore the inverse images of the Schur functions under the characteristic map. By the Jacobi-Trudi formula, these are the

$$\chi^\lambda = \det(1_{\lambda_i - i+j})_{1\le i,j\le n}.$$

FROBENIUS CHARACTER FORMULA 1.6.6. *The irreducible characters of the permutation group $\mathcal{S}_n$ are the χ^λ, where λ runs through the set of partitions of size n. Furthermore, their values on the different conjugacy classes of $\mathcal{S}_n$ are equal to the coordinates of the Newton functions with respect to the base of Schur functions. That is, for partitions μ with size n,*

$$p_\mu = \sum_{|\lambda|=n} \chi_\mu^\lambda s_\lambda.$$

REMARK 1.6.7. It is sometimes more convenient to express this result in the following form: the integer χ_μ^λ is equal to the coefficient of the monomial $x^{\lambda+\delta}$ in the antisymmetric polynomial $a_\delta p_\mu$. This reformulation is an immediate consequence of the fact that the polynomial $a_{\lambda+\delta}$ contains a unique monomial x^α with α strictly decreasing, namely $x^{\lambda+\delta}$, with coefficient 1.

PROOF. In order to prove the first assertion above, it suffices by taking into account the preceding remarks to verify that $\chi^\lambda(1) > 0$. Then the inverse image of p_μ under the characteristic map is equal to z_μ times the function δ^μ, equal to one on the conjugacy class associated to μ, and zero elsewhere. Therefore

$$(s_\lambda, p_\mu) = (\chi^\lambda, z_\mu\delta^\mu) = \chi_\mu^\lambda,$$

the number of elements of this conjugacy class being precisely $n!/z_\mu$. This proves the character formula since, in particular,

$$\chi^\lambda(1) = (s_\lambda, p_1^n) = K_\lambda > 0,$$

as it was necessary to show. □

COROLLARY 1.6.8. *The degree of the irreducible representation of S_n with character χ^λ is equal to the number K_λ of standard tableaux with shape λ, given by the Frame-Robinson-Thrall formula.*

EXERCISE 1.6.9. Establish the character table of S_5:

λ/μ	11111	2111	221	311	32	41	5
11111	1	−1	1	1	−1	−1	1
2111	4	−2	0	1	1	0	−1
221	5	−1	1	−1	−1	1	0
311	6	0	−2	0	0	0	1
32	5	1	1	−1	1	−1	0
41	4	2	0	1	−1	0	−1
5	1	1	1	1	1	1	1

EXERCISE 1.6.10. Show that the Frobenius character formula may be inverted to yield

$$s_\lambda = \sum_{|\mu|=n} z_\mu^{-1} \chi_\mu^\lambda p_\mu.$$

EXERCISE 1.6.11. Show that under the characteristic map the elementary symmetric function e_n corresponds to the character ε of the sign representation of S_n, which to each permutation associates its signature. Verify that under the characteristic map, the involution ω on symmetric functions corresponds to multiplication by ε.

EXERCISE 1.6.12. If ϕ is the unit character of S_1, show that ϕ^n is the character of the regular representation of S_n. Deduce therefore that the multiplicity of an irreducible representation in the regular representation is equal to its degree, which is a general fact.

The fact that the characteristic map is an isomorphism similarly permits the decomposition of the restriction of a representation to a subgroup S_{n-1} of S_n:

BRANCHING RULE 1.6.13. *Let λ be a partition of weight n. Then*

$$\mathrm{Res}_{S_{n-1}}^{S_n} \chi^\lambda = \sum_{\lambda \in \mu\otimes 1} \chi^\mu.$$

Recall that $\mu \otimes 1$ denotes the set of partitions which are obtained by adjoining a cell to μ. By the preceding rule, the restriction of the representation with character χ^λ to S_{n-1} is simply the sum of the irreducible representations associated to the partitions obtained by removing a cell from λ. Similarly, we give a statement traditionally called *Young's rule*:

YOUNG'S RULE 1.6.14. *Let $\mu = (\mu_1, \dots, \mu_l)$ be a partition of weight n, and let N^μ be the representation of S_n obtained by induction of the trivial representation* *from the Young subgroup $S_{\mu_1} \times \cdots \times S_{\mu_l}$ of S_n. Then the multiplicity in N^μ of the irreducible representation with character χ^λ is equal to the Kostka number $K_{\lambda\mu}$.*

Moreover, we may deduce a combinatorial interpretation of the coefficients χ^λ_μ, under an analogy following Pieri's formulas. Let us call a *ribbon* a connected set θ of cells which does not contain a two-by-two square. Its height $h(\theta)$ is defined to be one less than the number of rows occupied by the ribbon.

PROPOSITION 1.6.15. *For all partitions λ and integers k, we have*

$$s_\lambda p_k = \sum_\theta (-1)^{h(\theta)} s_{\lambda+\theta},$$

the sum running over the ribbons θ of length k such that $\lambda + \theta$ is a partition.

Here, the sum signifies the superposition of diagrams, which are supposed to be disjoint; see the figures above.

EXAMPLE 1.6.16. In particular, the Newton functions are alternating sums of Schur functions associated to hooks:

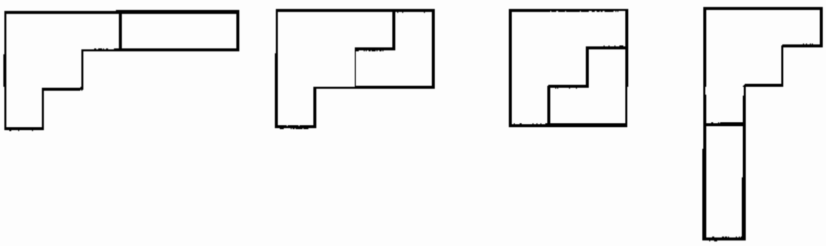
$$p_k = \sum_{i+j=k-1} (-1)^j s_{(i|j)}.$$

PROOF. By the Jacobi expression of Schur functions, it suffices to consider the product

$$a_{\lambda+\delta} p_k = \sum_{i=1}^n a_{\lambda+k\varepsilon_i+\delta}.$$

We reach the desired conclusion by reordering $\lambda + k\varepsilon_i + \delta$, which gives a sequence of the form $\lambda + \theta + \delta$, where θ is a ribbon of length k, with a number of inversions equal to $h(\theta)$. ☐

EXAMPLE 1.6.17. We have $s_{321}p_3 = s_{621} - s_{441} - s_{333} + s_{321111}$, where the different terms of the sum correspond to the following diagrams, in which ribbons of respective heights 0, 1, 1, and 2 appear.



EXERCISE 1.6.18. Show that $\chi^{n,1}(w) = s_1 - 1$, where s_1 is the number of fixed points of the permutation $w \in S_{n+1}$. In order to generalize this observation, we reuse the notation of exercise 1.6.2. Deduce from the Jacobi-Trudi formulas the identity

$$\sum_{m\in\mathbb{Z}} s_{m,\alpha} = h(1) D_{e(-1)}(s_\alpha),$$

where we assume the notations $h(t)$ and $e(t)$ for the generating functions introduced in the proof of proposition 1.1.5. We observe that if $n + |\alpha| = |\beta|$, we have

$$\chi_\beta^{n,\alpha} = \left(\sum_{m \in \mathbb{Z}} s_{m,\alpha}, p_\beta \right) = (s_\alpha, e(-1) D_{h(1)}(p_\beta)).$$

Show that if $w \in \mathcal{S}_{n+|\alpha|}$ is a permutation having s_k cycles of length k, then

$$\chi^{n,\alpha}(w) = \sum_{\lambda,\mu} \frac{(-1)^{l(\lambda)}}{z_\lambda} \chi^\alpha_{\lambda\cup\mu} \binom{s}{m(\mu)} = P_\alpha(s),$$

where $\lambda \cup \mu$ is the partition obtained by the union of the parts of λ and of μ. This partition therefore satisfies the equality $m(\lambda \cup \mu) = m(\lambda) + m(\mu)$. In particular, the polynomial $P_\alpha(s)$ is independent of n, the greatest part of the considered partition. For example, verify that

$$P_m(s) = Q_m(s) - Q_{m-1}(s) \quad \text{and} \quad P_{1^m}(s) = \sum_{k \geq 0} (-1)^k R_{m-k}(s),$$

where we respectively set

$$Q_m(s) = \sum_{|\mu|=m} \binom{s}{m(\mu)}, \quad R_m(s) = \sum_{|\mu|=m} (-1)^{m-l(\mu)} \binom{s}{m(\mu)}.$$

EXERCISE 1.6.19. Show that the symmetric function $h_k(x_1^l, \ldots, x_n^l)$ is a linear combination of Schur functions with coefficients equal to zero or one. Show that if the coefficient of s_λ in this expression is nonzero, then λ decomposes into horizontal ribbons of length k.

1.6.5. The Murnaghan-Nakayama Rule. We return to the Frobenius character formula. An immediate induction permits deducing the following result. Let us call a *multiribbon tableau* a tableau T numbered in increasing fashion, in the large sense, along its rows and columns, in such a way that the collection of its cells numbered with a given integer form a ribbon. We define its height $h(T)$ to be the sum of the heights of the ribbons from which it is formed.

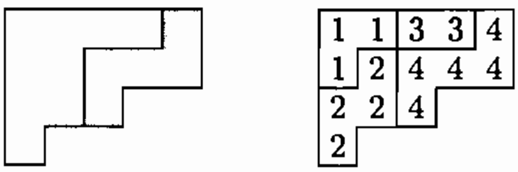


FIGURE 22. A ribbon and a multiribbon tableau

MURNAGHAN-NAKAYAMA RULE 1.6.20. *For each partition λ, the value of the irreducible character χ^λ on the conjugacy class of $\mathcal{S}_n$ associated to the partition μ is equal to*

$$\chi^\lambda_\mu = \sum_T (-1)^{h(T)},$$

where the sum runs over the set of multiribbon tableaux T with shape λ and weight μ.

EXERCISE 1.6.21. Recall that the product of the characters of two representations is the character of their tensor product. If

$$\chi^\lambda \chi^\mu = \sum_\nu g_\nu^{\lambda\mu} \chi^\nu,$$

show that the integers $g_\nu^{\lambda\mu}$ are symmetric in λ, μ, and ν.[10]

EXERCISE 1.6.22. If λ is a partition, let $c(\lambda)$ be the length of its principal diagonal. Let μ be a partition, of the same size as λ, which has for its parts the lengths of the diagonal hooks of λ; in other words, $\mu_i = h_{i,i}$ for $1 \le i \le c$. Verify that

$$\chi_\mu^\lambda = (-1)^{|\lambda| + \binom{c(\lambda)+1}{2}}.$$

Show further that if $\chi_\nu^\lambda \neq 0$, then $\nu \le \mu$.

1.6.6. Hopf Algebra Structure. By a *graded Hopf algebra* over a commutative unitary ring A we mean a graded A-module $H = \bigoplus_{n \ge 0} H_n$, together with graded A-module morphisms (A being uniquely graded of degree zero)

1. $m\colon H \otimes H \to H$, the *multiplication*. The multiplication is associative, and under it H has the structure of a ring:

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \quad \text{on} \quad H \otimes H \otimes H;$$

2. $e\colon A \to H$, the *unit*, such that $e(1)$ is the unit of H;
3. $m^*\colon H \to H \otimes H$, the *comultiplication*. The comultiplication is co-associative, in the sense that

$$(m^* \otimes \mathrm{id}) \circ m = (\mathrm{id} \otimes m^*) \circ m^*,$$

and we similarly require that it be a ring morphism;
4. $e^*\colon H \to A$, the *co-unit*, such that $(e^* \otimes \mathrm{id}) \circ m^* = (\mathrm{id} \otimes e^*) \circ m^* = \mathrm{id}$.

EXAMPLE 1.6.23. The ring $H = \Lambda$ of symmetric polynomials with integral coefficients in a countable family of indeterminates, graded by the degree, is endowed with a graded Hopf algebra structure in the following fashion: m is its ordinary multiplication, e its unit, and e^* is the projection on the constant term. Finally, the comultiplication m^* is defined by introducing two countable families of indeterminates, say $(y_i)_{i \ge 1}$ and $(z_j)_{j \ge 1}$. If $u \in H$, we obtain by evaluating u at the indeterminates $y_1, z_1, y_2, z_2, \ldots$ a symmetric function in each of the two families $(y_i)_{i \ge 1}$ and $(z_j)_{j \ge 1}$, hence an element $m^*(u) \in H \otimes H$. It is immediate that we then have a graded Hopf algebra structure. For example,

$$m^*(h_n) = \sum_{k+l=n} h_k \otimes h_l,$$

$$m^*(p_n) = p_n \otimes 1 + 1 \otimes p_n.$$

This last identity expresses the fact that the sums of powers are *primitive elements* of Λ.

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[10]We know relatively little about tensor products of irreducible representations of permutation groups. In particular, we do not have a combinatorial description of multiplicities analogous to the Littlewood-Richardson rule.

EXERCISE 1.6.24. Prove the identity

$$m^*(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu.$$

where the $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

EXAMPLE 1.6.25. The ring $H = R$ of the characters of the permutation groups similarly has a graded Hopf algebra structure. The multiplication m is defined by induction. The comultiplication is given by restriction: if χ_ρ is the character of a representation ρ of $\mathcal{S}_n$, we set

$$m^*(\chi_\rho) = \sum_{k+l=n} \mathrm{Res}^{\mathcal{S}_n}_{\mathcal{S}_k \times \mathcal{S}_l} \chi_\rho.$$

The unit r is defined to be the unit of R, and the co-unit e^* by projection on R^0. The only difficulty in showing that in fact we have a Hopf algebra structure is to verify that the comultiplication is a ring morphism. This last is a consequence of Mackey's Theorem mentioned above.[11]

There is no difficulty in making proposition 1.6.3 precise in the following fashion:

PROPOSITION 1.6.26. *The characteristic map* ch *is a graded Hopf algebra isomorphism.*

1.6.7. Specht Modules. The Frobenius character formula implies that the degree of the irreducible representation of $\mathcal{S}_n$ with character χ^λ is equal to the number K_λ of standard tableaux with shape λ. Observing this fact, we are led to the idea of directly realizing these representations by making the symmetric group act not directly on the tableaux, but on certain polynomials associated to them.

Let λ be a partition of size n and T a tableau of shape λ, numbered by the successive integers $1, \dots, n$, without any monotonicity condition. In this section, we will consider only tableaux of this type. To such a tableau T, we associate the polynomials

$$Q_T = \prod_{k=1}^n x_k^{l(k)-1}, \quad P_T = \prod_{i <_T j} (x_i - x_j),$$

where $l(k)$ is the index of the row of T containing the integer k, and $i <_T j$ signifies that i is found above j in the same column of T. The symmetric group $\mathcal{S}_n$ acts by permuting the variables, just as it acts on tableaux by the simple permutation of its entries: if $w \in \mathcal{S}_n$, we have

$$Q_{w(T)} = wQ_T, \quad \text{and} \quad P_{w(T)} = wP_T.$$

Moreover, taking into account the expression of a Vandermonde determinant, we easily verify that

$$P_T = b_T Q_T \quad \text{if} \quad b_T = \sum_{w \in C(T)} \varepsilon(w) w,$$

[11] For more details, I recommend the book of A. Zelevinsky [**100**]. Beyond its alternative proofs of numerous properties of symmetric functions, including the Littlewood-Richardson rule, one finds there a significant amplification, notably in the direction of the complex representations of the linear groups over finite fields $\mathrm{GL}(n, \mathbb{F}_q)$.

if $\mathcal{C}(T)$ denotes the subgroup of $\mathcal{S}_n$ consisting of the permutations preserving each column of T. We similarly denote by $\mathcal{L}(T)$ the subgroup of $\mathcal{S}_n$ of permutations preserving the rows of T, and we set $a_T = \sum_{w\in\mathcal{L}(T)} w$. The sums a_T and b_T must be considered as elements of the algebra $\mathbb{C}[\mathcal{S}_n]$ of the symmetric group. Their product $c_T = b_T a_T$ is called the *Young symmetrizer*.

We denote by N^λ the vector space spanned by the polynomials Q_T as T runs through the set of tableaux with shape λ. This space carries a natural action of the symmetric group. Since being given Q_T is equivalent to being given the content of each row of T, independently of the order, the $\mathcal{S}_n$-module N^λ may be identified with the representation induced from the trivial representation of the subgroup $\mathcal{L}(T)$ of $\mathcal{S}_n$. Similarly, let S^λ be the submodule of N^λ spanned by the polynomials P_T, where T runs through the set of tableaux with shape λ. The S^λ are called *Specht modules* [**71**].

THEOREM 1.6.27. *As λ runs through the set of partitions of n, the Specht modules S^λ form a complete set of irreducible representations of $\mathcal{S}_n$, pairwise inequivalent. Moreover, the character of S^λ is precisely χ^λ.*

PROOF. The key is the following

LEMMA 1.6.28. *Let λ and μ be two partitions of n, and let T and S be two tableaux with respective shapes λ and μ, numbered with consecutive integers. Then either $\lambda > \mu$, $\lambda = \mu$ and there exist $v \in \mathcal{C}(T)$ and $w \in \mathcal{L}(T)$ such that $vT = wS$, or there exist two integers appearing in the same column of T and the same row of S.*

PROOF OF THE LEMMA. Suppose that we are not in the last case. Because the integers of the first row of S are in distinct columns of T, one may therefore arrange them in the first row of T by operating within the columns of T. More generally, for all i, we may permute the entries in the columns of T in such a way as to make the entries of the ith row of S appear on the ith row of T. In particular, $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i, whence $\lambda > \mu$. If we are not in the first case, then $\lambda = \mu$. After having permuted each column of T, we then obtain a diagram having the same content as S on each row, so that we are in the second case. □

CONCLUSION. If we are in the second case of the lemma, we may write

$$b_T Q_S = b_T w Q_S = b_T v Q_T = \varepsilon(v) P_T.$$

On the other hand, if we are in the third case of the lemma, there must necessarily exist a transposition $t \in \mathcal{C}(T) \cap \mathcal{L}(S)$, and therefore

$$b_T Q_S = -b_T t Q_S = -b_T Q_S = 0.$$

This implies that S^λ is irreducible. In fact, if S is a tableau with shape λ, we see that $b_T Q_S$ is always proportional to λ, and therefore

$$b_T N^\lambda = b_T S^\lambda = \mathbb{C} P_T.$$

If S^λ is the direct sum of two submodules A and B, this equation becomes $\mathbb{C}P_T = b_T A \oplus b_T B$, therefore P_T is an element of A or of B, say A. But then, since the permuted P_T generate S^λ, we have $B = 0$.

Suppose now that λ precedes μ in the lexicographic order. We are then necessarily in the last case of the lemma, and hence after what we have already shown,

$$b_T N^\mu = b_T S^\mu = 0.$$

This guarantees that the representations S^λ and S^μ of $\mathcal{S}_n$ are inequivalent. The Specht modules therefore make up a complete set of irreducible representations of $\mathcal{S}_n$.

Finally, the same argument implies that the decomposition of N^λ into irreducible representations is of the form

$$N^\lambda = S^\lambda \oplus \left(\bigoplus_{\mu > \lambda} n_{\lambda\mu} S^\mu \right),$$

for certain integers $n_{\lambda\mu}$. Because N^λ is the $\mathcal{S}_n$-module induced by the trivial representation of the subgroup $\mathcal{L}(T)$ of $\mathcal{S}_n$, we know in particular its character, the image of which under the characteristic map is the symmetric function h_λ. Formula 1.2.9 then implies, by induction, that the character of S^λ is precisely χ^λ. $\square$

THEOREM 1.6.29. *As T runs through the set of standard tableaux with shape λ, the polynomials P_T form a basis of the Specht module S^λ.*

PROOF. By definition S^λ is generated by the polynomials P_T, where we may restrict ourselves to the tableaux T with entries which increase, from top to bottom, along each column. Order these tableaux by lexicographic order on the sequence of their entries, read from top to bottom and from left to right, reading the columns successively. We will show by induction, for this total order, that each polynomial P_S is a linear combination of polynomials P_T, with T standard. Knowing that the number of such polynomials is precisely the dimension of S^λ, the theorem will follow.

If the tableau S is not standard, then there exist two successive columns $a_1 < \cdots < a_l$ and $b_1 < \cdots < b_m$ of S, and an integer r, such that $a_i < b_i$ if $i < r$, but $a_r > b_r$. Suppose that these two columns are situated as far left as possible in S. Denote by H the group of permutations of the integers $b_1, \ldots, b_r, a_r, \ldots, a_l$, and K the subgroup of H which is the product of the groups of permutations of $b_1, \ldots, b_r$ and $a_r, \ldots, a_l$. Finally, suppose we have a system U of representatives of right cosets of K in H, other than that of the identity.

LEMMA 1.6.30. $P_S = -\sum_{u \in U} \varepsilon(u) P_{u(S)}$.

PROOF OF THE LEMMA. For $I \subset \mathcal{S}_n$, denote by $d_I = \sum_{w \in I} \varepsilon(w) w$, so that, in particular, $b_S = d_{\mathcal{L}(S)}$. Since K is a subgroup of $\mathcal{L}(S)$, it remains to verify that $d_H P_S = 0$. This will be a consequence of the identity $d_{HC(S)} Q_S = 0$. In fact, each element of $HC(S)$ may be written as a product of an element of H and an element of $C(S)$ in $\#(H \cap C(S)) = \#K$ different ways, hence

$$\#K \times d_{HC(S)} Q_S = d_H d_{C(S)} Q_S = d_H P_S.$$

If $w \in HC(S)$, let i be the smallest integer such that $w(a_i)$ and $w(b_i)$ are distinct from $a_1, \ldots, a_{r-1}$ and $b_{r+1}, \ldots, b_m$. Denote by $t(w)$ the transposition of the integers a_i and b_i, and $t^*(w) = wt(w)w^{-1}$ that of the integers $w(a_i)$ and $w(b_i)$. Then set $w^* = wt(w) = t^*(w)w$. Since $t(w), t^*(w) \in H$, we have $w^* \in HC(S)$, and the map $w \mapsto w^*$ in an involution of $HC(S)$ which changes the signature. We derive

$$d_{HC(S)} = \sum_{w \in HC(S)} \varepsilon(w^*) w^* Q_S = -\sum_{w \in HC(S)} \varepsilon(w) wt(w) Q_S = -d_{HC(S)} Q_S,$$

whence $d_{HC(S)} Q_S = 0$, since the transpositions $t(w)$, belonging to $\mathcal{L}(S)$, do not affect Q_S. $\square$

CONCLUSION. It remains to verify that in the right-hand side of the identity of the preceding lemma, the tableaux $u(S)$ which appear, after one has ordered the entries of each of their columns, are strictly the smallest in S with respect to the considered order. Then the columns which precede the column $a_1, \ldots, a_l$ are unchanged. Since the integers $a_1, \ldots, a_{r-1}$ of this same column C are unchanged as well, the sequence formed by the r smallest entries of $u(C)$ is bounded above by the sequence $a_1 < \cdots < b_i < \cdots < a_{r-1}$ for some integer $i < r$. And this latter sequence is strictly bounded above by the sequence $a_1 < \cdots < a_r$. Hence $u(S) < S$ for all $u \in U$, and the preceding lemma guarantees, by induction, that P_S is a linear combination of the polynomials P_T for T standard. □

REMARK 1.6.31. It would have been simpler in the proof to verify that the polynomials P_T, for T standard and with shape λ, are linearly independent. However, the lemma above, which gives the *Garnir relations*, gives in principle a means of calculating the matrices of the representations of the symmetric group on the Specht modules, in the base of the polynomials associated to the standard tableaux. Furthermore, for this it suffices to calculate the action on simple transpositions. Then the transposition of integers i and $i+1$ in a tableau T poses a problem only if these two integers lie in the same row of T, which makes the algorithm relatively easy to implement.

EXAMPLE 1.6.32. For the partition $(3, 2)$, we find the five standard tableaux

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, T_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, T_4 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, T_5 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array},$$

with associated polynomials as follows:

$$\begin{aligned} P_{T_1} &= (x_1 - x_4)(x_2 - x_5), \quad P_{T_2} = (x_1 - x_3)(x_2 - x_5), \\ P_{T_3} &= (x_1 - x_3)(x_2 - x_4), \quad P_{T_4} = (x_1 - x_2)(x_3 - x_5), \\ P_{T_5} &= (x_1 - x_2)(x_3 - x_4), \end{aligned}$$

By applying the algorithm above, one verifies that the action of the simple transpositions s_i, for $1 \leq i \leq 4$, of S_5 on the Specht module associated to the partition $(3, 2)$, is given by the following matrices M_i, in the basis formed by the five standard tableaux above:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

EXERCISE 1.6.33. Show that the Young symmetrizers are idempotents of the algebra of the symmetric group, at least up to a coefficient: there exists a strictly positive integer n_T such that $c_T^2 = n_T c_T$. What is the significance of the integer n_T?

1.7. Kostka-Foulkes Polynomials

Modern developments in the theory of symmetric functions have focused attention on what one calls the "q-analogues" of those functions. The Kostka-Foulkes polynomials, which are a polynomial extension of the Kostka numbers, are an important example of these analogues, and we study these in a detailed fashion.

1.7.1. The Kostant Formula. The definition of the Kostka-Foulkes polynomials rests on an expression of the Jacobi-Trudi formula which is slightly different from that which was given in 1.2.13. Define on the ring Λ_n some endomorphisms r_{ij} for $i < j$ in the following way. Recall first that Λ_n has for a base the set of symmetric functions h_λ, where λ runs through the set of partitions of length at most n. We then set

$$r_{ij} h_\lambda = h_{\lambda+\varepsilon_i-\varepsilon_j}.$$

In this expression, ε_i denotes the ith vector of the canonical base of $\mathbb{Z}^n$. Furthermore, we adopt the following convention: if the vector $\lambda + \varepsilon_i - \varepsilon_j$ has a negative component, or in other words if $\lambda_j = 0$, we set $r_{ij} h_\lambda = 0$. These operators, which clearly commute with each other, are connected by relations of the form $r_{ij} r_{jk} = r_{ik}$ for $i < j < k$.

PROPOSITION 1.7.1. *For all partitions λ we have the identity*

$$s_\lambda = \prod_{i<j} (1 - r_{ij}) h_\lambda.$$

PROOF. By the Jacobi-Trudi formula,

$$s_\lambda = \sum_{w \in S_n} \varepsilon(w) h_{\lambda_1 - 1 + w(1)} \cdots h_{\lambda_n - n + w(n)}.$$

For each permutation $w \in S_n$, we introduce integers n_{ij} for $i < j$, equal to one if $w(i) > w(j)$ and zero otherwise. These integers determine the permutation w, since

$$w(i) - i = \sum_{j>i} n_{ij} - \sum_{j<i} n_{ji}.$$

Moreover, $l(w) = \sum_{i<j} n_{ij}$. Similarly, note that these integers are not independent: if the integers n_{ij} and n_{jk} are equal, they must also be equal to n_{ik}. These relations characterize the families n_{ij} associated to permutations.

Then the stated formula may be written, after collecting terms,

$$s_\lambda = \sum_{n_{ij} \in \{0,1\}} \prod_{i<j} (-r_{ij})^{n_{ij}} h_\lambda.$$

It remains then to verify that in this formula, the contribution of the families of integers n_{lm} which do not correspond to permutations is globally zero. Suppose for example that there exist integers $i < j < k$ such that $n_{ij} = n_{jk} = 1$ and $n_{ik} = 0$. The contribution of these integers to the preceding formula is equal to $r_{ij} r_{jk}$, therefore equal to r_{ik}. If we set $n'_{ij} = n'_{jk} = 0$, $n'_{ik} = 1$, and $n'_{lm} = n_{lm}$ otherwise, we obtain a family of integers, the contribution of which is exactly opposite to that of the family n_{lm}. The proposition then follows easily. $\square$

We note that the preceding formula is also valid if one replaces λ by any n-tuple α of natural numbers, with the usual rules of rectification of Schur functions when α is not a partition. In fact, this also holds for the Jacobi-Trudi formulas, and the preceding proof extends without modification.

This formula similarly implies that $r_{ij}s_\lambda = s_{\lambda+\varepsilon_i-\varepsilon_j}$. Moreover, the formula inverts to become

$$h_\lambda = \prod_{i<j}(1-r_{ij})^{-1}s_\lambda.$$

We therefore define the *partition function* $\mathcal{P}$, which is a function on $\mathbb{Z}^n$ taking integral values, by the identity

$$\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)^{-1} = \sum_{\alpha\in\mathbb{Z}^n}\mathcal{P}(\alpha)x^\alpha.$$

In other words, $\mathcal{P}(\alpha)$ is the number of ways of writing α as a sum of vectors of the form $\varepsilon_i - \varepsilon_j$ for $i<j$. If we recall that by corollary 1.2.9, $K_{\mu\lambda}$ is the coefficient of s_μ in h_λ, the formula above leads to the following expression for the Kostka numbers:[12]

KOSTANT MULTIPLICITY FORMULA 1.7.2.

$$K_{\lambda\mu} = \sum_{w\in\mathcal{S}_n}\varepsilon(w)\mathcal{P}(w(\lambda+\delta)-\mu-\delta).$$

We may then define a "q-analogue" $\mathcal{P}_q$ of the partition function, where q is an indeterminate, by the identity

$$\prod_{1\le i<j\le n}\left(1-q\frac{x_i}{x_j}\right)^{-1} = \sum_{\alpha\in\mathbb{Z}^n}\mathcal{P}_q(\alpha)x^\alpha.$$

This leads directly to the definition of the "q-analogues" of the Kostka numbers, which are the *Kostka-Foulkes polynomials*:

$$K_{\lambda\mu}(q) = \sum_{w\in\mathcal{S}_n}\varepsilon(w)\mathcal{P}_q(w(\lambda+\delta)-\mu-\delta).$$

EXERCISE 1.7.3. Show that if $\lambda\ge\mu$, the polynomial $K_{\lambda\mu}(q)$ is monic with degree $n(\mu)-n(\lambda)$, where $n(\mu)=\sum_i(i-1)\mu_i$. In particular, $K_{\lambda\lambda}=1$. Show that if the relation $\lambda\ge\mu$ does not hold, then $K_{\lambda\mu}=0$.

1.7.2. Hall-Littlewood Functions. To the Kostka-Foulkes polynomials correspond "q-analogues" of the complete symmetric functions, called Hall-Littlewood functions:[13] for each partition μ, we set

$$Q_\mu(q) = \prod_{i<j}(1-qr_{ij})^{-1}s_\mu = \sum_\lambda K_{\lambda\mu}(q)s_\lambda.$$

These functions interpolate between the complete symmetric functions (which are obtained for $q=1$) and the Schur functions (for $q=0$). They satisfy the following relation, called the *Morris lemma*.

[12]This formula is. in fact, significant in a much more general context, under which, correctly interpreted, it gives the weight multiplicities in the irreducible modules of complex semisimple Lie algebras. See, for example, J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics **9**, Springer-Verlag 1972.

[13]These symmetric functions play an important role as characters of irreducible representations of linear groups over finite fields [**66**].

LEMMA 1.7.4. *If the integer r is greater than or equal to the largest part of μ, we have*

$$Q_{r,\mu}(q) = \sum_{k\geq 0} q^k \sum_{\nu} K_{\nu\otimes k,\mu}(q) s_{r+k,\nu}.$$

In this expression, $K_{\nu\otimes k,\mu}(q)$ is the sum of the $K_{\lambda\mu}$ for $\lambda \in \nu\otimes k$ (see the statement of Pieri's formulas 1.2.5). Moreover, $(r+k,\nu)$ is not necessarily a partition, and may need to be rectified.

PROOF. By definition of the Hall-Littlewood functions, we have

$$Q_{r,\mu}(q) = \prod_{l>1}(1-qr_{1l})^{-1} \prod_{1<i<j}(1-qr_{ij})^{-1} s_{r,\mu},$$

which expands to

$$Q_{r,\mu}(q) = \sum_{k\geq 0} q^k \sum_{k_2+\cdots+k_n=k} r_{12}^{k_2}\cdots r_{1n}^{k_n} \sum_{\mu} K_{\lambda\mu}(q) s_{r,\lambda}.$$

By making the same cancellations as in the proof of the second Pieri formula 1.2.5, we deduce the stated formula. □

1.7.3. The Charge Statistic. The Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$ takes as its value, for $q=1$, the number of semistandard tableaux with shape λ and weight μ. One might therefore imagine that each of these tableaux must naturally correspond to a monomial of some certain degree of this polynomial. This degree is given by the *charge*, a quantity that we now define.

Let m be a word in the alphabet $\mathbb{N}^*$. Suppose first that its weight is a partition. We decompose it into *pieces* in the following fashion: isolate the letter 1 which is furthest to the right, then the letter 2 immediately to its left—if there is not one, return to the right-hand end of the word m and take the 2 immediately to the left, and continue. We obtain then a subword m_1 of m, then a second subword m_2 in applying the process again to the remaining letters, and so on.

Each of the subwords m_j is then formed by pairwise distinct letters, making a sequence of consecutive integers since the weight of m is a partition. To each of the letters of m_j, we then associate a weight in the following way: the letter 0 has weight 0, and if the letter i has weight l, then $i+1$ has weight l or $l+1$ according to whether it lies to the left or to the right of i.

DEFINITION 1.7.5. *The charge $c(m)$ is the sum of the weights of the pieces of the word m. If T is a semistandard tableau with weight $\mu(T)$ a partition, its charge $c(T)$ is that of its word $m(T)$.*

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-3} 3 & 5 & \multicolumn{3}{c}{} \\ \cline{1-2} 4 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{l} m(T) = 43523411122 \\ m_1 = 43521, \quad m_2 = 3412, \quad m_3 = 12 \\ c(T) = 1+4+1 = 6 \end{array}$$

FIGURE 23. Charge of a tableau

EXERCISE 1.7.6. Verify that the charge is invariant under Knuth equivalence. Show that the charge of a tableau with weight μ consisting of a single row is equal to $n(\mu)$.

One may similarly describe the charge in the following way. Let T be a semistandard tableau, the weight of which is a partition. Say that a *cyclage* is a transformation of T into another such tableau S, such that there exists a letter $n > 1$ and a tableau U for which

$$m(T) \sim nm(U) \quad \text{and} \quad m(s) \sim m(U)n.$$

In other words, we choose a corner of T, extract from it by columns to obtain a tableau U and a letter n. Then, so long as $n \neq 1$, we reinsert n by rows in order to yield S.

EXAMPLE 1.7.7. Beginning with the tableau T above, by choosing for corner the end of the second row, we obtain $n = 3$ and the following transformation:

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \longrightarrow S = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

EXERCISE 1.7.8. Show that $\lambda(T) \geq \lambda(S)$, and that $c(T) = c(S) + 1$.

After a certain number of cyclages, we necessarily obtain a row tableau. The number of operations necessary is the *cocharge* of T, that is, the difference

$$co(T) = n(\mu) - c(T)$$

if T is of weight μ. The notion of cyclage permits the definition of a partial order on the set of tableaux with given weight, under which the row tableau is the minimal element.

We conclude this section with a lemma which we need in the proof of theorem 1.7.13.

LEMMA 1.7.9. *If $m = p1^r q$, where the words p and q do not contain the letter 1, then $c(m) = c(qp1^r) + |q|$, where $|q|$ is the length of the word q.*

PROOF. By induction, it suffices to show that

$$c(np1^r q) = c(p1^r qn) + 1$$

if n is a letter different from 1. But we may move from $m = np1^r q$ to $m' = p1^r qn$ by a cyclic permutation. By construction, the subwords m_j are then unchanged, except the one which contains the letter n, and this one passes from the leftmost to the rightmost point. Doing so adds one to its weight, but the weights of the other letters are unchanged. □

1.7.4. Pieri's Formula for Tableaux. Before proving the principal result of this course concerning Kostka-Foulkes polynomials, namely theorem 1.7.13, we need briefly to revisit Pieri's formula. At the level of tableaux, proposition 1.5.20 and corollary 1.5.21 yield the following particular case:

PROPOSITION 1.7.10. *Let λ be a partition and k an integer. Suppose that T is a tableau such that its shape μ is an element of $\lambda \otimes k$. Then there exists a unique tableau U with shape λ, and a unique row v of length k, such that T is the rectification of the product Uv (respectively vU).*

Note that the product Uv may be obtained row by row, instead of inserting the letters of v into U successively: express U as the product $u^1 \cdots u^l$ of its rows, and rectify $u^1 v$ into a tableau with two rows $v^1 t^1$, and then do the same successively, $u^i v^{i-1}$ into $v^i t^i$. This process finally gives the factorization into rows $v^d t^d \cdots t^1$ of the product tableau $T = Uv$.

Consider therefore the product of two rows u and v. If uv is a tableau, it is because u dominates the left factor of v of the same length, in the sense that each of its letters is greater than the corresponding letter of v. If not, let h be the length of the longest left factor of u which dominates the left factor corresponding to v. After inserting $v_1, \dots, v_h$ in u, the insertion of v_{h+1} corresponds to the following scheme:

$$\begin{matrix} v_1 \cdots v_h u_{h+1} \cdots u_m \cdots \\ u_1 \cdots u_h \end{matrix} \quad \mapsto \quad \begin{matrix} v_1 \cdots v_h u_{h+1} \cdots v_{h+1} \cdots \\ u_1 \cdots u_h u_m \end{matrix} .$$

LEMMA 1.7.11. *Let v, v', u, u' be rows such that the words uv and $u'v'$ are not tableaux but have the same rectification. Then the longest left factor of v dominated by u is identical to the longest left factor of v' dominated by u'.*

PROOF. Let $u''v''$ be the rectification of uv and $u'v'$. By the preceding discussion, this tableau has the form

$$\begin{matrix} v_1 \cdots v_h u_{h+1} \cdots v_{h+1} \cdots \\ u_1 \cdots u_h u_m \end{matrix} .$$

Moreover, we may recover the product $u'v'$ by removing from this tableau a certain number of cells from its first row, then by applying the inverse procedure of insertion to those from the second row. Then u_m bumps out a cell lying to the right of u_{h+1}, then u_h bumps out v_h, and so on. As a consequence, u' has $u_1 \cdots u_h$ and v' has $v_1 \cdots v_h$ for left factors of length h. Therefore the longest left factor of v' dominated by u' is of length at least h, and the lemma follows by symmetry. $\square$

COROLLARY 1.7.12. *Let U and U' be tableaux having the same number of rows. Let v and v' be rows such that Uv and $U'v'$ are not tableaux, but have the same rectification. Suppose further that $v = 1^r w$ and that neither w nor U contain 1. Then $v' = 1^r w'$, and neither w' nor U' contain 1.*

PROOF. Factor into rows $U = u^l \cdots u^1$ and $U' = u'^l \cdots u'^1$. Form the products Uv and $U'v'$ row by row: $u^i v^{i-1} \sim v^i s^i$ and $u'^i v'^{i-1} \sim v'^i s'^i$, obtaining the tableau $T = v^l s^l \cdots s^1 = v'^l s'^l \cdots s'^1$. Therefore $v^l = v'^l$, and by repeatedly using the preceding lemma, we deduce that u and u' must have the same left factor of length r. $\square$

1.7.5. The Foulkes Conjecture. The following result, which was conjectured by Foulkes, was established by A. Lascoux and M.-P. Schützenberger [**57**]:

THEOREM 1.7.13. *For all partitions λ and μ,*

$$K_{\lambda\mu}(q) = \sum_{\substack{\lambda(T)=\lambda, \\ \mu(T)=\mu}} q^{c(T)}.$$

PROOF. We remark first that this statement is equivalent to the following expression of modified Hall-Littlewood functions:

$$Q_\mu(q) = \sum_{\mu(T)=\mu} q^{c(T)} s_{\lambda(T)}.$$

We then proceed by induction on the length of μ. We add to μ another part greater than or equal to those it already has. By proposition 1.7.4, we have

$$Q_{r,\mu}(q) = \sum_{k\geq 0} \sum_{\nu} \sum_{\substack{\mu(T)=\mu,\\ \lambda(T)\in\nu\otimes k}} q^{c(T)+k} s_{r+k,\nu}.$$

Then by proposition 1.7.10, if a tableau T has for its shape an element of $\nu \otimes k$, there exists a tableau S with shape ν, and a row v with length $|v| = k$, uniquely determined by T, such that T is the rectification of the product vS. The preceding formula may therefore be written

$$Q_{r,\mu}(q) = \sum_{S,v|\mu(vS)=\mu} q^{c(vS)+|v|} s_{r+|v|,\lambda(S)}.$$

It is convenient to suppose that all of these tableaux are numbered from an alphabet of integers strictly greater than one. If S and v are as above, then consider a word $m = S1^r v$: it has weight (r,μ) and shape $(r+|v|, \lambda(S))$ if the shape of a word is the succession of the lengths of the rows of which it is the product. Moreover, by lemma 1.7.9, its charge takes the value $c(m) = c(vS1^r) + |v| = c(vS) + |v|$. Then

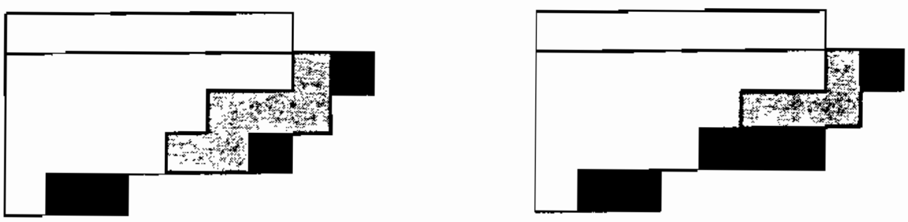
$$Q_{r,\mu}(q) = \sum_m q^{c(m)} s_{\lambda(m)},$$

the sum running over the set of words of the preceding form. It therefore remains to show, in order to conclude, that we may restrict ourselves to the words m which derive from tableaux.

If $m = S1^r v$ is not the word of a tableau, we may suppose that $\lambda(m) + \delta$ has pairwise distinct components—otherwise $s_{\lambda(m)} = 0$. Set $s = k + r$. Then there exists a unique integer p such that $\nu_{p+1} - p \leq s < \nu_p - p$, in which case

$$s_{\lambda(m)} = (-1)^p s_{\nu_1-1,\ldots,\nu_p-1,s+p,\nu_p+1,\ldots} = (-1)^p s_\rho.$$

In terms of Ferrers diagrams, the partition ν is deduced from ρ by adding a connected ribbon along the top of its first p rows, and omitting the first row of ρ. In the left diagram in the figure below, we have indicated in gray the connected ribbon which is added to ρ to form the partition ν.



Furthermore, the rectification T of m has for its shape τ an element of $\nu \otimes s$, which is obtained by adding cells to the diagram of ν, at most one per column. In the diagram above, we have indicated these cells in black.

Suppose for example that $\tau_{p+1} \leq s + p$. Then consider the partition ν' which is derived from ρ by adding to it a connected ribbon along the top of its first $p - 1$

rows only. If $s' - s = |v| - |v'|$, the right-hand diagram above shows that τ is also an element of $\nu' \otimes s'$. Similarly, if $\tau_{p+1} > s+p$, the partition ν' is derived this time from ρ by adding a connected ribbon along the top of its first $p+1$ rows.

This shows, thanks to proposition 1.7.10, that the tableau T is the rectification of the product of a tableau S' with shape ν', and a row u' of length s'. Moreover, by lemma 1.7.12, u' must have shape $1^r v'$. Associating to the word $m = S1^r v$ the word $m' = S'1^r v'$, it is easy to see that we have defined an involution on the set of words of this shape which are not words of tableaux. The global contribution of these words to $Q_{\tau,\mu}(q)$ is therefore zero, and the theorem follows. □

EXERCISE 1.7.14. Establish the table of Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$ of weight five:

$\lambda\backslash\mu$	11111	2111	221	311	32	41	5
11111	1	0	0	0	0	0	0
2111	$q+q^2+q^3+q^4$	1	0	0	0	0	0
221	$q^2+q^3+q^4+q^5+q^6$	$q+q^2$	1	0	0	0	0
311	$q^3+q^4+2q^5+q^6+q^7$	$q+q^2+q^3$	q	1	0	0	0
32	$q^4+q^5+q^6+q^7+q^8$	$q^2+q^3+q^4$	$q+q^2$	q	1	0	0
41	$q^6+q^7+q^8+q^9$	$q^3+q^4+q^5$	q^2+q^3	$q+q^2$	q	1	0
5	q^{10}	q^6	q^4	q^3	q^2	q	1

EXERCISE 1.7.15. Given two partitions λ and μ, determine the unique tableau with shape λ, weight μ, and (maximal) charge $n(\mu) - n(\lambda)$.

EXERCISE 1.7.16. If T is a tableau, shift its first row by one toward the right, and place the integer 1 in the hole thus created. Verify that one obtains a tableau with the same charge. Show that it remains so if one shifts the k first rows and places the integers $1, \ldots, k$ in the holes. Deduce that for any partitions λ, μ, and ν,

$$K_{\lambda+\nu,\mu+\nu}(q) \geq K_{\lambda,\mu}(q),$$

in the sense that each coefficient of the polynomial on the left is greater than or equal to the coefficient of the same degree of the polynomial on the right.

1.8. How the Symmetric Group Acts on Tableaux

1.8.1. Existence of an Action. Littlewood's Theorem 1.4.1 implies that for all permutations w, there exist as many semistandard tableaux with shape λ and weight μ as with shape λ and weight $w(\mu)$. However, to make an explicit one-to-one correspondence between these tableaux is to go much further. The appropriate action of the symmetric group on semistandard tableaux is given by the following result [**56**]:

THEOREM 1.8.1. *There exists one and only one action of the group of permutations on words which satisfies the following properties:*

1. *if m is a word of weight α, and if w is a permutation of the letters of m, then $w(m)$ is of weight $w(\alpha)$;*
2. *if $m \sim (P,Q)$, then $w(m) \sim (w(P),Q)$ under the Knuth correspondence;*
3. *if I is an interval which is stable under w, and if $m_{|I}$ is the word obtained by retaining only those letters of m belonging to I, then $w(m)_{|I} = w(m_{|I})$.*

EXERCISE 1.8.2. Let a and b be two consecutive letters, and let w be their transposition. If m is a word in these two letters, of weight (r, s), and $m \sim (P, Q)$, then determine P. Show that the first entry in the second row of Q is the first a of m preceded by a b. Show a similar statement for successive entries in this second row by disregarding previously isolated pairs ba. Deduce that $w(m)$ is obtained by isolating the pairs ba as above, and by replacing the word $a^p b^q$ remaining by $a^q b^p$.

This exercise makes explicit the action of a simple transposition s_i on a word m: isolate the subword formed from the letters i and $i+1$, determine its image as above, and replace its successive letters in m where they lay before. The action of any permutation may then be deduced.

EXERCISE 1.8.3. Deduce a proof of the preceding theorem by verifying that the action of simple transpositions is compatible with the braid relations.

EXAMPLE 1.8.4. We calculate the action of the permutation $w = s_2 s_3 s_1$ on the tableau T_0 with word $m(T_0) = 342331224$. By applying s_1, s_2, and s_3 successively, we obtain the words $m(T_1) = 342331114$, $m(T_2) = 342341114$, $m(T_3) = 342241114$, which correspond to the following tableaux:

$$\begin{array}{cccc} 1&2&2&4\\ 2&3&3&\\ 3&4&& \end{array} \xrightarrow{s_1} \begin{array}{cccc} 1&1&1&4\\ 2&3&3&\\ 3&4&& \end{array} \xrightarrow{s_3} \begin{array}{cccc} 1&1&1&4\\ 2&3&4&\\ 3&4&& \end{array} \xrightarrow{s_2} \begin{array}{cccc} 1&1&1&4\\ 2&2&4&\\ 3&4&& \end{array}$$

FIGURE 24. Action of the symmetric group on tableaux

EXERCISE 1.8.5. Let T be a tableau with shape λ and weight μ. If k is an integer, we transform T in the following fashion: add one to all of its entries, insert the integer one k times into the resulting tableau, and then apply the cyclic permutation which permits recovering a tableau whose weight is a partition. Show that the tableau obtained has shape $\lambda \cup k$, weight $\mu \cup k$, and the same charge as T. Deduce that for all partitions ν,

$$K_{\lambda\cup\nu,\mu\cup\nu}(q) \geq K_{\lambda,\mu}(q),$$

where $\lambda \cup \nu$ is the partition obtained by correctly ordering the union of the parts of λ and of ν.[14]

1.8.2. Decomposition into Atoms. This action of the symmetric group permits the extension of the definition of the charge to tableaux T having a weight α which is not a partition. In fact, if w is a permutation such that $w(\alpha)$ is a partition, we may define the charge of T to be that of the tableau $w(T)$, which does not depend on the choice of w, if there is any ambiguity. Equivalently, we will say that $T \to S$ is a cyclage if $w(T) \to w(S)$ is one, and the cocharge is again the number of cyclages necessary in order to transform T into a row tableau. The permutations then induce isomorphisms of partially ordered sets.

[14]This result is proved in a note of G.-N. Han, *Croissance des polynômes de Kostka*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 269–272.

EXAMPLE 1.8.6. We begin with the following tableau T of weight 1221, charge 3, and cocharge 4. The permutation s_1s_2 transforms T into a tableau S with the same cocharge.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \longrightarrow \quad S = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} .$$

The set of possible cyclages beginning with the tableau T, where the cocharge decreases from top to bottom, is the following:

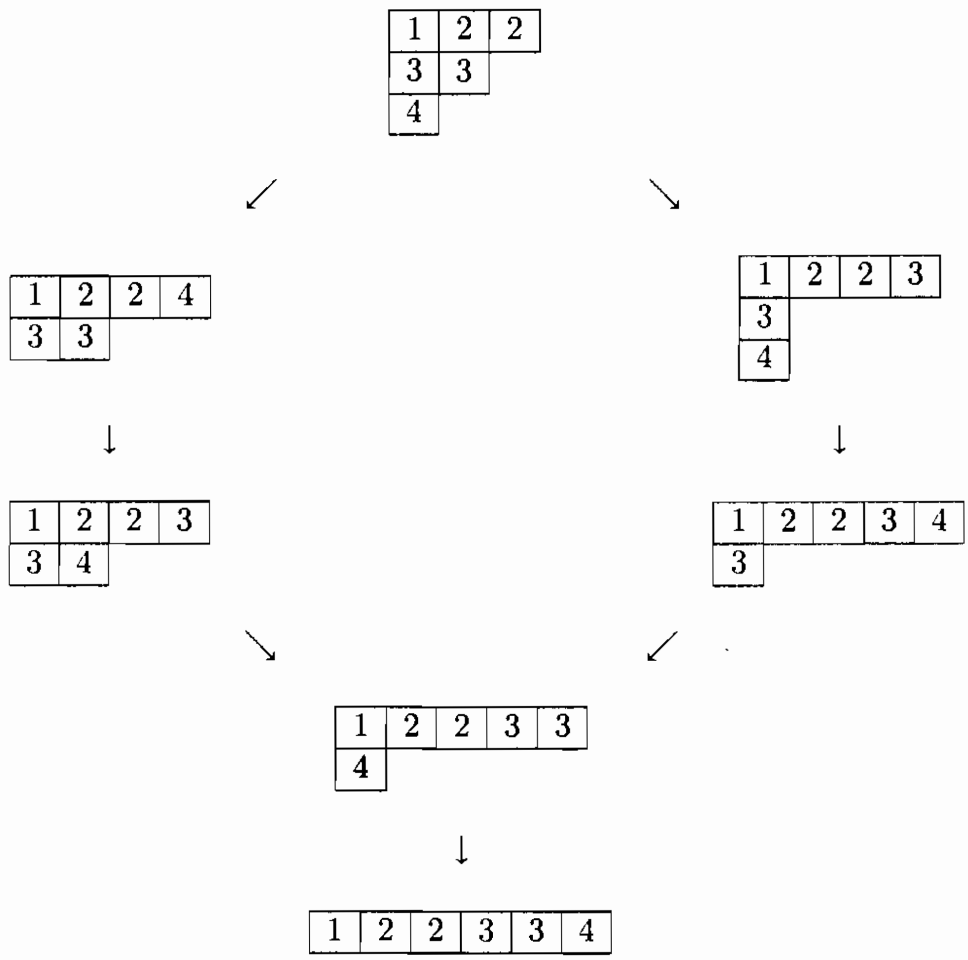


FIGURE 25. Cyclages

Suppose that $\alpha_1 > \alpha_2$, and set $\beta = \alpha - \varepsilon_1 + \varepsilon_2$. We obtain an injection from the set of tableaux with weight α to the set of tableaux with weight β by replacing the rightmost 1 of a tableau T into a 2.

EXERCISE 1.8.7. Show that this operation conserves the cocharge.

If μ and ν are two partitions such that $\mu \geq \nu$, they are connected by a chain of partitions ν^k such that we move from ν^k to ν^{k+1} by sliding one cell of a row of the diagram of ν^k to a lower row. Up to permutation, we have then recovered the preceding situation, which gives an injection from the set of tableaux of weight ν^k to those of weight ν^{k+1}. Hence, taken in total, we have an injection from the set of tableaux of weight μ to those of weight ν, compatible with the ordered structures of these sets, and which in particular respects the cocharge.

EXERCISE 1.8.8. Show that this injection does not depend on the chain of partitions chosen to connect μ to ν.

The set of semistandard tableaux thus decomposes into a disjoint union of *atoms*, where the atom $\mathcal{A}(\mu)$ associated to the partition μ is the set of tableaux of weight μ which do not derive, via the preceding injections, from tableaux of weight $\nu < \mu$. For the sake of convenience, we introduce the polynomials

$$K^{*}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1}) = \sum_{\substack{\lambda(T)=\lambda,\\ \mu(T)=\mu}} q^{co(T)},$$

which describes the statistic of the cocharge rather than of the charge. These polynomials then decompose as

$$K^{*}_{\lambda\mu}(q) = \sum_{\nu\leq\mu} L_{\lambda\mu}(q), \quad \text{with} \quad L_{\lambda\mu}(q) = \sum_{\substack{T\in\mathcal{A}(\nu),\\ \lambda(T)=\lambda}} q^{co(T)}.$$

EXERCISE 1.8.9. Establish the table of polynomials $L_{\lambda\mu}(q)$ in weight five

$\lambda\backslash\mu$	11111	2111	221	311	32	41	5
11111	q^{10}	0	0	0	0	0	0
2111	$q^7+q^8+q^9$	q^6	0	0	0	0	0
221	$q^6+q^7+q^8$	q^5	q^4	0	0	0	0
311	$q^5+q^6+q^7$	q^4+q^5	0	q^3	0	0	0
32	q^5+q^6	q^4	q^3	0	q^2	0	0
41	q^4	q^3	0	q^2	0	q	0
5	0	0	0	0	0	0	1

From the decomposition into atoms, we deduce, for example, the following property of Kostka-Foulkes polynomials immediately:

PROPOSITION 1.8.10. *If $\mu \geq \nu$, then $K^{*}_{\lambda\mu}(q) \geq K^{*}_{\lambda\nu}(q)$ for all partitions λ. In other words,*

$$q^{n(\nu)} K_{\lambda\mu}(q^{-1}) \geq q^{n(\mu)} K_{\lambda\nu}(q^{-1}).$$

It is possible to obtain some more general comparisons in the following way. To each finite set P with a partial order one may associate a Möbius function μ_P, in such a way that for every collection of numbers α_e and β_e, $e \in P$, the relations

$$\alpha_e = \sum_{f\leq e} \beta_f \quad \text{and} \quad \beta_e = \sum_{f\leq e} \mu_P(f,e)\alpha_f$$

are equivalent [**90**]. Consider, for example, the set of partitions of size n, under the dominance order. The Möbius function for this set takes only the values -1, 0, 1, and one may determine them explicitly.[15]

EXERCISE 1.8.11. Show that if $\sigma \leq \rho$ are two partitions, and if $[\sigma,\rho]$ denotes the interval that they define, then [**55**]

$$\sum_{\sigma\leq\nu\leq\rho} \mu_{[\sigma,\rho]}(\nu) K^{*}_{\lambda\mu}(q) \geq 0.$$

[15]See T. Brylawski, *The lattice of integer partitions*, Discrete Math. **6** (1973), 201–219.

CHAPTER 2

Schubert Polynomials

This second chapter begins with an introduction to the Bruhat order on symmetric groups, of which we give several different characterizations. We move next to a consideration of the particular classes of permutations called Grassmannian, vexillary, and 321-avoiding.

We then define the Schubert polynomials, introduced by Lascoux and Schützenberger in terms of divided differences. We make connections to the Yang-Baxter equation, Hecke algebras of symmetric groups, and the configurations of Fomin and Kirillov. The latter provide a particularly efficient way to make Schubert polynomials explicit, and to obtain several of the most important of their properties simply.

The lattice path method of Gessel and Viennot permits us to study the Schubert polynomials associated to the different classes of permutations which we introduce. Although no longer symmetric polynomials, certain of these act very similarly to Schur functions, and in particular admit descriptions in terms of tableaux.

Finally, we end with the multiplicative properties of Schubert polynomials, which remain poorly understood, but are connected in a subtle way with the combinatorics of the Bruhat order, and with reduced decompositions of permutations. The Monk formula, which expresses the product of a Schubert function with an indeterminate, permits us to reach a new formulation of the Littlewood-Richardson rule as well as a counting method for the number of reduced words of a permutation.

2.1. Permutations and the Bruhat Order

We first define and briefly study some different objects that one may associate to a permutation. We then define the Bruhat order on symmetric groups and give several characterizations of it.

2.1.1. Diagrams. The *length* of a permutation $w \in S_n$ is the cardinality of the set of its inversions

$$I(w) = \{1 \leq i < j \leq n, \quad w(i) > w(j)\}.$$

This length is also the cardinality of the *diagram*, sometimes called the *Rothe diagram*,

$$D(w) = \{(i,j),\ 1 \leq i,j \leq n,\ w(i) > j,\ w^{-1}(j) > i\}.$$

This diagram is obtained by permuting by w^{-1} the columns of $I(w)$. Graphically, we may construct it in the following fashion: it is the complement of the hooks with vertices $(i, w(i))$, $1 \leq i \leq n$. These NW (northwest) vertices are the points of the *graph* $G(w)$. We note in passing that, following this description, $D(w^{-1})$ is simply the transpose, or reflection, of $D(w)$ across the diagonal. Also, the diagram *has the NW (northwest) property* if it contains the cells (i,j) and (k,l) with $i < k$ and $j > l$, then it also contains the cell (i,l).

The sequence of the numbers of the points of the diagram in successive rows is the *code* $c(w)$, sometimes called the *Lehmer code.* The partition derived by sorting the components of the code is the *shape* $\lambda(w)$ of w. The *rank function* r_w associates to each point (i,j) the number of points of the graph $G(w)$ lying in the rectangle $\{1,\ldots,i\} \times \{1,\ldots,j\}$:

$$r_w(i,j) = \#\{k \le i,\ w(k) \le j\}.$$

Finally, the *essential set* is the set of the southeast corners of the diagram:

$$\mathrm{Ess}(w) = \{(i,j) \in D(w),\ (i+1,j),(i,j+1),(i+1,j+1) \notin D(w)\}.$$

EXAMPLE 2.1.1. The permutation $w = 3741652 \in S_7$, diagrammed in figure 1, has code $c(w) = 2520210$, shape $\lambda(w) = 52221$, and essential set

$$\mathrm{Ess}(w) = \{(2,6),(3,2),(5,5),(6,2)\}.$$

We have indicated in the figure the values of the rank function on the essential points.

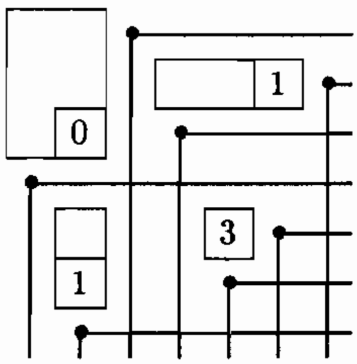


FIGURE 1. Diagram of the permutation $w = 3741652 \in S_7$

PROPOSITION 2.1.2. *A permutation w is determined by its code, therefore by its diagram, and also by the restriction of its rank function to the essential set.*

PROOF. Knowing the code $c(w)$, we may reconstruct w by recurrence: $w(k)$ is the $(c_k(w)+1)$-th integer of the sequence $1,\ldots,n$, omitting the integers $w(1),\ldots,$ $w(k-1)$.

Suppose now that we are given the function r_w on $\mathrm{Ess}(w)$ and determine $w(k)$ by induction. For $k = 1$, the point $(1, w(1))$ is immediately to the right of the rightmost essential point on which the rank function is zero, where if such a point does not exist, then $w(1) = 1$. Then we may proceed as follows in order to determine $w(k)$. By the inductive hypothesis, we know the rank function on the first $k-1$ rows. We may then identify the possible essential points (i,j) with $i \ge k$, for which

$$r_w(i,j) = r_w(k-1,j).$$

Then $w(k)$ must coincide with the first available value to the right of these points. If such a point does not exist, then it is simply the smallest available value. $\square$

We note that, by the construction of w from its code, $w(i) \leq w(i+1)$ if and only if $c_i(w) \leq c_{i+1}(w)$. Moreover, we always have $c_i(w) \leq n - i$, or, in other words, $\lambda(w) \subset \delta$. Conversely, every sequence of n natural numbers satisfying these inequalities is the code of a permutation.

2.1.2. The Bruhat Order. The symmetric group $\mathcal{S}_n$ is generated by the simple transpositions s_i for $1 \leq i \leq n-1$, permutations which exchange i and $i+1$ and fix the other integers. These simple transpositions satisfy the following relations:

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

The last relations are called *braid relations*.

A permutation w may be written as a product of at least $l(w)$ number of simple transpositions s_i. In the case of equality, we call such a product a *reduced decomposition*. The decomposition into a product of transpositions induces two natural partial orders on permutations. The *weak Bruhat order* is defined in the following fashion: one says that v precedes w if $l(w) = l(v) + 1$ and if there exists a simple transposition s_i such that $w = vs_i$. More generally, we write $v \preceq w$ if v and w are connected by a chain of permutations, each of which precedes the next—in other words, if there exists a reduced decomposition of w for which a left factor is a reduced decomposition of v. A reduced decomposition may therefore be identified with a chain, beginning with the identity, of permutations which are consecutive with respect to the weak Bruhat order.

This weak order has the advantage of being conveniently visualized: if we place in $\mathbb{R}^n$ the set of points $(w(1), \ldots, w(n))$, $w \in \mathcal{S}_n$, and if we connect by a segment the points corresponding to two permutations, one preceding the other, we obtain a polytope Π_{n-1} of dimension $n-1$, which is called a *permutohedron*.[1] For example, Π_3, represented in figure 2, has eight square faces and eight hexagonal faces.

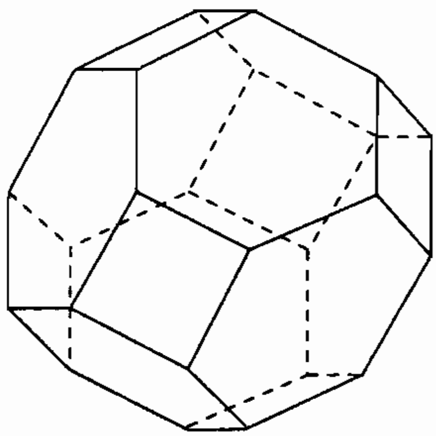


FIGURE 2. The permutohedron Π_3

[1]The permutohedron is what one calls a *zonotope*, the affine projection of a hypercube, and one may describe its faces simply. For a study of these polytopes, and in particular of the permutohedron, see the book by I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, 1994.

The *strong Bruhat order*, or *Ehresmann-Bruhat order*, has the same definition as the weak order, the difference being that we allow any transpositions, not necessarily simple: v precedes w if $l(w) = l(v) + 1$ and there exist integers $i < j$ such that $w = vt_{ij}$, where the transposition t_{ij} simply exchanges the integers i and j—in particular, $s_i = t_{i,i+1}$.

More generally, we write $v \leq w$ if there exists a sequence of permutations connecting v to w, each preceding the following. Unless we explicitly state the contrary, it is the strong Bruhat order that we use in the sequel when we employ the unadorned term *Bruhat order*.

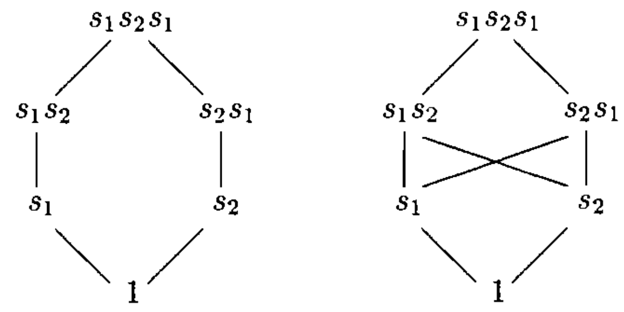


FIGURE 3. Weak and strong Bruhat orders on S_3

The strong Bruhat order is characterized by the following property, called the *subword property*:

PROPOSITION 2.1.3. *Let $s_{i_1} \dots s_{i_l}$ be a reduced decomposition of a permutation w. Then $v \leq w$ if and only if there exists a subsequence $(j_1, \dots, j_m)$ of $(i_1, \dots, i_l)$ such that $s_{j_1} \dots s_{j_m}$ is a reduced decomposition of v.*

PROOF. It suffices to show that $v^{-1}w$ is a transposition and $l(v) < l(w)$ if and only if there exists an integer k such that $v = s_{i_1} \dots \widehat{s_{i_k}} \dots s_{i_l}$. If $v = wt_{ij}$ with $i < j$ and $l(v) < l(w)$, then the pair (i, j) is in the set $I(w)$ of inversions of w. Then, knowing the reduced decomposition of w, we may deduce its inversions:

LEMMA 2.1.4. *The set of inversions of the permutation $w = s_{i_1} \dots s_{i_l}$, of length l, is the following:*

$$I(w) = \{s_{i_l} \cdots s_{i_{h+1}}(i_h, i_h + 1),\ 1 \leq h \leq l\}.$$

PROOF OF THE LEMMA. One verifies that if u is a permutation and m is an integer such that $l(us_m) = l(u) + 1$, then

$$I(us_m) = s_m I(u) \cup \{(m, m+1)\}.$$

The conclusion follows by induction. □

CONCLUSION. By the lemma, there exists an integer k such that

$$(i, j) = s_{i_l} \cdots s_{i_{k+1}}(i_k, i_k + 1).$$

Then this implies immediately that

$$t_{ij} = (s_{i_l} \cdots s_{i_{k+1}})s_{i_k}(s_{i_l} \cdots s_{i_{k+1}})^{-1} = s_{i_l} \cdots s_{i_{k+1}} s_{i_k} s_{i_{k+1}} \cdots s_{i_l},$$

whence we have the stated expression for v. The same calculation taken in reverse implies the converse. □

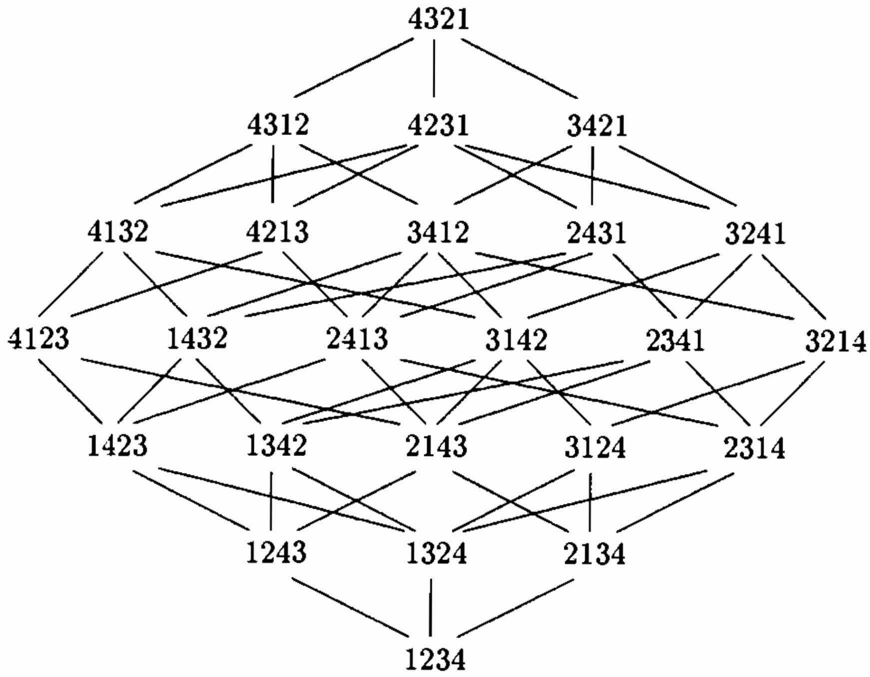


FIGURE 4. Strong Bruhat order on $\mathcal{S}_4$

If $s_{a_1} \cdots s_{a_l}$ is a reduced decomposition for w, we say that the word $a_1 \cdots a_l$ is a *reduced word* of w, and we denote by $R(w)$ the set of reduced words.

It is possible to determine the set of reduced decompositions of w beginning with one of them. To do so, associate to $w \in \mathcal{S}_n$ the graph $\mathcal{G}(w)$, the vertices of which are the reduced words of w, two vertices being connected if one may pass from one to the other by commuting two simple transpositions (hence by exchanging two digits, or indices, for which the difference is at least two), or by applying a braid relation.

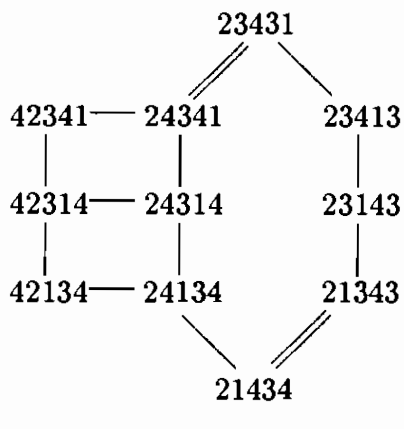


FIGURE 5. Graph $\mathcal{G}(w)$ of reduced words of $w = 31542 \in \mathcal{S}_5$

EXAMPLE 2.1.5. The permutation $w = 31542 \in \mathcal{S}_5$ has 11 reduced decompositions. Its graph is represented in figure 5, with vertices consisting of the different reduced words. The double lines represent connections derived from the braid relations.

An important property of the graph of reduced words of a permutation is its connectedness. This property signifies that, given a reduced decomposition of a permutation, one may deduce all the others by commuting simple transpositions with nonconsecutive indices, or by using the braid relations.

PROPOSITION 2.1.6. *For all permutations $w \in \mathcal{S}_n$, the graph $\mathcal{G}(w)$ of reduced words is connected.*

The proof will proceed by induction on the length of w, with the aid of the following lemma:

EXCHANGE LEMMA 2.1.7. *If $i_1 \cdots i_l$ and $j_1 \cdots j_l$ are two reduced words of w, then there exists an integer k such that*

$$w = s_{j_1} s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_l},$$

where the hat over s_{i_k} indicates that this transposition does not appear.

PROOF OF THE LEMMA. By the description of $I(w^{-1})$ as a function of a reduced decomposition, given in the preceding lemma, there exists an integer k such that

$$(j_1, j_1 + 1) = s_{i_1} \cdots s_{i_{k-1}} (i_k, i_k + 1).$$

Hence $s_{j_1} s_{i_1} \cdots s_{i_{k-1}} = s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$, which implies the result. □

PROOF OF THE PROPOSITION. We show, by induction on l, that one may pass from $i_1 \cdots i_l$ to $j_1 \cdots j_l$ by commutations or braid relations. Let k be the integer given by the exchange lemma. By the inductive hypothesis, one may pass from $i_1 \cdots i_l$ to $j_1 i_1 \cdots \widehat{i_k} \cdots i_l$ by operating on the first $l-1$ terms only, and then from $j_1 i_1 \cdots \widehat{i_k} \cdots i_l$ to $j_1 \cdots j_l$ by operating on the last $l-1$ terms.

The condition on this last statement, however, is that $k < l$. If the condition is violated, suppose first that i_1 and j_1 are not consecutive. Then one may pass from $i_1 \cdots i_l$ to $i_1 j_1 \cdots \widehat{i_k} \cdots i_l$ by operating on the last $l-1$ terms, then to $j_1 i_1 \cdots \widehat{i_k} \cdots i_l$ by commuting the two first terms, and one concludes similarly.

Finally, if i_1 and j_1 are consecutive, we again apply the exchange lemma to $i_1 \cdots i_l$ and $j_1 i_1 \cdots i_{l-1}$, obtaining a new reduced word of the form $i_1 j_1 i_1 \cdots \widehat{i_h} \cdots i_{l-1}$. We pass then from $i_1 \cdots i_l$ to $i_1 j_1 i_1 \cdots \widehat{i_h} \cdots i_{l-1}$ by operating on the $l-1$ last terms, then from this to $j_1 i_1 j_1 \cdots \widehat{i_h} \cdots i_{l-1}$ by a braid relation. We conclude as above. □

EXERCISE 2.1.8. Show that, due to the exchange lemma, if $s_{i_1} \cdots s_{i_m}$ is a nonreduced decomposition of a permutation w, then there exist integers $p < q$ such that $w = s_{i_1} \cdots \widehat{s_{i_p}} \cdots \widehat{s_{i_q}} \cdots s_{i_l}$. Deduce that $v \leq w$ if and only if w is derived from v by successive products by transpositions, on the right or left, each of these products augmenting the length, not necessarily by one.

REMARK 2.1.9. The diagram of a permutation permits us to determine simply a reduced decomposition, in the following way. We begin by numbering the cells of $D(w)$ consecutively on each row, from right to left, starting with the number of the row. We then read the rows from top to bottom, obtaining the following reduced decomposition:

$$w = s_{c_1} \cdots s_1 \cdots s_{c_k+k-1} \cdots s_k \cdots s_{c_{n-1}+n-2} \cdots s_{n-1}.$$

Reflecting $D(w)$ across the diagonal to obtain $D(w^{-1})$, we may obtain a new reduced word for w by reading backwards the correspondingly obtained word for w^{-1}.

We verify the preceding identity by induction on the number of inversions. Let k be the greatest integer such that $c_k > 0$, and v the permutation with code $(c_1, \ldots, c_{k-1}, 0, \ldots, 0)$: we must verify that $w = w'$, where $w' = v s_{c_k+k-1} \cdots s_k$. By the construction of a permutation from its code, we must have $w(i) = v(i) = w'(i)$ if $i < k$. Furthermore, $w'(k) = v(c_k + k)$, and from the code of v, this permutation is increasing beginning with index k. Hence $v(c_k + k)$ is the $(c_k + 1)$-th term of the sequence $1, \ldots, n$, omitting $v(1), \ldots, v(k-1)$, that is, $w(1), \ldots, w(k-1)$. This is therefore $w(k)$. Finally, w and w' are increasing beyond index k; they are therefore equal.

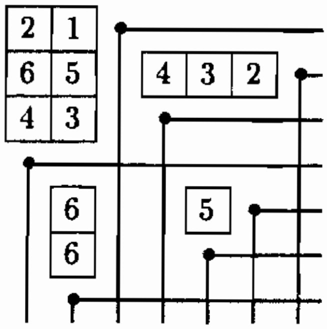


FIGURE 6. A reduced word of $w = 3741652 \in S_7$: $w = s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_4 s_3 s_6 s_5 s_6$

One may also characterize the Bruhat order in terms of *keys*.

DEFINITION 2.1.10. *If $w \in S_n$ is a permutation, its* key $K(w)$ *is the semistandard tableau with shape $\delta = (n-1, \ldots, 1, 0)$, in which the i-th column consists of the $n - i + 1$ integers $w(1), \ldots, w(n-i+1)$, placed in increasing order from top to bottom.*

PROPOSITION 2.1.11. *We have $v \le w$ if and only if $K(v) \le K(w)$, in the sense that each entry of the first of the tableaux is less than or equal to the corresponding entry of the second.*

PROOF. In one direction, it suffices to show that if $w = v t_{ij}$ with $v(i) < v(j)$, then $K(v) \le K(w)$, which is elementary.

For the other direction, we suppose that $v \ne w$ and proceed by induction on the smallest integer k such that $v(k) \ne w(k)$. Since $v(i) = w(i)$ for $i < k$, we have $w(k) = v(j)$ for an integer $j > k$, and $v(j) > v(k)$ since $K(v) \le K(w)$. Therefore let $u = v t_{jk}$ so that $v < u$ and $u(i) = v(i)$ for $i \le k$. Since we always have $K(u) \le K(w)$, this becomes $v < u \le w$ by the inductive hypothesis. $\square$

$$K(v) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 3 & \\ \cline{1-4} 3 & 3 & 6 & & \\ \cline{1-3} 5 & 6 & & & \\ \cline{1-2} 6 & & & & \\ \cline{1-1} \end{array} \quad \le K(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 2 & 4 & 6 & \\ \cline{1-4} 4 & 4 & 6 & & \\ \cline{1-3} 5 & 6 & & & \\ \cline{1-2} 6 & & & & \\ \cline{1-1} \end{array}$$

FIGURE 7. Comparison of $v = 326154$ and $w = 462153$ in S_6

Finally, the Bruhat order may be expressed just as simply on rank functions:

PROPOSITION 2.1.12. *Let v and w be two permutations. The inequality $v \leq w$ is satisfied if and only if $r_v \geq r_w$, in the sense that for all pairs (p, q), we have*

$$r_v(p, q) \geq r_w(p, q).$$

PROOF. By definition, $r_v(p, q)$ is the number of integers less than or equal to q in the $(n-p+1)$-th column of $K(v)$. Consequently, $r_v \geq r_w$ if and only if for every integer q, each column of $K(v)$ contains at least as many integers bounded above by q as the column corresponding to $K(w)$. But this is equivalent to $K(v) \leq K(w)$, therefore to $v \leq w$, by the preceding proposition. □

REMARK 2.1.13. We conclude this introduction to the Bruhat order by noting that the Bruhat order is *Eulerian*, in the sense that its Möbius function (see the end of the first chapter) may be expressed as follows:[2]

$$\mu_{\mathcal{S}_n}(u, v) = (-1)^{l(v)-l(u)} \quad \text{if } u \leq v.$$

2.2. Some Classes of Permutations

We now show how to use these different notions in the study of two large classes of permutations: the permutations called *vexillary* (and the two subclasses consisting of dominant permutations and Grassmannian permutations) on one hand, and on the other those that we call 321-avoiding.

2.2.1. Vexillary Permutations. The notion of the code of a permutation allow us to consider permutations generalizing partitions in at least two ways.

DEFINITION 2.2.1. *A permutation is called* dominant *if its code is a partition, that is, if $c_1(w) \geq \cdots \geq c_n(w)$.*

EXERCISE 2.2.2. Show that a permutation is dominant if and only if its diagram is the diagram of a partition. In particular, its inverse is also dominant. In a similar manner, verify that a permutation is dominant if and only if its rank function is zero on its essential set. Still further, show that a permutation is dominant if and only if there does not exist a triple of integers $i < j < k$ such that $w(i) < w(k) < w(j)$.

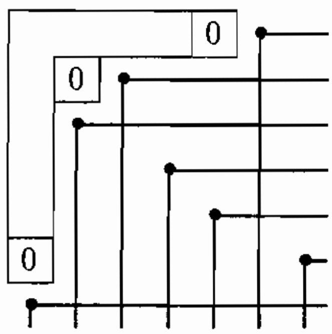


FIGURE 8. Dominant permutation $w = 6324571 \in \mathcal{S}_7$

DEFINITION 2.2.3. *A permutation w is called* Grassmannian *if there exists an integer r such that $c_1(w) \leq \cdots \leq c_r(w)$ and $c_i(w) = 0$ for $i > r$. In other words, $w(i) < w(i+1)$ if $i \neq r$: a Grassmannian permutation is a permutation having a unique* descent.

[2]See D. N. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. École Norm. Sup. **4** (1971), 393–398.

EXERCISE 2.2.4. Describe the characteristic properties of the diagrams of Grassmannian permutations.

EXERCISE 2.2.5. A permutation is called *bigrassmannian* if it is Grassmannian and its inverse is Grassmannian as well. What are the diagrams of bigrassmannian permutations? Describe these permutations completely.[3]

Dominant permutations and bigrassmannian permutations are special cases of a larger class of permutations, called *vexillary.*[4]

DEFINITION 2.2.6. *A permutation is called* vexillary *if its diagram, up to a permutation of its rows and columns, is the diagram of a partition.*

PROPOSITION 2.2.7. *A permutation w is vexillary if and only if there does not exist a sequence $i < j < k < l$ such that $w(j) < w(i) < w(l) < w(k)$.*

PROOF. The diagram of a permutation w may be transformed into a partition by exchanging its rows and columns if and only if these rows (or columns) may be totally ordered by inclusion. We find that the contrary situation occurs if and only if there exist points $(i, w(j))$ and $(k, w(l))$ of the diagram $D(w)$, with, for example, $i < k$, such that $(i, w(l))$ and $(k, w(j))$ are not elements of the same diagram.

Necessarily, $w(j) < w(l)$, since if not then $(i, w(l))$ will be in $D(w)$, by the NW property. Then our first two hypotheses are equivalent respectively to the inequalities $w(i) > w(j)$, $j > i$, and $w(k) > w(l)$, $l > k$, and the two following to $w(i) < w(l)$ and $w(k) < w(j)$. Hence $i < j < k < l$ and $w(j) < w(i) < w(l) < w(k)$. □

We may similarly characterize the property of a permutation's being vexillary in terms of its essential set.

PROPOSITION 2.2.8. *A permutation is vexillary if and only if its essential set lies on a piecewise linear curve always oriented SW-NE.*

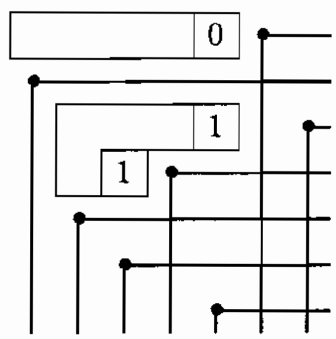


FIGURE 9. Vexillary permutation $w = 6174235 \in \mathcal{S}_7$

This property of having a SW-NE direction must be understood in the broad sense. For example, the essential points of Grassmannian permutations, which are evidently vexillary, lie on a horizontal line.

[3]Bigrassmannian permutations play a very important role in the Bruhat order, for they form bases of the symmetric groups in the sense that for two permutations $v, w \in \mathcal{S}_n$, we have $v \leq w$ if and only if every bigrassmannian permutation dominated by v is also dominated by w. See A. Lascoux and M.-P. Schützenberger, *Treillis et bases des groupes de Coxeter*, Electron. J. Combin. **3** (1996), no. 2, Research paper 27.

[4]From the Latin *vexillum*, ensign, standard of the Roman armies. This terminology is due to A. Lascoux and M.-P. Schützenberger.

PROOF. Suppose that w has two essential points (i,j) and (k,l) such that $i<k$ and $j<l$. Then $w(i+1)\leq j$ and $w^{-1}(j+1)\leq i$, and $(w(i+1),w^{-1}(j+1))\in D(w)$. On the other hand, neither $(k,w(i+1))$ nor $(w^{-1}(j+1),l)$ is a point of the diagram, and the rows with indices $w^{-1}(j+1)$ and k of $D(w)$ are then incomparable: v is not vexillary.

Conversely, if w is not vexillary, $D(w)$ contains two points (i,j) and (k,l), with $i<k$ and $j<l$, such that (i,l) and (k,j) are outside of $D(w)$. The connected components of the diagram which contain these points are then disjoint, and each contains at least one essential point to the SE of (i,j) and (k,l). But a pair of such points is directed NW-SE, strictly so. □

DEFINITION 2.2.9. *We define the* flag *of a vexillary partition w, starting from its code $c(w)$, in the following fashion. If $c_i(w)\neq 0$, let e_i be the greatest integer $j\geq i$ such that $c_j(w)\geq c_i(w)$. The flag $\phi(w)$ is then the sequence of integers e_i, ordered to be increasing. We remark that if $\lambda(w)=(l_1^{m_1}\cdots l_k^{m_k})$, with $l_1>\cdots>l_k>0$, then we may also write $\phi(w)=(f_1^{m_1}\cdots f_k^{m_k})$, with $f_1\leq\cdots\leq f_k$.*

PROPOSITION 2.2.10. *A vexillary permutation is completely determined by its shape and its flag.*

PROOF. The essential observation is the following: if $c_i(w)\leq c_j(w)$ for $i<j$, then $c_i(w)\leq c_k(w)$ for all k between i and j. If not, the kth row of the diagram $D(w)$ will be strictly contained in the ith, therefore in the jth, because w was supposed to be vexillary. But if a cell of the kth row is not in $D(w)$, then the corresponding cell of the jth row is not, and we have reached a contradiction.

We show then how to reconstruct the code of w beginning with its shape and its flag. The preceding remark implies first that the indices of the rows of $D(w)$ of length l_1 form a segment. By definition of the flag, the lowest among these is of index f_1, which determines the indices of the rows on which the code has the value l_1. We suppose then that the indices of the rows on which the code takes values strictly greater than l_i are known, and we determine those on which the code takes the value l_i.

The lowest row is necessarily of index f_i, and by the remark above we are constrained to assign the value l_i of the code to the rows immediately preceding for which the code has not yet been determined. Indeed, if one leaves a hole on a row of index h, it will have assigned to it a code strictly smaller than l_i, so that the row h will have above it a row with code taking the value k_i, and below it a row on which the code is greater than or equal to l_i, which is not possible. □

EXERCISE 2.2.11. If we write $\phi(w^{-1})=(g_1^{p_1}\cdots g_k^{p_k})$, show that the essential set is

$$\mathrm{Ess}(w)=\{(f_k,g_1),(f_{k-1},g_2),\ldots,(f_1,g_k)\}.$$

Further show that the values of the rank function of w on its essential points are given by

$$r_w(f_{k+1-i},g_i)=g_i-l_{k+1-i}.$$

2.2.2. 321-Avoiding Permutations. The class of permutations that we now define and study was introduced in [**3**].

DEFINITION 2.2.12. *A permutation w is called* 321-avoiding *if there does not exist a triple of integers $i<j<k$ such that $w(i)>w(j)>w(k)$.*

In particular, a permutation is 321-avoiding if and only if its inverse has the same property.

PROPOSITION 2.2.13. *If $w \in S_n$, the following are equivalent:*

1. *w is 321-avoiding;*
2. *its diagram $D(w)$ possesses the SE property: if $D(w)$ contains the points (i,k) and (j,l), with $i<j$ and $k>l$, then it contains the point (j,k); and*
3. *if $c_i(w), c_j(w) > 0$ with $i<j$, and $c_k(w)=0$ for $i<k<j$, then*

$$c_i(w) - c_j(w) < j - i.$$

PROOF. Suppose that (i,k), $(j,l) \in D(w)$, with $i<j$ and $k>l$, but that $(j,k) \notin D(w)$. This implies that $l < w(j) \le k$ or $i < w^{-1}(k) \le j$, and by replacing w by w^{-1} if necessary, we may suppose that we are in the first case. But then $i < j < w^{-1}(l)$ and $w(i) > w(j) > l$, whence w is not 321-avoiding, which shows that the first assertion implies the second.

If the second is satisfied, let $i<j$ be such that $c_i(w), c_j(w) > 0$, but $c_k(w)=0$, for $i<k<j$. Then the rows i and j of $D(w)$ consist of cells for which the column indices form respectively increasing sequences (A,B) and (B,C), with, if $a \in A$, $i < w^{-1}(a) < j$. Therefore $c_i(w) - c_j(w) \le \#A < j-i$, and the third assertion is established.

Finally, if w is not 321-avoiding, let $i<j<k$ be such that $w(i) > w(j) > w(k)$, with i maximal and j minimal for this choice of i. Then, if $i<l<j$, $w(l) < w(i)$ by the maximality of i, and $w(l) < w(j)$ by the minimality of j. Taking into account the one with index $w(j)$, we therefore have $j-i$ columns containing a cell of the diagram in the i-th row and not in the j-th row. Hence

$$c_i(w) \ge c_j(w) + j - i.$$

Since, for $i > l > m$, we cannot have $w(i) > w(l) > w(m)$ by the minimality of j, we have $c_l(w) = 0$ for $i<l<j$. We then reach a contradiction with the third assertion. $\square$

The second of the preceding properties may be interpreted in the following fashion. If one omits from an $n \times n$ square the rows and the columns which do not appear in the diagram of w, and then symmetrize with respect to a column, a skew partition $\lambda(w)/\mu(w)$ is obtained. More precisely, it will be convenient to embed canonically this skew partition in $\mathbb{Z} \times \mathbb{Z}$ as follows. If $k_1 < \cdots < k_l$ is the sequence of indices of the nonzero components of the code of w, the skew partition associated to w will be the set of cells (i,j), with $1 \le i \le l$, such that

$$i - k_i - c_{k_i} < j \le i - k_i.$$

EXERCISE 2.2.14. Let $c = (c_1, \ldots, c_n)$ be the code of a permutation, and $k_1 < \cdots < k_l$ the sequence of indices of its nonzero components. Then construct a path in the plane joining the point $(0,0)$ to the point (n,n) in the following fashion. Beginning from the origin, move first toward the right to the point with x-coordinate $c_{k_1} + k_1 - 1$. Then move up to the point with y-coordinate k_1. Then move right again to the x-coordinate $c_{k_2} + k_2 - 1$, up to the y-coordinate k_2, and so on.

Show that c is the code of a 321-avoiding permutation if and only if the associated path keeps a NW-SE orientation and remains above the diagonal. Deduce then from the solution of the ballot problem (formula 1.4.11 of the first chapter) for two candidates that the number of 321-avoiding permutations in $\mathcal{S}_n$ is given by the n-th *Catalan number*,

$$\gamma_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is similarly possible to characterize the 321-avoiding permutations in terms of reduced decompositions.

PROPOSITION 2.2.15. *The 321-avoiding permutations are the permutations for which all of the reduced decompositions may be derived from one to the other without using braid relations—in other words, by allowing only commutation of simple transpositions with nonconsecutive indices.*

PROOF. If $w \in \mathcal{S}_n$ has two reduced words, one of which may be derived from the other by using at least one braid relation, then it has in particular a reduced decomposition of the form $w = us_is_js_iv$ with $|j-i| = 1$. But $s_is_js_i$ then gives rise to an increasing triple of integers with decreasing images, which necessarily remains in w.

Conversely, suppose that there exist $i < j < k$ such that $w(i) > w(j) > w(k)$, and j being fixed, choose i maximal and k minimal. Then the permutation

$$v = ws_i \dots s_{j-1}s_{k-1}\cdots s_{j+1}$$

is of length $l(v) = l(w) - (j-i-1) - (k-j-1)$. Since it satisfies the inequalities $v(j-1) > v(j) > v(j+1)$, the permutation $u = vs_js_{j-1}s_j$ is of length

$$l(u) = l(v) - 3 = l(w) - (k-i+1).$$

Finally, a reduced decomposition of u gives a reduced decomposition of w, where the braid $s_js_{j-1}s_j = s_{j-1}s_js_{j-1}$ appears. □

If $w \in \mathcal{S}_n$ is 321-avoiding and of length l, let $a = a_1\cdots a_l \in R(w)$. All of the reduced words of w may then be obtained from a, by commuting only nonconsecutive letters. One may interpret the forbidden commutations by means of a total order on the set $P = \{p_1, \dots, p_l\}$, defined by

$$p_i < p_j \quad \text{if} \quad i < j \quad \text{and} \quad |a_i - a_j| = 1.$$

We then obtain the following result:

PROPOSITION 2.2.16. *If $w \in \mathcal{S}_n$ is 321-avoiding, its reduced words are in one-to-one correspondence with the total orders on P compatible with its partial order. If $p_{i_1} < \cdots < p_{i_l}$ is a total order, the reduced word associated to it is simply $a_{i_1}\cdots a_{i_l}$.*

Recall that above we associated, to each 321-avoiding permutation w, a skew partition $\lambda(w)/\mu(w) \subset \mathbb{Z}\times\mathbb{Z}$. Number its cells with the negative of the content $c(i,j) = j-i$. We find then, by reading from top to bottom and from left to right, the reduced decomposition canonically associated to the diagram of w (remark 2.1.9). Furthermore, the cells numbered with consecutive integers, which correspond to the forbidden commutations, are those which correspond to cells having a common side.

The NW-SE order on $\mathbb{Z}\times\mathbb{Z}$, defined by $(i,k) \leq (j,l)$ if $i \leq j$ and $k \leq l$, therefore induces on $\lambda(w)/\mu(w)$ a partial order isomorphic to the one on P. A compatible total order may then identified simply with a standard numbering of $\lambda(w)/\mu(w)$. Knowing moreover, by proposition 1.5.20 of the first chapter, that the

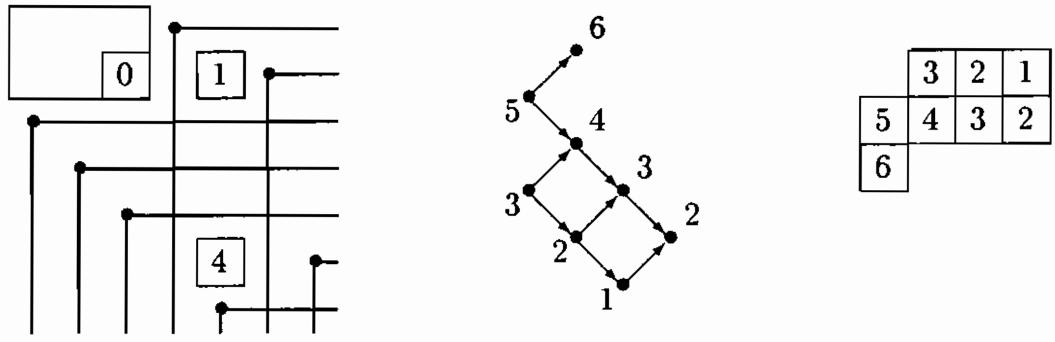


FIGURE 10. 321-avoiding permutation: $w = 4612375 \in \mathcal{S}_7$, P, and $\lambda(w)/\mu(w)$.

number of such skew tableaux with rectification a standard tableau with shape ν is the corresponding Littlewood-Richardson coefficient, we deduce in particular [**3**]:

COROLLARY 2.2.17. *If w is 321-avoiding, the number of its reduced decompositions is given by the following formula:*

$$\#R(w) = \sum_{\nu} c^{\lambda(w)}_{\mu(w),\nu} K_{\nu}.$$

We will see in section 2.8 that the number of reduced decompositions of a permutation may always be written as a sum of degrees of certain representations of the symmetric group (in other words, of integers K_ν) which are naturally associated to it. This result will be a consequence of the remarkable combinatorial properties of Schubert polynomials, polynomials which it is time to introduce.

2.3. Schubert Polynomials

Schubert polynomials are defined in an inductive manner, starting from the one of highest degree, which is a simple product of linear forms. From the nilCoxeter algebra and the Yang-Baxter equation, we obtain numerous properties of these polynomials.

2.3.1. Divided Differences. Newton's *divided differences* act on polynomials with n variables. We denote by ∂_i the operator of degree -1 defined, for $1 \le i < n$, by

$$(\partial_i P)(x_1, \dots, x_n) = \frac{P(\dots, x_i, x_{i+1}, \dots) - P(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

More compactly, $\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$. The space of symmetric polynomials in x_i and x_{i+1} is both the kernel and the image of this operator. The divided differences satisfy relations which are very similar to those satisfied by the simple transpositions:

$$\begin{aligned} \partial_i^2 &= 0, \\ \partial_i\partial_j &= \partial_j\partial_i \quad \text{if } |i-j| > 1 \\ \partial_i\partial_j\partial_i &= \partial_j\partial_i\partial_j \quad \text{if } |i-j| = 1. \end{aligned}$$

DEFINITION 2.3.1. *If $a = a_1 \cdots a_l$ is a reduced word of a permutation w, the operator $\partial_{a_1} \cdots \partial_{a_l}$ depends only on w. This is an immediate consequence of the relations satisfied by the divided difference operators and of the connectedness of* *the graph $\mathcal{G}(w)$ of the reduced words of w. We denote by ∂_w this operator,*[5] *which is homogeneous of degree $-l(w)$.*

Since divided differences have square zero, the operator associated as above to a nonreduced word $a_1 \cdots a_l$ is zero. Indeed, let i be the greatest integer such that $a_1 \cdots a_i$ is reduced, and u the corresponding permutation. Then $v = us_{a_{i+1}}$ is of length $i-1$, and if $b_1 \ldots b_{i-1}$ is a reduced word for it, $b_1 \ldots b_{i-1}a_{i+1}$ is a reduced word for u. This implies

$$\partial_{a_1} \cdots \partial_{a_l} = \partial_u \partial_{a_{i+1}} \cdots \partial_{a_l} = \partial_{b_1} \cdots \partial_{b_{i-1}} \partial_{a_{i+1}} \partial_{a_{i+1}} \cdots \partial_{a_l} = 0.$$

As a consequence, for any two permutations u and v,

$$\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } l(uv) = l(u) + l(v), \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.3.2. *If w_0 is a permutation of maximal length in S_n, then*

$$\partial_{w_0} = a_\delta^{-1} \sum_{w \in S_n} \varepsilon(w) w.$$

PROOF. Each operator ∂_v may be written in the form

$$\partial_v = \sum_w f_{v,w} w,$$

where $f_{v,w}$ is a rational function. Since the image of ∂_{w_0} consists of symmetric polynomials, we have $u^{-1}\partial_{w_0} = \partial_{w_0}$ for each permutation u. Therefore

$$f_{w_0,w} = \varepsilon(u) f_{w_0,uw} = \varepsilon(ww_0) f_{w_0,w_0},$$

and one verifies that $f_{w_0,w_0} = \varepsilon(w_0) a_\delta^{-1}$ by using for example the reduced decomposition of w_0 which may be read from its diagram. ☐

REMARK 2.3.3. We do not make use of the fact that the divided differences have square zero in the definition of the operators ∂_w. If we define *isobaric divided differences* π_i by the identity $\pi_i P = \partial_i(x_i P)$, the braid relations and the commutation relations remain satisfied, and for these we have $\pi_i^2 = \pi_i$. We may then associate to every permutation w an operator π_w of degree zero.

Consider two sets $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of indeterminates, and their partial resultant

$$\Delta(x, y) = \prod_{i+j \leq n} (x_i - y_j).$$

DEFINITION 2.3.4. *To each permutation $w \in S_n$ is associated the double Schubert polynomial*

$$\mathfrak{S}_w(x, y) = \partial_{w^{-1}w_0} \Delta(x, y),$$

[5]Divided difference operators have been introduced in a more general context independently by Demazure [**10**] and Bernstein-Gelfand-Gelfand [**2**] (and even before in a manuscript of C. Chevalley, which for many years remained unedited [**9**]). In this situation, they are most often referred to as *Demazure operators*, and they are operators on the cohomology ring of a generalized flag variety $X = G/P$, where G is a complex semisimple Lie group and P is a parabolic subgroup. Among these varieties figure the Grassmannians and the complete flag varieties, which we consider in the third chapter, and the cohomology rings of which we study in detail.

where the divided differences are taken with respect to the variable x. The simple Schubert polynomials are the specializations

$$\mathfrak{S}_w(x) = \mathfrak{S}_w(x,0) = \partial_{w^{-1}w_0} x^\delta.$$

The definition of these polynomials is due to A. Lascoux and M.-P. Schützenberger [**58**][6], and dates from 1982. The composition properties of divided difference operators immediately implies that

$$\partial_u \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wu^{-1}} & \text{if } l(wu^{-1}) = l(w) - l(u), \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2. The Yang-Baxter Equation and Configurations. Before beginning the study of properties of Schubert polynomials, we develop an approach to these polynomials proposed by S. Fomin and A. Kirillov [**15, 14, 17**], first taking detour via Hecke algebras of the symmetric group.

If a and b are complex numbers, we define the *Hecke algebra* $\mathcal{H}^n_{a,b}$ by generators and relations. The generators number $n-1$, and we denote them by $u_1, \ldots, u_{n-1}$. The relations are the following:

$$\begin{aligned} u_i^2 &= au_i + b, \\ u_i u_j &= u_j u_i && \text{if } |i-j| > 1 \\ u_i u_j u_i &= u_j u_i u_j && \text{if } |i-j| = 1. \end{aligned}$$

EXERCISE 2.3.5. Show that as a complex vector space, $\mathcal{H}^n_{a,b}$ always has a basis indexed by permutations.

For example, $\mathcal{H}^n_{0,1} = \mathbb{C}[\mathcal{S}_n]$ is the algebra of the symmetric group, while $\mathcal{H}^n_{1,0}$ is the algebra generated by the isobaric divided differences. In what follows we most often use $\mathcal{H}^n_{0,0}$, the algebra generated by the divided differences, which is called the *nilCoxeter algebra*.

For simplicity we denote by $\mathcal{H} = \mathcal{H}^n_{0,0}$ the nilCoxeter algebra and by $u_1, \ldots, u_{n-1}$ its generators. If we associate to each generator u_i the simple transposition s_i of $\mathcal{S}_n$, then as a vector space, $\mathcal{H}$ has a basis which consists of permutations. Moreover, the product in $\mathcal{H}$ of two permutations is given by

$$u \cdot v = \begin{cases} uv & \text{if } l(uv) = l(u) + l(v), \\ 0 & \text{otherwise.} \end{cases}$$

If x is an indeterminate, set $h_i(x) = 1 + xu_i$. The relations satisfied by the generators of $\mathcal{H}$ translate into the following:

$$\begin{aligned} h_i(x)h_i(y) &= h_i(x+y), \\ h_i(x)h_j(y) &= h_j(y)h_i(x) && \text{if } |i-j| > 1 \\ h_i(x)h_j(x+y)h_i(y) &= h_j(y)h_i(y+x)h_j(x) && \text{if } |i-j| = 1. \end{aligned}$$

This last relation is called the *Yang-Baxter equation*.

---

[6]We note that one may develop the same formalism by replacing divided differences by isobaric divided differences and defining Grothendieck polynomials to take the place of Schubert polynomials. Grothendieck polynomials possess very similar properties: just as the Schubert polynomials, as we see in the third chapter, permit a description of the cohomology ring of flag varieties, Grothendieck polynomials permit a description of Grothendieck rings, which are rings of isomorphism classes of virtual vector bundles. For more details, consult for example [**54**] and the article by W. Fulton and A. Lascoux [**24**].

We call a *configuration* a family C of n continuous strands which cut each vertical line at a unique point, with the restriction that each pair of strands crosses at most once, always transversally, and at distinct x-coordinates. We adopt the convention that an isotopy, that is, a continuous deformation, preserving these properties and the order of the strand crossings, does not change the configuration.

If w is the permutation corresponding to the n strands, the configuration C codes a reduced decomposition of w, which may be read as follows. If $a_1, \ldots, a_l$ are the heights of the successive crossings, read from left to right (the height being the number of strands lying below the crossing, plus one), then $w = s_{a_1} \cdots s_{a_l}$. This is clearly a reduced decomposition since two strands are constrained to cross no more than one time.

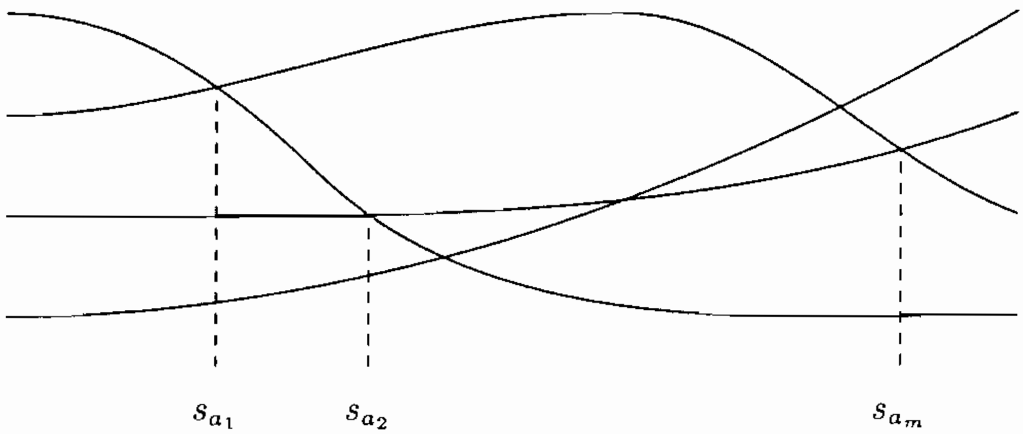


FIGURE 11. A configuration

We now associate to each strand an indeterminate, which we call its *weight*. Suppose that the i-th crossing of C involves strands of weight x_k and x_l, the second corresponding to the strand with the greatest slope—in other words, the second strand passes above the first when we move from the left to the right. We then set

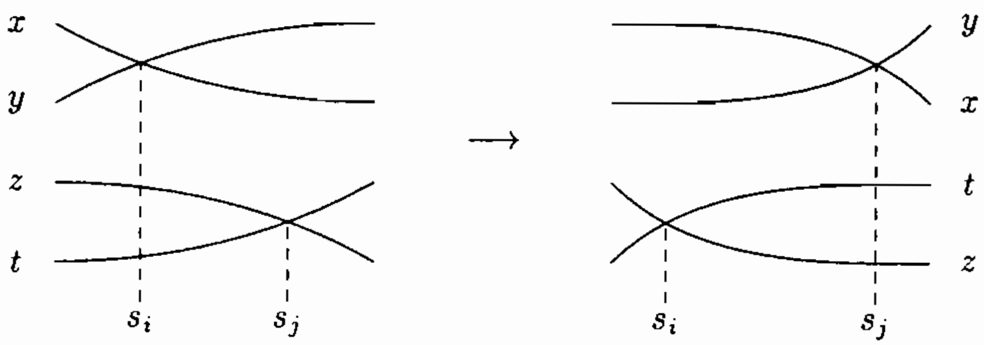
$$\phi(C) = h_{a_1}(x_{k_1} - x_{l_1}) \cdots h_{a_m}(x_{k_m} - x_{l_m}).$$

This is a polynomial with coefficients in the nilCoxeter algebra $\mathcal{H}$.

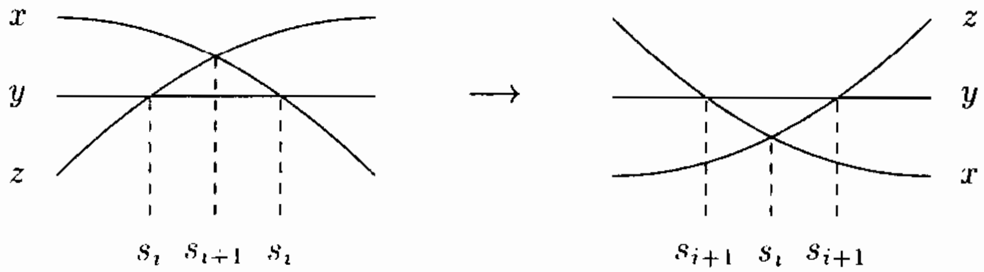
LEMMA 2.3.6. *The weights of the strands being fixed, the polynomial $\phi(C)$ depends only on the permutation w which corresponds to the configuration C.*

PROOF. By the connectedness of the graph $\mathcal{G}(w)$ of the reduced words of w, guaranteed by proposition 2.1.6, it suffices to show that $\phi(C)$ remains unchanged when one applies to C the deformations which correspond to the elementary operations on the reduced decompositions.

Permuting two simple transpositions with nonconsecutive indices corresponds to making the following deformation:



The associated polynomials are the same by the second identity of the preceding lemma: $h_i(x-y)h_j(z-t) = h_j(z-t)h_i(x-y)$, since $|i-j|>1$. Similarly, a braid relation between consecutive indices corresponds to a deformation of the following type:



Here, it is the Yang-Baxter relation

$$h_i(y-z)h_{i+1}(x-z)h_i(x-y) = h_{i+1}(x-y)h_i(x-z)h_{i+1}(y-z)$$

which guarantees that the associated polynomial is not modified. □

One may extend this construction to configurations for which the strands are divided into segments with distinct weights. The associated polynomial remains unchanged if one deforms such a configuration as above, with the additional condition that one never traverses a point of a strand which separates two segments with distinct weights.

2.3.3. A Particular Configuration. Fix an integer n, and consider the configuration C_{Sch} in figure 12 for which the strands, with the exception of the two diagonals, consist of two segments with distinct weights.

The polynomial associated to this configuration is, by definition,

$$\phi(C_{\mathrm{Sch}}) = \prod_{d=2-n}^{n-2} \prod_{\substack{i-j=d,\\ i+j\le n}} h_{i+j-1}(x_i - y_j),$$

where one must keep the order of the factors, since they do not commute. Deform C_{Sch} in the following way: slide the downward diagonal all the way to the left of the diagram, then the following strand a little less far, in order that its first crossing remains to the right of the last crossing of the preceding, and repeat the process for all of the strands. This deformation does not change the associated polynomial, which is then

$$\phi(C_{\mathrm{Sch}}) = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j).$$

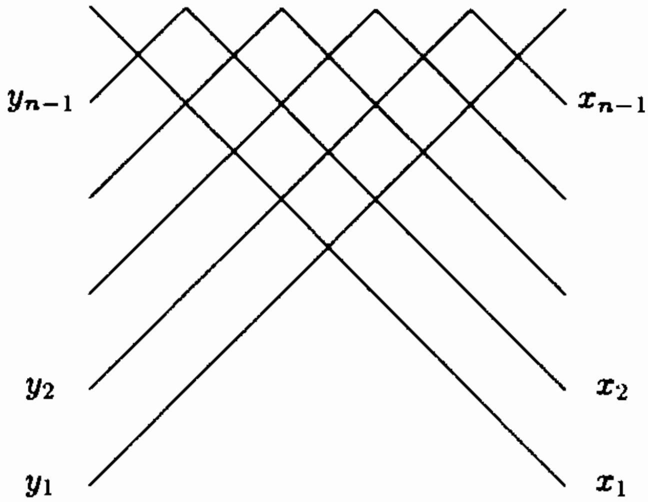


FIGURE 12. The configuration C_{Sch}

THEOREM 2.3.7. *If one decomposes $\phi(C_{\mathrm{Sch}})$ in $\mathcal{H}[x,y]$ as*

$$\phi(C_{\mathrm{Sch}}) = \sum_{w \in S_n} \phi_w(C_{\mathrm{Sch}}) w,$$

then $\phi_w(C_{\mathrm{Sch}})$ is none other than the double Schubert polynomial $\mathfrak{S}_w(x,y)$.

PROOF. It is certainly the case if $w = w_0$; indeed, if w_0 is the permutation associated to the configuration C_{Sch}, then

$$\phi_{w_0}(C_{\mathrm{Sch}}) = \prod_{i+j \leq n} (x_i - y_j) = \mathfrak{S}_{w_0}(x,y).$$

It remains to show that $\partial_i \phi(C_{\mathrm{Sch}}) = \phi(C_{\mathrm{Sch}}) u_i$, the divided differences acting on the variable x. This is equivalent to the relation

$$\partial_i \phi_{ws_i}(C_{\mathrm{Sch}}) = \phi_w(C_{\mathrm{Sch}}) \quad \text{if } l(ws_i) = l(w) + 1,$$

which defines by recurrence the Schubert polynomials. This relation will be a consequence of the following lemmas. Set

$$H_i(x) = h_{n-1}(x) \cdots h_{i+1}(x) h_i(x).$$

Then $H_i(x) = H_{i+1}(x) h_i(x)$, and $h_i(x)$ commutes with $H_j(y)$ if $j > i+1$. We similarly note that $h_i(x)$ and $h_i(-x)$ are inverses of each other.

LEMMA 2.3.8. *The following identities are satisfied:*

1. $H_i(x) H_i(y) = H_i(y) H_i(x)$, *and*
2. $H_i(x) H_{i+1}(y) - H_i(y) H_{i+1}(x) = (x-y) H_i(x) H_{i+1}(y) u_i$.

PROOF OF THE LEMMA. The first item is demonstrated by descending induction on i. We have successively

$$\begin{aligned} H_i(x)H_i(y) &= H_{i+1}(x) h_i(x) H_{i+2}(y) h_{i+1}(y) h_i(y) \\ &= H_{i+1}(x) H_{i+2}(y) h_i(x) h_{i+1}(y) h_i(y-x) h_i(x) \\ &= H_{i+1}(x) H_{i+2}(y) h_{i+1}(y-x) h_i(y) h_{i+1}(x) h_i(x) \\ &= H_{i+1}(y) H_{i+2}(x) h_i(y) h_{i+1}(x) h_i(x) \\ &= H_i(y) H_i(x), \end{aligned}$$

where we pass from the second to the third line via the Yang-Baxter equation, then to the following line by the inductive hypothesis. The second item is then deduced immediately. □

LEMMA 2.3.9. *The following identities are satisfied:*

1. $h_i(x-y) = H_{i+1}^{-1}H_i^{-1}(y)H_i(x)h_{i+1}(y)$,
2. $h_{n-1}(x-y_{n-1})\cdots h_i(x-y_i) = H_{n-1}^{-1}(y_{n-1})\cdots H_i^{-1}(y_i)\times H_i(x)H_{i+1}(y_i)\cdots H_{n-2}(y_{n-1})$.

PROOF OF THE LEMMA. The first item is an immediate consequence of the first item of the preceding lemma. The second is deduced by descending induction on i:

$$\begin{aligned}
&h_{n-1}(x-y_{n-1})\cdots h_i(x-y_i)\\
&\quad= H_{n-1}^{-1}(y_{n-1})\cdots H_{i+1}^{-1}(y_{i+1})H_{i+1}(x)H_{i+2}(y_{i+1})\cdots H_{n-2}(y_{n-1})h_i(x-y_i)\\
&\quad= H_{n-1}^{-1}(y_{n-1})\cdots H_{i+1}^{-1}(y_{i+1})H_{i+1}(x)h_i(x-y_i)H_{i+2}(y_{i+1})\cdots H_{n-2}(y_{n-1})\\
&\quad= H_{n-1}^{-1}(y_{n-1})\cdots H_i^{-1}(y_i)H_i(x)H_{i+1}(y_i)\cdots H_{n-2}(y_{n-1}),
\end{aligned}$$

the passage from the penultimate to the last line thanks to the first item of the lemma. □

CONCLUSION. By using the second item of the preceding lemma, we may rewrite the polynomial

$$\phi(C_{\text{Sch}}) = \prod_{i=1}^{n-1}\prod_{j=n-i}^{1} h_{i+j-1}(x_i-y_j)$$

in the form $\phi(C_{\text{Sch}}) = \mathfrak{S}(y)^{-1}\mathfrak{S}(x)$, where we have set

$$\mathfrak{S}(x) = H_1(x_1)H_2(x_2)\cdots H_{n-1}(x_{n-1}).$$

We are then reduced to checking that $\partial_i\mathfrak{S}(x) = \mathfrak{S}(x)u_i$. But this is an immediate consequence of the second item of lemma 2.3.8. □

2.4. Some Properties of Schubert Polynomials

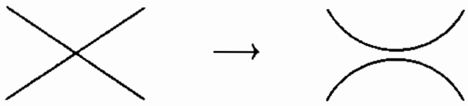
From the formalism that we develop in this section, a certain number of remarkable properties of Schubert polynomials follows directly. We begin with a diagrammatic method convenient for making these polynomials explicit, something which their definition does not necessarily allow.

2.4.1. How to Calculate a Schubert Polynomial. Consider one more time the expression

$$\phi(C_{\text{Sch}}) = \prod_{i=1}^{n-1}\prod_{j=n-i}^{1} h_{i+j-1}(x_i-y_j).$$

Each of the terms of the product corresponds to a crossing in the configuration C_{Sch}: it is a sum of two terms, the first of which is equal to one. If we expand, we may then code each of the resulting terms by the set of crossings for which we have kept the second factor. Graphically, we may transform C_{Sch} by deleting the other crossings as in the following figure.

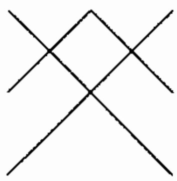
We obtain then a family of configurations, corresponding to certain decompositions of permutations into products of simple transpositions. Those which



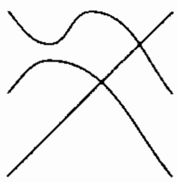
correspond to nonreduced decompositions have a zero contribution. We restrict ourselves therefore only to the others, and the preceding theorem may be transcribed as follows.

THEOREM 2.4.1. *If $w \in \mathcal{S}_n$, the Schubert polynomial $\mathfrak{S}_w(x, y)$ is the sum of the contributions of the configurations for which the associated permutation is precisely w.*

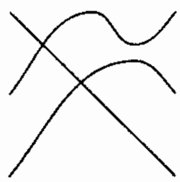
The description given by this theorem allows us to calculate the Bruhat order very naturally, since one passes from the configuration associated to a permutation w to a configuration associated to a permutation v immediately consecutive to w under this order, by creating a crossing between two strands which did not cross each other. We have given below the example of the group $\mathcal{S}_3$.



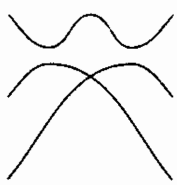
$\mathfrak{S}_{s_1 s_2 s_1} = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$



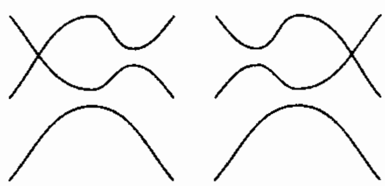
$\mathfrak{S}_{s_1 s_2} = (x_1 - y_1)(x_2 - y_1)$



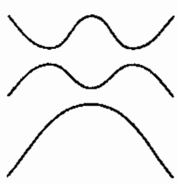
$\mathfrak{S}_{s_2 s_1} = (x_1 - y_1)(x_1 - y_2)$



$\mathfrak{S}_{s_1} = x_1 - y_1$



$\mathfrak{S}_{s_2} = x_1 - y_2 + x_2 - y_1$



$\mathfrak{S}_1 = 1$

FIGURE 13. Schubert polynomials for $\mathcal{S}_3$

2.4.2. Symmetries and Stability. If we reflect a configuration associated to a permutation w by applying a symmetry with respect to a vertical line, we obtain a configuration associated to the inverse permutation. Moreover, the contributions of the different crossings remain the same if we replace x_i with $-y_i$ and y_j with $-x_j$.

COROLLARY 2.4.2. *For all permutations $w \in \mathcal{S}_n$,*

$$\mathfrak{S}_{w^{-1}}(x,y) = \mathfrak{S}_w(-y,-x) = \varepsilon(w)\mathfrak{S}_w(y,x).$$

We note moreover that $\partial_i \mathfrak{S}_w = 0$ if and only if $l(ws_i) > l(w)$, the divided difference acting on x. This signifies that $\mathfrak{S}_w$ is symmetric in x_i and x_{i+1}. Taking the preceding corollary into account, this becomes

COROLLARY 2.4.3. *The polynomial $\mathfrak{S}_w(x,y)$ is symmetric in x_i and x_{i+1} if and only if $w(i) < w(i+1)$, and symmetric in y_j and y_{j+1} if and only if $w^{-1}(j) < w^{-1}(j+1)$.*

EXERCISE 2.4.4. Show that

$$\mathfrak{S}_{s_i}(x,y) = x_1 + \cdots + x_i - y_1 - \cdots - y_i.$$

Show that if r is the greatest descent of w, that is, the greatest integer such that $w(r) > w(r+1)$, and s is the greatest descent of w^{-1}, then $\mathfrak{S}_w(x,y)$ is a polynomial in $x_1, \dots, x_r$ and $y_1, \dots, y_s$ only.

Schubert polynomials moreover possess some remarkable properties of stability. We denote by i_n the embedding of $\mathcal{S}_n$ in $\mathcal{S}_{n+1}$ defined by adjoining the fixed point $n+1$. A configuration associated to the permutation $i_n(w)$ is obtained by adjoining to the configuration associated to w a strand over the highest row, such that this strand crosses no other. All of the configurations associated to $i_n(w)$ may be obtained by adjunction of this strand, which does not modify the contributions in the associated polynomials. As a consequence,

COROLLARY 2.4.5. *For all permutations $w \in \mathcal{S}_n$, we have*

$$\mathfrak{S}_w = \mathfrak{S}_{i_n(w)}.$$

We therefore define $\mathcal{S}_\infty$ as the set of bijections of $\mathbb{N}^*$ which fix almost all of the points; this is the union of the permutation groups $\mathcal{S}_n$ via the natural inclusions $i_n\colon \mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$. To every element $w \in \mathcal{S}_\infty$ is then associated a Schubert polynomial which does not depend on the representative chosen in a given symmetric group. More generally, if $u \in \mathcal{S}_m$ and $v \in \mathcal{S}_n$, denote by $u \times v$ the permutation $(u(1), \dots, u(m), m+v(1), \dots, m+v(n))$, an element of $\mathcal{S}_{m+n}$. Denote further by 1_m the identity of $\mathcal{S}_m$. Because the first m strands do not cross the last n in a configuration associated to $u \times v$, we reach the following factorization:

COROLLARY 2.4.6. *For all $u \in \mathcal{S}_m$ and $v \in \mathcal{S}_n$, we have*

$$\mathfrak{S}_{u\times v} = \mathfrak{S}_u \times \mathfrak{S}_{1_m \times v}.$$

2.4.3. The Cauchy Formula for Schubert Polynomials. Recall that in the proof of theorem 2.3.7 we obtained a factorization $\phi(C_{\mathrm{Sch}}) = \mathfrak{S}^{-1}(y)\mathfrak{S}(x)$ in the algebra $\mathcal{H}[x,y]$. If we make $y = 0$ and then $x = 0$, we obtain from corollary 2.4.2 the identities

$$\mathfrak{S}(x) = \sum_{w\in\mathcal{S}_n} \mathfrak{S}_w(x)w \quad \text{and} \quad \mathfrak{S}(y)^{-1} = \sum_{w\in\mathcal{S}_n} \varepsilon(w)\mathfrak{S}_{w^{-1}}(y)w.$$

It suffices to multiply these together in order to yield the following decomposition:

PROPOSITION 2.4.7. *For all permutations* $w \in \mathcal{S}_n$,

$$\mathfrak{S}_w(x,y) = \sum_{\substack{w=v^{-1}u,\\ l(w)=l(u)+l(v)}} \mathfrak{S}_u(x)\mathfrak{S}_v(-y).$$

In particular, if $w = w_0$, we obtain an identity for Schubert polynomials which is analogous to the Cauchy formula.

COROLLARY 2.4.8. *We have the following identity:*

$$\prod_{i+j\leq n} (x_i - y_j) = \sum_{w\in\mathcal{S}_n} \mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(-y).$$

EXERCISE 2.4.9. By applying a well chosen divided difference to the preceding formula, recover the second Cauchy formula for Schur functions.

2.5. Simple Schubert Polynomials

We momentarily leave the general consideration of Schubert polynomials in order to consider simple Schubert polynomials. After showing that these are polynomials with positive coefficients, we study the space generated by these polynomials, and show in particular that the symmetric group acts on this space via the regular representation. We conclude this study with the proof, by Fomin and Stanley, of a conjecture of Macdonald concerning the principal specialization of a Schubert polynomial.

2.5.1. Calculation. For the simple Schubert polynomials, it suffices to expand the identity

$$\mathfrak{S}(x) = \phi(C)_{|y=0} = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i),$$

in order to obtain the following result:

THEOREM 2.5.1. *For all permutations* $w \in \mathcal{S}_n$,

$$\mathfrak{S}_w(x) = \sum_{a\in R(w)} \sum_{b\in C(a)} x_{b_1}\cdots x_{b_l},$$

where $C(a)$ *is the set of increasing words* b *compatible with* a*, that is, of the same length and such that for all* i, $b_i \leq a_i$ *and* $b_{i+1} > b_i$ *if* $a_{i+1} > a_i$.

A remarkable consequence of this result is that a simple Schubert polynomial is a sum of monomials with positive coefficients.[7]

EXERCISE 2.5.2. Describe the configurations associated to a Grassmannian permutation w, and show that they are in natural correspondence with the semi-standard tableaux with shape $\lambda(w)$ numbered with integers between one and the greatest descent of w.

---

[7] A different method for making the different monomials of a Schubert polynomial explicit was proposed by A. Kohnert [**67**]. The validity of Kohnert's conjecture has recently been proved by R. Winkel, *Diagram rules for the generation of Schubert polynomials*, J. Combin. Theory Ser. A **86** (1999), 14–48. See [**67**] as well for still other rules.

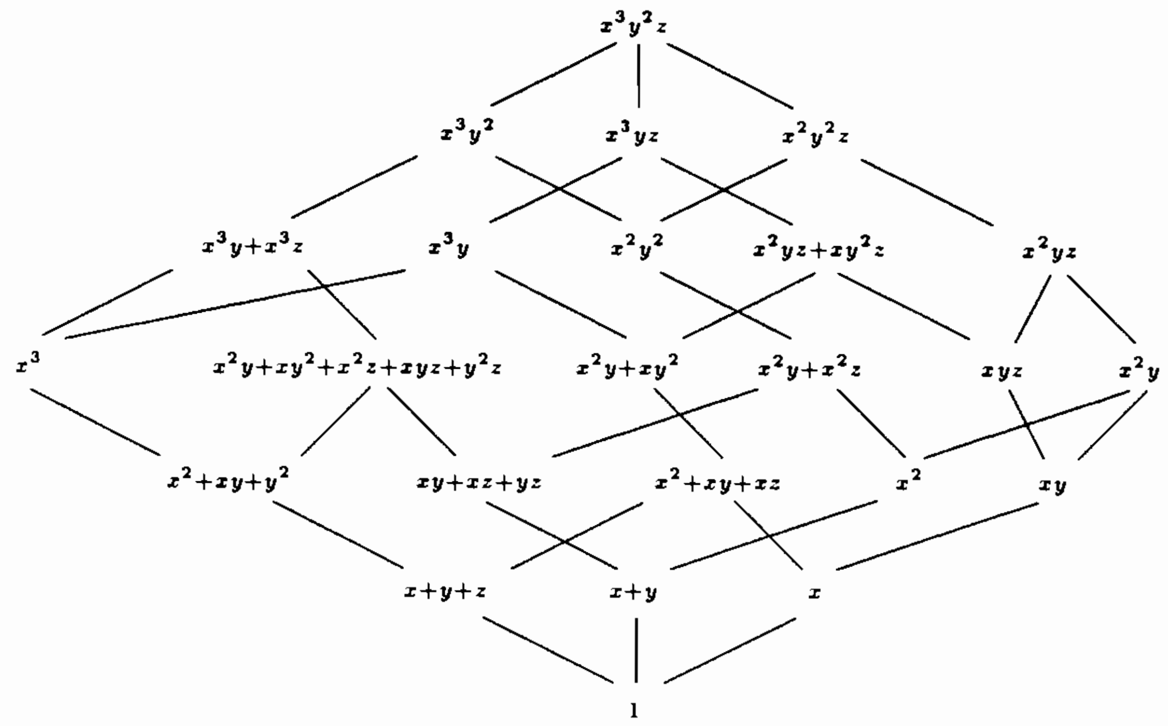


FIGURE 14. Simple Schubert polynomials for S_4

2.5.2. Bases. In the ring $\mathcal{P}_n$ of polynomials with integral coefficients in n variables, we denote by $\mathcal{H}_n$ the subgroup generated by the monomials x^α with $\alpha \subset \delta$, or $\alpha_i \leq n - i$, for all i between 1 and n. This group is of rank $n!$ and contains the simple Schubert polynomials.

PROPOSITION 2.5.3. *The Schubert polynomials $\mathfrak{S}_w(x)$, as w runs through S_n, form an integral base of $\mathcal{H}_n$.*

PROOF. Beginning with a relation $\sum_w s_w \mathfrak{S}_w(x) = 0$, we obtain the equality $s_v = 0$ by applying the operator ∂_v and keeping only the terms of degree zero. Every element P of $\mathcal{H}_n$ is then a linear combination of Schubert polynomials, with rational coefficients. But if $P(x) = \sum_w p_w \mathfrak{S}_w(x)$ is a polynomial with integral coefficients, the same application of ∂_v shows that p_v is an integer. □

Denote by $\mathcal{P}_\infty$ the increasing union of the $\mathcal{P}_n$.

PROPOSITION 2.5.4. *As w runs through S_∞, the $\mathfrak{S}_w(x)$ form an integral base of $\mathcal{P}_\infty$. More precisely, the $\mathfrak{S}_w(x)$ form an integral base of $\mathcal{P}_n$ as w runs through the permutations for which the greatest descent is less than or equal to n.*

PROOF. The first assertion is a consequence of the preceding proposition. Let $P = \sum_{w \in S_\infty} p_w \mathfrak{S}_w \in \mathcal{P}_n$. For all $m > n$, we have $\partial_m P = 0$, which implies that $p_w = 0$ if $l(ws_m) = l(w) - 1$. Then $p_w = 0$ for all permutations w with greatest descent greater than n. □

PROPOSITION 2.5.5. *The multiplication in $\mathcal{P}_n$ induces a canonical isomorphism of the tensor product $\mathcal{H}_n \otimes \Lambda_n$ with $\mathcal{P}_n$.*

PROOF. By permuting the variables if necessary, we may replace $\mathcal{H}_n$ by the submodule of $\mathcal{P}_n$ generated by the monomials x^α with $\alpha_i < i$. Note then that in

$\mathcal{P}_n$, the identities

$$e_k(x_1,\dots,x_{n-1}) = \sum_{j=0}^{k}(-1)^j x_n^j e_{k-j}(x_1,\dots,x_n),$$

imply that $\Lambda_n[x_n] = \Lambda_{n-1}[x_n]$. By induction on n, this permits us to assert that every polynomial $P \in \mathcal{P}_n$ may be written in the form

$$P = \sum_{k,j} h_{k,j} l_{k,j} x_n^k,$$

with $h_{k,j} \in \mathcal{H}_{n-1}$ and $l_{k,j} \in \Lambda_n$. But the identity

$$0 = \prod_{i=1}^{n}(x_n - x_i) = \sum_{h=0}^{n}(-1)^j x_n^{n-j} e_j(x_1,\dots,x_n)$$

then implies the existence of a similar expression for P, where k remains strictly less than n. This guarantees that the natural map from $\mathcal{H}_n \otimes \Lambda_n$ to $\mathcal{P}_n$ is surjective.

For injectivity, we consider a dependence relation

$$\sum_{\alpha\subset\delta} l_\alpha x^\alpha = 0,$$

where $l_\alpha \in \Lambda_n$, and we show that $l_\alpha = 0$ by descending induction on $|\alpha|$ and, for $|\alpha|$ given, by descending induction on the lexicographic order. Let $\beta \subset \delta$. Multiply the preceding identity by $x^{\delta-\beta}$, and antisymmetrize. Then we have

$$\sum_{\alpha\subset\delta} l_\alpha a_{\delta-\beta+\alpha} = 0.$$

But if $a_{\delta-\beta+\alpha} \neq 0$, then $|\alpha| \geq |\beta|$, and if moreover $|\alpha| = |\beta|$, then there exists a permutation w such that $\delta - \beta + \alpha = w(\delta)$. But then β precedes α in the lexicographic order, and the inductive hypothesis then guarantees that $l_\beta = 0$. $\square$

COROLLARY 2.5.6. *Let $I\Lambda_n$ be the ideal of $\mathcal{P}_n$ generated by the homogeneous elements of Λ_n of strictly positive degree. Then $\mathcal{P}_n/I\Lambda_n \simeq \mathcal{H}_n$.*

2.5.3. Action of the Symmetric Group. We define a symmetric bilinear form on $\mathcal{P}_n$ by setting

$$\langle P, Q\rangle = \partial_{w_0}(PQ)(0), \quad \text{if } P, Q \in \mathcal{P}_n.$$

If R is symmetric without constant term,

$$\langle PR, Q\rangle = \partial_{w_0}(PQR)(0) = R(0)\partial_{w_0}(PQ)(0) = 0.$$

The kernel of this bilinear form then contains the ideal $I\Lambda_n$. Moreover, the divided differences are self-adjoint operators with respect to this form:

$$\begin{aligned} \langle \partial_i P, Q\rangle &= \partial_{w_0}(\partial_i P \times Q)(0) \\ &= \partial_{w_0 s_i}\partial_i(\partial_i P \times Q)(0) \\ &= \partial_{w_0 s_i}(\partial_i P \times \partial_i Q)(0) \\ &= \langle P, \partial_i Q\rangle, \end{aligned}$$

taking into account the fact that $\partial_i P$, being symmetric in x_i and x_{i+1}, is a scalar for ∂_i. More generally, this immediately implies the identity $\langle \partial_u P, Q\rangle = \langle P, \partial_{u^{-1}} Q\rangle$, for all permutations u.

PROPOSITION 2.5.7. *The bilinear form* $\langle P, Q\rangle = \partial_{w_0}(PQ)(0)$ *defines a scalar product on* $\mathcal{H}_n$, *and the dual basis of the Schubert polynomials consists of the polynomials*

$$\mathfrak{S}_v^* = \varepsilon(v) w_0 \mathfrak{S}_{w_0 v}, \quad v \in \mathcal{S}_n.$$

In other words, the Schubert polynomials form a self-dual basis of $\mathcal{H}_n$, up to signs and the order of the variables. The polynomials $w_0\mathfrak{S}_{w_0 v}$ must however be considered as elements of the quotient $\mathcal{P}_n/I\Lambda_n$, which we know is isomorphic to $\mathcal{H}_n$.

PROOF. Let u and v be two permutations. Then

$$\begin{aligned}\langle \mathfrak{S}_u, \mathfrak{S}_v^*\rangle &= \langle \mathfrak{S}_{w_0 u^{-1}} x^\delta, \mathfrak{S}_v^*\rangle \\ &= \varepsilon(v)\langle x^\delta, \partial_{u w_0} w_0 \mathfrak{S}_{w_0 v}\rangle \\ &= \varepsilon(vu)\langle x^\delta, w_0 \partial_{w_0 u} \mathfrak{S}_{w_0 v}\rangle = \delta_{u,v}.\end{aligned}$$

Indeed, the identity $w_0\partial_i w_0 = -\partial_{n-i}$ implies that $w_0\partial_w w_0 = \varepsilon(w)\partial_{w_0 w w_0}$ for all permutations w, which permits passage from the penultimate to the last line. And the fact that

$$\partial_u \mathfrak{S}_v = \partial_u \partial_{v^{-1} w_0} \mathfrak{S}_{w_0} = \begin{cases} \mathfrak{S}_{vu^{-1}} & \text{if } l(vu^{-1}) = l(v) - l(u), \\ 0 & \text{otherwise} \end{cases}$$

permits the conclusion. □

The symmetric group acts on $\mathcal{P}_n$, hence on the quotient $\mathcal{P}_n/I\Lambda_n = \mathcal{H}_n$, by permutation of variables, and the preceding proposition permits the identification of this representation of $\mathcal{S}_n$. Recall that if G is a finite group, its *regular representation* is defined by the natural action, for example on the left, of G on the group algebra $\mathbb{C}[G]$. The character χ_{reg} of the regular representation is particularly simple, since if $g \in G$, then

$$\chi_{\mathrm{reg}}(g) = \begin{cases} 0 & \text{if } g \neq e, \\ \#G & \text{if } g = e, \end{cases}$$

e designating the identity element of G. Moreover, the regular representation contains each of the irreducible representations of G, with multiplicities equal to the dimensions of these representations.

COROLLARY 2.5.8. *The action of* $\mathcal{S}_n$ *in* $\mathcal{P}_n/I\Lambda_n = \mathcal{H}_n$ *by permutation of variables is isomorphic over* $\mathbb{C}$ *to the regular representation.*[8]

PROOF. We determine the character χ of this representation. The preceding proposition permits us to write

$$\chi(v) = \partial_{w_0}\left(\sum_{w\in\mathcal{S}_n} \mathfrak{S}_w(v(x))\varepsilon(w) w_0 \mathfrak{S}_{w_0 w}(x)\right).$$

[8]This statement is a particular case of the following result, due to Chevalley. Let G be a finite group acting linearly on a vector space V of finite dimension. An element of G is called a pseudoreflection if it acts trivially on a hyperplane of V. The group G acts naturally on the algebra of polynomial functions on V, and if G is generated by pseudoreflections, the quotient of this algebra by the ideal generated by the G-invariant functions without constant term is then isomorphic to the regular representation of G. See, for example, Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie*, chap. 5, p. 107.

But by the Cauchy identity for Schubert polynomials 2.4.8, this sum is equal to

$$\prod_{1\leq i<j\leq n}(x_{v(i)}-x_j)=0 \quad \text{if } v\neq \mathrm{id}.$$

Because, moreover, $\chi(\mathrm{id})=\dim\mathcal{H}_n=n!$, the character χ is clearly that of the regular representation of $\mathcal{S}_n$, which guarantees that the corresponding representations are isomorphic over $\mathbb{C}$. □

2.5.4. Harmonic Polynomials. A slightly different interpretation of $\mathcal{H}_n$ makes use of *harmonic polynomials*. These are the elements of $\mathcal{P}_n$ annihilated by all of the symmetric derivations (in particular by the Laplacian), that is, all the symmetric polynomials without constant term, in $\partial_1,\dots,\partial_{n-1}$.

The space $\mathcal{H}_n^*$ of harmonic polynomials is evidently stable under all the derivations. This space contains the Vandermonde $a_\delta=\prod_{i<j}(x_i-x_j)$, since its symmetric derivations are antisymmetric polynomials of degree strictly smaller than $n(n-1)/2$, hence annihilated. The derivations of the Vandermonde are also therefore harmonic polynomials. We write

$$\Delta^\alpha=\frac{1}{\alpha!}d^\alpha a_\delta$$

for all $(n-1)$-tuples of integers $\alpha=(\alpha_1,\dots,\alpha_{n-1})$, where $\alpha!=\alpha_1!\cdots\alpha_{n-1}!$, and d^α designates the monomial in the partial derivatives (not the divided differences!) associated to α.

LEMMA 2.5.9. *The polynomials Δ^α, for $\alpha\subset\delta$, form a rational basis of $\mathcal{H}_n^*$.*

PROOF. Harmonic polynomials are annihilated by the symmetric derivations, hence by the ideal that they generate in the algebra of differential operators with constant coefficients. Then, by proposition 2.5.5, such an operator may be written, modulo the ideal generated by the symmetric derivations, as a linear combination of the derivations d^β, for $\beta\subset\delta$. Taylor's formula then guarantees that the map which associates to a harmonic polynomial P the $n!$-tuple of integers $d^\beta P(0)/\beta!$, for $\beta\subset\delta$, is injective.

We note that $d^\beta\Delta^\alpha(0)=d^{\alpha+\beta}a_\delta(0)$ is nonzero if and only if there exists a permutation w such that $\alpha+\beta=w(\delta)$. But then, β is bounded above by $\delta-\alpha$ under the dominance order. This implies that the images of the polynomials Δ^α, $\alpha\subset\delta$, under the preceding map, form a basis of $\mathbb{Q}^{n!}$. The lemma then follows immediately. □

We define a map $\phi\colon\mathcal{H}_{n,\mathbb{Q}}\to\mathcal{H}^*_{n,\mathbb{Q}}$ by setting

$$\phi(P)=d(P)a_\delta,$$

where $d(P)$ designates the polynomial in the partial derivatives which results from P by replacing each variable by the corresponding partial derivative. The symbol $\mathbb{Q}$ as an index signifies that here we consider polynomials with rational coefficients.

PROPOSITION 2.5.10. *The map ϕ is an isomorphism.*

PROOF. By the preceding lemma, ϕ is surjective. Moreover, $\phi(x^\delta)\neq 0$. Injectivity follows, since if P is a nonzero element of $\mathcal{H}_n$, there exists by proposition 2.5.5 a polynomial Q such that PQ is equal, modulo Λ_n, to a nonzero multiple of x^δ. But then $\phi(P)\phi(Q)=\phi(PQ)\neq 0$, hence $\phi(P)\neq 0$. □

EXERCISE 2.5.11. The Specht modules, to which we devoted section 1.6.7 of the first chapter. are naturally embedded in the space of harmonic polynomials. Indeed, let T be a tableau numbered by consecutive integers $\{1,\ldots,n\}$. To each integer i, we associate the integer α_i which is the sum of the lengths of the columns lying strictly to the left of the integer i in tableau T. Show that the polynomial P_T, which we associated to T in section 1.6.7, is proportional to the harmonic polynomial Δ^α.

2.5.5. Interpolation. We return one last time to the identity

$$\phi(C_{\mathrm{Sch}}) = \mathfrak{S}^{-1}(y)\mathfrak{S}(x),$$

obtained in the proof of theorem 2.3.7. If we multiply on the left by $\mathfrak{S}(y)$ and expand, we obtain, for all permutations w,

$$\mathfrak{S}_w(x) = \sum_{\substack{w=uv,\\ l(w)=l(u)+l(v)}} \mathfrak{S}_u(y)\mathfrak{S}_v(x,y) = \sum_v [\partial_v \mathfrak{S}_w(y)]\mathfrak{S}_v(x,y).$$

By linearity this implies, taking into account the preceding proposition, the following interpolation formula, in which y must be considered as a family of parameters.

INTERPOLATION FORMULA 2.5.12. *For each polynomial $P \in \mathcal{P}_\infty$, we have*

$$P(x) = \sum_{w\in\mathfrak{S}_\infty} [\partial_w P(y)]\mathfrak{S}_w(x,y).$$

EXERCISE 2.5.13. Show that if $P \in \mathcal{P}_\infty$, we may restrict ourselves to the permutations for which the greatest descent is less than or equal to n. Show that for $n = 1$, we recover the classical interpolation formula of Newton

$$P(x) = \sum_{l\geq 0}\sum_{i=1}^{l+1}\left(\frac{P(y_i)}{\prod_{1\leq j\neq i\leq l+1}(y_i - y_j)}\right)(x-y_1)\cdots(x-y_l).$$

2.5.6. Principal Specialization. As the Schur functions, the Schubert polynomials exhibit some remarkable expressions when they are specialized at geometric series. We are speaking of the principal specialization, which we already encountered in an example with the Schur functions. In order to obtain this specialization, we take a look one last time at the calculations carried out in the nilCoxeter algebra.

LEMMA 2.5.14. *In the nilCoxeter algebra, if q is an indeterminate, we have the identity*

$$\mathfrak{S}(1,q,\ldots,q^{n-1}) = \prod_{k=0}^{+\infty}\prod_{j=1}^{n-1} h_j(q^k(1-q^j)),$$

the order of the factors being the decreasing order of the indices.

PROOF. We show by induction on n that for every integer i,

$$\mathfrak{S}(q^i,q^{i+1},\ldots,q^{n+i-1}) = \mathfrak{S}(q^{i+1},\ldots,q^{n+i}) \times \prod_{j=n-1}^{1} h_j(q^i(1-q^j)).$$

The lemma will be achieved by making i tend toward infinity. Observe first that since the Yang-Baxter relations are homogeneous, it suffices to show the preceding identity for $i = 0$. We begin with the right-hand term

$$\begin{aligned}
\mathfrak{S}(q, \dots, q^n) \times \prod_{j=n-1}^{1} h_j(1-q^j)& \\
&= H_1(q) \cdots H_{n-1}(q^{n-1}) h_{n-1}(1-q^{n-1}) \cdots h_1(1-q) \\
&= H_2(q) h_1(q) \cdots H_{n-1}(q^{n-2}) h_{n-2}(q^{n-2}) \\
&\qquad \times h_{n-1}(1) h_{n-2}(1-q^{n-2}) \cdots h_1(1-q) \\
&= H_2(q) \cdots H_{n-1}(q^{n-2}) h_1(q) \cdots h_{n-2}(q^{n-2}) \\
&\qquad \times h_{n-1}(1) h_{n-2}(1-q^{n-2}) \cdots h_1(1-q).
\end{aligned}$$

Having arrived at this point, we use a braid relation on the three terms lying between the two last sets of ellipses, then commute the terms obtained at the two extremities, the rightmost and leftmost possible respectively, always remaining on the right of the product of H factors. Then we begin again, up to the point where we obtain

$$\begin{aligned}
\mathfrak{S}(q, \dots, q^n) \times \prod_{j=n-1}^{1} h_j(1-q^j)& \\
&= H_2(q) \cdots \left[H_{n-1}(q^{n-2}) h_{n-1}(1-q^{n-2}) \cdots h_2(1-q)\right] \\
&\qquad \times h_1(1) h_2(q) \cdots h_{n-1}(q^{n-2}) \\
&= H_2(1) H_3(q) \cdots H_{n-1}(q^{n-3}) h_1(1) h_2(q) \cdots h_{n-1}(q^{n-2}) \\
&= H_2(1) h_1(1) H_3(q) h_2(q) \cdots H_{n-1}(q^{n-3}) h_{n-1}(q^{n-2}) \\
&= \mathfrak{S}(1, q, \dots, q^{n-1}).
\end{aligned}$$

Here, we passed from the second to the third line by applying the inductive hypothesis, taken for the interval $2, \dots, n$, to the expression between square brackets, then to the fourth line via commutations. □

Let $a = a_1 \cdots a_l$ be a reduced word of a permutation. Set $\varepsilon_i(a) = 1$ if $a_i < a_{i+1}$ and $\varepsilon_i(a) = 0$ otherwise, then $\mu(a) = \sum_i i\varepsilon_i(a)$. The following result, conjectured by Macdonald [**67**, p. 91], is due to Fomin and Stanley [**17**].

PROPOSITION 2.5.15. *For all permutations $w \in \mathcal{S}_n$,*

$$\mathfrak{S}_w(1, q, \dots, q^{n-1}) = \sum_{a \in R(w)} q^{\mu(a)} \frac{(1-q^{a_1}) \cdots (1-q^{a_l})}{(1-q) \cdots (1-q^l)}.$$

PROOF. It suffices to show that if the product of the preceding lemma is expanded, the coefficient c_a of $u_{a_1} \cdots u_{a_l}$ is that which appears in the statement of the proposition. Now

$$\prod_{i=1}^{l} (1-q^{a_i})^{-1} c_a = \sum_{k_j \geq k_{j+1} + \varepsilon_j(a)} q^{k_1 + \cdots + k_l} = q^{\mu(a)} \prod_{j=1}^{l} (1-q^j)^{-1},$$

which may be checked with a simple calculation. □

COROLLARY 2.5.16. *For all permutations $w \in \mathcal{S}_n$,*

$$\mathfrak{S}_w(1, 1, \dots, 1) = \frac{1}{l!} \sum_{a \in R(w)} a_1 \cdots a_l.$$

2.6. Flagged Schur Functions

Among the Schubert polynomials we encounter not only the ordinary and skew Schur functions, but also some functions a little more general, namely the multi-Schur functions and flagged Schur functions, which correspond to vexillary permutations.

2.6.1. Determinants and Paths. We show first how a remarkably simple and clever method, introduced by Gessel and Viennot in an unpublished manuscript [**29**] allows us to recover the fact that a skew Schur function is a sum of monomials associated to semistandard skew tableaux with corresponding shape, and also greatly to extend this result.[9]

The idea is the following: consider a directed graph, that is, a finite set of vertices connected in pairs by directed edges which do not meet except at their endpoints. Suppose that this graph does not contain a cycle, or in other words a (directed, as in everything that follows) closed path.

If M and N are two vertices of the graph, we denote by $\mathcal{C}(M, N)$ the set of paths from M to N. More generally, if $M_\bullet$ and $N_\bullet$ are two sequences of the same cardinality n of vertices of the graph, we denote by $\mathcal{C}(M_\bullet, N_\bullet)$ the set of n-tuples $C = (C_1, \dots, C_n)$ of paths C_i from the vertex M_i to the vertex N_i. Similarly, we denote by $\mathcal{C}_0(M_\bullet, N_\bullet)$ the subset formed by the k-tuples of paths which are pairwise disjoint.

Suppose now that each edge is weighted by an indeterminate, and associate to each path (or n-tuple of paths) C the monomial, denoted x^C, which we call its weight, obtained by taking the product of the indeterminates associated to each of its edges. We may then associate to a set $\mathcal{C}$ of paths the generating function

$$G(\mathcal{C}) = \sum_{C \in \mathcal{C}} x^C.$$

DEFINITION 2.6.1. *We say that two sequences $M_\bullet$ and $N_\bullet$ of n vertices of the graph are* compatible *if for every n-tuple of paths $C_1, \dots, C_n$ such that C_i joins M_i to $N_{w(i)}$, w being a permutation distinct from the identity, then two of the paths must necessarily meet.*

PROPOSITION 2.6.2. *If the sequences of n points $M_\bullet$ and $N_\bullet$ are compatible, then*

$$G(\mathcal{C}_0(M_\bullet, N_\bullet)) = \det(G(\mathcal{C}(M_i, N_j)))_{1 \le i,j \le n}.$$

PROOF. We first note that the determinant may be written

$$\det(G(\mathcal{C}(M_i, N_j)))_{1 \le i,j \le n} = \sum_{w \in \mathcal{S}_n} \varepsilon(w) G(\mathcal{C}(M_\bullet, w(N_\bullet))).$$

Now let w be a permutation, and $C \in \mathcal{C}(M_\bullet, w(N_\bullet))$ a family of n paths $C_1, \dots, C_n$ which are not pairwise disjoint. We define the sign of $\mathcal{C}$ to be the signature of w. Let i be the smallest integer such that C_i meets another path, and j the smallest

[9]This path method has since been repeated with success. We find applications other than those which we describe, for example, in the following articles: J. Stembridge, *Nonintersecting paths, Pfaffians, and plane partitions*, Adv. Math. **83** (1990), 96–131, and *The enumeration of totally symmetric plane partitions*, Adv. Math. **111** (1995), 227–243, as well as A. M. Hamel and I. P. Goulden, *Planar decomposition of tableaux and Schur function determinants*, European J. Combin. **16** (1995), 461–477, and K. Ueno, *Lattice path proof of the ribbon determinant formula for Schur functions*, Nagoya Math. J. **124** (1991), 55–59.

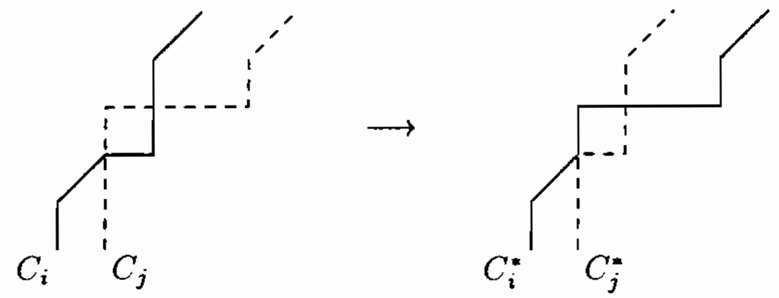


FIGURE 15. The involution $C \to C^*$

integer such that C_i meets C_j. Let C_i^- and C_j^- be the parts of the paths which precede their first point of intersection, and C_i^+ and C_j^+ their remaining parts. We then define a family C^* of paths by setting $C_k^* = C_k$ if $k \neq i$, and

$$C_i^* = C_i^- \cup C_j^+, \quad C_j^* = C_j^- \cup C_i^+.$$

Then the map $C \to C^*$ is an involution on the set of families of pairwise nondisjoint paths which changes sign and preserves weight. The proposition follows. □

2.6.2. First Application. We take for a graph a rectangle in $\mathbb{Z} \times \mathbb{N}^*$, with edges either horizontal directed segments $(i, j) \to (i+1, j)$ weighted by x_j or vertical directed segments $(i, j) \to (i, j+1)$ weighted by 1. Let (a_i, b_i) be the coordinates of M_i, (c_i, d_i) those of N_i. The two sequences are compatible if we suppose that $a_i > a_{i+1}$, $b_i \leq b_{i+1}$, $c_i > c_{i+1}$, and $d_i \leq d_{i+1}$. Moreover,

$$G(\mathcal{C}(M_i, N_j)) = h_{c_j - a_i}(x_{b_i}, \ldots, x_{d_j}).$$

We then encode an n-tuple C of paths into a skew tableau U in the following fashion: if C_i contains the horizontal segment $(l, h) \to (l+1, h)$, we set $U_{i, l+i} = h$. This skew tableau is therefore of the form λ/μ, where $\lambda_i = a_i + i - 1$ and $\mu_i = c_i + i - 1$, and the fact that C_i joins vertices with respective y-values b_i and d_i means that for all j, $b_i \leq U_{ij} \leq d_i$. Finally, the fact that the paths of C are pairwise disjoint precisely translates to the fact that U is semistandard.

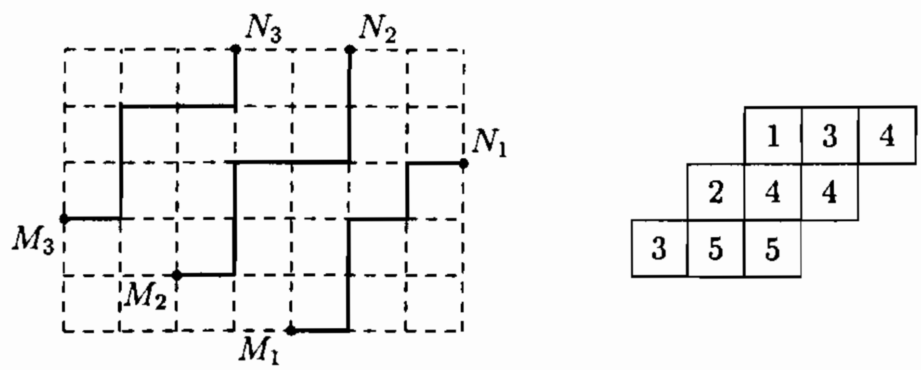


FIGURE 16. Paths encoded in a skew tableau

COROLLARY 2.6.3. *Let $\lambda \supset \mu$ be partitions and b and d two weakly increasing sequences of n integers with $n \geq l(\lambda)$. Then*

$$\det(h_{\lambda_i - \mu_j - i + j}(x_{b_j}, \ldots, x_{d_i}))_{1 \leq i,j \leq n} = \sum_U x^{\mu(U)},$$

where the sum runs over the set of semistandard skew tableaux U with shape λ/μ, numbered on the ith row of integers ranging between b_i and d_i. In particular, the flagged Schur functions expand into the form

$$s_{\lambda/\mu}(X_{d_1},\ldots,X_{d_n}) = \sum_T x^{\mu(T)},$$

where the sum runs over the set of semistandard skew tableaux T with shape λ/μ for which the entries of the ith row are bounded above by d_i.

If the integers d_i are greater than the number of indeterminates, we recover the ordinary skew Schur functions and Littlewood's Theorem 1.4.1.

2.6.3. Second Application. Keeping the edges of the preceding application, we now additionally allow diagonal edges $(i,j) \to (i+1, j+1)$, weighted by y_j, where the y_k, $k \in \mathbb{N}$, are another family of indeterminates.

As before, we associate to a family of paths a skew tableau of the shape λ/μ, of which the ith row determines the path C_i as follows: it is the sequence of y-values of the origins of the successive horizontal or diagonal segments of the path, these last being identified with a "prime."

We then obtain a one-to-one correspondence between families of pairwise disjoint paths and skew tableaux with shape λ/μ over an alphabet $1 < 1' < 2 < 2' < \cdots$, having the following properties: they are numbered in an increasing fashion along the rows and columns, strictly increasing along the rows with respect to the subalphabet $1' < 2' < 3' < \cdots$, strictly increasing along the columns with respect to the subalphabet $1 < 2 < 3 < \cdots$; and finally, they are numbered in the i-th row with integers h such that $b_i \le h \le d_i$, or h' such that $b_i \le h' < d_i$. We call these *semistandard skew bitableaux* and denote by $\mathcal{T}(\lambda/\mu, b_\bullet, d_\bullet)$ the set of these skew bitableaux U to which are naturally associated two weights $\mu(U)$ and $\mu'(U)$.

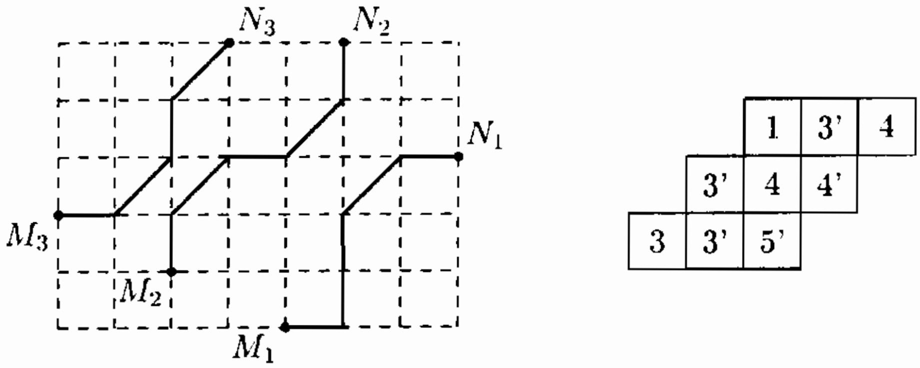


FIGURE 17. Paths encoded into a skew bitableau

DEFINITION 2.6.4. *If $X = (x_i)_{i\in I}$ and $Y = (y_j)_{j\in J}$ are two finite families of indeterminates, we define the polynomials $e_k(X-Y)$ and $h_k(X-Y)$ to be the infinite series*

$$\sum_{k\in\mathbb{Z}} t^k e_k(X-Y) = \sum_{k\in\mathbb{Z}} t^k h_k(Y-X) = \prod_{i\in I}(1+tx_i) / \prod_{j\in J}(1-ty_j).$$

The multi-Schur functions *are then the determinants*

$$s_{\lambda/\mu}(X-Y) = \det(h_{\lambda_i-\mu_j-i+j}(X-Y))_{1\le i,j\le n}.$$

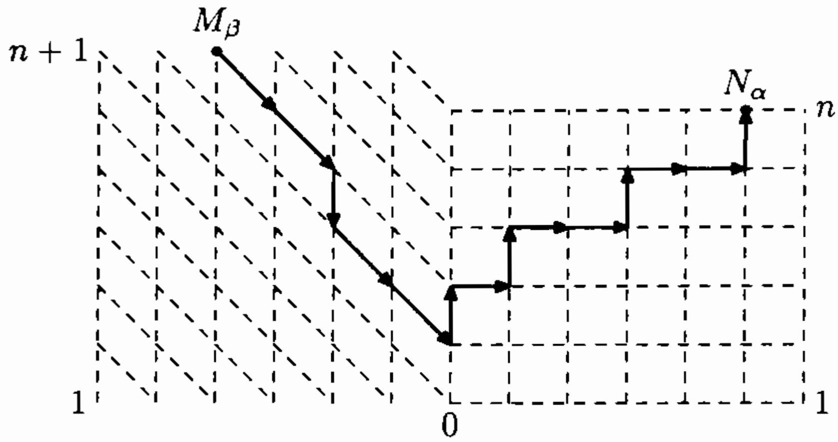
If we choose $b_i = 1$, and if d_j is strictly greater than the number of indeterminates in the families X and Y, the preceding proposition translates into the following identity:

COROLLARY 2.6.5. *Let $\lambda \supset \mu$ be partitions and X and Y two families of n and m variables respectively, with $n, m \geq l(\lambda)$. Then*

$$s_{\lambda/\mu}(X - Y) = \sum_{U \in \mathcal{T}_{m,n}(\lambda/\mu)} x^{\mu(U)} y^{\mu'(U)},$$

where $\mathcal{T}_{m,n}(\lambda/\mu)$ is the set of semistandard skew bitableaux with shape λ/μ, numbered from $1, \ldots, n$ and $1', \ldots, m'$.

EXERCISE 2.6.6. Consider the graph below in $\mathbb{Z} \times \mathbb{N}^*$. The vertical edges are rising with x-value zero or positive and are falling with negative x-value. These are weighted with the integer 1. The diagonal edges $(-l, h+1) \to (-l+1, h)$ and horizontal edges $(l, h) \to (l+1, h)$ are weighted by x_h.



For $\alpha, \beta \geq 0$, consider the vertices $N_\alpha = (\alpha, n)$ and $M_\beta = (-\beta - 1, n+1)$. Show that the generating functions of the set of paths joining these vertices is

$$G(C(M_\beta, N_\alpha)) = s_{(\alpha|\beta)}(x_1, \ldots, x_n).$$

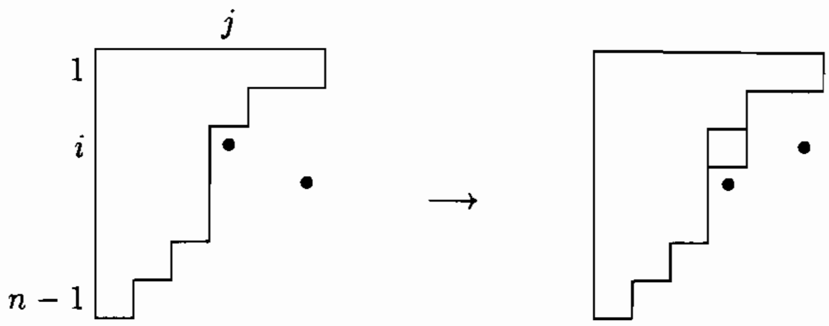
Deduce then a new proof of Giambelli's formula 1.2.16.

2.6.4. Dominant and Grassmannian Permutations. After this long digression, we return to Schubert polynomials. We show first that the expression of $\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leq n}(x_i - y_j)$ as a product of differences of the variables x and y extends to the set of dominant permutations.

PROPOSITION 2.6.7. *For all dominant permutations w with shape $\lambda(w)$, we have*

$$\mathfrak{S}_w(x, y) = \prod_{(i,j) \in \lambda(w)} (x_i - y_j).$$

PROOF. We proceed by induction on the size of $\lambda(w) \subset \delta$. Let j be the greatest integer such that $\lambda(w)_{n-k} = k$ for $1 \leq k < j$, and let $i = \lambda(w)^*_j + 1$. Then ws_i is dominant, and $\lambda(ws_i)$ is obtained by adjoining to $\lambda(w)$ the cell (i, j). We note that $\lambda(w)_i = \lambda(w)_{i+1}$. Graphically, the passage from w to ws_i may be seen in the following fashion, where the • denote the vertices of the graph of w on the left, and of ws_i on the right:



By the inductive hypothesis, we may then write

$$\begin{aligned}\mathfrak{S}_w(x,y) &= \partial_i \mathfrak{S}_{ws_i}(x,y)\\ &= \partial_i\Big[(x_i - y_j) \prod_{(p,q)\in\lambda(w)} (x_p - y_q)\Big]\\ &= \prod_{(p,q)\in\lambda(w)} (x_p - y_q),\end{aligned}$$

the last equality coming from the fact that, since $\lambda(w)_i = \lambda(w)_{i+1}$, the preceding product is a scalar for ∂_i since it is symmetric in x_i and x_{i+1}. □

For the Grassmannian permutations, we first treat the case of simple Schubert polynomials. Those for double Schubert polynomials will be a particular case of theorem 2.6.9.

PROPOSITION 2.6.8. *If w is a Grassmannian permutation, and if r is its unique descent, then*

$$\mathfrak{S}_w(x) = s_{\lambda(w)}(x_1, \ldots, x_r).$$

PROOF. We have $\lambda(w) = (w(r) - r, \ldots, w(1) - 1)$. Let w_0^r be the element of $\mathcal{S}_r$ of maximal length. Then, if we denote by δ^r the sequence $(r-1, \ldots, 1, 0)$, the permutation ww_0^r is dominant with shape $\lambda(w) + \delta^r$. Moreover, $l(w) = l(ww_0^r) + l(w_0^r)$, so by the preceding proposition,

$$\mathfrak{S}_w(x) = \partial_{w_0^r} \mathfrak{S}_{ww_0^r}(x) = \partial_{w_0^r} x^{\lambda(w)+\delta^r} = s_{\lambda(w)}(x_1, \ldots, x_r),$$

this last equality being a consequence of proposition 2.3.2. □

2.6.5. Vexillary Permutations. We now show that the Schubert polynomials associated to vexillary permutations are the flagged Schur functions. Given a vexillary permutation w, we assume the notation of definition 2.2.9 and of exercise 2.2.11 for the shape and the flag of w and its inverse. Moreover, we denote by X_i the family of indeterminates $(x_1, \ldots, x_i)$. We introduce the polynomials

$$s_{\lambda/\mu}(X_{a_1} - Y_{b_1}, \ldots, X_{a_m} - Y_{b_m}) = \det(h_{\lambda_i - \mu_j - i + j}(X_{a_i} - Y_{b_i}))_{1\le i,j\le m},$$

which generalize the multi-Schur functions from section 2.6.3.

THEOREM 2.6.9. *If the permutation w is vexillary, then*

$$\mathfrak{S}_w(x,y) = s_{\lambda(w)}(\underbrace{X_{f_1} - Y_{g_k}}_{m_1}, \ldots, \underbrace{X_{f_k} - Y_{g_1}}_{m_k}).$$

COROLLARY 2.6.10. *If w is a vexillary permutation with shape $\lambda(w)$ and with flag $\phi(w) = (\phi_1, \dots, \phi_m)$, we have*

$$\mathfrak{S}_w(x) = s_{\lambda(w)}(X_{\phi_1}, \dots, X_{\phi_m}).$$

PROOF. We assume the notation below:

$$\lambda(w) = (l_1^{m_1} \dots l_k^{m_k}),$$
$$\phi(w) = (f_1^{m_1} \dots f_k^{m_k}),$$
$$\phi(w^{-1}) = (g_1^{n_1} \dots g_k^{n_k}).$$

We proceed by induction on the greatest integer j such that the code of w begins with the sequence $l_1^{m_1} \dots l_{j-1}^{m_{j-1}}$. We have that $j = k+1$ if and only if w is dominant. Note that $f_i = m_1 + \cdots + m_i$ for $i < j$.

Since the permutation w is vexillary, its code takes the value l_j for the indices ranging between $f_j - m_j + 1$ and f_j. Now consider the permutation

$$v = w(s_{f_j - m_j} \cdots s_{m_1 + \cdots + m_{j-1} + 1}) \cdots (s_{f_j - 1} \cdots s_{m_1 + \cdots + m_j}),$$

the product of w with the cycles taking $f_j - m_j + 1$ to $m_1 + \cdots + m_{j-1} + 1$, and so on, finally taking f_j to $m_1 + \cdots + m_j$. Its shape and flag are the following:

$$\lambda(v) = (l_1^{m_1} \dots (l_j + f_j - m_1 - \cdots - m_j)^{m_j} \dots l_k^{m_k}),$$
$$\phi(v) = (f_1^{m_1} \dots (m_1 + \cdots + m_j)^{m_j} \dots f_k^{m_k}),$$
$$\phi(v^{-1}) = (g_1^{n_1} \dots g_j^{n_j + f_j - m_1 - \cdots - m_j} \dots g_k^{n_k}).$$

Moreover, the code of w is increasing between the indices $m_1 + \cdots + m_{j-1} + 1$ and f_j, hence we have the relation

$$\mathfrak{S}_w = (\partial_{f_j - m_j} \cdots \partial_{m_1 + \cdots + m_{j-1} + 1}) \cdots (\partial_{f_j - 1} \cdots \partial_{m_1 + \cdots + m_j}) \mathfrak{S}_v.$$

But by the inductive hypothesis, we know that

$$\mathfrak{S}_v = s_{\lambda(v)}(\underbrace{X_{f_1} - Y_{g_k}, \dots}_{m_1}, \underbrace{X_{m_1 + \cdots + m_j} - Y_{g_{k+1-j}}, \dots}_{m_j}, \underbrace{X_{f_k} - Y_{g_1}}_{m_k}).$$

The proof of the following elementary lemma is left to the reader:

LEMMA 2.6.11. *Let λ be a partition and a and b two sequences of integers.*

1. *If $\lambda_{i+1} = \cdots = \lambda_j$, with $i < j$, then*

$$s_\lambda(\dots, X_{a_j - j + i - 1} - Y_{b_i}, \dots, X_{a_j - 1} - Y_{b_j}, X_{a_j} - Y_{b_j}, \dots)$$
$$= s_\lambda(\dots, X_{a_j} - Y_{b_i}, \dots, X_{a_j} - Y_{b_i}, X_{a_j} - Y_{b_j}, \dots).$$

2. *If $a_j \neq a_i$ for $j \neq i$, then*

$$\partial_{a_i} s_\lambda(\dots, X_{a_i} - Y_{b_i}, \dots) = s_{\lambda - \varepsilon_i}(\dots, X_{a_i + 1} - Y_{b_i}, \dots).$$

The first item of the lemma allows us to write

$$\mathfrak{S}_v = s_{\lambda(v)}(\dots, X_{m_1 + \cdots + m_{j-1} + 1} - Y_{g_{k+1-j}}, \dots, X_{m_1 + \cdots + m_j} - Y_{g_{k+1-j}}, \dots).$$

We are then in a position to apply the second item of the lemma in order to deduce $\mathfrak{S}_w$; we obtain

$$\mathfrak{S}_w = s_{\lambda(w)}(\dots, X_{f_j - m_j + 1} - Y_{g_{k+1-j}}, \dots, X_{f_j} - Y_{g_{k+1-j}}, \dots)$$
$$= s_{\lambda(w)}(\underbrace{X_{f_1} - Y_{g_k}, \dots}_{m_1}, \underbrace{X_{f_j} - Y_{g_{k+1-j}}, \dots}_{m_j}, \underbrace{X_{f_k} - Y_{g_1}}_{m_k}),$$

the last rewriting possible due to the first item of the preceding lemma. This concludes the induction. □

EXERCISE 2.6.12. Assuming the notation above, set $\lambda(w^{-1}) = (p_1^{n_1} \dots p_k^{n_k})$. Show that if w is dominant,

$$l_i + f_i = p_{k+1-i} + g_{k+1-i}$$

for all integers i. By remarking that these quantities remain unchanged when w is transformed as in the preceding proposition, show that these equalities are satisfied for all vexillary permutations.

EXERCISE 2.6.13. Deduce from theorem 2.6.9 and corollary 2.4.2 a duality theorem for flagged Schur functions.

2.6.6. 321-Avoiding Permutations. If $w \in S_n$ is a 321-avoiding permutation, recall that we have canonically associated to it, in paragraph 2.2.2, a skew partition $\lambda(w)/\mu(w)$, embedded in $\mathbb{Z} \times \mathbb{Z}$ and weighted by the opposite of the content. A reduced word for w may then be obtained by prolonging the partial order on this set to a total order. This amounts to numbering $\lambda(w)/\mu(w)$ with consecutive integers, in a standard fashion, and the corresponding reduced word is read by reading the opposite of the contents according to this total order.

	4	3
	5	4
7	6	

$\lambda(w) - \mu(w)$

	1	4
	3	5
2	6	

$a = 475346$

	1	3
	2	4
2	6	

$b = 122346$

FIGURE 18. Reduced decomposition and compatible word for $w = 12563847$.

If a is such a word, and $b \in C(a)$ is a compatible word, number $\lambda(w)/\mu(w)$ by placing b_i in the cell numbered i. The conditions $b_i \le b_{i+1}$ imply that this numbering is increasing along rows, and the conditions $b_i < b_{i+1}$ when $a_i < a_{i+1}$ imply that it is strictly increasing along columns. To the pair (a, b) is then associated a semistandard skew tableau with shape $\lambda(w)/\mu(w)$.

Inversely, if T is such a tableau, number the cells of $\lambda(w)/\mu(w)$ according to the increasing order of its entries, with cells having the same entry being numbered from left to right and from bottom to top. We then obtain a total order on $\lambda(w)/\mu(w)$, hence a reduced word a of w. Moreover, the word b obtained by reading the entries of T in the increasing order is compatible with a.

These two procedures are inverses of each other. Moreover, the conditions $b_i \le a_i$ indicate that the tableau T corresponding to the pair (a, b) is numbered on its jth row with integers less than or equal to k_j, where $k_1 < \dots < k_m$ is the sequence of indices of the nonzero components of the code $c(w)$. Theorem 2.5.1 then implies the following result:

PROPOSITION 2.6.14. *If $w \in S_n$ is 321-avoiding, $k_1 < \dots < k_m$ is the sequence of indices of the nonzero components of its code, and $\lambda(w)/\mu(w)$ is the associated skew partition, then*

$$\mathfrak{S}_w(x) = s_{\lambda(w)/\mu(w)}(X_{k_1}, \dots, X_{k_m}) = \sum_T x^{\mu(T)},$$

the sum running over the set of semistandard skew tableaux with shape $\lambda(w)/\mu(w)$ for which the entries on the jth row are bounded above by k_j.

2.7. Multiplication of Schur Polynomials

The general rule for multiplication of Schubert polynomials, a rule which generalizes that of Littlewood and Richardson for Schur functions, is unknown. At most we have an analogue of Pieri's formulas [**58**], the simplest case of which is called the Monk formula. Throughout this section, we consider only simple Schubert polynomials.

2.7.1. Monk's Formula. Recall first of all that

$$\mathfrak{S}_{s_m}(x) = x_1 + \cdots + x_m = e_1(X_m).$$

THEOREM 2.7.1. *For all permutations $w \in S_\infty$, and for all natural numbers m,*

$$\mathfrak{S}_w \times \mathfrak{S}_{s_m} = \sum_{\substack{j \leq m < k, \\ l(wt_{jk}) = l(w)+1}} \mathfrak{S}_{wt_{jk}}.$$

PROOF. Let v be a permutation of length $l(v) = l(w) + 1$, and $a = a_1 \cdots a_l$ a reduced word for v. A divided difference acts on a product according to the following variant of the Leibniz formula:

$$\partial_i(PQ) = \partial_i P.s_i Q + P.\partial_i Q.$$

Since $\mathfrak{S}_{s_m}$ is of degree one, we deduce that

$$\partial_v(\mathfrak{S}_w \mathfrak{S}_{s_m}) = \sum_{p=0}^{l} \partial_{a_1} \cdots \partial_{a_{p-1}} \partial_{a_{p+1}} \cdots \partial_{a_l} \mathfrak{S}_w \times \partial_{a_p}(s_{a_{p+1}} \cdots s_{a_l} \mathfrak{S}_{s_m}).$$

In this expression, the divided differences applied to $\mathfrak{S}_w$ give a nonzero result if and only if the word $a_1 \cdots a_{p-1} a_{p+1} \cdots a_l$ belongs to $R(w)$. In this case, w precedes v under the Bruhat order, and we have $v = wt_{jk}$ with $\{j, k\} = s_{a_1} \cdots s_{a_{p+1}} \{a_p, a_p{+}1\}$, by lemma 2.1.4. But then,

$$\partial_{a_p}(s_{a_{p+1}} \cdots s_{a_l} \mathfrak{S}_{s_m}) = \partial_{a_p} \sum_{i=1}^{m} x_{s_{a_{p+1}} \cdots s_{a_l}(i)}$$

is nonzero if and only if one and only one of the indeterminates x_{a_p} and $x_{a_{p+1}}$ appears in this sum. Now the first is obtained when $i = j$, the second when $i = k$, and we deduce then that $\partial_{a_p}(s_{a_{p+1}} \cdots s_{a_l} \mathfrak{S}_{s_m})$ is nonzero if and only if $j \leq m < k$, in which case it is equal to one. Monk's formnla follows from the interpolation formula 2.5.12. □

EXERCISE 2.7.2. Deduce from Monk's formula that if $a_1 < \cdots < a_l$,

$$\mathfrak{S}_{s_{a_1}} \cdots \mathfrak{S}_{s_{a_l}} = \sum_w \mathfrak{S}_w,$$

the sum running over the set of permutations, necessarily 321-avoiding, which have for a reduced word a permutation of $a_1 \cdots a_l$. We do not have such a formula when the sequence $a_1, \ldots, a_l$ is not strictly increasing.

EXERCISE 2.7.3. Extend Monk's formula to double Schubert polynomials in the following way. For simple Schubert polynomials, the formula may be rewritten

$$x_m \mathfrak{S}_u = \sum_{\substack{q>m,\\ l(ut_{mq})=l(u)+1}} \mathfrak{S}_{ut_{mq}} - \sum_{\substack{q<m,\\ l(ut_{qm})=l(u)+1}} \mathfrak{S}_{ut_{qm}}.$$

Show that for double Schubert polynomials, we have

$$(x_m - y_{u(m)})\mathfrak{S}_u(x,y) = \sum_{\substack{q>m,\\ l(ut_{mq})=l(u)+1}} \mathfrak{S}_{ut_{mq}}(x,y) - \sum_{\substack{q<m,\\ l(ut_{qm})=l(u)+1}} \mathfrak{S}_{ut_{qm}}(x,y).$$

2.7.2. The Pieri Formula. It is possible to generalize Monk's formula into a product formula for a Schubert polynomial with a elementary symmetric function in the first m variables.

DEFINITION 2.7.4. *Let w be a partition and ζ a cycle of length $p+1$. We say that $w\zeta$ is a m*-left lift of degree *p of w if*

$$\zeta = t_{i_1,q} \cdots t_{i_p,q} = (qi_p \cdots i_2 i_1),$$

where the integers $i_1, \ldots, i_p \leq m < q$ are such that $w(q) > w(i_1) > \cdots > w(i_p)$, and furthermore. $l(w\zeta) = l(w) + p$.

For a cycle, the notation $\zeta = (qi_p \cdots i_2 i_1)$ signifies that ζ sends each of the integers inside the parentheses to the integer immediately to its right, except that the rightmost integer is send to the leftmost integer. Graphically, the passage from w to $w\zeta$ appears as follows, where the $\times$ are the points of the graph of w and the $\bullet$ are the corresponding points of the graph of $w\zeta$. Visually, this clearly corresponds to a left lift if we turn the figure a quarter turn counterclockwise.

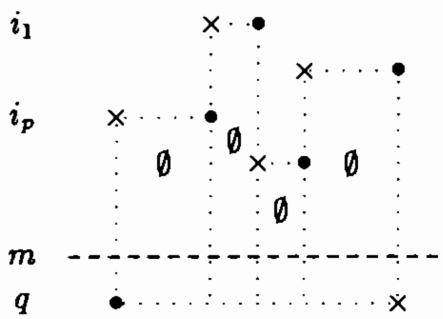


FIGURE 19. Left lifts

More generally, the permutation v is a m-left lift of degree p of w if, when one writes

$$v = w\zeta_1 \cdots \zeta_h,$$

where $\zeta_1, \ldots, \zeta_h$ are pairwise disjoint cycles, then each $w\zeta_i$ is a m-left lift of degree p_i of w, and $p = p_1 + \cdots + p_h = l(v) - l(w)$. We denote by $S_{m,p}(w)$ the set of these permutations.

THEOREM 2.7.5. *For all permutations $w \in S_\infty$, and all pairs m, p of integers, we have a product formula*

$$\mathfrak{S}_w \times e_p(x_1, \ldots, x_m) = \sum_{v \in S_{m,p}(w)} \mathfrak{S}_v.$$

PROOF. Recall that $X_i = (x_1, \ldots, x_i)$. We proceed by double induction on p and m, by using the following elementary identity:

$$e_p(X_m) = e_p(X_{m-1}) + x_m e_{p-1}(X_{m-1}).$$

Note first of all that the Monk formula is equivalent to the identity

$$x_m \mathfrak{S}_u = \sum_{\substack{q>m, \\ l(ut_{mq})=l(u)+1}} \mathfrak{S}_{ut_{mq}} - \sum_{\substack{q<m, \\ l(ut_{qm})=l(u)+1}} \mathfrak{S}_{ut_{qm}}.$$

By induction, we then obtain the following formula:

$$e_p(X_m)\mathfrak{S}_w = \sum_{v \in S_{m-1,p}(w)} \mathfrak{S}_v + \sum_{\substack{q>m, \\ u \in S_{m-1,p-1}(w)}} \mathfrak{S}_{ut_{mq}} - \sum_{\substack{q<m, \\ u \in S_{m-1,p-1}(w)}} \mathfrak{S}_{ut_{qm}}.$$

Then let $v \in S_{m,p}(w) \backslash S_{m-1,p}(w)$. Decompose $w^{-1}v$ into a product of cycles. Among them one must have the expression

$$\zeta = t_{j_1 q} \cdots t_{j_h q},$$

with $j_1, \ldots, j_h \leq m < q$, $w(q) > w(j_1) > \cdots > w(j_h)$, and there must exist an index i such that $j_i = m$. But then, we may write

$$\zeta = (t_{j_1 q} \cdots t_{j_{i-1} q})(t_{j_{i+1} q} \cdots t_{j_h m}) t_{mq},$$

and this shows that $v = ut_{mq}$, with $u \in S_{m-1,p-1}(w)$. Then, all the m-left lifts of degree p of w appear in one (and only one) of the two first sums of the right-hand side of the preceding identity.

Inversely, let $v \in S_{m-1,p}(w) \backslash S_{m,p}(w)$. This time, one of the cycles of the product $w^{-1}v$ may be written

$$\zeta = t_{j_1 m} \cdots t_{j_h m},$$

with $j_1, \ldots, j_h < m$, $w(m) > w(j_1) > \cdots > w(j_h)$. But it suffices then to replace ζ by $\zeta' = t_{j_1 m} \cdots t_{j_{h-1} m}$, in order to show that $v = ut_{qm}$ with $u \in S_{m-1,q-1}(w)$. The permutation v then disappears thanks to the last term of the identity above.

Moreover, if $u \in S_{m-1,p-1}(w)$ is such that $l(ut_{mq}) = l(u) + 1$ with $q > m$, but $ut_{mq} \notin S_{m,p}(w)$, consider the cycles of $w^{-1}v$ containing m and q. Write these respectively as $\zeta = t_{i_1 m} \cdots t_{i_r m}$ and $\zeta' = t_{j_1 q} \cdots t_{j_s q}$. Then

$$\zeta\zeta' = (t_{i_1 m} \cdots t_{i_{r-1} m})(t_{j_1 q} \cdots t_{j_s q} t_{i_r q}) t_{i_r m},$$

and since the inequality $l(ut_{mq}) = l(u) + 1$ translates into $w(j_s) > w(i_r)$, this shows that we may write $ut_{mq} = u' t_{i_r m}$ with $u' \in S_{m-1,p-1}(w)$. Hence ut_{mq} disappears in the identity above.

It remains to verify that these two types of cancellations make all of the terms of the last sum of this identity disappear. This is clearly the case, since the two preceding types correspond to the products ut_{qm} for which q is a fixed point of u or not. □

REMARK 2.7.6. It is not difficult to establish an analogous formula for products of Schubert polynomials with complete symmetric functions $h_p(x_1, \dots, x_m)$, the left lifts being replaced by the m-right lifts, which are defined by reflecting the preceding figure with respect to the horizontal line of index m.

2.7.3. Transitions. We mentioned in the proof of the Pieri formula for Schubert polynomials that Monk's formula is equivalent to the identity

$$x_m \mathfrak{S}_u = \sum_{\substack{k>m,\\ l(ut_{mk})=l(u)+1}} \mathfrak{S}_{ut_{mk}} - \sum_{\substack{j<m,\\ l(ut_{jm})=l(u)+1}} \mathfrak{S}_{ut_{jm}}.$$

Let r be the greatest descent of the permutation $w \in \mathcal{S}_\infty$, and s the greatest integer such that $w(s) < w(r)$. We apply the preceding formula to $u = wt_{rs}$, and find

$$\mathfrak{S}_w = x_r \mathfrak{S}_u + \sum_{v \in S(w,r)} \mathfrak{S}_v,$$

where $S(w,r)$ is the set of permutations of the form $ut_{jr} = wt_{rs}t_{jr}$ with $j < r$ and of the same length as w. We call such an expression a *transition*, and we denote it by $T(w,r)$. We say *maximal transition* when r is the greatest descent of w.

The conditions for the existence of transitions are easy to visualize with the graphs of the considered permutations. Given r, there must exist an integer $k > r$ such that $u = wt_{rk}$ is of length $l(w) - 1$. This means that $w(k) < w(r)$ and that the rectangle with vertices $(k, w(k))$ and $(r, w(r))$ contains no other point of the graph of w. Moreover, there must not exist another integer $k' > r$ such that $wt_{rk}t_{rk'}$ is of length $l(w)$; in other words, the regions $]w(r), \infty[\times [r, k]$ and $]k, \infty[\times [w(k), w(r)]$ are both empty of the points of the graph. In particular, r is necessarily a descent of w.

Now let $v \in S(w,r)$. There exists an integer $j < r$ such that $v = ut_{jr}$ has the same length as w, which means that $w(j) < w(k)$ and that the rectangle with vertices $(r, w(k))$ and $(j, w(j))$ contain no other point of the graph than this last. If j is maximal, the rectangle $[1, w(j)[\times [j, r]$ is disjoint from the graph; we say that v is *extremal*.

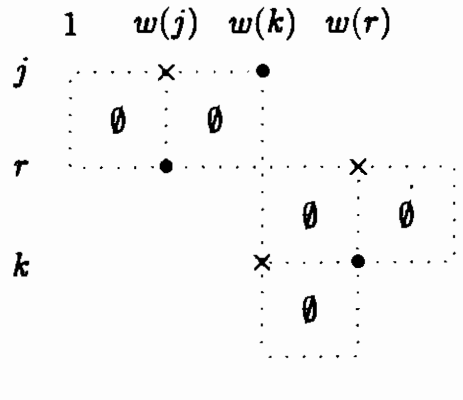


FIGURE 20. Extremal transition

LEMMA 2.7.7. *For all permutations w, we have $\lambda(w^{-1})^* \geq \lambda(w)$, with equality if and only if w is vexillary. Moreover, if $T(w,r)$ is a transition, and if $v \in S(w,r)$, then*

$$\lambda(w^{-1})^* \geq \lambda(v^{-1})^* \geq \lambda(v) \geq \lambda(w),$$

the last of these inequalities being an equality if and only if v is extremal.

PROOF. Denote be $L_1, \dots, L_h$ the nonempty rows of the diagram of w, ordered so that their lengths decrease. For all subsets H of $\{1, \dots, h\}$, we denote by $n(H)$ the number of columns meeting all of the rows L_j, $j \in H$. The, for all integers j,

$$\lambda(w)_1 + \cdots + \lambda(w)_j = \sum_{i=1}^{j} \sum_{i \in H} n(H),$$

$$\lambda(w^{-1})_1^* + \cdots + \lambda(w^{-1})_j^* = \sum_{i=1}^{j} \sum_{\#H \geq i} n(H).$$

If H is fixed, it appears in the right-hand term of the first identity, with coefficient $\#H \cap \{1, \dots, j\}$, and in the second with a coefficient $\min(\#H, j)$. The first being bounded above by the second, the inequality $\lambda(w^{-1})^* \geq \lambda(w)$ follows. Moreover, equality means that $n(H)$ is nonzero if and only if H is of the form $\{1, \dots, i\}$, which means that the rows of the diagram of $D(w)$ are totally ordered by inclusion, hence that w is vexillary.

Now let $T(w, r)$ be a transition, and $v \in S(w, r)$. Beginning with w, with the same notations as in the beginning of the lemma, the permutation v may be obtained by setting $v(j) = w(k)$, $v(r) = w(j)$, and $v(k) = w(r)$. As a consequence, the codes of v and w coincide, with the exception of their components of index j and r, which satisfy

$$c_r(v) \leq c_j(w), \quad c_r(w) \leq c_j(v).$$

This implies the inequality $\lambda(v) \leq \lambda(w)$. Moreover, in the case of equality, we must necessarily have $c_r(v) = c_j(w)$ and $c_r(w) = c_j(v)$, which is equivalent to the fact that v is extremal. We remark now that this entire discussion is invariant under reflection across the diagonal of the graphs, so that if v is obtained by transition starting with w, then v^{-1} is similarly obtained beginning with w^{-1}. In particular, we have $\lambda(v^{-1}) \leq \lambda(w^{-1})$, and we conclude thanks to the first part of the lemma. □

2.7.4. Multiplication of Schur Functions. To each permutation $w \in \mathcal{S}_\infty$, we associate a tree $A(w)$ consisting of permutations, with root w, in the following fashion. If w is vexillary, we simply set $A(w) = \{w\}$. If not, we consider the maximal transition $T(w, r)$, and connect w to the permutations $v \in S(w, r)$, to which we apply the same procedure. We do so if $S(w, r)$ is nonempty, for if not, we begin by replacing w by $1 \times w$.

Note that all of the permutations which appear in this tree have their greatest descent bounded above by that of w. Since their shape itself is restricted by the preceding lemma, it is easy to deduce that the tree $A(w)$ is finite.

PROPOSITION 2.7.8. *If $r_m \colon \mathcal{P}_\infty \to \mathcal{P}_m$ is the restriction homomorphism, and if m is less than or equal to the smallest descent of $w \in \mathcal{S}_\infty$, then*

$$r_m(\mathfrak{S}_w) = \sum_{v \in A(w)} s_{\lambda(v)}(x_1, \dots, x_m).$$

PROOF. For vexillary permutations, this is none other than proposition 2.6.8. The general case is deduced by induction via the maximal transition of w. □

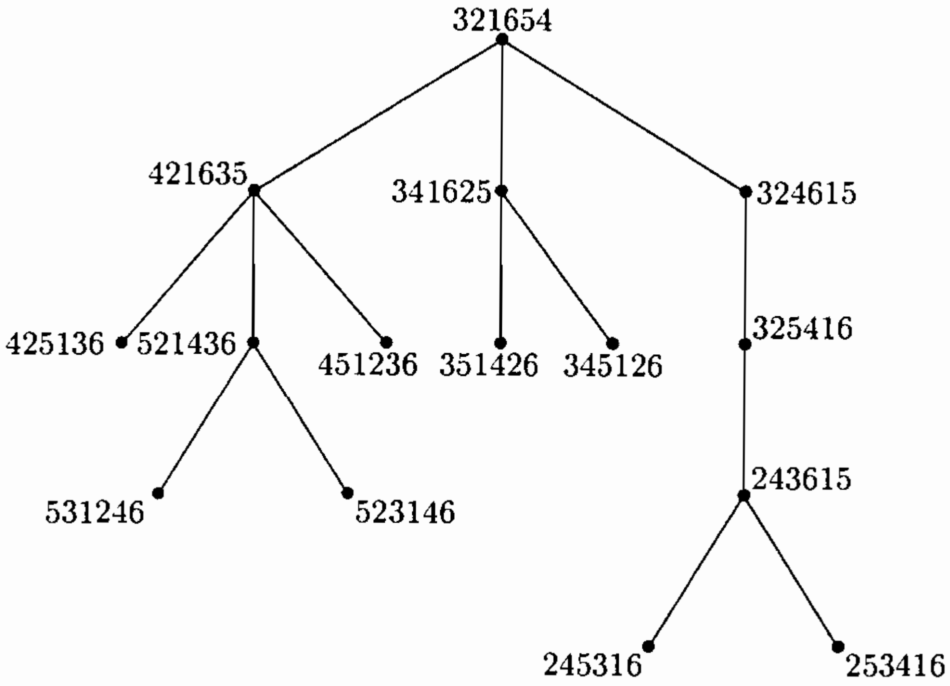


FIGURE 21. The tree $A(w)$ of $w = 321654$

We immediately deduce a new formulation of the Littlewood-Richardson rule which counts among its advantages that it may be extended to the product of an arbitrary number of Schur functions.

COROLLARY 2.7.9. *Let μ and ν be partitions, and u and v the Grassmannian permutations which are associated to them. Then*

$$s_\mu \times s_\nu = \sum_{w \in A(u \times v)} s_{\lambda(w)}.$$

EXERCISE 2.7.10. Apply this formula to the calculation of $s_{21} \times s_{21}$, and compare with the original version of the Littlewood-Richardson rule. Despite appearances, it is actually the corollary above which algorithms use to calculate products of Schur functions most efficiently.

2.8. Enumeration of Reduced Words

A remarkable combinatorial application of Schubert polynomials concerns the reduced words of permutations. We have already mentioned that for a 321-avoiding permutation, the number of reduced decompositions may be expressed as a certain sum of integers K_ν. It is this result that we will extend to the set of permutations.

2.8.1. The Number of Reduced Decompositions. First, the tree of a permutation allows us to obtain the *stable symmetric part* of the corresponding Schubert polynomial.

PROPOSITION 2.8.1. *If $w \in \mathcal{S}_\infty$, we have that $r_m(\mathfrak{S}_{1_m \times w}) = F_w(x_1, \ldots, x_m)$ for all m, where*

$$F_w = \sum_{v \in A(w)} s_{\lambda(v)} \in \mathcal{P}_\infty.$$

PROOF. This is a direct consequence of proposition 2.7.8 on one hand, and on the other the fact that the set of shapes of permutations of $A(1 \times w)$ is the same as that of the shapes of the permutations of $A(w)$. □

We note that F_w is then a Schur function if and only if w is vexillary. On the other hand, if $l = l(w) \leq m$, theorem 2.5.1 implies that the coefficient of $x_1 \cdots x_l$ in $\mathfrak{S}_{1_m \times w}$ is equal to the cardinality of the set $R(w)$ of reduced words of w. Since, by theorem 1.4.1 of the first chapter, the coefficient of this same monomial in s_λ is the number K_λ of standard tableaux with shape λ, we have reached the following result:

COROLLARY 2.8.2. *The number of reduced decompositions of a permutation w is*

$$\#R(w) = \sum_{v \in A(w)} K_{\lambda(v)}.$$

In particular, if w is vexillary, and only in this case,

$$\#R(w) = K_{\lambda(w)}.$$

EXERCISE 2.8.3. Let w_0 be the permutation of maximal length in S_n, and denote its length by $N = n(n-1)/2$. The permutation $1_m \times w_0$ is then vexillary. Deduce from theorem 2.6.9, from proposition 2.6.3, and from corollary 2.5.16, that as a runs through $R(w_0)$, the sum of the products $(m+a_1)\cdots(m+a_N)$ is equal to the product of $N!$ by the number of semistandard tableaux with shape δ for which the entries on the ith row are bounded above by $m+i$. If T is such a tableau, by replacing entry t_{ij} by $m+i+1-t_{ij}$ we obtain a plane partition with shape δ, of height bounded above by $m+1$. This procedure induces a bijection between these sets of tableaux and plane partitions. The number of these plane partitions has been calculated by Proctor [**75**], so obtain finally the identity [**16**]

$$\sum_{a \in R(w_0)} (m+a_1)\cdots(m+a_N) = N! \prod_{1 \leq i < j \leq n} \frac{2m+i+j-1}{i+j-1}.$$

2.8.2. Balanced Tableaux. A reduced decomposition of the permutation w_0 of maximal length of S_n may be interpreted as a maximal chain of permutations under the weak Bruhat order. Each is composed of the preceding and a transposition which creates only a supplementary inversion. Suppose then that the kth inversion switches those of the integers i_k and j_k, with $i_k < j_k$. Since all of the pairs of integers $i < j$ between 1 and n appear one and only one time, we may encode the reduced decomposition under consideration into a tableau B with shape δ, where the integer k is placed in the cell $(i_k, n+1-j_k)$.

Denote by b_{ij}, $i+j \leq n$, the entries of the tableau B, and set $\bar{\imath} = n+1-i$. How do we translate to B the fact that the tableau was derived from a reduced decomposition of w_0? For each triple $i < j < k$, the inversion of the pair (i,k) must have taken place between the inversions of the pairs (i,j) and (j,k). In other words, $b_{i\bar{k}}$ must be contained between $b_{i\bar{j}}$ and $b_{j\bar{k}}$.

Then the hook of B with vertex $(i,\bar{k})$, omitting this vertex, is the disjoint union of the pairs $(i,\bar{j})$ and $(j,\bar{k})$, j varying between i and k. Under the preceding hypothesis, the number of entries of the hook with vertex $(i,\bar{k})$ which are less than or equal to $b_{i\bar{k}}$ is exactly the height of this hook. A tableau of any shape which satisfies this property is called a ***balanced tableau*** [**12**]. The preceding discussion implies the following result.

THEOREM 2.8.4. *This correspondence defines a bijection between the reduced decompositions of w_0 and the balanced tableaux with shape δ.*

EXAMPLE 2.8.5. For $n = 4$, the reduced decomposition $w = s_2s_3s_1s_2s_3s_1$ corresponds to the chain

$$(1234) \to (1324) \to (1342) \to (3142) \to (3412) \to (3421) \to (4321),$$

which successively creates the inversions of the pairs 23, 24, 13, 14, 12, and 34. Hence we have the following balanced tableau:

4	3	5
2	1	
6		

FIGURE 22. Balanced tableau for the decomposition $w_0 = s_2s_3s_1s_2s_3s_1$

Moreover, corollary 2.8.2 indicates that the number of reduced decompositions of w_0 is equal to the number of standard tableaux with shape δ, but to make explicit a natural correspondence between these two families of objects is a very delicate problem. We indicate only the method, which requires the definition of the operators of *promotion* and *evacuation* of Schützenberger [**83**].

Let T be a standard tableau with shape δ, and denote the size of its partition by N. We associate to T a standard tableau ∂T in the following fashion: we remove the greatest integer of the tableau, leaving a cell empty, in which we place the greatest of the two integers lying to its left and above it. This frees up a new cell of T, to which we apply the same procedure, and we repeat until we reach a NW corner, in which we place a zero. The tableau ∂T is obtained by adding a one to all of its entries.

The tableau $p(T) = \partial^N T$ is then that which we obtain after having applied this transformation as many times as T has cells. On the other hand, we could also omit at each step the addition of one to the entries of the tableaux $\partial^j T$ obtained; then negative or zero integers appear, in an order coded by the standard tableau $e(T)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array} \quad \longmapsto \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array}$$

FIGURE 23. Promotion, evacuation: $T \mapsto p(T), e(T)$

Note that for this example $p(T)$ is simply the transpose of the original tableau T. One may show that this is a general fact. In particular, the promotion operator is an involution, and the evacuation operator is one as well.

These operations furnish the desired connection between standard tableaux and reduced decompositions in the following way. Denote by q_k the number of the column of the greatest entry of the tableau $\partial^{k-1}T$.

THEOREM 2.8.6. *The map which associates to T the word $q_1 \cdots q_N$ is a bijection from the set of standard tableaux with shape δ to the set of reduced words of w_0.*

2.8.3. The Edelman-Greene Correspondence. We defined in the preceding paragraph a correspondence between reduced words of maximal length and tableaux, which reminds us of the Knuth correspondence. We may indeed define a correspondence between reduced words and certain pairs of tableaux, by replacing the plactic ring by a ring called *nilplactic* and defined to be the $\mathbb{Z}$-module of equivalence classes of words of the alphabet of natural numbers, modulo the Knuth relations, except that the relations $xyx \sim yxx$ and $yyx \sim yxy$ are replaced, for $y = x+1$, by the braid relation $yxy \sim xyx$. The product is defined as in the plactic ring, by juxtaposition of words.

We may also define an insertion procedure, called the Coxeter-Knuth procedure, corresponding to these modified relations. This procedure is identical to Knuth insertion, except that when we insert x in a row where the pair $x, x+1$ already appears, we leave the row unchanged and pass to the following line with $x+1$. This procedure is reversible if one knows the starting cell. Just as for the Knuth correspondence, we may deduce the following correspondence:

EDELMAN-GREENE CORRESPONDENCE 2.8.7. *The Coxeter-Knuth insertion procedure induces a one-to-one correspondence $m \mapsto (P^*(m), Q^*(m))$ between reduced words and pairs of tableaux with the same shape, the second of which is standard and the first of which is strictly increasing along its rows and columns.*

We note that there exists a single tableau with shape δ, numbered with integers smaller than n, and which is strictly increasing along its rows and columns. We obtain then again a correspondence between reduced decompositions of w_0 and standard tableaux with shape δ.

Moreover, as in the case of the plactic equivalence, two reduced words m and m' are equivalent in the sense of Coxeter-Knuth if and only if $P^*(m) = P^*(m')$. Hence two equivalent reduced words are necessarily reduced decompositions of the same permutation. To each permutation $w \in \mathcal{S}_n$ corresponds then a certain number of nilplactic equivalence classes, defined by a family $\mathcal{P}(w)$ of tableaux P, and the words of the class of P are in correspondence with the standard tableaux with the same shape. We then reach the following expression for the number of reduced words of w:

$$\#R(w) = \sum_{P \in \mathcal{P}(w)} K_{\lambda(P)},$$

from which we recover corollary 2.8.2 in a slightly different form.

CHAPTER 3

Schubert Varieties

The geometric chapter of this course begins with Grassmannians and their Plücker embeddings. We define Schubert varieties, a family of subvarieties of Grassmannians indexed by partitions. The study of the intersection properties of Schubert varieties reveals a remarkable formal analogy with the multiplication of Schur functions. The standard monomials allow us to describe the ideal of a Schubert variety and to study its singularities.

We then define the Chern classes of a complex vector bundle on a differentiable variety and explain their connection with symmetric functions. In making the link to fundamental classes of Schubert varieties of Grassmannians, we will prove the formula of Thom and Porteous for fundamental classes of degeneracy loci, where the rank of a morphism between vector bundles is bounded above by a given integer. We go on to give some enumerative applications of the formula.

For complete flag varieties, it is possible to undertake a study similar to that which we take toward Grassmannians. The Schubert varieties are indexed this time by permutations, and their fundamental classes are represented by Schubert polynomials. We then characterize simply the permutations which define nonsingular Schubert varieties.

The formalism introduced for complete flag varieties allows us to conclude with a proof of a theorem of Fulton, which constitutes a vast generalization of the formula of Thom and Porteous. This time, the theorem concerns morphisms between vector bundles endowed with structures respectively of flags of subbundles and of quotient bundles, under certain conditions on rank. The fundamental class of the corresponding degeneracy locus is then given by a Schubert polynomial in the Chern classes of the involved bundles.

3.1. Grassmannians

3.1.1. Grassmannians as Algebraic Varieties. We denote by $\mathbb{G}_{m,n}$ the set of linear subspaces of dimension m, therefore of codimension n in $\mathbb{C}^{m+n}$. This set is a complex *Grassmannian*.

The complex linear group $\mathrm{GL}(m+n,\mathbb{C})$ acts transitively on $\mathbb{G}_{m,n}$, as does the unitary group U_{m+n}, whence we have some identifications

$$\mathbb{G}_{m,n} \simeq \mathrm{GL}(m+n,\mathbb{C})/P_{m,n} \simeq U_{m+n}/U_m \times U_n,$$

where we denote by $P_{m,n}$ the subgroup of $\mathrm{GL}(m+n,\mathbb{C})$ which stabilizes the subspace of $\mathbb{C}^{m+n}$ spanned by the first m vectors of the canonical basis.[1] For $m = 1$,

[1]Grassmannians, like complete flag varieties which we encounter a little further on, are examples of ***generalized flag varieties***, which are projective varieties obtained as quotients of a complex semisimple Lie group by a parabolic subgroup [**50**].

we obtain in particular the set $\mathbb{P}(\mathbb{C}^{m+1})$ of lines in $\mathbb{C}^{m+1}$, which is the complex projective space of dimension m, denoted $\mathbb{P}^m$.

If $V \in \mathbb{G}_{m,n}$, let $v_1, \dots, v_m$ be a basis of this space. Extend this basis to a basis of $\mathbb{C}^{m+n}$ with vectors $v_{m+1}, \dots, v_{m+n}$. We may then define local coordinates in a neighborhood of V as follows. Denote by $V^\perp$ the space spanned by $v_{m+1}, \dots, v_{m+n}$ and suppose that $W \in \mathbb{G}_{m,n}$ satisfies $W \cap V^\perp = \{0\}$. Then this space W admits a unique basis consisting of vectors $w_1, \dots, w_n$ of the form

$$w_i = v_i + \sum_{j=1}^{n} x_{ij} v_{m+j}, \quad 1 \le i \le m.$$

In other words, W is spanned by the rows of a unique $m \times (m+n)$ matrix of the form

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 & x_{11} & \dots & x_{1n} \\ 0 & 1 & \dots & \dots & 0 & x_{21} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & x_{m1} & \dots & x_{mn} \end{pmatrix}$$

The x_{ij} then form a system of local coordinates in a neighborhood of V, a neighborhood which is isomorphic to $\mathbb{C}^{mn}$. Moreover, for every point W of this neighborhood, if one specifies analogous local coordinates for a neighborhood of W, the equations of change of basis are affine, therefore *a fortiori* polynomials. The Grassmannian $\mathbb{G}_{m,n}$ is then endowed with the structure of a complex algebraic variety.

3.1.2. The Plücker Embedding. If $W \in \mathbb{G}_{m,n}$, the exterior power $\wedge^m W$ is a line in $\wedge^m \mathbb{C}^{m+n}$, whence we have a map

$$\varphi \colon \mathbb{G}_{m,n} \to \mathbb{P}(\wedge^m \mathbb{C}^{m+n}).$$

Suppose that W is spanned by m vectors which we represent as rows of a $m \times (m+n)$ matrix

$$\begin{pmatrix} x_{11} & x_{12} & \dots & \dots & x_{1,m+n} \\ x_{21} & x_{22} & \dots & \dots & x_{2,m+n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & \dots & x_{m,m+n} \end{pmatrix}.$$

Then the homogeneous coordinates of $\varphi(W)$ in $\mathbb{P}(\wedge^m \mathbb{C}^{m+n})$ are the minors of order m of this matrix, which we denote

$$P_{i_1,\dots,i_m} = \det(x_{p,i_q})_{1 \le p,q \le m}, \quad i_1 < \dots < i_m.$$

These minors are the *Plücker coordinates* of W. In what follows we use this notation even when $i_1, \dots, i_m$ is not an increasing sequence.

PROPOSITION 3.1.1. *The map φ is an embedding, called the Plücker embedding.*

PROOF. We must show that φ, along with its differential at each point, is injective. Therefore let $V \in \mathbb{G}_{m,n}$. As we have already done, choose a basis $v_1, \dots, v_m$ of V and extend it to a basis of $\mathbb{C}^{m+n}$ with vectors $v_{m+1}, \dots, v_{m+n}$, forming a basis of a space $V^\perp$. Suppose that $W \in \mathbb{G}_{m,n}$ is spanned by vectors, the coordinates of which with respect to the preceding basis are the rows of a matrix of the form

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 & x_{11} & \dots & x_{1n} \\ 0 & 1 & \dots & \dots & 0 & x_{21} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 & x_{m1} & \dots & x_{mn} \end{pmatrix}.$$

We then obtain, among the Plücker coordinates, $P_{1,\dots,i-1,m+j,i+1,\dots,m} = x_{ij}$. This implies on one hand that the differential of φ is injective at V, and on the other that φ itself is injective on an open set of $\mathbb{G}_{m,n}$ consisting of subspaces for which $V^\perp$ is a supplement. But two elements of $\mathbb{G}_{m,n}$ always have a common supplement, and therefore φ is injective. $\square$

The Plücker embedding thus permits the realization of the Grassmannian $\mathbb{G}_{m,n}$ as a subvariety of a projective space. We establish that it is an algebraic subvariety, that is, it is defined as the vanishing set of certain polynomials, in this case quadratic, in the Plücker coordinates.

If $J_{m,n}$ is the set of strictly increasing m-tuples of integers between 1 and $m+n$, we denote by $\mathbb{C}[P_J, J \in J_{m,n}]$ the ring of polynomials in the Plücker coordinates. We designate $\mathcal{I}(\mathbb{G}_{m,n})$ to be the ideal consisting of homogeneous polynomials vanishing identically on $\mathbb{G}_{m,n}$.

EXERCISE 3.1.2. Show that $\mathcal{I}(\mathbb{G}_{m,n})$ does not contain a polynomial of degree one. In other words, the Grassmannian $\mathbb{G}_{m,n}$ is not contained in a hyperplane of $\mathbb{P}(\wedge^m\mathbb{C}^{m+n})$. We say then that the Grassmannian is nondegenerate.

On the other hand, we make explicit a whole family of quadratic equations, the *Plücker relations*.

PLÜCKER RELATIONS 3.1.3. *Let $i_1, \dots, i_m$ and $j_1, \dots, j_m$ be two sets of integers between* 1 *and $m+n$, and l an integer between* 1 *and m. Then, identically on $\mathbb{G}_{m,n}$, we have the relation*

$$\sum_{w \in S/S' \times S''} \varepsilon(w) P_{i_1,\dots,i_{l-1},w(i_l),\dots,w(i_m)} P_{w(j_1),\dots,w(j_l),j_{l+1},\dots,j_m} = 0,$$

where S is the group of permutations of the symbols $i_l, \dots, i_m, j_1, \dots, j_l$, S' that of $i_l, \dots, i_m$, and S'' that of $j_1, \dots, j_l$.

PROOF. Given vectors a_i, b_j, c_k of $\mathbb{C}^m$, consider the sum

$$\sum_{w \in S/S' \times S''} \varepsilon(w) \det(a_1, \dots, a_{l-1}, b_{w(i_l)}, \dots, b_{w(i_m)}) \\ \times \det(b_{w(j_1)}, \dots, b_{w(j_l)}, c_{l+1}, \dots, c_m).$$

This sum is an alternating multilinear form in $b_{i_l}, \dots, b_{i_m}$, $b_{j_1}, \dots, b_{j_l}$, which are $m+1$ vectors in $\mathbb{C}^m$. The sum is then identically zero, since $\wedge^{m+1}\mathbb{C}^m = 0$. The Plücker relations are then deduced by specializing to the column vectors of the $m \times (m+n)$ matrix of which the minors of order m are the Plücker coordinates. $\square$

EXAMPLE 3.1.4. In particular, for $l = m$, we obtain the relations

$$P_{i_1,\dots,i_m} P_{j_1,\dots,j_m} = \sum_{k=1}^{m} (-1)^{k-1} P_{i_1,\dots,i_{m-1},j_k} P_{i_m,j_1,\dots,\widehat{j_k},\dots,j_m}.$$

EXERCISE 3.1.5. If $m = n = 2$, we obtain a single equation,

$$P_{1,2}P_{3,4} - P_{1,3}P_{2,4} + P_{1,4}P_{2,3} = 0.$$

From this equation deduce that the Plücker embedding realizes $\mathbb{G}_{2,2}$ as a quadratic hypersurface of $\mathbb{P}^5$.

THEOREM 3.1.6. *The Plücker relations completely determine the Grassmannian and generate the ideal $\mathcal{I}(\mathbb{G}_{m,n})$.*

PROOF. Denote for the moment by $\mathcal{J}_{m,n}$ the ideal generated by the Plücker relations. We show first that this ideal determines the Grassmannian as a set, that is, that the locus of common zeros of elements of $\mathcal{J}_{m,n}$ is precisely $\mathbb{G}_{m,n}$.

We then consider a point of $\mathbb{P}(\wedge^m\mathbb{C}^{m+n})$ with homogeneous coordinates satisfying the Plücker relations, and we show that it is in the image of the Plücker embedding. First let $P_{i_1,\dots,i_m}$ be a nonzero coordinate, which by homogeneity we may suppose is equal to one. Set

$$x_{pq} = P_{i_1,\dots,i_{p-1},q,i_{p+1},\dots,i_m} \quad \text{for} \quad 1 \le p \le m,\ 1 \le q \le m+n.$$

The columns with indices $i_1, \dots, i_m$ of this matrix form an identity matrix, and its rows span therefore a subspace W of $\mathbb{C}^{m+n}$ of dimension m. Denote by $Q_{j_1,\dots,j_m}$ the Plücker coordinates of W. Then

$$Q_{i_1,\dots,i_{p-1},q,i_{p+1},\dots,i_m} = P_{i_1,\dots,i_{p-1},q,i_{p+1},\dots,i_m},$$

for any indices p and q. But, taking into account the Plücker relations, these coordinates determine all the others. Indeed, these are satisfied by decreasing induction on the number of common indices of the m-tuples $i_1, \dots, i_m$ and $j_1, \dots, j_m$, using the relations of the example above. This proves the first part of the theorem.

Moreover, under these conditions, $\mathcal{J}_{m,n}$ cannot be very different from $\mathcal{I}(\mathbb{G}_{m,n})$. Indeed, consider a ring of polynomials $\mathbb{C}[Z_0, \dots, Z_N]$, and $\mathcal{J}$ a homogeneous ideal of this ring (homogeneous in the sense that it contains the homogeneous components of each of its elements). The locus of common zeros of the polynomial elements of $\mathcal{J}$ is an algebraic subvariety $X = X(\mathcal{J})$ of $\mathbb{P}^N$, and we have the following fundamental result [**37**]:

HILBERT'S NULLSTELLENSATZ 3.1.7. *Let $\mathcal{J}$ be a homogeneous ideal of a polynomial ring $\mathbb{C}[Z_0, \dots, Z_N]$. Let $X = X(\mathcal{J}) \subset \mathbb{P}^N$ be the set of common zeros of the elements of this ideal, and $\mathcal{I} = \mathcal{I}(X)$ the ideal of polynomials vanishing on X. Then, if X is not empty, $\mathcal{I}$ is the* radical ideal *of $\mathcal{J}$:*

$$\mathcal{I} = \mathrm{rad}(\mathcal{J}) = \{P \in \mathcal{I},\ \exists k > 0, P^k \in \mathcal{J}\}.$$

To prove the theorem, then, it remains to verify that $\mathcal{J}_{m,n}$ is equal to its radical, which is the object of Theorem 3.3.4. □

DEFINITION 3.1.8. *With the notation of Hilbert's Nullstellensatz above, we call the quotient $R[X] = \mathbb{C}[Z_0, \dots, Z_N]/\mathcal{I}$ the* coordinate ring *of the variety X.*

3.2. Schubert Varieties of Grassmannians

Projective spaces contain linear subspaces, which are the simplest subvarieties and are defined by incidence relations. Moreover, these subspaces admit a cellular decomposition and therefore determine the cohomology groups of the projective spaces. In Grassmannians, a similar role will be played by the Schubert varieties:

they are defined likewise by incidence relations, encoded by certain partitions. But their geometry, as we will see, is infinitely richer.

3.2.1. Schubert Cells and Schubert Varieties.

Fix a *complete flag*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{n+m} = \mathbb{C}^{n+m},$$

which is our flag of reference. It is a strictly increasing sequence of vector subspaces of $\mathbb{C}^{n+m}$, where V_i is of dimension i.

Let λ be a partition contained in an $m \times n$ rectangle, that is, a decreasing sequence of integers $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. We associate to it the *Schubert cell*

$$\Omega_\lambda = \{W \in \mathbb{G}_{m,n},\ \dim(W \cap V_j) = i \text{ if } n + i - \lambda_i \leq j \leq n + i - \lambda_{i+1}\},$$

as well as the *Schubert variety*

$$X_\lambda = \{W \in \mathbb{G}_{m,n},\ \dim(W \cap V_{n+i-\lambda_i}) \geq i,\ 1 \leq i \leq m\}.$$

For example, $X_\varnothing = \mathbb{G}_{m,n}$ and $X_{m\times n}$ is the point V_m. For a partition with only one nonzero part, we have a *special Schubert variety*,

$$X_k = \{W \in \mathbb{G}_{m,n},\ W \cap V_{n+1-k} \neq 0\}.$$

Another example: let $\lambda(p,q)$ be the partition such that its diagram is the complement of a $p \times q$ rectangle in an $m \times n$ rectangle. Then

$$X_{\lambda(p,q)} = \{W \in \mathbb{G}_{m,n},\ V_{m-p} \subset W \subset V_{m+q}\} \simeq \mathbb{G}_{p,q}.$$

Theorem 3.3.4 shows that the $X_{\lambda(p,q)}$ are the only nonsingular Schubert varieties.

REMARK 3.2.1. When we fix a basis $v_1, \ldots, v_{n+m}$ of $\mathbb{C}^{m+n}$ which respects the flag of reference, that is, such that $V_i = \langle v_1, \ldots, v_i \rangle$ for all i, we have a distinguished point of the Schubert cell Ω_λ, namely

$$W^\lambda = \langle v_{n+1-\lambda_1}, \ldots, v_{n+m-\lambda_m} \rangle.$$

If B is the subgroup of $\mathrm{GL}(m+n,\mathbb{C})$ which stabilizes the flag of reference, the Schubert cell Ω_λ is homogeneous under the action of B and coincides with the orbit of W^λ in the Grassmannian.

EXERCISE 3.2.2. One may also describe the Schubert variety X_λ by the incidence conditions $\dim(W \cap V_{n+i-j}) \geq i$, where (i,j) runs through the set of cells of the diagram of λ. Show that it suffices to consider the conditions given by the corners of the diagram.

PROPOSITION 3.2.3. *For all partitions $\lambda \subset m \times n$,*

1. *the Schubert variety X_λ is an algebraic subvariety of $\mathbb{G}_{m,n}$, of which Ω_λ is dense open set contained in the set of nonsingular points;*
2. $\Omega_\lambda \simeq \mathbb{C}^{mn-|\lambda|}$;
3. $X_\lambda = \overline{\Omega_\lambda} = \coprod_{\mu\supset\lambda} \Omega_\mu$;
4. $X_\lambda \supset X_\mu$ *if and only if* $\lambda \subset \mu$.

PROOF. The dimension of $W \cap V_i$ is bounded below by j if and only if the rank of the map $W \subset \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}/V_i$ is bounded above by $m - j$. In local coordinates, this is expressed as the vanishing of minors of order $m - j + 1$ of the matrix representing this map, therefore by polynomial equations. The Schubert variety X_λ, being defined by such incidence conditions, is therefore an algebraic subvariety of $\mathbb{G}_{m,n}$.

If $W \in \mathbb{G}_{m,n}$, the sequence of dimensions of the intersections $W \cap V_i$ runs from 0 to m, increasing at each step by at most one. There therefore exist m jumps, that we denote by $n+i-\mu_i$, where μ is a partition contained in an $m \times n$ rectangle. This shows that

$$\mathbb{G}_{m,n} = \coprod_{\mu \in m \times n} \Omega_\mu.$$

Moreover, if the dimension of $W \cap V_{n+i-\lambda_i}$ is bounded below by i, the first i jumps in dimension have taken place before $n+i-\lambda_i$, which is therefore greater than or equal to $n+i-\mu_i$. As a consequence,

$$X_\lambda = \coprod_{\mu \supset \lambda} \Omega_\mu.$$

We then choose a basis $v_1, \ldots, v_{m+n}$ of $\mathbb{C}^{m+n}$ such that $V_i = \langle v_1, \ldots, v_i \rangle$ for each i. If $W \in \Omega_\lambda$, this space admits a unique basis consisting of vectors of the form

$$w_i = v_{n+i-\lambda_i} + \sum_{\substack{1 \leq j \leq n+i-\lambda_i, \\ j \neq n+k-\lambda_k,\ k \leq i}} x_{ij} v_j,$$

for $1 \leq i \leq m$, and the parameters x_{ij} then determine an isomorphism from Ω_λ to $\mathbb{C}^{mn-|\lambda|}$. More precisely, in the system of local coordinates naturally defined in a neighborhood of W^λ, this realizes Ω_λ as a coordinate subspace.

We note that the points of Ω_λ are spaces spanned by the rows of a matrix of the form

$$\begin{pmatrix} * & \ldots & * & 0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\ * & \ldots & * & * & \ldots & * & 0 & \ldots & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ * & \ldots & * & * & \ldots & * & \ldots & * & 0 & \ldots & 0 \end{pmatrix},$$

where the asterisk the furthest to the right of the ith row lies in column $n+i-\lambda_i$ and is nonzero—but not necessarily equal to 1. It is therefore clear that if $\mu \supset \lambda$, then $\Omega_\mu \subset \overline{\Omega_\lambda}$. Indeed, we may continuously vary the coefficients of a matrix of the preceding type in order to obtain, at the limit, any matrix of type corresponding to the partition μ. Therefore $\Omega_\lambda \subset X_\lambda \subset \overline{\Omega_\lambda}$, and since X_λ is closed, $X_\lambda = \overline{\Omega_\lambda}$. The proposition is proved. □

By the preceding proposition, the incidence of Schubert varieties corresponds simply to the inclusion of the corresponding partitions. Moreover, the Ω_λ form a cellular decomposition of the Grassmannian: the fundamental classes[2] of their closures X_λ therefore form a basis of the integral cohomology of $\mathbb{G}_{m,n}$. We denote by $\sigma_\lambda = [X_\lambda]$ the *Schubert class* associated to a partition $\lambda \subset m \times n$.

COROLLARY 3.2.4. *For all partitions λ contained in an $m \times n$ rectangle, the Schubert class σ_λ is an element of $H^{2|\lambda|}(\mathbb{G}_{m,n})$, and we have the decomposition*

$$H^*(\mathbb{G}_{m,n}) = \bigoplus_{\lambda \subset m \times n} \mathbb{Z}\sigma_\lambda.$$

[2]See the appendix for a succinct presentation of some notions of algebraic topology used in the sequel.

This corollary permits, in particular, the determination of the rank of different cohomology spaces of the Grassmannian. We now introduce the *Poincaré polynomial*

$$P_q(\mathbb{G}_{m,n}) = \sum_{k\geq 0} q^k \operatorname{rank} H^{2k}(\mathbb{G}_{m,n}).$$

COROLLARY 3.2.5. *The Poincaré polynomial of the Grassmannian may be expressed in the following form:*

$$P_q(\mathbb{G}_{m,n}) = \frac{(1-q)(1-q^2)\cdots(1-q^{m+n})}{(1-q)\cdots(1-q^m)(1-q)\cdots(1-q^n)}.$$

PROOF. Let q be an integer greater than or equal to 2, and denote by $\mathbb{K}_q$ the field of q elements. Each vector subspace of dimension m of $\mathbb{K}_q^{m+n}$ admits a unique basis consisting of rows of a matrix of the form below, where the 1 of the ith row lies in column $n+i-\lambda_i$:

$$\begin{pmatrix} * & \dots & * & 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \dots & \dots & \dots \\ * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & 0 & * & \dots & * & 0 & * & * & 1 & 0 & \dots & 0 \end{pmatrix}.$$

Such a matrix has $|\lambda|$ free entries, which each may take q values in $\mathbb{K}_q$. Hence we have the identity

$$P_q(\mathbb{G}_{m,n}) = \#\mathbb{G}_{m,n}(\mathbb{K}_q),$$

where the right-hand term denotes the number of subspaces of dimension m of $\mathbb{K}_q^{m+n}$.[3] Then this number is equal to the number of free families of m vectors in $\mathbb{K}_q^{m+n}$, or $(q^{m+n}-1)\cdots(q^{m+n}-q^{m-1})$, divided by the number of bases of $\mathbb{K}_q^m$, which is $(q^m-1)\cdots(q^m-q^{m-1})$. □

EXERCISE 3.2.6. Show that the duality $\delta\colon W \in \mathbb{G}_{m,n} \to W^\perp \in \mathbb{G}_{n,m}$ is an isomorphism, and that $\delta^*(\sigma_\lambda) = \sigma_{\lambda^*}$, where λ^* is the conjugate partition of λ.

In order to determine the multiplicative structure of the integral cohomology ring (appendix A.2), it is now necessary to determine the products $\sigma_\lambda \cup \sigma_\mu$, and then (appendix A.4) to understand the intersection properties of Schubert cells and Schubert varieties.

3.2.2. Intersection of Schubert Varieties. We keep the notation of the preceding section, where we fixed a flag of reference

$$0 = V_0 \subset \cdots \subset V_{m+n} = \mathbb{C}^{m+n}$$

and a basis $v_1, \dots, v_{m+n}$ which respects this flag, that is, such that $V_i = \langle v_1, \dots, v_i\rangle$. We now consider the flag defined by

$$V_i' = \langle v_{m+n-i+1}, \dots, v_{m+n}\rangle,$$

which we call the dual flag. To it correspond Schubert cells and Schubert varieties which we denote by Ω_λ' and X_λ'. Since the connected group $\mathrm{GL}(m+n,\mathbb{C})$ acts transitively on the set of complete flags, two Schubert varieties associated to the same partition, and in particular X_λ and X_λ', have the same fundamental class σ_λ.

[3]The preceding identity is a very particular case of the famous conjectures of André Weil, who gives precisely this example in *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497–508.

We saw in the proof of proposition 3.2.3 that if $W \in \Omega_\lambda$, this space admits a unique basis consisting of rows of an $m \times (m+n)$ matrix of the form

$$\begin{pmatrix} * & \dots & * & 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \dots & \dots & \dots \\ * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & 0 & * & \dots & * & 0 & * & * & 1 & 0 & \dots & 0 \end{pmatrix},$$

where the 1 of the ith row appears in the column with index $n+i-\lambda_i$. In the same manner, if $W \in \Omega'_\mu$, it admits a unique basis consisting of rows of an $m \times (m+n)$ matrix of the form

$$\begin{pmatrix} 0 & \dots & 0 & 1 & * & * & 0 & * & \dots & * & 0 & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * \\ \dots & \dots & \dots & 0 & \dots & \dots & 0 & 0 & \dots & 0 & 1 & * & \dots & * \end{pmatrix},$$

and this time the 1 of the ith row appears in the column with index $\mu_{m+1-i}+i$.

Suppose then that $W \in \Omega_\lambda \cap \Omega'_\mu$ simultaneously admits a basis of rows of the two preceding matrices. In order for the first row of the first to be a linear combination of rows of the second, it is necessary that $\mu_m + 1 \leq n + 1 - \lambda_1$, that is, $\lambda_1 + \mu_m \leq n$. More generally, for the ith row of the first matrix to be a linear combination of rows of the second, but not of the first $i-1$, it is necessary that $\lambda_i + \mu_{m+1-i} \leq n$. In other words, $\mu \subset \widehat{\lambda}$, where $\widehat{\lambda}$ denotes the complementary partition of λ in an $m \times n$ rectangle.

We use in the following statement the Kronecker symbol δ, which is equal to one if its two arguments are identical and is zero otherwise.

PROPOSITION 3.2.7. *Let λ and μ be two partitions contained in an $m \times n$ rectangle, and suppose that $|\lambda| + |\mu| = mn$. Then*

$$\sigma_\lambda \cup \sigma_\mu = \delta_{\mu,\widehat{\lambda}}.$$

PROOF. By the preceding remarks,

$$\Omega_\eta \cap \Omega'_\tau \neq 0 \implies \eta \subset \widehat{\tau}.$$

Taking the incidence relations stated in proposition 3.2.3 into account, this implies that for $|\lambda|+|\mu| = mn$, the intersection $X_\lambda \cap X'_\mu$ is empty if $\mu \neq \widehat{\lambda}$, hence $\sigma_\lambda \cup \sigma_\mu = 0$. For the same reasons, if $\mu = \widehat{\lambda}$, then $X_\lambda \cap X'_\mu = \Omega_\lambda \cap \Omega'_\mu$, and this set reduces to the point

$$W^\lambda = \langle v_{n+1-\lambda_1}, \dots, v_{n+m-\lambda_m} \rangle.$$

It remains to verify that the intersection of Ω_λ and Ω'_μ is transverse to W^λ. This is clear, since in the natural coordinates of $\mathbb{G}_{m,n}$ in which we express this point, Ω_λ and Ω'_μ correspond to coordinate subspaces. ☐

We say that the classes σ_λ and $\sigma_{\widehat{\lambda}}$ are *dual*. If $x \in H^*(\mathbb{G}_{m,n})$, its decomposition into Schubert classes is given by

$$x = \sum_{\lambda \in m \times n} (x \cup \sigma_{\widehat{\lambda}}) \sigma_\lambda.$$

This allows us to determine the product of a Schubert class with the class of a special Schubert variety, a product given by *Pieri's formula*. For this we adopt the same notation as in the formula of the same name 1.2.5 of the first chapter.

PIERI'S FORMULA 3.2.8. *If $\lambda \subset m \times n$ is a partition, and k is an integer between 1 and n, then*

$$\sigma_\lambda \cup \sigma_k = \sum_{\substack{\nu \subset m\times n,\\ \nu \in \lambda \otimes k}} \sigma_\nu .$$

PROOF. By the preceding proposition, we must show that if $|\lambda| + |\mu| = nm - k$, then $\sigma_\lambda \cup \sigma_\mu \cup \sigma_k = 1$ when the condition

$$n - \lambda_m \geq \mu_1 \geq n - \lambda_{m-1} \geq \mu_2 \geq \cdots \geq n - \lambda_1 \geq \mu_m$$

is satisfied, and $\sigma_\lambda \cup \sigma_\mu \cup \sigma_k = 0$ otherwise. We assume from now on that for all i, we have $\lambda_i + \mu_{n+1-i} \leq n$, since otherwise $\sigma_\lambda \cup \sigma_\mu = 0$. Set

$$\begin{array}{rclcl} A_i & = & \langle v_1, \ldots, v_{n+i-\lambda_i}\rangle & = & V_{n+i-\lambda_i} \\ B_i & = & \langle v_{\mu_{m+1-i}+i}, \ldots, v_{m+n}\rangle & = & V'_{n+m+1-i-\mu_{m+1-i}}, \\ C_i & = & \langle v_{\mu_{m+1-i}-i}, \ldots, v_{n+i-\lambda_i}\rangle & = & A_i \cap B_i . \end{array}$$

The condition above is then satisfied *if and only if* $C_1, \ldots, C_m$ are direct summands, that is, if and only if their dimension C is of dimension $m + k$. We note in passing that

$$C = \bigcap_i (A_i + B_{i+1}).$$

Then if $W \in X_\lambda \cap X'_\mu$, we have $\dim(W \cap A_i) \geq i$ and $\dim(W \cap B_i) \geq m - i + 1$. This implies that for all i, $W \subset A_i + B_{i+1}$. Indeed, if this sum is not the entire space $\mathbb{C}^{m+n}$, A_i and B_{i+1} are direct summands, so

$$\dim(W \cap (A_i + B_{i+1})) \geq i + (m - i) = m,$$

which implies the inclusion. Hence $W \subset C$.

Now let L be a subspace of dimension $n + 1 - k$ of $\mathbb{C}^{m+n}$, and consider the associated special Schubert variety

$$X_k(L) = \{W \in \mathbb{G}_{m,n},\ W \cap L \neq 0\}.$$

If the condition above is not satisfied, $\dim C \leq m + k - 1$ and we may choose L meeting C trivially. But then $X_\lambda \cap X'_\mu \cap X_k(L) = \varnothing$, hence $\sigma_\lambda \cup \sigma_\mu \cup \sigma_k = 0$.

On the other hand, if $\dim C = m + k$, the intersection $C \cap L$ is in general a line $\mathbb{C}c$. But then, the components $c_1, \ldots, c_m$ of c on $C_1, \ldots, C_m$ are necessarily in W and therefore form a basis. The triple intersection $X_\lambda \cap X'_\mu \cap X_k(L)$ therefore reduces to a point, and it is easy to check that this intersection is transverse. Hence $\sigma_\lambda \cup \sigma_\mu \cup \sigma_k = 1$. $\square$

Recall that in the first chapter we denoted by Λ_m the ring of symmetric polynomials in m variables with integral coefficients.

COROLLARY 3.2.9. *The map*

$$\phi_{m,n} \colon \Lambda_m \longrightarrow H^*(\mathbb{G}_{m,n}),$$

which to the Schur function s_λ associates the Schubert class σ_λ if $\lambda \subset m \times n$, and zero otherwise, is a surjective morphism of rings.

COROLLARY 3.2.10. *Each Schubert class of the Grassmannian $\mathbb{G}_{m,n}$ may be expressed in terms of classes of special Schubert varieties using Giambelli's formula, where we set $\sigma_k = 0$ when $k > n$:*

$$\sigma_\lambda = \det(\sigma_{\lambda_i - i + j})_{1 \le i,j \le m}.$$

COROLLARY 3.2.11. *The product of Schubert classes is given by*

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \in m \times n} c^\nu_{\lambda\mu} \sigma_\nu,$$

where the integers $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients.

PROOF. These two corollaries are consequences of the fact that Pieri's formulas completely determine the ring structure of Λ_m and similarly of $H^*(\mathbb{G}_{m,n})$.[4] □

EXERCISE 3.2.12. Show that the kernel of $\phi_{m,n}$ is the ideal of Λ_m generated by the complete symmetric functions h_k for $k > n$, and then also for k running only between $n+1$ and $n+m$. Deduce that $H^*(\mathbb{G}_{m,n})$ is isomorphic to the quotient of $\mathbb{Z}[h_1, \dots, h_n]$ by the relations

$$\det(h_{j-i+1})_{1 \le i,j \le n+k} = 0, \quad 1 \le k \le m,$$

where we make $h_l = 0$ if $l > n$.

EXERCISE 3.2.13. We similarly have an isomorphism

$$H^*(\mathbb{G}_{m,n}) \simeq \Lambda_m \otimes \Lambda_n / I\Lambda_{m+n},$$

where $I\Lambda_{m+n}$ denotes the ideal generated by the homogeneous elements of Λ_{m+n} of strictly positive degree. Indeed, denote by x a set of m indeterminates, y another set of n indeterminates. One may show by induction on k that the relations $e_j(x, y) = 0$, $j \le k$ are equivalent to the relations $h_j(x) = (-1)^j e_j(y)$, $j \le k$, and then deduce the isomorphism above.

Recall that the degree of a subvariety of a projective space (appendix A.4) may be defined as the number of its intersection points with a linear subspace of complementary dimension, in general position. The degree of a Schubert variety, considered as a subvariety of a projective space thanks to the Plücker embedding, is given by the following corollary.

COROLLARY 3.2.14. *The Schubert variety $X_\lambda \subset \mathbb{P}^{\binom{m+n}{m}-1}$ has for degree the number $K_{\widehat{\lambda}}$ of standard tableaux with shape $\widehat{\lambda}$, the complementary partition of λ in an $m \times n$ rectangle. In particular,*

$$\deg(\mathbb{G}_{m,n}) = \frac{0!1!\cdots(n-1)!}{m!(m+1)!\cdots(m+n-1)!}(mn)!.$$

[4]The preceding proof of the Pieri formula in its geometric form is essentially due to Hodge [40]. He remarks there that it is possible to deduce a determinantal expression of Schubert classes in terms of cells of special Schubert varieties (Giambelli's formula above), and that this suffices in principle to determine the product of any two Schubert classes. We quote the conclusion of Hodge's article: "To obtain the intersection of any two Schubert varieties, we merely have to express the first in the determinantal form. . . and then use Pieri's formula to calculate the intersection with the second. The problem is then, essentially, one of elementary algebra." I hope that the first chapter of this book has at least been convincing about one thing: that this problem of elementary algebra is anything but elementary.

PROOF. We keep the notation from the beginning of this section. A point $W \in \mathbb{G}_{m,n}$ has its first Plücker coordinate $P_{1,2,\dots,m}$ equal to zero if and only if it contains a vector which is a linear combination of the last n vectors of the chosen basis of $\mathbb{C}^{m+n}$, that is, if and only if $W \cap V'_n \neq 0$. In other words, the Schubert variety X'_1 is the intersection of $\mathbb{G}_{m,n}$ with a hyperplane of $\mathbb{P}(\wedge^m \mathbb{C}^{m+n})$.

This implies, if φ is the Plücker embedding and h is the hyperplane class (appendix A.4) of the corresponding projective space, that $\varphi^* h = \sigma_1$. As a consequence, to intersect with a hyperplane of $\mathbb{P}(\wedge^m \mathbb{C}^{m+n})$ amounts to intersecting with a Schubert variety of type X_1 in the Grassmannian $\mathbb{G}_{m,n}$. Taking Pieri's formula 3.2.8 into account, this becomes

$$\deg(X_\lambda) = \sigma_\lambda \cup \sigma_1^{mn-|\lambda|} = K_{\hat{\lambda}},$$

and the corollary is proved. □

3.3. Standard Monomials

We now return to the ideals of Schubert varieties, and in particular finish the proof of theorem 3.1.6. We saw that the monomials in the Plücker coordinates, and their restrictions to Schubert varieties, admit between them an enormous number of relations, beginning with the Plücker relations. The object of the theory of standard monomials is to extract bases of these relations.

3.3.1. Coordinate Rings of Schubert Varieties. Consider the lexicographic order on the set of increasing m-tuples $J = (j_1 < \cdots < j_m)$: $J < J'$ if there exists an integer k such that $j_l = j'_l$ for $l < k$, but $j_k < j'_k$.

DEFINITION 3.3.1. *If $M = P_{J^1} \cdots P_{J^h}$ is a monomial in the Plücker coordinates. suppose that $J^1 \leq \cdots \leq J^h$. Then M is represented by a tableau*

$$\begin{bmatrix} j_1^1 & \dots & j_m^1 \\ \dots & \dots & \dots \\ j_1^h & \dots & j_m^h \end{bmatrix},$$

and we say that M is a standard monomial *if this tableau, which is strictly increasing along its rows, is increasing in the broad sense along its columns.*

EXAMPLE 3.3.2. If $m = n = 2$, the unique Plücker relation is written

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

and expresses a nonstandard monomial as a linear combination of standard monomials. This is then an illustration of the following result:

PROPOSITION 3.3.3. *Modulo the Plücker relations, every monomial in the Plücker coordinates is a linear combination of standard monomials.*

PROOF. We consider the lexicographic order on monomials of the same degree, according to which $M = P_{Q^1} \cdots P_{Q^h} < P_{R^1} \cdots P_{R^h}$ if there exists an integer k such that $Q^l = R^l$ for $l < k$ but $Q^k < R^k$. In order to prove the proposition, it suffices to verify that a nonstandard monomial M is a linear combination, modulo the Plücker relations, of strictly smaller monomials under the lexicographic order.

If M is nonstandard, the corresponding tableau contains two consecutive rows of the form

$$\begin{bmatrix} j_1 & \dots & j_m \\ k_1 & \dots & k_m \end{bmatrix}, \quad \text{with } k_1 < \dots < k_l < j_l < \dots < j_m$$

for a certain integer l. But then the Plücker relation 3.1.3,

$$\sum_{w \in S/S' \times S''} \varepsilon(w) P_{j_1,\dots,j_{l-1},w(j_l),\dots,w(j_m)} P_{w(k_1),\dots,w(k_l),j_{k+1},\dots,k_m} = 0,$$

gives exactly the desired relation. □

On a Schubert variety X_λ, a certain number of Plücker coordinates are identically zero, namely the P_J, where $J = (j_1 < \dots < j_m)$ is such that there exists an integer k for which $j_k > i_k = n + k - \lambda_k$. Indeed, P_J is a minor of a matrix of the form

$$\begin{pmatrix} * & \dots & * & 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & * & \dots & * & 0 & \dots & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & * & \dots & * & \dots & * & 0 & \dots & 0 \end{pmatrix},$$

where the asterisk furthest to the right on the kth row lies in column $n + k - \lambda_k$. If $j_k > i_k$, the last $m - k + 1$ rows of the matrix of which P_J is the determinant are related. In other words, if M is standard but not MP_I, where $I = (i_1, \dots, i_m)$, then M is identically zero on the Schubert variety X_λ.

THEOREM 3.3.4. *Let $\mathcal{R}_{m,n} = \mathbb{C}[P_J, J \in J_{m,n}]/\mathcal{J}_{m,n}$, where $\mathcal{J}_{m,n}$ is the ideal generated by the Plücker relations. Then*

1. *the standard monomials form a basis of $\mathcal{R}_{m,n}$;*
2. *the ideal $\mathcal{J}_{m,n}$ is radical, hence equal to $\mathcal{I}(\mathbb{G}_{m,n})$, and $\mathcal{R}_{m,n}$ is the coordinate ring of the Grassmannian;*
3. *more generally, let $\lambda \subset m \times n$ be a partition, to which is associated the m-tuple I defined by $i_k = n + k - \lambda_k$. Then the standard monomials M such that MP_I remains standard form a basis of the coordinate ring $\mathcal{R}_\lambda$ of the Schubert variety X_λ.*

PROOF. We first show the last assertion by decreasing induction on λ, showing that the standard monomials M such that MP_I remains standard, or at least their images in $\mathcal{R}_\lambda$, constitutes a basis. By the proposition above, and the preceding remarks, it suffices to show that they are independent. Consider then a dependence relation

$$\sum_M a_M M = 0 \quad \text{on } X_\lambda.$$

Since P_I is nonzero on X_λ, we may suppose—by simplifying P_I if necessary—that a monomial M_0 not divisible by P_I appears in this relation with nonzero coefficient. The corresponding tableau is written

$$\begin{bmatrix} * & \dots & * \\ \dots & \dots & \dots \\ * & \dots & * \\ j_1 & \dots & j_m \end{bmatrix}, \quad \text{with } j_k = n + k - \mu_k \le i_k = n + k - \lambda_k.$$

In other words, the partition μ thus defined strictly contains λ, hence $X_\mu \subset X_\lambda$. As a consequence, the dependence relation may be restricted to X_μ: certain monomials being zero, there remains a relation

$$\sum_{M,\ MP_J \text{ standard}} a_M M = 0 \quad \text{on} \quad X_\mu.$$

By the inductive hypothesis this relation is trivial, and since $M_0 P_J$ is evidently standard, this implies that $a_{M_0} = 0$, a contradiction!

In particular, the standard monomials generate, as a vector space, the coordinate ring of the Grassmannian. The first assertion is therefore a consequence of the second. We now prove the second. Let P be a polynomial such that for some integer $k > 0$, $P^k \in \mathcal{J}_{m,n}$. It is necessary to show that, in fact, $P \in \mathcal{J}_{m,n}$. Modulo $\mathcal{J}_{m,n}$, one may write P as a linear combination of standard monomials,

$$P = p_0 M_0 + \sum_{M < M_0} p_M M,$$

with $p_0 \neq 0$ if the combination is nonempty. We then have

$$P^k = p_0^k M_0^k + \sum_{N < M_0^k} q_N N,$$

where the monomials N are *a priori* not necessarily standard. However, by the proof of proposition 3.3.3, we may write these monomials N modulo $\mathcal{J}_{m,n}$ as linear combinations of smaller standard monomials. Since $\mathcal{J}_{m,n} \subset \mathcal{I}(\mathbb{G}_{m,n})$, the preceding theorem implies that $p_0 = 0$, and $P \in \mathcal{J}_{m,n}$. □

3.3.2. Postulation. We denote by $r_\lambda(k)$ the dimension of the space of elements of degree k of $\mathcal{R}_{\widehat{\lambda}}$, where $\widehat{\lambda}$ denotes the complementary partition of λ in an $m \times n$ rectangle. We call this the *postulation* of the corresponding Schubert variety.

COROLLARY 3.3.5. *For all partitions* λ,

1. *the integer* $r_\lambda(k)$ *is equal to the number of increasing chains of length* $k+1$ *of partitions* $\varnothing \subset \lambda^0 \subset \cdots \subset \lambda^k = \lambda$, *and hence is also equal to the number of plane partitions with support* λ *and height bounded above by* $k+1$;
2. *there is a recurrence relation*

$$r_\lambda(k+1) - r_\lambda(k) = \sum_{\mu \subset \lambda} r_\mu(k);$$

3. *if* $a_{\lambda,j}$ *denotes the number of strictly increasing chains of partitions* $\varnothing \subsetneq \lambda^0 \subsetneq \cdots \subsetneq \lambda^j = \lambda$, *then*

$$r_\lambda(k) = \sum_{j \geq 0} a_{\lambda,j} \binom{k}{j};$$

4. r_λ *is then a polynomial of degree* $|\lambda|$, *and*

$$r_\lambda(k) = K_\lambda \frac{k^{|\lambda|}}{|\lambda|!} + o(k^{|\lambda|}).$$

PROOF. The first item is a consequence of the fact, stated in the preceding theorem, that the space of elements of degree k of $\mathcal{R}_{\widehat{\lambda}}$ has a basis consisting of standard monomials M of degree k such that MP_I remains standard, where $i_k = k + \lambda_{m+1-k}$. If we write the entries of the corresponding tableau M in the form

$$j_{pq} = q + \lambda^{p-1}_{m+1-q},$$

we then obtain the stated correspondence with chains of partitions contained in λ.

Furthermore, such a chain of partitions yields by superposition a plane partition, whence the correspondence with the set of plane partitions with support λ and height at most $k+1$. Moreover, the recurrence relation of the second item is obtained by partitioning the set of partition chains of given length according to their penultimate term.

The third item is deduced from the first by considering a weakly increasing chain as a strictly increasing chain of partitions endowed with multiplicities. This clearly implies that r_λ is a polynomial, the dominant term of which is given by the number of maximal strictly increasing chains between $\varnothing$ and λ. These maximal chains correspond to standard tableaux with shape λ, and the last item follows. $\square$

The postulation of $X_{\hat{\lambda}}$ is determined by the *Hilbert series*

$$\mathcal{P}_\lambda(t) = \sum_{k\geq 0} r_\lambda(k)t^k.$$

Since r_λ is a polynomial of degree $|\lambda|$, the identity $(1-t)^{-l-1} = \sum_{k\geq 0}\binom{k+l}{l}t^k$ implies that we may write

$$\mathcal{P}_\lambda(t) = \frac{p_\lambda(t)}{(1-t)^{|\lambda|+1}},$$

where the polynomial $p_\lambda(t) = \sum_{j\geq 0} a_{\lambda,j}t^j(1-t)^{|\lambda|-j}$ has degree less than or equal to $|\lambda|$. Moreover, $p_\lambda(1) = K_\lambda$ is equal to the number of standard tableaux with shape λ. The coefficients of p_λ similarly have a combinatorial interpretation [88]:

PROPOSITION 3.3.6. *We have $p_\lambda(t) = \sum_{i\geq 0} p_{\lambda,i}t^i$, where $p_{\lambda,i}$ is the number of standard tableaux with shape λ such that for exactly i integers j, the integer $j+1$ lies in a row of index strictly less than the index of the row where the integer j lies.*

PROOF. Consider a plane partition P with shape λ, interpreted as a numbering of λ which is weakly decreasing along its rows and columns. We may then associate to it a standard tableau T with shape λ by numbering the cells according to the decreasing values of their entries and, for the same entry, from top to bottom and from left to right.

This correspondence is not injective. We try then to understand, given a standard tableau T, of which plane partitions P of height at most $k+1$ it is the image. We denote these by $c_1, \ldots, c_l$, where $l = |\lambda|$, the cells of T being numbered according to the increasing values of their entries. If $D(\lambda)$ is the Ferrers diagram of λ, such a plane partition is given by a decreasing function

$$f\colon D(\lambda) \longrightarrow \{1, \ldots, k+1\},$$

such that if $f(c_i) = f(c_{i+1})$, then c_i lies in a row with index smaller than that of c_{i+1}, or is to the left on the same row. For each i we set $\varepsilon_i = 0$ if this condition holds, and $\varepsilon_i = 1$ otherwise. The function f must satisfy $f(c_i) \geq f(c_{i+1}) + \varepsilon_i$. In other words, if $e_i = \varepsilon_i + \cdots + \varepsilon_{l-1}$,

$$1 \leq f(c_l) \leq f(c_{l-1}) - e_{l-1} \leq \cdots \leq f(c_1) - e_1 \leq k+1-e_1.$$

We then have exactly $\binom{k+l-e_1}{l}$ possible choices for f, where $e_1 = e(T)$ is the number of integers j in T for which $j+1$ lies in a row with index strictly smaller than that of the row where j lies. Hence we have

$$r_\lambda(k) = \sum_T \binom{k+l-e(T)}{l},$$

the sum running over the standard tableaux T with shape λ. The proposition then follows. □

EXAMPLE 3.3.7. We have $p_{32}(t) = 1 + 3t + t^2$, by virtue of the five following standard tableaux:

T	$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$	$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$	$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$
$e(T)$	0	1	1	1	2

EXERCISE 3.3.8. Show that in the statement of the preceding proposition, we may replace *row* by *column*. Deduce that $p_\lambda = p_{\lambda^*}$.

EXERCISE 3.3.9. Determine $r_{m\times n}$ with the aid of results from the first chapter.

EXERCISE 3.3.10. Show that p_λ is a polynomial of degree $|\lambda| - d_\lambda$, where d_λ is the number of diagonals $i + j$ =constant meeting the diagram of λ. Determine the coefficients of the terms of degree zero and one. Verify that the polynomial p_λ is unimodular.

3.4. Singularities of Schubert Varieties

Which Schubert varieties are smooth, and which singular? What are the singular loci of the singular ones? And how may we understand these singularities? These questions, which have given rise to a great deal of work, have nevertheless been only partially answered. The case of Grassmannians, to which this section is devoted, is certainly the one which is the best understood. Later we examine the situation for complete flag varieties; however, for these we do not obtain results as precise as those which we now establish.

3.4.1. The Singular Loci of Schubert Varieties. In order to determine the singular locus of a Schubert variety, we have for example the following "Jacobian criterion" [**37**]:

PROPOSITION 3.4.1. *Let $X \subset \mathbb{C}^N$ be an affine algebraic variety with ideal $\mathcal{I}_X$. Let x be a point of X, and $\mathcal{N}_x$ the space of linear forms on $\mathbb{C}^N$ generated by the differentials in x of elements of $\mathcal{I}_x$. Then the dimension of $\mathcal{N}_x$ is always less than or equal to the codimension of X, and is equal if and only if x is a nonsingular point of X.*

Before applying this criterion to Schubert varieties, some remarks are required. First, since such a variety X_λ is stable under the action of the group B which fixes the flag of reference 3.2.1, so will be its singular locus. This locus must then be a union of Schubert varieties:

$$\mathrm{Sing}(X_\lambda) = \bigcup_{\mu\in S(\lambda)} X_\mu.$$

This singular locus being closed, the set $S(\lambda)$ is an ideal under inclusion: if a partition ν belongs to it, all of the partitions containing ν belong to it also. Of course, $S(\lambda)$ is formed of partitions which contain λ. Finally, again because it is homogeneous under the action of B, a Schubert cell Ω_μ, with $\mu \supset \lambda$, lies in the singular locus of X_λ if and only if its point of reference W^μ is a singular point of the cell.

In a neighborhood of W^μ, a subspace W is of dimension m is generated by the rows of a unique matrix $(x_{ij})_{1\le i\le m,\ 1\le j\le m+n}$ such that

$$x_{k,n+k-\lambda_k} = 1, \quad \text{and} \quad x_{l,n+k-\lambda_k} = 0 \ \text{ if } \ l \neq k.$$

The x_{ij} such that $1 \le i \le m$, and $1 \le j \le m+n$ which are distinct from $n+k-\lambda_k$, $1 \le k \le m$, form a local coordinate system on the Grassmannian, in a neighborhood $U_\lambda \simeq \mathbb{C}^{mn}$ of W^μ. We denote by e_{ij} the canonical basis of this last space.

Recall that the Schubert variety X_λ is defined by a family of incidence conditions $\dim(W \cap V_{n+i-\lambda_i}) \ge i$, or again

$$\dim(W + V_{n+i-\lambda_i}) \le m+n-\lambda_i \quad \text{for} \quad 1 \le i \le m.$$

In the affine open set U_λ, such a condition may be expressed by the vanishing of the minors of order $m+n-\lambda_i+1$ of the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ x_{1,1} & x_{1,2} & \cdots & x_{1,n+i-\lambda_i} & \cdots & x_{1,n+m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n+i-\lambda_i} & \cdots & x_{m,n+m} \end{pmatrix}.$$

These conditions are themselves equivalent to the vanishing of the minors of order $m-i+1$ of the matrix formed from the last m rows and the last $m-i+\lambda_i$ columns of the preceding matrix.

LEMMA 3.4.2. *The set $M(\lambda)$ of these minors generate the ideal of the Schubert variety X_λ in the open affine set U_λ.*

PROOF. Indeed, let Q be a polynomial in the variables x_{ij}, identically vanishing on X_λ. Since the x_{ij} are particular Plücker coordinates, we may consider Q as a polynomial in these Plücker coordinates. Since Q vanishes on X_λ, theorem 3.3.4 implies that, modulo the Plücker relations, Q belongs to the ideal generated by the Plücker coordinates P_J such that there exists an integer k for which $j_k > n+k-\lambda_k$.

The minor P_J may then be expanded into a sum of products of minors taken on the columns $j_1, \dots, j_{k-1}$ on one hand, and on the columns $j_k, \dots, j_m$ on the other. Since these minors are elements of $M(\lambda)$, the lemma is proved. □

We return then to the preceding matrix. Since $\mu_i \ge \lambda_i$, the matrix has at most $m-i$ entries equal to 1, and these are found in rows and columns which are pairwise distinct. There then exist minors of order $m-i+1$ on which a derivative at the origin is nonzero, only if $\mu_{i+1} \le \lambda_i$. And these minors are obtained by adjoining a column, say of index r, and a row, of index s, to the minor of order $m-i$ where the 1 is found. The only nonzero derivative of this minor is then obtained with respect to the variable x_{rs}.

The cotangent subspace generated by the first derivatives at the point W^μ of the elements of the ideal of X_λ is then the subspace generated by the vectors e_{rs}, for which there exists an integer i between 1 and m such that

$$\mu_{i+1} \leq \lambda_i \leq \mu_i,\ 1 \leq r \leq i, \quad \text{and} \quad n+i-\lambda_i < s \leq n+m,$$

s being furthermore distinct from the $n+i-\mu_{i+1}, \ldots, n+m-\mu_m$. By the preceding proposition, W^μ is a smooth point of X_λ if and only if these pairs number $|\lambda|$, the codimension of X_λ in $\mathbb{G}_{m,n}$.

Let $i < j$ be two integers such that $\mu_{i+1} \leq \lambda_i \leq \mu_i$ and $\mu_{j+1} \leq \lambda_j \leq \mu_j$ and they are consecutive among those integers which have this property. For $i < r \leq j$, the integer r may take exactly λ_j values, and the preceding smoothness condition then implies that $\lambda_r = \lambda_j$. In other words, X_λ is singular on Ω_μ if and only if there exist an integer j such that $\mu_{j+1} \leq \lambda_j \leq \mu_j$, an integer $k < j$ such that $\lambda_k > \lambda_j$, and $\lambda_h > \mu_{h+1}$ for $k \leq h < j$.

The partitions μ which are minimal with respect to this property are obtained by choosing an integer k such that $\lambda_k > \lambda_{k+1}$, then by setting $\mu_{k+1} = \lambda_k + 1$, and by completing μ in minimal fashion in order to obtain a partition containing λ. In other words, one adjoins to λ a ribbon of unit width around each of its corners, as in the following example.

EXAMPLE 3.4.3. In $\mathbb{G}_{5,5}$, we have $\operatorname{Sing}(X_{44221}) = X_{55521} \cup X_{44333}$, these two components being irreducible, of respective codimensions 5 and 4 in X_{44221}, corresponding to the following diagrams:

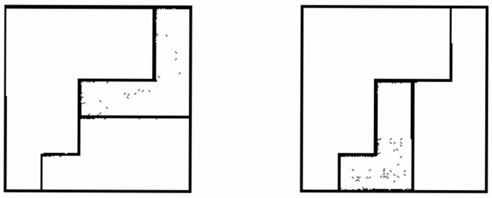


FIGURE 1. $\operatorname{Sing}(X_{44221}) = X_{55521} \cup X_{44333}$ in $\mathbb{G}_{5,5}$

We have then established the following result:

THEOREM 3.4.4. *Let $T(\lambda)$ be the set of partitions defined as follows: if the cell $c = (i,j)$ is a corner of λ, with $1 \leq i < m$ and $1 \leq j < n$, we add to the diagram of λ the cell $(i+1, j+1)$, and complete a partition from it in a minimal way. Then*

$$\operatorname{Sing}(X_\lambda) = \bigcup_{\mu \in T(\lambda)} X_\mu$$

is the decomposition of the singular locus of the Schubert variety X_λ into irreducible components.

REMARK 3.4.5. This implies in particular that $\operatorname{Sing}(X_\lambda)$ is always of codimension at least three in X_λ. Moreover, the only smooth Schubert varieties are those which correspond to partitions which have diagrams complementary, in an $m \times n$ rectangle, to smaller rectangles. We have seen that these varieties are none other than sub-Grassmannians.

3.4.2. Desingularizations of Schubert Varieties. A very different way to determine the singularities of Schubert varieties is to construct their *desingularizations*. A desingularization of X_λ is a proper morphism

$$f_\lambda \colon Z_\lambda \longrightarrow X_\lambda,$$

where the variety Z_λ is nonsingular, such that f_λ restricts to an isomorphism over a dense Zariski open set of X_λ, that is, over the complement of a proper algebraic subvariety, which is however not necessarily reduced to the singular locus. We say that such a morphism is *birational*.

Before giving a general procedure for the construction of such desingularizations, we begin with a simple example of a Schubert cycle associated to a rectangular partition:

$$X_{p\times q} = \{W \in \mathbb{G}_{m,n},\ \dim(W \cap V_{n+p-q}) \geq p\}.$$

This incidence condition means that the intersection $W \cap V_{n+p-q}$ contains a space U of dimension p which is uniquely determined when W is an element of the Schubert cell $\Omega_{p\times q}$. This leads us to introduce the set

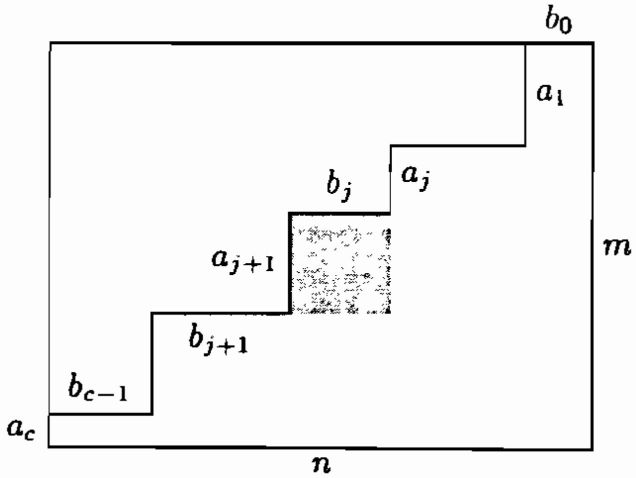
$$Z_{p\times q} = \{(W,U) \in \mathbb{G}_{m,n} \times \mathbb{G}_{p,m+n-p},\ U \subset W \cap V_{n+p-q}\}.$$

The projection on the second factor makes of $Z_{p\times q}$ a fibration of Grassmannians, on the Grassmannian of subspaces of dimension p of V_{n+p-q}; it is therefore a smooth projective variety. On the other hand, the projection on the first factor sends $Z_{p\times q}$ surjectively onto $X_{p\times q}$, and restricts to an isomorphism above the cell $\Omega_{p\times q}$. It is therefore a desingularization of $X_{p\times q}$.

In order to extend this type of construction to any Schubert variety, we introduce some notation. If λ is a partition, we denote the coordinates of the *creux* of its diagram as

$$(a_1 + \cdots + a_{i-1} + 1,\ b_i + \ldots b_{c-1} + 1), \quad 1 \leq i \leq c.$$

We define a_c and b_0 in such a fashion that $a_1 + \cdots + a_c = m$ and $b_0 + \cdots + b_{c-1} = n$.



By exercise 3.2.2, the Schubert variety X_λ is then determined solely by the incidence conditions

$$\dim(W \cap V^i) \geq a_1 + \cdots + a_i, \quad 1 \leq i < c,$$

where we set $V^i = V_{a_1+\cdots+a_i+b_0+\cdots+b_{i-1}}$. We denote $X_\lambda = X_\lambda(V^\bullet)$ for convenience, where $V^\bullet$ denotes the partial flag

$$0 \subset V^1 \subset \cdots \subset V^{c-1} \subset V^c = \mathbb{C}^{m+n}.$$

We now show how to construct a desingularization of X_λ by induction on the number c of creux of the diagram of λ, according to a procedure described by A. Zelevinsky [**99**]. If $c = 1$, the partition λ is empty and the corresponding Schubert variety is reduced to a point—and therefore smooth! If $c = 2$, then λ is a rectangular partition and we have seen above how to desingularize X_λ.

Choose first a creux of λ, say the jth. Consider the partition μ obtained by making the creux disappear as in the figure above: μ therefore has precisely one fewer creux than λ. Let Y_λ then be the variety of pairs (W, U^1), where W is of dimension m and U^1 of dimension $\dim V^j + a_j$, with $V^j \subset U^1 \subset V^{j+1}$, and also satisfying the incidence conditions

$$\begin{aligned} \dim(W \cap V^i) &\geq a_1 + \cdots + a_i, \quad i \neq j,\ j+1, \\ \dim(W \cap U^1) &\geq a_1 + \cdots + a_{j+1}. \end{aligned}$$

In other words, W is an element of the Schubert variety $X_\mu(U^\bullet_{(1)})$ associated to the partition μ and to the following partial flag $U^\bullet_{(1)}$:

$$0 \subset V^1 \subset \cdots \subset V^{j-1} \subset U^1 \subset V^{j+2} \subset \cdots \subset V^c = \mathbb{C}^{m+n}.$$

The projection on the first factor then sends Y_λ to the Schubert variety X_λ. Indeed, the preceding incidence conditions imply that

$$\dim(W \cap V^{j+1}) \geq \dim(W \cap U^1) \geq a_1 + \cdots + a_{j+1},$$

$$\dim(W \cap V^j) \geq \dim(W \cap U^1) - \operatorname{codim}_{U^1} V_j \geq a_1 + \cdots + a_j.$$

Moreover, this projection on the first factor is an isomorphism over the cell Ω_λ. Indeed, suppose that we have $\dim(W \cap V^j) = a_1 + \cdots + a_j$ and $\dim(W \cap V^{j+1}) = a_1 + \cdots + a_{j+1}$. Then if $(W, U^1) \in Y_\lambda$, the incidence conditions imply that $W \cap V^{j+1} = W \cap U^1$, hence $U^1 \supset V^j + W \cap V^{j+1}$. For simple reasons of dimensionality, this inclusion is then an equality.

By the inductive hypothesis, we have desingularizations

$$f_{\mu,U^1} : Z_\mu(U^\bullet_{(1)}) \longrightarrow X_\mu(U^\bullet_{(1)}),$$

and we then define the variety

$$Z_\lambda = \{(Z, U^1),\ \ Z \in Z_\mu(U^\bullet_{(1)})\}.$$

The projection on the second factor of Z_λ is a locally trivial fibration of smooth varieties over a Grassmannian, and is therefore a nonsingular variety. Furthermore, the morphism f_λ defined by

$$f_\lambda(Z, U^1) = f_{\mu,U^1}(Z)$$

sends Z_λ to the Schubert variety X_λ, and is therefore a desingularization.

We stress that this construction procedure for Z_λ by induction depends on an order o chosen on the set of creux of λ, which we will denote Z^o_λ. Moreover, this desingularization may be described as the variety of subspaces $U^1, \ldots, U^c$, of dimensions depending on λ and o, under certain incidence conditions. These conditions are moreover easy to visualize graphically. The induction essentially acts by introducing the auxiliary subspace U^1 such that $V^j \subset U^1 \subset V^{j+1}$, and moves the incidence conditions to the flag

$$0 \subset V^1 \subset \cdots \subset V^{j-1} \subset U^1 \subset V^{j+2} \subset \cdots \subset V^c = \mathbb{C}^{m+n}.$$

By repeating this procedure, we establish that the set of incidence relations are conveniently represented by a triangular diagram, as illustrated in the example below.

EXAMPLE 3.4.6. Consider a partition λ having a diagram with four corners. For the orders 12345 and 43152 of its creux, we obtain the incidence diagrams below.

$$\begin{array}{c}
0 \subset V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 \\
\cap\,\cup \quad /\!/ \quad /\!/ \quad /\!/ \quad /\!/ \\
U^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 \\
\cap\,\cup \quad /\!/ \quad /\!/ \quad /\!/ \\
U^2 \subset V^3 \subset V^4 \subset V^5 \\
\cap\,\cup \quad /\!/ \quad /\!/ \\
U^3 \subset V^4 \subset V^5 \\
\cap\,\cup \quad /\!/ \\
U^4 \subset V^5 \\
\cap\,\cup \\
U^5
\end{array}
\qquad
\begin{array}{c}
0 \subset V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 \\
\backslash\!\backslash \quad \backslash\!\backslash \quad \backslash\!\backslash \quad \cap\,\cup \quad /\!/ \\
0 \subset V^1 \subset V^2 \subset U^1 \subset V^5 \\
\backslash\!\backslash \quad \backslash\!\backslash \quad \cap\,\cup \quad \backslash\!\backslash \\
0 \subset V^1 \subset U^2 \subset V^5 \\
\cap\,\cup \quad /\!/ \quad /\!/ \\
U^3 \subset U^2 \subset V^5 \\
\backslash\!\backslash \quad \cap\,\cup \\
U^3 \subset U^4 \\
\cap\,\cup \\
U^5
\end{array}$$

FIGURE 2. Desingularizations of Schubert varieties

We now resume the preceding discussion:

THEOREM 3.4.7. *For all orders o of the set of creux of λ, let Z^o_λ be the variety described above, consisting of subspaces $(U^1, \ldots, U^c)$ satisfying certain incidence conditions. Then the projection on the last factor makes Z^o_λ a desingularization of the Schubert variety X_λ.*

REMARK 3.4.8. This freedom on the order o may be precious. Indeed, a desingularization is often more interesting when its fibers, above the points where it is not an isomorphism, are of small dimension. And we are particularly interested in small desingularizations, that is, in desingularizations $f\colon Z \longrightarrow X$, such that for all strictly positive integers k, the set of points $x \in X$ for which the fiber $f^{-1}(x)$ is of dimension at least k, is itself of codimension strictly greater than $2k$. Zelevinsky has proved the following result:

THEOREM 3.4.9. *For all partitions λ, there exists an order o on the set of its creux such that Z^o_λ is a small desingularization of X_λ.*

EXERCISE 3.4.10. For a Schubert variety associated to a rectangular partition, we have *a priori* two distinct desingularizations. Verify that if one of them is a small desingularization, then the other is not.

3.5. Characteristic Classes and Degeneracy Loci

This section is devoted to the construction of Chern classes, which are cohomology classes naturally attached to a complex vector bundle on a compact differentiable variety (in fact, over any topological space). We show in particular how these classes are connected to Schubert varieties of Grassmannians, and we deduce the formula of Thom and Porteous.

The properties of the cohomology groups which we use are briefly presented in an appendix. One may similarly use [**42**] or [**68**] to complement what follows concerning bundles and their characteristic classes. For the meaning of notions from differential and algebraic geometry that are necessary for us, we recommend the reader, for example, to [**34**].

3.5.1. Chern Classes. Let L be a complex line bundle on a compact differentiable variety X of dimension n. It is possible to associate to it an integral cohomology class, its *first Chern class*

$$c_1(L) \in H^2(X) = H^2(X, \mathbb{Z})$$

(for what follows, we systematically omit the symbol $\mathbb{Z}$), in such a way that the following properties are satisfied:

1. *functoriality*: if $f\colon Y \to X$ is continuous, then $c_1(f^*L) = f^*c_1(L)$;
2. *additivity*: if L and M are two complex line bundles on X, then
$$c_1(L \otimes M) = c_1(L) + c_1(M);$$
3. *normalization*: if $\mathcal{O}(-1)$ is the tautological line bundle on the complex projective space $\mathbb{P}^n$, then $c_1(\mathcal{O}(-1)) = -h$ is the negative of the hyperplane class (appendix A.4).

An immediate consequence of the additive property is that the first Chern class of the trivial bundle $X \times \mathbb{C}$ is zero. The first Chern class of the dual bundle of a line bundle L is, for the same reasons, $c_1(L^*) = -c_1(L)$. In particular, over $\mathbb{P}^n$, the hyperplane class h is the first Chern class of the bundle denoted $\mathcal{O}(1)$, the dual to the tautological line bundle. Above a line $d \in \mathbb{P}^n$, its fiber $\mathcal{O}(1)_d$ is the space of linear forms on d, and may be identified with the quotient of $(\mathbb{C}^{n+1})^*$ by the subspace T_d of linear forms which are zero on d. We then have the exact sequence of vector bundles on $\mathbb{P}^n$:

$$0 \longrightarrow T \longrightarrow (\mathbb{C}^{n+1})^* \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

A line bundle L on X always admits a finite family $s_0, \dots, s_N$ of global sections of class $\mathcal{C}^\infty$ which simultaneously vanish at no point. These sections allow us to define a map

$$\begin{array}{rccl} \Phi\colon X & \longrightarrow & \mathbb{P}^N, \\ x & \longmapsto & [s_0(x) : \cdots : s_N(x)], \end{array}$$

where in brackets we denote the homogeneous coordinates on projective space.

Let α be a linear form on the line $\Phi(x)$, which is the restriction of the linear form $(\alpha_0, \dots, \alpha_N)$ on $\mathbb{C}^{N+1}$. Then the element $\alpha_0 s_0(x) + \cdots + \alpha_N s_N(x)$ of the fiber L_x depends only on α, and this defines a canonical isomorphism between the fiber of $\mathcal{O}(1)$ at $\Phi(x)$ and that of L at x. Hence we have an identification $L = \Phi^*\mathcal{O}(1)$.

In particular, by functoriality, the first Chern class of L is completely determined by Φ; we have $c_1(L) = \Phi^*(h)$. Moreover, h may be identified, via

Poincaré duality, with the fundamental class of any hyperplane. If such a hyperplane H is defined by the linear form $(\alpha_0, \dots, \alpha_N)$, consider the global section $s = \alpha_0 s_0 + \cdots + \alpha_N s_N$ of L. The preimage of H under Φ is the zero locus X_s of the section s. In a neighborhood of a point x of X_s, a neighborhood in which we suppose L to be trivialized, this section s is defined by a function f with complex values. And if $df(x) \neq 0$, the map from X_s to H obtained by restriction of Φ is a submersion; we say that $s(X)$ meets the zero section transversally at the point x. It follows (appendix A.3) that

$$c_1(L) = \Phi^*(h) = \Phi^*[H] = [\Phi^{-1}(H)] = [X_s]$$

when $s(X)$ meets the zero section of L transversally at every point. Under this hypothesis, X_s is an oriented smooth subvariety of X, of real codimension equal to 2. We revisit this notion of transversality shortly.

More generally, it is possible to associate to a complex vector bundle E of rank e on a compact differentiable variety X a *total Chern class* $c(E) \in H^*(X)$, in such a way that the following properties are satisfied:

1. *functoriality*: if $f\colon Y \to X$ is continuous, then $c(f^*E) = f^*c(E)$;
2. *additivity*: if E and F are two complex vector bundles on X, we have the Whitney addition formula
$$c(E \oplus F) = c(E) \cup c(F);$$
3. *normalization*: if L is a line bundle, then $c(L) = 1 + c_1(L)$.

One may decompose the total Chern class into the sum

$$c(E) = \sum_k c_k(E), \quad \text{with } c_k(E) \in H^{2k}(X).$$

The Whitney formula may then be written, since the cohomology classes of even degree commute, as

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \cup c_j(F).$$

EXAMPLE 3.5.1. Suppose that the vector bundle E is the C^∞ direct sum of e line bundles $L_1, \dots, L_e$ (we say then that it is *topologically split*), and write $x_i = c_1(L_i)$. Then, by the Whitney formula,

$$c_k(E) = c_k(x_1, \dots, x_e)$$

is the kth elementary symmetric function in these first Chern classes.

The example above implies that the theory of Chern classes is determined by the case of line bundles. In fact, by virtue of the following result, that we establish a little further on (see proposition 3.8.1 and the remark which follows), one may always act as if a vector bundle E is the direct sum of line bundles. In other words, one may interpret the Chern classes $c_k(E)$ as the elementary symmetric functions in the "virtual" elements $x_1, \dots, x_e$ of $H^2(X)$, its *Chern roots*. A polynomial in $x_1, \dots, x_e$ makes sense only when it is symmetric, in which case we may express it in terms of elementary symmetric functions and interpret it as a cohomology class over X associated to E and to the given polynomial.

SPLITTING PRINCIPLE 3.5.2. *For all complex vector bundles E on a variety X, there exists a continuous map $f\colon Y \to X$ such that the inverse image f^*E is split, and such that the induced map of cohomology $f^*\colon H^*(X) \to H^*(Y)$ is injective.*

An immediate consequence of this principle is that $c_k(E) = 0$ if $k > e$. More generally, we may associate to a vector bundle E and to a partition λ a *characteristic class* $s_\lambda(E)$, defined for example in terms of Chern classes just as the Jacobi-Trudi formula expresses the Schur function in terms of elementary symmetric functions:

$$s_\lambda(E) = \det(c_{\lambda_i^* - i + j}(E))_{1 \le i,j \le \lambda_1}.$$

In this determinant, the products are evidently given by the cup product, for which we omit the symbol $\cup$. If E is split, we obtain the corresponding Schur function in the first Chern classes of the line bundles of which E is the sum, and this in turn implies that $s_\lambda(E) = 0$ when $l(\lambda) > e$.

3.5.2. Gysin Morphisms. It is also possible to define directly the Chern classes, in the following fashion. Consider the variety $Y = \mathbb{P}(E^*)$ of hyperplanes of E, which is a fibration on X with fiber $\mathbb{P}^{e-1}$. This variety is endowed with a line bundle $\mathcal{O}_E(1)$, the restriction of which to each fiber of the projection $\pi \colon Y \to X$ is the bundle $\mathcal{O}(1)$.

Set $\zeta = c_1(\mathcal{O}_E(1))$, and let $i \colon F \simeq \mathbb{P}^{e-1} \hookrightarrow Y$ denote the inclusion of a fiber of π. Then $i^*\zeta = h$, the cohomology class of a hyperplane of $\mathbb{P}^{e-1}$. We note that the induced map of cohomology $\pi^* \colon H^*(X) \to H^*(Y)$ makes $H^*(Y)$ into a $H^*(X)$-module, and we may define a Gysin morphism of degree $-2e+2$ (appendix A.3):

$$\pi_* \colon H^*(Y) \longrightarrow H^*(X).$$

PROPOSITION 3.5.3. *The ring $H^*(Y)$ is a free $H^*(X)$-module with base 1, ζ, ..., ζ^{e-1}. In other words, every cohomology class y over Y may be written in a unique way in the form*

$$y = \sum_{i=0}^{e-1} \pi^* y_i \cup \zeta^i, \quad \textit{with } y_i \in H^*(X).$$

The Gysin morphism is then given by $\pi_*(y) = y_{e-1}$.

We note that the Gysin morphism π_* satisfies $\pi_*\zeta^j = 0$ if $j < e-1$, for simple reasons of dimension. On the other hand, $\pi_*\zeta^{e-1} = 1$. Indeed, let α and β be cohomology classes over X, such that $\alpha \cup \beta$ is the class of a point. Then $\pi^*(\alpha \cup \beta) = \pi^*\alpha \cup \pi^*\beta$ is the class of a fiber of π, therefore $\pi^*\alpha \cup \pi^*\beta \cup \zeta^{e-1}$ is the class of a point of Y. The projection formula then implies that $\pi_*\zeta^{e-1} = 1$. This remark leads immediately to the assertion of uniqueness in the proposition above.

We consider over Y the surjective map $\pi^*E \twoheadrightarrow \mathcal{O}_E(1)$, the kernel of which is the tautological hyperplane bundle H, of rank $e-1$. The Whitney formula implies the equality $c(\pi^*E) = c(H)(1+\zeta)$. The vanishing of $c_e(H)$ then translates into the identity

$$\zeta^e = \pi^*c_1(E) \cup \zeta^{e-1} + \cdots + (-1)^{e-1}\pi^*c_e(E).$$

By the preceding proposition, this identity completely determines the Chern classes of E. One sometimes takes it to be a definition.

REMARK 3.5.4. We give a slightly different interpretation of the preceding proposition. Denote by $-x_1, \dots, -x_{e-1}$ the Chern roots of the bundle H, and let $-x_e = \zeta$. The Chern roots of E are then $-x_1, \dots, -x_e$, and every symmetric polynomial in $x_1, \dots, x_e$ may then be interpreted as a characteristic class of E, in particular as a cohomology class over X.

Now let P be a polynomial in $x_1, \ldots, x_e$ with coefficients in $H^*(X)$, and which is symmetric only in $x_1, \ldots, x_{e-1}$. Then $P(x_1, \ldots, x_e)$ defines a cohomology class over Y, and we have the identity

$$\pi_*(P(x_1, \ldots, x_e)) = (\partial_1\partial_2 \cdots \partial_{e-1}P)(x_1, \ldots, x_e).$$

We remark first of all that the right-hand side is symmetric in $x_1, \ldots, x_e$ and so defines a cohomology class over X. Moreover, the symmetric polynomials in $x_1, \ldots, x_e$ are scalars for the operator $\partial_1\partial_2 \cdots \partial_{e-1}$. By the preceding proposition, it then suffices to verify the above identity for $P = x_e^k$ with $k < e$, which presents no difficulty.

3.5.3. The Gauss-Bonnet Formula. We saw that the first Chern class of a line bundle may be interpreted concretely, under certain conditions, as a fundamental class of the zero locus of a global section. We now obtain some analogous interpretations for vector bundles of any rank.

We begin with the Grassmannian $\mathbb{G}_{m,n}$. We consider on this variety the *tautological bundle* T of rank m, for which the fiber above a subspace W of $\mathbb{C}^{m+n}$ is W itself. In an analogous fashion, the *quotient bundle* Q, of rank n, has for fiber above W the quotient $\mathbb{C}^{m+n}/W$. This quotient bundle plays the role of the bundle $\mathcal{O}(1)$ for vector bundles of rank greater than one.

PROPOSITION 3.5.5. *The kth Chern class $c_k(Q)$ of the quotient bundle on the Grassmannian is equal to the class σ_k of a special Schubert variety of codimension k.*

PROOF. By the duality theorem 3.2.7, we must verify that for all partitions $\lambda \subset m \times n$ of size $|\lambda| = mn - k$, we have

$$c_k(Q) \cup \sigma_k = \begin{cases} 1 & \text{if } \lambda = \lambda(1, k), \\ 0 & \text{otherwise,} \end{cases}$$

where we recall that $\lambda(1, k)$ denotes the complement of an $1 \times k$ rectangle in the rectangle $m \times n$.

We note first of all that if $|\lambda| = mn - k$ but $\lambda \neq \lambda(1, k)$, we necessarily have the inequality $\lambda_m \geq n - k + 1$. Consequently, if $W \in X_\lambda$, then $\dim(W \cap V_{m+k-1}) \geq m$, or in other words $W \subset V_{m+k-1}$. The Schubert variety X_λ is then contained in the Grassmannian $\mathbb{G} = \mathbb{G}_{m,k-1}$ of subspaces of dimension m of V_{m+k-1}.

On $\mathbb{G}$, we have an exact sequence of fiber bundles

$$0 \longrightarrow V_{m+k-1}/W \longrightarrow \mathbb{C}^{n+m}/W \longrightarrow \mathbb{C}^{n+m}/V_{m+k-1} \longrightarrow 0.$$

The restriction $Q_{\mathbb{G}}$ of the quotient bundle to $\mathbb{G}$ is then an extension of the trivial bundle by a bundle of rank $k-1$. Hence $c_k(Q_{\mathbb{G}}) = 0$ by the Whitney formula. If j is the inclusion of $\mathbb{G}$ in $\mathbb{G}_{m,n}$, the projection formula (appendix A.3) gives

$$c_k(Q) \cup \sigma_\lambda = j_*(j^*(c_k(Q) \cup [X_\lambda])) = 0,$$

since $j^*(c_k(Q)) = c_k(Q_{\mathbb{G}}) = 0$. On the other hand,

$$X_{\lambda(1,k)} = \{W \in \mathbb{G}_{m,n}, \quad V_{m-1} \subset W \subset V_{m-k}\}$$

may be identified with the projective space $\mathbb{P} = \mathbb{P}(V_{m+k}/V_{m-1})$, of dimension k. We denote by i their natural isomorphism. On this space $\mathbb{P}$ is defined the tautological line bundle $\mathcal{O}(-1)$, and a quotient bundle $Q_{\mathbb{P}}$. Moreover, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) = W/V_{m-1} \longrightarrow V_{m+k}/V_{m-1} \longrightarrow Q_{\mathbb{P}} \longrightarrow 0.$$

As a consequence, the total Chern class of $Q_{\mathbb{P}}$ is $c(Q_{\mathbb{P}}) = (1-h)^{-1}$. And since the restriction of Q to $X_{\lambda(1,k)}$ is none other than $i^*Q_{\mathbb{P}}$, we have

$$c_k(Q) \cup \sigma_{\lambda(1,k)} = i_*(c_k(Q_{\mathbb{P}})) = 1.$$

The proposition is then proved. □

Before deriving the consequences of this statement, we need to introduce some definitions relative to problems of transversality. We have already made use of the following definition:

DEFINITION 3.5.6. *Two subspaces U and V of a vector space W are called* transverse *if* $\operatorname{codim}(U \cap V) = \operatorname{codim} U + \operatorname{codim} V$.

DEFINITION 3.5.7. *We say that a map $f\colon X \longrightarrow Y$ of class C^∞ is* transverse *to a smooth subvariety Z of Y, if for all $x \in f^{-1}(Z)$, the spaces $df(T_xX)$ and $T_{f(x)}Z$ are transverse in $T_{f(x)}Y$.*

Now let F be a complex vector bundle of rank f on X. In what follows we consider sections s of F transverse to the zero section in the preceding sense. For a family of sections $s = (s_1, \ldots, s_f)$ of global sections of class C^∞ of F, the notion of transversality is a little more delicate. We consider the degeneracy loci, for $0 \leq i - j < f$,

$$D_{i,j}(s) = \{x \in X,\ \dim \langle s_1(x), \ldots, s_i(x)\rangle \leq i - j\}.$$

If we trivialize F above an open set U of X, the sections $s_1, \ldots, s_i$ define a map

$$s_U\colon U \longrightarrow M_{f,i},$$

the space of $f \times i$ matrices with complex coefficients. In $M_{f,i}$ are defined the degeneracy loci $M_{f,i}^k$, formed, for $k < \min(f, i)$, of matrices whose rank is less than or equal to k. These loci form a chain of algebraic subvarieties of $M_{f,i}$. We leave it to the reader to establish the following lemma:

LEMMA 3.5.8. *If $k \leq \min(f, i)$, the locus $M_{f,i}^k$ of $i \times f$ matrices of rank bounded above by k is irreducible, of codimension $(i-k)(f-k)$, and in addition is smooth outside $M_{f,i}^{k-1}$.*

We say that the sections $s_1, \ldots, s_f$ are transverse if for all i, and in every trivialization of F above an open set U, the map s_U is transverse to the subvarieties $M_{f,i}^k \backslash M_{f,i}^{k-1}$ of $M_{f,i}$. This is a local property which it suffices, in a neighborhood of a point, to verify in a particular trivialization.

We say that sections are *generic* when such hypotheses of transversality are verified. One may show that the space of C^∞ global sections of F, the generic sections—whatever hypotheses of transversality are meant by this term—form a dense subset, the intersection of a countable family of dense open subsets. This is the *Transversality Theorem of Thom*.

Having made such hypotheses of transversality, the degeneracy locus $D_{i,j}(s)$ is smooth of codimension $j(f-i+j)$ outside of $D_{i,j+1}(s)$ and $D_{i-1,j}(s)$. Proceeding by induction just as for algebraic subvarieties of a complex algebraic variety (appendix A.3), we deduce that it is possible to associate to it a fundamental class

$$[D_{i,j}(s)] \in H^{2j(f-i+j)}(X).$$

Now we complete our global sections $s_1, \ldots, s_f$ of F into a family $s_1, \ldots, s_N$ of sections such that the vectors $s_1(x), \ldots, s_N(x)$ generate the fiber F_x at every point $x \in X$. We then obtain a map

$$\Phi\colon X \longrightarrow \mathbb{G}_{N-f,f}$$

which associates to the point x the subspace of linear combinations of $s_1, \ldots, s_N$ which vanish at x. In the same fashion as in the case of line bundles, this construction gives a natural identification $F = \Phi^* Q$. Then for all integers k, we have

$$c_k(F) = \Phi^* c_k(Q) = \Phi^*(\sigma_k).$$

But the corresponding Schubert variety is

$$X_k = \{W \in \mathbb{G}_{N-f,f}, \quad W \cap V_{f+1-k} \neq 0\},$$

hence $\Phi^{-1}(X_k)$ is the set of points of X where the sections $s_1, \ldots, s_{f-k+1}$ do not simultaneously vanish—that is, $D_{f-k+1,1}(s)$. For generic sections, the restriction

$$\Phi\colon D_{f-k+1,1}(s) \longrightarrow X_k$$

is a submersion, which implies the following result:

PROPOSITION 3.5.9. *If s is a generic family of sections of F,*

$$c_k(F) = [D_{f-k+1,1}(s)].$$

For $k = f$, we obtain a formula often called[5] the

GAUSS-BONNET FORMULA 3.5.10. *If s is a global section of F, transverse to the zero section, and if $X_s = s^{-1}(0)$, then*

$$c_f(F) = [X_s].$$

REMARK 3.5.11. In an algebraic context, in other words if one considers algebraic sections of an algebraic complex vector bundle, then things are simpler, in a manner of speaking. Indeed, in order to satisfy statements of the type above, it suffices to suppose that the degeneracy loci $D_{i,j}(s)$ are of "expected" codimension, that is, $j(f - i + j)$ (and this for each of their irreducible components, the set of which may actually be empty) [**46**]. It is necessary, however, to ensure that the degeneracy loci are locally defined by the vanishing of certain minors of matrices representing the considered sections. And these minors may vanish with certain multiplicities (equal to one in the transverse case), and one must take these into account in the definition of the fundamental class of degeneracy loci in order that the preceding formulas may remain valid.

3.5.4. The Thom-Porteous Formula. The preceding result may be extended to the degeneracy locus $D_{i,j}(s)$, which is the inverse image under Φ of the Schubert variety $X_{j\times(f-i+j)}$. Giambelli's formula then implies the following result:

PROPOSITION 3.5.12. *If s is a generic family of sections of F,*

$$[D_{i,j}(s)] = s_{j\times(f-i+j)}(F) = \det(c_{f-i+j-p+q}(F))_{1\leq p,q\leq j}.$$

[5]Properly speaking, the Gauss-Bonnet formula refers rather to the identity between the topological Euler-Poincaré characteristic of a compact complex variety M, of dimension n, and the integer $c_n(M)$ obtained by evaluating the nth Chern class of the tangent bundle of M at its fundamental class. See in [**34**] how this identity may be derived from the formula which follows.

More generally, consider a morphism $\phi\colon E \to F$ between vector bundles of respective ranks e and f, and suppose the morphism to be transverse—in the sense that for each open set U of X in which E and F are simultaneously trivialized, the induced map $\phi_U \to M_{f,e}$ is transverse to the loci formed by matrices of fixed rank. Then for all integers k, the degeneracy locus

$$D_k(\phi) = \{x \in X, \quad \text{rank } \phi_x \leq k\}$$

is smooth of codimension $(e-k)(f-k)$ outside of $D_{k-1}(\phi)$, and we may associate to it a fundamental class $[D_k(\phi)]$.

René Thom remarked at the beginning of the fifties that this class depends only on the characteristic classes of the bundles involved. Indeed, without entering into the details, we have seen that in the differentiable category one may always obtain the bundles E and F as inverse images of quotient bundles on Grassmannians. And the cohomology ring of a Grassmannian is generated by the characteristic classes of its only quotient bundle, by proposition 3.5.5.

But it was necessary to wait several years before Porteous [**72**] obtained the explicit formula which follows. If $x_1, \ldots, x_e$ and $y_1, \ldots, y_f$ are the Chern roots of the bundles E and F, and if λ is a partition, the multi-Schur function $s_\lambda(Y - X)$ (see definition 2.6.4) is symmetric in x as well as in y. The function may then be expressed in terms of the characteristic classes of E and F, and as a consequence defines a cohomology class over X, denoted $s_\lambda(F - E)$.

THOM-PORTEOUS FORMULA 3.5.13. *Under the hypotheses above,*

$$[D_k(\phi)] = s_{(f-k)\times(e-k)}(F - E) = \det(c_{f-k-i+j}(F - E))_{1\leq i,j\leq e-k}.$$

REMARK 3.5.14. One may decompose this class into a product of characteristic classes of E and F. By using the expansion

$$c_k(F - E) = \sum_{i+j=k} (-1)^j c_i(F) s_j(E),$$

one obtains from the Jacobi-Trudi formula the expression

$$\begin{aligned} s_{p\times q}(F - E) &= \det(c_{q-i+j}(F - E))_{1\leq i,j\leq p} \\ &= \sum_\alpha \det(c_{\alpha_i - i+j}(F))_{1\leq i,j\leq p} \prod_k (-1)^{q-\alpha_k} h_{q-\alpha_k}(E), \end{aligned}$$

the sum running over the p-tuples α of integers. Hence we have the remarkable formula

$$s_{p\times q}(F - E) = \sum_{\lambda\subset p\times q} (-1)^{|\lambda|} s_{\widehat{\lambda}}(F) s_{\lambda^*}(E),$$

the last sum running over the partitions λ contained in the $p \times q$ rectangle, where we denote by $\widehat{\lambda}$ the complement of λ in this rectangle.

PROOF. The idea is to consider $Y = \mathbb{G}_{e-k,k}(E)$, the Grassmannian of subspaces of codimension k of the bundle E, where we denote by π the projection on X. On this variety, just as on an ordinary Grassmannian, we have a tautological bundle T of rank $e-k$ and a quotient bundle Q of rank k, with morphisms

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[dr, "\psi"'] & \pi^*E \arrow[r] \arrow[d, "\pi^*\phi"] & Q \arrow[r] & 0 \\ & & \pi^*F & & \end{tikzcd}$$

We have denoted by $\psi\colon T \to \pi^*F$ the induced morphism, the composition of the inclusion of T in π^*E and the inverse image of ϕ on Y. Then

$$D_0(\phi) = \{(x, W) \in Y, \quad W \subset \ker \phi_x\}.$$

This implies, in particular, that the restriction of π to $D_0(\phi)$ induces an isomorphism above $D_k(\phi)\backslash D_{k-1}(\phi)$.[6] Under the hypothesis of transversality, we have

$$[D_k(\phi)] = \pi_*[D_0(\phi)].$$

Moreover, we know by the Gauss-Bonnet formula that $[D_0(\phi)]$ is the maximal Chern class of the bundle $T^* \otimes \pi^*F$, since ϕ is a global section transverse to the zero section. Then Cauchy's second formula 1.4.6 precisely permits the decomposition of the maximal Chern class of a tensor product according to the characteristic classes of the bundles involved. By induction, this yields

$$[D_0(\phi)] = c_{(e-k)f}(T^* \otimes \pi^*F) = \sum_{\mu \subset f \times (e-k)} (-1)^{|\mu|} s_{\hat{\mu}}(\pi^*F) s_{\mu^*}(T).$$

We note that the characteristic classes of the vector bundles T, Q, and E are related by the identity $\pi^*c(E) = c(T)c(Q)$, so that

$$s_m(T) = \sum_{p+q=m} (-1)^p c_p(Q) \pi^* s_q(E).$$

If we express $s_{\mu^*}(T)$ in terms of the classes $s_m(T)$ using the Jacobi-Trudi formula, then in terms of the Chern classes of Q and of E via the identity above, we may arrange for products of Chern classes of Q, of indices ranging between 0 and k, the number of terms of these products being the length of μ^*, which is bounded above by $e-k$. But for simple reasons of dimension, π being a fibration of relative dimension $k(e-k)$, we have

$$\pi_* \left(\prod_{i=1}^{k} c_i(Q)^{m_i} \right) = 0 \quad \text{if} \quad \sum_{i=1}^{k} i m_i < k(e-k).$$

When we apply the Gysin morphism π_* to the preceding expression, only the term $c_k(Q)\pi^* s_{m-k}(E)$ of $s_m(T)$ remains. Moreover, we have

$$\pi_* \left(c_k(Q)^{e-k} \right) = 1.$$

Indeed, proposition 3.5.5 and Pieri's formula (or the Gauss-Bonnet formula applied to $Q^{\oplus e-k}$, since a generic section of this bundle is zero at a unique point of the Grassmannian, and this without multiplicity) show that this is certainly the case when X is a point. The general case is deduced by reasoning as we did for a

[6] We remark that this construction is exactly the same as that of the desingularizations of special Schubert cycles.

fibration of projective spaces in the remarks which follow proposition 3.5.3. Taking these different observations into account, we then have

$$\begin{aligned}\pi_* s_{\mu^*}(T) &= \det(\pi_* s_{\mu_i^* - i + j}(T))_{1 \le i,j \le f-k} \\ &= \det(s_{\mu_i^* - i + j - k}(E))_{1 \le i,j \le e-k} \times (-1)^{k(e-k)}.\end{aligned}$$

In particular, we obtain nonzero terms only for the partitions μ such that $\mu_{e-k}^* \ge k$. If to such a partition μ we associate the partition $\lambda \subset (f-k) \times (e-k)$ such that $\lambda_i^* = \mu_i^* - k$ for $1 \le i \le e-k$, we finally obtain

$$\pi_*[D_0(\phi)] = \sum_{\lambda \subset (f-k) \times (e-k)} (-1)^{|\lambda|} s_{\hat{\lambda}}(F) s_{\lambda^*}(E) = s_{(f-k)\times(e-k)}(F - E),$$

which is the stated formula. □

EXAMPLE 3.5.15. In the Grassmannian $\mathbb{G}_{m,n}$, the Schubert variety $X_{p\times q}$ is the set of subspaces W such that $\dim(W \cap V_{n+p-q}) \ge p$. It is therefore the degeneracy locus $D_{m-p}(\phi)$, where ϕ is the composition

$$\phi \colon T \hookrightarrow V \twoheadrightarrow V/V_{n+p-q},$$

where T denotes as usual the tautological bundle on $\mathbb{G}_{m,n}$. The Thom-Porteous formula and proposition 3.5.5 then give

$$[X_{p\times q}] = s_{p\times q}(-T) = s_{p\times q}(Q) = \sigma_{p\times q},$$

in accordance with Giambelli's formula. We see, in passing, that the time it took to discover the Thom-Porteous formula was not unrelated to the degree of ignorance into which the works of certain Italian geometers fell.

EXAMPLE 3.5.16. An example which was studied by the classical geometers consists of considering a matrix P of size $e \times f$, containing as entries homogeneous polynomials $P_{i,j}$ in $n+1$ variables, with $\deg P_{i,j} = a_i + b_j > 0$. With this degree condition, the minors of the matrix P are homogeneous polynomials. For $r < \min(e, f)$, the condition $\operatorname{rank} P \le r$ then defines a subvariety $D_r(P)$ of projective space $\mathbb{P}^n$. This subvariety may be interpreted as the degeneracy locus $D_r(\mathcal{P})$, where $\mathcal{P}$ is the induced morphism of vector bundles on $\mathbb{P}^n$:

$$\mathcal{P} \colon \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_e) \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(-b_f).$$

For a generic matrix P of polynomials, the degeneracy locus $D_r(P)$ is then of codimension $(e-r)(f-r)$, moreover irreducible, and the Thom-Porteous formula gives

$$\deg D_r(P) = \det(c_{f-r+j-i})_{1 \le i,j \le e-r},$$

where the integers c_k are defined by the formal identity

$$\sum_{k \ge 0} c_k t^k = \prod_{j=1}^{f} (1 + b_j t) \Big/ \prod_{i=1}^{e} (1 - a_i t).$$

In the simplest example, where $e = f = r+1$, the degeneracy locus $D_r(P)$ is the hypersurface of the equation $\det P = 0$, and the determinant is of degree $a_1 + \cdots + a_e + b_1 + \cdots + b_f$.

REMARK 3.5.17. With some additional work, the same methods permit establishing a generalization of the Thom-Porteous formula, due to Kempf and Laksov [**46**]. One again considers a morphism $\phi: E \to F$ of vector bundles, but this time one supposes that E is endowed with a flag of subbundles

$$0 \subset E_1 \subset \cdots \subset E_k = E, \quad \text{with} \quad \operatorname{rank} E_i = e_i.$$

One considers the degeneracy locus $D_{r_\bullet}(\phi)$, defined by the conditions

$$\operatorname{rank}(E_i \xrightarrow{\phi} F) \leq r_i, \quad 1 \leq i \leq k.$$

Suppose that $0 < e_1 - r_1 < \cdots < e_k - r_k$ and $0 \leq r_1 < \cdots < r_k < f$, and define the partition λ, the parts of which are the integers $f - r_i$ with multiplicity

$$n_i = (e_i - r_i) - (e_{i-1} - r_{i-1}).$$

Then, under suitable hypotheses of genericity,

$$[D_{r_\bullet}] = s_\lambda(\underbrace{F - E_1}_{n_1}, \ldots, \underbrace{F - E_k}_{n_k}).$$

As before, these last classes are defined by the polynomials denoted in the same way, except that we replace the variables by the Chern roots of the corresponding bundles. This formula of Kempf-Laksov is in fact a very particular case of Fulton's theorem that we establish in section 3.8.3. See also example 3.8.9.

3.5.5. Some Enumerative Applications. Historically, what we call *Schubert calculus* stems from problems of enumerative geometry. The methods that Schubert and his contemporaries used with as much spirit as little caution [**81**] have been formalized and rendered rigorous—for certain of them—only thanks to the contribution of ideas coming from algebraic topology [**94, 13**]. In particular, it was necessary to interpret enumerative problems as calculations of intersections in cohomology rings. The simplest problems, of which we give several examples, often led to intersection problems in the Grassmannian, for which we now know the multiplicative structure of the cohomology ring very precisely.

Number of Linear Subspaces Defined by Incidence Conditions. The number of lines of $\mathbb{P}^3$ meeting four lines in general position is equal to two. Indeed, the lines of $\mathbb{P}^3$ meeting a given line form a Schubert variety of codimension one in $\mathbb{G}_{2,2}$, with class σ_1. In general, four such varieties intersect transversally in a number of points equal to $\sigma_1^4 = 2$.

Similarly, the number of lines of $\mathbb{P}^4$ meeting six planes in general position is equal to five. More generally, the number of linear subspaces of dimension k in $\mathbb{P}^l$ meeting $N_{k,l} = (k+1)(l-k)$ subspaces of dimension $l - k - 1$ in general position is equal to

$$\deg \mathbb{G}_{k+1,l-k} = \frac{0!1!\cdots k!}{(l-k)!\cdots l!} N_{k,l}!.$$

Degeneracy Loci of Matrices. Recall that $M_{e,f}$ denotes the space of $e \times f$ matrices with complex entries, and $M_{e,f}^k$ the locus of matrices with rank bounded above by $k < \min(e, f)$. Since this locus is invariant under multiplication by a nonzero constant, we may consider its image $\mathbb{P}(M_{e,f}^k)$ in the projective space $\mathbb{P}(M_{e,f})$. Then

$$\deg \mathbb{P}(M_{e,f}^k) = \frac{0!1!\cdots(e-k-1)!}{k!\cdots(e-1)!} \frac{f!\cdots(e+f-k-1)!}{(f-k)!\cdots(e+f-2k-1)!}.$$

Indeed, denote by x_{ij} for $1 \le i \le e$ and $1 \le j \le f$ the homogeneous coordinates of $\mathbb{P}(M_{e,f})$. We may consider x_{ij} as a linear form on the matrices, hence a global section of the line bundle $\mathcal{O}(1)$. The matrix of these linear forms then defines a morphism of vector bundles on $\mathbb{P}(M_{e,f})$,

$$\phi\colon \mathcal{O}^{\oplus e} \to \mathcal{O}(1)^{\oplus f},$$

where $\mathcal{O}$ is the trivial line bundle. Moreover, $D_k(\phi) = \mathbb{P}(M^k_{e,f})$. The Thom-Porteous formula then gives the identity

$$[\mathbb{P}(M^k_{e,f})] = s_{(f-k)\times(e-k)}(\mathcal{O}(1)^{\oplus f}).$$

But by homogeneity, for each partition λ, adopting the notation from paragraph 1.4.3 of the first chapter, we have

$$s_\lambda(\mathcal{O}(1)^{\oplus f}) = s_\lambda(\underbrace{1,\dots,1}_{f})h^{|\lambda|} = K_\lambda(f)h^{|\lambda|},$$

where h is the hyperplane class. To conclude, it remains only to apply corollary 1.4.11 from the first chapter.

Fano Schemes. Let $X \subset \mathbb{P}^n$ be a hypersurface defined by a polynomial P_X in $n+1$ variables which is homogeneous of degree d. Let $F_k(X)$ be the set of linear subspaces of dimension k of $\mathbb{P}^n$ which are contained in X: this set is a subset of the Grassmannian $\mathbb{G}_{k+1,n-k}$, which we call the Fano scheme of the hypersurface X.

It is convenient to understand the scheme in the following way. A polynomial of degree d on $\mathbb{P}^n$ may be viewed as an element of the symmetric power $S^d(\mathbb{C}^{n+1})^*$. The polynomial P_X then defines a global section s_X of the symmetric power S^dT^* of the dual of the tautological bundle, of which the value on a subspace W is simply the restriction of P_X to W. In particular, this section vanishes on W if and only if $\mathbb{P}(W) \subset X$. For a general hypersurface X, s_X will be transverse to the zero section, and the Gauss-Bonnet formula 3.5.10 implies that

$$[F_k(X)] = c_{\binom{k+d}{d}}(S^dT^*).$$

We note that the isomorphism $\mathbb{G}_{m,n} \simeq \mathbb{G}_{n,m}$ induced by duality makes the quotient bundle on $\mathbb{G}_{n,m}$ correspond with the dual of the tautological bundle on $\mathbb{G}_{m,n}$. Proposition 3.5.5 and the exercise which precedes corollary 3.2.14 imply then that $c_k(T^*) = \sigma_{1^k}$ is the Schubert class associated to the partition having k nonzero parts, all equal to one. In order to obtain the expression for $[F_k(X)]$ on Schubert classes, it is then sufficient to know how to express the maximal Chern class of the symmetric power of a bundle E as a function of its characteristic classes. Unfortunately, we do not have such an expression, except for the cases where $d = 2$ or $k = 1$.

In small dimensions, we may avoid this difficulty with the help of the splitting principle. Suppose for example that $n = d = 3$, which is to say that X is a cubic surface in $\mathbb{P}^3$. We need only consider the case $k = 1$, for which we must express $c_4(S^3E)$ as a function of the characteristic classes of the bundle E, of rank two. A little calculation then gives

$$c_4(S^3E) = 18s_{3,1}(E) + 27s_{2,2}(E).$$

If one applies this formula to the dual of the tautological bundle on $\mathbb{G}_{2,2}$, the first term disappears, and the second gives 27 times the class of a point. Consequently, *a generic cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.*[7]

In general, it is similarly possible to obtain the degree of a Fano scheme as a certain coefficient of an explicit polynomial, given as the product of linear forms. Indeed, denote by $x_0, \dots, x_k$ the Chern roots of the bundle T^*, so that

$$c_{\binom{k+d}{d}}(S^d T^*) = \prod_{a_0+\cdots+a_k=d} (a_0x_0 + \cdots + a_kx_k).$$

We denote by $P_{d,k}$ this polynomial, and suppose for simplicity that $F_k(X)$ is of dimension zero, that is, a finite set, for a generic hypersurface of degree d in $\mathbb{P}^n$.

THEOREM 3.5.18. *When it is finite, the number of linear subspaces of dimension k contained in a generic hypersurface of degree d in $\mathbb{P}^n$ is equal to the coefficient of the monomial $x_0^n x_1^{n-1} \cdots x_k^{n-k}$ in the product of the polynomial $P_{d,k}$ by the Vandermonde determinant $a_\delta = \prod_{0 \le i < j \le k}(x_i - x_j)$.*

PROOF. The desired number is the coefficient of $P_{d,k}$, when we write it as a linear combination of Schur polynomials, on

$$s_{(k+1)\times(n-k)} = e_{k+1}^{n-k} = (x_0 \cdots x_k)^{n-k}.$$

Indeed, this Schur function represents the class of a point of the Grassmannian. It then suffices to show that for all symmetric polynomials P, and all partitions μ, the coefficient of P on the Schur polynomial s_μ is equal to the coefficient of $x^{\mu+\delta}$ in the product of P with the Vandermonde a_δ. By linearity, it suffices to verify this when P is itself a Schur polynomial s_λ. Then $s_\lambda a_\delta = a_{\lambda+\delta}$ by definition, and this last polynomial evidently contains a unique monomial x^α for which α is strictly decreasing, that is, $x^{\lambda+\delta}$, the coefficient of which has the value one. □

In the case of lines, that is, for $k = 1$, this result is due to Van der Waerden [**95**]. One may deduce from the result, for example, that a generic quintic in $\mathbb{P}^5$ contains 2875 lines, that a generic quartic in $\mathbb{P}^7$ contains 3, 297, 280 planes, and a generic cubic in $\mathbb{P}^8$ contains 1, 812, 768, 336 linear subspaces of dimension three.

3.6. Flag Varieties

Rather than consider subspaces of fixed dimension in a complex vector space—hence Grassmannians—we now treat chains of vector subspaces, flags. And we are led from complete flag varieties to the study of the Schubert varieties which are associated, this time, to permutations. In particular, we will find that their fundamental class is given by a Schubert polynomial.

3.6.1. Projective Algebraic Structure. Let $\mathbb{F}_n$ be the set of complete flags of vector subspaces of $\mathbb{C}^n$,

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n.$$

[7]One can indeed prove a more precise result: all smooth cubic surfaces contain exactly 27 distinct lines. The configuration of these lines has been studied classically by Steiner and Schläffli (1858). One can show that its automorphism group, of cardinality 51840, is isomorphic to the Weyl group of the exceptional complex simple Lie algebra of type E_6.

We denote by $W_\bullet$ such a complete flag. The complex linear group $\mathrm{GL}(n,\mathbb{C})$ acts transitively on $\mathbb{F}_n$, as does the unitary group U_n. Hence we have identifications

$$\mathbb{F}_n \simeq \mathrm{GL}(n,\mathbb{C})/B \simeq U_n/(U_1)^n,$$

where B denotes the subgroup of upper triangular matrices of $\mathrm{GL}(n,\mathbb{C})$. This subgroup is the stabilizer of the canonical flag of $\mathbb{C}^n$.

Given a fixed complete flag $V_\bullet$, choose a basis $e_1,\dots,e_n$ of $\mathbb{C}^n$ such that V_i is generated by $e_1,\dots,e_i$. One may define local coordinates on $\mathbb{F}_n$ in a neighborhood of $V_\bullet$ in the following way: if V_i' is the space generated by $e_{n+1-i},\dots,e_n$, and if $W_\bullet \in \mathbb{F}_n$ is such that for all i, $W_i \cap V_{n-i}' = 0$, then there exists a unique basis of $\mathbb{C}^n$ consisting of vectors $f_1,\dots,f_n$ of the form

$$f_i = e_i + \sum_{j>i} x_{ij} e_j, \quad 1 \le i \le n,$$

and such that for all i, $f_1,\dots,f_i$ is a basis of W_i. In other words, there exists a unique $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & x_{12} & \cdots & \cdots & \cdots & x_{1n} \\ 0 & 1 & \cdots & \cdots & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 & x_{n-1,n} \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

the first i rows of which generate W_i, for all i.

We then obtain a system of local coordinates in a neighborhood of $V_\bullet$, isomorphic to $\mathbb{C}^{n(n-1)/2}$. We may in this way endow $\mathbb{F}_n$ with the structure of a complex algebraic variety of dimension $n(n-1)/2$.

Just as with Grassmannians, flag varieties $\mathbb{F}_n$ are projective varieties. Indeed, $\mathbb{F}_n$ is a closed subvariety of the product $\mathbb{G}_{1,n-1} \times \mathbb{G}_{2,n-2} \times \cdots \times \mathbb{G}_{n-1,1}$ of the Grassmannians corresponding to each of the members of a complete flag. By using the different Plücker embeddings $\varphi_i\colon \mathbb{G}_{i,n-i} \to \mathbb{P}^{N_i-1}$, where $N_i = \binom{n}{i}$, we deduce from this an embedding

$$\psi_n\colon \mathbb{F}_n \to \mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_{n-1}-1}.$$

EXERCISE 3.6.1. Let A and B be two complex vector spaces. Show that the map

$$\mathbb{P}(A) \times \mathbb{P}(B) \to \mathbb{P}(A \otimes B),$$

which associates to the lines generated by vectors $a \in A$ and $b \in B$ the line generated by $a \otimes b \in A \otimes B$, is an embedding, the *Segre embedding*.

The preceding exercise shows that the variety $\mathbb{F}_n$ may be embedded in the projective space $\mathbb{P}^{N_1 \cdots N_{n-1}-1}$, of which it is an algebraic subvariety.

3.6.2. Schubert Varieties of a Flag Variety. Let us again fix a flag of reference $V_\bullet$ of subspaces of $\mathbb{C}^n$, and a basis $e_1,\dots,e_n$ such that the first i elements generate V_i. We denote by $V_\bullet'$ the *dual flag*, such that V_j' is generated by the last j elements of the preceding basis.

To a permutation $w \in \mathcal{S}_n$, we associated in section 2.1.1 of the second chapter a rank function r_w, given by

$$r_w(p,q) = \#\{i \le p, \quad w(i) \le q\}.$$

We then define the *Schubert cell*

$$\Omega_w = \{W_\bullet \in \mathbb{F}_n, \quad \dim(W_p \cap V_q) = r_w(p,q), \quad 1 \le p, q \le n\}.$$

Its closure in $\mathbb{F}_n$ is the *Schubert variety* X_w associated to the permutation w. We note that Ω_w contains the flag $W_\bullet^w$ such that

$$W_i^w = \langle e_{w(1)}, \dots, e_{w(i)} \rangle.$$

EXERCISE 3.6.2. Show that $\mathbb{F}_n$ is the disjoint union of the Schubert cells Ω_w. In other words, given two flags $V_\bullet$ and $W_\bullet$, the function

$$r(p,q) = \dim(V_p \cap W_q)$$

is always the rank function of a certain permutation.

By reasoning as we did for the Schubert cells of Grassmannians, we verify that an element $W_\bullet \in \Omega_w$ is defined by a unique matrix $(x_{ij})_{1 \le i,j \le n}$ (in the sense that W_i is generated by the first i rows of this matrix) such that

$$x_{i,w(i)} = 1, \quad \text{and} \quad x_{i,j} = 0 \quad \text{if} \quad j > w(i) \quad \text{or} \quad i > w^{-1}(j).$$

In other words, for each integer i we write a hook of zeros with summit $(i, w(i))$, this summit excepted where we write 1 instead, and we leave the remaining entries undetermined. On the ith row, the number of indeterminates is the number of columns with indices $j < i$ in which we were not constrained to write a zero, which is the case if $w(j) < w(i)$. The number of indeterminates is then equal to the number of inversions $l(w)$, and we obtain an isomorphism

$$\Omega_w \simeq \mathbb{C}^{l(w)}.$$

More precisely, we have local coordinates in a neighborhood of $W_\bullet^w$, which is the flag defined by the matrix of the permutation w. In a neighborhood of $W_\bullet^w$, a flag $W_\bullet$ is indeed defined by a unique matrix obtained by placing a 1 at coordinates $(i, w(i))$ and zeros below, the remaining entries being undetermined. The Schubert cell Ω_w appears then as a subspace of coordinates.

EXAMPLE 3.6.3. If $w = 365142 \in S_6$, an element of Ω_w is defined by a unique matrix of the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 \\ * & * & 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

PROPOSITION 3.6.4. *For all permutations $w \in S_n$, the Schubert variety*

$$X_w = \coprod_{v \le w} \Omega_v$$

is the disjoint union of the Schubert cells associated to the permutations bounded above by w under the Bruhat order. Consequently,

$$X_w = \{W_\bullet \in F_n, \quad \dim(W_p \cap V_q) \ge r_w(p,q), \quad 1 \le p, q \le n\},$$

and the Schubert varieties are algebraic subvarieties of $\mathbb{F}_n$.

PROOF. We temporarily denote by Y_w the algebraic subvariety of $\mathbb{F}_n$ defined by the incidence conditions above. Since Y_w is closed, we have $\Omega_w \subset X_w \subset Y_w$. But by proposition 2.1.12 from the second chapter, we have $v \leq w$ if and only if $r_v \geq r_w$. This implies that

$$Y_w = \coprod_{v \leq w} \Omega_v.$$

In order to establish that $Y_w = X_w$, it then suffices to show that $\Omega_v \subset X_w$ if $v \leq w$. By induction, it is enough to check the condition when $v = wt_{jk}$ is of length $l(v) = l(w) - 1$. This is nothing but an elementary exercise which we leave to the reader. □

COROLLARY 3.6.5. *We have the inclusion $X_v \subset X_w$ if and only if $v \leq w$.*

For Schubert cells of Grassmannians, we remarked that the incidence conditions were not necessarily independent. It is the same with flag varieties, and one may prove the following proposition:

PROPOSITION 3.6.6. *A Schubert variety X_w is determined by the incidence relations*

$$\dim(W_p \cap V_q) \geq r_w(p, q) \quad \textit{for} \quad (p, q) \in \mathrm{Ess}(w),$$

corresponding to the essential points of the permutation w. On other hand, one cannot dispense with any of these conditions.

EXAMPLE 3.6.7. For the permutation above, the given conditions for the essential points are simply those which require that the intersections $W_2 \cap V_5$, $W_3 \cap V_4$, and $W_5 \cap V_2$ be nontrivial. Attention: the preceding proposition does not extend to open Schubert cells.

3.6.3. The Cohomology Ring of a Flag Variety. We denote by σ_w the Poincaré dual of the fundamental class of the Schubert variety $X_{w_0 w}$, where w_0 denotes the permutation of maximal length in $\mathcal{S}_n$. The preceding section implies that the Schubert cells define a cellular decomposition of $\mathbb{F}_n$. Its integral cohomology groups may then be deduced:

PROPOSITION 3.6.8. *The cohomology of $\mathbb{F}_n$ admits for a base the classes $\sigma_w \in H^{2l(w)}(\mathbb{F}_n)$, duals of the fundamental classes of the Schubert varieties:*

$$H^*(\mathbb{F}_n) = \bigoplus_{w \in \mathcal{S}_n} \mathbb{Z}\sigma_w.$$

Just as in the case of Grassmannians, one may then deduce a remarkable factorization of the Poincaré polynomial

$$P_q(\mathbb{F}_n) = \sum_{k \geq 0} q^k \operatorname{rank} H^{2k}(\mathbb{F}_n).$$

COROLLARY 3.6.9. *The Poincaré polynomial of the variety of complete flags has the following expression:*

$$P_q(\mathbb{F}_n) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}.$$

PROOF. By reasoning exactly as in the case of Grassmannians, which was the object of corollary 3.2.5, one checks that for all integers q equal to at least two,

$$P_q(\mathbb{F}_n) = \#\mathbb{F}_n(\mathbb{K}_q)$$

is equal to the number of complete flags in a vector space of dimension n over the field $\mathbb{K}_q$ with q elements. This number is the quotient of the number of bases $v_1, \dots, v_n$ of $\mathbb{K}_q^n$, namely $(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$, by the number of bases compatible with a given flag. The bases compatible with the flag defined by $v_1, \dots, v_n$ are those formed by vectors $w_i = \sum_{j\le i} t_{ij}v_j$, with $t_{ii} \neq 0$, and that number is then equal to the product $(q-1)q(q-1)\cdots q^{n-1}(q-1)$. The corollary then follows. $\square$

Just as we did for Grassmannians, we study the multiplicative structure of the cohomology ring of $\mathbb{F}_n$ by introducing *dual* Schubert cells and varieties. For this we consider the dual flag $V'_\bullet$, and set

$$\Omega'_w = \{W_\bullet \in \mathbb{F}_n,\ \ \dim(W_p \cap V'_q) = r_{w_0w}(p,q),\ \ 1 \le p, q \le n\}.$$

We denote the closure of this Schubert cell by X'_w.

We have isomorphisms $\Omega'_w \simeq \mathbb{C}^{n(n-1)/2-l(w)}$, which are obtained for the cells Ω_w by verifying that a flag $W_\bullet \in \Omega'_w$ is defined by a unique matrix such that the entries with coordinates $(i, w(i))$ are equal to 1, the entries to the left or below these are equal to zero, and the remaining entries are undetermined.

EXAMPLE 3.6.10. If $w = 365142 \in \mathcal{S}_6$, an element of Ω'_w is defined by a unique matrix of the form

$$\begin{pmatrix} 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The connected group $\mathrm{GL}(n,\mathbb{C})$ acts transitively on flags. It then also acts transitively on the set of Schubert varieties associated to different flags for the same permutation. The varieties X'_w and X_{w_0w} then have the same fundamental class σ_w. As in the case of Grassmannians, we being by establishing a duality result.

PROPOSITION 3.6.11. *If the permutations $u, v \in \mathcal{S}_n$ have the same length, then*

$$\sigma_u \cup \sigma_{w_0v} = \delta_{u,v}.$$

PROOF. We show first that given two permutations u and v, the intersection $\Omega_u \cap \Omega'_v$ may be nonempty only if $v \le u$. In order to do so we consider a flag $W_\bullet$ in this intersection, if there exists one. It is simultaneously defined by two matrices U and V such that the first has zero entries to the right of the points with coordinates $(i, u(i))$ and the second has zero entries to the left of the points $(j, v(j))$. For each integer k, we define the integer $i_{1,k}$ by the equality

$$u(i_{1,k}) = \min(u(1), \dots, u(k)).$$

In order that the row vector of index $i_{1,k}$ of U be a linear combination of the first k rows of V, it is necessary that

$$u(i_{1,k}) \ge \min(v(1), \dots, v(k)) = v(j_{1,k}).$$

More generally, we define integers $i_{h,k}$ by the recurrence relation

$$u(i_{h,k}) = \min\{u(1), \ldots, u(k)\}\backslash\{u(i_{1,k}), \ldots, u(i_{h-1,k})\}.$$

The integers $j_{h,k}$ are similarly defined for the permutation v. One then verifies in a similar fashion that $u(i_{h,k}) \geq v(j_{h,k})$, for if not, the first h rows of U will be linear combinations of the rows of V with indices $j_{l,k}$ for $l < h$, and therefore cannot be independent. Then these inequalities may be rewritten in terms of keys. Indeed, the kth column of $K(u)$ consists of the entries $u(i_{1,k}), \ldots, u(i_{k,k})$, in this order from top to bottom. The preceding inequalities may then be written as $K(v) \leq K(u)$, and are equivalent to $u \leq v$ by proposition 2.1.11 of the second chapter. In particular, if $\Omega_u \cap \Omega'_v \neq \varnothing$, we must have $l(u) \leq l(v)$, and in the case of equality, necessarily $u = v$. But

$$X_u \cap X'_u = \Omega_u \cap \Omega'_u = \{W^u_\bullet\},$$

and this intersection is clearly transverse. □

In the same way that in the Grassmannian the multiplication of Schubert classes is entirely determined by Pieri's formula, the multiplicative structure of $H^*(\mathbb{F}_n)$ is determined by Monk's formula, analogous to proposition 3.2.8. The dual Schubert variety corresponding to a simple transposition is

$$X'_{s_i} = \{W_\bullet \in \mathbb{F}_n, \quad W_i \cap V'_i \neq 0\}.$$

This is the inverse image under the natural projection of $\mathbb{F}_n$ on $\mathbb{G}_{i,n-i}$ of a Schubert variety of codimension one. We simply denote by σ_i its fundamental class in $\mathbb{G}_{i,n-i}$.

MONK'S FORMULA 3.6.12. *For all permutations $w \in \mathcal{S}_n$, and all integers $i < n$, we have*

$$\sigma_w \cup \sigma_i = \sum_{\substack{j \leq i < k, \\ l(wt_{jk}) = l(w)+1}} \sigma_{wt_{jk}}.$$

PROOF. By the preceding proposition, we must verify that for a permutation v of length $l(w)+1$, we have $\sigma_{w_0 v} \cup \sigma_w \cup \sigma_i = 1$ if $v = wt_{jk}$ with $j \leq i < k$, and that $\sigma_{w_0 v} \cup \sigma_w \cup \sigma_i = 0$ otherwise. Therefore we consider the intersection of the varieties X_v and X'_w.

If this intersection is nonempty, we must have that $w \leq v$, and in this case, this intersection consists of the flags $W^v_\bullet$ and $W^w_\bullet$, and of the intersection of the open cells $\Omega_v \cap \Omega'_w$. We then obtain precisely the flags $W_\bullet$ defined by a basis $f_1, \ldots, f_n$ such that $f_l = e_l$ if $l \neq j, k$, and such that f_j and f_k are linear combinations of e_j and e_k. In particular, $W_l = W^w_l$ if $l < j$ or $l \geq k$. A generic Schubert variety

$$X_{s_i}(L) = \{W_\bullet \in \mathbb{F}_n, \quad W_i \cap L \neq 0\},$$

where L is a subspace of dimension $n - i$, does not then cut $X_v \cap X'_w$ if $i < j$ or $i \geq k$. And this implies that $\sigma_{w_0 v} \cup \sigma_w \cup \sigma_i = 0$. On the other hand, if $j \leq i < k$, and for L generic, $X_v \cap X'_w$ cuts $X_{s_i}(L)$ transversally at a unique point. Therefore $\sigma_{w_0 v} \cup \sigma_w \cup \sigma_i = 1$. □

Recall that we denoted by $\mathcal{P}_n$ the ring of polynomials with integral coefficients in n variables. By proposition 2.5.4 of the second chapter, $\mathcal{P}_n$ has a base consisting of the Schubert polynomials $\mathfrak{S}_w$, where $w \in \mathcal{S}_\infty$ runs through the set of permutations such that the largest descent is less that or equal to n—a set which contains $\mathcal{S}_n$.

COROLLARY 3.6.13. *The map* $j_n : \mathcal{P}_n \to H^*(\mathbb{F}_n)$, *defined by*

$$j_n(\mathfrak{S}_w) = \begin{cases} \sigma_w & \text{if } w \in \mathcal{S}_n, \\ 0 & \text{otherwise,} \end{cases}$$

is a surjective morphism of rings.

PROOF. By Monk's formula, j_n is certainly a morphism of rings if one restricts it to products of Schubert polynomials associated to simple transpositions. But since $x_i = \mathfrak{S}_{s_i} - \mathfrak{S}_{s_{i-1}}$, every polynomial is a linear combination of such products. □

COROLLARY 3.6.14. *If* $u, v \in \mathcal{S}_n$, *and if in* $\mathcal{P}_n$ *we decompose the product of the corresponding Schubert polynomials as follows,*

$$\mathfrak{S}_u \times \mathfrak{S}_v = \sum_{w \in \mathcal{S}_\infty} c^w_{u,v} \mathfrak{S}_w,$$

then in the cohomology ring $H^*(\mathbb{F}_n)$, *we have the identity*

$$\sigma_u \cup \sigma_v = \sum_{w \in \mathcal{S}_n} c^w_{u,v} \sigma_w.$$

An immediate consequence of this last corollary is that the coefficients $c^w_{u,v}$ may be interpreted as the intersection multiplicities of certain Schubert varieties, and are then nonnegative integers—something we do not know how to prove combinatorially! In the preceding corollary, however, the presentation of the cohomology ring of the flag variety as a quotient of a polynomial ring rests on a simple formal analogy. The object of the following section is to give this some geometric content.

3.6.4. The Borel Presentation and Schubert Polynomials. Instead of deducing the cohomology ring of the flag variety from its cellular decomposition defined by Schubert cells, we may determine the ring by remarking that a flag variety may be constructed with the help of a chain of fibrations of projective spaces, and that we can handle such a fibration with proposition 3.5.3.

We note first that over $\mathbb{F}_n$, we may consider an element of dimension i of a flag $W_\bullet$ as the fiber of a tautological bundle $\mathcal{W}_i$ of rank i. Hence we have quotient line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$, where $\mathcal{L}_i = \mathcal{W}_i/\mathcal{W}_{i-1}$. We denote by $x_i = -c_1(\mathcal{L}_i)$ the negative of the first Chern class. This allows us to define a new morphism $k_n : \mathcal{P}_n \to H^*(\mathbb{F}_n)$ and to obtain the *Borel presentation* of the cohomology ring of $\mathbb{F}_n$:

PROPOSITION 3.6.15. *The morphism* k_n *is surjective, with kernel the ideal* $I\Lambda_n$ *of* $\mathcal{P}_n$ *generated by the homogeneous elements of* Λ_n *of strictly positive degree. Hence we have an isomorphism of rings*

$$H^*(\mathbb{F}_n) = \mathcal{P}_n / I\Lambda_n.$$

PROOF. We denote by Y_i the space of flags of $\mathbb{C}^n$ with the form

$$0 = W_0 \subset W_1 \subset \cdots \subset W_i \subset \mathbb{C}^n.$$

We denote by $\mathcal{W}_j$, for $j \leq i$, the tautological bundles defined on this variety, and set $x_j = -c_1(\mathcal{W}_j/\mathcal{W}_{j-1})$. Then $Y_1 = \mathbb{P}^{n-1}$, $Y_{n-1} = \mathbb{F}_n$, and as a bundle on Y_{i-1}, the variety Y_i may be identified with $\mathbb{P}(\mathcal{W}_n/\mathcal{W}_{i-1})$. This allows us to express $H^*(Y_i)$ as the $H^*(Y_{i-1})$-module generated by x_i, modulo a relation which requires that the

Chern class of degree $n-i+1$ of the bundle $\mathcal{W}_n/\mathcal{W}_i$, which is of rank $n-i$, be zero. Then the exact sequences

$$0 \to \mathcal{L}_i \to \mathcal{W}_n/\mathcal{W}_{i-1} \to \mathcal{W}_n/\mathcal{W}_i \to 0$$

give, by induction,

$$c(\mathcal{W}_n/\mathcal{W}_i) = \prod_{j\leq i}(1+x_j)^{-1} = \sum_{k\geq 0}(-1)^k h_k(x_1,\dots,x_i).$$

Finally, we then obtain the cohomology of $\mathbb{F}_n$ as the quotient of $\mathcal{P}_n$ by the relations $h_i(x_1,\dots,x_{n+1-i}) = 0$, for $1 \leq i \leq n$. But the ideal generated by these relations is exactly $I\Lambda_n$, which it is easy to verify beginning with the recurrence relations

$$h_i(x_1,\dots,x_n) = h_i(x_1,\dots,x_{n+1-i}) + \sum_{j<i} x_{n+1-j}h_{i-1}(x_1,\dots,x_{n+1-j}).$$

The proposition then follows. □

REMARK 3.6.16. The preceding proposition gives a description of the cohomology of a flag variety very analogous to that of the cohomology of the Grassmannian which was signaled in the second exercise after corollary 3.2.10. More generally, given a strictly increasing sequence of integers $m = (0 = m_0 < \cdots < m_h = n)$, one may define a *partial (or incomplete) flag variety* $\mathbb{F}_{m_\bullet}$: these are the chains of subspaces of $\mathbb{C}^n$ which have their dimensions specified by the sequence $m_\bullet$. Set $n_i = m_i - m_{i-1}$, $1 \leq i \leq h$. One may then verify that

$$H^*(\mathbb{F}_{m_\bullet}) \simeq \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_h}/I\Lambda_n.$$

Moreover, we may determine the Gysin morphisms associated to the different projections $\pi_{m_\bullet} : \mathbb{F}_n \to \mathbb{F}_{m_\bullet}$. Under the preceding identification

$$\pi_{m_\bullet} = \partial_{w_{m_\bullet}}$$

is the divided difference associated to the permutation $w_{m_\bullet} = w_{0,n_1} \times \cdots \times w_{0,n_h}$, the product of the permutations of maximal length of the different symmetric groups $\mathcal{S}_{n_1},\dots,\mathcal{S}_{n_h}$. We may then deduce, for example, the cohomology of the Grassmannians from that of the complete flags: for the corresponding projection, the Gysin morphism gives the Schubert varieties of the Grassmannian as the image of the Schubert varieties of the flag variety associated to the Grassmannian permutations—and it vanishes on the Schubert varieties of the non-Grassmannian permutations.

The quotient $\mathcal{P}_n/I\Lambda_n$ is isomorphic, by proposition 2.5.3 of the second chapter, to the submodule $\mathcal{H}_n$ of $\mathcal{P}_n$ generated by the Schubert polynomials associated to the permutations of $\mathcal{S}_n$, or again by the monomials x^α with $\alpha \subset \delta$.

COROLLARY 3.6.17. *The monomials x^α with $\alpha \subset \delta$, that is, with $\alpha_j \leq n-j$, in the Chern classes $x_i = -c_1(\mathcal{L}_i)$, form a base of $H^*(\mathbb{F}_n)$.*

Having identified the cohomology ring $H^*(\mathbb{F}_n)$ with a ring in which the Schubert polynomials naturally "live," we would certainly like to know how these compare to the fundamental classes of Schubert varieties.

THEOREM 3.6.18. *For all permutations $w \in \mathcal{S}_n$, the image of the Schubert polynomial $\mathfrak{S}_w$ in $H^*(\mathbb{F}_n) \simeq \mathcal{P}_n/I\Lambda_n \simeq \mathcal{H}_n$ coincides with the Schubert class σ_w.*

PROOF. It suffices to prove that for all permutations $w \in \mathcal{S}_n$, and all integers $i < n$, we have

$$\partial_i(\sigma_w) = \begin{cases} \sigma_{ws_i}, & \text{if } w(i) > w(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathbb{F}_n^i$ the variety formed by pairs of flags $(W_\bullet, W'_\bullet)$ for which $W_j = W'_j$ if $j \neq i$. The two natural projections of $\mathbb{F}_n^i$ on $\mathbb{F}_n$ are fibrations of projective lines: more precisely, we have the identification $\mathbb{F}_n^i = \mathbb{P}(\mathcal{W}_{i+1}/\mathcal{W}_{i-1})$. We then denote by δ_i the composition

$$\delta_i := H^*(\mathbb{F}_n) \xrightarrow{p_2^*} H^*(\mathbb{F}_n^i) \xrightarrow{p_{1*}} H^*(\mathbb{F}_n).$$

The statement is then a consequence of the two following lemmas. □

LEMMA 3.6.19. *Let $u \in \mathcal{S}_n$. If $u(i) < u(i+1)$, then the image of $p_2^{-1}(X_u)$ is contained in X_u. On the other hand, if $u(i) > u(i+1)$, and if we denote by Δ the diagonal of $\mathbb{F}_n^i$, then p_1 realizes an isomorphism of $p_2^{-1}(\Omega_u)\backslash\Delta$ with Ω_{us_i}. Consequently,*

$$\delta_i(\sigma_u) = \begin{cases} \sigma_{us_i} & \text{if } u(i) > u(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

PROOF OF THE LEMMA. Let $W'_\bullet$ be a flag, and $w'_1, \dots, w'_n$ a basis which respects the flag. An element of $p_1(p_2^{-1}(W'_\bullet))$ is a flag $W_\bullet$ such that all of its members coincide with those of $W'_\bullet$, except perhaps in dimension i. Such a flag then has an adapted basis $w_1, \dots, w_n$, with $w_j = w'_j$ if $j \neq i, i+1$, and w_i, w_{i+1} linear combinations of w'_i, w'_{i+1}. More precisely, if $W_\bullet \neq W'_\bullet$, we may take $w_i = w'_{i+1} + a_i w'_i$, and $w_{i+1} = w'_i$.

Suppose then that $W'_\bullet \in \Omega_u$. If $u(i) < u(i+1)$, an elementary verification shows that a flag of the preceding form satisfies the incidence conditions which define X_u. On the other hand, if $u(i) < u(i+1)$, we choose local coordinates for $W'_\bullet$ over $\mathbb{F}_n$, which allow us to represent a flag by an invertible $n \times n$ matrix, as we did in the preceding section. We pass from $W'_\bullet$ to $W_\bullet \in p_1(p_2^{-1}(W'_\bullet) - \Delta)$ by modifying the rows i and $i+1$ in the following fashion:

$$\begin{array}{ccccccc} & u(i) & & & & u(i+1) & \\ i & 1 & 0 & \cdots & 0 & 0 & \cdots \\ i+1 & 0 & * & \cdots & * & 1 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccccc} & u(i) & & & & u(i+1) & \\ i & * & * & \cdots & * & 1 & 0 \\ i+1 & 1 & 0 & \cdots & 0 & 0 & \cdots \end{array}$$

We then obtain very precisely the description of the matrices which represent the flags of the open cell Ω_{us_i}. □

LEMMA 3.6.20. *For all $i < n$, we have $\delta_i = \partial_i$ as operators on $\mathcal{H}_n$.*

PROOF OF THE LEMMA. Recall that its projections on $\mathbb{F}_n$ identify $\mathbb{F}_n^i$ with the variety of lines of the bundle of rank two $\mathcal{W}_{i+1}/\mathcal{W}_{i-1}$. If we introduce the variety

$\mathbb{G}_n^i$ of complete flags with the exception of the term in dimension i, we have a commutative diagram

$$\begin{tikzcd} & \mathbb{F}_n^i \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{F}_n \arrow[dr, "q"'] & & \mathbb{G}_n. \arrow[dl, "q"] \\ & \mathbb{G}_n^i & \end{tikzcd}$$

Since they play a symmetric role in proposition 3.6.15, we may permute the variables in the statement of corollary 3.6.17: if one chooses for the two last variables x_i and x_{i+1}, it follows that every element of $H^*(\mathbb{F}_n)$ may be written $P+x_iQ$, where P and Q are polynomials in which neither x_i nor x_{i+1} appears. This implies that we may write $P = q^*P'$ and $Q = q^*Q'$, where P' and Q' are cohomology classes of $\mathbb{G}_n^i$, and the commutativity of the diagram above assures that $p_2^*P = p_1^*P$ and $p_2^*Q = p_1^*Q$.

Consider then the inverse image $p_2^*(P + x_iQ)$. Relative to the projection p_1, $p_2^*x_i$ is the negative of the first Chern class of the tautological line bundle. We then know the action of the Gysin morphism—which we made explicit following proposition 3.5.3. In this situation,

$$p_{1*}p_2^*(P + x_iQ) = Q = \partial_i(P + x_iQ).$$

The lemma, and the theorem at the same time, are therefore proved. □

REMARK 3.6.21. Consider the natural embeddings

$$i_n \colon \mathbb{F}_n \hookrightarrow \mathbb{F}_{n+1}.$$

If $w \in \mathcal{S}_n$, we may consider this permutation in $\mathcal{S}_{n+1}$ and associate to it a Schubert class that we denote $\sigma_{w\times 1} \in H^*(\mathbb{F}_{n+1})$. We may then verify that if $v \in \mathcal{S}_{n+1}$, then

$$i_n^*(\sigma_v) = \begin{cases} \sigma_w & \text{if } v = w \times 1, \\ 0 & \text{if } v \text{ does not come from } \mathcal{S}_n. \end{cases}$$

Moreover, we obtain a morphism $p_n \colon \mathcal{H}_{n+1} \to \mathcal{H}_n$ by sending x_{n+1} to zero and without touching the other variables. Hence we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}_{n+1} \arrow[r] \arrow[d, "p_n"'] & H^*(\mathbb{F}_{n+1}) \arrow[d, "i_n^*"] \\ \mathcal{H}_n \arrow[r] & H^*(\mathbb{F}_n). \end{tikzcd}$$

One may then show that for a permutation $w \in \mathcal{S}_\infty$ for which the greatest descent is equal to k, the Schubert polynomial $\mathfrak{S}_w$ is *the unique polynomial* $P_w \in \mathcal{P}_k$ such that the image in $\mathcal{H}_n$ coincides, for all sufficiently large integers n, with the Schubert class $\sigma_w \in H^*(\mathbb{F}_n)$—where for n large enough, we consider w as an element of $\mathcal{S}_n$. The simple Schubert polynomials may then be characterized as the only polynomials which, for all sufficiently large n, give the cohomology class of the corresponding Schubert varieties in the flag varieties $\mathbb{F}_n$.

REMARK 3.6.22. The cohomology ring $H^*(\mathbb{F}_n) \simeq \mathcal{H}_n$ is naturally endowed with an action of the symmetric group $\mathcal{S}_n$, acting in $\mathcal{H}_n$ by permutation of variables. We saw in corollary 2.5.8 that this representation is isomorphic to the regular representation of $\mathcal{S}_n$. In particular, its character has the expression

$$\chi(H^*(\mathbb{F}_n)) = \sum_{|\lambda|=n} K_\lambda \chi^\lambda.$$

Moreover, this action respects the natural grading of $H^*(\mathbb{F}_n)$ by the degree. One may then show that the characters of the action of $\mathcal{S}_n$ on each of the components of this graded ring are given by the identity

$$\sum_k q^k \chi(H^{2k}(\mathbb{F}_n)) = \sum_{|\lambda|=n} K_{\lambda,1^n}(q)\chi^\lambda,$$

where $K_{\lambda,1^n}$ is a Kostka-Foulkes polynomial. It is similarly possible to obtain all of the Kostka-Foulkes polynomials by a variant of this construction: in place of considering all flag varieties, one considers only those which are fixed by a given unipotent (that is, such that all of its eigenvalues are equal to one) endomorphism. The Jordan decomposition shows that the conjugacy classes of these endomorphisms, under the action of the linear group, are in correspondence with the partitions of size n. If we fix such a partition μ, the cohomology ring of the corresponding flag variety $\mathbb{F}_\mu$ again has an action of the symmetric group, and its graded character is given by the Kostka-Foulkes polynomials [**27**]:

$$\sum_k q^k \chi(H^{2k}(\mathbb{F}_\mu)) = \sum_{|\lambda|} K_{\lambda\mu}(q)\chi^\lambda.$$

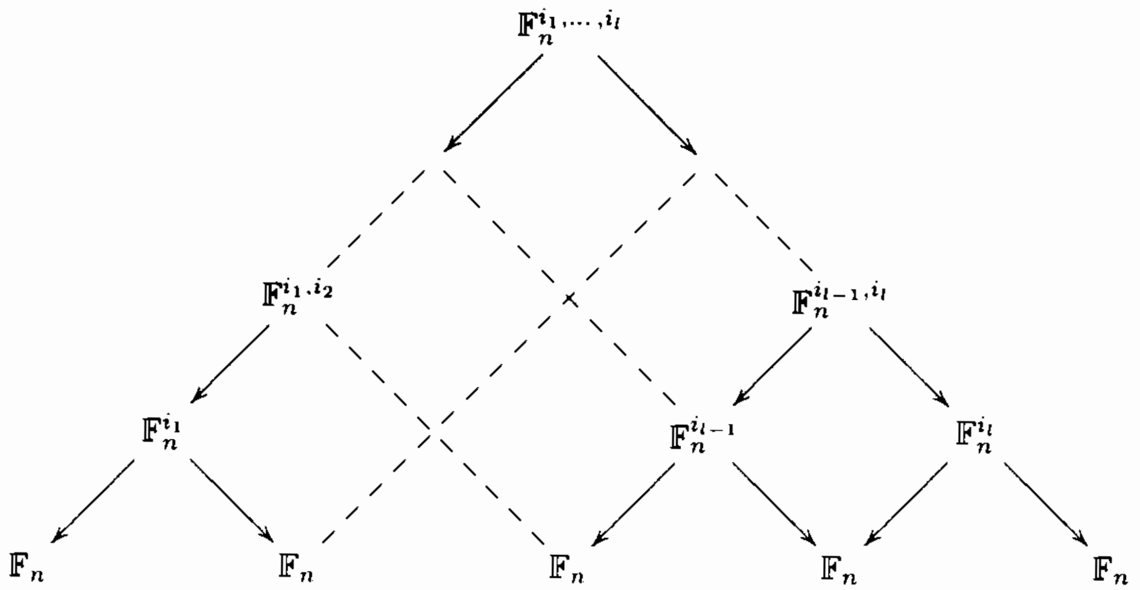
3.7. Singularities of Schubert Varieties, *reprise*

We now characterize, as we did in the case of Grassmannians, the permutations which correspond to nonsingular Schubert varieties in complete flag varieties, and we will attempt to describe the singular loci of those which do not. We begin, however, by first giving a simple construction of desingularizations of Schubert varieties.

3.7.1. Desingularizations of Schubert Varieties. A simple construction of desingularizations of Schubert varieties follows directly from lemma 3.6.19. Indeed, we introduce the variety

$$\mathbb{F}_n^{i_1,\dots,i_l} = \{W_\bullet^j \in \mathbb{F}_n,\ \ W_k^j = W_k^{j-1} \ \text{ if }\ k \neq i_j,\ \ 1 \leq j \leq l\}.$$

We denote by $p_0, \dots, p_l$ the $l+1$ projections of this variety to $\mathbb{F}_n$. We use the commutative diagrams



PROPOSITION 3.7.1. *Let $u \in S_n$ be a permutation, and suppose that*

$$l(us_{i_1}\cdots s_{i_l}) = l(u) + l.$$

Then the projection p_l defines a birational morphism from the variety $p_0^{-1}(X_u)$ to the Schubert variety $X_{us_{i_1}\cdots s_{i_l}}$.

PROOF. For $l = 1$, this is none other than lemma 3.6.19. The induction on l is then immediate, since each of the commutative squares of the diagram above is a cartesian diagram of fibrations of projective lines. For $l = 2$ for example, knowing that the inverse image of X_u in $\mathbb{F}_n^{i_1}$ is sent birationally to $X_{us_{i_1}}$, we have that the inverse image of X_u in $\mathbb{F}_n^{i_1,i_2}$ is also sent birationally onto the inverse image of $X_{us_{i_1}}$ in $\mathbb{F}_n^{i_2}$. And this last variety is itself sent birationally to the Schubert variety $X_{us_{i_1}s_{i_2}}$. □

If the Schubert variety X_u is smooth, then p_0^{-1} is smooth as well, since it is obtained from X_u by a sequence of fibrations of projective lines. In particular, if u is the identity permutation, then X_u reduces to the flag of reference $V_\bullet$ and hence is smooth. Now set $Z_{i_1,\dots,i_l} = p_0^{-1}(V_\bullet)$, that is,

$$Z_{i_1,\dots,i_l} = \{W_\bullet^j \in \mathbb{F}_n^{i_1,\dots,i_l},\quad W_k^0 = V_k,\quad 1 \le k \le n\}.$$

COROLLARY 3.7.2. *If $s_{i_1}\cdots s_{i_l}$ is a reduced decomposition of the permutation w, then the last projection*

$$p_l\colon Z_{i_1,\dots,i_l} \to X_w$$

defines a desingularization of the Schubert variety X_w.

This construction is a particular case of that of *Bott-Samelson schemes* **[36, 10, 25]**. We note that, contrary to the case of Grassmannians, one does not know how to construct small resolutions of Schubert varieties in complete flag varieties.

3.7.2. Singular Loci of Schubert Varieties. As we did in the case of Grassmannians, we now determine the singular Schubert varieties in complete flag varieties, and attempt to describe their singular locus. The strategy is the same, to translate into combinatorial terms the jacobian criterion of smoothness 3.4.1.

Since a Schubert variety is stable under the action of the group B which fixes the flag of reference, so is its singular locus, which must be a union of Schubert varieties:

$$\mathrm{Sing}(X_w) = \bigcup_{v \in S(w)} X_v.$$

The singular locus being closed, the set $S(w)$ is an ideal under the Bruhat order: if a permutation v belongs to it, all the permutations u such that $u \leq v$ also belong to it. Clearly, $S(w)$ consists of permutations v such that $v \leq w$. Finally, since it is homogeneous under the action of B, a Schubert cell Ω_v with $v \leq w$ is contained in the singular locus of X_w if and only if its point of reference $W_\bullet^v$ is a singular point.

In a neighborhood of $W_\bullet^v$, for each flag $W_\bullet$, there exists a unique matrix $(x_{ij})_{1 \leq i,j \leq n}$, the first h rows of which generate W_h for each h from 1 to n, and which is such that

$$x_{i,v(i)} = 1, \quad \text{and} \quad x_{i,j} = 0 \ \text{ if } \ i > v^{-1}(j).$$

The indeterminates x_{ij}, for $1 \leq i \leq n$ and $j \neq v(1), \ldots, v(i)$ then form a system of local coordinates in a neighborhood of $W_\bullet^v$. We denote by e_{ij} the basis corresponding to the cotangent space of the flag variety at the given point, and we set $f_{ij} = e_{i,v(j)}$ for $i < j$.

The Schubert variety X_w is defined by the family of incidence conditions

$$\dim(W_p \cap V_q) \geq r_w(p,q),$$

or also

$$\dim(W_p + V_q) \leq p + q - r_w(p,q),$$

for $1 \leq p, q \leq n$. Such a condition may be expressed as the vanishing of minors of order $p + q - r_w(p,q) + 1$ of the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ x_{1,1} & x_{1,2} & \cdots & x_{1,q} & \cdots & x_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{p,1} & x_{p,2} & \cdots & x_{p,q} & \cdots & x_{p,n} \end{pmatrix}.$$

It is known that, as in the case of Grassmannians, these minors generate the ideal of X_w.[8] Their vanishing is equivalent to the vanishing of the minors of order $p - r_w(p,q) + 1$ of the matrix formed from the last p rows and the $n - q$ last columns of the preceding matrix. Then this matrix has exactly $p - r_v(p,q)$ entries equal to one, which are placed in pairwise distinct rows and columns.

But $v \leq w$ by hypothesis, therefore $r_v \geq r_w$ by proposition 2.1.12 from chapter two. Then there exist minors of order $p - r_w(p,q) + 1$ for which the derivative is nonzero at the origin if and only if $r_w(p,q) = r_v(p,q)$. Moreover, these minors are obtained by adding a column, say with index i, and a row, with index j, to the minor of order $p - r_w(p,q)$ where these 1's were placed. And the only nonzero

[8] See, for example, V. Lakshmibai V. and C. S. Seshadri, *Geometry of G/P. V*, J. Algebra **100** (1986), 462–557.

derivative of this minor is obtained with respect to the variable x_{ij}, which must satisfy the following conditions:

$$i \leq p, \quad j > q, \quad v(i) \leq q, \quad v^{-1}(j) > p.$$

The cotangent subspace of the flag variety, generated by the first derivatives at the point $W_\bullet^v$ of the elements of the ideal of X_w, is then generated by the vectors f_{ij} such that there exist integers p, q with $i \leq p < j$ and $v(i) \leq q < v(j)$ for which $r_w(p,q) = r_v(p,q)$. But if the pair (p,q) satisfies these inequalities, we have $r_{vt_{ij}}(p,q) = r_v(p,q) + 1$, and $r_{vt_{ij}}(p,q) = r_v(p,q)$ otherwise. The preceding condition is then precisely equivalent to the relation $r_w \geq r_{vt_{ij}}$. We have then established the following result:[9]

PROPOSITION 3.7.3. *The Schubert variety X_w is singular on the subvariety X_v, where $v \leq w$, if and only if*

$$l(w) < m(w,v) := \#\{i < j, \quad vt_{ij} \leq w\}.$$

In particular, we may derive from this result a smoothness criterion for Schubert varieties. Indeed, since the singular locus of X_w is closed, it is empty if and only if it does not contain the flag of reference, which corresponds to $v = \mathrm{id}$. Then X_w is smooth if and only if

$$l(w) = m(w) := \#\{i < j, \quad t_{ij} \leq w\}.$$

If we set $p_w(i) = \min(\max(w(1), \ldots, w(i)), \max(w^{-1}(1), \ldots, w^{-1}(i)))$, we easily check that the equality

$$m(w) = \#\{i < j \leq p_w(i)\}$$

is satisfied.

EXERCISE 3.7.4. Show that all of the Schubert varieties X_w are smooth for $w \in \mathcal{S}_3$. For $w \in \mathcal{S}_4$, verify that the only singular Schubert varieties are obtained from the permutations $w = 3412$ and $w = 4231$, with

$$\mathrm{Sing}(X_{3412}) = X_{1324}, \qquad \mathrm{Sing}(X_{4231}) = X_{2143}.$$

It is remarkable that the preceding criterion, which hardly appears useful, may be transformed into an astonishingly simple condition.[10]

THEOREM 3.7.5. *The Schubert variety X_w is smooth if and only if there does not exist a quadruple of integers $i < j < k < l$ such that*

$$w(l) < w(j) < w(k) < w(i) \quad \text{or} \quad w(k) < w(l) < w(i) < w(j).$$

We will speak of a configuration of type (a) and of type (b), respectively. In particular, if X_w is smooth, the nonexistence of a configuration of type (b) means, by proposition 2.2.8 of the second chapter, that the permutation $w_0 w$ is vexillary.

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[9]The integer $m(w,v)$ is none other than the dimension of the Zariski tangent space of X_w at the point $W_\bullet^v$.

[10]In the case of Grassmannians, we have seen that the only smooth Schubert varieties are the subgrassmannians. For complete flag varieties, one can show that the smooth Schubert varieties may be constructed by series of fibrations of Grassmannians [**78**].

PROOF. We begin with some preliminaries and with a procedure for reducing w into simpler permutations. If $f = w^{-1}(1)$, we may write $w = vs_1 \dots s_{f-1}$, where v is a permutation of length $l(v) = l(w) - f + 1$, such that $v(1) = 1$. The graph of v is deduced from that of w by replacing the fth row by the first, and by moving the rows of index smaller than f up a row. In particular, if v has a configuration of type (a) or (b), then w must also.

Further observe that $w \geq t_{ij}$, where $i < j$, if and only if the rectangles

$$R_{ij}^{+} = \{(p, q),\ \ p \leq i,\ q \geq j\} \quad \text{and} \quad R_{ij}^{-} = \{(p, q),\ \ p \geq j,\ q \leq i\}$$

each contain at least one point of the graph of w. Then if $1 < j \leq f$, there exists a unique integer $i(j) < j$ such that $w(i(j)) > j$ but $j > w(l)$ for $l < i(j)$. And this implies that the transposition $t_{i(j),j}$ is dominated by w, but not by v. Indeed, the rectangle $R_{i(j),j}^{+}$ contains a unique point of the graph of w, which is removed when we pass to v. We say that such a transposition is of type R^{+} for w. In particular, the existence of these transpositions implies that

$$m(w) - m(v) \geq f - 1 = l(w) - l(v),$$

which permits by induction reaching the *a priori* inequality $m(w) \geq l(w)$.

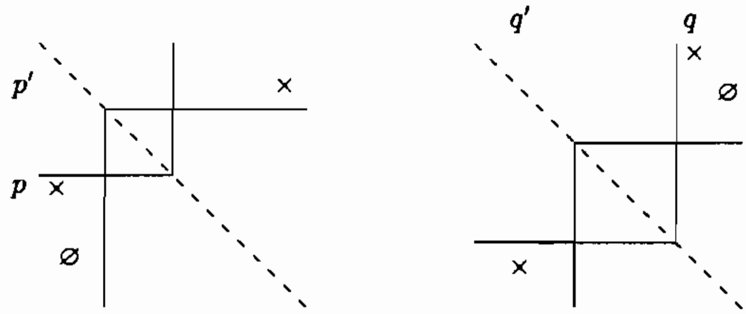
Of course, we may also reason the same with w^{-1}. If $g = w(1)$, it is possible to write $w = s_1 \cdots s_{g-1}u$, with u of length $l(u) = l(w) - g + 1$, and $u(1) = 1$. We derive again that $m(w) - m(u) \geq l(w) - l(u)$, reasoning as above, up to a diagonal symmetry.

We now prove the theorem by induction. Suppose first that there exists a quadruple of integers $i < j < k < l$ which is a configuration of type (a) or (b) for w. In the first case, we have

$$m(w) = m(wt_{jk}) \geq l(wt_{jk}) > l(w),$$

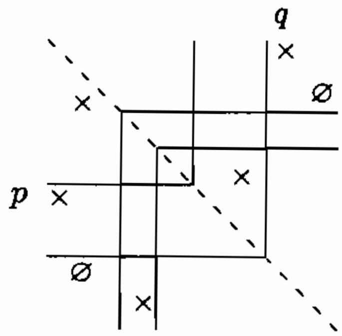
since $p_{wt_{jk}} = p_w$. The Schubert variety X_w is then singular. In the second case, up to applying the reductions above a certain number of times, we may suppose that $i = w(k) = 1$. Moreover, suppose for example that $w(1) \geq w^{-1}(1) = k$. Then $t_{2,k}$ is dominated by w but not by v (except if $w(k-1) = 2$, in which case one may replace k by $k-1$ and reason again). Since $t_{2,k}$ is not of type R^{+}, this implies that $m(w) - m(v) > l(w) - l(v)$, hence that $m(w) > l(w)$ and X_w is again singular.

Conversely, suppose that X_w is singular and make the preceding reductions: write $w = vs_1 \cdots s_{f-1}$ and $w = s_1 \cdots s_{g-1}u$. Our inductive reasoning allows us to suppose that X_v and X_u are both smooth, hence that $m(v) = l(v)$ and $m(u) = l(u)$. Set $f = w^{-1}(1), g = w(1)$, and suppose for example that $f \leq g$. There exists then a transposition t_{kf} which is not of type R^{+} for w, so a transposition t_{lg} which is not of type R^{+} for w^{-1}. Graphically, this corresponds to the following configurations in the graph of w:



Suppose that the points of the graph which appear in these diagrams are pairwise distinct. By superposing these two figures, we obtain a configuration of type (b).

In the other case, the two points lying at the top right of the two diagrams are merged, but not the two others. We then examine $w(p-1)$. By hypothesis, it is greater than q'. If it is greater than q, we obtain a configuration of type (b), and one may then suppose that $q' < w(p-1) < q$. But then, since necessarily $w^{-1}(2) < p-1$, the point $(w^{-1}(2), 2)$ makes a configuration of type (a) appear:



The theorem is proved. □

3.8. Degeneracy Loci and Schubert Polynomials

We now consider Schubert varieties no longer of a simple complex vector space, but of a bundle. More generally, we study the degeneracy loci of morphisms between vector bundles endowed with flags. And we generalize the Thom-Porteous formula by showing that, if there is no pathology, the fundamental class of such a locus is given by a certain Schubert polynomial. This is Fulton's Theorem.

3.8.1. Cohomology of Relative Flag Varieties. Let E be a complex vector bundle of rank e on a compact differentiable variety X. We denote by $\mathbb{F}(E)$ the flag variety relative to this bundle, and by π its natural projection on X. This is a fibration of ordinary flag varieties $\mathbb{F}_e$, the fiber above a point $x \in X$ being the variety of complete flags of the fiber corresponding to E,

$$0 = W_0 \subset W_1 \subset \cdots \subset W_e = E_x.$$

As in the case of ordinary flag varieties—which corresponds to the case where X is a point—one may consider each W_i as defining a tautological bundle $\mathcal{W}_i$ on $\mathbb{F}(E)$. Set $x_i = c_1(\mathcal{W}_i/\mathcal{W}_{i-1})$. Proposition 3.6.15 extends in the following way:

PROPOSITION 3.8.1. *As an $H^*(X)$-module, the cohomology ring $H^*(\mathbb{F}(E))$ is the quotient of $H^*(X)[x_1, \dots, x_e]$ by the relations*

$$e_k(x_1, \dots, x_e) = c_k(E), \quad 1 \le k \le e.$$

Moreover, the Gysin morphism π_* is then identified with the Jacobi symmetrizer ∂_{w_0}, if we denote by $w_0 = w_{0,e}$ the permutation of maximal length of S_e; for all polynomials P in $x_1, \dots, x_e$ with coefficients in $H^*(X)$, we have

$$\pi_*(P(x_1, \dots, x_e)) = (\partial_{w_0} P)(x_1, \dots, x_e).$$

PROOF. The variety $\mathbb{F}(E)$ is built of a sequence of fibrations of projective lines, so we proceed by induction, beginning with proposition 3.6.15. More precisely, denote by $\mathcal{W}_{e-1}$ the tautological bundle of hyperplanes on $\mathbb{P}(E^*)$. Then the variety

$\mathbb{F}(\mathcal{W}_{e-1})$ of complete flags of $\mathcal{W}_{e-1}$ is identified naturally with $\mathbb{F}(E)$, and we have a commutative diagram

$$\begin{tikzcd} \mathbb{F}(E) \arrow[r, "\pi"] \arrow[d, equal] & X \\ \mathbb{F}(\mathcal{W}_{e-1}) \arrow[r, "\alpha"] & \mathbb{P}(E^*). \arrow[u, "\beta"'] \end{tikzcd}$$

The description of the cohomology of $\mathbb{F}(E)$ follows immediately. Moreover, by proposition 3.6.15 and the inductive hypothesis,

$$\pi_* = \beta_*\alpha_* = \partial_1 \cdots \partial_{e-1}\partial_{w_{0,e-1}} = \partial_{w_{0,e}},$$

which was to be proved. □

REMARK 3.8.2. The preceding statement immediately implies the splitting principle 3.5.2. Indeed, the inverse image in cohomology

$$\pi^* \colon H^*(X) \to H^*(\mathbb{F}(E))$$

is injective, and π^*E is endowed with a complete flag of subbundles

$$0 = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_e = \pi^*E.$$

In particular, we have exact sequences $0 \to \mathcal{W}_{i-1} \to \mathcal{W}_i \to \mathcal{L}_i \to 0$. Then such a sequence is always split in the differentiable sense (for example, we may endow $\mathcal{W}_i$ with a hermitian metric and consider the orthogonal complement $\mathcal{M}_i$ of $\mathcal{W}_{i-1}$ in $\mathcal{W}_i$; then $\mathcal{W}_i = \mathcal{W}_{i-1} \oplus \mathcal{M}_i$, and the line bundle $\mathcal{M}_i$ is isomorphic to $\mathcal{L}_i$). By induction, we deduce that π^*E is indeed split.

EXERCISE 3.8.3. Let λ be partition of length at most e. The preceding proposition implies the formula

$$\pi_*(x^{\lambda+\delta}) = s_\lambda(E),$$

where $\delta = (e-1, \ldots, 1, 0)$ is the minimal strictly decreasing sequence of length e. Then let $\mathbb{G}_{m,n}(E)$ be the variety of subspaces of dimension m and codimension n of E. Let p be its projection on X, and T and Q its tautological vector and quotient bundles. By remarking that

$$\mathbb{F}(E) = \mathbb{F}(T) \times_{\mathbb{G}_{m,n}(E)} \mathbb{F}(Q),$$

the set of pairs of flags of T and Q which have the same projection on $\mathbb{G}_{m,n}(E)$, verify that the Gysin morphism ρ_* is given by the formulas

$$\rho_*(s_\alpha(Q)s_\beta(T)) = s_\gamma(E),$$

where $\gamma = (\alpha_1 - m, \ldots, \alpha_n - m, \beta_1, \ldots, \beta_m)$ [**73**]—if γ is not a partition, we apply the usual rules of rectification of Schur functions.

EXERCISE 3.8.4. Extend the preceding proposition to partial flag varieties.

3.8.2. Relative Degeneracy Loci. Suppose that E is endowed with a complete flag E^* of quotient bundles

$$E = E^e \twoheadrightarrow E^{e-1} \twoheadrightarrow \cdots \twoheadrightarrow E^0 = 0.$$

To each permutation $w \in \mathcal{S}_e$, we may then associate the degeneracy locus

$$X_w(E^\bullet) = \{W_\bullet \in \mathbb{F}(E), \quad \operatorname{rank}(W_p \to E^q) \leq r_w(p,q), \quad 1 \leq p,q \leq e\}.$$

If we denote by V_{n-p} the kernel of the projection $E \twoheadrightarrow E^p$, these different conditions are equivalent to the inequalities

$$\begin{aligned}\dim(W_p \cap V_q) &\geq p - r_w(p, n-q)\\ &= \#\{i \leq p, \quad w(i) \geq n-q+1\} = r_{w_0 w}(p,q).\end{aligned}$$

This implies that $X_w(E^\bullet)$ is a subvariety of codimension $l(w)$ of $\mathbb{F}(E)$, the intersection of which with each fiber of the projection $\pi\colon \mathbb{F}(E) \to X$ is the Schubert variety X'_w defined by the flag $V_\bullet$ of E.

EXAMPLE 3.8.5. Return to the complete flag variety $\mathbb{F}_n$, with a fixed flag of reference $V_\bullet$. This amounts to being given a trivial complete flag of quotient bundles

$$V = V_n \twoheadrightarrow V_n/V_1 \twoheadrightarrow \cdots \twoheadrightarrow V_n/V_{n-1} \twoheadrightarrow 0.$$

Moreover, we have over $\mathbb{F}_n$ a tautological flag of bundles $W_\bullet$, and $X_w(W_\bullet)$ is the Schubert variety $X_{w_0 w}$. The theorem which follows is then a generalization of theorem 3.6.18.

THEOREM 3.8.6. *The Poincaré dual of the fundamental class of $X_w(E^\bullet)$ is the double Schubert polynomial*

$$\begin{aligned}[X_w(E^\bullet)] &= \mathfrak{S}_w(-x_1, \ldots, -x_e; -y_1, \ldots, -y_e),\\ &= \mathfrak{S}_{w^{-1}}(y_1, \ldots, y_e; x_1, \ldots, x_e)\end{aligned}$$

in the Chern classes $x_i = c_1(W_i/W_{i-1})$ and $y_j = c_1(\ker(E^j \twoheadrightarrow E^{j-1}))$.

PROOF. Lemmas 3.6.19 and 3.6.20 extend without change to the relative case and allow us to proceed by induction. It suffices therefore to treat the case of the permutation of maximal length w_0. The corresponding degeneracy locus is defined by the simultaneous vanishing of the induced morphisms $W_p \to E^{e-p}$, $1 \leq p \leq e-1$. However, these morphisms are not independent; the composition of the arrow $W_p \to E^{e-p}$ with $j_p\colon E^{e-p} \twoheadrightarrow E^{e-p-1}$ may be obtained by composing the inclusion $i_p\colon W_p \to W_{p+1}$ with the arrow $W_{p+1} \to E^{e-p-1}$. We are then led to introduce the bundle

$$\mathcal{K} = \ker\left(\bigoplus_{p=1}^{e-1} \operatorname{Hom}(W_p, E^{e-p}) \to \bigoplus_{p=1}^{e-2} \operatorname{Hom}(W_p, E^{e-p-1})\right),$$

the kernel of the morphism defined as follows: to a family of maps $u_p\colon W_p \to E^{e-p}$, where $1 \leq p \leq e-1$, we associate the maps

$$v_p = j_p \circ u_p - u_{p+1} \circ i_p, \quad 1 \leq p \leq e-2.$$

This morphism is surjective, so that $\mathcal{K}$ has rank $e(e-1)/2$, and $X_{w_0}(E^\bullet)$ is the locus of zeros of the section of $\mathcal{K}$ induced by the identity of E. The Gauss-Bonnet formula 3.5.10 then gives

$$[X_{w_0}(E^\bullet)] = c_{\binom{e}{2}}(\mathcal{K}) = \prod_{i+j \leq e} (y_i - x_j),$$

which, thanks to a small calculation with the help of the splitting principle, may be verified. The theorem is then proved for w_0, hence for every permutation. □

3.8.3. A Theorem of Fulton. More generally, suppose that the bundle E is endowed only with a partial flag of quotient bundles

$$E = E^h \twoheadrightarrow \cdots \twoheadrightarrow E^0 = 0,$$

where E^i is of rank e_i. Consider then the variety $\mathbb{F}_{m_\bullet}(E)$ of partial flags of E of the form

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_k = f^*E,$$

the dimensions of the members of which form a strictly increasing sequence of integers $m_\bullet = (0 = m_0 < \cdots < m_k = e)$. We denote by f the projection of $\mathbb{F}_{m_\bullet}(E)$ on X.

One would like to impose some rank conditions

$$\operatorname{rank}(W_p \to f^*E^q) \leq r(p,q) \quad \text{for} \quad 1 \leq p \leq k \text{ and } 1 \leq q \leq h,$$

for a given function r, and study the corresponding degeneracy loci. However, it may happen, depending on the choice of the function r, that such conditions are not sufficiently precise, or in fact are redundant, or even contradictory. Suppose for example that one has some complete flags of the same length. We have seen that in this case, only those functions which are rank functions of permutations may be effectively realized (exercise 3.6.2). A degeneracy locus defined by a rank function r which is not of this type then decomposes, in the generic situation, into irreducible components defined by the "good" rank conditions. More precisely, we obtain an irreducible component for each maximal element of the set of permutations w such that $r_w \leq r$. These considerations motivate the following definition:

DEFINITION 3.8.7. *We say that a function* $r\colon \{1, \dots, h\} \times \{1, \dots, k\} \to \mathbb{N}$ *is* admissible *if there exists a permutation* $w \in S_\infty$ *such that all of its essential points are of the form* (m_p, e_q)*, and such that* $r(p,q) = r_w(m_p, e_q)$ *for all pairs* p,q*. We note that in these conditions, a permutation* w *is necessarily increasing on each interval* $]m_{p-1}, m_p]$*, and similarly that* w^{-1} *is increasing on the intervals* $]e_{q-1}, e_q]$*.*

We consider then on $\mathbb{F}_{m_\bullet}(E)$ the degeneracy locus $X_r(E^\bullet)$, consisting of the flags $W_\bullet \in \mathbb{F}_{m_\bullet}(E)$ satisfying the rank conditions

$$\operatorname{rank}(W_p \to f^*E^q) \leq r(p,q) \quad \text{for} \quad 1 \leq p \leq k \quad \text{and} \quad 1 \leq q \leq h.$$

Let $\pi_{m_\bullet}\colon \mathbb{F}(E) \to \mathbb{F}_{m_\bullet}(E)$ be the natural projection. Under the preceding conditions, the inverse of image of $X_r(E^\bullet)$ under $\pi_{m_\bullet}$ coincides with the degeneracy locus $X_w(E^\bullet)$, and the projection of this locus on $X_r(E^\bullet)$ is a fibration of products of complete flag varieties. We then have

$$\pi^*_{m_\bullet}[X_r(E^\bullet)] = [X_w(E^\bullet)] = \mathfrak{S}_w(x,y),$$

the double Schubert polynomial given by the preceding proposition.

Since w (resp., w^{-1}) is increasing on each interval $]m_{p-1}, m_p]$ (resp., $]e_{q-1}, e_q]$), the polynomial $\mathfrak{S}_w$ is, by corollary 2.4.3 of chapter two, symmetric on these intervals in the variable x (resp., y). If we consider $x_{m_{p-1}+1}, \dots, x_{m_p}$ as Chern roots of the quotient W_p/W_{p-1}, and similarly $y_{e_{q-1}+1}, \dots, y_{e_q}$ as those of the kernel $\ker(E^q \twoheadrightarrow E^{q-1})$, it is then possible to express $\mathfrak{S}_w(x,y)$ in the form of a polynomial

$$P_{r,e_\bullet,q_\bullet}(c(\pi^*_{m_\bullet} W_\bullet), c(\pi^*_{m_\bullet} E^\bullet))$$

in the Chern classes of these bundles. And because $\pi_{m_\bullet}$, which is a fibration of complete flag varieties, induces by proposition 3.8.1 an injection of cohomology, we have

$$[X_r(E^\bullet)] = P_{r,e_\bullet,q_\bullet}(c(W_\bullet), c(E^\bullet)),$$

the Schubert polynomial associated to the permutation w, in the negatives of the Chern roots of the quotients W_p/W_{p-1} and of the kernels $\ker(E^q \twoheadrightarrow E^{q-1})$.

Finally, consider the following situation. Let $\phi\colon F \to G$ be a morphism between vector bundles of ranks f and g on X. Suppose that F is endowed with a flag of subbundles $F_\bullet$, such that the ith member F_i is of rank f_i for $1 \leq i \leq h$. Suppose similarly that G is endowed with a flag of quotient bundles $G^\bullet$, such that the jth member G_j of rank g_j for $1 \leq j \leq k$. Moreover, let

$$r\colon \{1, \dots, f\} \times \{1, \dots, g\} \to \mathbb{N}$$

be an admissible function, associated to a permutation w, and $X_r(F_\bullet, G^\bullet, \phi)$ the corresponding degeneracy locus.

Set $H = F \oplus G$. The quotient flag $G^\bullet$ of G gives a quotient flag $H^\bullet$ of H, due to the projection $p_2\colon H \twoheadrightarrow G$. Moreover, the flag $F_\bullet$ of F gives a flag $H_\bullet$ of subbundles of H, due to the injection $\mathrm{id} \times \phi\colon F \to H$. This flag defines a map

$$u\colon X \to \mathbb{F}_{f_\bullet, f+g}(H)$$

of X into a variety of partial flags of H, and the degeneracy locus $X_r(F_\bullet, G^\bullet, \phi)$ is the inverse image of $X_r(H^\bullet)$ under u. If we suppose that the morphism ϕ is generic, the restriction of u to $X_r(F_\bullet, G^\bullet, \phi)$ will generically be a submersion. The fundamental class of $X_r(F_\bullet, G^\bullet, \phi)$ will then be the inverse image of that of $X_r(H^\bullet)$. We immediately deduce the expression of this fundamental class as a Schubert polynomial in the Chern roots of the bundles involved; this is the theorem of Fulton below [**21**].

In this statement, we are freed from the signs appearing in theorem 3.8.6 by changing w to w^{-1} and by appealing to corollary 2.4.2 of the second chapter.

THEOREM 3.8.8. *Let $\phi\colon F \to G$ be a generic morphism between complex vector bundles F and G of ranks f and g, endowed respectively with a flag of subbundles*

$$F_\bullet = (0 \subset F_1 \subset \cdots \subset F_h = F),$$

of ranks $f_1, \dots, f_h$, and with a flag of quotient bundles of ranks $g_k, \dots, g_1$,

$$G^\bullet = (G = G_k \twoheadrightarrow \cdots \twoheadrightarrow G_1 \twoheadrightarrow 0).$$

Let $r\colon \{1, \dots, h\} \times \{1, \dots, k\} \to \mathbb{N}$ be an admissible function, associated to a permutation $w^{-1} \in S_\infty$. Then the Poincaré dual of the fundamental class of the corresponding degeneracy locus is

$$[X_r(F_\bullet, G^\bullet, \phi)] = \mathfrak{S}_w(x_1, \dots, x_g; y_1, \dots, y_f) = P_{r,g_\bullet,f_\bullet}(c(G^\bullet), c(F_\bullet)),$$

the Schubert polynomial associated to the permutation w, in the Chern roots $x_{g_{j-1}+1}, \dots, x_{g_j}$ of the kernel bundles $\ker(G_j \twoheadrightarrow G_{j-1})$, and $y_{f_{i-1}+1}, \dots, y_{f_i}$ of the quotient bundles F_i/F_{i-1}.

EXAMPLE 3.8.9. Suppose that $h = k$ and consider the rank conditions given by a vexillary permutation w. If we assume the notation of definition 2.2.9 and of exercise 2.2.11 for the shape and the flag of w and of its inverse, these rank conditions, become, by the same exercise, the following:

$$\operatorname{rank}(F_{k+1-i} \to G_i) \leq g_i - l_{k+1-i} \quad \text{for } 1 \leq i \leq k.$$

Under suitable conditions of genericity, the preceding theorem and theorem 2.6.9 of the second chapter show that we obtain the fundamental class of the corresponding degeneracy locus as a determinant in certain characteristic classes of $F_\bullet$ and $G^\bullet$ associated to a multi-Schur function:

$$[X_r(F_\bullet, G^\bullet, \phi)] = s_{\lambda(w)}(\underbrace{X_{f_1} - Y_{g_k}}_{m_1}, \dots, \underbrace{X_{f_k} - Y_{g_1}}_{m_k}).$$

We recover, as particular cases of this formula, the formula of Thom and Porteous as well as that of Kempf and Laksov.

EXAMPLE 3.8.10. Recall example 3.5.16 of a matrix P of homogeneous polynomials $P_{i,j}$ of degrees $a_i + b_j$. Instead of a simple condition on the global rank of this matrix, consider the conditions given by the rank function of a permutation $w \in S_{e+f}$. Let $X_w(P)$ be the corresponding locus. Then, for a generic matrix of polynomials, this locus is irreducible of codimension $l(w)$, and

$$\deg X_w(P) = \mathfrak{S}_w(a_1, \dots, a_e, \underbrace{0, \dots, 0}_{f}; -b_1, \dots, -b_f, \underbrace{0, \dots, 0}_{e}).$$

REMARK 3.8.11. The double Schubert polynomials then represent the degeneracy loci between morphisms of vector bundles, one endowed with a flag $F_\bullet$ of subbundles, and the other with a flag $G^\bullet$ of quotient bundles. One may imagine the following more general situation: we are given two families of vector bundles $F^\bullet$ and $G^\bullet$, with morphisms

$$F_i \xrightarrow{r_i} F_{i+1} \quad \text{and} \quad G_j \xrightarrow{s_j} G_{j-1}$$

where we no longer suppose that those in the first group are injective and that those in the second are surjective. Now let $\phi\colon F = F_h \to G = G_k$ be a morphism. By imposing rank conditions on the induced morphisms

$$F_i \xrightarrow{r_i} \cdots \xrightarrow{r_{h-1}} F_h \xrightarrow{\phi} G_k \xrightarrow{s_k} \cdots \xrightarrow{s_{j+1}} G_j,$$

we define degeneracy loci which, under suitable conditions of transversality, are represented in homology by certain polynomials in characteristic classes of the original bundles. These polynomials, of which the Schubert polynomials are a special case, were introduced by Fulton [**23**] and are called *universal Schubert polynomials*.

A Brief Introduction to Singular Homology

In this appendix we briefly review the fundamental properties of singular homology, as well as some topological particulars of algebraic varieties which are used in this course. General references on this subjects are not lacking; one might consult for example [**31**, **42**, **85**] for more details.

A.1. Singular Homology

The theory of singular homology associates to a topological space X a family of commutative groups $H_q(X, \mathbb{Z}) = H_q(X)$, where $q \in \mathbb{N}$. We quickly recall the steps. Denote by e_0 the origin of $\mathbb{R}^q$, by $e_1, \ldots, e_q$ its canonical basis, and by $\Delta_q = (e_0, \ldots, e_q)$ the convex hull of these $q+1$ points. A *singular q-simplex* of a topological space X is then a continuous map $u\colon \Delta_q \to X$. We denote by $S_q(X)$ the group of their formal linear combinations with integer coefficients and say that this is the group of *q-chains*.

Now we introduce a *boundary operator* on $S_q(X)$, denoted ∂_q, by linearly extending the following definition:

$$\partial_q(u) = \sum_{i=0}^{q} (-1)^i u^{(i)},$$

where the *ith face* $u^{(i)}$ of u is the singular $(q-1)$-simplex defined by the composition

$$u^{(i)}\colon \Delta_{q-1} = (e_0, \ldots, e_{q-1}) \xrightarrow{\text{affine}} (e_0, \ldots, \widehat{e_i}, \ldots, e_q) \subset \Delta_q \xrightarrow{u} X,$$

the first of these arrows being the restriction of the unique affine map from $\mathbb{R}^{q-1}$ to $\mathbb{R}^q$ which sends the q-tuple $(e_0, \ldots, e_{q-1})$ to $(e_0, \ldots, \widehat{e_i}, \ldots, e_q)$. An easy calculation shows that $\partial \circ \partial = 0$, which justifies the following definition:

DEFINITION A.1.1. *The qth group of singular homology of X is the quotient of the group of* cycles *by the group of* boundaries*:*

$$H_q(X) = \ker \partial_q / \operatorname{im} \partial_{q+1}.$$

More generally, if A is a subset of X and $S_q(A)$ is the group of q-chains with values in A, the boundary operator sends $S_q(A)$ into $S_{q-1}(A)$, which then induces an operator which we denote in the same way

$$\partial_q\colon S_q(X)/S_q(A) \to S_{q-1}(X)/S_{q-1}(A).$$

DEFINITION A.1.2. *The qth homology group relative to (X, A) is the quotient*

$$H_q(X, A) = \ker \partial_q / \operatorname{im} \partial_{q+1}.$$

These constructions are functorial: if $f\colon X \to Y$ is a continuous map, and if $f(A) \subset B$, then by composition we obtain maps from $S_q(X)$ to $S_q(Y)$ and from $S_q(A)$ to $S_q(B)$. These maps commute with the boundary operator, whence we have an induced direct image map of homology

$$f_*\colon H_q(X,A) \to H_q(Y,B).$$

In particular, the inclusions $i\colon (A,\varnothing) \to (X,\varnothing)$ and $j\colon (X,\varnothing) \to (X,A)$ induce the arrows

$$H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A).$$

The fundamental properties of homology are the six following statements, called *Eilenberg-Steenrod axioms*. They completely characterize singular homology on topological spaces called *triangularizable*:

1. if X is a point, then $H_0(X) = \mathbb{Z}$ and $H_q(X) = 0$ for $q > 0$;
2. if f is the identity map on a topological space X, then f_* is also the identity;
3. if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then $(g \circ f)_* = g_* \circ f_*$;
4. if $f, g\colon X \to Y$ are two *homotopic* continuous maps, then $f_* = g_*$;
5. for all pairs (X, A), where A is a subset of a space X, we have a *long exact sequence*
$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \to \ldots;$$
6. finally, if U is a subset of A with closure contained in the interior of A, then U may be *excised*, or in other words the map
$$H_q(X-U, A-U) \to H_q(X,A)$$
induced by inclusion is an isomorphism.

These statements call for several remarks:

- In degree 0, axiom 1 generalizes as follows. For all topological spaces X, we have
$$H_0(X) = \mathbb{Z}^C,$$
where C is the set of arc-connected components of X. In particular, this group is a group of finite type when X is a compact variety. It can be shown that under this hypothesis, it is the same for the other homology groups.
- Axioms 2 and 3 express the fact that for each integer $q \in \mathbb{N}$, the correspondence $X \rightsquigarrow H_q(X)$ defines a *covariant functor* from the category of topological spaces to the category of commutative groups.
- An immediate consequence of axiom 4 is that a *contractible* topological space X (in other words, a space in which the identity map is homotopic to a constant map) has the same singular homology as a point:
$$H_0(X) = \mathbb{Z}, \quad H_q(X) = 0 \quad \text{if} \quad q > 0.$$
- In axiom 5, a long exact sequence is a sequence of morphisms
$$\cdots \to E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1} \to \ldots$$
such that $\ker f_i = \operatorname{im} f_{i-1}$. The operator $\partial\colon H_q(X,A) \to H_{q-1}(A)$ is defined in the following fashion. If $c \in H_q(X,A)$, let C be a representative of c in $S_q(X)$. Then $\partial_q C \in S_{q-1}(A)$ is a cycle, and its class ∂c in $H_{q-1}(A)$ depends only on c.

EXERCISE A.1.3. Let U be a convex neighborhood of the origin in $\mathbb{R}^n$, and let $q > 1$. Show that

$$H_q(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = H_q(U, U - \{0\}) = H_{q-1}(U - \{0\}) = H_{q-1}(S_{n-1}),$$

where S_{n-1} is the unit sphere in $\mathbb{R}^n$. The first of these isomorphisms is obtained by excising $\mathbb{R}^n - \overline{U}$; the second by using a long exact sequence and the fact that U is contractible; and the third by a homotopy argument. By decomposing S_n into two hemispheres, which are contractible and have intersection S_{n-1}, show that $H_{q-1}(S_{n-1}) = H_q(S_n)$. Deduce that

$$\begin{aligned} &H_0(S_n) = H_n(S_n) = \mathbb{Z}, && H_q(S_n) = 0 && \text{if } q \neq 0, n, \\ &H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \mathbb{Z}, && H_q(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = 0 && \text{if } q \neq n. \end{aligned}$$

A.2. Singular Cohomology

The group of *singular q-cochains* of a space X is defined as the dual group $S^q(X) = \operatorname{Hom}_{\mathbb{Z}}(S_q(X), \mathbb{Z})$. We defined the *coboundary operator*

$$\delta_q \colon S^q(X) \to S^{q+1}(X)$$

as the transpose of the boundary operator δ_{q-1}.

DEFINITION A.2.1. *The qth singular cohomology group is the quotient of the* cocycles *by the* coboundaries

$$H^q(X) = \ker \delta_q / \operatorname{im} \delta_{q-1}.$$

More generally, if A is a subset of X, we denote by $S^q(X, A)$ the group of elements of $S^q(X)$ which vanish on $S^q(A)$, and similarly define the relative cohomology group $H^q(X, A)$.

If $f \colon X \to Y$ is a continuous map such that $f(A) \subset B$, the induced map of cohomology is this time an inverse image morphism

$$f^* \colon H^q(Y, B) \to H^q(X, A).$$

The Eilenberg-Steenrod axioms for singular cohomology are the following:

1. if X is a point, then $H^0(X) = \mathbb{Z}$ and $H^q(X) = 0$ for $q > 0$;
2. if f is the identity map on a topological space X, then f^* is also the identity;
3. if $f \colon X \to Y$ and $g \colon X \to Y$ are continuous, then $(g \circ f)^* = f^* \circ g^*$;
4. if $f, g \colon X \to Y$ are two homotopic continuous maps, then $f^* = g^*$;
5. for every pair (X, A), where A is a subset of a space X, we have a *long exact sequence*
$$\cdots \to H^q(X, A) \to H^q(X) \to H^q(A) \xrightarrow{\delta} H^{q+1}(X, A) \to \ldots;$$
6. finally, if U is a subset of A with closure contained in the interior of A, then U may be *excised*, or in other words the map
$$H^q(X, A) \to H^q(X - U, A - U)$$
induced by inclusion is an isomorphism.

As for homology, axiom 1 extends in degree 0 to any topological space whatsoever in the following way: $H^0(X)$ is the group of locally constant functions on X with values in $\mathbb{Z}$.

Statements 2 and 3 express the fact that for each integer $q \in \mathbb{N}$, the correspondence $X \rightsquigarrow H^q(X)$ defines a *contravariant functor* from the category of topological spaces to the category of commutative groups.

We similarly note that the duality $S^q(X) \times S_q(X) \to \mathbb{Z}$ passes to the quotient to yield a *Kronecker product* $H^q(X) \times H_q(X) \to \mathbb{Z}$. It may be shown that this product is nondegenerate. In particular, $H^q(X)$ and $H_q(X)$ have the same rank.

The essential advantage of cohomology over homology is that it is possible to define on the direct sum

$$H^*(X) = \bigoplus_{q \geq 0} H^q(X)$$

a structure of a graded ring: there exists a bilinear operation

$$\cup\colon H^p(X) \otimes H^q(X) \to H^{p+q}(X)$$

called the *cup product*. Furthermore, the direct sum

$$H_*(X) = \bigoplus_{q \geq 0} H_q(X)$$

is then endowed with the structure of a graded $H^*(X)$-module: there exists an operation

$$\cap\colon H^p(X) \otimes H_q(X) \to H_{q-p}(X)$$

called the *cap product* which is compatible with the cup product.

A.3. The Fundamental Class and Poincaré Duality

Let X be a connected differentiable variety of dimension n. One may show that $H_q(X) = 0$ if $q > n$, while

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } X \text{ is compact and orientable,} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, when X is compact and orientable, choosing an orientation amounts to choosing a generator of $H_n(X)$. As a consequence, a differentiable variety which is compact, connected, and oriented, of dimension n, possesses a *fundamental class*, denoted $[X]$, such that

$$H_n(X) = \mathbb{Z}[X].$$

The cap product operation with the fundamental class defines a morphism

$$\bullet \cap [X]\colon H^q(X) \to H_{n-q}(X).$$

The *Poincaré duality* theorem guarantees that this morphism is an isomorphism. We give two applications:

1. *Gysin morphisms.* If $f\colon Y \to X$ is a continuous map between compact, connected, orientable differentiable varieties, Y being of dimension x and X of dimension n, one may define a direct image morphism f_* of cohomology by the composition

   $$f_*\colon H^q(Y) \simeq H_{m-q}(Y) \xrightarrow{f_*} H_{m-q}(X) \simeq H^{q-(m-n)}(X).$$

   There is also a *projection formula*

   $$f_*(f^*\alpha \cup \beta) = \alpha \cup f_*(\beta).$$

2. *The fundamental class of a subvariety.* If Y is a connected, closed, oriented differentiable variety of dimension m in X, we denote by $[Y]$ the image of the fundamental class under the composition

$$H_m(Y) \to H_m(X) \simeq H^{n-m}(X),$$

where the difference is the codimension of Y in X. These fundamental classes behave well with respect to the direct image and the inverse image: if the map $f\colon Z \to X$ is smooth (in the sense in which its differential is of constant rank, therefore that it is a submersion over its image), then $f^*[Y] = [f^{-1}(Y)]$, and if $g\colon X \to Z$ sends Y to a subvariety of Z of dimension strictly less than m, then $g_*[Y] = 0$, while if g restricts to a degree d covering over its image, then $g_*[Y] = d[g(Y)]$.

It is possible to extend the notion of a fundamental class to closed subsets of a compact differentiable variety which are not too singular. One of the possible methods consists of using *Borel-Moore homology*, defined for locally compact spaces, in the following fashion: if X is homeomorphic to a closed subset of an affine space $\mathbb{R}^n$, we set

$$\overline{H}_q(X) = H^{n-q}(\mathbb{R}^n, \mathbb{R}^n - X).$$

One may show that this does not depend on the choice of embedding of X in an affine space, and that the affine space may be replaced by any oriented differentiable variety in which X may be embedded. If X is itself a connected, oriented differentiable variety of dimension n, we then have

$$\overline{H}_q(X) = H^{n-q}(X).$$

In particular, if X is also compact, its Borel-Moore homology is equal to its singular homology. However, even if X is not compact, the identity $\overline{H}_n(X) = H^0(X)$ implies that X admits a fundamental class $[X] \in \overline{H}_n(X)$ corresponding to the constant function equal to 1 in $H^0(X)$ ([**5**], or [**22**, Appendix B]).

Another advantage of Borel-Moore homology is the following. If X may be embedded in an oriented variety M of dimension m, and if U is an open subset of X, then it may be identified with a closed subset of $N = M - (X - U)$, whence we have the morphisms

$$\overline{H}_q(X) = H^{m-q}(M, M - X) \to H^{m-q}(N, N - U) = \overline{H}_q(U),$$

called localization morphisms. We then have the long exact sequences

$$\cdots \to \overline{H}_q(X - U) \to \overline{H}_q(X) \to \overline{H}_q(U) \to \overline{H}_{q-1}(X - U) \to \cdots.$$

On the other hand, the direct image for Borel-Moore homology is defined only if we restrict to proper maps (under which the inverse image of a compact set is compact).

We now consider a compact complex algebraic variety X of complex dimension n. Its complex structure confers a natural orientation on it, so that it has a fundamental class $[X] \in H_{2n}(X)$.

An algebraic subvariety of X is a closed subset Y which is locally defined by algebraic equations. It is endowed with the structure of a complex variety outside of its singular points, which form an algebraic subvariety $\mathrm{Sing}(Y) \subset Y$ of complex codimension at least one, so of real codimension at least two. Suppose moreover that T is *irreducible*, in the sense that $Y - \mathrm{Sing}(Y)$ is a connected smooth variety, say of dimension m. An induction on m together with the long exact sequences above give that $\overline{H}_q(Y) = 0$ if $q > 2m$. By applying that to $\mathrm{Sing}(Y)$, and thanks to the same long exact sequences, one may deduce that

$$\overline{H}_{2m}(Y) \simeq \overline{H}_{2m}(Y - \mathrm{Sing}(Y)) \simeq \mathbb{Z},$$

which allows us to associate to Y a fundamental class $[Y]$, the image of the class of $Y - \mathrm{Sing}(Y)$ under the isomorphism above. We denote in the same fashion its image in $H^{2n-2m}(X)$.[11]

Under certain conditions, the fundamental classes of algebraic subvarieties of a complex variety X generate its cohomology ring. This is the case, for example, if X has a *cellular decomposition* in which the closures of the cells are algebraic subvarieties. By cellular decomposition of a complex variety X one means a finite partition $X = \coprod_{i\in I} Y_i$, where the *cells* Y_i are isomorphic to affine spaces $\mathbb{C}^{n_i}$ and satisfy the *boundary conditions*

$$\overline{Y_i} - Y_i = \coprod_{j\in J_i} Y_j,$$

with $n_j < n_i$ if $j \in J_i$. In this situation, the closures $\overline{Y_i}$ have fundamental classes $[\overline{Y_i}] \in H^{2c_i}(X)$, where $c_i = n - n_i$, and we have

$$H^{2q}(X) = \bigoplus_{c_i = q} \mathbb{Z}[\overline{Y_i}], \qquad H^{2q+1}(X) = 0.$$

EXAMPLE A.3.1. Let $\mathbb{P}^n$ be a complex projective space of lines of $\mathbb{C}^{n+1}$. It is then a compact complex algebraic variety of dimension n. Suppose we then give a complete flag

$$0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_{n+1} = \mathbb{C}^{n+1}.$$

Then $Y_i = \mathbb{P}(V_{i+1})\backslash\mathbb{P}(V_i) \simeq \mathbb{C}^i$, and $\overline{Y_i} - Y_i = Y_{i-1} \coprod \cdots \coprod Y_0$. We then obtain a cellular decomposition of $\mathbb{P}^n$, and deduce that its nonzero cohomology groups are the

$$H^{2q}(\mathbb{P}^n) = \mathbb{Z}[\overline{Y_{n-q}}], \quad 0 \le q \le n,$$

$[\overline{Y_q}]$ being the fundamental class of a linear projective subspace of codimension q. We note that the real case is clearly different, for the cohomology of a real projective space has torsion.

A.4. Intersection of Algebraic Subvarieties

Let Y and Y' be irreducible algebraic subvarieties of a compact and connected complex variety X. Their intersection may be written

$$Y \cap Y' = \bigcup_{i\in I} Z_i,$$

where the Z_i are irreducible algebraic subvarieties of X. Suppose that this intersection is *transverse*, which means that for all $i \in I$,

- the codimension of Z_i is the sum of the codimensions c and c' of Y and Y', in which case we say that the intersection is *proper*; and

[11] One could just as well invoke some classical arguments in order to justify the existence of a fundamental class of an irreducible algebraic subvariety. For example, one may consider the possibility of triangulating it in a fashion compatible with its singularities, or still yet, the fact that the singular set of real codimension at least two allows the integration of differential forms as on a smooth subvariety, so that one may invoke the De Rham theorems.

- a generic point of Z_i is a smooth point simultaneously in Z_i, Y, and Y', and the tangent spaces of this point are such that

$$T_z Z_i = T_z Y \cap T_z Y' \subset T_z X.$$

Under this hypothesis of transversality, it can be shown that the cup product of fundamental classes is none other than the fundamental class of the intersection:

$$[Y] \cup [Y'] = \sum_{i \in I} [Z_i] \in H^{2c+2c'}(X).$$

More generally, if the intersection if supposed only to be proper, there exist *intersection multiplicities* $m_i > 0$ such that

$$[Y] \cup [Y'] = \sum_{i \in I} m_i [Z_i].$$

The localization morphisms of Borel-Moore homology, for example, allow us to check that these intersection multiplicities depend only on the local intersection of Y and Y' in Z_i.

EXAMPLE A.4.1. We return to the example of complex projective space $\mathbb{P}^n$, and denote by h the generator of $H^2(\mathbb{P}^n)$ defined as the fundamental class of a hyperplane H. We call this class the *hyperplane class*. Since the connected group $\mathrm{PGL}(n+1, \mathbb{C})$ acts transitively on hyperplanes, this class does not depend on the chosen hyperplane. As a consequence, if we consider q transverse hyperplanes $H_1, \ldots, H_q$, which is to say in this situation that their intersection is a linear subspace L of codimension q, we have that

$$h^q = [H_1] \cup [H_2] \cup \cdots \cup [H_q] = [H_1 \cap \cdots \cap H_q] = [L].$$

Since $[L]$ is a generator of $H^{2q}(\mathbb{P}^n)$, we derive an isomorphism of graded rings

$$H^*(\mathbb{P}^n) \simeq \mathbb{Z}[h]/h^{n+1}.$$

In other words, the cohomology ring of the complex projective space may be identified with the ring of polynomials of one variable with integral coefficients, truncated at degree $n+1$.

Now let X be an algebraic subvariety of $\mathbb{P}^n$ of codimension c. We define its *degree* by the identity $[X] = \deg(X) h^c$. The equality

$$\deg(X) h^n = [X] \cup h^{n-c}$$

can be interpreted as follows. The degree of X is the number of intersection points with any linear subspace of $\mathbb{P}^n$ of dimension c which cuts it transversally. More generally, two subvarieties X and Y of complementary dimensions which meet transversally have precisely $\deg(X)\deg(Y)$ intersection points.

In particular, two curves C, D in $\mathbb{P}^2$, defined by respective polynomials c and d, meet at exactly cd points when their intersection is transverse, and in at most cd points in general, at least when their intersection is finite. More precisely, it is possible to associate to each of their intersection points p a multiplicity, that is, a strictly positive integer $m_{C,D}(p)$, equal to one if and only if C and D are transverse at p, in such a fashion that

$$\sum_{p \in C \cap D} m_{C,D}(p) = cd.$$

This is *Bezout's theorem*.

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