UNICITY OF GRADED COVERS OF THE CATEGORY $\mathcal{O}$ OF BERNSTEIN-GELFAND-GELFAND

MICHAEL ROTTMAIER AND WOLFGANG SOERGEL

ABSTRACT. We show that the standard graded cover of the well-known category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand can be characterized by its compatibility with the action of the center of the enveloping algebra

1. INTRODUCTION

Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be a complex semi-simple Lie-algebra with a choice of a Borel and Cartan subalgebra. In [BG76] Bernstein, Gelfand and Gelfand introduced the so called category $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ of representations of $\mathfrak{g}$. Later on Beilinson and Ginzburg [BG86] argued that it is natural to study $\mathbb{Z}$-gradings of category $\mathcal{O}$, see also [BGS96, Soe95]. In this article we introduce the notion of a graded cover, a generalization of the notion of a $\mathbb{Z}$-grading which seemed to be more natural to us, and prove the following uniqueness theorem for graded covers of $\mathcal{O}$, to be explained in more detail in the later parts of this introduction.

Theorem 1.1 (**Uniqueness of graded covers of $\mathcal{O}$**). (1) *Category $\mathcal{O}$ admits a graded cover compatible with the action of the center of the universal enveloping algebra of $\mathfrak{g}$;*
(2) *If two graded covers of $\mathcal{O}$ are both compatible with the action of the center of the universal enveloping algebra, they are cover-equivalent.*

1.2. Graded covers of category $\mathcal{O}$ which are compatible with the action of the center have already been constructed in [Soe90], [Soe92b] and [BGS96]. The main point of this article is to show that compatibility with the action of the center already determines a graded cover of $\mathcal{O}$ up to cover-equivalence. To our knowledge this statement is already new in the case of $\mathbb{Z}$-gradings [BGS96, 4.3], although to see this case treated the reader could skip most of the paper and go directly to the proof of 7.3. In the body of the paper, we mainly investigate the notion of a graded cover.

1.3. An analogous theorem holds with the same proof for the modular versions of category $\mathcal{O}$ considered in [Soe00, RSW13]. It is for this case, that the generalization from gradings to graded covers is really needed. An analogous theorem also holds for the category of all Harish-Chandra modules over a complex connected reductive algebraic group, considered as a real Lie group. If we interpret our Harish-Chandra modules as bimodules over the enveloping algebra in the usual way and restrict to the subcategories of objects killed from the left by a fixed power of a given maximal ideal in the center of the enveloping algebra, the same proof in conjunction with [Soe92a] will work. To deduce the general case, some additional arguments are needed to justify passing to the limit, which can be found in the diploma thesis of M. Rottmaier.

Definition 1.4. An abelian category in which each object has finite length will henceforth be called an artinian category.

Definition 1.5. By a **graded cover** of an artinian category $\mathcal{A}$ we understand a triple $(\tilde{\mathcal{A}}, v, \epsilon)$ consisting of an abelian category $\tilde{\mathcal{A}}$ equipped with a strict automorphism [1] "shift the grading", an exact functor $v : \tilde{\mathcal{A}} \to \mathcal{A}$ "forget the grading" and an isotransformation of functors $\epsilon : v \stackrel{\sim}{\Rightarrow} v[1]$, such that the following hold:

(1) For all $M, N \in \tilde{\mathcal{A}}$ the pair (v, ϵ) induces an isomorphism on the homomorphism groups $\bigoplus_{i \in \mathbb{Z}} \tilde{\mathcal{A}}(M, N[i]) \stackrel{\sim}{\to} \mathcal{A}(vM, vN)$;
(2) Given $M \in \mathcal{A}$, $N \in \tilde{A}$ and an epimorphism $M \twoheadrightarrow vN$ there exists $P \in \tilde{A}$ and a morphism $vP \to M$ such that the composition $vP \to vN$ comes from an epimorphism $P \twoheadrightarrow N$ in $\tilde{\mathcal{A}}$.

Remark 1.6. The main difference to the concept of a $\mathbb{Z}$-grading in the sense of [BGS96, 4.3] is that for our graded covers we don't ask for any kind of positivity or semisimplicity. In particular, if we start with a grading and "change all degrees to their negatives", we would always get a graded cover again, but most of the time this would not be a grading anymore. If A is an left-artinian ring with a $\mathbb{Z}$-grading, for which in the article we ask no positivity condition whatsoever, then the forgetting of the grading on finitely generated graded modules $v : \tilde{A}\text{-Modf}^{\mathbb{Z}} \to A\text{-Modf}$ with the obvious ϵ always is a graded cover, see 2.1. In 3.2 we will show that the opposed category of a graded cover is a graded cover of the opposed category.

Remark 1.7. We would like to know whether condition (2) for general N will follow, when we ask it only to hold for $N = 0$.

Remark 1.8. We will try to strictly follow a notation, where calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$ denote categories, roman capitals $F, G, \ldots$ denote functors between or objects of our categories, and little greek letters $\tau, \epsilon, \ldots$ denote transformations. The only exception is the "forgetting of grading" in all its variants, to be denoted by the small letter v although it is a functor.

Definition 1.9. We say that a graded cover $(\tilde{\mathcal{O}}, v, \epsilon)$ of the BGG-category $\mathcal{O}$ is **compatible with the action of the center** $Z \subset \mathrm{U}(\mathfrak{g})$ iff the following holds: Given an object $\tilde{M} \in \tilde{\mathcal{O}}$ and a maximal ideal

$\chi \subset Z$ such that $\chi^n(v\tilde{M}) = 0$ for some $n \in \mathbb{N}$, the induced morphism

$$Z/\chi^n \to \mathrm{End}_{\mathfrak{g}}(v\tilde{M}), \quad z + \chi^n \mapsto (z\cdot)$$

is homogeneous for the grading on $\mathrm{End}_{\mathfrak{g}}(v\tilde{M})$ induced by the pair (v, ϵ) and the natural grading on Z/χ^n induced by the Harish-Chandra-homomorphism as explained next in 1.10.

1.10 (**The natural grading on Z/χ^n**). Let $S = \mathrm{S}(\mathfrak{h})$ be the symmetric algebra of our Cartan. The Weyl group W acts on it in a natural way. We have the Harish-Chandra isomorphism $Z \xrightarrow{\sim} S^W$. For any maximal ideal $\lambda \subset S$ let W_λ be its isotropy group and $Y = Y(\lambda) \subset S$ be the W_λ-invariants and put $\chi = \lambda \cap Z$ and $\mu = \lambda \cap Y$. Now general results in invariant theory [Liu09, Ex. 3.18.] tell us, that the natural maps are in fact isomorphisms

$$Z_\chi^\wedge \xrightarrow{\sim} (S_\lambda^\wedge)^{W_\lambda} \xleftarrow{\sim} Y_\mu^\wedge$$

from the completion of the invariants to the invariants of the completion, leading to isomorphisms $Z/\chi^n \xrightarrow{\sim} Y/\mu^n$. Moving the obvious W-invariant grading on S with the comorphism of the translation by λ, we obtain a W_λ-invariant grading on S, which induces a grading on Y with $\mu = Y^{>0}$ its part of positive degree. This way we get a natural grading on Y/μ^n. The reader may easily check that the induced grading on Z/χ^n doesn't depend on the choice of λ. We call it the **natural grading on** Z/χ^n.

Definition 1.11. Let $(\tilde{\mathcal{A}}, \tilde{v}, \tilde{\epsilon})$ and $(\hat{\mathcal{A}}, \hat{v}, \hat{\epsilon})$ be graded covers of an artinian category $\mathcal{A}$. An **cover-equivalence** is a triple (F, π, ϵ), where $F : \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ is an additive functor and $\epsilon : [1]F \overset{\sim}{\Rightarrow} F[1]$ and $\pi : \hat{v}F \overset{\sim}{\Rightarrow} \tilde{v}$ are isotransformations of functors such that the following diagram of isotransformations of functors commutes:

$$\begin{tikzcd}
\hat{v}[1]F \arrow[r, Rightarrow, "\epsilon", "\sim"'] \arrow[d, Rightarrow, "\hat{\epsilon}"', "\wr"] & \hat{v}F[1] \arrow[r, Rightarrow, "\pi", "\sim"'] & \tilde{v}[1] \arrow[d, Rightarrow, "\wr"', "\tilde{\epsilon}"] \\
\hat{v}F \arrow[rr, Rightarrow, "\pi", "\sim"'] & & \tilde{v}
\end{tikzcd}$$

Two covers are said to be **cover-equivalent** iff there is a cover-equivalence from one to the other. We will show in 6.1 that this is indeed an equivalence relation.

1.12. This generalizes the definition of equivalence of gradings given in [BGS96, 4.3.1.2]. We will show in 5.1 that given a left-artinian ring A every graded cover of A-Modf is cover-equivalent to the graded cover given by a $\mathbb{Z}$-grading on A. The question when two graded covers of this type are cover-equivalent to each other is discussed in 5.2.

2. Graded covers of artinian categories

Example 2.1. Let A be a left-artinian ring. Given any $\mathbb{Z}$-grading $\tilde{}$ on A and let $(\tilde{A}\text{-Modf}^{\mathbb{Z}},\, v,\, \epsilon)$ be the category of finitely generated $\mathbb{Z}$-graded left $\tilde{A}$-modules with morphisms homogeneous of degree 0 and (v, ϵ) the natural forgetting of the grading. Then $(\tilde{A}\text{-Modf}^{\mathbb{Z}},\, v,\, \epsilon)$ is a graded cover of A-Modf. To check the second condition in definition 1.5, let $M \twoheadrightarrow N$ be a surjection of a not necessarily graded module onto a graded module. A generating system of M gives a generating system of N. Take the nonzero homogeneous components of its elements. It is possible to choose preimages of these components in M in such a way, that they generate M. Then a suitable graded free A-module P with its basis vectors going to these preimages will do the job.

Remark 2.2. Every graded cover of an artinian category is also artinian, as the length can get only bigger when we apply an exact functor which doesn't annihilate any object.

Definition 2.3. Given a graded cover $(\tilde{\mathcal{A}}, v, \epsilon)$ of an artinian category $\mathcal{A}$, a **$\mathbb{Z}$-graded lift** or for short **lift** of an object $M \in \mathcal{A}$ is a pair $(\tilde{M}, \varphi)$ with $\tilde{M} \in \tilde{\mathcal{A}}$ and $\varphi : v\tilde{M} \xrightarrow{\sim} M$ an isomorphism in $\mathcal{A}$.

Lemma 2.4. *If for an indecomposable object there exists a lift, then this lift is unique up to isomorphism and shift.*

Proof. Given an indecomposable object $M \in \mathcal{A}$, its endomorphism ring $\mathcal{A}(M, M)$ is a local ring. Suppose there are two lifts $(\tilde{M}, \varphi)$, $(\hat{M}, \psi)$ of M. We can then decompose the identity morphism of M into homogeneous components. Because the non-units in $\mathcal{A}(M, M)$ form its maximal ideal, at least one of the homogeneous components has to be a unit, i.e. an isomorphism. $\square$

Lemma 2.5. *Given a graded cover $(\tilde{\mathcal{A}}, v, \epsilon)$ of an artinian category $\mathcal{A}$ a non-trivial object $0 \neq M \in \tilde{\mathcal{A}}$ is never isomorphic to its shifted versions $M[i]$ for $i \neq 0$.*

Proof. It is enough to prove the statement for simple objects. From now on let $M \in \tilde{A}$ be simple. Suppose there is an isomorphism $M \xrightarrow{\sim} M[i]$ for some $i \neq 0$. Then the endomorphism ring of $vM \in \mathcal{A}$ is given by twisted Laurent series over a skew-field, more precisely $\mathcal{A}(vM, vM)$ is of the form $K^{\sigma}[X, X^{-1}]$ where the skew-field $K = \tilde{\mathcal{A}}(M, M)$ is the endomorphism ring of M in $\tilde{\mathcal{A}}$, $\sigma : K \to K$ an automorphism of skew-fields and $cX = X\sigma(c)$ for all $c \in K$. Obviously 0 and 1 are the only idempotents in $K^{\sigma}[X, X^{-1}]$, so $vM \in \mathcal{A}$ is an indecomposable object. On the other hand it has finite length by assumption. Thus, by a version of Fittings lemma, all elements in the endomorphism ring $\mathcal{A}(vM, vM)$ have to be either units or nilpotent, and this is just not the case. $\square$

Lemma 2.6. *Given a graded cover $(\tilde{\mathcal{A}}, v, \epsilon)$ of an artinian category $\mathcal{A}$, forgetting of the grading induces a bijection of sets*

$$(\mathrm{irr}\tilde{\mathcal{A}})/\mathbb{Z} \xrightarrow{\sim} \mathrm{irr}\mathcal{A}$$

between the isomorphism classes of simple objects in the graded cover modulo shift and the isomorphism classes of simple objects in $\mathcal{A}$.

Proof. First we show that each epimorphism $vM \twoheadrightarrow L$ in $\mathcal{A}$, where $M \in \tilde{\mathcal{A}}$ is simple and $L \in \mathcal{A}$ is non-zero, has to be an isomorphism. By definition of a graded cover we find an object $N \in \tilde{\mathcal{A}}$ such that vN maps epimorphically onto $\ker(vM \twoheadrightarrow L)$, i.e. there is a resolution $vN \xrightarrow{\varphi} vM \twoheadrightarrow L$ of L in $\mathcal{A}$ by objects having a $\mathbb{Z}$-graded lift. We can decompose φ into homogeneous components $\varphi = \sum_{i\in\mathbb{Z}} \varphi_i$ and each summand $\varphi_i : N \to M[i]$ has to be either trivial or an epimorphism in $\tilde{\mathcal{A}}$. Using 2.5, the non-trivial summands give an epimorphism

$$(\varphi_i) : N \twoheadrightarrow \bigoplus_{i\in\mathbb{Z},\ \varphi_i\neq 0} M[i]$$

in $\tilde{\mathcal{A}}$. This has to stay an epimorphism when we forget the grading and postcompose with the morphism $\bigoplus_{\varphi_i\neq 0} vM \twoheadrightarrow vM$ adding up the components. So if φ is non-zero, it is a surjection. Thus we have shown that the forgetting of the grading induces a map $(\mathrm{irr}\tilde{\mathcal{A}})/\mathbb{Z} \to \mathrm{irr}\mathcal{A}$. To see that it is surjective, take $L \in \mathrm{irr}\mathcal{A}$. By our assumptions there is an $M \in \tilde{\mathcal{A}}$ together with an epimorphism $vM \twoheadrightarrow L$; consider the set of all objects in $\tilde{\mathcal{A}}$ which map, after forgetting the grading, epimorphically onto L and choose among them an object $M \in \tilde{\mathcal{A}}$ of minimal length. If M is not simple, there is a non-trivial sub-object $K \subset M$ with non-trivial quotient and we get a short exact sequence

$$0 \to vK \to vM \to vC \to 0.$$

Then the restriction of $vM \twoheadrightarrow L$ to vK must be trivial, otherwise it contradicts the minimal length assumption on M; therefore vC has to map epimorphically onto L again contradicting the minimal length assumption on M. We conclude that $M \in \tilde{A}$ was already simple. The proof of injectivity is left to the reader. □

Proposition 2.7. *Projective objects do lift.*

Remark 2.8. Using the stability of graded covers by passing to the opposed categories 3.2, we easily deduce that injective objects do lift as well.

Proof. It is enough to prove the statement for indecomposable projective objects. Let P be one of those. It is known that P admits a unique simple quotient, which in turn by 2.6 admits a graded lift, so that we can write

$$P \twoheadrightarrow vL$$

with $L \in \tilde{A}$ a simple object. By our definition of a graded cover we can find an epimorphism $vM \twoheadrightarrow P$ such that the composition $vM \twoheadrightarrow P \twoheadrightarrow vL$ comes from a morphism $M \to L$. Assume now in addition, that M has minimal length for such a situation. If we can show that vM is indecomposable we are done, because P is projective and thus the morphism $vM \twoheadrightarrow P$ splits. Suppose $vM \cong A \oplus B$. Then one summand, say A, has to map epimorphically onto L. If B is not zero, then B also has a simple quotient $\pi : B \twoheadrightarrow vE$ and we get an epimorphism $\psi : vM \twoheadrightarrow vL \oplus vE$. We can decompose the composition

$$\lambda = \mathrm{pr}_2 \circ \psi : vM \to vL \oplus vE \twoheadrightarrow vE$$

into homogeneous components $\sum_{i\in\mathbb{Z}} \lambda_i$. If there was a non-zero $\lambda_i : M \to E[i]$ with $E[i] \ncong L$, then $\ker \lambda_i$ would also surject onto L and $v(\ker \lambda_i)$ would surject onto vL and thus onto P, contradicting our assumption of minimal length. So we may assume our epimorphism ψ is obtained by forgetting the grading from an epimorphism $(\tilde{\lambda}, \tilde{\varphi}) : M \to L \oplus L$. But then again $v(\ker \tilde{\lambda})$ will still surject onto vL, contradicting our assumption of minimal length. Thus vM is indecomposable and the split epimorphism $vM \twoheadrightarrow P$ has to be an isomorphism. □

Corollary 2.9. *Let $(\tilde{\mathcal{A}}, v, \epsilon)$ be a graded cover of an artinian category $\mathcal{A}$ and suppose $\mathcal{A}$ has enough projective objects. Then forgetting the grading induces a bijection of sets*

$$(\mathrm{inProj}\tilde{\mathcal{A}})/\mathbb{Z} \xrightarrow{\sim} \mathrm{inProj}\mathcal{A}$$

between the isomorphism classes of indecomposable projective objects in the graded cover modulo shift and the isomorphism classes of indecomposable projective objects in $\mathcal{A}$.

Proof. It is well known that if an artinian category $\mathcal{A}$ has enough projectives, taking the projective cover gives a bijection between the set of isomorphism classes of simple objects in $\mathcal{A}$ and the set of isomorphism classes of indecomposable projective objects in $\mathcal{A}$. So both sides are in bijection to the corresponding sets of isomorphism classes of simple objects. Since each projective admits a lift by 2.7, and since such a lift clearly is again projective, the statement follows from the existence and unicity statement about lifts of simple objects 2.6. □

Corollary 2.10. *Let A be a left-artinian ring with a $\mathbb{Z}$-grading. Then:*

(1) *There exists a complete system of primitive pairwise orthogonal idempotents in A such that all its elements are homogeneous;*
(2) *If $(1_x)_{x\in I}$ is such a system, $(\tilde{M}, \varphi)$ a graded lift of A considered as a left A-module, and $\tilde{A}$ the lift of A given by the $\mathbb{Z}$-grading, then there exists a map $n : I \to \mathbb{Z}$ along with an isomorphism of graded left $\tilde{A}$-modules*

$$\tilde{M} \xrightarrow{\sim} \bigoplus_{x\in I} \tilde{A} 1_x [n(x)]$$

Proof. It is easy to see that every homomorphism $\tilde{A} \to \tilde{A}[i]$ of graded left A-Modules is the multiplication from the right with an element of A homogeneous of degree i. There is a direct sum decomposition $\tilde{A} \cong \bigoplus_{x\in I} P_x$ into indecomposable objects in $\tilde{A}$-Modf$^{\mathbb{Z}}$, and its summands are projective. The corresponding idempotent endomorphisms of $\tilde{A}$ are right multiplications with some idempotents $1_x \in A$, homogeneous of degree zero. Forgetting the grading on the P_x we get indecomposable projective A-modules by 2.9, and thus our family $(1_x)_{x\in I}$ is a full set of primitive orthogonal idempotents in A. For the second statement let $\tilde{M} = \bigoplus_{y\in J} Q_y$ be a direct sum decomposition into indecomposable objects in $\tilde{A}$-Modf$^{\mathbb{Z}}$. Again its summands are projective, so by 2.9 they stay indecomposable when we forget the grading. So there is a bijection $\sigma : I \xrightarrow{\sim} J$ with $vP_x \cong vQ_{\sigma(x)}$ and by the uniqueness of lifts 2.4 we find $P_x[n(x)] \cong Q_{\sigma(x)}$ for suitable $n(x) \in \mathbb{Z}$. □

3. Alternative definition of graded covers

3.1. The following proposition establishes the relation to the concept of a $\mathbb{Z}$-grading as introduced in [BGS96]. It also ensures our concept of graded cover to be stable upon passing to the opposed categories. Apart from that, this section is not relevant for the rest of this article.

Proposition 3.2. *Let $\mathcal{A}$ be an artinian category. A triple $(\tilde{\mathcal{A}}, v, \epsilon)$ consisting of an abelian category $\tilde{\mathcal{A}}$ equipped with a strict automorphism $[1]$, an exact functor $v : \tilde{\mathcal{A}} \to \mathcal{A}$ and an isotransformation of functors $\epsilon : v \xrightarrow{\sim} v[1]$ is a graded cover of $\mathcal{A}$ if and only if the following hold:*

1. *For all $M, N \in \tilde{\mathcal{A}}$ the pair (v, ϵ) induces isomorphisms of extensions $\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^j_{\tilde{\mathcal{A}}}(M, N[i]) \xrightarrow{\sim} \mathrm{Ext}^j_{\mathcal{A}}(vM, vN)$;*
2. *Every irreducible object in $\mathcal{A}$ admits a graded lift.*

Proof. Let us first show that every graded cover has these two properties. For the second property, this follows from 2.6. For the first property, we use

$$\mathrm{Ext}^j_{\mathcal{A}}(vM, vN) = \varinjlim_{Q} \mathrm{Hot}^j_{\mathcal{A}}(Q, vN)$$

where Q runs over the system of all resolutions $Q \to vM$ of vM. Our condition (2) on a graded cover ensures that if we take all resolutions $P \to M$ in $\tilde{\mathcal{A}}$, the resolutions $vP \to vM$ will be cofinal in the system of all resolutions of vM and thus give the same limit. The claim follows. Now let us show to the contrary that our two properties ensure both conditions of the definition of a graded cover 1.5. The first condition is obvious. To show the second condition, we may use pull-backs and induction to restrict to the case $\ker(M \twoheadrightarrow vN)$ is simple. Then by assumption this kernel admits a graded lift, so we arrive at a short exact sequence

$$vL \hookrightarrow M \twoheadrightarrow vN$$

Now use the isomorphism $\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^1_{\tilde{\mathcal{A}}}(N, L[i]) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}}(vN, vL)$ to write the class e of the above extension as a finite sum of homogeneous components $e = \sum_{i=a}^{b} e_i$. We then get a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=a}^{b} L[i] \arrow[r, hook] & \bigoplus_{i=a}^{b} E_i \arrow[r, two heads] & \bigoplus_{i=a}^{b} N \\
\bigoplus_{i=a}^{b} L[i] \arrow[r, hook] \arrow[u, equal] & P \arrow[r, two heads] \arrow[u] & N \arrow[u, "\Delta"']
\end{tikzcd}$$

with a right pullback square and by forgetting the grading another commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=a}^{b} vL \arrow[r, hook] \arrow[d, "\Sigma"] & vP \arrow[r, two heads] \arrow[d] & vN \arrow[d, equal] \\
vL \arrow[r, hook] & M \arrow[r, two heads] & vN
\end{tikzcd}$$

with a left pushout square. This finishes the proof. $\square$

4. Lifting functors to $\mathbb{Z}$-graded covers

Definition 4.1. A **$\mathbb{Z}$-category** is a category $\tilde{A}$ together with a strict autoequivalence [1] which we call "shift of grading". A **$\mathbb{Z}$-Functor** between $\mathbb{Z}$-categories is a pair (F, ϵ) consisting of a functor F between the underlying categories and an isotransformation $\epsilon : F[1] \overset{\sim}{\Rightarrow} [1]F$.

Definition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be artinian categories, $F : \mathcal{A} \to \mathcal{B}$ an additive functor and let $(\tilde{\mathcal{A}}, v, \epsilon)$ and $(\tilde{\mathcal{B}}, w, \eta)$ be graded covers of $\mathcal{A}$ and $\mathcal{B}$ respectively. A **$\mathbb{Z}$-graded lift of** F is a triple $(\tilde{F}, \pi, \epsilon)$, where $\tilde{F} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ is an additive $\mathbb{Z}$-functor and $\pi : w\tilde{F} \overset{\sim}{\Rightarrow} Fv$ and $\epsilon : [1]\tilde{F} \overset{\sim}{\Rightarrow} \tilde{F}[1]$ are isotransformations of functors, such that the following diagram of isotransformations of functors commutes:

$$\begin{tikzcd}
w[1]\tilde{F} \arrow[r, Rightarrow, "\epsilon", "\sim"'] \arrow[d, Rightarrow, "\eta"', "\wr"] & w\tilde{F}[1] \arrow[r, Rightarrow, "\pi", "\sim"'] & Fv[1] \arrow[d, Rightarrow, "\wr"', "\epsilon"] \\
w\tilde{F} \arrow[rr, Rightarrow, "\pi", "\sim"'] & & Fv
\end{tikzcd}$$

4.3. By definition, a cover-equivalence as defined in 1.11 between two graded covers of a given artinian category is a graded lift of the identity functor in the sense of 4.2.

4.4. Take two $\mathbb{Z}$-graded left-artinian rings $\tilde{A}$ and $\tilde{B}$ and in addition a B-A-bimodule X of finite length as left B-module. Then obviously the functor $F = X\otimes_A : A\text{-Modf} \to B\text{-Modf}$ admits a graded lift $\tilde{F} : \tilde{A}\text{-Modf}^{\mathbb{Z}} \to \tilde{B}\text{-Modf}^{\mathbb{Z}}$ if and only if X admits a $\mathbb{Z}$-grading making it into a graded $\tilde{B}$-$\tilde{A}$-bimodule $\tilde{X}$.

5. Comparing graded covers of module categories

Proposition 5.1. *Let A be a left artinian ring and $(\tilde{\mathcal{A}}, v, \epsilon)$ a graded cover of A-Modf. Then there exists a $\mathbb{Z}$-grading $\tilde{}$ on A such that $\tilde{A}$-Modf$^{\mathbb{Z}}$ is cover-equivalent to $\mathcal{A}$.*

Proof. By 2.7 there exists a lift $(\tilde{M}, \varphi)$ of A in $\tilde{\mathcal{A}}$. By assumption we obtain isomorphisms

$$\bigoplus_i \tilde{\mathcal{A}}(\tilde{M}, \tilde{M}[i]) \xrightarrow{\sim} \operatorname{End}_A(A) \xleftarrow{\sim} A^{\mathrm{opp}}$$

Here the left map comes from forgetting the grading and the right map from the action by right multiplication. We leave it to the reader to check that this grading on A will do the job. □

Proposition 5.2. *Let A be a left-artinian ring and let $\tilde{}$ and $\hat{}$ be two $\mathbb{Z}$-gradings on A. Then the following statements are equivalent:*

1. *The $\mathbb{Z}$-graded covers $\tilde{A}$-Modf$^{\mathbb{Z}}$ and $\hat{A}$-Modf$^{\mathbb{Z}}$ of A-Modf are cover-equivalent;*
2. *There exists a $\mathbb{Z}$-grading on the abelian group A making it a graded $\hat{A}$-$\tilde{A}$-bimodule $\hat{\tilde{A}}$;*
3. *For each complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in A, homogeneous for the grading $\hat{A}$, there exist a unit $u \in A^\times$ and a function $n : I \to \mathbb{Z}$ such that the homogeneous elements of $\tilde{A}$ in degree i are given by*
$$\tilde{A}_i = \textstyle\bigoplus_{x,y\in I} u1_x\hat{A}_{n(x)-n(y)+i}1_y u^{-1}$$
4. *There exist a complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in A, homogeneous for the grading $\hat{A}$, a unit $u \in A^\times$ and a function $n : I \to \mathbb{Z}$ such that the homogeneous elements of $\tilde{A}$ in degree i are given by*
$$\tilde{A}_i = \textstyle\bigoplus_{x,y\in I} u1_x\hat{A}_{n(x)-n(y)+i}1_y u^{-1}$$

Proof. (1) ⇔ (2) follows from 4.4. Next we prove (2) ⇒ (3). Our graded bimodule $\hat{\tilde{A}}$ with $\mathrm{id} : \hat{v}\hat{\tilde{A}} \xrightarrow{\sim} A$ is a $\mathbb{Z}$-graded lift in $\hat{A}$-Modf$^{\mathbb{Z}}$ of the left A-module A. By 2.10, for each complete system of pairwise orthogonal idempotents $1_x \in A$, homogeneous for $\hat{A}$, there exist integers $n(x)$ along with an isomorphism

$$\psi : \hat{\tilde{A}} \xrightarrow{\sim} \textstyle\bigoplus_{x\in I} \hat{A}1_x[n(x)]$$

of graded left $\hat{A}$-modules. Here both sides, when considered as ungraded left A-modules, admit obvious natural isomorphisms to the left A-module A. In these terms ψ has to correspond to the right multiplication by a unit $u \in A^\times$. Now certainly $h \mapsto \psi h \psi^{-1}$ is an isomorphism between the endomorphism rings of these graded left $\hat{A}$-modules and with $a \mapsto u^{-1}au$ in the lower horizontal we get a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_A\left(\tilde{\hat{A}}\right) & \stackrel{\sim}{\rightarrow} & \mathrm{End}_A\left(\bigoplus_{x\in I} \hat{A}1_x[n(x)]\right) \\ \uparrow\wr & & \wr\uparrow \\ A & \stackrel{\sim}{\rightarrow} & A \end{array}$$

Here End_A means endomorphism rings of ungraded modules, but with the grading coming from the grading on our modules, and the vertical arrows are meant to map $a \in A$ to the multiplication by a from the right modulo the obvious natural isomorphisms mentioned above. In particular, the vertical maps are not compatible but rather "anticompatible" with the multiplication. Nevertheless, the lower horizontal has to be homogeneous for the gradings induced by the vertical isomorphisms from the upper horizontal and from that we deduce

$$u^{-1}\tilde{A}_i u = \bigoplus\nolimits_{x,y} 1_x \hat{A}_{n(x)-n(y)+i} 1_y$$

To prove (3) $\Rightarrow$ (4), just recall that by 2.10 we can always find a complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in A, which are homogeneous for the grading $\hat{A}$. To finally check (4) $\Rightarrow$ (2), just equip A with the grading $\tilde{A}$ for which the right multiplication by u as a map $(\cdot u) : \tilde{\hat{A}} \stackrel{\sim}{\rightarrow} \bigoplus_x \hat{A}1_x[-n(x)]$ is homogeneous of degree zero. $\square$

6. Cover equivalence is an equivalence relation

6.1. Clearly cover-equivalence of graded covers is a reflexive relation. We are now going to show it is also symmetric and transitive, so it is indeed an equivalence relation on the set of graded covers of a fixed artinian category.

Lemma 6.2. *Any graded lift $\tilde{F}$ of an equivalence F of artinian categories is again an equivalence of categories.*

Proof. Since a direct sum of morphisms of abelian groups is an isomorphism if and only if the individual morphisms are isomorphisms, a graded lift of a fully faithful additive functor is clearly fully faithful itself. We just have to show that if F is an equivalence of categories, then $\tilde{F}$ is essentially surjective. So take an object $\tilde{B} \in \tilde{\mathcal{B}}$. By assumption there is an object $A \in \mathcal{A}$ with an isomorphism $FA \stackrel{\sim}{\rightarrow} w\tilde{B}$. By the definition of graded cover and 3.2, there are $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{A}}$ with an epimorphism and a monomorphism $v\tilde{X} \twoheadrightarrow A \hookrightarrow v\tilde{Y}$. Applying F we find an epimorphism and a monomorphism $w\tilde{F}\tilde{X} \twoheadrightarrow w\tilde{B} \hookrightarrow w\tilde{F}\tilde{Y}$. Now if $\lambda_i : \tilde{F}\tilde{X}[i] \to \tilde{B}$ for i runnig through a finite set $I \subset \mathbb{Z}$ of degrees are the homogeneous components of the first map, then they together define the left epimorphism of a sequence

$$\bigoplus_{i\in I} \tilde{F}\tilde{X}[i] \twoheadrightarrow \tilde{B} \hookrightarrow \bigoplus_{j\in J} \tilde{F}\tilde{Y}[j]$$

in $\tilde{\mathcal{B}}$. The left monomorphism is constructed dually. But the composition in this sequence has to come from a morphism in $\tilde{\mathcal{A}}$, and the image of this morphism is the looked-for object of $\tilde{\mathcal{A}}$ essentially going to $\tilde{B}$ under our functor $\tilde{F}$. □

6.3 (**Symmetry of cover-equivalence**). In particular, given a cover-equivalence (F, π) of graded covers the functor F is always an equivalence of categories. Given a quasiinverse (G, η) with $\eta : \mathrm{Id} \stackrel{\sim}{\Rightarrow} FG$ an isotransformation, from $\pi : \hat{v}F \stackrel{\sim}{\Rightarrow} \tilde{v}$ we get as the composition $\hat{v} \stackrel{\sim}{\Rightarrow} \hat{v}FG \stackrel{\sim}{\Rightarrow} \tilde{v}G$ or more precisely $(\pi G)(\hat{v}\eta)$ an isotransformation $\tau : \hat{v} \stackrel{\sim}{\Rightarrow} \tilde{v}G$. Similarly from $\epsilon : [1]F \stackrel{\sim}{\Rightarrow} F[1]$ we get a unique $\epsilon : [1]G \stackrel{\sim}{\Rightarrow} G[1]$ such the composition

$$[1] \stackrel{\sim}{\Rightarrow} [1]FG \stackrel{\sim}{\Rightarrow} F[1]G \stackrel{\sim}{\Rightarrow} FG[1] \stackrel{\sim}{\Rightarrow} [1]$$

with our adjointness η at both ends and the old and the newly to be defined ε in the middle ist the identity transformation. Then one may check that (G, τ, ϵ) is also a cover-equivalence.

6.4 (**Transitivity of cover-equivalence**). Let finally be given artinian categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, additive functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, graded covers $(\tilde{\mathcal{A}}, v, \epsilon), (\tilde{\mathcal{B}}, w, \eta), (\tilde{\mathcal{C}}, u, \theta)$ of $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and graded lifts $(\tilde{F}, \pi, \varepsilon)$ of F and $(\tilde{G}, \tau, \varepsilon)$ of G. Then $(\tilde{G}\tilde{F}, (G\pi)(\tau\tilde{F}), \varepsilon)$ is a lift of GF, where we leave the definition of the last ε to the reader. In particular the relation of cover-equivalence is transitive.

7. Proof of the Main Theorem

7.1 (**Push-forward of graded covers**). Let $(\tilde{\mathcal{A}}, \tilde{v}, \tilde{\epsilon})$ be a graded cover of an artinian category $\mathcal{A}$ and let $E : \mathcal{A} \stackrel{\approx}{\to} \mathcal{B}$ be an equivalence of categories. Then obviously $(\tilde{\mathcal{A}}, E\tilde{v}, E(\tilde{\epsilon}))$ is a graded cover of $\mathcal{B}$. We call it the "push-forward" of our graded cover of $\mathcal{A}$ along E. Obviously two graded covers of $\mathcal{A}$ are cover-equivalent iff their pushforwards are cover-equivalent as graded covers of $\mathcal{B}$.

7.2. Recall the block decomposition of $\mathcal{O}$. It is enough to prove theorem 1.1 for each block $\mathcal{O}_\lambda$ of $\mathcal{O}$.

Theorem 7.3 (**Uniqueness of graded covers of $\mathcal{O}$**). *If two graded covers $(\tilde{\mathcal{O}}_\lambda, \tilde{v}, \tilde{\epsilon})$ and $(\hat{\mathcal{O}}_\lambda, \hat{v}, \hat{\epsilon})$ of a block $\mathcal{O}_\lambda$ of category $\mathcal{O}$ are both compatible with the action of the center, they are cover-equivalent.*

Proof. Take a system $(P_x)_{x \in I}$ of representatives for the isomorphism classes of indecomposable projectives in $\mathcal{O}_\lambda$ and let $P = \bigoplus_{x \in I} P_x$ be their sum, a minimal projective generator $P \in \mathcal{O}_\lambda$. For $A := \mathrm{End}_{\mathfrak{g}}(P)$ the functor $M \mapsto \mathrm{Hom}_{\mathfrak{g}}(P, M)$ is an equivalence $E : \mathcal{O}_\lambda \stackrel{\approx}{\to} \mathrm{Modf}\text{-}A$. By 7.1 it is enough to check the pushforwards of our graded covers along E are cover-equivalent. Next choose $\mathbb{Z}$-graded lifts $\tilde{P}_x \in \tilde{\mathcal{O}}_\lambda$ and $\hat{P}_x \in \hat{\mathcal{O}}_\lambda$ of our projectives $P_x \in \mathcal{O}_\lambda$. Put $\tilde{P} := \bigoplus_{x \in I} \tilde{P}_x$. By

5.1 the pushforward of our first graded cover $E\tilde{v} : \tilde{\mathcal{O}}_\lambda \to \text{Modf-}A$ is cover-equivalent to the graded cover given by the $\mathbb{Z}$-grading $\tilde{A}$ on A determined by the canonical isomorphism

$$\bigoplus_i \tilde{\mathcal{O}}_\lambda\left(\tilde{P}, \tilde{P}[i]\right) \xrightarrow{\sim} A$$

Similarly, the pushforward of our second graded cover $E\hat{v} : \hat{\mathcal{O}}_\lambda \to \text{Modf-}A$ is cover-equivalent to the graded cover given by the $\mathbb{Z}$-grading $\hat{A}$ on A constructed analogously. By 7.1 we just need to check the two $\mathbb{Z}$-gradings $\tilde{A}$ and $\hat{A}$ on A lead to cover-equivalent graded covers of Modf- A. For this we use our criterion 5.2 along with more detailed information on the structure of A. More precisely, to avoid indices, let us choose in addition an antidominant projective $Q \in \mathcal{O}_\lambda$. The Struktursatz from [Soe90] tells us that the functor $\mathbb{V} : \mathcal{O}_\lambda \to \text{Modf-End}_{\mathfrak{g}}(Q)$ given by $P \mapsto \text{Hom}_{\mathfrak{g}}(Q, P)$ is fully faithful on projective objects. Furthermore the Endomorphismensatz from [Soe90] tells us that the action of the center Z on Q gives an epimorphism $Z \twoheadrightarrow \text{End}_{\mathfrak{g}}(Q)$. In addition it tells us, that given a maximal ideal $\chi \subset Z$ and $n \in \mathbb{N}$ with $\chi^n Q = 0$, the kernel of the induced surjection $Z/\chi^n \twoheadrightarrow \text{End}_{\mathfrak{g}}(Q)$ is a homogeneous ideal for the natural grading on Z/χ^n defined in 1.10. Now recall the $\mathbb{Z}$-graded lifts $\tilde{P}_x$ and $\hat{P}_x$ of our projectives $P_x \in \mathcal{O}_\lambda$. Also choose $\mathbb{Z}$-graded lifts $\tilde{Q} \in \tilde{\mathcal{O}}_\lambda$ and $\hat{Q} \in \hat{\mathcal{O}}_\lambda$ of our antidominant projective Q. Set

$$\tilde{\mathbb{V}}\tilde{P}_x := \mathcal{O}_\lambda(\tilde{Q}, \tilde{P}_x) := \bigoplus_{i \in \mathbb{Z}} \tilde{\mathcal{O}}_\lambda(\tilde{Q}, \tilde{P}_x[i])$$

and similarly define the $\mathbb{Z}$-graded space $\hat{\mathbb{V}}\hat{P}_x := \mathcal{O}_\lambda(\hat{Q}, \hat{P}_x)$ for each $x \in I$. Then, because our graded covers are compatible with the action of the center, the Endomorphismensatz tells us that these are finitely generated $\mathbb{Z}$-graded modules over Z/χ^n. Furthermore, forgetting the grading both are canonically identified with the same indecomposable module $\mathbb{V}P_x := \mathcal{O}_\lambda(Q, P_x)$ over Z/χ^n. Thus by 2.4 for each $x \in I$ there exist a number $n(x) \in \mathbb{Z}$ along with an isomorphism

$$\psi_x : \hat{\mathbb{V}}\hat{P}_x[n(x)] \xrightarrow{\sim} \tilde{\mathbb{V}}\tilde{P}_x$$

of graded Z/χ^n-modules. Forgetting the grading this isomorphism induces an automorphism $v(\psi_x)$ of the Z/χ^n-module $\mathbb{V}P_x$. Using fully-faithfulness of the functor $\mathbb{V} := \text{Hom}_{\mathfrak{g}}(Q, \ )$ on projectives, the direct sum of these automorphisms gives us an automorphism u of our minimal projective generator P alias a unit $u \in A^\times$. We claim this is precisely the unit we need in order to apply our equivalence criterion 5.2(4) with $1_x \in A$ the projections onto the P_x and $n(x)$ as above. Well, for this it will be sufficient to show

$$1_x \tilde{A}_i 1_y = u 1_x \hat{A}_{n(x)-n(y)+i} 1_y u^{-1}$$

for all x, y, i. In the following, it will be convenient to abbreviate $n(x) = n_x$. By definition, it is equivalent to check the equality of the images of the natural maps in the left vertical of the commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{O}}_\lambda(\tilde{P}_y, \tilde{P}_x[i]) \arrow[r, "\sim"] \arrow[d] & \mathrm{Mod}_Z^{\mathbb{Z}}(\tilde{\mathbb{V}}\tilde{P}_y, \tilde{\mathbb{V}}\tilde{P}_x[i]) \arrow[d] \\
\mathcal{O}_\lambda(P_y, P_x) \arrow[r, "\sim"] & \mathrm{Mod}_Z(\mathbb{V}P_y, \mathbb{V}P_x) \\
\hat{\mathcal{O}}_\lambda(\hat{P}_y, \hat{P}_x[n_x - n_y + i]) \arrow[r, "\sim"] \arrow[u, "\mathrm{int}\, u"] & \mathrm{Mod}_Z^{\mathbb{Z}}(\hat{\mathbb{V}}\hat{P}_y[n_y], \hat{\mathbb{V}}\hat{P}_x[n_x + i]) \arrow[u, "v(\psi_x)\circ ? \circ v(\psi_y)^{-1}"']
\end{tikzcd}$$

However this equality of images is clear in the right vertical, by the very definition of the ψ_x, and thus follows for the left vertical. $\square$

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