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Graded Ring Theory

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FOREWORD

This book is aimed to be a 'technical' book on graded rings. By 'technical' we mean that the book should supply a kit of tools of quite general applicability, enabling the reader to build up his own further study of non-commutative rings graded by an arbitrary group. The body of the book, Chapter A, contains: categorical properties of graded modules, localization of graded rings and modules, Jacobson radicals of graded rings, the structure theory for simple objects in the graded sense, chain conditions, Krull dimension of graded modules, homogenization, homological dimension, primary decomposition, and more .....

One of the advantages of the generality of Chapter A is that it allows direct applications of these results to the theory of group rings, twisted and skew group rings and crossed products. With this in mind we have taken care to point out on several occasions how certain techniques may be specified to the case of strongly graded rings. We tried to write Chapter A in such a way that it becomes suitable for an advanced course in ring theory or general algebra, we strove to make it as self-contained as possible and we included several problems and exercises.

Other chapters may be viewed as an attempt to show how the general techniques of Chapter A can be applied in some particular cases, e.g. the case where the gradation is of type Z. In compiling the material for Chapters B and C we have been guided by our own research interests. Chapter B deals with commutative graded rings of type Z and we focus on two main topics: artihmetically graded domains, and secondly, local conditions for Noetherian rings.

In Chapter C we derive some structural results relating to the graded properties of the rings considered. The following classes of graded rings receive special attention: fully bounded Noetherian rings, birational extensions of commutative rings, rings satisfying polynomial identities, and Von Neumann regular rings. Here the basic idea is to derive results of ungraded nature from graded information. Some of these sections lead naturally to the study of sheaves over the projective spectrum Proj(R) of a positively graded ring, but we did not go into these topics here. We refer to [125] for a non-commutative treatment of projective geometry, i.e. the geometry of graded P.I. algebras.

Chapter D presents the theory of filtered rings and modules and we focus in particular on the gradation associated to a filtration.

In the sections entitled 'Comments, References, Exercises' we included references to interesting related topics that have not been treated in this book.

A remark about references: only references to a chapter different from the chapter containing the reference contain the chapter's indication A, B, C or D; references within the same chapter only mention the section numbers.

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Francine did a nice job typing this book and we thank her for the quality of her work as well as for her patience.

CHAPTER A: SOME GENERAL TECHNIQUES IN THE THEORY OF GRADED RINGS

I: Graded Rings and Modules

I.1. Graded Rings and the Category of Graded Modules

Let G be a multiplicative group with identity element e. A ring R is said to be a graded ring of type G if there is a family of additive subgroups of R, say $\{R_\sigma, \sigma \in G\}$, such that $R = \bigoplus_{\sigma\in G} R_\sigma$ and $R_\sigma R_\tau \subset R_{\sigma\tau}$ for all $\sigma,\tau \in G$, where we denoted by $R_\sigma R_\tau$ the set of all finite sums of products $r_\sigma r'_\tau$ with $r_\sigma \in R_\sigma$, $r'_\tau \in R_\tau$. The elements of $\bigcup_{\sigma\in G} R_\sigma$ are called homogeneous elements of R, a nonzero $r \in R_\sigma$ is said to be homogeneous of degree σ, and we write $\deg r = \sigma$. Any nonzero $r \in R$ has a unique expression as a sum of homogeneous elements, $r = \sum_{\sigma\in G} r_\sigma$, where r_σ is nonzero for a finite number of σ in G.

The nonzero elements r_σ in the decomposition of r are called the homogeneous components of r.

Let R be a graded ring of type G. If $\pi : G \to H$ is an epimorphism of groups then the ring S = R with gradation $S_\alpha = \bigoplus_{\sigma\in\pi^{-1}(\alpha)} R_\sigma$, for all $\alpha \in H$, is a graded ring of type H. This ring will be denoted by $R_{(H)}$. If $\theta : H \to G$ is a monomorphism of groups, the ring $R^{(H)}$ with gradation $(R^{(H)})_\sigma = R_{\theta(\sigma)}$ for all $\sigma \in H$ is a graded ring of type H. In case $G = Z$, $H = \{nd, n \in Z\}$ then $R^{(H)}$ is denoted by $R^{(d)}$ i.e. $R^{(d)} = \bigoplus_{n\in Z} R_{nd}$.

If G is commutative and R is a graded ring of type G then the ring S = R with gradation defined by $S_\sigma = R_{\sigma^{-1}}$ for all $\sigma \in G$, is called the opposite graded ring.

I.1.1. Proposition If R is a graded ring of type G then R_e is a subring of R and $1 \in R_e$.

Proof : From $R_eR_e \subset R_e$ it follows immediately that R_e is a subring of R. Let $1 = \sum_{\sigma\in G} r_\sigma$ be the homogeneous decomposition of $1 \in R$. Pick $\tau \in G$ and $\lambda_\tau \in R_\tau$, then $\lambda_\tau = 1.\lambda_\tau = \sum_{\sigma\in G} r_\sigma\lambda_\tau$, with $r_\sigma\lambda_\tau \in R_{\sigma\tau}$. Consequently $r_\sigma\lambda_\tau = 0$ holds for all $\sigma \neq e$ in G. It follows that $r_\sigma\lambda = 0$ for all $\sigma \neq e$ in G and for all $\lambda \in R$. Therefore $1 = r_e \in R_e$.

I.1.2. Examples

1. Let G be a group, A a ring. The group ring $R = A[G]$ is a graded ring of type G where $R_\sigma = A_\sigma$ for all $\sigma \in G$.
2. Any ring R may be considered as a graded ring of type G, for any group G, by putting $R_e = R$, $R_\sigma = 0$ for $\sigma \neq e$ in G. Such a ring is said to be trivially G-graded.
3. Let R be a graded ring of type G. Let R° be the opposite ring for R i.e. R° has the same underlying additive group as R but multiplication in R° is defined by the rule $x \circ y = yx$ for x, y R°. Putting $(R^\circ)_\sigma = R_{\sigma^{-1}}$ makes R° into a graded ring of type G.
4. Let R be any ring and let $\varphi : R \to R$ an injective ring homomorphism The ring of twisted polynomials $S = R[x,\varphi]$ is obtained by adding a variable X to R and define multiplication by $\varphi(r)X = Xr$ for all $r \in R$ and extend it to polynomials in X (writing coefficients on the left hand side of the powers of X). Obviously S is graded of type Z with $S_i = 0$ if $i < 0$ and $S_i = \{a X^i, a \in R\}$ if $i \geq 0$. If φ is an automorphism of R then it is clear that $R[X, X^{-1},\varphi]$ may be constructed in a similar way.

5. Let k be a field, X an indeterminate, and let k(X) be the field of rational functions in X over k. Although k(X) can only be graded of type Z in the trivial way, it is very well possible to define non-trivial gradations of type Z/nZ on k(X), for every $n \in \mathbb{N}$. Indeed;

if $\bar{i} \in \mathbb{Z}/n\mathbb{Z}$, put $k(X)_{\bar{i}} = X^i k(X^n) = \{X^i \dfrac{f(X^n)}{g(X^n)}, g \neq 0\}$.

It is easily seen that, if i and j have the same image mod n then $k(X)_{\bar{i}} = k(X)_{\bar{j}}$, hence the gradation is well defined. Straightforward verification learns that $k(X)$ is graded of type $\mathbb{Z}/n\mathbb{Z}$.

For a ring R we write R-mod for the category of left R-modules whereas mod-R denotes the category of right R-modules.

Let R be a graded ring of type G, an $M \in$ R-mod is said to be a graded left R-module if there is a family $\{M_\sigma, \sigma \in G\}$ of additive subgroups of M such that $M = \bigoplus_{\sigma \in G} M_\sigma$ and $R_\sigma M_\tau \subset M_{\sigma\tau}$ for all $\sigma, \tau \in G$. Elements of $h(M) = \bigcup_{\sigma \in G} M_\sigma$ are called homogeneous elements of M. If $m \neq 0$ is in M_σ for some $\sigma \in G$ then m is homogeneous of degree σ and we write : $\deg m = \sigma$. Any nonzero $m \in M$ has a unique decomposition $m = \sum_{\sigma \in G} m_\sigma$ where all but a finite number of the m_σ are zero.
Unless otherwise stated module will mean left module. A submodule N of M is a graded submodule if $N = \bigoplus_{\sigma \in G} (N \cap M_\sigma)$, or equivalently, if for any $x \in N$ the homogeneous components of x are again in N.

I.1.3. Example. Let R be a graded ring, M a graded R-module and let N be a submodule of M. Let $(N)_g$ be the submodule of M generated by $N \cap h(M)$. It is easily verified that $(N)_g$ is maximal amongst submodules of N which are graded submodules of M.

Consider graded R-modules M and N. An R-linear $f : M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$, if $f(M_\sigma) \subset M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree τ build an additive subgroup $\mathrm{HOM}_R(M,N)_\tau$ of $\mathrm{Hom}_R(M,N)$. It is clear that $\mathrm{HOM}_R(M,N) = \bigoplus_{\tau \in G} \mathrm{HOM}_R(M,N)_\tau$ is a graded abelian group of type G.

Composition of a graded morphism $f : M \to N$ of degree $\sigma \in G$ with a graded morphism $g : N \to P$ of degree $\tau \in G$ yields a graded morphism $g \circ f$ of degree $\sigma\tau$. If N is a submodule of M then M/N may be made into a graded module by putting : $(M/N)_\sigma = M_\sigma + N/N$. With this definition, the canonical R-linear projection $M \to M/N$ is a graded morphism of degree e (left and right!). The category $R\text{-gr}_G$ consists of graded R-modules of type G and the morphisms are taken to be the graded morphisms of degree e. In the sequel we shall write R-gr if there is no ambiguity possible about G. The category R-gr possesses direct sums and products. Furthermore if $(M_\alpha, f_{\alpha\beta}; \alpha, \beta \in I)$ is some inductive system, resp. projective system of objects in R-gr, then the module $\varinjlim_\alpha M_\alpha \in$ R-mod, resp. $\varprojlim_\alpha M_\alpha \in$ R-mod may be graded by putting $(\varprojlim_\alpha M_\alpha)_\sigma = \varinjlim_\alpha (M_\alpha)_\sigma$, resp. $(\varprojlim_\alpha M_\alpha)_\sigma = \varprojlim_\alpha (M_\alpha)_\sigma$ for all $\sigma \in G$.

If M, N $\in$ R-gr then we write $\mathrm{Hom}_{R\text{-gr}}(M,N)$ for the graded morphisms of degree e from M to N. For any $f \in \mathrm{Hom}_{R\text{-gr}}(M,N)$, both Ker f and Coker f are in R-gr. Indeed, since Im f is a graded submodule of N it follows that Coker f = N/Im f = $\bigoplus_{\sigma\in G} (N_\sigma + \mathrm{Im}\, f)/\mathrm{Im} f$ is the cokernel of f in R-gr. It is not hard to verify that R-gr is an abelian category which satisfies Grothendieck's axioms Ab 3, Ab 4, Ab $3^\star$ and Ab $4^\star$, cf [102]. Since also Ab 5 is easily verified, we will say that R-gr is a Grothendieck category.

The forgetful functor, U : R-gr $\to$ R-mod, (U for "ungrading", unusual perhaps, but preferable to "degrading") associates to M the underlying ungraded R-module. The forgetful functor U has a right adjoint which associates to M $\in$ R-mod the graded R-module ${}^GM = \bigoplus_{\sigma\in G} {}^\sigma M$ where each ${}^\sigma M$ is a copy of M written $\{{}^\sigma x,\ x \in M\}$, with R-module structure defined by : $r \star {}^\tau x = {}^{\sigma\tau}(rx)$ for each $r \in R_\sigma$. If $f : M \to N$ is R-linear then

${}^{G}f : {}^{G}M \to {}^{G}N$ is given by ${}^{G}f({}^{\sigma}x) = {}^{\sigma}f(x)$, and it is clear that ${}^{G}f$ has degree $e \in G$.

The functor $M \to {}^{G}M$, from R-mod $\to$ R-gr is exact. Note that $U({}^{G}M)$ need not be a direct sum of copies of M since the component ${}^{\sigma}M$, $\sigma \in G$, is not an R-submodule of ${}^{G}M$ (but it is an R_e-submodule of course).

I.1.4. Notation .

We will write $U(M) = \underline{M}$ but we omit this for R since it will always be clear from the context whether R is being considered as a graded ring or not.

One of the standard problems concerning R-gr that we will be dealing with may be phrased as follows : if M has property P as an object of R-gr, is it true then that $\underline{M}$ has property P in R-mod. The converse problem is usually far more easier to handle.

If $M \in R\text{-gr}_G$ then for $\sigma \in G$ we define the σ-suspension $M(\sigma)$ of M to be the graded module obtained from $\underline{M}$ by putting $M(\sigma)_\tau = M_{\tau\sigma}$. So we defined a functor T_σ : R-gr $\to$ R-gr, $T_\sigma(M) = M(\sigma)$ and this functor may be characterized by :

1° $T_\sigma o T_\tau = T_{\sigma\tau}$, for all $\sigma, \tau \in G$

2° $T_\sigma \text{ o } T_{\sigma^{-1}} = T_{\sigma^{-1}} \text{ o } T_\sigma$ = Id, for all $\sigma \in G$

3° $U \text{ o } T_\sigma = U$, for all $\sigma \in G$

It follows that, for any $\sigma \in G$, T_σ defines an equivalence of categories. In a straightforward way one proves that $\{R(\sigma), \sigma \in G\}$ is a set of generators of R-gr and that R-gr is a Grothendieck category with enough injective objects, cf. [102] .

An $F \in$ R-gr is said to be gr-free if F has a basis of homogeneous elements, or equivalently $F \cong \bigoplus_{\sigma \in S} R(\sigma)$, where S is a subset of G. Since any $M \in$ R-gr is isomorphic to a quotient of a gr-free object in R-gr and since the latter are certainly projective as objects of R-gr, it follows that R-gr has enough projective objects.

I.1.5. Example. Let H be a subgroup of G and let $S \subset G$ be a section for G modulo H i.e. a set of representatives for the right H-cosets of G. Let R be a graded ring of type G and $M \in R\text{-gr}_G$. For any $\sigma \in S$ we put $M^{(\sigma,H)} = \bigoplus_{h \in H} M_{h\sigma}$. Then $M^{(\sigma,H)}$ is a graded $R^{(H)}$-module with gradation : $(M^{(\sigma,H)})_h = M_{h\sigma}$ for all $h \in H$. It is clear that $M = \bigoplus_{\sigma \in S} M^{(\sigma,H)}$ and for any $\sigma \in S$, $M \to M^{(\sigma,H)}$ defines a functor : $R\text{-gr} \to R^{(H)}\text{-gr}$. In particular if $G = Z$ and $H = \{nd, n \in 7)$ with $d > 0$ then we may choose $S = \{0,1,\dots,d-1\}$. For $0 \le k \le d-1$ we write $R^{(k, d)} = R^{(k, H)}$.

I.1.6. Remark If R is a graded ring of type G then we can define the category gr-R of graded right R-modules. If M, N $\in$ gr-R then an R-linear $f : M \to N$ is said to be graded of degree τ, $\tau \in G$, if $f(M_\sigma) \subset M_{\tau\sigma}$ for all $\sigma \in G$. Note that the side on which τ is written is determined by the fact that f should be right R-linear. The σ-suspension of $M \in$ gr-R should then be defined as follows $(\sigma)M$ is determinded by $\underline{M} \in$ mod-R with gradation $((\sigma)M)_\tau = M_{\sigma\tau}$; again,the side on which σ acts is determined by the fact that $(\sigma)M$ should now be a graded right R-module ! Consequently if f is both a left and right graded morphism of two-sided graded R-modules then f must have central degree i.e. deg $f \in Z(G)$.

I.2. Elementary Properties of Graded Modules

Throughout this section R is a graded ring of type G for some fixed group G, so we omit notational references to G in what follows.

I.2.1. Lemma. Let M; N, P $\in$ R-gr and consider the following commutative diagram of R-morphisms :

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "f"'] & & N \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

where f has degree e. If g (resp. h) has degree e then there exists a graded morphism h' (resp. g') of degree e, such that f = g o h' (resp. f = g' o h).

Proof We give the proof in case deg g = e.
Pick $x \in M_\sigma$. From $h(x) = \sum_{\sigma \in G} h(x)_\sigma$ it follows that $h' : M \to N$ may be defined by putting $h'(x) : h(x)_\sigma$.

I.2.2. Corollary. Consider P $\in$ R-gr; then P is a projective object in R-gr if and only if $\underline{P}$ is a projective R-module.
Proof If P is a projective in R-gr then it is a direct summand of a gr-free object F. Therefore $\underline{P}$ is projective in R-mod. Conversely, suppose $\underline{P}$ is projective in R-mod. Since P $\in$ R-gr there is a gr-free object F $\in$ R-gr such that $F \underset{f}{\to} P \to 0$ is exact in R-gr. Projectivity of $\underline{P}$ entails the existence of an R-linear map $g : \underline{P} \to \underline{F}$ such that $f \circ g = 1_{\underline{P}}$. According to Lemma I.2.1. there exists a graded morphism $g' : P \to F$ of degree e such that $f \circ g' = 1_P$. Thus the exact sequence $F \underset{f}{\to} P \to 0$, splits and therefore P is a projective object of R-gr.
We will sometimes refer to projective objects of R-gr as gr-projective graded modules. So by the above corollary gr-projective is nothing but

graded plus projective.

I.2.3. Corollary. Let $0 \to N \to M$ be exact in R-gr. Then N is a direct summand of M if and only if $\underline{N}$ is a direct summand of $\underline{M}$.

A graded left R-submodule I of R is called (an homogeneous or) a graded left ideal of R. If I is two-sided then it is called a graded ideal of R.

I.2.4. Lemma. Let $E \in$ R-gr. The following statements are equivalent :

1. E is an injective object of R-gr.
2. The functor $HOM_R(-, E)$ is exact.
3. For every graded left ideal I of R, the canonical inclusion $i : I \to R$ gives rise to a surjective morphism $HOM_R(i, 1_E) : HOM_R(R,E) \to HOM_R(I,E)$

Proof. $1 \Leftrightarrow 2$. By definition E is injective in R-gr if and only if the functor $Hom_{R\text{-}gr}(-,E) = HOM_R(-, E)_0$ is exact. Moreover E is injective in R-gr if and only if $E(\sigma)$ is injective for every $\sigma \in G$. Since a morphism $f : M \to E$ of degree σ may be viewed as a morphism of degree e, $M \to E(\sigma)$, the foregoing remarks prove the desired equivalence.

$2 \Rightarrow 3$. Obvious.

$3 \Rightarrow 1$. The proof is similar to the proof of Baer's theorem in the ungraded case.

I.2.5. Corollary. If $E \in$ R-gr is such that $\underline{E}$ is an injective R-module then E is injective in R-gr. Injective objects in R-gr will be said to be gr-injective.

I.2.6. Remarks

1. If E is gr-injective then $\underline{E}$ need not be injective. For example, let k be any field and consider $R = k[X, X^{-1}]$ where X is a variable commuting with k and with gradation given by $R_i = \{a X^i, a \in k\}$ for

all $\in \mathbb{Z}$. Check that R is injective in R-gr but not in R-mod.

2. If $L \in$ R-gr is such that $\underline{L}$ is a free R-module then L need not be gr-free. For example, take $R = \mathbb{Z} \times \mathbb{Z}$ with trivial gradation. Let L be R but endowed with the following gradation : $L_0 = \mathbb{Z} \times \{0\}$, $L_1 = \{0\} \times \mathbb{Z}$ and $L_i = 0$ for $i \neq 0, 1$. Obviously L is not gr-free.

The projective dimension of an R-module $\underline{M}$ will be denoted by $\mathrm{p.dim}_R(\underline{M})$. If $M \in$ R-gr then the projective dimension of M in the category R-gr will be denoted by $\mathrm{gr.p.dim}_R(M)$. We have the following direct consequence of Corollary I.2.2. :

<u>I.2.7. Corollary</u>. If $M \in$ R-gr then $\mathrm{gr.p.dim}_R(M) = \mathrm{p.dim}_R(\underline{M})$

<u>I.2.8. Lemma</u>. Let $M \in$ R-gr, N a graded submodule of M. If N is essential in M as a subobject of M in R-gr, then $\underline{N}$ is essential in $\underline{M}$.

<u>Proof</u>. Recall that a subobject of M is said to be an essential subobject or M is an essential extension of N, if N intersects all nonzero subobjects of M non-trivially. Hence, the fact that N is gr-essential in M means that for each $x \neq 0$ in $h(M)$ there exists an $a \in h(R)$ such that $ax \in N - \{0\}$. If $x \in M$ is nonzero then we may decompose x as $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$ where the x_σ are the homogeneous components of x of degree σ. The proof goes by induction on n, establishing that for each $x \in \underline{M}$ there is an $a \in h(R)$ such that $a\,x \in \underline{N}$. For $n = 1$ this is obvious from gr-essentiality of N in M. Put $y = x_{\sigma_1} + \ldots + x_{\sigma_{n-1}}$. Since $x_{\sigma_n} \neq 0$ there is an $a \in h(R)$ such that, $ax_{\sigma_n} \in N$ and $x_{\sigma_n} \neq 0$. Therefore $ax - ax_{\sigma_n} = ay$ has at most $n-1$ homogeneous components in M. Now, if $ay = 0$, then $ax \in N$. If $ay \neq 0$ then the induction hypothesis provides us with a $b \in h(R)$ such that $b\,a\,y \in N$ and $b\,a\,y \neq 0$. Thus $b\,a\,x = b\,a\,x_{\sigma_n} + b\,a\,y \in N$. But $b\,a\,x = 0$ would

entail $b\,a\,x_{\sigma_n} = b\,a\,y = 0$, contradicting the choice of b, therefore $b\,a\,x \neq 0$.

<u>I.2.9. Remark.</u> Let $M \in R$-gr. and let N be a small graded submodule of M (<u>small</u> is the notion dual to essential), then N need not be small in M. For example, in the graded ring $R = k[X]$, where k is a field, (X) is small in the graded sense but not in R-mod.

<u>I.2.10. Lemma.</u> Let $M, N \in R$-gr. The group $HOM_R(M,N)$ consists of all $f \in Hom_R(\underline{M}, \underline{N})$ for which there exist a finite subset of G, F say, such that :

(*) $f(M_\sigma) \subset \sum_{\varphi \in F} N_{\sigma\varphi}$, for all $\sigma \in G$.

<u>Proof.</u> Suppose that $f \in HOM_R(M,N)$ then there exist $\sigma_1, \sigma_2, \ldots, \sigma_n \in G$ such that $f = f_{\sigma_1} + \ldots + f_{\sigma_n}$ and $f_{\sigma_i} \in HOM_R(M,N)_{\sigma_i}$, $i = 1,\ldots,n$. Clearly, f satisfies (*). Conversely, suppose $f \in Hom_R(\underline{M}, \underline{N})$ satisfies (*) for some finite set $F \subset G$. Look at an $x_\sigma \in M_\sigma$, by (*) it follows that $f(x_\sigma) = \sum_{\varphi \in F} y_{\sigma,\sigma\varphi}$ for unique elements $y_{\sigma,\sigma\varphi} \in N_{\sigma\varphi}$, $\varphi \in F$. For any $\varphi \in F$ we define $f_\varphi(x_\sigma) = y_{\sigma,\sigma\varphi}$. It is obvious that $f_\varphi \in HOM_R(M,N)_\varphi$ and $f = \sum_{\varphi \in F} f_\varphi$ thus $f \in HOM_R(M,N)$.

<u>I.2.11. Corollary.</u> We have that $HOM_R(M,N) = Hom_R(\underline{M},\underline{N})$ in each of the following cases :

1° The group G is finite

2° $\underline{M}$ is finitely generated.

<u>Proof.</u>

1° Trivial

2° Let $\underline{M}$ be generated by $m_1, \ldots, m_r$. It is not restrictive to suppose that $m_1, \ldots, m_r$ are homogeneous and nonzero, of degree $\alpha_1, \ldots, \alpha_r$ resp.

Consider $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$, then : for $i = 1,\dots,r$, $f(m_i) = \sum_{j=1}^{t_i} n_{\sigma_{ij}}$ with $n_{\sigma_{ij}} \in N_{\sigma_{ij}}$, $\sigma_{ij} \in G$. For each i, $1 \le i \le r$ we put :

$F_i = \{\alpha_i^{-1}\sigma_{i1}, \alpha_i^{-1}\sigma_{i2},\dots, \alpha_i^{-1}\sigma_{it_i}\}$ and $F = \bigcup_{i=1}^{r} F_i$.

We will now check that $f(M_\sigma) \subset \sum_{\varphi \in F} N_{\sigma\varphi}$ for all $\sigma \in H$

For $x_\sigma \in M_\sigma$ we may write :

$x_\sigma = \sum_{i=1}^{r} r_i m_i$ with $r_i \in h(R)$ and, $\deg r_i = \sigma\alpha_i^{-1}$. Therefore it follows that $f(x_\sigma) = \sum_{i=1}^{r} r_i f(m_i) = \sum_{i=1}^{r}\sum_{j=1}^{t_i} r_i n_{\sigma_{ij}}$. However $\deg r_i n_{\sigma_{ij}} = \sigma\alpha_i^{-1}\sigma_{ij}$ and therefore :

$r_i n_{\sigma_{ij}} \in N_{\sigma\varphi}$ where $\varphi = \alpha_i^{-1}\sigma_{ij} \in F$ for all $1 \le i \le r$.

Remark If M is not finitely generated, then it may happen that $\mathrm{HOM}_R(M,N) \ne \mathrm{Hom}_R(M,N)$. For example, let $R = \bigoplus_{n\in Z} R_n$ be a graded ring of type Z such that $R_n \ne 0$, $\forall\, n \in Z$. Then there exists an element $(a_n)_{n\in Z} \in \prod_{n\in Z} R_n$ which is not in $\bigoplus_{n\in Z} R_n$ Put $M = R^{(Z)}$ and define $f \in \mathrm{Hom}_R(M,R)$ by putting $f((x_n)_n) = \sum_{i\in Z} x_i a_i$. By Lemma I.2.10, clearly $f \notin \mathrm{HOM}_R(M,R)$.

If $R = \bigoplus_{\sigma\in G} R_\sigma$ is a graded ring, then R is called left gr-Noetherian if every left graded ideal is finitely generated (for details see A.II.3.). The derived functors of the left exact functors $\mathrm{Hom}_R(.,.)$ resp. $\mathrm{HOM}_R(.,.)$ by $\mathrm{Ext}_R^n(.,.)$ resp. $\mathrm{EXT}_R^n(.,.)$. Then we have :

I.2.12. Corollary. If R is a graded left Noetherian ring and M is a finitely generated graded R-module, then, for every $n \ge 0$: $\mathrm{EXT}_R^n(M,N) = \mathrm{Ext}_R^n(M,N)$ for every graded R-module N.

Proof. Since R is gr. left Noetherian, M has a finite resolution :

$\dots F_2 \to F_1 \to F_0 \to M \to 0$

where the F_i are free graded R-module and also finitely generated.

Now Corollary I.2.11. finishes the proof.

Remark. If the group G is finite, we have $\mathrm{EXT}^n_R(M, N) = \mathrm{Ext}^n_R(M, N)$ for every $n \geq 0$, and $M, N \in R\text{-gr}$.

To conclude this section we study tensor products of graded modules Throughout $R = \bigoplus_{\sigma \in G} R_\sigma$ is a graded ring of type G; wherever Z-gr is being considered we assume that Z is equipped with the trivial G-gradation.

Definition. Let $M \in \text{gr-}R$, $N \in R\text{-gr}$. The abelian group $M \underset{R}{\oplus} N$ may be graded by putting $(M \underset{R}{\otimes} N)_\sigma$, $\sigma \in G$, equal to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$, $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of Z-gr hence defined will be called the graded tensor product of M and N. An $N \in R\text{-gr}$ is said to be gr-flat if the functor $\_ \underset{R}{\otimes} N : \text{gr-}R \to Z\text{-gr}$ is exact (it is in general only right exact).

Let us list some elementary properties, the proof of which is similar to the proof of the corresponding statements in the ungraded case.

I.2.13. Proposition. Let R and S be graded rings of type G. Consider $M \in \text{gr-}R$, $N \in R\text{-gr-}S$, then $M \underset{R}{\otimes} N$ is a graded right S-module.

I.2.14. Proposition. Let $M \in \text{gr-}R$, $P \in \text{gr-}S$, $N \in R\text{-gr-}S$, then there is a natural graded isomorphism :

$$\mathrm{HOM}_S(M \underset{R}{\otimes} N, P) \simeq \mathrm{HOM}_R(M, \mathrm{HOM}_S(N,P)).$$

I.2.15. Proposition. Let $P \in \text{gr-}R$, $N \in S\text{-gr}$, $M \in S\text{-gr-}R$, then there is a canonical graded morphism Φ,

$$\Phi : P \underset{R}{\otimes} \mathrm{HOM}_S(M,N) \to \mathrm{HOM}_S(\mathrm{HOM}_R(P,M), N),$$

defined by $\Phi(p \otimes f)(g) = (f \circ g)(p)$, for $p \in P$, $f \in \mathrm{HOM}_S(M,N)$, $g \in \mathrm{HOM}_R(P, M)$. If P is finitely generated and projective then Φ is an isomorphism.

I.2.16. Proposition. If $P \in$ gr-R, $N \in S$-gr, $M \in R$-gr-S, then there exists a canonical graded morphism Ψ,

$$\Psi : \mathrm{HOM}_R(P,M) \underset{S}{\otimes} N \to \mathrm{HOM}_R(P, M \underset{S}{\otimes} N),$$

defined by $\Psi(f \otimes n)(p) = f(p) \otimes n$. If P is finitely generated and projective, then Ψ is an isomorphism.

I.2.17. Remark. If G is abelian then for any $M \in$ gr-R, $N \in R$-gr we have $M(\sigma) \underset{R}{\otimes} N(\tau) = (M \underset{R}{\otimes} N)(\sigma\tau)$ for $\sigma, \tau \in G$.

I.2.18. Proposition. Let $M \in R$-gr, then M is gr-flat if and only if $\underline{M}$ is flat in R-mod.

Proof. That flatness of $\underline{M}$ entails gr-flatness of M is obvious. Conversely, as in [5], it is easy to show that M is the inductive limit in R-gr of gr-projective modules. Then, by Corollary I.2.2., $\underline{M}$ is the inductive limit of projective modules, hence $\underline{M}$ is flat.

I.2.19. Corollary. The gr-flat dimension of $M \in R$-gr, denoted by gr-$\mathrm{wdim}_R M$ (defined as the corresponding ungraded concept which is denoted by wdim_R), is equal to $\mathrm{w\,dim}_R \underline{M}$.

An exact sequence $0 \to M' \to M \to M'' \to 0$ in R-gr is said to be gr-pure if for any $N \in$ gr-R the sequence $0 \to N \underset{R}{\otimes} M' \to N \underset{R}{\otimes} M \to N \underset{R}{\otimes} M'' \to 0$ is exact in $\mathbb{Z}$-gr.

I.2.20. Corollary. The exact sequence $0 \to M' \to M \to M'' \to 0$ in R-gr is gr-pure if and only if the corresponding exact sequence $0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$ in R-mod is a pure sequence.

Proof. The idea of the proof is modeled upon ungraded techniques i.e. a result of [62] entails that the gr-purity of $0 \to M' \to M \to M'' \to 0$ implies that this sequence is an inductive limit of a filtered set of exact splitting sequences of the form : $0 \to M' \to M' \oplus P^{(i)} \to P^{(i)} \to 0$ where each $P^{(i)}$ is finitely presented in R-gr. The properties of the latter sequence imply the corresponding ungraded analogues for

$0 \to \underline{M}' \to \underline{M}' \oplus \underline{P}^{(i)} \to \underline{P}^{(i)} \to 0.$

Therefore by the result of loc. cit. the latter sequence is pure in R-mod and the same will hold for the limit sequence $0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$

Conversely, purity in R-mod obviously implies gr-purity in R-gr.

I.3. Strongly Graded Rings

A graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$ of type G is said to be a strongly graded ring of type G if $R_\sigma R_\tau = R_{\sigma\tau}$ for all σ, $\tau \in G$. If $G = Z$ then a Z-graded ring is strongly graded if and only if $RR_1 = R$, if and only if $R_1R = R$. Note also that a commutative ring can only be strongly graded by an abelian group.

I.3.1. Examples.

a. Let G be a group, A a ring. The group algebra $R = A[G]$ is a strongly graded ring of type G.

b. If R is a strongly graded ring of type G and if $\pi : G \to H$ is an epimorphism then $R_{(H)}$ is strongly graded. If H' is a subgroup of G then $R^{(H')}$ is also strongly graded.

c. If R is strongly graded then so is R/I, for every graded ideal I of R.

I.3.2. Lemma. The G-graded ring R is strongly graded if and only if $1 \in R_\sigma R_{\sigma^{-1}}$ for all $\sigma \in G$.

Proof. Suppose $1 \in R_\sigma R_{\sigma^{-1}}$ holds for all $\sigma \in G$. For any $\tau \in G$ it follows then that :

$R_{\sigma\tau} = 1.R_{\sigma\tau} \subset R_\sigma R_{\sigma^{-1}} R_{\sigma\tau} \subset R_\sigma R_\tau$, hence $R_{\sigma\tau} = R_\sigma R_\tau$.

The converse is even more obvious.

Note : It is clear that the foregoing may be strengthened to : R is strongly graded if $1 \in R_{\sigma_i \sigma_i^{-1}}$ for some set of generators $\sigma_i, i \in I$, of G.

I.3.3. Corollary. If R is strongly graded then, for all $\sigma \in G$, R_σ is finitely generated and projective in R_e-mod and also in mod-R_e (e the neutral element of G).

Proof. From $R_{\sigma^{-1}} R_\sigma = R_e$ it follows that R_σ contains a finitely generated R_e-module L_σ such that $R_{\sigma^{-1}} L_\sigma = R_e$. Then $R_\sigma R_{\sigma^{-1}} L_\sigma = R_\sigma R_e = R_\sigma$ yields yields : $L_\sigma = R_\sigma$; hence R_σ is finitely generated in R -mod (similarly as a right R_e-module). We now establish that R_σ is a projective left R_e-module. If $1 = \Sigma\, u_i v_i$ with $u_i \in R_{\sigma^{-1}}$, $v_i \in R_\sigma$ then right multiplication by u_i defines an R_e-morphism $\mu_i : R_\sigma \to R_e$ such that for each $r \in R_\sigma$ we have $r = \Sigma_i\, \mu_i(r) v_i$. Therefore by the well-known "Basis Lemma", R_σ is projective in R_e-mod (similar in mod-R_e).

For a graded ring R of type G we may define a functor $R \underset{R_e}{\otimes} - : R_e\text{-mod} \to R\text{-gr}$ as follows : for an $M \in R_e$-mod, define $(R \underset{R_e}{\otimes} M)_\sigma = R_\sigma \underset{R_e}{\otimes} M$ for all $\sigma \in G$.

On the other hand one can define the restriction functor $(-)_e : R\text{-gr} \to R_e\text{-mod}$ by putting $(M)_e = M_e$ while a morphism $f : M \to N$ in R-gr restricts to $(f)_e = $ $= f_e : M_e \to N_e$.

I.3.4. Theorem. If R is a graded ring of type G then the following statements are equivalent to one another :

1. R is a strongly graded ring.
2. Every graded R-module is strongly graded.
3. The functors $R \underset{R_e}{\otimes} -$ and $(-)_e$ define an equivalence between the Grothendieck categories R-gr and R_e-mod.

Proof. In proving $1 \Leftrightarrow 2$ we only have to establish $1 \Rightarrow 2$. For $\sigma, \tau \in G$ we have : $M_{\sigma\tau} = R_e M_{\sigma\tau} = R_\sigma R_{\sigma^{-1}} M_{\sigma\tau} \subset R_\sigma M_\tau$, hence $M_{\sigma\tau} = R_\sigma M_\tau$, for any $M \in R$-gr.

Again, in proving $2 \Leftrightarrow 3$ it will suffice to establish $2 \Rightarrow 3$.

Take $M \in R$-gr and let $\delta_M = R \underset{R_e}{\otimes} M_e \to M$ be the canonical R-linear morphism given by $\delta_M(r \otimes m_e) = r\, m_e$, $r \in R$, $m_e \in M_e$ and note that it is graded of

degree e. It is easily seen that δ_M is a natural transform of the composition functor $(R \underset{R_e}{\otimes} -) \circ (-)_e$ to the identity functor on R-gr. It is clear that δ_M is an epimorphism. If $K = \mathrm{Ker}\,\delta_M$ then K is a graded submodule of $R \underset{R_e}{\otimes} M_e$ and $K_e = \mathrm{Ker}(\delta_M)_e$ where $(\delta_M)_e : R_e \underset{R_e}{\otimes} M_e$ is an isomorphism, hence $K_e = 0$ and also $K_\sigma = R_\sigma . K_e = 0$, i.e. $K = 0$.
Conversely, if $M \in R_e$-mod let $\alpha_M : M \to (R \underset{R_e}{\otimes} M)_e$ be given by $\alpha_M(x) = 1 \otimes x$, $x \in M$. It is clear that α_M defines a natural transform between the identity functor on R_e -mod and the composition functor $(-)_e \circ (R \underset{R_e}{\otimes} -)$.

<u>I.3.5. Corallary</u>. Let R be strongly graded ring of type G and let M, N $\in$ R-gr. If $f : M \to N$ is a morphism in R-gr, then f is a monomorphism, epimorphism, isomorphism if and only if $f_\sigma : M_\sigma \to N_\sigma$ is resp. a monomorphism, epimorphism isomorphism in R_e-mod for some $\sigma \in G$.

<u>Proof</u>. Since the functor T_σ : R-gr $\to$ R-gr is an equivalence it follows that f is monic, epic, isomorphism if and only if $T_\sigma(f) : M(\sigma) \to N(\sigma)$ is resp. monic, epic, isomorphism.
Now $M(\sigma)_e = M_\sigma$, $N(\sigma)_e = N_\sigma$, hence application of Theorem I.3.4. yields the result.

An R-bimodule ${}_RM_R$ is <u>invertible</u> if there exists an R-bimodule ${}_RN_R$ such that $M \underset{R}{\otimes} N \simeq R$, $N \underset{R}{\otimes} M \simeq R$ as R-bimodules.

<u>I.3.6. Proposition</u>. The following properties hold for a strongly graded ring R :

1. For every $M \in$ R-gr and all $\sigma, \tau \in G$ the canonical morphism $R \underset{R_e}{\otimes} M_\sigma \to M(\sigma)$ defined by $r \otimes x \to rx$ is an isomorphism.
2. For every $\sigma, \tau \in G$, the canonical morphism $R_\sigma \underset{R_e}{\otimes} R_\tau \to R_{\sigma\tau}$ is an isomorphism of R_e-bimodules.

3. For every $\sigma \in G$, R_σ is an invertible R_e-bimodule.

Proof.

1. Direct from Theorem I.3.4.

2. By the first part, $R \otimes_{R_e} R_\tau \to R(\tau)$ is an isomorphism, hence $(R \otimes_{R_e} R_\tau)_\sigma =$ $= R(\tau)_\sigma = R_{\sigma\tau}$; therefore $R_\sigma \otimes_{R_e} R_\tau \simeq R_{\sigma\tau}$.

3. Immediate from 2.

I.3.7. Corollary. Let R be strongly graded, $M \in R\text{-gr}$, then $M = 0$ if and only if $M_\sigma = 0$ for some $\sigma \in G$.

I.3.8. Corollary.

In a strongly graded ring R every graded left ideal I is generated as a left ideal by I_e.

Proof By Theorem I.3.4., 2., I is strongly graded, hence $I_\sigma = R_\sigma I_e$ and thus : $I = R\, I_e$.

I.3.9. Corollary: Let $\varphi : R \to S$ be a graded ring morphism, of degree e, between strongly graded rings R and S. Then φ is injective (surjective, bijective) if and only if the restriction $\varphi_e : R_e \to S_e$ is injective (surjective, bijective).

Proof. If φ_e is injective then by Corollary I.3.8. it follows that φ is injective too. If φ_e is surjective then $\varphi(R_{\sigma^{-1}}) \subset S_{\sigma^{-1}}$ and $\varphi(R_e) = S_e$ entails $S_e = \varphi(R_{\sigma^{-1}})\, \varphi(R_\sigma) \subset S_{\sigma^{-1}}\, \varphi(R_\sigma)$. Therefore $S_{\sigma^{-1}}\, \varphi(R_\sigma) = S_e$ and thus $S_\sigma = S_\sigma\, S_e = S_\sigma S_{\sigma^{-1}}\, \varphi(R_\sigma)$ yielding $S_\sigma = \varphi(R_\sigma)$ for $\sigma \in G$.

Let us now focus on some methods for constructing strongly graded rings Let A be any ring; write Z(A) for the center of A, U(A) for the groups of units of A, Aut(A) for the group of ring automorphisms of A and

Inn(A) for the subgroup of Aut(A) consisting of inner automorphisms of A. Consider an A-bimodule M and α, $\beta \in$ Aut(A). Let ${}_{\alpha}M_{\beta}$ be the A-bimodule defined as follows : equip the additive group underlying M with an A-bimodule structure given by $a.x = \alpha(a)x$, $x.a = x\,\beta(a)$ for $x \in M$, $a \in A$. Identify M and ${}_1M_1$.

<u>I.3.10. Lemma</u>. If α, β, $\gamma \in$ Aut(A) then :

1. The map $x \to \gamma(x)$ determines an isomorphism of A-bimodules : ${}_{\alpha}A_{\beta} \to {}_{\gamma\alpha}A_{\gamma\beta}$.
2. The map $x \otimes y \to x\,\alpha(y)$ determines an isomorphism of A-bimodules :

   ${}_1A_{\alpha} \underset{A}{\otimes} {}_1A_{\beta} \to {}_1A_{\alpha\beta}$
3. There is equivalence between $\alpha\beta^{-1} \in$ Inn(A) and ${}_1A_{\alpha} \cong {}_1A_{\beta}$ as A-bimodules
4. If δ : Aut(A) $\to$ Pic(A) is given by $\delta(\alpha) = [\,{}_1A_{\alpha}\,]$, then the following sequence of groups is exact :

   $1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \underset{\delta}{\to} \mathrm{Pic}(A)$

<u>Proof.</u> 1. and 2 are obvious.

3. Suppose that $f : {}_1A_{\alpha} \to {}_1A_{\beta}$ is an A-bimodule isomorphism. Left A-linearity of f yields that $f(x) = xu$ where $u = f(1) \in U(A)$. Moreover $f(\alpha(a)) = f(1.a) = u\,\beta(a)$. On the other hand $f(\alpha(a)) = f(\alpha(a).1) = \alpha(a)u$. Hence it follows that $\alpha(a)u = u\beta(a)$ or $\alpha(a) = u\,\beta(a)u^{-1}$.

   Let us denote $u\,x\,u^{-1}$ by $\alpha_u(x)$, then α_u is an inner automorphism of A and $\alpha = \alpha_u \circ \beta$, thus $\alpha\beta^{-1} \in$ Inn(A). Conversely, suppose that $\alpha\beta^{-1}(a) = uau^{-1}$ for some $u \in U(A)$.

   If we define $f : {}_1A_{\alpha} \to {}_1A_{\beta}$ by $f(x) = xu$ for all $x \in {}_1A_{\alpha}$, then it is clear that f is an A-bimodule isomorphism.
4. A direct consequence of 2. and 3.

<u>I.3.11. Lemma</u>. Let P be an invertible A-bimodule, then :

1. $A \simeq \mathrm{End}_A(P_A)$ and $A \simeq \mathrm{End}_A({}_AP)^{o}$.

2. $Z(A) \simeq \mathrm{End}_{A-A}(P)$, where by $\mathrm{End}_{A-A}(P)$ we denote the group of all A-bimodule homomorphisms of P.
3. If $P \simeq A$ as left A-modules then $P \simeq {}_1A_\alpha$ as A-bimodules for some $\alpha \in \mathrm{Aut}(A)$.
4. P is left (right) finitely generated projective A-module.
5. The dual $P^\star = \mathrm{Hom}_A(P_A, A)$ is an invertible A-bimodule and $P \otimes_A P^\star \simeq A$, $P^\bullet \otimes_A P \simeq A$ as A-bimodules.
6. If M, N $\in$ A-mod., then the homomorphism
   $\alpha_P : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(P \otimes_A M, P \otimes_A N)$, given by $\alpha_P(f) = 1 \otimes f$ is an isomorphism.
7. If M, N are A-bimodules, then the homomorphism
   $\alpha_P : \mathrm{Hom}_{A-A}(M,N) \to \mathrm{Hom}_{A-A}(P \otimes_A M, P \otimes_A N)$, given by $\alpha_P(f) = 1 \otimes f$, is an isomorphism.

Proof

1. By assumption there exists an A-bimodule Q and the A-bimodule isomorphisms $P \otimes_A Q \overset{\varphi}{\to} A$, $Q \otimes_A P \overset{\psi}{\to} A$. Let $\theta : A \to \mathrm{End}(P_A)$ be defined by $\theta(a)(x) = ax$. Let $x_i \in P$, $y_i \in Q$, $i = 1, 2, \ldots, n$ be such that $1 = \varphi(\sum_{i=1}^{n} x_i \otimes y_i) = \sum_{i=1}^{n} \varphi(x_i \otimes y_i)$.
   If $\theta(a) = 0$ then $a = a.1 = \sum_{i=1}^{n} \varphi(axi \otimes yi) = 0$, therefore θ is an injective homomorphism. On the other hand, if $f \in \mathrm{End}(P_A)$ then $f \otimes 1_\theta : P \otimes_A Q \to P \otimes_A Q$ is right A-linear. There exists $a_0 \in A$ such that the following diagram is commutative.

$$\begin{tikzcd} P \otimes_A Q \arrow[r, "\varphi"] \arrow[d, "f\otimes 1_Q"] & A \arrow[d, "\rho(a_0)"] \\ P \otimes_A Q \arrow[r, "\varphi"] & A \end{tikzcd}$$

   where $\rho(a_0)(a) = a_0 a$. If we define $g : P \to P$ by putting $g(p) = a_0 p$ for all $p \in P$, it is clear that $f \otimes 1_Q = g \otimes 1_Q$. From $(P \otimes_A Q) \otimes_A P \simeq P \otimes_A (Q \otimes_A P) \simeq$ $\simeq P \otimes_A A \simeq P$ it follows then that $f = g$ and consequently θ is surjective.

2. We define: $\theta\ Z(A) \to \mathrm{End}_{A-A}(P)$, $\theta(a)(x) = ax$, $a \in Z(A)$, $x \in P$. By the statement 1., θ is injective. Let $f \in \mathrm{End}_{A-A}(P)$; there exists $a \in A$ such that $f(x) = ax$ for all $x \in P$. As f is a left A-homomorphism too, we have $f(bx) = b\,f(x)$, or $(ab)x = (ba)x$ for all $x \in P$. Since $1 = \sum_{i=1}^{n} \varphi(x_i \otimes y_i)$ then $ab = \sum_{i=1}^{n} \varphi(ab\, x_i \otimes y_i) = \sum_{i=1}^{n} \varphi\,(ba\, x_i \otimes y_i) = ba$. Hence $a \in Z(A)$.

3. Suppose $f : P \to A$ is a left A-linear isomorphism and let $a \in A$. We define a left A-linear isomorphism $g: P \to P$ by putting $g(p) = f^{-1}(f(p)a)$ for all $p \in P$. Since $A \simeq \mathrm{End}_A\,({}_AP)^o$, there exists an $\alpha(a) \in A$ such that $g(p) = p\,\alpha(a)$, in other words $f(p\alpha(a)) = f(p)\,a$ for all $p \in P$. Evidently $\alpha \in \mathrm{Aut}(A)$ and the foregoing may then be phrased as follows : $f : {}_1P_\alpha \to A$ is an A-bimodule isomorphism.

4,5. First we prove that we may choose the A-bimodule isomorphisms $P \otimes_A Q \xrightarrow{\varphi} A$ and $Q \otimes_A P \xrightarrow{\psi} A$ such that the diagrams

$$(1)\quad \begin{tikzcd} P \otimes_A Q \otimes_A P \arrow[r, "\varphi \otimes 1_P"] \arrow[d, "1_P \otimes \psi"] & A \otimes_A P \arrow[d, "\alpha"] \\ P \otimes_A A \arrow[r, "\beta"] & P \end{tikzcd} \quad \text{and (2)} \quad \begin{tikzcd} Q \otimes_A P \otimes_A Q \arrow[r, "\psi \otimes 1_Q"] \arrow[d, "1_Q \otimes \varphi"] & A \otimes_A Q \arrow[d, "\alpha'"] \\ Q \otimes_A A \arrow[r, "\beta"] & Q \end{tikzcd}$$

are commutative (here α, β, α', β' are the obvious maps).

Indeed, suppose we want the first diagram to commute. Since all maps are bimodule isomorphisms, there exists an A-bimodule automorphism $u : P \to P$, such that $\beta \circ (1_P \otimes g) = u \circ \alpha \circ (f \otimes 1_P)$. By the statement 2. there exists $c \in Z(A)$ such that $u(x) = cx$ for all $x \in P$. As u is an automorphism then c is an invertible element. Hence we replace φ by $c.\varphi$ and the first square is commutative.

We now prove that the second diagram commutes, i.e. $\alpha' o(\psi \otimes 1_Q) - \beta' o(1_Q \otimes \varphi) = 0$ Indeed, let $q, q' \in Q, p \in P$. Then $(\alpha' o(\psi \otimes 1_Q))(q \otimes p \otimes q') = \psi(q \otimes p)q'$, $(\beta' o(1_Q \otimes \varphi))(q \otimes p \otimes q') = q\,\varphi(p \otimes q')$. We denote $d = \psi(q \otimes p)q' - q\varphi(p \otimes q')$; $d \in Q$. Let $Q \otimes_A P \xrightarrow{\psi} A$ and $p' \in P$. Since ψ is an A-bimodule morphism, we have $\psi(\psi(q \otimes p)\, q' \otimes p') = \psi(q \otimes p).\psi(q' \otimes p') = \psi(q \otimes p.\psi(q' \otimes p'))$. Since the first diagram is commutative, then : $p\,\psi(q' \otimes p') = \varphi(p \otimes q')\, p'$

and therefore :

$\psi(\psi(q \otimes p)\, q' \otimes p') = \psi(q \otimes \varphi(p \otimes q')p') = \psi(q.\varphi(p \otimes q') \otimes p')$.

Hence it is clear that $\psi(d \otimes p') = 0$ and hence $d = 0$.

Therefore $\alpha' o(\psi \otimes 1_Q) - \beta' o(1_Q \otimes \varphi) = 0$

Now we prove the statement 4. Since $P \otimes_A Q \xrightarrow{\varphi} A$ is an isomorphism then $1 = \sum_{i=1}^{n} \varphi\,(p_i \otimes q_i)$ where $p_i \in P$, $q_i \in Q$.

Define $P \underset{g}{\overset{f}{\rightleftarrows}} A^n$ by $f(p) = (\psi(q_i \otimes p))_{i=1,2,\ldots,n}$ and $g((a_i)) = \sum_{i=1}^{n} p_i a_i$.

Then $(g \text{ o } f)(p) = g((\psi(q_i \otimes p)_{i=1}^{n}) = \sum_{i=1}^{n} p_i\,\psi\,(q_i \otimes p) = \sum_{i=1}^{n} \varphi(p_i \otimes q_i)p = p$

since the diagram (1) is commutative. Thus P is finitely generated and projective.

We prove the statement 5. Note that φ induces a bimodule homomorphism $Q \xrightarrow{\theta} \mathrm{Hom}_A(P_A, A)$, $\theta(q)(p) = \psi(q \otimes p)$. If $\theta(q) = 0$ then $\psi(q \otimes p) = 0$ for all $p \in P$. Since $1 = \sum_{i=1}^{n} \varphi(p_i \otimes q_i)$, $p_i \in P$, $q_i \in Q$, then $q = \sum_{i=1}^{n} q.\varphi(p_i \otimes q_i) =$
$= \sum_{i=1}^{n} \psi(q \otimes p_i)q_i = 0$ (as the diagram (2) is commutative). Hence θ is injective. To establish surjectivity let $f \in \mathrm{Hom}_A(P_A, A)$. If $p \in P$ then $f(p) = f(\sum_{i=1}^{n} \varphi(p_i \otimes q_i)p) = f(\sum_{i=1}^{n} .p_i.\psi(q_i \otimes p)) = \sum_{i=1}^{n} f(p_i)\,\psi\,(q_i \otimes p) =$
$= \sum_{i=1}^{n} \psi(f(p_i).q_i \otimes p) = \psi(\sum_{i=1}^{n} f(p_i).q_i \otimes p)$.

If we denote $q = \sum_{i=1}^{n} f(p_i)q_i$, then $f(p) = \psi(q \otimes p) = \theta(q)(p)$ and hence $\theta(q)=f$.

The statements 6. and 7. follow from the fact that the functor $M \to P \otimes_A M$ is an equivalence.

Recall that $\mathrm{Pic}(A)$ is the group of isomorphism classes $[P]$ of invertible A-bimodules. The group law is : $[P].[Q] = [P \otimes_A Q]$. By lemma I.3.11. we have that $[P]^{-1} = [\mathrm{Hom}_A(P_A, A)] = [\mathrm{Hom}_A({}_AP, A)]$.

I.3.12. Lemma Let A be a ring. There exists a canonical group homomorphism $\mathrm{Pic}(A) \xrightarrow{\theta} \mathrm{Aut}(Z(A))$.

Proof. Let $[P] \in \mathrm{Pic}(A)$ and $c \in Z(A)$. We have the homomorphism of bimodules $P \to P$, $x \to xc$.

Since $A \simeq \mathrm{End}_A(P_A)^o$ (Lemma I.3.11.1). There exists a unique element $\alpha_P(c)$ such that $\alpha_P(c)x = xc$ for all $x \in P$. Clearly that $\alpha_P : Z(A) \to Z(A)$ is an automorphism and $\theta : \mathrm{Pic}(A) \to \mathrm{Aut}(Z(A))$, $\theta([P]) = \alpha_P$, is a group homomorphism.

Let G be a group (not necessary commutative) and $\Phi : G \to \mathrm{Pic}(A)$ a group homomorphism. Put $\Phi(\sigma) = [P_\sigma]$ for all $\sigma \in G$. By a factor set associated to Φ we mean a family $f = \{f_{\sigma,\tau};\ \sigma,\tau \in G\}$ where $f_{\sigma,\tau};\ P_\sigma \otimes P_\tau \to P_{\sigma\tau}$ is an isomorphism of bimodules for all $\sigma,\tau \in G$ and the following diagrams (3) :

$$\begin{tikzcd}
P_\sigma \otimes P_\tau \otimes P_\theta \arrow[r, "1 \otimes f_{\tau,\theta}"] \arrow[d, "f_{\sigma,\tau} \otimes 1"'] & P_\sigma \otimes P_{\tau\theta} \arrow[d, "f_{\sigma,\tau\theta}"] \\
P_{\sigma\tau} \otimes P_\theta \arrow[r, "f_{\sigma\tau,\theta}"] & P_{\sigma\tau\theta}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_\gamma \otimes P_e \arrow[r, "1 \otimes \alpha"] \arrow[dr, "f_{\gamma,e}"'] & P_\gamma \otimes A \arrow[d] \\
 & P_\gamma
\end{tikzcd}$$

$$\begin{tikzcd}
P_e \otimes P_\gamma \arrow[r, "\alpha \otimes 1"] \arrow[dr, "f_{e,\gamma}"'] & A \otimes P_\gamma \arrow[d] \\
 & P_\gamma
\end{tikzcd}$$

are commutative for every $\sigma,\tau,\theta \in G$,

where $\alpha : R_e \to A$ is a given A-bimodule isomorphism.

We denote by $F_s(\Phi)$ the set of factor set associated to Φ

If $f \in F_s(\Phi)$ then we denote by

$A < f, \Phi, G > = \bigoplus_{\sigma \in G} P_\sigma$ with multiplication of elements in $A < f, \Phi, G >$ defined by the rule :

$x.y = f_{\sigma,\tau}(x \otimes y)$ for $x \in P_\sigma$, $y \in P_\tau$.

I.3.13. Proposition. $A < f, \Phi, G >$ is a strongly graded ring with identity and $A < f, \Phi, G >$ contains a subring isomorphic to A; i.e. $P_e \simeq A$ (e is the identity of G).

Conversely, if $R = \bigoplus_{\sigma \in G} R_\sigma$ is a strongly graded ring of type G there exist a group homomorphism $\Phi : G \to \mathrm{Pic}(R_e)$ and a factor set $f \in F_s(\Phi)$ such that $R \simeq R_e < f, \Phi, G >$.

Proof. Clearly $A < f, \Phi, G >$ is a ring. Since $f_{e,e} : P_e \otimes_A P_e \to P_e$ is an A- bimodule isomorphism, P_e is a subring of $A < f, \Phi, G >$. Since $\Phi(e) = [A] = [P_e]$, we have that $P_e \simeq A$ as A-bimodules. There exists $p \in P_e$ such that $P_e = Ap = pA$ and $ap = pa$ for all $a \in A$. Since $f_{e,e}$ is an isomorphism we can write $f_{e,e}(p \otimes p) = cp$ for some $c \in A$. Since P_e is an invertible bimodule, then $c \in Z(A)$ and c is a unit. If we put $p_0 = c^{-1}p$, then $f_{e,e}(p_0 \otimes p_0) = p_0$, and the map $A \to P_e$, $a \to ap_0$ is a ring isomorphism. Furthermore, p_0 is the identity of $A < f, \Phi, G >$. Indeed, if $x \in P_\sigma$ there exists $y \in P_\sigma$ such that $x = f_{e,\sigma}(p_0 \otimes y)$ and then $f_{e,\sigma}(p_0 \otimes x) = f_{e,\sigma}(p_0 \otimes f_{e,\sigma}(p_0 \otimes y)) = f_{e,\sigma}(f_{e,e}(p_0 \otimes p_0) \otimes y) = f_{e,\sigma}(p_0 \otimes y) = x$. Hence $p_0.x = x$ for all $x \in A < f, \Phi, G >$.

Similarly we have $x.p_0 = x$ for every $x \in A < f, \Phi, G >$.

Therefore, p_0 is the identity element of $A < f, \Phi, G >$.

The second statement follows from the proposition I.3.6.

Let Φ, Φ' : $G \to \mathrm{Pic}(A)$ two groups homomorphisms and $f \in F_s(\Phi)$, $f' \in F_s(\Phi')$ two factor sets associated to Φ, respectively to Φ'. Put $\Phi(\sigma) = [P_\sigma]$,$\Phi'(\sigma) =$ $= [P'_\sigma]$, $\sigma \in G$.

A morphism of f in f' is a family $\alpha = (\alpha_\sigma)_{\sigma\in G}$ where $\alpha_\sigma : P_\sigma \to P'_\sigma$ are A-bimodule homomorphisms and the following diagram

$$(4)\qquad \begin{array}{ccc} P_\sigma \otimes_A P_\tau & \xrightarrow{\ f_{\sigma,\tau}\ } & P_{\sigma\tau} \\ \Big\downarrow{\scriptstyle \alpha_\sigma \otimes \alpha\tau} & & \Big\downarrow{\scriptstyle \alpha_{\sigma\tau}} \\ P'_\sigma \otimes_A P'_\tau & \xrightarrow{\ f'_{\sigma,\tau}\ } & P'_{\sigma\tau} \end{array}$$

is commutative for every $\sigma,\tau \in G$.

I.3.14. Proposition. The map $\alpha = \bigoplus_{\sigma\in G} \alpha_\sigma$ is a graded ring homomorphism $A < f, \Phi, G > \to < f', \Phi', G >$.

If α_e is surjective then $\alpha(1) = 1$. Moreover α is an isomorphism if and only if α_σ is an isomorphism for every $\sigma \in G$.

Proof. If $x \in P_\sigma$, $y \in P_\tau$ then $\alpha(xy) = \alpha(xy) = \alpha(f_{\sigma,\tau}(x \otimes y)) =$ $= \alpha_{\sigma,\tau}(f_{\sigma,\tau}(x \otimes y)) = f'_{\sigma,\tau}(\alpha_\sigma(x) \otimes \alpha_\tau(y)) = \alpha_\sigma(x).\alpha_\tau(y) = \alpha(x).\alpha(y)$, and therefore α is a ring homomorphism. If p_0 is the identity element for $A < f, \Phi, G >$ then $f_{e,e}(p_0 \otimes p_0) = p_0$ and $P_e = Ap_0 = p_0A$ and $ap_0 = p_0a$ for every $a \in A$. We denote $p'_0 = \alpha(p_0) = \alpha_e(p_0)$. Since the diagram (4) is commutative, $f'(p_0' \otimes p_0') = p'_0$. Since α_e is surjective then $P'_e = Ap_0' = p'_0 A$. Clearly p'_0 is the identity of $A < f', \Phi, G >$.

The second statement is obvious.

I.3.15. Remark. From this proposition we have in particular that the ring $A < f, \Phi, G >$ is independent of the choice of the family $(P_\sigma)_{\sigma\in G}$ such that $\Phi(\sigma) = [P_\sigma]$

Indeed; if $\Phi(\sigma) = [P'_\sigma]$ then there exist A-bimodule isomorphisms $\alpha_\sigma : P'_\sigma \to P_\sigma$. If we denote by $f'_{\sigma,\tau} = \alpha_{\sigma\tau}^{-1} \circ f_{\sigma,\tau} \circ (\alpha_\sigma \otimes \alpha_\tau)$, where $f = (f_{\sigma,\tau})_{\sigma,\tau}$, $f \in F_s(\Phi)$ then clearly $f = (f'_{\sigma,\tau})_{\sigma\ \tau \in G}$ is a factor set. By the proposition I.3.14. we deduce that $A < f, \Phi, G > \simeq A < f', \Phi, G >$.

Let $\Phi : G \to Pic(A)$ be a group homomorphism. Put $\Phi(\sigma) = [P_\sigma]$. By Lemma I.3.12 we have the homomorphisms

$$G \overset{\Phi}{\to} Pic(A) \overset{\theta}{\to} Aut\ (Z(A)$$

Hence Φ defines an action of G on U(Z(A). If $\sigma \in G$, $c \in Z(A)$, then we define $\sigma(c)$ such that $\sigma(c)x = xc$ for every $x \in P_\sigma$.

Then we can consider the group $Z^2(G, U(Z(A))$ (the group of cocycles of second degree) i.e. the group of factor sets over G. An element $q \in Z^2(G,U(Z($ is a function $q : GxG \to U(Z(A))$, $q(\sigma,\tau) = q_{\sigma,\tau}$, such that $q_{\sigma,\tau\theta} \cdot \sigma(q_{\tau,\theta}) = q_{\sigma\tau,\theta} \cdot q_{\sigma,\tau}$ for every $\sigma,\tau,\theta \in G$. Also we can consider the group $B^2(G,U(Z(A))$ (the group of coboundaries), i.e. the group of all functions $q : G \times G \to U(Z(A))$, $q\ (\sigma,\tau) = q_{\sigma,\tau}$, for which there is a function $d : G \to U(Z(A))$, $d(\sigma) = d_\sigma$ such that $q_{\sigma,\tau} = d_{\sigma\tau}\ d_\sigma^{-1}\sigma(d_\tau)^{-1}$.

We have that $B^2(G, U(Z(A))$ is a subgroup of $Z^2(G, U(Z(A)))$ and we denote by $H^2\ (G, U(Z(A)))$ the factor group $Z^2(G, U(Z(A)))/B^2(G, U(Z(A)))$.

I.3.16. Theorem. Let $\Phi : G \to Pic(A)$ a group homomorphism. Put $\Phi(\sigma) = [P_\sigma], \sigma \in ($

i) If $f = (f_{\sigma,\tau})_{\sigma,\tau \in G}$, $f \in F_s(\Phi)$ and $q = (q_{\sigma,\tau})_{\sigma,\tau \in G}$, $q \in Z^2(G,U(Z(A)$ then $qf = (q_{\sigma,\tau} \cdot f_{\sigma,\tau})_{\sigma,\tau \in G}$ is a factor set associated to Φ.

ii) Conversely, if $f, g \in F_s(\Phi)$, then there exists $q \in Z^2(G, U(Z(A)))$ such that $g = qf$.

Furthermore, $q \in B^2(G, U(Z(A)))$ if and only if there exists a graded ring A-isomorphism from $A < f, \Phi, G >$ to $A < qf, \Phi, G >$.

Proof.

i) We only have to verify that the diagram

$$\begin{array}{ccc} P_\sigma \otimes P_\tau \otimes P_\theta & \xrightarrow{1\otimes g_{\tau,\theta}} & P_\sigma \otimes P_{\tau\theta} \\ \downarrow{\scriptstyle g_{\sigma,\tau}\otimes 1} & & \downarrow{\scriptstyle g_{\sigma,\tau\theta}} \\ P_{\sigma\tau}\otimes P_\theta & \xrightarrow{g_{\sigma\tau,\theta}} & P_{\sigma\tau\theta}, \end{array} \tag{5}$$

is commutative.

Indeed, if $x \in P_\sigma$, $y \in P_\tau$, $z \in P_\theta$, we have $(g_{\sigma\tau,\theta}\circ(g_{\sigma,\tau}\otimes 1))(x\otimes y\otimes z) =$

$= (q_{\sigma\tau,\theta}\cdot q_{\sigma,\tau})(f_{\sigma\tau,\theta}\circ(f_{\sigma,\tau}\otimes 1))\ (x\otimes y\otimes z)$ and$(g_{\sigma,\tau\theta}\circ(1\otimes g_{\tau,\theta}))(x\otimes y\otimes) =$

$= g_{\sigma,\tau\theta}(x\otimes g_{\tau,\theta}(y\otimes z)) = g_{\sigma,\tau\theta}\ (x\otimes q_{\tau,\theta}\cdot f_{\tau,\theta}(y\otimes z)) =$

$= g_{\sigma,\tau\theta}(x.q_{\tau,\theta}\otimes f_{\tau,\theta}\ (y\otimes z)) = g_{\sigma,\tau\theta}\ (\sigma(q_{\tau,\theta}).x\otimes f_{\tau,\theta}\ (y\otimes z)) =$

$= q_{\sigma,\tau\theta}\cdot\sigma(q_{\tau,\theta})\cdot(f_{\sigma,\tau\theta}\ \circ 1\otimes f_{\tau,\theta})\ (x\otimes y\otimes z).$

Since $f = (f_{\sigma,\tau})$ is a factor set, $g_{\sigma,\tau\theta}\circ(1\otimes g_{\tau,\theta}) = g_{\sigma\tau,\theta}\circ(g_{\sigma,\tau}\otimes 1)$.

ii) If $f, g \in F_s(\Phi)$, $f = (f_{\sigma,\tau})_{\sigma,\tau\in G}$ and $g = (g_{\sigma,\tau})_{\sigma,\tau\in G}$ we put $q_{\sigma,\tau} = g_{\sigma,\tau}\circ f^{-1}_{\sigma,\tau}$. Clearly, $q_{\sigma,\tau}$ is a A-bimodule automorphism of $P_\sigma\otimes P_\tau$. By Lemma I.3.11.2. $q_{\sigma,\tau}$ is the multiplication by an element $q_{\sigma,\tau}\in U(Z(A))$. Hence $g_{\sigma,\tau} = q_{\sigma,\tau}\cdot f_{\sigma,\tau}$. Because the diagram (5) is commutative, we obtain that $q = (q_{\sigma,\tau})_{\sigma,\tau\in G}$ is an element from $Z^2(G, U(Z(A)))$.

Suppose now that $q \in B^2\ (G, U(Z(A)))$. Then there exists a function $d : G \to U(Z(A))$, $d(\sigma) = d_\sigma$ such that $q_{\sigma,\tau} = d_{\sigma\tau}\, d_\sigma^{-1}\, \sigma(d_\tau)^{-1}$. We define $\alpha : A < f, \Phi, G > \to A < qf, \Phi, G >$ by $\alpha(x) = d_\sigma x$ for every $x \in P_\sigma$.

If $x \in P_\sigma$, $y \in P_\tau$ then $\alpha(xy) = d_{\sigma\tau}\, f_{\sigma,\tau}\ (x\otimes y) =$

$= q_{\sigma,\tau}\, d_\sigma\ \sigma(d_\tau)^{-1}\, f_{\sigma,\tau}(x\otimes y) = q_{\sigma,\tau}\, f_{\sigma,\tau}\ (d_\sigma x\otimes d_\tau y) = \alpha(x).\alpha(y).$

On the other hand, if $x \in P_\sigma$, $a \in A$, we have $\alpha(ax) = a\alpha(x)$ and hence α is A-linear. Clearly α is an isomorphism, hence a graded ring isomorphi

Conversely, we suppose that there exists a graded ring A-isomorphism $\alpha : A < f, \Phi, G > \to A < qf, \Phi, G >$, where $q \in Z^2(G, U(Z(A)))$. Then we can write $\alpha = \bigoplus_{\sigma \in G} \alpha_\sigma$ where α_σ is the A-bimodule automorphism of P_σ. There exist $d_\sigma \in U(Z(A))$ such that $\alpha_\sigma(x) = d_\sigma x$ for every $x \in P_\sigma$. Because α commutes with multiplication we obtain that $q_{\sigma,\tau}\ d_\sigma\ \alpha(d_\tau)\ f_{\sigma,\tau}\ (x \otimes y) = d_{\sigma\tau}\ f_{\sigma,\tau}\ (x \otimes y)$ for every $x \in P_\sigma$, $y \in P_\tau$. Hence $q_{\sigma,\tau}\ d_\sigma\ \alpha\ (d_\tau) = d_{\sigma\tau}$ or $q_{\sigma,\tau} = d_{\sigma\tau}\ d_\sigma^{-1}\alpha(d_\tau)^{-1}$. Hence $q \in B^2(G, U(Z(A)))$.

I.3.17. Corollary. Let $f \in F_s(\Phi)$. Then the map $q \to qf$ of $Z^2(G, U(Z(A))) \to F_s(\Phi)$ is bijective.

Proof. Directly from Theorem I.3.16, i).

If $\Phi : G \to \mathrm{Pic}(A)$ is a group homomorphism, we can consider the set of strongly graded rings $P = \{ A < f, \Phi, G > \}_{f \in F_s(\Phi)}$. We denote by $C(A,\Phi)$ the set $P/\approx$, where $D, D' \in P$, $D \approx D'$ if and only if D and D' are graded A-isomorphic.

I.3.18. Corollary. Let $f \in F_s(\Phi)$. Then the map $\hat{q} \to qf$ of $H^2(G, U(Z(A))) \to C(A, \Phi)$ is bijective.

Proof. We apply Theorem I.3.16.,ii).

We denote by $Z^1(G, U(Z(A)))$ the set of cocycles of first degree, i.e. the set of maps $d : G \to U(Z(A))$ such that $\sigma\, d(\tau)\ d(\sigma)\ d(\sigma\tau)^{-1} = 1$.

Let $\Phi : G \to \mathrm{Pic}(A)$ be a group homomorphism and $f \in F_s(\Phi)$. We denote $S = A < f, \Phi, G >$. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a strongly graded ring of type G such that $R_e \cong A$. We denote by $\mathrm{Algr}_A\ (R,S)$ the set of G-graded ring homomorphisms which are also A-linear.

I.3.19. Proposition. Let $\varphi \in \mathrm{Algr}_A(R,S)$.

i) φ is an isomorphism.

ii) The map $Z^1(G,U(Z(A))) \to \mathrm{Algr}_A(R,S)$, given by $d \to d.\varphi$ (where $d = (d_\sigma)_{\sigma\in G}$, $d \in Z^1(G,U(Z(A)))$ and $(d.\varphi)(x) = d_\sigma\, \varphi(x)$ for every $x \in S_\sigma$, $\sigma \in G$, is bijective.

iii) $Z^1(G,U(Z(A)) \simeq \mathrm{Algr}_A\,(S,S)$.

Proof.

i) Suppose that $\varphi(x) = 0$, $x \in R_\sigma$. Let $y \in R_{\sigma^{-1}}$. Then $\varphi(xy) = \varphi(x).\varphi(y)=0$. Since $x\,y \in R_e \simeq A$ and φ is A-linear we have $0 = \varphi\,(xy) = (xy)\varphi(1)=(xy).1$. Hence $xy = 0$. Since $R_{\sigma^{-1}}$ is invertible, $x = 0$ and therefore φ is injective. Since $1 \in R_{\sigma^{-1}} R_\sigma$ we have $1 \in \varphi\,(R_{\sigma^{-1}})\,\varphi(R_\sigma)$ and hence there exists $x_i \in \varphi(R_{\sigma^{-1}})$, $y_i \in \varphi\,(R_\sigma)$ such that $1 = \sum_i x_i y_i$. If $y \in S_\sigma$ then $yx_i \in S_e$ and then $(yx_i)y_i \in \varphi\,(R_\sigma)$ (φ is A-linear).

Hence $y = \sum_i (yx_i)\, y_i \in \varphi(R_\sigma)$, or $\varphi(R_\sigma) = S_\sigma$. Therefore φ is surjective.

ii) It is clear that the map $d \to d.\varphi$ is injective.

If $\psi \in \mathrm{Algr}_A\,(R,S)$ then $d = \psi\,\varphi^{-1} \in \mathrm{Alg}_A(S,S)$. By the statement i), $\theta = \bigoplus_{\sigma\in G} \theta_\sigma$, where d_σ is an A-bimodule isomorphism of S_σ. By Lemma I.3.11. there exists an element $d_\sigma \in U(Z(A))$ such that the map θ_σ is just multiplication by the element d_σ. Since θ is a ring homomorphism, the system $d = (d_\sigma)_{\sigma\in G}$ is an element of $Z^1(G,U(A)))$.

iii) We apply ii)

Now consider the following commutative diagram

$$(6) \qquad \begin{tikzcd} G \arrow[rr, "u"] \arrow[dr, "F"'] & & G' \arrow[dl, "F'"] \\ & \mathrm{Pic}(A) & \end{tikzcd}$$

Where G, G' are groups and u, F, F' are group homomorphisms.

If $f' = (f'_{\sigma',\tau'})_{\sigma',\tau' \in G'}$ is an element of $F_s(F')$, then it is clear that the system $f = (f'_{u(\sigma),u(\tau)})_{\sigma,\tau \in G}$ is a factor set associated to F; i.e. $f \in F_s(F)$.

I.3.20. Examples.

1) Let T be a ring and $A \subset T$ a subring of T, having the same identity element. We denote by $\mathrm{Inv}(T,A)$ the group of all invertible two-sided A-submodules of T, where a two-sided A-submodule X of T is invertible in T if and only if $XY = YX = A$ for some two-sided A-submodule of T. It is clear that every invertible two-sided A-submodule X of T is an invertible A-bimodule. Hence there exists a canonical group homomorphism :

$$\mathrm{Inv}(T,A) \to \mathrm{Pic}(A) \quad , \quad X \to [X].$$

Let G be a group and $\varphi : G \to \mathrm{Inv}(T,A)$ a group homomorphism. We put $\varphi(\sigma) = X_\sigma$ for every $\sigma \in G$. We obtain the group homomorphism $\Phi = \mathrm{can} \circ \varphi$.

$$G \xrightarrow{\varphi} \mathrm{Inv}(T,A) \xrightarrow{\mathrm{can}} \mathrm{Pic}(A) .$$

If $\sigma,\tau \in G$ then the maps $f_{\sigma,\tau} : X_\sigma \otimes X_\tau \to X_{\sigma\tau}$, $f_{\sigma,\tau}(x \otimes y) = xy$, $x \in X_\sigma$, $y \in X_\tau$, define a factor set associated to Φ. Then we can define the strongly graded ring $\mathcal{X}(\varphi) = A < f, \Phi, G >$. This ring is called <u>the generalized Rees ring of type G associated to φ.</u>

Hence $\mathcal{X}(\varphi) = \bigoplus_{\sigma\in G} X_\sigma$ with the canonical multiplication.

2) Let A be a ring and $\varphi : G \to \mathrm{Aut}(A)$ a group homomorphism. We have the group homomorphisms

$$G \xrightarrow{\varphi} \mathrm{Aut}(A) \xrightarrow{\delta} \mathrm{Pic}(A), \text{ where } \delta(\alpha) = [{}_1A_\alpha] .$$

We denote by $\Phi = \delta \circ \varphi$ and we consider the A-bimodule isomorphism

$f_{\sigma,\tau} : {}_1A_\sigma \otimes {}_1A_\tau \to {}_1A_{\sigma\tau}$, $f_{\sigma,\tau}(x \otimes y) = x.\sigma(y)$.

It is easily seen that $f = (f_{\sigma,\tau})_{\sigma,\tau \in G}$ is a factor set associated to Φ. We denote by $A < \varphi, G >$ the strongly graded rings $A < f,\Phi,G >$.

We have $A < \varphi, G > = \bigoplus_{\sigma\in G} {}_1A_\sigma$, where the multiplication is defined by the rule : $x.y = x\sigma(y)$, where $x \in {}_1A_\sigma$, $y \in {}_1A_\tau$.

3) Consider a ring A, a group G and a group extension of U(A) by G i.e. an exact sequence of groups.

(ε) $1 \to U(A) \to H \overset{\pi}{\to} G \to 1.$

Suppose that the conjugation action of H on U(A) defined by (ε) can be extended to an action $\alpha : H \to \mathrm{Aut}(A)$ (i.e. if $u \in U(A)$ then $\alpha(u)\, a = uau^{-1}$, $a \in A$).

For each $\sigma \in G$ we choose a represtentative $\varphi_\sigma \in \pi^{-1}(\sigma)$ and then we define the A-bimodule $P_\sigma = {}_1A_{\alpha(\rho_\sigma)}$.

Exactness of (ε) entails $u = \varphi_\sigma\varphi_\tau\, \varphi_{\sigma\tau}^{-1} \in U(A)$.

The previous results provide us with A-bimodule isomorphisms :

$${}_1A_{\alpha(\rho_\sigma)} \otimes_A {}_1A_{\alpha(\rho_\tau)} \xrightarrow{f_{\sigma,\tau}} {}_1A_{\alpha(\rho_\sigma)\alpha(\rho_\tau)} = {}_1A_{\alpha(\rho_\sigma\rho_\tau)} \xrightarrow{g_{\sigma,\tau}} {}_1A_{\alpha(\rho_{\sigma\tau})}$$

where $f_{\sigma,\tau}(x\otimes y) = x\,(\alpha(\rho_\sigma)(y))$ and $g_{\sigma,\tau}(z) = zu = z(\rho_\sigma\rho_\tau\rho_{\sigma\tau}^{-1})$.

Let $h_{\sigma,\tau}$ be the A-bimodule isomorphism $g_{\sigma,\tau} \circ f_{\sigma,\tau}$ i.e. $h_{\sigma,\tau}(x\otimes y) = x(\alpha(\rho_\sigma)(y)\ \rho_\sigma\rho_\tau\rho_{\sigma\tau}^{-1}$.

From the hypothesis it follows that $\Phi : G \to \mathrm{Pic}(A)$ defined by $\Phi(\sigma) = [\,{}_1A_{\alpha(\rho_\sigma)}]$, is a group homomorphism and the family $h = (h_{\sigma,\tau})_{\sigma,\tau \in G}$ is a factor set associated to Φ. The ring $A < h, \Phi, G >$ will be denoted by $A[\varepsilon, \alpha]$, it is a strongly graded ring of type G with : $A[\varepsilon,\alpha]_\sigma = {}_1A_{\alpha(\rho_\sigma)}$ and the product of $x \in {}_1A_{\alpha(\rho_\sigma)}$, $y \in {}_1A_{\alpha(\rho_\tau)}$ is defined by $x.y = x(\alpha(\rho_\sigma)(y))\rho_\sigma\rho_\tau\rho_{\sigma\tau}^{-1}$.

The strongly graded ring $A[\varepsilon,\alpha]$ is called the crossed product associated to A, the exact sequence (ε) and the morphism α.

<u>I.3.21. Lemma.</u> If $u \in U(R)\cap R_\sigma$ for some $\sigma \in G$, then $u^{-1} \in R_{\sigma^{-1}}$.

Proof. We may write $u^{-1} = \sum_{\tau \in G} v_\tau$ with $v_\tau \in R_\tau$ and all but a finite number of the elements v_τ are zero. From $1 = \sum_{\tau \in G} u\, v_\tau$ it follows that $v_\tau = 0$ unless $\tau = \sigma^{-1}$ and in that case $u\, v_{\sigma^{-1}} = 1 = v_{\sigma^{-1}} u$ i.e. $u^{-1} \in R_{\sigma^{-1}}$.

Using the degree map $\deg : h(U(R)) \to G$ we obtain a sequence of group homomorphisms :

$$\varepsilon < R > \ : \ 1 \to U(R_e) \to h(U(R)) \xrightarrow{\deg} G \to e$$

which is exact at each place except possibly at G.

Define a group homomorphism $\alpha_R : h(U(R)) \to \mathrm{Aut}(R_e)$ by $\alpha_R(u)(y) = uyu^{-1}$ for $u \in h(U(R))$, $y \in R_e$.

I.3.22. Lemma. If $\varepsilon < R >$ is exact then :

1. The ring R is strongly graded
2. If $u_\sigma \in h(U(R)) \cap R_\sigma$ then there is an isomorphism of R_e-bimodules : $\theta_\sigma : {}_1(R_e)_{\alpha_R(u_\sigma)} \to R_\sigma$, given by $\theta_\sigma(\lambda) = \lambda u_\sigma$.

Proof.

1. Exactness of $\varepsilon < R >$ yields the existence of a unit $u_\sigma \in G$. Since $u_\sigma^{-1} \in R_{\sigma^{-1}}$, it follows that $1 \in R_{\sigma^{-1}} R_\sigma$ for all $\sigma \in G$.
2. Obvious.

Note. Let R be a strongly graded ring of type G such that the sequence $\varepsilon < R >$ is exact, then :

1. For every subgroup H of G, the sequence $\varepsilon < R^{(H)} >$ is exact.
2. If H is a normal subgroup of G then the sequence $\varepsilon < R_{(G/H)} >$ is exact.

The foregoing results may now be combined in :

I.3.23. Theorem (Crossed Product) Let R be a graded ring of type G such that the sequence $\varepsilon < R >$ is exact, then R is isomorphic to the crossed product $R_e\, [\, \varepsilon < R >, \alpha_R]$.

I.3.24. Corollary. If R is $\mathbb{Z}$-graded and if we suppose that $h(U(R)) \cap R_1 \neq \emptyset$ then R is strongly graded, the sequence $\varepsilon < R >$ is exact and $R \cong R_0[X, X^{-1}, \varphi]$ where $\varphi : R_0 \to R_0$ is an automorphism, X is an indeterminate of degree one and multiplication is given by $Xa = \varphi(a)X$.

An arbitrary ring S is said to have the left I.B.N. property (Invariant Basis Number) if free left R_e-modules of different rank cannot be isomorphic e.g. commutative rings, local rings, left Noetherian rings, etc... have the left I.B.N. Property. We refer to P. Cohn, [23], for more properties of these rings. We use the I.B.N. property in the following :

I.3.25. Lemma. Let R be a strongly graded ring such that R_e has the left IBN property and such that finitely generated projective left R_e-modules are free. Then $\varepsilon < R >$ is exact.

Proof. Since R_σ is a left projective R_e-module we find $R_\sigma \simeq R_e^{(I)}$ in R_e-mod for some set I. From $R_{\sigma^{-1}} \otimes_{R_e} R_\sigma \simeq R_e$ it follows then that $(R_{\sigma^{-1}})^{(I)} \simeq R_e$ in R_e-mod.

But $R_{\sigma^{-1}} \simeq R_e^{(J)}$ in R_e-mod for some set J. Hence the IBN property for R_e entails that card $(I \times J) = 1$.

Consequently $R_\sigma \simeq R_e$ in R_e-mod. By Lemma I.3.14.2. there exists $\alpha_\sigma \in \mathrm{Aut}(R_e)$ such that $R_\sigma \simeq {}_1(R_e)_{\alpha_\sigma}$ as R_e-bimodules. Let $\theta : {}_1(R_e)_{\alpha_\sigma} \to R_\sigma$ be a bimodule isomorphism. Putting $u_\sigma = \theta(1)$ yields $R_\sigma = R_e u_\sigma = u_\sigma R_e$ and from $R_{\sigma^{-1}} R_\sigma =$ $= R_\sigma R_{\sigma^{-1}} = R_e$ we may derive :

$u_\sigma R_e . R_e u_{\sigma^{-1}} = R_e$ and $u_{\sigma^{-1}} R_e . R_e u_\sigma = R_e$. Hence there exist $\lambda, \mu \in R_e$ such that $1 = u_\sigma \lambda u_{\sigma^{-1}}$, and $1 = u_{\sigma^{-1}} \mu u_\sigma$. It follows that each u_σ is an invertible element and therefore $\varepsilon < R >$ is exact at G as well.

I.3.26. Corollary. Let R be a strongly graded ring such that R_e is a semi-local ring, in the sense that $R/J(R)$ is semisimple Artinian, then $R \cong R_e [\varepsilon < R >, \alpha_R]$.

Proof. Apply Lemma I.3.15. and I.3.20.

I.3.27. Corollary. If R is a strongly graded ring such that R_e is a left principal ideal domain then $R \simeq R_e[\varepsilon < R >, \alpha_R]$ and R is a left graded principal ideal ring. Moreover if $G = Z$ then $R \simeq R_o[X, X^{-1}, \varphi]$ where X is a variable of degree one and φ is an automorphism of R_o.

Proof. The first statement is clear from the lemma. Recall that a graded ring is a left graded principal ideal ring if every left ideal is principal. Now if L is a graded left ideal of R then by Corollary I.3.8., $R_\sigma I_e = I_\sigma$ for all $\sigma \in G$. But $I_e = R_e a$, hence $I = Ra$. The last statement is evident from Corollary I.3.24.

We now proceed to study in some detail the graded ideals of strongly graded rings. If R is a strongly graded ring of type G, then $\mathrm{Mod}(R_1,R)$ will denote the family of all two-sided R_1-submodules of R. If $A \in \mathrm{Mod}(R_1,R)$ and $\sigma \in G$ then put $A^\sigma = R_{\sigma^{-1}} A R_\sigma$. It is clear that $A^\sigma \in \mathrm{Mod}(R_1,R)$; we say that A^σ is the σ-conjugate of A.

I.3.28. Lemma. Fixing $\sigma \in G$, the map $\varphi_\sigma : \mathrm{Mod}(R_1,R) \to \mathrm{Mod}(R,R)$, $A \to A^\sigma$ is bijective and it preserves inclusions, sums, intersections and products in R of elements of $\mathrm{Mod}(R_1,R)$.

Proof. That φ_σ is bijective is clear, actually the map $A \to A^{\sigma^{-1}}$ is its inversive. For $\sigma,\tau \in G$ we get $(A^\sigma)^\tau = A^{\sigma\tau}$ for all $A \in \mathrm{Mod}(R_1,R)$. It is not hard to see that φ^σ preserves sums and inclusions. Now consider $A, B \in \mathrm{Mod}(R_1,R)$ then $A \cap B = R_\sigma R_{\sigma^{-1}}(A \cap B)\, R_\sigma R_{\sigma^{-1}} \subset R_\sigma(A^\sigma \cap B^\sigma)\, R_{\sigma^{-1}} \subset$
$\subset (A^\sigma)^{\sigma^{-1}} \cap (B^\sigma)^{\sigma^{-1}}$, hence $A \cap B = R_\sigma(A^\sigma \cap B^\sigma)R_{\sigma^{-1}}$ and thus $(A \cap B)^\sigma =$
$= A^\sigma \cap B^\sigma$.

In a similar way one checks that φ_σ preserves products.

We say that $A \in \mathrm{Mod}(R_1, R)$ is G-invariant if $A^\sigma = A$ for all $\sigma \in G$. It is straightforward to check that any ideal of R is a G-invariant element of $\mathrm{Mod}(R_1,R)$.

I.3.29. Corollary. Let R be strongly graded of type G, and let I be a graded ideal of R. Then I_e is an ideal of R_1, which is G-invariant and $I = RI_e = I_eR$. The correspondence $I \to I_e$ defines a bijection between the set of graded ideals of R and the set of G-invariant ideals of R_e.

Proof. For any $\tau \in G$, $R_{\tau^{-1}} I_e R_\tau \subset I$ and thus $R_{\tau^{-1}} I_e R_\tau \subset I_e$ or $I_e^\tau \subset I_e$. Pick any $\sigma \in G$, then : $I_e = (I_e^\sigma)^{\sigma^{-1}} \subset I_e^{\sigma^{-1}} \subset I_e$, hence $I_e = I_e^{\sigma^{-1}}$ and $I_e^\sigma = I_e$.

By Corollary I.3.8., $I = RI_e = I_eR$. Conversely, let J be any G-invariant ideal of R_e then $RJ = JR$ is a graded ideal of R is easily seen. The statement about the correspondence $I \to I_e$ is also easily checked.

I.3.30. Lemma. Let R be a strongly graded ring of type G.

1. If M is a simple left R_e-module then for every $\sigma \in G$, $R_\sigma \otimes_{R_e} M$ is a simple left R_e-module.
2. Putting $I = \mathrm{Ann}_{R_e}(M)$, $J = \mathrm{Ann}_{R_e}(R_\sigma \otimes_{R_e} M)$ then : $I = J^\sigma$.

Proof. Let $N \subset R_\sigma \otimes_{R_e} M$ be a left R_e-submodule, then since $R_{\sigma^{-1}}$ is a projective R_e-bimodule it follows that $R_{\sigma^{-1}} \otimes_{R_e} N \subset R_{\sigma^{-1}} \otimes_{R_e} R_\sigma \otimes_{R_e} M$. By Proposition I.3.6. it follows that $R_{\sigma^{-1}} \otimes_{R_e} N \subset M$ thus $R_{\sigma^{-1}} \otimes_{R_e} N = M$ or $R_{\sigma^{-1}} \otimes_{R_e} N = 0$. The latter case obviously entails that $N = 0$ while the first case yields $N = R_\sigma \otimes_{R_e} M$.

2. Consider $a \in R_\sigma$, $i \in I$, $b \in R_{\sigma^{-1}}$, then $a\, i\, b \in I^{\sigma^{-1}}$ and on the other hand $(a\, i\, b)\, (R_\sigma \underset{R_e}{\otimes} M) = a \otimes ib\, R_\sigma M$.

Now $b\, R_\sigma\, M \subset M$ and $r \in \mathrm{Ann}_{R_1}(M)$, thus $i\, b\, R_\sigma M = 0$, consequently $abi \in \mathrm{Ann}_{R_e}(R_\sigma \underset{R_e}{\otimes} M)$. Thus $I^{\sigma^{-1}} \subset J$ and $I \subset J^\sigma$.

Using the fact that $M = R_{\sigma^{-1}} \underset{R_e}{\otimes} (R_\sigma \underset{R}{\otimes} M)$ we also obtain $J \subset I^{\sigma^{-1}}$, hence $J^\sigma \subset I$. Combination of both inclusions yields $I = J^\sigma$.

<u>I.3.31. Lemma.</u> Let R be a strongly graded ring of type G.
If P is a prime ideal of R_e then P^σ is a prime ideal of R_e for all $\sigma \in G$.

<u>Proof.</u> If I and J are ideals of R_e such that $I\, J \subset P^\sigma$ then $(IJ)^{\sigma^{-1}} =$
$= I^{\sigma^{-1}}\, J^{\sigma^{-1}} \subset (P^\sigma)^{\sigma^{-1}} = P$, hence $I^{\sigma^{-1}} \subset P$ say.
Then $I \subset P^\sigma$ follows.

By Prim (-) we denote the subset of Spec (-) consisting of the primitive ideals of the ring under consideration.

<u>I.3.32. Corollary.</u> If R is a strongly graded ring of type G then the Jacobson radical of R_e, $J(R_e)$, and the nilradical of R_e, $\mathrm{rad}(R_e)$, are both G-invariant.

<u>Proof.</u> Since $J(R_e) = \cap\, \{P,\ P \in \mathrm{Prim}\, R_e\}$, and $\mathrm{rad}(R_e) = \cap\, \{P,\ P \in \mathrm{Spec}(R_e)\}$ we only have to apply Lemma I.3.30. and Lemma I.3.31.

The following theorem is a generalization of the classical "Clifford theorem' cf. [88] theorem 2.16 pag. 281. The proof we give here closely resembles the proof of theorem 2.16. in loc. cit.

<u>I.3.33. Theorem.</u> Let R be a strongly graded ring of type G, where G is a finite group. If $\underline{M}$ is simple in R-mod then $\underline{M}$ contains a simple R_e-

submodule W such that ${}_{R_e}M = \sum_{\sigma\in G} R_\sigma W$. In particular M is semisimple in R_e-mod.

Put $H = \{\sigma \in G,\ R_\sigma \otimes_{R_e} W \cong W\} = \{\sigma \in G,\ R_\sigma W = W\}$ and let M_W be the sum of all R_e-submodules X of $\underline{M}$ such that $X \cong W$. Then H is a subgroup of G and M_W is a simple $R^{(H)}$-module such that $M \cong R \otimes_{R^{(H)}} M_W$.

Proof. If $x \neq 0$ in $\underline{M}$ then $\underline{M} = Rx = (\sum_{\sigma\in G} R_\sigma)x = \sum_{\sigma\in G} R_\sigma x$. Since each R_σ, $\sigma \in G$, is a finitely generated R_e-module it follows that $\underline{M}$ is a finitely generated R_e-module, hence there is a maximal proper R_e-submodule K of $\underline{M}$.

Put $K_0 = \bigcap_{\sigma\in G} R_\sigma K$. Clearly $R_\sigma K_0 \subset K_0$ for all $\sigma \in G$. Therefore, K_0 is an R-submodule of $\underline{M}$ and then $K_0 \neq M$ implies $K_0 = 0$. Thus we find an exact sequence in R_e-mod : $0 \to \underline{M} \to \bigoplus_{\sigma\in G} \underline{M} / R_\sigma K$.

But each R_σ is an invertible R_e-bimodule, therefore each $R_\sigma K$ is also a maximal proper R_e-submodule of $\underline{M}$. In other words, $\bigoplus_{\sigma\in G} \underline{M}/R_\sigma K$ is a semisimple R_e-module and then also $\underline{M}$ is a semisimple R_e-module. From this we deduce the existence of a simple R_e-submodule of $\underline{M}$, W say. Clearly $M = RW$ entails that $M = \sum_{\sigma\in G} R_\sigma W$. Again, R_σ being an invertible R_e-bimodule, it follows that each $R_\sigma W$ is a simple R_e-module. The canonical multiplication map $R_\sigma \otimes_{R_e} W \to R_\sigma W$ is an isomorphism because $R_\sigma W$ is simple in R_e-mod and R_σ is invertible. The second statement of the theorem follows now immediately.

I.4. Graded Division Rings.

A graded ring of type G is said to be a graded division ring if every nonzero homogeneous element is invertible. Clearly if R is a graded division ring then R_e is a skewfield.

I.4.1. Lemma. Let R be a graded ring of type G, then the following statements are equivalent :

1. R has no graded left ideals different from o, R.
2. R has no graded right ideals different from o, R.
3. R is a graded division ring.

Proof. Straightforward.

I.4.2. Theorem.

Let R be a graded division ring.

1. If $G' = \{ g \in G, R_g \neq 0 \}$ then G' is a subgroup of G and R is a domain if G' is ordered.
2. The ring $R^{(G')} = \bigoplus_{\sigma \in G'} R_\sigma$ is a strongly G'-graded ring.
3. $R^{(G')} = R_e [\varepsilon < R > , \alpha_R]$ where $< \varepsilon >$ is the following exact sequence of groups :

   $$1 \to U(R_e) \to \bigcup_{\sigma \in G'} R_\sigma^\star \to G' \to 1,$$

   where $R_\sigma^\star = R_\sigma - \{0\}$ for all $\sigma \in G$, and α_R is the group morphism

   $$\alpha_R : \bigcup_{\sigma \in G'} R_\sigma^\star \to \mathrm{Aut}\ R_e,$$

   given by : $\alpha_R(u)(x) = u\, x\, u^{-1}$, for $x \in R_e$, $u \in \bigcup_{\sigma \in G'} R_\sigma^\star$.

Proof.

1. The first statement is obvious. It is also clear that there are no homogeneous zerodivisors and under the assumption of 1. we only have

to consider the situation where $(1 + a_{\sigma_1} + \ldots + a_{\sigma_n})\ (1 + b_{\tau_1} + \ldots + b_{\tau_m}) = 0$ where $1 < \sigma_1 < \ldots < \sigma_n$ and $1 < \tau_1 < \ldots < \tau_m$. Then $a_{\sigma_n} b_{\tau_m} = 0$ entails $a_{\sigma_n} = 0$ or $b_{\tau_m} = 0$. After reducing the length of the decompositions a contradiction is reached.

2., 3. are immediate consequences of Section A.I.3.

A strongly G-graded ring $R = R_e\,[\varepsilon < R >, \alpha_R]$ may be written as $R = R_e[\,X_\sigma, \alpha_\sigma, c_{\sigma,\tau}; \tau,\sigma \in G\,]$ where the X_σ ,$\sigma \in G$, are units of R and multiplication in R is defined by the rules :

1° For all $\sigma \in G$, $X_\sigma\, a = \alpha_\sigma(a)\, X_\sigma$ for all $a \in R_e$ where $\alpha_\sigma \in \mathrm{Aut}\, R_e$, for all $\sigma \in G$.

2° For all $\sigma,\tau \in G$, $X_\sigma X_\tau = c_{\sigma,\tau}\, X_{\sigma\tau}$ where the $c_{\sigma,\tau} \in U\,(R_e)$ satisfy the following relation :

$$c_{\sigma,\tau}\, c_{\sigma\tau,\gamma} = \alpha_\sigma(c_{\tau,\gamma})\, c_{\sigma,\tau\gamma} \text{ for all } \sigma,\tau,\gamma \in G.$$

I.4.3. Corollary. If R is a graded division ring of type $\mathbb{Z}$ then either $R = R_0$, when the gradation is trivial, or else $R \cong R_0\,[X, X^{-1}, \varphi]$ where $\varphi \in \mathrm{Aut}\, R_0$, and X is an indeterminate such that $Xa = \varphi(a)X$ for all $a \in R_0$.

A graded division ring of type $\mathbb{Z}$ is a left and right principal ideal domain.

Proof. The first statement is obvious. The ring of twisted polynomials $S = R_0[\,X,\varphi\,]$ is a left and right principal ideal domain, then R will have the same property because it is obtained from S by localizing S at $\{1,X,X^2,\ldots\}$.

I.4.4. Remark. The equivalent of I.4.3. is false for graded division rings of type G, for certain G, e.g. if G is finite abelian then k [G] is not left and right principal. In the commutative case some of these problems may be viewed in the light of the theory of arithmetically graded rings, which we will return to later.

I.4.5. Lemma. Let R be a graded division algebra of type G.

1. R is strongly graded if and only if $G = G'$.
2. If $G = G'$ then the set $H = \{ \sigma \in G$, there exists $x \neq 0$ in $Z(R) \cap R_\sigma\}$ is a subgroup of $Z(G)$, the center of G.
3. If $G = G'$ is abelian then $Z(R)$ is strongly graded of type H.

Proof. 1 and 3 are obvious.

2. If $\tau \in H$, choose $x_\tau \in Z(R) \cap R_\tau$ such that $x_\tau \neq 0$.
For any $\sigma \in G$ and every $a \in R_\sigma$ we have $x_\tau\ a = a\ x_\tau$.
Since $R_\sigma \neq 0$ and since there are no nonzero homogeneous zero divisors, it follows that $\sigma\tau = \tau\sigma$ i.e. $\tau \in Z(G)$.
That H is indeed a subgroup is readily checked.

I.4.6. Proposition. Let R be a strongly graded division ring of type G.

1. For $\sigma \in H$ it is necessary that $\sigma \in Z(G)$ and α_σ is inner.
2. If G is abelian then $Z(R) = (Z(R))_e\,[Y_\sigma, d_{\sigma,\tau};\ \sigma,\tau \in H]$, where $d : H \times H \to U(Z(R)_e)$ represents an element of $\bar{d} \in H^2(H, U(Z(R)_e))$ describing up to isomorphism the structure of the crossed product $Z(R)$.

Proof.

1. If $\sigma \in H$ then there is a nonzero $c^{(\sigma)} X_\sigma \in Z(R)$ for some $c^{(\sigma)} \in R_e$.
For every $a \in R_e$ we obtain : $c^{(\sigma)}X_\sigma\ a = c^{(\sigma)}\alpha_\sigma(a)\ X_\sigma = a\ c^{(\sigma)}\ X_\sigma$,
hence since all occuring elements are homogeneous : $\alpha_\sigma(a) = (c^{(\sigma)})^{-1} a\ c^{(\sigma)}$
i.e. α_σ is inner and induced by $(c^{(\sigma)})^{-1}$.

2. That $Z(R)$ is of that form follows from I.4.5.3. and the crossed product theorem. Because of the remarks preceding I.4.3. it follows that $d_{\sigma,\tau}$ does indeed determine an element of $H^2(H, U((Z(R))_e))$. If the Y_σ are replaced by $u^{(\sigma)}Y_\sigma$ for certain $u^{(\sigma)} \in U((Z(R))_e)$ the

$d_{\sigma,\tau}$ will be replaced by an equivalent 2.cocycle, therefore the structure of the crossed product $Z(R)$ is up to isomorphism determined by $\{\alpha_\sigma, \sigma \in G\}$ and $\bar{d}$.

I.4.7. Proposition. Let R be a strongly graded division ring of type G such that R_e is commutative, then :

1. $R \cong R_e[X_\sigma, c_{\sigma,\tau}; \sigma\ \tau \in G]$ where $c_{\sigma,\tau}$ represents an element $c \in H^2(G, U(R_e))$.
2. If G is abelian then $Z(R) \cong (Z(R))_e[Y_\sigma, d_{\sigma,\tau}, \sigma,\tau \in H]$ where $d_{\sigma,\tau}$ represents an element d of $H^2(H,U((Z(R))_e))$ which is equivalent to $\mathrm{res}_H\ c$ in $H^2(H,U(R_e))$.

Proof.

1. Commutativity of R_e allows to use cohomological notation for $c_{\sigma,\tau}$ satisfying $c_{\sigma,\tau}\ c_{\sigma\tau\ \gamma} = \alpha_\sigma(c_{\tau,\gamma})\ c_{\sigma,\tau\gamma}$ for all $\sigma,\tau,\gamma \in G$.
2. We only have to establish the second statement. Now $Y_\sigma = c^{(\sigma)}\ Y_\sigma$ for some $c^{(\sigma)} \in R_e$ hence, for $\sigma,\tau \in H$:

$$\begin{aligned} d_{\sigma,\tau} = Y_\sigma Y_\tau Y_{\sigma\tau}^{-1} &= c^{(\sigma)}\ X_\sigma\ c^{(\tau)} X_\tau\ X_{\sigma\tau}^{-1}\ (c^{(\sigma\tau)})^{-1} \\ &= c^{(\sigma)}\ \alpha_\sigma(c^{(\tau)})\ X_\sigma X_\tau\ X_{\sigma\tau}^{-1}\ (c^{(\sigma\tau)})^{-1} \\ &= c^{(\sigma)}\ \alpha_\sigma(c^{(\tau)})\ c_{\sigma,\tau}\ (c^{(\sigma\tau)})^{-1} \\ &= c^{(\sigma)}\ \alpha_\sigma(c^{(\tau)})\ (c^{(\sigma\tau)})^{-1}\ c_{\sigma,\tau} \end{aligned}$$

Put $x_{\sigma,\tau} = c^{(\sigma)}\ \alpha_\sigma\ (c^{(\tau)})\ (c^{(\sigma\tau)})^{-1}$ for all $(\sigma,\tau) \in H \times H$.

From $d_{\sigma,\tau} = x_{\sigma,\tau}\ c_{\sigma,\tau}$ and $x_{\sigma,\tau}$ obviously representing the trivial element of $H^2(H,U(R_e))$ it follows that d is equivalent to $\mathrm{res}_H c$ in $H^2(H,U(R_e))$.

I.4.8. Remark.

These techniques enable one to construct homogeneous elements in Z(R) of degree different from e provided one can find a subgroup H of Z(G)

such that the α_σ, $\sigma \in H$, are inner and such that $\mathrm{res}_H c$ is equivalent in $H^2(H, U(R_e))$ to an H-invariant cocycle.
For example in the Z-graded case $R = R_e[X, X^{-1}, \varphi]$ it is clear that, in order to have a non-trivially graded center, some power of φ is inner if R_e is a skewfield i.e. $\varphi^n = 1$ if R_e is a commutative field.

I.5. Graded Endomorphism Rings.

Let R be a graded ring of type G, $M \in R\text{-gr}$. Then $END_R(M) = HOM_R(M,M)$ is a graded ring of type G. Indeed it is easily seen that composition of graded endomorphisms of M yields a multiplication making the graded abelian group $HOM_R(M,M)$ into a graded ring.

However, due to the fact that we will work with left modules and left operations, this multiplication is defined as follows : $g.f = f \circ g$.

Consider M, N in R-gr. We say that N divides M in R-gr if N is isomorphic to a graded direct summand of M i.e. if and only if there exist $f \in \mathrm{Hom}_{R\text{-gr}}(M,N)$, $g \in \mathrm{Hom}_{R\text{-gr}}(N,M)$ such that $f \circ g = 1_M$.

We say N weakly divides M in R-gr if it divides the direct sum $M^{(t)}$ of a finite number of t copies of M. Evidently N weakly divides M in R-gr if and only if there exist : $f_1, \ldots, f_t \in \mathrm{Hom}_{R\text{-gr}}(M,N)$ and

$g_1, \ldots, g_t \in \mathrm{Hom}_{R\text{-gr}}(N,M)$ such that $1_M = f_1 \circ g_1 + \ldots + f_t \circ g_t$.

We say that the graded modules M, N are weakly isomorphic in R-gr, (notation $M \sim N$) if each of them weakly divides the other in R-gr. Equivalantly $M \sim N$ if and only if there exist natural numbers n, m and graded modules M', N' of type G such that $M \oplus M' \cong N^m$, $N \oplus N' \cong M^n$.

An $M \in R\text{-gr}$ is said to be weakly G-invariant if $M \sim M(\sigma)$ for all $\sigma \in G$.

I.5.1. Theorem. Let $M \in R\text{-gr}$. Then the graded ring $END_R(M)$ is strongly graded of type G if and only if M is weakly G-invariant.

Proof. $END_R(M)$ is strongly graded if and only if we have $1_M \in (END_R(M))_\tau (END_R(M))_{\tau^{-1}}$ for all $\tau \in G$, if and only if there exist $g_1, \ldots, g_n \in END_R(M)_\tau$ and $f_1, \ldots, f_n \in (END_R(M))_{\tau^{-1}}$ such that :

$$(\star) \quad 1_M = \sum_{i=1}^{n} g_i . f_i = \sum_{i=1}^{n} f_i \circ g_i .$$

Now, $(\mathrm{END}_R(M))_\tau = \mathrm{HOM}_{R\text{-gr}}(M(\sigma), M(\tau\sigma))$, and

$$(\mathrm{END}_R(M))_{\tau^{-1}} = \mathrm{HOM}_{R\text{-gr}}(M(\tau\sigma), M(\sigma)), \text{ for all } \sigma \in G.$$

The relation (★) is therefore equivalent to the fact that $M(\sigma)$ weakly divides $M(\sigma\tau)$ for all $\sigma,\tau \in G$. Since G is a group, this last condition is equivalent to M being weakly G-invariant.

I.5.2. Corollary.

Let R be a strongly graded ring of type G and let M be a left R_e-module. Put $M^\sigma = R_\sigma \otimes_{R_e} M$, for all $\sigma \in G$. Then $\mathrm{END}_R(R \otimes_{R_e} M)$ is strongly graded if and only if $M \sim M^\sigma$ in the category R_e-mod, for all $\sigma \in G$.

Proof. By Theorem I.3.4. it follows that : $R \otimes_{R_e} M \sim (R \otimes_{R_e} M)(\sigma)$ in R-gr if and only if $M \sim M^\sigma$ in R_e-mod. This holds for every $\sigma \in G$ and hence the Theorem proves the corollary.

An $M \in$ R-gr is said to be G-invariant if for all $\sigma \in G$, $M \cong M(\sigma)$ in R-gr.

I.5.3. Corollary. A non-zero $M \in$ R-gr is G-invariant if and only if the sequence :

$$\varepsilon_R < M > : 1 \to U(\mathrm{End}_{R\text{-gr}}(M)) \to U(\mathrm{END}_R(M)) \to G \to 1$$

is exact. In that case we have that $\mathrm{END}_R(M)$ is strongly graded and isomorphic to the crossed product $\mathrm{End}_{R\text{-gr}}(M)[\varepsilon_R < M >, \alpha_M]$.

Proof. The statement follows immediately from Theorem I.5.1. and the theory expounded in Section I.3.

Let us now focus on matrix rings over graded rings and see how these rings can be made into graded rings. If R is a graded ring of type G, $n \in \mathbb{N}$, then $M_n(R)$ denotes the ring of n by n matrices with entries from R. Fix $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$. To $\lambda \in G$ we associate the following additive subgroup of $M_n(R)$:

$$M_n(R)_\lambda\ (\bar{\sigma}) = \begin{pmatrix} R_{\sigma_1\lambda\sigma_1^{-1}} & R_{\sigma_1\lambda\sigma_2^{-1}} & \cdots\cdots\cdots & R_{\sigma_1\lambda\sigma_n^{-1}} \\ R_{\sigma_2\lambda\sigma_1^{-1}} & R_{\sigma_2\lambda\sigma_2^{-1}} & \cdots\cdots\cdots & R_{\sigma_2\lambda\sigma_n^{-1}} \\ \vdots & \vdots & & \vdots \\ R_{\sigma_n\lambda\sigma_1^{-1}} & R_{\sigma_n\lambda\sigma_2^{-1}} & \cdots\cdots\cdots & R_{\sigma_n\lambda\sigma_n^{-1}} \end{pmatrix}$$

It is easy to check that $M_n(R)$ equipped with the gradation of type G given by $\underset{\lambda\in G}{\oplus} M_n(R)_\lambda\ (\bar{\sigma})$ is a graded ring, which we will denote by $M_n(R)\ (\bar{\sigma})$, containing R as a graded subring. Taking $\sigma_1 = \ldots = \sigma_n = e$ yields $M_n(R)\ (\bar{e})$ which we denote simply by $M_n(R)$ whenever there cannot arise any ambiguity whether the latter ring is considered as a graded object or not.

I.5.4. Lemma. Let $M \in R\text{-gr}$ be gr-free with homogeneous basis $e_1,\ldots,e_n$, say $\deg e_i = \sigma_i$. Then $\mathrm{END}_R(M) \cong M_n(R)(\bar{\sigma})$ where $\bar{\sigma} = (\sigma_1,\ldots,\sigma_n)$.

Proof. If $f \in (\mathrm{END}_R(M))_\lambda$, $\lambda \in G$, then $f(M_\sigma) \subset M_{\sigma\lambda}$ for all $\sigma \in G$. Consequently $f(e_i) \in M_{\sigma_i\lambda}$ for $i = 1,\ldots,n$. So we may write $f(e_i) = \sum_{j=i}^n a_{ij}e_j$ with $\deg a_{ij} = \sigma_i\,\lambda\,\sigma_j^{-1}$. The matrix (a_{ij}) associated to f is in $M_n(R)_\lambda\ (\bar{\sigma})$.

I.5.5. Corollaries.

1. If φ is a permutation of the set $\bar{\sigma}$ then $M_n(R)\ (\bar{\sigma}) \simeq M_n(R)\ (\varphi(\bar{\sigma}))$.

2. Writing $\bar{\sigma}.\tau$ for the set $\{\sigma_1\tau,\ldots,\sigma_n\tau\}$ we get :
   $M_n(R)(\bar{\sigma}.\tau) \simeq M_n(R)(\bar{\sigma})$.

Proof.

1. Is obvious.

2. Let $M \in R\text{-gr}$ be gr-free with homogeneous basis $e_1,\ldots,e_n$, say $\deg e_i =$

$= \sigma_i \in G$. From Lemma I.5.4. we obtain :
$M_n(R)(\bar{\sigma}) \simeq END_R(M)$. Consider $M(\tau^{-1})$; it is gr-free with homogeneous basis $e_1,\ldots,e_n$ where $\deg e_i = \sigma_i\tau$, $i = 1,\ldots,n$. Since $END_R(M) \cong END_R M(\tau^{-1})$ and, using Lemma I.5.4., $M_n(R)(\bar{\sigma}\tau) \cong END_R(M(\tau^{-1}))$, we have the statement.

<u>I.5.6. Theorem.</u> Consider a strongly graded ring R and a subset $\{\sigma_1,\ldots\sigma_n\} = \bar{\sigma}$ of G. The graded ring $M_n(R)(\bar{\sigma})$ is a strongly graded ring of type G.

<u>Proof.</u> Consider the gr-free $M = R(\sigma_1) \oplus\ldots\oplus R(\sigma_n)$. Since R is strongly graded, Theorem I.5.1. entails that $R \sim R(\sigma)$ for all $\sigma \in G$; consequently: $M(\sigma) = R(\sigma_1)(\sigma) \oplus\ldots\oplus R(\sigma_n)(\sigma) = R(\sigma.\sigma_1) \oplus\ldots\oplus R(\sigma\sigma_n)$. By transitivity of $\sim$, $R(\sigma_i) \sim R(\sigma\sigma_i)$ for all $\sigma \in G$ follows.
Hence we obtain that : $\overset{n}{\underset{i=1}{\oplus}} R(\sigma_i) \sim \overset{n}{\underset{i=1}{\oplus}} R(\sigma\sigma_i)$ or $M \sim M(\sigma)$ for all $\sigma \in G$. Again, because of Theorem I.5.1., weakly G-invariance for M entails that $END_R(M)$ is a strongly graded ring. Lemma I.5.4. finishes the proof of the theorem.

<u>I.5.7. Corollary.</u> Let R be a graded ring of type G such that $\varepsilon < R >$ is an exact sequence (see Section I.3.). For any $n \in \mathbb{N}$ and any $\bar{\sigma} = \{\sigma_1,\ldots,\sigma_n\} \subset G$, the sequence $\varepsilon < M_n(R)(\bar{\sigma}) >$ is exact. In this case, $M_n(R)(\bar{\sigma})$ is the crossed product $(M_n(R)(\bar{\sigma}))_e\,[\varepsilon < M_n(R)(\bar{\sigma}) >, \alpha_{M_n(R)}]$

<u>Proof.</u> Let $M \in R$-gr be gr-free and such that :

$M_n(R)(\bar{\sigma}) = END_R(M)$, where $\bar{\sigma}$ is obtained from :

$M \cong R(\sigma_1) \oplus\ldots\oplus R(\sigma_n)$.

Exactness of $\varepsilon < R >$ entails $R \cong R(\sigma)$ for all $\sigma \in G$, hence $M \cong M(\sigma)$ for all $\sigma \in G$. Now Corollary I.5.3. entails the result stated.

A graded ring R of type G is said to be gr-semisimple if R-gr is a semisimple category i.e. any object of R-gr is semisimple (i.e. if $M \in R$-gr then M is a direct sum of gr-simple modules). It is clear that R is <u>gr-semisimple</u> if and only if : (*) $R = L_1 \oplus \ldots \oplus L_n$, where the L_i, $i = 1,\ldots,n$ are minimal graded left ideals of R. Clearly, if R is gr-semisimple then R_e is semisimple. A graded ring R of type G is <u>gr -simple</u> if it has a decomposition (*) but with $\mathrm{HOM}_R(L_i,L_j) \neq 0$ for every $i,j = 1,\ldots,n$, equivalently : there exists a $\sigma_{ij} \in G$ such that $L_j \cong L_i(\sigma_{ij})$. A gr-simple graded ring R is said to be <u>gr-uniformly simple</u> if R if R admits the decomposition (*) but with $L_i \cong L_j$ in R-gr for any i,j.

<u>I.5.8. Theorem.</u> (The graded version of Weddenburn's Theorem.)
Let R be a ring of type G. Then the following statements are equivalent :

1. R is gr-simple (resp. gr-uniformly simple).
2. There exists a graded division ring D and a $\bar{\sigma} = (\sigma,\ldots,\sigma_n) \in G^n$ such that $R \cong M_n(D)(\bar{\sigma})$ (resp. $R \cong M_n(D)$).

<u>Proof.</u> Let $M \in$ R-gr be gr-simple. Since R is gr-simple there exist $\tau_1,\ldots,\tau_n \in G$ such that $R \cong M(\tau_1) \oplus \ldots \oplus M(\tau_n)$.
Put $D = \mathrm{END}_R(M)$, then D is a gr-division ring.
Then $R \cong \mathrm{END}_R(R) \cong \mathrm{END}_R\,(M(\tau_1) \oplus \ldots \oplus M(\tau_n))$.
Since $\mathrm{END}_R(M(\tau_i)) \cong \mathrm{END}_R(M) = D$ for all $i = 1,\ldots,n$ it follows from the foregoing that $R \cong M_n(D)(\bar{\sigma})$ for some $\bar{\sigma} = (\sigma_1,\ldots,\sigma_n) \in G^n$. The uniformly simple statement also follows.

<u>I.5.9. Remarks.</u>

1. If R is gr-simple then R_e need not be a simple ring. As noted before R_e is semisimple.
2. If R is gr-uniformly simple then R_e is a simple ring.
3. If R is gr-simple and suppose that there exists a simple object in R-gr which is G-invariant, then R is gr-uniformly simple.

In case G is an ordered group we may derive some extra results in this vein, we refer to Section II.1. , in particular Lemma II.1.5., Proposition II.1.6. and the remarks in II.1.7.. For more results about gr-simple and gr-semisimple objects of R-gr in case R is Z graded, we refer to Section II.6.

I.6. Graded Rings of Fractions.

Recall that if R is any ring and S is a multiplicatively closed subset of R such that $1 \in S$, $0 \notin S$ then the left ring of fractions $S^{-1}R$, with respect to S, exists if and only if R satisfies the left Ore conditions with respect to S :

0_1 : If $s \in S$, $r \in R$ are such that $rs = 0$ then there exists an $s' \in S$ such that $s'r = 0$.

0_2 : For $r \in R$, $s \in S$ there exist $r' \in R$, $s' \in S$ such that $s'r = r's$.

If R is a left Noetherian ring then 0_1 always holds and only has to verify 0_2.

If the Ore conditions with respect to S are being satisfied then :

$S^{-1}R = \{ \frac{a}{s}, a \in R, s \in R, s \in S \}$, where operations are defined by

$\frac{x}{s} + \frac{y}{t} = \frac{ax + by}{u}$, where $a, b \in R$ are such that $u = as = bt \in S$.

$\frac{x}{s} \cdot \frac{y}{t} = \frac{x_1 . y}{t_1 . s}$, where $t_1 \in S$, $x_1 \in R$ are such that $t_1x = x_1t$.

Recall also that every $M \in R\text{-mod}$ allows the construction of a module of fractions $S^{-1}M$ which is a left $S^{-1}R$-module actually $S^{-1}M \cong S^{-1} R \underset{R}{\otimes} M$.

I.6.1. Lemma. Let R be a graded ring of type G and let S be a multiplicatively closed subset of R contained in h(R), i.e. consisting of homogeneous elements, then R satisfies the left Ore conditions with respect to S if and only if :

0_1^g : If $r \in h(R)$, $s \in S$ are such that $rs = 0$ then there is an $s' \in S$ such that $s'r = 0$

0_2^g : For any $r \in h(R)$, $s \in S$, there exist $r' \in h(R)$, $s' \in S$ such that $s'r = r's$.

Proof. Clearly 0_1 and 0_2 imply 0_1^g and 0_2^g. Conversely, consider $s \in S$, $r \in R$, where $r = r_{\sigma_1}+\ldots+r_{\sigma_n}$ with $r_{\sigma_i} \in h(R)_{\sigma_i}$, $\sigma_i \in G$. If $n = 1$ then the left Ore conditions for r, s are seen to hold because O_1^g and O_2^g hold. Now we proceed by induction on n, supposing O_1 and O_2 may be verified for $r \in R$ having a homogeneous decomposition of length less than n. By assumption there exist $r^1 \in R$, $s^1 \in S$ such that $s^1(r_{\sigma_1}+\ldots+r_{\sigma_{n-1}}) = r^1 s$, and $s^2 \in S$, $r^2 \in R$ such that $s^2 r_{\sigma_n} = r^2 s$. By O_1^g and O_2^g there exist $u \in S$, $v \in h(R)$ such that $us^1 = vs^2 = t$, and $t \in S$ is non-zero. Then $tr = (ur^1 + vr^2)s$, hence O_2 holds also if r has a decomposition of length n. Furthermore, if $as = 0$ with $a = a_{\sigma_1} +\ldots+ a_{\sigma_n}$, $a_{\sigma_n} \in R_{\sigma_i}$, then $a_{\sigma_n} s = 0$ and $(a_{\sigma_1}+\ldots+a_{\sigma_{n-1}})s = 0$. By the induction hypothesis we may pick a $t_1 \in S$ such that $t_1(a_{\sigma_1}\ldots a_{\sigma_{n-1}}) = 0$. Now from $t_1 a_{\sigma_n} s = 0$ and O_1^g it follows that there exist a $t_2 \in S$ such that $t_2 t_1 a_{\sigma_n} = 0$. Putting $s' = t_2 t_1 \in S$ we see that $s'a = 0$ ($s' \neq 0$ since $0 \notin S$).

If the graded ring R satisfies the left Ore conditions with respect to some multiplicatively closed $S \subset h(R)$ then we can define a gradation on $S^{-1}R$ by putting $(S^{-1}R)_\lambda = \{ \frac{a}{s}, s \in S, a \in R$ such that $\lambda = (\deg s)^{-1} \deg a\}$.

<u>I.6.2. Proposition.</u> $S^{-1}R$ is a graded ring of type G.

<u>Proof.</u> If $\frac{x}{s}, \frac{y}{t} \in (S^{-1}R)_\lambda$ then $\lambda = (\deg s)^{-1}\deg x = (\deg t)^{-1} \deg y$. Putting $u = as = bt \in S$ we get $\frac{x}{s} + \frac{y}{t} = \frac{ax + by}{u}$,

Hence :
$$\deg\left(\frac{ax + by}{u}\right) = (\deg u)^{-1} \deg(ax) = (\deg s)^{-1} (\deg a)^{-1} (\deg a) (\deg x) = (\deg s)^{-1} \deg x$$

Therefore $(S^{-1}R)_\lambda$ is an additive subgroup of $S^{-1}R$, for each $\lambda \in G$. In a similar way one verifies $(S^{-1}R)_\lambda (S^{-1}R)_\mu \subset (S^{-1}R)_{\lambda\mu}$. Obviously, $S^{-1}R = \sum_{\lambda\in G} (S^{-1}R)_\lambda$. The common denominator theorem yields that the sum is direct.

<u>I.6.3. Corollary.</u> If R is a strongly graded ring satisfying the left

Ore conditions with respect to $S \subset h(R)$ then $S^{-1}R$ is a strongly graded ring.

<u>Proof.</u> Clearly $(S^{-1}R)_\sigma (S^{-1}R)_{\sigma^{-1}} = (S^{-1}R)_e$ follows from $R_{\sigma^{-1}} R_\sigma =$ $= R_\sigma R_{\sigma^{-1}} = R_e$.

I.7. Jacobson Radicals of Graded Rings.

Although a theorem of G. Bergman's states that the Jacobson radical of a Z-graded ring is a graded ideal, many problems of an intrinsically graded nature may be dealt with by using a graded version of the Jacobson radical and, in particular, a graded version of Nakayama's Lemma.

Let G be a graded ring of type G. An $S \in R$-gr is said to be gr-simple if o and S are its only graded submodules. An $M \in R$-gr is gr-semisimple if M is direct sum of gr-simple modules, see also Section I.5., in particular I.5.8..

I.7.1. Lemma. If S is gr-simple then for each $\sigma \in G$, $S_\sigma = 0$ or S_σ is a simple R_e-module.

Proof. If $x \in S_\sigma$ and $x \neq 0$ then $Rx = S$, hence $R_e x = S_\sigma$. Consequently, S_σ is a simple R_e-module.

A graded submodule N of $M \in R$-gr is said to be gr-maximal in M if M/N is a gr-simple R-module. In other words, N is gr-maximal in M if and only if N is proper and for all $m \in h(M)$, $N + Rm = M$. Note that a gr-maximal N in M is not necessarily such that $\underline{N}$ is maximal in $\underline{M}$.

I.7.2. Lemma. If S is gr-simple then there exists a graded left ideal I of R which is gr-maximal, and there is an element $\sigma \in G$, such that $S \cong (R/I)(\sigma)$.

Proof. Take $x \in h(S)_\tau$, $x \neq 0$. The map $\alpha : R \to S(\tau)$ defined by $\alpha(r) = rx$ is graded of degree e. Since $S(\tau)$ is gr-simple too, α is surjective i.e. $S(\tau) \cong R/I$ with I gr-maximal is R. Putting $\sigma = \tau^{-1}$ yields $S \cong (R/I)(\sigma)$.

I.7.3. Definition. Consider $M \in R$-gr. The Jacobson graded radical of M, denoted by $J^g(M)$, is the intersection of all gr-maximal submodules of M, taking $J^g(M)=M$ if M possesses no proper gr-maximal submodules.

I.7.4. Lemma. Let $M \in R\text{-gr}$, then :

1. If M is of finite type then $J^g(M) \neq M$ or $M = 0$.
2. $J^g(M) = \cap \{ \text{Ker } f,\ f \in \underset{R\text{-gr}}{\text{Hom}}(M, S),\ S \text{ gr-simple}\}$
   $= \cap \{ \text{Ker } f,\ f \in \text{HOM}_R(M,S),\ S \text{ gr-simple} \}$
3. If $f \in \text{HOM}_R(M,N)$ then $f(J^g(M)) \subset J^g(N)$.
4. $J^g({}_RR) = \cap \{ \text{Ann}_R S,\ S \text{ gr-simple in } R\text{-gr} \}$
5. $J^g({}_RR)$ is a graded ideal of R.
6. $J^g({}_RR)$ is the largest proper graded ideal such that any $a \in h(R)$, for which the image in $R/J^g({}_RR)$ is invertible, is itself invertible.
7. $J^g({}_RR) = J^g(R_R)$.

Proof. Along the lines of proof of the ungraded equivalents, let us establish only 6. here.

Let $\pi : R \to R/J^g({}_RR)$ be the canonical epimorphism. Suppose $a \in h(R)$ is such that $\pi(a)$ is invertible. If $Ra \neq R$ then there exists a gr-maximal left ideal M of R such that $Ra \subset M$. Since $J^g({}_RR) \subset M$ and $\pi(M) = \pi(R)$ it follows that $R = M$, a contradiction. Hence $Ra = R$ and therefore $ba = 1$ for some $b \in h(R)$. From $1 = \pi(b)\,\pi(a)$ it follows that $cb = 1$ for some $c \in h(R)$ and therefore b is invertible in R. Moreover $ba = 1$ then implies that a is invertible in R. Now let I be a graded proper ideal of R such that any $a \in h(R)$ such that $\pi(a)$ is invertible in $\pi_I(R)$ (where $\pi_I : R \to R/I$ is the canonical epimorphism) is invertible in R. If $I \not\subset J^g({}_RR)$ then there is a gr-maximal left ideal M of R such that $I \not\subset M$. From $I + M = R$ it follows that $1 = a + b$ with $a \in I$, $b \in M$ and it is clear that we may assume that $a, b \in R_e$.

Because $\pi_I(b)$ is invertible, b is invertible in R, but then $M = R$ yields a contradiction.

In view of 5 and 7. in the above lemma we write $J^g(R)$ $(= J^g({}_RR) = J^g(R_R))$

for the Jacobson graded radical of R.

From 6. in the above lemma we retain that $J^g(R)$ is the largest proper graded ideal of R such that $1 + ar$ is a unit for all $a \in J^g(R) \cap R_e$, $r \in R_e$. In other words, $J^g(R)$ is the largest proper graded ideal of R such that its intersection with R_e is in $J(R_e)$.

I.7.5. Lemma. (Graded version of Nakayama's Lemma).
If $M \in R$-gr is finitely generated then $J^g(R)M \neq M$.

Proof. From I.7.4. and similar to the ungraded case.

I.7.6. Proposition. If R is a strongly graded ring of type G then $J^g(R) \cap R_e = J(R_e)$, where $J(R_e)$ is the Jacobson radical of the ring R_e.

Proof. If I is left gr-maximal in R then I_e is a maximal left ideal of R_e, (cf. Corollary I.3.8. in Section I.3.).
Since $I \cap R_e = I_e$ and since every maximal left ideal of R_e extends to a proper graded left ideal of R, it follows that $J^g(R) \cap R_e = J(R_e)$.

An $M \in R$-gr is said to be left gr.-Noetherian, resp. left gr.-Artinian if M satisfies the ascending, resp. descending, chain condition for graded left R-submodules of M. Clearly, a gr.-simple M in R-gr is both left gr.-Noetherian and left gr.-Artinian.

I.7.7. Theorem. (Graded version of Hopkins' theorem).
If R is left gr-Artinian then R is left gr.-Noetherian.

Proof. Formally similar to the proof of the ungraded equivalent. One shows first that $R/J^g(R)$ is a semisimple object of R-gr and that $(J^g(R))^n=0$ for some positive $n \in \mathbb{Z}$.
For each i, $1 \le i \le n-1$, $(J^g(R))^i / J^g(R))^{i+1}$ is annihilated by $J^g(R)$, hence an $R/J^g(R)$-module. It follows that $(J^g(R))^i / (J^g(R))^{i+1}$ is semisimple in R-gr and because it is also left gr-Artinian it has to be

of finite type, hence left gr-Noetherian. It is then also clear that ${}_R R$ is left gr-Noetherian in R-gr, hence R is left gr-Noetherian.

<u>I.7.8. Remark.</u> A left gr.-Artinian ring R need not be left Artinian. For example $k[X,X^{-1}]$, where k is a field and X a variable of degree 1, is gr.-Artinian and not Artinian.

Next we include a short disgression in the ungraded theory which is necessary in order to present some of the deeper results concerning the Jacobson radical and the symplectic radical of graded rings of type Z.

<u>I.7.9. Proposition.</u> Let S be a ring containing a subring R such that S is as a left R-module generated by a finite set of R-normalizing elements, $a_1,\ldots, a_n$ say (i.e. $Ra_i = a_iR$ for $i = 1,\ldots n.$). If $M \in$ S-mod is a simple S-module then ${}_R M$ is semisimple and finitely generated.

<u>Proof.</u> If $0 \neq x \in M$ then $M = Sx = \sum_{i=1}^{n} Ra_ix$ and thus ${}_R M$ has finite type. If $N \subset M$ is a maximal proper R-submodule then $N_i = (N : a_i) = \{x \in M, a_ix \in N\}$ is clearly an R-submodule of M. We claim that N_i is a maximal proper R-submodule of M too. Indeed, if $x \in M- N_i$ then $a_ix \notin N$ and $M = N+Ra_ix$ or $N + a_iRx = M$. If $y \in M$ then $a_iy = z + a_iu$ where $z \in N$, $u \in R\,x$, hence $a_i(y-u) = z \in N$ and $y - u \in (N : a_i)$. The latter means that $y \in (N:a_i) + Rx$, or it follows that $(N:a_i)$ is a maximal R-submodule of M. Note that not all N_i are equal to M since otherwise N = M.
Put $L = \bigcap_{i=1}^{n} N_i$. If $L \neq 0$ then $SL \neq 0$. By definition of L (it is easily checked that) it follows that $SL \subset N$, since $S = \sum_{i=1}^{n} Ra_i$, $L = \bigcap_{i=1}^{n} N_i$. The simplicity of ${}_S M$ yields SL = M hence N=M, a contradiction. Therefore L = 0 follows. The nonzero M/N_i are simple in R-mod and we obtain ${}_R M = \oplus M/N_i$, thus ${}_R M$ is semisimple in R-mod.

I.7.10. Corollary. If J is a primitive ideal of S then $I = J \cap R$ is a finite intersection of primitive ideals of R.

Let us just mention the following elaboration of the arguments in the foregoing proof.

I.7.11. Lemma. Let S be a ring containing a subring R such that S is generated as a left R-module by a finite set of R-centralizing elements of S, then :

1. If $M \in S$-mod is simple then ${}_R M$ is semisimple of finite length and the simple R-submodules of ${}_R M$ are isomorphic to one-another.
2. If $N \in R$-mod is simple then $S \otimes_R N$ is semisimple of finite length in R-mod.
3. We have $J(S) \cap R \supset J(R)$ and equality holds for example in each of the following cases :
   a. $S \otimes_R -$ does not annihilate a simple R-module
   b. R is rationally closed in S, i.e. if $x \in R$ and $x^{-1} \in S$ then $x^{-1} \in R$.

Proof. cf. [8] . In this preprint G. Bergman asks whether equality holds generally. For this and related results we may refer to a.o. Robson's work on the theory of normalizing extensions, cf. [9] ,[48].

We introduce the following notation; if $M \in R$-mod then $J_R(M)$, the Jacobson radical of M in R-mod, is the intersection of the proper maximal R-submodules of M.

I.7.12. Lemma. Let S be a ring containing a subring R such that S is free R-module generated by a finite set $\{a_1,\dots,a_n\}$ such that each a_i, $i = 1,\dots,n$, commutes with R in S. Then $J_R(M) = M \cap J_S(S \otimes_R M)$, for every $M \in R$-mod.

Proof. cf. [8].

I.7.13. Proposition. Let S be a ring containing a subring R such that S is generated as a left R-module by a finite number of R normalizing elements, $a_1,\ldots,a_n$ say. If J is a maximal ideal of S then $J \cap R$ is a finite intersection of maximal ideals of R.

Proof. The proof is based upon the ideas of proof used in Proposition I.7.9. but adapted to the bimodule situation; let us just present a blue print of the proof. It is not hard to see that J is contained in a proper maximal R-bimodule in S (using Zorn's lemma). If K is maximal as an R-bimodule in S then : $(K:a_i)_l = \{ s \in S,\ s\,a_i \in K \}$ and $(K : a_i)_r = \{ s \in S,\ a_i s \in K \}$ are also maximal R-bimodules in S. Furthermore any such K intersects R in a maximal ideal of R as is easily verified. Now put $L_l = \bigcap_{i=1}^{n} (K:a_i)_l$ and $L_r = \bigcap_{i=1}^{n} (K:a_i)_r$, $L = L_r \cap L_l$.

Since J is an ideal of S, $L \supset J$ follows, whereas on the other hand $S\,L \subset K$, $LS \subset K$ is evident from the construction of L. Thus SLS is an ideal of S, containing J and contained in K. By maximality of J, SLS = J and L = J follows. Finally, this entails $R \cap J = R \cap L = \bigcap_{i=1}^{n} \subset (R\cap(K:a_i)_l) \cap \cap (R \cap (K:a_i)_r)]$, which is a finite intersection of maximal ideals of R.

We now apply this in case R is a graded ring of type $\mathbb{Z}$. The following results stern from the author's paper [84], but the basic idea comes from G. Bergman's [8].

I.7.14. Theorem. Let R be a graded ring of type Z/nZ for some $n \neq 0$ in $\mathbb{N}$, let $M \in$ R-gr. Then, if $x \in$ R-gr. Then, if $x \in J_R(M)$ decomposes as $x_0 +\ldots+ x_{n-1}$, we have $nx_j \in J_R(M)$ for each $j \in Z/nZ$.

Proof. Let ω be a primitive n^{th}- root of unity in $\mathbb{C}$ and consider $S = R \otimes_Z Z[\omega]$. We consider S as a graded ring of type Z/nZ by putting $S_j = R_j \otimes_Z Z[\omega]$ for all $j \in Z/nZ$. If $M \in$ R-gr then on $S \otimes_R M$ we consider the gradation defined by putting $(S \underset{R}{\otimes} M)_j = \sum_{i+k=j} S_i \otimes M_k$ and then $S \underset{R}{\otimes} M \in S\text{-gr}$. Write $\bar{M} = S \underset{R}{\otimes} M$.

Now we define $\sigma : S \to S$ by putting $\sigma\left(\sum_{j\in Z/nZ} s_j\right) = \sum_{j\in Z/nZ} \omega^j s_j$ where $s_j \in S_j$. Obviously σ is an automorphism.

If we define $\bar{\sigma} : \bar{M} \to \bar{M}$ by putting $\bar{\sigma}\left(\sum_{j\in Z/nZ} m_j\right) = \sum_{j\in Z/nZ} \omega^j m_j$, then $\bar{\sigma}(s_j m_i) = \omega^{i+j} s_j m_i = \sigma(s_j)\bar{\sigma}(m_i)$, where $s_j \in S_j$ and $m_i \in \bar{M}_i$.

Hence $\bar{\sigma}$ is a σ-automorphism of $\bar{M}$ and it is clearly also a graded morphism of degree $o \in Z/nZ$.

Consider $x \in J_S(\bar{M})$ and let $g : \bar{M} \to T$ be an S-linear map to any simple S-module T. We show that $g(\bar{\sigma}(x)) = 0$. The automorphism σ defines a "restriction of scalars" functor $\sigma_\star$: S-mod $\to$ S-mod, and $\sigma_\star(T)$ is simple in S-mod. Consider $g' = g \circ \bar{\sigma} = \bar{M} \to \bar{M} \to \sigma_\star(T)$ as a set map ($\sigma_\star(T)$ and T are identified as sets).

From $g'(sz) = g(\sigma(s)\,\bar{\sigma}(z)) = \sigma(s)\, g'(z)$ with $s \in S$, $z \in \bar{M}$ it follows that g' is an S'-linear map in the module structure of $\sigma_\star(T)$. Then $x \in J_S(\bar{M})$ entails $g(x) = 0$ or $\bar{\sigma}(x) \in J_S(\bar{M})$ follows. Thus $\bar{\sigma}(J_S(\bar{M})) \subset J_S(\bar{M})$. Now applying Lemma I.7.12 we obtain $J_R(M) = M \cap J_S(\bar{M})$.

If we calculate $t = \sum_{j\in Z/nZ} \omega^{-ij}(\bar{\sigma})^j (x_0 + \ldots + x_{n-1})$, for some fixed $i \in Z/nZ$, then we find $t = nx_i$. But $t \in M \cap J_S(\bar{M})$ since it is fixed under the automorphisms $\bar{\sigma}^j$, $j \in \{1,\ldots n-1\}$. Therefore $n\, x_i \in J_S(M)$ for all i, follows.

I.7.15. Corollary. Let R be a graded ring of type Z and let $M \in R\text{-gr}$, then $J_R(M)$ is a graded submodule of M. In particular, if R = M, we obtain G. Berman's result that J(R) is a graded ideal of R.

Proof. The Z-gradation induces a Z/nZ gradation for each $n > 0$. Consider $a = a_1+\ldots+a_h \in J_R(M)$ with $1 < h$. Take $n > h-1$, then the Z/nZ-homogeneous components of a will be exactly the $a_1,\ldots,a_h$. By the theorem it follows that $n\, a_i \in J_R(M)$ for all i. The same argument holds if n is replaced by $n + 1$ but then $(1 + n)a_i - na_i \in J_R(M)$.

I.7.16. Remark. If R is Z-graded then $J(R) \subset J^g(R)$.
Indeed, if $a \in J(R) \cap R_0$ then $1+a$ is a unit of R. Since $J(R)$ is graded, the maximality property of $J^g(R)$ (see I.7.4., 6) implies $J(R) \subset J^g(R)$.

I.7.17. Corollary. Let R be a graded ring of type Z and let M be a simple object of R-gr then $J_R(M) = 0$.

I.7.18. Corollary. Let A be an arbitrary ring and let S be a simple A-module, then $J_{A[X]}(S[X]) = 0$, where X is a variable commuting with A and $S[X] = S \otimes_A A[X]$.

Proof. The theorem yields that $M = J_R(S[X])$ is a graded submodule of $S[X]$ (writing $R = A[X]$). We have that $M = M_0 \oplus M_1 \oplus \ldots \oplus M_i \oplus \ldots$ with $X M_i \subset M_{i+1}$. But each M_i is isomorphic to an A-submodule of S Consequently $M_i = 0$ for all $i = 0, 1, \ldots$.

I.7.19. Corollary. Let R be a graded ring of type Z.
If M is maximal left ideal of R then M_g is the intersection of all maximal left ideals containing it.

Proof. Put $X = R/M_g$. Then $J_R(X)$ is graded. So there is a graded left ideal Y of R such that $M_g \subset Y \subset M$ which maps to $J_R(X)$ under the canonical map $R \to X$. Hence $Y = M_g$ and $J_R(X) = 0$ follows.

I.7.20. Corollary. Let R a graded ring of type Z. If I is a primitive ideal of R then I_g is the intersection of the primitive ideals of R containing it.

I.7.21. Proposition. 1. Let A be an arbitrary ring, T a variable commut with A, then $J(A[T]) = I[T]$ where

$I = J(A[T]) \cap A$ is a nil ideal of A.

2. Let A be an algebra over the field k and let $k(T)$ be the field of rational functions in one variable over k.

Then $J(A \otimes_k k(T)) = I \otimes_k k(T)$ where $I = A \cap J(A \otimes_k k(T))$ is a nil ideal of A (S. Amitsur).

Proof.

1. Using the fact that $A[T]$ has an automorphism sending T to $T + 1$, the proof of 1. becomes an easy exercise.

2. (Following G. Bergman, [8]). That $I \otimes k(T) \subset J(A \otimes_k k(T))$ is obvious Conversely, any element of $J(A \otimes_k k(T))$ can be written as $p(T)^{-1}(a_m T^m + ..$ where $p(T) \in k[T]$ is nonzero and $a_i \in A$. Now $k(T)$ is not $\mathbb{Z}$-graded but it can be $\mathbb{Z}/n\mathbb{Z}$-graded by putting $k(T)_i = T^i k(T^n)$ for $n > 0$, $i \in \mathbb{Z}/n\mathbb{Z}$ Looking at the induced gradation of $A \otimes_k k(T)$ and applying Theorem I.7.14 we obtain for all $n > m$: $n a_i T^i \in J(A \otimes_k k(T))$; thus $a_i T^i \in J(A \otimes_k k($ and therefore $a_i \in I$.

Pick $x \in I$, then $(1 + xT)^{-1} = 1 - xT + x^2T^2 - \dots$ in $A \otimes_k k((T))$. However the coefficients have to ly in a finite dimensional k-subspace of A i.e x is algebraic over k. If x were not nilpotent then the polynomial equat satisfied by x would give rise to a nonzero idempotent element in the Jacobson radical, a contradiction.

In view of proposition I.7.13 we may produce statements similar to I.7.1 and I.7.18., a two sided version of I.7.19. and a analogon for I.7.21.; with respect to the symplectic radical $J^s(R)$ of a graded ring. Let us just mention the main result and refer to [84] for more detail.

I.7.22. Proposition.

Let R be a graded ring of type Z. Then :

1. $J^s(R) = \cap \{M, M \text{ maximal ideal of } R\}$ is graded.

2. If A is any ring, X a variable commuting with A then :

$J^s(A[X]) = I[X]$, $J^s(A[X, X^{-1}]) = I'[X, X^{-1}]$ for ideals I, I' of A.

I.7.23. Corollary.

Let R be a graded ring of type Z. If M is a maximal ideal of R then M_g is the intersection of the maximal ideals containing M_g. In particular any gr-maximal ideal is an intersection of maximal ideals.

I.7.24. Example. Let k be a field, $R = k[X]$ then the intersection of all maximal ideals of R is zero. Note that M_g is not necessarily gr-maximal if M is maximal; indeed $M = (X-1)$ yields $M_g = 0 \subset (X)$.

I.8. Almost strongly graded rings

I.8.1. Definition. The ring R is called an almost strongly graded ring over the group G if there exists a decomposition sum (not necessarily direct sum) :

(1) $R = \sum_{\sigma \in G} R_\sigma$ (as additive groups) into additive subgroups R_σ, $\sigma \in G$, such that $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$.

If the decomposition sum (1) is direct, then R is a strongly graded ring as in I.3.

In [24], Dade calls an almost strongly graded ring : Clifford system.

I.8.2. Remarks.

1) If $R = \sum_{\sigma \in G} R_\sigma$ is an almost strongly graded ring, then R/I is also an almost strongly graded ring, for all two-sided ideals I of R.
In particular if $R = \oplus_{\sigma \in G} R_\sigma$ is a strongly graded ring, then R/I is an almost strongly graded ring for all two-sided ideals I of R. (not necessarily graded ideals !).

2) If $R = \sum_{\sigma \in G} R_\sigma$ is an almost strongly graded ring, then for all $\sigma \in G$, R_σ is an invertible R_e-bimodule. In particular, we obtain that R_σ is a left (right) finitely generated R_e- module.
Moreover R_σ is a projective R_e-module (left and right).

3) If R is an almost strongly graded ring, then there exists a strongly graded ring T and an ideal I such that $R \cong T/I$.
Indeed we can define the strongly graded ring $T = \oplus_{\sigma \in G} T_\sigma$ where $T_\sigma = R_\sigma$ for all $\sigma \in G$. The multiplication of R given by maps :

$$R_\sigma \times R_\tau \to R_{\sigma\tau} ,$$

Induces the multiplication of T given by maps.

$T_\sigma \times T_\tau \to T_{\sigma\tau}$

The application $T \overset{\varphi}{\to} R$, $\varphi((x_\sigma)_{\sigma\in G}) = \sum_{\sigma\in G} x_\sigma$ is clearly a surjective ring homomorphism.

I.8.3. Notations.

1) If $M \in R$-mod, then by $\mathcal{L}_R(M)$ (resp. $\mathcal{L}_{R_e}$ (M) we denote the lattice of R-submodules of M(resp R_e-submodules of M).

2) By $K.\dim_R M$ (resp. $K.\dim_{R_e} M$) we denote the Krull dimension of M over the ring R (resp. over the ring R_e).

3) By $G.\dim_R M$ (resp. $G.\dim_{R_e} M$) we denote the Gabriel dimension of M over the ring R (resp. over the ring R_e).

Throughout this section $R = \sum_{\sigma\in G} R_\sigma$ is an almost strongly graded ring over a finite group G.

Let $M \in R$-mod and N be an R_e-submodule of M. We denote by $N^\star = \bigcap_{\sigma\in G} R_\sigma N$.

Clearly $N^\star$ is the largest R-submodule of M contained in N.

Remark Define $N^{\star\star} = \sum_{\sigma\in G} R_\sigma N$. This is the smallest R-submodule of M such that $N \subseteq N^{\star\star}$.

I.8.4. Lemma. If N is an essential R_e-submodule of M, then so is $N^\star$; (the group G is finite throughout).

Proof. If $X \subset M$ is R_e-submodule then $R_{\sigma^{-1}} X \neq 0$ (because $R_{\sigma^{-1}}$ is invertible). Hence $N \cap R_{\sigma^{-1}} X \neq 0$ and $R_\sigma(N \cap R_{\sigma^{-1}} X) \neq 0$. Since $R_\sigma(N \cap R_{\sigma^{-1}} X) =$ $= R_\sigma N \cap X_e$ it follows that $R_\sigma N \cap X \neq 0$, thus $R_\sigma N$ is an essential R_e-submodule of M.

Since G is finite, $N^\star$ is an essential R_e-submodule.

I.8.5. Lemma. Let $M \in R$-mod and L be an R_e-submodule of M. Then

i) $\mathcal{L}_{R_e}(M/R_\sigma L) \simeq \mathcal{L}_{R_e}(M/L)$

ii) $\mathcal{L}_{R_e}(R_\sigma N) \simeq \mathcal{L}_{R_e}(N)$

Proof.

i) Let $X/L \in \mathcal{L}_{R_e}(M/L)$. Clearly $X/L \to R_\sigma X/R_\sigma L$ is an isomorphism of lattices. The inverse isomorphism is $Y/R_\sigma L \to R_{\sigma^{-1}} Y/L$ where $Y/R_\sigma L \in \mathcal{L}_{R_e}(M/R_\sigma L)$.

The second statement is clear since R_σ is an invertible R_e-bimodule.

I.8.6. Lemma. $M \in R$-mod. Then M contains an R_e-submodule N maximal with respect to $N^\star = 0$

Proof. Let $\{N_i, i \in I\}$ be a chain of R_e-submodules of M such that $N_i^\star = 0$ for each $i \in I$. If $(\bigcup_{i\in I} N_i)^\star \neq 0$ then there is an $x \in (\bigcup_{i\in I} N_i)^\star$, $x \neq 0$, i.e. $R x \subset (\bigcup_{i\in I} N_i)^\star \subset \bigcup_{i\in I} N_i$. Hence $R_\sigma x \subset \bigcup_{i\in I} N_i$ for all $\sigma \in G$.

Since R_σ is a left finitely generated R_e-module, it follows that for all $\sigma \in G$ there exists an $i_o \in I$ such that $R_\sigma x \subset N_{i_o}$ for all $\sigma \in G$. Thus $Rx \subset N_{i_o}$ and $N_{i_o}^\star \neq 0$, contradiction.

Hence $(\bigcup_{i\in I} N_i)^\star = 0$ and so Zorn's lemma can be applied to yield the existence of an N as desired.

If $M \in R$-mod, we let $\mathrm{rank}_R M$, $\mathrm{rank}_{R_e} M$ denote the Goldie dimension of M over these rings.

I.8.7. Lemma. Let ord $G = n$ and $\mathrm{rank}_R M = m$. If N is an R_e-submodule maximal with respect to $N^\star = 0$, then $\mathrm{rank}_{R_e}(M/N) \leq m$ and $\mathrm{rank}_{R_e} M \leq mn$.

Proof. Let $A_1,\ldots,A_t$ be R_e-submodules of M strictly containing N whose sum is direct modulo N.

Then $A_i^\star \neq 0$ for each $1 \leq i \leq t$. If $t > m$ then for some i :

$(\sum_{j\neq i} A_j^\star) \cap A_i^\star \neq 0.$

Because $(\sum_{j\neq i} A_j \cap A_i)^\star \supseteq (\sum_{j\neq i} A_j^\star) \cap A_i^\star$ it follows that $(\sum_{j\neq i} A_j \cap A_i)^\star \neq 0.$

and so $\sum_{j\neq i} A_j \cap A_i \supsetneq N$, a contradiction. Thus $\mathrm{rank}_{R_e}(M/N) \leq m$.

By Lemma I.8.5. we have that $\mathrm{rank}_{R_e}(M/R_\sigma N) \leq m$ for all $\sigma \in G$. Because

$0 = N^\star = \bigcap_{\sigma\in G} R_\sigma N$ we have : $0 \to M \to \bigoplus_{\sigma\in G} M/R_\sigma N$, and so $\mathrm{rank}_{R_e} M \leq m.n.$

I.8.8. Corollary. If R has the finite Goldie dimension then R_e has finite Goldie dimension.

Moreover, if $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly graded ring, then R has finite Goldie dimension if and only if R_e has finite Goldie dimension.

Proof. For the first statement we aplly Lemma I.8.7.. Now, suppose that $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly graded ring. Since R_σ is a projective finitely generated R_e-module for each $\sigma \in G$, then R_σ has finite Goldie-dimension as an R_e-module. Because G is finite, R has finite Goldie dimension as an R_e-module. Hence R has finite Goldie dimension as an R-module.

I.8.9. Lemma. Let $M \in R$-mod and $N \subseteq M$ be an R_e-submodule of M such that $N^\star = 0$. Then :

i) M is a left Noetherian R_e-module if and only if M/N is left Noetherian R_e-module.

ii) $\mathrm{K.dim}_{R_e} M = \mathrm{K.dim}_{R_e}(M/N)$ if either side exists.

iii) $\mathrm{G.\,dim}_{R_e} M = \mathrm{G.dim}_{R_e}(M/N)$ if either side exists.

Proof. Note that M embeds in $\bigoplus_{\sigma\in G} M/R_\sigma N$. The result now follows from Lemma I.8.5.

<u>I.8.10. Theorem.</u> Let $M \in R$-mod. Then M is left Noetherian if and only if ${}_{R_e}M$ is left Noetherian. In particular R is left Noetherian if and only if R_e is left Noetherian.

<u>Proof.</u> Suppose that ${}_RM$ is Noetherian. By Lemma I.8.9. it is sufficient to show that M/N is a left Noetherian R_e-module where $N \subseteq M$ is a maximal R_e-submodule with respect to $N^\star = 0$.
Let $M_1/N \subset M_2/N \subset \dots$ be an ascending chain of non-zero R-submodules of M/N. Then $M_1 \supsetneq N$ and hence $0 \neq M_1^\star \subset M_1$. Since $M_1^\star$ is R-submodule of M it follows by Noetherian induction that $M/M_1^\star$ a left Noetherian R_e-module. Thus the chain must terminate and hence M/N is a left Noetherian R_e-module. The converse is clear.

<u>I.8.11. Lemma.</u> Let $M \in R$-mod and suppose that $K.\dim_{R_e} M$ exists. Then $K.\dim_R M = K.\dim_{R_e} M$.

<u>Proof.</u> Clearly, $K.\dim_R M \leq K.\dim\, {}_{R_e}M$. Let $K.\dim_R M = \alpha$. By induction on α we show that $K.\dim_{R_e} M \leq \alpha$. It is easy to reduce the problem to the case where M is α-critical. Let $N \subseteq M$ be a maximal R_e-module with respect to $N^\star = 0$. If $X \supsetneq N$, then $X^\star \neq 0$ and hence $K.\dim_R M/X^\star < \alpha$ and so $K.\dim_{R_e} M/X^\star < \alpha$, by introduction. Thus $K.\dim_{R_e} M/X < \alpha$ and hence $K.\dim_{R_e} M/N \leq \alpha$. Therefore $K.\dim_{R_e} M \leq \alpha$ by Lemma I.8.9.

<u>I.8.12. Theorem.</u> Let $M \in R$-mod. Then $K.\dim_R M = K.\dim_{R_e} M$ if either side exists. In particular , $K.\dim R = K.\dim_{R_e} R$ if either side exists and R is right Artinian if and only if R_e is right Artinian.

<u>Proof.</u> Follows from Lemma I.8.11.

<u>I.8.13. Corollary</u>. If ${}_RM$ is α-critical, then ${}_{R_e}M$ is K-homogeneous and it contains an essential direct sum of at most $n = \text{ord } G$, α-critical R_e-modules.

(Here M is called K-homogeneous if for all R_e-submodules $N \subset M$, $N \neq 0$ we have $K.\dim_{R_e} N = K.\dim_{R_e} M$).

Proof. By Theorem I.8.12., $K.\dim_{R_e} M = \alpha$.
Let $0 \neq L \subset M$ be an R_e-submodule of L. By Lemma I.8.5., ii), we have $K.\dim_{R_e} L = K.\dim_{R_e} R_\sigma L$ for all $\sigma \in G$.
Hence $K.\dim_{R_e}(RL)$. Since M is α-critical then $K.\dim_R(RL) = \alpha$ and hence $K.\dim_{R_e}(RL) = \alpha$ (by Theorem I.8.12.). Hence $K.\dim_{R_e} L = \alpha$.
Thus M is K-homogeneous over the ring R_e.
Now let $K \subset M$ be a α-critical R_e-submodule of M. Because R_σ is an invertible R_e-bimodule, $R_\sigma K$ is an α-critical R_e-module for all $\sigma \in G$. Thus $RK = \sum_{\sigma\in G} R_\sigma K$ is a sum of at most $n = \text{ord } G$, α-critical R_e-modules. Since M is α-critical over the ring R, $K.\dim_R M/RK \lneq \alpha$ and thus $K.\dim_{R_e}(M/RK) < \alpha$.
If RK is not essential in M as an R_e-module, then $RK \cap L = 0$ for some nonzero R_e-submodule L of M. Thus $K.\dim_{R_e} L \leq K.\dim_{R_e} M/RK < \alpha$. Because M is K-homogeneous as an R_e-module we obtain a contradiction.
Thus RK is essential in M as an R_e-module.

I.8.14. Theorem. Let $M \in R\text{-mod}$; then $G.\dim_R M = G.\dim_{R_e} M$ if either side exists. In particular $G.\dim R = G.\dim R_e$ is either side exists.

Proof. The same proof as for the theorem 2.2. in [10].

We now turn to the investigation of prime ideals in almost strongly graded rings. Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring. If $A \subset R_e$ is a two-sided ideal of R_e we define the σ-conjugate of A ($\sigma \in G$) by : $A^\sigma = R_{\sigma^{-1}} A R_\sigma$. Clearly A^σ is a two-sided ideal of R_e. Moreover, if Q is a prime ideal in R_e, then Q^σ is a prime ideal in R_e, for any $\sigma \in G$.

If $A^\sigma = A$ for all $\sigma \in G$, A is said to be a <u>G-invariant</u> ideal. We say that the G-invariant ideal A of R_e is G-<u>prime</u> if and only if $B_1B_2 \subset A$ for G-invariant ideals B_i of R_e implies that $B_1 \subset A$ or $B_2 \subset A$.

<u>I.8.15. Lemma</u>. Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring. If P is a prime ideal in R then $P \cap R_e$ is a G-prime ideal in R_e.

Conversely, if $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly graded ring and Q is a G-prime ideal in R_e, then there exists at least one prime ideal P of R such that $P \cap R_e = Q$.

<u>Proof.</u> If P is an ideal in R, then $Q = P \cap R_e$ is a G-invariant ideal of R_e. Suppose now that $B_1B_2 \subset Q$, where B_i are G-invariant ideals of R_e. Because B_1 and B_2 are G-invariant ideals we have $RB_1 = B_1R$ and $RB_2 = B_2R$. Now $B_1B_2 \subset Q$ yields $B_1B_2 \subset P$ and then $R(B_1B_2)R \subset P$ yields $(RB_1R)(RB_2R) \subset P$ hence $RB_1R \subset P$ or $RB_2R \subset P$. Thus $B_1 \subset Q$ or $B_2 \subset Q$.
Conversely, we suppose that Q is a G-prime ideal in R_e. Since Q is G-invariant we have $RQ = QR$. Because R is a strongly graded ring, it follows that $RQ \cap R_e = Q$.
Hence, by Zorn's Lemma, we may select an ideal P of R, maximal with respect to $P \cap R_e = Q$. If I, J are ideals of R properly larger then P, then $I \cap R_e$ and $J \cap R_e$ are G-invariant ideals of R_e properly larger then Q. Since Q is G-prime, this yields $(I \cap R_e)(J \cap R_e) \not\subset Q$ and hence $IJ \not\subset P$. Thus P is prime.

<u>I.8.16. Theorem.</u> Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring (G is a finite group). If P is a prime ideal of R then there exists a prime ideal Q of R_e such that : $P \cap R_e = \bigcap_{\sigma\in G} Q^\sigma$

Moreover, Q is minimal over the ideal $P \cap R_e$.

Proof. Since the ring R/P is also an almost strongly graded ring, we may assume that $P = 0$. We view R as an R_e-R-bimodule. Choose an R_e-R-subbimodule Y maximal with respect to $Y^* = 0$. Hence $\bigcap_{\sigma\in G} R_\sigma Y = 0$.

First we note that $a R_e b \subset Y$ for $a \in R_e$, $b \in R$ implies that $a \in Y$ or $b \in Y$. Suppose that $a, b \notin Y$. Because Y is a maximal R_e-R-subbimodule with respect to $Y^* = 0$ it follows that $(R_e aR + Y)^* \neq 0$ and $(R_e\, b\, R + Y)^* \neq 0$. Put $I = (R_e a\, R + Y)^*$, $J = (R_e b\, R + Y)^*$. Clearly I and J are two-sided ideals of R. Since $I \neq 0$, $J \neq 0$ and R is prime we have $IJ \neq 0$.

On the other hand, $IJ \subset (R_e\, a\, R + Y)\, J \subset R_e\, a\, RJ + Y\, J \subset R_e\, a\, J + Y \subset$
$\subset R_e\, a\, (R_e\, b\, R + Y) + Y \subset R_e\, a\, R_e\, b\, R + Y \subset Y$.

Hence $0 \neq IJ \subset Y$. Therefore $0 \neq IJ \subset Y^*$, contradiction.

Put $Q = Y \cap R_e$. By the preceding statement we have that Q is a prime ideal in R_e.

Now, $\bigcap_{\sigma\in G} Q^\sigma = \bigcap_{\sigma\in G} R_{\sigma^{-1}}\, Q\, R_\sigma \subseteq \bigcap_{\sigma\in G} R_{\sigma^{-1}}\, Y\, R_\sigma \subseteq \bigcap_{\sigma\in G} R_{\sigma^{-1}} Y = 0$.

Hence $\bigcap_{\sigma\in G} Q^\sigma = 0$.

Let Q' be a prime ideal minimal over the ideal $P \cap R_e$. Since $Q' \supset \bigcap_{\sigma\in G} Q^\sigma$ it follows that $Q' \supset \prod_{\sigma\in G} Q^\sigma$. Hence there exists $\sigma \in G$ such that $Q' \supset Q^\sigma \supset P \cap R_e$. Since Q^σ is prime, $Q' = Q^\sigma$; hence $Q = (Q^\sigma)^{\sigma^{-1}} = (Q')^{\sigma^{-1}}$

Clearly $(Q')^{\sigma^{-1}}$ is a prime ideal minimal over the ideal $P \cap R_e$.

I.8.17. Corollary. Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a strongly graded ring (G is a finite group) and $Q \subset R_e$ a prime ideal of R_e. Then there exists a prime ideal P of R such that $P \cap R_e = Q$ if and only if Q is G-invariant.

I.8.18. Corollary. Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a strongly graded ring and Q be a prime ideal of R_e. Then there exists a prime ideal P of R such that Q is one of the minimal primes over $P \cap R_e$.

<u>Proof.</u> Put $A = \bigcap_{\sigma \in G} Q^{\sigma}$. Clearly A is G-invariant and a G-prime ideal in R_e. Now there exists a prime ideal P such that $P \cap R_e = A$. Clearly Q is minimal prime ideal over the ideal $P \cap R_e$.

<u>I.8.19. Corollary.</u> Let $R = \sum_{\sigma \in G} R_{\sigma}$ be an almost strongly graded ring (G is finite group). Then $\mathrm{rad}(R_e) = \mathrm{rad}(R) \cap R_e$.

I.9. Exercises. Comments. References.

1. Let A be a commutative ring and R an A-algebra graded of type G such that R_e is an A-algebra. Then R_σ is an A-bimodule for all $\sigma \in G$. If S is any graded A-algebra of type H, show that the A-algebra $R \underset{A}{\otimes} S$ has a natural $G \times H$-gradation.

2. With notations as in 1. Show that $R \underset{A}{\otimes} S$ is strongly graded if R and S are strongly graded.

3. Let R be a strongly graded ring of type G such that

   i) $R_e \subset Z(R)$

   ii) G is the direct product of subgroups H and K.

   iii) For all $\sigma \in H$, $\tau \in K$ and for all $x \in R_\sigma$, $y \in R_\tau$ we have that $xy = yx$.

Show that $R \cong R^{(H)} \underset{R_e}{\otimes} R^{(K)}$.

4. Let A be a commutative ring, $G = H \times K$ a direct product of groups show that $A[H] \underset{A}{\otimes} A[K] \cong A[H \times K]$.
Can this be generalized to crossed products ?

5. Let R be a graded ring of type G and let H be a subgroup of G. Consider $a \in R^{(H)}$. Show that a is invertible in R if it invertible in $R^{(H)}$.

6. Let R be a graded ring of type G and suppose that there exists a family $\{H_i, i \in I\}$ of subgroups of G such that $G = \bigcup_{i \in I} H_i$ and suppose furthermore that the H_i, $i \in I$, form a directed system with respect to inclusion. Show that $R = \bigcup_{i \in I} R^{(H_i)}$.

7. Let R be a graded ring of type G where G is an abelian torsion free group. Show that $R \cong \varinjlim_{\alpha} T_\alpha$ where T_α is a graded ring of type Z^n for some $n \in \mathbb{N}$ (depending on α).

8. Let R be a graded ring of type G and consider $M \in$ R-mod. If M is injective then ${}^{G}M$ is injective in the category R-gr.

9. Let R be a graded ring of type G and consider $M \in$ R-gr. Show that M is a direct summand of ${}^{G}M$ in R-gr.

10. If G is a finite group and R is strongly graded of type G then $J_g(R) \subset J(R)$.
Hint. Use Clifford's theorem.

11. A ring S is called <u>projective relative to the subring R</u> provided each exact sequence of S-modules :

$$0 \to M' \to M \to M'' \to 0,$$

which splits when considered as a sequence in R-mod; is split in S-mod.
<u>Lemma</u> (The crossed product version of Maschke's lemma).
Let R be a strongly graded ring of type G where G is a finite group of order n say. Suppose that n a unit in R and $U(R) \cap R_\sigma \neq \emptyset$ for every $\sigma \in G$. Then R is projective relative to the subring R_e.
<u>Proof</u>. Let $N \subset M$ in R-mod be such that N is an R_e-direct summand of ${}_{R_e}M$.
Pick $u_\sigma \in U(R) \cap R_\sigma$ for all $\sigma \in G$.
Define an R-linear $f : M \to N$ as follows :

$$f(m) = \frac{1}{n} \sum_{\sigma \in G} u_\sigma g(u_\sigma^{-1} m),$$

where g is an R_e-linear map splitting $0 \to N \to M$.

Check that f is R-linear and $f|N$ is the identity of N.

12. Let R and G be as in the lemma in 11. Show that, if R_e is a semisimple Artinian ring, then R is semisimple Artinian too.

13. Let R be strongly graded of type G, where G is a finite group, and suppose that $U(R) \cap R_\sigma \neq \emptyset$ for all $\sigma \in G$. Let H be a subgroup of G such that its order is a unit of R. Show that, if M is an irreducible

R-module, then the restricted $R^{(H)}$-module ${}_{R(H)}M$ is completely reducible of finite length over the ring $R^{(H)}$.

Hint : Apply Clifford's theorem and Maschke's lemma.

14. Let R and G be as in 13 and suppose moreover that for each $u_\sigma \in U(R) \cap R_\sigma$ the automorphism of R_e given by $\lambda \to u_\sigma^{-1} \lambda u_\sigma$ is an outer automorphism. Show that R is simple if R_e is simple.

15. Let R and G be as in 13. Pick an $u_\sigma \in U(R) \cap R_\sigma$ for reach $\sigma \in G$. For any $M \in R$-mod, define :

$$\varphi : \mathrm{Hom}_{R_e}(M, R_e) \to \mathrm{Hom}_R(M,R)$$

by $\varphi(f) = \tilde{f}$ where $\tilde{f}(m) = \sum_{\sigma\in G} u_\sigma f(u_\sigma^{-1}m)$

Prove the following statements

i) φ is a Z-isomorphism

ii) $R \cong \mathrm{Hom}_{R_e}(R, R_e)$ as left R-modules.

iii) If R_e is left self-injective then R is left self-injective.

16. Let R be strongly graded by a finite group G.

If R_e is a quasi Fröbenius ring then so is R.

17. Check whether in a left gr-Artinian ring the graded prime ideals are gr-maximal.

18. i) Let A be a commutative graded ring of type Z.

If A is gr-Artinian then A is isomorphic to a (unique) finite direct product of gr-Artinian gr-local rings.

ii) Check i) in case A is graded of type G, G arbitrary.

iii) Try to extend i) ii) to the non-commutative case.

19. Let R be a strongly graded ring of type G.

If E is an injective object in R-gr then E_σ is an R_e-injective module for all $\sigma \in G$. If R_e is a left Noetherian ring than E is an R_e-injective module.

20. Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring over the group G. Put : $D = C(R_e,R) = \{r \in R \mid r\, a = ar \text{ for all } a \in R_e\}$ (it is called the centralizer of R_e in R). Let $\sigma \in G$; since $R_\sigma R_{\sigma^{-1}} = R_e$ there exist $x_i \in R_\sigma$, $y_i \in R_{\sigma^{-1}}$ such that $\sum_{i=1}^{n} x_i\, y_i = 1$. Prove that :

a) If $z \in D$ then $\sum_{i=1}^{n} x_i\, z\, y_i \in D$.

b) The map $\varphi_\sigma : D \to D$, $\varphi_\sigma(z) = \sum_{i=1}^{n} x_i\, z\, y_i$ is a ring automorphism.

c) $\varphi_\sigma(z).x = xy$ for all $x \in R_\sigma$ and $z \in D$

d) The map $\theta: G \to \mathrm{Aut}(D)$, $\theta(\sigma) = \varphi_\sigma$ is a group homomorphism.

e) By d) θ defines on action of G on the ring D. Prove that $D^G = Z(R_e)$ where $D^G = \{ z \in D \mid \varphi_\sigma(z) = z \text{ for all } \sigma \in G \}$.

f) If we suppose that $R = R_e.D$ then prove that there exists a group homomorphism

$$\alpha : G \to \mathrm{Aut}(R),\ \alpha(\sigma)\ (a z) = a\, \varphi_\sigma(z)$$

for all $a \in R_e$ and $z \in D$ (i.e. α defines an action of G on R).

g) If A is a commutative ring and R_e is an Azumaya A-algebra prove that $R = R_e.D$. Moreover if $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly graded ring then D is a strongly graded ring, where $D_\sigma = R_\sigma \cap D$ for all $\sigma \in G$.

21. Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a strongly graded ring, $M \in R$-mod and $N = \bigoplus_{\sigma\in G} N_\sigma$ a graded R-module. If $\sigma \in G$ we define the group homomorphisms:

$\alpha : \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_{R_e}(N_\sigma, M)$, given by $\alpha(f) = f \circ i_\sigma$ where $f \in \mathrm{Hom}_R(N,M)$ and where $i_\sigma : N_\sigma \to N$ is the canonical inclusion homomorphism;

$\beta : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R_e}(M, N_\sigma)$, given by $\beta(g) = \pi_\sigma\ g$ where $\pi_\sigma : N \to N_\sigma$ is the canoncial projection.

a. Prove that α and β are injective.

b. If G is a finite group then prove that α and β are bijective.

Hint. From $R_\tau R_{\tau^{-1}} = R_e$ for all $\tau \in G$ it follows that $1 = \sum_{i\in I(\tau)} x_i^\tau y_i^\tau$ where $x_i^\tau, y_i^\tau \in R_\tau, R_{\tau^{-1}}$ resp. , and $I(\tau)$ is a finite set. If $\alpha(f) = 0$ then $f(n_\sigma) = 0$ for all $n_\sigma \in N_\sigma$. But $f(n_\tau) = \sum_{i\in J(\sigma^{-1}\tau)} x_i^{\sigma^{-1}\tau} f(y_i^{\sigma^{-1}\tau} n_\tau) = 0$ and $y_i^{\sigma^{-1}} n_\tau \in N_\sigma$ yield = 0. Consider $f_\sigma \in \mathrm{Hom}_{R_e}(N_\sigma, M)$.

Define : $f : N \to M$ by $f(n) = \sum_{\tau\in G} f(n_\tau)$ where $n = \sum_{\tau\in G} n_\tau$ and $f(n_\tau) = $

$= \sum_{i\in J(\sigma^{-1}\tau)} x_i^{\sigma^{-1}\tau} f_\sigma (y^{\sigma^{-1}\tau} n_\tau)$.

Check that f is R-linear and also check that $\alpha(f) = f_\sigma$.

22. Let R be a strongly graded ring of type G such that R_e is an A-algebra. Prove : a. $\mathrm{proj.dim}_{(R_e)^e} (R_e) \leq \mathrm{proj.dim}_{R^e}(R)$ where $R^e = R \otimes_A R^o, (R_e)^e = R_e \otimes_A R_e^{\,o}$.

b. If R is a separable A-algebra then R_e is a separable A-algebra.

23. Let R be a strongly graded ring of type G and let D be $\mathrm{End}_{R_e}(R)$. Show that D is a strongly graded ring of type G with gradation given by :

$D_\sigma = \{ f \in D, f(R_\tau) \subset R_{\sigma\tau}$ for all $\tau \in G \}$, for all $\sigma \in G$.

24. Let R be a graded ring of type G and consider $M \in R$-gr. Put

$X = \bigoplus_{\sigma\in G} M(\sigma)$, $Y = \prod_{\sigma\in G} M(\sigma)$.

Show that X and Y are G-invariant objects of R-gr.

25. If $M \in$ R-gr is G-invariant, show that $E^g(M)$, the injective hull of M in R-gr, is G-invariant.

26. If R is a graded ring of type G and left gr-Artinian then $J^g(R) \subset J(R)$. If $G = Z$ then $J^g(R) = J(R)$.

27. Let R be a graded ring of type G and $P \in$ R-gr a finitely generated projective module. The trace ideal of P, $\tau(P) = \sum_{f\in \mathrm{Hom}_R(P,R)} \mathrm{Im} f$, is a graded ideal.

28. G is a u.p. group if for any two nonempty finite subsets E, F of G, there exists at least one element in E.F which is uniquely presented in the form xy with $x \in E$, $y \in F$. All ordered groups are u.p. groups but not all u.p. groups can be ordered, cf. [88].

a. Show that a minimal prime ideal of a G-graded ring R is homogeneous if G is a u.p. group.

b. Let A be a ring with unit, H a normal subgroup of a group G such that G/H is a u.p. group show that rad $(A[G]) = (\mathrm{rad}(A[G]) \cap A[H])\, A[G]$.

c. Let R be graded by the u.p. group G. Let I be an ideal of R and $a \in I$ an element such that its decomposition $a = a_{\sigma_1} + \ldots + a_{\sigma_n} \neq 0$ has minimal length. If $b\, a_{\sigma_j} c = 0$ for some $1 \le j \le n$, $b \in R_\sigma$, $c \in R_\tau$, then $b\, a_{\sigma_i} c = 0$ for all $1 \le i \le n$.

d. The locally nilpotent radical of a ring R, graded by a u.p. group G is a graded ideal, cf. [56].

e. Let A be a ring with unity and H a central subgroup of a group G such that G/H is a u.p. group.

Then $L(R[H]).R[G] = L(R[G])$, where L denotes the locally nilpotent radical.

f. Let G be a u.p. group. Let A be an U-semisimple ring where U denotes the upper nil radical. Then every crossed product of A by G is U-semisimple.

29. If G is a u.p. group then in every E.F where E and F are nonempty finite subsets of G one of which is not a singleton, there are at least two elements uniquely presented in a form x.y with $x \in E$, $y \in F$. cf. A. Strojnowski, A note on u.p. groups, Comm. in Algebra 8(3), 1980, 231-234.

Chapter I may be considered as an extensive work-out of part of the contents of [83]. Considering arbitrary gradations of type G we present a more general theory here.
Let us refer to some basic papers by E.C. Dade, [24] , [25] and a paper by the authors, [82] .

More results about crossed products, twisted group rings and group rings have appeared in papers dealing with the theory of group rings. Let us mention : D. Passman, [88] , [89] , M. Lorenz [67] and also [70] . Related result may be found in J. Roseblade's, [95] .
The representation theory of Artinian algebras, cf. papers by M. Auslander and others, has a graded equivalent i.e. representations of left gr. Artinian algebras. The latter has been bought to an attention by R. Gordon [42] ; we did not go into these matters here.

In the section of Jacobson radicals the importance of finite normalizing extensions is evident. The results on almost strongly graded rings we included in Section I.8. generalize similar results concerning normalizing extensions, for the latter we refer to J.C. Robson [9], J.C. Robson and A. Heinicke [48], and J. Bit-David [10].
Some of the exercises have been extracted from a paper by E. Jespers, J. Krempa, E. Puczylowski,[56] .

II: Some General Techniques (in the Theory of Graded Rings)

II.1. Gradation by Ordered Groups.

Let us recall some definitions and basic properties of ordered groups.

An ordered group, or an O-group, is a group G together with a subset S (the set of positive elements) of G such that the following conditions are being fulfilled :

OG_1 . $e \notin S$

OG_2 . If $a \in G$ then either $a \in S$, $a = e$, or $a^{-1} \in S$.

OG_3 . If $a, b \in S$ then $ab \in S$.

OG_4 . For any $a \in G$, $a S a^{-1} \subset S$. Actually it is easily verified that this is equivalent to $aS a^{-1} = S$ for $a \in G$.

Write $b < a$ if $b^{-1} a \in S$. By OG_4 this is equivalent to $ab^{-1} \in S$. From OG_3 it follows that $<$ is a transitive relation OG, and OG_2 imply that for any $a, b \in G$ we have either $b < a$, $b = a$, or $a < b$. If $b < a$ and $c \in G$ then $(cb)^{-1} (ca) = b^{-1}a \in S$, hence $cb < ca$ and in a similar way $bc < ac$ follows. Combining this with the transitivity of $<$ we find that $b < a$ and $a < c$ entails $bd < ac$. If $b < a$ then $ab^{-1} \in S$ hence $(a^{-1})^{-1} b^{-1} \in S$ and thus $a^{-1} < b^{-1}$.

Conversely, if the elements of G are lineary ordered with respect to a relation $<$ such that $b < a$ implies $bc < ac$ for all $c \in G$, then $S = \{ x \in G, e < x \}$ satisfies OG_{1-4}.

II.1.1. Examples

1. Any torsion-free nilpotent group is ordered.
2. Every free group is an ordered group.

For these examples cf. [88], Lemma 1.6. and Corollary 2.8.

In the sequel of this section we suppose throughout that G is an O-group.

Let R be a graded ring of type G. An $M \in R\text{-gr}$ is said to be <u>left-limited</u> <u>(resp. right limited)</u> if there is a $\sigma_o \in G$ such that $M_\sigma = 0$ for all $\sigma < \sigma_o$ (resp. $\sigma > \sigma_o$). If $M_\sigma = 0$ for each $\sigma < e$ then M is said to be <u>positively graded</u> and if $M_\sigma = 0$ for all $\sigma > e$ then M is <u>negatively graded</u>. It is clear that a strongly graded ring of type G cannot be left (or right) limited (note that finite groups and groups containing torsion elements cannot be ordered).

<u>II.1.2. Lemma</u> . Let R be a left limited graded ring of type G and consider $M \in R\text{-gr}$. The following properties hold :

1. If M is finitely generated then M is left limited.
2. If M is left limited then there exists a gr-free R-module F which is left limited and such that there is an exact sequence $F \to M \to 0$. in R-gr.
3. M is finitely presented in R-gr if and only if $\underline{M}$ is finitely presented in R-mod.

<u>Proof</u>. Since M is finitely generated there is a finitely generated gr-free $F \in R\text{-gr}$ such that $F \to M \to 0$ is an exact sequence in R-gr. Since R is left limited and F being finitely generated, it follows that F is left limited and so is M.

2. Let $\sigma_o \in G$ be such that $R_\sigma = 0$ for $\sigma < \sigma_o$ and let $\tau_o \in G$ be such that $M_\tau = 0$ for all $\tau < \tau_o$. If $x \neq 0$ is in $h(M)$ then $\deg x \geq \tau_o$. To x there corresponds a morphisms of degree e, $\varphi_x\colon R(\sigma_x^{-1}) \to M$, given by $\varphi_x(1) = x$, where $\sigma_x = \deg x$. For any $\sigma < \sigma_o\tau_o$ we have $\sigma\tau_o^{-1} < \sigma_o$ and on the other hand $\tau_o \leq \sigma_x$ yields $\sigma_x^{-1} \leq \tau_o$, hence $\sigma\,\sigma_x^{-1} \leq \sigma\,\tau_o^{-1} < \sigma_o$.
Therefore $R_{\sigma\sigma_x^{-1}} = 0$, thus $R(\sigma_x^{-1})_\sigma = 0$ for all $\sigma < \sigma_o\tau_o$.

Now, putting $\varphi = \bigoplus_{x\in h(M)} \varphi_x$ yields an exact sequence in R-gr :

$\bigoplus_{x\in h(M)} R(\sigma_x^{-1}) \to M \to 0$ and the first term in the sequence is obviously a gr-free and left limited graded module.

3. Obvious.

II.1.3. Lemma. Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a positively graded ring and let $M \in$ R-gr be a left limited module. Then, if we write R_+ for the ideal $\bigoplus_{\sigma>e} R_\sigma$ of R, $R_+M = M$ if and only if $M = 0$.

Proof. If M were non-zero then we could pick $\sigma_0 \in G$ such that $M_{\sigma_0} \neq 0$ but $M_\sigma = 0$ for all $\sigma < \sigma_0$.
Clearly $M_{\sigma_0} \cap R_+M = 0$, thus $R_+M \neq M$.

II.1.4. Proposition. Let R be graded of type G where G is an O-group. Then :

1. A graded ideal P of R is prime if and only if for $a, b \in h(R)$ such that $a R b \subset P$ it follows that a or b is in P.

2. R is a domain if and only if R has no homogeneous zero divisors.

Proof.

1. If P is a prime ideal then clearly the property mentioned holds. Conversely suppose $a R b \subset P$ implies a or b in P in case $a, b \in h(R)$ and consider $x R y \subset P$ for certain $x, y \in R$. We may decompose $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$, $y = y_{\tau_1} + \ldots + y_{\tau_m}$ with $\sigma_1 < \ldots < \sigma_n$ and $\tau_1 < \ldots < \tau_m$. Then $x R y \subset P$ yields $x_{\sigma_n} R y_{\tau_m} \subset P$, thus x_{σ_n} or y_{τ_m} is in P. Repeating this argument leads to the statement 1.

2. One implication is obvious and the other follows again by introducing ordered decompositions.

Consider a graded division ring D of type G as in Section I.4. It is now possible, for ordered G, to give a better description of the possible G-gradations on $M_n(D)$ of the type $M_n(D)(\bar{\sigma})$, $\bar{\sigma} = (\sigma_1,\ldots,\sigma_n) \in G^n$ as described in Section I.5.. With these notations :

II.1.5. Lemma. Let σ be the smallest positive element of G for which $D_\sigma \neq 0$. Then to any $\bar{\sigma} \in G^n$ there corresponds a $\bar{\tau} \in G^n$ such that $e \leq \tau_1 \leq \ldots \leq \tau_n < \sigma$ such that $M_n(D)(\bar{\sigma}) \cong M_n(D)(\bar{\tau})$ as graded rings.

Proof. Apply Corollary I.5.5. (2) and use the ordering on G.

II.1.6. Proposition. Let D be a graded division ring of type Z and let $1 = \sigma$. The number of graded isomorphism classes of the $M_n(D)(\bar{d})$, $\bar{d} \in Z^n$ is finite.

Proof. Let $X_{n,1}$ be the set $\{ \bar{d} \in Z^n,\ 0 \leq d_1 \leq \ldots \leq d_n < 1 \}$. On $X_{n,1}$ introduce the following equivalence relation : $\bar{d} \sim \bar{d}'$ if and only if there exist t, $q_1, \ldots, q_n \in \mathbb{Z}$ and σ a permutation on $\{ 1, \ldots, n \}$ such that $d_i + t = 1q_i + d'_{\sigma(i)}$, $i = 1, \ldots, n$. Let $C_n(D)$ be $X_{n,1}/\sim$ and let $c_n(D)$ be the cardinality of $C_n(D)$, which is finite since 1 is finite. Clearly $\bar{d} \sim \bar{d}'$ yields $M_n(D)(\bar{d}) \cong M_n(D)(\bar{d}')$. Conversely if the isomorphism holds consider $V \in D\text{-gr}$, $V = D(-d_1) \oplus \ldots \oplus D(-d_n)$, $W \in D\text{-gr}$, $W = D(-d'_1) \oplus \ldots \oplus D(-d'_n)$. Then $END_D V = END_D W$. By Corollary I.4.3., D is a principal ideal domain and this entails the existence of an isomorphism of degree zero : $\alpha : D \to D$, and an α-isomorphism of degree zero : $f : V(-p) \to W$ for some integer p. It follows that there exists an α-isomorphism:

$D(-d_1-p) \oplus \ldots \oplus D(-d_n - p) \cong D(-d'_1) \oplus \ldots \oplus D(-d'_n)$.

Choose $\overline{d''} \in Z^n$ such that $\overline{d+p} = \overline{1q} + \overline{d''}$ and with $d''_i < 1$ for $c = 1,\ldots n$. Then for any $m \in Z$ there is a D-isomorphism of degree zero : $D \cong D(-1m)$. The existence of f then entails the existence of an α-isomorphism of degree zero :

$D(-d_1'') \oplus \ldots \oplus D(-d_n'') \cong D(-d_1') \oplus \ldots \oplus D(-d_n')$.

Because $D_1 = D_2 = \ldots = D_{l-1} = 0$, there exists a $\sigma \in S_n$ such that $d_i'' = d'_{\sigma(i)}$ for all i, $1 \leq i \leq n$.

II.1.7. Remark

Let D be a graded division ring of type Z such that $l = 1$, then $C_n(D) = 1$. If $l = 2$ then $C_n(D) = 1 + [\frac{n}{2}]$.

II.1.8. Proposition. Let R be a graded ring of type G where G is an O-group, then Z(R) is graded of type G.

Proof. Let $z \in Z(R)$ and decompose z as $z_{\sigma_1} + \ldots + z_{\sigma_n}$ with $\sigma_1 < \ldots < \sigma_n$. For $x \in R_\sigma$ we have $zx = xz$ hence $z_{\sigma_1}x + \ldots + z_{\sigma_n}x = x z_{\sigma_1} + \ldots + x z_{\sigma_n}$. Now $\sigma_1\sigma < \ldots \sigma_n\sigma$ and $\sigma\sigma_1 < \ldots < \sigma\sigma_n$, therefore $z_{\sigma_n}x = x z_{\sigma_n}$. Consequently $z_{\sigma_n} \in Z(R)$ follows. Repeating this argument we finally end up showing that z_{σ_i} is in Z(R) for all $i = 1,\ldots,n$, i.e. Z(R) is graded of type G.

II.I.9. Remark. If G is abelian then the result of II.I.8. is true even if G is not ordered !

II.2. Internal Homogenization.

Let G be an O-group throughout the section; R is a graded ring of type G. Consider $M \in$ R-gr and let $X \subset \underline{M}$ be an R-submodule. An $x \in X$ may be written in a unique way as $x_{\sigma_1} + \ldots + x_{\sigma_n}$ with $\sigma_1 < \ldots < \sigma_n$. Write $X^\sim$, resp. $X_\sim$, for the submodule of $\underline{M}$ generated by x_{σ_n}, resp. x_{σ_1}, for all $x \in X$, i.e. $X^\sim$ is generated by the homogeneous components of highest degree appearing in elements of X. With these notations :

II.2.1. Lemma.

1. $X^\sim$ and $X_\sim$ are graded submodules of M.
2. We have that $X = X^\sim$ if and only if X is a graded submodule of M.
3. If $X \subset Y \subset \underline{M}$ then $X^\sim \subset Y^\sim$ and $X_\sim \subset Y_\sim$.
4. We have $X = 0$ if and only if $X^\sim = 0$, if and only if $X_\sim = 0$.
5. If L is a left ideal of R and $\underline{N}$ an R-submodule of $\underline{M}$ then $L^\sim N^\sim \subset (LN)^\sim$ and $L_\sim N_\sim \subset (LN)_\sim$. Therefore if L is an ideal then $L^\sim$ and $L_\sim$ are ideals.

Proof. The assertions 1-4 are easy enough to prove.

5. Consider $\tilde{a} \in h(L^\sim)$, $\tilde{x} \in h(N^\sim)$ and let $a \in L$, $x \in N$ be such that $\tilde{a}, \tilde{x}$ are the components of highest degree in a, x resp. .
If $\tilde{a}\,\tilde{x} \neq 0$ then it is the homogeneous component of highest degree of $a\,x \in LN$ hence $\tilde{a}\,\tilde{x} \in LN$ hence $\tilde{a}\,\tilde{x} \in (L\,N)^\sim$.

If $M \in$ R-gr and $X \subset \underline{M}$ is an R-submodule then $(X)_g$ will denote the submodule of M generated by $X \cap h(M)$. One easily verifies that $(X)_g$ is maximal amongst submodules of $\underline{X}$ which are graded submodules of M. Let us write $\mathrm{Spec}_g(-)$ for the subset of Spec(-), consisting of graded prime ideals of the ring under consideration.

II.2.2. Lemma. Let G be an O-group and let R be a graded ring of type G. Then the following properties hold :

1. If I is an ideal of R then $(I)_g$ is a graded ideal of R.
2. If P is a graded ideal then $P \in \mathrm{Spec}_g(R)$ if and only if for a, b $\in h(R)$ such that $a R b \subset P$, $a \in P$ or $b \in P$ follows
3. If $P \in \mathrm{Spec}(R)$ then $(P)_g \in \mathrm{Spec}_g R$.
4. If $P \in \mathrm{Spec}(R)$ is minimal then $(P)_g = 0$ or $P \in \mathrm{Spec}_g(R)$.
5. If I is a graded ideal then rad $I = \cap \{ P, P \in \mathrm{Spec}(R), P \supset I \}$ is again a graded ideal.

Proof.

1. Obvious.
2. Proposition II.14., 1.

3, 4, 5. Follow immediately from 2.

For $M \in R$-gr we introduce $M^+ = \bigoplus_{\sigma \geq e} M_\sigma$, $M^- = \bigoplus_{\sigma \leq e} M_\sigma$.

It is clear that R^+ and R^- are graded subrings of R and that M^+ is in R^+-gr, M^- is in R^--gr. If N, $M \in R$-gr are such that $N \subset M$ then we have : $(M/N)^+ = M^+/N^+$ and $(M/N)^- = M^-/N^-$.

II.2.3. Proposition.

Let R be a graded ring of type $\mathbb{Z}$. Consider R-submodules $X \subset Y \subset \underline{M}$ for some $M \in R$-gr.

The following statements are equivalent :

1. $X = Y$.
2. $X^\sim = Y^\sim$ and $X \cap M^- = Y \cap M^-$
3. $X_\sim = Y_\sim$ and $X \cap M^+ = Y \cap M^+$.

Proof. $1 \Leftrightarrow 2$. Take $y \in Y$ and decompose it as $y = y_{\sigma_1} + \ldots + y_{\sigma_m}$ with $\sigma_1 < \ldots \sigma_m$. If $\sigma_m \leq e$ then $y \in M^-$ and thus $y \in X$. If $\sigma_m > e$ then

$Y^{\sim} = X^{\sim}$ yields that there exists $x \in X$ such that $x = x_{\tau_1} + \ldots + x_{\tau_{n-1}} + y_{\sigma_m}$ with $\tau_1 < \ldots < \tau_{n-1} < \sigma_m$. It follows that $y - x \in Y$ has a homogeneous decomposition in which the largest degree appearing is less than σ_m.

Clearly, after a finite number of steps we find $x^1, \ldots, x^k \in X$ such that $y - (x^1 + \ldots + x^k) \in Y \cap M^- = X \cap M^-$ and therefore $y \in X$. Equivalence of 1 and 3 is established in a similar way.

II.2.4. Corollary. If M is left limited (resp. right limited) then $X = Y$ if and only if $X^{\sim} = Y^{\sim}$ (note : we already assumed $X \subset Y$).

II.3. Chain Conditions for Graded Modules.

Let R be a graded ring of type G; $M \in R\text{-gr}$ is said to be left gr-Noetherian, resp. gr-Artinian (see section I.7.), if M satisfies the ascending, resp. descending, chain condition for graded submodules of M. It is straightforward to verify that M is left gr-Noetherian if and only if each non-empty family of graded submodules of M has a maximal element. Dually, M is left gr-Artinian if and only if each non-empty family of graded submodules of M has a minimal element, or if and only if each intersection of graded submodules may be reduced to a finite intersection.

II.3.1. Proposition. Let R be a graded ring of type Z. Let $M \in R\text{-gr}$ be left limited. Then M is left gr-Noetherian, resp. gr-Artinian, if and only if $\underline{M}$ is left Noetherian, resp. left Artinian, in R-mod.

Proof. Apply Corollary II.2.4..

II.3.2. Lemma. Let R be a graded ring of type G. If $M \in R\text{-gr}$ is left gr-Noetherian (gr-Artinian) then M_σ is left Noetherian (Artinian) in R_e-mod, for all $\sigma \in G$.

Proof. If $N \subset M_\sigma$ is an R_e-submodule then $N = RN \cap M_\sigma$. To an ascending chain $N_1 \subset N_2 \subset \dots \subset N_n \subset \dots$, of submodules of M_σ there corresponds an ascending chain $RN_1 \subset \dots \subset RN_n \subset \dots$, of graded submodules of M. Since M is left gr-Noetherian $RN_n = RN_{n+1} = \dots$ for some n, follows. Thus $N_n = N_{n+1} = \dots$.
The proof in the Artinian case is similar.

II.3.3. Corollary. Let R be graded by a finite group G. If $M \in R\text{-gr}$ is left gr-Noetherian (Artinian) then $\underline{M}$ is left Noetherian, (Artinian), in R-mod.

Proof. By the lemma, M_σ is a Noetherian R_e-module. Since σ is finite, $\bigoplus_{\sigma\in G} M_\sigma$ is Noetherian in R_e-mod, hence in R-mod.

The same proof works in the Artinian case.

II.3.4. Proposition. Let R be a graded ring of type Z . If $M \in$ R-gr is left gr-Noetherian, then :

1. $\underline{M}^+$ is left Noetherian in R^+-mod.
2. $\underline{M}^-$ is left Noetherian in R^--mod.

Conversely, if M^+, M^- are left Noetherian in R^+-mod, resp. R^--mod, then M is left gr.-Noetherian.

Proof.

1. By Porposition II.3.1. it is sufficient to show that M^+ is left gr-Noetherian in R^+-gr. Let N be a graded R^+-submodule of M^+. Since RN is finitely generated as a graded R-module we assume that $RN = Rx_{\sigma_1} + \ldots + Rx_{\sigma_k}$ with $x_{\sigma_i} \in h(N)$, $i = 1, 1, \ldots, k$.

Put $\sigma = \max.\{\sigma_i,\ i = 1,\ldots k\}$ and let $y_\tau \in h(N)_\tau$ be such that $\tau \geq \sigma$. There exist $\lambda_1,\ldots,\lambda_k \in h(R)$ such that $y_\tau = \lambda_1 x_{\sigma_1} + \ldots + \lambda_k x_{\sigma_k}$. Since $\tau \geq \sigma_i$ for all i it follows that $\lambda_i \in R^+$ for $i = 1, \ldots k$.

Apply Lemma II.3.2. and derive that $\bigoplus_{e\leq\gamma\leq\sigma} M_\gamma$ is finitely generated in R_e-mod, say by $\{y_1,\ldots,y_s\}$.

Since $N = (\bigoplus_{e\leq\gamma\leq\sigma} N_\gamma) \oplus (\bigoplus_{\tau\geq\sigma} N_\tau)$ it is clear that $\{x_{\gamma_1},\ldots, x_{\sigma_k}, y_1\ldots y_s\}$ generates N over R^+; therefore M^+ is a left gr-Noetherian R-module.

2. Similar to 1.

Now assume M^+, M^- to be left Noetherian in R^+-mod, resp. R^--mod and consider and ascending chain, $N_1 \subset N_2 \subset \ldots \subset N_p \subset \ldots$, of graded R-submodules of M. From this we get ascending chains :

$$N_1^+ \subset N_2^+ \subset \dots \subset N_p^+ \subset \dots \subset M^+$$

$$N_1^- \subset N_2^- \subset \dots \subset N_p^- \subset \dots \subset M^-$$

The assumptions amount to $N_k^+ = N_{k+1}^+ = \dots$ and $N_k^- = N_{k+1}^- = \dots$ for some k; then by Proposition II.2.3., $N_k = N_{k+1} = \dots$ follows

II.3.5. Theorem.

Let R be a graded ring of type Z, let $M \in$ R-gr. The following assertions are equivalent :

1. M is left gr.-Noetherian
2. $\underline{M}$ is a left Noetherian R-module.

Proof. Since $2 \Rightarrow 1$ is obvious let us show $1 \Rightarrow 2$.

Consider an ascending chain of R-submodules of $\underline{M}$, $X_1 \subset \dots \subset X_p \subset \dots$, say. By proposition II.3.4. there is an $n_o \in \mathbb{N}$ such that $M^- \cap X_i = M^- \cap X_{i+1} = \dots$ and $\tilde{X_i} = \tilde{X}_{i+1} = \dots$ for all $i \geq n_o$. By proposition II.2.3.(2.) it follows that $X_i = X_{i+1} = \dots$ for $i \geq n_o$.

II.3.6. Lemma. Let R be a strongly graded ring of type G and $M \in$ R-gr, then the following properties are equivalent :

1. M_e is a left Noetherian R_e-module
2. M is left gr-Noetherian

Proof. Immediate from Theorem I.3.4.

II.3.7. Lemma. Let R be a strongly graded ring of type Z.

If R_o is a left Noetherian ring then R is a left Noetherian ring.

Proof. From Lemma II.3.6. it follows that R is left gr.-Noetherian.

From Theorem II.3.5. it follows that R is a left Noetherian ring.

A group G is polycyclic-by-finite if there exists a finite series $\{e\} = G_o \subset G_1 \subset \dots \subset G_n = G$ of subgroups of G such that each G_{i-1}

is normal in G_i and G_i/G_{i-1} is either finite of cyclic for each i.

II.3.8. Theorem. Let R be a strongly graded ring of type G where G is a polycyclic-by-finite group. If R_e is a left Noetherian ring then R is a left Noetherian ring.

Proof. Consider the normal series $\{e\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$. If n = 1 then $G = G_1$ is finite or cyclic and from Corollary II.3.3. of Lemma II.3.7. we deduce that R is left Noetherian. We proceed by induction on n, assuming that the statement holds for normal series of length less than n. Put $H = G_{n-1}$, $G' = G/H$ and let $\pi : G \to G'$ be the canonical homomorphism. Consider $R_{(G')} = S$ (see introduction of I.1.). If e' is the neutral element of G' then $S_{e'} = \underset{\sigma \in H}{\oplus} R_\sigma = R^{(H)}$. Both $R_{(G')}$ and $R^{(H)}$ are strongly graded rings and as $R^{(H)}$ is left Noetherian (induction), whereas G' is finite or cyclic, it follows that S is a left Noetherian ring.

II.3.9. Corollary. Let A be a left Noethrian ring, $\varphi : A \to A$ a ring automorphism. Then $A[X, \varphi]$ and $A[X, X^{-1}, \varphi]$ are left Noetherian.

Proof. Put $S = A[X, X^{-1}, \varphi]$. Clearly S is strongly graded of type Z, thus the theorem applies. Since $A[X, \varphi] = S^+$, Proposition II.3.4. entails that $A[X, \varphi]$ is left Noetherian.

II.3.10. Application. The group ring A[G] of a polycyclic-by-finite group over a left Noetherian ring A is a left Noetherian ring. A similar result holds for crossed products.

II.3.11. Example. Let R and S be any two rings and let there be given bimodules $M \in$ R-mod-S, $N \in$ S-mod-R.

Consider the ring $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix} = \{ \begin{pmatrix} r & m \\ n & s \end{pmatrix}, r \in R, m \in M, n \in N, s \in S \}$,

with the obvious addition but with multiplication defined by
$\begin{pmatrix} r & m \\ n & s \end{pmatrix}\begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} rr' & rm'+ms' \\ nr'+sn' & ss' \end{pmatrix}$.

Define a Z-gradation on T by putting $T_{-1} = \begin{pmatrix} o & o \\ N & o \end{pmatrix}$, $T_o = \begin{pmatrix} R & o \\ o & S \end{pmatrix}$, $T_1 = \begin{pmatrix} o & M \\ o & o \end{pmatrix}$ and $T_n = 0$ if $n \neq -1, 0, 1$.

Left graded ideals of T are of the form : $\begin{pmatrix} I & M' \\ N' & J \end{pmatrix}$, where I and J are left ideals in R and S respectively, while M' is an R-submodule of M, N' an S-submodule of N, such that $MJ \subset M'$, $NI \subset N'$. We then may formulate the following statement : T is left Noetherian (Artinian) if and only if R and S are left Noetherian (Artinian) and M is finitely generated in R-mod, N is finitely generated in S-mod.

Let G be an arbitrary group, H a subgroup of G and $\{\sigma_i, i \in I\}$ a set of representatives for the right H-cosets of G. Let R be a graded ring of type G, let $M \in$ R-gr. For each $i \in I$ we put $M^{(\sigma_i,H)} = \bigoplus_{h\in H} M_{h\sigma_i}$

It is clear that $M^{(\sigma_i,H)}$ is a graded $R^{(H)}$-module with gradation given by $(M^{(\sigma_i,H)})_h = M_{h\sigma_i}$ for all $h \in H$.

<u>II.3.12. Lemma</u>. Let P be a graded $R^{(H)}$-submodule of $M^{(\sigma_i,H)}$ then $RP \cap M^{(\sigma_i,H)} = P$.

<u>Proof.</u> If $y \in h(RP \cap M^{(\sigma_i,H)})$ then $y = \sum_{k=1}^{n} \lambda_k x_k$ with $\lambda_k \in h(R)$, $x_k \in h(P)$, $k = 1\ldots n$. Put $\tau_k = \deg x_k$, $k = 1,\ldots,n$. Then $x_k \in (M^{(\sigma_i,H)})_{\tau_k} = M_{\tau_k\sigma_i}$. If we write $\deg y = \tau$ in $M^{(\sigma_i,H)}$ then $y \in M_{\tau\sigma_i}$. It follows from all this that : $\tau\sigma_i = (\deg \lambda_k).\tau_k\sigma_i$ for all $k = 1,\ldots,n$.

On the other hand $\deg(\lambda_k) = \tau\tau_k^{-1}$ for all $k = 1,\ldots,n$.

Hence $\lambda_k \in R^{(H)}$ for all $k = 1,\ldots,n$ and therefore $y \in P$ i.e. $RP \cap M^{(\sigma_i,H)} \subset P$.

<u>II.3.13. Corollary</u>. If M is a left gr-Noetherian R-module then for every $i \in I$, $M^{(\sigma_i,H)}$ is a left gr.-Noetherian $R^{(H)}$-module.

II.3.14. Corollary. If H is of finite index in G then the following statements are equivalent :

1. M is a left gr.-Noetherian R-module
2. $M^{(\sigma_i,H)}$ is left gr-Noetherian in $R^{(H)}$-mod for every $i \in I$.

Proof. $1 \Rightarrow 2$ is just II.3.13.

$2 \Rightarrow 1$. Since I is finite and since $M \cong \bigoplus_{i\in I} M^{(\sigma_i,H)}$ it follows that M is a left gr.-Noetherian $R^{(H)}$-module but then M is certainly left gr.-Noetherian in R-mod.

II.3.15. Corollary. If R is a left gr-Noetherian ring then the ring $R^{(H)}$ is left gr.-Noetherian.

II.4. Invertible Ideals. The Rees Ring and the Generalized Rees Ring.

Let R be any ring, I an ideal of R. The Rees ring associated to I is the ring $R(I) = R + IX + \dots + I^n X^n + \dots \subset R[X]$ which obviously is a graded subring of $R[X]$ isomorphic to $R \oplus I \oplus I^2 \oplus \dots \oplus I^n \oplus \dots$

A ring T is an overring of R if $R \subset T$ and $1_T = 1_R$. An ideal I of R is an invertible ideal if there is an overring T of R containing an R-bimodule J such that $JI = IJ = R$.

II.4.1. Example. Let R be a left Noetherian prime ring and let $x \in R$ be a normalizing element (i.e. $xR = Rx$). By Goldie's theorems the ideal $I = Rx = xR$ is an invertible ideal of R with inverse $x^{-1}R = Rx^{-1}$ in the ring of quotients of R.

If I is an invertible ideal of R (in the overring T) then we can define the generalized Rees ring of type Z, to be $\check{R}(I) = \bigoplus_{n \in Z} I^n X^n$ which is a graded subring of $T[X, X^{-1}]$.

Note that, with notations as before, $\check{R}(I)^+ = R(I)$.

II.4.2. Proposition. If R is left Noetherian and the ideal I of R is invertible then $\check{R}(I)$ and $R(I)$ are left Noetherian rings.

Proof. Clearly $\check{R}(I)$ is strongly graded of type Z, thus by Lemma II.3.7., $\check{R}(I)$ is left Noetherian and by Proposition II.3.4. $R(I)$ is left Noetherian.

Let I be an ideal of R. On an $M \in R$-mod we consider the I-adic filtration; $(\star)$: $M = M_0 \supset M_1 \supset \dots \supset M_n \supset \dots$ (More about filtrations is in the final appendix in this book). Note that this filtration has the property $IM_n \subset M_{n+1}$, for all n. The module $R(M) = M_0 \oplus \dots \oplus M_n \oplus \dots$ is called the Rees module associated to M and the filtration $(\star)$.

It is clear that $R(M)$ is a graded $R(I)$-module.

The filtration $(\star)$ is said to be I-fitting if there is a $n \in \mathbb{N}$ such that

$IM_n = M_{n+1}$ for all $n \geq p$. As an example one may consider the filtration $M_n = I^n M$ for all $n \geq 0$. If $M = M_0 \supset M_1 \supset \ldots \supset M_n \supset \ldots$ is an I-fitting filtration then for any submodule N of M we may define an I-fitting filtration on M/N by putting $(M/N)_n = M_n + N/N$.

II.4.3. Lemma. Let R be a left Noetherian ring and let I be an ideal of R. Consider a left Noetherian $M \in$ R-mod and equip it with the I-adic filtration $(\star)$ $M = M_0 \supset M_1 \supset \ldots \supset M_n \supset \ldots$.

The following statements are equivalent :

1. The filtration $(\star)$ is I-fitting.
2. R(M) is an R(I)-module of finite type.

Proof. $1 \Rightarrow 2$. If $IM_n = M_{n+1}$ for all $n \geq p$ for a certain $p \geq 0$ then R(M) is generated by $M_0 \oplus M_1 \oplus \ldots \oplus M_p$ as an R(I)-module. The latter is clearly a finitely generated R-module, whence the statement follows.

$2 \Rightarrow 1$. Let R(M) be generated by elements $x_1, \ldots, x_n$ of degree $d_1, \ldots, d_n$ resp. Put $p = \max \{ d_i, i=1,\ldots n \}$ and check that $M_{p+1} = IM_p$.

II.4.4. Lemma. Let R be a ring and I an ideal of R such that R(I) is left Noetherian. Let $M \in$ R-mod be filtered by the I-adic filtration $(\star)$ $M_0 = M \supset M_1 \supset \ldots \supset M_n \supset \ldots$.

If M is finitely generated and $(\star)$ is I-fitting then for every R-submodule N of M, the induced filtration :

$(\star)_N : N = N \cap M_0 \supset N \cap M_1 \supset \ldots \supset N \cap M_n \supset \ldots$ is also I-fitting.

Proof. By the foregoing lemma R(M) is a finitely generated R(I)-module hence a left Noetherian R(I)-module. Now R(N) is an R(I)-submodule of R(M) and therefore R(N) is finitely generated as an R(I)-module. Lemma II.4.3. implies that $(\star)_N$ is I-fitting.

II.4.5. Corollary. (Strong Artin-Rees property) Let R be a ring, I an ideal of R, such that R(I) is left Noetherian. Let M be a finitely generated R-module containing the R-submodules K and L, then there is a $p \in \mathbb{N}$ such that for all $n \geq p$:

$$I^{n-p}(I^p K \cap L) = I^n K \cap L.$$

Proof. Put $M_i = I^i M$. From Lemma II.4.3. we derive that R(M) is a left Noetherian R(I)-module. Thus $\bigoplus_{n\geq 0} (I^n K \cap L)$ is a finitely generated R(I)-submodule of R(M). By Lemma II.4.4. the filtration $\{ I^n K \cap L, n \geq 0 \}$ is I-fitting. The result follows.

II.4.6. Corollary. Let R be a left Noetherian ring. If I is an invertible ideal of R then I has the strong Artin-Rees property.

II.4.7. Remark. Let R be a left Noetherian ring and let I be an ideal generated be central elements. We claim that R(I) is a left Noetherian ring. Indeed, suppose $a_1, \ldots, a_n \in Z(R)$ generate I, then we have a canonical homomorphism $\varphi : R[X_1, \ldots, X_n] \to R(I)$ defined by $\varphi(X_i) = a_i X$. Obviously φ is surjective, thus R(I) is left Noetherian and by Corollary II.4.5. it follows that I has the strong Artin-Rees property.

II.5. Krull Dimension of Graded Rings.

The Krull dimension of ordered sets has been defined by P. Gabriel and Rentschler, in [33], for finite ordinal numbers, and G.Krause generalized the notion to other ordinal numbers cf. [59]. Let us recall some definitions and elementary facts.

Let $(E, \leq)$ be an ordered set. For $a, b \in E$ we write $[a, b]$ for the set $\{ x \in E, a \leq x \leq b \}$ and we put $\Gamma(E) = \{ (a,b), a \leq b \}$.
By transfinite recurrence we define on $\Gamma(E)$ the following filtration :

$\Gamma_{-1}(E) = \{(a,b), a=b\}$

$\Gamma_0(E) = \{ (a,b) \in \Gamma(E), [a,b]$ is Artinian $\}$

supposing $\Gamma_\beta(E)$ has been defined for all $\beta < \alpha$, then

$\Gamma_\alpha(E) = \{ (a,b) \in \Gamma(E), \forall\, b \geq b_1 \geq \ldots \geq b_n \geq \ldots \geq a$ there is an $n \in \mathbb{N}$ such that $[b_{i+1}, b_i] \in \Gamma_\beta(E)$ for all $i \geq n \}$.

We obtain an ascending chain $\Gamma_{-1}(E) \subset \Gamma_0(E) \subset \ldots \subset \Gamma_\alpha(E) \subset \ldots$.
There exists an ordinal ξ such that $\Gamma(E)_\xi = \Gamma(E)_{\xi+1} = \ldots$.
If there exists an ordinal α such that $\Gamma(E) = \Gamma(E)_\alpha$ then E is said to have Krull dimension. The smallest ordinal with the property that $\Gamma_\alpha(E) = \Gamma(E)$ will be called the Krull dimension of E and we denote it by K dim E.

II.5.1. Lemma. Let E, F be ordered sets and let : $E \to F$ be a strictly increasing map. If F has Krull dimension then E has Krull dimension and K dim E $\leq$ Kdim F. (cf. [43]).

II.5.2. Lemma. If E, F are ordered sets with Krull dimension then $E \times F$ has Krull dimension and Kdim ExF = sup (Kdim E, Kdim F)
(Note that $E \times F$ has the product ordering).

If $\underline{A}$ is an arbitrary abelian category and M is an object of $\underline{A}$ then we consider the set E_M of all subobjects of M in $\underline{A}$ ordered by inclusion. If E_M has Krull dimension then M is said to have Krull dimension and we denote it by $\mathrm{Kdim}_{\underline{A}}$ M or simply Kdim M if no ambiguity can arise. In this case we may reformulate the definition of Krull dimension as follows:

If M = 0; Kdim = -1; if α is an ordinal and Kdim M $\nless \alpha$ then Kdim M = α provided there is no infinite descending chain : $M \supset M_0 \supset M_1 \supset \dots$ of subobjects M_i of M in $\underline{A}$ such that for i = 1, ..., Kdim $(M_{i-1}/M_i) \nless \alpha$.

An object M of $\underline{A}$ having Kdim M = α is said to be <u>α-critical</u> if $\mathrm{Kdim}(M/M') < \alpha$ for every non-zero subobject M' of M.

For example, M is o-critical if and only if M is a simple object of $\underline{A}$. Also, it is obvious that any nonzero subobject of an α-critical object is again α-critical.

<u>II.5.3. Lemma.</u> If M is an object of $\underline{A}$ and N is a subobject of M then Kdim M = sup { Kdim(M/N), Kdim N } and equality holds provided either side exists.

<u>Proof.</u> Using Lemma II.5.2., cf [43].

<u>II.5.4. Lemma.</u> Every Noetherian object of $\underline{A}$ has Krull dimension.

<u>Proof.</u> See Proposition 1.3. [43].

<u>II.5.5. Lemma.</u> Suppose that $\underline{A}$ is an abelian category allowing infinite direct sums. If an object M of $\underline{A}$ has Krull dimension then M cannot contain an infinite direct sum of subobjects. In particular M has finite Goldie dimension.

<u>II.5.6. Lemma.</u> Suppose that $\underline{A}$ is an abelian category allowing infinite direct sums and suppose the object M of $\underline{A}$ has Krull dimension.

Put $\alpha = \sup \{ 1 + \mathrm{Kdim}_{\underline{A}} (M/N);$ N an essential subobject of M$\}$. Then $\alpha \geq$ KdimM.

Proof. For II.5.5., II.5.6. we refer to [43] .

Let us now come back down to earth and put $\underline{A}$ = R-mod. The Krull dimension of M in R-mod is called the left Krull dimension of M and we denote it by Kdim_R M. The Krull dimension of R considered as a left R-module is called the left dimension of R and we denote it by Kdim R. If R is a graded ring, putting $\underline{A}$ = R-gr we define $\mathrm{Kdim}_{R\text{-gr}}$ M as before (if it exists for that particular M $\in$ R-gr). The purpose of this paragraph is to compare Kdim_R (M) and $\mathrm{Kdim}_{R\text{-gr}}$ (M) for M $\in$ R-gr.

II.5.7. Lemma. Let R be a graded ring of type Z and consider a left (or right) limited M $\in$ R-gr, then :

1. M has Krull dimension if and only if $\underline{M}$ has Krull dimension and in this case we have : $\mathrm{Kdim}_{R\text{-gr}}$ M = Kdim_R M.
2. If M is α-critical then $\underline{M}$ is α-critical.

Proof.

1. Follows from Corollary II.2.4.
2. If $\underline{X} \neq 0$ is any submodule of $\underline{M}$ then $\mathrm{Kdim}_{R\text{-gr}} (M/X^{\sim}) < \alpha$.

By proposition II.2.3. and Corollary II.2.4. and Lemma II.5.1. we have $\mathrm{Kdim}_R (\underline{M}/\underline{X}) \leq \mathrm{Kdim}_{R\text{-gr}} (M/X^{\sim})$. Consequently , $\underline{M}$ is α-critical.

II.5.8. Proposition.

Let R be a prositively graded ring of type Z. For M $\in$ gr the following properties hold :

1. M has Krull dimension if and only if $\underline{M}$ has Krull dimension and if this is the case then $\mathrm{Kdim}_{R\text{-gr}}$ M = Kdim_R M.

2. If M is α-critical then $\underline{M}$ is α-critical.

Proof. For $\sigma \in S$, the positive elements of G, we define $M_{>\sigma} = \bigoplus_{\tau>\sigma} M_\tau$.

If is clear that $M_{>\sigma}$ is a graded submodule of M and $M/M_{>\sigma}$ is right limited. From the foregoing lemma we retain : $\mathrm{Kdim}_{R\text{-gr}} M_{>\sigma} = \mathrm{Kdim}_R (\underline{M_{>\sigma}})$, and also $\mathrm{Kdim}_{R\text{-gr}} (M/M_{>\sigma}) = \mathrm{Kdim}_R (\underline{M}/\underline{M_{>\sigma}})$.

Application of Lemma II.5.3. yields the statement.

2. Consider a nonzero submodule $\underline{X}$ of $\underline{M}$. For some $\sigma \in G, X \cap M_{>\sigma} \neq 0$. Combining the fact that M is α-critical with the result of the foregoing lemma we deduce that $\mathrm{Kdim}_{R\text{-gr}} (M_{>\sigma}/X \cap M_\sigma) < \alpha$. Now there exists a strictly increasing map from the lattice of submodules of $\underline{M}/\underline{X}$ into the set-product of the lattices of submodules of $\underline{M_{>\sigma}}/\underline{X \cap M_{>\sigma}}$ and of $\underline{M}/\underline{M_{>\sigma}}$, given by

$\underline{Y}/\underline{X} \to ((\underline{Y \cap M_{>\sigma}}) / (\underline{X \cap M_{>\sigma}}), (\underline{Y + M_{>\sigma}}) / \underline{M_{>\sigma}})$.

By Lemma II.5.2., and because $\mathrm{Kdim}_R (\underline{M}/\underline{M_{>\sigma}}) < \alpha$, it follows that $\mathrm{Kdim}_R (\underline{M}/\underline{X}) < \alpha$; hence $\underline{M}$ is α-critical.

II.5.9. Lemma. Let R be a left Noetherian graded ring of type Z, and let $M \in R\text{-gr}$ be a uniform object. Suppose that either M is limited (left or right) or that R is positively graded. Then $\underline{M}$ is uniform in R-mod.

Proof. Theorem 2.7. of [43] states that an object from any abelian category, having Krull dimension α, contains a nonzero subobject which is α-critical. Since R is left Noetherian, we may apply Lemma II.5.4. and the foregoing remark to deduce that M contains a graded R-submodule N which is α-critical. Uniformity of M entails that M is an essential extension of N in R-gr. then $\underline{M}$ is an essential extension of $\underline{N}$ (see I.2.8.) Lemma II.5.7. and Proposition II.5.8. yield that $\underline{N}$ is an α-critical

R-module, hence it is uniform and therefore $\underline{M}$ is uniform too.

II.5.10. Theorem. Let R be a graded ring of type Z. If $M \in$ R-gr is a uniform object of R-gr then $\underline{M}$ is a uniform R-module.

Proof. For $\sigma \in G$ let $M_{\geq\sigma}$ stand for $\bigoplus_{\tau\geq\sigma} M_\tau$. Consider a nonzero submodule $\underline{X} \subset \underline{M}$ such that for some $\sigma \in G$ we have $\underline{X} \cap \underline{M_{\geq\sigma}} = 0$. Then $\underset{\sim}{X} \cap M_{\geq\sigma} = 0$. Indeed if $x \in h(\underset{\sim}{X} \cap M_{\geq\sigma})$ then there is a $y \in X$ such that $y = y_{\sigma_1}+\ldots+y_{\sigma_m}$ with $x = y_{\sigma_1}$ and $\sigma_1 < \ldots \sigma_m$. Since $\sigma \leq \sigma_1$ it follows that $y \in M_{\geq\sigma}$, hence $y = 0$ thus $(\underset{\sim}{X})_\tau = 0$ for all $\tau \geq \sigma$. Now $\underset{\sim}{X} \neq 0$ and $\underset{\sim}{X}$ is uniform in R-gr, so we may use Lemma II.5.9. and deduce that $\underline{\underset{\sim}{X}}$ is a uniform R-module. Consequently $\underline{M}$ is a uniform R-module. Let us suppose now that $X \cap M_{\geq\sigma} \neq 0$ for any nonzero submodule X of $\underline{M}$ and for all $\sigma \in G$. We proceed to show that M^+ is a uniform R^+-module. In view of Lemma II.5.9. it will be sufficient to establish uniformity of M^+ in R^+-gr. Pick $y, x \in h(M^+)$, nonzero elements with $\deg x = \sigma$, $\deg y = \tau$ and $\tau \leq \sigma$. Uniformity of M yields $R\,x \cap R\,y \neq 0$ and by our assumptions $Rx \cap Ry \cap M_{\geq\sigma} \neq 0$. Pick $z \neq 0$ in $h(Rx \cap Ry \cap M_{\geq\sigma})$ and write $z = \lambda x = \mu y$ for $\lambda,\mu \in h(R)$; note $\lambda,\mu \in R^+$. Thus $z \in R^+x \cap R^+y$ and therefore M^+ is uniform in R^+-gr.

Finally, if $\underline{X}$, $\underline{Y}$ are nonzero submodules of $\underline{M}$ then $\underline{X} \cap \underline{M}^+ \neq 0$ and $\underline{Y} \cap M^+ \neq 0$ yield $\underline{X} \cap \underline{Y} \cap \underline{M}^+ \neq 0$; hence $\underline{X} \cap \underline{Y} \neq 0$.

II.5.11. Corollary. If R is as in II.5.10. is moreover left Noetherian then the Goldie dimension of M in R-gr equals the Goldie dimension of $\underline{M}$ in R-mod.

Proof. Direct from the theorem and Lemma I.2.8.

II.5.12. Lemma. If R is as in II.5.10, and $N \in$ R-gr is a left Noetherian module that N has a composition series $N \supset N_1 \supset \ldots \supset N_n = 0$, where N_{i-1}/N_i is a critical module for each $1 \leq i \leq n$.

Proof. As in the ungraded case, cf. [43].

II.5.13. Lemma. Let R be as in II.5.10. and let $M \in R\text{-gr}$ have Krull dimension α. Then $\operatorname{Kdim}_{R_e} M_\sigma \leq \alpha$ for every $\sigma \in G$.

Proof. For any submodule N_σ of M_σ we have that $N_\sigma = RN_\sigma \cap M_\sigma$; then apply Lemma II.5.1.

II.5.14. Lemma. Let R be a graded ring of type G. Consider an α-critical $M \in R\text{-gr}$ then $\operatorname{Kdim}_{R^+} M^+ \leq \alpha + 1$ and $\operatorname{Kdim}_{R^-} M^- \leq \alpha + 1$.

Proof. Pick $x \neq 0$ in $h(M^+)$. We intend to show by transfinite induction on α that $\operatorname{Kdim}_{R^+}(M^+/R^+x) \leq \alpha$. If $\alpha = 0$ then M is a simple object R-gr and in this case the assertion follows from the structure of simple objects (proof in section II.6.). If $\alpha \neq 0$ then $Rx \neq 0$ yields $\operatorname{Kdim}_{R\text{-gr}}(M/rx) < \alpha$ and the induction hypothesis, combined with Lemma II.5.3., Lemma II.5.12., implies that $\operatorname{Kdim}_{R^+}(M/Rx)^+ \leq \alpha$.

Since $(M/Rx)^+ = M^+/(Rx)^+$ and $R^+x \subset (Rx)^+$ we find the exact sequence :

$0 \to (Rx)^+/R^+x \to M^+/R^+x \to M^+/(Rx)^+ \to 0$.

If $\deg x = \tau$ then $(Rx)^+/R^+x \cong (\bigoplus_{\gamma \in [\tau^{-1}, e]} R_\gamma x)/R_e x$ as R_e-modules. By Lemma II.5.13., it follows that $\operatorname{Kdim}_{R^+}(Rx)^+/R^+x \leq \alpha$. Then Lemma II.5.6. applied to the above exact sequence yields : $\operatorname{Kdim}_{R^+} M^+ \leq \alpha + 1$. In a similar way, $\operatorname{Kdim}_{R^-} M^- \leq \alpha + 1$ follows.

II.5.15. Corollary. If a graded R-module N is left Noetherian and such that $\operatorname{Kdim}_{R\text{-gr}} M = \alpha$, then : $\operatorname{Kdim}_{R^+} M^+ \leq \alpha + 1$, $\operatorname{Kdim}_{R^-} M^- \leq \alpha + 1$.

Proof. Directly form the lemma, Lemma II.5.12. and Lemma II.5.3.

II.5.16. Theorem. Let R be as before and let M be left Noetherian in R-gr with $\operatorname{Kdim}_{R\text{-gr}} M = \alpha$, then $\alpha \leq \operatorname{Kdim}_R \underline{M} \leq \alpha + 1$.

Proof. Clearly $\alpha \le \mathrm{Kdim}_R \underline{M}$. By Lemma II.5.2. and Proposition II.2.3. we obtain :

$$\mathrm{Kdim}_R \underline{M} \le \sup(\mathrm{Kdim}_{R\text{-gr}} M, \mathrm{Kdim}_{R^-} M^-).$$

Then the above corollary finishes the proof.

II.5.17. Corollary. Let R be as before and let $M \in R\text{-gr}$ be left Noetherian and α-critical. Suppose that $\mathrm{Kdim}_R \underline{M} = \alpha + 1$ then :

1. $\underline{M}$ is an $(\alpha + 1)$-critical R-module.
2. $M^+(M^-)$ is an $(\alpha + 1)$-critical R^+-(R^-)-module.

Proof. Let $\underline{X} \neq 0$ be a submodule of M.

Obviously $\mathrm{Kdim}_{R^+} M^+ = \mathrm{Kdim}_{R^-} M^- = \alpha + 1$.

If $X \cap M_{\le\sigma} = 0$ then $X^\sim \cap M_{\le\sigma} = 0$. Considering that $\mathrm{Kdim}_{R\text{-gr}} M(\sigma) = \alpha$ for every $\sigma \in G$, we may assume that $X \cap M^- \neq 0$.

Then $\mathrm{Kdim}_{R\text{-gr}} M/X^\sim < \alpha$ and $\mathrm{Kdim}_{R^-} M^- \le \alpha$ (see the proof of Lemma II.5.14.).

By proposition II.2.3. and Corollary II.2.4. we obtain $\mathrm{Kdim}_R \underline{M} \le \alpha$; hence $\underline{M}$ is $(\alpha + 1)$-critical because of the foregoing results.

If $M \in R\text{-gr}$ is left Noetherian and α-critical then one expects that $\mathrm{Kdim}_R M = \alpha$ implies that $\underline{M}$ is α-critical.

In the commutative case this is indeed the case :

II.5.18. Proposition. Let R be a commutative graded ring of type Z and let $M \in R\text{-gr}$ be α-critical. Then $\underline{M}$ is an α-critical or an $(\alpha+1)$-critical R-module.

Proof. M is α-critical in R-gr if and only if there exists a graded ideal I of R such that for any $x \in h(M)$, $x \neq 0$, we have $\mathrm{Ann}_R\, x = I$ and $\mathrm{K\text{-}dim}_{R\text{-gr}} R/I = \alpha$. Let $y \in M$, $y \neq 0$, then $y = y_{\sigma_1} + \ldots + y_{\sigma_n}$ with $\sigma_1 < \ldots < \sigma_n$. It is clear that $I \subset \mathrm{Ann}_R\, y$. If $\lambda \in \mathrm{Ann}_R\, y$, say $\lambda = \lambda_{\tau} + \ldots + \lambda_{\tau_m}$ with

$\tau_1 < \ldots < \tau_m$, then $\lambda y = 0$ yields $\lambda_{\tau_1} y_{\sigma_1} = 0$ i.e. $\lambda_{\tau_1} \in I$. Then $\lambda_{\tau_1} y_{\sigma_2} = \ldots = \lambda_{\tau_1} y_{\sigma_n} = 0$ hence $\lambda_{\tau_1} \in \mathrm{Ann}_R\, y$.

Furthermore, we obtain $\lambda_{\tau_2},\ldots, \lambda_{\tau_m} \in I$ and $\mathrm{Ann}_R\, y = I$. Since R/I is a domain, R/I is α-critical or $(\alpha+1)$-critical as an R-module i.e. $\underline{M}$ is α-critical or $(\alpha+1)$-critical in R-mod.

Let us now focus on Krull dimension for strongly graded rings. We use notations of II.3.12. and subsequent results.

II.5.19. Proposition. Let R be a graded ring of type G, where G is arbitrary. If $M \in R\text{-gr}$ has Krull dimension then for each $i \in I$, $M^{(\sigma_i,H)}$ has Krull dimension in $R^{(H)}\text{-gr}$ and $\mathrm{Kdim}_{R^{(H)}\text{-gr}}\, M^{(\sigma_i,H)} \le \mathrm{Kdim}_{R\text{-gr}}\, M$. If H has finite index in G then $\mathrm{Kdim}_{R\text{-gr}}\, M = \sup_{i\in I} \{ \mathrm{Kdim}_{R^{(H)}\text{-gr}}\, M^{(\sigma_i,H)} \}$.

Proof. The first statement follows from Lemma II.3.12. and its corollaries For the second statement, note first that $\sup_{i\in I} \{\mathrm{Kdim}_{R^{(H)}\text{-gr}}\, M^{(\sigma_i,H)}\} \le \mathrm{Kdim}_{R\text{-gr}}\, M$. On the other hand, the lattice of graded submodules of M maps into the product of lattices of graded submodules of $M^{(\sigma_i,H)}$, $i\in I$, as follows : $N \to (N^{(\sigma_i,H)})_{i\in I}$. This map is strictly increasing and we may apply Lemma II.5.1.

II.5.20. Corollary. If the graded ring R has Krull dimension in R-gr then, for every subgroup H of finite index in G, the ring $R^{(H)}$ has Krull dimension and $\mathrm{Kdim}_{R\text{-gr}}\, R = \mathrm{Kdim}_{R^{(H)}\text{-gr}}\, R^{(H)}$

II.5.21. Corollary. If R is graded by a finite group G if M has Krull dimension in R-gr then $\underline{M}$ has Krull dimension in R-mod and :

$$\mathrm{Kdim}_R\, \underline{M} = \mathrm{Kdim}_{R\text{-gr}}\, M = \sup_{\sigma\in G} \{ \mathrm{Kdim}_{R_e}\, M_\sigma \} .$$

Proof. Applying Proposition II.5.19. to $H = \{e\}$ then $\mathrm{Kdim}_{R\text{-gr}} M = \sup_{\sigma\in G} \{ \mathrm{Kdim}_{R_e} M_\sigma \}$. Since R-submodules of M are also R_e-submodules, $\mathrm{Kdim}_R M \leq \mathrm{Kdim}_{R_e} M$.

Finiteness of G entails : $\mathrm{Kdim}_{R_e} M = \sup_{\sigma\in G} \{\mathrm{Kdim}\, M_\sigma\}$.

Hence $\mathrm{Kdim}_R M \leq \mathrm{Kdim}_{R\text{-gr}} M$.

On the other hand it is clear that $\mathrm{Kdim}_{R\text{-gr}} M \leq \mathrm{Kdim}_R M$.

II.5.22. Lemma. Let R be a strongly graded ring of type G and let $M \in R\text{-gr}$ be left Noetherian. Then $\mathrm{Kdim}_{R\text{-gr}} M = \mathrm{Kdim}_{R_e} M_e$

Proof. Immediately from Theorem I.3.4.

II.5.23. Lemma. Let R be a strongly graded ring of type Z. If R_e is left Noetherian and has Krull dimension, α say, then $\mathrm{Kdim}\, R \leq \alpha + 1$.

Proof. Because of Lemma II.5.22. and Theorem II.5.16.

II.5.24. Theorem. Let R be a strongly graded ring of type G where G is a polycyclic-by-finite group with normalising series $\{e\}=G_0 \subset G_1 \subset \ldots \subset G_s = G$. Suppose R_e is left Noetherian and has Krull dimension α then $\alpha \leq \mathrm{Kdim}\, R \leq \alpha + s$.

Proof. If $s = 1$ then G is finite or cyclic. Then by Corollary II.5.21. or Lemma II.5.23. the statement follows, i.e. $\mathrm{Kdim}\, R \leq \alpha + 1$. We proceed by induction, assuming that for s smaller than n the statement holds.

Put $H = G_{n-1}$, $G' = G/H$ and let $\pi : G \to G'$ be the canonical epimorphism. Consider $R_{(G')}$ and, if we let e' be the neutral element of G' then $(R_{(G')})_{e'} = R^{(H)}$. Applying the induction hypothesis to $R^{(H)}$ yields that : $\mathrm{Kdim}\, R^{(H)} \leq \alpha + s - 1$, hence $\mathrm{Kdim}\, R \leq \alpha + s - 1 + s = \alpha + s$, follows.

II.5.26. Corollary. Let A be a left Noetherian ring with $\mathrm{Kdim}\, A = \alpha$. Let G be polycyclic-by-finite, then the group ring A[G] (similar for any crossed product of G over A) has Krull dimension and $\alpha \le \mathrm{Kdim}\, A[G] \le \alpha + s$ where s is as in the theorem. Actually it is easily seen that we may in fact take $s = h(G)$ (the Hirsch number, i.e. the number of cyclic non-finite G_i/G_{i-1} in the normal series).

If A is a ring and φ is an automorphism of A then we consider the skew polynomial rings $A[X,\varphi]$ and $A[X, X^{-1},\varphi]$. If $M \in R$-mod then $M[X,\varphi]$ stands for $R[X,\varphi] \underset{R}{\otimes} M$ in $R[X,\varphi]$ - mod and similarly for $M[X,X^{-1},\varphi]$.
Put $M[X,\varphi]_i = \{X^i \otimes m,\ m \in M\}$ for $i \ge 0$, then $M[X,\varphi] = \underset{i\in \mathbb{N}}{\oplus} M[X,\varphi]_i$ is a graded $R[X,\varphi]$-module. Similarly, $M[X,X^{-1},\varphi]$ is a graded $R[X,X^{-1},\varphi]$-modul

II.5.27. Theorem. If M is left Noetherian in A-mod and if $\mathrm{Kdim}_A M = \alpha$ then $\mathrm{Kdim}_{A[X,X^{-1},\varphi]} M[X, X^{-1},\varphi] \le \alpha + 1$.

Proof. Since $A[X, X^{-1},\varphi]$ is strongly graded of type Z we may apply Theorem II.5.16. and Lemma II.5.22.

II.5.28. Theorem. Let $M \in A$-mod.

1. $M[X,\varphi]$ has Krull dimension if and only if M is left Noetherian and then $\mathrm{Kdim}_{A[X,\varphi]} M[X,\varphi] = \mathrm{Kdim}_R M + 1$.
2. If M is left Noetherian and α-critical then $M[X,\varphi]$ is $(\alpha+1)$-critical.

Proof.

1. Put $S = A[X,\varphi]$, $N = M[X,\varphi]$. If M is left Noetherian then N is left Noetherian in S-mod and 1. follows.
   Conversely, assume that N has Krull dimension in S-mod and that M is not left Noetherian, i.e. there exists an infinite ascending chain $M_o \subset M_1 \subset \dots \subset M_n \subset \dots$
   Put : $K = 1 \otimes M_1 + X \otimes M_2 + \dots + X^k \otimes M_{k+1} + \dots$
   $L = 1 \otimes M_o + X \otimes M_1 + \dots + X^k \otimes M_k + \dots$

Obviously $L \subset K$ are graded submodules of N and $X K \subset L$. Let π_i be the canonical map $M_i \to M_i/M_{i-1}$, $i=1,2\ldots$.

Note that M_i/M_{i-1} for all $i = 1,2,\ldots$, is an S-module by :

$(a_0 + a, X + \ldots)\ \pi_i(m) = \pi_i(\varphi^{1-i}(a_0)m)$, for every $m \in M_i$.

Define $\theta : K/L \to M_1/M_0 \oplus M_2/M_1 \oplus \ldots$, as follows :

if $f = 1 \otimes m, + X \otimes m_2 + \ldots \in K$, put

$$\theta(f \bmod L) = (\pi_1(m_1), \pi_2(m_2), \ldots).$$

It is straightforward to check that θ is an S-isomorphism. Since N has Krull dimension, so does K/L and thus the latter has finite Goldie dimension (see Lemma II.5.5.) but this contradicts the fact that θ is an isomorphism.

2. Assume that M is α-critical. Since N is a graded S-module, we only have to establish that N is $(\alpha+1)$-critical in S-gr and then both the equalities in 1 and 2 will follow from this.

Consider the infinite strictly decreasing sequence :

$$N \supset X N \supset X^2 N \supset \ldots \supset X^i N \supset \ldots$$

where $X^iN/X^{i+1}N \cong N/XN \cong M$.

Obviously $\mathrm{Kdim}_S N \geq \alpha + 1$. If $z \in h(N)$, $z \in h(N)$, $z = X^k \otimes m \neq 0$. has degree k then we consider the following sequence in S-gr :

$(\star)\quad N \supset XN \supset \ldots \supset X^kN \supset Sz$,

where $\mathrm{Kdim}_S\ \alpha^iN/X^{i+1}N = \alpha$ and $\mathrm{Kdim}_S\ X^kN/Sz$ is $\mathrm{Kdim}((M/Rm)[X,\varphi])$. Since M is α-critical it follows that $\mathrm{Kdim}((M/Rm)[X,\varphi] \leq \alpha$. Lemma II.5.3. apllied to the sequence $(\star)$ entails now : $\mathrm{Kdim}(N/Sz) \leq \alpha$ and therefore N is $(\alpha + 1)$-critical

II.6. The structure of Simple Objects in R-gr.

In this section the graded rings considered are exclusively graded of type Z. Recall that an $S \in$ R-gr is said to be gr-simple or simple in R-gr if 0 and S are its only graded submodules.

II.6.1. Lemma. If S is gr-simple then for each $i \in Z$, $S_i = 0$ or S_i is a simple R_0-module.

Proof. If $S_i \neq 0$ for certain $i \in Z$, say $x \neq 0$ in S_i, then $Rx = S$ and thus $R_0 x = S_i$. Consequently S_i is a simple R_0-module.

II.6.2. Remarks.

1. If R is positively graded and $S \in$ R-gr is gr-simple then $S = S_j$ for some $j \in Z$. Indeed, if both S_i and S_j are nonzero, say $x_i \neq 0$ in S_i and $x_j \neq 0$ in S_j, then $x_i = rx_j$ and $x_j = sx_i$ for certain $r, s \in h(R)$. Depending whether $i > j$ or $j > i$, one of these relations is impossible whence $i = j$ follows.

2. If S is left-(or right) limited and gr-simple then $\underline{S}$ is simple in R-mod.

II.6.3. Lemma. Let S be gr-simple. Then $D = \mathrm{END}_R(S)$ is a gr-division ring and if $D_0 \neq D$ then $\underline{S}$ is 1-critical in R-mod.

Proof. Consider a nonzero graded morphism of degree n, $f : S \to S$, then f may be represented by a morphism of degree zero $f : S \to S(n)$. But both S and S(n) are simple in R-gr, thus $\bar{f}$ is an isomorphism in R-gr i.e. invertible. In order to prove the second statement we first establish that $\underline{S/X}$ has finite length, for every nonzero R-submodule $\underline{X}$ of $\underline{S}$. Since $X \neq 0$, $X_\sim \neq 0$ hence $X_\sim = S$. If $\underline{X} \cap \underline{S}^+ = 0$ then $X_\sim \cap S^+ = 0$ entails $S^+ = 0$ but then S is right-limited and Remark II.6.2.2. then yields that $\underline{S}$ is simple.

So we may assume that $\underline{X} \cap \underline{S}^+ \neq 0$. Now let $\underline{X}_1 \supset \underline{X}_2 \supset \ldots \supset \underline{X}$ be a descending chain of submodules of $\underline{S}$ containing $\underline{X}$. We then obtain descending chains of R^+-submodules of $\underline{S}^+$:

$\underline{X}_1 \cap \underline{S}^+ \supset \underline{X}_2 \cap \underline{S}^+ \supset \ldots \supset \underline{X}_n \cap \underline{S}^+ \supset \ldots \supset \underline{X} \cap \underline{S}^+$, and

$(\underline{X}_1 \cap \underline{S}^+)^{\sim} \supset (\underline{X}_2 \cap \underline{S}^+)^{\sim} \supset \ldots \supset (\underline{X}_n \cap S^+)^{\sim} \supset \ldots \supset (\underline{X} \cap S^+)^{\sim}$,

where the operation $(\ )^{\sim}$ is carried out in S^+. We claim that the only graded submodules of S^+ are the $S_{\geq p}$ with $p \geq 0$; the proof of this claim is part 1 of the following Theorem II.6.4..

Write $(\underline{X} \cap \underline{S}^+)^{\sim} = S_{\geq p}$, $(\underline{X}_k \cap \underline{S})^{\sim} = S_{\geq p_k}$ where $p_1 \leq p_2 \leq \ldots \leq p_k \leq \ldots \leq p$. It follows that for some j, $p_j = p_{j+1} = \ldots$, hence $(\underline{X}_j \cap \underline{S}^+)^{\sim} = (\underline{X}_{j+1} \cap \underline{S}^+)^{\sim} = \ldots$ By Proposition II.2.3. and Corollary II.2.4. it follows that $\underline{X}_j = \underline{X}_{j+1} = \ldots$ Therefore $\underline{S}/\underline{X}$ is left Artinian whereas by Theorem II.3.5., $\underline{S}$ is left Noetherian, thus $\underline{S}/\underline{X}$ has finite length. If D has nontrivial gradation, then $D \cong D_0[X, X^{-1}, \varphi]$ by Corollary I.4.3. The endomorphism of $\underline{S}$ represented by $1-X$ will be a non-invertible injective morphism and thus we obtain the strictly descending chain of R-submodules in $\underline{S}$:

$S \supset (1-X)\underline{S} \supset (1-X)^2 \underline{S} \supset \ldots.$

Consequently $\underline{S}$ is not left Artinian, $\mathrm{Kdim}_R \underline{S} = 1$ and the above implies that $\underline{S}$ is 1-critical.

II.6.4. Theorem. Let S be gr-simple, then :

1. Every graded R^+-submodule of S^+ (similarly for $\bar{R}$-submodules of S^-) is of the form $S_{\geq p}$ for some $p \geq 0$.
2. The R^+-submodules of $\underline{S}^+$ and the R^--submodules of $\underline{S}^-$ are principal submodules.
3. Every R-submodule of $\underline{S}$ is principal.
4. $\underline{S}$ is either simple in R-mod or a 1-critical R-module.
5. The intersection of all maximal R-submodules of $\underline{S}$ is zero.

Proof. Let $M = \bigoplus_{i\geq 0} M_i$ be a graded R^+-submodule of S^+. Let p be minimal in $\mathbb{N}$ such that $M_p \neq 0$. By Lemma II.6.1., $M_p = S_p$ and then $R^+S_p = S_{\geq p}$ follows from $RS_p = S$. Therefore, $M \supset R^+M_p = S_{\geq p}$ follows, thus $M = S_{\geq p}$.

2. Easy enough.

3. Consider a nonzero R-submodule $\underline{X} \subset \underline{S}$. If $\underline{X} \cap \underline{S}^+ = 0$ then $(\underline{X} \cap \underline{S}^+)_{\sim}=0$ but also $X_{\sim} \cap S^+ = 0$. Indeed $x \in h(X_{\sim} \cap S^+)$ is nonzero only if there is a nonzero $y \in \underline{X}$ such that $y = y_{i_1}+\ldots+y_{i_m}$ with $\deg y_{i_j} = i_j$ and Since $\deg x \geq 0$ it follows that $y \in \underline{S}^+$ hence $y \in \underline{S}^+$ hence $y \in \underline{X} \cap \underline{S}^+$, contradiction. It is thus clear that $X_{\sim} \neq S$ but this contradicts gr. simplicity of S. Now assume $\underline{X} \cap \underline{S}^+ \neq 0$. Then we obtain $(\underline{X} \cap \underline{S}^+)^{\sim} =$
$= S_{\geq p}$ for some $p \in \mathbb{N}$ (where $\sim$ is defined in S^+). Pick a nonzero $x_p \in S_p$. There exists $y \in \underline{X} \cap \underline{S}^+$ such that $y = y_{i_1} + \ldots + y_{i_m} = x_p$ and $i_1<\ldots<i_m$ This means that $(\underline{X} \cap \underline{S}^+)^{\sim} \supset (\underline{R^+y})^{\sim} \supset (\underline{R^+y})^{\sim} \supset Rx_p = S_{\geq p}$, hence $(\underline{X} \cap \underline{S}^+)^{\sim} = (\underline{R^+y})^{\sim}$ and therefore $\underline{X} \cap \underline{S}^+ = \underline{R^+y}$.
On the other hand we have : $\underline{X} \cap \underline{S}^+ \supset \underline{R^+y}$, hence $\underline{X} \cap \underline{S}^+ = \underline{Ry} \cap \underline{S}^+$. Since $\underline{Ry} \neq 0$, $(\underline{Ry})_{\sim} = S = (\underline{X})_{\sim}$. Now the results of Section II.2. yield that $\underline{X} = \underline{Ry}$.

4. and 5. For the proof of these statements we need some preliminary facts, presented here as sublemma's. Any $M \in R\text{-gr}$ may be viewed as a graded left module over $T = \mathrm{END}_R(M)$ and the graded ring $B^g(M) = \mathrm{END}_T(M)$ is called the graded bi-endomorphism ring M. There is a canonical graded ring morphism $\varphi : R \to B^g(M)$, defined by $\varphi(r)(x) = rx$ for all $x \in M$.

Sublemma 1. (The graded version of the density theorem)
If M is gr-semisimple (i.e. a direct sum of gr-simple objects) and if $x_1,\ldots,x_n \in h(M)$, $\alpha \in h(B^g(M))$, then there is an $r \in h(R)$ such that $\alpha(x_i) = rx_i$ for $i = 1,\ldots,n$.

Sublemma 2. Let L be minimal as a graded left ideal of R. Then either $L^2 = 0$ or $L = Re$ where e is an idempotent in $h(R)$. Moreover, the ring eRe is a gr-division ring.

Conversely, if R is a semiprime ring and e an idempotent in $h(R)$ such that eRe is a gr-division ring, then $L = Re$ is minimal as a graded left ideal of R. If eRe is a field then $\underline{L}$ is minimal as a left ideal.

Proof. Similar to the ungraded equivalents.

Let us now return to the proof of the theorem.

4. Since $\underline{S}$ is finitely generated in R-mod, $D = \mathrm{END}_R S = \mathrm{End}_R S$ is a gr-division ring. If $\underline{D}$ is not a field then Lemma II.6.3. implies that $\underline{S}$ is 1-critical. If $D = D_0$ is a field, let P be the annihilator of $\underline{S}$ in R. It is clear that P is a graded prime ideal and because $\underline{S}$ is an R/P-module we reduce the proof to the case $P = 0$. According to sublemma 1, R is dense in $B^g(S) = T$. But S is simple in T-gr and isomorphic to a graded left ideal of T, L say. Since $L^2 = L$ and $\mathrm{END}_T(L) \cong \mathrm{END}_T(S) \cong D$, we may use the second sublemma to deduce that $\underline{L}$ hence $\underline{S}$ is a simple T-module. Choose $x, y \in S$ and $x \neq 0$; with $x = x_{i_1} + \ldots + x_{i_n}$, $\deg x_{i_j} = i_j$, $i_1 < \ldots i_n$. Simplicity of $\underline{S}$ in T-mod entails $y = \alpha x$ for certain $\alpha \in T$, say $\alpha = \alpha_{j_1} + \ldots + \alpha_{j_m}$, $j_1 < \ldots < j_m$. By the first sublemma we obtain $r_{j_k} \in h(R)$ such that $\alpha_{j_k} x_{i_l} = r_{j_k} x_{i_l}$ for $k = 1,\ldots m$, $l = 1\ldots n$. Thus $\alpha_{j_k} x = r_{j_k} x$ and hence $y = \alpha x = (\sum_{k=1}^{m} r_{j_k})x$, proving that $\underline{S}$ is simple in R-mod.

5. By 4. it will suffice to consider the case where $D = \mathrm{END}_R(S)$ is a gr-division ring with nontrivial gradation. By sublemma 1, a maximal T-submodule of $\underline{S}$ is also a maximal R-submodule. Therefore it is actually sufficient to show that the Jacobson radical $J(\underline{S})$ of $\underline{S}$ over T is zero. But $\underline{S}$ is a projective T-module of finite type, thus :

$END_R(\underline{S}/J(\underline{S})) \cong END_T(S)/J(END_T(S)) \cong D/J(D)$.

For a gr-division ring one easily checks that $J(D) = 0$ (see also Theorem I.7.9.). So if $J(\underline{S})$ were nonzero then $\underline{S}/J(\underline{S})$ would be a T-module of finite length, but then $END_T(\underline{S}/J(\underline{S})) \cong D$ entails that D is a field and therefore trivially graded, a contradiction.

II.6.5. Corollaries.

1. If R is graded of type Z then $J(R) \subset J^g(R)$.
2. If R is positively graded then $J^g(R) = J(R_0) + R_+$.

Proof.

1. See Remark I.7.16. of Theorem II.6.4.,5.
2. If $S \in$ R-gr is gr-simple then $R_+S = 0$ since $R_+S \neq S$, hence $R_+ \subset J^g(R)$ follows. By Remark II.6.2., 1., we have that $S = S_{n_0}$ for some n_0, where S_{n_0} is a simple R_0 module. Thus $J(R_0)S_{n_0} = 0$ or $J(R_0) + R_+ \subset J^g(R)$. Now $J^g(R) = J_0 + R_+$. If S_0 is a simple R_0-module, then we take $S \in$ R-gr to be S_0 equipped with the trivial gradation. Clearly S is gr-simple and $J_0S_0 = J_0S = 0$ follows, hence $J_0 \subset J(R_0)$.

II.6.6. Lemma. If R is a strongly graded ring of type Z and S is gr-simple then S_0 is a simple R_0-module and $S \cong R \otimes_{R_0} S_0$.

Proof. See Theorem I.3.4.

We apply this to modules of polynomials over simple objects :

II.6.7. Theorem. Let R be an arbitrary ring and let S be a simple R-module. Consider $S[X] \in R[X]$-mod and $S[X,X^{-1}]$ over $R[X,X^{-1}]$ the following properties hold :

1. $\underline{S[X,X^{-1}]}$ is a 1-critical $R[X,X^{-1}]$-module.
2. Every $R[X,X^{-1}]$- submodule of $\underline{S[X,X^{-1}]}$ is principal.
3. The maximal $R[X,X^{-1}]$ - submodules of $\underline{S[X,X^{-1}]}$ intersect in 0.

4. $\underline{S[X]}$ is 1-critical in $R[X]$-mod.
5. Every $R[X]$-submodule of $\underline{S[X]}$ is principal.
6. The maximal $R[X]$ - submodules of $\underline{S[X]}$ intersect in 0.

Proof.

Since $R[X,X^{-1}]$ is strongly graded of type Z, $S[X,X^{-1}]$ is gr-simple in $R[X,X^{-1}]$-gr. Thus 1,2.,3. follow from Theorem II.6.4.

4. Since $S[X] = S[X,X^{-1}]^+$ we may apply Corollary II.5.17. to obtain that $\underline{S[X]}$ is 1-critical in $R[X]$-mod.

5. Direct from Theorem II.6.4.,3.

6. Put $D = \mathrm{END}_R(S)$, $B^g(S) = \mathrm{END}_D(S)$. Note that $B^g(S)$ is Von Neumann regular. As a $B^g(S)$-module, S is isomorphic to a minimal left ideal L of $B^g(S)$. The canonical $\rho : R \to B^g(S)$, extends to a morphism $R[X] \to B^g(S)[X]$. Evidently, $S[X]$ is isomorphic to $L[X]$ in $B^g(S)[X]$-mod and the latter is a direct summand in $B^g(S)[X]$. Since $B^g(S)$ is regular, a classical result of Amitsur yields that $J(B^g(S)[X]) = 0$ (actually we may use gr-regularity of $B^g(S)$ and Theorem I.7.9.!)

The proof may now be finished by showing that a maximal $B^g(S)[X]$-submodule $\underline{M}$ of $S[X]$ is also maximal as $R[X]$ - submodules.

Pick $s(X) = s_0 + s_1X + \ldots + s_j X^j \in \underline{S[X]} - \underline{M}$.

Then $\underline{M} + \underline{B^g(S)s(X)} = \underline{S[X]}$. Given $b_0 + b_1 X + \ldots + b_m X^m \in \underline{B^g(S)[X]}$, there exist $a_0,\ldots,a_m \in R$ such that $a_0s_i = b_0s_i,\ldots, a_ms_i = b_ms_i$ for each $0 \le i \le j$. Thus $\underline{B^g(S)\ s(X)} = \underline{R[X]\ s(X)}$ and hence $\underline{M} + \underline{R[X]\ s(X)} = \underline{S[X]}$.

II.6.8. Corollary. Let R an arbitrary ring. If $M \in$ R-mod has finite length n then every $R[X]$ - submodule of $\underline{M[X]}$ and every $R[X,X^{-1}]$ - submodule of $\underline{M[X,X^{-1}]}$ may be generated by less that n elements.

Proof. If n = 1 then $\underline{M}$ is simple and the statement then follows from the foregoing theorem. We now proceed by induction on n, assuming the statement to be true for modules of length at most n-1. Consider a maximal left submodule $\underline{M}_1$ of $\underline{M}$. The exact sequence in R-mod :

$$0 \to \underline{M}_1 \to \underline{M} \to \underline{M/M_1} \to 0,$$

gives rise to an exact sequence in R[X]-mod :

$$0 \to \underline{M_1[X]} \to \underline{M[X]} \to \underline{(M/M_1)[X]} \to 0 .$$

If $\underline{N}$ is an R[X] - submodule of $\underline{M[X]}$ then $\underline{N} \cap \underline{M_1[X]}$ is generated by less than n-1 elements , whereas $\underline{N/N} \cap \underline{M_1[X]}$ is a submodule of $(M/M_1)[X]$ and it may thus be generated by a single element. Consequently $\underline{N}$ may be generated by (less than) n elements.

The second statement follows in formally the same way.

If D is a gr-division ring, let d(D) be the minimal positive integer for which there exist nonzero homogeneous elements in D of that degree. Putting d = d(D) we have :

II.6.9. Proposition. Let R be a graded ring of type Z and let S be a gr-simple, then :

1. $\underline{S}$ is a semisimple R_0-module.
2. If $D = END_R(S)$ then the number of isotopic components of the R_0-module $\underline{S}$ cannot exceed d.

Proof.

1. Easy (see Lemma II.6.1.)
2. If α is nonzero in D_d then $\alpha : S \to S$ is an R-automorphism of degree d. Thus, for any $n \in Z$, the mapping $S_n \to S_{d+n}$ defined by $x \to \alpha(x)$ is an R_0-isomorphism. It is clear that statement 2.is a direct consequence of this.

II.6.10. Example.

Let K be a(trivially graded) field and let $V = \bigoplus_{n\geq 0} V_n$ be a graded K-vector space where $V_n = K(n) \oplus \ldots \oplus K(n)$, summation over (n + 1)-terms.

Put $R = \mathrm{END}_K(V)$ and identify K with a subring of R_0. It is not much of a problem to verify that V is gr-simple in R-gr. But as V is left limited as a graded R-module we obtain that $\underline{V}$ is a simple R-module. If $n \neq n'$ then $\dim_K V_n \neq \dim_K V_{n'}$; a fortiori $V_n \not\cong V_n$, in R -mod. Therefore, V has infinitely may isotopic components in R_0-mod.

II.7. Primary Decomposition.

In this section R will always be a graded ring of type G where G is an ordered group. If $\underline{M}$ is a nonzero R-module then a prime ideal P of R is said to be associated to M if there exists a nonzero submodule $\underline{M}'$ of $\underline{M}$ such that $P = \mathrm{Ann}_R\underline{M}' = \mathrm{Ann}_R\underline{M}''$ for every submodule $\underline{M}''$ of $\underline{M}'$. Write Ass $\underline{M}$ for the set of prime ideals of R associated to $\underline{M}$. If R is a commutative ring then $P \in \mathrm{Spec}(R)$ is associated to $\underline{M}$ if and only if $P = \mathrm{Ann}_R x$ for some $x \neq 0$ in $\underline{M}$.

II.7.1. Lemma. Suppose $M \neq 0$ in R-gr. If R is left gr-Noetherian then Ass $\underline{M} \neq \emptyset$.

Proof. For any graded submodule $N \subset M$, $\mathrm{Ann}_R N$ is a graded left ideal of R. Consider the set $\mathcal{P} = \{\mathrm{Ann}_R N,\ N \neq 0$ a graded submodule of $M\}$. Obviously $\mathcal{P} \neq \Phi$. Let $P = \mathrm{Ann}_R N_0$ be a maximal element of $\mathcal{P}$. If $a R b \subset P$ with $a \in R$, $b \in R-P$ then $RbN_0 \neq 0$ in N_0. It is equally obvious to see that $\mathrm{Ann}_R(N_0) \subset \mathrm{Ann}_R(RbN_0)$ and hence $P = \mathrm{Ann}_R(N_0) = \mathrm{Ann}_R(RbN_0)$. Now $a\, RbN_0 = 0$ implies that $a \in \mathrm{Ann}_R(RbN_0)$ and therefore $a \in P$. Thus $P \in \mathrm{Ass}\underline{M}$.

II.7.2. Remark. Without further assumptions on R and G it is not true that R is left Noetherian if it is left gr-Noetherian. For example $G = Z^{(J)}$ where J is infinite, is an ordered group. Let k be a field. The group ring $R = k[G]$ is a strongly graded ring of type G and since k is a field, hence Noetherian, R is left gr-Noetherian. On the other hand R is not a left Noetherian ring if G does not satisfy the ascending chain condition on subgroups, (see Lemma 2.2. p. 420, [88]).

II.7.3. Theorem. Let $0 \neq M \in$ R-gr and suppose $P \in$ Ass $\underline{M}$, then :

1. P is a graded prime ideal of R.
2. There exists an $x \in h(M)$, $x \neq 0$, such that $P = \mathrm{Ann}_R(Rx)$.

Proof.

1. $P = \mathrm{Ann}_R(M')$ for some R-submodule $0 \neq M' \subset M$.

Take $\lambda \in P$, $x \in M'$, say $\lambda = \lambda_{\sigma_1} + \ldots + \lambda_{\sigma_m}$ with $\sigma_1 < \ldots < \sigma_m$ and $x = x_{\tau_1} + \ldots + x_{\tau_n}$ with $\tau_1 < \ldots < \tau_m$ for $\sigma_1, \ldots \sigma_m, \tau_1, \ldots \tau_n \in G$.

Choose $a \in h(R)$. From $\lambda\, a \in P$ we get that $\lambda\, a\, x = 0$ hence $\lambda_{\sigma_n}\, a\, x_{\tau_n} = 0$.

For any $b \in h(R)$ we have $\lambda\, b\, x = 0$ i.e. $\lambda_{\sigma_m}\, b\, x_{\tau_{n-1}} + \lambda_{\sigma_{m-1}}\, b x_{\tau_n} = 0$ if $\sigma_m\, \gamma\, \tau_{n-1} = \lambda_{\sigma_{m-1}}\, \gamma\tau_n$ with $\gamma = \deg\ b$, or else $\lambda_{\sigma_m}\, b\, x_{\tau_{n-1}} = 0$.

In these equalities we may replace b by $b\, \lambda_{\sigma_m}\, a$: if $\lambda_{\sigma_m}\, b\, x_{\tau_{n-1}} + \lambda_{\sigma_{m-1}}\, b\, x_{\tau_n} = 0$ then $\lambda_{\sigma_m}\, b\, \lambda_{\sigma_m}\, a\, x_{\tau_{n-1}} = 0$, hence $(\lambda_{\sigma_m} R)^2\, x_{\tau_{n-1}} = 0$; on the other hand, the equality $\lambda_{\sigma_m}\, b\, x_{\tau_{n-1}} = 0$ also entails that : $(\lambda_{\sigma_m} R)^2\, x_{\tau_{n-1}} = 0$.

It follows that $(\lambda_{\sigma_m} R)^p\, x_{\tau_i} = 0$ for a certain $p \in \mathbb{N}$ (depending on τ_i), for all τ_i and since the number of components of x is finite we may pick one $p \in \mathbb{N}$ for all x_{τ_i}. Consequently, $(\lambda_{\sigma_m} R)^p Rx = 0$, thus $\lambda_{\sigma_m} R \subset P$.

It now follows that $\lambda - \lambda_{\sigma_m} \in P$ and we may repeat this argumentation and deduce $\lambda_{\sigma_i} \in P$ for $i = 1, \ldots, m$. So, P is graded.

2. Take $0 \neq x \in \underline{M}'$. Then $P = \mathrm{Ann}_R(Rx)$, with $x = x_{\tau_1} + \ldots + x_{\tau_n}, \tau_1 < \ldots < \tau_n$.

Put $J_i = \mathrm{Ann}_R(R\, x_{\tau_i})$ $i = 1, \ldots, x$. One easily verifies that $P = J_1 \cap \ldots \cap J_n$ and thus $P = J_i$ for some i.

Therefore $P = \mathrm{Ann}_R(Rx_i)$.

With notations as introduced earlier we have :

II.7.4. Theorem. Let A be any ring, $M \in A$-mod, then :

$\mathrm{Ass}_{A[X]}\, \underline{M[X]} = \{P[X],\ P \in \mathrm{Ass}\ M\}$, and,

$\mathrm{Ass}_{A[X,X^{-1}]}\, \underline{M[X,X^{-1}]} = \{\ P[X,X^{-1}]\ ,\ P \in \mathrm{Ass}\ M\ \}$.

Proof. If $P \in \mathrm{Ass}\ M$ then $P = \mathrm{Ann}_A M'$ for some nonzero $M' \subset M$. It is clear that $P[X] = \mathrm{Ann}_{A[X]} \underline{M'[X]}$. Let $m'X^\nu$ be a nonzero element of $h(M'[X])$. Since we have that $P = \mathrm{Ann}_A Am'$ it follows that $P[X] = \mathrm{Ann}_{A[X]} (\underline{A[X]m'X^\nu})$, hence $P[X] \in \mathrm{Ass}_{A[X]} (\underline{M[X]}.)$

Conversely, consider $Q \in \mathrm{Ass}_{A[X]}(\underline{M[X]})$.

There is a nonzero homogeneous element mX^ν such that $Q = \mathrm{Ann}_{A[X]} (\underline{A[X]mX^\nu})$ If $P = \mathrm{Ann}_A(\underline{Am})$ then $Q = P[X]$, P is prime and $P \in \mathrm{Ass}_A M$.

The second statement of the theorem follows in a similar way. $\square$

Let $\underline{N} \subset \underline{M}$ in R-mod. We say that $\underline{N}$ is a primary submodule of $\underline{M}$ if $\mathrm{Ass}(\underline{M/N})$ is a singleton, and we say that $\underline{N}$ is a P-primary submodule of $\underline{M}$ if $\mathrm{Ass}(\underline{M/N}) = \{P\}$. Recall from [129], ch. 4, that, if R is a commutative Noetherian ring, and $\underline{M}$ is an R-module of finite type then $\underline{N}$ is in primary in $\underline{M}$ if and only if $\underline{N}$ is a classical primary submodule.

II.7.5. Lemma. Let R be graded of type G, G an O-group, and suppose that R is left gr-Noetherian. Consider $M \in \mathrm{Spec}(R)$. The following statements are equivalent.

1. 0 is a P-primary submodule of M.
2. The set $(0 : P)_M = \{ m \in M,\ Pm = 0 \}$ is an essential submodule of M and P contains any graded ideal which annihilates a nonzero graded submodule of M.

Proof. $1 \Rightarrow 2$. If $X \neq 0$ is a graded submodule of M then by 1 and Lemma II.7.1. : $\mathrm{Ass}(\underline{X}) = \{P\}$. Hence $X \cap (0 : P)_M \neq 0$. If I is a graded ideal of R annihilating a nonzero graded submodule $M' \subset M$ then from $\mathrm{Ass}\ \underline{M} = \{P\}$ it follows that there is an $M'' \subset M'$, $M'' \not\subset 0$, and $P = \mathrm{Ann}_R(M'')$. However $IM'' = 0$ yields $I\ M'' = 0$, hence $I \subset P$.

$2 \Rightarrow 1$. Because R is left gr-Noetherian, we may apply Lemma II.7.1. to find a $Q \in \mathrm{Ass}\ \underline{M}$. By the assumptions $Q \subset P$ follows. Now consider a nonzero submodule $M' \subset M$ such that $Q = \mathrm{Ann}_R(M')$. Since $M' \cap (0:P)_M \neq 0$, $P \subset \mathrm{Ann}_R(M' \cap (0:P)_M)$ follows, thus $P \subset Q$ and hence $P = Q$.

II.7.6. Lemma. Let $M \in R\text{-gr}$ and let $\underline{N} \subset \underline{X} \subset \underline{M}$ be R-submodules. If $\underline{X}/\underline{N}$ is essential in $M/(N)_g$ then $X^{\sim}/(N)_g$, resp. $X_{\sim}(N)_g$ is essential in $N/(N)_g$. In particular, if $\underline{X}$ is a left essential submodule of $\underline{M}$ then $X^{\sim}$ and $X_{\sim}$ are left essential in M.

Proof. Take $x \in h(M)-(N)_g$ and let $\bar{x}$ be its image in $M/(N)_g$. Since $x \notin N$, $\hat{x} = x \bmod N$ is nonzero in $\underline{M}/\underline{N}$. The assumptions assure the existence of a $\lambda \in R$ such that $\lambda\,\hat{x} \in \underline{X}/\underline{N}$, $\lambda\,\hat{x} \neq 0$. Write $\lambda = \lambda_{\sigma_1}+\ldots+\lambda_{\sigma_n}$, $\sigma_1 < \ < \sigma_n$ Then $\lambda x = \lambda_{\sigma_1}x + \ldots + \lambda_{\sigma_n}x$. If $\lambda_{\sigma_n}x \in \underline{N}$ then we may replace λ by $\mu = \lambda-\lambda_{\sigma_n}$ while $\lambda\hat{x} = \mu\hat{x}$, so we may assume that $\lambda_{\sigma_n}x \notin \underline{N}$, $\lambda_{\sigma_n}x \neq 0$. On the other hand, $\lambda\,\hat{x} \in \underline{X}/\underline{N}$ yields that $\lambda\,x \in \underline{X}$ hence $\lambda_{\sigma_n}x \in X^{\sim}$. But $\lambda_{\sigma_n}x \notin (N)_g$, therefore $\lambda_{\sigma_n}\bar{x} \in X^{\sim}/(N)_g$ is nonzero. Lemma I.2.8. finishes the proof for this case. The statement about $X_{\sim}/(N)_g$ may be established in a similar way.

II.7.7. Theorem. Let R be a left Noetherian graded ring of type G. Let $M \in R\text{-gr}$ and suppose that $\underline{N} \subset \underline{M}$ is a P-primary submodule of $\underline{M}$. Then $(N)_g$ is a $(P)_g$-primary submodule of M.

Proof. Define $\underline{X}$ by $\underline{X}/\underline{N} = (0:P)_{\underline{M}/\underline{N}}$; clearly the latter is a left essential R-submodule of $\underline{M}/\underline{N}$ (Lemma II.7.5.). The foregoing lemma entails that $X^{\sim}/(N)_g$ is left essential in $M/(N)_g$. From $P\,\underline{X} \subset \underline{N}$, we deduce that that $(P)_g\,X^{\sim} \subset (N)_g$, hence $X^{\sim}/(N)_g \subset (0:P)_{M/(N)_g}$ and therefore the latter is left essential in $M(N)_g$. Consider a graded ideal I of R which annihilates a nonzero $Y/(N)_g$ in $M/(N)_g$.
Then $IY^{\sim} \subset (N)_g$ and since $Y \neq (N)_g$ we have $(N)_g \neq Y^{\sim}$ i.e. $Y^{\sim}/(N)_g \neq 0$. Therefore we may assume that I annihilates a nonzero graded submodule

$Y^{\sim}(N)_g$ of $M/(N)_g$. As before we then deduce that $IY^{\sim} \subset N$, so we arrive at the conclusion that $I(Y^{\sim} + N/N) = 0$ with $Y^{\sim} + N/N \neq 0$. The hypotheses imply $I \subset P$, thus $I \subset (P)_g$ follows. Now we have reduced the proof to an application of Lemma II.7.5.

II.7.8. Corollary. Let A be any left Noetherian ring, M a nonzero left A-module. Then 0 is a P-primary submodule of M if and only if 0 is $P[X]$ - primary in $M[X]$, if and only if 0 is $P[X,X^{-1}]$ -primary in $M[X,X^{-1}]$.

Proof. A consequence of II.7.7. and II.7.4.

We say that a finite family $(N_i)_{i \in J}$ of primary submodules of M is a primary decomposition for the submodule N of M if $N = \bigcap_{i \in J} N_i$. The decomposition is said to be reduced if the following conditions are met :

RD_1. $\bigcap_{j \neq i} N_j \not\subset N_i$ for all $i \in J$.

RD_2. If $\mathrm{Ass}(M/N_i) = P_i$ then the ideals P_i are two-by-two disjoint.

II.7.9. Lemma. Let R be a graded ring of type G, G an O-group, and let $M \in R$-gr be finitely generated. If R is left gr-Noetherian then each graded submodule N of M admits a reduced primary decomposition.

Proof. Since M/N has finite type, it is a left gr-Noetherian module. There exist graded submodules $N_1,\dots,N_k \subset M$ such that $N = N_1 \cap N_2 \cap \dots \cap N_k$ and such that the M/N_i are gr-uniform, for all $i = 1,\dots,k$. Uniformity of M/N_i in R-gr entails, because of Lemma II.7.5., that $\mathrm{Ass}(M/N_i) = \{P_i\}$, $i = 1,\dots,k$. Since the obtained decomposition is finite it is not hard to verify that we may very well reduce it to a reduced decomposition.

II.7.10. Remark. The lines of prove used in the above lemma follow closely the proof of the corresponding ungraded statement (where R is left Noetherian). More detail about primary decomposition may be found in [129] ch. 4.

II.7.11. Corollary. Let R be a left Noetherian graded ring, and suppose $M \in R\text{-gr}$ is finitely generated. Let N be a graded submodule of M such that $N = \cap \{N_i, i \in J\}$ is a primary decomposition of N in M. Then the following properties also hold :

1. $N = \cap \{(N_i)_g, i \in J\}$ is a primary decomposition of N in M.
2. If the primary decomposition $N = \cap \{N_i, i \in J\}$ is reduced then $N = \cap \{N_i)_g, i \in J\}$ is reduced and for any $i \in J$, $\mathrm{Ass}(M/N_i) =$ $= \mathrm{Ass}(M/(N_i)_g)$.

Proof.

1. Follows from Theorem II.7.7.
2. We have $\mathrm{Ass}(M/N_i) = \mathrm{Ass}(M/(N_i)_g)$ for all $i \in J$.

Supose that $\bigcap_{j\neq i} (N_j)_g \subset (N_i)_g$, then $N = \bigcap_{j\neq i} (N_j)_g$.

There exists an injective homomorphism, $\varphi : M/N \to \bigoplus_{j\neq i} M/(N_j)_g$.

We conclude that :

$$\mathrm{Ass}(M/N) \subset \mathrm{Ass}\left(\bigoplus_{j\neq i} M/(N_j)_g\right) = \bigcup_{j\neq i} \mathrm{Ass}(M/(N_j)_g)$$

$$= \bigcup_{j\neq i} \mathrm{Ass}(M/N_j) ,$$

yielding a contradiction. Therefore the decomposition $N = \cap \{(N_i)_g, i \in J\}$ is reduced.

By a classical P-primary submodule of $M \in R\text{-mod}$ we mean a submodule N of M which is P-primary and such that $P^n M \subset N$ for some $n \in \mathbb{N}$. Theorem II.7.7. implies that for a classical P-primary submodule N of M, $(N)_g$ is a classical $(P)_g$-primary submodule of M.

Corollary II.7.11. still holds if "primary" is replaced by "classical primary". That, for a left Noetherian ring R and an R-module M of finite type, 0 need not have a classical primary decomposition in M, may be justly termed "old hat".

II.8. <u>External Homogenization. Homological Dimension for Graded Rings.</u>

In this section, R is a graded ring of type Z.

The ring $R[T]$ may be graded as follows : $\deg T = 1$, $R[T]_n = \{\sum_{i+j=n} r_i T^j,\ r_i \in R_i\}$. Starting from $M \in R\text{-gr}$ we may construct $M[T] \in R[T]\text{-gr}$. Actually $M[T] = R[T] \otimes_R M$ where the tensor product is graded in the usual way.

If $x \in M$ is of the form $x_{-m} + \ldots + x_o + \ldots + x_n$ then we associate to it, $x^\star \in M[T]$, given by :

$x^\star = x_{-m} T^{m+n} + x_{1-m} T^{m+n-1} + \ldots + x_o T^n + \ldots + x_n$.

We say that $x^\star$ is the (external) <u>homogenized</u> of x. If $u \in h(M[T])$, say $u = u_{-k} T^{k+j} \ldots + u_o T^j + \ldots + u_j$, with $u_i \in M$, $i = k,\ldots,j$.

Then $u_\star = u_{-k} + \ldots + u_o + \ldots + u_j \in M$ is said to be the <u>dehomogenized</u> of u. The following properties are easily verified.

i) If $f \in h(R[T])$, $u \in h(M[T])$ then $(fu)_\star = f_\star u_\star$.

ii) If $u, v \in h(M[T])$ have the same degree then $(u+v)_\star = u_\star + v_\star$

iii) If $x \in M$, $(x^\star)_\star = x$.

iv) If $u \in h(M[T]$, then $(u_\star)^\star T^k = u$, where $k = \deg u - \deg(u_\star)^\star$.

For $\rho \in R$, $m \in M$ let $d(\rho)$ or $d(m)$ be the highest degree appearing in a homogeneous decomposition for ρ or m, then :

v) $\rho^\star m^\star = T^k(\rho m)^\star$, where $k = d(\rho) + d(\rho) + d(m) - d(\rho m)$

vi) If $x, y \in M$ are such that $d(x) > d(y)$, then $T^k(x+y)^\star = x^\star + T^l y^\star$ where $l = d(x) - d(y)$ and $k = d(x) - d(x+y)$.

If $\underline{N}$ is an R-submodule of M then $N^\star$ stands for the graded submodule of $M[T]$ generated by the $n^\star$, $n \in \underline{N}$. We say that $N^\star$ is the <u>homogenized of $\underline{N}$</u>. Note that each $x \in (N^\star)$ is of the form $T^\nu n^\star$ for some $n \in \underline{N}$.

To a graded $R[T]$ - submodule L of $M[T]$ we associate $\underline{L_\star} = \{u_\star,\ u \in h(L)\}$. The properties mentioned above entail that $\underline{L_\star}$ is an R-submodule of $\underline{M}$.

The correspondence $\underline{N} \to N^\star$ satisfies the following :

$1^\star$. $\underline{N} = (\underline{N}^\star)_\star$.

$2^\star$. $\underline{N}^\star \cap M = (N)_g$.

$3^\star$. If $\underline{L} \subsetneq \underline{N}$ then $L^\star \subsetneq N^\star$.

$4^\star$. If $N \subset M$ in R-gr then $\underline{N} = N^\star[T]$.

$5^\star$. If $\underline{L}$ is a left ideal of R then $(\underline{L}\ \underline{N})^\star = (\underline{L})^\star\ (\underline{N})^\star$.

$6^\star$. If $x \in M$ then $(\underline{N} : x)^\star = N^\star : x^\star$.

$7^\star$. $(\bigcap_{i\in I} \underline{N}_i)^\star = \bigcap_{i\in I} (\underline{N}_i)^\star$.

The correspondence $L \to L_\star$ satisfies :

$1_\star$. $(L_\star)^\star \supset L$

$2_\star$. If $L \subset L'$ then $L_\star \subset L'_\star$.

$3_\star$. $(\bigcap_{i\in I} L_i)_\star = \bigcap_{i\in I} (L_i)_\star$

$4_\star$. If L is a graded left ideal of $R[T]$, then $(JL)_\star = J_\star L_\star$.

$5_\star$. If $u \in h(M[T]$ then $[L:u]_\star = L_\star : u_\star$.

$6_\star$. $(\sum_{i\in I} L_i)_\star = \sum_{i\in I} (L_i)_\star$.

Note that in combination with $(\ \ )_\star$ we omit to write the $(\underline{\ \ })$ indicating that we are looking at an ungraded module. From the above properties it is evident that we may define a functor $E : R[T] - gr \to$ R-mod, given by $E(M) = R[T] / (T-1) \underset{R[T]}{\otimes} M \cong M/(T-1)) M$, for $M \in R[T]$ - gr. With the notation we have :

<u>II.8.1. Lemma.</u>

1. E is an exact functor.
2. If $\underline{N} \in$ R-mod (of finite type, of finite presentation), then there is an $M \in R[T]$-gr (of finite type, of finite presentation) such that $E(M) \cong N$.
3. If R has negative grading and if P is a projective R-module, then there exists a projective Q in $R[T]$-gr such that $E(Q) \cong P$.

Proof.

1. Right exactness of E is obvious. Take $M \in R[T]$ - gr and consider $M' \subset M$ in R-gr. Left exactness of E will follow if we can show that $M' \cap (T-1)M = (T-1)M'$.

If $x \in M' \cap (T-1)M$ then $x = (T-1)m \in M'$ with $m \in M$, where $m = m_{-t} + \ldots + m_0 + \ldots + m_s$ where $\deg m_i = 0$, $i = -t, \ldots, 0, \ldots s$ and we may take $m_{-t} \neq 0$, $m_s \neq 0$. Now $(T-1)m \in M'$ yields :

$-m_{-t} + Tm_{-t} - m_{1-t} + Tm_{1-t} + Tm_{1-t} \cdots -m_s + Tm_s \in M'$.

Since M' is graded $m_{-t} \in M'$ follows; then $m_{1-t} \in M'$ follows, and so on till we find that $m \in M'$ i.e. $x \in (T-1)M'$.

2. We have $N = R^{(V)}/L$, for some set V. The $R[T]$ - module $R^{(V)}[T]$ is gr-free. Put $M = R^{(V)}[T] / L^{\star}$. One easily verifies that $M(T-1)M \cong N$, and also M is finitely generated (presented) if N is finitely generat

(presented).

3. If P is projective then $R^{(V)} \cong P \oplus Q$ for some set V. Let $R^{(V)}$ be graded in the obvious way. An R-basis $\{ e_v, v \in V \}$ for $R^{(V)}$ also yields an $R[T]$ - basis for $R^{(V)}[T]$. Write $e_v = x_v + y_v$, $x_v \in P$, y_v (we still write e_v for the isomorphic copy in $P \oplus Q$).

Chosing $e_v = (0, 0, \ldots, 0, 1, 0 \ldots 0)$ we see that $\deg e_v = 0$ for all $v \in V$, thus $d(x_v + y_v) = d(x_v) = 0$ or else $d(x_v + y_v) = d(y_v) = 0$

In both cases $(x_v + y_v)^{\star} = x_v^{\star} + y_v^{\star}$ and hence $e_v = x_v^{\star} + y_v^{\star}$.

Thus $P^{\star} + Q^{\star} = (R^{(V)})^{\star} = R^{(V)}[T]$. Since $P^{\star} \cap Q^{\star} = 0$ it follows tha $R^{(V)}[T] = P^{\star} \oplus Q^{\star}$, therefore $R^{(V)}[T] / Q^{\star}$ is projective in $R[T]$-gr (or in $R[T]$ - mod) , and it is clear that $E(R^{(V)}[T] / Q^{\star}) \cong R^{(V)}/($

II.8.2. Theorem. Let R be a graded ring of type Z, then gr.gl. dim R $\leq$ gl.dim $R \leq 1 +$ gr.gl. dim R.

Proof. The first inequality is just Corollary I.2.7. . If $M \in R[T]$-gr then the following sequence is exact in $R[T]$ - mod :

$$0 \to M[T] \xrightarrow{m_{T-1}} M[T] \xrightarrow{s} M \to 0.$$

where m_{T-1} is multiplication on the right by T-1, and s is given by $s(\sum_i m_i T^i) = \sum_i m_i$. If $\underline{N} \in$ R-mod then let $M \in R[T]$-gr be such that $\underline{N} \cong \underline{E(M)}$. Exactness of E entails that $p.\dim_R N \le gr.p.\dim_{R[T]} M \le$ $\le gr.gl\dim R[T]$, hence $gl.\dim R \le 1 + gr.gl.\dim R$.

II.8.3. Lemma. If R is strongly graded of type Z, then : $gl.\dim R_0 \le$ $\le gl.\dim R \le 1 + gl.\dim R_0$.

Proof. By Theorem I.3.4. , $gr.gl.\dim R = gl\dim R_0$.

II.8.4. Corollary. If R is strongly graded of type Z^n then $gl\dim R \le$ $\le n + gl\dim R_e$.

Proof. If n = 1, apply Lemma II.8.3.. We proceed by introduction on n. Consider $H \subset Z^n$ such that $H \cong Z^{n-1}$, $Z^n/H \cong Z$. If $\pi : Z^n \to Z^n/H$ is the canonical epimorphism, consider $R_{(Z^n/H)}$ where $(R_{(Z^n/H)})_{e'} \cong Z$. Since $Z^n/H \cong Z$ we may apply II.8.3. to deduce that :

$gl.\dim R_{(Z^n/H)} \le 1 + gl.\dim R^{(H)}$.

The induction hypothesis implies that : $gl.\dim R^{(H)} \le n - 1 + gl.\dim R_e$. Combining both results yields $gl.\dim R \le n + gl.\dim R_e$.

II.8.5. Lemma. Let R be a left Noetherian positively graded ring and assume $gl.\dim R < \infty$. Let $\underline{P} \in$ R-mod be projective and of finite type, then there exist projective R_0-modules of finite type, Q and Q' say, such that : $P \oplus (R \otimes_{R_0} Q) \cong R \otimes_{R_0} Q'$.

Proof. Let $M \in R[T]$ - gr be such that $E(M) \cong P$, and choose M of finite type. Since $gl.\dim R < \infty$ we obtain the following exact sequence in $R[T]$

$(\star) : 0 \to Q_n \to \dots \to Q_1 \to Q_0 \to M \to 0$ where $Q_0, Q_1, \dots, Q_n$ are projective and of finite type in $R[T]$ - gr. The graded version of Nakayama's lemma may be used to deduce that each projective module of finite type in $R[T]$ - gr is of the form $R[T] \otimes_{R_0} S$ for some projective R_0-module of finite type S. Putting, $Q_i = R[T] \otimes_{R_0} S_i$ in $(\star)$ and applying E to $(\star)$ yields in exact sequence in R-mod,

$$E(\star) : 0 \to E(Q_n) \xrightarrow{d_n} E(Q_{n-1}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} E(Q_1) \xrightarrow{d_1} E(Q_0) \xrightarrow{d_0} P \to 0$$

where $E(Q_i) \cong Q_i/(T-1)\, Q_i \simeq R \otimes_{R_0} S_i$. Projectivity of P entails :

$P \oplus \operatorname{Ker} d_0 \cong R \otimes_{R_0} S_0$, $\operatorname{Ker} d_0 \oplus \operatorname{Ker} d_1 \cong R \otimes_{R_0} S_1, \dots$

$R \otimes_{R_0} S_{n-1} \cong \operatorname{Ker} d_{n-1} \oplus R \otimes_{R_0} S_n$. So the statement follows.

II.8.6. Theorem. Let R be a left Noetherian ring with gl dim $R < \infty$. Let P be a projective left $R[T_1,\dots,T_n]$ -module of finite type. There exist projective R-modules of finite type Q, Q' such that

$Q[T_1,\dots,T_n] \oplus P \cong Q'[T_1,\dots,T_n]$.

Proof. Apply Lemma II.8.5. to the ring $R[T_1,\dots,T_n]$.

Let us now point out some further applications of these homogenization techniques :

Homogenization of Presheaves and Sheaves.

The results included in this section allow to relate certain sheaves of (graded) rings over Spec and Proj. In this book we do not want to go into the "geometrical" aspects of Proj; the interested reader may consult [125] for some projective "non-commutative algebraic geometry". The general theory of sheaves and presheaves is well documented in [35], [123].

Let X be a fixed topological space and let Open (X) be the category of open subsets of X with morphisms being given by the "containing" relations Let R be a graded ring over X i.e. a sheaf of graded rings over X such that the sheaf restriction morphisms $\rho_V^U : R(U) \to R(V)$, is graded of degree zero for each pair $V \subset U$ in Open (X). We introduce the following notations : $\pi(R)$, respectively $\sigma(R)$ is the category of presheaves, respectively sheaves, of modules over the sheaf R ; $g\pi(R)$, respectively $g\sigma(R)$, will be the category of presheaves, respectively sheaves, of graded modules over the sheaf R, and in this case the morphisms in the category are presheaf, respectively sheaf, morphisms which are locally of degree zero. The functor "forgetting gradations" is denoted by : $\underline{\quad} : g\pi(R) \to \pi(R)$.

If R is a graded ring then the ring of polynomials, $R[T]$ say, is defined as follows : $R[T](U) = R(U)[T]$ for every $U \in \text{Open}(X)$, where the gradation on $R(U)[T]$ is as in section A.II.2., and the restriction morphism $(\rho^\star)_V^U : R(U)[T] \to R(V)[T]$ is given by :
$$(\rho^\star)_V^U(T^{m+n+p}x_{-m}+\ldots+T^{n+p}x_0+\ldots+T^p x_n) = T^{m+n+p}\rho_V^U(x_{-m})+\ldots+T^p\rho_V^U(x_n)$$
where x_i is homogeneous in $R(U)$ of degree i. It is not hard to verify that $R[T]$ is a sheaf, i.e. $R[T]$ is a graded ring. For any Module $M \in g\pi(R)$ the Module of polynomials $M[T]$ may be defined similarly and we have that $M[T] \in \pi(R[T])$. Now let $N \in \pi(R)$ be a subpresheaf of M. For all $U \in \text{Open}(X)$ put $N^\star(U) = N(U)^\star \subset M(U)[T]$ where $N(U)^\star$ is the homogenization of $N(U)$, as discussed in A.II.2. If $V \subset U$ in Open (X) then $(N^\star)_V^U$ is given by :
$$(N^\star)_V^U(T^{m+n+p}x_{-m}+\ldots+T^{n+p}x_0+\ldots+T^p x_n)$$
$$= T^{m+n+p}M_V^U(x_{-m})+\ldots+T^{n+p}M_V^U(x_0)+\ldots+T^p M_V^U(x_n)$$
where M_V^U is the restriction morphism for M corresponding to $V \subset U$, and

x_i is homogeneous of degree i in $N(U)$. It is clear that $N^\star$ is a graded subpresheaf of $M[T]$.

II.8.7. Proposition. With notations as above, if $N \in \sigma(R)$ then $N^\star \in g\sigma\ (R[T])$.

Proof.

1. Let $U \in$ Open (X) and $\{U_i, i \in I\}$ an open covering of U. If $n \in N(U)$ is such that $(N^\star)^U_{U_i}(n^\star) = 0$ for all $i \in I$ then $o = (M^\star)^U_{U_i}(n^\star) = (M^U_{U_i}(n))^\star = (N^U_{U_i}(n))^\star$. Since $(x^\star)_\star = x$ holds for any $x \in N(U)$, it follows that $N^U_{U_i}(n) = 0$ for all $i \in I$. Since N is a separated presheaf $n = 0$ and $n^\star = 0$ follows.

2. Consider $n_i' \in h(N^\star(U_i))$ such that $(N^\star)^{U_i}_{U_i \cap U_j}(n_i') = (N^\star)^{U_i}_{U_i \cap U_j}(n_j')$

Each $n_i' = T^{\nu_i}\, n_i^\star$ for some $n_i \in N(U_i)$.

We obtain :

$$(\star)\quad T^{\nu_i}\,(N^{U_i}_{U_i \cap U_j}(n_i))^\star = (N^\star)^{U_i}_{U_i \cap U_j}(T^{\nu_i}\, n_i^\star) = (N^\star)^{U_j}_{U_i \cap U_j}\,(T^{\nu_j}\, n_j^\star)$$
$$= T^{\nu_j}\,(N^{U_j}_{U_i \cap U_j}(n_j))^\star$$

Dehomogenization yields $N^{U_i}_{U_i \cap U_j}(n_i) = N^{U_j}_{U_i \cap U_j}(n_j)$ and since N is a sheaf it is possible to find an $n \in N(U)$ such that $N^U_{U_i}(n) = n_i$ for all $i \in I$. The degree of $(N^{U_i}_{U_i \cap U_j}(n_i))^\star$ equals the degree of the homogeneous component of highest degree in $N^{U_i}_{U_i \cap U_j}(n_i)$. The above equality entails that $\deg\,(N^{U_i}_{U_i \cap U_j}(n_i))^\star = \deg\,(N^{U_i}_{U_i \cap U_j}(n_j))^\star$ and consequently $(\star)$ entails $\nu = \nu_i = \nu_j$ for all $i, j \in I$.

Setting $n' = T^\nu\, n^\star$ provides an element of $N^\star(U)$ mapping to the given n_i' under the corresponding $(N^\star)^U_{U_i}$.

If the $n_i' \in N^{\star}(U_i)$ are not necessarily homogeneous the proof may be completed by checking the homogeneous decomposition of the n_i''s.

If L is a graded subpresheaf of $M[T]$ then $L_{\star}(U) = L(U)_{\star}$ for each $U \in \mathrm{Open}(X)$, and $(L_{\star})_V^U(x)$ with $x \in L_{\star}(U)$ equal to $(L_V^U(y))_{\star}$ for some $y \in h(L(U))$ such that $y_{\star} = x$, defines the dehomogenized presheaf $L_{\star}$.

II.8.8. Proposition.

With notations as above, if X is a compact topological space and $L \in g\sigma(R[T])$ then $L_{\star} \in \sigma(R)$.

Proof. Straightforward. Compactness of X is not needed to prove that $L_{\star}$ is a separated presheaf; in checking the sheaf axiom one has to reduce the problem to a finite covering however. Compactness cannot be dropped; for a routine example cf. [63].

If $i : \sigma(R) \to \pi(R)$ is the canonical inclusion then there exists a sheafification functor $\underline{\underline{a}} : \pi(R) \to \sigma(R)$ which is the left adjoint of i. This functor is the reflector with respect to the Giraud subcategory $\sigma(R)$ of the Grothendieck category $\pi(R)$; actually in [123] it is made clear how $\underline{\underline{a}}$ may be viewed as a localization functor. Let $\underline{\underline{a}}'$ be the sheafification functor $\pi(R[T]) \to \sigma(R[T])$.

II.8.9. Lemma. If $M \in g\,\pi(R[T])$ then $\underline{\underline{a}}'(M) \in g\,\sigma(R[T])$.

Proof. Easy ($\underline{\underline{a}}'$ is given by certain direct and inverse consecutive limits).

II.8.10. Theorem.

1. Let $N \in \pi(R)$ be a subpresheaf of an $M \in g\,\pi(R)$ then $\underline{\underline{a}}'(N^{\star}) = (\underline{\underline{a}}(N))^{\star}$.
2. Let X be a compact topological space and let $N \in g\,\pi(R[T]$ be a graded subpresheaf of $M[T]$ with $M \in \pi(R)$, then $(\underline{\underline{a}}'(N))_{\star} = \underline{\underline{a}}(N_{\star})$.

Proof.

1. Because of the foregoing, both $(\underline{\underline{a}}(N))^\star$ and $\underline{\underline{a}}'(N^\star)$ are in $g\,\sigma(R\,[T])$ and therefore it will be sufficient to establish isomorphisms between the stalks, and to this end it will be sufficient to establish that $(N^\star)_x = (N_x)^\star$.

Given $a \in h((N^\star)_x)$, there is a neighbourhood of x, U say, and an element $y \in h(M^\star(U))$ representing a, say $y = y_{-m}\,T^{m+n+p} + \ldots + y_x\,T^p$ with $y_i \in M(U)$ degree i. Define $f : (N^\star)_x \to (N_x)^\star$ by sending a to $M_x^U(y_{-m})\,T^{m+n+p} + \ldots + M_x^U(y_n)\,T^p$. One easily checks that f does not depend on the choice of U and y.

If $f(a) = 0$ then $M_x^U(y_i) = 0$ for all i ; hence there -s a neighbourhood $W \subset U$ of x such that $M_W^U(y_i) = 0$ for all i. Replacing U by W in the definition of f, gives a = 0 and thus f is injective.

If on the other hand $y \in h(N_x)^\star)$ then y is of the form $y = T^\nu(y')^\star$ for some $y' \in N_x$, say $y' = y'_{-m} + \ldots + y'_n$ where y'_i is of degree i. For each i there is a neighbourhood U_i of x and an $x_i \in h(M(U_i))$ representing y_i. Put $U = \cap_i U_i$ and consider $b' = M_U^{U-m}(y_{-m}) + \ldots + M_U^{U_n}(y_n)$ then $b = T^p(b')^\star \in h(N^\star(U))$ yields a preimage for y.

2. Again it will suffice to prove the isomorphism $(N_\star)_x \cong (N_x)_\star$ for each $x \in X$. For an $a \in (N_\star)_x$ there is a neighbourhood U of x and a $y \in N_\star(U)$ representing a. Pick $z \in h(N(U))$ such that $z_\star = y$ and define $f : (N_\star)_x \to (N_x)_\star$ by $f(a) = (N_x^U(z))_\star$. It is easily verified that this definition is independent of the choices made.

If $f(a) = 0$ then $N_x^U(z) = 0$ and it follows that z, hence y, represents zero, i.e. a = 0 and f is injective.

On the other hand, if $y' \in (N_x)_\star$ then $y' = z'_\star$ for some $z' \in h(N_x)$. Pick a $v \in h(N(U))$ representing z and put $a = (N_\star)_x^U(v_\star)$; then $f(a) = y$, hence f is surjective.

The above theory can be modified so as to obtain a similar theory in the general case of Giraud subcategories of R-mod and corresponding subcategories of R-gr. The obvious application of the above techniques will be the cases where X is Spec R or Proj R[T]; actually if R is positively graded then Proj(R[T] $\simeq$ Proj (R) U Spec(R), where Spec(R) is then homomorphic to the open subset of Proj R described by the ideal generated by T.

II.9. Torsion Theories over Graded Rings.

Let us recall some definitions and basic elements of general torsion theory.

Consider a Grothendieck category b (more general one may consider abelian categories). A torsion theory in b is a couple (T,F) of nonvoid classes of objects of b such that the following properties hold.

TT 1. : $T \cap F = \{0\}$, 0 denites the initial object for b.

TT 2. : T is closed under homomorphic images in b.

TT 3. : F is closed under taking subobjects in b.

TT 4. : For an $M \in b$ there is a subobject $t(M)$ of M such that $t(M) \in T$, $M/t(M) \in F$.

A torsion theory is said to be hereditary if T is closed under taking subobjects as well. We will only use hereditary torsion theories here.

A kernel functor on b is a left exact subfunctor of the identity in b.

A kernel functor κ is said to be idempotent if and only if $\kappa(M/\kappa(M))=0$ for $M \in b$.

Elementary results in torsion theory yield that there is a one-to-one correspondence between idempotent kernel functors κ on b and hereditary torsion theories in b . These also correspond bijectively to idempotent Gabriel- filters (or, -topologies) in some chosen generator G for b.

Such a Gabriel-filter in G will then consist of the subobjects L of G such that $G/L \in T$ i.e. $\kappa(G/L) = G/L$ for the corresponding kernel functor. Indeed, the torsion class of a torsion theory associated to an idempotent kernel functor κ is given as the class of $M \in b$ such that $\kappa(M) = M$, while $M \in F$ if and only if $\kappa(M) = 0$. For more detail about torsion theory cf. [32] , [36] , [40] .

Notation . In the sequel , (T,F) will always be a hereditary torsion theory in b and κ will be the corresponding idempotent kernel functor on b .

Evidently we wish to apply these techniques to b = R-gr in case R is a graded ring of type G. Since R-gr is indeed a Grothendieck category this presents no fundamental problems. However R is not a generator for R-gr so the theory developped for R-mod does not completely generalise to the graded case. Moreover we will encounter (solvable) problems in relating torsion theories in R-gr and torsion theories in R-mod.

II.9.1. Lemma. Let R be a graded ring of type G, and let (T,F) be a torsion theory in R-gr.

The following statements are equivalent :

1. If $M \in T$ then $M(\sigma) \in T$ for every $\sigma \in G$.
2. If $N \in F$ then $N(\tau) \in F$ for every $\tau \in G$.

Proof. Straightforward.

II.9.2. Example. Let R be a graded ring of type Z which is positively graded. Consider for T the class of $M \in$ R-gr such that $M_i = 0$ if $i < 0$ and take F to be the class of $M \in$ R-gr such that $M_i = 0$ if $i \geq 0$. It is easily verified that (T,F) is a hereditary torsion theory which does not have the properties mentioned in Lemma II.9.1.. On the other hand it is easy to generate torsion theories wich do satisfy these properties by adding to T all objects obtained as σ suspensions of the original objects with respect to all $\sigma \in$ G.

A torsion theory satisfying the equivalent conditions of Lemma II.9.1. is said to be a rigid torsion theory. A kernel functor is said to be rigid if the corresponding torsion theory is. For any torsion theory (T,F) in R-gr we have that $\mathrm{Hom}_{R\text{-gr}}(M,N) = 0$ for every $M \in T$, $N \in F$. However it should be noted that $\mathrm{HOM}_R(M,N) = 0$ for all $M \in T$, $N \in F$ is exactly equivalent to (T,F) being a rigid torsion theory.

II.9.3. Lemma. A kernel functor κ on R-gr is rigid if and only if $\kappa(M(\sigma)) = \kappa(M)(\sigma)$ for all $M \in$ R-gr, $\sigma \in G$.

Proof. If κ is rigid then $\kappa(M)(\sigma) \subset (M(\sigma))$ is obvious. On the other hand $(M/\kappa(M))(\sigma) = M(\sigma)/\kappa(M)(\sigma)$ is κ-torsion free i.e. in F. Therefore $\kappa(M(\sigma)/\kappa(M)(\sigma)) = 0$. Conversely, if $M \in T$ then $\kappa(M) = M$ i.e. $\kappa(M(\sigma)) = M(\sigma)$ follows. Thus $M(\sigma) \in T$ for all $\sigma \in G$.

A non-empty set of graded left ideals of R, say $\mathcal{L}$, is said to be a graded filter if the following properties hold :

F.1. If $L \in \mathcal{L}$ and L_1 is a graded left ideal of R such that $L \in L_1$ then $L_1 \in \mathcal{L}$.

F.2. If $L_1 L_2 \in \mathcal{L}$ then $L_1 \cap L_2 \in \mathcal{L}$.

F.3. If $L \in \mathcal{L}$ then $[L:x] \in \mathcal{L}$ for all $x \in \mathcal{L}(R)$.

F.4. If $L_1 \in \mathcal{L}$ and $[L:x] \in \mathcal{L}$ for all $x \in h(L_1)$ then $L \in \mathcal{L}$.

Since R is not a generator for R-gr we cannot expect that graded filters in R and (hereditary) torsion theories in R-gr correspond bijectively. However we do have the following :

II.9.4. Lemma. There is a bijective correspondence between the following sets (!) :

1. (Hereditary) rigid torsion theories in R-gr.
2. Rigid (idempotent) kernel functors on R-gr.
3. Graded filters in R.

Proof. A trivial modification of the proof of the similar statement in the ungraded cases.

The sequel of this section is concerned with the characterization of kernel functors on R-gr which can be induced by kernel functors on R-mod. An idempotent kernel functor $\underline{\kappa}$ on R-mod is said to be graded if the filter $\mathcal{L}(\underline{\kappa})$ of $\underline{\kappa}$ possesses a cofinal set of graded left ideals.

II.9.5. Lemma. Let R a graded ring of type G and let $\underline{\kappa}$ be a graded kernel functor on R-mod. If $M \in$ R-gr then $\underline{\kappa}(\underline{M})$ and $\underline{M}/\underline{\kappa}(\underline{M})$ are graded left R-modules and the canonical morphism $\underline{M} \to \underline{M}/\underline{\kappa}(\underline{M})$ may be considered as a graded morphism of degree zero. Furthermore, $\underline{\kappa}$ induces a kernel functor κ on R-gr wich is rigid and idempotent. The torsion class $\mathcal{T}(\kappa)$ consists of $N \in$ R-gr such that $\underline{\kappa}(\underline{N}) = \underline{N}$ whereas $\mathcal{F}(\kappa)$ consists of $N \in$ R-gr such that $\underline{\kappa}(\underline{N}) = 0$.
If $\mathcal{L}(\kappa)$ is the graded filter of κ in R then we have : $\mathcal{L}(\kappa) = L_g(R) \cap \mathcal{L}(\underline{\kappa})$, where $L_g(R)$ denotes the set of graded left ideals of R.

Proof. If we establish that $\underline{\kappa}(\underline{M})$ is graded then it will follow immediately that $\underline{M}/\underline{\kappa}(\underline{M})$ is graded and that $\underline{M} \to \underline{M}/\underline{\kappa}(\underline{M})$ may be viewed as a graded morphism of degree zero. Since $\underline{\kappa}$ is graded, $x \in \underline{\kappa}(\underline{M})$ entails that $Lx = 0$ for some $L \in \mathcal{L}(\underline{\kappa}) \cap L_g(R)$. Write $x = \sum_{\sigma\in G} x_\sigma$, $L = \sum_{\sigma\in G} L_\sigma$. Then $L_\sigma x = 0$ for each $\sigma \in G$ yields $L_\sigma x_\tau = 0$ for all $\sigma,\tau \in G$. Thus $L x_\tau = 0$ for all $\tau \in G$ and hence $x_\tau \in \underline{\kappa}(\underline{M})$ for all $\tau \in G$, i.e. $\underline{\kappa}(\underline{M})$ is a graded submodule of M, which we will therefore denote by $\underline{\kappa}(M)$. Furthermore, if $\mathcal{T}(\kappa)$ and $\mathcal{F}(\kappa)$ are as stated then it is straightforward to verify T.T.1. till T.T.4. and so $(\mathcal{T}(\kappa), \mathcal{F}(\kappa))$ defines a torsion theory in R-gr which is easily seen to be rigid. The kernel functor κ corresponding to this torsion theory is a rigid kernel functor on R-gr, such that $\kappa(M) = \underline{\kappa}(M)$ holds for every $M \in$ R-gr. The statement $\mathcal{L}(\kappa) = L_g(R) \cap \mathcal{L}(\underline{\kappa})$ follows easily from this.

II.9.6. Theorem. Let R be a graded ring of type G. There is a bijective correspondence, $\underline{\kappa} \leftrightarrow \kappa$, between :

1° graded kernel functors $\underline{\kappa}$ on R-mod.

2° rigid kernel functors κ on R-gr.

Proof. Consider a rigid kernel functor κ on R-gr and let $\mathcal{L}(\kappa)$ be its graded filter in R. We define $\mathcal{L}_1 = \{ L$ a left ideal of R, $L_g \in \mathcal{L}(\kappa) \}$, and it is straightforward to check that $\mathcal{L}_1$ is a filter. Let $\underline{\kappa}_1$ be the kernel functor on R-mod associated to $\mathcal{L}_1$. By the definition of $\mathcal{L}_1$ it is clear that $\mathcal{L}_1$ it is clear that $\mathcal{L}_1 \cap L_g(R) = \mathcal{L}(\kappa)$. We claim that $\underline{\kappa}_1(\underline{M}) = \underline{\kappa(M)}$. Indeed, for a rigid kernel functor, like κ, $x \in \kappa(M)$ is equivalent to $\mathrm{ann}_R x \in \mathcal{L}(\kappa)$, thus $\underline{\kappa(M)} \subset \underline{\kappa}_1(\underline{M})$. Conversely, if $x \in \underline{\kappa}_1(\underline{M})$ then $\mathrm{ann}_R\, x \in \mathcal{L}_1$ i.e. $(\mathrm{ann}_R x)_g \in \mathcal{L}(\kappa)$. Writing, $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$ for the homogeneous decomposition of x, we have a $x_{\sigma_i} = 0$ for all $i = 1,\ldots,n$, for every homogeneous $a \in h(\mathrm{ann}_R x)$. Hence $x_{\sigma_i} \in \kappa(M)$ and thus $x \in \underline{\kappa(M)}$ Since $\mathcal{L}_1$ obviously has a cofinal set of graded left ideals, it is obvious from Lemma II.9.5. that we may write $\underline{\kappa}_1 = \underline{\kappa}$, it induces κ on R-gr, and so the correspondence $\underline{\kappa} \leftrightarrow \kappa$ is indeed bijective.

As in any Grothendieck category, we define, the torsion theory cogenerated in R-gr by an $M \in$ R-gr by putting $T(\kappa_M)$ equal to the class of $N \in$ R-gr such that $\mathrm{Hom}_{R\text{-gr}}(N, E^g(M)) = 0$, where $E^g(M)$ is the injective hull of M in R-gr. If κ_M is rigid then $N \in T(\kappa_M)$ entails $\mathrm{HOM}_R(N,E^g(M))=0$ or, $\mathrm{Hom}_{R\text{-gr}}(N, \bigoplus_{\sigma\in G} E^g(M)(\sigma)) = 0$.

Hence the rigid kernel functors amongst the κ_M are associated to the torsion theories cogenerated by a suspension-invariant $M \in$ R-gr. Let $\kappa_M(\sigma)$ denote the kernel functor associated to the torsion theory on R-gr cogenerated by $M(\sigma)$, $\sigma \in G$. Recall that $\kappa = \inf\{\kappa_i, i \in I\}$ is by definition the torsion theory obtained by taking for $T(\kappa)$ the intersection of the $T(\kappa_i)$, $i \in I$. We will write $\kappa = \wedge\{\kappa_i, i \in I\}$ in this case. With this notation it is then clear that $\wedge\{\kappa_M(\sigma), \sigma \in G\}$ is a rigid kernel functor which corresponds to the torsion theory cogenerated by $E^g(\bigoplus_{\sigma\in G} M(\sigma))$.

We define $\kappa_1 \leq \kappa_2$ via the lattice of hereditary torsion theories in R-gr as follows : $\kappa_1 \leq \kappa_2$ if and only if $T(\kappa_1) \subset T(\kappa_2)$. It follows that a rigid torsion theory κ satisfies $\kappa \leq \kappa_M$, $M \in$ R-gr, if and only if $\kappa \leq \wedge \{\kappa_M(\sigma), \sigma \in G\}$. The kernel functor $\wedge \{\kappa_M(\sigma), \sigma \in G\}$ will be called the <u>rigid kernel functor associated (or cogenerated by) to $M \in$ R-gr; it will be denoted by κ^r_M</u>.

<u>II.9.7. Theorem.</u> Let R be a graded ring of type G and consider $M \in$ R-gr. Let $\kappa_{\underline{M}}$ be the kernel functor on R-mod cogenerated by $\underline{M} \in$ R-mod, then

1. $\mathcal{L}(\kappa_{\underline{M}}) = \{J$, left ideal of R such that that for all $0 \neq x \in M$ and $a \in R$, $(J:a) \not\subset \mathrm{ann}_R\, x$. $\}$
2. $\mathcal{L}\ (\kappa^r_M) = \{\ L \in L_g(R)$ such that for all $0 \neq x \in h(M)$ and $a \in h(R)$, $(L:a) \not\subset \mathrm{ann}_R x\}$.
3. $\mathcal{L}(\kappa_{\underline{M}}) \cap L_g(R) = \mathcal{L}(\kappa^r_M)$.
4. If $X \in$ R-gr then $\underline{\kappa^r_M(X)} = \kappa_{\underline{M}}(\underline{X})$.

<u>Proof.</u> By definition of κ^r_M we have :

$\mathcal{L}(\kappa^r_M) = \{\ L \in L_g(R),\ \mathrm{Hom}_R(R/L,\ E^g(M)) = 0\ \}$

Let $L \in \mathcal{L}(\kappa^r_M)$ and suppose there exists $x \neq 0$ in $h(M)$ together with an $a \in h(R)$ such that $(L:a) \subset \mathrm{ann}_R x$. Since $(L:a) \in \mathcal{L}(\kappa^r_M)$ it then follows that $\mathrm{ann}_R x \in \mathcal{L}(\kappa^r_M)$. The latter entails the existence of a nonzero f in $\mathrm{HOM}_R(R/\mathrm{ann}_R x,\ E^g(M))$ and this contradicts $R/\mathrm{ann}_R x \in T(\kappa^r_M)$. Conversely, if $\mathrm{HOM}_R(R/L,\ E^g(M)) \neq 0$ for some $L \in L_g(R)$ having the property that for all $x \neq 0$ in $h(M)$ and all $a \in h(R)$, $(L:a) \not\subset \mathrm{ann}_R x$, then there exists a graded morphism $f : R/L \to E^g(M)$ which is not the zero map. Since M id gr-essential in $E^g(M)$ there exists an $\bar{a} \in R/L$, represented by $a \in h(R)$ such that $x = f(\bar{a}) \neq 0$ is in $h(M)$. Clearly $(L:a) \subset \mathrm{ann}_R x$, a contradiction. This proves 2 and that $\mathcal{L}(\kappa_{\underline{M}})$ is as stated in 1. is well known. Further, the inclusion $\mathcal{L}(\kappa_{\underline{M}}) \cap L_g(R) \subset \mathcal{L}(\kappa^r_M)$ is obvious.

For the converse inclusion consider $L \in \mathcal{L}(\kappa^r_M)$, $x \neq 0$ in M and $a \in R$. Decompose x as $x_{\sigma_1} + \ldots + x_{\sigma_n}$, $\sigma_i \in G$, $i = 1,\ldots,n$, and $a = a_{\tau_1} + \ldots + a_{\tau_m}$ $\tau_j \in G$, $j = 1,\ldots,m$. If $x_{\sigma_n} \neq 0$ then there is a $\lambda_1 \in h(R)$ such that $\lambda_1 a_{\tau_1} \in L$ and $\lambda_1 x_{\sigma_n} \neq 0$. Similarly, there is a $\lambda_2 \in h(R)$ such $\lambda_2\lambda_1 a_{\tau_2} \in L$ and $\lambda_2\lambda_1 x_{\sigma_n} \neq 0$. Using a recurrence argument we end up with an element $\lambda_m \in h(R)$ such that $\lambda_m \ldots \lambda_2 \lambda_1 a_{\tau_m} \in L$ and $\lambda_m \ldots \lambda_2 \lambda_1 x_n \neq 0$. Put $\lambda = \lambda_m \ldots \lambda_2 \lambda_1$, then $\lambda x \neq 0$ while $\lambda a \in L$, hence $(L:a) \not\subset \mathrm{ann}_R x$ i.e. $L \in \mathcal{L}(\kappa_{\underline{M}})$. This proves 3. An immediate consequence of 3. is that $\underline{\kappa^r_M (X)} \subset \kappa_{\underline{M}} (\underline{X})$; let us prove the converse.

Since $X/\kappa^r_M(M)$ is κ^r_M-torsion free there exists a monomorphism $X/\kappa^r_M(X) \to \prod_{i\in I} (\prod_{\sigma\in G} E^g(M)(\sigma))$ (direct products in R-gr).
Since $E^g(M)(\sigma) \subset E(M)$ for all $\sigma \in G$, there is thus a monomorphism of $X/\kappa^r_M(\alpha)$ into a product of copies of E(M). Consequently $\underline{X/\kappa^r_M(X)}$ is $\kappa_{\underline{M}}$-torsion free. Hence $\kappa_{\underline{M}}(\underline{X}) = \underline{\kappa^r_M(X)}$.

<u>II.9.10. Remark.</u> In general for arbitrary $M \in$ R-gr, $\kappa_{\underline{M}}$ is not a graded kernel functor.

Let us now try to bring this abstract theory down to earth by constructing some explicite examples of well behaved torsion theories in R-gr.

Consider a multiplicatively closed subset S of R. To S we associate a kernel functor $\kappa_{\underline{S}}$ on R-mod given by its filter $\mathcal{L}(\kappa_{\underline{S}})$ where :
$\mathcal{L}(\kappa_{\underline{S}}) = \{$ L left ideal of R such that $(L:r) \cap S \neq \emptyset$ for all $r \in R$ $\}$.

<u>II.9.11. Proposition</u>. Let R be a graded ring of type G, S a multiplicatively closed subset of h(R) not containing zero. Then $\kappa_{\underline{S}}$ is a graded kernel functor on R-mod, i.e. to S there also corresponds a rigid

kernel functor κ_S on R-gr with graded filter by :

$\mathcal{L}(\kappa_S) = \{ L \in L_g(R),\ (L:r) \cap S \neq \emptyset$ for all $r \in h(R) \}$.

Proof. Take $L \in \mathcal{L}(\underline{\kappa}_S)$ i.e. $(L:r) \cap S \neq \emptyset$ for all $r \in R$. If r is in $\mathcal{L}(R)$ this yields that $s\, r \in L$ for some $s \in S$, hence $s\, r \in L_g$ and $(L_g : r) \cap S \neq \emptyset$. If r is not homogeneous, write $r = r_{\sigma_1} + \ldots + r_{\sigma_n}$. Pick $s_n \in (L : r_{\sigma_n}) \cap S$, $s_{n-1} \in (L : s_n r_{\sigma_{n-1}}) \cap S, \ldots$, $s_1 \in (L:s_2 \ldots s_n r_{\sigma_1}) \cap S$. If we put s equal to $s_1 s_2 \ldots s_n$ then $s\, r \in L_g$ and $s \in S$ i.e. $L_g \in \mathcal{L}(\underline{\kappa}_S)$, or in other words $\underline{\kappa}_S$ is graded. The other statements follows directly from Theorem II.9.7.

II.9.12. Remark. If S satisfies the left Ore conditions then $\kappa_S(M) = \{ m \in M,\ s\, m = 0$ for some $s \in S \}$ for any $M \in$ R-gr.

Let us now introduce the graded Lambek torsion theory, it is the torsion theory in R-gr cogenerated by R in R-gr, i.e. we set $\mathcal{T} = \{ M \in$ R-gr, $\mathrm{HOM}_R(M, E^g(R)) = 0 \}$. Its graded filter is given by $\mathcal{L} = \{ L \in L_g(R),\ (L:a) \not\subset \mathrm{ann}_R b$ for all $a,b \in h(R)$, $b \neq 0 \}$.
This definition parallels the definition of Lambek's torsion theory in R-mod which is the torsion theory cogenerated by R in R-mod.

II.9.13. Note. Let R be a gr-field of type Z which is not trivially graded. The kernel functor associated to Lambek's torsion theory in R-mod is not a graded kernel functor.

Now let R be a graded ring of type G where G is an ordered group. For $M \in$ R-gr define the singular submodule Z(M) of M to be $\{ m \in M,\ \mathrm{ann}_R x$ is an essential left ideal of R $\}$.

II.9.14. Lemma. Let R be a graded ring of type G, G being an ordered group . If $M \in$ R-gr then Z(M) is a graded R-submodule of M.

Proof. Pick $x \neq 0$ in $Z(M)$ and decompose x as $x_{\sigma_1}+\ldots+x_{\sigma_n}$ with $\sigma_1<\ldots<\sigma_n$, assuming $x_{\sigma_i} \neq 0$ for $i = 1,\ldots,n$. Put $L = \mathrm{ann}_R x$, then $L^{\sim} \subset \mathrm{ann}_R\, x_{\sigma_n}$. Since L is an essential left ideal, $L^{\sim}$ is an essential left ideal because of Lemma II.7.6. But there $\mathrm{ann}_R x_{\sigma_n}$ is an essential left ideal, hence $x_{\sigma_n} \in Z(M)$ follows. The lemma then follows by recurrence on n.

It may easily be checked that Z defines a rigid preradical (terminology of [102]) on R-gr. The associated radical Z_G satisfies $Z_G(M)/Z(M) = Z(M/Z(M))$. To a radical there corresponds an idempotent kernel functor, cf. [36]. The kernel functor κ_G on R-gr corresponding to the radical Z_G is rigid and we refer to κ_G as being Goldie's graded kernel functor. The following is an interesting application of this construction.

II.9.15. Proposition. Let A be an arbitrary ring, $M \in$ A-mod. then $Z_{A[X]}(M[X]) = Z_A(M)[X]$, and also :

$Z_{A[X,X^{-1}]}(M[X,X^{-1}] = Z_A(M)[X,X^{-1}]$.

Proof. Let us sketch the proof of the first statement, the second statement follows in exactly the same way.

Proof. Consider $m X^n \in Z_A(M)[X]_n$ with $m \neq 0$. Since $m \in Z_A(M)$ it follows that $\mathrm{ann}_R m$ is an essential left ideal of R, entailing that $(\mathrm{ann}_R m)[X]$is an essential left ideal of $A[X]$. But $\mathrm{ann}_{A[X]}\, m X^n = (\mathrm{ann}_A m)[X]$, thus $m X^n \in Z_{A[X]}(M[X])$ and consequently $Z_A(M)[X]$ is contained in $Z_{A[X]}(M[X])$. In view of Lemma II.9.14. the converse inclusion will follow if we show that each homogeneous element of $Z_{A[X]}(M[X])$ is in $Z_A(M)[X]$. If $m X^n$ is in $Z_{A[X]}(M[X])$, then $\mathrm{ann}_{A[X]} m X^n = (\mathrm{ann}_A m)[X]$ and therefore it follows immediately that $(\mathrm{ann}_A m)[X]$ is an essential left ideal of $A[X]$ and that $\mathrm{ann}_A m$ is an essential left ideal of A i.e. $m \in Z_A(M)$ and

$m X^n \in Z_A(M)[X]$.

Yet another example of a torsion theory in R-gr is obtained if one lets (T,F) be the torsion theory in R-gr generated by the class of simple objects in R-gr. It is an easy exercise to verify that this defines a rigid torsion theory.

Recall that a torsion theory in R-mod is symmetric if its associated filter in R allows a cofinal system of ideals of R. Similarly we now say that a rigid kernel functor κ (or rigid torsion theory) on R-gr is symmetric (in R-gr) it its graded filter has a cofinal set consisting of graded ideals of R.

II.9.16. Lemma. Let R be a graded ring of type G. Then a kernel functor κ on R-gr is symmetric if and only if $\underline{\kappa}$ on R-mod is symmetric.

Proof. Suppose that κ is symmetric. If $L \in \mathcal{L}(\underline{\kappa})$ then $L_g \in \mathcal{L}(\kappa)$, hence L_g contains an ideal I of R such that $I \in \mathcal{L}(\kappa)$. But then also $I \in \mathcal{L}(\kappa)$ Suppose, conversely, that $\underline{\kappa}$ is symmetric. If $L \in \mathcal{L}(\kappa)$ then $L \in \mathcal{L}(\underline{\kappa})$ and thus there exists an ideal J of R such that $J \subset L$ and $J \in \mathcal{L}(\underline{\kappa})$. But then J_g is an ideal of R and $J_g \in \mathcal{L}(\kappa)$ by Theorem II.9.7. Since L contains J_g as well, the lemma follows.

Let R be a left Noetherian ring an let $\underline{\kappa}$ be a kernel functor on R-mod. To $\underline{\kappa}$ we may associate a symmetric kernel functor $\underline{\kappa}^o$ given by the filter : $\mathcal{L}(\underline{\kappa}^o) = \{$ L left ideal of R such that $L \supset I$, I an ideal of R and $I \in \mathcal{L}(\underline{\kappa})$ $\}$.

Since R is left Noetherian the above does indeed define a filter in R and one may verify that $\underline{\kappa}^o$ is the largest symmetric kernel functor on R-mod which is smaller than $\underline{\kappa}$.

II.9.17. Proposition. Let R be an left Noetherian graded ring of type G.. Let $\underline{\kappa}$ be a graded kernel functor on R-mod introducing the rigid kernel functor κ on R-gr. Then $\underline{\kappa}^0$ is a graded kernel functor on R-mod and the rigid κ^0 induced on R-gr is the largest symmetric kernel functor on R-gr smaller than κ (what justifies the notation κ^0).

Proof. Straightforward.

Concluding this section we characterise torsion theories in R-gr in case R is strongly graded of type G. Let us introduce the following.

II.9.18. Definition. If R is a graded ring of type G then a kernel functor γ on R_e-mod is G-stable, or stable under conjugation, if $\gamma(R_\sigma \otimes_{R_e} M) = R_\sigma \otimes_{R_e} \gamma(M)$ for every $M \in R_e$-mod.

II.9.19. Proposition. Let R be a strongly graded ring of type G, then :

1. There is a bijective correspondence between (idempotent) kernel functors on R-gr and (idempotent) kernel functors on R_e-mod.
2. In the bijective correspondence of 1. the rigid kernel functors on R-gr correspond bijectively to the G-stable kernel functors on R_e-mod.

Proof. To an idempotent kernel functor κ on R-gr we associate the functor $\kappa_e : R_e\text{-mod} \to R_e\text{-mod}$ given by $\kappa_e(X) = (\kappa(R \otimes_{R_e} X))_e$, for $X \in$ R-mod.

By Theorem I.3.4., one easily sees that κ_e is an idempotent kernel functor. Using the same theorem one easily sees that to an idempotent kernel functor κ' on R-mod one may associate κ on R-gr defined by $\kappa(M) = R \otimes_{R_e} \kappa'(M_e)$ for all $M \in$ R-gr.

II.10. Graded Rings and Modules of Quotients.

The general theory of localization in a Grothendieck category may be applied so as to obtain objects of quotients with respect to kernel functors. In the case of R-gr the interesting point is that the objects of quotients with respect to rigid kernel functors relate in a very direct sense to certain objects of quotients constructed in R-mod. The exposition of this link between localization in R-mod and localization in R-gr is the subject of this section. Some of the results are really intrinsic for the graded situation, the interested reader may convince himself on this point by comparing these results to general results concerning reflectors, cf. [123], or relative localization, cf. [122], or the use of the Gabriel-Popescu theorem as in [127].

Throughout this section R will be a graded ring of type G, κ will be an idempotent kernel functor on R-gr with associated torsion theory (T,F) and with associated graded filter $\mathcal{L}(\kappa)$ in R.

If $M \in$ R-gr, we define the <u>object of quotients in R-gr with respect to κ</u> of M, to be the graded R-module $Q^g_\kappa(M) = \varinjlim_{L \in \mathcal{L}(\kappa)} HOM_R(L, M/\kappa(M))$.

As in the ungraded situation, one easily establishes that $Q^g_\kappa(R)$ is in a natural way a graded ring of type G whereas for every $M \in$ R-gr, $Q^g_\kappa(M)$ turns out to be a graded $Q^g_\kappa(R)$-module i.e. $Q^g_\kappa(M) \in Q^g_\kappa(R)$-gr. Let $\underline{\kappa}$ be the graded kernel functor on R-mod inducing κ on R-gr (see Theorem II.9.6.). Then for every $M \in$ R-gr we also have $Q_{\underline{\kappa}}(M)$, the localization of $\underline{M}$ at $\underline{\kappa}$ in R-mod. Let us polish up our memory and recall that $\underline{\kappa}$ (resp. κ) is said to be of <u>finite type</u> if for every $L \in \mathcal{L}(\underline{\kappa})$ (resp. $\mathcal{L}(\kappa)$) there exists an $L' \in \mathcal{L}(\underline{\kappa})$ (resp. $\mathcal{L}(\kappa)$) such that $L' \subset L$ and L' is finitely generated in R-mod (resp. in R-gr).

II.10.1. Lemma. Let $\underline{\kappa}$ be a graded kernel functor on R-mod introducing the rigid kernel functor κ on R-gr. Then κ has finite type if and only if $\underline{\kappa}$ has finite type.

Proof. Obviously $\underline{\kappa}$ will have finite type if κ has finite type. Conversely, if $\underline{\kappa}$ has finite type consider $L \in \mathcal{L}(\kappa)$. There is an $L' \in \mathcal{L}(\kappa)$ which is finitely generated and such that $L' \subset L$. The left ideal generated by the finite number of homogeneous components of a finite set of generators of the left ideal L', K say, is in L since L is a graded left ideal of R. But K is graded and $K \in \mathcal{L}(\kappa)$ since $K \in \mathcal{L}(\underline{\kappa}) \cap L_g(R)$, thus $\mathcal{L}(\kappa)$ is of finite type.

II.10.2. Proposition. Let $\underline{\kappa}$ be a graded kernel functor of finite type on R-mod. If $M \in$ R-gr then : $\underline{Q_\kappa^g(M)} \cong Q_\kappa(\underline{M})$

Proof. By definition : $Q_\kappa^g(M) = \varinjlim_{L \in \mathcal{L}(\kappa)} \mathrm{HOM}_R(L, M/\kappa(M))$.

But, since $\underline{\kappa}$, hence κ, has finite type, we obtain :

$$\begin{aligned} Q_\kappa^g(M) &= \varinjlim_{L \in \mathcal{L}(\kappa)'} \mathrm{HOM}_R(L, M/\kappa(M)) \\ &= \varinjlim_{L \in \mathcal{L}(\kappa)} \mathrm{Hom}_R(L, M/\kappa(M)) \end{aligned}$$

Where $\mathcal{L}(\kappa)'$ is the subset of $\mathcal{L}(\kappa)$ consisting of the left ideals of finite type. Therefore, as ungraded modules :

$Q_\kappa^g(M) \cong \underline{Q_\kappa}(\underline{M})$.

II.10.3. Corollary. Let R be a graded ring of type G such that R is left gr-Noetherian and consider $M \in$ R-gr. Then $\underline{Q_\kappa^g(M)} \cong Q_\kappa(\underline{M})$ holds for every graded kernel functor $\underline{\kappa}$ on R-mod.

Proof. Every graded left ideal of R is finitely generated. Let us now proceed to obtain a more constructive (in R-mod) method of determining graded objects of quotients. Let $\underline{\kappa}$ be a graded kernel functor on R-mod and consider $M \in R\text{-gr}$. For each $\sigma \in G$ put

$$N_\sigma = \{ x \in Q_{\underline{\kappa}}(\underline{M}) \text{ , there is } L \in \mathcal{L}(\kappa) \text{ such that } L_\tau x \subset (M/\underline{\kappa}(M))_{\tau\sigma} \text{ for all } \tau \in G \}.$$

(Note that $M/\underline{\kappa}(M)$ is indeed a graded module). The N_σ are subgroups of the additive groups of $Q_{\underline{\kappa}}(\underline{M})$. The sum $\sum_{\sigma\in G} N_\sigma$ is direct. Indeed if we can find a relation : $x_{\sigma_1} = x_{\sigma_2} + \ldots + x_{\sigma_n}$ with $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n \in G$ then we also find $L_i \in \mathcal{L}(\kappa)$ such that :

$$(L_i)_\tau x_{\sigma_i} \subset (M/\underline{\kappa}(M))_{\tau\sigma_i} \text{ , } i = 1, \ldots , n.$$

Take $L = \bigcap_{i=1}^n L_i$, then on one hand $L_\tau x_{\sigma_1} \subset (M/\underline{\kappa}(M))_{\tau\sigma_1}$ but on the other hand $L_\tau x_{\sigma_1} = L_\tau(x_{\sigma_2} + \ldots + x_{\sigma_n}) \subset \sum_{i=2}^n (M/\underline{\kappa}(M))_{\tau\sigma_i}$.

However, $(M/\underline{\kappa}(M))_{\tau\sigma_1} \cap (\sum_{i=2}^n (M/\underline{\kappa}(M))_{\tau\sigma_i} = 0$,

therefore $L_\tau x_{\sigma_1} = 0$ for all $\tau \in G$, hence $L x_{\sigma_1} = 0$.

Since $L \in \mathcal{L}(\kappa)$ this would imply $x_{\sigma_1} \in \underline{\kappa}(M/\underline{\kappa}(M)) = 0$

Furthermore, if $a \in R_\lambda$ and $x \in N_\sigma$ then $a x \in N_{\lambda\sigma}$.

So, $\bigoplus_{\sigma\in G} N_\sigma$ is a graded R-module; we will denote it by $g Q_{\underline{\kappa}}(\underline{M})$.

<u>II.10.4. Proposition.</u> Let R be a graded ring of type G and let $\underline{\kappa}$ be a graded kernel functor on R-mod. Then $g\, Q_{\underline{\kappa}}(R)$ is a graded ring containing $R/\underline{\kappa}(R)$ as a graded subring. The graded ring structure of $g\, Q_{\underline{\kappa}}(R)$ is the unique ring stucture compatible with its graded R-module structure. For every $M \in R\text{-gr}$, $g\, Q_{\underline{\kappa}}(\underline{M})$ is a graded $g\, Q_{\underline{\kappa}}(R)$-module.

<u>Proof.</u> It is fairly obvious that $gQ_{\underline{\kappa}}(R)$ is a graded left R-module contained in $Q_{\underline{\kappa}}(R)$ and containing $R/\underline{\kappa}(R)$ as a graded R-submodule. Pick $x, y \in gQ_{\underline{\kappa}}(R)$ and consider $xy \in Q_{\underline{\kappa}}(R)$ (this is a ring by general localization theory in R-mod) without loss of generality we may assume

$x \in (g\,\underline{Q}_\kappa(R))_\sigma$, $y \in (g\underline{Q}_\kappa(R))_\tau$ for some $\sigma,\tau \in G$. Let I, J $\in \mathcal{L}(\kappa)$ be such that $I_\lambda\, x \subset (R/\underline{\kappa}(R))_{\lambda\sigma}$, $J_\mu\, y \subset (R/\underline{\kappa}(R))_{\mu\tau}$ for all $\lambda,\mu \in G$. Let $\bar{J}$ be the image of J in $R/\underline{\kappa}(R)$ and put $L = (\bar{J} : x)_R \cap I$, i.e. $L = \{ a \in I$ such that $a\,x \in \bar{J} \}$. Then $L \in \mathcal{L}(\kappa)$. Indeed , if $b \in I$ then $(L:b) \supset (J:y)$ where y is a representative for $bx \in R/\underline{\kappa}(R)$.

We obtain : $L_\theta\, xy \in (R/\underline{\kappa}(R))_{\theta\sigma\tau}$, and thus $xy \in (g\underline{Q}_\kappa(R))_{\sigma\tau}$.

It follows that $g\underline{Q}_\kappa(R)$ is a graded ring and this ring structure is determined by the graded R-module structure. Actually this ring structure is induced by the ring structure of $\underline{Q}_\kappa(R)$ and the latter is uniquely determined by its R-module structure; consequently the ring structure on $g\underline{G}_\kappa(R)$ is uniquely determined by the R-module structure (note that $\underline{Q}_\kappa(g\underline{Q}_\kappa(R)) = \underline{Q}_\kappa(R)$). Formally similar argumentation yields that $g\underline{Q}_\kappa(\underline{M})$ is a graded $g\underline{Q}_\kappa(R)$-module as desired.

II.10.5. Theorem. Let $\underline{\kappa}$ be a graded kernel functor on R-mod and let κ be the rigid kernel functor induced on R-gr.

For every $M \in$ R-gr we have that $Q^g_\kappa(M) \cong g\underline{Q}_\kappa(\underline{M})$.

Proof. It is well known that :

$$\underline{Q}_\kappa(\underline{M}) = \varinjlim_{L\in\mathcal{L}(\kappa)} \mathrm{Hom}\,(L, M/\underline{\kappa}(M)) = \varinjlim_{L\in\mathcal{L}(\kappa)} \mathrm{Hom}\,(\underline{L}, \underline{M/\kappa(M)}).$$

(the latter equality follows since $\underline{\kappa}$ is graded).

The construction of $g\underline{Q}_\kappa(\underline{M})$ is such that $x \in (g\underline{Q}_\kappa(\underline{M}))_\sigma$ if and only if x may be represented by a graded morphism of degree σ, $f_x : L \to M/\underline{\kappa}(M)$ for some $L \in \mathcal{L}(\kappa)$ i.e. $f_x \in \mathrm{HOM}_R\,(L, M/\kappa(M))_\sigma$.

II.10.6. Corollary. Q^g_κ is a covariant left exact endofunctor in R-gr. Moreover Q^g_κ is functorial with respect to graded morphisms of arbitrary degree.

Proof. We only have to verify the second statement. Consider $N, M \in R\text{-gr}$ and a graded morphism of degree $\sigma \in G$, $f : N \to M$. Then $\underline{Q}_\kappa(f) : \underline{Q}_\kappa(N) \to \underline{Q}_\kappa(M)$ is R linear. If $x \in (Q_\kappa^g(N))_\lambda$, $\lambda \in G$, then there exists an $I \in \mathcal{L}(\kappa)$ such that $I_\mu\, x \subset (N/\kappa(N))_{\mu\lambda}$ for all $\mu \in G$. Thus for all $\mu \in G$ we have
$f(I_\mu x) \subset (M/\kappa(M))_{\mu\lambda\sigma}$.
From $f(I_\mu x) = \underline{Q}_\kappa(f)(I_\mu x) = I_\mu\, \underline{Q}_\kappa(f)(x)$ we obtain that $I_\mu\, \underline{Q}_\kappa(f)(x) \in (M/\kappa(M))_{\mu\lambda\sigma}$ for all $\mu \in G$.
Hence $\underline{Q}_\kappa(f)(x) \in Q_\kappa^g(M)_{\lambda\sigma}$ and it follows that $\underline{Q}_\kappa(f)$ is actually a graded morphism of degree σ from $Q_\kappa^g(N)$ to $Q_\kappa^g(M)$ (using Theorem II.10.5., Proposition II.10.2.).
We now focus on some practical properties of graded localization.

II.10.7. Proposition. Let R be a graded ring of type G. Let $M, N, S \in R\text{-gr}$ be such that $N \subset M$ and $\kappa(M/N) = \dot{M}/N$ for some rigid kernel functor κ on R-gr. Any graded morphism f of degree $\sigma \in G$, $f : N \to Q_\kappa^g(S)$, extends in a unique way to a graded morphism, h of degree σ, $h : M \to Q_\kappa^g(S)$.

Proof. Let $\underline{\kappa}$ be the graded kernel functor on R-mod, inducing κ on R-gr
Consider the following diagram :

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, dashed, "h"] & M/N \arrow[r] & 0 \\
 & Q_\kappa^g(S) \arrow[r, hook] & \underline{Q}_\kappa(S) & &
\end{tikzcd}$$

Since $\underline{\kappa}(\underline{M/N}) = \underline{\kappa(M/N)} = \underline{M/N}$, f extends to a unique R-linear $h : \underline{M} \to \underline{Q}_\kappa(S)$.
If $x \in M_\lambda$, $\lambda \in G$ then the assumption on M/N entails that $L\, x \subset N$ for some $L \in \mathcal{L}(\kappa)$. Hence $L_\mu\, x \subset N_{\mu\lambda}$ and therefore :
$L_\mu h(x) = h(L_\mu x) = f(L_\mu x) \subset Q_\kappa^g(S)_{\mu\lambda\sigma}$. So for $a \in L_\mu$ we have $ah(x) \in Q_\kappa^g(S)_{\mu\lambda\sigma}$, thus there exists an ideal $L' \in \mathcal{L}(\kappa)$ such that :
$L'_\theta(a\, h(x)) \subset (S/\kappa(S))_{\theta\mu\lambda\sigma}$ for all $\theta \in G$.

Denoting L'L by L" we obtain $L'' \in \mathcal{L}(\kappa)$ such that $L''_\tau\, h(x) \in (S/\kappa(S))_{\tau\lambda\sigma}$ for all $\tau \in G$; hence $h(x) \in Q^g_\kappa(S)_{\lambda\sigma}$ because of Theorem II.10.5. Thus h is as stated in the proposition.

At this point recall that for a kernel functor $\underline{\kappa}$ on R-mod $Q_{\underline{\kappa}}(\underline{M})$, for $\underline{M} \in$ R-mod, may be characterized by the following exact sequence in R-mod :

$$0 \to \underline{M}/\underline{\kappa}(\underline{M}) \to Q_{\underline{\kappa}}(M) \xrightarrow[\pi(\underline{M})]{} \underline{\kappa}(E(\underline{M})/\underline{M}) \to 0$$

where $E(\underline{M})$ is an injective hull of $\underline{M}$ in R-mod and $\pi(M) : E(\underline{M}) \to E(\underline{M})/\underline{M}$ the canoncial epimorphism . In the graded situation, if we consider $M \in$ R-gr, then we have the following exact diagram in R-mod :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{M} & \longrightarrow & E(\underline{M}) & \xrightarrow{\pi(\underline{M})} & E(\underline{M})/\underline{M} & \longrightarrow & 0 \\
 & & \uparrow{=} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & \underline{E^g(M)} & \longrightarrow & \underline{E^g(M)/M} & \longrightarrow & 0
\end{array}$$

where the bottom row may be derived from the corresponding exact row in R-gr. With these notations we have :

II.10.8. Proposition. Let R be a graded ring of type G. Let κ be a rigid kernel functor on R-gr and take $M \in$ R-gr. Then, $Q^g_\kappa(M) \in$ R-gr is characterized by the following exact sequence in R-gr :

$$0 \to M/\kappa(M) \longrightarrow Q^g_\kappa(M) \xrightarrow[\pi^g(M)]{} \kappa(E^g(M)/M) \to 0$$

where $E^g(M)$ is an injective hull of M in R-gr and $\pi^g(M)$ is the canonical epimorphism $E^g(M) \to E^g(M)/M$.

Proof. Mere verification.

With notations and assumptions as before we have :

II.10.9. Proposition. Consider $N \in$ R-gr, $M \in Q^g_\kappa(R)$-gr. If a map $f : M \to Q^g_\kappa(N)$ is R-linear and graded of degree σ then it is also $Q^g_\kappa(R)$-linear and graded of degree σ.

Proof. Easy from Theorem II.10.5. and the characterization of $g\underline{Q}_{\kappa}$.

Recall the following definition. A kernel functor on R-mod, $\underline{\kappa}$ say, has property T or is a T-functor or the corresponding torsion theory is said to be perfect, if one of the following equivalent conditions is fulfilled.

P.T.1. $\underline{Q}_{\kappa}$ is exact and commutes with direct sums.

P.T.2. For all $\underline{M} \in$ R-mod, $\underline{Q}_{\kappa}(\underline{M}) = \underline{Q}_{\kappa}(R) \underset{R}{\otimes} \underline{M}$.

P.T.3. For all $L \in \mathcal{L}(\underline{\kappa})$, $\underline{Q}_{\kappa}(R) = \underline{Q}_{\kappa}(R)\ \underline{j}_{\kappa}(L)$ where $\underline{j}_{\kappa}$ is the canonical morphism $R \to \underline{Q}_{\kappa}(R)$.

P.T.4. The quotient category with respect to $\underline{\kappa}$ (in the sense of [32]) coincides with $\underline{Q}_{\kappa}(R)$-mod.

P.T.5. Every $\underline{Q}_{\kappa}(R)$-module is $\underline{\kappa}$-torsion free as an R-module.

For more information concerning perfect torsion theories cf. [36]. The condition of P.T.3. entails that any kernel functor having property T has finite type. Let us now investigate the graded version of the above properties.

II.10.10. Theorem. Let R be a graded ring of type G. Let κ a rigid kernel functor on R-gr with associated graded kernel functor $\underline{\kappa}$ on R-mod.

The following statements are equivalent :

1. For all $L \in \mathcal{L}(\kappa)$, $Q_{\kappa}^{g}(R)\ j_{\kappa}\ (L) = Q_{\kappa}^{g}(R)$, where $j_{\kappa}: R \to Q_{\kappa}^{g}(R)$ is the canonical graded localizing morphism.
2. For every $\underline{M} \in Q_{\kappa}^{g}(R)$-mod, $\underline{\kappa}(\underline{M}) = 0$
3. For every $\underline{M} \in Q_{\kappa}^{g}(R)$-gr, $\kappa(M) = 0$.
4. The functors Q_{κ}^{g} and $Q_{\kappa}^{g}(R) \underset{R}{\otimes} -$ are naturally equivalent in R-gr.

Proof. $1 \Rightarrow 2$. Take $\underline{M} \in Q_{\kappa}^{g}(R)$-mod and suppose $x \in \underline{\kappa}(\underline{M})$. Then $L\,x = 0$ for some $L \in \mathcal{L}(\kappa)$ hence $Q_{\kappa}^{g}(R)(Lx) = 0$ but then $Q_{\kappa}^{g}(R)\ j_{\kappa}(L)\ x = 0$ entails $x = 0$, by 1.

2. ⇒ 3. Obvious from Theorem II.9.6.

3. ⇒ 4. Let $M \in R\text{-gr}$. The canonical R-linear $\underline{M} \to Q_{\underline{\kappa}}(\underline{M})$ factorizes as follows :

$$\underline{M} \xrightarrow{\alpha} Q_{\underline{\kappa}}(R) \underset{R}{\otimes} \underline{M} \longrightarrow Q_{\underline{\kappa}}(\underline{M}).$$

Since $Q_\kappa^g(M)$ is a $Q_\kappa^g(R)$-module, the above sequence restricts to a sequence of graded morphisms :

$$M \xrightarrow{\alpha^g} Q_\kappa^g(R) \underset{R}{\otimes} M \longrightarrow Q_\kappa(M)$$

where $\alpha^g(m) = 1 \otimes m$ and $\beta^g(\sum_i{}' q_i \otimes m_i) = \sum_i{}' q_i m_i$. The assumptiom 3. entails that : $\kappa(Q_\kappa^g(R) \underset{R}{\otimes} M) = 0$ hence $\operatorname{Ker} \alpha^g \supset \kappa(M)$. That $\kappa(M) \supset \operatorname{Ker} \alpha^g$ is clear because $\operatorname{Ker} \alpha \subset \underline{\kappa}(M)$; thus $\operatorname{Ker} \alpha^g = \kappa(M)$. Then β^g is a monomorphism. Because of Proposition II.10.9. β^g is $Q_\kappa^g(R)$-linear, hence $\operatorname{Im} \beta^g$ is a graded $Q_\kappa^g(R)$- submodule of $Q_\kappa^g(M)$. Note that β^g is a graded morphism of degree $e \in G$! Since $\operatorname{Im} \beta^g$ contains $M/\kappa(M)$ it follows that $Q_\kappa^g(M)/\operatorname{Im}\beta^g$ is κ-torsion and a graded $Q_\kappa^g(R)$-module. Thus by 3. we obtain that β^g is surjective, hence an isomorphism in R-gr.

4. ⇒ 1. For every $L \in \mathcal{L}(\kappa)$ we have $Q_\kappa^g(L) = Q_\kappa^g(R)$ because R/L is κ-torsion. From 4. it follows then that :

$$Q_\kappa^g(R) = Q_\kappa^g(L) = \beta^g(Q_\kappa^g(R) \underset{R}{\otimes} L) = Q_\kappa^g(R)\, j_\kappa(L).$$

<u>II.10.11. Definition.</u> A rigid kernel functor κ on R-gr is said to have <u>property T</u> if Q_κ^g is exact and commutes with direct sums.

<u>II.10.12. Corollary.</u> Let κ be a rigid kernel functor on R-gr with associated kernel functor $\underline{\kappa}$ on R-mod, then the following assertions are equivalent :

1. κ has property T.
2. $\underline{\kappa}$ has property T.

If this is the case then $\underline{Q_\kappa^g(M)} = Q_{\underline{\kappa}}(\underline{M})$

Proof.

1. $\Rightarrow$ 2. From the foregoing theorem we retain that, for all $L \in \mathcal{L}(\kappa)$, L contains a graded left ideal of finite type L' which is still in $\mathcal{L}(\kappa)$. By Proposition II.10.2. we have : $\underline{Q^g_\kappa(M)} = Q_{\underline{\kappa}}(\underline{M})$ for all $M \in R$-gr. If now $H \in \mathcal{L}(\underline{\kappa})$ then $H_g \in \mathcal{L}(\kappa)$ by definition of $\underline{\kappa}$. Thus $Q^g_\kappa(R)\ j_\kappa(H_g) = Q^g_\kappa(R)$ and from the above it is then obvious that $Q_{\underline{\kappa}}(R)\ j_{\underline{\kappa}}(H) = Q_{\underline{\kappa}}(R)$ follows i.e. $\underline{\kappa}$ has property T.

2. $\Rightarrow$ 1. If $\underline{\kappa}$ has property T then in particular $\underline{\kappa}$ has finite type and then Lemma II.10.1. yields that κ has finite type on R-gr. Proposition II.10.2. then implies that $\underline{Q^g_\kappa(R)} = Q_{\underline{\kappa}}(R)$ and one easily deduces from this that κ has property T.

Some particular results concerning localization of graded rings of type Z will be given in Section C.I.1., after we have deduced the graded version of Goldie's theorem(s).

II.11. Injective Modules, Prime Kernel Functors and Localization at Prime Ideals.

In this section we use the notations of Section II.9. and in particular Theorem II.9.7. is very useful. For any $M \in$ R-gr, $E^g(M)$ denotes the injective hull of M in R-gr; it is clear that $E^g(M(\sigma)) = E^g(M)(\sigma)$ for all $\sigma \in G$ and consequently : Ass M = Ass $(E^g(M)(\sigma))$ for $\sigma \in G$ (terminology of II.7). The following Lemma is a slight strenghtening of a statement of Section II.9.(one of the remarks II.9.7.).

II.11.1. Lemma. Let R be a graded ring of type G.
If $M \in$ R-gr is shift invariant i.e. $M \cong M(\sigma)$ for all $\sigma \in G$ then κ_M is rigid. Every rigid kernel functor κ on R-gr is of the form κ_M for some shift invariant $M \in$ R-gr.

Proof. The first statement is easily checked. Conversely, suppose κ is a rigid kernel functor on R-gr. That $\kappa = \kappa_N$ for some $N \in$ R-gr is clear by general torsion theory. Now for any $\sigma \in G$ we obtain, for every $L \in$ R-gr :

$$\begin{aligned}\kappa_{N(\sigma)}(L) &= \cap \{ \operatorname{Ker} f,\ f \in \operatorname{Hom}_{R\text{-gr}}(L, E^g(N(\sigma))) \} \\ &= \cap \{ (\operatorname{Ker} f)(\sigma),\ f \in \operatorname{Hom}_{R\text{-gr}}(L(\sigma^{-1}), E^g(N)) \} \\ &= (\kappa_N(L(\sigma^{-1})))(\sigma) = \kappa_N(L).\end{aligned}$$

Hence $\kappa_{N(\sigma)} = \kappa_N$ for all $\sigma \in G$. This implies that $\kappa_N = \wedge \{\kappa_{N(\sigma)}, \sigma \in G\} =$
$= \kappa_{\underset{\sigma}{\oplus} N(\sigma)}$ and we may take $M = \underset{\sigma}{\oplus} N(\sigma)$.

Recall from II.9. (see Theorem II.9.7.) that to $M \in$ R-gr there corresponds a rigid kernel functor κ_M^r defined by $\kappa_M^r = \wedge \{\kappa_M(\sigma), \sigma \in G\}$, which is such that $\mathcal{L}(\kappa_{\underline{M}}) \cap L_g(R) = \mathcal{L}(\kappa_M^r)$ and also, for $X \in$ R-gr : $\kappa_M^r(X) = \kappa_{\underline{M}}(\underline{X})$.

It follows that κ_M^r is the largest rigid kernel functor on R-gr such that $\kappa_M^r(M) = 0$.

An $M \in$ R-gr is said to be a graded supporting module for the kernel functor κ on R-gr if $\kappa(M) = 0$ but $\kappa(M/N) = M/N$ for every graded submodule N of M (deleting "graded" everywhere yields the definition of a supporting module for a kernel functor $\underline{\kappa}$ on R-mod, utilized in [40]). A kernel functor κ on R-gr is said to be graded prime if it is cogenerated by a graded supporting module. It is clear that if $M \in$ R-gr is such that $\kappa_{\underline{M}}$ is a graded prime kernel functor on R-mod with supporting module $\underline{M}$ then κ_M is a graded prime kernel functor on R-gr with supporting module M. Warning : remember Remark II.9.10. i.e. $M \in$ R-gr does not make $\kappa_{\underline{M}}$ graded! By definition of cogeneration in R-gr (see II.9.) it is clear that κ_M depends on $E^g(M)$, not on M. Therefore let us investigate $E^g(M)$ just a little more. Let $E(\underline{M})$ be an injective hull of $\underline{M}$ in R-mod. Suppose $M \in$ R-gr is such that the singular radical $Z(M)$ equals zero.

Put $E'_\sigma = \{ x \in E(\underline{M})$, $L_\tau x \subset M_{\tau\sigma}$ for $\tau \in G$, for some graded essential left ideal L of R $\}$.

Obviously $E' = \underset{\sigma}{\oplus} E'_\sigma$ is in R-gr and it contains M as a graded essential R-submodule.

II.11.2. Lemma. $M \in$ R-gr be such that $Z(M) = 0$, then E' is the largest R-submodule of $E(\underline{M})$ with the property that E' is graded and containing M as a graded submodule.

Proof. Suppose we have $M \subset N$ in R-gr such that $\underline{M} \subset \underline{N} \subset E(\underline{M})$. If $x \in N_\sigma$ then $(\underline{M} : x)$ is a graded left ideal of R which is also essential. Since $(M:x)_\tau\, x \subset M \cap N_{\tau\sigma} = M_{\tau\sigma}$, $x \in E'$ follows.

II.11.3. Proposition. If $M \in$ R-gr is such that $Z(M) = 0$ then $E^g(M) \cong E'$ in R-gr.

Proof. In an arbitrary Grothendieck category, injective hulls of objects may be characterized as being maximal essential extensions of those objects.

However, in R-gr, $E^g(M)$ is a gr-essential of M, then by Lemma I.2.8., $\underline{E^g(M)}$ is an essential extension of $\underline{M}$ in R-mod i.e. we may assume $\underline{E^g(M)} \subset E(\underline{M})$. The foregoing lemma then yields $E^g(M) \subset E'$ and the maximality of $E^g(M)$ with respect to being gr-essential then yields $E' = E^g(M)$.

II.11.4. Remark. A kernel functor κ in R-gr is cogenerated by M if it is cogenerated by $E^g(M)$, if and only if $\underline{\kappa}$ is cogenerated by $E(\underline{M})$.

We say that a rigid kernel functor κ on R-gr is a rigid-prime kernel functor if $\kappa = \wedge \{ \kappa_1(\sigma), \sigma \in G \}$, where κ_1 is a graded prime kernel functor on R-gr, i.e. $\kappa = \kappa^r_M$ for some $M \in$ R-gr which is a graded supporting module for κ_M.

Let $\underline{\kappa}$ be an arbitrary kernel functor on R-mod, then we denote by $(\underline{\kappa})_g$ the rigid kernel functor on R-gr given by the graded filter $\mathcal{L}((\underline{\kappa})_g) = \mathcal{L}(\underline{\kappa}) \cap L_g(R)$. Note that $\underline{(\underline{\kappa})_g} \leq \underline{\kappa}$ but equality only holds if $\underline{\kappa}$ is a graded kernel functor on R-mod. By Theorem II.9.7. it follows that for $M \in$ R-gr, $(\kappa_{\underline{M}})_g = \kappa^r_M$. For $X \in$ R-gr we introduce the following notations :

$Q^g_M(X) = Q^g_{\kappa^r_M}(X), Q_{\underline{M}}(X) = Q_{\kappa_{\underline{M}}}(X)$.

Since $\kappa^r_M(X) = \kappa_M(X)$, $Q^g_M(X) \subset \underline{Q_M}(X)$.

Since $\kappa_{\underline{M}}$ need not be a graded kernel functor on R-mod the following modification of the techniques of Proposition II.10.4. is necessary.

By $g\, Q_{\underline{M}}(\underline{X})$ we now mean the graded module $\bigoplus_{\sigma\in G} Y_\sigma$, where

$Y_\sigma = \{ x \in Q_{\underline{M}}(\underline{X})$, there is an $L \in \mathcal{L}(\kappa^r_M)$ such that $L_\tau x \subset (X/\kappa_M(X))_{\tau\sigma}$, for all $\tau \in G \}$.

II.11.5. Theorem. Let R be a graded ring of type G. Then with notations as before :

1. $g\, Q_{\underline{M}}(R)$ is a graded ring containing $R/\kappa_M(R)$ as a graded subring

2. For every $X \in R$-gr, $g\,Q_{\underline{M}}(\underline{X})$ is a graded $g\,Q_{\underline{M}}(R)$-module. Moreover : for all $X \in R$-gr :

   $\underline{Q_M^g(X)} = g\,Q_{\underline{M}}(\underline{X})$.

Proof.

1. As in Proposition II.10.4.
2. Consider the following diagram in R-mod :

$$\begin{tikzcd} 0 \arrow[r] & \underline{X/\kappa_M(X)} \arrow[rr, "\alpha_X"] \arrow[dr, "\beta_X"'] & & \underline{Q_M^g(X)} \arrow[r, hook] & Q_{\underline{M}}(\underline{X}) \\ & & g\,Q_{\underline{M}}(\underline{X}) \arrow[urr, hook] & & \end{tikzcd}$$

If $x \in Q_M^g(X)_\sigma$ then, because coker α_X is $\kappa_{\underline{M}}$-torsion, we have $L_\tau x \subset (X/\kappa_M(X))_{\tau\sigma}$ for every $\tau \in G$, i.e. $x \in g\,Q_{\underline{M}}(X)^\tau$. On the other hand, the construction of $g\,Q_{\underline{M}}(\underline{X})$ is such that coker β_X is κ_M^r-torsion in R-gr. Proposition II.10.7. and and the fact that $\underline{Q_M^g(X)}$ is an essential extension of $\underline{X/\kappa_M(X)}$ in R-mod, entail that $g\,Q_{\underline{M}}(\underline{X}) \subset Q_M^g(X)$, hence

$g\,Q_{\underline{M}}(\underline{X}) = \underline{Q_M^g(X)}$.

Now consider a graded prime ideal P of a left Noetherian graded ring R of type G. Put M = R/P and write $\underline{\kappa}_P = \kappa_{\underline{R/P}}$, $\kappa_P^r = \kappa_{R/P}^r$. In Section C.I.3. it will be shown that $E^g(R/P)$ is a P-cotertiary graded R-module. In the Z-graded case we will be able to deduce a more complete result, see Section C.I.1., here in the general G-graded case we have :

II.11.6. Proposition.

Let R be a left Noetherian ring graded of type G and let $P \in \mathrm{Spec}_g(R)$, then $\underline{\kappa}_P$ and κ_P^r may be described by the following filters :

1. $\mathcal{L}(\underline{\kappa}_P)$ = { J left ideal of R, $(J:a) \cap G(P) \neq \emptyset$ for all $a \in R$ }

   = { J left ideal of R, J maps to an essential left ideal of R/P }

2. $\mathcal{L}(\kappa_P^r)$ = { J graded left ideal of R such that J maps to an essential graded left ideal of R/P }

Proof. By Theorem II.9.7. and the fact that $(\kappa_{\underline{R/P}})_g = \kappa_P^r$ it follows that 2. is an immediate consequence of 1.

1. If $(J:a) \cap G(P) \neq \emptyset$ for all $a \in R$, consider $\bar{x} \neq 0$ in R/P, where $\bar{x}$ is the image of some $x \in R-P$. Pick $s \in G(P) \cap (J:a)$. Hence $sa \in J$ and $s\,x \neq 0$, therefore $(J:a) \not\subset \mathrm{ann}_R X$.
By II.9.7. (1), $J \in \mathcal{L}(\underline{\kappa}_P)$ follows.
Conversely, suppose that $J \in \mathcal{L}(\underline{\kappa}_P)$, and $a \in R$. Then $(J:a) \in \mathcal{L}(\underline{\kappa}_P)$. Put $L = (J:a)$. Hence $L \not\subset \mathrm{ann}_R\bar{x}$ for all $\bar{x} \in R/P$, $\bar{x} \neq 0$. The image $\bar{L}$ of L in R/P is an essential left ideal of R/P. Indeed, if $\bar{s} \in R/P$, $\bar{s} \neq 0$ then from $(L:s) \not\subset \mathrm{ann}_R\,\bar{s}$ it follows that there exists a $\lambda \in (L:s)$ such that $\lambda \notin \mathrm{ann}_R\bar{s}$, hence $\lambda\,s \in L$ and $\lambda\,\bar{s} \neq 0$. Since R/P is a left Noetherian prime ring, an essential left ideal has to contain a regular element, $\bar{t}$ say, i.e. $\bar{t} \in \bar{L}$. But a representative t for $\bar{t}$ is then (by definition of G(P)) in G(P) and thus $L \cap G(P) \neq \emptyset$.

II.11.7. Remark.

1. The foregoing proposition states that $\underline{\kappa}_P$ is the kernel functor associated to the multiplicative set G(P) (in the sense of Section I.6., but ungraded). The exact graded analoge can only be included after we derived a graded version of Goldie's theorems. Therefore we will include this result in Section C.I.1.

Exercise. Let R and S be two graded rings and $f : R \to S$ an homomorphism of graded rings $(f(1) = 1)$.
Then f is an epimorphism of rings if and only if f is an epimorphism in the category of graded rings.

CHAPTER B: COMMUTATIVE GRADED RINGS

I: Some Commutative Algebra Revisited

I.1. Graded Fields.

Throughout part B of this book all rings are commutative associative rings with unit unless otherwise stated. Moreover all graded rings will be graded of type Z. Some more general results, in particular for graded rings of type G where G is torsion free abelian, will be hinted at in the exercises. The results of Section A.I.4. entail that graded fields of type Z are of the form k or $k[T,T^{-1}]$ where k is a field, T a variable of degree $t \in Z$. We will write gr-field for graded field.

Let R be a gr-field, let K be its field of fractions and let Ω be an algebraic closure of K. An $\alpha \in \Omega$ is gr-algebraic over R if the minimal monic polynomial of α over K has the form, $\star$: $X^n + a_{n-1} X^{n-1} + \ldots + a_o$, with $a_i \in h(R)$ and $\deg(a_i) - \deg(a_{i+1}) = c \in Z$ for all $i = 0,\ldots, n-1$. This condition states exactly that this minimal polynomial, f(X) say, is a homogeneous element of $R[X]$ when this ring is considered with the gradation given by $R[X]_n = \sum_{i+jc=n} R_i X^j$ for all $n \in Z$.

A graded ring S containing R as a graded subring is said to be gr-algebraic over R if every element of S is gr-algebraic over R.

I.1.1. Lemma. Let $\alpha \in S$ be gr-algebraic over R. If g(X) is a monic polynomial in $R[X]$ with minimal degree in X such that $g(\alpha) = 0$, then g(X) is a minimal polynomial for α over K.

Proof. If $g(X) = h_1(X)h_2(X)$ in $R[X]$ then the leading coefficients of h_1 and h_2 are units in R and thus in h(R). Since R is a gr-field we may assume that $h_1(X)$ and $h_2(X)$ are both monic polynomials. The minimality assumption on $\deg_X g(X)$ entails that either h_1 or h_2 equals 1.

Hence $g(X)$ is irreducible in $R[X]$. However R is a factorial ring because $R \cong R_0[T,T^{-1}]$ with R_0 a field, therefore $g(X)$ will also remain irredicuble in $K[X]$. Thus $g(X)$ is a minimal monic polynomial for α over K.

I.1.2. Remark. An $\alpha \in \Omega$ is gr-algebraic over R if it is integral over R and the monic polynomial of minimal degree satisfied by α has the form $\star$. This justifies the definition of gr-algebraic. Indeed, a weaker definition asserting only the existence of a polynomial of the form $(\star)$, satisfied by α, will not lead to the desired results.

I.1.3. Corollary. If $\alpha \in \Omega$ is gr-algebraic over R then the subring $R[\alpha]$ of Ω is a gr-field.

Proof. Let $f(X)$ be the minimal monic polynomial over R satisfied by α, $f(X)$ is of the form $(\star)$ by assumption. By the above lemma $R[\alpha] \cong R[X]/(f(X)$ as rings. If $R[X]$ is graded by giving X degree c, where c is the constant in Z appearing in $(\star)$, then $(f(X))$ is a graded ideal so we obtain a gradation on $R[\alpha]$ such that $\deg \alpha = c$ and R is a graded subring of $R[\alpha]$. Let us proceed to show that $(f(X))$ is a gr-maximal ideal of $R[X]$. If $h(X)$ is any homogeneous element of $R[X]$ then it follows from Lemma I.1.1. that either $K[X](h(X), f(X)) = K[X]$ or $K[X](h(X), f(X)) = K[X](f(X))$. In the first case $(h(X), f(X)) = R[X]$ follows ; in the second case $r\,h(X) \in (f(X))$ for some $r \in R$. Hence $r_i\,h(X) \in (f(X))$ for each homogeneous component r_i of r. Since R is a gr-field, each nonzero r_i is invertible in R, hence $h(X) \in (f(X))$ follows. So $(f(X))$ is a gr-maximal ideal of $R[X]$ and thus $R[\alpha] \cong R[X]/(f(X))$ is a gr-field.

I.1.4. Lemma. Let $\alpha \in \Omega$ be gr-algebraic over R and put $\deg \alpha = c_\alpha$. Let $\beta \in \Omega$ be gr-algebraic over R of degree c_β. Then β is gr-algebraic over $R[\alpha]$ and of degree c_β with respect to its minimal polynomial over $R[\alpha]$.

Proof. Let h be a monic polynomial of minimal degree in X satisfied by β over $R[\alpha]$. Since $R[\alpha]$ is a gr-field, cf. Corollary I.1.3., $R[\alpha][X]$ is a factorial ring and thus h is irreducible and it is also the minimal monic polynomial for β over $K(\alpha)$, where $K(\alpha)$ is the field of fractions of $R[\alpha]$. If $g \in R[X]$ is the minimal monic polynomial for β over R then g is homogeneous in $R[X]$ when this ring is considered with the gradation determined by $\deg X = c_\beta$. Embed $R[X]$ into $R[\alpha][X]$ in the obvious way. In $R[\alpha][X]$ we have that $g = h\xi + \mu$. But $\mu(\beta) = 0$ entails $\mu = 0$ by the minimality condition on $\deg_X h$. The gradation on $R[\alpha][X]$ being determinied by the gradation of $R[\alpha]$ and by $\deg X = c_\beta$, it follows from $g = h\xi$ that h and ξ are homogeneous in $R[\alpha][X]$ i.e. h has the form (*) over $R[\alpha]$ with the same constant $c_\beta \in Z$. By definition β is gr-algebraic over $R[\alpha]$ of degree c_β.

I.1.5. Corollary. The subset of Ω consisting of the gr-algebraic elements over R forms a graded ring, actually it is a gr-field.

Proof. If α satisfies of polynomial of the form (*) with constant $c \in Z$ then we write $\deg(\alpha) = c$; this is justified by the above lemma. Put
$E(R)_c = \{ \alpha \in \Omega,\ \alpha$ gr-algebraic over R and $\deg(\alpha) = c \}$.
Take $\alpha, \beta \in E(R)$ and consider $R[\alpha, \beta]$ in Ω. From Lemma I.1.4. it follows that $R[\alpha, \beta]$ is a gr-field and thus $\alpha + \beta$ is homogenous of degree c in $R[\alpha, \beta]$. If $\sum_{i=0}^n r_i (\alpha+\beta)^i = 0$ is the evaluation of the minimal monic polynomial for $\alpha+\beta$ over R then we may select terms of highest degree in the gradation of $R[\alpha,\beta]$ and obtain : $\sum_{j+ic=N} r_{i,j} (\alpha+\beta)^i = 0$, where $r_{i,j}$ is the homogeneous component of degree j of r_i. Up to multiplication by a unit of R we obtain a monic polynomial $\sum_{j+ic=N} r_{i,j} X^i$ of the form * which is satisfied by $\alpha+\beta$.

Hence $\alpha+\beta \in E(R)_c$. In much the same way it is possible to show that, if $\alpha \in E(R)_c$, $\beta \in E(R)_d$ then $\alpha\beta \in E(R)_{c+d}$. That $E(R) = \bigoplus_{c\in Z} E(R)_c$ is a graded ring and a gr-field is clear.

I.1.6. Remarks.

1. $E(R)$ is called the gr-algebraic closure of R in Ω.

2. Since graded modules over gr-fields are gr-free it is possible to use the rank (we will not write gr-rank since there is no ambiguity possible). For example : if the degree of the minimal polynomial (of form ★) $f(x)$ for a gr-algebraic $\alpha \in \Omega$, over R, is equal to n, then $n = [R[\alpha] : R]$. We leave to the reader the verification of some very elementary properties of gr-algebraic extensions, (the proof of these properties is just like the ungraded case), which we will use freely in the sequel.

Now consider a field extension L of K with $[L:K] = n$. Put $E_L(R)_c = \{ \alpha \in L, \alpha$ is gr-algebraic over R, $\deg(\alpha) = c \}$, and $E_L(R) = \bigoplus_{c\in Z} E_L(R)_c$. Then $E_L(R)$ is a gr-field and we refer to it as the gr-algebraic closure of R in L. It is clear that $[E_L(R) : R] \leq [L:K] = n$, but equality need not hold generally. A finite field extension L of K, $n = [L:K]$, is graded realised if $n = [E_L(R):R]$. If we can find a $t' \in Z$ such that $R = R_0[T,T^{-1}]$, with $\deg T = t$, may be changed to R', which is the ring R with a new gradation given by $\deg T = t'$, such that $[E_L(R'):R'] = n$ then we say that L is graded realisable. We will return to these concepts in Section I.3. First let us now present the graded versions of two well-known theorems in commutative algebra.

I.1.7. Theorem. (Graded version of Zariski's theorem).

Let R be a gr-field and let S be a gr-field extension of R generated as an R-ring by a finite number of homogeneous elements, $x_1,\dots,x_n$ say. Then S is a gr-algebraic extension of R and S is finitely generated in R-gr.

Proof. If $n = 1$ then $R[x_1]$ can only be a gr-field if x_1^{-1} is a polynomial $\sum_{i=0}^{m} r_i x_1^i$ with : $\deg(r_i) + i \deg(x_1) = -\deg(x_1)$. The monic polynomial $f(x)$ deduced from $x_1(\sum_{i=0}^{m} r_i x_1^i) = 1$ has the form $\star$ and it is easily seen to be the minimal polynomial of x_1 over K. Thus x_1 is gr-algebraic over R, and by Remark I.1.6., 2 we have $[R[x_1] : R] = \deg_X f(x)$. We proceed by induction on n, assuming the statements to be true for gr-field extensions generated by less that n elements. As in the ungraded case one first proves that $R[x_1,\dots,x_n]$ is an integral extension of R and then, knowing that $R[x_1,\dots,x_n]$ is a gr-field it is easy to establish that it is a gr-algebraic extension of R.

The above theorem may be used in deriving the following graded version of Hilbert's Nulstellensatz.

I.1.8. Theorem. If R is a gr-field, consider $S = R[X_1,\dots,X_n]$, the polynomial ring in n variables graded by putting $S_n = \sum_{i+\nu_1+\dots+\nu_n=m} R_i X_1^{\nu_1}\dots X_n^{\nu_n}$.

Every proper graded ideal I of S has a zero $(x_1,\dots,x_n)$ where the x_i lie in a suitable gr-algebraic extension of R.

Proof. As in the ungraded case, utilizing Theorem I.1.7.. Note that this is not the projective-nulstellensatz used commonly in projective algebraic geometry. Basically this is due to the fact that the variable T appears in R together with its inverse so that Proj $R[\underline{X}]$ may be considered as an open subspace of Proj $R_0[T,\underline{X}]$ i.e. the complement to the hyperplane, $T = 0$.

I.2. Integral Extensions of Graded Rings.

Consider a graded ring R which is a subring of a commutative domain S (not necessarily graded).

I.2.1. Definition. An $s \in S$ is gr-algebraic over R of degree c if $R[s] \simeq R[X] / \operatorname{Ker} \pi_s$, where $R[X]$ is graded by putting $R[X]_n = \sum_{i+jc=n} R_i X^j$, and where $\pi_s : R[X] \to S$ is the ring morphism defined by $\pi_s(X) = s$, $\pi_s(r) = r$, $r \in R$. i.e. $\operatorname{Ker} \pi_s$ is generated by homogeneous polynomials. We say that s is gr-integral over R of degree c, if s is gr-algebraic of degree c and $\operatorname{Ker} \pi_s$ contains a monic polynomial. It should be clear that this definition is different from : s satisfies a homogeneous monic polynomial in $R[X]$! Indeed, even in the case where R is a gr-field the "right" definition should imply that the minimal monic polynomial of a gr-integral element is homogeneous .

I.2.2. Lemma. If s is a gr-integral over R then there is a homogeneous monic polynomial in $\operatorname{Ker} \pi_s$.

Proof. Let $X^m + \sum_{i=0}^{m-1} a_i X^i = f(X)$, $a_i \in R$, be a monic polynomial in $\operatorname{Ker} \pi_s$. If s is of degree c, give X degree s, and selecting homogeneous components of degree mc in f(X) yields a homogeneous monic polynomial in $\operatorname{Ker} \pi_s$.

I.2.3. Lemma. Let $s \in S$ be such that $R[s]$ (constructed within S) is a graded ring such that s is homogeneous of degree c. If $t \in S$ is graded integral over R of degree d, then t is graded integral over $R[s]$ of degree d.

Proof. By our hypothesis $\pi_t : R[X] \to S$ has graded kernel $\operatorname{Ker} \pi_t$ (note that $R[X]$ is now graded such that X has degree d !) and it contains a monic homogeneous polynomial. We have to establish similar properties for $\pi_t : R[X] \to S$, given by $\pi_t(\lambda) = \lambda$ for $\lambda \in R[s]$, $\pi_t(X) = t$. Let $h(X)$ be the monic polynomial in Ker π_t. Then also $h(X) \in$ Ker π_t; we now proceed to show that the homogeneous components of any $f(X) \in$ Ker π_t will again be in Ker π_t, this being the case, $h(X) \in$ Ker π_t will finish the proof.

First let us point out that if R' is any domain, $f(X), g(X) \in R[X]$ such that $\deg_X f(X) \leq \deg_X g(X)$, then there exists $\alpha \in R'$ such that $\alpha g(X) = f(X)\, q(X) + r(X)$ with $q(X) \in R'[X]$ and $\deg_X r(X) < \deg_X f(X)$.

Part 1. Take $g(X) \in$ Ker π_t such that $g(X)$ has minimal degree in X as such. Taking the homogeneous component of highest degree appearing, will be denoted by putting an index $(-)_H$ while taking components of lowest degree will be denoted by $(-)_M$.

Case 1. $g(X)_H \in (R[s][X])_0$.

If also $g(X)_M \in (R[s][X])_0$ then $g(X)$ is homogenous and then there is nothing to prove. So let us assume the statement to be true if $g(X)_M \in (R[s][X])_1$ for $0 \geq 1 > k$. Next consider the case where $g(X)_M \in (R[s][X])_k$. Let $\alpha \in R[s]$ be such that $\alpha\, h(X) = g(X)\, q(X) + r(X)$. Then $\alpha h(t) = 0$ and $g(t) = 0$ yield $r(t) = 0$ but then $\deg_X r(X) < \deg_X g(X)$ entails $r(X) = 0$ and $\alpha h(X) = g(X) q(X)$. Since $h(X)$ is homogeneous this yields $\alpha_M h(X) = g_M(X)\, q_M(X)$ thus again $g_M(t) = 0$ or $q_M(t) = 0$. Suppose that $q_M(t) = 0$. Then $\deg_X q_M(X) \geq \deg_X g_M(X)$.

Therefore there is an $\alpha' \in R[s]$ such that $\alpha' q_M(X) = g(X) q'(X)$, hence $\alpha'_M q_M(X) = g_M(X)\, q'_M(X)$. If $g_M(X) \notin R[s]$ then we may repeat the procedure and since now $\deg_X q'_M(X) < \deg_X q_M(X)$ we will reach the situation where $\deg_X q'_M(X) \leq \deg_X g_M(X)$, thus $g_M(t) = 0$. On the other hand, if $g_M(X) \in R[s]$ then $\alpha_M h(X) = g_M(X)\, q_M(X)$ yields (putting $g_M(X) = \beta$) :

(★) $\beta\ \alpha h(X) = g(X)(\beta\ q_H(X) + \ldots + \alpha_M\ h(X))$

If $h(X)$ has degree r in X then $\alpha_M\ X^r$ appears in $\alpha_M h(X)$ and this term cannot cancel out in $\beta\ q_H + \ldots + \alpha_M h(X)$ (where $\alpha_M h(X) = \beta q_M(X)$), because β is homogeneous and terms different from $\beta\ q_M\ (X)$ yield terms in X of gradation degree different from the degree of $\alpha_M\ X^r$! Therefore, comparing degrees in (★) learns that $g(X) \in R[s]$, contradiction. So far we have proved that $g_M(t) = 0$. But then $(g-g_M)(t) = 0$ and $g(X)-g_M(X) \in (R[s][X])_1$ with $0 > 1 > k$.

The induction hypothesis then yields that the homogeneous parts of $g(X)-g_M(X)$ i.e. the homogeneous components of $g(X)$ are in Ker π_t.

<u>Case 2</u> : $g(X)_H \in (R[s][X])_i$ with $i \geq 0$.

If $i = 0$ then we are in Case 1, so we proceed by induction on i. Replacing M by H in the proof for Case 1 yields a completely similar proof.

<u>Part 2</u> : Now consider $p\ (X) \in$ Ker π_t of arbitrary degree in X. Let $p_1(X)$ be the sum of the homogeneous components of $p(X)$ which are in Ker π_t. If $p(X)-p_1(X) = 0$ then we are done, otherwise $q(X) = p(X)-p_1(X)$ is in Ker π_t and no components in $q(X)$ is in Ker π_t. Let $q(X)$ be as in Part 1. then $\deg_X q(X) \geq \deg_X g(K)$, hence we may select $\gamma \in R[s]$ such that $\gamma\ q(X) = g(X)q_r\ (X)+r(X)$ with $\deg_X r(X) < \deg_X g(X)$. Then $q(t) = 0 = r(t)$ implies that $r(X) = 0$ hence $\gamma_M q_M(X) = g_M(X)\ q_1(X)$. The choice of $q(X)$ and the foregoing part of this proof yield that $g_M(t) = 0$ hence $\gamma_M(t) = 0$ hence $\gamma_M\ q_M(t) = 0$ or $q_M(t) = 0$ but this contradicts $q_M(X) \notin$ Ker π_t. ■

<u>I.2.4. Corollary</u>. Let R be a graded ring which is a subring of the domain S. Put $I(R)_c = \{\ x \in S,\ x$ graded integral of degree c over $R\ \}$ and write $I(R) = \bigoplus_{c \in Z} I(R)_c$. Then $I(R)$ is a graded ring, which is called the <u>graded integral closure</u> of R in S. If S is graded, then an $s \in S_n$ which is gr-integral over R is necessarily of degree n. This shows that the gradation of $I(R)$ is induced by the gradation of S in this case.

I.3. Graded Valuation Rings in Graded Fields.

First let us recall some facts about primes in rings.
Primes in rings and algebras over fields have been introduced in [111], [107], [108] and this theory yields a unified approach to several techniques used in trying to extend valuation theory to first commutative rings and later also non-commutative rings.

Let R be any ring with unit. A couple (P,R') is a prime of R if R' is a subring of R, P is a prime ideal of R' and if $x R' y \subset P$ with $x, y \in R$ then $x \in P$ or $y \in P$. In the commutative case this coincides with the primes studied by F. Van Oystaeyen in [110],[111]. The ideal P is called the kernel of the prime. The set of kernels of primes of a ring R is denoted by Prim(R). To a finite subset F of R associate D(F) = $= \{ P \in \mathrm{Prim}(R), P \cap F = \phi \}$. The sets D(F) form a basis for a topology on Prim(R). Prim thus defines a contravariant functor from the category of commutative rings to the category of topological spaces, and this functor is a natural transformation of the Spec functor. The kernel of a prime does not determine the prime completely; however if only the kernel of a prime is specified, say P, then it will be understood that we are considering the prime (P,R^P) where R^P is the idealizer of the set P in R. A prime P of R is said to be semirestricted if for all $x \in R-R^P$ there exist nonzero elements λ and μ in R^P suh that $\lambda x \mu \in R^P$. These primes seem to be the right generalization of valuation rings ; actually if R is a field the primes of R do correspond to valuation rings of R. For a detailed account of the theory of primes we refer to J. Van Geel's thesis [107]. For a commutative ring R, the following lemma will be rather useful :

I.3.1. Lemma.

1. If P is the kernel of a prime of R then R^P is integrally closed in R.

2. The integral closure of a subring S of R in R is the intersection of all R for all primes (P,R) of R such that $S \subset R$.

Proof. cf. [107].

Now we return to the case where R is a gr-field. A graded subring V of R is said to be a gr-valuation ring R if for every homogeneous $x \in R$ either x or x^{-1} is in V.

I.3.2. Lemma. If V is a gr-valuation ring of the gr-field R then the graded ideals of V are linearly ordered by inclusion.

Proof. Easy. As in the ungraded case.

I.3. Lemma. If $a_1,\ldots,a_n \in h(R)-V$ then there is an i such that $a_i^{-1} a_j \in V$ for all $j = 1,\ldots,n$.

Proof. Look at the ideals A_j generated by a_j^{-1} in V. Because of Lemma I.3.2. we may pick an i such that A_i is minimal amongst the A_j, $j = 1,\ldots,n$. Then $a_i^{-1} a_j \in V$ because otherwise $a_j^{-1} = (a_j^{-1}a_i)\, a_i^{-1}$ yields $A_j = A_i$, hence $r\, a_j^{-1} = a_i^{-1}$ for some $r \in V$ and again $a_i^{-1} a_j \in V$ follows.

From Lemma I.3.2. it is evident that a gr-valuation ring is a gr-local ring ; let M_V denote the unique gr-maximal ideal of V. In Lemma I.3.3. it is then clear that $a_i^{-1} a_j \in M_V$ if and only if $A_i \subsetneq A_j$.

I.3.4. Proposition. If V is a gr-valuation ring of the gr-field R then (M_V, V) is a semi restricted prime of R.

Proof. Pick $a = a_1 + \ldots + a_n \in R-V$, where $\deg a_i = m_i$.
Suppose $a_{i_1},\ldots,a_{i_t} \in R-V$. By Lemma I.3.3. we may select an i_j, $1 \le j \le t$, such that $x = a_{i_j}^{-1}a = a_{i_j}^{-1}a_1 +\ldots+ 1 + \ldots a_{i_j}^{-1}a_n$ is in V. Suppose that $x \in M_V$ then, taking parts of degree zero, we obtain $1 \in M_V$ a contradiction. Therefore $a_{i_j}^{-1} a \in V-M_V$ and thus (M_V,V) is semirestricted (that it is indeed a prime of R is obvious !).

I.3.5. Proposition. If (P, R^P) is a prime of the gr-field R then (P_g, R_g^P) is a prime of R too. Moreover (P_g, R_g^P) is a gr-valuation ring of R with gr-maximal ideal P_g.

Proof. Take $x \in h(R)$. Since $x\, x^{-1} \in R^P$ it follows that either $x \in R^P$ or $x^{-1} \in R^P$ and therefore either $x^{-1} \in R_g^P$. Consequently R_g^P is a gr-valuation ring of R. Now P_g obviously contains the ideal generated by the $y \in h(R)$ such that $y^{-1} \notin R_g^P$, thus it follows that P_g is the maximal graded ideal of R_g^P.

I.3.6. Remark. It is not necessarily so that $P \cap R_g^P = P_g$. Indeed if this were the case then (P, R^P) would dominate $(P_g, R\ \ )$ in the sense of [107]. However a semirestricted prime is maximal with respect to the domination relation i.e. $P = P_g$ and $R^P = R_g^P$ would follow.

Due to Lemma I.3.1. it is natural to try to apply primes to the study of integrality and integral extensions.

I.3.7. Theorem. Let S be a graded domain with graded field R. Consider a gr-field extension T over R and let $\bar{S}$ be the integral closure of S in T. Then $\bar{S}$ is a graded subring of T and thus $\bar{S}$ equals the gr-integral closure of S in T.

Proof. By Lemma I.3.1., 2., $\bar{S}$ is the intersection of the domains of primes of T containing S, say $\bar{S} = \cap R'$. Since S is graded $\cap R' = \cap R'_g$ follows. Hence $\bar{S}$ is a graded ring.

I.3.8. Corollary. The use of primes yields an easy proof of the fact that the integral closure of a graded domain in its field of fractions is again a graded ring (a well-known fact in commutative algebra, cf. [129]).

Extending further on the results of Sections I.1. and I.2. :

I.3.9. Theorem. Let R be a gr-field with field of fractions K. Let L be a finite field extension of K with $[L:K] = n$.

Then the following statements are equivalent :

1. L is graded realisable over R.
2. If $R = k[T,T^{-1}]$, where k is field, then the integral closure $\tilde{R}$ of R in L is a graded ring containing R as a graded subring provided the gradation of R is changed by giving T a suitable degree.

Proof. By notation : $K = k(T)$, where $k = R_0$ is a field

$1 \Rightarrow 2$. The gradation of R may be changed by putting $\deg T = t'$ such that $[E_L(R):R] = n$, (note that over a gr-field $E_L(R) = I_L(R)$) Consequently the field of fractions of $E_L(R)$ is L. Theorem I.3.7. entails that the integral closure $\overline{E_L(R)}$ in L is graded , but since $E_L(R)$ is a gr-field (cf. remarks following I.1.6.) it has to be integrally closed in its field of fractions. Thus $\tilde{R} = E_L(R)$ is a graded ring.

$2 \Rightarrow 1$. In the suitable gradation on R we obtain that $\tilde{R}$ is a graded ring containing R as a graded subring. Any $x \in h(\tilde{R})$ satisfies a homogeneous $f(X) \in R[X]$ (where $R[X]$ is graded by putting $\deg X = \deg x$.) which may be assumed to be monic since R is a gr-field. It is then clear that such a polynomial of minimal degree in X will also be the minimal polynomial for x over K i.e. $x \in E_L(R)$ and x has degree c as an element of $E_L(R)$.

I.3.10. Corollary. Theorem I.3.9. extends I.3.8. in that it allows graded realisable extension fields L of K. In particular, if $L = F(T)$ where F is a finite extension of the field $k = R_0$ then the integral closure of R in L is a graded ring, this particular case is the generalization of Theorem 11 mentioned in [129] .

I.3.11. Examples. Let R be $k[T,T^{-1}]$ with deg T = t; K = k(T).

1. Put $L = K(\alpha)$ with $\alpha^2 = T$. Clearly L is graded realizable if T is given even degree. However if t is odd then $E_L(R) = R$;
2. Put $L = K(\alpha)$ with $\alpha^2 = T + 1$; then L is not graded realizable;
3. put $L = K(\alpha)$ with $\alpha^2 + T\alpha = T^3$; then L is not graded realizable.

Moreover $E_L(R') = R'$ for every admissible change of gradation on the underlying ring R. This is of course due to the fact that the polynomial $X^2 + TX - T^3$ cannot be homogeneous in $R[X]$, whatever deg X may be.

I.3.12. Remark. The gr-valuation rings of a fixed gr-field R form a dense subset of Prim(R), with the induced Zariski topology. This dense subset may be considered to be the compact Riemann surface $\underline{\nu}$ associated to R. Note that a copy of the Riemann surface of the field R_0, $\underline{\nu}_0$ say, may may be found in $\underline{\nu}$ by considering the gr-valuation rings $O_0[T,T^{-1}]$, where O_0 is a valuation ring of R_0.

Let us focus on the relation between gr-valuations and valuations. This relation becomes evident through the study of certain valuation functions.

I.3.13. Theorem. Let V a gr-valuation ring of the gr-field R. To V there corresponds a valuation function $v: R^* \to \Gamma$ for some ordered group Γ. This function v extends to a valuation $v: K^* \to \Gamma$ of the field of fractions K of R. If $x \in R$ then $x \in V$ if and only if $v(x) \geq 0$.

Proof. On $h(R^*)$, where $R^* = R - \{0\}$, we define an equivalence relation as follows. For a, $b \in h(R^*)$ write $a \underset{V}{\sim} b$ if there exist $x_1, x_2 \in V$ such that $a = x_1 b$ and $b = x_2 a$. Let Γ be the totally ordered group obtained from the equivalence classes with respect to $\underset{V}{\sim}$. It is obvious how we define a valuation function $v : h(R^*) \to \Gamma$. Let us check that r does define a "valuation function" as desired on the whole of R^*.

If $a \in R^\star$, say $a = a_1 + \ldots + a_n$ with deg $a_i = d_i$, $d_1 < \ldots < d_n$. Put $v(a) = \min_j \{ v(a_j), j = 1,\ldots,n \}$. Let us verify that for any $a, b \in R^\star$ we have $v(ab) = v(a) + v(b)$; the other properties of a valuation function are even more easily verified.

Write $b = b_1 + \ldots + b_m$ with deg $b_j = e_j$, $j = 1,\ldots,m$.

Let $a_{i_1},\ldots,a_{i_r}$, resp. $b_{j_1},\ldots,b_{j_s}$ be the homogeneous components of a resp. b, where the valuation reaches its lowest value and assume that $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_s$. We have to check whether $v(ab) = v(a_{i_1}) + v(b_{j_1})$.

Clearly , $(ab)_{i_1 + j_1} = a_{i_1} b_{j_1} + \sum_{k+l=i_1+j_1} a_k b_l$.

If $a_k b_l$ with $k \neq i_1$, $l \neq j_1$, has the same valuation as $a_{i_1} b_{j_1}$ then either $v(a_k) = v(a_{i_1})$ and $v(b_l) = v(b_{j_1})$, or $v(a_k) = v(b_{j_1})$ and $v(b_l) = v(a_{i_1})$, by the choice of i_1 and j_1. The latter implies $v(a_k) \leq v(a_{i_1})$ hence $v(a_k) = v(a_{i_1})$ and $v(b_{j_1}) = v(a_{i_1})$; therefore the second possibility reduces to the first.

Now $v(a_k) = v(a_{i_1})$ entails $k > i_1$ whereas $v(b_l) = v(b_{j_1})$ entails $l > j_1$; this contradicts $k + l = i_1 + j_1$. Consequently $a_{i_1} b_{j_1}$ is the unique term in the decomposition of $(ab)_{i_1+j_1}$ with minimal valuation.

Hence : $v((ab)_{i_1+j_1}) = v(a_{i_1} b_{j_1}) = v(a_{i_1}) + v(b_{j_1})$.

That the above valuation value is minimal amongst the valuations of the homogeneous components of ab is obvious, so by definitions v on $R^\star$ it follows that $v(ab) = v(a_{i_1}) + v(b_{j_1}) = v(a)+v(b)$. From this it is also clear that $V = \{ x \in R^\star, v(x) \geq 0 \} \cup \{0\}$. Let M be the gr-maximal ideal of V. It is easy to obtain from v an explicite valuation $\tilde{v} : K^\star \to \Gamma$; however let us include a direct proof of the fact that $Q_M(V)$ is a valuation ring of K

(Q_M denotes the localization functor with respect to V-M). Pick $z \in K$, say $z = (x_1 +\ldots+ x_r)(d_1 +\ldots+d_s)^{-1}$ for certain $n_1,\ldots,n_r$, $d_1,\ldots,d_s \in h(V)$. By Lemma I.3.3. there is a $\xi \in h(R)$ such that $z = (\xi n_1+\ldots+\xi n_r)(\xi d_1+\ldots+\xi d_0)^{-1}$ and such that ξn_i, $\xi d_j \in h(V) - \{0\}$ while some ξn_i or ξd_j is equal to 1. Thus either the nominator or the denominator of z, in this representation, lies outside M and this means that either z or z^{-1} is in $Q_M(V)$. That the valuation associated to this valuation ring is exactly $\tilde{v}$ may be easily checked.

I.3.14. Proposition. Let R be a gr-field and let V be a gr-valuation ring of R with gr-maximal ideal M. Then M does not contain $V_{<0}$ unless either V is trivially graded and then so is R, or $V = k[T]$ or $V = k[T^{-1}]$ (where $R = k[T,T^{-1}]$).

Proof. Take $y \in V_{-1}$ with $1 > 0$ and suppose $v(y) = m > 0$. If there is an $\alpha \in V_0$ such that $v(\alpha) = n \neq 0$ then $v(\alpha^{-m} y^n) = 0$ i.e. $\alpha^{-m} y^n$ is a unit of V but this contradicts $V_{-1n} \subset M$ unless $n \leq 0$. But $M \cap V_0 = 0$ implies $M \cap k = 0$ hence $k \subset V$ and $V_0 \supset k$. Thus $M \supset V_{<0}$ entails $V_0 = k$ or V, and hence R, is trivially graded. Suppose $V_0 = k$. If there is an $x \neq 0$ in V_{t1} for some $t > 0$ then we obtain : $y^t x \in V_0 = k$. Hence, since $y^t x \neq 0$, $y^t x$ is a unit of V but since y is not a unit, by assumption, $V_{t1} = 0$ follows for all $t > 0$. Now from $V_1^t \subset V_{t1} = 0$ it follows that $V_t = 0$ for all $t > 0$.

If V is not negatively graded then the foregoing implies that $y = 0$ for all $y \in V_{-1}$ with $1 > 0$ i.e. V is positively graded. In this case $T^{-1} \notin V$ hence $T \in V$ i.e. $k[T] \subset V$. Since $k[T]$ is a gr-valuation ring and a principal ideal ring one easily checks that $V = k[T]$. Similarly if V is negatively graded thet $V = k[T^{-1}]$ follows.

I.3.15. Corollary. A non-trivially graded valuation ring V of a gr-field R possesses elements of negative valuation and of non-zero degree.

The construction in Theorem I.3.13. shows that a graded valuation has value group Z if and only if the associated valuation of the field of fractions has value group Z.

If this is the case, we may say that the gr-valuation is a discrete gr-valuation.

I.3.16. Proposition. Let V be a gr-valuation ring of the gr-field R and let v be the associated valuation function. Then v is discrete if and only if V is Noetherian.

Proof. If V is Noetherian then so is $Q_M(V)$ and then $Q_M(V)$ is a discrete valuation ring of K, hence v is discrete.

For the converse it will suffice to check that if $I \subsetneq J \subset M$ are graded ideals of V then $Q_M(V)I \subsetneq Q_M(V)J$.

Indeed, if $Q_M(V)I = Q_M(V)J$ then any $x \in h(J)$ may be multiplied by some $c \in V-M$ such that $cx \in I$.

Write $c = c_1+\ldots+c_n$ with $\deg c_i = t_i$, $t_1 <\ldots< t_n$ and $c_i \notin M$. From $cx \in I$, $c_i x \in I$ follows ; but c_i is a unit of V hence $x \in I$.

Because of Proposition I.3.4. a gr-valuation ring V is integrally closed in R, but also in K. Therefore a discrete gr-valuation ring will be a Noetherian integrally closed domain i.e. also a Krull domain. The $\operatorname{Kdim}_{V\text{-gr}} V=1$, hence V has classical Krull dimension at most two (cf. A.II.5.). A prime ideal P of height one is either M, or such that $P_g = 0$. So if $P \neq M$ has $\operatorname{ht}(P) = 1$ then $Q_p(V)$ is obtained as a localization of R and at the prime ideal RP of R and therefore $Q_p(V)$ is a discrete valuation ring (but its valuation is not the extension of a gr-valuation of R!).

Conclusion : a discrete gr-valuation ring is a Noetherian Krull domain (and a regular gr-local ring) of dimension at most 2. Results about the class group and other arithmetical features are included in the next section.

II: Arithmetically Graded Rings

II.1. Graded Krull domains. Class groups.

Throughout this section R is commutative domain with field of fractions K. We say that R is a Krull domain if there is a family of discrete rank one valuation rings of K, $\{ V_i, i \in I \}$ say, such that $R = \bigcap_{i \in I} V_i$ and such that for any nonzero $r \in R$ we have that r is a unit in V_i for all but a finite number of $i \in I$. Let us recall some of the well-known properties of Krull domains :

II.1.1. Proposition. Let R be a Krull domain in K. Then R is integrally closed. If k is a subfield of K then $R \cap k$ is a Krull domain in k. If L is a finite extension of K then the integral closure of R in L is a Krull domain in L. The polynomial ring $R[X_i, i \in I]$, as well as the ring of formal power series $R[[X]]$, is a Krull domain.

II.1.2. Proposition. If $R = \bigcap_{i \in I} V_i$ and J is a subset of I then $\bigcap_{j \in J} V_j$ is a Krull domain. If S is a multiplicatively closed subset of R then $Q_S(R) = \bigcap_{j \in J} V_j$ (where $J = \{ j \in I, S \subset U(V_j) \}$) is a Krull domain.

II.1.3. Proposition. The domain R is a Krull domain if and only if the following conditions are satisfied :

1. For every prime ideal P of height one in R, the localized ring $Q_P(R)$ is a discrete rank one valuation ring.
2. We have $R = \bigcap Q_P(R)$, the intersection ranging over all prime ideals of height one of R.
3. Any nonzero $r \in R$ is contained in at most a finite number of prime ideals of height one.

For more details on Krull domains we refer to Krull [60], Zariski and Samuel [129], Nagata [77], Bourbaki [11] and Fossum [29]. It is noteworthy that there do exist Krull domains possessing height one prime ideals which are not of finite type, cf. Eakin and Heinzer [26]. On the other hand, if R is a Noetherian normal domain then R is a Noetherian Krull domain.

II.1.4. Theorem (Krull - Akizuki). If R is a Noetherian domain of Krull dimension 1 then every intermediate ring between R and its integral closure in K is also Noetherian.

II.1.5. Remark. If R has Krull dimension two then the integral closure of R in K is Noetherian but there may exist non-Noetherian intermediate rings. We will return to this situation in the graded case.

II.1.6. Theorem (Mori-Nagata). Let S be the integral closure of the Noetherian domain R then :

1. S is a Krull domain
2. Only a finite number of prime ideals of S ly over a given prime ideal of R.
3. If $P \in \operatorname{Spec} S$, $p = P \cap R$ then $Q(S/P)$ is a finite dimensional field extension of $Q(R/p)$.

Proof. For the proof of this very interesting theorem we refer to Fossum [29].

It is a known consequence of a theorem of I. Beck that a flat extension of the Krull domain $\bigcap_{i \in I} V_i$ in K is a subintersection $\bigcap_{j \in J} V_j$, $J \subset I$. An R submodule of K, I say, is called a fractional ideal of R if there exists $r \neq o$ in R such that $r I \subset R$. If I is a fractional ideal of R, put $I^{\star} = [R:I] = \{ \lambda \in K, \lambda I \subset R \}$ and write $I^{\star\star}$ for $(I^{\star})^{\star}$.

If $I = I^{**}$ then I is said to be a divisorial ideal of R. A fractional ideal I of R such that there exists a fractional ideal J of R such that $IJ = R$ is said to be invertible. Any invertible ideal of R is divisorial.

If R is a Krull domain then the set D(R) of divisorial ideals may be made into a group by introducing the product $I.J = (IJ)^{**}$. The subgroup of principal (divisorial) ideals is denoted by P(R) and the quotient group $Cl(R) = D(R)/P(R)$ is called the divisor class group of R. It is easily seen that the invertible fractional ideals of R from a group Inv(R) containing P(R), and Inv(R) / P(R) is isomorphic to the Picard group, Pic(R), of R which is defined to the group of isomorphism classes of projective R-modules of rank 1, with respect to the sensor product. The group D(R) is isomorphic to the free abelian group generated by the prime ideals of height 1. Also, P(R) is isomorphic to U(K)/U(R) via the mapping $\mathrm{div} : U(K) \to P(R)$ given by $\mathrm{div}(\lambda) = \mathrm{div}(\lambda R)$ = the divisor of prime ideals of height one containing λR.

For a Krull domain R the property $Cl(R) = 0$ is equivalent to R being a factorial ring i.e. if every element in R is, up to multiplication by a unit, uniquely a finite product of irreducible elements. Let us mention the following result of Storch [103].

II.1.7. Proposition. For a Krull domain R, the following statements are equivalent :

1. Cl(R) is a torsion group.
2. Every subintersection of R is a ring of quotients.
3. If $f, g \in R$ then there is an $n \in \mathbb{N}$ such that $Rf^n \cap Rg^n$ is a principal ideal.
4. If P is a prime ideal of height 1, then P^n is principal for some $n \in \mathbb{N}$.

The following result gives the relation between divisor class groups of subintersections of R

II.1.8. Theorem (Nagata). Let R be a Krull domain and let S be a subintersection of R, say $R = \bigcap_{i \in I} V_i$, $S = \bigcap_{j \in J} V_j$ with $J \subset X'(R)$, the set of prime ideals of height one of R, then :

1. $\pi : Cl(R) \to Cl(S)$ is a surjection.
2. Ker π is generated by the classes of the prime ideals of height one not in J.

II.1.9. Corollary.

1. If S is a multiplicatively closed set in R then $Cl(R) \to Cl(Q_S(R))$ is surjective and the kernel is generated by the classes of prime ideals which meet S.
2. If S is generated by prime elements then $Cl(R) \to Cl(Q_S(R))$ is bijective; in particular if $Q_S(R)$ is factorial then R is factorial.

Let us now turn to the case of graded domains i.e. from hereon we assume that R is a graded commutative domain with graded field of fractions K^g and field of fractions K. We write X = Spec(R), $X_g = \mathrm{Spec}_g(R)$ the set of graded prime ideals of R, $X^1 = \{ P \in X, \mathrm{ht}\,(P) \leq 1 \}$, $X^1_g = \{ P \in X_g, P \in X^1\}$. A gr-fractional ideal of R is a graded R-submodule of K^g, I say, such that $d\, I \subset R$ for some nonzero $d \in R$; obviously one may assume that $d \in h(R)-\{0\}$. The elements of D(R) which are graded form the subgroup of graded divisors $D_g(R)$. The subgroup of graded principal divisors is denoted by $P_g(R)$. the graded divisor class group of R is defined to be $Cl_g(R) = D_g(R)/P_g(R)$ The graded Picard group of R, $\mathrm{Pic}_g(R)$, is defined to be the group of isomorphism (not necessarily degree zero !) classes of graded projective R-modules of rank 1. Similarly, $\mathrm{Inv}_g(R)$ denotes the group of invertible graded fractional ideals of R. It is easy to verify : $\mathrm{Pic}_g(R) = \mathrm{Inv}_g(R)/P_g(R)$, and consequently we may consider Pic(R) and $Pic_g(R)$ as subgroups od Cl(R) and $Cl_g(R)$ respectively.

II.1.10 Definition. A graded domain R is said to be a gr-Krull domain if there exists a family of discrete gr-valuation rings V_i, $i \in I$, such that :

1. $R = \bigcap_{i \in I} V_i$
2. For all $x \in h(K^g) - \{0\}$, $\{ i \in I, x$ not a unit of $V_i \}$ is a finite set.

II.1.11. Lemma. Let $I \subset K^g$ be a nonzero graded fractional ideal of R, then $I^\star \subset K^g$ and $I^\star$ is graded. Furthermore $I^\star = \bigcap_{x \in T} Rx^{-1}$, where $T = [R:I] \cap h^\star(K^g)$. In particular, if I is invertible then $I^{-1} = [R:I]$ is graded.

Proof. Since I is nonzero, $I \cap R \neq 0$ follows. Pick $s \in h^\star(I \cap R)$. If $t \in h(I^\star)$ then $t\,s \in R$ hence $t = r\,s^{-1} \in K^g$ for some r in R. It follows that $I^\star \subset K^g$. Now, for arbitrary $t \in I^\star$ we have $t\,I \subset R$. So, if we write $t = t_1 + \ldots + t_n$ with t_i of degree d_i being the homogeneous components of t in K^g, we find $t_i x \in R$ for $i = 1,\ldots,n$. Since the latter holds then for all $x \in h(I)$ we obtain $t_i I \subset R$ or $t_i \in I^\star$. This establishes that $I^\star$ is graded. The remaining statements of the lemma are now easily checked.

II.1.12. Proposition. The graded domain R is a gr-Krull domain if and only if it is a Krull domain in K.

Proof. It is clear that if R is a Krull domain then it will also be a gr-Krull domain ; let us now establish the converse. Since K^g is of the form $k[X,X^{-1}]$ it is a Krull domain in $K = k(X)$. Let $\{\omega_j\}_{j \in J}$ be the family of valuations of K describing K^g in K. Let v_i be the (graded) valuation of K^g corresponding to v_i as in Theorem I.3.13.. Extend v_i

to a valuation $\tilde{v}_i$ on K by putting $\tilde{v}_i(ab^{-1}) = v_i(a)-v_i(b)$.
Consider the family of valuation rings of K, $\{W_l\}_{l\in I\cup J}$ consisting of the valuation rings associated to $\{\omega_j\}_{j\in J} \cup \{\tilde{v}_i\}_{i\in I}$. Obviously the W_k are discrete rank one valuation rings of K. and

$$\bigcap_{l\in I\cup J} W_l = (\bigcap_{i\in I} W_i) \cap K^g = \bigcap_{i\in I} (W_i \cap K^g) = \bigcap_{i\in I} V_i = R.$$

If $x = ab^{-1}$ is arbitrary but nonzero in K then the sets $\{ i \in I, \tilde{v}_i(x) \neq 0\}$ and $\{j \in J, \omega_j(x) \neq 0 \}$ are finite, therefore $\{ l \in I \cup J$, x is not a unit of $W_l \}$ is finite. □

Before turning to the study of special graded Krull domains like gr-Dedekind rings, we include some general properties of graded Krull domains.

<u>II.1.13. Lemma.</u> If R is a gr-Krull domain then $Cl(R) \cong Cl_g(R)$ and $Pic(R) \cong Pic_g(R)$.

<u>Proof.</u> We know that $K^g = k[X,X^{-1}]$ (we may assume that R is non-trivially graded because otherwise there is nothing to prove) where k is a field. By Nagata's theorem Cl(R) is generated by the prime ideals of height one containing an homogeneous nonzero element of R. If P is such a prime ideal then $P_g \neq 0$ hence $P = P_g$ i.e. P is a graded prime ideal. Consequently $Cl(R) \simeq Cl_g(R)$. The statement about $Pic_g(R)$ is an easy exercise, left to the reader.
Some consequences of this lemma have been included in the exercises.
Let us focus on positively graded Krull domains for a moment.

<u>II.1.14. Proposition</u>. Let R be a positively graded factorial Noetherian domain. Then the following properties hold :

1. R_0 is factorial
2. For each $n \in \mathbb{N}$, R_n is a reflexive R_0-module.

Proof.

1. Select a, b $\in R_0$ such that $ab \neq 0$. Then, for some $c \in R$, $Ra \cap Rb = Rc$. It follows immediately from this that c must be homogeneous of degree zero, hence $R_0 c = R_0 a \cap R_0 b$. If the intersection of principal ideals is principal then it follows that R_0 is factorial.

2. Let p be a prime ideal of R_0, of height one. Then :

$$Q_{R_0-p}(R_n) = R_n \underset{R_0}{\otimes} Q_{R_0-p}(R_0) \hookrightarrow (Q_{R_0-p}(R))_n$$

Taking the intersection over all $p \in X^1(R_0)$ yields :

$$\bigcap_{p\in X'(R_0)} Q_{R_0-p}(R_n) \hookrightarrow \left(\bigcap_{p\in X^1(R_0)} Q_{R_0-p}(R)\right)_n = R_n .$$

Since R_n is obviously a finitely generated and torsion free R_0-module, $R_n = \bigcap_{p\in X^1(R_0)} Q_{R_0-p}(R_n)$ yields that R_n is a reflexive R_0-module.

II.1.15. Example (cf. [29]). Let r, s, $t \in \mathbb{N}$ be pairwise relatively prime. Let R be a Krull domain. Then $R[x,y,z]$ with $x^r + y^s + z^t = 0$ is a Krull domain and if R is a regular local ring then $R[x,y,z]$ is factorial. Actually if projective R-modules are free then we obtain $Cl(R) \cong Cl(R[x,y,z])$. It is clear that $R[x,y,z]$ may be graded by putting $R[x,y,z]_n = \Sigma R_i x^{\nu_1} y^{\nu_2} z^{\nu_3}$ where $n = i + \nu_1 st + \nu_2 rt + \nu_3 rs$. (If R is trivially graded then $R_i = 0$ if $i \neq 0$ and $R_0 = R$).

II.1.16. Example. Let C be a Noetherian Krull domain and let M be a finitely generated C-module. Construct a positively graded ring $S_C(M)$ as follows : $(S_C(M))_0 = C$, $(S_C(M))_1 = M$, ..., $S_C(M))_n = n^{th}$-symmetric power of M,... .

A result going back to P. Samuel , [98], states that if each $(S_C(M))_n$ is a reflexive C-module then $S_C(M)$ is a Noetherian Krull domain and $Cl(C) \cong Cl(S_C(M))$. Moreover if M is an invertible C-module then one may define the Z-graded $S'_C(M)$ as before but with $(S'_C(M))_{-n}$ equal to the n^{th} symmetric power of M^{-1}, for $n \geq 0$. Checking along the lines of [29] Theorem 10.II. and Proposition 10.7.c; yields that $S'_C(M)$ is a graded Krull domain too. Note however that the proof given in [29] of the fact that $Cl(C) \to Cl(S_C(M))$ is injective fails for $S'_C(M)$. The study of the kernel of the map $Cl(R_0) \to Cl(R)$ for the Z-graded ring R will be returned to in the following section.

II.2. Generalized Rees Rings and Gr-Dedekind Rings.

This section extends, in the commutative case, some problems and results hinted at in Section A.II.4.

Recall that a graded commutative domain is called a gr-principal ideal ring if every graded ideal is principal. A graded domain is said to be a gr-Dedekind ring if every graded ideal is a projective module.

II.2.1. Theorem. Let R be a graded domain. THe following statements are equivalent.

1. R is a gr-Dedekind ring.
2. R is a Krull domain and nonzero graded prime ideals are maximal graded ideals.
3. R is Noetherian and integrally closed in its field of fractions K, and nonzero graded prime ideals are gr-maximal.
4. Every graded ideal of R is invertible.
5. Every graded ideal of R is (in a unique way) a product of graded prime ideals.
6. R is Noetherian and gr-integrally closed (in the sense of II.1.2.), and every nonzero graded prime ideal of R is gr- maximal.
7. The graded fractional ideals of R form a multiplicative group.
8. R is Noetherian and each localization R_M of R at a gr-maximal ideal M of R is a principal ideal ring.
9. R is Noetherian and each graded localization R_p^g of R at a graded prime ideal P of R is a gr-principal ideal ring.
10. R is Noetherian and each graded localization R_M^g of R at a gr-maximal ideal M of R is a gr-principal ideal ring.
11. All graded R-modules which are graded divisible are gr-injective. Recall that $N \in R\text{-gr}$ is graded divisible if $ax = b$ with $a \in h(R)$, $b \in h(N)$ has a solution in N.

Proof. Most implications may be established in a way similar to the proofs of the corresponding statements in the ungraded case, taking care to insert the standard tricks concerning gradation when necessary. In particular 6 ⇒ 5 is a further refinement of 3 ⇒ 5 which is the straightforward "graded version" of the analogous ungraded statement. 6 ⇒ 3 stems from the fact that the integral closure of a graded domain in its field of fractions is graded, see Theorem I.3.7., Corollary I.3.8. Let us give a more detailed proof only for the implication 7 ⇒ 6.

If I is a graded ideal of R then by 7, I is invertible and hence of finite type. Thus R is gr-Noetherian, hence Noetherian. Now suppose $x \in h(K^g)$ satisfies $x^m + a_{m-1}x^{m-1} + \ldots + a_0 = 0$ with $\deg a_i - \deg a_{i+1} = \deg x$, $i = 0,\ldots,m-1$, and $a_i \in R$. Then x^m is in the fractional graded ideal J of R generated by $1, x, \ldots, x^{m-1}$. It is clear thet $J^2 \subset J$ follows from $a_i \in R$ for $i = 0,\ldots,m-1$. Therefore $J^2 = J$ follows from $R \subset J$. However J is invertible by assumption, thus $J = R$ and it has been proven that R is gr-integrally closed in K^g (and then also in K). Consider $P \neq 0$ in $\mathrm{Spec}_g(R)$, and take $a \in h(R)-P$. Write $H = P+Ra$, $H' = H^{-1}P$. Obviously $P \subset H'$. On the other hand if $y \in H'$ then $a\,y \in HH^{-1}P = P$ yields $y \in P$. This entails that $H' = P$ and $HP = P$ whence $H = R$ by 7. Consequently P is a gr-maximal ideal.

The equivalence 1 ⇔ 3 yields that the notions of gr-local gr-Dedekind ring and discrete gr-valuation ring are equivalent (in view of the results of I.3.) .

An easy elaboration of this fact yields :

II.2.2. Proposition. Let V be a gr-valuation ring of the gr-field R, then the following statements are equivalent :

1. V is a discrete gr-valuation ring.
2. V is a gr-Dedekind ring.

3. V is a gr-principal ideal domain.

4. The gr-maximal ideal M of V is generated by a simple homogeneous element of V.

5. V is a Krull domain with $Cl(V) = 0$.

6. V is factorial

7. V is a gr-Krull domain with $Cl^g(V) = 0$

8. V satisfies the ascending chain condition for principal ideals.

9. V is Noetherian.

10. V is a Krull domain and $ht(M) = 1$.

II.2.3. Lemma. If R is a gr-Dedekind ring then R_0 is a Dedekind ring.

Proof. Let I' be an ideal of R_0, generated by $e_1,\dots,e_n$ say. Since $Re_1+\dots+Re_n$ is a projective R-module, RI' is a direct summand of a gr-free R-module L (see A.I.), generated by elements $v_1,\dots,v_n$ of degree zero. It is clear that L_0 is a free R_0-module and that $(RI')_0 = I'$ is a direct summand of L_0. Hence I' is a projective R_0-module and therefore R_0 is a Dedekind domain.

II.2.4. Lemma. If R is a gr-Dedekind ring then there is an $e \in \mathbb{N}$ such that $R = \bigoplus_{k\in Z} R_{ek}$ with $R_e \neq 0$.

Proof. Easy.

II.2.4. Lemma. If R is a gr-Dedekind ring then every fractional graded ideal of R can be generated by two homogeneous elements, one chosen arbitrarily in the ideal.

Proof. Let I be a graded fractional ideal of R. An argument similar to the ungraded case yields that I may be generated by two elements a and b, $a \in h(I)$ and b not necessary homogeneous. Write $b = b_1 +\dots+b_n$, $\deg(b_i)=t_i$,

$t_1 < \ldots < t_n$. Since $b_i \in I$ we have : ($\star$) $b_i = x_i a + y_i b_1 + \ldots + y_i b_i + \ldots y_i b_n$ for each $i = 1,\ldots,n$ for certain x_i, $y_i \in h(R)$. If $b_i = x_i a$ then we may replace b by $b - b_i$ and repeat the argument. If $b \neq b_i \neq x_i a$ then comparing degrees in ($\star$) yields either $y_i = 0$ which contradicts $b_i \neq x_i a$, or $y_i = 1$ and $x_i a + b_1 + \ldots + \hat{b}_i + \ldots + b_n = 0$. In the latter case $b - b_i \in Ra$ hence we may replace b by b_i. Finally we obtain that I is generated by a and some b_i.

Now we determine the structure of gr-Dedekind rings, first in the strongly graded case i.e. when $RR_1 = R$.

<u>II.2.5. Lemma.</u> Let R be a gr-Dedekind ring which is strongly graded and let P be a graded prime ideal of R. Then $P_0 \subset P^n$ entails $n = 1$.

<u>Proof</u>. Since R is a gr-Dedekind ring $P \neq P^n$ if $n > 1$. Therefore there exists a $t \in Z$ such that $P_t \not\subset P^n$. But then $R_{-t}\, P_t \subset P_0 \subset P^n$ yields $PR_{-t}\, R\, P_t \subset P^n$, hence $R_{-t} \subset P$. However $(R_{-1})^t \subset R_{-t}$ then implies $R_{-1} \subset P$ but $R\, R_{-1} = R$ contradicts $P \neq R$. Consequently if $n > 1$, then $P_0 \not\subset P^n$.

<u>II.2.6. Proposition</u>. Let R be a strongly graded gr-Dedekind ring, then :

1. For each graded prime ideal P of R, $P = RP_0$.
2. For every graded ideal of R is generated by its part of degree zero.
3. If R_0 is a principal ideal ring then R is a gr-principal ideal ring and $R \cong R_0\,[X,X^{-1}]$ where X is an indeterminate.

<u>Proof.</u> 1. and 2. hold for any strongly graded ring, cf. 1.I.3. 3. Immediate from 2. and the results of Section A.I.3.

<u>II.2.7. Theorem.</u> Let R be a strongly graded gr-Dedekind ring then there exists a fractional ideal I of R_0 such that $R = \check{R}_0(I)$ (for the definition of the generalized Rees ring, see A.II.4.) There is a canonical group epimorphism $\pi : Cl(R_0) \to Cl(R)$, the kernel of π is exactly the subgroup of $Cl(R_0)$ generated by the class of I. Moreover , π is an isomorphism

if and only if I is a principal ideal and in this case $R \cong R_0[X,X^{-1}]$. Also, every $\check{R}_0(I)$ for some fractional ideal I of R_0 is a gr-Dedekind ring.

Proof. By A.II.4., Proposition A.II.4.2., the rings $\check{R}_0(I)$ are all Noetherian rings, so to show that $\check{R}_0(I)$ is gr-Dedekind it will suffice to establish that it is integrally closed in its field of fractions, whilst graded prime ideals are gr-maximal.

Let $\bar{S}$ be the integral closure of $S = \check{R}_0(I)$ in its field of fractions $Q(S)$. Suppose $y \in (\bar{S} - S)_n$ for some $n \in Z$. Then $S_{-n}\, y \subset \bar{S}$ but $S_{-n}\, y \not\subset S$, because $SS_{-n} = S$ for all $n \in Z$. Therefore there is a $z \in (\bar{S} - S)_0$. If z satisfies $T^n + a_{n-1}\, T^{n-1} + \ldots + a_0$ with $a_i \in S$ then it also satisfies $T^n + (a_{n-1})_0\, T^{n-1} + \ldots + (a_0)_0 = 0$ with $(a_i)_0 \in S_0$. Since $S_0 = R_0$ is a Dedekind ring, $z \in S_0$ follows, a contradiction. Therefore $\bar{S} = S$. Let us now show that for any prime ideal P_0 of R_0, SP_0 is a gr-maximal ideal of S.

Indeed S/SP_0 is a strongly graded ring of type Z with $(S/SP_0)_0 = R_0/P_0$ being a field. Thus S/SP_0 is a gr-field and hence SP_0 is gr-maximal in S. So far we have established that every generalized Rees ring $\check{R}_0(I)$ is a gr-Dedekind ring. Conversely if R is a gr-Dedekind ring which is strongly graded then it is clear that, in order to invert all elements of $h(R)$ it suffices to invert the elements of $R_0 - \{0\}$. Take $x_{-1} \in R_{-1} - \{0\}$. Then $x_{-1}^{-1} \in K^g$ yields that $K^g = K_0[x_{-1}, x_{-1}^{-1}]$ where K_0 is the field of fractions of R_0. Let J be the largest fractional ideal of R_0 such that $J\, x_{-1} \subset R$ and I the largest (fractional) ideal of R_0 such that $I\, x_{-1}^{-1} \subset R$. Then : $Jx_{-1} = R_{-1}$, $I\, x_{-1}^{-1} = R_1$. From $R_{-1}\, R_1 = R_0$ it follows now that $IJ = JI = R_0$ i.e. $J = I^{-1}$. Similarly, if I_2 is the ideal of R_0 given by $(R_2 : x_1^2)$ then $I_2 I^{-1} \subset I$ follows from $R_{-1}\, R_2 \subset R_1$, hence $I_2 \subset I^2$. On the other hand, $I^2\, x_1^2 = (Ix_1)^2 \subset R_2$ yields $I^2 \subset I_2$. Repeating this argument, we obtain $R \cong \check{R}_0(I)$.

Finally , let $\pi : Cl(R_0) \to Cl(R)$ be defined as the class map obtained from mapping an ideal of R_0 to its extension in R. That π is surjective follows from Proposition II.2.6.. Certainly the class of I, $\bar{I}$ say, is contained in Ker π since $\check{R}_0(I).X^{-1} = \check{R}_0(I).I$.

If H is an ideal of R_0 such that RH is principal then write RH = Rh for some $h \in h(RH)$, of degree t say. Taking parts of degree n yields (RH) = $= HI^nX^n = R_{n-t}h'$ where $h' = hX^t$ (we identified R and $\check{R}_0(I)$ here). Consequently : $HI^n = I^{n-t}h$ and $\bar{H}$ is in the subgroup of $Cl(R_0)$ generated by the class of I. Finally if $I = R_0 i$ is principal then $R \cong R_0 [iX,(iX)^{-1}]$ and $Cl(R) \cong ClR_0)$. Conversely if π is also injective then $RI = RX^{-1}$ entails that I is principal.

<u>II.2.8. Proposition.</u> If $R = \check{R}_0(I) \cong \check{R}_0(J)$ then I and J represent the same element of $Cl(R_0)$. Conversely, if I and J are in the same class then $\check{R}_0(I) \cong \check{R}_0(J)$.

<u>Proof</u>. Without loss of generality we may assume that I and J are R_0- integral ideals. From $RI = RX^{-1}$ we obtain $I = R_1X^{-1} = JYX^{-1}$ because $R_1 = JY$ where $\check{R}_0(J) = \sum_{n\in Z} J^nY^n$. Therefore $IJ^{-1} = R_0Y^{-1}$.

Conversely, if I = Jz for some $z \in R_0$ then :

$$\check{R}_0(I) = \sum_{n\in Z} (Jz)^n X^n = \sum_{n\in Z} J^n (zX)^n \cong \sum_{n\in Z} J^n Y^n = \check{R}_0(J).$$

<u>II.2.9. Remark</u>. A gr-Dedekind ring R which is positively graded is the form k [X] where $R_0 = k$ is a field and X a variable.

<u>Proof</u>. R_+ is a nonzero graded prime ideal of R hence it is gr-maximal. Therefore $R_0 = R/R_+$ is a gr-field hence a field. So R_+ is the unique graded prime ideal of R hence R is a gr-principal ideal ring. If $R_+ = Ra$ then $R = R_0 [a]$ and it is clear that a is transcendental over R_0 since deg $a \neq 0$.

Recall that $R^{(e)}$, $e \in Z$, is the $\underset{m\in Z}{\oplus} R_{me}$ with the gradation $(R^{(e)})_n = R_{en}$. It turns out that in the abscence of the strongly graded property, i.e. if $RR_1 \neq R$, then for some $e \in \mathbb{N}$ the structure of $R^{(e)}$ is nice.

II.2.10. Lemma. If R is a gr-Dedekind ring then $R^{(e)}$ is a gr-Dedekind ring for any nonzero $e \in \mathbb{N}$.

Proof. By Corollary A.II.3.13., $R^{(e)}$ is Noetherian. We may consider $\underline{R}^{(e)}$ as a subring of $\underline{R}$, neglecting the gradation, and the graded field of fractions $Q^g(R^{(e)})$ of $R^{(e)}$ may be considered as a subring of $Q^g(R)$. Therefore an $x \in Q^g(R^{(e)})$ which is integral over $R^{(e)}$ is integral over R; thus in R. Obviously $Q^g(R^{(e)}) = S^{-1}R^{(e)}$ where $S = R_0 - \{0\}$. Hence $Q^g(R^{(e)}) = Q^g(R)^{(e)}$ and $R \cap Q^g(R)^{(e)} = R^{(e)}$. This yields that $x \in R^{(e)}$. Moreover the correspondence $P \to P^{(e)}$ defines a bijective correspondes between $\mathrm{Spec}_g(R)$ and $\mathrm{Spec}_g(R^{(e)})$, the inverse correspondence being given by $Q \to \mathrm{rad}(R(\sum_{m\in Z} Q_{me}))$. Therefore, the graded prime ideals of $R^{(e)}$ are gr-maximal. The lemma now follows from Theorem II.2.1,3.

II.2.11. Lemma. If R is a gr-Dedekind ring and P is a graded prime ideal of R then the graded ring of fractions at P is obtained by localizing at the multiplicatively closed set $R_0 - P_0$ and the localized ring is a discrete gr-valuation ring.

Proof. By Theorem II.2.1. it is clear that $Q^g_P(R)$ is a discrete gr-valuation ring. Now R/P is a graded field. If $\bar{y} \in (R/P)_n$ is nonzero then there is an $\bar{y}' \in (R/P)_{-n}$ which is also nonzero and if y and y' are representatives for $\bar{y}$ and $\bar{y}'$ in R then it suffices to invert $yy' \in R_0$ in order to invert y. This proves the statement in case R/P is non-trivially graded. If R/P is trivially graded then P contains all R_n with $n \neq 0$.

Then the gr-maximal ideal M of $Q_P^g(R)$ also contains all $(Q_P^g(R))_n$ with $n \neq 0$ (note that in this case $R-P = R_0-P_0$!). By Proposition I.3.14. this is impossible unless $Q_P^g(R)$ is positively or negatively graded. But then R is positively or negatively graded hence of the form $k[X]$ because of Remark II.2.9. . In this case $P = (X)$ and the statement of the lemma is then trivial.

<u>II.2.12. Theorem.</u> Let R be a gr-Dedekind ring, then there exists an $e \in \mathbb{N}$ and a fractional ideal I of R_0 such that $R^{(e)} = \check{R}_0(I)$.

<u>Proof.</u> If $RR_1 = R$ then $e = 1$ and Theorem II.2.7. holds . If $RR_1 \neq R$ write $RR_1 = P_1^{\nu_1} \ldots P_n^{\nu_n}$. Since R/P is a gr field for every graded prime ideal $P \neq 0$ of R it follows that each such P contains $\underset{m \in Z}{\oplus} R_{me+r}$ for some $e \in \mathbb{N}$, some r, $0 < r < e$. The argument used in the proof of Lemma II.2.11. states that P cannot contain all R_t with $t \neq 0$, hence $e \neq 0$. In other words for each P_i containing R_1 we find an $e_i \gneq 0$ in $\mathbb{N}$ such that $P_i^{(e_i)}$ does not contain $(R^{(e_i)})_1$. Let e be the smallest common multiple of the e_i $i = 1 \ldots n$. On the other hand, if $P \not\supset R_1$ then $P \not\supset R_m$ for any $m > 0$ and $P^{(e)} \not\supset (R^{(e)})_1$. By Lemma II.2.10 it follows that $R^{(e)}$ is a gr-Dedekind ring and since $R^{(e)}(R^{(e)})_1 = R^{(e)}$ follows from the fact that $(R^{(e)})_1$ is not contained in any graded prime ideal of $R^{(e)}$ it follows that Theorem II.2.7. applies to $R^{(e)}$; whence the statement of the theorem follows.

Before turning to the study of extensions of gr-Dedekind rings let us introduce the graded R-length of an $M \in R$-gr. Let R be any graded domain and let $E \in R$-gr be finitely generated. Denote by $tE \in R$-gr the submodule of E consisting of the torsion elements. Then $M = E/tE$ is a finitely generated torsion-free graded module. Let $\{v_1,\ldots,v_n\}$ be a maximal set of homogeneous elements of M among a given set of homogeneous generators

$\{y_1,\ldots,y_m\}$ such that $\{v_1,\ldots,v_n\}$ is linearly independent. $L = Rv_1 \oplus\ldots\oplus Rv_n$ is gr-free maximal submodule of M. Let's see that the dimension of L, is uniquely determined. For each $y_j \in \{y_1,\ldots,y_m\}$, there exist elements $a_j, b_{1j},\ldots,b_{nj} \in h(R)$ not all 0, such that

$$a_j y_j + b_{1j} v_1 + \ldots + b_{nj} v_n = 0.$$

Evidently, $a_j \neq 0$, which proves that $a_j y_j \in L$. Let $a = a_1 \ldots a_m$. We have a monomorphism of aM into L. Let $u_1,\ldots,u_s$ be another maximal set of linearly independent homogeneous elements of M and suppose $s > r$. Then the elements $au_1,\ldots,au_s$ are not linearly indenendent and there exist elements $c_j \in h(R)$, $j = 1,\ldots,s$ not all such that $\sum_{j=1}^{s} c_j\, a\, u_j = a \sum_{j=1}^{s} c_j\, u_j = 0$. This yields $\sum_{j=1}^{s} c_j u_j = 0$, contradiction. Similarly, by interchanging the roles of the v_i's and the u_j's, one proves that the case $r > s$ is impossible too. So $r = s$.

II.2.13. Definition. The dimension of the gr-free module L , just defined, is called the graded R-length of the finitely generated module E. If E is not finitely generated, define its graded R-length as the supremum of the graded ranks of the elements of the family of finitely generated submodules of E.

II.2.14. Lemma. Let R be a graded noetherian ring without zero divisors in which every nonzero graded prime ideal is graded maximal and M a finitely generated graded R-torsion module. Then the graded R-length of M is finite.

Proof. As M is a graded torsion module, each associated graded prime ideal of M is nonzero, hence maximal (for informations about the graded associated prime ideals of a graded module, cf. section A.II.7.) . The lemma follows then from the fact that, since the category of graded R-modules is a Grothendieck category, the Jordan-Hölder theorem holds in it.

II.3.15. Lemma. Let R be a graded ring, T a graded R-module, (T_i) an ascending, filtering family of submodules of T, such that their union is T. Then the graded length of T on R is equal to the supremum of the graded lengths of the T_i on R.

II.2.16. Lemma. Let R be a graded Noetherian ring without zero divisors such that every nonzero graded prime ideal of R is graded maximal, M a torsion-free graded R-module of finite graded rank n, and a a nonzero homogeneous element of R. Then R/aR is a graded R-module of finite length and we have :

$$\text{length } (M/aM) \leq n.\text{length } (R/aR).$$

II.2.17. Theorem. ("graded" Krull-Akizuki) Let R be a graded noetherian ring without zero divisors such that every nonzero graded prime ideal is graded maximal, K^g its graded field of fractions L^g a graded extension of K^g of finite degree and S a graded subring of L^g containing R. Then S is Noetherian, and every nonzero graded prime ideal of S is gr-maximal.

The proofs of the preceding assertions are rather similar to the ungraded equivalents (cf. [13]). Caution ! graded free modules are graded and free but the converse does not hold : as a consequence the graded rank can be different from the rank !

From the above theorem we obtain the following two corollaries.

II.2.18. Corollary. The graded integral closure of a gr-Dedekind ring in a graded extension of finite degree of its graded field of fractions is a gr-Dedekind ring.

II.2.19. Corollary. Let R be a gr-Dedekind ring and S a multiplicative closed subset of $h^\star(R)$. Then the graded localization of R at S, $Q_S^g(R)$, is a gr-Dedekind ring.

II.2.20. Remark. If R is a generalized Rees ring, say $R = \check{R}_0(I)$, then the integral closure of R, $\bar{R}$, in an extension of the field of fractions is a generalized Rees ring too. This follows from the fact that, since $R_{-1} R_1 =$ $= R_0, 1 \in (\bar{R})_1(\bar{R})_{-1}$ and hence $(\bar{R})_1(\bar{R})_{-1} = (\bar{R})_0$.
If we denote by $K = K_0 [X,X^{-1}]$ and $L = L_0 [Y,Y^{-1}]$ the graded fields of fractions of R and $\bar{R}$ respectively and if we put $\bar{R} = (\check{\bar{R}})_0(J)$, then the class of $\bar{R}I$ in $C(\bar{R})$ is equal to the class of J^e, where $e = \dim(L_0 [Y,Y^{-1}] : L_0 [X,X^{-1}])$ this is verified in a straightforward way.

II.2.21. The Graded Norm.

Let R be a gr-Dedekind ring in Q(t) where t is a variable over the rational numbers Q. Let S be the graded integral closure of R in a finite extension field L of Q(t). We may suppose that L is the field of fractions of S, otherwise we restrict L (note the connection to graded realizable fields). Let P be a graded prime ideal of S and put $p = R \cap P$. We obtain the following inclusion of graded fields :

$$R/p = (R/p)_0 [X,X^{-1}] \to S/P = (S/P)_0 [Y,Y^{-1}] ,$$

hence deg X ≥ deg Y. (We assume that the graded fields have been written down such that deg X ≥ 0, deg Y ≥ 0 (i.e. up to interchanging X and X^{-1}, Y and Y^{-1} if necessary). So we find a natural number e_p such that $X = \lambda Y^{e_p}$ with $\lambda \in (S/P)_0$. Clearly this e_p equals the rank $[(S/P)_0[Y,Y^{-1}] : (S/P)_0[X,X^{-1}]]$. The norm of P_0 in S_0 is the number of elements in the residue field $S_0/P_0 = (S/P)_0$ which is a finite extension of the prime field R_0/P_0. It is clear that addition of X,X^{-1} made S/P into an infinite ring, however it makes sense to define the graded norm of P as follows $N(P) = N_0(P_0)e_p, N(P)^n =$ $= N(P)^n$ and if $I = P_1^{v_1} \ldots P_r^{v_r}$ then we put $N(I) = \prod_{i=1}^r N(P_i)^{v_i}$.

II.2.22. The graded ζ function of the (graded) integral closure of a graded Dedekind ring in Q(t).

Definition. $\zeta_g(s) = \sum_I N(I)^{-s}$, (s a complex variable) where the summation is over graded integral ideals of S.

The definition of the norm implies that $\zeta_g(s)$ is dominated by $\zeta_0(s)$, where $\zeta_0(s)$ is the zêta function of the part of degree 0.

II.2.23. Proposition. For Re(s) > 1 we have :

1. $\zeta_g(s)$ converges absolutely and almost uniformly.
2. $\zeta_g(s) = \prod_{P \in \mathrm{Spec}_g(S)} (1-N(P)^{-1})^{-1}$
3. $\zeta_g(s) = \zeta_0(s) . \prod_{e_p \neq 1} \dfrac{1-N(P_0)^{-s}}{1-(N(P_0)e_p)^{-s}}$

Proof.

1.,2. Slight modification of the proof in the ungraded case, cf. for example [99]. Note that the product appearing in 3 is finite.

3. Obvious from 2.

Let us end this section with some results on the class group.

II.2.24. Proposition. If R is a gr-Dedekind ring then there exists $e \in \mathbb{N}$ such that the following diagram of group homomorphisms is commutative.

$$\begin{tikzcd} 1 \arrow[r] & <\bar{I}> \arrow[r] & Cl(R_0) \arrow[r] \arrow[dr] & Cl(R^{(e)}) \arrow[r] \arrow[d] & 1 \\ & & & Cl(R) & \end{tikzcd}$$

Where $\bar{I}$ is the class of I in $Cl(R_0)$ and I is the fractional ideal of R_0 describing the structure of $R^{(e)}$.

Proof. Take e and I as in Theorem II.2.12. Identify $R^{(e)}$ with a subring of R by forgetting the gradation. Extension of ideals from R_0 to $R^{(e)}$, "from" $R^{(e)}$ to R and from R_0 to R defines the arrows in the diagram. One easily checks commutativity of the diagram.

If $RR_1 \neq R$ then we refer to the ideal R_1R_{-1} of R_0 as being the <u>discriminator</u> of the gr-Dedekind ring R and it will be denoted by $\delta(R)$.

<u>II.2.15. Lemma.</u> Let R be a gr-Dedekind ring and κ be an idempotent kernel functor on the category of graded R-modules. Then, if S denotes $Q^g_\kappa(R)$, S is a gr-Dedekind ring with

$$Cl(S) = Cl(R)/ < \{\bar{P},\ P \in L(\kappa)\} >$$

where $L(\kappa)$ is the filter of κ, and P is graded prime.

<u>Proof.</u> Since R is a gr-Dedekind ring, κ has property T(A II.9) and the localization Q^g_κ is perfect. For every ideal L of S, we then have $L = S.(L\cap R)$ and therefore we obtain an epimorphism,

$$\gamma = Cl(R) \to Cl(S).$$

If J is an ideal of R such that $SJ = Sa$ for some homogeneous $a \in S$, then there exists an ideal $L \in L(\kappa)$ such that $La \subset J$. The graded Dedekind property yields $La = J.H$ where H is such that $SH = S$ i.e. $H \in L(\kappa)$. H may be written as $H = P_1^{\nu_1} \dots P_n^{\nu_n}$ and each P_i $i = 1,\dots,n$ belongs to $L(\kappa)$. So $J = H^{-1}$ La and $\bar{J} = \bar{H}^{-1}\bar{L}$. Now $\bar{L}$ and $\bar{H}$ belong to $< \{ \bar{P},\ \bar{P}$ is a graded prime belonging to $L(\kappa)\}>$, thus $\bar{J}$ belongs to it too, hence Ker γ equals this subgroup.

<u>II.2.26. Corollary.</u> Let R be a gr-Dedekind ring and S a multiplicative closed subset of $h^\star(R)$. Then : $Cl(Q^g_S(R)) = Cl(R)/<\{\bar{P}$, P graded prime and $P \cap S \neq \Phi\}>$.

<u>II.2.27. Corollary</u>. Let R be a gr-Dedekind ring and I a graded ideal of R. Then : $Cl(Q^g_I(R)) = Cl(R)/< \bar{P}_1,\dots,\bar{P}_s >$ where $P_1,\dots,P_s$ are the graded prime ideals containing I.

<u>Proof.</u> In this case $L(\kappa_I)$ contains only a finite number of graded prime ideals containing I.

II.2.28. Corollary. We have a commutative diagram.

$$\begin{tikzcd}
 & Cl(R^{(n)}) \arrow[r] & Cl(R) \arrow[r] & Cl\left(Q^g_{\delta(R)}(R)\right) = Cl(R)/<\bar{P}_1,\dots,\bar{P}_S> \arrow[d] \\
Cl(R_o) \arrow[ur] \arrow[r] & Cl(Q^g_{\delta(R)}(R_o)) = Cl(R_o)/<\bar{p}_1,\dots,\bar{p}_S> \arrow[rr] & & Cl\left(Q^g_{RR_1}(R)\right) = \left(Cl(R_o)/<\bar{p}_1,\dots,\bar{p}_S>\right)/<\bar{I}_{\delta(R)}>
\end{tikzcd}$$

where $P_1,\dots,P_S$ are the graded prime ideals containing R_1; $p_1,\dots,p_S$ are the prime ideals of R_o containing $\delta(R)$ i.e. $p_i = P_i \cap R_o$, and $\bar{I}_{\delta(R)}$ is the class of the fractional ideal of $Q^g_{\delta(R)}(R_o)$ defining the generalized Rees ring structure of $Q^g_{RR_1}(R)$.

Proof. Apply corollary II.2.27. with $I = RR_1$ and note that $Q^g_{\delta(R)}(R) \cong Q_{RR_1}(R)$ as graded rings because of Lemma II.2.11.

II.2.29. Proposition. Let R be a gr-Dedekind ring such that there is an $N \in \mathbb{N}$ such that the generalized Rees ring $R^{(N)}$ has the property that $Cl(R^{(N)}) \to Cl(R)$ is injective. Then

$$Cl(Q^g_{RR_1}(R)) \cong Cl(R_o)/<\bar{p}_1,\dots,p_S,\ \bar{I}>,$$

where I is the ideal of R_o determining the structure of the generalized Rees-ring $R^{(N)}$ and p_i, $i = 1,\dots,S$, are the primitive factors of the discriminator in R_o.

Proof. In Corollary II.2.28., we write $I_{\delta(R)} = Q^g_{\delta(R)}(I')$ for some ideal I' of R_o. Commutativity of the diagram yields that I' is in the kernel of the composition $Cl(R_o) \to Cl(R^{(N)}) \to Cl(R) \to Cl(Q^g_{\delta(R)}(R))$. Hence, if we write $I' = p_1^{\nu_1}\dots p_S^{\nu_S} q_1^{\omega_1}\dots q_t^{\omega_t}$, then $RI' = P_1^{e_1\nu_1}\dots P_S^{e_S\nu_S'} Q_1^{\omega_1}\dots Q_t^{\omega_t}$ (where e_j, $j = 1,\dots,s$ are the ramification indices of $p_1,\dots,p_S$ respectively) and $SI' = (SQ_1)^{\omega_1}\dots(SQ_t)^{\omega_t}$ where S denotes $Q^g_{\delta(R)}(R)$.

Consequently, $(SQ_1)^{\omega_1} \ldots (SQ_t)^{\omega_t}$ is a product of the $P_1,\ldots,P_s$ up to principal R-ideals. Since $Cl(R^{(N)}) \to Cl(R)$ is injective it follows that the class of $q_1^{\omega_1} \ldots q_t^{\omega_t}$ is in the group generated by $\bar{p}_1,\ldots,\bar{p}_s$, $\bar{I}$; whence : $\bar{I}' \in \langle \bar{p}_1,\ldots,\bar{p}_s,\bar{I} \rangle$. Therefore :

$$(Cl(R_0)/ \langle \bar{p}_1,\ldots,p_s \rangle) / \langle \bar{I}_{S(R)} \rangle = Cl(R_0) / \langle \bar{p}_1,\ldots,\bar{p}_s, \bar{I} \rangle.$$

II.3. References, Comments, Exercises.

II.3.1. References.

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II.3.2. Exercises.

1. Let R be positively graded Noetherian and suppose that R_0 is a field. Let $M \in R$-gr be finitely generated. Then M is free if and only if $Q_{R-R_+}(M)$ is projective.

2. Let R be a positively graded Noetherian ring and let $M \in R$-gr be finitely generated. Then M is free if and only if $R_0 \otimes_R M$ is a free R_0-module and $\mathrm{Tor}_1(R_0,M) = 0$.

3. Let R be a positively graded Krull domain such that R_0 is a field k. Consider a field extension k' of k. If $R' = R \otimes_k k'$ is a Krull domain then R' is a faithfully flat R-module and $\mathrm{Cl}(R) \to \mathrm{Cl}(R')$ is injective.

4. (I. Beck) Let A be the integral closure of the Noetherian integral domain B and let $\{a_1,\ldots,a_r\}$ be a subset of A; then $B:(B:\sum_j B a_j) \subset A:(A:\Sigma Aa_j)$. Deduce from this fact that a divisorial (prime) ideal of A intersects B in a divisorial (prime) ideal.

5. Consider Krull domains A and B, $A \subset B$. We say that P D E (Pas d'éclatement = no blowing up) is satisfied if $P \in (\mathrm{Spec}\, B)^1$ entails $\mathrm{ht}(P \cap A) \leq 1$. Condition P.D.E is equivalent to demanding that principal ideals of A map to principal ideals of B under the map (div)i induced by the inclusion $i : A \to B$.

   Show that PDE holds in any of the following cases :

   a. B is a flat A-module

   b. B is an integral extension of A

   c. B is a subintersection of A

   Deduce the graded analo of the above statements.

6. Examples. (Cf. [29] and also P. Samuel's Tata lectures. Let k be a field of characteristic different from z.

   (a) If F is a non-degenerate quadratic form in $k[X_1,X_2,X_3]/(F)$ $Cl(k[X_1,X_2,X_3]/(F)$ is cyclic of order 2 or else $k[X_1,X_2,X_3]/(F)$ is factorial. If k is algebraically closed then $Cl(k[X_1,X_2,X_3]/(F)) = Z/2Z$.

   (b) If F is a non-degenerate quadratic form in $k[X_1,X_2,X_3,X_4]$ then $Cl(k[X_1,X_2,X_3,X_4]/(F))$ is either infinite cyclic or zero. It is infinite cycle for example in the case where k is algebraically closed.

   (c) If F is a non-degenerate quadratic form in $k[X_1,\ldots,X_n]$ with $n \geq 5$, then $k[X_1,\ldots,X_n]/(F)$ is a factorial ring.

   Exercise. $Q[X_1,X_2,X_3,X_4]/(X_1^2 + X_2^2 + X_3^2 + 2X_4^2)$ is a factorial graded ring. However the form $X_1^2 + X_2^2 + 2X_3^2 + 2X_4^2$ yields a graded ring $Q[X_1,\ldots,X_4]/X_1^2 + X_2^2 + 2X_3^2 + 2X_4^2$ with class group Z.

7. Let D be a commutative graded domain. Check whether the following conditions are equivalent.

   a. D is a gr-Dedekind ring

   b. All graded ideals of D are divisorial.

8. Let C be a Krull domain and let I be a divisorial ideal of C. Consider the ring $S_C(I)$ defined in II.1.16. Check whether the kernel of $Cl(C) \to Cl(S'_C(I))$ is generated by the class of I.

9. Let C be a Krull domain and let M be a divisorial C-module. Is the injective hull of M in C-mod again divisorial. What about maximal rings of quotients ? What about localizations at torsion theories. Call a torsion theory divisorial if the associated filter of ideals

has a filter basis consisting of divisorial ideals. Is the localization of a divisorial module at a divisorial torsion theory divisorial over the localized ring ?

10. Let R be a commutative graded semiprime ring such that 0 and 1 are the only idempotent elements of R. Show that the invertible elements of R are homogeneous.

II.3.3. Comments

1. If G is a torsion free abelian group satisfying the ascending chain condition on cyclic subgroups then the theory of G-graded Krull domains parallels the Z-graded theory. The zealous reader may derive most of the results in Section II.3. for the G-graded case, using the tools of chapter A.

2. The generalized Rees rings introduced here may be further generalized to the non-commutative case, e.g. the P.I. case, yielding an interesting new class of rings. This class of rings consists of subrings of certain crossed products $R[G,\varphi,c]$ where $\varphi : G \to \mathrm{Aut}(R)$ is a group morphism, $c : G \times G \to U(Z(R))$ a 2-cocycle and when R is usually a "nice" P.I. ring like a tame order, an HNP ring, a maximal order over a Krull domain, ... For all this we refer to a recent paper by L. Le Bruyn and F. Van Oystaeyen, [64].

III: Local Conditions for Noetherian Graded Rings

In this chapter, all rings considered will be commutative and Noetherian unless explicitely mentioned otherwise. All graded rings considered will be of type Z.

III.1. Injective dimension of graded rings.

Let us introduce the following terminology : a ring R is called a gr-local ring if it has exactly one maximal graded ideal (these ideals are said to be gr-maximal) in the set of proper graded ideals of R. If $R = \bigoplus_{i \in Z} R_i$ is gr-local with M the maximal graded ideal then R_0 is a local ring and $M \cap R_0$ is the maximal ideal of R_0.

Let P be a prime ideal of R and let S be the multiplicatively closed set $S = h(R)-P$. If $M \in R$-gr we denote the graded module of the fractions $Q^g_S(M)$ by $Q^g_{R-P}(M)$. If $Q = (P)_g$ then : $Q^g_{R-P}(M) = Q^g_{R-Q}(M)$. Clearly $Q^g_{R-P}(R)$ is gr-local with the maximal graded ideal $(P)_g Q^g_{R-P}(P)$. Let $k^g(R)$ denote the graded division ring $Q^g_{R-P}(R)/(P)^g Q^g_{R-P}(R)$.

III.1.1. Lemma. Let P be a prime ideal of R, then :

1. For any $M \in R$-gr, $Q_{R-P}(\underline{M}) = 0$ if and only if $Q_{R-(P)_g}(\underline{M}) = 0$ if and only if $Q^g_P(M) = 0$.
2. Put supp $M = \{ P \in \text{Spec } R,\ Q_{R-P}(\underline{M}) = 0 \}$, then $P \in$ supp M if and only if $(P)_g \in$ supp M.
3. If P is a graded prime ideal then : $E^g_{Q^g_{R-P}(R)}(k^g(P)) \cong E^g_R(R/P)$
4. Let P be a graded prime ideal and let $A_i = \{x \in E^g(R/P),\ P^i x = 0 \}$ then the following properties hold :

   a. A_i is a graded R-submodule of $E^g(R/P)$ and $E^g(R/P) = \bigcup_{i \geq 1} A_i$.

b. A_{i+1}/A_i is a graded $k^g(P)$-module

c. $A_1 \cong k^g(P)$.

5. Any injective object in R-gr is a direct sum of some $E^g(R/P)(n)$, for some graded prime ideals P of R and some $n \in Z$.
6. Denote by $\text{gr-inj.dim}_R M$ the injective dimension of M in the category R-gr. If $S \subset h(R)$ is any multiplicatively closed set $M \in$ R-gr, then $\text{gr. inj. dim}_{S^{-1}R} S^{-1}M \leq \text{gr. inj. Dim}_R M$

Proof. 1., 2., 3. are easy consequences of the foregoing chapters;4., 5., 6., follow as in the ungraded case.

Let $M \in$ R-gr. and consider minimal injective resolutions in R-gr and R-mod resp. :

$$0 \to M \to E_0{}^g \to E_1{}^g \to \dots$$

$$0 \to M \to E_0 \to E_1 \to \dots$$

For a graded prime ideal P of R, let $\mu_n^g(P,M)$, resp $\mu_n(P,M)$, be the number of copies of $E_R^g(R/P)$ in E_n^g, resp. of $E_R(R/P)$ in E_n.

III.1.2. Propositions. Let $M \in$ R-gr, P be a graded prime ideal of R. For notational convenience we write Q^g for the localization functor on R-gr associated to h(R)-P, then :

1. The group $Q^g\ \text{Ext}_R^n(R/P,M)$ is a free graded $k^g(P)$-module of rank $\mu_n^g(P,M)$.
2. $\mu_i^g(P,M) = \mu_i(P,M)$
3. $\text{inj. dim}_{Q^g(R)} Q^g(M) = \text{gr. inj dim}_{Q^g(R)} Q^g(M)$
4. If $S \subset h(R)$ is any multiplicatively closed set such that $P \cap S = \emptyset$ then $\mu_n^g(P,M) = \mu_n(S^{-1}P, S^{-1}M)$ for all $n \geq 0$.

Proof. We have :

$\text{Ext}_R^i(R/P,M) = \text{EXT}_R^i(R/P,M) = \text{HOM}_R^i(R/P,Q_i^g) = \text{Hom}_R(R/P,\ Q_i^g)$, hence

$Q^g\text{Ext}_R^i(R/P,M) = Q^g\text{Hom}_R(R/P,Q_i^g) = \text{Hom}_{Q^g(R)}(k^g(P),\ Q_i^g)$

and the latter is a free $k^g(P)$-module.

If P_1 is a graded prime ideal of R such that $P_1 \neq P$ then $Q^g\mathrm{Hom}(R/P, E_R^g(P_1))=0$, hence $Q^g\mathrm{Hom}_R(R/P, Q_i^g)$ is a free $k^g(P)$-module of rank equal to $\mu_i^g(P,M)$. Since $\mathrm{Ext}_R^i(R/P,M)$ is a free $k(P)$-module of rank $\mu_i(P,M)$ whilst $Q\mathrm{Ext}(R/P,M)$ is the localization of $Q^g\mathrm{Ext}(R/P,M)$ at P (Q the localization in R-mod associated to P). 2) follows immediately and 3) follows from 2).
For the statement 4) we apply the statement 1).

III.1.3. Proposition. Let $M \in R$-gr, P a graded prime ideal of R and $a \in h(R)$ a nonzero divisor on M and R, then for all $n \geq 0$:

$$\mu_n^g(P/aR,\ M/aM) = \mu_{n+1}^g\ (P;M)$$

Proof. Let $0 \longrightarrow M \longrightarrow E_0^g \xrightarrow{d_0} E_1^g \xrightarrow{d_{-1}} E2^g \longrightarrow \cdots$
be a minimal injective resolution of M on R-gr and $N = d_0(E_0{}^g)$
If $K \in R$-gr is arbitrary, then $\mathrm{Hom}_R\ (R/aR,\ K) \simeq (K:a) = \{\ x \in K \mid ax = 0\}$
From the commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E_0{}^g & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_a} & & \downarrow{\scriptstyle \varphi_a} & & \downarrow{\scriptstyle \varphi_a} & & \\ 0 & \longrightarrow & M & \longrightarrow & E_0{}^g & \longrightarrow & N & \longrightarrow & 0 \end{array} \qquad \text{where } \varphi_a(x) = ax$$

we have $(N:a) \simeq M/aM$ so $\mathrm{Hom}(R/aR,N) \simeq M/aM$.
From the exact sequence

$$0 \longrightarrow R \xrightarrow{\varphi_a} R \longrightarrow R/aR \longrightarrow 0$$

we deduce that $\mathrm{Ext}_R^i(R/aR,\ M) = 0$ for all $i > 1$.
This implies that the sequence :

(1): $0 \longrightarrow \mathrm{Hom}_R(R/aR,N) \longrightarrow \mathrm{Hom}_R\ (R/aR,\ E_1{}^g) \longrightarrow \mathrm{Hom}_R\ (R/aR,\ E_2{}^g) \longrightarrow \ldots,$
is exact. Because $\mathrm{Hom}_R\ (R/aR,\ E_i{}^g) \simeq (E_i{}^g:a)$ is an injective module over the ring R/aR we have that (1) is a minimal injective resolution of the R/aR-module $M/aM \simeq \mathrm{Hom}_R\ (R/aR,N)$.

Now, if P is a graded prime ideal such that $a \in P$, then it is easily verified that $\mathrm{Hom}_R(R/aR, E(R/P))$ is the injective envelope of the R/aR-module $R/P = (R/aR)/(P/aR)$. Therefore, from (1) we obtain that $\mu_i(P/aR, M\ M) = \mu_{i+1}(P,M)$.

III.1.4. Proposition. Let P be a prime ideal of R such that $P \neq (P)_g$ then for all $n \in \mathbb{N}$ and $M \in R\text{-gr}$ we obtain : $\mu_n((P)_g,M) = \mu_{n+1}(P,M)$.

Proof. By Proposition III.1.2.4. we have : $\mu_n((P)_g,M) = (Q^g((P)_g),Q^g(M))$ while $\mu_{n+1}(P,M) = \mu_{n+1}(Q^g(P),Q^g(M))$. So we may replace R and M by $Q^g(R)$ and $Q^g(M)$ resp., i.e. we may suppose that R is gr-local and $(P)_g$ gr-maximal ideal of R. In that case P is a maximal ideal of R. Since $R/(P)_g$ is a graded division ring, there exists $a \in R$ such that $P = (P)_g+Ra$.

From the exact sequence :

$$(\star)\quad 0 \longrightarrow R/(P)_g \xrightarrow{m_a} R/(P)_g \longrightarrow R/P \longrightarrow 0$$

where m_a denotes multiplication by a, we obtain : $\ldots \to \mathrm{Ext}^n(R/(P)_g,M \xrightarrow{m_a}$

$$\longrightarrow \mathrm{Ext}^n(R/(P)_g,M) \longrightarrow \mathrm{Ext}^{n+1}(R/P,M) \longrightarrow \ \ldots \ \to \mathrm{Ext}^{n+1}(R/(P)_g,M) \xrightarrow{m_a}$$

$$\longrightarrow \mathrm{Ext}^{n+1}(R/(P)_g,M) \longrightarrow \ \ldots$$

Note that $\mathrm{Ext}^n(R/(P)_g,M) = \mathrm{EXT}^n(R/(P)_g, M)$ is a graded $R/(P)_g$-module, hence $\mathrm{Ext}^n(R/(P)_g,M$ is a free graded $R/(P)_g$ module. Consequently $m_a : \mathrm{Ext}^n(R/(P)_g,M) \longrightarrow \mathrm{Ext}^n(R(P)_g,M)$ is monomorphic.

The long exact sequence obtained above yields a short exact sequence:

$$0 \to \mathrm{Ext}^n(R/(P)_g,M) \xrightarrow{m_a} \mathrm{Ext}^n(R(P)_g,M) \longrightarrow \mathrm{Ext}^{n+1}(R/P,M) \longrightarrow 0$$

In this sequence $V = \mathrm{Ext}^n(R(P)_g,M)$ is free graded of rank $\mu_n((P)_g,M)$, whereas $V' = \mathrm{Ext}^{n+1}(R/P,M)$ is a vector space of dimension $\mu_{n+1}(P,M)$ over the field R/P. In $R/(P)_g$-mod we have the exact sequence :

$$(\star\star)\quad 0 \to V \xrightarrow{m_a} V \to V' \to 0$$

Since $V \cong (R/(P)_g)^{\mu_n((P)_g,M)}$ we deduce from $(\star)$ and $(\star\star)$ that

$V' \cong (R/P)^{\mu_n((P)_g,M)}$. On the other hand $V' = (R/P)^{\mu_{n+1}(P,M)}$, therefore $\mu_n((P)_g,M) = \mu_{n+1}(P,M)$. □

III.1.5. Corollary. In the situation of III.1.4., $\mu_n((P)_g,M) = 0$ if and only if $\mu_{n+1}(P,M) = 0$.

III.1.6. Corollary. Let P be a graded prime ideal of R.
The minimal injective resolution of $E^g(R/P)$ in R-mod is the following :

$$0 \longrightarrow E^g(R/P) \longrightarrow E(R/P) \longrightarrow \bigoplus_{P'} E(R/P') \longrightarrow 0$$

where the direct sum is over all prime ideals $P' \neq P$ of R such that $(P')_g = P$.

Proof. The resolution has the form :

$$0 \longrightarrow E^g(R/P) \longrightarrow E(R/P) \longrightarrow Q_1 \longrightarrow 0$$

where $Q_1 = \sum_{P' \in \mathrm{Spec}\, R} E(R/P')^{\mu_1(P',E^g(R/P))}$. Applying Proposition III.1.4. and Corollary III.1.5. we get $Q_1 = \bigoplus_{P'} E(R/P')$ where $(P')_g = P$. □

III.1.7. Corollary. If $M \in$ R-gr with gr. inj. $\dim_R M = n$, then

$$n \leq \text{inj. } \dim_R \underline{M} \leq n + 1$$

Proof. Directly from the proposition 1.4.

III.1.8. Remark. If M is a maximal ideal of R which is graded then $E^g_R(R/M) = E_R(R/M)$, e.g. if $R = k[X_1,\dots,X_n]$ and $M=(X_1,\dots,X_n)$. (k is a field).

III.1.9. Lemma. Let R be a graded ring and suppose $P \subset Q$ are distinct graded primes with no other graded prime ideal between them. If M is a graded finitely generated module then $\mu^g_n(P,M) \neq 0$ implies $\mu^g_{n+1}(Q,M) \neq 0$.

Proof. By Proposition III.1.2.4. we may assume $R = Q^g_{R-Q}(R)$. Choose $a \in h(R)$ such that $a \in Q$, $a \notin P$. We denote $A = R/P$ and $B = A/aA = R/(a,P)$.

Clearly, B is an object of finite length in R-gr and a is a nonzero divisor on A. The exact sequence : $0 \longrightarrow A \xrightarrow{\varphi_a} A \longrightarrow B \longrightarrow 0$ yields an exact sequence : $\mathrm{Ext}_R^n(A,M) \xrightarrow{\varphi_a} \mathrm{Ext}_R^n(A,M) \longrightarrow \mathrm{Ext}_R^{n+1}(B,M)$. By hypothesis R is gr-local with Q as the gr-maximal ideal and $a \in Q$, then $\mathrm{Ext}_R^{n+1}(B,M) \neq 0$. Now, by induction on l(B) (l denoting length), and using the right exactness of functors $\mathrm{Ext}_R^{n+1}(.,M)$ we obtain that $\mathrm{Ext}_R^{n+1}(R/Q, M) \neq 0$, and hence $\mu_{n+1}^g(Q,M) \neq 0$.

III.1.10. Corollary. Let R be a commutative Noetherian ring with left limited grading. If $M \in$ R-gr is a finitely generated graded module then : gr. inj. $\dim._R M$ = inj. $\dim_R \underline{M}$.

Proof. Put n = gr. inj. $\dim._R M < \infty$. If inj. $\dim_R \underline{M} \neq n$, then inj.$\dim_R \underline{M} = n+1$. Now $\mu_{n+1}(P,M) \neq 0$ for some prime ideal P of R, so by the foregoing results P is maximal. Clearly P is not graded and by Proposition III.1.4. it follows that $\mu_n((P)_g,M) \neq 0$ hence $\mu_n^g((P)_g,M) \neq 0$. We claim that $(P)_g$ is graded maximal, indeed if not, then $(P)_g \subset P_1$, but if P_1 is graded then $\mu_{n+1}^g(P_1,M) \neq 0$ by Lemma III.1.9., contradiction. Since R has left limited gradation it follows that (P) is a maximal ideal, i.e. $P = (P)_g$, contradiction. Consequently, inj. $\dim._R \underline{M} = n$.

III.1.11. Corollary. Let R be a commutative Noetherian ring with left limited grading. Let $M \in$ R-gr. be a finitely generated graded module with gr.inj.dim. M = n, $n < \infty$. If $0 \longrightarrow M \longrightarrow E_0^g \longrightarrow E_1^g \longrightarrow \dots \longrightarrow E_n^g \longrightarrow 0$ is a minimal injective resolution of M in R-gr then E_n^g is injective in R-mod.

Proof. Let P be a graded prime ideal such that $\mu_n^g(P,M) \neq 0$. By Lemma III.1.9. we obtain that P is a gr-maximal ideal. Because R has left limited grading P is a maximal ideal. By Remark III.1.8. it follows that $E_R^g(R/P)$ is an injective R-module. Therefore E_n^g is injective as an R-module.

In extending from R to $R[X]$ we need the following lemma where R has trivial gradation and $R[X]$ is graded as usual.

III.1.12. Lemma. If P is a graded prime ideal of $R[X]$ then either $P = pR[X]$ or $P = pR[X] + (X)$, where $p = P \cap R$.

Proof. Since $P \supset pR[X]$ and $R[X]/pR[X] = (R/P)[X]$ we may assume $p = 0$, i.e. R is a domain. Let K be the field of fractions of R. Since $P \cap R = 0$, $PK[X]$ is a graded prime ideal of $K[X]$ i.e. $PK[X] = 0$ or $PK[X] = X$. □

III.1.13. Proposition. Let R be a Noetherian commutative ring and take $P \in \operatorname{Spec} R$ Put $E = E_R(R/P)$. Consider the ring $R[X]$ graded corresponding to the degree in X. Denote $M = E[X] = E \otimes_R R[X]$.
The minimal injective resolution of M in the category $R[X]$-gr is :

$$0 \longrightarrow M \longrightarrow E^g_R(R[X]/PR[X]) \longrightarrow E^g(R[X]/PR[X] + (X)) \longrightarrow 0$$

and the minimal injective resolution of M in $R[X]$-mod is :

$$0 \longrightarrow \underline{M} \longrightarrow E_{R[X]}(R[X]/PR[X]) \longrightarrow \bigoplus_{P'} E_{R[X]}(R[X]/P') \longrightarrow 0$$

the direct sum being taken over the prime ideals P' of $R[X]$ such that $P' \cap R = P$.

Proof. Consider the minimal injective resolution of M in $R[X]$ - gr :

$$0 \longrightarrow M \longrightarrow I^g_0 \longrightarrow I^g_1 \longrightarrow I^g_2 \longrightarrow \dots$$

Since $\operatorname{Ass} M = \{PR[X]\}$, $I^g_0 = E^g(R[X]/PR[X])$. Taking into account Proposition III.1.3., we obtain :

$$\mu_{i+1}(PR[X] + (X),M) = \mu_i(PR[X] + (X), E[X]/XE(X)) = \mu_i(P,E)$$

Since $\mu_i(P,E) = 1$ if $i = 0$ and $\mu_i(P,E) = 0$ if $i > 0$, we have $\mu_1(PR[X] + (X),M) = 1$ and $\mu_i(PR[X] + (X),M) = 0$ for all $i > 1$.
Let P' be a prime ideal of $R[X]$ such that $\mu_i(P',M) > 0$. Then necessarily $P' \in \operatorname{Supp.} M$ and therefore $P' \cap R \supset P$. If $P' \cap R \neq P$, put $P' \cap R = Q$. By Lemma III.1.12. it follows that $P' = QR[X]$ or $P' = QR[X] + (X)$.

In case $P' = Q\,R[X]$ we have :

$\mu_i(P',M) = \mu_i\,(Q\,R[X], E[X]) = \dim_{k(P)}(\mathrm{Ext}^i_{R[X]}\,(R[X]/QR[X],\ E[X])$

However,

$\mathrm{Ext}^i_{R[X]}\,(R[X]/QR[X],\ E[X]) = \mathrm{Ext}^i_R\,(R/Q,E) \otimes_R R[X]$.

and as $Q \neq P$ we obtain $\mathrm{Ext}^i_R\,(R/Q,E) = 0$ for all $i \geq 0$.

Thus $\mu_i\,(P',M) = 0$ for all $i \geq 0$.

In case $P' = Q\,R[X] + (X)$, we have an exact sequence

$$0 \longrightarrow R[X]/Q\,R[X] \xrightarrow{m_X} R[X]/QR[X] \longrightarrow R[X]/P' \longrightarrow 0$$

Yielding a long exact sequence :

$$\ldots \longrightarrow \mathrm{Ext}^i(R[X]/Q\,R[X],\ M) \xrightarrow{m_X} \mathrm{Ext}^i\,(R[X],\ Q\,R[X],\ M) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^{i+1}\,(R[X]/P',M) \longrightarrow E^{i+1}(R[X]/Q\,R[X],\ M) \xrightarrow{m_X} \mathrm{Ext}^{i+1}(R[X]/QR[X],\ M) \rightarrow$$

(all Ext are $\mathrm{Ext}_{R[X]}$).

From the above we deduce : $\mathrm{Ext}^i\,(R[X]/P',M) = 0$ for all $i \geq 1$.

Finally this amounts to $I_1^g = E^g\,(R[X]/PR[X] + (X))$ and $I_k^g = 0$ for all $k > 1$.

For the second statement of the proposition, let the minimal injective resolution for $\underline{M}$ in $R[X]$-mod be given by :

$$0 \longrightarrow \underline{M} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots$$

It is clear that $I_0 = E_{R[X]}\,(R[X]/PR[X])$. From Proposition III.1.2. and III.1.4. it follows that I_1 contains $\bigoplus_{P'} E_{R[X]}\,(R[X]/P')$ where P' varies in the set of prime ideals of $R[X]$ such that $(P')_g = PR[X]$ or $P' = PR[X] + (X)$. Therefore $(P')_g = PR[X]$ if and only if $P' \cap R = P$. Indeed if $P' \cap R = P$ and $P' \neq (P')_g$ then $ht(P') = 1 + ht(P')_g$, thus if $(P')_g \neq PR[X]$ then $ht(P')_g \geq 1 + ht(PR[X]) = 1 + ht(P)$. Thus $ht(P') \geq 2 + ht(P)$. On the other hand, since R is Noetherian and $P' \cap R = P$, $ht(P') \leq 1 + ht(P)$, contradiction. Since the converse is obvious, this proves the statement.

Let P_1 be a prime ideal of $R[X]$ which is not graded and such that $P_1 \cap R \supsetneq P$. Then $(P_1)_g \supsetneq PR[X]$ and $(P_1)_g \neq PR[X] + (X)$ because $(P_1)_g = PR[X] + (X)$ would imply $P_1 \cap R = (P_1)_g \cap R = P$.

From Proposition III.1.4. we retain that $\mu_i((P_1)_g,M) = \mu_{i+1}(P_1,M)$.

However we have already established $\mu_i((P_i)_g,M) = 0$ for all $i \geq 0$, hence $\mu_i\ (P_1,M) = 0$ for all $i \geq 1$. This yields that $I_1 = \underset{P'}{\oplus} E_{R[X]}(R[X]/P')$ where P' is as stated. If I_2 were nonzero then there exists a prime ideal P_2 of $R[X]$ such that $\mu_2(P_2,M) \neq 0$.

If P_2 is graded then from Proposition III.1.2. we deduce $\mu_2^g(P_2,M) \neq 0$ and thus $I_2^g \neq 0$, contradiction.

If P_2 is not graded, then we deduce from Proposition III.1.4. that $\mu_2(P_2,M) = \mu_1((P_2)_g,M) \neq 0$. In this case it is necessary that $(P_2)_g = PR[X]+ (X)$, hence $P_2 \cap R = (P_2)_g \cap R = P$. Since $P_2 \neq (P_2)_g$, $ht(P_2) = 1 + ht((P_2)_g) = 2 + ht(PR[X]) = 2 + ht(P)$. But from $P_2 \cap R = P$, $ht(P_2) \leq 1 + ht(P)$ follows hence we reach a contradiction.

Therefore $\mu_2(P_2,M) = 0$ and $I_2 = 0$ follows.

<u>III.1.14. Corollary</u>. Let R be a commutative Noetherian ring, E injective in R-mod, then :

$$\text{gr. inj. dim.}_{R[X]}\, E[X] + 1 = \text{inj. dim.}_{R[X]}\, E[X].$$

III.2. Regular, Gorenstein and Cohen-Macaulay Rings.

Let R be a commutative Noetherian ring and M a nonzero finitely generated R-module. We put : $V(M) = \text{supp.}(M) = V(\text{Ann}_R M) = \{ P \in \text{Spec } R, \text{Ann}_R M \subset P \}$. Recall that the Krull dimension of M, denoted by $K.\dim_R M$ is defined to be the supremum of the lengths of chains of prime ideals of V(M) if this supremum exists , and ∞ if not (the reader may verify that $K.\dim_R M$ coincides with the Krull dimension of M defined in section II.5.).

We have $ht_M P = K.\dim_{Q_{R-P}(R)} (Q_{R-P}(M))$.

If I is an ideal of R such that $IM \neq M$ then the least r for which $\text{Ext}^r_R(R/I,M) \neq 0$ will be called the grade of I on M, denoted by : grade (I,M); grade (I,M) is exactly the common length of all maximal sequences contained in I (see Lemma III.3.3. of § 3).

If R is a local ring with maximal ideal Ω, then M is said to be a Cohen-Macaulay module or a C.M.-module if grade $(\Omega,M) = K.\dim_R M$. A not necessarily local ring is said to be a Cohen-Macaulay ring or C.M-ring if $Q_{R-P}(R)$ is a Cohen-Macaulay $Q_{R-P}(R)$-module and in that case **M** is a C.M-module if $Q_{R-P}(M)$ is a C.M.-$Q_{R-P}(R)$ module. It is easy to verify using Proposition III.1.2. that M is a C.M.-module if and only if for each maximal ideal $\Omega \in V(M)$, $\mu_i(\Omega,M) = 0$ whenever $i < ht_M(\Omega)$. We say that M is a Gorenstein-module if for every maximal ideal $\Omega \in V(M)$, $\mu_i(\Omega,M) = 0$ if and only if $i \neq ht_M(\Omega)$. Proposition III.1.2. entails that M is a Gorenstein-module if and only if $Q_{R-P}(M)$ is a Gorenstein $Q_{R-P}(R)$-module for all $P \in V(M)$.

A Gorenstein ring R is a ring which is a Gorenstein-module when considered as a module over itself. For a detailed study of these classes of rings we refer to Bass, [5]. First we give some local - global theorems in the graded case.

III.2.1. Theorem. Let R be a commutative Noetherian graded ring and let $M \in R\text{-gr}$ be finitely generated. The following statements are equivalent.

1. M is a Cohen - Macaulay module.
2. For any graded prime ideal P of $V(M)$, $Q_{R-P}(M)$ is a Cohen-Macaulay $Q_{R-P}(R)$-module.

Proof. $1 \Rightarrow 2$ is trivial. To prove the converse let us assume first that $Q_{R-P}(M)$ is a C.M. $Q_{R-P}(R)$-module for every graded $P \in V(M)$. Let $P' \in V(M)$. If $(P')g = P'$ then we are done. So suppose that $P' \neq (P')_g$. In this case we have that $ht_H P' = ht_M(P')_g + 1$. Choose $i < ht_M(P')$. Then $i-1 < ht_M((P')_g)$, hence $\mu_{i-1}((P')_{g,M}) = 0$. By Proposition III.1.4. it follows then that $\mu_i(P',M) = 0$ and therefore M is a C.M.-module. □

III.2.2. Theorem. Let R be a graded commutative Noetherian ring and let $M \in R\text{-gr}$ be finitely generated. The following statements are equivalent :

1. M is a Gorenstein-module
2. For each graded $P \in V(M)$, $Q_{R-P}(M)$ is a Gorenstein $Q_{R-P}(R)$ module.

Proof. $1 \Rightarrow 2$. Easy. For the converse, suppose that $Q_{R-P}(M)$ is a Gorenstein $Q_{R-P}(R)$-module for each graded $P \in V(M)$. Let P' be arbitrary in V(M). If $P' = (P')_g$ then the assertion is clear. Supposing that $P' \neq (P')_g$ and $i \neq ht_M(P')$, we obtain $i-1 \neq ht_M((P')_g)$. Since $Q_{R-(P')_g}(M)$ is a Gorenstein $Q_{R-(P')_g}(R)$-module it follows that :

$$\mu_{i-1}((P')_g M) = \mu_{i-1}\ (Q_{R-(P')_g}((P')_g),\ Q_{R-(P')_g}(M)) = 0$$

From Proposition III.1.4. we deduce that $\mu_i(P',M) = 0$. In a similar way it may be deduced from $\mu_i(P',M) = 0$ that $i \neq ht_M(P')$ hence M is a Gorenstein module. □

A local ring with maximal ideal Ω is said to be regular if $\text{gl. dim } R < \infty$ or equivalently, if $\text{K.dim } R = \dim_{R/\Omega}(\Omega/\Omega^2)$. If R is not a local ring, then R is called a regular ring if for every maximal ideal $\Omega, Q_{R-\Omega}$ is regular.

III.2.3. Corollary. Let R be a commutative Noetherian ring and M a finitely generated R-module. If M is a Cohen-Macaulay (resp. Gorenstein) module then $M[X]$ is a Cohen - Macaulay (resp. Gorenstein) R-module.

Proof. Let $P \in V(M[X])$ be a graded prime ideal; then $\mathrm{Ann}_{R[X]} M[X] \subset P$. If we write $p = P \cap R$ then $p \in V(M)$. By Lemma III.1.11. we have $P = pR[X]$ or $P = pR[X] + (X)$. If $P = pR[X]$ we have $\mu_i(P, M[X]) = \mu_i(P,M)$ and if $P = pR[X] + (X)$ we have by Proposition III.1.3. that $\mu_i(p,M) = \mu_{i+1}(P, M[X])$.
Moreover $ht_{M[X]} P = ht_M P$ if $P = p[X]$ and $ht_{M[X]} P = 1 + ht_M P$ if $P = pR[X] + (X)$. Now the corollary following from Theorem III.2.1. and III.2.2. finishes the proof.

III.2.4. Theorem. Let R be a graded commutative Noetherian ring then the following statements are equivalent :

1. R is a regular ring.
2. $Q_{R-P}(R)$ is regular for every graded prime ideal P of R.

Proof. The implication $1 \Rightarrow 2$ is obvious.
$2 \Rightarrow 1$. If P' is an arbirary prime ideal of R then the ring $Q_{R-(P')_g}(R)$ is regular i.e. gl.dim. $Q_{R-(P')_g}(R) < \infty$. Since $Q_{R-P'}(R)$ is the localization of $S = Q_{R-(P')_g}(R)$ at the prime ideal generated by P' in it, it follows that we may assume that we have chosen R to be gr.local with gr.maximal ideal $(P')_g$. By hypothesis gl. dim $S = n < \infty$. Consider a finitely generated R-module M. By Proposition III.1.2. we have that $\text{inj. dim}_S\, Q_{R-(P')_g}(M) \leq n$ and hence gr. inj. $\dim_R M \leq n$. Therefore gr.gl.dim $R < \infty$ and Corollary III.1.10. yields gl.dim. $R < \infty$, hence gl. dim $Q_{R-P'}(R) < \infty$.

Our following result shows that regular and positively graded rings are close to being polynomial rings over regular rings.

III.2.5. Proposition. Let R be a positively graded regular ring with a unique graded maximal ideal Ω , then R_0 is a regular local ring and $R \cong R_0 [X_1,\ldots,X_k]$, the X_i, $i = 1,\ldots,k$ being indeterminates over R_0, which are taken to be homogeneous elements of positive degree.

Proof. We proceed by induction on $n = K.\dim R$. If $n = 0$ then $R = R_0$ is a field. If $n > 0$, choose a nonzero homogeneous element x of positive degree in $\Omega - \Omega^2$ (if there is no such x then $R = R_0$ follows). Put $R' = R'/(x)$, $\Omega' = \Omega/(x)$. Now R' is a graded regular ring and $K.\dim R' = n-1$. The induction hypothesis yields : $(R')_0$ is a regular local ring and $R' \cong (R')_0 [Y_1,\ldots,Y_k]$ with $\deg Y_i > 0$, $i = 1,\ldots,k$. Since $(x) \cap R_0 = 0$ it follows that $(R')_0 \cong R_0$.
So if $K.\dim R_0 = d$ then $d + k = n-1$. Choose representatives $U_1,\ldots,U_k$ in R for the images of $Y_1 \ldots Y_k$ in R', and put:$U_{k+1} = x$. One easily checks that $R = R_0 [U_1,\ldots,U_{k+1}]$. Therefore we have an epimorphic graded ring homomorphism $\Psi : R_0 [X_1,\ldots,X_{k+1}] \to R_0 [U_1,\ldots,U_{k+1}] = R$. However, the fact that $K.\dim R = k + d + 1 = K.\dim R_0 [X_1,\ldots,X_{k+1}]$ yields that Ψ is an isomorphism. □

III.2.6. Remarks.

1. If R is a positively graded regular ring, then R_0 is regular. Indeed, if ω is a maximal ideal in R_0 then $\Omega = \omega \oplus R_+$ is a maximal graded ideal. Since the localization $Q_{R-\Omega}(R)$ is regular and also positively graded (note that it coincides with the graded ring of quotients $Q^g_{R-\Omega}(R)$) it follows from the foregoing that $(Q_{R-\Omega}(R))_0$ is regular. However $(Q_{R-\Omega}(R))_0 = Q_{R_0-\omega}(R_0)$. It follows that R_0 is regular too.

2. See also J. Matijevic [7]. If $R = \bigoplus_{i\geq o} R_i$ is Gorenstein, hence Cohen-Macaulay, then it is not necessarily true that R_o is Gorenstein or C.M.. Indeed, put $T = k[X,Y] / (X^2, XY)$, where R is any field. Put $Q_1 = (Y)$, $Q_2 = (X)$, then the image of $Q_1 \cap Q_2$ in T is zero. Let W, V be variables of degree 1 and put $S = T[W], Q_1^e = Q_1S$, $Q_2^e = Q_2S + WS$ and $I = Q_1^e \cap Q_2^e$. Put $U = S/I$, $R = U[V] / (xV + yV, V^2)$ where x, y denote the images of X, Y resp. This construction yields a graded ring R with K.dim R = 1 and $R_o = T$. However T is not a C.M-ring.
If Ω denotes the unique maximal graded ideal of R then $Q_{R-\Omega}(R)$ is Gorenstein.

III.3. Graded Rings and M-sequences.

R is a commutative graded ring throughout this section. Such a ring R is said to be completely projective or is said to have property C.P. if for any graded ideal I and for any finite set of graded prime ideals $P_1,\ldots,P_n$ with $h(I) \subset P_1 \cup \ldots \cup P_n$ it follows that I is contained in at least one P_i. It is not hard to verify that, if R is completely projective then so is $S^{-1}R$ for every multiplicatively closed subset S of $h(R)$. It is also straightforward to check that epimorphic images of a completely projective ring have property C.P. too.

III.3.1. Lemma. Let R be a commutative ring such that all graded prime ideals of R are in Proj R, then R is completely projective.

Proof. Assume $I \not\subset P_i$, $1 \le i \le n$ and assume the p_i are not comparable. If $n = 1$ then $h(I) \not\subset P_1$ follows. We proceed by induction on n. The induction hypothesis yields $h(I \cap P_i) \not\subset \bigcup_{j\neq i} P_j$, $1 \le i \le n$ hence there is an $a_i \in h(I \cap P_i)$ such that $a_i \notin P_j$ for each $j \neq i$. Let $x_i = \prod_{j\neq i} a_j$. Clearly $x_i \in I$ and $x_i \in P_j$ for $j \neq i$ but $x_i \notin P_i$. Since no graded prime ideal of R contains R_+ it is easy to see that we may choose $\deg x_i > 0$ for each i, $1 \le i \le n$. Put $d_i = \deg x_i$ and $d = d_1 \cdot \ldots \cdot d_n$, $y_i = x_i^{d/d_i}$ Then $\deg y_i = d$ and it is obvious that $y = y_1 + \ldots + y_n$ is a homogeneous element such that $y \in I$ and $y \notin P_1 \cup \ldots \cup P_n$. $\square$

III.3.2. Examples.

1. If R is a positively graded ring such that R_0 is a field then R is completely projective.

2. Let R be a gr-local ring with gr-maximal ideal Ω such that R/Ω is a non trivial graded gr- division ring. Then R has property C.P. . Indeed, if P is a graded prime ideal then $P \subset \Omega$. Since R/Ω is non-trivially graded, there is a homogeneous t of positive degree, $t \notin \Omega$, hence $t \notin P$.

2. Let O_v be a discrete valuation ring with maximal ideal ω_v generated by t say. Put $R = O_v[X]/(tX)$ i.e. in R we have $t\,x = 0$. Then (t,x) is a graded ideal and every homogeneous element of (t,x) is a zero-divisor; however $t + x$ is not a zero-divisor. If $P_1,\ldots,P_n$ are the prime ideals associated to R, then these are graded and clearly $h(I) \subset P_1 \cup \ldots \cup P_n$ but $I \not\subset P_i$ for each i, $1 \le i \le n$. So $O_v[X]$ does not have property C.P. .

Let R be a commutative Noetherian ring and M a finitely generated R-module. A sequence $a_1,\ldots,a_n$ in R is said to be an M-sequence if $(a_1,\ldots,a_n)M \neq M$ and for each $i = 1,\ldots,n$, a_i is not an annihilator for the module $M/(a_1,\ldots,a_{i-1})M$. If I is an ideal of R such that $IM \neq M$, then we shall denote by grade (I,M) the common length of all maximal M-sequences contained in I.

<u>III.3.3. Lemma</u>. Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Suppose that I is an ideal of R such that $IM \neq M$. Then grade (I,M) is the smallest number r such that $\mathrm{Ext}_R^r(R/I,M) \neq 0$.

<u>Proof.</u> First we show by induction on r that if $a_1, a_2,\ldots,a_r$ is an M-sequence contained in I then $\mathrm{Ext}_R^i(R/I, M) = 0$ for $0 \le i \le r$, and $\mathrm{Ext}_R^r(R/I,M) = \mathrm{Hom}(R/I, M/(a_1,\ldots,a_r)M)$.

Indeed, from the exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi_{a_1}} M \longrightarrow M/a_1M \longrightarrow 0$$

We derive the exact sequence :

$$\mathrm{Ext}^{r-1}(R/I,M) \xrightarrow{\varphi_{a_1}} \mathrm{Ext}^{r-1}(R/I,M) \longrightarrow \mathrm{Ext}^{r-1}(R/I,M/a_1M) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^r(R/I,M) \xrightarrow{\varphi_{a_1}} \mathrm{Ext}^r(R/I,M) \longrightarrow \ldots$$

By the the induction hypothesis : $\mathrm{Ext}^{r-1}(R/I,M) = 0$ and the homomorphism $\varphi_{a_1} = 0$. Hence $\mathrm{Ext}^{r-1}(R/I, M/a_1M) \simeq \mathrm{Ext}^r(R/I,M)$.

By induction : $Ext^{r-1}(R/I, M/a_1M) \simeq Hom(R/I, M/(a_1 \dots a_r)M)$ and therefore $Ext^r(R/I,M) \simeq Hom(R/I,M/(a_1 \dots a_r)M)$. If $Hom(R/I,M') \neq 0$ then the module $M'' = \{x \in M', \ Ix = 0\}$ is not zero. Therefore $Ass\ M'' \neq \emptyset$ i.e. there exist $P \in Ass\ M'' \subset Ass\ M'$ such that $I \subset P$, and therefore the sequence $\{a_1,\dots,a_n\}$ is an M-sequence of maximal length.

III.3.4. Corollary. Let R be a commutative Noetherian ring which is graded and completely projective. Let $M \in R\text{-gr}$ be finitely generated and I a graded ideal of R such that $IM \neq M$. If grade $(I,M) = m$, then there exists in I an M-sequence of length m, consisting of homogeneous elements. Moreover any M-sequence formed by homogeneous elements of I has the same length grade (I,M).

Proof. For $m = 0$ the assertion is clear. Let $m \geq 1$ and let $P_1,\dots,P_s$ be graded prime ideals associated to M. Since grade $(I,M) > 0$, $I \not\subset P_1 \cup \dots \cup P_s$ and because of Lemma III.3.1. $h(I) \not\subset P_1 \cup \dots \cup P_s$. Hence there exists $f_1 \in h(I)$ such that $f_1 \not\subset P_1 \cup \dots \cup P_s$, therefore f_1 is a non-annihilator of M. Let M_1 be the graded R-module M/f_1M, then grade $(I,M) = m-1$ and we apply the induction hypothesis. The second statement may also be proved using a similar induction argument. □

III.3.5. Corollary. Let R be a commutative Noetherian graded ring having property C.P. Let P be a prime ideal of R having height n. Then there exist homogeneous elements $a_1,\dots,a_n$ in R such that P is minimal over $Ra_1 + \dots + Ra_n$.

Proof. For $n = 0$ there is nothing to prove, so we may assume $n > 0$ and proceed by induction on n. Let $\{Q_1,\dots,Q_k\}$ be the set of minimal prime ideals of R and as R is graded, each one of the Q_i, $i = 1,\dots,k$ is graded. Since $ht(P) \geq 1$, P is not contained in any Q_i and therefore Lemma III.3.1. entails $h(P) \not\subset Q_1 \cup \dots \cup Q_k$. Pick $a_1 \in h(P) - \bigcup_{i=1}^{k} Q_i$.

In R/Ra_1, P/Ra_1 is a graded prime ideal with $ht(P/Ra_1) \leq n-1$. The induction hypothesis yields that P/Ra_1 is minimal over $(\bar{a}_2,\ldots,\bar{a}_n)$, where the a_i, $i = 2,\ldots,n$ are homogeneous elements of R/Ra_1. Choosing representatives a_i for $\bar{a}_i$ in R it is obvious that P is minimal over $(a_1,\ldots,a_n)$. □

Consider a gr-local ring R with maximal graded ideal Ω and suppose that R is completely projective. Let $M \in R$-gr have finite type. The ring $\bar{R} = R/\mathrm{ann}_R M$ is a gr.-local ring with maximal graded ideal $\bar{\Omega} = \Omega/\mathrm{ann}_R M$ and $\bar{R}$ has property C.P. Write n for the Krull dimension of M, then $n = \mathrm{K.dim}\, \bar{R} = ht(\bar{\Omega})$. By Corollary III.3.5., there exist homogeneous elements $\bar{a}_1,\ldots,\bar{a}_n \in \bar{\Omega}$ such that $\bar{\Omega}$ is minimal over $\bar{R}\,\bar{a}_1 +\ldots+\bar{R}\,\bar{a}_n$ in $\bar{R}$. Our hypothesis yields that $M/a_1M +\ldots+a_nM = M/\bar{a}_1M+\ldots+\bar{a}_nM$. Furthermore $\bar{R}/\bar{R}\bar{a}_1+\ldots+ \bar{R}\bar{a}_n$ is a gr-Artinian ring, hence $M/\bar{a}_1M+\ldots+\bar{a}_n$ is a gr.Artinian $(\bar{R}/\bar{R}\bar{a}_1+\ldots+\bar{R}\bar{a}_n)$-module. On the other hand we have that $\bar{R}/\bar{A}\bar{a}_1+\ldots+\bar{R}\bar{a}_n =$ $= R/\mathrm{ann}_R M + Ra_1+\ldots+Ra_n$, thus it follows that $M/a_1M+\ldots+a_nM$ is a gr.Artinian $(R/\mathrm{ann}_R M + Ra_1+\ldots+Ra_n)$-module; consequently, the latter module actually is gr.Artinian in R-gr.

<u>A system of homogeneous parameters</u> for the module M with $\mathrm{K.dim}\, M = n$ is a set of elements $a_1,\ldots,a_n \in h(\Omega)$ such that $M/a_1M+\ldots+a_nM$ is Artinian in R-gr (hence of finite length in R-gr!).

<u>III.3.6. Remarks.</u>

1. As in the ungraded case one may establish that every M-system may be included in a system of homogeneous parameters for M.
2. If R is positively graded and such that R_0 is a field then there exists a system of homogeneous parameters for R_+. However if R_0 is not a field but a local ring with maximal ideal ω then the graded ideal $\Omega = \omega + R_+$ may not have a system of homogeneous parameters, as the following example shows.

III.3.7. Example. Let V be a discrete valuation ring with maximal ideal ω_V and uniform parameter t. Let $R = V[X]/(tX)$; then R is Noetherian and positively graded. Moreover R is a gr.local ring with Krull dimension 1. One easily verifies that there cannot exist a homogeneous $a_1 \in (\omega_V + R_+)/(tX)$ such that $\{a_1\}$ is a system of homogeneous parameters.

In the light of the foregoing observations we now suppose that the graded Noetherian ring R has the form $R_0 \oplus R_1 \oplus \dots$, where R_0 is a field. Then R is gr.local with gr. maximal ideal R_+.

III.3.8. Proposition. Let $M \in R$-gr have finite type and let $\{a_1,\dots,a_n\} \in h(R_+)$. A necessary and sufficient condition for $a_1,\dots,a_n$ to be a system of homogeneous parameters for M is that M has finite type as an object of $R_0[a_1,\dots,a_n]$-gr while n is minimal as such.

Proof. If $a_1,\dots,a_n$ is a system of parameters for M then $\bar{M} = M/a_1M+\dots+a_nM$ is Artinian in R-gr. Since $\bar{M}$ has finite type its graded is left limited. On the other hand since $\bar{M}$ is Artinian in R-gr and since R is positively graded, the grading of $\bar{M}$ is also right limited. Consequently $\bar{M}$ has to be a finite dimensional R_0-vector space. Nakayama's lemma then yields that M is finitely generated as an $R_0[a_1,\dots,a_n]$- module. Conversely, if we start from the assumption that M is a finitely generated $R_0[a_1,\dots,a_n]$-module, then $M/a_1M+\dots+a_nM$ is a finite dimensional R_0-vector space thus Artinian in R-gr. Next let us check minimality of n as such. Indeed if $\{b_1,\dots,b_m\}$ is a set of homogeneous elements such that M is a finitely generated $R_0[b_1,\dots,b_m]$-module then M/b_1M is Artinian in R-gr, i.e. of finite length too. But then $K.\dim M \le n$ excludes the possibility $m < n$. □

IV.3.9. Proposition. Let $M \in R\text{-gr}$ have finite type. If $a_1,\dots,a_n$ is a system of homogeneous parameters for M then the elements $a_1,\dots,a_n$ are algebraically independent over R_0.

Proof. Letting $a_1,\dots,a_n$ be such a system of parameters, consider the ring $S = R_0[X_1,\dots,X_n]$ with gradation defined by putting $\deg X_i = \deg a_i$, $i = 1,\dots,n$. Specializing $X_i \to a_i$ defines a surjective ring homomorphism of degree 0, $\varphi : R_0[X_1,\dots,X_n] \to R_0[a_1,\dots,a_n]$. We have that $K.\dim R_0[a_1,\dots,a_n] = K.\dim M = n$. However if $\operatorname{Ker}\varphi \neq 0$ then the Krull dimension should have dropped, so φ is an isomorphism. □

III.3.10. Proposition. Let $M \in R\text{-gr}$ be finitely generated. The following statements are equivalent.

1. M is a Cohen-Macaulay module.
2. If $a_1,\dots,a_n$ is a system of homogeneous parameters for M then M is a free (graded free) $R_0[a_1,\dots,a_n]$-module.
3. Every system of homogeneous parameters for M is an M-sequence.

Proof.

$1 \Rightarrow 2$. Put $S = R_0[a_1,\dots,a_n]$. By Proposition III.3.9., S is a regular ring with $K.\dim S = n$. Since M is a Cohen-Macaulay module, $\operatorname{grade}(R_+,M) = K.\dim M = n$ and by Corollary III.3.3. there exists an M-sequence $b_1,\dots,b_n$ with $b \in h(R)$ $i = 1,\dots,n$. For $i = 1,\dots,n$ put $M_{(i)} = M/b_1M+\dots+b_iM$ and put $M_{(o)} = M$. Considering the fact that S is a regular ring, we may derive from the exactness of

$$0 \longrightarrow M_{(i)} \xrightarrow{m_{i+1}} M_{(i)} \to M_{(i+1)} \longrightarrow 0,$$

(where m_{i+1} denotes multiplication by b_{i+1}) that $p.\dim_S M_{(i+1)} = 1+p.\dim_S M_{(i)}$

On the other hand, $M_{(n)} = M/b_1M+\dots+b_n M$ is an S-module of finite length. Therefore $M_{(n)}$ is a finitely dimensional R_0-vector space. Because S is gr-local we have that $p.\dim_S M_{(n)} = n = p.\dim_S M_{(o)}$. Now $p.\dim_S M_{(n)} = n$

yields $p.\dim_S M_{(o)} = p.\dim_S M = 0$, consequently M is a free S-module.

$2 \Rightarrow 3$, as well as $3 \Rightarrow 1$ are rather easy. □

Supplementay Bibliography.

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Exercises.

1. Let R be a graded ring of type Z. Prove that if for any graded prime ideal P_1 $Q_{R-P}(R)$ is Noetherian then for any prime ideal $Q, Q_{R-Q}(R)$ is Noetherian.

2. Suppose that R is graded Noetherian ring and $\underline{a}$ is a graded ideal in R. For each $n \in Z$ we define $(R^{gr(a)})_n = \varprojlim_r (R/\underline{a}^r)_n$ and we denote by $R^{gr(\underline{a})} = \bigoplus_{n \in Z} (R^{gr(\underline{a})})_n$. $R^{gr(\underline{a})}$ is a graded ring; this ring is called the graded $\underline{a}$-adic completion of R. Prove that :

   1) The ring $R^{gr(\underline{a})}$ is Noetherian

   2) $R^{gr(a)}$ is flat R-module.

   3) If we denote by $i : R \to R^{gr(\underline{a})}$ the canonical homomorphism then $i(a)R^{gr(\underline{a})}$ is contained in the graded Jacobson radical of $R^{gr(\underline{a})}$

   4) If $\underline{a}$ is gr-maximal ideal then $R^{gr(\underline{a})}$ is gr-local with $i(a)R^{gr(\underline{a})}$ as the gr-maximal ideal.

   5) If $\underline{a}$ is contained in the graded Jacobson radical of R then the canonical homomorphism $i : R \to R^{gr(\underline{a})}$ is injective. Moreover if $\hat{R}$ is the $\underline{a}$-adic completion of R then $R \subset R^{gr(\underline{a})} \subset \hat{R}$ and $\hat{R}$ is the $\underline{a}\, R^{gr(\underline{a})}$ -completion of $R^{gr(\underline{a})}$

3. Let R be a graded Noetherian ring and P a graded prime ideal. Prove that the ring $END_R(E_R^g(R/P))$ is isomorphic to the graded completion of the ring $Q_{R-P}^g(R)$ at its graded maximal ideal.

4. Let R be a local graded ring with M the gr-maximal ideal. Suppose that R is graded M-completed (i.e. $R = R^{gr(M)}$). Write : $E = E_R^g(R/M)$. Prove that :
   1) If X is gr-Noetherian R module then $HOM_R(X,E)$ is gr-Artinian and if Y is a gr-Artinian R-module then $HOM_R(Y,E)$ is gr-Noetherian R-module.
   2) The functor $HOM_R(.,E)$ is a duality between the category of gr-Noethberian modules and the category of gr-Artinian modules.

5. Let $R = \bigoplus_{i \in Z} R_i$ be a graded ring of type Z and T is an indeterminate with deg T = 1. Then we can form the graded (T)-adic completion of R [T]; we denote by gr R [[T]] the (T)-adic completion of R[T]. Prove that :
   1) For every $n \in Z$, $(gr\ R\ [[T]])_n = \{ \sum_{i=n}^{-\infty} a_i T^{n-i},\ a \in R_i \}$
   2) If $R_i = 0$ for every $i < 0$, then $gr\ R\ [[T]] = R\ [T]$.
   3) If R is Noetherian then gr R [[T]] is Noetherian and gr R [[T]] is a flat R-module.

6. Let R be a graded Noetherian ring. The following statements are equivalent :
   1) The ring R is Gorenstein
   2) inj.dim $Q_{R-P}(R) < \infty$ for all graded prime ideals P.
   3) inj. dim $Q_{R-P}^g(R) < \infty$ for all graded prime ideals P.

7. Let R be a gr-local Noetherian ring with gr-maximal ideal m. Prove that :

1) Every graded finitely generated R-module $M = \bigoplus_{i\in Z} M_i$ has a minimal resolution, i.e. a resolution :

$$\cdots \longrightarrow L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

where $L_0, L_1, L_2, \ldots$ are graded free modules and $\operatorname{Ker} d_i \subset \underline{m}\, L_i$ for all $i \geq 0$.

2) $\operatorname{Tor}_L^R(R/\underline{m},M) \simeq L_i/\underline{m}\, L_i$

3) If for all $i < 0$ $R_i = 0$ and $M_i = 0$ then prove that the p-th component of the graded vector space $\operatorname{Tor}_i^R(R+\underline{m},M)$ is zero for $p < i$.

4) $\text{p.dim}_R M = n \Leftrightarrow L_n \neq 0$ and $L_i = 0$ for all $i > n$

8. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$ be a graded Noetherian ring.

1) Prove that there exists homogeneous elements $a_1,\ldots,a_j \in R$ with $k_i = \deg a_i$ such that R is generated as an R_0-algebra by $a_1, a_2,\ldots,a_s$.

2) Suppose that R_0 is an Artinian ring. If $M = M_0 \oplus M_1 \oplus \ldots$ is a graded finitely generated R-module then for all $i \geq 0$, M_i is an R_0-module of finite length.

3) If we denote by $l(M_n)$ the length of M_n in R_0-mod, we can consider the power series

$$P(M,T) = \sum_{n=0}^{\infty} l(M_n)\, T^n \in Z[[T]].$$

Prove that $P(M,T)$ is a rational function in T of the form $f(T)/\prod_{i=1}^{s}(1-T^{k_i})$, where $f(T) \in Z[T]$.

4) If moreover $k_i = 1$, $i = 1,\ldots,s$, prove that for all sufficiently large n, $l(M_n)$ is a polynomial in n.

9. Let R be a graded Noetherian ring of type $Z^n (n \geq 1)$. If M is a finitely generated graded R-module then the following statements are equivalent :

1) M is Cohen-Macaulay (resp. Gorenstein) module.

2) For any graded prime ideal P of $V(M)$, $Q_{R-P}(M)$ is Cohen-Macaulay (resp. Gorenstein) $Q_{R-P}(R)$-module

10. Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded Noetherian ring where $R_0 = k$ is a field. We put $\underline{m} = \bigoplus_{i\geq 1} R_i$. Let $M \bigoplus_{i\in Z} M_i$ be a graded R-module. We define $M^\star = HOM_R\ (M,k)$ and call it the graded R-dual of M. Prove that :

1) $M^\star$ is a graded R-module with $\{ Hom_k\ (M_{-n},k)\ \}_{n\in Z}$ as its grading.

2) $M = M^{\star\star}$ if and only if $\dim_k M_i < \infty$ for all $i \in Z$.

3) If we denote by A_k the subcategory of all graded R-modules $M = \bigoplus_{i\in Z} M_i$ such that $\dim_k(M_i) < \infty$ for all $i \in Z$; prove that the contravariant functor $(.)^\star = HOM_k(.,k)$ is a duality over this category.

4) If M is graded Noetherian (resp. Artinian) R-module then $M^\star$ is graded Artinian (resp. Noetherian) R-module.

5) If L is graded free R-module and finitely generated then $L^\star$ is an injective R-module.

6) If M is graded finitely generated R-module and if

$$\ldots \longrightarrow L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

is free minimal resolution of M in R-gr (see exercises 7) then

$0 \longrightarrow M^\star \longrightarrow L_0^\star \longrightarrow L_1^\star \longrightarrow L_2^\star \longrightarrow \ldots$ is an injective minimal resolution of $M^\star$.

CHAPTER C:
STRUCTURE THEORY FOR GRADED RINGS OF TYPE Z
I: Non-Commutative Graded Rings

I.1. Rings of Fractions and Goldie's Theorems.

Throughout R stands for a non-commutative ring, which is graded of type Z. For information about rings of fractions we refer to Section A.I.6., for full detail on graded rings of quotients and localization, see Section A.II.9. and consequent.

A graded ring R having finite Goldie dimension in R-gr and satisfying the ascending chain condition on graded left annihilators is called a graded Goldie ring (sometimes written as gr-Goldie ring). If R is trivially graded then this reduces to the definition of a Goldie ring as it is used e.g. in [49]. Now a theorem known as "Goldie's theorem" asserts that a ring R has a semisimple Artinian (resp. simple Artinian) classical ring of fractions if and only if R is a semisimple (resp. prime) Goldie ring. Now, exactly as in the ungraded case one proves that if R has a graded ring of fractions which is gr-Artinian and gr-simple then R is a prime gr-Goldie ring. The converse does not hold, as the following example shows :

I.1.1. Example. Let k be a field and let R be the polynomial ring k [X,Y] subjected to the relation $XY = YX = 0$. Put $R_n = kX^n$ if $n \geq 0$ and $R_m = kY^m$ if $m \leq 0$. Clearly, each $x \neq 0$ in h(R) with deg $x \neq 0$ is non-regular. Moreover the ideal (X,Y) is essential in R but it does not contain a regular homogeneous element. This shows that, although R is a semisimple gr-Goldie ring, it does not allow a gr-semisimple gr-Artinian ring of fractions.

I.1.2. Lemma. Let R be a graded ring satisfying the ascending chain condition for graded left annihlators, then the left singular radical of R is nilpotent.

Proof. Write J for the left singular radical of R; note that J is a graded ideal. If S is a subset of R let $l(S)$ be the left annihilator of S in R. Since there is an ascending chain $l(J) \subset l(J^2) \subset \ldots \subset l(J^n) \subset \ldots$, the hypotheses imply that $l(J^n) = l(J^{n+1})$ happens for some $n \in \mathbb{N}$. If J^{n+1} is nonzero then $a\, J^n \neq 0$ for some $a \in h(R)$ and we may choose a such that $l(a)$ is maximal with respect to that property. If $b \in J \cap h(R)$ then $l(b) \cap Ra \neq 0$ because $l(b)$ is an essential left ideal of R. Thus there exists $c \in h(R)$ such that $ca \neq 0$ and $cab = 0$. So $l(a) \subsetneq l(ab)$; the hypothesis on a entails that $ab\, J^n = 0$. Consequently, since J is graded, $a\, J^{n+1} = 0$ i.e. $a \in l(J^{n+1}) = l(J^n)$, a contradiction. Therefore $J^{n+1} = 0$ follows.

I.1.3. Theorem. Let R be a semisiple graded ring.
The following statements are equivalent.
1. R is a gr-Goldie ring.
2. R is a Goldie ring.

Proof.
2. $\Rightarrow$ 1. Easy.
1. $\Rightarrow$ 2 From Lemma I.1.2. it follows that the left singular radical, $Z(R)$, is zero. Consider the graded maximal quotient ring , $Q^g_{max}(R)$, of R and the maximal quotient ring, $Q_{max}(R)$, of R. Since $Z(R) = 0$ we obtain :
$R \subset Q^g_{max}(R) \subset Q_{max}(R)$. It is not hard to see that $Q^g_{max}(R)$ is a gr-regular ring in Von Neumann's sense (see Section I.5. for more detail on these rings) whereas $Q_{max}(R)$ is a regular ring. Since R has finite Goldie dimension in R-gr, $Q^g_{max}(R)$ has finite Goldie dimension in $Q^g_{max}(R)$-gr. Therefore (see also results of Section A.) $Q^g_{max}(R)$ is a gr-semisimple gr-Artinian ring and in particular $Q^g_{max}(R)$ is left Noetherian. Since

$Q_{max}(R)$ is an essential extension of R in R-mod, it is also an essential extension of $Q^g_{max}(R)$ in $Q^g_{max}(R)$-mod. Now $Q^g_{max}(R)$ is left Noetherian, hence it has finite Goldie dimension, therefore $Q_{max}(R)$ has finite Goldie dimension too. That R has finite Goldie dimension in R-mod follows.

I.1.4. Lemma. Let R be a semisimple graded ring satisfying the ascending chain conditions on graded left annihilators. If I is a graded left (or right) ideal such that every element of $h(I)$ is nilpotent, then $I = 0$.

Proof. For $a \in h(R)$ there is equivalence between the statements :

1° every element of $h(Ra)$ is nilpotent.

2° every element of $h(aR)$ is nilpotent.

Let us prove the case where I is a right ideal.

If $I \neq 0$ then $h(I) \neq 0$ and we may choose $a \in h(I)$, $a \neq 0$, such that $\mathrm{ann}^l_R(a)$ is maximal amongst left annihilators of elements of $h(I)$ which are nonzero. Take $\lambda \in h(R)$ such that $a\lambda \neq 0$. By assumption, there exists a $t > 0$ such that $(a\lambda)^t = 0$ and $(a\lambda)^{t-1} \neq 0$. From the inclusion $\mathrm{ann}^l_R(a) \subset \mathrm{ann}^l_R\,(a\lambda)^{t-1}$, with $(a\lambda \in h(I)$, it follows that $\mathrm{ann}^l_R(a) =$ $= \mathrm{ann}^l_R\,(a\lambda)^{t-1}$. Therefore $a\lambda a = 0$. Thus $aRa = 0$ but this contradicts the fact that R is a semiprime ring.

I.1.5. Corollary. Let R be a semiprime graded ring satisfying the ascending chain condition on graded left annihilators. If I is a left (or right) nil-ideal then $I = 0$.

Proof. If $I \neq 0$ then $I^\sim \neq 0$. If $a \neq 0$ is in $h(I^\sim)$ then there is an $x = x_1 + \ldots x_n$ in I such that $x_n = a$, $\deg x_1 < \ldots < \deg x_n$. Since x is nilpotent, a is nilpotent. So $I^\sim$ is a left nil-ideal and it is graded, so the lemma entails $I^\sim = 0$, contradiction.

I.1.6. Theorem. Let R be a semiprime gr-Goldie ring satisfying one (or more) of the following properties :

1. R has regular central homogeneous elements of positive degree.
2. R is positively graded and minimal prime ideals of R do not contain R_+.
3. R is positively graded and there exist regular homogeneous elements of positive degree.
4. R is nilgraded i.e. every homogeneous element of nonzero degree is nilpotent.

Then R has a gr-semisimple gr-Artinian ring of fractions.

Proof. Let L be a nonzero graded left ideal of R. We claim that the hypothesis 1.,2.,3., lead to the existence of an element of positive degree in $h(L)$, which is not a nilpotent element. Indeed, suppose 1. By the foregoing lemma, there exists an $a \in h(L)$ such that $a^k \neq 0$, $k = 1,2\dots$. Pick m such that : $m \deg s + \deg a > 0$. Then $b = s^m a \in h(L)$ is not nilpotent and $\deg b > 0$. Now suppose 2 . If L is not as claimed, then the graded left ideal $L' = L_1 \oplus L_2 \oplus \dots$ is a nil left ideal. By the lemma we have $L' = 0$ and then $L = L_0$. In that case $R_+L = 0$ and therefore L is contained in every minimal prime ideal. Hence $L = 0$, contradiction. Now suppose 3. Then we have :

$$0 \neq sL \subset L_t \oplus L_{t+1} \oplus \dots,$$

where $t = \deg s > 0$. This leads to the existence of a homogeneous element of positive degree which is non-nilpotent in $L_t \oplus L_{t+1} \oplus \dots$ The hypothesis 4. Leads to the existence of a non-nilpotent element of $h(L)$ which has degree zero. The essential part of the proof is now to establish that a graded essential left ideal of R contains a regular homogeneous element. Under the assumptions 1.,2., 3., we find that there is an $a_1 \in h(L)$ with $\deg a_1 > 0$ and $a_1^\nu \neq 0$ $\nu = 1,\dots,\dots$. For some

$n \in \mathbb{N}$: $\mathrm{Ann}(a_1^n) = \mathrm{Ann}(a_1^{n+1}) = \dots$. Put $b_1 = a_1^n$, then $\deg b_1 > 0$ and $\mathrm{Ann}(b_1) = \mathrm{Ann}(b_1^\nu)$. If $\mathrm{Ann}(b_1) \neq 0$ then there is an homogeneous $b_2 \in L \cap \mathrm{Ann}(b_1)$ with $\deg b_2 > 0$ and $\mathrm{Ann}(b_2) = \mathrm{Ann}(b_2^\nu)$, $\nu = 1, 2, \dots$. Proceeding in this way we obtain a direct sum $Rb_1 \oplus Rb_2 \oplus \dots$ of nonzero graded left ideals and since the Goldie dimension of R is finite, there exists an $r \in \mathbb{N}$ such that :

$$\mathrm{Ann}(b_1) \cap \dots \cap \mathrm{Ann}(b_r) = 0$$

Let $d_i = \deg b_i > 0$, $1 \le i \le r$ and $d = d_1 \dots d_r$. Then $c = b_1^{d/d_1} + \dots + b_r^{d/d_r}$ is a homogeneous element of L with $\deg c = d$. However $\mathrm{Ann}\, c = \cap_{i=1}^r \mathrm{Ann}(b_i^{d/d_i}) =$

$= \cap_{i=1}^r \mathrm{Ann}(b_i) = 0$; so Rc is an essential graded left ideal, c is a regular element and $\deg c > 0$. Under the assumption 4, L contains a regular homogeneous element of degree 0. From here on the proof will be similar to the proof of Goldie's theorem in the ungraded case. □

I.1.7. Corollary. A prime positively graded gr-Goldie ring R admits a gr-simple gr-Artinian ring of fractions.

I.1.8. Corollary. Let R be a semisimple gr-Goldie ring. If R is left - and right - limited then R admits a gr-semisimple gr-Artinian ring of fractions.

I.1.9. Corollary Let R be a left and right Noetherian graded ring which is either positively graded or commutative. Let P be any prime ideal of R then either : $ht(P) = ht((P)_g)$ or $ht(P) = ht((P)_g) + 1$.

Proof. Suppose that Q is a prime ideal of R such that $(P)_g \subset Q \subset P$. Up to passing to $R/(P)_g$ we may assume that $(P)_g = 0$ and hence that R is a prime ring.

Let S be the set of regular homogeneous elements of R. Clearly, R being Noetherian prime and graded, R is a graded Goldie ring hence, the foregoing proposition implies that $S^{-1}R$ exists and that it is a gr-simple gr-Artinian ring. From our results on the Krull dimension we retain : K.dim $S^{-1} R \le 1$. On the other hand $S^{-1}Q$ and $S^{-1}P$ are nonzero prime ideals of $S^{-1}R$, therefore $S^{-1}Q = S^{-1}P$. This means, because of the right Noetherian property that $sP \subset Q$ for some $s \in S$, $s \neq o$ hence $P = Q$ or $s \in Q$; $s \in Q$ would imply $s \in (P)_g = 0$ the latter is impossible hence $P = Q$ is the only remaining possibility. Now if $P = P_g$ then $ht(P) = ht(P_g)$ so suppose that $P \neq P_g$ and assume that $ht(P) = n < \infty$. For $n = 1$, For $n = 1$, P_g is a minimal prime ideal, i.e. ht $P_g = 0$. We proceed by induction on n. Let $P_o \subset P_1 \subset \ldots \subset P_n = P$ be an ascending chain of length n of prime ideals contained in P. By the induction hypothesis we have $ht(P_{n-1}) = n-1$ if $P_{n-1} = (P_{n-1})_g$ but in this case $P_g = P_{n-1}$ and also $ht(P) = ht(P_g)+1$. On the other hand if $P_{n-1} \neq (P_{n-1})_g$ then $ht(P_{n-1}) = ht((P_{n-1})_g) + 1$. Now $P_g \neq (P_{n-1})_g$ since otherwise P_{n-1} would be properly contained in P and $P_{n-1} \supset P_g$, so ht $(P_g) > ht((P_{n-1})_g)$. Thus $ht(P_g) \ge n-1$ and $ht(P) \le ht(P_g)+1$. □

<u>I.1.10.Corollary</u>. Let R be a Noetherian commutative ring. If P is a graded prime ideal with $ht(P) = n$, then there exists a chain of graded prime ideals $= P_o \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = P$.

<u>Proof.</u> The statement is true for $n = 1$. If $ht(P) = n>1$ then there is a chain of distinct prime ideals : $Q_o \subsetneqq Q_1 \subsetneqq \ldots \subset Q_n = P$.
If Q_{n-1} is graded then application of the induction hypothesis finishes the proof. Suppose that Q_{n-1} is not graded and then the foregoing corollary yields that $ht((Q_{n-1})_g) = n-2$. The induction hypothesis asserts that we may find a chain of distinct graded prime ideals
$P_o \subset P_1 \subset \ldots \subset P_{n-2} = (Q_{n-1})_g$. Consider the graded ring $R/(Q_{n-1})_g = \bar{R}$ and denote by $\bar{P}, \bar{Q}_{n-1}$ the image of P, Q_{n-1} in $\bar{R}$. Choose a nonzero homogeneous element $\bar{a}$ of $\bar{P}$, hence $\bar{a} \in \bar{Q}_{n-1}$. Let $\bar{P}_{n-1}$ be a minimal prime

ideal containing $\bar{R}\bar{a}$. Then $\bar{P}_{n-1}$ is graded and $\bar{P}_{n-1} \neq \bar{P}$ because otherwise $ht(\bar{P}) \leq 1$ follows from the principal ideal theorem. Putting $\bar{P}_{n-1} = P_{n-1}/(Q_{n-1})_g$ we find a chain of distinct prime ideals :
$P_0 \subset P_1 \subset \dots \subset P_{n-1} \subset P_n = P$. □

I.1.11. Corollary. Let R be a Noetherian commutative graded ring and let I be an ideal of R, then :

$$ht(I) \leq ht(I^{\sim}),\ ht(I) \leq ht(I_{\sim}).$$

Proof. Since $ht(I) = ht(rad(I))$ and $(rad(I))^{\sim} \subset rad(I^{\sim})$ we may reduce the proof to the case where I is a semiprime ideal, $I = P_1 \cap \dots \cap P_s$, P_i prime ideals of R. Then $P_1 \dots P_s \subset I$ yields $P_1^{\sim}.P_2^{\sim} \dots P_s^{\sim} \subset I^{\sim}$. Let Q be a prime ideal containing $I^{\sim}$, such that $ht(I^{\sim}) = ht(Q)$. For some i, $P_i^{\sim} \subset Q$. But $I \subset P_i$ hence we may assume that $I = P$ is a prime ideal in proving the statement. If P is graded then $P = P^{\sim}$. If P is not graded then we have : $ht(P) = ht(P_g) + 1$. Clearly $P_g \subset P^{\sim}$ and $P_g \neq P^{\sim}$. It results from this that $ht(P_g) < ht(P^{\sim})$ and therefore $ht(P) \leq ht(P^{\sim})$. In a similar way, $ht(I) \leq ht(I_{\sim})$ may be established.

If one is interested in having a nice ring of quotients i.e. not being particular about it not being a classical ring of fractions, then one may avoid extra hypotheses as in Proposition I.1.4. and derive another graded version of Goldie's theorem. Write $E^g(R)$ for the injective hull of R in R-gr.

I.1.12. Proposition. Let R be a graded prime Goldie ring then $E^g(R)$ is a gr-simple gr-Artinian ring such that the canonical ring morphism $R \to E^g(R)$ is a left flat ring epimorphism.

Proof. By Theorem I.1.3. the injective hull $E(R)$ of R in R-mod is a simple Artinian ring and one easily verifies (using Goldie's theorems) that this makes $E^g(R)$ a Goldie ring. Injectivity of $E^g(R)$ in R-gr entails that $E^g(R)$ is a graded regular ring (in Von Neumann's sense). The proof that $E^g(R)$ is gr-simple gr-Artinian is now similar to the ungraded case. However, $E^g(R)$ is the localization of R at the graded torsion theory corresponding to the filter generated by the graded essential left ideals of R (see also A.II.9. etc.). The fact that $E^g(R)$ is gr-simple Artinian then entails that this torsion theory is graded perfect (i.e. corresponds to a graded kernel functor satisfying property T, see A.II.10.). The last statement of the proposition then follows by general localization theory.

I.1.13.Corollary. Let $h(R)^r$ be the multiplicatively closed set of regular elements in $h(R)$. Put $E(R) = Q$, $E^g(R) = Q^g$ and let Q_h be the graded ring of fractions obtained by inverting the elements of $h(R)^r$. We have canonical ring homomorphisms : $R \to Q_h \to Q^g \to Q$. Each localization involved is a perfect localization (property T). We have that $Q_h = Q^g$ if and only if graded essential left ideals of R contain homogeneous regular elements.

Proof. R satisfies the left Ore condition with respect to the set of regular elements of R because R is a Goldie ring. Because of Proposition A.II.10. ..., it follows that R also satisfies the left Ore conditions with respect to $h(R)^r$. All assertions of the corollary follow from this and the foregoing results.

I.1.14. Theorem. Let R be a positively graded left Noetherian ring and let P be a graded prime ideal of R. Then $\kappa_P^r = \kappa_{h(P)}$ where $\kappa_{h(P)}$ is the rigid kernel functor on R-gr associated to the multiplicative system $h(G(P)) = h(R) \cap G(P)$. If R satisfies the left Ore conditions with respect to G(P) then it also satisfies these conditions with respect to h(G(P)).

Proof. From Chapter A, Proposition II.11.6. , we retain that $\mathcal{L}(\kappa_P^r)$ consists of the graded left ideals J of R such that the image of J in R/P is an essential left ideal of R/P. By Theorem I.1.6. it follows that $J \cap h(G/P)) \neq \emptyset$; the first statement follows. Consider $s \in h(G(P))$, $r \in h(R)$ and look at (Rs : r). Write $r = r_1+\ldots+r_n$ with $\deg r_1<\ldots<\deg r_n$, and put $L = \bigcap_{i=1}^{n} (Rs : r_i) \subset (Rs:r)$. Then L is a graded left ideal, and L is in $\mathcal{L}(\kappa_P)$ because Rs and $(Rs:r_i)$ are in $\mathcal{L}(\kappa_P)$ for all $1 \leq i \leq n$. Since L is graded, $L \in \mathcal{L}(\kappa_P^r)$. Therefore there exist $s_1 \in h(G(P))$ and $r_1 \in R$ such that $s_1 r = r_1 s$.

Since R is left Noetherian this establishes that R satisfies the left Ore conditions with respect to h(G(P)).

I.2. Graded Rings Satisfying Polynomial Identities.

For the theory of rings satisfying polynomial identities our basic references are the following books : N. Jacobson [53], C. Procesi [90]; L. Rowen [96]. Let us just mention here some results that apply most directly to this section.

Let C be a commutative ring and let R be a C-algebra. Let C $\{X_s\}$ be the free C-algebra in uncountably many variables. If $f(X_s) \in C\{X_s\}$ then only finitely many variables appear in $f(X_s)$ and by "abus de langage" we say that $f(X_s)$ is a polynomial. We say that $f(X_s)$ is a polynomial identity for R if any substitution of $r_1, \ldots, r_n \in R$ for the variables of $f(X_s)$ makes the polynomial zero. The C-algebra R is said to be a P.I. algebra if it satisfies a polynomial identity $f(X_s)$ which is not contained in $I\{X_s\}$ for some ideal I of C.
The standard polynomial of degree n is given by :

$$S_n(X_1, \ldots, X_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, X_{\sigma(1)} \cdots X_{\sigma(n)},$$

where S_n is the symmetric group on n elements and $\varepsilon(\sigma)$ is the signature of σ. Obviously $S_2(x_1, x_2) = x_1x_2 - x_2x_1$ yields a polynomial identity satisfied by all commutative rings. However, not every P.I. algebra satisfies a standard identity, but :

I.2.1. Theorem. (S. Amitsur) If R is a P.I. algebra over C then R satisfies the polynomial identity $S_n(X_1, \ldots, X_n)^n$, for suitable n and m. Note that R satisfies thus an identity with coefficients +1 or -1.

The structure of prime P.I. algebras may now be described accurately in the following :

I.2.2. Theorem. (E. Posner). If R is a prime P.I. algebra then :

1. R has a classical ring of fractions, Q(R) say.
2. Q(R) is a central simple algebra
3. Q(R) is a central extension of R, i.e. $Q(R) = R\, Z_{Q(R)}(R)$.
4. Q(R) satisfies all polynomial identities satisfied by R.

In the above theorem one intuitively hopes that the center of Q(R) is the field of fractions of the center of R. This is actually the case and it follows from a result of E. Formanek on central polynomials. First note that Posner's theorem enables us to embed prime P.I. algebras in matrix rings over commutative fields, just by splitting Q(R) (see Section A.I.1.). This fact highlights the importance of the study of polynomial identities satisfied by n x n matrices over a commutative, preferable infinite, field. A polynomial $g(X_s)$ in $C\{X_s\}$ is said to be a central polynomial for n x n matrices with coefficients in C if every evaluation of $g(X_s)$ in some $M_n(A)$, where A is any commutative C-algebra yields central elements. Both E. Formanek and I. Razmyslov constructed explicit central polynomials (Razmyslov's polynomial is even multilinear) As a consequence one may deduce :

<u>I.2.3. Proposition</u>. Let R be a semisimple P.I. algebra. Then nonzero ideals of R intersect the center non-trivially.

<u>I.2.4. Corollary</u>. If R is a prime P.I. algebra then, $Z(Q(R)) = Q(Z(R))$.

<u>I.2.5. Convention</u>. If C is not important or more precisely, if we consider Z-algebras we will usually speak about P.I. rings (not : algebras).

The <u>Formanek center</u> F(R) of a ring R satisfying the identities of n x n matrices is the subring of Z(R) obtained by evaluating all central polynomials for R without constant term. This F may be viewed as a right semi exact functor and R is an Azumaya algebra of constant rank

n^2 over $Z(R)$ if and only if $R\,F(R) = R$ holds.

The p.i. degree of a prime P.I. algebra R will be $\sqrt{N}$ where $N = \dim_{Z(Q(R))} Q(R)$. If P is a prime ideal of R then we define p.i. deg P = p.i. deg R/P.

I.2.6. Theorem (M. Artin). If R satisfies the identities of $n \times n$ matrices then R is an Azumaya algebra of constant rank n^2 over its center if and only if for every prime ideal P of R, p.i. deg $P = n$.

For extensions, applications etc. ... of the very interesting theorem above we refer to [3], [90], ...
Let us now return to the situation where R is graded of type Z and continue with notations as in I.1.13..

I.2.7. Proposition. If R is a prime graded P.I. ring then $Q_h = Q^g$ i.e. Q^g is a gr-simple gr-Artinian ring of fractions of R.

Proof. Put $C = Z(R)$. If there exists a $c \in h(C)$ with deg $c > o$ then Proposition I.1.4., 1. applies, since c is regular. If there is a $c \in h(C)$ with deg $c < o$ then the graded ring of fractions of R at $\{1,c,c^2,\ldots\}$, Q^g_c say, embeds in Q^g. So we have ring homomorphisms :

$$R \to Q^g_c \to Q_h \to Q^g \to Q.$$

Clearly, Q^g_c is again a prime graded Goldie ring and c^{-1} with deg $c^{-1} > o$ is in the center of Q^g_c. Again by Proposition I.1.4., 1, it follows that Q^g_c has a gr-semisimple gr-Artinian ring of fractions which obviously has to coincide with $Q_h = Q^g$. So we are left with the problem $C = C_o$. By the graded version of Wedderburn's theorem (see A.I.5.), $Q^g \cong M_n(\Delta)(\underline{d})$ for some graded division ring, $n \in N$ and $\underline{d} \in Z^n$. By Posner's theorem, Q is finite dimensional over the field of fractions of C and thus it follows that for every $x \in h(D)$ we have an equation $c_o x^m + \ldots + c_{m-1}\, x + c_m = 0$ with $c_i \in C$, $i = 0,\ldots,m$. But $C = C_o$ implies deg $x = 0$ hence $D = D_0$, and the latter entails that P is a skewfield, thus $Q^g = Q$ follows.

In case D is a non-trivially graded division algebra and also a P.I. ring then the results of A.I.4. entail that $D = D_0[X, X^{-1}, \varphi]$ where D_0 is a skewfield and φ is an automorphism of D_0 some power of which is an inner automorphism (cf. also G. Cauchon [17]). Put $m^2 = [D_0 : Z(D_0)]$ $k = Z(D_0)^{\varphi}$ the fixed field of φ in $Z(D_0)$. The center $Z(D)$ equals $k[T, T^{-1}]$ where $T = \lambda X^e$, some $e \in \mathbb{N}$, $\lambda \in D_0^{\star}$. Since D is an Ore domain it has a skewfield of fractions $Q(D) = D_0(X, \varphi)$ having $k(T)$ for its center. Furthermore one easily verifies that : $[D_0(X,\varphi):k(T)] = [D:k[T,T^{-1}]] = (me)^2$.

<u>I.2.8. Theorem</u>. (Graded version of Posner's theorem).
A graded prime ring R is a P.I.ring if and only if R is a graded order of $M_n(K)(\underline{d})$ for some gr-field K, and some $\underline{d} \in Z^n$, $n \in \mathbb{N}$.

<u>Proof.</u> Clearly, each (graded) order of $M_n(K)(\underline{d})$ is a P.I. ring. Conversely, if R is a prime graded P.I. ring then by Proposition I.2.7. it follows that $Q_h = Q^g$ is a gr-simple gr-Artinian ring of fractions of R. By the graded version of Wedderburns's theorem (cf. A.I.5.), $Q^g = M_{n'}(D)(\underline{d}')$ for some $n' \in \mathbb{N}$, $d' \in Z^{n'}$. If $D = D_0$ then the theorem follows from $Q^g = Q(R)$ since the latter can be split by some trivially graded commutative field. Now assume $D \neq D_0$ is a non-trivially graded division algebra. The remarks preceding the theorem imply that

$$Q(D) = D \underset{k[T,T^{-1}]}{\otimes} k(T) \cong D_0(X,\varphi)$$

Consider a maximal commutative graded subring of D, L say. It is not hard to verify that L is a gr-field and that is also maximal as a commutative subring of D.

Write $Q(L) = k(T) \otimes_{k[T,T^{-1}]} L$; this is a commutative subring of $Q(D)$.
If $y \in Q(D)$ commutes with $Q(L)$ then for some $s \in k[T,T^{-1}]$, $sy \in L$ because L is a maximal commutative subring of D. Consequently, $Q(L)$ is a maximal commutative subfield of $Q(D)$, hence a splitting field.
Since $D \otimes_{k[J,J^{-1}]} L$ is a prime ring, we get :

$$Q(D \otimes_{k[T,T^{-1}]} L) = Q(D) \otimes_{k(T)} Q(L) \cong M_n(Q(L)).$$

Now, $D \otimes_{k[T,T^{-1}]} L$ is a gr-c.s. a. with center L hence it is isomorphic to some $M_k(\Delta)(\underline{d})$ for some graded division algebra Δ, $k \in \mathbb{N}$, $\underline{d} \in Z^k$.
Central localization at the prime ideal 0 yields $= M_k(Q(\Delta)) \cong M_n(Q(L))$, so it follows immediately that $k = n$ and $Q(\Delta) = Q(L)$ hence Δ is commutative or $\Delta = L$. Finally we have established that $M_n(L) \cong D \otimes_{k[T,T^{-1}]} L$ becomes an isomorphism of graded rings if the gradation chosen on $M_n(L)$ is described by $\underline{d} \in Z^n$.

I.2.9. Remark. Posner's theorem implies that every (graded) P.I. ring can be embedded as an order in a full matrix ring $M_n(K)$ over a commutative field K. There exists an infinite number of non-isomorphic Z-gradations on $M_n(K)$ extending the trivial gradation of K. Unless R is itself trivially graded none of these gradations will extend the gradation of R. The above graded version of Posner's theorem may thus be thought of as a statement about the extension of the gradation of R to some "nice" subring of $M_n(K)$.

I.2.10. Proposition. If R is a graded P.I. ring then p.i. deg R/P = = p.i. deg R/P_g for every $P \in \operatorname{Spec} R$.

Proof. That p.i. deg $R/P \leq$ p.i. deg R/P_g is obvious. Now if for some $n \in \mathbb{N}$ the standard identity S_{2n} vanishes on R/P, then the ideal generated

be the evaluations of S_{2n} in h(R) is a graded ideal of R because S_{2n} is multilinear, thus if this ideal is in P then it is in P_g. Therefore S_{2n} also vanishes on R/P_g and therefore :
p.i. deg $R/P_g \leq$ p.i. deg R/P.

I.2.11. Corollary. Let R be a P.I. ring satisfying the identities of n x n matrices and let R be graded of type Z. The radical $\mathrm{rad}(I_n)$ of the central kernel I_n (cf. [96], for more details, I_n = ideal generated by the evaluations in R of a multilinear central polynomial for n x n matrices) is a graded ideal of R.

Proof. Prime ideals containing I_n are exactly those having p.i. degree < n (cf. [90]). Hence $P \supset I_n$ if and only if $P_g \supset I_n$, follows from the foregoing proposition.

I.2.12. Corollary. (Graded version of M. Artin's Theorem).A graded P.I. ring R is an Azumaya algebra of constant rank over Z(R), if and only if every graded prime ideal has p.i. degree equal to p.i. deg R, and if and only if for every maximal ideal M of R the p.i. degree of M_g equals p.i. deg R.

Proof. Straightforward from I.2.10. and I.2.6. . .
Note that in general graded maximal ideals, maximal graded ideals and the graded parts M_g of maximal ideals M are very poorly related. Let us give a trivial example to show that Corollary I.2.12 does allow to economize a lot when checking p.i. degree's of prime ideals in order to verify whether a graded P.I. ring R is an Azumaya algebra of constant rank or not. Take $R = D_0[X,\varphi]$, φ an automorphism of the skewfield D_0; then R is Azumaya if and only if p.i. deg(X) = p.i. deg R and it is known that this happens only if for some $\lambda \in U(D_0)$, λX is central in R.

I.2.13. Proposition. If R is a graded P.I. ring such that its center is a gr-field then R is an Azumaya algebra of constant rank.

Proof. Let f be a multilinear central polynomial for R.
Clearly, f cannot vanish for every homogeneous substitution i.e. there exist $u_1,\dots,u_k \in h(R)$ such that $0 \neq f(u_1,\dots,u_k) \in h(C)$ where $C=Z(R)$. Since homogeneous elements of C are invertible by hypothesis, it follows that $f(u_1,\dots,u_k)$ of rank n^2.

I.2.14. Corollary. Let G a finite abelian group of automorphisms of the finite dimensional skewfield D_0. The iterated twisted polynomial ring $D_0[X_\sigma, X_\sigma^{-1}, \sigma \in G]$ is an Azymaya algebra.

Proof. A rather straightforward modification of the techniques used in foregoing results.

Let us conclude this section with a result concerning splitting fields and splitting gr-fields, first an example :

I.2.15. Example. Consider $D = \mathbb{C}[X,X^{-1},\varphi]$ where φ is complex conjugation. Then $Z(D) = \mathbb{R}[X^2,X^{-2}]$ and $[D:Z(D)] = 4$. Clearly $L = \mathbb{C}[X^2,X^{-2}]$ and $K = \mathbb{R}[X,X^{-1}]$ are splitting fields for D in the sense that $D \otimes_{Z(D)} L \cong M_2(L)(\underline{d})$ for certain $\underline{d} \in Z^2$ determined by the natural gradation of the tensor product (similar for K). Also, $\mathbb{C}(X^2)$ and $\mathbb{R}(X)$ are splitting fields of $\mathbb{C}(X,\varphi)$ and both are graded realizable fields (in the sense of B.I.3.) over $\mathbb{R}(X^2)$. On the other hand $\mathbb{R}(X+i)$ is a maximal commutative subfields of $\mathbb{C}(X,\varphi)$ which is not graded realizable.

I.2.16. Corollary. Let D be a graded division P.I. algebra then Q(D) has graded realizable splitting fields.

Proof. Write $D = D_0[X,X^{-1},\varphi]$, $Q(D) = D_0(X,\varphi)$. It is easily checked that a maximal graded commutative subring of D, L say, yields a realisable splitting field Q(L) of Q(D).

I.3. Fully Bounded Graded Rings.

Throughout this section R is a graded ring of type Z which is supposed to be left Noetherian. In this and the following section we introduce classes of graded rings generalizing in different directions the class of graded P.I. rings, studied in Section I.2. . We use results and results and notations from A.II.7., A.II.9., A.II.11.,...

Let $\mathcal{E}^g(R)$ be the set of equivalence classes up to shifts and isomorphisms of the indecomposable injectives in R-gr. Let $\Phi^g : \mathcal{E}^g(R) \to \mathrm{Spec}_g(R)$ be the map defined by associating to a class $\bar{E}$ the graded prime ideal $\mathrm{Ass}(E)$, where E represents $\bar{E}$.

I.3.1. Lemma. If $P \in \mathrm{Spec}_g(R)$ then $E^g(R/P)$ is P-cotertiary. In particular, the map Φ^g is surjective.

Proof. Decompose $E^g(R/P)$ as a direct sum of indecomposable injectives in R-gr, say $E^g(R/P) = E_1 \oplus \ldots \oplus E_n$. From Proposition I.1.8. we ratain that $E^g(R/P)$ is gr-simple left gr-Artinian, hence $E^g(R/P) = L_1 \oplus \ldots \oplus L_r$ where the L_i are minimal graded left ideals of $E^g(R/P)$ such that for each $i, j \in \{1,\ldots,r\}$ there is an $n_{i,j} \in Z$ such that $L_i \simeq L_j(n_{ij})$. By the Krull-Remak-Schmidt theorem we obtain : $r = n$ and $E_i = L_{\pi(i)}$, for some permutation π of $\{1,\ldots,n\}$. Therefore we have :

$\mathrm{Ass}(E_i) = \mathrm{Ass}(L_{\pi(i)}) = \mathrm{Ass}(L_{\pi(j)}(n_{\pi(i),\pi(j)})) = \mathrm{Ass}(L_{\pi(j)} = \mathrm{Ass}(E_j)$

This yields that $E^g(R/P)$ is P-cotertiary.

Recall that a prime ring is bounded if every essential left ideal contains a nonzero ideal. A ring is fully bounded if each of its prime factor rings is bounded. A prime graded ring R is said to be graded bounded if each gr-essential left ideal contains a nonzero graded ideal. A graded ring R is graded fully bounded if every R/P, $P \in \mathrm{Spec}_g(R)$, is graded bounded.

I.3.2. Theorem. Let R be a left Noetherian graded ring.
The following properties are equivalent :

1. Φ^g is bijective
2. For every $P \in \mathrm{Spec}_g(R)$, R/P is graded bounded.
3. Each cotertiary graded R-module M is isotypic in R-gr i.e. $E^g(M)$ is a direct sum of equivalent indecomposable injectives in R-gr.

Proof.

$1 \Rightarrow 3$ If $M \in$ R-gr is cotertiary and $E^g(M) = \bigoplus_i E_i$ where the E_i are indecomposable injective in R-gr then $\mathrm{Ass}(M) = \mathrm{Ass}(E_i)$ implies that all E_i are equivalent (up to shift and isomorphism) by 1.

$3 \Rightarrow 1$. If E, E' are indecomposable injectives in R-gr such that $P =$ $= \mathrm{Ass}(E) = \mathrm{Ass}(E')$ then $E \oplus E'$ is P-cotertiary and by 3. then E and E' are equivalent.

$1 \Rightarrow 2$. First note that if Φ^g is bijective for R then the corresponding map $\Phi^g(\bar{R})$ is bijective for each graded epimorphic image $\bar{R}$ of R. Hence we may assume that R is prime and $P = 0$. If the implication $1 \Rightarrow 2$ fails, let L be a gr-essential left ideal of R maximal with respect to the property of not containing a nonzero graded ideal. Obviously L has to be gr-irreducible and therefore $\mathrm{Ass}(R/L)$ consists of a single element which clearly has to be the zero ideal. It will now suffice to construct an indecomposable injective E different from $E^g(R/L)$ in R-gr, such that $\mathrm{Ass}(E) = \{0\}$. If we decompose $E^g(R)$ as $E_1 \oplus \ldots \oplus E_n$, where E_i is indecomposable injective in R-gr for each $i = 1,\ldots,n$, then it is clear that $\mathrm{Ass}(E_i) = 0$ for each i. Since R is non-singular no element of $E^g(R)$

is annihilated by L whereas $E^g(R/L)$ certainly contains such nonzero elements. So, putting $E = E_1$, E and $E^g(R/L)$ cannot be equivalent.

$2 \Rightarrow 1$. By Theorem A.II.7.3. we obtain that $P = \mathrm{Ass}(E)$, where E is an indecomposable injective in R-gr, is equal to $\mathrm{ann}_R(Rx)$ for some $x \neq 0$ in h(E). If we put $L = \mathrm{ann}_R x$ then L is a gr-irreducible graded left ideal containing P. Now if L/P is a gr-essential left ideal of R/P then L/P contains a graded ideal $J/P \neq 0$ of R/P. However $J \supset P$ and $JRx \subset Lx = 0$ contradicts $\mathrm{ann}_R(Rx) = P$, therefore L/P is not gr-essential. The latter leads to the existence of a graded left ideal $K \supsetneq P$ such that $P = L \cap K$. Writing K as an intersection of gr-irreducible graded left ideals $K = K_1 \cap \ldots \cap K_r$, we obtain :

$E = E^g(Rx) \cong E^g((R/L)(n)) \subset E^g(R/K_1) \oplus \ldots \oplus E^g(R/K_r) = E^g(R/P)$.

Consequently, we have an injective graded morphism of degree zero : $E(n) \to E^g(R/P)$ for some of $n \in \mathbb{Z}$. From Lemma I.3.1. it thus follows that E is equivalent to one of the indecomposable injectives in the decomposition of $E^g(R/P)$.

A theorem of G. Cauchon states that a left Noetherian ring is fully left bounded if and only if it satisfies P. Gabriel's condition H. We now proceed to prove the graded version of this result.

<u>I.3.3. Lemma.</u> For a left Noetherian graded ring R the following statements are equivalent :

1. If L is a graded left ideal of R then there exists a finite set $\{x_1,\ldots,x_n\} \subset h(R)$ such that $\overset{\circ}{L} = (L:x_1) \cap \ldots \cap (L:x_n)$, where $\overset{\circ}{L}$ denotes the largest graded ideal of R contained in L.
2. For every finitely generated $M \in$ R-gr there exists a finite set $\{m_1,\ldots,m_r\} \subset h(M)$ such that :
   $\mathrm{ann}_R(M) = \mathrm{ann}_R(m_1) \cap \ldots \cap \mathrm{ann}_R(m_r)$.

Proof.

$1 \Rightarrow 2$. Let $g_1,\ldots,g_s \in h(M)$ generate M, then $:\mathrm{ann}_R(M) = \overset{\circ}{L}$ where $L = \bigcap_{i=1}^{s} \mathrm{ann}_R(g_i)$. By 1 we have $\overset{\circ}{L} = \bigcap_{j=1}^{n} (\bigcap_{i=1}^{s} \mathrm{ann}_R(g_i):x_j)$ for some $\{x_1,\ldots,x_n\} \subset h(R)$. Thus $\overset{\circ}{L} = \bigcap_{i,j} \mathrm{ann}_R(x_j g_i)$ and the assertion follows.

$2 \Rightarrow 1$. Since $\mathrm{ann}_R(R/L) = \overset{\circ}{L}$ it follows from 2 that $\overset{\circ}{L} = \mathrm{ann}_R(\bar{x}_1) \cap \ldots \cap \mathrm{ann}_R(\bar{x}_n)$ for some $\bar{x}_i \in h(R/L)$. Now we have that $\mathrm{ann}_R(\bar{x}_i) = (L:x_i)$ for all i, where x_i represents $\bar{x}_i$, and so 1. follows.

The ungraded version of the equivalent conditions in the above lemma yields P. Gabriel's condition (H). If a graded left Noetherian ring R satisfies condition (H) then it also satisfies the conditions mentioned in Lemma I.3.3., which we will refer to as condition $(H)^g$.

I.3.4. Proposition. Let R be a graded left Noetherian ring satisfying condition $(H)^g$ then R is graded fully bounded.

Proof. Let E represent a class of $\mathcal{E}^g(R)$ with $P = \mathrm{Ass}(E)$. By Theorem A. II.7.3., $P = \mathrm{ann}_R(Rm)$ for some $m \in h(E)$. If $(H)^g$ is satisfied then $P = \mathrm{ann}_R(Rm) = \mathrm{ann}_R(x_1) \cap \ldots \cap \mathrm{ann}_R(x_n)$ and it follows that $R/P \to Rx_1(\nu_1) \oplus \ldots \oplus Rx_n(\nu_n)$ for some $(\nu_1,\ldots,\nu_n) \in Z^n$. Consequently : $E^g(R/P) \to \bigoplus_{i=1}^{n} E_i$, where $E_i = E^g(Rx_i)(\nu_i)$ is equivalent to E for each $i \in \{1,\ldots,n\}$. Foregoing results yield that P determines E up to shifts and isomorphisms i.e. Φ^g is bijective or R is graded fully left bounded.

Note that a left Noetherian graded fully bounded ring need not be fully bounded. For example, consider $R = \Delta[X,\varphi]$ where φ is an automorphism of the skewfield Δ, X a variable and multiplication given by $Xa = a^{\varphi}X$. In $\Delta[X,\varphi]$ every graded left ideal is two-sided but it is fully left bounded if and only if some power of φ is inner in Δ.

I.3.5. Remarks.

1. If R is a positively graded fully left bounded left Noetherian ring, then R_0 is left Noetherian fully left bounded.

2. If R is strongly graded then it is graded fully left bounded left Noetherian if and only if R_0 is fully bounded left Noetherian.

3. A left Noetherian graded ring R is left gr-Artinian if and only if R is graded fully bounded and all graded prime ideals are gr-maximal. If R is positively graded then R is left gr-Artinian if and only if R_0 is left Artinian and R_+ is nilpotent.

Now we turn to a torsion theoretic description of graded fully left bounded rings and its applications.

I.3.6. Lemma. Let R be a left Noetherian graded ring and let κ be a rigid kernel functor on R-gr.
Let $C^g(\kappa)$ be the set of graded left ideals of R which are maximal amongst graded left ideals of R not in $\mathcal{L}(\kappa)$. Consider $\kappa_1 = \inf\{\kappa_{K/I}, I \in C^g(\kappa)\}$. Then $\kappa = \kappa_1^r$.

Proof. If $L \in C^g(\kappa)$ then $\kappa(R/L) = 0$ thus $\kappa \leq \kappa_1$ and $\kappa \leq \kappa_1^r$ (see also A. II.9.7.). Conversely if $\kappa_2 \geq \kappa$ is such that $\kappa_2^r \neq \kappa$ then there exists a graded left ideal J of R, $J \in \mathcal{L}(\kappa_2) - \mathcal{L}(\kappa)$. Since R is left Noetherian there is a $J_1 \in C^g(\kappa)$ such that $J \subset J_1$. Now : $\kappa_2(R/J_1) = R/J_1$ implies that $\kappa_2 \geq \kappa_{R/J_1}$ hence $\kappa_2 \geq \kappa_1^r$. It follows that $\kappa = \kappa_1^r$.

I.3.7. Theorem. A graded left Noetherian ring R is graded fully left bounded if and only if every rigid kernel functor on R-gr is symmetric.

Proof. Suppose R is graded fully left bounded and let κ be a rigid kernel functor on R-gr. By the lemma it will suffice to establish that $\kappa_{R/L}^r$ is symmetric, for each $L \in C^g(\kappa)$. Now L is gr-irreducible by definition, hence bijectivity of Φ^g entails that $\kappa^r_{R/I} = \kappa^r_{R/P}$ for some $P \in \mathrm{Spec}_g(R)$. By Proposition A.II.11.6.(2.) and in view of Theorem I.3.2. it is clear that graded fully boundedness of R implies that $\kappa^r_{R/P}$ is symmetric for every $P \in \mathrm{Spec}_g(R)$.

Conversely, suppose now that $\kappa^r_{R/P} = \kappa^r_P$ is symmetric for every $P \in \mathrm{Spec}_g(R)$. If $L \in \mathcal{L}(\kappa^r_P)$ then L mod P is an essential graded left ideal of R/P and every graded left ideal of R with the latter property is in $\mathcal{L}(\kappa^r_P)$. The symmetry of κ^r_P therefore implies that each graded essential left ideal I of R/P contains a graded ideal of R/P which is nonzero i.e. each R/P, $P \in \mathrm{Spec}_g(R)$ is graded bounded. Then Theorem I.3.2. entails that R is graded fully left bounded. □

I.3.8. Corollary. Let R be a left Noetherian positively graded ring, which is graded fully left boundêd, then for every $P \in \mathrm{Spec}_g(R)$ we have : $\kappa^r_P = \kappa_{R-P}$ and $\kappa_{h(P)} = \kappa_{h(R-P)}$ (notation of A.II.9., and I.1.).

Proof. A straightforward consequence of the above, combined with Theorem I.1.10.

I.3.9. Proposition. Let R be a left Noetherian graded fully bounded ring, then the following properties hold :

1. If $M \in R\text{-gr}$ is κ-torsion for some rigid kernel functor on R-gr, then $\mathrm{Ass}(M) \subset \mathcal{L}(\kappa)$.

2. Every graded filter $\mathcal{L}(\kappa)$ of a rigid kernel functor κ on R-gr has the form $\mathcal{L} = \{L$, graded left ideal of R such that there exist $P_o,\dots,P_n \in \mathcal{L}(\kappa) \cap \mathrm{Spec}_g(R)$ such that $P_n.P_{n-1}\dots P_o \subset L\}$.

Proof.

1. Easy.

2. Since $P_n.P_{n-1}\dots P_o \in \mathcal{L}(\kappa)$, $\mathcal{L}(\kappa) \supset \mathcal{L}$ is obvious. For the converse, first note that we may restrict to finitely generated $L \in \mathcal{L}(\kappa)$ because R is

left Noetherian. Now for any finitely generated $M \in R$-gr there exists a chain $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ in R-gr such that each M_i is a tertiary graded submodule of M_{i+1} and M_{i+1}/M_i is annihilated by its associated graded prime ideal (to check this statement one may follow the lines of the proof of the ungraded analoge). So if we start with some finitely generated graded left ideal L in $\mathcal{L}(\kappa)$, we find a sequence of graded left ideals of $R, L_0 = L \subset L_1 \subset \ldots \subset L_n = R$ such that L_r/L_{r-1} is annihilated by a graded prime ideal P_r i.e. $L \supset P_0 \ldots P_x$. Since each L_r/L_{r-1} is an epimorphic image of a graded submodule of R/L, it follows that$(\kappa L_r/L_{r-1}) = L_r/L_{r-1}$ for $r = 1, \ldots, n+1$. From 1 it follows then that $P_{r-1} \in \mathcal{L}(\kappa)$ for $r = 1, \ldots n + 1$.

I.3.10. Proposition. Let R be a left Noetherian graded ring. Consider a symmetric rigid kernel functor κ on R-gr having property (T) and such that ideals I of R extend to ideals $Q^g_\kappa(R)\, j_\kappa(I)$ of $Q^g_\kappa(R)$. Then R is graded fully bounded if and only if $Q^g_\kappa(R)$ is graded fully bounded.

Proof. Straightforward, note that $\underline{Q^g_\kappa(R)} = \underline{Q_\kappa}(R)$ by Corollary A.II.10.12.

I.3.11. Proposition. A graded ring is left Noetherian graded fully bounded if and only if $M_n(R)(\underline{d})$ is left Noetherian graded fully bounded for some (hence all) $n \in \mathbb{N}$, $\underline{d} \in Z^n$.

Proof. The gradation on $M_n(R)(\underline{d})$ described by $\underline{d} \in Z^n$ is as in Section A.I.5. . That E is left Noetherian if and only if the (ungraded) ring $M_n(R)$ is left Noetherian is well known. Assume that $M_n(R)(\underline{d})$ is graded fully bounded for some $n \in \mathbb{N}$, $\underline{d} \in Z^n$, and consider $P \in \mathrm{Spec}_g(R)$ and a gr-essential graded left ideal L of R/P. Obviously, $M_n(L)(\underline{d})$ is a gr-essential graded left ideal of $M_n(R/P)(\underline{d})$ and therefore it contains a nonzero graded ideal $M_n(I)(\underline{d})$ of $M_n(R/P)(\underline{d})$. Thus $L \supset I \neq 0$, where I is a graded ideal of R/P. Conversely, assume that R is graded fully bounded. A graded prime ideal Q of $M_n(R)(\underline{d})$ is necessarily of the form $M_n(P)(\underline{d})$ for some graded prime ideal P of R. Since $M_n(R)(\underline{d})/M_n(P)(\underline{d}) \cong$ $\cong M_n(R/P)(\underline{d})$ we only have to show that $M_n(R/P)(\underline{d})$ is bounded. If we apply Proposition I.1.8. to R/P we obtain that $Q^g = E^g(R/P)$ is obtained by localizing R/P at the rigid kernel functor κ on R/P-gr where κ has graded filter $\mathcal{L}(\kappa)$ generated by the gr-essential graded left ideals of R/P. Now κ is symmetric by Theorem I.3.7. and Q^g is gr-simple left gr-Artinian (cf. I.1.8.). It is easily seen that $M_n(Q^g)(\underline{d}) = M_n(R/P)(\underline{d}) \underset{R/P}{\otimes} Q^g =$ $= Q^g_\kappa(M_n(R/P)(\underline{d})$ is also obtained by localizing $M_n(R/P)(\underline{d})$ at κ' in $M_n(R/P)(\underline{d})$-gr where κ' is the rigid kernel functor with graded filter $\mathcal{L}(\kappa')$ given by the $M_n(H)(\underline{d})$, $H \in \mathcal{L}(\kappa)$.

However $M_n(Q^g)(\underline{d})$ is gr-simple left gr-Artinian hence graded bounded and κ' is clearly symmetric since κ was symmetric. So if L' is a gr-essential graded left ideal of $S = M_n(R/P)(\underline{d})$ then $Q^g_{\kappa'}(S)\ j_{\kappa'}\ (L') = Q^g_{\kappa'}(S)$ and thus $J'.1 \in j_{\kappa'}(L')$ for some graded ideal $J' \in \mathcal{L}(\kappa')$. Since S in prime, $j_{\kappa'} : S \to Q^g_{\kappa'}(S)$ is injective; therefore the definition of κ' entails that $J' = M_n(J)(\underline{d})$ for some graded ideal J of R/P and $J' \subset L'$. Hence $M_n(R/P)(\underline{d})$ is graded bounded.

Let R be an arbitrary graded ring, the polynomial ring R [T] , where T commutes with R, is considered as a graded ring of type Z via :

$$R[T]_n = \sum_{i+j=n} R_i T^j.$$

<u>I.3.12. Theorem</u>. Let R be a graded ring. The following statements are equivalent :

1. R is fully left bounded
2. R [T] is graded fully left bounded.

Proof. If P is a graded prime ideal of $R[T]$ such that $T \in P$ then $P = p + (T)$ where $p \in \mathrm{Spec}_g(R)$, $P = R \cap R$. On the other hand, if $T \notin P$ then $P_\star$ (see A.II.8.) is a proper prime ideal of R. Moreover we may assume the rings R and $R[T]$ to be left Noetherian and prime and it will suffice to establish the boundedness condition.

$1 \Rightarrow 2$. Consider $P \in \mathrm{Spec}_g(R[T])$. If $T \in P$ then $R[T]/P = R/P$ where $p = P \cap R$. A graded essential left ideal L of $R[T]/P$ will therefore contain an ideal I of $R[T]/P$. Since L is graded it will also contain the graded ideal of $R[T]/P$ generated by the homogeneous parts of elements of I, i.e. $R[T]/P$ is graded bounded. In the other situation $T \notin P$, $L_\star$ is an essential left ideal of $R/P_\star$ as is easily checked.
By 1. $L_\star$ contains an ideal I of $R/P_\star$. Put $J = \{l \in R[T]/P,\ l_\star \in I\}$. the properties of $(-)^\star$ and $(-)_\star$ yield that J is an ideal of $R[T]/P$ such that $J_\star \subset L_\star$ i.e. $J \subset (J_\star)^\star \subset (L_\star)^\star$. Since $R[T]$ is left Noetherian. $(L_\star)^\star$ is finitely generated hence there exists an $n \in \mathbb{N}$ such that $T^n(L_\star)^\star \subset L$. So $T^n J \subset L$, but T^nJ is an ideal of $R[T]$, therefore $R[T]$, is graded bounded.

$2 \Rightarrow 1$. Take $p \in \mathrm{Spec}(R)$, then $p^\star$ is a graded prime ideal of $R[T]$ such that $T \notin p^\star$. Let $L \supset p$ be a left ideal of R mapping to an essential left ideal of R/p. Thus $L^\star \supsetneq p^\star$ and if $N \supsetneq p^\star$ is a graded left ideal of $R[T]$ then $N_\star \supsetneq p$ implies $L \cap N_\star \supsetneq p$ or $(L \cap N_\star)^\star \supsetneq p^\star$.
Since, for some $n \in \mathbb{N}$ we have : $T^n(L^\star \cap (N_\star)^\star) \subset L^\star \cap N$ with $T \notin p^\star$, it follows that $L^\star \cap N \supsetneq p^\star$. By 2., $L^\star$ has to contain an ideal J of $R[T]$ which is graded and $J \not\subset p^\star$. By 2., $L^\star$ has to contain an ideal J of $R[T]$ which is graded and $J \not\subset p^\star$.
Finally this implies that $L \supset J_\star$ where $J_\star$ is an ideal of R such that $J_\star \not\subset p$ and thus R/p is bounded.

I.3.13. Proposition. Let R be a positively graded left Noetherian graded fully left bounded ring.

The following properties are equivalent :

1. R is fully left bounded
2. For every $P \in \operatorname{Proj}(R)$, $Q^g(R/P) = M_n(\Delta)(\underline{d})$ where $\underline{d} \in Z^n$ and Δ is a graded skewfield such that either $\Delta = \Delta_0$ or $\Delta = \Delta_0[X, X^{-1}, \varphi]$ where φ is an automorphism of Δ_0 such that φ^e is inner for some $e \in \mathbb{N}$, and where Δ_0 satisfies condition C_0 ;C_0: For every $m \in \mathbb{N}$, $M_m(\Delta_0)$ is algebraic over its center.

Proof. Straightforward, using the fact that $\Delta_0[X, \varphi]$ is fully left bounded if and only if φ^e is inner for some $e \in \mathbb{N}$ and Δ_0 satisfies C_0 (a result of G. Cauchon).

I.4. Birational Graded Algebras.

Birational extensions of rings, in particular Zariski central rings, have been introduced and studied in [113] and [114], [115]. We will recall briefly some of the definitions and basic properties and afterwards specify to the Z-graded case. In view of the conventions introduced in Section A.II.9. we call pre-kernel functor, any left exact subfunctor κ of the identity on R-mod (it differs from a kernel functor because it may be non-idempotent). To κ there corresponds a filter of left ideals of R, $\mathcal{L}(\kappa)$ say, and κ is a kernel functor exactly then when $\mathcal{L}(\kappa)$ is an idempotent filter i.e. if L is a sub-left ideal of I, $I \in \mathcal{L}(\kappa)$, such that I/L is κ-torsion then $L \in \mathcal{L}(\kappa)$. A pre-kernel functor is bilateral if $\mathcal{L}(\kappa)$ allows a cofinal set of ideals. Consequently, a bilateral kernel functor is just a symmetric kernel functor in the sense of A.II.9. If R is left Noetherian then a bilateral pre-kernel functor is a symmetric kernel functor if and only if $\mathcal{L}(\kappa)$ is multiplicatively closed. In general a pre-kernel functor κ on R-mod is said to be radical if and only if it is bilateral and $\mathrm{rad}(J) \in \mathcal{L}(\kappa)$ implies $J \in \mathcal{L}(\kappa)$ for any ideal J of R. Again, it is easily checked that, if R is a left Noetherian ring, a pre-kernel functor is radical exactly then when it is a symmetric kernel functor. In the absence of the Noetherian condition however, radicality does not imply symmetry and vice versa. In the left Noetherian case a symmetric kernel functor κ is completely determined by the set $\{P \in \mathrm{Spec}(R), P \notin \mathcal{L}(\kappa)$ and P is maximal with respect to this property$\}$. If R is not left Noetherian a similar statement fails, moreover, the supremum of bilateral pre-kernel functors (in the lattice of pre-kernel functors on R-mod) need not be bilateral. We aim to avoid these problems by restricting attention to radical pre-kernel functors. Here are the two basic examples of radical pre-kernel functors we will use in the sequel :

1° Consider κ_I, I an ideal of R, given by $\mathcal{L}(\kappa_I)$ = {L left ideal of R such that rad(J) $\supset$ I}.

2° Consider κ_{R-P}, where P is a prime ideal of R, given by the filter : $\mathcal{L}(\kappa_{R-P})$ = {L left ideal of R, L contains an ideal J of R such that J $\not\subset$ P}.

In case R is a graded ring of type Z (or any G) all definitions given above have natural graded equivalents. We leave to the reader the task of finding formulations of these concepts in terms of rigid kernel functors on R-gr, etc...

Here we will consider birational extensions of commutative rings only, for a slightly more general set up cf. [115]. Let C be a commutative ring and let A be a C-algebra.

I.4.1. Definition. We say that A is a birational C-algebra if there exist non-empty open sets U $\subset$ Spec(A), V $\subset$ Spec(C) such that the following conditions hold :

1. If P $\in$ Spec(A) is such that P $\cap$ C $\in$ V then P $\in$ U.
2. The correspondence P $\to$ f^{-1}(P), where f:C $\to$ A defines the algebra structure defines a topological homomorphism U $\cong$ V.

We say that A is a globally birational C-algebra if U = Spec(A). Note that any semiprime P.I. algebra A is a birational Z(A)-algebra where U may be taken to be the open set consisting of prime ideals of A of maximal p.i. degree, (see I.2. for more detail).
Of course, Azumaya algebras present a nice class of examples of globally birational algebras.
Let us fix notations as follows : Y = Spec(A), X = Spec(C), U = Y(I) for some ideal I of A, V = X(I') for some ideal I' of C. We summarize some elementary properties in the following :

I.4.2. Lemma. Let A be a birational C-algebra.

1. We have $U = Y(AI')$ i.e. $\mathrm{rad}(I) = \mathrm{rad}(AI')$.
2. For any ideal J of A there exists an ideal J' of C such that $Y(IJ) = Y(I) \cap Y(J) \simeq X(I') = X(I'J')$ and thus $\mathrm{rad}(IJ) = \mathrm{rad}(AI'J') = \mathrm{rad}(AI') \cap \mathrm{rad}(AJ') = \mathrm{rad}(I) \cap \mathrm{rad}(AJ') = \mathrm{rad}(IJ')$.
3. If $H \subset I$ is an ideal of A then there exists an ideal $H' \subset I'$ of C such that $\mathrm{rad}(H) = \mathrm{rad}(BH')$; $Y(H) \simeq X(H')$.
4. If A is a globally birational C-algebra then $AI' = A$.

Note that the homeomorphism $Y(I) \simeq X(I')$ is in fact given by $Y(H) \simeq X((H \cap C)I')$ for all $H \subset I$; this is due to the commutativity of C here, in general one has to distinguish between birational extensions and Zariski extensions (the latter satisfying the forementioned property). The following theorem enables us to localize at certain radical pre-kernel functors even in the absence of the Noetherian condition.

I.4.3. Theorem. Let A be a birational C-algebra, and let κ be a radical pre-kernel functor on A-mod such that $I \in \mathcal{L}(\kappa)$. Let $\mathcal{L}(\kappa')$ be the filter generated by the set $\{H', H'$ an ideal of C such that $X(H') \simeq Y(H \cap I)$ for some ideal H of B, $H \in \mathcal{L}(\kappa)\}$. If κ' is idempotent then κ is idempotent.

I.4.4. Corollary. If C is Noetherian and κ is a pre-kernel functor on A-mod such that $I \in \mathcal{L}(\kappa)$ then κ will be symmetric if it is radical.

Let us now focus on special pre-kernel functors.

I.4.5. Proposition. Let A be a birational C-algebra.

1. If $H \subset I$ is an ideal of A then $(\kappa_H)' = \kappa_{H'}$ where $(\kappa_H)'$ is obtained from κ_H as indicated in I.4.3., and H' is associated to H as in I.4.2.(3).

2. If $P \in Y(I)$ then $(\kappa_{A-P})' = \kappa_{C-p}$ where $p = A \cap P$.

I.4.6. Corollary.

1. Since C is commutative and $\kappa_{H'}$ and κ_{C-p} are radical, they are also symmetric kernel functors. Therefore, if H is an ideal of A contained in I or if P is a prime ideal of A in $Y(I)$ then κ_H and κ_{A-P} are idempotent i.e. kernel functors.

2. If A is a globally birational C-algebra then to every radical pre-kernel functor κ on A-mod there corresponds a radical pre-kernel functor κ' on C-mod. If κ' is idempotent then κ is idempotent.

All the above results are special cases of the results in [115] . To an arbitrary ring homomorphism $\varphi : R \to S$ there corresponds a lattice homomorphism $\bar{\varphi}$: S-ker $\to$ R-ker where to a kernel functor κ' on S-mod we correspond $\bar{\varphi}(\kappa')$ on R-mod given by its torsion class as follows : $M \in$ R-mod is $\bar{\varphi}(\kappa')$-torsion if $\varphi_*(M)$ is κ'-torsion.

I.4.7. Proposition. Let the C-algebra A contains C in its center. Consider a kernel functor κ' on C-mod and put $\bar{\kappa} = \bar{f}(\kappa')$.

1. If κ' has property T then $Q_{\bar{\kappa}}$ and $Q_{\kappa'}$ coincide on A-mod and $\bar{\kappa}$ has property T.
2. For arbitrary κ', $Q_{\bar{\kappa}}(A) \cong Q_{\kappa'}(A)$.

Proof. Special case of Proposition 2.7. in [115].

I.4.8. Corollary. Let A be a birational C-algebra. Let κ be a radical pre-kernel functor such that $I \in \mathcal{L}(\kappa)$ and let κ' be the corresponding pre-kernel functor on C-mod. If κ' is idempotent then $\bar{f}(\kappa') = \kappa$.

The following proposition sums up some further results of [115] .

I.4.9. Proposition. Let A be a birational C-algebra.

1. Any radical pre-kernel functor κ on A-mod such that $I \in \mathcal{L}(\kappa)$ is idempotent hence symmetric. For all $P \in Y(I)$, κ_{A-P} is a symmetric kernel functor having property T. If A is globally birational then Spec(A) has a basis for the Zariski-topology consisting of T-sets. i.e. open sets $Y(J)$ such that κ_J has property T.

2. Let κ be a radical kernel functor such that $I \in \mathcal{L}(\kappa)$ and suppose κ has property T. Then any ideal J of A extends to an ideal $Q_\kappa(A)j_\kappa(J)$ of $Q_\kappa(A)$, where $j_\kappa : A \to Q_\kappa(A)$ is the canonical localizing ring morphism.

3. Let κ be a symmetric kernel functor such that $I \in \mathcal{L}(\kappa)$ and such that κ' has property T, then $Q_\kappa(A)$ is a globally birational $Q_{\kappa'}(C)$-algebra.

From hereon R is a ring with unit and C denotes the center of R. The ring R is said to be Zariski central if the Zariski topology of Spec R = X has a basis of open sets $X(S) = \{P \in X,\ P \not\supset S\}$ where S varies through the subsets of C; equivalently : R is Zariski central if $\mathrm{rad}(I) = \mathrm{rad}(R(I \cap C))$ for every ideal of R. Examples of Zariski central rings are, semisimple Artinian rings, Azumaya algebras, rings $R = A[X,\varphi,\delta]$ where is a simple ring and where the center of R is not a field : in particular $A[X,\varphi]$ is Zariski central.

I.4.10. Lemma. If R is a Zariski central ring then the filters :

$\mathcal{L}(I) = \{L$ left ideal of R, $L \supset J$ ideal of R, $\mathrm{rad}(J) \supset I\}$

$\mathcal{L}(R-P) = \{L$ left ideal of R, $L \supset J$ ideal of R, $J \not\subset P$,

are idempotent filters for every ideal I of R and every prime ideal P of R.

Because of the above lemma we are able to localize at κ_I, κ_{R-P}, the kernel functors on R-mod associated to $\mathcal{L}(I)$, $\mathcal{L}(R-P)$ resp., even if R is not necessarily left Noetherian. If κ is a symmetric kernel functor on R-mod then let $\mathcal{L}(\kappa^C)$ be the filter generated by the ideals I of C such that $RI \in \mathcal{L}(\kappa)$. The following statements are equivalent, for some $M \in$ R-mod :

1. M is κ-torsion
2. ${}_CM$ is κ^C-torsion

The class of Zariski central rings is closed under taking epimorphic images.

I.4.11. Theorem. Let κ be a symmetric kernel functor on R-mod, R a Zariski central ring, and suppose that κ^C has property T then has κ has property T. Moreover :

1. $Q_\kappa(R)$ is a central extension of R and $Q_\kappa(R)$ and $Q_{\kappa^C}(R)$ are isomorphic rings
2. For every $M \in$ R-mod the C-modules $Q_\kappa(M)$ and $Q_{\kappa^C}({}_CM)$ are isomorphic.
3. If I is an ideal of R then $Q_\kappa(R)\, j_\kappa(I)$ is an ideal of $Q_\kappa(R)$, where $j_\kappa : R \to Q_\kappa(R)$ is the canoncial morphism.
4. There is a one-to-one correspondence between prime ideals of $Q_\kappa(R)$ and prime ideals of R which are not in $\mathcal{L}(\kappa)$.

I.4.12. Lemma. If D is a graded division ring then D is a Zariski central ring.

Proof. $D = D_0[X,X^{-1},\varphi]$ for some skewfield D_0 and an automorphism φ of D_0, cf. A. Theorem 6.3., unless $D = D_0$ is a skewfield and in the latter case the statement is trivial. Now $S = D_0[X,\varphi]$ is Zariski central, cf. [31] and localizing S at the Ore set $\{1,X,X^2,\ldots\}$ yields that D is Zariski central too. □

Now suppose that C is a graded commutative ring and let R be a graded C-algebra.

I.4.13. Definition. R is said to be a graded birational (Gr-birational) C-algebra if it is a birational C-algebra and the open sets Y(I) and

$X(I')$ are given by graded ideals I and I' of A and C respectively.

I.4.14. Proposition. Let R be a Gr-birational C-algebra, let κ be a radical kernel functor on R-mod and let κ' be the radical kernel functor on C-mod associated to κ. Then κ is graded (rigid) if and only if κ' is graded (rigid).

Proof. Suppose κ is graded. If $L \in \mathcal{L}(\kappa)$ then L contains a graded left ideal $J \in \mathcal{L}(\kappa)$. Then $J \cap C \in \mathcal{L}(\kappa')$ is a graded ideal of C, hence, since the ideals $J \cap C$ with $J \in \mathcal{L}(\kappa)$ from a basis for $\mathcal{L}(\kappa')$ it follows that κ' is graded. Conversely suppose that κ' is graded. Let $L \in \mathcal{L}(\kappa)$. Then $X(L \cap C) \simeq Y(L)$ yields $L \cap C \supset J'$ for some graded ideal J' of C, $J' \in \mathcal{L}(\kappa')$. From $L \supset A\,J'$ with $J' \in \mathcal{L}(\kappa')$ it follows that $\mathrm{rad}(AJ') \in \mathcal{L}(\kappa)$. The radicality of κ entails $AJ' \in \mathcal{L}(\kappa)$ and so L contains a graded ideal which is again in $\mathcal{L}(\kappa)$ i.e. κ is graded. The statements about rigidity are easily checked.

We leave to the reader the task of reformulating the properties of birational algebras mentioned before in the graded situation. The proofs of these graded versions arevery similar to the proofs of the results in [115] referred to, keeping in mind that the radical of a graded ideal is again graded. We skip these technical details and turn to Zariski central graded rings.

A graded ring R is said to be a GZ-ring if it is a Zariski central ring. A graded ring R is said to be a ZG ring if for every graded ideal I of R we have that $\mathrm{rad}(I) = \mathrm{rad}(R(I \cap C))$. A GZ-ring is a ZG-ring; we will deal with GZ-rings here, most results may also be obtained for the larger class of ZG-ring.

I.4.15. Theorem. Let R be a graded Zariski central ring and let P be a graded prime ideal of R. Then :

1. κ_{R-P} coincides on R-gr with κ_{C-p} where $p = P \cap C$ i.e. $\kappa_{R-P}(M) = \kappa_{C-p}({}_C M)$ for every $M \in$ R-gr.
2. κ_{R-P} has property T and $Q^g_{R-P}(M) \cong Q^g_{C-p}(M)$ for all $M \in$ R-gr.
3. If J is a graded ideal of R then $Q^g_{R-P}(R)\, j_{R-P}(J)$ is a graded ideal of $Q^g_{R-P}(R)$ (here $j_{R-P} : R \to Q^g_{R-P}(R)$ is the canonical graded morphism of degree o).
4. There is a one-to-one correspondence between graded prime ideals P of R such that $P \in \mathcal{L}(\kappa_{R-P})$, and proper graded prime ideals of $Q^g_{R-P}(R)$.

Consequently, if I is a graded ideal of R (hence rad(I) is a graded ideal of R too), then :

$$Q^g_{R-P}(R) j_{R-P}(\mathrm{rad}(I)) = \mathrm{rad}\ (Q^g_{R-P}(R) j_{R-P}(I)).$$

Proof.

1. Since κ_{R-P} is induced on R-gr by $\underline{\kappa}_{R-P}$ and since $\underline{\kappa}_{R-P}$ and $\underline{\kappa}_{C-p}$ coincide on R-mod it follows that κ_{R-P} and κ_{C-p} coincide on R-gr. Consequently, κ_{R-P} may be regarded as being the kernel functor on R-gr associated to the central (Ore) set h(C-p).

2. From 1 it follows that , $I \in \mathcal{L}(\kappa_{R-P})$ if and only if there is an $x \in I \cap h(C-p)$, but then $x^{-1} \in Q^g_{R-P}(R)$ follows from $x^{-1} \in Q_{R-P}(R)$ and $(Rx)x^{-1} \subset R$ with $Rx \in \mathcal{L}(\kappa_{R-P})$. Thus $Q^g_{R-P}(R)\ j_{R-P}(I) = Q^g_{R-P}(R)$, where $j_{R-P} : R \to Q^g_{R-P}(R)$ is the canonical epimorphism of degree 0. For $M \in$ R-gr, $Q_{R-P}(\underline{M}) \cong Q_{C-p}(\underline{M})$ follows from Zariski centrality of R. Since both $\underline{\kappa}_{R-P}$ and $\underline{\kappa}_{C-p}$ are graded and of finite type it follows that $Q^g_{R-P}(M) \cong Q_{R-P}(\underline{M}) \cong Q_{C-p}(\underline{M}) \cong Q^g_{C-p}(M)$

3. Let J be any graded ideal of R and take $x \in Q^g_{R-P}(R) j_{R-P}(J) Q^g_{R-P}(R)$. Write $x = \sum_i{}' q_i j_i q_i'$ with $q_i, q_i' \in h(Q^g_{R-P}(R))$, $j_i \in h(J)$. We may fix an

element $c \in h(C-p)$ such that $cq' \in j_{R-P}(R)$ for every i. Since c is central in $Q^g_{R-P}(R)$, we get : $cx \in Q^g_{R-P}(R)j_{R-P}(J)$ or $x \in Q^g_{R-P}(R)j_{R-P}(J)$ by left multiplication with $c^{-1} \in Q^g_{R-P}(R)$.

4. For all graded prime ideals Q of R we have that $Q^e = Q^g_{R-P}(R)j_{R-P}(Q)$ is an ideal of $Q^g_{R-P}(R)$, which is non-proper if and only if $Q \in \mathcal{L}(R-P)$. It is easily checked that Q^e is a graded ideal and that $Q^e \cap j_{R-P}(R) =$ $= j_{R-P}(Q)$ if $Q \notin \mathcal{L}(R-P)$. If I is an ideal of $Q^g_{R-P}(R)$ which is not in Q^e then $I \cap j_{R-P}(R)$ is an ideal of $j_{R-P}(R)$ which is not in $j_{R-P}(Q)$ because $Q^g_{R-P}(R)(I \cap j_{R-P}(R)) = I$ by property T for κ_{R-P}.
Therefore, if I and J are ideals in $Q^g_{R-P}(R)$ such that $IJ \subset Q^e$ then $(I \cap j_{R-P}(R))(J \cap j_{R-P}(R)) \subset Q^e \cap j_{R-P}(R) = j_{R-P}(Q)$ and this entails that $I \cap j_{R-P}(R)$ or $J \cap j_{R-P}(R)$ is contained in $j_{R-P}(Q)$ since the latter is a prime ideal. This I or J is contained in Q^e. Combination of the foregoing properties yields that every prime ideal of $Q^g_{R-P}(R)$ equals Q^e for some graded prime ideal Q of R. The second statement in 4. is a trivial consequence of the first. □

I.4.16. Remarks.

1. The same holds for ZG-rings.

2. In the same way similar properties may be established for any kernel functor associated to a central multiplicative set. In particular, such properties hold on a basis for the Zariski topology of Spec R, indeed one may consider κ_{Rc}, $c \neq 0$ in C, which is associated to $\{1, c, c^2, \ldots\}$.

3. From the general theory of graded localizations it follows that :
if R is prime then $Q^g_\kappa(R)$ is prime for any rigid kernel functor on R-gr,
if κ has property T then $Q^g_\kappa(R)$ is left Noetherian if R is Noetherian.

I.4.17. Proposition. Let R be a positively graded ZG-ring. Then R_0 is a Zariski central algebra.

<u>Proof.</u> Let I_0 be an arbitrary ideal of R_0. Clearly C_0 is in the center of R_0. If P_0 is a prime ideal of R_0 such that $P_0 \supset R_0(I_0 \cap C_0)$ then $P_0 + R_+$ is a prime ideal of R such that :

$$P_0 + R_+ \supset \mathrm{rad}(R_0(I_0 \cap C_0)+R_+) = \mathrm{rad}(R((R_0(I_0 \cap C_0) + R_+) \cap C))$$

Now, $(R_0(I_0 \cap C_0)+R_+) \cap C \supset RI_0R \cap C$ is easily checked by comparing parts of arbitrary degree; therefore

$$P_0 + R_+ \supset \mathrm{rad}(R(RI_0R \cap C)) = \mathrm{rad}(RI_0R),$$

hence $P_0 \supset I_0$. $\square$

The first main advantage of Zariski central rings is that, even in the non left Noetherian case, Proj is having the nice properties one can hope for , cf. [125] .

I.5. Graded Von Neumann Regular Rings.

In this section we study graded rings satisfying Von Neumann's regularity condition on homogeneous elements. Part of the theory is presented in the general case of graded rings of type G. We do not go into the theory of Gr-V-rings (V for Villamayor) because the less controlable behaviour of gr-injective modules makes this theory diverge in many different directions.

A graded ring R of type G is said to be Gr-regular if for every homogeneous $x \in h(R)$ there exists y in R such that $x = xyx$. Obviously if $x \in R_\sigma$ then we may replace y by its part $y_{\sigma^{-1}}$ of degree $\sigma^{-1} \in G$.

I.5.1. Proposition. For a graded ring of type G the following statements are equivalent :

1. R is Gr-regular.
2. Every principal left (right) graded ideal is generated by a homogeneous idempotent element.
3. Every finitely generated left (right) graded ideal is generated by a homogeneous idempotent element.

Proof. As in the ungraded case, cf. [41].

I.5.2. Remarks.

1. Homogeneous idempotents have degree $e \in G$ (e : the neutral element of G); however arbitrary idempotents need not be homogenous.
2. If for some $\sigma \in G$, $R_\sigma \neq 0$ then $R_{\sigma^{-1}} \neq 0$ and $R_{\sigma^{-1}} R_\sigma \neq 0$.
3. The ring R_e is a regular ring.
4. In a Gr-regular ring each left (right) graded ideal is generated as a left (right) ideal by its part of degree e.
5. Graded left (right) ideals are idempotent. Graded ideals are semiprime.

I.5.3. Corollary. If R is a strongly graded ring of type G then R is Gr-regular if and only if R_e is regular.

Proof. Apply Theorem A.I.3.4. and Proposition I.5.1..

I.5.4. Remarks.

1. If R is Gr-regular and prime while G is ordered or abelian, then the center $Z(R)$ is Gr-regular and without zero-divisors, hence a graded field of type G. If R is moreover a P.I. ring then it will be an Azumaya algebra (use the technique of I.2.10, I.2.11.)

2. If P is a finitely generated projective graded module over a Gr-regular ring R of type G, then $\mathrm{End}_R P$ is gr-regular.

3. A graded ring R of type G is Gr-regular if and only if all graded left modules are gr-flat (i.e. flat).

4. Let A be a graded finitely generated left module over a Gr-regular ring R and let B be a graded projective left R-module, $f \in \mathrm{HOM}_R(A,B)$, and let C be any finitely generated graded submodule of B, then $f^{-1}(C)$ is finitely generated and a direct summand.

5. Let $B_1,\ldots,B_m$ be finitely generated graded submodules of a graded projective module A, then $B_1 \cap \ldots \cap B_m$ is finitely generated. For the proof of 4. and 5. see Lemma's 2.1. and 2.2. in [41].

I.5.5. Proposition. Let J be a graded ideal of the Gr-regular ring R of type G. Let f_1, f_2 be orthogonal homogeneous idempotent of R/J, then there exist e_1 and e_2, orthogonal homogeneous idempotents in R such that $e_1 \bmod J = f_1$, $e_2 \bmod J = f_2$. If $f_1 + f_2 = 1$ then e_1 and e_2 may be chosen such that $e_1 + e_2 = 1$.

Proof. This statement only affects degree e, and since R_e is regular and $(R/J)_e = R_e/J_e$, this is exactly Proposition 2.18 of [41].

I.5.6. Definition. Recall that a regular ring R is said to be abelian regular if R has no nonzero nilpotent elements, or equivalently : if all idempotent elements of R are central. Let us term the graded ring R of type G Gr-abelian regular if all homogeneous idempotents are central, i.e. if and only if R_e is abelian regular.

Let us recall the following easy but useful lemma :

I.5.7. Lemma. Let R be any semiprime ring. For any idempotent element e of R the following statements are equivalent :

1. e is central in R.
2. e commutes with every idempotent element of R.
3. Re is an ideal of R.
4. $(1-e)Re = 0$.
5. $e\,R(1-e) = 0$.

I.5.8. Proposition. For a Gr-regular ring R the following conditions are equivalent :

1. R is Gr-abelian regular.
2. For every graded prime ideal P of R, R/P is a Gr-division ring.
3. R has no nonzero homogeneous nilpotent elements.
4. All graded left (right) ideals of R are ideals.
5. Every nonzero graded left ideal contains a nonzero central homogeneous idempotent.

Proof. Easy, along the lines of Theorem 3.2. in [41].

In order to deduce from the Gr-abelian regularity condition statements about arbitrary nilpotent elements we need that G is an ordered group. First we note :

I.5.9. Proposition. Let G be an ordered group and let R be Gr-regular of type G. Then R is a left and right nonsingular ring.

Proof. Let us first establish that each $x \neq 0$ in $h(R)$ is nonsingular. Indeed, suppose $Jx = 0$ for some essential left ideal J of R. Then $JxR = 0$ yields $JeR = 0$ for some idempotent $i \in R_e$ such that $x\,R = i\,R$. Hence $J \subset R(1-i)$ but this contradicts $J \subset Ri \neq 0$. Now take $x \neq 0$ arbitrary in R, say $x = x_{\sigma_1}+\ldots+x_{\sigma_n}$ with $\sigma_1 >\ldots > n$. Any $j \in J$ may be written as $j_{\tau_1}+\ldots+j_{\tau_m}$ with $\tau_1 >\ldots> \tau_m$ and $jx = 0$ yields that $j_{\tau_1} x_{\sigma_1} = 0$. Consequently $J^{\sim} x_{\sigma_1} = 0$. Since $x_{\sigma_1} \neq 0$ by assumption, it will suffice to show that $J^{\sim}$ is essential as a left ideal of R, ($J^{\sim}$ is defined in Section A.II.2.) Since J is an essential left ideal of R, we have that for any $y \in h(R)$ there exists an $r \in R$ such that $0 \neq ry \in J$. Writing $r = r_{\gamma_1}+\ldots+r_{\gamma_t}$ with $\gamma_1 >\ldots > \gamma_t$, we obtain $r_{\gamma_i} y \in J^{\sim}$ where γ_i is the highest degree for which $r_{\gamma_i} y \neq 0$. Consequently $J^{\sim}$ intersects every graded left ideal of R in a nontrivial way, i.e. $J^{\sim}$ is gr-essential. Now Lemma A.I.2.8. yields that $J^{\sim}$ is an essential left ideal of R, so $J^{\sim} x_{\sigma_1} = 0$ leads to a contradiction and so R is left nonsingular
A symmetrical argumentation may be used to derive that R is also right nonsingular.

I.5.10. Proposition. Let G be an ordered group and let R be a Gr-regular ring of type G. Then the conditions of Proposition I.5.8. being equivalent to R is Gr-abelian regular, are also equivalent to :

6. R has no nonzero nilpotent elements.
7. Every nonzero graded left ideal of R contains a nonzero central idempotent.

Proof. That 6. is equivalent to 3 in Proposition I.5.8. is obvious when G is ordered. Furthermore, condition 5 in Proposition I.5.8. obviously implies 7. The implication, $7 \Rightarrow 1$ follows from Proposition I.5.9. by the arguments used in proving Theorem 3.2. in [41].

I.5.11. Corollary. Let R be Gr-regular of type G where G is an ordered group. If R is gr-abelian regular then :

1. All idempotent elements of R are central.
2. Every minimal graded prime ideal is completely prime.

Proof.

1. If i is idempotent in R then (1-i)R i consists of nilpotent elements so, if R is Gr-abelian then (1-i)Ri = 0 by Proposition I.5.10. and then i is central because R is semiprime and because of Lemma I.5.7. .
2. If P is a minimal graded prime ideal of R then R/P is a gr-division ring because of Proposition I.5.8., 2. . Since G is ordered, gr-division rings of type G cannot have zero-divisors.

The structure of Gr-abelian rings may be completely described in the strongly graded case for arbitrary groups. We need :

I.5.12. Lemma. Let A be an abelian regular ring. Then every invertible A-bimodule P is free as a left (or right) A-module.

Proof. Suppose P, Q are A-bimodules such that :

$$P \otimes_A Q \cong A \cong Q \otimes_A P$$

We then know that P and Q are finitely generated projective left (right) A-modules. The structure theory for finitely generated projective modules over a regular ring, cf. [41], yields that we may write

$P = Rx_1 \oplus \ldots \oplus Rx_n$.

Define a morphism $f : A \to P$ by $f(a) = ax_1 + ax_2 + \ldots + ax_n$. (f is a left A-morphism). We have that :

$\mathrm{Ker}\, f = \{a \in A,\ f(a) = 0\} = \bigcap_{i=1}^{n} \mathrm{ann}_A(x_i)$. Since A is abelian regular, each left ideal is an ideal, hence $Ra \subset aR$. So if $ax_i = 0$ then $a\,Rx_i = 0$; therefore any $a \in \mathrm{Ker} f$ would annihilate P. But P is faithful since it is invertible, hence f is injective. Since Q is projective as a right A-module we have the exact sequence :

$$0 \to Q \otimes_A A \xrightarrow{1 \otimes f} Q \otimes_A P \simeq A$$

The map $1 \otimes f$ is left A-linear and so we obtain $0 \to {}_AQ \xrightarrow{u} {}_AA$. Now Q is finitely generated as a left A-module, hence there exists an idempotent i in the center of A such that ${}_AQ \cong Ai$. From $(1-i)Ai = 0$ it follows that $(1-i)Q = 0$ but Q is faithful as a left A-module since it is invertible, hence $1-i = 0$ and ${}_AQ \cong {}_AA$ follows. Similarly ${}_AP \cong {}_AA$ follows.

I.5.12. Theorem. Let R be a strongly graded ring of type G. The following statements are equivalent :

1. R is Gr-abelian regular.
2. R is the crossed product $R_e\,[\,\varepsilon\,\langle R\rangle\,,\alpha_R]$ where R_e is an abelian regular ring (for the structure of the crossed products see Section A.I.3.).

Proof.

$2 \Rightarrow 1$. Follows immediately from Corollary I.5.3. and then from Definition I.5.6.

$1 \Rightarrow 2$ From the lemma above and the results of Section A.I.3. in particular Lemma A.I.3.20.

I.5.13. Corollary. Let R be a strongly graded ring of type Z then the following statements are equivalent :

1. R is Gr-abelian regular.
2. $R = R_0\,[X, X^{-1}, \varphi]$ where φ is an automorphism of the abelian regular ring R_0.

I.5.14. Example. If R is graded of type Z but not strongly graded then there exist Gr-abelian regular rings, which need not be of the form

$R_0\ [X,X^{-1},\varphi]$. Let k be a field and for each $n \in \mathbb{N}$ consider $k\,[X_{(n)},X_{(n)}^{-1}]$ where the gradation on $k\,[X_{(n)}, X_{(n)}^{-1}]$ is defined by putting $\deg X_{(n)} = n$. Consider $B = \prod_{n\in\mathbb{N}} k\,[X_{(n)}, X_{(n)}^{-1}]$ and let A be the subring of B consisting of all $b = (b_1, b_2,\dots, b_n,\dots)$ in B which become eventually constant series. It is clear that A is Gr-abelian regular and A_0 is the regular subring of $k^{\mathbb{N}}$ consisting of the eventually constant series. Clearly, $A \neq A_0\,[X,X^{-1}]$ for any possible X one might think of.

Let R be a graded ring of type G; we define :

$N(R) = \sum_i R\,i\,R(1-i)R$, where i runs through the set of all idempotent elements of R ;

$N^g(R) = \sum_j Rj\,R(1-j)R$, where j runs through the set of idempotent homogeneous elements of R;

S(R) = the subring (without identity perhaps) generated by the nilpotent elemenst of R;

$S^g(R)$ = the graded subring of R generated by the homogeneous nilpotent elements of R.

I.5.15. Theorem. Let R be graded of type G where G is an ordered group and suppose that R is a Gr-regular ring . Then $N^g(R) = N(R) = S(R) = S^g(R)$ and $R/N^g(R)$ is gr-abelian regular.

Proof. If $f \in R/N^g(R)$ is an idempotent homogeneous element then there exists an idempotent homogeneous $j \in R$ which lifts f. Since $jR(i-j) \subset N^g(R)$ it follows that $0 = f(R/N^g(R))(1-f)$ and thus f is central i.e. $R/N^g(R)$ is Gr-abelian regular. By Proposition I.5.10.,6., we obtain $S^g(R) \subset S(R) \subset N_g(R) \subset N(R)$. If E is an idempotent, $E \in N(R)$, then its image in $R/N^g(R)$ is central because of Corollary I.5.10. . Therefore $ER(I-E) \subset N^g(R)$, hence $N(R) \subset N^g(R)$ and $N(R) = N^g(R)$.

Let us now proceed to show that $N^g(R) \subset S^g(R)$. Let j be an idempotent homogeneous elements of R, then :

$$RjR(1-j)R = j\,R\,j\,R(1-j)R + (1-j)\,R\,j\,R(1-j)\,R$$
$$\subset j\,R(1-j)R + (1-j)R\,j\,R.$$

Moreover : $j\,R(1-j)R = j\,R(1-j)(1-j)\,R\,j + j\,R(1-j)\,R(1-j) \subset S^g(R)$

Similarly : $(1-j)\,R\,j\,R \subset S^g(R)$.

Therefore $\sum_j R\,j\,R(1-j)R = N^g(R) \subset S^g(R)$.

In the sequel R will be graded of <u>type Z.</u>

<u>I.5.16. Proposition</u>. If R is Gr-regular of type Z then $J(R) = 0$.

<u>Proof.</u> In Section A.I.7. we established that the Jacobson radical of a Z-graded ring is a graded ideal. As a graded ideal of a Gr-regular ring $J(R)$ then has to contain an idempotent element or else $J(R) = 0$. Since the first is impossible the latter is a fact.

<u>I.5.17. Proposition.</u> Let R be a Gr-regular ring of type Z such that for each prime ideal p of R_o there exists a graded prime ideal P of R such that $P_o = p$.

Then the following properties hold :

1. For all $n \in Z$, $R_n\,R_{-n} = R_{-n}\,R_n$
2. For all $n \in Z$, $RR_n = RR_{-n}$ is an ideal of R.

<u>Proof.</u>

1. In a Gr-regular ring each left (right) graded ideal is generated as a left (right) ideal of R by its part of degree zero. Thus $RR_1 = RI_o$ where $I_o = R_{-1}\,R_1$ is an ideal of R_o. From $RR_1 = RR_{-1}R_1$ it follows that $R_1 = R_1R_{-1}\,R_1 = J_o\,R_1$ where $J_o = R_1\,R_{-1}$. Hence : $J_o = R_1R_{-1}.\ R_1R_{-1} = R_1I_o\,R_{-1}$ and $R_1I_o = J_oR_1$, $R_{-1}\,J_o = I_o\,R_{-1}$. Since R_o is regular, I_o and J_o are semiprime ideals. If p is a prime ideal of R_o containing I_o

then the hypothesis on R implies that there is a graded prime ideal P of R such that $P_0 = p$ i.e. P contains $R\, I_0\, R$. Consequently $p = P_0$ contains $R_1\, I_0 R_{-1} = J_0$ and thus we obtain $J_0 \subset I_0$. Similarly, by symmetry, we find $I_0 \subset J_0$ and $I_0 = J_0$. In formally the same way it can be shown that $R_n\, R_{-n} = R_{-n}\, R_n$ for all $n \in Z$.

2. Consider $RR_1\, R_n$ for some positive $n \in Z$.
Clearly : $I_0\, R_n = R_1\, I_0\, R_{-1}\, R_n \subset R_1\, I_0\, R_{n-1}$ and thus $RR_1\, R_n \subset RR_1\, I_0\, R_{n-1} =$ $= R\, I_0^2\, R_{n-1} = RI_0\, R_{n-1} = RR_1\, R_{n-1}$. Repetition of this argumentation yields that $RR_1 R_n \subset RR_1$ for all $n \geq 0$. A similar calculation entails that $RR_{-1}\, R_m \subset RR_{-1}$ for all $m \leq 0$ in Z. But since $RR_1 = RR_{-1}R_1 = RR_1\, R_{-1} =$ $= RR_{-1}$ it follows that RR_1 is an ideal of R. Now note that :
$R_n\, R_1 = (RR_1)_{n+1} = (R_1 R)_{n+1} = R_1\, R_n$ holds for all $n \in Z$.
To prove that RR_n is an ideal we proceed by induction on n (we write the proof for $n > 0$, the case $n < 0$ is similar). Suppose we know that RR_m is an ideal for each m, $0 < m < n$. If for some m, RR_m is an ideal then RR_m is as right ideal generated by its part of degree zero i.e. $RR_m = R_{-m}R_m\, R = R_m\, R_{-m}\, R = R_m\, R$ follows. Consider $RR_n\, R_t$ for some $t \in Z$. If $-n < t < n$ then RR_t is an ideal, hence $RR_t = R_t R$ yields $R_l\, R_t = R_t\, R_l$ for all $l \in Z$. Hence $RR_n\, R_t \subset RR_n$. If $t > n$ then $RR_n\, R_t = RR_{-n}\, R_n\, R_t = RR_n\, R_{-n}\, R_t \subset RR_n\, R_{t-n}$. Repetition of this argument leads to $RR_n\, R_t \subset RR_n\, R_{t_0}$ where $t_0 = t \bmod n$ i.e. $t_0 < n$, hence $RR_n R_t \subset RR_n$ as before. If $t < -n$ then interchanging the role of n and -n yields : $RR_n\, R_t \subset RR_{-n}\, R_{t_0} = RR_n\, R_{t_0} \subset RR_n$. Therefore RR_n is an ideal of R.

<u>I.5.18. Corollary.</u> If R is Gr-abelian regular then it satisfies the condition of Proposition I.5.17.

Proof. If p is a prime ideal of R_0 then R p is a graded left ideal of R hence it is an ideal. Since $(Rp)_0 = p$ it is clear that there exists a graded prime ideal P of R such that $P_0 = p$.

I.5.19. Proposition. If R is a left Noetherian Gr-regular ring of type Z, then it is a gr-semisimple gr-Artinian ring. Moreover, R_0 is semi-simple Artinian and R_i is a semi-simple R_0-module for each $i \in Z$.

Proof. If R is not gr-semisimple then there is a graded left ideal of R which is not a direct summand, hence it cannot be finitely generated. The latter contradicts the left Noetherian hypothesis. The results of Section A.II.6. entail that R_i is a semisimple R_0 - module for each $i \in Z$. The decomposition of $R = S^{(1)} \oplus \ldots \oplus S^{(n)}$ into gr-simple gr-Artinian rings yields that $S_0^{(1)} \oplus \ldots \oplus S_0^{(n)}$ is a decomposition of R_0 into simple Artinian rings. Note also that in this case $R_i = S_i^{(1)} \oplus \ldots \oplus S_i^{(n)}$, where $S_i^{(j)}$ is a simple R_0-module or zero depending on whether i is divisible by e_j or not, where e_j is the lowest positive degree appearing in the gradation of the gr-simple ring $S^{(j)}$.

If $M \in R$-gr then $M^+ = \bigoplus_{n\geq 0} M_n$ (notation of Section A.II.2., in particular Proposition A.II.2.3.). We investigate graded prime ideals of R and R^+ and also the left Ore conditions with respect to these prime ideals.

I.5.20. Lemma. Let P be a graded prime ideal of the Gr-regular ring R of type Z. If $P \supset N^g(R)$ then P^+ is a graded prime ideal of R^+.

Proof. Since R/P is a prime epimorphic image of the Gr-abelian regular ring $R/N^g(R)$ it follows that R/P is a gr-division ring i.e. either a skewfield Δ if R/P is trivially graded, or else of the form $\Delta[X, X^{-1}, \varphi]$ as mentioned in A.I.4. So R^+/P^+ is either Δ or $\Delta[X, \varphi]$ i.e. a prime ring in either case.

I.5.21. Proposition. In the situation of Lemma I.5.20. $h(G(P^+)) = = h(G(P))^+ = h(G(P)) \cap R^+$, and R^+ satisfies the left Ore condition with respect to $h(G(P^+))$ if and only if R satisfies the left Ore condition with respect to $h(G(P))$.

Proof. If R/P is trivially graded then all statements are easily deduced from the fact R/P is a skewfield.
So now suppose $R/P \cong \Delta[X,X^{-1},\varphi]$. Since G(P) is exactly the set of elements mapping to left regular element of R/P it follows that $G(P) = R-P$ $h(G(P)) = h(R-P)$, and the first statements are easily verified. Assume $\deg X > 0$ and choose a representative Y in h(R) for X. Obviously $Y \in h(G(P))^+$ and for any $s \in h(G(P))$, $Y^m s \in h(G(P))^+$ for some m large enough. If $r \in h(R)$, $s \in h(G(P))^+$ are such that $rs = 0$ then the left Ore conditions with respect to $h(G(P))$ entails that $s'r = 0$ for some $s' \in h(G(P))$ and thus $Y^m s'r = 0$ for some $Y^m s' \in h(G(P))^+$. Furthermore, for given $r \in h(R)$, $s \in h(G(P))^+$ we find $r' \in h(R)$, $s' \in h(G(P))$ such that $s'r = r's$, hence $Y^m s'r = (Y^m r')s$.
Consequently, if R satisfies the left Ore conditions with respect to $h(G(P))$ then R^+ satisfies these conditions with respect to $h(G(P))^+$. Conversely, if $r \in h(R)$, $s \in h(G(P))$ are such that $rs = 0$ then $Y^m rsY^1 = 0$ where m,1 are chosen such that $m\deg Y + \deg r > 0$, $1 \deg Y + \deg s > 0$.
The left Ore conditions with respect to $h(G(P))^+$ yield $s'Y^m r = 0$ for some $s' \in h(G(P))^+$, putting $s'' = s'Y^m$ we found $s'' \in h(G(P))$ such that $s''r = 0$. Also, for given $s \in h(G(P))$, $r \in h(R)$ we find again m, 1 such that $Y^m r \in R^+$, $Y^1 s \in h(G(P))^+$. Therefore we obtain : $s'Y^m = r'Y^1 s$, for some $r' \in h(R^+)$, $s' \in h(G(P))^+$ and since $s'Y^m \in h(G(P))$ this finishes the proof. □

I.5.22. Corollary. If P is a graded prime ideal of the Gr-regular ring R such that $P \supset N^g(R)$ then R^+ satisfies the left Ore condition with respect to $h(G(P^+))$.

Proof. R satisfies the left Ore condition with respect to $h(G(P))$ if and only if $R/\kappa_P(R)$ satisfies the left Ore condition with respect to the image of P, where κ_P is the rigid kernel functor associated to $h(G(P))$ (see A.II.9.). But $\kappa_P(R) = P$; indeed, if $p \in h(P)$ then $p = pyp$ yields $(1-py)p = 0$ with $1-py \in G(P)$! Furthermore $R/\kappa_P(R)$ is of the form Δ or $\Delta[X,X^{-1},\varphi]$ for some skewfield Δ. Evidently R/P satisfies the Ore conditions with respect to zero. The statement is then a direct consequence of Proposition I.5.21.

I.5.23. Remark. In a way formally similar to the above we can prove that a Gr-regular ring R of type Z satisfies the left and right Ore conditions with respect to $G(P)$, and $h(G(P))$, for any graded prime ideal P of R containing $N^g(R)$.

I.5.24. An example : the Weyl-algebra.

Let k be any field and let $S = k\{x,y\}$ be the specialization of the free algebra in two indeterminates defined by the relation $XY-YX = 1$. Put $\deg X = 1$, $\deg Y = -1$. Then S is a graded ring such that $S_0 = k[XY]$. The total graded ring of fractions of S is then given by $Q^g(S) = k(YX)[X,X^{-1},\varphi]$ where φ is the automorphism of $k(XY)$ defined by $\varphi(XY) = 1 + XY$. If char $k \neq 0$. then φ has finite order and then $Q^g(S)$ is a finite dimensional gr-division algebra over its center $k(XY)^{\varphi}[T,T^{-1}]$ where $T = X^n$, some $n \in \mathbb{N}$. If char $k = 0$ then S is the first Weyl-algebra i.e. a simple Ore domain which is not a regular ring. Therefore we see that $Q^g(S)$ is a simple domain which is Gr-regular and even Gr-abelian regular but certainly not a regular ring. It is well-known that

$S \cong k[X][y, \delta]$ where δ denotes the "derivative with respect to X". It may be surprizing to see how this differential polynomial ring has a graded ring of fractions which is is a crossed product. The gradation on S shows that the part of degree zero of a simple and graded domain need not be simple ! One can calculate furthermore that the subrings $k[X^nY^m]$ with $(n,m) = 1$ are maximal graded commutative subrings of S and these are also maximal as commutative subrings of S.

Finally let us mention some statements about the incompatibility of gradations and common Von Neumann regularity. A non - trivially graded ring is said to be nilgraded if R_n consist of nilpotent elements for all $n \neq 0$; If R is not nilgraded then we say that the gradation of R is reduced, or simply that R is reduced.

I.5.25. Proposition If R is a reduced graded ring then R is regular only if R is trivially graded.

Proof. Suppose $0 \neq x_n \in R_n$ is not nilpotent, say $n > 0$. If R is regular then there exists $y \in R$ such that $(1+x)_n\, y(1+x_n) = 1+x_n$.
Write $y = y_{i_1}+\ldots+y_{i_m}$ where $i_1>\ldots>i_m$. Then y_{i_m} is the part of minimal degree in the left hand side hence $y_{i_m} = 1$, $i_m = 0$ and $i_1>\ldots>i_m = 0$ follows. Straightforward calculations yield :

$$2x_n+y_{i_1}+\ldots+y_{i_m}+x_ny_{i_1}+\ldots+y_{i_1}x_n+\ldots = 1+x_n$$

Since all y_{i_k} have positive degree it follows easily that :
$y_{i_k}=0$ if $i_k < n, y_n = -x_n$, $y_{n+m} = 0$ for $0 < m < n$, $y_{2n} = x_n^2$ and so on, i.e.
$(1+x_n)(1-x_n+x_n^2-x_n^3+\ldots)(1+x_n) = 1+x_n$.
Since the homogeneous decomposition of y is finite it follows that x_n must be nilpotent, contradiction. The only possibility left is that R is trivially graded. $\square$

I.5.26. Proposition. Let R be a reduced graded ring and consider the graded ring $S = R[X,X^{-1}]$ where X commutes with R and the gradation on S is defined by $S_m = \sum_{i \in Z} R_i X^{m-i}$. If S is gr-regular then R is trivially graded and regular.

Proof. Suppose $0 \neq x_n \in R_n$ is not nilpotent and consider $1 + x_n X^{-n} \in S_o$. By regularity of S_o we obtain :
$(1 + x_n X^{-n})(1 + z_i X^{-i} + \ldots)(1 + x_n X^{-n}) = 1 + x_n X^{-n}$ for some $z_i \in R_i$.
Now first use the degree in X in order to get rid of the terms in X^i with $i \notin nZ$ and then to derive that $i > 0$ for all appearing terms, Then using the gradation of S one easily calculates, as in the foregoing proof, that $z_n = -x_n$, $z_{2n} = x_n^2$ a.s.o. leading to the fact that x_n has to be nilpotent i.e. $R = R_o = S_o$ and R is regular.

Let us include some observations about graded rings of type Z with Gr-regular center.
If graded (left) ideals are idempotent then R is said to be Gr-fully (left) idempotent. A Gr-fully (left) idempotent ring has a Gr-regular center. A graded ring R such that each graded principal ideal is generated by a homogeneous central idempotent is said to be Gr-biregular So if R is Gr-biregular then R_o is biregular, R is Gr-fully left idempotent and Z(R) is Gr-regular. The following result is an easy adaptation of Proposition 1.2. and 1.3. in [85].

I.5.27. Theorem. Let R be a graded ring with center C, then the following conditions are equivalent :
1. C is Gr-regular and for each $M \in \Omega_g(R)$, $M = R(M \cap C)$ where $\Omega_g(R)$ is the set of r-maximal ideals of R.
2. R is Gr-fully left idempotent and a Zariski extension of C (cf. I.4. for Zariski extensions).

3. R is Gr-fully idempotent and a Zariski extension of C.
4. R is Gr-biregular.
5. For every $m \in \Omega_g(C)$, $Q^g_{C-m}(R)$ is gr-simple.
6. R is a Zariski extension of C and $Q^g_{R-M}(R)$ is gr-simple for all $M \in \Omega_g(R)$.

If moreover, R is finitely generated as a C-module then R is a Zariski extension of C is and only if R satisfies the left and right Ore conditions with respect to G(P) for all graded prime ideals P of R.

I.5.28. Theorem. Let R be a graded ring with regular center C, which is finitely generated as a C-module then the following statements are equivalent :

1. R is semiprime
2. R is Gr-regular
3. R is Gr-biregular
4. R is a Zariski extension of C and all graded prime ideals of R are idempotent.
5. R is an Azumaya algebra over C.

Proof. Easy modification of 2.6. in [85]. Note that already the weaker homogeneous conditions on C make R into an Azumaya algebra. This is of course related to the graded version of the Artin-Procesi theorem for P.I. rings as expounded in Section I.4.

I.5.29 Remark. The above may be used in describing the "graded Brauer group" of a commutative Gr-regular ring C. However, the study of graded Azumaya algebras over C reduces easily to the study of graded Azumaya algebras over the stalks at gr-maximal ideals of C. The latter stalks are graded fields of the form $k[X,X^{-1}]$ and hence the study of the graded Brauer groups of C in the sense of [117] reduces to the study of the graded Brauer groups of graded fields for which we can refer to [12] .

I.6. Exercises, Comments, References.

1. Exercises.

1. Let R be a strongly graded ring of type G, where G is a finite group. Show that : rad $R_e = R_e \cap \text{rad } R$.

2. If R as in 1 is semiprimitive then R_e is semiprimitive. The converse holds if the order of G is invertible in R_e.

3. Let R be strongly graded of type G, $|G| = n < \infty$, and assume that R_e is semi-prime. Show that R has a unique maximal nilpotent ideal N and $N^n = 0$. Moreover, if R_e has no n-torsion then R is semiprime.

4. Let R be strongly graded by a finite group G of order n. The injective hull of ${}_R R$ in R-mod is graded; actually $E(R)_\sigma$ is the injective hull of R_σ in R_e-mod. Apply this to derive the following :

   a) The maximal quotient ring $Q_{max}(R)$ is strongly graded of type G.

   b) $(Q_{max}(K))_e = Q_{max}(R_e)$

   c) If R is a crossed product then $Q_{max}(R)$ is a crossed product.

5. Let R be as in 4. If R is a semiprime left Goldie ring then R_e is a semiprime left Goldie ring. In this case, the classical ring of quotients of R, Q say, is strongly graded of type G and $Q_e = Q(R_e)$.

6. Let R be as in 4, and assume that R_e is a semiprime left Goldie ring then $Q_{max}(R) = E(R)$ is a quasi -Frobenius ring. Moreover $Q_{max}(R) = S^{-1}R$ where S is the set of all non-zero divisors of R_e.

7. Let R be a graded ring of type Z and consider $M \in$ R-gr. Let $s_R(M)$ resp. $s_R^g(M)$ be the socle resp. the graded socle of M. Show that

$s_R(M) \subset s_R^g(M)$.

8. Let R be a graded ring of type Z/pZ, where p is a prime number, and let M be a graded R-module. Suppose that M contains a finite number of non-isomorphic simple R-submodules $N_1,\ldots,N_r$. Put $p_i = \text{char } D_i$, where $D_i = \text{End}_R(N_i)$, $i = 1,\ldots,r$, and assume that $p \neq p_i$ for all i. If $x \in s_R(M)$ has homogeneous decomposition $x = x_{i_1} + \ldots + x_{i_k}$, then $p\, x_{i_j} \in s_R(M)$ for all $j \in \{o, p-1\}$.

9. Using 8, prove the following result :
If R is graded of type Z, $M \in R$-gr, then $s_R(M)$ is a graded submodule of M.

10. Deduce from 9 a new proof for the fact that if $M \in R$-gr is gr-simple then M is either a simple R-module or a 1-critical R-module.

11. Let R be a graded ring of type Z and consider a projective graded R module P. Show that the following statements are equivalent :
    a) Homomorphic images of P are flat.
    b) Finitely generated submodules of P are direct-summands.
    c) For every submodule P' of P and every right ideal I of R, $IP \cap P' = I P'$.

    If one of these conditions holds then P is said to be a gr-regular module.

12. With terminology as in 11. The graded projective R-module P is regular if P is isomorphic in R-gr to a direct sum $I_1(n_1) \oplus \ldots \oplus I_r(n_r)$ where $n_1,\ldots,n_r \in Z$ and I_j is a gr-principal left ideal of R, which is gr-regular, for each $j = 1,\ldots,r$.

13. Let R be a reduced graded ring, then a gr-regular module P is regular if and only if it is trivially graded.

14. If $M \in R$-gr then $\mathrm{END}_R(M)$ is Gr-regular (Von Neumann) if for every $f \in h(\mathrm{END}_R(M))$, Ker f and Im f are direct summand of M.

15. Let R be a graded ring of type Z. The following properties are equivalent :

    a) R is gr-semisimple gr-Artinian.

    b) There exists an infinitely generated gr-free module F such that $\mathrm{END}_R(F)$ is a Gr-regular ring.

16. Let $P \in R$-gr be a finitely generated gr-regular R-module, then $\mathrm{END}_R(P)$ is a Gr-regular ring.

17. Let R be a commutative graded ring of type Z. If P is a graded projective R-module such that $\mathrm{END}_R(P)$ is a Gr-regular ring, then P is a gr-regular module.

18. For a graded R-progenerator P, the following statements are equivalent :

    a) R is a Gr-regular ring

    b) P is a gr-regular R-module.

    c) $\mathrm{END}_R(P)$ is a Gr-regular ring.

    d) P is a gr-regular $\mathrm{END}_R(P)$-module.

2. Comments.

The theory of graded P.I. rings, in particular the projective spectrum Proj may be further studied in a geometrical context cf. the theory of non-commutative projective algebraic varieties in [125]. The results in Chapter C may also be used in studying (graded) maximal orders over graded domains in particular graded Krull domains, cf. [18], [64], both in the P.I. case or in the more general gr-Goldie case. If G is a torsion free abelian group satisfying the ascending chain condition on cyclic subgroups then a gr-maximal order Λ in some c.s.a. Σ is a maximal order over its center which is a (graded) Krull domain, cf [64], hence a non-commutative Krull ring in the sense of [18].
The notion of a generalized Rees ring may be extended by using the class group of a maximal order over a Krull domain instead of the Picard group of a maximal order over a Dedekind domain.
These recent developments may be found in :
L. Le Bruyn, F. Van Oystaeyen, [64].
The notions of GZ-rings (or ZG-rings) and gr-birationality may also be applied to non-commutative projective geometry, cf [125], and also the theory of gr-orders over gr-Dedekind rings because a gr-maximal order (over a gr-Dedekind ring) in some c.s.a. turns out to be a GZ-ring.
We did not include these applications here but we hope that the references above establish contact with some of the up-to-date research in this area.

3. References for Chapter C.

G. Cauchon [17]; M. Chamarie [18]; E. Formanek [28] ; A. Goldie [39]; I. Herstein [49]; L. Le Bruyn, F. Van Oystaeyen [64]; C. Năstăsescu, F. Van Oystaeyen [82],[84] ; C. Procesi [90] ; Y. Razmyslov [91] ; L. Rowen [96]; F. Van Oystaeyen [110],[115], [118], [119] ; A. Verschoren, F. Van Oystaeyen [124], [125], [126].

For Picard groups of orders we refer to A.Fröhlich [31] .
The generalization of the Picard group introduced by A. Verschoren in [128] allows the application to orders over Krull domains mentioned in the comments.

CHAPTER D:
FILTERED RINGS AND MODULES
I: The Category of Filtered Modules

I.1. Definition. An associative ring R is said to be a filtered ring if there is an ascending chain $\{F_nR, n \in Z\}$ of additive subgroups of R satisfying the following conditions : $1 \in F_0R$, $F_nR.F_mR \subset F_{n+m}R$ for any $(n,m) \in Z^2$. The family of these subgroups, denoted by FR, is called the filtration of R.

Note that the definition implies that F_0R is a subring of R.

I. 2. Examples.

1° Any ring R can be made into a filtered ring by means of the trivial filtration , i.e. $F_nR = 0$ if $n < 0$, $F_nR = R$ if $n \geq 0$.

2° Let I be an ideal of R. The I-adic filtration is obtained by putting $F_nR = R$ for $n \geq 0$ while $F_nR = I^{-n}$ for $n < 0$.

3° Let R be any ring and let $\varphi : R \to R$ be an injective ring homomorphism, $\delta : R \to R$ a φ-derivation i.e. a group homomorphism for the additive structure of R such that $\delta(xy) = \delta(x)y + x^{\varphi}\,\delta(y)$ for $x,y \in R$. The ring of skew polynomials $R[X,\varphi,\delta]$ is obtained from R by adjoining a variable X and defining multiplication according to : $Xb = r^{\varphi}X + \delta(r)$, for $r \in R$, in such a way that R becomes a subring of $R[X,\varphi,\delta]$. (Note that the action of endomorphisms is written exponent-wise). The fact that φ is injective allows to introduce the the degree function (denoted by : deg) in the usual way and so we may define the degree - filtration by: $F_nR[X,\varphi,\delta] = \{P \in R[X,\varphi,\delta]$, $\deg P \leq n\}$ hence $F_0R[X,\varphi,\delta] = R$.

I. 3. Definition. Let R be a filtered ring. A left R module M is called a filtered module if there exists an ascending chain $\{F_nM, n \in Z\}$

of additive subgroups of M such that $F_nR.F_mM \subset F_{n+m}M$ for any $(n,m) \in Z^2$. The family $\{F_nM, n \in Z\}$ is the <u>filtration</u> of M. Obviously, if R is a filtered ring $_RR$ (resp. R_R) is a filtered left (resp. right) R-module.

If $F_0R = R$, then F_iR is a two-sided ideal for any $i \leq 0$ and F_iM are submodules of M.

If M is a filtered left R-module then its filtration F may have one of the following properties :

(E) F is <u>exhaustive</u> if $M = \bigcup_{n \in Z} F_nM$.

(D) F is <u>discrete</u> if there is an $n_0 \in Z$ such that $F_iM = 0$ for all $i < n_0$.

(S) F is <u>separated</u> if $\bigcap_{n \in Z} F_nM = 0$.

In the filtration of R is trivial and $M \in$ R-mod, then any ascending chain of submodules of M defines a filtration for M; in this case the <u>trivial filtration of M</u> is given by $F_nM = 0$ if $n < 0$, $F_nM = M$ if $n \geq 0$.

<u>I. 4. Remarks.</u>

1. If the filtration FR of the ring R is exhaustive then R is a topological ring with the family $(F_nR)_{n \in Z}$ for a fundamental system of neighbourhoods of 0 in R.

Moreover, if the filtration FM of the module M is exhaustive then M is a topological R-module with the family $(F_nM)_{n \in Z}$ for a fundamental system of neighbourhoods of 0 in M.

2. If the filtration FR is exhaustive, then

$$(\bar{0}) = \bigcap_{n \in Z} F_nM$$

is a submodule of M.

Indeed, if $\lambda \in R$ and $x \in \bigcap_{n \in Z} F_nM$, there exists a $p \in Z$ such that $\lambda \in F_pR$. As $x \in F_{n-p}M$ then $\lambda x \in F_pR.F_{n-p}M \subset F_nM$ (arbitrary n). Therefore $\lambda x \in \bigcap_{n \in Z} F_nM$.

I. 5. Definition. Let R and S be filtered rings. An $M \in$ R-mod-S is said to be a filtered R-S module if there exists an ascending chain $\{F_nM,\ n \in Z\}$ of additive subgroups such that : $F_nR.F_mM \subset F_{n+m}M$ for all $n,m \in Z$, and $F_mM.F_tS \subset F_{m+t}M$ for all $n,\ t \in Z$.

Let R be a filtered ring, M and N filtered left R-modules. An R-homomorphism $f \in \mathrm{Hom}_R(M,N)$ is said to have degree p if $f(F_iM) \subset F_{i+p}N$ for all $i \in Z$. Homomorphisms of finite degree form a subgroup $\mathrm{HOM}_R(M,N)$ of $\mathrm{Hom}_R(M,N)$, homomorphisms of degree p form a subgroup $F_p\mathrm{HOM}_R(M,N)$ of $\mathrm{HOM}_R(M,N)$.

One easily checks :

1. If $p \leq q$ then : $F_q\ \mathrm{HOM}_R(M,N) \subset F_q\ \mathrm{HOM}_R(M,N)$.
2. $\mathrm{HOM}_R(M,N) = \bigcup_{p \in Z} F_p\mathrm{HOM}_R(M,N)$.
3. If $f : M \to N$ has degree p and $g : N \to P$ has degree q then $g \circ f$ has degree $p + q$.

These properties allow us to introduce the category R-filt of filtered left R-modules where the morphisms are the homomorphisms in $\mathrm{HOM}_R(-,-)$ of degree 0. If $M,N \in$ R-filt then $F_0\mathrm{HOM}_R(M,N)$ will simply be denoted by $\mathrm{Hom}_{FR}\ (M,N)$. Clearly R-filt is an additive category and furthermore if $f \in \mathrm{Hom}_{FR}\ (M,N)$ then Ker f and Coker f are in R-filt. Indeed, Ker f has the induced filtration F_i Ker f = Ker f $\cap\ F_iM$, whereas Coker f is filtered by putting F_i Coker f = (Im f + F_iN)/ Im f. In particular it follows that monomorphisms and epimorphisms in R-filt coincide with the injective resp. surjective morphisms of R-filt.

Arbitrary direct sums as well as direct products exist in R-filt. Indeed, if $(M_i)_{i \in I}$ is a family of objects of R-filtration $F_p\ (\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} F_p\ M_i$ is the direct sum and the module $\prod_{i \in I} M_i$ with the filtration $F_p(\prod_{i\in I} M_i) = \prod_{i\in I} F_p M_i$ is the direct product in the category R-filt. Moreover, if (M_i, φ_{ij}), with i, j in some index set I, is an inductive (projective) system in R-filt then its inductive (projective) limit exists in R-filt; it is the R-module $\varinjlim M_i$ ($\varprojlim M_i$) with filtration $F_p \varinjlim M_i = \varinjlim F_p M_i$, ($F_p \varprojlim M_i = \varprojlim F_p M_i$).

I.6. Remarks.

1. Let $\{M_i, i\in I\}$ be a family of objects from R-filt such that their filtrations FM_i, $i\in I$, are exhaustive, then the filtration of $\bigoplus_I M_i$ is exhaustive. This property fails for the direct product $\prod_I M_i$! The similar result however does hold for inductive or projective limits.

2. R-filt is preabelian (cf. [46]) but not abelian. Indeed, take $M \neq 0$ in R-mod and put $F_iM = 0$ for all $i \in Z$. Let F'M be another filtration on M and denote G, resp. H, the filtered modules from M using FM resp. F'M. The identity morphism of M is in $\mathrm{Hom}_{FR}(G,H)$ but not in $\mathrm{Hom}_{FR}(H,G)$ and thus the identity of M is a bijective mapping which fails to be an isomorphism.

3. In a similar way as before one defines the category R-filt-S for a pair of filtered rings R and S.

Before introducing some special functors in R-filt let us point out that, if R is a filtered ring with filtration FR, we have a functor D : R-mod $\to$ R-filt which associates to $M \in$ R-mod the R-module M with filtration $F_nM = F_nR.M$. This filtration of M is called the induced filtration. Note that $F_nD(M) = F_nR.M = M$ for any $n \geq 0$.
Dually, we can define the functor $\tilde{D}$: R-mod $\to$ R-filt which associates to every module M, the module M with filtration $F_nM = \{x\in M, F_{-n}R.x=0\}$ $(n \in Z)$. F_nM is a subgroup and $F_pR.F_nM \subseteq F_{p+n}M$. This filtration of

M is called the <u>co-induced filtration</u>. Note that $F_n\bar{D}(M) = 0$ for any $n \leq 0$. Since $1 \in F_oR$, the co-induced filtration is discrete.

<u>Particular case.</u>

Let I be a two-sided ideal. If for the ring R we consider the I-adic filtration, then $F_{-n}D(M) = I^nM$ and $F_n\bar{D}(M) = \mathrm{ann}_M I^n$ for any $n \geq o$.

<u>The suspension functor.</u> For $n \in Z$ we define the n-th suspension functor T_n : R-filt $\to$ R-filt, which associates to $M \in$ R-filt FM the filtered module $M(n)$ with filtration $F_iM(n) = F_{i+n}M$ hence $F_iT_n(M) = F_{i+n}(M)$. Suspension is characterized by : $T_n \circ T_m = T_{n+m}$, $T_o = \mathrm{Id}$ (the identity functor). In particular, every T_n is an equivalence of categories which commutes with direct sums, products, inductive and projective limits. It is easily verified that the following holds : for $p \in Z$, and for M, N $\in$ R-filt :

$$F_p\, \mathrm{HOM}_R(M,N) = \mathrm{Hom}_{FR}\,(M(-p),N) = \mathrm{Hom}_{FR}\,(M,N(p)).$$

<u>The exhaustion functor.</u> Let R have filtration FR. If we put $R' = \bigcup_{n\in Z} F_nR$ then R' is a filtered subring of R. If $M \in$ R-filt, then, clearly $M' = \bigcup_{n\in Z} F_nM$ is a filtered R'-submodule and the filtration FM' given by $F_nM' = F_nM$ for all $n \in Z$ is an exhaustive filtration. In this way we obtain a functor : R-filt $\to$ R'-filt carrying M into the corresponding M', which is called the <u>exhausting functor</u>.

II. Complete Filtered Modules. The Completion Functor

Let $M \in R$-filt. A sequence $(m_i)_{i\in \mathbb{N}}$ of elements of M is said to be a Cauchy -sequence if for every $p \in \mathbb{N}$ there is an $N(p) \in \mathbb{N}$ such that $m_s - m_t \in F_{-p} M$ for all $s,t \geq N(p)$. Obviously, in this definition it is sufficient to have $x_s - x_{s+1} \in F_{-p}$ for any $s \geq N(p)$. A sequence $(m_i)_{i\in N}$ converges to $m \in M$ if for every $p \in \mathbb{N}$ there exists $N(p) \in \mathbb{N}$ such that $m_s - m \in F_{-p}M$ whenever $s \geq N(p)$.
If the filtration FM is separated then all sequences converge to a unique limit.

II.1. Definition. A filtered module $M \in R$-filt is said to be complete if FM is separated and all Cauchy sequences converge in M.

II.2. Remarks.

1. Obviously $M \in R$-filt is complete if and only if T_nM is complete, $n \in Z$.

2. If the filtration of M is discrete, then M is complete.

3. According to definition II.1. one deduces that any finite direct sum of complete modules is also a complete module.

Now let $M \in R$-filt and consider the projective system
$\{M/F_pM, \Pi_{pq}: M/F_pM \to M/F_qM, p \leq q\}$.
Let us put $\hat{M} = \varprojlim M/F_pM$. If $\Pi_p = \hat{M} \to M/F_pM$ are the canonical morphisms we note $F_p \hat{M} = \mathrm{Ker}\, \Pi_p$. Since Π_{pq} are surjective then the applications Π_p are surjective. If $u_p : M \to M/F_pM$ are the canonical morphisms, then all elements of $\hat{M}$ have the following form : $\xi = (u_p(x_p))\ p \in Z$ with $x_p \in M$ and $x_p - x_{p+1} \in F_{p+1}M$ for all $p \in Z$.
Then $F_p\hat{M} = \{\xi = (u_p(x_p))_{p\in Z}, x_p \in F_pM\}$.
It follows that $F_p\hat{M} = \varprojlim_{r\leq p} F_pM/F_rM$.

Let $\varphi_M : M \to \hat{M}$ be the canonical morphism : $\varphi_M(x) = (u_p(x))_{p\in Z}$.
Obviously, $\Pi_p \circ \varphi_M = u_p$ for all $p \in Z$.
The family of subgroups $\{F_p\hat{M}\}_{p\in Z}$ is a filtration for the abelian group $\hat{M}$.

II.3. Proposition. The abelian group $\hat{M}$ with filtration $\{F_p\hat{M}\}_{p\in Z}$ has the following properties :

1. Ker $\varphi_M = \bigcap_{i\in Z} F_iM$. Particularly FM is separated if and only if φ_M is injective.

2. If the filtration FM is exhaustive then the filtration $F\hat{M}$ is exhaustive.

3. For all $p \in Z$, $F_{p+1}\hat{M}/F_p\hat{M} \simeq F_{p+1}M/F_pM$.

4. $\hat{M}$ is a complete abelian group.

5. $\varphi_M(M)$ is dense in $\hat{M}$.

6. The module M is complete if and only if φ_M is an isomorphism.

Proof.

1. Obvious.

2. Let $\xi = (u_p(x_p))_{p\in Z}$ be an element of $\hat{M}$ If $p_0\in Z$, for the element x_{p_0} there exists a $k \in Z$ such that $x_{p_0} \in F_kM$. If $k \le p_0$ then $x_{p_0} \in F_{p_0}M$ and therefore $\xi \in F_{p_0}\hat{M}$. If $p_0 \le k$, we get $x_k \in F_kM$ and consequently $\xi \in F_k\hat{M}$.

3. Let $\xi = (u_i(x_i))_{i\in Z}$ be an element of $F_{p+1}\hat{M}$. Therefore $x_{p+1} \in F_{p+1}M$. Since $x_p - x_{p+1} \in F_{p+1}M$, then $x_p \in F_{p+1}M$. Let us define the application $F_{p+1}\hat{M}/F_p\hat{M} \to F_{p+1}M/F_pM$, $\hat{\xi} \to (u_p)$. One may easily deduce that this application is an isomorphism.

4. Obviously $\bigcap_{p\in Z} F_p\hat{M} = 0$. Let $(\xi_i)_{i\ge1}$ be a Cauchy sequence of $\hat{M}$. This sequence is equivalent to a sequence $(\eta_i)_{i\ge1}$ with the following property

$\eta_{i+1} - \eta_i \in F_{-i} \hat{M}$ for any $i \geq 0$. Indeed, there exists a natural N_i such that $\xi_r - \xi_s \in F_{-i} \hat{M}$ for $r,s \geq N_i$. We get the ascending sequence $N_1 < N_2 < \dots$. Let $\eta_i = \xi_{N_i}$; the sequence $(\eta_i)_{i\geq 1}$ satisfies the required condition. Therefore in order to show that $\hat{M}$ is complete it will be sufficient to establish that all sequences $(\xi_i)_{i\geq 1}$, with the property $\xi_{i+1} - \xi_i \in F_{-i} \hat{M}$, converge. Let $\xi_i = (u_p(x_p^i))_{p\in Z}$ $(i\geq 1)$.
So $x_{p+1}^i - x_p^i \in F_{p+1} M$, $\forall\, p \in Z\ \forall\, i \geq 1$. As $\xi_{i+1} - \xi_i \in F_{-i} \hat{M}$, then $x_{-i}^{i+1} - x_{-i}^i \in F_{-i} M$ for all $i\geq 1$. As $x_{p+1}^{i+1} - x_p^{i+1} \in F_{p+1} M$, then for $p = -i-1$ we get $x_{-i}^{i+1} - x_{-(i+1)}^{i+1} \in F_{-i} M$ and so $x_{-i}^i - x_{-(i+1)}^{i+1} \in F_{-i} M$.
If we denote $\xi = (y_i)_{i\in Z}$ where $y_{-i} = u_{-i}\ (x_{-i}^i)$ for $i\geq 1$ and $y_j = u_j(x_{-1}^1)$ for $j\geq 0$ it follows that $\xi \in \hat{M}$. On the other hand, $\xi - \xi_i \in F_{-i} \hat{M}$. for $i \geq 1$, therefore $\xi = \lim\limits_{i\to\infty} \xi_i$. Consequently $\hat{M}$ is complete.

5. Let $\xi = (u_p(x_p))_{p\in Z}$ be an element of $\hat{M}$. Let us denote $\xi_i = \varphi_M(x_{-i})$ for all $i\geq 1$. As $\Pi_{-i}\ (\xi - \xi_i) = 0$ one deduces $\xi - \xi_i \in F_{-i}\hat{M}$, therefore $\xi = \lim\limits_{i\to\infty} \xi_i$, consequently $\varphi_M(M)$ is dense in $\hat{M}$.

6. If M is complete, then FM is separated, and so φ_M is injective. Let $\xi \in \hat{M}$; as $\varphi_M(M)$ is dense in $\hat{M}$, then there exists a sequence $(x_n)_{n\geq 1}$, $x_n \in M$, such that $\xi = \lim\limits_{n\to\infty} \varphi_M(x_n)$. Since $\varphi_M\ (F_iM) = F_i \hat{M} \cap \varphi_M(M)$ and φ_M is injective, then $(x_n)_{n\geq 1}$ is a Cauchy sequence of M. As M is complete, there exists an $x = \lim\limits_{n\to\infty} x_n$. Then $\xi = \varphi_M(x)$ and therefore φ_M is surjective.

Let us assume now that the filtration of the ring R is exhaustive.

If $M \in R$-filt and FM is exhaustive, then M is a topological R-module (Remark 1.4.). In this case $\hat{M}$ is the completed of M, in the topological sense . Since $R \times M \to \hat{M}$, $(a,x) \to ax$ is continuous,
it may be extended to a continuous application : $\hat{R}\times \hat{M} \to \hat{M}$. Since

$F_iR.\ F_jM \subset F_{j+i}M$, we get $F_i\hat{R}.F_j\hat{M} \subset F_{i+j}\hat{M}$. If $M = R^R$, then $\hat{R}$ is a filtered ring with filtration $\{F_i\hat{R}\}_{i\in Z}$. $\hat{M}$ is also a filtered $\hat{R}$-module with filtration $\{F_i\hat{M}\}$.

II.4. Remarks.

1. If we denote $M' = \bigcap_{p\in Z} F_pM$, then clearly M/M' is separated for the quotient filtration and $\hat{M} = M/M'$. Let us assume now that M has separated filtration. According to Proposition II.3., the completed $\hat{M}$ may be obtained by Cantor's method, that is by means of classes of equivalent Cauchy sequences.

2. If $F_0R = R$, then the multiplication on $\hat{R}$ may be defined as follows : if $\xi, \eta \in \hat{R}$ with $\xi = (u_p(a_p))_{p\in Z}$ and $\eta = (u_p(b_p))_{p\in Z}$ then $\xi\eta = (u_p(a_pb_p))_{p\in Z}$. The subgroups $F_p\hat{R}$, $p\in Z$, are two-sided ideals of $\hat{R}$. Moreover, if $F_0M = M$, then the module structure of $\hat{M}$ is given by the following equality : if $\alpha = (u_p(m_p))_{p\in Z}$ is an element of $\hat{M}$, then :

$$\xi\alpha = (u_p(a_pm_p))_{p\in Z}$$

Let R'-filt be the category of filtered modules with exhaustive filtration; then the application $M \to \hat{M}$ defines a functor c : R'-filt $\to$ $\hat{R}$-filt which is said to be the completion functor.

(If $f : M \to N$ is an isomorphism of the category R'-filt, $c(f) = \hat{f} : \hat{M} \to \hat{N}$ is the following morphism: if $\xi = (\Pi_p^M(x_p))_{p\in Z}$ is an element of $\hat{M}$ then $\hat{f}(\xi) = (\Pi_p^N(f(x_p))_{p\in Z})$.

III: Filtration and Associated Gradation

Let R be a filtered ring , $M \in$ R-filt. Consider the abelian groups : $G(R) = \bigoplus_{i \in Z} F_iR/F_{i-1}R$, $G(M) = \bigoplus_{i \in Z} F_iM/F_{i-1}M$. If $x \in F_pM$ then x_p denotes denotes the image of x in $G(M)_p = F_pM/F_{p-1}M$. If $a \in F_iR$, $x \in F_jM$ then define $a_ix_j = (ax)_{i+j}$ and extend it to a Z-bilinear mapping $\mu : G(R) \times G(M) \to G(M)$. Taking $M = R$, μ makes G(R) into a graded ring and in general G(M) is thus made into a graded G(R)-module. Let $f \in \mathrm{Hom}_{FR}(M,N)$ for some $M, N \in$ R-filt; then f induces canonical mappings $f_i : F_iM/F_{i-1}M \to F_iN/F_{i-1}N$, given by $f_i(x_i) = (f(x))_i$ for $x \in Z$. Putting $G(f) = \bigoplus_{i \in Z} f_i$ defines a morphism of G(R)-modules. All of these amounts to the statement that G : R-filt $\to$ G(R)-gr is a functor. We have the following :

D.3.1. Proposition.

The functor G has the following properties :

1. If $M \in$ R-filt and FM is exhaustive and separated then $M = 0$ if and only if $G(M) = 0$.
2. If $n \in Z$ and $M \in$ R-filt, then $G(T_nM) = T_n(G(M))$.
3. If $M \in$ R-filt and FM is discrete then G(M) is left limited.
4. If $M \in$ R-filt and both FR and FM are exhaustive then $G(M) = G(\hat{M})$.
5. The functor G commutes with direct sums, products and inductive limits.

Proof. Assertions 1, 2, 3, are clear. For assertion 4 we use Proposition II.3.. In order to prove statement 5, let $(M_\alpha)_{\alpha \in J}$ be an arbitrary family of objects of R-filt, and put $M = \bigoplus_{\alpha \in J} M_\alpha$.

We have :

$$G(M)_n = F_n(M)/F_{n-1}(M) = \bigoplus_{\alpha \in J} F_n(M_\alpha)/ \bigoplus_{\alpha \in J} F_{n-1}(M_\alpha)$$

$$= \bigoplus_{\alpha \in J} F_n M_\alpha/F_{n-1} M_\alpha = \bigoplus_{\alpha \in J} G(M_\alpha)_n.$$

Hence $G(M) = \bigoplus_{\alpha \in J} G(M_\alpha)$. In a similar way we may establish that G commutes with direct products and inductive limits. □

III.2. Remark. If R is a graded ring and $M \in R$-gr then we can define an exhaustive and separated filtration on R (resp. M) by means of the subgroups $F_pR = \bigoplus_{i \le p} R_i$, (resp. $F_pM = \bigoplus_{i \le p} M_i$). Let us denote the obtained filtered ring (resp. filtered module) by R' (resp. M'). It is straightforward to prove that $G(R') \cong R$, $G(M') \cong M$.

Similarly, the subgroups $F'_pR = \bigoplus_{i \ge -p} R_i$ (resp. $F'_pM = \bigoplus_{i \ge -p} M_i$) define a filtration on R (resp. M). Denoting by R'' (resp. M'') the obtained filtered ring (resp. module) then again $G(R'') \cong R$ and $G(M'') \cong M$.

In studying the effect of G on morphisms we need :

III.3. Definition. Let $M, N \in R$-filt. A filtered morphism $f : M \to N$ is said to be strict if $f(F_pM) = \operatorname{Im} f \cap F_pN$ for each $p \in \mathbb{Z}$.

A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in R-filt is strict exact if it is an exact sequence in R-mod such that both f and g are strict morphisms in R-filt.

It is clear that a filtered morphism $f : M \to N$ is an isomorphism if and only if f is bijective and f is strict.

III.3. Theorem. Let R be a filtered ring and consider the following 0-sequence in R- filt :

$$(\star) \quad L \xrightarrow{f} M \xrightarrow{g} N,$$

as well as the sequence $G(\star)$ in $G(R)$-gr :

$$G(L) \xrightarrow{G(f)} G(M) \xrightarrow{G(g)} G(N).$$

Then :

1. If $(\star)$ is strict exact then $G(\star)$ is exact.

2. If $G(\star)$ is exact and FM is exhaustive then g is strict.
3. If $G(\star)$ is exact while FL is complete and FM is separated then f is strict.
4. If $G(\star)$ is exact and FM is discrete then f is strict.
5. If L is complete, FM is exhaustive and separated, or if FM is exhaustive and discrete then $(\star)$ is exact.

Proof.

1. Clearly $G(g) \circ G(f) = 0$. Look at an $x \in F_pM$ which is such that $G(g)x_p=0$. This says $(g(x))_p = 0$ and therefore $g(x) \in F_{p-1}N$. Now since g is strict there must exist an $x' \in F_{p-1}M$ such that $g(x) = g(x')$ i.e. $x-x' = f(y)$ with $y \in F_pL$, but then $G(f)(y_p) = x_p$ i.e. $\mathrm{Im}\, G(f) = \mathrm{Ker}\, G(g)$.

2. Let $y \in F_pN \cap \mathrm{Im} g$ and $y \notin F_{p-1}N$. There exists an $x \in M$ such that $g(x) = y$ and since FM is exhaustive we may assume that $x \in F_{p+s}M$ for some $s \geq 0$. If $s = 0$ then we are done. If $s > 0$ then $G(g)(x_{p+s}) = 0$ and the fact that $G(\star)$ is exact implies that $x_{p+s} = G(f)(z_{p+s})$ for some $z \in F_{p+s}L$. Thus $x - f(z) \in F_{p+s-1}M$ and $y = g(x) = g(x-f(z)) = g(x')$ with $x' \in F_{p+s-1}M$. Repeating this procedure we finally get that there is an $\mu \in F_pM$ such that $y = G(u)$.

3. Take $y \in F_pM \cap \mathrm{Im} f$. Exactness of $G(\star)$ yields : $G(g)(y_p) = g(y_p) = 0$, therefore $y_p = G(f)(x_p^{(p)})$ for some $x^{(p)} \in F_pL$. Hence $y-f(x^{(p)}) \in \mathrm{Im} f \cap F_{p-1}M$. By induction we find a sequence $x^{(p)}, x^{(p-1)}, \dots, x^{(p-s)}$ with $x^{(p-s)} \in F_{p-s}L$ such that :

$$y - f(x^{(p)}) - \dots - f(x^{(p-s)}) \in \mathrm{Im} f \cap F_{p-s-1}M.$$

Completeness of FL allows to define $x = \sum_{s=0}^{\infty} x^{(p-s)} \in F_pL$.

Hence $y - f(x) = y - \lim_{s\to\infty} f(x^{(p)} + x^{(p-1)} + \dots + x^{(p-s)}) = 0$, the latter since FM is separated, and it follows from this that $y \in f(F_pL)$.

That $f(F_pL) \subset F_pM \cap \mathrm{Im}\, f$ is obvious.

4. Along the lines of 3.

5. Strict exactness of $(\star)$ implies exactness of $G(\star)$ because of 1. Conversely suppose that $G(\star)$ is exact and let $y \in M$, $y \neq 0$ be such that $g(y) = 0$. Since FM is exhaustive we have $y \in F_pM$ and $y \notin F_{p-1}$ for some $p \in Z$. We obtain thus : $G(g)(y_p) = 0$ and so $y_p = G(f)(x_p{}^{(p)}) = f(x^{(p)})_p$ for some $x^{(p)} \in F_pL$.

Consequently : $y - f(x^{(p)}) \in F_{p-1}M$. By induction we obtain $x^{(p)},\dots,x^{(p-s)}$ with $x^{(p-s)} \in F_{p-s}\, L$ and such that :

$$y-f(x^{(p)} + \dots + x^{(p-s)}) \in F_{p-s-1}M.$$

If FM is discrete then for some index s, $F_{p-s-1}\, M = 0$; hence $y = f(x^{(p)} + \dots + x^{(p)})$ and thus $(\star)$ is exact. If FM is complete then we may take $x = \sum_{s=0}^{\infty} x^{(p-s)}$ and get $y = f(x)$. The fact that f and g are strict morphisms will follow from 2. and 3.

III.5. Corollary. Let $f : M \to N$ be a morphism in R-filt and suppose that FM and FN are separated and exhaustive, while FM is also complete. Then : G(f) is bijective if and only if f is an isomorphism in R-filt.

III.6. Corollary. Let $f \in \mathrm{Hom}_{FR}\,(M,N)$. If the filtration FM is separated and exhaustive then :

G(f) is injective if and only if f is injective and strict.

III.7. Corollary. Let $f \in \mathrm{Hom}_{FR}\,(M,N)$. Assume that one of the following hypotheses is satisfied :

1) M is complete, FN is separated and exhaustive.

2) FM is discrete and exhaustive.

Then : G(f) is surjective if and only if f is surjective and f is strict.

Now consider a morphism $f : M \to N$ of the category R-filt. f is said to have the <u>Artin-Rees property</u> if for every $n \in Z$ there exists a natural number $h(n) \in Z$ such that :

$$f(F_nM) \supseteq \operatorname{Im} f \cap F_{h(n)}N.$$

One can see that if f is strict then f has the Artin-Rees property.

<u>III.8. Theorem.</u> Consider in R-filt the exact sequence :

$$L \xrightarrow{f} M \xrightarrow{g} N$$

where the filtrations FL, FM and FN are exhaustive. If the morphisms f and g have the Artin-Rees property then the sequence

$$\hat{L} \xrightarrow{\hat{f}} \hat{M} \xrightarrow{\hat{g}} \hat{N}$$

is exact.

<u>Proof.</u> First we consider the case where L is a submodule with filtration induced by M and N = M/L with quotient filtration, that is $F_iN = F_iM + L/L$ $(i \in Z)$. Thus we have the exact sequence

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{\pi} M/L \longrightarrow 0.$$

Since the projective limit functor is left exact, it is sufficient to prove that $\hat{\pi}$ is surjective. We have $\widehat{M/L} = \varprojlim_i (M/L + F_i)$.

Let $\eta = (\pi_p^{M/N}(y_p)\ , p \in Z)$ be an element of $\widehat{M/L}$.

$(\pi_p^{M/N} : M \to M/L + F_pM$ is the canonical morphism). Let $s \in Z$ and denote $x_s = y_s$. Since $y_{s-1} - y_s \in L + F_sM$, we may write $y_{s-1} - y_s = u_1+v_1$ where $u_1 \in L$ and $v_1 \in F_sM$. Let $x_{s-1} = y_{s-1} - u_1$. It is clear that $x_{s-1} - x_1 \in F_sM$. Since $y_{s-2} - y_{s-1} \in L + F_{s-1}M$, then $y_{s-2} - y_{s-1} = u_2+v_2$ where $u_2 \in L$, $v_2 \in F_{s-1}M$. Let $x_{s-2} = y_{s-2}-u_2-u_1$. It is clear that $x_{s-2} - x_{s-1} = v_2 \in F_{s-1}M$. Following this procedure we can construct for every $t \le s$ the element x_t with the property $x_{t-1} - x_t \in F_tM$ and $\pi(x_t) = y_t$. If for every $t \ge s$ we let $x_t = x_s$, then the sequence $\xi = (\pi_t(x_t))_{t \in Z}$ is an element of $\hat{M}$. Clearly, $\hat{\pi}(\xi) = \eta$.

Consider the following diagram :

$$L \xrightarrow{f_1} L/\mathrm{Ker}\, f \xrightarrow{f_2} \mathrm{Im}\, f = \mathrm{Ker}\, g \xrightarrow{f_3} M \xrightarrow{g_1} M/\mathrm{Ker}\, g \xrightarrow{g_2} \mathrm{Im}\, g \xrightarrow{g_3} N$$

where $f = f_3 \circ f_2 \circ f_1$, $g = g_3 \circ g_2 \circ g_1$; f_1, g_1 are the canonical surjections ; f_2, g_2 are bijective and f_3, g_3 are the canonical injections. The filtrations of L/Ker f and M/Ker g are $F_i(L/\mathrm{Ker}\, f) = F_iL + \mathrm{Ker}\, f/\mathrm{Ker}\, f$, $F_i(M/\mathrm{Ker}\, g) = F_iM + \mathrm{Ker}\, g/\mathrm{Ker}\, g$. The filtrations of Im f and Im g are $F_i\ (\mathrm{Im}\, f) = \mathrm{Im}\, f\ \cap F_iM$, $F_i\ (\mathrm{Im}\, g) = \mathrm{Im}\, g \cap F_iN$. We have : $\hat{f} = \hat{f}_3 \circ \hat{f}_2 \circ \hat{f}_1$ $\hat{g} = \hat{g}_3 \circ \hat{g}_2 \circ \hat{g}_1$ and have that the sequence :

$$0 \longrightarrow \widehat{\mathrm{Ker}\, g} \xrightarrow{\hat{f}_3} \hat{M} \xrightarrow{\hat{g}_1} \widehat{M/\mathrm{Ker}\, g} \longrightarrow 0$$

is exact, $\hat{f}_1$ is surjective and $\hat{g}_3$ is injective. Since f and g have the Artin-Rees property, f_2 and g_2 are isomorphisms of topological modules and therefore $\hat{f}_2$ and $\hat{g}_2$ are isomorphisms. Hence the sequence

$\hat{L} \xrightarrow{\hat{f}} \hat{M} \xrightarrow{\hat{g}} \hat{N}$ is exact.

IV: Free and Finitely Generated Objects of R-filt

Let R be a filtered ring, then $L \in$ R-filt is <u>filt-free</u> if it is free in R-mod and has a basis $(x_i)_{i\in J}$ consisting of elements with the property that there exists a family $(n_i)_{i\in J}$ of integers such that :

$$F_pL = \sum_{i\in J} F_{p-n_i} R.x_i = \bigoplus_{i\in J} F_{p-n_i} R.x_i.$$

Note that for any $i \in J$, $x_i \in F_{n_i}L$ and $x_i \notin F_{n_i-1}L$.

We say that $(x_i, n_i)_{i\in J}$ is a <u>filt-basis</u> for L. Next lemma follows from this definition and Theorem III.4.

<u>IV.1. Lemma.</u> Let R be a filtered ring, $L \in$ R-filt, then :

1° L is filt-free with filt-basis $(x_i, n_i)_{i\in J}$ if and only if $L \cong \bigoplus_{i\in J} R(-n_i)$.

2° If L is filt-free with filt-basis $(x_i, n_i)_{i\in J}$ then G(L) is a free graded module in G(R)-gr with homogeneous basis $\{(x_i)_{n_i}, i\in J\}$.

3° If G(L) is a free object in G(R)-gr with homogeneous basis $\{(x_i)_{n_i}, i\in J\}$ and if FL is discrete, then L is filt-free in R-filt with filt-basis $(x_i, n_i)_{i\in J}$.

4° If $M \in$ G(R)-gr is free graded then there exists a filt-free $L\in$R-filt such that $G(L) \cong M$.

5° Let $L \in$ R-filt be filt-free with filt-basis $(x_i, n_i)_{i\in J}$, let $M\in$ R-filt. If $f_i\ \{x_i, i\in J\} \to M$ is a function such that $f(x_i) \in F_{s+n_i}M$, then there is a unique filtered morphism of degree s, $g:L \to M$ which extends f.

6° Let $M \in$ R-filt and suppose that L is filt-free. If $g:G(L) \to G(M)$ is a graded morphism of degree s then there is a filtered morphism $f : L \to M$ of degree s such that $G(f) = g$.

7° Let $L \in$ R-filt be filt-free, then FL is exhaustive (separated) if and only if FR is exhaustive (separated). If FR is discrete and $\{n_i \in Z,\ i \in J\}$ is bounded below then FL is discrete. If J is finite and FR is complete then FL is complete.

8° If FR is exhaustive and $M \in$ R-filt is such that FM is exhaustive, then there is a free resolution of M in R-filt.

$$(\star)\quad \dots \longrightarrow L_2 \xrightarrow{f_2} L_1 \xrightarrow{f_1} L_o \xrightarrow{f_o} M \longrightarrow 0$$

where every L_j is filt-free and every f_j is a strict morphism. Moreover if FR and FM are discrete then we assume that every FL_j, $j \geq 0$, is also discrete.

9° If FR is exhaustive and complete and if G(R) is left Noetherian and G(M) being generated as a left graded R-module then we may select the L_j in the resolution $(\star)$ to be finitely generated too.
Note that an $M \in$ R-filt is said to be <u>finitely generated</u> or <u>filt-f.g.</u> if there is a finite family $(x_i, n_i)_{i \in J}$ with $x_i \in M$ and $n_i \in Z$, such that $F_p M = \sum_{i=1}^{n} F_{p-n_i} R.x_i$.

<u>IV.2. Remarks</u>.

1° If $L \in$ R-filt-free as well as finitely generated in R-mod then R is <u>filt-f.g.</u>

2° If $M \in$ R-filt is such that FM is exhaustive then M is filt-f.g. if and only if there exists a filt-free L which is finitely generated, and a strict epimorphism $\pi : L \to M$.

<u>IV.3. Proposition</u>. Let R be a filtered complete ring and let $M \in$ R-filt be such that FM is separated and exhaustive. Then M is filt-f.g.if and only if G(M) is finitely generated as a graded G(R)-module. Furthermore, if G(M) may be generated by n homogeneous elements, then M may be generated as an R-module by less than n generators.

Proof. From the remark IV.2.2° it follows that, if M is filt-f.g, then G(M) is finitely generated. Conversely, let G(M) be generated by homogeneous generators $x^{(i)}_{p_i}$, $1 \leq i \leq n$. Consider $L \in$ R-filt, L filt free with filt-basis $(y_i, p_i)_i$. Define $f : L \to M$ by $f(y_i) = x^{(i)}$. Since $G(f) : G(L) \to G(M)$ is an epimorphism, Theorem III.4.5. yields that f is a strict epimorphism and then Remark IV.2.2° entails that M is filt-f.g and generated by $x^{(i)}$ as an R-module. □

IV.4. Corollary. Let $M \in$ R-filt, such that FM is separated and exhaustive and suppose that FR is complete. If G(M) is left Noetherian as an object of G(R)-gr, then M is left Noetherian too. Moreover, we have : $\mathrm{K.dim}_R M \leq \mathrm{K.dim}_{F(R)} G(M)$. If $\mathrm{K.dim}_R M = \mathrm{K.dim}_{G(R)} G(M) = \alpha$ and G(M) is α-critical, then M is an α-critical R-module.

Proof. Let N be a submodule of M equipped with the induced filtration, $F_n N = F_p M \cap N$. Since $G(N) \subset G(M)$, G(N) is finitely generated. Proposition IV.3. entails that N is finitely generated, so M is left Noetherian. Consider submodules N and P of M, $N \subset P$, both equipped with the induced filtration, and assume that $G(N) = G(P)$. If $\{x^{(i)}_{p_i}, 1 \leq i \leq n\}$ is a set of homogeneous generators for G(N) where $x^{(i)} \in N$ for all i, $1 \leq i \leq n$, then, again from proposition IV.3., we infer that $\{x^{(i)}, 1 \leq i \leq n\}$ generates N as well as P. Thus $N = P$. This yields $\mathrm{K.dim}_R M \leq \mathrm{K.dim}_{G(R)} G(M)$. In order to establish our last claim, let $N \neq 0$, $N \subset M$ and filter M/N by putting : $F_p(M/N) = (N + F_p M)/N$. Clearly, we obtain an exact sequence in G(R)-gr :

$$0 \longrightarrow G(N) \longrightarrow G(M) \longrightarrow G(M/N) \longrightarrow 0.$$

Since $G(N) \neq 0$, the Krull dimension of G(M/N) is less than α , hence

$$\mathrm{K.dim}_R\, M/N \leq \mathrm{K.dim}_{G(R)} G(M/N) < \alpha.$$

This shows exactly that M is an α-critical module. □

IV.5. Corollary. Let R be a complete filtered ring such that $G(R)$ is a left Noetherian ring, then :

1° R is left Noetherian and $K.\dim R \le K.\dim G(R)$

2) $F_0 R$ is left Noetherian and $K.\dim F_0R \le K.\dim G(R) + 1$.

Proof

1. Follows from the foregoing corollary.

2. $F_0 R$ is a subring of R and it is clearly complete; Corollary IV.4. together with the results of Section A.II.5. finish the proof. □

IV.6. Proposition. Let $M \in R$-filt with FM being exhaustive. Assume that submodules of M are closed, i.e. if $N \subset M$ then $N = \bigcap_{p\in Z} (N + F_pM)$.

Consider submodules $N \subset P \subset M$, then :

1. If $G(N) = G(P)$ then $N = P$. In particular if the Krull dimension of $G(M)$ is well defined then the Krull dimension of M is defined and $K.\dim_R M \le K.\dim_{G(R)} G(M)$.

2. If $K.\dim_R M = K.\dim_{G(R)} G(M) = \alpha$ and if $G(M)$ is α-critical then M is an α-critical module.

3. If $G(M)$ may be generated by n homogeneous generators then M may be generated by n generators.

Proof.

1. Take $x \in P$, then $x \in F_iM$ for some i, $x \notin F_{i-1}M$. Because :
$x_i \in G(P)_i = G(N)_i = (P \cap F_iM) + F_{i-1}M/F_{i-1}M = (N \cap F_iM) + F_{i-1}M/F_{i-1}M$
there is a $y_1 \in N \cap F_iM$ such that $x-y_1 \in F_{i-1}M$. Hence $x \in y_1 + P \cap F_{i-1}M$.
Repeating this process we and up with elements $y_1, y_2,\dots,y_o \in N$ such that $x-(y_1 + y_2 +\dots+y_s)$ is in $F_{i-s} M \cap P$, $s > 0$.
Thus $x \in N + F_{i-s}M$ for $s > 0$. It follows that $x \in \bigcap_{p\in Z} (N+F_pM) = N$, whence $N = P$ follows.

2. Proceed as in the proof of Corollary.

3. Let $\{ x^{(i)}_{p_i}, 1 \leq i \leq n \}$ be a family of homogeneous generators for $G(M)$. Write M' for the submodule of M generated by the elements $x^{(i)}$, $1 \leq i \leq n$. Obviously $x^{(i)}_{p_i} \in G(M')_{p_i} = (M' \cap F_{p_i}M)/(M' \cap F_{p_i-1}M)$. Therefore $G(M') = G(M)$ and by 1. this yields $M' = M$.

IV.7. Remark. The above proposition is generally applied in case FM is exhaustive and discrete.

V: The I-adic Filtration

The I-adic filtration is very important for the study of commutative rings (see [D.1]).

The I-adic non commutative filtration is useful, for example in the study of Lie Algebras or for the study of the integral group ring Z(G) where G is a nilpotent group (see exercises).

Let R be a ring, I an ideal of R, $M \in$ R-mod. Using the I-adic filtration as introduced before, we obtain the I-adic completion $\hat{M}$ of M as $\hat{M} = \varprojlim_n M/I^nM$.

Let $i : M \to \hat{M}$ be the canonical filtered morphism, $\{\widehat{I^nM}\}_{n\geq 0}$ is the filtration of $\hat{M}$. In case $M = R$, then $R \to \hat{R}$ is a ring homomorphism, by means of which we regard $\hat{R}$ as an R-bimodule.

The $\hat{R}$- linear $\alpha : \hat{R} \otimes_R M \to \hat{M}$, $\alpha(\xi \otimes m) = \xi\, i(m)$ has the property that the diagram :

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr] & \hat{R} \otimes_R M \arrow[d, "\alpha"] \\ & \hat{M} \end{tikzcd} \qquad \text{where } j(m) = 1 \otimes m$$

is commutative.

We shall now present some of the important properties of the I-adic filtration.

1° If M is of finite type then α is surjective. In particular $\hat{R}\, i(M) = \hat{M}$ and $\hat{M}$ is an $\hat{R}$module of finite type.

Indeed, there exists a free module L of finite type such that

$$L \xrightarrow{\pi} M \longrightarrow 0 \qquad (\pi \text{ surjective})$$

We have the commutative diagram

$$\begin{tikzcd} \hat{R}\otimes_R L \arrow[r, "1\otimes\pi"] \arrow[d, "\beta"] & \hat{R}\otimes_R M \arrow[r] \arrow[d, "\alpha"] & 0 \\ \hat{L} \arrow[r, "\hat{\pi}"] & \hat{M} \arrow[r] & 0 \end{tikzcd}$$

The morphism $\hat{\pi}$ is surjective, since π is strict.

As the completion functor commutes with finite direct sums, then β is an isomorphism. Hence α is surjective.

2. $\hat{I}$ is an ideal and $\hat{I}^n \subset \widehat{I^n}$ for all $n \geq 0$. Indeed, it is sufficient to proof that if $\xi_1, \xi_2,\ldots,\xi_n \in \hat{I}$, then $\xi_1, \xi_2,\ldots,\xi_n \in \widehat{I^n}$. Let $\xi_i = (\pi_p(a_p^i))_{p\in\mathbb{N}}$ (π_p: $R \to R/I^p$ is the canonical morphism). Since $\xi_i \in \hat{I}$ then $a_1^i \in I$ for $1 \leq i \leq n$. On the other hand, since $a_2^i - a_1^i \in I$ we have $a_2^i \in I$. But $a_3^i - a_2^i \in I^2$ yields $a_3^i \in I$, $1 \leq i \leq n$. Finally, we obtain $a_n^i \in I$, for all $1 \leq i \leq n$. Hence $a_n^1\ a_n^2 \ldots a_n^n \in I^n$ and therefore $\xi_1\ \xi_2\ldots\xi_n \in \widehat{I^n}$.

3. $\hat{I}$ is contained in the Jacobson radical of $\hat{R}$.
Indeed let $\xi \in \hat{I}$. As $\xi^n \in \hat{I}^n \subset \widehat{I^n}$, $\lim\limits_{n\to\infty} \xi^n = 0$ and therefore the series $1 + \xi + \xi^2 + \ldots$ converges. We have $(1-\xi)^{-1} = 1 + \xi + \xi^2 + \ldots$ and therefore ξ belongs to the Jacobson radical.

4. If I is of finite type as a left and as a right ideal, then :
i) $\hat{I}^n = \hat{R}\ I^n = I^n\ \hat{R}$ for all $n \geq 0$. In particular $\widehat{I^n} = \hat{I}^n$.
ii) If M is an R-module of finite type then :

$\widehat{I^nM} = I^n\hat{M} = \hat{I}^n\ \hat{M}$, for all $n \geq 0$.

<u>Proof.</u> For the assertion i) we apply 1.
ii) Since I^nM is of finite type we have :

$\widehat{I^nM} = \hat{R}\, i\, (I^nM) = \hat{R}\, I^n\, i(M) = I^n\, \hat{R}i(M) = I^n\, \hat{M}$.

The ideal I is said to satisfy the <u>left Artin-Rees property</u> if for any finitely generated left R-module M, any submodule N of M and any natural number n, there exists an integer $h(n) \geq 0$ such that $I^{h(n)}\, M \cap N \subset I^n\, N$. In other words I satisfies the left Artin-Rees property if the I-adic topology of N coincides with the topology induced in N by the I-adic topology of M

5) If I has the Artin-Rees property then for every module M of finite type :

$\operatorname{Ker} i = \bigcap_{n\geq 1} I^nM = \{x \in M,\ (1-r)x = 0 \text{ for an arbitrary } r \in I\}$

In particular, if I is contained in the Jacobson radical of R, then $\bigcap_{n\geq 1} I^nM = 0$.

Obviously we have the inclusion :

$\{x \in M, (1-r)x = 0 \text{ for an arbitrary } r \in I\} \subseteq \bigcap_{n\geq 1} I^nM$.

Conversely, if $x \in \bigcap_{n\geq 1} I^nM$, then because I has the Artin-Rees property, there exists a $n \in \mathbb{N}$ such that $I^n\, M \cap Rx \subset I(Rx) = Ix$. Hence $Rx \subset Ix$ and therefore there exists an $r \in I$ such that $x = rx$; therefore $(1-r)x = 0$.

6) If I has the Artin-Rees property and R is a domain then $\bigcap_{n\geq 1} I^n = 0$

7) If R is a left Noetherian ring, I an ideal of R satisfying the left Artin-Rees property, then the following statements are equivalent :

i) I is contained in the Jacobson radical of R.

ii) If $M \in R\text{-mod}$ is finitely generated then the submodules of M are closed in the I-adic topology.

i) $\Rightarrow$ ii) We apply 5)

ii) $\Rightarrow$ i) Let A be a maximal left ideal. Assume that $I \not\subset A$. Then $R = I + A$. Hence $R = A + I.R = A + I(I+A) = A + I^2 = A + I^2(I+A) = A + I^3 = \ldots = A + I^n = \ldots$

Since A is closed, $A = \bigcap_{n\geq 1} (A + I^n)$ and therefore $A = R$; contradiction. Hence $I \subset A$ finishes the proof.

8) Let R be a left Noetherian ring and I an ideal satisfying the left Artin-Rees property.

i) The functor taking M to $\hat{M}$ is exact in the category of left R-modules of finite type.

ii) If $M \in R$-mod is finitely generated then $\hat{M} \simeq \hat{R} \otimes_R M$.

iii) $\hat{R}$ is a right flat R-module.

Proof.

For i) we apply Theorem III.8.

ii) There exists an exact sequence

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{\pi} M \longrightarrow 0$$

where L is a free module of finite type and K is of finite type.

From this we deduce the following commutative and exact diagram :

$$\begin{array}{ccccccccc}
 & & \hat{R}\otimes K & \longrightarrow & \hat{R}\otimes L & \longrightarrow & \hat{R}\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \hat{K} & \xrightarrow{\hat{i}} & \hat{L} & \xrightarrow{\hat{\pi}} & \hat{M} & \longrightarrow & 0
\end{array}$$

Since β is an isomorphism and γ is surjective, it follows that α is injective and α is an isomorphism.

Obviously , ii) $\Rightarrow$ iii)

9) Let R be a left Noetherian ring. Assume that I is contained in the Jacobson radical of R and has the left Artin-Rees property. Then $\hat{R}$ is faithfully flat.

Proof. It is sufficient to prove that $\hat{R} \otimes_R M = 0$ implies $M = 0$. Since $\hat{R}$ is a right flat R-module, it is sufficient to consider the case $\hat{M} = \hat{R} \otimes_R M = 0$. Since $\hat{M}/I\hat{M} \simeq M/IM$ we obtain $M/IM = 0$, hence $M = IM$, therefore $M = 0$.

10) Let R be a right and left Noetherian ring. Assume that I has the right and left Artin-Rees property. If $a \in R$ is a nonzero divisor in R, then the image of a in $\hat{R}$ is also a nonzero-divisor.

Proof. The homomorphism $R \xrightarrow{\varphi_a} R$, $\varphi_a(\lambda) = \lambda a$ and $R \xrightarrow{\psi_a} R$, $\psi_a(\lambda) = a\lambda$ are injective. Then we apply 8), iii).

Definition. Let R be a ring and $a \in R$. We say that a is a normalizing element of R if $aR = Ra$. The set of normalizing elements of R is denoted by $N(R)$.

Definition. Let R be a ring. The elements $a_1, a_2, \ldots, a_n$ are a normalizing set of elements of R if

i) $a_1 \in N(R)$

ii) $a_i + (a_1, a_2, \ldots, a_{i-1}) \in N(R/(a_1, a_2, \ldots, a_{i-1})$ for $i = 2, \ldots n$.

The elements $a_1, a_2, \ldots, a_n$ form a centralizing set of elements of R if

i) $a_1 \in Z(R)$ (the center of the ring R)

ii) $a_i + (a_1, \ldots, a_{i-1}) \in Z(R/(a_1, a_2, \ldots, a_{i-1})$ for $i = 2, \ldots, n$.

A generating set of an ideal I of R which is also a normalizing set of elements of R is called a normalizing set of generators of I. Similarly one defines a centralizing set of generators for an ideal.

V. 1. Proposition. Let R be a left Noetherian ring, I an ideal generated by a central system, then I satisfies the left Artin-Rees property.

Proof. Let N be a submodule of a finitely generated $M \in R$-mod. Let $s \in \mathbb{N}$ and consider a submodule M' of M maximal with the property that $M' \cap N = I^s N$. This choice makes M/M' into an essential extension of N/I^sN, where $I^s(N/I^sN) = 0$. If we are able to establish that under the circumstances M/M' may be annihilated by some I^t then $I^t(M/M') = 0$ yields $I^tM \cap N \subset I^sN$. Note also that it is sufficient to do this for $s = 1$, because for arbitrary s it will follow from this case applied to $IN, \ldots, I^{s-1}N$ instead of N together with the conjunction of the inclusions thus obtained at each step. If $I = Rc_1 + \ldots + Rc_n$, where $(c_1, \ldots, c_n)$ is a central system, let $\mu : M \to M$ be the map given by $m \to c_1 m$ for all $m \in M$. Since M is left Noetherian, there exists $r \in \mathbb{N}$ such that $\operatorname{Ker} \mu^r = \operatorname{Ker} \mu^{r+k}$ for all $k \in \mathbb{N}$ i.e. $\operatorname{Im} \mu^r \cap \operatorname{Ker} \mu^r = 0$. However N is contained in $\operatorname{Ker} \mu^r$ and N is an essential submodule of M, therefore $\operatorname{Im} \mu^r = 0$. Let t be the smallest natural number such that $\mu^t = 0$, then μ gives rise to an injective R-morphism $M/\operatorname{Ker} \mu \to \operatorname{Ker} \mu^{t-1}$. To prove our original claim it will therefore be sufficient to prove that both $\operatorname{Ker} \mu$ and $\operatorname{Ker} \mu^{t-1}$ can be annihilated by large powers of I. Now c_1 being central, $\operatorname{Ker} \mu$ is an R/Rc_1-module and we apply induction on n to deduce that $I^p \operatorname{Ker} \mu = 0$ for some $p \in \mathbb{N}$. Since there is an injective morphism : $\operatorname{Ker} \mu^m/\operatorname{Ker} \mu \to \operatorname{Ker} \mu^{m-1}$, for $1 \le m \le t$, the fact that $I^q \operatorname{Ker} \mu^{t-1} = 0$ for some $q \in \mathbb{N}$ willfollow by induction on t. $\square$

V.2. Remark. It is easy to see that (exactly as in Proposition V.1.) one may deduce the following assertion : Let R be a ring and I be an ideal of R which is generated by elements of N(R) (I is not necessarily generated by a normalizing set of generators). Then I has the Artin-Rees property.

V.3. Proposition. Let R be a right Noetherian ring and let J be an ideal and denote $\bar{R} = R/J$.

i) If $J \subset I$ is an ideal and $\bar{I} = I/J$, then the completion of $\bar{R}$ for the $\bar{I}$ - adic topology is identified with $\hat{R}/\hat{R}J$.

ii) If I is generated by a centralizing system $(a_1, a_2, \ldots, a_n)$ then $\hat{I}$ is generated by the centralizing system $(i(a_1), i(a_2), \ldots, i(a_n))$ where $i : \hat{R}$ is the canonical morphism.

Proof.

i) Since $\bar{I}^n = J + I^n/J$, then the completion of the left R-module $\bar{R}$ for the I-adic topology is identified with the completion of $\bar{R}$ for the $\bar{I}$ - adic-topology.

From the exact sequence of left R-modules :

$$0 \longrightarrow J \longrightarrow R \xrightarrow{\pi} \bar{R} \longrightarrow 0$$

we obtain the exact sequence of left $\hat{R}$- modules :

$$0 \longrightarrow \hat{J} \longrightarrow \hat{R} \xrightarrow{\hat{\pi}} \hat{\bar{R}} \longrightarrow 0$$

But $\hat{\pi}$ is a ring homomorphism. Since $\hat{J} = \hat{R}J$ it follows that $\hat{\bar{R}} \simeq \hat{R}/\hat{R}J$.

ii) If $I = (a_1)$ then since $a_1 \in Z(R)$ and $i(R)$ is dense in $\hat{R}$, $i(a_1) \in Z(\hat{R})$. Then we procede by recurrence, taking into account i).

V.4. Lemma. If I is an ideal of a ring R, such that R/I is left Noetherian and I = (a) is generated by a single element of the center of R, then $\hat{R}$ is left Noetherian.

Proof. We see that $G_I(R)$ is isomorphic to the quotient ring of the ring R/I [X] . Since R/I is a Noethetian ring, R/I [X] is a Noetherian ring. Hence $G_I(R)$ is a Noetherian ring. Since $G_{\hat{I}}(\hat{R}) \simeq G_I(R)$ (from Corollary IV.5.) it follows that $\hat{R}$ is a left Noetherian ring.

V.5. Theorem. Let R be a ring and I an ideal with a centralizing set of generators If R is left Noetherian, then $\hat{R}$ is left Noetherian.

Proof. Suppose that $I = (a_1, a_2, \ldots, a_n)$ where $a_1,a_2,\ldots,a_n$ is a centralizing set of generators. The proof is given by induction on the number of generators of I.

If $I = (a_1)$ then $\hat{R}$ is left Noetherian by Lemma V.4.. From statement 8) and Proposition V.3. it follows that there exists an exact sequence :

$$0 \longrightarrow \hat{R}i(a_1) \longrightarrow \hat{R} \longrightarrow \hat{\bar{R}} \longrightarrow 0 \qquad (*)$$

where $\bar{R} = R/Ra_1$ and the $i(a_1),\dots,i(a_n)$ are a centralizing set of generators of $\hat{I}$. From (*) it follows that $\hat{\bar{R}}$ is left Noetherian by the induction hypothesis.

Hence $\hat{R}/\hat{R}i(a_1)$ is a left-Noetherian ring. If we denote by J the ideal $J = \hat{R}i(a_1)$ then from Lemma V.4. the J-adic completion $\hat{\hat{R}}$ of $\hat{R}$ is a left Noetherian ring. Because $J \subset \hat{I}$ we have a commutative diagram

$$\begin{tikzcd}
\hat{\hat{R}} = \varprojlim_n \hat{R}/J^n \arrow[r, "\alpha"] & \varprojlim_n \hat{R}/\hat{I}^n = \hat{R} \\
\hat{R} \arrow[u, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

where α, i and j are the canonical morphisms. Since j is an isomorphism, it follows that α is a surjective morphism. Since $\hat{\hat{R}}$ is left Noetherian, $\hat{R}$ is left Noetherian.

VI: The Functor $\mathrm{HOM}_R(-,-)$. Filt-projective (-injective) Modules

The properties of HOM_R mentioned earlier make it into a functor $(R\text{-filt})^2 \to Z\text{-filt}$; and it is clear that, by definition of the filtration in $\mathrm{HOM}_R(-,-)$, $F\,\mathrm{HOM}_R(M,N)$ is exhaustive for any $M, N \in R\text{-filt}$.

VI.1. Lemma. Let R be a filtered ring and let $M, N \in R\text{-filt}$ be such that M is a finitely generated object while $F\,N$ is exhaustive, then $\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$.

Proof. Let $\{x^{(i)}, n_i\}$ be a system of generators for M and take $f \in \mathrm{Hom}_R(M,N)$. There exists $s \in Z$ such that $f(x^{(i)}) \in F_{n_i+s}N$ for any $1 \le i \le n$. It follows from this that $f \in F_s\,\mathrm{HOM}_R(M,N)$ and hence $f \in \mathrm{HOM}_R(M,N)$. Note that, if FN is not exhaustive and if N' is the exhaustion of N then

$$\mathrm{HOM}_R(M,N) = \mathrm{HOM}_R(M,N') = \mathrm{Hom}_R(M,N')$$

VI.2. Remark. In general $\mathrm{HOM}_R(M,N) \neq \mathrm{Hom}_R(M,N)$. For example let R be a discretely and exhaustively filtered ring such that $R = F_iR$ for all i when $F_i(R) \neq 0$. Take M to be filt-free with filt-basis $\{x^{(i)},0\}$ with $1 \le i \le \infty$, and take $N = R$. Define $f : M \to N$ by $f(x^{(i)}) = a^{(i)}$ where $a^{(i)} \notin F_iR$. If the degree of f is p, let $i_0 \in Z$ be such that $F_iN = 0$ for any $i < i_0$. Consider $i < i_0 - p$, hence $f(x^{(i)}) = a^{(i)} = 0$, contradiction.

VI.3. Proposition. Let $M, N \in R\text{-filt}$, then :

1. If FM is exhaustive and FN is separated, then $F\,\mathrm{HOM}_R(M,N)$ is separated
2. If M is a finitely generated R-module and if FM is exhaustive and FN is discrete, then $F\,\mathrm{HOM}_R(M,N)$ is discrete.
3. If FM is exhaustive and FN is complete then $F\,\mathrm{HOM}_R(M,N)$ is complete.

Proof.

1. Let $f \in \bigcap_p F_p\, HOM_R\,(M,N)$. Then $f(F_pM) \subset \bigcap_s F_{p+s}\, N = 0$. Therefore $f(M) = f^p(\bigcup_p F_pM) = 0$.

2. Let $x^{(1)},\ldots,x^{(n)}$ be generators for M. Since FM is exhaustive, there is an $r_o \in Z$ such that $x^{(i)} \in F_{r_o}M$ for all $i \in \{1,\ldots,n\}$. Since $F_iN = 0$ for all $i < n_o$ for some $n_o \in Z$, it follows that for any $f \in F_i\, HOM_R(M,N)$ where $i < n_o - r_o$ we have $f(x^{(i)}) = 0$, hence $F_i\, HOM_R\,(M,N) = 0$ if $i < n_o - r_o$

3. Let $q < p \in Z$ and consider the projective system

$$\begin{tikzcd}[column sep=small] & HOM_R\,(M,N) \arrow[dl, "\varphi_p"'] \arrow[dr, "\varphi_q"] & \\ HOM_R(M,N)/F_pHOM_R(M,N) \arrow[rr, "\pi_{pq}"] & & HOM_R(M,N)/F_q\, HOM_R(M,N). \end{tikzcd}$$

We claim that $HOM_R(M,N) = \varprojlim_p HOM_R\,(M,N)/F_p\, HOM_R(M,N) = X$.

Indeed, let $(f^{(p)}, p \in Z)$ be an element of X such that $\pi_{pq}\,(f^{(p)}) = f^{(q)}$ for $p < q$. Let $f^{(p)} = \varphi_p(g^{(p)})$, with $g^{(p)} \in HOM_R\,(M,N)$. If $p < q$ then $g^{(p)}-g^{(q)} \in F_q\, HOM_R(M,N)$ and therefore $(g^{(p)}-g^{(q)})(F_sM) \subset F_{s+q}N$ for $s \in Z$. Since FM is exhaustive, for any $x \in M$ there exists some $t \in Z$ such that $x \in F_tM$ and thus $g^{(p)}(x)-g^{(q)}(x) \in F_{t+q}N$. Consequently the sequence $\{q^{(p)}(x),\ p \in Z\}$ is a Cauchy sequence in N. Completeness of N entails that the mapping $f : M \to N$ defined by $f(x) = \lim_{p\to\infty} g^{(p)}x$, is well defined. If $x \in F_tM$, then : $g^{(p)}(x) \in g^{(q)}(x) + F_{t+q}N$ for $p \leq q$ implies that $f(x) \in g^{(q)}(x) + F_{t+q}N$. Therefore $(f-g^{(q)})(F_tM) \subset F_{t+q}\,N$, or $f-g^{(q)} \in F_qHOM_R(M,N)$. Hence $f \in HOM_R(M,N)$ and $\varphi_q(f) = f^{(q)}$ for the selected q . □

VI.4. Corollary. Let $M \in R$-filt be such that FM is exhaustive and complete then :

i) The ring $HOM_R(M,M)$ is complete

ii) The ring $Hom_{FR}(M,M)$ is complete.

<u>Proof.</u>

i) Directly from proposition VI.3.3.

ii) Since $Hom_{FR}(M,M) = F_0 HOM_R(M,M)$ we see that $Hom_{FR}(M,M)$ is closed in $HOM_R(M,M)$. $Hom_{FR}(M,M)$ is complete.

We now consider filt-projective modules. We say that a $P \in R$ filt is <u>filt-projective</u> if the following diagram in R-filt, where f is a strict morphism,

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ M \arrow[r, "f"] & M'' \arrow[r] & 0 \end{tikzcd}$$

may be completed by a filtered morphism such that the diagram is commutative. If P is filt-projective, the canonical mapping $HOM(1_P, f) : HOM_R(P,M) \to HOM_R(P,M'')$ is an epimorphism.

<u>VI.5. Proposition.</u>Let R be a filtered ring, such that FR is exhaustive. Suppose $P \in$ R-filt is such that FP is exhaustive. Then the following statements hold :

1. P is filt-projective if and only if P is a direct summand in R-filt of a filt-free object.

2. If P is filt projective, then G(P) is projective in G(R)-gr.

3. If FR is complete and if P is finitely generated filt-projective then FP is complete.

<u>Proof.</u>

1. Let L be filt-free with filt-basis $(x^{(i)}, n_i)_{i \in J}$. Consider the following diagram in R-filt :

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow g & & \\ M & \xrightarrow{f} & M'' & \longrightarrow & 0 \end{array}$$

where f is a strict epimorphism. Since $x^{(i)} \in F_{n_i}L$, it follows that $g(x^{(i)}) \in F_{n_{i_0}} M'' = f(F_{n_i} M)$. Hence there exits $y^{(i)} \in F_{n_i}M$ such that $g(x^{(i)}) = f(y^{(i)})$. Lemma IV.1. 5° yields the existence of a filtered morphism of degree 0, $h : L \to M$ such that $h(x^{(i)}) = y^{(i)}$. Clearly $g = f \circ h$, hence L is filt-projective. If P is any filt-projective object then there exists a filt-free object L and a strict epimorphism $L \xrightarrow{f} P \longrightarrow 0$ (see Lemma IV.1.8°). Projectivity of P implies the existence of a morphism $g' : P \to L$ of degree 0, with $f \circ g' = 1_P$. We have $g'(F_iP) \subset F_iL \cap \operatorname{Im} g'$, while conversely $x \in F_iL \cap \operatorname{Im} g'$ implies $x = g'(y) \in F_iL$ i.e. $f(x) = f \circ g'(y) = y \in F_iP$.

Therefore $x \in g'(F_iP)$ or $g'(F_iP) = F_iL \cap \operatorname{Im} g'$ follows. The latter means that g' is strict. Note that in R-mod we have :

$L = \operatorname{Im} g' \oplus \operatorname{Ker} f$. Now we claim that :

$F_iL = F_i \operatorname{Im} g' \oplus F_i \operatorname{Ker} f = (\operatorname{Im} g' \cap F_iL) \oplus (\operatorname{Ker} f \cap F_iL)$.

Indeed, if $x \in F_iL$ then $x = y+z$ with $y \in \operatorname{Im} g'$ and $z \in \operatorname{Ker} f$. Hence $f(y) = f(x) \in F_iP$ and, while $y \in \operatorname{Im} g'$, we get that $g' \circ f(y) = y$ or $y \in g'(F_iP) = F_iL \cap \operatorname{Im} g'$. On the other hand, $z = x-y \in F_iL \cap \operatorname{Ker} f$ and thus $L = \operatorname{Im} g \oplus \operatorname{Ker} f'$ in R-filt. Finally since g' is a strict monomorphism and since we have $\operatorname{Im} g' \cong P$ in R-filt it follows that P is a direct summand of a filt-free object.

The converse implication is easy.

2. Directly from 1 and the properties of the functor G

3. The construction of L in 1 shows that, in case P is finitely generated

in R-filt, we may choose L to be finitely generated too and we have that $L = P \oplus Q$ for some $Q \in$ R-filt. Since R is complete, L is complete and so a Cauchy-sequence $(x_n)_{n\in\mathbb{N}}$ in P converges to some $x \in L$. Write $x = p + q$ with $p \in P$, $q \in Q$. Since FP is exhaustive we have $p \in F_kP$ for some $k \in Z$ and it is clear that $x_n - p$ will be in F_tL for t large enough i.e. $q = \lim\limits_{\vec{n}} (x_n - p) = 0$ and $x \in P$. That P is complete follows. □

If $M,N \in$ R-filt, we introduce a natural map. $\varphi = \varphi(M,N)$,

$$\varphi : G(HOM_R(M,N)) \to HOM_{G(R)}(G(M),G(N)),$$

as follows : for $f \in F_p\, HOM_R(M,N)$, $x \in F_qM$ put $\varphi(f_p)(x_q) = f(x)_{p+q}$.

VI.6. Lemma. $\varphi(M,N)$ is a monomorphism Moreover, $\varphi(M,N)$ is an isomorphism if M is filt-projective.

Proof. Fixing M, $N \in$ R-filt we write φ instead of $\varphi(M,N)$. If $\varphi(f_p) = 0$, then $f(x)_{p+q} = 0$ for every $x \in F_qM$, $q \in Z$. It follows that $f(F_qM) \subset F_{p+q-1}N$ for any $q \in Z$ i.e. $f \in F_{p-s}\, HOM_R(M,N)$. Consequently $f_p = 0$. In order to prove the second statement, first assume that M is filt-free and let $(x^{(i)}, p_i)_{i\in J}$ be a filt-basis for M. In this case $\{x^{(i)}_{p_i}, i \in J\}$ is a homogeneous basis for G(M). Letting $g \in HOM_{G(R)}(G(M), G(N))_p$, we obtain $g(x^{(i)}_{p_i}) = y^{(i)}_{p+p_i}$. Define $f : M \to N$ by putting $f(x^{(i)}) = y^{(i)}$, with this definition it is clear that $f \in F_pHOM_R(M,N)$ and also that $\varphi(f_p) = g$. This establishes surjectivity of φ. If M is filt-projective then there is a filt-free L such that $L = M \oplus M'$ in R-filt. We know that HOM_R commutes with finite direct sums therefore we may use the previous part of the proof. □

VI.7. Definition. Let FR be an exhaustive filtration of R. A filt-injective object is an object $M \in$ R-filt such that, if I is a left ideal of R equiped with the induced filtration then any filtered morphism $f : I \to M$ extends to a filtered morphism $R \to M$.

Remarks.

1. Let FR be exhaustive and complete and suppose that G(R) is left Noetherian. If M is R-filt injective then it is injective in R-mod, because of Lemma VI.1.

2. Let FR be exhaustive. If $Q \in$ R-filt is injective in R-mod then it is injective in R-filt.

VI.9. Theorem. Let FR be exhaustive and let $Q \in$ R-filt be complete. If G(Q) is injective in G(R)-gr then Q is filt-injective.

Proof. Equip the left ideal I of R, with the filtration $F_pI = F_pR \cap I$. Since $i : I \to R$ is a strict morphism we obtain the following commutative diagram :

$$\begin{array}{ccc} G(HOM_R(R,Q)) & \xrightarrow{G\ HOM_R\ (i,1_Q)} & G(HOM_R(I,Q)) \\ \downarrow{\scriptstyle \varphi(R,Q)} & & \downarrow{\scriptstyle \varphi(I,Q)} \\ HOM_{G(R)}(G(R),G(Q)) & \xrightarrow{HOM_{G(R)}(G(i),G(1_Q))} & HOM_{G(R)}(G(I),G(Q)) \end{array}$$

Since G(i) is a monomorphism, injectivity of G(Q) entails that $HOM_{G(R)}(G(i),G(1_Q))$ is an epimorphism . Hence φ (I,Q) is an epimorphism too, therefore an isomorphism. Thus G $HOM_R(i,1_Q)$ is an isomorphism, while F $HOM_R(R,Q)$ and F $HOM_R(I,Q)$ are exhaustive and complete, therefore by Theorem III.4., $HOM_R(i,1_Q)$ is epimorphic. □

We will say that $x \in R$ is topologically nilpotent if the sequence $\{ x^n, n \in \mathbb{N} \}$ is a Cauchy sequence converging to 0. Recall the following lemma about topological rings :

VII.1. Lemma. Let R be a complete topological ring and consider a fundamental set of neighbourhoods of zero, consisting of additive subgroups. If $\varphi : R \to S$ is a ring homomorphism such that every $x \in \operatorname{Ker} \varphi$ is topologically nilpotent, then every idempotent element of $\operatorname{Im} \varphi$ may be lifted to R.

Proof. Let $e \in \operatorname{Im} \varphi$ be idempotent, $e = \varphi(x)$ for some $x \in R$. Since $\varphi(x^2-x) = 0$, $a = x^2-x \in \operatorname{Ker} \varphi$. Put $b = x + \lambda(1-2x)$ with $\lambda \in R$, and determine λ such that $b^2 = b$ while λ commutes with a. We get $(\lambda^2-\lambda)(1+4a)+a=0$. Put :

$$(\star) \qquad \lambda = \frac{1}{2}\left(1-(1+4a)^{\frac{1}{2}}\right) = \frac{1}{2}\sum_{1\le k<\infty} (-1)^{k-1}\, C_{2k}^{k}\, a^k$$

where C_{2k}^{k} is the coefficient of $1^k(-1)^k$ in the binomial expansion of $(1-1)^{2k} = 0$ i.e. $\frac{1}{2}\, C_{2k}^{k}$ is an integer. Therefore λ may be viewed as a power series with integer coefficients which is easily seen to be a convergent one. Hence $(\star)$ determines λ and it commutes with a. Consequently b is idempotent. Furthermore , $a \in \operatorname{Ker} \varphi$ yields $\lambda \in \operatorname{Ker} \varphi$ and thus $\varphi(b) = \varphi(x) = e$. □

VII.2. Lemma. Let R be a filtered ring, $M \in R$-filt such that FM is exhaustive and complete. Suppose that $f : M \to M$ is a filtered morphism such that $G(f)^2 = G(f)$. Then there exists a strict morphism $g : M \to M$ in R-filt satisfying : $g^2 = g$ and $G(g) = G(f)$.

Proof. Consider the morphism :

$$\varphi : \mathrm{HOM}_{FR}(M,M) \to \mathrm{Hom}_{G(R)\text{-}gr}(G(M), G(M))$$

given by $\varphi(f) = G(f)$; φ is a ring homomorphism. We see that Ker φ = $= F_{-1}HOM_R(M,M) = \{ f:M \to M, f(F_pM) \subset F_{p-1}M, \forall p \in Z \}$. Hence if $f \in$ Ker φ then $f^n \in F_{-n} HOM_R(M,M)$ and we find that $\lim_{n\to\infty} f^n = 0$. Hence, by Lemma VII.1., every idempotent in Im φ can be lifted. If only remains to be proved that any idempotent f in $Hom_{FR}(M,M)$ is strict.

Suppose that $f(x) \in F_pM \cap$ Im f. Then $f(f(x)) = f(x)$ so that $f(x) \in f(F_pM)$, hence f is strict.

<u>VII.3. Remark.</u> If $M \in$ R-filt is such that FM is exhaustive and complete then $F_{-1} HOM_R(M,M) \subset J(HOM_R(M,M))$. (J = Jacobson radical). Indeed if $f \in F_{-1} HOM_R(M,M)$ then f is topologically nilpotent and so 1-f has an inverse $\sum_{0\leq n<\infty} f^n$.

<u>VII.4. Corollary.</u> If M is as in the foregoing lemma then any countable set of orthogonal idempotent elements in Im φ can be lifted to $Hom_{R\text{-filt}}(M,M)$.

<u>VII.5. Theorem.</u> Let R be a filtered ring such that FR is exhaustive. Let P_g be projective in G(R)-gr and suppose that, either FR is discrete and the gradation of P_g is left limited or P_g is finitely generated while FR is complete. Then there is a filt-projective module $P \in$ R-filt such that $G(P) = P_g$. If $M \in$ R-filt, then for any morphism $g : P_g \to G(M)$ of degree p there is a filtered morphism $f : P \to M$ of degree p such that $g = G(f)$.

<u>Proof.</u> The hypotheses imply that there is a free object L_g in G(R)-gr and a morphism $h : L_g \to L_g$ of degree zero, such that $h^2 = h$ and Im $h = P_g$. If P_g is finitely generated then L_g may be chosen to be finitely generated too. Assume that $L_g = G(L)$, where $L \in$ R-filt is filt-free. If P_g has left limited grading, then L_g may be chosen so that it has left-limited grading too and we suppose FL to be discrete since FR is discrete. In the case

where P_g is finitely generated and FR is complete, we obtain that FL is complete. By Lemma D.7., there exists a strict filtered morphism $f : L \to L$ with $f^2 = f$ and $G(f) = h$. Put $P = \mathrm{Im}\, f$. Since f is strict it follows that $G(P) = P_g$ and that P is filt-projective as required. Moreover if P is finitely generated then FP is complete. To prove the remaining statement we may write $g : G(P) \to G(M)$ and up to taking a suitable suspension we may assume $\deg g = 0$. We have $L = P \oplus Q$ and hence there is a graded morphism $h : G(L) \to G(M)$ such that $h| G(P) = g$. However $h = G(k)$ for some filtered morphism $k : L \to M$. Putting $f = h|P$ we arrive at $g = G(f)$. □

VII.6. Corollary. Let R be a filtered ring such that FR is exhaustive. Let $P \in$ R-filt be such that FP is separated and exhaustive. Suppose that FR is discrete and G(P) is left limited or G(P) is finitely generated while FR is complete. If G(P) is projective in G(R)-gr, then P is filt-projective.

Proof. By the Theorem VII.5. we may select a projective object $Q \in$ R-filt such that $G(Q) \simeq G(P)$ (Q may be chosen in both cases such that FQ is complete and exhaustive); let $g : G(Q) \to G(P)$ denote this isomorphism There exists a filtered morphism $f : Q \to P$ such that $G(f) = g$. We can apply Corollary II.5. and deduce that f is an isomorphism.

VII.7. Corollary. Let R be a filtered ring such that FR is exhaustive and discrete and let $M \in$ R-filt be such that FM too is exhaustive and discrete. Then we have :

$$\mathrm{p.dim}_R M \leq \mathrm{p.dim}_{G(R)} G(M)$$

Proof. By Lemma IV.1.8°, there exists a free resolution for M in R-filt:

$$\cdots \longrightarrow L_{n-1} \xrightarrow{f_{n-1}} L_{n-2} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{f_1} L_o \xrightarrow{f_o} M \longrightarrow 0$$

where FL_j are discrete for all j. Write Q for Ker f_{n-1}, where $n = \mathrm{p.dim}_{G(R)}\, G(M)$, then G(Q) is a projective G(R)-module with left limited grading. By the theorem we may select a projective object $Q \in R\text{-filt}$ such that $G(Q) \cong G(Q')$, let $g : G(Q') \to G(Q)$ denote this isomorphism. There exists a filtered morphism $f : Q' \to Q$ such that $G(f) = g$. Now we end up with the situation where Q' is filt-projective, FR is discrete (note that FQ' turns out to be discrete just as well) while $G(f)$ is an isomorphism, therefore we may apply Corollary III.5. and deduce that f is an isomorphism, whence it follows that Q is filt projective and therefore projective in R-mod. □

VII.8. Corollary. Let R be a filtered ring with exhaustive and discrete filtration, then $\text{gl.dim}\, R \le \text{gr.gl.dim}G(R)$.

Proof. Equip a left R-module M with the filtration $F_iM = F_iR.M$, then FM is exhaustive and discrete. By the foregoing corollary we have : $\text{p.dim}\, M \le \text{p.dim}\, G(M) \le \text{gr.gl.dim}\, G(R)$. □

VII.9. Corollary. Let R be a graded ring, then : $\text{gr.gl.dim}\, R \le \text{gl.dim}\, \underline{R}$. If the gradation of R is left limited then : $\text{gr.gl.dim}\, R = \text{gl.dim}\, \underline{R}$.

Proof. The second statement follows from the foregoing corollary applied to the ring R filtered with the associated filtration. The first statement from the graded theory in Chapter A.

VII.10. Corollary. Let R be a filtered ring such that FR is exhaustive and complete, let $M \in R\text{-filt}$ be such that FM is exhaustive. Suppose that $G(R)$ and $G(M)$ are left Noetherian, then : $\text{p.dim}_R M \le \text{p.dim}_{G(R)} G(M)$.

Proof. Apply Lemma IV.1. and Theorem VII.5. and use the argumentation of the proof of Corollary VII.7. □

VII.11. Corollary. Let R be a filtered ring such that FR is exhaustive and complete. Suppose that $G(R)$ is left Noetherian. We have $\text{gl.dim}\, R \le \text{gr.gl.dim}\, G(R)$.

Proof. Let M be a finitely generated left R-module and take $x_1, \ldots, x_m$ to be a set of generators. Filter M with the filtration : $F_iM = \sum_{k=1}^{n} F_iR.x_k$. Doing this we obtain a finitely generated filtered module with an exhaustive filtration. Corollary VII.9. finishes the proof. □

VIII: Weak (Flat) Dimension of Filtered Modules

Let us start this section with a basic lemma on tensor products of filtered modules. If $M \in$ R-filt, $N \in$ filt-R then the Z-module $N \otimes_R M$ has a filtration : $F_p\,(N \otimes_R M)$ is the Z-submodule of $N \otimes_R M$ generated by all elements of type $n \otimes m$ where $n \in F_tN$, $m \in F_sM$ and $p = s+t$. This construction yields a functor :

$$\underset{R}{\otimes} : \text{filt-R R x R-filt} \to \text{Z-filt.}$$

VIII.1. Lemma. With notations as above, the following statements are true :

1° If FM and FN are exhaustive then so is $F(N \underset{R}{\otimes} M)$.

2° If FM and FN are discrete then so is $F(N \underset{R}{\otimes} M)$.

3° For all $m, n \in Z$: $N(n) \underset{R}{\otimes} M(m) = (N \underset{R}{\otimes} M)(m+n)$.

4° In the category R-filt we have the following isomorphism $R \underset{R}{\otimes} M \cong M$; similarly $N \otimes_R R \cong N$ in filt-R.

5° This functor $\underset{R}{\otimes}$ commutes with direct sums and inductive limits.

6° For $M \in$ filt-R, $N \in$ R-filt-S, $P \in$ filt-S, we have the following isomorphism in Z-filt : $HOM_S(M \underset{R}{\otimes} N,P) \cong HOM_R\,(M, HOM_S(N,P))$.

Proof.

Statements 1°, 2°, 3° are clear.

4° Consider the R-module morphism, $\varphi : N \underset{R}{\otimes} R \to N$, $\varphi(x.r) = xr$. Its inverse is given by $\psi : N \to N \underset{R}{\otimes} R$, $\psi(x) = x \otimes 1$. Now, if $x \otimes r \in F_p\,(N \underset{R}{\otimes} R)$ with $x \in F_iM$, $r \in F_jR$, $i+j = p$, then $\varphi(x \otimes r) = x\,r \in F_iN.F_jR \subset F_pN$. In a similar way one establishes that ψ is a morphism in filt-R.

5° Let $N = \underset{\alpha \in A}{\oplus} N_\alpha$, $M = \underset{\beta \in B}{\oplus} M_\beta$, then

$$\underset{(\alpha,\beta) \in A x B}{\oplus} N_\alpha \underset{R}{\otimes} M \cong N \underset{R}{\otimes} M. \text{ Indeed, let}$$

$$\varphi : N \underset{R}{\otimes} M \to \bigoplus_{(\alpha,\beta)\in A.B} (N_\alpha \underset{R}{\otimes} M_\beta)$$

be the R-module morphism which is given by :

$$\varphi((\sum_{\alpha\in A} x_\alpha) \otimes (\sum_{\beta\in B} y_\beta)) = \sum_{(\alpha,\beta)\in A\times B} x_\alpha \otimes y_\beta$$

Then it is well known that φ is an isomorphism of abelian groups. Let $x \otimes y \in F_p (N \underset{R}{\otimes} M)$ where $x \in F_iN$, $y \in F_jN$ and $i + j = p$. If $x = \sum_{\alpha\in A} x_\alpha$, $y = \sum_{\beta\in B} y_\beta$ then $x_\alpha \in F_iN_\alpha$ and $y_\beta \in F_j M_\beta$. It follows immediately that $x_\alpha \otimes y_\beta \in F_p (N_\alpha \underset{R}{\otimes} M_\beta)$, hence $\varphi(x \otimes y) \in F_p (\bigoplus_{(\alpha,\beta)\in A\times B} N_\alpha \otimes M_\beta)$. In a similar way one establishes that φ^{-1} is actually a morphism in Z-filt. The assertion for inductive limits may be proved in an equally straightforward way .

6° Define

$$\varphi : HOM_S(M \underset{R}{\otimes} N,P) \to HOM (M,HOM_S(N,P))$$

by : $(\varphi(f)(x))(y) = f(x \otimes y)$, and define

$$\psi : HOM_R(M,HOM_S(N,P)) \to HOM_S (M \underset{R}{\otimes} N,P)$$

by : $\psi(g)(x \otimes y) = (g(x))(y)$. If $f \in F_p HOM_S (M \underset{R}{\otimes} N,P)$, $x \in F_tM$, $y \in F_rN$ then it follows from $x \otimes y \in F_{t+r} M \underset{R}{\otimes} N$ that $f(x \otimes y) \in F_{p+t+r}P$= Hence $\varphi(f)(x) \in F_{p+t} HOM_S(N,P)$ and thus $\varphi(f) \in F_p HOM_R (M,HOM_S(N,P))$. This expresses that φ is a morphism in Z-filt. Along the same lines one verifies that ψ is a morphism of degree 0 and ψ is the inverse of φ. □

Let $N \in$ R-filt, $M \in$ filt-R. Define a graded morphism

$$\varphi = \varphi(N,M) : G(M) \underset{G(R)}{\otimes} G(N) \to G(M \otimes N),$$ by putting

$\varphi (m_s \otimes n_t) = (m \otimes n)_{s+t}$. Obviously this is well defined and it is clear that φ is surjective.

<u>VIII.2. Lemma.</u> If either M or N is a free object in filt-R or R-filt respectively then $\varphi(M,N)$ is an isomorphism.

<u>Proof.</u> Knowing that $\otimes_R$ and G commute with direct sums, the proof may be reduced to the case where $M = R(n)_R$. But then Lemma VIII.1. allows us to reduce the proof further to the case $M = R_R$. In this case we have the following commutative diagram of graded morphisms, vertical arrows representing isomorphisms :

$$\begin{tikzcd}
G(R_R) \underset{G(R)}{\otimes} G(N) \arrow[r, "\varphi(M,R)"] & G(R \underset{R}{\otimes} N) \\
G(N) \arrow[u] \arrow[r, "G(N)"] & G(N) \arrow[u]
\end{tikzcd}$$

So, $\varphi(M,R)$ is also an isomorphism. □

<u>VIII.3. Lemma.</u> Let R be a filtered ring and let $M \in R$-filt be such that FR and FM are discrete and exhaustive. If $G(M)$ is a flat object in $G(R)$-gr then M is a flat R-module.

<u>Proof.</u> If J is a right ideal of R, equip it with the filtration $F_pJ = F_pR \cap J$. Let $i : J \to R$ be the canonical inclusion morphisms ; note that i is a strict morphism. Application of Theorem III.4. yields that $G(i)$ is injective, and so we obtain the following commutative diagram :

$$\begin{tikzcd}
G(J) \underset{G(R)}{\otimes} G(M) \arrow[r, "G(i) \otimes 1_{G(M)}"] \arrow[d, "\varphi(M,J)"] & G(R) \underset{G(R)}{\otimes} G(M) \arrow[d, "\varphi(M,R)"] \\
G(J \underset{R}{\otimes} M) \arrow[r, "G(i \otimes 1_M)"'] & G(R \underset{R}{\otimes} M)
\end{tikzcd}$$

The fact that $\varphi(M,R)$ is an isomorphism, whereas $G(i) \otimes 1_{G(M)}$ is a monomorphism, entails that $\varphi(M,J)$ is a monomorphism, hence an isomorphism. Then $G(i \otimes 1_M)$ has to be monomorphic and repeated application of Theorem III.4. allows to derive from this that $F(J \underset{R}{\otimes} M)$ is discrete. Hence, $i \otimes 1_M$ is a strict monomorphism and it follows that M is flat. □

VIII.4. Corollary. Let R be a filtered ring, $M \in R$-filt, such that FR are discrete and exhaustive. Then we have :

$$\text{w.dim}_R\, M \leq \text{gr.w.dim}_{G(R)}\, G(M).$$

Proof. Put $n = \text{gr.w.dim}_{G(R)}\, G(M)$. Consider an exact sequence :

$$(*) \quad 0 \longrightarrow L \xrightarrow{f} L_{n-1} \longrightarrow \dots \longrightarrow L_1 \xrightarrow{f_1} L_o \xrightarrow{f_o} M \longrightarrow 0\ ,$$

where $L_o, \dots, L_{n-1}$ are free objects, while FL_i, $i = 0,\dots,n-1$, and FL are exhaustive and discrete. By exactness of the functor G we derive from (*) an exact sequence :

$$0 \longrightarrow G(L) \longrightarrow G(L_{n-1}) \longrightarrow \dots \longrightarrow G(L_o) \longrightarrow G(M) \longrightarrow 0$$

Our assumptions imply that G(L) is a flat object of G(R)-gr and the foregoing lemma entails that L is flat.
Therefore $\text{w.dim}_R\, M \leq n$. □

VIII.5. Corollary. For a filtered ring R with FR being exhaustive and discrete we have :

$$\text{gl.w.dim}\, R \leq \text{gr.gl.w.dim}\, G(R).$$

Proof. Put n = gr.gl. w.dim G(R). Any $M \in R$-mod may be filtered by the filtration $F_pM = F_pR.M$ which is easily seen to be exhaustive and discrete. The inequality of Corollary VII.4. thus holds for any M with the induced filtration, and this proves our claim. □

VIII.6. Corollary. Let R be a graded ring, then :

$\text{gr.gl. w. dim.} R \leq \text{gl.w.dim} \underline{R}$

If moreover R is left limited then :

$\text{gr.gl.w dim} R = \text{gl.w.dim} \underline{R}$.

Proof. Easy combination of foregoing results.

VIII.7. Corollary. Let R be a positively graded ring and assume that R_0 is a Von Neumann regular ring :

$\text{gl.w.dim} R = \text{left-w.dim}_R R_0 = \text{right - w.dim}_R R_0$.

Proof. Take $M \in R\text{-mod}$ and equip it with the trivial filtration $F_pM = 0$ for $p < 0$, $F_pM = M$ for $p \geq 0$. Endow R with the filtration $F_pR = \bigoplus_{i \leq p} R_i$. In this setting M is a filtered R-module and clearly $G(M)_p = 0$ for all $p \neq 0$. Therefore G(M) is annihilated by the ideal $R_+ = \bigoplus_{n \geq 1} R_n$. From Corollary VIII.4. , we obtain :

$\text{w.dim}_R M \leq \text{w.dim}_R G(M) \leq \text{l.w.dim}_R R_0 + R_0 + \text{l.w.dim}_{R_0} G(M)$

and the latter is just $\text{l.w.dim}_R R_0$. It follows that $\text{gl.w.dim} R = \text{l.w.dim}_R R_0$. The other equality may be established in exactly the same way. □

VIII.8. Corollary. Let R be a filtered ring such that FR is discrete and exhaustive. Let $N \in R\text{-filt}$, $M \in \text{filt-}R$ be such that FM and FN are discrete and exhaustive, then :

$\text{Tor}_n^{G(R)}(G(M),G(N)) = 0$ implies $\text{Tor}_n^R (M,N) = 0$.

Proof. Consider a free resolution of N :

$$\cdots \to L_2 \xrightarrow{f_2} L_1 \xrightarrow{f_1} L_0 \xrightarrow{f_0} N \to 0,$$

where F_iL is discrete and exhaustive, for all i, and where $f_0, f_1, f_2, \ldots$ are strict morphisms. Consider the following commutative diagrams, the vertical arrows of which represent isomorphisms :

$$\begin{array}{ccccccc}
\cdots \longrightarrow G(M) \underset{G(R)}{\otimes} G(L_2) & \longrightarrow & G(M) \underset{G(R)}{\otimes} G(L_1) & \longrightarrow & G(M) \underset{G(R)}{\otimes} G(L_0) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow G(M \underset{R}{\otimes} L_2) & \longrightarrow & G(M \underset{R}{\otimes} L_1) & \longrightarrow & G(M \underset{R}{\otimes} L_0) & \longrightarrow & 0
\end{array}$$

We have :

$$G_1^{\star,n}(M \underset{R}{\otimes} L_\star) = H_n G(M \underset{R}{\otimes} L_\star) = H_n\,(G(M) \underset{G(R)}{\otimes} G(L_\star)) = \mathrm{Tor}_n^{G(R)}(G(M),G(N)).$$

If $\mathrm{Tor}_n^{G(R)}(G(M),G(N)) = 0$ then $G^{\star,n}(M \underset{R}{\otimes} L_\star) = 0$ and, since $F(M \underset{R}{\otimes} L_\star)$ is exhaustive and discrete ,

$$\mathrm{Tor}_n^R\,(M,N) = H_n(M \underset{R}{\otimes} L_\star) = 0. \quad \square$$

IX: Exercises. Comments. References

1. Let R be a filtered ring with the filtration F_iR such that $F_0R = R$, $M \in R$-filt with the filtration F_iM, such that $F_0M = M$ and $N \subset M$ a submodule equiped with the induced filtration. Let $\varphi_M : M \to \hat{M}$ be the canonical morphism. We shall denote by N' the adherence of $\varphi_M(N)$ in $\hat{M}$, that is $N' = \bigcap_{i \in Z} (\varphi_M(N) + F_i \hat{M})$. Prove that :
   1) $\hat{N} \simeq N'$ and $\varphi_M^{-1}(N') = \bigcap_{i \in Z} (N + F_iM)$.
   2) If N,P are submodules of M then $(N \cap P)' = N' \cap P'$ and $(N+P)' = N'+P'$.
   3) Let $f:M \to N$ be a morphism in R-filt. If G is a submodule of N and we denote by $H = f^{-1}(G)$ then $f^{-1}(G') = H'$.
   4) If N,P are submodules such that P is finitely generated then $(N:P)' = (N':P')$.
   5) If I is a finitely generated left ideal, then $(N:I)' = (N':I')$.

2. Let I be a two-sided ideal of a ring R and E be a left R-module; we shall denote by $G_I(E)$ the graded module associated to the filtration. co-induced by I-adic filtration (i.e. $\bar{G}_I(E) = \bigoplus_{n=1}^{\infty} E_n/E_{n-1}$ where $E_n = \mathrm{Ann}_E I^n$). $\bar{G}_I(E)$ is a $G_I(R)$-module ($G_I(R) = \bigoplus_{n=0}^{\infty} I^{n-1}/I^n$); then
   1) The function sending an R-submodule F of E to the $G_I(R)$-submodule $\bar{G}_I(F)$ of $\bar{G}_I(E)$ is strict if $E = \bigcup_{n\geq 1} \mathrm{Ann}_E I^n$.
   2) $\mathrm{Ann}_E I$ is an essential $G_I(R)$-submodule of $\bar{G}_I(E)$.

3. Let R be a ring with filtration F_iR such that $F_0R = R$. Let P be a two-sided ideal of R. If $G(P)$ is a prime ideal in $G(R)$ then the adherence $P = \bigcap_{i \geq 0} (P+F_iR)$ of P is a prime ideal. In particular if the filtration of R is separated and $G(R)$ is a prime ring then R is a prime ring.

4. Let R be a ring with filtration F_iR such that $F_oR = R$ and F_iR is separated. If $G(R)$ is a domain then R is a domain.

5. Let $M = Z$ with the p-adic filtration ($p \in Z$, $p \neq o$) and $N = Z$ with filtration $F_iN = Z$, $\forall\ i \leq o$. Then the morphism 1_Z : $M \to N$ is bijective but it is not strict. Moreover $G(1_Z) : G(M) \to G(N)$ is injective nor surjective.

6. Let R be a filtered ring with filtration F_iR such that $F_oR = R$ and let M be a finitely generated R-module. Consider on M the induced filtration $F_iM = F_iR.M$. Then :
   1) $G(M)$ is a finitely generated $G(R)$-module.
   2) $\hat{M}$ is a finitely generated $\hat{R}$- module.
   3) $\hat{F_iM} = \hat{F_iR}.\hat{M} = \hat{F_iR}.\varphi_M(M)$ and $\hat{M} = \hat{R}\ \varphi_M(M)$
   ($\varphi_M : M \to \hat{M}$ is the canonical morphism).
   (Hint : There exists a free module L finitely generated and a surjective homomorphism $L \to M \to 0$; further on we use the properties of the completion functor).

7. Let $M, N, P \in R$-filt and consider the strict morphisms $M \xrightarrow{f} N \xrightarrow{g} P$. If f is surjective (resp. g is injective) then $g \circ f$ is strict.

8. Let $M, N \in R$-filt and let $f : M \to N$ be a morphism in R-filt.
   1) If f is strict then $\hat{f}: \hat{M} \to \hat{N}$ is strict
   2) If f has the Artin-Rees property then $\hat{f} : \hat{M} \to \hat{N}$ has the Artin-Rees property.

9. 1) If R is a ring, show that if I is the ideal of $R[X_1, X_2,\dots,X_n]$ that is generated by X_i, then the I-adic completion of $R[X_1,X_2,\dots,X_n]$ is the power series ring $R[[X_1,X_2,\dots,X_n]]$.
   2) If R is a Noetherian ring, show that $R[[X_1,\dots,X_n]]$ is a Noetherian ring.

10. Let R be a left Noetherian ring and I an ideal generated by n elements belonging to the center of the ring R. Consider the I-adic filtration on R. Then :

    1) The associated graded ring $G_I(R)$ is left Noetherian

    2) The completed $\hat{R}$ is a left Noetherian ring.

11. 1) Let R be a ring (not necessarily Noetherian) and I an ideal generated by central elements. Then I satisfies the Artin-Rees property for left Noetherian modules.

    2) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of Noetherian R-modules , then $0 \longrightarrow \hat{M}' \longrightarrow \hat{M} \longrightarrow \hat{M}'' \longrightarrow 0$ is exact.

12. 1) Assume that R is a left and right Noetherian ring and I an ideal which satisfies the left and right Artin-Rees properties. For any finitely graded left R-module M we have :

    $\mathrm{Tor}_n^R(R/I,M) \simeq \mathrm{Tor}^R{}_n(R/I,\hat{M})$, $n \geq 0$

    ($\hat{M}$ is the completion with respect to the I-adic filtration)

    2) The following statements are equivalent for a map $f : M \to N$ of finitely generated R-modules :

    a) The map $f_i = \mathrm{Tor}_i^R(R/I,M) \to \mathrm{Tor}_i^R(R/I,N)$ induced by f is an isomorphism for all $i \geq 0$.

    b) f_0 is an isomorphism and f_1 is an epimorphism.

    c) f induces an isomorphism $\hat{f} : \hat{M} \to \hat{N}$

13. Let R be a filtered complete ring and $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ an exact sequence in R-filt where f and g are strict and such that FM', FM, FM'' are separated and exhaustive.

    1) If M is filt.f.g then M'' is filt.f.g.

    2) If M' and M'' are filt.f.g then M is filt.f.g.

14. See [66] .

1) Let $U(g)$ be the universal enveloping algebra of a finite dimensional solvable Lie Algebra g over an algebraically closed field K of characteristic zero and let I be an ideal of $U(g)$. Then I has a normalizing set of generators.

2) Moreover, if g is a nilpotent Lie Algebra, then I has a centralizing set of generators.

(Hint 1)) Consider $U(g)$ as a left g-module by setting $x \bullet r = xr-rx = [x,r]$, $x \in g$, $r \in U(g)$, where xr is the product of x and r in $U(g)$. Let $U(g)_n$ be the R-subspace of $U(g)$ spanned by the m-fold products of elements of g where $0 \le m \le n$. Then $U(g)_n$ is a finite-dimensional space and is a submodule of $U(g)$. Also $U(g) = \bigcup_{n\ge 0} U(g)_n$. Let $0 \ne x \in I$ and let $g(x)$ denote the g-submodule of $U(g)$ generated by x. From above $g(x)$ is finite-dimensional, so by Lie's theorem ([D.2.], pag 60) $\exists\, 0 \ne y_1 \in g(x)$ such than $K y_1$ is a submodule of $U(g)$. Thus $y_1 \in N(U(g)) \cap I$. Suppose now that $y_1, y_2, \dots, u_i$ is a normalizing set of elements of $U(g)$ which belong to I. Let $\overline{U(g)} = U(g)/J$ where $J = (y_1, y_2, \dots, y_i)$. If $J \ne I$ then, by applying the above argument again, $\exists\, \hat{o} \ne \hat{y}_{i+1} \in N(\overline{U(g)}) \cap I$. The set $\{y_1, y_2, \dots, y_{i+1}\}$ is a normalizing set of elements of I. Since $U(g)$ is left and right Noetherian I has a normalizing set of generators.

15. If G is a finitely generated and nilpotent group then the ideal I of the ring $Z[G]$, where $I = \langle 1-g,\ g \in G \rangle$ has the Artin-Rees property. (Hint : G has a central series $G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$ such that each quotient G_i/G_{i+1} is cyclic. The proof continues by induction on the minimal length n of such a series.

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Index