DEGENERATIONS OF FLAG AND SCHUBERT VARIETIES TO TORIC VARIETIES

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Abstract. In this paper, we prove the degenerations of Schubert varieties in a minuscule G/P, as well as the class of Kempf varieties in the flag variety $SL(n)/B$, to (normal) toric varieties. As a consequence, we obtain that determinantal varieties degenerate to (normal) toric varieties.

Introduction

In this paper, we carry out the proof of the results announced in [21]. Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k. Fix a maximal torus T in G, a Borel subgroup $B \supset T$. Let W be the Weyl group of G relative to T. Let $Q \supseteq B$ be a parabolic subgroup of classical type, say $Q = \bigcap_{i=1}^{r} P_{k_i}$, where P_{k_i}, $1 \leq i \leq r$, is a maximal parabolic subgroup of classical type (see [26] for the definition of a parabolic subgroup of classical type). Let $W(Q)$ be the Weyl group of Q. For $w \in W/W(Q)$, let $X(w)(=\overline{BwQ} \pmod{Q}$ with the canonical reduced structure of a scheme) denote the Schubert variety in G/Q, associated to w. Given $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}_+^r$, the notion of "standard Young tabeaux" on $X(w)$ of type $\mathbf{m}$ was introduced in [24] (see also [22]), and an explicit basis for $H^0(X(w), L^{\mathbf{m}})$ (where $L^{\mathbf{m}} = L_{k_1}^{m_1} \otimes \cdots \otimes L_{k_r}^{m_r}$, L_{k_i} being the ample generator of $Pic(G/P_{k_i})$), indexed by standard Young tableaux of type $\mathbf{m}$, was constructed in [24] (see also [22]). If $G = SL(n)$ and $Q = B$, then this notion in fact, coincides with the classical Hodge-Young notion of standard Young tableaux on the flag manifold $SL(n)/B$ (cf [13]). The explicit nature of this basis has led to very many interesting (and important) geometric and representation theoretic consequences – such as the determination of the singular locus of a Schubert variety ([27], [19], [20]), generalization of the Littlewood–Richardson rule (cf. [29], [30]), etc. In this paper using this basis we show that the flag variety $SL(n)/B$, and the Schubert varieties in a minuscule G/P degenerate to (normal) toric varieties. (Of course, such a degeneration of a projective variety affords easier computations of those invariants of the variety which are preserved under flat deformations, such as

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the Hilbert polynomial). Such a degeneration is carried out as follows. Let

$$R = \bigoplus_{\mathbf{a}} H^0(G/Q, L^{\mathbf{a}}), \quad R(w) = \bigoplus_{\mathbf{a}} H^0(X(w), L^{\mathbf{a}}).$$

It is shown in [17] (see also [34]) that the map

$$\bigoplus_{\mathbf{a}} \bigotimes_{i} S^{a_i} H^0(G/Q, L_i) \to R$$

is surjective, and its kernel I is a multigraded ideal, generated by $\bigcup_{|\mathbf{a}|=2} I_{\mathbf{a}}$ (here, for $\mathbf{a} = (a_1, \ldots, a_r), |\mathbf{a}| = \sum a_i$). We then use the explicit nature of the straightening relations (cf. [26], [15]) to construct a flat family whose general fiber is R, and whose special fiber is $\mathcal{R}_{\mathcal{L}}$, the algebra associated to a finite distributive lattice $\mathcal{L}$ (namely $k[\mathcal{L}]/I(\mathcal{L})$, where $I(\mathcal{L})$ is the ideal generated by all binomials of the form $xy - (x \wedge y)(x \vee y)$, with $x, y \in \mathcal{L}$ non-comparable).

In [11], it is shown that $\mathcal{R}_{\mathcal{L}}$ (where $\mathcal{L}$ is a finite distributive lattice) is a normal domain; on the other hand, one knows (see [8] for example) that a prime binomial ideal is toric (here, by a binomial, we mean a polynomial with at most two terms). This then gives the required degeneration. We also give (cf. Section 4) a short and direct proof of the result that $I(\mathcal{L})$ (where $\mathcal{L}$ is a finite distributive lattice) is a toric ideal. This is proved by showing that the ideal associated to a chain product lattice is toric.

Our results extend to Schubert varieties in a minuscule G/P, and also to the class of Kempf varieties in $SL(n)/B$. As a consequence of our results for Schubert varieties in a minuscule G/P, we obtain the degeneration of the determinantal variety D_n (the subvariety in $\mathcal{M}_{r\times s}$ - the space of all $r \times s$ matrices with entries in k for some $r, s > n$, defined by the vanishing of all $(n+1) \times (n+1)$ minors), degenerates to a toric variety.

We also construct reduced Gröbner bases for Schubert varieties in $SL(n)/Q$, and these Gröbner bases descend to Gröbner bases for the corresponding toric varieties. Here, it should be pointed out (cf. [7]) that a Gröbner basis for a variety X (rather for the ideal defining X) determines a flat family whose general fiber is X, and the special fiber is a monomial scheme, i.e. a scheme defined by monomials. The analugous concept (in Computational Algebra) for degenerating a variety X to a toric variety is the SAGBI bases (cf. [35], [38]). In fact, using SAGBI theory, Sturmfels (cf. [38]) has proved the degeneration of the Grassmannian to a toric variety, and Conca-Herzog-Valla (cf. [4]) have proved the degeneration of certain normal scrolls to toric varieties (as a particular case of this, they obtain a degeneration of the determinantal variety D_1 in $\mathcal{M}_{2,c}$ to a toric variety). For $k = \mathbb{C}$, the degeneration of the Bott-Samelson scheme of G/B (for any semisimple G) into a toric variety is proved in [10] (refer to [6] for the definition of the Bott-Samelson scheme of G/B).

The sections are organized as follows.

Sections 1, 2, 3 deal with generalities on Gröbner bases, Distributive lattices, Toric ideals respectively. In Section 4, we give a direct and short proof of the result that $I(\mathcal{L})$, $\mathcal{L}$ being a finite distributive lattice is a toric ideal. In Section 5, we carry out the details on degenerations to toric varieties. Section 6 is on generalities on G/Q. In Section 7, we carry out the degenerations of a minuscule G/P and its Schubert varieties to toric varieties. Section 8 is on generalities on the flag variety $SL(n)/B$. In Section 9, we construct Gröbner bases for unions of Schubert varieties in $SL(n)/B$, and as an application, we construct Gröbner basis for a variety of complexes. In Section 10, we carry out the degeneration of $SL(n)/B$ to a toric variety. In Section 11, we carry out the degenerations of Kempf varieties in $SL(n)/B$ to toric varieties. In Section 12, we carry out the degenerations of the determinantal varieties to toric varieties.

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1. Generalities on Gröbner bases

Let k be a field, and consider the ring $k[x_1, \dots, x_n]$ of polynomials in n variables $x_1, \dots, x_n$. We recall below some generalities on Gröbner bases; for more details one may refer to [5], [7].

Definition 1.1. A total order $\preceq$ on the set of monomials in $k[x_1, \dots, x_n]$ is called a *monomial order* if for given monomials $\underline{m}$, $\underline{m}_1$, $\underline{m}_2$, with $\underline{m} \neq 1$, $\underline{m}_1 \prec \underline{m}_2$ implies $\underline{m}_1 \prec \underline{m} \cdot \underline{m}_1 \prec \underline{m} \cdot \underline{m}_2$.

For the rest of this Section, a fixed monomial order $\preceq$ is considered.

If f is a nonzero polynomial in $k[x_1, \dots, x_n]$, then the greatest monomial (with respect to $\preceq$) occuring in f is called the *initial monomial of* f, and we denote it by $in(f)$; the coefficient of $in(f)$ is called the initial coefficient of f. For a family of polynomials $\mathcal{F} \subset k[x_1, \dots, x_n]$, the ideal generated by its elements will be denoted by $\langle \mathcal{F} \rangle$, and the set of the initial monomials of all polynomials in $\mathcal{F}$ will be denoted by $in(\mathcal{F})$.

Definition 1.2. Let $I \subset k[x_1, \dots, x_n]$ be an ideal. A finite set of polynomials $\mathcal{F} \subset I$ is called a *Gröbner basis for I with respect to* $\preceq$ if $\langle in(\mathcal{F}) \rangle = \langle in(I) \rangle$.

Definition 1.3. A *minimal Gröbner basis for I with respect to* $\preceq$ is a Gröbner basis $\mathcal{F}$ for I with respect to $\preceq$ such that the initial coefficients of the elements in $\mathcal{F}$ are all 1, and for any $f \in \mathcal{F}$, $in(f) \notin \langle in(\mathcal{F} \setminus \{f\}) \rangle$.

Definition 1.4. A *reduced Gröbner basis for I with respect to* $\preceq$ is a Gröbner basis $\mathcal{F}$ for I with respect to $\preceq$ such that the initial coefficients of the elements in $\mathcal{F}$ are all 1, and for any $f \in \mathcal{F}$, none of the monomials present in f lies in $\langle in(\mathcal{F} \setminus \{f\}) \rangle$.

Proposition 1.5. *Any Gröbner basis for I generates I as an ideal.*

In the case when I is the defining ideal of an algebraic variety X, a Gröbner basis for I will be also called a *Gröbner basis for X*.

Proposition 1.6. *A nonzero ideal $I \subset k[x_1, \ldots, x_n]$ has a unique reduced Gröbner basis (with respect to a given monomial order).*

1.7. Lexicographic order. Assume that the variables $x_1, \ldots, x_n$ are toatally ordered as follows: $x_1 < \cdots < x_n$. A monomial $\underline{m}$ of degree r in the polynomial ring $k[x_1, \ldots, x_n]$ will be written in the form $\underline{m} = x_{i_1} \ldots x_{i_r}$, with $1 \leq i_1 \leq \cdots \leq i_r \leq n$. The *lexicographic order* on the set of monomials $\underline{m} \in k[x_1, \ldots, x_n]$ is denoted by $\preceq_{lex}$, and defined as follows: $x_{i_1} \ldots x_{i_r} \prec_{lex} x_{j_1} \ldots x_{j_s}$ if and only if either $r < s$, or $r = s$ and there exists an $l < r$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} < j_{l+1}$. It is easy to check that $\preceq_{lex}$ is a monomial oder.

2. Generalities on distributive lattices

Definition 2.1. A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y, defined by:

$$x \vee y \geq x,\ x \vee y \geq y,\ \text{and if } z \geq x \text{ and } z \geq y,\ \text{then } z \geq x \vee y,$$
$$x \wedge y \leq x,\ x \wedge y \leq y,\ \text{and if } z \leq x \text{ and } z \leq y,\ \text{then } z \leq x \wedge y.$$

It is easy to check that the operations $\vee$ and $\wedge$ are commutative and associative.

Definition 2.2. An element $z \in \mathcal{L}$ is called the *zero* of $\mathcal{L}$, denoted by 0, if $z \leq x$ for all x in $\mathcal{L}$. An element $z \in \mathcal{L}$ is called the *one* of $\mathcal{L}$, denoted by 1, if $z \geq x$ for all x in $\mathcal{L}$.

Definition 2.3. Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$.

Definition 2.4. Two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isomorphic* if there exists a bijection $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ such that, for all $x, y \in \mathcal{L}_1$,

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y) \text{ and } \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

Definition 2.5. A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \vee z) \tag{1}$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \wedge z). \tag{2}$$

2.6. An important example. Given an integer $n \geq 1$, $\mathcal{C}(n)$ will denote the chain $\{1 < \cdots < n\}$, and for $n_1, \ldots, n_d > 1$, $\mathcal{C}(n_1, \ldots, n_d)$ will denote the chain product lattice $\mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_d)$ consisting of all d-tuples $(i_1, \ldots, i_d)$, with $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d$. For $(i_1, \ldots, i_d)$, $(j_1, \ldots, j_d)$ in $\mathcal{C}(n_1, \ldots, n_d)$, we define

$$(i_1, \ldots, i_d) \leq (j_1, \ldots, j_d) \iff i_1 \leq j_1, \ldots, i_d \leq j_d \,.$$

We have

$$(i_1, \ldots, i_d) \vee (j_1, \ldots, j_d) = (\max\{i_1, j_1\}, \ldots, \max\{i_d, j_d\})$$
$$(i_1, \ldots, i_d) \wedge (j_1, \ldots, j_d) = (\min\{i_1, j_1\}, \ldots, \min\{i_d, j_d\}).$$

$\mathcal{C}(n_1, \ldots, n_d)$ is a finite distributive lattice, and its zero and one are $(1, \ldots, 1)$, $(n_1, \ldots, n_d)$ respectively.

Note that there is a total order $\triangleleft$ on $\mathcal{C}(n_1, \ldots, n_d)$ extending $<$, namely the lexicographic order, defined by $(i_1, \ldots, i_d) \triangleleft (j_1, \ldots, j_d)$ if and only if there exists $l < d$ such that $i_1 = j_1, \ldots, i_l = j_l, i_{l+1} < j_{l+1}$. Also note that two elements $(i_1, \ldots, i_d) \triangleleft (j_1, \ldots, j_d)$ are non-comparable with respect to $\leq$ if and only if there exists $1 < h \leq d$ such that $i_h > j_h$.

Sometimes we denote the elements of $\mathcal{C}(n_1, n_2, \ldots, n_d)$ by $x_{i_1 \ldots i_d}$, with $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d$.

2.7. The lattice of all subsets of the set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{B}(n)$, and called the *Boolean algebra of rank n*. Note that $\mathcal{B}(n)$ is isomorphic to $[\mathcal{C}(2)]^n$.

One has the following (see [1]):

Theorem 2.8. *Any finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of finite rank, and hence, in particular, to a sublattice of a finite chain product.*

3. Generalities on toric varieties

3.1. Let $T = (k^*)^m$ be the m-dimensional torus. Let M be the character group $(=\mathrm{Hom}_{alg.gp.}(T, \mathbb{G}_m))$ of T. Then M can be identified with $\mathbb{Z}^m$. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a subset of $\mathbb{Z}^m$. Consider the map

$$\pi_{\mathcal{A}} : \mathbb{Z}_+^n \to \mathbb{Z}^m, \quad \mathbf{u} = (u_1, \ldots, u_n) \mapsto u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n.$$

Let $k[\mathbf{x}] = k[x_1, \ldots, x_n]$, $k[\mathbf{t}^{\pm 1}] = k[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$.

The map $\pi_{\mathcal{A}}$ induces a homomorphism of semigroup algebras

$$\hat{\pi_{\mathcal{A}}} : k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}], \qquad x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

Definition 3.2. The kernel of $\hat{\pi}$ is denoted by $I_{\mathcal{A}}$ and called the *toric ideal* associated to $\mathcal{A}$. A variety of the form $\mathcal{V}(I_{\mathcal{A}})$, the affine variety of the zeroes in k^n of $I_{\mathcal{A}}$, is called an *affine toric variety.*

Note that a toric ideal is prime.

Remark 3.3. Consider the action of T on k^n given by $\mathbf{t}e_i = \mathbf{t}^{\mathbf{a}_i}e_i$ (here, $e_i,\ 1 \leq i \leq n$ are the standard basis vectors of k^n). Then $\mathcal{V}(I_{\mathcal{A}})$ is simply the Zariski closure of the T-orbit through $(1, 1, \ldots, 1)$.

Remark 3.4. In the above definition, we do not require $\mathcal{V}(I_{\mathcal{A}})$ to be normal. Using [16], we have that $\mathcal{V}(I_{\mathcal{A}})$ is normal if and only if the semi subgroup S of M generated by $\mathcal{A}$ is saturated (here, S is said to be saturated, if for $\mathbf{a} \in M$, $r\mathbf{a} \in S \implies \mathbf{a} \in S, \forall r \in \mathbb{Z}_+$ (cf. [16])).

Recall the following (see [38]).

Proposition 3.5. *The toric ideal $I_{\mathcal{A}}$ is spanned as a k-vector space by the set of binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^n \text{ with } \pi_{\mathcal{A}}(\mathbf{u}) = \pi_{\mathcal{A}}(\mathbf{v})\}. \tag{3}$$

(Here, a *binomial* is a polynomial with at most two terms.)

3.6. An example. Let us fix the integers $n_1, \ldots, n_d > 1$, and let $n = \Pi_{i=1}^d n_i$, $m = \sum_{i=1}^d n_i$. Let $\mathbf{e}_1^l, \ldots, \mathbf{e}_{n_l}^l$ be the unit vectors in $\mathbb{Z}^{n_l}$, for $1 \leq l \leq d$. For $1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d$, define

$$\mathbf{a}_{\xi_1 \ldots \xi_d} = \mathbf{e}_{\xi_1}^1 + \cdots + \mathbf{e}_{\xi_d}^d \in \mathbb{Z}^{n_1} \oplus \cdots \oplus \mathbb{Z}^{n_d}$$

and let

$$\mathcal{A}_{n_1, \ldots, n_d} = \{\mathbf{a}_{\xi_1 \ldots \xi_d} \mid 1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d\}.$$

The corresponding map

$$\pi_{\mathcal{A}} : \mathbb{Z}_+^{n_1 \cdot \cdots \cdot n_d} \to \mathbb{Z}^{n_1 + \cdots + n_d}$$

is defined as follows: for $1 \leq l \leq d$ and $1 \leq i_l \leq n_l$ fixed, the $(n_1 + \cdots + n_{l-1} + i_l)$-th coordinate of $\pi_{\mathcal{A}}(\mathbf{u})$ is given by $\sum u_{\xi_1 \ldots \xi_{l-1} \xi_l \xi_{l+1} \ldots \xi_d}$, the sum being taken over the elements $(\xi_1, \ldots, \xi_{l-1}, \xi_l, \xi_{l+1}, \ldots, \xi_d)$ of $\mathcal{C}(n_1, \ldots, n_d)$ with $\xi_l = i_l$. We call this subset the l-th *slice* of $\mathcal{C}(n_1, \ldots, n_d)$ defined by i_l, and denote it by $\{\xi_l = i_l\}$. The components (or *entries*) of an element $\mathbf{u} \in \mathbb{Z}^{n_1 \cdot \cdots \cdot n_d}$ are indexed by the elements $(i_1, \ldots, i_d)$ of $\mathcal{C}(n_1, \ldots, n_d)$. If $(j_1, \ldots, j_d) \in \{\xi_l = i_l\}$, sometimes we also say that $u_{j_1 \ldots j_d}$ itself belongs to the slice $\{\xi_l = i_l\}$.

The map $\pi_{\mathcal{A}}$ induces the map

$$\hat{\pi_{\mathcal{A}}} : k[x_{11 \ldots 1}, \ldots, x_{\xi_1 \xi_2 \ldots \xi_d}, \ldots, x_{n_1 n_2 \ldots n_d}] \to k[t_{11}, \ldots, t_{1n_1}, \ldots, t_{d1}, \ldots, t_{dn_d}]$$

given by

$$x_{\xi_1 \ldots \xi_d} \mapsto t_{1\xi_1} \ldots t_{d\xi_d}, \text{ for } 1 \leq \xi_1 \leq n_1, \ldots, 1 \leq \xi_d \leq n_d.$$

4. The algebra associated to a distributive lattice

Definition 4.1. Given a finite lattice $\mathcal{L}$, the *ideal associated to* $\mathcal{L}$, denoted by $I(\mathcal{L})$, is the ideal of the polynomial ring $k[\mathcal{L}]$ generated by the set of binomials

$$\mathcal{G}_{\mathcal{L}} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{L} \text{ non-comparable}\}.$$

By Theorem 2.8, a finite distributive lattice $\mathcal{L}$ may be identified with a sublattice of a finite chain product lattice . Hence it inherits a total order extending the given partial order. In turn, this total order induces the lexicographic order on the monomials in $k[\mathcal{L}]$, as in 1.7.

The following theorem shows that the ideal associated to a chain product lattice is toric.

Theorem 4.2. 1) *We have* $I(\mathcal{C}(n_1, \dots, n_d)) = I_{\mathcal{A}_{n_1,\dots,n_d}}$.
2) *The set of binomials*

$$\mathcal{G} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{C}(n_1, \dots, n_d) \textit{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{C}(n_1, \dots, n_d))$ *with respect to the lexicographic order.*

Proof. Let $\mathcal{C} = \mathcal{C}(n_1, \dots, n_d)$ and $\mathcal{A} = \mathcal{A}_{n_1,\dots,n_d}$. Let $f \in I_{\mathcal{A}}$; by Proposition 3.5, there exist $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{Z}_+^n$ with $\pi_{\mathcal{A}}(\mathbf{u}_i) = \pi_{\mathcal{A}}(\mathbf{v}_i)$, and $c_i \in k^*$, $1 \leq i \leq s$ such that

$$f = \sum_{i=1}^{s} c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) \tag{4}$$

for some $s \geq 1$, with the property that s is the smallest integer ≥ 1 such that f can be expressed as a linear combination of s binomials in the set (3). Now we rewrite f as

$$f = \sum_{i=1}^{s} a_i \mathbf{x}^{\mathbf{u}_i} + \sum_{i=1}^{s} b_i \mathbf{x}^{\mathbf{v}_i}, \quad a_i, b_i \in k.$$

Then none of the coefficients $a_1, \dots, a_s, b_1, \dots, b_s$ is zero. Indeed, suppose that $a_i = 0$ for some $1 \leq i \leq s$. This implies that there exists $j \in \{1, \dots, s\}$, $j \neq i$ such that either $c_j = c_i$ and $\mathbf{v}_j = \mathbf{u}_i$, or $c_j = -c_i$ and $\mathbf{u}_j = \mathbf{u}_i$. In the first case we have

$$c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + c_j(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = c_i(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi_{\dashv}(\mathbf{u}_j) = \pi_{\mathcal{A}}(\mathbf{v}_i). \tag{5}$$

In the second case we have

$$c_i(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) + c_j(\mathbf{x}^{\mathbf{u}_j} - \mathbf{x}^{\mathbf{v}_j}) = c_i(\mathbf{x}^{\mathbf{v}_j} - \mathbf{x}^{\mathbf{v}_i}), \quad \pi_{\mathcal{A}}(\mathbf{v}_j) = \pi_{\mathcal{A}}(\mathbf{v}_i). \tag{6}$$

But (4), (5) and (6) imply that f can be written as a linear combination of $s-1$ binomials in the set (3), contradicting the choice of s. Thus $a_i \neq 0$, $1 \leq i \leq s$, and similarly $b_i \neq 0$, $1 \leq i \leq s$. This shows that $in(f) = in(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i})$ for some $1 \leq i \leq s$. Let us write

$$\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i} = \mathbf{x}^{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}),$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_+^n$, with $\pi_{\mathcal{A}}(\mathbf{u}) = \pi_{\mathcal{A}}(\mathbf{v})$ and $supp(\mathbf{u}) \cap supp(\mathbf{v}) = \emptyset$. Let us suppose that $\mathbf{x}^{\mathbf{u}} \succeq_{lex} \mathbf{x}^{\mathbf{v}}$, i.e. $in(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) = \mathbf{x}^{\mathbf{u}}$ and $in(f) = in(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}) = \mathbf{x}^{\mathbf{u}_i}$. Let $x_{i_1 \ldots i_d}$ be the smallest variable appearing in $\mathbf{x}^{\mathbf{u}}$, i.e. $(i_1, \ldots, i_d)$ is the smallest element of $supp(\mathbf{u})$ with respect to $\lhd$. Then $\mathbf{x}^{\mathbf{v}}$ contains a variable $x_{k_1 \ldots k_d}$, with $(k_1, \ldots, k_d) \lhd (i_1, \ldots, i_d)$. Since $\pi_{\mathcal{A}}(\mathbf{u}) = \pi_{\mathcal{A}}(\mathbf{v})$, the sum of the entries in every slice is the same for both $\mathbf{u}$ and $\mathbf{v}$. In particular, since all the entries of $\mathbf{u}$ in the slices $\{\xi_1 = i\}$, with $1 \leq i < i_1$, are 0 (by the choice of $(i_1, \ldots, i_d)$), all the entries of $\mathbf{v}$ in these slices must also be 0. This implies that $(k_1, \ldots, k_d) \in \{\xi_1 = i_1\}$. Let $1 < h \leq n_1$ such that $k_1 = i_1, \ldots, k_{h-1} = i_{h-1}, k_h < i_h$. Then the sum of the elements of $\mathbf{v}$ in the slice $\{\xi_h = k_h\}$ is nonzero, which implies that $\{\xi_h = k_h\} \cap supp(\mathbf{u}) \neq \emptyset$. Let $(j_1, \ldots, j_d)$ be an element in this intersection. We have $(i_1, \ldots i_d) \lhd (j_1, \ldots, j_d)$ (by the definition of $(i_1, \ldots i_d)$), and since $i_h > k_h = j_h$, we conclude that $(i_1, i_2, \ldots, i_d)$ and $(j_1, \ldots, j_d)$ are non-comparable. Thus we obtain that $\mathbf{x}^{\mathbf{u}}$ is divisible by $x_{i_1 \ldots i_d} x_{j_1 \ldots j_d}$. Hence $\mathbf{x}^{\mathbf{u}_i}$ is also divisible by $x_{i_1 \ldots i_d} x_{j_1 \ldots j_d}$. Therefore $in(f)$ is divisible by the initial term of an element of the set

$$\mathcal{G} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{C} \text{ non-comparable}\}$$

of generators of the ideal $I(\mathcal{C})$. Since $\mathcal{G} \subset I_{\mathcal{A}}$, it follows that $\mathcal{G}$ is a Gröbner basis for $I_{\mathcal{A}}$. In particular it is a set of generators for this ideal. Thus $\mathcal{G}$ generates both $I(\mathcal{C})$ and $I_{\mathcal{A}}$, which implies the equality of the two ideals. □

Theorem 4.3. *Let $\mathcal{L}$ be a finite distributive lattice . Then*
1) *The ideal $I(\mathcal{L})$ is toric.*
2) *The set of binomials*

$$\mathcal{G}_{\mathcal{L}} = \{xy - (x \wedge y)(x \vee y) \mid x, y \in \mathcal{L} \textit{ non-comparable}\}$$

is a Gröbner basis for $I(\mathcal{L})$ with respect to the lexicographic order.

Proof. By Theorem 2.8, we may assume that $\mathcal{L}$ is a sublattice of $\mathcal{C}(n_1, \ldots, n_d)$, for some $n_1, \ldots, n_d \geq 1$. Let us denote $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$, $\mathcal{A} = \mathcal{A}_{n_1, \ldots, n_d}$ and $\mathcal{G} = \mathcal{G}_{\mathcal{C}}$. Note that $\mathcal{G}_{\mathcal{L}}$ is the subset of $\mathcal{G}$ consisting of all binomials in $\mathcal{G}$ involving only the variables from $\mathcal{L}$. Let us denote

$$\mathcal{G}_{\mathcal{L}} = \{f_1, \ldots, f_r\}, \quad \mathcal{G} \setminus \mathcal{G}_{\mathcal{L}} = \{g_1, \ldots, g_s\}$$

Let $g_i = xy - (x \wedge y)(x \vee y)$, with $x, y \in \mathcal{C}$ non-comparable, $1 \leq i \leq s$; then at least one of x and y does not belong to $\mathcal{L}$ ($\mathcal{L}$ being a sublattice of $\mathcal{C}$, $x, y \in \mathcal{L}$ would imply $x \wedge y, x \vee y \in \mathcal{L}$, so g_i would involve only variables from $\mathcal{L}$).

Let $\mathcal{A}_{\mathcal{L}} \subset \mathcal{A}$ be given by the elements in $\mathcal{A}$ indexed by the elements of $\mathcal{L}$, and let f be an element of

$$I_{\mathcal{A}_{\mathcal{L}}} = \ker\left(\hat{\pi_{\mathcal{A}}}\Big|_{k[\mathcal{L}]}\right) = (\ker \hat{\pi_{\mathcal{A}}}) \cap k[\mathcal{L}] = I_{\mathcal{A}} \cap k[\mathcal{L}].$$

In the course of the proof of Theorem 4.2, we saw that $in(f)$ is divisible by the initial term of a binomial in $\mathcal{G}$, and since $f \in k[\mathcal{L}]$, this binomial must be one of the f_i's, i.e. an element of $\mathcal{G}_{\mathcal{L}}$. Since $\mathcal{G}_{\mathcal{L}} \subset I_{\mathcal{A}_{\mathcal{L}}}$, it follows that $\mathcal{G}_{\mathcal{L}}$ is a Gröbner basis for $I_{\mathcal{A}_{\mathcal{L}}}$, hence $\mathcal{G}_{\mathcal{L}}$ generates $I_{\mathcal{A}_{\mathcal{L}}}$. Therefore $I(\mathcal{L}) = I_{\mathcal{A}_{\mathcal{L}}}$, and the stated assertions follow now. □

Remark 4.4. (1) In [11], it is shown that $\mathcal{R}_{\mathcal{L}} = k[\mathcal{L}]/I(\mathcal{L})$ ($\mathcal{L}$ being a finite distributive lattice) is a normal domain; on the other hand, one knows (see [8] for example) that prime binomial ideal is toric (here, by a binomial, we mean a polynomial with at most two terms). Thus the result that $I(\mathcal{L})$ is toric ($\mathcal{L}$ being a finite distributive lattice) may also be concluded using [11] and [8].

(2) In fact, given a finite lattice $\mathcal{L}$, $I(\mathcal{L})$ is toric if and only if $\mathcal{L}$ is distributive.

5. Lattices and flat deformations

Let H be a finite lattice. Let R be a k-algebra with generators $\{p_\alpha \mid \alpha \in H\}$.

Definition 5.1. A monomial $p_{\alpha_1} \dots p_{\alpha_r}$ is said to be standard if $\alpha_1 \geq \cdots \geq \alpha_r$.

Suppose that the standard monomials form a k-basis for R. Given any nonstandard monomial $\underline{n}$, the expression

$$\underline{n} = \sum c_i \underline{s}_i, \qquad c_i \in k^* \tag{7}$$

for $\underline{n}$ as a sum of standard monomials will be referred as a *straightening relation*. Let $P = k[x_\alpha, \alpha \in H]$, and consider the surjective map

$$\pi : P \to R, \qquad x_\alpha \mapsto p_\alpha.$$

Let us denote $\ker \pi$ by I.

For $\alpha, \beta \in H$ with $\alpha > \beta$, we set

$$]\beta, \alpha[= \{\gamma \in H \mid \alpha > \gamma > \beta\}.$$

Theorem 5.2. *Let $S = P/I(H)$, where $I(H)$ is as in Definition 4.1. Suppose that I is generated as an ideal by elements of the form $x_\tau x_\phi - \sum c_{\alpha\beta} x_\alpha x_\beta$ (where τ, ϕ are non-comparable, and $\alpha \geq \beta$). Further suppose that in the straightening relation*

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta, \tag{8}$$

we have

1) *$p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs on the right-hand side of (8) with coefficient 1.*
2) *$\tau, \phi \in]\beta, \alpha[$, for every pair (α, β) appearing on the right-hand side of (8).*

3) *There exists an embedding* $H \hookrightarrow \mathcal{C}$, *where* $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_d)$ *for some* $n_1, \ldots, n_d \geq 1$, *such that* $\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta$, *for every* (α, β) *on the right-hand side of (8).*

Then there exists a flat deformation whose special fiber is S *and general fiber is* R.

Proof. Let $\theta \in H$. Let $\theta = (a_1, \ldots, a_d) \in \mathcal{C}$ under the identification of H with a sublattice of $\mathcal{C}$, given by Theorem 2.8. We fix an integer $N \gg 0$, and let

$$N_\theta = \sum_{r=1}^{d} N^{d-r} a_r$$

be the integer $a_1 \ldots a_d$ in the N-ary representation.

For τ, ϕ non-comparable, let

$$f_{\tau,\phi} = x_\tau x_\phi - \sum c_{\alpha\beta} x_\alpha x_\beta.$$

Then hypothesis implies that I is generated by $\{f_{\tau,\phi} \mid (\tau, \phi) \in Q\}$, where $Q = \{(\tau, \phi) \mid \tau, \phi \in H \text{ non-comparable}\}$.

Let $A = k[t]$, and $P_A = A[x_\alpha, \alpha \in H]$. For $(\tau, \phi) \in Q$, we define the element $f_{\tau,\phi,t}$ in P_A as

$$f_{\tau,\phi,t} = x_\tau x_\phi - \sum c_{\alpha\beta} x_\alpha x_\beta t^{N_\alpha + N_\beta - N_\tau - N_\phi}.$$

Note that $N_{\tau\vee\phi} + N_{\tau\wedge\phi} = N_\tau + N_\phi$; for, if $\tau = (i_1, \ldots, i_d)$, $\phi = (j_1, \ldots, j_d)$, we have $\tau \vee \phi = (k_1, \ldots, k_d)$, $\tau \wedge \phi = (l_1, \ldots, l_d)$, with $k_r = \max\{i_r, j_r\}$, $l_r = \min\{i_r, j_r\}$ for $1 \leq r \leq d$.

Also note that for any other (α, β) on the right-hand side of (8), by hypothesis $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathcal{C}$ and $\alpha > \tau \vee \phi$, and $\beta < \tau \vee \phi$. Let s be the smallest integer $\leq d$ such that $\alpha_s > k_s$. Then the hypotheses that $\beta < \tau \vee \phi$, and $\alpha \dot{\cup} \beta = \tau \dot{\cup} \phi$ imply that $\beta_r = l_r$, $1 \leq r \leq s$. Hence $\alpha_s + \beta_s > k_s + l_s = i_s + j_s$. Thus for $(\alpha, \beta) \neq (\tau \vee \phi, \tau \wedge \phi)$, we have $N_\alpha + N_\beta > N_\tau + N_\phi$.

Let $\mathcal{I}$ be the ideal in P_A generated by $\{f_{\tau,\phi,t} \mid (\tau, \phi) \in Q\}$, and $\mathcal{R} = P_A/\mathcal{I}$.

Claim. (*a*) $\mathcal{R}$ *is* $k[t]$*-free.*

(*b*) $\mathcal{R} \otimes_{k[t]} k[t, t^{-1}] \simeq R[t, t^{-1}]$.

(*c*) $\mathcal{R} \otimes_{k[t]} k[t]/(t) \simeq S$.

Proof. We have

$$\mathcal{R} \otimes_{k[t]} k[t]/(t) = P_A/\langle \mathcal{I} + (t) \rangle = S$$

This proves (*c*). Let $B = k[t, t^{-1}]$, and $P_B = B[x_\alpha, \alpha \in H]$. Let $\tilde{I}$(resp. $\tilde{\mathcal{I}}$) be the ideal in P_B generated by $\{f_{\tau,\phi} \mid (\tau, \phi) \in Q\}$(resp. $\{f_{\tau,\phi,t} \mid (\tau, \phi) \in Q\}$. We have

$$P_B/\tilde{I} \simeq R[t, t^{-1}] \tag{9}$$

$$P_B/\tilde{\mathcal{I}} \simeq \mathcal{R} \otimes_{k[t]} k[t, t^{-1}] \tag{10}$$

The automorphism

$$P_B \simeq P_B, \qquad x_\alpha \mapsto t^{N_\alpha} x_\alpha$$

induces an isomorphism

$$P_B/\tilde{I} \simeq P_B/\tilde{\mathcal{I}} \tag{11}$$

From (9), (10), (11) we obtain (b). Finally, it remains to show (a). Let $X_\alpha = \bar{x}_\alpha$ (in $\mathcal{R} = P_A/\mathcal{I}$), $P_\alpha = t^{N_\alpha} X_\alpha$ and

$$\mathcal{M} = \{P_{\alpha_1} \dots P_{\alpha_r} \mid \alpha_1 \geq \cdots \geq \alpha_r,\ r \in \mathbb{Z}_+\}.$$

We shall now show that $\mathcal{M}$ is a $k[t]$-basis for $\mathcal{R}$.

First we prove the linear independence. By base change, $\{P_{\alpha_1} \dots P_{\alpha_r} \mid \alpha_1 \geq \cdots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$-basis for $R[t, t^{-1}]$. Denoting the isomorphism $P_B/\tilde{I} \simeq R[t, t^{-1}]$ by φ, we have $\{\varphi^{-1}(p_{\alpha_1} \dots p_{\alpha_r}) \mid \alpha_1 \geq \cdots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$-basis for $\mathcal{R}[t^{-1}]$. For a monomial $\underline{m} = p_{\tau_1} \dots p_{\tau_r}$ (in $R[t, t^{-1}]$), we have $\varphi^{-1}(\underline{m}) = t^{-N_{\underline{m}}} X_{\tau_1} \dots X_{\tau_r}$ (where $N_{\underline{m}} = \sum_{i=1}^r N_{\tau_i}$)$= uX_{\tau_1} \dots X_{\tau_r}$, where $u = t^{-N_{\underline{m}}}$ is a unit in $k[t, t^{-1}]$. Thus we obtain $\{X_{\alpha_1} \dots X_{\alpha_r} \mid \alpha_1 \geq \cdots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$-basis for $\mathcal{R}[t^{-1}]$. Hence $\{P_{\alpha_1} \dots P_{\alpha_r} \mid \alpha_1 \geq \cdots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is a $k[t, t^{-1}]$-basis for $\mathcal{R}[t^{-1}]$. In particular, we obtain that $\{P_{\alpha_1} \dots P_{\alpha_r} \mid \alpha_1 \geq \cdots \geq \alpha_r, r \in \mathbb{Z}_+\}$ is linearly independent over $k[t, t^{-1}]$, and hence over $k[t]$.

Next we prove the generation. Let $F = X_{\tau_1} \dots X_{\tau_r}$ be any monomial in $\mathcal{R}$, where $\tau_r \trianglelefteq \dots \trianglelefteq \tau_1$. Further, we suppose that there exists an i such that $\tau_i \not\geq \tau_{i+1}$. Let us denote τ_i, τ_{i+1} by τ, ϕ respectively. Then using the relation

$$X_\tau X_\phi = \sum c_{\alpha\beta} X_\alpha X_\beta t^{N_\alpha + N_\beta - N_\tau - N_\phi} \quad ,$$

we obtain

$$F = \sum a_i F_i \quad ,$$

where $F_i \preceq_{lex} F$ for each i. Hence, by induction, we obtain that F_i is a $k[t]$-linear combination of elements of $\mathcal{M}$. Now (a) follows. This completes the proof of the claim. □

Now claim implies that $\mathcal{R}$ is a flat family over $k[t]$, whose fiber over 0 is S and whose fiber over any $t - u$, $u \in k^*$ is R. □

In view of the above Theorem, Theorem 4.3, and Remark 4.4, we obtain the following

Theorem 5.3. *With hypotheses as in Theorem* 5.2*, assume that H is a distributive lattice. Then there exists a flat deformation whose general fiber is R and special fiber is a normal toric algebra (by a toric algebra we mean a quotient of a polynomial algebra by a toric ideal).*

6. Generalites on G/Q

Let $G_{\mathbb{Z}}$ be a semisimple, simply connected Chevalley group scheme over $\mathbb{Z}$ (see [37] for basic facts on Chevalley groups). We fix a maximal torus subgroup scheme $T_{\mathbb{Z}}$, and a Borel subgroup scheme $B_{\mathbb{Z}}$ containing $T_{\mathbb{Z}}$. We talk of roots, weights, etc., with respect to $T_{\mathbb{Z}}$ and $B_{\mathbb{Z}}$. We denote the root system by R, and the set of positive (resp. simple) roots by R^+ (resp. S). The Weyl group scheme $N(T_{\mathbb{Z}}/T_{\mathbb{Z}})$ (where $N(T_{\mathbb{Z}})$ is the normalizer of $T_{\mathbb{Z}}$) is a constant group scheme, and hence we talk about the Weyl group W of $G_{\mathbb{Z}}$.

Let $Q_{\mathbb{Z}}$ be a parabolic subgroup scheme of $G_{\mathbb{Z}}$. Let W_Q be the Weyl group of $Q_{\mathbb{Z}}$, and W^Q the set of minimal representatives in W of W/W_Q. For $\tau \in W^Q$, let $X_{\mathbb{Z}}(\tau) = \overline{B_{\mathbb{Z}}\tau Q_{\mathbb{Z}}}$ (mod $Q_{\mathbb{Z}}$) be the Schubert subscheme of $G_{\mathbb{Z}}/Q_{\mathbb{Z}}$ associated to τ.

For a field k, we denote the objects G_k, T_k, B_k, etc. obtained by the base change $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$ by just G, T, B, etc..

6.1. The $\mathbb{Z}$-module $V_{\mathbb{Z},w}$. Let δ be a dominant integral weight, and let V_δ be the irreducible G-module (over $\mathbb{Q}$) with highest weight δ. We fix a highest weight vector e in V_δ. Let Q be the stabilizer of e in G. Let $U_{\mathbb{Z}}$ be the Kostant $\mathbb{Z}$-form of U, the universal envelopping algebra of $\mathfrak{g} = \operatorname{Lie}(G)$, the Lie algebra of G. Let $U_{\mathbb{Z}}^{\pm}$ be the $\mathbb{Z}$-subalgebra of $U_{\mathbb{Z}}$ spanned by $X_\alpha^n/n!$ (resp. $X_{-\alpha}^n/n!$), $\alpha \in R^+$, $n \in \mathbb{Z}^+$ (here, for $\beta \in R$, X_β denotes the element of the Chevalley basis of $\mathfrak{g}$, associated to β). Let $V_{\mathbb{Z}} = U_{\mathbb{Z}}e$. For $w \in W^Q$, representing w by a $\mathbb{Z}$-valued point of $N(T_{\mathbb{Z}})$, the vector $w \cdot e$ is well-determined up to a factor ± 1. We set $V_{\mathbb{Z},w} = U_{\mathbb{Z}}^+ we$. Then it is well known that $V_{\mathbb{Z},w}^*$ (the $\mathbb{Z}$-dual of $V_{\mathbb{Z},w}$) is isomorphic to $H^o(X_{\mathbb{Z}}(w), L_{\mathbb{Z}}(\delta))$, where $L_{\mathbb{Z}}(\delta)$ is the line bundle on $G_{\mathbb{Z}}/Q_{\mathbb{Z}}$ associated to δ.

Next we recall some generalities on G/P (cf. [23]), where P is a maximal parabolic subgroup with associated fundamental weight ω.

Definition 6.2. Let w_1, $w_2 \in W^P$, and let $X(w_2)$ be a Schubert divisor in $X(w_1)$. Then $X(w_2)$ is called a *moving divisor in* $X(w_1)$, and is said to be *moved by* α, if $w_2 = s_\alpha w_1$, α a *simple* root.

Lemma 6.3. *Let α be a simple root. Let $\phi \in W^P$. Then*

(1) *$X(\phi)$ is a moving divisor in $X(s_\alpha\phi)$ moved by α if and only if $(\phi(\omega), \alpha^*) > 0$.*
(2) *$X(s_\alpha\phi)$ is a moving divisor in $X(\phi)$ moved by α if and only if $(\phi(\omega), \alpha^*) < 0$.*

Lemma 6.4. *Let $\tau, \phi \in W^P$. Let $X(\phi)$ be a moving divisor in $X(\tau)$ moved by a simple root α. Then for any Schubert subvariety $X(w)$ of $X(\tau)$ we have, either $X(w) \subset X(\phi)$, or there exists a moving divisor $X(w')$ in $X(w)$ moved by α such that $X(w') \subset X(\phi)$.*

7. Degenerations of Schubert varieties in minuscule G/P to toric varieties

Let G, T, B, W, etc. be as in Section 6. Let $X(T)$ be the character group of T. Let P be a maximal parabolic subgroup with associated fundamental weight ω. Let L be the ample generator of $\mathrm{Pic}\,(G/P)$. Throughout this Section we shall assume that ω is minuscule, i.e. $(\omega, \alpha^*) \leq 1$, for all $\alpha \in R^+$ (here $(\ ,\ )$ is a W-invariant inner product on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, and $(\omega, \alpha^*) = 2\frac{(\omega,\alpha)}{(\alpha,\alpha)}$).

Let V_ω be the irreducible G-module (over $\mathbb{Q}$) with highest weight ω. We fix a highest weight vector e in V_ω. For $w \in W^P$, we make a canonical choice of the extremal weight vector in V_ω of weight $w(\omega)$, as given by Proposition 7.2 below. We first recall the following

Lemma 7.1. (*cf.* [28]) (1) *Let $w \in W^P$. Let $X(w')$ be a divisor in $X(w)$, say, $w' = s_\alpha w$ for some $\alpha \in R^+$. Then α is simple.*

(2) *Let $X(w_i)$, $i = 1\,, 2$ be two divisors in $X(w)$. Let $w_1 = s_\beta w$, $w_2 = s_\gamma w$. Then s_β and s_γ commute.*

(3) *Let $w, \tau \in W^P$. Then $w \geq \tau \iff w(\omega) \leq \tau(\omega)$.*

Proposition 7.2. *Let $w \in W^P$, and $w = s_r \dots s_1$ be a reduced expression for w. Then the vector $X_{-\alpha_r} \dots X_{-\alpha_1} e$ is an extremal weight vector in V_ω of weight $w(\omega)$. Further, it depends only on w and not on the reduced expression chosen.*

Proof. (by induction on $\dim X(w)$)

If $\dim X(w) = 0$, then $w = \mathrm{id}$, and the assertion is clear. Let then $\dim X(w) \geq 1$. Let $s_{i_1} \dots s_{i_r}$, $s_{j_1} \dots s_{j_r}$ be two reduced expressions for w. Let $\phi = s_{i_2} \dots s_{i_r}$, $\tau = s_{j_2} \dots s_{j_r}$. For simplicity of notation, let us denote α_{i_1}, α_{j_1} by just α, β (note that α, $\beta \in S$). We have (cf. Lemma 7.1, (2)) $s_\alpha s_\beta = s_\beta s_\alpha$. Let $\theta = s_\alpha s_\beta w$. Then $\phi = s_\beta \theta$, $\tau = s_\alpha \theta$. Let $Q_\phi = X_{-\alpha_{i_2}} \dots X_{-\alpha_{i_r}} e$, $Q_\tau = X_{-\alpha_{j_2}} \dots X_{-\alpha_{j_r}} e$. Then we have, by induction hypothesis, that Q_ϕ and Q_τ extremal weight vectors of weight $\phi(\omega)$, $\tau(\omega)$ respectively; further, they are uniquely determined by ϕ and τ respectively. Now $Q_\phi = X_{-\beta} Q_\theta$, $Q_\tau = X_{-\alpha} Q_\theta$ (where Q_θ is the uniquely determined extremal weight vector in V_ω of weight $\theta(\omega)$, as guaranteed by induction hypothesis). Hence

$$X_{-\alpha} Q_\phi = X_{-\alpha} X_{-\beta} Q_\theta = X_{-\beta} X_{-\alpha} Q_\theta = X_{-\beta} Q_\tau.$$

(note that $X_{-\alpha} Q_\phi$ (resp. $X_{-\beta} Q_\tau$) is nonzero, since $(\phi(\omega), \alpha^*) > 0$ (resp. $(\tau(\omega), \beta^*) > 0$) (cf. Lemma 6.3). The required result now follows. □

Definition 7.3. For $w \in W^P$, we define Q_w to be uniquely determined extremal weight vector in V_ω of weight $w(\omega)$ as given by Proposition 7.2. We define $V_{\mathbb{Z},w} = U^+_{\mathbb{Z}} Q_w$.

Remark 7.4. We have $H^0(X_{\mathbb{Z}}(w), L_{\mathbb{Z}}) = V^*_{\mathbb{Z},w}$, the $\mathbb{Z}$-dual of $V_{\mathbb{Z},w}$.

Remark 7.5. The weight ω being minuscule, it is well known that $\{\mathbf{Q}_w \mid w \in W\}$ is a $\mathbb{Z}$-basis for $V_{\mathbb{Z}}(= U_{\mathbb{Z}}e)$.

Definition 7.6. We define $\{\mathbf{P}_w \mid w \in W^P\}$ as the $\mathbb{Z}$-basis of $H^0(G_{\mathbb{Z}}/P_{\mathbb{Z}}, L_{\mathbb{Z}})$ (= the $\mathbb{Z}$-dual of $V_{\mathbb{Z}}$) dual to $\{\mathbf{Q}_w \mid w \in W^P\}$.

In view of Definition 7.6 and Remark 7.4, we have

Theorem 7.7. *Let $\tau \in W^P$. Then $\{\mathbf{P}_w \mid \tau \geq w\}$ is a $\mathbb{Z}$-basis for $H^0(X_{\mathbb{Z}}(\tau), L_{\mathbb{Z}})$.*

Lemma 7.8. *Let $\lambda, \mu \in W^P$, where $\lambda > \mu$. Further, let $\lambda = s_r \dots s_1\mu$ (s_i being simple reflections), with $l(\lambda) = l(\mu) + r$. Then $\mathbf{P}_\mu = (-1)^r X_{-\alpha_1} \dots X_{-\alpha_r}\mathbf{P}_\lambda$.*

Proof. Let $\langle\, , \rangle$ denote the canonical pairing on $H^0(G_{\mathbb{Z}}/P_{\mathbb{Z}}, L_{\mathbb{Z}}) \times V_{\mathbb{Z}}$. We have, by $\mathfrak{g}$-invariance of $\langle\, , \rangle$

$$\begin{aligned}\langle X_{-\alpha_1} \dots X_{-\alpha_r}\mathbf{P}_\lambda\, ,\, \mathbf{Q}_\mu\rangle &= (-1)^r\langle \mathbf{P}_\lambda\, ,\, X_{-\alpha_r} \dots X_{-\alpha_1}\mathbf{Q}_\mu\rangle \\ &= (-1)^r\langle \mathbf{P}_\lambda\, ,\, \mathbf{Q}_\lambda\rangle = (-1)^r\end{aligned}$$

(note that $\mathbf{Q}_\lambda = X_{-\alpha_r} \dots X_{-\alpha_1}\mathbf{Q}_\mu$, cf. Definition 7.3). The result now follows from this. $\square$

For a field k, let us denote the canonical image of $\mathbf{P}_w$ in $H^0(G/P, L)$ by p_w, $w \in W^P$. We recall below some results from [28], [36].

Definition 7.9. A monomial $p_{\tau_1} \dots p_{\tau_r}$, $\tau_i \in W^P$, is standard on $X(\tau)$ if
$\tau \geq \tau_1 \geq \dots \geq \tau_r$.

Theorem 7.10. (*cf.* [36]) (1) *Let $\tau \in W^P$. Then $p_w\big|_{X(\tau)} \neq 0 \iff \tau \geq w$. Further, $\{p_w \mid \tau \geq w\}$ is a k-basis for $H^0(X(\tau), L)$.*

(2) *The standard monomials on $X(\tau)$ of degree r form a basis of $H^0(X(\tau), L^r)$.*

Lemma 7.11. (*cf.* [36]) *Let $p_\tau p_\phi$ be a nonstandard monomial on G/P, and let the corresponding straightening relation be given by*

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

Then $\alpha \geq \tau \vee \phi$, $\beta \leq \tau \wedge \phi$, for all α, β with $c_{\alpha\beta} \neq 0$.

We next recall (cf. [33], [12])

Proposition 7.12. *W^P is a distributive lattice.*

Lemma 7.13. *Let $\tau, \phi \in W^P$ be two non-comparable elements. Let $\lambda = \tau \vee \phi$, $\mu = \tau \wedge \phi$. Then $\tau(\omega) + \phi(\omega) = \lambda(\omega) + \mu(\omega)$.*

Proof. Let

$$\tau(\omega) = \omega - \sum_{i=1}^{n} a_i\alpha_i, \quad \phi(\omega) = \omega - \sum_{i=1}^{n} b_i\alpha_i,$$

where $n = \mathrm{rank}G$, α_i's being simple roots. Then it is easily seen that

$$\lambda(\omega) = \omega - \sum_{i=1}^{n} k_i\alpha_i, \quad \mu(\omega) = \omega - \sum_{i=1}^{n} l_i\alpha_i,$$

where $k_i = \max\{a_i, b_i\}$, $l_i = \min\{a_i, b_i\}$. Hence we obtain $\tau(\omega) + \phi(\omega) = \lambda(\omega) + \mu(\omega)$. $\square$

Lemma 7.14. *Let the notations be as in Lemma 7.13. Then for $\theta, \delta \in W^P$, $\theta \geq \delta$ such that $\theta(\omega) + \delta(\omega) = \tau(\omega) + \phi(\omega)$, we have $\tau, \phi \in\,]\delta, \theta[$, i.e., $\theta \geq \lambda$, and $\delta \leq \mu$.*

Proof. By hypothesis, we have,

$$\theta(\omega) - \tau(\omega) = \phi(\omega) - \delta(\omega)$$

Claim. $\theta(\omega) - \tau(\omega) < 0$.

If possible, let us assume that $\theta(\omega) - \tau(\omega) \geq 0$. The assumption implies that $\phi(\omega) - \delta(\omega) \geq 0$. Hence we obtain

$$\tau(\omega) \leq \theta(\omega),\ \delta(\omega) \leq \phi(\omega)$$

This implies (cf. Lemma 7.1)

$$\phi \leq \delta \leq \theta \leq \tau$$

which is not possible, since by hypothesis, τ and ϕ are not comparable. Hence our assumption is wrong and the Claim follows, and the required result follows from Lemma 7.1. $\square$

Lemma 7.15. *Let the notations be as in Lemma 7.13. Then in the straightening relation*

$$p_\tau p_\phi = \sum c_{\theta\delta} p_\theta p_\delta \tag{12}$$

either $\theta > \lambda$, or $\theta = \lambda$, and $\delta = \mu$.

Proof. For any pair (θ, δ) on the right-hand side of (12), we have (cf. Lemma 7.14), $\theta \geq \lambda$. Hence either $\theta > \lambda$, or $(\theta, \delta) = (\lambda, \mu)$ (by weight considerations; note $\theta(\omega) + \delta(\omega)(= \tau(\omega) + \phi(\omega)) = \lambda(\omega) + \mu(\omega)$ (in view of Lemma 7.13), and hence $\delta = \mu$). $\square$

Proposition 7.16. *Let $\tau, \phi \in W^P$ two non-comparable elements. Then in the straightening relation (12), $p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs with coefficient ± 1.*

Proof. As in Lemma 7.13, let us denote $\tau \vee \phi$ and $\tau \wedge \phi$ by λ and μ respectively. Now λ is $>$ both τ and ϕ. Hence, the restriction of the left-hand side of (12) to $X(\lambda)$ is non zero. This implies that the right-hand side of (12) restricts to a nonzero standard sum on $X(\lambda)$. This together with Lemma 7.15 and Theorem 7.10, (1) implies that $p_\lambda p_\mu$ does occur on the right-hand side with a nonzero coefficient. Thus we obtain

$$p_\tau p_\phi = a p_\lambda p_\mu, \quad a \in k^*, \text{ on } X(\lambda). \tag{13}$$

In fact, the above relation holds even over $\mathbb{Z}$, i.e. $\mathbf{P}_\tau \mathbf{P}_\phi = a \mathbf{P}_\lambda \mathbf{P}_\mu$ on $X_{\mathbb{Z}}(\lambda)$, so that going reduction modulo p, for any prime p, we obtain (13). Hence we conclude $a = \pm 1$. $\square$

We next prove some lemmas to be used later in this section.

Lemma 7.17. *Let τ, ϕ be two noncomparable elements of W^P, and $\lambda = \tau \vee \phi$. Let $\lambda = s_{\beta_1} \dots s_{\beta_s} \tau$, $\lambda = s_{\gamma_1} \dots s_{\gamma_t} \phi$, where $\beta_i, \gamma_j \in S$, $l(\lambda) = l(\tau) + s$, and $l(\lambda) = l(\phi) + t$. Then*

(1) *$\beta_i \neq \gamma_j$, for $1 \leq i \leq s$, $1 \leq j \leq t$.*
(2) *$s_{\beta_i} s_{\gamma_j} = s_{\gamma_j} s_{\beta_i}$, for $1 \leq i \leq s$, $1 \leq j \leq t$.*

Proof. (by induction on t)

Let $\lambda_1 = s_{\gamma_1} \lambda$. We have $\lambda_1 \geq \phi$, and $\lambda_1 \not\geq \tau$ (since $\lambda_1 < \lambda = \tau \vee \phi$). Hence, by Lemma 6.4, we obtain that $s_{\gamma_1} \tau \leq \lambda_1$. Let us denote $\tau_1 = s_{\gamma_1} \tau$.

Claim. $\lambda_1 = \tau_1 \vee \phi$

We have, clearly, $\lambda_1 \geq \tau_1$, $\lambda_1 \geq \phi$. Let $\lambda' = \tau_1 \vee \phi$. Then $\lambda' \leq \lambda_1$. Let ρ be the bigger of $\{\lambda', s_{\gamma_1} \lambda'\}$. Then $\rho \geq s_{\gamma_1} \tau_1 (= \tau)$ and $\rho \geq \phi$. Hence $\rho \geq \lambda$. Also $\rho \leq \lambda$, since $\lambda' \leq \lambda$, $s_{\gamma_1} \lambda' \leq \lambda$. Hence $\rho = \lambda$. This implies $\lambda' < s_{\gamma_1} \lambda' = \lambda$. Hence we obtain $\lambda' = s_{\gamma_1} \lambda = \lambda_1$, and the Claim follows.

Let $\theta_i = s_{\beta_i} \dots s_{\beta_s} \tau$, $1 \leq i \leq s$, $\theta_{s+1} = \tau$. Since $\lambda_1 \not\geq \tau$, we have $\lambda_1 \not\geq \theta_i$; we conclude (in view of Lemma 6.4) that $s_{\gamma_1} \theta_i \leq \lambda_1$, $1 \leq i \leq s$ (note that $\theta_i \leq \lambda$). Also note that $\gamma_1 \neq \beta_i$ (since $s_{\gamma_1} \theta_i \leq \lambda_1$, while $s_{\beta_i} \theta_i (= \theta_{i+1}) \not\leq \lambda_1$). Thus $X(s_{\gamma_1} \theta_i)$ and $X(s_{\beta_i} \theta_i) (= X(\theta_{i+1}))$ are two distinct Schubert divisors in $X(\theta_i)$. Hence we deduce (in view of Lemma 7.1) that s_{β_i} and s_{γ_1} commute. Thus we obtain

$$\begin{gathered} \beta_i \neq \gamma_1, \ 1 \leq i \leq s, \\ s_{\beta_i} s_{\gamma_1} = s_{\gamma_1} s_{\beta_i}, \ 1 \leq i \leq s, \end{gathered} \tag{I}$$

$\lambda_1 = s_{\gamma_1} \lambda = s_{\gamma_1} s_{\beta_1} \dots s_{\beta_s} \tau = s_{\beta_1} \dots s_{\beta_s} s_{\gamma_1} \tau = s_{\beta_1} \dots s_{\beta_s} \tau_1$, and $l(\lambda_1) = l(\tau_1) + s$ (since $l(\lambda_1) = l(\lambda) - 1$, and $l(\tau_1) = l(\tau) - 1$). Also, $\lambda_1 = s_{\gamma_2} \dots s_{\gamma_t} \phi$, and $\lambda_1 = \tau_1 \vee \phi$. Hence, by induction on t, we obtain that

$$\begin{gathered} \beta_i \neq \gamma_j, \ 1 \leq i \leq s, \ 2 \leq j \leq t \\ s_{\beta_i} s_{\gamma_j} = s_{\gamma_j} s_{\beta_i}, \ 1 \leq i \leq s, \ 2 \leq j \leq t \end{gathered} \tag{II}$$

The result now follows from (I) and (II) (note that when $t = 1$, we have $\lambda_1 = \phi$, and the result in this case follows from (I)). $\square$

Lemma 7.18. *Let notations be as in Lemma 7.17. Let $\mu = \tau \wedge \phi$. Then $s_{\gamma_1} \dots s_{\gamma_t}\mu = \tau$, $s_{\beta_1} \dots s_{\beta_s}\mu = \phi$, $\lambda(\tau) = l(\mu) + t$, $l(\phi) = l(\mu) + s$.*

Proof. As seen in the proof of Lemma 7.17, $s_{\gamma_1}\tau \leq \lambda_1$, $\tau \not\leq \lambda_1$. Hence $\tau > s_{\gamma_1}\tau$ (cf. Lemma 6.4). Let $\tau_1 = s_{\gamma_1}\tau$. Then we have (cf. proof of Lemma 7.17)

$$\lambda_1 = \tau_1 \vee \phi,\ \lambda_1 = s_{\gamma_2} \dots s_{\gamma_t}\phi,\ \lambda_1 = s_{\beta_1} \dots s_{\beta_s}\tau_1.$$

Repeating the argument in the above paragraph, we obtain $\tau_1 > s_{\gamma_2}\tau_1$. Thus proceeding, we arrive at $\tau = s_{\gamma_1}s_{\gamma_2} \dots s_{\gamma_t}\delta$, with $l(\tau) = l(\delta) + t$.

Now $\lambda = s_{\beta_1} \dots s_{\beta_s}\tau = s_{\beta_1} \dots s_{\beta_s}s_{\gamma_1} \dots s_{\gamma_t}\delta$. this implies that

$$\delta(\omega) - \lambda(\omega) = \sum_{i=1}^{s} \beta_i + \sum_{j=1}^{t} \gamma_j.$$

Also,

$$\tau(\omega) - \lambda(\omega) = \sum_{i=1}^{s} \beta_i, \quad \phi(\omega) - \lambda(\omega) = \sum_{j=1}^{t} \gamma_j.$$

Hence

$$\delta(\omega) - \lambda(\omega) = \tau(\omega) - \lambda(\omega) + \phi(\omega) - \lambda(\omega),$$

i.e.

$$\delta(\omega) + \lambda(\omega) = \tau(\omega) + \phi(\omega).$$

This, together with Lemma 7.13 implies that $\delta(\omega) = \mu(\omega)$, and hence $\delta = \mu$, and $\tau = s_{\gamma_1} \dots s_{\gamma_t}\mu$. Switching the roles of τ and ϕ, we obtain $\phi = s_{\beta_1} \dots s_{\beta_s}\mu$, with $l(\phi) = l(\mu) + s$. $\square$

To carry out the flat deformation as described in Section 5, we need the fact that $p_{\tau\vee\phi}p_{\tau\wedge\phi}$ occurs with coefficient 1 in the straightening relation for $p_\tau p_\phi$. Towards proving this, we first carry out the discussion on $H^0(G/P, L^2)$.

7.19. Standard monomial basis for $H^0(G/P, L^2)$. By Theorem 7.10, $\{p_\theta p_\delta \mid \theta, \delta \in W^P, \theta \geq \delta\}$ is a basis for $H^0(G/P, L^2)$. To arrive at this basis (as seen in the beginning of this Section), one first starts out with the weight ω, constructs a basis for $H^0(G/P, L)$; then one constructs the standard monomials to obtain a basis for $H^0(G/P, L^m), m \in \mathbb{Z}^+$. One may also start with $m\omega$ straight away, and carry out a similar construction. For our purpose, we need this approach also for $m = 2$, and a comparative study of the two bases.

7.20. Construction of a basis for $V_{2\omega}$. Let $V_{2\omega}$ be the irreducible G-module (over $\mathbb{Q}$) with highest weight 2ω. Let us fix a highest weight vector e in $V_{2\omega}$. Let $U_{\mathbb{Z}}$, $U_{\mathbb{Z}}^{+}$ be as in 6.1. Let $V_{\mathbb{Z}} = U_{\mathbb{Z}}e$.

Proposition 7.21. *Let $w \in W^P$, and $w = s_r \dots s_1$ be a reduced expression for w. Then the vector $X_{-\alpha_r}^{(2)} \dots X_{-\alpha_1}^{(2)} e$ is an extremal weight vector in $V_{2\omega}$ of weight $w(2\omega)$. Further, it depends only on w, and not on the reduced expression chosen.*

The proof is similar to that of Proposition 7.2.

Definition 7.22. For $w \in W^P$, we define $\mathbf{E}_w$ to be the uniquely determined extremal weght vector in $V_{2\omega}$ of weight $w(2\omega)$, as given by Proposition 7.21. We define $V_{\mathbb{Z},w} = U_{\mathbb{Z}}^{+}\mathbf{E}_w$.

Remark 7.23. We have $H^0(X_{\mathbb{Z}}(w), L_{\mathbb{Z}}^2) = V_{\mathbb{Z},w}^*$, the $\mathbb{Z}$-dual of $V_{\mathbb{Z},w}$.

Given $\theta, \delta \in W^P$ such that $\theta \geq \delta$, let $\theta = s_{\beta_r} \dots s_{\beta_1}\delta$, $\beta_i \in R^+$, where $r = l(\theta) - l(\delta)$. Then, in view of Lemma 7.1 , $\beta_i \in S$. Set $\mathbf{E}_{\theta,\delta} = X_{-\beta_r} \dots X_{-\beta_1}\mathbf{E}_\delta$.

Remark 7.24. By similar considerations as in [26], Remark 3.8, $\mathbf{E}_{\theta,\delta}$ depends only on θ and δ; it is independent of the path from θ to δ. For $\theta = \delta$, we denote $\mathbf{E}_{\delta,\delta}$ by just $\mathbf{E}_\delta$.

Theorem 7.25. (1) *The set $\{\mathbf{E}_{\theta,\delta} \mid \theta, \delta \in W^P, \theta \geq \delta\}$ is a $\mathbb{Z}$-basis for $V_{\mathbb{Z}}$.*
(2) *$\mathbf{E}_{\theta,\delta}$ is a weight vector of weight $\theta(\omega) + \delta(\omega)$.*
(3) *Let $w \in W^P$. Then $\{\mathbf{E}_{\theta,\delta} \mid w \geq \theta \geq \delta\}$ is a $\mathbb{Z}$-basis for $V_{\mathbb{Z},w}$.*

Proof. The proof is carried out in the same spirit as in [26]. In fact, this case can be considered as a "multimplicity 2 case", i.e. (the highest weight, α^*) ≤ 2, for all $\alpha \in R^+$. Further, every pair $(\theta, \delta) \in W^P \times W^P$, with $\theta \geq \delta$, is an admissible pair (refer to [26] for the definition of an admissible pair). $\square$

Definition 7.26. Let $\{\mathbf{A}_{\theta,\delta} \mid \theta, \delta \in W^P, \theta \geq \delta\}$ be the basis of $V_{\mathbb{Z}}^*$ (=the $\mathbb{Z}$-dual of $V_{\mathbb{Z}}$) dual to $\{\mathbf{E}_{\theta,\delta} \mid \theta, \delta \in W^P, \theta \geq \delta\}$. For any field k, let $a_{\theta,\delta} = \mathbf{A}_{\theta,\delta} \otimes 1$.

As an immediate consequence of Theorem 7.25, we have

Theorem 7.27. (1) *The set $\{a_{\theta,\delta} \mid \theta, \delta \in W^P, \theta \geq \delta\}$ is a k-basis for $H^0(G/P, L^2)$.*
(2) *$a_{\theta,\delta}$ is a weight vector of weight $-(\theta(\omega) + \delta(\omega))$.*
(3) *Let $w \in W^P$. Then $a_{\theta,\delta}\Big|_{X(w)} \neq 0 \iff w \geq \theta$.*
(4) *$\{a_{\theta,\delta} \mid w \geq \theta \geq \delta\}$ is a k-basis for $H^0(X(w), L^2)$.*

Lemma 7.28. *Let the notations be as in Lemma 7.13. Then the multiplicity of the weight $-(\lambda(\omega) + \mu(\omega))$ in $H_{\mathbb{Z}}^0(X(\lambda), L_{\mathbb{Z}}^2)$ is 1.*

Proof. First observe that the assertion in the Lemma is equivalent to the assertion that the multiplicity of the weight $\lambda(\omega)+\mu(\omega)$ in $V_{\mathbb{Z},\lambda}$ is 1. Let us denote $\lambda(\omega)+\mu(\omega)$ by χ. Then we have (cf. Theorem 7.25), multiplicity of χ in $V_{\mathbb{Z}} = \#\{(\theta,\delta),\theta,\delta \in W^P,\ \theta \geq \delta \mid \theta(\omega)+\delta(\omega) = \lambda(\omega)+\mu(\omega)\}$. Now $\lambda(\omega)+\mu(\omega) = \tau(\omega)+\phi(\omega)$ (cf. Lemma 7.13), and hence $\theta(\omega)+\delta(\omega) = \tau(\omega)+\phi(\omega)$. This implies (cf. Lemma 7.14) $\theta \geq \lambda$, $\delta \leq \mu$. The vector $\mathbf{E}_{\theta,\delta}$ belongs to $V_{\mathbb{Z},\lambda}$ if and only if $\theta \leq \lambda$ (cf. Theorem 7.25). Thus the only $\mathbf{E}_{\theta,\delta} \in V_{\mathbb{Z},\lambda}$ of weight χ is $\mathbf{E}_{\lambda,\mu}$. The result now follows from this. $\square$

7.29. Let us fix a highest weight vector u_0 in V_ω, and let us take e of 7.20 to be $u_0 \otimes u_0$, rather $\pi(u_0 \otimes u_0)$ under the canonical projection $\pi : V_\omega \otimes V_\omega \to V_{2\omega}$. Then for $\theta \in W^P$, $\mathbf{E}_{\theta,\theta}$ can be taken to be $\pi(\mathbf{Q}_\theta \otimes \mathbf{Q}_\theta)$, $\mathbf{Q}_\theta$ being as in Definition 7.3 (note that $\mathbf{Q}_\theta$ is an extremal weight vector in V_ω of weight $\theta(\omega)$). In the discussion below, for $u, v \in V_\omega$, we shall denote $\pi(u \otimes v)$ by just uv. Similarly, for $f, g \in H^0(G/P, L)$, we shall denote the image of $f \otimes g$ under $H^0(G/P, L) \otimes H^0(G/P, L) \to H^0(G/P, L^2)$ by just fg.

7.30. Comparative study of $\{\mathbf{A}_{\theta,\delta}\}$ and $\{\mathbf{P}_\theta\mathbf{P}_\delta\}$.

Proposition 7.31. *Let notations be as in Lemma* 7.13. *Then we have* $\mathbf{A}_{\lambda,\mu} = \mathbf{P}_\lambda\mathbf{P}_\mu$, *in* $H^0(X_{\mathbb{Z}}(\lambda), L^2_{\mathbb{Z}})$ *(here* $\mathbf{P}_\lambda$, $\mathbf{P}_\mu$ *are as in Definition* 7.6*).*

Proof. With the convention as in 7.29, we see easily that for $\lambda = \mu$, $\mathbf{A}_{\lambda,\lambda} = \mathbf{P}_\lambda\mathbf{P}_\lambda$. Let now $\lambda = s_{\alpha_r}\dots s_{\alpha_1}\mu$, $r = l(\lambda) - l(\mu)$, and α_i simple, $1 \leq i \leq r$ (cf. Lemma 7.1). We have $\mathbf{P}_\mu = (-1)^r X_{-\alpha_1}\dots X_{-\alpha_r}\mathbf{P}_\lambda$ (cf. Lemma 7.8). On the other hand, by our construction $\mathbf{E}_{\lambda,\mu} = X_{-\alpha_r}\dots X_{-\alpha_1}\mathbf{E}_{\mu,\mu}$. Hence we obtain
$< \mathbf{P}_\lambda\mathbf{P}_\mu, \mathbf{E}_{\lambda,\mu} > = < \mathbf{P}_\lambda\mathbf{P}_\mu, X_{-\alpha_r}\dots X_{-\alpha_1}\mathbf{E}_{\mu,\mu} > = (-1)^r < X_{-\alpha_1}\dots X_{-\alpha_r}(\mathbf{P}_\lambda\mathbf{P}_\mu), \mathbf{E}_{\mu,\mu} >$ (by $\mathfrak{g}$-invariance of $<,>$). Now writing $X_{-\alpha_1}\dots X_{-\alpha_r}(\mathbf{P}_\lambda\mathbf{P}_\mu)$ as

$$\sum (X_{-\beta_1}\dots X_{-\beta_l}\mathbf{P}_\lambda)(X_{-\gamma_1}\dots X_{-\gamma_m}\mathbf{P}_\mu),$$

the only relevant term is $(X_{-\alpha_1}\dots X_{-\alpha_r}\mathbf{P}_\lambda)\mathbf{P}_\mu$, which is simply $(-1)^r\mathbf{P}_\mu\mathbf{P}_\mu$. Thus we obtain $< \mathbf{P}_\lambda\mathbf{P}_\mu, \mathbf{E}_{\lambda,\mu} >= 1$. This implies that in the $\mathbb{Z}$-linear combination for $\mathbf{P}_\lambda\mathbf{P}_\mu$ in terms of the $\{\mathbf{A}_{\theta,\delta}\}$'s, the vector $\mathbf{A}_{\lambda,\mu}$ occurs with coefficient 1. This together with the fact (cf. Lemma 7.28) that the multiplicity of the weight $-(\lambda(\omega)+\mu(\omega))$ in $H^0(X_{\mathbb{Z}}(\lambda), L^2_{\mathbb{Z}})$ is 1, implies the required result. $\square$

Lemma 7.32. *With notations as in Lemma* 7.13*, we have*

$$\mathbf{P}_\tau\mathbf{P}_\phi = \mathbf{P}_\lambda\mathbf{P}_\mu \text{ in } H^0(X_{\mathbb{Z}}(\lambda), L^2_{\mathbb{Z}}).$$

Proof. We have (cf. Lemma 7.15 and Proposition 7.16)

$$\mathbf{P}_\tau\mathbf{P}_\phi = a\mathbf{P}_\lambda\mathbf{P}_\mu \text{ on } X_{\mathbb{Z}}(\lambda),$$

where $a = \pm 1$. To show that $a = 1$, we proceed as follows. Let $\lambda = s_{\beta_1}\dots s_{\beta_s}\tau = s_{\gamma_1}\dots s_{\gamma_t}\phi$, $\beta_i, \gamma_j \in S$, $l(\lambda) = l(\tau) + s = l(\phi) + t$. Then we have (in view of Lemmas

7.17 and 7.18) that $s_{\beta_i}s_{\gamma_j} = s_{\gamma_j}s_{\beta_i}$, $1 \le i \le s$, $1 \le j \le t$, and $\lambda = s_{\beta_1} \dots s_{\beta_s} s_{\gamma_1} \dots s_{\gamma_t}\mu$. Now

$$\begin{aligned}\langle \mathbf{P}_\tau \mathbf{P}_\phi \,,\, \mathbf{E}_{\lambda,\mu}\rangle &= \langle \mathbf{P}_\tau \mathbf{P}_\phi \,,\, X_{-\beta_1} \dots X_{-\beta_s} X_{-\gamma_1} \dots X_{-\gamma_t} \mathbf{E}_\mu \rangle \\ &= (-1)^{s+t} \langle X_{-\gamma_t} \dots X_{-\gamma_1} X_{-\beta_s} \dots X_{-\beta_1} (\mathbf{P}_\tau \mathbf{P}_\phi) \,,\, \mathbf{E}_\mu \rangle\end{aligned}$$

(by invariance of $\langle \,,\, \rangle$).

Let $D_\gamma = X_{-\gamma_t} \dots X_{-\gamma_1}$, $D_\beta = X_{-\beta_s} \dots X_{-\beta_1}$. Then $D_\beta D_\gamma = D_\gamma D_\beta$ (since $s_{\beta_i}s_{\gamma_j} = s_{\gamma_j}s_{\beta_i}$, $1 \le i \le s$, $1 \le j \le t$ (cf. Lemma 7.17)). We have

$$D_\gamma D_\beta(\mathbf{P}_\tau \mathbf{P}_\phi) = \mathbf{P}_\tau D_\gamma D_\beta(\mathbf{P}_\phi) + (D_\gamma \mathbf{P}_\tau)(D_\beta \mathbf{P}_\phi) + (D_\beta \mathbf{P}_\tau)(D_\gamma \mathbf{P}_\phi) + (D_\gamma D_\beta(\mathbf{P}_\tau))\mathbf{P}_\phi.$$

Now $\mathbf{E}_\mu$ being an extremal weight vector in $V_{\mathbb{Z},2\omega}$ of weight $2\mu(\omega)$, the only term in the above sum contributing a nonzero value to $\langle D_\gamma D_\beta(\mathbf{P}_\tau \mathbf{P}_\phi) \,,\, \mathbf{E}_\mu\rangle$ is the term $D_\gamma \mathbf{P}_\tau D_\beta \mathbf{P}_\phi = (-1)^{s+t} \mathbf{P}_\mu \mathbf{P}_\mu$ (note that $D_\gamma \mathbf{P}_\tau = (-1)^s \mathbf{P}_\mu$, $D_\beta \mathbf{P}_\phi = (-1)^t \mathbf{P}_\mu$ (cf. Lemma 7.8)). Hence we obtain that

$$\langle \mathbf{P}_\tau \mathbf{P}_\phi \,,\, \mathbf{E}_{\lambda,\mu}\rangle = 1.$$

Hence we obtain that in the $\mathbb{Z}$-linear combination for $\mathbf{P}_\tau \mathbf{P}_\phi$ in terms of $\{\mathbf{A}_{\theta,\delta}\}$'s, the vector $\mathbf{A}_{\lambda,\mu}$ occurs with coefficient 1. This together with the fact (cf. Lemma 7.28) that the multiplicity of the weight $-(\lambda(\omega) + \mu(\omega))$ in $H^0(X_{\mathbb{Z}}(\lambda), L_{\mathbb{Z}}^2)$ is 1, implies the required result. $\square$

Combining Proposition 7.16 and Lemma 7.32, we obtain

Proposition 7.33. *Let notations be as in Lemma* 7.13. *Then in the straightening relation*

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

$p_\lambda p_\mu$ *occurs with coefficient* 1.

Theorem 7.34. *Let X be a Schubert variety in a minuscule G/P. Then X degenerates to a normal toric variety Y. Further, the Gröbner basis for X as constructed in* [9] *descends to a Gröbner basis for Y.*

Proof. Let $X = X(w)$. We set

$$R(w) = \bigoplus_{\mathbf{m} \ge 0} H^0(X(w), L^{\mathbf{m}}),$$

where L is the ample generator of $\operatorname{Pic}(G/P)$.

Then $R(w)$ has algebra generators given by $\{p_\tau \mid \tau \in W^P, w \ge \tau\}$. Letting

$$H_w = \{\tau \in W^P \mid w \ge \tau\}$$

we have that H_w is a distributive lattice (being a sublattice of the distributive lattice W^P) . All of the hypotheses of Theorem 5.3 hold in view of the results of this section (the hypothesis regarding the embedding of H_w in a chain product lattice holds in view of weight considerations). $\square$

As a special case of Theorem 7.34, we have

Theorem 7.35. *Let X be a Schubert variety in the Grassmannian. Then X degenerates to a normal toric variety Y. Further, the Gröbner basis for X as constructed in* [9] *descends to a Gröbner basis for Y.*

8. Generalities on $SL(n)/Q$

By way of generalizing the results of Section 7 to other G/P's, and more generally, to G/Q's, (and their Schubert varieties), we prove the degeneration of $SL(n)/B$ to a normal toric variety (and also the degenerations of a certain class of Schubert varieties in $SL(n)/B$, namely the class of Kempf varieties, cf. [8]). The rest of the paper is devoted to proving these.

Let $G = SL(n)$, B the Borel subgroup of G consisting of the upper triangular matrices, and T the maximal torus consisting of diagonal matrices. Let R be the set of roots of G relative to T, and S the set of simple roots of R relative to B. We shall index the elements of S as in [2]. Let $P_1, \ldots, P_{n-1}$ be the maximal parabolic subgroups of G containing B. Let W be the Weyl group of G with respect to T. For a parabolic subgroup $Q \supset B$, let W_Q be the Weyl group of Q, and W^Q the set of minimal representatives in W of W/W_Q. For a maximal parabolic subgroup P_i, W_{P_i} and W^{P_i} will be denoted by just W_i and W^i respectively. For $w \in W^Q$, let $X(w)$ be the Schubert variety in G/Q defined to be the Zariski closure, $\overline{BwQ}$ (mod Q) of the orbit of w in G/Q, endowed with the canonical reduced scheme structure. Recall the partial order on W^Q: if $\tau_1, \tau_2 \in W^Q$, then $\tau_1 \geq \tau_2$ if and only if $X(\tau_1) \supseteq X(\tau_2)$. Let

$$Q = \bigcap_{t=1}^{r} P_{k_t}.$$

Definition 8.1. (cf. [26]). A *Young tableau in W/W_Q of type* $\mathbf{m} = (m_1, \ldots, m_r)$, where $m_i \geq 0$, $1 \leq i \leq r$, is a sequence $\lambda = (\lambda_{ij})$ with $\lambda_{ij} \in W^{k_i}$, $1 \leq j \leq m_i$, $1 \leq i \leq r$.

Definition 8.2. (cf. [26]). A Young tableau $\lambda = (\lambda_{ij})$ is said to be a *Young tableau on a Schubert variety* $X(\phi) \subset G/Q$ if $\phi_i \geq \lambda_{ij}$ for all $1 \leq i \leq r$, $1 \leq j \leq m_i$, where $X(\phi_i)$ is the projection of $X(\phi)$ under $G/Q \to G/P_{k_i}$.

Definition 8.3. (cf. [26]). A Young tableau $\lambda = (\lambda_{ij})$ on $X(\phi)$, $\phi \in W/W_Q$, is said to be *standard* on $X(\phi)$ if there exists a sequence $\theta = (\theta_{ij})$ (called a *defining sequence for* λ) so that

1) $\theta_{ij} \in W/W_Q$, $1 \leq i \leq r$, $1 \leq j \leq m_i$,
2) each θ_{ij} is a lifting of λ_{ij} under $W/W_Q \to W/W_{k_i}$,
3) $X(\phi) \supseteq X(\theta_{11}) \supseteq X(\theta_{12}) \supseteq \cdots \supseteq X(\theta_{1m_1}) \supseteq X(\theta_{21}) \supseteq \cdots \supseteq X(\theta_{rm_r})$ (in G/Q).

More generally, a Young tableau $\lambda = (\lambda_{ij})$ is said to be *standard on a union of Scubert varieties* $Z = \bigcup_{i=1}^{t} X(\phi_i)$ in G/Q, if λ is standard on $X(\phi_i)$, for some i,

$1 \leq i \leq t$. If $m_t = 0$ for any t, $1 \leq t \leq r$, the family $\{\theta_{tj} \mid 1 \leq j \leq m_t\}$ is understood to be empty.

Remark 8.4. For $X(w) = G/Q$, the above definition of standard Young tableau coincides with the the classical notion.

Let L_i be the ample generator of $Pic(G/P_i)$. One knows that the extremal weight vectors in $H^0(G/P_i, L_i)$ give a k-basis for $H^0(G/P_i, L_i)$, which we shall denote $\{p_\tau \mid \tau \in W^i\}$. (Note that G/P_i is simply the Grassmannian $G_{i,n}$ of i-dimensional subspaces of k^n, and p_τ, $\tau \in W^i$, are simply the Plücker coordinates.) Given a Young tableau $\lambda = (\lambda_{ij})$ of type $\mathbf{m}$, we shall set

$$p(\lambda) = \prod_{i=1}^{r} \prod_{j=1}^{m_i} p_{\lambda_{ij}}.$$

(Note that $p(\lambda) \in H^0(G/Q, L^{\mathbf{m}})$, where $L^{\mathbf{m}} = L_{k_1}^{m_1} \otimes \cdots \otimes L_{k_r}^{m_r}$). Such a monomial will be called *standard on* Z if λ is standard on Z.

We recall the following (cf. [26]):

Theorem 8.5. *Let G, Q, Z as above. Then the standard monomials on Z of degree $\mathbf{m} = (m_1, \ldots, m_r)$ form a basis of $H^0(Z, L^{\mathbf{m}})$, where $L^{\mathbf{m}} = L_{k_1}^{m_1} \otimes \cdots \otimes L_{k_r}^{m_r}$.*

8.6. We make some identifications which we will use throughout the rest of this Section. As $W(SL(n))$ is the symmetric group on n letters S_n, the reflections in W are precisely the elements (i, j) of S_n which switch the i-th and j-th positions. For the maximal parabolic subgroup

$$P_i = \left\{ A \in G \;\middle|\; A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \text{ where 0 is the } (n-i) \times i\text{- zero matrix} \right\},$$

W_{P_i} can be identified with the subgroup of W generated by the reflections $\{(j, j+1), j \neq i\}$. Then W^i can be identified with $I_{i,n}$ $(= \{(j_1, \ldots, j_i) \mid 1 \leq j_1 < \cdots < j_i \leq n\})$. Further, if $\tau_1, \tau_2 \in I_{i,n}$, say $\tau_1 = (l_1, \ldots, l_i)$ and $\tau_2 = (j_1, \ldots, j_i)$, then the induced order in W^i is the following: $\tau_1 \geq \tau_2$ if and only if $l_k \geq j_k$ for $1 \leq k \leq i$.

Remark 8.7. (cf.[26]). Let $X(\phi)$ be a Schubert variety in G/Q, $\phi \in W^Q$. Let λ be a standard Young tableau on $X(\phi)$. Then there exists a (minimal) defining tableau for λ; we have (cf. [26]) the minimal tableau is unique, and does not depend on ϕ. This unique minimal tableau for λ will be denoted $\lambda^{\min}$.

8.8. Not only do the standard monomials of type $\mathbf{m}$ on $Z = \bigcup X(\phi_i)$ provide a basis for $H^0(Z, L^{\mathbf{m}})$, they also satisfy a type of "straightening law" which will be important for us.

We first need to observe a property satisfied by the Plücker coordinates, namely, given $\tau, w \in W^i$, $\tau \leq w$ if and only if $p_\tau\Big|_{X(w)} \neq 0$

Let $S = S_{m_1} \times \cdots \times S_{m_r}$. If $\lambda = (\lambda_{ij})$ is a $m_1 + \cdots + m_r$-tuple of elements, where $\lambda_{ij} \in W^{k_i}$, $1 \leq j \leq m_i$, $1 \leq i \leq r$, and $\sigma \in S$, $\sigma = (\sigma_1, \ldots, \sigma_r)$, then by λ^σ we denote the expression obtained by permuting the first m_1 entries according to σ_1, the next m_2 according to σ_2, etc.

Definition 8.9. Let

$$M = \{\lambda = (\lambda_{ij}) \mid \lambda_{ij} \in W^{k_i}, 1 \leq j \leq m_i, 1 \leq i \leq r\}.$$

We define a partial order $\geq_L$ on M as follows: let $\lambda = (\lambda_{ij})$, $\mu = (\mu_{ij})$ in M, and let us write $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m)$, where $m = m_1 + \cdots + m_r$; we say $\lambda \geq_L \mu$ if and only if either $\lambda = \mu$, or there exists $1 \leq t < m$ such that $\lambda_i = \mu_i$, $1 \leq i < t$ and $\lambda_t > \mu_t$ (note that for any $1 \leq i \leq m$, $\lambda_i, \mu_i \in W^P$ for some maximal parabolic P, and $\geq$ on W^P denotes the Bruhat order).

Recall the following (cf. [14]):

Proposition 8.10. *Let $Z = \bigcup_{i=1}^{l} X(\phi_i)$ be a union of Schubert varieties in G/Q. Let $p(\lambda)$ be a nonzero, nonstandard monomial on Z of degree $\mathbf{m} = (m_1, \ldots, m_r)$, and*

$$p(\lambda) = \sum_{i=1}^{N} a_i p(\tau_i), \qquad a_i \in k^* \tag{14}$$

be the expression of $p(\lambda)$ as a sum of standard monomials on Z. Then for every i, $\tau_i >_L \lambda^\sigma$, $\sigma \in S$.

A relation as (14) will be referred to as a *straightening relation on Z*.

9. Gröbner bases for unions of Schubert varieties in $SL(n)/Q$

Let Q be a parabolic subgroup of $SL(n)$. Let $Q = \cap_{i=1}^{r} P_{k_i}$, P_{k_i}, $1 \leq i \leq r$ being the maximal parabolic subgroups containing Q. We shal denote $W^{P_{k_i}}$ (cf. Section 6) by just W^{k_i}. Let $X(w)$ be a Schubert variety in G/Q. For $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}_+^r$, let $|\mathbf{a}| = a_1 + \cdots + a_r$ and $(R_w)_{\mathbf{a}} = H^0(X(w), L^{\mathbf{a}})$, where $L^{\mathbf{a}} = L_{k_1}^{a_1} \otimes \cdots \otimes L_{k_r}^{a_r}$. Now define

$$R_w = \bigoplus_{\mathbf{a} \in \mathbb{Z}_+^r} H^0(X(w), L^{\mathbf{a}})$$

For $X(w) = G/Q$, we shall denote R_w(resp. $(R_w)_{\mathbf{a}}$) by just R(resp. $R_{\mathbf{a}}$). Recall the following (cf. [17]):

Theorem 9.1. 1) *For each* $\mathbf{a} \in \mathbb{Z}_+^r$ *the canonical map*

$$\theta_{\mathbf{a}} : \bigotimes_{i=1}^{r} \mathcal{S}^{a_i}(H^0(G/Q, L_{k_i}) \to R_{\mathbf{a}}$$

is surjective. Let $I_{\mathbf{a}}$ *be its kernel.*

2) *Let* I *be the kernel of the canonical map*

$$\theta : \bigoplus_{\mathbf{a}\in\mathbb{Z}_+^r} \bigotimes_{i=1}^{r} \mathcal{S}^{a_i}(H^0(G/Q, L_{k_i})) \to R,$$

Then I *is multigraded, generated by* $I_2 = \bigcup_{|\mathbf{a}|=2} I_{\mathbf{a}}$.

If $\underline{n}$ is a nonstandard monomial of degree $\mathbf{a}$, then, by Theorem 9.1, $\underline{n}$ can be written in a unique way as a linear combination of standard monomials of degree $\mathbf{a}$, modulo the ideal I:

$$\underline{n} = \sum_{i=1}^{t} c_i \underline{s}_i \pmod{I}, \qquad c_i \in k^*. \tag{15}$$

We refer to (15) as a *straightening relation*. Denote

$$f_{\underline{n}} = \underline{n} - \sum_{i=1}^{t} c_i \underline{s}_i, \tag{16}$$

$$\mathcal{F}_{\mathbf{a}} = \{f_{\underline{n}} \mid \underline{n} \text{ is a nonstandard monomial of degree } \mathbf{a}\},$$

$$\mathcal{F} = \bigcup_{\mathbf{a}} \mathcal{F}_{\mathbf{a}} \qquad \mathcal{F}_2 = \bigcup_{|\mathbf{a}|=2} \mathcal{F}_{\mathbf{a}}.$$

Clearly, $\mathcal{F}_{\mathbf{a}} \subset I_{\mathbf{a}}$ and $\mathcal{F} \subset I$.

9.2. Let $H = \bigcup_{i=1}^{r} W^{k_i}$. We define a partial order $\succeq$ on the H as follows: Given $\tau_1 = (a_1, \dots, a_r)$, $\tau_2 = (b_1, \dots, b_s)$, where $r, s \in \{k_1, \dots, k_r\}$,

$$\tau_1 \succeq \tau_2 \iff r \le s \text{ and } a_t \ge b_t \text{ for } 1 \le t \le r.$$

It is easily seen that $(H, \succeq)$ is a distributive lattice with 1, and 0.

We now extend $\succeq$ to a total order on H, also denoted by $\succeq$. This induces a total order on the set $\{p_\tau \mid \tau \in H\}$: $p_{\tau_1} \prec p_{\tau_2} \iff \tau_1 \succ \tau_2$ (we have taken the order for a specific purpose). Thus, according to our convention, a monomial of degree r in the polynomial ring $k[p_\tau, \tau \in H]$ will be written in the form $p_{\tau_1} \dots p_{\tau_r}$, with $\tau_1 \succeq \cdots \succeq \tau_r$. The above total order on the p_τ's induces the lexicographic order on monomials, as in 1.7.

In view of Proposition 8.10, we have

Theorem 9.3. *Let*

$$f_{\underline{n}} = \underline{n} - \sum_{i=1}^{t} c_i \underline{s}_i, \qquad c_i \in k^*$$

as in (16), be a typical element in $\mathcal{F}$. Then $in(f_{\underline{n}}) = \underline{n}$, i.e. $\underline{s}_i \prec_{lex} \underline{n}$, for all i.

Theorem 9.4. *We have $in(\mathcal{F}) = in(I)$.*

Proof. Since $in(\mathcal{F}) \subset in(I)$, and $in(\mathcal{F})$ consists of all the nonstandard monomials, it is enough to prove that the leading monomial of any element $f \in I$ is nonstandard. Assume this is not true, and let f be an element of I such that $in(f)$ is a standard monomial. Let $\underline{s}_0, \underline{s}_1, \dots, \underline{s}_t$ be all the standard monomials, including $in(f) = \underline{s}_0$, and $\underline{n}_1, \dots, \underline{n}_l$ all the nonstandard monomials appearing in f, so that f is written as

$$f = a_0 \underline{s}_0 + \sum_{i=1}^{t} a_i \underline{s}_i + \sum_{j=1}^{l} b_j \underline{n}_j, \quad a_0, a_i, b_j \in k^*.$$

Consider the polynomial $f' = f - \sum_{j=1}^{l} b_j f_{\underline{n}_j}$ ($f_{\underline{n}_j}$ being as in (16)). Then $in(f') = \underline{s}_0$, since $in(f) = \underline{s}_0$, $in(f_{\underline{n}_j}) = \underline{n}_j$, and $\underline{n}_j \prec_{lex} \underline{s}_0$, for all j. Therefore f' is a nontrivial linear comination of standard monomials, since the coefficient of $\underline{s}_0$ in its expression is $a_0 \neq 0$. On the other hand, $f' \equiv 0 \pmod{I}$, contradicting the linear independence of standard monomials. Hence our assumption is wrong, and $in(f)$ is nonstandard. $\square$

Theorem 9.5. *$\mathcal{F}_2$ is the reduced Gröbner basis for I with respect to the lexicographic order.*

Proof. We have to show that $\langle in(\mathcal{F}_2)\rangle = \langle in(I)\rangle$. In view of Theorem 9.4, in order to show that $\mathcal{F}_2$ is a Gröbner basis, it is enough to show that $\langle in(\mathcal{F}_2)\rangle = \langle in(\mathcal{F})\rangle$. Since $\langle in(\mathcal{F}_2)\rangle \subset \langle in(\mathcal{F})\rangle$, it suffices to show that for any nonstandard monomial $\underline{n}$, $in(f_{\underline{n}}) \in \langle in(\mathcal{F}_2)\rangle$, i.e. $\underline{n} \in \langle in(\mathcal{F}_2)\rangle$.

Let $\underline{n} = p_{\tau_1} \dots p_{\tau_r}$ be nonstandard of degree r. Then there exists an i such that $p_{\tau_i} p_{\tau_{i+1}}$ is nonstandard. Since $\underline{n} \in \langle p_{\tau_i} p_{\tau_{i+1}}\rangle$ and $p_{\tau_i} p_{\tau_{i+1}} = in(f_{p_{\tau_i} p_{\tau_{i+1}}})$, we conclude that $\underline{n} \in \langle in(\mathcal{F}_2)\rangle$. Hence $\mathcal{F}_2$ is a Gröbner basis for I. The fact that $\mathcal{F}_2$ is reduced can be easily seen from the form of its elements. $\square$

Let $w \in W^Q$, and $X(w)$ the Schubert variety in G/Q corresponding to w. Under $\pi_i : G/Q \to G/P_{k_i}$, let $\pi_i(X(w)) = X(w^{(i)})$, where $w^{(i)} \in W^{k_i}$. For $\tau \in W^{k_i}$, $1 \leq i \leq r$, we have $p_\tau\big|_{X(w^{(i)})} \neq 0$ if and only if $w^{(i)} \geq \tau$.

We have (cf. [17], [34]):

Theorem 9.6. 1) *The restriction maps $H^0(G/Q, L^{\mathbf{a}}) \to H^0(X(w), L^{\mathbf{a}})$, $\mathbf{a} \in \mathbb{Z}_+^r$, are surjective.*

2) *The kernel* $\mathfrak{a}(w)$ *of the epimorphism* $R \to R(w)$ *is multigraded say* $\mathfrak{a}(w) = \bigoplus_{\mathbf{a}} \mathfrak{a}(w)_{\mathbf{a}}$*, and generated by* $\{p_\tau \mid \tau \in W^{k_i}, 1 \le i \le r, w^{(i)} \not\ge \tau\}$*, i.e.* $\mathfrak{a}(w)$ *is generated by* $\bigoplus_{|\mathbf{a}|=1} \mathfrak{a}(w)_{\mathbf{a}}$.

9.7. Denote $H_w = \{\tau \in W^{k_i} \mid 1 \le i \le r,\ w^{(i)} \ge \tau\}$, and $A(w) = k[p_\tau, \tau \in H_w]$. Consider the canonical epimorphism $A \to A(w)$, and denote its kernel by $J(w)$; then $J(w)$ is generated by $\{p_\tau, \tau \in H \setminus H_w\}$. By Theorem 9.6 we obtain an epimorhism $A(w) \to R(w)$, whose kernel is $I + J(w) \pmod{J(w)}$, where, recall that I is the kernel of the epimorphism $A \to R$. We shall denote this kernel by $I(w)$; thus $R(w) = A(w)/I(w)$.

For an element $f \in A(w)$, we shall denote its image in $R(w)$ by f^w. For a monomial $\underline{n}$ which is nonstandard on $X(w)$, let $\underline{n}^w = \sum c_i \underline{s}_i^w$ be the expression for $\underline{n}^w$ as a sum of standard monomials on $X(w)$. Let us denote $f_{\underline{n}^w} = \underline{n}^w - \sum c_i \underline{s}_i^w \in A(w)$, and set

$$\mathcal{F}_{\mathbf{a}}^w = \{f_{\underline{n}^w} \mid \underline{n}^w \text{ is a nonstandard monomial on } X(w) \text{ of degree } \mathbf{a}\},$$
$$\mathcal{F}^w = \bigcup_{\mathbf{a}} \mathcal{F}_{\mathbf{a}}^w, \qquad \mathcal{F}_2^w = \bigcup_{|\mathbf{a}|=2} \mathcal{F}_{\mathbf{a}}^w.$$

Clearly, $\mathcal{F}_{\mathbf{a}}^w \subset I(w)_{\mathbf{a}}$ and $\mathcal{F}^w \subset I(w)$.

In view of Theorems 9.1, part 2), and 9.6, part 2), we have

Theorem 9.8. *The multigraded ideal* $I(w)$ *is generated by* $\mathcal{F}_2^w$.

Using Proposition 8.10, we obtain:

Theorem 9.9. *Let*

$$f_{\underline{n}^w} = \underline{n}^w - \sum_{i=1}^{t} c_i \underline{s}_i^w, \qquad c_i \in k^*$$

be a typical element in $\mathcal{F}_{\mathbf{a}}^w$*, where* $\underline{n}^w$ *is a nonstandard monomial and* $\underline{s}_i^w$ *are standard monomials on* $X(w)$*, all of degree* $\mathbf{a}$*. Then* $in(f_{\underline{n}}^w) = \underline{n}$*, i.e.* $\underline{s}_i^w \prec_{lex} \underline{n}^w$*, for all* i*.*

Proceeding as in Theorem 9.4, we obtain:

Theorem 9.10. *We have* $in(\mathcal{F}^w) = in(I(w))$*.*

Theorem 9.11. $\mathcal{F}_2^w$ *is the reduced Gröbner basis for* $I(w)$ *with respect to the lexicographic order.*

Proof. We need to show that $\langle in(\mathcal{F}_2^w\rangle = \langle in(I(w))\rangle$. In view of theorem 9.10, in order to prove that $\mathcal{F}_2^w$ is a Gröbner basis, it is enough to prove that $\langle in(\mathcal{F}_2^w)\rangle = \langle in(\mathcal{F}^w)\rangle$. Since $\langle in(\mathcal{F}_2^w)\rangle \subset \langle in(\mathcal{F}^w)\rangle$, it suffices to show that for any nonstandard monomial $\underline{n}^w$, $in(f_{\underline{n}^w}) \in \langle in(\mathcal{F}_2^w)\rangle$, i.e. $\underline{n}^w \in \langle in(\mathcal{F}_2^w)\rangle$. Let $\underline{n}^w = x_{\tau_1}^w \dots x_{\tau_r}^w$, where $x_{\tau_i}^w$ is $p_{\tau_i}\Big|_{X(w)}$. Then, in view of Theorem 9.6, there exists a pair (i,j) such that (τ_i, τ_j) is not standard on $X(w)$. Hence $\underline{n}^w \in \langle x_{\tau_1}^w x_{\tau_j}^w\rangle$; now, $x_{\tau_i}^w x_{\tau_j}^w = in(f_{p_{\tau_i} p_{\tau_j}}^w)$ (note that

$f^w_{p_{\tau_i}p_{\tau_j}} = p_{\tau_i}p_{\tau_j} - \sum c_{\alpha\beta}p_\alpha p_\beta$, where $p_{\tau_i}p_{\tau_j} = \sum c_{\alpha\beta}p_\alpha p_\beta$ is the straightening relation for $p_{\tau_i}p_{\tau_j}$ on $X(w)$). Thus we obtain $\underline{n}^w \in \langle in(\mathcal{F}^w_2)\rangle$. $\square$

Let $Z = \bigcup_{i=1}^r X(w_i)$ be a union of Schubert varieties in G/Q. Let $H_Z = \bigcup_{i=1}^r H_{w_i}$ and $A_Z = k[p_\tau, \tau \in H_Z]$. Consider the canonical epimorphism $A \to A_Z$ and denote its kernel by J_Z; then J_Z is generated by $\{p_\tau \mid \tau \in H \setminus H_Z\}$. Using the surjective map $R \to R_Z = \bigoplus_{\mathbf{a}} H^0(Z, L^{\mathbf{a}})$, we obtain an epimorphism $A_Z \to R_Z$ whose kernel is $I + J_Z$, which we denote by I_Z. For an element $f \in A_Z$, we shall denote its image in R_Z by f^Z. For a monomial $\underline{n}$ which is nonstandard on Z, let $\underline{n}^Z = \sum c_i \underline{s}_i^Z$ be the expression for $\underline{n}^Z$ as a sum of standard monomials on Z. Let us denote $f_{\underline{n}^Z} = \underline{n}^Z - \sum c_i \underline{s}_i^Z \in A_Z$, and set

$$\mathcal{F}^Z_{\mathbf{a}} = \{f_{\underline{n}^Z} \mid \underline{n}^Z \text{ is a nonstandard monomial on } Z \text{ of degree } \mathbf{a}\},$$
$$\mathcal{F}^Z = \bigcup_{\mathbf{a}} \mathcal{F}^Z_{\mathbf{a}}, \qquad \mathcal{F}^Z_2 = \bigcup_{|\mathbf{a}|=2} \mathcal{F}^Z_{\mathbf{a}}.$$

Noting that Theorem 9.6 holds when $X(w)$ is replaced by Z (cf. [34], [17]), and proceeding as above, we obtain:

Theorem 9.12. 1) *$in(\mathcal{F}^Z) = in(I_Z)$.*
2) *$\mathcal{F}^Z_2$ is the reduced Gröbner basis for I_Z with respect to the lexicographic order.*

9.13. Application to Variety of Complexes. Let $V_1 \ldots, V_{r+1}$ be a sequence of vector spaces of dimension n_i, $1 \le i \le r+1$. Let

$$X = \bigoplus_{1 \le i \le r} \mathrm{Hom}(V_i, V_{i+1})$$

be the affine space whose coordinate ring is the polynomial ring $A = k[Y^{(1)} \ldots, Y^{(r)}]$, where $Y^{(i)}$ denotes an $n_{i+1} \times n_i$ matrix of inderminates, for each $1 \le i \le r$. Let $\mathcal{C} \subset X$ be the closed subscheme of "complexes", i.e.

$$\mathcal{C} = \{(f_1, \ldots, f_r) \mid V_1 \xrightarrow{f_1} \ldots V_r \xrightarrow{f_r} V_{r+1}, f_i \text{ linear}, f_{i+1}f_i = 0, 1 \le i \le r-1\}$$

In other words, $\mathcal{C}$ is defined by the quadratic forms given by the entries of $Y^{(i+1)}Y^{(i)}$, $1 \le i \le r-1$. It is shown in [32] that $\mathcal{C}$ can be identified with $B^- e_{\mathrm{id}} \cap Z$, where Z is a union of Schubert varieties in $SL(n)/Q$, $n = n_1 + \cdots + n_{r+1}$, and $Q = \bigcap_{i=1}^r P_{m_i}$, $m_i = n_1 + \ldots, n_i$ (here B^- denotes the "opposite Borel subgroup" in $SL(n)$ consisting of all the lower triangular matrices, and e_{id} denotes the coset Q in G/Q). Hence we obtain a "standard monomial basis" for $k[\mathcal{C}]$, the coordinate ring of $\mathcal{C}$ (this basis consists of just the restrictions of standard monomials on Z; note also that $p_\tau\Big|_{\mathcal{C}} = 1$ for $\tau = \mathrm{id}$ in W^{m_i}, $1 \le i \le r$, $\mathcal{C}$ having been identified with $B^- e_{\mathrm{id}} \cap Z$). Hence we obtain the reduced Gröbner basis for $\mathcal{C}$ with respect to the lexicographic order.

10. Degeneration of $SL(n)/Q$ to a toric variety

In this section we show that the flag variety $SL(n)/Q$ degenerates to a toric variety. Let $Q=\bigcap_{i=1}^{r} P_{k_i}$, $H=\bigcup_{i=1}^{r} W^{k_i}$. We follow the notations of Section 9. As in Section 9, let $R=\bigoplus_{\mathbf{a}} H^0(SL(n)/Q, L^{\mathbf{a}})$. We shall show that all the hypotheses of Theorem 5.3, hold for R.

R is generated as an algebra by $\{p_\tau, \tau \in H\}$.

We take the canonical partial order on H, namely, given $\tau,\phi \in H$, say $\tau = (i_1,\dots,i_c)$, $\phi=(j_1,\dots,j_d)$,

$$\tau \geq \phi \iff c \leq d \text{ and } i_t \geq j_t,\ 1 \leq t \leq c.$$

From Theorem 9.1, we have that the quadratic relations among the p_τ's generate all other relations. Let

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta \tag{17}$$

be a typical quadratic relation, where we suppose that τ and ϕ are two non-comparable elements of H, and for each (α,β) on the right-hand side, $\alpha \geq \beta$. Then, by Proposition 8.10 we have

$$\alpha > \tau, \phi \tag{18}$$

(in fact we also have $\beta < \tau,\phi$). Further, we have (by weight considerations, for example)

$$\tau \dot{\cup} \phi = \alpha \dot{\cup} \beta. \tag{19}$$

Towards proving the crucial result of this section (Proposition 10.4 below), we need the following two lemmas:

Lemma 10.1. *Let $\tau=(i_1,\dots,i_c)$, $\phi=(j_1,\dots,j_d)$, $\lambda=\tau\vee\phi$, $\mu=\tau\wedge\phi$. Then $\lambda=(k_1,\dots,k_c)$, $\mu=(l_1,\dots,l_d)$, where for $t\leq c$, $k_t=\max\{i_t,j_t\}$, $l_t=\min\{i_t,j_t\}$, and for $t>c$, $l_t=j_t$.*

Proof. We just need to check that $(k_1,\dots,k_c)$ and $(l_1,\dots,l_d)$ belong to H, i.e. the k_i's are distinct, and the l_j's are distinct. The fact that the k_i's are distinct is clear, since for $r<s\leq c$, $\max\{i_r,j_r\}<\max\{i_s,j_s\}$. Regarding the l_i's, it is clear that $l_r\neq l_s$ for $r<s\leq c$, for similar reasons as above. Also, we have $l_r=j_r$, $r>c$. Hence it suffices to check that $\{l_1,\dots,l_c\}\cap\{j_{c+1},\dots,j_d\}=\emptyset$. But this is obvious, since $l_c\leq j_c<j_{c+1}<\cdots<j_d$. $\square$

Lemma 10.2. *Let $\alpha=(a_1,\dots,a_c)\in W^c$, and $\beta=(b_1,\dots,b_d)\in W^d$. Let $\alpha\geq\beta$. Let $\theta=(\alpha,\beta)^{min}$, say $\theta=(\theta_1,\theta_2)$, $\theta_i\in W^Q$, $i=1,2$, where $Q=P_c\cap P_d$ (cf. Remark 8.7). Then*

$$\theta_1^{(c)}=(a_1,\dots,a_c),\qquad \theta_1^{(d)}=(a_1,\dots,a_c,s_1,\dots,s_e),$$
$$\theta_2^{(c)}=(b_1,\dots,b_c),\qquad \theta_2^{(d)}=(b_1,\dots,b_d),$$

where $e = d - c$, and $\{s_1, \ldots, s_e\}$ is the subset of $\{b_1, \ldots, b_d\}$ with the property that the complement of $\{s_1, \ldots, s_e\}$ in $\{b_1, \ldots, b_d\}$ is the largest c-tuple which is $\leq \alpha$.

The proof is immediate from the definition of $(\alpha, \beta)^{\min}$.

Remark 10.3. Let notations be as in Lemma 10.1. Let $\theta = (\lambda, \phi)^{\min}$, say $\theta = (\theta_1, \theta_2)$. Then we have (by Lemma 10.2)

(1) $\theta_1 \geq \theta_2$.

(2) $\theta_1^{(c)} = \lambda$.

(3) $\theta_2^{(d)} = \phi$.

In particular we have

$$(*) \qquad p_\tau\Big|_{X(\theta_1)} \neq 0, \quad p_\phi\Big|_{X(\theta_1)} \neq 0.$$

Proposition 10.4. *$p_{\tau\vee\phi}p_{\tau\wedge\phi}$ occurs on the right-hand side of (17) with coefficient 1.*

Proof. Let notations be as above. Then, in view of (18), for any (α, β) on the right-hand side of (17), we have

$$(20) \qquad \alpha \geq \tau \vee \phi.$$

Further, if $\alpha = \tau \vee \phi$, then in view of (19) we have $\beta = \tau \wedge \phi$. Hence we conclude that for any p_α appearing on the right-hand side of (17) such that $\alpha \neq \tau \vee \phi$

$$p_\alpha\Big|_{X(\theta_1)} = 0.$$

Now we restrict (17) to $X(\theta_1)$. Then, in view of $(*)$ in Remark 10.3, the restriction of the left-hand side of (17) to $X(\theta_1)$ is nonzero, while on the right-hand side of (17), $p_\alpha p_\beta\Big|_{X(\theta_1)} = 0$, if $(\alpha, \beta) \neq (\tau \vee \phi, \tau \wedge \phi)$. Hence we obtain

$$p_\tau p_\phi = c p_{\tau\vee\phi} p_{\tau\wedge\phi}, \qquad \text{on } X(\theta_1).$$

In order to prove that $c = 1$, we compare the coefficients of the monomial $\underline{m} = x_{i_1 1} \ldots x_{i_c c} x_{j_1 1} \ldots x_{j_d d}$ on both sides (note that a Plücker coordinate $p_{a_1,\ldots,a_s}$ (being the $s \times s$ minor of the generic $n \times n$ matrix (x_{ij}) with row indices $a_1, \ldots, a_s$ and column indices $1, \ldots, s$) is a polynomial in the x_{ij}'s). Let $p_\alpha p_\beta$ appearing on the right-hand side of (17) be such that $\alpha \neq \tau \vee \phi$. This imlies $\alpha > \tau \vee \phi$, and $\beta < \tau \wedge \phi$. Let $\alpha = (\alpha_1, \ldots, \alpha_c)$ and $\beta = (\beta_1, \ldots, \beta_d)$. We have $(\alpha_1, \ldots, \alpha_c) > (k_1, \ldots, k_c)$. Let t be the smallest integer $\leq c$ such that $\alpha_t > k_t$. This implies, in view of (19)

$$\alpha_p = k_p, \quad \beta_p = l_p, \quad p < t,$$
$$k_t \notin \{\alpha_1, \ldots, \alpha_c\}, \quad k_t, l_t \in \{\beta_1, \ldots, \beta_d\}.$$

Hence in the expression for $p_\alpha p_\beta$ as a polynomial in the x_{ij}'s, $x_{k_t t} x_{l_t t}$ will not be a factor in any of the monomials. Hence the monomial $\underline{m}$ does not occur in $p_\alpha p_\beta$. Now the term $x_{i_1 1} \ldots x_{i_c c}$ (resp. $x_{j_1 1} \ldots x_{j_d d}$) being the product of the diagonal entries in

p_τ (resp. p_ϕ), it occurs with coefficient 1 in p_τ (resp. p_ϕ). Hence $\underline{m}$ occurs with coefficient 1 in $p_\tau p_\phi$. This, together with the fact that $\underline{m}$ does not occur in any $p_\alpha p_\beta$, $\alpha \neq \tau \vee \phi$, implies that $\underline{m}$ should occur with coefficient 1 in $p_{\tau\vee\phi}p_{\tau\wedge\phi}$ (note that in $p_{\tau\vee\phi}p_{\tau\wedge\phi}$ the monomial $\underline{m}$ is realized as the product $\underline{m}_1\underline{m}_2$, where $\underline{m}_1$ (resp. $\underline{m}_2$) is the product of the diagonal entries in $p_{\tau\vee\phi}$ (resp. $p_{\tau\wedge\phi}$) Now $\underline{m}$ occurs with coefficient 1 on the left-hand side of (17). Hence $\underline{m}$ should appear with coefficient 1 in $p_{\tau\vee\phi}p_{\tau\wedge\phi}$. From this, it follows that $c = 1$. $\square$

10.5. To an element $\tau = (i_1, \dots, i_d) \in H$, we associate the number

$$n_\tau = \sum_{t=1}^{d} N^{d-t} i_t,$$

where $N \gg 0$. We now carry out the flat deformation as described in Section 5, and obtain (using [11])

Theorem 10.6. *The flag variety $SL(n)/Q$ degenerates to a (normal) toric variety Y. Further, the Gröbner basis for $SL(n)/Q$ as constructed in Section 9 descends to a Gröbner basis for Y.*

11. Degenerations of Kempf varieties to toric varieties

We first recall (cf. [18]) the definition of Kempf varieties, in $SL(n)/B$. For defining these varieties, we need to consider the projections $\pi_r : G/B \to G/P$, $\pi_l : B\backslash G \to P\backslash G$, where P is a given parabolic subgroup of $G = SL(n)$. For $w \in W$, we shall denote by $X(w)_r$ (resp. $X(w)_l$) the Schubert variety in G/B (resp. $B\backslash G$). Let P_1 be the maximal parabolic subgroup corresponding to α_1, and π_l be the projection $B\backslash G \to P_1\backslash G$.

Definition 11.1. (cf. [18]) A Schubert variety $X(w)_r$ in G/B is a Kempf variety if

(1) $\pi_l\Big|_{X(w)_l} : X(w)_l \to ImX(w)_l$ is equidimensional.

(2) $X(w)_l \cap B\backslash P_1$ is irreducible.

(3)Let $X(w)_l \cap B\backslash P_1 = Y(w')_l$. Then $Y(w')_r$ is a Kempf variety in P_1/B.

Remark 11.2. Note that $SL(n)/B$ itself is a Kempf variety.

Remark 11.3. It is possible to define a Kempf variety $X(w)$ purely algebraically, in terms of a reduced expression for w, as well as purely combinatorially, in terms of the permutation corresponding to w (cf. [18]).

Remark 11.4. Recall (see ([31]) for example) that for $SL(n)/P$ (P being a maximal parabolic subgroup), we have the nice phenomenon that a monomial $\mathbf{m}$ standard on G/P when restricted to a Schubert variety X, either vanishes on X or remains standard on X. This no longer holds when P is not a maximal parabolic subgroup. For example, in $SL(3)/B$, the monomial p_2p_{13} while being standard on $SL(3)/B$ is not standard on $X(w)$ where w is the permutation (312) (since the minimal defining tableau for $\lambda = \{(2),(13)\}$ is given by $\{(231),(132)\}$, and $X(w) \not\supseteq X(231)$). In fact, it is easy to see that on $X(w)$, $p_2 p_{13} = p_3 p_{12}$.

It is shown in [14] that a Kempf variety has the property that any standard monomial on $SL(n)/B$, when restricted to $X(w)$, either remains standard on $X(w)$, or is identically zero . Hence proceeding as in Section 10, we obtain

Theorem 11.5. *Let X be a Kempf variety.Then X degenerates to a (normal) toric variety Y. Further, the Gröbner basis for X as constructed in Section* 9 *descends to a Gröbner basis for Y.*

12. Degenerations of determinantal varieties

Let D_n be a determinantal variety, the subvariety in the space $\mathcal{M}_{m,s}$ of $m \times s$ matrices with entries in k, for some $m, s > n$, defined by the vanishing of all $(n+1) \times (n+1)$ minors. It is known (cf. [25]) that D_n can be identified with the opposite big cell, $B^- e_{\mathrm{id}} \cap X(w)$ of $X(w)$, where $X(w)$ is the Schubert variety $X(w)$ in $G_{s,m+s} (= SL_{m+s}/P_s$, notations being as in 8.6) given by

$$w = (n+1, n+2, \dots, s, m+s+1-n, m+s+2-n, \dots, m+s)$$

(here B^- is the subgroup in $SL(n)$ consisting of all the upper triangular matrices).

Let $R = k[D_n]$ and $\check{H}_w = H_w \setminus \{(1,\dots,s)\}$ where $H_w = \{\tau \in I_{s,m+s} \mid \tau \leq w\}$. Then R is a quotient of the polynomial ring $k[x_\alpha, \alpha \in \check{H}_w]$. The relations are simply induced from those on $X(w)$. To be very precise

$$R = k[x_\alpha, \alpha \in \check{H}_w]/I$$

where I is generated by the straightening relations of the following form. Let τ, ϕ be two noncomparable elements in $\check{H}_w$. Let

$$p_\tau p_\phi = \sum c_{\alpha\beta} p_\alpha p_\beta$$

be the straightening relation for $p_\tau p_\phi$ on $X(w)$. Restricting this relation to D_n, we obtain a relation

$$X_\tau X_\phi = X_{\tau\vee\phi} X_{\tau\wedge\phi} + \sum c_{\alpha\beta} X_\alpha X_\beta,$$

where X_β is understood as 1 if $\beta = (1,\dots,s)$ (here, for $\alpha \in \check{H}_w$, $X_\alpha = x_\alpha (mod\ I)$). We carry out the flat deformation as in Section 5; to each $\tau \in \check{H}_w$, we associate the number

$$m_\tau = \sum_{t=1}^{s} N^{m-t}(i_t - t).$$

We first observe that given τ, ϕ two non-comparable elements in $\check{H}_w$, $\tau \wedge \phi >$ id $= (1,\dots,s)$ (since in $G_{s,m+s}$, there is a unique one-dimensional Schubert variety, namely $X(\theta)$, $\theta=(1,\dots,s-1,s+1)$. Also, for the same reasons, we have, $\check{H}_w$ is a distributive lattice (the element 0 being $(1,\dots,s-1,s+1)$). The proof of Theorem 5.3 goes through exactly as in Section 5. (Note that while proving the generation of $\mathcal{R}$ as a $k[t]$-module by $\mathcal{S}$, if $F=X_{\tau_1}\dots X_{\tau_r}$ is such that $\tau_i\not\geq\tau_{i+1}$ for some i, then in the relation

$$X_\tau X_\phi=\sum c_{\alpha\beta}X_\alpha X_\beta t^{m_\alpha+m_\beta-m_\tau-m_\phi},$$

where $\tau=\tau_i$, $\phi=\tau_{i+1}$, we can have a term of the form $X_\alpha t^{m_\alpha-m_\tau-m_\phi}$ (corresponding to $\beta=(1,\dots,s)$). Nevertheless, we have,

$$X_{\tau_1}\dots X_{\tau_{i-1}}X_\alpha X_{\tau_{i+2}}\dots X_{\tau_r}\preceq_{lex}F.$$

Thus $F=\sum a_iF_i$, with each $F_i\preceq_{lex}F$. The rest of the argument is as in Section 5. Thus we obtain (as in Section 10)

Theorem 12.1. *The determinantal variety D_n in $\mathcal{M}_{r,s}$ degenerates to a (normal) toric variety Y. Further, the Gröbner basis for D_n as constructed in* [9] *descends to a Gröbner basis for Y.*

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