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A NEW FAMILY OF IRREDUCIBLE REPRESENTATIONS OF A_n

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0. **Introduction.** For a simple Lie algebra L over the complex numbers $\mathbb{C}$ all irreducible representations admitting a highest weight have been constructed and characterized for example in [3, 6]. In [1] Bouwer considered the family of all irreducible representations of L admitting at least one one-dimensional weight space (this includes, of course, all those having a highest weight space) and showed, by construction, that this is a strictly larger class of representations. A complete characterization of this family of irreducible representations requires more information about existence. In this paper we shall construct and study a large new family of irreducible representations having a one-dimensional weight space.

1. **The Lie Algebra** A_n. The Lie algebra A_n consists of all complex square matrices of order $n+1$ having zero trace with the usual matrix addition and commutation product. Using the notation of [2] a Cartan subalgebra H of A_n is the (maximal abelian) subalgebra of diagonal matrices in A_n. Letting w_i denote the projection of any square matrix of order $n+1$ onto its (i, i)th component then the set of all roots Δ of A_n with respect to H is $\{w_i-w_j \mid i\neq j,\ i,j=1, 2, \ldots, n+1\}$. A simple set of roots Δ^{++} is $\{w_i-w_{i+1} / i=1, 2, \ldots, n\}$ and ordering the roots Δ with respect to Δ^{++} the set of positive roots of A_n is $\Delta^+=\{w_i-w_j \mid 1\leq i<j\leq n+1\}$. For each $i=1, 2, \ldots, n$ we set $h_i=E(i, i)-E(i+1, i+1)$ and for each $\xi=w_i-w_j\in\Delta$ we set $x_\xi=E(i,j)$ (where $E(k, l)$ denotes the matrix of order $n+1$ having 1 in (k, l)th position and zero elsewhere). The elements x_ξ for each $\xi\in\Delta$ is in the ξ root space of A_n with respect to H. A linear basis of A_n is given by

$$\{h_i, x_\xi \mid i = 1, 2, \ldots, n; \qquad \xi\in\Delta\}$$

The commutation product in A_n is completely described by

$$
\begin{aligned}
&[h_i, h_j] = 0 && \text{for } i, j = 1, 2, \ldots, n\\
&[h_i, x_\xi] = \xi(h_i)x_\xi && \text{for } i = 1, \ldots, n \text{ and } \xi\in\Delta\\
(1)\qquad &[x_\xi, x_\eta] = h_i+h_{i+1}+\cdots+h_{j-1} && \text{for } -\eta = \xi = w_i-w_j\in\Delta^+\\
& = -h_i-h_{i+1}-\cdots-h_{j-1} && \text{for } \eta = -\xi = w_i-w_j\in\Delta^+\\
& = (\delta_{jk}-\delta_{li})x_{\xi+\eta} && \text{for } \eta\neq\xi \text{ with}\\
& && \quad \xi = w_i-w_j; \qquad \eta = w_k-w_l.
\end{aligned}
$$

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2. **Construction of Representations of A_n.** Let V denote a complex vector space with basis $\{v(\mathbf{k}) \mid \mathbf{k}=(k_1, \ldots, k_n) \in \mathbb{Z}^n\}$. Fix a complex parameter s and a linear functional $\lambda \in H^*$ and define

$$\rho(h_i)v(\mathbf{k}) = (\lambda(h_i)-k_{i-1}+2k_i-k_{i+1})v(\mathbf{k}) \qquad \text{for} \quad i = 1, 2, \ldots, n$$

$$\rho(x_\xi)v(\mathbf{k}) = (s-\lambda(h_1+\cdots+h_{i-1})-k_{i-1}+k_i)v(\mathbf{k}+\boldsymbol{\xi}) \tag{2}$$

$$\rho(x_{-\xi})v(\mathbf{k}) = (s-\lambda(h_1+\cdots+h_{j-1})-k_{j-1}+k_j)v(\mathbf{k}-\boldsymbol{\xi}) \qquad \text{where} \quad \xi = w_i-w_j \in \Delta^+$$

(Note i) $\boldsymbol{\xi} \equiv$ the n-tuple having 1 in the i, $i+1, \ldots, j-1$ components and 0 elsewhere

(ii) by convention $h_0=0$ and $k_0=k_{n+1}=0$)

By direct computations one can verify that ρ preserves the commutation products (1) and hence extending ρ linearly to A_n we have a representation of A_n on the vector space V. Since for each $\mathbf{k}=(k_1, \ldots, k_n) \in \mathbb{Z}^n$ the vector $v(\mathbf{k})$ belongs to the $\lambda+\sum_{i=1}^n k_i(w_i-w_{i+1})$ weight space of this representation the weight lattice consists of $\{\lambda+\sum l_i(w_i-w_{i+1}) \mid l_i \in \mathbb{Z}\}$ and each weight space is one dimensional.

Now for a fixed linear functional $\lambda \in H^*$ if $s \in \mathbb{C}$ such that

$$s \notin \bigcup_{i=0}^{n} (\mathbb{Z}+\lambda(h_1+\cdots+h_i))$$

the above representation is irreducible. In fact this restriction on s insures that each of the scalar coefficients in (2) is non-zero and hence the representation is cyclic, generated by any basis vector $v(\mathbf{k})$. Now for any non-zero vector $v \in V$ the subrepresentation generated by v contains at least one basis vector since each basis vector belongs to a distinct weight space. Therefore V is generated by any non-zero vector.

If, on the other hand, $\lambda \in H^*$ is fixed and $s \in \bigcup_{i=0}^n (\mathbb{Z}+\lambda(h_1+\cdots+h_i))$, for definiteness suppose $s=\lambda(h_1+\cdots+h_{i-1})+m$, then the subspace W of V generated by $\{v(k_1, \ldots, k_n) \mid k_{i-1}-k_i \geq m\}$ is a proper subrepresentation. Thus we have the following

PROPOSITION 1. *To each complex scalar s and each linear functional $\lambda \in H^*$ we have constructed a representation which we shall denote $V_{s,\lambda}$ of A_n having a weight lattice $\{\lambda+\sum_{i=1}^n l_i(w_i-w_{i+1}) \mid l_i \in \mathbb{Z}\}$. This representation is irreducible iff $s \notin \bigcup_{i=0}^n (\mathbb{Z}+\lambda(h_1+\cdots+h_i))$.*

We now wish to analyze the equivalence classes of these representations. If $\lambda, \lambda' \in H^*$ such that $\lambda'-\lambda \notin \sum_{i=1}^n \mathbb{Z}(w_i-w_{i+1})$ then the representations $V_{s,\lambda}$ and $V_{t,\lambda'}$ have different weight lattices and hence are not equivalent.

If, on the other hand, $\lambda'-\lambda=\sum_{i=1}^{n} l_i(w_i-w_{i+1})$ where $l_i\in\mathbb{Z}$ for all i then the map

$$\phi: V_{s,\lambda}\to V_{t,\lambda'}$$

defined for each $(k_1,\ldots,k_n)\in\mathbb{Z}^n$ by

$$\phi(v(k_1,\ldots,k_n)) = v(k_1-l_1,\ldots,k_n-l_n)$$

yields an equivalence between $V_{s,\lambda}$ and $V_{t,\lambda'}$ provided $t=s+l_1$. Thus we have

PROPOSITION 2. *Every representation $V_{t,\lambda'}$ defined above is equivalent to exactly one representation $V_{s,\lambda}$ where $\lambda=\sum_{i=1}^{n}\rho_i(w_i-w_{i+1})$ with $0\le\operatorname{Re}\rho_i<1$.*

3. **New Irreducible Representations of other Simple Lie Algebras.** We now make use of the representations which we have constructed for A_n in order to obtain new irreducible representations of simple Lie algebras other than the A_n-series.

Each weight space of the representation $V_{s,\lambda}$ is a one-dimensional representation of $C(A_n)$, the centralizer of the Cartan subalgebra H of A_n in the universal enveloping algebra $U(A_n)$. Thus, for example, the map $\gamma: C(A_n)\to\mathbb{C}$ determined by

$$\rho(c)v(\mathbf{0}) = \gamma(c)v(\mathbf{0}) \qquad (c\in C(A_n))$$

is an algebra homomorphism.

Now consider an arbitrary simple Lie algebra L whose system of roots Δ contains a "complete" subsystem Δ_0 isomorphic to the root system of A_n. If $H(L)$ denotes a fixed Cartan subalgebra of L and $C(L)$ denotes the centralizer of $H(L)$ in the universal enveloping algebra $U(L)$ of L then $C(L)$ contains an isomorphic copy of $C(A_n)$. In [5] we have shown that the algebra homomorphism γ defined above can be trivially extended to an algebra homomorphism $\hat\gamma: C(L)\to\mathbb{C}$. Using the construction in [4] we know that there exists a unique maximal left ideal $M_{\hat\gamma}$ of $U(L)$ containing $\ker\hat\gamma$. Provided the parameter $s\notin\bigcup_{i=0}^{n}(\mathbb{Z}+\lambda(h_1+\cdots+h_i))$, we claim the left regular representation of L on $U/M_{\hat\gamma}$ is a standard representation of L of order n. Conditions (iii) and (iv) of definition 3.1 in [1] are obviously satisfied thus it suffices to show that for each simple root $\alpha\in\Delta_0$ the α-ladder through $\hat\lambda=\hat\gamma\downarrow H(L)$ is doubly infinite and for each positive root $\beta\in\Delta$ with $\beta\notin\Delta_0$, $\hat\lambda+\beta$ is not a weight of $U(L)/M_{\hat\gamma}$.

Now for each simple root $\alpha\in\Delta_0$

$$\hat\gamma(X_{-\alpha}^n X_\alpha^n) = \gamma(X_{-\alpha}^n X_\alpha^n) = \text{coefficient of } \rho(X_{-\alpha}^n X_\alpha^n)v(\mathbf{0}) \neq 0$$

(due to the condition on s). Similarly $\hat\gamma(X_\alpha^n X_{-\alpha}^n)\neq 0$. Thus $X_{-\alpha}^n, X_\alpha^n\notin M_{\hat\gamma}$ for all $n\in\mathbb{Z}$ which implies that $\hat\lambda+n\alpha$ is a weight of $U(L)/M_{\hat\gamma}$ for all $n\in\mathbb{Z}$. For any positive root $\beta\in\Delta$, $\beta\notin\Delta_0$ every element of $U(L)$ having mass β belongs to $M_{\hat\gamma}$ (cf. Theorem 4.4 [1]). Thus $\hat\lambda+\beta$ is not a weight of $U(L)/M_{\hat\gamma}$.

The root systems of the simple Lie algebras B_k, C_k and D_k each contain complete subsystems of roots isomorphic to the root system of A_n for $n \leq k-1$. The root systems of the exceptional simple Lie algebras G_2, F_4, E_6, E_7 and E_8 contain complete subsystems of roots isomorphic to the root system of A_n for $n \leq 1, 2, 5, 6$ and 7 respectively. Thus we have

PROPOSITION 3. *There exist standard irreducible representations of order less than or equal to n for the simple Lie algebras* B_{n+1}, C_{n+1}, *and* D_{n+1}. *The exceptional simple Lie algebras* G_2, F_4, E_6, E_7, *and* E_8 *admit standard irreducible representations of order less than or equal to* 1, 2, 5, 6, *and* 7 *respectively.*

BIBLIOGRAPHY

1. I. Z. Bouwer, *Standard Representations of Simple Lie Algebras*, Canad. J. Math. **70** (1968) 344–361.

2. H. Freudenthal, H. de Vries, *Linear Lie Groups*, London-New York: Academic Press 1969.

3. Harish-Chandra, *Some applications of the universal enveloping algebra of a semi-simple Lie algebra*, Trans. Amer. Math. Soc. **70** (1951), 28–99.

4. F. W. Lemire, *Weight Spaces and Irreducible Representations of Simple Lie Algebras*, Proc. Amer. Math. Soc. **22** (1969), 192–197.

5. F. W. Lemire, *One-dimensional Representations of the Cycle Subalgebra of a Semi-simple Lie Algebra*, Canad. Math. Bull. **13** (1970), 463–467.

6. Séminaire Sophus Lie, *Théorie des algébres de Lie Topologie des groupes Lie*, Paris: École Norm. Sup. 1954–55.