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Commuting Limits with Colimits

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1. INTRODUCTION

Let $\mathfrak{C}$ be a category, let I, J be two index categories and let $F: I \times J \to \mathfrak{C}$ be a functor. Then for each $j \in |J|$ we have a functor $F_j : I \to \mathfrak{C}$ and we may take the limit $R_j = \varprojlim_I F_j$. We may then take the colimit $\varinjlim_J R_j$ and refer to the double limit as

$$\varinjlim_J \varprojlim_I F.$$

Similarly we may construct $\varprojlim_I \varinjlim_J F$ and there is a natural transformation (of functors $\mathfrak{C}^{I \times J} \to \mathfrak{C}$)

$$\omega = \omega_F : \varinjlim_J \varprojlim_I F \to \varprojlim_I \varinjlim_J F. \tag{1.1}$$

The object of this paper is to study conditions under which ω in (1.1) is a natural equivalence. The transformation ω, together with the question when ω is an equivalence, have already been considered by Roos [8], under the heading of the distributivity of $\lim_{\leftarrow}$ with respect to $\lim_{\rightarrow}$, but he worked in a special category (of faisceaux or topos) and showed that distributivity holds there when one distributes limits with respect to sums, no matter what the functor F. In this work we are primarily concerned to show that, if $\mathfrak{C}$ is abelian and I, J are suitable, then ω is an equivalence for certain functors F. For, clearly, if $\mathfrak{C}, I, J$ are fairly general, then it cannot be the usual case that ω is an equivalence. Consider, for example, the following situation. Let $\mathfrak{Ab}$ be the category of abelian groups and let $D(i, j)$ be a doubly-indexed family of non-zero objects of $\mathfrak{Ab}$, $0 \leqslant i < \infty$, $0 \leqslant j < \infty$. Let $I^{\text{opp}} = J$ be the ordered set of nonnegative integers and let $F : I \times J \to \mathfrak{Ab}$ be given by

$$F(i_0, j_0) = \bigoplus_{\substack{i \leqslant i_0 \\ j \leqslant j_0}} D(i,j).$$

Then $F(i_0, j_0)$ projects onto $F(i_1, j_0)$ if $i_1 < i_0$, and $F(i_0, j_0)$ embeds in $F(i_0, j_1)$ if $j_0 < j_1$. For this functor F it is clear that

$$\varinjlim_J \varprojlim_I F = \bigoplus_j \prod_i D(i,j),$$

$$\varprojlim_I \varinjlim_J F = \prod_i \bigoplus_j D(i,j),$$

and ω is a monomorphism but not an isomorphism.

Thus F must be substantially restricted in order to achieve an equivalence ω in (1.1). Nevertheless, it turns out that two distinct cases when ω is an equivalence arise in the very foundations of spectral sequence theory, and it was the study of these two cases which led to the considerations which occupy the main part of this paper. We now describe these cases.

First, we recall from [4] how an exact couple

$$(1.2)\qquad \begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

in an abelian category $\mathfrak{A}$ gives rise to a diagram

$$(1.3)\qquad \begin{tikzcd} D_m \arrow[r, "\beta_{m,n}"] & E_{m,n} \arrow[r, "\gamma_{m,n}"] & D_n \arrow[d, tail, "\nu_n"] \\ D \arrow[u, two heads, "\eta_m"] \arrow[r, "\beta"] & E \arrow[r, "\gamma"] & D \end{tikzcd}, \quad D_n = \alpha^n D,$$

and hence to a commutative square

$$(1.4)\qquad \begin{tikzcd} E_{m,n+1} \arrow[r, two heads, "\sigma_{m,n+1}"] \arrow[d, tail, "\rho_{m,n}"'] & E_{m+1,n+1} \arrow[d, tail, "\rho_{m+1,n}"] \\ E_{m,n} \arrow[r, two heads, "\sigma_{m,n}"] & E_{m+1,n} \end{tikzcd}$$

Moreover, if $E_n = E_{nn}$ and $d_n = \beta_{nn}\gamma_{nn}$, then (E_n, d_n) is the spectral sequence associated with (1.2). It turns out that the limit of the spectral sequence, E_∞, is given by

$$E_\infty = \varinjlim_m \varprojlim_n (E_{mn}\,;\rho_{mn}, \sigma_{mn}) = \varprojlim_n \varinjlim_m (E_{mn}\,;\rho_{mn}, \sigma_{mn}) \tag{1.5}$$

(Theorem 4.13 of [*4*]), so that in this case ω is an isomorphism. We remark that, in fact, (1.4) and (1.5) are valid for any spectral sequence in $\mathfrak{A}$ without it being necessary to postulate an exact couple lying behind the spectral sequence.

Second let

$$\cdots X^{p-1} \subseteq X^p \subseteq \cdots \subseteq X \tag{1.6}$$

be a filtration of the object X in the abelian category $\mathfrak{A}$, and, for $q \leqslant p$, let $X_q{}^p$ be the quotient object $X_q{}^p = X^p/X^q$. Then there is a commutative square

$$(1.7)\quad \begin{tikzcd} X_q^p \arrow[r, tail, "\xi^p_{(q)}"] \arrow[d, "\xi_q^{(p)}"'] & X_q^{p+1} \arrow[d, "\xi_q^{(p+1)}"] \\ X_{q+1}^p \arrow[r, tail, "\xi^p_{(q+1)}"] & X_{q+1}^{p+1} \end{tikzcd}\quad ,\ q+1 \leq p.$$

It then turns out (Theorem 5.26 of [*5*]) that

$$\omega : \varinjlim_p \varprojlim_q (X_q{}^p;\, \xi_q{}^{(p)}, \xi^p_{(q)}) \cong \varprojlim_q \varinjlim_p (X_q^p\,;\, \xi_q^{(p)}, \xi^p_{(q)}). \tag{1.8}$$

Indeed the common double limit, $X^\infty_{-\infty}$, is precisely the object obtained in completing the filtration (1.6). However it will be noted that there is an essential algebraic distinction between (1.4) and (1.7) for whereas, in (1.4), we take limits over *monics* and colimits over *epics*, in (1.7) we take limits over *epics* and colimits over *monics*. On the other hand, (1.4) and (1.7) do have in common the property of being bicartesian (pull-backs and push-outs). This property, which is not possessed by the counterexample in $\mathfrak{Ab}$ given earlier, is basic to the arguments of this paper. The property, while restrictive, is by no means unreasonable. In the general setting of the functor $F : I \times J \to \mathfrak{C}$ we are asking that, if $\phi : i_2 \to i_1$ in I, $\psi : j_1 \to j_2$ in J, then

$$(1.9)\quad \begin{tikzcd} F_{i_2 j_1} \arrow[r, "\psi_{i_2}"] \arrow[d, "\phi_{j_1}"'] & F_{i_2 j_2} \arrow[d, "\phi_{j_2}"] \\ F_{i_1 j_1} \arrow[r, "\psi_{i_1}"] & F_{i_1 j_2} \end{tikzcd}$$

generalizing (1.4) and (1.7), be bicartesian. But (1.9) is the F-image of the square

$$(1.10) \qquad \begin{tikzcd} i_2 \times j_1 \arrow[r, "1 \times \psi"] \arrow[d, "\phi \times 1"] & i_2 \times j_2 \arrow[d, "\phi \times 1"] \\ i_1 \times j_1 \arrow[r, "1 \times \psi"] & i_1 \times j_2 \end{tikzcd}$$

in $I \times J$, which, of course, is bicartesian. The main results of this paper give hypotheses on the square (1.9) and on the categories I and J which ensure that (when $\mathfrak{C}$ is abelian) ω in (1.1) is an isomorphism.

The plan of the paper is as follows. In section 2 we define the natural transformation ω (assuming only that the limits exist and that I, J are connected) and show that the definition is selfdual. We also compare the main problem with the situation of commuting two limits or two colimits. These situations, of course, present no problem and the commutability of limits (and colimits) is exploited in section 3. In that section we describe the restrictions to be placed on the categories I and J. Apart from the requirement that they be small (to avoid foundational difficulties) these restrictions are those discussed by Artin and Grothendieck ([*1*], [*2*]) among others to ensure good properties of limits; we use their terminology of *quasi-filtered* and quasi-cofiltered categories. We also introduce in this section a notion of cofinality which is a very natural extension of the classical notion for directed sets; generalizations and specializations of this notion of cofinality certainly exist in literature and folklore[1].

In section 4 we consider the square (1.9) which we embed in what we call the *key diagram* (4.3). A general proposition is proved (Proposition 4.4) giving conditions on the diagram under which ω is an isomorphism; and special cases, including, of course, the two cases discussed above, are deduced from it. In the final section we further specialize the category $\mathfrak{C}$ to be a category of modules. This enables us to relax the requirements on the functor F, for we can exploit the fact that, in a category of modules, the direct limit (= colimit) functor, with respect to a quasi-filtered category, is exact. It thus turns out that we may concentrate attention on the problem of when the inverse limit functor is exact. We base ourselves here on ([*3*], [*6*]) and introduce the notion of the *Mittag-Leffler property* for functors of quasi-cofiltered categories, generalizing the usual definition. We prove the natural generalization of Theorem 3.8 of [*3*], and Proposition 13.2.2 of [*6*] allowing

[1] Dold (unpublished) has exploited a very general form of this notion in studying cohomology theories on the category of topological spaces.

the domain category of our functors to be a *special* quasi-cofiltered category (although special, such a category is more general than a countable codirected set) and the range category to be the category of groups. Thus we prove that if F, G, H are such functors and

$$F \rightarrowtail G \twoheadrightarrow H$$

is exact, and if F has the Mittag-Leffler property, then

$$\varprojlim F \rightarrowtail \varprojlim G \twoheadrightarrow \varprojlim H$$

is exact (Theorem 5.5). This theorem allows us to establish conditions on the functor $F : I \times J \to \mathfrak{C}$ ($\mathfrak{C}$ a category of modules), considerably more general than those of section 4, under which ω in (1.1) is an isomorphism. These conditions are embodied in Theorem 5.12.

2. The Limit-Switching Transformation

We first recall some familiar notions and establish notation.

Let I be an index category and, for any category $\mathfrak{C}$, let $P = P(I, \mathfrak{C})$ be the embedding functor of $\mathfrak{C}$ into the functor category $\mathfrak{C}^I$,

$$P : \mathfrak{C} \to \mathfrak{C}^I.$$

Lemma 2.1. *A functor $F : \mathfrak{C} \to \mathfrak{D}$ induces a functor $F^I : \mathfrak{C}^I \to \mathfrak{D}^I$ such that $(F_1F_2)^I = F_1{}^IF_2{}^I$ and $F^IP = PF$. A natural transformation $\pi : F \to G : \mathfrak{C} \to \mathfrak{D}$ induces a natural transformation*

$$\pi^I : F^I \to G^I : \mathfrak{C}^I \to \mathfrak{D}^I$$

such that $\pi^IP = P\pi$, and $\pi^IS^I = (\pi S)^I$, $T^I\pi^I = (T\pi)^I$ for any $S : \mathfrak{C}' \to \mathfrak{C}$, $T : \mathfrak{D} \to \mathfrak{D}'$.

Given any category $\mathfrak{E}$ and index categories I, J we may identify by canonical isomorphism the categories

$$\mathfrak{E}^{I \times J} \cong (\mathfrak{E}^I)^J \cong (\mathfrak{E}^J)^I. \tag{2.2}$$

Indeed, if Ω is the second isomorphism in (2.2), then plainly

$$\Omega P^J(I, \mathfrak{E}) = P(I, \mathfrak{E}^J). \tag{2.3}$$

Our convention in what follows is always to write $\mathfrak{E}^{I \times J}$ for any of the three categories in (2.2) and then to understand that all embedding functors are related to the category $\mathfrak{E}$. Then Lemma 2.1 implies

PROPOSITION 2.4. *A functor $F : \mathfrak{E}^J \to \mathfrak{E}$ induces a functor $F^I : \mathfrak{E}^{I \times J} \to \mathfrak{E}^I$ such that*

$$F^I P^J = PF$$

where $P : \mathfrak{E} \to \mathfrak{E}^I$ is the embedding functor.

PROPOSITION 2.5. *A natural transformation $\pi : F \to G : \mathfrak{E}_1{}^J \to \mathfrak{E}_2{}^J$ induces a natural transformation $\pi^I : F^I \to G^I : \mathfrak{E}_1^{I \times J} \to \mathfrak{E}_2^{I \times J}$ such that $\pi^I P_1{}^J = P_2{}^J \pi$, where $P_i : \mathfrak{E}_i \to \mathfrak{E}_i{}^I$, $i = 1, 2$, are the embedding functors.*

We also have the trivial

PROPOSITION 2.6. *Let $P : \mathfrak{E} \to \mathfrak{E}^I$, $\underset{\sim}{Q} : \mathfrak{E} \to \mathfrak{E}^J$ be embedding functors. Then $P^J \underset{\sim}{Q} = \underset{\sim}{Q}^I P$.*

We now suppose I and J to be connected so that P and $\underset{\sim}{Q}$ are full embeddings. We further suppose that P has a right adjoint (right inverse) $R : \mathfrak{E}^I \to \mathfrak{E}$ and that $\underset{\sim}{Q}$ has a left adjoint (left inverse) $L : \mathfrak{E}^J \to \mathfrak{E}$. Thus there are natural transformations $\tau : PR \to 1 : \mathfrak{E}^I \to \mathfrak{E}^I$, $\pi : 1 \to \underset{\sim}{Q}L : \mathfrak{E}^J \to \mathfrak{E}^J$ and

$$RP = 1 : \mathfrak{E} \to \mathfrak{E}, \qquad \tau P = 1, \qquad R\tau = 1, \tag{2.7}$$

$$L\underset{\sim}{Q} = 1 : \mathfrak{E} \to \mathfrak{E}, \qquad \pi \underset{\sim}{Q} = 1, \qquad L\pi = 1. \tag{2.8}$$

From the naturality of τ, π we infer immediately

PROPOSITION 2.9. (i) *For any $\phi : H \to PX$, $\psi : PY \to K$, H, $K \in \mathfrak{E}^I$, X, $Y \in \mathfrak{E}$ we have*

$$PR\phi = \phi \circ \tau_H , \qquad \psi = \tau_K \circ PR\psi.$$

(ii) *For any $\phi : H \to \underset{\sim}{Q}X$, $\psi : \underset{\sim}{Q}Y \to K$, H, $K \in \mathfrak{E}^J$, X, $Y \in \mathfrak{E}$ we have*

$$\phi = \underset{\sim}{Q}L\phi \circ \pi_H , \qquad \underset{\sim}{Q}L\psi = \pi_K \circ \psi.$$

Moreover, the adjunction maps

$$\eta_R : \mathfrak{E}^I(PY, K) \cong \mathfrak{E}(Y, RK), \qquad \eta_L : \mathfrak{E}(LH, X) \cong \mathfrak{E}^J(H, \underset{\sim}{Q}X)$$

are given by

$$\begin{aligned} \eta_R &= R, \qquad \eta_R^{-1}(\zeta) = \tau_K \circ P\zeta, \qquad \zeta : Y \to RK \\ \eta_L^{-1} &= L, \qquad \eta_L(\theta) = P\theta \circ \pi_H , \qquad \theta : LH \to X. \end{aligned} \tag{2.10}$$

We will write

$$\mathbf{P} \overset{\boldsymbol{\tau}\cdot}{\vdash} \mathbf{R}, \qquad \mathbf{L} \overset{\cdot\boldsymbol{\pi}}{\vdash} \mathbf{Q} \tag{2.11}$$

to express the adjunction relations (2.7), (2.8).

PROPOSITION 2.12. $P^J \overset{\tau\cdot^J}{\vdash} R^J$, $L^I \overset{\cdot\pi^I}{\vdash} \underset{\sim}{Q}^I$.

We now consider the two functors

$$LR^J, RL^I : \mathfrak{C}^{I\times J} \to \mathfrak{C}. \tag{2.13}$$

In view of Proposition 2.12 it is legitimate to write

$$LR^J = \varinjlim_J \varprojlim_I , \qquad RL^I = \varprojlim_I \varinjlim_J .$$

Since $\tau^J : P^J R^J \to 1$, it follows that

$$RL^I \tau^J : RL^I P^J R^J \to RL^I,$$

but

$$\begin{aligned} RL^I P^J R^J &= RPLR^J && \text{by Proposition 2.4} \\ &= LR^J && \text{by (2.7).} \end{aligned}$$

Thus

$$\omega = RL^I \tau^J : LR^J \to RL^I : \mathfrak{C}^{I\times J} \to \mathfrak{C}. \tag{2.14}$$

Similarly,

$$\omega' = LR^J \pi^I : LR^J \to RL^I : \mathfrak{C}^{I\times J} \to \mathfrak{C}. \tag{2.15}$$

Plainly ω and ω' are defined by dual procedures. The main theorem of this section is

THEOREM 2.16. $\omega = \omega'$.

We first prove a preliminary lemma. Denote the embedding $P^J \underset{\sim}{Q} = \underset{\sim}{Q}^I P : \mathfrak{C} \to \mathfrak{C}^{I\times J}$ by $\bar{P}$. Then let $A : \mathfrak{C} \to \mathfrak{C}^I$, $B : \mathfrak{C} \to \mathfrak{C}^J$ be functors and $\lambda : LB \to RA : \mathfrak{C} \to \mathfrak{C}$ a natural transformation.

LEMMA 2.17. *Let* $\mu = \underset{\sim}{Q}^I \tau A \circ \bar{P}\lambda \circ P^J \pi B : P^J B \to \underset{\sim}{Q}^I A : \mathfrak{C} \to \mathfrak{C}^{I\times J}$. *Then*

$$\lambda = RL^I \mu = LR^J \mu.$$

Proof of lemma. We have

$$\begin{aligned} \mu &= \underset{\sim}{Q}^I \tau A \circ \bar{P}\lambda \circ P^J \pi B \\ &= \underset{\sim}{Q}^I(\tau A \circ P\lambda) \circ \pi^J PB \text{ by Proposition 2.5.} \end{aligned}$$

Thus $L^I\mu = \tau A \circ P\lambda$, $RL^I\mu = \lambda$ by (2.8) and (2.7). Similarly $LR^J\mu = \lambda$.

Proof of theorem. Consider

$$P^J R^J \xrightarrow{P^J \pi R^J} \bar{P}LR^J \underset{\bar{P}\omega'}{\overset{\bar{P}\omega}{\rightrightarrows}} \bar{P}RL^I \xrightarrow{\underset{\sim}{Q}^I \tau L^I} \underset{\sim}{Q}^I L^I.$$

Then

$$\begin{aligned} \underset{\sim}{Q}^I \tau L^I \circ \bar{P}\omega \circ P^J \pi R^J &= \underset{\sim}{Q}^I(\tau L^I \circ PRL^I \tau^J) \circ \pi^I P^J R^J \text{ by Proposition 2.5} \\ &= \underset{\sim}{Q}^I L^I \tau^J \circ \pi^I P^J R^J \text{ by Proposition 2.9 (i)} \\ &= \pi^I \circ \tau^J \text{ by the naturality of } \pi^I \text{ (or } \tau^J\text{).} \end{aligned}$$

We obtain, by the dual argument, the same equality as above with ω' replacing ω. But in any case we may invoke Lemma 2.17 to infer that

$$\omega = LR^J(\pi^I \circ \tau^J) = LR^J \pi^I = \omega'.$$

Thus we have established the existence of a natural self-dual limit-switching transformation $\omega = \omega' : LR^J \to RL^I$, or, as we may write it,

$$\omega_F = \omega'_F : \varinjlim_J \varprojlim_I F \to \varprojlim_I \varinjlim_J F, \qquad F \in \mathfrak{C}^{I \times J}$$

In the next sections we will consider under what conditions on $\mathfrak{C}, I, J, F$ it will happen that ω_F is an isomorphism. We close this section by rendering precise the fact that we may always commute two limits or two colimits. (Of course, this is only a special case of the fact that right adjoints commute with limits, and left adjoints commute with colimits.)

Thus (to concentrate on limits) we now suppose that, as before, $P \overset{\tau \cdot}{\longmapsto} R$, and, moreover,

$$\underset{\sim}{Q} \overset{\kappa \cdot}{\longmapsto} S, \qquad S : \mathfrak{C}^J \to \mathfrak{C}, \qquad \kappa : \underset{\sim}{Q}S \to 1 : \mathfrak{C}^J \to \mathfrak{C}^J.$$

Then we may show (see [7])

Proposition 2.18. *$\bar{P} \overset{(\kappa^I \circ \underset{\sim}{Q}^I \tau S^I)\cdot}{\longmapsto} RS^I$, $\bar{P} \overset{(\tau^J \circ P^J \kappa R^J)\cdot}{\longmapsto} SR^J$, and there is a natural equivalence*

$$SR^J \kappa^I : RS^I \to SR^J$$

with inverse $RS^I \tau^J$.

If we write χ for the natural equivalence $\chi = SR^J \kappa^I$, then we have

$$\chi_F : \varprojlim_I \varprojlim_J F \cong \varprojlim_J \varprojlim_I F, \qquad F \in \mathfrak{C}^{I \times J}. \tag{2.19}$$

There is, of course, a dual result for colimits.

3. Quasifiltered and Quasicofiltered Categories

In this section we are concerned with the restrictions we intend to impose on the index categories I, J of the preceding section. Of course, they are to be small non-empty connected categories. We say that J is *quasi-filtered* if it also has the following two properties[2]:

(3.1) given
$$\begin{tikzcd} & {} \\ {} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & {} \end{tikzcd}$$
in J, we may find
$$\begin{tikzcd} & {} \arrow[dr, "\gamma"] & \\ {} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & {} \\ & {} \arrow[ur, "\delta"'] & \end{tikzcd}$$
in J, with $\gamma\alpha = \delta\beta$

(3.2) given $\overset{\rho}{\underset{\sigma}{\rightrightarrows}}$ in J, we may find $\overset{\rho}{\underset{\sigma}{\rightrightarrows}} \xrightarrow{\eta}$ in J with $\eta\rho = \eta\sigma$.

By dualizing these requirements (we will refer to the duals as (3.1)d, (3.2)d) we get the notion of a *quasi-cofiltered* category. We write, for brevity, qf, qf' for quasi-filtered, quasi-cofiltered. *Examples of qf-categories*: (i) any directed set; (ii) any small connected category with terminal object; (iii) any small connected category cocomplete with respect to finite diagrams.

We wish to consider certain full subcategories J_0 of a qf-category J. First, however, we make an elementary but important observation showing that J resembles a directed set quite closely.

Proposition 3.3. *Let $j_1, j_2 \in |J|$. Then there is a diagram*
$$\begin{tikzcd} j_1 \arrow[dr] & \\ & j \\ j_2 \arrow[ur] & \end{tikzcd}$$
in J.

Proof. Since J is connected there is certainly a diagram
$$k^1 \to k^2 \leftarrow k^3 \to k^4 \cdots k^{2n-1} \to k^{2n} \leftarrow k^{2n+1} \tag{3.4}$$
in J with $k^1 = j_1$, $k^{2n+1} = j_2$. If $n = 1$, we are home. If $n > 1$ we use (3.1) to reduce the length of the chain (3.4). For we find
$$\begin{tikzcd} & k_2 \arrow[dr] & \\ k_3 \arrow[ur] \arrow[dr] & & l \\ & k_4 \arrow[ur] & \end{tikzcd}$$

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[2] This terminology is essentially due to Grothendieck, but we insist that quasi-filtered and quasi-cofiltered categories be connected. See ([1], [2]).

in J and hence construct

$$k^1 \to l \leftarrow k^5 \cdots k^{2n-1} \to k^{2n} \leftarrow k^{2n+1}$$

in J.

The full subcategory J_0 is then to be *cofinal* in J. That is to say, given $j \in |J|$, we require a morphism

$$j \to j_0$$

in J with $j_0 \in |J_0|$.

PROPOSITION 3.4. *Let J_0 be a non-empty full subcategory of J with the property that if $\phi : j_0 \to j$ in J and $j_0 \in |J_0|$ then $\phi \in J_0$. Then J_0 is cofinal in J.*

Proof. Let $k \in |J|$. Choose any $j_0 \in |J_0|$. By Proposition 3.3 there exists a diagram

$$k \to j_1 \leftarrow j_0$$

in J. But then $j_1 \in |J_0|$ by hypothesis, so that J_0 is cofinal in J.

It is clear that the property attributed to J_0 in this proposition is more special than cofinality: take, for example, any infinite subsequence of an infinite sequence. This is cofinal but only right-hand sections enjoy the special property.

We omit the proof of the following proposition.

PROPOSITION 3.5. *A cofinal subcategory of a qf-category is qf. A cofinal subcategory of a cofinal subcategory is cofinal.*

We now suppose J_0 cofinal in the qf-category J; let $E : J_0 \subseteq J$ be the embedding, inducing

$$E^* : \mathfrak{C}^J \to \mathfrak{C}^{J_0}.$$

If $\underset{\sim}{Q} : \mathfrak{C} \to \mathfrak{C}^J$, $\underset{\sim}{Q}_0 : \mathfrak{C} \to \mathfrak{C}^{J_0}$ are the embeddings of Section 2, then

$$\underset{\sim}{Q}_0 = E^* \underset{\sim}{Q}. \tag{3.6}$$

We further suppose the existence of a left adjoint to $\underset{\sim}{Q}_0$,

$$L_0 \overset{\cdot \pi_0}{\longmapsto} \underset{\sim}{Q}_0, \qquad L_0 : \mathfrak{C}^{J_0} \to \mathfrak{C}, \qquad \pi_0 : 1 \to \underset{\sim}{Q}_0 L_0 : \mathfrak{C}^{J_0} \to \mathfrak{C}^{J_0}$$

$$L_0 \underset{\sim}{Q}_0 = 1, \qquad \pi_0 \underset{\sim}{Q}_0 = 1, \qquad L_0 \pi_0 = 1.$$

Set $L = L_0 E^* : \mathfrak{C}^J \to \mathfrak{C}$. We prove

THEOREM 3.7. *There exists a unique $\pi : 1 \to \underset{\sim}{Q} L : \mathfrak{C}^J \to \mathfrak{C}^J$ such that $E^* \pi = \pi_0 E^*$ and*

$$L \overset{\cdot \pi}{\longmapsto} \underset{\sim}{Q}.$$

In other words, colimits over J coincide with colimits over J_0. However we should be cautious over using this imprecise formulation since we do not postulate that functors $J_0 \to \mathfrak{C}$ can be extended to J. For example, let

$$J = \begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "\gamma"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \cdot \\ & \cdot \arrow[ur, "\delta"'] & \end{tikzcd}, \quad J_0 = \begin{tikzcd}[column sep=small] \cdot \arrow[dr, "\gamma"] & \\ & \cdot \\ \cdot \arrow[ur, "\delta"'] & \end{tikzcd}, \quad \mathcal{E} = \begin{tikzcd} \cdot \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \cdot \end{tikzcd}, \quad F_0(\gamma) = \phi, \; F_0(\delta) = \psi.$$

$$\phi \neq \psi$$

Then plainly F_0 has no extension to $F : J \to \mathfrak{C}$.

Proof of theorem. Let $F : J \to \mathfrak{C}$ be a functor. Write F_i for $F(i)$, $i \in |J|$, and, by abuse, write $\phi : F_i \to F_j$ for $F(\phi) : F_i \to F_j$ where $\phi : i \to j$ in J. Also write L for $L(F)$ and π_i for $(\pi_0)_{FE}(i)$, $i \in |J_0|$, so that $\pi_i : F_i \to L$, $i \in |J_0|$. Then

$$\pi_j \phi = \pi_i \qquad \text{for all} \quad \phi : i \to j \quad \text{in} \quad J_0, \tag{3.8}$$

and we have the universal property: if $\rho_i : F_i \to M$, $i \in |J_0|$, such that $\rho_j \phi = \rho_i$ for all ϕ in J_0, then there exists a unique $\rho : L \to M$ with $\rho_i = \rho \pi_i$. Of course, ρ is given by $\rho = L_0(\rho_0)$, where $\rho_0 : FE \to Q_0 M$ is defined by $\rho_0(i) = \rho_i$.

The main burden of the theorem is that we may define $\pi_i : F_i \to L$, uniquely, over the whole of J so that (3.8) then also holds in J and the universal property extends to morphisms $\rho_0 : F \to QM$.

We now establish these facts. Let $i \in |J|$. Then, there exists $\phi : i \to j$ in J with $j \in |J_0|$ and we set

$$\pi_i = \pi_j \phi : F_i \to L. \tag{3.9}$$

Plainly this agrees with the given π_i if $i \in |J_0|$, by (3.8). Also, since (3.8) is to hold throughout J, this is the only possible value to assign to π_i. Thus the question of uniqueness is disposed of, and it remains to show that, with the definition (3.9), (3.8) does in fact hold throughout J—and that the universal property also holds.

In fact we show that the definition (3.9) is independent of the choice of ϕ. Suppose also given $\psi : i \to k$ with $k \in |J_0|$. By (3.1) and the cofinality of J_0 in J it follows that there is a diagram

$$\begin{tikzcd}[column sep=small] & j \arrow[dr, "\alpha"] & \\ i \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & & l \\ & k \arrow[ur, "\beta"'] & \end{tikzcd}$$

with $l \in | J_0 |$ and $\alpha\phi = \beta\psi$. But since J_0 is full, α and β are in J_0 so $\pi_j = \pi_l\alpha : F_j \to L$, $\pi_k = \pi_l\beta : F_k \to L$. Thus[3]

$$\pi_j\phi = \pi_l\alpha\phi = \pi_l\beta\psi = \pi_k\psi.$$

Now let $\phi : i \to j$ in J. Then there exists $\psi : j \to k$ in J with k in $| J_0 |$ and, by (3.9)

$$\pi_j = \pi_k\psi, \qquad \pi_i = \pi_k\psi\phi.$$

Thus $\pi_i = \pi_j\phi$ and (3.8) is established throughout J.

To establish the universal property we suppose $\rho_i : F_i \to M$, for all $i \in | J |$ with $\rho_j\phi = \rho_i$ for all $\phi : i \to j$ in J. We must show that $\rho_i = \rho\pi_i$ for all $i \in | J |$. Given such an i and $\phi : i \to j$ with $j \in | J_0 |$, then

$$\begin{aligned} \rho_i &= \rho_j\phi \\ &= \rho\pi_j\phi, && \text{by the universal property of } L \text{ in } J_0 \\ &= \rho\pi_i, && \text{by definition of } \pi_i . \end{aligned}$$

The naturality of π is evident so that the theorem is completely proved.

So far no use has been made of condition (3.2), so that all our results in this section are valid even if (3.2) does not hold in J. We now, however, make our main application of qf-categories and in this application we make essential use of (3.2).

Let $\kappa : A \to B : J \to \mathfrak{C}$ be a natural transformation. Thus for each $\psi : j \to k$ in J we have a commutative diagram (in obvious notation, as already employed)

$$\text{(3.10)} \qquad \begin{array}{ccc} A_j & \xrightarrow{\psi} & A_k \\ \downarrow{\scriptstyle \kappa_j} & & \downarrow{\scriptstyle \kappa_k} \\ B_j & \xrightarrow{\psi} & B_k \end{array}$$

Let $L \overset{\cdot\pi}{\longmapsto} Q$ as in §2 (and Theorem 3.7). Then we obtain, for each $i \in | J |$, the commutative diagram

$$\text{(3.11)} \qquad \begin{array}{ccc} A_i & \xrightarrow{\pi_i} & LA \\ \downarrow{\scriptstyle \kappa_i} & & \downarrow{\scriptstyle L\kappa} \\ B_i & \xrightarrow{\pi_i} & LB \end{array}$$

[3] Notice that we may have $j = k$.

We call the passage from (3.10) to (3.11) the *L-process*; there is, of course, a dual *R-process*.

We assert

THEOREM 3.12. *Let J be a qf-category. If* (3.10) *is a push-out diagram for all ϕ in J, then* (3.11) *is a push-out diagram for all $i \in |J|$. That is, the L-process preserves push-outs.*

Before proving the theorem we draw attention to the special case when J is a directed set. In that case there is an easy proof, applying (2.19)d and Theorem 3.7. We fix $i \in |J|$ and let J_0 be the directed subset of J consisting of all $j \geqslant i$. Let Γ be the category represented by

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd}$$

and let $F : \Gamma \times J_0 \rightarrow \mathfrak{C}$ be the functor such that, if $j \leqslant k$ in J_0, then

$$(3.13) \qquad F[\Gamma \times (j \leqslant k)] = \begin{tikzcd} A_i \arrow[rr] \arrow[d, "\kappa_i"] \arrow[dr, "1"] & & A_j \arrow[dr] & \\ B_i \arrow[dr] & A_i \arrow[rr] \arrow[d, "\kappa_i"] & & A_k \\ & B_i & & \end{tikzcd}$$

Then J_0 is cofinal in J and the push-out in just $\varinjlim_{\Gamma}$. Thus

$$\varinjlim_{\Gamma} \varinjlim_{J_0} F \cong \varinjlim_{J_0} \varinjlim_{\Gamma} F,$$

which, in view of Theorem 3.7, says that the push-out of

$$\begin{tikzcd} A_i \arrow[r, "\pi_i"] \arrow[d, "\kappa_i"] & LA \\ B_i & \end{tikzcd}$$

is just (3.11) provided

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "\kappa_i"] & A_j \arrow[d, "\kappa_j"] \\ B_i \arrow[r] & B_j \end{tikzcd}$$

is a push-out for all $j \geqslant i$.

However for an arbitrary qf-category we cannot define the functor F above (the horizontal maps in (3.13) are not well-defined) and we must modify the procedure.

To this end we construct, for a fixed $i \in | J |$, a new category J_i, called[4] the *category of objects of* J *under* i. An object of J_i is a morphism $\varphi : i \to j$ in J and a morphism $\theta : \varphi \to \psi$ in J_i is a morphism $\theta : j \to k$ such that the diagram

$$\begin{tikzcd} & j \arrow[d, "\theta"] \\ i \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & k \end{tikzcd}$$

commutes. Let J_0 stand, as before, for the full subcategory of J such that $j \in | J_0 |$ if and only if there is a morphism $i \to j$ in J, and let J be, as usual, a qf-category. Then there is a functor

$$E_i : J_i \to J$$

given by $E_i(\varphi) = j$, $E_i(\theta) = \theta$, and E_i factors through the embedding $E : J_0 \subseteq J$. Let $E^* : \mathfrak{C}^J \to \mathfrak{C}^J$. be induced by E_i and let

$$Q_i = E_i^* Q : \mathfrak{C} \to \mathfrak{C}^{J_i}, \qquad F_i = FE_i \qquad \text{for any} \quad F : J \to \mathfrak{C}.$$

Suppose (compare Theorem 3.7) the existence of a left adjoint to Q_i, which is a full embedding: thus,

$$L_i \xrightarrow{\;\cdot\pi_i\;} Q_i, \qquad L_i : \mathfrak{C}^{J_i} \to \mathfrak{C}, \qquad \pi_i : 1 \to Q_i L_i : \mathfrak{C}^{J_i} \to \mathfrak{C}^{J_i}.$$

Set[5] $L = L_i E^* : \mathfrak{C}^J \to \mathfrak{C}$. We prove

---

[4] It is also called a "comma category". Other notations are i/J and (i, J).

[5] Note the slight change of notation in the use of the symbol π_i. To avoid misunderstanding we use ordinary functorial notation in the proof of Theorem 3.14 (compare the proof of Theorem 3.7).

THEOREM 3.14. *There exists a unique* $\pi : 1 \to \underset{\sim}{Q}L : \mathfrak{C}^J \to \mathfrak{C}^J$ *such that* $E_i^* \pi = \pi_i E_i^*$ *and*

$$L \cong \underset{\sim}{Q}.$$

In other words,

$$\varinjlim_J F = \varinjlim_{J_i} F_i , \qquad F \in \mathfrak{C}^J.$$

Proof. In view of Theorem 3.7 (and Proposition 3.6) it suffices to take $J = J_0$. Then to prove (3.15) we only need to show that if L, together with the morphisms $\pi_\varphi : F(j) \to L$, is the colimit of F_i, then (i) $\pi_\varphi = \pi_j$ is independent of the choice of $\varphi : i \to j$, and (ii) L, together with the morphisms π_j, is the colimit of F.

Now $\pi_\varphi = \pi_\psi F(\theta)$ where $\theta : \varphi \to \psi$ in J_i. Given $\varphi, \varphi' : i \to j$, choose $\eta : j \to k$,

$$i \underset{\varphi'}{\overset{\varphi}{\rightrightarrows}} j \xrightarrow{\eta} k$$

such that $\eta\varphi = \eta\varphi'$ (by (3.2)). Then $\pi_\varphi = \pi_{\eta\varphi} F(\eta) = \pi_{\eta\varphi'} F(\eta) = \pi_{\varphi'}$, establishing (i) above. Certainly $\pi_j = \pi_k F(\theta)$ for all $\theta : j \to k$ in J_0. For let $\varphi : i \to j$ in J (this exists by definition of J_0 and J_i) and let $\psi = \theta\varphi$. Then $\pi_\varphi = \pi_\psi F(\theta)$ so $\pi_j = \pi_k F(\theta)$. Thus it only remains to show that if $\sigma_j : F(j) \to M$, $j \in |J_0|$, with $\sigma_j = \sigma_k F(\theta)$ then there exists a unique $\sigma : L \to M$ with $\sigma\pi_j = \sigma_j$. Set $\sigma_\varphi = \sigma_j$ for $\varphi : i \to j$ in $|J_i|$. This defines σ_φ for all φ in $|J_i|$ and $\sigma_\varphi = \sigma_\psi F(\theta)$ for $\theta : \varphi \to \psi$ in J_i. The conclusion (ii) now follows from the fact that $\varinjlim_{J_i} F_i = L$.

We return now to the issue of Theorem 3.12. We again take Γ to be the category

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \\ {} \end{tikzcd}$$

Then given $\kappa : A \to B : J \to \mathfrak{C}$ and $i \in |J|$ we define $F : \Gamma \times J_i \to \mathfrak{C}$ to be the functor such that, if $\theta : \varphi \to \psi$ in J_i, then

$$(3.16) \qquad F(\Gamma \times \theta) = \begin{tikzcd} A_i \arrow[rr, "\varphi"] \arrow[d, "\kappa_i"] \arrow[ddr, "1"] & & A_j \arrow[ddr, "\theta"] & \\ B_i \arrow[ddr, "1"] & & & \\ & A_i \arrow[rr, "\psi"] \arrow[d, "\kappa_i"] & & A_k \\ & B_i & & \end{tikzcd}$$

Then, using (2.19)d,

$$\varinjlim_{I} \varinjlim_{J_i} F = \varinjlim_{J_i} \varinjlim_{I} F$$

which, in the light of Theorem 3.14, asserts just Theorem 3.12.

Example 3.17. The importance of condition (3.2) may be brought out by the following counterexample where J is $\rightrightarrows$. Thus we consider, with $\mathfrak{C} = \mathfrak{Ab}$, the square

$$\begin{tikzcd} O \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A \arrow[d, "\kappa"] \\ B \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & A \oplus B \end{tikzcd}$$

Here $A = \mathbf{Z} = (a)$, $B = \mathbf{Z} = (b)$, $\kappa(a) = (a, 0)$, $\varphi_1(b) = (0, b)$, $\varphi_2(b) = (a, b)$. Clearly we get a push-out with φ_1 or φ_2. The equalizer of φ_1 and φ_2 is plainly the projection $A \oplus B \to B$. But the diagram

$$\begin{tikzcd} O \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ B \arrow[r, "1"] & B \end{tikzcd}$$

is obviously not a push-out. We can apply Theorem 3.12 if we replace J by the category

$$\begin{tikzcd} \cdot \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & \cdot \arrow[r, "\beta"] & \cdot \end{tikzcd}$$

where $\beta\alpha_1 = \beta\alpha_2$. This means that we look at diagrams

$$\begin{tikzcd} A \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & B \arrow[r, "\eta"] & C \end{tikzcd}$$

in $\mathfrak{C}$ with $\eta\varphi_1 = \eta\varphi_2$. Then we construct the equalizer "relative to η". That is, we seek equalizers of φ_1, φ_2 which factor through η, postulating the universal property with respect to such equalizers. The "limit" diagram then retains the push-out property.

For later applications we will also need

PROPOSITION 3.18. *If J is a qf-category, so is J_i for any $i \in |J|$.*

Proof. J_i is obviously small if J is small. Also J_i is connected. For given $\phi, \psi \in |J_i|$ we invoke (3.1) to find θ, χ with $\theta\phi = \chi\psi$,

$$\begin{tikzcd}
& {} \arrow[dr, "\theta"] & \\
{} \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & & {} \\
& {} \arrow[ur, "\chi"'] &
\end{tikzcd}$$

But then $\theta : \phi \to \theta\phi$, $\chi : \psi \to \theta\phi$ in J_i.

Thus it remains to prove (3.1), (3.2) for J_i. We suppose given $\kappa, \lambda, \mu \in |J_i|$ and $\alpha : \kappa \to \lambda$, $\beta : \kappa \to \mu$. Thus we have, in J, $\kappa : i \to k$, $\lambda : i \to l$, $\mu : i \to m$ and $\alpha\kappa = \lambda$, $\beta\kappa = \mu$,

$$\begin{tikzcd}
& & l \\
i \arrow[urr, "\lambda"] \arrow[r, "\kappa"] \arrow[drr, "\mu"'] & k \arrow[ur, "\alpha"'] \arrow[dr, "\beta"] & \\
& & m
\end{tikzcd} \quad \text{in } J.$$

By (3.1) for J we may find $\gamma : l \to n$, $\delta : m \to n$ with $\gamma\alpha = \delta\beta$. Then we have

$$\begin{tikzcd}
& \lambda \arrow[dr, "\gamma"] & \\
\kappa \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \nu \\
& \mu \arrow[ur, "\delta"'] &
\end{tikzcd}, \quad \nu = \gamma\lambda = \delta\mu,$$

in J_i establishing (3.1) for J_i. As to (3.2) we suppose given $\rho, \sigma : \kappa \to \lambda$ with κ, λ as above. Then

$$k \underset{\sigma}{\overset{\rho}{\rightrightarrows}} l \quad \text{in } J.$$

By (3.2) for J we may find $\eta : l \to m$ with $\eta\rho = \eta\sigma$. But then $\eta : \lambda \to \eta\lambda$ in J_i, so the proposition is proved.

4. Commuting Limits in an Abelian Category

We will be assuming later in this section that $\mathfrak{C}$ is an abelian category and will be examining when the canonical map

$$\omega_F : \varinjlim_J \varprojlim_I F \to \varprojlim_I \varinjlim_J F, \qquad F \in \mathfrak{C}^{I \times J}, \tag{4.1}$$

is an isomorphism. We first, however, fix notation, which will differ somewhat from that of Section 2 in view of our special purposes. We suppose throughout that the appropriate limits and colimits exist.

Let, then, $F \in \mathfrak{C}^{I \times J}$ and let $\phi : i_2 \to i_1$ in I, $\psi : j_1 \to j_2$ in J. Then we have the commutative square

$$
\text{(4.2)} \qquad
\begin{tikzcd}
F_{i_2 j_1} \arrow[r, "\psi_{i_2}"] \arrow[d, "\phi_{j_1}"] & F_{i_2 j_2} \arrow[d, "\phi_{j_2}"] \\
F_{i_1 j_1} \arrow[r, "\psi_{i_1}"'] & F_{i_1 j_2}
\end{tikzcd}
$$

where $F_{ij} = F(i, j)$, $\phi_j = F(\phi \times 1_j)$, $\psi_i = F(1_i \times \psi)$.

We may now carry out the L-process on (4.2), with respect to J, followed by the R-process with respect to I; alternatively we may proceed in the opposite order. We obtain the *key diagram*:

(4.3)

$$
\begin{tikzcd}
F_{i_2 j_1} \arrow[r, "\psi_{i_2}"] \arrow[d, "\phi_{j_1}"] \arrow[dr, phantom, "1"] & F_{i_2 j_2} \arrow[d, "\phi_{j_2}"] \\
F_{i_1 j_1} \arrow[r, "\psi_{i_1}"'] & F_{i_1 j_2}
\end{tikzcd}
$$

(R) ↙ ↘ (L)

$$
\begin{tikzcd}
R_{j_1} \arrow[r, "R\psi"] \arrow[d, "\tau_{i_1 j_1}"] \arrow[dr, phantom, "2"] & R_{j_2} \arrow[d, "\tau_{i_1 j_2}"] \\
F_{i_1 j_1} \arrow[r, "\psi_{i_1}"'] & F_{i_1 j_2}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F_{i_2 j_1} \arrow[r, "\pi_{i_2 j_1}"] \arrow[d, "\phi_{j_1}"] \arrow[dr, phantom, "3"] & L_{i_2} \arrow[d, "L\phi"] \\
F_{i_1 j_1} \arrow[r, "\pi_{i_1 j_1}"'] & L_{i_1}
\end{tikzcd}
$$

(L) ↓ ↓ (R)

$$
\begin{tikzcd}
R_{j_1} \arrow[r, "\pi_{j_1}"] \arrow[d, "\tau_{i_1 j_1}"] \arrow[dr, phantom, "4"] & LR \arrow[d, "L\tau_{i_1}"] \\
F_{i_1 j_1} \arrow[r, "\pi_{i_1 j_1}"'] & L_{i_1}
\end{tikzcd}
\xrightarrow{\quad (\omega) \quad}
\begin{tikzcd}
R_{j_1} \arrow[r, "R\pi_{j_1}"] \arrow[d, "\tau_{i_1 j_1}"] \arrow[dr, phantom, "5"] & RL \arrow[d, "\tau_{i_1}"] \\
F_{i_1 j_1} \arrow[r, "\pi_{i_1 j_1}"'] & L_{i_1}
\end{tikzcd}
$$

The arrows labelled (L), (R) simply indicate L- and R-processes. On the other hand, the horizontal passage at the bottom of (4.3) is a genuine map

$$\begin{pmatrix} 1 & \omega \\ 1 & 1 \end{pmatrix}$$

of commutative squares, by Proposition 2.9, (i) or (ii).

PROPOSITION 4.4. *Let $\mathfrak{C}$ be an abelian category. Then*

(i) *If ④ is exact and ⑤ is cartesian, then ④ is cartesian and ω is monic.*

(ii) *If ④ is cocartesian and ⑤ is exact, then ⑤ is cocartesian and ω is epic.*

(iii) *If ④ is cocartesian and ⑤ is cartesian, then both are bicartesian and ω is an isomorphism.*

Proof. We have the commutative diagram

$$(4.5)\qquad \begin{tikzcd} R_{j_1} \arrow[r, "\{\pi,\tau\}"] \arrow[d, equal] & LR \oplus F_{i_1j_1} \arrow[r, "\langle L\tau,\ \pi\rangle"] \arrow[d, "\omega \oplus 1"] & L_{i_1} \arrow[d, equal] \\ R_{j_1} \arrow[r, "\{R\pi,\tau\}"'] & RL \oplus F_{i_1j_1} \arrow[r, "\langle\tau,\ -\pi\rangle"'] & L_{i_1} \end{tikzcd}$$

Then under hypothesis (i) it is obvious that ④ is cartesian and that $\omega \oplus 1$ is monic, so that ω is monic. (ii) follows similarly and (iii) simply puts (i) and (ii) together.

We will apply Proposition 4.4 (iii) to two situations. We will suppose I to *be qf', J to be qf and ① to be bicartesian for fixed i_1, j_1 and variable ϕ, ψ.*

COROLLARY 4.6. *Suppose, in addition, that*

(a) *ψ_{i_1} is epic for all ψ;*

(b) *ϕ_{j_1} is monic for all ϕ.*

Then ω_F in (4.1) *is an isomorphism.*

Proof. Since ① is bicartesian and I is qf', it follows from Theorem 3.12d that ② is cartesian. But ψ_{i_1} is epic, so ② is bicartesian. It now follows from Theorem 3.12, since J is qf, that ④ is cocartesian. By a similar argument, using the fact that ϕ_{j_1} is monic, it follows that ⑤ is cartesian, so we may apply Proposition 4.4 (iii) to infer that ω_F is an isomorphism.

Example 4.7. It was shown in [3] how a spectral sequence $\{E_n, d_n\}$

arising from an exact couple in an abelian category $\mathfrak{A}$ gives rise to a functor $E : \mathbf{Z} \times \mathbf{Z} \to \mathfrak{A}$ yielding a bicartesian square

$$\begin{tikzcd} E_{m,n+1} \arrow[r, two heads] \arrow[d, tail] & E_{m+1,n+1} \arrow[d, tail] \\ E_{m,n} \arrow[r, two heads] & E_{m+1,n} \end{tikzcd} \tag{4.8}$$

with $E_n = E_{nn}$. Corollary 4.6 yields the conclusion

$$\varinjlim_m \varprojlim_n E_{mn} = \varprojlim_n \varinjlim_m E_{mn} ;$$

this common double limit is precisely E_∞.

For our second corollary we need some notation. An object $G \in |\mathfrak{C}^J|$ is called a *prefiltration* [5] if there exists a monic $G \rightarrowtail QA$ in $\mathfrak{C}^J$. This condition implies that $G\psi$ is monic for every ψ in J; the converse also holds if $\mathfrak{C}$ is a category of modules. Similarly we have the notion of a *precofiltration*; $H \in \mathfrak{C}^I$ is a precofiltration if there exists an epic $PB \twoheadrightarrow H$ in $\mathfrak{C}^I$. Then $H\phi$ is epic for every ϕ in I; and the converse also holds if $\mathfrak{C}$ is a category of modules and I is *special* in the sense of Section 5 (e.g., if I is a countable codirected set).

We recall from Section 3 the category J_j of objects of J under j; the obvious dual notion is that of the category I_i of objects of I over i. We are now ready to state the corollary; again I is qf', J is qf and (1) is bicartesian for fixed i_1, j_1 and variable ϕ, ψ.

COROLLARY 4.9. *Suppose, in addition, that*

(a) ψ_i *is a prefiltration for every* ψ *and every* $i \to i_1$ *in* I_{i_1},

(b) ϕ_j *is a precofiltration for every* ϕ *and every* $j_1 \to j$ *in* J_{j_1}.

Then ω_F *in* (4.1) *is an isomorphism.*

Proof. As in Corollary 4.6, (2) is cartesian. Hypothesis (b), however, implies that $\tau_{i_1 j_2}$ is epic, so (2) is bicartesian, whence (4) is cocartesian. Similarly (5) is cartesian, so we again apply Proposition 4.4 (iii).

Example 4.10. Let (see [5])

$$\cdots X^{p-1} \subseteq X^p \subseteq \cdots \subseteq X, \qquad -\infty < p < \infty, \tag{4.11}$$

be a filtration of X, an object in the abelian category $\mathfrak{A}$. We may write this as

$$\mu : G \rightarrowtail PX$$

where $G : \mathbf{N} \to \mathfrak{A}$, $P : \mathfrak{A} \to \mathfrak{A}^{\mathbf{N}}$, $\mathbf{N}$ being the ordered set of integers, $G(p) = X^p$, and P is the embedding functor. We may also write $\mu^p : X^p \rightarrowtail X$, $\xi^p : X^p \rightarrowtail X^{p-1}$ for the inclusion maps. The annihilating cofiltration of (4.11) is then

$$X \overset{\epsilon_p}{\twoheadrightarrow} X_p \overset{\xi_p}{\twoheadrightarrow} X_{p-1} \twoheadrightarrow \cdots$$

where ϵ_p annihilates μ^p and ξ_p is induced by ξ^p. Let us also write $X_q{}^p$, $q \leq p$, for the cokernel of the composite inclusion $X^q \subseteq X^p$. Then there is a bicartesian square

$$(4.12) \qquad \begin{tikzcd} X_q^p \arrow[r, tail, "\xi^p_{(q)}"] \arrow[d, "\xi_q^{(p)}"'] & X_q^{p+1} \arrow[d, "\xi_q^{(p+1)}"] \\ X_{q+1}^p \arrow[r, tail, "\xi^p_{(q+1)}"'] & X_{q+1}^{p+1} \end{tikzcd}$$

if $p \geq q + 1$, induced by the maps ξ^p or ξ_p (thus depending *only* on G and not on $\mu : G \to PX$). However, the *existence* of μ, that is, the fact that G is a prefiltration, guarantees that $(\xi^p_{(q)})$ is a prefiltration, filtering X_q; of course, $(\xi_q^{(p)})$ is a precofiltration, cofiltering X^p. Thus we may apply Corollary 4.9 to yield the conclusion that

$$\varinjlim_p \varprojlim_q X_q{}^p = \varprojlim_q \varinjlim_p X_q{}^p;$$

this common double limit $X^{\hat{}}_{\infty}$, is the (filtered) object obtained by completing the filtration (4.11) according to [5]. As remarked, in a category of modules the conclusion may be drawn without postulating μ, since we only needed the existence of μ to guarantee that $(\xi^p_{(q)})$ is a prefiltration.

The fact that the square (4.12) only exists for $p \geq q + 1$ naturally has no effect on the argument. We have not thought it necessary to insist, in stating the corollaries, on such an evident generalization. Plainly it is sufficient that the functor F be defined on a full subcategory S of $I \times J$ with the property:

(a) for each $i \in |I|$, there exists $J(i)$ cofinal in J with $i \times J(i) \subseteq S$;

(b) for each $j \in |J|$, there exists $I(j)$ cofinal in I with $I(j) \times j \subseteq S$.

Of course, this by no means exhausts the possibility of generalization in this direction.

Remark 4.13. Axiom *AB*-6 of Grothendieck is also expressible as a statement that, for a certain type of functor F, the transformation ω_F is an isomorphism. We note that if ① is a bicartesian square of monics and if L is an exact functor (for this AB-5 is sufficient), then $(\omega) : ④ \rightarrow ⑤$ is a morphism of cartesian squares. Axiom AB-6 implies that if, in addition, the objects F_{ij} constitute a system of subobjects of an object A in $\mathfrak{C}$ then ω (and hence (ω)) is an isomorphism. We note that we may then deduce from the main diagram that if ② is becartesian, so is ⑤. We will devote the next section to a study of the situation (present in a category of modules) when L is exact.

5. Commuting Limits in a Category of Modules

In this section we continue to base our arguments on the key diagram (4.3) but we suppose that $\mathfrak{C}$ is a category of modules. Precisely we require that $\mathfrak{C}$ be embedded exactly in a category of modules and satisfy AB-5, but we will use the terminology of modules, in particular allowing our objects to possess elements.

We have immediately

Proposition 5.1. *Let $\mathfrak{C}$ be a category of modules. Let I be qf′, let J be qf, and let ① in* (4.5) *be cartesian. Then ω_F in* (4.1) *is monic.*

Proof. In a category of modules L is an exact functor. Now let $\phi : i_2 \rightarrow i_1$ in I and let

$$G'_\phi, \quad G_\phi, \quad G''_\phi : J_{i_1} \rightarrow \mathfrak{C}$$

be the functors given by

$$\begin{aligned} G'_\phi(\psi) &= F_{i_2 j_1}, \quad \psi \in |J_{i_1}|, \quad \psi : j_1 \rightarrow j_2 ; \\ G_\phi(\psi) &= F_{i_1 j_1} \oplus F_{i_2 j_2}, \\ G_\phi(\theta) &= 1 \oplus \theta_{i_2}, \quad \theta : \psi \rightarrow \psi' \quad \text{in} \quad J_{i_1} ; \\ G''_\phi(\psi) &= F_{i_1 j_2}, \\ G''_\phi(\theta) &= \theta_{i_1} . \end{aligned}$$

We then have natural transformations $\mu : G'_\phi \rightarrow G_\phi$, $\epsilon : G_\phi \rightarrow G''_\phi$, given by

$$\mu(\psi) = \{\phi_{j_1}, \psi_{i_2}\}, \quad \epsilon(\psi) = \langle \psi_{i_1}, \quad -\phi_{j_2} \rangle, \quad \text{and ① is cartesian}$$

if and only if

$$G'_\phi \xrightarrow{\mu} G_\phi \xrightarrow{\epsilon} G''_\phi \tag{5.2}$$

is left exact. Applying L to (5.2) we then find that

$$LG'_\Phi \xrightarrow{L\mu} LG_\Phi \xrightarrow{L\epsilon} LG''_\Phi$$

is left exact; this, however, is precisely the statement that ③ is cartesian. More generally, we see that the L-process preserves exact, cartesian, co-cartesian and bicartesian squares. Thus we infer that if ① is cartesian then ③ is cartesian, so ⑤ is cartesian. Similarly ② is cartesian so ④ is cartesian. We now apply Proposition 4.4 (i).

Remark. The conditions governing the monicity of ω_F in Proposition 5.1 are clearly too strong. For example it is readily seen that if $D(p, q), p \geqslant 0$, $q \geqslant 0$, is a doubly indexed system of modules then we have a limit-switching morphism

$$\omega : \bigoplus_p \prod_q D(p, q) \to \prod_q \bigoplus_p D(p, q), \tag{5.3}$$

which is monic (but not in general epic). On the other hand ω may be obtained from the diagram

$$\begin{tikzcd} X_{m,n} \arrow[r, tail] \arrow[d] & X_{m+1,n} \arrow[d] \\ X_{m,n-1} \arrow[r, tail] & X_{m+1,n-1} \end{tikzcd}, \quad X_{m,n} = \bigoplus_{\substack{p \leq m \\ q \leq n}} D(p,q)$$

which is not, in general, cartesian. In fact our purpose in enunciating Proposition 5.1 is really to pinpoint the problem as to when ω is an isomorphism. Proposition 5.1 shows that, *in a category of modules, ω is an isomorphism if ψ_{i_1} is epic for all ψ, and ① is bicartesian.* More generally, it shows that the essential question is whether the R-process will preserve bicartesian squares. This motivates our next discussion.

We say that the qf'-category I is *special* if (i) $|I|$ is countable and (ii) given $i_1, i_2 \in |I|$, there exists φ in I with range i_1, such that $\theta\varphi = \bar\theta\varphi$ for any $\theta, \bar\theta : i_1 \to i_2$. We then say that φ *equalizes* I (i_1, i_2). Of course, (ii) is a strengthening of (3.2)d; in particular, if the set I (i_1, i_2) is finite for all $i_1, i_2 \in |I|$, then (ii) follows (by induction) from (3.2)d. Notice that every countable codirected set is a special qf'-category; on the other hand, the important category

$$\cdot \xrightarrow{\kappa} \cdot \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \cdot, \qquad \varphi_1\kappa = \varphi_2\kappa,$$

is a special qf'-category which is not a codirected set.

LEMMA 5.4. *Let I be a special qf'-category. Then given $1, 2,..., k \in |I|$, $\exists\, m \in |I|$ such that $I(m, i) \neq \varnothing$, $i = 1, 2,..., k$. Moreover, for any such m, $\exists\, \rho : n \to m$ in I such that, for any i, $1 \leqslant i \leqslant k$, and any $\theta, \bar{\theta} : m \to i$, we have $\theta\rho = \bar{\theta}\rho$. That is, ρ equalizes $I(m, i)$, $1 \leqslant i \leqslant k$.*

Proof. The first assertion does not require that I be special and follows immediately from Proposition 3.3d by induction on k.

Since I is special there exists $\varphi_i : n_i \to m$ equalizing $I(m, i)$, $i = 1, 2,..., k$. It now follows from 3.1d by induction on k, that there exist $\rho_i : n \to n_i$, $i = 1, 2,..., k$, with $\varphi_1\rho_1 = \varphi_2\rho_2 = \cdots = \varphi_k\rho_k : n \to m$. This morphism is then a suitable ρ.

Let I be a qf'-category, and let $F : I \to \mathfrak{G}$ be a functor to the category of groups. We then say (see [3], [6]) that F has the *Mittag-Leffler property* if, given any $i \in |I|$, there exists $\phi : j \to i$ in I such that

$$\phi\psi(F_k) = \phi(F_j)$$

for all k and all $\psi : k \to j$ in I. Of course this definition is equally applicable when the range of F is a category of modules[6]; so, too, then is the following theorem (generalizing Proposition 13.2.2 of [6]).

THEOREM 5.5. *Let I be a special qf'-category and let*

$$F \overset{\kappa}{\rightarrowtail} G \overset{\lambda}{\twoheadrightarrow} H$$

be an exact sequence of functors $I \to \mathfrak{G}$. Then if F has the Mittag-Leffler property

$$RF \overset{R\kappa}{\rightarrowtail} RG \overset{R\lambda}{\twoheadrightarrow} RH \tag{5.6}$$

is exact.

Proof. The point at issue is the epicity of $R\lambda$. To simplify the proof we take κ, $R\kappa$ to be inclusions. Indeed, the sequence (5.6) is obtained by restricting the exact sequence

$$\prod_{i\in|I|} F_i \overset{\Pi\kappa_i}{\rightarrowtail} \prod_{i\in|I|} G_i \overset{\Pi\lambda_i}{\twoheadrightarrow} \prod_{i\in|I|} H_i\,;$$

the element $\{h_i\}$ then belongs to RH if and only if $\phi(h_i) = h_j$ for all $\phi : i \to j$ in I and we must find $\{g_i\}$ in $\prod_{i\in|I|} G_i$ such that $\lambda_i(g_i) = h_i$ and $\phi(g_i) = g_j$

[6] It could also be applied when the range of F is an arbitrary abelian category, but we have no proof of Theorem 5.5 in this generality.

for all ϕ. We enumerate the objects of I as $1, 2, 3, \ldots$, and we make the inductive hypothesis that we have constructed $g_i \in G_i$, $i \leqslant k-1$, so that

(i) $\lambda_i(g_i) = h_i$, $i \leqslant k-1$,

(ii) $\phi(g_i) = g_j$, $\phi : i \to j$, $i, j \leqslant k-1$,

(iii) for every $i \in |I|$, $\exists\, g_i' \in G_i$ with $\lambda_i(g_i') = h_i$ and $\phi(g_i') = g_j$, $\phi : i \to j$, $j \leqslant k-1$.

(The hypothesis is vacuous for $k = 1$, save for the observation that $\Pi\lambda_i$ is epic)

By Lemma 5.4 there exists $\rho : n \to m$ which equalizes $I(m, i)$, $1 \leqslant i \leqslant k$, none of $I(m, i)$ being empty. Let $\omega : u \to n$ be such that

$$\omega\psi(F_l) = \omega(F_u), \qquad \text{for all} \quad \psi : l \to u \quad \text{in} \quad I; \tag{5.7}$$

such a morphism ω exists because F has the Mittag-Leffler property. Then by hypothesis

$$g_i = \nu_i \rho\omega(g_u'), \qquad \text{for any} \quad \nu_i : m \to i, \qquad 1 \leqslant i \leqslant k-1. \tag{5.8}$$

We set

$$g_k = \nu_k \rho\omega(g_u'), \qquad \text{for a fixed but arbitrary} \quad \nu_k : m \to k. \tag{5.9}$$

Then

(i) $\lambda_k(g_k) = \nu_k\rho\omega\lambda_u(g_u') = \nu_k\rho\omega(h_u) = h_k$,

(ii) if $\phi : i \to j$ in I, $i, j \leqslant k$, then $\phi(g_i) = \phi\nu_i\rho\omega(g_u')$, $g_j = \nu_j\rho\omega(g_u')$.

But $\nu_j\rho = \phi\nu_i\rho$, by choice of ρ, so $\phi(g_i) = g_j$.

(iii) Thus it remains to define suitable elements g_u''. Given i, we may apply Proposition 3.3d and Lemma 5.4 to find

$$\begin{tikzcd}[row sep=small] & & u \\ w \arrow[r, "\gamma"] & v \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & \\ & & i \end{tikzcd}$$

in I, where γ equalizes $I(v, j)$, $j \leqslant k$. Now $\lambda_u(g_u') = \lambda_u\beta\gamma(g_w')(=h_u)$ so that

$$\begin{gathered} g_u'\{\beta\gamma(g_w')\}^{-1} \in F_u \\ g_u' = \beta\gamma(g_w') f_u, \qquad f_u \in F_u . \end{gathered} \tag{5.10}$$

By (5.7) we infer $f_w \in F_w$ with

$$\omega\beta\gamma(f_w) = \omega(f_u),$$

so that, by (5.10)

$$\omega(g_u') = \omega\beta\gamma(g_w' f_w). \tag{5.11}$$

We set $g''_i = \alpha\gamma(g'_w f_w)$. Then $\lambda_i(g''_i) = \alpha\gamma\lambda_w(g'_w f_w) = \alpha\gamma(h_w) = h_i$. Finally if $\phi : i \to j$ in I with $j \leqslant k$, then

$$\phi(g''_i) = \phi\alpha\gamma(g'_w f_w); \qquad \text{but} \quad \phi\alpha\gamma = \nu_j\rho\omega\beta\gamma, \qquad \text{by choice of } \gamma.$$

Thus

$$\begin{aligned} \phi(g''_i) &= \nu_j\rho\omega\beta\gamma(g'_w f_w) \\ &= \nu_j\rho\omega(g'_u) \qquad \text{by (5.11)} \\ &= g_j \qquad \text{by (5.8) or (5.9).} \end{aligned}$$

This completes the inductive step: it also completes the proof of the theorem. For we have constructed an element $\{g_i\}$ in $\prod_i G_i$ with the property that $\lambda_i(g_i) = h_i$ and $\varphi(g_i) = g_j$ for any φ in I; this is guaranteed at the stage $\max(i, j)$ of the construction.

Theorem 5.5 enables us to prove the main theorem of this section.

THEOREM 5.12. *Let $F : I \times J \to \mathfrak{C}$ be a functor such that*

(i) *I is a special qf'-category:*

(ii) *J is a qf-category:*

(iii) *$\mathfrak{C}$ is a category of modules.*

Let $F_{j_1} : I \to \mathfrak{C}$ have the Mittag-Leffler property for a fixed $j_1 \in |J|$. Then if 1 *in the key diagram* (4.3) *is bicartesian, for fixed i_1, j_1 and variable φ, ψ,*

$$\omega_F : \varinjlim_J \varprojlim_I F \cong \varprojlim_I \varinjlim_J F.$$

Proof. If we compare the proof of Proposition 5.1, and take the dual viewpoint, then the bicartesian property of 1 is reflected in the exact sequence of functors

$$F'_\psi \rightarrowtail F_\psi \twoheadrightarrow F''_\psi \tag{5.13}$$

where $\psi : j_1 \to j_2$ in J, and $F'_\psi, F_\psi, F''_\psi : I_{i_1} \to \mathfrak{C}$, I_{i_1} being as usual the category of objects of I over i_1. Moreover the R-process from 1 to 2 is then translated into the application of the limit functor to (5.13).

We now make the additional assumption that $I(i, i_1)$ is countable for all i. Now by Proposition 3.18d I_{i_1} is a qf'-category. It is special since I is special and $|I_{i_1}| = \bigcup_{i \in |I|} I(i, i_1)$. Moreover F'_ψ is given by

$$F'_\psi(\varphi) = F_{j_1}(i_2), \qquad \varphi : i_2 \to i_1 \text{ in } I,$$

$$F'_\psi(\theta) = F_{j_1}(\theta), \qquad \theta \text{ in } I_{i_1} \qquad \text{(and hence in } I\text{)},$$

so plainly F'_ψ has the Mittag-Leffler property for all ψ if F_{j_1} has the Mittag-Leffler property. We may therefore apply Theorem 5.5 to infer that the limit functor is exact on (5.13). This means that, in (4.3), the square ② is bicartesian and hence, as in the proof of Proposition 5.1, so is ④. We already know that ⑤ is cartesian, so we get our conclusion by applying Proposition 4.4 (iii).

It remains to avoid the assumption that $I(i, i_1)$ is countable for all i. We base ourselves on the following proposition.

PROPOSITION 5.14. *Let I be a special qf'-category. Then there exists a subcategory $\bar{I}$ of I such that*

(i) $|\bar{I}| = |I|$;

(ii) *$\bar{I}$ is countable*;

(iii) *$\bar{I}$ is a special qf'-category*;

(iv) *for any $F : I \to \mathfrak{E}$, $\varprojlim \bar{F} = \varprojlim F$, where $\bar{F} = F \mid \bar{I}$*;

(v) *if $F : I \to \mathfrak{G}$ has the Mittag-Leffler property, so has $\bar{F}$.*

Proof. We construct $\bar{I}$ as follows. For each non-empty $I(i, j)$ we pick one morphism $\xi_{ij} : i \to j$ and one equalizer, η_{ij}, of $I(i, j)$; for convenience we choose $\xi_{ii} = 1$. Let $\bar{I}$ be the smallest subcategory of I containing all ξ_{ij} and all η_{ij}. Then certainly $|\bar{I}| = |I|$ and $\bar{I}$ is countable since $|I|$ is countable. It remains to verify (iii), (iv), (v).

Plainly $\bar{I}$ is connected, for any two objects connected by an arrow in I are connected by an arrow in $\bar{I}$. Now consider[7]

$$(5.15) \qquad \begin{tikzcd} \cdot \arrow[dr, "\varphi"] & \\ & \cdot \\ \cdot \arrow[ur, "\psi"'] & \end{tikzcd}$$

in $\bar{I}$. Since I is qf' there exists

$$\begin{tikzcd} & \cdot \arrow[dr, "\bar{\varphi}"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \cdot \\ & \cdot \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

in I (in fact with $\bar{\varphi}\alpha = \bar{\psi}\beta$). Thus by construction of $\bar{I}$ there exists

$$\begin{tikzcd} & \cdot \arrow[dr, "\varphi"] & \\ \cdot \arrow[ur, "\bar{\alpha}"] \arrow[dr, "\bar{\beta}"'] & & \cdot \\ & \cdot \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

[7] The argument holds even if (5.15) is only a diagram in I, not Ī.

in $\bar{I}$, though we can no longer claim that $\bar{\varphi}\bar{\alpha} = \bar{\psi}\bar{\beta}$. However, again invoking the construction of $\bar{I}$, we may equalize $\bar{\varphi}\bar{\alpha}$ and $\bar{\psi}\bar{\beta}$ with η in $\bar{I}$. Then

$$\begin{tikzcd}
& \cdot \arrow[dr, "\bar{\varphi}"] & \\
\cdot \arrow[ur, "\bar{\alpha}\eta"] \arrow[dr, "\bar{\beta}\eta"'] & & \cdot \\
& \cdot \arrow[ur, "\bar{\psi}"'] &
\end{tikzcd}$$

is a commutative diagram in $\bar{I}$, proving (3.1)d for $\bar{I}$.

Clearly η_{ij} equalizes $\bar{I}(i, j)$ since it equalizes $I(i, j)$ so (3.2)d holds in $\bar{I}$ and (iii) is established.

Now let $R = \varprojlim F$. Thus there exists, for each $i \in |I|$, a morphism $\tau_i : R \to F_i$ with $\varphi\tau_i = \tau_j$ for $\varphi : i \to j$ in I; and (R, τ_i) enjoys the universal property in I. We must show that (R, τ_i) enjoys the universal property in $\bar{I}$. Thus we suppose given $\sigma_i : S \to F_i$ with $\bar{\varphi}\sigma_i = \sigma_j$ for $\bar{\varphi} : i \to j$ in $\bar{I}$; it is then sufficient to show that also $\varphi\sigma_i = \sigma_j$ for any $\varphi : i \to j$ in I (recall that if there is $\varphi : i \to j$ in I then there is $\bar{\varphi} : i \to j$ in $\bar{I}$). Now let $\eta \in \bar{I}$ equalize φ and $\bar{\varphi}$ so that $\varphi\eta = \bar{\varphi}\eta$, and let $\eta : k \to i$. Then $\eta\sigma_k = \sigma_i$ and

$$\sigma_j = \bar{\varphi}\sigma_i = \bar{\varphi}\eta\sigma_k = \varphi\eta\sigma_k = \varphi\sigma_i\,.$$

This establishes (iv).

Finally let $F : I \to \mathfrak{G}$ have the Mittag-Leffler property and let $i \in |I|$. Then there exists $\varphi : j \to i$ in I such that $\varphi\rho(F_m) = \varphi(F_j)$ for any $\rho : m \to j$ in I. Let $\bar{\varphi} : j \to i$ in $\bar{I}$ and let $\eta : k \to j$ in $\bar{I}$ equalize φ and $\bar{\varphi}$. Then, for any $\xi : l \to k$ in $\bar{I}$,

$$\bar{\varphi}\eta\xi(F_l) = \varphi\eta\xi(F_l) = \varphi(F_j) = \varphi\eta(F_k) = \bar{\varphi}\eta(F_k).$$

This proves that $\bar{F}$ has the Mittag-Leffler property and hence establishes the proposition.

To complete the proof of Theorem 5.12, we replace I at the outset by $\bar{I}$. Then if $\bar{F} = F \mid \bar{I} \times J$, it is plain that $L^{\bar{I}}\bar{F} = L^{I}F$ and moreover

$$\bar{R}^{J}\bar{F} = R^{J}F \qquad \text{(Proposition 5.14, (iv)).}$$

Thus ω_F is left unchanged by the restriction to $\bar{I}$, that is, $\omega_{\bar{F}} = \omega_F$. Moreover $\bar{F}_{j_1}$ has the Mittag-Leffler property (Proposition 5.14 (v)); and trivially $\bar{I}(i, i_1)$ is countable since $\bar{I}$ is countable. Thus on replacing I by $\bar{I}$ we prove that ω_F is an isomorphism and I may be restored in the conclusion. This completes the proof of Theorem 5.12.

Remarks.

(i) Theorem 5.5, in the case in which I is a countable codirected set[8], appeared as (3.8) of [*3*], the proof in that case being attributed to Dieudonné and Grothendieck ([*6*], Proposition 1.3.2.2), where codirected sets having countable cofinal subsets are considered. Assertion (3.6), (3.7) and (3.9) of [*3*], which also relate to the Mittag-Leffler property, all remain true when I is a special qf'-category. Indeed, assertions (3.6), (3.7) hold for any category I.

(ii) Reverting to Theorem 5.5, it is plain that the Mittag-Leffler property holds trivially if all morphisms $F(\varphi)$, φ in I, are epic. The conclusion of Theorem 5.5 thus holds if I has a cofinal special qf'-subcategory all of whose morphisms are mapped by F to epimorphisms. Thus Theorem 5.12 applies when θ_{λ_1} is epic for all θ in I_{λ_1} or in a cofinal subcategory of I_{λ_1}. On the other hand, we may improve Corollary 4.9 in a category of modules by simply dropping condition (a), without any special assumptions on I.

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[8] Actually functors to abelian groups are considered in [*3*] in view of the author's applications (in particular (3.9)).