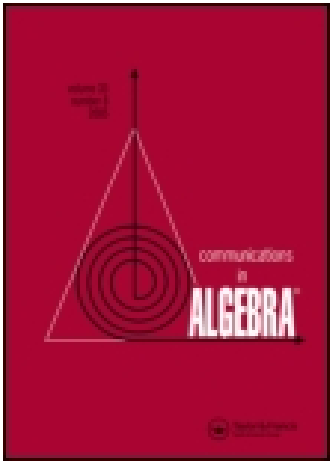

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On the structure of relative hopf modules

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ON THE STRUCTURE OF RELATIVE HOPF MODULES

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Let A be a Hopf algebra over a field k. In this paper we study the notions of (A, B)-Hopf modules for an A-comodule algebra B and $[C, A]$-Hopf modules for an A-module coalgebra C.

In [2], Sweedler has proved for any Hopf algebra H, that the existence of an integral $x : H \longrightarrow k$ with $x(1) = 1$ is equivalent to the complete reducibility of all H-comodules. He also has reduced the structure theorem for H-Hopf modules. Here we give an extension of these results.

Throughout we freely use the terminology and results of [2]. All vector spaces will be over a field k. Map always means k-linear map, and the unadorned tensor product $V \otimes W$ is understood to be $V \otimes_k W$. We use the sigma notation. Thus, if C is a coalgebra, we write $\Delta(c) = \Sigma\, c_{(1)} \otimes c_{(2)}$, for $c \in C$. If M is a right C-comodule with comodule structure map $\rho : M \longrightarrow M \otimes C$, we write for $m \in M$,

$$\rho(m) = \Sigma\, m_{(0)} \otimes m_{(1)}.$$

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<u>Throughout this paper</u> A <u>is a Hopf algebra with antipode</u> S.

Let B be an algebra and a right A-comodule. The comodule structure map will be denoted by $\rho_B : B \longrightarrow B \otimes A$ and for $\rho_B(b)$ we write $\Sigma\, b_{(0)} \otimes b_{(1)}$. B is called a right A-<u>comodule algebra</u> if ρ_B is an algebra map.

A is itself a right A-comodule algebra via $\Delta : A \longrightarrow A \otimes A$. More generally, if B is a subalgebra and a right coideal of A then B becomes a right A-comodule algebra. The ground field k has a trivial right A-comodule algebra structure given by

$$u_A : k \longrightarrow A \simeq k \otimes A.$$

<u>Definition</u>. Let B be a right A-comodule algebra. M is called a right (A, B)-<u>Hopf module</u> if M is a right A-comodule and a right B-module such that the following diagram commutes

$$\begin{tikzcd}
M \otimes B \arrow[r, "\omega"] \arrow[d, "\rho_M \otimes \rho_B"] & M \arrow[r, "\rho"] & M \otimes A \\
M \otimes A \otimes B \otimes A \arrow[rr, "I \otimes T \otimes I"] & & M \otimes B \otimes A \otimes A \arrow[u, "\omega_M \otimes M_A"']
\end{tikzcd}$$

(ω_M is the B-module action of M, ρ_M is the A-comodule structure map of M, M_A is the multiplication in A, T is the twist map).

The diagram can be expressed as

$$\rho_M(mb) = \Sigma\, m_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)}$$

for all $m \in M$, $b \in B$.

We note that B is itself a right (A, B)-Hopf module via ρ_B and $M_B : B \otimes B \longrightarrow B$.

Theorem 1. Let B be a right A-comodule algebra where there is a right A-comodule map $\phi : A \longrightarrow B$ with $\phi(1_A) = 1_B$. Then every right (A, B)-Hopf module is injective as an A-comodule.

Proof. Let M be a right (A, B)-Hopf module. If $M \otimes A$ has the right A-comodule structure given by $I \otimes \Delta : M \otimes A \longrightarrow (M \otimes A) \otimes A$ then the comodule structure map $\rho_M : M \longrightarrow M \otimes A$ is an A-comodule map. We show that there is an A-comodule map $\lambda : M \otimes A \longrightarrow M$ with $\lambda\rho_M = I$. Thus M is injective since it is isomorphic to a direct summand of $M \otimes A$, an injective A-comodule.

Define $\lambda : M \otimes A \longrightarrow M$ as the composite

$$M \otimes A \xrightarrow{\rho \otimes I} M \otimes A \otimes A \xrightarrow{I \otimes S \otimes I} M \otimes A \otimes A \xrightarrow{I \otimes M_A} M \otimes A$$

$$\xrightarrow{I \otimes \phi} M \otimes B \xrightarrow{\omega} M$$

so that $\lambda(m \otimes a) = \Sigma\, m_{(0)}\phi(S(m_{(1)})a)$ for $m \in M$, $a \in A$.

For any $m \in M$,

$$\lambda\rho_M(m) = \lambda(\Sigma\, m_{(0)} \otimes m_{(1)}) = \Sigma\, m_{(0)}\phi(S(m_{(1)})m_{(2)})$$

$$= \Sigma\, m_{(0)}\varepsilon(m_{(1)})\phi(1_A) = m$$

so that $\lambda\rho_M$ is the identity of M.

Next we claim λ is an A-comodule map.

$$\rho_M\lambda(m \otimes a) = \rho_M(\Sigma\, m_{(0)}\phi(S(m_{(1)})a))$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(2)})a)_{(0)} \otimes m_{(1)}\phi(S(m_{(2)})a)_{(1)}$$

The condition that ϕ be a right A-comodule map is exactly $\rho_B\phi = (\phi \otimes I)\Delta_A$ or for $a \in A$, $\Sigma\, \phi(a)_{(0)} \otimes \phi(a)_{(1)} = \Sigma\phi(a_{(1)})\otimes a_{(2)}$. Since the antipode S is an anti-algebra map the above expression equals

$$\begin{aligned}
&\Sigma\, m_{(0)}\phi(S(m_{(3)})a_{(1)}) \otimes m_{(1)}S(m_{(2)})a_{(2)} \\
&= \Sigma\, m_{(0)}\phi(S(m_{(2)})a_{(1)}) \otimes \varepsilon(m_{(1)})a_{(2)} \\
&= \Sigma\, m_{(0)}\phi(S(m_{(1)})a_{(1)}) \otimes a_{(2)} \\
&= (\lambda \otimes I)(I \otimes \Delta_A)(m \otimes a).
\end{aligned}$$

Thus λ is an A-comodule map. q. e. d.

In case $B = k$, the above result reduces to [2, LEMMA 14.0.2].

<u>Corollary</u>. The following statements concerning a right A-comodule algebra B are equivalent:

(i) B is an injective A-comodule.

(ii) There is a right A-comodule map $\phi : A \longrightarrow B$ with $\phi(1_A) = 1_B$.

<u>Proof</u>. Consider the diagram of right A-comodules

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "u_A"] \arrow[d, "u_B"] & A \arrow[dl, dashed, "\phi"] \\
& B &
\end{tikzcd}$$

If B is an injective A-comodule then the diagram can be completed by an A-comodule map ϕ to a commutative diagram. Thus we have that (i) implies (ii).

Since B may be regarded as a right (A, B)-Hopf module it follows from Theorem 1 that (ii) implies (i). q. e. d.

Let A, A' be Hopf algebras and $f : A' \longrightarrow A$ be a Hopf algebra map. Then A' becomes a right A-comodule algebra via

$$A' \xrightarrow{\Delta} A' \otimes A' \xrightarrow{I \otimes f} A' \otimes A.$$

<u>Theorem 2</u>. Let $f : A' \longrightarrow A$ be a surjective Hopf algebra map. If there is a right A-comodule map $\phi : A \longrightarrow A'$ with $\phi(1_A)=1_{A'}$, then we have:

(1) A' is injective as a right A-comodule.

(2) For any left A-comodule V, the canonical map

$$A' \square_A V \xrightarrow{f \otimes I} A \square_A V \simeq V$$

is surjective, where $\square_A$ denotes the cotensor product over A.

<u>Proof</u>. (1) is clear by Theorem 1 and thus we need only show (2). Since f is an A-comodule map, $\operatorname{Ker} f$ becomes a right (A, A')-Hopf module in a natural way. Thus we have from Theorem 1 that $\operatorname{Ker} f$ is an injective A-comodule. This implies that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow A' \xrightarrow{f} A \longrightarrow 0$$

is a split exact sequence of right A-comodules. Cotensoring over A by V yields the exact sequence

$$0 \longrightarrow (\operatorname{Ker} f) \square_A V \longrightarrow A' \square_A V \longrightarrow A \square_A V \longrightarrow 0.$$

q. e. d.

<u>Remark</u>. The above Theorem shows that for surjective Hopf algebra map $f : A' \longrightarrow A$, f is right coflat if and only if it is right faithfully coflat.

We return to the first setting where B is a right A-comodule algebra.

Define the A-invariant subspace of B to be the set

$$B_0 = \{ b \in B \mid \rho_B(b) = b \otimes 1_A \}.$$

It is clear that B_0 is a subalgebra of B.

Let V be a right B_0-module. Then $V \otimes_{B_0} B$ is a right B-module in the usual way. It is also a right A-comodule with comodule structure map $\rho : v \otimes_{B_0} b \longmapsto \Sigma\, v \otimes_{B_0} b_{(0)} \otimes b_{(1)}$ (this is well defined). One easily checks that $V \otimes_{B_0} B$ is a right (A, B)-Hopf module.

Let M be a right (A, B)-Hopf module. Define the set

$$M_0 = \{ m \in M \mid \rho_M(m) = m \otimes 1_A \}.$$

For any $m \in M_0$ and $b \in B_0$ we have $mb \in M_0$ and thus M_0 is a right B_0-module. Define

$$\alpha : M_0 \otimes_{B_0} B \longrightarrow M$$

by $\alpha(m \otimes_{B_0} b) = mb$ for $m \in M_0$, $b \in B$. It is then an (A, B)-Hopf map, that is, an A-comodule map and a B-module map.

Theorem 3. Let B be a right A-comodule algebra. If there is a right A-comodule map $\phi : A \longrightarrow B$ which is an algebra map then for every right (A, B)-Hopf module M,

$$\alpha : M_0 \otimes_{B_0} B \longrightarrow M$$

is an isomorphism of (A, B)-Hopf modules.

Proof. Let $P : M \longrightarrow M$ be the composite

$$M \xrightarrow{\rho} M \otimes A \xrightarrow{I \otimes S} M \otimes A \xrightarrow{I \otimes \phi} M \otimes B \xrightarrow{\omega} M.$$

Explicitly $P(m) = \Sigma\, m_{(0)}\phi(S(m_{(1)}))$.
We claim $P(M) \subset M_0$:

$$\begin{aligned}\rho_M P(m) &= \Sigma\, m_{(0)}\phi(S(m_{(2)}))_{(0)} \otimes m_{(1)}\phi(S(m_{(2)}))_{(1)} \\ &= \Sigma\, m_{(0)}\phi(S(m_{(3)})) \otimes m_{(1)}S(m_{(2)}) \\ &= \Sigma\, m_{(0)}\phi(S(m_{(2)})) \otimes \varepsilon(m_{(1)})1_A \\ &= \Sigma\, m_{(0)}\phi(S(m_{(1)})) \otimes 1_A = P(m) \otimes 1_A.\end{aligned}$$

Thus P is in fact a map $M \longrightarrow M_0$.

Define $\beta : M \longrightarrow M_0 \otimes_{B_0} B$ by

$$\beta(m) = \Sigma\, P(m_{(0)}) \otimes_{B_0} \phi(m_{(1)}).$$

We will show $\alpha\beta = I$ and $\beta\alpha = I$:

$$\begin{aligned}\alpha\beta(m) &= \alpha(\Sigma\, m_{(0)}\phi(S(m_{(1)})) \otimes_{B_0} \phi(m_{(2)})) \\ &= \Sigma\, m_{(0)}\phi(S(m_{(1)}))\phi(m_{(2)}) \\ &= \Sigma\, m_{(0)}\phi(S(m_{(1)})m_{(2)}) = m.\end{aligned}$$

For $m \in M_0$, $b \in B$,

$$\begin{aligned}\beta\alpha(m \otimes_{B_0} b) &= \beta(mb) = \Sigma\, P(mb_{(0)}) \otimes_{B_0} \phi(b_{(1)}) \\ &= \Sigma\, mb_{(0)}\phi(S(b_{(1)})) \otimes_{B_0} \phi(b_{(2)})\end{aligned}$$

since $\Sigma\, b_{(0)}\phi(S(b_{(1)})) \in B_0$ for any $b \in B$

$$= \Sigma\, m \otimes_{B_0} b_{(0)}\phi(S(b_{(1)}))\phi(b_{(2)})$$
$$= \Sigma\, m \otimes_{B_0} b_{(0)}\phi(\varepsilon(b_{(1)})1_B) = m \otimes_{B_0} b. \qquad \text{q. e. d.}$$

In case $B = A$ and $\phi = I$, the above Theorem reduces to [2, THEOREM 4.1.1].

We dualize Theorem 1, 2 and 3.

Let C be a coalgebra which is a right A-module. C is a right A-module coalgebra if the following hold for all $c \in C$, $a \in A$:

(1) $\Delta(ca) = \Sigma\, c_{(1)}a_{(1)} \otimes c_{(2)}a_{(2)}$

(2) $\varepsilon(ca) = \varepsilon(c)\varepsilon(a)$.

A is itself a right A-module coalgebra via $M_A : A \otimes A \longrightarrow A$. The ground field k has a trivial right A-module coalgebra structure.

Let N be a right C-comodule and a right A-module. N is called a right $[C, A]$-Hopf module if the following folds for all $n \in N$, $a \in A$:

$$\rho(na) = \Sigma\, n_{(0)}a_{(1)} \otimes n_{(1)}a_{(2)}.$$

Suppose that there exists a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon_A\psi = \varepsilon_C$. For any right $[C, A]$-Hopf module N, define $\lambda : N \longrightarrow N \otimes A$ as the composite

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{I \otimes \psi} N \otimes A \xrightarrow{I \otimes \Delta} N \otimes A \otimes A$$
$$\xrightarrow{I \otimes S \otimes I} N \otimes A \otimes A \xrightarrow{\omega \otimes I} N \otimes A$$

so that $\lambda(n) = \Sigma\, n_{(0)}S(\psi(n_{(1)})_{(1)}) \otimes \psi(n_{(1)})_{(2)}$. If $N \otimes A$ has the right A-module structure given by $(N \otimes A) \otimes A \xrightarrow{I \otimes M} N \otimes A$ then λ is an A-module map with $\omega\lambda = I$. Thus N is a projective A-module since it is isomorphic to a direct summand of $N \otimes A$, a free A-module.

We summarize this in the following theorem :

Theorem 4. Let C be a right A-module coalgebra where there is a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon\psi = \varepsilon$. Then every right [C, A]-Hopf module is a projective A-module.

Remarks. If C is finite dimensional then $\psi(C)$ is a non-zero finite dimensional right ideal of A so that A must be finite dimensional ([2], p.107). In case $C = k$, the above Theorem reduces to [2, THEOREM 5.1.8].

We state without proof the dual of Corollary of Theorem 1 and Theorem 2 :

Corollary 1. Let C be a right A-module coalgebra. The following are equivalent :

(i) C is a projective A-module.

(ii) There is a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon\psi = \varepsilon$.

Corollary 2. Let H be a Hopf algebra and A a Hopf subalgebra. If there is a right A-module map $\psi : H \longrightarrow A$ with $\varepsilon\psi = \varepsilon$ then we have :

(1) H is a projective A-module.

(2) For any left A-module V, the canonical map

$$V \simeq A \otimes_A V \longrightarrow H \otimes_A V$$

is injective.

Let C be a right A-module coalgebra. If A^+ denotes the kernel of $\varepsilon : A \longrightarrow k$ then CA^+ is a coideal of C. Hence

$\bar{C} = C/CA^+$ has a unique coalgebra structure such that the projection $p : C \longrightarrow \bar{C}$ is a coalgebra map.

Let N be a right $[C, A]$-Hopf module. Then the map p induces the right $\bar{C}$-comodule structure of N

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{I \otimes p} N \otimes \bar{C}.$$

NA^+ is then a $\bar{C}$-subcomodule of N. Thus $\bar{N} = N/NA^+$ has a unique comodule structure $\bar{\rho} : \bar{N} \longrightarrow \bar{N} \otimes \bar{C}$ making the projection $\pi : N \longrightarrow \bar{N}$ a $\bar{C}$-comodule map, that is, $\bar{\rho}\pi = (\pi \otimes p)\rho_N$.

Note that we have $\pi(na) = \pi(n)\varepsilon(a)$, for $n \in N$, $a \in A$.

Since C has the left $\bar{C}$-comodule structure induced by

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{p \otimes I} \bar{C} \otimes C$$

for any right $\bar{C}$-comodule W, $W \square_{\bar{C}} C$ is defined and it is a right $[C, A]$-Hopf module via

$$W \square_{\bar{C}} C \xrightarrow{I \otimes \Delta} W \square_{\bar{C}} C \otimes C$$
$$W \square_{\bar{C}} C \otimes A \xrightarrow{I \otimes \omega} W \square_{\bar{C}} C.$$

For any right $[C, A]$-Hopf module N, define $\alpha : N \longrightarrow \bar{N} \otimes C$ be the composite

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{\pi \otimes I} \bar{N} \otimes C.$$

It is easy to see that $\alpha(N) \subset \bar{N} \square_{\bar{C}} C$. Thus α is in fact a map $N \longrightarrow \bar{N} \square_{\bar{C}} C$. α is then a $[C, A]$-Hopf module map.

In these terms Theorem 3 can be dualized as follows :

<u>Theorem 5</u>. Let C be a right A-module coalgebra. If there is a right A-module map $\psi : C \longrightarrow A$ which is a coalgebra map then

for every right $[C, A]$-Hopf module N,

$$\alpha : N \longrightarrow \overline{N} \,\square_{\overline{C}}\, C$$

is an isomorphism of $[C, A]$-Hopf modules.

Since the proof of Theorem 3 is not so easily dualized we include a proof of Theorem 5.

Proof. Let $Q : N \longrightarrow N$ denote the composite

$$N \xrightarrow{\ \rho\ } N \otimes C \xrightarrow{\ I \otimes \psi\ } N \otimes A \xrightarrow{\ I \otimes S\ } N \otimes A \xrightarrow{\ \omega\ } N$$

so that $Q(n) = \Sigma\, n_{(0)} S(\psi(n_{(1)}))$ for $n \in N$.

For $n \in N$ and $a \in A$,

$$\begin{aligned} Q(na) &= \Sigma\, n_{(0)} a_{(1)} S(\psi(n_{(1)} a_{(2)})) \\ &= \Sigma\, n_{(0)} a_{(1)} S(\psi(n_{(1)}) a_{(2)}) \\ &= \Sigma\, n_{(0)} a_{(1)} S(a_{(2)}) S(\psi(n_{(1)})) \\ &= \Sigma\, n_{(0)} \varepsilon(a) S(\psi(n_{(1)})) = Q(n\varepsilon(a)). \end{aligned}$$

Hence Q vanishes on NA^+. Thus there is a map $\overline{Q}$ making

$$\begin{tikzcd} N \arrow[rr, "Q"] \arrow[dr, "\pi"'] & & N \\ & \overline{N} \arrow[ur, "\overline{Q}"'] & \end{tikzcd}$$

commute. In particular, if we define $Q_0 : C \longrightarrow C$ by $Q_0(c) = \Sigma\, c_{(1)} S(\psi(c_{(2)}))$ then Q_0 factors through $\overline{C}$, that is, there is a map $\overline{Q}_0 : \overline{C} \longrightarrow C$ with $Q_0 = \overline{Q}_0 p$. Note that we have

$$\omega_C (Q_0 \otimes \psi) \Delta_C = I_C .$$

Let $\beta : \overline{N} \,\square_{\overline{C}}\, C \longrightarrow N$ denote the composite

$$\overline{N} \,\square_{\overline{C}}\, C \xrightarrow{\text{inclusion}} \overline{N} \otimes C \xrightarrow{\overline{Q} \otimes \psi} N \otimes A \xrightarrow{\omega} N.$$

For any $n \in N$

$$\beta\alpha(n) = \beta(\Sigma\, \pi(n_{(0)}) \otimes n_{(1)})$$

$$= \Sigma\, n_{(0)} S(\psi(n_{(1)}))\psi(n_{(2)})$$

$$= \Sigma\, n_{(0)} S(\psi(n_{(1)})_{(1)})\psi(n_{(1)})_{(2)}$$

$$= \Sigma\, n_{(0)} \varepsilon\psi(n_{(1)}) = \Sigma\, n_{(0)} \varepsilon(n_{(1)}) = n.$$

For any $\pi(n) \in \overline{N}$, $c \in C$

$$\alpha\omega_N(\overline{Q} \otimes \psi)(\pi(n) \otimes c)$$

$$= (\pi \otimes I)\rho(Q(n)\psi(c))$$

$$= (\pi \otimes I)(\Sigma\, Q(n)_{(0)}\psi(c)_{(1)} \otimes Q(n)_{(1)}\psi(c)_{(2)})$$

$$= (\pi \otimes I)(\Sigma\, n_{(0)} S(\psi(n_{(3)}))\psi(c_{(1)}) \otimes n_{(1)} S(\psi(n_{(2)}))\psi(c_{(2)})$$

$$= \Sigma\, \pi(n_{(0)})\varepsilon(n_{(3)})\varepsilon(c_{(1)}) \otimes n_{(1)} S(\psi(n_{(2)}))\psi(c_{(2)})$$

$$= \Sigma\, \pi(n_{(0)}) \otimes n_{(1)} S(\psi(n_{(2)}))\psi(c)$$

$$= \Sigma\, \pi(n_{(0)}) \otimes Q_0(n_{(1)})\psi(c).$$

Let $\Sigma\, \pi(n) \otimes c \in \overline{N} \,\square_{\overline{C}}\, C$, thus

$$(\overline{\rho} \otimes I)(\Sigma\, \pi(n) \otimes c) = (I \otimes p \otimes I)(I \otimes \Delta)(\Sigma\, \pi(n) \otimes c).$$

Since $\overline{\rho}\pi = (\pi \otimes p)\rho_N$ we have

$$(\pi \otimes p \otimes I)(\rho_N \otimes I)(\Sigma\, n \otimes c) = (I \otimes p \otimes I)(I \otimes \Delta)(\Sigma\, \pi(n) \otimes c).$$

Applying $(I \otimes \omega_C)(I \otimes \overline{Q}_0 \otimes \psi)$ to this, we have

$$\Sigma\, \pi(n_{(0)}) \otimes Q_0(n_{(1)})\psi(c) = \Sigma\, \pi(n) \otimes c.$$

Thus we have shown that $\alpha\beta$ is the identity on $\overline{N} \,\square_{\overline{C}}\, C$. q.e.d.

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