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Confluence and Koszulity

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A lattice-theoretic approach to Koszulity is well known. This paper gives a lattice-theoretic approach to polynomiality, i.e., existence of a P.B.W. basis in Priddy's sense. This provides a straightforward proof for Priddy's theorem. Polynomiality is interpreted in terms of confluence in noncommutative computational algebra. The basic notion of a reduction operator is investigated in some detail. The reduction algebras are introduced to study the confluence of two reduction operators in terms of representation theory. The meet and the join of two reduction operators are constructed in an algorithmic way. A geometric characterization of the confluence is obtained.

Contents.

1. INTRODUCTION

We are concerned with (noncommutative) *quadratic* algebras. To define a quadratic algebra A, we have to take a finite dimensional vector space V with $r = \dim(V) \geq 1$ and a subspace E (the subspace of *relations* of A) of

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$V \otimes V$. The ground field will be denoted by K. In this paper, we study A (or equivalently E) through a choice of an *ordered* basis of V. Let us fix such an ordered basis $X = X^{(1)}$. The elements of X are called *generators*. The set $X^{(2)}$ of the noncommutative monomials of degree 2 in the generators is *lexicographically* ordered. By a triangular process (Section 2), it is possible to divide $X^{(2)}$ in two parts: the monomials called *reduced* w.r.t. E and the monomials called *nonreduced* w.r.t. E. Moreover, a unique basis of E is formed by all the relations of the following form (we shall frequently identify elements of $V \otimes V$ with equalities)

$$\textit{a nonreduced monomial} = \textit{a reduced expression}, \tag{1.1}$$

where the reduced expression on the right-hand side of the relation is a linear combination of reduced monomials which are *less* than the nonreduced monomial on the left-hand side. These relations give rise in an obvious manner to a projection (i.e., an idempotent endomorphism) S of $V \otimes V$ which is called the *X-reduction operator* of A (or E). The relations (1.1) are called the *relations defining* S. Reduction operators are the basic objects of this paper. The reduction operator S is defined on $V \otimes V$. More generally, a reduction operator can be defined on any finite dimensional vector space endowed with an ordered basis.

An n-degree monomial ($n \geq 3$) in the generators is called reduced w.r.t. E if each submonomial of degree 2 is reduced w.r.t. E. The unit and the generators are considered to be reduced monomials. The classes of all the reduced monomials generate the vector space A. If these classes are linearly independent, the algebra A is called *polynomial* and the basis thus obtained is a *PBW-basis* of A in Priddy's sense [9]. The terms of polynomiality and PBW-basis are frequently used, e.g., for algebras derived from quantum groups [8]. We prefer to say that the algebra A is *X-confluent*. In fact, the polynomiality of A is equivalent to saying that the X-reduction operator S is confluent, i.e., that the following equality holds

$$\dots S_1 S_2 S_1 = \dots S_2 S_1 S_2, \tag{1.2}$$

where $S_1 = S \otimes 1_V$, $S_2 = 1_V \otimes S$, and 1_V is the identity of V. It is noteworthy that the single equality (1.2) encodes all the Bergman confluence conditions [5]. The reduction operators S_1 and S_2 defined on $V \otimes V \otimes V$ are compositions of many Bergman reductions but they are sufficient to analyse the situation (see the end of Section 4 for a precise statement).

More generally, any two X-reduction operators T and U are said to be confluent if

$$\dots TUT = \dots UTU. \tag{1.3}$$

So the previous definition means that S_1 and S_2 are confluent. Some different characterizations for confluence of T and U are given in the text, from various points of view, such as lattice theory (Section 2), computational algebra (Section 3), representation theory of finite dimensional algebras (Section 4), and elementary projective geometry (Section 8). We must also notice that the evaluations on each side of (1.2) are reduced expressions obtained from a finite number of operators which is bounded once the number of generators is known. So (1.2) gives an algorithm for determining the confluence of S. The same holds for T and U.

In our terminology, Priddy's theorem asserts that if A is X-confluent, then A is Koszul. We offer two new proofs of Priddy's theorem. One proof (Section 5) consists in constructing an homotopy for the Koszul complex; this homotopy is based on the algebra representation which is set up in Section 4. The other proof (Section 2) uses a well-known distributivity criterion for Koszulity [3, 4]. This criterion means that some lattices generated by certain vector subspaces are distributive. By the triangular process, we have a distributivity criterion on *reduction operators*. In fact, reduction operators from a lattice via the one-to-one correspondence $T \mapsto \mathrm{Ker}(T)$ with subspaces (for some algorithmic constructions related to this lattice, see Section 6). Then, Priddy's theorem is a direct consequence of the following result.

THEOREM 1.1. *The sublattice generated by reduction operators* $T_1, \ldots, T_n$ $(n \geq 3)$ *which are pairwise confluent is distributive*.

A proof of this theorem is outlined as follows. The map $T \mapsto \mathrm{Im}(T)$ is strictly increasing, and pairwise confluence implies that its restriction to the sublattice is a lattice morphism. Hence the sublattice is isomorphic to its image, which is distributive in an obvious manner: all the $\mathrm{Im}(T)$ form a Boolean algebra. Note that there exists a converse to Theorem 1.1 (Section 7). On the other hand, a geometric proof of Theorem 1.1 is given in Section 8.

The dependence of the confluence on the choice of the ordered basis X will be examined elsewhere. We just say here that an algebra can be confluent for a basis and cannot be for another one, and there exists some Koszul algebras which are not confluent for *any* basis. The latter statement holds for the Sklyanin algebra in three generators (a proof of the Koszulity for Sklyanin algebras is given in [10]).

Notations and Conventions. Throughout this paper, K is a field and V is any finite dimensional vector space over K with $r = \dim(V) \geq 1$. For any integer $n \geq 0$, we let $V^{(n)} = V^{\otimes n}$, so we have $V^{(0)} = K, V^{(1)} = V$. The bases of any finite-dimensional vector space are always considered as *totally ordered*. Let X be a (totally ordered) basis of V. The elements of X

are called the *X-generators* of V. The noncommutative monomials in the X-generators are called X-monomials. We let $X^{(0)} = \{1\}$, $X^{(1)} = X$, and for $n \geq 2$, $X^{(n)}$ denotes the basis of $V^{(n)}$ which is the *lexicographically* ordered set of X-monomials of degree n. For example, if $r = 2$ and X is the basis $x < y$ of V, $X^{(2)}$ is the basis $xx < xy < yx < yy$ of $V^{(2)}$. Let E be a vector subspace of $V^{(2)} = V \otimes V$. We denote by $I(E)$ the two-sided ideal of the tensor algebra $\mathrm{Tens}(V)$ which is generated by E. It is naturally graded by the subspaces $I(E)_n$ which are given by $I(E)_0 = I(E)_1 = 0$ and

$$I(E)_n = \sum_{i+j+2=n} V^{(i)} \otimes E \otimes V^{(j)}, \qquad n \geq 2. \tag{1.4}$$

The algebra $A = \mathrm{Tens}(V)/I(E)$ is called a *quadratic algebra* on V, and E is the *subspace of relations* of the quadratic algebra A. The algebra A is graded by the subspaces $A_n = V^{(n)}/I(E)_n$. The *dual* quadratic algebra of A is the quadratic algebra $A^!$ on V^* with $E^\perp$ as subspace of relations. We have $I(E^\perp)_n = J(E)_n^\perp$, where $J(E)_0 = K$, $J(E)_1 = V$, and

$$J(E)_n = \bigcap_{i+j+2=n} V^{(i)} \otimes E \otimes V^{(j)}, \qquad n \geq 2. \tag{1.5}$$

The identity operator of V is denoted by 1_V, while the operator of V which is vanishing everywhere is denoted by 0_V.

2. A LATTICE-THEORETIC APPROACH TO PRIDDY'S THEOREM

Let X be a basis of V. We denote by $\leq$ the total order of X. For any nonzero element a of V, $hg(a)$ denotes the highest X-generator occurring in the linear decomposition of a in the basis X. We extend the relation $<$ to V in the following manner. For a and b in V with $a \neq 0$, we have $b < a$ if either $b = 0$ or $hg(b) < hg(a)$.

DEFINITION 2.1. A linear map $T : V \to V$ is called an X-reduction operator on V if the following hold:

(i) $T^2 = T$,

(ii) for each X-generator a, either $T(a) = a$ or $T(a) < a$.

The generator a is called *T-reduced* or *T-nonreduced* according to $T(a) = a$ or $T(a) < a$. The subspace generated by the T-reduced (T-nonreduced) generators is denoted by $\mathrm{Red}(T)$ ($\mathrm{Nred}(T)$). We have $V =$

$\mathrm{Red}(T) \oplus \mathrm{Nred}(T)$. If a is an element of V, there are three cases:

(1) $a \in \mathrm{Red}(T)$. Then, $T(a) = a$.

(2) $a \notin \mathrm{Red}(T)$ and $hg(a) \in \mathrm{Red}(T)$. Then, $hg(T(a)) = hg(a)$.

(3) $hg(a) \in \mathrm{Nred}(T)$. Then, $T(a) < a$.

In particular, $T(a) < a$ if $a \in \mathrm{Nred}(T)$ with $a \neq 0$. This implies $\mathrm{Nred}(T) \cap \mathrm{Im}(T) = 0$. As the inclusion $\mathrm{Red}(T) \subseteq \mathrm{Im}(T)$ is obvious, we obtain the equality $\mathrm{Red}(T) = \mathrm{Im}(T)$. Moreover, $1_V - T$ induces a linear isomorphism from $\mathrm{Nred}(T)$ onto $\mathrm{Ker}(T)$. The following lemma will be often used.

LEMMA 2.2. *Let T and U be two X-reduction operators.*

(i) *If $a < b$ in V, then $T(a) < b$.*

(ii) *If* $\mathrm{Ker}(T) \supseteq \mathrm{Ker}(U)$, *then* $\mathrm{Red}(T) \subseteq \mathrm{Red}(U)$.

Proof. (i) This is clear by considering the above three cases.

(ii) Let a be a generator which is T-reduced and U-nonreduced. Then $T(a - U(a)) = 0$, hence $T(U(a)) = a$. But $U(a) < a$, so (i) gives a contradiction. ∎

The matrix relative to X of an X-reduction operator is called a *reduction matrix*. The reduction matrices do not depend on the basis X. In fact, an $r \times r$ matrix M with entries in K is a reduction matrix if and only if the three following hold:

(i) M is upper triangular with 0 or 1 as diagonal entries.

(ii) The entries locates above a diagonal entry 1 (in the same column) all vanish.

(iii) The entries located on the right of a diagonal entry 0 (in the same row) all vanish.

The following theorem describes the X-reduction operators by means of their kernels. The set of the subspaces of the vector space V is denoted by $\mathscr{L}(V)$, while the set of the X-reduction operators on V is denoted by $\mathscr{L}_X(V)$.

THEOREM 2.3. *The map $\theta_X : \mathscr{L}_X(V) \to \mathscr{L}(V)$ sending T to* $\mathrm{Ker}(T)$ *is a bijection.*

Proof. Lemma 2.2(ii) shows immediately that θ_X is injective. For the surjectivity, we need a triangular process which is described as follows. Let E be a subspace of V, $E \neq 0$. Suppose $\mathscr{F}$ is a finite generating set of the vector space E, and the linear decomposition of each $a \in \mathscr{F}$ relative to the basis X is known. From these data, the triangular process provides constructively a basis $\mathscr{B}$ of E which does not depend on the choice of $\mathscr{F}$.

Precisely, we choose a_1 in $\mathscr{F}$ containing the highest X-generator (denoted by α_1) which occurs in the decompositions of all the a in $\mathscr{F}$. Up to a normalization, we can write

$$a_1 = \alpha_1 - \sum \alpha < \alpha_1,$$

where the sum means a linear combination of X-generators which are less than α_1. Next, we eliminate α_1 in $\mathscr{F} \setminus \{a_1\}$. We denote again by $\mathscr{F}$ the generating set thus obtained. If all the elements in $\mathscr{F} \setminus \{a_1\}$ vanish, the process stops. If not, we choose a_2 in $\mathscr{F} \setminus \{a_1\}$ as a_1 in $\mathscr{F}$ at the beginning, and we eliminate α_2 in $\mathscr{F} \setminus \{a_2\}$. Obviously, $\alpha_1 > \alpha_2$ and

$$a_1 = \alpha_1 - \sum \alpha < \alpha_1 \qquad \text{and} \qquad \alpha \neq \alpha_2,$$

with a clear meaning for the sum. At the end of the process, we obtain X-generators $\alpha_1 > \cdots > \alpha_n$ and a basis $\mathscr{B} = (a_1, \dots, a_n)$ of E such that

$$a_i = \alpha_i - \sum \alpha < \alpha_i \qquad \text{and} \qquad \alpha \neq \alpha_{i+1}, \dots, \alpha_n, \qquad 1 \leq i \leq n.$$

Denoting by F the subspace of V generated by the X-generators which are distinct from all the α_i, we define the linear operator T on the X-generators α by

$$T(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in F \\ \alpha_i - a_i & \text{if } \alpha = \alpha_i. \end{cases} \tag{2.1}$$

Then T is an X-reduction operator on V such that $\operatorname{Red}(T) = F$ and $\operatorname{Ker}(T) = E$. ∎

Note that $\mathscr{B}$ is determined by T, and so it does not depend on $\mathscr{F}$. Conversely $\mathscr{B}$ determines T. We say that $\mathscr{B}$ is the basis of $\operatorname{Ker}(T)$ *defining* T.

The bijection θ_X allows us to carry over the (opposite) lattice structure from $\mathscr{L}(V)$ to $\mathscr{L}_X(V)$. Let us recall that $\mathscr{L}(V)$ is a lattice for inclusion of subspaces, and meet (join) is intersection (sum) of subspaces. For any X-reduction operators T and U such that $\operatorname{Ker}(T) \supseteq \operatorname{Ker}(U)$, we let $T \leq U$. Note that $T \leq U$ is equivalent to the algebraic equality $TU = T$. Moreover, if $T \leq U$, then T and U commute (because $\operatorname{Im}(T) \subseteq \operatorname{Im}(U)$ by Lemma 2.2(ii)). Endowed with this order, $\mathscr{L}_X(V)$ is a lattice whose meet and join are denoted by $\wedge$ and $\vee$, respectively. So we have

$$\begin{cases} \operatorname{Ker}(T \wedge U) = \operatorname{Ker}(T) + \operatorname{Ker}(U) \\ \operatorname{Ker}(T \vee U) = \operatorname{Ker}(T) \cap \operatorname{Ker}(U). \end{cases} \tag{2.2}$$

The reason of reversing the order by θ_X is that 0_V and 1_V become respectively the lowest and the highest reduction operator. We are now interested in the map $T \mapsto \mathrm{Im}(T)$.

A subspace of V is called *X-generated* if it is generated by X-generators. We can identify an X-generated subspace to the set of X-generators which generates it. The set $\mathscr{P}_V(X)$ of the X-generated subspaces is a sublattice of $\mathscr{L}(V)$ which can be identified to the Boolean algebra of the subsets of X. We introduce the map

$$\pi_X : \mathscr{L}_X(V) \to \mathscr{P}_V(X)$$
$$T \mapsto \mathrm{Red}(T) = \mathrm{Im}(T).$$

Lemma 2.2(ii) shows that π_X is *strictly increasing*. In particular, $\pi_X^{-1}(F)$ is an antichain for any X-generated subspace F. Before proving surjectivity of π_X, we give a definition.

DEFINITION 2.4. An X-reduction operator T is said to be trivial if one of the following equivalent properties holds

(i) $\mathrm{Ker}(T) = \mathrm{Nred}(T)$,

(ii) $\mathrm{Ker}(T)$ is X-generated,

(iii) the matrix of T relative to X is diagonal.

PROPOSITION 2.5. *The set $\mathscr{L}_X^{tri}(V)$ of the trivial X-reduction operators is a sublattice of $\mathscr{L}_X(V)$, and π_X induces a lattice isomorphism from $\mathscr{L}_X^{tri}(V)$ onto $\mathscr{P}_V(X)$. In particular, π_X is surjective.*

Proof. Assume that T and U are trivial. By relations (2.2), $\mathrm{Ker}(T \wedge U)$ and $\mathrm{Ker}(T \vee U)$ are X-generated, thus $T \wedge U$ and $T \vee U$ are trivial. On the other hand, $\mathrm{Red}(T \wedge U)$ is the complement (in sense of $\mathscr{P}_V(X)$) of $\mathrm{Nred}(T) + \mathrm{Nred}(U)$, hence $\mathrm{Red}(T \wedge U) = \mathrm{Red}(T) \cap \mathrm{Red}(U)$. Analogously, we have $\mathrm{Red}(T \vee U) = \mathrm{Red}(T) + \mathrm{Red}(U)$. So, the restriction of π_X to $\mathscr{L}_X^{tri}(V)$ is a lattice morphism. If F is a X-generated subspace of V and if E is its complement in the sense of $\mathscr{P}_V(X)$, the projection T such that $\mathrm{Im}(T) = F$ and $\mathrm{Ker}(T) = E$ is the only trivial X-reduction operator whose image is F. ∎

The *dimension* of an X-reduction operator T is the number of T-reduced generators, and is denoted by $\dim(T)$. We have $\dim(T) = \dim(\mathrm{Im}(T)) = r - \dim(\mathrm{Ker}(T))$ and consequently

$$\dim(T \wedge U) + \dim(T \vee U) = \dim(T) + \dim(U). \qquad (2.3)$$

An easy exercise in lattice theory states that a strictly increasing lattice morphism is one-to-one. As there exists at least a nondiagonal reduction matrix if the number r of generators is >1, we see that π_X is *not* a lattice morphism in this case. We are going to examine this "defect" more closely.

Let T and U be any X-reduction operators. To abbreviate later notations, we let $P=(T,U)$. Since the map π_X is order-preserving, we have $\operatorname{Red}(T\wedge U)\subseteq \operatorname{Red}(T)\cap\operatorname{Red}(U)$ and $\operatorname{Red}(T)+\operatorname{Red}(U)\subseteq\operatorname{Red}(T\vee U)$. Therefore, we get two X-generated subspaces, denoted by Obs^P_{red} and Obs^P_{amb}, which are uniquely determined by the basic relations

$$\begin{cases}\operatorname{Red}(T)\cap\operatorname{Red}(U)=\operatorname{Red}(T\wedge U)\oplus \operatorname{Obs}^P_{red}\\ \operatorname{Red}(T\vee U)=(\operatorname{Red}(T)+\operatorname{Red}(U))\oplus\operatorname{Obs}^P_{amb}.\end{cases}\tag{2.4}$$

An X-generator belonging to $\operatorname{Red}(T)\cap\operatorname{Red}(U)$ $(\operatorname{Nred}(T)\cap\operatorname{Nred}(U))$ is called P-reduced (P-ambiguous). The X-generators in $\operatorname{Obs}^P_{red}$ or $\operatorname{Obs}^P_{amb}$ are called *P-obstructions*. The latter terminology is taken from [1], while the term ambiguous is taken from [5]. Equation (2.3) shows that $\operatorname{Obs}^P_{red}$ and $\operatorname{Obs}^P_{amb}$ have the *same* dimension. This dimension is called the *confluence defect* of T and U (or of the pair P). It is denoted by $\operatorname{def}_c(P)$. We arrive at the central notion of this paper.

DEFINITION 2.6. Two X-reduction operators T and U are called confluent if their confluence defect vanishes.

Our first examples are provided by the following proposition.

PROPOSITION 2.7. *Let T and U be two X-reduction operators.*

(i) *If T and U commute, they are confluent and*

$$\begin{cases}T\wedge U=TU\\ T\vee U=T+U-TU.\end{cases}\tag{2.5}$$

In particular, the commutativity assumption holds in case T and U are comparable for $\leq$, or in case they are trivial.

(ii) *If T and U are confluent with* $\operatorname{Red}(T)\subseteq\operatorname{Red}(U)$, *then $T\leq U$. In particular, if* $\operatorname{Red}(T)=\operatorname{Red}(U)$ *and $T\neq U$, T and U are not confluent.*

Proof. (i) It is well known that $TU=UT$ is a projection such that $\operatorname{Ker}(TU)=\operatorname{Ker}(T)+\operatorname{Ker}(U)$ and $\operatorname{Im}(TU)=\operatorname{Im}(T)\cap\operatorname{Im}(U)$. So $\operatorname{Ker}(TU)=\operatorname{Ker}(T\wedge U)$ and $\operatorname{Im}(TU)\supseteq\operatorname{Im}(T\wedge U)$, which yields the equality $T\wedge U=TU$ for dimensional reason. Moreover $\operatorname{Obs}^P_{red}$ vanishes, which implies the confluence of T and U. Reasoning in a similar fashion when $T+U-TU$ replaces TU, the second equality of (2.5) is obtained. The first particular case has already been observed. The second one is clear because two diagonal reduction matrices commute.

(ii) We have $\mathrm{Red}(T) = \mathrm{Red}(T \wedge U)$ by (2.4). Along with $T \wedge U \leq T$, this implies $T \wedge U = T$ forcing $T \leq U$. ∎

There is a *duality* for reduction operators which is drawn from Koszul duality of quadratic algebras. One of the benefits of this duality is to cut in half some proofs. Let V^* be the dual space of V. If T is any projection operator on V, we let

$$T^! = 1_{V^*} - T^*,$$

where T^* is the transpose of T. Then $T^!$ is a projection operator on V^* such that $\mathrm{Ker}(T^!) = (\mathrm{Ker}\, T)^\perp$, $\mathrm{Im}(T^!) = (\mathrm{Im}\, T)^\perp$, and $T^{!!}$ is naturally identified to T. We denote by $X^!$ the dual basis of X endowed with the *reverse* order. In other words, if X is $x_1 < \cdots < x_r$, then $X^!$ is $x_r^* < \cdots < x_1^*$. If F is an X-generated subspace of V, $F^\perp$ is an $X^!$-generated subspace of V^* whose generators are the dual of the generators not belonging to F.

PROPOSITION 2.8. (i) *Let T be an X-reduction operator on V. Then $T^!$ is an $X^!$-reduction operator on V^* and* $\mathrm{Red}(T^!) = \mathrm{Red}(T)^\perp$. *In particular,* $\dim(T^!) = r - \dim(T)$.

(ii) *The map*

$$\kappa_X : \mathscr{L}_X(V) \to \mathscr{L}_{X^!}(V^*)$$
$$T \mapsto T^!$$

is an anti-isomorphism of lattices.

Proof. (i) Let M and $M^!$ be the respective matrices of T and $T^!$ relative to X and $X^!$. The transformation which sends M to $M^!$ consists of the symmetry with respect to the anti-diagonal followed by the exchange between the 0's and the 1's for the diagonal entries and the change of sign for the nondiagonal ones. So $M^!$ is a reduction matrix with the good reduced generators.

(ii) κ_X is a bijection, and κ_X and its inverse are order-reversing. ∎

Remark 2.9. Taking a closer look at duality on reduction matrices, we may view κ_X as a Fourier transform. Let us define the *support*(*spectrum*) of an X-reduction operator T as the subspace generated by the nonreduced (reduced) X-generators a such that $T(a) \neq 0$ (a occurs in the decomposition of $T(b)$ for some nonreduced generator b). Then κ_X exchanges support and spectrum, as the usual Fourier transform. This suggests the following question: Does there exist a differential calculus on $\mathscr{L}_X(V)$ which would be compatible with κ_X? (Also: What is the role of the reduction operators with one-dimensional support?) The situation is really new because there is *no* linear (or algebra) structure on $\mathscr{L}_X(V)$: sum or composition of two X-reduction operators is not an X-reduction operator in general.

The effect of duality on the pair $P = (T, U)$ is easily described. For brevity, we shall use the notations

$$\mathrm{Red}(P) = \mathrm{Red}(T) \cap \mathrm{Red}(U), \qquad \mathrm{Amb}(P) = \mathrm{Nred}(T) \cap \mathrm{Nred}(U),$$

and $\mathrm{Nred}(P)$ and $\mathrm{Namb}(P)$ will be the respective complements of $\mathrm{Red}(P)$ and $\mathrm{Amb}(P)$. Then we have the following

$$\mathrm{Red}(P^!) = (\mathrm{Namb}(P))^{\perp}, \qquad \mathrm{Amb}(P^!) = (\mathrm{Nred}(P))^{\perp}. \quad (2.6)$$

Applying passage to orthogonal subspaces in (2.4) and comparing the result with the same equations (2.4) for $P^!$, we see that the reduced (ambiguous) $P^!$-obstructions are dual of the ambiguous (reduced) P-obstructions. In particular, $\mathrm{def}_c(P^!) = \mathrm{def}_c(P)$ and $P^!$ is confluent if and only if P is. Our purpose is now to study pairwise confluence of many reduction operators.

DEFINITION 2.10. A non-empty set P of X-reduction operators is said to be 2-ply confluent if the elements of P are pairwise confluent.

In Section 3, a k-ply confluence for $k \geq 3$ will be introduced and it will turn out to be weaker than 2-ply confluence. For the time being, Proposition 2.7(i) shows that any commuting set of X-reduction operators is 2-ply confluent. Moreover if S, T, U are pairwise commuting X-reduction operators, then S commutes with $T \wedge U$ and $T \vee U$. So the sublattice generated by a commuting set is a commuting set. This generation property extends to pairwise confluence, but the proof is less obvious.

PROPOSITION 2.11. *Let P be a* 2*-ply confluent set of X-reduction operators. The sublattice generated by P is* 2*-ply confluent.*

Proof. Suppose that S, T, U are pairwise confluent. By duality, it suffices to prove that S is confluent with $T \wedge U$. This fact is a direct consequence of the following inequality which holds for *any* reduction operators:

$$\begin{aligned}\mathrm{def}_c(S, T \wedge U) &+ \mathrm{def}_c(S \vee T, S \vee U) \\ &\leq \mathrm{def}_c(S, T) + \mathrm{def}_c(S, U) + \mathrm{def}_c(T, U). \qquad (2.7)\end{aligned}$$

In order to prove (2.7), we begin with the inequality $S \vee (T \wedge U) \leq (S \vee T) \wedge (S \vee U)$ which holds in any lattice. Using (2.4), we deduce

$$\mathrm{Red}(S \vee (T \wedge U)) \oplus \mathrm{Obs}_{red}^{S \vee T, S \vee U} \subseteq (\mathrm{Red}(S \vee T) \cap \mathrm{Red}(S \vee U)).$$

Denoting by L and R the left-hand side and the right-hand side of this inclusion and using repeatedly (2.4), we develop L and R in order to arrive at

$$L = \mathrm{Red}(S) \oplus (\mathrm{Nred}(S) \cap \mathrm{Red}(T \wedge U)) \oplus \mathrm{Obs}_{amb}^{S,T \wedge U} \oplus \mathrm{Obs}_{red}^{S \vee T, S \vee U},$$

$$R \subseteq \mathrm{Red}(S) \oplus (\mathrm{Nred}(S) \cap \mathrm{Red}(T \wedge U)) \oplus \left(\mathrm{Nred}(S) \cap \mathrm{Obs}_{red}^{T,U}\right)$$
$$\oplus \left(\mathrm{Obs}_{amb}^{S,T} + \mathrm{Obs}_{amb}^{S,U}\right).$$

Comparing, we get

$$\mathrm{Obs}_{amb}^{S,T \wedge U} \oplus \mathrm{Obs}_{red}^{S \vee T, S \vee U} \subseteq \left(\mathrm{Obs}_{amb}^{S,T} + \mathrm{Obs}_{amb}^{S,U}\right)$$
$$\oplus \left(\mathrm{Nred}(S) \cap \mathrm{Obs}_{red}^{T,U}\right), \qquad (2.8)$$

and relation (2.7) follows. ∎

The lattice $\mathscr{L}_X(V)$ of the X-reduction operators is *not* distributive (it is well known that the lattice of the subspaces of V is not distributive) but it contains distributive sublattices. The following theorem states essentially that, for the sublattices, *pairwise confluence implies distributivity*. Later on, a geometric set-up of confluence will give a more transparent proof of this implication (see Section 8).

THEOREM 2.12. (i) *Let P be a* 2*-ply confluent sublattice of $\mathscr{L}_X(V)$. Then the restriction of π_X to P is a lattice isomorphism of P onto its image. In particular, P is a finite distributive lattice with* $\mathrm{Card}(P) \le 2^r$.

(ii) *Let P be a* 2*-ply confluent subset of $\mathscr{L}_X(V)$. Then the sublattice generated by P is distributive and P is finite with* $\mathrm{Card}(P) \le 2^r$. *Moreover if P is a maximal* 2*-ply confluent set, P is a sublattice isomorphic to $\mathscr{P}_V(X)$.*

Proof. (i) Relations (2.4) show that the restriction of π_X to P is a lattice morphism. Thus the first assertion comes from the (already mentioned) fact that a strictly increasing lattice morphism is one-to-one. The Boolean algebra $\mathscr{P}_V(X)$ is a finite distributive lattice with 2^r elements, so the second assertion follows.

(ii) All is a consequence of (i) and the preceding proposition, except the isomorphism assertion which will be proved geometrically in Section 8. ∎

Proposition 2.7(i) provides two examples. Firstly $\mathscr{L}_X^{tri}(V)$ is a maximal 2-ply confluent (and even, maximal commuting) set. Secondly, any chain P in $\mathscr{L}_X(V)$ is 2-ply confluent and $\pi_X(P)$ is a chain with same length. (Conversely, if $P = (T_1, \dots, T_n)$ is 2-ply confluent and satisfies $\mathrm{Red}(T_1) \subset \dots \subset \mathrm{Red}(T_n)$, then P is the chain $T_1 < \dots < T_n$).

We are now ready to specialize our framework to quadratic algebras. We freely use the general notations of the end of Section 1. Let A be a quadratic algebra on V whose subspace of relations is E. According to [3], A is Koszul if and only if for each $n > 3$ the sublattice of $\mathscr{L}(V^{(n)})$ generated by the subspaces $V^{(i-1)} \otimes E \otimes V^{(n-i-1)}$, $1 \leq i \leq n-1$, is *distributive* (the distributivity is clear for $n = 2, 3$). Let X be a basis of V. For brevity, an $X^{(n)}$-reduction operator on $V^{(n)}$ is called an X-reduction operator. Let S be the X-reduction operator on $V^{(2)}$ such that $\operatorname{Ker}(S) = E$. We let

$$S_n = 1_{V^{(n-1)}} \otimes S, \qquad n \geq 1.$$

It is a general fact that the tensor product of two reduction operators is a reduction operator on the tensor product of the spaces endowed with the lexicographically ordered tensor product basis. So S_n is an X-reduction operator on $V^{(n+1)}$. Tensoring on the right by an appropriate identity, we may consider S_n as an X-reduction operator on $V^{(m)}$ for $m > n$. With this convention, we can regard the operators S_i, $1 \leq i \leq n-1$, on the *same* space $V^{(n)}$. The images of these operators under the lattice anti-isomorphism θ_X are the subspaces $V^{(i-1)} \otimes E \otimes V^{(n-i-1)}$, $1 \leq i \leq n-1$. In particular, $I(E)_n = \operatorname{Ker}(S_1 \wedge \cdots \wedge S_{n-1})$, thus the projection $S_1 \wedge \cdots \wedge S_{n-1}$ induces a linear isomorphism from A_n onto $\operatorname{Red}(S_1 \wedge \cdots \wedge S_{n-1})$. In other words, the classes of the X-monomials of degree n which are $S_1 \wedge \cdots \wedge S_{n-1}$-reduced form a *basis* of the vector space A_n. This basis is called the *canonical* X-basis of A_n. In Section 6, we shall see an algorithm for the computation of $T \wedge U$ and $T \vee U$ from T and U. So the canonical X-basis of A_n can be computed step by step as n increases. In the same manner, $1_V - S_1 \vee \cdots \vee S_{n-1}$ induces a linear isomorphism from $\operatorname{Nred}(S_1 \vee \cdots \vee S_{n-1})$ onto $J(E)_n$.

DEFINITION 2.13. With the above notations, the algebra A (or the subspace E) is said to be X-confluent if $S_1 = S \otimes 1_V$ and $S_2 = 1_V \otimes S$ are confluent on $V^{(3)}$.

Clearly, A is X-confluent if and only if $A^!$ is $X^!$-confluent. Suppose A is X-confluent. For every $i \geq 1$, S_i and S_{i+1} are confluent. Moreover S_i and S_j commute if $|i - j| > 1$. So the operators S_i, $1 \leq i \leq n-1$, are pairwise confluent. An easy induction on n shows

$$\operatorname{Red}(S_1 \wedge \cdots \wedge S_{n-1}) = \bigcap_{i=1}^{n-1} V^{(i-1)} \otimes \operatorname{Red}(S) \otimes V^{(n-i-1)}, \qquad n \geq 3.$$

This means that the canonical X-basis of A_n is the set of the classes of the monomials whose submonomials of degree 2 are all S-reduced. (Conversely, this fact for $n = 3$ exactly says that A is confluent.) By duality, we have also

$$\mathrm{Nred}(S_1 \vee \cdots \vee S_{n-1}) = \bigcap_{i=1}^{n-1} V^{(i-1)} \otimes \mathrm{Nred}(S) \otimes V^{(n-i-1)}, \qquad n \geq 3.$$

On the other hand, using Theorem 2.12 and the lattice anti-isomorphism θ_X, we see that A is Koszul. The result thus obtained is the promised lattice theoretic formulation of Priddy's theorem.

THEOREM 2.14. *Any X-confluent quadratic algebra is Koszul.*

Let E be generated by monomials. This means that S is trivial. Accordingly, S_1, S_2 are trivial, hence confluent. Thus we recover the fact due to Fröberg [7] that any quadratic algebra with monomial relations is Koszul. The algorithm set up in the next section will provide many other examples.

3. CONFLUENCE FROM A COMPUTATIONAL VIEWPOINT

In computational algebra, confluence is a basic concept which is defined at the level of sets. This concept is useful within various algebraic frameworks. A standard example in commutative algebra is provided by Gröbner bases [2]. The aim of this section is to show that the confluence of two reduction operators as defined in the previous section is a particular case of the general concept. Here the algebraic framework consists of a pair (or more generally a finite set) P of X-reduction operators on a finite dimensional vector space V endowed with a basis X. This runs parallel to Gröbner bases. For the latter, P is a finite subset of the (commutative) polynomial algebra $K[x_1, \ldots, x_r]$, and reductions act on $K[x_1, \ldots, x_r]$ itself. In our context, the elements of P belong to the lattice $\mathscr{L}_X(V)$ which is contained in the (noncommutative) algebra of the linear operators on V, and reductions act on V. In both cases, the order $x_1 < \cdots < x_r$ is of crucial importance and the finiteness assumption on P can be removed.

We begin with a brief review of confluence in computational algebra (see [2] for a thorough set-up). Let $\to$ be a *reduction relation* on a non-empty set M. This means that $\to$ is a relation on M such that it is excluded to have simultaneously $a \to b$ and $b \to a$. We denote by $\overset{*}{\to}$ the reflexive-transitive closure of $\to$. In other words, we have $a \overset{*}{\to} a$ for any $a \in M$, and $a_0 \to \cdots \to a_n$ is shortened in $a_0 \overset{*}{\to} a_n$. An element a of M is a *normal form* (or it is in normal form) if a is $\to$ -maximal in M, and it is a normal form of $b \in M$ if we have furthermore $b \overset{*}{\to} a$. The notation $a \downarrow b$ means that there exists $c \in M$ with $a \overset{*}{\to} c$ and $b \overset{*}{\to} c$ (or $a \overset{*}{\to} c \overset{*}{\leftarrow} b$ for short).

DEFINITION 3.1. Let $\to$ be a reduction relation on M. Then $\to$ is said

(i) to be confluent if $b \overset{*}{\leftarrow} a \overset{*}{\to} c$ implies $b \downarrow c$ for all a, b, c in M,

(ii) to be locally confluent if $b \leftarrow a \to c$ implies $b \downarrow c$ for all a, b, c in M,

(iii) to have unique normal form if $b \overset{*}{\leftarrow} a \overset{*}{\to} c$ with b and c in normal form implies $b = c$ for all a, b, c in M.

Newman's lemma states that the three properties in this definition are equivalent if the reduction relation $\to$ is *Noetherian*. It is also worth remarking that each element of M has at least one normal form if $\to$ is Noetherian. We can now return to our context.

Throughout the rest of this section, $P = (T_1, \dots, T_n)$ is an n-tuple ($n \geq 2$) of X-reduction operators on V. We shall use the notations

$$\operatorname{Red}(P) = \bigcap_{i=1}^{n} \operatorname{Red}(T_i), \qquad \operatorname{Amb}(P) = \bigcap_{i=1}^{n} \operatorname{Nred}(T_i),$$

and $\operatorname{Nred}(P)$, $\operatorname{Namb}(P)$ will be the respective complements (in sense of the Boolean algebra of the X-generated subspaces) of $\operatorname{Red}(P)$, $\operatorname{Amb}(P)$. An X-generator belonging to $\operatorname{Red}(P)$ ($\operatorname{Amb}(P)$) is called P-reduced (P-ambiguous). We define a relation $\underset{P}{\to}$ on V as follows. Let a, b be in V. We have $a \underset{P}{\to} b$ if there exists i, $1 \leq i \leq n$, such that the two following conditions hold:

(i) a does not belong to the subspace $\operatorname{Red}(T_i)$,

(ii) $b = T_i(a)$.

In particular, a does not belong to $\operatorname{Red}(P)$. Moreover, we can uniquely write $a = a_1 + a_2$ with $a_1 \in \operatorname{Red}(T_i)$, $a_2 \in \operatorname{Nred}(T_i)$, and we have $a_2 \neq 0$ by (i). Then (ii) becomes $b = a_1 + T_i(a_2)$, with $T_i(a_2) < a_2$.

LEMMA 3.2. *The so-defined relation* $\underset{P}{\to}$ *is a Noetherian reduction relation on* V.

Proof. Suppose $a \underset{P}{\to} b$ with the above notations. We denote respectively by A and B the sets of the non-P-reduced X-generators occurring in the linear decomposition of a and b relatively to the basis X. Note that A

is non-empty. Let C be the set of the elements x of A on which T_i acts nonidentically, i.e., belonging to $\mathrm{Nred}(T_i)$. Then (i) means that C is non-empty and (ii) implies

$$B \subseteq (A \setminus C) \cup \{x \notin C; x < \max(C)\},$$

where $\max(C)$ is the greatest element of the subset C of the totally ordered set X. We abbreviate all this by $A \underset{E}{\rightarrow} B$ where E is the set of the non-P-reduced X-generators. In fact, the latter relation concerns the subsets of E. So $a \underset{P}{\rightarrow} b$ implies $A \underset{E}{\rightarrow} B$. To prove the result, it suffices to show the following.

LEMMA 3.3. *Let E be a totally ordered finite set. On the set of subsets of E, the relation $\underset{E}{\rightarrow}$ is defined as follows. For A and B contained in E, we have $A \underset{E}{\rightarrow} B$ if A is non-empty and there exists a non-empty subset C of A such that the following holds*

$$B \subseteq (A \setminus C) \cup \{x \notin C; x < \max(C)\}, \tag{3.1}$$

where $\max(C)$ *is the greatest element of the subset C. Then $\underset{E}{\rightarrow}$ is a Noetherian reduction relation.*

Proof. Suppose $A \underset{E}{\rightarrow} B$ and $B \underset{E}{\rightarrow} A$. There exist $C \neq \varnothing$ contained in A and $D \neq \varnothing$ contained in B satisfying (3.1) and the following

$$A \subseteq (B \setminus D) \cup \{x \notin D; x < \max(D)\}. \tag{3.2}$$

By (3.1), we have $\max(C) \in A$ and $\max(C) \notin B$. Thus (3.2) implies $\max(C) < \max(D)$. Reversing the roles of A and B, we arrive at a contradiction. So $\underset{E}{\rightarrow}$ is a reduction relation. Note that the empty set is the only normal form of any non-empty subset A. In order to prove that $\underset{E}{\rightarrow}$ is Noetherian, we proceed by induction on $e = \mathrm{Card}(E)$. The case $e = 1$ is obvious. We assume that $\mathrm{Card}(E) = e \geq 2$ and the property is true for any integer $< e$. Let A and B be two non-empty subsets of E such that $A \underset{E}{\rightarrow} B$. Then $\max(B) \leq \max(A)$ by (3.1). Let us examine the "bad" case $\max(A) = \max(B) = \mu$. We let $\overline{E} = \{x \in E; x < \mu\}$, $\overline{A} = A \cap \overline{E}$, $\overline{B} = B \cap \overline{E}$, $\overline{C} = C \cap \overline{E}$. Relation (3.1) shows that $\mu \in A \setminus C$. In particular, every element of C is $< \mu$, implying $\overline{C} = C$. Furthermore, $\overline{A}$ is non-empty (otherwise, we would have $A = \{\mu\} = C$ and a contradiction) and it is easy to check that $\overline{A} \underset{\overline{E}}{\rightarrow} \overline{B}$. More generally, from a chain

$$A_0 \underset{E}{\rightarrow} \cdots \underset{E}{\rightarrow} A_k,$$

with $\max(A_0) = \cdots = \max(A_k)$, we deduce

$$\overline{A_0} \underset{\overline{E}}{\to} \cdots \underset{\overline{E}}{\to} \overline{A_k}.$$

By induction, k is $\leq S$ where S is the supremum of the lengths of all the chains w.r.t. the relation $\underset{\overline{E}}{\to}$. Indeed S is finite because $\underset{\overline{E}}{\to}$ is Noetherian and $\overline{E}$ is finite. Accordingly, for *any* chain $A_0 \underset{E}{\to} \cdots \underset{E}{\to} A_k$ with $k = eS + e - 1$, we have necessarily $A_k = \emptyset$. Therefore $\underset{E}{\to}$ is Noetherian. ∎

Our purpose now is to investigate the Noetherian reduction relation $\underset{P}{\to}$ in order to obtain some useful characterizations of its confluence (Theorem 3.6 below). We begin by noticing that the set of the normal forms is $\mathrm{Red}(P)$. To go further, we need some definitions.

DEFINITION 3.4. A linear operator S on V is called a P-composition if $S = T_{i_s} \dots T_{i_1}$ with $s \geq 1, 1 \leq i_k \leq n$, and $i_k \neq i_{k+1}$, $1 \leq k \leq s - 1$. A P-composition $S = T_{i_s} \dots T_{i_1}$ is contained in a P-composition $S' = T_{j_t} \dots T_{j_1}$ if $t \geq s$ and $i_k = j_k$ for $1 \leq k \leq s$.

Clearly, $a \overset{*}{\underset{P}{\to}} b$ if and only if there exists a P-composition S such that $S(a) = b$. A P-composition S is said to *reduce* $a \in V$ if $S(a)$ belongs to $\mathrm{Red}(P)$. Then, any P-composition containing S reduces a. An element b of V is a normal form of a if and only if $b = S(a)$, where S is a P-composition reducing a. A P-composition S satisfies the inclusion $\mathrm{Red}(P) \subseteq \mathrm{Im}(S)$. We say that S is *complete* if this inclusion is an equality. It is equivalent to say that S reduces every element (or every generator) of V. If S is complete, we have $T_i S = S$, $1 \leq i \leq n$. To construct a complete P-composition S, we proceed as follows. We start by choosing a P-composition S_1 reducing the first generator. Then there is a P-composition S_2 containing S_1 and reducing the two first generators, and we pursue inductively to get S. A similar construction shows that any P-composition is contained in a complete P-composition. If S runs over the set of the complete P-compositions, then for each $a \in V, S(a)$ runs over the set of the normal forms of a.

PROPOSITION 3.5. *A complete P-composition S is an X-reduction operator such that* $\mathrm{Red}(S) = \mathrm{Red}(P)$ *and* $S \geq T_1 \wedge \cdots \wedge T_n$.

Proof. Clearly $S^2 = S$ and $S(a) = a$ for every a in $\mathrm{Red}(P)$. Writing $S = T_{i_s} \dots T_{i_1}$, we choose a non-$P$-reduced X-generator a. There is an integer k, $1 \leq k \leq s$ such that $T_{i_{k-1}}(a) = \cdots = T_{i_1}(a) = a$ and $T_{i_k}(a) < a$. Then Lemma 2.2(i) shows that $S(a) < a$. So S is an X-reduction operator such that $\mathrm{Red}(S) = \mathrm{Red}(P)$. For any non-$P$-reduced X-generator a, the following holds

$$a - S(a) = \sum_{k=1}^{s} T_{i_{k-1}} \dots T_{i_1}(a) - T_{i_k} \dots T_{i_1}(a).$$

In particular, $a - S(a)$ belongs to $\sum_{i=1}^{n} \mathrm{Ker}(T_i) = \mathrm{Ker}(T_1 \wedge \cdots \wedge T_n)$. Thus $\mathrm{Ker}(S)$ is contained in $\mathrm{Ker}(T_1 \wedge \cdots \wedge T_n)$ as expected. ∎

Generalizing the first relation (2.4), we define the X-generated subspace Obs_{red}^{P} by

$$\mathrm{Red}(P) = \mathrm{Red}(T_1 \wedge \cdots \wedge T_n) \oplus \mathrm{Obs}_{red}^{P}. \tag{3.3}$$

The X-generators of Obs_{red}^{P} are called reduced P-obstructions. It is worth noticing the following induction formula (the proof is left to the reader),

$$\mathrm{Obs}_{red}^{P} = \mathrm{Obs}_{red}^{T_j, \wedge P_j} \oplus \left(\mathrm{Red}(T_j) \cap \mathrm{Obs}_{red}^{P_j}\right), \qquad 1 \le j \le n, \tag{3.4}$$

where P_j is the $(n-1)$-tuple obtained from P by deleting T_j and $\wedge P_j$ is the meet of the operators of P_j (we let $\mathrm{Obs}_{red}^{T} = 0$ for any X-reduction operator T). This formula allows us, if necessary, to delete possible repetitions in the n-tuple P.

THEOREM 3.6. *The following are equivalent.*

(i) *The reduction relation* $\underset{P}{\to}$ *is confluent.*

(ii) *All the complete P-compositions coincide.*

(iii) *For every distinct $i, j, 1 \le i, j \le n$, there exists a P-composition containing T_i and T_j.*

(iv) $T_1 \wedge \cdots \wedge T_n$ *is a complete P-composition.*

(v) $\mathrm{Obs}_{red}^{P} = 0$.

Proof. By Newman's lemma, $\underset{P}{\to}$ is confluent if and only if any $a \in V$ has unique normal form, i.e., $S(a) = S'(a)$ for any complete P-compositions S and S', whence the equivalence (i) ⇔ (ii). The implication (ii) ⇒ (iii) is clear, and the converse (iii) ⇒ (i) comes from the local confluence and Newman's lemma. Supposing (ii) and denoting by S the unique complete P-composition, we see that S contains each T_i, implying $\mathrm{Ker}(S) \supseteq \mathrm{Ker}(T_i)$. Thus we have $\mathrm{Ker}(S) \supseteq \mathrm{Ker}(T_1 \wedge \cdots \wedge T_n)$ which, along with the reverse inclusion seen in Proposition 3.5, gives $S = T_1 \wedge \cdots \wedge T_n$. So (iv) holds. Supposing (iv) and recalling Proposition 3.5, we have $\mathrm{Red}(T_1 \wedge \cdots \wedge T_n) = \mathrm{Red}(P)$, hence (v). Finally, if we have (v) and if S is a complete P-composition, we know that $S \ge T_1 \wedge \cdots \wedge T_n$ and $\mathrm{Red}(S) = \mathrm{Red}(P)$. Using $\mathrm{Red}(S) = \mathrm{Red}(T_1 \wedge \cdots \wedge T_n)$, we conclude that $S = T_1 \wedge \cdots \wedge T_n$, hence (ii). ∎

DEFINITION 3.7. The n-tuple $P = (T_1, \dots, T_n)$ is said to be confluent if the equivalent conditions of the previous theorem hold. The dimension of Obs^P_{red} is called the confluence defect of P and is denoted by $\mathrm{def}_c(P)$.

In case $n = 2$, the confluence defect and the confluence of the pair P agree with the definitions given in the previous section. Keep the assumption $n = 2$. Then there is exactly one complete P-composition containing T_i, $i = 1, 2$. Indeed, if we denote it by R^P_i, we necessarily have

$$R^P_1 = \dots T_1 T_2 T_1, \qquad R^P_2 = \dots T_2 T_1 T_2, \tag{3.5}$$

where the number of factors on the right-hand sides is sufficiently large to reduce everything. Clearly, $2r$ factors are sufficient. As a corollary of the previous theorem, we get:

THEOREM 3.8. *The X-reduction operators T_1 and T_2 are confluent if and only if the following holds*

$$R^P_1 = R^P_2. \tag{3.6}$$

In this case, $T_1 \wedge T_2 = R^P_1 = R^P_2$.

Equality (3.6) provides an algorithm deciding confluence of T_1 and T_2. Since this equality holds once evaluated on the non-ambiguous X-generators, it suffices to test it on the ambiguous X-generators. This algorithm will be commonly used in the examples. We want now to generalize the 2-ply confluence.

DEFINITION 3.9. Let Q be any set of X-reduction operators with $\mathrm{Card}(Q) \geq 2$. Let k be an integer such that $2 \leq k \leq \mathrm{Card}(Q)$. We say that Q is k-ply confluent (or shortly k-confluent) if any subset of k (distinct) operators in Q is confluent. (If $\mathrm{Card}(Q) = n$, the n-confluence of Q is the same as the confluence of Q).

PROPOSITION 3.10. *Let Q be any set of X-reduction operators with* $\mathrm{Card}(Q) \geq 3$. *Let k, l be such that* $2 \leq k < l \leq \mathrm{Card}(Q)$. *Then k-confluence of Q implies l-confluence of Q.*

Proof. We use Theorem 3.6(iii). Suppose that Q' is a subset of l operators in Q and T, U are distinct in Q'. Then T and U belong to some subset $Q'' \subset Q'$ with $\mathrm{Card}(Q'') = k$. By k-confluence assumption, there exists a Q''-composition S containing T and U. But S is a Q'-composition. Thus Q' is confluent. ∎

In particular, if the n-tuple P is pairwise confluent, P is confluent. Proposition 3.11 below shows that the converse is false for $n \geq 3$. As we shall see in Section 7, some natural additional assumptions yield a converse for $n = 3$. We end this section by a sketch of duality for the n-tuples.

We define the dual n-tuple of P by $P^! = (T_1^!, \ldots, T_n^!)$. Suppose $S : V \to V$ is a linear map such that $S^! = 1_{V^*} - S^*$ is a $P^!$-composition. Then we can write $S = 1_V - (1_V - T_{i_1}) \ldots (1_V - T_{i_s})$ with $s \geq 1$, $1 \leq i_k \leq n$, and $i_k \neq i_{k+1}$, $1 \leq k \leq s-1$. From $\mathrm{Red}(P^!) \subseteq \mathrm{Im}(S^!)$, we deduce $\mathrm{Im}(S) \subseteq \mathrm{Namb}(P)$. The $P^!$-composition $S^!$ is complete if and only if the latter inclusion is an equality. If $S^!$ is complete, S an X-reduction operator such that $\mathrm{Red}(S) = \mathrm{Namb}(P)$ and $S \leq T_1 \vee \cdots \vee T_n$. The X-generated subspace Obs^P_{amb} is defined by

$$\mathrm{Red}(T_1 \vee \cdots \vee T_n) = \mathrm{Namb}(P) \oplus \mathrm{Obs}^P_{amb}. \tag{3.7}$$

The X-generators of Obs^P_{amb} are called ambiguous P-obstructions. The dual ones of the latter are the reduced $P^!$-obstructions. With analogous notations, the dual formula of (3.4) is

$$\mathrm{Obs}^P_{amb} = \mathrm{Obs}^{T_j, \vee P_j}_{amb} \oplus \left(\mathrm{Nred}\left(T_j\right) \cap \mathrm{Obs}^{P_j}_{amb}\right), \qquad 1 \leq j \leq n. \tag{3.8}$$

If $P^!$ is confluent, P is said to be *dual*-confluent. Since $\mathrm{def}_c(P^!) = \dim(\mathrm{Obs}^P_{amb})$, P is dual-confluent if and only if $\mathrm{Obs}^P_{amb} = 0$. In case $n = 2$, we have $\mathrm{def}_c(P^!) = \mathrm{def}_c(P)$, but it is no longer true for $n \geq 3$ (Section 7). In other words, 2-confluence is self-dual, but not k-confluence for $k \geq 3$. Using Theorem 3.6(v) or its dual statement, the following result is easily proved.

PROPOSITION 3.11. *If the n-tuple P has a lowest* (*highest*) *element*, *P is confluent* (*dual-confluent*). *In particular*, *any finite sublattice is confluent and dual-confluent*.

4. SOME PRIMITIVE ORTHOGONAL IDEMPOTENTS

Let $P = (T_1, T_2)$ be a pair of X-reduction operators on V. The aim of this section is to complete the projections R_i^P defined by (3.5), in order to obtain two decompositions of the identity, namely

$$1_V = R_i^P + \Lambda_i^P + \Gamma_1^P + \Gamma_2^P, \qquad i = 1, 2. \tag{4.1}$$

For that, we introduce a finite dimensional algebra denoted by $\mathscr{A}_n^r$. The parameter n is the number of factors in R_i^P (sufficiently large to reduce everything). In case P is confluent, the two decompositions (4.1) are the same and can be obtained from another algebra denoted by $\mathscr{A}_n^c$. In fact, $\mathscr{A}_n^c$ is a quotient of $\mathscr{A}_n^r$ by a relation which expresses confluence in an abstract way. The algebra $\mathscr{A}_n^c$ is known as the nil-Hecke algebra of the dihedral group, while $\mathscr{A}_n^r$ seems to be new. In both cases, these algebras have two generators s_1 and s_2, which will be "represented" by T_1 and T_2, respectively. The following notations will be convenient to define these algebras and their idempotents. For $k > 0$, $[k]s_1$ refers to the braided product $\dots s_1 s_2 s_1$ with k factors, and $[k]s_2$ has the same meaning after the exchange of the indices. For $i = 1$ or 2, $s_i[k]$ indicates that the braided product begins by s_i on the left-hand side. In the previous expressions, s_i may be replaced by $1 - s_i$ if we make systematically the same replacements for all the factors. These notations will coincide with the unit 1 of the algebra if $k = 0$.

Definition 4.1. For any integer $n \geq 2$, the associative K-algebra $\mathscr{A}_n^r$ defined by the two generators s_1 and s_2 subjected to the relations

$$s_i^2 = s_i, \qquad [n+1]s_i = [n]s_i, \qquad i = 1, 2, \tag{4.2}$$

is called the reduction algebra of degree n.

Clearly, $1, s_1, s_2, s_2 s_1, s_1 s_2, \dots, [n]s_1, [n]s_2$ constitute a linear basis of $\mathscr{A}_n^r$, so $\dim(\mathscr{A}_n^r) = 2n + 1$. From now on, i equals 1 or 2. We introduce the following elements of $\mathscr{A}_n^r$:

$$\sigma_i = [n]s_i, \tag{4.3}$$

$$\lambda_i = (1 - s_i)[n], \tag{4.4}$$

$$\gamma_1 = (1 - s_2) \sum [k]s_1, \tag{4.5}$$

$$\gamma_2 = (1 - s_1) \sum [k]s_2. \tag{4.6}$$

In the two last formulas, k is *odd* and varies between 1 and $n - 1$. More explicit formulas can be written. In case n is odd (the case n even is left to the reader), they are

$$\lambda_i = 1 - s_1 - s_2 + s_2 s_1 + s_1 s_2 - \cdots + [n-1]s_1 + [n-1]s_2 - \sigma_i \tag{4.7}$$

$$\gamma_1 = s_1 - s_2 s_1 + s_1 s_2 s_1 - \cdots - [n-1]s_1 \tag{4.8}$$

$$\gamma_2 = s_2 - s_1 s_2 + s_2 s_1 s_2 - \cdots - [n-1]s_2. \tag{4.9}$$

In both cases, these formulas imply immediately the decomposition

$$1 = \sigma_i + \lambda_i + \gamma_1 + \gamma_2. \tag{4.10}$$

In order to shorten the verification of subsequent formulas, it is worth noticing the existence of two *involutions* for $\mathscr{A}_n^r$. The first involution is the algebra involution ω defined by the exchange of the two generators. Clearly, ω permutes the two σ_i (the two λ_i, the two γ_i). The second involution is the Koszul involution κ. It is a linear (not algebra) involution and is defined as follows. For any element a, we let $\tilde{a} = 1 - a$. Developing $\tilde{s}_i[n+1]$ as in (4.7), we observe that the two elements $\tilde{s}_i$ satisfy the relations (4.2) in the *opposite* algebra $\mathscr{A}_n^{r,\,opp}$ of the algebra $\mathscr{A}_n^r$. Then κ is the unique algebra morphism from $\mathscr{A}_n^r$ to $\mathscr{A}_n^{r,\,opp}$ sending s_i to $\tilde{s}_i$. In fact, κ is an involution, hence an isomorphism. For each i, κ exchanges σ_i and λ_i, and also it exchanges γ_1 and γ_2. The latter fact comes from the relations $\kappa((1-s_2)([k]s_1)) = (1-s_1)([k]s_2)$. Basic information about the finite dimensional algebra $\mathscr{A}_n^r$ is given by the following.

PROPOSITION 4.2. *We fix i to be equal to* 1 *or* 2. *The elements* $\sigma_i, \lambda_i, \gamma_1, \gamma_2$ *are primitive orthogonal idempotents. They are not central, but* $\sigma_1 + \lambda_1 = \sigma_2 + \lambda_2$ *and* $\gamma_1 + \gamma_2$ *are central.*

Proof. Our first task is to verify that $\sigma_i, \lambda_i, \gamma_1, \gamma_2$ are orthogonal idempotents. Through the involution ω, we can suppose $i = 1$. We have obviously $s_i\sigma_1 = \sigma_1$, and consequently $\sigma_1^2 = \sigma_1$. Using κ, we deduce $\lambda_1 s_i = 0$ and $\lambda_1^2 = \lambda_1$. Orthogonality of σ_1 and λ_1 is immediate from (4.3)–(4.4). Note that s_1 commutes with σ_1 and λ_1, but $\sigma_1 s_2 = \sigma_2$ and (applying κ) $s_2\lambda_1 = \sigma_2 - \sigma_1$. So σ_1 and λ_1 are not central and their sum is central. Next, we have the following identities which will be useful later:

$$([k]s_1)\gamma_1 = [k]s_1 - \sigma_1, \qquad 1 \le k \le n. \tag{4.11}$$

It suffices to verify the case $k = 1$, which is straightforward. For $k = n$, we obtain $\sigma_1\gamma_1 = 0$, hence $\gamma_2\lambda_1 = 0$ by application of κ. From (4.3)–(4.6), we have directly $\gamma_1\lambda_1 = 0$ and $\sigma_1\gamma_2 = 0$. Using $([k]s_i)\sigma_1 = \sigma_1$, we draw $\gamma_i\sigma_1 = 0$, hence $\lambda_1\gamma_i = 0$ by κ. Moreover, $\gamma_1^2 = \gamma_1$ comes from (4.5) and $s_1\gamma_1 = s_1 - \sigma_1$. We apply again κ to get $\gamma_2^2 = \gamma_2$. Since $\gamma_i s_i = \gamma_i$, the idempotent γ_i is not central.

It remains to check that our idempotents are primitive. It is equivalent to check that each left module in the direct sum

$$\mathscr{A}_n^r = \mathscr{A}_n^r\sigma_i \oplus \mathscr{A}_n^r\lambda_i \oplus \mathscr{A}_n^r\gamma_1 \oplus \mathscr{A}_n^r\gamma_2 \tag{4.12}$$

is *indecomposable*. We first introduce the four simple left modules $S_{11}, S_{00}, S_{10}, S_{01}$ as follows: S_{ij} is the field K with the action $s_1.1 = i$, $s_2.1 = j$. Note that ω leaves S_{11} and S_{00} invariant and exchanges S_{10}, S_{01}. The module $\mathscr{A}_n^r\sigma_i$ is evidently isomorphic to S_{11}. The relations $s_i\lambda_i = 0$, $s_2\lambda_1 = -s_1\lambda_2 = \sigma_2 - \sigma_1$ show that $\mathscr{A}_n^r\lambda_i = K\lambda_i \oplus K(\sigma_1 - \sigma_2)$. Furthermore, $K(\sigma_1 - \sigma_2)$ is the unique one-dimensional submodule (isomorphic to S_{11}). Thus $\mathscr{A}_n^r\lambda_i$ is indecomposable and its top composition factor is S_{00}. Next, thanks to the involution ω, we can limit our study to $\mathscr{A}_n^r\gamma_1$. Relation

(4.12) implies that $\dim(\mathscr{A}_n^r\gamma_1) = n - 1$. We let

$$u_k = ([k]s_1)\gamma_1, \qquad 1 \le k \le n-1. \tag{4.13}$$

Identities (4.11) show that these elements form a *basis* of $\mathscr{A}_n^r\gamma_1$. Now the following lemma is easily proved.

LEMMA 4.3. *The nonzero submodules of $\mathscr{A}_n^r\gamma_1$ are*

$$\bigoplus_{l=k}^{n-1} Ku_l, \qquad 1 \le k \le n-1.$$

Therefore, the module $\mathscr{A}_n^r\gamma_1$ has a unique composition series, and its composition factors are S_{10} and S_{01} alternatively. Thus $\mathscr{A}_n^r\gamma_1$ is indecomposable, with S_{10} as the top composition factor. ∎

The second assertion of Proposition 4.2 means that the two blocks (i.e., indecomposable two-sided ideals) of $\mathscr{A}_n^r$ are $B_1 = \mathscr{A}_n^r\sigma_i \oplus \mathscr{A}_n^r\lambda_i$ and $B_2 = \mathscr{A}_n^r\gamma_1 \oplus \mathscr{A}_n^r\gamma_2$. The most interesting one is B_2. A description of B_2 as the algebra of a quiver with relations can be derived from the basis (4.13) and its analogue in $\mathscr{A}_n^r\gamma_2$; in particular, B_2 is Frobenius (these properties will not be used later).

DEFINITION 4.4. For any integer $n \ge 2$, the associative K-algebra $\mathscr{A}_n^c$ defined by the two generators s_1 and s_2 subjected to the relations

$$s_i^2 = s_i, \qquad [n]s_1 = [n]s_2, \tag{4.14}$$

is called the confluence algebra of degree n.

Clearly, there exists a unique algebra (epi)morphism $\varphi : \mathscr{A}_n^r \to \mathscr{A}_n^c$ such that $\varphi(s_i) = s_i$. The kernel of φ is one-dimensional, generated by $\sigma_1 - \sigma_2$. A decomposition of $\mathscr{A}_n^c$ into indecomposable modules is

$$\mathscr{A}_n^c = \mathscr{A}_n^c\sigma \oplus \mathscr{A}_n^c\lambda \oplus \mathscr{A}_n^c\gamma_1 \oplus \mathscr{A}_n^c\gamma_2, \tag{4.15}$$

where $\sigma = \varphi(\sigma_i)$ and $\lambda = \varphi(\lambda_i)$. Here the second summand is one-dimensional, but the list of the four simple modules is the same as for $\mathscr{A}_n^r$. The idempotents γ_i are not central, while σ and λ are central. Thus $\mathscr{A}_n^c$ has *three* blocks: the two first summands in (4.15) and the sum of the two last ones (the latter block is isomorphic via φ to B_2 defined previously). Up to a change of sign of the generators, $\mathscr{A}_n^c$ is the nil-Hecke algebra of the dihedral group [6].

We are now ready to obtain the decompositions (4.1). Choose n as being a common number of factors to the two R_i^P, that is, $R_1^P = [n]T_1, R_2^P = [n]T_2$. Since R_i^P is a complete P-composition, we have $[n+1]T_i = [n]T_i$,

$i = 1, 2$. Let ρ^P be the algebra morphism from $\mathscr{A}_n^r$ to $\mathrm{Hom}_K(V, V)$ such that $\rho^P(s_i) = T_i$. Then P is confluent if and only if ρ^P factorizes through φ in an algebra morphism from $\mathscr{A}_n^c$ to $\mathrm{Hom}_K(V, V)$. Furthermore, $R_i^P = \rho^P(\sigma_i), \Lambda_i^P = \rho^P(\lambda_i), \Gamma_1^P = \rho^P(\gamma_1), \Gamma_2^P = \rho^P(\gamma_2)$ are four orthogonal idempotent endomorphisms of V satisfying (4.1). In particular, we have

$$V = \mathrm{Im}\big(R_i^P\big) \oplus \mathrm{Im}\big(\Lambda_i^P\big) \oplus \mathrm{Im}\big(\Gamma_1^P\big) \oplus \mathrm{Im}\big(\Gamma_2^P\big). \tag{4.16}$$

We want to give a description of the summands thus obtained. This description provides an explicit basis of each summand. We know that $\mathrm{Im}(R_i^P) = \mathrm{Red}(P)$. The relation $\Lambda_i^P = (1_V - T_i)[n]$ and the sketch of duality at the end of the previous section show that $1_V - \Lambda_i^P$ is an X-reduction operator such that $\mathrm{Red}(1_V - \Lambda_i^P) = \mathrm{Namb}(P)$. Accordingly Λ_i^P induces an isomorphism from $\mathrm{Amb}(P)$ onto $\mathrm{Im}(\Lambda_i^P)$. Note that $\mathrm{Ker}(\Lambda_i^P) = \mathrm{Namb}(P)$ along with the decomposition (4.1) in orthogonal idempotents leads to

$$\mathrm{Namb}(P) = \mathrm{Red}(P) \oplus \mathrm{Im}\big(\Gamma_1^P\big) \oplus \mathrm{Im}\big(\Gamma_2^P\big). \tag{4.17}$$

Next, introduce the subspaces

$$E_1 = (1_V - T_2)\big(\mathrm{Red}(T_1) \cap \mathrm{Nred}(T_2)\big),$$
$$E_2 = (1_V - T_1)\big(\mathrm{Nred}(T_1) \cap \mathrm{Red}(T_2)\big).$$

Then $1_V - T_2$ induces an isomorphism from $\mathrm{Red}(T_1) \cap \mathrm{Nred}(T_2)$ onto E_1, with an analogous property for $1_V - T_1$. Furthermore, relation $\gamma_1(1 - s_2) = 1 - s_2 - \lambda_2$ implies that Γ_1^P is identity on E_1. In the same manner, Γ_2^P is identity on E_2. So $E_i \subseteq \mathrm{Im}(\Gamma_i^P)$. Comparing (4.17) with

$$\begin{aligned}\mathrm{Namb}(P) = \mathrm{Red}(P) &\oplus \big(\mathrm{Red}(T_1) \cap \mathrm{Nred}(T_2)\big)\\ &\oplus \big(\mathrm{Nred}(T_1) \cap \mathrm{Red}(T_2)\big),\end{aligned}$$

we conclude that $\mathrm{Im}(\Gamma_i^P) = E_i$.

In the two decompositions (4.16), the summands coincide respectively, except perhaps the second ones. Consequently, if $\mathrm{Im}(\Lambda_1^P) = \mathrm{Im}(\Lambda_2^P)$, T_1 and T_2 are confluent. Another easy consequence of (4.16) is the following description of the subspace of the elements which have a unique normal form w.r.t. $\underset{P}{\rightarrow}$:

$$\mathrm{Ker}\big(R_1^P - R_2^P\big) = \mathrm{Namb}(P) \oplus \big(\mathrm{Im}\big(\Lambda_1^P\big) \cap \mathrm{Im}\big(\Lambda_2^P\big)\big). \tag{4.18}$$

The reader can likewise verify that $\mathrm{Im}(\Lambda_1^P) \cap \mathrm{Im}(\Lambda_2^P) = \mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2)$. The latter equality is equivalent to $(1_V - \Lambda_1^P) \vee (1_V - \Lambda_2^P) = T_1 \vee T_2$ which in turn is dual of $R_1^P \wedge R_2^P = T_1 \wedge T_2$. Note also the decomposition

$$\operatorname{Ker}(T_1) + \operatorname{Ker}(T_2) = \operatorname{Im}\left(R_1^P - R_2^P\right) \oplus \operatorname{Im}\left(\Lambda_i^P\right) \oplus \operatorname{Im}\left(\Gamma_1^P\right) \oplus \operatorname{Im}\left(\Gamma_2^P\right), \tag{4.19}$$

whose first summand is the intersection of $\operatorname{Ker}(T_1) + \operatorname{Ker}(T_2)$ with $\operatorname{Red}(P)$. A simple calculation from (4.19) gives $\operatorname{def}_c(P) = \dim(\operatorname{Im}(R_1^P - R_2^P))$. On the other hand, dualizing (4.1) is easy from the relation $\rho^{P^!} = \tau \cdot \rho^P \cdot \kappa$, where κ is the Koszul involution and τ is the transposition $f \mapsto f^*$.

In case P is confluent, we can let $R^P = R_i^P$ and $\Lambda^P = \Lambda_i^P$, so the two decompositions (4.1) coincide with

$$1_V = R^P + \Lambda^P + \Gamma_1^P + \Gamma_2^P. \tag{4.20}$$

With the aid of Theorem 3.8 and its dual version, we have

$$R^P = T_1 \wedge T_2, \qquad 1_V - \Lambda^P = T_1 \vee T_2. \tag{4.21}$$

In particular, the sublattice generated by some pairwise confluent X-reduction operators is contained in the subalgebra of $\operatorname{Hom}_K(V, V)$ generated by these operators.

In the preceding section, we have associated a reduction relation $\underset{P}{\to}$ to the pair $P = (T_1, T_2)$. Each step of reduction w.r.t. $\underset{P}{\to}$ consists in applying T_i $(i = 1$ or $2)$ on the whole element obtained from the previous step. Dealing with general associative algebras, Bergman [5] considered another way to reduce. We are going to show that this way to reduce is in fact "equivalent" to ours. In our framework, Bergman's reduction theory works out as follows: each step of reduction is called an elementary reduction and consists in applying T_i only on *one* X-generator occurring in the element obtained from the previous step (obviously this generator is T_i-nonreduced). This defines a relation on V denoted by $\underset{BP}{\to}$ (B stands for Bergman). With the same notation as in the preceding section $\underset{BP}{\to}$ implies $\underset{E}{\to}$, thus is a Noetherian reduction relation. The set of the normal forms w.r.t. $\underset{BP}{\to}$ is $\operatorname{Red}(P)$. A reduction acting on the whole element is a composition of elementary reductions, therefore $\overset{*}{\underset{P}{\to}}$ implies $\overset{*}{\underset{BP}{\to}}$. Evidently the converse of the latter implication is false, but we have the two following facts. Firstly, suppose a belongs to $\operatorname{Namb}(P)$ and b is a normal form of a w.r.t. $\underset{BP}{\to}$. Through the definition of elementary reductions, we see that $a - b$ belongs to $E_1 \oplus E_2$. Then the decomposition (4.17) joint to the relations $\operatorname{Im}(\Gamma_i^P) = E_i$ shows that b is *unique* and coincides with the normal form of a w.r.t. $\underset{P}{\rightarrow}$. Secondly, let a be an ambiguous X-generator. There are only two elementary reductions acting on a, namely T_1 and T_2. Furthermore, $T_i(a)$ belongs to Namb(P) and we can apply the first fact. Thus a has at most two normal forms w.r.t. $\underset{BP}{\rightarrow}$, which are the same as w.r.t. $\underset{P}{\rightarrow}$. Finally, we claim that confluence of $\underset{BP}{\rightarrow}$ is equivalent to confluence of $\underset{P}{\rightarrow}$. Indeed, the second fact shows that confluence of $\underset{BP}{\rightarrow}$ implies confluence of $\underset{P}{\rightarrow}$. Using local confluence, the converse comes from the fact that any elementary reduction can be completed in a reduction w.r.t. $\underset{P}{\rightarrow}$.

5. A HOMOTOPY OF THE KOSZUL COMPLEX USING AN IDEMPOTENT

In this section, we present a proof of Priddy's theorem based on the construction of an explicit homotopy for the Koszul complex. This construction will use the material of the previous section. More precisely, the homotopy will be a by-product of the idempotent γ_1 and the relation (4.11) for $k = 1$ will play an essential role. Moreover, all the pairs of reduction operators considered here will be confluent, thus we shall use the decomposition (4.20). We take up the notations of the end of Section 2. Let A be a quadratic algebra on V whose subspace of relations is E. Let S be the X-reduction operator on $V^{(2)}$ such that $\mathrm{Ker}(S) = E$. Throughout this section, the algebra A is X-confluent. For each $n \geq 2$, the X-reduction operators $S_i^{(n)} = 1_{V^{(i-1)}} \otimes S \otimes 1_{V^{(n-i-1)}}, 1 \leq i \leq n-1$, are pairwise confluent. Introduce the following $X^{(n)}$-generated subspaces of $V^{(n)}$ when $n \geq 2$:

$$\begin{cases} \mathrm{Red}^{(n)} = \displaystyle\bigcap_{i=1}^{n-1} V^{(i-1)} \otimes \mathrm{Red}(S) \otimes V^{(n-i-1)} \\ \mathrm{Amb}^{(n)} = \displaystyle\bigcap_{i=1}^{n-1} V^{(i-1)} \otimes \mathrm{Nred}(S) \otimes V^{(n-i-1)}. \end{cases} \tag{5.1}$$

The monomials belonging to $\mathrm{Red}^{(n)}$ or $\mathrm{Amb}^{(n)}$ are respectively called *totally reduced* or *totally ambiguous*. We also consider the following idempotent endomorphisms of $V^{(n)}$

$$\begin{cases} R^{(n)} = S_1^{(n)} \wedge \cdots \wedge S_{n-1}^{(n)} \\ \Lambda^{(n)} = 1_{V^{(n)}} - S_1^{(n)} \vee \cdots \vee S_{n-1}^{(n)}. \end{cases} \tag{5.2}$$

We extend these notations by letting $\mathrm{Red}^{(1)} = \mathrm{Amb}^{(1)} = V$ and $R^{(1)} = \Lambda^{(1)} = 1_V$. Then the images of $R^{(n)}$ and $\Lambda^{(n)}$ are respectively $\mathrm{Red}^{(n)}$ and $J(E)_n$. Furthermore $R^{(n)}$ induces an isomorphism from A_n onto $\mathrm{Red}^{(n)}$, and $\Lambda^{(n)}$ induces an isomorphism from $\mathrm{Amb}^{(n)}$ onto $J(E)_n$.

The Koszul complex of A is graded by the total degree denoted by n. We fix $n \geq 2$. The differential of the homogeneous part of degree n is the sequence of the following linear maps [4]

$$A_k \otimes J(E)_{n-k} \xrightarrow{d_{n-k}} A_{k+1} \otimes J(E)_{n-k-1}, \qquad -1 \leq k \leq n. \quad (5.3)$$

For $0 \leq k \leq n-1$, the map d_{n-k} is the restriction to $A_k \otimes J(E)_{n-k}$ of the linear map defined on $A_k \otimes V^{(n-k)}$ by

$$\bar{a} \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{n-k} \mapsto \overline{a \otimes v_1} \otimes v_2 \otimes \cdots \otimes v_{n-k},$$
$$a \in V^{(k)},\ v_i \in V.$$

Using the isomorphisms induced by the projections $R^{(k)}$, the complex (5.3) becomes the following (we use the same notation for the differential):

$$\mathrm{Red}^{(k)} \otimes J(E)_{n-k} \xrightarrow{d_{n-k}} \mathrm{Red}^{(k+1)} \otimes J(E)_{n-k-1}, \qquad -1 \leq k \leq n. \quad (5.4)$$

This time, for $0 \leq k \leq n-1$, d_{n-k} is the restriction to $\mathrm{Red}^{(k)} \otimes J(E)_{n-k}$ of the linear map $R^{(k+1)} \otimes 1_{V^{(n-k-1)}}$ defined by $V^{(n)}$. The advantage of the complex (5.4) is that the objects are subspaces of the *same* space $V^{(n)}$ and the differential is defined as a sequence of restrictions of endomorphisms of $V^{(n)}$.

Our aim is to construct a homotopy of the complex (5.4), that is, a sequence of linear maps $h_{n-k} : \mathrm{Red}^{(k)} \otimes J(E)_{n-k} \to \mathrm{Red}^{(k-1)} \otimes J(E)_{n-k+1}$, $1 \leq k \leq n$, such that

$$d_{n-k+1}h_{n-k} + h_{n-k-1}d_{n-k} = 1_{Red^{(k)} \otimes J(E)_{n-k}}, \qquad 0 \leq k \leq n, \quad (5.5)$$

with $h_{-1} = h_n = 0$. We define h_{n-k}, $1 \leq k \leq n$, from the following pair of X-reduction operators on $V^{(n)}$:

$$P_k = \left(R^{(k)} \otimes 1_{V^{(n-k)}}, 1_{V^{(k-1)}} \otimes \left(1_{V^{(n-k+1)}} - \Lambda^{(n-k+1)}\right)\right), \qquad 1 \leq k \leq n. \quad (5.6)$$

Observing that $P_k = (S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}, S_k^{(n)} \vee \cdots \vee S_{n-1}^{(n)})$ and using Proposition 2.11, we see that P_k is *confluent*. Then we define h_{n-k} as the restriction of the idempotent $\Gamma_1^{P_k}$ to $\mathrm{Red}^{(k)} \otimes J(E)_{n-k}$. From (4.4), we deduce the relation $\gamma_1 = (1 - s_2)s_1(\dots)$ which is represented by

$$\Gamma_1^{P_k} = \left(1_{V^{(k-1)}} \otimes \Lambda^{(n-k+1)}\right)\left(R^{(k)} \otimes 1_{V^{(n-k)}}\right)(\dots).$$

Thus the image of h_{n-k} is indeed contained in $\mathrm{Red}^{(k-1)} \otimes J(E)_{n-k+1}$. It remains to prove the identities (5.5).

First, notice that each of the two pairs

$$P_1 = \left(1_{V^{(n)}}, 1_{V^{(n)}} - \Lambda^{(n)}\right), \qquad P_n = \left(R^{(n)}, 0_{V^{(n)}}\right)$$

is formed of commuting operators. Furthermore, if s_1 and s_2 commute, we have $\gamma_1 = s_1 - s_2 s_1$. Therefore, h_0 is identity on $\mathrm{Red}^{(n)}$ and h_{n-1} is the restriction of $\Lambda^{(n)}$ to $V \otimes J(E)_{n-1}$. Then the verifications of (5.5) for $k = 0$ and $k = n$ are immediate.

Now suppose that $1 \leq k \leq n - 1$. Applying the relation $s_1\gamma_1 = s_1 - \sigma$ to the pair P_k, we obtain

$$\left(R^{(k)} \otimes 1_{V^{(n-k)}}\right)\Gamma_1^{P_k} = \left(R^{(k)} \otimes 1_{V^{(n-k)}}\right) - R^{P_k}. \tag{5.7}$$

We want to show that (5.5) is exactly the restriction of (5.7) to $\mathrm{Red}^{(k)} \otimes J(E)_{n-k}$. Clearly, the restriction of the left-hand side is $d_{n-k+1}h_{n-k}$ and the restriction of $R^{(k)} \otimes 1_{V^{(n-k)}}$ is identity. In order to prove that the restriction of R^{P_k} is $h_{n-k-1}d_{n-k}$, we must link the pairs P_k and P_{k+1} together. This lies on the following.

LEMMA 5.1. *The reduction operators $R^{(k)} \otimes 1_{V^{(n-k)}}$ and $1_{V^{(n)}} - \Lambda^{P_{k+1}}$ commute, and their composite coincides with R^{P_k}.*

Proof. We have $R^{(k)} \otimes 1_{V^{(n-k)}} = S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}$ and $P_{k+1} = (S_1^{(n)} \wedge \cdots \wedge S_k^{(n)}, S_{k+1}^{(n)} \vee \cdots \vee S_{n-1}^{(n)})$. Using the fact that $S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}$ commutes with the operators of the pair P_{k+1} and keeping in mind that the latter pair is confluent, we see that $S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}$ commutes with the join U of this pair, hence the first assertion. On the other hand, the $S_i^{(n)}$ generate a distributive sublattice thanks to Theorem 2.12(ii). In particular, the following holds

$$\left(S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}\right) \wedge U = \left(S_1^{(n)} \wedge \cdots \wedge S_{k-1}^{(n)}\right) \wedge \left(S_k^{(n)} \vee S_{k+1}^{(n)} \vee \cdots \vee S_{n-1}^{(n)}\right),$$

hence the second assertion. ∎

Therefore, the relation $R^{P_k} = (1_{V^{(n)}} - \Lambda^{P_{k+1}})(R^{(k)} \otimes 1_{V^{(n-k)}})$ indicates that the restriction of R^{P_k} is equal to the restriction of $1_{V^{(n)}} - \Lambda^{P_{k+1}}$. Now, we need to make explicit the pair P_{k+1}:

$$P_{k+1} = \left(R^{(k+1)} \otimes 1_{V^{(n-k-1)}}, 1_{V^{(k)}} \otimes \left(1_{V^{(n-k)}} - \Lambda^{(n-k)}\right)\right).$$

Since σ and γ_2 factorize on the right by s_2, $R^{P_{k+1}}$ and $\Gamma_2^{P_{k+1}}$ vanish on $\mathrm{Red}^{(k)} \otimes J(E)_{n-k}$. Thus the restriction of R^{P_k} to this subspace is the same as the restriction of $\Gamma_1^{P_{k+1}}$. The relation $\gamma_1 = \gamma_1 s_1$ gives $\Gamma_1^{P_{k+1}} = (\Gamma_1^{P_{k+1}})(R^{(k+1)} \otimes 1_{V^{(n-k-1)}})$ whose restriction is $h_{n-k-1} d_{n-k}$, as we wished.

Remark 5.2. Using the isomorphisms $\Lambda^{(k)} : \mathrm{Amb}^{(k)} \to J(E)_k$, we can transform the complex (5.4) into the following (same notation for the differential):

$$\mathrm{Red}^{(k)} \otimes \mathrm{Amb}^{(n-k)} \xrightarrow{d_{n-k}} \mathrm{Red}^{(k+1)} \otimes \mathrm{Amb}^{(n-k-1)}, \qquad -1 \leq k \leq n. \tag{5.8}$$

The objects of the complex (5.8) are $X^{(n)}$-generated subspaces of $V^{(n)}$, but the differential and the homotopy are quite involved. An advantage of this complex is that it brings out the duality between totally reduced monomials and totally ambiguous monomials. Recall that the duality between the bases X and $X^!$ exchanges these monomials. After applying the X-$X^!$ duality and the flip on each tensor product which separates Red and Amb, the differential of (5.8) for the dual algebra $A^!$ becomes an arrow on the objects of the complex (5.8), but in the *reverse* direction. The reader can verify that this arrow is *not* a homotopy of the complex (5.8), even up to a factor (it is sufficient to suppose that A is an usual commutative polynomial algebra in two variables). This situation contrasts strongly with the case of the quadratic algebras determined by a solution of the quantum Yang–Baxter equation satisfying a quadratic relation [11]. In fact, in the latter case, a duality exists too, but the reverse-directed arrow provided by the duality *is* a homotopy up to a factor.

6. AN ALGORITHM FOR REPAIRING THE NONCONFLUENCE

A pair $P = (T_1, T_2)$ of *nonconfluent* X-reduction operators on V is given through the matrices of T_1 and T_2 relative to the basis X. The algorithm below finds the matrices of $T_1 \wedge T_2$ and $T_1 \vee T_2$ from these data. Furthermore, a *confluent* pair $\tilde{P} = (\tilde{T}_1, \tilde{T}_2)$ is constructed. (In case P is confluent, the algorithm is clear from relations (4.21) and we have $\tilde{P} = P$.) Our algorithm will not be presented in a formal or semi-formal programming language but we think that the constructions will be comprehensive enough to perform this task. The first construction consists in selecting an increasing sequence $\alpha_1, \dots, \alpha_s$ of P-ambiguous X-generators. It will turn out later that these generators are the ambiguous P-obstructions. Define the α_j inductively on $j = 1, \dots, s$. Precisely, α_1 is the lowest ambiguous generator such that $(R_1^P - R_2^P)(\alpha_1) \neq 0$. For $j > 1$, α_j is the lowest ambiguous generator such that $\alpha_j > \alpha_{j-1}$ and $(R_1^P - R_2^P)(\alpha_j)$ is linearly independent of the elements $(R_1^P - R_2^P)(\alpha_k)$, $1 \leq k < j$. The construction stops when the elements $(R_1^P - R_2^P)(\alpha_j)$, $1 \leq j \leq s$, form a basis of $\mathrm{Im}(R_1^P - R_2^P)$. In particular, $s = \mathrm{def}_c(P)$.

Next, we construct an X-reduction operator $\tau_\wedge^P$ by successive approximation (Proposition 6.1 below). It is the essential part of our algorithm. In fact (see the relation (6.5)), $\tau_\wedge^P$ solves the ambiguity $R_1^P \neq R_2^P$ and provides the computation of $T_1 \wedge T_2$. By duality, we shall get the computation of $T_1 \vee T_2$.

PROPOSITION 6.1. *There exists a unique strictly descending chain $\tau_0 > \tau_1 > \cdots > \tau_s$ of X-reduction operators which verifies the two following conditions*

(i) $\dim(\tau_j) = r - j$, $0 \leq j \leq s$.

(ii) *For any j, $1 \leq j \leq s$, one can find an X-generator v_j and $c_j \in K$ such that*

$$\tau_{j-1}\left[\left(R_1^P - R_2^P\right)(\alpha_j)\right] = c_j\left[v_j - \tau_j(v_j)\right]. \tag{6.1}$$

Proof. We set $\tau_0 = 1_V$. Define v_1 as the highest X-generator occurring in $(R_1^P - R_2^P)(\alpha_1)$ and c_1 as its coefficient. (Note that these definitions are necessary from (6.1) and the fact that τ_1 is an X-reduction operator.) Then (6.1) defines uniquely the X-reduction operator τ_1 of dimension $r - 1$. Furthermore v_1 is the unique τ_1-nonreduced generator. Suppose that $\tau_1, \dots, \tau_t$ are constructed for some t, $1 \leq t < s$, as in the proposition and suppose that $v_1, \dots, v_t$ are the τ_t-nonreduced generators. Applying τ_t to (6.1) for $1 \leq j \leq t$ and remarking that $\tau_t\tau_{j-1} = \tau_t\tau_j = \tau_t$, we deduce that $(R_1^P - R_2^P)(\alpha_j)$, $1 \leq j \leq t$, form a basis of $\mathrm{Ker}(\tau_t)$. Accordingly, $(1_V - \tau_t)(a)$ with $a = (R_1^P - R_2^P)(\alpha_{t+1})$ decomposes in this basis. This fact along with the definition of α_{t+1} shows that $\tau_t(a) \neq 0$. Denote by v_{t+1} the highest X-generator of $\tau_t(a)$ and by c_{t+1} its coefficient. Let S be an X-reduction operator such that $\dim(S) = r - t - 1$, $\tau_t > S$, and $\tau_t(a) = c_{t+1}[v_{t+1} - S(v_{t+1})]$. Then v_{t+1} is S-nonreduced and $S(v_{t+1})$ is uniquely determined. Furthermore we have

$$S(v_j) = S\left(\tau_t(v_j)\right), \qquad 1 \leq j \leq t. \tag{6.2}$$

Since the v_j are τ_t-nonreduced, (6.2) implies that they are also S-nonreduced. On the other hand, the $S(v_j)$ are uniquely determined by (6.2) because $v_1, \dots, v_t$ do not occur in $\tau_t(v_j)$. Thus the unicity of S is proved. For the existence, we define τ_{t+1} on the basis X as follows: $\tau_{t+1}(\alpha) = \alpha$ for α different of all the v_j, $1 \le j \le t+1$; $\tau_{t+1}(v_{t+1})$ is defined by (6.1) for $j = t+1$; and replacing S by τ_{t+1} we use (6.2). Then τ_{t+1} has the required properties. ∎

A closer look at the above proof shows that the v_j and the $\tau_j(v_k)$ for $1 \le k \le j$ belong to $\operatorname{Red}(P)$. We shall see that in fact the v_j are the reduced P-obstructions. So the relations (6.1) associate a reduced P-obstruction to each ambiguous P-obstruction (but notice that the sequence $v_1, \dots, v_s$ is not increasing in general). Define the X-reduction operator $\tau_\wedge^P$ as

$$\tau_\wedge^P = \tau_s, \tag{6.3}$$

so that the following holds

$$\operatorname{Ker}\left(\tau_\wedge^P\right) = \operatorname{Im}\left(R_1^P - R_2^P\right). \tag{6.4}$$

In particular, $\tau_\wedge^P R_1^P = \tau_\wedge^P R_2^P$. Furthermore, denoting by W the subspace of V generated by $v_1, \dots, v_s$, we have $\operatorname{Nred}(\tau_\wedge^P) = W$ and $\tau_\wedge^P(W) \subseteq \operatorname{Red}(P) \ominus W$. Letting $R = \tau_\wedge^P R_1^P$, we easily check that R is an X-reduction operator such that $\operatorname{Red}(R) = \operatorname{Red}(P) \ominus W$. The definition of R shows also that $\operatorname{Ker}(R_1^P)$ and $\operatorname{Ker}(\tau_\wedge^P)$ are contained in $\operatorname{Ker}(R)$. Using (4.19) and (6.4), we deduce the inclusion $\operatorname{Ker}(T_1 \wedge T_2) \subseteq \operatorname{Ker}(R)$ which in turn is an equality for a dimensional reason. So we have obtained

$$\tau_\wedge^P R_1^P = \tau_\wedge^P R_2^P = T_1 \wedge T_2. \tag{6.5}$$

Comparing $\operatorname{Red}(R)$ to $\operatorname{Red}(T_1 \wedge T_2) = \operatorname{Red}(P) \ominus \operatorname{Obs}_{red}^P$, we get

$$\operatorname{Nred}\left(\tau_\wedge^P\right) = \operatorname{Obs}_{red}^P. \tag{6.6}$$

In other words, $v_1, \dots, v_s$ are the reduced P-obstructions.

It remains to dualize the previous construction. Letting

$$\tau_\vee^P = \left(\tau_\wedge^{P^!}\right)^!, \tag{6.7}$$

we have by duality

$$\operatorname{Ker}\left(\tau_{\vee}^{P}\right) = \operatorname{Ker}\left(R_1^P - R_2^P\right), \tag{6.8}$$

$$\operatorname{Red}\left(\tau_{\vee}^{P}\right) = \operatorname{Obs}_{amb}^{P}, \tag{6.9}$$

$$\Lambda_1^P\left(1_V - \tau_{\vee}^{P}\right) = \Lambda_2^P\left(1_V - \tau_{\vee}^{P}\right) = 1_V - T_1 \vee T_2. \tag{6.10}$$

Actually, a direct (i.e., without using the dual space of V) computation of $T_1 \vee T_2$ exists and lies upon the following facts. First, it suffices to compute $T_1 \vee T_2(\alpha)$ and for any ambiguous generator α which is not an obstruction. For such α and in view of relation (6.8), we can write linear decompositions

$$\left(R_1^P - R_2^P\right)(\alpha) = \sum_{\beta < \alpha} c_\beta^\alpha \left(R_1^P - R_2^P\right)(\beta), \tag{6.11}$$

where β runs over the set of the ambiguous obstructions which are less than α. Denote by $\beta_1, \dots, \beta_s$ the ambiguous obstructions in the increasing order. The decompositions (6.11) imply that the $(R_1^P - R_2^P)(\beta_j)$, $1 \leq j \leq s$, form a basis of $\operatorname{Im}(R_1^P - R_2^P)$. Furthermore, the comparison of (6.11) with the definition of the α_j shows inductively that $\beta_1 = \alpha_1, \dots, \beta_s = \alpha_s$. So the α_j defined at the beginning of the algorithm are the ambiguous obstructions, as we claimed. Once the coefficients c_β^α are determined (using (6.11), there is an algorithm to do that), we apply (6.10) to obtain the desired computation:

$$(T_1 \vee T_2)(\alpha) = \alpha - \Lambda_1^P(\alpha) + \sum_{\beta < \alpha} c_\beta^\alpha \Lambda_1^P(\beta). \tag{6.12}$$

The formulas (6.4)–(6.10) are valid if P is confluent by letting $\tau_{\wedge}^{P} = 1_V$ and $\tau_{\vee}^{P} = 0_V$ in this case. So the X-reduction operator $\tau_{\vee}^{P}$ measures the extent of departure from confluence, as the Lie bracket $T_1T_2 - T_2T_1$ measures the extent of departure from commutativity. Notice also that the formulas (6.5) and (6.10) can be inversed as

$$\tau_{\wedge}^{P} = (T_1 \wedge T_2) I_{Red(P)} + I_{Nred(P)}, \tag{6.13}$$

$$\tau_{\vee}^{P} = I_{Amb(P)} T_1 \vee T_2, \tag{6.14}$$

where, for any X-generated subspace F of V, I_F denotes the trivial X-reduction operator such that $\operatorname{Red}(I_F) = F$. The reader can likewise verify that $\tau_{\vee}^{P} \leq \tau_{\wedge}^{P}$. $T_1 \wedge T_2 \leq \tau_{\wedge}^{P}$, $\tau_{\vee}^{P} \leq T_1 \vee T_2$. The relation $\operatorname{Red}(T_i) + \operatorname{Red}(\tau_{\wedge}^{P}) = V$ shows that T_i and $\tau_{\wedge}^{P}$ are confluent and $T_i \vee \tau_{\wedge}^{P} = 1_V$. From $\tau_{\wedge}^{P}(\operatorname{Red}(T_i)) \subseteq \operatorname{Red}(T_i)$, we draw $T_i \tau_{\wedge}^{P} T_i = \tau_{\wedge}^{P} T_i$, hence

$$T_i \wedge \tau_{\wedge}^{P} = \tau_{\wedge}^{P} T_i. \tag{6.15}$$

By duality, T_i and $\tau^P_\vee$ are confluent, $T_i \wedge \tau^P_\vee = 0_V$, and

$$T_i \vee \tau^P_\vee = \tau^P_\vee + T_i\left(1_V - \tau^P_\vee\right). \tag{6.16}$$

We are ready to define $\tilde{P} = (\tilde{T}_1, \tilde{T}_2)$. Noting that $T_i, \tau^P_\wedge, \tau^P_\vee$ are pairwise confluent, we let

$$\tilde{T}_i = \left(T_i \wedge \tau^P_\wedge\right) \vee \tau^P_\vee = \left(T_i \vee \tau^P_\vee\right) \wedge \tau^P_\wedge, \qquad i = 1, 2. \tag{6.17}$$

The pairwise confluence just mentioned gives

$$\operatorname{Red}\left(\tilde{T}_i\right) = \operatorname{Red}(T_i) \ominus \operatorname{Obs}^P_{red} \oplus \operatorname{Obs}^P_{amb}, \tag{6.18}$$

$$\operatorname{Red}(\tilde{P}) = \operatorname{Red}(P) \ominus \operatorname{Obs}^P_{red} \oplus \operatorname{Obs}^P_{amb}, \tag{6.19}$$

$$\operatorname{Amb}(\tilde{P}) = \operatorname{Amb}(P) \ominus \operatorname{Obs}^P_{amb} \oplus \operatorname{Obs}^P_{red}. \tag{6.20}$$

In particular, $\dim(\tilde{T}_i) = \dim(T_i)$. Deriving $(T_i \vee \tau^P_\vee)\tau^P_\wedge(T_i \vee \tau^P_\vee) = \tau^P_\wedge(T_i \vee \tau^P_\vee)$ from (6.16), (6.15), and $\tau^P_\vee T_i = 0$, we get

$$\tilde{T}_i = \tau^P_\vee + \tau^P_\wedge T_i\left(1_V - \tau^P_\vee\right). \tag{6.21}$$

It is remarkable that the latter expressions are *multiplicative*, that is, the following holds

$$\tilde{T}_1\tilde{T}_2 = \tau^P_\vee + \tau^P_\wedge T_1 T_2\left(1_V - \tau^P_\vee\right), \tag{6.22}$$

along with the same when the indices 1 and 2 are permuted. The multiplicativity extends to any monomial (i.e., braided product) of degree > 0 in the T_i, leading to

$$R^{\tilde{P}}_i = \tau^P_\vee + \tau^P_\wedge R^P_i\left(1_V - \tau^P_\vee\right). \tag{6.23}$$

Introducing (6.5) in the latter equality, we see that $\tilde{P}$ is *confluent* and

$$\tilde{T}_1 \wedge \tilde{T}_2 = \tau^P_\vee + (T_1 \wedge T_2)\left(1_V - \tau^P_\vee\right) \tag{6.24}$$

(from this, it is easy to deduce $\tilde{T}_1 \wedge \tilde{T}_2 = (T_1 \wedge T_2) \vee \tau^P_\vee$).

Now we investigate what $\tilde{P}$ brings to the representation ρ^P of the reduction algebra $\mathscr{A}^r_n$ (Section 4). Since $\tilde{P}$ is confluent, $\rho^{\tilde{P}}$ factorizes through φ in a representation $\bar{\rho}^{\tilde{P}}$ of the confluence algebra $\mathscr{A}^c_n$. Note that $\rho^P(\mathscr{A}^r_n)$ and $\bar{\rho}^{\tilde{P}}(\mathscr{A}^c_n)$ are the subalgebras of $\operatorname{Hom}_K(V, V)$ generated by T_1, T_2 and $\tilde{T}_1, \tilde{T}_2$, respectively. Assume that P is not confluent and there is a relation between the T_i (we use the notations of Section 4 for the braided products), namely

$$c_0 1_V + \sum_{k=1}^{n} \left(c_{k1}[k]T_1 + c_{k2}[k]T_2 \right) = 0. \tag{6.25}$$

The multiplicativity of (6.21) implies

$$\begin{aligned} c_0 1_V &+ \sum_{k=1}^{n} \left(c_{k1}[k]\tilde{T}_1 + c_{k2}[k]\tilde{T}_2 \right) \\ &= \left(c_0 + \sum_{k=1}^{n} (c_{k1} + c_{k2}) \right) \tau_\vee + c_0 (1_V - \tau_\wedge). \end{aligned} \tag{6.26}$$

Next, observe that there exists a P-reduced (P-ambiguous) generator a. Applying (6.25) to a, we find that $c_0 + \sum_{k=1}^{n}(c_{k1} + c_{k2}) = 0$ ($c_0 = 0$), thus the $\tilde{T}_i$ satisfy the same relation as the T_i. Accordingly, there is a (unique) algebra morphism Φ^P such that the following diagram commutes

$$\begin{tikzcd} \mathscr{A}_n^r \arrow[r, "\varphi"] \arrow[d, "\rho^P"] & \mathscr{A}_n^c \arrow[d, "\bar{\rho}^{\tilde{P}}"] \\ \rho^P(\mathscr{A}_n^r) \arrow[r, "\Phi^P"] & \bar{\rho}^{\tilde{P}}(\mathscr{A}_n^c) \end{tikzcd} \tag{6.27}$$

Notice that Φ^P is surjective. For any braided product f of degree > 0, we have

$$\Phi^P(f) = \tau_\vee^P + \tau_\wedge^P f \left(1_V - \tau_\vee^P \right). \tag{6.28}$$

In case P is confluent, Φ^P is the identity and the diagram (6.27) reduces to the factorization of ρ^P through φ.

LEMMA 6.2. *The kernel of* Φ^P *is* $K(R_1^P - R_2^P)$. *Furthermore, if P is not confluent,* φ *induces an isomorphism from* $\mathrm{Ker}(\rho^P)$ *onto* $\mathrm{Ker}(\bar{\rho}^{\tilde{P}})$.

Proof. The second assertion follows from the first one by the snake lemma. For the first assertion, we can assume the nonconfluence of P. We have already seen that $R_1^P - R_2^P$ belongs to the kernel of Φ^P. Suppose that f belongs to $\mathrm{Ker}(\Phi^P)$. Using the block decomposition of the algebra $\mathscr{A}_n^r$ determined in Section 4, we write

$$f = c_1 R_1^P + c_2 \Lambda_1^P + c_3 \left(R_1^P - R_2^P \right) + g, \tag{6.29}$$

where c_1, c_2, c_3 are in K and g belongs to $\rho^P(B_2)$ (B_2 is the block generated by γ_1 and γ_2). Applying Φ^P to this relation, we get

$$0 = c_1 R^{\tilde{P}} + c_2 \Lambda^{\tilde{P}} + \Phi^P(g). \tag{6.30}$$

The orthogonal decomposition $1_V = R^{\tilde{P}} + \Lambda^{\tilde{P}} + \Gamma_1^{\tilde{P}} + \Gamma_2^{\tilde{P}}$ along with the fact that $\mathrm{Red}(\tilde{P})$ and $\mathrm{Amb}(\tilde{P})$ are nonzero shows that $c_1 = c_2 = 0$, hence $\Phi^P(g) = 0$. As the braided products occurring in g have a degree > 0, the formula (6.28) provides the following relation in which c belongs to K:

$$c\tau_\vee^P + \tau_\wedge^P g\left(1_V - \tau_\vee^P\right) = 0. \tag{6.31}$$

On the other hand, $\tau_\wedge^P = \mathrm{Id}$ and $\tau_\vee^P = 0$ on the subspace $\mathrm{Im}(\Gamma_1^P) \oplus \mathrm{Im}(\Gamma_2^P)$, because the latter is contained in $\mathrm{Namb}(P)$. Putting that in (6.31), we obtain that g vanishes on $\mathrm{Im}(\Gamma_1^P) \oplus \mathrm{Im}(\Gamma_2^P)$. Thanks to the orthogonal decomposition relative to P, g vanishes on $\mathrm{Im}(R_1^P) \oplus \mathrm{Im}(\Lambda_1^P)$. Thus $g = 0$ and f reduces to $c_3(R_1^P - R_2^P)$. ∎

7. CONFLUENCE BEYOND TWO REDUCTION OPERATORS

We begin to sum up some facts obtained in Section 3.

PROPOSITION 7.1. *Let $P = (T_1, \ldots, T_n)$ be an n-tuple $(n \geq 3)$ of pairwise confluent X-reduction operators on V. Then P is confluent, dual-confluent, and the sublattice generated by P is distributive.*

This section deals with the converse of Proposition 7.1. Our first remark is that the converse is *false* for $n = 4$. Indeed, if T and U are two nonconfluent X-reduction operators, then $P = (T, U, T \wedge U, T \vee U)$ is a distributive sublattice, and Proposition 3.11 shows that P is confluent and dual-confluent. Compared to this, the next result is rather surprising.

THEOREM 7.2. *The converse of Proposition 7.1 is true for $n = 3$.*

Proof. Suppose that $P = (T_1, T_2, T_3)$ is confluent, dual-confluent, and generates a distributive sublattice. It is sufficient to prove that T_1 and T_2 are confluent. The dual-confluence of P means that $\mathrm{Obs}_{amb}^P = 0$. Therefore the formula (3.8) implies that $\mathrm{Nred}(T_3) \cap \mathrm{Obs}_{amb}^{T_1, T_2} = 0$. We let $E = \mathrm{Red}(T_3) \cap \mathrm{Obs}_{amb}^{T_1, T_2}$. It remains to prove that $E = 0$. The same formula (3.8) shows that T_3 is confluent with $T_1 \vee T_2$, so that we have

$$\mathrm{Red}\left(T_3 \wedge (T_1 \vee T_2)\right) = \mathrm{Red}(T_3) \cap \mathrm{Red}\left(T_1 \vee T_2\right). \tag{7.1}$$

Using the second basic relation (2.4) and developing the right-hand side of (7.1), we can write

$$\mathrm{Red}\left(T_3 \wedge (T_1 \vee T_2)\right) = \left[\mathrm{Red}(T_1 \wedge T_3) + \mathrm{Red}(T_2 \wedge T_3)\right] \oplus F \oplus E, \tag{7.2}$$

where F is some X-generated subspace. In addition, the second basic relation (2.4) provides again

$$\operatorname{Red}\left[(T_1 \wedge T_3) \vee (T_2 \wedge T_3)\right] = \left[\operatorname{Red}(T_1 \wedge T_3) + \operatorname{Red}(T_2 \wedge T_3)\right] \oplus \operatorname{Obs}_{amb}^{T_1 \wedge T_3, T_2 \wedge T_3}. \tag{7.3}$$

Applying distributivity, we see that the subspace $\operatorname{Obs}_{amb}^{T_1 \wedge T_3, T_2 \wedge T_3}$ coincides with $F \oplus E$. But, through Lemma 7.3 below (for $n = 3$ and $j = 3$), the confluence of P implies that this subspace vanishes, forcing $E = 0$. ∎

LEMMA 7.3. *Let $P = (T_1, \dots, T_n)$ be a confluent n-tuple $(n \geq 3)$ of X-reduction operators on V. For $j = 1, \dots, n$, the $(n$-1$)$-tuple*

$$Q_j = \left(T_1 \wedge T_j, \dots, T_{j-1} \wedge T_j, T_{j+1} \wedge T_j, \dots, T_n \wedge T_j\right)$$

is confluent.

Proof. In any case (P confluent or not), the basic relations $\operatorname{Red}(T_k) \cap \operatorname{Red}(T_j) = \operatorname{Red}(T_k \wedge T_j) \oplus \operatorname{Obs}_{red}^{T_k, T_j}$ show that there exists some X-generated subspace G such that

$$\begin{aligned}\operatorname{Red}(P) &= \left(\bigcap_{k \neq j} \operatorname{Red}(T_k \wedge T_j)\right) \oplus G \\ &= \operatorname{Red}(T_1 \wedge \cdots \wedge T_n) \oplus \operatorname{Obs}_{red}^{Q_j} \oplus G.\end{aligned} \tag{7.4}$$

(The second equality comes from relation (3.3) once applied to Q_j.) Next, compare (7.4) with (3.3) to get

$$\operatorname{Obs}_{red}^{P} = \operatorname{Obs}_{red}^{Q_j} \oplus G. \tag{7.5}$$

Clearly the confluence of Q_j is a consequence of the confluence of P. ∎

EXAMPLE 7.4. We use the geometric interpretation which will be presented in the following section. Let T_1, T_2, T_3 be three points, i.e., three X-reduction operators of dimension one. We denote by Σ_α the canonical stratum containing T_1, and we suppose that T_2 and T_3 are distinct in the same canonical stratum Σ_β with $\alpha \neq \beta$. The geometric criterion of the confluence says that T_1 is confluent with T_2 and T_3, but T_2 and T_3 are not confluent. So the pairwise confluence of $P = (T_1, T_2, T_3)$ fails. On the other hand, $\operatorname{Red}(P) = K\alpha \cap K\beta = 0$ implies that P is confluent. The geometric interpretation of $\wedge$ and $\vee$ shows that $T_1 \wedge T_2 = T_1 \wedge T_3 = 0$ (i.e., the empty set) and $T_2 \vee T_3$ is the projective line generated by T_2 and T_3. Therefore, the sublattice generated by P is distributive if and only if T_1 is *not* the point at infinity of the projective line generated by T_2 and T_3.

Suppose that the latter condition holds. Then the confluent triple P is not dual-confluent by Theorem 7.2. In particular, $\operatorname{def}_c(P^!) \neq \operatorname{def}_c(P)$ in this case. Now suppose that T_1 is the point at infinity of the projective line generated by T_2 and T_3. Then $\operatorname{Red}(T_1 \vee T_2 \vee T_3) = K\alpha \oplus K\beta = \operatorname{Namb}(P)$, thus P is dual-confluent.

8. GEOMETRIC INTERPRETATION

It is well known that the lattice $\mathscr{L}(V)$ of the linear subspaces of V can be identified to the lattice of the projective subspaces of a projective space. This geometric interpretation is carried over to the lattice $\mathscr{L}_X(V)$ of the X-reduction operators as follows. An X-reduction operator of dimension one (i.e., having exactly one reduced X-generator) is called a *point*. The set of the points is denoted by $\mathbf{P}_X(V)$. Any X-reduction operator T is identified to the set of the points S such that $S \leq T$. In this interpretation, the relation $\leq$ becomes the inclusion, and the operators 0_V and 1_V become $\varnothing$ and $\mathbf{P}_X(V)$, respectively. The dimension of the projective subspace T is $\dim(T) - 1$. The meet $\wedge$ is interpreted as the intersection, whereas the join $T_1 \vee T_2$ is the union of all of the projective lines S_1S_2 generated by distinct points S_1 and S_2 belonging to T_1 and T_2, respectively.

For any X-generator α, we introduce the notation

$$\Sigma_\alpha = \{S \in \mathbf{P}_X(V);\ \text{the reduced generator of } S \text{ is } \alpha\}. \tag{8.1}$$

A point S belonging to Σ_α expresses as follows on the generators,

$$S(\beta) = \begin{cases} \alpha & \text{if } \beta = \alpha \\ 0 & \text{if } \beta < \alpha \\ c_\beta \alpha & \text{if } \beta > \alpha, \end{cases} \tag{8.2}$$

where $c_\beta \in K$. In an obvious manner, Σ_α is a vector space whose linear combinations are defined as usual on any generator *different* from α, and zero vector is the trivial point (i.e., trivial as X-reduction operator). If the ordered basis X is $x_1 < \cdots < x_r$ and if $\alpha = x_j$, the dimension of Σ_α is $r - j$. The projective lines in $\mathbf{P}_X(V)$ are described by the following.

LEMMA 8.1. *Let $P = (S_1, S_2)$ be a pair of two distinct points of Σ_α. The projective line S_1S_2 is the union of the affine line joining S_1 and S_2 in the vector space Σ_α (called the* affine part *of S_1S_2) and a point (called the* point at infinity *of S_1S_2). The point at infinity coincides with $\tau_\vee^P$ and its reduced generator is $> \alpha$.*

Proof. We shall apply the algorithm of Section 6 to P. For $i = 1, 2$ and $\beta \in X$, write $S_i(\beta) = c_{\beta,i}\alpha$, where $c_{\beta,i} = 0$ if $\beta < \alpha$ and $c_{\alpha,i} = 1$. Since $S_1 \wedge S_2 = 0$ and $\mathrm{Red}(P) = K\alpha$, the first relation (2.4) shows that α is the unique reduced P-obstruction. Denote by ω the corresponding ambiguous obstruction. By the second relation (2.4), α and ω are the reduced generators of $S_1 \vee S_2$. As $R_i^P = S_i$, the definition of the ambiguous generators given in Section 6 says that ω is the lowest generator $\beta > \alpha$ such that $c_{\beta,1} \neq c_{\beta,2}$. To compute $S_1 \vee S_2$, fix a generator β. Observing that

$$(S_1 - S_2)(\beta) = \frac{c_{\beta,1} - c_{\beta,2}}{c_{\omega,1} - c_{\omega,2}}(S_1 - S_2)(\omega),$$

$\Lambda_1^P(\beta) = \beta - c_{\beta,1}\alpha$ and $\Lambda_1^P(\omega) = \omega - c_{\omega,1}\alpha$, we can use (6.12) to obtain

$$(S_1 \vee S_2)(\beta) = \left(\frac{c_{\omega,1}c_{\beta,2} - c_{\omega,2}c_{\beta,1}}{c_{\omega,1} - c_{\omega,2}}\right)\alpha + \left(\frac{c_{\beta,1} - c_{\beta,2}}{c_{\omega,1} - c_{\omega,2}}\right)\omega. \quad (8.3)$$

Next let S be a point on S_1S_2. The reduced generator of S is α or ω. In the first case, we have $S(\alpha) = \alpha$ and $S(\omega) = \lambda\alpha$, so the application of $S = S(S_1 \vee S_2)$ to (8.3) shows that S runs over the affine line joining S_1 and S_2 in Σ_α when λ runs over K. In the second case, we have $S(\alpha) = 0$ and $S(\omega) = \omega$; then applying S to (8.3), we see that S necessarily coincides with $\tau_\vee^P$ (remember how (6.11) comes from (6.8)). ∎

Now consider a projective line $S_1S_2, S_i \in \Sigma_{\alpha_i}$ with $\alpha_1 < \alpha_2$. Let S be a third point on this line. Lemma 8.1 implies that S does not belong to Σ_{α_1}. In other words, S_2 is the point at infinity and the affine line joining S_1 to S in Σ_{α_1} is the affine part. An immediate consequence of Lemma 8.1 is the following.

LEMMA 8.2. *The projective subspace $\overline{\Sigma}_\alpha$ of $\mathbf{P}_X(V)$ generated by Σ_α is the union of the Σ_β, $\beta \geq \alpha$.*

This lemma can be restated as follows: the family of the Σ_α, $\alpha \in X$, is a *stratification* of $\mathbf{P}_X(V)$ endowed with the Zariski topology.

DEFINITION 8.3. The stratification Σ_α, $\alpha \in X$, is called the canonical stratification of $\mathbf{P}_X(V)$. The strata of this stratification are said to be canonical. Denoting X by $x_1 < \cdots < x_r$, the chain

$$\varnothing \subset \overline{\Sigma}_{x_r} \subset \overline{\Sigma}_{x_{r-1}} \subset \cdots \subset \overline{\Sigma}_{x_1} = \mathbf{P}_X(V)$$

is called the canonical flag of $\mathbf{P}_X(V)$.

Let T be a nonzero X-reduction operator, that is, a nonempty projective subspace of $\mathbf{P}_X(V)$. If the reduced generators of T are $\alpha_1, \ldots, \alpha_k$ in the increasing order, the stratification induced on T by the canonical stratification is formed by the $T \cap \Sigma_{\alpha_j}$, $1 \le j \le k$. The induced stratum $T \cap \Sigma_{\alpha_j}$ has the dimension $k - j$. The first induced stratum (i.e., for $j = 1$) generates T and is called the *affine part* of T, whereas the union of the other induced strata is called the *part at infinity* of T. Our next purpose is to give some geometric characterizations for the confluence.

Consider two X-reduction operators T_1 and T_2. They are confluent if and only if we have $\mathrm{Red}(T_1 \vee T_2) = \mathrm{Red}(T_1) + \mathrm{Red}(T_2)$. Geometrically, this is equivalent to saying that every point of each line S_1S_2 with $S_i \in T_i$ is in the same canonical stratum as a point of T_1 or T_2. Setting aside the obvious cases for which the latter condition holds, we can claim.

PROPOSITION 8.4. *Two X-reduction opreators T_1 and T_2 are confluent if and only if for any two distinct points S_1, S_2 in a* same *canonical stratum and respectively belonging to T_1 and T_2, the point at infinity of the projective line S_1S_2 is located in the same canonical stratum as a point of T_1 or T_2.*

Using the condition $\mathrm{Red}(T_1 \wedge T_2) = \mathrm{Red}(T_1) \cap \mathrm{Red}(T_2)$, we similarly obtain the dual characterization which will turn out to be more useful for the applications.

PROPOSITION 8.5. *Two X-reduction operators T_1 and T_2 are confluent if and only if any induced stratum of T_1 and any induced stratum of T_2 which are located in a same canonical stratum, necessarily intersect* (*shortly*: *any canonical stratum intersecting T_1 and T_2 intersects their intersection*)

For example, two distinct points are confluent if and only if they are not in the same canonical stratum.

COROLLARY 8.6. (i) *The confluence center of $\mathscr{L}_X(V)$, i.e., the set of the X-reduction operators which are confluent with any X-reduction operator, identifies geometrically with the canonical flag of $\mathbf{P}_X(V)$.*

(ii) *The commutation center of $\mathscr{L}_X(V)$, i.e., the set of the X-reduction operators which commute with any X-reduction operator, is formed of 0_V and 1_V.*

Proof. (i) It is an immediate consequence of Proposition 8.5.

(ii) Let T be in the commutation center. In particular, T belongs to the confluence center, that is, T is trivial and its reduced generators form a connected chain ending by the last generator x_r. Suppose T is distinct from 0_V and 1_V. There exist two generators $\alpha_1 < \alpha_2$ such that the first one is T-nonreduced and the second one is T-reduced. Then the point S defined by $S(\alpha_1) = S(\alpha_2) = \alpha_1$ and $S(\beta) = 0$ for the other generators does not commute with T. ∎

A family $S_1, \ldots, S_r$ of r points such that $S_j \in \Sigma_{x_j}$ for $j = 1, \ldots, r$ is called a *basic family* of $\mathbf{P}_X(V)$. Let T be an X-reduction operator. The basic family $S_1, \ldots, S_r$ is said to be *compatible* with T if for any T-reduced generator x_j the point S_j belongs to T. In this case, T is generated by the points S_j belonging to it. Rewriting Proposition 8.5, we see that T_1 and T_2 are confluent if and only if there exists a basic family compatible with T_1 *and* T_2. The latter characterization extends to pairwise confluence.

THEOREM 8.7. *The X-reduction operators $T_1, \ldots, T_n$ are pairwise confluent if and only if there exists a basic family compatible with each T_i, $i = 1, \ldots, n$.*

Proof. The "if" part is clear from the already treated case $n = 2$. For the "only if" part, we make an induction on n. Suppose that the property is true for less than n operators ($n \geq 3$). Let $T_1, \ldots, T_n$ be pairwise confluent and let $(S_j)_{1 \leq j \leq r}$ be a basic family compatible with $T_1, \ldots, T_{n-1}$. Fix a canonical stratum Σ_{x_j}. If Σ_{x_j} does not intersect T_n, we keep the point S_j. If Σ_{x_j} intersects only T_n, we take S_j in T_n. Examine the remaining case: Σ_{x_j} intersects T_n and some other operators, call them $T_{i_1}, \ldots, T_{i_k}$, $1 \leq i_1 < \ldots < i_k \leq n - 1$. Obviously S_j belongs to $T_{i_1} \wedge \cdots \wedge T_{i_k}$, but it does not belong to T_n in general. However, Proposition 2.11 implies that T_n is confluent with $T_{i_1} \wedge \cdots \wedge T_{i_k}$, hence we can take S_j in the intersection by Proposition 8.5. The new family $(S_j)_{1 \leq j \leq r}$ thus obtained is a basic family compatible with $T_1, \ldots, T_n$. ∎

It would be interesting to have a purely geometric proof of the previous result, i.e., without using Proposition 2.11 (notice that Proposition 2.11 is a direct consequence of Theorem 8.7). It is also worth noticing that Theorem 8.7 allows us to recover the implication *pairwise confluence* $\Rightarrow$ *distributivity*. Indeed, according to [3], $T_1, \ldots, T_n$ generate a distributive sublattice if and only if there exist r points generating the projective space $\mathbf{P}_X(V)$ such that each T_j is generated by some of those points (in fact, the latter statement is the projective version of the equivalence given in Section 4.5 of [3]). Furthermore, it is easy to construct counterexamples to the converse of the above implication: n points which are projectively independent in a same canonical stratum generate a distributive sublattice which is not 2-ply confluent. Another interest in basic families is to make precise the results dealing with maximal 2-ply confluent sublattices as stated in Theorem 2.12(ii). Recall that π_X is the map $T \mapsto \operatorname{Red}(T) = \operatorname{Im}(T)$ from $\mathscr{L}_X(V)$ onto the Boolean algebra $\mathscr{P}_V(X)$ of the X-generated subspaces.

PROPOSITION 8.8. *Let $\mathscr{T}$ be a 2-ply confluent sublattice ($\mathscr{T}$ is a finite set according to Theorem 2.12(i)). The following are equivalent.*

(i) *$\mathscr{T}$ is a maximal 2-ply confluent sublattice.*

(ii) *$\mathscr{T}$ is generated by a basic family.*

(iii) $\pi_X(\mathscr{T}) = \mathscr{P}_V(X)$.

(iv) *The lattice $\mathscr{T}$ is isomorphic to $\mathscr{P}_V(X)$.*

Proof. Suppose (i) and choose through Theorem 8.7 a basic family (S_j) compatible with each element of $\mathscr{T}$. Let T be in $\mathscr{T}$. If $\mathrm{Red}(S_j) \cap \mathrm{Red}(T) = 0$, then S_j is confluent with T by the first relation (2.4). If not, the point S_j belongs to T by compatibility, hence is again confluent with T. Since $\mathscr{T}$ is also a maximal 2-ply confluent subset, we see that $\mathscr{T}$ contains the basic family (S_j). Therefore, $\mathscr{T}$ contains the sublattice generated by this basic family. But the other inclusion comes from the fact that each T is generated by the S_j belonging to it. Thus we have (ii).

Suppose (ii), i.e., $\mathscr{T}$ is generated by a basic family (S_j). Let E be a subspace of V generated by some X-generators $x_{j_1}, \dots, x_{j_k}$. Then $S_{j_1} \vee \cdots \vee S_{j_k}$ belongs to $\mathscr{T}$. Since the family (S_j) is 2-ply confluent hence dual-confluent, we deduce from (3.7) that $\mathrm{Red}(S_{j_1} \vee \cdots \vee S_{j_k}) = E$. Thus we have (iii).

Theorem 2.12(i) says that π_X is an isomorphism from $\mathscr{T}$ on its image, hence the implication (iii) $\Rightarrow$ (iv) and also the converse by cardinality. Finally, suppose (iii) and consider U confluent with any element T of $\mathscr{T}$. By assumption, there exists U' in $\mathscr{T}$ such that $\mathrm{Red}(U') = \mathrm{Red}(U)$. As U and U' are confluent, the latter equality implies that $U = U'$ (Proposition 2.7(ii)). Thus U is in $\mathscr{T}$ and (i) holds. ∎

Let $\mathscr{T}$ be a maximal 2-ply confluent sublattice and let (S_j) be a basic family generating $\mathscr{T}$. If S is any point of $\mathscr{T}$, it belongs to some canonical stratum Σ_{x_j}. But S and S_j being two confluent points in the same stratum, they coincide. So the points $S_1, \dots, S_r$ are the *only* points of $\mathscr{T}$. In particular, we have a one-to-one correspondence between the basic families and the maximal 2-ply confluent sublattices. Note that the latter always contain the confluence center, i.e., the canonical flag. We now want to investigate the maximal commuting sublattices.

We begin to state a commutation criterion for two points. Suppose that S_1 and S_2 are two distinct points and denote by α_1 and α_2 their respective reduced generators. If the generators are the same, the points do not commute (they are not confluent). If $\alpha_1 < \alpha_2$, a simple calculation shows that the points commute if and only if $S_1(\alpha_2) = 0$. We have already seen in Section 2 that the sublattice $\mathscr{L}_X^{tri}(V)$ of the trivial X-reduction operators is maximal commuting.

PROPOSITION 8.9. *The sublattice $\mathscr{L}_X^{tri}(V)$ is the unique maximal commuting sublattice $\mathscr{T}$ such that $\pi_X(\mathscr{T}) = \mathscr{P}_V(X)$ (or, it is equivalent to say, such that the lattice $\mathscr{T}$ is isomorphic to $\mathscr{P}_V(X)$).*

Proof. Let $\mathscr{T}$ be as in the proposition. In particular, $\mathscr{T}$ is a 2-ply confluent sublattice. Using Proposition 8.8, we claim that $\mathscr{T}$ is generated by a basic family (S_j). Since S_j commutes with $S_{j+1}, \ldots, S_r$, the above criterion implies that S_j is trivial. ∎

The description of the other maximal commuting sublattices seems less clear. Choose a commuting sublattice $\mathscr{T}$ generated by $T_1, \ldots, T_n$ such that one of the T_i is not trivial (for instance, take $n = 2$ and two commuting non-trivial points). Then $\mathscr{T}$ is contained in a maximal commuting sublattice $\mathscr{T}'$. The previous result shows that $\pi_X(\mathscr{T}')$ is *strictly* contained in $\mathscr{P}_V(X)$. Therefore, the 2-ply confluent sublattice $\mathscr{T}'$ is not maximal.

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