Indag. Mathem., N.S., 3 (1), 69–77 March 30, 1992

Intersections of double cosets in algebraic groups

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Communicated by Prof. T.A. Springer at the meeting of October 28, 1991

ABSTRACT

This paper deals with intersections of double cosets on a connected algebraic group. Under appropriate hypotheses, we show that such intersections are transversal and are smooth irreducible varieties. We also show that the closure of the intersection is the intersection of the closures.

INTRODUCTION

In this note we will prove several results concerning the intersection of double cosets in algebraic groups. Let G be a connected (affine) algebraic group and let H, K and L be closed connected subgroups of G satisfying the following conditions: (i) $H \cap K$ is connected and $\mathrm{Lie}(H) + \mathrm{Lie}(K) = \mathrm{Lie}(G)$; (ii) the sets $H \setminus G/L$ of (H, L) double cosets and $K \setminus G/L$ of (K, L) double cosets are finite. We consider double coset intersections $HxL \cap KxL$. We show that each such double coset intersection is a smooth irreducible variety and that the closure of the intersection is the intersection of the closures.

In particular, let G be a reductive group, let B be a Borel subgroup of G and let T be a maximal torus of B. Let P, Q and R be parabolic subgroups of G containing B and let R^- be the parabolic subgroup containing T which is opposite to R. We consider double coset intersections of the form $PxQ \cap R^-xQ$. We give a nice parametrization of the set of these double coset intersections and we describe the partial order on the set of these intersections given by inclusion of the closures of the intersections.

Although all of our proofs are elementary, the results seem to be new and they are quite useful (see [8]).

1. INTERSECTIONS OF DOUBLE COSETS

Our basic reference for algebraic geometry and algebraic groups is the book by Borel [1]. All algebraic varieties are taken over an algebraically closed field F. If X is an irreducible algebraic variety, then X_{sm} denotes the set of smooth points of X; X_{sm} is a non-empty open subvariety of X.

First we review some standard results on transversality. The following result is a standard result in intersection theory (see [7, p. 57] or [2, p. 221]):

LEMMA 1.1. *Let X be a smooth irreducible affine variety and let Y and Z be irreducible subvarieties of X. If S is an irreducible component of $Y \cap Z$, then*

$$\dim S \geq \dim Y + \dim Z - \dim X.$$

Now let Y and Z be irreducible subvarieties of the smooth irreducible affine variety X. Let $x \in Y \cap Z$. We say that Y and Z *intersect transversally at* x if $x \in Y_{sm} \cap Z_{sm}$ and if $T_x(Y) + T_x(Z) = T_x(X)$. We say that Y and Z *intersect transversally* if the intersection is transversal at every point of $Y \cap Z$.

We have the following well-known consequence of Lemma 1.1.

PROPOSITION 1.2. *Let X, Y and Z be as above, let $x \in Y \cap Z$ and assume that Y and Z intersect transversally at x. Then*: (1) *x belongs to a unique irreducible component S of $Y \cap Z$ and x is a smooth point of S; and* (2) $\dim S = \dim Y + \dim Z - \dim X$.

PROOF. Let S be an irreducible component of $Y \cap Z$. Then we have

$$\begin{aligned}\dim S &\leq \dim T_x(S)\\ &\leq \dim(T_x(Y) \cap T_x(Z))\\ &= \dim T_x(Y) + \dim T_x(Z) - \dim(T_x(Y) + T_x(Z))\\ &= \dim Y + \dim Z - \dim X\\ &\leq \dim S.\end{aligned}$$

Proposition 1.2 now follows immediately.

We will need the following (known) technical lemma:

LEMMA 1.3. *Let D be a closed connected subgroup of the connected algebraic group M and let $\pi : M \to M/D$ denote the canonical projection. If X is an irreducible subset of M/D, then $\pi^{-1}(X)$ is irreducible.*

PROOF. It is known that π is an open map. Let U and V be non-empty relatively open subsets of $\pi^{-1}(X)$. Since π is an open map, $\pi(U)$ and $\pi(V)$ are non-empty relatively open subsets of X. Thus $\pi(U) \cap \pi(V) \neq \emptyset$. Let

$x \in \pi(U) \cap \pi(V)$ and let $Y = \pi^{-1}(x)$. Then Y is irreducible and $Y \cap U$ and $Y \cap V$ are non-empty, relatively open subsets of Y. Thus there exists $y \in Y \cap U \cap V$. This shows that $\pi^{-1}(X)$ is irreducible.

THEOREM 1.4. *Let G be a connected algebraic group and let H, K and L be closed connected subgroups of G. Assume that $H \cap K$ is connected and that* $\mathrm{Lie}(H) + \mathrm{Lie}(K) = \mathrm{Lie}(G)$. *Then for every $x \in G$, the double cosets HxL and KxL intersect transversally at x and the intersection $HxL \cap KxL$ is irreducible. Thus $HxL \cap KxL$ is a smooth irreducible locally closed subvariety of G and* $\dim(HxL \cap KxL) = \dim HxL + \dim KxL - \dim G$.

PROOF. It follows easily from the condition $\mathrm{Lie}(H) + \mathrm{Lie}(K) = \mathrm{Lie}(G)$ that HxL and KxL intersect transversally at every point of $HxL \cap KxL$. Thus $HxL \cap KxL$ is a smooth subvariety of G and each irreducible component has dimension equal to $\dim HxL + \dim KxL - \dim G$.

It remains to prove that the intersection $HxL \cap KxL$ is irreducible. Let $M = xLx^{-1}$. It will suffice to prove that $HM \cap KM$ is irreducible. Let $E = \{(h,k) \in H \times K \mid hk^{-1} \in M\}$ and define $\sigma : E \to H$ by $\sigma(h,k) = h^{-1}$. Let $A = \sigma(E)$. We claim that

(1-1) $\quad HM \cap KM = AM.$

Let $h^{-1} \in A$. Then $h^{-1}M \subset HM$ and there exists $k \in K$ such that $kh^{-1} \in M$, so that $h^{-1}M \subset KM$. Therefore $AM \subset HM \cap KM$. Now let $y \in HM \cap KM$. Then $y = h^{-1}m_1 = k^{-1}m_2$, with $h \in H$, $k \in K$, and $m_1, m_2 \in M$. Hence $hk^{-1} \in M$ and consequently $h^{-1} \in A$, so that $y \in AM$. This proves (1-1).

To prove that $HM \cap KM = AM$ is irreducible, it will suffice to prove that E is irreducible. We let $H \times K$ act on G by $(h,k) \cdot g = hgk^{-1}$. Let $\mu : H \times K \to G$ denote the orbit map at $1 \in G$: we have $\mu(h,k) = hk^{-1}$. The isotropy subgroup $(H \times K)_1$ is equal to $\{(h,k) \in H \times K \mid h = k\}$. Set $D = (H \times K)_1$. Then D is isomorphic to $H \cap K$. In particular, D is irreducible. Since $\mathrm{Lie}(H) + \mathrm{Lie}(K) = \mathrm{Lie}(G)$, the differential $d\mu_{(1,1)} : T_{(1,1)}(H \times K) \to T_1(G)$ is surjective. This implies that μ induces an isomorphism $\tau : (H \times K)/D \to HK$ of $H \times K$-varieties and that HK is open in G. Thus $HK \cap M$ is a non-empty open subvariety of the irreducible variety M. In particular, $HK \cap M$ is an irreducible variety. Consequently $\tau^{-1}(HK \cap M)$ is an irreducible subvariety of $(H \times K)/D$. Let $\pi : H \times K \to (H \times K)/D$ be the canonical projection. It follows from Lemma 1.3 that

$$E = \mu^{-1}(HK \cap M) = \pi^{-1}(\tau^{-1}(HK \cap M))$$

is irreducible. This proves Theorem 1.4.

Let $X = G/L$. The geometric picture becomes clearer if we take intersections of H-orbits and K-orbits on X. In this case, Theorem 1.4 gives:

COROLLARY 1.5. Let *G, H, K and L satisfy the hypotheses of Theorem* 1.4 *and let $X = G/L$. For every $x \in X$, the orbits $H \cdot x$ and $K \cdot x$ intersect transversally* *and the intersection* $H\cdot x\cap K\cdot x$ *is a smooth irreducible subvariety of* X *with*

$$\dim(H\cdot x\cap K\cdot x)=\dim H\cdot x+\dim K\cdot x-\dim X.$$

2. CLOSURES OF DOUBLE COSET INTERSECTIONS

If X is a subset of the algebraic group G, then $cl(X)$ denotes the Zariski closure of X in G. Let H and L be closed subgroups of G. We define a partial order on the set $H\backslash G/L$ of double cosets as follows: if $\mathscr{H}', \mathscr{H}\in H\backslash G/L$, then $\mathscr{H}'\leq\mathscr{H}$ if and only if $\mathscr{H}'\subset cl(\mathscr{H})$. Thus $cl(\mathscr{H})=\bigcup_{\mathscr{H}'\leq\mathscr{H}}\mathscr{H}'$.

THEOREM 2.1. *Let* G, H, K *and* L *satisfy the conditions of Theorem* 1.4. *Assume further that* $H\backslash G/L$ *and* $K\backslash G/L$ *are finite. If* $\mathscr{H}\in H\backslash G/L$ *and* $\mathscr{K}\in K\backslash G/L$, *then*

$$cl(\mathscr{H}\cap\mathscr{K})=cl(\mathscr{H})\cap cl(\mathscr{K}).$$

Furthermore $cl(\mathscr{H})_{sm}\cap cl(\mathscr{K})_{sm}\subset cl(\mathscr{H}\cap\mathscr{K})_{sm}$.

PROOF. It is clear that

$$cl(\mathscr{H}\cap\mathscr{K})\subset cl(\mathscr{H})\cap cl(\mathscr{K}).$$

Let $\mathscr{E}$ denote the (finite) set of all pairs $(\mathscr{H}',\mathscr{K}')\in(H\backslash G/L)\times(K\backslash G/L)$ such that $\mathscr{H}'\leq\mathscr{H}$, $\mathscr{K}'\leq\mathscr{K}$ and $\mathscr{H}'\cap\mathscr{K}'\neq\emptyset$. Then

$$cl(\mathscr{H})\cap cl(\mathscr{K})=\coprod_{(\mathscr{H}',\mathscr{K}')\in\mathscr{E}}\mathscr{H}'\cap\mathscr{K}'=\bigcup_{(\mathscr{H}',\mathscr{K}')\in\mathscr{E}}cl(\mathscr{H}'\cap\mathscr{K}').$$

Since each intersection $\mathscr{H}'\cap\mathscr{K}'$ is an irreducible subvariety of G, it follows that the irreducible components of $cl(\mathscr{H})\cap cl(\mathscr{K})$ are the maximal elements among the sets

$$cl(\mathscr{H}'\cap\mathscr{K}'),\quad(\mathscr{H}',\mathscr{K}')\in\mathscr{E}.$$

It follows from 1.1 that each irreducible component of $cl(\mathscr{H})\cap cl(\mathscr{K})$ has dimension $\geq\dim\mathscr{H}+\dim\mathscr{K}-\dim G$. If $(\mathscr{H}',\mathscr{K}')\in\mathscr{E}$ and $(\mathscr{H}',\mathscr{K}')\neq(\mathscr{H},\mathscr{K})$, then it follows from Theorem 1.4 that

$$\begin{aligned}\dim cl(\mathscr{H}'\cap\mathscr{K}')&=\dim(\mathscr{H}'\cap\mathscr{K}')\\&=\dim\mathscr{H}'+\dim\mathscr{K}'-\dim G\\&<\dim\mathscr{H}+\dim\mathscr{K}-\dim G,\end{aligned}$$

so that $cl(\mathscr{H}'\cap\mathscr{K}')$ is not an irreducible component of $cl(\mathscr{H})\cap cl(\mathscr{K})$. Thus we see that $cl(\mathscr{H}\cap\mathscr{K})$ is the unique irreducible component of $cl(\mathscr{H})\cap cl(\mathscr{K})$, so that $cl(\mathscr{H})\cap cl(\mathscr{K})=cl(\mathscr{H}\cap\mathscr{K})$.

Now let $x\in cl(\mathscr{H})_{sm}\cap cl(\mathscr{K})_{sm}$. Then there exists $(\mathscr{H}',\mathscr{K}')\in\mathscr{E}$ such that $x\in\mathscr{H}'\cap\mathscr{K}'$. By Theorem 1.1, $\mathscr{H}'$ and $\mathscr{K}'$ intersect transversally at x. Clearly $T_x(\mathscr{H}')\subset T_x(cl(\mathscr{H}))$ and $T_x(\mathscr{K}')\subset T_x(cl(\mathscr{K}))$. It follows immediately that $cl(\mathscr{H})$ and $cl(\mathscr{K})$ intersect transversally at x. Thus x is a smooth point of $cl(\mathscr{H})\cap cl(\mathscr{K})=cl(\mathscr{H}\cap\mathscr{K})$. This completes the proof of Theorem 2.1.

3. DOUBLE COSET INTERSECTIONS FOR PARABOLIC SUBGROUPS

For the rest of the paper, G will denote a reductive group. Let B and B^- be opposite Borel subgroups of G and let $T=B\cap B^-$; T is a maximal torus of G. Let $\Phi=\Phi(G,T)$ be the set of roots of G relative to T, let $\Phi^+=\Phi(B,T)$ be the set of positive roots determined by B and let $\Delta\subset\Phi^+$ be the corresponding set of simple roots. Let $W=W(T)$ be the Weyl group and let $S=\{s_\alpha \mid \alpha\in\Delta\}$ be the set of simple reflections. Then $W=(W,S)$ is a Coxeter group. Let l be the length function on W and let $\leq$ denote the Bruhat order on W. We let w_0 denote the longest element of W. We have $B^-=w_0Bw_0$.

LEMMA 3.1. Let $x\in W$. (1) $cl(BxB)=\coprod_{w\leq x} BwB$ *(where the symbol* $\coprod$ *denotes the disjoint union) and* $\dim BxB=l(x)+\dim B$. (2) $cl(B^-xB)=\coprod_{w\geq x} B^-wB$ *and*

$$\dim B^-xB = l(w_0)-l(x)+\dim B = \dim G - l(x).$$

(3) *Let* $y\in W$. *Then the intersection* $BxB\cap B^-yB$ *is non-empty if and only if* $y\leq x$.

PROOF. The result (1) is standard. Since $B^-=w_0Bw_0$, (2) follows easily from (1). The result (3) is in [5, Cor. 1.2].

THEOREM 3.2. *Let* $x,y\in W$ *with* $y\leq x$. *Then* $BxB\cap B^-yB$ *is a smooth irreducible subvariety of* G *and*

$$\dim(BxB\cap B^-yB) = l(x)-l(y)+\dim B.$$

Moreover

$$cl(BxB\cap B^-yB) = cl(BxB)\cap cl(B^-yB) = \coprod_{y\leq w'\leq w\leq x} BwB\cap B^-w'B. \tag{3-1}$$

PROOF. The proof of the first conclusion follows from Theorem 1.4. (The dimension formula is also in [5]). The proof of (3-1) follows from Theorem 2.1 and Lemma 3.1.

Now we wish to generalize the results of Theorem 3.2 to the case of parabolic subgroups. For each subset $J\subset S$, let W_J denote the subgroup of W generated by J and let w_J be the longest element of W_J. We let $P_J=BW_JB$ denote the "standard" parabolic subgroup of G corresponding to J. Let $J^*=\{\alpha\in\Delta \mid s_\alpha\in J\}$ denote the subset of Δ corresponding to J and let $D_J=\{d\in W \mid d(J^*)\subset\Phi^+\}$ be the set of minimal representatives for the left cosets of W modulo W_J. If $I\subset S$, we set $D(J,I)=D(J)^{-1}\cap D_I$. Then $D(J,I)$ is the set of minimal double coset representatives for the (W_J,W_I) double cosets of W. See [4, §64].

The following lemma is immediate (and well-known):

LEMMA 3.3. *Let* $J\subset S$ *and let* $C_J=D_Jw_J=\{dw_J \mid d\in D_J\}$. *Then each left coset* wW_J *contains a unique element of* C_J. *If* $c\in C_J$ *and if* $w\in W_J$, *then* $l(cw)=l(c)-l(w)$. *Furthermore, we have* $C_J=w_0D_J=\{c\in W \mid c(J^*)\subset -\Phi^+\}$.

More generally, we have:

LEMMA 3.4. *Let J and I be subsets of S and let $C(J,I)=C(J)^{-1}\cap C_I$. Then every (W_J, W_I) double coset of W contains a unique element of $C(J,I)$. If $c\in C(J,I)$, $w\in W_J$ and $w'\in W_I$, then $l(c)\geq l(wcw')$ with equality only if $c=wcw'$.*

PROOF. Let $X=W_JwW_I$ be a (W_J, W_I) double coset and let $c\in X$ be such that $l(c)\geq l(x)$ for every $x\in X$. It follows easily from Lemma 3.3 that $c\in C_J^{-1}\cap C_I$. In order to prove the lemma, it will suffice to prove that $|C(J,I)|=|D(J,I)|=|W_I\setminus W/W_J|$. We note that

$$w_0C(J,I) = w_0C_J^{-1}\cap w_0C_I = w_0C_J^{-1}\cap D_I.$$

Let $J_0=w_0Jw_0$. Then $w_0C_J^{-1}=w_0D_J^{-1}w_0=D_{J_0}^{-1}$. Thus we see that

$$(3\text{-}2)\qquad w_0C(J,I) = D_{J_0}^{-1}\cap D_I = D(J_0,I),$$

so that $|C(J,I)|=|D(J_0,I)|$. Thus it will suffice to prove that $|D(J_0,I)|=|D(J,I)|$, or equivalently that $|W_{J_0}\setminus W/W_I|=|W_J\setminus W/W_I|$. If $w\in W$, then $W_{J_0}wW_I=w_0W_Jw_0wW_I$. Hence the map $X\mapsto w_0X$ is a bijection from $W_{J_0}\setminus W/W_I$ to $W_J\setminus W/W_I$. This proves 3.4.

The set $C(J,I)$ is the set of *maximal double coset representatives* for (W_J, W_I) double cosets. A slightly different description of the set $C(J,I)$ of maximal double coset representatives is given in [3, Thm. 1.2].

We define the "Bruhat order" $\leq$ on the set $P\setminus G/Q=P_J\setminus G/P_I$ of (P,Q) double cosets by:

$$PxQ\leq PyQ \quad \text{if and only if} \quad PxQ\subset cl(PyQ).$$

LEMMA 3.5. *Let I and J be subsets of S and let $C=C(J,I)$.* (1) *The map $w\mapsto P_JwP_I$ from W to the set $P_J\setminus G/P_I$ of (P_J,P_I) double cosets of G is constant on (W_J,W_I) double cosets of W and induces a bijection from $W_J\setminus W/W_I$ to $P_J\setminus G/P_I$.* (2) *If $c\in C$, then*

$$P_JcP_I = \coprod_{w\in W_JcW_I} BwB$$

and

$$\begin{aligned} cl(P_JcP_I) &= cl(BcB) \\ &= \coprod_{w\in W,\, w\leq c} BwB \\ &= \coprod_{c'\in C,\, c'\leq c} P_Jc'P_I. \end{aligned}$$

Thus the map $c\mapsto P_JcP_I$ is an order preserving bijection from C to $P_J\setminus G/P_I$. (We give C the partial order induced from the Bruhat order on W.)

PROOF. The proof of (1) is in [4, § 65]. The proof of (2) follows from (1) and from 3.1.

Now let I, J and K be subsets of S. Let $P=P_J$, let $Q=P_I$ and let $R=P_K$. We let R^- be the unique parabolic subgroup of G containing T which is opposite to R. Let $K_0=w_0Kw_0$. Then $K_0\subset S$ and $R^-=w_0P_{K_0}w_0$. We wish to study intersections of (P,Q) double cosets and (R^-,Q) double cosets. It is clear that $\mathrm{Lie}(P)+\mathrm{Lie}(R^-)=\mathrm{Lie}(G)$ and it is a standard result that the intersection of any two parabolic subgroups of G is connected, so that $P\cap R^-$ is connected. Thus Theorems 1.4 and 2.5 apply to double coset intersections $PxQ\cap R^-xQ$.

We define the Bruhat order on the set $R^-\backslash G/Q$ of R^- double cosets as above. The following lemma describes the parametrization and the Bruhat order on the set of (R^-,Q) double cosets:

LEMMA 3.6. (1) *Let $D=D(K,I)$. Then the map $d\mapsto R^-dQ$ is an order reversing bijection from D to $R^-\backslash G/Q$.* (2) *If $d\in D$, then $R^-dQ=\coprod_{w'\in W_KdW_I}B^-w'B$ and*

$$\begin{aligned}
cl(R^-dQ) &= cl(B^-dB)\\
&= \coprod_{w'\geq d} B^-w'B\\
&= \coprod_{d'\in D,\, d'\geq d} R^-d'Q.
\end{aligned}$$

PROOF. Let $c\in C(K_0,I)$. Then $P_{K_0}cQ=w_0R^-w_0cQ$. Hence it follows from Lemma 3.5 that the map $c\mapsto R^-w_0cQ$ is an order preserving bijection from $C(K_0,I)$ to $R^-\backslash G/Q$. It follows from (3-2) that $w_0C(K_0,I)=D(K,I)$. The proof of (1) now follows easily. The proof of (2) follows from (1).

THEOREM 3.7. *Let $(c,d)\in C\times D$.*

(1) *Let $A(c,d)=\{(x,y)\in(W_JcW_I)\times(W_KdW_I)\mid d\leq y\leq x\leq c\}$. Then*

$$PcQ\cap R^-dQ=\coprod_{(x,y)\in A(c,d)} BxB\cap B^-yB.$$

In particular $PcQ\cap R^-dQ\neq\emptyset$ if and only if $d\leq c$.

(2) *$BcB\cap B^-dB$ is a dense open subset of $PcQ\cap R^-dQ$.*

(3) *$PcQ\cap R^-dQ$ is a smooth irreducible variety of dimension $l(c)-l(d)+\dim B$.*

(4) *Let $\mathscr{K}(c,d)=\{(c',d')\in C\times D\mid d\leq d'\leq c'\leq c\}$. Then*

$$\begin{aligned}
cl(PcQ\cap R^-dQ) &= cl(PcQ)\cap cl(R^-dQ)\\
&= cl(BcB)\cap cl(B^-dB)\\
&= \coprod_{(c',d')\in\mathscr{K}(c,d)} Pc'Q\cap R^-d'Q\\
&= \coprod_{d\leq w'\leq w\leq c} BwB\cap B^-w'B.
\end{aligned}$$

(5) *$cl(PcQ)_{sm}\cap cl(R^-dQ)_{sm}\subset cl(PcQ\cap R^-dQ)_{sm}$.*

PROOF. The proof of (1) follows easily from 3.1, 3.5 and 3.6. If $(x,y) \in A(c,d)$, then

$$\dim(BxB \cap B^{-}yB) = l(x) - l(y) + \dim B \leq l(c) - l(d) + \dim B,$$

and we get equality only if $(x,y) = (c,d)$. Thus

$$\dim(PcQ \cap R^{-}dQ) = \dim(BcB \cap B^{-}dB) = l(c) - l(d) + \dim B.$$

Since $PcQ \cap R^{-}dQ$ is irreducible, (2) now follows from (1). The proofs of (3), (4), and (5) follow easily from (1), (2), Theorem 1.4, Theorem 2.1, Lemma 3.5 and Lemma 3.6.

Let $\mathscr{D} = \mathscr{D}(J,K,I)$ denote the set of all double coset intersections $PgQ \cap R^{-}hQ$. Let

$$\Gamma = \Gamma(J,K,I) = \{(c,d) \in C \times D \mid d \leq c\}.$$

For each $(c,d) \in \Gamma$, let $\mathscr{O}(c,d) = PcQ \cap R^{-}dQ$. Then $\mathscr{D} = \{\mathscr{O}(c,d) \mid (c,d) \in \Gamma\}$. We define a partial order on $\mathscr{D}$ by:

$$\mathscr{O}(c',d') \leq \mathscr{O}(c,d) \quad \text{if and only if} \quad \mathscr{O}(c',d') \subset cl(\mathscr{O}(c,d)).$$

We define a partial order on Γ by:

$$(c',d') \leq (c,d) \quad \text{if and only if} \quad d \leq d' \leq c' \leq c.$$

The result below follows from Theorem 3.7.

COROLLARY 3.8. *The map $(c,d) \mapsto \mathscr{O}(c,d)$ from Γ to $\mathscr{D}$ is an isomorphism of posets.*

3.9. The case $P = R$. We now consider the case in which $J = K$ or, equivalently, $P = R$. Thus $P^{-} = R^{-}$ and we look at double coset intersections of the form $PxQ \cap P^{-}yQ$, where P and P^{-} are opposite parabolic subgroups. We let $D = D(J,I)$ and $C = C(J,I)$. We have the following lemma:

LEMMA 3.9.1. *Let $d_1, d_2 \in D$ and let c_1 and c_2 be the maximal double coset representatives for (respectively) $W_J d_1 W_I$ and $W_J d_2 W_I$. Thus $W_J d_i W_I \cap C = \{c_i\}$, $i = 1, 2$. The following conditions are equivalent:*

(1) $d_1 \leq d_2$.

(2) $d_1 \leq c_2$.

(3) $c_1 \leq c_2$.

(4) *There exists $y \in W_J d_1 W_I$ and $x \in W_J d_2 W_I$ such that $y \leq x$.*

PROOF. It is clear that (2)$\Leftrightarrow$(4), (1)$\Rightarrow$(2) and (3)$\Rightarrow$(2).

(2)$\Rightarrow$(1). By [4, Prop. 64.38], we may write $c_2 = w'd_2w''$, with $w' \in W_J$, $w'' \in W_I$ and $l(c_2) = l(w') + l(d_2) + l(w'')$. Since $d_1 \leq c_2$, it follows easily from standard properties of the Bruhat order that we can write $d_1 = w_1 x w_2$, with $w_1 \leq w'$, $x \leq d_2$, $w_2 \leq w''$ and $l(d_1) = l(w_1) + l(x) + l(w_2)$. It will suffice to show that $w_1 = 1 = w_2$. Assume that $l(w_1) > 0$ and let $w_1 = s_1 \cdots s_r$ with each s_i in J and $l(w_1) = r$. Then $s_1 d_1 = s_2 \cdots s_r x w_2$. Since $d_1 \in D(J)^{-1}$, we see that $l(s_1 d_1) = l(d_1) + 1 = 1 + r + l(x) + l(w_2)$. On the other hand, we have

$l(s_2 \cdots s_r x w_2) \leq r-1+l(x)+l(w_2)$. This gives a contradiction, which shows that $w_1 = 1$. A similar argument shows that $w_2 = 1$.

(1)$\Rightarrow$(3). It will suffice to show that, for every $x \in W_J d_1 W_I$, we have $x \leq c_2$. The proof is by induction on $m(x) = l(x) - l(d_1)$. Since $d_1 \leq d_2 \leq c_2$, we are OK for $m(x) = 0$. If $x \in W_J d_1 W_I$, $x \neq d_1$, then we may write $x = w_1 d_1 w_2$ with $w_1 \in W_J$, $w_2 \in W_I$ and $l(x) = l(w_1) + l(d_1) + l(w_2) > l(d_1)$. It follows easily from this that (at least) one of the following two conditions holds: (i) there exists $s \in J$ such that $sx < x$; or (ii) there exists $s \in I$ such that $xs < x$. Assume that condition (i) holds. By the inductive hypothesis, we may assume that $sx < c_2$. By definition of c_2, we have $sc_2 < c_2$. It now follows from a standard property of the Bruhat order (see [6, Lemma 7.4]) that $x \leq c_2$. Similarly if condition (ii) holds, then $x \leq c_2$. This completes the induction.

Since we have a natural bijection between $D = D(J,I)$ and $C = C(J,I)$, we can parametrize the set $\mathscr{D} = \mathscr{D}(J,J,I)$ of double coset intersections by a subset of $D \times D$. If $(d_1, d_2) \in D \times D$ then it follows from Theorem 3.7 and Lemma 3.9.1 that $Pd_1Q \cap P^- d_2 Q \neq \emptyset$ if and only if $d_2 \leq d_1$. Let $\Gamma_0 = \{(d_1, d_2) \in D \times D \mid d_2 \leq d_1\}$. For each $(d_1, d_2) \in \Gamma_0$, let $\mathscr{O}(d_1, d_2) = Pd_1 Q \cap P^- d_2 Q$. Then $\mathscr{D} = \{\mathscr{O}(d_1, d_2) \mid (d_1, d_2) \in \Gamma_0\}$. We define a partial order on Γ_0 by: $(d_1', d_2') \leq (d_1, d_2)$ if and only if $d_2 \leq d_2' \leq d_1' \leq d_1$. We define the partial order on $\mathscr{D}$ as above. Then we have:

PROPOSITION 3.9.2. *The map $(d_1, d_2) \mapsto \mathscr{O}(d_1, d_2)$ from Γ_0 to $\mathscr{D}$ is an isomorphism of posets.*

The proof follows immediately from Lemma 3.9.1 and Corollary 3.8.

3.10. *An example.* (See [8].) Assume that the root system Φ is irreducible. Let $\alpha^* = \sum_{\alpha \in \Delta} n_\alpha(\alpha^*)\alpha$ denote the highest root. Let $\beta \in \Delta$ be such that $n_\beta(\alpha^*) = 1$ and let $J = S \setminus \{s_\beta\}$. Set $I = K = J$, so that $P = Q = R$. Let $D = D(J,J)$. It is shown in [8] that there exists a positive integer m and an order preserving bijection $\sigma : D \to \{1, \ldots, m\}$. Using this bijection, we may identify the set Γ_0 defined above with $\mathscr{A}_m = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq i \leq m\}$. If we define a partial order on $\mathscr{A}_m$ by requiring σ to be an isomorphism of posets, then we have: $(i,j) \leq (k,l)$ if and only if $l \leq j \leq i \leq k$. Thus the poset $\mathscr{A}_m$ is isomorphic to the poset $\mathscr{D}$ of double coset intersections.

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