Journal of Algebra **239**, 513–548 (2001)
doi:10.1006/jabr.2000.8689, available online at http://www.idealibrary.com on IDEAL®

Dualizing Complexes over Noncommutative Local Rings

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Communicated by J. T. Stafford

Received December 16, 1999

We prove an existence theorem for dualizing complexes over noncommutative noetherian complete semilocal algebras, which generalizes Van den Bergh's existence theorem in the graded case. Using the dualizing complex, noncommutative versions of Bass theorem and the no-holes theorem are proved. We also prove that noetherian complete semilocal algebras satisfying polynomial identities are catenary.

Key Words: semilocal algebra; PI algebra; local cohomology; dualizing complex; Bass theorem; no-holes theorem

0. INTRODUCTION

The noncommutative version of a dualizing complex was introduced by Yekutieli in 1991 [Ye1]. It was studied further by Jørgensen [Jo1, Jo2, Jo3], Miyachi [Mi], Van den Bergh [VdB], and Yekutieli [Ye2, YZ1, YZ2]. Existence of dualizing complexes allows one to prove various properties. Here are a few examples: (a) Serre duality for noncommutative projective schemes [YZ1]; (b) existence of exact, finitely partitive, symmetric dimension functions [YZ2]; (c) purity of the characteristic varieties [YZ2]. The

0021-8693/01 $35.00

main existence theorem is due to Van den Bergh [VdB, 6.3]: a noetherian connected graded algebra A has a balanced dualizing complex if and only if A satisfies the χ condition and has finite cohomological dimension. As a consequence, noetherian connected graded PI algebras have balanced dualizing complexes. (A PI algebra is an algebra satisfying a polynomial identity.) If A is filtered and the associated graded ring has a balanced dualizing complex, then A has a rigid dualizing complex [VdB, 8.6].

The first part of the paper is to prove an existence theorem in the ungraded semilocal case.

THEOREM 0.1. *Let* $(A, \mathfrak{m})$ *and* $(T, \mathfrak{n})$ *be noetherian complete semilocal algebras such that there is a Morita duality between them. Suppose that*

(1) $\Gamma_{\mathfrak{m}}$ *and* $\Gamma_{\mathfrak{n}^\circ}$ *have finite cohomological dimension*,

(2) A *and* T *satisfy the* χ *condition*, *and*

(3) $A_0 = A/\mathfrak{m}$ *is weakly symmetric*.

Then there is a bifinite, Cdim-*symmetric*, *pre-balanced dualizing complex over* (A, T).

The terminology will be explained in Sections 1 and 2. The proof of 0.1 follows from the outline of Van den Bergh's proof in [VdB]. A version of 0.1 was also proved independently by Chan [Cha]. In the PI case the hypotheses of 0.1 can be checked, whence the following holds.

COROLLARY 0.2. *Let* A *be a noetherian complete semilocal PI algebra. Then there is an Auslander*, *bifinite*, Cdim-*symmetric*, Kdim-*Macaulay*, *pre-balanced dualizing complex over* (A, T) *for some noetherian complete semilocal PI algebra* T.

The second part of this paper contains some applications of dualizing complexes. We prove the Auslander–Buchsbaum formula, Bass theorem, and the no-holes theorem in the ungraded case.

THEOREM 0.3. *Let* A *and* T *be noetherian local algebras and let* R *be a pre-balanced dualizing complex over* (A, T). *Suppose* M *is a nonzero noetherian left* A*-module*.

(1) (*Auslander–Buchsbaum formula*) *If* M *has finite projective dimension* pd M, *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A,$$

where depth M *is the depth of* M.

(2) (*Bass Theorem*) *If* M *has finite injective dimension* id M, *then*

$$\operatorname{id} M = \operatorname{depth} A.$$

(3) (*No-Holes Theorem*) *For every integer i,*

$$\textit{the Bass number } \mu^i(M) \neq 0 \quad \textit{if and only if} \quad \operatorname{depth} M \leq i \leq \operatorname{id} M.$$

The definitions of the Bass number and the depth are given in Section 5. By symmetry, these assertions hold for right T-modules. The proof of 0.3 is similar to Jørgensen's proof in the graded case [Jo1] and similar to the proofs in the commutative case given in [Fo, Ro]. If A is a noetherian local PI ring, then it follows from 0.2 and 0.3 that the Auslander–Buchsbaum formula, Bass theorem, and the no-holes theorem hold for A (see 6.2). Another immediate corollary is that the conclusions of 0.3 hold for noetherian local AS–Gorenstein algebras (see 5.13). We also prove the following by using the Auslander property of a dualizing complex.

PROPOSITION 0.4. *Every noetherian complete semilocal PI algebra is catenary.*

Let us mention the main difference between the graded case and the ungraded one. In the graded case, the basic ingredients of constructing dualizing complexes are local cohomology and graded Matlis duality (i.e., graded vector space duality). In the ungraded case, the graded Matlis duality is replaced by the Morita duality. We have seen in [WZ] that given a complete local noetherian ring A, usually there is another complete local noetherian ring T such that A and T are in Morita duality (see also 7.6). Given an (A, T)-bimodule M, the left Morita dual of M is not necessarily isomorphic to the right Morita dual of M, even when $A = T$. So we need to prove some technical lemmas which are obvious in the graded or the commutative case. The hypothesis that A_0 is weakly symmetric (see 0.1(3)) is used in our proof, and we do not know whether this can be avoided.

The dualizing complex is an object in a derived category, so we will work with complexes and derived categories. Basic definitions and properties related to derived categories can be found in [Ha] and in [Jo1, Jo3, WZ]. This paper may be viewed as a sequel of [WZ], from which many notations are retained. One motivation of this paper is to show that every noetherian complete semilocal PI algebra admits a dualizing complex (see 0.2). The Auslander property of the dualizing complexes over PI algebras was basically proved in [WZ, 1.5].

This paper is organized as follows. Section 1 contains some definitions. In Sections 2 and 3 we prove some preliminary results (see 2.9 and 3.6) which will be used in the proof of the existence theorem in Section 4. Noncommutative versions of the Auslander–Buchsbaum formula, Bass theorem, and the no-holes theorem are given in Section 5. Section 6 contains some consequences in the PI case. In particular, we prove 0.4. Also we show that if A is a noetherian semilocal PI algebra, then its global dual Krull dimension is equal to its Krull dimension. Finally some examples are discussed in Section 7.

1. SOME DEFINITIONS

Throughout the paper A is an algebra over a fixed base field k, $\mathfrak{m}$ is the Jacobson radical of A, and $A_0 = A/\mathfrak{m}$. If A_0 is artinian, A is called *semilocal*. If A_0 is an artinian simple algebra, then A is called *local*. By A being *complete* we mean that it is complete with respect to the $\mathfrak{m}$-adic topology. Unless otherwise stated we are working with left modules. Let A° be the opposite ring of A and $A^e = A \otimes A^\circ$, where $\otimes = \otimes_k$. We use the term A°-modules for right A-modules and A^e-modules for (A, A)-bimodules. Usually noetherian and artinian mean two-sided noetherian and two-sided artinian, respectively.

Let A-Mod denote the category of A-modules, and let Mod-$A = A^\circ$-Mod. We say an A-module is *finite* if it is finitely generated over A. We refer to [Ha] and [Jo1, Jo3, WZ, Ye1, YZ2] for basic notations about complexes of modules and derived categories. For example, $D^-_f(A)$ is the derived category of right-bounded complexes of A-modules with finite (or noetherian, when A is left noetherian) cohomological modules. Given any complex X, define

$$\sup X = \sup\{i \mid h^i(X) \neq 0\} \quad \text{and} \quad \inf X = \inf\{i \mid h^i(X) \neq 0\},$$

where $h^i(X)$ is the ith cohomology of X.

Let B and C be two algebras. For $X \in K(A \otimes B^\circ)$ and $Y \in K(A \otimes C^\circ)$, $\operatorname{Hom}_A(X, Y)$ is a complex in $K(B \otimes C^\circ)$ where the nth term is $\operatorname{Hom}^n_A(X, Y) = \Pi_p \operatorname{Hom}_A(X^p, Y^{n+p})$ and the differential $d^n_{\operatorname{Hom}_A(X,Y)}$ is $(f_p)_p \rightsquigarrow (f_{p+1} \cdot d^p_X + (-1)^{n+1} d^{p+n}_Y \cdot f_p)_p$. This induces a bi-$\partial$-functor,

$$\operatorname{Hom}_A(-, -)\colon K(A \otimes B^\circ)^{op} \times K(A \otimes C^\circ) \to K(B \otimes C^\circ).$$

Similarly, for $X \in K(B \otimes A^\circ)$ and $Y \in K(A \otimes C^\circ)$, we have $X \otimes_A Y$ in $K(B \otimes C^\circ)$, where the nth term is $X \otimes^n_A Y = \coprod_{p+q=n} X^p \otimes_A Y^q$ and the differential $d^n_{X \otimes_A Y}$ is $\coprod_{p+q=n}(d^p_X \otimes 1_{Y^q} + (-1)^n 1_{X^p} \otimes d^q_Y)$. This induces a bi-$\partial$-functor,

$$-\otimes_A -\colon\ K(B \otimes A^\circ) \times K(A \otimes C^\circ) \to K(B \otimes C^\circ).$$

The right-derived functor of Hom is denoted by R Hom, and the left-derived functor of $\otimes$ is denoted by ${}^L\otimes$. The Ext and Tor groups are defined by

$$\operatorname{Ext}^i_A(X, Y) := h^i \operatorname{R Hom}_A(X, Y) \quad \text{and} \quad \operatorname{Tor}^A_i(X, Y) := h^{-i}(X {}^L\otimes_A Y).$$

DEFINITION 1.1 [Ye1, 3.3; YZ2, 1.1]. Let A be a left noetherian algebra and let T be a right noetherian algebra. An object $R \in D^b(A \otimes T^\circ)$ is called a *dualizing complex over* (A, T) if it satisfies the following three conditions:

(1) R has finite injective dimension over A and T°.

(2) R has noetherian cohomologies over A and T°.

(3) The canonical morphisms $T \longrightarrow \mathrm{R}\operatorname{Hom}_A(R, R)$ and $A \longrightarrow \mathrm{R}\operatorname{Hom}_{T^\circ}(R, R)$ are isomorphisms in $D(T^e)$ and in $D(A^e)$, respectively.

In case $A = T$, we say that R is a *dualizing complex over* A.

Associated to R, we define the two contravariant functors

$$F := \mathrm{R}\operatorname{Hom}_A(-, R)\colon D(A) \longrightarrow D(T^\circ)$$

and

$$F^\circ := \mathrm{R}\operatorname{Hom}_{T^\circ}(-, R)\colon D(T^\circ) \longrightarrow D(A),$$

which induce a duality between the triangulated categories $D_f(A)$ and $D_f(T^\circ)$, restricted to a duality between $D^b_f(A)$ and $D^b_f(T^\circ)$ [YZ2, 1.3]. The dualizing complex was called the *cotilting bimodule complex* by Miyachi in [Mi], where he developed Morita duality theory for derived categories.

Throughout this paper when we say R is a dualizing complex over (A, T) we always assume that A is left noetherian and T is right noetherian. One might impose extra conditions on the dualizing complexes. Let R be a dualizing complex over (A, T) and let M be an A-module. The *grade* (or *j-number*) of M with respect to R is

$$j(M) = \inf\{q \mid \operatorname{Ext}^q_A(M, R) \neq 0\}.$$

The grade of a T°-module is defined similarly.

DEFINITION 1.2 [Ye2, 1.2; YZ2, 2.1]. A dualizing complex R over (A, T) is called *Auslander* if

(1) For every finite A-module M, every q, and every T°-submodule $N \subset \operatorname{Ext}^q_A(M, R)$ one has $j(N) \geq q$.

(2) The same holds after exchanging A and T°.

The *canonical dimension* with respect to a dualizing complex R is defined to be

$$\operatorname{Cdim} M = -j(M)$$

for all finite A-(or T°)-modules M. We say R is Cdim-*symmetric* if for every (A, T)-bimodule M finite on both sides, one has $\operatorname{Cdim}_A M = \operatorname{Cdim} M_T$. Let δ be a dimension function defined on left A-modules. We say R is *(left)* δ*-Macaulay* if $\operatorname{Cdim} M = \delta\, M$ for all noetherian A-modules M.

By [YZ2, 2.10], if R is Auslander then Cdim is an exact, finitely partitive dimension function.

DEFINITION 1.3. A dualizing complex R over (A, T) is *bifinite* if the following conditions hold:

(1) For every A-bimodule M finite on both sides, $\mathrm{Ext}^q_A(M, R)$ is finite on both sides.

(2) The same holds after A and T° are exchanged.

DEFINITION 1.4 [Ye1, 4.5]. Let $(A, \mathfrak{m})$ and $(T, \mathfrak{n})$ be semilocal. A dualizing complex R over (A, T) is called *pre-balanced* if

(1) $\mathrm{Ext}^i_A(A/\mathfrak{m}, R) = \mathrm{Ext}^i_{T^\circ}(T/\mathfrak{n}, R) = 0$ for all $i \neq 0$, and

(2) $\mathrm{Ext}^0_A(A/\mathfrak{m}, R)$ and $\mathrm{Ext}^0_{T^\circ}(T/\mathfrak{n}, R)$ are artinian on both sides.

The following two definitions are related to the existence of dualizing complexes.

DEFINITION 1.5. Let A be a left noetherian semilocal ring. We say A satisfies the *left χ condition* if $\mathrm{Ext}^i_A(A_0, M)$ is an A_0-module of finite length for every i and every finite A-module M. The right χ condition is defined similarly. If, moreover, A is noetherian, we say A satisfies the χ condition when A satisfies the left and the right χ conditions.

Stafford and Zhang showed that noetherian semilocal PI algebras satisfy the χ condition [SZ, 3.5].

DEFINITION 1.6. Let B be an artinian ring. We say B is *weakly symmetric* if

(1) for any bimodules ${}_CM_B$ and ${}_LN_B$ of finite length on both sides $\mathrm{Hom}_B(M, N)$ is of finite length on both sides, and

(2) the above condition holds after B and B°, are exchanged.

It was proved in [WZ, 7.3 and 7.4] that artinian PI algebras and stratiform simple artinian algebras (e.g., the Weyl skew fields and the q-skew fields) are weakly symmetric.

2. LOCAL COHOMOLOGY

Let $(A, \mathfrak{m})$ be a left noetherian semilocal algebra. We say $\mathfrak{m}$ has the *left AR property* (AR stands for Artin–Rees) if for every left ideal $I \subset A$, there is a positive integer n such that $I \cap \mathfrak{m}^n \subset \mathfrak{m}I$ [CH, p.140]. By [MR, 4.2.2] $\mathfrak{m}$ has the left AR property if and only if the injective hulls of all simple A-modules are $\mathfrak{m}$-torsion, and if and only if the injective hulls of $\mathfrak{m}$-torsion modules are $\mathfrak{m}$-torsion. If A is FBN or if A is noetherian and complete, then $\mathfrak{m}$ has the left and right AR property (see [CH, 11.3] and [Ja, 1.1]). If A is noetherian and PI, then the injective hulls of simple modules are even artinian [Va, Theorem A].

DEFINITION 2.1. Let $(A, \mathfrak{m})$ be a left noetherian semilocal ring.

(1) For $M \in A$-Mod, the $\mathfrak{m}$-*torsion functor* $\Gamma_{\mathfrak{m}}$ is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n x = 0, \text{for } n \gg 0\}.$$

(2) The derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$ is defined on the derived category $D^+(A)$. We define the *ith local cohomology* of $X \in D^+(A)$ to be

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \mathrm{R}^i\Gamma_{\mathfrak{m}}(X).$$

(3) The *local cohomological dimension* of an A-module M is defined to be

$$lcd(M) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}.$$

(4) The *cohomological dimension* of $\Gamma_{\mathfrak{m}}$ is defined to be

$$cd(\Gamma_{\mathfrak{m}}) = \sup\{lcd(M) \mid \text{for all } A\text{-modules M}\},$$

which is also called the *left (local) cohomological dimension of* A and is denoted by $cd(A)$.

Obviously, $\Gamma_{\mathfrak{m}}(M) = \varinjlim \mathrm{Hom}_A(A/\mathfrak{m}^n, M)$, which implies that

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \varinjlim \mathrm{Ext}^i_A(A/\mathfrak{m}^n, X)$$

for all $X \in D^+(A)$. Since $\mathrm{H}^i_{\mathfrak{m}}$ commutes with direct limits, we have

$$cd(A) = \sup\{lcd(M) \mid \text{for all finite } A\text{-modules } M\}.$$

If $cd(A)$ is finite, then $cd(A) = lcd({}_A A)$.

Remark 2.2. For any algebra A with an ideal $I \subset A$, one may define the local cohomology $\mathrm{R}\Gamma_I$ in the same way as above. In 2.7, we use local cohomologies of A^e-modules defined for the non-noetherian ring A^e and for the ideal $I = \mathfrak{m}^e$ which is not the Jacobson radical of A^e.

Not every noetherian local ring satisfies the χ condition [WZ, 9.4 and 9.6]. If A satisfies the left χ condition, then $\mathrm{Ext}^i_A(A/\mathfrak{m}^n, X)$ is of finite length for all $X \in D^b_f(A)$, all i, and all n [WZ, 2.3]. Moreover, if the injective hulls of all simple A-modules are artinian, then $\mathrm{H}^i_{\mathfrak{m}}(X)$ is artinian for every i and every $X \in D^+_f(A)$ [WZ, 6.3]. The next statement is a converse of [WZ, 6.3].

LEMMA 2.3. *Let A be a left noetherian semilocal ring. Suppose that injective hulls of all simple A-modules are artinian. If $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for every i and every finite A-module M, then A satisfies the left χ condition.*

Proof. Let $M \to I$ be a minimal injective resolution of M. Then $\mathrm{Ext}^i_A(A_0, M) \cong \mathrm{soc}(I^i)$, where $\mathrm{soc}(-)$ denotes the socle of $-$. Hence it suffices to show that $\mathrm{soc}(I^i)$ is artinian. The assertion holds for $i = 0$ because $\mathrm{soc}(I^0) = \mathrm{soc}(M)$. We now assume that $\mathrm{soc}(I^n)$ is artinian and prove that $\mathrm{soc}(I^{n+1})$ is artinian. Since the injective hulls of simple A-modules are artinian, $\Gamma_{\mathfrak{m}}(I^n)$ is artinian and so is its image in $\Gamma_{\mathfrak{m}}(I^{n+1})$. This together with the hypothesis $\mathrm{H}^{n+1}_{\mathfrak{m}}(M)$ being artinian implies that $K := \Gamma_{\mathfrak{m}}(I^{n+1}) \cap \ker(I^{n+1} \to I^{n+2})$ is artinian. By the minimality of I, $\mathrm{soc}(I^{n+1})$ is a submodule of K, and hence is artinian. The assertion follows by induction. ∎

In the rest of this section, we assume that $(A, \mathfrak{m})$ is a left noetherian semilocal algebra and $(T, \mathfrak{n})$ is a right noetherian semilocal algebra where $\mathfrak{n}$ is the Jacobson radical of T. Let $\Gamma_{\mathfrak{n}^\circ}$ be the $\mathfrak{n}$-torsion functor on T°-modules (namely, right T-modules) and let $\mathrm{R}\Gamma_{\mathfrak{n}^\circ}$ be the right-derived functor of $\Gamma_{\mathfrak{n}^\circ}$. The graded versions of the following lemmas can be found in [VdB].

LEMMA 2.4. (1) *Suppose $\mathfrak{m}$ has the left AR property. If E is A-injective, then $\Gamma_{\mathfrak{m}}(E)$ is A-injective.*

(2) *If E is $A \otimes T^\circ$-injective, then $\Gamma_{\mathfrak{m}}(E)$ is T°-injective.*

Proof. (1) This follows from the fact that the injective hulls of $\mathfrak{m}$-torsion modules are $\mathfrak{m}$-torsion.

(2) Let E be $A \otimes T^\circ$-injective. By the adjoint isomorphism

$$\mathrm{Hom}_{T^\circ}(-, \mathrm{Hom}_A(M, E)) \cong \mathrm{Hom}_{A\otimes T^\circ}(M \otimes -, E), \tag{2.4.1}$$

$\mathrm{Hom}_A(M, E)$ is T°-injective for all A-modules M. Since T is right noetherian, direct limits of T°-injectives are T°-injective. The assertion follows from $\Gamma_{\mathfrak{m}}(E) = \varinjlim_n \mathrm{Hom}_A(A/\mathfrak{m}^n, E)$. ∎

Remark 2.5. (1) Spaltenstein [Sp] and Böckstedt and Neeman [BN] defined two triangulated subcategories of $K(A)$, $K^s(\mathrm{Inj}\, A)$ and $K^s(\mathrm{Proj}\, A)$, which consist of *special* complexes of injective (respectively, projective) A-modules. For any (unbounded) complex X of A-modules, there exist a *special* injective and a *special* projective resolution of X; namely, there exist quasi-isomorphisms $X \to I$ and $P \to X$ for some $I \in K^s(\mathrm{Inj}\, A)$ and some $P \in K^s(\mathrm{Proj}\, A)$. The derived functor $\mathrm{R}\,\mathrm{Hom}(-, -)$ can be calculated either by replacing the first variable with a special projective resolution or by replacing the second variable with a special injective resolution. Similarly the local cohomolgy is defined for unbounded complexes by special injective resolutions. Another way of defining local cohomology of unbounded complexes is to use the homotopy colimit of $\mathrm{R}\,\mathrm{Hom}_A(A/\mathfrak{m}^n, -)$.

(2) If $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension then, by [Ha, I.5.3γ], the derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$ is defined for any complexes of A-modules, and it can be computed by using $\Gamma_{\mathfrak{m}}$-acyclic resolutions. We will use this fact several times.

LEMMA 2.6. *Suppose that* $\mathfrak{m}$ *has the left AR property. If* $X \in D^+(A)$ *and all the cohomologies of* X *are* $\mathfrak{m}$*-torsion, then* $\mathrm{R}\Gamma_{\mathfrak{m}}(X) \cong X$ *functorially in* $D^+(A)$.

If $\Gamma_{\mathfrak{m}}$ *has finite cohomological dimension, then the same statement is true for all* $X \in D(A)$.

Proof. Let $X \in D^+_{\mathrm{tor}}(A)$, where tor is the full subcategory of A-Mod consisting of $\mathfrak{m}$-torsion modules (the definition of $D^*_{\mathrm{tor}}(A)$ is given in [Ha, 1.4]). By [Ha, I.4.6], since tor is thick and injective hulls of $\mathfrak{m}$-torsion modules are $\mathfrak{m}$-torsion, X is quasi-isomorphic to a left-bounded complex I consisting of injective modules which are $\mathfrak{m}$-torsion. Hence

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X) \cong \Gamma_{\mathfrak{m}}(I) = I \cong X.$$

Now suppose $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension and $X \in D_{\mathrm{tor}}(A)$. As in the last paragraph all $\mathfrak{m}$-torsion modules are $\Gamma_{\mathfrak{m}}$-acyclic. Let I be a special injective resolution of X [BN, 2.4]. Let $I^{\bullet\bullet}$ be a *Cartan–Eilenberg injective resolution of* I. We refer to [We, Sect. 5.7] for the definition of the Cartan–Eilenberg injective resolution and the related material. Truncating $I^{\bullet\bullet}$ at the second index equal to $cd(\Gamma_{\mathfrak{m}}) + 1$, we still denote the new double complex by $I^{\bullet\bullet}$. Hence there is still an augmentation $I \to I^{\bullet 0}$ which induces a morphism from $\Gamma_{\mathfrak{m}}(I)$ to the total complex $\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))$. Note that each component of $I^{\bullet\bullet}$ is $\Gamma_{\mathfrak{m}}$-acyclic. Let us consider two spectral sequences associated to the double complex $\Gamma_{\mathfrak{m}}(I^{\bullet\bullet})$. The first spectral sequence (see, e.g., [We, p.150]) is

$$^{I}E_2^{pq} := h^p(\mathrm{H}^q_{\mathfrak{m}}(I)) \Rightarrow h^{p+q}(\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))). \tag{2.6.1}$$

Since every term in I is injective, $h^p(\mathrm{H}^q_{\mathfrak{m}}(I)) = 0$ for $q > 0$ and $h^p(\mathrm{H}^0_{\mathfrak{m}}(I)) = h^p(\Gamma_{\mathfrak{m}}(I)) = \mathrm{H}^p_{\mathfrak{m}}(I)$. Thus $\Gamma_{\mathfrak{m}}(I)$ is quasi-isomorphic to $\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))$. The second spectral sequence is

$$^{II}E_2^{pq} := \mathrm{H}^p_{\mathfrak{m}}(h^q(I)) \Rightarrow h^{p+q}(\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))). \tag{2.6.2}$$

Since the cohomologies of X and I are $\mathfrak{m}$-torsion, $\mathrm{H}^p_{\mathfrak{m}}(h^q(I)) = 0$ for $p > 0$ and $\mathrm{H}^0_{\mathfrak{m}}(h^q(I)) = h^q(I)$. Therefore I is quasi-isomorphic to $\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))$. It follows that $\mathrm{R}\Gamma_{\mathfrak{m}}(I)$ and I have the same cohomologies. Hence,

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(I) \cong I \cong X$$

in $D(A)$. ∎

LEMMA 2.7. *Let* $\mathfrak{m}^e = A \otimes \mathfrak{n}^\circ + \mathfrak{m} \otimes T^\circ$. *Then*

$$\mathrm{R}\Gamma_{\mathfrak{m}^e}(X) = \mathrm{R}\Gamma_{\mathfrak{n}^\circ} \cdot \mathrm{R}\Gamma_{\mathfrak{m}}(X) = \mathrm{R}\Gamma_{\mathfrak{m}} \cdot \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X)$$

for all $X \in D^+(A \otimes T^\circ)$. *If both* $\Gamma_{\mathfrak{m}}$ *and* $\Gamma_{\mathfrak{n}^\circ}$ *have finite cohomological dimension, then the above is true for* $X \in D(A \otimes T^\circ)$.

Proof. Since $\mathfrak{m}^e = A \otimes \mathfrak{n}^\circ + \mathfrak{m} \otimes T^\circ$ is a finitely generated left ideal of $A \otimes T^\circ$, $\{(\mathfrak{m}^e)^n \mid n \geq 1\}$ defines a bounded Gabriel topology on $A \otimes T^\circ$, and $\{\mathfrak{m}^n \otimes T^\circ + A \otimes \mathfrak{n}^n \mid n \geq 1\}$ is another basis [St, p.150]. So $\Gamma_{\mathfrak{m}^e}$ is a left exact torsion functor defined over $(A \otimes T^\circ)$-modules. For any $(A \otimes T^\circ)$-module N, we have

$$\Gamma_{\mathfrak{m}^e}(N) = \Gamma_{\mathfrak{n}^\circ}(\Gamma_{\mathfrak{m}}(N)) = (\Gamma_{\mathfrak{n}^\circ} \cdot \Gamma_{\mathfrak{m}})(N) = (\Gamma_{\mathfrak{m}} \cdot \Gamma_{\mathfrak{n}^\circ})(N).$$

By 2.4, $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ preserve the injectivity, so we have the equalities of derived functors,

$$\mathrm{R}\Gamma_{\mathfrak{m}^e}(X) = \mathrm{R}\Gamma_{\mathfrak{n}^\circ} \cdot \mathrm{R}\Gamma_{\mathfrak{m}}(X) = \mathrm{R}\Gamma_{\mathfrak{m}} \cdot \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X),$$

for all $X \in D^+(A \otimes T^\circ)$. This shows the first assertion. If $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension, $\mathrm{R}\Gamma_{\mathfrak{m}}(X)$ and $\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X)$ (even for unbounded complexes X) can be computed by quasi-isomorphic complexes consisting of $\Gamma_{\mathfrak{m}}$- and $\Gamma_{\mathfrak{n}^\circ}$-acyclic modules, respectively [Ha, I.5.3γ]. Therefore the second assertion holds. ∎

LEMMA 2.8. *Assume that* A *satisfies the left* χ *condition. Let* $X \in D^+(A \otimes T^\circ)$. *If* X *has all its cohomologies finite on both sides, then all* $\mathrm{H}^i_{\mathfrak{m}}(X)$ *are torsion over both sides.*

If $\Gamma_{\mathfrak{m}}$ *has finite cohomological dimension, then the same statement is true for* $X \in D(A \otimes T^\circ)$.

Proof. If ${}_AM_T$ is a bimodule finite (i.e., noetherian) on both sides, then $\mathrm{Ext}^i_A(A/\mathfrak{m}^n, M)$ is of finite length as a left A-module [WZ, 2.3]. Since $\mathrm{Ext}^i_A(A/\mathfrak{m}^n, M)$ is also noetherian on the right, Lenagan's lemma [GW, 7.10] implies that it is of finite length as a right T-module. Hence it is $\mathfrak{n}^\circ$-torsion. It follows that $\mathrm{H}^i_{\mathfrak{m}}(M)$ is torsion on both sides for all i.

If $X \in D^+(A \otimes T^\circ)$, we take an injective resolution $I \in D^+(A \otimes T^\circ)$ of X and let $I^{\bullet\bullet}$ be a Cartan–Eilenberg injective resolution I. Then we will consider the double complex $\Gamma_{\mathfrak{m}}(I^{\bullet\bullet})$ (without the truncation). If X is unbounded and $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension, as in 2.6, let I be a special injective resolution of X in $K(A \otimes T^\circ)$ and let $I^{\bullet\bullet}$ be a Cartan–Eilenberg injective resolution I. We truncate $I^{\bullet\bullet}$ at the second index equal to $cd(\Gamma_{\mathfrak{m}}) + 1$, and we denote the resulting double complex by $I^{\bullet\bullet}$ also. In both cases we consider two spectral sequences associated to the double complex $\Gamma_{\mathfrak{m}}(I^{\bullet\bullet})$. From the first spectral sequence (2.6.1) one obtains that

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X) \cong \Gamma_{\mathfrak{m}}(I) \cong \mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet})).$$

From the second spectral sequence (2.6.2) one obtains that $\mathrm{Tot}(\Gamma_{\mathfrak{m}}(I^{\bullet\bullet}))$ has cohomologies torsion on both sides (using the fact proved in the first paragraph). Therefore $\mathrm{R}\Gamma_{\mathfrak{m}}(X)$ has cohomologies torsion on both sides. ∎

The following result is the ungraded version of [VdB, 4.7].

THEOREM 2.9. *Let A and T° be left noetherian semilocal algebras as always. Suppose that*

(1) *A satisfies the left χ condition and $\mathfrak{m}$ has the left AR property, and*

(2) *the same holds for T°.*

If $X \in D^+(A \otimes T^\circ)$ has cohomologies finite on both sides, then

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X) = \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X).$$

Suppose further that $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension. Then the same statement is true for all $X \in D(A \otimes T^\circ)$.

Proof. This follows from 2.6, 2.7, and 2.8.

3. LOCAL DUALITY

In this section, we prove a version of the local duality theorem [VdB, 5.1]. We start with Morita duality. Let A and T be semilocal algebras with a bimodule ${}_AE_T$ which induces a Morita duality between A and T. We refer to [AF, Xu1] for the definition and basic properties of Morita duality and to [WZ, Sect. 7] for some related results.

By [Va, Theorem B] and [WZ, 8.5], if A is a noetherian complete semilocal PI algebra, then there exists a Morita duality between A and some noetherian PI algebra T. If A is a noetherian complete semilocal algebra with $\dim_k A_0 < \infty$, then A has a Morita self-duality (which means that $T = A$) [Ja, 2.7]. However, we do not know in general (even in the PI case) where there is a Morita self-duality over A.

Let us fix some notations. From now on E *denotes an* (A, T)*-bimodule which induces a Morita duality between* A *and* T. We do not assume that A is left noetherian and T is right noetherian until Theorem 3.6. By [Xu1, 2.7] E is an injective cogenerator as an A-module and as a T°-module. By the definition of Morita duality the left and the right module structures of E induce canonical isomorphisms $A \to \mathrm{Hom}_{T^\circ}(E, E)$ and $T^\circ \to \mathrm{Hom}_A(E, E)$. Define

$$(-)^* := \mathrm{Hom}_A(-, {}_AE_T)\colon A - \mathrm{Mod} \to \mathrm{Mod} - T$$

to be the duality functor from left A-modules to right T-modules, and

$$(-)^\vee := \mathrm{Hom}_{T^\circ}(-, {}_AE_T)\colon \mathrm{Mod} - T \to A - \mathrm{Mod}$$

to be the duality functor from right T-modules to left A-modules.

An A-module M is called *E-reflexive* if the canonical morphism $M \to (M^*)^\vee$ is an isomorphism. If A and T° are left noetherian, then E is artinian on both sides [Xu1, 2.6(12)]. In this case it is well known that both artinian and noetherian A-modules are E-reflexive. Denote by $D_{\mathrm{ref}}(A)$ the full subcategory of $D(A)$ consisting of objects with E-reflexive cohomologies. These definitions can be made for the T°-module similarly. Since E is injective (on both sides), dualities $(-)^*$ and $(-)^\vee$ can be extended to homotopy and derived categories. For example, for $X \in K(A)$ and $Y \in K(T^\circ)$, we have $X^* = \mathrm{Hom}_A(X, {}_AE_T)$ and $Y^\vee = \mathrm{Hom}_{T^\circ}(Y, {}_AE_T)$. Obviously, we have the following.

LEMMA 3.1. *Let ${}_AE_T$ be a bimodule that induces a Morita duality between A and T. Let $X \in K(A)$ or $\in D(A)$. Then $X[n]^* \cong X^*[-n]$ and $h^{-n}(X^*) \cong h^n(X)^*$ functorially.*

The following holds because $A^* = E_T$ and $E^* = T$.

LEMMA 3.2. *Let ${}_AE_T$ be a bimodule that induces a Morita duality between A and T. If M is a projective A-module, then M^* is an injective T°-module. If M is an artinian injective A-module, then M^* is a noetherian projective T°-module.*

PROPOSITION 3.3. *Let ${}_AE_T$ be a bimodule that induces a Morita duality between A and T.*

(1) *Functors $(-)^*\colon D_{\mathrm{ref}}(A) \to D_{\mathrm{ref}}(T^\circ)$ and $(-)^\vee\colon D_{\mathrm{ref}}(T^\circ) \to D_{\mathrm{ref}}(A)$ induce a duality between these two derived categories.*

(2) *For any $X, Y \in D_{\mathrm{ref}}(A)$ and any i, $\mathrm{Ext}^i_A(X, Y) \cong \mathrm{Ext}^i_{T^\circ}(Y^*, X^*)$.*

(3) *If either $X \in D^-_{\mathrm{ref}}(A)$ or $Y \in D^+_{\mathrm{ref}}(A)$, $\mathrm{R}\,\mathrm{Hom}_A(X, Y) \cong \mathrm{R}\,\mathrm{Hom}_{T^\circ}(Y^*, X^*)$.*

Proof. (1) For any $X \in K_{\mathrm{ref}}(A)$, there is a functorial morphism

$$X \longrightarrow (X^*)^\vee = \mathrm{R}\,\mathrm{Hom}_{T^\circ}(\mathrm{R}\,\mathrm{Hom}_A(X, {}_AE_T), {}_AE_T).$$

Since $h^i(X)$ is E-reflexive, we have canonical isomorphisms

$$h^i(X) \cong ((h^i(X))^*)^\vee \cong (h^{-i}(X^*))^\vee \cong h^i((X^*)^\vee).$$

So $X \longrightarrow (X^*)^\vee$ is a quasi-isomorphism in $K_{\mathrm{ref}}(A)$ and the assertion follows.

(2) By (1), $\mathrm{Hom}_{D(A)}(X, Y) \cong \mathrm{Hom}_{D(T^\circ)}(Y^*, X^*)$ for any $X, Y \in D_{\mathrm{ref}}(A)$. It follows that for any i,

$$\begin{aligned}\mathrm{Ext}^i_A(X, Y) &\cong \mathrm{Hom}_{D(A)}(X, Y[i]) \cong \mathrm{Hom}_{D(T^\circ)}(Y[i]^*, X^*)\\ &\cong \mathrm{Hom}_{D(T^\circ)}(Y^*[-i], X^*) \cong \mathrm{Hom}_{D(T^\circ)}(Y^*, X^*[i])\\ &\cong \mathrm{Ext}^i_{T^\circ}(Y^*, X^*).\end{aligned}$$

(3) By the symmetry between A and T°, we only prove the case when $X \in D^-_{\mathrm{ref}}(A)$. So we may assume that X is a right-bounded complex of projective A-modules. Hence $\mathrm{R}\,\mathrm{Hom}_A(X, Y) = \mathrm{Hom}_A(X, Y)$. By the definitions of $\mathrm{Hom}_A(X, Y)$ and $(-)^*$, there is a canonical morphism $\mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_{T^\circ}(Y^*, X^*)$.

If X is a right-bounded complex of projective A-modules, then X^* is a left-bounded complex of injective T°-modules (see 3.2). Hence $\mathrm{R}\,\mathrm{Hom}_{T^\circ}(Y^*, X^*) = \mathrm{Hom}_{T^\circ}(Y^*, X^*)$. Therefore there is a canonical morphism

$$\mathrm{R}\mathrm{Hom}_A(X,Y)=\mathrm{Hom}_A(X,Y)\to \mathrm{Hom}_{T^\circ}(Y^*,X^*)=\mathrm{R}\mathrm{Hom}_{T^\circ}(Y^*,X^*).$$

By (2) this is an isomorphism after taking cohomologies. Hence the assertion follows. ∎

LEMMA 3.4. *Let A be a left noetherian semilocal algebra with finite left local cohomological dimension. Let C be another algebra. For any $X \in D(A^e)$ and $Y \in D(A \otimes C^\circ)$, the following holds functorially in $D(A \otimes C^\circ)$:*

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X^L \otimes_A Y) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(X)^L \otimes_A Y.$$

Proof. Since $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension, $\mathrm{R}\Gamma_{\mathfrak{m}}$ (defined over $D(A)$, or $D(A^e)$, or $D(A \otimes C^\circ)$) can be computed via complexes consisting of $\Gamma_{\mathfrak{m}}$-acyclic modules (see the proof of [Ha, I.5.3γ]). Let $X \to I$ be a special injective resolution of X in $K(A^e)$ and let $P \to Y$ be a special projective resolution of Y in $K(A \otimes C^\circ)$ [BN, 2.4]. Then, $X^L \otimes_A Y \cong I \otimes_A P$. Every module in the complex $I \otimes_A P$ is a direct sum of A-injective modules. Since A is left noetherian, the complex is A-injective and $\Gamma_{\mathfrak{m}}$-acyclic over $A \otimes C^\circ$. Hence

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X^L \otimes_A Y) \cong \Gamma_{\mathfrak{m}}(I \otimes_A P).$$

Since $\Gamma_{\mathfrak{m}}$ commutes with direct sums, and when P^q is projective, $\Gamma_{\mathfrak{m}}(I^p \otimes_A P^q) \cong \Gamma_{\mathfrak{m}}(I^p) \otimes_A P^q$, we have

$$\Gamma_{\mathfrak{m}}(I \otimes_A P) \cong \Gamma_{\mathfrak{m}}(I) \otimes_A P$$

in $K(A \otimes C^\circ)$. Therefore

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X^L \otimes_A Y) \cong \Gamma_{\mathfrak{m}}(I \otimes_A P) \cong \Gamma_{\mathfrak{m}}(I) \otimes_A P \cong \mathrm{R}\Gamma_{\mathfrak{m}}(X)^L \otimes_A Y$$

in $D(A \otimes C^\circ)$. ∎

LEMMA 3.5. *Let A, B, C, and T be algebras and let ${}_A\overline{E}_T$ be a bimodule injective on the left. For $X \in D(B \otimes C^\circ)$, $Y \in D(A \otimes B^\circ)$, the following holds functorially in $D(C \otimes T^\circ)$:*

$$\mathrm{R}\,\mathrm{Hom}_B(X, \mathrm{R}\,\mathrm{Hom}_A(Y, \overline{E})) \cong \mathrm{R}\,\mathrm{Hom}_A(Y^L \otimes_B X, \overline{E}).$$

Proof. This is a generalized version of (2.4.1). It is routine to check this from the definition with special attention to the sign of the differential maps. ∎

THEOREM 3.6 (Local Duality Theorem). *Let $(A, \mathfrak{m})$ be a left noetherian semilocal algebra, let $(T, \mathfrak{n})$ be a right noetherian semilocal algebra, and let ${}_AE_T$ be a bimodule which induces a Morita duality between them. Suppose that $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension.*

(1) $R\Gamma_{\mathfrak{m}}(A)^*$ *has finite injective dimension in* $D(A)$.

(2) *For an object* $X \in D(A \otimes C^\circ)$ *where* C *is another algebra, the following holds functorially in* $D(C \otimes T^\circ)$:

$$R\Gamma_{\mathfrak{m}}(X)^* \cong \mathrm{RHom}_A(X, R\Gamma_{\mathfrak{m}}(A)^*).$$

(3) *For an object* $X \in D(A \otimes T^\circ)$, *the following holds functorially in* $D(T^e)$:

$$R\Gamma_{\mathfrak{n}^\circ}(X)^* \cong \mathrm{R\,Hom}_T(R\Gamma_{\mathfrak{n}^\circ}(T), X^*).$$

Proof. (2) By 3.4 and 3.5, we have

$$\begin{aligned} R\Gamma_{\mathfrak{m}}(X)^* &= \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{m}}(X), {}_AE_T) = \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{m}}(A \,{}^L\!\otimes_A X), {}_AE_T) \\ &\cong \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{m}}(A)\,{}^L\!\otimes_A X, {}_AE_T) \\ &\cong \mathrm{R\,Hom}_A(X, \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{m}}(A), {}_AE_T)) \\ &\cong \mathrm{R\,Hom}_A(X, R\Gamma_{\mathfrak{m}}(A)^*). \end{aligned}$$

(1) Taking $C = k$ and X to be concentrated in degree zero, (2) together with the hypothesis that $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension implies that $R\Gamma_{\mathfrak{m}}(A)^*$ has finite injective dimension in $D(A)$.

(3) By 3.4 and 3.5, we have

$$\begin{aligned} R\Gamma_{\mathfrak{n}^\circ}(X)^* &= \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{n}^\circ}(X), {}_AE_T) = \mathrm{R\,Hom}_A(R\Gamma_{\mathfrak{n}^\circ}(X \,{}^L\!\otimes_T T), {}_AE_T) \\ &\cong \mathrm{R\,Hom}_A(X \,{}^L\!\otimes_T R\Gamma_{\mathfrak{n}^\circ}(T), {}_AE_T) \\ &\cong \mathrm{R\,Hom}_T(R\Gamma_{\mathfrak{n}^\circ}(T), \mathrm{R\,Hom}_A(X, {}_AE_T)) \\ &\cong \mathrm{R\,Hom}_T(R\Gamma_{\mathfrak{n}^\circ}(T), X^*). \end{aligned}$$

∎

Similar statements hold for T.

THEOREM 3.7. *Assume the hypotheses of* 3.6.

(1) $R\Gamma_{\mathfrak{n}^\circ}(T)^\vee$ *has finite injective dimension in* $D(T^\circ)$.

(2) *For an object* $X \in D(C \otimes T^\circ)$, *where* C *is another algebra, the following holds functorially in* $D(A \otimes C^\circ)$:

$$\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X)^\vee \cong \mathrm{R}\operatorname{Hom}_{T^\circ}(X, \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee).$$

(3) *For an object* $X \in D(A \otimes T^\circ)$, *the following holds functorially in* $D(A^e)$:

$$\mathrm{R}\Gamma_{\mathfrak{m}}(X)^\vee \cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), X^\vee).$$

COROLLARY 3.8. *Assume the hypotheses of* 3.6.

(1) $T \cong \mathrm{R}\operatorname{Hom}_A({}_AE_T, \mathrm{R}\Gamma_{\mathfrak{m}}(A)^*)$ *as objects in* $D(T^e)$.

(2) $A \cong \mathrm{R}\operatorname{Hom}_{T^\circ}({}_AE_T, \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee)$ *as objects in* $D(A^e)$.

(3) $A \cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), A)$ *as objects in* $D(A^e)$.

(4) $T \cong \mathrm{R}\operatorname{Hom}_T(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T), T)$ *as objects in* $D(T^e)$.

Proof. This follows from 3.6 and 3.7 by taking $X = {}_AE_T$. ∎

PROPOSITION 3.9. *Let* $(A, \mathfrak{m})$ *and* $(T, \mathfrak{n})$ *be noetherian complete semilocal algebras, and suppose that* ${}_AE_T$ *induces a Morita duality between them. Suppose that* $\Gamma_{\mathfrak{m}}$ *and* $\Gamma_{\mathfrak{n}^\circ}$ *have finite cohomological dimension and that* A *and* T *satisfy the* χ *condition.*

(1) $A \cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), \mathrm{R}\Gamma_{\mathfrak{m}}(A))$ *as objects in* $D(A^e)$;

(2) $T \cong \mathrm{R}\operatorname{Hom}_T(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T), \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T))$ *as objects in* $D(T^e)$.

Proof. Since both A and T are noetherian complete semilocal, $\mathfrak{m}$ and $\mathfrak{n}^\circ$ have the AR property. By 2.9, $\mathrm{R}\Gamma_{\mathfrak{m}}(A) = \mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A)$ in $D(A^e)$. Let I be an A^e-injective resolution of A. Then $\mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A) = \Gamma_{\mathfrak{m}^\circ}(I)$, which consists of an $\mathfrak{m}^\circ$-torsion injective A°-module. Hence

$$\mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A), A) = \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A), \mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A)).$$

Now as objects in $D(A^e)$ we have

$$\begin{aligned}\mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), \mathrm{R}\Gamma_{\mathfrak{m}}(A)) &\cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A), \mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A))\\ &\cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}^\circ}(A), A)\\ &\cong \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), A) \cong A,\end{aligned}$$

where the last isomorphism is 3.8(3). This proves (1). The proof of (2) is similar. ∎

4. EXISTENCE THEOREM

In this section, we prove an existence theorem for dualizing complexes analogous to [VdB, 6.3]. We continue to assume as in 3.9 that A and T are noetherian complete semilocal algebras and that ${}_AE_T$ induces a Morita duality between them. Consider the contravariant functors

$$F := (-)^* \cdot \mathrm{R}\Gamma_{\mathfrak{m}}(-) = \mathrm{R}\operatorname{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(-), {}_AE_T)\colon D(A) \longrightarrow D(T^\circ)$$

and

$$G := (-)^\vee \cdot \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(-) = \mathrm{R}\operatorname{Hom}_{T^\circ}(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(-), {}_AE_T)\colon D(T^\circ) \longrightarrow D(A),$$

which are compositions of the top two maps and the bottom two maps, respectively, of the following diagram:

$$\begin{tikzcd}
& D(A) \arrow[dr, "(-)^*"] & \\
D(A) \arrow[ur, "\mathrm{R}\Gamma_{\mathfrak{m}}(-)"] & & D(T^\circ) \arrow[dl, "\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(-)"] \\
& D(T^\circ). \arrow[ul, "(-)^\vee"] &
\end{tikzcd}$$

By 3.6 and 3.7, we have

$$F \cong \mathrm{R}\operatorname{Hom}_A(-, \mathrm{R}\Gamma_{\mathfrak{m}}(A)^*) \qquad \text{and} \qquad G \cong \mathrm{R}\operatorname{Hom}_{T^\circ}(-, \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee).$$

Since $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension, the image of $F|_{D^b(A)}$ is contained in $D^b(T^\circ)$ and the image of $G|_{D^b(T^\circ)}$ is contained in $D^b(A)$.

For $X \in D_f(A)$, there exists a natural morphism $\mathrm{R}\Gamma_{\mathfrak{m}}(X) \to X$ in $D(A)$. This induces a morphism $X^* \to \mathrm{R}\Gamma_{\mathfrak{m}}(X)^*$. Since the cohomologies of X^* are artinian T°-modules, by 2.6, $\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X^*) \cong X^*$. Thus by 3.3, $(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X^*))^\vee \cong (X^*)^\vee \cong X$ in $D(A)$. Hence we obtain a natural morphism,

$$\eta_X\colon (\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(X)^*))^\vee \longrightarrow X.$$

Since this morphism is functorial, we obtain a natural transformation $\eta\colon GF \to id_{D_f(A)}$.

Similarly, we have a natural transformation $\xi\colon FG \to id_{D_f(T^\circ)}$.

THEOREM 4.1. *Let $(A, \mathfrak{m})$ and $(T, \mathfrak{n})$ be noetherian complete semilocal algebras, and suppose that ${}_AE_T$ induces a Morita duality between them. Suppose that*

(i) *$\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension,*
(ii) *A and T satisfy the (left and right) χ condition, and*
(iii) *$A_0 = A/\mathfrak{m}$ is weakly symmetric.*

Then

(1) *F and G induce a duality between $D_f(A)$ and $D_f(T^\circ)$, restricting to a duality between $D_f^b(A)$ and $D_f^b(T^\circ)$;*

(2) $\mathrm{R}\Gamma_{\mathfrak{m}}(A)^* \cong \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee$ *as objects in* $D(A \otimes T^\circ)$;

(3) $\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*$ *is a dualizing complex over* (A, T).

Proof. (1) By 3.6(1), ${}_A\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*$ and $\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee_T$ have finite injective dimension. So both GF and FG are way-out functors in both directions [Ha, I.7.6]. By a dual version of [Ha, I.7.1], it suffices to prove that $GF(A) \cong A$ as objects in $D(A)$ and $FG(T) \cong T$ as objects in $D(T^\circ)$.

Since A_0 is weakly symmetric, by 4.2 below, $\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*$ is in $D_f^b(A)$ and $D_f^b(T^\circ)$. By 2.9, $\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*)$. By the definition of η_A, we have the following commutative diagram:

$$\begin{tikzcd}
GF(A) \arrow[r, "\simeq"] \arrow[d] & (\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*))^\vee \arrow[r, "\simeq"] \arrow[d] & (\mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*))^\vee \arrow[d] \\
A \arrow[r, "\simeq"] & (\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(A^*))^\vee \arrow[r, "\simeq"] & (\mathrm{R}\Gamma_{\mathfrak{m}}(A^*))^\vee
\end{tikzcd}$$

Thus it suffices to prove that $(\mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*))^\vee \longrightarrow (\mathrm{R}\Gamma_{\mathfrak{m}}(A^*))^\vee$ is an isomorphism in $D(A)$. This is true by reading the following commutative diagram in $D(A^e)$,

$$\begin{tikzcd}
(\mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*))^\vee \arrow[r] \arrow[d, "\simeq"'] & (\mathrm{R}\Gamma_{\mathfrak{m}}(A^*))^\vee \arrow[d, "\simeq"'] \\
\mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), (\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*)^\vee) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), (A^*)^\vee) \arrow[d, "\simeq"'] \\
\mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), \mathrm{R}\Gamma_{\mathfrak{m}}(A)) \arrow[r] & \mathrm{R}\operatorname{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), A)
\end{tikzcd}$$

where the isomorphisms in the columns are obtained from 3.7(3) and 3.3(1), and the isomorphism in the bottom row is given in the proof of 3.9. Therefore $GF \cong id_{D_f(A)}$.

Similarly, one can show that $FG(T) \cong T$ and hence $FG \cong id_{D_f(T^\circ)}$. This proves (1).

To prove (2) and (3) we need 4.3, which will be proved later.

(2) By a dual version of 4.3(3) for the functor G, as objects in $D(A \otimes T^\circ)$, we have

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathfrak{m}}(A)^* &\cong \mathrm{R}\operatorname{Hom}_{T^\circ}(T, \mathrm{R}\Gamma_{\mathfrak{m}}(A)^*) \cong \mathrm{R}\operatorname{Hom}_A(G(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*), G(T)) \\
&\cong \mathrm{R}\operatorname{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)^*)^\vee, \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee) \\
&\cong \mathrm{R}\operatorname{Hom}_A(A, \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee) \\
&\cong \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(T)^\vee.
\end{aligned}$$

(3) We check the three conditions in the definition of dualizing complex. By 3.6 and part (2), $R\Gamma_{\mathfrak{m}}(A)^*(\cong R\Gamma_{\mathfrak{n}^\circ}(T)^\vee)$ has finite injective dimension on both sides. By 2.3, $R\Gamma_{\mathfrak{m}}(A)^* \in D_f(T^\circ)$ and $R\Gamma_{\mathfrak{n}^\circ}(T)^\vee \in D_f(A)$. By 4.3, we have

$$R\operatorname{Hom}_{T^\circ}(R\Gamma_{\mathfrak{m}}(A)^*, R\Gamma_{\mathfrak{m}}(A)^*) \cong R\operatorname{Hom}_A(A, A) \cong A \qquad \text{in } D(A^e)$$

and

$$R\operatorname{Hom}_A(R\Gamma_{\mathfrak{n}^\circ}(T)^\vee, R\Gamma_{\mathfrak{n}^\circ}(T)^\vee) \cong R\operatorname{Hom}_{T^\circ}(T, T) \cong T \qquad \text{in } D(T^e).$$

Thus $R\Gamma_{\mathfrak{m}}(A)^*(\cong R\Gamma_{\mathfrak{n}^\circ}(T)^\vee)$ is a dualizing complex over (A, T). ∎

Lemma 4.2 below is the only place where we use the hypothesis that A_0 is weakly symmetric. If 4.2 holds without this hypothesis, then so does 4.1.

LEMMA 4.2. *Assume the hypotheses of* 4.1. *Then* $R\Gamma_{\mathfrak{m}}(A)^*$ *is in* $D_f^b(A)$ *and* $D_f^b(T^\circ)$.

Proof. By [WZ, 6.3] and 2.9, the local cohomology $R^q\Gamma_{\mathfrak{m}}(A)$ is artinian on both sides. Since $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension, $R\Gamma_{\mathfrak{m}}(A) \in D^b(A^e)$. Now it suffices to show that if $X \in D^b(A^e)$ has cohomologies artinian on both sides, then X^* has cohomologies finite on both sides. By the induction on the length of X, it suffices to show the following: if M is an (A, A)-bimodule artinian on both sides then M^* is finite on both sides. That was proved in [WZ, 7.6], which uses the hypothesis that A_0 is weakly symmetric. ∎

The following lemma is needed in the proof of parts (2) and (3) of 4.1. Here are some temporary notations. Let $D^b_{l.f}(A \otimes C^\circ)$ be the full subcategory of $D^b(A \otimes C^\circ)$ consisting of complexes whose cohomologies are finite A-modules. Define $D^b_{r.f}(C \otimes T^\circ)$ similarly with the same condition on the right T-module structures.

LEMMA 4.3. *Assume the hypotheses of* 4.1. *Let C be another algebra.*

(1) *The functors* $F\colon D^b_{l.f}(A \otimes C^\circ) \longrightarrow D^b_{r.f}(C \otimes T^\circ)$ *and* $G\colon D^b_{r.f}(C \otimes T^\circ) \longrightarrow D^b_{l.f}(A \otimes C^\circ)$ *induce a duality.*

(2) *For all* $X, Y \in D_f^b(A)$ *and all integers* i,

$$\operatorname{Ext}^i_A(X, Y) \cong \operatorname{Ext}^i_{T^\circ}(F(Y), F(X)).$$

If $X \in D^b_{l.f}(A^e)$ *and* $Y \in D^b_{l.f}(A \otimes C^\circ)$, *then the above isomorphism is an* $A \otimes C^\circ$*-module isomorphism.*

(3) *For all* $X, Y \in D_f^b(A)$,

$$R\operatorname{Hom}_A(X, Y) \cong R\operatorname{Hom}_{T^\circ}(F(Y), F(X)).$$

If $X \in D_{l.f}(A^e)$ *and* $Y \in D_{l.f}(A \otimes C^\circ)$, *then the above isomorphism is in* $D(A \otimes C^\circ)$.

Proof. (1) It is clear that if $X \in D^b_{l.f}(A \otimes C^\circ)$ then $F(X) \in D^b_{r.f}(C \otimes T^\circ)$. Similarly, if $W \in D^b_{r.f}(C \otimes T^\circ)$ then $G(W) \in D^b_{l.f}(A \otimes C^\circ)$. Hence F and G in part (1) are well defined.

We still use the notations F, G, η, and ξ for these derived categories. If $X \in D(A \otimes C^\circ)$, then we have a natural morphism $\eta_X\colon GF(X) \longrightarrow X$ in $D(A \otimes C^\circ)$. By 4.1(1), $GF(X) \to X$ is an isomorphism restricted to $D(A)$. Hence it is an isomorphism in $D^b_{l.f}(A \otimes C^\circ)$. This shows that $G \cdot F \cong id_{D^b_{l.f}(A \otimes C^\circ)}$.

Similarly, $FG \cong id_{D^b_{r.f}(C \otimes T^\circ)}$.

(2) It follows from 4.1(1) that

$$\operatorname{Hom}_{D(A)}(X, Y) \cong \operatorname{Hom}_{D(T^\circ)}(F(Y), F(X)).$$

If $X \in D^b_{l.f}(A^e)$ and $Y \in D^b_{l.f}(A \otimes C^\circ)$, by part (1) the above morphism commutes with $(A \otimes C^\circ)$-module actions. Hence as abelian groups or as $A \otimes C^\circ$-modules,

$$\begin{aligned}\operatorname{Ext}^i_A(X, Y) &\cong \operatorname{Hom}_{D(A)}(X, Y[i]) \cong \operatorname{Hom}_{D(T^\circ)}(F(Y[i]), F(X)) \\ &\cong \operatorname{Hom}_{D(T^\circ)}(F(Y)[-i], F(X)) \cong \operatorname{Hom}_{D(T^\circ)}(F(Y), F(X)[i]) \\ &= \operatorname{Ext}^i_{T^\circ}(F(Y), F(X)).\end{aligned}$$

(3) Replace X and Y by bounded below injective resolutions in $K(A^e)$ and in $K(A \otimes C^\circ)$, still denoted by X and Y, respectively. Then $\mathrm{R}\operatorname{Hom}_A(X, Y) \cong \operatorname{Hom}_A(X, Y)$. By the definitions of Hom and F, we have a canonical morphism,

$$\operatorname{Hom}_A(X, Y) \to \operatorname{Hom}_{T^\circ}((\Gamma_{\mathfrak{m}}(Y))^*, (\Gamma_{\mathfrak{m}}(X))^*) = \operatorname{Hom}_{T^\circ}(F(Y), F(X)),$$

in $K(A \otimes C^\circ)$.

Let $F(X) \to Z$ be an injective resolution of $F(X)$ in $K(A \otimes T^\circ)$. Then we have the canonical morphisms in $K(A \otimes C^\circ)$

$$\operatorname{Hom}_A(X, Y) \to \operatorname{Hom}_{T^\circ}(F(Y), F(X)) \to \operatorname{Hom}_{T^\circ}(F(Y), Z),$$

which induces canonical morphisms in $D(A \otimes C^\circ)$,

$$\begin{aligned}\mathrm{R}\operatorname{Hom}_A(X, Y) \cong \operatorname{Hom}_A(X, Y) &\to \operatorname{Hom}_{T^\circ}(F(Y), F(X)) \\ &\to \operatorname{Hom}_{T^\circ}(F(Y), Z) \\ &\cong \mathrm{R}\operatorname{Hom}_{T^\circ}(F(Y), F(X)).\end{aligned}$$

Taking the ith cohomology, we get that the composition of the morphisms on the cohomology is the isomorphism given in part (2). This proves (3). ∎

Now we are ready to prove 0.1.

Proof of Theorem 0.1. We need to show that the dualizing complex given in 4.1 is bifinite, Cdim-symmetric, and pre-balanced.

Let R be the dualizing complex given in 4.1(3). Let M be an A-bimodule noetherian on both sides. By 3.6(2),

$$\mathrm{Ext}_A^q(M, R) = \mathrm{H}_{\mathfrak{m}}^{-q}(M)^*. \tag{4.3.1}$$

By 2.8, $\mathrm{H}_{\mathfrak{m}}^q(M)$ is torsion on both sides. Since ${}_AE_T$ induces a Morita duality and since T is (right) noetherian, ${}_AE$ is artinian. Since E contains every simple A-module as a submodule, the injective hull of a simple A-module is artinian. By [WZ, 6.3], $\mathrm{H}_{\mathfrak{m}}^q(M)$ is left artinian. By the symmetry proved in 2.9, $\mathrm{H}_{\mathfrak{m}}^q(M)$ is right artinian. By [WZ, 7.6], $\mathrm{H}_{\mathfrak{m}}^q(M)^*$ is noetherian on both sides and therefore R is bifinite.

The Cdim-symmetry follows from 2.9 and (4.3.1), and the pre-balanced property follows from the definition of $\mathrm{H}_{\mathfrak{m}}^q$ and (4.3.1). ∎

COROLLARY 4.4. *Let A be a noetherian complete semilocal algebra such that $A/\mathfrak{m}$ is finite-dimensional over k. Suppose A satisfies the χ-condition and has finite left and right cohomological dimension. Then there is a bifinite,* Cdim-*symmetric, pre-balanced dualizing complex over A.*

Proof. By [Ja, 2.7] there is a Morita self-duality over A. The hypotheses of 0.1 hold, and the assertion follows from 4.1. Note that A_0 is weakly symmetric because it is finite-dimensional over the base field k. ∎

5. HOMOLOGICAL IDENTITIES

In this section we prove Bass theorem and the no-holes theorem for noetherian complete local algebras. The ideas come from [Jo1, 4.6 and 4.8], where Jørgensen proved these two theorems in the connected graded case. As pointed out by Jørgensen, these ideas originate from the commutative proofs (see [Fo, Ro]). First we review some facts about projective and injective dimensions.

Each $X \in K^-(A)$ has a free (in particular, projective and flat) resolution. The resolution can be chosen to consist of modules vanishing above $\sup X$. If A is left noetherian, then each $X \in K_f^-(A)$ has a finite projective resolution, and if, further, A is semiperfect (e.g., complete), then each $X \in K_f^-(A)$ has a finite minimal projective resolution. The projective dimension of X is

$$\operatorname{pd} X = \min_Y\{-\min\{i \mid Y^i \neq 0\}\},$$

where Y ranges over all projective resolutions of X.

Each $X \in K^+(A)$ has a minimal injective resolution, and the resolution can be chosen to consist of modules vanishing below $\inf X$. The injective dimension of X is

$$\operatorname{id} X = \min_I \{\max\{i \mid I^i \neq 0\}\},$$

where I ranges over all injective resolutions of X. If I is a minimal injective resolution of X, then $\operatorname{id} X = \max\{i \mid I^i \neq 0\}$. It is clear that, for $X \in D^+(A)$,

$$\operatorname{id} X = \sup\{i \mid \operatorname{Ext}^i_A(M, X) \neq 0 \text{ for some } M \in A\text{-Mod}\}.$$

If A is left noetherian we only need to consider all finite A-modules M in the above formula.

Similarly for $X \in D^-(A)$,

$$\operatorname{pd} X = \sup\{i \mid \operatorname{Ext}^i_A(X, M) \neq 0 \text{ for some } M \in A\text{-Mod}\}.$$

If A is left noetherian and $\operatorname{pd} X < \infty$, then

$$\operatorname{pd} X = \max\{i \mid \operatorname{Ext}^i_A(X, A) \neq 0\}.$$

If A is semilocal and X has a minimal projective resolution Y, then $\operatorname{pd} X = -\inf\{i \mid Y^i \neq 0\}$.

LEMMA 5.1. *Let $(A, \mathfrak{m})$ be a left noetherian semilocal ring and $A_0 = A/\mathfrak{m}$.*

(1) *If $X \in D^-_f(A)$, then* $\operatorname{pd} X = \sup(\operatorname{R Hom}_A(X, A_0))$.

(2) *If $X \in D^-_f(A)$, then* $\sup X = -\inf(\operatorname{R Hom}_A(X, A_0))$.

(3) *If M is a nonzero finite A-module, then*

$$\operatorname{Ext}^i_A(M, A_0) \neq 0 \qquad \textit{if and only if} \qquad 0 \le i \le \operatorname{pd} M.$$

Proof. (1) It suffices to show that if $\operatorname{Ext}^i_A(X, M) \neq 0$ for some i and some finite A-module M, then $\operatorname{Ext}^i_A(X, A_0) \neq 0$. Without loss of generality we may assume X is a complex of projective finite A-modules and $i = 0$, i.e., $\operatorname{Ext}^0_A(X, M) \neq 0$. Let f: $X^0 \to M$ be a map representing a nonzero element in $\operatorname{Ext}^0_A(X, M)$. Then f induces a map f': $N := X^0/\operatorname{im}(X^{-1}) \to M$. Since $f \neq 0$ in $\operatorname{Ext}^0_A(X, M)$, f' cannot factor through the map ∂: $N \to X^1$. In particular, ∂ does not split.

We claim that the surjection p: $N \to N/\mathfrak{m}N$ induces a nonzero element in $\operatorname{Ext}^0_A(X, N/\mathfrak{m}N)$. Otherwise p factors through ∂: $N \to X^1$, say $p = \delta\partial$ for some δ: $X^1 \to N/\mathfrak{m}N$. Since X^1 is projective δ lifts to a map ϕ: $X^1 \to N$. Hence

$$p = \delta\partial = p\phi\partial.$$

The Nakayama lemma implies that $\phi\partial$ is surjective. Since N is finite, $\phi\partial$ is an isomorphism and thus ∂ splits. This yields a contradiction, and therefore $\mathrm{Ext}^0_A(X, N/\mathfrak{m}N) \neq 0$. Finally $N/\mathfrak{m}N$ is a direct summand of a finite direct sum of copies of A_0. Thus $\mathrm{Ext}^0_A(X, A_0) \neq 0$, and the assertion follows.

(2) By a shifting, we assume that $\sup(X) = 0$. Further assume that X is a complex of projective finite A-modules and $X^i = 0$ for all $i > 0$. Hence $\inf(\mathrm{R}\,\mathrm{Hom}_A(X, A_0)) \geq 0$. To see $\inf = 0$ we consider a nonzero map $f\colon X^0 \to h^0(X) \to A_0$. Then f is a nonzero element in $\mathrm{Ext}^0_A(X, A_0)$. The assertion follows.

(3) It is clear that $\mathrm{Hom}_A(M, A_0) \neq 0$. If $\operatorname{pd} M = 0$, then it is done. If $\operatorname{pd} M > 0$, we consider a short exact sequence

$$0 \to N \to P \to M \to 0$$

for some finite projective module P. The long exact sequence of $\mathrm{Ext}^i(-, A_0)$ shows that

$$\mathrm{Ext}^{i+1}(M, A_0) \cong \mathrm{Ext}^i(N, A_0) \qquad \text{for all} \quad i \geq 1,$$

and that

$$0 \to \mathrm{Hom}(M, A_0) \to \mathrm{Hom}(P, A_0) \to \mathrm{Hom}(N, A_0) \to \mathrm{Ext}^1(M, A_0) \to 0$$

is exact. By induction it remains to show that the map $\mathrm{Hom}(P, A_0) \to \mathrm{Hom}(N, A_0)$ is not surjective. If it is surjective, then the map $\mathrm{Hom}(P, N/\mathfrak{m}N) \to \mathrm{Hom}(N, N/\mathfrak{m}N)$ is also surjective because $N/\mathfrak{m}N$ is a direct summand of a finite direct sum of copies of A_0. Now the argument in the proof of (1) shows that the map $N \to P$ splits, which contradicts the hypothesis $\operatorname{pd} M > 0$. ∎

If A is semiperfect, or X has a minimal projective resolution, then the proof may be simplified by using a minimal projective resolution of X.

Let $(A, \mathfrak{m})$ be a left noetherian semilocal algebra and let $A_0 = A/\mathfrak{m}$. For $X \in D^+(A)$ and $i \in \mathbb{Z}$, the *ith Bass number* is defined to be

$$\mu^i(X) = \text{length of the } A_0\text{-module } \mathrm{Ext}^i_A(A_0, X).$$

In general $\mu^i(X)$ could be infinite. The *depth* of X is defined to be

$$\operatorname{depth} X = \inf \mathrm{R}\,\mathrm{Hom}_A(A_0, X) = \inf\{i \mid \mu^i(X) \neq 0\}.$$

By [WZ, 6.2], depth $X = \inf \mathrm{R}\Gamma_{\mathfrak{m}}(X)$. So we may also take this as a definition of depth. The A_0*-injective dimension* of X [WZ, 5.6] is

$$\mathrm{id}^0 X = \sup \mathrm{R}\,\mathrm{Hom}_A(A_0, X) = \sup\{i \mid \mu^i(X) \neq 0\}.$$

If A is noetherian, semilocal, and PI, then $\mathrm{id}\, X = \mathrm{id}^0\, X$ for all $X \in D^b_f(A)$ [WZ, 5.7].

In the rest of this section we assume that A and T satisfy the following hypotheses.

HYPOTHESES 5.2. (1) $(A, \mathfrak{m})$ *is a left noetherian semilocal algebra*,

(2) $(T, \mathfrak{n})$ *is a right noetherian semilocal algebra, and*

(3) R *is a pre-balanced dualizing complex over* (A, T).

Let

$$F(-) = \mathrm{R}\operatorname{Hom}_A(-, R)\colon D(A) \to D(T^\circ)$$

and

$$F^\circ(-) = \mathrm{R}\operatorname{Hom}_{T^\circ}(-, R)\colon D(T^\circ) \to D(A).$$

By [YZ2, 1.3(2)], (F, F°) is a duality between triangulated categories $D_f(A)$ and $D_f(T^\circ)$.

LEMMA 5.3. *Assume Hypotheses* 5.2. *The functors* $\operatorname{Ext}^0_A(-, R)$ *and* $\operatorname{Ext}^0_{T^\circ}(-, R)$ *induce a duality between the category of finite-length modules over* A *and the category of finite-length modules over* T°.

In particular, $\operatorname{Ext}^0_A(A_0, R)$ *is a direct sum of all simple* T°*-modules with positive multiplicities*.

Proof. It is clear from the definition of "pre-balanced" that these two functors are exact on modules of finite length. Note that these $\operatorname{Ext}^0(-, R)$'s are restrictions of F and F° on the finite-length modules over A and over T°, respectively. By duality one sees that both functors preserve the length of the modules. Therefore the assertions follow. ∎

It follows from this lemma that if A is local then so is T. We need a few more lemmas to prove Bass theorem. The graded versions of these lemmas can be found in [Jo1, Jo3].

LEMMA 5.4. *Assume Hypotheses* 5.2. *For any nonzero* $X \in D^+_f(A)$, *we have*

(1) $\operatorname{depth} X = -\sup F(X)$, *where* $F(X) \in D^-_f(T^\circ)$;

(2) $\operatorname{depth} X < \infty$; *and*

(3) $\operatorname{id}^0 X = \operatorname{pd}_{T^\circ} F(X)$.

Proof. (1) Since $X \in D^+_f(A)$, $F(X) \in D^-_f(T^\circ)$ [Mi, 2.6]. By duality,

$$\mathrm{R}\operatorname{Hom}_A(A_0, X) \cong \mathrm{R}\operatorname{Hom}_{T^\circ}(F(X), F(A_0)).$$

By 5.3, $F(A_0) \cong \operatorname{Ext}^0_A(A_0, R)$ is a direct sum of all simple T°-modules with positive multiplicities. This implies that $\inf \mathrm{R}\operatorname{Hom}_{T^\circ}(F(X), F(A_0)) = \inf \mathrm{R}\operatorname{Hom}_{T^\circ}(F(X), T_0)$. The assertion follows from the definition of depth and 5.1(2).

(2) Follows from the duality and (1).

(3) Similar to the proof of (1) (use 5.1(1) instead of 5.1(2)). ∎

In the situation of 0.1 and 4.1, we have

$$\mathrm{id}_A R = \mathrm{id}\, R_T = 0,$$

which follows from (4.3.1). If, furthermore, $Y \in D_f^+(T^\circ)$ has finite projective dimension, say $\mathrm{pd}\, Y$, then the A-module complex $\mathrm{R}\,\mathrm{Hom}_{T^\circ}(Y, R)$ has injective dimension at most $\mathrm{pd}\, Y$. Combining this with 5.4(1), we obtain that for $X \in D_f^+(A)$,

$$\mathrm{id}\, X = \mathrm{id}\, F^\circ F(X) \le \mathrm{pd}\, F(X) = \mathrm{id}^0 X \le \mathrm{id}\, X,$$

which implies that $\mathrm{id}\, X = \mathrm{id}^0\ X$. We suspect that this holds under Hypotheses 5.2. We will prove this equality in a special case, i.e., when A is local (see 5.6(2)).

LEMMA 5.5. *Let A be a left noetherian local algebra and let $X \in D_f^-(A)$, $Y \in D_f^+(A)$ be nonzero. If* $\mathrm{pd}\, X < \infty$ *and* $\sup Y < \infty$, *then*

$$\sup \mathrm{R}\,\mathrm{Hom}_A(X, Y) = \mathrm{pd}\, X + \sup Y.$$

Proof. This is a generalization of [Jo3, 3.3], and the proof is similar to that of [Jo3, 3.3].

The inequality $\le$ is obvious. To prove the inequality $\ge$, replace Y with the truncation

$$\sigma_{\le s}(Y) = \cdots \to Y^{s-2} \to Y^{s-1} \to \ker d^s \to 0 \to \cdots,$$

where $s = \sup Y$, and we may assume that $Y^m = 0$ when $m > s$. Then $h^s(Y)$ is a nonzero finite homomorphic image of Y^s. Let S be a simple homomorphic image of Y^s. Thus we have an exact sequence of complexes

$$0 \to Z \to Y \to S[-s] \to 0.$$

Consider the induced long exact sequence

$$\cdots \to \mathrm{Ext}_A^{d+s}(X, Y) \to \mathrm{Ext}_A^{d+s}(X, S[-s]) \to 0 \to \cdots,$$

where $d = \mathrm{pd}\ X$. By 5.1(1),

$$\mathrm{Ext}_A^{d+s}(X, S[-s]) = \mathrm{Ext}_A^{d}(X, S) \ne 0.$$

Therefore $\mathrm{Ext}_A^{d+s}(X, Y) \ne 0$, and the assertion follows. ∎

LEMMA 5.6. *Assume Hypotheses* 5.2. *Assume that A is local.*

(1) *If $X \in D_f^b(A)$, $Y \in D_f^+(A)$ are nonzero and* $\mathrm{id}^0\, Y < \infty$, *then*

$$\sup \mathrm{R}\,\mathrm{Hom}_A(X, Y) = \mathrm{pd}\, F(Y) + \sup F(X) = \mathrm{id}^0\, Y - \mathrm{depth}\, X.$$

(2) *For any $X \in D_f^+(A)$,* $\mathrm{id}\, X = \mathrm{id}^0\, X$.

Proof. (1) By duality,

$$\sup \mathrm{R}\,\mathrm{Hom}_A(X, Y) = \sup \mathrm{R}\,\mathrm{Hom}_{T^\circ}(F(Y), F(X)).$$

Hence the first identity follows from 5.4(3) and 5.5 and the second from 5.4.

(2) It is clear that $\mathrm{id}\, X \geq \mathrm{id}^0\, X$. To prove $\leq$, we may assume that $\mathrm{id}^0\, X$ is finite. For any finite A-module M, the identity in (1) shows that

$$\sup \mathrm{R}\,\mathrm{Hom}_A(M, X) = \mathrm{id}^0 X - \operatorname{depth} M \leq \mathrm{id}^0\, X.$$

Hence $\mathrm{id}\, X \leq \mathrm{id}^0\, X$. ∎

Next is a noncommutative version of Bass theorem, which can be viewed as a dual statement of the Auslander–Buchsbaum formula (see [Jo3, 3.2; WZ, 3.3]).

THEOREM 5.7. *Assume Hypotheses* 5.2, *and assume* A *is local. If* $X \in D_f^b(A)$ *is nonzero with* $\mathrm{id}\, X < \infty$, *then* $\mathrm{id}\, X = \operatorname{depth} A + \sup X$.

Proof. By 5.6,

$$\sup X = \sup \mathrm{R}\,\mathrm{Hom}_A(A, X) = \mathrm{id}\, X - \operatorname{depth} A,$$

which is equivalent to the identity we want. ∎

The Auslander–Buchsbaum formula can also be derived from 5.7.

THEOREM 5.8. *Assume Hypotheses* 5.2, *and assume* A *is local. If* $X \in D_f^b(A)$ *is nonzero with* $\mathrm{pd}\, X < \infty$, *then*

$$\mathrm{pd}\, X + \operatorname{depth} X = \operatorname{depth} A = \operatorname{depth} T.$$

Proof. Since $X \in D_f^b(A)$, $F(X) \in D_f^b(T^\circ)$. By 5.4 and 5.6,

$$\mathrm{id}\, F(X) = \mathrm{id}^0 F(X) = \mathrm{pd}\, F^\circ F(X) = \mathrm{pd}\, X < \infty.$$

By 5.4(1), depth $X = -\sup F(X)$; and by 5.7 (for complex $F(X)$ of T°-modules),

$$-\sup F(X) = \operatorname{depth} T - \mathrm{id}\, F(X).$$

Hence

$$\operatorname{depth} X = \operatorname{depth} T - \mathrm{pd}\, X.$$

Let $X = A$ in the above identity; we obtain that depth $T = \operatorname{depth} A$. The assertion follows. ∎

In [WZ, 6.10] we showed that the left little finitistic projective dimension of A is bounded by the depth of A. A dual statement holds in the setting of 5.7. We define the left little finitistic injective dimension of A to be

$$\operatorname{lfID} A = \sup\{\operatorname{id} M \mid M \text{ finite } A\text{-module with } \operatorname{id} M < \infty\}.$$

COROLLARY 5.9. *Assume the hypotheses of* 5.7. *Then* $\operatorname{lfID} A \leq \operatorname{depth} A < \infty$.

It follows from 5.6 and 5.7 that if there is a nonzero finite A-module M with finite injective dimension, then

$$\operatorname{depth} A = \operatorname{id} M \geq \operatorname{depth} N$$

for all finite A-modules N. A noncommutative version of Bass question is stated as follows: if a noetherian local algebra A has a nonzero finite A-module of finite injective dimension, then A is Cohen–Macaulay (in the sense that $\mathrm{H}^i_{\mathfrak{m}}(A)$ vanishes except for one i). We will not discuss here the relationship between Bass theorem and local homological conjectures (for noncommutative local rings) such as Bass's question, small Cohen–Macaulay module conjecture, etc.

To prove the no-holes theorem we need the following lemma.

LEMMA 5.10. *Let A be a left noetherian semilocal algebra. If $X \in D^-_f(A)$ is nonzero with* $\operatorname{Ext}^i_A(X, A_0) = 0$ *for some i, then there exist $Y, Z \in D^-_f(A)$ such that $X \cong Y \oplus Z$ in $D(A)$ and*

$$\sup Z < -i < -\operatorname{pd} Y.$$

Proof. We may assume X is a complex of finite projective A-modules and assume that $i = 0$; i.e., $\operatorname{Ext}^0_A(X, A_0) = 0$. Consider the complex at the position $i = 0$,

$$\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots.$$

Let $N = X^0/\operatorname{im}(X^{-1})$. Since $\operatorname{Ext}^0(X, A_0) = 0$ and $N/\mathfrak{m}N$ is a finite direct sum of simples, $\operatorname{Ext}^0(X, N/\mathfrak{m}N) = 0$. By the proof of 5.1(1), the map $N \to X^1$ splits. Hence $X^1 \cong N \oplus Y^1$ and $N = X^0/\operatorname{im}(X^{-1})$ is projective. This implies that $X^0 \cong N \oplus \operatorname{im}(X^{-1})$ and $\operatorname{im}(X^{-1})$ is projective, and that $X^{-1} = Z^{-1} \oplus \operatorname{im}(X^{-1})$ and Z^{-1} is projective. Therefore X is isomorphic to the complex

$$\cdots \to X^{-2} \to Z^{-1} \to 0 \to Y^1 \to X^2 \to \cdots.$$

We denote

$$Z = \cdots \to X^{-2} \to Z^{-1} \to 0 \to 0 \to 0 \to \cdots$$

and

$$Y = \cdots \to 0 \to 0 \to 0 \to Y^1 \to X^2 \to \cdots,$$

then $X = Y \oplus Z$ and $\operatorname{pd} Y < 0$ and $\sup Z < 0$. ∎

Part (1) of the following is a dual version of 5.10.

PROPOSITION 5.11. *Assume Hypotheses* 5.2.

(1) *If* $X \in D_f^+(A)$ *is nonzero with* $\mu^i(X) = 0$ *for some* i, *then there exist* $Y, Z \in D_f^+(A)$ *such that* $X \cong Y \oplus Z$ *in* $D(A)$ *and*

$$\mathrm{id}^0 Y < i < \operatorname{depth} Z.$$

(2) *If* $X \in D_f^+(A)$ *is nonzero and indecomposable* (*i.e.*, $X \neq Y \oplus Z$ *for nonzero* Y, Z *in* $D(A)$), *then*

$$\mu^i(X) \neq 0 \qquad \textit{if and only if} \quad \operatorname{depth} X \leq i \leq \mathrm{id}^0 X.$$

(3) *If* M *is a nonzero, indecomposable, and finite* A-*module*, *then*

$$\mu^i(M) \neq 0 \qquad \textit{if and only if} \quad \operatorname{depth} M \leq i \leq \mathrm{id}^0 M.$$

Proof. (2) and (3) are consequence of (1), so it suffices to show (1). Since $X \in D_f^+(A)$, $F(X) \in D_f^-(T^\circ)$. The condition $\mu^i(X) = 0$ is equivalent to the condition $\operatorname{Ext}^i(F(X), T_0) = 0$. Applying 5.10 to the complex $F(X)$ of T°-modules, we obtain $F(X) \cong V \oplus W$ and $\sup W < -i < -\operatorname{pd} V$. Then $X \cong F^\circ F(X) \cong F^\circ(V) \oplus F^\circ(W)$. By 5.4(1,3) one sees that depth $F^\circ(W) = -\sup W > i$ and $\mathrm{id}^0 F^\circ(V) = \operatorname{pd} V < i$. ∎

Next is a noncommutative version of the no-holes theorem, which can be viewed as a dual version of 5.1(3). Note that if A is local, then $\operatorname{id} M = \mathrm{id}^0 M$ by 5.6(2).

THEOREM 5.12. *Assume Hypotheses* 5.2. *Let* A *be local*. *For any finite* A-*module* M,

$$\mu^i(M) \neq 0 \qquad \textit{if and only if} \quad \operatorname{depth} M \leq i \leq \operatorname{id} M.$$

Proof. One implication is clear. For the other implication we assume that i is between $\operatorname{depth} M$ and $\operatorname{id} M$ and want to show that $\mu^i(M) \neq 0$. Suppose that $\mu^i(M) = 0$. By 5.11(1), there exist $Y, Z \in D_f^+(A)$ such that $M \cong Y \oplus Z$ in $D(A)$ and $\operatorname{id} Y < i < \operatorname{depth} Z$. Obviously, $Y \cong h^0(Y) =: M_1$ and $Z \cong h^0(Z) := M_2$. Hence $M \cong M_1 \oplus M_2$ and $\operatorname{id} M_1 < i < \operatorname{depth} M_2$. The hypothesis $\operatorname{depth} M \leq i \leq \operatorname{id} M$ implies that both M_1 and M_2 are nonzero. On the other hand, by 5.6,

$$\sup \mathrm{R}\operatorname{Hom}_A(M_2, M_1) = \operatorname{id} M_1 - \operatorname{depth} M_2 \leq -2.$$

Since $\operatorname{id} M_1 -$ $\operatorname{depth} M_2$ is a finite number, $\mathrm{R}\operatorname{Hom}_A(M_2, M_1)$ is nonzero. Hence

$$\sup \mathrm{R}\operatorname{Hom}_A(M_2, M_1) \geq 0,$$

which yields a contradiction. ∎

Theorem 0.3 follows immediately from 0.1, 5.7, and 5.12. As stated, Bass theorem and the no-holes theorem may apply to non-complete local rings. Recall that a noetherian local algebra A is *AS–Gorenstein* (here AS stands for Artin and Schelter) if

(1) A has finite left and right injective dimension d,

(2) $\operatorname{Ext}^i(A_0, A) \cong \begin{cases} 0 & \text{if } i \neq d \\ A_0 & \text{if } i = d. \end{cases}$

COROLLARY 5.13. *Let $(A, \mathfrak{m})$ be a noetherian local algebra which is AS–Gorenstein of finite injective dimension d. Let M be a nonzero finite A-module.*

(1) *If M has finite injective dimension, then* $\operatorname{id} M = d$.

(2) *If M has finite injective dimension, then* $\operatorname{pd} M + \operatorname{depth} M = d$.

(3) $\mu^i(M) \neq 0$ *if and only if* $\operatorname{depth} M \leq i \leq \operatorname{id} M$.

Proof. (1) Since A has finite injective dimension, A is a dualizing complex over A (so $T = A$). By the AS–Gorenstein condition, $A[d]$ is a pre-balanced dualizing complex. By 5.7, $\operatorname{id} M = \operatorname{depth} A$. The AS–Gorenstein condition implies that $\operatorname{depth} A = d$. (2) and (3) follow from 5.8 and 5.12. ∎

6. PI ALGEBRAS

In this section we list some corollaries for PI rings. First of all, noetherian semilocal PI algebras satisfy the hypotheses of 0.1.

LEMMA 6.1. *Let A be a noetherian semilocal PI algebra with Jacobson radical $\mathfrak{m}$.*

(1) *A satisfies the χ condition* [*SZ*, 3.5].

(2) *$\mathfrak{m}$ has the AR property* [*CH*, 11.3].

(3) *A (and $\Gamma_{\mathfrak{m}}$) has finite cohomological dimension* [*WZ*, 1.5].

(4) *A_0 is weakly symmetric* [*WZ*, 7.3].

(5) *If A is complete, then there is a complete noetherian semilocal PI algebra T such that A and T are in Morita duality* [*Va*; *WZ*, 1.4].

Here are some obvious consequences of results in Sections 2, 3, and 4.

(1) Let A and T be two noetherian semilocal PI algebras. If $X \in D(A \otimes T^\circ)$ has finite cohomologies over both A and T°, then $\mathrm{R}\Gamma_{\mathfrak{m}}(X) \cong \mathrm{R}\Gamma_{\mathfrak{n}^\circ}(X)$.

(2) Let A be a noetherian complete semilocal PI algebra. Then

$$A \cong \mathrm{R}\,\mathrm{Hom}_{A^\circ}(\mathrm{R}\Gamma_{\mathfrak{m}}(A), \mathrm{R}\Gamma_{\mathfrak{m}}(A)) \cong \mathrm{R}\,\mathrm{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(A), \mathrm{R}\Gamma_{\mathfrak{m}}(A))$$

as objects on $D(A^e)$.

(3) Let A and T be two noetherian complete semilocal PI algebras and let E be an (A, T)-bimodule which induces a Morita duality between A and T. Then there is a bifinite, Cdim-symmetric, pre-balanced dualizing complex over (A, T).

Proof of Corollary 0.2. It remains to show that the dualizing complex $R = \mathrm{R}\Gamma_{\mathfrak{m}}(A)^*$ given in 0.1 is Auslander and $\mathrm{Cdim}\, M = \mathrm{Kdim}\, M$ for all neotherian A-modules M. By the definition of Cdim, we have

$$\begin{aligned}\mathrm{Cdim}\, M &= -j(M) = -\inf\{i \mid \mathrm{Ext}^i_A(M,R) \neq 0\}\\ &= -\inf\{i \mid \mathrm{Ext}^i_{T^\circ}(T, \mathrm{R}\Gamma_{\mathfrak{m}}(M)^*) \neq 0\} = -\inf\{i \mid h^i(\mathrm{R}\Gamma_{\mathfrak{m}}(M)^*) \neq 0\}\\ &= \sup\{i \mid h^i(\mathrm{R}\Gamma_{\mathfrak{m}}(M)) \neq 0\} = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}\\ &= lcd(M).\end{aligned}$$

By [WZ, 1.5], $lcd(M) = \mathrm{Kdim}\, M$. Hence R is Kdim–Macaulay. Next we prove the Auslander property. Let i be any integer and let N be any T°-submodule of $\mathrm{Ext}^i_A(M, R)$. Then

$$\begin{aligned} j(N_T) &= -lcd(N_T) = -\mathrm{Kdim}\, N_T\\ &\geq -\mathrm{Kdim}\,\mathrm{Ext}^i_A(M, R)_T = -\mathrm{Kdim}\,(\mathrm{H}^{-i}_{\mathfrak{m}}(M))^*_T\\ &\geq i,\end{aligned}$$

where the last $\geq$ is [WZ, 8.3(1c)]. By symmetry the same assertion holds after we exchange A and T°. Therefore R is Auslander. ∎

The next corollary is Bass theorem and the no-holes theorem for noetherian local PI algebras. The Auslander–Buchsbaum formula in the PI case was already proved in [WZ, 4.7].

COROLLARY 6.2. *Let A be a noetherian local PI algebra.*

(1) *If $X \in D^b_f(A)$ is nonzero with* $\mathrm{id}\, X < \infty$, *then* $\mathrm{id}\, X = \mathrm{depth}\, A + \sup X$.

(2) *For any finite A-module M,*

$$\mu^i(M) \neq 0 \qquad \textit{if and only if} \quad \mathrm{depth}\, M \leq i \leq \mathrm{id}\, M.$$

Proof. First we assume A is complete. By 0.2, there is a pre-balanced dualizing complex over (A, T) for some noetherian local PI algebra. The assertion follows from 5.7 and 5.12.

If A is not complete, let $\widehat{A}$ be the completion of A with respect to the Jacobson radical $\mathfrak{m}$. Hence the assertion holds for $\widehat{A}$. We use this to show that the assertions hold for A. Let $X \in D^b_f(A)$ be a complex. The completion $\widehat{X} \in D^b_f(\widehat{A})$ is well defined [WZ, Sect. 4]. By [WZ, 5.3], $\operatorname{depth} X = \operatorname{depth} \widehat{X}$ and $\mu^i(X) = \mu^i(\widehat{X})$, and by [WZ, 5.7], $\operatorname{id} X = \operatorname{id} \widehat{X}$. Since $\widehat{A}$ is faithfully flat over A, $\sup X = \sup \widehat{X}$. Therefore the assertions hold for A. ∎

The Krull dimension of an A-module M, denoted by $\operatorname{Kdim} M$, is defined to be the deviation of the lattice of submodules of M. We refer to [MR, Chap. 6] for the details. Albu and Vámos define the dual Krull dimension of M to be the deviation of the dual lattice of the lattice of submodules of M. The (left) global Krull dimension of A is defined to be the supremum of the Krull dimensions of all (left) A-modules with Krull dimension, and the (left) global dual Krull dimension is defined in a similar way [AlV]. The next result is similar to a result of Albu and Vámos [AlV, 3.2] in the commutative case.

COROLLARY 6.3. *If A is a noetherian semilocal PI algebra, the* (*global*) *Krull dimension of A is equal to the global dual Krull dimension of A.*

Proof. It follows from [AlV, 2.1] that the global Krull dimension of A is equal to the Krull dimension of A and the global dual Krull dimension of A is equal to the dual Krull dimension of the injective hull E of $A/\mathfrak{m}$. By [WZ, 1.3], $\operatorname{Kdim} A = \operatorname{Kdim} \widehat{A}$, where $\widehat{A}$ is the completion of A. By [Va], the injective hull E of $A/\mathfrak{m}$ has a natural $\widehat{A}$-module structure such that the lattice of A-submodules of E and that of $\widehat{A}$-submodules coincide. Hence the (global) Krull dimension and the global dual Krull dimension are preserved under the completion. So we may assume A is complete.

By 0.2, there is a noetherian complete semilocal PI algebra T such that (a) there is a Morita duality between A and T, and (b) there is an Auslander, Kdim–Macaulay dualizing complex R over (A, T). The Kdim–Macaulay property implies that

$$\operatorname{Kdim}_A A = \inf R = \operatorname{Kdim} T_T.$$

It follows from [AlV, 2.1 and 3.1] that the global dual Krull dimension of A is equal to the Krull dimension of T_T. Hence the assertion follows. ∎

In the rest of this section we prove 0.4; i.e., every noetherian complete semilocal PI algebra is catenary.

Recall that a ring A is *catenary* if, for any prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of A, all the saturated chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length. Let δ be an exact dimension function on left A-modules (in our application $\delta = \mathrm{Kdim}$). A module M is called δ-*pure* if $\delta(N) = \delta(M)$ for every nonzero submodule N of M. We say δ is *weakly symmetric* if there is a dimension function δ' on A°-modules such that $\delta_A(J/I) = \delta'_{A^\circ}(J/I)$ for any pair of two-sided ideals $I \subset J \subset A$. In the application A is PI and $\delta = \mathrm{Kdim}$. Then δ is even *symmetric*, meaning that for every A-bimodule M noetherian on both sides, $\delta_A M = \delta M_A$.

As in [GL], Spec A is said to have *normal separation* provided that for any pair of prime ideals $\mathfrak{p} \subsetneq \mathfrak{q}$, the factor $\mathfrak{q}/\mathfrak{p}$ contains a nonzero normal element of $A/\mathfrak{p}$. We say that *Tauvel's height formula* holds in A with respect to δ provided for all primes $\mathfrak{p}$.

$$\text{height } \mathfrak{p} + \delta(A/\mathfrak{p}) = \delta(A).$$

In [YZ2], Yekutieli generalized Gabber's Maximality Principal [YZ2, 2.19], then using the ideas from [GL], proved the following two theorems in the case $A = T$ [YZ2, 2.22 and 2.23]. Similar results were proved by Goodearl and Lenagan in [GL, 1.4 and 1.6] for Auslander regular Cohen–Macaulay rings. The proofs of the following are the same as the proofs of [YZ2, 2.22 and 2.23].

THEOREM 6.4. *Let A be a noetherian algebra with an Auslander dualizing complex over (A, T) for some right noetherian algebra T. Suppose that* Cdim_A *is weakly symmetric. Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be prime ideals of A with height $\mathfrak{q}/\mathfrak{p} = 1$. If there exists an element $a \in \mathfrak{q} - \mathfrak{p}$ that is normal modulo $\mathfrak{p}$, then* $\mathrm{Cdim}\, A/\mathfrak{p} = \mathrm{Cdim}\, A/\mathfrak{q} + 1$.

THEOREM 6.5. *Let A be a noetherian algebra with an Auslander dualizing complex over (A, T) for some right noetherian algebra T. Suppose that* Cdim_A *is weakly symmetric. If* Spec *A is normally separated, then A is catenary. If, in addition, A is prime, then Tauvel's height formula holds with respect to* Cdim_A.

COROLLARY 6.6. *Let A be a noetherian PI complete semilocal algebra. Then A is catenary. If A is prime, then Tauvel's height formula holds with respect to* Kdim.

Proof. By 0.2, there exists an Auslander, Kdim–Macaulay dualizing complex R over (A, T) for some noetherian complete semilocal PI algebra T. The dimension function here is Cdim = Kdim, which is symmetric. For every PI algebra A, Spec A is clearly normally separated. Hence the assertion follows by 6.5. ∎

Remark 6.7. The condition "complete" in 6.6 is necessary. The example of Nagata in [Na, Appendix, Example 2, p. 203] shows that there are non-catenary, commutative noetherian local rings.

As in the commutative case Gorenstein local PI algebras are catenary. An algebra is called Gorenstein if it has finite left and right injective dimension. If a noetherian local PI algebra is Gorenstein, then it is also AS–Gorenstein and Auslander–Gorenstein [SZ]. The proof of the following is left to the interested reader.

COROLLARY 6.8. *Every noetherian Gorenstein local PI algebra is catenary.*

7. EXAMPLES

In this section we use Xue's examples [Xu1, 12.1 and 12.9] to show that

(1) there is a left artinian local algebra A such that there is no dualizing complex over (A, T) for any right noetherian algebra T;

(2) there is an artinian local algebra A which does not admit a dualizing complex, but there is a dualizing complex over (A, T) for some right artinian local algebra T.

LEMMA 7.1. *Let A be a left artinian algebra and let $X \in D^+(A)$.*

(1) $\inf X = \operatorname{depth} X = \inf\{i \mid \operatorname{Ext}^i_A(A_0, X) \neq 0\}$.

(2) $\operatorname{id} X = \operatorname{id}^0 X = \sup\{i \mid \operatorname{Ext}^i_A(A_0, X) \neq 0\}$.

Proof. We may assume X is a minimal complex of injective modules. Then

$$\inf X = \inf\{i \mid X^i \neq 0\} \qquad \text{and} \qquad \operatorname{id} X = \sup\{i \mid X^i \neq 0\}.$$

For every i with $X^i \neq 0$, there is a nonzero map $f\colon A_0 \to \operatorname{soc}(X^i)$. By the minimality of X, f gives rise to a nonzero element in $\operatorname{Ext}^i(A_0, X)$. The assertions follow. ∎

LEMMA 7.2. *Let A be a left artinian local algebra and let $X \in D^+(A)$.*

(1) *If M is a nonzero finite A-module, the* $\inf X = \inf\{i \mid \operatorname{Ext}^i_A(M, X) \neq 0\}$.

(2) *If M is a nonzero finite A-module and* $\operatorname{id} X < \infty$, *then* $\operatorname{id} X = \sup\{i \mid \operatorname{Ext}^i_A(M, X) \neq 0\}$.

(3) *If* $\operatorname{id} X < \infty$, $\operatorname{id} X = \sup X$.

(4) *If M is an A-module with finite injective dimension, then its injective dimension is zero.*

Proof. (1) and (2) follow by the induction on the length of M. (3) follows from (2) by letting $M = A$, and (4) is a special case of (3). ∎

If an (A, T)-bimodule E induces a Morita duality between A and T and if A is left artinian and T is right artinian, then E is a pre-balanced dualizing complex over (A, T). The next proposition is a partial converse in the artinian local case.

PROPOSITION 7.3. *Let A be a left artinian local algebra with a dualizing complex R over (A, T) for some right noetherian algebra T.*

(1) $\operatorname{id}_A R = \operatorname{id} R_T = \inf R$.

(2) *Let $E = h^m(R)$ where $m = \inf R$; then E is an injective module on both sides.*

(3) *T is right artinian local and E induces a Morita duality between A and T.*

Proof. (1) Let N be any nonzero finite T°-module. By [YZ2, 1.7] there is a convergent spectral sequence

$$E_2^{pq} := \operatorname{Ext}_A^p(\operatorname{Ext}_{T^\circ}^{-q}(N, R), R) \Rightarrow N. \tag{7.3.1}$$

Let $m = \inf R$ and $n = \operatorname{id}_A R = \sup R$ and let

$$i = \inf\{j \mid \operatorname{Ext}_{T^\circ}^j(N, R) \neq 0\} \qquad \text{and} \qquad s = \sup\{j \mid \operatorname{Ext}_{T^\circ}^j(N, R) \neq 0\}.$$

Then the (possible) nonzero terms of the E_2-page of the spectral sequence are

$$\begin{array}{ccccc}
E^{m,i}(N) & E^{m+1,i}(N) & \cdots & E^{n-1,i}(N) & E^{n,i}(N) \\
E^{m,i+1}(N) & E^{m+1,i+1}(N) & \cdots & E^{n-1,i+1}(N) & E^{n,i+1}(N) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
E^{m,s-1}(N) & E^{m+1,s-1}(N) & \cdots & E^{n-1,s-1}(N) & E^{n,s-1}(N) \\
E^{m,s}(N) & E^{m+1,s}(N) & \cdots & E^{n-1,s}(N) & E^{n,s}(N),
\end{array} \tag{7.3.2}$$

where $E^{p,q}(N) = \operatorname{Ext}_A^p(\operatorname{Ext}_{T^\circ}^q(N, R), R)$. Since $\operatorname{Ext}_{T^\circ}^i(N, R)$ and $\operatorname{Ext}_{T^\circ}^s(N, R)$ are nonzero, by 7.2(1,2), four corner terms in the above rectangle (7.3.2) are nonzero. Clearly $E^{m,s}(N)$ and $E^{n,i}(N)$ will survive in the E_∞-page. Since (7.3.1) is convergent, this means that $m = s$ and $n = i$. Therefore $m = s = n = i$. As a consequence, $\operatorname{id} R_T = \operatorname{id}_A R = \inf R$.

(2) Follows from (1).

(3) By definition and (2), ${}_A E_T$ induces a Morita duality between A and T. Since ${}_A E$ is left noetherian, by duality T is right artinian. By the lattice isomorphism between ideals of A and ideals of T, A is local if and only if T is. ∎

COROLLARY 7.4. *Let A be an artinian local algebra. Then every dualizing complex over A is given by a Morita self-duality over A.*

EXAMPLE 7.5. This example is taken from [Xu1, 12.1]. Let $k \subset K$ be a field extension with an algebra homomorphism $f: K \to K$ such that $\dim K_{f(K)} = \infty$. Let ${}_K M_K$ be the (K, K)-bimodule with underlying space K such that ${}_K M = {}_K K$ as left K-module and that, as right K-module, $m \cdot a = mf(a)$ for all $m \in M$ and $a \in K$. Let A be the trivial extension $K \oplus M$. Then A is a left artinian local algebra without Morita duality. By 7.3(3), for any right noetherian algebra T, there is no dualizing complex over (A, T).

EXAMPLE 7.6. This example is taken from [Xu1, 12.9]. As given in [Sc, pp.214–218], there is a division ring extension $D \subset K$ such that $\dim K_D = 2, \dim {}_D K = 3$ and D is isomorphic to K as rings (also see [DRS] and [Xu2]). Consider the trivial extension $A = K \oplus M$ where the bimodule structure of ${}_K M_K$ is defined as the following: ${}_K M = {}_K K$ as left K-module and $m \cdot a = m\sigma(a)$ for any $m \in M$ and $a \in K$ where $\sigma: K \longrightarrow D$ is the isomorphism. So, in A, we have the following multiplication: $(\alpha, m)(\beta, n) = (\alpha\beta, \alpha n + m\sigma(\beta))$. Hence A is an artinian local algebra. According to [Xu1, 12.9] A has a Morita duality, namely, there is an (A, T)-bimodule E which induces a Morita duality between A and T for some right artinian ring T. Therefore E is a dualizing complex over (A, T). On the other hand, by [Xu1, 12.9], A does not have Morita self-duality. By 7.4, there is no dualizing complex over A.

ACKNOWLEDGMENTS

This research was finished during the first author's visit to the Department of Mathematics at the University of Washington supported by a research fellowship from the China Scholarship Council, and he thanks these two institutions for their hospitality and support. The second author was supported in part by the National Science Foundation and a Sloan Research Fellowship. The authors thank the referee for her/his careful reading of the manuscript and valuable comments.

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