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Free Hopf algebras generated by coalgebras

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Introduction

If H is a commutative or cocommutative Hopf algebras over a field k, it is well known that the antipode of H is of order 2. If H is a finite dimensional Hopf algebra over k, then the antipode of H is a bijection. Is the antipode of a Hopf algebra always a bijection? In this paper we construct some Hopf algebras whose antipodes are not bijective. In order to do so, we introduce the concept of free Hopf algebras generated by coalgebras.

Let C be a coalgebra over a field k. The free Hopf algebra $(H(C), i)$ generated by C is characterized by the following universal property:

(1) $i: C \to H(C)$ is a coalgebra map

(2) $\mathrm{Hom}(i, H): \mathrm{Hopf}(H(C), H) \to \mathrm{Coalg}(C, H)$ is a bijection for any Hopf algebra H.

$H(C)$ is constructed in § 1. One of our main results is the following

THEOREM. *The antipode of $H(C)$ is bijective if and only if the $\bar{k}$-coalgebra $\bar{k} \otimes C$ is pointed, where $\bar{k}$ is the algebraic closure of k.*

Some important consequences are obtained as corollaries to this theorem.

It is interesting to consider the algebra structure of $H(C)$ and to give a k-basis for $H(C)$ explicitly. We present a partial answer to this problem in Chapter III. In Chapter IV we consider the corresponding problem in the category of commutative algebras. The definition of the free commutative Hopf algebra $H_c(C)$ generated by a coalgebra C is similar to that of $H(C)$. In the category of commutative algebras, there is an interesting relation between norms and antipodes. This relation leads to a simple construction of $H_c(C)$. A consequence of this construction is

THEOREM. *A commutative bialgebra H has an antipode if and only if the grouplike elements of H are invertible in H.*

This is a generalization of [5, Prop. 9.2.5] in the category of commutative algebras.

Throughout this paper, we shall adopt the terminology and utilize theorems in [5].

Chapter I. Free Hopf algebras; basic concepts

In this chapter we construct the free Hopf algebra $H(C)$ generated by a coalgebra C and characterize it in the category of algebras.

§1. The construction of $H(C)$

Let k be a field. If A is an algebra and C is a coalgebra over k, then their structure maps are denoted by

$$A\otimes A \xrightarrow{\mu} A, \qquad k \xrightarrow{\eta} A$$

$$C \xrightarrow{\Delta} C\otimes C, \qquad C \xrightarrow{\varepsilon} k.$$

The multiplication $f*g=\mu\circ(f\otimes g)\circ\Delta$ defines an algebra structure on $\mathrm{Hom}(C, A)$. The unit is $\eta\circ\varepsilon$. In particular $C^*=\mathrm{Hom}(C, k)$ is an algebra. The algebra $\mathrm{Hom}(C, A)$ is functorial in C and A. $G(C)$ denotes the set of all grouplike elements of C. If A is finite dimensional, then A^* is a coalgebra. The tensor algebra $T(C)$ has a natural bialgebra structure.

If G is a group, the group algebra $k[G]$ has a natural bialgebra structure.

Let H be a bialgebra over k. If the identity 1_H is invertible in the algebra $\mathrm{End}(H)$, H is said to be a Hopf algebra and $S=(1_H)^{-1}$ is said to be the antipode of H.

$\mathrm{Alg}(A, B)$, $\mathrm{Coalg}(A, B)$, $\mathrm{Bialg}(A, B)$ and $\mathrm{Hopf}(A, B)$ denote the set of algebra maps from A to B etc.

Now let us begin to construct $H(C)$. Let C be a coalgebra over k. Let $(V_i)_{i\geqq 0}$ be a sequence of coalgebras as follows

$$V_0=C, \qquad V_{i+1}=V_i^{op}.$$

Let $V=\sum_{i=0}^{\infty} V_i$ be the direct sum of coalgebras. Let $S\colon V\to V^{op}$ be the coalgebra map $(x_0, x_1, x_2, \cdots)\to(0, x_0, x_1, x_2, \cdots)$. S induces a bialgebra map $S\colon T(V)\to T(V)^{op}$. Let I be the 2-sided ideal of $T(V)$ generated by $\sum x_{(1)}S(x_{(2)})-\varepsilon(x)1$ and $\sum S(x_{(1)})x_{(2)}-\varepsilon(x)1$ for $x\in V$. Then it is easy to see that

$$\Delta(I)\subset I\otimes T(V)+T(V)\otimes I$$

$$\varepsilon(I)=0$$

$$S(I)\subset I.$$

$H(C)=T(V)/I$ is therefore a bialgebra and S induces a bialgebra map $S\colon H(C)\to H(C)^{op}$. Let $i\colon C\to H(C)$ be the composite $C=V_0\to V\to T(V)\to H(C)$. Then we have

LEMMA 1. *S is an antipode of $H(C)$ and*

$$\mathrm{Hom}(i, H): \mathrm{Hopf}(H(C), H) \longrightarrow \mathrm{Coalg}(C, H)$$

is a bijection for any Hopf algebra H.

PROOF. The intersection of the kernels of $1*S-\eta\circ\varepsilon$ and $S*1-\eta\circ\varepsilon$ is a subalgebra of $H(C)$ which contains V. Since V generates $H(C)$ as an algebra, we have $1*S=S*1=\eta\circ\varepsilon$. Let H be a Hopf algebra. We construct the inverse of $\mathrm{Hom}(i, H)$. Let $f: C\to H$ be a coalgebra map. Let $f_i: V_i\to H$ be a coalgebra map as follows:

$$f_0=f$$

$$f_{i+1}=f_i^{-1} \text{ in the algebra } \mathrm{Hom}(V_i, H).$$

$(f_i)_{i\geqq 0}$ determines a coalgebra map $V\to H$ and it induces a bialgebra map $T(V)\to H$. This map is zero on I by the construction of $(f_i)_{i\geqq 0}$. Therefore a bialgebra map $\bar{f}: H(C)\to H$ is induced. It is clear that the correspondence $f\mapsto\bar{f}$ is an inverse of $\mathrm{Hom}(i, H)$.

DEFINITION 2. $H(C)$ is said to be the free Hopf algebra generated by C.

§ 2. $H(C)$ as an algebra

Let C be a coalgebra and A be an algebra over k.

DEFINITION 3. $L(C, A)$ is the set of sequences $(f_i)_{i\geqq 0}$ of elements of $\mathrm{Hom}(C, A)$ which satisfy the following condition:

$$f_{i+1}=f_i^{-1} \text{ in } \mathrm{Hom}(C, A) \text{ if } i \text{ is even}$$

$$f_{i+1}=f_i^{-1} \text{ in } \mathrm{Hom}(C^{op}, A) \text{ if } i \text{ is odd}.$$

PROPOSITION 4. $\mathrm{Alg}(H(C), A)\to L(C, A)$

$$f\longmapsto (f\circ S^j\circ i)_{j\geqq 0}$$

is a bijection, where S is the antipode of $H(C)$.

PROOF. It is clear that $(f\circ S^j\circ i)_{j\geqq 0}$ belongs to $L(C, A)$. We construct the inverse map. Let $(f_i)_{i\geqq 0}$ be an element of $L(C, A)$. Let $V=\sum_{i=0}^{\infty} V_i$ be as in § 1. Then $f_i: V_i\to A$ determines a linear map $V\to A$ and it induces an algebra map $T(V)\to A$. This map is zero on I by the definition of $L(C, A)$. Hence an algebra map $\bar{f}: H(C)\to A$ is induced. It is clear that the correspondence $(f_i)_{i\geqq 0}\mapsto\bar{f}$ is the required inverse map.

COROLLARY 5. $\mathrm{Hom}(i, A): \mathrm{Alg}(H(C), A)\to\mathrm{Hom}(C, A)$ *is injective.*

PROOF. $(f_i)_{i\geqq 0}\mapsto f_0: L(C, A)\to\mathrm{Hom}(C, A)$ is injective.

COROLLARY 6. *The coalgebra structure on $H(C)$ is unique such that $H(C)$ is a bialgebra and that $i: C\to H(C)$ is a coalgebra map.*

COROLLARY 7. *The following statements are equivalent:*

(a) *The antipode of $H(C)$ is a bijection.*

(b) *For any algebra A and for any element $(f_i)_{i\geqq 0}$ of $L(C, A)$, f_0 is invertible in* $\mathrm{Hom}(C^{op}, A)$.

PROOF. $S: H(C)\to H(C)^{op}$ is a bijection if and only if

$$\mathrm{Alg}(S, A): \mathrm{Alg}(H(C), A) \longrightarrow \mathrm{Alg}(H(C)^{op}, A)$$

is a bijection for any algebra A. If we identify $\mathrm{Alg}(H(C), A)$ with $L(C, A)$, then we have

$$\mathrm{Alg}(S, A): L(C, A) \longrightarrow L(C, A^{op})$$
$$(f_0, f_1, f_2, \cdots) \longmapsto (f_1, f_2, \cdots).$$

This map is a bijection if and only if the condition (b) is satisfied.

COROLLARY 8. *Let K/k be a field extension. Let $H_K(K\otimes C)$ denote the free Hopf algebra over K generated by the K-coalgebra $K\otimes C$. There exists a natural isomorphism of Hopf algebras over K:*

$$K\otimes H(C) \approx H_K(K\otimes C).$$

PROOF. Let A be a K-algebra. Then we have

$$\mathrm{Hom}_K(K\otimes C, A) \approx \mathrm{Hom}_k(C, A) \text{ as } k\text{-algebras}.$$

It follows that $L_K(K\otimes C, A) \approx L(C, A)$. So we have

$$\mathrm{Alg}_K(K\otimes H(C), A) \approx \mathrm{Alg}_k(H(C), A) \approx L(C, A)$$
$$\approx L_K(K\otimes C, A) \approx \mathrm{Alg}_K(H_K(K\otimes C), A)$$

natural in C and A. Therefore we have

$$K\otimes H(C) \approx H_K(K\otimes C) \text{ as } K\text{-algebras}.$$

But then, from the commutativity of

$$\begin{array}{ccc} & K\otimes C & \\ \swarrow & & \searrow \\ K\otimes H(C) & \approx & H_K(K\otimes C) \end{array}$$

and Corollary 6, it follows that the isomorphism:

$$K\otimes H(C) \approx H_K(K\otimes C)$$

commutes also with Δ and ε.

COROLLARY 9. *$i: C\to H(C)$ is injective.*

PROOF. We may assume $C\neq 0$. Let g be an element of C such that $\varepsilon(g)=1$. Then we have $C=kg+C^+$, where $C^+=\mathrm{Ker}(\varepsilon)$. Let $A=k+C^+$ be the following algebra:

$$(a+x)(b+y)=ab+(ay+bx) \qquad \text{for } a, b \in k \text{ and } x, y \in C^+ .$$

Put $f_i : ag+x \mapsto a+(-1)^i x$, $C \to A$, where $a \in k$ and $x \in C^+$. Since we have

$$\Delta(g) \equiv g \otimes g \bmod C^+ \otimes C^+$$

$$\Delta(x) \equiv g \otimes x + x \otimes g \bmod C^+ \otimes C^+ \qquad \text{for } x \in C^+ ,$$

$(f_i)_{i \geqq 0}$ belongs to $L(C, A)$. Therefore f_0 is factorized as

$$C \longrightarrow H(C) \longrightarrow A .$$

Because f_0 is injective, so is i.

COROLLARY 10. *If C is a finite dimensional coalgebra, then $L(C, A)$ is the set of sequences $(x_i)_{i \geqq 0}$ of elements of $C^* \otimes A$ which satisfy the following condition:*

$$x_{i+1} = x_i^{-1} \text{ in } C^* \otimes A \qquad \text{if } i \text{ is even}$$

$$x_{i+1} = x_i^{-1} \text{ in } C^* \otimes A^{op} \quad \text{if } i \text{ is odd} .$$

PROOF. We have $\mathrm{Hom}(C, A) \approx C^* \otimes A$ as k-algebras.

Chapter II. The antipode of $H(C)$

In this chapter we prove the main result about the antipode of $H(C)$.

§ 3. Case $C = M_m(k)^*$

Let $M_m(k)$ be the $m \times m$ matrix algebra over k. $C = M_m(k)^*$ is a coalgebra. The purpose of this section is to prove

THEOREM 11. *If $m > 1$, then the antipode of $H(M_m(k)^*)$ is not bijective.*

LEMMA 12. *Let A be a k-algebra. Then $L(M_m(k)^*, A)$ can be identified with the set of sequences $(X_i)_{i \geqq 0}$ of elements of $M_m(A)$ which satisfy the following condition:*

$$X_{i+1} = X_i^{-1} \qquad \text{if } i \text{ is even}$$

$${}^tX_{i+1} = ({}^tX_i)^{-1} \qquad \text{if } i \text{ is odd} ,$$

where ${}^tX$ denotes the transpose of X.

PROOF. Because of $M_m(k) \otimes A \approx M_m(A)$, this is an immediate consequence of Corollary 10.

PROOF OF THEOREM 11. By Corollary 7, it suffices to find an algebra A and an element $(X_i)_{i \geqq 0}$ of $L(M_m(k)^*, A)$ such that ${}^tX_0$ is not invertible in $M_m(A)$. Let t be a transcendental element over k. Let $k(t)\{y, z\}$ denote the free $k(t)$-algebra generated by y and z. Let J be the 2-sided ideal of $k(t)\{y, z\}$ generated by $yz - t^2 zy - t$. Put $B = k(t)\{y, z\}/J$. We claim B is non zero. Indeed let

$$V=\sum_{i=-\infty}^{\infty} k(t)e_i$$

be a free $k(t)$-module with basis $\{e_i\}$. Let y and z be elements of $\mathrm{End}_{k(t)}(V)$ as follows

$$\begin{aligned}
&y(e_i)=e_{i-1}, \quad z(e_i)=f(i)e_{i+1}\\
&f(n)=t+t^3+\cdots+t^{2n-1} \qquad \text{for } n>0\\
&f(0)=0\\
&f(-n)=-t^{-1}-t^{-3}-\cdots-t^{-2n+1} \qquad \text{for } n>0.
\end{aligned}$$

Then it is clear that $yz-t^2zy-t=0$.

Now under the $k(t)$-algebra automorphism of $k(t)\{y, z\}$:

$$y\longmapsto ty, \qquad z\longmapsto t^{-1}z$$

the element $yz-t^2zy-t$ is invariant. So a $k(t)$-algebra automorphism of B is induced. Let

$$A=B\,\sharp\,k(t)[\langle x\rangle]$$

be the smash product over $k(t)$, where $\langle x\rangle$ denotes the free group generated by x. A has the following properties:

(1) A is a non zero $k(t)$-algebra.

(2) A contains elements y and z such that

$$yz-t^2zy-t=0.$$

(3) A contains an invertible element x such that

$$x^{-1}yx=ty, \qquad x^{-1}zx=t^{-1}z.$$

Let A be such an algebra. Let $(a_i)_{i\geq 0}$ be a sequence of elements of $k(t)$ defined by

$$a_0=1, \qquad a_na_{n+1}=\Big(\sum_{i=0}^{n} t^{2i}\Big)^{-1}.$$

Now we set

$$X_{2n}=a_{2n}\begin{pmatrix} x & t^{2n}y & 0\\ t^{2n}z & (t^{4n+1}zy+\sum_{i=0}^{2n} t^{2i})x^{-1} & 0\\ 0 & 0 & I_{m-2}\end{pmatrix}$$

$$X_{2n+1}=a_{2n+1}\begin{pmatrix} (t^{4n+3}zy+\sum_{i=0}^{2n+1} t^{2i})x^{-1} & -t^{2n+1}y & 0\\ -t^{2n+1}z & x & 0\\ 0 & 0 & I_{m-2}\end{pmatrix},$$

where I_{m-2} denotes the unit of $M_{m-2}(A)$. Then it is easy to see that $(X_i)_{i\geq 0}$

belongs to $L(M_m(k)^*, A)$. Since we have

$$\begin{pmatrix} 1 & 0 & 0 \\ -yx^{-1} & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix} {}^tX_0 = \begin{pmatrix} x & z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix},$$

${}^tX_0$ is not invertible in $M_m(A)$.

REMARK 13. If k is of characteristic zero, A and $(X_i)_{i\geqq 0}$ can be constructed more easily. There exists a non zero k-algebra A which contains elements y and z such that

$$yz - zy = 1.$$

For example let $V = \sum_{i=-\infty}^{\infty} ke_i$ be a vector space with basis $\{e_i\}$. Let y and z be elements of $\mathrm{End}(V)$ as follows

$$y(e_i) = e_{i-1}, \qquad z(e_i) = ie_{i+1}.$$

Then we have $yz - zy = 1$. Let A be such an algebra. Put

$$a_{2n} = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}, \qquad a_0 = 1$$

$$a_{2n+1} = \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdots (2n+1)}, \qquad a_1 = 1$$

$$X_{2n} = a_{2n} \begin{pmatrix} 1 & y & 0 \\ z & 2n+yz & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix}$$

$$X_{2n+1} = a_{2n+1} \begin{pmatrix} 2n+1+yz & -y & 0 \\ -z & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix}.$$

Then it is easily verified that $(X_i)_{i\geqq 0}$ belongs to $L(M_m(k)^*, A)$. Because we have

$$\begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix} {}^tX_0 = \begin{pmatrix} 1 & z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix},$$

${}^tX_0$ is not invertible.

§ 4. Coradicals and antipodes

Let $(C_i)_{i \geqq 0}$ be the coradical filtration of a k-coalgebra C. Let A be an algebra.

LEMMA 14. *For any element f of* Hom(C, A), *f is invertible if and only if $f|C_0$ is invertible in* Hom(C_0, A).

PROOF. Assume $f|C_0$ is invertible in Hom(C_0, A). There is an element g of Hom(C, A) such that

$$f*g=\eta\circ\varepsilon=g*f \qquad \text{on } C_0 .$$

If $f*g$ and $g*f$ are both invertible in Hom(C, A), then f is invertible. So we may assume that $f=\eta\circ\varepsilon$ on C_0. Put

$$h=\eta\circ\varepsilon-f .$$

Because $h=0$ on C_0, we have $h^{n+1}=0$ on C_n, where h^{n+1} is $(n+1)$-th power of h in the algebra Hom(C, A). Therefore $\sum_{n=0}^{\infty} h^n$ can be well defined. This is an inverse of f.

PROPOSITION 15. *The antipode of $H(C)$ is bijective if and only if the antipode of $H(C_0)$ is bijective.*

PROOF. Let A be an algebra. We have shown that the natural algebra map: Hom$(C, A)\rightarrow$Hom(C_0, A) induces a surjection: Reg$(C, A)\rightarrow$Reg(C_0, A), where Reg(C, A) is the group of units in Hom(C, A). It follows that the natural map: $L(C, A)\rightarrow L(C_0, A)$ is surjective.

Suppose the antipode of $H(C)$ is bijective. Let $(f_i)_{i \geqq 0}$ be an element of $L(C_0, A)$. There exists an element $(g_i)_{i \geqq 0}$ of $L(C, A)$ such that $g_i|C_0=f_i$. Since g_0 is invertible in Hom(C^{op}, A), f_0 is invertible in Hom(C_0^{op}, A). Therefore the antipode of $H(C_0)$ is bijective. The converse is similarly proved.

LEMMA 16. *Let $C=\sum C_\alpha$ be a direct sum of coalgebras. Then the antipode of $H(C)$ is bijective if and only if the antipode of $H(C_\alpha)$ is bijective for any α.*

PROOF. For any algebra A, we have

$$\mathrm{Hom}(C, A) \approx \prod \mathrm{Hom}(C_\alpha, A) \qquad \text{as algebras},$$

from which it follows that $L(C, A)\approx\prod L(C_\alpha, A)$. The lemma follows immediately from Corollary 7, if we notice that $L(C, A)$ is non-empty. Indeed the algebra map $\eta\circ\varepsilon: H(C)\rightarrow A$ belongs to $L(C, A)$.

PROPOSITION 17. *The antipode of $H(C)$ is bijective if and only if the antipode of $H(R)$ is bijective for any simple subcoalgebra R of C.*

PROOF. $C_0=\sum R$.

§ 5. Main results

Let $\bar{k}$ be the algebraic closure of k.

THEOREM 18. *Let C be a coalgebra over k. The antipode of $H(C)$ is bijective if and only if the $\bar{k}$-coalgebra $\bar{k}\otimes C$ is pointed.*

PROOF. Let S be the antipode of $H(C)$. S is bijective if and only if $\bar{k}\otimes S$ is bijective. But then $\bar{k}\otimes S$ is the antipode of $\bar{k}\otimes H(C)\approx H_{\bar{k}}(\bar{k}\otimes C)$. Hence $\bar{k}\otimes S$ is bijective if and only if the antipode of $H_{\bar{k}}(R)$ is bijective for any simple subcoalgebra R of $\bar{k}\otimes C$. But then R^* is a finite dimensional simple algebra over $\bar{k}$. So R^* is of the form $M_n(\bar{k})$. If $n>1$, then the antipode of $H_{\bar{k}}(R)$ is not bijective. If $n=1$, then R is cocommutative and so is $H_{\bar{k}}(R)$. Therefore the antipode of $H_{\bar{k}}(R)$ is bijective. Henceforth the antipode of $H(C)$ is bijective if and only if all simple subcoalgebras of $\bar{k}\otimes C$ are 1-dimensional, i. e. $\bar{k}\otimes C$ is pointed.

DEFINITION 19. A coalgebra C is said to be separable if R^* is a separable algebra in the sense of [2, p. 90] for any simple subcoalgebra R of C.

COROLLARY 20. *Let C be a separable coalgebra. Then the antipode of $H(C)$ is bijective if and only if the coradical of C is cocommutative.*

PROOF. By Proposition 17, we may assume C is simple. Because C^* is a separable algebra, $\bar{k}\otimes C^*$ has no radical. So the coradical of $\bar{k}\otimes C$ is $\bar{k}\otimes C$ itself. Therefore $\bar{k}\otimes C$ is pointed if and only if $\bar{k}\otimes C$ is cocommutative.

COROLLARY 21. *If k is a perfect field, then the antipode of $H(C)$ is bijective if and only if the coradical of C is cocommutative.*

COROLLARY 22. *If D is a finite dimensional central simple algebra, the antipode of $H(D^*)$ is bijective if and only if $D=k$.*

Chapter III. The algebra structure of $H(C)$

In this chapter we consider the algebra structure of $H(C)$ and give a k-basis of $H(C)$ for some special C.

§ 6. The fundamental theorem

Let C and D be coalgebras, A be an algebra and H be a bialgebra over k.

LEMMA 23. *If $f\in\mathrm{Hom}(C, A)$ and $g\in\mathrm{Hom}(D, A)$ are invertible, then the map: $x\otimes y\mapsto f(x)g(y)$ is invertible in* $\mathrm{Hom}(C\otimes D, A)$.

PROOF. $x\otimes y\mapsto g^{-1}(y)f^{-1}(x)$ is an inverse.

COROLLARY 24. *If $f\in\mathrm{Hom}(C, A)$ is invertible, then the algebra map $\bar{f}: T(C)\to A$ induced by f is invertible.*

PROOF. We have $T(C)=\sum_{n=0}^{\infty}\overset{n}{\otimes}C$ as a coalgebra.

LEMMA 25. *If* $f \in \mathrm{Alg}(H, A)$ *is invertible in* $\mathrm{Hom}(H, A)$, *then the inverse* f^{-1} *belongs to* $\mathrm{Alg}(H, A^{op})$.

The proof is similar to that of [**5**, Proposition 4.0.1].

Let C be a coalgebra with coradical C_0. Put

$$C = C_0 \oplus V$$

in the category of k-vector spaces. Let $B = T(V) \amalg H(C_0)$ be the direct sum of $T(V)$ and $H(C_0)$ in the category of k-algebras. Let $i: C \to B$ be the linear map induced by $V \to T(V) \to B$ and $C_0 \to H(C_0) \to B$.

LEMMA 26. *There exists a unique coalgebra structure on* B *such that* B *is a bialgebra and that* $i: C \to B$ *is a coalgebra map.*

PROOF. By the definition of direct sums, we have

$$\mathrm{Alg}(T(V) \amalg H(C_0), A) \approx \mathrm{Hom}(V, A) \times \mathrm{Alg}(H(C_0), A),$$

for any algebra A. Let $\Delta \in \mathrm{Alg}(B, B \otimes B)$ and $\varepsilon \in \mathrm{Alg}(B, k)$ be defined by

$$((i \otimes i) \circ (\Delta \mid V),\ (j \otimes j) \circ \Delta) \in \mathrm{Hom}(V, B \otimes B) \times \mathrm{Alg}(H(C_0), B \otimes B)$$

and

$$(\varepsilon \mid V, \varepsilon) \in \mathrm{Hom}(V, k) \times \mathrm{Alg}(H(C_0), k)$$

respectively, where $j: H(C_0) \to B$ is the canonical map. Then it is easy to see that (Δ, ε) is the unique coalgebra structure on B which satisfies the condition in Lemma 26.

LEMMA 27. *The bialgebra* B *has an antipode.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad i \quad} & B \\ \uparrow & & \uparrow{\scriptstyle j} \\ C_0 & \xrightarrow{\qquad} & H(C_0). \end{array}$$

Because j is invertible and $C_0 \to H(C_0)$ is a coalgebra map, i is invertible. So the algebra map $\bar{i}: T(C) \to B$ induced by i is invertible. Let P and Q be the inverse of $\bar{i}$ and j respectively. Then the diagram:

$$\begin{array}{ccc} T(C) & \xrightarrow{\quad P \quad} & B^{op} \\ \uparrow & & \uparrow{\scriptstyle Q} \\ C_0 & \xrightarrow{\qquad} & H(C_0) \end{array}$$

commutes and P and Q are algebra maps. Let $S \in \mathrm{Alg}(B, B^{op})$ be defined by $(P \mid T(V), Q) \in \mathrm{Alg}(T(V), B^{op}) \times \mathrm{Alg}(H(C_0), B^{op})$. Then we have $S \circ \bar{i} = P$ and $S \circ j = Q$. Put

$$M = \operatorname{Ker}(1*S-\eta\circ\varepsilon)\cap\operatorname{Ker}(S*1-\eta\circ\varepsilon).$$

Since $S: B\to B^{op}$ is an algebra map, M is a subalgebra of B. If we notice that $\bar{i}$ and j are also coalgebra maps, the equalities $S\circ\bar{i}=P$ and $S\circ j=Q$ mean that M contains the images of $\bar{i}$ and j. A fortiori M contains the image of $T(V)\to B$. So the identity $1_B: B\to B$ is factorized by $M\to B$. Hence we have $M=B$. Therefore S is an antipode of B.

LEMMA 28. *For any Hopf algebra H*

$$\operatorname{Hom}(i, H): \operatorname{Hopf}(B, H) \longrightarrow \operatorname{Coalg}(C, H)$$

is a bijection.

PROOF. It is clear from the definition.

PROPOSITION 29. *Let C be a coalgebra with coradical C_0. Write $C=C_0\oplus V$ as a vector space. Let $T(V)\amalg H(C_0)$ be the direct sum in the category of algebras. Then the element of* $\operatorname{Alg}(T(V)\amalg H(C_0), H(C))$ *determined by*

$$(i|V, i|C_0)\in\operatorname{Hom}(V, H(C))\times\operatorname{Coalg}(C_0, H(C))$$

is an algebra isomorphism, where $i: C\to H(C)$ is the canonical coalgebra map.

LEMMA 30. *Let (H_α) be a family of bialgebras. Let $\amalg H_\alpha$ be the direct sum as algebras.*

(1) *$\amalg H_\alpha$ has a unique coalgebra structure such that it becomes a bialgebra and that the natural maps $H_\alpha\to\amalg H_\alpha$ are all bialgebra maps.*

(2) *Then $\amalg H_\alpha$ is the direct sum of (H_α) in the category of bialgebras.*

(3) *If each H_α has an antipode, then $\amalg H_\alpha$ has an antipode.*

PROPOSITION 31. *Let $C=\sum C_\alpha$ be a direct sum of coalgebras. $\amalg H(C_\alpha)$ has a natural Hopf algebra structure. Let $i: C\to\amalg H(C_\alpha)$ be the coalgebra map determined by the natural maps: $C_\alpha\to H(C_\alpha)\to\amalg H(C_\alpha)$. Then $(\amalg H(C_\alpha), i)$ is the free Hopf algebra generated by C.*

PROOF. It is clear.

THEOREM 32. *Let (R_α) be the set of all simple subcoalgebras of C. Write $C=V\oplus(\sum R_\alpha)$ as a vector space. Let $(\amalg H(R_\alpha))\amalg T(V)$ be the direct sum in the category of algebras. Then the element of* $\operatorname{Alg}((\amalg H(R_\alpha))\amalg T(V), H(C))$ *determined by $(i|R_\alpha)\in\operatorname{Coalg}(R_\alpha, H(C))$ and $(i|V)\in\operatorname{Hom}(V, H(C))$ is an algebra isomorphism, where $i: C\to H(C)$ is the canonical coalgebra map.*

COMMENT 33. The coalgebra structure of $H(C)$ is determined by that of C and the algebra structure of $H(C)$ in view of Corollary 6. If we are concerned with the algebra structure of $H(C)$, it suffices to consider the case C is simple. If furthermore C is cocommutative, then C^* is a finite field extension of k. In the following two sections we study this case. The general case is an open problem.

§ 7. Case C is pointed

Let C be a coalgebra and $G(C)$ be the set of grouplike elements of C. Then $kG(C)$ is a subcoalgebra of C. Let $\langle G(C)\rangle$ be the free group generated by $G(C)$. Then the group algebra $k[\langle G(C)\rangle]$ has a natural Hopf algebra structure. It is easy to see the natural map: $kG(C)\to k[\langle G(C)\rangle]$ satisfies the universal property for free Hopf algebras. So we have

LEMMA 34. $H(kG(C))=k[\langle G(C)\rangle]$.

THEOREM 35. *Let C be a pointed coalgebra. Let $G(C)\cup B$ be a k-basis for C. Put $X=(G(C)\times\{\pm 1\})\cup B$. Let Y be the set of finite sequences $(x_1, \cdots, x_n)$ of elements of X and of length $\geqq 0$ such that (x_i, x_{i+1}) is not of the form $((g, \pm 1), (g, \mp 1))$. We put*

$$i(g, \pm 1)=i(g)^{\pm 1} \qquad \text{for } g\in G(C)$$
$$\bar{x}=i(x_1)\cdots i(x_n) \qquad \text{for } x=(x_1, \cdots, x_n)\in Y,$$

where $i: C\to H(C)$ is the canonical map. Then $\{\bar{x}\,;\, x\in Y\}$ forms a k-basis for $H(C)$.

PROOF. We have

$$\begin{aligned} H(C) &= H(kG(C))\amalg T(kB) \\ &= k[\langle G(C)\rangle]\amalg T(kB) \end{aligned}$$

as algebras. It is well known that $k[\langle G(C)\rangle]\amalg T(kB)$ has such a k-basis as described in the theorem.

REMARK 36. *A pointed bialgebra has an antipode if and only if its grouplike elements are all invertible.*

PROOF. This is an application of Lemma 14 and a generalization of [**5**, Proposition 9.2.5].

COMMENT 37. *If C has a cocommutative coradical, then the $\bar{k}$-algebra $\bar{k}\otimes C$ is pointed.* Indeed let $\{C_i\}_{i\geqq 0}$ be the coradical filtration of C. Then $\{\bar{k}\otimes C_i\}_{i\geqq 0}$ defines a filtered coalgebra structure on $\bar{k}\otimes C$. So $\bar{k}\otimes C_0$ contains the coradical of $\bar{k}\otimes C$ by [**5**, Proposition 11.1.1]. Because $\bar{k}\otimes C_0$ is cocommutative, the coradical of $\bar{k}\otimes C$ is cocommutative. Hence $\bar{k}\otimes C$ is pointed. So $\bar{k}\otimes H(C)\approx H_{\bar{k}}(\bar{k}\otimes C)$ has a $\bar{k}$-basis described in Theorem 35. From this point of view we study the structure of $H(C)$ in the next section.

§ 8. A basis for $H(C)$ for some C

We give a k-basis for $H(C)$ and the multiplication table for coalgebras C which satisfy the following conditions:

(Z1) There is a finite field extension K/k with a k-galois Hopf algebra

H such that K is a free H-module of rank one.

(Z2) There is a pointed k-coalgebra D and a K-coalgebra isomorphism $j: K\otimes C \approx K\otimes D$.

(Z3) $K\otimes C$ is a left H-module by $h(x\otimes y)=hx\otimes y$ for $h\in H$, $x\in K$ and $y\in C$. Under the identification $j: K\otimes C\approx K\otimes D$, we have $HD\subset D$.

Let's recall the definition of galois Hopf algebras. Let K/k be a finite field extension and H be a pointed cocommutative Hopf algebra over k. H is said to be a k-galois Hopf algebra of K/k if K is a left H-module algebra over k such that

$$k=K^H$$

$$\dim_k(H)=[K:k].$$

First we consider what coalgebras satisfy the condition (Z). We are chiefly concerned with the case $C=K^*$, where K is a finite field extension of k.

PROPOSITION 38. *Let K/k be a finite field extension which satisfies one of the following conditions. Then K^* satisfies* (Z).

(1) *K/k is normal modular in the sense of* [**3**].

(2) *K/k is separable.*

PROOF. Case (1). K/k has a k-galois Hopf algebra H [**1**, p. 250].

LEMMA 39. *If H is a k-galois Hopf algebra of K/k, the K-linear map:*

$$K\otimes H \longrightarrow K\otimes K^*$$

induced by the H-module structure: $H\rightarrow \mathrm{End}(K)\approx K\otimes K^*$ is a K-coalgebra isomorphism.

PROOF. The module algebra structure on K is defined by a k-algebra map

$$K \longrightarrow \mathrm{Hom}(H, K).$$

It is known [**5**, Theorem 10.1.1] that this induces a K-algebra isomorphism

$$K\otimes K\approx \mathrm{Hom}_K(K\otimes H, K).$$

If we apply the functor $\mathrm{Hom}_K(-, K)$, we get the K-coalgebra isomorphism in the lemma.

LEMMA 40. *$K\otimes H$ and $K\otimes K^*$ are left H-modules by*

$$h(a\otimes g)=\sum h_{(1)}a\otimes h_{(2)}g$$

$$h(a\otimes b^*)=ha\otimes b^*,$$

for $g, h\in H$, $a\in K$ and $b^*\in K^*$. Then the isomorphism

$$K\otimes H\approx K\otimes K^*$$

defined in Lemma 39 *is H-linear.*

PROOF. It is easy.

By the above two lemmas, the conditions (Z2) and (Z3) are satisfied. Now by the hypothesis K is of the form

$$K_0 \otimes K_1 \otimes \cdots \otimes K_n ,$$

where K_0/k is a finite galois extension with group G and $K_i = k[x_i]$ is purely inseparable over k. If H_i is a k-galois Hopf algebra of K_i/k, then $H_0 \otimes H_1 \otimes \cdots \otimes H_n$ is a k-galois Hopf algebra of K/k. If K_i is a free H_i-module of rank one, then K is a free $H_0 \otimes \cdots \otimes H_n$-module of rank one. The following lemma completes the proof of Proposition 38 in case (1).

LEMMA 41. *If K/k is a finite galois extension with group G or purely inseparable with one generator x, then K/k has a k-galois Hopf algebra H such that K is a free H-module of rank one.*

PROOF. If K/k is galois, we can take $H = k[G]$. Then a normal basis of K/k is a basis of K over H. Suppose k is of characteristic $p > 0$ and $K = k[x]$ is purely inseparable over k. Let $X^{p^n} - a$ be the minimal polynomial of x. We may assume $n > 0$. By [1, p. 251] K/k has the following k-galois Hopf algebra:

$$H = ky_0 + \cdots + ky_{p^n-1}$$

$$\Delta(y_m) = \sum y_i \otimes y_{m-i}$$

$$\varepsilon(y_m) = \delta_{0m}$$

$$y_i y_j = \binom{i+j}{i} y_{i+j}$$

$$y_0 = 1$$

$$y_0(x) = x, \quad y_1(x) = 1, \quad y_2(x) = \cdots = y_{p^n-1}(x) = 0 .$$

Put $z = (x-1)^{-1}$. If we apply y_m to $(x-1)z = 1$, then we have

$$(x-1)y_m(z) + y_{m-1}(z) = 0 .$$

It follows $y_m(z) = (-1)^m (x-1)^{-m-1}$. Hence $\{y_m(z)\}$ are linearly independent over k. Thus z is a basis of K over H.

PROOF OF PROPOSITION 38 (continued). Case (2). Let K/k be a finite separable extension. Let L/k be a finite galois extension with group G which contains K. $L \otimes K^*$ is an L-coalgebra. Then

LEMMA 42. $G(L \otimes K^*) = \mathrm{Alg}_k(K, L)$.

PROOF. $G(L \otimes K^*) = G(\mathrm{Hom}_L(L \otimes K, L)) = \mathrm{Alg}_L(L \otimes K, L) = \mathrm{Alg}_k(K, L)$.

LEMMA 43. *For any $g \in G$ the diagram*

$$\begin{array}{ccc} L\otimes K^* & \supset & \mathrm{Alg}\,(K, L) \\ \big\downarrow g\otimes 1 & & \big\downarrow \mathrm{Alg}\,(K, g) \\ L\otimes K^* & \supset & \mathrm{Alg}\,(K, L) \end{array}$$

commutes.

If we notice that $L\otimes K^* = L\cdot \mathrm{Alg}\,(K, L)(\because [K: k] = \# \mathrm{Alg}\,(K, L))$, the proof of Proposition 38 is complete in case (2) by the existence of a normal basis for L/k.

Let C be a coalgebra which satisfies (Z). Let K, H, D and j be as in (Z). In particular D is a left H-module.

LEMMA 44. *The action $H\otimes D \to D$ is a k-coalgebra map.*

PROOF. If M and N are left $K\# H$-modules, then $M\underset{K}{\otimes} N$ is a left $K\# H$-module by

$$h(m\otimes n) = \sum h_{(1)}(m)\otimes h_{(2)}(n)$$

for $h\in H$, $m\in M$ and $n\in N$, by the cocommutativity of H. Now $K\otimes C$ is a left $K\# H$-module and the maps

$$\Delta : K\otimes C \longrightarrow (K\otimes C)\underset{K}{\otimes}(K\otimes C)$$

$$\varepsilon : K\otimes C \longrightarrow K$$

are $K\# H$-linear. Since $j: K\otimes C \approx K\otimes D$ is a $K\# H$-module isomorphism and commutes with Δ and ε, the maps

$$\Delta : K\otimes D \longrightarrow (K\otimes D)\underset{K}{\otimes}(K\otimes D)$$

$$\varepsilon : K\otimes D \longrightarrow K$$

are also $K\# H$-linear. This means that the maps

$$\Delta : D \longrightarrow D\otimes D$$

$$\varepsilon : D \longrightarrow k$$

are H-linear. This is equivalent to Lemma 44.

LEMMA 45. *Let E and F be coalgebras and A be an algebra over k. The natural isomorphism*

$$\mathrm{Hom}\,(E\otimes F, A) \approx \mathrm{Hom}\,(F, \mathrm{Hom}\,(E, A))$$

is a k-algebra isomorphism. If E is cocommutative, then

$$\mathrm{Hom}\,(E, A)^{op} = \mathrm{Hom}\,(E, A^{op})\,,$$

and we can identify

$$L(E\otimes F, A) \approx L(F, \mathrm{Hom}\,(E, A))\,.$$

PROOF. It is trivial.

LEMMA 46. *$H(D)$ has a unique left H-module algebra structure such that*

$i: D \to H(D)$ is H-linear.

PROOF. Since H is cocommutative, we have natural isomorphisms

$$\begin{aligned} \mathrm{Alg}\,(H(D), \mathrm{Hom}\,(H, H(D))) &\approx L(D, \mathrm{Hom}\,(H, H(D))) \\ &\approx L(H \otimes D, H(D)) \\ &\approx \mathrm{Alg}\,(H(H \otimes D), H(D))\,. \end{aligned}$$

The coalgebra map $H \otimes D \to D$ in Lemma 44 induces a bialgebra map $H(H \otimes D) \to H(D)$. This can be identified with a k-algebra map $p: H(D) \to \mathrm{Hom}\,(H, H(D))$. p is a unique algebra map which makes the following diagram commutative

$$\begin{tikzcd} H(D) \arrow[r, "p"] & \mathrm{Hom}\,(H, H(D)) \\ D \arrow[u, "i"] \arrow[r, "q"] & \mathrm{Hom}\,(H, D)\,, \arrow[u, "\mathrm{Hom}\,(H, i)"'] \end{tikzcd}$$

where q is the H-module action on D. By the uniqueness it is clear that p defines an H-module structure on $H(D)$.

Define the action of H on $K \otimes H(C)$ and $K \otimes H(D)$ as

$$h(a \otimes x) = ha \otimes x$$

$$h(a \otimes y) = \sum h_{(1)}a \otimes h_{(2)}y$$

for $h \in H$, $a \in K$, $x \in H(C)$ and $y \in H(D)$. $K \otimes H(C)$ and $K \otimes H(D)$ clearly become $K \# H$-modules and H-module algebras over k.

LEMMA 47. *The K-coalgebra isomorphism*

$$K \otimes H(C) \approx H_K(K \otimes C) \overset{H_K(j)}{\approx} H_K(K \otimes D) \approx K \otimes H(D)$$

commutes with the action of H.

PROOF. Let A be a K-algebra. $\mathrm{Hom}\,(H, A)$ is a k-algebra. If we define

$$(af)(h) = \sum h_{(1)}(a) f(h_{(2)})$$

for $a \in K$, $f \in \mathrm{Hom}\,(H, A)$ and $h \in H$, then $\mathrm{Hom}\,(H, A)$ becomes a K-algebra, by the cocommutativity of H. Suppose A has a left H-module algebra structure with $p: A \to \mathrm{Hom}\,(H, A)$ its structure map. Then p is a K-algebra map if and only if A is a $K \# H$-module. Now $K \otimes H(C)$ and $K \otimes H(D)$ are K-algebras and H-module algebras over k such that they are $K \# H$-modules. Hence their H-module algebra structure maps are K-algebra maps. Thus the commutativity of the diagram

$$\begin{tikzcd} K \otimes H(C) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Hom}\,(H, K \otimes H(C)) \arrow[d, phantom, "\wr"] \\ K \otimes H(D) \arrow[r] & \mathrm{Hom}\,(H, K \otimes H(D)) \end{tikzcd}$$

follows immediately from Corollaries 5 and 8 and the assumption (Z3).

COROLLARY 48. *We have* $H(C)\approx(K\otimes H(D))^H$ *as k-algebras.*

K is a left H-module. So K^* is a right H-module. If we put

$$hx^*=x^*S(h)$$

for $h\in H$ and $x^*\in K^*$, where S is the antipode of H, then K^* is a left H-module.

LEMMA 49. (1) *Under the natural k-algebra isomorphism*

$$K\otimes H(D)\approx \mathrm{Hom}(K^*, H(D)),$$

$(K\otimes H(D))^H$ *corresponds to* $\mathrm{Hom}_H(K^*, H(D))$.

(2) K^* *is a free H-module of rank one.*

PROOF. (1) f belongs to $\mathrm{Hom}(K^*, H(D))^H$ if and only if

$$\sum h_{(2)}\circ f\circ S(h_{(1)})=\varepsilon(h)f$$

for any $h\in H$. This is equivalent to $h\circ f=f\circ h$.

(2) We have $K\approx H$ as left H-modules. So we have $K^*\approx H^*$ as right H-modules. But H^* is a free H-module of rank one [5, Lemma 16.0.1].

COROLLARY 50. $\mathrm{Hom}_H(K^*, H(D))$ *is a subalgebra of the k-algebra* $\mathrm{Hom}(K^*, H(D))$ *and we have*

$$H(C)\approx \mathrm{Hom}_H(K^*, H(D))$$

as k-algebras.

THEOREM 51. *Let C be a coalgebra which satisfies* (Z). *Let D, K, H and j be as in* (Z). *Let* a^* *be a basis of* K^* *over H. Put*

$$\Delta(a^*)=\sum_i g_i(a^*)\otimes h_i(a^*),$$

where g_i *and* h_i *are some elements of H. Then there exists an isomorphism of k-vector spaces*

$$P: H(C)\approx H(D)$$

such that we have

$$P(xy)=\sum_i g_i(P(x))h_i(P(y))$$

$$P(1)=\varepsilon(a^*)1$$

for $x, y\in H(C)$.

PROOF. It suffices to put

$$P(x)=x(a^*)$$

for $x\in H(C)\approx \mathrm{Hom}_H(K^*, H(D))$

COMMENT 52. A k-basis for $H(D)$ is given in Theorem 35. So Theorem 51 gives a multiplication table for $H(C)$.

Chapter IV. Free commutative Hopf algebras

First we notice a role of norms in the theory of commutative bialgebras. Then we construct the free commutative Hopf algebra generated by a coalgebra by a quite different method from that of the preceding chapters.

§ 9. A role of norms

Let A be a fixed finite dimensional algebra over k. For any commutative algebra B let

$$N_B : A \otimes B \longrightarrow B$$

be the norm function of the B-algebra $A \otimes B$ [**2**, p. 136].

LEMMA 53. (1) *N_B is natural in B*

(2) *We have*

$$N(xy) = N(x)N(y)$$
$$N(1) = 1$$

for $x, y \in A \otimes B$

(3) *$x \in A \otimes B$ is invertible if and only if $N(x) \in B$ is invertible.*

(4) *Let B and B' be two commutative algebras. We define*

$$t : (A \otimes B) \otimes (A \otimes B') \longrightarrow A \otimes (B \otimes B')$$
$$(x \otimes a) \otimes (y \otimes b) \longmapsto xy \otimes (a \otimes b)\,.$$

Then we have

$$N_B(u) \otimes N_{B'}(v) = N_{B \otimes B'}(t(u \otimes v))$$

for any $u \in A \otimes B$ and $v \in A \otimes B'$.

PROOF. (1) N_B can be factorized as

$$A \otimes B \longrightarrow \mathrm{End}\,(A) \otimes B \approx M_n(B) \xrightarrow{\det} B\,.$$

These are natural maps. (2) and (3) are classical.

(4) Write

$$u = \sum x_i \otimes a_i \qquad \text{and} \qquad v = \sum y_j \otimes b_j\,.$$

Then we have

$$\begin{aligned} t(u \otimes v) &= \sum x_i y_j \otimes a_i \otimes b_j \\ &= (\sum x_i \otimes a_i \otimes 1)(\sum y_j \otimes 1 \otimes b_j) \\ N(t(u \otimes v)) &= (N(u) \otimes 1)(1 \otimes N(v)) \\ &= N(u) \otimes N(v)\,. \end{aligned}$$

Let X be a subset of a commutative algebra B. The ring of quotients of B with respect to the multiplicatively closed subset of B generated by X

is denoted by $B[X^{-1}]$.

LEMMA 54. *Let H be a commutative bialgebra. Let X be a subset of H consisting of grouplike elements. Then the ring of quotients $H[X^{-1}]$ has a unique coalgebra structure such that $H[X^{-1}]$ is a bialgebra and that the natural map $H \to H[X^{-1}]$ is a bialgebra map.*

PROOF. It is clear from the universal property for $H[X^{-1}]$.

In the following we shall identify

$$A \otimes B = \mathrm{Hom}(A^*, B)$$

as k-algebras.

PROPOSITION 55. *Let H be a commutative bialgebra. If $f \in A \otimes H = \mathrm{Hom}(A^*, H)$ is a coalgebra map, then the norm $N_H(f) \in H$ is a grouplike element. Hence $H[N(f)^{-1}]$ is a bialgebra. The coalgebra map $f: A^* \to H[N(f)^{-1}]$ is invertible.*

PROOF. $\Delta \circ f = (f \otimes f) \circ \Delta$ is equivalent to

$$(1 \otimes \Delta)(f) = t(f \otimes f),$$

where $t: (A \otimes H) \otimes (A \otimes H) \to A \otimes (H \otimes H)$ is defined in Lemma 53 (4). So we have

$$N(f) \otimes N(f) = N(t(f \otimes f)) = \Delta(N(f)),$$

by Lemma 53 (1) and (4). Now $\varepsilon \circ f = \varepsilon$ is equivalent to

$$(1 \otimes \varepsilon)(f) = 1.$$

Hence we have $1 = N(1) = \varepsilon(N(f))$. Thus $N(f)$ is grouplike. The element f of $A \otimes H[N(f)^{-1}]$ is invertible because $N(f)$ is invertible in $H[N(f)^{-1}]$ (Lemma 53 (3)). This completes the proof.

§ 10. An interpretation of $G(S(A^*))$

Let A be a finite dimensional algebra. The symmetric algebra $S(A^*)$ has a natural bialgebra structure. There is a bijection from the set of grouplike elements $G(S(A^*))$ onto the set of norm functions on A.

DEFINITION 56. N is said to be a norm function on A if it satisfies the following conditions.

(1) For any commutative algebra B a function

$$N_B : A \otimes B \longrightarrow B$$

is defined.

(2) We have

$$N_B(xy) = N_B(x) N_B(y) \qquad \text{and} \qquad N_B(1) = 1$$

for $x, y \in A \otimes B$

(3) N_B is natural in B.

The set of all norm functions on A is denoted by $NF(A)$.

LEMMA 57. *Let N be a norm function on A. Let B and B' be two commutative algebras. Let $t: (A\otimes B)\otimes(A\otimes B')\rightarrow A\otimes(B\otimes B')$ be defined as in Lemma* 53 (4). *Then we have*

$$N_{B\otimes B'}(t(x\otimes y))=N_B(x)\otimes N_{B'}(y)$$

for any $x\in A\otimes B$ and $y\in A\otimes B'$.

LEMMA 58. *Let H be a commutative bialgebra. If*

$$f\in A\otimes H=\mathrm{Hom}\,(A^*, H)$$

is a coalgebra map, then $N(f)$ is a grouplike element of H for any norm function N on A.

The proofs are similar to those of Lemma 53 and Proposition 55.

Let

$$i\in A\otimes S(A^*)=\mathrm{Hom}\,(A^*, S(A^*))$$

be the natural injection. This is a coalgebra map. So $N(i)$ is a grouplike element of $S(A^*)$ for any norm function N on A.

PROPOSITION 59. *The map $N\mapsto N(i)$ is a bijection from $NF(A)$ onto $G(S(A^*))$.*

PROOF. Let g be a grouplike element of $S(A^*)$. For any commutative algebra B, we have

$$A\otimes B=\mathrm{Hom}\,(A^*, B)=\mathrm{Alg}\,(S(A^*), B)\,.$$

Under this identification put

$$\begin{aligned} N_g: A\otimes B &\longrightarrow B \\ f &\longmapsto f(g)\,. \end{aligned}$$

N_g is clearly natural in B. Since $i: A^*\rightarrow S(A^*)$ is a coalgebra map, this induces an algebra map

$$\mathrm{Hom}\,(A^*, B)\longleftarrow \mathrm{Hom}\,(S(A^*), B)\,.$$

Hence we have

$$N_g(xy)=x*y(g)=x(g)y(g)=N_g(x)N_g(y)$$

$$N_g(1)=\varepsilon(g)=1$$

for $x, y\in A\otimes B$. So N_g is a norm function. $g\mapsto N_g$ is clearly the inverse of $N\mapsto N(i)$.

REMARK 60. In the following two sections we only use the special norm function given in § 9.

§ 11. Free commutative Hopf algebras

Let $S(V)$ be the symmetric algebra on a vector space V. If C is a coalgebra, then $S(C)$ has a natural bialgebra structure. For any finite dimensional subcoalgebra D of C, we shall associate with it a grouplike element of $S(C)$. Since D^* is a finite dimensional algebra and $S(C)$ is a commutative algebra, we can define the norm function

$$N: D^* \otimes S(C) \longrightarrow S(C)$$

as in § 9. Let $i: D \to S(C)$ be the natural injection. Then by Proposition 55, $g(D) = N(i)$ is a grouplike element of $S(C)$ and the natural map

$$D \longrightarrow S(C)[g(D)^{-1}]$$

is invertible.

THEOREM 61. *Let C be a coalgebra. Let $\{D_\alpha\}$ be a family of finite dimensional subcoalgebras such that $\sum D_\alpha$ contains the coradical of C. Let X be a set of grouplike elements of $S(C)$ which contains $g(D_\alpha)$ for any α. Then*

(1) *$S(C)[X^{-1}]$ has an antipode and*

(2) *For any commutative Hopf algebra H, we have*

$$\mathrm{Hopf}\,(S(C)[X^{-1}], H) \approx \mathrm{Coalg}\,(C, H)\,.$$

PROOF. Let R be a simple subcoalgebra of C. Then R is contained in some D_α [**5**, Proposition 8.0.3]. Hence the canonical map $R \to S(C)[X^{-1}]$ is invertible because it is factorized by the invertible map $D_\alpha \to S(C)[X^{-1}]$. By Lemma 14, the natural map $i: C \to S(C)[X^{-1}]$ is invertible and so is the algebra map $\bar{i}: T(C) \to S(C)[X^{-1}]$ induced by i. But the inverse of $\bar{i}$, which is also an algebra map, is factorized by $T(C) \to S(C)$. Hence the natural map $j: S(C) \to S(C)[X^{-1}]$ has an inverse P which is an algebra map. Then it is clear that

$$P(x) = x^{-1}$$

for $x \in X$. Therefore P induces an algebra map from $S(C)[X^{-1}]$ to $S(C)[X^{-1}]$ which is clearly an antipode of $S(C)[X^{-1}]$. (2) is clear by the definition.

DEFINITION 62. The free commutative Hopf algebra $H_c(C)$ generated by a coalgebra C is defined by the universal property (2) in Theorem 61.

COROLLARY 63. *Let C be a coalgebra. Let $\{R_\alpha\}$ be the family of all simple subcoalgebras. Then we have*

$$H_c(C) = S(C)[g(R_\alpha)^{-1}\,;\,\alpha]\,.$$

COROLLARY 64. *If C has a finite dimensional coradical C_0, then we have*

$$H_c(C) = S(C)[g(C_0)^{-1}]\,.$$

§ 12. An application

Let H be a commutative bialgebra. For any finite dimensional subcoalgebra D of H let $g(D)$ be the norm of the natural injection $D \to H$, which is viewed as an element of the H-algebra $D^* \otimes H$. Then $g(D)$ is a grouplike element of H and the natural map

$$D \longrightarrow H[g(D)^{-1}]$$

is invertible. Just as in the preceding section we have

THEOREM 65. *Let H be a commutative bialgebra over k. Let $\{D_\alpha\}$ be a family of finite dimensional subcoalgebras of H such that $\sum D_\alpha$ contains the coradical of H. Let X be a set of grouplike elements of H which contains $g(D_\alpha)$ for any α. Then*

(1) *$H[X^{-1}]$ has an antipode and*

(2) *For any commutative Hopf algebra H', we have*

$$\mathrm{Hopf}\,(H[X^{-1}], H') \approx \mathrm{Bialg}\,(H, H')\,.$$

DEFINITION 66. The free commutative Hopf algebra $HP_c(H)$ generated by a commutative bialgebra H is defined by the universal property (2).

COROLLARY 67. *Let $\{R_\alpha\}$ be the family of all simple subcoalgebras of a commutative bialgebra H. Then we have*

$$HP_c(H) = H[g(R_\alpha)^{-1}\,;\,\alpha]\,.$$

COROLLARY 68. *If a commutative bialgebra H has a finite dimensional coradical H_0, then we have*

$$HP_c(H) = H[g(H_0)^{-1}]\,.$$

COROLLARY 69. *A commutative bialgebra H has an antipode if and only if its grouplike elements are all invertible.*

The last corollary is a generalization of [**5**, 9.2.5] in the category of commutative algebras.

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