INTEGRABLE $\mathfrak{sl}(\infty)$-MODULES AND CATEGORY $\mathcal{O}$ FOR $\mathfrak{gl}(m|n)$

CRYSTAL HOYT, IVAN PENKOV, VERA SERGANOVA

Abstract. We introduce and study new categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of integrable $\mathfrak{g} = \mathfrak{sl}(\infty)$-modules which depend on the choice of a certain reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k} \subset \mathfrak{g}$. The simple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ are tensor modules as in the previously studied category $\mathbb{T}_{\mathfrak{g}}$ [DPS]; however, the choice of $\mathfrak{k}$ provides for more flexibility of nonsimple modules in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ compared to $\mathbb{T}_{\mathfrak{g}}$. We then choose $\mathfrak{k}$ to have two infinite-dimensional diagonal blocks, and show that a certain injective object $\mathbf{K}_{m|n}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ realizes a categorical $\mathfrak{sl}(\infty)$-action on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$, the integral category $\mathcal{O}$ of the Lie superalgebra $\mathfrak{gl}(m|n)$. We show that the socle of $\mathbf{K}_{m|n}$ is generated by the projective modules in $\mathcal{O}^{\mathbb{Z}}_{m|n}$, and compute the socle filtration of $\mathbf{K}_{m|n}$ explicitly. We conjecture that the socle filtration of $\mathbf{K}_{m|n}$ reflects a "degree of atypicality filtration" on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$. We also conjecture that a natural tensor filtration on $\mathbf{K}_{m|n}$ arises via the Duflo–Serganova functor sending the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ to $\mathcal{O}^{\mathbb{Z}}_{m-1|n-1}$. We prove a weaker version of this latter conjecture for the direct summand of $\mathbf{K}_{m|n}$ corresponding to finite-dimensional $\mathfrak{gl}(m|n)$-modules.

Mathematics subject classification (2010): Primary 17B65, 17B10, 17B55.
Key words: super category $\mathcal{O}$, integrable $\mathfrak{sl}(\infty)$-module, Duflo–Serganova functor, socle filtration, injective module.

1. Introduction

Categorification has set a trend in mathematics in the last two decades and has proved important and useful. The opposite process of studying a given category via a combinatorial or algebraic object such as a single module has also borne ample fruit. An example is Brundan's idea from 2003 to study the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$ of finite-dimensional integral modules over the Lie superalgebra $\mathfrak{gl}(m|n)$ via the weight structure of the $\mathfrak{sl}(\infty)$-module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$, where $\mathbf{V}$ and $\mathbf{V}_*$ are the two nonisomorphic defining (natural) representations of $\mathfrak{sl}(\infty)$. Using this approach Brundan computes decomposition numbers in $\mathcal{F}^{\mathbb{Z}}_{m|n}$ [B]. An extension of Brundan's approach was proposed in the work of Brundan, Losev and Webster in [BLW], where a new proof of the Brundan–Kazhdan–Lusztig conjecture for the category $\mathcal{O}$ over the Lie superalgebra $\mathfrak{gl}(m|n)$ is given. (The first proof of the Brundan–Kazhdan–Lusztig conjecture for the category $\mathcal{O}$ over the Lie superalgebra $\mathfrak{gl}(m|n)$ was given by Cheng, Lam and Wang in [CLW].) The same approach was also used by Brundan and Stroppel in [BS], where the algebra of endomorphisms of a projective generator in $\mathcal{F}^{\mathbb{Z}}_{m|n}$ is described as a certain diagram algebra and the Koszulity of $\mathcal{F}^{\mathbb{Z}}_{m|n}$ is established.

All three authors have been supported in part by DFG Grant PE 980/6-1. The first and third authors been partially supported by BSF Grant 2012227. The third author has been also supported by NSF grant DMS-1701532.

The representation theory of the Lie algebra $\mathfrak{sl}(\infty)$ is of independent interest and has been developing actively also for about two decades. In particular, several categories of $\mathfrak{sl}(\infty)$-modules have been singled out and studied in detail, see [DP, PStyr, DPS, PS, Nam].

The category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ from [DPS] has been playing a prominent role: its objects are finite-length submodules of a direct sum of several copies of the tensor algebra $T(\mathbf{V}\oplus\mathbf{V}_*)$. In [DPS] it is proved that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is a self-dual Koszul category, in [SS] it has been shown that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has a universality property, and in [FPS] $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has been used to categorify the Boson-Fermion Correspondence.

Our goal in the present paper is to find an appropriate category of $\mathfrak{sl}(\infty)$-modules which contains modules relevant to the representation theory of the Lie superalgebras $\mathfrak{gl}(m|n)$. For this purpose, we introduce and study the categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, where $\mathfrak{g}=\mathfrak{sl}(\infty)$ and $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}$ containing the diagonal subalgebra and consisting of finitely many blocks along the diagonal. The Lie algebra $\mathfrak{k}$ is infinite dimensional and is itself isomorphic to the commutator subalgebra of a finite direct sum of copies of $\mathfrak{gl}(n)$ (for varying n) and copies of $\mathfrak{gl}(\infty)$. When $\mathfrak{k}=\mathfrak{g}$, this new category coincides with $\mathbb{T}_{\mathfrak{g}}$. A well-known property of the category $\mathbb{T}_{\mathfrak{g}}$ states that for every $\mathbf{M}\in\mathbb{T}_{\mathfrak{g}}$, any vector $m\in\mathbf{M}$ is annihilated by a "large" subalgebra $\mathfrak{g}'\subset\mathfrak{g}$, i.e. by an algebra which contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra $\mathfrak{s}\subset\mathfrak{g}$. For a general $\mathfrak{k}$ as above, the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ has the same simple objects as $\mathbb{T}_{\mathfrak{g}}$ but requires the following for a nonsimple module $\mathbf{M}$: the annihilator in $\mathfrak{k}$ of every $m\in\mathbf{M}$ is a large subalgebra of $\mathfrak{k}$. This makes the nonsimple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ more "flexible" than in those of $\mathbb{T}_{\mathfrak{g}}$, the degree of flexibility being governed by $\mathfrak{k}$.

In Section 3, we study the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ in detail, one of our main results being an explicit computation of the socle filtration of an indecomposable injective object $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ (where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two Young diagrams), see Theorem 20. An effect which can be observed here is that with a sufficient increase in the number of infinite blocks of $\mathfrak{k}$, the layers of the socle filtration of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ grow in a "self-similar" manner. This shows that $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is an intricate extension of the category $\mathbb{T}_{\mathfrak{g}}$ within the category of all integrable $\mathfrak{g}$-modules.

In Section 4, we show that studying the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ achieves our goal of improving the understanding of the integral category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$. More precisely, we choose $\mathfrak{k}$ to have two blocks, both of them infinite. Then we show that the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is a categorification of an injective object $\mathbf{K}_{m|n}$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. In order to accomplish this, we exploit the properties of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ as a category, and not just as a collection of modules. The object $\mathbf{K}_{m|n}$ of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ can be defined as the complexified reduced Grothendieck group of the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$, endowed with an $\mathfrak{sl}(\infty)$-module structure (categorical action of $\mathfrak{sl}(\infty)$). For $m,n\geq 1$, $\mathbf{K}_{m|n}$ is an object of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, but not of $\mathbb{T}_{\mathfrak{g}}$. We prove that the socle of $\mathbf{K}_{m|n}$ as an $\mathfrak{sl}(\infty)$-module is the submodule generated by classes of projective $\mathfrak{gl}(m|n)$-modules in $\mathcal{O}^{\mathbb{Z}}_{m|n}$. Moreover, we conjecture that the socle filtration of $\mathbf{K}_{m|n}$ (which we already know from Section 3) arises from filtering the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ according to the degree of atypicality of $\mathfrak{gl}(m|n)$-modules. We provide some partial evidence toward this conjecture.

We also show that the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$ of finite-dimensional integral $\mathfrak{gl}(m|n)$-modules categorifies a direct summand $\mathbf{J}_{m|n}$ of $\mathbf{K}_{m|n}$ which is nothing but an injective hull in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of Brundan's module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$, see Corollary 28. (Note that the module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$ is an injective object of $\mathbb{T}_{\mathfrak{g}}$, but is not injective in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ when $\mathfrak{k}$ has two (or more) infinite blocks.)

Finally, we conjecture that a natural filtration on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ defined via the Duflo–Serganova functor $DS : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m-1|n-1}$ categorifies the tensor filtration of $\mathbf{K}_{m|n}$, i.e. the coarsest filtration of $\mathbf{K}_{m|n}$ whose successive quotients are objects of $\mathbb{T}_{\mathfrak{g}}$. We have a similar conjecture for the direct summand $\mathbf{J}_{m|n}$ of $\mathbf{K}_{m|n}$, and we provide evidence for this conjecture in Proposition 42.

2. Acknowledgements

We would like to thank two referees for their extremely thorough and thoughtful comments.

3. New categories of integrable $\mathfrak{sl}(\infty)$-modules

3.1. **Preliminaries.** Let $\mathbf{V}$ and $\mathbf{V}_*$ be countable-dimensional vector spaces with fixed bases $\{v_i\}_{i\in\mathbb{Z}}$ and $\{v_j^*\}_{j\in\mathbb{Z}}$, together with a nondegenerate pairing $\langle\cdot,\cdot\rangle : \mathbf{V}\otimes\mathbf{V}_* \to \mathbb{C}$ defined by $\langle v_i, v_j^*\rangle = \delta_{ij}$. Then $\mathfrak{gl}(\infty) := \mathbf{V}\otimes\mathbf{V}_*$ has a Lie algebra structure such that

$$[v_i\otimes v_j^*, v_k\otimes v_l^*] = \langle v_k, v_j^*\rangle v_i\otimes v_l^* - \langle v_i, v_l^*\rangle v_k\otimes v_j^*.$$

We can identify $\mathfrak{gl}(\infty)$ with the space of infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero entries, where the vector $v_i\otimes v_j^*$ corresponds to the matrix E_{ij} with 1 in the i,j-position and zeros elsewhere. Then $\langle\cdot,\cdot\rangle$ corresponds to the trace map, and its kernel is the Lie algebra $\mathfrak{sl}(\infty)$, which is generated by $e_i := E_{i,i+1}$, $f_i := E_{i+1,i}$ with $i\in\mathbb{Z}$. One can also realize $\mathfrak{sl}(\infty)$ as a direct limit of finite-dimensional Lie algebras $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$. In contrast to the finite-dimensional setting, the exact sequence

$$0\to\mathfrak{sl}(\infty)\to\mathfrak{gl}(\infty)\to\mathbb{C}\to 0$$

does not split, and the center of $\mathfrak{gl}(\infty)$ is trivial.

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. The representations $\mathbf{V}$ and $\mathbf{V}_*$ are the defining representations of $\mathfrak{g}$. The tensor representations $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$, $p,q\in\mathbb{Z}_{\geq 0}$ have been studied in [PStyr]. They are not semisimple when $p,q>0$; however, each simple subquotient of $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ occurs as a submodule of $\mathbf{V}^{\otimes p'}\otimes\mathbf{V}_*^{\otimes q'}$ for some p',q'. The simple submodules of $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ can be parameterized by two Young diagrams $\boldsymbol{\lambda},\boldsymbol{\mu}$, and we denote them $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Recall that the *socle* of a module $\mathbf{M}$, denoted $\operatorname{soc}\mathbf{M}$, is the largest semisimple submodule of $\mathbf{M}$. The *socle filtration* of $\mathbf{M}$ is defined inductively by $\operatorname{soc}^0\mathbf{M} := \operatorname{soc}\mathbf{M}$ and $\operatorname{soc}^i\mathbf{M} := p_i^{-1}(\operatorname{soc}(\mathbf{M}/(\operatorname{soc}^{i-1}\mathbf{M})))$, where $p_i : \mathbf{M}\to\mathbf{M}/(\operatorname{soc}^{i-1}\mathbf{M})$ is the natural projection. We also use the notation $\overline{\operatorname{soc}}^i\mathbf{M} := \operatorname{soc}^i\mathbf{M}/\operatorname{soc}^{i-1}\mathbf{M}$ for the layers of the socle filtration.

Schur-Weyl duality for $\mathfrak{sl}(\infty)$ implies that the module $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ decomposes as

$$\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q} = \bigoplus_{|\boldsymbol{\lambda}|=p,|\boldsymbol{\mu}|=q} (\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))\otimes(Y_{\boldsymbol{\lambda}}\otimes Y_{\boldsymbol{\mu}}), \tag{3.1}$$

where $Y_{\boldsymbol{\lambda}}$ and $Y_{\boldsymbol{\mu}}$ are irreducible S_p- and S_q-modules, and $\mathbb{S}_{\boldsymbol{\lambda}}$ denotes the Schur functor corresponding to the Young diagram (equivalently, partition) $\boldsymbol{\lambda}$. Each module $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ is indecomposable and its socle filtration is described in [PStyr]. Moreover, Theorem 2.3 of [PStyr] claims that

$$\overline{\operatorname{soc}}^k(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',|\boldsymbol{\gamma}|=k} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \tag{3.2}$$

where $N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma}$ are the standard Littlewood-Richardson coefficients. In particular, $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ has simple socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. It was also shown in [PStyr, Theorem 2.2] that the socle of $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ equals the intersection of the kernels of all contraction maps

$$\Phi_{ij} : \mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q} \to \mathbf{V}^{\otimes (p-1)} \otimes \mathbf{V}_*^{\otimes (q-1)} \tag{3.3}$$

$$v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* \mapsto \langle v_j^*, v_i \rangle v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_q^*$$

A $\mathfrak{g}$-module is called a *tensor module* if it is isomorphic to a submodule of a finite direct sum of $\mathfrak{sl}(\infty)$-modules of the form $\mathbf{V}^{\otimes p_i} \otimes \mathbf{V}_*^{\otimes q_i}$ for $p_i, q_i \in \mathbb{Z}_{\geq 0}$. The category of tensor modules $\mathbb{T}_{\mathfrak{g}}$ is by definition the full subcategory of $\mathfrak{g}$-mod consisting of tensor modules [DPS]. A finite-length $\mathfrak{g}$-module $\mathbf{M}$ lies in $\mathbb{T}_{\mathfrak{g}}$ if and only if $\mathbf{M}$ is integrable and satisfies the large annihilator condition [DPS]. Recall that a $\mathfrak{g}$-module $\mathbf{M}$ is called *integrable* if $\dim\{m, x\cdot m, x^2\cdot m, \ldots\} < \infty$ for any $x \in \mathfrak{g}$, $m \in \mathbf{M}$. A $\mathfrak{g}$-module is said to satisfy the *large annihilator condition* if for each $m \in \mathbf{M}$, the annihilator $\mathrm{Ann}_{\mathfrak{g}} m$ contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra of $\mathfrak{g}$.

The modules $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$, $p, q \in \mathbb{Z}_{\geq 0}$ are injective in the category $\mathbb{T}_{\mathfrak{g}}$. Moreover, every indecomposable injective object of $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to an indecomposable direct summand of $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$ [DPS]. Consequently, by (3.1), an indecomposable injective in $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ for some $\boldsymbol{\lambda}, \boldsymbol{\mu}$.

The category $\mathbb{T}_{\mathfrak{g}}$ is a subcategory of the category $\widetilde{Tens}_{\mathfrak{g}}$, which was introduced in [PS] as the full subcategory of $\mathfrak{g}$-mod whose objects $\mathbf{M}$ are defined to be the integrable $\mathfrak{g}$-modules of finite Loewy length such that the algebraic dual $\mathbf{M}^* = \mathrm{Hom}_{\mathbb{C}}(\mathbf{M}, \mathbb{C})$ is also integrable and of finite Loewy length. The categories $\mathbb{T}_{\mathfrak{g}}$ and $\widetilde{Tens}_{\mathfrak{g}}$ have the same simple objects $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ [PS, DPS]. The indecomposable injective objects of $\widetilde{Tens}_{\mathfrak{g}}$ are (up to isomorphism) the modules $(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*$, and $\mathrm{soc}(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^* \cong \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ [PS]. A recent result of [CP2] shows that the Grothendieck envelope $\overline{Tens}_{\mathfrak{g}}$ of $\widetilde{Tens}_{\mathfrak{g}}$ is an ordered tensor category, and that any injective object in $\overline{Tens}_{\mathfrak{g}}$ is a direct sum of indecomposable injectives from $\widetilde{Tens}_{\mathfrak{g}}$.

3.2. **The categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.** In this section, we introduce new categories of integrable $\mathfrak{sl}(\infty)$-modules. This is motivated in part by the applications to the representation theory of the Lie superalgebras $\mathfrak{gl}(m|n)$.

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ with the natural representation denoted $\mathbf{V}$. Consider a decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_r, \tag{3.4}$$

for some vector subspaces $\mathbf{V}_i$ of $\mathbf{V}$. Let $\mathfrak{l}$ be the Lie subalgebra of $\mathfrak{g}$ preserving this decomposition. Then $\mathfrak{k} := [\mathfrak{l}, \mathfrak{l}]$ is isomorphic to $\mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, where each $\mathfrak{k}_i$ is isomorphic to $\mathfrak{sl}(n_i)$ or $\mathfrak{sl}(\infty)$.

Definition 1. Denote by $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ the full subcategory of $\widetilde{Tens}_{\mathfrak{g}}$ consisting of modules $\mathbf{M}$ satisfying the large annihilator condition as a module over $\mathfrak{k}_i$ for all $i = 1, \ldots, r$. By $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we denote the full subcategory of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ consisting of finite-length modules.

Both categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are abelian symmetric monoidal categories with respect to the usual tensor product of $\mathfrak{g}$-modules. Two categories $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\bar{\mathfrak{k}}}$ are equal if $\mathfrak{k}$ and $\bar{\mathfrak{k}}$ have finite corank in $\mathfrak{k} + \bar{\mathfrak{k}}$, so we will henceforth assume without loss of generality that each $\mathbf{V}_i$ in decomposition (3.4) is infinite dimensional. Note that $\mathbb{T}_{\mathfrak{g},\mathfrak{g}} = \mathbb{T}_{\mathfrak{g}}$.

We define the functor $\Gamma_{\mathfrak{g},\mathfrak{k}} : \widetilde{Tens}_{\mathfrak{g}} \to \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ by taking the maximal submodule lying in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$. Then

$$\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_r}, \tag{3.5}$$

where the union is taken over all finite corank subalgebras $\mathfrak{s}_1 \subset \mathfrak{k}_1, \ldots, \mathfrak{s}_r \subset \mathfrak{k}_r$.

Lemma 2. *Let $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ be as in Definition 1.*

1. *The simple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are isomorphic to $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.*
2. *The functor $\Gamma_{\mathfrak{g},\mathfrak{k}}$ sends injective modules in $\widetilde{Tens}_{\mathfrak{g}}$ to injective modules in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$.*
3. *The category $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ has enough injective modules.*
4. *The indecomposable injective objects of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are isomorphic to $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$.*

Proof.

1. The category $\mathbb{T}_{\mathfrak{g}}$ is a full subcategory of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$, which are both full subcategories of $\widetilde{Tens}_{\mathfrak{g}}$. Since the categories $\mathbb{T}_{\mathfrak{g}}$ and $\widetilde{Tens}_{\mathfrak{g}}$ have the same simple objects $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, the claim follows.
2. This follows from the definition of $\Gamma_{\mathfrak{g},\mathfrak{k}}$, since $\mathrm{Hom}_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(X, \Gamma_{\mathfrak{g},\mathfrak{k}}(Y)) = \mathrm{Hom}_{\widetilde{Tens}_{\mathfrak{g}}}(X, Y)$ for all $X \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $Y \in \widetilde{Tens}_{\mathfrak{g}}$.
3. Every module $\mathbf{M}$ in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ can be embedded into $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}^{**})$, which is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$, since $\mathbf{M}^{**}$ is injective in $\widetilde{Tens}_{\mathfrak{g}}$ [PS].
4. This follows from (1) and (2), since $(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*$ is an indecomposable injective object of $\widetilde{Tens}_{\mathfrak{g}}$, and consequently $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ is an indecomposable injective object of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ with $\mathrm{soc}\,\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*) \cong \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

$\square$

Remark 3. It will follow from Corollary 12 that the indecomposable injective objects $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ are objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. Consequently, $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ have the same indecomposable injectives.

3.3. **The functor** R **and Jordan-Hölder multiplicities.** In this section, we calculate the Jordan-Hölder multiplicities of the indecomposable injective objects of the categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. One of the main tools we use for this computation is the functor R, which we will now introduce.

Let

$$\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}, \quad \mathfrak{g}' = \mathfrak{g} \cap \mathfrak{gl}(\mathbf{V}'), \quad \mathfrak{k}' = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_{r-1}. \tag{3.6}$$

Let $(\mathbf{V}_r)_* \subset \mathbf{V}_*$ be the annihilator of $\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}$ with respect to the pairing $\langle \cdot, \cdot \rangle$. We have $\mathfrak{g}' \cong \mathfrak{sl}(\infty)$ and $\mathfrak{k}' \subset \mathfrak{g}'$.

Define a functor R from the category $\mathfrak{g}$–mod of all $\mathfrak{g}$-modules to the category $\mathfrak{g}'$–mod by setting

$$\mathrm{R}(\mathbf{M}) = \mathbf{M}^{\mathfrak{k}_r}.$$

It follows from the definition that after restricting to $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ we have a functor $\mathrm{R} : \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} \to \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$.

Lemma 4. *The following diagram of functors is commutative:*

$$\begin{tikzcd} \mathfrak{g}-\mathrm{mod} \arrow[r, "\mathrm{R}"] \arrow[d, "\Gamma_{\mathfrak{g},\mathfrak{k}}"'] & \mathfrak{g}'-\mathrm{mod} \arrow[d, "\Gamma_{\mathfrak{g}',\mathfrak{k}'}"'] \\ \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} \arrow[r, "\mathrm{R}"] & \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'} \end{tikzcd}.$$

Proof. By (3.5) we have

$$\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_r}$$

for any $\mathfrak{g}$-module $\mathbf{M}$. Then

$$\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M})) = (\bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_r})^{\mathfrak{k}_r} = \bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_{r-1}\oplus\mathfrak{k}_r} = \bigcup(\mathrm{R}(\mathbf{M}))^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_{r-1}} = \Gamma_{\mathfrak{g}',\mathfrak{k}'}(\mathrm{R}(\mathbf{M})).$$

$\square$

Lemma 5. *If $\boldsymbol{\lambda},\boldsymbol{\mu}$ are Young diagrams, then*

$$\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*.$$

Proof. Since $\mathrm{R}(\mathbf{V}) = \mathbf{V}'$, we have the decompositions

$$\mathbf{V} = \mathrm{R}(\mathbf{V})\oplus\mathbf{V}_r, \quad \mathbf{V}_* = \mathrm{R}(\mathbf{V}_*)\oplus(\mathbf{V}_r)_*.$$

We also have the identity

$$\mathbb{S}_{\boldsymbol{\lambda}}(V\oplus W) = \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}}\mathbb{S}_{\boldsymbol{\mu}}(V)\otimes\mathbb{S}_{\boldsymbol{\nu}}(W), \tag{3.7}$$

which holds for all vector spaces V and W. These imply

$$\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma},\boldsymbol{\gamma}'} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}'} \mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\gamma}}(\mathbf{V}_r)\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*))\otimes\mathbb{S}_{\boldsymbol{\gamma}'}((\mathbf{V}_r)_*).$$

By definition

$$\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*) = \mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*),\mathbb{C}),$$

and it follows from (3.2) that

$$\dim\mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\gamma}}(\mathbf{V}_r)\otimes\mathbb{S}_{\boldsymbol{\gamma}'}((\mathbf{V}_r)_*),\mathbb{C}) = \delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}'},$$

$\delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}$ being Kronecker's delta. Therefore,

$$\mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*),\mathbb{C}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*.$$

$\square$

Lemma 6. *If $0\to A\to B\to C\to 0$ is an exact sequence of modules in $\widetilde{Tens}_{\mathfrak{g}}$, then the dual exact sequence $0\to C^*\to B^*\to A^*\to 0$ splits.*

Proof. This follows from the fact that C^* is injective in $\widetilde{Tens}_{\mathfrak{g}}$. $\square$

Lemma 7. *The functor $\mathrm{R}:\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}\to\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$ sends an indecomposable injective object to an injective object.*

Proof. Let $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$. Then by Lemma 4 we have

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \Gamma_{\mathfrak{g}',\mathfrak{k}'}(\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)),$$

and hence by Lemma 5

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*). \tag{3.8}$$

Therefore, $\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$ is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$. Every indecomposable injective object in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ is isomorphic to $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*)$ for some simple object $\mathbf{L} = \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, and by Lemma 6, $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*)$ is a direct summand of $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$. Since the functor R is left exact, $\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*))$ is a direct summand of $\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$. Hence, $\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*))$ is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$. □

Lemma 8. *Let $\mathbf{V} = V_n \oplus \mathbf{W}$ and $\mathbf{V}_* = V_n^* \oplus \mathbf{W}_*$ be decompositions with $\dim V_n = n$, $\mathbf{W}^\perp = V_n^*$ and $\mathbf{W}_*^\perp = V_n$. Let $\mathfrak{s}$ be the commutator subalgebra of $\mathbf{W} \otimes \mathbf{W}_*$. Let $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$ be a module such that all its simple constituents are of the form $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$. Then the length of $\mathbf{M}^{\mathfrak{s}}$ in the category of $\mathfrak{sl}(n)$-modules equals the length of $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. It follows from (3.7) and the fact that $\mathbb{S}_{\boldsymbol{\lambda}}(V_n)$ and $\mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$ are nonzero (since $\dim V_n \geq |\boldsymbol{\lambda}|, |\boldsymbol{\mu}|$) that

$$(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^{\mathfrak{s}} = \mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*).$$

The description of the layers of the socle filtration of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ in (3.2) shows that the length of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ equals the length of $\mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$. Furthermore, since the socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ coincides with the set of vectors annihilated by all contraction maps (see (3.3)), and the set of vectors in $\mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$ annihilated by all contraction maps is the simple $\mathfrak{sl}(n)$-module $V_n^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, we obtain $(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^{\mathfrak{s}} = V_n^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. It then follows from left exactness that the functor $(\cdot)^{\mathfrak{s}}$ does not increase the length.

Let $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$, and let $k(\mathbf{M})$ be the maximum of $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ over all simple constituents $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbf{M}$. We proceed by proving the statement by induction on $k(\mathbf{M})$ with the obvious base case $k(\mathbf{M}) = 0$. Consider an exact sequence

$$0 \to \mathbf{M} \to \mathbf{I} \to \mathbf{N} \to 0,$$

where $\mathbf{I}$ is an injective hull of $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g}}$. From the description of the socle filtration of an injective module in $\mathbb{T}_{\mathfrak{g}}$ (see (3.2)), we have $k(\mathbf{N}) < k(\mathbf{M})$. Therefore, the length $l(\mathbf{N})$ of $\mathbf{N}$ equals the length $l(\mathbf{N}^{\mathfrak{s}})$ of $\mathbf{N}^{\mathfrak{s}}$ by the induction assumption. On the other hand, since $\mathbf{I}$ is injective and hence isomorphic to a direct sum of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$, the length of $\mathbf{I}$ equals the length of $\mathbf{I}^{\mathfrak{s}}$. Now if $l(\mathbf{M}^{\mathfrak{s}}) < l(\mathbf{M})$, then

$$l(\mathbf{N}^{\mathfrak{s}}) \geq l(\mathbf{I}^{\mathfrak{s}}) - l(\mathbf{M}^{\mathfrak{s}}) > l(\mathbf{I}) - l(\mathbf{M}) = l(\mathbf{N}),$$

which is a contradiction. □

Corollary 9. *Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ as in Lemma 8, and let $\mathbf{M} \in \widetilde{\mathbb{T}}_{\mathfrak{g}}$ be a module such that all its simple constituents are of the form $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$. Then $\mathbf{M} = U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}}$.*

Proof. Since $\mathbf{M}$ is a direct limit of modules of finite length it suffices to prove the statement for $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$. This can be easily done by induction on the length of $\mathbf{M}$. Indeed, consider an exact sequence $0 \to \mathbf{N} \to \mathbf{M} \to \mathbf{L} \to 0$ with simple $\mathbf{L}$. Lemma 8 implies that $0 \to \mathbf{N}^{\mathfrak{s}} \to \mathbf{M}^{\mathfrak{s}} \to \mathbf{L}^{\mathfrak{s}} \to 0$ is also exact, because the functor $(\cdot)^{\mathfrak{s}}$ is left exact and $l(\mathbf{L}^{\mathfrak{s}}) = l(\mathbf{M}^{\mathfrak{s}}) - l(\mathbf{N}^{\mathfrak{s}})$. Now if $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} \neq \mathbf{M}$ then, since $U(\mathfrak{g})\mathbf{N}^{\mathfrak{s}} = \mathbf{N}$ by the induction assumption, we obtain $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} = \mathbf{N}$. This implies $\mathbf{M}^{\mathfrak{s}} = \mathbf{N}^{\mathfrak{s}}$, and hence $l(\mathbf{L}^{s}) = 0$, which contradicts Lemma 8. □

Lemma 10. *For any $\mathbf{M} \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we have $U(\mathfrak{g})\mathrm{R}(\mathbf{M}) = \mathbf{M}$.*

Proof. Recall the definition of $k(\mathbf{M})$ from the proof of Lemma 8, and recall the decomposition (3.4). Let $\mathbf{U}$ be a subspace of $\mathbf{V}$, and $\mathbf{U}_*$ be a subspace of $\mathbf{V}_*$ such that $\mathbf{V}_r \subset \mathbf{U}$ and $(\mathbf{V}_r)_* \subset \mathbf{U}_*$, each of codimension $k(\mathbf{M})$.

Denote by $\mathfrak{l} \subset \mathfrak{g}$ the commutator subalgebra of $\mathbf{U} \otimes \mathbf{U}_*$, and by $\mathrm{Res}_{\mathfrak{l}}$ the restriction functor from $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ to $\widetilde{\mathbb{T}}_{\mathfrak{l}}$. The identity (3.7) implies that $k(\mathrm{Res}_{\mathfrak{l}} \mathbf{M}) = k(\mathbf{M})$. By Corollary 9 with $\mathfrak{g} = \mathfrak{l}$ and $\mathfrak{s} = \mathfrak{k}_r$, we get $\mathbf{M} = U(\mathfrak{l})\mathrm{R}(\mathbf{M})$. The statement follows. $\square$

Lemma 11. *The functor* $\mathrm{R} : \mathbb{T}_{\mathfrak{g},\mathfrak{k}} \to \mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$ *is exact and sends a simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *to the corresponding simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$*, and hence induces an isomorphism between the Grothendieck groups of* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *and* $\mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$*.*

Proof. Since $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is in fact an object of $\mathbb{T}_{\mathfrak{g}}$, the statement about simple modules follows by the argument concerning contraction maps from the proof of Lemma 8.

Since R is left exact, we have the inequality

$$l(\mathrm{R}(\mathbf{M})) \leq l(\mathbf{M}). \tag{3.9}$$

Thus, to prove exactness of R it suffices to show that R preserves the length, i.e. $l(\mathbf{M}) = l(\mathrm{R}(\mathbf{M}))$. We prove this by induction on $l(\mathbf{M})$. Consider an exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathbf{N} \to \mathbf{M} \to \mathbf{L} \to 0,$$

such that $\mathbf{L}$ is simple. By the induction hypothesis we have $l(\mathrm{R}(\mathbf{N})) = l(\mathbf{N})$. If we assume that $l(\mathrm{R}(\mathbf{M})) < l(\mathbf{M})$, then $l(\mathrm{R}(\mathbf{M})) = l(\mathbf{N})$ and so $\mathrm{R}(\mathbf{N}) = \mathrm{R}(\mathbf{M})$. But then by Lemma 10, we have $\mathbf{N} = \mathbf{M}$, which is a contradiction. $\square$

Corollary 12. *For any* $\boldsymbol{\lambda}, \boldsymbol{\mu}$*, the module* $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$ *has finite length. Hence, the module* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} := \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ *has finite length and is an object of the category* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$*.*

Proof. It was proven in [DPS] that $\Gamma_{\mathfrak{g},\mathfrak{g}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$ has finite length in $\widetilde{\mathbb{T}}_{\mathfrak{g}}$ (see the proof of Proposition 4.5 in [DPS] and note that the functor $\Gamma_{\mathfrak{g},\mathfrak{g}}$ is denoted by $\mathcal{B}$ in [DPS]). Using (3.8), the first claim follows by induction on the number r of components in the decomposition of $\mathbf{V}$. For the second claim, observe that Lemma 6 implies $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is isomorphic to a direct summand of the module $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^*)$. $\square$

Lemma 13. *Let* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *denote an injective hull of the simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *in* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$*, and let* $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *denote an injective hull of* $\mathrm{R}(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$ *in* $\mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$*. Then*

$$\mathrm{R}(\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{J}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Proof. We have $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ and $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$. Let

$$\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^*), \quad \mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbb{S}_{\boldsymbol{\mu}}(\mathrm{R}(\mathbf{V})) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathrm{R}(\mathbf{V}_*))^*).$$

Then we have

$$\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{I}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}, \quad \mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{J}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}. \tag{3.10}$$

Indeed, using Lemma 6, we can deduce from (3.2) that

$$(\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^* = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'})^*,$$

and then by applying $\Gamma_{\mathfrak{g},\mathfrak{k}}$ to both sides we obtain (3.10).

By (3.8), we have

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\gamma} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\gamma} \mathbf{Q}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Let $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ denote the complexified Grothendieck group of the additive subcategory of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ generated by indecomposable injective modules. Then $\{[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ both form a basis for $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$. Let $A = (A^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\lambda}',\boldsymbol{\mu}'})$ be the change of basis matrix on $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ given by (3.10) which expresses $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in terms of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. The same matrix A expresses $\mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in terms of $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ by (3.10).

The functor R induces a linear operator from $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ to $\mathfrak{I}_{\mathfrak{g}',\mathfrak{k}'}$ which is represented by the matrix A with respect to both bases $\{[\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$. Hence, the matrix which represents R with respect to the bases $\{[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ is again A as $A = AA(A^{-1})$. □

Corollary 14. *The Jordan-Hölder multiplicities of the indecomposable injective modules* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *are given by*

$$[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}] = \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\gamma_1,\dots,\gamma_r} N^{\boldsymbol{\lambda}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\lambda}'} N^{\boldsymbol{\mu}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\mu}'}.$$

Proof. After applying the functor R to the module $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ $(r-1)$ times, we obtain a direct sum of injective modules in the category $\mathbb{T}_{\mathfrak{g}}$. The multiplicity of each indecomposable injective in this sum is thus determined by applying the matrix A^{r-1} to $[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]$. The Jordan-Hölder multiplicities of an indecomposable injective module in $\mathbb{T}_{\mathfrak{g}}$ are also given by the matrix A (see 3.2). Therefore,

$$[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}] = \sum (A^r)^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} [\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}].$$

□

3.4. **The socle filtration of indecomposable injective objects in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.** In this section, we describe the socle filtration of the injective objects $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.

We consider the restriction functor

$$\mathrm{Res}_{\mathfrak{k}} : \mathbb{T}_{\mathfrak{g},\mathfrak{k}} \to \mathbb{T}_{\mathfrak{k}},$$

where $\mathbb{T}_{\mathfrak{k}}$ denotes the category of integrable $\mathfrak{k}$-modules of finite length which satisfy the large annihilator condition for each $\mathfrak{k}_i$ (recall (3.4)). Note that simple objects of $\mathbb{T}_{\mathfrak{k}}$ are outer tensor products of simple objects of the categories $\mathbb{T}_{\mathfrak{k}_i}$ for each $\mathfrak{k}_i$, $i = 1, \dots, r$, (recall that $\mathfrak{k}_i \cong \mathfrak{sl}(\infty)$); we will use the notation

$$\mathbf{V}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} := \mathbf{V}_1^{\boldsymbol{\lambda}_1,\boldsymbol{\mu}_1} \boxtimes \cdots \boxtimes \mathbf{V}_r^{\boldsymbol{\lambda}_r,\boldsymbol{\mu}_r}.$$

Injective hulls of simple objects in $\mathbb{T}_{\mathfrak{k}}$ will be denoted by $\mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$, and they are also outer tensor products of injective $\mathfrak{k}_i$-modules:

$$\mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} := \left(\mathbb{S}_{\boldsymbol{\lambda}_1}(\mathbf{V}_1) \otimes \mathbb{S}_{\boldsymbol{\mu}_1}(\mathbf{V}_1)_*\right) \boxtimes \cdots \boxtimes \left(\mathbb{S}_{\boldsymbol{\lambda}_r}(\mathbf{V}_r) \otimes \mathbb{S}_{\boldsymbol{\mu}_r}(\mathbf{V}_r)_*\right).$$

Recall that for every object $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we denote by $k(\mathbf{M})$ the maximum of $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ for all simple constituents $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbf{M}$. Similarly for every object $\mathbf{X}$ in $\mathbb{T}_{\mathfrak{k}}$ we denote by $c(\mathbf{X})$ the maximum of $|\boldsymbol{\lambda}_1| + \cdots + |\boldsymbol{\lambda}_r| + |\boldsymbol{\mu}_1| + \cdots + |\boldsymbol{\mu}_r|$ for all simple constituents $\mathbf{V}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$ of $\mathbf{X}$. It follows from Corollary 14 that

$$k(\mathbf{M}) = k(\operatorname{soc}\mathbf{M}), \quad c(\mathbf{X}) = c(\operatorname{soc}\mathbf{X}). \tag{3.11}$$

The identities

$$k(\mathbf{M}\otimes\mathbf{N}) = k(\mathbf{M}) + k(\mathbf{N}), \quad c(\mathbf{X}\otimes\mathbf{Y}) = c(\mathbf{X}) + c(\mathbf{Y}). \tag{3.12}$$

follow easily from the Littlewood–Richardson rule, and we leave their proof to the reader.

Lemma 15. *The restriction functor* $\mathrm{Res}_{\mathfrak{k}}$ *maps the category* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *to the category* $\mathbb{T}_{\mathfrak{k}}$*, and it maps* $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ *to an injective module. Furthermore, we have the identity*

$$c(\mathrm{Res}_{\mathfrak{k}}\,\mathbf{M}) = k(\mathbf{M}).$$

Proof. After applying identity (3.7) r-times to $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$, we get

$$\mathrm{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \cong \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} \mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}.$$

This implies the first and the second assertions of the lemma. Identity (3.11) implies that it is sufficient to prove the last assertion for $\mathbf{M} = \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$. Hence, this assertion follows from the above computation. □

Conjecture 16. *Suppose* $\mathrm{Ext}^k_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) \neq 0$*. Then* $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'| = |\boldsymbol{\mu}|-|\boldsymbol{\mu}'| = k$*.*

Remark 17. For $\mathfrak{k}=\mathfrak{g}$, this was proven in [DPS]. Proving this conjecture would imply that the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is Koszul. We prove the case $k=1$.

Proposition 18. *Suppose* $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) \neq 0$*. Then* $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'| = |\boldsymbol{\mu}|-|\boldsymbol{\mu}'| = 1$*.*

Proof. Since $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is isomorphic to a simple constituent of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, we know by Corollary 14 that $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'| = |\boldsymbol{\mu}|-|\boldsymbol{\mu}'| = s \geq 1$. It remains to show that $s=1$. We will do this in two steps.

First, we show that $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \neq 0$ implies $s=1$. Consider a nonsplit short exact sequence in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$

$$0 \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*) \to \mathbf{M} \to \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \to 0. \tag{3.13}$$

Let $\varphi : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes\mathfrak{g} \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ be a cocyle which defines this extension. By Lemma 15, the module $\mathrm{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$ is injective in $\mathbb{T}_{\mathfrak{k}}$, and therefore the sequence (3.13) splits over $\mathfrak{k}$. Without loss of generality we may assume that $\varphi(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes\mathfrak{k}) = 0$. Then the cocycle condition implies that $\varphi : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes(\mathfrak{g}/\mathfrak{k}) \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ is a nonzero homomorphism of $\mathfrak{k}$-modules. Consequently, the image of φ contains a simple submodule in the socle of $\mathrm{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$. By Lemma 15, we have

$$\mathrm{soc}\,\mathrm{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) = \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}.$$

In particular,

$$c(V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}) = |\boldsymbol{\lambda}_1|+\dots+|\boldsymbol{\lambda}_r|+|\boldsymbol{\mu}_1|+\dots+|\boldsymbol{\mu}_r| = |\boldsymbol{\lambda}|+|\boldsymbol{\mu}|$$

for every simple submodule $V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$ of $\mathrm{soc}\,\mathrm{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$. Therefore,

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes(\mathfrak{g}/\mathfrak{k})) \geq |\boldsymbol{\lambda}|+|\boldsymbol{\mu}|,$$

and so (3.12) implies

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}) + c(\mathfrak{g}/\mathfrak{k}) \geq |\boldsymbol{\lambda}|+|\boldsymbol{\mu}|.$$

Since $\mathfrak{g}/\mathfrak{k} \cong \bigoplus_{i\neq j}(\mathbf{V}_i\otimes(\mathbf{V}_j)_*)$, we have

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}) = |\boldsymbol{\lambda}'|+|\boldsymbol{\mu}'|, \quad c(\mathfrak{g}/\mathfrak{k}) = 2,$$

and thus $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'|+|\boldsymbol{\mu}|-|\boldsymbol{\mu}'|=2s\leq 2$. This yields $s=1$.

Assume now to the contrary that $s\geq 2$. Set

$$\mathbf{X}=(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))/\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$$

and consider the long exact sequence of Ext

$$\cdots\to\mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{X})\to\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})\to\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))\to\dots.$$

Since $s\geq 2$, $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is not isomorphic to a submodule of $\operatorname{soc}\mathbf{X}$, so $\mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{X})=0$, and by the already considered case when $s=1$, we have

$$\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))=0.$$

Hence, $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})=0$, which is a contradiction. □

Corollary 19. *Suppose that $\mathbf{M}\in\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ has a simple socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and the multiplicity of $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ in $\overline{\operatorname{soc}}^k\mathbf{M}$ is nonzero. Then $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'|=|\boldsymbol{\mu}|-|\boldsymbol{\mu}'|=k$.*

Proof. This follows by induction on $|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|$. By Proposition 18, the module $\mathbf{M}/\operatorname{soc}\mathbf{M}$ embeds into a direct sum of injective indecomposable modules $\bigoplus\mathbf{I}^{\boldsymbol{\gamma},\boldsymbol{\nu}}$ with simple socles $\mathbf{V}^{\boldsymbol{\gamma},\boldsymbol{\nu}}$ satisfying $|\boldsymbol{\lambda}|-|\boldsymbol{\gamma}|=|\boldsymbol{\mu}|-|\boldsymbol{\nu}|=1$, and by induction each $\mathbf{I}^{\boldsymbol{\gamma},\boldsymbol{\nu}}$ satisfies our claim. If the multiplicity of $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is nonzero in $\overline{\operatorname{soc}}^k\mathbf{M}=\overline{\operatorname{soc}}^{k-1}(\mathbf{M}/\operatorname{soc}\mathbf{M})\subset\overline{\operatorname{soc}}^{k-1}(\bigoplus\mathbf{I}^{\boldsymbol{\gamma},\boldsymbol{\nu}})$, then $|\boldsymbol{\gamma}|-|\boldsymbol{\lambda}'|=|\boldsymbol{\nu}|-|\boldsymbol{\mu}'|=k-1$. The result follows. □

Finally, by combining Corollary 14 and Corollary 19 we obtain the following.

Theorem 20. *The layers of the socle filtration of an indecomposable injective $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ satisfy*

$$\overline{\operatorname{soc}}^k\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}\cong\bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\bigoplus_{|\gamma_1|+\cdots+|\gamma_r|=k}N^{\boldsymbol{\lambda}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\lambda}'}N^{\boldsymbol{\mu}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\mu}'}\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},$$

where r is the number of (infinite) blocks in $\mathfrak{k}$ (see (3.4)).

Example 21. Consider an injective hull of the adjoint representation of $\mathfrak{sl}(\infty)$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ in the case that $\mathfrak{k}$ has k (infinite) blocks. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ each consist of one box, and $\operatorname{soc}V^{\boldsymbol{\lambda},\boldsymbol{\mu}}=\mathfrak{sl}(\infty)$ and $\overline{\operatorname{soc}}^1V^{\boldsymbol{\lambda},\boldsymbol{\mu}}=\mathbb{C}^k$, the trivial representation of dimension k. The self-similarity effect mentioned in the introduction amounts here to the increase of the dimension of $\overline{\operatorname{soc}}^1$ by 1 when the number of blocks of $\mathfrak{k}$ increases by 1.

Remark 22. Let's observe that the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is another example of an ordered tensor category as defined in [CP1]. Indeed, the set I in the notation of [CP1] can be chosen as the set of pairs of Young diagrams $(\boldsymbol{\lambda},\boldsymbol{\mu})$, and then the object X_i for $i=(\boldsymbol{\lambda},\boldsymbol{\mu})$ equals $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

4. $\mathfrak{sl}(\infty)$-modules arising from category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$

For the remainder of this paper, we let $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2$ be the commutator subalgebra of the Lie algebra preserving a fixed decomposition $\mathbf{V}=\mathbf{V}_1\oplus\mathbf{V}_2$ such that both $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are isomorphic to $\mathfrak{sl}(\infty)$ ($r=2$ in (3.4)).

4.1. **Category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$.** Let $\mathcal{O}_{m|n}$ denote the category of $\mathbb{Z}_2$-graded modules over $\mathfrak{gl}(m|n)$ which when restricted to $\mathfrak{gl}(m|n)_{\bar{0}}$, belong to the BGG category $\mathcal{O}_{\mathfrak{gl}(m|n)_{\bar{0}}}$ [M, Section 8.2.3]. This category depends only on a choice of simple roots for the Lie algebra $\mathfrak{gl}(m|n)_{\bar{0}}$, and not for all of $\mathfrak{gl}(m|n)$. We denote by $\mathcal{O}^{\mathbb{Z}}_{m|n}$ the Serre subcategory of $\mathcal{O}_{m|n}$ consisting of modules with integral weights. Any simple object in $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is isomorphic to $L(\lambda)$ (the unique simple quotient of the Verma module $M(\lambda)$) for some $\lambda \in \Phi$, where Φ denotes the set of integral weights. Any object in the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ has finite length.

We denote by $\mathcal{F}^{\mathbb{Z}}_{m|n}$ the Serre subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ consisting of finite-dimensional modules. Let $\Pi : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m|n}$ be the parity reversing functor. We define the *reduced Grothendieck group* $K_{m|n}$ (respectively, $J_{m|n}$) to be the quotient of the Grothendieck group of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ (respectively, $\mathcal{F}^{\mathbb{Z}}_{m|n}$) by the relation $[\Pi M] = -[M]$. The elements $[L(\lambda)]$ with $\lambda \in \Phi$ (respectively, $\lambda \in \Phi^+$) form a basis for $K_{m|n}$ (respectively, $J_{m|n}$).

We introduce an action of $\mathfrak{sl}(\infty)$ on $\mathbf{K}_{m|n} := K_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ following Brundan [B]. Our starting point is to define the translation functors E_i and F_i on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$. Consider the invariant form $\mathrm{str}(XY)$ on $\mathfrak{gl}(m|n)$ and let X_j, Y_j be a pair of $\mathbb{Z}_2$-homogeneous dual bases of $\mathfrak{gl}(m|n)$ with respect to this form. Then for two $\mathfrak{gl}(m|n)$-modules V and W we define the operator

$$\Omega : V \otimes W \to V \otimes W,$$

$$\Omega(v \otimes w) := \sum_j (-1)^{p(X_j)(p(v)+1)} X_j v \otimes Y_j w,$$

where $p(X_j)$ denotes the parity of the $\mathbb{Z}_2$-homogeneous element X_j. It is easy to check that $\Omega \in \mathrm{End}_{\mathfrak{gl}(m|n)}(V \otimes W)$. Let U and U^* denote the natural and conatural $\mathfrak{gl}(m|n)$-modules. For every $M \in \mathcal{O}^{\mathbb{Z}}_{m|n}$ we let $\mathrm{E}_i(M)$ (respectively, $\mathrm{F}_i(M)$) be the generalized eigenspace of Ω in $M \otimes U^*$ (respectively, $M \otimes U$) with eigenvalue i. Then, as it follows from [BLW], the functor $\cdot \otimes U^*$ (respectively, $\cdot \otimes U$) decomposes into the direct sum of functors $\oplus_{i\in\mathbb{Z}} \mathrm{E}_i(\cdot)$ (respectively, $\oplus_{i\in\mathbb{Z}} \mathrm{F}_i(\cdot)$). Moreover, the functors E_i and F_i are mutually adjoint functors on $\mathcal{O}^{\mathbb{Z}}_{m|n}$. We will denote by e_i and f_i the linear operators which the functors E_i and F_i induce on $\mathbf{K}_{m|n}$.

If we identify e_i and f_i with the Chevalley generators $E_{i,i+1}$ and $F_{i+1,i}$ of $\mathfrak{sl}(\infty)$, then $\mathbf{K}_{m|n}$ inherits the natural structure of a $\mathfrak{sl}(\infty)$-module. This follows from [B, BLW]. Another proof can be obtained by using Theorem 3.11 of [CS] and (4.2) below. Weight spaces with respect to the diagonal subalgebra $\mathfrak{h} \subset \mathfrak{sl}(\infty)$ correspond to the complexified reduced Grothendieck groups of the blocks of $\mathcal{O}^{\mathbb{Z}}_{m|n}$.

Let $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$, and let $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n}$ denote the subspace generated by the classes $[M(\lambda)]$ of all Verma modules $M(\lambda)$ for $\lambda \in \Phi$. Let furthermore $\mathbf{\Lambda}_{m|n} \subset \mathbf{J}_{m|n}$ denote the subspace generated by the classes $[K(\lambda)]$ of all Kac modules $K(\lambda)$ for $\lambda \in \Phi^+$ (for the definition of a Kac module see for example [B]). Then $\mathbf{T}_{m|n}$ is an $\mathfrak{sl}(\infty)$-submodule isomorphic to $\mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$ and $\mathbf{\Lambda}_{m|n}$ is a submodule of $\mathbf{T}_{m|n}$ isomorphic to $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ [B]. To see this, let $\{v_i\}_{i\in\mathbb{Z}}$ and $\{w_i\}_{i\in\mathbb{Z}}$ be the standard dual bases in $\mathbf{V}$ and $\mathbf{V}_*$ (i.e. $\mathfrak{h}$-eigenbases in $\mathbf{V}$ and $\mathbf{V}_*$), and let $\bar{\lambda} := \lambda + (m-1, \ldots, 1, 0|0, -1, \ldots, 1-n)$,

$$m_\lambda := v_{\bar{\lambda}_1} \otimes \cdots \otimes v_{\bar{\lambda}_m} \otimes v^*_{-\bar{\lambda}_{m+1}} \otimes \cdots \otimes v^*_{-\bar{\lambda}_{m+n}}.$$

The map $[M(\lambda)] \mapsto m_\lambda$ establishes an isomorphism $\mathbf{T}_{m|n} \cong \mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$, and restricts to an isomorphism

$$\begin{aligned} \mathbf{\Lambda}_{m|n} &\cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_* \\ [K(\lambda)] &\mapsto k_\lambda := v_{\bar\lambda_1} \wedge \cdots \wedge v_{\bar\lambda_m} \otimes v^*_{-\bar\lambda_{m+1}} \wedge \cdots \wedge v^*_{-\bar\lambda_{m+n}}. \end{aligned}$$

Lemma 23. *The $\mathfrak{sl}(\infty)$-module $\mathbf{K}_{m|n}$ satisfies the large annihilator condition as a module over $\mathfrak{k}_1$ and $\mathfrak{k}_2$, that is, $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{K}_{m|n}) = \mathbf{K}_{m|n}$.*

Proof. Note that an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$ satisfies the large annihilator condition over $\mathfrak{k}_1$ and $\mathfrak{k}_2$ if and only if for each $x \in \mathbf{M}$, we have $e_i x = f_i x = 0$ for all but finitely many $i \in \mathbb{Z}$. Indeed, if $e_i x = f_i x = 0$ for all but finitely many $i \in \mathbb{Z}$, then the subalgebra generated by the e_i, f_i that annihilate x contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra. The other direction is also clear.

Since the classes of simple $\mathfrak{gl}(m|n)$-modules $[L(\lambda)]$ form a basis of $\mathbf{K}_{m|n}$, we just need to show that for each $L(\lambda)$ we have $\mathrm{E}_i(L(\lambda)) = \mathrm{F}_i(L(\lambda)) = 0$ for almost all $i \in \mathbb{Z}$. However, since $\mathbf{T}_{m|n}$ satisfies the large annihilator condition, we know that the analogous statement is true for $M(\lambda)$. Therefore, since $L(\lambda)$ is a quotient of $M(\lambda)$, the exactness of the functors E_i and F_i implies the desired statement for $L(\lambda)$. □

If we consider the Cartan involution σ of $\mathfrak{sl}(\infty)$, $\sigma(e_i) = -f_i$, $\sigma(f_i) = -e_i$, we obtain

$$\langle gx, y\rangle = -\langle x, \sigma(g)y\rangle \tag{4.1}$$

for all $g \in \mathfrak{sl}(\infty)$. If $\mathbf{X}$ is a $\mathfrak{sl}(\infty)$-module, we denote by $\mathbf{X}^\vee$ the twist of the algebraic dual $\mathbf{X}^*$ by σ. Note that $(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^\vee = \mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}}$. Hence, if $\mathbf{X}$ is a semisimple object of finite length in $\widetilde{Tens}_{\mathfrak{g}}$, then $\mathbf{X}^\vee$ is an injective hull of $\mathbf{X}$ in $\widetilde{Tens}_{\mathfrak{g}}$.

Let $\mathcal{P}_{m|n}$ denote the semisimple subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ which consists of projective $\mathfrak{gl}(m|n)$-modules, and let $P_{m|n}$ denote the reduced Grothendieck group of $\mathcal{P}_{m|n}$. The $\mathfrak{sl}(\infty)$-module $\mathbf{P}_{m,n} := P_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ is the socle of $\mathbf{T}_{m,n}$ [CS, Theorem 3.11]. Note that for any projective module $P \in \mathcal{P}_{m|n}$ the functor $\mathrm{Hom}_{\mathfrak{gl}(m|n)}(P, \cdot)$ on $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is exact, and for any module $M \in \mathcal{F}_{m|n}$ the functor $\mathrm{Hom}_{\mathfrak{gl}(m|n)}(\cdot, M)$ on $\mathcal{P}_{m|n}$ is exact. Moreover, we have the dual bases in $\mathbf{K}_{m|n}$ and $\mathbf{P}_{m|n}$ given by the classes of irreducible modules and indecomposable projective modules, respectively.

Consider the pairing $\mathbf{K}_{m|n} \times \mathbf{P}_{m|n} \to \mathbb{C}$ defined by

$$\langle [M], [P]\rangle := \dim \mathrm{Hom}_{\mathfrak{gl}(m|n)}(P, M).$$

Since the functors E_i and F_i are adjoint, we have

$$\langle e_i x, y\rangle = \langle x, f_i y\rangle$$

and

$$\langle f_i x, y\rangle = \langle x, e_i y\rangle,$$

for all $i \in \mathbb{Z}$, $x \in \mathbf{K}_{m|n}$, $y \in \mathbf{P}_{m|n}$. Thus, there is an embedding of $\mathfrak{sl}(\infty)$-modules

$$\Psi : \mathbf{K}_{m|n} \hookrightarrow \mathbf{P}^\vee_{m|n} \tag{4.2}$$

given by $[M] \mapsto \langle [M], \cdot\rangle$.

Theorem 24. *The $\mathfrak{sl}(\infty)$-module $\mathbf{K}_{m|n}$ is an injective hull in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of the semisimple module $\mathbf{P}_{m|n}$. Furthermore, there is an isomorphism*

$$\mathbf{K}_{m|n} \cong \bigoplus_{|\boldsymbol{\lambda}|=m, |\boldsymbol{\mu}|=n} \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \otimes (Y_{\boldsymbol{\lambda}} \otimes Y_{\boldsymbol{\mu}})$$

where $Y_{\boldsymbol{\lambda}}$, $Y_{\boldsymbol{\mu}}$ are irreducible modules over S_m and S_n respectively, and $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is an injective hull of the simple module $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. Consequently, the layers of the socle filtration of $\mathbf{K}_{m|n}$ are given by

$$\overline{\operatorname{soc}}^k \mathbf{K}_{m|n} \cong \bigoplus_{|\boldsymbol{\lambda}|=m, |\boldsymbol{\mu}|=n} (\overline{\operatorname{soc}}^k \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^{\oplus(\dim Y_{\boldsymbol{\lambda}} \dim Y_{\boldsymbol{\mu}})}$$

where

$$\overline{\operatorname{soc}}^k \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \bigoplus_{|\gamma_1|+|\gamma_2|=k} N^{\boldsymbol{\lambda}}_{\gamma_1,\gamma_2,\boldsymbol{\lambda}'} N^{\boldsymbol{\mu}}_{\gamma_1,\gamma_2,\boldsymbol{\mu}'} \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Proof. The module $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$ is an injective hull of the semisimple module $\mathbf{P}_{m|n}$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, so it suffices to show that the image of $\mathbf{K}_{m|n}$ under the embedding (4.2) equals $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$. The fact that $\Psi(\mathbf{K}_{m|n}) \subset \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$ follows from Lemma 23. Herein, we will identify $\mathbf{K}_{m|n}$ with its image $\Psi(\mathbf{K}_{m|n}) = \operatorname{span}\{\langle l_\lambda, \cdot\rangle \mid \lambda \in \Phi\}$, where $l_\lambda := [L(\lambda)]$.

Now $\operatorname{soc}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})) = \mathbf{P}_{m|n}$, since $\mathbf{P}_{m|n}$ is semisimple, and $\operatorname{soc}\mathbf{T}_{m|n} = \mathbf{P}_{m|n}$ by [CS, Theorem 3.11]. Therefore, since $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n} \subset \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$, we have $\operatorname{soc}\mathbf{K}_{m|n} = \mathbf{P}_{m|n}$.

We will show that $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$. To accomplish this, we use the existence of the dual bases $p_\lambda := [P(\lambda)] \in \mathbf{P}_{m|n}$ and $l_\lambda \in \mathbf{K}_{m|n}$, where $L(\lambda)$ denotes the irreducible $\mathfrak{gl}(m|n)$-module with highest weight $\lambda \in \Phi$ and $P(\lambda)$ is a projective cover of $L(\lambda)$.

Fix $\omega \in \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^{\vee}_{m|n})$. To prove that $\omega \in \mathbf{K}_{m|n} = \operatorname{span}\{\langle l_\lambda, \cdot\rangle \mid \lambda \in \Phi\}$, it suffices to show that $\omega(p_\lambda) = 0$ for almost all $\lambda \in \Phi$. For each $q, r \in \mathbb{Z}$, with $q < r$, we let $\mathfrak{g}_{q,r} := \mathfrak{g}^-_q \oplus \mathfrak{g}^+_r$, where $\mathfrak{g}^-_q$ is the subalgebra of $\mathfrak{g}$ generated by e_i, f_i for $i < q$ and $\mathfrak{g}^+_r$ is the subalgebra of $\mathfrak{g}$ generated by e_i, f_i for $i > r$. By the annihilator condition, ω is $\mathfrak{g}_{q,r}$-invariant for suitable q and r. Fix such q and r. Then since ω is $\mathfrak{g}_{q,r}$-invariant, it suffices to show that $p_\lambda \in \mathfrak{g}_{q,r}\mathbf{P}_{m|n}$ for almost all $\lambda \in \Phi$.

If $p_\lambda \in \mathbf{P}_{m|n} \cap (\mathfrak{g}_{q,r}\mathbf{T}_{m|n})$, then $p_\lambda \in \mathfrak{g}_{q,r}\mathbf{P}_{m|n}$. Indeed, for any $\mathfrak{g}_{q,r}$-module $\mathbf{M}$ we have

$$\mathfrak{g}_{q,r}\mathbf{M} = \bigcap_{\varphi \in \operatorname{Hom}_{\mathfrak{g}_{q,r}}(\mathbf{M},\mathbb{C})} \ker\ \varphi.$$

Now any $\mathfrak{g}_{q,r}$-module homomorphism $\varphi : \mathbf{P}_{m|n} \to \mathbb{C}$ lifts to a $\mathfrak{g}_{q,r}$-module homomorphism $\varphi : \mathbf{K}_{m|n} \to \mathbb{C}$, since the trivial module $\mathbb{C}$ is injective in the full subcategory of $\mathfrak{g}_{q,r}$-mod consisting of integrable finite-length $\mathfrak{g}_{q,r}$-modules satisfying the large annihilator condition [DPS]. Hence, the claim follows.

For each $\lambda \in \Phi$ we define $\operatorname{supp}(\bar{\lambda})$ to be the multiset $\{\bar{\lambda}_1, \ldots, \bar{\lambda}_m, -\bar{\lambda}_{m+1}, \ldots, -\bar{\lambda}_{m+n}\}$, where

$$\bar{\lambda} := \lambda + (m-1, \ldots, 1, 0|0, -1, \ldots, 1-n).$$

The set of $\lambda \in \Phi$ such that $\operatorname{supp}(\bar{\lambda}) \cap (\mathbb{Z}_{<(q-m-n)} \cup \mathbb{Z}_{>(r+m+n)}) = \emptyset$ is finite. Hence, to finish the proof of the theorem, it suffices to show the following.

Lemma 25. *If $\operatorname{supp}(\bar{\lambda}) \cap \mathbb{Z}_{<(q-m-n)} \neq \emptyset$, then $p_\lambda \in \mathfrak{g}^-_q\mathbf{T}_{m|n}$. Similarly, if $\operatorname{supp}(\bar{\lambda}) \cap \mathbb{Z}_{>(r+m+n)} \neq \emptyset$, then $p_\lambda \in \mathfrak{g}^+_r\mathbf{T}_{m|n}$.*

Proof. We will prove the first statement; the proof of the second statement is similar. We can write $p_\lambda = \sum_\nu c_\nu m_\nu$, where each $c_\nu \in \mathbb{Z}_{>0}$ and $m_\nu = [M(\nu)]$ is the class of the Verma module $M(\nu)$ over $\mathfrak{gl}(m|n)$ of highest weight $\nu \in \Phi$.

We claim that $\mathrm{supp}(\bar{\nu}) \cap \mathbb{Z}_{<q} \neq \emptyset$ for every m_ν which occurs in the decomposition of p_λ. Indeed, recall that $P(\lambda)$ is a direct summand in the induced module $\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} P^0(\lambda)$, where $P^0(\lambda)$ is a projective cover of the simple $\mathfrak{gl}(m|n)_{\bar{0}}$-module with highest weight λ. Now

$$[P^0(\lambda)] = \sum_{w\in\mathcal{W}} b_{w\cdot\lambda}[M^0(w\cdot\lambda)], \tag{4.3}$$

where $M^0(\mu)$ denotes the Verma module over $\mathfrak{gl}(m|n)_{\bar{0}}$ with highest weight μ, $\mathcal{W}$ denotes the Weyl group of $\mathfrak{gl}(m|n)_{\bar{0}}$ and $w\cdot\lambda$ denotes the $\rho_{\bar{0}}$-shifted action of $\mathcal{W}$. The isomorphism of $\mathfrak{gl}(m|n)$-modules

$$M(\mu) \cong \mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}\oplus\mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} M^0(\mu)$$

implies that

$$\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} M^0(\mu) \cong \mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}\oplus\mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} (M^0(\mu)\otimes U(\mathfrak{gl}(m|n)_1).$$

Therefore, $\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} M^0(\mu)$ admits a filtration by Verma modules $M(\mu+\gamma)$ where γ runs over the set of weights of $U(\mathfrak{gl}(m|n)_1)$. Since $\mathrm{supp}(\gamma) \subset \{-m-n,\dots,m+n\}$ for every γ, we have

$$|(\overline{\mu+\gamma})_i - \bar{\mu}_i| \leq m+n.$$

Combining this with (4.3) we obtain that for each $i \leq m+n$, $|\bar{\nu}_i - \bar{\lambda}_{w(i)}| < m+n$, for some $w \in \mathcal{W}$. The claim follows.

Following the notations of Lemma 47 from the appendix, we set

$$\mathbf{W}_1 = \mathrm{span}\{v_i, \mid i<q\}, \quad \mathbf{W}_2 = \mathrm{span}\{v_j, \mid j \geq q\}.$$

Then $\mathfrak{g}_q^- = \mathfrak{sl}(\mathbf{W}_1) = \mathfrak{s}$. By above, every m_ν occurring in the decomposition of p_λ is contained in $\mathbf{Y}_{m|n}$. Hence $p_\lambda \in \mathbf{Y}_{m|n}$. Since we also have $p_\lambda \in \mathrm{soc}\,\mathbf{T}_{m|n}$, Lemma 47 implies that $p_\lambda \in \mathfrak{g}_q^-\mathbf{T}_{m|n}$. □

Hence, $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}_{m|n}^\vee)$, and the description of the socle filtration now follows from Theorem 20. □

4.2. **The symmetric group action on $\mathbf{K}_{m|n}$.** Recall that we have a natural action of the product of symmetric groups $S_m \times S_n$ on $\mathbf{T}_{m|n}$, which commutes with the $\mathfrak{sl}(\infty)$-module structure on $\mathbf{T}_{m|n}$. Moreover, it follows from [DPS, Sect. 6] that

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{T}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n]. \tag{4.4}$$

A similar result is true for $\mathbf{K}_{m|n}$:

Proposition 26.

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n].$$

Proof. Recall that $\mathbf{P}_{m|n}$ is the socle of $\mathbf{K}_{m|n}$ by Theorem 24. Every $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$ maps the socle to the socle, hence we have a homomorphism

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) \to \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}). \tag{4.5}$$

Let $\mathbf{K}'_{m|n} = \mathbf{K}_{m|n}/\mathbf{P}_{m|n}$. By Theorem 20, for every simple module $\mathbf{V}^{\lambda,\mu}$ we have

$$[\mathbf{K}'_{m|n} : \mathbf{V}^{\lambda,\mu}][\mathbf{P}_{m|n} : \mathbf{V}^{\lambda,\mu}] = 0.$$

Therefore, every $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$ such that $\varphi(\mathbf{P}_{m|n}) = 0$ is identically zero, since for such φ the socle of $\operatorname{im}\varphi$ is zero. In other words, homomorphism (4.5) is injective. The surjectivity follows from the fact that every $\varphi : \mathbf{P}_{m|n} \to \mathbf{P}_{m|n} \hookrightarrow \mathbf{K}_{m|n}$ extends to $\tilde{\varphi} : \mathbf{K}_{m|n} \to \mathbf{K}_{m|n}$ by the injectivity of $\mathbf{K}_{m|n}$. □

4.3. **The Zuckerman functor $\Gamma_{\mathfrak{gl}(m|n)}$ and the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$.** Let us recall the definition of the derived Zuckerman functor. A systematic treatment of the Zuckerman functor for Lie superalgebras can be found in [S]. Assume that M is a finitely generated $\mathfrak{gl}(m|n)$-module which is semisimple over the Cartan subalgebra of $\mathfrak{gl}(m|n)$. Let $\Gamma_{\mathfrak{gl}(m|n)}(M)$ denote the subspace of $\mathfrak{gl}(m|n)_0$-finite vectors. Then $\Gamma_{\mathfrak{gl}(m|n)}(M)$ is a finite-dimensional $\mathfrak{gl}(m|n)$-module, and hence $\Gamma_{\mathfrak{gl}(m|n)}$ is a left exact functor from the category of finitely generated $\mathfrak{gl}(m|n)$-modules, semisimple over the Cartan subalgebra, to the category $\mathcal{F}_{m|n}$ of finite-dimensional modules. The corresponding right derived functor $\Gamma^i_{\mathfrak{gl}(m|n)}$ is called the *i-th derived Zuckerman functor*. Note that $\Gamma^i_{\mathfrak{gl}(m|n)}(X) = 0$ for $i > \dim \mathfrak{gl}(m|n)_0 - (m+n)$. We are interested in the restriction of this functor

$$\Gamma^i_{\mathfrak{gl}(m|n)} : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{F}^{\mathbb{Z}}_{m|n}.$$

Let us consider the linear operator $\gamma : \mathbf{K}_{m|n} \to \mathbf{J}_{m|n}$ given by

$$\gamma([M]) = \sum_i (-1)^i [\Gamma^i_{\mathfrak{gl}(m|n)} M].$$

This operator is well defined as for any short exact sequence of $\mathfrak{gl}(m|n)$-modules

$$0 \to N \to M \to L \to 0,$$

we have the Euler characteristic identity

$$\gamma([M]) = \gamma([N]) + \gamma([L]).$$

It is well known that $\Gamma^i_{\mathfrak{gl}(m|n)}$ commutes with the functors $\cdot \otimes U$ and $\cdot \otimes U^*$, and with the projection to the block $(\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi$ with a fixed central character χ. Therefore, γ is a homomorphism of $\mathfrak{sl}(\infty)$-modules.

Proposition 27. *The homomorphism γ is given by the formula*

$$\gamma = \sum_{s \in S_m \times S_n} \operatorname{sgn}(s) s, \tag{4.6}$$

where the action of s on $\mathbf{K}_{m|n}$ is defined in Proposition 26.

Proof. By Proposition 26, it suffices to check the equality (4.6) on vectors in $\mathbf{T}_{m|n}$, which amounts to checking that for all Verma modules $M(\lambda)$

$$\gamma([M(\lambda)]) = \sum_{s \in S_m \times S_n} \operatorname{sgn}(s)[M(s \cdot \lambda)], \tag{4.7}$$

where $s \cdot \lambda = s(\lambda + \rho) - \rho$ and $\rho = (m-1, \dots, 0|0, -1, \dots, 1-n)$.

Consider the functor Res_0 of restriction to $\mathfrak{gl}(m|n)_0$. This is an exact functor from the category of finitely generated $\mathfrak{gl}(m|n)$-modules, semisimple over the Cartan subalgebra, to the similar category of $\mathfrak{gl}(m|n)_0$-modules. It is clear from the definition of $\Gamma^i_{\mathfrak{gl}(m|n)}$ that

$$\mathrm{Res}_0 \,\Gamma^i_{\mathfrak{gl}(m|n)} = \Gamma^i_{\mathfrak{gl}(m|n)_0} \,\mathrm{Res}_0\,. \tag{4.8}$$

Recall that every Verma module $M(\lambda)$ over $\mathfrak{gl}(m|n)$ has a finite filtration with successive quotients isomorphic to Verma modules $M^0(\mu)$ over $\mathfrak{gl}(m|n)_0$. Hence by (4.8) it suffices to check the analogue of (4.7) for even Verma modules:

$$\gamma^0([M^0(\lambda)]) = \sum_{s\in S_m\times S_n} \mathrm{sgn}(s)[M^0(s\cdot\lambda)], \tag{4.9}$$

where γ^0 is the obvious analogue of γ. To prove (4.9) we observe that $[M^0(\lambda)] = [M^0(\lambda)^\vee]$ where $X^\vee$ stands for the contragredient dual of X.

It is easy to compute $\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee$. Let $\mathfrak{t}$ denote the Cartan subalgebra of $\mathfrak{gl}(m|n)$, and let $\mathfrak{n}_0^+$, $\mathfrak{n}_0^-$ be the maximal nilpotent ideals of the Borel and opposite Borel subalgebras of $\mathfrak{gl}(m|n)_0$, respectively. From the definition of the derived Zuckerman functor, the following holds for any $\mu\in\Phi^+$

$$\mathrm{Hom}_{\mathfrak{gl}(m|n)_0}(L^0(\mu), \Gamma^i_{\mathfrak{gl}(m|n)_0} M) \simeq \mathrm{Ext}^i(L^0(\mu), M),$$

where the extension is taken in the category of modules semisimple over $\mathfrak{t}$. If $M = M^0(\lambda)^\vee$, then M is cofree over $U(\mathfrak{n}_0^+)$ and therefore

$$\mathrm{Ext}^i(L^0(\mu), M^0(\lambda)^\vee) \simeq \mathrm{Hom}_{\mathfrak{t}}(H_i(\mathfrak{n}_0^-, L^0(\mu)), \mathbb{C}_\lambda).$$

Now we apply Kostant's theorem to conclude that

$$\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee = \begin{cases} L^0(\mu) & \text{if } \mu = s\cdot\lambda \text{ for } s\in S_m\times S_n,\ l(s)=i,\\ 0 & \text{otherwise.}\end{cases}$$

Here μ is the only dominant weight in $(S_m\times S_n)\cdot\lambda$ and hence s is unique. Moreover, if $\lambda+\rho$ is a singular weight then $\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee = 0$ for all i. Combining this with the Weyl character formula

$$[L^0(\mu)] = \sum_{s\in S_m\times S_n} \mathrm{sgn}(s)[M^0(s\cdot\mu)]$$

we obtain (4.9), and hence the proposition. $\square$

Corollary 28. *We have $\mathbf{J}_{m|n} = \gamma(\mathbf{K}_{m|n})$ and $\mathbf{K}_{m|n} = \mathbf{J}_{m|n}\oplus\ker\gamma$. In particular, $\mathbf{J}_{m|n}$ is an injective hull of $\mathbf{\Lambda}_{m|n} \cong \Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$.*

Recall that $\mathbf{\Lambda}_{m|n}\subset\mathbf{J}_{m|n}$ denotes the subspace generated by the classes of all Kac modules. Let $\mathcal{Q}_{m|n}$ denote the additive subcategory of $\mathcal{F}^{\mathbb{Z}}_{m|n}$ which consists of projective finite-dimensional $\mathfrak{gl}(m|n)$-modules, and let $Q_{m|n}$ denote the reduced Grothendieck group of $\mathcal{Q}_{m|n}$. It was proven in [CS, Theorem 3.11] that $\mathbf{Q}_{m|n} := Q_{m|n}\otimes_{\mathbb{Z}}\mathbb{C}$ is the socle of the module $\mathbf{\Lambda}_{m|n}$, implying that $\mathbf{Q}_{m|n}\cong\mathbf{V}^{(m)^\perp,(n)^\perp}$, where $\perp$ indicates the conjugate partition. Corollary 28 implies the following.

Corollary 29. *$\mathbf{J}_{m|n}$ is an injective hull of $\mathbf{Q}_{m|n}$, and the socle filtration of $\mathbf{J}_{m|n}$ is*

$$\overline{\mathrm{soc}}^i\mathbf{J}_{m|n} \cong \left(\mathbf{V}^{(m-i)^\perp(n-i)^\perp}\right)^{\oplus(i+1)}.$$

4.4. **The Duflo–Serganova functor and the tensor filtration.** In this section, we discuss the relationship between the Duflo–Serganova functor and submodules of the $\mathfrak{sl}(\infty)$-modules $\mathbf{K}_{m|n}$ and $\mathbf{J}_{m|n}$.

Let $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be a finite-dimensional contragredient Lie superalgebra. For any odd element $x \in \mathfrak{a}_{\bar{1}}$ which satisfies $[x, x] = 0$, the *Duflo–Serganova functor* DS_x is defined by

$$\begin{aligned} DS_x :\ \mathfrak{a} - \mathrm{mod} &\to \mathfrak{a}_x - \mathrm{mod} \\ M &\mapsto \ker_M x/xM, \end{aligned}$$

where $\ker_M x/xM$ is a module over the Lie superalgebra $\mathfrak{a}_x := \mathfrak{a}^x/[x, \mathfrak{a}]$ (here $\mathfrak{a}^x$ denotes the centralizer of x in $\mathfrak{a}$) [DS]. In what follows we set

$$M_x := DS_x(M).$$

The Duflo–Serganova functor DS_x is a symmetric monoidal functor, [DS], see also Proposition 5 in [Ser].

It is known that the functor DS is not exact, nevertheless it induces a homomorphism ds_x between the reduced Grothendieck groups of the categories $\mathfrak{a}$-mod and $\mathfrak{a}_x$-mod defined by $ds_x([M]) = [M_x]$. (Recall that "reduced" indicates passage to the quotient by the relation $[\Pi M] = -[M]$, where Π is the parity reversing functor.) This follows from the following statement, see Section 1.1 in [GS].

Lemma 30. *For every exact sequence of $\mathfrak{a}$-modules*

$$0 \to M_1 \xrightarrow{\psi} M_2 \xrightarrow{\varphi} M_3 \to 0$$

there exists an exact sequence of $\mathfrak{a}_x$-modules

$$0 \to E \to DS_x(M_1) \xrightarrow{DS_x(\psi)} DS_x(M_2) \xrightarrow{DS_x(\varphi)} DS_x(M_3) \to \Pi E \to 0,$$

for an appropriate $\mathfrak{a}_x$-module E.

Proof. Set $E := \mathrm{Ker}(DS_x(\psi))$, $E' := \mathrm{Coker}(DS_x(\varphi))$, and consider the exact sequence

$$0 \to E \to DS_x(M_1) \to DS_x(M_2) \to DS_x(M_3) \to E' \to 0.$$

The odd morphism $\psi^{-1}x\varphi^{-1} : DS_x(M_3) \to DS_x(M_1)$ induces an isomorphism $E' \to \Pi E$. $\square$

In [HR] the existence of the homomorphism ds_x was proven for finite-dimensional modules.

Remark 31. If $0 \to C_1 \to \cdots \to C_k \to 0$ is a complex of $\mathfrak{a}$-modules with odd differentials, the Euler characteristic of this complex is defined as the element $\sum_{i=1}^{k}[C_i]$ in the reduced Grothendieck group. If H_i denotes the i-th cohomology group, then

$$\sum_{i=1}^{k}[C_i] = \sum_{i=1}^{k}[H_i].$$

The absence of the usual sign follows from the relation $[\Pi M] = -[M]$ and the fact that the differentials are odd. For example, for an acyclic complex $0 \to X \to \Pi X \to 0$ the Euler characteristic is zero.

Let $\mathfrak{a} = \mathfrak{gl}(m|n)$ and suppose $\operatorname{rank} x = k$. Then $\mathfrak{a}_x \cong \mathfrak{gl}(m-k|n-k)$. Let $\mathcal{O}^{ind}_{m|n}$ be the category whose objects are direct limits of objects in $\mathcal{O}_{m|n}$. Then by Lemma 5.2 in [CS] the restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor

$$DS_x : \ \mathcal{O}_{m|n} \to \mathcal{O}^{ind}_{m-k|n-k}.$$

Lemma 32. *The functor $DS_x : \ \mathcal{O}^{\mathbb{Z}}_{m|n} \to (\mathcal{O}^{\mathbb{Z}}_{m-k|n-k})^{ind}$ commutes with translation functors.*

Proof. Recall that U is the natural $\mathfrak{gl}\,(m|n)$-module. Since DS is a monoidal functor, we have a canonical isomorphism

$$(M \otimes U)_x \simeq M_x \otimes U_x.$$

Moreover, a direct computation shows that U_x is isomorphic to the natural $\mathfrak{gl}(m-k|n-k)$-module. We will use these observations to show that there is a canonical isomorphism

$$\mathrm{E}_i(M_x) \simeq (\mathrm{E}_i(M))_x. \tag{4.10}$$

Recall the notations of Section 3.1. Define the homomorphism of $\mathfrak{gl}(m|n)$-modules

$$\omega_{m|n} : \mathbb{C} \to \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n), \quad 1 \mapsto \sum (-1)^{p(X_j)} X_j \otimes Y_j.$$

We have $DS_x(\omega_{m|n}) = \omega_{m-k|n-k}$. Consider the composition

$$\Omega : M \otimes U \xrightarrow{1\otimes\omega_{m|n}\otimes 1} M \otimes \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n) \otimes U \xrightarrow{r_M \otimes l_U} M \otimes U,$$

where $r_M : M \otimes \mathfrak{gl}(m|n) \to M$ is the morphism of right action, and $l_U : \mathfrak{gl}(m|n) \otimes U \to U$ is the morphism of left action. The morphism $DS_x(\Omega) : M_x \otimes U_x \to M_x \otimes U_x$ is defined in a similar manner in the category of $\mathfrak{gl}(m-k|n-k)$-modules. Recall that

$$\mathrm{E}_i(M) = \{v \in M \otimes U \mid (\Omega - i)^N v = 0 \quad \text{for some } N > 0\};$$

similarly

$$\mathrm{E}_{\mathrm{i}}(M_x) = \{v \in M_x \otimes U_x \mid (DS_x(\Omega) - i)^N v = 0 \quad \text{for some } N > 0\}.$$

This implies the existence of the isomorphism (4.10) as desired.

The proof for F_i is similar. ◻

We are going to strengthen the result of [CS] by proving the following proposition.

Proposition 33. *The restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor*

$$DS_x : \ \mathcal{O}_{m|n} \to \mathcal{O}_{m-k|n-k}.$$

To prove the proposition we first consider the case when $k = 1$.

Lemma 34. *If $k = 1$, then the restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor*

$$DS_x : \ \mathcal{O}_{m|n} \to \mathcal{O}_{m-1|n-1}.$$

Proof. By Theorem 5.1 in [CS] we may assume without loss of generality that x is a generator of the root space $\mathfrak{gl}(m|n)_\alpha$ for some $\alpha = \pm(\varepsilon_i - \delta_j)$. Moreover, we can choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{gl}(m|n)$ so that α is a simple root. Let M be an object in the category $\mathcal{O}_{m|n}$ and M^μ denote the weight space of weight μ. The set of all weights of M is denoted by $\operatorname{supp} M$. Let $x_\mu : M^\mu \to M^{\mu+\alpha}$ be the restriction of x as an operator on M. Then

$$M_x = \oplus_{\mu \in \operatorname{supp} M} M^\mu_x \quad \text{where} \quad M^\mu_x = \ker x_\mu / x_{\mu-\alpha}(M^{\mu-\alpha}).$$

Let us first check that all weight multiplicities of M_x are finite with respect to the Cartan subalgebra $\mathfrak{h}_x := \ker \varepsilon_i \cap \ker \delta_j$ of $\mathfrak{g}_x$. We have to show that for any $\nu \in \mathfrak{h}_x^*$

$$\sum_{\mu \in \operatorname{supp} M, \mu|_{\mathfrak{h}_x} = \nu} \dim M_x^\mu < \infty. \tag{4.11}$$

Note that $\dim M_x^\mu \neq 0$ implies $(\mu, \alpha) = 0$, by $\mathfrak{sl}(1|1)$-representation theory. If $(\mu', \alpha') = 0$ and $\mu|_{\mathfrak{h}_x} = \mu'|_{\mathfrak{h}_x}$, then $\mu - \mu' \in \mathbb{C}\alpha$. Denote by Δ_s the set of simple roots of $\mathfrak{b}$. Since M is an object of $\mathcal{O}_{m|n}$, M has a finite filtration by highest weight modules. Therefore it suffices to consider the case when M is a highest weight module. Let λ be the highest weight of M. Then every $\mu \in \operatorname{supp} M$ has the form $\lambda - \sum_{\beta \in \Delta_s} k_\beta \beta$ for some $k_\beta \in \mathbb{Z}_{\geq 0}$ satisfying $k_\alpha \leq 1 + \sum_{\beta \in \Delta_S \setminus \alpha} k_\beta$. Therefore, for any $\mu \in \operatorname{supp} M$ the set $(\mu + \mathbb{C}\alpha) \cap \operatorname{supp} M$ is finite. Hence, for any $\nu \in \mathfrak{h}_x^*$ the set of $\mu \in \operatorname{supp} M$ such that $\mu|_{\mathfrak{h}_x} = \nu$ and $(\mu, \alpha) = 0$ is finite. Since all weight spaces of M are finite dimensional, this implies (4.11).

To finish the proof we observe that Lemma 32 implies $\mathrm{E}_i(M_x) = \mathrm{F}_i(M_x) = 0$ for almost all $i \in \mathbb{Z}$. Now for each $i \in \operatorname{supp}(\bar{\lambda})$, at least one of the $\mathrm{E}_i, \mathrm{E}_{i+1}, \mathrm{F}_i, \mathrm{F}_{i+1}$ does not annihilate $L_{\mathfrak{g}_x}(\lambda)$. Together this implies that the set S_M of all weights λ satisfying $[M_x : L_{\mathfrak{g}_x}(\lambda)] \neq 0$ is a finite set. On the other hand, since M_x has finite weight multiplicities, every simple constituent occurs in M_x with finite multiplicity. Hence M_x has finite length. □

Proof. Now we prove Proposition 33 by induction on $\operatorname{rank}(x) = k$. By Theorem 5.1 in [CS], x is B_0-conjugate to $x_1 + \cdots + x_k$, where $x_i \in \mathfrak{gl}(m|n)_{\alpha_i}$ for some linearly independent set of mutually orthogonal odd roots $\beta_1, \ldots, \beta_k$. So without loss of generality we may suppose that $x = x_1 + \cdots + x_k$. Let $y = x_1 + \cdots + x_{k-1}$. Choose $h_y \in \mathfrak{h}_{x_k}$ and $h_{x_k} \in \mathfrak{h}_y$ such that $\alpha(h_y), \alpha(h_{x_k}) \in \mathbb{Z}$ for all roots α of $\mathfrak{gl}(m|n)$, $[h_y, y] = y$ and $[h_{x_k}, x_k] = x_k$. Assume that $M \in \mathcal{O}_{m|n}$ and $\operatorname{supp} M \in \lambda + Q$, where Q is the root lattice. Then $\operatorname{ad} h_y - \lambda(h_y)$ and $\operatorname{ad} h_{x_k} - \lambda(h_{x_k})$ define a $\mathbb{Z} \times \mathbb{Z}$-grading on M and the differentials y and x_k form a bicomplex. Moreover, M_x is nothing but the cohomology $\bigoplus_r H^r(y + x_k, M)$ of the total complex.

Consider the second term

$$E_2^{p,q}(M) = H^p(x_k, H^q(y, M))$$

of the spectral sequence of this bicomplex. By the induction assumption $M_y \in \mathcal{O}_{m-k+1|n-k+1}$, and in particular, $H^q(y, M) \neq 0$ for finitely many q. The induction assumption implies that $H^p(x_k, H^q(y, M)) \in \mathcal{O}_{m-k|n-k}$ does not vanish for finitely many p. This yields $\bigoplus_{p,q} E_2^{p,q}(M) \in \mathcal{O}_{m-k|n-k}$. Since $\bigoplus_r H^r(y + x_k, M)$ is a subquotient of $\bigoplus_{p,q} E_2^{p,q}(M)$, we obtain

$$M_x = \bigoplus_r H^r(y + x_k, M) \in \mathcal{O}_{m-k|n-k}.$$

□

Next note that the restriction of DS_x to $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is a well-defined functor

$$\mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m-k|n-k}.$$

Since DS_x is a well-defined functor from $\mathcal{O}^{\mathbb{Z}}_{m|n}$ to $\mathcal{O}^{\mathbb{Z}}_{m-k|n-k}$ we see that $ds_x : K_{m|n} \to K_{m-k|n-k}$ is a well-defined group homomorphism.

Lemma 35. *If $x = x_1 + \cdots + x_k$ with commuting $x_1, \ldots, x_k$ of rank 1, then on $K_{m|n}$ we have the identity*

$$ds_x = ds_{x_k} \circ \cdots \circ ds_{x_1}.$$

Proof. We retain the notation of the proof of Proposition 33. Clearly, it suffices to check that

$$ds_x = ds_{x_k} \circ ds_y,$$

where $y = x_1 + \cdots + x_{k-1}$. The Euler characteristic of the E_s-terms of the spectral sequence from the proof of Proposition 33 remains unchanged for $s \geq 2$:

$$[\bigoplus_{p,q} E_2^{p,q}(M)] = [\bigoplus_{p,q} E_s^{p,q}(M)].$$

As the spectral sequence converges to $[M_x]$, we obtain

$$ds_{x_k} \circ ds_y([M]) = [\bigoplus_{p,q} E_2^{p,q}(M)] = [M_x] = ds_x([M]).$$

□

For the category of finite-dimensional modules the above statement is proven in [HR].

Proposition 36. *The complexification $ds_x : \mathbf{K}_{m|n} \to \mathbf{K}_{m-k|n-k}$ is a homomorphism of $\mathfrak{sl}(\infty)$-modules, as is its restriction $ds_x : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ to the $\mathfrak{sl}(\infty)$-submodule $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$.*

Proof. This follows from the fact that the Duflo–Serganova functor commutes with translation functors, see Lemma 32. □

Remark 37. Note that in [HR] the ring $J_{m|n}$ is denoted by $\mathcal{J}_G$ where $G = GL(m|n)$.

Let $X_{\mathfrak{a}} = \{x \in \mathfrak{a}_{\bar{1}} : [x, x] = 0\}$, and let

$$\mathcal{B}_{\mathfrak{a}} = \{B \subset \Delta_{iso} \mid B = \{\beta_1, \ldots, \beta_k \mid (\beta_i, \beta_j) = 0,\ \beta_i \neq \pm\beta_j\}\} \tag{4.12}$$

be the set of subsets of linearly independent mutually orthogonal isotropic roots of $\mathfrak{a}$. Then the orbits of the action of the adjoint group $G_{\bar{0}}$ of $\mathfrak{a}_{\bar{0}}$ on $X_{\mathfrak{a}}$ are in one-to-one correspondence with the orbits of the Weyl group $\mathcal{W}$ of $\mathfrak{a}_{\bar{0}}$ on $\mathcal{B}_{\mathfrak{a}}$ via the correspondence

$$B = \{\beta_1, ..., \beta_k\} \mapsto x = x_{\beta_1} + \cdots + x_{\beta_k} \in X_{\mathfrak{a}}, \tag{4.13}$$

where each $x_{\beta_i} \in \mathfrak{a}_{\beta_i}$ is chosen to be nonzero [DS, Theorem 4.2].

Lemma 38. *Let $\mathfrak{a} = \mathfrak{gl}(m|n)$. Fix $x \in X_{\mathfrak{a}}$ and set $k = |B_x|$, where $B_x \in \mathcal{B}_{\mathfrak{a}}$ corresponds to x. The homomorphism $ds_x : J_{m|n} \to J_{m-k|n-k}$ depends only k, and not on x.*

Proof. This follows from the description of ds_x given in [HR, Theorem 10], using the fact that supercharacters of finite-dimensional modules are invariant under the Weyl group $\mathcal{W} = S_m \times S_n$ of $\mathfrak{gl}(m|n)$. If $B_1, B_2 \in \mathcal{B}$ with $|B_1| = |B_2|$ then there exists $w \in \mathcal{W}$ satisfying: $\pm\beta \in w(B_1)$ if and only if $\pm\beta \in B_2$. So if $f \in J_{m|n}$ we have that

$$ds_{x_1}(f) = f|_{\beta_1^1,\ldots,\beta_k^1=0} = w(f)\,|_{w(\beta_1^1),\ldots,w(\beta_k^1)=0} = w(f)\,|_{\beta_1^2,\ldots,\beta_k^2=0} = f|_{\beta_1^2,\ldots,\beta_k^2=0} = ds_{x_2}(f).$$

□

Note that Lemma 38 does not hold if we replace $J_{m|n}$ with $K_{m|n}$.

Remark 39. Since the homomorphism $ds_x : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ does not depend on x, we denote it by ds^k, where $|B_x| = k$, and we let $ds := ds^1$.

Now we introduce a filtration of an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$, whose layers are tensor modules.

Definition 40. The *tensor filtration* of an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$ is defined inductively by

$$\operatorname{tens}^0 \mathbf{M} := \operatorname{tens} \mathbf{M} := \Gamma_{\mathfrak{g},\mathfrak{g}}(\mathbf{M}), \qquad \operatorname{tens}^i \mathbf{M} := p_i^{-1}(\operatorname{tens}(\mathbf{M}/(\operatorname{tens}^{i-1} \mathbf{M}))),$$

where $p_i : \mathbf{M} \to \mathbf{M}/(\operatorname{tens}^{i-1} \mathbf{M})$ is the natural projection.

We also use the notation $\overline{\operatorname{tens}}^i\mathbf{M} = \operatorname{tens}^i \mathbf{M}/\operatorname{tens}^{i-1} \mathbf{M}$.

Note that $\operatorname{tens} \mathbf{M}$ is the maximal tensor submodule of $\mathbf{M}$.

Example 41. The socle of $\mathbf{J}_{1|1}$ is isomorphic to the adjoint module of $\mathfrak{sl}(\infty)$, and $\overline{\operatorname{soc}}^1\mathbf{J}_{1|1} = \mathbb{C} \oplus \mathbb{C}$. Note that this is a special case of Example 21 in the case that $\mathfrak{k}$ has two infinite blocks.

Consider now the tensor filtration of $\mathbf{J}_{1|1}$. This filtration also has length 2, $\operatorname{tens} \mathbf{J}_{1|1} = \mathbf{\Lambda}_{1|1} \cong \mathbf{V} \otimes \mathbf{V}_*$ and $\overline{\operatorname{tens}}^1\mathbf{J}_{1|1} \cong \mathbb{C}$. The module $\mathbf{J}_{1|1}$ admits a nice matrix realization. Indeed, we can identify the $\mathfrak{sl}(\infty)$-module $\mathbf{\Lambda}_{1|1}$ with the matrix realization of $\mathfrak{gl}(\infty)$ (see Section 3.1), and then extend it by the diagonal matrix D which has entries $D_{ii} = 1$ for $i \geq 1$ and 0 elsewhere. The action of $\mathfrak{sl}(\infty)$ in this realization of $\mathbf{J}_{1|1}$ is the adjoint action.

Proposition 42. *For each k, let $ds^k : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ be the homomorphism induced by the Duflo–Serganova functor (see Remark 39). Set $t := 1 + \min\{m, n\}$ and let $\mathbf{M}_k^t := \ker ds^k$. Consider the filtration of $\mathfrak{sl}(\infty)$-modules*

$$\mathbf{M}_1^t \subset \mathbf{M}_2^t \subset \cdots \subset \mathbf{M}_t^t = \mathbf{J}_{m|n}.$$

Then $\mathbf{M}_1^t = \mathbf{\Lambda}_{m|n}$ and $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t \cong \Lambda^{m-k}\mathbf{V} \otimes \Lambda^{n-k}\mathbf{V}_*$. This filtration is the tensor filtration of $\mathbf{J}_{m|n}$, that is, $\operatorname{tens}^{k-1} \mathbf{J}_{m|n} = \ker ds^k$.

Proof. In the proof we let m and n vary. It follows from [HR, Theorems 17 and 20] that for every $m, n \in \mathbb{Z}_{>0}$ the map $ds : \mathbf{J}_{m|n} \to \mathbf{J}_{m-1|n-1}$ is surjective and the kernel is spanned by the classes of Kac modules. So we have an exact sequence of $\mathfrak{sl}(\infty)$-modules

$$0 \to \mathbf{\Lambda}_{m|n} \to \mathbf{J}_{m|n} \xrightarrow{ds} \mathbf{J}_{m-1|n-1} \to 0.$$

Thus, we obtain the following diagram of $\mathfrak{sl}(\infty)$-modules for each $l = |m - n|$, in which the horizontal arrows represent the map ds.

$$\begin{array}{ccccccccccc}
\twoheadrightarrow & \mathbf{M}_5^5 & \twoheadrightarrow & \mathbf{M}_4^4 & \twoheadrightarrow & \mathbf{M}_3^3 & \twoheadrightarrow & \mathbf{M}_2^2 & \twoheadrightarrow & \mathbf{M}_1^1 \\
& \cup & & \cup & & \cup & & \cup & & \cup \\
\twoheadrightarrow & \mathbf{M}_4^5 & \twoheadrightarrow & \mathbf{M}_3^4 & \twoheadrightarrow & \mathbf{M}_2^3 & \twoheadrightarrow & \mathbf{M}_1^2 & \twoheadrightarrow & 0 \\
& \cup & & \cup & & \cup & & \cup & & \\
\twoheadrightarrow & \mathbf{M}_3^5 & \twoheadrightarrow & \mathbf{M}_2^4 & \twoheadrightarrow & \mathbf{M}_1^3 & \twoheadrightarrow & 0 & & \\
& \cup & & \cup & & \cup & & & & \\
\twoheadrightarrow & \mathbf{M}_2^5 & \twoheadrightarrow & \mathbf{M}_1^4 & \twoheadrightarrow & 0 & & & & \\
& \cup & & \cup & & & & & & \\
\twoheadrightarrow & \mathbf{M}_1^5 & \twoheadrightarrow & 0 & & & & & &
\end{array}$$

By induction we get $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t \cong \mathbf{M}_1^{t-k} = \mathbf{\Lambda}_{m-k|n-k}$, and by [B], $\mathbf{\Lambda}_{m-k|n-k} \cong \Lambda^{m-k}\mathbf{V} \otimes \Lambda^{n-k}\mathbf{V}_*$. Hence, the first claim follows.

For the second claim, suppose for sake of contradiction that for some k, the module $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t$ is not the maximal tensor submodule of $\mathbf{J}_{m|n}/\mathbf{M}_k^t$. By projecting to $\mathbf{J}_{m-k|n-k}$, we obtain that $\mathbf{M}_1^t$ is not the maximal tensor submodule of $\mathbf{J}_{m|n}$, for some m, n. Since

$\mathbf{M}_1^t = \mathbf{\Lambda}_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ is injective in the category $\mathbb{T}_{\mathfrak{g}}$ [DPS], this implies that $\operatorname{soc} \mathbf{J}_{m|n}$ is larger than $\operatorname{soc} \mathbf{M}_1^t$, which is a contradiction since $\operatorname{soc} \mathbf{J}_{m|n} = \operatorname{soc} \mathbf{\Lambda}_{m|n} = \mathbf{P}_{m|n}$. □

In the rest of this subsection, we fix x to be a generator of the root space corresponding to $\delta_j - \varepsilon_i$. We denote by $ds_{ij} : \mathbf{K}_{m|n} \to \mathbf{K}_{m-1|n-1}$ the $\mathfrak{sl}(\infty)$-module homomorphism ds_x.

Proposition 43. *We have*

$$\bigcap_{i,j} \ker ds_{ij} = \mathbf{T}_{m|n}.$$

Proof. It follows from [HR] that $ds_{ij}[M] = 0$ if and only if $e^{\varepsilon_i} - e^{\delta_j}$ divides the supercharacter $\operatorname{sch} M$ of M. Hence, $[M]$ lies in the intersection of kernels of all ds_{ij} if and only if $\prod_{i,j}(e^{\varepsilon_i} - e^{\delta_j})$ divides $\operatorname{sch} M$. This means that $\operatorname{sch} M$ is a linear combination of supercharacters induced from the parabolic subalgebra $\mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_1$. Therefore, $\operatorname{sch} M$ is a linear combination of supercharacters of Verma modules. □

Proposition 44. *We have* $\operatorname{tens} \mathbf{K}_{m|n} = \mathbf{T}_{m|n}$. *Moreover,* $\mathbf{K}_{m|n}$ *has an exhausting tensor filtration of length* $\min(m,n)+1$.

Proof. Obviously $\operatorname{tens} \mathbf{K}_{m|n} \supset \mathbf{T}_{m|n}$. Assume that $\operatorname{tens} \mathbf{K}_{m|n} \neq \mathbf{T}_{m|n}$. Then since $\mathbf{T}_{m|n}$ is injective in $\mathbb{T}_{\mathfrak{g}}$ the socle of $\operatorname{tens} \mathbf{K}_{m|n}$ is larger than the socle of $\mathbf{T}_{m|n}$, but this is a contradiction since $\operatorname{soc} \mathbf{T}_{m|n} = \operatorname{soc} \mathbf{K}_{m|n}$. The second claim can be proven by induction on $\min(m,n)$, since $\mathbf{K}_{m|n}/\mathbf{T}_{m|n}$ is isomorphic to a submodule of $\mathbf{K}_{m-1|n-1}^{\oplus mn}$ via the map $\oplus_{ij} ds_{ij}$. □

4.5. **Meaning of the socle filtration.** Now we will define a filtration on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$. For a $\mathfrak{gl}(m|n)$-module M, let

$$X_M = \{x \in X_{\mathfrak{gl}(m|n)} \mid DS_x(M) \neq 0\},$$

and let $X^k_{\mathfrak{gl}(m|n)}$ be the subset of all elements in $X_{\mathfrak{gl}(m|n)}$ of rank less than or equal to k. We define $[\mathcal{O}^{\mathbb{Z}}_{m|n}]^k$ to be the full subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ consisting of all modules M such that $X_M \subset X^k_{\mathfrak{gl}(m|n)}$. Note that $[\mathcal{O}^{\mathbb{Z}}_{m|n}]^k$ is not an abelian category. Furthermore, we define $[\mathcal{O}^{\mathbb{Z}}_{m|n}]^k_-$ to be the full subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ consisting of all modules M such that

$$X_M \cap \mathfrak{gl}(m|n)_{-1} \subset X^k_{\mathfrak{gl}(m|n)}.$$

Let $\mathbf{K}^k_{m|n}$ denote the complexification of the subgroup in $\mathbf{K}_{m|n}$ generated by the classes of modules lying in $[\mathcal{O}^{\mathbb{Z}}_{m|n}]^k$, and let $(\mathbf{K}^k_{m|n})_-$ be defined similarly for the category $[\mathcal{O}^{\mathbb{Z}}_{m|n}]^k_-$. Since both categories are invariant under the functors E_i and F_i, both $\mathbf{K}^k_{m|n}$ and $(\mathbf{K}^k_{m|n})_-$ are $\mathfrak{sl}(\infty)$-submodules of $\mathbf{K}_{m|n}$.

Conjecture 45. $\mathbf{K}^k_{m|n} = \operatorname{soc}^{k+1} \mathbf{K}_{m|n}$ *and* $(\mathbf{K}^k_{m|n})_- = \operatorname{tens}^{k+1} \mathbf{K}_{m|n}$.

Here we prove a weaker statement. Recall that $\mathcal{O}^{\mathbb{Z}}_{m|n}$ has block decomposition:

$$\mathcal{O}^{\mathbb{Z}}_{m|n} = \bigoplus (\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi,$$

where $(\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi$ is the subcategory of modules admitting generalized central character χ. The complexified reduced Grothendieck group of $(\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi$ coincides with the weight subspace $(\mathbf{K}_{m|n})_\chi$. The degree of atypicality of χ is defined in [DS]. In [CS] it is proven that $(\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi \subset [\mathcal{O}^{\mathbb{Z}}_{m|n}]^k$ if the degree of atypicality of χ is not greater than k. Note that the degree of atypicality of the highest weight χ of the irreducible $\mathfrak{sl}_\infty$-module $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is equal to $m-|\boldsymbol{\lambda}|=n-|\boldsymbol{\mu}|$ and the degree of atypicality of any weight of $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is not less than the degree of atypicality of the highest weight. Combining this observation with the description of the socle filtration of $\mathbf{K}_{m|n}$ we obtain the following.

Proposition 46. $\operatorname{soc}^{k+1}\mathbf{K}_{m|n}$ *is the submodule in* $\mathbf{K}_{m|n}$ *generated by weight vectors of weights with degree of atypicality less or equal to* k. *Therefore we have* $\operatorname{soc}^{k+1}\mathbf{K}_{m|n}\subset\mathbf{K}^k_{m|n}$.

5. Appendix

In this section, we prove the technical lemma used in Lemma 25, which in turn is needed for the proof of Theorem 24.

Consider decompositions $\mathbf{V}=\mathbf{W}_1\oplus\mathbf{W}_2$ and $(\mathbf{V})_*=(\mathbf{W}_1)_*\oplus(\mathbf{W}_2)_*$ such that $\mathbf{W}_1^\perp=(\mathbf{W}_2)_*$ and $\mathbf{W}_2^\perp=(\mathbf{W}_1)_*$. Denote by $\mathfrak{s}$ the subalgebra $\mathfrak{sl}(\mathbf{W}_1)$ of $\mathfrak{g}$. Let $\mathbf{T}_{m|n}=\mathbf{V}^{\otimes m}\otimes\mathbf{V}_*^{\otimes n}$, and let $\mathbf{Y}_{m|n}$ be the intersection with $\mathbf{T}_{m|n}$ of the ideal generated by $\mathbf{W}_1\oplus(\mathbf{W}_1)_*$ in the tensor algebra $T(\mathbf{V}\oplus\mathbf{V}_*)$. Then $\mathbf{T}_{m|n}$ considered as an $\mathfrak{s}$-module admits the decomposition

$$\operatorname{Res}_{\mathfrak{s}}\mathbf{T}_{m|n}=(\mathbf{W}_2^{\otimes m}\otimes(\mathbf{W}_2)_*^{\otimes n})\oplus\mathbf{Y}_{m|n}.$$

Lemma 47. *We have*

$$(\operatorname{soc}\mathbf{T}_{m|n})\cap\mathbf{Y}_{m|n}\subset\mathfrak{s}\mathbf{Y}_{m|n}.$$

Proof. Note that $\mathbf{Y}_{m|n}$ is an object of $\widetilde{\mathbb{T}}_{\mathfrak{s}}$ and

$$\mathfrak{s}\mathbf{Y}_{m|n}=\bigcap_{\varphi\in\operatorname{Hom}_{\mathfrak{s}}(\mathbf{Y}_{m|n},\mathbb{C})}\ker\ \varphi. \tag{5.1}$$

Let τ denote a map from $\{1,\dots,m+n\}$ to $\{1,2\}$. Denote by $\mathbf{T}^\tau_{m|n}$ the subspace of $\mathbf{T}_{m|n}$ spanned by $v_1\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes u_{m+n}$ with $v_i\in\mathbf{W}_{\tau(i)}$ and $u_j\in(\mathbf{W}_{\tau(j)})_*$. Clearly,

$$\operatorname{Res}_{\mathfrak{s}}\mathbf{T}_{m|n}=\bigoplus_\tau\mathbf{T}^\tau_{m|n},$$

and we have an $\mathfrak{s}$-module isomorphism

$$\mathbf{T}^\tau_{m|n}\cong\mathbf{W}_1^{\otimes p(\tau)}\otimes\mathbf{W}_2^{\otimes(m-p(\tau))}\otimes(\mathbf{W}_1)_*^{\otimes q(\tau)}\otimes(\mathbf{W}_2)_*^{\otimes(n-q(\tau))},$$

where

$$p(\tau):=|\tau^{-1}(1)\cap\{1,\dots,m\}|,\quad q(\tau):=|\tau^{-1}(1)\cap\{m+1,\dots,m+n\}|.$$

Furthermore,

$$\mathbf{Y}_{m|n}=\bigoplus_{p(\tau)+q(\tau)>0}\mathbf{T}^\tau_{m|n}.$$

Recall from [PStyr, Theorem 2.1] that

$$\operatorname{soc}\mathbf{T}_{m|n}=\bigcap_{1\le i\le m,m<j\le m+n}\ker\Phi_{ij},$$

where Φ_{ij} is defined in (3.3). For $r=1,2$, let $\Phi_{ij}^{\mathbf{W}_r}:\mathbf{T}_{m|n}\to\mathbf{T}_{m-1|n-1}$ be defined by

$$v_1\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes u_{m+n}\mapsto\langle u_j,v_i\rangle^{\mathbf{W}_r}v_1\otimes\cdots\otimes\widehat{v_i}\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes\widehat{u_j}\otimes\cdots\otimes u_{m+n},$$

where $\langle\cdot,\cdot\rangle^{\mathbf{W}_r}$ is defined on homogeneous elements by

$$\langle u_j, v_i\rangle^{\mathbf{W}_r} := \begin{cases} \langle u_j, v_i\rangle \text{ if } u_j, v_i \in \mathbf{W}_r \\ 0 \text{ otherwise.} \end{cases}$$

Next, recall from [DPS] that $\mathrm{Hom}_{\mathfrak{s}}(\mathbf{W}_1^{\otimes p} \otimes (\mathbf{W}_1)_*^{\otimes q}, \mathbb{C}) = 0$ if $p \neq q$, and if $p = q$, is spanned by compositions of contractions $\Phi^{\mathbf{W}_1}_{1,j_1} \dots \Phi^{\mathbf{W}_1}_{p,j_p}$ for all possible permutations $j_1, \dots, j_p$. Using (5.1) we can conclude that $\mathfrak{s}\mathbf{Y}^\tau_{m|n} = \mathbf{Y}^\tau_{m|n}$ if $p(\tau) \neq q(\tau)$, whereas if $p = p(\tau) = q(\tau)$ we have

$$\mathfrak{s}\mathbf{Y}^\tau_{m|n} = \bigcap_{i_1,\dots,i_p,j_1,\dots,j_p \in \tau^{-1}(1)} \ker \Phi^{\mathbf{W}_1}_{i_1,j_1} \dots \Phi^{\mathbf{W}_1}_{i_p,j_p}.$$

Observe that

$$\Phi_{ij} = \Phi^{\mathbf{W}_1}_{ij} + \Phi^{\mathbf{W}_2}_{ij}. \tag{5.2}$$

We claim that if $y = \sum_\tau y_\tau \in \mathbf{Y}_{m|n}$ and $\Phi_{ij}(y) = 0$ for all i, j, then $y_\tau \in \mathfrak{s}\mathbf{T}^\tau_{m|n}$ for all τ. The statement is trivial for every τ such that $p(\tau) \neq q(\tau)$. Now we proceed to prove the claim in the case $p(\tau) = q(\tau) = p$ by induction on p.

Let $p = 1$ and consider τ' with $p(\tau') = 1 = q(\tau')$. Let $i \leq m$ and $j > m$ be such that $\tau'(i) = \tau'(j) = 1$. Note that $\Phi_{i,j}(y'_\tau) \in (\mathbf{W}_2^{\otimes m-1} \otimes (\mathbf{W}_2)_*^{\otimes n-1})$ and for $\tau \neq \tau'$ we have $\Phi_{i,j}(y_\tau) \in Y_{m-1|n-1}$. Therefore, $\Phi_{i,j}(y_{\tau'}) = \Phi^{\mathbf{W}_1}_{i,j}(y_{\tau'}) = 0$ and hence $y_{\tau'} \in \mathfrak{s}\mathbf{T}^{\tau'}_{m|n}$.

Now consider $y_{\tau'}$ such that $p(\tau') = p = q(\tau')$. Let $i_1, \dots, i_p \leq m$ and $j_1, \dots j_p > m$ such that $\tau'(i) = \tau'(j) = 1$. We would like to show that

$$\Phi^{\mathbf{W}_1}_{i_1,j_1} \dots \Phi^{\mathbf{W}_1}_{i_p,j_p}(y_{\tau'}) = \Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) = 0. \tag{5.3}$$

Note that τ' has the property

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) \ \in \ \mathbf{W}_2^{\otimes m-p} \otimes (\mathbf{W}_2)_*^{\otimes n-p}. \tag{5.4}$$

Suppose that τ'' also has property (5.4). Then $(\tau'')^{-1}(1) \subset (\tau')^{-1}(1)$, and if $\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau''}) \neq 0$, then $\tau''(i_r) = \tau''(j_r)$ for all $r = 1, \dots, p$. For every such $\tau'' \neq \tau'$ we have $p(\tau'') = q(\tau'') := l < p$. Let $\{i_{r_1}, \dots, i_{r_l}, j_{r_1}, \dots, j_{r_l}\} = (\tau'')^{-1}(1)$. Then by induction assumption $y_{\tau''} \in \mathfrak{s}\mathbf{T}^{\tau''}_{m|n}$ and hence

$$\Phi^{\mathbf{W}_1}_{i_{r_1},j_{r_1}} \dots \Phi^{\mathbf{W}_1}_{i_{r_l},j_{r_l}}(y_{\tau''}) = \Phi_{i_{r_1},j_{r_1}} \dots \Phi_{i_{r_l},j_{r_l}}(y_{\tau''}) = 0.$$

But then

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau''}) = 0,$$

which implies

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) = 0.$$

Now (5.3) follows, and this implies $y_{\tau'} \in \mathfrak{s}\mathbf{T}^\tau_{m|n}$. □

References

[B] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$*, J. Amer. Math. Soc. 16 (2003), no. 1, 185–231.

[BLW] J. Brundan, I. Losev, B. Webster, *Tensor Product Categorifications and the Super Kazhdan–Lusztig Conjecture*, Int. Math. Res. Notices 20 (2017), 6329–641.

[BS] J. Brundan, C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup* J. Eur. Math. Soc. 14 (2012), 373–419.

[CLW] S.J. Cheng, N. Lam, W. Wang, *Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras* Duke Math. J. 164 (2015), 617–695

[CP1] A. Chirvasitu, I. Penkov, *Ordered tensor categories and representations of the Mackey Lie algebra of infinite matrices*, arXiv:1512.08157.

[CP2] A. Chirvasitu, I. Penkov, *Representation categories of Mackey Lie algebras as universal monoidal categories*, arXiv:1710.00976.

[CS] K. Coulembier, V. Serganova, *Homological invariants in category O for the general linear superalgebra*, Trans. Amer. Math. Soc. 369 (2017), no. 11, 7961–7997.

[DP] I. Dimitrov, I. Penkov, *Weight modules of direct limit Lie algebras*, IMRN 1999, no. 5, 223–249.

[DPS] E. Dan-Cohen, I. Penkov, V. Serganova, A Koszul category of representations of finitary Lie algebras, Advances in Mathematics 289 (2016), 250–278.

[DS] M. Duflo, V. Serganova, *On associated variety for Lie superalgebras*, arXiv:Math/0507198.

[FPS] I. Frenkel, I. Penkov, V. Serganova, *A categorification of the boson-fermion correspondence via representation theory of* $\mathfrak{sl}(\infty)$, Comm. Math. Phys. 341 (2016) no.3, 911–931 .

[GS] M. Gorelik, V. Serganova, *On DS functors for affine Lie superalgebras*, arXiv:1711.10149.

[HR] C. Hoyt, S. Reif, *Grothendieck rings for Lie superalgebras and the Duflo-Serganova functor*, arXiv:1612.05815.

[M] I.M. Musson, *Lie superalgebras and enveloping algebras*, Graduate Studies in Mathematics, vol. 131, 2012.

[Nam] T. Nampaisarn, *Categories O for Dynkin Borel subalgebras of root-reductive Lie algebras*, arXiv:1706.05950.

[PS] I. Penkov, V. Serganova, Categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules, Contemp. Math. 557, AMS, 2011, 335–357.

[PStyr] I. Penkov, K. Styrkas, *Tensor representations of classical locally finite Lie algebras*, in Developments and Trends in Infinite-Dimensional Lie Theory, Progress in Mathematics 288, Birkhäuser, 2011, 127–150.

[SS] S. Sam, A. Snowden, *Stability patterns in representation theory*, Forum of Mathematics, Sigma, vol. 3, e11, 2015.

[S] J. C. Santos, *Zuckerman functors for Lie superalgebras*, J. of Lie theory, 9 (1999), 61–112.

[Ser] V. Serganova, *Representations of Lie Superalgebras*, Lecture notes in Perspectives in Lie Theory, Ed. F. Callegaro, G. Carnovale, F. Caselli, C. De Concini, A. De Sole, Springer, 2017, 125–177.

Crystal Hoyt
Department of Mathematics, ORT Braude College & Weizmann Institute, Israel
e-mail: crystal@braude.ac.il

Ivan Penkov
Jacobs University Bremen, Campus Ring 1, 28759, Bremen, Germany
e-mail: i.penkov@jacobs-university.de

Vera Serganova
Department of Mathematics, University of California Berkeley, Berkeley CA 94720, USA
e-mail: serganov@math.berkeley.edu