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Connected Graded Gorenstein Algebras with Enough Normal Elements

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We generalize [12, 1.1 and 1.2] to the following situation.

Theorem 1. *Let A be a connected graded noetherian algebra of injective dimension d such that every nonsimple graded prime factor ring of A contains a homogeneous normal element of positive degree. Then:*

(*1*) *A is Auslander–Gorenstein and Cohen–Macaulay.*
(*2*) *A has a quasi-Frobenius quotient ring.*
(*3*) *Every minimal prime ideal P is graded and* GKdim $A/P = d$.
(*4*) *If, moreover, A has finite global dimension, then A is a domain and a maximal order in its quotient division ring.*

To prove the above we need the following result, which is a generalization of [3, 2.46(ii)].

Theorem 2. *Let A be a connected graded noetherian AS-Gorenstein algebra of injective dimension d. Then:*

(*1*) *The last term of the minimal injective resolution of A_A is isomorphic to a shift of A^*.*

(*2*) *For every noetherian graded A-module M, $\underline{\mathrm{Ext}}^d(M, A)$ is finite dimensional over k.*

0. INTRODUCTION

Let k be a field. A k-algebra A is called *connected* if $A = \bigoplus_{i \geq 0} A_i$, $A_i A_j \subset A_{i+j}$, and $A_0 = k$. In this paper we will only consider connected left and right noetherian algebras and graded modules except for the proof

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of Theorem 3.2. If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a noetherian (left and/or right) A-module, we simply say M is *finite*. For every integer n, $M_{\geq n}$ denotes the submodule $\bigoplus_{i \geq n} M_i$. Let $\mathfrak{m}$ be the unique maximal graded ideal $A_{\geq 1}$ of A. The *trivial A-bimodule $A/\mathfrak{m}$* is denoted by k because it is isomorphic to k as a vector space. Given a graded module M, the degree shift $s(M)$ is defined by $s(M)_n = M_{n+1}$ and $s^l(M)$ is denoted by $M(l)$ for all $l \in \mathbb{Z}$. A graded module $M = \bigoplus_i M_i$ is called *left bounded* (respectively *right bounded*) if $M_i = 0$ for all $i \ll 0$ (respectively $i \gg 0$). We say M is *locally finite* if $\dim M_i < \infty$ for all i, where dim is the dimension of a vector space. Every finite graded A-module is left bounded and locally finite.

If M and N are two left (or right) A-modules, we use $\operatorname{Hom}^d(M, N)$ to denote the set of all A-module homomorphisms $h: M \to N$ such that $h(M_i) \subset N_{i+d}$. We set $\underline{\operatorname{Hom}}(M, N) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}^d(M, N)$ and denote the corresponding derived functors by $\underline{\operatorname{Ext}}^i(M, N)$. Given any A-module M, the *j-number* of M is defined by

$$j(M) = \min\left\{i \mid \underline{\operatorname{Ext}}^i(M, A) \neq 0\right\} \in \mathbb{N} \cup \{\infty\}.$$

In particular, if $M = 0$, then $j(M) = \infty$. If A is a noetherian ring with finite left and right injective dimension and M is a nonzero left or right A-module, then $j(M) < \infty$. By [16, Lemma A], if A has finite left and right injective dimension, then the left injective dimension is equal to the right injective dimension. We will write this common integer as injdim A. Note that if M is a finite graded right (respectively left) A-module, then $\underline{\operatorname{Ext}}^i(M, A)$ is a finite graded left (respectively right) A-module for each i. An algebra A is called *Auslander–Gorenstein* if A has finite left and right injective dimension and, for every finite graded A-module M and for every graded A-submodule $N \subset \underline{\operatorname{Ext}}^i(M, A)$, $j(N) \geq i$; A is called *Cohen–Macaulay* if, for every finite graded A-module M, $j(M) + \operatorname{GKdim} M = \operatorname{GKdim} A < \infty$, where GKdim is the Gelfand–Kirillov dimension. A connected algebra A is called *AS-Gorenstein* (Artin–Schelter Gorenstein) if A has finite injective dimension d and

$$\underline{\operatorname{Ext}}^i(k, A) = 0 \quad \text{for } i \neq d \quad \text{and} \quad \underline{\operatorname{Ext}}^d(k, A) \cong k(e) \quad \text{for some } e \in \mathbb{Z}, \tag{E1}$$

where k is viewed as a either left or right A-module. If A is Auslander–Gorenstein, then A is AS-Gorenstein [7, 6.3]. By induction and (E1), if F is a finite dimensional left or right A-module, then $\underline{\operatorname{Ext}}^d(F, A) \cong F^*(e)$ as graded k-vector spaces, where F^* is the *graded k-linear dual* $\bigoplus_n \operatorname{Hom}_k(F_{-n}, k)$. For more information about the above definitions and related results see [3], [7], and [12].

Stafford and the author proved the following result for PI (polynomial identity) rings [12, 1.1 and 1.2]. As usual clKdim denotes the classical Krull dimension and Kdim denotes the Krull (Rentschler–Gabriel) dimension.

THEOREM 0.1 [12]. *Let A be a connected noetherian PI algebra of injective dimension d. Then*:

(1) *A is Auslander–Gorenstein and Cohen–Macaulay.*

(2) GKdim A = Kdim A = clKdim A = injdim A.

(3) *If, moreover, A has finite global dimension, then A is a domain and a maximal order in its quotient division ring.*

Recent studies on quantum groups and deformations of commutative algebras suggest that we should generalize the above result to non-PI quantized algebras. As we saw from [5] and other papers on quantum groups, many quantized algebras are not PI, but satisfy the property defined next. If, for every nonsimple graded prime factor ring A/P, there is a nonzero homogeneous normal element in $(A/P)_{\geq 1}$, then we say A has *enough normal elements*. If A has a sequence of normal elements $\{x_1, \dots, x_n\} \subset A_{\geq 1}$ [namely, the image of x_i in $A/(x_1, \dots, x_{i-1})$ is normal for all i] such that $A/(x_1, \dots, x_n)$ is finite dimensional, then A has enough normal elements. The prime spectrum Spec A is called *normally separated* if, for any pair of primes $P \subsetneq Q$, Q/P contains a nonzero normal element of A/P [5, 1.5]. The prime spectrum of a PI ring is normally separated [8, 13.6.4]. Many quantized algebras are non-PI, but their spectra are normally separated [5]. By definition, if Spec A is normally separated, then A has enough normal elements. If $\underline{\mathrm{Ext}}^i(k_A, M)$ is finite dimensional over k for all $i \geq 0$ and for all finite graded right A-modules M, we say A satisfies χ [4, Definition 3.7]. The condition χ is equivalent to the AS-Gorenstein condition when A has finite injective dimension (see [15, 4.3] and Theorem 0.3). By [4, 8.12(2)], connected algebras with enough normal elements satisfy the condition χ. If, moreover, A has finite global dimension, then A is a domain [12, p. 1024]. The main result of this paper is the following.

THEOREM 0.2. *Let A be a connected noetherian algebra of injective dimension d. Suppose that A has enough normal elements. Then*:

(1) *A is Auslander–Gorenstein and Cohen–Macaulay.*

(2) *A has a quasi-Frobenius ungraded quotient ring.*

(3) *For every finite graded left or right A-module M,* GKdim M = Kdim $M < \infty$; *for every two-sided graded ideal $I \subset A$,* GKdim A/I = clKdim A/I.

(4) *Every minimal prime ideal P is graded and* GKdim $A/P = d$.

(5) *If, moreover, A has finite global dimension, then A is a domain and a maximal order in its quotient division ring.*

The key step is to prove Theorem 0.2(1) and our basic idea is to modify the proof of [12, 3.10]. The difficulty here is to show that if A is an AS-Gorenstein algebra, then

$$\underline{\mathrm{Ext}}^i(M, A) \cong \underline{\mathrm{Ext}}^i(M^\sigma, A) \tag{E2}$$

as graded k-vector spaces for all finite A/P-modules M and for graded algebra automorphisms $\sigma \in \mathrm{Aut}(A/P)$, where P is an ideal of A. Note that if σ is a graded algebra automorphism of A, then (E2) holds for any graded algebra A [Lemma 2.1(1)]. Given a right A-module M and $x \in M$, x is called $\mathfrak{m}$-*torsion* if $x\mathfrak{m}^n = 0$ for some n. The set of torsion elements of M forms a submodule, which is denoted by τM. A graded module M is called $\mathfrak{m}$-*torsion* (respectively $\mathfrak{m}$-*torsion-free*) if $\tau M = M$ (respectively $\tau M = 0$). If M is finite, then τM is the largest finite dimensional submodule of M. By using the recent results in [4], [14], and [15] we can show the following.

THEOREM 0.3. *Let A be a connected noetherian algebra of injective dimension d. Suppose that $\underline{\mathrm{Ext}}^i(k_A, A)$ and $\underline{\mathrm{Ext}}^i({}_Ak, A)$ are finite dimensional for all i. Then*

(1) *A is AS-Gorenstein, i.e., $\underline{\mathrm{Ext}}^i(k_A, A) = \underline{\mathrm{Ext}}^i({}_Ak, A) = 0$ for $i \neq d$ and $\underline{\mathrm{Ext}}^d(k_A, A) \cong \underline{\mathrm{Ext}}^d({}_Ak, A) \cong k(e)$ for some integer e.*

(2) *A satisfies χ.*

(3) *The last term of the minimal injective resolution of A_A (or ${}_AA$) is $A^*(e)$.*

(4) *$\underline{\mathrm{Ext}}^d(M, A)$ is finite dimensional for all finite graded left and right A-modules M and $\underline{\mathrm{Ext}}^d(M, A) \cong \underline{\mathrm{Ext}}^d(\tau M, A) \cong (\tau M)^*(e)$ as graded k-vector spaces.*

Theorem 0.3(1) was also proved in [12, 3.8] and [6, 3.5]. By using Theorem 0.3, the local cohomology introduced in [14] and [4, Sect. 7], and the Serre duality [15], we can prove (E2).

1. PROOF OF THEOREM 0.3

LEMMA 1.1. *Let A be a connected algebra with finite injective dimension. Suppose that F_i are nonzero finite dimensional graded right A-modules. Then $\underline{\mathrm{Ext}}^i(F_i, A) = 0$ for all $i \le p$ (respectively for all $i \ge p$) if and only if $\underline{\mathrm{Ext}}^i(k_A, A) = 0$ for all $i \le p$ (respectively for all $i \ge p$).*

Proof. By using a long exact sequence we see that if $\underline{\mathrm{Ext}}^i(k_A, A) = 0$, then $\underline{\mathrm{Ext}}^i(F_i, A) = 0$ for each i. Conversely, we suppose $\underline{\mathrm{Ext}}^i(F_i, A) = 0$ for all $i \le p$. If $\underline{\mathrm{Ext}}^i(k_A, A) \neq 0$ for some $i \le p$, we may assume i is minimal amongst such values. Since F_i is finite dimensional, we have a short exact sequence

$$0 \to K \to F_i \to k_A(l) \to 0$$

for some l and some submodule $K \subset F_i$. The short exact sequence above yields an exact sequence

$$\to \underline{\mathrm{Ext}}^{i-1}(K, A) \to \underline{\mathrm{Ext}}^i(k_A(l), A) \to \underline{\mathrm{Ext}}^i(F_i) \to .$$

The left term is zero because $\underline{\mathrm{Ext}}^{i-1}(k_A, A) = 0$ and the right term is zero by the hypothesis. Hence the middle term is zero, a contradiction. Therefore $\underline{\mathrm{Ext}}^i(k_A, A) = 0$ for all $i \le p$. The proof of the other case is similar. ∎

In the proof of Theorem 0.3 below and in the next section we will use the notion of noncommutative projective scheme introduced in [4]. Let $\mathrm{Gr}\, A$ be the category of graded right A-modules. Let $\mathrm{Tor}\, A$ be the subcategory of $\mathrm{Gr}\, A$ consisting of $\mathfrak{m}$-torsion right A-modules and let $\mathrm{QGr}\, A$ denote the quotient category $\mathrm{Gr}\, A/\mathrm{Tor}\, A$. The canonical functor from $\mathrm{Gr}\, A$ to $\mathrm{QGr}\, A$ is denoted by π. The functor π has a right adjoint functor Γ: $\mathrm{QGr}\, A \to \mathrm{Gr}\, A$ (in [4, 2.2.2)] the right adjoint functor of π was ω and then in [4, Sect. 4] it was proved that $\Gamma \cong \omega$). We use script letter $\mathscr{M}$ for the object $\pi(M)$. The triple $(\mathrm{QGr}\, A, \mathscr{A}, s)$ is called the *projective scheme* of A and is denoted by $\mathrm{Proj}\, A$, where $\mathscr{A} = \pi(A_A)$ and s is the automorphism of $\mathrm{QGr}\, A$ defined by the degree shift. For more details about $\mathrm{Proj}\, A$, see [4], and for basic properties about quotient category, see [9].

Proof of Theorem 0.3. (1) In this proof $\mathrm{E}^i(M)$ denotes $\underline{\mathrm{Ext}}^i(M, A)$ for any left or right A-module M. Since both $j(k_A)$ and $j({}_A k)$ are finite, there are i and j (maybe the same) such that $\mathrm{E}^i(k_A) \neq 0$ and $\mathrm{E}^j({}_A k) \neq 0$. We claim that $\mathrm{E}^i(k_A) = \mathrm{E}^i({}_A k) = 0$ except for one i. If not, there are two distinct integers l and r such that $\mathrm{E}^l({}_A k) \neq 0$ and $\mathrm{E}^r(k_A) \neq 0$. Without loss of generality, we may assume that $l < r$ and that $l = \min\{i \mid \mathrm{E}^i({}_A k) \neq 0\}$ and $r = \max\{i \mid \mathrm{E}^i(k_A) \neq 0\}$. By Lemma 1.1, we have $\mathrm{E}^l(\mathrm{E}^r(k_A)) \neq 0$. By [12, (3.8.1)] there is a convergent spectral sequence

$$E_2^{p,q} := \underline{\mathrm{Ext}}^p\big(\underline{\mathrm{Ext}}^q(M, A), A\big) \quad \Rightarrow \quad \mathbb{H}^{p-q}(M), \tag{E3}$$

where $\mathbb{H}^{p-q}(M) = 0$ if $p \neq q$ and $\mathbb{H}^0(M) = M$. The bidegree of the rth differential of (E3) is $(r, r-1)$. Let $M = k$ in (E3). We have a table of

$E_2^{p,q}$ terms,

$$
\begin{array}{cccccc}
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \mathrm{E}^{l,r}(k) & \cdots & \mathrm{E}^{d,r}(k), \\
\vdots & \vdots & \vdots & \vdots & \mathrm{E}^{i,j}(k) & \vdots \\
0 & \cdots & 0 & \mathrm{E}^{l,0}(k) & \cdots & \mathrm{E}^{d,0}(k)
\end{array}
\tag{E4}
$$

where $\mathrm{E}^{p,q}(k)$ denotes $\underline{\mathrm{Ext}}^p(\underline{\mathrm{Ext}}^q(k_A, A), A)$. From (E4) we see that every boundary map passing through $\mathrm{E}^{l,r}(k)$ is zero, whence $\mathrm{E}^{l,r}(k)$ is the term $E_\infty^{l,r}$. Since $l < r$, this is zero, a contradiction. Therefore we have proved our claim that there is an integer $p \le d$ such that $\mathrm{E}^p(k_A) \ne 0$, $\mathrm{E}^p({}_Ak) \ne 0$, and $\mathrm{E}^i(k_A) = \mathrm{E}^i({}_Ak) = 0$ for all $i \ne p$. Let F be the finite dimensional left A-module $\mathrm{E}^p(k_A)$. By (E3), $\mathrm{E}^p({}_AF) \cong k_A$. For every finite dimensional left A-module F', we can prove by induction that $\dim \mathrm{E}^p(F') = \dim \mathrm{E}^p({}_Ak) \dim F'$ because $\mathrm{E}^i({}_Ak) = 0$ for all $i \ne p$. Hence

$$\dim \mathrm{E}^p({}_Ak) \dim F = \dim \mathrm{E}^p(F) = \dim k_A = 1.$$

Therefore $\dim F = 1$ and $F \cong {}_Ak(e)$, i.e., $\mathrm{E}^p(k_A) \cong {}_Ak(e)$ for some e. So we have

$$\mathrm{E}^p({}_Ak) \cong \mathrm{E}^p\big({}_Ak(e)\big)(e) \cong \mathrm{E}^p(F)(e) \cong k_A(e).$$

Next we will prove that $p = d$. Since A is a noetherian ring with finite injective dimension, the complex A satisfies the conditions (i) and (ii) in [14, 3.3]. Since A is projective as either left or right A-module, the complex A satisfies the condition [14, 3.3(iii)] and hence A is a dualizing complex. We have proved above that $\mathrm{E}^i(k) = 0$ for all $i \ne p$ and $\mathrm{E}^p(k) = k(e)$, whence the complex $A(-e)[p]$ satisfies the condition [14, 4.4(i)]. Here in general $M^{\cdot}[n]$ denotes the shift of a complex $M^{\cdot}$ by n. By [14, 4.5 and p. 61], $A(-e)[p]$ is a prebalanced dualizing complex. By [14, 4.10 and 4.13], A has a balanced dualizing complex $A(\phi, -e)[p]$ for some graded algebra automorphism ϕ (see [14, p. 48] for the definition of $A(\phi, -e)[p]$). By [14, 4.18], the local duality theorem holds. By [14, (4.17)] or equivalently [15, 4.2.1], we have graded k-vector space isomorphisms

$$\underline{\mathrm{Ext}}^{p-q}(M, A)^*(e) \cong \underline{\mathrm{Ext}}^{-q}(M, A(-e)[p])^* \cong \underline{\mathrm{H}}_{\mathfrak{m}}^q(M), \tag{E5}$$

where $\underline{\mathrm{H}}_{\mathfrak{m}}^q(M) = \lim_{n\to\infty} \underline{\mathrm{Ext}}^q(A/\mathfrak{m}^n, M)$. Letting $q < 0$ in (E5), we obtain $\underline{\mathrm{Ext}}^{p-q}(M, A) = 0$. Hence the injective dimension of A is at most p and thus $p = d$.

(2) Follows from (1) and [15, 4.3].

(3) By [15, 4.2 and 4.3], $\mathscr{A}(-e)[d-1]$ is a dualizing complex for $X := \operatorname{Proj} A$ and X is classical Cohen–Macaulay [15, Definition 2.4]. In particular,

$$\begin{aligned}\underline{\operatorname{Ext}}^i(\mathscr{M}, \mathscr{A}(-e)) &\cong \underline{\operatorname{Ext}}^{i-(d-1)}(\mathscr{M}, \mathscr{A}(-e)[d-1]) \\ &\cong \underline{\mathrm{H}}^{(d-1)-i}(X, \mathscr{M})^* = 0\end{aligned}$$

for all $i > d-1$. Hence the injective dimension of $\mathscr{A}(-e)$ (and of $\mathscr{A}$) is at most $d-1$. Suppose the minimal injective resolution of A_A is

$$0 \to A \to I^0 \to \cdots \to I^d \to 0.$$

By [4, p. 234] the functor π: $\operatorname{Gr} A \to \operatorname{QGr} A$ is exact and the right adjoint functor Γ is left exact. Hence the minimal injective resolution of $\mathscr{A}$ is

$$0 \to \mathscr{A} \to \mathscr{I}^0 \to \cdots \to \mathscr{I}^d \to 0,$$

where $\mathscr{I}^i = \pi(I^i)$. We refer to [4, Sect. 4.5] for some basic facts about quotient category and injective object. Since the injective dimension of $\mathscr{A}$ is at most $d-1$, $\mathscr{I}^d = 0$. Equivalently, I^d is an $\mathfrak{m}$-torsion injective and consequently I^d is a direct sum of shifts of A^*. By [4, 7.7(1)], $\underline{\operatorname{Ext}}^d(k, A) \cong k(e)$ implies $I^d \cong A^*(e)$ and we have proved (3). Further, by [4, 7.7(1)], $\underline{\operatorname{Ext}}^i(k, A) = 0$ implies that I^i is $\mathfrak{m}$-torsion-free for all $i < d$ and hence $\mathscr{A}$ has injective dimension $d-1$.

(4) By the proof of (1), $A(\phi, -e)[d]$ is a balanced dualizing complex over A for some graded algebra automorphism ϕ. Letting $q = 0$ in (E5) and taking k-linear dual of (E5), we have

$$\underline{\operatorname{Ext}}^d(M, A) \cong \lim_{n \to \infty} \underline{\operatorname{Hom}}(A/\mathfrak{m}^n, M)^*(e) \cong (\tau M)^*(e)$$

as graded k-vector spaces. Hence $\underline{\operatorname{Ext}}^d(M, A)$ is finite dimensional for all finite graded modules M. Similarly, $\underline{\operatorname{Ext}}^d(\tau M, A) \cong (\tau M)^*(e)$. Therefore $\underline{\operatorname{Ext}}^d(M, A) \cong \underline{\operatorname{Ext}}^d(\tau M, A)$. ∎

COROLLARY 1.2. *Let A be a connected noetherian algebra with finite injective dimension. If $\underline{\operatorname{Ext}}^i(k_A, A) = 0$ for all $i \neq l$ and $\underline{\operatorname{Ext}}^l(k_A, A) \cong k(e)$, then A is AS-Gorenstein and l is the injective dimension of A.*

Proof. By the hypotheses, we have $\underline{\operatorname{Ext}}^p(\underline{\operatorname{Ext}}^q(k_A, A), A) = 0$ for all $q \neq l$. By (E3), we have $\underline{\operatorname{Ext}}^p(\underline{\operatorname{Ext}}^l(k_A, A), A) = 0$ for all $p \neq l$ and $\underline{\operatorname{Ext}}^l(\underline{\operatorname{Ext}}^l(k_A, A), A) = k_A$. Since $\underline{\operatorname{Ext}}^l(k_A, A) = {}_A k(e)$, $\underline{\operatorname{Ext}}^i({}_A k, A) = 0$ for all $i \neq l$ and $\underline{\operatorname{Ext}}^l({}_A k, A) = k_A(e)$. Therefore the hypotheses of Theorem 0.3 hold and the statement follows from Theorem 0.3(1). ∎

COROLLARY 1.3. *Let A be a connected noetherian AS-Gorenstein algebra of injective dimension d and e as in* (E1). *Let M be a finite graded right A-module and $\mathscr{M} = \pi(M)$. Then*:

(1) $\underline{\mathrm{Ext}}^i(M, A) \cong \underline{\mathrm{Ext}}^i(\mathscr{M}, \mathscr{A})$ *for all $i \le d-2$.*

(2) *There is an exact sequence of graded k-vector spaces*

$$0 \to \underline{\mathrm{Ext}}^{d-1}(M, A) \to \underline{\mathrm{Ext}}^{d-1}(\mathscr{M}, \mathscr{A}) \to M^*(e) \to (\tau M)^*(e) \to 0.$$

Proof. For the proof of (1), see [4, 8.1(5)].

(2) Applying $\underline{\mathrm{Ext}}^{\cdot}(-, A)$ to the short exact sequence

$$0 \to M_{\ge n} \to M \to M/M_{\ge n} \to 0$$

and using the fact $\underline{\mathrm{Ext}}^{d-1}(M/M_{\ge n}, A) = 0$, we obtain an exact sequence

$$\begin{aligned} 0 &\to \underline{\mathrm{Ext}}^{d-1}(M, A) \to \underline{\mathrm{Ext}}^{d-1}(M_{\ge n}, A) \\ &\to \underline{\mathrm{Ext}}^{d}(M/M_{\ge n}, A) \to \underline{\mathrm{Ext}}^{d}(M, A) \to \underline{\mathrm{Ext}}^{d}(M_{\ge n}, A) \to 0. \quad \text{(E6)} \end{aligned}$$

By Theorem 0.3(4), $\underline{\mathrm{Ext}}^d(M_{\ge n}, A) = (\tau M_{\ge n})^*(e) = 0$ for all $n \gg 0$, $\underline{\mathrm{Ext}}^d(M, A) = (\tau M)^*(e)$, and $\underline{\mathrm{Ext}}^d(M/M_{\ge n}, A) = (M/M_{\ge n})^*(e)$. Let n go to the infinity. Then the second term in (E6) becomes $\underline{\mathrm{Ext}}^{d-1}(\mathscr{M}, \mathscr{A})$ and the third term becomes $M^*(e)$. Hence (2) follows. ∎

2. SIMILAR MODULES

Let M and N be two graded left (or right) A-modules. We say M is *similar* to N and write $M \sim N$ if

(S1) $M \cong N$ as graded k-vector spaces and

(S2) $\underline{\mathrm{Ext}}^i(M, A) \cong \underline{\mathrm{Ext}}^i(N, A)$ as graded k-vector spaces for all i.

Recall that the *Hilbert series* of a graded, left bounded, locally finite module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is defined to be the formal power series

$$H_M(t) = \sum_i \dim M_i \, t^i.$$

The condition (S1) is equivalent to $H_M(t) = H_N(t)$ and (S2) is equivalent to $H_{\underline{\mathrm{Ext}}^i(M, A)}(t) = H_{\underline{\mathrm{Ext}}^i(N, A)}(t)$. If A is a connected noetherian algebra with finite global dimension, then $H_M(t)$ is determined by $\sum_i (-1)^i H_{\underline{\mathrm{Ext}}^i(M, A)}(t)$ [13, 2.3]. Hence in this case (S2) implies (S1). It is unclear if (S2) implies (S1) in general. It is easy to construct two finite A-modules M and N such that (S1) holds and (S2) fails. If M has a proper graded submodule $\tilde{M} \subset M$ such that $\tilde{M}$ is similar to $M(-l)$ for some $l > 0$, then we say M has a *proper similar submodule*. If for every $\mathfrak{m}$-torsion-free module, M, there is a submodule $N \subset M$ such that N has a proper similar submodule, then we say A satisfies the *similar submodule condition* (or SSC for short). Graded PI algebras satisfy SSC as we now prove. Let A be a connected noetherian PI algebra and M be a finite graded right A-module. Then there is a nonzero submodule $N \subset M$ such that N is isomorphic to a uniform right ideal of A/P for some graded prime ideal P [12, 2.1]. Thus there exists a proper submodule of N isomorphic (and hence similar) to a shift of N. Therefore A satisfies SSC. In the same way we can prove that graded FBN rings satisfy SSC.

Let σ be a graded algebra automorphism of A. For every graded right A-module M, we define an A-module structure on the twisted module M^σ by $m \cdot a = m\sigma(a)$. Then $M \mapsto M^\sigma$ defines an invertible functor from $\operatorname{Gr} A$ to itself. Since every graded projective module is free, A^σ is free and hence $A^\sigma \cong A$ as graded right A-modules.

LEMMA 2.1. (1) *Let A be a graded noetherian algebra and σ a graded algebra automorphism of A. Then $M \sim M^\sigma$.*

(2) *Let A be an AS-Gorenstein noetherian algebra with finite injective dimension and let M and N be finite dimensional graded right A-modules. Then $M \sim N$ if and only if $M \cong N$ as graded k-vector spaces.*

Proof. (1) By definition, $M^\sigma \cong M$ as graded k-vector spaces. For every i,

$$\underline{\operatorname{Ext}}^i(M^\sigma, A) \cong \underline{\operatorname{Ext}}^i(M, A^{\sigma^{-1}}) \cong \underline{\operatorname{Ext}}^i(M, A).$$

Hence $M^\sigma \sim M$.

(2) If $M \cong N$ as graded k-vector spaces, $M^* \cong N^*$ as graded k-vector spaces. Let d be the injective dimension of A. By Theorem 0.3(4), $\underline{\operatorname{Ext}}^d(M,A) \cong \underline{\operatorname{Ext}}^d(N, A)$. For every $i \neq d$, $\underline{\operatorname{Ext}}^i(M, A) = \underline{\operatorname{Ext}}^i(N, A) = 0$. Hence $M \sim N$. ∎

If A satisfies SSC, then we can use induction on modules effectively. First we prove some good properties of GK-dimension. Let A be a connected noetherian algebra and M a finite graded A-module. Let $f_M(n) = \dim M_n$ for all n. The GK-dimension of M is equal to

$$\operatorname{GKdim} M = \varlimsup_{n \to \infty} \log_n \Bigl(\sum_{i \leq n} f_M(i) \Bigr). \tag{E7}$$

Since $\dim(-)$ is additive, (E7) implies that GKdim is exact, i.e.,

$$\operatorname{GKdim} M = \max\{\operatorname{GKdim} N, \operatorname{GKdim} N/N\}$$

for all $N \subset M$. Let $f(n)$ be a function from $\mathbb{Z}$ to $\mathbb{N}$. If there exist an integer t and polynomial functions $p_1(n), \ldots, p_t(n) \in \mathbb{Q}[n]$ such that $f(n) = p_s(n)$ for all $n \equiv s \pmod{t}$, then $f(n)$ is called a *multi-polynomial function*. Define

$$\deg f(n) = \max\{\deg p_s(n) \mid s = 1, \ldots, t\}.$$

LEMMA 2.2. *Let A be a connected noetherian algebra satisfying SSC and M be a finite graded A-module. Then*:

(1) *$f_M(n)$ is a multi-polynomial of n for $n \gg 0$ and* $\operatorname{GKdim} M = \deg f_M(n) + 1 < \infty$.

(2) *$K\dim M \geq GK\dim M$.*

Suppose N is isomorphic to M as graded k-vector spaces and the sequence

$$0 \to K_1 \to N(-l) \to M \to L \to K_2 \to 0$$

is exact, for some $l > 0$. Then

(3) $\operatorname{GKdim} M \leq \operatorname{GKdim} L + 1$. *If moreover $K_1 = K_2 = 0$, then* $\operatorname{GKdim} M = \operatorname{GKdim} L + 1$.

Remark. This lemma is similar to [12, 6.1]. Note that there is a gap in [12, p. 1022, 1.16] for the inequality $\operatorname{Kdim} M \leq \operatorname{GKdim} M$. However, that does not affect [12, 6.2] (and hence other theorems in [12]) because $\operatorname{Kdim} M \leq \operatorname{GKdim} M$ holds for Auslander−Gorenstein and Cohen−Macaulay rings [see the proof of Theorem 3.1(2)].

Proof. (1) and (2): We modify the proof of [12, 6.1]. If $\operatorname{Kdim} M = 0$, then M is finite dimensional and the statements are obvious. Now suppose the statements hold for all modules of Kdim $< \alpha$ for some $\alpha > 0$. Let $\operatorname{Kdim} M = \alpha$ for a finite graded A-module M and we will prove the statements for M. By the noetherian property and the exactness of GKdim and Kdim, it suffices to show (1) and (2) for a nonzero submodule of M. Since A satisfies SSC, we may assume that M is Kdim-critical and has a proper similar submodule, i.e., there is a finite graded A-module $\tilde{M}$ such that $\tilde{M} \sim M$ and $\tilde{M}(-l) \subset M$ for some $l > 0$. Then we have an exact sequence

$$0 \to \tilde{M}(-l) \to M \to \overline{M} \to 0,$$

where $\overline{M} \cong M/\tilde{M}(-l)$. Since M is Kdim-critical, $\operatorname{Kdim} \overline{M} < \operatorname{Kdim} M = \alpha$. By induction hypothesis, $f_{\overline{M}}(n)$ is a multi-polynomial function for $n \gg 0$. Since $\tilde{M} \sim M$, $f_M(n) = f_{\tilde{M}}(n)$ and hence $f_{\overline{M}}(n) = f_M(n) - f_M(n-l)$. Thus $f_M(n)$ is a multi-polynomial function for $n \gg 0$ with degree equal to $\deg f_{\overline{M}}(n) + 1$. By (E7), $\operatorname{GKdim}(M) = \deg f_M(n) + 1$. Since $\deg f_M(n) = \deg f_{\overline{M}}(n) + 1$, $\operatorname{GKdim} M = \operatorname{GKdim} \overline{M} + 1$. As a consequence of this equality and the induction hypothesis,

$$\operatorname{Kdim} M \geq \operatorname{Kdim} \overline{M} + 1 \geq \operatorname{GKdim} \overline{M} + 1 = \operatorname{GKdim} M.$$

(3) By the additivity of vector space dimension we have

$$f_M(n) - f_M(n-l) = f_M(n) - f_N(n-l) = f_L(n) - f_{K_1}(n) - f_{K_2}(n) \\ \le f_L(n).$$

Hence $\deg f_M(n) \le \deg f_L(n) + 1$ and by (1), GKdim $M \le$ GKdim $L + 1$. If $K_1 = K_2 = 0$, $f_M(n) - f_M(n-l) = f_L(n)$. Hence $\deg f_M(n) = \deg f_L(n) + 1$ and by (1), GKdim $M =$ GKdim $L + 1$. ∎

Next we will show that AS-Gorenstein algebras with enough normal elements satisfy SSC. Let P be a graded ideal of A and x be a regular normal element in $(A/P)_{\ge 1}$. Then x induces a graded algebra automorphism σ by $xa = \sigma(a)x$. Let M be a graded A/P-module. The twisted module M^σ is defined by $m \cdot a = m\sigma(a)$. Then $M \mapsto M^\sigma$ defines an invertible functor from Gr A/P to itself. It induces an invertible functor $\mathscr{M} \mapsto \mathscr{M}^\sigma$ from QGr A/P to QGr A/P. It is easy to see that $\pi(A/P)^\sigma \cong \pi(A/P)$ because $(A/P)^\sigma \cong A/P$.

PROPOSITION 2.3. (1) *Suppose that A is a connected noetherian AS-Gorenstein algebra of injective dimension d. Let P be an ideal of A, σ be a graded algebra automorphism of A/P, and M be a finite right A/P-module. Then M^σ is similar to M.*

(2) *Let A be a connected noetherian algebra of injective dimension d. Suppose that A has enough normal elements. Then A is AS-Gorenstein and satisfies SSC.*

Proof. (1) By [15, 4.3], $\mathscr{A}(-e)$ is the dualizing sheaf for $X =$ Proj A and X is classical Cohen–Macaulay, i.e., $\underline{\mathrm{Ext}}^i(\mathscr{M}, \mathscr{A}(-e)) \cong (\underline{\mathrm{H}}^{d-1-i}(X, \mathscr{M}))^*$ for all i and $\mathscr{M}$. Let Y be the projective scheme Proj A/P. Since M is an A/P-module, by [4, 8.3(3)], $\mathrm{H}^{d-1-i}(X, \mathscr{M}) \cong \mathrm{H}^{d-1-i}(Y, \mathscr{M})$. Hence we have graded k-vector space isomorphisms

$$\underline{\mathrm{Ext}}^i(\mathscr{M}, \mathscr{A}) \cong \underline{\mathrm{H}}^{d-1-i}(X, \mathscr{M}(-e))^* \cong \underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}(-e))^*.$$

Similarly, $\underline{\mathrm{Ext}}^i(\mathscr{M}^\sigma, \mathscr{A}) \cong \underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}^\sigma(-e))^*$. Since σ is an automorphism of A/P, we have $\pi(A/P)^{\sigma^{-1}} \cong \pi(A/P)$ and hence $\underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}(-e)) \cong \underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}^\sigma(-e))$. By Corollary 1.3(1) and above, we have graded k-vector space isomorphisms

$$\underline{\mathrm{Ext}}^i(M, A) \cong \underline{\mathrm{Ext}}^i(\mathscr{M}, \mathscr{A}) \cong \underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}(-e))^* \\ \cong \underline{\mathrm{H}}^{d-1-i}(Y, \mathscr{M}^\sigma(-e))^* \cong \underline{\mathrm{Ext}}^i(M^\sigma, A)$$

for $i \leq d-2$. If $i = d-1$, we still have $\underline{\mathrm{Ext}}^{d-1}(\mathscr{M}, \mathscr{A}) \cong \underline{\mathrm{Ext}}^{d-1}(\mathscr{M}^{\sigma}, A)$. It is easy to see that $(M^{\sigma})^*(e) \cong M^*(e)$ and $(\tau M^{\sigma})^*(e) \cong (\tau M)^*(e)$. By Corollary 1.3(2), we have $\underline{\mathrm{Ext}}^{d-1}(M, A) \cong \underline{\mathrm{Ext}}^{d-1}(M^{\sigma}, A)$. By Theorem 0.3(4), $\underline{\mathrm{Ext}}^{d}(M, A) \cong \underline{\mathrm{Ext}}^{d}(M^{\sigma}, A)$. Therefore $M \sim M^{\sigma}$.

(2) By [4, 8.12(2)], A and A^{op} satisfy χ and hence $\underline{\mathrm{Ext}}^{i}(k_A, A)$ and $\underline{\mathrm{Ext}}^{i}({}_Ak, A)$ are finite dimensional. By Theorem 0.3(1), A is AS-Gorenstein. Let N be a finite torsion-free graded A-module. There is a Kdim-critical submodule $M \subset N$ such that (i) $\mathrm{ann}(M_A) = P$ is a prime ideal of A and (ii) M is a fully faithful A/P-module. By the hypothesis, there is a nonzero normal element $x \in (A/P)_{\geq 1}$. Hence Mx is a proper submodule of M and $Mx \cong M^{\sigma}(-l)$, where $l = \deg x$ and $xa = \sigma(a)x$. By (1), $Mx \cong M^{\sigma}(-l) \sim M(-l)$ and (2) follows. ∎

It is not difficult to construct a connected noetherian algebra A and a finite graded right A-module M such that $\mathrm{Kdim}\, M < \mathrm{GKdim}\, M$. By Lemma 2.2(2), such a graded ring does not satisfy SSC. On the other hand, some connected algebras without enough normal elements satisfy SSC. The following can be proved by using the structure of algebras and the proof is omitted. For details on AS-regular algebras, see [2] and [3], and on the Sklyanin algebra, see [10].

PROPOSITION 2.4. *Connected AS-regular algebras of dimension three and the Sklyanin algebra of dimension four satisfy SSC.*

3. PROOF OF THEOREM 0.2

THEOREM 3.1. *Let A be a connected noetherian AS-Gorenstein algebra of injective dimension d. Suppose that A satisfies SSC. Then*:

(1) *A is Auslander–Gorenstein and Cohen–Macaulay and* $\mathrm{GKdim}\, A = \mathrm{injdim}\, A$.

(2) *For every finite graded A-module M*, $\mathrm{GKdim}\, M = \mathrm{Kdim}\, M < \infty$.

Proof. (1) We will use the proof of [12, 3.10] with some modifications. First we replace Kdim by GKdim and second we use only graded modules as in [12, 6.2]. As in the proof of [12, 3.10], it suffices to prove that the following properties hold for all finite graded left and right A-modules M.

(a) $j(M) + \mathrm{GKdim}\, M = d$;

(b) $\mathrm{GKdim}\, \underline{\mathrm{Ext}}^{j}(M, A) = \mathrm{GKdim}\, M$, where $j = j(M)$;

(c) For all $j(M) \leq i \leq d$, $\mathrm{GKdim}\, \underline{\mathrm{Ext}}^{i}(N, A) \leq d - i$ for all finite graded modules N.

The inequality in (c) is equivalent to $\mathrm{GKdim}\, \underline{\mathrm{Ext}}^{i}(N, A) \leq \min\{\mathrm{GKdim}\, N, d-i\}$ because $\underline{\mathrm{Ext}}^{i}(N, A) = 0$ when $\mathrm{GKdim}\, N < d - i$

[see (a)]. We induce on GKdim M. If GKdim $M = 0$, then M is finite dimensional. (a) and (b) are obvious and (c) holds by Theorem 0.3(4). Suppose (a), (b), and (c) hold for all modules with GKdim $< i$. We now consider a finite module M with $\operatorname{GKdim}(M) = i$. By using the same arguments as in the proof of [12, 3.10] we see that if (a) and (b) can be proved for the modules M_1 and M_2 with GKdim $\le i$, and $0 \to M_1 \to M \to M_2 \to 0$ is exact, then (a) and (b) also hold for M. By the noetherian property and the fact above, to prove that (a) and (b) hold for a noetherian module M it suffices to show (a) and (b) hold for some nonzero submodule of M. By the SSC hypothesis, we may assume that M has a proper similar submodule, i.e., there is an exact sequence

$$0 \to \tilde{M}(-l) \to M \to \overline{M} \to 0 \tag{E8}$$

for some $\tilde{M}$ similar to M and for some $l > 0$, where $\overline{M} = M/\tilde{M}(-l)$. By Lemma 2.2(3), GKdim $\overline{M}$ = GKdim $M - 1$. Applying $\underline{\operatorname{Ext}}^{\cdot}(-, A)$ to (E8) we have an exact sequence

$$\to \underline{\operatorname{Ext}}^j(\overline{M}, A) \to \underline{\operatorname{Ext}}^j(M, A) \to \underline{\operatorname{Ext}}^j\big(\tilde{M}(-l), A\big) \to \underline{\operatorname{Ext}}^{j+1}(\overline{M}, A) \to . \tag{E9}$$

If $j < d - i$, by induction hypothesis (a), the left and right terms of (E9) are zero. Since $\tilde{M} \sim M$, $\underline{\operatorname{Ext}}^j(M, A) \cong \underline{\operatorname{Ext}}^j(\tilde{M}, A)$ as graded k-vector spaces. Since $\underline{\operatorname{Ext}}^j(M, A)$ is left bounded, (E9) implies that $\underline{\operatorname{Ext}}^j(M, A) = 0$. Thus $j(M) \ge d - i$. If GKdim $\underline{\operatorname{Ext}}^{d-i}(M, A) < i$, then, by induction hypothesis (c), GHdim $\underline{\operatorname{Ext}}^s(M, A) < i$ for all s. Applying induction hypotheses (a) and (c) to the modules $\underline{\operatorname{Ext}}^s(M, A)$ for all s, we obtain that the GK-dimension of $\underline{\operatorname{Ext}}^p(\underline{\operatorname{Ext}}^q(M, A), A)$ is less than i for all p, q. Then the spectral sequence (E3) implies that GKdim $M < i$, a contradiction. Hence GKdim $\underline{\operatorname{Ext}}^{d-i}(M, A) = i$ and, consequently, $j(M) = d - i$. Thus we have proved (a) and (b). It remains to prove (c). By the noetherian property and the long exact sequence on $\underline{\operatorname{Ext}}^{\cdot}(-, A)$ and the SSC hypothesis, we may assume that N has a proper similar submodule and that there is a short exact sequence similar to (E8) for N. Letting $j = d - i$ and $M = N$ in (E9), we obtain an exact sequence

$$\to \underline{\operatorname{Ext}}^{d-i}(N, A) \to \underline{\operatorname{Ext}}^{d-i}\big(\tilde{N}(-l), A\big) \to \underline{\operatorname{Ext}}^{d-i+1}(\overline{N}, A) \to . \tag{E10}$$

By induction hypothesis (c), GKdim $\underline{\operatorname{Ext}}^{d-i+1}(\overline{N}, A) \le i - 1$. Since $N \sim \tilde{N}$ as chosen, $\underline{\operatorname{Ext}}^i(N, A) \cong \underline{\operatorname{Ext}}^i(\tilde{N}, A)$ as graded k-vector spaces. Applying Lemma 2.2(3) to (E10) we have GKdim $\underline{\operatorname{Ext}}^{d-i}(N, A) \le i$. Therefore (c) follows and we have finished our proof of (1).

(2) By (1), GKdim $M = d - j(M) =: \delta(M)$, and by [7, 4.5], GKdim is finitely partitive in the sense of [8, 8.3.17]. Hence by [8, 8.3.18], GKdim M $\ge$ Kdim M. Combining this inequality with Lemma 2.2(2), (2) follows. ∎

(1) and (2) of the following theorem were also proved in [1] under some weaker hypotheses and a part of (1) was proved in [7, 5.3]. Recall that a ring is called *quasi-Frobenius* if it is left and right artinian and self-injective.

THEOREM 3.2. *Let A be a connected noetherian, Auslander–Gorenstein, and Cohen–Macaulay algebra of injective dimension d. Then*:

(1) *A has a quasi-Frobenius ungraded quotient ring.*

(2) *For every minimal prime ideal P*, $\operatorname{GKdim} A/P = d$.

(3) *If, moreover, A has finite global dimension, then A is a domain and a maximal order in its quotient division ring.*

Proof. (1) and (2). By [7, 3.1 and 5.8], A is Auslander–Gorenstein and Cohen–Macaulay as an ungraded algebra, i.e., the Auslander–Gorenstein and the Cohen–Macaulay conditions hold for finite ungraded A-modules. Let $N := N(A)$ be the intersection of all prime ideals of A, which is called the *prime radical* of A. By [8, 8.3.14], N is left and right invariant with respect to GKdim in the sense of [8, 6.8.13]. In particular N is left and right weakly invariant with respect to GKdim (for definition, see [8, 6.8.13]). By the Cohen–Macaulay condition, $\operatorname{GKdim} M = \operatorname{GKdim} A$ for all nonzero submodules $M \subset A$, i.e., A is homogeneous with respect to GKdim in the sense of [8, 6.8.8]. By [8, 6.8.15], A has a left and right artinian quotient ring Q. Let I be an ideal of a ring R and let $\mathscr{C}(I)$ denote the set of elements in R which are regular in R/I. By [8, 4.1.3 and 4.1.4], we obtain $\mathscr{C}(0) = \mathscr{C}(N)$, QN is the prime radical of Q, and $Q/QN \cong Q(A/N)$, where $Q(A/N)$ is the quotient ring of the semiprime noetherian ring A/N. Let P be a minimal prime ideal of A. Since N is nilpotent, P/N is a minimal prime of A/N. Suppose that $\operatorname{GKdim}(A/P) < d$. By [8, 6.8.14(ii) and 6.8.15], there is a regular element $c \in \mathscr{C}(0) = \mathscr{C}(N)$ such that $c \in P$. Thus $A/P \otimes Q(A/N) = 0$ and then $\operatorname{Hom}(A/P, A/N) = 0$. This contradicts the fact that P/N is a minimal prime of A/N. Therefore $\operatorname{GKdim}(A/P) = d$ and by the Cohen–Macaulay property, $\operatorname{Hom}(A/P, A) \neq 0$.

It remains to show that Q is self-injective. Since Q is a localization of A, $n := \operatorname{injdim} Q \leq \operatorname{injdim} A$. Assume on the contrary that $n > 0$. Since $Q/QN \cong Q(A/N)$, there is a one-to-one correspondence between minimal prime ideals of A and prime ideals of Q via $P \leftrightarrow PQ$. Since Q is artinian, every simple Q-module M has a finite direct sum isomorphic to Q/PQ for some minimal prime ideal $P \subset A$. Hence, for some l,

$$\operatorname{Hom}(M^{\oplus l}, Q) \cong \operatorname{Hom}(Q/PQ, Q) \cong \operatorname{Hom}(A/P \otimes Q, A \otimes Q) \neq 0$$

because $\mathrm{Hom}(A/P, A) \neq 0$. Consequently, $\mathrm{Hom}(M, Q) \neq 0$. Therefore $\mathrm{Hom}(L, Q) \neq 0$ for every nonzero finite Q-module L. By the spectral sequence (E3), which holds for ungraded rings [7, 2.2], $E_2^{0,n} := \mathrm{Hom}(\mathrm{Ext}^n(L, Q), Q) = 0$. This implies that $\mathrm{Ext}^n(L, Q) = 0$ for all finite modules L. Hence the injective dimension of Q is less than n, a contradiction. Therefore $n = 0$ and Q is self-injective.

(3) Follows from [11, 2.10]. ∎

Now we are ready to finish our proof of Theorem 0.2.

Proof of Theorem 0.2. If A has enough normal elements, then, by Proposition 2.3, A is AS-Gorenstein and satisfies SSC. Hence most of Theorem 0.2 follows from Theorems 3.1 and 3.2. It remains to show that $\mathrm{GKdim}\, A/I = \mathrm{clKdim}\, A/I$ for all graded ideals $I \subset A$ and that every minimal prime ideal of a connected algebra is graded.

We prove the second statement first. It is easy to see that graded prime is prime and that for every nonnilpotent element x, there is a graded prime ideal P such that $x^n \notin P$ for all n. Hence the intersection of all graded minimal prime ideals is the prime radical N. However, every (ungraded) minimal prime ideal must appear in the intersection. Therefore every minimal prime ideal is graded.

By [8, 6.4.5] and Theorem 3.1(2), $\mathrm{clKdim}\, A/I \leq \mathrm{Kdim}\, A/I = \mathrm{GKdim}\, A/I$. We show next that $\mathrm{clKdim}\, A/I \geq \mathrm{GKdim}\, A/I$ for all graded ideals I. Since A is noetherian and every minimal prime ideal of A/I is graded as proved in the last paragraph, we only need to prove the inequality when I is a graded prime ideal. Pick a nonzero normal and regular element $x \in (A/I)_{\geq 1}$. By induction we have

$$\begin{aligned} \mathrm{clKdim}\, A/I &\geq \mathrm{clKdim}\, A/(I + x) + 1 \geq \mathrm{GKdim}\, A/(I + x) + 1 \\ &= \mathrm{GKdim}\, A/I. \end{aligned}$$

Therefore $\mathrm{clKdim}\, A/I = \mathrm{GKdim}\, A/I$. ∎

Several families of quantum algebras listed in [5] can be constructed by localizing some special normal elements in connected noetherian algebras with enough normal elements. The base connected algebras have also finite global dimension (or finite injective dimension). Hence by Theorem 0.2 these algebras are Auslander−Gorenstein and Cohen−Macaulay. By [1], the Auslander−Gorenstein and Cohen−Macaulay properties are preserved under localization. If, in addition, the prime spectra of these algebras are normally separated, then [5, 1.6] implies that these algebras are *catenary* in the sense that, for any two prime ideals $P \subset Q$ of A, all saturated chains of prime ideals between P and Q has the same length. In particular the following is a consequence of [5, 1.6] and Theorem 0.2.

COROLLARY 3.3. *Let A be a connected noetherian algebra with finite injective dimension. Suppose that* Spec *A is normally separated. Then A is catenary.*

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