A Koszul category of representations of finitary Lie algebras

Elizabeth Dan-Cohen[a,1,*], Ivan Penkov[a,2], Vera Serganova[b,3]

[a] *Jacobs University Bremen, Campus Ring 1, 28759 Bremen, Germany*
[b] *University of California Berkeley, Berkeley CA 94720, USA*

Abstract

We find for each simple finitary Lie algebra $\mathfrak{g}$ a category $\mathbb{T}_{\mathfrak{g}}$ of integrable modules in which the tensor product of copies of the natural and conatural modules are injective. The objects in $\mathbb{T}_{\mathfrak{g}}$ can be defined as the finite length absolute weight modules, where by absolute weight module we mean a module which is a weight module for every splitting Cartan subalgebra of $\mathfrak{g}$. The category $\mathbb{T}_{\mathfrak{g}}$ is Koszul in the sense that it is antiequivalent to the category of locally unitary finite-dimensional modules over a certain direct limit of finite-dimensional Koszul algebras. We describe these finite-dimensional algebras explicitly. We also prove an equivalence of the categories $\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ corresponding respectively to the orthogonal and symplectic finitary Lie algebras $o(\infty)$, $sp(\infty)$.

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*Corresponding author. Fax: +49 421 200-3103
Email addresses: `elizabeth.dancohen@gmail.com` (Elizabeth Dan-Cohen), `i.penkov@jacobs-university.de` (Ivan Penkov), `serganov@math.berkeley.edu` (Vera Serganova)

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1. Introduction

The classical simple complex Lie algebras $sl(n)$, $o(n)$, $sp(2n)$ have several natural infinite-dimensional versions. In this paper we concentrate on the "smallest possible" such versions: the direct limit Lie algebras $sl(\infty) := \varinjlim (sl(n))_{n\in\mathbb{Z}_{>2}}$, $o(\infty) := \varinjlim (o(n))_{n\in\mathbb{Z}_{\geq 3}}$, $sp(\infty) := \varinjlim (sp(2n))_{n\in\mathbb{Z}_{\geq 2}}$. From a traditional finite-dimensional point of view, these Lie algebras are a suitable language for various stabilization phenomena, for instance stable branching laws as studied by R. Howe, E.-C. Tan and J. Willenbring [**HTW**]. The direct limit Lie algebras $sl(\infty)$, $o(\infty)$, $sp(\infty)$ admit many characterizations: for instance, they represent (up to isomorphism) the simple finitary (locally finite) complex Lie algebras [**B**, **BSt**]. Alternatively, these Lie algebras are the only three locally simple locally finite complex Lie algebras which admit a root decomposition [**PStr**].

Several attempts have been made to build a basic representation theory for $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$. As the only simple finite-dimensional representation of $\mathfrak{g}$ is the trivial one, one has to study infinite-dimensional representations. On the other hand, it is still possible to study representations which are close analogs of finite-dimensional representations. Such a representation should certainly be integrable, i.e. it should be isomorphic to a direct sum of finite-dimensional representations when restricted to any simple finite-dimensional subalgebra.

The first phenomenon one encounters when studying integrable representations of $\mathfrak{g}$ is that they are not in general semisimple. This phenomenon has been studied in [**PStyr**] and [**PS**], but it had not previously been placed within a known more general framework for non-semisimple categories. The main purpose of the present paper is to show that the notion of Koszulity for a category of modules over a graded ring, as defined by A. Beilinson, V. Ginzburg and W. Soergel in [**BGS**], provides an excellent tool for the study of integrable representations of $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$.

In this paper we introduce the category $\mathbb{T}_\mathfrak{g}$ of tensor $\mathfrak{g}$-modules. The objects of $\mathbb{T}_\mathfrak{g}$ are defined at first by the equivalent abstract conditions of Theorem 3.4. Later we show in Corollary 4.6 that the objects of $\mathbb{T}_\mathfrak{g}$ are nothing but finite length submodules of a direct sum of several copies of the tensor algebra T of the natural and conatural representations. In the finite-dimensional case, i.e. for $sl(n)$, $o(n)$, or $sp(2n)$, the appropriate tensor algebra is a cornerstone of the theory of finite-dimensional representations (Schur-Weyl duality, etc.). In the infinite-dimensional case, the tensor algebra T was studied by Penkov and K. Styrkas in [**PStyr**]; nevertheless its indecomposable direct summands were not understood until now from a categorical point of view.

We prove that these indecomposable modules are precisely the indecomposable injectives in the category $\mathbb{T}_{\mathfrak{g}}$. Furthermore, the category $\mathbb{T}_{\mathfrak{g}}$ is Koszul in the following sense: $\mathbb{T}_{\mathfrak{g}}$ is antiequivalent to the category of locally unitary finite-dimensional modules over an algebra $\mathcal{A}_{\mathfrak{g}}$ which is a direct limit of finite-dimensional Koszul algebras (see Proposition 5.1 and Theorem 5.5).

Moreover, we prove in Corollary 6.4 that for $\mathfrak{g} = sl(\infty)$ the Koszul dual algebra $(\mathcal{A}_{\mathfrak{g}}^{!})^{\mathrm{opp}}$ is isomorphic to $\mathcal{A}_{\mathfrak{g}}$. This together with the main result of [**PStyr**] allows us to give an explicit formula for the Ext group between any two simple objects of $\mathbb{T}_{\mathfrak{g}}$ when $\mathfrak{g} = sl(\infty)$. For the cases of $\mathfrak{g} = o(\infty)$ and $\mathfrak{g} = sp(\infty)$ we discover another interesting fact: the algebras $\mathcal{A}_{o(\infty)}$ and $\mathcal{A}_{sp(\infty)}$ are isomorphic. This yields an equivalence of categories $\mathbb{T}_{o(\infty)} \simeq \mathbb{T}_{sp(\infty)}$, which is Corollary 6.11.

In summary, the results of the present paper show how the non-semisimplicity of tensor modules arising from the limit process $n \to \infty$ falls strikingly into the general Koszul pattern discovered by Beilinson, Ginzburg and Soergel. This enables us to uncover the structure of the category of tensor representations of $\mathfrak{g}$.

Since the present paper has been submitted there have been several developments. First, in [**SS**] the categories $\mathbb{T}_{\mathfrak{g}}$ have been studied from a different perspective. In particular, it is shown there that these categories satisfy important universality properties in the class of abelian symmetric monoidal categories.

In [**PS1**] categories of tensor modules have been introduced for a larger class of infinite-dimensional Lie algebras, and it has been shown that these categories are equivalent to $\mathbb{T}_{\mathfrak{g}}$ for appropriate $\mathfrak{g}$. In [**Sr**] results from the present paper are generalized to the case of classical Lie superalgebras.

Finally, in [**FPS**] the category $\mathbb{T}_{sl(\infty)}$ has been used to categorify the boson-fermion correspondence.

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2. Preliminaries

The ground field is $\mathbb{C}$. By S_n we denote the n-th symmetric group, and by $\mathbb{C}[S_n]$ its group algebra. The sign $\otimes$ stands for $\otimes_{\mathbb{C}}$, and the sign $\supsetplus$ stands for the semidirect sum of Lie algebras. We denote by $(\,\cdot\,)^*$ the algebraic dual, i.e. $\mathrm{Hom}_{\mathbb{C}}(\,\cdot\,,\mathbb{C})$.

Let $\mathfrak{g}$ be one of the infinite-dimensional simple finitary Lie algebras, $sl(\infty)$, $o(\infty)$, or $sp(\infty)$. Here $sl(\infty) = \varinjlim sl(n)$, $o(\infty) = \varinjlim o(n)$, $sp(\infty) = \varinjlim sp(2n)$, where in each direct limit the inclusions can be chosen as "left upper corner" inclusions. We consider the "exhaustion" $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ to be fixed, taking $\mathfrak{g}_n = sl(n)$ for $\mathfrak{g} = sl(\infty)$, $\mathfrak{g}_n = o(2n)$ or $\mathfrak{g}_n = o(2n+1)$ for $\mathfrak{g} = o(\infty)$, and $\mathfrak{g}_n = sp(2n)$ for $\mathfrak{g} = sp(\infty)$. By G_n we denote the adjoint group of $\mathfrak{g}_n$ It is clear that $\{G_n\}$ forms a direct system and defines an ind-group $G = \varinjlim G_n$. As mentioned in the introduction, the Lie algebras $sl(\infty)$, $o(\infty)$, and $sp(\infty)$ admit several equivalent intrinsic descriptions, see for instance [**B**, **BSt**, **PStr**].

It is clear from the definition of $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$ that the notions of semisimple or nilpotent elements make sense: an element $g \in \mathfrak{g}$ is *semisimple* (respectively, *nilpotent*) if g is semisimple (resp., nilpotent) as an element of $\mathfrak{g}_n$ for some n. In [**NP**, **DPS**], Cartan subalgebras of $\mathfrak{g}$ have been studied. In the present paper we need only the notion of a *splitting Cartan subalgebra* of $\mathfrak{g}$: this is a maximal toral (where *toral* means consisting of semisimple elements) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}$ is an $\mathfrak{h}$-weight module, i.e.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h)g \text{ for all } h \in \mathfrak{h}\}$. The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq 0\}$ is the *set of* $\mathfrak{h}$*-roots* of $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$ and M is a $\mathfrak{g}$-module, M is an $\mathfrak{h}$*-weight module* if

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^\alpha,$$

where $M^\alpha := \{m \in M \mid h \cdot m = \alpha(h)m \text{ for all } h \in \mathfrak{h}\}$.

By V we denote the natural representation of $\mathfrak{g}$; that is, $V = \varinjlim V_n$, where V_n is the natural representation of $\mathfrak{g}_n$. We set also $V_* = \varinjlim V_n^*$; this is the conatural representation of $\mathfrak{g}$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, $V \simeq V_*$, whereas $V \not\simeq V_*$ for $\mathfrak{g} = sl(\infty)$. Note that V_* is a submodule of the algebraic dual

$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of V. Moreover, $\mathfrak{g} \subset V \otimes V_*$, and $sl(\infty)$ can be identified with the kernel of the contraction $\phi : V \otimes V_* \to \mathbb{C}$, while

$$
\begin{aligned}
\mathfrak{g} &\simeq \Lambda^2(V) \subset V \otimes V = V \otimes V_* &\qquad& \text{for } \mathfrak{g} = o(\infty),\\
\mathfrak{g} &\simeq S^2(V) \subset V \otimes V = V \otimes V_* &\qquad& \text{for } \mathfrak{g} = sp(\infty).
\end{aligned}
$$

Let $\tilde{G}$ be the subgroup of $\mathrm{Aut}\, V$ consisting of those automorphisms for which the induced automorphism of V^* restricts to an automorphism of V_*. Then clearly $G \subset \tilde{G} \subset \mathrm{Aut}\, \mathfrak{g}$, and moreover $\tilde{G} = \mathrm{Aut}\, \mathfrak{g}$ for $\mathfrak{g} = o(\infty)$, $sp(\infty)$ [**BBCM**, Corollary 1.6 (b)]. For $\mathfrak{g} = sl(\infty)$, the group $\tilde{G}$ has index 2 in $\mathrm{Aut}\, \mathfrak{g}$: the quotient $\mathrm{Aut}\, \mathfrak{g}/\tilde{G}$ is represented by the automorphism

$$g \mapsto -g^t$$

for $g \in sl(\infty)$ [**BBCM**, Corollary 1.2 (a)].

It is essential to recall that if $\mathfrak{g} = sl(\infty)$, $sp(\infty)$, all splitting Cartan subalgebras of $\mathfrak{g}$ are $\tilde{G}$-conjugate, while there are two $\tilde{G}$-conjugacy classes for $\mathfrak{g} = o(\infty)$. One class comes from the exhaustion of $o(\infty)$ as $\varinjlim o(2n)$, and the other from the exhaustion of the form $\varinjlim o(2n+1)$. For further details we refer the reader to [**DPS**]. Here are the explicit forms of the root systems of $\mathfrak{g}$:

$$
\begin{aligned}
&\{\epsilon_i - \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = sl(\infty),\ \mathfrak{g}_n = sl(n),\\
&\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \cup \{\pm 2\epsilon_i \mid i \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = sp(\infty),\ \mathfrak{g}_n = sp(2n),\\
&\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = o(\infty),\ \mathfrak{g}_n = o(2n),\\
&\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \cup \{\pm\epsilon_i \mid i \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = o(\infty),\ \mathfrak{g}_n = o(2n+1).
\end{aligned}
$$

Our usage of $\epsilon_i \in \mathfrak{h}^*$ is compatible with the standard usage of ϵ_i as a linear function on $\mathfrak{h} \cap \mathfrak{g}_n$ for all $n > i$.

In the present paper we study integrable $\mathfrak{g}$-modules M for $\mathfrak{g} \simeq sl(\infty)$, $o(\infty)$, $sp(\infty)$. By definition, a $\mathfrak{g}$-module M is *integrable* if $\dim\{m, g \cdot m, g^2 \cdot m, ...\} < \infty$ for all $g \in \mathfrak{g}$, $m \in M$. More generally, if M is any $\mathfrak{g}$-module, the set $\mathfrak{g}[M]$ of M-locally finite elements in $\mathfrak{g}$, that is

$$\mathfrak{g}[M] := \{g \in \mathfrak{g} \mid \dim\{m, g \cdot m, g^2 \cdot m, \dots\} < \infty \text{ for all } m \in M\},$$

is a Lie subalgebra of $\mathfrak{g}$. This follows from the analogous fact for finite-dimensional Lie algebras, discovered and rediscovered by several mathematicians [**GQS**, **F**, **K**]. We refer to $\mathfrak{g}[M]$ as the *Fernando-Kac subalgebra* of M.

By $\mathfrak{g}$-mod we denote the category of all $\mathfrak{g}$-modules, and following the notation of [**PS**], we let $\mathrm{Int}_{\mathfrak{g}}$ denote the category of integrable $\mathfrak{g}$-modules. We have the functor

$$\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \to \mathrm{Int}_{\mathfrak{g}}$$

which takes an arbitrary $\mathfrak{g}$-module to its largest integrable submodule.

3. The category $\mathbb{T}_{\mathfrak{g}}$

If $\gamma \in \operatorname{Aut}\mathfrak{g}$ and M is a $\mathfrak{g}$-module, let M^γ denote the $\mathfrak{g}$-module twisted by γ: that is, M^γ is equal to M as a vector space, and the $\mathfrak{g}$-module structure on M^γ is given by $\gamma(g) \cdot m$ for $m \in M^\gamma$ and $g \in \mathfrak{g}$.

Definition 3.1.
1. A $\mathfrak{g}$-module M is called an *absolute weight module* if M is an $\mathfrak{h}$-weight module for every splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.
2. A $\mathfrak{g}$-module M is called *$\tilde{G}$-invariant* if for any $\gamma \in \tilde{G}$ there is a $\mathfrak{g}$-isomorphism $M^\gamma \simeq M$.
3. A subalgebra of $\mathfrak{g}$ is called *finite corank* if it contains the commutator subalgebra of the centralizer of some finite-dimensional subalgebra of $\mathfrak{g}$.

Proposition 3.2. *Any absolute weight $\mathfrak{g}$-module is integrable.*

Proof. Let M be an absolute weight $\mathfrak{g}$-module. Every semisimple element h of $\mathfrak{g}$ lies in some splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and since M is an $\mathfrak{h}$-weight module, we see that h acts locally finitely on M. As $\mathfrak{g}$ is generated by its semisimple elements, the Fernando-Kac subalgebra $\mathfrak{g}[M]$ equals $\mathfrak{g}$, i.e. M is integrable. $\square$

We define the category of absolute weight modules as the full subcategory of $\mathfrak{g}$-mod whose objects are the absolute weight modules. Proposition 3.2 shows that the category of absolute weight modules is in fact a subcategory of $\mathrm{Int}_{\mathfrak{g}}$.

Lemma 3.3. *For each n one has $\tilde{G} = G \cdot \tilde{G}'_n$, where*

$$\tilde{G}'_n := \{\gamma \in \tilde{G} \mid \gamma(g) = g \text{ for all } g \in \mathfrak{g}_n\}.$$

Proof. Let $\mathfrak{g}$ be $o(\infty)$ or $sp(\infty)$, and let $\gamma \in \tilde{G}$. Fix a basis $\{w_i\}$ of V_n. There exists $\gamma'' \in G$ such that $(\gamma'')^{-1}(\gamma(w_i)) = w_i$ for all $1 \leq i \leq 2n$. Since $\mathfrak{g} \subset V \otimes V$ and $\mathfrak{g}_n = \mathfrak{g} \cap (V_n \otimes V_n)$, we see that $(\gamma'')^{-1}\gamma \in \tilde{G}'_n$.

For $\mathfrak{g} = sl(\infty)$, the analogous statement is as follows. In this case one has $\mathfrak{g}_n = \mathfrak{g} \cap (V_n \otimes V_n^*)$. Fix dual bases $\{w_i\}$ and $\{w_i^*\}$ of V_n and V_n^*, respectively. Then for any $\gamma \in \tilde{G}$, there is an element $\gamma'' \in G$ such that $(\gamma'')^{-1}(\gamma(w_i)) = w_i$ and $(\gamma'')^{-1}(\gamma(w_i^*)) = w_i^*$ for each $1 \leq i \leq n$. Therefore $(\gamma'')^{-1}\gamma \in \tilde{G}'_n$. □

Theorem 3.4. *The following conditions on a $\mathfrak{g}$-module M of finite length are equivalent:*

1. *M is an absolute weight module.*
2. *M is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and M is $\tilde{G}$-invariant.*
3. *M is integrable and $\operatorname{Ann}_{\mathfrak{g}} m$ is finite corank for all $m \in M$.*

Proof. Let us show that (1) implies (3). We already proved in Proposition 3.2 that a $\mathfrak{g}$-module M satisfying (1) is integrable. Furthermore, it suffices to prove that $\operatorname{Ann}_{\mathfrak{g}} m$ is finite corank for all $m \in M$ under the assumption that the $\mathfrak{g}$-module M is simple. This follows from the observation that a finite intersection of finite corank subalgebras is finite corank.

Fix a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$; let $\mathfrak{b} = \mathfrak{h} \ni \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$ whose set of roots (i.e. positive roots) is denoted by Δ^+. For each positive root α, let e_α, h_α, f_α be a standard basis for the corresponding root $sl(2)$-subalgebra. Fix additionally a nonzero $\mathfrak{h}$-weight vector $m \in M$.

Choose a set of commuting mutually orthogonal positive roots $Y \subset \Delta^+$. The set of semisimple elements $\{h_\alpha + e_\alpha \mid \alpha \in Y\}$ is $\tilde{G}$-conjugate to the set $\{h_\alpha \mid \alpha \in Y\}$, and can thus be extended to a splitting Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$. Since M is an absolute weight module, there is a nonzero $\mathfrak{h}'$-weight vector $m' \in M$. As M is simple, it must be that $m \in U(\mathfrak{g}) \cdot m'$. Moreover, one has $m \in U(\mathfrak{g}_n) \cdot m'$ for some n. For almost all $\alpha \in Y$, h_α and e_α commute with $\mathfrak{g}_n$, in which case m is an eigenvector for $h_\alpha + e_\alpha$. Thus $e_\alpha \cdot m$ is a scalar multiple of m. Since M is integrable, e_α acts locally nilpotently, and we conclude that $e_\alpha \cdot m = 0$ for all but finitely many α. By considering the set $\{h_\alpha + f_\alpha \mid \alpha \in Y\}$ in place of $\{h_\alpha + e_\alpha \mid \alpha \in Y\}$, we see that $f_\alpha \cdot m = 0$ for all but finitely many α, and hence $e_\alpha \cdot m = f_\alpha \cdot m = 0$ for all but finitely many $\alpha \in Y$.

We now consider separately each of the three possible choices of $\mathfrak{g}$. For $\mathfrak{g} = sl(\infty)$, we may assume that the simple roots of $\mathfrak{b}$ are of the form $\{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$. We first choose the set of commuting mutually orthogonal positive roots to be $Y_1 = \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$ and obtain in this way that $e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0$ for almost all odd indices i. By choosing the set of commuting mutually orthogonal positive roots as $Y_2 = \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$, we have $e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0$ for almost all even indices i, hence for almost all i. Since it contains $e_{\epsilon_i - \epsilon_{i+1}}$ and $f_{\epsilon_i - \epsilon_{i+1}}$ for almost all i, the subalgebra $\mathrm{Ann}_{\mathfrak{g}}\, m$ is a finite corank subalgebra of $\mathfrak{g} = sl(\infty)$.

For $\mathfrak{g} = o(\infty)$, one may assume that the set of simple roots of $\mathfrak{g}$ is $\{-\epsilon_1 - \epsilon_2\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$. In this case in addition to the two sets $Y_1 = \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$ and $Y_2 = \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$, one considers also the set of commuting mutually orthogonal positive roots $Y_3 = \{-\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$. For $\mathfrak{g} = sp(\infty)$, the set of simple roots can be chosen as $\{-2\epsilon_1\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$, and one considers the following three sets of commuting mutually orthogonal positive roots:

$$\begin{aligned} Y_1 =& \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\} \\ Y_2 =& \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\} \\ Y_3 =& \{-2\epsilon_i \mid i \in \mathbb{Z}_{>0}\}. \end{aligned}$$

In both cases $\mathrm{Ann}_{\mathfrak{g}}\, m$ contains e_α, f_α for all but finitely many $\alpha \in Y_1 \cup Y_2 \cup Y_3$. Hence we conclude that the subalgebra $\mathrm{Ann}_{\mathfrak{g}}\, m$ is a finite corank subalgebra of $\mathfrak{g}$; that is, (1) implies (3).

Next we prove that (3) implies (2).

We first show that a $\mathfrak{g}$-module M satisfying (3) is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Fix a finite set $\{m_1, \dots, m_s\}$ of generators of M. Let $\mathfrak{g}'_n$ be the commutator subalgebra of the centralizer in $\mathfrak{g}$ of $\mathfrak{g}_n$. There exists a finite corank subalgebra that annihilates $m_1, \dots, m_s$, and hence $\mathfrak{g}'_n$ annihilates $m_1, \dots, m_s$ for some n. Let $\mathfrak{h}'_n$ be a splitting Cartan subalgebra of $\mathfrak{g}'_n$. Obviously M is semisimple over $\mathfrak{h}'_n$. One can find k and a Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{g}_k$ such that $\mathfrak{h} = \mathfrak{h}'_n + \mathfrak{h}_k$ is a splitting Cartan subalgebra of $\mathfrak{g}$. (If $\mathfrak{g} = o(\infty)$ or $sp(\infty)$ one can choose $k = n$; if $\mathfrak{g} = sl(\infty)$, one can set $k = n + 1$). Since M is integrable, M is semisimple over $\mathfrak{h}_k$. Hence M is semisimple over $\mathfrak{h}$.

To finish the proof that (3) implies (2), we need to show that M is $\tilde{G}$-invariant. For each n one has $\tilde{G} = G \cdot \tilde{G}'_n$ by Lemma 3.3. Fix $\gamma \in \tilde{G}$

and $m \in M$. Then for some n, the vector m is fixed by $\mathfrak{g}'_n$. We choose a decomposition $\gamma = \gamma''\gamma'$ so that $\gamma' \in \tilde{G}'_n$ and $\gamma'' \in G$. We then set $\gamma(m) := \gamma''(m)$, and note that the action of G on M is well defined because M is assumed to be integrable. This yields a well-defined $\tilde{G}$-module structure on M since, for any other decomposition $\gamma = \bar{\gamma}''\bar{\gamma}'$ as above, one has $(\bar{\gamma}'')^{-1}\gamma'' = \bar{\gamma}'(\gamma')^{-1} \in \tilde{G}'_n \cap G = \{\gamma \in G \mid \gamma(g) = g \text{ for all } g \in \mathfrak{g}_n\}$ which must preserve m.

Fix now $\gamma \in \tilde{G}$ and consider the linear operator

$$\varphi_\gamma : M^\gamma \to M, \quad m \mapsto \gamma^{-1}(m).$$

We claim that φ_γ is an isomorphism. For this we need to check that $g \cdot \varphi_\gamma(m) = \varphi_\gamma(\gamma(g) \cdot m)$ for any $g \in \mathfrak{g}$ and $m \in M$. We have $g \cdot \varphi_\gamma(m) = g \cdot (\gamma^{-1}(m))) = \varphi_\gamma(\gamma(g \cdot \gamma^{-1}(m)))$, hence it suffices to check that $\gamma(g \cdot \gamma^{-1}(m)) = \gamma(g) \cdot m$ for every $g \in \mathfrak{g}$ and $m \in M$. After choosing a decomposition $\gamma = \gamma''\gamma'$ such that $\gamma'' \in G$ and γ' fixes m, g and $g \cdot m$, all that remains to check is that

$$\gamma''(g \cdot (\gamma'')^{-1}(m)) = \gamma''(g) \cdot m$$

for all $g \in \mathfrak{g}$. This latter equality is the well-known relation between the G-module structure on M and the adjoint action of G on $\mathfrak{g}$.

To complete the proof of the theorem we need to show that (2) implies (1). What is clear is that (2) implies a slightly weaker statement, namely that M is a weight module for any splitting Cartan subalgebra belonging to the same $\tilde{G}$-conjugacy class as the given splitting Cartan subalgebra $\mathfrak{h}$. For $\mathfrak{g} = sl(\infty), sp(\infty)$, this proves (1), as all splitting Cartan subalgebras are conjugate under $\tilde{G}$.

Consider now the case $\mathfrak{g} = o(\infty)$. In this case there are two $\tilde{G}$-conjugacy classes of splitting Cartan subalgebras [**DPS**]. Note that if M is semisimple over every Cartan subalgebra from one $\tilde{G}$-conjugacy class, then (3) holds as follows from the proof of the implication (1) $\Rightarrow$ (3). Furthermore, the proof that a $\mathfrak{g}$-module of finite length M satisfying (3) is a weight module for some splitting Cartan subalgebra involves a choice of $\mathfrak{g}_n$. For $\mathfrak{g} = o(\infty)$ there are two different possible choices, namely $\mathfrak{g}_n = o(2n)$ and $\mathfrak{g}_n = o(2n+1)$, which in turn produce splitting Cartan subalgebras from the two $\tilde{G}$-conjugacy classes. This shows that in each $\tilde{G}$-conjugacy class there is a splitting Cartan subalgebra of $\mathfrak{g}$ for which M is a weight module, and hence we may conclude that (2) implies (1) also for $\mathfrak{g} = o(\infty)$. □

Corollary 3.5. *Let M be a module satisfying the conditions of Theorem 3.4. Then $M = \bigcup_{n>0} M^{\mathfrak{g}'_n}$, where $\mathfrak{g}'_n$ is the commutator of the centralizer of $\mathfrak{g}_n$ in $\mathfrak{g}$.*

Proof. Any submodule of M with the property that the annihilator of each of its elements contains $\mathfrak{g}'_n$ for some n is clearly contained in $\bigcup_{n>0} M^{\mathfrak{g}'_n}$. Condition (3) of Theorem 3.4 states that $\operatorname{Ann}_{\mathfrak{g}} m$ is finite corank for all $m \in M$, which is to say that M has this property. □

Corollary 3.6. *Let $\mathfrak{g} = o(\infty)$ and M be a finite length $\mathfrak{g}$-module which is an $\mathfrak{h}$-weight module for all splitting Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ in either of the two $\tilde{G}$-conjugacy classes. Then M is an $\mathfrak{h}$-weight module for all splitting Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let M be a finite length $\mathfrak{g}$-module which is a weight module for all splitting Cartan subalgebras in the $\tilde{G}$-conjugacy class of $\mathfrak{h}$. Then M is integrable, by the same argument as in the proof of Proposition 3.2. Finally, (3) holds by the same proof as that of (1)⇒(3) in Theorem 3.4. □

We denote by $\mathbb{T}_{\mathfrak{g}}$ the full subcategory of $\mathfrak{g}$-mod consisting of finite length modules satisfying the equivalent conditions of Theorem 3.4. Then $\mathbb{T}_{\mathfrak{g}}$ is an abelian category and a monoidal category with respect to the usual tensor product of $\mathfrak{g}$-modules, and $\mathbb{T}_{\mathfrak{g}}$ is a subcategory of the category of absolute weight modules. In addition, for $\mathfrak{g} = sl(\infty)$, $\mathbb{T}_{\mathfrak{g}}$ has an involution

$$(\,\cdot\,)_* : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{g}},$$

which one can think of as "restricted dual." Indeed, in this case any outer automorphism $w \in \operatorname{Aut} \mathfrak{g}$ induces the autoequivalence of categories

$$\begin{aligned} w_{\mathfrak{g}} : \mathbb{T}_{\mathfrak{g}} &\to \mathbb{T}_{\mathfrak{g}} \\ M &\mapsto M^w. \end{aligned}$$

Since, however, any object of $\mathbb{T}_{\mathfrak{g}}$ is $\tilde{G}$-invariant, the functor $w_{\mathfrak{g}}$ does not depend on the choice of w and is an involution, i.e. $w_{\mathfrak{g}}^2 = \mathrm{id}$. We denote this involution by $(\,\cdot\,)_*$ in agreement with the fact that it maps V to V_*. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, we define $(\,\cdot\,)_*$ to be the trivial involution on $\mathbb{T}_{\mathfrak{g}}$.

4. Simple objects and indecomposable injectives of $\mathbb{T}_{\mathfrak{g}}$

Next we describe the simple objects of $\mathbb{T}_{\mathfrak{g}}$. For this we need to recall some results about tensor representations from [**PStyr**].

By T we denote the tensor algebra $T(V \oplus V_*)$ for $\mathfrak{g} = sl(\infty)$, and $T(V)$ for $\mathfrak{g} = o(\infty)$, $sp(\infty)$. That is, we have

$$T := \bigoplus_{p \geq 0,\ q \geq 0} T^{p,q} \qquad\qquad \text{for } \mathfrak{g} = sl(\infty),$$

and

$$T := \bigoplus_{p \geq 0} T^{p} \qquad\qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty),$$

where $T^{p,q} := V^{\otimes p} \otimes (V_*)^{\otimes q}$ and $T^p := V^{\otimes p}$. In addition, we set

$$T^{\leq r} := \bigoplus_{p+q \leq r} T^{p,q} \qquad\qquad \text{for } \mathfrak{g} = sl(\infty),$$

and

$$T^{\leq r} := \bigoplus_{p \leq r} T^{p} \qquad\qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

By a *tensor module* we mean any $\mathfrak{g}$-module isomorphic to a subquotient of a finite direct sum of copies of $T^{\leq r}$ for some r.

By a *partition* we mean a non-strictly decreasing finite sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$. The empty partition is denoted by 0.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ and a classical finite-dimensional Lie algebra $\mathfrak{g}_n$ of rank $n \geq s$, the irreducible $\mathfrak{g}_n$-module $(V_n)_\mu$ with highest weight μ is always well defined. Moreover, for a fixed μ and growing n, the modules $(V_n)_\mu$ are naturally nested and determine a unique simple ($\mathfrak{g} = \varinjlim \mathfrak{g}_n$)-module $V_\mu := \varinjlim (V_n)_\mu$. For $\mathfrak{g} = sl(n)$, there is another simple $\mathfrak{g}$-module naturally associated to μ, namely $(V_\mu)_*$.

In what follows we will consider pairs of partitions for $\mathfrak{g} = sl(\infty)$ and single partitions for $\mathfrak{g} = o(\infty)$, $sp(\infty)$. Given $\lambda = (\lambda^1, \lambda^2)$ for $\mathfrak{g} = sl(\infty)$, we set $\tilde{V}_\lambda := V_{\lambda^1} \otimes (V_{\lambda^2})_*$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$ and for a single partition λ, the

$\mathfrak{g}$-module $\tilde{V}_\lambda$ is similarly defined: we embed $\mathfrak{g}$ into $sl(\infty)$ so that both the natural $sl(\infty)$-module and the conatural $sl(\infty)$-module are identified with V as $\mathfrak{g}$-modules, and define $\tilde{V}_\lambda$ as the irreducible $sl(\infty)$-module V_λ corresponding to the partition λ as defined above. Then $\tilde{V}_\lambda$ is generally a reducible $\mathfrak{g}$-module.

It is easy to see that for $\mathfrak{g} = sl(\infty)$,

$$T = \bigoplus_{\lambda} d_\lambda \tilde{V}_\lambda \tag{4.1}$$

where $\lambda = (\lambda^1, \lambda^2)$, $d_\lambda := d_{\lambda^1} d_{\lambda^2}$, and d_{λ^i} is the dimension of the simple S_n-module corresponding to the partition λ^i for $n = |\lambda^i|$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, Equation (4.1) also holds, with λ taken to stand for a single partition. Both statements follow from the obvious infinite-dimensional version of Schur-Weyl duality for the tensor algebra T considered as an $sl(\infty)$-module (see for instance [**PStyr**]). Moreover, according to [**PStyr**, Theorems 3.2, 4.2],

$$\operatorname{soc}(\tilde{V}_\lambda) = V_\lambda \tag{4.2}$$

for $\mathfrak{g} = o(\infty)$, $sp(\infty)$, while $\operatorname{soc}(\tilde{V}_\lambda)$ is a simple $\mathfrak{g}$-module for $\mathfrak{g} = sl(\infty)$ [**PStyr**, Theorem 2.3]. Here $\operatorname{soc}(\,\cdot\,)$ stands for the socle of a $\mathfrak{g}$-module. We set $V_\lambda := \operatorname{soc}(\tilde{V}_\lambda)$ also for $\mathfrak{g} = sl(\infty)$, so that (4.2) holds for any $\mathfrak{g}$. It is proved in [**PStyr**] that $\tilde{V}_\lambda$ (and consequently $T^{\leq r}$) has finite length.

It follows also from [**PStyr**] that any simple tensor module is isomorphic to V_λ for some λ. In particular, every simple subquotient of T is also a simple submodule of T.

For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, we set $\#\mu := s$ and $|\mu| := \sum_{i=1}^{s} \mu_i$. In the case of $\mathfrak{g} = sl(\infty)$, when $\lambda = (\lambda^1, \lambda^2)$, we set $\#\lambda := \#\lambda^1 + \#\lambda^2$ and $|\lambda| := |\lambda^1| + |\lambda^2|$.

We are now ready for the following lemma.

Lemma 4.1. *Let $\mathfrak{g} = sl(\infty)$ and $\lambda = (\lambda^1, \lambda^2)$ with $\#\lambda = k > 0$. Then $(V_k)_{\lambda^1} \otimes (V_k^*)_{\lambda^2}$ generates $\tilde{V}_\lambda$.*

Let $\mathfrak{g} = o(\infty)$, $sp(\infty)$, and let λ be a partition with $\#\lambda = k > 0$. Then the $sl(V_k)$-submodule $(V_k)_\lambda$ of $\tilde{V}_\lambda$ generates $\tilde{V}_\lambda$.

Proof. Set $M := \tilde{V}_\lambda$. Let $\mathfrak{g} = sl(\infty)$. Then $M = V_{\lambda^1} \otimes (V_*)_{\lambda^2}$, and let $M_n := (V_n)_{\lambda^1} \otimes (V_n^*)_{\lambda^2}$. It is easy to check that the length of M_n as a $\mathfrak{g}_n$-module stabilizes for $n \geq k$, and moreover it coincides with the length of M;

a formula for the length of M is implied by [**PStyr**, Theorem 2.3]. Hence $(V_k)_{\lambda^1} \otimes (V_k^*)_{\lambda^2}$ generates M.

For $\mathfrak{g} = o(\infty)$, $sp(\infty)$ the length of the $sl(V_n)$-module $(V_n)_\lambda$ considered as a $\mathfrak{g}_n$-module equals the length of M as a $\mathfrak{g}$-module when $2k \leq \dim V_n$ (see Theorems 3.3 and 4.3 in [**PStyr**]). Hence $(V_k)_\lambda$ generates M. $\square$

Theorem 4.2. *A simple absolute weight $\mathfrak{g}$-module is a simple tensor module.*

Proof. Let M be a simple absolute weight $\mathfrak{g}$-module. Then M is integrable by Proposition 3.2, and it also satisfies Theorem 3.4 (3). Fix $0 \neq m \in M$ and choose k such that the commutator subalgebra $\mathfrak{g}_k'$ of the centralizer of $\mathfrak{g}_k$ annihilates m. In the orthogonal case we assume that $\mathfrak{g}_k = o(2k)$. We will prove that M is the unique simple quotient of a parabolically induced module for a parabolic subalgebra $\mathfrak{p}$ of the form $\mathfrak{p} = \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{m}$ is the nil-radical of $\mathfrak{p}$ and $\mathfrak{l}$ is a locally reductive subalgebra. Define $\mathfrak{p} \subset \mathfrak{g}$ as follows:

- If $\mathfrak{g} = sl(\infty)$, we identify $\mathfrak{g}$ with the subspace of traceless elements in $V_* \otimes V$. Consider the decomposition $V = V_k \oplus V'$, where V_k is the natural $\mathfrak{g}_k$-module and V' is the natural $\mathfrak{g}_k'$-module. Furthermore, $V_* = V_k^\perp \oplus (V')^\perp$, where $(V')^\perp = V_k^*$ and $V_k^\perp = V_*'$. We define the subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ to be equal the traceless part of $V_k^* \otimes V_k \oplus V_*' \otimes V'$, and we set $\mathfrak{m} := V_k^* \otimes V'$.

- If $\mathfrak{g} = o(\infty)$, we use the identification $\mathfrak{g} \simeq \Lambda^2(V)$. Let $V_k \subset V$ be the copy of the natural representation of $\mathfrak{g}_k$. Consider the decomposition $V_k = W \oplus W^*$ for some maximal isotropic subspaces W, W^* of V_k and set $V' = V_k^\perp$. Then $\mathfrak{p} := \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{l} := W^* \otimes W \oplus \Lambda^2(V')$ and $\mathfrak{m} := W \otimes V' \supsetplus \Lambda^2(W)$.

- If $\mathfrak{g} = sp(\infty)$, we use the identification $\mathfrak{g} \simeq S^2(V)$. Then V_k, W, W^* and V' are defined in the same way as for $\mathfrak{g} = o(\infty)$, and $\mathfrak{p} := \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{l} := W^* \otimes W \oplus S^2(V')$ and $\mathfrak{m} := W \otimes V' \supsetplus S^2(W)$.

Note that $\mathfrak{g}_k'$ is a subalgebra of finite codimension in $\mathfrak{l}$. In the orthogonal and symplectic cases $\mathfrak{l} = gl(W) \oplus \mathfrak{g}_k'$. If $\mathfrak{g} = sl(\infty)$, then $\mathfrak{l} = sl(V_k) \oplus \tilde{\mathfrak{g}}_k$, where $\tilde{\mathfrak{g}}_k$ is the centralizer of $\mathfrak{g}_k$ in $\mathfrak{g}$:

$$\tilde{\mathfrak{g}}_k = \{-\frac{\operatorname{tr} X}{k} \operatorname{Id}_{V_k} \oplus X \mid X \in V_*' \otimes V'\}.$$

Clearly $\tilde{\mathfrak{g}}_k$ is isomorphic to the Lie algebra $V_*' \otimes V'$.

We claim that the $\mathfrak{m}$-invariant part of M, denoted $M^{\mathfrak{m}}$, is nonzero. Note that $\mathfrak{m}$ is abelian for $\mathfrak{g} = sl(\infty)$. For $\mathfrak{g} = o(\infty)$ or $sp(\infty)$, we have a decomposition $\mathfrak{m} = \mathfrak{m}_1 \niplus \mathfrak{m}_2$ such that $\mathfrak{m}_2 = [\mathfrak{m}_1, \mathfrak{m}_1]$ is a finite-dimensional abelian subalgebra: $\mathfrak{m}_2 = \Lambda^2(W)$ in the orthogonal case and $\mathfrak{m}_2 = S^2(W)$ in the symplectic case. Since M is integrable, $\mathfrak{m}_2$ acts locally nilpotently on M. Hence without loss of generality we may assume that $\mathfrak{m}_2 \cdot m = 0$. We put $\mathfrak{m}_1 := \mathfrak{m}$ in the case $\mathfrak{g} = sl(\infty)$.

Next observe that $U(\mathfrak{m})\cdot m = S^{\cdot}(\mathfrak{m}_1)\cdot m$ and that $S^{\cdot}(\mathfrak{m}_1)$ is isomorphic as a $\mathfrak{g}'_k$-module to a direct sum of $(\tilde{V}')_\lambda$ for some (infinite) set of λ satisfying $\#\lambda \leq k$. By Lemma 4.1, there exists a finite-dimensional subspace $X \subset \mathfrak{m}_1$ such that $S^{\cdot}(X)$ generates $S^{\cdot}(\mathfrak{m}_1)$ as a $\mathfrak{g}'_k$-module. Since M is integrable, X acts locally nilpotently on M. Hence $S^{>p}(X) \cdot m = 0$ for some p. This, together with our assumption that $\mathfrak{g}'_k \cdot m = 0$, allows us to conclude $S^{>p}(\mathfrak{m}) \cdot m = 0$, which in turn implies $M^{\mathfrak{m}} \neq 0$.

Since M is irreducible, it is generated by $M^{\mathfrak{m}}$ and is therefore the unique irreducible quotient of the parabolically induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{m}}$. Furthermore, the irreducibility of M implies the irreducibility of $M^{\mathfrak{m}}$ as an $\mathfrak{l}$-module (otherwise a proper submodule of $M^{\mathfrak{m}}$ would generate a proper submodule of M). Note also that the argument of the previous paragraph implies that as a $\mathfrak{g}'_k$-module $M^{\mathfrak{m}}$ is isomorphic to a subquotient of $S^{\cdot}(\mathfrak{m}_1)$; that is, $M^{\mathfrak{m}}$ is isomorphic to a subquotient of a finite direct sum of some tensor powers of V'.

Let us first consider the case $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. Recall that $M^{\mathfrak{m}}$ is irreducible as an $\mathfrak{l}$-module and is a tensor module over $\mathfrak{g}'_k$. This, together with the integrability of $M^{\mathfrak{m}}$ as an $\mathfrak{l}$-module, implies the existence of an isomorphism of $\mathfrak{l}$-modules $M^{\mathfrak{m}} \simeq L \otimes (V')_\nu$, where L is some irreducible finite-dimensional $gl(W)$-module and $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ is some partition. Let $(\mu_1, \ldots, \mu_k)$ denote the highest weight of L with respect to some Borel subalgebra of $\mathfrak{b}_k$ of $gl(W)$. Consider a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{b}_k \subset \mathfrak{b} \subset \mathfrak{p}$. Without loss of generality, we may assume that the roots of $\mathfrak{b}$ are

$$\begin{aligned} &\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = o(\infty), \\ &\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\} \cup \{2\epsilon_i \mid i \in \mathbb{Z}_{>0}\} && \text{for } \mathfrak{g} = sp(\infty). \end{aligned}$$

The roots of $\mathfrak{b}_k$ will then be

$$\begin{aligned} &\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} && \text{for } \mathfrak{g} = o(\infty), \\ &\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} \cup \{2\epsilon_i \mid 0 < i \leq k\} && \text{for } \mathfrak{g} = sp(\infty). \end{aligned}$$

Observe that M is a highest weight module with respect to $\mathfrak{b}$, and its highest weight equals $\lambda := \mu_1\epsilon_1 + \cdots + \mu_k\epsilon_k + \nu_1\epsilon_{k+1} + \cdots + \nu_r\epsilon_{k+r}$. Furthermore, the integrability of M as a $\mathfrak{g}$-module implies that $\mu_k \geq \nu_1$ and all μ_i are integers. In other words, the weight λ can be identified with the partition $(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_r)$. Next, consider the simple tensor $\mathfrak{g}$-module V_λ (where λ is considered as a partition), and note that both M and V_λ are simple $\mathfrak{g}$-modules with the same highest weight with respect to $\mathfrak{b}$. Therefore M and V_λ are isomorphic as $\mathfrak{g}$-modules.

Now let $\mathfrak{g} = sl(\infty)$. Then by the same argument as above we see that $M^{\mathfrak{m}}$ is isomorphic to $L_1 \otimes L_2$, where L_1 is a simple finite-dimensional $sl(V_k)$-module and L_2 is a simple integrable $\tilde{\mathfrak{g}}_k$-module. Since, however, L_2 is isomorphic to a submodule of the tensor algebra $T(V')$ as a $\mathfrak{g}'_k$-module, we check immediately that as a $\mathfrak{g}'_k$-module L_2 must be isomorphic to $(V')_\nu$ for some partition $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$. There is a $\tilde{\mathfrak{g}}_k$-submodule of $(V')^{\otimes|\nu|}$ (i.e. a tensor module of $\tilde{\mathfrak{g}}_k \cong V'_* \otimes V'$) with the same restriction to $\mathfrak{g}'_k$ as L_2; abusing notation slightly, we denote it also by $(V')_\nu$. Next, using the inclusions

$$sl(V_k) \oplus \mathfrak{g}'_k \subset \mathfrak{l} \subset gl(V_k) \oplus \tilde{\mathfrak{g}}_k$$

and the fact that $gl(V_k) \oplus \tilde{\mathfrak{g}}_k$ is a direct sum of $\mathfrak{l}$ and the abelian one-dimensional Lie algebra (namely the center of $gl(V_k)$), we conclude that $M^{\mathfrak{m}}$ must be isomorphic to the restriction to $\mathfrak{l}$ of a $gl(V_k) \oplus \tilde{\mathfrak{g}}_k$-module of the form $L \otimes (V')_\nu$, where the $gl(V_k)$-module L is simple and uniquely determined up to isomorphism. Denote by $\mu = (\mu_1, \ldots, \mu_k)$ the highest weight of L. It is easy to check in this case that the integrability of M as a $\mathfrak{g}$-module implies that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ are nonpositive integers. Consider the pair of partitions

$$\lambda := ((\nu_1, \nu_2, \ldots, \nu_r), (-\mu_k, \ldots, -\mu_1))$$

and the corresponding tensor $\mathfrak{g}$-module V_λ. Then we clearly have an isomorphism of $\mathfrak{p}$-modules $V_\lambda^{\mathfrak{m}} \simeq M^{\mathfrak{m}}$. Therefore, being the unique irreducible quotients of the corresponding parabolically induced modules, M and V_λ are isomorphic as $\mathfrak{g}$-modules. $\square$

Remark 4.3. In [**PS**] certain categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are introduced and studied in detail. The simple objects of both $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are the same as the simple objects of $\mathbb{T}_{\mathfrak{g}}$, and in fact these three categories form the following chain:

$$\mathbb{T}_{\mathfrak{g}} \subset \mathrm{Tens}_{\mathfrak{g}} \subset \widetilde{\mathrm{Tens}}_{\mathfrak{g}}.$$

However, the objects of the categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ generally have infinite length. In the present paper we will not make use of the categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, and refer the interested reader to [**PS**].

Let us denote by $\mathcal{C}$ the category of $\mathfrak{g}$-modules which satisfy Condition (3) of Theorem 3.4. Consider the functor $\mathcal{B}$ from $\mathrm{Int}_{\mathfrak{g}}$ to $\mathcal{C}$ given by

$$\mathcal{B}(M) = \bigcup_{n>0} M^{\mathfrak{g}_n'}.$$

It is clear that $\mathcal{B}$ does not depend on the choice of fixed exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$.

Lemma 4.4. *For any $M \in \mathrm{Int}_{\mathfrak{g}}$, the module $\mathcal{B}(\Gamma_{\mathfrak{g}}(M^*))$ is injective in the category $\mathcal{C}$. Furthermore, any finite length injective module in the category $\mathcal{C}$ is injective in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. First, let us note that $\mathcal{B}$ is a right adjoint to the inclusion functor $\mathcal{C} \subset \mathrm{Int}_{\mathfrak{g}}$. To see this, consider that the image of any homomorphism from a module $M \in \mathcal{C}$ to a module $Y \in \mathrm{Int}_{\mathfrak{g}}$ is automatically contained in $\mathcal{B}(Y)$. Since it is a right adjoint to the inclusion functor, $\mathcal{B}$ takes injective modules to injective modules, and the lemma follows from the fact that $\Gamma_{\mathfrak{g}}(M^*)$ is injective for any integrable $\mathfrak{g}$-module M, which is [**PS**, Proposition 3.2]. The second statement is clear. □

Proposition 4.5. *For each r, the module $T^{\leq r}$ is injective in the category of absolute weight modules and in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. We consider the case $\mathfrak{g} = sl(\infty)$, and note that the other cases are similar. It was shown in [**PS**] that $(T^{q,p})^*$ is an integrable $\mathfrak{g}$-module. We will show $\mathcal{B}\big((T^{q,p})^*\big)$ is a finite-length module, and furthermore that it has a direct summand isomorphic to $T^{p,q}$. Since any direct summand of an injective module is itself injective, it will follow immediately that $T^{p,q}$ is injective in the category $\mathbb{T}_{\mathfrak{g}}$.

We start with calculating $\big((T^{q,p})^*\big)^{\mathfrak{g}'_n}$. Consider the decomposition

$$V = V_n \oplus V', \quad V_* = V_n^* \oplus V'_*,$$

where V' and V'_* are respectively the natural and conatural $\mathfrak{g}'_n$-modules. If we use the notation

$$T_n^{r,s} := V_n^{\otimes r} \otimes (V_n^*)^{\otimes s}, \quad (T')^{r,s} := (V')^{\otimes r} \otimes (V'_*)^{\otimes s},$$

then we have the following isomorphism of $\mathfrak{g}_n \oplus \mathfrak{g}'_n$-modules

$$T^{q,p} \simeq \bigoplus_{r \le q, s \le p} \big((T')^{r,s} \otimes T_n^{q-r,p-s}\big)^{\oplus b_{r,s}},$$

where $b_{r,s} = \binom{q}{r}\binom{p}{s}$.

Therefore

$$\big((T^{q,p})^*\big)^{\mathfrak{g}'_n} \simeq \bigoplus_{r \le q, s \le p} \mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,s}, \mathbb{C}) \otimes \big(T_n^{p-s,q-r}\big)^{\oplus b_{r,s}}.$$

Since $\mathfrak{g}'_n \simeq \mathfrak{g}$, we can use the results of [**PStyr**]. In particular,

$$\mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,s}, \mathbb{C}) = \begin{cases} 0 & \text{if} \quad r \neq s \\ \mathbb{C}^{r!} & \text{if} \quad r = s. \end{cases}$$

The degree $r!$ appears for the following reason. For any $\sigma \in S_r$ we define $\varphi_\sigma \in \mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,r}, \mathbb{C})$ by

$$\varphi_\sigma(v_1 \otimes \cdots \otimes v_r \otimes u_1 \otimes \cdots \otimes u_r) = \prod_{i=1}^{r} \langle u_i, v_{\sigma(i)} \rangle.$$

Then φ_σ for all $\sigma \in S_r$ form a basis in $\mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,r}, \mathbb{C})$. Thus we obtain

$$\big((T^{q,p})^*\big)^{\mathfrak{g}'_n} \simeq \bigoplus_{r \le \min(p,q)} \big(T_n^{p-s,q-r}\big)^{\oplus b_{r,r} r!},$$

which implies

$$\mathcal{B}\big((T^{q,p})^*\big) \simeq \bigoplus_{r \le \min(p,q)} \big(T^{p-s,q-r}\big)^{\oplus b_{r,r} r!}.$$

Hence the statement. $\square$

Corollary 4.6. 1. *$\tilde{V}_\lambda$ is injective in $\mathbb{T}_\mathfrak{g}$.*

2. *$\tilde{V}_\lambda$ is an injective hull of V_λ in $\mathbb{T}_\mathfrak{g}$.*
3. *Every indecomposable injective module in $\mathbb{T}_\mathfrak{g}$ is isomorphic to $\tilde{V}_\lambda$ for some λ.*
4. *Every module $M \in \mathbb{T}_\mathfrak{g}$ is isomorphic to a submodule of the direct sum of finitely many copies of $T^{\leq r}$ for some r.*
5. *A $\mathfrak{g}$-module M is a tensor module if and only if $M \in \mathbb{T}_\mathfrak{g}$.*

Proof. 1. Each module $\tilde{V}_\lambda$ is a direct summand of $T^{\leq r}$ for some r, and a direct summand of an injective module is injective.

2. Any indecomposable injective module is an injective hull of its socle, and $\mathrm{soc}(\tilde{V}_\lambda) = V_\lambda$ by (4.2).
3. Every indecomposable injective module in $\mathbb{T}_\mathfrak{g}$ has a simple socle, which must be isomorphic to V_λ for some λ by Theorem 4.2.
4. Let $M \in \mathbb{T}_\mathfrak{g}$. Then $\mathrm{soc}(M)$ admits an injective homomorphism into a direct sum of finitely many copies of $T^{\leq r}$ for some r. Since the latter is injective in $\mathbb{T}_\mathfrak{g}$, this homomorphism factors through the inclusion $\mathrm{soc}(M) \hookrightarrow M$. The resulting homomorphism must be injective because its kernel has trivial intersection with $\mathrm{soc}(M)$.
5. A tensor module is by definition a subquotient of a direct sum of finitely many copies of $T^{\leq r}$ for some r, hence it is clearly finite length. Furthermore, any subquotient of an absolute weight module must be an absolute weight module, so any tensor module must be in $\mathbb{T}_\mathfrak{g}$. The converse was seen in (4).

$\square$

5. Koszulity of $\mathbb{T}_\mathfrak{g}$

For $r \in \mathbb{Z}_{\geq 0}$, let $\mathbb{T}_\mathfrak{g}^r$ be the full abelian subcategory of $\mathbb{T}_\mathfrak{g}$ whose simple objects are submodules of $T^{\leq r}$. Then $\mathbb{T}_\mathfrak{g} = \varinjlim \mathbb{T}_\mathfrak{g}^r$. Moreover, $T^{\leq r}$ is an injective cogenerator of $\mathbb{T}_\mathfrak{g}^r$. Consider the finite-dimensional algebra $\mathcal{A}_\mathfrak{g}^r := \mathrm{End}_\mathfrak{g} T^{\leq r}$ and the direct limit algebra $\mathcal{A}_\mathfrak{g} = \varinjlim \mathcal{A}_\mathfrak{g}^r$.

Let $\mathcal{A}_\mathfrak{g}^r$-mof denote the category of unitary finite-dimensional $\mathcal{A}_\mathfrak{g}^r$-modules, and $\mathcal{A}_\mathfrak{g}$-mof the category of locally unitary finite-dimensional $\mathcal{A}_\mathfrak{g}$-modules.

Proposition 5.1. *The functors $\mathrm{Hom}_\mathfrak{g}(\,\cdot\,, T^{\leq r})$ and $\mathrm{Hom}_{\mathcal{A}_\mathfrak{g}^r}(\,\cdot\,, T^{\leq r})$ are mutually inverse antiequivalences of the categories $\mathbb{T}_\mathfrak{g}^r$ and $\mathcal{A}_\mathfrak{g}^r$-mof.*

Proof. Consider the opposite category $(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}$. It has finitely many simple objects and enough projectives, and any object has finite length. Moreover, $T^{\leq r}$ is a projective generator of $(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}$. By a well-known result of Gabriel [**G**], the functor

$$\mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(T^{\leq r},\,\cdot\,) = \mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,,T^{\leq r}) : (\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}} \to \mathcal{A}_{\mathfrak{g}}^r\text{-mof}$$

is an equivalence of categories.

We claim that $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(\,\cdot\,,T^{\leq r})$ is an inverse to $\mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,,T^{\leq r})$. For this it suffices to check that $\mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(T^{\leq r},\,\cdot\,)$ is a right adjoint to $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(\,\cdot\,,T^{\leq r})$, i.e. that

$$\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, \mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(T^{\leq r}, M)) \simeq \mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, T^{\leq r}), M)$$

for any $X \in \mathcal{A}_{\mathfrak{g}}^r$-mof and any $M \in \mathbb{T}_{\mathfrak{g}}^r$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, \mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(T^{\leq r}, M)) &= \mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, \mathrm{Hom}_{\mathfrak{g}}(M, T^{\leq r}))\\
&\overset{\Psi}{\simeq} \mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r\otimes U(\mathfrak{g})}(X\otimes M, T^{\leq r})\\
&= \mathrm{Hom}_{U(\mathfrak{g})\otimes\mathcal{A}_{\mathfrak{g}}^r}(M\otimes X, T^{\leq r})\\
&\overset{\Theta}{\simeq} \mathrm{Hom}_{\mathfrak{g}}(M, \mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, T^{\leq r}))\\
&= \mathrm{Hom}_{(\mathbb{T}_{\mathfrak{g}}^r)^{\mathrm{opp}}}(\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, T^{\leq r}), M),
\end{aligned}$$

where $\Psi(\varphi)(x\otimes m) = \varphi(x)(m)$ and $(\Theta(x)(m))(\psi) = \psi(m\otimes x)$ for $x \in X$, $m \in M$, $\varphi \in \mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(X, \mathrm{Hom}_{\mathfrak{g}}(M, T^{\leq r}))$, and $\psi \in \mathrm{Hom}_{U(\mathfrak{g})\otimes\mathcal{A}_{\mathfrak{g}}^r}(M\otimes X, T^{\leq r})$. $\square$

In order to relate the category $\mathcal{A}_{\mathfrak{g}}$-mof with the categories $\mathcal{A}_{\mathfrak{g}}^r$-mof for all $r \geq 0$, we need to establish some basic facts about the algebra $\mathcal{A}_{\mathfrak{g}}$. Note first that by [**PStyr**] $\mathrm{Hom}_{sl(\infty)}(T^{p,q}, T^{r,s}) = 0$ unless $p - r = q - s \in \mathbb{Z}_{\geq 0}$, and for $\mathfrak{g} = o(\infty)$, $sp(\infty)$, $\mathrm{Hom}_{\mathfrak{g}}(T^p, T^q) = 0$ unless $p - q \in 2\mathbb{Z}_{\geq 0}$. Furthermore, put

$$(\mathcal{A}_{\mathfrak{g}})_i^{p,q} = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i}) \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}})_i^{p} = \mathrm{Hom}_{\mathfrak{g}}(T^{p}, T^{p-2i}) \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

Then one can define a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}_{\mathfrak{g}}^r$ by setting

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p+q\leq r} (\mathcal{A}_{\mathfrak{g}})_i^{p,q} \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p\leq r} (\mathcal{A}_{\mathfrak{g}})_i^{p} \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

It also follows from the results of [**PStyr**] that

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p+q\leq r} \mathrm{End}_{\mathfrak{g}}(T^{p,q}) = \bigoplus_{p+q\leq r} \mathbb{C}[S_p \times S_q] \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p\leq r} \mathrm{End}_{\mathfrak{g}}(T^{p}) = \bigoplus_{p\leq r} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

Hence $(\mathcal{A}_{\mathfrak{g}}^r)_0$ is semisimple.

In addition, we have

$$(\mathcal{A}_{\mathfrak{g}})_i^{p,q}(\mathcal{A}_{\mathfrak{g}})_j^{r,s} = 0 \text{ unless } p = r - j,\ q = s - j \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}})_i^{p}(\mathcal{A}_{\mathfrak{g}})_j^{r} = 0 \text{ unless } p = r - 2j \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

This shows that for each r,

$$\bar{\mathcal{A}}_{\mathfrak{g}}^r := \bigoplus_{p+q>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p,q} \qquad \text{for } \mathfrak{g} = sl(\infty)$$

or

$$\bar{\mathcal{A}}_{\mathfrak{g}}^r := \bigoplus_{p>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p} \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty)$$

is a $\mathbb{Z}_{\geq 0}$-graded ideal in $\mathcal{A}_{\mathfrak{g}}$ such that $\mathcal{A}_{\mathfrak{g}}^r \oplus \bar{\mathcal{A}}_{\mathfrak{g}}^r = \mathcal{A}_{\mathfrak{g}}$. Hence each unitary $\mathcal{A}_{\mathfrak{g}}^r$-module X admits a canonical $\mathcal{A}_{\mathfrak{g}}$-module structure with $\bar{\mathcal{A}}_{\mathfrak{g}}^r X = 0$, and thus becomes a locally unitary $\mathcal{A}_{\mathfrak{g}}$-module. This allows us to claim simply that

$$\mathcal{A}_{\mathfrak{g}}\text{-mof} = \varinjlim (\mathcal{A}_{\mathfrak{g}}^r\text{-mof}).$$

Moreover, Proposition 5.1 now implies the following.

Corollary 5.2. *The functors* $\mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,, T)$ *and* $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}}(\,\cdot\,, T)$ *are mutually inverse antiequivalences of the categories* $\mathbb{T}_{\mathfrak{g}}$ *and* $\mathcal{A}_{\mathfrak{g}}$-mof.

We now need to recall the definition of a Koszul ring. See [**BGS**], where this notion is studied extensively, and, in particular, several equivalent definitions are given. According to Proposition 2.1.3 in [**BGS**], a $\mathbb{Z}_{\geq 0}$-graded ring A is *Koszul* if A_0 is a semisimple ring and for any two graded A-modules M and N of pure weight $m, n \in \mathbb{Z}$ respectively, $\mathrm{ext}^i_A(M, N) = 0$ unless $i = m - n$, where ext^i_A denotes the ext-group in the category of $\mathbb{Z}$-graded A-modules.

In the rest of this section we show that $\mathcal{A}_{\mathfrak{g}}^r$ is a Koszul ring.

We start by introducing the following notation: for any partition μ, we set

$$\mu^+ := \{\text{partitions } \mu' \mid |\mu'| = |\mu| + 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\},$$
$$\mu^- := \{\text{partitions } \mu' \mid |\mu'| = |\mu| - 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\}.$$

For any pair of partitions $\lambda = (\lambda^1, \lambda^2)$, we define

$$\lambda^+ := \{\text{pairs of partitions } \eta \mid \eta^1 \in {\lambda^1}^+, \eta^2 = \lambda^2\},$$
$$\lambda^- := \{\text{pairs of partitions } \eta \mid \eta^1 = \lambda^1, \eta^2 \in {\lambda^2}^-\}.$$

Lemma 5.3. *For any simple object* V_λ *of* $\mathbb{T}_{\mathfrak{g}}$*, there is an exact sequence*

$$0 \to V_\lambda^+ \to V \otimes V_\lambda \to V_\lambda^- \to 0,$$

where

$$V_\lambda^+ = \bigoplus_{\eta\in\lambda^+} V_\eta$$
$$V_\lambda^- = \bigoplus_{\eta\in\lambda^-} V_\eta.$$

Moreover, $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$*.*

Proof. We will prove the statement for $\mathfrak{g} = sl(\infty)$. The other cases are similar. The fact that the semisimplification of $V\otimes V_\lambda$ is isomorphic to $V_\lambda^+ \oplus V_\lambda^-$ follows from the classical Pieri rule.

To prove the equality $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$, observe that

$$V\otimes V_\lambda \subset V\otimes \tilde{V}_\lambda = T^{|\lambda^1|+1,|\lambda^2|} \cap (V\otimes \tilde{V}_\lambda).$$

On the other hand [**PStyr**, Theorem 2.3] implies directly that

$$V_\lambda^+ = \mathrm{soc}(T^{|\lambda^1|+1,|\lambda^2|}) \cap (V\otimes \tilde{V}_\lambda).$$

Hence $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$.

It remains to show that the quotient $(V\otimes V_\lambda)/V_\lambda^+$ is semisimple. This follows again from [**PStyr**, Theorem 2.3], since all simple subquotients of $V_\lambda^- = (V\otimes V_\lambda)/V_\lambda^+$ lie in $\mathrm{soc}^1(T^{|\lambda^1|+1,|\lambda^2|})$. □

Proposition 5.4. *If* $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu) \neq 0$*, then*

$$|\mu^1| - |\lambda^1| = |\mu^2| - |\lambda^2| = i \qquad \textit{for } \mathfrak{g} = sl(\infty)$$

and

$$|\mu| - |\lambda| = 2i \qquad \textit{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

Proof. Let $\mathfrak{g} = sl(\infty)$. We will prove the statement by induction on $|\mu|$. The base of induction $\mu = (0,0)$ follows immediately from the fact that $V_{(0,0)} = \mathbb{C}$ is injective. We assume $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu) \neq 0$. Without loss of generality we may assume that $|\mu^1| > 0$. Then there exists a pair of partitions η such that $\mu \in \eta^+$. Since V_μ is a direct summand of V_η^+, we have $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\eta^+) \neq 0$.

Consider the short exact sequence from Lemma 5.3

$$0 \to V_\eta^+ \to V \otimes V_\eta \to V_\eta^- \to 0.$$

The associated long exact sequence implies that either $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V \otimes V_\eta) \neq 0$ or $\mathrm{Ext}^{i-1}_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\eta^-) \neq 0$. In the latter case, the inductive hypothesis implies that

$$|\eta^1| - |\lambda_1| = (|\eta^2| - 1) - |\lambda^2| = i - 1.$$

The condition in the statement of the proposition follows, as $|\eta^1| = |\mu^1| - 1$ and $|\eta^2| = |\mu^2|$.

Now assume that $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V \otimes V_\eta) \neq 0$. Let

$$0 \to V_\eta \to M_0 \to M_1 \to \dots$$

be a minimal injective resolution of V_η in $\mathbb{T}_\mathfrak{g}$. By the inductive hypothesis, $\mathrm{Ext}^j_{\mathbb{T}_\mathfrak{g}}(V_\nu, V_\eta) \neq 0$ implies

$$|\eta^1| - |\nu^1| = |\eta^2| - |\nu^2| = j. \tag{5.1}$$

We claim that by the minimality of the resolution, $\tilde{V}_\nu$ appears as a direct summand of M_j only if (5.1) holds, that is $M_j = \oplus \tilde{V}_\nu$ for some set of ν such that $|\nu^1| = |\eta^1| - j$ and $|\nu^2| = |\eta^2| - j$. Indeed, otherwise the sequence

$$\mathrm{Hom}_\mathfrak{g}(V_\nu, M_{j-1}) \to \mathrm{Hom}_\mathfrak{g}(V_\nu, M_j) \to \mathrm{Hom}_\mathfrak{g}(V_\nu, M_{j+1})$$

would be exact, and replacing M_j by $M_j/\tilde{V}_\nu$, and M_{j+1} by $M_{j+1}/\tilde{V}_\nu$ or M_{j-1} by $M_{j-1}/\tilde{V}_\nu$, we would obtain a "smaller" resolution.

Furthermore, since the functor $V \otimes (\cdot)$ is obviously exact (vector spaces are flat), the complex

$$0 \to V \otimes V_\eta \to V \otimes M_0 \to V \otimes M_1 \to \dots$$

is an injective resolution of $V \otimes V_\eta$. Thus $\mathrm{Hom}_\mathfrak{g}(V_\lambda, V \otimes M_i) \neq 0$ implies $|\lambda^1| = |\eta^1| - i + 1$ and $|\lambda^2| = |\eta^2| - i$, and the proof for $\mathfrak{g} = sl(\infty)$ is complete.

The proof for $\mathfrak{g} = o(\infty)$, $sp(\infty)$ is similar, and we leave it to the reader. $\square$

Recall that any $\mathfrak{g}$-module W has a well-defined socle filtration

$$0 \subset \mathrm{soc}^0(W) = \mathrm{soc}(W) \subset \mathrm{soc}^1(W) \subset \cdots$$

where $\mathrm{soc}^i(W) := \pi_{i-1}^{-1}(\mathrm{soc}(W/\mathrm{soc}^{i-1}(W))$ and $\pi_{i-1} : W \to W/\mathrm{soc}^{i-1}(W)$ is the projection. Similarly, any $\mathcal{A}_{\mathfrak{g}}$-module X has a radical filtration

$$\cdots \subset \mathrm{rad}^1(X) \subset \mathrm{rad}^0(X) = \mathrm{rad}(X) \subset X$$

where $\mathrm{rad}(X)$ is the joint kernel of all surjective $\mathcal{A}_{\mathfrak{g}}$-homomorphisms $X \to X'$ with X' simple, and $\mathrm{rad}^i(X) = \mathrm{rad}(\mathrm{rad}^{i-1}(X))$.

Note furthermore that the Ext's in the category $\mathbb{T}_{\mathfrak{g}}$ differ essentially from the Ext's in $\mathfrak{g}$-mod. In particular, as shown in [**PS**], $\mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\mu)$ is uncountable dimensional whenever nonzero, whereas $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g}}}(V_\lambda, V_\mu)$ is always finite dimensional by Corollary 5.2. Here are two characteristic examples.

1. Consider the exact sequence of $\mathfrak{g}$-modules

$$0 \to V \to (V_*)^* \to (V_*)^*/V \to 0.$$

   The $\mathfrak{g}$-module $(V_*)^*/V$ is trivial, and any vector in $\mathrm{Ext}^1_{sl(\infty)}(\mathbb{C}, V)$ determines a unique 1-dimensional subspace in $(V_*)^*/V$. On the other hand, $\mathrm{Ext}^1_{\mathbb{T}_{sl(\infty)}}(\mathbb{C}, V) = 0$ by Proposition 5.4.

2. Each nonzero vector of $\mathrm{Ext}^1_{sl(\infty)}(\mathbb{C}, sl(\infty))$ corresponds to a 1-dimensional trivial quotient of $\mathrm{soc}^1((sl(\infty)_*)^*)$ (see [**PS**]). The nonzero vectors of the 1-dimensional space $\mathrm{Ext}^1_{\mathbb{T}_{sl(\infty)}}(\mathbb{C}, sl(\infty))$ on the other hand correspond to the unique 1-dimensional quotient of $\mathrm{soc}^1((sl(\infty)_*)^*)$ which determines an absolute weight module, namely $\widetilde{sl(\infty)}/sl(\infty) = (V \otimes V_*)/sl(\infty)$.

The following is the main result of this section.

Theorem 5.5. *The ring $\mathcal{A}^r_{\mathfrak{g}}$ is Koszul.*

Proof. According to [**BGS**, Proposition 2.1.3], it suffices to prove that unless $i = m - n$, one has $\mathrm{ext}^i_{\mathcal{A}^r_{\mathfrak{g}}}(M, N) = 0$ for any pure $\mathcal{A}^r_{\mathfrak{g}}$-modules M, N of weights m, n respectively. We will prove that unless $i = m - n$, one has $\mathrm{ext}^i_{\mathcal{A}_{\mathfrak{g}}}(M, N) = 0$ for any simple pure $\mathcal{A}_{\mathfrak{g}}$-modules M, N of weights m, n respectively. Since any $\mathcal{A}^r_{\mathfrak{g}}$-module admits a canonical $\mathcal{A}_{\mathfrak{g}}$-module structure, it will follow that $\mathrm{ext}^i_{\mathcal{A}^r_{\mathfrak{g}}}(M, N) = 0$ for any simple pure $\mathcal{A}^r_{\mathfrak{g}}$-modules M, N

of weights m, n respectively unless $i = m - n$. The analogous statement for arbitrary $\mathcal{A}^r_{\mathfrak{g}}$-modules of pure degree will also follow, since all such modules are semisimple.

Let X_λ (respectively, $\tilde{X}_\lambda$) be the $\mathcal{A}_{\mathfrak{g}}$-module which is the image of V_λ (resp., $\tilde{V}_\lambda$) under the antiequivalence of Corollary 5.2. Then $\tilde{X}_\lambda$ is a projective cover of the simple module X_λ. Proposition 5.4 implies that $\mathrm{Ext}^i_{\mathcal{A}_{\mathfrak{g}}}(X_\mu, X_\lambda) = 0$ unless $|\mu^1| - |\lambda^1| = |\mu^2| - |\lambda^2| = i$. We consider a minimal projective resolution of X_μ

$$\cdots \to P^1 \to P^0 \to 0 \tag{5.2}$$

and claim that it must have the property $P^i \simeq \oplus \tilde{X}_\nu$ for some set of ν with $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. This follows from the similar fact for a minimal injective resolution of V_μ in $\mathbb{T}_{\mathfrak{g}}$ (see the proof of Proposition 5.4) and the antiequivalence of the categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathcal{A}_{\mathfrak{g}}$-mof.

On the other hand, by [**PStyr**] if V_ν is a simple constituent of $\mathrm{soc}^i(\tilde{V}_\mu)$, or if under the antiequivalence X_ν is a simple constituent of $\mathrm{rad}^i\,\tilde{X}_\mu$, then $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. Therefore we see that in the above resolution the image of $\mathrm{rad}^j(P^i)$ lies in $\mathrm{rad}^{j+1}(P^{i-1})$. Now it is clear that we can endow the resolution (5.2) with a $\mathbb{Z}$-grading by setting the degree of X_μ to be an arbitrary integer n. Indeed, one should assign to each simple $(\mathcal{A}_{\mathfrak{g}})_0$-constituent of P^i which lies in $\mathrm{rad}^j(P^i)$ and not in $\mathrm{rad}^{j+1}(P^i)$ the degree $n + i + j + 1$. This immediately implies that $\mathrm{ext}^i_{\mathcal{A}_{\mathfrak{g}}}(X_\mu, X_\lambda) = 0$ unless the difference between the weights of X_λ and X_μ is i. $\square$

6. On the structure of $\mathcal{A}_{\mathfrak{g}}$

It is a result of [**BGS**] that for any r the Koszulity of $\mathcal{A}^r_{\mathfrak{g}}$ implies that $\mathcal{A}^r_{\mathfrak{g}}$ is a quadratic algebra generated by $(\mathcal{A}^r_{\mathfrak{g}})_0$ and $(\mathcal{A}^r_{\mathfrak{g}})_1$. That is, $\mathcal{A}^r_{\mathfrak{g}} \simeq T_{(\mathcal{A}^r_{\mathfrak{g}})_0}((\mathcal{A}^r_{\mathfrak{g}})_1)/(R^r)$, where (R^r) is the two-sided ideal generated by some $(\mathcal{A}^r_{\mathfrak{g}})_0$-bimodule R^r in $(\mathcal{A}^r_{\mathfrak{g}})_1 \otimes_{(\mathcal{A}^r_{\mathfrak{g}})_0} (\mathcal{A}^r_{\mathfrak{g}})_1$. Moreover, it is easy to see that $\mathcal{A}_{\mathfrak{g}}$ is isomorphic to the quotient $T_{(\mathcal{A}_{\mathfrak{g}})_0}((\mathcal{A}_{\mathfrak{g}})_1)/(R)$, where $R = \varinjlim R^r$. In this section we describe $(\mathcal{A}_{\mathfrak{g}})_1$ and R.

In what follows we fix inclusions $S_n \subset S_{n+1}$ such that S_{n+1} acts on the set $\{1, 2, \ldots, n+1\}$ and S_n is the stabilizer of $n+1$. We start with the following lemma.

Lemma 6.1. *If* $\mathfrak{g} = sl(\infty)$, *then* $\mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-1,q-1})$ *as a left module over*

$\mathbb{C}[S_{p-1} \times S_{q-1}]$ *is generated by the contractions*

$$\phi_{i,j} : T^{p,q} \to T^{p-1,q-1},$$
$$v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \mapsto \langle v_i, w_j \rangle (v_1 \otimes \cdots \hat{v}_i \cdots \otimes v_p \otimes w_1 \otimes \cdots \hat{w}_j \cdots \otimes w_q).$$

If $\mathfrak{g} = o(\infty)$ *or* $sp(\infty)$, *then* $\mathrm{Hom}_{\mathfrak{g}}(T^p, T^{p-2})$ *as a left module over* $\mathbb{C}[S_{p-2}]$ *is generated by the contractions*

$$\psi_{i,j} : T^p \to T^{p-2},$$
$$v_1 \otimes \cdots \otimes v_p \mapsto \langle v_i, v_j \rangle (v_1 \otimes \cdots \hat{v}_i \cdots \otimes \cdots \hat{v}_j \cdots \otimes v_p),$$

where $\langle \cdot\,, \cdot \rangle$ *stands for the symmetric bilinear form on* V *for* $\mathfrak{g} = o(\infty)$, *and the symplectic bilinear form on* V *for* $\mathfrak{g} = sp(\infty)$.

Proof. Let $\mathfrak{g} = sl(\infty)$ and $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-1,q-1})$. Theorem 3.2 in [**PStyr**] claims that $\mathrm{soc}(T^{p,q}) = \cap_{i \leq p, j \leq q} \ker \phi_{i,j}$; moreover, the same result implies that $\mathrm{soc}(T^{p,q}) \subset \ker \varphi$. Define

$$\Phi : T^{p,q} \to \bigoplus_{i \leq p, j \leq q} T^{p-1,q-1}$$

as the direct sum $\bigoplus_{i,j} \phi_{i,j}$. Then there exists $\alpha : \bigoplus_{i \leq p, j \leq q} T^{p-1,q-1} \to T^{p-1,q-1}$ such that $\varphi = \alpha \circ \Phi$. But $\alpha = \bigoplus_{i,j} \alpha_{i,j}$ for some $\alpha_{i,j} \in \mathbb{C}[S_{p-1} \times S_{q-1}]$. Therefore $\varphi = \sum_{i,j} \alpha_{i,j} \phi_{i,j}$. This proves the lemma for $\mathfrak{g} = sl(\infty)$.

We leave the proof in the cases $\mathfrak{g} = o(\infty)$, $sp(\infty)$ to the reader. $\square$

Let $\mathfrak{g} = sl(\infty)$. Recall that $(\mathcal{A}_{\mathfrak{g}})_i^{p,q} = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i})$ and that $(\mathcal{A}_{\mathfrak{g}})_0^{p,q} = \mathbb{C}[S_p \times S_q]$.

Lemma 6.2. *Let* $\mathfrak{g} = sl(\infty)$.

1. $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ *is isomorphic to* $\mathbb{C}[S_p \times S_q]$ *as a right* $(\mathcal{A}_{\mathfrak{g}})_0^{p,q}$*-module, and the structure of a left* $(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}$*-module is given by left multiplication via the fixed inclusion*

$$(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1} = \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q] = (\mathcal{A}_{\mathfrak{g}})_0^{p,q}.$$

2. *We have*

$$(\mathcal{A}_{\mathfrak{g}})_1 \otimes_{(\mathcal{A}_{\mathfrak{g}})_0} (\mathcal{A}_{\mathfrak{g}})_1 = \bigoplus_{p,q} ((\mathcal{A}_{\mathfrak{g}})_1^{p-1,q-1} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}} (\mathcal{A}_{\mathfrak{g}})_1^{p,q}),$$

where $(\mathcal{A}_{\mathfrak{g}})_1^{p-1,q-1} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}} (\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ *is isomorphic to* $\mathbb{C}[S_p \times S_q]$*. Moreover,* $(\mathcal{A}_{\mathfrak{g}})_1^{p-1,q-1} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}} (\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ *is a* $(\mathbb{C}[S_{p-2} \times S_{q-2}], \mathbb{C}[S_p \times S_q])$*-bimodule via the embeddings* $\mathbb{C}[S_{p-2} \times S_{q-2}] \subset \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q]$*.*

Proof. It is clear that all contractions $\phi_{i,j} \in (\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ can be obtained from $\phi_{p,q}$ via the right $\mathbb{C}[S_p \times S_q]$-module structure of $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$. Thus by Lemma 6.1, as a $\mathbb{C}[S_p \times S_q]$-bimodule, $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ is generated by the single contraction $\phi_{p,q}$. Moreover, $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ is a free right $\mathbb{C}[S_p \times S_q]$-module of rank 1. Indeed, if for some $a_\sigma \in \mathbb{C}$

$$\sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma = 0,$$

then for all $v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \in T^{p,q}$

$$\begin{aligned} 0 &= \sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma (v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q) \\ &= \sum_{\substack{\sigma = (\sigma_1, \sigma_2) \\ \in S_p \times S_q}} a_\sigma \langle v_{\sigma_1(p)}, w_{\sigma_2(q)} \rangle (v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(p-1)} \otimes w_{\sigma_2(1)} \otimes \cdots \otimes w_{\sigma_2(q-1)}), \end{aligned}$$

and hence $a_\sigma = 0$ for all $\sigma \in S_p \times S_q$. Finally, for any $\sigma \in S_{p-1} \times S_{q-1}$ we have

$$\sigma \phi_{p,q} = \phi_{p,q} \sigma.$$

This implies part (1). Part (2) is a direct corollary of part (1). $\square$

Lemma 6.3. *Let* $\mathfrak{g} = sl(\infty)$*. Let* $S \simeq S_2 \times S_2$ *denote the subgroup of* $S_p \times S_q$ *generated by* $(p, p-1)_l$ *and* $(q, q-1)_r$*, where* $(i,j)_l$ *and* $(i,j)_r$ *stand for the transpositions in* S_p *and* S_q*, respectively. Then* $R = \bigoplus_{p,q} R^{p,q}$*, where*

$$R^{p,q} = (triv \boxtimes sgn \, \oplus \, sgn \boxtimes triv) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p \times S_q],$$

and triv and sgn denote respectively the trivial and sign representations of S_2*.*

Proof. The statement is equivalent to the equality of $R^{p,q}$ and the right $\mathbb{C}[S_p \times S_q]$-module

$$(1+(p,p-1)_l)(1-(q,q-1)_r)\mathbb{C}[S_p \times S_q] \oplus (1-(p,p-1)_l)(1+(q,q-1)_r)\mathbb{C}[S_p \times S_q].$$

We have the obvious relations in $\mathcal{A}_{sl(\infty)}$

$$\begin{aligned}\phi_{p-1,q-1}\phi_{p,q} &= \phi_{p-1,q-1}\phi_{p,q}(p,p-1)_l(q,q-1)_r,\\ \phi_{p-1,q-1}\phi_{p,q}(p,p-1)_l &= \phi_{p-1,q-1}\phi_{p,q}(q,q-1)_r.\end{aligned}$$

Therefore $R^{p,q}$ contains the module

$$(1+(p,p-1)_l)(1-(q,q-1)_r)\mathbb{C}[S_p\times S_q]\oplus(1-(p,p-1)_l)(1+(q,q-1)_r)\mathbb{C}[S_p\times S_q],$$

which has dimension $\frac{p!q!}{2}$. On the other hand, it is easy to see that

$$\begin{aligned}\dim R^{p,q} &= \dim\left((\mathcal{A}_\mathfrak{g})_1^{p-1,q-1}\otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}}(\mathcal{A}_\mathfrak{g})_1^{p,q}\right)-\dim(\mathcal{A}_\mathfrak{g})_2^{p,q}\\ &= \frac{(p-1)!(q-1)!p!q!}{(p-1)!(q-1)!}-\frac{p!q!}{2}\\ &= \frac{p!q!}{2}.\end{aligned}$$

Hence the statement. □

Corollary 6.4. *Let $\mathfrak{g} = sl(\infty)$. Then $\mathcal{A}_\mathfrak{g}^r$ is Koszul self-dual, i.e. $\mathcal{A}_\mathfrak{g}^r \simeq ((\mathcal{A}_\mathfrak{g}^r)^!)^{\mathrm{opp}}$. Furthermore, $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$, where $\mathcal{A}_\mathfrak{g}^! := \varinjlim\, (\mathcal{A}_\mathfrak{g}^r)^!$.*

Proof. By definition, we have $(\mathcal{A}_\mathfrak{g}^r)^! = T_{(\mathcal{A}_\mathfrak{g}^r)_0}((\mathcal{A}_\mathfrak{g}^r)_1^*)/(R^{r\perp})$, where $(\mathcal{A}_\mathfrak{g}^r)_1^* = \mathrm{Hom}_{(\mathcal{A}_\mathfrak{g}^r)_0}((\mathcal{A}_\mathfrak{g}^r)_1,(\mathcal{A}_\mathfrak{g}^r)_0)$, [**BGS**]. Note that $((A_\mathfrak{g})_1^{p,q})^*$ is a $((A_\mathfrak{g})_0^{p,q},(A_\mathfrak{g})_0^{p-1,q-1})$-bimodule. Moreover, Lemma 6.2 (1) implies an isomorphism of bimodules

$$((A_\mathfrak{g})_1^{p,q})^*\simeq\mathbb{C}[S_p\times S_q].$$

Hence we have an isomorphism of $((\mathcal{A}_\mathfrak{g}^r)^!)_0^{\mathrm{opp}}$-bimodules

$$((\mathcal{A}_\mathfrak{g}^r)^!)_1^{\mathrm{opp}}\simeq(\mathcal{A}_\mathfrak{g}^r)_1.$$

One can check that $R^\perp = \bar{R}$, where $\bar{R} := \oplus\bar{R}^{p,q}$, and the modules $\bar{R}^{p,q}$ are defined via the decomposition of $(\mathcal{A}_\mathfrak{g})_0^{p,q}$-modules

$$(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1}\otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}}(\mathcal{A}_\mathfrak{g})_1^{p,q} = R^{p,q}\oplus\bar{R}^{p,q}.$$

Therefore $((\mathcal{A}^r)_\mathfrak{g}^!)^{\mathrm{opp}}\simeq T_{(\mathcal{A}_\mathfrak{g}^r)_0}((\mathcal{A}_\mathfrak{g}^r)_1)/(\bar{R}^r)$. Now consider the automorphism σ of $\mathbb{C}[S_p\times S_q]$ defined for all p and q by $\sigma(s,t) = sgn(t)(s,t)$ for all $s\in S_p$,

$t \in S_q$. Recall that $(\mathcal{A}_\mathfrak{g})_0 = \bigoplus_{p,q} \mathbb{C}[S_p \times S_q]$. Extend σ to an automorphism of $T_{(\mathcal{A}_\mathfrak{g})_0}((\mathcal{A}_\mathfrak{g})_1)$ by setting $\sigma(x) = x$ for any $x \in (\mathcal{A}_\mathfrak{g})_1$. One immediately observes that $\sigma(R^{p,q}) = \bar{R}^{p,q}$, hence σ induces an isomorphism $\mathcal{A}_\mathfrak{g}^r \simeq ((\mathcal{A}_\mathfrak{g}^r)^!)^{\mathrm{opp}}$, and clearly also an isomorphism $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$. □

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, we set $\mu^\perp := (s = \#\mu, \#(\mu_1 - 1, \mu_2 - 1, \ldots), \ldots)$, or in terms of Young diagrams, $\mu^\perp$ is the conjugate partition obtained from μ by interchanging rows and columns.

Corollary 6.5. *Let $\mathfrak{g} = sl(\infty)$, and for a pair of partitions $\nu = (\nu^1, \nu^2)$ take $\nu^\perp := (\nu^1, (\nu^2)^\perp)$. Then $\dim \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu)$ equals the multiplicity of $V_{\lambda^\perp}$ in $\mathrm{soc}^i(\tilde{V}_{\mu^\perp}) / \mathrm{soc}^{i-1}(\tilde{V}_{\mu^\perp})$, as computed in [**PStyr**, Theorem 2.3].*

Proof. The statement follows from [**BGS**, Theorem 2.10.1] applied to $\mathcal{A}_\mathfrak{g}^r$ for sufficiently large r. Indeed, this result implies that $\mathrm{Ext}_{\mathcal{A}_\mathfrak{g}}((\mathcal{A}_\mathfrak{g})_0, (\mathcal{A}_\mathfrak{g})_0)$ is isomorphic to $(\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$ as a graded algebra. Moreover, the simple $\mathcal{A}_\mathfrak{g}$-module X_λ (which is the image of V_λ under the antiequivalence of Corollary 5.2) is isomorphic to $(\mathcal{A}_\mathfrak{g})_0 \mathbb{Y}_\lambda$, where $\mathbb{Y}_\lambda$ is the product of Young projectors corresponding to the partitions λ^1 and λ^2. This follows immediately from the fact that $\mathbb{Y}_\lambda$ is a primitive idempotent in $(\mathcal{A}_\mathfrak{g})_0$ and hence also in $\mathcal{A}_\mathfrak{g}$, see for example [**CR**, Theorem 54.5]. The projective cover $\tilde{X}_\lambda$ of X_λ is isomorphic to $\mathcal{A}_\mathfrak{g} \mathbb{Y}_\lambda$. Therefore we have

$$\dim \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu) = \dim \mathrm{Ext}^i_{\mathcal{A}_\mathfrak{g}}(X_\mu, X_\lambda) = \dim \mathbb{Y}_\lambda (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}_i \mathbb{Y}_\mu.$$

By Corollary 6.4,

$$\dim \mathbb{Y}_\lambda (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}_i \mathbb{Y}_\mu = \dim \mathbb{Y}_{\lambda^\perp} (\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\mu^\perp}.$$

Furthermore, $\dim \mathbb{Y}_{\lambda^\perp} (\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\mu^\perp}$ equals the multiplicity of $X_{\lambda^\perp}$ in the module $\mathrm{rad}^{i-1} \tilde{X}_{\mu^\perp} / \mathrm{rad}^i \tilde{X}_{\mu^\perp}$ [**CR**, Theorem 54.15]), which coincides with the multiplicity of $V_{\lambda^\perp}$ in $\mathrm{soc}^i(\tilde{V}_{\mu^\perp}) / \mathrm{soc}^{i-1}(\tilde{V}_{\mu^\perp})$. □

Corollary 6.6. *The blocks of the category $\mathbb{T}_{sl(\infty)}$ are parametrized by $\mathbb{Z}$. In particular,*

1. *V_λ and V_μ belong to the block $\mathbb{T}_{sl(\infty)}(i)$ for $i \in \mathbb{Z}$ if and only if $|\lambda^1| - |\lambda^2| = |\mu^1| - |\mu^2| = i$.*
2. *Two blocks $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(j)$ are equivalent if and only if $i = \pm j$.*

Proof. 1. The fact that $\tilde{V}_{(\mu^1,\mu^2)}$ is an injective hull of $V_{(\mu^1,\mu^2)}$, together with Theorem 2.3 in [**PStyr**], implies that $\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_{(\mu^1,\mu^2)}, V_{(\lambda^1,\lambda^2)}) \neq 0$ iff $\mu^1 \in (\lambda^1)^+$ and $\mu^2 \in (\lambda^2)^+$. More precisely, Theorem 2.3 in [**PStyr**] computes the multiplicities of the constituents of the socle of $\tilde{V}_\lambda/V_\lambda$, and a simple module has nonzero $\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}$ with V_λ if and only if it is isomorphic to a submodule of $\tilde{V}_\lambda/V_\lambda$. Consider the minimal equivalence relation on pairs of partitions for which (λ^1,λ^2) and (μ^1,μ^2) are equivalent whenever $\mu^1 \in (\lambda^1)^+$ and $\mu^2 \in (\lambda^2)^+$. It is a simple exercise to show that then $\lambda = (\lambda^1,\lambda^2)$ and $\mu = (\mu^1,\mu^2)$ are equivalent if and only if $|\lambda^1| - |\lambda^2| = |\mu^1| - |\mu^2|$. The first assertion follows.

2. The functor $(\,\cdot\,)_*$ establishes an equivalence of $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(-i)$. To see that $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(j)$ are inequivalent for $i \neq \pm j$, assume without loss of generality that $i > 0$, $j \geq 0$. Then the isomorphism classes of simple injective objects in $\mathbb{T}_{sl(\infty)}(i)$ are parametrized by the partitions of i, since $\{V_{(\lambda^1,0)} \mid |\lambda^1| = i\}$ represents the set of isomorphism classes of simple injective objects in $\mathbb{T}_{sl(\infty)}(i)$. As the sets $\{V_{(\lambda^1,0)} \mid |\lambda^1| = i\}$ and $\{V_{(\lambda^1,0)} \mid |\lambda^1| = j\}$ have different cardinalities for $i \neq j$ except the case $i = 1, j = 0$, the assertion follows in other cases. Each of the blocks $\mathbb{T}_{sl(\infty)}(0)$ and $\mathbb{T}_{sl(\infty)}(1)$ has a single simple injective module, up to isomorphism. However, V has nontrivial extensions by both $V_{((2),(1))}$ and $V_{((1,1),(1))}$, whereas $\mathbb{C}$ has a nontrivial extension only by $V_{((1),(1))}$. This completes the proof.

$\square$

Now we proceed to describing the structure of $\mathcal{A}_\mathfrak{g}$ for $\mathfrak{g} = o(\infty)$ and $sp(\infty)$. Recall that $(\mathcal{A}_\mathfrak{g})^p_i = \mathrm{Hom}_\mathfrak{g}(T^p, T^{p-2i})$. and $(\mathcal{A}_\mathfrak{g})^p_0 = \mathbb{C}[S_p]$. Let $S_{p-2} \subset S_p$ denote the stabilizer of p and $p-1$, and let $S' \subset S_p$ be the subgroup generated by the transposition $(p-1, p)$.

Lemma 6.7. *We have*

$$(\mathcal{A}_\mathfrak{g})^p_1 \simeq triv \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \textit{for } \mathfrak{g} = o(\infty)$$

and

$$(\mathcal{A}_\mathfrak{g})^p_1 \simeq sgn \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \textit{for } \mathfrak{g} = sp(\infty).$$

In both cases left multiplication by $\mathbb{C}[S_{p-2}]$ is well defined, as S' centralizes S_{p-2}.

Proof. Lemma 6.1 implies that the contraction $\psi_{p-1,p}$ generates $(\mathcal{A}_{\mathfrak{g}})_1^p$ as a right $\mathbb{C}[S_p]$-module. Then the statement follows from the relation

$$\psi_{p-1,p} = \pm\psi_{p-1,p}(p, p-1),$$

where the sign is $+$ for $\mathfrak{g} = o(\infty)$ and $-$ for $\mathfrak{g} = sp(\infty)$. $\square$

Corollary 6.8. *Let $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. Then*

$$(\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p \simeq L_{\mathfrak{g}} \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p],$$

where $S \simeq S_2 \times S_2$ is the subgroup generated by $(p, p-1)$ and $(p-2, p-3)$ and

$$L_{\mathfrak{g}} = \begin{cases} triv & \text{for } \mathfrak{g} = o(\infty) \\ sgn \boxtimes sgn & \text{for } \mathfrak{g} = sp(\infty). \end{cases}$$

To describe R, write $R = \bigoplus_p R^p$, where $R^p \subset (\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p$.
We will need the following decompositions of S_4-modules:

$$triv \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = X_{(2,1,1)} \oplus X_{(2,2)} \oplus X_{(4)}, \tag{6.1}$$
$$(sgn \boxtimes sgn) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = X_{(3,1)} \oplus X_{(2,2)} \oplus X_{(1,1,1,1)}. \tag{6.2}$$

Lemma 6.9. *Let $S'' \subset S_p$ be the subgroup isomorphic to S_4 that fixes 1, 2,..., $p-4$. Then*

$$R^p \simeq X_{(2,1,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = o(\infty),$$

and

$$R^p \simeq X_{(3,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = sp(\infty).$$

Proof. Let us deal with the case of $o(\infty)$. We consider the following Young projectors in $S'' \simeq S_4$

$$\mathbb{Y}_{(2,1,1)} = (1+(p-1,p))(1-(p,p-2)-(p,p-3)-(p-2,p-3)+(p,p-2,p-3)+(p,p-3,p-2)),$$

$\mathbb{Y}_{(2,2)} = (1 + (p, p-1))(1 + (p-2, p-3))(1 - (p-2, p))(1 - (p-1, p-3)),$
and

$$\mathbb{Y}_{(4)} = \sum_{s \in S''} s.$$

By Equation (6.1) we have

$$R^p \subset (\mathcal{A}_\mathfrak{g})_1^{p-2} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-2}} (\mathcal{A}_\mathfrak{g})_1^p = \mathbb{Y}_{(2,1,1)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(2,2)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(4)}\mathbb{C}[S_p].$$

By direct inspection one can check that

$$\begin{aligned}
\psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(2,1,1)} &= 0, \\
\psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(2,2)} &= 2\psi_{p-3,p-2}\psi_{p-1,p} - 2\psi_{p-3,p}\psi_{p-1,p-2}, \\
\psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(4)} &= 4\psi_{p-3,p-2}\psi_{p-1,p}.
\end{aligned}$$

The statement follows for $o(\infty)$.

We leave the case of $sp(\infty)$ to the reader. □

Corollary 6.10. $\mathcal{A}_{sp(\infty)} \simeq \mathcal{A}_{o(\infty)}$.

Proof. We use the automorphism σ of $\mathbb{C}[S_p]$ which sends s to $sgn(s)s$. □

Corollary 6.11. *The categories $\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ are equivalent.*

In [**Sr**] a tensor functor $\mathbb{T}_{o(\infty)} \to \mathbb{T}_{sp(\infty)}$ establishing an equivalence of tensor categories is constructed using the Lie superalgebra $osp(\infty, \infty)$.

Proposition 6.12. *$\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ have two inequivalent blocks $\mathbb{T}_\mathfrak{g}^{ev}$ and $\mathbb{T}_\mathfrak{g}^{odd}$ generated by all V_λ with $|\lambda|$ even and odd, respectively.*

Proof. Due to the previous corollary it suffices to consider the case $\mathfrak{g} = o(\infty)$. As follows from [**PStyr**], $\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ if and only if $\mu \in \lambda^{++}$, where

$$\begin{aligned}
\lambda^{++} := \{&\text{partitions } \lambda' \,|\lambda_i \leq \lambda'_i \text{ for all } i, \, |\lambda'| = |\lambda| + 2, \\
&\qquad \lambda'_j \neq \lambda_j \text{ and } \lambda'_k \neq \lambda_k \text{ for } j \neq k \text{ implies } \lambda_j \neq \lambda_k\}.
\end{aligned}$$

Note that the partitions in λ^{++} are those which arise from λ via the Pieri rule for tensoring with $S^2(V)$. Consider the minimal equivalence relation on partitions for which λ and μ are equivalent whenever $\mu \in \lambda^{++}$. One can check that there are exactly two equivalence classes which are determined by the parity of $|\lambda|$.

To show that $\mathbb{T}_{\mathfrak{g}}^{ev}$ and $\mathbb{T}_{\mathfrak{g}}^{odd}$ are not equivalent observe that all simple injective modules in $\mathbb{T}_{\mathfrak{g}}$ correspond to partitions μ with $\mu_1 = \cdots = \mu_s = 1$, or equivalently are isomorphic to the exterior powers $\Lambda^s(V)$ of the standard module. If $s \geq 1$ then $\Lambda^s(V)$ has nontrivial extensions by two non-isomorphic simple modules, namely $V_{(3,1,\ldots,1)}$ and $V_{(2,1,1,\ldots,1)}$. The trivial module on the other hand has a nontrivial extension by only $S^2(V) = V_{(2)}$. Therefore $\mathbb{T}_{\mathfrak{g}}^{ev}$ contains a simple injective module admitting a nontrivial extension with only one simple module, whereas $\mathbb{T}_{\mathfrak{g}}^{odd}$ does not contain such a simple injective module. $\square$

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Elizabeth Dan-Cohen is an assistant in mathematics at FAU Erlangen-Nürnberg. She obtained her Ph.D. in 2008 at the University of California, Berkeley. She did postdoctoral work at Rice University and Jacobs University Bremen, and spent one year as assistant professor at Louisiana State University.

Ivan Penkov is professor of mathematics at Jacobs University Bremen. He received his Ph.D. in 1987 at the Steklov Mathematical Institute. From 1991 to 2004 he was professor at the University of California, Riverside.

Vera Serganova is professor of mathematics at University of California, Berkeley. She received her Ph.D. in 1988 at St. Petersburg State University.