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Cuspidal representations of $\mathfrak{sl}(n+1)$

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Abstract

In this paper we study the subcategory of cuspidal modules of the category of weight modules over the Lie algebra $\mathfrak{sl}(n+1)$. Our main result is a complete classification and an explicit description of indecomposable cuspidal modules.

MSC: 17B10

Keywords: Lie algebra; Indecomposable representations; Quiver; Weight modules

1. Introduction

The category of weight representations has attracted considerable mathematical attention in the last thirty years. General weight modules have been extensively studied by G. Benkart, D. Britten, S. Fernando, V. Futorny, A. Joseph, F. Lemire, and others (see e.g. [2,4,5,10,12]). Following their works, in 2000 O. Mathieu, [21], established the classification of all simple weight modules with finite-dimensional weight spaces over reductive Lie algebras. An important role in this classification plays the category $\mathcal{C}$ of all cuspidal modules, i.e. weight modules on which all root vectors of the Lie algebra act bijectively. This role is due to the parabolic induction theorem of Fernando and Futorny. The theorem states that every simple weight module M with finite-dimensional weight spaces over a reductive finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to

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the unique simple quotient of a parabolically induced generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} S$, where S is a cuspidal module over the Levi component of $\mathfrak{p}$. The Fernando–Futorny result naturally initiates the study of the category $\mathcal{C}$ as a necessary step towards the study of all weight modules with finite weight multiplicities.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra. A result of Fernando implies that $\mathcal{C}$ is non-trivial for Lie algebras $\mathfrak{g}$ of type A and C only. In the symplectic case the category $\mathcal{C}$ is semisimple (see [6]). In the present paper we focus on the remaining case, i.e. $\mathfrak{g} = \mathfrak{sl}(n+1)$. The main result is a classification of the indecomposable modules in this case.

There are two major differences between the two algebra types that make the study of the cuspidal modules much harder in the A-type case. First, in the $\mathfrak{sl}$-case, the translation functor does not provide equivalence of the subcategories $\mathcal{C}^\chi$ of $\mathcal{C}$ for all central characters χ. This leads to a consideration of three essentially different central character types: non-integral, regular integral and singular. Second, in many cases a simple cuspidal module has a non-trivial self-extension. Because of this the category of cuspidal modules does not have projective and injective objects, and one has to use a certain completion $\bar{\mathcal{C}}$ of $\mathcal{C}$. The most interesting central character type is, without a doubt, the regular integral one, because in this case there are n up to isomorphism simple objects in every block of $\mathcal{C}^\chi$. A convenient way to approach this case is to use methods and results from the quiver theory. Our main result can be formulated as follows

Theorem 1.1.

(a) *Every singular and non-integral block of the category $\mathcal{C}$ is equivalent to the category of finite-dimensional modules over the algebra of power series in one variable.*
(b) *Every regular integral block of $\mathcal{C}$ is equivalent to the category of locally nilpotent modules over the quiver*

$$\begin{tikzcd}
\bullet \arrow[loop above, "y"] \arrow[r, shift left, "x"] & \bullet \arrow[l, shift left, "x"] \arrow[r, shift left, "y"] & \bullet \arrow[l, shift left, "y"] \arrow[r, shift left, "x"] & \cdots \arrow[l, shift left, "x"] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above]
\end{tikzcd}$$

with relations $xy = yx = 0$.

The above quiver is special biserial and hence tame (see [9]). It was originally studied by Gelfand and Ponomarev in [18] in order to classify indecomposable representations of the Lorentz group.

Another important aspect of the category $\mathcal{C}$ is the geometric realization of its objects. The simple objects in $\mathcal{C}$, as well as their injective hulls in $\bar{\mathcal{C}}$, can be realized with the aid of sections of vector bundles on the projective space $\mathbb{P}^n$. These realizations are especially helpful for the explicit calculations of extensions of simple modules in the category $\mathcal{C}$.

The results in the present paper make a first step towards the study of other interesting category: the category $\mathcal{B}$ of all weight $\mathfrak{g}$-modules with uniformly bounded weight multiplicities. The study of the category $\mathcal{B}$ was initiated in [19], where the case of $\mathfrak{g} = \mathfrak{sp}(2n)$ was completely solved. The remaining case, i.e. $\mathfrak{g} = \mathfrak{sl}(n+1)$, will be treated in a future work.

Another natural problem is to extend the study of cuspidal modules to the category of generalized weight modules, i.e. modules that decompose as direct sums of generalized weight spaces on which $h - \lambda(h)$ act locally finitely for every weight λ and h in the Cartan subalgebra of $\mathfrak{g}$. The category of generalized cuspidal modules have the same simple objects as the category of simple weight modules, but the indecomposables are different. In fact, it follows from Theorem 4.12 that the category of generalized cuspidal modules is wild, so it should be studied using other methods. On the other hand, the geometric constructions we obtain in this paper can be easily generalized and lead to examples of generalized cuspidal modules that are generalized weight modules but not weight modules (see Remark 4.3).

The organization of the paper is as follows. In Section 2 we introduce the main notions and with the aid of the translation functor, reduce the general central character case to a specific set of central characters (namely, those that correspond to multiples of ε_0). In Section 3 we prove some preparatory statements and, in particular, consider the case of $\mathfrak{sl}(2)$. Let us mention that the classification of the indecomposable weight modules over $\mathfrak{sl}(2)$ with scalar action of the Casimir operator is usually attributed to Gabriel [13] and can be found in [7, §7.8.16]. The general case was treated in [8]. In Section 4 we calculate extensions between simple cuspidal modules with non-integral and singular central characters. In Section 5 we extend the category $\mathcal{C}$ of cuspidal modules to $\bar{\mathcal{C}}$ by adding injective limits, and construct injective objects in blocks with non-integral and singular central characters. The case of a regular integral central character is treated in Section 6, where we use translation functors [3] from singular to regular blocks to construct injective modules in the regular case. In the last section we provide an explicit realization of all indecomposable cuspidal modules. We expect that the description of the indecomposables of $\mathcal{C}$ will be useful for studying other categories of weight modules, including the category $\mathcal{B}$ of bounded modules.

2. Cuspidal representations

In this paper the ground field is $\mathbb{C}$, and $\mathfrak{g} = \mathfrak{sl}(n+1)$. All tensor products are assumed to be over $\mathbb{C}$ unless otherwise stated. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $(\,,\,)$ the Killing form on $\mathfrak{g}$. The induced form on $\mathfrak{h}^*$ will be denoted by $(\,,\,)$ as well. For our convenience, we fix a basis $\{\varepsilon_0, \dots, \varepsilon_n\}$ in $\mathbb{C}^{n+1}$, such that $\mathfrak{h}^*$ is identified with the subspace of $\mathbb{C}^{n+1}$ spanned by the simple roots $\alpha_1 = \varepsilon_0 - \varepsilon_1, \dots, \alpha_n = \varepsilon_{n-1} - \varepsilon_n$. By γ we denote the projection $\mathbb{C}^{n+1} \to \mathfrak{h}^*$ with one-dimensional kernel $\mathbb{C}(\varepsilon_0 + \cdots + \varepsilon_n)$. By $Q \subset \mathfrak{h}^*$ and $\Lambda \subset \mathfrak{h}^*$ we denote the root lattice and the weight lattice of $\mathfrak{g}$, respectively. The basis $\{\omega_1, \dots, \omega_n\}$ of Λ consists of the fundamental weights $\omega_i := \gamma(\varepsilon_0 + \cdots + \varepsilon_{i-1})$ for $i = 1, \dots, n$. Let $U := U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Denote by $Z := Z(U(\mathfrak{g}))$ the center of U and let $Z' := \mathrm{Hom}(Z, \mathbb{C})$ be the set of all central characters (here Hom stands for homomorphisms of unital $\mathbb{C}$-algebras). By $\chi_\lambda \in Z'$ we denote the central character of the irreducible highest weight module with highest weight λ. Recall that $\chi_\lambda = \chi_\mu$ iff $\lambda + \rho = w(\mu + \rho)$ for some element w of the Weyl group W, where, as usual, ρ denotes the half-sum of the positive roots. We say that $\chi = \chi_\lambda$ is *regular* if the stabilizer of $\lambda + \rho$ in W is trivial (otherwise χ is called *singular*), and that $\chi = \chi_\lambda$ is *integral* if $\lambda \in \Lambda$. We say that two weights λ and $\nu \in \lambda + \Lambda$ are in the same Weyl chamber if for any positive root α such that $(\lambda, \alpha) \in \mathbb{Z}$, $(\lambda, \alpha) \in \mathbb{Z}_{\geqslant 0}$ if and only if $(\mu, \alpha) \in \mathbb{Z}_{\geqslant 0}$. Finally recall that λ is *dominant integral* if $(\lambda, \alpha) \in \mathbb{Z}_{\geqslant 0}$ for all positive roots α.

A $\mathfrak{g}$-module M is a *generalized weight module* if $M = \bigoplus_{\mu \in \mathfrak{h}^*} M^{(\mu)}$, where

$$M^{(\mu)} := \bigl\{ m \in M \bigm| \bigl(h - \mu(h)\,\mathrm{Id}\bigr)^N m = 0, \text{ for every } h \in \mathfrak{h}, \text{ and some } N = N(h, m) \bigr\}.$$

A generalized weight $\mathfrak{g}$-module M is called a *weight module* if $M^{(\mu)} = M^\mu$, where

$$M^\mu := \bigl\{ m \in M \bigm| hm = \mu(h)m, \text{ for every } h \in \mathfrak{h} \bigr\}.$$

By definition the *support of* M, $\operatorname{supp} M$, is the set of weights $\mu \in \mathfrak{h}^*$ such that $M^\mu \neq 0$. A weight $\mathfrak{g}$-module M is *cuspidal* if M is finitely generated, all M^μ are finite-dimensional, and $X : M^\mu \to M^{\mu+\alpha}$ is an isomorphism for every root vector $X \in \mathfrak{g}_\alpha$. Denote by $\mathcal{C}$ the category of all cuspidal $\mathfrak{g}$-modules.

It is clear that, if M is a cuspidal module, then $\mu \in \operatorname{supp} M$ implies $\mu + Q \subset \operatorname{supp} M$. Hence for every cuspidal module M one can define $s(M) \subset \mathfrak{h}^*/Q$ as the image of $\operatorname{supp} M$ under the natural projection $\mathfrak{h}^* \to \mathfrak{h}^*/Q$. As M is finitely generated, $s(M)$ is a finite set.

It is not difficult to see that a submodule and a quotient of a cuspidal module are cuspidal. Hence the category $\mathcal{C}$ is an abelian category. It is also clear that every cuspidal module has finite Jordan–Hölder series. Since the center Z of U preserves weight spaces, it acts locally finitely on the cuspidal modules. For every central character $\chi \in Z'$ let $\mathcal{C}^\chi$ denote the category of all cuspidal modules M with generalized central character χ, i.e. such that for some $n(M)$, $(z - \chi(z))^{n(M)} = 0$ on M for all $z \in Z$. It is clear that every cuspidal module M is a direct sum of finitely many $M_i \in \mathcal{C}^{\chi_i}$. Furthermore, if $\operatorname{Ext}_{\mathcal{C}}$ stands for the extension functor in the category $\mathcal{C}$, then $\operatorname{Ext}_{\mathcal{C}}(M, N) \neq 0$ implies $s(M) \cap s(N) \neq \varnothing$. Therefore every $M \in \mathcal{C}^\chi$ is a direct sum $M = \bigoplus_{\bar{\nu} \in s(M)} M[\bar{\nu}]$, where, for $\bar{\nu} := \nu + Q \in \mathfrak{h}^*/Q$, $M[\bar{\nu}]$ denotes the maximal submodule of M such that $s(M[\bar{\nu}]) = \{\bar{\nu}\}$. Thus, one can write

$$\mathcal{C} = \bigoplus_{\chi \in Z',\, \bar{\nu} \in \mathfrak{h}^*/Q} \mathcal{C}^\chi_{\bar{\nu}},$$

where $\mathcal{C}^\chi_{\bar{\nu}}$ is the category of all modules M in $\mathcal{C}^\chi$ such that $s(M) = \{\bar{\nu}\}$.

In this section we describe the simple cuspidal $\mathfrak{g}$-modules following the classification of Mathieu in [21]. We formulate Mathieu's result in convenient for us terms.

We fix an $\mathfrak{h}$-eigenbasis of the natural representation of $\mathfrak{g}$. Let E_{ij}, $i, j = 0, \dots, n$ denote the elementary matrices. Define a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1$ is spanned by E_{i0} for all $i > 0$, $\mathfrak{g}_{-1}$ is spanned by E_{0i} for all $i > 0$, and $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$. The subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a maximal parabolic subalgebra in $\mathfrak{g}$. If $G = \mathrm{SL}(n+1)$ and $P \subset G$ is the subgroup with the Lie algebra $\mathfrak{p}$, then G/P is isomorphic to $\mathbb{P}^n$. Every point t in $\mathbb{P}^n$ can be represented by its homogeneous coordinates $[t_0, \dots, t_n]$. Then $\mathfrak{g}$ defines an algebra of vector fields on $\mathbb{P}^n$ via the map

$$E_{ij} \mapsto t_i \frac{\partial}{\partial t_j}. \tag{2.1}$$

By E we denote the Euler vector field $\sum_{i=0}^n t_i \frac{\partial}{\partial t_i}$.

Let $\mathcal{U}$ be the affine open subset of $\mathbb{P}^n$ consisting of all points $[t_0, \dots, t_n]$ such that $t_0 \neq 0$. Introduce local coordinates $x_1, \dots, x_n$ on $\mathcal{U}$ by setting $x_i := \frac{t_i}{t_0}$ for all $i = 1, \dots, n$. The ring $\mathcal{O} = \mathbb{C}[x_1, \dots, x_n]$ of regular functions on $\mathcal{U}$ is naturally a $\mathfrak{g}$-module. We say that M is a $(\mathfrak{g}, \mathcal{O})$-*module* if M is both a $\mathfrak{g}$-module and an $\mathcal{O}$-module, and

$$g(fm) - f(gm) = g(f)m$$

for any $m \in M$, $g \in \mathfrak{g}$, $f \in \mathcal{O}$. In particular, $\mathcal{O}$ is a $(\mathfrak{g}, \mathcal{O})$-module. If M and N are $(\mathfrak{g}, \mathcal{O})$-modules, then $M \otimes_{\mathcal{O}} N$ is a $(\mathfrak{g}, \mathcal{O})$-module as well.

For $\mu \in \mathbb{C}^{n+1}$, let $|\mu|$ denote $\mu_0 + \cdots + \mu_n$ and t^μ denote $t_0^{\mu_0} \dots t_n^{\mu_n}$. Then the vector space $t^\mu \mathbb{C}[t_0^{\pm 1}, \dots, t_n^{\pm 1}]$ has a natural structure of a $(\mathfrak{g}, \mathcal{O})$-module. Let

$$\mathcal{F}_\mu := \{ f \in t^\mu \mathbb{C}[t_0^{\pm 1}, \dots, t_n^{\pm 1}] \mid Ef = |\mu| f \}.$$

One can check that $\mathcal{F}_\mu$ is a $(\mathfrak{g}, \mathcal{O})$-module. It is not hard to see that $\mathcal{F}_\mu$ is an irreducible cuspidal $\mathfrak{g}$-module iff $\mu_i \notin \mathbb{Z}$ for all $i = 0, \dots, n$, in that case all weight spaces are one dimensional. It is also clear from the construction that $\mathcal{F}_\mu \cong \mathcal{F}_{\mu'}$ if $\mu - \mu' \in Q$.

Now we recall a slightly more general construction of a cuspidal module (see [23]). Let V_0 be a finite-dimensional P-module. Then V_0 induces a vector bundle on $\mathbb{P}^n$ which we denote by $\mathcal{V}_0$. The space of its sections $\Gamma(\mathcal{U}, \mathcal{V}_0)$ on the affine open set $\mathcal{U} \subset G/P$ is another example of a $(\mathfrak{g}, \mathcal{O})$-module. Since $\mathcal{V}_0$ is trivial on $\mathcal{U}$ one can identify $\Gamma(\mathcal{U}, \mathcal{V}_0)$ with $\mathcal{O} \otimes V_0$. Define a new $\mathfrak{g}$-module

$$\mathcal{F}_\mu(V_0) = \mathcal{F}_\mu \otimes_{\mathcal{O}} \Gamma(\mathcal{U}, \mathcal{V}_0).$$

Again, one easily checks that $\mathcal{F}_\mu(V_0)$ is cuspidal iff $\mu_i \notin \mathbb{Z}$.

Remark 2.1. If V is a $\mathfrak{g}$-module, then $\Gamma(\mathcal{U}, \mathcal{V}) \cong \mathcal{O} \otimes V$ as a $\mathfrak{g}$-module, and we have the following isomorphisms of $\mathfrak{g}$-modules

$$\begin{aligned} \mathcal{F}_\mu(V_0) \otimes V &\cong \mathcal{F}_\mu(V_0) \otimes_{\mathcal{O}} \Gamma(\mathcal{U}, \mathcal{V}) \cong \mathcal{F}_\mu \otimes_{\mathcal{O}} \Gamma(\mathcal{U}, \mathcal{V}_0) \otimes_{\mathcal{O}} \Gamma(\mathcal{U}, \mathcal{V}) \\ &\cong \mathcal{F}_\mu \otimes_{\mathcal{O}} \Gamma(\mathcal{U}, \mathcal{V} \otimes \mathcal{V}_0) \cong \mathcal{F}_\mu(V_0 \otimes V). \end{aligned}$$

Remark 2.2. An exact sequence of P-modules

$$0 \to V_0 \to V_1 \to V_2 \to 0$$

induces the exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathcal{F}_\mu(V_0) \to \mathcal{F}_\mu(V_1) \to \mathcal{F}_\mu(V_2) \to 0.$$

Recall that χ_λ denotes the central character of the simple highest weight $\mathfrak{g}$-module with highest weight λ. For simplicity we put $\mathcal{C}^\lambda := \mathcal{C}^{\chi_\lambda}$ and $\mathcal{C}_{\bar{\nu}}^\lambda := \mathcal{C}_{\bar{\nu}}^{\chi_\lambda}$.

Lemma 2.3. *Let V_0 be an irreducible P-module with highest weight τ. Then $\mathcal{F}_\mu(V_0)$ admits a central character χ_λ, where $\lambda = \gamma(|\mu|\varepsilon_0) + \tau$. Moreover, $s(\mathcal{F}_\mu(V_0))$ coincides with the class of $\gamma(\mu) + \tau$ in $\mathfrak{h}^*/Q$.*

Proof. As a $\mathfrak{g}_0$-module $\mathcal{F}_\mu(V_0)$ is isomorphic to

$$\mathcal{F}_\mu \otimes V_0 = x_1^{\mu_1} \dots x_n^{\mu_n} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes V_0.$$

We will consider $\mathfrak{g}$ as a Lie subalgebra of $\mathcal{A}_n \otimes \mathrm{End}(V_0)$, where $\mathcal{A}_n$ is the algebra of polynomial differential operators in $\mathbb{C}[x_1, \dots, x_n]$. The embedding $\mathfrak{g} \to \mathcal{A}_n \otimes \mathrm{End}(V_0)$ is defined by the formulae

$$E_{0i} \mapsto \frac{\partial}{\partial x_i}, \qquad E_{ij} \mapsto x_i \frac{\partial}{\partial x_j} \otimes 1 + 1 \otimes E_{ij}, \quad i, j > 0, i \neq j,$$

$$E_{00} - E_{ii} \mapsto \left(-|\mu| - E' - x_i \frac{\partial}{\partial x_i} \right) \otimes 1 + 1 \otimes (E_{00} - E_{ii}), \quad i > 0,$$

$$E_{i0} \mapsto \left(-x_i\left(E' + |\mu|\right)\right) \otimes 1 - \sum_{j \neq i} x_j \otimes E_{ij} + x_i \otimes (E_{00} - E_{ii}),$$

where $E' := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. This embedding can be extended to a homomorphism $\chi_{\mu,\tau} : U(\mathfrak{g}) \to \mathcal{A}_n \otimes \mathrm{End}(V_0)$. Note now that the formulae defining $\chi_{\mu,\tau}$ depend only on $|\mu|$ and τ. Hence $\chi_{\nu,\tau} = \chi_{\mu,\tau}$ if $|\nu| = |\mu|$. Let $\nu = |\mu|\varepsilon_0$. Then $\mathcal{F}_\nu(V_0)$ is not cuspidal and contains a $\mathfrak{b}$-singular vector $t_0^{|\mu|} \otimes v_0$, where v_0 is a highest weight vector of V_0 and $\mathfrak{b}$ is the standard Borel subalgebra of $\mathfrak{g}$. Hence $\mathcal{F}_\nu(V_0)$ contains a submodule L with the highest weight $\gamma(\nu) + \tau = \lambda$. It is not difficult to see that L is a faithful module over $\mathcal{A}_n \otimes \mathrm{End}(V_0)$. Since L admits a central character χ_λ, $\chi_{\mu,\tau}(z) = \chi_{\nu,\tau}(z) = \chi_\lambda(z)$ for any z in Z. The second statement of the lemma is trivial. □

The formulae in the proof of the previous lemma depend only on $|\mu|$ and the representation V_0 of $[\mathfrak{g}_0, \mathfrak{g}_0] \simeq \mathfrak{sl}(n) \subset \mathfrak{g}$. Therefore we obtain the following corollary.

Corollary 2.4. *If $(\tau, \alpha_i) = (\tau', \alpha_i)$ for $i = 2, \ldots, n$ and $\mu' - \mu \in Q + (\tau - \tau', \alpha_1)\varepsilon_0$, then $\mathcal{F}_\mu(V_0)$ is isomorphic to $\mathcal{F}_{\mu'}(V_0')$, where τ and τ' are the highest weights of V_0 and V_0' respectively.*

The following theorem is proven in [21].

Theorem 2.5. *Suppose that M is an irreducible cuspidal $\mathfrak{g}$-module and χ be the central character of M. Then $\chi = \chi_\lambda$ where λ is integral dominant when restricted to the subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. If χ is non-integral or singular integral, then M is isomorphic to $\mathcal{F}_\mu(V_0)$ for some $\mu \in \mathbb{C}^{n+1}$ and an irreducible P-module V_0.*

This theorem gives a complete description of the simple cuspidal modules with singular or non-integral central characters. We provide a description of the simple cuspidal modules with regular integral central character (also obtained by Mathieu) in Section 6.

Theorem 2.5 implies:

Corollary 2.6. *Let ρ be the half-sum of the positive roots and $\lambda + \rho$ be a non-integral or a singular integral weight. Then $\mathcal{C}_{\bar{\nu}}^\lambda$ is not empty iff $(\lambda, \omega_n) \notin (\nu, \omega_n) + \mathbb{Z}$, $(\lambda, \omega_i - \omega_{i+1}) \notin (\nu, \omega_i - \omega_{i+1}) + \mathbb{Z}$ for $i = 1, \ldots, n-1$, and $(\lambda, \omega_1 - \omega_2) \notin -(\nu, \omega_1) + \mathbb{Z}$. If $\mathcal{C}_{\bar{\nu}}^\lambda$ is not empty it has exactly one up to an isomorphism simple object.*

Proof. Recall that $\lambda = \gamma(|\mu|\varepsilon_0) + \tau$, where τ is the highest weight of V_0. As follows from Corollary 2.4, we can assume that $(\tau, \alpha_1) = 0$. We fix $\mu \in \mathfrak{h}^*$ so that $\varphi := \gamma(\mu) + \tau$ is a representative of $\bar{\nu}$ in $\mathfrak{h}^*/Q$, i.e. $\bar{\varphi} = \bar{\nu}$. One has the following relations between μ, λ and φ

$$(\varphi, \alpha_1) = \mu_0 - \mu_1, \qquad (\varphi, \alpha_i) = \mu_{i-1} - \mu_i + (\lambda, \alpha_i), \quad i = 2, \ldots, n, \ (\lambda, \alpha_1) = |\mu|.$$

The above relations imply that

$$\mu_n(n+1) + (\varphi, n\alpha_n + \cdots + 2\alpha_2 + \alpha_1) - (\lambda, n\alpha_n + \cdots + 2\alpha_2 + \alpha_1) = 0.$$

Using that $n\alpha_n + \cdots + 2\alpha_2 + \alpha_1 = (n+1)\omega_n$, we obtain

$$\mu_n = (\lambda - \varphi, \omega_n), \qquad \mu_{n-1} = (\lambda - \varphi, \omega_{n-1} - \omega_n), \ldots,$$
$$\mu_1 = (\lambda - \varphi, \omega_1 - \omega_2), \qquad \mu_0 = (\lambda, \omega_1 - \omega_2) + (\varphi, \omega_1).$$

Since $\mathcal{F}_\mu(V_0)$ is cuspidal iff $\mu_i \notin \mathbb{Z}$ for all $i = 0, \ldots, n$, the proof is completed. $\square$

Recall now the definition of translation functors. Let V be a finite-dimensional $\mathfrak{g}$-module and $\eta, \lambda \in \mathfrak{h}^*$ be such that $\tau = \lambda - \eta$ is in the support of V. Let $\mathfrak{g}^\kappa$-mod denote the category of $\mathfrak{g}$-modules which admit a generalized central character χ_κ. The translation functor $T_\eta^\lambda : \mathfrak{g}^\eta\text{-mod} \to \mathfrak{g}^\lambda\text{-mod}$ is defined by $T_\eta^\lambda(M) = (M \otimes V)^{\chi_\lambda}$, where $(M \otimes V)^{\chi_\lambda}$ stands for the direct summand of $M \otimes V$ admitting generalized central character χ_λ. Assume in addition that τ belongs to the W-orbit of the highest weight of V, the stabilizers of $\eta + \rho$ and $\lambda + \rho$ in the Weyl group coincide and $\nu + \rho, \lambda + \rho$ lie in the same Weyl chamber. Then $T_\eta^\lambda : \mathfrak{g}^\eta\text{-mod} \to \mathfrak{g}^\lambda\text{-mod}$ defines an equivalence of categories (see [3]).

Lemma 2.7. *With the above notations, if M is cuspidal, then $T_\eta^\nu(M)$ is cuspidal and $s(M) + s(V) = s(T_\eta^\nu(M))$.*

Proof. First we observe that $M \otimes V$ is a semisimple $\mathfrak{h}$-module with finite weight multiplicities. Let $X \in \mathfrak{g}_\alpha$. Since M is cuspidal the action of X is free, therefore the action of X on $M \otimes V$ is free as well. Since the multiplicities of all weight spaces in M are the same, the weight multiplicities of $M \otimes V$ are all the same as well. Therefore $M \otimes V$ is cuspidal. Since $(M \otimes V)^{\chi_\nu}$ is a direct summand of $M \otimes V$, $(M \otimes V)^{\chi_\nu}$ is also cuspidal. The second statement is obvious. $\square$

Lemma 2.8. *If the category $\mathcal{C}^\chi$ is not empty, then it is equivalent to $\mathcal{C}^{\gamma(t\varepsilon_0)}$ for some $t \in (\mathbb{C} \backslash \mathbb{Z}) \cup \{0, -1, \ldots, -n\}$.*

Proof. Since $\mathcal{C}^\chi$ is not empty, Theorem 2.5 implies that $\chi = \chi_\lambda$ where λ is dominant when restricted to $[\mathfrak{g}_0, \mathfrak{g}_0] \cong \mathfrak{sl}(n)$. If λ itself is integral dominant it is well known that $\mathfrak{g}^\lambda$-mod is equivalent to $\mathfrak{g}^0$-mod (see [3] for instance). Thus, by Lemma 2.7, $\mathcal{C}^\lambda$ is equivalent to $\mathcal{C}^0$.

Now consider the case when the central character of $\chi = \chi_\lambda$ is not integral. Using the action of the Weyl group, one can choose $\lambda = \gamma(t\varepsilon_0) + \tau$, where τ is some integral dominant weight and $t \notin \mathbb{Z}$. One can easily see that $\eta + \rho := \gamma(t\varepsilon_0) + \rho$ and $\lambda + \rho$ are both regular and belong to the same Weyl chamber. Therefore T_η^λ defines an equivalence of the categories $\mathcal{C}^\eta$ and $\mathcal{C}^\lambda$ and hence the lemma holds for a non-integral central character.

Finally, let us consider the case when $\chi = \chi_\lambda$ is singular integral. The conditions on λ in Theorem 2.5 ensure that one can choose λ satisfying $(\lambda + \rho, \varepsilon_0 - \varepsilon_k) = 0$ for exactly one $0 < k \leqslant n$. Let us put $\eta = \gamma(-k\varepsilon_0)$, $\tau = \lambda - \eta$. Then the stabilizers of $\eta + \rho$ and $\lambda + \rho$ coincide, $\eta + \rho$ and $\lambda + \rho$ belong to the same Weyl chamber and τ is a regular integral weight. Hence τ is in the W-orbit of the highest weight of some finite-dimensional $\mathfrak{g}$-module V. Therefore T_η^λ defines an equivalence of the categories $\mathcal{C}^\eta$ and $\mathcal{C}^\lambda$, and the lemma holds in this case as well. $\square$

Denote by σ the antiautomorphism of $\mathfrak{g}$ defined by $\sigma(X) = X^t$. For any weight module $M = \bigoplus_{\mu \in \mathfrak{h}^*} M^\mu$ one can construct a new module $M^\vee := \bigoplus_{\mu \in \mathfrak{h}^*} (M^\mu)^*$ with $\mathfrak{g}$-action defined by the formula $\langle Xu, m\rangle = \langle u, \sigma(X)m\rangle$ for any $X \in \mathfrak{g}, u \in (M^\mu)^*$, and $m \in M^\nu$. Then ${}^\vee : \mathcal{C} \to \mathcal{C}$ is a contravariant exact functor, which maps $\mathcal{C}^\chi_{\bar\nu}$ to itself. If χ is singular or non-integral $\mathcal{C}^\chi_{\bar\nu}$ has only one simple object and therefore $L^\vee \cong L$ for any simple module L in $\mathcal{C}^\chi_{\bar\nu}$. We show in Section 6 (Lemma 6.8) that every simple module L in $\mathcal{C}^0_{\bar\nu}$ can be obtained as a unique simple submodule and a unique simple quotient in $T^\lambda_\eta(L')$ for some singular η and simple module L' in $\mathcal{C}^\eta$. Since ${}^\vee$ commutes with T^λ_η we obtain that $L^\vee \cong L$ for any simple module L in $\mathcal{C}$.

3. Extensions between cuspidal modules

Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We consider the functors Ext in the category of $\mathfrak{s}$-modules that are semisimple over $\mathfrak{h}$. If M and N are two $\mathfrak{s}$-modules that are semisimple over $\mathfrak{h}$, then $\mathrm{Ext}^i_{\mathfrak{s},\mathfrak{h}}(M, N)$ can be expressed in terms of relative Lie algebra cohomology. In particular,

$$\mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(M, N) \cong H^1\bigl(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N)\bigr),$$

where the right-hand side is the corresponding relative cohomology group (see [11, Sections 1.3 and 1.4] for instance). For a sake of completeness we recall the definition of $H^1(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N))$. The set of 1-cocycles $C^1(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N))$ is the subspace of all $c \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{s}, \mathrm{Hom}_{\mathbb{C}}(M, N))$ such that

$$c(\mathfrak{h}) = 0, \qquad c\bigl([g_1, g_2]\bigr) = \bigl[g_1, c(g_2)\bigr] - \bigl[g_2, c(g_1)\bigr] \tag{3.1}$$

for any $g_1, g_2 \in \mathfrak{s}$. A 1-cocycle c is a coboundary if $c(g) = [g, \varphi]$ for some $\varphi \in \mathrm{Hom}_{\mathfrak{h}}(M, N)$. Denote by $B^1(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N))$ the space of all coboundaries. Then

$$H^1\bigl(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N)\bigr) := C^1\bigl(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N)\bigr)/B^1\bigl(\mathfrak{s}, \mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(M, N)\bigr).$$

For any multiplicative subset X of $U(\mathfrak{s})$ we denote the localization of $U(\mathfrak{s})$ relative to X by $U_X(\mathfrak{s})$. For any $\mathfrak{s}$-module M, $\mathcal{D}_X M := U_X(\mathfrak{s}) \otimes_{U(\mathfrak{s})} M$ denotes the localization of M relative to X.

Lemma 3.1. *Let $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$ or $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_{-1}$, M be a simple cuspidal $\mathfrak{g}$-module, and $\mu \in \operatorname{supp} M$. Then $\mathrm{End}_{\mathfrak{s}}(M) \cong \mathrm{End}_{\mathbb{C}} M^\mu$ and $\mathrm{Ext}^1_{\mathfrak{s},\mathfrak{h}}(M, M) = 0$.*

Proof. Let $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$ (the case $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_{-1}$ is treated in the same way). Let $X_1, \dots, X_n$ be an $\mathrm{ad}_{\mathfrak{h}}$-eigenbasis of $\mathfrak{g}_1$. Since M is cuspidal, the action of $X_1, \dots, X_n$ is invertible. Therefore the localization $\mathcal{D}_X M$ of M relative to $X := \{X_1, \dots, X_n\} \subset U(\mathfrak{s})$ is isomorphic to M. In other words, M is a module over $U_X(\mathfrak{s})$. Moreover, if μ is a weight of M, then M is generated by M^μ as $U_X(\mathfrak{s})$-module. Hence M is isomorphic to the induced module $U_X(\mathfrak{s}) \otimes_{U(\mathfrak{h})} M^\mu$. Therefore

$$\mathrm{End}_{\mathfrak{s}}(M) \cong \mathrm{End}_{U_X(\mathfrak{s})}(M) \cong \mathrm{Hom}_{U(\mathfrak{h})}\bigl(M^\mu, M\bigr) \cong \mathrm{End}_{\mathbb{C}}\bigl(M^\mu\bigr).$$

Thus, we have an isomorphism $\mathrm{End}_{\mathfrak{s}}(M) \cong \mathrm{End}_{\mathbb{C}} M^\mu$. To prove the second statement note that any $\mathfrak{s}$-module M' (semisimple over $\mathfrak{h}$) which can be included in an exact sequence of $\mathfrak{s}$-modules

$$0 \to M \to M' \to M \to 0$$

is a module over $U_X(\mathfrak{s})$ since all X_i are invertible. Since

$$U_X(\mathfrak{s}) \otimes_{U(\mathfrak{h})} M^\mu \simeq \mathbb{C}\big[X_1^{\pm 1}, \dots, X_n^{\pm 1}\big] \otimes_{\mathbb{C}} M^\mu,$$

M is free over $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, the exact sequence splits over $U_X(\mathfrak{s})$, and therefore over $\mathfrak{s}$ as well. $\square$

The following lemma is used in the next section.

Lemma 3.2. *Let M be a simple cuspidal $\mathfrak{g}$-module and c be a 1-cocycle in $C^1(\mathfrak{g}, \mathfrak{h}; \operatorname{End}(M))$. Then there exists $\psi \in \operatorname{End}_{\mathfrak{h}}(M)$ such that for any $g_1 \in \mathfrak{g}_1$ and $g_0 \in \mathfrak{g}_0$,*

$$c(g_1) = [g_1, \psi], \qquad [g_0, \psi] - c(g_0) \in \operatorname{End}_{\mathfrak{g}_1}(M).$$

Proof. The first identity follows directly from the second statement of Lemma 3.1 applied to $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$, since the restriction of c on $\mathfrak{s}$ is a coboundary. To obtain the second statement, use the identity

$$c\big([g_0, g_1]\big) = \big[g_0, c(g_1)\big] - \big[g_1, c(g_0)\big].$$

Then

$$\big[[g_0, g_1], \psi\big] = \big[g_0, [g_1, \psi]\big] - \big[g_1, c(g_0)\big]$$

implies

$$\big[g_1, [g_0, \psi] - c(g_0)\big] = 0$$

for any $g_1 \in \mathfrak{g}_1$. Hence $[g_0, \psi] + c(g_0) \in \operatorname{End}_{\mathfrak{g}_1}(M)$. $\square$

Example 3.3. Let $\mathfrak{g} = \mathfrak{sl}(2)$ and M and N be two simple cuspidal $\mathfrak{g}$-modules. If $\operatorname{Ext}^1_{\mathfrak{g}}(M, N) \neq 0$, then M and N are in the same block of $\mathcal{C}$. Therefore by Corollary 2.6 and Theorem 6.2, $M \cong N$. Moreover, M is isomorphic to $\mathcal{F}_\mu$ for some $\mu \in \mathbb{C}^2$. In particular, the weight multiplicities of M are equal to 1. Let $\{X, H, Y\}$ be the standard $\mathfrak{sl}(2)$-basis, and $c \in \operatorname{Hom}_{\mathfrak{h}}(\mathfrak{g}, \operatorname{End}(M))$ be a 1-cocycle. Then $c(H) = 0$, and, by Lemma 3.1 one may assume without loss of generality that $c(X) = 0$, since one can add a coboundary $d(m)$ such that $c(X) = [X, m]$. Then $[X, c(Y)] = 0$ and therefore $c(Y) \in \operatorname{End}_{\mathfrak{s}}(M)$, where $\mathfrak{s} = \mathbb{C}H \oplus \mathbb{C}X$. Then, again by Lemma 3.1, $c(Y) = bX^{-1}$ for some $b \in \mathbb{C}$. It is straightforward to check that $c(H) = c(X) = 0$ and $c(Y) = bX^{-1}$ imply (3.1). Now let us check that c is not trivial if $b \neq 0$. Indeed, assume the contrary and let $c(g) = [g, \varphi]$ for some $\varphi \in \operatorname{End}_{\mathfrak{h}}(M)$. But then $[X, \varphi] = 0$, and again by Lemma 3.1, φ is a scalar map. Hence $c = 0$.

Thus, $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N) = 0$ if M and N are not isomorphic, and $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N) = \mathbb{C}$ if M and N are isomorphic.

4. The case of singular or non-integral central character

In this section we compute $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M,N)$ for $n \geqslant 2$ and irreducible cuspidal modules M and N admitting a singular or a non-integral central character χ. If $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M,N) \neq 0$ then M and N must belong to the same block of $\mathcal{C}$, therefore by Corollary 2.6, M is isomorphic to N. Lemma 2.8 implies that it suffices to calculate $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M,M)$ for the case $M = \mathcal{F}_\mu$ since any block $\mathcal{C}^\lambda_{\bar{\nu}}$ is equivalent to the one with irreducible object $\mathcal{F}_\mu$. The main result of this section is that for any cuspidal module $\mathcal{F}_\mu$, $H^1(\mathfrak{s},\mathfrak{h};\mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)) = \mathbb{C}$. Note that, the latter cohomology group describes the space of infinitesimal deformations of $\mathcal{F}_\mu$ in $\mathcal{C}$ with the same support. On the other hand, the family $\mathcal{F}_{\mu+s(\varepsilon_0+\cdots+\varepsilon_n)}$ provides a one-parameter deformation with desired properties, hence the dimension of $H^1(\mathfrak{s},\mathfrak{h};\mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu))$ is at least one. The difficult part is to show that the dimension is not bigger.

Lemma 4.1. *Let*

$$z = t_1\frac{\partial}{\partial t_1} + \cdots + t_n\frac{\partial}{\partial t_n} - nt_0\frac{\partial}{\partial t_0},$$

and in particular, $\mathfrak{g}_0 \cong \mathfrak{sl}(n) \oplus \mathbb{C}z$. *Then*

$$\mathcal{F}_\mu = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}^k_\mu, \tag{4.1}$$

where $\mathcal{F}^k_\mu$ is the z-eigenspace corresponding to the eigenvalue $|\mu| + (n+1)(k-\mu_0)$. Moreover, each $\mathcal{F}^k_\mu$ is a simple cuspidal $\mathfrak{sl}(n)$-module isomorphic to $\mathcal{F}_{(\mu_1+k,\mu_2,\dots,\mu_n)}$.

Proof. A straightforward calculation shows that

$$\mathcal{F}^k_\mu = t_0^{\mu_0-k}t_1^{\mu_1+k}\dots t_n^{\mu_n}\mathbb{C}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1},\dots,\left(\frac{t_n}{t_1}\right)^{\pm 1}\right].$$

The $\mathfrak{sl}(n)$-module isomorphism $\mathcal{F}^k_\mu \cong \mathcal{F}_{(\mu_1+k,\mu_2,\dots,\mu_n)}$ follows directly from (2.1). □

Lemma 4.2. *For $n \geqslant 2$ and $u = \log(t_0t_1\dots t_n)$ let $N := \mathcal{F}_\mu \oplus u\mathcal{F}_\mu$ be the $\mathfrak{g}$-module with action induced by the correspondence* (2.1). *Then N is a non-trivial self-extension of $\mathcal{F}_\mu$. The cocycle defining this extension is given by the formulae $c(E_{ij}) = \frac{t_i}{t_j}$, $0 \leqslant i \neq j \leqslant n$.*

Proof. It is obvious that N contains a submodule $\mathcal{F}_\mu$ and $N/\mathcal{F}_\mu \cong \mathcal{F}_\mu$. It remains to check that this extension does not split. We will prove the statement by induction on n.

Let us start with $\mathfrak{g} = \mathfrak{sl}(2)$ and show that this self-extension is non-trivial for almost all μ. Indeed, it is sufficient to show that the Casimir operator Ω does not act as a scalar on M. The Casimir operator of $\mathfrak{sl}(2)$ can be written in the following form

$$\Omega = E_{01}E_{10} + E_{10}E_{01} + \frac{(E_{00}-E_{11})^2}{2}.$$

Then for $f \in \mathcal{F}_\mu$,

$$\Omega(uf) = u\Omega(f) + \left(E_{01}\frac{t_1}{t_0} + \frac{t_0}{t_1}E_{10} + E_{10}\frac{t_0}{t_1} + \frac{t_1}{t_0}E_{01}\right)f.$$

But

$$E_{01}\frac{t_1}{t_0} + \frac{t_0}{t_1}E_{10} + E_{10}\frac{t_0}{t_1} + \frac{t_1}{t_0}E_{01} = 2 + 2E = 2\bigl(1 + |\mu|\bigr).$$

Hence this self-extension is non-trivial for $|\mu| \neq -1$.

We now apply induction on n. Since (4.1) is the $\mathfrak{g}_0$-decomposition of $\mathcal{F}_\mu$, the restriction of c on $\mathfrak{sl}(n) \subset \mathfrak{g}_0 \subset \mathfrak{sl}(n+1)$ is non-trivial because it is not trivial on the component $\mathcal{F}_\mu^k$ for almost all k. Hence c is not a trivial cocycle. □

Remark 4.3. One can easily generalize the construction in Lemma 4.2 and obtain a family of non-trivial self-extensions of $\mathcal{F}_\mu$ in the category of generalized weight modules. Indeed, for $(u_0, \dots, u_n) \in \mathbb{C}^{n+1}$ we define $N(u_0, \dots, u_n) := \mathcal{F}_\mu \oplus u\mathcal{F}_\mu$ for $u = \sum_{i=0}^n u_i \log t_i$. Then it is easy to check that $N(u_0, \dots, u_n)$ is a non-trivial self extension of $\mathcal{F}_\mu$ and is a generalized weight module but not a weight module unless $u_0 = \cdots = u_n$.

Lemma 4.4. *Let V_0 be a simple finite-dimensional $\mathfrak{g}_0$-module, and $n \geqslant 2$. A simple cuspidal module $\mathcal{F}_\mu(V_0)$ has a non-trivial self-extension defined by the cocycle $c(E_{ij}) = \frac{t_i}{t_j}$ for all $i \neq j$, $0 \leqslant i, j \leqslant n$.*

Proof. By using Lemmas 4.1 and 2.2 we obtain the following decomposition of $\mathcal{F}_\mu(V_0)$ as a $\mathfrak{g}_0$-module

$$\mathcal{F}_\mu(V_0) = \bigoplus_{k\in\mathbb{Z}} \bigl(\mathcal{F}_\mu^k \otimes V_0\bigr).$$

As in the proof of Lemma 4.2 it suffices to check that the following sequence of $\mathfrak{g}_0$-modules

$$0 \to \mathcal{F}_\mu^k \otimes V_0 \to \bigl(\mathcal{F}_\mu^k \oplus u\mathcal{F}_\mu^k\bigr) \otimes V_0 \to \mathcal{F}_\mu^k \otimes V_0 \to 0$$

does not split. That is a consequence of the following general fact. □

Lemma 4.5. *Let M, N and L be modules over a Lie algebra $\mathfrak{a}$. If*

$$0 \to M \to N \to L \to 0 \tag{4.2}$$

does not split, then for any finite-dimensional $\mathfrak{a}$-module V the sequence

$$0 \to M \otimes V \to N \otimes V \to L \otimes V \to 0$$

does not split either.

Proof. Assume that the latter sequence splits. Then

$$0 \to M \otimes V \otimes V^* \to N \otimes V \otimes V^* \to L \otimes V \otimes V^* \to 0$$

also splits. Suppose that $p : L \otimes V \otimes V^* \to N \otimes V \otimes V^*$ is a splitting map. Denote by i the natural embedding $L \to L \otimes V \otimes V^*$ and by j the natural projection $N \otimes V \otimes V^* \to N$. Then $j \circ p \circ i$ is a splitting map for (4.2). □

Remark 4.6. Lemma 4.4 still holds for $n = 1$ and $|\mu| \neq -1$ (see the proof of Lemma 4.2). In the special case of $n = 1$ and $|\mu| = -1$ one can easily check that the cocycle c is trivial. Indeed, $c(g) = [g, \varphi]$, for $\varphi \in \operatorname{End}_{\mathfrak{h}}(M)$ defined by $\varphi(t^\mu) = \varphi(\mu)t^\mu$, where the function $\varphi(\mu)$ can be found inductively using $\varphi(\mu_0, \mu_1) - \varphi(\mu_0 + 1, \mu_1 - 1) = \frac{1}{\mu_1}$. Nevertheless, in this special case, we still have a non-trivial cocycle (see Example 3.3).

Lemma 4.7. *Let $c \in \operatorname{Hom}_{\mathfrak{h}}(\mathfrak{g}, \operatorname{End}_{\mathbb{C}}(\mathcal{F}_\mu))$ be defined by the formulae $c(E_{ii}) = 0$, $c(E_{ij}) = b_{ij}\frac{t_i}{t_j}$ for some $b_{ij} \in \mathbb{C}$ if $i \neq j$. Then c is a 1-cocycle iff there exists $b \in \mathbb{C}$ such that $b_{ij} = b$ for all $i \neq j$.*

Proof. Let $b = b_{0n}$ and $c'(E_{ij}) = c(E_{ij}) - b\frac{t_i}{t_j}$. Then c' is a cocycle. On the other hand, for any $k \neq 0$ and $k \neq n$,

$$c'(E_{kn}) = c'\big([E_{k0}, E_{0n}]\big) = -\left[E_{0n}, (b_{k0} - b)\frac{t_k}{t_0}\right] = 0$$

and for any $k \neq n$

$$0 = c'\big([E_{kn}, E_{nk}]\big) = \big[E_{kn}, c'(E_{nk})\big] = \left[E_{kn}, (b_{nk} - b)\frac{t_n}{t_k}\right] = b_{nk} - b.$$

Hence $c'(E_{nk}) = c'(E_{jn}) = 0$ for all $j, k \neq n$. Then

$$c'(E_{jk}) = c'\big([E_{jn}, E_{nk}]\big) = 0. \qquad \square$$

Lemma 4.8. *Let $n \geqslant 2$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and $\mathcal{F}_\mu$ be cuspidal. The restriction map*

$$r : H^1\big(\mathfrak{g}, \mathfrak{h}; \operatorname{End}_{\mathbb{C}}(\mathcal{F}_\mu)\big) \to H^1\big(\mathfrak{p}, \mathfrak{h}; \operatorname{End}_{\mathbb{C}}(\mathcal{F}_\mu)\big)$$

is injective.

Proof. Let $c \in C^1(\mathfrak{g}, \mathfrak{h}; \operatorname{End}_{\mathbb{C}}(\mathcal{F}_\mu))$ be such that $c(\mathfrak{p})$ is a coboundary. Without loss of generality we can choose c so that $c(\mathfrak{p}) = 0$. Then

$$\big[\mathfrak{g}_1, c(\mathfrak{g}_{-1})\big] \subset \big[\mathfrak{g}_{-1}, c(\mathfrak{g}_1)\big] + c(\mathfrak{g}_0) = 0$$

implies

$$\big[\mathfrak{g}_1, c(E_{0k})\big] = 0$$

for all k. Since $E_{k0} \in \mathfrak{g}_1$, $E_{k0}c(E_{0k})$ commutes with the action of $\mathfrak{g}_1$. Moreover it commutes with the action $\mathfrak{h}$ since it maps every weight space to itself. Therefore $E_{k0}c(E_{0k}) \in \mathrm{End}_{\mathfrak{s}}(\mathcal{F}_\mu)$ for $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$, and the first statement of Lemma 3.1 implies

$$c(E_{0k}) = b_k E_{k0}^{-1}$$

for some constant $b_k \in \mathbb{C}$. Choose $i \geqslant 1$ and $i \neq k$. Then

$$c(E_{0i}) = -c\big([E_{ki}, E_{0k}]\big) = -\big[E_{ki}, b_k E_{k0}^{-1}\big] = 0.$$

That proves $c(E_{0i}) = 0$ for all $1 \leqslant i \leqslant n$. Hence $c = 0$. □

Let $F(\mu)$ be the set of functions $\varphi : \mu + Q \to \mathbb{C}$. Then one can identify $F(\mu)$ with the space $\mathrm{End}_{\mathfrak{h}}(\mathcal{F}_\mu)$ by the formula

$$\varphi\big(t^\lambda\big) = \varphi(\lambda) t^\lambda.$$

A function $\varphi(\lambda)$ that depends only on its i-th coordinate λ_i will be often written as $\varphi(\lambda_i)$.

Lemma 4.9. *Let $\mathcal{F}_\mu$ be cuspidal and $c \in \mathrm{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_1, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu))$. Then*

$$c(E_{i0}) = E_{i0}\phi$$

for some $\phi \in F(\mu)$ such that $\phi(\lambda) = \phi(\lambda_0)$ (i.e. ϕ depends only on the first coordinate λ_0 of $\lambda \in \mu + Q$). Moreover, there exists some $\zeta(\lambda) = \zeta(\lambda_0)$ such that $c(E_{i0}) = [E_{i0}, \zeta]$.

Proof. Lemma 4.1 implies

$$\mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{g}_1, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)\big) \cong \bigoplus_{k,l \in \mathbb{Z}} \mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{g}_1 \otimes \mathcal{F}_\mu^k, \mathcal{F}_\mu^l\big).$$

By comparing the eigenvalues of z one verifies that

$$\mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{g}_1 \otimes \mathcal{F}_\mu^k, \mathcal{F}_\mu^l\big) = 0$$

if $l \neq k + 1$. We claim that

$$\mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{g}_1 \otimes \mathcal{F}_\mu^k, \mathcal{F}_\mu^{k+1}\big) = \mathbb{C}. \tag{4.3}$$

This follows from Remarks 2.1 and 2.2 applied to $\mathfrak{g}' = \mathfrak{sl}(n) \subset \mathfrak{g}_0$ and the corresponding parabolic subalgebra $\mathfrak{p}'$ of $\mathfrak{g}'$. Denote by $V(\eta)$ the simple highest weight $\mathfrak{p}'$-module with highest weight η. Using the isomorphism of $\mathfrak{g}'$-modules

$$\mathfrak{g}_1 \otimes \mathcal{F}_\mu^k \cong \mathcal{F}_\mu^k(\mathfrak{g}_1),$$

and the exact sequence of $\mathfrak{p}'$-modules

$$0 \to V(\varepsilon_2) \to \mathfrak{g}_1 \to V(\varepsilon_1) \to 0,$$

we obtain the following exact sequence

$$0 \to \mathcal{F}_\mu^k\big(V(\varepsilon_2)\big) \to \mathcal{F}_\mu^k \otimes \mathfrak{g}_1 \to \mathcal{F}_\mu^k\big(V(\varepsilon_1)\big) = \mathcal{F}_\mu^{k+1} \to 0.$$

Since $\operatorname{Hom}_{\mathfrak{g}_0}(\mathcal{F}_\mu^k(V(\varepsilon_2)), \mathcal{F}_\mu^{k+1}) = 0$, the last exact sequence implies (4.3).

Obviously the map $E_{i0} \otimes v \to E_{i0}v$ defines a $\mathfrak{g}_0$-module homomorphism $\mathfrak{g}_1 \otimes \mathcal{F}_\mu \to \mathcal{F}_\mu$. Therefore any non-zero homomorphism $c \in \operatorname{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes \mathcal{F}_\mu, \mathcal{F}_\mu)$ can be written in the form

$$c(E_{i0} \otimes v) = \phi(\mu_0 - k)E_{i0}v$$

where $v \in \mathcal{F}_\mu^k$. That implies the first statement of the lemma.

To prove the second statement note that the equation $c(E_{i0}) = [E_{i0}, \zeta]$ is equivalent to the following functional equation

$$-(\lambda_0 - 1)\zeta(\lambda_0 - 1) + \lambda_0\zeta(\lambda_0) = \phi(\lambda_0).$$

Such ζ can be easily found inductively since λ_0 is never 0, as $\mu_0 \notin \mathbb{Z}$. $\square$

Lemma 4.10. *If $\mathcal{F}_\mu$ is cuspidal, then* $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu, \mathcal{F}_\mu) = \mathbb{C}$.

Proof. By Lemma 4.2 it suffices to prove that $\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu, \mathcal{F}_\mu) \leqslant 1$.

Let $c \in C^1(\mathfrak{g}, \mathfrak{h}, \operatorname{End}_{\mathbb{C}}(\mathcal{F}_\mu))$. As follows from Lemma 3.2, we may assume that there is $\psi \in F(\mu)$ such that for all $g_0 \in \mathfrak{g}_0$, and $g_1 \in \mathfrak{g}_1$,

$$c(g_1) = [g_1, \psi], \qquad [g_0, \psi] - c(g_0) \in \operatorname{End}_{\mathfrak{g}_1}(\mathcal{F}_\mu).$$

Let $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$. Note that $\frac{t_i}{t_j} \in \operatorname{End}_{\mathfrak{g}_1}(\mathcal{F}_\mu)$ if $i, j > 0$ and

$$\frac{t_j}{t_i}\big([E_{ij}, \psi] + c(E_{ij})\big) \in \operatorname{End}_{\mathfrak{s}}(\mathcal{F}_\mu).$$

Therefore, by the first statement of Lemma 3.1, for any $i \neq j$, $1 \leqslant i, j \leqslant n$, there is a constant $b_{ij} \in \mathbb{C}$ such that

$$[E_{ij}, \psi] - c(E_{ij}) = b_{ij}\frac{t_i}{t_j}.$$

But then $c'(E_{ij}) := c(E_{ij}) - [E_{ij}, \psi] = b_{ij}\frac{t_i}{t_j}$ is a cocycle on $\mathfrak{g}_0$. Lemmas 4.1 and 4.7 imply that $b = b_{ij}$ for some constant b, and thus $c_{|\mathfrak{g}_0}$ is equivalent to the cocycle $b\frac{t_i}{t_j}$ modulo some coboundary. Therefore, we may assume that

$$c(E_{ij}) = b\frac{t_i}{t_j},$$

for $1 \leqslant i \neq j \leqslant n$. Then $[c(E_{ij}), \mathfrak{g}_1] = 0$, and one has

$$\bigl[g_0, c(g_1)\bigr] = c\bigl([g_0, g_1]\bigr)$$

for all $g_0 \in \mathfrak{g}_0$ and $g_1 \in \mathfrak{g}_1$. By Lemma 4.9, this implies

$$c(g_1) = [\zeta, g_1]$$

for some $\zeta = \zeta(\lambda_0)$. But $[\mathfrak{g}_0, \zeta] = 0$. Therefore, the cocycle

$$c'(g) = c(g) + [g, \zeta]$$

defines the same cohomology class as c and

$$c'(E_{ij}) = b\frac{t_i}{t_j}, \qquad c'(E_{i0}) = 0$$

for $i \neq j$, $1 \leqslant i, j \leqslant n$. Therefore, $\dim H^1(\mathfrak{p}, \mathfrak{h}; \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)) = 1$ and, by Lemma 4.8, $\dim H^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)) \leqslant 1$. $\square$

Theorem 4.11. *Let M be a simple cuspidal $\mathfrak{g}$-module with singular or non-integral central character χ and let N be any simple cuspidal module. Then $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N) = 0$ if N is not isomorphic to M and $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, M) = \mathbb{C}$.*

Proof. The theorem follows from Lemmas 2.8 and 4.10 since any block of $\mathcal{C}$ with non-integral or singular central character is isomorphic to a block with unique simple module $\mathcal{F}_\mu$ for some μ. $\square$

Theorem 4.12. *If $\mathcal{F}_\mu$ is cuspidal, then $\mathrm{Ext}^1_{\mathfrak{g}}(\mathcal{F}_\mu, \mathcal{F}_\mu) = \mathbb{C}^{n+1}$.*

Proof. If M is a self-extension of $\mathcal{F}_\mu$, then $\mathfrak{h}$ acts locally finitely on M and therefore M is a generalized weight module. On the other hand, we have an isomorphism

$$\mathrm{Ext}^1_{\mathfrak{g}}(A, B) \simeq H^1\bigl(\mathfrak{g}, \mathrm{Hom}_{\mathbb{C}}(A, B)\bigr)$$

for any $\mathfrak{g}$-modules A and B (see [11]). Since in our case the extension is a generalized weight module, we can assume without loss of generality that the cocycle defining it is $\mathfrak{h}$-invariant, i.e. $c \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu))$.

Use the same notations as in the proof of Lemma 3.1 and let $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}_1$. Since $\mathcal{F}_\mu$ is free over $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ we may assume that $c(X_i) = 0$ for all $i \leqslant n$. Since $0 = c([h, X_i]) = [c(h), X_i]$ we obtain that $c(h) \in \mathrm{End}_{U_X(\mathfrak{s})}(\mathcal{F}_\mu)$ for all $h \in \mathfrak{h}$. But $\mathrm{End}_{U_X(\mathfrak{s})}(\mathcal{F}_\mu) = \mathbb{C}$, so $c(h)$ is a constant for each $h \in \mathfrak{h}$. Let $u_i = c(E_{ii} - E_{00})$, then, as we explained already in Remark 4.3, the linear functional $c'(X) = X(u)$ where $u = \sum_{i=1}^n u_i \log t_i$ defines a non-trivial cocycle $c' \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu))$. Moreover, $c'(X_i) = 0$ and $c'(E_{ii} - E_{00}) = u_i \in \mathbb{C}$. Let $c'' = c - c'$. Then $c''(h) = 0$ for any $h \in \mathfrak{h}$, and therefore $c'' \in C^1(\mathfrak{g}, \mathfrak{h}, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu))$. By Theorem 4.11 we have $H^1(\mathfrak{g}, \mathfrak{h}, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)) = \mathbb{C}$, and hence $H^1(\mathfrak{g}, \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\mu)) = \mathbb{C}^{n+1}$. Theorem 4.12 is proven. $\square$

5. An extension $\bar{\mathcal{C}}$ of $\mathcal{C}$ and the structure of the category $\mathcal{C}_{\bar{\nu}}^{\chi}$ for non-integral or singular χ

Let $\mathcal{A}$ be an abelian category and P be a projective generator in $\mathcal{A}$. It is a well-known fact (see, for example, [17] Exercise 2, Section 2.6) that the functor $\mathrm{Hom}_{\mathcal{A}}(P, M)$ provides an equivalence of $\mathcal{A}$ and the category of right modules over the ring $\mathrm{Hom}_{\mathcal{A}}(P, P)$. In case when every object in $\mathcal{A}$ has a finite length and each simple object has a projective cover, one reduces the problem of classifying indecomposable objects in $\mathcal{A}$ to the similar problem for modules over a finite-dimensional algebra (see [15,14]). In many cases when $\mathcal{A}$ does not have projective modules it is possible to consider a certain completion of $\mathcal{A}$ and reduce the case to the category of modules over some pointed algebra. We use this strategy to study the category $\mathcal{C}$ of cuspidal modules. However, we use injective modules instead of projectives and exploit the existence of the duality functor $^{\vee}$ in $\mathcal{C}$. We prefer this consideration since in this case we avoid taking projective limits and introducing topology. Another advantage of this approach is that the center of $U(\mathfrak{g})$ acts locally finitely on injective limits of cuspidal modules.

Let $\bar{\mathcal{C}}$ be the full subcategory of all weight modules consisting of $\mathfrak{g}$-modules M which have countable dimension and whose finitely generated submodules belong to $\mathcal{C}$. It is not hard to see that every such M has an exhausting filtration $0 \subset M_1 \subset M_2 \subset \cdots$ such that each $M_i \in \mathcal{C}$. It implies that the action of the center Z of the universal enveloping algebra U on M is locally finite and we have a decomposition

$$\bar{\mathcal{C}} = \bigoplus_{\chi \in Z',\, \bar{\nu} \in \mathfrak{h}^*/Q} \bar{\mathcal{C}}_{\bar{\nu}}^{\chi},$$

defined in the same way as for $\mathcal{C}$.

Before we proceed with studying blocks of $\bar{\mathcal{C}}$ let us formulate a general result. Let R be a unital $\mathbb{C}$-algebra and let $\bar{\mathcal{A}}$ be an abelian category of R-modules satisfying the following conditions:

- $\bar{\mathcal{A}}$ contains finitely many up to isomorphism simple objects $L_1, \ldots, L_n$ such that $\mathrm{End}_R(L_i) = \mathbb{C}$.
- $\bar{\mathcal{A}}$ contains indecomposable injective modules $I_1, \ldots, I_n$ such that $\mathrm{Hom}_R(L_i, I_j) = 0$ if $i \neq j$, and $\mathrm{Hom}_R(L_i, I_i) = \mathbb{C}$.
- Let $\mathcal{A}$ be the subcategory of $\bar{\mathcal{A}}$ which consists of all objects in $\bar{\mathcal{A}}$ of finite length. Assume that every module M in $\bar{\mathcal{A}}$ has an increasing exhausting filtration

$$0 = F^0(M) \subset F^1(M) \subset \cdots \subset F^k(M) \subset \cdots \tag{5.1}$$

  such that $F^k(M) \in \mathcal{A}$ for all k.
- Finally, assume that there exists an involutive contravariant exact faithful functor $^{\vee} : \mathcal{A} \to \mathcal{A}$ such that

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R\big(N^{\vee}, M^{\vee}\big).$$

Let $I := I_1 \oplus \cdots \oplus I_n$ and $\mathcal{E} := \mathrm{End}_R(I)$. Define a functor Φ from $\mathcal{A}$ to $\mathcal{E}$-mod by

$$\Phi(M) := \mathrm{Hom}_R\big(M^{\vee}, I\big). \tag{5.2}$$

Theorem 5.1. *The functor Φ establishes an equivalence of the category $\mathcal{A}$ and the category of all finite-dimensional $\mathcal{E}$-modules.*

Proof. The functor Φ is exact as follows from the injectivity of I. It is straightforward that $\Phi(L_1), \dots, \Phi(L_n)$ are pairwise non-isomorphic one-dimensional $\mathcal{E}$-modules. Therefore Φ maps a simple object to a one-dimensional $\mathcal{E}$-module, hence an object of finite length to a finite-dimensional $\mathcal{E}$-module.

Next we will show that if V is a simple finite-dimensional $\mathcal{E}$-module, then $V \cong \Phi(L_i)$. The conditions imposed on the category $\bar{\mathcal{A}}$ ensure that I has a filtration $0 = F^0(I) \subset F^1(I) \subset \cdots \subset F^k(I) \subset \cdots$ such that $F^1(I)$ is a maximal semisimple submodule in I (in fact, $F^1(I) \cong L_1 \oplus \cdots \oplus L_n$) and $F^k(I)/F^{k-1}(I)$ is semisimple for all $k > 0$. Let

$$\mathcal{E}' := \{\phi \in \mathcal{E} \mid \phi(F^1(I)) = 0\}. \tag{5.3}$$

It is easy to check that $\mathcal{E}'$ is a two sided ideal in $\mathcal{E}$, and

$$\mathcal{E}/\mathcal{E}' \cong \mathrm{Hom}_R(L_1 \oplus \cdots \oplus L_n, I) \simeq \mathrm{End}_R(L_1 \oplus \cdots \oplus L_n). \tag{5.4}$$

Moreover, any $\phi \in \mathcal{E}'$ is locally nilpotent, because $\phi^k(F^k(I)) = 0$. Hence $c + \phi$ is invertible for any non-zero $c \in \mathbb{C}$. Therefore the only eigenvalue of ϕ in V is zero, and in particular, every $\phi \in \mathcal{E}'$ acts nilpotently on V. That implies $(\mathcal{E}')^N(V) = 0$. By the simplicity of V, $\mathcal{E}'(V) = 0$. Now the statement follows directly from (5.4).

Now consider the natural isomorphism

$$\mathrm{Hom}_R\big(M, \mathrm{Hom}_{\mathcal{E}}(F, I)\big) \cong \mathrm{Hom}_{\mathcal{E}}\big(F, \mathrm{Hom}_R(M, I)\big)$$

for any $\mathcal{E}$-module F and $M \in \bar{\mathcal{A}}$. If F is a finite-dimensional $\mathcal{E}$-module, it is not difficult to see by induction on $\dim F$ that $\mathrm{Hom}_{\mathcal{E}}(F, I)$ has a finite length as an R-module and hence lies in $\mathcal{A}$. Therefore for any $M \in \mathcal{A}$ we have

$$\mathrm{Hom}_R\big(\big(\mathrm{Hom}_{\mathcal{E}}(F, I)\big)^\vee, M\big) \cong \mathrm{Hom}_R\big(M^\vee, \mathrm{Hom}_{\mathcal{E}}(F, I)\big) \cong \mathrm{Hom}_{\mathcal{E}}\big(F, \mathrm{Hom}_R\big(M^\vee, I\big)\big).$$

Thus, the functor Ψ from the category of finite-dimensional $\mathcal{E}$-modules to $\mathcal{A}$ defined by

$$\Psi(F) = \big(\mathrm{Hom}_{\mathcal{E}}(F, I)\big)^\vee$$

is the right adjoint of Φ. It is obvious that $\Psi(\Phi(L_i)) \simeq L_i$ for all i. That implies Theorem 5.1. □

Let χ be a non-integral or singular central character and $\bar{\nu} \in \mathfrak{h}^*/Q$ be such that $\mathcal{C}_{\bar{\nu}}^{\chi}$ is not empty. By Corollary 2.6, there is exactly one up to isomorphism simple object in $\mathcal{C}_{\bar{\nu}}^{\chi}$, which is isomorphic to $\mathcal{F}_\mu(V_0)$ for suitable μ and V_0. Define the $\mathfrak{g}$-modules (see Lemma 4.4)

$$\mathcal{F}_\mu^{(m)} := \mathcal{F}_\mu \oplus u\mathcal{F}_\mu \oplus \cdots \oplus u^m \mathcal{F}_\mu, \qquad \mathcal{F}_\mu^{(m)}(V_0) := \mathcal{F}_\mu^{(m)} \otimes_{\mathcal{O}} \Gamma(U, \mathcal{V}_0).$$

For $n \geqslant 2$ or $|\mu| \neq -1$ the action of $\mathfrak{g}$ on $\mathcal{F}_\mu^{(m)}$ is the standard one. For $n = 1$ and $|\mu| = -1$ we set $X(u^m \otimes f) := u^m \otimes Xf$, $H(u^m \otimes f) := u^m \otimes Hf$, and $Y(u^m \otimes f) := u^m \otimes Yf + u^{m-1} \otimes X^{-1} f$, where $f \in \mathcal{F}_\mu$. Note that the standard action of $\mathfrak{g}$ in the latter case would lead to semisimple modules $\mathcal{F}_\mu^{(m)}$ (see Remark 4.6). In the proofs of the results in this section we assume that the action is standard, i.e. $n \geqslant 2$ or $|\mu| \neq -1$. However, it is not hard to check that all results remain valid in the exceptional case as well. The details are left to the reader.

Lemma 5.2. *$\mathcal{F}_\mu^{(m)}(V_0)$ is an indecomposable module.*

Proof. First, note that $\mathcal{F}_\mu^{(m)}(V_0)$ has a filtration

$$0 \subset \mathcal{F}_\mu(V_0) \subset \mathcal{F}_\mu^{(1)}(V_0) \subset \cdots \subset \mathcal{F}_\mu^{(m)}(V_0).$$

To check that $\mathcal{F}_\mu^{(m)}(V_0)$ is indecomposable it is sufficient to check that $\mathcal{F}_\mu^{(m)}(V_0)$ has a unique irreducible submodule. We prove this by induction on m. Assume that $\mathcal{F}_\mu^{(m-1)}(V_0)$ has a unique irreducible submodule, and let $\mathcal{F}_\mu^{(m)}(V_0)$ have an irreducible submodule $L \neq \mathcal{F}_\mu(V_0)$. Then $L \cap \mathcal{F}_\mu^{(m-1)}(V_0) = 0$ and thus

$$\mathcal{F}_\mu^{(m)}(V_0) \cong L \oplus \mathcal{F}_\mu^{(m-1)}(V_0).$$

But in this case

$$\mathcal{F}_\mu^{(1)}(V_0) \cong \mathcal{F}_\mu^{(m)}(V_0)/\mathcal{F}_\mu^{(m-2)}(V_0) \cong \mathcal{F}_\mu(V_0) \oplus L.$$

However, by Lemma 4.4, $\mathcal{F}_\mu^{(1)}(V_0)$ is indecomposable. Contradiction. □

Lemma 5.3. $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(m)}(V_0)) = \mathbb{C}$.

Proof. We again apply induction on m. For $m = 0$ the statement follows from Theorem 4.11. We use now the exact sequence

$$0 \to \mathcal{F}_\mu^{(m-1)}(V_0) \to \mathcal{F}_\mu^{(m)}(V_0) \to \mathcal{F}_\mu(V_0) \to 0.$$

Since $\operatorname{Hom}_{\mathfrak{g}}(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(k)}(V_0)) = \mathbb{C}$ for all k, and $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(m-1)}(V_0)) = \mathbb{C}$, by the inductive assumption, the corresponding long exact sequence of Ext starts with

$$0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(m)}(V_0)\big) \to \mathbb{C} \to \cdots.$$

Therefore, $\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(m)}(V_0)) \leqslant 1$. On the other hand, Lemma 5.2 implies that $\mathcal{F}_\mu^{(m+1)}(V_0)$ is a non-trivial extension of $\mathcal{F}_\mu(V_0)$ by $\mathcal{F}_\mu^{(m)}(V_0)$. Hence $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(m)}(V_0)) = \mathbb{C}$. □

A natural example of a module in $\bar{\mathcal{C}}_{\bar{\nu}}^{\chi}$ is

$$\bar{\mathcal{F}}_\mu(V_0) := \varinjlim \mathcal{F}_\mu^{(m)}(V_0) = \bigoplus_{m \geqslant 0} u^m \mathcal{F}_\mu(V_0).$$

Lemma 5.4. *$\bar{\mathcal{F}}_\mu(V_0)$ is an indecomposable injective object in $\bar{\mathcal{C}}_{\bar{\nu}}^{\chi}$, and* $\mathrm{End}_{\mathfrak{g}}(\bar{\mathcal{F}}_\mu(V_0)) = \mathbb{C}[[\frac{\partial}{\partial u}]]$.

Proof. $\bar{\mathcal{F}}_\mu(V_0)$ is indecomposable since it contains a unique simple submodule. The latter follows from Lemma 5.2. To verify the endomorphism identity note that

$$\mathrm{End}_{\mathfrak{g}}\big(\bar{\mathcal{F}}_\mu(V_0)\big) = \varprojlim \mathrm{End}_{\mathfrak{g}}\big(\mathcal{F}_\mu^{(k)}(V_0)\big) \text{ and } \mathrm{End}_{\mathfrak{g}}\big(\mathcal{F}_\mu^{(k)}(V_0)\big) \cong \mathbb{C}\left[\frac{\partial}{\partial u}\right] \Big/ \left(\frac{\partial^k}{\partial u^k}\right).$$

To prove the injectivity it suffices to show that $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)) = 0$.[1] Assume the opposite. Let $c \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g} \otimes \mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0))$ be a non-trivial cocycle that induces an exact sequence

$$0 \to \bar{\mathcal{F}}_\mu(V_0) \to M \to \mathcal{F}_\mu(V_0) \to 0.$$

Pick $m \in M$ such that $m \notin \bar{\mathcal{F}}_\mu(V_0)$ and let $M' := U(\mathfrak{g})m$. Since M' is finitely generated, $M' \cap \bar{\mathcal{F}}_\mu(V_0) = \mathcal{F}_\mu^{(k)}(V_0)$ for some k. Since $\mathcal{F}_\mu(V_0)$ is simple, we have the following exact sequence

$$0 \to \mathcal{F}_\mu^{(k)}(V_0) \to M' \to \mathcal{F}_\mu(V_0) \to 0.$$

If we identify M with $\mathcal{F}_\mu(V_0) \oplus \bar{\mathcal{F}}_\mu(V_0)$ as a vector space, then the action of $g \in \mathfrak{g}$ on M is given by $g(m_1, m_2) = (gm_1, c(g)m_1 + gm_2)$. Since $M' = \mathcal{F}_\mu(V_0) \oplus \mathcal{F}_\mu^{(k)}(V_0)$ is $\mathfrak{g}$-invariant, $c \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g} \otimes \mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(k)}(V_0))$. Now consider the exact sequence

$$0 \to \mathcal{F}_\mu^{(k)}(V_0) \to \bar{\mathcal{F}}_\mu(V_0) \xrightarrow{\varphi} \bar{\mathcal{F}}_\mu(V_0) \to 0,$$

where $\varphi = \frac{\partial^k}{\partial u^k}$. This sequence leads to the long exact sequence

$$\begin{gathered} 0 \to \mathrm{Hom}_{\mathfrak{g}}\big(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(k)}(V_0)\big) = \mathbb{C} \to \mathrm{Hom}_{\mathfrak{g}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big) = \mathbb{C} \to \\ \mathrm{Hom}_{\mathfrak{g}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big) = \mathbb{C} \to \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \mathcal{F}_\mu^{(k)}(V_0)\big) = \mathbb{C} \to \\ \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big) \to \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big) \to \cdots \end{gathered}$$

and therefore the map

$$\varphi : \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big) \to \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}\big(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)\big)$$

is injective. But by our construction $\varphi(c) = 0$. Thus, we obtain contradiction with our assumption that c is non-trivial. □

Lemma 5.4 and Theorem 5.1 imply the following

[1] Indeed, if $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(\mathcal{F}_\mu(V_0), \bar{\mathcal{F}}_\mu(V_0)) = 0$, then $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, \bar{\mathcal{F}}_\mu(V_0)) = 0$ for any cuspidal module M. Since any module in $\bar{\mathcal{C}}_{\bar{\nu}}^{\chi}$ is an injective limit of cuspidal ones, $\mathrm{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, \bar{\mathcal{F}}_\mu(V_0)) = 0$ for any module M in $\bar{\mathcal{C}}_{\bar{\nu}}^{\chi}$.

Theorem 5.5. *Let χ be a non-integral or singular central character and $\bar{\nu} \in \mathfrak{h}^*/Q$ be such that $\mathcal{C}_{\bar{\nu}}^{\chi}$ is not empty. Then $\mathcal{C}_{\bar{\nu}}^{\chi}$ is equivalent to the category of finite-dimensional modules over the algebra of power series in one variable.*

Corollary 5.6. *Let χ be a non-integral or singular central character and $\bar{\nu} \in \mathfrak{h}^*/Q$ be such that $\mathcal{C}_{\bar{\nu}}^{\chi}$ is not empty. Every indecomposable module in $\mathcal{C}_{\bar{\nu}}^{\chi}$ is isomorphic to $\mathcal{F}_{\mu}^{(k)}(V_0)$ for some non-negative integer k.*

Proof. Every finite-dimensional $\mathbb{C}[[\frac{\partial}{\partial u}]]$-module has trivial action of the maximal ideal of $\mathbb{C}[[\frac{\partial}{\partial u}]]$. By the Jordan decomposition theorem every finite-dimensional indecomposable $\mathbb{C}[[\frac{\partial}{\partial u}]]$-module is isomorphic to $\mathbb{C}[\frac{\partial}{\partial u}]/(\frac{\partial^k}{\partial u^k})$. □

6. The structure of the category $\mathcal{C}_{\bar{\nu}}^{\chi}$ for regular integral χ

The goal of this section is to prove the following

Theorem 6.1. *Let $\mathfrak{g} = \mathfrak{sl}(n+1)$. Every regular integral block of $\mathcal{C}$ is equivalent to the category of locally nilpotent modules over the quiver $\mathcal{Q}_n$ (where n is the number of vertices)*

$$\begin{tikzcd}
\bullet \arrow[loop above, "y"] \arrow[r, shift left, "x"] & \bullet \arrow[l, shift left, "x"] \arrow[r, shift left, "y"] & \bullet \arrow[l, shift left, "y"] \arrow[r, shift left, "x"] & \cdots \arrow[l, shift left, "x"] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above]
\end{tikzcd}$$

with relations $xy = yx = 0$.

Lemma 2.8 implies that if χ is regular integral then $\mathcal{C}_{\bar{\nu}}^{\chi}$ is equivalent to $\mathcal{C}_{\bar{\nu}_1}^{0}$ for suitable $\bar{\nu}_1$. Thus we may assume that $\chi = 0$. First we describe the simple objects in $\mathcal{C}_{\bar{\nu}}^{0}$ following §11 in [21]. For our convenience we slightly change the description provided in [21] by using homogeneous coordinates instead of local coordinates on $\mathbb{P}^n$. Let $\mu \in \mathbb{C}^{n+1}$, $\widehat{\Omega}^k$ be the space of k-forms on $\mathbb{C}^{n+1}$, and

$$\widehat{\Omega}^k(\mu) := t^{\mu}\mathbb{C}\big[t_0^{\pm 1}, \dots, t_n^{\pm 1}\big] \otimes_{\mathbb{C}[t_0,\dots,t_n]} \widehat{\Omega}^k,$$
$$\Omega^k(\mu) := \big\{\omega \in \widehat{\Omega}^k(\mu) \bigm| L_E(\omega) = |\mu|\omega,\ i_E(\omega) = 0\big\},$$

where i_E denotes the contraction with the Euler vector field E, and L_E denotes the Lie derivative. The space $\Omega^k(\mu)$ is a $\mathfrak{g}$-module with $\mathfrak{g}$-action defined by the Lie derivative. In this section we assume that all $\mu_i \notin \mathbb{Z}$. Then $\Omega^k(\mu)$ is a cuspidal module; and it is simple if $|\mu| \neq 0$.

In this section we assume $|\mu| = 0$. Then the de Rham differential $d : \Omega^k(\mu) \to \Omega^{k+1}(\mu)$ is well defined as $L_E = d \circ i_E + i_E \circ d = 0$. Furthermore, it is not difficult to see that $\mu_i \notin \mathbb{Z}$ imply that the de Rham complex is exact. Let

$$L_k := d\big(\Omega^{k-1}(\mu)\big) = \operatorname{Ker} d \cap \Omega^k(\mu).$$

The following two results are proven in [21].

Theorem 6.2. *$L_1, \dots, L_n$ are all up to isomorphism simple objects in $\mathcal{C}_{\overline{\gamma(\mu)}}^{0}$.*

Lemma 6.3. *Let $|\mu| = 0$. Then $\Omega^0(\mu) \cong L_1$, $\Omega^n(\mu) \cong L_n$. If $k = 1, \dots, n-1$, then $\Omega^k(\mu)$ is an indecomposable $\mathfrak{g}$-module, i.e. the following exact sequence*

$$0 \to L_k \to \Omega^k(\mu) \to L_{k+1} \to 0 \tag{6.1}$$

does not split.

Our next step is to construct indecomposable injectives in $\bar{\mathcal{C}}^0$. We will do it by applying translation functors to injectives in singular blocks.

Following the construction of $\mathcal{F}_\mu^{(m)}$ in the previous section, for an arbitrary $m > 0$ define

$$\Omega^k(\mu)^{(m)} := \Omega^k(\mu) \oplus u\Omega^k(\mu) \oplus u^2\Omega^k(\mu) \oplus \cdots \oplus u^m\Omega^k(\mu),$$

where $u = \log(t_0 \dots t_n)$. Then define a module

$$\bar{\Omega}^k(\mu) = \mathbb{C}[u]\Omega^k(\mu)$$

in $\bar{\mathcal{C}}$. Then $\bar{\Omega}^k(\mu)$ has an obvious filtration

$$0 \subset \Omega^k(\mu) \subset \Omega^k(\mu)^{(1)} \subset \Omega^k(\mu)^{(2)} \subset \cdots \subset \Omega^k(\mu)^{(m)} \subset \cdots. \tag{6.2}$$

For every object M in $\bar{\mathcal{C}}$ and a finite-dimensional $\mathfrak{g}$-module V, the module $M \otimes V$ is in $\bar{\mathcal{C}}$ as well. Since the center of $U(\mathfrak{g})$ acts locally finitely on M, one can define M^{χ_λ} as the subspace of M on which all elements of the center lying in $\operatorname{Ker}\chi_\lambda$ act locally nilpotently. The following is a well-known fact (see [3]).

Lemma 6.4. *For every injective module M in $\bar{\mathcal{C}}$ and a finite-dimensional $\mathfrak{g}$-module V, the modules $M \otimes V$, M^{χ_λ}, and $(M \otimes V)^{\chi_\lambda}$ are injective.*

Proof. It is enough to show that $M \otimes V$ and M^{χ_λ} are injective. The injectivity of $M \otimes V$ follows from the isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}(X, M \otimes V) \cong \operatorname{Hom}_{\mathfrak{g}}\big(X \otimes V^*, M\big).$$

Since $\bullet \otimes V^*$ and $\operatorname{Hom}_{\mathfrak{g}}(\bullet, M)$ are both exact, $\operatorname{Hom}_{\mathfrak{g}}(\bullet, M \otimes V)$ is also exact.

The injectivity of M^{χ_λ} follows from the fact that M^{χ_λ} is a direct summand in M. □

Lemma 6.5. *Let $|\mu| = 0$. Then the modules $\Omega^k(\mu)^{(m)}$ and $\bar{\Omega}^k(\mu)$ are indecomposable modules with unique irreducible submodules. The same holds for any non-trivial quotients of $\Omega^k(\mu)^{(m)}$ and $\bar{\Omega}^k(\mu)$ as well.*

Proof. We prove the statement for $\Omega^k(\mu)^{(m)}$ by induction on m using the filtration (6.2). We reason as in the proof of Lemma 5.4. It suffices to prove the statement for $\Omega^k(\mu)^{(1)}$.

Suppose that L is a simple submodule of $\Omega^k(\mu)^{(1)}$ and $L \neq L_k$. Then $L \cap \Omega^k(\mu) = 0$ by Lemma 6.3, hence the image of L under the natural projection $\Omega^k(\mu)^{(1)} \to \Omega^k(\mu)$ is L_k (since $\Omega^k(\mu)$ has only one simple submodule and it is $L(k)$). This implies that $L(k) \oplus L = L_k + uL_k$ is a submodule of $\Omega^k(\mu)^{(1)}$, which, as one can easily check, is not true.

Now let $M := \Omega^k(\mu)^{(1)}/L_k$ and $p : \Omega^k(\mu)^{(1)} \to M$ be the natural projection. Then $p(L_{k+1}) \subset M$ is a simple submodule. Suppose that there is another simple submodule L. Then the image of L in $\Omega^k(\mu)$ under the natural projection $M \to \Omega^k(\mu)$ must be L_k. This again implies that $L_k + uL_k$ is a submodule of $\Omega^k(\mu)^{(1)}$, which leads to a contradiction. The cases $k = 1$ and $k = n$ are similar to the general case. □

Corollary 6.6. *There exists a unique filtration*

$$0 = F^0 \subset F^1 \subset F^2 \subset F^3 \subset \cdots$$

of $\bar{\Omega}^k(\mu)$ such that all quotients F^i/F^{i-1} are simple. Furthermore, $F^i/F^{i-1} \cong L_1$ if $k = 0$, and $F^i/F^{i-1} \cong L_n$ if $k = n$. If $1 \leqslant k \leqslant n-1$, then $F^i/F^{i-1} \cong L_k$ for odd i and $F^i/F^{i-1} \cong L_{k+1}$ for even i.

Lemma 6.7. $\mathrm{Hom}_{\mathfrak{g}}(\bar{\Omega}^k(\mu), \bar{\Omega}^l(\mu)) = 0$ *if $k \neq l$, and* $\mathrm{End}_{\mathfrak{g}}(\bar{\Omega}^k(\mu)) = \mathbb{C}[[\frac{\partial}{\partial u}]]$.

Proof. Let $\phi \in \mathrm{Hom}_{\mathfrak{g}}(\bar{\Omega}^k(\mu), \bar{\Omega}^l(\mu))$ and $\phi \neq 0$. Then $\mathrm{Im}\,\phi$ contains a simple submodule $L_l \subset \bar{\Omega}^l(\mu)$. Hence $\bar{\Omega}^k(\mu)$ contains a simple subquotient isomorphic to L_l and (6.1) implies $l = k$ or $k+1$. On the other hand, by Corollary 6.6, $\bar{\Omega}^k(\mu)/\mathrm{Ker}\,\phi$ contains a simple subquotient isomorphic to L_{k-1}. Hence $\bar{\Omega}^l(\mu)$ has a simple subquotient isomorphic to L_{k-1}. Therefore $\mathrm{Hom}_{\mathfrak{g}}(\bar{\Omega}^k(\mu), \bar{\Omega}^l(\mu)) \neq 0$ implies $k = l$. To prove the second statement use Corollary 6.6. Since any endomorphism preserves the filtration, $\mathrm{End}_{\mathfrak{g}}(\bar{\Omega}^k(\mu))$ is the projective limit of $\mathrm{End}_{\mathfrak{g}}(F^m) = \mathbb{C}[\frac{\partial}{\partial u}]/(\frac{\partial^m}{\partial u^m})$. □

Let V be the span of the functions $t_0, t_1, \dots, t_n$ and consider V as the natural $(n+1)$-dimensional $\mathfrak{g}$-module. For $k = 1, \dots, n$ we have the following sequence

$$0 \to \Omega^k(\mu) \xrightarrow{\theta} \Omega^{k-1}(\mu - \varepsilon_0) \otimes V \xrightarrow{\sigma} \Omega^{k-1}(\mu) \to 0,$$

where $\theta = \sum i_{\frac{\partial}{\partial t_i}} \otimes t_i$ and $\sigma = \sum t_i \otimes \frac{\partial}{\partial t_i}$. Obviously θ and σ are $\mathfrak{g}$-equivariant. The direct computation shows that $\sigma \circ \theta = i_E = 0$. Furthermore, assume that $|\mu| = 0$ and consider the component of the above exact sequence corresponding to the trivial generalized central character. The resulting sequence is

$$0 \to \Omega^k(\mu) \xrightarrow{\varphi} S^k \xrightarrow{\psi} \Omega^{k-1}(\mu) \to 0, \tag{6.3}$$

where $S^k := (\Omega^{k-1}(\mu - \varepsilon_0) \otimes V)^{\chi_0}$.

Lemma 6.8. *The sequence* (6.3) *is exact for $k = 1, \dots, n$. Moreover, S^k is an indecomposable module with unique simple submodule and unique simple quotient, both isomorphic to L_k.*

Proof. Note that $\Omega^k(\mu) \simeq \mathcal{F}_{\mu - k\varepsilon_0}(V(\varepsilon_1 + \cdots + \varepsilon_k))$, where $V(\eta)$ is the irreducible P-module with highest weight η. By Remark 2.1,

$$\Omega^{k-1}(\mu - \varepsilon_0) \otimes V \cong \mathcal{F}_{\mu - k\varepsilon_0}\big(V(\varepsilon_1 + \cdots + \varepsilon_{k-1}) \otimes V\big).$$

Now use the exact sequence of P-modules

$$\begin{aligned}&0\to V(2\varepsilon_1+\cdots+\varepsilon_{k-1})\oplus V(\varepsilon_1+\cdots+\varepsilon_k)\to V(\varepsilon_1+\cdots+\varepsilon_{k-1})\otimes V\\&\quad\to V(\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_{k-1})\to 0\end{aligned}$$

which induces the following exact sequence of $\mathfrak{g}$-modules

$$\begin{aligned}&0\to \mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(2\varepsilon_1+\cdots+\varepsilon_k)\bigr)\oplus\mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(\varepsilon_1+\cdots+\varepsilon_k)\bigr)\\&\quad\to\mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(\varepsilon_1+\cdots+\varepsilon_{k+1})\bigr)\otimes V\to\mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_{k-1})\bigr)\to 0.\end{aligned}$$

But

$$\begin{gathered}\mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_{k-1})\bigr)=\Omega^{k-1}(\mu),\qquad \mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(\varepsilon_1+\cdots+\varepsilon_k)\bigr)=\Omega^k(\mu),\\ \bigl(\mathcal{F}_{\mu-k\varepsilon_0}\bigl(V(2\varepsilon_1+\cdots+\varepsilon_k)\bigr)\bigr)^{\chi_0}=0.\end{gathered}$$

Hence (6.3) is an exact sequence.

Now we will show that S^k has a unique irreducible submodule isomorphic to L_k (this will imply the indecomposability of S^k as well). Since the functor $^\vee$ preserves tensor products with finite-dimensional modules and maps a simple module to itself, the irreducibility of $\Omega^{k-1}(\mu-\varepsilon_0)$ implies $(S^k)^\vee=S^k$. By Lemma 6.3, $\Omega^k(\mu)$ and $\Omega^{k-1}(\mu)$ are indecomposable. If S^k has another irreducible submodule then, by the indecomposability of $\Omega^k(\mu)$, this submodule is isomorphic to L_{k-1}. But then since $(S^k)^\vee=S^k$, S^k must have an irreducible quotient isomorphic to L_{k-1}, which is impossible due to the indecomposability of $\Omega^{k-1}(\mu)$. Finally, again by duality, S^k has a unique irreducible quotient isomorphic to L_k. $\square$

Recall that $\Omega^{k-1}(\mu-\varepsilon_0)$ is a simple cuspidal module with singular central character $\chi_{-k\varepsilon_0}$, and $\bar{\Omega}^{k-1}(\mu-\varepsilon_0)$ is an indecomposable injective in this singular block. Set $I^k:=(\bar{\Omega}^{k-1}(\mu-\varepsilon_0)\otimes V)^{\chi_0}$ for $k=1,\dots,n$. The exact sequence (6.3) leads to the following exact sequence

$$0\to\bar{\Omega}^k(\mu)\xrightarrow{i_k}I^k\xrightarrow{p_k}\bar{\Omega}^{k-1}(\mu)\to 0.\tag{6.4}$$

Lemma 6.9. *The module I^k is an injective object in $\bar{\mathcal{C}}^0_{\overline{\gamma(\mu)}}$, and it has a unique simple submodule, which is isomorphic to L_k.*

Proof. The injectivity of I^k follows from Lemma 6.4. To prove that I^k has a unique simple submodule isomorphic to L_k recall that $T^\eta_\chi\circ T^\chi_\eta=\mathrm{Id}\oplus\mathrm{Id}$ if η is singular and χ is regular (see [3]). In our case $\eta=\chi_{-k\varepsilon_0}$ and $\chi=\chi_0$. The exact sequence (6.3) implies that

$$\bigl(L_k\otimes V^*\bigr)^{\chi_{-k\varepsilon_0}}=\Omega^{k-1}(\mu-\varepsilon_0)$$

and

$$\bigl(L_i\otimes V^*\bigr)^{\chi_{-k\varepsilon_0}}=0$$

if $i\neq k$. Thus, we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak g}(L_k, I_k) &= \mathrm{Hom}_{\mathfrak g}\big(L_k, \big(\bar{\Omega}^{k-1}(\mu-\varepsilon_0)\otimes V\big)^{\chi_0}\big)\\ &= \mathrm{Hom}_{\mathfrak g}\big(\big(L_k\otimes V^*\big)^{\chi-k\varepsilon_0}, \bar{\Omega}^{k-1}(\mu-\varepsilon_0)\big) = \mathbb{C}\end{aligned}$$

and for $i \neq k$

$$\begin{aligned}\mathrm{Hom}_{\mathfrak g}(L_i, I_k) &= \mathrm{Hom}_{\mathfrak g}\big(L_i, \big(\bar{\Omega}^{k-1}(\mu-\varepsilon_0)\otimes V\big)^{\chi_0}\big)\\ &= \mathrm{Hom}_{\mathfrak g}\big(\big(L_i\otimes V^*\big)^{\chi-k\varepsilon_0}, \bar{\Omega}^{k-1}(\mu-\varepsilon_0)\big) = 0. \qquad \square\end{aligned}$$

Corollary 6.10. $\mathrm{Hom}_{\mathfrak g}(\bar{\Omega}^k(\mu), I^l) = 0$ *if* $k \neq l, l-1$, *and* $\mathrm{Hom}_{\mathfrak g}(I^k, I^l) = 0$ *if* $k \neq l, l \pm 1$.

Proof. The statements follow from (6.4) and Lemma 6.7. $\square$

Using (6.4), for $k = 1, \dots, n-1$, define $\psi_k \in \mathrm{Hom}_{\mathfrak g}(I^{k+1}, I^k)$ by setting $\psi_k := i_k \circ p_{k+1}$.

Corollary 6.6 implies that $\bar{\Omega}^{k-1}(\mu)/L_{k-1}$ has a submodule isomorphic to L_k. Since I^k is injective and has a submodule isomorphic to L_k, there is a homomorphism $s_k \colon \bar{\Omega}^{k-1}(\mu)/L_{k-1} \to I^k$. Using the exact sequence (6.4) and Corollary 6.6, one can easily prove the existence of an exact sequence

$$0 \to \bar{\Omega}^{k-1}(\mu)/L_{k-1} \xrightarrow{s_k} I^k \xrightarrow{t_k} \bar{\Omega}^k(\mu)/L_k \to 0.$$

We assume that $L_0 = 0$, so that the above exact sequence is valid for $k = 1$.

Define $\varphi_k \in \mathrm{Hom}_{\mathfrak g}(I^k, I^{k+1})$ by $\varphi_k := s_{k+1} \circ t_k$. It is not hard to verify that

$$\varphi_{k+1} \circ \varphi_k = \psi_k \circ \psi_{k+1} = 0.$$

Finally, introduce $\xi \in \mathrm{End}_{\mathfrak g}(I^1)$ by $\xi := s_1 \circ p_1$ and $\eta \in \mathrm{End}_{\mathfrak g}(I^n)$ by $\eta := i_n \circ t_n$ (for the latter we use that $\bar{\Omega}^n(\mu)/L_n \cong \bar{\Omega}^n(\mu)$). One can check that

$$\xi \circ \psi_1 = \varphi_1 \circ \xi = \psi_{n-1} \circ \eta = \eta \circ \varphi_{n-1} = 0.$$

Let $I := I^1 \oplus \dots \oplus I^n$ and $\mathcal{E} := \mathrm{End}_{\mathfrak g}(I)$. Let $e_1, \dots, e_n$ be the standard idempotents in $\mathcal{E}$ and let $\mathcal{R}$ be the radical of $\mathcal{E}$. Then $\mathcal{R}$ defines a filtration of I

$$0 \subset \mathcal{R}^1(I) \subset \mathcal{R}^2(I) \subset \cdots,$$

such that $\mathcal{R}^m(I) = \mathrm{Ker}\,\mathcal{R}^m$. The quotients $\mathcal{S}^m(I) := \mathcal{R}^m(I)/\mathcal{R}^{m-1}(I)$ are semisimple over $\mathcal{E}$ and therefore over $\mathfrak g$ (see Theorem 5.1). Moreover, $\mathcal{S}^1(I) \simeq L_1 \oplus L_2 \oplus \dots \oplus L_n$.

Lemma 6.11. $I^k/L_k \cong (\bar{\Omega}^k(\mu)/L_k) \oplus \bar{\Omega}^{k-1}(\mu)$, *for* $k = 1, \dots, n$.

Proof. The exact sequence (6.4) leads to the following exact sequence

$$0 \to \bar{\Omega}^k(\mu)/L_k \xrightarrow{w_k} I^k/L_k \xrightarrow{u_k} \bar{\Omega}^{k-1}(\mu) \to 0. \tag{6.5}$$

We will show that (6.5) splits. Recall that $s_k : \bar{\Omega}^{k-1}(\mu)/L_{k-1} \to I^k$ is an injection. Using Corollary 6.6 we obtain the following exact sequence

$$0 \to L_k \to \bar{\Omega}^{k-1}(\mu)/L_{k-1} \to \bar{\Omega}^{k-1}(\mu) \to 0.$$

Therefore one can construct an injective map

$$v_k : \bar{\Omega}^{k-1}(\mu) \cong \big(\bar{\Omega}^{k-1}(\mu)/L_{k-1}\big)/L_k \to I^k/L_k.$$

We claim that the composite $u_k \circ v_k : \bar{\Omega}^{k-1}(\mu) \to \bar{\Omega}^{k-1}(\mu)$ is an isomorphism. We first note that $u_k \circ v_k$ is injective for $k \leqslant n-1$. Indeed, we have that v_k is injective, $\operatorname{Ker} u_k$ has a unique simple submodule isomorphic to L_{k+1}, and all irreducible constituents of $\operatorname{Im} v_k$ are isomorphic to L_k or L_{k-1}. Hence, $\operatorname{Ker} u_k$ intersects $\operatorname{Im} v_k$ trivially. In the case $k = n$, we notice that $\operatorname{Ker} u_n$ has a unique simple submodule isomorphic to L_n and $\operatorname{Im} v_n$ has a unique maximal submodule isomorphic to L_{n-1}. Thus, again $\operatorname{Im} v_n \cap \operatorname{Ker} u_n = 0$. Now note that $\operatorname{Im}(u_k \circ v_k)$ has infinite length, hence, by Corollary 6.6, $u_k \circ v_k$ is surjective. Therefore, (6.5) splits. $\square$

Corollary 6.12. *If $k > 1$, then $\mathcal{S}^k(I) \cong L_1^{\oplus 2} \oplus \cdots \oplus L_n^{\oplus 2}$.*

With a slight abuse of notation we will denote the images of φ_i, ψ_i, ξ and η under the natural projection $\mathcal{E} \to \mathcal{E}/\mathcal{R}^m$ by the same letters.

Theorem 6.13. *The set $\{\xi, \eta, \varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}\}$ forms a basis of $\mathcal{R}/\mathcal{R}^2$ and generates $\mathcal{R}/\mathcal{R}^m$ for any $m > 0$.*

Proof. It is clear from their construction that $\xi, \eta, \varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}$ are linearly independent. Since $\dim \mathcal{R}/\mathcal{R}^2 = 2n$, by Corollary 6.12, $\xi, \eta, \varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}$ form a basis of $\mathcal{R}/\mathcal{R}^2$. Similarly, $\dim \mathcal{R}^m/\mathcal{R}^{m+1} = 2n$ for $m > 1$ as well. Set $\phi_k := \psi_k \circ \varphi_k$ for $k = 1, \ldots, n-1$, and $\bar{\phi}_k := \varphi_{k-1} \circ \psi_{k-1}$ for $k = 2, \ldots, n$. Then using Corollary 6.12 and dimension calculations we verify that $\xi^{2p}, \eta^{2p}, \phi_1^p, \ldots, \phi_{n-1}^p, \bar{\phi}_2^p, \ldots, \bar{\phi}_n^p$ form a basis of $\mathcal{R}^{2p}/\mathcal{R}^{2p+1}$, and $\xi^{2p+1}, \eta^{2p+1}, \varphi_1 \circ \phi_1^p, \ldots, \varphi_{n-1} \circ \phi_{n-1}^p, \psi_1 \circ \bar{\phi}_2^p, \ldots, \psi_{n-1} \circ \bar{\phi}_n^p$ form a basis of $\mathcal{R}^{2p+1}/\mathcal{R}^{2p+2}$. $\square$

Let $\mathcal{B}$ denote the subalgebra in $\mathcal{E}$ generated by $e_1, \ldots, e_n, \xi, \eta, \varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}$. Denote by $\mathcal{D}$ the subcategory of finite-dimensional $\mathcal{B}$-modules on which $\mathcal{R} \cap \mathcal{B}$ acts nilpotently.

Corollary 6.14. *Let $|\mu| = 0$ and $\mu_i \notin \mathbb{Z}$ for all $i = 0, \ldots, n$. Then the category $\mathcal{C}^0_{\overline{\gamma(\mu)}}$ is equivalent to the category $\mathcal{D}$.*

Proof. As follows from Theorem 6.13, $\mathcal{B}/(\mathcal{R}^m \cap \mathcal{B}) \cong \mathcal{E}/\mathcal{R}^m$. Since a finite-dimensional module over $\mathcal{E}$ is a module over $\mathcal{E}/\mathcal{R}^m$ for some m, the category $\mathcal{D}$ coincides with the category of finite-dimensional $\mathcal{E}$-modules. Thus, the statement follows from Theorem 5.1. $\square$

The correspondence

$$\sum_{i=1}^{\frac{n}{2}} (\varphi_{2i-1} + \psi_{2i-1}) \mapsto x, \qquad \xi + \sum_{i=1}^{\frac{n}{2}-1} (\varphi_{2i} + \psi_{2i}) + \eta \mapsto y, \quad \text{for even } n$$

$$\sum_{i=1}^{\frac{n-1}{2}}(\varphi_{2i-1}+\psi_{2i-1})+\eta\mapsto x,\qquad \xi+\sum_{i=1}^{\frac{n-1}{2}}(\varphi_{2i}+\psi_{2i})\mapsto y,\quad \text{for odd } n$$

establishes an equivalence of the category $\mathcal{D}$ and the locally nilpotent representations of the quiver $\mathcal{Q}_n$. Hence Theorem 6.1 is proven.

Remark 6.15. Note that $\mathcal{B}$ is a graded quadratic algebra. It is natural to ask if $\mathcal{B}$ is Koszul (in the sense of [1]). We conjecture that the answer to this question is positive. A strong indication that this conjecture is true is that the numerical criterion (Lemma 2.11.1 [1]) holds. Indeed, the matrix $B(t)=P(\mathcal{B},t)$ is given by the formula

$$b_{ij}=0\quad \text{if } |i-j|>1,$$

$$b_{ij}=\frac{t}{1-t^2}\quad \text{if } |i-j|=1,$$

$$b_{ii}=\frac{1+t^2}{1-t^2}\quad \text{if } i\neq 1,n,$$

$$b_{11}=b_{nn}=\frac{1+t+t^2}{1-t^2}.$$

The matrix $C(t)=P(\mathcal{B}^!,t)$ is a symmetric matrix defined by

$$c_{ij}=\frac{t^{j-i}(1+t^{2i-1})(1+t^{2(n-j)+1})}{1-t^{2n}}\quad \text{if } i\leqslant j.$$

Then one can check that

$$P(\mathcal{B},t)P^t(\mathcal{B}^!,-t)=1.$$

7. Explicit description of the indecomposable objects in $\mathcal{C}_{\bar{\nu}}^{\chi}$ for regular integral χ

In this section we parameterize and explicitly describe all indecomposable objects in the category $\mathcal{C}_{\bar{\nu}}^{\chi}$. In particular we show that every indecomposable can be obtained by applying natural combinatorial operations (gluing and polymerization) to subquotients of the modules $\Omega^k(\mu)^{(m)}$.

7.1. The quiver $\mathcal{Q}_n$ and its indecomposable representations

The quiver $\mathcal{Q}_n$ defines a special biserial algebra the theory of which is well established. The classification of the indecomposable representations of special biserial algebras is usually attributed to Gelfand–Ponomarev. In [18] they considered a special case but the proof can be generalized as shown in [20] and [22]. A good overview of the representation theory of special biserial quivers can be found for example in [9]. In what follows we describe the two types of indecomposable objects of $\mathcal{Q}_n$ following [18] and [20].

First, we note that the problem of classifying indecomposable representations of $\mathcal{Q}_n$ is equivalent to the similar problem for two linear operators x and y in a graded vector space $V=V_1\oplus\cdots\oplus V_n$ satisfying the conditions:

(C0) $x(V_i) \subseteq V_{\pi_1(i)}$, $y(V_i) \subseteq V_{\pi_2(i)}$, where π_1 and π_2 are the permutations $(12)(34)\dots$ and $(23)(45)\dots$, respectively;
(C1) $xy = yx = 0$;
(C2) x and y are nilpotent.

Strings. The first type of indecomposable representations of $\mathcal{Q}_n$ is parameterized by string quivers. More precisely, a *string quiver* is a quiver $Q = (Q_v, Q_a)$, for which $Q_v = \{v_1, \dots, v_k\}$, there are no arrows between v_i and v_j for $|i - j| \geqslant 2$, and for every i, $2 \leqslant i \leqslant k$, there is exactly one arrow connecting v_{i-1} and v_i. The arrow connecting v_{i-1} and v_i will be denoted by $\overrightarrow{v_{i-1}v_i}$ if its head is v_{i-1}, and by $\overleftarrow{v_{i-1}v_i}$, otherwise. To each vertex $v_i \in Q_v$ we attach a *label* $l(v_i)$, $1 \leqslant l(v_i) \leqslant n$, such that: $l(v_i) = \pi_1(l(v_{i-1}))$ whenever $\overrightarrow{v_{i-1}v_i} \in Q_a$, and $l(v_i) = \pi_2(l(v_{i-1}))$ whenever $\overleftarrow{v_{i-1}v_i} \in Q_a$. A *graded string* S is a pair (Q, l) of a string quiver Q and a labeling $l : Q_v \to \{1, \dots, n\}$ compatible with the grading condition (C0) as described above.

Example 7.1. Let $n = 3$ and $k = 6$. An example of a graded string is pictured below. The numbers above the vertices are the labels of the string:

$$\overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \longleftarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$$

Every graded string $S = (Q, l)$ determines in a natural way a *string representation* $I(S)$ of $\mathcal{Q}_n$: we attach one-dimensional spaces $\mathbb{C}e_i$ to each vertex v_i, and set $I(S) = \bigoplus_{j=1}^n V_j$, where $V_j := \bigoplus_{l(v_i)=j} \mathbb{C}e_i$. We also set $x(e_i) = e_{i+1}$ if $\overrightarrow{v_i v_{i+1}} \in Q_a$ and $y(e_{i+1}) = e_i$ if $\overleftarrow{v_i v_{i+1}} \in Q_a$. All remaining $x(e_j)$ and $y(e_j)$ are zeroes.

Bands. We consider quivers Q whose sets of vertices $Q_v = \{v_1, \dots, v_k\}$ form a regular k-polygon ($k > 2$); the adjacent vertices v_{i-1} and v_i are connected by exactly one arrow. As in the case of strings we denote this arrow by either $\overrightarrow{v_{i-1}v_i}$ or $\overleftarrow{v_{i-1}v_i}$ (we set $v_{k+1} := v_1$). We again define labeling $l : Q_v \to \{1, \dots, n\}$ compatible with the grading condition (C0) with compatibility conditions identical to those for the strings. Such labeled quivers will be called *graded polygons*. Note that every graded polygon P can be "unfolded" at a sink v_i to a graded string $S(P, v_i)$. Namely, the graded string $S(P, v_i)$ has a set of vertices $\{v_i, \dots, v_1, v_2, ..v_{i-1}, \bar{v}_i\}$ and the same arrows as P, except that the arrow $\overrightarrow{v_{i-1}v_i}$ or $\overleftarrow{v_{i-1}v_i}$ is replaced by $\overrightarrow{v_{i-1}\bar{v}_i}$ or $\overleftarrow{v_{i-1}\bar{v}_i}$, respectively.

For a graded directed[2] polygon $P = (Q, l)$ with k vertices, a non-zero complex number λ, and a positive integer r we define the *band representation* $I(P, \lambda, r)$ of $\mathcal{Q}_n$ as follows. To each vertex v_i we attach an r-dimensional vector space U_i, and set $I(P, \lambda, r) := \bigoplus_{j=1}^n V_j$, where $V_j := \bigoplus_{l(v_i)=j} U_i$. We define $x : U_i \to U_{i+1}$ and $y : U_{i+1} \to U_i$ to be isomorphisms whenever $\overrightarrow{v_i v_{i+1}} \in Q_a$ and $\overleftarrow{v_i v_{i+1}} \in Q_a$, respectively (in other words, if the vertices are numbered clockwise, then the x's are directed clockwise, while the y's – counterclockwise). In all other cases $x_{|U_j} = 0$ and $y_{|U_j} = 0$. In addition, we require that the matrix of the composition $d_k \dots d_2 d_1 : U_1 \to U_1$ of all isomorphisms $d_i : U_i \to U_{i+1}$, $d_i = x$ or y^{-1}, is represented by a single Jordan block $J_r(\lambda)$ in an appropriate basis of U_1 with non-zero eigenvalue λ. It is easy to check that $d_i \dots d_1 d_k \dots d_{i-1}$ and $d_k \dots d_2 d_1$ are similar, and thus the choice of U_1 is irrelevant, i.e. $I(P, \lambda, r)$ is indeed uniquely determined by the triple (P, λ, r).

[2] Notice that a graded polygon is directed if and only if not all arrows go clockwise or counterclockwise.

One should note that not every band representation $I(P,\lambda,r)$ is indecomposable. In order $I(P,\lambda,r)$ to be indecomposable, P has to be a graded directed polygon with no rotational symmetry (a rotational symmetry of P is a rotation of the plane which preserves the quiver and the labels of P).

Proposition 7.2. *(See [18,20].) Every indecomposable representation of $\mathcal{Q}_n$ is isomorphic either to a string module $I(S)$ for some (unique) graded string S, or to a band module $I(P,\lambda,r)$ for some (unique) triple (P,λ,r), where P is a graded directed polygon with no rotational symmetry, λ is a non-zero complex number, and r is a positive integer.*

7.2. Operations on strings and bands

In this subsection we introduce three operations on the set of $\mathfrak{g}$-modules which help us to describe the string and band modules in an alternative way. Namely, we reduce the case of a general indecomposable representation of $\mathcal{Q}_n$ to the case of a string whose arrows have the same direction. We follow the terminology and notation of [16].

Gluing. Let A_1 and A_2 be vector spaces, A_1' and A_2' be isomorphic subspaces of A_1 and A_2, respectively, and let $\sigma : A_1' \to A_2'$ be an isomorphism. Set $D_\sigma := \{(a,\sigma(a)) \mid a \in A_1'\}$. Then the *gluing of A_1 and A_2 relative to σ* is the quotient space $A_1 \oplus A_2/D_\sigma$. The notion of gluing easily extends for $\mathfrak{g}$-modules. In the case of 1-dimensional spaces (respectively, simple $\mathfrak{g}$-modules) A_1' and A_2', the gluing $A_1 \oplus A_2/D_\sigma$ does not depend on a choice of σ, because $A_1 \oplus A_2/D_\sigma \cong A_1 \oplus A_2/D_{\lambda\sigma}$ for any non-zero $\lambda \in \mathbb{C}$. In such cases we will write $A_1 \oplus A_2/D(A_1')$ (note that $D(A_1')$ is the diagonal embedding of $A_1' \cong A_2'$ in $A_1 \oplus A_2$).

Dual gluing. In the dual setting we start with two pairs of spaces $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$ and an isomorphism $\sigma : A_1/A_1' \to A_2/A_2'$. Then *the dual gluing of A_1 and A_2 relative to σ* is the subspace $\{(a_1,a_2) \in A_1 \oplus A_2 \mid \sigma(\bar{a}_1) = \bar{a}_2\}$ of $A_1 \oplus A_2$ (here $\bar{a}_i = a_i + A_i'$). In the case of weight $\mathfrak{g}$-modules we will use the following alternative form of the dual gluing of A_1 and A_2: $(A_1^\vee \oplus A_2^\vee/D_\sigma)^\vee$.

Polymerization. Let A be a vector space and $A_1 \neq A_2$ be two isomorphic subspaces of A. Fix an isomorphism $\sigma : A_1 \to A_2$. The *polymerization of p copies of A relative to σ* is by definition the vector space $A^{(p)}(\lambda,\sigma) := A^{\oplus p}/A_\sigma^\lambda$ where A_σ^λ is the submodule of $A^{\oplus p}$ consisting of $(\sigma(a_1) - \lambda a_1, \sigma(a_2) - \lambda a_2 - a_1, \ldots, \sigma(a_p) - \lambda a_p - a_{p-1})$. The polymerization is also well defined for a $\mathfrak{g}$-module A and two isomorphic submodules A_1 and A_2 of A.

Notice that for a graded string S with a set of vertices $\{v_1,\ldots,v_k\}$, $I(S)$ can be obtained by gluing $I(S_1)$ and $I(S_2)$ at a sink v_i where S_1 and S_2 have sets of vertices $\{v_1,\ldots,v_i\}$ and $\{v_{i+1},\ldots,v_n\}$, respectively. We similarly represent $I(S)$ by dual gluing of $I(S_1)$ and $I(S_2)$ at a source v_i. In both cases we will write $S = S_1S_2$. We also may represent a band representation $I(P,\lambda,r)$ as a polymerization of the unfolded graded string representation $I(S(P,v_i))$ for any sink v_i of P.

7.3. Explicit description of the indecomposables in $\mathcal{C}_{\bar{\nu}}^{\chi}$

As discussed in the beginning of the previous section, we may restrict our attention to the category $\mathcal{C}^0_{\overline{\gamma(\mu)}}$ for $|\mu| = 0$ and $\mu_i \notin \mathbb{Z}$.

In order to describe explicitly all indecomposables in $\mathcal{C}^0_{\overline{\gamma(\mu)}}$ we use Theorem 6.1 and combine it with results in Sections 7.1 and 7.2. We provide the description in three steps. With small abuse of notation we will denote the indecomposable objects (defined up to an isomorphism) of $\mathcal{C}^0_{\overline{\gamma(\mu)}}$ by $I(S)$ and $I(P,\lambda,r)$ as well.

Homogeneous strings. Here we list all indecomposables $I(S)$ that correspond to *homogeneous* graded strings S, i.e. such that all arrows have the same direction. There are exactly two homogeneous graded strings with m vertices whose leftmost vertex is labeled by s, $1 \leqslant s \leqslant n$: the string $X_m(s)$ where all arrows go in the left-to-right direction; and the string $Y_m(s)$ where all arrows go in the right-to-left direction. With the aid of Lemma 6.5 and Corollary 6.6 and using a case-by-case verification we easily find an explicit realization of $I(X_m(s))$ and $I(Y_m(s))$ as subquotients of $\Omega^k(\mu)^{(r)}$ for suitable k and r. Alternatively, we may use quotients of $\Omega^k(\mu)^{(r)}$ and $(\Omega^k(\mu)^{(r)})^\vee$. For example if m is even and s is odd, then $I(X_m(s)) \cong (\Omega^s(\mu)^{(\frac{m}{2}+1)})^\vee / L_s$. The details are left to the reader.

Arbitrary strings. In the case of an arbitrary graded string S we first represent S as a product of homogeneous strings and then apply gluing and dual gluing to find $I(S)$. More explicitly, if $S = X_m(s)S'$ (or, respectively, $S = Y_m(s)S'$), where S' is a graded string with a right-to-left (respectively, left-to-right) leftmost arrow, then

$$I\big(X_m(s)S'\big) = \big(I\big(X_m(s)\big) \oplus I\big(S'\big)\big)/D(L_{(\pi_1)^m(s)}),$$
$$I\big(Y_m(s)S'\big) = \big(\big(I\big(Y_m(S)\big)^\vee \oplus I\big(S'\big)^\vee\big)/D(L_{(\pi_2)^m(s)})\big)^\vee,$$

respectively.

Bands. Following the description of all bands in Section 7.1 and using polymerization we can easily present a band representation $I(P,\lambda,r)$ as a polymerization of a string representation. Namely, for any sink v_i of P and an isomorphism $\sigma_i : \mathbb{C}v_i \to \mathbb{C}\bar{v}_i$, we have $I(P,\lambda,r) \cong I(S(P,v_i))^{(r)}(\lambda,\sigma_i)$.

7.4. *Socle series of the indecomposables*

Recall that the socle filtration of a module M is the increasing filtration $0 = \mathrm{soc}_0 M \subset \mathrm{soc}_1 M \subset \cdots \subset \mathrm{soc}_s M = M$ uniquely defined by the property that $\mathrm{soc}_i M/\mathrm{soc}_{i-1} M$ is the maximal semisimple submodule of $M/\mathrm{soc}_{i-1} M$. Here we list all semisimple quotients $\mathrm{soc}_i M/\mathrm{soc}_{i-1} M$ in the socle series $0 = \mathrm{soc}_0 M \subset \mathrm{soc}_1 M \subset \cdots \subset \mathrm{soc}_s M = M$ of any indecomposable M in $\mathcal{C}^0_{\overline{\gamma(\mu)}}$ (s is the Loewy length of M).

For a graded string or a graded directed polygon Q denote by $\mathrm{Sink}(Q)$ the set of sinks of Q. Set $Q^{(1)} := Q$ and for $i > 1$ let $Q^{(i)}$ be the quiver obtained from $Q^{(i-1)}$ by removing all sinks and all arrows whose tails are sinks of $Q^{(i-1)}$.

Proposition 7.3.

(i) *Let S be a graded string. Then for $i \geqslant 1$,*

$$\mathrm{soc}_i I(S)/\mathrm{soc}_{i-1} I(S) \cong \bigoplus_{v \in \mathrm{Sink}(S^{(i)})} L_{l(v)}.$$

(ii) *Let P be a graded directed polygon. Then for $i \geqslant 1$,*

$$\mathrm{soc}_i I(P,\lambda,r)/\mathrm{soc}_{i-1} I(P,\lambda,r) \cong \bigoplus_{v \in \mathrm{Sink}(P^{(i)})} L_{l(v)}^{\oplus r}.$$

Example 7.4. The components of the socle series of the string module $I(S)$ corresponding to the string S described in Example 7.1 are as follows:

$$\mathrm{soc}_1 I(S) \cong L_1 \oplus L_2, \qquad \mathrm{soc}_2 I(S)/\mathrm{soc}_1 I(S) \cong L_1 \oplus L_2,$$
$$\mathrm{soc}_3 I(S)/\mathrm{soc}_2 I(S) \cong L_2, \qquad \mathrm{soc}_4 I(S)/\mathrm{soc}_3 I(S) \cong L_1.$$

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