QUANTUM BINARY POLYHEDRAL GROUPS
AND THEIR ACTIONS ON QUANTUM PLANES

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ABSTRACT. We classify quantum analogues of actions of finite subgroups G of $SL_2(k)$ on commutative polynomial rings $k[u, v]$. More precisely, we produce a classification of pairs (H, R), where H is a finite dimensional Hopf algebra that acts inner faithfully and preserves the grading of an Artin-Schelter regular algebra R of global dimension two. Remarkably, the corresponding invariant rings R^H share similar regularity and Gorenstein properties as the invariant rings $k[u, v]^G$ in the classical setting. We also present several questions and directions for expanding this work in noncommutative invariant theory.

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0. INTRODUCTION

Let k be an algebraically closed field of characteristic zero, unless stated otherwise. This work is part of a program which extends classical invariant theory to a noncommutative setting. We depart from the traditional situation of finite groups acting linearly on $k[u, v]$ by considering finite dimensional Hopf algebras acting on noncommutative analogues of $k[u, v]$. The latter algebras are called *Artin-Schelter (AS) regular* algebras of global dimension 2. Previous work [12], [14], [15], [16] demonstrates that there is a rich invariant theory in this context.

The goal of this paper is to classify noncommutative analogues of linear actions of finite subgroups of $SL_2(k)$ on AS regular algebras of global dimension 2 and study the resulting rings of invariants. The *homological determinant* generalizes the determinant associated to a linear group action [11] [15], and so we only consider actions with trivial homological determinant. With this assumption, the rings of invariants turn out to have good homological properties (AS Gorenstein). In fact, they are often isomorphic to the coordinate rings of Kleinian singularities. The finite subgroups of $SL_2(k)$ were classified by Felix Klein, and their invariant subrings play an important role in classical invariant theory, representation theory, and algebraic geometry. Understanding their noncommutative analogues is an important contribution to noncommutative invariant theory.

Let us discuss in more detail the noncommutative structures mentioned above. Naturally, one can view quantum analogues of finite subgroups of $SL_2(k)$ (or *quantum binary polyhedral groups*) as finite subgroups of the quantum group $SL_q(2)$ for $q \in k^\times$. Following Drinfeld, we interpret finite subgroups of $SL_q(2)$ as finite dimensional Hopf quotients of the coordinate Hopf algebra $\mathcal{O}_q(SL_2(k))$. The problem of classifying finite dimensional Hopf quotients of $\mathcal{O}_q(SL_2(k))$ has been examined thoroughly in the literature (see e.g.[5], [24], [25]). The novelty of our work is that we address the question of when these non(co)commutative finite subgroups of $SL_2(k)$ act on AS regular algebras R of global dimension 2.

Since we assumed k is algebraically closed, the AS regular algebras R of global dimension 2, generated in degree one, are isomorphic to either

$$k_J[u,v] := k\langle u,v\rangle/(vu - uv - u^2), \qquad \text{or} \qquad k_q[u,v] := k\langle u,v\rangle/(vu - quv),$$

where $q \in k^\times$ [Example 1.2]. These algebras are also known as *quantum planes*.

We say that a finite dimensional Hopf algebra H acts on an algebra R if R is a *left H-module algebra* [Definition 1.3]. The following statements are <u>standing assumptions</u> for this article.

Hypothesis 0.1. *(1) Let $H \neq k$ be a finite-dimensional Hopf algebra.*

(2) Let R be an Artin-Schelter (AS) regular algebra of global dimension 2 that is generated in degree one, that is, R is isomorphic to either $k_J[u,v]$ or $k_q[u,v]$.

(3) Let H act on R inner faithfully [Definition 1.5], while preserving the grading of R.

The hypotheses below are <u>used selectively</u> throughout this work.

Hypothesis 0.2. *Assume Hypothesis 0.1 with the additional condition:*

(4) The H-action on R has trivial homological determinant [Definition 1.7].

Hypothesis 0.3. *Assume Hypothesis 0.2 with the additional condition:*

(5) The Hopf algebra H (with antipode S) is semisimple, and hence $S^2 = Id_H$ [17, Theorem 3].

Conditions (1)-(3) generalize faithful, linear actions of finite groups G on $k[u,v]$ to the quantum setting. We view condition (4) as analogous to the condition that $G \subseteq SL_2(k)$ in the following sense. In the classical setting G is realized as a subgroup of $GL_2(k)$, and the usual determinant induces a group homomorphism, $\mathsf{det}\colon G \hookrightarrow GL_2(k) \to k^\times$. Note that det is not intrinsic to G, but depends on the action of G on $k[u,v]$. The *homological determinant* of a Hopf algebra action on a graded k-algebra R is a useful generalization of the map det. Although not needed for all our results, semisimplicity (Condition (5)) will be used for a large part of this paper, in particular for our results on the regularity and the geometry of the resulting invariant rings R^H [Theorem 0.6 and Proposition 0.7].

Now we state our main theorem. We say that a Hopf algebra is *nontrivial* if it is both noncommutative and noncocommutative.

Theorem 0.4 (Lemma 4.1, Theorems 4.5, 5.2, 6.2)**.** *Let R be an Artin-Schelter regular algebra of global dimension 2. The finite dimensional Hopf algebras H which act inner faithfully on R, and satisfy Hypotheses 0.2, are classified in Table 1. In particular, we have the following statements.*

(a) If R is commutative, then H is cocommutative.

(b) If R is non-PI, then H is both commutative and cocommutative.

(c) If R is noncommutative and PI, then there are nontrivial Hopf algebras acting inner faithfully on R.

Furthermore, if H is non-semisimple, then $R \cong k_q[u,v]$ where q is an n-th root of unity for $n \geqslant 3$.

Notation. $[\tilde{\Gamma}, \Gamma, C_n, D_{2n}, U]$ Let $\tilde{\Gamma}$ denote a finite subgroup of $SL_2(k)$, Γ denote a finite subgroup of $PSL_2(k)$, C_n denote a cyclic group of order n, and D_{2n} denote a dihedral group of order $2n$. Let $\mathrm{ord}(q)$ denote the order of q, for $q \in k^\times$ a root of unity. We also write $R = k\langle U \rangle / I$ where $U = ku \oplus kv$ and I is the two sided ideal generated by the relation.

AS regular alg. R of gldim 2	f.d. Hopf algebra(s) H acting on R	Result number(s)
$k[u,v]$	$k\tilde{\Gamma}$	4.5(a1), 5.2(b1,b3)
$k_{-1}[u,v]$	kC_n for $n \geq 2$	5.2(b1,b2,b3)
	kD_{2n}	4.5(a2)
	$(kD_{2n})^\circ$	5.2(b4)
	$\mathcal{D}(\tilde{\Gamma})^\circ, \tilde{\Gamma}$ nonabelian	4.5(a3) and 4.6
$k_q[u,v]$, q root of 1, $q^2 \neq 1$		
if U non-simple	kC_n for $n \geq 2$	5.2(b1,b3)
	$(T_{q,\alpha,n})^\circ$	6.2(c1)
if U simple, $\mathrm{ord}(q)$ odd	H in $1 \to (k\tilde{\Gamma})^\circ \to H^\circ \to \mathfrak{u}_q(\mathfrak{sl}_2)^\circ \to 1$;	6.2(c2);
if U simple, $\mathrm{ord}(q)$ even, $q^4 \neq 1$	H in $1 \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$;	6.2(c3) and 6.22;
if U simple, $q^4 = 1$	H in $1 \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$ or $1 \to (k\Gamma)^\circ \to H^\circ \to \frac{\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}{(e_{12}-e_{21}e_{11}^2)} \to 1$	6.2(c3) and 6.24
$k_q[u,v]$, q not a root of 1	$kC_n, n \geq 2$	5.2(b1,b3)
$k_J[u,v]$	kC_2	5.2(b1)

Table 1: Summary of Theorem 0.4

When U is a simple left H-module, we can fit H into the following diagram, where the rows are exact sequences of Hopf algebras.

$$
\begin{array}{ccccccccc}
k & \longrightarrow & \mathcal{O}(G) & \longrightarrow & \mathcal{O}_q(SL_2(k)) & \longrightarrow & \overline{\mathcal{O}_q(SL_2(k))} & \longrightarrow & k \\
& & \downarrow & & \downarrow & & \downarrow & & \\
k & \longrightarrow & (kG')^\circ & \longrightarrow & H^\circ & \longrightarrow & K & \longrightarrow & k
\end{array}
$$

Theorem no.	q	G	G'	K
4.5(a1)	1	$SL_2(k)$	$\tilde{\Gamma}$ nonabelian	k
4.5(a2)	-1	$PSL_2(k)$	C_n	kC_2
4.5(a3)	-1	$PSL_2(k)$	Γ nonabelian	kC_2
6.2(c2)	$\mathrm{ord}(q)$ is odd	$SL_2(k)$	$\tilde{\Gamma}$	$\overline{\mathcal{O}_q(SL_2(k))} \cong \mathfrak{u}_q(\mathfrak{sl}_2)^\circ$
6.2(c3)	$\mathrm{ord}(q)$ is even, $q^4 \neq 1$	$PSL_2(k)$	Γ	$\overline{\mathcal{O}_q(SL_2(k))} \cong \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$
6.2(c3)	$\mathrm{ord}(q)$ is even, $q^4 = 1$	$PSL_2(k)$	Γ	$\overline{\mathcal{O}_q(SL_2(k))} \cong \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$ or $\frac{\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}{(e_{12}-e_{21}e_{11}^2)}$

Diagram-Table 2: For H-actions with U a simple H-module

Note that part (a) of Theorem 0.4 is proved in both [8, Proposition 0.7] and [9, Theorem 1.3] via different techniques. Moreover, the most interesting (nontrivial) Hopf algebra actions occur on the PI algebras $k_q[u,v]$ for q a root of unity.

Next, we study the rings of invariants arising from the Hopf algebra actions appearing in Theorem 0.4, again with the aim of generalizing results on actions of finite subgroups of $SL_2(k)$ on $k[u,v]$ in classical invariant theory. Watanabe's theorem implies that if G is a finite subgroup of $SL_2(k)$, then $k[u,v]^G$ is Gorenstein [30, Theorem 1]. By [15, Theorem 0.1], if the pair (H,R) satisfies Hypothesis 0.3 (with H semisimple), then R^H is *Artin-Scheler Gorenstein* [Definition 1.1]; we extend this result to the non-semisimple case.

Proposition 0.5 ([15], Propositions 6.8, 6.14, 6.26). *Let (H,R) be a pair as in Theorem 0.4. Then the invariant subring R^H is Artin-Scheler Gorenstein.*

The Shepherd-Todd-Chevalley theorem implies $k[v_1,\ldots,v_n]^G$ is not regular if G acts via a subgroup of $SL_n(k)$. Our next result gives a sufficient condition for the invariant subrings from Proposition 0.5 <u>not</u> to be Artin-Scheler regular. Note that this result holds for actions on AS regular algebras of arbitrary global dimension.

Theorem 0.6 (Theorem 2.3). *Let H be a semisimple Hopf algebra, and R be a noetherian connected graded Artin-Scheler regular algebra equipped with an H-module algebra structure. If $R^H \neq R$ and the homological determinant of the H-action on R is trivial, then R^H is not Artin-Scheler regular.*

This theorem fails if H is non-semisimple; see Lemma 6.7 (Case: $l = m = n$).

Finally, we consider the McKay correspondence in the context of Hopf algebra actions on Artin-Scheler regular algebras. As in the classical setting, the McKay quiver of the H-action on R has vertices indexed by the isomorphism classes of irreducible representations of H, and hence records information about the representations of H. The McKay quivers of the actions in Theorem 0.4 are given in the final result.

Proposition 0.7 (Proposition 7.1). *Assume Hypothesis 0.3 for a semisimple Hopf algebra H acting on an Artin-Scheler regular algebra $R = k\langle U\rangle/(r)$ of global dimension two, with trivial homological determinant. Then, the McKay quiver $Q(H,U)$ is either of type A, D, E, L, or DL.*

Classically, only types A, D, E appear for the McKay quivers of finite subgroups of $SL_2(k)$ (acting on $k[u,v]$). In particular, the McKay quiver of type DL comes from the dihedral group action on $k_{-1}[u,v]$ in Theorem 0.4. However, in the classical setting, the trivial determinant condition precludes dihedral group actions. Hence, by replacing the algebra $k[u,v]$ with a noncommutative algebra, and by extending the notion of determinant, our noncommutative version of the McKay correspondence involves more groups. Further study of this correspondence is the subject of future work.

This paper is organized as follows. We provide background material on Artin-Scheler regular algebras, Hopf algebra actions, and the homological determinant in Section 1. As trivial homological determinant is a vital condition in our work, Section 2 is devoted to the study of Hopf actions on Artin-Scheler regular algebras under this condition. For example, we show how to detect if the homological determinant of an H-action on an Artin-Scheler regular algebra of dimension 2 is trivial [Theorem 2.1]. In Section 3, we prove some elementary results on self-dual H-modules which are used later in the paper. Sections 4, 5, and 6 are dedicated to the proof of Theorem 0.4 in the cases where H is semisimple and noncommutative, H is commutative (so, semisimple), and H is non-semisimple, respectively. Moreover, we compute the McKay quivers of our classified semisimple Hopf algebra actions in Section 7. In Section 8, we suggest several questions for further study.

1. Background material

Here, we provide background material for Artin-Scheler regular algebras, for Hopf algebra actions on graded algebras, and for the homological determinant of such actions.

1.1. **Artin-Schelter regularity.** An algebra R is said to be *connected graded* if $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ with $R_i \cdot R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. The *Hilbert series* of R is defined to be $\sum_{i \in \mathbb{N}} (\dim_k R_i) t^i$. As mentioned in the introduction, we consider a class of (noncommutative) graded algebras that serve as noncommutative analogues of commutative polynomial rings. These algebras are defined as follows.

Definition 1.1. Let R be a connected graded algebra. Then, R is *Artin-Schelter (AS) regular* if it satisfies the conditions below:

(1) R has finite global dimension;
(2) R has finite Gelfand-Kirillov dimension;
(3) R is *Artin-Schelter Gorenstein*, that is
　　(a) R has finite injective dimension $d < \infty$, and
　　(b) $\mathrm{Ext}_R^i(k, R) = \delta_{i,d} \cdot k(\ell)$ for some $\ell \in \mathbb{Z}$ called the AS index of R. Here, $k(\ell)$ is the ℓ-th shift.

Example 1.2. Since k is algebraically closed, the AS regular algebras of global dimension two that are generated in degree one are listed below (up to isomorphism):

(i) the *Jordan plane*: $k_J[u, v] := k\langle u, v \rangle / (vu - uv - u^2)$, and
(ii) the *skew polynomial ring*: $k_q[u, v] := k\langle u, v \rangle / (vu - quv)$ for $q \in k^\times$.

1.2. **Hopf actions and homological determinant.** We adopt the usual notation for the Hopf structure of a Hopf algebra H, namely $(H, m, u, \Delta, \epsilon, S)$. We also adopt Sweedler's notation for the comultiplication: $\Delta(h) = \sum h_1 \otimes h_2$ for $h \in H$. Note that every finite dimensional Hopf algebra has bijective antipode, so the Hopf algebras appearing in this paper all have bijective antipodes. Moreover, let H° denote its *Hopf dual*, which is just the k-linear dual H^* of H since H is finite dimensional. Consider a sequence of Hopf algebra maps

$$k \to L \overset{\iota}{\to} H \overset{\pi}{\to} \bar{H} \to k$$

where ι is injective and π is surjective. We say the sequence is *exact* if either of the following equivalent conditions hold:

(i) $\ker \pi = HL^+$, where $L^+ = \ker \epsilon$ is the augmentation ideal;
(ii) $L = H^{\mathrm{co}\pi} = \{h \in H \mid (\pi \otimes \mathrm{id})\Delta(h) = 1 \otimes h\}$.

We say that H' is a *normal* Hopf subalgebra of H if

$$\mathrm{ad}_l(h)(f) := \sum h_1 f S(h_2) \in H' \qquad \text{and} \qquad \mathrm{ad}_r(h)(f) := \sum S(h_1) f h_2 \in H'$$

for all $f \in H'$ and $h \in H$. The above two conditions are equivalent since we assume that Hopf algebras have bijective antipode.

Given a left H-module M, we denote the H-action by $\cdot : H \otimes M \to M$. Similarly for a Hopf algebra K, given a right K-comodule M, we denote the K-coaction by $\rho : M \to M \otimes K$. If H is finite dimensional, then M is a left H-module if and only if M is a right H°-comodule.

Now, we recall basic facts about Hopf algebra actions; refer to [23] for further details.

Definition 1.3. Let H be a Hopf algebra H and R be a k-algebra. We say that H *acts on R* (from the left), or R is a *left H-module algebra*, if R is a left H-module, if $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$, and if $h \cdot 1_R = \epsilon(h)1_R$, for all $h \in H$, and for all $a, b \in R$.

The *invariant subring* of such an action is defined to be

$$R^H = \{a \in R \mid h \cdot a = \epsilon(h)a, \ \forall h \in H\}.$$

Dually, we say that a Hopf algebra K *coacts on R* (from the right), or R is a *right K-comodule algebra*, if R is a right K-comodule, if $\rho(1_R) = 1_R \otimes 1_K$, and if $\rho(ab) = \rho(a)\rho(b)$, for all $a, b \in R$. The *coinvariant subring* of such a coaction is given as follows:

$$R^{\mathrm{co}K} = \{a \in R \mid \rho(a) = a \otimes 1_K\}.$$

Here is an example of a Hopf coaction on an AS regular algebra, which will be used in Sections 4 and 6.

Example 1.4. Consider the *quantum special linear group* $\mathcal{O}_q(SL_2(k))$ for $q \in k^\times$, generated by e_{11}, e_{12}, e_{21}, e_{22}, subject to relations:

$$e_{12}e_{11} = qe_{11}e_{12}, \quad e_{21}e_{11} = qe_{11}e_{21}, \quad e_{22}e_{12} = qe_{12}e_{22}, \quad e_{22}e_{21} = qe_{21}e_{22},$$
$$e_{21}e_{12} = e_{12}e_{21}, \quad e_{22}e_{11} = e_{11}e_{22} + (q - q^{-1})e_{12}e_{21}, \quad e_{11}e_{22} - q^{-1}e_{12}e_{21} = e_{22}e_{11} - qe_{12}e_{21} = 1.$$

The coalgebra structure and antipode are given by $\Delta(e_{ij}) = \sum_{m=1}^2 e_{im} \otimes e_{mj}$, $\epsilon(e_{ij}) = \delta_{ij}$, and

$$S(e_{11}) = e_{22}, \quad S(e_{12}) = -qe_{12}, \quad S(e_{21}) = -q^{-1}e_{21}, \quad S(e_{22}) = e_{11}.$$

We have that $\mathcal{O}_q(SL_2(k))$ coacts on $k_q[u, v]$ from the right by

$$\rho(u) = u \otimes e_{11} + v \otimes e_{21} \quad \text{and} \quad \rho(v) = u \otimes e_{12} + v \otimes e_{22}.$$

We want to restrict ourselves to H-actions that do not factor through 'smaller' Hopf algebras.

Definition 1.5. [4] Let M be a left H-module. We say that M is an *inner faithful H-module*, or H *acts inner faithfully* on M, if $IM \neq 0$ for every nonzero Hopf ideal I of H.

Dually, let N be a right K-comodule. We say that N is an *inner faithful K-comodule*, or K *coacts inner faithfully* on N, if for any proper Hopf subalgebra $K' \subsetneq K$, $\rho(N)$ is not in $N \otimes K'$.

Let U be any left H-submodule of R that generates R as an algebra. Then H acts on R inner faithfully if and only if H acts on U inner faithfully. Here is a useful lemma pertaining to inner faithfulness.

Lemma 1.6. *Suppose H is finite dimensional and let $K = H^\circ$. Let U be a left H-module, so it is also a right K-comodule with coaction ρ. Then the following conditions hold.*

    (a) *The H-action on U is inner faithful if and only if the induced K-coaction on U is inner faithful.*

    (b) *Let C be the smallest subcoalgebra of K such that $\rho(U) \subseteq U \otimes C$. Then U is an inner faithful K-comodule if and only if K is generated as an algebra by $\bigcup_{n \geq 0} S^n(C)$.* □

We define the homological determinant of an H-action on an algebra R below. We refer to [15, Section 2] for background on local cohomology modules.

Definition 1.7. Let H be a finite dimensional Hopf algebra acting on a connected graded noetherian AS Gorenstein algebra R. Suppose that the H-action on R preserves the grading of R. Let $d = \text{injdim}(R)$ which is finite since R is AS Gorenstein. We denote by $\mathfrak{m}$ the maximal graded ideal of R^H consisting of all elements with positive degree, and $H_{\mathfrak{m}}^d(R)$ the d-th local cohomology of R with respect to $\mathfrak{m}$. The lowest degree nonzero homogeneous component of $H_{\mathfrak{m}}^d(R)^*$ is 1-dimensional, and for it, we choose a basis element $\mathfrak{e}$. Then there is an algebra homomorphism $\eta : H \to k$ such that the right H-action on $H_{\mathfrak{m}}^d(R)^*$ is given by $\eta(h)\mathfrak{e}$ for all $h \in H$.

    (1) The composite map $\eta \circ S : H \to k$ is called the *homological determinant* of the H-action on R, denoted by $\text{hdet}_H R$.

    (2) We say that $\text{hdet}_H R$ is *trivial* if $\eta \circ S = \epsilon$ [15, Definition 3.3].

Dually, if K coacts on R from the right, then K coacts on $k\mathfrak{e}$ and $\rho(\mathfrak{e}) = \mathfrak{e} \otimes \mathsf{D}^{-1}$ for some grouplike element D in K.

    (3) The *homological codeterminant* of the K-coaction on R is defined to be $\text{hcodet}_K R = \mathsf{D}$.

    (4) We say that $\text{hcodet}_K R$ is *trivial* if $\text{hcodet}_K R = 1_K$ [15, Definition 6.2].

Let V be a k-vector space and G be a finite subgroup of $GL(V)$ acting linearly on $R = k[V]$. Then $\text{hdet}_{kG} R = \det$ where $\det : kG \to k$ is the determinant map on $GL(V)$, restricted to G, then extended linearly to kG. Consequently, $\text{hdet}_{kG} R$ is trivial if and only if $G \subseteq SL(V)$, see [12]. Moreover in [15], the authors assume that H is semisimple, yet the definition of homological (co)determinant does not require that H is semisimple.

2. Hopf actions with trivial homological determinant

In this section, we discuss several results pertaining to Hopf actions on graded algebras with trivial homological determinant. In the case of Hopf algebras acting on AS regular algebras of global dimension 2, we can express the homological determinant concretely.

Theorem 2.1. *Let $R = k\langle U \rangle/(r)$ be an AS regular algebra of global dimension 2 generated in degree one and let H be a Hopf algebra that acts on R. Then, $\mathrm{hdet}_H R$ is equal to the map $H \to \mathrm{End}_k(M)$ where M is the 1-dimensional H-module $(kr)^*$. Consequently, $\mathrm{hdet}_H R$ is trivial if and only if kr is the trivial H-module.*

Proof. First, we construct an H-invariant free resolution of the left R-module $_R k$. Since k is algebraically closed, we may assume that $r = vu - quv + \eta u^2$ for some $q \in k^\times$ and $\eta \in k$, where $\{u, v\}$ is a basis of the k-vector space U [Example 1.2]. Since R is an H-module algebra, the multiplication map $\mu : R \otimes R \to R$ is an H-homomorphism. This induces an H-homomorphism $\phi : R \otimes U \to R$. By using the chosen basis, we have an H-homomorphism $d_1 : (R \otimes ke_u) \oplus (R \otimes ke_v) \to R$ induced by the multiplication map where $e_u := u \in U$ and $e_v := v \in U$. This gives rise to a partial resolution of $_R k$,

$$(R \otimes ke_u) \oplus (R \otimes ke_v) \xrightarrow{d_1} R \xrightarrow{d_0} k \longrightarrow 0.$$

Since R has global dimension 2, the kernel of the map d_1 is a free module of rank 1, denoted by Re_r with basis element e_r. Using the relation r, we have that e_r is identified with element $(v + \eta u) \otimes e_u - qu \otimes e_v$ in $(R \otimes ke_u) \oplus (R \otimes ke_v)$. By definition, kr is a left H-module. Hence, ke_r is a left H-module which is isomorphic to kr. The H-module structure on Re_r is equivalent to the H-module structure on $R \otimes ke_r$. Now we have an H-equivariant free resolution of $_R k$,

$$0 \to R \otimes ke_r \to (R \otimes ke_u) \oplus (R \otimes ke_v) \to R \to k \to 0.$$

Applying $\mathrm{Hom}_R(-, k)$ to the resolution above, we obtain an H-module isomorphism

$$\mathrm{Ext}_R^2(k, k) \cong \mathrm{Hom}_k(ke_r, k) =: (ke_r)^*.$$

Thus the H-action on $\mathrm{Ext}_R^2(k, k)$ is the H-action on $(ke_r)^*$. By [15, Lemma 5.10(c)], the homological determinant can be computed as the H-action on $\mathrm{Ext}_R^2(k, k)$, so the result follows, and the consequence is clear. $\qquad\square$

The following lemma could be useful for future work.

Lemma 2.2. *Let $R = k\langle U \rangle/(r_1, \cdots, r_s)$ be a Koszul AS regular algebra of global dimension d, generated in degree one, and let H be a Hopf algebra that acts on R. If $\mathrm{hdet}_H R$ is trivial, then $U^{\otimes d}$ contains a copy of the trivial H-module k.*

Proof. Recall that the Koszul dual $R^!$ of R is defined to be $k\langle U^* \rangle/(r_1, \ldots, r_s)^\perp$. Since R is AS regular, $R^!$ is isomorphic to

$$\mathrm{Ext}_R^*(k, k) = k \oplus \mathrm{Ext}_R^1(k, k) \oplus \cdots \oplus \mathrm{Ext}_R^d(k, k).$$

Here, $\mathrm{Ext}_R^1(k, k) \cong U^*$. Since $R^!$ is Koszul, it is generated in degree one. Thus $\mathrm{Ext}_R^d(k, k)$ is isomorphic to a quotient of $(U^*)^{\otimes d}$ by some k-vector space. Since $\mathrm{hdet}_H R$ is trivial, we have that $\mathrm{Ext}_R^d(k, k) \cong k$ as H-modules by [15, Lemma 5.10(c)]. Thus, k is a quotient module of the H-module $(U^*)^{\otimes d}$, which is equivalent to our desired result. $\qquad\square$

Our main application of Hopf actions on algebras with trivial homological determinant is illustrated in the following theorem.

Theorem 2.3. *Let H be a semisimple Hopf algebra. Suppose that R is a noetherian connected graded AS regular algebra and H acts on R. If $R^H \neq R$ and $\mathrm{hdet}_H R$ is trivial, then R^H is not AS regular.*

By [15, Theorem 0.1], R^H is always AS Gorenstein under the hypotheses above. On the other hand, we set some notation to prove the theorem above. The following lemma is proved in [26].

Lemma 2.4. [26, Sections 2 and 3] *Let A be a noetherian AS regular algebra. The Hilbert series $H_A(t)$ is equal to $\frac{1}{p(t)}$, where $p \in \mathbb{Z}[t]$ satisfies*

(a) $p(0) = 1$,

(b) [26, Proposition 3.1(4)] $\deg p(t) = \ell$, *where ℓ is the AS index of A,*

(c) [26, Theorem 2.4(2)] *the leading coefficient of $p(t)$ is 1 or -1.* □

When we are working with more than one algebra, let us use $p_A(t)$ to denote the polynomial $p(t)$. For a graded module M, let $M(n)$ denote the shift of M by degree n. The following lemma is well-known and follows from [14, Lemma 1.10].

Lemma 2.5. *Let A be a noetherian connected graded algebra, and let B be a noetherian connected graded subalgebra of A. Suppose that*

(i) $B \neq A$;

(ii) *A and B are AS regular algebras of the same global dimension; and*

(iii) $_BA$ *and A_B are finitely generated left and right B-modules respectively.*

Then the following hold.

(a) *Both $_BA$ and A_B are graded free B-modules.*

(b) *Suppose that $H_A(t) = \frac{1}{p_A(t)}$ and $H_B(t) = \frac{1}{p_B(t)}$. Then, $p_A(t) \mid p_B(t)$, and $\deg p_A(t) < \deg p_B(t)$.*

Proof. (a) This follows from [14, Lemma 1.10(b)].

(b) By part (a), A is a finitely generated free graded left B-module. Hence, we can write $_BA = \bigoplus_{n=0}^{w} B(n)^{i_n}$ for a finite sequence of non-negative integers $i_0, i_1, \cdots, i_w$. Then

$$\frac{1}{p_A(t)} = H_A(t) = \left(\sum_{n=0}^{w} i_n t^n \right) H_B(t) = \frac{\sum_{n=0}^{w} i_n t^n}{p_B(t)},$$

which implies that $p_B(t) = p_A(t) \left(\sum_{n=0}^{w} i_n t^n \right)$ and $p_A(t) \mid p_B(t)$. By Lemma 2.4(a), $p_A(0) = p_B(0) = 1$, so we have that $i_0 = 1$. Since $B \neq A$, we get that $\sum_{n=0}^{w} i_n t^n \neq 1$. Thus $\deg(\sum_{n=0}^{w} i_n t^n) > 0$. Therefore $\deg p_A(t) < \deg p_B(t)$. □

The following lemma follows from [15, Section 3].

Lemma 2.6. *Let A be a noetherian AS Gorenstein algebra, and let H be a semisimple Hopf algebra acting on A. Suppose $B := A^H$ is AS Gorenstein. Then, we have the following statements:*

(a) $\ell_A \leq \ell_B$, *where ℓ_A and ℓ_B denote the AS indexes of A and B, respectively; and*

(b) $\text{hdet}_H A$ *is trivial if and only if $\ell_A = \ell_B$.*

(c) *Let*

$$k \to H' \to H \to H'' \to k$$

be an exact sequence of semisimple Hopf algebras. Let $C = A^{H'}$ and let H'' act on C naturally. If $\text{hdet}_H A$ is trivial, then so is $\text{hdet}_{H''} C$.

Proof. Let $\ell := \ell_A$ be the AS index of A. We combine the proof of (a) and of (b) as follows.

We use the convention introduced in [15, Section 3], in particular, we assume that H acts on A on the right. Let Y denote the local cohomology module $H_{\mathfrak{m}}^d(A)^*$, where $\mathfrak{m} = A_{\geq 1}$ and $d = \text{injdim} A$. By [15, Lemma 1.7], Y is isomorphic to $H_{\mathfrak{m}'}^d(A)^*$, where $\mathfrak{m}'$ is the maximal graded ideal of A^H, since A is finite over A^H as left and right modules. We also denote by Ω, the A^H-module $Y \cdot \int$ where $\int$ is an integral in H. Then by [15, Lemma 3.2(b)], $\Omega[d]$ is the balanced dualizing complex of A^H. If $\text{hdet}_H A$ is trivial, then by [15,

Lemma 3.5(c,d)], $Y^H \cong \Omega \cong A^H(-\ell)$ as right A^H-modules. Therefore, by [15, Lemma 1.6], the AS index of A^H is equal to ℓ. This proves one implication of part (b).

Conversely, assume that $\mathrm{hdet}_H A$ is not trivial. By [15, Lemma 3.5(f)], $\mathfrak{e} \notin Y^H$, where $\mathfrak{e}$ is a generator of Y. Thus, the lowest degree of any nonzero elements in Y^H is larger than ℓ. Since $Y^H[d]$ $(= \Omega[d])$ is the balanced dualizing complex over B, we have that $Y^H[d] \cong {}^\sigma B^1(-\ell_B)[d]$ by [15, Lemmas 3.2(b) and 1.6]. Hence $Y^H \cong B(-\ell_B)$ as right B-modules. Thus the lowest degree of any nonzero elements in $B(-\ell_B)$ is larger than ℓ. So $\ell_B > \ell$. This proves the other implication of (b), as well as part (a).

(c) Since $\mathrm{hdet}_H A$ is trivial, we have that $B := A^H$ is AS Gorenstein. Since H' is a Hopf subalgebra of H, $\mathrm{hdet}_{H'} A$ is trivial. Consequently, C is AS Gorenstein. Clearly, $B = A^H = C^{H''}$. By part (a), $\ell_A \leq \ell_C \leq \ell_B$. By part (b), $\ell_A = \ell_B$, and hence $\ell_C = \ell_B$. The assertion follows from part (b). □

Now, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Proceed by contradiction and assume that $B := R^H$ is AS regular. By the hypotheses of Theorem 2.3, R is noetherian and AS regular. The hypothesis in Lemma 2.5(iii) holds because H is a semisimple Hopf algebra [23, Theorem 4.4.2]. Since R and B have the same injective dimension, they have the same global dimension when both have finite global dimension. By Lemma 2.5(b), $\deg p_R(t) < \deg p_B(t)$. Hence, $\ell_R = \deg p_R(t) < \deg p_B(t) = \ell_B$ due to Lemma 2.4(b). By Lemma 2.6(b), $\mathrm{hdet}_H R$ is not trivial, thus producing a contradiction. □

Next, we recall results from [8] on finite dimensional Hopf actions on AS regular algebras of global dimension 2, sometimes with trivial homological determinant. These results are used later, especially in Section 6.

Proposition 2.7. *Let H be a finite dimensional Hopf algebra. Let R be an AS regular algebra of global dimension 2, that is to say, $k_q[u,v]$ or $k_J[u,v]$. Suppose that H acts on R inner faithfully and preserves the grading of R.*

(a) *[8, Theorem 5.10] If R is either $k_J[u,v]$ or $k_q[u,v]$ for q not a root of unity, then H is a group algebra.*

(b) *[8, Theorem 0.1 and Remark 4.4] If $R = k_q[u,v]$ with $q = \pm 1$ and if the H-action on R has trivial homological determinant, then H is semisimple.*

(c) *Assume Hypothesis 0.2 with H non-semisimple. Then $R = k_q[u,v]$ for q a root of unity with $q \neq \pm 1$.*

Proof. Part (c) follows from parts (a,b). □

The following result provides the classification of finite dimensional, cocommutative Hopf algebras H acting on R satisfying Hypothesis 0.2. Recall that H is a finite group algebra in this case. Moreover, if $R = k[u,v]$, then G is a finite subgroup of $SL_2(k)$, so we cover the cases where R is noncommutative below.

Proposition 2.8. *Let R be a noncommutative AS regular algebra of dimension 2 and kG be a finite group algebra acting on R with trivial homological determinant. Then, assuming that $G \neq 1$, we have the following statements.*

(a) *If $R \simeq k_J[u,v]$, then $G = C_2 = \langle \sigma \mid \sigma^2 \rangle$ with $\sigma(z) = -z$ for all $z \in R_1$.*

(b) *If $R \simeq k_q[u,v]$ and $q \neq \pm 1$, then $G = C_n = \langle \sigma \mid \sigma^n \rangle$ with $\sigma(u) = \zeta u$, $\sigma(v) = \zeta^{-1} v$, where ζ is a primitive n-th root of unity.*

(c) *If $R \simeq k_{-1}[u,v]$, then either*
* $G = C_n$ *with action as in part (b),*
* $G = C_2 = \langle \sigma \mid \sigma^2 \rangle$ *with $\sigma(u) = v$ and $\sigma(v) = u$, or*
* $G = D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, \sigma\tau\sigma\tau \rangle$ *with $\sigma(u) = \zeta u$, $\sigma(v) = \zeta^{-1} v$ and $\tau(u) = v$, $\tau(v) = u$, where ζ is a primitive n-th root of unity.*

Proof. Suppose that $\sigma \in G$ acts on R by $\sigma(u) = a_{11}u + a_{21}v$ and $\sigma(v) = a_{12}u + a_{22}v$ for some $a_{ij} \in k$. Since k is algebraically closed, we can assume that the relation r of R is of the form $vu - quv - \eta u^2$ for $q \in k^\times$ and $\eta = 0, 1$. Moreover,

$$
\begin{aligned}
\sigma(r) = \quad & ((1-q)a_{11}a_{12} - \eta a_{11}^2)u^2 + (a_{12}a_{21} - qa_{11}a_{22} - \eta a_{11}a_{21})uv \\
& + (a_{11}a_{22} - qa_{12}a_{21} - \eta a_{11}a_{21})vu + ((1-q)a_{21}a_{22} - \eta a_{21}^2)v^2.
\end{aligned}
$$

By Theorem 2.1, the trivial homological condition implies that $\sigma(r) = r$. Hence we have the equations

$$
\begin{aligned}
(1-q)a_{11}a_{12} - \eta a_{11}^2 &= -\eta & (1-q)a_{21}a_{22} - \eta a_{21}^2 &= 0 \\
a_{11}a_{22} - qa_{12}a_{21} - \eta a_{11}a_{21} &= 1 & a_{12}a_{21} - qa_{11}a_{22} - \eta a_{11}a_{21} &= -q.
\end{aligned}
$$

(a) Suppose that $q = \eta = 1$. The top two equations yield $a_{11}^2 = 1$ and $a_{21} = 0$. The bottom two equations then yield $a_{11}a_{22} = 1$, so $a_{11} = a_{22} = \pm 1$. Now $\sigma^m(v) = ma_{11}^{m-1}a_{12}u + a_{11}^m v$. Since σ has finite order and $a_{11} \neq 0$, we see that $a_{12} = 0$. This proves the claim in part (a).

(b) Suppose that $\eta = 0$ and $q \neq \pm 1$. Since $q \neq 1$, the top two equations yield $a_{11}a_{12} = 0$ and $a_{21}a_{22} = 0$. Since $1 - q^2 \neq 0$, the bottom two equations also give $a_{11}a_{22} = 1$ and $a_{12}a_{21} = 0$, so $a_{12} = a_{21} = 0$. Since G has finite order, it is cyclic with the action on R as shown in case (b).

(c) Lastly, suppose that $\eta = 0$ and $q = -1$. First, observe that if the equations $a_{11}a_{22} = 1$ and $a_{12} = a_{21} = 0$ still hold, then $G = C_n$ as in part (b).

The bottom two equations both become $a_{11}a_{22} + a_{12}a_{21} = 1$, so we have more solutions. From the top two equations, one of the a_{ij} is zero. If $a_{12} = 0$ (resp. $a_{21} = 0$) then $a_{11}a_{22} = 1$, so $a_{21} = 0$ (resp. $a_{12} = 0$). So, for ζ a primitive n-th root of unity, we have that σ_ζ is an automorphism of R in this case, where $\sigma_\zeta(u) = \zeta u$ and $\sigma_\zeta(v) = \zeta^{-1}v$.

If $a_{11} = 0$ or $a_{22} = 0$, then $a_{12}a_{21} = 1$. So, $a_{12} = \lambda$, but up to base change, $a_{12} = a_{21} = 1$. Thus we have another automorphism τ of $R = k_{-1}[u, v]$ with $\tau(u) = v$ and $\tau(v) = u$. Now suppose that τ_α with $\tau_\alpha(u) = \alpha v$ and $\tau_\alpha(v) = \alpha^{-1}u$ is in G for $\alpha \in k^\times$. Then $\tau_\alpha \tau =: \sigma_\alpha$, which implies that α must be a root of unity. This shows that $G = \langle \tau \rangle$ or $G = \langle \sigma_\zeta, \tau \rangle$. This proves the claim in (c). $\qquad\square$

Remark 2.9. It is worth noting that $SL_2(k)$ does not contain any finite subgroup isomorphic to a dihedral group D_{2n}. In other words, there is no inner faithful action of D_{2n} on $k[u, v]$ having trivial homological determinant. In contrast, by part (c) of the above proposition, there does exist such an action on the (noncommutative) skew polynomial ring $k_{-1}[u, v]$. This phenomenon should be further investigated. See Remark 7.2, for instance.

3. Self-dual modules

We study self-dual modules of Hopf algebras in this section, objects which are relevant to the material in the previous section (see Corollary 3.3), and play a crucial role in the proof of part (a) of Theorem 0.4 in Section 4.

Given a left A-module M, there is a natural right A-module structure on $M' := \mathrm{Hom}_k(M, k)$, which is defined by

$$(f \bullet a)(m) = f(am)$$

for all $f \in M'$, $a \in A$ and $m \in M$. If H is a Hopf algebra with antipode S, then for every finite dimensional left H-module M, we define a left H-module structure on M' by

(E3.0.1) $$a * f = f \bullet S(a)$$

for all $a \in H$ and $f \in M'$. Then M' becomes a left H-module, which is now denoted by M^*.

Definition 3.1. Let U denote a left H-module. We say U is *self-dual* if $U \cong U^*$ as left H-modules.

We have the following basic results.

Lemma 3.2. *Let U and V be simple left H-modules. Then the following statements hold.*

> *(a) $U \cong V^*$ if and only if the trivial H-module k is a quotient module of $U \otimes V$.*
>
> *(b) If $U \cong V^*$, then the quotient map $U \otimes V \to k$ is unique up to a scalar.*
>
> *(c) U is self-dual if and only if k is a quotient H-module of $U^{\otimes 2}$ and if and only if k is an H-submodule of $U^{\otimes 2}$.*

Proof. (a) By $\mathrm{Hom} - \otimes$ adjointness [6, Lemma 1.1], there is an isomorphism

$$\mathrm{Hom}_H(U \otimes V, k) \cong \mathrm{Hom}_H(U, \mathrm{Hom}_k(V, k)) = \mathrm{Hom}_H(U, V^*).$$

The rightmost term is nonzero if and only if $U \cong V^*$, as U and V are simple H-modules. The leftmost term is nonzero if and only if k is a quotient module of $U \otimes V$. The assertion follows.

(b) The assertion follows from the fact that $\dim_k \mathrm{Hom}_H(U, V^*) = \dim_k \mathrm{Hom}_H(U, U) = 1$.

(c) The first assertion follows from part (a) by setting $V = U$. It is easy to see that U is self-dual if and only if U^* is self-dual. The second assertion follows by applying $(-)^*$. $\qquad\square$

By [13, Corollary 6], if H is semisimple and has a nontrivial self-dual simple module, then the dimension of H is even.

Corollary 3.3. *Let H be a finite-dimensional Hopf algebra acting inner faithfully on an AS regular algebra $R = k\langle U \rangle/(r)$ of global dimension 2. Suppose that U is a simple H-module and that $\mathrm{hdet}_H R$ is trivial. Then*

> *(a) k is an H-submodule of $U^{\otimes 2}$,*
>
> *(b) k is a quotient H-module of $U^{\otimes 2}$, and*
>
> *(c) U is a self-dual H-module.*

Proof. Applying Theorem 2.1, one sees that $U^{\otimes 2}$ contains k as an H-submodule. The rest follows from Lemma 3.2. $\qquad\square$

Another application of self-duality pertains to the antipode S of H.

Lemma 3.4. *Let H is a finite dimensional Hopf algebra. Suppose that H has an algebra decomposition $H = C \oplus I$ where C and I are ideals of H so that $C \cong M_n(k)$. Then, $S(C) \subset C$ if and only if C is self-dual as a left H-module (viewing C as H/I).*

Proof. Viewing C as a left H-module, we have that $IC = 0$. Then $C' := \mathrm{Hom}_k(C, k)$ has a right H-module structure and $C'I = 0$. The dual H-module C^* is a left H-module defined by (E3.0.1). Thus $S^{-1}(I)C^* = 0$. Since S is an anti-automorphism of H, we have that $S^{-1}(C)$ and $S^{-1}(I)$ are ideals of H such that $H = S^{-1}(C) \oplus S^{-1}(I)$.

If $S(C) \subset C$ (or equivalently, $S^{-1}(C) = C = S(C)$ since C is finite dimensional and S is bijective), then $S^{-1}(I) = I$ by the fact that I is uniquely determined by C. Hence $IC^* = 0$. Since $\dim C^* = \dim C$ and C is a simple k-algebra, $C^* \cong C$ as left H-modules. So, C is self-dual.

Conversely, if C is self-dual, then $IC^* = IC = 0$, which implies that $C'S(I) = 0$. Since $I = r.\mathrm{ann}_H(C')$, $S(I) \subset I$. As a consequence $S(I) = I$, which implies that $S(C) = C$. $\qquad\square$

Note that Lemma 3.4 can be applied when H is semisimple, in which case H is a direct sum of simple rings.

Now, we turn our attention to nondegenerate invariant bilinear forms. Let U be a simple left H-module. By $\mathrm{Hom} - \otimes$ adjointness, there is a nondegenerate invariant bilinear form

$$\langle \cdot, \cdot \rangle : U^* \times U \to k \quad \text{satisfying} \quad \langle h_1 \cdot u, h_2 \cdot v \rangle = \epsilon(h)\langle u, v \rangle$$

for all $h \in H$ and all $u \in U^*$ and $v \in U$. Here, $\Delta(h) = \sum h_1 \otimes h_2$. Since k is algebraically closed,

$$\dim_k \mathrm{Hom}_H(U^* \otimes U, k) = \dim_k \mathrm{Hom}_H(U^*, U^*) = 1.$$

This implies that the nondegenerate invariant bilinear form on $U^* \times U$ is unique up to scalar multiplication. Thus, if U is self-dual, there is a nondegenerate invariant bilinear form on U

$$\langle \cdot, \cdot \rangle : U \times U \to k \quad \text{satisfying} \quad \langle h_1 \cdot u, h_2 \cdot v \rangle = \epsilon(h) \langle u, v \rangle$$

for all $h \in H$ and all $u, v \in U$.

A self-dual simple H-module admits a nondegenerate invariant bilinear form which is either symmetric or skew-symmetric [13, Theorem 3]. Therefore, we proved the following lemma.

Lemma 3.5. [13, Theorem 3] *Let H be a semisimple Hopf algebra and U be a self-dual simple left H-module. Then there is a nondegenerate invariant bilinear form, unique up to scalar multiplication,*

$$\langle \cdot, \cdot \rangle : U \times U \to k.$$

Furthermore, such a nondegenerate invariant bilinear form is either symmetric or skew-symmetric. □

In particular, we use this lemma to study self-dual left H-modules of dimension 2.

Lemma 3.6. *Let H be a semisimple Hopf algebra and U be a 2-dimensional self-dual simple left H-module. Then, there is a basis of U, say $\{u, v\}$, such that the trivial module k, as a direct summand in $U^{\otimes 2}$, has a basis element of the form $u^2 + v^2$ or $uv - vu$.*

Proof. Let $V = U^*$. Then V is a 2-dimensional self-dual H-module. By Lemma 3.5, we can choose a bilinear form $\langle \cdot, \cdot \rangle : V \times V \longrightarrow k$ which induces a morphism $\vartheta : V \otimes V \longrightarrow k$ of H-modules where k is the trivial H-module. Consider the dual morphism $\vartheta^* : k \longrightarrow (V \otimes V)^* \simeq V^* \otimes V^* \simeq U^{\otimes 2}$. This map is split since H is semisimple, that is, $\vartheta^*(1)$ generates a trivial direct summand of $U^{\otimes 2}$.

We now show that we can choose a basis such that $\vartheta^*(1)$ has the desired form. By Lemma 3.5, $\langle \cdot, \cdot \rangle$ is either symmetric or skew-symmetric, so by linear algebra, we can choose a basis $\{v_1, v_2\}$ of V such that $\vartheta(v_i \otimes v_j) = \delta_{ij}$ (if $\langle \cdot, \cdot \rangle$ is symmetric) or $\vartheta(v_i \otimes v_j) = (-1)^i (\delta_{ij} - 1)$ (if $\langle \cdot, \cdot \rangle$ is skew-symmetric). Therefore $\vartheta^*(1) = v_1^* \otimes v_1^* + v_2^* \otimes v_2^*$ or $\vartheta^*(1) = v_1^* \otimes v_2^* - v_2^* \otimes v_1^*$ as desired. □

4. Proof of Theorem 0.4: H is noncommutative and semisimple

The goal of this section is to prove Theorem 0.4 in the case where H is a semisimple Hopf algebra and the H-module U of $R = k\langle U \rangle / (r)$ is simple. The first step is to show that H is noncommutative precisely when U is not a direct sum of two 1-dimensional simple left H-modules.

Lemma 4.1. *Let H be a finite dimensional Hopf algebra that acts on R as above. Then, we have the following statements.*

    (a) *The Hopf algebra H is commutative if and only if the 2-dimensional left H-module U is a direct sum of two 1-dimensional simple H-modules.*

    (b) *We have that H is noncommutative if and only if U is not a direct sum of two 1-dimensional simple H-modules.*

    (c) *Suppose that H is semisimple. Then, H is noncommutative if and only if U is simple.*

Proof. (a) If H is commutative, then H is semisimple and all of its simple modules are 1-dimensional. Therefore U is a direct sum of two 1-dimensional simple H-modules. Conversely, assume that H acts on a direct sum of 1-dimensional left H-modules $U = T_1 \oplus T_2$. Then, each element in H acts on T_i as a scalar multiplication, and hence the Hopf ideal $[H, H]$ acts as zero on T_i. Thus, the H-action on U factors through $[H, H]$, and H is commutative.

(b) This is equivalent to (a).

(c) Suppose H is semisimple. Then U must be simple as it is not a direct sum of two 1-dimensional simple H-modules. Then, part (c) is equivalent to part (b). □

Now we specify the AS regular algebras R of global dimension two that occur in Theorem 0.4, when H is noncommutative and semisimple.

Proposition 4.2. *Assume Hypothesis 0.3. If H is noncommutative, then the algebra R is isomorphic to either $k[u, v]$ or $k_{-1}[u, v]$.*

Proof. Note that R is an AS regular algebra of global dimension two, generated in degree one, of the form $k\langle U \rangle / (r)$ for some $r \in U^{\otimes 2}$. By Lemma 4.1(c), U is simple. By Corollary 3.3, U is self-dual. Also, by Lemma 3.6, r is either $u^2 + v^2$ (which is equivalent to $r = uv + uv$ after a base change) or $uv - vu$, so the assertion follows. $\qquad\square$

We require the following results pertaining to an anti-homomorphism of a matrix coalgebra. Let $C = M_n(k)$ be a matrix coalgebra over an algebraically closed field k. Let $\tau : C \longrightarrow C$ denote the transpose, and $c_N : C \longrightarrow C$ denote conjugation by $N \in GL_n(k)$. Let S and T be k-linear automorphisms of C. We say that S is *equivalent* to T if $S = c_N^{-1} \circ T \circ c_N$ for some $N \in GL_n(k)$. Note that τ and c_M obey the following commutation law

(E4.2.1) $$\tau \circ c_{M^{-1}} = c_{M^T} \circ \tau.$$

We establish the following lemma.

Lemma 4.3. *Let $M \in GL_n(k)$, then $c_M \circ \tau$ is equivalent to $c_{P^T M P} \circ \tau$ for any $P \in GL_n(k)$.*

Proof. It suffices to show that $c_{P^{-1}}^{-1} \circ c_M \circ \tau \circ c_{P^{-1}} = c_{P^T M P} \circ \tau$. Using (E4.2.1) we have that

$$c_{P^{-1}}^{-1} \circ c_M \circ \tau \circ c_{P^{-1}} \;=\; c_P \circ c_M \circ c_{P^T} \circ \tau \;=\; c_{P^T M P} \circ \tau.$$

$\qquad\square$

Proposition 4.4. *Let $C = M_2(k)$ and $S : C \longrightarrow C$ be an anti-automorphism of order 2. Then S is equivalent to $c_W \circ \tau$ where*

$$W \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Since $S : C \longrightarrow C$ is an anti-automorphism, the composition $\tau \circ S : C \longrightarrow C$ is an algebra automorphism. By the Skolem-Noether theorem, the automorphism $\tau \circ S$ is inner, so we can write $\tau \circ S = c_{W^{-1}}$ for some $W \in GL_2(k)$. Using (E4.2.1), we obtain $S = \tau \circ c_{W^{-1}} = c_{W^T} \circ \tau$. Since S has order 2, we obtain, again using (E4.2.1), that

$$1 \;=\; S^2 \;=\; c_{W^T} \circ \tau \circ c_{W^T} \circ \tau \;=\; c_{W^T} \circ c_{W^{-1}} \circ \tau \circ \tau \;=\; c_{W^{-1} W^T}.$$

In other words, $W^{-1} W^T \in Z(C)$, so $W^{-1} W^T$ is a scalar matrix. Furthermore $\det\left(W^{-1} W^T\right) = 1$, from which we conclude that $W^{-1} W^T = \pm I_{2 \times 2}$. Finally the assertion follows by applying Lemma 4.3. $\qquad\square$

Now, we prove Theorem 0.4 in the case that H is noncommutative and semisimple.

Theorem 4.5. *Let H be a noncommutative, semisimple Hopf algebra acting on an algebra R satisfying Hypothesis 0.3. Then, $R = k_{\pm 1}[u, v]$ and $U = ku \oplus kv$ is a simple H-module. Moreover, we have the following statements.*

(a1) If $R = k[u, v]$ and H is cocommutative, then $H \cong k\tilde{\Gamma}$ where $\tilde{\Gamma}$ is a non-abelian binary polyhedral group.

(a2) If $R = k_{-1}[u, v]$ and H is cocommutative, then $H \cong kD_{2n}$ for $n \geq 3$.

(a3) If H is noncocommutative, then $R = k_{-1}[u, v]$ and H° is a finite dimensional Hopf quotient of $\mathcal{O}_{-1}(SL_2(k))$.

Proof. The first assertion follows from Lemma 4.1(c) and Proposition 4.2. The 2-dimensional left H-module U is simple due to Lemma 4.1. To proceed, we classify the $K := H^\circ$-coactions on R which induce H-actions satisfying our hypotheses.

Since the homological determinant of the H-action on R is trivial, we have by Corollary 3.3 that U is self-dual. Let C be the smallest sub-coalgebra of K so that $\rho(U) \subseteq U \otimes C$. Note that $C \cong M_2(k)$ as coalgebras, and U is a simple right C-comodule. In particular, $C \cong U^{\oplus 2}$ as right C-comodules. Since U is self-dual, C is also self-dual. Now, the antipode S of H satisfies $S(C) \subseteq C$ by Lemma 3.4. Let $\{e_{ij}\}_{i,j=1,2}$ be a k-vector space basis for C. Then by Lemma 1.6(b), $K = k\langle e_{ij} \mid i,j = 1,2\rangle$. Next, we determine the possibilities for S and the relations between the e_{ij}. This will completely determine K as a Hopf algebra.

Since H is semisimple and char$(k) = 0$, we have that $S^2 = 1$, so we can apply a version of Proposition 4.4 to the coalgebra C. Hence we can assume that $S = c_W \circ \tau$, as an anti-automorphism of the coalgebra C, where $W = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ for $p^2 = 1$. Let E denote the coalgebra matrix units $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$. Note that

$$(E4.5.1) \qquad \Delta(e_{ij}) = \sum_{m=1}^{2} e_{im} \otimes e_{mj}.$$

By the antipode axiom and (E4.5.1), we get that

$$\sum_{m=1}^{2} S(e_{im})e_{mj} \;=\; \sum_{m=1}^{2} e_{im}S(e_{mj}) \;=\; \epsilon(e_{ij}) \;=\; \delta_{ij}.$$

Therefore,

$$S(E)E \;=\; (WE^TW^{-1})E \;=\; I \;=\; E(WE^TW^{-1}) \;=\; ES(E),$$

which is equivalent to:

$$\begin{pmatrix} e_{22} & p^{-1}e_{12} \\ pe_{21} & e_{11} \end{pmatrix} \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} e_{22} & p^{-1}e_{12} \\ pe_{21} & e_{11} \end{pmatrix}.$$

Hence, $\{e_{ij}\}$ satisfies the following equations:

$$(E4.5.2) \qquad\qquad e_{22}e_{11} + p^{-1}e_{12}e_{21} \;=\; 1,$$

$$(E4.5.3) \qquad\qquad e_{22}e_{12} + p^{-1}e_{12}e_{22} \;=\; 0,$$

$$(E4.5.4) \qquad\qquad e_{11}e_{21} + pe_{21}e_{11} \;=\; 0,$$

$$(E4.5.5) \qquad\qquad e_{11}e_{22} + pe_{21}e_{12} \;=\; 1,$$

$$(E4.5.6) \qquad\qquad e_{11}e_{22} + pe_{12}e_{21} \;=\; 1,$$

$$(E4.5.7) \qquad\qquad e_{12}e_{11} + p^{-1}e_{11}e_{12} \;=\; 0,$$

$$(E4.5.8) \qquad\qquad e_{21}e_{22} + pe_{22}e_{21} \;=\; 0,$$

$$(E4.5.9) \qquad\qquad e_{22}e_{11} + p^{-1}e_{21}e_{12} \;=\; 1.$$

Equations (E4.5.5) and (E4.5.6), along with Equations (E4.5.6) and (E4.5.9) yield the following two equations:

$$(E4.5.10) \qquad\qquad e_{12}e_{21} \;=\; e_{21}e_{12},$$

$$(E4.5.11) \qquad\qquad e_{11}e_{22} - e_{22}e_{11} \;=\; (p^{-1} - p)e_{12}e_{21}.$$

Observe that Equations (E4.5.2)−(E4.5.4), (E4.5.7), (E4.5.8), (E4.5.10), (E4.5.11) yield precisely the relations of the quantum group $\mathcal{O}_{-p^{-1}}(SL_2(k))$ from Example 1.4. Therefore, for $p^2 = 1$,

$$\mathcal{O}_{-p^{-1}}(SL_2(k)) \twoheadrightarrow K = k\langle C\rangle.$$

Thus, the Hopf algebras H in the theorem can be expressed as Hopf duals of finite dimensional Hopf quotients of $\mathcal{O}_{\pm 1}(SL_2(k))$.

If $p = -1$, then $\mathcal{O}_{-p^{-1}}(SL_2)$ is commutative. Hence $K = k\langle C \rangle$ is commutative. Thus, $H = K^\circ = k\tilde{\Gamma}$, a group algebra for $\tilde{\Gamma}$ a nonabelian finite subgroup of $SL_2(k)$. Again, by Proposition 4.2, R is isomorphic to either $k[u, v]$ or $k_{-1}[u, v]$. The classical result for $k[u, v]$ [27, Theorem 3.6.17.I], and the last case of Proposition 2.8(c) for $k_{-1}[u, v]$ complete these cases. In other words, we have statements (a1) and (a2), respectively.

Now consider the $p = 1$ case; we show that $R = k_{-1}[u, v]$. By way of contradiction suppose that $R = k[u, v]$. For the coaction $\rho(U) \subseteq U \otimes C$, choose u and v so that:

$$\rho(u) = u \otimes e_{11} + v \otimes e_{21} \quad \text{and} \quad \rho(v) = u \otimes e_{12} + v \otimes e_{22}.$$

Now, writing $[a, b] := ab - ba$, we have that

$$0 = \rho([u, v]) = u^2 \otimes [e_{11}, e_{12}] + uv \otimes ([e_{11}, e_{22}] + [e_{21}, e_{12}]) + v^2 \otimes [e_{21}, e_{22}].$$

Hence, by considering the term u^2, we have $0 = e_{11}e_{12} - e_{12}e_{11}$. Equation (E4.5.7) then implies that $e_{11}e_{12} = e_{12}e_{11} = 0$. Similarly, by considering the terms uv and v^2, one sees that $e_{21}e_{22} = e_{22}e_{21} = 0$ and $e_{11}e_{22} - e_{22}e_{11} = 0$. Applying the antipode S, we obtain that $e_{11}e_{21} = e_{21}e_{11} = e_{12}e_{22} = e_{22}e_{12} = 0$. Together with the relation $e_{12}e_{21} = e_{21}e_{12}$ in $\mathcal{O}_{-p^{-1}}(SL_2)$, we have that K is commutative, which yields a contradiction. Therefore, $R \cong k_{-1}[u, v]$. This gives case (a3). $\qquad\square$

Finite dimensional noncommutative Hopf quotients of $\mathcal{O}_{-1}(SL_2)$ are described explicitly in [5, Theorem 5.19]; we restate their classification in our context below.

Corollary 4.6. *The Hopf algebras H appearing in case (a3) of Theorem 4.5 are duals of the finite dimensional Hopf quotients of $\mathcal{O}_{-1}(SL_2)$. That is, H° is isomorphic to exactly one of the Hopf algebras $\mathcal{B}(\tilde{I})$, $\mathcal{A}(\tilde{\Gamma})$ or $\mathcal{B}(\tilde{\Gamma})$, all of which we denote by $\mathcal{D}(\tilde{\Gamma})$ (as in the Introduction). Here, $\tilde{I}$ is the binary icosahedral group of order 120, and $\tilde{\Gamma}$ is either the binary tetrahedral group of order 24, the binary octahedral group of order 48, or the binary dihedral group of order $4n$ for $n \geq 2$.* $\qquad\square$

When $\tilde{\Gamma} = BD_{4n}$, further results are given in [21]. By the remarks after [21, Definition 3.3], we have that $\mathcal{A}(BD_{4m})$ (which is $\mathcal{A}_{4m}$ in [21]) for $m \geq 3$, and $\mathcal{B}(BD_{4m})$ (which is $\mathcal{B}_{4m}$ in [21]) for $n \geq 2$, are nontrivial. On the other hand, $\mathcal{A}(BD_8)$ is isomorphic to $(kBD_8)^\circ$, hence is commutative. Therefore, not all Hopf deformations of binary polyhedral groups are nontrivial.

Proposition 4.7. *Let (H, R) satisfy Hypothesis 0.3. If H is noncommutative, then $\dim H$ is even.*

Proof. Since H is semisimple and noncommutative, the H-module U is simple by Lemma 4.1(c). Since the H-action has trivial homological determinant, U is self-dual by Corollary 3.3(c). By [13, Theorem 4], $\dim H$ is even. $\qquad\square$

This leads to the following conjecture.

Conjecture 4.8. *Let R be an AS regular algebra generated in degree one of Gelfand-Kirillov dimension d. Suppose that H is a semisimple Hopf algebra acting inner faithfully on R such that R_1 is a simple H-module. Then, $\dim H$ is divisible by d' for some $2 \leq d' \leq d$.*

5. Proof of Theorem 0.4: H is commutative (and semisimple)

The goal of this section is to prove Theorem 0.4 in the case where a semisimple Hopf algebra H acts on an algebra R satisfying Hypothesis 0.2. We assume that $R = k\langle U \rangle / (r)$ with U a non-simple left H-module, so by Lemma 4.1(c), H is commutative. See Theorem 5.2 below for the main classification result in this setting. First, we need the following lemma.

Lemma 5.1. *Let H be a finite dimensional Hopf algebra and $K := H^\circ$.*

(a) If T is a 1-dimensional right K-comodule, then $T \cong kg$ for some grouplike element $g \in G(K)$.

(b) If K coacts on $R = k\langle U \rangle/(r)$ with trivial homological determinant, then $kr \cong k1_K$ as K-comodules.

Proof. (a) Take a nonzero basis element t of T. Now $\rho(t) = t \otimes g$, and by coassociativity,

$$t \otimes \Delta(g) = (1 \otimes \Delta) \circ \rho(t) = (\rho \otimes 1) \circ \rho(t) = t \otimes g \otimes g.$$

Hence, $\Delta(g) = g \otimes g$.

(b) This follows from Theorem 2.1. $\qquad\square$

Theorem 5.2. *Assume Hypothesis 0.3, i.e. H be a semisimple Hopf algebra acting on an AS regular algebra $R = k\langle U \rangle/(r)$ of global dimension 2, with trivial homological determinant. Assume that the left H-module U is non-simple so H is commutative by Lemma 4.1(c). The pairs (H, R) that occur are given as follows:*

(b1) $(kC_2,\ R)$ where $R = k_J[u,v]$ or $k_q[u,v]$, and the action of the generator $\sigma \in C_2$ on R is defined by $\sigma(f) = (-1)^{\deg f} f$ for all homogeneous elements $f \in R$;

(b2) $\left(kC_2,\ k\langle u,v\rangle/(u^2 + v^2)\right)$, and the action of the generator $\sigma \in C_2$ is defined by $\sigma(u) = u, \sigma(v) = -v$;

(b3) $(kC_n, k_q[u,v])$ for $n \geq 3$, and the action of a generator $\sigma \in C_n$ is defined by $\sigma(u) = \zeta u$ and $\sigma(v) = \zeta^{-1}v$, for some primitive n-th root of unity ζ;

(b4) $\left((kD_{2n})^\circ,\ k\langle u,v\rangle/(u^2 + v^2)\right)$ for $n \geq 3$, and if $\{p_x\}_{x \in D_{2n}}$ is the dual basis of D_{2n}, then $p_x \cdot u = \delta_{x,g}u$ and $p_x \cdot v = \delta_{x,h}v$. Here, $D_{2n} = \langle g, h \mid g^2 = h^2 = 1,\ (gh)^n = 1 \rangle$.

Note that algebra $k\langle u,v\rangle/(u^2 + v^2)$ is isomorphic to $k_{-1}[u,v]$.

Proof. Recall that $H \neq k$. The left H-module U is isomorphic to $T_1 \oplus T_2$ where T_i is a 1-dimensional left H-module by Lemma 4.1(a). To prove (b1) - (b4), recall that since H is commutative and k is algebraically closed of characteristic zero, we have that H is isomorphic to the dual of a group algebra $(kG)^\circ$ [23, Theorem 2.3.1]. In this case, we consider R as a G-graded algebra. Let K denote kG, the Hopf dual of H and let ρ denote the K-coaction on R.

(b1) Assume that $U \cong T \oplus T$ for some 1-dimensional right K-comodule $T \ncong k$. Pick a basis u for the first copy of T, and v for the second. By Lemma 5.1(a), $\rho(u) = u \otimes g$ and $\rho(v) = v \otimes g$, for some $g \in G(K)$. Since the relation of the AS regular algebra R of global dimension two is of the form $r = au^2 + buv + cvu + dv^2$, we have that $\rho(r) = r \otimes g^2$.

Since $\mathrm{hdet}_H R$ is trivial, we have that $g^2 = 1$ by Lemma 5.1(b). Also, $\deg(u) = \deg(v) = g$, so K is generated by g and whence $K = k\langle g \mid g^2 \rangle = kC_2$ by Lemma 1.6. Thus, $H = (kC_2)^\circ \cong kC_2$.

Since R is any AS regular algebra of dimension 2, and $H \cong kC_2 = k\langle \sigma \rangle$ for some generator σ of C_2, the only possible action of H on R with $U \cong T \oplus T$ is given by $\sigma(u) = -u$ and $\sigma(v) = -v$. This also follows from Proposition 2.8.

(b2) Assume that $U \cong k \oplus T$ where $T \ncong k$. Pick a basis u for k, and v for T. Then $\rho(u) = u \otimes 1$ and $\rho(v) = v \otimes g$, for some $g \in G(K)$. Note that

$$\rho(u^2) = u^2 \otimes 1, \quad \rho(uv) = uv \otimes g, \quad \rho(vu) = vu \otimes g, \quad \rho(v^2) = v^2 \otimes g^2.$$

Since $\mathrm{hdet}_H R$ is trivial and $g \neq 1$, and since R is a domain, the relation r of R must be of the form $u^2 + \alpha v^2$ for some $\alpha \neq 0$. By Lemma 5.1(b), $g^2 = 1$. Rescaling this relation yields the desired algebra, $R = k\langle u,v\rangle/(u^2 + v^2)$. Again by Lemma 1.6, $K = k\langle g \mid g^2 = 1\rangle = kC_2$. Thus, $H = (kC_2)^\circ \cong kC_2$. Note that $R \cong k_{-1}[u,v]$ and the non-diagonal action of H on R is given in the second case $G = C_2$ in Proposition 2.8(c). Equivalently, the action of $H = k\langle \sigma \rangle$ on $k\langle u,v\rangle/(u^2 + v^2)$ is given by $\sigma(u) = u$ and $\sigma(v) = -v$.

(b3) Assume that $U \cong T_1 \oplus T_2$ for some non-isomorphic 1-dimensional left H-modules T_i, where $T_1 \otimes T_2 \cong k$. Pick a basis u for T_1, and v for T_2. By Lemma 5.1(a), $\rho(u) = u \otimes g$ and $\rho(v) = v \otimes h$, for some $g, h \in G(K)$.

Here, $g \neq h$, else we are in case (b1). Note that

$$\rho(u^2) = u^2 \otimes g^2, \quad \rho(uv) = uv \otimes gh, \quad \rho(vu) = vu \otimes hg, \quad \rho(v^2) = v^2 \otimes h^2.$$

Since $T_1 \otimes T_2 \cong k$, and $T_1 \otimes T_2 \cong kg \otimes kh$ as K-comodules, we have that $gh = 1 = hg$. This implies that $g^2 \neq 1$ and $h^2 \neq 1$. Hence, the relation r of R lies in $(T_1 \otimes T_2) \oplus (T_2 \otimes T_1)$. Rescaling r implies that $R = k_q[u, v]$ for some $q \in k^{\times}$.

Note that $h = g^{-1}$, so by Lemma 1.6, K is generated by the grouplike element g and $G(K) = \langle g \mid g^n \rangle = C_n$ for some $n \geq 3$. Thus, $H = (kC_n)^{\circ} \cong kC_n$. By Proposition 2.8, we have that the action of $H = k\langle \sigma \rangle$ on R is given by $\sigma(u) = \zeta u$ and $\sigma(v) = \zeta^{-1} v$, where ζ is a primitive n-th root of unity.

(b4) Assume that $U \cong T_1 \oplus T_2$, where $T_1 \otimes T_2 \not\cong k$. Pick a basis u for T_1, and v for T_2. Again by Lemma 5.1(a), $\rho(u) = u \otimes g$ and $\rho(v) = v \otimes h$, for some $g, h \in G(K)$. Here, $g \neq h$, else we are in case (b1). Note that

$$\rho(u^2) = u^2 \otimes g^2, \quad \rho(uv) = uv \otimes gh, \quad \rho(vu) = vu \otimes hg, \quad \rho(v^2) = v^2 \otimes h^2.$$

Since $T_1 \otimes T_2 \not\cong k$, we have that $gh \neq 1$ and $hg \neq 1$. Since $\text{hdet}_H R$ is trivial and R is a domain, we get that $g^2 = h^2 = 1$ by Lemma 5.1(b). After rescaling, $R = k\langle u, v \rangle / (u^2 + v^2)$ as desired. Note that by Lemma 1.6, K is generated by g and h. Since these are both grouplike elements, we have $K = kG(K)$. Let n denote an integer such that $(gh)^n = 1$, so we have the following relations $g^2 = h^2 = (gh)^n = 1$. Thus $G(K)$ is a quotient of the dihedral group D_{2n}. By classical group theory, any quotient group of D_{2n} is of the form D_{2m} for some $m \mid n$. Without loss of generality, $G(K) = D_{2n}$, or equivalently, K is the group algebra kD_{2n}. Therefore $H = (kD_{2n})^{\circ}$.

Furthermore, the right K-coaction on U yields a left H-action on U as follows. Let $\{p_x\}_{x \in D_{2n}}$ be the dual basis of the group algebra $K = kD_{2n}$, which serves as a basis of H. The right coaction of u is defined as $\rho(u) = u \otimes g \in U \otimes K$, which implies that the left action of p_x on $u \in U$ is given by: $p_x \cdot u = \langle p_x, g \rangle u = \delta_{x,g} \cdot u$. Likewise, the left coaction $\rho(v) = v \otimes h \in U \otimes K$ implies that: $p_x \cdot v = \langle p_x, h \rangle v = \delta_{x,h} \cdot v$. $\qquad \square$

6. Proof of Theorem 0.4: H is non-semisimple

The main result of this section is to prove Theorem 0.4 in the case that H is non-semisimple. We set the following notation for the rest of the section.

Notation 6.1. We set q to be a root of unity with $q^2 \neq 1$. Let l be order of q, let m be the order of q^2. Note that $l = m$ if l is odd, and $l = 2m$ if l is even.

Theorem 6.2. *Assume Hypothesis 0.2 so that H is a non-semisimple Hopf algebra that acts on an Artin-Schelter regular algebra $R = k\langle U \rangle / (r)$ of global dimension 2 with trivial homological determinant. Then the pair (H, R) arises in one of the following cases.*

    (c1) $(T_{q,\alpha,n}^{\circ}, k_q[u, v])$ where $T_{q,\alpha,n}$ is a generalized Taft algebra for q a root of unity with $q^2 \neq 1$ (Definition 6.4). In this case, the left H-module U is not semisimple as an H-module.

If in addition, U is a simple left H-module, then $R = k_q[u, v]$ for q a root of unity with $q^2 \neq 1$, and one of the following pairs occur.

    (c2) Assume that the order of q is odd. We have $(H, k_q[u, v])$, where H° is an $(k\tilde{\Gamma})^{\circ}$-extension of the dual of the Frobenius-Lusztig kernel $\mathfrak{u}_q(\mathfrak{sl}_2)$, with $\tilde{\Gamma}$ a finite subgroup of $SL_2(k)$. Uniqueness of this extension is discussed in Proposition 6.12. Moreover, H° is coquasitriangular, quasicommutative, not pointed, and $\dim_k H = \dim_k H^{\circ} = |\tilde{\Gamma}| \cdot l^3$.

    (c3) Assume that the order of q is even. We have $(H, k_q[u, v])$, where H° is an $(k\Gamma)^{\circ}$-extension of either

        • *the dual of the double Frobenius-Lusztig kernel $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)$ (Definition-Theorem 6.16) if $q^4 \neq 1$, or*

        • $u_{2,q}(\mathfrak{sl}_2)^{\circ}$ *or the 8-dimensional quotient $u_{2,q}(\mathfrak{sl}_2)^{\circ}/(e_{12} - e_{21}e_{11}^2)$ if $q^4 = 1$.*

Here, Γ is a finite subgroup of $PSL_2(k)$. Moreover, H° is not pointed. If $q^4 \neq 1$, then $\dim_k H = 2|\Gamma|(l/2)^3 = 2|\Gamma|m^3$.

The invariant subring $R^H \cong R^{\mathrm{co}\ H^\circ}$ in each case is AS Gorenstein.

Proof. The proof is based on analysis of these three cases, (c1) - (c3), which are discussed in Sections 6.1-6.3, respectively. Namely, see Proposition 6.6 for (c1); see Proposition 6.12 for (c2); and see Proposition 6.22 and Remark 6.24 for (c3). Moreover, Proposition 2.7(c) verifies the statement before (c2). The AS Gorenstein condition is established in Lemma 6.7 and Propositions 6.8, 6.14, 6.26. $\qquad\square$

6.1. U **non-simple.** In this section, we assume that H is non-semisimple, and for $R = k\langle U\rangle/(r)$, we have that U is a non-simple left H-module. We classify such pairs (H, R) in Proposition 6.6 under the condition that the homological determinant is trivial. The coaction of H° on R is also provided here. Moreover, we describe the invariant subrings R^H in Lemma 6.7 and Proposition 6.8.

Let K denote the Hopf dual H° of H. Furthermore, the non-simple K-comodule U is not the direct sum of two 1-dimensional simple modules. (Recall that if U is the direct sum of two 1-dimensional H-modules, then H is commutative [Lemma 4.1]. Hence H is semisimple, yielding a contradiction.) In this case we also call U non-semisimple. For any 2-dimensional non-semisimple K-comodule U, there is a non-split short exact sequence of K-comodules

$$0 \to T_1 \to U \to T_2 \to 0,$$

where T_1 and T_2 are 1-dimensional K-comodules. By Lemma 5.1(a), $T_i \cong kg_i$ for some grouplike elements $g_i \in G(K)$, for $i = 1, 2$.

To classify the algebras $R = k\langle U\rangle/(r)$ that occur in Theorem 6.2(c1), pick a basis $\{b_1, b_2\}$ of U so that $b_1 \in T_1$ and $b_2 \in U \setminus T_1$. We get that

(E6.2.1) $$\rho(b_1) = b_1 \otimes g_1,$$

(E6.2.2) $$\rho(b_2) = b_2 \otimes g_2 + b_1 \otimes x$$

for some nonzero $x \in K$. (This holds because $\overline{b_2} \equiv b_2 \mod T_1$ is a basis of T_2, so $\rho(\overline{b_2}) = \overline{b_2} \otimes g_2$. This is equivalent to $\rho(b_2) = b_2 \otimes g_2 + b_1 \otimes x$.) By the coassociativity of ρ, that is to say $(\rho \otimes 1) \circ \rho = (1 \otimes \Delta) \circ \rho$, we have that

(E6.2.3) $$\Delta(x) = x \otimes g_2 + g_1 \otimes x.$$

Lemma 6.3. *Retain the notation above. Assume that*

$$r := a_{11}b_1^2 + a_{12}b_1b_2 + a_{21}b_2b_1 + a_{22}b_2^2 = 0$$

is the quadratic relation of R. Then, the following statements hold.

(a) $a_{22} = 0$.
(b) $a_{12}a_{21} \neq 0$. So, from now on we assume that $a_{21} = 1$.
(c) $g_1g_2 = g_2g_1$.
(d) *The homological codeterminant of the K-coaction on R is trivial if and only if $g_1g_2 = 1$.*
(e) *If the K-coaction on R has trivial homological codeterminant, then $a_{12} \neq -1$. Moreover, after a basis change of $\{b_1, b_2\}$, we have that $a_{11} = 0$ and $xg_1 + a_{12}g_1x = 0$.*

Proof. Since $\rho : R \to R \otimes K$ defines a comodule algebra, we have by (E6.2.1) - (E6.2.2) that

$$\rho(b_1^2) = (b_1^2) \otimes g_1^2,$$
$$\rho(b_1b_2) = (b_1^2) \otimes g_1x + (b_1b_2) \otimes g_1g_2,$$
$$\rho(b_2b_1) = (b_1^2) \otimes xg_1 + (b_2b_1) \otimes g_2g_1,$$
$$\rho(b_2^2) = (b_1^2) \otimes x^2 + (b_1b_2) \otimes xg_2 + (b_2b_1) \otimes g_2x + (b_2^2) \otimes g_2^2.$$

Since r is a relation of R, we have that $\rho(r) = 0$.

(a) Proceed by contradiction. If $a_{22} \neq 0$, then we may assume that $a_{22} = 1$. In this case, $\{b_1^2, b_1b_2, b_2b_1\}$ are linearly independent. Using the computation above, we have that

$$0 = \rho(r) = \rho(a_{11}b_1^2 + a_{12}b_1b_2 + a_{21}b_2b_1 + b_2^2)$$
$$= (b_1^2) \otimes (a_{11}g_1^2 + a_{12}g_1x + a_{21}xg_1 + x^2 - a_{11}g_2^2)$$
$$+ (b_1b_2) \otimes (a_{12}g_1g_2 + xg_2 - a_{12}g_2^2) + (b_2b_1) \otimes (a_{21}g_2g_1 + g_2x - a_{21}g_2^2).$$

Therefore,

$$a_{11}g_2^2 = a_{11}g_1^2 + a_{12}g_1x + a_{21}xg_1 + x^2,$$
$$a_{12}g_2^2 = a_{12}g_1g_2 + xg_2,$$
$$a_{21}g_2^2 = a_{21}g_2g_1 + g_2x.$$

Since g_1 and g_2 are invertible, the last two equations can be simplified to $x = a_{12}(g_2 - g_1) = a_{21}(g_2 - g_1)$. Hence, K is generated by grouplike elements g_1 and g_2, and consequently, K is a group algebra which is semisimple, yielding a contradiction. Therefore, $a_{22} = 0$.

(b) Since R is a domain, the relation r is not a product of two factors of degree one. Therefore $a_{12}a_{21} \neq 0$ as $a_{22} = 0$. For simplicity we may assume that $a_{21} = 1$ from now on.

(c) By parts (a) and (b), $a_{22} = 0$ and $a_{21} = 1$. An easy computation shows that

$$0 = \rho(r) = \rho(a_{11}b_1^2 + a_{12}b_1b_2 + b_2b_1)$$
$$= (b_1^2) \otimes (a_{11}g_1^2 + a_{12}g_1x + xg_1 - a_{11}g_2g_1) + (b_1b_2) \otimes (a_{12}g_1g_2 - a_{12}g_2g_1).$$

This implies that

(E6.3.1) $$a_{12}g_1g_2 = a_{12}g_2g_1,$$

(E6.3.2) $$a_{11}g_2g_1 = a_{11}g_1^2 + a_{12}g_1x + xg_1.$$

By part (b), $a_{12} \neq 0$. Therefore $g_1g_2 = g_2g_1$.

(d) We now compute ρ on the free algebra generated by b_1 and b_2. Recall that $a_{22} = 0$ and $a_{21} = 1$. Using (E6.3.1)-(E6.3.2), we have that

$$\rho(r) = \rho(a_{11}b_1^2 + a_{12}b_1b_2 + b_2b_1)$$
$$= (b_1^2) \otimes (a_{11}g_2g_1) + (b_1b_2) \otimes (a_{12}g_2g_1) + (b_2b_1) \otimes g_2g_1$$
$$= r \otimes g_2g_1.$$

Thus, the homological codeterminant of K-coaction on $k\langle b_1, b_2 \rangle$ is $(g_2g_1)^{-1}$ by Theorem 2.1. The assertion follows.

(e) By way of contradiction, assume that $a_{12} = -1$. Then, the relation r becomes $b_2b_1 = b_1b_2 - a_{11}b_1^2$. If $a_{11} \neq 0$, then R is isomorphic to $k_J[u, v]$. By Proposition 2.7(a), H is a group algebra. Hence K is semisimple, yielding a contradiction. If $a_{11} = 0$, then R is isomorphic to $k[b_1, b_2]$. Again by Proposition 2.7(b), H (and so K) is semisimple, yielding a contradiction. Therefore, $a_{12} \neq -1$. Since $a_{12} \neq -1$, by change a basis (replacing b_2 by $b_2 + (1 + a_{12})^{-1}a_{11}b_1$), we may assume that $a_{11} = 0$. The last assertion follows from (E6.3.2). □

Thus if the K-coaction on R has trivial homological determinant, then $R = k_q[u, v]$ for some $q \in k^\times$. We define a family of Hopf algebras that coacts on $k_q[u, v]$.

Definition 6.4. Here, we define a *generalized Taft algebra* $T_{q,\alpha,n}$. Let q be a root of unity such that the order m of q^2 is larger than 1. Let $\alpha \in k$, and let n be a positive integer divisible by the order of q. Let $T_{q,\alpha,n}$ be an algebra generated by g, g^{-1} and x subject to the relations:

$$gg^{-1} = g^{-1}g = 1, \quad xg = qgx, \quad g^n = 1, \quad x^m = \alpha(g^m - g^{-m}).$$

Here, α is either 0 or 1: if $q^m \neq 1$, then $\alpha = 0$; and if $q^m = 1$, then α could be 0 or 1. The k-vector space dimension of $T_{q,\alpha,n}$ is mn as it has a basis $\{g^i x^j \mid 0 \leq i \leq n, 0 \leq j \leq m\}$.

Finally, $T_{q,\alpha,n}$ becomes a Hopf algebra with the following coalgebra structure and antipode:

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g^{-1} + g \otimes x, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0, \quad S(g) = g^{-1}, \quad S(x) = -qx.$$

Remark 6.5. Write T for $T_{q,\alpha,n}$. The coradical filtration of T is given by

$$\begin{aligned}
C_0(T) &= kG(T) = \bigoplus_{i=0}^{n-1} kg^i, \\
C_1(T) &= kG(T) \oplus kG(T)x &&= kG(T) \oplus xG(T), \\
C_s(T) &= \bigoplus_{j=0}^{s} kG(T)x^j &&= \bigoplus_{j=0}^{s} x^j kG(T)
\end{aligned}$$

for all $s \leq m - 1$. As a consequence, if I is a nonzero Hopf ideal of T, then $I \cap C_1(T) \neq 0$ by [23, Theorem 5.3.1].

Let us recall briefly the quantum binomial theorem. Let p be a scalar. Let $\binom{r}{s}_p$ denote the p-binomial coefficient $\prod_{i=0}^{s-1} \dfrac{1 - p^{r-i}}{1 - p^{i+1}}$. If $YX = qXY$, then

$$(X + Y)^n = \sum_{s=0}^{n} \binom{n}{s}_p X^s Y^{n-s},$$

which is called the quantum binomial theorem. If p is a primitive n-th root of unity, then $\binom{n}{s}_p = 0$ for all $s = 1, 2, \cdots, n-1$. In this case, $(X + Y)^n = X^n + Y^n$. We will apply these formulas soon. Now we classify the pairs (H, R) arising in Theorem 6.2 with U a non-semisimple left H-module.

Proposition 6.6. *Assume Hypothesis 0.2, and assume that U is not semisimple as an H-module. Then, $R = k_q[u, v]$ for some root of unity q with $q^2 \neq 1$, and $H^\circ \cong T_{q,\alpha,n}$. The coaction of $T_{q,\alpha,n}$ on R is determined by*

(E6.6.1) $$\rho(u) = u \otimes g,$$

(E6.6.2) $$\rho(v) = v \otimes g^{-1} + u \otimes x.$$

Proof. Let $K = H^\circ$ and retain the notation used in Lemma 6.3. Let $u = b_1$ and $v = b_2$. By Lemma 6.3(e), $q := -a_{12} \neq -1$, and $R = k_q[u, v]$. By Proposition 2.7(c), q is a root of unity, $q \neq \pm 1$. Hence, $m > 1$.

Let $g = g_1$. By Lemma 6.3(d), $g_2 = g^{-1}$. Then the K-coaction on $k_q[u, v]$ is determined by (E6.2.1) and (E6.2.2). Since K-coaction is inner faithful, K is generated by $g^{\pm 1}$ and x. By Lemma 6.3(e), $xg = qgx$. Since K is finite dimensional, the order of g, denoted by n, is finite. Using the relation $xg = qgx$, one sees that n is divisible by the order of q: $(xg)g^{n-1} = qgxg^{n-1} = q^n g^n x$ implies that $x = q^n x$.

We now show that $K \cong T_{q,\alpha,n}$. By coassociativity,

$$\Delta(x) = x \otimes g^{-1} + g \otimes x.$$

It is easy to see that $(x \otimes g^{-1})(g \otimes x) = q^2 (g \otimes x)(x \otimes g^{-1})$. By using the quantum binomial theorem and the fact that q^2 is a primitive m-th root of unity, we get that

$$\Delta(x^m) = x^m \otimes g^{-m} + g^m \otimes x^m$$

Hence,

$$\Delta(g^m x^m) = (g^m x^m) \otimes 1 + g^{2m} \otimes g^m x^m \quad \text{and} \quad (g^m x^m)g^{2m} = g^{2m}(g^m x^m).$$

If $g^m x^m$ is not in $G(K)$, then by [29, Theorem 0.2], GKdim $K \geq 1$, yielding a contradiction. Therefore $g^m x^m \in G(K)$. An easy computation shows that $g^m x^m = \alpha(g^{2m} - 1)$ as $g^m x^m$ is also $(g^{2m}, 1)$-primitive for some $\alpha \in k$. Equivalently, $x^m = \alpha(g^m - g^{-m})$.

Rescaling implies that α is either 0 or 1. At this point, we have shown that K satisfies all of the relations, and the coalgebra structure, and the antipode of $T_{q,\alpha,n}$. Therefore, there is a surjective Hopf algebra homomorphism $\pi : T_{q,\alpha,n} \to K$. Let $\{K_i\}$ be the coradical filtration of K. Since $kG(T) = kG(K) = K_0 = \bigoplus_{i=0}^{n-1} kg^i$, $\{1, g, \cdots, g^{n-1}\}$ are linearly independent. The Hopf algebra structure on K induces a K_0-Hopf module structure on K_1/K_0. By [23, Theorem 1.9.4], K_1/K_0 is a free K_0-module. Since $gx \in K_1/K_0$ is a nonzero coinvariant, $\{gx, g^2x, \cdots, g^n x\}$ is linear independent modulo K_0. This shows that π is injective when restricted to $C_1(T)$ (as defined in Remark 6.5). Therefore π is injective by [23, Theorem 5.3.1]. The assertion follows. $\qquad\square$

Next, we compute the invariant subring R^H for the pair (H, R) in the proposition above. Recall Notation 6.1. First, we require the following lemma.

Lemma 6.7. *Suppose that $n = l$ and $K := T_{q,0,l}$ coacts on $R = k_q[u, v]$ with coaction given by (E6.6.1)-(E6.6.2). Then*

$$R^{coK} = \begin{cases} k[u^m, v^m] & \text{if } l = m \\ k[a, b, c]/(b^2 - ac) \text{ where } a = u^{2m}, \ b = u^m v^m, \ c = v^{2m} & \text{if } l = 2m, \end{cases}$$

which are both (AS-)Gorenstein.

Proof. Note that $(v \otimes g^{-1})(u \otimes x) = q^2(u \otimes x)(v \otimes g^{-1})$. By (E6.6.1)-(E6.6.2) and the quantum binomial theorem, we have for any integers i and j,

$$\rho(u^i) = u^i \otimes g^i,$$

$$\rho(v^i) = \sum_{s=0}^{i} \binom{i}{s}_{q^2} (u \otimes x)^s (v \otimes g^{-1})^{i-s},$$

$$\rho(u^i v^j) = (u^i \otimes g^i) \sum_{s=0}^{j} \binom{j}{s}_{q^2} (u \otimes x)^s (v \otimes g^{-1})^{j-s}$$

$$= (u^i v^j \otimes g^{i-j}) + (u^i \otimes g^i) \sum_{s=1}^{j} \binom{j}{s}_{q^2} (u \otimes x)^s (v \otimes g^{-1})^{j-s}.$$

Case 1: $l = m$. In this case, l is odd. Then

$$\begin{aligned}\rho(u^m) &= u^m \otimes g^m & &= u^m \otimes 1,\\ \rho(v^m) &= \sum_{s=0}^{m} \binom{m}{s}_{q^2}(u \otimes x)^s (v \otimes g^{-1})^{m-s} = u^m \otimes x^m + v^m \otimes g^{-m} & &= v^m \otimes 1.\end{aligned}$$

Thus $u^m, v^m \in R^{coK}$. If $f = \sum_{i+j=d} c_{ij} u^i v^j$ is any homogeneous element in R^{coK}, then $\rho(f) = f \otimes 1$. If $c_{ij} \neq 0$, then $\rho(f) = f \otimes 1$ implies that $g^{i-j} = 1$, or $i \equiv j \mod m$. Hence, we can rewrite f as

$$f = \sum_{j=0}^{m-1} u^j v^j f_j(u^m, v^m)$$

where $f_j(u^m, v^m)$ is a polynomial of u^m and v^m. In this setting,

$$\rho(f) = \sum_{j=0}^{m-1} \left\{ (u^j v^j \otimes 1) + (u^j \otimes g^j) \sum_{s=1}^{j} \binom{j}{s}_{q^2} (u \otimes x)^s (v \otimes g^{-1})^{j-s} \right\} (f_j(u^m, v^m) \otimes 1) = f \otimes 1,$$

which forces

$$\sum_{j=0}^{m-1} \{ (u^j \otimes g^j) \sum_{s=1}^{j} \binom{j}{s}_{q^2} (u \otimes x)^s (v \otimes g^{-1})^{j-s} \} (f_j(u^m, v^m) \otimes 1) = 0.$$

Let $j > 0$ be the maximal integer such that $f_j(u^m, v^m) \neq 0$. By looking at the largest degree of x in the second tensor component, we have that

$$(u^j \otimes g^j)(u \otimes x)^j (f_j(u^m, v^m) \otimes 1) = 0$$

which implies that $f_j(u^m, v^m) = 0$, a contradiction. Therefore $f_j(u^m, v^m) = 0$ for all $j \neq 0$, and hence $f \in k[u^m, v^m]$. The assertion follows.

<u>Case 2</u>: $l = 2m$. Then $g^{2m} = 1$ (but $g^m \neq 1$). As in Case 1, one sees that $\rho(u^m) = u^m \otimes g^m$ and $\rho(v^m) = v^m \otimes g^{-m}$. Hence u^{2m}, v^{2m}, and $u^m v^m$ are in $R^{\mathrm{co}K}$. As in the proof of Case 1, if $f = \sum_{i+j=d} c_{ij} u^i v^j$ and if $c_{ij} \neq 0$, then $g^{i-j} = 1$, or $2m$ divides $i - j$. Hence we can rewrite f as

$$f = \sum_{j=0}^{m-1} u^j v^j f_j(u^{2m}, v^{2m}, u^m v^m)$$

where $f_j(u^{2m}, v^{2m}, u^m v^m)$ is a polynomial of u^{2m}, v^{2m} and $u^m v^m$. The rest of the argument is similar to the proof of Case 1. $\square$

Now we consider the case where $n \neq l$.

Proposition 6.8. *Consider the pair (H, R) under the hypothesis of Theorem 6.2(c1). By Proposition 6.6, $H° \cong T_{q,\alpha,n} =: K$. Suppose that K coacts on $R = k_q[u, v]$ for $n \neq l$ with coaction given by (E6.6.1)-(E6.6.2). Then, $R^H \cong R^{\mathrm{co}K} \cong k[a, b, c]/(ac - b^{n/m})$, which is (AS-)Gorenstein.*

Proof. Retain the notation used in Lemma 6.7. It is clear that there is a surjective Hopf homomorphism $K \to K/(g^l - 1) \cong T_{q,0,l}$. So $R^{\mathrm{co}K}$ is a subring of $R^{\mathrm{co}T_{q,0,l}}$. We will show that there is an induced K-coaction on $R^{\mathrm{co}T_{q,0,l}}$ and $R^{\mathrm{co}K} = (R^{\mathrm{co}T_{q,0,l}})^{\mathrm{co}K}$. Again, we divide the proof into two cases as in Lemma 6.7.

<u>Case 1</u>: $l = m$. By Lemma 6.7, $R^{\mathrm{co}T_{q,0,l}} = k[u^m, v^m]$. By computation,

$$\rho(u^m) = u^m \otimes g^m,$$
$$\rho(v^m) = u^m \otimes x^m + v^m \otimes g^{-m} \quad = u^m \otimes \alpha(g^m - g^{-m}) + v^m \otimes g^{-m}.$$

Hence, $k[u^m, v^m]$ is a K_0-comodule algebra, where K_0 is the subalgebra of K generated by g^m. Then, K_0 is the cyclic group algebra kC_w where $w = n/m$, and $\rho(v^m - \alpha u^m) = (v^m - \alpha u^m) \otimes g^{-m}$. Hence $k[u^m, v^m]^{\mathrm{co}K_0} = k[a, b, c]/(ac - b^w)$ where $a = (u^m)^w$, $b = u^m(v^m - \alpha u^m)$ and $c = (v^m - \alpha u^m)^w$. Therefore

$$R^{\mathrm{co}K} = (R^{\mathrm{co}T_{q,0,l}})^{\mathrm{co}K_0} = k[a, b, c]/(ac - b^w).$$

<u>Case 2</u>: l is even and $l = 2m$. By Definition 6.4, $\alpha = 0$ and $x^m = 0$. Then, we have $\rho(u^m) = u^m \otimes g^m$ and $\rho(v^m) = v^m \otimes g^{-m}$. Consequently,

$$\rho(u^{2m}) = u^{2m} \otimes g^{2m}, \quad \rho(u^m v^m) = u^m v^m \otimes 1, \quad \rho(v^{2m}) = v^{2m} \otimes g^{-2m}.$$

Let K_0 be the subalgebra generated by g^{2m}. Then $K_0 = kC_w$ where $w = n/2m$. It is clear that

$$(R^{\mathrm{co}T_{q,o,l}})^{\mathrm{co}K_0} = (k[u^{2m}, u^m v^m, v^{2m}])^{\mathrm{co}K_0} = k[a, b, c]/(ac - b^{2w}),$$

where $a = (u^{2m})^w$, $b = u^m v^m$ and $c = (v^{2m})^w$. $\square$

6.2. **U simple; order of q odd.** Here, we classify the pairs (H, R) in Theorem 0.4 for H non-semisimple and U a simple left H-module. We will show that H is a finite Hopf algebra quotient of $\mathcal{O}_q(SL_2(k))$ for q a root of unity with $q^2 \neq 1$. We assume that $l := \mathrm{ord}(q)$ is odd. In this case, $l = m = \mathrm{ord}(q^2)$.

Recall the presentation of the quantum special linear group $\mathcal{O}_q(SL_2(k))$ generated by $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ from Example 1.4. Let L be the subalgebra of $\mathcal{O}_q(SL_2(k))$ generated by $\{e_{ij}^m\}$ for $1 \leq i, j \leq 2$.

Lemma 6.9. [7, Proposition III.3.1] *Retain the notation above. We have the following statements.*

(a) *L is a Hopf subalgebra of $\mathcal{O}_q(SL_2(k))$.*

(b) L is in the center of $\mathcal{O}_q(SL_2(k))$.

(c) $L \cong \mathcal{O}(SL_2(k))$ as Hopf algebras. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we define a key object for the study of H (or of H°, in particular).

Definition-Theorem 6.10. [7, Theorems III.7.10-11] The kernel of the quantum analogue of the Frobenius map, $SL_q(2) \to SL_1(2)$, is the finite dimensional Hopf algebra:

$$\mathcal{O}_q(SL_2(k))/(\mathcal{O}_q(SL_2(k))L^+) = \mathcal{O}_q(SL_2(k))/(e_{11}^m - 1,\ e_{12}^m,\ e_{21}^m,\ e_{22}^m - 1).$$

This is isomorphic to the dual $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ of the *Frobenius-Lusztig kernel* of $\mathfrak{sl}_2$ at q. Here, $\dim_k \mathfrak{u}_q(\mathfrak{sl}_2)^\circ = m^3$.

To classify the pairs (H, R) in Theorem 6.2(c2), let $K := H^\circ$, and let C be the smallest subcoalgebra of K so that $\rho(U) \subseteq U \otimes C$. Here, U is simple, so the coalgebra C is also simple and isomorphic to the matrix coalgebra $M_2(k)$. Since the H-action on R has trivial homological determinant, we have that C is S-invariant by Corollary 3.3 and Lemma 3.4. Now [25, Theorem 1.5] implies that K is a quotient Hopf algebra of $\mathcal{O}_{-\omega^{-1}}(SL_2(k))$ where ω is a root of unity with $\mathrm{ord}(\omega^2) = \mathrm{ord}(S^2|_C) > 1$. Let $q := -\omega^{-1}$. Now to classify $K = H^\circ$ for (H, R) in Theorem 6.2(c2), it suffices to classify finite Hopf algebra quotients of $\mathcal{O}_q(SL_2(k))$, for q a root of unity, not equal to ± 1. We proceed as follows.

If $\mathcal{O}_q(SL_2(k))$ is generated by $\{e_{ij}\}_{1 \leq i,j \leq 2}$, then we have an inclusion of a central Hopf subalgebra $\mathcal{O}(SL_2(k))$ into $\mathcal{O}_q(SL_2(k))$ given by $E_{ij} \xmapsto{\iota_1} e_{ij}^m$. Moreover, consider the following diagram:

$$
\begin{array}{ccccccccc}
k & \longrightarrow & \mathcal{O}(SL_2(k)) & \xhookrightarrow{\iota_1} & \mathcal{O}_q(SL_2(k)) & \xtwoheadrightarrow{\pi_1} & \mathfrak{u}_q(\mathfrak{sl}_2)^\circ & \longrightarrow & k \\
& & \downarrow{\scriptstyle\eta_1} & & \downarrow{\scriptstyle\eta_2} & & \downarrow{\scriptstyle\eta_3} & & \\
k & \longrightarrow & K' & \xhookrightarrow{\iota_2} & K & \xtwoheadrightarrow{\pi_2} & \mathfrak{u}_q(\mathfrak{sl}_2)^\circ/I & \longrightarrow & k.
\end{array}
$$

Diagram 3

Here, $K' \cong (k\tilde{\Gamma})^\circ$, for $\tilde{\Gamma}$ a finite subgroup of $SL_2(k)$, as $\mathrm{im}(\eta_1)$ is a commutative finite Hopf algebra quotient of $\mathcal{O}(SL_2(k))$.

Lemma 6.11. *The Hopf ideal I of $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ appearing in Diagram 3 is (0).*

Proof. Let $x_{ij} \in K$ be the image of $e_{ij} \in \mathcal{O}_q(SL_2(k))$. We will abuse notation and let e_{ij} denote also its image in $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$. If $e_{12} \in I$, then by [24, Proposition 4.9] we have $x_{12} = 0$. This is a contradiction since x_{12} is a generator for K, so we conclude that $e_{12} \notin I$. Similarly, $e_{21} \notin I$. By [24, Theorem 4.2], any nonzero Hopf ideal I of $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ contains e_{12} or e_{21}. Therefore, $I = 0$. $\qquad\qquad\square$

Therefore we have proved part of the following proposition.

Proposition 6.12. *Assume Hypothesis 0.2 with H non-semisimple and assume that the left H-module U is simple. We have that H° is a finite Hopf algebra quotient of $\mathcal{O}_q(SL_2(k))$ that coacts on $k_q[u, v]$ for q a root of unity with $q^2 \neq 1$.*

If $\mathrm{ord}(q)$ is odd, then H° fits into the following exact sequence of Hopf algebras:

(E6.12.1) $$k \to (k\tilde{\Gamma})^\circ \to H^\circ \to \mathfrak{u}_q(\mathfrak{sl}_2)^\circ \to k,$$

for $\tilde{\Gamma}$ a finite subgroup of $SL_2(k)$. The uniqueness of the $(k\tilde{\Gamma})^\circ$-extension by $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ is given by Table 4 below. Moreover, H° is coquasitriangular, quasicommutative, not pointed, and $\dim_k H = \dim_k H^\circ = |\tilde{\Gamma}|l^3 = |\tilde{\Gamma}|m^3$.

$\tilde{\Gamma} \leq SL_2(k)$	$K = H^\circ$ unique?
C_n	Yes if and only if $(m,n) = 1$
BD_{4n}	Yes
E_6	Yes if and only if $(m,3) = 1$
E_7	Yes
E_8	Yes

Table 4: The uniqueness of H°

Proof. By Proposition 2.7(c), R must be isomorphic to $k_q[u,v]$ where q is a root of unity, not equal to ± 1. By [8, Proposition 5.5], H° is a Hopf algebra quotient of $\mathcal{O}_q(SL_2(k))$ that coacts on $k_q[u,v]$. The Hopf algebra H° is coquasitriangular, quasicommutative, and is not pointed [24, Lemma 5.1 and Theorem 5.6]. We see from Diagram 3 that $\dim H^\circ = |\tilde{\Gamma}| \cdot m^3$.

By [24, Theorem 4.11], the $(k\tilde{\Gamma})^\circ$-extensions by $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ are in one-to-one correspondence with extensions $\lambda : \mathbb{Z} \to \chi(\tilde{\Gamma})$ of $\kappa : m\mathbb{Z} \to \chi(\tilde{\Gamma})$. Here, $\kappa(mz) = 1$ for all $z \in \mathbb{Z}$ by [24, Equation 4].

$$\mathbb{Z} \quad \overset{\lambda}{\searrow} \\ \Big\uparrow \qquad\quad \\ m\mathbb{Z} \overset{\kappa}{\longrightarrow} \chi(\tilde{\Gamma})$$

Note that $\chi(C_n) \cong C_n$ as follows. If $C_n = \langle \sigma \mid \sigma^n = 1 \rangle$, then $\chi(C_n) = \{\chi_j\}_{1 \leq j \leq n}$ where $\chi_j(\sigma) = \zeta^j$ for ζ some n-th root of unity. Considering this group isomorphism, we have that $\lambda(1) = \zeta^j$ for some j. For λ to be an extension of κ, we require that $1 = \kappa(m) = \lambda(m) = \zeta^{jm}$. Therefore, λ is unique if and only if n is coprime to m. In this case, λ is the trivial map.

Moreover, $\chi(BD_{4n}) \cong C_4$ as follows. Say that $BD_{4n} = \langle \sigma, \tau \mid \sigma^{2n} = 1, \tau^2 = \sigma^n, \sigma\tau = \sigma^{-1}\tau \rangle$. Then, $\chi(BD_{4n}) = \{\chi_j\}_{0 \leq j \leq 3}$ where $\chi_j(\sigma, \tau) = ((-1)^j, (\sqrt{-1})^j)$. Now if λ defined by $\lambda(1) = ((-1)^j, (\sqrt{-1})^j)$ is an extension of κ, then $(1,1) = \kappa(m) = \lambda(m) = ((-1)^{mj}, (\sqrt{-1})^{mj})$. Since m is odd, $j = 0$. Therefore λ is the trivial map, and is unique.

The character groups of E_6, E_7, and E_8 are isomorphic to C_3, C_2, and C_1 respectively. Hence, we apply the argument from the cyclic group case to yield the result. $\square$

Now we study the invariant subring $R^H \cong R^{\mathrm{co}H^\circ}$ for the pair (H, R) in the proposition above. By Equation (E6.12.1), $R^{\mathrm{co}H^\circ} = (R^{\mathrm{co}\,\mathfrak{u}_q(\mathfrak{sl}_2)^\circ})^{\mathrm{co}(k\tilde{\Gamma})^\circ}$ for $R = k_q[u,v]$ with q a root of unity for $q^2 \neq 1$.

Lemma 6.13. *The coinvariant subring $R^{\mathrm{co}\,\mathfrak{u}_q(\mathfrak{sl}_2)^\circ}$ of $R = k_q[u,v]$ is isomorphic to $k[u^m, v^m]$.*

Proof. By the quantum binomial theorem, the coaction of $\mathcal{O}_q(SL_2(k))$ on $k_q[u,v]$ (from Example 1.4) yields:

$$\text{(E6.13.1)} \qquad \rho(u^m) = u^m \otimes e_{11}^m + v^m \otimes e_{21}^m, \quad \rho(v^m) = u^m \otimes e_{12}^m + v^m \otimes e_{22}^m.$$

Let $\bar{\rho} : R \to R \otimes \mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ be the induced coaction of $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ$ on $k_q[u,v]$. Then by the formulas above, $\bar{\rho}(u^m) = u^m \otimes 1$ and $\bar{\rho}(v^m) = v^m \otimes 1$. Hence, $k[u^m, v^m] \subset R^{\mathrm{co}\,\mathfrak{u}_q(\mathfrak{sl}_2)^\circ}$. Consider the factor $\mathfrak{u}_q(\mathfrak{sl}_2)^\circ/(e_{21})$, which is isomorphic to the Hopf algebra $T_{q,0,m}$ in Definition 6.4. Now by Lemma 6.7, $R^{\mathrm{co}\,\mathfrak{u}_q(\mathfrak{sl}_2)^\circ} \subseteq R^{\mathrm{co}\,T_{q,0,m}} \subseteq k[u^m, v^m]$, so the assertion holds. $\square$

Proposition 6.14. *Let (H, R) be a pair satisfying Hypothesis 0.2 in the setting of Theorem 6.2(c2). Then, the invariant subring R^H of R is (AS-)Gorenstein.*

Proof. From the sequence, $k \to (k\tilde{\Gamma})^\circ \to H^\circ \to \mathfrak{u}_q(\mathfrak{sl}_2)^\circ \to k$, we have that

$$R^H = R^{\operatorname{co} H^\circ} = \left(R^{\operatorname{co} \mathfrak{u}_q(\mathfrak{sl}_2)^\circ}\right)^{\operatorname{co}(k\tilde{\Gamma})^\circ}.$$

By Lemma 6.13, $R^{\operatorname{co} \mathfrak{u}_q(\mathfrak{sl}_2)^\circ} \cong k[u^m, v^m]$ where $l = m$ is the order of q (or order of q^2). By Lemma 6.9, the subalgebra of $\mathcal{O}_q(SL_2(k))$ generated by $\{e_{ij}^m\}$ is isomorphic to $\mathcal{O}(SL_2(k))$. We see from (E6.13.1) that $(k\tilde{\Gamma})^\circ$ coacts inner faithfully on $k[u^m, v^m]$. Equivalently $\tilde{\Gamma}$ acts faithfully on $k[u^m, v^m]$. Since $\tilde{\Gamma}$ is a finite subgroup of $SL_2(k)$, we conclude by [30, Theorem 1] that R^H is (AS-)Gorenstein. $\square$

6.3. U **simple; order of** q **even.** We continue to classify the pairs (H, R) in Theorem 0.4 for H non-semisimple and U a simple left H-module. By Proposition 2.7(c), we have that $R \cong k_q[u, v]$ where q is a root of unity with $q^2 \neq 1$. Recall Notation 6.1. We assume here that $2m = l =: \operatorname{ord}(q)$ is even. By [8, Proposition 5.5], H° is a finite Hopf algebra quotient of $\mathcal{O}_q(SL_2(k))$ that coacts on $k_q[u, v]$. We will show that H is as described in Theorem 6.2 (c3). Since U is a two-dimensional simple H-module, $K := H^\circ$ is not pointed.

As $l = 2m$, consider the subalgebra N of $\mathcal{O}_q(SL_2(k))$ generated by $\{e_{ij}^m e_{st}^m\}$ for $1 \leq i, j, s, t \leq 2$.

Lemma 6.15. *Retain the notation above. We have the following statements.*

(a) *N is a normal Hopf subalgebra of $\mathcal{O}_q(SL_2(k))$.*
(b) *$N \cong \mathcal{O}(PSL_2(k))$ as Hopf algebras.*

Proof. Both parts follows from [6, I.7.7] and routine computations. $\square$

Now we define a key object for the study of H (or of H° in particular). This algebra was first introduced in [28]; we refer to it by a different name.

Definition-Theorem 6.16. [28, Definition 5.4.1] Let q be a root of unity with $q^2 \neq 1$, with $\operatorname{ord}(q) = 2m$ even. Consider the quotient Hopf algebra:

$$\mathcal{O}_q(SL_2(k))/(\mathcal{O}_q(SL_2(k))N^+) = \mathcal{O}_q(SL_2(k))/(e_{11}^{2m} - 1, e_{12}^m, e_{21}^m, e_{22}^{2m} - 1).$$

We denote this Hopf algebra by $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$, and we call this the *dual of the double Frobenius-Lusztig kernel of $\mathfrak{sl}_2$ at q*. Moreover, $\dim_k \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ = 2m^3$ by [28, Section 5.5].

To study the Hopf algebra H in the pair (H, R) in Theorem 6.2(c3), we proceed as follows. Again, we assume that $\operatorname{ord}(q) = 2m$ is even. If $\mathcal{O}_q(SL_2(k))$ is generated by $\{e_{ij}\}_{1 \leq i,j \leq 2}$, then we have an inclusion of a normal Hopf subalgebra $\mathcal{O}(PSL_2(k))$ into $\mathcal{O}_q(SL_2(k))$ given by $E_{ij}E_{st} \overset{\iota_1}{\mapsto} e_{ij}^m e_{st}^m$. Moreover, we can consider the following diagram:

$$
\begin{array}{ccccccccc}
k & \longrightarrow & \mathcal{O}(PSL_2(k)) & \overset{\iota_1}{\hookrightarrow} & \mathcal{O}_q(SL_2(k)) & \overset{\pi_1}{\twoheadrightarrow} & \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ & \longrightarrow & k \\
 & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\
k & \longrightarrow & K' & \overset{\iota_2}{\hookrightarrow} & K & \overset{\pi_2}{\twoheadrightarrow} & \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ/I & \longrightarrow & k.
\end{array}
$$

Diagram 5

Here, $K' \cong (k\Gamma)^\circ$, for Γ a finite subgroup of $PSL_2(k)$, as $\operatorname{im}(\eta_1)$ is a commutative, finite Hopf algebra quotient of $\mathcal{O}(PSL_2(k))$. As before, we will denote $x_{ij} = \eta_2(e_{ij})$ and use e_{ij} to denote also their images in $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$. We will show that I in the diagram above is (0) when $q^4 \neq 1$. Consider the next four lemmas.

Lemma 6.17. *Let W be the Hopf algebra $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$. Then, we have the following statements.*

(a) $W \cong k\langle e_{11}, e_{12}, e_{21}\rangle/(R)$ where the relation ideal (R) is generated by

$$e_{12}e_{11} - qe_{11}e_{12}, \quad e_{21}e_{11} - qe_{11}e_{21}, \quad e_{21}e_{12} - e_{12}e_{21}, \quad e_{11}^{2m} - 1, \quad e_{12}^{m}, \quad e_{21}^{m},$$

and W has a k-linear basis

$$\{e_{11}^{i}e_{12}^{j}e_{21}^{l} \mid 0 \leq i \leq 2m-1, \ 0 \leq j \leq m-1, \ 0 \leq l \leq m-1\}.$$

(b) W is an $\mathbb{N}$-graded algebra with $\deg(e_{11}) = 0$ and $\deg(e_{12}) = \deg(e_{21}) = 1$.

(c) Any ideal of W is $\mathbb{N}$-graded.

Proof. (a) Using the relations $e_{11}^{2m} = 1$ and $e_{11}e_{22} - q^{-1}e_{12}e_{21} = 1$ (see Example 1.4), one can write e_{22} in terms of e_{11}, e_{12}, and e_{21}. Hence W is generated by e_{11}, e_{12}, e_{21}. All relations listed above are satisfied by W. Hence, there is a surjective algebra homomorphism from $k\langle e_{11}, e_{12}, e_{21}\rangle/(R) \to W$. By Definition-Theorem 6.16, $\dim W = 2m^3$. Moreover, it is clear that $k\langle e_{11}, e_{12}, e_{21}\rangle/(R)$ is spanned by monomials

$$\{e_{11}^{i}e_{12}^{j}e_{21}^{l} \mid 0 \leq i \leq 2m-1, 0 \leq j \leq m-1, 0 \leq l \leq m-1\}.$$

So, the assertions follow.

(b) Since all relations are homogeneous, W is graded.

(c) For any element f in W, let f_i denote the degree i component of f. Then $f = f_0 + f_1 + \cdots + f_{2m-1}$. Let η be the conjugation by e_{11}, namely, $\eta : f \to e_{11}^{-1}fe_{11}$ for all $f \in W$. Using relations $e_{12}e_{11} - qe_{11}e_{12} = e_{21}e_{11} - qe_{11}e_{21} = 0$, one sees that $\eta(f_i) = q^i f_i$ for all i. Applying η^j to f we have

$$\eta^{j}(f) = \eta^{j}(f_0 + f_1 + \cdots + f_{2m-1}) = f_0 + (q^{j})^{1}f_1 + \cdots + (q^{j})^{2m-1}f_{2m-1}.$$

If f is in an ideal I of W, so is $\eta^{j}(f) = e_{11}^{-j}fe_{11}^{j}$ for all j. Thus we have $f_0 + (q^{j})^{1}f_1 + \cdots + (q^{j})^{2m-1}f_{2m-1} \in I$ for all j. Taking $j = 0, \cdots, 2m-1$, we obtain that

$$M\left(f_0, f_1, \ldots, f_{2m-2}, f_{2m-1}\right)^{T} \in I^{\oplus n},$$

where M is the $(2m) \times (2m)$-matrix $\left((q^{i})^{j}\right)_{0 \leq i,j \leq 2m-1}$.

Since $q^i \neq q^l$ for any $0 \leq i < l \leq 2m-1$, M is invertible and $f_i \in I$ for all i. Therefore, I is graded. $\square$

Lemma 6.18. *Let B be a Hopf quotient of W. Let y_{ij} denote the image of e_{ij} in B.*

(a) *If $y_{12} \neq 0$ (or $y_{21} \neq 0$), then $y_{11}^{k} \neq 1$ for all $0 < k < 2m$.*

(b) *If $y_{12} \neq 0$, then $y_{12}^{k} \neq 0$ for all $0 < k < m$.*

(c) *Let I be nonzero Hopf ideal of the Hopf quotient $B = W/(e_{21})$. Then $y_{12} \in I$, or equivalently, $e_{12} \in I + (e_{21})$.*

Proof. (a) Observe that $y_{12}(y_{11}^{k} - 1) = (q^{k}y_{11}^{k} - 1)y_{12}$. If $y_{11}^{k} = 1$ for some $0 < k < 2m$, the above equation implies that $0 = y_{12}(q^{k} - 1)$. Since $q^{k} - 1 \neq 0$, we have that $y_{12} = 0$, a contradiction.

(b) In the proof below we will use the image of e_{22}, denoted by y_{22}. By the definition of W, $y_{22} = y_{11}^{-1}(1 + qy_{12}y_{21})$. Let $N \geq 2$ be the minimal positive integer such that $y_{12}^{N} = 0$. Suppose $N < m$. Then we have

$$0 = \Delta(y_{12}^{N}) = (y_{11} \otimes y_{12} + y_{12} \otimes y_{22})^{N} = \sum_{i=0}^{N} \binom{N}{i}_{q^{-2}} y_{12}^{i}y_{11}^{N-i} \otimes y_{22}^{i}y_{12}^{N-i}.$$

Using the hypothesis $y_{12}^{N} = 0$, the above equation becomes

$$0 = \sum_{i=1}^{N-1} \binom{N}{i}_{q^{-2}} y_{12}^{i}y_{11}^{N-i} \otimes y_{22}^{i}y_{12}^{N-i}.$$

Now the elements $\{\binom{N}{i}_{q^{-2}} y_{12}^{i}y_{11}^{N-i} \mid i = 1, \cdots, N-1\}$ are nonzero homogeneous elements of distinct degrees. Hence, these are linearly independent. Thus $y_{22}^{i}y_{12}^{N-i} = 0$ for all $i = 1, \cdots, N-1$. Since y_{22} is invertible, $y_{12}^{N-i} = 0$, a contradiction. Therefore $N = m$ and the assertion follows.

(c) Consider the Hopf quotient $\bar{B} = B/I = W/(e_{21}, I)$. Suppose $y_{12} \notin I$ (or $y_{12} \neq 0$ in $\bar{B}$). The coradical $\bar{B}_0$ of $\bar{B}$ is spanned by the grouplike elements $1, y_{11}, \ldots, y_{11}^{2m-1}$. By part (a), these are distinct, we have $\bar{B}_0 \simeq kC_{2m}$. Now $\bar{B}$ is pointed, so the coradical filtration is given by (c.f. [23, Theorem 5.4.1])

$$\bar{B}_i \quad = \quad \bar{B}_0 \oplus \bar{B}_0 y_{12} \oplus \cdots \oplus \bar{B}_0 y_{12}^i$$

Here, $y_{12} \notin k(y_{11} - y_{11}^{-1})$, which follows since $y_{12}^m = 0$ for $m > 1$. By part (b), none of $y_{12}, \ldots, y_{12}^{m-1}$ are zero, so we see that $\bar{B}_{m-1} = \bar{B}$ so $\dim_k (\bar{B}) = \dim_k (\bar{B}_{m-1}) = 2m^2$. However, $\dim_k (B) = 2m^2$ as well, hence $I = 0$, a contradiction. Therefore, $y_{12} \in I$. $\qquad\square$

Lemma 6.19. *Suppose $q^4 \neq 1$. Let I be a Hopf ideal in W. Then one of the following must occur:*

$$I \subseteq J^2, \quad e_{12} \in I, \quad e_{21} \in I,$$

where J denotes the Jacobson radical of W.

Proof. Suppose that $I \nsubseteq J^2$. By the k-linear basis of W given in Lemma 6.17(a), we have $(e_{12}) \cap (e_{21}) = (e_{12}e_{21}) \subseteq J^2$. So we can assume, without loss of generality, that $I \nsubseteq (e_{21})$. This means that $I + (e_{21})$ is a nonzero Hopf ideal of $B := W/(e_{21})$. By Lemma 6.18(c), $e_{12} \in I + (e_{21})$ that is, $e_{12} + e_{21}p \in I$ for some $p \in W$. If $p = 0$, then we are done. If not, we apply Lemma 6.17(c) to obtain $e_{12} + e_{21}p_0 = (e_{12} + e_{21}p)_1 \in I$ where p_0 is the degree 0 component of p.

Next we use the hypothesis $q^4 \neq 1$. Since p_0 is generated by e_{11}, $S^2(p_0) = p_0$, so

$$S^2(e_{12} + e_{21}p_0) \quad = \quad q^2 e_{12} + q^{-2} e_{21}p_0.$$

Since $q^4 \neq 1$, we have that $e_{12} = \frac{1}{q^2 - q^{-2}}(S^2(e_{12} + e_{21}p_0) - q^{-2}(e_{12} + e_{21}p_0)) \in I$. We are done. $\qquad\square$

Lemma 6.20. *Let π_2 be the map in Diagram 5. If $q^2 \neq 1$, then $\pi_2(x_{12}) = 0$ if and only if $\pi_2(x_{21}) = 0$.*

Proof. Let $\{p_g\}_{g \in \Gamma}$ denote the dual basis for $K' = (k\Gamma)^\circ$. We will use the following fact repeatedly. If $x_{ij}p_g = 0$, then

$$x_{i1}p_h \otimes x_{1j}p_{h'} \quad = \quad 0$$
$$x_{i2}p_h \otimes x_{2j}p_{h'} \quad = \quad 0$$

for all $h, h' \in G$ such that $hh' = g$. Indeed, the terms on the left above sum to $\Delta(x_{ij}p_g)(p_h \otimes p_{h'}) = 0$ and are eigenvectors of $S^2 \otimes 1$ with different eigenvalues: $1, q^2$ for $i = 1$; and $q^{-2}, 1$ for $i = 2$.

Suppose that $\pi_2(x_{12}) = 0$. By exactness, we have that $x_{12} \in K(K')^+$, so we can write $x_{12} = \sum_{g \in G \setminus \{1\}} y_q p_g$, where $y_g \in K$. Thus, $x_{12}p_1 = 0$. We show below that $x_{21}p_1 = 0$, which is equivalent to $\pi_2(x_{21}) = 0$. The converse follows by symmetry.

If $x_{12} = 0$, then x_{11} is grouplike and K is generated by $x_{11}, x_{11}^{-1}x_{21}$. Since $x_{11}^{-1}x_{21}$ is skew-primitive, we see that K is pointed, which is a contradiction. This shows that $x_{12} \neq 0$, so we must have $x_{12}p_g \neq 0$ for some $g \in \Gamma$. From $x_{12}p_1 = 0$, we get $x_{12}p_g \otimes x_{22}p_{g^{-1}} = 0$ which implies $x_{22}p_{g^{-1}} = 0$. Using this we get $x_{21}p_{g^{-2}} \otimes x_{12}p_g = 0$ so $x_{21}p_{g^{-2}} = 0$.

Now if $x_{21}p_h = 0$ for some $h \in \Gamma$ (for example, $h = g^{-2}$), then $x_{22}p_h \otimes x_{21}p_1 = 0$. We have $x_{21}p_1 = 0$ or $x_{22}p_h = 0$. In the first case, we are done, so assume $x_{22}p_h = 0$. This gives $x_{21}p_{hg^{-1}} \otimes x_{12}p_g = 0$ which implies $x_{21}p_{hg^{-1}} = 0$.

The last two paragraphs show that $x_{21}p_{g^{-i}} = 0$ for $i \geqslant 2$. Since Γ is finite, this shows that $x_{21}p_1 = 0$ which completes the proof. $\qquad\square$

Corollary 6.21. *If $q^4 \neq 1$, then the Hopf ideal I of $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$ appearing in Diagram 5 is (0).*

Proof. First note that $q^4 \neq 1$ implies $q^2 \neq 1$, so we can use Lemmas 6.19 and 6.20. If $e_{12} \in I$, then $\pi_2(x_{12}) = 0$. By Lemma 6.20, we also have $\pi_2(x_{21}) = 0$, so $e_{21} \in I$. This shows there are really only two

cases in Lemma 6.19, namely $I \subseteq J^2$ or $e_{12} \in I$. We show that $I = 0$ in the first case, and the second case leads to a contradiction.

Assume $I \subseteq J^2$. Let $\{C_j\}$ denote the coradical filtration of $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)$. By [23, Proposition 5.2.9], $C_s = \left(J^{s+1}\right)^\perp$, so we have that $I^\perp \supseteq \left(J^2\right)^\perp = C_1$. By [28, Section 5], we see that C_1 generates $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)$, hence $I = 0$.

Assume $e_{12} \in I$. As we observed at the beginning of the proof, this implies $\pi_2(x_{12}) = \pi_2(x_{21}) = 0$. Then $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ / I \simeq kF$ for some finite cyclic group F, since it is generated by a single grouplike element $\pi_2(x_{11})$. Therefore K is a Hopf algebra extension

$$k \longrightarrow (k\Gamma)^\circ \longrightarrow K \longrightarrow kF \longrightarrow k.$$

By [22, Proposition 1.5] (or [20, page 571]), the algebra K is a crossed product. So, by [23, Proposition 7.4.2(2)] it is semisimple. This is a contradiction.

$\square$

Therefore, we have proved the following proposition.

Proposition 6.22. *Assume Hypothesis 0.2 with H non-semisimple and the left H-module U is simple. Here, H° is a finite Hopf algebra quotient of $\mathcal{O}_q(SL_2(k))$ that coacts on $k_q[u,v]$ for q a root of unity with $q^4 \neq 1$.*

Assume that the $\mathrm{ord}(q) =: l = 2m$ is even. Then, H° fits into the following exact sequence of Hopf algebras:

(E6.26.1) $k \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to k,$

for Γ a finite subgroup of $PSL_2(k)$. Moreover, $\dim_k H = \dim_k H^\circ = 2|\Gamma|m^3 = 2|\Gamma|(l/2)^3$. $\square$

Now, we study the invariant subring $R^H \cong R^{\mathrm{co}\, H^\circ}$ for the pair (H, R) in the proposition above. By (E6.26.1), $R^{\mathrm{co}\, H^\circ} = (R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ})^{\mathrm{co}\,(k\Gamma)^\circ}$ for $R = k_q[u,v]$ with q a root of unity for $q^4 \neq 1$.

Lemma 6.23. *The coinvariant subring $R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$ of $R = k_q[u,v]$ is isomorphic to $k[a,b,c]/(b^2 - ac)$ where $a = u^{2m}$, $b = u^m v^m$, $c = v^{2m}$. Here, the coaction has trivial homological codeterminant.*

Proof. By the quantum binomial theorem, $u^{2m}, u^m v^m, v^{2m} \in R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$. Consider the factor $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ/(e_{21})$, which is isomorphic to the Hopf algebra $T_{q,0,2m}$ in Definition 6.4. Now by Lemma 6.7,

$$R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ} \subseteq R^{\mathrm{co}\, T_{q,0,2m}} \subseteq k[a,b,c]/(b^2 - ac)$$

where $a = u^{2m}$, $b = u^m v^m$, $c = v^{2m}$. So the assertion holds. $\square$

Now we deal with the case where $q^4 = 1$.

Remark 6.24. If $q^4 = 1$, then the Hopf ideal I of $W = \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$ appearing in Diagram 5 is equal to $J_\lambda = (e_{12} - \lambda e_{21} e_{11}^2)$ for some $\lambda \in k^*$ or (0). By using similar arguments to the $q^4 \neq 1$ case, it is easy to check that $W/J_\lambda \cong W/J_1$ for all $\lambda \neq 0$. So under the hypothesis of U being simple, there are two possibilities for $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ/I$ in Diagram 5, either $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ$ or $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ/(e_{12} - e_{21} e_{11}^2)$. Note that the Hopf algebra $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ/(e_{12} - e_{21} e_{11}^2)$ is the unique 8-dimensional non-semisimple non-pointed Hopf algebra [25, Theorem 3.5].

Lemma 6.25. *If $q^4 = 1$, then $T := \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ / \left(e_{12} - e_{21} e_{11}^2\right)$ coacts on $R = k_q[u,v]$. In this case, $R^{\mathrm{co}\, T} = R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$.*

Proof. The Hopf algebra quotients $\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to T$ and $T \to T/(e_{12}, e_{21})$ induce the following chain

$$R^{\mathrm{co}\, \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ} \subseteq R^{\mathrm{co}\, T} \subseteq R^{\mathrm{co}(T/(e_{12}, e_{21}))}$$

of coinvariant subrings. Now $T/(e_{12}, e_{21}) \simeq kC_4$ coacts diagonally on R, so $R^{\mathrm{co}(T/(e_{12}, e_{21}))} = R^{(4)}$, the fourth Veronese subalgebra of R. Therefore, it suffices to show that for each $f \in R^{(4)}$ coinvariant under the

T-coaction, we have $f \in R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$. Suppose that $f \in R^{(4)}$ and f is T-coinvariant. By Lemma 6.23, $R^{(4)}$ is generated by $1, uv^3$ and u^3v as an $R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$-module, so we can write $f = f_0 + f_1 uv^3 + f_2 u^3 v$ for some $f_i \in R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$. Since f is T-coinvariant, so is $f - f_0$, that is, $\rho(f - f_0) = (f - f_0) \otimes 1 = (f_1 uv^3 + f_2 u^3 v) \otimes 1$. Now

$$
\begin{aligned}
\rho(f - f_0) &= (f_1 \otimes 1)\,\rho(uv)\rho(v^2) + (f_2 \otimes 1)\,\rho(u^2)\rho(uv) \\
&= -(f_1 \otimes 1)\,\rho(v^2)\rho(uv) + (f_2 \otimes 1)\,\rho(u^2)\rho(uv) \\
&= \left((f_2 u^2 - f_1 v^2) \otimes e_{11}^2\right)\left(u^2 \otimes e_{11}e_{12} + v^2 \otimes e_{21}e_{22} + uv \otimes 1\right) \\
&= (f_2 u^2 - f_1 v^2)\,u^2 \otimes e_{11}^3 e_{12} + q^{-1}(f_2 u^2 - f_1 v^2)\,v^2 \otimes e_{11}e_{21} + (f_2 u^2 - f_1 v^2)\,uv \otimes e_{11}^2.
\end{aligned}
$$

Since $\{e_{11}^3 e_{12},\ e_{11}e_{21},\ e_{11}^2,\ 1\}$ are linearly independent, we have that $f_1 uv^3 + f_2 u^3 v = 0$. Hence $f = f_0 \in R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}$. This completes the proof. $\qquad\square$

Proposition 6.26. *Let (H, R) be a pair satisfying Hypothesis 0.2 in the setting of Theorem 6.2(c3). Then, the invariant subring R^H of R is (AS-)Gorenstein.*

Proof. Recall that H° fits into an exact sequence $k \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to k$ of Hopf algebras, so

$$
R^H = R^{\mathrm{co}\ H^\circ} = \left(R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}\right)^{\mathrm{co}\ (k\Gamma)^\circ}.
$$

If $q^4 = 1$, then H° can also fit into an exact sequence $k \to (k\Gamma)^\circ \to H^\circ \to T \to k$; see Lemma 6.25. Lemma 6.25 also shows that

$$
R^H = R^{\mathrm{co}\ H^\circ} = \left(R^{\mathrm{co}\ T}\right)^{\mathrm{co}\ (k\Gamma)^\circ} = \left(R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}\right)^{\mathrm{co}\ (k\Gamma)^\circ}.
$$

In both cases, we have $R^{\mathrm{co}\ \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ} \cong k\left[u^{2m}, v^{2m}, u^m v^m\right]$ with induced coaction of $(k\Gamma)^\circ$ given by

$$
\begin{aligned}
\bar\rho\left(u^{2m}\right) &= u^{2m} \otimes e_{11}^{2m} + v^{2m} \otimes e_{21}^{2m} + 2uv \otimes e_{11}^m e_{21}^m \\
\bar\rho\left(v^{2m}\right) &= u^{2m} \otimes e_{12}^{2m} + v^{2m} \otimes e_{22}^{2m} + 2uv \otimes e_{12}^m e_{22}^m \\
\bar\rho\left(u^m v^m\right) &= u^{2m} \otimes e_{11}^m e_{12}^m + v^{2m} \otimes e_{21}^m e_{22}^m + uv \otimes \left(e_{11}^m e_{22}^m + (-1)^m e_{21}^m e_{12}^m\right).
\end{aligned}
$$

If m is even (resp. odd), then the above coaction of $(k\Gamma)^\circ$ on $k\left[u^{2m}, v^{2m}, u^m v^m\right]$ coincides with the coaction given in Lemma 6.27 below for $\nu = 1$ (resp. $\nu = -1$). By Lemma 6.27(b), we conclude that $R^H \simeq k\left[u^{2m}, v^{2m}, u^m v^m\right]^{\mathrm{co}\ (k\Gamma)^\circ}$ is AS Gorenstein. $\qquad\square$

Lemma 6.27. *Consider the coaction of $\mathcal{O}(PSL_2(k)) \cong k[\alpha_{ij}\alpha_{kl} \mid i, j, s, t = 1, 2]$ on $S = k[x^2, y^2, xy]$ below:*

$$
\begin{aligned}
\tau_\nu\left(x^2\right) &= x^2 \otimes \alpha_{11}^2 + y^2 \otimes \alpha_{21}^2 + 2xy \otimes \alpha_{11}\alpha_{21} \\
\tau_\nu\left(y^2\right) &= x^2 \otimes \alpha_{12}^2 + y^2 \otimes \alpha_{22}^2 + 2xy \otimes \alpha_{12}\alpha_{22} \\
\tau_\nu(xy) &= x^2 \otimes \alpha_{11}\alpha_{12} + y^2 \otimes \alpha_{21}\alpha_{22} + xy \otimes (\alpha_{11}\alpha_{22} + \nu\alpha_{21}\alpha_{12})
\end{aligned}
$$

where $\nu = \pm 1$.

(a) *Let τ'_ν be the coaction of $\mathcal{O}_\nu(SL_2(k)) = k\langle \alpha_{ij}^{(\nu)} \mid i, j = 1, 2 \rangle$ on $k_\nu[x, y]$ given by*

$$
\tau'_\nu(x) = x \otimes \alpha_{11}^{(\nu)} + y \otimes \alpha_{21}^{(\nu)} \quad \text{and} \quad \tau'_\nu(y) = x \otimes \alpha_{12}^{(\nu)} + y \otimes \alpha_{22}^{(\nu)}.
$$

The induced coaction of $\mathcal{O}(PSL_2(k)) = k\left[\alpha_{ij}^{(\nu)}\alpha_{kl}^{(\nu)} \mid i, j, k, l = 1, 2\right]$ on $k_\nu[x, y]^{(2)} \simeq S$ is equal to τ_ν.

(b) *Let Γ be a finite subgroup of $PSL_2(k)$, so that τ_ν induces a coaction of $(k\Gamma)^\circ$ on S. Then $S^{\mathrm{co}\ (k\Gamma)^\circ} = k_\nu[x, y]^{\mathrm{co}\ \tilde{H}}$ where $\tilde{H}$ is a Hopf algebra extension of $k\mathbb{Z}_2$ by $(k\Gamma)^\circ$. In particular, the $\tilde{H}$-coaction on $k_\nu[x, y]$ has trivial homological codeterminant, so $S^{\mathrm{co}\ (k\Gamma)^\circ}$ is AS Gorenstein.*

Proof. Part (a) follows from direct computation. For part (b), we assume $\nu = -1$. The argument for $\nu = 1$ is similar and will be omitted. Now $(k\Gamma)^\circ$ is a finite dimensional Hopf algebra quotient $\mathcal{O}(PSL_2)/I$. Since $I \subset \mathcal{O}(PSL_2) \subset \mathcal{O}_{-1}(SL_2)$, we can consider the Hopf ideal $\tilde{I}$ of $\mathcal{O}(SL_2)$ generated by I. Then $\mathcal{O}_{-1}(SL_2)/\tilde{I} := \tilde{H}$ is a finite dimensional Hopf algebra quotient of $\mathcal{O}_{-1}(SL_2)$ such that

$$k \longrightarrow (k\Gamma)^\circ \longrightarrow \tilde{H} \longrightarrow k\mathbb{Z}_2 \longrightarrow k$$

so $k_{-1}[x,y]^{\mathrm{co}\,\tilde{H}} = \left(k_{-1}[x,y]^{\mathbb{Z}_2}\right)^{\mathrm{co}\,(k\Gamma)^\circ} = S^{\mathrm{co}\,(k\Gamma)^\circ}$. Using the quantum determinant relation, we have

$$\tau_{-1}(xy + yx) = xy \otimes (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}) + yx \otimes (\alpha_{21}\alpha_{12} + \alpha_{22}\alpha_{11}) = (xy + yx) \otimes 1.$$

so by Theorem 2.1, the $\tilde{H}$-coaction has trivial homological codeterminant. Finally by [15, Theorem 0.1] $S^{\mathrm{co}\,(k\Gamma)^\circ}(\cong k_{-1}[x,y]^{\mathrm{co}\,\tilde{H}})$ is AS Gorenstein. $\qquad\square$

7. McKay quivers

Let H be a semisimple Hopf algebra and U be a distinguished H-module. The *(left) McKay quiver* $Q(H,U)$ is the quiver whose vertices are indexed by the isomorphism classes of irreducible H-modules $\{S_i\}$ with $m_{ij} := \mathrm{Hom}(S_i, U \otimes S_j)$ arrows from S_i to S_j. Suppose $R = k\langle U \rangle/(r)$ is an H-module algebra as in Theorem 0.4, then we associate to (H,R) the McKay quiver $Q(H,U)$.

Proposition 7.1. *Let H be a semisimple Hopf algebra acting on an AS regular algebra $R = k\langle U\rangle/(r)$ of global dimension two under Hypotheses 0.3. Then the McKay quiver $Q(H,U)$ is one of the types as listed in Table 6 below. Refer to Theorems 4.5 and 5.2 for the description of H in the cases (a1) - (b4).*

Here, $\tilde{\Gamma}$ denotes a nonabelian finite subgroup of $SL_2(k)$.

	H		$Q(H,U)$ is of type:
(a1)	$k\tilde{\Gamma}$	$\tilde{\Gamma}$ of type D_n, E_6, E_7, E_8, resp.	$\widetilde{\mathrm{D}}_n$, $\widetilde{\mathrm{E}}_6$, $\widetilde{\mathrm{E}}_7$, $\widetilde{\mathrm{E}}_8$ resp.
(a2)	kD_{2n}		$\begin{cases} \widetilde{\mathrm{D}}_{\frac{n+4}{2}}, \text{ if } n \text{ even} \\ \widetilde{\mathrm{DL}}_{\frac{n+1}{2}}, \text{ if } n \text{ odd} \end{cases}$
(a3)	$\mathcal{D}(\tilde{\Gamma})^\circ$	$\tilde{\Gamma}$ of type D_n, E_6, E_7, E_8, resp.	$\widetilde{\mathrm{D}}_n$, $\widetilde{\mathrm{E}}_6$, $\widetilde{\mathrm{E}}_7$, $\widetilde{\mathrm{E}}_8$ resp.
(b1)	kC_2		$\widetilde{\mathrm{A}}_1$
(b2)	kC_2		$\widetilde{\mathrm{L}}_1$
(b3)	kC_n	$n \geq 3$	$\widetilde{\mathrm{A}}_{n-1}$
(b4)	$(kD_{2n})^\circ$		$\widetilde{\mathrm{A}}_{2n-1}$

Table 6: McKay Quivers for (H,R) in Theorem 0.4 with H semisimple

Proof. Cases (a1), (b1), and (b3) are well known. Case (b4) follows from direct computation. For case (a2), since U is a faithful irreducible 2-dimensional D_{2n}-module, the McKay quiver $Q(kD_{2n}, U)$ can be found in [10, pages 320 and 325].

For case (b2), let S_1 and S_{-1} be the trivial and sign representations of C_2, respectively, so that $U = S_1 \oplus S_{-1}$. Then, $S_{\pm 1} \otimes U = S_1 \oplus S_{-1}$. Hence $Q(kC_2, U)$ is of type $\tilde{\mathrm{L}}$ as pictured below.

Finally in case (a3), dualizing the result of [5, Lemma 5.15] gives an algebra isomorphism $H \cong k\tilde{\Gamma}$, where $\tilde{\Gamma}$ is a non-abelian binary dihedral group. Therefore, the irreducible representations of H and $\tilde{\Gamma}$ have the same dimensions. By basic combinatorial considerations (alternatively, a character theoretic argument is given in the proof of [5, Remark 5.23]), the McKay quiver of $\mathcal{D}(\tilde{\Gamma})^\circ$ coincides with that of $\tilde{\Gamma}$ if $\tilde{\Gamma}$ is exceptional. The binary dihedral case follows from [21, Proposition 3.9].

□

Remark 7.2. Note that all of the McKay quivers $Q(H, U)$ in the proposition above are of Dynkin type $\widetilde{A}$-$\widetilde{D}$-$\widetilde{E}$, except in cases (b2) and (a2) for n odd. In the latter case, Q is of type $\widetilde{DL}$.

Quiver of type $\widetilde{L}_1$ ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ Quiver of type $\widetilde{DL}$
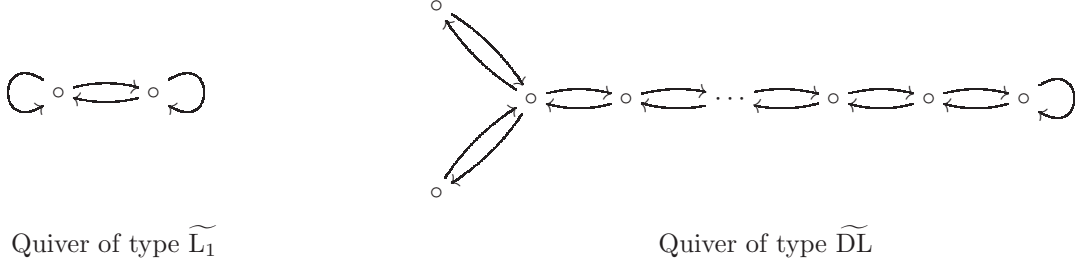
The quiver of type $\widetilde{DL}$ also arises in other studies of a quantum McKay correspondence. For instance, see the work of Malkin, Ostrik, and Vybornov [19] on quiver varieties; there, type $\widetilde{DL}$ is referred to as type T.

8. QUESTIONS FOR FURTHER STUDY

We conclude by listing several open questions that merit further study. First, one natural extension of our main classification result, Theorem 0.4, is to obtain a similar classification result for Artin-Schelter regular algebras of global dimension 3. These algebras were classified by Artin, Schelter, Tate and Van den Bergh, [1, 2, 3]. It is likely that new methods beyond those developed in this paper are needed for this problem. We state this task as follows.

Question 8.1. Let A be an Artin-Schelter regular algebra of global dimension 3. What are the finite dimensional Hopf algebras that act inner faithfully on and preserve the grading of A with trivial homological determinant?

Moreover, recall Proposition 0.5: we have that R^H is AS Gorenstein, for all H-actions on R in Theorem 0.4. This suggests the following question pertaining to Hopf actions on AS regular algebras of global dimension n. Recall that Hopf coaction is dual to Hopf action, that is to say, a Hopf algebra H acts on an algebra R from the left if and only if its dual Hopf algebra $K := H^\circ$ coacts on R from the right.

Question 8.2. Let K be a finite dimensional Hopf algebra quotient of the standard quantum special linear group $\mathcal{O}_q(SL_n(k))$ that coacts on the skew polynomial ring $k_q[v_1, \cdots, v_n]$ naturally. Is the co-invariant subring $k_q[v_1, \cdots, v_n]^{\mathrm{co}K}$ Artin-Schelter Gorenstein?

Observe that we assume that all H-actions on R in this work have trivial homological determinant. This begs the question of whether a classification result could be obtained for H-actions on R with *arbitrary* homological determinant. Naturally, this would be a quantum analogue of such a result for actions of finite subgroups of $GL_2(k)$ on $k[u, v]$ in classical invariant theory. This prompts several questions, some of which are given below.

Question 8.3. Consider the following questions.

(a) What are the finite dimensional Hopf algebras H that act on an Artin-Schelter regular algebra R (of global dimension two), with the action having arbitrary homological determinant?
(b) For the actions in part (a), when are the corresponding invariant rings R^H Artin-Schelter regular? When are they Artin-Schelter Gorenstein?

More generally for the first question in part (b), we ask if there is a version of the Shephard-Todd-Chevalley theorem for finite dimensional Hopf actions on AS regular algebras. This task has been addressed for finite group actions on skew polynomial rings [16].

On the other hand, in the view of a noncommutative McKay correspondence in our context, there are many questions that can be posed. For example, it would be useful to have an analogue of the following theorem of Auslander.

Theorem 8.4. *(Auslander)* [18, Theorems 5.15] *Let G be a finite subgroup of $SL_n(k)$ and let S be the polynomial ring $k[v_1, \ldots, v_n]$. Then, $S \# G$ is isomorphic to $\mathrm{End}_{S^G}(S)$ as rings.*

This result was vital for establishing an equivalence between the category of finitely generated projective modules over $k[u, v] \# G$ and the category of $k[u, v]^G$-direct summands of $k[u, v]$, for a finite subgroup G of $SL_2(k)$. Note that the objects of the latter category are precisely the maximal Cohen-Macaulay modules over $k[u, v]^G$. Thus, we have the question below.

Question 8.5. Given a finite dimensional Hopf algebra H acting on an Artin-Schelter regular algebra R, satisfying Hypothesis 0.3, do we have that $R \# H$ is isomorphic to $\mathrm{End}_{R^H}(R)$ as rings?

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