Quantum Deformations of GL_n

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1. Introduction and Statements of Results

One of the best-known examples of "quantum group" (cf. [13]) is the well-known one-parameter family $\mathcal{O}(GL_n(q))$ of deformations of the ring of functions on GL_n. Recently several people have constructed other deformations. A first example, given by Dipper and Donkin (see [4]), is another one-parameter deformation of GL_n. Gerstenhaber and Schack (see [6]) studied the infinitesimal deformations of GL_n and noted that they depend on more parameters. Maltsiniotis in [7] and Takeuchi in [15] constructed two-parameter deformations of GL_2 and GL_n, respectively. A family depending on $1 + \binom{n}{2}$ parameters has been constructed independently by Sudbery (see [14]), Reshetikhin (see [12]), and the authors.

We show here that this family can be obtained, and characterized, by a construction of Manin; see [8]. We also explain how the algebras in the family are twists of $\mathcal{O}(GL_n(q))$ by 2-cocycles.

Our Theorem 1 below answers in a very special case the following general problem. Suppose we are given two quadratic graded algebras A and B together with a non-degenerate pairing $A_1 \times B_1 \to k$. Let H be the bialgebra resulting from Manin's construction (see (3), (4) below) using generating sets $x = (x_i)$ and $y = (y_i)$ which are dual bases for A_1 and B_1 for the given pairing. (The structure of H depends only on the algebras and the pairing, not on the particular dual bases chosen.) *Under what circumstances does H have the Hilbert function of a polynomial ring in n^2 commuting variables, or even have other good properties like regularity or Koszulness?* This seems to be a difficult question in general. The origin of this paper was Manin's asking us whether this can ever happen when A and B are regular algebras of dimension 3 on 3 generators coming from translation on an elliptic curve, i.e., Sklyanin algebras, those of "type A" in [1], cf. [2]. After a few negative experiments in that direction we decided to look at the simpler, more usual quantum

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type A's and B's considered below in the relations (1) and were happy to discover Theorem 1, which generalizes a result of Manin; see [9, Theorem 1.14].

The following special case of the general problem seems interesting. Take $B = k[x_1, \cdots, x_n]$ to be the algebra of polynomials in n commuting variables. For which A do we obtain a good H in the above sense? Here the pairing does not matter because the automorphism group $\mathrm{GL}_n(k)$ of B acts transitively on the bases of B_1. Our Theorem 1 shows that *among the A's of type* (1) *considered in this paper*, the only ones are the "classical" quantum spaces $A = A(q)$ in which $q_{ji} = q$ for $j > i$. Are there any others?

Let I be a finite set, and n be its cardinality. Let $x = (x_i)$ and $y = (y_i)$ be two families of elements indexed by I. Let $A = k\langle x\rangle$ and $B = k\langle y\rangle$ be associative algebras generated over a field k by these families, with defining relations of the form

$$(1) \qquad x_j x_i = q_{ji} x_i x_j \quad \text{and} \quad y_j y_i = p_{ij} y_i y_j\,, \qquad \text{for } i, j \in I\,.$$

Here $p = (p_{ij})$ and $q = (q_{ij})$ are $I \times I$-matrices with entries from k^* which satisfy the following relations of "multiplicative antisymmetry":

$$(2) \qquad p_{ij}p_{ji} = 1\,, \quad p_{ii} = 1 \quad ; \quad q_{ij}q_{ji} = 1\,, \quad q_{ii} = 1\,.$$

Note that in the relations (1), p_{ij} plays the role analogous to the one played by q_{ji}.

Following Manin (see [8]), we use A and B and their generating families x and y to construct a bialgebra H which is a "quantum matrix semigroup." As algebra, $H = k\langle u\rangle$ is generated by a family of elements $u = (u_{ij})$ indexed by $I \times I$. The defining relations for H are the ones imposed by the condition that the two maps

$$(3) \qquad x_i \mapsto x'_i := \sum_j u_{ij} \otimes x_j \quad \text{and} \quad y_j \mapsto y'_j := \sum_i y_i \otimes u_{ij}$$

extend to homomorphisms of algebras

$$(4) \qquad A \to H \otimes A \qquad B \to B \otimes H\,.$$

(Here and in the sequel, all tensor products are over k.) The comultiplication homomorphism $H \to H \otimes H$ is given by

$$(5) \qquad u_{ij} \mapsto \sum_m u_{im} \otimes u_{mj}\,,$$

and the counit $H \to k$ by

$$(6) \qquad u_{ij} \mapsto \delta_{ij} \quad \text{(Kronecker delta)}\,.$$

The defining relations for H are homogeneous, in fact quadratic, in the generators u_{ij}; we grade H by giving the u_{ij} degree 1.

THEOREM 1. *The algebra H defined by condition* (4) *has the same Hilbert function as the commutative polynomial algebra in n^2 variables, i.e.*,

$$\dim H_m = \binom{n^2+m-1}{m} \quad \text{for all } m \in \mathbb{Z}\,,$$

if and only if there is a total ordering of the index set I and an element $\lambda \neq -1$ in k such that

$$q_{ji} = \lambda p_{ji} \quad \textit{for all } j > i\,. \tag{7}$$

If so, H is defined by the $\binom{n^2}{2}$ relations

$$u_{j\beta}u_{i\alpha} = \begin{cases} \dfrac{p_{ji}}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} + (\lambda-1)p_{ji}u_{i\beta}u_{j\alpha} & ,\ \textit{if } j>i\,,\ \beta>\alpha \\[2ex] \lambda \dfrac{p_{ji}}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} & ,\ \textit{if } j>i\,,\ \beta \leqq \alpha \\[2ex] \dfrac{1}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} & ,\ \textit{if } j=i\,,\ \beta>\alpha\,. \end{cases} \tag{8}$$

We will sometimes refer to the relations (8) as the "(p, λ)-relations" for the $u_{i\alpha}$.

For the rest of this section, let $I = \{1, 2, \cdots, n\}$ with its usual order, let $p = (p_{ij})$ satisfy (2), and let $\lambda \in k^*$ (including the possibility $\lambda = -1$). Define $q = (q_{ij})$ by (2) and (7) and A and B by (1), as above.

THEOREM 2. *Let $H = H(p, \lambda)$ be the algebra defined by the relations* (8). *Then H is a bialgebra with comultiplication* (5) *and counit* (6). *For fixed λ, all $H(p, \lambda)$ are canonically isomorphic as coalgebras. The algebras A and B are left and right comodules for H via* (3). *If we order the generators lexicographically* ($u_{j\beta} > u_{i\alpha} \Leftrightarrow j > i$ *or* $j = i$ *and* $\beta > \alpha$), *then the ordered monomials* (*monomials* $uu'u''\cdots$ *with* $u \leqq u' \leqq u'' \leqq \cdots$) *form a k-base for H. Moreover H is a right and left Noetherian domain which is regular in the sense of* [1].

We postpone the proofs of Theorems 1 and 2 to Sections 2 and 3.

To get at the determinant and antipode for H we consider the quadratic algebras $A^! = k\langle \xi \rangle$ and $B^! = k\langle \eta \rangle$ dual to A and B in the sense of Manin (see [8]), generated by $\xi = (\xi_i)$ and $\eta = (\eta_i)$, $i \in I$, with the defining relations orthogonal to (1), viz.

$$\xi_i^2 = 0\,, \quad \xi_j\xi_i = -q_{ij}\xi_i\xi_j\,; \quad \eta_i^2 = 0\,, \quad \eta_j\eta_i = -p_{ji}\eta_i\eta_j\,. \tag{9}$$

For each subset $J \subset I$, let

$$(10) \qquad \xi_J = \prod_{j \in J} \xi_j \quad \text{and} \quad \eta_J = \prod_{j \in J} \eta_j ,$$

the products taken in increasing order. Then (ξ_J), $J \subset I$, is a k-base for $A^!$ and (η_J), $J \subset I$, is a k-base for $B^!$.

As shown in [8], there are natural homomorphisms of algebras

$$(11) \qquad A^! \to A^! \otimes H \quad , \quad B^! \to H \otimes B^!$$

induced by

$$(12) \qquad \xi_k \mapsto \sum_{j \in I} \xi_j \otimes u_{jk} \quad , \quad \eta_j \mapsto \sum_{k \in I} u_{jk} \otimes \eta_k .$$

An easy calculation gives the effect of these on the elements ξ_J and η_J:

LEMMA 1. *If J, K are subsets of I, let $[J, K]$ denote the set of bijections $f\colon J \xrightarrow{\sim} K$. Then*

$$(13) \qquad \xi_K \mapsto \sum_J \xi_J \otimes U_{JK} \quad , \quad \eta_J \mapsto \sum_K U_{JK} \otimes \eta_K ,$$

where

$$(14) \qquad U_{JK} = \sum_{f \in [J,K]} \sigma(p,f) \prod_{j \in J} u_{j,fj} = \sum_{g \in [K,J]} \sigma(q^{-1}, g) \prod_{k \in K} u_{gk,k} ,$$

the products taken in increasing order, and the "p-sign" $\sigma(p,f) \in k^*$ of a bijection $f\colon J \xrightarrow{\sim} K$ being given by

$$(15) \qquad \sigma(p,f) := \prod_{\substack{j<j' \\ fj>fj'}} (-p_{fj,fj'}) .$$

In fact, the sums (14) are equal term-wise: with $g = f^{-1}$ the equality of the f term on the left with the g term on the right involves only the middle commutation rule (8) for the u's, not the top one, and is therefore straightforward to check.

Let

$$(16) \qquad D = U_{II} = \sum_{\pi \in S_n} \sigma(p, \pi) \prod_{j \in I} u_{j,\pi j}$$

be the "determinant element" of H_n which gives the action of H on the one-dimensional spaces $A^!_n = k\xi_I$ and $B^!_n = k\eta_I$.

THEOREM 3. (a) *The element D is normalizing; more precisely,*

$$D u_{jk} = \frac{P_k}{P_j} u_{jk} D, \quad \textit{for all } j, k \in I, \tag{17}$$

where

$$P_j := \lambda^j \prod_{m=1}^{n} p_{jm}. \tag{18}$$

(b) *Let $G = H\langle D^{-1}\rangle$. In G, the matrix $u = (u_{jk})$ has a (right and left) inverse matrix $v = (v_{jk})$ given by*

$$v_{jk} := \frac{\beta_k}{\beta_j} D^{-1} U_{\hat{k},\hat{j}} = \frac{\gamma_j}{\gamma_k} U_{\hat{k},\hat{j}} D^{-1}, \tag{19}$$

where $\hat{k}$ denotes $I - \{k\}$, and

$$\beta_j := \prod_{m=j+1}^{n} (-q_{jm}) \quad \textit{and} \quad \gamma_j := \prod_{m=1}^{j-1} (-p_{jm}).$$

(c) *There is an antiautomorphism ψ of G such that $\psi(u_{ij}) = v_{ij}$, $\psi(v_{ij}) = \lambda^{j-i} u_{ij}$, $\psi(D) = D^{-1}$, which serves as an antipode making $G = G(p, \lambda)$ into a Hopf algebra.*

Parts (a) and (b) of Theorem 3 are consequences of the lemma. The images of the identities

$$\xi_{\hat{k}} \xi_k = \beta_k \xi_I \quad \text{and} \quad \eta_j \eta_{\hat{j}} = \gamma_j \eta_I \tag{20}$$

yield the equalities

$$D = \sum_j \frac{\beta_j}{\beta_k} U_{\hat{j},\hat{k}} u_{jk} = \sum_k u_{jk} \frac{\gamma_k}{\gamma_j} U_{\hat{j},\hat{k}}. \tag{21}$$

From this, and similar consideration of the equations $\xi_{\hat{j}} \xi_k = 0$ and $\eta_j \eta_{\hat{k}} = 0$ which hold for $j \neq k$, one proves (a) and (b), using

$$\beta_j \gamma_j = (-1)^{n-1} \lambda^{-n} P_j. \tag{22}$$

We have sketched here the direct approach to parts (a) and (b) of Theorem 3 in order to record the formulas involved. Using the twisting method described in Section 3, all parts of Theorem 3 can be reduced to the case of $\mathcal{O}(\mathrm{GL}_n(q))$, where they are known. We thank Du, Parshall, and Wang for pointing out an error in our original version of (c). For the case of $\mathcal{O}(\mathrm{GL}_n(q))$, part (c) is proved in Parshall and Wang (see [11]) and in Takeuchi (see [16]). The extension to twists is carried out at the end of Section 3.

2. Proof of Theorem 1

We choose an arbitrary ordering of the index set I. The relations (1) with $i < j$ define A and B and the monomials $x_i x_j$ and $y_i y_j$ with $i \leqq j$ form bases for A_2 and B_2. Treating the u_{ij} as indeterminates we find for x' as in (3)

$$x'_j x'_i - q_{ji} x'_i x'_j = \sum_{\alpha \leqq \beta} \Phi_{ij\alpha\beta}(u) \otimes x_\alpha x_\beta \,, \tag{23}$$

where

$$\Phi_{ij\alpha\beta}(u) = \begin{cases} u_{j\alpha} u_{i\beta} + q_{\beta\alpha} u_{j\beta} u_{i\alpha} - q_{ji} u_{i\alpha} u_{j\beta} - q_{ji} q_{\beta\alpha} u_{i\beta} u_{j\alpha} & , \text{if } \alpha < \beta \\ u_{j\alpha} u_{i\alpha} - q_{ji} u_{i\alpha} u_{j\alpha} & , \text{if } \alpha = \beta \,. \end{cases} \tag{24}$$

Similarly

$$y'_\alpha y'_\beta - p_{\beta\alpha} y'_\beta y'_\alpha = \sum_{i \leqq j} y_i y_j \otimes \Phi_{ij\alpha\beta}(u) \,, \tag{25}$$

where

$$\Psi_{ij\alpha\beta}(u) = \begin{cases} u_{j\alpha} u_{i\beta} - p_{\beta\alpha} u_{j\beta} u_{i\alpha} + p_{ji} u_{i\alpha} u_{j\beta} - p_{ji} p_{\beta\alpha} u_{i\beta} u_{j\alpha} & , \text{if } i < j \\ u_{i\alpha} u_{i\beta} - p_{\beta\alpha} u_{i\beta} u_{i\alpha} & , \text{if } i = j \,. \end{cases} \tag{26}$$

Thus, for any ordering of the index set I, the algebra H is defined by the $\binom{n^2}{2}$ relations

$$\begin{cases} \Phi_{ij\alpha\beta}(u) = 0 & \text{for } i < j \text{ and } \alpha \leqq \beta \text{, and} \\ \Psi_{ij\alpha\beta}(u) = 0 & \text{for } i \leqq j \text{ and } \alpha < \beta \,. \end{cases} \tag{27}$$

On the other hand, we have $\Phi_{ij\alpha\beta}(u) = 0 = \Psi_{ij\alpha\beta}(u)$ for all $i, j, \alpha, \beta \in I$, because (23) holds for all i, j, (25) holds for all α, β, and the ordering of I was arbitrary.

Since H is defined by the $\binom{n^2}{2}$ relations (27), we have $\dim H_2 \geqq n^4 - \binom{n^2}{2} = \binom{n^2+1}{2}$, with equality if and only if the relations (27) are linearly independent. To check independence we subtract Ψ from Φ obtaining the relation

$$(q_{\beta\alpha} + p_{\beta\alpha}) u_{j\beta} u_{i\alpha} - (q_{ji} + p_{ji}) u_{i\alpha} u_{j\beta} - (q_{ji} q_{\beta\alpha} - p_{ji} p_{\beta\alpha}) u_{i\beta} u_{j\alpha} = 0 \qquad \text{for } i \neq j \,, \alpha \neq \beta \,. \tag{28}$$

LEMMA 2. *We have* $\dim H_2 = \binom{n^2+1}{2}$ *if and only if* $q_{ji} + p_{ji} \neq 0$ *for all* $i \neq j$. *When this is so, then, for every ordering of the index set* I, *the relations* (29a, b, c) *below are defining relations for* H *and the ordered monomials* $u_{i\alpha}u_{j\beta}$ *with* $(i, \alpha) \leqq (j, \beta)$ *form a basis for* H_2.

Proof: If $p_{ji} + q_{ji} = 0$ for some $i \neq j$, then $\Phi_{ijij}(u) = \Psi_{ijij}(u)$ and the relations (27) are therefore dependent. Suppose $p_{ji} + q_{ji} \neq 0$ for all $i \neq j$, and suppose $(j, \beta) > (i, \alpha)$. From (28) we find

$$(29a) \qquad u_{j\beta}u_{i\alpha} = \frac{q_{ji} + p_{ji}}{q_{\beta\alpha} + p_{\beta\alpha}} u_{i\alpha}u_{j\beta} + \frac{q_{ji}q_{\beta\alpha} - p_{ji}p_{\beta\alpha}}{q_{\beta\alpha} + p_{\beta\alpha}} u_{i\beta}u_{j\alpha}\,, \quad \text{if } j > i\,, \beta \neq \alpha\,,$$

and the relations $\Phi_{ij\alpha\alpha}(u) = 0$ and $\Psi_{ii\alpha\beta}(u) = 0$ give

$$(29b) \qquad u_{j\beta}u_{i\alpha} = q_{ji}u_{i\alpha}u_{j\beta}\,, \quad \text{if } j > i\,, \alpha = \beta$$

$$(29c) \qquad u_{j\beta}u_{i\alpha} = p_{\beta\alpha}^{-1}u_{i\alpha}u_{j\beta}\,, \quad \text{if } j = i\,, \alpha > \beta\,.$$

These relations show that in each case the monomial $u_{j\beta}u_{i\alpha}$ on the left, which is not in lexicographical order, is equal to a linear combination of ordered monomials. Each monomial on the left occurs in exactly one of the $\binom{n^2}{2}$ relations (29), so those relations are independent. Hence they span the space of quadratic relations, $\dim H_2 = \binom{n^2+1}{2}$, and the ordered monomials form a base for H_2 as claimed.

Suppose now that H has the same Hilbert function as the algebra of polynomials in n^2 variables. By the lemma, the ordered (with respect to our chosen ordering of I) quadratic monomials, form a basis of H_2. It follows that for all r, H_r is spanned by the ordered monomials of degree r, and the Hilbert function shows that the ordered monomials form a basis.

We wish to show that there is an ordering of I and an element λ in k such that

$$(30) \qquad q_{ji} = \lambda p_{ji} \quad \text{for } j > i.$$

Since $p_{ji} + q_{ji} \neq 0$ for $i \neq j$ by the lemma, such a λ will be different from -1 as required for Theorem 1, if $n \geqq 2$. If $n \leqq 1$ then take $\lambda = 0$. If $n = 2$ we can define λ by (30). Suppose therefore $n \geqq 3$. Let the first three indices be denoted by 1, 2, 3. Using the relations (29) repeatedly, beginning with the replacements indicated by the parentheses, we reduce the "overlap"

$$(31) \qquad (u_{33}u_{32})u_{21} - u_{33}(u_{32}u_{21})$$

to a linear combination of ordered monomials. The coefficient of $u_{22}u_{31}u_{33}$ in the result turns out to be

$$(32) \qquad \frac{(q_{32}q_{21} - p_{32}p_{21})(q_{32}p_{31} - p_{32}q_{31})}{p_{31}p_{32}(q_{21} + p_{21})(q_{31} + p_{31})}\,,$$

and this coefficient vanishes because the ordered monomials are linearly independent. Hence its numerator vanishes. We set

$$\lambda_{ji} := \frac{q_{ji}}{p_{ji}} \tag{33}$$

and conclude that one of the two relations

$$\lambda_{32} = \lambda_{21}^{-1} \quad \text{or} \quad \lambda_{32} = \lambda_{31}$$

must hold. Note also that $\lambda_{ij} = \lambda_{ji}^{-1}$ and $\lambda_{ii} = 1$, by (2). Since our ordering is arbitrary, the same dichotomy must hold for every triple of distinct indices:

$$\lambda_{kj} = \lambda_{ij} \quad \text{or} \quad \lambda_{ki} \quad \text{for } i \neq j \neq k \neq i\,. \tag{34}$$

Permuting the indices 1, 2, 3 we find

$$\begin{aligned} \lambda_{32} &= \lambda_{12} \quad \text{or} \quad \lambda_{31} \\ \lambda_{31} &= \lambda_{21} \quad \text{or} \quad \lambda_{32} \\ \lambda_{21} &= \lambda_{31} \quad \text{or} \quad \lambda_{23}\,. \end{aligned} \tag{35}$$

Set $\lambda = \lambda_{21}$. The three dichotomies (35) imply that $\lambda_{32} = \lambda^{\pm 1}$ and that $\lambda_{31} = \lambda^{\pm 1}$. Hence $\lambda_{ji} = \lambda^{\pm 1}$ for every pair ij of distinct indices. If $\lambda = \lambda^{-1}$ then the condition $\lambda_{ji} = \lambda$ for $j > i$ holds for every ordering. Suppose $\lambda \neq \lambda^{-1}$. We define

$$j > i \Leftrightarrow \lambda_{ji} = \lambda\,. \tag{36}$$

Then for all i, j exactly one of the following three relations holds

$$j > i \quad , \quad j = i \quad , \quad \text{or} \quad i > j\,.$$

Moreover,

$$k > i \quad \text{and} \quad j > i \Rightarrow k > i\,;$$

otherwise $\lambda_{kj} = \lambda_{ji} = \lambda_{ik} = \lambda$, which contradicts (34). Hence the relation $j > i$ is a total order on I. This concludes the proof of the "only if" part of Theorem 1.

The relations (8) are obtained by making the substitutions

$$q_{ji} = \begin{cases} \lambda p_{ji} & , \text{if } j > i \\ 1 & , \text{if } j = i \\ \lambda^{-1} p_{ji} & , \text{if } j < i \end{cases}$$

and using $p_{ij} = p_{ji}^{-1}$ and $p_{ii} = 1$ in (29). The lemma implies that (8) is a full set of relations. Therefore the "if" part of Theorem 1 follows from Theorem 2. In the next section we explain how "twisting" reduces the problem to the well-known case of $\mathcal{O}(M_n(q))$. However we point out here that in view of the special nature of the relations (8), the general case follows from the case $n = 3$ and the case $n = 3$ can be proved by means of a number of disagreeable computations (reducing overlaps). By the twisting method of the next section, it suffices to do these computations in the special case that $p_{ij} = 1$ for all i, j.

By the Diamond lemma (see [3]), one has only to show that for every triple of double indices in descending order $(k, \gamma) > (j, \beta) > (i, \alpha)$, the overlap

$$(u_{k\gamma}u_{j\beta})u_{i\alpha} - u_{k\gamma}(u_{j\beta}u_{i\alpha}) \tag{37}$$

gives zero when it is reduced by means of (8) to a linear combination of ordered monomials. By the nature of (8) the reduction of (37) will involve only $u_{l\delta}$ with $l \in \{i, j, k\}$ and $\delta \in \{\alpha, \beta, \gamma\}$, and its pattern will depend only on the sets $\{i, j, k\}$ and $\{\alpha, \beta, \gamma\}$ as ordered sets. Since any ordered set of one, two, or three elements is isomorphic to a subset of $\{1, 2, 3\}$, the case $I = \{1, 2, 3\}$ is all that needs to be considered. By various symmetries one can then reduce to a small number of cases. We have carried them out and so have Parshall and Wang; see [11] for a discussion and sample computation.

3. Proof of Theorem 2

We begin with a discussion of a general process of "twisting by 2-cocyles." Let M be a multiplicative abelian group with unit element 1, and let $R = \sum_{m \in M} R_m$ be a k-algebra graded by M. Let c be a 2-cocycle of M with values in k^*, i.e., a 2-cochain satisfying the identity

$$c(m, n)c(l, mn) = c(lm, n)c(l, m)\,. \tag{38}$$

Assume in addition that $c(1, 1) = 1$. Then we can *twist R by c* to obtain a new M-graded algebra R' as follows. By definition, R' is canonically isomorphic to R as graded vector space; we denote the isomorphism by $a' \leftrightarrow a$. The multiplication of homogeneous elements $a' \in R'_m$ and $b' \in R'_n$ is defined by

$$a'b' = c(m, n)(ab)'\,. \tag{39}$$

The condition (38) ensures the associativity of this product, and since $c(1, 1) = 1$ the map $a \to a'$ is an isomorphism on subrings $R_1 \xrightarrow{\sim} R'_1$.

If $c = \delta f$ is a coboundary, i.e., if

$$c(m, n) = \delta f(m, n) := \frac{f(m)f(n)}{f(mn)}$$

for some function $f\colon M \to k^*$, then R' is isomorphic to R via $a' \leftrightarrow f(m)a$ for $a \in A_m$. Thus the isomorphism class of the twisted algebra depends only on the cohomology class of c.

If S is a subset of R, we denote by S' the corresponding subset of R'. The following lemma is obvious in view of the fact that in (39) the coefficient $c(m, n)$ is invertible.

LEMMA 3. *Let S be a set of homogeneous elements of R.*

(i) *We have $k\langle S\rangle' = k\langle S'\rangle$. In particular, if S generates R, then S' generates R'.*
(ii) *Let J be the two-sided ideal generated by S. Then J' is the two-sided ideal generated by S', and $(R/J)' = R'/J'$ via $(a + J)' = a' + J'$.*

Now suppose M is the free abelian group generated by elements t_i, $i \in I$. The free associative algebra on variables u_{ij} has two gradings by M which we will call the left grading and the right grading. These are defined by the rule that u_{ij} has left degree t_i and right degree t_j. The (p, λ)-relations (8), which we repeat here for convenience,

$$(40)\qquad u_{j\beta}u_{i\alpha} = \begin{cases} \dfrac{p_{ji}}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} + (\lambda - 1)p_{ji}u_{i\beta}u_{j\alpha} & , \text{if } j > i\,,\ \beta > \alpha \\[2ex] \lambda\, \dfrac{p_{ji}}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} & , \text{if } j > i\,,\ \beta \leqq \alpha \\[2ex] \dfrac{1}{p_{\beta\alpha}}\, u_{i\alpha}u_{j\beta} & , \text{if } j = i\,,\ \beta > \alpha\,, \end{cases}$$

are left homogeneous of degree $t_jt_i = t_it_j$ and right homogeneous of degree $t_\alpha t_\beta = t_\beta t_\alpha$. Consequently our algebra H inherits left and right gradings by M from the free algebra, and we can twist H either on the left or on the right by 2-cocycles on M.

If c is a 2-cocycle on M, we define

$$(41)\qquad \rho_c(m, n) = \frac{c(m, n)}{c(n, m)}\quad , \quad r_{ij}(c) = \frac{c(t_i, t_j)}{c(t_j, t_i)} = \rho_c(t_i, t_j)\,.$$

PROPOSITION 1. (i) *The map $c \mapsto r(c)$ induces a bijection from $H^2(M, k^*)$ to the set of multiplicatively antisymmetric matrices p (those satisfying the relations (2)).*

(ii) *The function ρ_c is bimultiplicative and alternating. It depends only on the cohomology class of c, and its matrix with respect to the basis $\{t_i\}$ of M is $r(c) = (r_{ij}(c))$.*

Proof: (i) We have $r_{ij}(cc') = r_{ij}(c)r_{ij}(c')$. So the map r is a homomorphism with respect to component-wise multiplication of matrices. Since $r(\delta f) = 1$ for every 1-cochain f, r factors through $H^2(M, k^*)$. To show surjectivity, define for each multiplicatively antisymmetric matrix $p = (p_{ij})$ a function c_p by

$$c_p(\prod t_i^{m_i}, \prod t_j^{n_j}) := \prod_{i<j} p_{ij}^{m_i n_j}. \tag{42}$$

This c_p is the unique bimultiplicative function $M \times M \to k^*$ such that

$$c_p(t_i, t_j) = \begin{cases} p_{ij} & \text{for } i < j \\ 1 & \text{for } i \geqq j. \end{cases}$$

Since bimultiplicative functions are 2-cocycles, c_p is a 2-cocycle, and by construction, $r(c_p) = p$. To show injectivity we must show that a 2-cocycle c such that

$$c(t_i, t_j) = c(t_j, t_i) \quad \text{for } i, j \in I$$

is a coboundary. This becomes obvious if we consider the corresponding group extension. The extension is central, and lifts of the generators t_i of M commute. Hence the extension is abelian, and since M is free abelian, the extension splits.

(ii) It is obvious that ρ_c depends only on the class of c, because a coboundary $\delta f(\mu, \nu) = f(\mu)f(\mu\nu)^{-1}f(\nu)$ is symmetric in μ and ν. Since each class contains a bimultiplicative cocycle (42), ρ_c is bimultiplicative. The remaining assertions are clear.

THEOREM 4. *Let $H = H(p, \lambda)$ be the algebra considered in Theorem 2. If we twist $H(p, \lambda)$ simultaneously on the left by c^{-1} and on the right by c we obtain $H(r(c)p, \lambda)$.*

Proof: By the lemma, the twist of $H(p, \lambda)$ by c^{-1} on the left is defined by the relations obtained from (40) by replacing each monomial $u_{k\gamma}u_{l\delta}$ by $c(t_k, t_l)u_{k\gamma}u_{l\delta}$. Similarly, to twist by c on the right, we replace $u_{k\gamma}u_{l\delta}$ *by* $c(t_\gamma, t_\delta)^{-1}u_{k\gamma}u_{l\delta}$. Doing both of these operations simultaneously does effectively replace p by $r(c)p$.

By the proposition and Theorem 4, for given λ, all $H(p, \lambda)$'s are twists of each other and hence have the same Hilbert function. To determine that function it therefore suffices to look at one of them. Putting $\lambda = q^2$ and defining p by $p_{ij} = q^{-1}$ for $i > j$ (and $p_{ii} = 1$, $p_{ij} = q$ for $i < j$) we obtain the well-known algebra $\mathcal{O}(M_n(q))$ defined by the relations

$$u_{j\beta}u_{i\alpha} = \begin{cases} u_{i\alpha}u_{j\beta} + (q - q^{-1})u_{i\beta}u_{j\alpha} & \text{if } j > i,\ \beta > \alpha \\ qu_{i\alpha}u_{j\beta} & \text{if } j > i,\ \beta = \alpha \\ qu_{i\alpha}u_{j\beta} & \text{if } j = i,\ \beta > \alpha \\ u_{i\alpha}u_{j\beta} & \text{if } j > i,\ \beta < \alpha. \end{cases} \tag{43}$$

This algebra has the Hilbert function of the polynomial algebra (see, e.g., [11]), so all $H(p, \lambda)$ do.

We now prove Theorem 2. The existence of the comultiplication in $H = H(p, \lambda)$ and the coaction of H on A and B is obvious from Theorem 1 for $\lambda \neq -1$, and hence, by the principle of preservation of algebraic identities, for $\lambda = -1$, too. These things can also be reduced by twists to the usual q-case (43). Indeed, given algebras H, A, B satisfying (3), (4), and (5), if we twist A by c^{-1}, twist H simultaneously by c^{-1} on the left and by c on the right, and twist B by c, then the twisted algebras H', A', B' also satisfy (3), (4), and (5).

Also, if we identify H and H' by means of the isomorphism of graded vector spaces $h \mapsto h'$ underlying the twists, then the comultiplications $\gamma : H \to H \otimes H$ and $\gamma' : H' \to H' \otimes H'$ are the same map. This was suggested to us by Brian Parshall who told us of the corresponding invariance proved in [5]; it has also been remarked by Reshetikhin; see [12]. To prove it we must show that $\gamma' h' = (\gamma h)'$ for $h \in H$. Since this is true for $h = u_{ij}$, we have only to show that the map $h' \mapsto (\gamma h)'$ preserves multiplication. For $m, n \in M$, let $H_{m,n}$ be the part of H of left degree m and right degree n. It follows from (5) that

$$\gamma H_{m,n} \subset \sum_{\nu \in M} H_{m,\nu} \otimes H_{\nu,n} . \tag{44}$$

Let $h_1 \in H_{m_1,n_1}$ and $h_2 \in H_{m_2,n_2}$. Twisting H as above changes the product $h_1 h_2$ by the scalar factor $c(m_1, m_2)^{-1} c(n_1, n_2)$. We must show that the product $(\gamma h_1)(\gamma h_2)$ changes by the same factor. This is true because, by (44), it is a sum of products indexed by pairs $(\nu_1, \nu_2) \in M \times M$, each of which changes by that factor, because for each (ν_1, ν_2),

$$c(m_1, m_2)^{-1} c(\nu_1, \nu_2) c(\nu_1, \nu_2)^{-1} c(n_1, n_2) = c(m_1, m_2)^{-1} c(n_1, n_2) .$$

The form of the relations (40) shows that H is spanned by the ordered monomials in the u_{ij}, and from the Hilbert function it follows that these monomials are linearly independent.

To prove the rest of Theorem 2 we use the fact that H can be obtained from k as a succession of Ore extensions, by adjoining the u_{ij} one at a time in lexicographic order. For each $(j, \beta) \in I \times I$, let $H(j, \beta)$ be the subalgebra of H generated by the $u_{i\alpha}$ with $(i, \alpha) < (j, \beta)$. The relations (40) show that $H(j, \beta)$ is spanned by the ordered monomials in that set of generators. The same is true if we consider the monomials which are ordered from right to left instead of from left to right (indeed, $H(p, \lambda)^{\mathrm{opp}} = H(1/p, 1/\lambda)$). Now fix a $(j, \beta) \in I \times I$ and, to simplify notation, let $R = H(j, \beta)$, $t = u_{j\beta}$, and $S = R\langle t \rangle$, so that $R \subset S$ is a step in the chain of subrings leading from k to H. Because of the linear independence of the ordered monomials, S is a free left and right R-module with basis $\{1, t, t^2, \cdots\}$. Taking into account the grading we have, for each n,

$$S_n = \bigoplus_{m=0}^{r} R_{r-m}t^m = \bigoplus_{m=0}^{r} t^m R_{r-m}\,. \tag{45}$$

It also follows from (40) that $tR \subset Rt + R$. For $a \in R$, write

$$ta = a^\sigma t + Da \tag{46}$$

with $a^\sigma \in R$ and $Da \in R$. Then $a \mapsto a^\sigma$ is a degree preserving automorphism of R and $a \mapsto Da$ is a σ-derivation ($D(ab) = (a^\sigma)Db + (Da)b$) which raises degrees by 1. In other words, $R \subset S$ is an Ore extension. To complete the proof of Theorem 2, we have only to show that the properties of being Noetherian, being a domain, and being regular, are preserved by Ore extension. For the first two this is standard (cf. Section 1.2.9 of [10]).

PROPOSITION 2. *Let R be a regular graded algebra of dimension d. Let $S = R\langle t\rangle$ be an Ore extension. Then S is regular of dimension $d + 1$.*

Proof: We must prove three things:

(a) $gk \dim S < \infty$
(b) $\sum_{n=0}^{\infty} \dim \operatorname{Tor}_n^S(k, k) < \infty$
(c) $\dim \operatorname{Ext}_S^n(k, S) = \delta_{n,d+1}$ (Kronecker delta).

By hypothesis, they are true if we replace S by R and $d + 1$ by d. Let $\delta > 0$ be the degree of t. (In our application, $\delta = 1$, but this is not necessary.) Then

$$S_n = R_n \oplus R_{n-\delta}t \oplus R_{n-2\delta}t^2 \oplus \cdots .$$

Hence $gk \dim S = (gk \dim R) + 1$. This takes care of (a).

Let I_R be the augmentation ideal in R, so $k = R/I_R$. The polynomial ring $k[t]$ is the quotient of S by the ideal $SI_R = I_RS$, and thus $k[t] = S \otimes_R k$ as left S-module and $k[t] = k \otimes_R S$ as right S-module. Let

$$0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to 0 \tag{$P.$}$$

be a minimal free resolution of k as left R-module. Then $S \otimes_R P.$ is a free resolution of $k[t]$ as left S-module, because S is free, hence flat, as left R-module. Thus for a right S-module M,

$$\operatorname{Tor}_r^S(M, k[t]) = \operatorname{Tor}_r^R(M, k)\,, \tag{47}$$

and for a left S-module N,

$$\operatorname{Ext}_S^r(k[t], N) = \operatorname{Ext}_R^r(k, N)\,. \tag{48}$$

The functor $N \mapsto \mathrm{Tor}^S_\cdot(k, N)$ applied to the exact sequence

$$(49) \qquad 0 \to k[t] \xrightarrow{t} k[t] \to k \to 0$$

yields a long exact homology sequence which shows, using (47) with $M = k$, that (b) is true.

The functor $M \mapsto \mathrm{Ext}^\cdot_S(M, S)$ applied to (49) yields a long exact cohomology sequence which shows using (48) with $N = S$ that $\mathrm{Ext}^r_S(k, S) = 0$ for $r \neq d$, $d + 1$, and which boils down to

$$(50) \qquad \begin{aligned} \cdots 0 \to \mathrm{Ext}^d_S(k, S) \to \mathrm{Ext}^d_S(k[t], S) \xrightarrow{\alpha} \mathrm{Ext}^d_S(k[t], S) \to \\ \mathrm{Ext}^{d+1}_S(k, S) \to 0 \to \cdots . \end{aligned}$$

By (48)

$$\mathrm{Ext}^d_S(k[t], S) = \mathrm{Ext}^d_R(k, S) = \mathrm{Ext}^d_R(k, \bigoplus_{n \geq 0} Rt^\nu) .$$

Since R is regular, $P_\cdot$ is finitely generated and the functor $N \mapsto \mathrm{Ext}^d_R(k, N)$ commutes with direct sums. Hence

$$\mathrm{Ext}^d_S(k[t], S) = \bigoplus_{n \geq 0} \mathrm{Ext}^d_R(k, R)t^\nu \approx k[t] .$$

This is a free $k[t]$-module of rank 1, via the right action of S on S. The map α in (50) is a $k[t]$-endomorphism of this module, so is multiplication by an element of $k[t]$.

To compute α we can use the resolution $S \otimes_R P_\cdot$. Let $\varphi : S \otimes_R P_\cdot \to S \otimes_R P_\cdot$ be a homomorphism of complexes inducing multiplication by t on $k[t] = H_0(S \otimes_R P_\cdot)$. Since t is homogeneous of degree δ we can suppose that φ is also. Since R is regular, P_d is free on one generator, e. Hence φ in dimension d is given by $\varphi(e) = a_\delta + a_0 t$ for some $a_\delta \in R_\delta$ and $a_0 \in R_0 = k$. Using φ to compute the map α, we find that α is given by multiplication by $a_0 t$.

Our problem is to show that $a_0 \neq 0$, for that will imply (c) via (50). To show $a_0 \neq 0$ we use φ to compute the endomorphism β of $\mathrm{Tor}^S_d(k[t], k[t]) \simeq k[t]$ which is induced by multiplication by t on the right-hand argument of Tor^S_d. We find that if $a_0 = 0$ then $\beta = 0$. But the kernel of β is a quotient of $\mathrm{Tor}^S_{d+1}(k[t], k)$ which is zero, because $k[t]$ as right S-module has projective dimension $\leq d$. Indeed, since S is free as right R-module, we get a free resolution of length d of the right S-module $k[t]$ by tensoring a resolution of the right R-module k with S.

Theorem 2 being proved, we now use the twisting method to reduce part (c) of Theorem 3 to the known case of $\mathcal{O}(\mathrm{GL}_n(q))$. This is done by the next proposition.

PROPOSITION 3. *Let $H = H(p, \lambda)$, and let $H' = H(rp, \lambda)$ be a twist of H, as in Theorem 4. Let D, D' be the determinant elements (16) in these bialgebras. If $H\langle D^{-1}\rangle$ has an antipode satisfying the conditions of Theorem 3(c), so does $H'\langle D'^{-1}\rangle$.*

Let M be the free abelian group with basis $\{t_i\}$ as above.

LEMMA 4. *Every class in $H^2(M, k^*)$ contains a cocycle c such that $c(\mu, \mu^{-1}) = 1$ for all $\mu \in M$.*

Proof: Order M in some way, e.g., lexicographically. Since M has no element of order 2, we can, given a cocycle c, define a 1-cochain $f\colon M \to k^*$ by

$$f(1) = 1$$

$$f(\mu) = \begin{cases} c(\mu, \mu^{-1}) & \text{if } \mu > \mu^{-1} \\ 1 & \text{if } \mu < \mu^{-1} \end{cases}$$

then $\delta f(\mu, \mu^{-1}) = c(\mu, \mu^{-1})$ for all μ, so $(c/\delta f)(\mu, \mu^{-1}) = 1$.

Thus we may view the bialgebra H' given in the statement of the proposition as the simultaneous twist of H by c^{-1} on the left and c on the right, by a cocycle c such that $c(\mu, \mu^{-1}) = 1$ for all $\mu \in M$. We do so, and we denote by $h \mapsto h'$ the canonical isomorphism of vector spaces $H \xrightarrow{\sim} H'$.

LEMMA 5. *Let* $\det u$ *denote the determinant element defined by* (16). *Then* $(\det u)' = \det(u')$ *in* H'.

Proof: The same argument which shows that the comultiplication $H \to H$ tensor H is unchanged by twisting can be used to show that the coaction $B^! \to H$ tensor $B^!$ is also unchanged, i.e., that with notation as in (14)

$$\eta'_J \to \sum_K U'_{JK} \otimes \eta'_K,$$

and in particular,

$$\eta'_I \to D' \otimes \eta'_I.$$

Since η'_I is a basis for $(B^!)_n$, the lemma follows.

From now on, we use the unambiguous notation $D = \det u$, $D' = \det u'$. Let $G = H\langle D^{-1}\rangle$, and let G' be the simultaneous twist of G by c^{-1} and c on left and right. The property $c(\mu, \mu^{-1}) = 1$ implies that $(g^{-1})' = (g')^{-1}$ for every invertible homogeneous element $g \in G$. Hence D' is invertible, and it follows immediately that $G' = H'\langle D'^{-1}\rangle$.

LEMMA 6. *Let $v = (v_{ij})$ denote the matrix inverse (19) of (u_{ij}). Then $v' = (u')^{-1}$.*

Proof: By (16) and (19), v_{jk} has left degree t_k^{-1} and right degree t_j^{-1}. Hence

$$\delta_{ik} = \sum (u_{ij}v_{jk})' = \sum_j c(t_i, t_k^{-1})^{-1} c(t_j, t_j^{-1}) u'_{ij} v'_{jk} = c(t_i, t_k)^{-1} \sum_j u'_{ij} v'_{jk},$$

which shows that

$$\delta_{ik} = \sum_j u'_{ij} v'_{jk} .$$

Suppose now that we know that there is an antiautomorphism $\psi : G \to G$ such that $\psi(u) = v$, $\psi(D) = D^{-1}$, and $\psi(v) = l^{-1}ul$, where l is the diagonal matrix with $l_{ii} = \lambda^i$. We will show that the same is true for G'. Since the elements u_{ij} satisfy the (p, λ)-relations (8), the elements u_{ij}^{opp} in the opposite ring satisfy the (p^{-1}, λ^{-1})-relations. So the existence of ψ implies that the elements v_{ij} satisfy the (p^{-1}, λ^{-1})-relations. By Theorem 4, the elements u'_{ij} satisfy the (p', λ)-relations, where $p'_{ij} = r_{ij}(c)p_{ij} = \rho_c(t_i, t_j)p_{ij}$. By the same argument, the elements v'_{ij} satisfy the (p'', λ^{-1})-relations, where $p''_{ij} = \rho_c(t_j^{-1}, t_i^{-1})p_{ij}^{-1}$. Since ρ_c is alternating and bimultiplicative, $\rho_c(t_j^{-1}, t_i^{-1}) = \rho_c(t_i, t_j)^{-1} = r_{ij}^{-1}(c)$. Hence $p'' = (p')^{-1}$ and it follows that there is an antihomorphism

$$\psi' : H' = k\langle u'\rangle \to k(v')$$

such that $\psi'(u') = v'$. Writing $g_1 \sim g_2$ for the relation $g_1k^* = g_2k^*$, we have $\psi'(h') \sim (\psi(h))'$ for all homogeneous elements $h \in H$ (proof by induction on the total degree of h). Hence

$$\psi'(D') \sim (\psi(D))' = (D^{-1})' = (D')^{-1} .$$

Thus $\psi'(D')$ is invertible, and so ψ' extends from H' to G'. Applying ψ' to the identity

$$\sum_j u'_{ij} v'_{jk} = \delta_{ik}$$

yields

$$\sum_j \psi'(v'_{jk}) v'_{ij} = \delta_{ik} .$$

This means that $\psi'(v')^t = 1$, or that

$$\psi'(v') = ((((u')^{-1})^t)^{-1})^t,$$

where t denotes "transpose." The same formula holds without the primes, in which case we know by hypothesis that the right side is lul^{-1}, i.e., its (i, j)-coefficient is $\lambda^{j-i}u_{ij}$. Since our twist preserves λ, the same is true after twisting.

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