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CANONICAL BASES FOR THE QUANTUM GROUP OF TYPE A_r AND PIECEWISE-LINEAR COMBINATORICS

ARKADY BERENSTEIN AND ANDREI ZELEVINSKY

Introduction. This work was motivated by the following two problems from the classical representation theory. (Both problems make sense for an arbitrary complex semisimple Lie algebra, but since we shall deal only with the A_r case, we formulate them in this generality.)

1. Construct a "good" basis in every irreducible finite-dimensional sl_{r+1}-module V_λ, which "materializes" the Littlewood-Richardson rule. A precise formulation of this problem was given in [3]; we shall explain it in more detail a bit later.

2. Construct a basis in every polynomial representation of GL_{r+1}, such that the maximal element w_0 of the Weyl group S_{r+1} (considered as an element of GL_{r+1}) acts on this basis by a permutation (up to a sign), and explicitly compute this permutation. This problem is motivated by recent work by John Stembridge [10] and was brought to our attention by his talk at the Jerusalem Combinatorics Conference, May, 1993.

We show that the solution to both problems is given by the same basis, obtained by the specialization $q = 1$ of Lusztig's *canonical* basis for the modules over the Drinfeld-Jimbo q-deformation U_r of $U(sl_{r+1})$. More precisely, we work with the basis dual to Lusztig's, and our main technical tool is the machinery of *strings* developed in [4]. The solution to Problem 1 appears below as Corollary 6.2, and the solution to Problem 2 is given by Proposition 8.8 and Corollary 8.9.

Now let us describe our results and their relationship with the preceding work in more detail. We start with Problem 1. The concept of "good bases" was introduced independently by K. Baclawski [1] and by I. M. Gelfand and A. Zelevinsky [6]. Technically speaking, in every irreducible sl_{r+1}-module V_λ there is a family of subspaces $V_\lambda(\beta; v)$ (their definition and its motivation can be found, e.g., in [3], [4], or in Section 6 below); a basis B_λ in V_λ is *good* if $B_\lambda \cap V_\lambda(\beta; v)$ is a basis in $V_\lambda(\beta; v)$ for every subspace $V_\lambda(\beta; v)$ in this family. As explained in [4], the results by G. Lusztig and M. Kashiwara allow us to construct good bases in the following way. First, each V_λ can be obtained by the specialization $q = 1$ from the corresponding irreducible U_r-module, which, with some abuse of notation, we shall denote by the same symbol V_λ. The concept of good bases generalizes to the U_r-modules, and a good basis in the U_r-module V_λ specializes to a good basis in the corresponding sl_{r+1}-module. Now consider the algebra $\mathscr{A} = \mathscr{A}_r$ over the field of rational functions $\mathbf{Q}(q)$, which is a q-deformation of the ring of regular

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functions on the maximal unipotent subgroup $N_+ \subset SL_{r+1}$. Every U_r-module V_λ has a canonical realization as a subspace of $\mathscr{A}$. There exists a basis B in $\mathscr{A}$ (for instance, the dual of Lusztig's canonical basis) such that $B_\lambda = B \cap V_\lambda$ is a good basis in V_λ for all λ. This basis B and the corresponding bases B_λ are the main objects of study in the present paper.

The basis B is naturally labeled by the so-called *A_r-partitions*. By an A_r-partition we mean a family of nonnegative integers $d = (d_{ij})_{(i,j) \in I_r}$, where the index set I_r consists of pairs of integers (i, j) such that $1 \leqslant i \leqslant j \leqslant r$. (The set I_r is in a natural bijection with the set of positive roots of type A_r, so we can think of A_r-partitions as partitions of weights into the sum of positive roots.) A labeling of B by A_r-partitions was already used by G. Lusztig (see, e.g., [8, Chapter 42]). We use a different approach which allows us to obtain much more explicit results. As an example, let us discuss the above-mentioned "materialization" of the Littlewood-Richardson rule. It is known (see, e.g., [3]) that for every three highest weights λ, μ, ν the multiplicity of V_μ in the tensor product $V_\lambda \otimes V_\nu$ is equal to the dimension of the subspace $V_\lambda(\mu - \nu; \nu) \subset V_\lambda$. So this multiplicity is equal to the number of A_r-partitions d such that the corresponding basis vector $b_d \in B$ lies in the subspace $V_\lambda(\mu - \nu; \nu) \subset V_\lambda \subset \mathscr{A}$. Corollary 6.2 below describes explicitly all such d, thus providing a combinatorial expression for the multiplicity of V_μ in $V_\lambda \otimes V_\nu$. This expression was earlier obtained in [3], where it was shown to be equivalent to the classical Littlewood-Richardson rule.

As for Problem 2 above, we put it in a more general context of studying various symmetries of the bases B_λ. There are several such symmetries, and we compute their action explicitly in terms of A_r-partitions. Our main technical tool is the study of certain continuous piecewise-linear transformations ("transition maps") acting in the space of A_r-partitions. This is what we mean by "piecewise-linear combinatorics" appearing in the title of this paper. Such combinatorics appeared already in Lusztig's work; we believe that it constitutes a natural combinatorial framework for the representation theory of quantum groups. This framework complements in a nice way the traditional machinery of Young tableaux.

Our solution of Problem 2 is a good illustration of the interaction between piecewise-linear and traditional combinatorics. First we compute in terms of A_r-partitions the action on B_λ of a certain involution $\eta_\lambda \colon V_\lambda \to V_\lambda$ that specializes at $q = 1$ to the action of w_0 (Theorem 7.2 below). Then we translate this description from the language of A_r-partitions to that of Young tableaux (actually, we use an equivalent language of Gelfand-Tsetlin patterns that suits better for our purposes). The final result (Theorem 8.2) is that under a natural parametrization of B_λ by Young tableaux of shape λ, the involution η_λ acts by the well-known Schützenberger involution. The combinatorial implications of these results are explored by J. Stembridge in [11].

The material is organized as follows. For the convenience of the reader we collect in Sections 1 and 2 the necessary material from [4]; all the facts we need about the structure of the Drinfeld-Jimbo algebra U_r and its irreducible modules V_λ are presented in Section 5. In Section 3 we compute the action on the basis B

of the natural group of symmetries isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. In Section 4 we compute explicitly the so-called *exponents* of the vectors from B; this calculation is crucial for our derivation of a "piecewise-linear Littlewood-Richardson rule." The rule itself appears in Section 6. In Section 7 we describe the "twist" of the basis vectors from B_λ under the action of three natural involutive automorphisms of U_r (these automorphisms together with the identity automorphism form another group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$). One of these twists is the involution η_λ mentioned above. Finally, in Section 8 we describe the relationship between A_r-partitions and Young tableaux; as a corollary, we show that w_0 acts on the canonical basis in every irreducible polynomial GL_{r+1}-module by means of the Schützenberger involution.

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1. The algebras $\mathscr{A}$ and U_+. We fix a positive integer r and let $\mathscr{A}$ denote the associative algebra with unit over the field of rational functions $\mathbf{Q}(q)$ generated by the elements $x_1, \dots, x_r$ subject to the relations

$$x_i x_j = x_j x_i \qquad \text{for } |i-j| > 1, \tag{1.1}$$

$$x_i^2 x_j - (q+q^{-1}) x_i x_j x_i + x_j x_i^2 = 0 \qquad \text{for } |i-j| = 1. \tag{1.2}$$

This is a *quantum deformation* (or q-deformation) of the algebra of polynomial functions on the group of upper unitriangular $(r+1) \times (r+1)$ matrices.

We also consider an isomorphic copy U_+ of $\mathscr{A}$ which is generated by $E_1, \dots, E_r$ satisfying the same relations as the x_i. This is a q-deformation of the universal enveloping algebra of the Lie algebra of nilpotent upper triangular $(r+1) \times (r+1)$ matrices.

Both algebras are graded by the semigroup Q_+ generated by simple roots $\alpha_1, \dots, \alpha_r$ of the root system of type A_r: we have $\deg x_i = \deg E_i = \alpha_i$. The homogeneous components of degree γ will be denoted by $\mathscr{A}(\gamma)$ and $U_+(\gamma)$.

These algebras are naturally dual to each other (as graded spaces), according to the following two propositions from [4].

Proposition 1.1 [4, Proposition 1.1]. *There exists a unique action $(E, x) \mapsto E(x)$ of the algebra U_+ on $\mathscr{A}$ satisfying the following properties:*

(a) *(Homogeneity) If $E \in U_+(\alpha), x \in \mathscr{A}(\gamma)$ then $E(x) \in \mathscr{A}(\gamma - \alpha)$;*

(b) *(Leibniz formula)*

$$E_i(xy) = E_i(x)y + q^{-(\gamma,\alpha_i)} x E_i(y) \qquad \textit{for } x \in \mathscr{A}(\gamma),\ y \in \mathscr{A};$$

(c) *(Normalization) $E_i(x_j) = \delta_{ij}$ for $i, j = 1, \dots, r$.*

PROPOSITION 1.2 [4, Proposition 1.2]. (a) *If* $\gamma \in Q_+ \backslash \{0\}$, *and* x *is a nonzero element of* $\mathscr{A}(\gamma)$ *then* $E_i(x) \neq 0$ *for some* $i = 1, \ldots, r$.

(b) *For every* $\gamma \in Q_+$ *the mapping* $(E, x) \mapsto E(x)$ *defines a nondegenerate pairing*

$$U_+(\gamma) \times \mathscr{A}(\gamma) \to \mathscr{A}(0) = \mathbf{Q}(q).$$

Here (γ, α) in the Leibniz formula is the usual scalar product on the weight space, so that $\|(\alpha_i, \alpha_j)\|$ is the Cartan matrix of type A_r.

2. The canonical basis in $\mathscr{A}$ and its string parametrizations. We recall from [4] that there is a distinguished basis B in $\mathscr{A}$ which is dual to Lusztig's canonical basis in U_+. This basis is a *string* basis in the terminology of [4].

It is shown in [4] that the elements of B can be labeled by certain integer sequences of length $(r(r+1))/2$ called *strings*. The string parametrization is associated to every reduced decomposition of w_0, the maximal element in the Weyl group. Let us reproduce the definition of strings from [4, Section 2]. Let x be a nonzero homogeneous element of $\mathscr{A}$. For each $i = 1, \ldots, r$ we set

$$l_i(x) = \max\{l \in \mathbf{Z}_+ \colon E_i^l(x) \neq 0\}; \tag{2.1}$$

we call $l_1(x), l_2(x), \ldots, l_r(x)$ the *exponents* of x. We shall use the following notation:

$$E_i^{(\mathrm{top})}(x) := E_i^{(l_i(x))}(x)$$

(here $E_i^{(l)}$ stands for the divided power, see [4]). Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a sequence of indices from $\{1, 2, \ldots, r\}$ such that no two consecutive indices are equal to each other. We associate to x and $\mathbf{i}$ a nonnegative integer vector $a(\mathbf{i}; x) = (a_1, \ldots, a_m)$ defined by

$$a_k = l_{i_k}(E_{i_{k-1}}^{(\mathrm{top})} E_{i_{k-2}}^{(\mathrm{top})} \cdots E_{i_1}^{(\mathrm{top})}(x)).$$

We call $a(\mathbf{i}; x)$ the *string* of x in direction $\mathbf{i}$. We abbreviate

$$E_{\mathbf{i}}^{(\mathrm{top})}(x) = E_{i_m}^{(\mathrm{top})} E_{i_{m-1}}^{(\mathrm{top})} \cdots E_{i_1}^{(\mathrm{top})}(x).$$

Note that $E_{\mathbf{i}}^{(\mathrm{top})}(x)$ is a nonzero homogeneous element of $\mathscr{A}$ of degree $\deg(x) - \sum_k a_k \alpha_{i_k}$.

Let $W = S_{r+1}$ be the Weyl group of type A_r. For each $w \in W$ we denote by $R(w)$ the set of all reduced decompositions of w, i.e., the set of sequences $\mathbf{i} = (i_1, i_2, \ldots, i_l)$ of the minimal possible length $l = l(w)$ such that w is equal to the product of simple reflections $s_{i_1} s_{i_2} \cdots s_{i_l}$. We are particularly interested in the reduced decompositions of w_0, the maximal element of W. We denote $m = l(w_0) = r(r+1)/2$.

PROPOSITION 2.1 [4, Theorems 2.3, 2.4]. *For every* $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in R(w_0)$, *we have* $E_{\mathbf{i}}^{(\mathrm{top})}(b) = 1$ *for all* $b \in B$. *Furthermore, the correspondence* $b \mapsto a(\mathbf{i}; b)$ *is a bijection between* B *and the semigroup* $C_{\mathbf{Z}}(\mathbf{i})$ *of all lattice points of some polyhedral convex cone* $C(\mathbf{i}) \subset \mathbf{R}_+^m$.

PROPOSITION 2.2. [4, Theorem 2.2]. *For every* $\mathbf{i}, \mathbf{i}' \in R(w_0)$ *there is a piecewise-linear automorphism* ${}_{\mathbf{i}'}T_{\mathbf{i}}\colon \mathbf{R}^m \to \mathbf{R}^m$ *preserving the lattice* $\mathbf{Z}^m$ *and such that* $a(\mathbf{i}'; b) = {}_{\mathbf{i}'}T_{\mathbf{i}}(a(\mathbf{i}; b))$ *for* $b \in B$.

The transition maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ can be computed as follows. It is well known that any two reduced decompositions of w_0 can be transformed into each other by a sequence of elementary transformations of two kinds:

$$(\mathbf{i}_1, i, j, \mathbf{i}_2) \mapsto (\mathbf{i}_1, j, i, \mathbf{i}_2) \qquad \text{for } |i-j| > 1, \tag{2.2}$$

$$(\mathbf{i}_1, i, j, i, \mathbf{i}_2) \mapsto (\mathbf{i}_1, j, i, j, \mathbf{i}_2) \qquad \text{for } |i-j| = 1. \tag{2.3}$$

PROPOSITION 2.3 [4, Theorem 2.7]. *Suppose* $\mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_p = \mathbf{i}'$ *are reduced decompositions of* w_0 *such that for every* $t = 1, \ldots, p$ *the transition from* $\mathbf{i}_{t-1}$ *to* $\mathbf{i}_t$ *is of the form (2.2) or (2.3).*

(a) *We have*

$${}_{\mathbf{i}'}T_{\mathbf{i}} = {}_{\mathbf{i}_p}T_{\mathbf{i}_{p-1}} \circ \cdots \circ {}_{\mathbf{i}_2}T_{\mathbf{i}_1}.$$

(b) *If the transition from* $\mathbf{i}_{t-1}$ *to* $\mathbf{i}_t$ *is of the form (2.2) with* i, j *occupying positions* $k, k+1$, *then* ${}_{\mathbf{i}_t}T_{\mathbf{i}_{t-1}}$ *leaves all the components of a string* a *except* a_k, a_{k+1} *unchanged and changes* (a_k, a_{k+1}) *to* (a_{k+1}, a_k).

(c) *If the transition from* $\mathbf{i}_{t-1}$ *to* $\mathbf{i}_t$ *is of the form (2.3) with* i, j, i *occupying positions* $k, k+1, k+2$, *then* ${}_{\mathbf{i}_t}T_{\mathbf{i}_{t-1}}$ *leaves all the components of a string* a *except* a_k, a_{k+1}, a_{k+2} *unchanged and changes* (a_k, a_{k+1}, a_{k+2}) *to*

$$(\max(a_{k+2}, a_{k+1} - a_k), a_k + a_{k+2}, \min(a_k, a_{k+1} - a_{k+2})).$$

The most important for us will be the following reduced decomposition of w_0:

$$\mathbf{i}(1) = (1; 2, 1; 3, 2, 1; \ldots; r, r-1, \ldots, 1).$$

We abbreviate $\Gamma := C_{\mathbf{Z}}(\mathbf{i}(1))$. This semigroup has the following explicit description.

PROPOSITION 2.4. [4, Theorem 2.5]. *The semigroup* Γ *is the set of all integer sequences*

$$(a_{11}; a_{22}, a_{12}; a_{33}, a_{23}, a_{13}; \ldots; a_{rr}, \ldots, a_{1r})$$

such that $a_{jj} \geqslant a_{j-1,j} \geqslant \cdots \geqslant a_{1j} \geqslant 0$ *for all* $j = 1, \ldots, r$.

Note that we have chosen a double indexation for the strings from Γ. Let $I = I_r = \{(i, j) : 1 \leqslant i \leqslant j \leqslant r\}$ be the index set for our numeration. It can be identified with the set of positive roots of type A_r via

$$(i, j) \mapsto \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j. \tag{2.4}$$

Let $\mathbf{Z}^I$ be the lattice of families $(d_{ij})_{(i,j) \in I}$ of integers indexed by I, and let $\mathbf{Z}^I_+ \subset \mathbf{Z}^I$ be the semigroup formed by all families (d_{ij}) of nonnegative integers. We call elements of $\mathbf{Z}^I_+$ *A_r-partitions* (in [7] they were called *multisegments*). We define a map $\partial \colon \Gamma \to \mathbf{Z}^I_+$ by

$$\partial(a)_{ij} = a_{ij} - a_{i-1,j}. \tag{2.5}$$

For $x \in \mathscr{A}$ we abbreviate $\partial(x) := \partial(a(\mathbf{i}(1); x))$. The following proposition is straightforward.

PROPOSITION 2.5. *The map ∂ is a semigroup isomorphism between Γ and $\mathbf{Z}^I_+$. Thus, the mapping $b \mapsto \partial(b)$ is a bijection between B and $\mathbf{Z}^I_+$. Furthermore, if x is a homogeneous element of $\mathscr{A}$ and $(d_{ij}) = \partial(x)$, then the degree of x is equal to $\sum_{(i,j) \in I} d_{ij}\alpha_{ij}$.*

According to Proposition 2.5, the elements of B can be labeled by A_r-partitions $d \in \mathbf{Z}^I_+$. For $d \in \mathbf{Z}^I_+$ we shall denote by b_d the element of B with $\partial(b_d) = d$.

Clearly, the inverse bijection $\partial^{-1} \colon \mathbf{Z}^I_+ \to \Gamma$ is given by

$$(\partial^{-1}(d))_{ij} = d_{1j} + d_{2j} + \cdots + d_{ij}. \tag{2.6}$$

3. The action of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on B. Let $x \mapsto x^*$ be the antiautomorphism of $\mathscr{A}$ such that $x_i^* = x_i$ for $i = 1, \ldots, r$. Let $x \mapsto \hat{x}$ be the antiautomorphism of $\mathscr{A}$ such that $\hat{x}_i = x_{r+1-i}$ for $i = 1, \ldots, r$. Clearly, both maps are involutions and commute with each other. Thus four transformations Id, $x \mapsto x^*$, $x \mapsto \hat{x}$, $x \mapsto \hat{x}^*$ form a group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Moreover, it follows from the results of Lusztig (see [8, Section 14.4]) that all these transformations preserve B. So they give an action of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on B. Using the parametrization of B by $\mathbf{Z}^I_+$ described above, we obtain the action of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on $\mathbf{Z}^I_+$. Three nontrivial elements of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ act on $\mathbf{Z}^I_+$ by the mutually commuting involutions $d \mapsto d^*$, $d \mapsto \hat{d}$, $d \mapsto \hat{d}^*$ defined by

$$b_d^* = b_{d^*}, \qquad \hat{b}_d = b_{\hat{d}}, \qquad \hat{b}_d^* = b_{\hat{d}^*}. \tag{3.1}$$

We shall describe these involutions quite explicitly. We start with $d \mapsto \hat{d}$, which turns out to be a permutation of the components d_{ij}. We shall use the notation $\hat{\imath} := r + 1 - i$.

PROPOSITION 3.1. *We have $\hat{d}_{ij} = d_{\hat{\jmath}\hat{\imath}}$ for all $d \in \mathbf{Z}^I_+$.*

Proof. For $d = (d_{ij}) \in \mathbf{Z}_+^I$ we denote temporarily by $d^\dagger$ the A_r-partition given by $d_{ij}^\dagger = d_{\hat{j}\hat{i}}$; so we have to prove that $\hat{d} = d^\dagger$.

We shall use the Poincaré-Birkhoff-Witt type bases in $\mathscr{A}$ constructed in [4]. For each $(i, j) \in I$ we set

$$x_{ij} = [x_i, \ldots [x_{j-2}, [x_{j-1}, x_j]] \ldots], \tag{3.2}$$

where $[x, y]$ for $x \in \mathscr{A}(\gamma), y \in \mathscr{A}(\gamma')$ is the *q-commutator* defined by

$$[x, y] = \frac{xy - q^{(\gamma,\gamma')}yx}{q - q^{-1}} \tag{3.3}$$

(see [4, (1.5)]); note that in [4] x_{ij} was denoted by $t_{i,j+1}$. The elements x_{ij} satisfy the following commutation relations which are special cases of [4], Proposition 3.11:

$$[x_{ij}, x_{i',j'}] = 0 \qquad \text{for } i < i' \leqslant j' \leqslant j, \tag{3.4}$$

$$[x_{ij}, x_{j+1,k}] = x_{i,k} \qquad \text{for } i \leqslant j < k. \tag{3.5}$$

Iterating (3.5), we get

$$x_{ij} = [\ldots [[x_i, x_{i+1}], x_{i+2}] \ldots, x_j]. \tag{3.6}$$

For $d = (d_{ij}) \in \mathbf{Z}_+^I$ we set

$$x^d = x_{11}^{d_{11}} x_{12}^{d_{12}} x_{22}^{d_{22}} \cdots x_{1r}^{d_{1r}} \cdots x_{rr}^{d_{rr}}. \tag{3.7}$$

Note that the order of factors here differs from that used in the definition of t^d in [4, Section 3], which was just the order in Proposition 2.4 above; however, (3.4) implies that the products taken in these two orders differ from each other only by a multiple which is a power of q. The advantage of the present order is clear from the following result.

LEMMA 3.2. *We have $\widehat{x^d} = x^{d^\dagger}$ for all $d \in \mathbf{Z}_+^I$.*

Proof. First of all, applying the antiautomorphism $x \mapsto \hat{x}$ to both sides of (3.2) and using (3.6) we see that $\widehat{x_{ij}} = x_{\hat{j}\hat{i}}$. It follows that $\widehat{x^d}$ is the product of the same factors as $x^{d^\dagger}$ but taken in the different order:

$$\widehat{x^d} = x_{11}^{d_{11}^\dagger} x_{12}^{d_{12}^\dagger} \cdots x_{1r}^{d_{1r}^\dagger} x_{22}^{d_{22}^\dagger} \cdots x_{2r}^{d_{2r}^\dagger} \cdots x_{rr}^{d_{rr}^\dagger}. \tag{3.8}$$

The orders in (3.7) and (3.8) differ as follows: the term with x_{ij} precedes the term with $x_{i'j'}$ in (3.7) but goes after it in (3.8) if and only if $i' < i \leqslant j < j'$. In view of (3.4), such terms commute with each other (the q-commutator in this case coincides with the ordinary commutator since the degrees of x_{ij} and $x_{i'j'}$ are orthogonal to each other). Thus, by interchanging some commuting terms in the product in (3.8) we can put the factors in the same order as in (3.7). This proves our lemma.

Consider the following linear order on $\mathbf{Z}_+^I$: we say that $d' \prec d$ if and only if $\partial^{-1}(d')$ precedes $\partial^{-1}(d)$ in the lexicographic order. Then the results in [4] imply that the expansion of x^d in the basis B has the following form:

$$x^d = q^{n(d)} b_d + \sum_{d' \prec d} c_{dd'} b_{d'}, \tag{3.9}$$

where $n(d)$ is some integer. (To prove (3.9) we notice that $\partial(x^d) = d$, which follows from [4, (3.5) and (3.8)], and then apply [4, Proposition 4.1].)

Now we can complete the proof of Proposition 3.1. Applying the map $x \mapsto \hat{x}$ to both sides of (3.9) and using Lemma 3.2, we get

$$x^{d^\dagger} = q^{n(d)} b_{\hat{d}} + \sum_{d' \prec d} c_{dd'} b_{\hat{d}'}. \tag{3.10}$$

Comparing (3.10) with the expansion of $x^{d^\dagger}$ given by (3.9), we conclude that $\hat{d} \preceq d^\dagger$. In other words, we have $\hat{d^\dagger} \preceq d$ for all $d \in \mathbf{Z}_+^I$. But the map $d \mapsto \hat{d^\dagger}$ is a weight-preserving bijection of the set $\mathbf{Z}_+^I$ with itself. Restricting this bijection to every weight component of $\mathbf{Z}_+^I$, we obtain a bijection of a *finite* linearly ordered set with itself such that $\hat{d^\dagger} \preceq d$ for all d. Clearly, such a bijection is the identity map, so we have $\hat{d^\dagger} = d$ for all d. Hence $\hat{d} = d^\dagger$. Proposition 3.1 is proved.

The involution $d \mapsto \hat{d}^*$ can be expressed in terms of the transition operators ${}_{\mathbf{i}'}T_{\mathbf{i}}$ introduced in Section 2. For every sequence $\mathbf{i}$ of indices from $[1, r]$ let $\widehat{\mathbf{i}^*}$ denote the sequence obtained from $\mathbf{i}$ by replacing each term i by $\hat{\imath} = r + 1 - i$ (leaving the terms in the same order). Clearly, if $\mathbf{i} \in R(w_0)$ then $\widehat{\mathbf{i}^*} \in R(w_0)$; in particular, $\widehat{\mathbf{i}(1)^*} \in R(w_0)$. Following [4], we denote $\widehat{\mathbf{i}(1)^*}$ by $\mathbf{i}(r)$, so

$$\mathbf{i}(r) = (r; r-1, r; \ldots; 1, 2, \ldots, r). \tag{3.11}$$

PROPOSITION 3.3. *We have*

$$\hat{d}^* = (\partial \circ {}_{\mathbf{i}(r)}T_{\mathbf{i}(1)} \circ \partial^{-1})(d)$$

for all $d \in \mathbf{Z}_+^I$.

Proof. We start with the following observation. Let $E \mapsto E^*$ and $E \mapsto \hat{E}$ be the antiautomorphisms of U_+ defined in the same way as the antiautomorphisms $x \mapsto x^*$ and $x \mapsto \hat{x}$ of $\mathscr{A}$, that is, $E_i^* = E_i$, $\hat{E}_i = E_{\hat{\imath}}$ for $i = 1, \ldots, r$. Then for every two homogeneous elements $E \in U_+$, $x \in \mathscr{A}$ of the same degree, we have

$$E(x) = E^*(x^*) = \hat{E}(\hat{x}) = \hat{E}^*(\hat{x}^*). \tag{3.12}$$

The first equality in (3.12) is proved in [4, Proposition 3.10]; the other equalities are proved in exactly the same way.

Now since both maps $x \mapsto \hat{x}^*$ and $E \mapsto \hat{E}^*$ are automorphisms, the last equality in (3.12) and the definition of strings imply that

$$a(\widehat{\mathbf{i}^*}; \hat{x}^*) = a(\mathbf{i}; x) \tag{3.13}$$

for all $\mathbf{i} \in R(w_0)$ and homogeneous $x \in \mathscr{A}$. Applying (3.13) to $\mathbf{i} = \mathbf{i}(r)$ and $b = b_d$ for $d \in \mathbf{Z}_+^I$, we obtain our proposition.

Remark. It can be shown that the involution $d \mapsto d^*$ coincides with the *multisegment duality* ζ, studied recently in [7]. The main result of [7] is an explicit formula for ζ. Comparing this with Propositions 3.1 and 3.3 yields an explicit formula for ${}_{\mathbf{i}(r)}T_{\mathbf{i}(1)}$, which should be helpful for understanding the linearity domains of this piecewise-linear map.

4. The exponents. In this section we compute the *exponents* (see (2.1) above) of the basis vectors from B.

THEOREM 4.1. *For every $d = (d_{ij}) \in \mathbf{Z}_+^I$ the exponents of the corresponding basis vector $b_d \in B$ are given by*

$$l_j(b_d) = \max_{1 \leqslant i \leqslant j} \left(\sum_{h=1}^{i} d_{hj} - \sum_{h=1}^{i-1} d_{h,j-1} \right) \tag{4.1}$$

for $j = 1, \dots, r$.

Proof. Let $a = (a_{11}; a_{22}, a_{12}; \dots; a_{rr}, \dots, a_{1r})$ be a string from Γ. By slight abuse of notation, we shall write $l_j(a)$ for $l_j(b_{\partial(a)})$. Taking into account (2.6), we see that (4.1) is equivalent to

$$l_j(a) = \max(a_{1j}, a_{2j} - a_{1,j-1}, a_{3j} - a_{2,j-1}, \dots, a_{jj} - a_{j-1,j-1}). \tag{4.2}$$

By the definition of strings, $l_j(a)$ is the first component of the string ${}_{\mathbf{i}}T_{\mathbf{i}(1)}(a)$ for any $\mathbf{i} \in R(w_0)$ which starts with j. So our strategy in proving (4.2) will be to choose some $\mathbf{i} \in R(w_0)$ starting with j, and to compute the first component of ${}_{\mathbf{i}}T_{\mathbf{i}(1)}(a)$ by using Proposition 2.3. In doing this, we can assume without loss of generality that $j = r$ (because in computing $l_j(a)$ we can just ignore the components a_{ik} of a with $k > j$).

For every sequence $\mathbf{i} = (i_1, \dots, i_l)$ of indices let $\mathbf{i}^* = (i_l, \dots, i_1)$ denote the same sequence written in the reverse order. Clearly, if $\mathbf{i} \in R(w_0)$ then $\mathbf{i}^* \in R(w_0)$ as well.

LEMMA 4.2. *The transformation ${}_{\mathbf{i}(1)^*}T_{\mathbf{i}(1)}$ can be decomposed into a composition of several transformations of the type described in Proposition 2.3 (b). For every string $a = (a_{11}; a_{22}, a_{12}; \dots; a_{rr}, \dots, a_{1r})$, we have*

$${}_{\mathbf{i}(1)^*}T_{\mathbf{i}(1)}(a) = (a_{11}, a_{22}, \dots, a_{rr}; a_{12}, a_{23}, \dots, a_{r-1,r}; \dots; a_{1,r-1}, a_{2r}; a_{1r}). \tag{4.3}$$

Proof of Lemma 4.2. We shall use the following notation from [4, Section 3]: for $i < j$ let $\overline{i, j}$ stand for the sequence $i, i+1, \ldots, j$, and $\overline{j, i}$ stand for the sequence $j, j-1, \ldots, i$. In this notation, we have

$$\mathbf{i}(1) = (1, \overline{2,1}, \ldots, \overline{r,1}), \qquad \mathbf{i}(1)^* = (\overline{1,r}, \overline{1,r-1}, \ldots, \overline{1,2}, 1).$$

Let us write $\mathbf{i}(1)$ as $\mathbf{i}(1) = (\mathbf{i}'(1), \overline{r,1})$, where $\mathbf{i}'(1)$ is the element of the same kind as $\mathbf{i}(1)$ but with r replaced by $r-1$. In order to transform $\mathbf{i}(1)$ into $\mathbf{i}(1)^*$, we first ignore the last group $\overline{r,1}$ in $\mathbf{i}(1)$ and transform $\mathbf{i}(1)$ to $(\mathbf{i}'(1)^*, \overline{r,1})$. Using induction on r, we can assume that this can be done by a chain of transformations of type (2.2), and that the corresponding transformation of strings is

$$_{(\mathbf{i}'(1)^*, \overline{r,1})}T_{\mathbf{i}(1)}(a) = (a_{11}, a_{22}, \ldots, a_{r-1,r-1}; a_{12}, a_{23}, \ldots, a_{r-2,r-1}; \ldots; a_{1,r-1};$$
$$a_{rr}, a_{r-1,r}, \ldots, a_{1r}). \tag{4.4}$$

It remains to transform

$$(\mathbf{i}'(1)^*, \overline{r,1}) = (\overline{1,r-1}, \overline{1,r-2}, \ldots, \overline{1,2}, 1, \overline{r,1})$$

into $\mathbf{i}(1)^*$. To do this, we move the term r of the last group $\overline{r,1}$ to the left, interchanging it with its left neighbors until it arrives at the end of the first group $\overline{1,r-1}$. Then we do the same thing with the terms $r-1, r-2, \ldots, 2$ of the last group, moving each of them to the end of the corresponding group $\overline{1,r-2}$, $\overline{1,r-3}, \ldots, 1$. This sequence of moves transforms the string in (4.4) into that in the right-hand side of (4.3), which completes the proof of Lemma 4.2.

Now we can finish the proof of Theorem 4.1. Let us write $\mathbf{i}(1)^*$ as $\mathbf{i}(1)^* = (\overline{1,r}, \mathbf{i}'(1)^*)$ and transform it into $(\overline{1,r}, \mathbf{i}'(1))$ by using the transformation $\mathbf{i}'(1)^* \mapsto \mathbf{i}'(1)$ inverse to that in Lemma 4.2 (with r replaced by $r-1$). Computing the inverse transformation to that given by (4.3) we see that $_{(\overline{1,r}, \mathbf{i}'(1))}T_{\mathbf{i}(1)^*}$ takes the string in (4.3) to the string $(a_{11}, a_{22}, \ldots, a_{rr}; a')$, where

$$a' = (a_{12}; a_{23}, a_{13}; \ldots; a_{r-1,r}, a_{r-2,r}, \ldots, a_{1r}).$$

Now let us recall that our goal is to prove (4.2). Using induction on r we can assume that (4.2) holds for $j = r-1$ and a replaced by a'. This gives

$$l_{r-1}(a') = \max(a_{1r}, a_{2r} - a_{1,r-1}, a_{3r} - a_{2,r-1}, \ldots, a_{r-1,r} - a_{r-2,r-1}). \tag{4.5}$$

Therefore we can find a reduced decomposition of w_0 of the form $(\overline{1,r}, \mathbf{i}')$ such that $\mathbf{i}'$ begins with $r-1$, and the string $_{(\overline{1,r}, \mathbf{i}')}T_{(\overline{1,r}, \mathbf{i}'(1))}(a_{11}, a_{22}, \ldots, a_{rr}; a')$ begins with $a_{11}, a_{22}, \ldots, a_{rr}, l_{r-1}(a')$. To conclude the proof, we concentrate on the first $r+1$ terms of $(\overline{1,r}, \mathbf{i}')$, which are $1, 2, \ldots, r-1, r, r-1$. We transform this sequence into

$r, 1, 2, \ldots, r-1, r$ by first performing the operation $(r-1, r, r-1) \mapsto (r, r-1, r)$ of type (2.3) and then moving the first term r to the left interchanging it with $r-2, r-3, \ldots, 1$. Using Proposition 2.3(c), (b) we see that these operations transform the string ${}_{(\overline{1,r},\mathbf{i}')}T_{\mathbf{i}(1)}$ which begins with $a_{11}, a_{22}, \ldots, a_{rr}, l_{r-1}(a')$ into the string whose first term is $\max(l_{r-1}(a'), a_{rr} - a_{r-1,r-1})$. By definition, this first term is equal to $l_r(a)$, which implies (4.2) since $l_{r-1}(a')$ is given by (4.5). This completes the proof of (4.2) and hence of Theorem 4.1.

Combining Theorem 4.1 with the results in Section 3, we obtain a formula for the exponents $l_i(b_d^*)$ in terms of the A_r-partition d.

COROLLARY 4.3. *For every $d = (d_{ij}) \in \mathbf{Z}_+^I$ the exponents of the basis vector $b_d^* \in B$ are given by*

$$l_i(b_d^*) = \max_{i \leqslant j \leqslant r} \left(\sum_{k=j}^{r} d_{ik} - \sum_{k=j+1}^{r} d_{i+1,k} \right) \tag{4.6}$$

for $i = 1, \ldots, r$.

Proof. It follows from (3.13) that

$$l_{\hat{i}}(\hat{x}^*) = l_i(x) \tag{4.7}$$

for $i = 1, \ldots, r$ and any homogeneous $x \in \mathscr{A}$. Therefore, we have $l_i(b_d^*) = l_{\hat{i}}(\hat{b}_d) = l_{\hat{i}}(b_{\hat{d}})$. We see that $l_i(b_d^*)$ is given by (4.1) with j replaced by $\hat{i}$ and d replaced by $\hat{d}$. Substituting into (4.1) the expressions for the components of $\hat{d}$ given by Proposition 3.1, we obtain (4.6).

5. The q-analog of sl_{r+1} and its irreducible modules. According to Drinfeld and Jimbo, the q-analog of the universal enveloping algebra of sl_{r+1} is the $\mathbf{Q}(q)$-algebra with unit U_r generated by the elements F_i, K_i, K_i^{-1}, E_i for $i = 1, \ldots, r$ subject to the following relations:

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1; \tag{5.1}$$

$$K_i F_j = q^{-(\alpha_i, \alpha_j)} F_j K_i, \qquad K_i E_j = q^{(\alpha_i, \alpha_j)} E_j K_i \qquad \text{for all } i, j; \tag{5.2}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}; \tag{5.3}$$

$$F_i F_j = F_j F_i, \qquad E_i E_j = E_j E_i \qquad \text{for } |i - j| > 1; \tag{5.4}$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0,$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \qquad \text{for } |i - j| = 1. \tag{5.5}$$

The algebra U_+ introduced above, can and will be identified with the $\mathbf{Q}(q)$-subalgebra of U_r generated by all E_i and 1.

Let us recall some well-known properties of finite-dimensional U_r-modules. All the proofs can be found in [8].

Let P be the weight lattice of the root system of type A_r, that is,

$$P = \{\beta \in Q_{\mathbf{Q}}\colon (\beta, \alpha) \in \mathbf{Z} \text{ for all } \alpha \in Q\},$$

where Q is the root lattice and $Q_{\mathbf{Q}} = Q \otimes \mathbf{Q}$. Let $P_+ \subset P$ be the semigroup of dominant weights, that is, the semigroup generated by fundamental weights $\omega_1, \ldots, \omega_r$, where we have $(\omega_i, \alpha_j) = \delta_{ij}$. For a U_r-module V the *weight component* $V(\beta)$ of weight $\beta \in P$ is defined as

$$V(\beta) = \{v \in V\colon K_i(v) = q^{(\beta, \alpha_i)}v,\ i = 1, \ldots, r\}. \tag{5.6}$$

We will only consider diagonalizable finite-dimensional U_r-modules V, i.e., such that V is the (direct) sum of its weight components. Every such module is completely reducible, and the classification of irreducible modules coincides with that for sl_{r+1}-modules (see [8], Chapter 6). Thus, to every $\lambda \in P_+$ there corresponds an irreducible finite-dimensional U_r-module V_λ with highest weight λ (so the component $V_\lambda(\lambda)$ is 1-dimensional and is annihilated by all the E_i). The modules V_λ are nonisomorphic and exhaust all irreducible diagonalizable finite-dimensional U_r-modules.

Now we fix a weight $\lambda = l_1\omega_1 + \cdots + l_r\omega_r \in P_+$. We shall use an explicit realization of V_λ as a subspace of $\mathscr{A}$ given by the following proposition.

PROPOSITION 5.1. (a) *The action of U_+ on $\mathscr{A}$ given by Proposition 1.1 extends to the action of the whole algebra U_r on $\mathscr{A}$ which is given by*

$$K_i(x) = q^{l_i - (\gamma, \alpha_i)}x, \tag{5.7}$$

$$F_i(x) = \frac{q^{l_i}xx_i^* - q^{(\gamma, \alpha_i) - l_i}x_i x}{q - q^{-1}} \tag{5.8}$$

for $x \in \mathscr{A}(\gamma)$, $i = 1, \ldots, r$.

(b) *The elements $x \in \mathscr{A}$ such that $E_i^{(l_i+1)}(x^*) = 0$ for $i = 1, \ldots, r$ form a U_r-submodule of the module $\mathscr{A}$ under the action in (a) (here $x \mapsto x^*$ is the anti-automorphism introduced in Section 3). This submodule is isomorphic to V_λ.*

This proposition can be deduced from the results in [8] in the following way. In [8], the irreducible module V_λ is realized as a quotient of the Verma module M_λ (see [8, 3.4.5 and Propositions 3.5.6, 6.3.4, 6.3.5]). A direct check shows that the U_r-module $\mathscr{A}$ described in (a) is obtained from M_λ by passing to the dual module and twisting it by the involutive automorphism φ of U_r given by $\varphi(E_i) = F_i$, $\varphi(F_i) = E_i$, $\varphi(K_i) = K_i^{-1}$ (for the definition of the twisting see [8, 3.4.4] or

Section 7 below). The same operation of passing to the dual and twisting by φ transforms the quotient V_λ of M_λ to the submodule of $\mathscr{A}$ described in (b). It remains to observe that this operation transforms V_λ to a module isomorphic to itself. We leave the details of this argument to the reader.

By some abuse of notation, we shall write

$$V_\lambda = \{x \in \mathscr{A}\colon E_i^{(l_i+1)}(x^*) = 0 \text{ for } i = 1, \ldots, r\}, \tag{5.9}$$

with the understanding that the action of U_r on V_λ is that in Proposition 5.1(a). Note that the weight components of V_λ are given by

$$V_\lambda(\beta) = V_\lambda \cap \mathscr{A}(\lambda - \beta). \tag{5.10}$$

In particular, the highest vector of V_λ is just $1 \in \mathscr{A}(0)$.

6. The canonical basis in V_λ. We retain the notation of the previous sections. So V_λ is an irreducible U_r-module with the highest weight $\lambda = l_1\omega_1 + \cdots + l_r\omega_r \in P_+$. We set $B_\lambda = B \cap V_\lambda$, where B is the canonical basis in $\mathscr{A}$. We recall from Section 2 that the vectors from B are labeled by A_r-partitions $d = (d_{ij}) \in \mathbf{Z}_+^I$. Combining (5.9) and (4.6), we get

$$B_\lambda = \left\{ b_d\colon \sum_{k=j}^{r} d_{ik} - \sum_{k=j+1}^{r} d_{i+1,k} \leqslant l_i \text{ for } 1 \leqslant i \leqslant j \leqslant r \right\}. \tag{6.1}$$

For every $\beta \in P$ and $\nu = n_1\omega_1 + \cdots + n_r\omega_r \in P_+$, we set

$$V_\lambda(\beta; \nu) = \{x \in V_\lambda(\beta)\colon E_i^{(n_i+1)}(x) = 0 \text{ for } i = 1, \ldots, r\}, \tag{6.2}$$

where $V_\lambda(\beta)$ is the component of weight β in V_λ.

PROPOSITION 6.1. *For every $\lambda, \nu \in P_+$, $\beta \in P$, the set $B_\lambda \cap V_\lambda(\beta; \nu)$ is a basis in $V_\lambda(\beta; \nu)$.*

Proof. As in [4, (4.1)], for every $\gamma \in Q_+$ and $\nu \in P_+$, we set

$$\mathscr{A}(\gamma; \nu) = \{x \in \mathscr{A}(\gamma)\colon E_i^{(n_i+1)}(x) = 0 \text{ for } i = 1, \ldots, r\}.$$

By [4, Proposition 4.6], each subspace $\mathscr{A}(\gamma; \nu) \subset \mathscr{A}$ is spanned by its intersection with B. Since B is invariant under the antiautomorphism $x \mapsto x^*$, every subspace $\mathscr{A}(\gamma; \nu)^*$ is also spanned by its intersection with B. This implies our statement since, in view of (5.9) and (5.10), we have

$$V_\lambda(\beta; \nu) = \mathscr{A}(\lambda - \beta; \nu) \cap \mathscr{A}(\lambda - \beta; \lambda)^*.$$

In view of Proposition 2.5 and (4.1), we can reformulate Proposition 6.1 as follows.

COROLLARY 6.2. *The space $V_\lambda(\beta; \nu)$ has as a basis the set of elements b_d, where d runs over all A_r-partitions satisfying three conditions:*

$$\sum_{i,j} d_{ij}\alpha_{ij} = \lambda - \beta, \tag{6.3}$$

$$\sum_{k=j}^{r} d_{ik} - \sum_{k=j+1}^{r} d_{i+1,k} \leqslant l_i \qquad \text{for } 1 \leqslant i \leqslant j \leqslant r. \tag{6.4}$$

$$\sum_{h=1}^{i} d_{hj} - \sum_{h=1}^{i-1} d_{h,j-1} \leqslant n_j \qquad \text{for } 1 \leqslant i \leqslant j \leqslant r. \tag{6.5}$$

Corollary 6.2 allows us to "materialize" the Littlewood-Richardson rule along the lines of [3]. To do this we notice that the specialization $q = 1$ makes V_λ into an irreducible sl_{r+1}-module with highest weight λ, which we will (with some abuse of notation) denote also by V_λ. More precisely, it is known (see [8, Chapter 22]) that all the matrix entries of the operators E_i and F_i (acting on V_λ) in the basis B_λ are rational functions in q regular at $q = 1$. Therefore, we can define the operators e_i and f_i by specializing the matrices of E_i and F_i at $q = 1$, and let h_i act on each weight component $V_\lambda(\beta)$ by multiplication by (β, α_i). Then the relations (5.1)–(5.5) imply that the e_i, f_i and h_i satisfy the commutation relations among the Cartan generators of sl_{r+1}. This makes the $\mathbf{C}$-space with the basis B_λ an sl_{r+1}-module.

Under this specialization, the subspace $V_\lambda(\beta; \nu) \subset V_\lambda$ (or rather its $\mathbf{C}$-form) becomes the space of vectors of weight β in V_λ annihilated by the operators $e_i^{n_i+1}$, $i = 1, \ldots, r$. It is well known that the dimension of this space is equal to the multiplicity of the irreducible sl_{r+1}-module $V_{\nu+\beta}$ in the tensor product $V_\lambda \otimes V_\nu$ (see [3]). Thus, Corollary 6.2 implies that this multiplicity is equal to the number of A_r-partitions d satisfying (6.3)–(6.5). This statement was established in [3] in a combinatorial way, essentially by showing that it is equivalent to the classical Littlewood-Richardson rule. Proposition 6.1 and Corollary 6.2 provide us with a representation-theoretic proof of this result.

Note also that the map $x \mapsto x^*$ induces an isomorphism of vector spaces $V_\lambda(\lambda - \gamma; \nu)$ and $V_\nu(\nu - \gamma; \lambda)$ for every $\lambda, \nu \in P_+$, $\gamma \in P$. Thus, the involution $d \mapsto d^*$ on A_r-partitions provides a bijective proof of the fact that our expressions for the multiplicity of $V_{\lambda+\nu-\gamma}$ in $V_\lambda \otimes V_\nu$ and the multiplicity of $V_{\lambda+\nu-\gamma}$ in $V_\nu \otimes V_\lambda$ give the same answer.

7. Twisting B_λ by the automorphisms of U_r. The commutation relations (5.1)–(5.5) imply that there exist three $\mathbf{Q}(q)$-linear involutive automorphisms φ,

ψ and η of the algebra U_r acting on the generators as follows:

$$\varphi(E_i) = F_i, \qquad \varphi(F_i) = E_i, \qquad \varphi(K_i) = K_i^{-1}; \tag{7.1}$$

$$\psi(E_i) = E_{\hat{i}}, \qquad \psi(F_i) = F_{\hat{i}}, \qquad \psi(K_i) = K_{\hat{i}}; \tag{7.2}$$

$$\eta(E_i) = F_{\hat{i}}, \qquad \eta(F_i) = E_{\hat{i}}, \qquad \eta(K_i) = K_{\hat{i}}^{-1}, \tag{7.3}$$

where $\hat{i} = r + 1 - i$. Clearly, these three automorphisms together with the identity automorphism form a group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. It turns out that each of these automorphisms induces some transformation ("twist") of the bases B_λ; in this section we shall compute this twist explicitly.

The general setup is as follows. If V is a module over an associative algebra U and σ is an automorphism of U, then the *twisted* U-module ${}^\sigma V$ is the same vector space V but with the new action $u * v = \sigma^{-1}(u)v, u \in U, v \in V$. Clearly, ${}^{\sigma\tau}V = {}^\sigma({}^\tau V)$ for every two automorphisms σ, τ of U. Furthermore, if V is a simple U-module, then so is ${}^\sigma V$. In particular, if $U = U_r$ and $V = V_\lambda$, then ${}^\sigma V_\lambda$ is isomorphic to $V_{\sigma(\lambda)}$ for some highest weight $\sigma(\lambda)$. Thus there exists an isomorphism of vector spaces $\sigma_\lambda \colon V_\lambda \to V_{\sigma(\lambda)}$ such that

$$\sigma_\lambda(uv) = \sigma(u)\sigma_\lambda(v), \qquad u \in U_r,\ v \in V_\lambda.$$

Clearly, σ_λ is unique up to a scalar multiple. It follows that the operator $\sigma_{\tau(\lambda)}\tau_\lambda$ is proportional to $(\sigma\tau)_\lambda$ for every two automorphisms σ and τ of U_r.

Returning to our situation, it follows at once from (7.1)–(7.3) that

$$\varphi(\lambda) = \psi(\lambda) = -w_0(\lambda), \qquad \eta(\lambda) = \lambda, \tag{7.4}$$

where w_0 is the maximal element of the Weyl group. We recall that every module V_λ is canonically realized as a subspace in $\mathscr{A}$, so that the highest vector in V_λ is 1. We denote by b_λ^{low} the lowest weight vector in V_λ, normalized by the condition that it lies in $B_\lambda = B \cap V_\lambda$. Now we normalize each of the maps $\varphi_\lambda, \psi_\lambda$ and η_λ by the requirement that

$$\varphi_\lambda(1) = b_{-w_0(\lambda)}^{\mathrm{low}}, \qquad \psi_\lambda(1) = 1, \qquad \eta_\lambda(1) = b_\lambda^{\mathrm{low}} \tag{7.5}$$

(of course, we also set Id_λ to be the identity map of V_λ).

PROPOSITION 7.1. (a) *Each of the maps φ_λ and ψ_λ sends B_λ to $B_{-w_0(\lambda)}$, while η_λ sends B_λ to B_λ.*

(b) *For every two (not necessarily distinct) elements σ, τ of the group $\{\mathrm{Id}, \varphi, \psi, \eta\}$, we have $(\sigma\tau)_\lambda = \sigma_{\tau(\lambda)}\tau_\lambda$.*

Part (a) of the proposition is proved in [8, Proposition 21.1.2]. Part (b) follows from (a) and the fact that $\sigma_{\tau(\lambda)}\tau_\lambda$ and $(\sigma\tau)_\lambda$ are always proportional to each other.

Using Proposition 7.1 (a), we shall write $\sigma_\lambda(b_d) = b_{\sigma_\lambda(d)}$ for all $\sigma \in \{\mathrm{Id}, \varphi, \psi, \eta\}$ and all $b_d \in B_\lambda$. So the map $d \mapsto \sigma_\lambda(d)$ on A_r-partitions is well defined for d such that $b_d \in B_\lambda$, that is, for d satisfying (6.4).

THEOREM 7.2. *The mappings φ_λ, ψ_λ and η_λ act on A_r-partitions as follows:*

$$\varphi_\lambda(d)_{j+1-i,j} = l_i - \left(\sum_{k=j}^{r} d_{ik} - \sum_{k=j+1}^{r} d_{i+1,k}\right); \tag{7.6}$$

$$\psi_\lambda(d)_{r+1-j,r+1-i} = d^*_{ij}; \tag{7.7}$$

$$\eta_\lambda(d)_{j+1-i,r+1-i} = l_j - \left(\sum_{h=1}^{i} d^*_{hj} - \sum_{h=1}^{i-1} d^*_{h,j-1}\right) \tag{7.8}$$

for all i, j such that $1 \leqslant i \leqslant j \leqslant r$.

Proof. First of all, let us show that (7.8) follows from (7.6) and (7.7). Indeed, in view of Proposition 7.1(b) and (7.4), we have $\eta_\lambda(d) = \varphi_{-w_0(\lambda)}\psi_\lambda(d)$, so (7.6) and (7.7) imply

$$\begin{aligned}\eta_\lambda(d)_{j+1-i,r+1-i} &= \varphi_{-w_0(\lambda)}\psi_\lambda(d)_{\hat{\imath}+1-\hat{\jmath},\hat{\imath}} \\ &= l_j - \left(\sum_{k=\hat{\imath}}^{r} \psi_\lambda(d)_{\hat{\jmath},k} - \sum_{k=\hat{\imath}+1}^{r} \psi_\lambda(d)_{\hat{\jmath}+1,k}\right) \\ &= l_j - \left(\sum_{k=\hat{\imath}}^{r} d^*_{\hat{k},j} - \sum_{k=\hat{\imath}+1}^{r} d^*_{\hat{k},j-1}\right).\end{aligned}$$

Substituting $h = \hat{k}$ in the last summation yields (7.8).

To prove (7.7), it is enough to show that the map $\psi_\lambda\colon V_\lambda \to V_{-w_0(\lambda)}$ is just the restriction to V_λ of the automorphism $x \mapsto \hat{x}^*$ of the algebra $\mathscr{A}$ (see Proposition 3.1 above). Remembering the definitions and the fact that $x \mapsto \hat{x}^*$ preserves B, it is easy to see that this map sends b_λ^{low} to $b_{-w_0(\lambda)}^{\mathrm{low}}$ and has the property $\widehat{ux}^* = \psi(u)\hat{x}^*$ for $u \in U_+$, $x \in \mathscr{A}$. It follows that $\psi_\lambda(x) = \hat{x}^*$, which implies (7.7).

It remains to prove (7.6). Fix $b = b_d \in B_\lambda \cap V_\lambda(\beta)$, and consider the strings

$$a = (a_{11}; a_{22}, a_{12}; \ldots; a_{rr}, \ldots, a_{1r}) = a(\mathbf{i}(1); b),$$

$$a^- = (a^-_{11}; a^-_{22}, a^-_{12}; \ldots; a^-_{rr}, \ldots, a^-_{1r}) = a(\mathbf{i}(1); \varphi_\lambda(b)).$$

By the definition (2.5), we have

$$d_{ij} = a_{ij} - a_{i-1,j}, \qquad \varphi_\lambda(d)_{j+1-i,j} = a^-_{j+1-i,j} - a^-_{j-i,j}, \tag{7.9}$$

so it will be enough for us to express the string a^- through a.

For $j = 1, \ldots, r$ we consider U_j as the subalgebra of U_r generated by the elements E_i, F_i and $K_i^{\pm 1}$ for $i = 1, \ldots, j$. We denote by $\mathbf{i}(1,j)$ the initial subword $(1; 2, 1; \ldots; j, j-1, \ldots, 1)$ of $\mathbf{i}(1)$ (so $\mathbf{i}(1)$ itself is now denoted $\mathbf{i}(1,r)$). Clearly, $\mathbf{i}(1,j) \in R(w_0(j))$ is a reduced decomposition of the element $w_0(j) \in W$ which can be identified with the maximal element of the Weyl group of U_j.

We set

$$b(j) = E_1^{(a_{1j})} E_2^{(a_{2j})} \cdots E_j^{(a_{jj})} \cdots E_1^{(a_{12})} E_2^{(a_{22})} E_1^{(a_{11})}(b);$$

in the notation of Section 2, we have $b(j) = E^{(\mathrm{top})}_{\mathbf{i}(1,j)}(b)$. We also set

$$b^-(j) = F_1^{(a^-_{1j})} F_2^{(a^-_{2j})} \cdots F_j^{(a^-_{jj})} \cdots F_1^{(a^-_{12})} F_2^{(a^-_{22})} F_1^{(a^-_{11})}(b). \tag{7.10}$$

In view of (7.1), we have

$$b^-(j) = \varphi_\lambda^{-1}(E^{(\mathrm{top})}_{\mathbf{i}(1,j)}(\varphi_\lambda(b))). \tag{7.11}$$

It follows that we can also write $b^-(j) = F^{(\mathrm{top})}_{\mathbf{i}(1,j)}(b)$, with the obvious meaning that each power of the type $F_i^{(a_i)}$ appearing in (7.10) is the maximal possible power of F_i which still produces a nonzero vector.

We shall show that for each j the vectors $b(j)$ and $b^-(j)$ are, respectively, the highest and lowest-weight vectors of the same irreducible U_j-submodule in V_λ. We need the following results from [4] which we reproduce here for the convenience of the reader.

LEMMA 7.3 [4, Theorem 2.2, Proposition 6.1]. *Let $b \in B, w \in W$ and $\mathbf{i} \in R(w)$. Then the element $E_{\mathbf{i}}^{(\mathrm{top})}(b)$ belongs to B, depends only on w (not on the choice of a reduced decomposition of w), and is annihilated by all E_i such that $l(ws_i) < l(w)$.*

Lemma 7.3 implies that $E_i b(j) = 0$ for $i = 1, \ldots, j$, so $b(j)$ is a highest-weight vector of some irreducible U_j-submodule of V_λ. Using (7.11), we see also that $b^-(j)$ is a lowest-weight vector of some irreducible U_j-submodule of V_λ. To show that $b(j)$ and $b^-(j)$ generate the same U_j-submodule, it is enough to see that $b(j) = E^{(\mathrm{top})}_{\mathbf{i}(1,j)}(b^-(j))$. In other words, we have to check the equality

$$E^{(\mathrm{top})}_{\mathbf{i}(1,j)} \circ F^{(\mathrm{top})}_{\mathbf{i}(1,j)} = E^{(\mathrm{top})}_{\mathbf{i}(1,j)}, \tag{7.12}$$

where both sides are considered as operators acting on B_λ. To prove (7.12) we first notice that $E_i^{(\mathrm{top})} \circ F_i^{(\mathrm{top})} = E_i^{(\mathrm{top})}$ for $i = 1, \ldots, r$; this follows from the commutation relations (5.2), (5.3) between E_i, F_i and K_i by a standard argument from the representation theory of sl_2. Next we prove that

$$E_{\mathbf{i}(1,j)}^{(\mathrm{top})} \circ F_i^{(\mathrm{top})} = E_{\mathbf{i}(1,j)}^{(\mathrm{top})} \tag{7.13}$$

for $i = 1, \ldots, j$. To see this, we notice that there exists a reduced decomposition of $w_0(j)$ starting with i, i.e., having the form $(i, \mathbf{i}')$; in view of Lemma 7.3, we have $E_{\mathbf{i}(1,j)}^{(\mathrm{top})} = E_{(i,\mathbf{i}')}^{(\mathrm{top})} = E_{\mathbf{i}'}^{(\mathrm{top})} \circ E_i^{(\mathrm{top})}$, so

$$E_{\mathbf{i}(1,j)}^{(\mathrm{top})} \circ F_i^{(\mathrm{top})} = E_{\mathbf{i}'}^{(\mathrm{top})} \circ E_i^{(\mathrm{top})} \circ F_i^{(\mathrm{top})} = E_{\mathbf{i}'}^{(\mathrm{top})} \circ E_i^{(\mathrm{top})} = E_{\mathbf{i}(1,j)}^{(\mathrm{top})},$$

proving (7.13). Finally, (7.13) obviously implies (7.12) since $E_{\mathbf{i}(1,j)}^{(\mathrm{top})}$ "swallows" all the factors $F_i^{(\mathrm{top})}$ occuring in $F_{\mathbf{i}(1,j)}^{(\mathrm{top})}$.

Since $b(j)$ and $b^-(j)$ are the highest- and lowest-weight vectors of the same irreducible U_j-module, their weights with respect to U_j are obtained from each other by the action of $w_0(j)$. To write down this statement explicitly, we need some notation. Let $P(j)$ be the weight lattice for U_j (so the lattice P is $P(r)$). To distinguish the fundamental weights and simple roots of U_j from those of U_r, we shall denote them by $\omega'_1, \omega'_2, \ldots, \omega'_j$ and $\alpha'_1, \alpha'_2, \ldots, \alpha'_j$. Clearly, the natural projection $p_j \colon P \to P(j)$ acts as follows:

$$p_j(\omega_i) = \begin{cases} \omega'_i & \text{if } 1 \leqslant i \leqslant j \\ 0 & \text{if } j < i \leqslant r. \end{cases} \tag{7.14}$$

We also have

$$w_0(j)(\omega'_i) = -\omega'_{j+1-i}, \quad w_0(j)(\alpha'_i) = -\alpha'_{j+1-i} \qquad \text{for } i = 1, \ldots, j. \tag{7.15}$$

Clearly, the U_j-weight of $b(j)$ is equal to

$$p_j(\beta) + \sum_{i=1}^{j} \left(\sum_{k=i}^{j} a_{ik} \right) \cdot \alpha'_i;$$

similarly, the U_j-weight of $b^-(j)$ is equal to

$$p_j(\beta) - \sum_{i=1}^{j} \left(\sum_{k=i}^{j} a_{ik}^- \right) \cdot \alpha'_i.$$

Therefore, we have the equality

$$p_j(\beta) + \sum_{i=1}^{j}\left(\sum_{k=i}^{j} a_{ik}\right)\cdot\alpha'_i = w_0(j)\left[p_j(\beta) - \sum_{i=1}^{j}\left(\sum_{k=i}^{j} a^-_{ik}\right)\cdot\alpha'_i\right]$$

$$= w_0(j)p_j(\beta) + \sum_{i=1}^{j}\left(\sum_{k=j+1-i}^{j} a^-_{j+1-i,k}\right)\cdot\alpha'_i. \tag{7.16}$$

Taking the scalar product of both sides of (7.16) with ω'_i for $i = 1, \ldots, j$, we obtain

$$\sum_{k=j+1-i}^{j} a^-_{j+1-i,k} = \sum_{k=i}^{j} a_{ik} + c_{ij}(\beta), \tag{7.17}$$

where

$$c_{ij}(\beta) = (p_j(\beta), \omega'_i) - (w_0(j)p_j(\beta), \omega'_i). \tag{7.18}$$

The rest of the proof is a formal calculation deducing (7.6) from (7.17), (7.18).

First, we rewrite (7.18) in a more convenient form. Using (7.15) and the fact that $w_0(j)$ preserves the scalar product in $P(j)$, we obtain

$$c_{ij}(\beta) = (p_j(\beta), \omega'_i + \omega'_{j+1-i}).$$

An easy check shows that

$$\omega'_i + \omega'_{j+1-i} = \sum_{h=1}^{i}(\alpha'_h + \alpha'_{h+1} + \cdots + \alpha'_{h+j-i}).$$

Using (7.14), we can rewrite (7.18) as

$$c_{ij}(\beta) = \left(\beta, \sum_{h=1}^{i}(\alpha_h + \alpha_{h+1} + \cdots + \alpha_{h+j-i})\right). \tag{7.19}$$

Now we subtract from (7.17) the similar equality obtained from it by replacing (i, j) with $(i-1, j-1)$. Using (7.19), we can rewrite the resulting equality as follows:

$$a^-_{j+1-i,j} = \sum_{k=i}^{j} a_{ik} - \sum_{k=i-1}^{j-1} a_{i-1,k} + c_{ij}(\beta) - c_{i-1,j-1}(\beta)$$

$$= a_{ij} - a_{i-1,i-1} + \sum_{k=i}^{j-1} d_{ik} + \left(\beta, \sum_{k=i}^{j}\alpha_k\right). \tag{7.20}$$

Subtracting from (7.20) the similar inequality obtained from it by replacing (i, j) with $(i+1, j)$, and taking into account (7.9), we obtain

$$\varphi_\lambda(d)_{j+1-i,j} = a_{ii} - a_{i-1,i-1} + \sum_{k=i}^{j-1} d_{ik} - \sum_{k=i+1}^{j} d_{i+1,k} + (\beta, \alpha_i). \tag{7.21}$$

The final step is to compute (β, α_i). In view of (5.10), we have

$$\beta = \lambda - \sum_{i=1}^{r} \left(\sum_{k=i}^{r} a_{ik} \right) \cdot \alpha_i,$$

which implies

$$(\beta, \alpha_i) = l_i + \sum_{k=i-1}^{r} a_{i-1,k} - 2\sum_{k=i}^{r} a_{ik} + \sum_{k=i+1}^{r} a_{i+1,k}$$

$$= l_i + a_{i-1,i-1} - a_{ii} - \sum_{k=i}^{r} d_{ik} + \sum_{k=i+1}^{r} d_{i+1,k}.$$

Substituting this expression into (7.21) and performing the obvious cancellation we obtain (7.6). Theorem 7.2 is proved.

8. The Schützenberger involution. In this section we give a combinatorial description of the involution η_λ in terms of Young tableaux and Gelfand-Tsetlin patterns. Here $\lambda = l_1\omega_1 + \cdots + l_r\omega_r$ is a fixed highest weight for sl_{r+1}. We associate with λ a partition $\Lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{r+1} \geqslant 0)$ of length $\leqslant r+1$, where λ_{r+1} is an arbitrary nonnegative integer, and $l_i = \lambda_i - \lambda_{i+1}$ for $i = 1, \ldots, r$. Let us recall some well-known combinatorial definitions. We identify Λ with its *diagram* (denoted by the same letter)

$$\Lambda = \{(i, j) \in \mathbf{Z} \times \mathbf{Z}\colon 1 \leqslant i \leqslant r+1,\ 1 \leqslant j \leqslant \lambda_i\}.$$

By an *A_r-tableau of shape* Λ we shall mean a map $\tau\colon \Lambda \to [1, r+1]$ satisfying the conditions

$$\tau(i, j+1) \geqslant \tau(i, j), \qquad \tau(i+1, j) > \tau(i, j) \tag{8.1}$$

for all $(i, j) \in \Lambda$; here and in the sequel we adopt the convention that $\tau(i, j) = +\infty$ for $i > r+1, j \geqslant 1$ or $1 \leqslant i \leqslant r+1, j > \lambda_i$. (In the combinatorial literature the tableaux satisfying (8.1) are often called *semistandard*.)

The *weight* of an A_r-tableau τ is an integral vector $\beta = (\beta_1, \ldots, \beta_{r+1})$ defined by $\beta_i = \#\tau^{-1}(i)$. We denote by the same letter β the sl_{r+1}-weight $\sum_{i=1}^{r} (\beta_i - \beta_{i+1})\omega_i$.

The language of A_r-tableaux is equivalent to that of Gelfand-Tsetlin patterns, which will be more convenient for us. By a *GT-pattern of highest weight* Λ we mean an array of integers $\pi = (\pi_{ij})_{1 \leqslant i \leqslant j \leqslant r+1}$ such that $\pi_{i,r+1} = \lambda_i$ for $i = 1, \ldots, r+1$ and

$$\pi_{i,j+1} \geqslant \pi_{ij} \geqslant \pi_{i+1,j+1} \tag{8.2}$$

for $1 \leqslant i \leqslant j \leqslant r$. Such a pattern is displayed as a triangular array

$$\pi = \begin{pmatrix} \pi_{14} & & \pi_{24} & & \pi_{34} & & \pi_{44} \\ & \pi_{13} & & \pi_{23} & & \pi_{33} & \\ & & \pi_{12} & & \pi_{22} & & \\ & & & \pi_{11} & & & \end{pmatrix}.$$

Clearly, the numbers in each row are weakly decreasing: $\pi_{1j} \geqslant \pi_{2j} \geqslant \cdots \geqslant \pi_{jj}$. The *weight* of a GT-pattern π is an integral vector $\beta = (\beta_1, \ldots, \beta_{r+1})$ defined by

$$\beta_1 + \beta_2 + \cdots + \beta_j = \pi_{1j} + \pi_{2j} + \cdots + \pi_{jj}$$

for $j = 1, \ldots, r+1$.

Let YT_Λ denote the set of all A_r-tableaux of shape Λ, and GT_Λ denote the set of all GT-patterns of highest weight Λ. It is well known that the sets YT_Λ and GT_Λ can be identified with each other by the following weight-preserving bijection $\tau \mapsto \pi(\tau) = (\pi_{ij})$:

$$\pi_{ij} = \#\{s\colon 1 \leqslant s \leqslant \lambda_i, \tau(i,s) \leqslant j\}; \tag{8.3}$$

indeed, it is easy to see that (8.3) transforms the conditions (8.1) to (8.2).

Now we introduce the Schützenberger involution $\eta\colon GT_\Lambda \to GT_\Lambda$. It was first defined in [9] by means of a beautiful combinatorial algorithm sometimes called the *evacuation*. We shall use an equivalent definition due to E. Gansner [5]. It is given in terms of the so-called Bender-Knuth involutive operators $t_1, t_2, \ldots, t_r$ acting on A_r-tableaux. Translating them into the language of GT-patterns, we arrive at the following definition. For $j = 1, \ldots, r$ and $\pi = (\pi_{ij}) \in GT_\Lambda$ we define the pattern $t_j(\pi) \in GT_\Lambda$ by

$$\begin{aligned} t_j(\pi)_{ik} &= \pi_{ik} \qquad \text{for } k \neq j, \\ t_j(\pi)_{ij} &= \min(\pi_{i,j+1}, \pi_{i-1,j-1}) + \max(\pi_{i+1,j+1}, \pi_{i,j-1}) - \pi_{ij}. \end{aligned} \tag{8.4}$$

(In view of (8.2), if we fix all the components π_{ik} with $k \neq j$, then each π_{ij} takes the values in the segment $[\max(\pi_{i+1,j+1}, \pi_{i,j-1}), \min(\pi_{i,j+1}, \pi_{i-1,j-1})]$; the transformation (8.4) is just the reflection of π_{ij} in the midpoint of this segment.) Clearly, each t_j is an involutive map $GT_\Lambda \to GT_\Lambda$.

As shown in [5], the Schützenberger involution η: $GT_\Lambda \to GT_\Lambda$ can be defined by

$$\eta := (t_1 \cdots t_r)(t_1 \cdots t_{r-1}) \cdots (t_1 t_2) t_1; \tag{8.5}$$

the fact that η is an involution follows readily from the obvious relations

$$t_j^2 = 1, t_i t_j = t_j t_i \qquad \text{for } |i-j| > 1.$$

(The group generated by the piecewise-linear automorphisms $t_1, \dots, t_r$ was studied in [2].)

We define the mapping ∂: $GT_\Lambda \to \mathbf{Z}_+^I$ by the formula

$$\partial(\pi)_{ij} = \pi_{i,j+1} - \pi_{ij} \qquad (1 \leqslant i \leqslant j \leqslant r). \tag{8.6}$$

Comparing (8.6) and (8.3) we see that

$$\partial(\pi(\tau))_{ij} = \#\{s\colon 1 \leqslant s \leqslant \lambda_i, \tau(i,s) = j+1\} \tag{8.7}$$

for any $\tau \in YT_\Lambda$; in the notation of [7], we have $\partial(\pi(\tau)) = d^{(1)}(\tau)$.

The following proposition is an easy consequence of (8.6) and (8.7) combined with (6.1) and (6.3) (cf. [7, (4.4)]).

PROPOSITION 8.1. *The map ∂ is a bijection of GT_Λ with the set $\{d \in \mathbf{Z}_+^I\colon b_d \in B_\lambda\}$. In other words, the canonical basis B_λ in V_λ can be parametrized by GT-patterns of highest weight Λ via $\pi \mapsto b_{\partial(\pi)}$. Furthermore, if $\pi \in GT_\Lambda$ has weight β then $b_{\partial(\pi)} \in V_\lambda(\beta)$, i.e., the parametrization $\pi \mapsto b_{\partial(\pi)}$ is weight preserving.*

Now we can state the main result of this section.

THEOREM 8.2. *For any $\pi \in GT_\Lambda$ we have*

$$\eta_\lambda(b_{\partial(\pi)}) = b_{\partial(\eta(\pi))}. \tag{8.8}$$

In other words, under the parametrization of basis vectors by GT-patterns, the involution η_λ acts on patterns as the Schützenberger involution.

In order to deduce Theorem 8.2 from Theorem 7.2, we shall translate the operations t_j into the language of A_r-partitions. We define the maps $R_1, R_2, \dots, R_r$: $\mathbf{Z}_+^I \to \mathbf{Z}_+^I$ in the following way. For $d = (d_{ij}) \in \mathbf{Z}_+^I$ and $j = 1, \dots, r$, we define

$R_j(d)$ by

$$R_j(d)_{ik} = d_{ik} \qquad \text{for } k \neq j, j-1;$$

$$R_j(d)_{ij} = \min(d_{ij}, d_{i-1,j-1}) + [d_{i,j-1} - d_{i+1,j}]_+; \tag{8.9}$$

$$R_j(d)_{i,j-1} = \min(d_{ij}, d_{i-1,j-1}) + [d_{i+1,j} - d_{i,j-1}]_+,$$

where $[x]_+ = \max(0, x)$. Here we use the convention that $d_{0,j-1}$ that can appear under the minimum sign in (8.9) is $+\infty$, and $[d_{j,j-1} - d_{j+1,j}]_+ = 0$. In particular, R_1 is just the identity map.

Note that each R_j is invertible, and the inverse map R_j^{-1} is given by similar formulas

$$R_j^{-1}(d)_{ik} = d_{ik} \qquad \text{for } k \neq j, j-1;$$

$$R_j^{-1}(d)_{ij} = \min(d_{ij}, d_{i,j-1}) + [d_{i-1,j-1} - d_{i-1,j}]_+; \tag{8.10}$$

$$R_j^{-1}(d)_{i-1,j-1} = \min(d_{ij}, d_{i,j-1}) + [d_{i-1,j} - d_{i-1,j-1}]_+$$

(with the conventions similar to the above). The check that the maps defined by (8.10) and (8.9) are indeed inverse to each other, is straightforward (it uses the obvious identity $[x]_+ - [-x]_+ = x$).

The most important for us will be the following composition of the maps R_j, which is reminiscent of the definition (8.5) of the Schützenberger involution η:

$$\rho := (R_1 \cdots R_r)(R_1 \cdots R_{r-1}) \cdots (R_1 R_2) R_1. \tag{8.11}$$

We shall express both η and the involution $d \mapsto d^*$ in terms of the map ρ. For this we need another map $\partial'\colon GT_\Lambda \to \mathbf{Z}_+^I$, whose definition is similar to (8.6):

$$\partial'(\pi)_{ij} = \pi_{ij} - \pi_{i+1,j+1} \qquad (1 \leqslant i \leqslant j \leqslant r). \tag{8.12}$$

Comparing (8.12) and (8.3), we see that

$$\partial'(\pi(\tau))_{ij} = \#\{s\colon 1 \leqslant s \leqslant \lambda_i, \tau(i,s) \leqslant j, \tau(i+1,s) \geqslant j+2\} \tag{8.13}$$

for any $\tau \in YT_\Lambda$; in the notation of [7], we have $\partial'(\pi(\tau)) = d^{(2)}(\tau)$.

A direct calculation shows that for every $\pi \in GT_\Lambda$ the A_r-partitions $d^{(1)} = \partial(\pi)$

and $d^{(2)} = \partial'(\pi)$ are related as follows (cf. [7, (4.5) and (4.6)]):

$$d_{ij}^{(2)} = l_i - \left(\sum_{k=j}^{r} d_{i,k}^{(1)} - \sum_{k=j+1}^{r} d_{i+1,k}^{(1)}\right); \tag{8.14}$$

$$d_{j+1-i,r+1-i}^{(1)} = l_j - \left(\sum_{h=1}^{i} d_{j+1-h,r+1-h}^{(2)} - \sum_{h=1}^{i-1} d_{j-h,r+1-h}^{(2)}\right). \tag{8.15}$$

THEOREM 8.3. (a) *The composition of ∂' with the Schützenberger involution $\eta: GT_\Lambda \to GT_\Lambda$ is equal to*

$$\partial'\eta = \rho\partial. \tag{8.16}$$

(b) *For every $d \in \mathbf{Z}_+^I$ the A_r-partitions $\rho(d)$ and d^* are related as follows:*

$$\rho(d)_{j+1-i,r+1-i} = d_{ij}^*. \tag{8.17}$$

Remark. We have already mentioned that $d \mapsto d^*$ coincides with the involution ζ from [7]. In fact, Theorem 8.3 is equivalent to Theorem 4.4 from [7], with ζ replaced by $d \mapsto d^*$.

Before proving Theorem 8.3, let us show that together with Theorem 7.2 it implies Theorem 8.2. Let $\pi \in GT_\Lambda$, and $d = \partial(\pi)$. By (8.16), $\partial'(\eta(\pi)) = \rho(d)$; so (8.17) gives

$$\partial'(\eta(\pi))_{j+1-i,r+1-i} = d_{ij}^*.$$

Substituting this expression into the right-hand side of (8.15), we obtain

$$\partial(\eta(\pi))_{j+1-i,r+1-i} = l_j - \left(\sum_{h=1}^{i} d_{hj}^* - \sum_{h=1}^{i-1} d_{h,j-1}^*\right).$$

Comparing this with (7.8), we see that $\eta_\lambda(b_d) = b_{\partial(\eta(\pi))}$, which proves Theorem 8.2.

Proof of Theorem 8.3(a). We include ∂ and ∂' into a family of mappings $\partial^\varepsilon: GT_\Lambda \to \mathbf{Z}_+^I$, where ε is a *sign vector* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$, each ε_j being either $+$ or $-$. For $1 \leqslant i \leqslant j \leqslant r$ we set

$$\partial^\varepsilon(\pi)_{ij} = \begin{cases} \partial'(\pi)_{ij} & \text{if } \varepsilon_j = + \\ \partial(\pi)_{ij} & \text{if } \varepsilon_j = -. \end{cases} \tag{8.18}$$

In particular, we have $\partial^{(-,-,\ldots,-)} = \partial, \partial^{(+,+,\ldots,+)} = \partial'$.

If a sign vector ε is such that $\varepsilon_j = -$, $\varepsilon_{j-1} = +$ then we denote by $s_j(\varepsilon)$ the sign vector obtained from ε by switching ε_j to $+$ and ε_{j-1} to $-$ (in particular, $s_1(\varepsilon)$ makes sense if $\varepsilon_1 = -$ and is then obtained from ε by switching this $-$ to $+$).

LEMMA 8.4. *If a sign vector ε is such that $s_j(\varepsilon)$ makes sense then we have*

$$\partial^{s_j(\varepsilon)} t_j = R_j \partial^{\varepsilon}. \tag{8.19}$$

Proof. Let $\pi = (\pi_{ij}) \in GT_\Lambda$, and $d = \partial^{\varepsilon}(\pi)$, so that

$$d_{ij} = \pi_{i,j+1} - \pi_{ij}, \qquad d_{i,j-1} = \pi_{i,j-1} - \pi_{i+1,j}. \tag{8.20}$$

It follows that

$$\pi_{i,j-1} - \pi_{i+1,j+1} = d_{i,j-1} - d_{i+1,j}. \tag{8.21}$$

Using (8.4), (8.20), (8.21), and (8.9), we get

$$\begin{aligned} \partial^{s_j(\varepsilon)} t_j(\pi)_{ij} &= t_j(\pi)_{ij} - t_j(\pi)_{i+1,j+1} \\ &= \min(\pi_{i,j+1}, \pi_{i-1,j-1}) + \max(\pi_{i+1,j+1}, \pi_{i,j-1}) - \pi_{ij} - \pi_{i+1,j+1} \\ &= \min(\pi_{i,j+1} - \pi_{ij}, \pi_{i-1,j-1} - \pi_{ij}) + [\pi_{i,j-1} - \pi_{i+1,j+1}]_+ \\ &= \min(d_{ij}, d_{i-1,j-1}) + [d_{i,j-1} - d_{i+1,j}]_+ = R_j(d)_{ij}. \end{aligned}$$

The equality $\partial^{s_j(\varepsilon)} t_j(\pi)_{i,j-1} = R_j(d)_{i,j-1}$ is proved in exactly the same way. Finally, for $k \neq j, j-1$, we have

$$\partial^{s_j(\varepsilon)} t_j(\pi)_{ik} = \partial^{s_j(\varepsilon)}(\pi)_{ik} = d_{ik} = R_j(d)_{ik}.$$

This completes the proof of the lemma.

To complete the proof of Theorem 8.3(a), we notice that the sign vector $(+,+,\ldots,+)$ can be obtained from $(-,-,\ldots,-)$ by the following chain of transformations:

$$(+,+,\ldots,+) = (s_1 \cdots s_r)(s_1 \cdots s_{r-1}) \cdots (s_1 s_2) s_1(-,-,\ldots,-). \tag{8.22}$$

Using this chain of transformations and applying Lemma 8.4 on each step, we obtain

$$\begin{aligned} &(R_1 \cdots R_r)(R_1 \cdots R_{r-1}) \cdots (R_1 R_2) R_1 \partial^{(-,-,\ldots,-)} \\ &\quad = \partial^{(+,+,\ldots,+)}(t_1 \cdots t_r)(t_1 \cdots t_{r-1}) \cdots (t_1 t_2) t_1, \end{aligned}$$

which proves (8.16).

Proof of Theorem 8.3(b). In view of Proposition 3.3, the involution $d \mapsto d^*$ is closely related to the transition operation ${}_{\mathbf{i}(r)}T_{\mathbf{i}(1)}$ on strings. We start with studying the transition from $\mathbf{i}(1)$ to $\mathbf{i}(r)$ in more detail. To do this, we include $\mathbf{i}(1)$ and $\mathbf{i}(r)$ into a family of reduced decompositions $\mathbf{i}(\varepsilon) \in R(w_0)$, labeled by the same sign vectors ε as in part (a) above. We shall write a reduced decomposition $\mathbf{i} \in R(w_0)$ as $\mathbf{i} = (\mathbf{i}^{(1)}; \mathbf{i}^{(2)}; \ldots; \mathbf{i}^{(r)})$, where each $\mathbf{i}^{(j)}$ is a sequence of indices of length j. We shall also use the notation $\overline{i, j}$ introduced in the proof of Lemma 4.2 above. In this notation, if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ is a sign vector then $\mathbf{i}(\varepsilon) \in R(w_0)$ is uniquely determined by the following properties:

(1) Each $\mathbf{i}(\varepsilon)^{(j)}$ has the form $\overline{k, k+j-1}$ for some k if $\varepsilon_j = +$, and $\overline{k, k-j+1}$ for some k if $\varepsilon_j = -$ (in particular, $\mathbf{i}(\varepsilon)^{(r)} = \overline{1, r}$ if $\varepsilon_r = +$, and $\mathbf{i}(\varepsilon)^{(r)} = \overline{r, 1}$ if $\varepsilon_r = -$).

(2) Let $j \geqslant 2$. If $\mathbf{i}(\varepsilon)^{(j)} = \overline{k, k+j-1}$ (i.e., $\varepsilon_j = +$), then we have

$$\varepsilon_{j-1} = + \quad \Rightarrow \quad \mathbf{i}(\varepsilon)^{(j-1)} = \overline{k+1, k+j-1},$$

$$\varepsilon_{j-1} = - \quad \Rightarrow \quad \mathbf{i}(\varepsilon)^{(j-1)} = \overline{k+j-1, k+1}.$$

If $\mathbf{i}(\varepsilon)^{(j)} = \overline{k, k-j+1}$ (i.e., $\varepsilon_j = -$), then we have

$$\varepsilon_{j-1} = + \quad \Rightarrow \quad \mathbf{i}(\varepsilon)^{(j-1)} = \overline{k-j+1, k-1},$$

$$\varepsilon_{j-1} = - \quad \Rightarrow \quad \mathbf{i}(\varepsilon)^{(j-1)} = \overline{k-1, k-j+1}.$$

In particular, we have $\mathbf{i}(1) = \mathbf{i}(-, -, \ldots, -)$, $\mathbf{i}(r) = \mathbf{i}(+, +, \ldots, +)$. It is easy to check that all $\mathbf{i}(\varepsilon)$ are indeed reduced decompositions of w_0 (for instance, using induction on r). Clearly, the map $\varepsilon \to \mathbf{i}(\varepsilon)$ is two-to-one: if ε and ε' differ only in the first component, then $\mathbf{i}(\varepsilon) = \mathbf{i}(\varepsilon')$.

Suppose $\mathbf{i} = (\mathbf{i}^{(1)}; \mathbf{i}^{(2)}; \ldots; \mathbf{i}^{(r)}) \in R(w_0)$ is such that for some j the pair $(\mathbf{i}^{(j-1)}; \mathbf{i}^{(j)})$ has the form $(\overline{k-j+1, k-1}; \overline{k, k-j+1})$. We denote by $s_j(\mathbf{i})$ the sequence obtained from $\mathbf{i}$ by replacing $(\mathbf{i}^{(j-1)}; \mathbf{i}^{(j)})$ with $(\overline{k, k-j+2}; \overline{k-j+1, k})$. It is clear that $s_j(\mathbf{i}) \in R(w_0)$ (this follows from the fact that both $(\overline{k-j+1, k-1}; \overline{k, k-j+1})$ and $(\overline{k, k-j+2}; \overline{k-j+1, k})$ are reduced decompositions of the same element in S_{r+1}). The operation $\mathbf{i} \mapsto s_j(\mathbf{i})$ is consistent with the operation $\varepsilon \mapsto s_j(\varepsilon)$ on sign vectors introduced above: the definitions imply at once that

$$s_j(\mathbf{i}(\varepsilon)) = \mathbf{i}(s_j(\varepsilon)), \tag{8.23}$$

whenever $s_j(\varepsilon)$ makes sense (we use the convention that s_1 is the identity operation on reduced decompositions). Comparing (8.22) and (8.23) we see that

$$\mathbf{i}(r) = (s_1 \cdots s_r)(s_1 \cdots s_{r-1}) \cdots (s_1 s_2) s_1(\mathbf{i}(1)). \tag{8.24}$$

Our next step is to lift the chain of transformations (8.24) to the level of strings. We shall write vectors from $\mathbf{R}^m$ as $x = (x^{(1)}; \ldots ; x^{(r)})$, where each $x^{(j)}$ belongs to $\mathbf{R}^j$.

PROPOSITION 8.5. *If $\mathbf{i} \in R(w_0)$ is such that $s_j(\mathbf{i})$ makes sense, then the transition map ${}_{s_j(\mathbf{i})}T_{\mathbf{i}}\colon \mathbf{R}^m \to \mathbf{R}^m$ depends only on j, not on the choice of $\mathbf{i}$. This map leaves unchanged all the components $x^{(k)}$ of $x \in \mathbf{R}^m$ except $x^{(j)}$ and $x^{(j-1)}$; furthermore, the $(j-1)$st and jth components of ${}_{s_j(\mathbf{i})}T_{\mathbf{i}}(x)$ depend only on $x^{(j)}$ and $x^{(j-1)}$.*

Proof. The proposition follows at once from the description of transition maps given by Proposition 2.3. Indeed, $s_j(\mathbf{i})$ is obtained from $\mathbf{i} = (\mathbf{i}^{(1)}; \ldots ; \mathbf{i}^{(r)})$ by transforming $(\mathbf{i}^{(j-1)}; \mathbf{i}^{(j)}) = (\overline{k-j+1, k-1}; \overline{k, k-j+1})$ to $(\overline{k, k-j+2}; \overline{k-j+1, k})$. In order to compute ${}_{s_j(\mathbf{i})}T_{\mathbf{i}}$ we have to decompose the transformation $(\overline{k-j+1, k-1}; \overline{k, k-j+1}) \mapsto (\overline{k, k-j+2}; \overline{k-j+1, k})$ into a sequence of transformations of type (2.2) and (2.3), and then compose the corresponding mappings in Proposition 2.3(b), (c). Clearly, this composition does not depend on k and has all the properties claimed in Proposition 8.5.

In view of Proposition 8.5, we shall denote the map ${}_{s_j(\mathbf{i})}T_{\mathbf{i}}\colon \mathbf{R}^m \to \mathbf{R}^m$ simply by T_j; in particular, T_1 is the identity map. Combining Proposition 2.3(a) and (8.24), we obtain

$$ {}_{\mathbf{i}(r)}T_{\mathbf{i}(1)} = (T_1 \cdots T_r)(T_1 \cdots T_{r-1}) \cdots (T_1 T_2) T_1. \tag{8.25} $$

The last step in the proof of Theorem 8.3(b) is to relate the maps T_j on strings with the maps R_j on A_r-partitions. We recall that the semigroup $\Gamma \subset \mathbf{R}^m$ consists of $x = (x^{(1)}; \ldots ; x^{(r)})$ such that for $j = 1, \ldots, r$ we have $x^{(j)} \in \mathbf{Z}_+^j$ and the components of $x^{(j)}$ are weakly decreasing: $x_1^{(j)} \geqslant x_2^{(j)} \geqslant \cdots \geqslant x_j^{(j)} \geqslant 0$ (see Proposition 2.4). For every sign vector ε we define the map $\partial^{\varepsilon}\colon \Gamma \to \mathbf{Z}_+^I$ by

$$ \partial^{\varepsilon}(x)_{ij} = \begin{cases} x_i^{(j)} - x_{i+1}^{(j)} & \text{if } \varepsilon_j = +; \\[2ex] x_{j+1-i}^{(j)} - x_{j+2-i}^{(j)} & \text{if } \varepsilon_j = -, \end{cases} \tag{8.26} $$

with the convention $x_{j+1}^{(j)} = 0$. Clearly, all ∂^{ε} are semigroup isomorphisms. In particular, $\partial^{(-,-,\ldots,-)}$ is the map ∂ in (2.5).

The following lemma is completely analogous to Lemma 8.4.

LEMMA 8.6. *If a sign vector ε is such that $s_j(\varepsilon)$ makes sense, then we have*

$$ \partial^{s_j(\varepsilon)} T_j = R_j \partial^{\varepsilon} \tag{8.27} $$

(as mappings $\Gamma \to \mathbf{Z}_+^I$).

Proof. An easy calculation using the strategy described in the proof of Proposition 8.5, shows that T_j acts on the components $x^{(j)}$ and $x^{(j-1)}$ in the following way:

$$
\begin{aligned}
T_j(x)_i^{(j)} &= \min(x_{i-1}^{(j-1)}, x_i^{(j-1)} + x_{j+1-i}^{(j)} - x_{j+2-i}^{(j)}),\\
T_j(x)_i^{(j-1)} &= \max(x_{i+1}^{(j)}, x_i^{(j)} + x_{j+1-i}^{(j-1)} - x_{j-i}^{(j-1)}).
\end{aligned}
\tag{8.28}
$$

(In fact, a part of this calculation was already done in the end of the proof of Theorem 4.1.)

Now (8.27) follows from (8.28), (8.26), and (8.9) by a straightforward calculation completely analogous to that in the proof of Lemma 8.4. We leave the details to the reader.

Now we can complete the proof of Theorem 8.3(b). As in the proof of part (a), using repeatedly Lemma 8.6, we deduce from (8.25) and (8.22) that

$$\partial^{(+,+,\dots,+)}{}_{\mathbf{i}(r)}T_{\mathbf{i}(1)} = \rho\partial. \tag{8.29}$$

Comparing (8.29) with Proposition 3.3, we see that for every $d \in \mathbf{Z}_+^I$ the A_r-partitions $\hat{d}^*$ and $\rho(d)$ are related by

$$\hat{d}^*{}_{ij} = \rho(d)_{j+1-i,j}.$$

It follows that

$$d_{ij}^* = \hat{d}^*{}_{r+1-j,r+1-i} = \rho(d)_{j+1-i,r+1-i},$$

which is the identity (8.17). Theorems 8.3 and 8.2 are proved.

Remark 8.7. The above calculations allow us to relate our transition maps ${}_{\mathbf{i}'}T_{\mathbf{i}}$ with Lusztig's piecewise-linear automorphisms $R_{\mathbf{i}}^{\mathbf{i}'}: \mathbf{R}^m \to \mathbf{R}^m$ (the definition of Lusztig's maps can be found, e.g., in [8, 42.1.3, 42.2.1]). Namely, let us define the linear automorphism $\delta: \mathbf{R}^m \to \mathbf{R}^m$ by the formula

$$\delta(x)_i^{(j)} = x_{j+1-i}^{(j)} - x_{j+2-i}^{(j)}$$

(so $\delta(x)$ is obtained from the vector $d = \partial(x) \in \mathbf{R}^I$ by arranging its components d_{ij} in the order $(d_{11}, d_{12}, d_{22}, \dots, d_{1r}, \dots, d_{rr})$; cf. (8.26)). Then an easy calculation using Lemma 8.6 shows that for any two sign vectors $\varepsilon, \varepsilon'$ we have

$$R_{\mathbf{i}(\varepsilon)}^{\mathbf{i}(\varepsilon')} = \delta \circ {}_{\mathbf{i}(\varepsilon')}T_{\mathbf{i}(\varepsilon)} \circ \delta^{-1}. \tag{8.30}$$

This formula shows in fact that the reduced decompositions $\mathbf{i}(\varepsilon)$ are rather special. It is almost obvious from the definitions that there exists a unique family of piecewise-linear automorphisms $\delta_{\mathbf{i}}: \mathbf{R}^m \to \mathbf{R}^m$ ($\mathbf{i} \in R(w_0)$) such that $\delta_{\mathbf{i}(1)} = \delta$ and

$$R_{\mathbf{i}}^{\mathbf{i}'} = \delta_{\mathbf{i}'} \circ {}_{\mathbf{i}'}T_{\mathbf{i}} \circ \delta_{\mathbf{i}}^{-1}$$

for all $\mathbf{i}$, $\mathbf{i}' \in R(w_0)$. In this notation, (8.30) means that $\delta_{\mathbf{i}(\varepsilon)} = \delta$ for every sign vector ε. In general, the maps $\delta_{\mathbf{i}}$ are not even linear; the simplest example is given by $\mathbf{i} = (1,3,2,1,3,2)$ for $r = 3$. Finding an explicit formula for $\delta_{\mathbf{i}}$ is an interesting problem in piecewise-linear combinatorics.

Our last result is an application of Theorem 8.2 to the classical representation theory obtained by the specialization $q = 1$. Having in mind potential combinatorial aplications, we prefer to speak about polynomial representations of the group GL_{r+1}. Let V_Λ be the polynomial representation of GL_{r+1} corresponding to a partition Λ. As an sl_{r+1}-module, V_Λ is just an irreducible module V_λ. Therefore, as explained in the end of Section 6, we can view B_λ as a basis in V_Λ. On the other hand, an element $w_0 \in S_{r+1}$ can be viewed as an element of GL_{r+1}, so it acts on V_Λ.

Proposition 8.8. *The action of w_0 on $V_\Lambda = V_\lambda$ is equal to $\varepsilon(\Lambda)\eta_\lambda$, where $\varepsilon(\Lambda) = \pm 1$. Therefore, $w_0(b_{\partial(\pi)}) = \varepsilon(\Lambda) b_{\partial(\eta(\pi))}$ for all $\pi \in GT_\Lambda$ (see (8.8)).*

Proof. Obviously, the action of w_0 on V_Λ is compatible with the sl_{r+1}-action in the following sense:

$$w_0(fx) = (w_0 f w_0^{-1})(w_0(x)) \tag{8.31}$$

for all $f \in sl_{r+1}$, $x \in V_\Lambda$. In view of (7.3), the automorphism $f \mapsto w_0 f w_0^{-1}$ of sl_{r+1} coincides with the one obtained from the automorphism η of U_r by the specialization $q = 1$. Comparing (8.31) and (7.5), we see that $w_0 = \varepsilon(\Lambda)\eta_\lambda$, where $\varepsilon(\Lambda)$ is the coefficient of proportionality between $w_0(1)$ and b_λ^{low}. Since both w_0 and η_λ are involutions, it follows that $\varepsilon(\Lambda) = \pm 1$, and we are done.

Corollary 8.9. *The number of A_r-tableaux of shape Λ invariant under the Schützenberger involution η, is equal to $|\mathrm{tr}(w_0, V_\Lambda)|$.*

The results in this section have interesting combinatorial consequences explored in [11].

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Department of Mathematics, Northeastern University, Boston, Massachusetts 02115, USA