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LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 365 (2003) 349–367
www.elsevier.com/locate/laa

Strongly nilpotent matrices and Gelfand–Zetlin modules

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Accepted 4 November 2002

Submitted by Y. Drozd

Abstract

Let X^n be the variety of $n \times n$ matrices, which $k \times k$ submatrices, formed by the first k rows and columns, are nilpotent for any $k = 1, \ldots, n$. We show, that X^n is a complete intersection of dimension $(n-1)n/2$ and deduce from it, that every character of the Gelfand–Zetlin subalgebra in $U(gl_n)$ extends to an irreducible representation of $U(gl_n)$.

Keywords: Nilpotent matrix; Regular sequence; Lie algebra representation

1. Introduction

1.1. Some notations

Throughout the paper we fix an algebraically closed field $\Bbbk$ of characteristic 0. All algebraic varieties are defined over $\Bbbk$ and endowed with Zariski topology (we use the word "variety" to denote closed or locally closed subsets in an affine space). The word "component" means "irreducible component" of a variety. All Lie and associative algebras are $\Bbbk$-algebras. The notation "dim" means the dimension over $\Bbbk$ of a $\Bbbk$-vector space or an algebraic variety. For an associative and commutative algebra Λ we denote by $\operatorname{Specm}\Lambda$ the variety of maximal ideals of Λ. If $I \subset \Lambda$ is an ideal, then $\mathsf{V}(I) \subset \operatorname{Specm}\Lambda$ denotes the set of all zeroes of I and if $C \subset \operatorname{Specm}\Lambda$, then $\mathsf{I}(C)$ denotes the ideal of elements of Λ, vanishing on C. For a set X by $1_X : X \to X$ we denote the identity mapping.

doi:10.1016/S0024-3795(02)00675-4

1.2. Motivations

Let L be a finite dimensional Lie algebra, U $(= U(L))$ be its universal enveloping algebra, $\overline{U}$ be the associated graded algebra, $u_1, \ldots, u_t \in U$ be mutually commuting elements, which generate an associative subalgebra $\Gamma \subset U$. For $\mu \in \operatorname{Specm} \Gamma$ denote by $\chi_\mu : \Gamma \to \mathbb{k}$ a $\mathbb{k}$-algebra homomorphism, such that $\operatorname{Ker} \chi_\mu = \mu$. We start from the following question: *when for any $\mu \in \operatorname{Specm} \Gamma$ there exists a simple U-module M $(= M_\mu)$, generated by $m \in M$, such that for any $\gamma \in \Gamma$ holds $\gamma m = \chi_\mu(\gamma) m$*? An obvious sufficient condition is the following: $\overline{u}_1, \ldots, \overline{u}_t$ form a regular sequence [1] in $\overline{U}$, where $\overline{u}_i$ is the class of $\overline{u}_i$ in $\overline{U}$ (see Section 3.1, Corollary 2). $\overline{U}$ is isomorphic to the polynomial algebra in $\dim L$ variables, hence it is a Cohen–Macaulay ring [3] and the condition above is equivalent to the fact, that *the dimension of any irreducible component of* $\mathsf{V}((\overline{u}_1, \ldots, \overline{u}_t)) \subset \operatorname{Specm} \overline{U}$ *equals* $\dim L - t$.

Originally the question raised in [1]. As L there was considered the full linear Lie algebra of $n \times n$-matrices gl_n $(= gl_n(\mathbb{k}))$, as Γ was considered *a Gelfand–Zetlin* (GZ) *subalgebra* and in this case the variety $\mathsf{X}^n = \mathsf{V}((\overline{u}_1, \ldots, \overline{u}_t))$ consists of so-called *strongly nilpotent matrices* (Section 1.3). It transforms the question of existence of some simple modules over gl_n into a linear algebraic question on dimensions of irreducible components of the matrix variety X^n.

1.3. Main results

We denote by $\{e_{ij}\}$, $1 \leqslant i, j \leqslant n$ a basis of gl_n of the matrix units, by $U_n = U(gl_n)$ the universal enveloping algebra of gl_n and by Z_n the center of U_n. For $m \leqslant n$ we consider U_m as a subalgebra in U_n by an inclusion $U_m \hookrightarrow U_n$, mapping $e_{ij} \mapsto e_{ij}$, $1 \leqslant i, j \leqslant m$. The GZ subalgebra Γ in $U = U_n$ is a commutative algebra, generated by the centers $Z_1, \ldots, Z_n \subset U$. Following [7] Z_m is the polynomial algebra in m variables $\{c_{im} \mid i = 1, \ldots, m\}$, where for $i \geqslant 2$

$$c_{im} = \sum_{1 \leqslant k_1, \ldots, k_i \leqslant m} e_{k_1 k_2} e_{k_2 k_3} \cdots e_{k_{i-1} k_i} e_{k_i k_1} \quad \text{and} \quad c_{1m} = \sum_{k=1}^{m} e_{kk}.$$

The algebra Γ also is the polynomial algebra in $n(n+1)/2$ variables $\{c_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$ [1]. A $U(gl_n)$-module, that in restriction on Γ splits in a direct sum of finite dimensional Γ-modules, is called *Gelfand–Zetlin* (GZ) *module*.

We denote by X_{ij} the class $\overline{e}_{ij}$ in $\overline{U}$ and identify $\overline{U}$ with the polynomial algebra in n^2 indeterminates X_{ij}, $1 \leqslant i, j \leqslant n$, denoted by $\mathbb{k}[(X_{ij})_n]$. Points of the variety $\mathsf{M}_n = \operatorname{Specm} \mathbb{k}[(X_{ij})_n]$ are $n \times n$-matrices over $\mathbb{k}$ and $x \in \mathsf{M}_n$ we write down as (x_{ij}), $1 \leqslant i, j \leqslant n$, or, if n is fixed, as (x_{ij}), where $x_{ij} \in \mathbb{k}$ is the ijth element of x.

[1] A sequence of elements $\lambda_1, \lambda_2, \ldots, \lambda_t$ of a commutative algebra Λ we call *regular*, if the class of λ_i is non-invertible and not a zero divisor in $\Lambda/(\lambda_1, \ldots, \lambda_{i-1})$ for any $i = 1, \ldots, t$.

For $m \leqslant n$ the canonical inclusion $\mathbf{i}_{mn} : \Bbbk[(X_{ij})_m] \to \Bbbk[(X_{ij})_n]$, $X_{ij} \mapsto X_{ij}$, $1 \leqslant i, j \leqslant m$ induces the projection $p_{nm} : \mathsf{M}_n \to \mathsf{M}_m$ onto $m \times m$-submatrix, formed by elements of m first rows and columns.

We denote by $X = (X_{ij})$, $1 \leqslant i, j \leqslant n$ a $n \times n$-matrix with indeterminate elements, by $\mathbb{1}_n$ denote the unit $n \times n$-matrix and introduce polynomials $\chi_{in} \in \Bbbk[(X_{ij})_n]$, $i = 1, \ldots, n$ as the coefficients of the characteristic polynomial of X $|t\mathbb{1}_n - X| = t^n - \chi_{1n}t^{n-1} + \chi_{2n}t^{n-2} - \cdots + (-1)^n\chi_{nn}$ and polynomials z_{in} as

$$z_{in} = \sum_{1 \leqslant k_1, \ldots, k_i \leqslant n} X_{k_1k_2}X_{k_2k_3} \cdots X_{k_{i-1}k_i}X_{k_ik_1} \quad \text{for } i \geqslant 2 \text{ and } z_{1n} = \sum_{k=1}^{n} X_{kk},$$

so $z_{in} = \overline{c}_{in}$. Let $\mathsf{N}_n \subset \mathsf{M}_n$ be the variety of nilpotent matrices. It is known, that $\mathsf{I}(\mathsf{N}_n) = (\chi_{1n}, \chi_{2n}, \ldots, \chi_{nn}) = (z_{1n}, z_{2n}, \ldots, z_{nn})$ (see [7] [2]), N_n is irreducible and $\dim \mathsf{N}_n = n^2 - n$.

A $n \times n$-matrix $x = (x_{ij})$, $1 \leqslant i, j \leqslant n$ we call *strongly nilpotent*, provided all its $m \times m$-submatrices (x_{ij}), $1 \leqslant i, j \leqslant m$ for $m = 1, \ldots, n$ are nilpotent. Strongly nilpotent $n \times n$-matrices form in M_n an affine variety $\mathsf{X}^n = \cap_{i=1}^n p_{ni}^{-1}(\mathsf{N}_i)$. Since $p_{ni}^{-1}(\mathsf{N}_i) = \mathsf{V}((\chi_{1i}, \chi_{2i}, \ldots, \chi_{ii}))$, there holds $\mathsf{X}^n = \mathsf{V}(I)$, where $I = (\chi_{ij}, 1 \leqslant i \leqslant j \leqslant n) = (z_{ij}, 1 \leqslant i \leqslant j \leqslant n)$. Especially, the ideal I is generated by $n(n+1)/2$ elements. It turns out, that X^n is a (geometric) complete intersection, or the algebra $\Bbbk[(X_{ij})_n]/I$ is a complete intersection in sense [3, p. 462]. Namely in Section 2.9 (see also Section 2.1, Proposition 1) will be proved.

Theorem 1. *If* C *is an irreducible component of* X^n, *then* $\dim \mathsf{C} = n(n-1)/2$.

Corollary 1. $\{z_{ij}, 1 \leqslant i \leqslant j \leqslant n\}$ *is a regular sequence in* $\Bbbk[(X_{ij})_n]$.

Proof. $\Bbbk[(X_{ij})_n]$ is Cohen–Macaulay [3] and the polynomials z_{ij} are homogeneous, hence the corollary follows from Theorem 1. □

Remark 1.1. The equality $\mathsf{I}(\mathsf{X}^n) = I$ fails, e.g. for $n = 3$ the ideal I is not radical one: it is easy to check $X_{12}X_{23}X_{31} \notin I$, but $(X_{12}X_{23}X_{31})^2 \in I$.

In Section 3.1 we prove the following statement about simple GZ modules.

Theorem 2. *Let* $\Gamma \subset U(gl_n)$ *be the GZ subalgebra,* $\chi : \Gamma \to \Bbbk$ *be a character of* Γ. *Then there exists a simple* gl_n*-module* M, *generated by an element* $m \in M$, *such that for any* $\gamma \in \Gamma$ *holds* $\gamma m = \chi(\gamma)m$.

[2] The statement $\mathsf{I}(\mathsf{N}_n) = (\chi_{1n}, \chi_{2n}, \ldots, \chi_{nn})$ is obvious. By inspection of the action of elementary matrices one can check, that z_{in} are constant on conjugacy classes. On diagonal matrices z_{in} turns out the ith elementary symmetrical polynomial and χ_{in} the ith power sum of the diagonal values. Since diagonalizable matrices are dense in M_n, the Newton formulas [4] gives $i\chi_{in} = z_{1n}\chi_{i-1n} - z_{2n}\chi_{i-2n} + \cdots + (-1)^{i-2}z_{i-1n}\chi_{1n} + (-1)^n z_{nn}$.

2. Strongly nilpotent matrices

2.1. Main result of this section

Denote $\mathsf{T}^n = \{(m_{ij}) \in \mathsf{M}_n \mid m_{ij} = 0 \text{ for } i \geqslant j\}$ and by $\mathbb{GL}_n \subset \mathsf{M}_n$ denote the group of all invertible matrices. Obviously, $\mathsf{T}^n \subset \mathsf{X}^n$ and $\dim \mathsf{T}^n = n(n-1)/2$. In this section we prove the following:

Proposition 1. *Let $\mathsf{X}^n = \cup_{k \in \mathbf{K}^n} \mathsf{C}_k^n$ be a minimal decomposition of X^n in irreducible components. There exist maps $\mathsf{e}_n : \mathsf{X}^n \to \mathbf{K}^n$ and $S_k^n : \mathsf{K}_k^n \to \mathbb{GL}_n$, where $\mathsf{K}_k^n = \mathsf{e}_n^{-1}(k)$, $k \in \mathbf{K}^n$ such that*

(1) *K_k^n is an open subset in C_k^n and S_k^n is a regular map*;
(2) *the transformation $F_k^n(u) = S_k^n(u)^{-1} u S_k^n(u)$, $u \in \mathsf{K}_k^n$ defines a regular embedding $F_k^n : \mathsf{K}_k^n \hookrightarrow \mathsf{T}^n$ and $F_k^n(\mathsf{K}_k^n)$ is open in T^n.*

Since $\dim \mathsf{T}^n = n(n-1)/2$, Theorem 1 follows from Proposition 1. Especially, T^n is a component of X^n.

The plan of the proof of Proposition 1 is as follows. The proof uses the induction on n and consists of two unequal parts. The smaller one (Section 2.9) contains the deduction of Proposition 1 from Lemma 2.6. The most part of the text (Sections 2.2–2.8) is devoted to the proof of Lemma 2.6, which is an analogue of Proposition 1 for the variety $\mathsf{Y} = p_{n,n-1}^{-1}(\mathsf{T}^{n-1})$. We enumerate irreducible components of Y by some set $\mathbf{I}_n^*$ (Section 2.6). Let $\mathsf{C}_{\mathbb{I}}$ be a component of Y, corresponding to $\mathbb{I} \in \mathbf{I}_*^n$. In Lemma 2.6 as in Proposition 1, we construct an open set $\mathsf{K}_{\mathbb{I}} \subset \mathsf{C}_{\mathbb{I}}$, together with corresponding maps $S_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \to \mathbb{GL}_n$ and $F_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \hookrightarrow \mathsf{T}^n$. Firstly (Lemma 2.5) for any $\mathbb{I} \in \mathbf{I}_n^*$ we construct an open set $\mathsf{V}(\mathscr{G}_{1,n}^{\mathbb{I}})$ in T^n and an isomorphism $\varphi_{\mathbb{I}} : \mathsf{V}(\mathscr{G}_{1,n}^{\mathbb{I}}) \to \mathsf{V}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}_I$, where $\mathsf{V}(\mathscr{K}_{\mathbb{I}})$ is a locally closed subset in M_n and $\mathsf{U}_{\mathbb{I}}$ is an affine subvariety in $\mathbb{GL}_n$. On other hand, for any component C of Y we construct (Lemma 2.6(4)) a unique $\mathbb{I} \in \mathbf{I}_n^*$, an open $\mathsf{K}_{\mathbb{I}} \subset \mathsf{C}$ together with an isomorphism of varieties $\Phi_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \to \mathsf{V}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})$, where $\mathsf{U}(D_{\mathbb{I}})$ is isomorphic to $\mathsf{U}_{\mathbb{I}}$ (Lemma 2.5(2) and Lemma 2.6(2)). The existence of $\varphi_{\mathbb{I}}$ and $\Phi_{\mathbb{I}}$ gives us $F_{\mathbb{I}}$.

$\Phi_{\mathbb{I}}$ maps a matrix $x \in \mathsf{K}_{\mathbb{I}}$ to a pair $(v, u) \in \mathsf{V}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})$, where v is some "canonical form of x with respect to conjugation" and $v = u^{-1}xu$ (an analogous property has $\varphi_{\mathbb{I}}$). This property is important in deducing of Proposition 1 from Lemma 2.6. We formalize such property of a map in the notion of admitted bijection (Sections 2.4 and 2.5). The sets of corresponding canonical forms we describe by graphs (Sections 2.2 and 2.3). The graphs, which arose in our proofs, are presented in Section 2.6.

2.2. Marked graphs

Let I be a subset of $\{1, \ldots, n\}$. We define *a graph* $\mathscr{G}$ with the set of *vertices* I (*over* I) as a subset $\mathscr{G} \subset I \times I$, where elements of $\mathscr{G}$ are called *arrows*. By $\mathscr{F}$ we denote *the full graph* $I \times I$. On the graphs over I are defined the operations of *union* and *difference* as sets, denoted by "+" and "−" correspondingly. For a graph $\mathscr{G}$ we denote by $\overline{\mathscr{G}}$ the graph $\mathscr{F} - \mathscr{G}$. The *support* of $\mathscr{G}$ is the minimal $|\mathscr{G}| \subset I$, such that $\mathscr{G} \subset |\mathscr{G}| \times |\mathscr{G}|$.

On $\mathscr{F}$ is defined a partial binary operation "$\circ$": for any $i, j, k \in I$ set $(j, k) \circ (i, j) = (i, k)$, on other pairs "$\circ$" is undefined. For $x \in \mathscr{F}$, $S \subset \mathscr{F}$ we introduce $x \circ S$, $S \circ x \subset \mathscr{F}$ as the sets of all possible products.

An arrow (i, j) also is displayed as $i \to j$. A sequence of arrows of $\mathscr{G}$ $i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k$ is called *a way of the length* $k - 1$ from i_1 to i_k and a sequence of arrows of $\mathscr{G}$ $i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k \to i_1$ is called a *cycle of the length k in* $\mathscr{G}$.

All the graphs $\mathscr{G}$ are supposed to be marked by a map $c(= c_{\mathscr{G}}) : \mathscr{G} \to \{a, *\}$. We denote $\mathscr{G}^a = c^{-1}(a)$, $\mathscr{G}^* = c^{-1}(*)$. For $x \in \mathscr{G}$ the notation x^a (x^*) means $c(x) = a$ ($c(x) = *$). For marked graphs $\mathscr{G}_1$, $\mathscr{G}_2$ we write $\mathscr{G}_1 \sim \mathscr{G}_2$, provided $\mathscr{G}_1$ and $\mathscr{G}_2$ have the same arrows and $\mathscr{G}_1 = \mathscr{G}_2$, provided $\mathscr{G}_1 \sim \mathscr{G}_2$ and $c_{\mathscr{G}_1} = c_{\mathscr{G}_2}$. The marking on $\mathscr{G}_1 - \mathscr{G}_2$ is defined by $\mathscr{G}_1$. If $\mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2$, then $c_{\mathscr{G}}|_{\mathscr{G}_2} = c_{\mathscr{G}_2}$ and $c_{\mathscr{G}}|_{\mathscr{G}_1 - \mathscr{G}_2} = c_{\mathscr{G}_1}|_{\mathscr{G}_1 - \mathscr{G}_2}$, so $\mathscr{G}_2$ has a priority (!). If $(I, \mathscr{G}, c_{\mathscr{G}})$ and $(I_1, \mathscr{G}_1, c_{\mathscr{G}_1})$ are marked graphs over I and I_1 correspondingly, then *an isomorphism* $f : \mathscr{G} \to \mathscr{G}_1$ is a bijection $f : I \to I_1$, such that $(f \times f)(\mathscr{G}) = \mathscr{G}_1$ and $c_{\mathscr{G}} = c_{\mathscr{G}_1} f$. If a marking on $\mathscr{G}$ is not defined, then we suppose $\mathscr{G}$ is endowed with the trivial marking $c(\mathscr{G}) = \{a\}$.

Later on the term "graph" is equivalent to "marked graph". All the graphs we consider (except $\mathscr{F}$) do not contain cycles of the length 1. Otherwise opposite stated, we assume $I = \{1, \ldots, n\}$.

2.3. Matrices, defined by graph

Let $\Bbbk^* = \Bbbk \setminus \{0\}$ and $\mathscr{G}$ be a graph. We define a variety $\mathsf{V}(\mathscr{G}) \subset \mathsf{M}_n$ as follows:

$$\mathsf{V}(\mathscr{G}) = \left\{ (m_{ij}) \in \mathsf{M}_n,\ \text{satisfying} \begin{cases} m_{ij} = 0 & \text{for } (j, i) \in \overline{\mathscr{G}} \\ m_{ij} \in \Bbbk & \text{for } (j, i) \in \mathscr{G}^a \\ m_{ij} \in \Bbbk^* & \text{for } (j, i) \in \mathscr{G}^* \end{cases} \right\}.$$

Obviously, $\mathsf{V}(\mathscr{G}_1) \subset \mathsf{V}(\mathscr{G}_2)$ iff $\mathscr{G}_1 \subset \mathscr{G}_2$ and $\mathscr{G}_1^a \subset \mathscr{G}_2^a$. If $\mathscr{G}_1 \sim \mathscr{G}_2$ and $\mathscr{G}_1^* \subset \mathscr{G}_2^*$, then $\mathsf{V}(\mathscr{G}_1)$ contains $\mathsf{V}(\mathscr{G}_2)$ as an open subset. By $\mathsf{N}(\mathscr{G})$ we denote $\mathsf{N}_n \cap \mathsf{V}(\mathscr{G})$. In particular, $\mathsf{V}(\mathscr{F}) = \mathsf{M}_n$, $\mathsf{N}(\mathscr{F}) = \mathsf{N}_n$.

Lemma 2.1

(1) *If* $\mathscr{G}$ *does not contain oriented cycles, then* $\mathsf{V}(\mathscr{G}) = \mathsf{N}(\mathscr{G})$.

(2) *Suppose* $a_1, \ldots, a_k \in \mathscr{G}$ *form a cycle in* $\mathscr{G}$, *that is the unique* (*up to cyclic permutations*) *cycle of the length* $\leqslant k$ *in* $\mathscr{G}$ *and* $a_1, \ldots, a_{k-1} \in \mathscr{G}^*$. *If* $a_k \in \mathscr{G}^a$, *then* $\mathsf{N}(\mathscr{G}) = \mathsf{N}(\mathscr{G} - \{a_k\})$, *otherwise* $\mathsf{N}(\mathscr{G}) = \emptyset$.

Proof. The statement (1) is obvious. Since $\mathsf{I}(\mathsf{N}_n) = (z_{1n}, z_{2n}, \ldots, z_{nn})$ (see Section 1.3) the statement (2) follows from $z_{kn}|_{\mathsf{N}(\mathscr{G})} = 0$. $\square$

2.4. Admitted bijections

For $S \subset \mathsf{M}_n$, $G \subset \mathbb{GL}_n$ we denote $S^G = \{s^g = g^{-1}sg \mid s \in S, g \in G\} \subset \mathsf{M}_n$. By $\bigsqcup$ we denote the disjoint union. Let $\mathscr{G}, \mathscr{G}_1, \ldots, \mathscr{G}_k$ be graphs, $\mathsf{U}_1, \ldots, \mathsf{U}_k$ be irreducible subvarieties in $\mathbb{GL}_n$. A bijection $\varphi : \mathsf{V}(\mathscr{G}) \to \bigsqcup_{i=1}^k \mathsf{V}(\mathscr{G}_i) \times \mathsf{U}_i$ is called *admitted* (shortly AB and AB's in plural), if for $\psi(= \psi(\varphi)) = \varphi^{-1}$ and all $\psi_i = \psi|_{\mathsf{V}(\mathscr{G}_i)\times \mathsf{U}_i}$, $i = 1, \ldots, k$ hold:

(1) $\psi_i : \mathsf{V}(\mathscr{G}_i) \times \mathsf{U}_i \to \mathsf{V}(\mathscr{G})$ is a regular embedding;
(2) if $\operatorname{Im} \psi_i = \mathsf{Q}_i$ and $\varphi|_{\mathsf{Q}_i} = (F_i, S_i)$, $(F_i = F_i(\varphi), S_i = S_i(\varphi))$, then $F_i : \mathsf{Q}_i \to \mathsf{V}(\mathscr{G}_i)$, $S_i : \mathsf{Q}_i \to \mathsf{U}_i$ are regular and for $v \in \mathsf{Q}_i$ holds $F_i(v) = v^{S_i(v)}$.

If for an AB $\varphi : \mathsf{V}(\mathscr{G}) \to \bigsqcup_{i=1}^k \mathsf{V}(\mathscr{G}_i) \times \mathsf{U}_i$ for all $i = 1, \ldots, k$ hold $\mathsf{U}_i = \{\mathbb{1}_n\}$, then we will omit U_ith and write $\varphi : \mathsf{V}(\mathscr{G}) \to \bigsqcup_{i=1}^k \mathsf{V}(\mathscr{G}_i)$. If, besides them, $k = 1$, then we consider $\varphi : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G}_1)$ as an identification. We need the following remarks.

Remark 2.1

(1) The conjugation keeps the nilpotency of a matrix, hence any AB $\varphi : \mathsf{V}(\mathscr{G}) \to \bigsqcup_{i=1}^k \mathsf{V}(\mathscr{G}_i) \times \mathsf{U}_i$ induces a bijection $\mathsf{N}(\mathscr{G}) \to \bigsqcup_{i=1}^k \mathsf{N}(\mathscr{G}_i) \times \mathsf{U}_i$. We also call it AB and denote by φ.
(2) Let $x \in \mathscr{G}^a$. Then there exist obvious canonical AB's $\varphi_x : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G} - \{x\}) \bigsqcup \mathsf{V}(\mathscr{G} + \{x\}^*)$ and $\varphi_x : \mathsf{N}(\mathscr{G}) \to \mathsf{N}(\mathscr{G} - \{x\}) \bigsqcup \mathsf{N}(\mathscr{G} + \{x\}^*)$.
(3) Let C be an irreducible component of $\mathsf{N}(\mathscr{G})$. Then there exists a unique $i = i(\mathsf{C})$, such that $\mathsf{C} \subset \overline{\psi(\mathsf{N}(\mathscr{G}_i) \times \mathsf{U}_i)}$, where the overline "$\overline{}$" means the closure in M_n in Zariski topology.
(4) Suppose all $\mathsf{N}(\mathscr{G}_i)$, $i = 1, \ldots, k$ are irreducible (it is true, if for all i holds $\mathsf{N}(\mathscr{G}_i) = \mathsf{V}(\mathscr{G}_i)$) and $\dim \mathsf{N}(\mathscr{G}_1) \times \mathsf{U}_1 = \cdots = \dim \mathsf{N}(\mathscr{G}_k) \times \mathsf{U}_k = d$. Then $\mathsf{N}(\mathscr{G})$ has exactly k irreducible components and they are d-dimensional of the form $\overline{\psi(\mathsf{N}(\mathscr{G}_i) \times \mathsf{U}_i)} \cap \mathsf{N}(\mathscr{G})$ (of the form $\overline{\psi(\mathsf{N}(\mathscr{G}_i) \times \mathsf{U}_i)}$, provided $\mathscr{G}^* = \emptyset$). In this case we say φ distinguishes components.

The remark (4) shows, how we can apply AB's to enumerate irreducible components of $\mathsf{N}(\mathscr{G})$.

2.5. Composition of admitted bijections

Let

$$\varphi : \mathsf{V}(\mathscr{G}) \to \bigsqcup_{i=1}^{k} \mathsf{V}(\mathscr{G}_i) \times \mathsf{U}_i, \quad \varphi_i : \mathsf{V}(\mathscr{G}_i) \to \bigsqcup_{j=1}^{k_i} \mathsf{V}(\mathscr{G}_{ij}) \times \mathsf{U}_{ij},$$
$$1 \leqslant i \leqslant k$$

be some AB's. Then it is easy to see, that for any $i = 1, \ldots, k$, $j = 1, \ldots, k_i$ the mapping $\mu_{ij} : \mathsf{U}_{ij} \times \mathsf{U}_i \to \mathsf{U}_{ij}\mathsf{U}_i$, $(u_{ij}, u_i) \mapsto u_{ij} \cdot u_i$, where '$\cdot$' is the product in

$\mathbb{GL}_n$, is bijective. A *composition* $\{\varphi_i\}_{1\leqslant i\leqslant k}\circ\varphi$ is the admitted bijection $\mathsf{V}(\mathscr{G})\to\bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{k_i}\mathsf{V}(\mathscr{G}_{ij})\times(\mathsf{U}_{ij}\mathsf{U}_i)$, which is a composition as mappings of φ and the obvious bijections

$$\left(\bigsqcup_{i=1}^{k}\varphi_i\right)\times 1_{\mathsf{U}_i}:\bigsqcup_{i=1}^{k}\mathsf{V}(\mathscr{G}_i)\times\mathsf{U}_i\to\bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{k_i}\mathsf{V}(\mathscr{G}_{ij})\times\mathsf{U}_{ij}\times\mathsf{U}_i$$

and

$$\bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{k_i}(1_{\mathsf{V}(\mathscr{G}_{ij})}\times\mu_{ij}):\bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{k_i}\mathsf{V}(\mathscr{G}_{ij})\times\mathsf{U}_{ij}\times\mathsf{U}_i$$
$$\to\bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{k_i}\mathsf{V}(\mathscr{G}_{ij})\times(\mathsf{U}_{ij}\mathsf{U}_i).$$

We use the following convention: if φ_i are defined only for $i\in S$, $S\subset\{1,\ldots,n\}$, then for $i\notin S$ we set φ_i to be an identification $\varphi_i=1_{\mathsf{v}(\mathscr{G}_i)}:\mathsf{V}(\mathscr{G}_i)\to\mathsf{V}(\mathscr{G}_i)$.

For $(i,j)\in\mathscr{F}$, $i\neq j$ we denote $\mathsf{U}((i,j))=\mathbb{1}_n+\mathsf{V}(\{(i,j)\})=\mathbb{1}_n+\Bbbk e_{ji}$. If $S=(x_1,\ldots,x_l)$ is a finite sequence, then by S° we denote the reverse sequence $(x_l,\ldots,x_1)$ and by $\mathsf{U}(S)=\mathsf{U}(x_1)\cdots\mathsf{U}(x_l)\subset\mathbb{GL}_n$.

Lemma 2.2

(1) *Let $\mathscr{G}$ be a graph $\mathscr{G}=\{x_1,\ldots,x_N\}$, such that for any sequence $x_{i_1},\ldots,x_{i_k}$, $1\leqslant i_1<\cdots<i_k\leqslant N$, $2\leqslant k\leqslant N$ holds $x_{i_1}\circ\cdots\circ x_{i_k}\notin\mathscr{G}$. Then $\mathsf{U}(\mathscr{G})$ is isomorphic as variety to $\mathsf{V}(\mathscr{G})\simeq\mathsf{U}(x_1)\times\cdots\times\mathsf{U}(x_N)$, $\iota_{\mathscr{G}}:u_1\cdots u_N\mapsto\sum_{i=1}^{N}(u_i-\mathbb{1}_n)\in\mathsf{V}(\mathscr{G})$, $u_i\in\mathsf{U}(x_i)$, $i=1,\ldots,N$ and $\dim\mathsf{U}(\mathscr{G})=N$.*

(2) *If $\mathsf{x}=\{x_1,\ldots,x_k\}\subset\mathscr{G}^a$, then there exists an* AB *$\varphi_{\mathsf{x}}:\mathsf{N}(\mathscr{G})\to\bigsqcup_{i=0}^{k}\mathsf{N}(\mathscr{G}_i)$, $\mathscr{G}_0=\mathscr{G}-\{x_1,\ldots,x_n\}$, $\mathscr{G}_i=(\mathscr{G}-\{x_1,\ldots,x_i\})+\{x_i^*\}$, $i=1,\ldots,k$.*

(3) *Let in assumption* (2) *above $\psi_{\mathsf{x}}=\varphi_{\mathsf{x}}^{-1}$, C be an irreducible component of $\mathsf{N}(\mathscr{G})$ and $\mathsf{C}\subset\overline{\psi_{\mathsf{x}}(\mathsf{N}(\mathscr{G}_i))}$ for some i, $i=0,\ldots,k$. Then the intersection $\psi_{\mathsf{x}}(\mathsf{N}(\mathscr{G}_i))\cap\mathsf{C}$ is open in C.*

Proof. (1) follows from the rule of the multiplication of matrices. The statement (2) we obtain by composition of admitted bijections $\varphi_{x_i}:\mathsf{N}(\mathscr{G}-\{x_1,\ldots,x_{i-1}\})\to\mathsf{N}(\mathscr{G}-\{x_1,\ldots,x_i\})\bigsqcup\mathsf{N}(\mathscr{G}_i)$ from Remark 2.1(2) for $i=1,\ldots,k$. If in assumption (3) $i=i(\mathsf{C})=0$, then $\psi_{\mathsf{x}}(\mathsf{N}(\mathscr{G}_0))\cap\mathsf{C}=\mathsf{C}$. If for $i=i(\mathsf{C})$, $1\leqslant i\leqslant k$ holds $x_i=(l_1,l_2)$, then $\psi_{\mathsf{x}}(\mathsf{N}(\mathscr{G}_i))\cap\mathsf{C}$ is defined in C by the inequality $X_{l_2l_1}\neq 0$, hence it is open in C. $\square$

Remark 2.2. Let $x=(k,i)$, $k\neq i\in I$. Then for any graphy $\mathscr{G}$ holds $\mathsf{V}(\mathscr{G})^{\mathsf{U}(x)}\subset\mathsf{V}(\mathscr{G}_1)$, where $\mathscr{G}_1=\mathscr{G}+x\circ\mathscr{G}+\mathscr{G}\circ x+x\circ\mathscr{G}\circ x$ and $\mathscr{G}_1^*=\mathscr{G}^*-\{x\circ\mathscr{G}+\mathscr{G}\circ x+x\circ\mathscr{G}\circ x\}$.

If $\mathscr{G}$ is a graph, $m = (i, j) \in \mathscr{G}^*$, $x = (k, i) \in \mathscr{F}$, $k \neq i, j$, then we call the pair (m, x)

(1) left generating, shortly LG (for the arrow $m \circ x$), provided $m \circ x \in \overline{\mathscr{G}}$ and $x \circ \mathscr{G} + \mathscr{G} \circ x \subset \mathscr{G}^a + \{m \circ x\}$ and in this case we denote $\mathscr{G}^{(m,x)} = \mathscr{G} + \{(m \circ x)^a\}$;
(2) left annihilating, shortly LA (for the arrow $m \circ x$), provided $m \circ x \in \mathscr{G}^a$ and $x \circ \mathscr{G} + \mathscr{G} \circ x \subset \mathscr{G}^a$ and in this case we denote $\mathscr{G}_{(m,x)} = \mathscr{G} - \{(m \circ x)\}$.

In both cases holds $x \circ \mathscr{G} \circ x \subset \mathscr{G} \circ x$. Obviously, (m, x) is LA in $\mathscr{G}^{(m,x)}$ and $(\mathscr{G}^{(m,x)})_{(m,x)} = \mathscr{G}$; analogously (m, x) is LG in $\mathscr{G}_{(m,x)}$ and $(\mathscr{G}_{(m,x)})^{(m,x)} = \mathscr{G}$.

Lemma 2.3
(1) *If* (m, x) *is* LG *in* $\mathscr{G}$, *then there exists a unique* AB $\gamma^{(m,x)} : \mathsf{V}(\mathscr{G}^{(m,x)}) \to \mathsf{V}(\mathscr{G}) \times \mathsf{U}(x)$.
(2) *If* (m, x) *is* LA *in* $\mathscr{G}$, *then there exists a unique* AB $\alpha_{(m,x)} : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G}_{(m,x)}) \times \mathsf{U}(x)$.

Proof. Suppose $m : i \to j$, $x : k \to i$. Consider in assumption of (2) a matrix $v \in \mathsf{V}(\mathscr{G})$. Since $v_{ij} \neq 0$, there exists a unique $u(v) \in \mathsf{U}(x)$, namely $u(v) = \mathbb{1}_n - ((v_{jk}/v_{ji})e_{ik})$, such that $(v^{u(v)})_{jk} = 0$. Moreover, the conditions on (m, x) show that $v^{u(v)} \in \mathsf{V}(\mathscr{G}_{(m,x)})$. We set $\alpha_{(m,x)}(v) = (v^{u(v)}, u(v))$. The inverse map is defined by $(v, u) \mapsto v^{(u^{-1})}$. The case (1) is analogous. $\square$

Let $\mathscr{G}$ be a graph and $S = \{(m_1, x_1), \ldots, (m_k, x_k)\} \subset \mathscr{G}^* \times \mathscr{F}$ be a sequence of pairs, $S_i = \{(m_1, x_1), \ldots, (m_i, x_i)\}$, $i = 0, 1, \ldots, k$. Then S is called LG (LA) for the arrows $\{m_1 \circ x_1, \ldots, m_k \circ x_k\}$ provided the pair (m_i, x_i) is LG in $\mathscr{G}^{S_{i-1}} = \mathscr{G} + \{(m_1 \circ x_1)^a, \ldots, (m_{i-1} \circ x_{i-1})^a\}$ (the pair (m_i, x_i) is LA in $\mathscr{G}_{S_{i-1}} = \mathscr{G} - \{m_1 \circ x_1, \ldots, m_{i-1} \circ x_{i-1}\}$) for any $i = 1, \ldots, k$. We set $\mathsf{U}(S) = \mathsf{U}(\{x_1, \ldots, x_k\})$, $\mathsf{U}(S^\circ) = \mathsf{U}(\{x_k, \ldots, x_1\})$ and $\mathsf{U}(\emptyset) = \{\mathbb{1}_n\}$. The following Lemma 2.4 describes the result of iterations of AB's from Lemma 2.3.

Lemma 2.4. *Let S be either* LA *or* LG *in* $\mathscr{G}$.

(1) *If S is* LG, *then there exists an* AB $\gamma^S : \mathsf{V}(\mathscr{G}^S) \to \mathsf{V}(\mathscr{G}) \times \mathsf{U}(S)$, *defined as the compostion of the* AB'*s* $\gamma^{(m_i,x_i)} : \mathsf{V}(\mathscr{G}^{S_i}) \to \mathsf{V}(\mathscr{G}^{S_{i-1}}) \times \mathsf{U}(x_i)$ *for* $i = 1, \ldots, k$.
(2) *If S is* LA, *then there exists an* AB $\alpha_S : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G}_S) \times \mathsf{U}(S^\circ)$, *defined as the composition of the* AB'*s* $\alpha_{(m_i,x_i)} : \mathsf{V}(\mathscr{G}_{S_{i-1}}) \to \mathsf{V}(\mathscr{G}_{S_i}) \times \mathsf{U}(x_i)$, $i = 1, \ldots, k$.

Remark 2.3. In an obvious way we can define the right version of the notions above. Namely, to obtain the definitions of right generating or right annihilating pair (x, m) (shortly RG, RA) for the arrow $x \circ m$ we should in the definitions of left generating and left annihilated pairs set $m = (j, i)$, $x = (i, k)$, change $m \circ x$ for $x \circ m$ and denote $\mathscr{G}^{(x,m)} = \mathscr{G} + \{(x \circ m)^a\}$, $\mathscr{G}_{(x,m)} = \mathscr{G} - \{(x \circ m)\}$. The “right” version of Lemma 2.3 gives us for RG (x, m) a unique AB $\gamma^{(x,m)} : \mathsf{V}(\mathscr{G}^{(x,m)}) \to \mathsf{V}(\mathscr{G}) \times \mathsf{U}(x)$ and for RA (x, m) gives a unique AB $\alpha_{(x,m)} : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G}_{(x,m)}) \times \mathsf{U}(x)$.

If $S = \{(x_1, m_1), \ldots, (x_k, m_k)\} \subset \mathscr{F} \times \mathscr{G}^*$ and $S_i = \{(x_1, m_1), \ldots, (x_i, m_i)\}$, $i = 0, 1, \ldots, k$, then we call S right generating (shortly RG) sequence for the arrows $\{x_1 \circ m_1, \ldots, x_k \circ m_k\}$ if (x_i, m_i) is a RG pair in $\mathscr{G}^{S_{i-1}} = \mathscr{G} + \{(x_1 \circ m_1)^a, \ldots, (x_{i-1} \circ m_{i-1})^a\}$. Analogously is defined a right annihilated (RA) sequence. The "right analogue" of Lemma 2.4 above is then the following: if S is RG, then there exists an AB $\gamma^S : \mathsf{V}(\mathscr{G}^S) \to \mathsf{V}(\mathscr{G}) \times \mathsf{U}(S)$, defined as the composition of AB $\gamma^{(m_i,x_i)} : \mathsf{V}(\mathscr{G}^{S_i}) \to \mathsf{V}(\mathscr{G}^{S_{i-1}}) \times \mathsf{U}(x_i)$ for $i = 1, \ldots, k$ and if S is RA, then there exists an AB $\alpha_S : \mathsf{V}(\mathscr{G}) \to \mathsf{V}(\mathscr{G}_S) \times \mathsf{U}(S^\circ)$, defined as the composition of $\alpha_{(m_i,x_i)} : \mathsf{V}(\mathscr{G}_{S_{i-1}}) \to \mathsf{V}(\mathscr{G}_{S_i}) \times \mathsf{U}(x_i)$, $i = 1, \ldots, k$.

2.6. *Examples of graphs*

Let "$\prec$" be a linear order on I and $\mathscr{G}_\prec$ be a graph over I, consisting of all $(i, j) \in \mathscr{F}$, such that $j \prec i$. For the usual order "$<$" and $i, j \in I$ we define a graph $\mathscr{G}_{i,j}$: $(i', j') \in \mathscr{G}_{i,j}$ iff $i \leqslant j' < i' \leqslant j$. If $i \geqslant j$, then $\mathscr{G}_{i,j}$ is empty and $\mathscr{G}_{1,n} = \mathscr{G}_<$. We will present a graph $\mathscr{G}$ over $\{1, \ldots, n\}$ by a diagram, consisting of "building" blocks of the form $\mathscr{G}_{ij}$ and solid and dotted arrows between blocks. In such diagram a brick block $\boxed{i \qquad j}$ denotes the graph $\mathscr{G}_{ij}$ over $\{i, \ldots, j\}$ with the trivial marking. A round block (i) means the one-point graph over $\{i\}$. Besides arrows in the diagram (both solid and dotted) are distinguished by its headers:

standard arrows $\lhd\!\!\relbar\!\!\relbar\!\!\relbar$, $\lhd\cdots\cdots$ and *transit* arrows $\longleftarrow$, $\leftarrow\cdots\cdots$.

A diagram of a graph $\mathscr{G}$ over I is defined by a partition of $\{1, \ldots, n\}$ into intervals and grouping the vertices in brick blocks and round blocks. Let $\mathscr{B}$ and $\mathscr{B}'$ be such blocks, $k \in \mathscr{B}$, $k' \in \mathscr{B}'$. Then $(k, k') \in \mathscr{G}$ iff either

(1) $\mathscr{B} = \mathscr{B}' = \mathscr{G}_{i,j}$ and $i \leqslant k' < k \leqslant j$ and then $(k, k') \in \mathscr{G}^a$,
(2) or there exists an arrow from $\mathscr{B}$ to $\mathscr{B}'$ and then (k, k') belongs to $\mathscr{G}^a$ or to $\mathscr{G}^*$, provided this arrow is solid or dotted correspondingly,
(3) or there exists a way of transit arrows from $\mathscr{B}$ to $\mathscr{B}'$ of length $\geqslant 2$, starting or ending with a solid arrow and in this case $(k, k') \in \mathscr{G}^a$.

Of course, such presentation of a graph is not unique.

Let $\mathbb{I}$ be an integral vector, $\mathbb{I} = (i, i_k, i_{k-1}, \ldots, i_0)$, where holds $0 < i \leqslant i_k < \cdots < i_1 < i_0 = n$. Such vector $\mathbb{I}$ we call *extensible*, provided $i_k > i$. The set of all these $\mathbb{I}$ we denote $\mathbf{I}(n)$ and the set of extensible $\mathbb{I}$ by $\mathbf{I}^e(n)$, $\mathbf{I}^*(n) = \mathbf{I}(n) \setminus \mathbf{I}^e(n)$. By $\mathbf{I}_k(n) \subset \mathbf{I}(n)$ ($\mathbf{I}_{\leqslant k}(n) \subset \mathbf{I}(n)$) we denote the set of all $\mathbb{I} \in \mathbf{I}(n)$ of the length $k + 2$

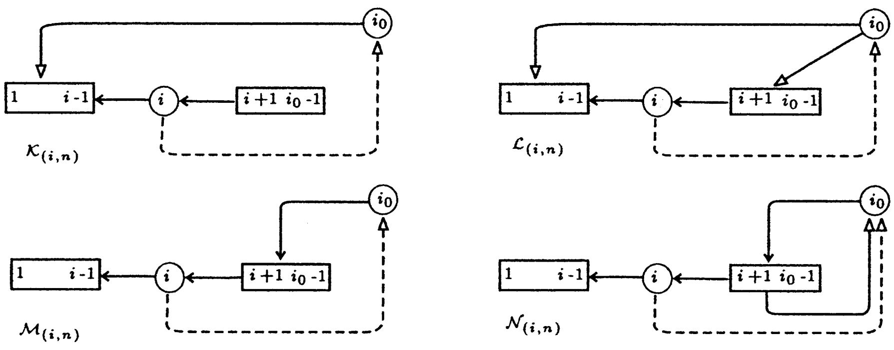
($\leqslant k+2$). An obvious sense have the notations $\mathbf{I}_k^e(n) = \mathbf{I}^e(n) \cap \mathbf{I}_k(n)$, $\mathbf{I}_k^*(n)$, etc. The cardinality of $\mathbf{I}^*(n)$ is 2^{n-1}. If $\mathbb{I} = (i, i_k, \dots, i_0) \in \mathbf{I}^e(n)$, then for $i_{k+1}, i_k > i_{k+1} \geqslant i$ we denote $\mathbb{I}[i_{k+1}] = (i, i_{k+1}, i_k, \dots, i_1, i_0) \in \mathbf{I}(n)$. If n is fixed, we will omit it in notations and write $\mathbf{I}^e$, $\mathbf{I}^*$, etc.

We define for any $\mathbb{I} \in \mathbf{I}^e$ the graphs $\mathscr{K}_{\mathbb{I}}$, $\mathscr{L}_{\mathbb{I}}$, $\mathscr{M}_{\mathbb{I}}$, $\mathscr{N}_{\mathbb{I}}$ and $\mathscr{T}_{\mathbb{I}}$ over $I = \{1, \dots, n\}$.

Remark 2.4. We extend the definitions of $\mathscr{K}_{\mathbb{I}}$, $\mathscr{L}_{\mathbb{I}}$, $\mathscr{M}_{\mathbb{I}}$ and $\mathscr{N}_{\mathbb{I}}$ on $\mathbf{I}^*$ by setting for $\mathbb{I} = (i, i_k, \dots, i_0) \in \mathbf{I}^e$ $\mathscr{K}_{\mathbb{I}[i]} = \mathscr{L}_{\mathbb{I}[i]} = \mathscr{M}_{\mathbb{I}[i]} = \mathscr{N}_{\mathbb{I}[i]} = \mathscr{K}_{\mathbb{I}}$.

Suppose first $k = 0$. In this case $\mathbb{I} = (i, n)$, $0 < i \leqslant n$, $\mathbf{I}_{\leqslant 0}^* = \{(n, n)\}$ and we set then $\mathscr{K}_{(n,n)} = \mathscr{L}_{(n,n)} = \mathscr{M}_{(n,n)} = \mathscr{N}_{(n,n)} = \mathscr{G}_{1,n}$. For $i < n$ we set

$$
\begin{aligned}
\mathscr{K}_{(i,n)} &= \mathscr{G}_{1,n} - \{(n, n-1), (n, n-2), \dots, (n-i)\} + \{(i, n)^*\},\\
\mathscr{L}_{(i,n)} &= \mathscr{G}_{1,n} - \{(n, i)\} + \{(i, n)^*\}\\
&= \mathscr{K}_{(i,n)} + \{(n, n-1)^a, \dots, (n, i+1)^a\},\\
\mathscr{M}_{(i,n)} &= \mathscr{G}_{1,n} + \{(i, n)^*\} = \mathscr{L}_{(i,n)} + \{(n, i)^a\},\\
\mathscr{N}_{(i,n)} &= \mathscr{G}_{1,n} + \{(i, n)^*, (i+1, n)^a, \dots, (n-1, n)^a\}\\
&= \mathscr{M}_{(i,n)} + \{(i+1, n)^a, \dots, (n-1, n)^a\}.
\end{aligned}
$$



The diagrams of the graphs $\mathcal{K}_{(i,n)}$, $\mathcal{L}_{(i,n)}$, $\mathcal{M}_{(i,n)}$, $\mathcal{N}_{(i,n)}$, $i \neq n$

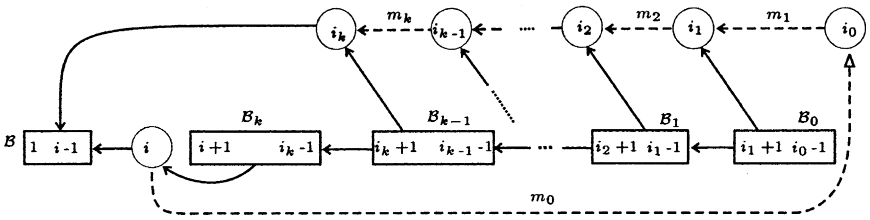
It will be convenient to use the notation $i_{-1} = i$. Diagrams of $\mathscr{K}_{\mathbb{I}}$, $\mathscr{L}_{\mathbb{I}}$, $\mathscr{M}_{\mathbb{I}}$, $\mathscr{N}_{\mathbb{I}}$ for $k \geqslant 1$ consist of brick blocks $\mathscr{B} = \mathscr{G}_{1,i-1}$, $\mathscr{B}_s = \mathscr{G}_{i_{s+1}+1, i_s-1}$, $0 \leqslant s \leqslant k-1$, $\mathscr{B}_k = \mathscr{G}_{i+1, i_k-1}$, round blocks i and i_s for $s = 0, 1, \dots, k$ and some extra arrows.

The graph $\mathscr{K}_{\mathbb{I}}$. Beside them the graph $\mathscr{K}_{\mathbb{I}}$ contains the following arrows:

$$
\begin{aligned}
&|\mathscr{B}_s| \times |\mathscr{B}|, \quad s = 0, \dots, k,\\
&\{i_s\} \times |\mathscr{B}|, \quad s = -1, \dots, k,
\end{aligned}
$$

$$\{i_s\} \times \{i_{s+1}\}, \quad s = -1, \dots, k-1,$$
$$|\mathscr{B}_s| \times |\mathscr{B}_t| \quad \text{for } s < t, \quad s, t = 0, \dots, k,$$
$$|\mathscr{B}_s| \times \{i_t\} \quad \text{for } s < t, \quad s, t = 0, \dots, k,$$
$$|\mathscr{B}_s| \times \{i\} \quad \text{for } s = 0, \dots, k,$$
$$\mathscr{K}^*_{\mathbf{I}} = \{m^*_{s+1} : i_s \to i_{s+1}, \quad s = -1, \dots, k-1\}.$$

The graph $\mathscr{K}_{\mathbf{I}}$ is directed (in spite of $\mathscr{L}_{\mathbf{I}}$, $\mathscr{M}_{\mathbf{I}}$, $\mathscr{N}_{\mathbf{I}}$).



The diagram of the graph $\mathcal{K}_{\mathbf{I}}$.

The graph $\mathscr{L}_{\mathbf{I}}$. $\mathscr{L}_{\mathbf{I}} = \mathscr{K}_{\mathbf{I}} + \{i_k\} \times |\mathscr{B}_k|$, $\mathscr{L}^*_{\mathbf{I}} = \mathscr{K}^*_{\mathbf{I}}$.

The graph $\mathscr{M}_{\mathbf{I}}$. $\mathscr{M}_{\mathbf{I}} = \mathscr{L}_{\mathbf{I}} + \{(i_k, i)^a\}$. The graph $\mathscr{M}_{\mathbf{I}}$ contains the unique cycle of the length $\leqslant k+2$, namely $i \to i_0 \to i_1 \to \cdots \to i_k \to i$.

The graph $\mathscr{N}_{\mathbf{I}}$. $\mathscr{N}_{\mathbf{I}} = \mathscr{M}_{\mathbf{I}} + \{i_{k-1}\} \times |\mathscr{B}_k|$ and $\mathscr{N}^*_{\mathbf{I}} = \mathscr{K}^*_{\mathbf{I}}$.

The graph $\mathscr{T}_{\mathbf{I}}$. The graph $\mathscr{T}_{\mathbf{I}}$ is isomorphic to the graph $\mathscr{K}_{\mathbf{I}}$, but $\mathscr{T}_{\mathbf{I}} \subset \mathscr{G}_{1,n}$. The diagram of $\mathscr{T}_{\mathbf{I}}$ contains the bricks $\mathscr{B}', \mathscr{B}'_1, \dots, \mathscr{B}'_k$ and $i' = i'_{-1}, i'_0, \dots, i'_k$, where $\mathscr{B}' = \mathscr{G}_{1,i-1}, \mathscr{B}'_s = \mathscr{G}_{i_{s+1}+s+2, i_s+s}, 0 \leqslant s \leqslant k-1, \mathscr{B}'_k = \mathscr{G}_{i+k+2, i_k+k}, i'_s = i+k-s, s = -1, 0, \dots, k$. The extra arrows are defined as in the case of $\mathscr{K}_{\mathbf{I}}$ by adding the dash "$'$" to $\mathscr{B}$th, ith and mth.

Remark 2.5. The isomorphism $f_{\mathbf{I}} : \mathscr{K}_{\mathbf{I}} \to \mathscr{T}_{\mathbf{I}}$ is given by the bijection $f_{\mathbf{I}} : I \to I$, which is uniquely defined by $f_{\mathbf{I}}(|\mathscr{B}|) = |\mathscr{B}'|$, $f_{\mathbf{I}}(|\mathscr{B}_s|) = |\mathscr{B}'_s|, s = 0, \dots, k, f_{\mathbf{I}}(i_l) = i'_l, l = -1, \dots, k$.

2.7. *The graphs* $\mathscr{K}_{\mathbf{I}}$, $\mathscr{T}_{\mathbf{I}}$ *and* T^n

For $\mathbf{I} \in \mathbf{I}^e$ we define a linear order "$\prec_{\mathbf{I}}$" on I as follows: on $|\mathscr{B}|$ and $|\mathscr{B}_j|$, $j = 0, \dots, k$ the order "$\prec_{\mathbf{I}}$" coincides with the usual one, besides elementwise hold:

$$|\mathscr{B}| \prec_{\mathbf{I}} i_k \prec_{\mathbf{I}} i_{k-1} \prec_{\mathbf{I}} \cdots \prec_{\mathbf{I}} i_1 \prec_{\mathbf{I}} i_0 \prec_{\mathbf{I}} i \prec_{\mathbf{I}} |\mathscr{B}_k| \prec_{\mathbf{I}} |\mathscr{B}_{k-1}| \prec_{\mathbf{I}} \cdots \prec_{\mathbf{I}} |\mathscr{B}_0|.$$

It is easy to see, $f_{\mathbf{I}}$ transforms the order "$\prec_{\mathbf{I}}$" in the usual one, $\mathscr{K}_{\mathbf{I}} \subset \mathscr{G}_{\prec_{\mathbf{I}}}$ and $\mathscr{G}_{\prec_{\mathbf{I}}} - \mathscr{K}_{\mathbf{I}} = R_{\mathbf{I}} = \cup_{j=0}^k R_{j\mathbf{I}}, R_{j\mathbf{I}} = C_{j\mathbf{I}} \times \{i_j\}, C_{j\mathbf{I}} = |\mathscr{B}_k| \cup \cdots \cup |\mathscr{B}_j| \cup \{i_{-1}, \dots, i_{j-2}\}$.

We set $G_{j\mathbf{i}} = C_{j\mathbf{i}} \times \{i_{j-1}\}$, $j = 0, \dots, k$, $G_{\mathbf{i}} = \cup_{j=0}^{k} G_{j\mathbf{i}}$ and define the order on $G_{\mathbf{i}}$ as follows: for $(x, i_s), (y, i_t) \in G_{\mathbf{i}}$ holds $(x, i_s) < (y, i_t)$ iff either $s < t$ or $s = t$ and $x > y$. It defines also the order on $R_{\mathbf{i}}$ and by the second coordinate on $S_{\mathbf{i}}$, where $S_{\mathbf{i}} = (\{m_0\} \times G_{0\mathbf{i}}, \{m_1\} \times G_{1\mathbf{i}}, \dots, \{m_k\} \times G_{k\mathbf{i}})$. Denote by $G'_{\mathbf{i}}$, $R'_{\mathbf{i}}$ and $S'_{\mathbf{i}}$ the sequences, obtained from $G_{\mathbf{i}}$, $R_{\mathbf{i}}$ and $S_{\mathbf{i}}$ by application of $f_{\mathbf{i}}$.

Lemma 2.5

(1) *The sequence* $S_{\mathbf{i}}$ *is* LG *in the graph* $\mathscr{K}_{\mathbf{i}}$ *for* $R_{\mathbf{i}}$. *Moreover,* $\mathscr{K}_{\mathbf{i}}^{S_{\mathbf{i}}} = \mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}}$, *where* $\mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}} \sim \mathscr{G}_{\prec_{\mathbf{i}}}$, *and* $(\mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}})^* = \mathscr{K}^*_{\mathbf{i}}$, *especially* $\mathsf{N}(\mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}}) = \mathsf{V}(\mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}})$ *is open in* $\mathsf{V}(\mathscr{G}_{\prec_{\mathbf{i}}})$.

(2) *The sequence* $S'_{\mathbf{i}}$ *is* LG *in the graph* $\mathscr{T}_{\mathbf{i}}$ *for* $R'_{\mathbf{i}}$ *and* $\mathscr{T}_{\mathbf{i}}^{S'_{\mathbf{i}}} = \mathscr{G}^{\mathbf{i}}_{1,n}$, *where* $\mathscr{G}^{\mathbf{i}}_{1,n} \sim \mathscr{G}_{1,n}$, $(\mathscr{G}^{\mathbf{i}}_{1,n})^* = \mathscr{T}^*_{\mathbf{i}}$, *especially* $\mathsf{N}(\mathscr{G}^{\mathbf{i}}_{1,n}) = \mathsf{V}(\mathscr{G}^{\mathbf{i}}_{1,n})$ *is open in* $\mathsf{V}(\mathscr{G}_{<}) = \mathsf{T}^n$.

(3) $\mathsf{U}(G_{\mathbf{i}})$ *is an affine space of dimension* $\sum_{j=0}^{k} i_j - (k+1)(i+1)$.

(4) *There exists an* AB $\varphi_{\mathbf{i}} : \mathsf{N}(\mathscr{G}^{\mathbf{i}}_{1,n}) \to \mathsf{N}(\mathscr{K}_{\mathbf{i}}) \times \mathsf{U}_{\mathbf{i}}$, *where* $\mathsf{U}_{\mathbf{i}} \simeq \mathsf{U}(G_{\mathbf{i}})$ *as a variety.*

Proof. Immediately can be checked, that $S_{\mathbf{i}}$ is left generating for $R_{\mathbf{i}}$. The equality $\mathscr{K}_{\mathbf{i}}^{S_{\mathbf{i}}} = \mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}}$ follows from $R_{\mathbf{i}} = \mathscr{G}_{\prec_{\mathbf{i}}} - \mathscr{K}_{\mathbf{i}}$ and from the fact, that AB's do not change the set of *-arrows. The openness $\mathsf{N}(\mathscr{G}^{\mathbf{i}}_{\prec_{\mathbf{i}}})$ in $\mathsf{V}(\mathscr{G}_{\prec_{\mathbf{i}}})$ is obvious, that proves (1). The statement (2) is analogous. The cardinality of $C_{t\mathbf{i}}$ is $(i_t - (i+1)) + (2t - k)$ and taking the sum by $t = 0, \dots, k$ we prove (3). Let $F_{\mathbf{i}}$ be $n \times n$-matrix, such that $(F_{\mathbf{i}})_{f_{\mathbf{i}}(i)i} = 1$, $i \in I$, all other elements of $F_{\mathbf{i}}$ are zero, so $\mathsf{V}(\mathscr{G}_{1,n})^{F_{\mathbf{i}}} = \mathsf{V}(\mathscr{G}_{\prec_{\mathbf{i}}})$. We set $U_{\mathbf{i}} = U(G'_{\mathbf{i}})F_{\mathbf{i}}$ and as $\varphi_{\mathbf{i}}$ from (4) we take the composition

$$\mathsf{N}(\mathscr{G}^{\mathbf{i}}_{1,n}) \xrightarrow{\gamma^{S'_{\mathbf{i}}}} \mathsf{N}(\mathscr{T}_{\mathbf{i}}) \times \mathsf{U}(G'_{\mathbf{i}}) \xrightarrow{q} \mathsf{N}(\mathscr{K}_{\mathbf{i}}) \times \mathsf{U}_{\mathbf{i}},$$

where $\gamma^{S'_{\mathbf{i}}}$ is the AB from Lemma 2.4(1) and $q((v, u)) = (v^{F_{\mathbf{i}}}, uF_{\mathbf{i}})$. $\square$

2.8. Irreducible components of $p^{-1}_{n,n-1}(\mathsf{T}^{n-1}) \cap \mathsf{N}_n$

We denote $p = p_{n,n-1}$. For $\mathbf{i} = (i, i_k, \dots, i_0) \in \mathbf{I}^e$ we introduce the sequences $D_{0\mathbf{i}} = \{i+1, \dots, i_0 - 1\} \times \{i\}$, $D_{j\mathbf{i}} = \{i_j\} \times \{i+1, \dots, i_j - 1\}$, $j = 1, \dots, k$ and $D_{\mathbf{i}} = D^{\circ}_{0\mathbf{i}} \cup D^{\circ}_{1\mathbf{i}} \cup \dots \cup D^{\circ}_{k\mathbf{i}}$.

Lemma 2.6

(1) *Let* $\mathsf{Y} = p^{-1}(\mathsf{T}^{n-1}) \cap \mathsf{N}_n \subset \mathsf{X}_n$. *Then* $\mathsf{Y} = \mathsf{N}(\mathscr{L}_n)$, *where* $\mathscr{L}_n = \mathscr{G}_{1,n} + \{1, \dots, n-1\} \times \{n\}$.

(2) $\dim \mathsf{U}(D_{\mathbf{i}}) = \dim \mathsf{U}(G_{\mathbf{i}})$.

(3) *For every* $k = 0, 1, \dots, n-1$ *there exists an* AB

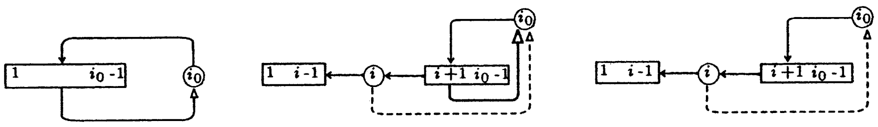
$$\Phi_k : \mathsf{N}(\mathscr{L}_n) \to \left(\coprod_{\mathbf{i} \in I^e_k} \mathsf{N}(\mathscr{L}_{\mathbf{i}}) \times \mathsf{U}(D_{\mathbf{i}}) \right) \coprod \left(\coprod_{\mathbf{i} \in I^*_{\leqslant k}} \mathsf{N}(\mathscr{K}_{\mathbf{i}}) \times \mathsf{U}(D_{\mathbf{i}}) \right).$$

(4) *Let* $\mathsf{Q}_{\mathbb{I}} = \mathsf{V}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})$, $\Phi_{\mathsf{Y}} = \Phi_{n-1}$, $\mathsf{K}_{\mathbb{I}} = \Phi_{\mathsf{Y}}^{-1}(\mathsf{Q}_{\mathbb{I}})$. *Then the* AB $\Phi_{\mathsf{Y}} : \mathsf{Y} \to \bigsqcup_{\mathbb{I}\in\mathbf{I}^*} \mathsf{Q}_{\mathbb{I}}$ *distinguishes components* (*Remark* 2.1(4)). *In particular*, $\mathsf{Y} = \cup_{\mathbb{I}\in\mathbf{I}^*} \mathsf{C}_{\mathbb{I}}$, *where* $\mathsf{C}_{\mathbb{I}} = \overline{\mathsf{K}_{\mathbb{I}}}$, *is a decomposition of* Y *in irreducible components.*
(5) *For any* $\mathbb{I} \in \mathbf{I}^*$ *holds* $\dim \mathsf{C}_{\mathbb{I}} = n(n-1)/2$.
(6) *For any* $\mathbb{I} \in \mathbf{I}^*$ *the set* $\mathsf{K}_{\mathbb{I}}$ *is open in* $\mathsf{C}_{\mathbb{I}}$.
(7) *For any* $\mathbb{I} \in \mathbf{I}^*$ *there exists a regular map* $S_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \to \mathbb{GL}_n$, *such that the map* $F_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \to \mathsf{T}^n$, *defined as* $F_{\mathbb{I}}(u) = S_{\mathbb{I}}(u)^{-1} u S_{\mathbb{I}}(u)$, $u \in \mathsf{K}_{\mathbb{I}}$ *is a regular embedding and* $F_{\mathbb{I}}(\mathsf{K}_{\mathbb{I}}) = \mathsf{N}(\mathscr{G}^{\mathbb{I}}_{1,n})$, *especially* $\operatorname{Im} F_{\mathbb{I}}$ *is open in* T^n.

Proof. Obviously, $p^{-1}(\mathsf{T}^{n-1}) = \mathsf{V}(\mathscr{L}_n + \{(n,n)\})$. For any $v \in \mathsf{X}^n$ holds $v_{nn} = 0$, that gives us (1). The statement (2) is checked immediately. The proof of the statement (3) consists of iterated compositions of AB's. We use below the agreement from Remark 2.4.

Step 1. There exists an AB $\varphi_{\mathscr{L}\mathscr{N}0} : \mathsf{N}(\mathscr{L}_n) \to \bigsqcup_{i=1}^{n} \mathsf{N}(\mathscr{N}_{(i,n)})$.

Proof. We apply Lemma 2.2(2) to $\mathsf{x} = ((1,n), \dots, (n-1,n))$. □



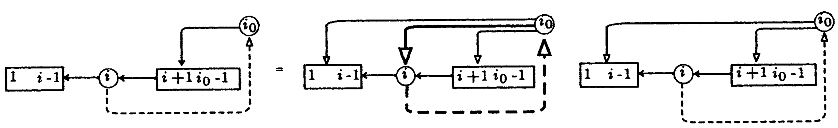
$\mathcal{L}_n$, $\mathcal{N}_{(i,n)}$ and $\mathcal{M}_{(i,n)}$. The bolder arrow shows $\{(i, i_0)\} \circ D_{0\,(i,n)}$

Step 2. There exists an AB $\varphi_{\mathscr{N}\mathscr{M}(i,n)} : \mathsf{N}(\mathscr{N}_{(i,n)}) \to \mathsf{N}(\mathscr{M}_{(i,n)}) \times \mathsf{U}(D_{0(i,n)})$, $i = 1, \dots, n$.

Proof. The case $i = n$ is obvious, otherwise the sequence $\{(i,n)\} \times D_{0(i,n)}$ is LA for $(i,n) \circ D_{0(i,n)} = \{i+1, \dots, i_0 - 1\} \times \{i_0\}$ and we apply Lemma 2.4(2). □

Step 3. $\mathsf{N}(\mathscr{M}_{(i,n)}) = \mathsf{N}(\mathscr{L}_{(i,n)})$, $i = 1, \dots, n$.

Proof. If $i = n$, then $(n,n) \in \mathbf{I}^*$ and $\mathscr{M}_{(n,n)} = \mathscr{L}_{(n,n)} = \mathscr{K}_{(n,n)}$. If $i < n$, then $i \to n \to i$ is the unique cycle of the length $\leqslant 2$ in $\mathscr{M}_{(i,n)}$ and we apply Lemma 2.1(2). □



From $\mathcal{M}_{(i,n)}$ to $\mathcal{L}_{(i,n)}$: the bolder arrows show the unique cycle of the length $\leqslant 2$.

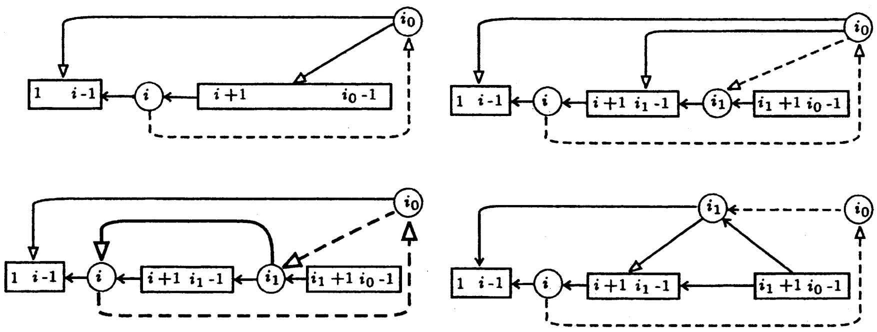


Fig. 1. The diagrams for the step $0 \mapsto 1$ (here exist solid arrows, incident to i_0).

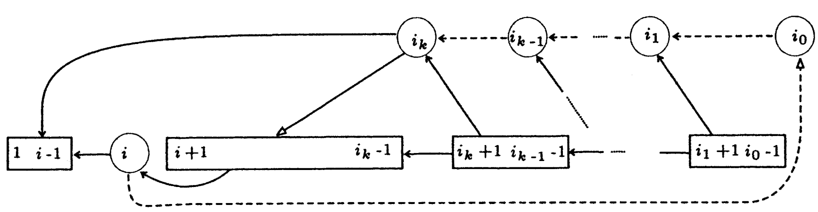
Step 4. Denote $\varphi_{\mathscr{M}\mathscr{L}(i,n)} : \mathsf{N}(\mathscr{M}_{(i,n)}) \to \mathsf{N}(\mathscr{L}_{(i,n)})$ the identity map. Then the composition of AB's $\{\varphi_{\mathscr{M}\mathscr{L}(i,n)}\}_{1\leqslant i\leqslant n} \circ \{\varphi_{\mathscr{N}\mathscr{M}(i,n)}\}_{1\leqslant i\leqslant n} \circ \varphi_{\mathscr{L}\mathscr{N}0}$ gives us the AB

$$\phi_0 : \mathsf{N}(\mathscr{L}_n) \to \bigsqcup_{\mathbb{I}\in\mathbf{I}_0} \mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}}) = \left(\bigsqcup_{\mathbb{I}\in\mathbf{I}_0^e} \mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})\right) \bigsqcup \left(\bigsqcup_{\mathbb{I}\in\mathbf{I}^*_{\leqslant 0}} \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})\right).$$

We construct an AB

$$\Phi_{k+1} : \mathsf{N}(\mathscr{L}_n) \to \left(\bigsqcup_{\mathbb{I}\in\mathbf{I}^e_{k+1}} \mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})\right) \bigsqcup \left(\bigsqcup_{\mathbb{I}\in\mathbf{I}^*_{\leqslant k+1}} \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})\right)$$

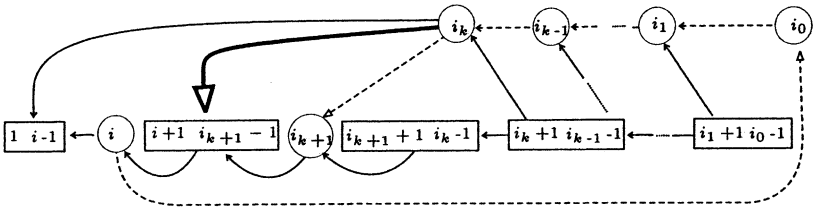
using induction on k. The base of induction Φ_0 is constructed in the step 4. [3]



Step of induction: we start from $\mathcal{L}_I$.

[3] The diagrams for the step $k \mapsto k+1$ are slightly different for $k = 0$ and $k > 0$ (see Fig. 1 in Section 2.8).

Step 5. For any $\mathbb{I} \in \mathbf{I}_k^e$ there exists an AB $\Phi_{\mathscr{L}\mathscr{N}\mathbb{I}} : \mathsf{N}(\mathscr{L}_{\mathbb{I}}) \to \bigsqcup_{i_{k+1}=i}^{i_k-1} \mathsf{N}(\mathscr{N}_{\mathbb{I}[i_{k+1}]})$.

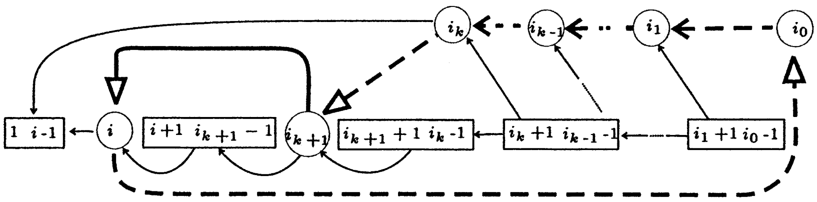


The diagram of $\mathcal{N}_{\mathbf{I}[i_{k+1}]}$: bolder arrow shows $D_{k+1\,\mathbf{I}[i_{k+1}]} \circ \{(i_k, i_{k+1})\}$.

Proof. It follows from Lemma 2.2(2), applied to $\mathsf{x} = ((i_k, i_k - 1), \ldots, (i_k, i + 1))$. □

Step 6. There exists an AB

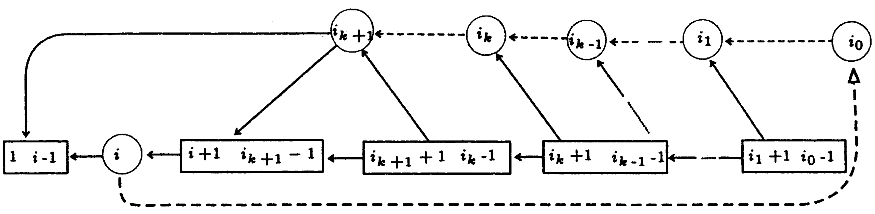
$$\Phi_{\mathscr{N}\mathscr{M}\mathbb{I}i_{k+1}} : \mathsf{N}(\mathscr{N}_{\mathbb{I}[i_{k+1}]}) \to \mathsf{N}(\mathscr{M}_{\mathbb{I}[i_{k+1}]}) \times \mathsf{U}(D^{\circ}_{k+1\,\mathbb{I}[i_{k+1}]}).$$



It is the diagram of $\mathcal{M}_{\mathbf{I}[i_{k+1}]}$: bolder arrows show the unique cycle of length $k+3$.

Proof. For $i_{k+1} = i$ it is obvious, otherwise $D_{k+1\,\mathbb{I}[i_{k+1}]} \times \{(i_k, i_{k+1})\}$ is RA for $D_{k+1\,\mathbb{I}[i_{k+1}]} \circ (i_k, i_{k+1}) = \{i_k\} \times \{i + 1, \ldots, i_{k+1} - 1\}$ and we apply the "right modified" Lemma 2.4(2) (see Remark 2.3). □

Step 7. $\mathsf{N}(\mathscr{M}_{\mathbb{I}[i_{k+1}]}) = \mathsf{N}(\mathscr{L}_{\mathbb{I}[i_{k+1}]}),\ i + 1 \leqslant i_{k+1} \leqslant i_k - 1, \mathsf{N}(\mathscr{M}_{\mathbb{I}[i]}) = \mathsf{N}(\mathscr{K}_{\mathbb{I}[i]})$.



Removing the arrow $i_{k+1} \to i$ from $\mathcal{M}_{\mathbf{I}[i_{k+1}]}$ we obtain $\mathcal{L}_{\mathbf{I}[i_{k+1}]}$.

Proof. If $i_{k+1} \neq i$, then $i \to i_0 \to \cdots \to i_k \to i_{k+1} \to i$ is the unique cycle of the length $\leqslant k+3$ in $\mathscr{M}_{\mathbb{I}[i_{k+1}]}$ and we apply Lemma 2.1(2) to $\mathscr{M}_{\mathbb{I}[i_{k+1}]}$. if $i_{k+1} = i$, then $\mathbb{I}[i_{k+1}] \in \mathbf{I}^*_{k+1}$ and $\mathsf{N}(\mathscr{M}_{\mathbb{I}[i_{k+1}]}) = \mathsf{N}(\mathscr{K}_{\mathbb{I}[i_{k+1}]})$.

Denote by $\Phi_{\mathscr{M}\mathscr{L}\mathbb{I}i_{k+1}} : \mathsf{N}(\mathscr{M}_{\mathbb{I}[i_{k+1}]}) \to \mathsf{N}(\mathscr{L}_{\mathbb{I}[i_{k+1}]})$ the identity map, $i \leqslant i_{k+1} \leqslant i_k - 1$. $\square$

Step 8. There exists on AB

$$\Phi_{k+1} : \mathsf{N}(\mathscr{L}_n) \to \left(\bigsqcup_{\mathbb{I} \in \mathbf{I}^e_{k+1}} \mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}}) \right) \bigsqcup \left(\bigsqcup_{\mathbb{I} \in \mathbf{I}^*_{\leqslant k+1}} \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}}) \right).$$

Proof. We construct Φ_{k+1} as composition of Φ_k and the family of AB's, which are identities on $\mathsf{N}(\mathscr{K}_{\mathbb{I}})$, $\mathbb{I} \in \mathbf{I}^*_{\leqslant k}$ and the composition

$$\{\Phi_{\mathscr{M}\mathscr{L}\mathbb{I}i_{k+1}}\}_{i \leqslant i_{k+1} < i_k} \circ \{\Phi_{\mathscr{N}\mathscr{M}\mathbb{I}i_{k+1}}\}_{i \leqslant i_{k+1} < i_k} \circ \Phi_{\mathscr{L}\mathscr{N}\mathbb{I}}$$

on $\mathscr{N}(\mathscr{L}_{\mathbb{I}})$, $\mathbb{I} \in \mathbf{I}^e_k$. The statement (3) is proved. $\square$

For $k = n-1$ holds $\mathbf{I}^e_k = \emptyset$, $\mathbf{I}^*_{\leqslant k} = \mathbf{I}^*$ and the statement (4) follows from (3) for $k = n-1$ and (5). To prove (5), we remark, that $\mathsf{V}(\mathscr{K}_{\mathbb{I}}) = \mathsf{N}(\mathscr{K}_{\mathbb{I}})$ (Lemma 2.1(1)) and $\dim \mathsf{U}(D_{\mathbb{I}}) = \dim \mathsf{U}(G_{\mathbb{I}})$ (2), hence $\dim \mathsf{K}_{\mathbb{I}} = \dim \mathsf{N}(\mathscr{K}_{\mathbb{I}}) + \dim \mathsf{U}(D_{\mathbb{I}}) = \dim \mathsf{V}(\mathscr{K}_{\mathbb{I}}) + \dim \mathsf{U}(G_{\mathbb{I}})$. Following $\mathscr{K}^{G_{\mathbb{I}}}_{\mathbb{I}} = \mathscr{G}^{\mathbb{I}}_{\prec_{\mathbb{I}}}$ (Lemma 2.5) the last sum equals $n(n-1)/2$, that gives us (5).

Let C be a component of Y and for some $\mathbb{I} \in \mathbf{I}_k$ both $(\mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})) \cap \mathsf{C}$ and $(\mathsf{N}(\mathscr{L}_{\mathbb{I}[i_{k+1}]}) \times \mathsf{U}(D_{\mathbb{I}[i_{k+1}]})) \cap \mathsf{C}$ are dense in C. Following Lemma 2.2(3), $(\mathsf{N}(\mathscr{L}_{\mathbb{I}[i_{k+1}]}) \times \mathsf{U}(D_{\mathbb{I}[i_{k+1}]})) \cap \mathsf{C}$ is open in $(\mathsf{N}(\mathscr{L}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})) \cap \mathsf{C}$, hence (6) follows from $\mathsf{C} \subset \mathsf{N}(\mathscr{L}_n)$.

Let $\varphi_{\mathbb{I}}$ be as defined in Lemma 2.5(4), $\iota_{\mathbb{I}} : \mathsf{N}(\mathscr{G}^{\mathbb{I}}_{1,n}) \hookrightarrow \mathsf{T}^n$ be the canonical (open) inclusion and $\Phi_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \to \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}})$ is the AB, induced by Φ_{Y} in restriction on $\mathsf{K}_{\mathbb{I}}$. Besides let us fix (following statement (2)) an isomorphism $\Phi_{D\mathbb{I}} : \mathsf{U}(D_{\mathbb{I}}) \to \mathsf{U}_{\mathbb{I}}$. Then we define $F_{\mathbb{I}}$ as composition

$$F_{\mathbb{I}} : \mathsf{K}_{\mathbb{I}} \xrightarrow{\Phi_{\mathbb{I}}} \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}(D_{\mathbb{I}}) \xrightarrow{1_{\mathsf{N}(\mathscr{K}_{\mathbb{I}})} \times \Phi_{D\mathbb{I}}} \mathsf{N}(\mathscr{K}_{\mathbb{I}}) \times \mathsf{U}_{\mathbb{I}} \xrightarrow{\varphi^{-1}_{\mathbb{I}}} \mathsf{N}(\mathscr{G}^{\mathbb{I}}_{1,n}) \xrightarrow{\iota_{\mathbb{I}}} \mathsf{T}^n.$$

The first three maps are bijective, hence $\operatorname{Im} F_{\mathbb{I}}$ is open in T^n. Denote for $u \in \mathsf{K}_I$ $\Phi_{\mathbb{I}}(u) = (F(u), S(u))$. Since $\Phi_{\mathbb{I}}$ and $\varphi_{\mathbb{I}}$ are AB's, for $u \in \mathsf{K}_n$ the map $S_{\mathbb{I}}$ can be defined as $S_{\mathbb{I}}(u) = S(u)\Phi_{D\mathbb{I}}(S(u))^{-1}$. $\square$

2.9. The proof of the main proposition

Proof. We prove Proposition 1 by induction on n. The base of induction is the case $n = 1$.[4] Let us prove the step of induction from $n-1$ to n. By induction hold $\mathsf{X}^{n-1} = \bigsqcup_{j \in \mathbf{K}^{n-1}} \mathsf{K}^{n-1}_j$, any $\mathsf{C}^{n-1}_j = \overline{\mathsf{K}^{n-1}_j}$ is a component of X^{n-1} and K^{n-1}_j is open

[4] For $n = 2$ holds $\mathsf{X}_2 = \mathsf{C}^2_1 \cup \mathsf{C}^2_2$, $\mathsf{C}^2_1 = \mathsf{V}((2,1))$, $\mathsf{C}^2_2 = \mathsf{V}((1,2))$, $\mathsf{K}^2_1 = \mathsf{C}^2_1$, $\mathsf{K}^2_2 = \mathsf{V}((1,2)^*)$, $S^2_1(\mathsf{K}^2_1) = \{\mathbb{1}_2\}$ and $S^2_2(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for any $u \in \mathsf{K}^2_2$.

in C_j^{n-1}. Then $\mathsf{X}^n = \bigsqcup_{j\in\mathbf{K}^{n-1}} \mathsf{Y}_j$, where $\mathsf{Y}_j = p^{-1}(\mathsf{K}_j^{n-1}) \cap \mathsf{N}_n = p^{-1}(\mathsf{K}_j^{n-1}) \cap \mathsf{X}^n$. For $j \in \mathbf{K}^{n-1}$ by $S_{\mathsf{Y}_j} : \mathsf{Y}_j \to \mathbb{GL}_n$ denote a regular map, such that

$$S_{\mathsf{Y}_j}(y) = \begin{pmatrix} S_j^{n-1}(p(y)) & 0 \\ 0 & 1 \end{pmatrix}$$

and by $F_{\mathsf{Y}_j} : \mathsf{Y}_j \to \mathsf{Y}$ denote such map, that $F_{\mathsf{Y}_j}(y) = S_{\mathsf{Y}_j}(y)^{-1} y S_{\mathsf{Y}_j}(y)$, $y \in \mathsf{Y}_j$, so the diagram

$$\begin{array}{ccc} \mathsf{Y}_j & \xrightarrow{F_{\mathsf{Y}_j}} & \mathsf{Y} \\ {\scriptstyle p|_{\mathsf{Y}_j}}\downarrow & & \downarrow{\scriptstyle p|_{\mathsf{Y}}} \\ \mathsf{K}_j^{n-1} & \xrightarrow{F_j^{n-1}} & \mathsf{T}^{n-1} \end{array}$$

is commutative. By induction F_j^{n-1} is an open embedding. F_{Y_j} is an open embedding as well: it maps bijectively Y_j onto $p^{-1}(F_j^{n-1}(\mathsf{K}_j^{n-1}))$. As $F_{\mathsf{Y}_j}(\mathsf{Y}_j) = \bigsqcup_{\mathbb{I}\in\mathbf{I}(n)^*} (F_{\mathsf{Y}_j}(\mathsf{Y}_j) \cap \mathsf{K}_{\mathbb{I}})$, every $F_{\mathsf{Y}_j}(\mathsf{Y}_j) \cap \mathsf{K}_{\mathbb{I}}$ is either empty or open dense in $\mathsf{K}_{\mathbb{I}}$.

Let C be a component of X^n. There exists a unique $j \in \mathbf{K}^{n-1}$, such that $p(\mathsf{C}) \subset \mathsf{C}_j^{n-1}$ and $\mathsf{K}_j^{n-1} \cap p(\mathsf{C})$ is open in $p(\mathsf{C})$.[5] Then $\mathsf{C} \cap \mathsf{Y}_j$ is open in C, moreover $\mathsf{C} \cap \mathsf{Y}_j$ is a component of Y_j. Since $F_{\mathsf{Y}_j}(\mathsf{Y}_j) \subset \mathsf{Y}$ is open, the component $F_{\mathsf{Y}_j}(\mathsf{C} \cap \mathsf{Y}_j)$ of $F_{\mathsf{Y}_j}(\mathsf{Y}_j)$ is open in a uniquely defined component $\mathsf{C}_{\mathbb{I}}$ of Y and from Lemma 2.6(5) follows, that $\dim \mathsf{C} = n(n-1)/2$. Hence if $F_{\mathsf{Y}_j}(\mathsf{Y}_j) \cap \mathsf{K}_{\mathbb{I}}$ is open dense in $\mathsf{K}_{\mathbb{I}}$, then $F_{\mathsf{Y}_j}^{-1}(\mathsf{K}_{\mathbb{I}})$ is open dense in $\overline{F_{\mathsf{Y}_j}^{-1}(\mathsf{K}_{\mathbb{I}})}$, which is a component of X^n. Thus we obtained the enumeration of the components of X^n by the set

$$\mathbf{K}^n = \{(j, \mathbb{I}) \mid j \in \mathbf{K}^{n-1}, \mathbb{I} \in \mathbf{I}^*(n), \text{ such that } F_{\mathsf{Y}_j}(\mathsf{Y}_j) \cap \mathsf{K}_{\mathbb{I}} \text{ is open in } \mathsf{K}_{\mathbb{I}}\},$$

where every such $k = (j, \mathbb{I})$ defines a component $\mathsf{C}_k^n = \overline{F_{\mathsf{Y}_j}^{-1}(\mathsf{K}_{\mathbb{I}})}$. We set $\mathsf{K}_k^n = F_{\mathsf{Y}_j}^{-1}(\mathsf{K}_{\mathbb{I}})$. It is open in $\mathsf{C}_k^n \cap \mathsf{Y}_j$, hence it is open in C_k^n. The map $S_k^n : \mathsf{K}_k^n \to \mathbb{GL}_n$ is defined as $S_k^n(u) = S_{\mathsf{Y}_j}(u) S_{\mathbb{I}}(u)$, $u \in \mathsf{K}_k^n$. Then $F_k^n(\mathsf{K}_k^n) = F_{\mathbb{I}} F_{\mathsf{Y}_j}(F_{\mathsf{Y}_j}^{-1}(\mathsf{K}_{\mathbb{I}})) = F_{\mathbb{I}}(F_{\mathsf{Y}_j}(\mathsf{Y}_j) \cap \mathsf{K}_{\mathbb{I}})$ is open in $F_{\mathbb{I}}(\mathsf{K}_{\mathbb{I}})$, hence (Lemma 2.6(7)) open in T^n. $\square$

Remark 2.6

(1) Propositions 1 and 2 show, that any C_k^n, $K \in \mathbf{K}^n$ is a rational variety.
(2) There exist $n!$ components of X^n of the form $\mathsf{V}(\mathscr{G}_{\prec})$ and 2^{n-1} components, which are components of Y_n. We do not know the cardinality of $\mathbf{K}^n$.

[5] In general $\overline{p(\mathsf{C})} = p(\mathsf{C})$ does not coincide with a component of X_{n-1}, e.g. X_3 contains a component C, $\mathsf{C} = \mathsf{V}(X_{ij}, i, j = 1, 2; X_{13}X_{31} + X_{23}X_{32})$. Then $p_{32}(\mathsf{C}) = \{0\}$, but the components of X_2 have the dimension 1.

3. An application to Lie algebras representations

3.1. Main results

Lemma 3.1. *Let U be the universal enveloping of a finite dimensional Lie algebra L, $\overline{U}$ be the associated graded algebra, for $u \in U$ $\overline{u}$ be the class of u in $\overline{U}$ and* deg *be the standard degree in U and in $\overline{U}$. If $\Gamma = \Bbbk[u_1, \dots, u_t]$ is a commutative subalgebra in U, such that $\overline{u}_1, \dots, \overline{u}_t$ form a regular sequence in $\overline{U}$ and for $\mu \in$* Specm *Γ $\chi_\mu : \Gamma \to \Bbbk$ is the homomorphism, such that* Ker *$\chi_\mu = \mu$, then for $f \in I_\mu = U(u_1 - \mu_1) + \cdots + U(u_t - \mu_t)$, where $\mu_i = \chi_\mu(u_i)$, there exist $f_i \in U$, $i = 1, \dots, t$, such that $f = f_1(u_1 - \mu_1) + \cdots + f_t(u_t - \mu_t)$ and for $d = \max_{i=1,\dots,t} \deg f_i u_i$ holds $d = \deg f$.*

Proof. We choose the presentation $f = f_1(u_1 - \mu_1) + \cdots + f_t(u_t - \mu_t)$ with the minimal possible d and suppose $d > \deg f$. Assume $d = \deg f_1 g_1 = \cdots = \deg f_r g_r$ and $\deg f_i g_i < d$, $i > r$. Let U^t $(\overline{U}^t)$ be the free U-module (the free $\overline{U}$-module) and $\vec{\mathsf{e}}_i = (\delta_{ij})_{j=1,\dots,t}$, $i = 1, \dots, t$ be their standard basis. For $\mathsf{s} = (s_1, \dots, s_t) \in U^t$ let $\overline{\mathsf{s}} = (\overline{s}_1, \dots, \overline{s}_r, 0, \dots, 0) \in \overline{U}^t$. Denote $\mathsf{f} = (f_1, \dots, f_t)$, $\mathsf{u} = (u_1, \dots, u_t)$, $\mathsf{m} = (\mu_1, \dots, \mu_t) \in U^t$ and define the $\Bbbk$-linear map $\cdot : U^t \times U^t \to U$ as $\mathsf{x} \cdot \mathsf{y} = \sum_{i=1}^{t} x_i y_i$ (analogously define $\cdot : \overline{U}^t \times \overline{U}^t \to \overline{U}$). Then $\mathsf{f} \cdot (\mathsf{u} - \mathsf{m}) = f$ and $\overline{\mathsf{f}} \cdot \overline{\mathsf{u} - \mathsf{m}} = \overline{\mathsf{f}} \cdot \overline{\mathsf{u}} = 0$, because $d > \deg f$. The sequence $\overline{u}_1, \dots, \overline{u}_r$ is regular, hence from the exactness of the Koszul complex follows, that for some $g_{ij} \in U$ holds $\overline{\mathsf{f}} = \sum_{1 \leqslant i \leqslant j \leqslant r} \overline{g}_{ij} \overline{\mathsf{g}_{ij}}$, where $\overline{\mathsf{g}_{ij}} = -\overline{u}_j \vec{\mathsf{e}}_i + \overline{u}_i \vec{\mathsf{e}}_j$. Let $\mathsf{g}_{ij}(\mathsf{m}) = -(u_j - \mu_j)\vec{\mathsf{e}}_i + (u_i - \mu_i)\vec{\mathsf{e}}_j$ for $1 \leqslant i \leqslant j \leqslant r$ and for other i, j we set $\mathsf{g}_{ij}(\mathsf{m}) = 0$, $g_{ij} = 0$. Then $\mathsf{f} = \sum_{1 \leqslant i \leqslant j \leqslant t} g_{ij} \mathsf{g}_{ij}(\mathsf{m}) + \mathsf{h}$ for some $\mathsf{h} = (h_1, \dots, h_t) \in U^t$, such that $\deg h_i \leqslant \deg f_i$ for all i and $\deg h_i < \deg f_i$ for $i \leqslant r$, hence $\max_{i=1,\dots,t} \deg h_i u_i < d$. Since u_ith mutual commute, there holds $\mathsf{g}_{ij}(\mathsf{m}) \cdot (\mathsf{u} - \mathsf{m}) = 0$ and then $\mathsf{h} \cdot (\mathsf{u} - \mathsf{m}) = f$, that contradicts the minimality of d. □

Corollary 2. *In assumptions of Lemma* 3.1 *for any $\mu \in$* Specm *Γ holds $I_\mu \neq U$ and there exists a simple U-module M, generated by $m \in M$, such that for any $\gamma \in \Gamma$ holds $\gamma m = \chi_\mu(\gamma) m$.*

Proof. If $I_\mu \neq U$, then we denote $N = U/I_\mu$, take a maximal submodule $N' \subset N$ and put $M = N/N'$. If $I_\mu = U$, then $1 \in I_\mu$. Following Lemma 3.1, for some $f_1, \dots, f_t \in U$ holds $1 = \sum_{i=1}^{t} f_i(u_i - \mu_i)$ and $\deg f_i u_i \leqslant \deg 1 = 0$, that is impossible. □

Theorem 2 follows from Corollaries 2 and 1 and the equalities $z_{ij} = \overline{c}_{ij}$, $1 \leqslant i \leqslant j \leqslant n$.

3.2. Concluding remarks

(1) Using Corollary 1 in [5] we obtain more results for GZ subalgebra Γ in gl_n, especially, *for fixed* $\mu \in \operatorname{Specm}\Gamma$ *there exists finitely many non-isomorphic* M, satisfying Corollary 2 (see also [2]), Γ is a maximal commutative subalgebra in U_n [6], etc.
(2) Analogous results seem to be true for other series of simple complex Lie algebras.
(3) Yu. Drozd remarked, that the results of Section 3.1 can be extended on a more wide class of PBW-algebras.

Acknowledgements

I would like to thank Yuriy Drozd, Slava Futorny and Volodymyr Mazorchuk for useful discussions. The first draft of this work was written while the author was an Alexander von Humboldt fellow, University Bielefeld, Germany. This research was partially supported by CRDF grant UM2-2094.

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