SET-THEORETICAL SOLUTIONS TO THE QUANTUM YANG-BAXTER EQUATION

PAVEL ETINGOF, TRAVIS SCHEDLER, ALEXANDRE SOLOVIEV

ABSTRACT. In the paper [Dr], V.Drinfeld formulated a number of problems in quantum group theory. In particular, he suggested to consider "set-theoretical" solutions of the quantum Yang-Baxter equation, i.e. solutions given by a permutation R of the set $X \times X$, where X is a fixed set. In this paper we study such solutions, which in addition satisfy the unitarity and nondegeneracy conditions. We discuss the geometric and algebraic interpretations of such solutions, introduce several constructions of them, and give their classification in group-theoretic terms.

0. Introduction

The quantum Yang-Baxter equation is one of the basic equations in mathematical physics, which lies in the foundation of the theory of quantum groups. This equation involves a linear operator $R:V\otimes V\to V\otimes V$, where V is a vector space, and has the form

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$
 in $\text{End}(V \otimes V \otimes V)$,

where R^{ij} means R acting in the i-th and j-th components.

In the last 15 years, many solutions of this equation were found and the related algebraic structures (Hopf algebras) have been intensively studied. However, these solutions were usually "deformations" of the identity solution. On the other hand, it is interesting to study solutions which are not obtained in this way. In [Dr], Drinfeld suggested to study the simplest class of such solutions – the so called settheoretical solutions. By definition, a set-theoretical solution is a solution for which V is a vector space spanned by a set X, and X is the linear operator induced by a mapping $X \times X \to X \times X$.

In this paper we study set-theoretical solutions of the quantum Yang-Baxter equation, satisfying additional conditions: invertibility, unitarity, and nondegeneracy. They turn out to have many beautiful properties. We discuss the geometric and algebraic interpretations of such solutions, introduce several constructions of them, and give their classification in terms of group theory.

The brief content of the paper is as follows.

Chapter 1 contains the background material. In Section 1.1 we give the main definitions and the simplest examples. We introduce the notion of a nondegenerate symmetric set, which is a set X with an invertible mapping $R: X^2 \to X^2$ satisfying the quantum Yang-Baxter equation and the nondegeneracy and unitarity conditions. We explain that if X is a nondegenerate symmetric set then the set X^n has a natural action of the symmetric group S_n , called the twisted action, which is, in general, different from the usual action by permutations. In Section 1.2 we

show that any nondegenerate symmetric set defines a coloring rule for collections of closed smooth curves in the plane, under which the number of colorings depends only on the number of curves involved, and not at the pattern of their intersections. This gives a topological interpretation of the notion of a nondegenerate symmetric set. In Section 1.3 we show that the twisted action of S_n on X^n for a nondegenerate symmetric set is conjugate to the action by permutations.

Chapter 2 introduces and studies the main algebraic structure associated to a nondegenerate symmetric set X – its structure group G_X . In Section 2.1 we show that G_X has two natural actions on X, which are conjugate to each other. In Sections 2.2,2.3 we show that the group G_X is naturally a subgroup of $Aut(X) \ltimes \mathbb{Z}^X$, such that the 1-cocycle defined by the projection $G_X \to \mathbb{Z}^X$ is bijective. Using this result, in Section 2.4 we show that nondegenerate symmetric sets, up to isomorphism, are in 1-1 correspondence with quadruples (G, X, ρ, π) , where G is a group, X is a set, ρ a left action of G on X, and π a bijective 1-cocycle of G with coefficients in \mathbb{Z}^X . In Sections 2.5-2.6 we show that there exists a unique, up to isomorphism, indecomposable nondegenerate symmetric set of order p, where p is a prime $-X = \mathbb{Z}/p\mathbb{Z}$, R(x,y) = (x+1,y-1). In Section 2.7 we prove solvability of the structure group. In Section 2.8 we apply the notion of the structure group of a nondegenerate symmetric set. to the study of decomposable nondegenerate symmetric sets. Finally, in Sections 2.9-2.10 we study the quantum algebras associated to a nondegenerate symmetric set by the Faddeev-Reshetikhin-Takhtajan-Sklyanin construction.

Chapter 3 introduces the main constructions of nondegenerate symmetric sets – linear, affine, multipermutation solutions, twisted unions, generalized twisted unions. In this chapter we classify such solutions, and study their properties. At the end we give the results of a computer calculation, which found all nondegenerate symmetric sets X with $|X| \leq 8$.

In Chapter 4 we consider power series solutions of the Yang-Baxter equation, which are a generalization of linear solutions. We show that a power series solution with a generic linear part is equivalent to a linear solution.

In the appendix we introduce the notion of a T-structure on an abelian group A, which is motivated by the definition of the map T in Proposition 2.2. We discuss the connection of T-structures with bijective 1-cocycles, in particular in the case of a cyclic group A.

Acknowledgments

We are grateful to Noam Elkies for useful discussions. The work of Pavel Etingof was supported by an NSF grant. Travis Schedler thanks MIT for hospitality.

1. Braided and symmetric sets

1.1. Definitions.

Let X be a nonempty set, and $S: X \times X \to X \times X$ be a bijection. We will denote the components of S by S_1 and S_2 (i.e. $S(x_1, x_2) = (S_1(x_1, x_2), S_2(x_1, x_2))$); they are binary operations on X. For positive integers i < n let the map $S^{ii+1}: X^n \to X^n$ be defined by $S^{ii+1} = id_{X^{i-1}} \times S \times id_{X^{n-i-1}}$.

Definition 1.1.

(i) A pair (X, S) is called nondegenerate if the maps $X \to X$ defined by $x \to S_2(x, y)$ and $x \to S_1(z, x)$ are bijections for any fixed $y, z \in X$.

(ii) A pair (X,S) is said to be a braided set if S satisfies the braid relation

$$(1.1) S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}.$$

(iii) A pair (X, S) is called involutive if

$$(1.2) S^2 = id_X$$

A braided set (X, S) which is involutive is called a symmetric set.

(iv) Pairs (X, S) and (X', S') are said to be isomorphic if there exists a bijection $\phi: X \to X'$ which maps S to S'.

The main objects of study in this paper are nondegenerate symmetric sets. Our main goal is to learn to construct them and to understand their properties. For brevity, nondegenerate symmetric sets will often be called "solutions" (meaning nondegenerate solutions of equations (1.1),(1.2)).

Examples. 1. Let X be any set, and S(x,y) = (y,x). Then (X,S) is a nondegenerate symmetric set. It is called "the trivial solution".

- 2. (Lyubashenko, see [Dr]) Let X be any set, and S(x,y)=(f(y),g(x)), where $f,g:X\to X$. Then: (X,S) is nondegenerate iff f,g are bijective; (X,S) is braided iff fg=gf; (X,S) is involutive iff $g=f^{-1}$ (in this case it is also braided, i.e. symmetric). In the last case (X,S) is called "a permutation solution". If f is a cyclic permutation, we will say that (X,S) is a cyclic permutation solution. It is clear that two permutation solutions are isomorphic if and only if the corresponding permutations are conjugate.
- 3. Let $(X, S_X), (Y, S_Y)$ be two solutions. Then $(X \times Y, S_X \times S_Y)$ is a solution, which is called the Cartesian product of X and Y.

Recall that the braid group B_n is generated by elements $b_i, 1 \le i \le n-1$, with defining relations

$$(1.3) b_i b_i = b_i b_i, |i - j| > 1; b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},$$

and that the symmetric group S_n is the quotient of B_n by the relations $b_i^2 = 1$. Therefore, we have the following obvious proposition.

Proposition 1.1. (i) The assignment $b_i \to S^{ii+1}$ extends to an action of B_n on X^n if and only if (X, S) is a braided set.

(ii) The assignment $b_i \to S^{ii+1}$ extends to an action of S_n on X^n if and only if (X,S) is a symmetric set.

This proposition explains our terminology.

Definition 1.2. The action of B_n (or S_n) on X_n defined by Proposition 1.1 will be called the twisted action.

Let $\sigma: X \times X \to X \times X$ be the permutation map, defined by $\sigma(x,y) = (y,x)$. Let $R = \sigma \circ S$. The map R is called the R-matrix corresponding to S. We have the following obvious proposition:

Proposition 1.2. (i) (X, S) is a braided set if and only if R satisfies the quantum Yang-Baxter equation

(1.4)
$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

and is a symmetric set if and only if in addition to (1.4) R satisfies the unitarity condition

$$(1.5) R^{21}R = 1.$$

A nice corollary of the properties of nondegenerate involutive pairs (X, S) is the following crossing symmetry property.

Let $X_1,...,X_n$ be sets, $Y=X_1\times...\times X_n$, and let $Q:Y\to Y$ be a map, $Q=(Q_1,...,Q_n)$. Suppose that the function $Q_i(x_1,...,x_n)$, regarded as a function of x_i when other x_j are fixed, is a bijection $X_i\to X_i$. In this case, define $Q^{t_i}:Y\to Y$ (the transposition of Q in the i-th component) by the condition: if $Q(x_1,...,x_n)=(y_1,...,y_n)$ then $Q^{t_i}(x_1,...,x_{i-1},y_i,x_{i+1},...,x_n)=(y_1,...,y_{i-1},x_i,y_{i+1},...,y_n)$.

Proposition 1.3. If (X, S) is nondegenerate and involutive, and $R = \sigma S$, then R^{t_i} is defined for i = 1, 2, and R has the crossing symmetry property

(1.6)
$$R^{t_1}(R^{21})^{t_1} = R^{t_2}(R^{21})^{t_2} = id_{X^2}.$$

In particular, R^{t_i} are bijections.

Proof. The statement of the proposition is equivalent to the equality

$$(1.7) |\{(k,l) \in X^2 : S(l,j) = (k,i), S(k,i') = (l,j')\}| = \delta_{ii'}\delta_{jj'}.$$

Let us check this equality. if S(l,j)=(k,i), S(k,i')=(l,j'), then by $S^2=1$ we have S(k,i)=(l,j), which by nondegeneracy implies i=i',j=j'. Conversely, if i=i',j=j', then there exist unique k,l such that conditions (1.7) are satisfied. \square

Remark. The operation t_i can be defined for all mappings Q, not necessarily such that Q_i is invertible as a function of x_i . To do this, we should regard Q not as a map of $X_1 \times ... \times X_n$ to itself, but as a linear operator on the vector space $V_1 \otimes ... \otimes V_n$, where V_i is the vector space spanned by X_i (for simplicity we assume that X_i are finite). In this case, Q^{t_i} can be defined to be the endomorphism of $V_1 \otimes ... \otimes V_i^* \otimes ... \otimes V_n$, obtained by dualizing the i-th component of Q. Under this definition, the crossing symmetry equations (1.6) make sense for any map R, and it is easy to show that if R satisfies the unitarity condition, then crossing symmetry is equivalent to nondegeneracy.

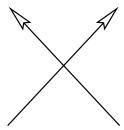
1.2. Colorings of flat links.

Nondegenerate symmetric sets turn out to have a nice geometric interpretation, which is given below. This interpretation is not new, but is a very simple special case of the theory of quantum invariants of links. This theory is described in detail in several textbooks, e.g. [Tu].

By a nondegenerate smooth curve in the plane we mean a parameterized curve $\gamma(t) = (x(t), y(t))$ such that the functions x, y are smooth, and their derivatives are never simultaneously zero. A nondegenerate smooth curve has a canonical orientation, defined by the direction of the tangent vector γ' .

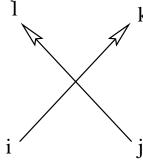
By a flat link we mean a finite collection of closed nondegenerate smooth curves in the plane.

It is clear that the only singularities of a generic flat link are simple crossings. Thus, from combinatorial point of view, a generic flat link is the same thing as an oriented flat graph, whose vertices are all 4-valent, and have the form



This graph is allowed to have closed edges, without any vertices on them. Let X be a set, and $S:X^2\to X^2$ a mapping.

Definition 1.3. An X-coloring of a generic flat link L is an assignment to every edge of the graph L of a color (an element of X), such that for any vertex of L of the form 1

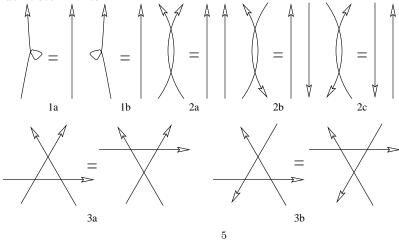


one has S(i,j) = (l,k).

For a general map S, it is not obvious why at least one coloring of L exists. However, we have the following proposition.

Proposition 1.4. If (X, S) is a finite nondegenerate symmetric set, then the number of X-colorings of L equals to $|X|^{n(L)}$, where |X| is the size of X, and n(L) the number of components in L.

Proof. If L consists of n(L) non-intersecting closed simple curves, then the result is clear (any component can have any color). However, it is well known that any generic flat link can be brought to this form by using a sequence of the following Reidemeister moves:



Thus, the only thing we have to prove is that the number of colorings does not change when either of these moves is applied.

The invariance of the number of colorings under move 1a follows from the statement

$$(1.8) |\{k \in X : S(i,k) = (j,k)\}| = \delta_{ij}.$$

This statement follows from nondegeneracy and involutivity. Indeed, suppose S(i,k) = (j,k), then $S^2 = 1$ implies S(j,k) = (i,k), which by nondegeneracy implies i = j. Thus, S(i,k) = (i,k). By nondegeneracy, k is unique if it exists. So it remains to show that for any i there exists k such that S(i,k) = (i,k). To do this, let k be defined by $S_1(i,k) = i$. By nondegeneracy, such a k exists. Then S(i,k) = (i,k'). Then S(i,k') = (i,k) (since $S^2 = 1$), and thus by nondegeneracy k = k'. Move 1b is treated analogously.

The invariance under move 2a follows directly from the involutivity. The invariance under moves 2b and 2c follows from the crossing symmetry.

The invariance under move 3a is exactly the braid relation. The invariance under move 3b follows from crossing symmetry and the braid relation. Indeed, we have to check the equality

$$(1.9) (R^{21})^{t_2}R^{13}(R^{32})^{t_2} = (R^{32})^{t_2}R^{13}(R^{21})^{t_2}.$$

This can be rewritten as

$$(1.10) ((R^{32})^{t_2})^{-1}(R^{21})^{t_2}R^{13} = R^{13}(R^{21})^{t_2}((R^{32})^{t_2})^{-1}.$$

Using crossing symmetry, we have $(R^{t_i})^{-1} = (R^{21})^{t_i}$, so (1.10) reduces to

$$(1.11) (R^{23})^{t_2}(R^{21})^{t_2}R^{13} = R^{13}(R^{21})^{t_2}(R^{23})^{t_2}.$$

Transposing the second component on both sides, we get

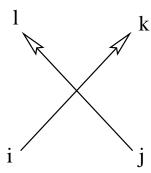
$$(1.12) R^{21}R^{23}R^{13} = R^{13}R^{23}R^{21},$$

which is the Yang-Baxter equation with 1 and 2 permuted. The proposition is proved. \Box

Proposition 1.4 has a rather trivial, but curious application. Suppose we have a system L of closed nondegenerate smooth curves with simple intersections in a space of dimension > 2, with orientations at intersection points (i.e. it is agreed which incoming edge at each vertex is the left incoming edge). For such a system we can compute the number of colorings as explained above.

Corollary 1.5. If the number of X-colorings of L is not equal to $|X|^{n(L)}$, then L cannot be put on the plane preserving orientations at vertices, without additional self-intersections.

Example. Consider the two-component graph L of the form



It is easy to see that the number of colorings of this graph equals to the number of fixed points of R on X^2 . So, if $R \neq id$, the number of colorings is less than $|X|^2$. Thus, L cannot be put on the plane without additional self-intersections (of course, this is obvious from the picture).

1.3. The isomorphism of the two S_n -actions.

Let (X, S) be a set with a mapping. Introduce the notation

(1.13)
$$S(x,y) = (g_x(y), f_y(x)).$$

Proposition 1.6. If S is involutive then

$$(1.14) f_{f_y(x)}g_x(y) = y.$$

If (X, S) is a braided set then

$$(1.15) f_y f_x = f_z f_t, \ g_x g_y = g_t g_z \ when \ S(x, y) = (t, z).$$

Proof. Straightforward. \square

Recall that in the previous chapter for any symmetric set (X, S) we defined the twisted action of S_n on X^n . We will need the following simple, but important result.

Proposition 1.7. If (X, S) is a symmetric set, then the map $J_n : X^n \to X^n$ given by the formula

$$J_n(x_1,...,x_n) = (f_{x_n} f_{x_{n-1}}...f_{x_2}(x_1),...,f_{x_n}(x_{n-1}),x_n).$$

satisfies the commutation relation

$$(1.17) J_n S^{ii+1} = \sigma^{ii+1} J_n.$$

Proof. We will prove this statement by induction in n. For n=2, the statement follows directly from the involutivity of S. So let us assume the statement for n=k, and prove it for n=k+1.

Observe that

$$(1.18) J_{k+1} = Q_{k+1} \circ (J_k \times id_X),$$

where $Q_n(x_1,...,x_n) = (f_{x_n}(x_1),...,f_{x_n}(x_{n-1}),x_n)$. Since Q_n commutes with σ^{ii+1} when i < n-1, formula (1.17) for i < k follows from the induction assumption. So it remains to prove the formula for i = k.

For i = k, the formula reduces to formulas (1.14),(1.15). \square

If in addition (X, S) is nondegenerate, the map J_n is obviously bijective. Therefore, we get

Corollary 1.8. If (X, S) is a nondegenerate symmetric set, then J_n conjugates the twisted action of S_n on X^n to the canonical action of S_n on X^n by permutations. Thus, the two actions of S_n are isomorphic.

Note that for a degenerate symmetric set the two actions of S_n may be non-isomorphic. For example, for any set X set S(x,y) = (x,y). Then (X,S) is a symmetric set, but it is degenerate for |X| > 1. As a result, the two actions of S_n are not isomorphic in this case, since the twisted action of S_n is trivial (all points are fixed).

2. The structure group

2.1. The structure group G_X and its actions on X.

Let X be a set and $S: X^2 \to X^2$ a mapping. It turns out to be very useful to introduce the group G_X generated by elements of X with defining relations

$$(2.1) xy = tz when S(x,y) = (t,z).$$

Definition 2.1. The group G_X is called the structure group of X.

Example. If (X, S) is the trivial pair (S(x, y) = (y, x)), then $G_X = \mathbb{Z}^X$ is the free abelian group generated by X.

One of the main properties of the structure group is the following:

Proposition 2.1. Suppose that (X, S) is nondegenerate. Then (X, S) is a braided set if and only if the following conditions are simultaneously satisfied:

- (i) the assignment $x \to f_x$ is a right action of G_X on X;
- (ii) the assignment $x \to g_x$ is a left action of G_X on X;
- (iii) the linking relation

$$f_{g_{f_y(x)}(z)}(g_x(y)) = g_{f_{g_y(z)}(x)}(f_z(y))$$

holds.

Proof. Conditions (i)-(iii) are exactly components 1-3 of the braid relation. \square

Proposition 2.2. (a) Suppose (X, S) is involutive, and the maps f_x are invertible and satisfy condition (i) of Proposition 2.1. Define the map $T: X \to X$ by the formula $T(y) = f_y^{-1}(y)$. Then one has $f_x^{-1}T = Tg_x$.

Suppose in addition g_x are invertible, so that (X, S) is nondegenerate and involutive. Then:

- (b) The map T is invertible. Thus, the left actions of G_X on X given by $x \to f_x^{-1}$, $x \to g_x$ are isomorphic to each other.
- (c) Condition (i) in Proposition 2.1 implies (ii) and (iii). Thus, (X, S) is symmetric if and only if the assignment $x \to f_x$ is a right action of G_X on X.

Proof. (a) We have

$$f_x^{-1}T(y) = f_x^{-1}f_y^{-1}(y) = f_{g_x(y)}^{-1}f_{f_y(x)}^{-1}(y) = f_{g_x(y)}^{-1}g_x(y) = Tg_x(y).$$

(b) It follows from nondegeneracy and involutivity (see the proof of Proposition 1.4, move 1a) that T is invertible, and $T^{-1}(z) = g_z^{-1}(z)$.

(c) The fact that (i) implies (ii) follows from (a) and (b). Let us prove that (i) implies (iii). Using (1.14), we can rewrite the linking relation in the form

$$(2.2) f_{f_{f_z f_y(x)}(z)}^{-1} f_{f_y(x)}^{-1}(y) = f_{f_{f_z(y)} f_{g_y(z)}(x)}^{-1} f_z(y).$$

Set $u = f_z f_y(x) = f_{f_z(y)} f_{g_y(z)}(x)$ (these two expressions are equal by (i)). Then (2.2) takes the form

$$f_{f_u^{-1}(z)}f_{f_z^{-1}(u)}^{-1} = f_u^{-1}f_z,$$

which is a direct consequence of (1.14) and (i). \square

2.2 The properties of the group G_X .

Now we will determine the structure of the group G_X for a nondegenerate symmetric set.

Let Aut(X) be the group of permutations of X, and \mathbb{Z}^X be the free abelian group spanned by X. We will denote the generator of \mathbb{Z}^X corresponding to $x \in X$ by t_x . Let $M_X = Aut(X) \ltimes \mathbb{Z}^X$ be the semidirect product, associated to the action of Aut(X) on \mathbb{Z}^X . The group M_X consists of elements of the form st, where $s \in Aut(X)$, $t \in \mathbb{Z}^X$, and we have the commutation relation $st_x = t_{s(x)}s$.

Consider the assignment

$$(2.3) x \to f_x^{-1} t_x.$$

Proposition 2.3. If (X, S) is a nondegenerate symmetric set then assignment (2.3) extends to a group homomorphism $G_X \to M_X$.

Proof. We have to show that

(2.4)
$$f_x^{-1}t_x f_y^{-1}t_y = f_u^{-1}t_u f_v^{-1}t_v \text{ when } S(x,y) = (u,v).$$

We have

$$f_x^{-1}t_xf_y^{-1}t_y=f_x^{-1}f_y^{-1}t_{f_y(x)}t_y,\ f_u^{-1}t_uf_v^{-1}t_v=f_u^{-1}f_v^{-1}t_{f_v(u)}t_v.$$

But we have $v = f_y(x)$, and by involutivity of S we have $f_v(u) = y$. Therefore, (2.4) follows from (1.15). \square

We will denote the constructed homomorphism $G_X \to M_X$ by ϕ_f . Thus, $\phi_f(x) = f_x^{-1} t_x$.

Analogously, we can construct a homomorphism $\phi_g: G_X \to M_X$ given by $\phi_g(x) = t_x g_x$. The homomorphisms ϕ_f, ϕ_g are conjugate in the following sense: if we denote by \hat{T} the automorphism of M_X induced by the permutation T of X, then \hat{T} conjugates ϕ_f to ϕ_g . Thus, it is enough for us to study the properties of ϕ_f . From now on we will denote it simply by ϕ .

Proposition 2.4. The homomorphism ϕ is injective.

Proof. Proposition 2.4 is an immediate corollary of Proposition 2.5 (see below). \Box

Let $\pi: G_X \to \mathbb{Z}^X$ be the map defined by $\pi(g) = t$ if $\phi(g) = st$, $s \in Aut(X)$, $t \in \mathbb{Z}^X$. We fix the structure of a left G_X -module on \mathbb{Z}^X induced from the assignment $x \to f_{\sigma}^{-1}$.

For convenience, we will write the group operation in \mathbb{Z}^X additively.

Proposition 2.5. (a) π is a 1-cocycle of G_X with coefficients in the G_X - module \mathbb{Z}^X , i.e. $\pi(g_1g_2) = g_2^{-1}\pi(g_1) + \pi(g_2)$. (b) π is bijective.

The proof of Proposition 2.5 is contained in the next section.

2.3. Proof of Proposition 2.5. Property (a) follows from the definition of the semidirect product and the fact that ϕ is a homomorphism. So, we have to prove (b). We will explicitly construct the map $h: \mathbb{Z}^X \to G_X$ inverse to π . Let $X^+ = \{t_x \in \mathbb{Z}^X | x \in X\}$, $X^- = \{-t_x \in \mathbb{Z}^X | x \in X\}$ and $Y = X^+ \cup X^-$. They are G_X -invariant subsets of \mathbb{Z}^X . For a nonnegative integer k consider the subset \mathbb{Z}^X_k of \mathbb{Z}^X consisting of all the elements that are sums of no more than k elements of Y. In particular, $\mathbb{Z}^X_0 = \{0\}$, $\mathbb{Z}^X_k \subset \mathbb{Z}^X_{k+1}$ and \mathbb{Z}^X_k form a covering of \mathbb{Z}^X . Similarly, let G_X^k denote the set of elements of G_X representable as a product $x_1...x_m$, where $x_1,...,x_m \in X \cup X^{-1} \subset G_X$ and $m \leq k$.

We want to define h inductively on each \mathbb{Z}_k^X in a compatible way. Let h(0) = 1, $h(t_x) = x$, $h(-t_x) = (g_x^{-1}(x))^{-1}$ for $x \in X$. In this way, h is defined on $\mathbb{Z}_1^X = Y$. For convenience, for $g \in G_X$ and $\xi \in \mathbb{Z}^X$, denote by $g * \xi$ the result of the action of g on ξ .

Lemma 2.6. For $\xi, \eta \in Y$ one has $h(h(\xi) * \eta)h(\xi) = h(h(\eta) * \xi)h(\eta)$.

Proof.

We have to consider 3 cases:

- (i) Both ξ and η belong to X^+ . Let $x = h(\xi), y = h(\eta)$. We want to show that $f_x^{-1}(y)x = f_y^{-1}(x)y$. Let $S(f_x^{-1}(y), x) = (z, y)$, then $S(z, y) = (f_x^{-1}(y), x)$, so $f_y(z) = x$ and hence $S(f_x^{-1}(y), x) = (f_y^{-1}(x), y)$.
- (ii) Only ξ belongs to X^+ , while $\eta \in X^-$. Then we have $h(\xi) = x \in X$, and $\eta = -t_{y'}$ for some $y' \in X$. We need to check that $(g_{f_x^{-1}(y')}^{-1}(f_x^{-1}(y')))^{-1}x = f_{g_{y'}^{-1}(y')}(x)(g_{y'}^{-1}(y'))^{-1}$. Let $z = g_{y'}^{-1}(y') = T^{-1}(y')$ (see the proof of Proposition 2.2 (a), (b)). Then $g_{f_x^{-1}(y')}^{-1}(f_x^{-1}(y')) = T^{-1}f_x^{-1}T(z) = g_x(z)$. So, the desired equality is just $(g_x(z))^{-1}x = f_z(x)z^{-1}$, which holds by the definition of G_X .
- (iii) Both ξ and η belong to X^- . This case is similar to (ii), so the proof is omitted. \square

Now, let us assume that h has already been defined for elements of \mathbb{Z}_k^X . Take $\eta \in \mathbb{Z}_{k+1}^X$, then $\eta = a + \xi$ for $a \in \mathbb{Z}_k^X$, $\xi \in Y$. Define $h(\eta) = h(h(a) * \xi)h(a)$.

Lemma 2.7. The map h is well-defined on each \mathbb{Z}_k^X , and thus on the whole \mathbb{Z}^X .

Proof. We proceed by induction on k. The lemma is certainly true for k=0,1. Suppose h is well-defined on \mathbb{Z}_{k-1}^X . For any $a\in\mathbb{Z}_{k-2}^X$ and $\xi,\eta\in Y$

$$h((a + \xi) + \eta) = h(h(a + \xi) * \eta)h(a + \xi) = h(h(h(a) * \xi)h(a) * \eta)h(h(a) * \xi)h(a).$$

We need to check that the last expression is symmetric in ξ, η and equal to h(a) when $\xi = -\eta$. Set $\xi' = h(a) * \xi$, $\eta' = h(a) * \eta$, then we arrive at the formula $h((a+\xi)+\eta) = h(h(\xi')*\eta')h(\xi')h(a)$ that is symmetric in ξ', η' (hence in ξ, η) by Lemma 2.6. If $\xi' = -\eta' = t_z$ for $z \in X$ we have

$$h(z*(-t_z))zh(a) = h(-t_{T(z)})zh(a) = (T^{-1}T(z))^{-1}zh(a) = h(a).$$

Lemma 2.7 is proved. \square

Lemma 2.8. The maps $h: \mathbb{Z}^X \to G_X$ and $\pi: G_X \to \mathbb{Z}^X$ are inverse to each other.

Proof. It suffices to check that:

- $(\mathbf{i})\pi \circ h(a) = a \text{ for } a \in \mathbb{Z}_k^X.$ $(\mathbf{ii})h \circ \pi(b) = b \text{ for } b \in G_X^k.$

The statements (i), (ii) are quite simple for k = 0, 1. Let us make an inductive step. For $a \in \mathbb{Z}_k^X$, $\xi \in Y$

$$\pi \circ h(a+\xi) = \pi(h(h(a) * \xi)h(a)) = h(a)^{-1} * \pi(h(h(a) * \xi)) + \pi(h(a)) = a + \xi,$$

provided $\pi \circ h(a) = a$. So, (i) is proved. Similarly, for $b \in G_X^k$, $y \in X \cup X^{-1}$

$$h\circ\pi(yb) = h(b^{-1}*\pi(y) + \pi(b)) = h(b^{-1}*(b*\pi(y)))b = yb,$$

provided $h \circ \pi(b) = b$. So, Lemma 2.8 is proved along with Proposition 2.5. \square

2.4. Classification of nondegenerate symmetric sets via groups with bijective 1-cocycles.

Definition 2.2. A bijective cocycle quadruple is a quadruple (G, X, ρ, π) , where G is a group, X is a set, $\rho: G \times X \to X$ a left action of G on X, and $\pi: G \to \mathbb{Z}^X$ a bijective 1-cocycle of G with coefficients in \mathbb{Z}^X , where G acts in \mathbb{Z}^X by ρ .

Theorem 2.9. Nondegenerate symmetric sets, up to isomorphism, are in 1-1 correspondence with bijective cocycle quadruples, up to isomorphism. This correspondence is given by $F(X) = (G_X, X, \rho, \pi)$, where ρ is the action of G_X on X by $x \to f_x^{-1}$, and π is the 1-cocycle of Proposition 2.5.

In fact, nondegenerate symmetric sets form a category, as well as bijective cocycle quadruples. Morphisms in these categories are just maps which preserve all structures. Namely, a morphism of nondegenerate symmetric sets $(X_1, S_1) \to (X_2, S_2)$ is a map $f: X_1 \to X_2$ such that $(f \times f)S_1 = S_2(f \times f)$. Similarly, a morphism of bijective cocycle quadruples $(G_1, X_1, \rho_1, \pi_1) \to (G_2, X_2, \rho_2, \pi_2)$ is a pair of maps $f: X_1 \to X_2, \phi: G_1 \to G_2$, where ϕ is a group homomorphism, and $\rho_2(\phi(g)) \circ f = f \circ \rho_1(g), \, \pi_2(\phi(g)) = f(\pi_1(g)) \text{ for any } g \in G_1 \text{ (in the last formula } f \text{ is extended to a group homomorphism } \mathbb{Z}^{X_1} \to \mathbb{Z}^{X_2}).$ Denote these categories by NSS and BCQ (by abbreviating the names). It is clear that the map F is not only a map but also a functor $NSS \to BCQ$. Theorem 2.9 can be strengthened as follows.

Theorem 2.10. The functor $F: NSS \to BCQ$ is an equivalence of categories.

Proof of Theorems 2.9,2.10. To prove Theorems 2.9, 2.10 it is necessary to construct the inverse functor to the functor F, i.e. learn to reconstruct (X,S) from the quadruple F(X,S).

Consider a bijective cocycle quadruple (G, X, ρ, π) . Let t_x be the generator of \mathbb{Z}^X corresponding to $x \in X$. Then we have a natural embedding $a: X \to G$ given by the formula $a_x = \pi^{-1}(t_x)$.

For any $x \in X$, define the map $f_x : X \to X$ by $f_x^{-1} = \rho(a_x)$. Define the map $g_x: X \to X$ by $g_x(y) = f_{f_u(x)}^{-1}(y)$.

Let us show that

(2.5)
$$a_x a_y = a_{g_x(y)} a_{f_y(x)}.$$

Indeed, we have

$$\pi(a_x a_y) = \rho(a_y)^{-1} \pi(a_x) + \pi(a_y) = \rho(a_y)^{-1} t_x + t_y = t_{f_y(x)} + t_y.$$

Similarly,

$$\pi(a_{g_x(y)}a_{f_y(x)}) = t_{f_{f_y(x)}g_x(y)} + t_{f_y(x)} = t_y + t_{f_y(x)}.$$

Since π is bijective, we have the desired equality $a_x a_y = a_{g_x(y)} a_{f_y(x)}$.

Let us show that g_x is invertible. For this purpose define the map $T: X \to X$ by the formula $T(x) = f_x^{-1}(x)$. It is enough for us to show that T is invertible, since by (2.5) (similarly to Proposition 2.2(a)) we have $f_x^{-1}T(y) = Tg_x(y)$.

Proof that T is invertible. We have $t_{T(x)} = t_{f_x^{-1}(x)} = \rho(a_x)(t_x) = \rho(a_x)(\pi(a_x))$. Since π is a 1-cocycle, $\rho(a_x)(\pi(a_x)) = -\pi(a_x^{-1})$. It is clear that the inverse of T is given by the map $T': X \to X$, $T'(x) = (\pi^{-1}(-t_x))^{-1}$ if the latter is well defined, i.e. we have to check that $T'(x) \in X$ for $x \in X$. One observes that by the definition of π $\pi(\rho(\pi^{-1}(-t_x))(x)\pi^{-1}(-t_x)) = 0$ and hence $(\pi^{-1}(-t_x))^{-1} = \rho(\pi^{-1}(-t_x))(x) \in X$. So, T is bijective.

Now define a map $S: X^2 \to X^2$ by the formula $S(x,y) = (g_x(y), f_y(x))$. By the construction, it is nondegenerate and involutive. The fact that the map S satisfies condition (i) of Proposition 2.1 follows from (2.5).

Thus, (X, S) is a nondegenerate symmetric set. In other words, we constructed a map $F': BCQ \to NSS$, by $F'(G, X, \rho, \pi) = (X, S)$.

It is clear that the map F' is a functor. Indeed, if $\mu: (G, X, \rho, \pi) \to (G', X', \rho', \pi')$ is a morphism of bijective cocycle quadruples, then it respects the assignment $x \to a_x$, and therefore respects f and g, so it defines a morphism $F'(\mu): F'(G, X, \rho, \pi) \to F'(G', X', \rho', \pi')$.

To complete the proof, we need to show that $F \circ F' = id$, $F' \circ F = id$.

The identity $F' \circ F = id$ is obvious. Let us prove that $F \circ F' = 1$.

Let (G, X, ρ, π) be a bijective cocycle quadruple, and $(X, S) = F'(G, X, \rho, \pi)$. We need to show that $G = G_X$, and ρ, π are defined in the standard way.

As we have seen, the map $X \to G$ defined by $x \to a_x$ extends to a homomorphism $a: G_X \to G$, which transforms ρ into the standard action of G_X on X ($x \to f_x^{-1}$) and the composition $\pi \circ a$ is the standard bijective 1-cocycle on G_X . Since π is bijective, we get that a is bijective, as desired.

Theorems 2.9, 2.10 are proved. \square

Remark 1. Groups with bijective 1-cocycles have a nice geometric interpretation, which was pointed out to us by R.Howe and G.Margulis. Namely, in the case of Lie groups, a bijective 1-cocycle on a simply connected group G with coefficients in some real representation is the same thing as a left-invariant affine structure on G as a manifold, which identifies G with an affine space. Of course, to admit such a structure, the group must be contractible topologically, i.e. solvable. We will show later that this is also the case for finite groups with a bijective 1-cocycles.

Remark 2. It is known that many solvable Lie groups admit left-invariant affine structures as above [Bu]. It was even believed that any solvable simply connected Lie group does (Milnor's conjecture), but it was recently disproved by D.Burde [Bu], who found a nilpotent group which has no such structures.

Remark 3. Bijective cocycles on finite groups were recently used by the first author and S.Gelaki to construct new examples of semisimple Hopf algebras [EG].

2.5. The canonical abelian subgroup Γ of G_X .

Let $\Gamma \subset G_X$ be the intersection $G_X \cap \mathbb{Z}^X$, where G_X and \mathbb{Z}^X are both regarded as subgroups of M_X .

It is clear that Γ is a normal subgroup of G_X . Indeed, Γ is the kernel of the homomorphism $\rho: G_X \to Aut(X)$, defined by the action of G_X on X.

Let $G_X^0 = G_X/\Gamma$, and $A = \mathbb{Z}^X/\Gamma$. The group G_X^0 is the image of G_X in Aut(X), so, in particular, it is finite if the set X is finite.

The 1-cocycle π is obviously equal to id on Γ , so it descends to a bijective 1-cocycle $\bar{\pi}: G_X^0 \to A$. This follows from the fact that $\pi(b\gamma) = \pi(b) + \gamma$, $b \in G_X$, $\gamma \in \Gamma$.

If X is finite, A is a finite abelian group, and $|A| = |G_X^0|$. The resulting quadruple $(G_X^0, A, \bar{\rho}, \bar{\pi})$ where $\bar{\rho}: G_X^0 \to Aut(A)$ descends from ρ motivates the following general definition:

Definition 2.3. Define a bijective cocycle datum to be a quadruple (G, A, ρ, π) where G is a group, A a G-module with the G-action given by the homomorphism $\rho: G \to Aut(A)$, and $\pi: G \to A$ a bijective 1-cocycle employing the action ρ .

In fact, given any bijective cocycle datum, if we choose any set X with a G-action $\rho': G \to Aut(X)$ and a map $\phi: X \to A$ carrying ρ' to ρ , then X carries the structure of a nondegenerate set. Explicitly, define $f_y(x) = \rho'(\pi^{-1}(y)^{-1})(x)$, $g_x(y) = f_{f_y(x)}^{-1}(y)$, and $S(x,y) = (g_x(y), f_y(x))$ for $x,y \in X$. Then (X,S) is a nondegenerate symmetric set. Let us therefore make the following definition.

Definition 2.4. Take any bijective cocycle datum (G, A, ρ, π) . Define a set-structure for the datum to be a triple (X, ρ', ϕ) where X, ρ', ϕ are as above.

Note that in any set-structure, if $\phi(X)$ does not generate G, then we could consider the subgroup $G' \subset G$ generated by X, and clearly $\pi(G')$ is generated by $\pi(X)$ yielding the smaller datum (G', A', ρ, π) . Thus, let us call a set-structure generating if $\phi(X)$ generates G; these are the kinds we will consider. Note that when $\phi(X)$ generates G freely the datum is merely the corresponding bijective cocycle quadruple to the induced solution (X, S).

Now we may generalize the notion of factoring by Γ as follows: Take any datum (G,A,ρ,π) together with a generating set-structure (X,ρ',ϕ) . Let $H=Ker(\rho')$, $K=\pi(H)$. Since X is generating, H is a subgroup of $Ker(\rho)$ so K is a subgroup of A since π is an injective homomorphism restricted to H. Then, form G/H, A/K. Now G/H is just the image of G in Aut(X). Then, let $\bar{\rho}:G/H\to Aut(A/K)$, $\bar{\pi}:G/H\to A/K$ be the maps descending from ρ,π . We arrive at the conclusion that $(G/H,A/K,\bar{\rho},\bar{\pi})$ is a bijective cocycle datum. Together with the same set structure (taking the images of ρ' and ϕ in G/H) we obtain the solution (X,S) which is isomorphic to the original solution since the actions $f_x^{-1}:X\to X, x\in X$ are the same. It is clear that this is the smallest such datum giving rise to this solution.

Call a generating set-structure faithful if G acts faithfully on X. Then we have the following result:

Proposition. Given a bijective cocycle datum (G, A, ρ, π) and a faithful generating set-structure (X, ρ', ϕ) , then (G, A, ρ, π) is isomorphic to $(G_X^0, A_X, \rho_X, \pi_X)$ where G_X^0, A_X, ρ_X and π_X are obtained from the solution (X, S) induced by the original datum and set-structure.

Proof. Indeed, it is clear by the construction in Section 2.4 that $xy = g_x(y)f_y(x)$ in G so G is the image of G_X under the homomorphism sending $x \in X \subset G_X$ to $\phi(x)$ in G. Since G acts faithfully on X, it must be isomorphic to the image of G_X in Aut(X), namely G_X^0 . The action ρ is determined by ρ' and hence is the same as ρ_X since $\phi(X)$ generates G, so it is easy to see that A and the cocycle π are the same as A_X and π_X .

There are two main ideas behind this development. First, we have seen that the bijective cocycle structure can be formulated without the choice of a set X, and this leads easily to nondegenerate symmetric sets once X is chosen. Secondly, we now may classify nondegenerate symmetric sets via faithful generating set-structures on such data, which is especially useful in the case where X is finite and hence so is G_X^0 . The usefulness of this formulation will be firmly established in the following section.

2.6. Indecomposable symmetric sets of prime order.

Definition 2.5. (a) A subset Y of a nondegenerate symmetric set X is said to be an invariant subset if $S(Y \times Y) \subset Y \times Y$.

- (b) An invariant subset $Y \subset X$ is said to be nondegenerate if $(Y, S|_{Y \times Y})$ is a nondegenerate symmetric set.
- (c) A nondegenerate symmetric set (X,S) is said to be decomposable if it is a union of two nonempty disjoint nondegenerate invariant subsets. Otherwise, (X,S) is said to be indecomposable.

For example, a permutation solution is indecomposable if and only if it is cyclic. **Remark.** If X is finite, then any invariant subset Y of X is nondegenerate. Indeed, the map $S|_{Y\times Y}$ is bijective, as $S^2=1$, and for any $y\in Y$, the maps $f_y,g_y:Y\to Y$ are injective (by nondegeneracy of X), hence bijective by the finiteness. Thus Y is nondegenerate.

Proposition 2.11. A nondegenerate symmetric set (X, S) is indecomposable if and only if G_X acts transitively on X.

Proof. Proposition 2.15 below implies that if G_X acts transitively on X then (X, S) is indecomposable. Conversely, if $x \to f_x^{-1}$ is not transitive, consider two complementary nonempty G_X -invariant subsets X_1 and X_2 . They are invariant under f_x for all x, and hence under T (since $T(y) = f_y^{-1}(y)$), so they are invariant under g_x for all x. Thus X_1 and X_2 are invariant subsets of X. It is clear that these subsets are nondegenerate. Thus (X, S) is decomposable. \square

Now we will classify finite indecomposable nondegenerate symmetric sets which have p elements, where p is a prime.

Theorem 2.12. Let (X, S) be an indecomposable nondegenerate symmetric set, and |X| = p, where p is a prime. Then (X, S) is isomorphic to the cyclic permutation solution $(\mathbb{Z}/p\mathbb{Z}, S_0)$, where $S_0(x, y) = (y - 1, x + 1)$.

Proof. Since by Proposition 2.11 the group G_X^0 acts transitively on X, its order is divisible by p. Since $G_X^0 \subset Aut(X)$, its order divides p!. Thus, $|G_X^0| = |A| = pn$, where n is coprime to p.

Thus, $A = \mathbb{Z}/p\mathbb{Z} \oplus A_0$, where $|A_0|$ is coprime to p.

The subgroup $A_0 \subset A$ is the group of all elements of order not divisible by p. Therefore, A_0 is G_X^0 -stable, and the cocycle $\bar{\pi}: G_X^0 \to A$ defines a 1-cocycle $\pi': G_X^0 \to \mathbb{Z}/p\mathbb{Z}$.

Set $H = (\pi')^{-1}(0) = \bar{\pi}^{-1}(A_0)$. It is easy to see that H is a subgroup of G_X^0 of order n.

Let H_x be the stabilizer of a point $x \in X$ in G_X^0 . This is a subgroup of G_X^0 of index p, i.e. of order n. We want to show that $H_x = H$ for all x. Since $|H_x| = |H|$, for this purpose it is enough to show that $\pi'(H_x) = \{0\}$.

We will show that H_x acts trivially on $A/A_0 = \mathbb{Z}/p\mathbb{Z}$. This implies that $\pi'|_{H_x}$ is simply a homomorphism $H_x \to \mathbb{Z}/p\mathbb{Z}$. This, in turn, implies that $\pi'|_{H_x} = 0$, since $|H_x| = n$ is coprime to p.

To show that H_x acts trivially on A/A_0 , it is enough to prove that the image $\bar{t}_x \in A/A_0$ of the element $t_x \in \mathbb{Z}^X$ is not zero. Indeed, in this case \bar{t}_x generates A/A_0 , while \bar{t}_x is by definition fixed by H_x .

Now we prove that $\bar{t}_x \neq 0$. Assume that $\bar{t}_x = 0$. Since G_X acts transitively on X, we get $\bar{t}_y = 0$ for all $y \in X$. Thus, the natural map $\mathbb{Z}^X \to A/A_0$ is zero. Contradiction.

Thus, we showed that $H_x = H$. Therefore, H acts trivially on X. This implies that $H = A_0 = \{1\}$, $G_X^0 = A = \mathbb{Z}/p\mathbb{Z}$, the action of G_X^0 on A is trivial, and $\bar{\pi} = id$.

To conclude the proof, it is enough to observe that $\bar{t}_x=\bar{t}_y$ for any $x,y\in X$. This follows from the fact that there exists $g\in G_X^0$ such that gx=y, while the action of G_X^0 on A is trivial. Thus, \bar{t}_x does not depend on x, and hence $a_x=\bar{\pi}^{-1}(\bar{t}_x)\in G_X^0$ does not depend on X. Therefore, (X,S) is a permutation solution. The theorem is proved. \square

2.7. Solvability of the structure group.

In this section we prove the solvability of the structure group of a finite nondegenerate symmetric set.

Let G be a finite group, p a prime divisor of |G|. Let us write $|G| = mp^k$ for a positive k and m coprime to p.

Definition 2.6. A subgroup H of G of order m is called a Hall p'-subgroup.

We will use the following theorem of Hall:

Theorem 2.13. [As] If a finite group G has a Hall p'-subgroup for each prime p dividing |G|, then it is solvable.

Theorem 2.14. The structure group G_X of a finite nondegenerate symmetric set is solvable.

Proof. It is enough to show that the finite group $G_X^0 = G_X/\Gamma$ is solvable. We shall make use of the 1-cocycle $\bar{\pi}: G_X^0 \to A$ defined in section 2.5. Let $|G_X^0| = mp^k$, m,p being coprime. Define $H_p = \bar{\pi}^{-1}(p^kA)$. Obviously, p^kA is invariant under the action of G_X^0 , so H_p is a subgroup of G_X^0 . The order of H_p equals the order of p^kA , that is, in turn, equals m. So, H_p is a Hall p'-subgroup of G_X^0 and we apply Lemma 2.13 to conclude that G_X^0 is solvable. \square

2.8. Extensions and the structure group.

Let (Z, S) be a nondegenerate symmetric set.

Definition 2.7. We will say that Z is a union of two nondegenerate invariant subsets X, Y if $X \cap Y = \emptyset$ and $Z = X \cup Y$ as a set.

Thus, Z can be represented as a union of two nonempty nondegenerate invariant subsets if and only if it is decomposable.

Proposition 2.15. If (Z, S) is a union of nondegenerate invariant subsets X and Y, then the map S defines bijections $X \times Y \to Y \times X$, and $Y \times X \to X \times Y$.

Proof. It is clear that S defines an automorphism of $X \times Y \cup Y \times X$. So we need to show that it does not map elements of $X \times Y$ to $X \times Y$, and the same for $Y \times X$.

Let $x \in X, y \in Y$, and assume that $S(x,y) = (x',y'), x' \in X, y' \in Y$. Thus, $g_x(y) = x'$. However, by nondegeneracy of X there exists $x'' \in X$ such that $g_x(x'') = x'$. This violates the nondegeneracy of Z. Contradiction. \square

Example. The most obvious example of a union is the trivial union, defined by S(x,y) = (y,x), and S(y,x) = (x,y). However, as we will see later, there are much more interesting ways of constructing unions.

Let $(X, S_X), (Y, S_Y)$ be nondegenerate symmetric sets. Denote by Ext(X, Y) (extensions of X by Y) the set of all decomposable solutions Z which are unions of X and Y. As we showed, an element $Z \in Ext(X, Y)$ is completely determined by the function $S_Z : X \times Y \to Y \times X$.

Let us write $S_Z(x,y)$ in the form $(g_x(y), f_y(x))$.

Proposition 2.16. If $Z \in Ext(X,Y)$ then the assignments $x \to g_x$, $y \to f_y^{-1}$ are actions of G_X on Y and of G_Y on X.

Proof. The statement follows from Proposition 2.1, since G_X, G_Y are obviously subgroups of G_Z . \square

The conclusion of Proposition 2.16 is clearly not sufficient for S_Z to define an extension. There is an additional, rather complicated linking condition between f_y and g_x . Thus, we will consider a special case of "one-sided extensions".

Definition 2.8. An element $Z \in Ext(X,Y)$ is called a right (respectively, left) extension of X by Y if $S_Z(x,y) = (y, f_y(x))$ (respectively, $S_Z(x,y) = (g_x(y),x)$) for $x \in X, y \in Y$. The set of right (left) extensions of X by Y will be denoted by $Ext_+(X,Y)$, $Ext_-(X,Y)$, respectively.

It is clear that $Ext_+(X,Y) = Ext_-(Y,X)$.

The following proposition gives a complete group-theoretic description of $Ext_+(X,Y)$.

Proposition 2.17. The formula $S_Z(x,y) = (y, f_y(x))$ defines an element $Z \in Ext_+(X,Y)$ if and only if the assignment $y \to f_y^{-1}$ defines an action of G_Y on (X,S_X) .

Proof. We have

(2.6)
$$S^{12}S^{23}S^{12}(x_1, x_2, y) = (y, f_y g_{x_1}(x_2), f_y f_{x_2}(x_1)),$$

and

(2.7)
$$S^{23}S^{12}S^{23}(x_1, x_2, y) = (y, g_{f_y(x_1)}f_y(x_2), f_{f_y(x_2)}f_y(x_1)).$$

Equating (2.6) and (2.7) shows that f_y preserves S_X . It is easy to check that other relations for Z do not impose any new restrictions. The proposition is proved. \square

Corollary 2.18. If $Z \in Ext_+(X,Y)$ then the group G_Z is isomorphic to $G_Y \ltimes G_X$, where the semidirect product is formed using the action of G_Y on X via $y \to f_y^{-1}$.

In particular, for the trivial union $Z = X \cup Y$ we have $G_Z = G_X \times G_Y$.

Finally, let us describe the set Ext(X,Y), where X,Y are trivial symmetric sets, i.e. S_X, S_Y are the permutation of components. In this case, $G_X = \mathbb{Z}^X$, and $G_Y = \mathbb{Z}^Y$.

Proposition 2.19. The formula $S_Z(x,y) = (g_x(y), f_y(x))$ defines an element $Z \in Ext(X,Y)$ if and only if

- (i) the assignments $x \to g_x$, $y \to f_y^{-1}$ are actions of \mathbb{Z}^X on Y and of \mathbb{Z}^Y on X;
- (ii) the homomorphisms $\rho_X : \mathbb{Z}^X \to Aut(Y)$, $\rho_Y : \mathbb{Z}^Y \to Aut(X)$ defined by (i) are invariant under \mathbb{Z}^Y , \mathbb{Z}^X , respectively (where the actions of \mathbb{Z}^X on Aut(X) and of \mathbb{Z}^Y on Aut(Y) are trivial, and the actions of \mathbb{Z}^X , \mathbb{Z}^Y on each other are by ρ_X , ρ_Y).

Proof. Straightforward. \square

2.9. The quantum algebras associated to a nondegenerate symmetric set, and their relation to the structure group.

Let (X,S) be a finite nondegenerate symmetric set. Let V be the complex vector space spanned by X. Let v_x be the vector in V corresponding to $x \in X$, and $E_{xy}: V \to V$ be the endomorphism of V defined by $E_{xy}v_z = \delta_{yz}v_x$.

Let $R = \sigma S$. We regard R as a linear operator $V \otimes V \to V \otimes V$.

Following Faddeev, Reshetikhin, Sklyanin, and Takhtajan [FRT], define two quadratic algebras over \mathbb{C} associated to X.

1. The quantized algebra of functions on V. This is the quadratic algebra Q_X with generators q_x , $x \in X$, and relations

$$(2.8) R^{12}q^{13}q^{23} = q^{23}q^{13},$$

where

$$(2.9) q := \sum_{x} v_x \otimes q_x \in V \otimes Q_X$$

2. The quantized algebra of functions on $\operatorname{End}(V)$. This is the quadratic algebra A_X with generators L_{xy} , $x, y \in X$, and relations

$$(2.10) R^{12}L^{13}L^{23} = L^{23}L^{13}R^{12},$$

where $L = \sum E_{xy} \otimes L_{xy}$, $E_{xy} \in End(V)$.

Remark. The algebra A_X is a special case of H_R of [FRT]; similar algebras were also studied in [Ha,Sch].

Observe that the relations of Q_X can be written in the form

$$(2.11) q_x q_y = q_{q_x(y)} q_{f_y(x)}.$$

Let G_X^+ be the set of elements of G_X representable as a product of the generators (without inverses). This set is a monoid. We have $G_X^+ = \bigcup_{n \geq 0} G_X^{+n}$, where G_X^{+n} is the set of elements representable as a product of n generators. It follows from

Proposition 2.5 that this is a
$$\mathbb{Z}_+$$
-grading of G_X^+ , and $|G_X^{+n}| = \binom{n+N-1}{n}$.

Equation (2.11) implies that the algebra Q_X is isomorphic to $\mathbb{C}[G_X^+]$. Similarly, the relations in A_X can be written in the form

(2.12)
$$L_{xz}L_{yt} = L_{g_x(y)g_z(t)}L_{f_y(x)f_t(z)}.$$

Thus, A_X is isomorphic to $\mathbb{C}[G_{X\times X}^+]$.

Thus, we have

Proposition 2.20. The Hilbert series of the quadratic algebras Q_X, A_X are equal to $(1-t)^{-N}, (1-t)^{-N^2}$.

Thus, the Hilbert series are always as in the classical case R=1, when we have usual polynomial algebras.

According to [FRT], the algebra A_X is a bialgebra, with coproduct and counit defined by

(2.13)
$$\Delta(L) = L^{12}L^{13}, \epsilon(L) = 1.$$

Moreover, the algebra Q_X is a left comodule over A_X , so that the coaction is an algebra homomorphism. This coaction is defined by

(2.14)
$$\Delta(q) = L^{12}q^{13},$$

and is the quantum analogue of the action of the monoid $\operatorname{End}(V)$ on the space V. Furthermore, the algebra A_X can actually be extended to a Hopf algebra. Namely, define the algebra \hat{A}_X to be generated by A_X and an additional set of generators $(L^{-1})_{xy}$, $x, y \in X$, with an additional defining relation

$$(2.15) LL^{-1} = L^{-1}L = 1,$$

where $L^{-1} := \sum E_{xy} \otimes (L^{-1})_{xy}$.

Proposition 2.21. The algebra \hat{A}_X is a Hopf algebra with the antipode defined by

(2.16)
$$\gamma(L) = L^{-1}, \gamma(L^{-1}) = L.$$

Proof. We need to check the antipode axiom and the fact that (2.16) extends to an antiautomorphism of \hat{A}_X . The first statement is immediate, and the second is checked by a direct computation. \square

In particular, $\gamma^2 = 1$. Thus, \hat{A}_X is an involutive Hopf algebra, which is a quantum analogue of the algebra of regular functions on the group GL_N .

Now let $Comod(\hat{A}_X)$ be the category of finite-dimensional comodules over \hat{A}_X . Since \hat{A}_X is a Hopf algebra, this category is a rigid tensor category (for a definition of a rigid tensor category, see e.g. [DM]). If R = 1, this category coincides with the category of finite-dimensional representations of GL_N .

Theorem 2.22. The category $Comod(A_X)$ is equivalent to the category $Rep(GL_N)$ of finite-dimensional representations of $GL_N(\mathbb{C})$ as a rigid tensor category.

The proof of Theorem 2.22 is contained in the next section.

2.10. Proof of Theorem 2.22.

Let $\mathcal{O}(Mat_N)$ be the bialgebra of polynomial functions on $Mat_N(\mathbb{C})$.

Proposition 2.23. There exists a coalgebra isomorphism $\eta: \mathcal{O}(Mat_N) \to A_X$.

Proof. Fix a labeling of elements of X by indices $\{1,...,N\}$, so that $X = \{x_1,...,x_N\}$. We have $\mathcal{O}(Mat_N) = \mathbb{C}[T_{ij}]$, where $T = (T_{ij})$ is a matrix of indeterminates. Define η by the formula

$$\eta(T_{i_1 j_1} ... T_{i_k j_k}) = L_{y_1 z_1} ... L_{y_k z_k},$$

where $(y_1,...,y_k) = J_k^{-1}(x_{i_1},...,x_{i_k}), (z_1,...,z_k) = J_k^{-1}(x_{j_1},...,x_{j_k}),$ where J_k was defined in Section 1.3. By Proposition 1.7, the map η is well defined, and is a linear isomorphism. It is easy to check that η also respects the coproduct. Thus, η is a coalgebra isomorphism. \square

Proposition 2.23 implies that for any irreducible representation W of GL(V) $(V = \mathbb{C}^N)$ which occurs in $V^{\otimes k}$ for some k, we can define its direct image – the corresponding comodule η_*W of A_X .

In particular, consider the 1-dimensional comodule $Det = \eta_*(\Lambda^N V)$. It is a 1-dimensional space $\mathbb{C}v$, with coaction given by $\Delta(v) = D \otimes v$, where D is an element of A_X . It is easy to see that this element is central and group-like (i.e. $\Delta(D) = D \otimes D$), and does not depend on the labeling of X. The element D is called the quantum determinant.

It is clear that D is not a zero divisor in A_X . Indeed, D is invertible in \hat{A}_X , and D^{-1} is the quantum determinant of $(L^{-1})^{t_1}$ (i.e. D^{-1} is obtained from $(L^{-1})^{t_1}$ in the same way as D is obtained from L). Therefore, it makes sense to consider the algebra $\tilde{A}_X = A_X \otimes_{\mathbb{C}[D]} \mathbb{C}[D, D^{-1}]$. This algebra inherits a bialgebra structure form A_X .

We claim that the algebras \hat{A}_X , \tilde{A}_X are isomorphic as bialgebras. Indeed, L is invertible in \tilde{A}_X , and $L^{-1} = D^{-1}M_{N-1}(L)$, where $M_{N-1}(L)$ is a polynomial of L of degree N-1 (the matrix of quantum minors). This allows to define an obvious isomorphism between the algebras \hat{A}_X , \tilde{A}_X , which is clearly an isomorphism of bialgebras.

Let $\mathcal{O}(GL_N)$ be the Hopf algebra of polynomial functions on $GL_N(\mathbb{C})$,

Proposition 2.24. There exists a coalgebra isomorphism $\hat{\eta}: \mathcal{O}(GL_N) \to \hat{A}_X$.

Proof. It is easy to see that the map η from Proposition 2.23 is $\mathbb{C}[D]$ -linear (for $\mathcal{O}(Mat_N)$, D denotes the usual determinant). Thus, $\hat{\eta}$ is obtained simply by tensoring η over $\mathbb{C}[D]$ with $\mathbb{C}[D, D^{-1}]$. \square

Proposition 2.24 implies that the functor η_* of direct image induces an equivalence of abelian categories $Rep(GL_N) \to Comod(\hat{A}_R)$. So, to prove Theorem 2.22, it is enough to introduce a tensor structure on this functor.

By definition, the tensor structure is a collection of functorial isomorphisms $J_{WU}: \eta_*(W) \otimes \eta_*(U) \to \eta_*(W \otimes U)$, satisfying the following compatibility condition: $J_{W \otimes UY}(J_{WU} \otimes 1) = (1 \otimes J_{UY})J_{WU \otimes Y}$ [DM].

 $J_{W\otimes U,Y}(J_{WU}\otimes 1)=(1\otimes J_{UY})J_{W,U\otimes Y}$ [DM]. Let us define J_{WU} for $W=V^{\otimes m}, U=V^{\otimes n}$. This should be an operator $J_{V^{\otimes m_{V}\otimes n}}:V^{\otimes m+n}\to V^{\otimes m+n}$. We set

(2.18)
$$J_{V \otimes^m V \otimes^n} = J_{m+n} (J_m^{-1} \otimes J_n^{-1}).$$

It is easy to see that $J_{V \otimes^m V \otimes^n}$ is an intertwiner, and that it satisfies the 2-cocycle condition. Further, by Proposition 1.7, $J_{V \otimes^m V \otimes^n}$ commutes with $S_m \times S_n$ (acting by permutations), so by Weyl duality it defines J_{WU} for any irreducible representations W, U which occur in $V^{\otimes m}$ for some m. Now set $J_{W'U'} = J_{WU}$ if $W' = W \otimes (Det)^{\otimes k}$, $U' = U \otimes (Det)^l$. This defines J_{WU} for any irreducible finite-dimensional representations W, U of GL_N . Since the category of finite-dimensional representations of GL_N is semisimple, we have defined J_{WU} for any objects W, U in $Rep(GL_N)$. We have

Proposition 2.25. The maps J_{WU} define a tensor structure on η_* .

Proof. Clear. \square

Theorem 2.22 is proved.

Remark. The proof of Theorem 2.22 shows that the bialgebra \hat{A}_X is obtained from $\mathcal{O}(GL_N)$ by twisting the multiplication, in the sense of Drinfeld.

3. METHODS OF CONSTRUCTION OF NONDEGENERATE SYMMETRIC SETS.

3.1. Linear and affine solutions.

In this section we will look for nondegenerate symmetric sets of the following form: X is an abelian group, and S is an affine linear transformation of $X \times X$. Such symmetric sets will be called *affine solutions*. Considering affine solutions was motivated by the results in [Hi].

We will start with considering a special case, when S is an automorphism of $X \times X$. In this case, an affine solution will be called a *linear solution*. For a linear solution, S has the form

$$(3.1) S(x,y) = (ax + by, cx + dy), \ a,b,c,d \in \text{End}X.$$

It is easy to check that for S of the form (3.1) the braid relation is equivalent to the equations [Hi]

$$a(1-a) = bac$$
, $d(1-d) = cdb$, $ab = ba(1-d)$, $ca = (1-d)ac$, $dc = cd(1-a)$,
(3.2) $bd = (1-a)db$, $cb - bc = ada - dad$.

It is also easy to see that the involutivity of S is equivalent to the equations

(3.3)
$$a^2 + bc = 1$$
, $cb + d^2 = 1$, $ab + bd = 0$, $ca + dc = 0$.

Finally, the nondegeneracy condition is obviously equivalent to the condition that b, c are invertible.

Proposition 3.1. If b, c are invertible, equations (3.2),(3.3) are equivalent to the equations

(3.4)
$$bab^{-1} = \frac{a}{a+1}, c = b^{-1}(1-a^2), d = \frac{a}{a-1}.$$

Proof. The second equation of (3.4) follows directly from (3.3). Also, (3.3) implies

(3.5)
$$a = -bdb^{-1}.$$

Therefore, multiplying the equation bd = (1 - a)db (which is in (3.2)) by b^{-1} on the right, we get

$$(3.6) -a = (1-a)d.$$

Since b, c are invertible, so is $bc = 1 - a^2$, so 1 - a is invertible. Thus, (3.6) implies the third equation of (3.4). Now the first equation of (3.4) follows from (3.5).

Conversely, substituting (3.4) into (3.2),(3.3), it is easy to show by a direct calculation that they are identically satisfied. \square

Corollary 3.2. A map S of the form (3.1) is a linear solution if and only if b, c are invertible, and (3.4) are satisfied. Thus, such solutions are in 1-1 correspondence with pairs (a,b) such that $bab^{-1} = \frac{a}{a+1}$.

Now consider general affine solutions. Then S has the form

(3.7)
$$S(x,y) = (ax + by + z, cx + dy + t), t, z \in X.$$

In this case, it is clear that the equations on a, b, c, d are the same as before. The only equation for z, t is obtained from the braid relation and has the form $t = -b^{-1}(1+a)z$. Thus, we get

Proposition 3.3. A map S of the form (3.7) is an affine solution if and only if b, c are invertible, (3.4) are satisfied, and $t = -b^{-1}(1+a)z$. Thus, such solutions are in 1-1 correspondence with triples (a,b,z) such that $bab^{-1} = \frac{a}{a+1}$.

Now consider examples of solutions of the equation

$$(3.8) bab^{-1} = \frac{a}{a+1}.$$

Example 1. [Hi] Let $X = \mathbb{Z}/n\mathbb{Z}$. Then $\operatorname{End} X = \mathbb{Z}/n\mathbb{Z}$, which is commutative, so equation (3.8) reads $a = \frac{a}{a+1}$, which is equivalent to $a^2 = 0$ (and b is any invertible element).

Example 2. Let $X = V^N$, where V is an abelian group. Then the algebra $Mat_N(\mathbb{Z})$ of integer matrices is mapped into EndX. Thus, it is enough for us to construct a solution of $bab^{-1} = \frac{a}{a+1}$ in $Mat_N(\mathbb{Z})$, such that $b \in GL_N(\mathbb{Z})$.

Let $a_{ij} = \delta_{i+1,j}$, and $b_{ij} = \begin{pmatrix} j \\ i \end{pmatrix}$. Then a, b satisfy (3.8). Indeed, this equation can be rewritten as ab = ba + aba, which at the level of matrix elements reduces to the well-known identity for binomial coefficients:

(3.9)
$$\binom{j}{i+1} = \binom{j-1}{i} + \binom{j-1}{i+1}.$$

We will use the following notation for this solution: $a = J_N$, $b = B_N$.

In fact, all solutions of (3.8) in $Mat_N(\mathbb{Z})$ can be obtained from J_N, B_N . Indeed, we have

Lemma 3.4. Let a,b be a solution of (3.8) in $Mat_N(\mathbb{C})$. Then a is nilpotent.

Proof. It follows from (3.8) that if λ is an eigenvalue of a then so is $\frac{\lambda}{\lambda+1}$. Therefore, if $\lambda \neq 0$, we get that a has infinitely many distinct eigenvalues. This is impossible, so $\lambda = 0$. \square

Thus, a is nilpotent. Then, by Jordan's theorem, a can be reduced, over \mathbb{Q} , to Jordan normal form: $a = J_{N_1} \oplus ... \oplus J_{N_K}$, where $J_{N_l} \in Mat_{N_l}(\mathbb{Z})$ are given by $(J_{N_l})_{ij} = \delta_{i+1,j}$. If a is of this form, then $b = b_0 A$, where A commutes with a, and $b_0 = B_{N_1} \oplus ... \oplus B_{N_K}$. Thus we have proved

Proposition 3.5. Any solution of (3.8) in $Mat_N(\mathbb{Z})$ with $b \in GL_N(\mathbb{Z})$ is conjugate under $GL_N(\mathbb{Q})$ to a solution of the form $a = J_{N_1} \oplus ... \oplus J_{N_K}$, $b = (B_{N_1} \oplus ... \oplus B_{N_K})A$, where [A, a] = 0.

Proposition 3.6. If $V = \mathbb{Z}/p\mathbb{Z}$, where p is a prime, and N < p, then any solution of (3.8) in EndX is conjugate to a solution of the form given in Proposition 3.5.

Proof. Let λ be an eigenvalue of a over the algebraic closure $\overline{\mathbb{Z}/p\mathbb{Z}}$. Then $\frac{\lambda}{\lambda+1}$ is also an eigenvalue. Therefore, if $\lambda \neq 0$, a has to have at least p distinct eigenvalues. Thus, $\lambda = 0$ and hence a is nilpotent. The rest of the proof is the same as for Proposition 3.5 (but instead of working over \mathbb{Q} we work over $\mathbb{Z}/p\mathbb{Z}$). \square

However, if $N \geq p$, other solutions are possible.

Example:
$$p = N = 2$$
, $a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3.2. Multipermutation solutions and equivariant fiber bundles.

Let (X,S) be a nondegenerate symmetric set. Then we can define another nondegenerate symmetric set (\bar{X},\bar{S}) , such that there exists a surjective morphism $\mu:(X,S)\to(\bar{X},\bar{S})$. This is done as follows.

For $x, y \in X$ we will write $x \sim y$ if $f_x = f_y$. Since $g_x = T^{-1}f_x^{-1}T$, in this case we also have $g_x = g_y$. It is clear that \sim is an equivalence relation. Let $\bar{X} = X/\sim$. For $x \in X$, let \bar{x} denote the image of x in \bar{X} .

By the definition, we have $x \sim y$ if and only if the images \bar{a}_x, \bar{a}_y of x, y in G_X^0 coincide. Thus, \bar{X} is naturally identified with the image of the map $\bar{a}: X \to G_X^0$ given by $\bar{a}_x = f_x^{-1}$.

Let A be the abelian group defined in Section 2.5. It is easy to see that $x \sim y$ if and only if $\bar{t}_x = \bar{t}_y$, where \bar{t}_x is the image of t_x under the homomorphism $\mathbb{Z}^X \to A$. This follows from the fact that $\bar{a}_x = \bar{\pi}^{-1}(\bar{t}_x)$, where $\bar{\pi}: G_X^0 \to A$ is the bijective 1-cocycle defined in Section 2.5. Thus, \bar{X} is naturally identified with the image of $X \subset \mathbb{Z}^X$ under the map $\mathbb{Z}^X \to A$.

The subset $\bar{X} \subset A$ is invariant under the action of G_X , i.e. under operators f_x^{-1} . Therefore, it is also invariant under g_x . Thus, according to the remark in Section 2.4, \bar{X} has a structure of a nondegenerate symmetric set, with $\bar{S}(\bar{x},\bar{y}) = (\overline{g_x(y)},\overline{f_y(x)})$, where $\bar{x},\bar{y} \in \bar{X}$, and x,y are any preimages of \bar{x},\bar{y} in X. We will call (\bar{X},\bar{S}) the retraction of (X,S), and denote it by Ret(X,S).

Definition 3.1. A solution (X, S) will be called a multipermutation solution of level n if n is the minimal nonnegative integer such that $|Ret^n(X, S)| = 1$ (i.e. $Ret^n(X, S)$ is finite of size 1).

A solution (X,S) is called irretractable if Ret(X,S)=(X,S) (i.e. if \sim is a trivial equivalence relation).

In particular, a multipermutation solution of level 0 is the trivial solution for |X| = 1, and a multipermutation solution of level 1 is a permutation solution. On the other hand, as we will see below, there exist irretractable affine solutions for |X| = 4.

In terms of the bijective cocycle datum (G_X^0, A, ρ, π) corresponding to (X, S) in the manner of Section 2.5, there is a nice formulation of retractions. Indeed, we have seen that \bar{X} is just the image of X in A, so the retraction is just given by a set-structure on (G_X^0, A, ρ, π) given by (\bar{X}, ρ, ϕ) where ϕ is the tautological embedding $\bar{X} \to A$, and the action ρ of G on A defines its action on X. We see

that the unique datum with a faithful generating set-structure for (X,S) is given by $\bar{G} = G_X^0/H$, $\bar{A} = A/K$ where $H = Ker(\rho)$, $K = \pi(H)$ along with the maps $\bar{\rho}, \bar{\pi}$ descending from ρ, π). Then, the faithful generating set-structure for $(\bar{G}, \bar{A}, \bar{\rho}, \bar{\pi})$ is given by $(X, \rho, \bar{\phi})$ where ρ is the action of G on X as before, which descends modulo H to an action of \bar{G} on X, and the map $\bar{\phi}: \bar{X} \to \bar{A}$ follows from the natural map $A \to A/K$.

Thus, given a datum (G, A, ρ, π) define its retraction by $Ret(G, A, \rho, \pi) = (G/H, A/K, \bar{\rho}, \bar{\pi})$ as above. We have seen that this definition corresponds to that for nondegenerate symmetric sets in the case of faithful generating set-structures. In view of this we have the following proposition:

Proposition. If the group A obtained from a nondegenerate symmetric set (X, S) as in Section 2.5 is finite and cyclic, then (X, S) is a multipermutation solution.

Proof. Assume |A| > 1. Then, since $|Aut(A)| = \phi(|A|) < |A|$, where ϕ is the Euler ϕ -function, $|Ker(\rho)| > 1$ and so (G_X^0, A, ρ, π) is retractable yielding a new datum with a smaller cyclic group \bar{A} . Inductively we find that for some n, $Ret^n(G_X^0, A, \rho, \pi) = (G', A', \rho', \pi')$ is trivial $(A' = G' = \{0\})$ since |A| was finite; clearly n = 0 if |A| = 1. This means that $Ret^n(X, S)$ is given by the trivial datum together with any faithful generating set-structure, which we find is just the trivial solution on |X| elements. Hence $|Ret^{n+1}(X, S)| = 1$. \square

Now we will consider solutions (Y, S_Y) such that $Ret(Y, S_Y)$ is a fixed solution (X, S). Such a solution (Y, S_Y) will be called a blow-up of (X, S).

Let BL(X,S) denote the category of all such solutions, where morphisms are homomorphisms of solutions which become identity under retraction.

Our goal is to describe this category in group-theoretical terms. Recall the following standard definition.

Definition 3.2. Let X be a set and G a group acting on X. A G-equivariant fiber bundle over X is a set Y equipped with a surjective map $p: Y \to X$, and an action ρ of G on Y which respects p and descends under p to the action of G on X.

Denote by Bun(X,G) the category of G-equivariant fiber bundles over X, where morphisms are G-invariant bundle mappings. Denote by $Bun_f(X,G_X)$ the full subcategory of $Bun(X,G_X)$ which consists of such bundles that $\rho(x) \neq \rho(y)$ for $x,y \in X \subset G_X, x \neq y$. Objects of Bun_f will be called *faithful* bundles.

Theorem 3.7. The categories BL(X,S) and $Bun_f(X,G_X)$ are equivalent. In particular, there is a 1-1 correspondence between isomorphism classes of blow-ups of size n, and isomorphism classes of faithful bundles of size n.

Proof. To prove the theorem, it is enough to construct two functors, $E:BL(X,S)\to Bun_f(X,G_X)$, and $E':Bun_f(X,G_X)\to BL(X,S)$, so that $E\circ E'=id$, $E'\circ E=id$. Let us construct E. Let $(Y,S_Y)\in BL(X,S)$. Then by definition $X=\bar{Y}$ and thus we have a natural surjective map $p:Y\to X$. Moreover, the group G_X acts on Y by $\rho(x)(y)=f_{\bar{x}}^{-1}(y)$, where $\bar{x}\in Y$ is any lifting of x to Y. As we explained, the action ρ respects the map p and descends to the standard action of G_X on X under p. Thus, $(Y,p,\rho)\in Bun(X,G_X)$. It is easy to see that in fact $(Y,p,\rho)\in Bun_f(X,G_X)$. Set $E(Y,S_Y)=(Y,p,\rho)$. It is clear that E is a functor.

Now let us construct E'. Let $(Y, p, \rho) \in Bun_f(X, G_X)$. For $y \in Y$, define $f_y : Y \to Y$ by $f_y^{-1} = \rho(p(y))$. Define $g_y : Y \to Y$ by the usual formula $g_y(z) = f_{f_z(y)}^{-1}(z)$. Set $S_Y(y, z) = (g_y(z), f_z(y))$. Then (Y, S_Y) is involutive.

Let us show that (Y, S_Y) is nondegenerate. It is clear that f_y is invertible. To show that g_y is invertible, it is enough to show that T is invertible, due to Proposition 2.2(a). Recall that $T(y) = f_y^{-1}(y)$. To show that T is invertible, it is enough to show that the equation $f_y^{-1}(y) = z$ has a unique solution for any z.

Let $z \in Y$ and $\bar{z} \in X$ be its equivalence class. Since the map T for X is invertible, we can find a unique $\bar{y} \in X$ such that $f_{\bar{y}}^{-1}(\bar{y}) = \bar{z}$. Now to solve the equation T(y) = z, is the same as to find an element $y \in p^{-1}(\bar{y})$ such that $f_{\bar{y}}^{-1}(y) = z$. Such an element exists and unique, since by definition $f_{\bar{y}}^{-1}$ induces an isomorphism of the fibers $p^{-1}(\bar{z})$ and $p^{-1}(\bar{y})$. Thus, (Y, S_Y) is nondegenerate.

Since ρ is an action of G_X on Y, the maps f_y satisfy condition (i) of Proposition 2.1. Therefore, by Proposition 2.2, (Y, S_Y) is a nondegenerate symmetric set, and $Ret(Y, S_Y) = (X, S)$, since $Y \in Bun_f$. Set $E'(Y, p, \rho) = (Y, S_Y)$. Clearly, E' is a functor.

The identities $E \circ E' = id$, $E' \circ E = id$ are obvious. The theorem is proved. \square

Let K_x denote the stabilizer of a point $x \in X$ in G_X .

Proposition 3.8. Let $(Y, S_Y) \in BL(X, S)$. Then Y is indecomposable if and only if X is indecomposable, and K_x acts transitively on $p^{-1}(x)$ for some $x \in X$.

Proof. Clear. \square

Remark. Theorem 3.7 and Proposition 3.8 reduce the classification of indecomposable solutions to classification of irretractable indecomposable solutions, modulo the group-theoretical question of classification of faithful bundles.

From now on and till the end of Chapter 3 we assume that the set X is finite.

3.3. Indecomposable multipermutation solutions of level 2.

Let (Y, S_Y) be an indecomposable multipermutation solution of level 2. Then $Ret(Y, S_Y) = (X, S)$, where $X = \mathbb{Z}/m\mathbb{Z}$, m > 1, and S(x, y) = (y - 1, x + 1).

Let us write Y as a direct product $Y = X \times Z$, where $Z = \{1, ..., n\}$. We may suppose that the natural map $p: Y \to X$ is the projection to the first component.

Now consider the group G_X . This group is the subgroup of $Aut(X) \ltimes \mathbb{Z}^X$ consisting of the elements $c^k(b_0,...,b_{m-1})$, where $c:X\to X$ is given by c(x)=x-1, and $b_i\in\mathbb{Z}$ are such that $\sum b_i=k \mod m$. In particular, the group K_x of elements stabilizing the point x is the group Γ of vectors $(b_0,...,b_{m-1})\in\mathbb{Z}^X$ such that $\sum b_i$ is divisible by m. (This is the same group Γ as we considered before).

As we know, blow-ups of X correspond to faithful G_X -equivariant fiber bundles on X. It is clear that equivariant fiber bundles on X correspond to Γ -actions on Z, via $Z \to G_X \times_{\Gamma} Z$.

Thus, it remains to classify transitive actions of Γ on finite sets Z of size n. Since Γ is a free abelian group, such actions correspond to sublattices L in Γ of index n ($Z = \Gamma/L$). When all such sublattices are found, one should separate those which define faithful bundles.

Example 1. m=2, n=2. In this case we have to classify transitive actions of Γ on a 2-element set, i.e. surjective maps $\Gamma \to \mathbb{Z}/2\mathbb{Z}$. Since Γ is the sublattice in \mathbb{Z}^2 generated by (1,1),(1,-1), we have 3 choices:

- 1. $(1,1) \to 1, (1,-1) \to 1.$
- 2. $(1,1) \to 0, (1,-1) \to 1.$
- 3. $(1,1) \to 1, (1,-1) \to 0.$

One can check that the first two choices define faithful bundles, while the third choice does not. Thus, there are two indecomposable multipermutation solutions of level 2 for |Y|=4 (one can show that choices 1 and 2 define non-equivalent solutions). The total number of indecomposable multipermutation solutions for |Y|=4 is three, since we also have the cyclic permutation solution. It turns out, however, that there are two more indecomposable solutions for |X|=4 – they are irretractable affine solutions (see below).

Example 2. m=3, n=2. In this case we have to classify transitive actions of Γ on a 2-element set, i.e. surjective maps $\Gamma \to \mathbb{Z}/2\mathbb{Z}$. Since Γ is the sublattice in \mathbb{Z}^3 spanned by (1,0,-1),(1,-1,0),(1,1,1), we have 7 choices:

```
1. (1,0,-1) \to 1, (1,-1,0) \to 0, (1,1,1) \to 0.
```

2.
$$(1,0,-1) \to 0, (1,-1,0) \to 1, (1,1,1) \to 0.$$

3.
$$(1,0,-1) \to 0, (1,-1,0) \to 0, (1,1,1) \to 1.$$

4.
$$(1,0,-1) \to 1, (1,-1,0) \to 1, (1,1,1) \to 0.$$

5.
$$(1,0,-1) \rightarrow 1, (1,-1,0) \rightarrow 0, (1,1,1) \rightarrow 1.$$

6.
$$(1,0,-1) \rightarrow 0, (1,-1,0) \rightarrow 1, (1,1,1) \rightarrow 1.$$

7.
$$(1,0,-1) \to 1, (1,-1,0) \to 1, (1,1,1) \to 1.$$

All of these choices but choice 3 define faithful bundles, which give rise to non-equivalent solutions. Thus, we get 6 indecomposable solutions for |Y| = 6.

Example 3. m=2, n=3. In this case we have to classify transitive actions of Γ on a 3-element set, i.e. surjective maps $\Gamma \to \mathbb{Z}/3\mathbb{Z}$. Since Γ is the sublattice in \mathbb{Z}^2 generated by (1,1),(1,-1), we have 8 choices:

- 1. $(1,1) \to 1, (1,-1) \to 1.$
- 2. $(1,1) \to 0, (1,-1) \to 1.$
- 3. $(1,1) \to 1, (1,-1) \to 0$.
- 4. $(1,1) \to 2, (1,-1) \to 2.$
- 5. $(1,1) \to 0, (1,-1) \to 2.$
- 6. $(1,1) \to 2, (1,-1) \to 0.$
- 7. $(1,1) \to 2, (1,-1) \to 1$.
- 8. $(1,1) \to 1, (1,-1) \to 2$.

All choices except 3 and 6 define faithful bundles, and we have the following isomorphisms between corresponding solutions: 1-4,2-5,7-8. Thus, we get 3 more indecomposable solutions for |Y| = 6.

The total number of indecomposable multipermutation solutions for |X|=6 is 6+3+1=10, since we also have the cyclic permutation solution. A computer calculation shows (see below) that these are all indecomposable solutions for |Y|=6

3.4. Twisted unions and generalized twisted unions.

Let X, Y be finite nondegenerate symmetric sets, and $Z = X \cup Y$ be their union.

Definition 3.3. Z is called a twisted union of X and Y if the map $S_Z: X \times Y \to Y \times X$ is given by the formula

$$(3.10) S_Z(x,y) = (g(y), f(x)),$$

where $g: Y \to Y$, $f: X \to X$ are permutations.

It follows from involutivity that for a twisted union,

(3.11)
$$S_Z(y,x) = (f^{-1}(x), g^{-1}(y)).$$

The classification of twisted unions is very simple.

Proposition 3.9. Formulas (3.10),(3.11) define a union of X and Y if and only if f preserves S_X and g preserves S_Y .

Proof. The nondegeneracy and involutivity of S_Z are automatic, and the braid relation is easily shown to be equivalent to the condition that f preserves S_X and g preserves S_Y . \square

Example 1. Any permutation solution is naturally a twisted union of cyclic permutation solutions.

Example 2. Let |X| = 1, and $Z = X \cup Y$ be a union. Then Z is obviously a twisted union.

At this point, it is easy for us to classify all solutions with $|X| \leq 3$. Indeed, any such indecomposable solution is a cyclic permutation solution by Theorem 2.12, and any decomposable one is a twisted union. However, already for |X|=4 there are unions which are not twisted unions. This encourages one to introduce the notion of a generalized twisted union.

Definition 3.4. A union Z of solutions X and Y is called a generalized twisted union of X and Y if the map $S_Z: X \times Y \to Y \times X$ is given by the formula

(3.12)
$$S_Z(x,y) = (g_x(y), f_y(x)),$$

where $g_x: Y \to Y$, $f_y: X \to X$ are permutations, such that the permutation $g_{f_y(x)}: Y \to Y$, $x \in X$, is independent of $y \in Y$, and the permutation $f_{g_x(y)}: X \to X$, $y \in Y$, is independent of $x \in X$.

For a generalized twisted union, we will write $g_{f_y(x)}$ and $f_{g_x(y)}$ as $g_{f_*(x)}$ and $f_{g_*(y)}$ (for $x \in X, y \in Y$).

It is easy to check that in a generalized twisted union, the permutation $g_{f_y^{-1}(x)}$ does not depend on x. Thus, we will denote these permutations by $g_{f_*^{-1}(x)}$ and $f_{g_*^{-1}(y)}$.

It follows from involutivity that in a generalized twisted union,

(3.13)
$$S_Z(y,x) = (f_{g_*^{-1}(y)}^{-1}(x), g_{f_*^{-1}(x)}^{-1}(y)),$$

It is clear that a twisted union is a special case of a generalized twisted union.

Proposition 3.10. Formulas (3.12),(3.13) define a generalized twisted union if and only if the following conditions are simultaneously satisfied:

- (i) The assignments $y \to f_y^{-1}$, $x \to g_x$ define left actions of G_Y on X and of G_X on Y.
- (ii) The map S_X commutes with $f_{g_*(y)} \times f_y$. The map S_Y commutes with $g_x \times g_{f_*(x)}$.

Proof. As in Proposition 3.9, involutivity and nondegeneracy are automatic. It is easy to check that the braid relation is equivalent to conditions (i) and (ii). \Box

Remark. Note that for a generalized twisted union, the assignments $y \to f_{g_*^{-1}(y)}^{-1}$ and $x \to g_{f_*^{-1}(x)}$ are actions of G_Y on X and G_X on Y. Let us call them the modified actions, as opposed to the standard actions $y \to f_y^{-1}$, $x \to g_x$. Thus, condition (ii) reads that S_X is invariant under the product of the standard and the modified action, and similarly for S_Y .

Example. Any multipermutation solution of level 2 is a generalized twisted union of indecomposable multipermutation solutions of level ≤ 2 .

3.5. Solutions for $|X| \leq 8$.

A computer program in C generated all solutions (S, X) up to isomorphism. Then, programs in Perl classified the solutions. Below we summarize the results in a table.

The following abbreviations will be used: s. = solutions; t.u. = twisted unions; g.t.u. = generalized twisted unions; id. = indecomposable; d. = decomposable; ir. = irretractable; a. = affine; mp. = multipermutation. The table gives the number of distinct maps up to isomorphism for $|X| \leq 8$:

X	s.	d.s.	t.u.	g.t.u.	id.s.	id.mp.s.	id.ir.s.	id.ir.a.
1	1	0	0	0	1	1	0	0
2	2	1	1	1	1	1	0	0
3	5	4	4	4	1	1	0	0
4	23	18	16	18	5	3	2	2
5	88	87	84	87	1	1	0	0
6	595	585	425	585	10	10	0	0
7	3456	3455	3270	3455	1	1	0	0
8	34528	34430	23856	34350	98	37	47	0

For $|X| \leq 7$, all decomposable solutions were found to be generalized twisted unions, and all indecomposables except for two affine solutions turned out to be multipermutation solutions.

However, for |X|=8, 47 solutions were found to be irretractable indecomposables, none of which are affine. The 14 solutions that are not irretractable and not multipermutation are clearly blow-ups of the two irretractable indecomposable solutions for |X|=4. Furthermore, 80 decomposable solutions for |X|=8 are not generalized twisted unions.

4. Power series solutions.

4.1. The definition of a power series solution.

The notion of a linear solution, introduced in Section 3.1, can be generalized, by defining the notion of a power series solution. This is done as follows.

Let K be a ring, and D^N be the formal N-dimensional polydisk over K. A formal mapping $D^N \to D^M$ is, by definition, a vector $\phi = (\phi_1, ..., \phi_M)$, where $\phi_i \in K[[x_1, ..., x_N]]$ are power series with zero free term.

We should remember that D^N is not a set but a formal scheme, and thus ϕ is not a mapping in the usual sense. To pass to sets and mappings, let I be a nilpotent commutative algebra over K. This means, I is an algebra over K (without unit), and any element of I is nilpotent. In this case, any formal power series $\psi \in K[[x_1,...,x_N]]$ with zero free term defines a mapping $I^N \to I$. Thus, a formal mapping $D^N \to D^M$ defines a mapping $I^N \to I^M$.

Let $X = I^N$. Let $S: X^2 \to X^2$ be a mapping, such that (X, S) is a nondegenerate symmetric set.

Definition 4.1. We will say that (X, S) is a power series solution if S is induced by a formal mapping $(D^N)^2 \to (D^N)^2$.

It is clear that a linear solution is a special case of a power series solution. Indeed, let I be any abelian group. Equip I with a \mathbb{Z} -algebra structure by defining its multiplication to be zero. Power series solutions for such I are the same thing as linear solutions, since all nonlinear terms in the power series are automatically

Let x, y be N-dimensional vectors of indeterminates, and $S: (D^N)^2 \to (D^N)^2$ be a formal mapping. The mapping S is given by $S(x,y) = (S_1(x,y), S_2(x,y))$, where $S_i(x,y)$ are formal mappings $(D^N)^2 \to D^N$.

Definition 4.2. S is called a universal power series solution over K if for any nilpotent K-algebra I the series S defines the structure of a nondegenerate symmetric set on $X = I^N$.

We have the following simple proposition.

Proposition 4.1. S is a formal power series solution if and only if it satisfies equations (1.1),(1.2), and the nondegeneracy condition: the series $S_1(x,*)$, $S_2(*,y)$ and invertible for fixed x, y.

Proof. The "if" statement is obvious. The "only if" statement: set $I = J/J^m$, where J is the ideal in $K[[z_1,...,z_l]]$ consisting of series with the zero free term, and m, l are arbitrary integers. Then the condition that (X, S) is a solution easily implies the claim. \square

Definition 4.2. Two universal power series solutions S, S' are said to be isomorphic if there exists an invertible formal mapping $\phi: D^N \to D^N$ such that $S' = (\phi \times \phi)S(\phi^{-1} \times \phi^{-1}).$

From now on we will be interested only in universal power series solutions, and drop the word "universal" in our discussions. For simplicity we will assume that K is a field.

4.2. Permutation power series solutions.

We are interested in classification of power series solutions, up to isomorphism. Consider first the special case of this problem: classification of permutation solutions. By definition, a permutation solution has the form $S(x,y) = (f(y), f^{-1}(x))$, where $f: D^N \to D^N$ is an invertible formal mapping. Two such solutions corresponding to formal mappings $f, f': D^N \to D^N$ are isomorphic if there exists an invertible formal mapping ϕ such that $f' = \phi f \phi^{-1}$. Thus, the problem of classifying permutation power series solutions is equivalent to classifying conjugacy classes in the group $Diff(D^N)$ of formal diffeomorphisms (=invertible formal mappings) of D^N into itself.

It is well known that this problem is "wild" (i.e. impossible to solve effectively) for N > 1. On the other hand, for N = 1, it is "tame", and the solution is well known. It is given by the following proposition (see [Ar]).

- **Proposition 4.2.** Let $f(x) = \sum_{m \geq 1} a_m x^m$, $a_1 \neq 0$. (i) If a_1 is not a root of unity, then f(x) is conjugate to the linear map f(x) = 0. a_1x .
- (ii) If a_1 is a root of unity of order k, then f(x) is conjugate to normal form (i), or to a map $f(x) = a_1x + x^{kr+1} + cx^{2kr+1}$, r > 1, and the parameters r, c are completely determined by f.

(iii) Normal forms (i),(ii) are never conjugate to each other.

Part (i) of Proposition 4.2 has the following multivariable generalization, due to Poincare (cf [Ar]).

We will say that a set of complex numbers $\{\lambda_1, ..., \lambda_N\}$ is resonance free if the equation $\lambda_i = \lambda_{i_1}...\lambda_{i_m}$ is not satisfied for $m \geq 2$ and $1 \leq i, i_1, ..., i_m \leq N$. For example, the set $\{\lambda\}$ is resonance free if and only if λ is not a root of unity.

Proposition 4.3. Let f(x) be a formal diffeomorphism of D^N , whose linear part has resonance free eigenvalues. Then f(x) is conjugate to its linear part.

4.3. Linearization of power series solutions.

Consider a power series solution given by a formal diffeomorphism $R:D^N\times D^N\to D^N\times D^N$. It is clear that the linear part R_1 of R (i.e. the collection of all linear terms of R) defines a linear solution. Proposition 4.3 gives rise to the question: when is a power series solution isomorphic to its linear part? In this section we will partially answer this question.

We will first consider the case when the linear part of R(x, y) is the permutation solution $R_1(x, y) = (b^{-1}x, by)$. Suppose that the lowest degree of terms in the series $R - R_1$ is m. Then we can write

(4.1)
$$R(x,y) = (b^{-1}x + P(x,y), by + Q(x,y)) \text{ mod degree m+1},$$

where P,Q are polynomials of degree m. The unitarity condition for R gives the following equation for P,Q:

(4.2)
$$P(x,y) = -b^{-1}Q(by, b^{-1}x).$$

Given (4.2), the quantum Yang-Baxter equation gives the following equation for Q:

(4.3)
$$bQ(b^{-1}x, z) - Q(x, bz) = bQ(y, z) - Q(by, bz).$$

Introduce a new function U(y,z) = bQ(y,z) - Q(by,bz). Then equation (4.3) can be rewritten in the form $U(b^{-1}x,z) = U(y,z)$, which implies that U(y,z) is independent of y: U(y,z) = U(z). Thus, we obtain the equation

(4.4)
$$bQ(y,z) - Q(by,bz) = U(z).$$

Proposition 4.4. If b has resonance free eigenvalues, then Q(y, z) is independent of y.

Proof.

Let $g \in GL_N(K)$, $h \in GL_M(K)$, and H the space of formal mappings $D^M \to D^N$. Let $A: H \to H$ be the linear operator defined by (Af)(x) = gf(x) - f(hx). Let λ_i be the eigenvalues of g, μ_j the eigenvalues of h.

Lemma. If $\lambda_i \neq \mu_{j_1}...\mu_{j_m}$ for any $m \geq 2$ then the operator A is invertible.

Proof of the Lemma. Let $H_m \subset H$ be the space of polynomials of degree m. It is clear that the eigenvalues of A on H_m are $\lambda_i - \mu_{i_1} ... \mu_{i_m}$, so they are not zero. The Lemma is proved.

Now take $M=2N,\ g=b,\ h=b\oplus b$. Then the Lemma implies that if the eigenvalues λ_i of b are resonance free, the operator $Q(y,z)\to bQ(y,z)-Q(by,bz)$ is invertible. Therefore, equation (4.4) for a fixed U(z) has a unique solution. This solution is obviously y-independent, since it does not change under rescalings of y. The Proposition is proved. \square

Corollary 4.5. If eigenvalues of b are resonance free then any power series solution solution R(x, y) whose linear part is $(bx, b^{-1}y)$ is isomorphic to its linear part (i.e. is a permutation solution with the same linear part).

Proof. We will prove the statement modulo terms of degree m+1 by induction in m Modulo terms of degree 2 the statement is a tautology. Suppose we know it modulo degree m, and want to prove it modulo degree m+1. By the induction assumption, R has the form (4.1). We proved that in this case Q(x,y) is independent on x and thus by (4.2) P(x,y) is independent on y. Thus, modulo terms of degree m+1 the solution R is a permutation solution, and thus by Poincare theorem (Proposition 4.3) it can be linearized modulo degree m+1. The Corollary is proved. \square

Now we will generalize Corollary 4.5 to the case of an arbitrary linear part $R_1(x,y) = (cx + dy, ax + by)$. For simplicity we will assume that charK = 0 or that N < charK.

Proposition 4.6. If the eigenvalues of b are resonance free, then any solution R with linear part $R_1(x,y) = (cx + dy, ax + by)$ is isomorphic to this linear part.

Proof. In our case, the classification of linear solutions is as in Proposition 3.5. In particular, we have the following important property: there exists a basis of the N-dimensional space (over the algebraic closure \bar{K}) in which the matrices b, c are upper triangular, and a, d strictly upper triangular.

Now consider R(x, y) of the form

(4.5)
$$R(x,y) = (cx + dy + P(x,y), ax + by + Q(x,y)) \text{ mod degree m+1},$$

where P,Q are polynomials of degree m. The unitarity condition for R gives the following equation for P,Q:

(4.6)
$$P(x,y) = -b^{-1}aQ(x,y) - b^{-1}Q(by,b^{-1}x).$$

Given (4.6), the quantum Yang-Baxter equation gives the following equation for Q:

$$a(a+1)^{-1}Q(x,y) - a(a+1)^{-1}Q(ax+by,cx+dy) + bQ(cx+dy) =$$

$$(4.7) bQ(y,z) + Q(x,ay+bz) - Q(ax+by,acx+ady+bz).$$

Lemma. If b has resonance free eigenvalues, then any solution of equation (4.7) has the form

(4.8)
$$Q(x,y) = aT(x) + bT(y) - T(ax + by),$$

where T is a suitable polynomial of degree m.

Proof of the Lemma. First of all, (4.8) is actually a solution of (4.7). Indeed, if we conjugate the linear solution R_1 by the permutation $x \to x - T(x)$, we will get exactly (4.8).

Denote by D(a,b) the dimension of the space of solutions of equation (4.7). Since the eigenvalues of b are resonance free, the assignment $T(x) \to aT(x) + bT(y) - T(ax + by)$ is injective. Thus, $D(a,b) \ge N \binom{N+m-1}{m}$ (the dimension of the space of polynomials T).

We will work in the aforementioned basis, in which a, b, c, d are upper triangular. Let b_{diag} be the diagonal part of b in this basis. We showed above that the lemma is true for a = 0, i.e. $D(0, b_{diag}) = N\binom{N+m-1}{m}$. So it is enough to show that $D(a,b) \leq D(0,b_{diag})$.

Set $g(t) = \text{diag}(1, t, t^2, ..., t^{N-1})$, and $b_t = g(t)^{-1}bg(t)$, $a_t = g(t)^{-1}ag(t)$. It is clear that $D(a_t, b_t) = D(a, b)$ for $t \neq 0$. On the other hand, since a, b are triangular, a_t, b_t are polynomial in t, and $a_0 = 0, b_0 = b_{diag}$. This gives us a desired inequality, since the space of solutions of a system of linear equations cannot get bigger when the system is deformed. The lemma is proved.

Now the Proposition easily follows. Indeed, conjugating R(x,y) by the permutation $x \to x + T(x)$, we come to the situation P = 0, Q = 0. Thus, the proposition can be proved in the same way as Corollary 4.5. Proposition 4.6 is proved. \square

APPENDIX: T-STRUCTURES AND BIJECTIVE 1-COCYCLES

In this appendix we will introduce the notion of a T-structure on an abelian group A, which is motivated by the definition of the map T in Proposition 2.2. We will show that any group G with an action ρ on A and a bijective 1-cocycle π into A defines such a structure, and that if A is cyclic then the T-structure completely determines G, ρ , and π .

Definition. A pair (A,T) of an abelian group A and a bijective map $T:A \to A$ is said to be a T-structure if for any $x \in A$, $k \in \mathbb{Z}$ one has

(A1)
$$T(kx) = kT^k(x).$$

Examples.

- 1. For any abelian group, T = id is a T-structure.
- 2. Let $A = \mathbb{Z}$. Then there are only two T-structures: T = id and $T(k) = (-1)^k k$. Indeed, let n_{\pm} be such that $T(n_{\pm}) = \pm 1$. Then $n_{\pm} \neq 0$ and $|n_{\pm}|T^{|n_{\pm}|}(\frac{n_{\pm}}{|n_{\pm}|}) = \pm 1$, so $n_{\pm} = 1$ or $n_{\pm} = -1$. If $n_{+} = 1$, $n_{-} = -1$ then $T(\pm 1) = \pm 1$ and $T(\pm k) = |k|T^{|k|}(\pm 1) = \pm k$ for k > 0, so T = id. Similarly, if $n_{+} = -1$, $n_{-} = 1$ then $T(k) = (-1)^k k$.
- 3. Let A be a ring with 1 (not necessarily commutative), and $c \in A$ an element such that the element 1 + cx is invertible for any $x \in A$. Define $T : A \to A$ by the formula $T(x) = x(1+cx)^{-1}$. Then T is a T-structure on A (as an additive group). Indeed, T is invertible (as $T^{-1}(y) = (1-yc)^{-1}y$), and it is easy to show by induction that $T^k(x) = x(1+kcx)^{-1}$, $T^{-k}(y) = (1-kyc)^{-1}y$, k > 0, which proves the claim.
- 4. This is a generalization of Example 2 to $A = \mathbb{Z}^n$. Define A_1 as a subset of \mathbb{Z}^n consisting of integer n-tuples $(x_1, ..., x_n)$ that are coprime, i.e. with the greatest common divisor equal to one. Denote by A_k the set of multiples of A_1 by a nonnegative integer k. It is easy to see that the sets A_k are pairwise non-intersecting sets whose union is the whole \mathbb{Z}^n . We claim that T-structures on \mathbb{Z}^n are labeled by permutations of A_1 of order two, i.e. by the maps $H: A_1 \to A_1$, such that $H^2 = id$. Indeed, starting with H we first define T on A_1 simply by the formula T(x) = H(-x), $x \in A_1$. So defined T trivially satisfies the relation $T(-x) = -T^{-1}(x)$, $x \in A_1$. Now, we extend T to each A_k , k nonnegative, by the formula $T(kx) = kT^k(x)$. It is easy to see that this defines a T-structure on \mathbb{Z}^n .

By reverting above argument and defining H(x) = T(-x), $x \in A_1$ we see that all the T-structures on \mathbb{Z}^n are of this form.

The simplest properties of T-structures are given by the following proposition.

Proposition A1. (i) If (A,T) is a T-structure and A is finite, then $T^{|A|} = id$.

- (ii) If (A,T) is a T-structure then (A,T^k) is a T-structure for any integer k.
- (iii) If (A,T) is a T-structure then (kA,\tilde{T}) is a T-structure for any integer k, \tilde{T} being the restriction of T to kA.
- *Proof.* (i) Applying (A1) for k = |A| + 1, we get $T((|A| + 1)x) = (|A| + 1)T^{|A| + 1}(x)$. Since |A| = 0 in A, the last equation yields $T(x) = T^{|A| + 1}(x)$, thus $T^{|A|} = id$.
- (ii) Since $T^{-1}(x) = -T(-x)$, it is easy to see that (A, T^{-1}) is a T-structure. So it suffices to prove the statement for k > 0. We do so by induction. For k = 1 the statement known, so assume it is known for k = n 1 and let us prove it for k = n. We have $T^n(mx) = T(T^{n-1}(mx)) = T(mT^{(n-1)m}(x)) = mT^m(T^{(n-1)m}(x)) = mT^{mn}(x)$, as desired.
- (iii) For any $x \in A$, $k \in \mathbb{Z}$, $T(kx) = kT^k(x)$, so T maps kA to kA. It follows that (kA, \tilde{T}) is a T-structure. \square

Remark. It is easy to generalize the statement (ii) of Proposition A1. Namely, any two T-structures (A, T_1) , (A, T_2) on the same group A such that $T_1T_2 = T_2T_1$ give rise to a new T-structure (T, A) with $T = T_1T_2$. The above observation can be applied to Example 3. Namely, let (A, T_1) , (A, T_2) be T-structures on the ring A, given by the formulas $T_1(x) = x(1 + c_1x)^{-1}$, $T_2(x) = x(1 + c_2x)^{-1}$, then $T(x) = T_1T_2(x) = x(1 + (c_1 + c_2)x)^{-1}$ defines a T-structure.

Now let us explain how to construct T-structures by examining their connection with bijective cocycle data.

Theorem A2. Let (G, A, ρ, π) be a bijective cocycle datum. Define $T: A \to A$ by the formula

(A2)
$$T(x) = \pi^{-1}(x) * x$$

for $x \in A$. Then, the pair (A, T) is a T-structure.

The proof of Theorem A2 follows from the Lemma stated below.

Define the product $x \circ y = \pi^{-1}(x) * y$ for $x, y \in A$. Then $T(x) = x \circ x$.

Lemma A3. One has

(A3)
$$(y+x) \circ z = (y \circ x) \circ (y \circ z)$$

for any $x, y \in A$.

Proof of the Lemma. One derives from the definition of a 1-cocycle that $\pi^{-1}(y + x) = \pi^{-1}(\pi^{-1}(y) * x)\pi^{-1}(y)$. So,

$$(y+x)\circ z=\pi^{-1}(y+x)*z=\pi^{-1}(\pi^{-1}(y)*x)*(\pi^{-1}(y)*z).$$

This implies $(y + x) \circ z = (y \circ x) \circ (y \circ z)$.

Proof of Theorem A2. Remark 1 in Section 2.4 implies that A has a natural structure of a nondegenerate symmetric set. The map T for this nondegenerate symmetric set, defined in Proposition 2.2, coincides with the one given by (A2). Thus

the bijectivity of T follows from Proposition 2.2(b). So, it remains to check that for any integer $k \geq 0$ one has

$$T(kx) = kT^{k}(x), T(-kx) = -kT^{-k}(x),$$

or, in terms of "o" product, it is enough to show that

into Lemma A2, we get $T^n(x) = (nx) \circ x$. \square

(A4)
$$kx \circ x = T^{k}(x), \qquad (-kx) \circ x = T^{-k}(x).$$

We will prove (A4) by induction. It is clear that (A4) holds for k=0. Suppose (A4) holds for k=n-1. Then, substituting y=-nx, z=x into Lemma A2, we get $(x-nx)\circ x=(-nT(x))\circ T(x)$, i.e. $T^{1-n}(x)=(-nT(x))\circ T(x)$. If we let y=T(x) we obtain $T^{-n}(y)=(-ny)\circ y$. Similarly, substituting y=(n-1)x, z=x

It is easy to check that examples 1-3 of T-structures are in fact obtained from bijective cocycle data by formula (A2). Indeed:

- 1. T = id is obtained from the datum (A, A, trivial action, id).
- 2. $T(k) = (-1)^k k$ for $A = \mathbb{Z}$ is obtained from the datum (G, A, ρ, π) where G is the group of affine transformations of \mathbb{Z} , $G = \{(a, b) : x \to ax + b | a = \pm 1, b \in \mathbb{Z}\}$, $\rho(a, b)x = ax$, $\pi(a, b) = 2ab + \frac{a-1}{2}$.
- 3. Let A be a ring with 1, and c an element of A. Introduce a new operation on A by $x \cdot y = x + y + xcy$. It is easy to check that this operation is associative, and 0 is the unit e with respect to it. Moreover, if 1 + cx is invertible for any $x \in A$ then (A, \bullet) is a group: for any $x \in A$ there exists an element $x^{-1} = -x(1 + cx)^{-1}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$. Define an action $\rho: (A, \bullet) \to \operatorname{End}(A)$ by $\rho(x)y = y(1 + cx)^{-1}$ and $\pi = id: (A, \bullet) \to A$. It is easy to check that $((A, \bullet), A, \rho, \pi)$ is a bijective cocycle datum, and that the corresponding map T is given by $T(x) = x(1 + cx)^{-1}$.

Remark. Note that Remark 1 in Section 2.4 implies that in the situation of example 3, A has a natural structure of a nondegenerate symmetric set. Moreover, it is easy to compute the map S explicitly, and the answer is

(A5)
$$S(x,y) = (y(1+cx+cxcy)^{-1}, x(1+cy)).$$

Note that this formula defines the structure of a nondegenerate symmetric set not only on A but also on any right ideal in A.

In each of these examples, one notes that $T^r = id$ where $r = |Im(\rho)|$, where $Im(\rho)$ denotes the image of G in Aut(A) under ρ . This is a consequence of the following general property:

Proposition A4. Let $H=Ker(\rho)$, $K=\pi(H)$ for a bijective cocycle datum and let T be the T-structure obtained from the datum. If A/K is finite, and r is any integer such that r(A/K)=0, then $T^r=id$.

Proof. If r is an integer such that r(A/K) = 0, then rA is a subset of K, and hence $T^r(x) = rx \circ x = x$ since $rx \in K$, for any $x \in A$. \square

Corollary A5. In the case when A/K is finite, set $r = |Im(\rho)|$, the cardinality of the image of G under ρ in Aut(A). Then, $T^r = id$. In particular, if A is cyclic, then $T^{gcd(\phi(|A|),|A|)} = id$ where ϕ is the Euler ϕ -function.

Proof. Since $r = |Im(\rho)| = |A/K|$, r(A/K) = 0 so this follows from the previous corollary. In the case of A cyclic, we have that $|Aut(A)| = \phi(|A|)$, |G| = |A| so $|Im(\rho)|$ divides $gcd(\phi(|A|), |A|)$. \square

Furthermore, when T comes from a bijective cocycle datum as before, then we can find a similar bijective cocycle datum that produces the map T^k , which was proven to be a T-structure in Proposition A1, (ii) for any integer k.

Proposition A6. (i) Given a bijective cocycle datum (G, A, ρ, π) and any integer k, there is a unique bijective cocycle datum given by $(G^{(k)}, A, \rho^{(k)}, \pi^{(k)})$ where $\rho^{(k)}((\pi^{(k)})^{-1}(x)) = \rho(\pi^{-1}(kx))$.

(ii) Let T be the natural T-structure induced by (G, A, ρ, π) (i.e. $T(x) = \rho(x)(x)$). Then the natural T-structure induced by $(G^{(k)}, A, \rho^{(k)}, \pi^{(k)})$ is given by T^k .

Proof. (i) Without loss of generality we may assume $G=(A, \bullet)$ and $\pi=id$ for a binary operation \bullet . We will use the notation $x*y=\rho(x)(y)$. Define $x\circ y=kx*y$ for $x,y\in A$ and set $\rho^{(k)}(x)(y)=x\circ y$. Then, define $x\odot y=y+(-T^k(y))\circ x$. Now, setting $G^{(k)}=(A,\odot)$ and $\pi^{(k)}=id$, we claim $(G^{(k)},A,\rho^{(k)},\pi^{(k)})$ is a bijective cocycle datum.

Indeed, (G, \odot) is associative:

$$(x \circledcirc y) \circledcirc z = x \circledcirc (y \circledcirc z) = z + (-T^k(z)) \circ y + ((-T^k(z + (-T^k(z)))) \circ y) \circ x.$$

Furthermore, G has inverses:

$$x \odot (-T^k(x)) = (-T^k(x)) + (-T^k(-T^k(x))) \circ x = -T^k(x) + x \circ x = 0,$$

and $-T^k(-T^k(x)) = x$. Thus $-T^k(x) = x^{-1}$ in $G^{(k)}$, and $G^{(k)}$ is a group. This implies that the cocycle condition is also satisfied: $x \odot y = y + (y^{-1}) * x$ where we take the inverse in $G^{(k)}$. Thus, the result is a bijective cocycle datum.

We note that this is the unique datum with the given $\rho^{(k)}$ since the operation \odot is obtained directly from $\rho^{(k)}$ given that the cocycle condition is satisfied.

(ii) It is clear that
$$x \circ x = T^k(x)$$
 since $kx * x = T^k(x)$. \square

In light of the above discussion, it makes sense to ask whether any map T comes from a bijective cocycle datum, and whether the map T determines this datum. The following theorem gives a positive answer to both questions in the case when the group A is cyclic.

Theorem A7. For a fixed cyclic group A, bijective cocycle data (G, A, ρ, π) are, up to isomorphism, in one to one correspondence with T-structures on A.

Proof. Let us construct the group G, representation $\rho: G \to End(A)$ and bijective 1-cocycle $\pi: G \to A$ from a given $T: A \to A$. This is done as follows.

(1) Pick a generator $1 \in A$. It defines a natural homomorphism $\mathbb{Z} \to A$ which defines a natural ring structure on A. Define the product " \circ " on A by the formula $y \circ z = zT^y(1)$, where we regard A as a ring. Since $T^{|A|} = id$, T^y makes sense and thus this operation is well defined.

Let us check that $(y+z) \circ w = (y \circ z) \circ (y \circ w)$. We have

$$(y+z)\circ w = wT^{y+z}(1), \ (y\circ z)\circ (y\circ w) = wT^{y}(1)T^{zT^{y}(1)}(1).$$

So it suffices to show that $T^z(u) = uT^{zu}(1)$ (we need the case $u = T^y(1)$), which was done in Proposition A1(ii).

(2) For $y, z \in A$ define $y \odot z = z + (-T(z)) \circ y = z + yT^{-T(z)}(1), \rho(y)z = y \circ z$. Denote by G the set A with the operation \odot .

It is easy to see that the map $\rho:G\to End(A)$ is multiplicative. Indeed, we must check

$$(y \odot z) \circ w = y \circ (z \circ w).$$

This can be rewritten as

$$(z \circ yT^{-T(z)}(1)) \circ (z \circ w) = y \circ (z \circ w).$$

Now we need only to show $T^z(1)T^{-T(z)}(1)=1$, which is true since $T^z(1)T^{-T(z)}(1)=T^z(1)T^{-zT^z(1)}(1)=T^{z-z}(1)=1$. This also proves that $\pi:G\to A$ satisfies the cocycle condition, since this amounts to proving z and -T(z) are inverses in G.

This implies that G is a group (the axioms of a group can be checked after application of π ; since π is bijective, the axioms must hold). Thus, we have constructed from T a bijective cocycle datum (G, A, ρ, π) .

Thus, we have shown that the map from bijective cocycle data to T-structures is surjective. To show that it is injective, we will show that if (A,T) is a T-structure obtained from (G,A,ρ,π) as in Theorem A2, and (G',A,ρ',π') is obtained from (A,T) as we just described, then $G'=G,\rho'=\rho,\pi'=\pi$ up to an isomorphism.

Recall that T is defined by $T(x) = \pi^{-1}(x) \circ x$. This implies that $T^n(x) = \pi^{-1}(nx) \circ x$. Thus $x \circ y = yT^x(1) = y(\pi^{-1}(x) * 1) = \pi^{-1}(x) * y$. Thus,

$$x \odot y = y + xT^{-T(y)}(1) = y + x\pi^{-1}(-\pi^{-1}(y) * y) * 1 = y + \pi^{-1}(y)^{-1} * x$$

This implies that $\pi^{-1}(x \odot y) = \pi^{-1}(x)\pi^{-1}(y)$, so $\pi^{-1}: G' \to G$ is a group isomorphism. This isomorphism clearly maps $\pi' = id$ to π . Also, $\rho'(x)y = \pi^{-1}(x) * y = \rho(\pi^{-1}(x))y$, so ρ' corresponds to ρ . The theorem is proved. \square

Corollary A8. If $A = \mathbb{Z}/p\mathbb{Z}$, where p is a prime, then any T-structure is the identity.

Proof. The group G must equal $\mathbb{Z}/p\mathbb{Z}$, and its action on A must be trivial, as there are no nontrivial actions. So π must be a group isomorphism. Thus, T=id. Alternatively, the theorem in conjunction with Corollary A5 immediately implies T=id. \square

Remark. This can also be proven directly as follows. Note that $T^k(1) \neq 0$ for all positive k since T(0) = 0. Thus the minimum positive j such that $T^j(1) = 1$ must be less than p and a factor of p since $T^p(1) = 1$, hence j = 1 and T is trivial since $T(x) = xT^x(1) = x$ for all $x \in A$.

Theorem A7 raises an interesting question of classification of T-structures on cyclic groups. Unfortunately, we could not find such a classification, even when $A = \mathbb{Z}/n\mathbb{Z}$ where n is a prime power.

Note, however, by the proposition in Section 3.2, that when A is cyclic T-structures correspond to multipermutation solutions. In terms of the map T this means that T is similarly retractable by taking A modulo $Ker(\rho)$, which by the above proposition (showing $x \circ 1 = T^x(1) = 1$ if $x \in Ker(\rho)$) is the subgroup consisting of all elements $x \in A$ such that $T^x = id$. This is the first step towards such a classification.

References

- [Ar] Arnold V.I., Geometric methods in the theory of ordinary differential equations, Springer, New York, 1988.
- [As] Aschbacher M., Finite group theory, Cambridge studies in advanced mathematics; 10, 1986.
- [Bu] Burde, D., Affine structures on nilmanifolds, Intern.J. of Math. 7 (1996), no. 4, 599-616.
- [DM] Deligne, P., and Milne, J., Tannakian categories, Lecture notes in math 900 (1982).
- [Dr] Drinfeld V.G., On some unsolved problems in quantum group theory, Lect. Notes Math. 1510 (1992), 1-8.
- [EG] Etingof P., and Gelaki, S., Method of construction of finite-dimensional triangular semisimple Hopf algebras, Math. Research Letters, to appear; math.QA/9806072 (1998).
- [FRT] Reshetikhin, N.Yu., Takhtadzhyan, L.A. and Faddeev, L.D., Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990), no. 1, 193–225.
- [Ha] Hayashi T., Quantum groups and quantum semigroups, J.Alg. 204 (1998), 224-254.
- [Sch] Schauenburg, P., Algebra-Berichte 66, Fisher, Munchen, 1997.
- [Hi] Hietarinta J., Permutation-type solutions to the Yang-Baxter and other simplex equations, q-alg 9702006 (1997).
- [Tu] Turaev V., Quantum invariants of knots and 3-manifolds, W.De Gruyter, Berlin, 1994.
- P. E.: DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
- T. S.: 2312 HARVARD YARD MAIL CENTER, CAMBRIDGE, MA 02138
- A. S.: Department of Mathematics, MIT, Cambridge, MA 02139

EMAIL ADDRESSES:

P. E.: etingof@math.harvard.edu
T. S.: schedler@fas.harvard.edu
A. S.: sashas@math.mit.edu