Advances in Mathematics 290 (2016) 453–482

Contents lists available at ScienceDirect

Advances in Mathematics

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Singular Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$

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ARTICLE INFO

Article history:
Received 2 September 2014
Received in revised form 15 November 2015
Accepted 4 December 2015
Available online 29 December 2015
Communicated by Roman Bezrukavnikov

MSC:
primary 17B67

Keywords:
Gelfand–Tsetlin modules
Gelfand–Tsetlin basis
Tableaux realization

ABSTRACT

The classical Gelfand–Tsetlin formulas provide a basis in terms of tableaux and an explicit action of the generators of $\mathfrak{gl}(n)$ for every irreducible finite-dimensional $\mathfrak{gl}(n)$-module. These formulas can be used to define a $\mathfrak{gl}(n)$-module structure on some infinite-dimensional modules – the so-called generic Gelfand–Tsetlin modules. The generic Gelfand–Tsetlin modules are convenient to work with since for every generic tableau there exists a unique irreducible generic Gelfand–Tsetlin module containing this tableau as a basis element. In this paper we initiate the systematic study of a large class of non-generic Gelfand–Tsetlin modules – the class of 1-singular Gelfand–Tsetlin modules. An explicit tableaux realization and the action of $\mathfrak{gl}(n)$ on these modules is provided using a new construction which we call derivative tableaux. Our construction of 1-singular modules provides a large family of new irreducible Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$, and is a part of the classification of all such irreducible modules for $n = 3$.

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http://dx.doi.org/10.1016/j.aim.2015.12.001
0001-8708/

1. Introduction

A recent major breakthrough in the representation theory was the classification of the irreducible weight modules with finite weight multiplicities of all finite-dimensional reductive complex Lie algebras. The classification result was obtained in two steps: first in [6], using parabolic induction functors, the problem was reduced to simple Lie algebras of type A and C; and then in [20], the classification was completed and an explicit realization of the irreducibles was obtained. Recall that M is a weight module of a Lie algebra $\mathfrak{g}$ with fixed Cartan subalgebra $\mathfrak{h}$ if M is $\mathfrak{h}$-diagonalizable, and that the dimension of the weight space $M_\lambda = \{v \in M \mid hv = \lambda(h)v \ (\forall h \in \mathfrak{h})\}$ is called the weight multiplicity of $\lambda \in \mathfrak{h}^*$.

On the other hand, the problem of classifying all irreducible weight modules (possibly with infinite weight multiplicities) is still largely open. A natural class of such modules consists of the so-called Gelfand–Tsetlin modules. The Gelfand–Tsetlin modules are defined by generalizing a classical construction of Gelfand and Tsetlin that provides a convenient basis for every simple finite-dimensional representation of a simple classical Lie algebra. The theory of general Gelfand–Tsetlin modules, especially for Lie algebras of type A, has attracted considerable attention in the last 30 years and has been studied in [2,3,13,21–23,27,25], among others.

In this paper we consider Gelfand–Tsetlin modules for Lie algebras of type A and for simplicity work with $\mathfrak{gl}(n)$ instead of $\mathfrak{sl}(n)$. The Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$ by definition are modules that admit a basis of common eigenvectors of a fixed maximal commutative subalgebra Γ of the universal enveloping algebra $U(\mathfrak{gl}(n))$ of $\mathfrak{gl}(n)$. The algebra Γ is called the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}(n))$ and has numerous applications that extend beyond the theory of Gelfand–Tsetlin modules. Gelfand–Tsetlin subalgebras were considered in [26] in connection with subalgebras of maximal Gelfand–Kirillov dimension in the universal enveloping algebra of a simple Lie algebra. Furthermore, these subalgebras are related to: general hypergeometric functions on the complex Lie group $GL(n)$, [14,15]; solutions of the Euler equation, [7,26]; and problems in classical mechanics in general, [16,17].

It is well known that the Gelfand–Tsetlin subalgebra Γ has a simple spectrum on any irreducible finite-dimensional module, that is, the characters of Γ separate the basis elements of such module. However, this property does not longer hold for infinite-dimensional Gelfand–Tsetlin modules, in which case Gelfand–Tsetlin characters may occur with multiplicities. These Gelfand–Tsetlin multiplicities are always finite by [24], and bounded by [9]. The so-called generic Gelfand–Tsetlin modules (for the explicit definition see Section 3) have the convenient property that all their Gelfand–Tsetlin multiplicities are 1. Gelfand–Tsetlin modules with multiplicities 1 have been studied in several papers. In [12], using the classical Gelfand–Tsetlin formulas, a Gelfand–Tsetlin basis was constructed for some of these modules. Later, in [18,19] the construction of [12] was extended to all such modules for $n = 3$. The existence of higher Gelfand–Tsetlin multiplicities is a major obstacle to study and explicitly construct tableaux-type basis for such modules in general. So far, the only known examples of Gelfand–Tsetlin modules with higher multiplicities and with basis of tableaux, are examples of Verma modules considered in [5].

In this paper we initiate the systematic study of non-generic Gelfand–Tsetlin modules, i.e. of singular Gelfand–Tsetlin modules. Singular modules are those for which the denominators in the Gelfand–Tsetlin formulas may vanish, and we focus on the case of 1-singular modules, that is, the case when only simple poles are allowed. In order to deal with the singularities of the coefficients we introduce a new type of tableaux – the derivative Gelfand–Tsetlin tableaux, or simply, the derivative tableaux. We expect that the derivative tableaux construction can be extended to more general singular Gelfand–Tsetlin modules using differential operators of higher order.

The main results of the present paper can be separated into two components:

(1) With the aid of derivative tableaux we construct a universal Gelfand–Tsetlin module for any 1-singular Gelfand–Tsetlin character. This is achieved by providing an explicit tableaux-type basis and defining the action of the generators of $\mathfrak{gl}(n)$ in the spirit of the original work of Gelfand and Tsetlin. Our construction generalizes the previous works on finite-dimensional and on generic Gelfand–Tsetlin modules. The universal 1-singular Gelfand–Tsetlin modules have finite length and many of their irreducible subquotients are examples of new irreducible singular Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules. We also obtain a sufficient condition for the universal module to be irreducible.

(2) We show that for any 1-singular Gelfand–Tsetlin character there exist at most two non-isomorphic irreducible Gelfand–Tsetlin modules with that character, and that the corresponding Gelfand–Tsetlin multiplicity of this character in any of the two modules is at most 2. We prove that, except for one case, every irreducible 1-singular Gelfand–Tsetlin module is a subquotient of a universal derivative tableaux Gelfand–Tsetlin module. This single case occurs when the universal module has two isomorphic irreducible subquotients. We conjecture that even in that single case the realization of the irreducibles as subquotients of universal derivative tableaux modules remain valid. This conjecture is true in the case of $\mathfrak{gl}(3)$ as we show in the forthcoming paper [11]. In particular, with the aid of derivative tableaux, we obtain a classification and explicit tableaux realization of all irreducible Gelfand–Tsetlin $\mathfrak{gl}(3)$-modules.

The paper is organized as follows. In Section 3 we recall the classical construction of Gelfand and Tsetlin and collect important properties for the generic Gelfand–Tsetlin modules. In this section we also rewrite the Gelfand–Tsetlin formulas in terms of permutations. In Section 4 we introduce our new construction – the derivative tableaux and prove that the space $V(T(\bar{v}))$ spanned by the (usual) tableaux $T(\bar{v}+z)$ and the derivative tableaux $\mathcal{D}T(\bar{v}+z)$ associated to a 1-singular vector $\bar{v}$ has a $\mathfrak{gl}(n)$-module structure. The proof that this $\mathfrak{gl}(n)$-module is a 1-singular Gelfand–Tsetlin module is included in Section 5, where explicit formulas for the action of the Gelfand–Tsetlin subalgebra on $V(T(\bar{v}))$ are obtained. In Section 6 we prove a sufficient condition for the irreducibility of $V(T(\bar{v}))$. In Section 7 we show that there are at most two Gelfand–Tsetlin modules associated with a fixed 1-singular Gelfand–Tsetlin character and show that, except for one case, every irreducible 1-singular Gelfand–Tsetlin module is a subquotient of $V(T(\bar{v}))$ for some $\bar{v}$. In the last section, the appendix, we prove some technical results that are needed in Section 4.

2. Conventions and notation

The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers m such that $m \geq a$. We fix an integer $n \geq 2$. By $\mathfrak{gl}(n)$ we denote the general linear Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ – the standard basis of $\mathfrak{gl}(n)$ of elementary matrices. We fix the standard triangular decomposition and the corresponding basis of simple roots of $\mathfrak{gl}(n)$. The weights of $\mathfrak{gl}(n)$ will be written as n-tuples $(\lambda_1, \dots, \lambda_n)$.

For a Lie algebra $\mathfrak{a}$ by $U(\mathfrak{a})$ we denote the universal enveloping algebra of $\mathfrak{a}$. Throughout the paper $U = U(\mathfrak{gl}(n))$. For a commutative ring R, by $\operatorname{Specm} R$ we denote the set of maximal ideals of R.

We will write the vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$ in the following form:

$$w = (w_{n1}, \dots, w_{nn}|w_{n-1,1}, \dots, w_{n-1,n-1}| \cdots |w_{21}, w_{22}|w_{11}).$$

For $1 \leq j \leq i \leq n$, $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero.

For $i > 0$ by S_i we denote the ith symmetric group. By (k, ℓ) we denote the transposition of S_n interchanging k and ℓ. Throughout the paper we set

$$G := S_n \times \cdots \times S_1.$$

Every element σ in G will be written as an n-tuple $(\sigma[n], \dots, \sigma[1])$ for permutations $\sigma[i] \in S_i$.

3. Gelfand–Tsetlin modules

3.1. Definitions

Recall that $U = U(\mathfrak{gl}(n))$. Let for $m \leqslant n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \,|\, i, j = 1, \dots, m\}$. Then we have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n,$$

which induces the chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of the universal enveloping algebras $U_m = U(\mathfrak{gl}_m)$, $1 \leq m \leq n$. Let Z_m be the center of U_m. Then Z_m is the polynomial algebra in the m variables $\{c_{mk} \,|\, k = 1, \dots, m\}$,

$$c_{mk} = \sum_{(i_1,\dots,i_k)\in\{1,\dots,m\}^k} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}. \tag{1}$$

Following [4], we call the subalgebra of U generated by $\{Z_m \,|\, m = 1, \dots, n\}$ the *(standard) Gelfand–Tsetlin subalgebra* of U and denote it by Γ. In fact, Γ is the polynomial algebra in the $\dfrac{n(n+1)}{2}$ variables $\{c_{ij} \,|\, 1 \leqslant j \leqslant i \leqslant n\}$ ([27]). Let Λ be the polynomial algebra in the variables $\{\lambda_{ij} \,|1 \leqslant j \leqslant i \leqslant n\}$.

Let $\imath : \Gamma \longrightarrow \Lambda$ be the embedding defined by $\imath(c_{mk}) = \gamma_{mk}(\lambda)$, where

$$\gamma_{mk}(\lambda) := \sum_{i=1}^{m} (\lambda_{mi} + m - 1)^k \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{mi} - \lambda_{mj}}\right). \tag{2}$$

Note that $\gamma_{mk}(\lambda)$ is a symmetric polynomial of m variables of degree k. Moreover, the image of $\imath$ coincides with the subalgebra of G-invariant polynomials in Λ ([27]) which we identify with Γ. Denote by K the field of fractions of Γ and by L the field of fractions of Λ. Note that Λ is the integral closure of Γ in L. Then we have a surjective map $\pi : \operatorname{Specm} \Lambda \to \operatorname{Specm} \Gamma$. If $\pi(\ell) = \mathsf{m}$ for some $\ell \in \operatorname{Specm} \Lambda$, then we write $\ell = \ell_{\mathsf{m}}$ and say that ℓ_{m} is *lying over* m.

Let $\mathcal{M}$ be the free abelian group generated by δ^{ij}, $1 \leqslant j \leqslant i \leqslant n-1$. Clearly, $\operatorname{Specm} \Lambda \simeq \mathbb{C}^{\frac{n(n+1)}{2}}$ and $\mathcal{M} \simeq \mathbb{Z}^{\frac{n(n-1)}{2}}$. Then G acts on $\operatorname{Specm} \Lambda$ via

$$\sigma(w) := (w_{n,\sigma^{-1}[n](1)}, \dots, w_{n,\sigma^{-1}[n](n)} | \dots | w_{1,\sigma^{-1}[1](1)}), \tag{3}$$

and $\mathcal{M}$ acts on $\operatorname{Specm} \Lambda$ by shifts: $\delta^{ij} \cdot \ell = \ell + \delta^{ij}$, $\delta^{ij} \in \mathcal{M}$. We have $L^G = K$, $\Lambda^G = \Gamma$ and $G = G(L/K)$ is the Galois group of the field extension $K \subset L$.

Definition 3.1. A finitely generated U-module M is called a *Gelfand–Tsetlin module (with respect to* Γ*)* if M splits into a direct sum of Γ-modules:

$$M = \bigoplus_{\mathsf{m} \in \operatorname{Specm} \Gamma} M(\mathsf{m}),$$

where

$$M(\mathsf{m}) = \{v \in M | \mathsf{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Identifying m with the homomorphism $\chi : \Gamma \to \mathbb{C}$ with $\operatorname{Ker} \chi = \mathsf{m}$, we will call m a *Gelfand–Tsetlin character* of M if $M(\mathsf{m}) \neq 0$, and $\dim M(\mathsf{m})$ – the *Gelfand–Tsetlin multiplicity of* m. The *Gelfand–Tsetlin support* of a Gelfand–Tsetlin module M is the set of all Gelfand–Tsetlin characters of M. We will often call Gelfand–Tsetlin character, Gelfand–Tsetlin multiplicity, and Gelfand–Tsetlin support simply character, multiplicity, and support, respectively.

Note that any irreducible Gelfand–Tsetlin module over $\mathfrak{gl}(n)$ is a weight module with respect to the standard Cartan subalgebra $\mathfrak{h}$ spanned by E_{ii}, $i = 1, \dots, n$. Also, any weight module with respect to $\mathfrak{h}$ with finite weight multiplicities is Gelfand–Tsetlin.

In particular, every highest weight module or, more generally, every module from the category $\mathcal{O}$ is a Gelfand–Tsetlin module. However, in general (except for $n = 2$), a weight module M with respect to $\mathfrak{h}$, even irreducible, need not be a Gelfand–Tsetlin module.

3.2. Preliminaries

In this subsection we collect some general results on Gelfand–Tsetlin modules. The first two theorems follow from the fact that U is a Galois order with respect to Γ, [8]. Another important fact used in the proofs is that U is free as a left and as a right Γ-module, [24].

The inequality in the next theorem follows by Lemma 4.1(c) and Theorem 4.12(c) in [9]. Note that, as proved in [9, §4], the set on the right hand side of the inequality does not depend on the choice of ℓ_{m}.

Theorem 3.2. *Let $M \neq 0$ be a Gelfand–Tsetlin $\mathfrak{gl}(n)$-module, and let $\mathsf{m} \in \operatorname{Specm}\Gamma$ be such that M is generated by $x \in M(\mathsf{m})$ and $\mathsf{m}x = 0$. Then, for each n in the support of M*

$$\dim M(\mathsf{n}) \leq |\{\phi \in \mathcal{M} \mid \pi(\phi\ell_{\mathsf{m}}) = \mathsf{n}\}|.$$

Theorem 3.3. *(See [9], Corollary 5.3.) Let $\mathsf{m} \in \operatorname{Specm}\Gamma$ and $Q_n = 1!2!\dots(n-1)!$ Then*

(i) *If M is a U-module generated by some $x \in M(\mathsf{m})$ (in particular for an irreducible module), then*

$$\dim M(\mathsf{m}) \leq Q_n.$$

(ii) *The number of isomorphism classes of irreducible U-modules N such that $N(\mathsf{m}) \neq 0$ is always nonzero and does not exceed Q_n.*

The theorem above shows that the elements of $\operatorname{Specm}\Gamma$ classify the irreducible $\mathfrak{gl}(n)$-modules (and, hence, irreducible $\mathfrak{sl}(n)$-modules) up to some finiteness.

The following result will be used in Sections 5 and 7.

Lemma 3.4. *Let $\mathsf{m} \in \operatorname{Specm}\Gamma$ and M be a $\mathfrak{gl}(n)$-module generated by a nonzero element $v \in M(\mathsf{m})$. Then M is a Gelfand–Tsetlin module.*

Proof. Recall that a commutative subalgebra A of some associative algebra B is a Harish-Chandra subalgebra, if for any $b \in B$, the A-bimodule AbA is finitely generated both as a left and as a right A-module (for details see [4]). By Corollary 5.4 and Proposition 7.2 in [8], the Gelfand–Tsetlin subalgebra Γ is a Harish-Chandra subalgebra of U.

Let $z_1, \dots, z_m$, $m = \frac{n(n+1)}{2}$, be a set of generators of Γ. Then there exist polynomials $f_j \in \mathbb{C}[x]$, $j = 1, \dots, m$ such that $f_j(z_j)v = 0$ for all j. Let z be any generator of Γ.

Since Γ is a Harish-Chandra subalgebra, there exist $\gamma_1, \dots, \gamma_s \in \Gamma$ such that for any large power N of z we can write

$$z^N E_{i,i+1} = \sum_{j=1}^{s} \gamma_i E_{i,i+1} g_j,$$

for any i and for some $g_j \in \Gamma$. Using this and the fact that there are only finitely many linearly independent elements in the set

$$\{z_i^{t_i} v | t_i \in \mathbb{Z}_{\geq 0}, i = 1, \dots, m\},$$

we conclude that $h_i(z)E_{i,i+1}v = 0$ for each i and some polynomial $h_i \in \mathbb{C}[x]$. We apply similar reasoning for $E_{i,i-1}v$. The computation for $E_{ii}v$ is trivial since $E_{ii} \in \Gamma$, which completes the proof. $\square$

3.3. Finite dimensional modules for $\mathfrak{gl}(n)$

In this section we recall a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite dimensional $\mathfrak{gl}(n)$-module.

Definition 3.5. The following array $T(v)$ with complex entries $\{v_{ij} : 1 \leq j \leq i \leq n\}$

$$\begin{array}{ccccccccc}
\boxed{v_{n1}} & & \boxed{v_{n2}} & & \cdots & & \boxed{v_{n,n-1}} & & \boxed{v_{nn}} \\
& \boxed{v_{n-1,1}} & & & \cdots & & & \boxed{v_{n-1,n-1}} & \\
& & & \cdots & \cdots & \cdots & & & \\
& & & \boxed{v_{21}} & & \boxed{v_{22}} & & & \\
& & & & \boxed{v_{11}} & & & &
\end{array}$$

is called a *Gelfand–Tsetlin tableau.*

We will sometimes consider a tableau $T(v)$ as an element in $\mathbb{C}^{\frac{n(n+1)}{2}} \simeq \operatorname{Specm}\Lambda$. However, we should note that $T(v + w) \neq T(v) + T(w)$.

A Gelfand–Tsetlin tableau $T(v)$ is called *standard* if:

$$v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}, \quad \text{for all } 1 \leq i \leq k \leq n-1.$$

Note that, for sake of convenience, the second condition above is slightly different from the original condition in [13].

Theorem 3.6. *(See [13,27].) Let $L(\lambda)$ be the finite dimensional irreducible module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$. Then there exists a basis of $L(\lambda)$ consisting of all standard tableaux $T(v)$ with fixed top row $v_{n1} = \lambda_1, v_{n2} = \lambda_2 - 1, \dots, v_{nn} = \lambda_n - n + 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the* Gelfand–Tsetlin formulas*:*

$$E_{k,k+1}(T(v)) = -\sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k+1}(v_{ki} - v_{k+1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})} \right) T(v + \delta^{ki}),$$

$$E_{k+1,k}(T(v)) = \sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k-1}(v_{ki} - v_{k-1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})} \right) T(v - \delta^{ki}),$$

$$E_{kk}(T(v)) = \left(\sum_{i=1}^{k}(v_{ki} + i - 1) - \sum_{i=1}^{k-1}(v_{k-1,i} + i - 1) \right) T(v),$$

where $T(v \pm \delta^{ki})$ is the tableau obtained by $T(v)$ adding ± 1 to the (k,i)-th entry of $T(v)$. If the new tableau $T(v \pm \delta^{ki})$ is not standard, then the corresponding summand of $E_{k,k+1}(T(v))$ or $E_{k+1,k}(T(v))$ is zero by definition. Furthermore, for $1 \leq s \leq r \leq n$,

$$c_{rs}(T(v)) = \gamma_{rs}(v)T(v), \tag{4}$$

where γ_{rs} are defined in (2).

One immediate consequence of the above theorem is that the algebra Γ acts semisimply on any finite-dimensional irreducible module $L(\lambda)$. Moreover, $L(\lambda)$ has a *simple spectrum*, that is, all Gelfand–Tsetlin multiplicities are 1.

3.4. Generic Gelfand–Tsetlin modules

Observing that the coefficients in the Gelfand–Tsetlin formulas in Theorem 3.6 are rational functions on the entries of the tableaux, it is natural to extend the Gelfand–Tsetlin construction to more general modules. In the case when all denominators are nonintegers, one can use the same formulas and define a new class of infinite dimensional $\mathfrak{gl}(n)$-modules: *generic* Gelfand–Tsetlin modules (cf. [4], Section 2.3).

Definition 3.7. A Gelfand–Tsetlin tableau $T(v)$ is called *generic* if $v_{rs} - v_{ru} \notin \mathbb{Z}$ for each $1 \leq s < u \leq r \leq n-1$.

Theorem 3.8. *(See §2.3 in [4] and Theorem 2 in [22].) Let $T(v)$ be a generic Gelfand–Tsetlin tableau. Denote by $V(T(v))$ the vector space with basis consisting of all Gelfand–Tsetlin tableaux $T(w)$ satisfying $w_{nj} = v_{nj}$, $w_{ij} - v_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n-1$.*

(i) *The vector space $V(T(v))$ has a structure of a $\mathfrak{gl}(n)$-module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand–Tsetlin formulas. The module $V(T(v))$ has finite length.*
(ii) *The action of the generators of Γ on the basis elements of $V(T(v))$ is given by (4).*
(iii) *The module defined in* (i) *is a Gelfand–Tsetlin module all Gelfand–Tsetlin multiplicities of which are* 1*.*

Note that since $T(v)$ is generic all denominators of $E_{k,k+1}(T(w))$ and $E_{k+1,k}(T(w))$, $T(w) \in V(T(v))$, are nonzero, so the condition that the summands corresponding to nonstandard tableaux are zero in Theorem 3.8 is not needed. By a slight abuse of notation we will denote the module constructed in Theorem 3.8 by $V(T(v))$ and will call it *the universal generic Gelfand–Tsetlin module associated with* $T(v)$. Note that $V(T(v))$ need not be irreducible. Because Γ has simple spectrum on $V(T(v))$ for $T(w)$ in $V(T(v))$ we may define the *irreducible* $\mathfrak{gl}(n)$*-module in* $V(T(v))$ *containing* $T(w)$ to be the subquotient of $V(T(v))$ containing $T(w)$ (see Theorem 3.8(i)). A basis for the irreducible subquotients of $V(T(v))$ can be described as follows.

Definition 3.9. Let $T(v)$ be a fixed Gelfand–Tsetlin tableau. For any $T(w) \in V(T(v))$, and for any $1 < r \leq n$, $1 \leq s \leq r$ and $1 \leq u \leq r-1$ we define:

$$\Omega^+(T(w)) := \{(r,s,u) \,|\, w_{rs} - w_{r-1,u} \in \mathbb{Z}_{\geq 0}\}$$

The following theorem is a subject of a direct verification. Details can be found in [10].

Theorem 3.10. *Let $T(v)$ be a generic tableau and let $T(w)$ be a tableau in $V(T(v))$. Then the following hold.*

(i) *The submodule of $V(T(v))$ generated by $T(w)$ has a basis*

$$\mathcal{N}(T(w)) := \{T(w') \in V(T(v)) | \Omega^+(T(w)) \subseteq \Omega^+(T(w'))\};$$

(ii) *The irreducible $\mathfrak{gl}(n)$-module in $V(T(v))$ containing $T(w)$ has a basis*

$$\mathcal{I}(T(w)) := \{T(w') \in V(T(v)) | \Omega^+(T(w)) = \Omega^+(T(w'))\}.$$

The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{N}(T(w))$ is given by the Gelfand–Tsetlin formulas. The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{I}(T(w))$ is given by the Gelfand–Tsetlin formulas with the correction that all tableau $T(w' \pm \delta^{ki})$ for which $\Omega^+(T(w' \pm \delta^{ki})) \neq \Omega^+(T(w))$ are omitted in the sums for $E_{k,k+1}(T(w'))$ and $E_{k+1,k}(T(w'))$.

3.5. Gelfand–Tsetlin formulas in terms of permutations

In this subsection we rewrite and generalize the Gelfand–Tsetlin formulas in Theorem 3.6 in convenient for us terms.

Recall the convention that for a vector $v = (v_{n1}, \dots, v_{nn}|\cdots|v_{11})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(v)$, we denote the corresponding to v Gelfand–Tsetlin tableau. Let us call v in $\mathbb{C}^{\frac{n(n+1)}{2}}$ *generic* if $T(v)$ is a generic Gelfand–Tsetlin tableau, and denote by $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$ the set of all generic vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$.

Let $\widetilde{S}_m$ denotes the subset of S_m consisting of the transpositions $(1, i)$, $i = 1, \dots, m$. For $\ell < m$, set $\Phi_{\ell m} = \widetilde{S}_{m-1} \times \cdots \times \widetilde{S}_\ell$. For $\ell > m$ we set $\Phi_{\ell m} = \Phi_{m\ell}$. Finally we let $\Phi_{\ell\ell} = \{\mathrm{Id}\}$. Every σ in $\Phi_{\ell m}$ will be written as a $|\ell - m|$-tuple of transpositions $\sigma[t]$ (recall that $\sigma[t]$ is the t-th component of σ). Also, we consider every $\sigma \in \Phi_{\ell m}$ as an element of $G = S_n \times \cdots \times S_1$ by letting $\sigma[t] = \mathrm{Id}$ whenever $t < \min(\ell, m)$ or $t > \max(\ell, m) - 1$.

Recall that the (left) action of $\Phi_{\ell m}$ on $\mathbb{C}^{\frac{n(n+1)}{2}}$ is given by (3).

Definition 3.11. Let $1 \le r < s \le n$. Set

$$\varepsilon_{rs} := \delta^{r,1} + \delta^{r+1,1} + \ldots + \delta^{s-1,1} \in \mathbb{C}^{\frac{n(n+1)}{2}}.$$

Furthermore, define $\varepsilon_{rr} = 0$ and $\varepsilon_{sr} = -\varepsilon_{rs}$.

Definition 3.12. For each generic vector w and any $1 \le r, s \le n$ we define

$$e_{rs}(w) := \begin{cases} \left(\prod_{j=r}^{s-2} \left(\dfrac{\prod_{t\neq 1}^{j+1} (w_{j1} - w_{j+1,t})}{\prod_{t\neq 1}^{j} (w_{j1} - w_{jt})} \right) \right) e_{s-1,s}(w), & \text{if} \quad r < s, \\ e_{s+1,s}(w) \left(\prod_{j=s+2}^{r} \left(\dfrac{\prod_{t\neq 1}^{j-2} (w_{j-1,1} - w_{j-2,t})}{\prod_{t\neq 1}^{j-1} (w_{j-1,1} - w_{j-1,t})} \right) \right), & \text{if} \quad r > s, \\ r - 1 + \sum_{i=1}^{r} w_{ri} - \sum_{i=1}^{r-1} w_{r-1,i}, & \text{if} \quad r = s, \end{cases}$$

where

$$e_{k,k+1}(w) := -\frac{\prod_{j=1}^{k+1} (w_{k1} - w_{k+1,j})}{\prod_{j\neq 1}^{k} (w_{k1} - w_{kj})}; \qquad e_{k+1,k}(w) := \frac{\prod_{j=1}^{k-1} (w_{k1} - w_{k-1,j})}{\prod_{j\neq 1}^{k} (w_{k1} - w_{kj})}.$$

It is not difficult to prove the following generic module version of Theorem 3.6

Proposition 3.13. *Let $v \in \mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$. The Gelfand–Tsetlin formulas for the generic Gelfand Tsetlin $\mathfrak{gl}(n)$-module $V(T(v))$ can be written as follows:*

$$E_{\ell m}(T(v+z)) = \sum_{\sigma\in\Phi_{\ell m}} e_{\ell m}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{\ell m})),$$

for $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ *with top row zero and* $1 \leq m \leq \ell \leq n$*.*

4. Derivative tableaux

In this section all Gelfand–Tsetlin tableaux $T(v)$, $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$, that we consider *will have fixed first row* $v_{nm} = a_m$, $m = 1, \ldots, n$. For this reason we will assume that the corresponding vectors v are in $\mathbb{C}^{\frac{n(n-1)}{2}}$. The goal is to introduce a new notion of tableaux, called derivative tableaux, and to define a module structure on the space spanned by all regular and derivative tableaux.

4.1. Singular tableaux

Definition 4.1. A vector $v \in \mathbb{C}^{\frac{n(n-1)}{2}}$ will be called *singular* if there exist $1 \leq s < t \leq r \leq n-1$ such that $v_{rs} - v_{rt} \in \mathbb{Z}$. The vector v will be called 1-*singular* if there exist i, j, k with $1 \leq i < j \leq k \leq n-1$ such that $v_{ki} - v_{kj} \in \mathbb{Z}$ and $v_{rs} - v_{rt} \notin \mathbb{Z}$ for all $(r, s, t) \neq (k, i, j)$.

From now on we fix (i, j, k) *such that* $1 \leq i < j \leq k \leq n-1$. By $\mathcal{H} = \mathcal{H}^k_{ij}$ we denote the hyperplane $v_{ki} - v_{kj} = 0$ in $\mathbb{C}^{\frac{n(n-1)}{2}}$ and let $\overline{\mathcal{H}} = \overline{\mathcal{H}}^k_{ij}$ be the subset of all v in $\mathbb{C}^{\frac{n(n-1)}{2}}$ such that $v_{tr} \neq v_{ts}$ for all triples (t, r, s) except for $(t, r, s) = (k, i, j)$. *From now on we fix* $\bar{v}$ *in* $\mathcal{H}$ *such that* $\bar{v}_{ki} = \bar{v}_{kj}$ *and all other differences* $\bar{v}_{mr} - \bar{v}_{ms}$ *are noninteger.* In other words, $\bar{v} \in \mathcal{H}$ and $\bar{v} + \mathbb{Z}^{\frac{n(n-1)}{2}} \subset \overline{\mathcal{H}}$. In particular, $\bar{v}$ is a 1-singular vector. A character χ and $\mathsf{n} = \operatorname{Ker}\chi$ are called *1-singular* if ℓ_{n} is 1-singular for one choice (hence for all choices) of ℓ_{n}. An indecomposable Gelfand–Tsetlin module M will be called 1-*singular Gelfand–Tsetlin module* if $M(\mathsf{n}) \neq 0$ for some 1-singular $\mathsf{n} \in \operatorname{Specm}\Gamma$.

Our first goal is to define a module $V(T(\bar{v}))$ whose support contains the image of $\{\bar{v}+z \mid z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$ under the map $\pi : \operatorname{Specm}\Lambda \to \operatorname{Specm}\Gamma$ defined in §3.1. This module will be our *universal* 1-*singular Gelfand–Tsetlin module.* In order to define $V(T(\bar{v}))$ we first introduce a family $\mathcal{V}_{\text{gen}}$ of generic Gelfand–Tsetlin modules.

4.2. Family of generic Gelfand–Tsetlin modules

Since for a generic v, $V(T(v)) = V(T(v'))$ whenever $v - v' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we may define $V(T(v))$ for v in the (generic) complex torus $T = \mathbb{C}_{\text{gen}}^{\frac{n(n-1)}{2}}/\mathbb{Z}^{\frac{n(n-1)}{2}}$. We will take a direct sum of such generic $V(T(v))$ by choosing representatives $w+\mathbb{Z}^{\frac{n(n-1)}{2}}$ of w in T as "close" as possible to $\bar{v}$ as follows.

Definition 4.2. For $w \in \mathbb{C}_{\text{gen}}^{\frac{n(n-1)}{2}}$, let $\lfloor Re(\bar{v} - w)\rfloor$ be the vector in $\mathbb{C}^{\frac{n(n-1)}{2}}$ whose (r, s)th component is $\lfloor Re(\bar{v}_{rs} - w_{rs})\rfloor$ (the integer part of the real part of $\bar{v}_{rs} - w_{rs}$). We set

$$\mathcal{S} := \left\{ w + \lfloor Re(\bar{v} - w) \rfloor \mid w \in \mathbb{C}_{\text{gen}}^{\frac{n(n-1)}{2}} \right\}.$$

Remark 4.3. The elements in $\mathcal{S}$ are as close as possible to $\bar{v}$ in the following sense. If $\bar{v}[w] := w + \lfloor Re(\bar{v} - w) \rfloor$, then $\lfloor Re(\bar{v}_{rs} - \bar{v}[w]_{rs}) \rfloor = 0$ for any r, s. Actually, if $u \in w + \mathbb{Z}^{\frac{n(n-1)}{2}}$, and $\lfloor Re(v_{rs} - u_{rs}) \rfloor = 0$ for any r, s, then we have $u = \bar{v}[w]$.

Now define the family of generic Gelfand–Tsetlin modules $\mathcal{V}_{\text{gen}} := \bigoplus_{v \in \mathcal{S}} V(T(v))$.

Denote by $\mathcal{F}$ the space of rational functions on $v_{\ell m}$, $1 \leq m \leq \ell \leq n$, with poles on the hyperplanes $v_{rs} - v_{rt} = 0$, and by $\mathcal{F}_{ij}$ the subspace of $\mathcal{F}$ consisting of all functions that are smooth on $\overline{\mathcal{H}}$. Then $\mathcal{F} \otimes \mathcal{V}_{\text{gen}}$ is a $\mathfrak{gl}(n)$-module with the trivial action on $\mathcal{F}$.

4.3. Derivative tableaux and definition of $V(T(\bar{v}))$

From now on by τ we denote the element in $S_{n-1} \times \cdots \times S_1$ such that $\tau[k]$ is the transposition (i, j) and all other $\tau[t]$ are Id. In particular $w \in \mathcal{H}$ if and only if $\tau(w) = w$.

Since $\bar{v}$ is a 1-singular vector, $\gamma_{rs}(\bar{v} + z) = \gamma_{rs}(\bar{v} + \tau(z))$ for any $1 \leq s \leq r \leq n$. Because of this we impose the conditions $T(\bar{v}+z) - T(\bar{v}+\tau(z)) = 0$ on the corresponding tableaux. Now we formally introduce new tableaux $\mathcal{D}_{ij}T(\bar{v} + z)$ for every $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ subject to the relations $\mathcal{D}_{ij}T(\bar{v} + z) + \mathcal{D}_{ij}T(\bar{v} + \tau(z)) = 0$ or, equivalently, $\mathcal{D}_{ij}T(u) + \mathcal{D}_{ij}T(\tau(u)) = 0$ for all u in $\bar{v} + \mathbb{Z}^{\frac{n(n-1)}{2}}$. We call $\mathcal{D}_{ij}T(u)$ *the derivative Gelfand–Tsetlin tableau* associated with u. Note that $\mathcal{D}_{ij}T(\bar{v} + z)$ are not new combinatorial objects, they should rather be considered as vectors in a vector space enlargement of the space of regular Gelfand–Tsetlin tableaux.

Definition 4.4. We set $V(T(\bar{v}))$ to be the vector space spanned by

$$\{T(\bar{v} + z),\, \mathcal{D}_{ij}T(\bar{v} + z) \mid z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}.$$

Remark 4.5. Note that $\{T(\bar{v} + z),\, \mathcal{D}_{ij}T(\bar{v} + z) \mid z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}$ is not a basis for $V(T(\bar{v}))$ since $T(\bar{v} + z) - T(\bar{v} + \tau(z)) = 0$ and $\mathcal{D}_{ij}T(\bar{v} + z) + \mathcal{D}_{ij}T(\bar{v} + \tau(z)) = 0$. A basis of $V(T(\bar{v}))$ is for example the set

$$\{T(\bar{v} + z), \mathcal{D}_{ij}T(\bar{v} + z') \mid z_{ki} \leq z_{kj}, z'_{ki} > z'_{kj}\}.$$

Set $\mathcal{V}' = V(T(\bar{v})) \oplus \mathcal{V}_{\text{gen}}$. Define *the evaluation map* $\text{ev}(\bar{v}) : \mathcal{F}_{ij} \otimes \mathcal{V}' \to \mathcal{V}'$, which is linear and

$$fT(v + z) \mapsto f(\bar{v})T(\bar{v} + z),\; f\mathcal{D}_{ij}T(\bar{v} + z) \mapsto f(\bar{v})\mathcal{D}_{ij}T(\bar{v} + z),$$

for $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, $f \in \mathcal{F}_{ij}$ and $v \in \mathcal{S}$.

Finally, let $\mathcal{D}_{ij}^{\bar{v}} : \mathcal{F}_{ij} \otimes V(T(v)) \to V(T(\bar{v}))$ be the linear map defined by

$$\mathcal{D}_{ij}^{\bar{v}}(fT(v + z)) = \mathcal{D}_{ij}^{\bar{v}}(f)T(\bar{v} + z) + f(\bar{v})\mathcal{D}_{ij}T(\bar{v} + z),$$

where $\mathcal{D}^{\bar{v}}_{ij}(f) = \frac{1}{2}\left(\frac{\partial f}{\partial v_{ki}} - \frac{\partial f}{\partial v_{kj}}\right)(\bar{v})$. In other words, this is the map $\mathcal{D}^{\bar{v}}_{ij} \otimes \mathrm{ev}(\bar{v}) + \mathrm{ev}(\bar{v}) \otimes \mathcal{D}^{\bar{v}}_{ij}$. This map certainly extends to a linear map $\mathcal{F}_{ij} \otimes \mathcal{V}_{\mathrm{gen}} \to V(T(\bar{v}))$ which we will also denote by $\mathcal{D}^{\bar{v}}_{ij}$. In particular, $\mathcal{D}^{\bar{v}}_{ij}(T(v+z)) = \mathcal{D}_{ij}(T(\bar{v}+z))$.

Remark 4.6. The operator $\mathcal{D}^{\bar{v}}_{ij} : \mathcal{V}_{\mathrm{gen}} \to V(T(\bar{v}))$ can be considered as a formal derivation. Namely, for $v \in \mathbb{C}_{\mathrm{gen}}^{\frac{n(n+1)}{2}}$ and $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, $\mathcal{D}^{\bar{v}}_{ij} T(v+z)$ is the formal limit of $\frac{T(v+z)-T(v+\tau(z))}{v_{ki}-v_{kj}}$ when $v \to \bar{v}$.

4.4. Module structure on $V(T(\bar{v}))$

From now on we set for convenience $\mathcal{D}^{\bar{v}} = \mathcal{D}^{\bar{v}}_{ij}$, $\mathcal{D}T(\bar{v}+z) = \mathcal{D}^{\bar{v}}_{ij}(T(v+z)) = \mathcal{D}_{ij}T(\bar{v}+z)$, $x = v_{ki}$, and $y = v_{kj}$.

We define the action of $\mathfrak{gl}(n)$ on the generators of $V(T(\bar{v}))$ as follows:

$$E_{rs}(T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}((x-y)E_{rs}(T(v+z)))$$
$$E_{rs}(\mathcal{D}T(\bar{v}+z')) = \mathcal{D}^{\bar{v}}(E_{rs}(T(v+z'))),$$

where v is a generic vector, $z, z' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ with $z' \neq \tau(z')$. One should note that $(x-y)E_{rs}(T(v+z))$ and $E_{rs}(T(v+z'))$ are in $\mathcal{F}_{ij} \otimes V(T(v))$, so the right hand sides in the above formulas are well defined.

The following proposition is proved in §A.2.

Proposition 4.7. *Let v be generic and $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$.*

(i) $\mathcal{D}^{\bar{v}}((x-y)E_{rs}T(v+z)) = \mathcal{D}^{\bar{v}}((x-y)E_{rs}T(v+\tau(z)))$ *for all z.*
(ii) $\mathcal{D}^{\bar{v}}(E_{rs}T(v+z)) = -\mathcal{D}^{\bar{v}}(E_{rs}T(v+\tau(z)))$ *for all z such that $\tau(z) \neq z$.*

Now, with the aid of Proposition 4.7 we define $E_{rs}F$ for any $F \in \mathcal{F}_{ij} \otimes V(T(\bar{v}))$. Then, using linearity we define gF for any $g \in \mathfrak{gl}(n)$ and $F \in \mathcal{F}_{ij} \otimes \mathcal{V}'$. It remains to prove that this well-defined action endows $V(T(\bar{v}))$, and hence $\mathcal{F}_{ij} \otimes \mathcal{V}'$, with a $\mathfrak{gl}(n)$-module structure.

Lemma 4.8. *Let $g \in \mathfrak{gl}(n)$.*

(i) $g(T(\bar{v}+z)) = \mathrm{ev}(\bar{v})g(T(v+z))$ *whenever $\tau(z) \neq z$.*
(ii) $\mathcal{D}^{\bar{v}}g(F) = g\mathcal{D}^{\bar{v}}(F)$ *if F and $g(F)$ are in $\mathcal{F}_{ij} \otimes \mathcal{V}_{\mathrm{gen}}$.*
(iii) $\mathcal{D}^{\bar{v}}((x-y)g(F)) = g(\mathrm{ev}(\bar{v})F)$ *if $F \in \mathcal{F}_{ij} \otimes \mathcal{V}_{\mathrm{gen}}$.*

Proof. Since $\mathcal{D}^{\bar{v}}$ is linear, it is enough to show each of the statements for $g = E_{rs}$ and $F = fT(v+z)$ with generic v and $f \in \mathcal{F}_{ij}$.

(i) Since $\tau(z) \neq z$, $e_{rs}(\sigma(v+z)) \in \mathcal{F}_{ij}$ for all $\sigma \in \Phi_{rs}$. Thus

$$\begin{aligned}E_{rs}(T(\bar v+z)) &= \mathcal{D}^{\bar v}((x-y)E_{rs}(T(v+z)))\\ &= \mathcal{D}^{\bar v}\left(\sum_{\sigma\in\Phi_{rs}}(x-y)e_{rs}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{rs}))\right)\\ &= \sum_{\sigma\in\Phi_{rs}}\mathrm{ev}(\bar v)\left(e_{rs}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{rs}))\right)\\ &= \mathrm{ev}(\bar v)E_{rs}(T(v+z)).\end{aligned}$$

(ii) Using (i) and the facts that $E_{rs}(T(v+z))$ is in $\mathcal{F}_{ij}\otimes\mathcal{V}$ and $E_{rs}(\mathcal{D}T(\bar v+z)) = \mathcal{D}^{\bar v}(E_{rs}(T(v+z)))$ we have

$$\begin{aligned}\mathcal{D}^{\bar v}E_{rs}(fT(v+z)) - E_{rs}\mathcal{D}^{\bar v}(fT(v+z)) &= \mathcal{D}^{\bar v}(f)\left(\mathrm{ev}(\bar v)E_{rs}(T(v+z)) - E_{rs}(T(\bar v+z))\right)\\ &= 0.\end{aligned}$$

(iii) Taking into consideration that $\mathrm{ev}(\bar v)\left((x-y)E_{rs}(T(v+z))\right)=0$, we have

$$\begin{aligned}\mathcal{D}^{\bar v}\left((x-y)E_{rs}(fT(v+z))\right) &= f(\bar v)\mathcal{D}^{\bar v}\left((x-y)E_{rs}(T(v+z))\right)\\ &= E_{rs}(\mathrm{ev}(\bar v)fT(v+z)). \qquad \square\end{aligned}$$

Proposition 4.9. *Let $g_1, g_2 \in \mathfrak{gl}(n)$. Then*

$$[g_1,g_2](T(\bar v+z)) = g_1(g_2(T(\bar v+z))) - g_2(g_1(T(\bar v+z))).$$

Proof. We have that $g_tT(\bar v+z) = \mathcal{D}^{\bar v}(G_t)$ where $G_t = (x-y)g_t(T(v+z))$ is in $\mathcal{F}_{ij}\otimes V(T(v))$ for $t=1,2$. Hence

$$\begin{aligned}g_1(g_2(T(\bar v+z))) - g_2(g_1(T(\bar v+z))) &= g_1\mathcal{D}^{\bar v}(G_2) - g_2\mathcal{D}^{\bar v}(G_1)\\ &= \mathcal{D}^{\bar v}(g_1G_2) - \mathcal{D}^{\bar v}(g_2G_1)\\ &= \mathcal{D}^{\bar v}\left((x-y)(g_1g_2-g_2g_1)T(v+z)\right)\\ &= (g_1g_2-g_2g_1)T(\bar v+z)\end{aligned}$$

by Lemma 4.8(i) and definitions. $\square$

The following proposition is proved in §A.3.

Proposition 4.10. *Let $g_1, g_2 \in \mathfrak{gl}(n)$ and $\tau(z)\neq z$. Then*

$$[g_1,g_2](\mathcal{D}T(\bar v+z)) = g_1(g_2(\mathcal{D}T(\bar v+z))) - g_2(g_1(\mathcal{D}T(\bar v+z))). \tag{5}$$

Combining Propositions 4.9 and 4.10 we obtain the following.

Theorem 4.11. *If $\bar{v}$ is a 1-singular vector in $\mathbb{C}^{\frac{n(n-1)}{2}}$, then $V(T(\bar{v}))$ is a $\mathfrak{gl}(n)$-module, with action of the generators of $\mathfrak{gl}(n)$ given by*

$$E_{rs}(T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}((x-y)E_{rs}(T(v+z)))$$
$$E_{rs}(\mathcal{D}T(\bar{v}+z')) = \mathcal{D}^{\bar{v}}(E_{rs}(T(v+z'))),$$

for any $z, z' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ with $z' \neq \tau(z')$.

4.5. Properties of the module $V(T(\bar{v}))$

In this subsection we summarize the main properties of the module $V(T(\bar{v}))$. Theorem 4.12 will be proven in §5.1 and Theorem 4.14 will be proven in §6.1.

Theorem 4.12. *The module $V(T(\bar{v}))$ is a 1-singular Gelfand–Tsetlin module. Moreover for any $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ and any $1 \leq r \leq s \leq n$ the following identities hold.*

$$c_{rs}(T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}((x-y)c_{rs}(T(v+z))) \tag{6}$$
$$c_{rs}(\mathcal{D}T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}(c_{rs}(T(v+z))), \text{ if } z \neq \tau(z). \tag{7}$$

The above theorem implies the following.

Corollary 4.13. *For any $z, z' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ with $z' \neq \tau(z')$, the action of the generators of Γ on $V(T(\bar{v}))$ is given by:*

$$c_{rs}(T(\bar{v}+z) = \gamma_{rs}(\bar{v}+z)T(\bar{v}+z)$$
$$c_{rs}(\mathcal{D}T(\bar{v}+z')) = \gamma_{rs}(\bar{v}+z')\mathcal{D}T(\bar{v}+z') + \mathcal{D}^{\bar{v}}(\gamma_{rs}(v+z'))T(\bar{v}+z').$$

Moreover, $\dim V(T(\bar{v}))(\mathsf{m}) \leq 2$ *for any* $\mathsf{m} \in \operatorname{Specm}\Gamma$.

Theorem 4.14. *The module $V(T(\bar{v}))$ is irreducible whenever $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$ for any $1 \leq t < r \leq n$, $1 \leq s \leq r$.*

5. Action of the Gelfand–Tsetlin subalgebra of $\mathfrak{gl}(n)$ on the module $V(T(\bar{v}))$

Recall that $\gamma_{rs}(w) = \sum_{i=1}^{r}(w_{ri}+r-1)^s \prod_{j\neq i}\left(1 - \frac{1}{w_{ri}-w_{rj}}\right)$ is a symmetric polynomial in $w_{r1}, \ldots, w_{rr}$ of degree s, and that $\{\gamma_{r1}, \ldots, \gamma_{rr}\}$ generate the algebra of symmetric polynomials in $w_{r1}, \ldots, w_{rr}$.

Also, recall that for a generic vector v and $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we have

$$c_{rs}(T(v+z)) \ = \gamma_{rs}(v+z)T(v+z),$$

where c_{rs} are the generators of Γ defined in (1). Recall the fixed set of representatives $\mathcal{S}$ from Definition 4.2.

Lemma 5.1. *Let v be any generic vector in $\mathcal{S}$ such that $v_{ki} = x$ and $v_{kj} = y$.*

(i) *If $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ is such that $|z_{ki} - z_{kj}| \geq n - m$ for some $0 \leq m \leq n$, then for each $1 \leq r \leq s \leq n - m$ we have:*
 (a) $c_{rs}(T(\bar{v} + z)) = \mathcal{D}^{\bar{v}}((x - y)c_{rs}(T(v + z)))$,
 (b) $c_{rs}(\mathcal{D}T(\bar{v} + z)) = \mathcal{D}^{\bar{v}}(c_{rs}(T(v + z)))$ *if* $z \neq \tau(z)$.

(ii) *If $1 \leq s \leq r \leq k$ and $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ then the action of c_{rs} on $T(\bar{v} + z)$ and $\mathcal{D}T(\bar{v} + z)$ is defined by the formulas in* (i).

Proof.

(i) By the hypothesis $|z_{ki} - z_{kj}| \geq n - m$, the coefficients that appear in the decompositions of the vectors $(x - y)E_{i_s,i_1}(T(v + z))$, $E_{i_s,i_1}(T(v + z))$ $(z \neq \tau(z))$, $(x - y)E_{i_t,i_{t+1}} \dots E_{i_s,i_1}(T(v + z))$ and $E_{i_t,i_{t+1}} \dots E_{i_s,i_1}(T(v + z))$ $(z \neq \tau(z))$ for $1 \leq t \leq s - 1$ are all in $\mathcal{F}_{ij}$.
 (a) For each $(i_1, \dots, i_s) \in \{1, \dots, r\}^s$ we have $(x - y)E_{i_s,i_1}(T(v + z)) \in \mathcal{F}_{ij} \otimes \mathcal{V}_{\text{gen}}$ and for each $1 \leq t \leq s-1$, we have $(x-y)E_{i_t,i_{t+1}} \dots E_{i_s,i_1}(T(v+z)) \in \mathcal{F}_{ij} \otimes \mathcal{V}_{\text{gen}}$. Then the statement follows from Lemma 4.8(ii).
 (b) If $z \neq \tau(z)$ then for each $(i_1, \dots, i_s) \in \{1, \dots, r\}^s$, $E_{i_s i_1}(T(v + z)) \in \mathcal{F}_{ij} \otimes \mathcal{V}_{\text{gen}}$ and for each $1 \leq t \leq s - 1$, $E_{i_t i_{t+1}} \dots E_{i_s i_1}(T(v+z)) \in \mathcal{F}_{ij} \otimes \mathcal{V}_{\text{gen}}$. Hence we can use Lemma 4.8(ii).

(ii) As $1 \leq s \leq r \leq k$ then every tableau that appears in the decomposition of $(x - y)E_{i_s i_1}(T(v + z))$, $E_{i_s i_1}(T(v + z))$ $(z \neq \tau(z))$, $(x - y)E_{i_t i_{t+1}} \dots E_{i_s i_1}(T(v + z))$ and $E_{i_t i_{t+1}} \dots E_{i_s i_1}(T(v + z))$ $(z \neq \tau(z))$ for $1 \leq t \leq s - 1$ has the same (k, i)th and (k, j)th entries. So, all of the listed vectors are in $\mathcal{F}_{ij} \otimes \mathcal{V}_{\text{gen}}$ and using Lemma 4.8(ii) we complete the proof. □

Lemma 5.2. *If $\tau(z) \neq z$ then we have the following identities.*

(i) $c_{k2}(T(\bar{v} + z)) = \gamma_{k2}(\bar{v} + z)T(\bar{v} + z)$.
(ii) $(c_{k2} - \gamma_{k2}(\bar{v} + z))\mathcal{D}T(\bar{v} + z) \neq 0$.
(iii) $(c_{k2} - \gamma_{k2}(\bar{v} + z))^2\mathcal{D}T(\bar{v} + z) = 0$.

Proof. If w is a generic vector then $c_{k2}T(w) = \gamma_{k2}(w)T(w)$ where $\gamma_{k2}(w)$ is a quadratic symmetric polynomial in variables $w_{k1}, \dots, w_{kk}$.

(i) By Lemma 5.1(ii), we have

$$\begin{aligned} c_{k2}(T(\bar{v} + z)) &= \mathcal{D}^{\bar{v}}((x - y)c_{k2}(T(v + z))) \\ &= \mathcal{D}^{\bar{v}}((x - y)\gamma_{k2}(v + z)T(v + z)) \\ &= \gamma_{k2}(\bar{v} + z)T(\bar{v} + z). \end{aligned}$$

(ii) Also by Lemma 5.1(ii) we have:

$$\begin{aligned} c_{k2}(\mathcal{D}T(\bar{v}+z)) &= \mathcal{D}^{\bar{v}}(c_{k2}(T(v+z))) \\ &= \mathcal{D}^{\bar{v}}(\gamma_{k2}(v+z)T(v+z)) \\ &= \mathcal{D}^{\bar{v}}(\gamma_{k2}(v+z))T(\bar{v}+z) + \gamma_{k2}(\bar{v}+z)\mathcal{D}T(\bar{v}+z) \end{aligned}$$

with $\mathcal{D}^{\bar{v}}(\gamma_{k2}(v+z)) = 2a(z_{ki} - z_{kj}) \neq 0$ where a is the coefficient of $(v_{ki} + z_{ki})^2$ in $\gamma_{k2}(v+z)$.

(iii) This part follows from (i) and (ii). □

Lemma 5.3. *Let Γ_{k-1} be the subalgebra of Γ generated by $\{c_{rs} : 1 \leq s \leq r \leq k-1\}$. If $z, z' \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ are such that $z_{rs} \neq z'_{rs}$ for some $1 \leq s \leq r \leq k-1$. Then, Γ_{k-1} separates the tableaux $T(\bar{v}+z)$ and $T(\bar{v}+z')$, that is, there exist $c \in \Gamma_{k-1}$ and $\gamma \in \mathbb{C}$ such that $(c-\gamma)T(\bar{v}+z) = 0$ but $(c-\gamma)T(\bar{v}+z') \neq 0$.*

Proof. By Lemma 5.1(ii), we know that the action of the generators of Γ_{k-1} on $T(\bar{v}+z)$ and $T(\bar{v}+z')$ is given by symmetric polynomials in the entries of rows $k-1, \dots, 1$. Assume the contrary, i.e. that $T(\bar{v}+z)$ and $T(\bar{v}+z')$ have the same character associated with the generators of Γ_{k-1}. Like in the generic case, the latter implies that one of the tableaux is obtained from the other by the action of an element in $S_{k-1} \times \cdots \times S_1$. But the difference of the entries on the rows $k-1, \dots, 1$ of $T(\bar{v}+z)$ and $T(\bar{v}+z')$ are not integers. Hence, the tableaux must have different characters which leads to a contradiction. □

For any $m \in \mathbb{Z}_{\geq 0}$ let R_m be the set of $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ such that $|z_{ki} - z_{kj}| = m$.

Lemma 5.4. *If $z \in R_m$ then there exists $\bar{z} \in R_{m+1}$ such that $T(\bar{v}+z)$ appears with non-zero coefficient in the decomposition of $E_{k+1,k-t}T(\bar{v}+\bar{z})$ for some $t \in \{0, 1, \dots, k-1\}$.*

Proof. If $\bar{v}_{ki} = \bar{v}_{kj} = x$ and $w = \bar{v} + z$ then there exist $l \in \mathbb{Z}$ such that $w_{ki} = x + m - l$ and $w_{kj} = x - l$. Let t be such that

$$\begin{cases} w_{ki} + 1 = w_{k-1,s_{k-1}}, & \text{for some} \quad 1 \leq s_{k-1} \leq k-1 \\ w_{k-1,s_{k-1}} + 1 = w_{k-2,s_{k-2}}, & \text{for some} \quad 1 \leq s_{k-2} \leq k-2 \\ \quad\vdots & \quad\vdots \qquad\qquad \vdots \\ w_{k-t+1,s_{k-t+1}} + 1 = w_{k-t,s_{k-t}}, & \text{for some} \quad 1 \leq s_{k-t} \leq k-t \\ w_{k-t,s_{k-t}} + 1 \neq w_{k-t-1,s}, & \text{for any} \quad 1 \leq s \leq k-t-1. \end{cases}$$

Then the coefficient of $T(\bar{v}+z)$ in the decomposition of $E_{k+1,k-t}T(\bar{v}+\bar{z})$ is not zero, where $\bar{z} = z + \delta^{ki} + \delta^{k-1,s_{k-1}} + \ldots + \delta^{k-t,s_{k-t}} \in R_{m+1}$. □

Remark 5.5. All tableaux that appear in the decomposition of $E_{k+1,k-t}T(\bar{v}+\bar{z})$ in the previous lemma are of the form $T(\bar{v}+\bar{z}+\sigma(\varepsilon_{k+1,k-t}))$ for $\sigma \in \Phi_{k+1,k-t}$. Furthermore, we have:

$$\begin{cases} \bar{z}+\sigma(\varepsilon_{k+1,k-t}) \in R_{m+2} & \text{if} \quad \sigma[k]=(1,j), \\ \bar{z}+\sigma(\varepsilon_{k+1,k-t}) \in R_{m} & \text{if} \quad \sigma[k]=(1,i), \\ \bar{z}+\sigma(\varepsilon_{k+1,k-t}) \in R_{m+1} & \text{if} \quad \sigma[k]\notin \{(1,i),(1,j)\}. \end{cases}$$

In particular, all tableaux $T(\bar{v}+\bar{z}+\sigma(\varepsilon_{k+1,k-t}))$ with $\bar{z}+\sigma(\varepsilon_{k+1,k-t}) \in R_m$ have the same entries in rows $k,\dots,n$ and two of these tableaux have at least one different entry in rows $1,\dots,k-1$. Therefore, by Lemma 5.3 all the tableaux $T(\bar{v}+\bar{z}+\sigma(\varepsilon_{k+1,k-t}))$ with $\bar{z}+\sigma(\varepsilon_{k+1,k-t}) \in R_m$ have different Gelfand–Tsetlin characters.

5.1. *Proof of Theorem 4.12*

Let $R_{\geq n} := \cup_{m\geq n} R_m$. For any $z \in R_{\geq n}$ consider the submodule W_z of $V(T(\bar{v}))$ generated by $T(\bar{v}+z)$. By Lemma 5.1(i)(a), $T(\bar{v}+z)$ is a common eigenvector of all generators of Γ and thus W_z is a Gelfand–Tsetlin module by Lemma 3.4. Then $W = \sum_{z\in R_{\geq n}} W_z$ is also a Gelfand–Tsetlin module. We first show that W contains all tableau $T(\bar{v}+z)$ for any $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$. Indeed, assume that $|z_{ki}-z_{kj}| = n-1$ and consider $T(\bar{v}+z)$. Then, by Lemma 5.4 there exists $z' \in R_n$ and a nonzero $x \in \mathfrak{gl}_{k+1}$ such that $xT(\bar{v}+z') = \sum_{t=0}^{N} a_t T(\bar{v}+z^{(t)})$, where $a_t \in \mathbb{C}$, $z^{(0)} = z$ and, $|z^{(t)}_{ki} - z^{(t)}_{kj}| \geq n-1$.

Following Remark 5.5, we assume that $N = 1$ and $z^{(0)}, z^{(1)} \in R_{n-1}$ without loss of generality since $z^{(m)}$ in $R_{\geq n}$ implies $T(\bar{v}+z^{(m)}) \in W$. The action of all generators $\{c_{rs}\}_{1\leq r\leq s\leq n-1}$ of Γ, except for the center of U, on $T(\bar{v}+z)$ and $T(\bar{v}+z^{(1)})$ is determined by (6). Let $c \in \Gamma$ be a central element and $(c-\gamma)T(\bar{v}+z') = 0$ for some complex γ. Then $(c-\gamma)xT(\bar{v}+z') = 0 = (c-\gamma)(a_0T(\bar{v}+z) + a_1T(\bar{v}+z^{(1)}))$. Recall that by Lemma 5.3, there exists $C \in \Gamma_{k-1}$ which acts with different scalars on $T(\bar{v}+z)$ and $T(\bar{v}+z^{(1)})$. Since C commutes with $(c-\gamma)$, both $T(\bar{v}+z)$ and $T(\bar{v}+z^{(1)})$ are in W. Moreover, $(c-\gamma)T(\bar{v}+z) = (c-\gamma)T(\bar{v}+z^{(1)}) = 0$. Hence, the action of Γ on any $T(\bar{v}+z)$ with $z \in R_{n-1}$ is given by (6). Moreover, $T(\bar{v}+z) \in W$ for any $z \in R_{n-1}$. Next we consider a tableau $T(\bar{v}+z)$ with $z \in R_{n-2}$. Again by Lemma 5.4 one finds a nonzero $y \in \mathfrak{gl}_{k+1}$ and $z' \in R_{n-1}$ such that $yT(\bar{v}+z')$ contains $T(\bar{v}+z)$ and at most one other tableau. For all generators of centers of $U(\mathfrak{gl}_m)$, $m \leq n-2$ the statement follows from Lemma 5.1. If c is in the center of U or in the center of $U(\mathfrak{gl}_{n-1})$ then it commutes with y. Choose $C \in \Gamma_{k-1}$ which separates the tableaux in the image $yT(\bar{v}+z')$ and which acts by a scalar on the tableau $T(\bar{v}+z')$. Applying the argument above we conclude that the action of Γ on any $T(\bar{v}+z)$ with $z \in R_{n-2}$ is determined by (6) and $T(\bar{v}+z) \in W$ for any $z \in R_{n-2}$. Continuing analogously with the sets $R_{n-3},\dots,R_0$ we show that any tableau $T(\bar{v}+z)$ belongs to W. Note that when $z \in R_0$, $\tau(z) = z$. In this case it will be the unique such tableau coming from some $T(\bar{v}+z')$ with $z' \in R_1$ and “separation” is not needed.

Consider the quotient $\overline{W} = V(T(\bar{v}))/W$. The vector $\mathcal{D}T(\bar{v}+z)+W$ of $\overline{W}$ is a common eigenvector of Γ by Lemma 5.1(i)(b) for any $z \in R_n$. We can repeat now the argument above substituting everywhere the tableaux $T(\bar{v}+z)$ by $\mathcal{D}T(\bar{v}+z)$. Hence, $\overline{W} = \sum_{z\in R_n} \overline{W}_z$, where $\overline{W}_z$ denotes the submodule of $\overline{W}$ generated by $\mathcal{D}T(\bar{v}+z)+W$. By Lemma 3.4 we conclude that $\overline{W}$ is a Gelfand–Tsetlin module. Therefore, $V(T(\bar{v}))$ is a Gelfand–Tsetlin module with action of Γ given by (6) and (7).

5.2. Example

Let $n = 3$, a be any complex number, $\bar{v} = (a, a, a|a, a|a)$ and $z = (m, n, k) \in \mathbb{Z}^3$. In this case the module $V(T(\bar{v}))$ has length 10. To describe the irreducible subquotients of $V(T(\bar{v}))$ we use the notation

$$L(D_1 \cup \ldots \cup D_k) = \operatorname{span}\{P(z) \,|\, z \text{ satisfies one of } D_1, \ldots, D_k\}$$

for sets of inequalities $D_1, \ldots, D_k$, where

$$P(z) := \begin{cases} T(\bar{v}+z), & \text{if} \quad m \leq n \\ \mathcal{D}T(\bar{v}+z), & \text{if} \quad m > n \end{cases}$$

The irreducible subquotients of $V(T(\bar{v}))$ can be separated into two groups:

(i) Eight (singular) irreducible Verma modules.

$$L_1 = L\left(\begin{cases} m \leq n \\ n \leq 0 \\ k \leq n \end{cases} \bigcup \begin{cases} m > n \\ m \leq 0 \\ k \leq n \end{cases}\right); L_3 = L\left(\begin{cases} m \leq n \\ n \leq 0 \\ k > n \end{cases} \bigcup \begin{cases} m > n \\ m \leq 0 \\ k > n \end{cases}\right);$$

$$L_5 = L\begin{pmatrix} m \leq n \\ m \leq 0 \\ n > 0 \\ k \leq m \end{pmatrix} \cong L\begin{pmatrix} m > n \\ n \leq 0 \\ m > 0 \\ k \leq n \end{pmatrix} = L_5';$$

$$L_2 = L\left(\begin{cases} m \leq n \\ m > 0 \\ k > n \end{cases} \bigcup \begin{cases} m > n \\ n > 0 \\ k > n \end{cases}\right); \ L_4 = L\left(\begin{cases} m \leq n \\ m > 0 \\ k \leq n \end{cases} \bigcup \begin{cases} m > n \\ n > 0 \\ k \leq n \end{cases}\right);$$

$$L_6 = L\begin{pmatrix} m \leq n \\ m \leq 0 \\ n > 0 \\ k > n \end{pmatrix} \cong L\begin{pmatrix} m > n \\ n \leq 0 \\ m > 0 \\ k > m \end{pmatrix} = L_6'.$$

The six non-isomorphic Verma modules are pictured on Fig. 1.

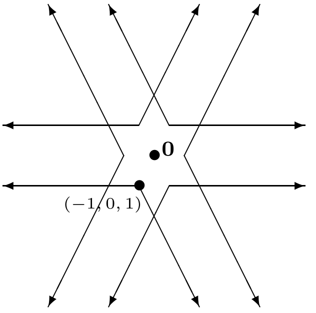

Fig. 1. Describes non-isomorphic Verma modules in a Gelfand–Tsetlin block of the trivial module for $\mathfrak{gl}(3)$.

(ii) Two isomorphic modules with infinite-dimensional weight spaces:

$$L_7 = L\begin{pmatrix} m \leq n \\ m \leq 0 \\ n > 0 \\ m < k \leq n \end{pmatrix} \cong L\begin{pmatrix} m > n \\ n \leq 0 \\ m > 0 \\ n < k \leq m \end{pmatrix} = L_7'.$$

Proposition 5.6. *The Loewy decomposition of $V(T(\bar{v}))$ has successive components:*

$$L_1,\ L_3 \oplus L_5,\ L_7,\ L_5' \oplus L_6,\ L_7',\ L_4 \oplus L_6',\ L_2$$

where L_1 is the socle of $V(T(\bar{v}))$.

Proof. A detailed proof of the proposition will be presented in [11]. Here we outline the important steps. We first show that L_1 is a submodule of $V(T(\bar{v}))$. Looking at the set of nonzero weights of L_1 we easily show that L_1 is the simple Verma module $L_{\mathfrak{b}_1}(\lambda_1)$ with highest weight $\lambda_1 = (-1, 0, 1)$ relative to the standard Borel subalgebra $\mathfrak{b}_1$, i.e. the one spanned by the standard Cartan subalgebra $\mathfrak{h}$ and the set $\{E_{12}, E_{23}, E_{13}\}$. We then show that $V(T(\bar{v}))/L_1$ has a submodule isomorphic to $L_3 \oplus L_5$ and looking at the sets of nonzero weights of L_3 and L_5 we prove that both of them are simple Verma modules $L_{\mathfrak{b}_3}(\lambda_3)$, $L_{\mathfrak{b}_5}(\lambda_5)$ relative to other Borel subalgebras $\mathfrak{b}_3$, $\mathfrak{b}_5$. We continue using the same reasoning. The nontrivial part is to show that $L_7 \simeq L_7'$ is irreducible. By direct verification we see that all basis elements $T(m, n, k)$ in L_7 are in $U \cdot T(0, 1, 1)$ and vice-versa, namely, that $T(0, 1, 1) \in U \cdot T(m, n, k)$ for $T(m, n, k) \in L_7$. This shows that L_7 is irreducible. Note that the correspondence $T(m, n, k) \to \mathcal{D}T(n, m, k)$ leads to an isomorphism between L_7 and L_7'. $\square$

Remark 5.7. An interesting observation is that the module L_7 (and hence L_7') can be described in terms of the localization functor D_{21} with respect to the Ore subset $\{E_{21}^k \mid k \in \mathbb{Z}_{\geq 0}\}$ of U (we refer the reader to [1] for a definition and properties of the functor D_{21}). More precisely we have the following exact sequence

$$0 \to L_7 \to (D_{21}L_5)/L_5 \to L_6 \to 0.$$

In fact, as we prove in [11], every irreducible Gelfand–Tsetlin $\mathfrak{gl}(3)$-module can be obtained as a subquotient of a twisted localized Verma module.

6. Irreducibility of $V(T(\bar{v}))$

6.1. Proof of Theorem 4.14

We introduce the following notation which will be used in this section only.

$$\begin{aligned} L &:= \{T(\bar{v}+z) | z \neq \tau(z)\}, \\ S &:= \{T(\bar{v}+z) | z = \tau(z)\}, \\ D &:= \{\mathcal{D}T(\bar{v}+z) | z \neq \tau(z)\}. \end{aligned}$$

Lemma 6.1. *Let $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ such that $z \neq \tau(z)$ and $w = \bar{v}+z$. If $w_{rs} - w_{r-1,t} \notin \mathbb{Z}_{\geq 0}$ for any r, s, t, then the module $V(T(\bar{v}))$ is generated by the two tableau $T(w)$ and $\mathcal{D}T(w)$.*

Proof. The action of $\mathfrak{gl}(n)$ on the elements from L is given by the classical Gelfand–Tsetlin formulas (see Proposition 3.13), and the conditions $w_{rs} - w_{r-1,t} \notin \mathbb{Z}_{\geq 0}$ imply that $\Omega^+(T(w)) = \emptyset$ (see Definition 3.9). Hence, by Theorem 3.10(i), the submodule generated by $T(w)$ contains $L \cup S$. Now, as $z \neq \tau(z)$, $\mathcal{D}T(w) \in D$ and the coefficient of $\mathcal{D}T(w + \sigma(\varepsilon_{rs}))$ in the decomposition of $E_{rs}(\mathcal{D}T(w))$ is $e_{rs}(\sigma(w))$. The condition $w_{rs} - w_{r-1,t} \notin \mathbb{Z}_{\geq 0}$ for any r, s, t implies again that $\Omega^+(\mathcal{D}T(w)) = \emptyset$. Thus, by Theorem 3.10, the submodule generated by the tableau $\mathcal{D}T(w)$ contains D. □

Lemma 6.2. *Let $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ such that $z \neq \tau(z)$ and $w = \bar{v}+z$. If $w_{rs} - w_{r-1,t} \notin \mathbb{Z}_{\geq -1}$ for any r, s, t, then $V(T(\bar{v}))$ is generated by $\mathcal{D}T(w)$.*

Proof. By Lemma 6.1 the module $V(T(\bar{v}))$ is generated by $\mathcal{D}T(w)$ and any tableau of the form $T(w')$ such that $w'_{rs} - w'_{r-1,t} \notin \mathbb{Z}_{\geq 0}$ for any r, s, t. Thus, it is enough to prove that we can generate one such $T(w')$ from $\mathcal{D}T(w)$. To prove this recall that the coefficient of $T(w + \varepsilon_{k-1,k})$ in the decomposition of $E_{k-1,k}(\mathcal{D}T(w))$ is $\mathcal{D}^{\bar{v}}(e_{k-1,k}(v+z))$. Now, since

$$\mathcal{D}^{\bar{v}}(e_{k-1,k}(v+z)) = -\frac{1}{2}\left(\frac{(z_{kj}-z_{ki})\prod_{t\neq i,j}^{k}(w_{k-1,1}-w_{k,t})}{\prod_{t\neq 1}^{k-1}(w_{k-1,1}-w_{k-1,t})}\right) \neq 0,$$

the tableau $T(w + \varepsilon_{k-1,k})$ satisfies the desired conditions. □

Lemma 6.3. *If $z \neq \tau(z)$ and $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$ for any $1 \leq t < r \leq n$, $1 \leq s \leq r$ then $T(\bar{v}+z)$ generates $V(T(\bar{v}))$.*

Proof. As $V(T(\bar{v}))$ is a Gelfand–Tsetlin module, the elements of the basis of $V(T(\bar{v}))$ can be separated by characters of Γ and by Lemma 5.2 we can separate different tableaux with the same Gelfand–Tsetlin character by the action of c_{k2}. And since a submodule of a Gelfand–Tsetlin module is a Gelfand–Tsetlin module, to prove the lemma it is sufficient to show that any tableau of $V(T(\bar{v}))$ appears with a nonzero coefficient in the decomposition of $gT(\bar{v}+z)$ for some $g \in U$.

We prove the lemma in three steps: first from $T(\bar{v}+z)$ we generate the set $L \cup S$, then from S we generate one tableau in D, and last, from this one tableau in D we generate the whole set D.

Step 1: All tableaux of $L \cup S$ are in the module $U \cdot T(\bar{v}+z)$ generated by $T(\bar{v}+z)$. The set $L \cup S$ can be obtain in this way because the action of $\mathfrak{gl}(n)$ on L is given by the classical Gelfand–Tsetlin formulas, so the coefficients in the formulas have numerators which are products of nonzero monomials of the form $\bar{v}_{rs} + z'_{rs} - \bar{v}_{r-1,t} - z'_{r-1,t}$, for some $z'_{rs}, z'_{r-1,t} \in \mathbb{Z}$.

Step 2: One tableau of D is in $U \cdot T(\bar{v}+z)$. If $z' = \tau(z')$ then, the coefficient of $\mathcal{D}T(\bar{v} + z' + \sigma(\varepsilon_{k+1,k}))$ on $E_{k+1,k}(T(\bar{v}+z'))$ is $\mathrm{ev}(\bar{v})((x-y)e_{k+1,k}(\sigma(v+z')))$, so in order to obtain an element of D from S we have to find some σ such that $\mathrm{ev}(\bar{v})((x-y)e_{k+1,k}(\sigma(v+z'))) \neq 0$. Take σ with $\sigma[k] = (1,i)$ and all other $\sigma[t] = \mathrm{Id}$. Then the numerator of $\mathrm{ev}(\bar{v})((x-y)e_{k+1,k}(\sigma(v+z')))$ is

$$\prod_{t=1}^{k-1} (\bar{v}_{ki} + z'_{ki} - \bar{v}_{k-1,t} - z'_{k-1,t}) \neq 0.$$

Step 3: All tableaux of D are in $U \cdot T(\bar{v}+z)$. Given $z' \neq \tau(z')$ and $\mathcal{D}T(\bar{v}+z') \in D$, the coefficient of $\mathcal{D}T(\bar{v}+z'+\sigma(\varepsilon_{rs}))$ on $E_{rs}(\mathcal{D}T(\bar{v}+z'))$ is $e_{rs}(\sigma(\bar{v}+z'))$. Now, using the lemma hypothesis, we see that $e_{rs}(\sigma(\bar{v}+z')) \neq 0$ for any σ implying that $\mathcal{D}T(\bar{v}+z')$ generates D. $\square$

Lemmas 6.3 and 6.2 together with Lemma 5.2 imply Theorem 4.14.

Remark 6.4. We conjecture that the condition $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$ is also necessary for the irreducibility of $V(T(\bar{v}))$. The conjecture is true for $n = 3$ as proved in [11].

7. Number of non-isomorphic irreducible modules associated with a singular character

Let $\mathsf{m} \in \operatorname{Specm}\Gamma$ and $\ell_{\mathsf{m}} \in \operatorname{Specm}\Lambda$ be such that $\pi(\ell_{\mathsf{m}}) = \mathsf{m}$. Then ℓ_{m} defines a vector $w = w_{\mathsf{m}}$. Assume that w_{m} is 1-singular, that is $w_{ki} - w_{kj} \in \mathbb{Z}$ for the fixed k, i, j, and all other differences of w_{tr} are noninteger.

Consider the module $M = U/U\mathsf{m}$. Since U is free as a left and as a right Γ-module ([24]) then, $M \neq 0$. Also M is a Gelfand–Tsetlin module by Lemma 3.4. Let $\mathsf{n} \in \operatorname{Specm}\Gamma$. Recall that by Theorem 3.2 we have

$$\dim M(\mathsf{n}) \leq |\{\phi \in \mathcal{M} \mid \pi(\phi\ell_{\mathsf{m}}) = \mathsf{n}\}|.$$

It is easy to see that the right hand side of the inequality above equals 2. Hence we proved the following.

Corollary 7.1. *All Gelfand–Tsetlin multiplicities of $M = U/U\mathsf{m}$ are at most 2. Moreover, if $w = w_{\mathsf{m}}$ and $w_{ki} = w_{kj}$, then $\dim M(\mathsf{m}) = 1$.*

Now we are ready to present irreducible 1-singular Gelfand–Tsetlin modules as subquotients of universal 1-singular modules.

Theorem 7.2. *Let $\mathsf{n} \in \operatorname{Specm}\Gamma$ be such that $w = \ell_{\mathsf{n}}$ is 1-singular and $w = \bar{v} + z$.*

(i) *There exist at most two non-isomorphic irreducible Gelfand–Tsetlin modules N_1 and N_2 such that $N_1(\mathsf{n}) \neq 0$ and $N_2(\mathsf{n}) \neq 0$.*
(ii) *If $w_{ki} = w_{kj}$ then there exists a unique irreducible Gelfand–Tsetlin module N with $N(\mathsf{n}) \neq 0$. This module appears as a subquotient of $V(T(\bar{v}))$.*
(iii) *If $w_{rs} - w_{r-1,t} \notin \mathbb{Z}$ for any $1 \leq t < r \leq n$, $1 \leq s \leq r$ then $V(T(\bar{v}))$ is the unique irreducible Gelfand–Tsetlin module N with such that $N(\mathsf{n}) \neq 0$.*

Proof. Let $X_{\mathsf{n}} = U/U\mathsf{n}$. We know that $X_{\mathsf{n}} = U/U\mathsf{n}$ is a Gelfand–Tsetlin module by Lemma 3.4. Furthermore, any irreducible Gelfand–Tsetlin module M with $M(\mathsf{n}) \neq 0$ is a homomorphic image of X_{n}, and $X_{\mathsf{n}}(\mathsf{n})$ maps onto $M(\mathsf{n})$. Since both spaces $X_{\mathsf{n}}(\mathsf{n})$ and $M(\mathsf{n})$ have additional structure as modules over certain algebra (see Corollary 5.3, [9]) then the projection $X_{\mathsf{n}}(\mathsf{n}) \to M(\mathsf{n})$ is in fact a homomorphism of modules. Taking into account that $\dim X_{\mathsf{n}}(\mathsf{n}) \leq 2$ by Corollary 7.1, we conclude that there exist at most two non-isomorphic irreducible N with $N(\mathsf{n}) \neq 0$. This proves part (i) of the theorem.

Recall that the Gelfand–Tsetlin multiplicities of $V(T(\bar{v}))$ are bounded by 2 by Corollary 4.13. Assume now that $w_{ki} = w_{kj}$. Then $\dim V(T(w))(\mathsf{n}) = 1$. But this is possible if and only if $\dim X_{\mathsf{n}}(\mathsf{n}) = 1$ by Theorem 3.2. Hence, there exists a unique irreducible quotient N of X_{n} (and of $V(T(\bar{v}))$) with $N(\mathsf{n}) \neq 0$ which implies part (ii). Finally, part (iii) follows from Theorem 4.14. $\square$

Remark 7.3. In order to complete the classification of irreducible 1-singular Gelfand–Tsetlin modules we need to find a presentation for two non-isomorphic irreducible modules N_1 and N_2 (if they exist) with $N_i(\mathsf{n}) \neq 0$, $i = 1, 2$, and some 1-singular n. We conjecture that both N_1 and N_2 appear as subquotients of $V(T(\bar{v}))$ for $w_{\mathsf{n}} = \bar{v} + z$. By Theorem 7.2(i) this is true if $V(T(\bar{v}))$ has two non-isomorphic irreducible subquotients with n in their support. On the other hand, if $V(T(\bar{v}))$ has two isomorphic irreducible subquotients, the conjecture remains open. Such irreducible modules have all Gelfand–Tsetlin multiplicities 1 (the case considered in [18,19]). Isomorphic subquotients of $V(T(\bar{v}))$ exist already in the case $n = 3$, a case in which the conjecture holds, as shown in [11].

Acknowledgments

D.G. gratefully acknowledges the hospitality and excellent working conditions at the São Paulo University and IHÉS, where part of this work was completed. V.F. is supported in part by CNPq grant (301320/2013-6) and by FAPESP grant (2014/09310-5). D.G. is supported in part by FAPESP grant (2011/21621-8) and by NSA grant H98230-13-1-0245. L.E.R. is supported by FAPESP grant (2012/23450-9). The authors are grateful to a referee for the careful reading of the manuscript and for making numerous valuable suggestions that improved the text.

Appendix A

Like in Section 4, in this appendix we assume that all tableaux $T(v)$ have fixed first row, hence the corresponding vectors v are in $\mathbb{C}^{\frac{n(n-1)}{2}}$. The goal of this appendix is to prove Propositions 4.7 and 4.10.

A.1. Useful identities

For a function $f = f(v)$ by f^τ we denote the function $f^\tau(v) = f(\tau(v))$.

The following lemma can be easily verified.

Lemma A.1. *Suppose $f \in \mathcal{F}_{ij}$ and $h := \frac{f-f^\tau}{x-y}$.*

(i) *If $f = f^\tau$, then $\mathcal{D}^{\bar{v}}(f) = 0$.*
(ii) *If $h \in \mathcal{F}_{ij}$, then $\mathrm{ev}(\bar{v})(h) = 2\mathcal{D}^{\bar{v}}(f)$.*
(iii) $\mathrm{ev}(\bar{v})(f) = \mathcal{D}^{\bar{v}}((x-y)f)$.

Lemma A.2. *Let f_m, g_m, $m = 1, \dots, t$, be functions such that $f_m, (x-y)g_m$, and $\sum_{m=1}^t f_m g_m$ are in $\mathcal{F}_{ij}$ and $g_m \notin \mathcal{F}_{ij}$. Assume also that $\sum_{m=1}^t f_m g_m^\tau = 0$. Then the following identities hold.*

(i) $2\sum_{m=1}^t \mathcal{D}^{\bar{v}}(f_m)\mathcal{D}^{\bar{v}}((x-y)g_m) = \mathcal{D}^{\bar{v}}\left(\sum_{m=1}^t f_m g_m\right)$.
(ii) $2\sum_{m=1}^t \mathcal{D}^{\bar{v}}(f_m)\mathrm{ev}(\bar{v})((x-y)g_m) = \mathrm{ev}(\bar{v})\left(\sum_{m=1}^t f_m g_m\right)$.

Proof. Set for simplicity $\bar{g}_m = (x-y)g_m$. For (i) we use Lemma A.1 and obtain

$$\begin{aligned}\mathcal{D}^{\bar{v}}\left(\sum_{k=1}^t f_m g_m\right) &= \mathcal{D}^{\bar{v}}\left(\sum_{m=1}^t f_m g_m + \sum_{m=1}^t f_m g_m^\tau\right)\\ &= \mathcal{D}^{\bar{v}}\left(\sum_{m=1}^t f_m \frac{\bar{g}_m - (\bar{g}_m)^\tau}{x-y}\right)\end{aligned}$$

$$= \sum_{m=1}^{t} \left(\mathcal{D}^{\bar{v}}(f_m) \mathrm{ev}(\bar{v}) \left(\frac{\bar{g}_m - (\bar{g}_m)^\tau}{x-y} \right) + \mathrm{ev}(\bar{v})(f_m) \mathcal{D}^{\bar{v}} \left(\frac{\bar{g}_m - (\bar{g}_m)^\tau}{x-y} \right) \right)$$

$$= 2 \sum_{m=1}^{t} \mathcal{D}^{\bar{v}}(f_m) \mathcal{D}^{\bar{v}}(\bar{g}_m).$$

For (ii) we use similar arguments. □

Given $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we define the following set:

$$\bar{\Phi}_{rs} = \{\sigma \in \Phi_{rs} \mid \tau(z + \sigma(\varepsilon_{rs})) = z + \sigma(\varepsilon_{rs})\}.$$

Also for any $(\sigma'_1, \sigma'_2) \in \bar{\Phi}_{rs} \times \bar{\Phi}_{\ell m}$ define

$$\Phi_{(\sigma'_1, \sigma'_2)} = \{(\sigma_1, \sigma_2) \in \Phi_{rs} \times \Phi_{\ell m} \mid \sigma_1(\varepsilon_{rs}) + \sigma_2(\varepsilon_{\ell m}) = \sigma'_1(\varepsilon_{rs}) + \sigma'_2(\varepsilon_{\ell m})\}.$$

Remark A.3. If $(\sigma'_1, \sigma'_2) \in \bar{\Phi}_{\ell m} \times \bar{\Phi}_{\ell m}$ then $\Phi_{(\sigma'_1, \sigma'_2)} \subseteq \bar{\Phi}_{\ell m} \times \bar{\Phi}_{\ell m}$. In fact, the condition $(\sigma'_1, \sigma'_2) \in \bar{\Phi}_{\ell m} \times \bar{\Phi}_{\ell m}$ fixes the k-th component of the permutations in $\Phi_{(\sigma'_1, \sigma'_2)}$.

Lemma A.4. *Let v be generic, $\ell \neq m$, $r \neq s$ and let $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ be such that $\sigma'_1 \in \bar{\Phi}_{rs}$, $\sigma'_2 \in \bar{\Phi}_{\ell m}$. Then*

$$\sum \big(e_{\ell m}(\sigma_2(v+z)) e_{rs}(\sigma_1(v+z+\sigma_2(\varepsilon_{\ell m}))) - e_{rs}(\sigma_1(v+z)) e_{\ell m}(\sigma_2(v+z+\sigma_1(\varepsilon_{rs}))) \big)$$
$$= 0,$$

where the sum above is over $(\sigma_1, \sigma_2) \in \Phi_{(\sigma'_1, \sigma'_2)}$.

Proof. The identity in the lemma follows by comparing the coefficient of $T(v + z + \sigma'_1(\varepsilon_{rs}) + \sigma'_2(\varepsilon_{\ell m}))$ on both sides of

$$E_{rs}(E_{\ell m} T(v+z)) - E_{\ell m}(E_{rs} T(v+z)) = [E_{rs}, E_{\ell m}] T(v+z). \tag{8}$$

Note that if $w = \sigma'_1(\varepsilon_{rs}) + \sigma'_2(\varepsilon_{\ell m})$ then

$$(w_{ki}, w_{kj}) \in \{(1,-1), (-1,1), (2,0), (-2,0), (0,2), (0,-2)\}$$

and in each of these six cases we have that the coefficient of $T(v + z + \sigma'_1(\varepsilon_{rs}) + \sigma'_2(\varepsilon_{\ell m}))$ in the right hand side of (8) is zero. □

For $\min(\ell, m) \leq k \leq \max(\ell, m) - 1$ and $1 \leq t \leq k$ we set

$$\Phi_{\ell m}(k, t) = \{\sigma \in \Phi_{\ell m} \mid \sigma[k] = (1, t)\}.$$

In most of the considerations in this section we will need $\Phi_{\ell m}(k,t)$ for $t=i$ and $t=j$ only. We set for convenience $\Phi_{\ell m}(i)=\Phi_{\ell m}(k,i)$ and $\Phi_{\ell m}(j)=\Phi_{\ell m}(k,j)$. The following lemma will be useful to prove that $V(T(\bar{v}))$ is a $\mathfrak{gl}(n)$-module.

Lemma A.5. *Let $z\in\mathcal{H}$ (equivalently, $\tau(z)=z$), $\sigma\in\Phi_{rs}$ and $w=v+z$. The function $e_{rs}(\sigma(w))$ of v has a simple pole on $\overline{\mathcal{H}}$ if $\min(r,s)\leq k\leq\max(r,s)-1$ and $\sigma\in\Phi_{rs}(i)\cup\Phi_{rs}(j)$. In all other cases $e_{rs}(\sigma(w))$ is in $\mathcal{F}_{ij}$, i.e. it is smooth on $\overline{\mathcal{H}}$.*

Proof. The case $r=s$ is trivial, since $e_{rr}(\sigma(w))\in\mathcal{F}_{ij}$ for any σ. Suppose now that $r<s$. Then the denominator of $e_{rs}(\sigma(w))$ is

$$\prod_{t=r}^{s-1}\left(\prod_{j\neq 1}^{t}(w_{t,\sigma^{-1}[t](1)}-w_{t,\sigma^{-1}[t](j)})\right),$$

which implies the lemma. The case $r>s$ is analogous. $\square$

Definition A.6. In the case when $\sigma\in\Phi_{\ell m}(i)\cup\Phi_{\ell m}(j)$ we define:

$$\tau\star\sigma:=\begin{cases}\tau\sigma\tau=\sigma\tau\sigma, & \text{if} \quad 1\notin\{i,j\}\\ \tau\sigma=\sigma\tau, & \text{if} \quad 1\in\{i,j\}\end{cases}$$

One easily shows that $\tau\star\sigma$ is well defined and that $\tau\star\sigma\in\Phi_{\ell m}(i)\cup\Phi_{\ell m}(j)$.

Lemma A.7. *Let v be generic, $z\in\mathbb{Z}^{\frac{n(n-1)}{2}}$, and $\sigma\in\Phi_{\ell m}$, $\ell\neq m$.*

(i) *If $\sigma\notin\Phi_{\ell m}(i)\cup\Phi_{\ell m}(j)$ then we have:*
  (a) $\mathrm{ev}(\bar{v})e_{\ell m}(\sigma(v+\tau(z)))=\mathrm{ev}(\bar{v})e_{\ell m}(\sigma(v+z))$ *and* $\mathcal{D}^{\bar{v}}e_{\ell m}(\sigma(v+\tau(z)))=-\mathcal{D}^{\bar{v}}e_{\ell m}(\sigma(v+z))$. *In particular,* $\mathcal{D}^{\bar{v}}e_{\ell m}(\sigma(v+z))=0$ *if* $\tau(z)=z$.
  (b) $\mathrm{ev}(\bar{v})((x-y)e_{\ell m}(\sigma(v+\tau(z)))=\mathrm{ev}(\bar{v})((x-y)e_{\ell m}(\sigma(v+z)))=0$ *and* $\mathcal{D}^{\bar{v}}((x-y)e_{\ell m}(\sigma(v+\tau(z))))=\mathcal{D}^{\bar{v}}((x-y)e_{\ell m}(\sigma(v+z)))$.

(ii) *If $\sigma\in\Phi_{\ell m}(i)\cup\Phi_{\ell m}(j)$ then $e_{\ell m}(\tau\star\sigma(v+z))=e_{\ell m}(\sigma\tau(v+z))$. In particular.*
  (a) *If* $\tau(z)\neq z$, $\mathrm{ev}(\bar{v})e_{\ell m}(\tau\star\sigma(v+\tau(z)))=\mathrm{ev}(\bar{v})e_{\ell m}(\sigma(v+z))$ *and* $\mathcal{D}^{\bar{v}}e_{\ell m}(\tau\star\sigma(v+\tau(z)))=-\mathcal{D}^{\bar{v}}e_{\ell m}(\sigma(v+z))$.
  (b) $\mathrm{ev}(\bar{v})((x-y)e_{\ell m}(\tau\star\sigma(v+\tau(z))))=-\mathrm{ev}(\bar{v})((x-y)e_{\ell m}(\sigma(v+z)))$ *and* $\mathcal{D}^{\bar{v}}((x-y)e_{\ell m}(\tau\star\sigma(v+\tau(z))))=\mathcal{D}^{\bar{v}}((x-y)e_{\ell m}(\sigma(v+z)))$.

Proof. Note that for any w in $\mathbb{C}^{\frac{n(n-1)}{2}}$ and $\sigma\in\Phi_{rs}(i)\cup\Phi_{rs}(j)$ we have $e_{rs}(\tau\star\sigma(w))=e_{rs}(\sigma\tau(w))$. Using this and Definition 3.12 we establish the four statements by direct verification. $\square$

A.2. Proof of Proposition 4.7

For part (i) we use

$$\mathcal{D}^{\bar{v}}((x-y)E_{rs}T(v+z)) = \sum_{\sigma\in\Phi_{rs}} \mathcal{D}^{\bar{v}}((x-y)e(\sigma(v+z)))\mathrm{ev}(\bar{v})T(v+z+\sigma(\varepsilon_{rs}))$$
$$+ \sum_{\sigma\in\Phi_{rs}} \mathrm{ev}(\bar{v})((x-y)e(\sigma(v+z)))\mathcal{D}T(\bar{v}+z+\sigma(\varepsilon_{rs})).$$

The same formula holds for $\mathcal{D}^{\bar{v}}((x-y)E_{rs}T(v+\tau(z)))$ after replacing z with $\tau(z)$ on the right hand side. If $\sigma \notin \Phi_{rs}(i)\cup\Phi_{rs}(j)$ then, $\tau(z+\sigma(\varepsilon_{rs})) = \tau(z)+\sigma(\varepsilon_{rs})$ which implies that $\mathcal{D}T(\bar{v}+z+\sigma(\varepsilon_{rs})) = -\mathcal{D}T(\bar{v}+\tau(z)+\sigma(\varepsilon_{rs}))$ and $\mathrm{ev}(\bar{v})T(v+z+\sigma(\varepsilon_{rs})) = \mathrm{ev}(\bar{v})T(v+\tau(z)+\sigma(\varepsilon_{rs}))$. Thanks to Lemma A.7(i)(b) the corresponding coefficients in the identity of part (i) are the same. In the case $\sigma\in\Phi_{rs}(i)\cup\Phi_{rs}(j)$ we have $\tau(z+\tau\star\sigma(\varepsilon_{rs})) = \tau(z)+\sigma(\varepsilon_{rs})$, so $\mathcal{D}T(\bar{v}+\tau(z)+\sigma(\varepsilon_{rs})) = -\mathcal{D}T(\bar{v}+z+\tau\star\sigma(\varepsilon_{rs}))$ and $\mathrm{ev}(\bar{v})T(v+\tau(z)+\sigma(\varepsilon_{rs})) = \mathrm{ev}(\bar{v})T(v+z+\tau\star\sigma(\varepsilon_{rs}))$ and, now by Lemma A.7(ii)(b) the coefficients are the same.

The proof of part (ii) is similar.

A.3. Proof of Proposition 4.10

First, we assume that $g_1 = E_{rs}$ and $g_2 = E_{\ell m}$. The case $r = s$ or $\ell = m$ follows by straightforward computations. For example, if $r = s$, due to the hypothesis $\tau(z)\neq z$, the functions $e_{rr}(v+z)$, $e_{rr}(v+z+\sigma(\varepsilon_{\ell m}))$ and $e_{\ell m}(\sigma(v+z))$ are in $\mathcal{F}_{ij}$. Hence we can apply $\mathcal{D}^{\bar{v}}$ and $\mathrm{ev}(\bar{v})$ to these functions in order to show that the corresponding coefficients in (5) coincide. Note that we need to use Lemma A.1(iii).

Assume now that $r\neq s$ and $\ell\neq m$. Since $\tau(z)\neq z$, we have $\overline{\Phi}_{rs}\subset\Phi_{rs}(i)\cup\Phi_{rs}(j)$ and $\overline{\Phi}_{\ell m}\subset\Phi_{\ell m}(i)\cup\Phi_{\ell m}(j)$. For convenience we will use the following convention for $\sigma_1\in\Phi_{rs}$ and $\sigma_2\in\Phi_{\ell m}$:

$$e_{rs}(\sigma_1) = e_{rs}(\sigma_1(v+z)), \qquad e_{rs}(\sigma_1,\sigma_2) = e_{rs}(\sigma_1(v+z+\sigma_2(\varepsilon_{\ell m}))),$$
$$e_{\ell m}(\sigma_2) = e_{\ell m}(\sigma_2(v+z)), \qquad e_{\ell m}(\sigma_1,\sigma_2) = e_{\ell m}(\sigma_2(v+z+\sigma_1(\varepsilon_{rs}))).$$

We also set $\bar{e}_{\ell m} = (x-y)e_{\ell m}, \bar{e}_{rs} = (x-y)e_{rs}$ and similarly introduce $\bar{e}_{rs}(\sigma_1,\sigma_2)$ and $\bar{e}_{\ell m}(\sigma_1,\sigma_2)$. Furthermore, we set

$$T(\sigma_1,\sigma_2) = T(v+z+\sigma_1(\varepsilon_{rs})+\sigma_2(\varepsilon_{\ell m})),$$

and

$$\mathcal{D}T(\sigma_1,\sigma_2) = \mathcal{D}T(\bar{v}+z+\sigma_1(\varepsilon_{rs})+\sigma_2(\varepsilon_{\ell m})).$$

Finally, for $\sigma_1\in\Phi_{rs}$ and $\sigma_2\in\Phi_{\ell m}$ we set:

$$L_1(\sigma_1,\sigma_2) = \begin{cases} \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2)e_{rs}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2)), \text{ if } \sigma_2\notin\overline{\Phi}_{\ell m} \\ \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2))\mathcal{D}^{\bar{v}}(\bar{e}_{rs}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2)), \text{ if } \sigma_2\in\overline{\Phi}_{\ell m}, \end{cases}$$

$$L_2(\sigma_1,\sigma_2) = \begin{cases} \mathcal{D}^{\bar{v}}(e_{rs}(\sigma_1)e_{\ell m}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2)), \text{ if } \sigma_1 \notin \overline{\Phi}_{rs} \\ \mathcal{D}^{\bar{v}}(e_{rs}(\sigma_1))\mathcal{D}^{\bar{v}}(\bar{e}_{\ell m}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2)), \text{ if } \sigma_1 \in \overline{\Phi}_{rs}, \end{cases}$$

$$R_1(\sigma_1,\sigma_2) = e_{\ell m}(\sigma_2)e_{rs}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2),$$

$$R_2(\sigma_1,\sigma_2) = e_{rs}(\sigma_1)e_{\ell m}(\sigma_1,\sigma_2)T(\sigma_1,\sigma_2).$$

Applying Lemma 4.8 we obtain

$$E_{rs}(E_{\ell m}(\mathcal{D}T(\bar{v}+z))) - E_{\ell m}(E_{rs}(\mathcal{D}T(\bar{v}+z))) = \sum_{\sigma_1,\sigma_2}(L_1(\sigma_1,\sigma_2) - L_2(\sigma_1,\sigma_2)) \quad (9)$$

and

$$\begin{aligned} [E_{rs},E_{\ell m}](\mathcal{D}T(\bar{v}+z)) &= \mathcal{D}^{\bar{v}}([E_{rs},E_{\ell m}]T(v+z)) \\ &= \mathcal{D}^{\bar{v}}\left(\sum_{\sigma_1,\sigma_2}(R_1(\sigma_1,\sigma_2) - R_2(\sigma_1,\sigma_2))\right). \end{aligned} \quad (10)$$

The sums in (9) and (10) run over $(\sigma_1,\sigma_2) \in \Phi_{rs} \times \Phi_{\ell m}$. Our goal is to show that the right hand sides of (9) and (10) coincide. By the definitions of L_i and R_i we have that $L_1(\sigma_1,\sigma_2) = \mathcal{D}^{\bar{v}}(R_1(\sigma_1,\sigma_2))$ if $\sigma_2 \notin \overline{\Phi}_{\ell m}$ and $L_2(\sigma_1,\sigma_2) = \mathcal{D}^{\bar{v}}(R_2(\sigma_1,\sigma_2))$ if $\sigma_1 \notin \overline{\Phi}_{rs}$. Next we show that $L_1(\sigma_1,\sigma_2') = \mathcal{D}^{\bar{v}}(R_1(\sigma_1,\sigma_2'))$ if $\sigma_1 \notin \Phi_{rs}(i) \cup \Phi_{rs}(j)$ and $\sigma_2' \in \overline{\Phi}_{\ell m}$. In this case we have $\tau(\bar{v}+z+\sigma_1(\varepsilon_{rs})+\sigma_2'(\varepsilon_{\ell m})) = \bar{v}+z+\sigma_1(\varepsilon_{rs})+\sigma_2'(\varepsilon_{\ell m})$ and, hence, $\mathcal{D}T(\sigma_1,\sigma_2') = 0$. Therefore,

$$\begin{aligned} L_1(\sigma_1,\sigma_2') &= \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2'))\mathcal{D}^{\bar{v}}(\bar{e}_{rs}(\sigma_1,\sigma_2')T(\sigma_1,\sigma_2')) \\ &= \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2'))\mathcal{D}^{\bar{v}}(\bar{e}_{rs}(\sigma_1,\sigma_2'))\mathrm{ev}(\bar{v})T(\sigma_1,\sigma_2') \\ &= \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2'))\mathrm{ev}(\bar{v})(e_{rs}(\sigma_1,\sigma_2'))\mathrm{ev}(\bar{v})T(\sigma_1,\sigma_2') \\ &= \mathcal{D}^{\bar{v}}(e_{\ell m}(\sigma_2')e_{rs}(\sigma_1,\sigma_2'))\mathrm{ev}(\bar{v})T(\sigma_1,\sigma_2') \\ &= \mathcal{D}^{\bar{v}}(R_1(\sigma_1,\sigma_2')). \end{aligned}$$

The first equality follows from the definition of L_1, while the second follows from $\mathcal{D}T(\sigma_1,\sigma_2') = 0$. The third equality follows from the fact that $\bar{e}_{rs}(\sigma_1,\sigma_2')$ and $e_{rs}(\sigma_1,\sigma_2)$ are in $\mathcal{F}_{ij}$ by Lemma A.5. The forth one is a consequence of $\mathcal{D}^{\bar{v}}e_{rs}(\sigma_1,\sigma_2') = 0$ (by Lemma A.7(i)(a)), and finally for the last equality we use $\mathcal{D}T(\sigma_1,\sigma_2') = 0$. Similarly one can show that $L_1(\sigma_1',\sigma_2) = \mathcal{D}^{\bar{v}}(R_1(\sigma_1',\sigma_2))$ if $\sigma_1' \in \overline{\Phi}_{rs}$ and $\sigma_2 \notin \Phi_{\ell m}(i) \cup \Phi_{\ell m}(j)$. We next note that if $\sigma_2' \in \overline{\Phi}_{\ell m}$ and $\sigma_1 \in \Phi_{rs}(i) \cup \Phi_{rs}(j)$ then either $\sigma_1 \in \overline{\Phi}_{rs}$ or $\tau \star \sigma_1 \in \overline{\Phi}_{rs}$; and if $\sigma_1' \in \overline{\Phi}_{rs}$ and $\sigma_2 \in \Phi_{\ell m}(i) \cup \Phi_{\ell m}(j)$ then either $\sigma_2 \in \overline{\Phi}_{\ell m}$ or $\tau \star \sigma_2 \in \overline{\Phi}_{\ell m}$. To complete the proof it is sufficient to prove that for $\sigma_1' \in \overline{\Phi}_{rs}$ and $\sigma_2' \in \overline{\Phi}_{\ell m}$ we have $L = \mathcal{D}^{\bar{v}}(R)$, where

$$L = \sum_{(\sigma_1,\sigma_2)\in\Phi_{(\sigma_1',\sigma_2')}}(L_1(\sigma_1,\sigma_2) + L_1(\tau\star\sigma_1,\sigma_2) - L_2(\sigma_1,\sigma_2) - L_2(\sigma_1,\tau\star\sigma_2)),$$

$$R = \sum_{(\sigma_1,\sigma_2)\in\Phi_{(\sigma_1',\sigma_2')}} \left(R_1(\sigma_1,\sigma_2) + R_1(\tau\star\sigma_1,\sigma_2) - R_2(\sigma_1,\sigma_2) - R_2(\sigma_1,\tau\star\sigma_2)\right).$$

We then note that $\sum_{\Phi_{(\sigma_1',\sigma_2')}} (R_1(\sigma_1,\sigma_2) - R_2(\sigma_1,\sigma_2)) = 0$ thanks to Lemma A.4.

We can simplify the expansion of L using various identities. We use first that $z + \tau\star\sigma_1(\varepsilon_{rs}) + \sigma_2(\varepsilon_{\ell m}) = z + \sigma_1(\varepsilon_{rs}) + \tau\star\sigma_2(\varepsilon_{\ell m})$, which implies $T(\tau\star\sigma_1,\sigma_2) = T(\sigma_1,\tau\star\sigma_2)$ and, in particular, $\mathrm{ev}(\bar v)T(\tau\star\sigma_1,\sigma_2) = \mathrm{ev}(\bar v)T(\sigma_1,\tau\star\sigma_2)$ and $\mathcal{D}T(\tau\star\sigma_1,\sigma_2) = \mathcal{D}T(\sigma_1,\tau\star\sigma_2)$. We also use that $\mathcal{D}T(\sigma_1,\sigma_2) = -\mathcal{D}T(\tau\star\sigma_1,\sigma_2)$ and $\mathrm{ev}(\bar v)T(\sigma_1,\sigma_2) = \mathrm{ev}(\bar v)T(\tau\star\sigma_1,\sigma_2)$. On the other hand, by Lemma A.7(ii)(b) we have $\mathcal{D}^{\bar v}(\bar e_{rs}(\sigma_1,\sigma_2)) = \mathcal{D}^{\bar v}(\bar e_{rs}(\tau\star\sigma_1,\sigma_2))$, $\mathcal{D}^{\bar v}(\bar e_{\ell m}(\sigma_1,\sigma_2)) = \mathcal{D}^{\bar v}(\bar e_{\ell m}(\sigma_1,\tau\star\sigma_2))$, $\mathrm{ev}(\bar v)(\bar e_{rs}(\sigma_1,\sigma_2)) = -\mathrm{ev}(\bar v)(\bar e_{rs}(\tau\star\sigma_1,\sigma_2))$, and $\mathrm{ev}(\bar v)(\bar e_{\ell m}(\sigma_1,\sigma_2)) = -\mathrm{ev}(\bar v)(\bar e_{\ell m}(\sigma_1,\tau\star\sigma_2))$. All these identities reduce L to twice the sum of

$$\begin{aligned}&\left(\mathcal{D}^{\bar v}(e_{\ell m}(\sigma_2))\mathcal{D}^{\bar v}(\bar e_{rs}(\tau\star\sigma_1,\sigma_2)) - \mathcal{D}^{\bar v}(e_{rs}(\sigma_1))\mathcal{D}^{\bar v}(\bar e_{\ell m}(\sigma_1,\tau\star\sigma_2))\right)\mathrm{ev}(\bar v)T(\sigma_1,\sigma_2)\\ &\quad + \left(\mathcal{D}^{\bar v}(e_{\ell m}(\sigma_2))\mathrm{ev}(\bar v)\bar e_{rs}(\tau\star\sigma_1,\sigma_2) - \mathcal{D}^{\bar v}(e_{rs}(\sigma_1))\mathrm{ev}(\bar v)\bar e_{\ell m}(\sigma_1,\tau\star\sigma_2)\right)\mathcal{D}T(\sigma_1,\sigma_2)\end{aligned}$$

over all $(\sigma_1,\sigma_2)\in\Phi_{(\sigma_1',\sigma_2')}$.

Let $\{(\sigma_1^{(p)},\sigma_2^{(p)})\}_{p=1}^t$ be the set of all distinct pairs of permutations in $\Phi_{(\sigma_1,\sigma_2)}$. For each $p = 1,\dots,t$ define the following functions:

$$\begin{aligned} f_{2p} &= e_{\ell m}(\sigma_2^{(p)}), & f_{2p-1} &= e_{rs}(\sigma_1^{(p)}),\\ g_{2p-1} &= e_{rs}(\tau\star\sigma_1^{(p)},\sigma_2^{(p)}), & g_{2p} &= -e_{\ell m}(\sigma_1^{(p)},\tau\star\sigma_2^{(p)}).\end{aligned}$$

Note that $e_{rs}(\tau\star\sigma_1^{(p)},\sigma_2^{(p)}) = e_{rs}(\sigma_1^{(p)},\sigma_2^{(p)})^\tau$ and $e_{\ell m}(\sigma_1^{(p)},\tau\star\sigma_2^{(p)}) = e_{\ell m}(\sigma_1^{(p)},\sigma_2^{(p)})^\tau$ thanks to Lemma A.7(ii). We finally apply Lemma A.2 to the set of functions f_p, g_p, $p = 1,\dots,2t$, and obtain $L = \mathcal{D}^{\bar v}(R)$. Note that the hypothesis $\sum_{p=1}^{2t} f_p g_p^\tau = 0$ of Lemma A.2 holds by Lemma A.4.

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