

## Graded Algebras of Global Dimension 3

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### 0. INTRODUCTION

We work with *graded algebras* and *graded modules* throughout this paper [12]. Let  $A = k \oplus A_1 \oplus A_2 \oplus \cdots$  be a finitely presented graded algebra over a field  $k$ . The algebra  $A$  will be called *regular* if it has the following properties:

(i)  $A$  has *finite global dimension*  $d$ : every graded  $A$ -module has projective dimension  $\leq d$ .

(ii)  $A$  has *finite  $gk$ -dimension*, i.e.,  $A$  has polynomial growth.

(iii)  $A$  is *Gorenstein*, meaning that  $\text{Ext}_A^q(k, A) = 0$  if  $q \neq d$ , and  $\text{Ext}_A^d(k, A) \cong k$ . (0.1)

This paper is a study of the regular algebras of global dimension three which are generated by elements of degree one.

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Our computations were done with the aid of a Symbolics Lisp machine using LISP, MACSYMA, and AFFINE. We would like to thank Woody Bledsoe, Robert Boyer, and Don Good at the University of Texas and Richard Zippel and Boris Katz at MIT for making Lisp machines available for our use. The version of MACSYMA used was our adaptation to COMMON LISP, and it is available through the Department of Energy. AFFINE is a collection of routines we implemented in COMMON LISP.

We assume throughout that the ground field  $k$  is algebraically closed, of characteristic zero, and we use the symbol  $GL_r$  to denote the general linear group. We also assume that the algebra  $A$  is generated by elements of degree 1 except in Sections 5 and 6, where arbitrary positive degrees are allowed. Most of the time,  $A$  will have global dimension 3. Note that in order for  $A$  to have finite global dimension, it is necessary and sufficient that the projective dimension of  $k$  as left or as right module be finite. (See the appendix to [10] for graded modules which are bounded below. For arbitrary graded modules, use the graded version of [2], Theorem 1.)

Regular algebras seem to have very nice properties, and are worth studying further. It turns out that conditions (ii) and (iii) hold in the most important examples of finite global dimension algebras, such as when  $A$  is a finite module over its center (1.16).

It is not difficult to describe the regular algebras of global dimension 2. They have the form

$$A = k[x, y]/(f), \quad (0.2)$$

where  $k[x, y]$  denotes the free ring and  $f = f(x, y)$  is one of the polynomials

$$yx - cxy \quad (c \neq 0), \quad \text{or} \quad yx - xy - x^2$$

(see [9]). If we drop the Gorenstein hypothesis, the only additional algebra is the one defined by the equation  $yx = 0$ , the case  $c = 0$  in the above equation. The algebra  $A = k[x, y]/(yx)$  has global dimension 2, but  $\text{Ext}^q(k, A) \neq 0$  for  $q = 1, 2$ . This algebra is also the only graded algebra of global dimension 2 and  $gk$ -dimension 2 which is not noetherian. (If  $c \neq 0$ , then (0.2) is an Ore extension determined by an automorphism [7], p. 439.) We conjecture that all regular algebras are noetherian.

When the global dimension is three, there are two cases to consider: Either  $A$  is generated by two elements satisfying two cubic relations, or else by three elements with three quadratic relations (1.5).

An example of the first type is the enveloping algebra of the Heisenberg algebra. It is generated by elements  $x, y, z$  satisfying the relations

$$[x, y] = z, [x, z] = [y, z] = 0. \quad (0.3)$$

This algebra is graded by assigning degree 1 to  $x, y$  and degree 2 to  $z$ . The first relation eliminates  $z$ , and the others define two cubic equations

$$\begin{aligned}yx^2 - 2xyx + x^2y &= 0, \\ y^2x - 2yxy + xy^2 &= 0\end{aligned}\tag{0.4}$$

relating  $x$  and  $y$ .

An example with three generators is the algebra defined by the equations

$$\begin{aligned}[x, y] - z^2 &= 0, \\ [y, z] - x^2 &= 0, \\ [z, x] - y^2 &= 0.\end{aligned}\tag{0.5}$$

This algebra is a finite module, of rank  $6^2$ , over its center.

After some preliminary work, the regular algebras are classified in Section 3 into 13 *types*, each type corresponding to an irreducible family of algebras. A generalized notion of *skew polynomial ring* is introduced in Section 6. We show that the property of being a skew polynomial ring is determined by a *j-invariant* (see Section 3) associated with a regular algebra  $A$ .

Two of the families, which we call *type A*, are especially interesting. Example (0.5) is of type *A*. We give a conjectural description of these families in Section 10, including a determination of the ones which are finite over their centers. However we can prove little about them at this time. The other types are easier to deal with, and conditions for finiteness over the center are described for some of them in Section 6 and 7.

## 1. DESCRIPTION OF THE RESOLUTION OF $k$

We write the algebra  $A$  as a quotient of a non-commutative polynomial ring,

$$A \approx k[x_1, \dots, x_{r_1}]/(f_1, \dots, f_{r_2}),\tag{1.1}$$

where  $r_1$  is the minimal number of homogeneous generators  $x_i$  for  $A$  as  $k$ -algebra, and  $r_2$  is the minimal number of homogeneous defining relations  $f_i$  for  $A$ , i.e., the minimal number of generators of the 2-sided ideal  $I = \ker(k[x] \rightarrow A)$ . Since  $A$  is assumed to be generated in degree 1, each  $x_i$  has degree 1. We may write the relations in the form

$$f_i = \sum m_{ij}x_j,\tag{1.2}$$

to obtain the matrix equation

$$f = Mx, \quad (1.3)$$

where

$$f = (f_1, \dots, f_{r_2})', \quad M = (m_{ij}), \quad x = (x_1, \dots, x_{r_1})'$$

have entries in  $k[x]$ . Then the sequence of left  $A$ -modules

$$A^{r_2} \xrightarrow{M} A^{r_1} \xrightarrow{x} A \longrightarrow {}_A k \rightarrow 0 \quad (1.4)$$

is exact.

**THEOREM (1.5).** *Let  $A$  be a regular algebra of global dimension 3, generated in degree 1. Then*

(i) *The relations  $f_i$  have the same degree, say  $s$ , and  $r_1 = r_2 (=r)$ . There are two possibilities for the pair  $(r, s)$ , namely  $(2, 3)$  and  $(3, 2)$ .*

(ii) *With a suitable choice of the relations  $f_i$ , the sequence (1.4) extends to a resolution*

$$0 \rightarrow A \xrightarrow{x'} A^r \xrightarrow{M} A^r \xrightarrow{x} A \longrightarrow {}_A k \rightarrow 0,$$

and the entries of  $x'M$  generate the ideal  $I$ . Hence  $x'M = (Qf)'$  for some  $Q \in Gl_r$ .

*Proof.* Choose a minimal resolution of the left  $A$ -module  $k$ , say

$$0 \rightarrow A^{r_3} \rightarrow A^{r_2} \rightarrow A^{r_1} \rightarrow A \rightarrow k \rightarrow 0, \quad (1.6)$$

where  $r_1, r_2$  are as above, while  $r_3$  may, a priori, be infinite. The ranks  $r_i$  are the Tor-dimensions

$$r_i = \dim_k \operatorname{Tor}_i^A(k, k). \quad (1.7)$$

It follows from the Gorenstein condition that  $\operatorname{Hom}(\cdot, A)$  applied to the sequence (1.6) yields a resolution of the right module  $k_A = \operatorname{Ext}_A^3(k, A)$ ,

$$0 \leftarrow k \leftarrow A^{r_3} \leftarrow \cdots \leftarrow A \leftarrow 0, \quad (1.8)$$

and minimality of (1.6) implies that (1.8) is also minimal. Since the Tor-dimensions are symmetric with respect to left and right, we find  $r_3 = 1$  and  $r_1 = r_2$ . Therefore (1.4) extends to a graded resolution

$$0 \rightarrow A \xrightarrow{x'} A^r \xrightarrow{M} A^r \xrightarrow{x} A \longrightarrow {}_A k \rightarrow 0, \quad (1.9)$$

for some  $y = (y_1, \dots, y_r)'$ . The exactness of (1.8) shows that  $\{y_1, \dots, y_r\}$  is a system of generators for  $\text{rad } A$ , hence that  $y = Px$  for some  $P \in Gl_r$ . In particular,  $\deg y_i = 1$  for each  $i$ . A change of basis in  $A^r$  replaces  $y'$  by  $x'$ , which completes the proof of (ii). Since  $\deg m_{ij} = \deg f_i - 1$ , the fact that the resolution is graded implies that  $f_i$  all have the same degree.

It remains to determine the possible pairs  $(r, s)$ , which is done by estimating the Hilbert function

$$a_n = \dim_k A_n \quad (1.10)$$

of  $A$ . The resolution (1.5)(ii) yields the recursion relation

$$a_n - ra_{n-1} + ra_{n-s} - a_{n-s-1} = 0 \quad (n \geq 1), \quad (1.11)$$

which can be used to compute  $a_n$ , starting with the initial conditions  $a_0 = 1$  and  $a_n = 0$  if  $n < 0$ .

Since  $A$  has finite  $gk$ -dimension, the growth of  $a_n$  is bounded by a polynomial. Also, we have the trivial inequalities

$$r \geq 2, \quad s \geq 2. \quad (1.12)$$

LEMMA (1.13). *The Hilbert function  $a_n$  determined by (1.11) has polynomial growth if and only if  $r + s \leq 5$ .*

This lemma is proved by showing that the characteristic polynomial

$$p(t) = t^{s+1} - rt^s + rt - 1 \quad (1.14)$$

of (1.11) has a real root  $t_0 > 1$  except in the cases listed. For example, if  $(r, s) = (3, 3)$  then  $p(1) = 0$ ,  $p'(1) < 0$ , and  $p(t) > 0$  if  $t \gg 0$ .

The case  $(r, s) = (2, 2)$  is impossible, because (1.11) then gives  $a_4 = 0$  and  $a_6 = 1$ . But since  $A$  is generated in degree 1,  $a_4 = 0$  implies  $a_n = 0$  for all  $n \geq 4$ , which is a contradiction. Taking into account (1.12), the remaining cases are the two listed in (1.5)(i). Examples are described in the Introduction.

The Hilbert functions in the two cases are easily determined. They are

$$\begin{aligned} r=3: \quad a_n &= \frac{1}{2}(n^2 + 3n + 2), \\ r=2: \quad a_n &= \begin{cases} \frac{1}{4}(n^2 + 4n + 4) & \text{if } n \text{ even,} \\ \frac{1}{4}(n^2 + 4n + 3) & \text{if } n \text{ odd.} \end{cases} \end{aligned} \quad (1.15)$$

One of our main interests is in studying algebras which are finite modules over their centers. For these algebras, we have

PROPOSITION (1.16). *Assume that  $A$  is a finite module over its center, and has finite global dimension  $d$ . Then*

- (i)  $A$  is regular.
- (ii) Let  $r_i = \dim \operatorname{Tor}_i(k, k)$ . Then  $r_i = r_{d-i}$ .

*Proof.* Since  $A$  is a finite module over its center  $Z$ , it is finite over a polynomial subring  $R$  of  $Z$ . Therefore it certainly has finite  $gk$ -dimension. The Gorenstein property follows from the same property [8] of the subring  $R$ . It is known that  $A$  is a Cohen–Macaulay, hence free,  $R$ -module [14, 3]. Moreover, every Cohen–Macaulay  $A$ -module is free [13]. Let  $\omega$  denote the relative dualizing module of  $A/R$ :

$$\omega = \operatorname{Hom}_R(A, R). \quad (1.17)$$

This two-sided  $A$ -module is clearly Cohen–Macaulay, hence free of rank 1 as left and as right module. Denote the restriction of an  $A$ -module to  $R$  by  $|_R$ . There are functorial  $R$ -isomorphisms

$$\operatorname{Ext}_A^q(M, \omega)|_R \cong \operatorname{Ext}_R^q(M|_R, R). \quad (1.18)$$

Setting  $M = k$  and using the Gorenstein property of  $R$ , we find that  $\operatorname{Ext}_A^q(k, M) \cong k$  if  $q = d$ , and  $= 0$  if  $q \neq d$ . Since  $\omega$  is free of rank 1 as left  $A$ -module, it follows that  $A$  is Gorenstein. Once this property has been proved, (ii) follows from properties of a minimal resolution of  $k$ , as in (1.6)–(1.8).

It is natural to ask for the possible Hilbert functions of regular algebras. A more detailed analysis than the one used above will be needed to determine them, but for an algebra of global dimension 4 which is finite over its center, the above method can be used. Gorenstein symmetry shows that the Tor-dimensions  $r_i$  are of the form

$$(r_0, \dots, r_4) = (1, r, 2r - 2, r, 1), \quad (1.19)$$

and polynomial growth determines the degrees of the defining relations. We state the result without proof:

PROPOSITION (1.20). *Let  $A$  be a graded algebra (1.1) of global dimension 4, generated in degree 1, which is finite over its center. Then one of the following holds:*

- (i)  $r_1 = 4$ ,  $r_2 = 6$ , and the defining equations have degree 2.
- (ii)  $r_1 = 3$ ,  $r_2 = 4$ , and there are two defining equations of degree 2 and two of degree 3.
- (iii)  $r_1 = r_2 = 2$ , and the defining equations have degrees 3 and 4.

EXAMPLES (1.21). The following rings illustrate the three cases:

(i) The commutative polynomial ring  $k[x] = k[x_1, x_2, x_3, x_4]$  in variables of degree 1.

(ii) The skew polynomial ring  $k[x_1, x_2, x_3, x_4]$ , where  $\deg(x_1, x_2, x_3, x_4) = (2, 1, 1, 1)$ , defined by

$$[x_2, x_1] = [x_3, x_1] = [x_4, x_1] = [x_4, x_2] = [x_4, x_3] = 0 \quad [x_3, x_2] = x_1.$$

(iii) The skew polynomial ring  $k[x]$ , where  $\deg(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$ , defined by

$$\begin{aligned} [x_2, x_3] &= [x_1, x_2] = [x_1, x_3] = [x_1, x_4] = 0 \\ [x_3, x_4] &= x_2, \quad [x_2, x_4] = x_1. \end{aligned}$$

## 2. THE ASSOCIATED REPRESENTATION OF $Gl_r$ .

For the rest of this paper, the algebra  $A$  will be assumed to have global dimension 3, unless the contrary is explicitly stated.

Let  $A$  be a regular algebra (1.5). So, there is a set of homogeneous generators  $\{x_i\}$  and defining equations  $\{f_i\}$  for  $A$  with  $Mx = f$ , and

$$x'M = (Qf)' \quad (2.1)$$

for some  $Q \in Gl_r(k)$ , where  $M$  is the matrix (1.2). Conversely, any  $r \times r$  matrix  $M$  in  $k[x]$  satisfying (2.1), whose entries are homogeneous of degree  $s$ , defines an algebra  $A$  and a complex (1.5)(ii) which is exact at the first three terms from the right. If (2.1) holds but (1.5)(ii) is not exact, we call  $A$  *degenerate*. Clearly, an algebra  $A$  which has a presentation satisfying (2.1) is regular if and only if the Hilbert function has the correct form (1.20) and the map  $x'$  is injective.

Let  $V = V_r$  denote the space of  $r$ -dimensional column vectors. The defining relations  $f_i$  of  $A$  can be viewed as elements of  $\otimes^s V^*$ , and the entries  $m_{ij}$  of the matrix  $M$  as elements of  $\otimes^{s-1} V^*$ , by viewing the polynomial ring  $k[x]$  as the tensor algebra on the dual space  $V^*$ . Note that there are natural bijective correspondences between the three vector spaces

- (i)  $r \times r$  matrices  $M$  with entries in  $\otimes^{s-1} V^*$ ,
  - (ii)  $r \times 1$  matrices  $f$  with entries in  $\otimes^s V^*$ , and
  - (iii) elements  $w$  of  $W = \otimes^{s+1} V^*$ ,
- (2.2)

given by

$$f = Mx, \quad w = x'Mx = x'f. \quad (2.3)$$

The matrix  $M$  can be recovered from  $w$  by factoring  $x_i$  out on the left and right. We will pass informally between these spaces.

The tensor  $w = w(A)$  has an intrinsic description in terms of the algebra  $A$ :

**PROPOSITION (2.4).** *Let  $A = k[x]/I$  be a regular algebra of global dimension 3. Let  $I_s$  denote the part of  $I$  of degree  $s$ , where  $s$  is as in (1.5). Then  $(V^*I_s \cap I_s V^*)$  is a 1-dimensional vector space, generated by  $w$  (2.3). Thus  $A$  determines  $w(A)$  up to a scalar factor, and conversely,  $w$  determines  $A$ .*

*Proof.* By Theorem (1.5),  $\dim I_s = r$ , and therefore  $V^*I_s$  and  $I_s V^*$  have dimension  $r^2$ . Clearly,  $I_{s+1} = V^*I_s + I_s V^*$ . The Hilbert function (1.20) for  $A$  predicts that  $\dim I_{s+1} = r^{s+1} - a_{s+1} = 2r^2 - 1$  in both cases. It follows that  $V^*I_s \cap I_s V^*$  has dimension 1. Formulas (1.3), (2.1) show that  $w$  is in this space.

Note that for an arbitrary basis  $f$  of  $I_s$ ,  $w$  will have the form

$$w = Lf = LMx \quad (Mx = f), \quad (2.5)$$

where  $L = (L_1, \dots, L_r)$  is a basis for  $V^*$ , and  $LM$  is a basis for  $I_s$ .

The natural operation of  $Gl_r$  on the left on  $V$  induces a right action on  $\otimes^l V^*$ . In order to distinguish this action from matrix multiplication, we denote it by

$$u, P \mapsto u \circ P, \quad u \in \otimes^l V^*, \quad P \in Gl_r. \quad (2.6)$$

Thus the symbol  $M \circ P$  denotes the matrix whose  $(i, j)$  entry is  $m_{ij} \circ P$ , while  $MP$  denotes the matrix product.

Since  $\{x_1, \dots, x_r\}$  is the standard basis of  $V^*$ , we have

$$Px = x \circ P, \quad (2.7)$$

for all  $P \in Gl_r$ . Since the entries of  $P$  are central, we also have the rule

$$(PN)' = N'P' \quad (2.8)$$

for all matrices  $N$  with entries in  $\otimes^l V^*$  and all  $P \in Gl_r$ .

We denote by  $L_Q$  the operation of  $Q'$  on the first factor of the tensor product  $W = \otimes^{s+1} V^*$ , i.e., in the notation of (2.3),

$$L_Q w = (x' \circ Q') Mx = x' Q Mx. \quad (2.9)$$

The equality of these expressions follows from (2.7) and (2.8):

$$x' \circ Q' = (x \circ Q')' = (Q'x)' = x'Q. \quad (2.10)$$



PROPOSITION (2.11). *Let  $\phi$  denote the automorphism of  $W$  induced by the permutation of indices  $\sigma: (v_1, \dots, v_{s+1}) \rightarrow (v_{s+1}, v_1, \dots, v_s)$ . Then Eq. (2.1) is equivalent to*

$$\phi w = L_Q w. \quad (2.12)$$

*Proof.* Transposing (2.1) and using (2.3) gives  $(x'M)' = QMx$ . This holds if and only if

$$x'(x'M)' = x'QMx = L_Q w.$$

Thus what has to be verified is that

$$\phi w = x'(x'M)'. \quad (2.13)$$

We write

$$w = \sum \omega_v x_v, \quad (2.14)$$

where  $v = (v_1 \cdots v_{s+1})$ ,  $x_v = x_{v_1} \cdots x_{v_{s+1}}$ , and  $\omega_v \in k$ . It is convenient to rewrite the multi-index  $v$  in the form  $v = (i, \mu, j)$ , where  $\mu = (\mu_1, \dots, \mu_{s-1})$ . Then the entries of  $M$  are

$$m_{ij} = \sum_{\mu} \omega_{i\mu j} x_{\mu}, \quad (2.15)$$

and

$$x'(x'M)' = \sum_{i,j} x_j x_i m_{ij} = \sum_v \omega_v x_j x_i x_{\mu} = \phi w,$$

as required.

In general, the isomorphism class of an algebra  $A$  defined by  $r$  equations of degree  $s$  is classified by an orbit for the action of  $Gl_r \times Gl_r$  on  $W$ , where the first factor represents a change of basis  $\{f_1, \dots, f_r\}$  for the ideal, and the second factor represents a change of variable. The action of a pair of matrices on the ideal basis is given by

$$f' = P'_1(f \circ P_2). \quad (2.16)$$

However, the fact that a regular algebra of global dimension 3 determines a 1-dimensional subspace of  $W$  by (2.4) reduces this group to the diagonal group. The action of a matrix  $P \in Gl_r$  results in

$$w' = w \circ P, \quad f' = P'(f \circ P), \quad M' = P'(M \circ P)P, \quad Q' = P'Q(P')^{-1}. \quad (2.17)$$

Using this action,  $Q$  may be put into Jordan form.

Another way to state the reduction is this:

LEMMA (2.18). *Let  $(P_1, P_2) \in GL_r \times GL_r$ . Assume that both  $f$  and  $f' = P_1'(f \circ P_2)$  satisfy identities of the form (2.1). Then either  $P_1 P_2^{-1}$  is a scalar matrix, or the algebra  $k[x]/(f)$  is degenerate.*

*Proof.* Since the diagonal group does act by (2.17), we may assume that  $P_2 = 1$ . Then  $f' = P_1'f$  and  $M' = P_1'M$ . By hypothesis, we have  $x'M = (Qf)'$ , and  $x'M' = (Q'f')'$ . Rewriting the second equation in terms of  $M, f$ , we obtain

$$x'P'M = (Q'P'f)'.$$

Thus  $w = x'f$  and  $w' = x'P'f$  are elements of  $V^*I_s \cap I_s V^*$ . If  $A$  is not degenerate, Proposition (2.4) implies that  $P$  is a scalar matrix.

COROLLARY (2.19). *The conjugacy class of the matrix  $Q$  is determined by the regular algebra  $A$ .*

### 3. CLASSIFICATION OF REGULAR ALGEBRAS OF DIMENSION 3

The classification of regular algebras of global dimension 3 is done by determining the solutions of (2.12) with  $Q$  in Jordan form. Let  $G_Q$  denote the stabilizer of the matrix  $Q$  for the action  $P, Q \rightarrow P'Q(P')^{-1}$ , i.e., the centralizer of  $Q'$ . Then by (2.17), the space  $W_Q$  of solutions of (2.12) with fixed  $Q$  is a representation of  $G_Q$ .

Consider the case that  $Q$  is diagonal, with diagonal entries  $\{\alpha_1, \dots, \alpha_r\}$ . We use the notation  $Q = \text{diag}(\alpha_1, \dots, \alpha_r)$ . Then  $G_Q$  contains the group  $T$  of diagonal matrices, and hence  $W_Q$  is a sum of weight spaces for  $T$ . In the notation of (2.14), the relation (2.12) is

$$\sum_v \omega_v x_{v_s+1} x_{v_1} \cdots x_{v_s} = \sum_v \alpha_{v_1} \omega_v x_{v_1} \cdots x_{v_s+1},$$

or

$$\omega_{\sigma^{-1}(v)} = \alpha_{v_1} \omega_v, \quad v = (v_1, \dots, v_{s+1}). \quad (3.1)$$

Therefore  $W_Q$  has a basis in which each element is a sum over an orbit of  $\sigma$ , of the form

$$x_v + \alpha_{v_1} x_{\sigma^{-1}(v)} + \alpha_{v_1} \alpha_{v_2} x_{\sigma^{-2}(v)} + \cdots. \quad (3.2)$$

If the orbit has order  $n$ , i.e.,  $\sigma^n(v) = v$ , then (3.1) gives us the condition

$$\omega_v = \alpha_{v_1} \cdots \alpha_{v_n} \omega_v \quad (3.3)$$

on the coefficients  $\omega_v$  of  $w$ . This implies

**PROPOSITION (3.4).** *Let  $w \in W_Q$ , with  $Q = \text{diag}(\alpha_1, \dots, \alpha_{r+1})$ . Let  $v$  be a multi-index whose  $\sigma$ -orbit has order  $n$ . Then  $\omega_v = 0$ , or else*

$$\alpha_{v_1} \cdots \alpha_{v_n} = 1. \quad (3.5)$$

If condition (3.5) holds, then (3.2) is an element of  $W_Q$ .

Suppose that  $r = 2$ , so that  $V^*$  has dimension 2 and  $W = \bigotimes^4 V^*$ . We use the notation  $(x, y)$  and  $(\alpha, \beta)$  in place of  $(x_1, x_2)$  and  $(\alpha_1, \alpha_2)$ . There are six orbits of  $\sigma$  on the set of multi-indices  $\{1, 2\}^4$ , and so  $W_Q$  has a basis of at most 6 weight vectors.

For example, one orbit is  $\{(1, 2, 1, 2), (2, 1, 2, 1)\}$ . The corresponding vector (3.2) is

$$w = xyxy + \alpha yxyx.$$

It is in  $W_Q$  if and only if the condition (3.5):  $\alpha\beta = 1$  holds. For this vector  $w$ , the associated matrix  $M$  and equations  $f$  defined by (2.3) are

$$M = \begin{pmatrix} 0 & yx \\ \alpha xy & 0 \end{pmatrix}, \quad f = \begin{pmatrix} yxy \\ \alpha yxy \end{pmatrix}.$$

The vectors (3.2) corresponding to the six orbits of  $\sigma$  are listed in Table (3.6), indexed by  $x$ -weights. In this table, the column labeled "condition" refers to (3.5):  $w \in W_Q$  if and only if this condition holds.

Now suppose  $r = 3$ , so that  $W = \bigotimes^3 V^*$ . We replace  $(x_1, x_2, x_3)$  and  $(\alpha_1, \alpha_2, \alpha_3)$  by  $(x, y, z)$  and  $(\alpha, \beta, \gamma)$ . Thus  $Q = \text{diag}(\alpha, \beta, \gamma)$ . There are eleven orbits of a  $\sigma$  on  $\{1, 2, 3\}^3$ , hence eleven possible basis vectors of

TABLE (3.6)  
Basis of  $W_Q$ ,  $r = 2$ ,  $Q = \text{diag}(\alpha, \beta)$

$w$	Vector	Weight	Condition
$w_0$	$y^4$	$(0, 4)$	$\beta = 1$
$w_1$	$y^3x + \beta y^2xy + \beta^2 yxy^2 + \beta^3 xy^3$	$(1, 3)$	$\alpha\beta^3 = 1$
$w_2$	$x^2y^2 + \alpha xy^2x + \alpha^2 y^2x^2 + \alpha^2 \beta yx^2y$	$(2, 2)$	$\alpha^2\beta^2 = 1$
$w'_2$	$xyxy + \alpha yxyx$	$(2, 2)$	$\alpha\beta = 1$
$w_3$	$x^3y + \alpha x^2yx + \alpha^2 xyx^2 + \alpha^3 yx^3$	$(3, 1)$	$\alpha^3\beta = 1$
$w_4$	$x^4$	$(4, 0)$	$\alpha = 1$

$W_Q$ . Each weight has one orbit, except for weight  $(1, 1, 1)$ , which has two. The corresponding vectors are listed in Table (3.8), indexed by their weights.

Let  $S^n V_r^*$  denote the space of symmetrized tensors. Thus  $S^4 V_2^*$  is the space of binary quartics, i.e., homogeneous forms of degree 4 in two variables  $x, y$ , and  $S^3 V_3^*$  is the space of ternary cubics. Each of these spaces has an essentially unique invariant rational function for the action of  $GL_r$ —its  $j$ -invariant (see [6; 16, pp. 406–7; 11, Section 4]). A ternary cubic form  $f(x, y, z)$  represents a cubic curve  $C: \{f=0\}$  in the projective plane  $\mathbf{P}^2$ . If  $C$  is smooth, then it is an elliptic curve whose isomorphism class is described by the value of  $j(f)$ , which is finite. Otherwise,  $j$  is infinite or indeterminate. Similarly, if  $f(x, y)$  is a binary quartic, then the  $j$ -invariant  $j(f)$  describes the isomorphism class of the elliptic curve  $z^2 = f(x, 1)$  provided that  $f(x, y)$  has distinct roots. If  $f$  has a multiple root,  $j$  is infinite or indeterminate.

We define the  $j$ -invariant  $j(w)$  on the space  $\bigotimes^4 V_2^*$  and  $\bigotimes^3 V_3^*$  to be  $j(S(w))$ , where  $S(w)$  is the symmetrization of  $w$ . In case  $j(S(w))$  is indeterminate, we assign the value  $\infty$  to  $j(w)$ .

It is natural to classify regular algebras of global dimension 3 into *types*, one type  $T$  for each irreducible component  $X_T$  of the variety

$$X = \bigcup_{Q \in Gl} W_Q \subset W \quad (3.7)$$

whose generic point  $w$  corresponds to a regular algebra. Thus  $A$  is of type  $T$  if  $w(A) \in X_T$ . Since some components of  $X$  intersect, special algebras may belong to more than one type. We allow this to happen. Also, a component  $X_T$  may contain special points  $w$  corresponding to degenerate algebras.

TABLE (3.8)  
Basis of  $W_Q$ ,  $r = 3$ ,  $Q = \text{diag}(\alpha, \beta, \gamma)$

$w$	Vector	Condition
$w_{300}$	$x^3$	$\alpha = 1$
$w_{030}$	$y^3$	$\beta = 1$
$w_{003}$	$z^3$	$\gamma = 1$
$w_{111}$	$xyz + \alpha yzx + \alpha \beta zxy$	$\alpha \beta \gamma = 1$
$w'_{111}$	$yxz + \beta xzy + \alpha \beta zyx$	$\alpha \beta \gamma = 1$
$w_{210}$	$xyx + \alpha yx^2 + \alpha \beta x^2 y$	$\alpha^2 \beta = 1$
$w_{120}$	$yxy + \beta xy^2 + \alpha \beta y^2 x$	$\alpha \beta^2 = 1$
$w_{201}$	$xzx + \alpha zx^2 + \alpha \gamma x^2 z$	$\alpha^2 \gamma = 1$
$w_{102}$	$zxx + \gamma xz^2 + \alpha \gamma z^2 x$	$\alpha \gamma^2 = 1$
$w_{021}$	$yzy + \beta zy^2 + \beta \gamma y^2 z$	$\beta^2 \gamma = 1$
$w_{012}$	$xyx + \gamma yz^2 + \beta \gamma z^2 y$	$\beta \gamma^2 = 1$

TABLE (3.9)

Regular Algebras,  $r = 2$  (Generic Forms)

Type	$Q$	$w$	$\dim X_T$	moduli	$j$
$A$	$(1, 1)$	$w_0 + aw_2 + bw'_2 + w_4$	6	2	<i>var.</i>
$E$	$(1, \zeta_3)$	$w_1 + w_4$	4	0	0
$H$	$(\zeta_8, -\zeta_8)$	$w_1 + w_3$	4	0	$12^3$
$S_1$	$(\alpha, \alpha^{-1})$	$w_2 + aw'_2$	5	2	$\infty$
$S_2$	$(\alpha, -\alpha^{-1})$	$w_2$	4	1	$\infty$
$S'_2$	$(1, -1)$	$w_2 + w_4$	4	0	$\infty$

THEOREM (3.10). *The types of regular algebras of global dimension 3, generated in degree 1, are listed in Tables (3.9) and (3.11). The generic algebra of each type is regular, and if  $A$  is any regular algebra, then  $w(A)$  is in the closure of the  $GL_r$ -orbit of one of the vectors  $w$  listed in these tables.*

The proof of this theorem is in Section 4.

The matrix  $Q$  of a generic member is diagonalizable, and the diagonalized matrix is used to represent the type. Its diagonal entries are listed. The vector  $w$  is described in terms of the bases (3.6), (3.8).

The coefficients  $\alpha, \beta, \gamma, a, b$  are arbitrary, and as always,  $\zeta_n$  denotes a primitive  $n$ th root of 1. Types  $E, H$  depend on the choice of a primitive root. Within each type, the coefficients of  $w$  have been normalized so that isomorphism classes occur finitely often. This normalization is rather arbitrary. The column labeled *moduli* gives the number of essential parameters, and  $j = j(w)$ .

TABLE (3.11)

Regular Algebras,  $r = 3$  (Generic Forms).

Type	$Q$	$w$	$\dim X_T$	moduli	$j$
$A$	$(1, 1, 1)$	$w_{300} + w_{030} + w_{003}$ $+ aw_{111} + bw'_{111}$	11	2	<i>var.</i>
$B$	$(1, 1, -1)$	$w_{210} + w_{120} + w_{102} + aw_{012}$	10	1	<i>var.</i>
$E$	$(\zeta_9, \zeta_9^4, \zeta_9^2)$	$w_{201} + w_{120} + w_{012}$	9	0	0
$H$	$(1, -1, \zeta_4)$	$w_{300} + w_{120} + w_{012}$	9	0	$12^3$
$S_1$	$(\alpha, \beta, (\alpha\beta)^{-1})$	$w_{111} + aw'_{111}$	10	3	$\infty$
$S'_1$	$(\alpha, \alpha^{-1}, 1)$	$w_{003} + w_{111} + aw'_{111}$	10	2	$\infty$
$S_2$	$(\alpha, -\alpha, \alpha^{-2})$	$w_{201} + w_{021}$	9	1	$\infty$

COROLLARY (3.12). *If  $A$  is a regular algebra of global dimension 3 and  $Q$  is the matrix (2.1), then  $\det Q = \zeta_n$ , where  $n = 1, 2, 3$  or 4.*

#### 4. PROOF OF THEOREM (3.10): CLASSIFICATION

The fact that the entries in Tables (3.9), (3.11) represent distinct types  $T$  follows from the form of the matrices  $Q$  and the dimensions of  $X_T$ . We omit the verification that the generic algebra of each type is regular. This is proved by verifying regularity for a special algebra. Then semi-continuity of cohomology implies the exactness of the general complex (1.9). For most types, representatives are exhibited in later sections. Thus it remains to show that every regular algebra is a specialization of one of the types listed, or equivalently, that there is no other component of  $X$  (3.7) whose generic member is regular.

Consider the closed subscheme  $X'$  of  $Gl_r \times W$  whose fibre over  $Q \in Gl_r$  is the space  $W_Q$ . It follows from Corollary (2.19) that the projection  $X' \rightarrow X$  is bijective at points which correspond to regular algebras, and so the types  $X_T$  correspond to irreducible components  $X'_T$  of  $X'$ . We denote the projection  $X' \rightarrow Gl_r$  by  $\pi$ .

LEMMA (4.1). *Let  $T$  be a type of regular algebras. Then*

$$(i) \quad \dim X_T = \dim X'_T \geq r^2.$$

$$(ii) \quad \text{If } Q \text{ is a generic point of } Y_T = \pi(X'_T), \text{ then } \pi^{-1}(Q) = W_Q.$$

$$(iii) \quad Y_T \text{ is closed in } Gl_r.$$

(iv) *Let  $Q$  be a generic point of  $Y_T$ , and let  $Q_0$  be a specialization of  $Q$ . If  $\dim W_{Q_0} \leq \dim W_Q$ , then equality holds and  $W_{Q_0} \subset X_T$ .*

*Proof.* The first assertion follows from the fact that the number of equations defining  $X'$  in  $Gl_r \times W$  is  $r^{s+1}$ . The second assertion is obvious, and it implies the third one. For, it shows that the closed subset  $X'_T \cap (Gl_r \times \{0\})$  of  $Gl_r \times \{0\}$  contains  $(Q, 0)$ , hence is equal to  $Y_T \times \{0\}$ , and so  $Y_T$  is closed. The last assertion follows from semi-continuity of fibre dimension and (iii).

We will need to check that certain algebras are degenerate.

LEMMA (4.2). *The algebra  $A$  is degenerate in each of the following cases:*

- (i)  $r=2, f_1=x^3, f_2$  arbitrary,
- (ii)  $r=3, f_1=x^2, f_2, f_3$  arbitrary,
- (iii)  $r=2, f'=(yxy+cx^3, xyx),$
- (iv)  $r=3, f'=(yz+bx^2, zx+cy^2, xy).$

*Proof.* In each case, there is no resolution (1.9) because  $\ker M$  is not generated by a linear vector of the required form. In case (i), the top row of  $M$  is  $(x^2, 0)$ , and so  $(x, 0) \in \ker M$ . Similarly,  $(x, 0, 0) \in \ker M$  in case (ii). In case (iii),

$$M = \begin{pmatrix} cx^2 & yx \\ xy & 0 \end{pmatrix},$$

and  $\ker M$  contains the vectors  $(x, y), (x^2, 0)$ . In case (iv),  $\ker M$  contains  $(x, y, z), (yz, 0, 0)$ .

For the classification, we first consider a regular algebra  $A$  such that  $Q$  is diagonalizable. We may then assume  $Q$  diagonal (2.17), so that  $w = w(A)$  is a linear combination of the basis vectors (3.6) or (3.8). In order for  $A$  to be non-degenerate,  $w$  must satisfy certain requirements, and the ones which we will use can be formalized as follows: Define the *reduced weight*  $\text{rwt}(w)$  of one of the basis vectors to be the vector of exponents of  $\alpha, \beta, \gamma$  in the condition of (3.6), (3.8). Thus

$$\begin{aligned} \text{rwt}(w_0) &= (0, 1), & \text{rwt}(w_4) &= (1, 0), & \text{rwt}(w'_2) &= (1, 1), \\ \text{rwt}(w_{300}) &= (1, 0, 0), & \text{rwt}(w_{030}) &= (0, 1, 0), & \text{rwt}(w_{003}) &= (0, 0, 1), \\ \text{and } \text{rwt}(w) &= wt(w) \text{ for the other vectors (3.6), (3.7).} \end{aligned} \quad (4.3)$$

Then we have

**LEMMA (4.4).** *Let  $A$  be a regular algebra, such that  $w = w(A)$  is a linear combination of the basis vectors (3.6) or (3.8). Then the sum of the reduced weights of the basis vectors appearing in  $w$  is at least  $(2, 2)$  if  $r=2$ , or  $(2, 2, 2)$  if  $r=3$ .*

*Proof.* If the condition is not satisfied and  $r=2$ , then, interchanging  $x, y$  if necessary, either one of the defining equations  $f_i$  for  $A$  is  $x^3$ , or else  $f' = (byxy + cx^3, axyx)$ . These algebras are degenerate, by (4.2). Similarly, if  $r=3$ , then permuting the variables as necessary results in  $f_i = x^2$  or else  $f' = (yz, zx, xy)$ . These algebras are also degenerate.

*Case 1.*  $r=2$  and  $Q$  diagonal. Then by Lemma (4.4), either  $A$  is of type

$S_2$ , or else at least two of the conditions (3.6) are satisfied. Interchanging  $\alpha$ ,  $\beta$  as necessary, we are left with the following subcases to consider:

$$(\alpha, \beta) = (\alpha, \pm \alpha^{-1}), \pm(1, 1), (\zeta_4, \zeta_4), (1, -1), (1, \zeta_3), (\zeta_8, -\zeta_8). \quad (4.5)$$

If  $Q$  is the scalar matrix  $\pm 1$  or  $\pm i$ , then  $G_Q = Gl_2$  (see Section 3). These values of  $Q$  give the decomposition of  $W$  into eigenspaces for  $\phi$ :

$$W = W_1 + W_{-1} + W_i + W_{-i}.$$

The  $Gl_2$  representations can be identified as follows (where  $S^2 = S^2 V^*$ , etc.):

$$\begin{aligned} W_1 &\approx S^4 + A^2 \otimes A^2: & \text{basis} \{w_0, \dots, w_4, w'_2\}, \\ W_{-1} &\approx \otimes^2 \otimes A^2: & \text{basis} \{w_1, w_2, w'_2, w_3\}, \\ W_{\pm i} &\approx S^2 \otimes A^2: & \text{basis} \{w_1, \omega_2, \omega_3\}. \end{aligned} \quad (4.6)$$

Since  $\otimes^2 \approx S^2 + A^2$ , this corresponds to the decomposition

$$\otimes^4 \approx S^4 + 2A^2 \otimes A^2 + 3S^2 \otimes A^2 \quad (4.7)$$

of  $W$  into irreducible  $Gl_2$ -representations [17], p. 127.

The case  $Q = 1$  is classified as type  $A$ . The coefficients in Table (3.9) have been normalized to give the standard representation of the binary quartic  $S(w) \in S^4$  in the form

$$x^4 + 6ux^2y^2 + y^4, \quad (4.8)$$

where

$$6u = 4a + 2b. \quad (4.9)$$

Every binary quartic with distinct roots is projectively equivalent to one of the form (4.8) [6], p. 166.

Assume that  $Q = -1$ . The  $Gl_2$ -orbits in  $\otimes^2 \otimes A^2$  are in 1-1 correspondence with the orbits in  $\otimes^2$ , which are represented by

$$x_1x_2 + cx_2x_1, \quad x_1^2 + x_1x_2 - x_2x_1, \quad x_1^2. \quad (4.10)$$

In this correspondence,  $w_2 \leftrightarrow [x_1, x_2]$ , and  $w'_2 \leftrightarrow 2(x_1x_2 + x_2x_1)$ . Hence the first orbit is classified as type  $S_1$ . The second is a specialization, and the third corresponds to a degenerate algebra.

The case  $Q = \zeta_4$  is not a distinct type by Lemma (4.1)(i), because  $\dim W_Q = 3 < r^2 = 4$ . These algebras are of type  $S_2$ .



The cases  $\alpha \neq \beta$  in (4.5) correspond to the remaining types, with coefficients in Table (3.9) normalized using  $G_Q$ .

*Case 2.*  $r=3$  and  $Q$  diagonal. Here Lemma (4.4) shows that either  $\det Q = \alpha\beta\gamma = 1$ , or else at least two of the conditions in the second column of Table (3.7) hold. Sorting through the possible configurations and permuting  $\alpha, \beta, \gamma$  as necessary leads to a list of subcases to consider (Table (4.11)). The verification is somewhat tedious, but uneventful.

Again, the coefficients in Table (3.1) have been normalized using  $G_Q$ . As in the case  $r=2$ , the scalar matrices  $Q=1, \zeta_3, \zeta_3^2$  correspond to the decomposition of  $W$  into eigenspaces for  $\phi$ . The associated  $Gl_3$  representations are

$$\begin{aligned} W_1 &\approx S_3 + \Lambda^3, \\ W_{\zeta} &\approx \text{adj} \otimes \Lambda^3, \end{aligned} \quad (4.12)$$

where  $S^3 = S^3(V_3^*)$ , etc., and where  $\text{adj}$  denotes the adjoint representation of  $Gl_3$  by conjugation on trace-zero matrices. This corresponds to the irreducible  $Gl_3$ -decomposition

$$\otimes^3 \approx S^3 + \Lambda^3 + 2 \text{adj} \otimes \Lambda^3.$$

The case  $Q=1$  is classified as type  $A$  in Table (3.11), with coefficients normalized to give the standard representation of a ternary cubic  $S(w) \in S^3$  in the form

$$x^3 + y^3 + z^3 + 6uxyz, \quad (4.13)$$

TABLE (4.11)

Subcases,  $r=3$ 

$\det Q$	$(\alpha, \beta, \gamma)$	Condition	Type or weights
1	$\alpha\beta\gamma = 1$	distinct	$S_1$
1	$(\alpha, \alpha, \alpha^{-2})$	$\alpha^2 \neq 1,$ $\alpha^3 \neq 1$	201, 111, 021
1	$(\alpha, \alpha^{-1}, 1)$	$\alpha^2 \neq 1$	$S'_1$
1	$(-1, -1, 1)$		201, 111, 021, 003
1	$(1, 1, 1)$		$A$
1	$(\zeta_3, \zeta_3, \zeta_3)$		all but three
-1	$(\alpha, -\alpha, \alpha^{-2})$	$\alpha^3 \neq 1$	$S_2$
-1	$(1, 1, -1)$		$B$
-1	$(\zeta_3, \zeta_3, -\zeta_3)$		210, 120, 102, 012
$\zeta_3^2$	$(1, \zeta_2, \zeta_4)$		$H$
$\zeta_3^2$	$(\zeta_9, \zeta_9^2, \zeta_9^4)$		$E$

where  $6u = 3a + 3b$ . Every nonsingular cubic is projectively equivalent to one of these ([16], p. 401).

The case  $Q = \zeta_3$  does not lead to a new type, because  $\dim W_Q = 8 < r^2 = 9$ . (4.1)(i). The algebras with  $Q = \zeta_3$  are of type  $S_1$ . Similarly, if  $Q = \text{diag}(\zeta_3, \zeta_3, -\zeta_3)$ , the conjugacy class  $C(Q)$  has dimension 4, and  $\dim W_Q = 4$ , a total of 8. This case belongs to type  $S_2$ .

If  $Q = \text{diag}(\alpha, \alpha, \alpha^{-2})$ , then  $G_Q = Gl_2 \times Gl_1$ , and the representation of  $Gl_2$  on  $W_Q$  is isomorphic to  $\otimes^2 V_2^*$ . This case is of type  $S_1$  because the generic orbits  $x_1x_2 + cx_2x_1$  (4.10) in  $\otimes^2 V_2^*$  correspond to  $w_{111} + cw'_{121}$  in  $W_Q$ . Similarly, if  $Q = \text{diag}(-1, -1, 1)$ , then the  $Gl_2$ -representation on  $W_Q$  is isomorphic to  $\otimes^2 V_2^* + \Lambda^2 V_2^*$ , and a generic orbit is represented by  $w_{111} + cw'_{111} + bw_{003}$ : type  $S_1$ . This completes the discussion of the case  $Q$  diagonal.

We next consider the case that  $Q$  is not diagonalizable. Then it is similar to a matrix of the form

$$Q = D + Nt, \quad (4.14)$$

where  $t$  is a variable.  $D$  is a diagonal matrix, and  $N = e_{12}$  or  $e_{12} + e_{23}$ , according to the case. We write Eq. (2.12) in the form

$$\psi w = Lw, \quad (4.15)$$

where  $\psi = \phi - L_D$ , and  $L = tL_N$ . Then we can solve (4.15) over the polynomial ring  $k[t]$ :  $w = w^0 + w^1t + \dots$ , where

$$\begin{aligned} \psi w^0 &= 0 \\ \psi w^1 &= Lw^0, \\ \psi w^2 &= Lw^1, \dots \end{aligned} \quad (4.16)$$

We may assume  $w^0 \neq 0$ , and then  $w$  is a primitive solution over the power series ring  $k[[t]]$ .

LEMMA (4.17).  $\dim W_Q \leq \dim(W_D \cap L^{-1}(\psi(W)))$ .

This lemma just restates the first two equations of (4.16):  $w^0 \in W_D = \ker \psi$  and  $Lw^0 \in \text{im } \psi$ .

Case 3.  $r = 2$ ,  $Q$  not diagonalizable. In this case,  $D = \text{diag}(\alpha, \alpha)$  is a scalar matrix. Since  $W_D \neq 0$ ,  $\alpha = \pm 1, \pm i$ . We estimate  $\dim W_Q$  using Lemma (4.17), by determining the vectors  $w^0 \in \ker \psi$  such that  $Lw^0 \in \text{im } \psi$ . Since  $\psi$  preserves weights while  $L$  shifts weights  $(i, j) \mapsto (i + 1, j - 1)$ , we may work with each weight vector (3.6) separately, except that if  $\alpha = \pm 1$ , a linear combination of  $w_2, w'_2$  is allowed.

Since  $\psi = \phi - \alpha$  is semi-simple, we can test membership in  $\text{im } \psi$  by projecting onto  $\ker \psi$ , which is the  $\alpha$ -eigenspace of  $\phi$  acting on  $W$ . Let  $\pi_\alpha$  denote this projection:

$$4\pi_\alpha(w) = w + \alpha^3\phi(w) + \alpha^2\phi^2(w) + \alpha\phi^3(w). \quad (4.18)$$

Then the condition (4.16) on  $w^0$  becomes

$$\psi(w^0) = 0, \quad \pi_\alpha(Lw^0) = 0. \quad (4.19)$$

This is easily checked for each of the basis vectors (3.8). The result is

$$\begin{aligned} \alpha = 1: & \quad w^0 = w^2 - 2w'_2, w_4, \\ \alpha = -1: & \quad w^0 = w_2, w_3, \\ \alpha = \pm i: & \quad w^0 = w_3. \end{aligned}$$

Thus  $\dim W_Q \leq 2$  if  $\alpha = \pm 1$ . It follows from Lemma (4.1)(i) that these cases belong to type  $S_1$ . The algebras turn out to be the following:

$$\begin{aligned} \alpha = 1: & \quad w = (w_2 - 2w'_2 + x^2yx - xyx^2) + aw_4, \\ \alpha = -1: & \quad w = (w_2 - x^2yx + xyz^2) + a(2w_3 - w_4). \end{aligned}$$

*Regular algebras,  $r = 2$ ,  $Q$  a Jordan 2-block.* (4.20)

It follows from Lemma (4.4) that the case  $\alpha = \pm i$  is degenerate.

*Case 4.*  $r = 3$ ,  $Q$  is a Jordan 3-block. Again,  $D$  is a scalar matrix, with  $\alpha = 1, \zeta_3, \zeta_3^2$ . The projection onto  $\ker \psi$  is given by

$$3\pi_\alpha(w) = w + \alpha^2\phi(w) + \alpha\phi^2(w). \quad (4.21)$$

In this case  $L$  mixes weights by the rule

$$(i, j, k) \rightarrow \{(i+1, j-1, k), (i, j+1, k-1)\}.$$

Computation of  $\pi_\alpha(L(w))$  for the basis vectors (3.8) leads to the following candidates for  $w^0$ :

$$\begin{aligned} \alpha = 1: & \quad w^0 = w_{300}, w_{111} - w'_{111}, w_{120} - 2w_{201}, \\ \alpha = \zeta_3: & \quad w^0 = w_{210}, w_{120} + w_{201}. \end{aligned} \quad (4.22)$$

Thus  $\dim W_Q \leq 2$  if  $\alpha = \zeta_3$  and  $\dim W_Q \leq 3$  if  $\alpha = 1$ . Lemma (4.1) shows that these algebras are types  $S_1, S'_1$ , respectively.

*Case 5.*  $r=3$ ,  $Q$  contains a Jordan 2-block. In this case  $N=e_{12}$ , and  $D=\text{diag}(\alpha, \alpha, \gamma)$ . Choose a weight, such as  $(0, 1, 2)$  for example. In order for  $w_{012}$  (3.8) to be a candidate for  $w^0$ , we must have  $\alpha\gamma^2=1$ . If so, we test

$$Lw_{012} = \gamma xz^2 \in W_{102} \quad (4.23)$$

for membership in  $\text{im } \psi$ . Taking the basis  $\{xz^2, zxz, z^2x\}$  for  $W_{102}$ , the matrix of the operator  $\psi$  is

$$\begin{pmatrix} -\alpha & & 1 \\ & 1 & -\gamma \\ & & 1 & -\gamma \end{pmatrix}. \quad (4.24)$$

The following lemma is a computation:

LEMMA (4.25). *Let  $a, b, c \in K$  satisfy  $abc=1$ , and  $a^{-1} + b^{-1} + c^{-1} \neq 0$ . Then the kernel of the operator*

$$\psi = \begin{pmatrix} -\alpha & & 1 \\ & 1 & -b \\ & & 1 & -c \end{pmatrix}$$

*has dimension 1, and the matrix*

$$\pi = (a^{-1} + b^{-1} + c^{-1})^{-1} \begin{pmatrix} bc & 1 & b \\ c & ac & 1 \\ 1 & a & ab \end{pmatrix}$$

*is the projection to  $\ker \psi$  which commutes with  $\psi$ . Hence  $\text{im } \psi = \ker \pi$ .*

Applying the operator  $\pi$  with  $(a, b, c) = (\alpha, \gamma, \gamma)$  shows that  $\pi Lw_{012} \neq 0$ ; hence  $w_{012}$  is ruled out. The special case  $\alpha^{-1} + 2\gamma^{-1} = 0$  implies  $\alpha^3 = 1/4$ ,  $\gamma^3 = -2$ . In this case  $\dim W_Q + \dim C(Q) \leq 2 + 6 = 8$ —not a new type. In fact, this case does not arise. Similarly, the weight vectors  $w_{030}$ ,  $w_{120}$ ,  $w_{021}$  are ruled out, leaving the following possibilities for  $w^0$ :

- (i)  $w_{300}, w_{201}, w_{102}, w_{003}: Lw = 0$ ,
  - (ii)  $w_{210}: Lw_{210} = \alpha w_{300}$ , if  $\alpha = \zeta_3$ ,
  - (iii)  $w_{111} - w'_{111}$ , if  $\alpha^2\gamma = 1$ .
- (4.26)

By Lemma (4.4), the vectors (4.26)(i), (ii) do not suffice for a regular algebra. Thus we must have  $\alpha^2\gamma = 1$ . This leaves the following cases:

$$\begin{aligned} D &= (\alpha, \alpha, \alpha^{-2}), \alpha \neq \pm 1, \zeta_3: \dim W_Q = 2, \\ (-1, -1, 1): \dim W_Q &= 3, \\ (\zeta_3, \zeta_3, \zeta_3): \dim W_Q &= 4, \\ (1, 1, 1): \dim W_Q &= 5. \end{aligned} \quad (4.27)$$

The first two cases are types  $S_1, S'_4$ , by Lemma (4.1). If  $D = (\zeta_3, \zeta_3, \zeta_3)$ , then  $\dim W_Q + \dim C(Q) \leq 4 + 4 = 8$ , not a new type. The case  $D = (1, 1, 1)$  is of type  $S_1$ . This is not easy to see directly, and we verified it using a computer, by calculating  $W_{Q(\lambda)}$  for the one-parameter family

$$Q(\lambda) = Q + ue_{21} + c_1\lambda e_{22} + c_2\lambda^2 e_{23} + c_3\lambda e_{31} + c_4\lambda^2(e_{33} - e_{22}),$$

$u$  being determined by the condition  $\det Q(\lambda) = 1$ .

## 5. COMPUTATION IN A GRADED ALGEBRA

Computation in a finitely generated graded algebra

$$A = k[x_1, \dots, x_r]/I \quad (5.1)$$

reduces immediately to linear algebra if generators for the ideal  $I$  are given. However, direct computation of  $I_n = k[x]_n \cap I$  gets out of hand quickly because of the exponential growth of  $k[x]$ . Since the algebras we are studying have polynomial growth, one may hope for faster algorithms. The method we use is to replace a polynomial  $\alpha \in k[x]$  by the earliest one, in lexicographic order, in its congruence class modulo  $I$ , using Bergman's *diamond lemma* [5]. This method is probably still exponential, but in practice it allows computation in fairly high degree. We will review it briefly to establish notation, though it is well known.

In this section, we allow the ordered set of variables  $\{x_1, \dots, x_r\}$  to have various degrees  $d_i \geq 1$ . Suppose given an arbitrary set of relations of the form

$$m_v = \phi_v, \quad v \in S, \quad (5.2)$$

where  $m_v$  is a monomial and  $\phi_v$  is a homogeneous polynomial of the same degree in  $\{x_1, \dots, x_r\}$ . Assume that the monomials occurring in  $\phi_v$  are earlier than  $m_v$  in the lexicographic ordering. We call (5.2) a set of *replacements*. Substitution of  $\phi_v$  for  $m_v$  in a monomial of the form  $um_vv$  replaces the monomial by an earlier polynomial. By repeating this procedure finitely often, every polynomial can be *reduced*, i.e., replaced by a *reduced polynomial*, meaning one in which no monomial of the form  $um_vv$  occurs.

The reduced monomials form a basis for the space of reduced polynomials.

By *overlap* of monomials  $m, m'$ , we mean a triple  $[u, v, w]$  of monomials of degrees  $\geq 1$  such that

$$m = uv, \quad m' = vw. \quad (5.3)$$

We omit the proof of the following proposition (see [5]):

**PROPOSITION (5.4).** *Given a set (5.2) of replacements, the following are equivalent:*

- (i) *Multiplication in the free ring, followed by reduction, induces a ring structure on the space of reduced polynomials.*
- (ii) *The result of reducing a given polynomial is unique.*
- (iii) *For each overlap  $[u, v, w]$  of  $m_\mu, m_\nu$  with  $\mu, \nu \in S$ , the results of reducing  $\phi_\mu w$  and  $u\phi_\nu$  in some arbitrary way are equal.*

We call a set (5.2) of replacements *complete* if the equivalent conditions of Proposition (5.4) hold. A complete set of replacements is *reduced* if each  $m_\nu$  and  $\phi_\nu$  is a reduced polynomial. If some  $\phi_\nu$  is not reduced, we can replace it by the corresponding reduced polynomial without affecting the validity of conditions (5.4)(i)–(iii), and if  $m_\nu$  is not reduced, then the replacement  $m_\nu = \phi_\nu$  is redundant.

Now let

$$A = k[x_1, \dots, x_r]/I \quad (5.5)$$

be a quotient of the free ring  $k[x_1, \dots, x_r]$ , and let  $\{f_\nu\}$ ,  $\nu \in S$ , be a generating set for  $I$ , as 2-sided ideal. For any polynomial  $f$ , we may write

$$f = c(m - \phi), \quad (5.6)$$

where  $m$  is the last monomial of  $f$  and  $c$  is its coefficient. Doing this for each  $f_\nu$ , we obtain a set (5.2) of replacements. Proposition (5.4)(iii) provides a systematic way to extend this set to a complete set, by checking overlaps  $[u, v, w]$  of  $m_\mu, m_\nu$  with  $\mu, \nu \in S$ . Let  $\alpha, \beta$  denote the results of reducing  $\phi_\mu w$  and  $u\phi_\nu$  in some way. If  $\alpha \neq \beta$ , then  $f = \alpha - \beta \in I$ , and we add the replacement obtained as in (5.6) to the set. Note that the degree of his new replacement is higher than the degree of  $m_\mu$  or  $m_\nu$ .

In practice, the number of overlaps to check can be reduced considerably if the Hilbert function  $a_n$  of  $A$  is known in advance. If all overlaps of degree less than  $n$  are consistent, then those of degree  $n$  are too, provided that the number of reduced monomials is  $a_n$ .

**COROLLARY (5.7).** *If (5.2) is a complete set of replacements, then the*

ring defined by (5.4)(i) is  $A = k[x_1, \dots, x_r]/I$ , where  $I$  is the ideal generated by  $\{f_v = m_v - \phi_v\}$ .

It is a very interesting problem to describe the algebras having a finite, complete set of replacements. Note that for a finite set (5.2), there are finitely many overlaps to check in order to verify completeness.

The number of replacements in each degree seemed to grow at most linearly, in our computations.

## 6. SKEW POLYNOMIAL RINGS

We retain the notation of the previous section. By *ordered monomial* in  $\{x_1, \dots, x_r\}$  we mean a monomial of the form  $x_1^{e_1} \cdots x_r^{e_r}$ . An *ordered polynomial* is a linear combination of ordered monomials. Consider a set of  $\frac{1}{2}r(r-1)$  replacements

$$x_j x_i = \phi_{ji}, \quad 1 \leq i < j \leq r, \quad (6.1)$$

where  $\phi_{ji}$  is an ordered polynomial of degree  $d_i + d_j$  whose monomials are earlier than  $x_j x_i$ , i.e., they begin with an element  $x_l$  where  $l < j$ .

Using (6.1), every polynomial can be reduced to an ordered one. The overlaps in this case are the triples  $[x_k, x_j, x_i]$ ,  $i < j < k$ . Thus Proposition (4.3)(iii) tells us that the set of replacements (6.1) is complete provided that the results of reducing  $\phi_{kj} x_i$  and  $x_k \phi_{ji}$  in some way are equal, for every triple of indices  $i < j < k$ . If so, the space of ordered polynomials becomes a ring which we call a *skew polynomial ring*.

EXAMPLES (6.2). Consider variables  $\{x, y\}$  of degrees 1, 2, respectively. Then the relation (6.1) has the form

$$yx = a_1 x^3 + a_2 xy, \quad a_1, a_2 \in k, \quad (6.3)$$

and there are no overlaps to check. The skew polynomial ring defined by (6.3) is an Ore extension of  $k[x]$ , in the sense of [7], p. 438. In case of three variables  $\{x, y, z\}$  of degree 1, the replacements have the form

$$\begin{aligned} yx &= a_1 x^2 + a_2 xy + a_3 xz, \\ zx &= b_1 x^2 + b_2 xy + b_3 xz + b_5 y^2 + b_6 yz, \\ zy &= c_1 x^2 + c_2 xy + c_3 xz + c_5 y^2 + c_6 yz, \end{aligned} \quad (6.4)$$

and the overlap  $[z, y, x]$  imposes conditions on the coefficients  $\{a_v, b_v, c_v\}$ .

Several applications of (6.4) are required to reduce  $(zy)x$  and  $z(yx)$ . This makes the locus of coefficients of skew polynomial rings fairly complicated.

It has several irreducible components. We will not write down the explicit equations.

EXAMPLE (6.5). The algebra  $A$  of type  $S'_2$ ,  $r = 2$ , is defined by equations

$$\begin{aligned}y^2x + xy^2 + x^3 &= 0, \\yx^2 - x^2y &= 0.\end{aligned}$$

The substitution  $yx = u$  exhibits  $A$  as a skew polynomial ring in the ordered variables  $\{x, u, y\}$ , with the replacements

$$\begin{aligned}yx &= u, \\ux &= x^2y, \\yu &= -xy^2 - x^3.\end{aligned}$$

But we have

PROPOSITION (6.6). *The algebra  $S'_2$  does not admit a structure of iterated Ore extension.*

The verification of this proposition requires some computation, and we omit it.

The following is a corollary of Anick's resolution [1], because that resolution has length  $n$ :

COROLLARY (6.7). (Anick) *Let  $A$  be a skew polynomial ring in  $n$  variables. Then the projective dimension of  ${}_A k$  is  $n$ .*

If  $A$  is a skew polynomial ring in 3 variables  $x, y, z$  of degree 1, then Anick's resolution [1] shows that  $r_1 = r_2 = 3$  and  $r_3 = 1$ . However, some of these rings are too degenerate to be regular in the sense of (1.6). For example, let  $A$  be the algebra defined by

$$zy = zx = yx = 0. \quad (6.8)$$

Then the resolution of  $k$  has the form

$$0 \rightarrow A \xrightarrow{(0 \ 0 \ z)} A^3 \xrightarrow{\begin{pmatrix} 0 & z & 0 \\ z & 0 & 0 \\ 0 & y & 0 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \rightarrow {}_A k \rightarrow 0. \quad (6.9)$$

Since the entries of  $(0, 0, z)$  are not linearly independent, (6.7) cannot be reduced to the form (1.5)(ii).



EXAMPLE (6.10). The following is a type  $A$  algebra which is skew polynomial, but which is not regular in the sense of Walker [15]. The algebra is given by

$$w = w_2 + w'_2 + w_4.$$

Thus the relations are

$$x^3 + xy^2 + y^2x + yxy = 0,$$

$$x^2y + yx^2 + xyx = 0.$$

We let  $u = xy + yx$ , to obtain the skew polynomial relations

$$yx = -xy + u,$$

$$ux = -x^2y,$$

$$yu = -x^3 - xy^2.$$

One can verify that the only graded automorphisms of this algebra are  $x \rightarrow \alpha x$  and  $y \rightarrow \varepsilon xy$ , where  $\varepsilon = \pm 1$ . This can be used to show that there are no normalizing elements of degree 2, and so the maximal ideal does not have a normalizing  $R$ -sequence.

THEOREM (6.11). *Let  $A$  be a regular algebra of global dimension 3, generated in degree 1 and let  $w = w(A)$ . Then  $A$  is a skew polynomial ring in 3 variables if and only if  $j(w) = \infty$ . If  $j = \infty$  and  $r = 3$  then  $A$  is generated by elements of degree 1, while if  $r = 2$  then  $A$  is generated by an ordered set of elements  $(x_1, x_2, x_3)$ , where  $\deg(x_1, x_2, x_3) = (1, 2, 1)$  and  $x_3x_1 = x_2$ .*

This theorem is proved in the next section.

## 7. PROOF OF THEOREM (6.11): SKEW POLYNOMIAL CONDITIONS

There are two parts to the proof of Theorem (6.11). One implication is done by

LEMMA (7.1). *Let  $A$  be a regular skew polynomial ring of global dimension 3, and let  $w = w(A)$ . If  $r = 2$ , then the binary quartic  $S(w)$  is zero, or else it has a double root. If  $r = 3$ , then  $S(w)$  is zero, or it represents a singular plane cubic curve.*

Then, to show that  $A$  is a skew polynomial ring if  $j = \infty$ , the main tool is

LEMMA (7.2). *A regular specialization of a skew polynomial ring, of global dimension 3 and generated in degree 1, is a skew polynomial ring.*

Once this lemma is proved, we need only verify that the generic points of the locus  $j = \infty$  in each type represent skew polynomial rings, which is easily done.

We do not know if the condition of being a skew polynomial ring is closed in the space of algebras of finite global dimension.

*Proof of Lemma (7.1)*

*Case 1.*  $r = 2$ . The Hilbert function (1.20) implies that if  $A$  is skew then it has two generators  $x, y$  of degree 1 and one generator  $u$  of degree 2. There are three orderings to consider:  $\{x, y, u\}$ ,  $\{x, u, y\}$  and  $\{u, x, y\}$ . We also know that  $u$  is a quadratic function of  $x, y$  because  $A$  is generated in degree 1. Therefore the replacement for  $yx$  must involve  $u$ . This rules out the first ordering.

In the second arrangement, the replacements have the form

$$\begin{aligned} yx &= a_1 x^2 + a_2 xy + a_3 u, & a_3 &\neq 0, \\ ux &= b_1 x^3 + b_2 x^2 y + b_3 xu + b_4 xy^2, \\ yu &= c_1 x^3 + c_2 x^2 y + c_3 xu + c_4 xy^2 + c_6 uy. \end{aligned} \quad (7.3)$$

An appropriate substitution of the form  $u \rightarrow *u + *xy + *x^2$  results in  $a_1 = a_2 = 0$ , and  $a_3 = 1$ .

LEMMA (7.4). *If  $a_3 = 1$  in (7.3), then  $b_4 = 0$  and  $b_3 = c_6$ .*

*Proof.* We compute the overlap  $[y, u, x]$  modulo  $xP$ , where  $P = k[x, y]$ . Note that  $ux \equiv 0$  (modulo  $xP$ ). The computation yields

$$0 = (yu)x - y(ux) = c_6 u^2 - b_3 u^2 - b_4 uy^2.$$

Hence  $b_4 = 0$  and  $b_3 = c_6$ .

Let  $f$  be the basis for  $I$  obtained from (7.3) by substituting  $yx$  for  $u$ ,

$$\begin{aligned} f_1 &= b_1 x^3 + b_2 x^2 y + b_3 xyx && + b_5 yx^2, \\ f_2 &= c_1 x^3 + c_2 x^2 y + c_3 xyx + c_4 xy^2 && + c_6 yxy + c_7 y^2 x, \end{aligned}$$

where  $b_5 = c_7 = -1$ . Let  $L = (p_1 x + p_2 y, q_1 x + q_2 y)$  be the vector of linear forms such that  $w = Lf$ , as in (2.5), and let  $M = (m_{ij})$  be the matrix such that  $Mx = f$ . The monomials occurring in  $M$  are

$$\begin{pmatrix} (x^2, xy, yx) & (x^2) \\ (x^2, xy, y^2) & (x^2, xy, yx) \end{pmatrix}. \quad (7.5)$$

The coefficient of  $y^2$  in  $m_{21}$  is  $c_7 \neq 0$ . Since  $f_1, f_2$  do not involve  $y^3$ , and  $LM$

is an ideal basis,  $q_2 = 0$ . It follows that  $x^2$  divides  $S(w) = S(LMx)$ , as required.

If the ordering of the variables is  $\{u, x, y\}$ , the replacements have the form

$$\begin{aligned} yx &= a_1 u + a_2 x^2 + a_3 xy & (a_1 \neq 0), \\ xu &= b_1 ux + b_2 uy, \\ yu &= c_1 ux + c_2 uy + c_4 x^3 + c_5 x^2 y + c_6 xy^2. \end{aligned} \quad (7.6)$$

LEMMA (7.7). *Let  $A$  be a skew polynomial ring defined by Eqs. (7.6).*

(i) *If  $b_2 = 0$ , then  $A$  is a skew polynomial ring with ordered variables  $\{x, u, y\}$ .*

(ii) *If  $b_2 \neq 0$ , then a substitution  $y \rightarrow y + *x$  results in  $c_2 = 0$ . In addition,  $c_4 = c_5 = c_6 = a_2 = 0$ .*

*Proof.* Assume that  $b_2 = 0$ . It is not possible to have  $b_1 = 0$  too, for if so, the overlap computation becomes

$$\begin{aligned} y(xu) &= 0, \\ (yx)u &= a_1 u^2 + \text{other terms.} \end{aligned}$$

Hence  $a_1 = 0$ , which is a contradiction. So  $b_1 \neq 0$ , and we can rewrite (7.6) in the form

$$\begin{aligned} yx &= \cdots, \\ ux &= b_1^{-1} xu, \\ yu &= c_1 b_1^{-1} xu + \cdots, \end{aligned} \quad (7.8)$$

which is in the form (7.3). The fact that  $A$  is a skew polynomial ring determines its Hilbert function, which implies that the overlap  $[y, u, x]$  is consistent. This proves (i).

Assume  $b_2 \neq 0$ . To prove that the coefficients  $c_4, c_5, c_6, a_2$  vanish, we compute the overlap  $[y, x, u]$ , first modulo  $uP$ . The coefficient of  $xy^3$  in

$$(yx)u - y(xu) \quad (7.9)$$

is  $-b_2 c_6$ . Since  $b_2 \neq 0$ , it follows that  $c_6 = 0$ . Since  $c_6 = 0$ , the coefficient of  $x^2 y^2$  in (7.9) is  $-b_2 c_5$ , hence  $c_5 = 0$ . Since  $c_5 = c_6 = 0$ , the coefficient of  $x^3 y$  in (7.9) is  $-b_2 c_4$ , hence  $c_4 = 0$ . A substitution  $y \rightarrow y + *x$  now changes  $c_2$  to  $c_2 - *b_2$ , and preserves the form (7.6). Thus  $c_2$  can be eliminated. Finally, we compute the coefficient of  $uy^2$  in (7.9). It is  $a_2 b_2^2$ , hence  $a_2 = 0$ . This proves part (ii) of the lemma.

Lemma (7.7)(i) reduces the verification of (7.1) to the previous arrangement if  $b_2 = 0$ . If  $b_2 \neq 0$ , then we may assume  $c_i = 0$ ,  $i > 1$  and  $a_2 = 0$ . In this case, the overlap  $[y, x, u]$  would force  $a_1 = 0$  if  $c_1 = 0$ , a contradiction. Thus  $a_1, c_1$  are not zero, and we normalize them to 1 by a change of variable  $y, u \mapsto *y, *u$ . We then reduce (7.9), obtaining

$$(1 + a_3 b_2) u^2 + (a_3 - 1) b_1 u x^2 + (a_3^2 - 1) b_2 u x y.$$

Thus there are two cases:

$$\begin{aligned} a_3 = 1, & & b_2 = -1, & & b_1 \text{ arbitrary,} \\ a_3 = -1, & & b_2 = 1, & & b_1 = 0. \end{aligned} \quad (7.10)$$

In the first case  $S(u) = 0$ , and the form of (7.6) shows that  $S(w) = 0$ . In the second case, the vector (2.5) is  $L = (y, x)$ , and  $S(w) = 2xy(x - y)^2$ . This polynomial has the required double root.

*Case 2.*  $r = 3$ . The replacements have the form (6.4).

LEMMA (7.11). *Either  $a_3 = 0$ , or else  $b_5 = b_6 = 0$ .*

*Proof.* We compute the overlap  $[z, y, x]$ , collecting coefficients of  $yz^2, y^2z$ . The result is

$$(zy)x - z(yx) = -a_3 b_6 yz^2 + (-a_3 b_5 + *b_6) y^2z + \cdots$$

The lemma follows.

Assume  $a_3 = 0$ . Let  $f$  be the generators of  $I$  corresponding to the replacements (6.4), and let  $M = (m_{ij})$  be the matrix  $Mx = f$ . The monomials occurring in  $M$  are

$$\begin{pmatrix} (x, y) & (x) & 0 \\ (x, z) & (x, y) & (x, y) \\ (x) & (x, y, z) & (x, y) \end{pmatrix},$$

and  $z$  has non-zero coefficient in  $m_{21}$  and  $m_{32}$ . Let  $L = (L_1, L_2, L_3)$  be the vector such that  $w = Lf$  (2.5). Since (6.4) does not involve  $z^2$ ,  $L_2, L_3$  do not involve  $z$ . Therefore  $w = LMx$  is linear in  $z$ , which shows that  $S(w)$  is singular at  $x = y = 0$ .

Assume  $a_3 \neq 0$ , so that  $b_5 = b_6 = 0$ . The monomials occurring in  $M$  are

$$\begin{pmatrix} (x, y) & (x) & (x) \\ (x, z) & (x) & (x) \\ (x) & (x, y, z) & (x, y) \end{pmatrix},$$

and as before,  $L_2, L_3$  do not involve  $z$ , which implies that  $L_1$  does involve  $z$ . Let  $M_i$  denote the  $i$ th column of  $M$ . Then  $LM_1$  has a non-zero  $zy$  term, while  $LM_2, LM_3$  do not. Therefore,  $f_1, f_2$  are linear combinations of  $LM_2, LM_3$ . Since neither involves  $yz$ , it follows that  $L_3$  does not involve  $y$ :

$$L = ((x, y, z), (x, y), (x)).$$

Therefore,  $x$  divides  $S(w) = S(LMx)$ , and the cubic is reducible. This completes the proof of Lemma (7.1).

*Proof of Lemma (7.2)*

*Case 1.*  $r = 2$ . We consider a family of regular algebras over  $k[[t]]$ . Let  $I$  denote the ideal of  $k[[t]][x, y]$  defining this family. Since the Hilbert function is constant on  $\text{Spec } k[[t]]$ , there is an ideal basis  $f_1, f_2 \in I_3$ . We can obtain such a basis from a basis over the field  $k((t))$  as follows: clear the denominator in  $f_1$  to make it integral and primitive. Then choose  $f_2$  to be integral and primitive (modulo  $f_1$ ). This may involve adding a multiple of  $f_1$  to  $f_2$ .

It will turn out that the ordering of variables  $\{x, u, y\}$  can always be used, so that over  $k((t))$  the relations can be put into the form (7.3), with  $a_1 = a_2 = b_4 = 0, a_3 = 1$ . Applying the above construction yields an integral basis for  $I_3$  of the form

$$\begin{aligned} f_1 &= b_1 x^3 + b_2 x^2 y + b_3 x y x + b_5 y x^2, \\ f_2 &= c_1 x^3 + c_2 x^2 y + c_3 x y x + c_4 x y^2 + c_5 y x^2 + c_6 y x y + c_7 y^2 x, \end{aligned} \quad (7.12)$$

with  $b_i, c_i \in k[[t]]$ . The monomials occurring in  $M$  are

$$\begin{pmatrix} (x^2, xy, yx) & (x^2) \\ (x^2, xy, yx, y^2) & (x^2, xy, yx) \end{pmatrix}.$$

The linear vector  $L$  such that  $w = Lf$  is obtained by clearing the denominator in the one over the field  $k((t))$ . From the fact that  $m_{21}$  involves  $y^2$  over  $k((t))$ , it follows, as in the proof of (7.4), that

$$L = (p_1 x + p_2 y, q_1 x), \quad (7.13)$$

and that we have

$$\begin{aligned} LM_1 &= *x^3 + *x^2 y + *x y x + q_1 c_7 x y^2 + p_2 b_1 y x^2 \\ &\quad + *y x y + p_2 b_5 y^2 x, \\ LM_2 &= (p_1 b_2 + q_1 c_2) x^3 + q_1 c_4 x^2 y + *x y x + p_2 b_2 y x^2. \end{aligned} \quad (7.14)$$

Therefore

$$LM_2 = (\text{const})f_1, \quad (7.15)$$

while  $LM_1$  is a linear combination of  $f_1, f_2$ .

We will now check that if an algebra  $A$  is of the above form and is regular, then  $b_5$  and  $c_7$  are not zero. It will follow that  $c_5$  can be eliminated from  $f_2$ , and that the algebra obtained by setting  $t=0$  is a skew polynomial ring of the form (7.3), with  $a_1=a_2=0, a_3=1$ .

The condition that  $A$  be regular implies that  $p_2, q_1$  are not zero. From  $c_7=0$  and (7.14), it follows that  $b_5=0$ . From  $b_5=0$ , we conclude  $c_7=c_4=b_2=0$ . If also  $b_3=0$ , the algebra is degenerate (4.2). If  $b_3 \neq 0$ , we kill  $b_1$  by a substitution of the form  $y \mapsto y + *x$ , and  $c_3$  by a substitution  $f_2 \mapsto f_2 + *f_1$ . Then it follows from (7.14), (7.15) that  $c_2=0, c_5=0$ . The non-zero coefficients are thus  $b_3, c_1, c_6$ . This algebra has defining equations  $xyx, cx^3+yxy$  and is degenerate (4.2).

*Case 2.  $r=3$ .* In this case, the replacements (6.4) over  $k((t))$  lead to integral equations over  $k[[t]]$  of the form

$$\begin{aligned} f_1 &= a_1x^2 + a_2xy + a_3xz + a_4yx, \\ f_2 &= b_1x^2 + b_2xy + b_3xz + b_4yx + b_5y^2 + b_6yz + b_7zx, \\ f_3 &= c_1x^2 + c_2xy + c_3xz + c_4yx + c_5y^2 + c_6yz + c_7zx + c_8zy. \end{aligned} \quad (7.16)$$

The monomials occurring in  $M$  are

$$\begin{pmatrix} (x, y) & (x) & (x) \\ (x, y, z) & (x, y) & (x, y) \\ (x, y, z) & (x, y, z) & (x, y) \end{pmatrix},$$

and the fact that the coefficient of  $z$  in  $m_{21}$  and  $m_{32}$  is not identically zero shows that  $L$  has the form

$$L = (p_1x + p_2y + p_3z, q_1x + q_2y, r_1x + r_2y).$$

Since the coefficient of  $y$  in  $m_{11}$  is not zero,  $f_1, f_2$  are linear combinations of  $LM_2, LM_3$ . We set  $t=0$ . If the resulting algebra is not a skew polynomial ring, then one of the coefficients  $a_4, b_7, c_8$  vanishes. If  $c_8=0$ , we can replace  $f_2$  by a linear combination to kill  $b_7$ . Thus  $c_8=0$  reduces to the case  $b_7=0$ . If  $a_4=0$ , then  $LM$  does not involve  $zy$ , hence  $c_8=0$ , and  $b_7=0$  again.

Assume that  $b_7=0$ . Then  $f_1, f_2$  do not involve  $zx$ , hence  $LM_2, LM_3$  also do not, and  $a_2=a_3=0$ . If also  $a_4=0$ , the algebra is degenerate (4.2). Thus  $a_4 \neq 0$ , and we can kill  $a_1$  by a substitution  $y \mapsto y + *x$ , and  $b_4, c_4$  by

adjusting  $f_2, f_3$ . Then  $LM$  does not involve  $zx$ , hence  $c_7 = 0$ . On the other hand, the coefficient of  $zy$  in  $LM_1$  is  $r_3 a_4 \neq 0$ . Thus  $c_8 \neq 0$ .

Since  $f_1$  does not involve  $z$ ,  $LM_3 = (\text{const})f_1 = (\text{const})yx$ . We have

$$\begin{aligned} LM_3 = & (q_1 b_3 + r_1 c_3) x^2 + (q_1 b_6 + r_1 c_6) xy \\ & + (q_2 b_3 + r_2 c_3) yx + (q_2 b_6 + r_2 c_6) y^2. \end{aligned} \quad (7.17)$$

Since the entries of  $L$  are linearly independent,  $b_6 = c_6 = 0$ .

Next,  $LM_2$  is a linear combination of  $f_1, f_2$ , and the coefficient of  $yz$  in  $LM_2$  is  $r_2 c_8$ . Since  $yz$  does not occur in  $f_i$  and  $c_8 \neq 0$ ,  $r_2 = 0$ . It follows from (7.17) that  $b_3 \neq 0$ , so we can kill  $b_1, b_2$  by a change of variable  $z \mapsto z + *y + *x$ , and also kill  $c_3$ , and then it follows from (7.17) that  $q_1 = 0$ . At this point,

$$LM_2 = r_1 c_2 x^2 + r_1 c_5 xy + \cdots, \quad (7.18)$$

hence  $c_2 = c_5 = 0$ . This algebra has ideal basis  $yx, xz + by^2, cx^2 + zy$ . Interchanging  $x, y$ , we obtain the degenerate algebra (4.2)(iv). This completes the proof of Lemma (7.2).

To finish the proof of Theorem (6.11), we have to verify that for each type the generic points of the locus  $j = \infty$  represent skew polynomial rings. We will check types  $A, B$ , and omit the verification for  $S_i$ .

The generic binary quartic with  $j = \infty$  has a double root, and can therefore be put into the form  $x^4 + x^2 y^2$ . Correspondingly, the generic algebra of type  $A$ ,  $r = 2$  can be put into the form  $w = aw_2 + bw'_2 + w_4$ , and

$$\begin{aligned} f_1 &= x^3 + axy^2 + ay^2x + byxy, \\ f_2 &= ax^2y + ayx^2 + bxyx. \end{aligned} \quad (7.19)$$

Substituting  $yx = u$  into  $f_i$ , we obtain the replacements

$$\begin{aligned} ux &= -x^2y - a^{-1}bxu, \\ yu &= -a^{-1}x^3 - xy^2 - a^{-1}buy, \end{aligned} \quad (7.20)$$

which have the required form (7.3).

The generic ternary cubic with  $j = \infty$  represents a nodal cubic, and can be put into the form  $x^3 + xyz + y^3$ . Correspondingly, the generic algebra of type  $A$ ,  $r = 3$  has  $w = w_{300} + w_{030} + aw_{111} + bw'_{111}$ , and

$$\begin{aligned} f_1 &= x^2 + ayz + bzy, \\ f_2 &= y^2 + azx + bxz, \\ f_3 &= axy + byx. \end{aligned} \quad (7.21)$$

This provides replacements of the required form (6.4).

For type  $B$ ,  $G_Q = Gl_2 \times Gl_1$ , and the  $Gl_2$ -representation on  $W_Q$  is isomorphic to  $S^3 V_2^* \oplus V_2^*$ . Let  $q$  be the image of  $w$  under the map  $S^3 V_2^* \oplus V_2^* \rightarrow S^4 V_2^*$ . It is easily seen that  $j(w) = j(q)$ . Hence there are two generic points of the locus  $j = \infty$ , corresponding to a double root of the binary cubic, and a coincidence of root of the binary cubic and the linear form. The corresponding algebras are represented by  $w = w_{210} + w_{102} + w_{012}$  and  $w = w_{210} + w_{120} + w_{102}$ , respectively, with equations

$$\begin{aligned} f_1 &= xy + yx - z^2, \\ f_2 &= x^2 - z^2, \\ f_3 &= xz - zx + yz - zy, \end{aligned} \tag{7.22}$$

and

$$\begin{aligned} f_1 &= xy + yx + y^2 - z^2, \\ f_2 &= x^2 + xy + yx, \\ f_3 &= xz - zx. \end{aligned} \tag{7.23}$$

The first is put into standard skew polynomial form (6.4) by the change of variable  $x \rightarrow x + y$ ,  $z \rightarrow x - y$ ,  $y \rightarrow z$ , and the second by the change of variable  $x \rightarrow x$ ,  $y \rightarrow y + z$ ,  $z \rightarrow z$ .

In each of the above cases, the overlap  $[y, u, x]$  or  $[z, y, x]$  must be checked. Once this is done, the ring is proved to be skew, and the Hilbert function (1.15) shows that it is a regular algebra.

## 8. CONDITIONS FOR FINITENESS: SKEW POLYNOMIAL CASE

In this section, we describe the regular algebras of types  $S_1$ ,  $S_2$ ,  $S'_2$ , with  $r = 2$ , which are finite modules over their center. This includes all the algebras having two generators which are skew-polynomial rings, except for those of type  $A$ . The results are summarized in the following theorem.

**THEOREM (8.1).** *Let  $A$  be a regular algebra of type  $S_1$ ,  $S_2$ , or  $S'_2$ , with  $r = 2$ . Then  $A$  is a finite module over its center if and only if it is isomorphic to one of the algebras listed in Table (3.9), and one of the following holds:*

- (i)  $A$  is of type  $S_1$ , and the roots of the polynomial

$$t^2 + at + \alpha \tag{8.2}$$

are distinct roots of unity,

- (ii)  $A$  is of type  $S_2$ , and  $\alpha$  is a root of unity, or

- (iii)  $A$  is of type  $S'_2$ .



Note that if the roots of (8.2) are roots of unity, then  $\alpha$  is also a root of unity. Looking over Table (3.9), we find the following corollary:

**COROLLARY (8.3).** *If a regular algebra of global dimension 3 with two generators is finite over its center, then the matrix  $Q$  (2.1) is an element of finite order of  $Gl_2$ .*

The proof of Theorem (8.1) consists of two parts: the verification that the algebras listed are finite over their centers, and the proof that the remaining algebras are not finite. We include the specializations of the forms listed in Table (3.9), so these must be gathered together from the discussion of Section 4. The specializations of type  $S_1$  which do not have the generic form are the non-diagonal forms (4.20), and the algebra

$$\alpha = -1, \quad w = w_2 + w_3, \quad (8.4)$$

which corresponds to the orbit  $[x, y] + x^2$  in  $\otimes^2$ , as in (4.10). All specializations of type  $S_2$  have the generic form.

*Finiteness.* Suppose that  $A$  is of type  $S_1$ , with the generic form (3.9). The equations defining  $A$  are

$$\begin{aligned} \alpha xy^2 + \alpha^2 y^2 x + \alpha xyxy &= 0, \\ x^2 y + \alpha yx^2 + \alpha xyx &= 0. \end{aligned} \quad (8.5)$$

To standardize notation, we interchange  $x$  and  $y$ , obtaining the complete set of replacements

$$\begin{aligned} y^2 x &= -\alpha yxy - \alpha xy^2, \\ yx^2 &= -\alpha xyx - \alpha x^2 y. \end{aligned} \quad (8.6)$$

Let  $\lambda, \lambda'$  be the roots of (8.2). The substitution

$$u = yx - \lambda xy \quad (8.7)$$

into (8.6) yields the skew polynomial relations

$$\begin{aligned} yx &= \lambda xy + u, \\ yu &= \lambda' uy, \\ ux &= \lambda' xu \end{aligned} \quad (8.8)$$

in the ordered set of variables  $\{x, u, y\}$ .

Assume that  $\lambda \neq \lambda'$  and  $\lambda^n = \lambda'^n = 1$ . We claim that then  $x^n, y^n, u^n$  are in the center  $Z = Z(A)$ . This will prove that  $A$  is finite over  $Z$ . Equation (8.8)

shows that  $u^n \in Z$  and that  $[y^n, u] = [x^n, u] = 0$ . It remains to show  $[y^n, x] = 0$  and  $[x^n, y] = 0$ , for which the proofs are the same. We will exhibit the first.

The element  $y^r x$  has the form

$$y^r x = a_r x y^r + b_r y x y^{r-1}, \quad (8.9)$$

where  $(a_0, b_0) = (1, 0)$  and  $(a_r, b_r)$  satisfies the recursive relation

$$\begin{pmatrix} 0 & -\alpha \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} a_r \\ b_r \end{pmatrix} = \begin{pmatrix} a_{r+1} \\ b_{r+1} \end{pmatrix}. \quad (8.10)$$

This is immediately verified by induction. The eigenvalues of the left hand matrix  $P$  of (8.10) are  $\lambda, \lambda'$ . Therefore  $P^n = 1$ , so  $(a_n, b_n) = (1, 0)$  and  $[y^n, x] = 0$ , as required.

Assume that  $A$  is of type  $S_2$  and  $\alpha^n = 1$ . Then interchanging  $x, y$ , we obtain the complete set of replacements

$$\begin{aligned} y^2 x &= \alpha x y^2, \\ y x^2 &= -\alpha x^2 y. \end{aligned} \quad (8.11)$$

It is easily seen that  $A$  is finite over the central subring generated by

$$x^{4n}, y^{2n}, (xy)^{2n} + (yx)^{2n}. \quad (8.12)$$

If  $A$  has type  $S'_2$ , then the replacements are

$$\begin{aligned} y^2 x &= -x y^2 - x^3, \\ y x^2 &= x^2 y. \end{aligned} \quad (8.13)$$

This algebra has rank  $2^2$  over the central subring generated by

$$x^2, y^4 + x^2 y^2, (xy)^2 + (yx)^2. \quad (8.14)$$

The method we use to prove that the remaining algebras are not finite over their centers is to find a suitable quotient  $\bar{A}$  which is not finite, and which can be analyzed easily. In every case except one, the quotient

$$\bar{A} = A/Ax^2A \quad (8.15)$$

does the job. The verifications are very similar, so we will do some examples which show the possible structures of  $\bar{A}$ .

Consider the case that  $A$  is of type  $S_1$ , as above, and  $a \neq 0$ . The defining equations for  $\bar{A}$  give a complete set of replacements

$$\begin{aligned}x^2 &= 0, \\xyx &= 0, \\y^2x &= -ayxy - \alpha xy^2.\end{aligned}\tag{8.16}$$

Note that these equations are bihomogeneous; hence  $\bar{A}$  is bigraded by degrees in  $x, y$ , and the nilradical  $\bar{N} = N(\bar{A})$  is generated by the residue of  $x$ . If  $\bar{A}$  is finite over its center  $\bar{Z}$ , then  $\bar{Z}$  must contain a homogeneous element of positive degree which is not nilpotent, i.e., a power of  $y$ . Now  $y^n x$  can be calculated by the recursive formula (8.9). So  $[y^n, x] = 0$  if and only if  $(1, 0)'$  is an eigenvector for  $P^n$ , with eigenvalue 1. Since  $(1, 0)'$  is not an eigenvector for  $P$  because  $(a_1, b_1) = (0, 1)$ , it follows that  $y^n$  is central only if  $P^n = 1$ , which happens precisely when the roots  $\lambda, \lambda'$  of (8.2) are distinct  $n$ th roots of unity. Otherwise, the center  $\bar{Z}$  is contained in  $k \oplus \bar{N}$ , and  $\bar{A}$  is not finite over  $\bar{Z}$ .

In case of a Jordan 2-block,  $\alpha = -1$  (4.20),  $\bar{A} = A/Ax^2A$  has the complete set of replacements

$$\begin{aligned}x^2 &= 0, \\y^2x &= bxyx + xy^2,\end{aligned}\tag{8.17}$$

where  $b = -1 - a$ . Assume that  $b \neq 0$ . The reduced basis for  $A_r$  consists of elements of the form  $(yx)^j y^k$  and  $x(yx)^j y^k$ , let us denote them by

$$z_0 = y^r, \quad z_1 = xy^{r-1}, \quad z_2 = yxy^{r-2}, \quad z_3 = xyxy^{r-3}, \dots, \tag{8.18}$$

suppressing notation for the degree  $n$ . It is not hard to compute the formulas

$$\begin{aligned}[z_{2j+1}, y] &= z_{2j+1} - z_{2j+2}, \\[z_0, y] &= 0, \\[z_{2j}, y] &= -z_{2j-1} + z_{2j} - jbz_{2j+1} \quad (j > 0).\end{aligned}\tag{8.19}$$

Let  $p = \sum_j c_j z_j$ , and  $[p, y] = \sum d_j z_j$ . Put  $c_j = 0$  if  $j > n$ . Then

$$\begin{aligned}d_0 &= 0, \quad d_1 = c_1 - c_2, \\d_{2j} &= -c_{2j-1} + c_{2j}, \\d_{2j+1} &= -jbc_{2j} + c_{2j+1} - c_{2j+2} \quad (j > 0).\end{aligned}\tag{8.20}$$

It follows that  $[p, y] = 0$  only if  $p = c_0 z_0 = c_0 y^r$ . Therefore the center  $\bar{Z}$  of  $\bar{A}$  is a subring of  $k[y]$ . Since  $\bar{A}$  has quadratic growth, it is not finite over  $\bar{Z}$ .

The exceptional case is that  $b = 0$ . In this case we use the quotient

$$\bar{A} = A/(AxA)^3, \quad (8.21)$$

for which the set of replacements

$$\begin{aligned} x^3 &= x^2 y x = x y x y x = 0, \\ y^2 x &= x y^2 - x^2 y, \\ y x^2 &= x^2 y \end{aligned} \quad (8.22)$$

is complete. Therefore if  $r \geq 4$ ,  $\bar{A}$  has the reduced basis

$$\begin{aligned} z_0 &= y^4, & z_1 &= x y^{r-1}, & z_2 &= y x y^{r-2}, & z_3 &= x^2 y^{r-2}, \\ z_4 &= x y x y^{r-3}, & z_5 &= y x y x y^{r-4}. \end{aligned} \quad (8.23)$$

The following brackets are easily computed for  $r = 2j$  even,

$$\begin{aligned} [z_0, x] &= -j z_3, & [z_1, x] &= z_4 - z_3, & [z_2, x] &= z_3 - z_4, \\ [z_i, x] &= 0 & \text{if } i &= 3, 4, 5. \end{aligned} \quad (8.24)$$

If  $p = \sum c_i z_i$  is central, then

$$[p, x] = (-j c_0 - c_1 + c_2) z_3 + (c_1 - c_2) z_4 = 0.$$

Hence  $c_0 = 0$ , which shows that  $p$  is in the nilradical  $\bar{N}$  of  $\bar{A}$ . Therefore  $\bar{Z}_{\text{even}} \subset k \oplus \bar{N}$ , and  $\bar{A}$  is not finite over  $\bar{Z}$ .

## 9. FINITENESS OF TYPES $B, E, H$

The algebras of types  $B, E, H$  are all finite over their centers. We verified this fact by computer, and list our results here for reference.

*Type E,  $r = 2$ .* This algebra is defined by the equations

$$\begin{aligned} y^3 + x^3 &= 0, \\ y^2 x + \zeta y x y + \zeta^2 x y^2 &= 0, \end{aligned} \quad (9.1)$$

where  $\zeta = \zeta_3$ . It has rank  $9^2$  over its center, which is polynomial ring generated by the elements

$$x^3, (yx - xy)^3, (yx^2 + \zeta^2 xyx + \zeta x^2 y)^3. \quad (9.2)$$

*Type H,  $r = 2$ .* This algebra is defined by the equations

$$\begin{aligned} -\zeta^3 y^3 + \zeta^2 yx^2 + \zeta xyx + x^2 y &= 0, \\ y^2 x - \zeta yxy + \zeta^2 xy^2 + \zeta^3 x^3 &= 0, \end{aligned} \quad (9.3)$$

where  $\zeta = \zeta_8$ . It has rank  $(16)^2$  over its central subring generated by elements of degrees 4, 8, 16.

*Type E,  $r = 3$ .* This algebra is defined by the equations

$$\begin{aligned} \zeta zx + \zeta^5 y^2 + xz &= 0, \\ \zeta^2 z^2 + yx + \zeta^4 xy &= 0, \\ zy + \zeta^7 yz + \zeta^8 x^2 &= 0, \end{aligned} \quad (9.4)$$

where  $\zeta = \zeta_9$ . This algebra is of rank  $2 \cdot 9^2$  over a central subring generated by elements of degrees 3, 6, 9.

*Type H,  $r = 3$ .* This algebra is defined by the equations

$$\begin{aligned} -y^2 + x^2 &= 0, \\ -\zeta z^2 + yx - xy &= 0, \\ zy + \zeta yz &= 0, \end{aligned} \quad (9.5)$$

where  $\zeta = \zeta_4$ . It has rank  $2 \cdot 8^2$  over the central subring generated by

$$x^4, (yx)^2 + (xy)^2, (zx)^4 + (xz)^4. \quad (9.6)$$

*Type B.* This is a 1-parameter family of algebras, all of which have rank  $4^2$  over their centers, and the centers are polynomial rings. Most algebras can be presented by equations corresponding to  $w$  as in Table (3.11). These equations are

$$\begin{aligned} -z^2 - y^2 + yx + xy &= 0, \\ -az^2 + yx + xy + x^2 &= 0, \\ azy + zx - ayz - xz &= 0. \end{aligned} \quad (9.7)$$

Two special algebras

$$\begin{aligned} w &= w_{210} + w_{102} + w_{012}, \\ w &= w_{210} + w_{012} \end{aligned} \quad (9.8)$$

have different defining equations.

The center of (9.7) is the polynomial ring generated by the elements

$$x^2, y^2, (zx)^2 + (xz)^2. \quad (9.9)$$

We do not have much insight into the structure of the algebras considered here, except that they are finite modules over their centers. The one case in which we have more information is for the algebra of type  $E$ ,  $r = 2$ :

EXAMPLE (9.10). A skew polynomial ring in four variables of which type  $E$ ,  $r = 2$ , is a quotient. Consider the algebra  $C$  defined by the ordered set of variables  $t, w, x, y$  of degrees  $(3, 2, 1, 1)$ , with the skew polynomial relations

$$\begin{aligned} yx &= xy + w, & yt &= \zeta^2 ty - \zeta^2 w^2, \\ yw &= \zeta^2 wy, & xt &= \zeta tx, \\ xw &= \zeta^2 wx - t, & wt &= \zeta^2 tw, \end{aligned}$$

where  $\zeta = \zeta_3$ . This algebra is of rank  $9^2$  over the central subring generated by the elements  $x^3, y^3, w^3, t^3$ , and  $A = C/(y^3 + x^3)$  is of type  $E$ .

## 10. CONJECTURAL DESCRIPTION OF ALGEBRAS OF TYPE $A$

We first consider the case that  $A$  has two generators:  $r = 2$ . In order to work with homogeneous equations, we introduce an extra coefficient  $c$  into the standard form (3.9) for type  $A$ , obtaining

$$w = cw_0 + cw_4 + aw_2 + bw'_2. \quad (10.1)$$

The equations defining  $A$  become

$$\begin{aligned} ay^2x + byxy + axy^2 + cx^3 &= 0, \\ cy^3 + ayx^2 + bxyx + ax^2y &= 0. \end{aligned} \quad (10.2)$$

Every point  $p = (a, b, c)$  in the projective plane  $\mathbf{P}^2$  determines an algebra, possibly a degenerate one.

The associated binary quartic  $S(w)$  has the form (4.8), where

$$6u = (4a + 2b)/c.$$

Its  $j$ -invariant is

$$j = \frac{12^3(3u^2 + 1)^3}{(3u + 1)^2(3u - 1)^2}. \quad (10.3)$$

It is unfortunately hard to find references giving this expression in the form

that we want. The normal form (4.8) for the binary quartic is derived in Clebsch [6], p. 166, and an invariant is exhibited on p. 168, formula (7). However, Clebsch's invariant is  $6j/(j-12^3)$ .

The algebra  $A$  has a central element of degree 4, found by computer:

$$\begin{aligned} \text{central}_4 = & b(c^2 - a^2) yxyx + a(a^2 - b^2) yx^2y \\ & - a(c^2 - a^2) x^2y^2 - c(a^2 - b^2) x^4. \end{aligned} \quad (10.4)$$

The existence of a central element is of practical importance, because the consequence of killing it can be understood, and doing so simplifies computation. Our conjectures are based on an analysis of the ring

$$\bar{A} = A/(\text{central}_4) \quad (10.5)$$

obtained in this way.

For generic values of  $(a, b, c)$ , the ring  $\bar{A}$  has two replacements in each degree  $n \geq 3$ . In degree 3 they are

$$\begin{aligned} cy^3 &= -ayx^2 - bxyx - ax^2y, \\ ay^2x &= -byxy - axy^2 - cx^3, \end{aligned} \quad (10.6)$$

and in higher degree  $n$  they have the form

$$\begin{aligned} s_1 yx^r yy &= t_1 yx^r xx + w_1, \\ s_2 yx^r yx &= t_2 yx^r xy + w_2, \end{aligned} \quad (10.7)$$

where  $r = n - 3$ , the coefficients  $s_i = s_i(n)$ ,  $t_i = t_i(n)$  are scalars, and  $w_i = w_i(n) \in x\bar{A}$ . We retain the coefficients  $s_i$  in order to avoid denominators.

The replacements (10.7) can be obtained in degree  $> 4$  by calculating the overlaps

$$[yx^r, y^2, x], \quad [yx^r, y^2, y]. \quad (10.8)$$

This calculation results in recursive formulas for  $s_i, t_i$ . We have

$$\begin{aligned} s_1(4) &= c(b^2 - a^2), & s_2(4) &= b(a^2 - c^2), \\ t_1(4) &= a(c^2 - a^2), & t_2(4) &= a(a^2 - b^2). \end{aligned} \quad (10.9)$$

Denote  $s_i(n)$ ,  $t_i(n)$ ,  $s_i(n+1)$ ,  $t_i(n+1)$  by  $s_i$ ,  $t_i$ ,  $s'_i$ ,  $t'_i$ , respectively. Then for  $n \geq 4$ ,

$$\begin{aligned} s'_1 &= s_1(bt_2 + as_2), & s'_2 &= s_1(at_2 + bs_2), \\ t'_1 &= -s_2(at_1 + cs_1), & t'_2 &= -s_2(ct_1 + as_1). \end{aligned} \quad (10.10)$$

For example, the first overlap (10.8), computed modulo  $x\bar{A}$ , is

$$\begin{aligned} as_1s_2(yx^ry^2)x - as_1s_2yx^r(y^2x) \\ = at_1s_2yx^{r+3} + (bs_1s_2(yx^ryx)y + as_1s_2yx^{r+1}y^2 + cs_1s_2yx^{r+3}) \\ = s_2(at_1 + cs_1)yx^{r+3} + s_1(bt_2 + as_2)yx^{r+1}y^2. \end{aligned}$$

This determines  $s'_1, t'_1$  as in (10.10).

The formulas are particularly simple for the ratios  $u_i(n) = t_i(n)/s_i(n)$ . Starting with the values

$$u_1(4) = \frac{a(c^2 - a^2)}{c(b^2 - a^2)}, \quad u_2(4) = \frac{a(b^2 - a^2)}{b(c^2 - a^2)}, \quad (10.11)$$

they are

$$u'_1 = -\frac{au_1 + c}{bu_2 + a}, \quad u'_2 = -\frac{cu_1 + a}{au_2 + b}. \quad (10.12)$$

In other words, if  $u_i(n) = u_i$ , then  $u_i(n+1) = u'_i$ .

The case of three variables,  $r=3$ , is very similar to the previous one. We homogenize the standard form (3.11) using the coefficient  $c$ , to obtain the vector

$$w = cw_{300} + cw_{030} + cw_{003} + aw_{111} + bw'_{111}, \quad (10.13)$$

and the defining equations

$$\begin{aligned} axy + byx + cz^2 &= 0, \\ ayz + bzy + cx^2 &= 0, \\ azx + bxz + cy^2 &= 0. \end{aligned} \quad (10.14)$$

The associated ternary cubic  $S(w)$  has the form (4.13), where

$$6u = (3a + 3b)/c,$$

and

$$j = \frac{-2^{12} \cdot 3^3 (u^3 - 1)^3 u^3}{(8u^3 + 1)^3}. \quad (10.15)$$

A reference for this formula is Weber [16, pp. 406–407]. In Weber's notation,

$$j = 2^{12} \cdot 3^3 \cdot S^3/D. \quad (10.16)$$



The algebra  $A$  has the central element

$$\begin{aligned} \text{central}_3 = & c(c^3 - b^3) y^3 + b(c^3 - a^3) yxz \\ & + a(b^3 - c^3) xyz + c(a^3 - c^3) x^3. \end{aligned} \quad (10.17)$$

Let  $\bar{A} = A/(\text{central}_3)$ . For generic values of the parameters, the ring  $\bar{A}$  has three replacements in each degree  $n \geq 2$ . In degree 2 they are

$$\begin{aligned} cz^2 &= -byx - axy, \\ bzy &= -ayz - cx^2, \\ azx &= -cy^2 - bxz, \end{aligned} \quad (10.18)$$

and for degree  $n \geq 3$  they have the form

$$\begin{aligned} s_1 yx^r yz &= t_1 yx^r xx + w_1, \\ s_2 yx^r yy &= t_2 yx^r xz + w_2, \\ s_3 yx^r yx &= t_3 yx^r xy + w_3, \end{aligned} \quad (10.19)$$

where  $r = n - 3$ , and the coefficients  $s_i = s_i(n)$  and  $t_i = t_i(n)$  are scalars. Here  $w_i \in x\bar{A}$  if  $n = 3$  and  $w_i \in x^2\bar{A}$  if  $n \geq 4$ . The coefficients can be calculated recursively by reducing the overlaps

$$[yx^r y, z, z], \quad [yx^r y, z, y], \quad [yx^r y, z, x]. \quad (10.20)$$

Let  $u_i(n) = t_i(n)/s_i(n)$ . Then starting with the initial values

$$(u_1(3), u_2(3), u_3(3)) = \left( \frac{a^2(c^3 - b^3)}{c^2(b^3 - a^3)}, \frac{b(a^3 - c^3)}{c(c^3 - b^3)}, \frac{a(b^3 - a^3)}{b(a^3 - c^3)} \right), \quad (10.21)$$

the recursive formula is

$$(u'_1, u'_2, u'_3) = \left( \frac{c^2 u_2 - abu_1}{b^2 u_3 - acu_2}, \frac{b^2 u_2 - acu_1}{a^2 u_3 - bcu_2}, \frac{a^2 u_2 - bcu_1}{c^2 u_3 - abu_2} \right). \quad (10.22)$$

**THEOREM (10.23).** *Suppose that  $a, b, c$  are not zero and that none of the coefficients  $s_i(n)$  and  $t_i(n)$  is zero. Then the algebra  $A$  is regular, and is not finite over its center.*

*Note.* Although it was not easy to see that there are algebras to which Theorem (10.23) applies, Tate has recently demonstrated that they do exist.

*Proof. Case 1,  $r = 2$ .* Let  $a_n$  denote the expected Hilbert function (1.15), and let  $a'_n$  denote the actual Hilbert function of  $A$ . There exists a regular algebra of type  $A$ , because the algebra  $a = 1, b = c = 0$  is a skew polynomial

ring, which is regular. It follows easily that the generic algebra of type  $A$  is regular, and that the Hilbert function of  $A$  satisfies the inequality

$$a'_n \geq a_n. \quad (10.24)$$

Also, let  $\bar{a}_n$  denote the Hilbert function of the algebra  $\bar{A} = A/(\text{central}_4)$ . Computation of the overlaps (10.8) does not show the replacements (10.7) form a complete list, but it is easily verified that the reduced monomials with respect to this list are the monomials

$$x^i, x^i y x^j, x^i y x^j y. \quad (10.25)$$

Since there are  $2n$  such monomials in degree  $n$ , we have the inequality

$$2n \geq \bar{a}_n. \quad (10.26)$$

Note that  $2n$  is the Hilbert function predicted by an exact sequence

$$0 \rightarrow A \xrightarrow{\text{central}_4} A \rightarrow \bar{A} \rightarrow 0, \quad (10.27)$$

provided that  $A$  is regular:

$$2n = a_n - a_{n-4}. \quad (10.28)$$

Now  $\text{central}_4$  may be a zero divisor in  $A$ , and if so, then (10.27) will not be exact on the left. The correct conclusion is the inequality

$$\bar{a}_n \geq a'_n - a'_{n-4}. \quad (10.29)$$

Moreover, if  $\text{central}_4$  is a zero divisor, then this inequality is strict for some  $n$ .

Assume that  $a'_r = a_r$  for all  $r < n$ . Then combining the above inequalities, we obtain

$$a_n - a_{n-4} \geq \bar{a}_n \geq a'_n - a'_{n-4} \geq a_n - a_{n-4}.$$

This shows that  $a'_n = a_n$ . By induction, each of the inequalities (10.24) is an equality, and  $\text{central}_4$  is not a zero divisor in  $A$ .

To complete the proof that  $A$  is regular, we still have to show that the map  $x'$  of (1.5)(ii) is injective. Now if  $\alpha \in \ker x'$ , then  $\alpha\beta = 0$  for all  $\beta \in A$  of positive degree. But we just saw that  $\text{central}_4$  is not a zero divisor. So  $\alpha = 0$ .

We now show that  $\bar{A}$  is not finite over its center by computing the com-

mutator  $[p, x]$ , where  $p$  is a linear combination of the elements (10.25) of some degree  $n$ . We may assume  $n \geq 4$ . Then

$$\begin{aligned} [x^n, x] &= 0, \\ [x^i y x^j, x] &= x^i y x^{j+1} + *, \\ [x^i y x^j y, x] &= u_2 x^i y x^{j+1} y + *, \end{aligned} \quad (10.30)$$

where  $u_2 = u_2(j+3)$ , and where the symbol  $*$  denotes earlier terms in the lexicographic ordering. Since the leading terms on the right side of (10.30) are all different, it follows that  $[p, x] = 0$  only if  $p$  is a multiple of  $x^n$ . But  $[x^n, y] = x^n y - y x^n$  is a reduced polynomial, hence is not zero. Therefore the center of  $\bar{A}$  has no element of degree  $\geq 4$ . So  $\bar{A}$  is not finite over its center.

*Case 2,  $r = 3$ .* The proof that  $A$  is regular parallels the previous case exactly, and so we omit it. However, the commutator calculation must be done more carefully. The reduced monomials with respect to the replacements (10.19) are

$$x^i, x^i z, x^i y x^j, x^i y x^j y, x^i y x^j z, \quad (10.31)$$

and the commutators with  $x$  are

$$\begin{aligned} [x^i, x] &= 0, \\ [x^i z, x] &= -(c/a) x^i y^2 - ((a+b)/a) x^i z, \\ [x^i y x^j, x] &= x^i y x^{j+1} - x^{i+1} y x^j, \\ [x^i y x^j y, x] &= u_3 x^i y x^{j+1} y - x^{i+1} y x^j y + *, \\ [x^i y x^j z, x] &= -((cu_2 + b)/a) x^i y x^{j+1} z - x^{i+1} y x^j z + *. \end{aligned} \quad (10.32)$$

Here, as before,  $u_i = u_i(j+3)$  and  $* \in x^{i+1} A$ .

Note that the dominant terms are all different. By assumption, the coefficients  $u_3, -c/a$  are defined and non-zero. If in addition  $cu_2(k) + b \neq 0$  for all  $k$ , we are done as before.

The possibility that  $cu_2(k) + b = 0$  for some  $k$  complicates the proof. The monomials  $x^i z, x^i y x^j, x^i y x^j y$  can still not appear in a polynomial which commutes with  $x$ , but we have to calculate the replacements (10.19) more carefully in order to eliminate the monomials  $x^i y x^j z$ . This calculation is rather complicated, and should be done on a computer. The result is

$$\begin{aligned} s_1 y x^r y z &= t_1 y x^r x x + p_1 x^{r+1} y x + q_1 x^{r+2} y, \\ s_2 y x^r y y &= t_2 y x^r x z + p_2 x^{r+1} y z + q_2 x^{r+2} x, \\ s_3 y x^r y x &= t_3 y x^r x y + p_3 x^{r+1} y y + q_3 x^{r+2} z. \end{aligned} \quad (10.33)$$

Using these formulas, the last commutators of (10.32) become

$$[x^i y x^{j'} z, x] = -((c u_2 + b)/a) x^i y x^{j+1} z - x^{i+1} y x^{j'} z \\ + * x^{i+j+1} y z + * x^{i+j+3}. \quad (10.34)$$

Let  $c_{j'}$  be the coefficient of  $x^i y x^{j'} z$  in  $[x^i y x^{j'} z, x]$ , where  $i' + j' = i + j + 1 = n - 1$ . If  $p$  has degree  $n$  then the relevant indices are  $0 \leq j \leq n - 2$ , and  $0 \leq j' \leq n - 1$ . The matrix  $(c_{j'})$  has the form

$$\begin{pmatrix} * & * & & & & \\ * & -1 & * & & & \\ & & -1 & & & \\ \vdots & & & \ddots & & * \\ * & & & & -1 & * \end{pmatrix}.$$

The rank of this matrix is at least  $n - 2$ . Therefore the space of polynomials of degree  $n$  with  $[p, x] = 0$  has dimension at most two, and it contains  $x^n$ . Since  $x^n$  is not central, the space of central polynomials of degree  $n$  has dimension at most one. So  $\bar{A}$ , which has quadratic growth, is not finite over its center.

Theorem (10.23) shows that the zeros of  $s_i(n)$  and  $t_i(n)$  contain all of the finite algebras. Unfortunately, each occurrence of a zero changes the replacements, which makes a uniform treatment of the cases difficult. This change in replacements seems to reflect a fundamental change in structure of the algebra, and is not simply an artifact of the method of computation.

Our computer evidence indicates that the zeros of  $u_2(n)$  contain all of the interesting loci. Let

$$D_n = \text{loci of zeros of } u_2(n) \text{ in } \mathbf{P}^2, \quad (10.35)$$

and let  $S$  denote the following set of points in  $\mathbf{P}^2$ :

$$S = \{(\zeta_r^i, \zeta_r^j, 1)\} \cup \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}, \quad 0 \leq i, j < r, \quad (10.36)$$

where  $\zeta_r$  is a primitive  $r$ th root of unity.

CONJECTURE (10.37). Let  $p = (a, b, c) \in \mathbf{P}^2$ , and let  $A = A_p$  be the associated algebra. Assume that  $c \neq 0$  if  $r = 3$ . Then

- (i)  $A$  is degenerate if and only if  $p \in S$ .

Assume in addition that  $A$  is regular.

- (ii)  $r = 2$ : If  $a = b$ , then  $A$  has rank  $3^2$  over its center and all higher  $n^2$  may occur as ranks.

- $r = 3$ . If  $a = b$ , then  $A$  has rank  $2^2$  over its center, and all higher  $n^2$  may occur as ranks.

(iii) The algebra  $A$  is finite over its center if and only if  $\bar{A}$  is. If  $A$  has rank  $n$  over its center, then  $x^n$  is central in  $\bar{A}$ .

(iv) If  $A$  is finite over its center, then  $\bar{A}$  has a finite complete set of replacements.

(v) The locus of points  $p \in \mathbf{P}^2$  for which  $A_p$  has rank  $n^2$  over its center and  $c \neq 0$  is of the form  $C_n - (S \cap C_n)$ , where  $C_n$  is a curve in  $\mathbf{P}^2$ . Moreover, if  $m \neq n$ , then

$$C_m \cap C_n \subset S.$$

(vi) Let  $D = \bigcup D_n$ ,  $C = \bigcup C_n$ ,  $L_a = \text{locus } (a=0)$ ,  $L_b = \text{locus } (b=0)$ . Then

$$D = C \cup L_a \cup L_b.$$

An interesting feature is that only one condition seems to be imposed by the requirement that the algebra be finite over its center. This is also the case in the examples of Section 8. A priori, one might expect the locus of finite algebras to have large codimension.

QUESTION (10.38). Let  $xR = k[[t]]$ , and let  $K = \text{Fract}(R)$ . Let  $A_R$  be a flat graded  $R$ -algebra such that  $A_K = A \otimes K$  and  $A_k = A \otimes k$  are regular. Assume that  $A_K$  is a finite algebra over its center  $Z(A_K)$ . It is true that  $A_k$  is finite over  $Z(A_k)$ ?

This seems quite likely. Assume that  $A$  is generated by homogeneous elements  $x_1, \dots, x_l$  of degrees  $(d_1, \dots, d_l)$ . Then the degree  $r$  part of  $Z(A)$  is

$$Z_r = \ker \left( A_r \xrightarrow{\sum [\ast, x_i]} \sum_i A_{r+d_i} \right).$$

Hence  $\dim_K(Z(A_K)_r) \leq \dim_k(Z(A_k)_r)$ . However, the hypothesis that  $A_k$  be regular is probably necessary.

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