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# Noncommutative Projective Schemes*

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An analogue of the concept of projective scheme is defined for noncommutative $\mathbb{N}$-graded algebras using the quotient category $\mathscr{C}$ of graded right $A$-modules modulo its full subcategory of torsion modules. We define proj $A = (\mathscr{C}, \mathscr{A}, s)$, where $\mathscr{A}$ is the object corresponding to the module $A_A$, and $s$ is the autoequivalence defined by the shift of degrees. The triples equivalent to proj $A$ for a right noetherian graded algebra $A$ are characterized in terms of a condition $\chi$ on extensions. 

*Contents.*



## 1. INTRODUCTION

The purpose of this paper is to study a version of projective geometry which can be defined for certain noncommutative rings. Some of our results were announced in [Ar]. We would like to thank Toby Stafford and Paul Smith for useful conversations on the subject and for many helpful comments.

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If $A$ is a commutative algebra which is finitely generated over a field $k$ then, as is well known, the geometric structure of the affine scheme Spec $A$ can be described in terms of the category of $A$-modules. For example, closed points of Spec $A$ correspond to isomorphism classes of simple modules. Similarly, a commutative graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is associated to a projective scheme Proj $A$, and the geometry of this scheme can be described in terms of the quotient category qgr $A =$ gr $A$/tors, where gr $A$ denotes the category of graded modules and tors denotes its subcategory of torsion modules. Thus rational points of Proj $A$ correspond to the classes of graded modules called *point modules*. A point module has the form $M = M_0 \oplus M_1 \oplus \cdots$ where $M_0$ generates $M$ and where $\dim_k M_n = 1$ for every $n \geqslant 0$ [ATV1, ATV2]. Since the module category is available when $A$ is not commutative, this observation provides a way to make a definition in a more general setting.

This idea of defining noncommutative geometry via the module category is of course not new. It has been suggested specifically by Manin [Ma, p. 83]. Also, the construction of the quotient category goes back to Serre, and it was studied in detail by Gabriel [Ga] in 1962. Clearly, projective geometry was one of the motivating examples for Gabriel's work. But the current interest in noncommutative rings generated by the discovery of quantum polynomial algebras such as the Skylyanin algebra [S1, S2, OF1, OF2, TV] suggest that it may be worthwhile to work out the special features of this particular case; hence this paper. This task has also been begun independently by Verevkin [V1, V2].

In his fundamental paper [Se], Serre proved a theorem which describes the quasi-coherent sheaves on a projective scheme in terms of graded modules as follows. Let $A$ be a finitely generated commutative graded $k$-algebra, and let $X$ be the associated projective scheme. Let coh $X$ denote the category of coherent sheaves on $X$, and let $\mathcal{O}_X(n)$ denote the $n$th power of the twisting sheaf on $X$ [Ha, p. 117]. Define a functor $\Gamma_*$:coh $X \to$ qgr $A$ by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d=-\infty}^{\infty} \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{O}_X(d)).$$

Serre's theorem [Se, Sect. 59, Prop. 7.8, p. 252; Ha, Prop II.5.15; EGA II, 3.3.5] asserts the following.

THEOREM. *Suppose that $A$ is generated over $k$ by elements of degree* 1. *Then $\Gamma_*$ defines an equivalence of categories* coh $X \to$ qgr $A$.

One of our purposes is to extend Serre's theorem to the noncommutative setting. Let $A$ be a graded algebra over a commutative noetherian ring $k$.

We define the associated projective scheme to be the pair $\operatorname{proj} A = (\operatorname{qgr} A, \mathscr{A})$, where $\operatorname{qgr} A$ is the quotient category introduced above and $\mathscr{A}$ is the object determined by the right module $A_A$. We also have the autoequivalence $s$ of $\operatorname{qgr} A$ defined by the shift of degrees in $\operatorname{gr} A$. Here the object $\mathscr{A}$ plays the role of the *structure sheaf* of $\operatorname{proj} A$ and $s$ the role of the *polarization* defined by the projective embedding. This definition is the same as is given by Verevkin.

For quantum polynomial rings in three variables, $\operatorname{proj} A$ is a quantum projective plane. The geometry of points and lines in these quantum planes was worked out by Van den Bergh, and it has been described in some detail [AV, Ar]. The study of the geometry of curves of higher degree is to a large extent unexplored.

It is important to note that Serre's theorem does not hold for all commutative graded algebras; i.e., the functor defined by $\Gamma_*$ need not be an equivalence. Our definition of $\operatorname{proj} A$ is compatible with the classical definition for commutative graded rings only under some additional hypotheses, such as that $A$ is generated in degree 1.

Having transformed Serre's theorem into the definition of $\operatorname{proj} A$, a fundamental problem now becomes to characterize those triples $X = (\mathscr{C}, \mathscr{A}, s)$ consisting of a $k$-linear abelian category $\mathscr{C}$, an object $\mathscr{A}$, and an autoequivalence $s$ of $\mathscr{C}$, which are equivalent to projective schemes $(\operatorname{proj} A, s)$ as defined above. This problem is solved in Section 4 for a restricted class of algebras.

The natural restrictions to put on the algebra $A$ are the ones which are customary in commutative algebraic geometry. Hence we may require that $A$ be a finitely generated $k$-algebra. But since finitely generated commutative algebras are noetherian, it seems acceptable to require that $A$ be right noetherian, and we make that assumption instead. (Verevkin works without this hypothesis.) This implies that $A$ is finitely generated, provided that $A_0$ is a finite $k$-module. In addition, there is another restriction to which we are led, and which we call $\chi_1$. This rather technical requirement is that $\underline{\operatorname{Ext}}^1(A/A_{\geqslant n}, M)$ be a bounded $k$-module for every finite right $A$-module $M$, and for $n \gg 0$. It is discussed in Section 3.

Given a triple $X = (\mathscr{C}, \mathscr{A}, s)$, we may mimic Serre's construction of a graded algebra. For an object $\mathscr{M}$ of $\mathscr{C}$, we define

$$\Gamma(\mathscr{M}) := \bigoplus_{d=-\infty}^{\infty} \operatorname{Hom}(\mathscr{A}, s^d(\mathscr{M})).$$

Then $\Gamma(\mathscr{A})$ is a graded $k$-algebra, and for every $\mathscr{M}$, $\Gamma(\mathscr{M})$ is a graded right module over $\Gamma(\mathscr{A})$ (see (4.0.3)). This provides a "representing functor" for the triple.

Theorem 4.5 describes the conditions on the triple which imply that $X$ is isomorphic to $(\operatorname{proj} A, s)$, where $A = \Gamma(\mathscr{A})_{\geqslant 0}$. Briefly, they are that $\mathscr{A}$ be a noetherian object, that $\operatorname{Hom}(\mathscr{A}, \mathscr{M})$ be a finite $k$-module for every object $\mathscr{M}$, and that $s$ be "ample" in a sense analogous to ampleness for invertible sheaves (4.2.1). When this is so, Theorem 4.5 asserts that $A$ is a right noetherian $k$-algebra and that $A$ satisfies $\chi_1$. Moreover, a right noetherian $k$-algebra $A$ which satisfies $\chi_1$ can be recovered up to torsion from $(\operatorname{proj} A, s)$ by $\Gamma$. Thus the technical condition $\chi_1$ turns out to be a natural requirement. The proof of this theorem follows the lines of Serre's original proof.

Section 7 contains an introduction to cohomology of Proj $A$, including a version of Serre's theorem on the finiteness of cohomology [Se, Sect. 66, Thm. 1, p. 259; Ha, Thm. III.5.2; EGA III, 2.2.1]. The condition $\chi$ appears again as a hypothesis for this theorem, but extended to higher $\underline{\operatorname{Ext}}$. We remark that Serre's proof is made by an explicit calculation of cohomology of $\mathscr{O}_X(n)$ on projective space. For noncommutative rings there is no ambient projective space in which to work, so a different proof is required. The one we give is based on an analysis of injective resolutions. We also review the notion of cohomological dimension, and we give bounds for the cohomological dimensions of certain rings in Section 8. But the problem of bounding cohomological dimension in terms of natural invariants such as the Gelfand–Kirillov dimension (GK-dimension) or the Krull dimension remains open.

Another antecedent for this paper is the idea of Van den Bergh (see [AV]) to study noncommutative polarizations of classical commutative projective schemes which are obtained by twisting an invertible sheaf by an automorphism $\sigma$ of $X$. In Section 6, we take up this idea again, and discuss an alternative construction of noncommutative scheme analogous to Grothendieck's definition of $\underline{\operatorname{Spec}}\, \mathscr{A}$. Let $Z$ be a commutative scheme of finite type over a field $k$, and let $\mathscr{A}$ be a coherent $\mathscr{O}_Z$-algebra. In this situation, the category coh $\mathscr{A}$ of coherent sheaves on $Z$ with a structure of right $\mathscr{A}$-module is available, and it provides a definition of "scheme" analogous to Grothendieck's notion of $\underline{\operatorname{Spec}}\, A$ [EGA II, 1.3.1]. One may pose the problem of describing the ample autoequivalences $s$ of coh $\mathscr{A}$. This question was considered in [AV] in the case that $\mathscr{A} = \mathscr{O}_Z$, and it leads to the notion of a $\sigma$-ample invertible sheaf. If $\sigma$ is an automorphism of $Z$, an invertible sheaf $\mathscr{L}$ is called *$\sigma$-ample* if it satisfies the usual conditions for ampleness, except that the functor $\cdot \otimes \mathscr{L}^{\otimes n}$ is replaced by $\cdot \otimes \mathscr{L} \otimes \mathscr{L}^{\sigma} \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}$. In Section 6, we complement the study of [AV] by showing that an autoequivalence of coh $\mathscr{A}$ is a "local Morita equivalence" defined by an invertible $(\mathscr{A}, \mathscr{A})$-bimodule.

## 2. Definition of Proj

In this paper $k$ will denote a noetherian commutative ring. A $\mathbb{Z}$-graded $k$-module $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is called *locally finite* if each component $V_n$ is a finitely generated $k$-module, and *left bounded* if $V_n = 0$ for all $n \ll 0$. The *left bound* of $V$ is the lowest degree in which $V$ has non zero elements. We allow the left bound to take the values $\pm\infty$, so that it is defined for all $M$. The terms *right bounded* and *right bound* are defined analogously, and $V$ is called *bounded* if it is both left and right bounded.

We will be working with *right noetherian graded* $k$-algebras $A$, and our $A$-modules will be $\mathbb{Z}$-graded right $A$-modules. An algebra is called *noetherian* if it is right and left noetherian. We will often omit the prefixes "$k$," "right," and "$\mathbb{Z}$." The definitions in this section are made to include arbitrary right noetherian, $\mathbb{Z}$-graded algebras. But Proposition 2.5 shows that for our purposes it is enough to consider the case that the algebra is $\mathbb{N}$-*graded*, i.e., that $A_n = 0$ for all $n < 0$. Beginning in Section 3, we restrict our attention to $\mathbb{N}$-graded algebras. A *connected* graded algebra $A$ is one which is $\mathbb{N}$-graded and such that $A_0 = k$.

Given a graded $k$-module $M$ and an integer $d$, the graded $k$-module $\bigoplus_{n \geqslant d} M_n$ is denoted by $M_{\geqslant d}$ and is called a *tail* of $M$. If $A$ is an $\mathbb{N}$-graded algebra and $M$ is a graded $A$-module, then the tail $M_{\geqslant d}$ is a submodule of $M$, and $A_{\geqslant d}$ is an ideal of $A$. In this case, we will also identify $M_d$ with the quotient module $M_{\geqslant d}/M_{\geqslant d+1}$.

Recall that a graded algebra $A$ is right noetherian if and only if it is "graded right noetherian," which means that every graded right ideal is finitely generated [NV, Thm. A.II.3.5]. Also, we have the following result [NV, Prop. A.II.3.4, Lemma A.II.3.2].

Proposition 2.1. *Let $A$ be a right noetherian, $\mathbb{Z}$-graded ring. Then its subrings $A_0$ and $A_{\geqslant 0}$ are also right noetherian, and if $M$ is a finitely generated right $A$-module, then $M_n$ is a finitely generated $A_0$-module for all $n$. If in addition $A_0$ is a finitely generated $k$-module, then $A$ is locally finite. In particular, a connected graded, right noetherian algebra is locally finite.* ∎

But note that $A_{\geqslant 0}$ may be noetherian, though $A$ is not. For example, let $k$ be a field and let $A$ be the graded algebra $k \oplus V$ with $\deg(V) = -1$, $\dim_k(V) = \infty$, and $V^2 = 0$, then $A$ is not noetherian, but $A_{\geqslant 0} = k$ is noetherian.

The *shift* operator $s$ on graded modules is defined as follows: $s(M)$ is the graded module such that $(s(M))_n = M_{n+1}$. We will often use the notation $M[d] := s^d(M)$ to denote the $d$th power of the shift operator.

A homomorphism $f\colon M \to N$ between graded modules is said to be of degree $d$ if $f(M_n) \subseteq N_{n+d}$ for all $n \in \mathbb{Z}$. If no degree is specified, the term homomorphism refers to one of degree zero. A homomorphism of degree $d$ can also be described as a homomorphism $f: M \to N[d]$.

We will use the following notation:

$$\text{Mod } A := \text{the category of right } A\text{-modules;}$$

$$\text{Gr } A := \text{the category of graded right } A\text{-modules.}$$

Morphisms in the category of graded modules are the homomorphisms of degree zero.

We say that an element $x$ of a graded module $M$ is *torsion* if $xA_{\geqslant s} = 0$ for some $s$. The torsion elements in $M$ form a graded $A$-submodule which we denote by $\tau(M)$ and call the *torsion submodule* of $M$. A module $M$ is called *torsion-free* if $\tau(M) = 0$ and *torsion* if $M = \tau(M)$. It is easy to check that $\tau(M)$ is the smallest submodule of $M$ such that $M/\tau(M)$ is torsion-free. The torsion modules form a subcategory for which we well use the following notation:

$$\text{Tors} := \text{the full subcategory of Gr } A \text{ of torsion modules.}$$

More care must be taken with the definition of torsion module if the ring $A$ is not right noetherian. In that case one must define $\tau(M)$ to be the smallest submodule of $M$ such that the quotient $M/\tau(M)$ is torsion-free. (This is one of the axioms of abstract torsion theory [St, Chap. 6]). We denote

$$\text{QGr } A := \text{the quotient category Gr } A/\text{Tors.}$$

We will modify the notation introduced above by using lower case to indicate that we are working with finitely generated $A$-modules, which we refer to simply as *finite* $A$-modules. Thus:

mod $A$ denotes the category of finite right $A$-modules,

gr $A$ the category of finite graded right $A$-modules,

tors the full subcategory of gr $A$ of torsion modules, and

qgr $A$ the quotient category gr $A$/tors.

The quotient construction works well because Tors is a *dense subcategory* of Gr $A$. This means that it is a full subcategory, and that if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in Gr $A$, then $M$ is torsion if and only if $M'$ and $M''$ are torsion. Similarly, tors is a dense subcategory of gr $A$.

In addition, torsion modules have the following property.

PROPOSITION 2.2. *Let $A$ be a right noetherian $\mathbb{N}$-graded ring, and let $M \to M'$ be an essential extension of graded $A$-modules. Then*

(1) *the right bounds of $M$ and $M'$ are equal, and*

(2) *if $M$ is a torsion module, so is $M'$.*

*Proof.* Recall that $M'$ is an essential extension if for every nonzero graded submodule $N' \subseteq M'$, $N' \cap M \neq 0$. Since $M'_{\geqslant n}$ is a submodule of $M'$, we have $M'_{\geqslant n} \neq 0$ if and only if $M'_{\geqslant n} \cap M \neq 0$. Then (1) follows. To prove (2), we consider the submodule $N' = m'A$ generated by a homogeneous element $m' \in M'$. Then $N'$ is an essential extension of $N := N' \cap M$. Since $A$ is noetherian, $N$ is finitely generated, and since it is also torsion, it is a right bounded module. By (1), $N'$ is also right bounded, hence torsion. ▌

The quotient category QGr $A$ can be described explicitly, as follows: We will often denote by $\pi$ the canonical functor from a category to its quotient category. The objects of QGr $A$ are the objects of Gr $A$, and

$$\mathrm{Hom}_{\mathrm{QGr}}(\pi(N), \pi(M)) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}}(N', M/\tau(M))$$

where $N'$ runs over the quasi-directed category of submodules of $N$ such that $N/N'$ is torsion.

Assume that $A$ is right noetherian, $\mathbb{N}$-graded. Let $M$ and $N$ be $A$-modules, and suppose that $N$ is a finite module. Then the tails $N_{\geqslant n}$ are cofinal among submodules $N'$ such that $N/N'$ is torsion. In this case,

$$\mathrm{Hom}(\pi(N), \pi(M)) = \lim_{n \to \infty} \mathrm{Hom}(N_{\geqslant n}, M). \tag{2.2.1}$$

This is true when $M$ is a finite module, because its torsion submodule $\tau(M)$ is then bounded. It follows in general from the fact that $\mathrm{Hom}(N, \cdot)$ commutes with direct limits when $A$ is right noetherian and $N$ is a finite module.

Since the subcategory Tors is dense, every graded module has a largest torsion submodule, and the category Gr $A$ has enough injectives, there is a section functor which we denote by $\omega$, from QGr $A$ to Gr $A$ which is right adjoint to $\pi$ in the sense that

$$\mathrm{Hom}(N, \omega(\mathcal{M})) \cong \mathrm{Hom}(\pi(N), \mathcal{M}). \tag{2.2.2}$$

for all $N \in \mathrm{Gr}\ A$ and $\mathcal{M} \in \mathrm{QGr}\ A$ [Po, Sect. 4.4]. Hence Tors is a *localizing subcategory* of Gr $A$. The functor $\pi$ is exact and the functor $\omega$ is left exact.

Given a module $M$, let $C_M$ denote the filtering category of maps $M \to M'$ whose kernel and cokernel are torsion modules, and set $\tilde{M} := \omega\pi(M)$. Then

$$\tilde{M} \cong \varinjlim_{C_M} M', \tag{2.2.3}$$

and (2.2.2) becoms $\mathrm{Hom}(N, \tilde{M}) \cong \mathrm{Hom}(\pi(N), \pi(M))$.

Since $\mathrm{Gr}\, A$ is an Ab 5-category with enough injective objects, the quotient category $\mathrm{QGr}\, A$ also has these properties. However, there are some advantages to working with $\mathrm{qgr}\, A$ because the objects in $\mathrm{qgr}\, A$ are noetherian, and most of the properties discussed in this paper can be carried over from $\mathrm{qgr}\, A$ to $\mathrm{QGr}\, A$.

PROPOSITION 2.3. *Let $A$ be a right noetherian graded algebra. Then* $\mathrm{QGr}\, A$ *is a locally noetherian category, and* $\mathrm{qgr}\, A$ *is the full subcategory of noetherian objects of* $\mathrm{QGr}\, A$. *Thus* $\mathrm{QGr}\, A$ *is determined up to equivalence by* $\mathrm{qgr}\, A$. ∎

All of these facts are standard torsion theory [Po, Ga].

The categories introduced above are $k$-linear. A *$k$-linear* abelian category $\mathscr{C}$ is one in which the bi-functor Hom takes its values in $\mathrm{Mod}\, k$. A $k$-linear structure on an abelian category is defined by a collection of ring homomorphisms $c_{\mathscr{M}}: k \to \mathrm{End}(\mathscr{M})$ for each object $\mathscr{M}$, so that for $f \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$, the two maps $fc_{\mathscr{M}}, c_{\mathscr{N}} f: k \to \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ are equal. Throughout this paper, functors between $k$-linear categories are assumed to be $k$-linear functors.

We will now review the notion of spectrum of a ring which is suited to our purposes. An ungraded algebra $A$ is determined as an endomorphism ring $\mathrm{End}(A_A)$ by the special object $A_A$ in $\mathrm{Mod}\, A$. For this reason it is natural to consider the pair $(\mathrm{Mod}\, A, A_A)$, when working with ungraded modules. This pair will be denoted by $\mathrm{Spec}\, A$, and will be called the *general spectrum* of $A$. If $A$ is right noetherian, we also denote the pair $(\mathrm{mod}\, A, A_A)$ by $\mathrm{spec}\, A$ and we will call it the *noetherian spectrum* of $A$. For simplicity, we may sometimes refer to either of these pairs simply as the *spectrum* of $A$.

Let $\mathscr{A}$ and $\mathscr{A}'$ be objects of two $k$-linear abelian categories $\mathscr{C}$ and $\mathscr{C}'$, respectively. A morphism of pairs $(\mathscr{C}, \mathscr{A}) \to (\mathscr{C}', \mathscr{A}')$ is a pair $(f, \theta)$ consisting of a $k$-linear functor $f: \mathscr{C} \to \mathscr{C}'$ and an isomorphism $\theta: f(\mathscr{A}) \xrightarrow{\sim} \mathscr{A}'$. We call the morphism *right exact* if $f$ preserves direct limits. In addition, we have the notion of *equivalence* of morphisms. A natural equivalence $(f, \theta) \to (f', \theta')$ is an isomorphism of functors $f \to f'$ which is compatible with $\theta$ and $\theta'$.

An algebra homomorphism $\phi: A \to B$ determines a morphism of pairs

$$(f, \theta): (\mathrm{Mod}\, A, A_A) \to (\mathrm{Mod}\, B, B_B),$$

where $f(M) = M \otimes_A B$ and $\theta$ is the canonical isomorphism $A \otimes_A B \xrightarrow{\sim} B$. Conversely, a morphism $(f', \theta')$ between these pairs determines a homomorphism $\phi: A \to B$; namely, $\phi$ is the induced map $\mathrm{End}(A_A) \to \mathrm{End}(B_B)$, and if the morphism is right exact, there is a unique natural equivalence $(f, \theta) \to (f', \theta')$ [Ro, Thm. 3.35]. The uniqueness follows from the fact that $A_A$ generates Mod $A$.

Following the custom in algebraic geometry, we reverse arrows when defining maps of spectra. Thus a *map*

$$F: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$$

is a natural equivalence class of right exact morphisms $(f, \theta): (\mathrm{Mod}\, A, A_A) \to (\mathrm{Mod}\, B, B_B)$, and $F$ is an *isomorphism* if $f$ is an equivalence of categories. As was noted above, a map of spectra induces an algebra homomorphism $\phi: A \to B$, and conversely, an algebra homomorphism $\phi: A \to B$ induces a map of spectra.

It follows from the above result (or because $A_A$ generates Mod $A$), that a natural equivalence between two maps $F, G: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is unique if it exists. This justifies calling a pair $X = (\mathscr{C}, \mathscr{A})$ consisting of an object $\mathscr{A}$ of a $k$-linear category $\mathscr{C}$ an *affine scheme* if there is a morphism of pairs $(f, \theta): X \to (\mathrm{Mod}\, A, A_A)$ such that $f$ is an equivalence of categories. A map $F: X' \to X$ between affine schemes is a natural equivalence class of right exact morphisms $(f, \theta): (\mathscr{C}, \mathscr{A}) \to (\mathscr{C}', \mathscr{A}')$, and $F$ is an *isomorphism* if $f$ is an equivalence. Similar considerations apply for noetherian spectra.

We now go back to the case that $A$ is a graded algebra. The category Gr $A$ has the shift operator $s$ as well as the special object $A_A$, and the algebra is determined by the triple (Gr $A$, $A_A$, $s$). To see this, we identify $A_d$ with $\mathrm{Hom}(A_A, s^d(A_A))$. Thus

$$A = \bigoplus_{d=-\infty}^{\infty} \mathrm{Hom}(A_A, s^d(A_A)).$$

The algebra structure is described by composition as follows: If $a \in A_i$ and $b \in A_j$, the product $ab \in A_{i+j}$ corresponds to the element $s^j(a) \circ b \in \mathrm{Hom}(A, s^{i+j}(A))$. So when working with graded modules, it is natural to introduce the triple (Gr $A$, $A_A$, $s$). Since QGr $A$ is a quotient category of Gr $A$, it inherits these two structures: the object $\mathscr{A}$ which is the image in QGr $A$ of $A_A$, and the shift operator $s$ on QGr $A$, which is the automorphism of the category QGr $A$ determined by the shift on Gr $A$.

We make the following definition. Let $A$ be a graded algebra. The triples (QGr $A$, $\mathscr{A}$, $s$) and (qgr $A$, $\mathscr{A}$, $s$) are called the *general projective scheme* and the *noetherian projective scheme* of $A$, respectively. In algebraic geometry, it is customary to omit the "polarization" defined by the shift operator $s$ from the structure, so we will use the following notation: We denote the pair $(\text{QGr } A, \mathscr{A})$ by $\text{Proj } A$ and the pair $(\text{qgr } A, \mathscr{A})$ by $\text{proj } A$. The shift operator will also be called the canonical *polarization* of $\text{Proj } A$ or of $\text{proj } A$. Thus the general projective scheme of $A$ may also be denoted by $(\text{Proj } A, s)$. If the context makes the meaning unambiguous, may simplify terminology by referring to any of the collections of data $(\text{Proj } A, s)$, $(\text{proj } A, s)$, $\text{Proj } A$, $\text{proj } A$ as the *projective scheme* of $A$.

A morphism of triples $(\mathscr{C}, \mathscr{A}, s) \to (\mathscr{C}', \mathscr{A}', s')$, where $\mathscr{A}$, $\mathscr{A}'$ are objects and $s$, $s'$ are autoequivalences of $\mathscr{C}$, $\mathscr{C}'$, respectively, is a triple $(f, \theta, \mu)$, where as before $f: \mathscr{C} \to \mathscr{C}'$ is a $k$-linear functor, $\theta: f(A) \xrightarrow{\sim} A'$ is an isomorphism, and where $\mu$ is an isomorphism of functors $f \circ s \to s' \circ f$. The morphism is called right exact if $f$ preserves direct limits. A natural equivalence of morphisms $(f_1, \theta_1, \mu_1) \to (f_2, \theta_2, \mu_2)$ is an isomorphism of functors $\varepsilon: f_1 \to f_2$ which is compatible with the rest of the data, meaning that

$$\theta_1 = \theta_2 \varepsilon(\mathscr{A}), \tag{2.3.1}$$

and for all objects $\mathscr{M}$,

$$(s'\varepsilon(M))\, \mu_1 = \mu_2(\varepsilon s(M)). \tag{2.3.2}$$

We define a *map*

$$F: \text{Proj } B \to \text{Proj } A$$

to be a natural equivalence class of right exact morphisms of pairs $(\text{QGr } A, \mathscr{A}) \to (\text{QGr } B, \mathscr{B})$, and a map of general schemes to be a natural equivalence class of morphisms of triples $(\text{QGr } A, \mathscr{A}, s_A) \to (\text{QGr } B, \mathscr{B}, s_B)$. In both cases, $F$ is called an *isomorphism* if the $k$-linear functor $f$ is an equivalence of categories. The analogous definitions are made for noetherian projective schemes.

There are two remarks to be made here. First of all, we can not rule out the existence of nontrivial automorphisms of the identity map. (Normally, $\mathscr{A}$ will not generate $\text{QGr } A$.) So we do not know that natural equivalences between two maps of projective schemes are uniquely determined. This point will be discussed again later (see Corollary 4.3 and Proposition 4.4). Second, the problem of determining all possible polarizations of $\text{Proj } A$ is difficult, even in the commutative case. One can not expect an easy solution to the problem of describing the graded algebras $B$ such that there exists an isomorphism between the unpolarized projective schemes $\text{Proj } A$ and $\text{Proj } B$.

Recall that the tensor product of $\mathbb{Z}$-graded right $A$-module and a $\mathbb{Z}$-graded left $A$-module has a natural structure of $\mathbb{Z}$-graded $k$-module. We use the symbol $\underline{\otimes}$ to denote this graded tensor product, and $\underline{\mathrm{Tor}}$ for its derived functors.

The next proposition collects together some elementary facts about the functors $\underline{\mathrm{Tor}}$.

PROPOSITION 2.4. *Let $A$ be a graded algebra, let $L$ be a graded left $A$-module, and let $N$ be a graded right $A$-module.*

(1) *The functors $\underline{\mathrm{Tor}}_i^A$ commute with direct limits.*

(2) *Suppose that $A$ is $\mathbb{N}$-graded, and let $l$, $l'$ denote the left bounds of $N$, $L$ respectively. Then for all $i$, the left bound of $\underline{\mathrm{Tor}}_i^A(N, L)$ is at least $l+l'$,*

(3) *Suppose that $A$ is right noetherian and $N$ is a finite module. If $L$ is left (or right) bounded, then so is $\underline{\mathrm{Tor}}_i^A(N, L)$.*

(4) *Suppose that $L$ is an $(A, B)$-bimodule, that $A$ and $B$ are right noetherian, and that $N_A$ and $L_B$ are finite modules. Then $\underline{\mathrm{Tor}}_i^A(N, L)$ is a finite right $B$-module.*

(5) *Suppose that $L$ is an $(A, B)$-bimodule. If $L_B$ is a torsion module, so is $\underline{\mathrm{Tor}}_i^A(N, L)_B$.*

(6) *Suppose that $A$ is $\mathbb{N}$-graded and that $N$ is a left bounded module. Denote by $l(n, i)$ the left bound of $\underline{\mathrm{Tor}}_i^A(N, A/A_{\geqslant n})$. Then for $i \geqslant 1$, $l(n, i)$ tends to infinity with $n$.*

*Analogous assertions to* (3)–(6) *hold when left and right are interchanged.*

*Proof.* Assertions (1)–(5) are proved by considering a resolution of $N$ by a sum of shifts of $A$. For the last assertion, we apply $\underline{\mathrm{Tor}}_i$ to the exact sequence

$$0 \to A_{\geqslant n} \to A \to A/A_{\geqslant n} \to 0. \tag{2.4.1}$$

Since $A$ is projective, $\underline{\mathrm{Tor}}_{i+1}^A(N, A/A_{\geqslant n}) \cong \underline{\mathrm{Tor}}_i^A(N, A_{\geqslant n})$ for all $i \geqslant 1$, and the canonical map $\underline{\mathrm{Tor}}_1^A(N, A/A_{\geqslant n}) \to N \otimes_A A_{\geqslant n}$ is injective. Since the left bound of $A_{\geqslant n}$ is at least $n$, the assertion follows by setting $L = A_{\geqslant n}$ in (2). ∎

We will now prove that Proj $A$ is isomorphic to Proj $A_{\geqslant 0}$ where $A_{\geqslant 0}$ denotes the subring $\bigoplus_{n \geqslant 0} A_n$. Hence when we study the projective scheme Proj $A$, we may assume $A$ is $\mathbb{N}$-graded.

PROPOSITION 2.5. *Let $\phi: A \to B$ be a homomorphism of graded $k$-algebras.*

(1) *The category* $\mathbf{T}_A$ *of graded B-modules whose restriction to A are torsion is a dense subcategory of* Gr $B$. *If the kernel and cokernel of* $\phi$ *are torsion right A-modules, then the extension of scalars* $\cdot \otimes_A B$ *defines an equivalence of categories* $\mathrm{QGr}\, A \to \mathrm{Gr}\, B/\mathbf{T}_A$.

(2) *If the kernel and cokernel of* $\phi$ *are right bounded, then* $\cdot \otimes_A B$ *defines an equivalence of categories* $\mathrm{QGr}\, A \to \mathrm{QGr}\, B$. *Hence the projective schemes* $(\mathrm{Proj}\, B, s)$ *and* $(\mathrm{Proj}\, A, s)$ *are isomorphic. If in addition B and A are right noetherian, then* $\cdot \otimes_A B$ *also defines an equivalence of categories* qgr $B$, *and hence the projective schemes* $(\mathrm{proj}\, B, s)$ *and* $(\mathrm{proj}\, A, s)$ *are isomorphic.*

*Proof.* If the kernel and cokernel of $\phi$ are right bounded, then a $B$-module is $B$-torsion if and only if its restriction to $A$ is $A$-torsion. Therefore the first part of (2) follows from (1), and the last part is an obvious corollary. Thus it suffices to prove (1).

For the rest of the proof, we will refer to $A$-torsion simply as torsion. It is immediate that $\mathbf{T}_A$ is a dense subcategory of Gr $B$. We have an exact sequence of $A$-bimodules

$$0 \longrightarrow T \longrightarrow A \xrightarrow{\phi} B \longrightarrow T' \longrightarrow 0, \qquad (2.5.1)$$

where $T$ and $T'$ are torsion. Tensoring this sequence with an arbitrary $A$-module $N$ and applying Proposition 2.4(5) shows that the canonical map $N \to N \otimes_A B$ is an isomorphism modulo torsion. It follows that $N \otimes_A B$ is torsion if $N$ is, and thus that $\cdot \otimes_A B$ defines the required functor $\mathrm{QGr}\, A \to \mathrm{Gr}\, B/\mathbf{T}_A$. In the other direction, restriction of scalars carries torsion to torsion by definition, and hence defines a functor $\mathrm{Gr}\, B/\mathbf{T}_A \to \mathrm{QGr}\, A$. As we have seen, the composition of these functors is isomorphic to the identity on QGr $A$. The remaining assertion, that the canonical map $M \otimes_A B \to M$ is an isomorphism modulo torsion for every $B$-module $M$, also follows, because the $A$-linear map $M \to M \otimes_A B$, which has just been shown to be an isomorphism modulo torsion, is its right inverse. ∎

With the exception of some parts of Proposition 3.1 and some examples given in Section 5, we assume that our graded algebras are $\mathbb{N}$-graded as well as right noetherian for the rest of this paper.

## 3. A Condition on Graded Ext

The next definitions are made to include an arbitrary $\mathbb{Z}$-graded algebra $A$, but after Proposition 3.1, we assume for the rest of the section that $A$ is right noetherian and $\mathbb{N}$-graded. Let $N$ and $M$ be two graded $A$-modules.

The derived functors of $\mathrm{Hom}(N, M)$, the groups $\mathrm{Ext}^i(N, M)$, take their values in $\mathrm{Mod}\, k$. We denote by $\underline{\mathrm{Hom}}(N, M)$ the graded $k$-module

$$\underline{\mathrm{Hom}}(N, M) = \bigoplus_{d=-\infty}^{\infty} \mathrm{Hom}(N, M[d]).$$

For $i \geqslant 0$, the derived functors of $\underline{\mathrm{Hom}}$ are the graded Ext groups

$$\underline{\mathrm{Ext}}^i(N, M) = \bigoplus_{d=-\infty}^{\infty} \mathrm{Ext}^i(N, M[d]).$$

We use a similar notation in the category $\mathrm{QGr}\, A$:

$$\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{M}) = \bigoplus_{d=-\infty}^{\infty} \mathrm{Hom}(\mathcal{N}, \mathcal{M}[d]).$$

The derived functors of $\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{M})$ will be discussed in Section 7.

The next proposition collects together some elementary facts about the functors $\underline{\mathrm{Ext}}$.

PROPOSITION 3.1. *Let $A$ be a graded algebra, and let $N$, $M$ be graded right $A$-modules.*

(1) *If $A$ is right noetherian and $N$ is a finite module, then:*

(a) *The "ungraded" Ext group, computed in the category* $\mathrm{Mod}\, A$, *is the $k$-module defined by removing the grading from $\underline{\mathrm{Ext}}^i(N, M)$.*

(b) *If $\{M_\nu\}$ is an inductive system of $A$-modules, then $\varinjlim_\nu \underline{\mathrm{Ext}}^i(N, M_\nu) \cong \underline{\mathrm{Ext}}^i(N, \varinjlim_\nu M_\nu)$.*

(c) *If $M$ is left (or right) bounded, so is* $\mathrm{Ext}^i(N, M)$.

(2) *If $A$ is $\mathbb{N}$-graded, $N$ is left bounded, with left bound $l$, and if the right bound of $M$ is $r$, then the right bound of $\underline{\mathrm{Ext}}^j(N, M)$ is at most $r - l$.*

(3) *If $A$ is right noetherian and locally finite and if $N$, $M$ are finite modules, then $\underline{\mathrm{Ext}}^i(N, M)$ is a locally finite $k$-module.*

(4) *Suppose that $A$ is a right noetherian and $B$ is a left noetherian graded ring, that $M$ is a graded $(B, A)$-bimodule and that $N_A$ and ${}_BM$ are finite modules. Then $\underline{\mathrm{Ext}}^i(N, M)$ is a finite left $B$-module.*

(5) *If $A$ is $\mathbb{N}$-graded and $M$ is right bounded, for all $j \geqslant 1$ the right bound of $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ tends to $-\infty$ as $n \to \infty$.*

*Proof.* Assertions (1)–(4) follow from a consideration of a resolution of $N$ by sums of shifts of $A$; (5) follows from (2) and the $\underline{\mathrm{Ext}}$ sequence associated to the exact sequence (2.4.1). ∎

From now on we assume that $A$ is right noetherian and $\mathbb{N}$-graded. As Proposition 3.1 shows, $\underline{\mathrm{Ext}}$ has good properties as a functor of the second variable. But the situation with respect to the first variable is more complicated. For example, there exist connected graded noetherian algebras $A$ over a field $k$ for which $\underline{\mathrm{Ext}}^1(k, A)$ is not bounded [SZ2, Thm. 2.3].

For any $A$-module $N$, there are exact sequences

$$0 \to N_{\geqslant n} \to N \to N/N_{\geqslant n} \to 0, \tag{3.1.1}$$

$$0 \to N_{\geqslant n+1} \to N_{\geqslant n} \to N_n \to 0, \qquad \text{and} \tag{3.1.2}$$

$$0 \to N_n \to N/N_{\geqslant n+1} \to N/N_{\geqslant n} \to 0, \tag{3.1.3}$$

and their associated $\underline{\mathrm{Ext}}$ sequences, the first, for example, being

$$\cdots \to \underline{\mathrm{Ext}}^j(N/N_{\geqslant n}, M) \to \underline{\mathrm{Ext}}^j(N, M) \to \underline{\mathrm{Ext}}^j(N_{\geqslant n}, M) \to \cdots$$

Since $A$ is projective, $\underline{\mathrm{Ext}}^j(A, M) = 0$ for every $M$ and every $j \geqslant 1$. Thus the $\underline{\mathrm{Ext}}$ sequence determined by setting $N = A$ in this sequence provides an exact sequence

$$0 \to \underline{\mathrm{Hom}}(A/A_{\geqslant n}, M) \to M \to \underline{\mathrm{Hom}}(A_{\geqslant n}, M) \to \underline{\mathrm{Ext}}^1(A/A_{\geqslant n}, M) \to 0, \tag{3.1.4}$$

and, for every $j \geqslant 1$, an isomorhism

$$\underline{\mathrm{Ext}}^j(A_{\geqslant n}, M) \cong \underline{\mathrm{Ext}}^{j+1}(A/A_{\geqslant n}, M). \tag{3.1.5}$$

DEFINITION 3.2. We say that $\chi_i^\circ(M)$ holds for an $A$-module $M$ if $\underline{\mathrm{Ext}}^j(A_0, M)$ is bounded for all $j \leqslant i$. If $\chi_i^\circ(M)$ holds for every finite $A$-module $M$, we say that $\chi_i^\circ$ holds for the graded algebra $A$, and if $\chi_i^\circ$ holds for every $i$, we say that $\chi^\circ$ holds for $A$.

Note that according to Proposition 3.1(1c), $\underline{\mathrm{Ext}}^j(A_0, M)$ is always left bounded if $M$ is left bounded. The condition $\chi_i^\circ(M)$ asks that there be a finite right bound as well.

The following proposition is elementary.

PROPOSITION 3.3. (1) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A$-modules. Then $\chi_i^\circ(M')$ and $\chi_i^\circ(M'')$ imply $\chi_i^\circ(M)$.*

(2) *$\chi_i^\circ$ holds for $A$ if and only if every finite, nonzero module $M$ as a nonzero submodule $M'$ such that $\chi_i^\circ(M')$ holds.* ∎

LEMMA 3.4. *Let $N$ be a bounded $A$-module, with left bound $l$ and right bound $r$, and let $M$ be an $A$-module. Denote by $\lambda$ and $\rho$ the minimum left bound and maximum right bound of $\underline{\mathrm{Ext}}^j(A_0, M)$ for $j \leqslant i$. Then for $j \leqslant i$, the* *left bound of* $\underline{\mathrm{Ext}}^j(N, M)$ *is at least* $\lambda - r$, *and the right bound is at most* $\rho - l$.

*Proof.* We first consider the case that $N$ is zero except in degree zero. In that case, $N$ is a quotient of a direct sum $F$ of copies of $A_0$. Then $\underline{\mathrm{Ext}}^i(F, M)$ is the graded product of copies of $\underline{\mathrm{Ext}}^j(A_0, M)$, so it is bounded by $\lambda$ and $\rho$. Also, the kernel $K$ of the map $F \to N$ is concentrated in degree zero. By induction on $i$, $\underline{\mathrm{Ext}}^j(K, M)$ is bounded by $\lambda$ and $\rho$ for $j < i$. It follows that $\underline{\mathrm{Ext}}^j(N, M)$ is bounded by $\lambda$ and $\rho$ for every $j \leqslant i$, as required. Next, if $N$ is zero except in degree $d$, then the bounds of $\underline{\mathrm{Ext}}^j(N, M)$ shift to $\lambda - d$ and $\rho - d$. The general case follows by induction on $l - r$, by an analysis of the exact sequence (3.1.1). ∎

PROPOSITION 3.5. *Let* $M$ *and* $N$ *be* $A$*-modules such that* $\chi_i^\circ(M)$ *holds.*

(1) *If* $N$ *is left bounded, then for every* $d$ *there exists an* $n_0$ *such that for all* $n_0$ *and all* $j \leqslant i$,

$$\underline{\mathrm{Ext}}^j(N/N_{\geqslant n_0}, M)_{\geqslant d} \cong \underline{\mathrm{Ext}}^j(N/N_{\geqslant n}, M)_{\geqslant d}.$$

*In particular,* $\underline{\mathrm{Ext}}^j(N/N_{\geqslant n}, M)$ *has a right bound independent of* $n$.

(2) *For every integer* $d$, *there is an integer* $n_0$ *such that for all* $n \geqslant n_0$ *and all* $j \leqslant i - 1$,

$$\underline{\mathrm{Ext}}^j(N_{\geqslant n_0}, M)_{\geqslant d} \cong \underline{\mathrm{Ext}}^j(N_{\geqslant n}, M)_{\geqslant d}.$$

(3) *If* $N$ *is left bounded, there is an integer* $l$ *such that*

$$\underline{\mathrm{Ext}}^j(N, M)_{\geqslant l} \cong \underline{\mathrm{Ext}}^j(N_{\geqslant n}, M)_{\geqslant l}$$

*for all* $n$ *and all* $j \leqslant i - 1$.

*Proof.* Lemma 3.4 shows that the right bound of $\underline{\mathrm{Ext}}^j(N_n, M)$ tends to $-\infty$ as $n \to \infty$. Because of this, assertions (1) and (2) follow from the $\underline{\mathrm{Ext}}$ sequences associated to the exact sequences (3.1.3) and (3.1.2). Since the right bound of $\underline{\mathrm{Ext}}^j(N/N_{\geqslant n}, M)$ is independent of $n \gg 0$ by (1), assertion (3) follows from the exact sequence (3.1.1). ∎

COROLLARY 3.6. (1) *If* $\chi_i^\circ(M)$ *holds then* $\underline{\mathrm{Ext}}^j(N, M)$ *is bounded for every bounded module* $N$ *and every* $j \leqslant i$.

(2) *Let* $I$ *be a right ideal such that for some* $n$, $A_{\geqslant n} \subseteq I \subseteq A_{\geqslant 1}$, *and let* $M$ *be a finite module. Then* $\chi_i^\circ(M)$ *holds if and only if* $\underline{\mathrm{Ext}}^j(A/I, M)$ *is bounded for* $j \leqslant i$,

(3) $\chi_i^\circ(M)$ *holds if and only if* $\lim_{n \to \infty} \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ *is right bounded for* $j \leqslant i$.

*Proof.* Assertion (1) follows directly from Lemma 2.4.

(2) Assume that $\underline{\mathrm{Ext}}^j(A/I, M)$ is bounded for $j \leqslant i$. We use induction on $i$. Let $K$ denote the kernel of the surjective map $A/I \to A_0$. There is an exact sequence $\underline{\mathrm{Ext}}^{i-1}(K, M) \to \underline{\mathrm{Ext}}^i(A_0, M) \to \underline{\mathrm{Ext}}^i(A/I, M)$. Since $A_{\geqslant n} \subseteq I$, $K$ is bounded. By induction and by (1), $\underline{\mathrm{Ext}}^{i-1}(K, M)$ is bounded, and $\underline{\mathrm{Ext}}^i(A/I, M)$ is bounded by hypothesis. Hence so is $\underline{\mathrm{Ext}}^i(A_0, M)$. This shows that $\chi_i^\circ(M)$ holds. The other implication follows from (1).

(3) If $\chi_i^\circ(M)$ holds, Proposition 3.5(1) shows that the limit is right bounded for $j \leqslant i$. Conversely, assume that the limit is right bounded. Applying $\underline{\mathrm{Ext}}$ to the exact sequence $0 \to A_{\geqslant 1}/A_{\geqslant n} \to A/A_{\geqslant n} \to A_0 \to 0$, we obtain an exact sequence

$$\underline{\mathrm{Ext}}^{j-1}(A_{\geqslant 1}/A_{\geqslant n}, M) \to \underline{\mathrm{Ext}}^j(A_0, M) \to \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M).$$

The limit of the left term is right bounded by Proposition 3.5(1) and induction, and the limit of the right term is right bounded by hypothesis. Hence the middle term is right bounded. ∎

The condition $\chi^\circ$ is satisfactory when $A$ is a locally finite algebra, but to include the general case, we need a refinement. Note that if $\chi_i^\circ(M)$ holds and if $j \leqslant i$, then according to Proposition 3.5(1), $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ is bounded and independent of $n \gg 0$. Note also that the left module structure of $A/A_{\geqslant n}$ makes $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ into a right $A$-module, and since $A/A_{\geqslant n}$ is annihilated by $A_{\geqslant n}$, so is $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$. In this way, $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ becomes a torsion right $A$-module.

DEFINITION 3.7. We say that $\chi_i(M)$ holds for an $A$-module $M$ if for all $d$ and all $j \leqslant i$, there is an integer $n_0$ such that $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ is a finite $A$-module, when $n \geqslant n_0$. If $\chi_i(M)$ holds for every finite $A$-module $M$, we say that $\chi_i$ holds for the graded algebra $A$, and if $\chi_i$ holds for every $i$ we say that $\chi$ holds for $A$.

*Remarks.* (1) In the definition, we could as well require that for $n \gg 0$, $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ be a finite module over the subring $A_0$. For, if $\chi_i(M)$ holds then it is a finite module over $A/A_{\geqslant n}$ and $A/A_{\geqslant n}$ is a finite $A_0$-module (see Proposition 2.1).

(2) Since $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ is torsion, the definition of $\chi_i(M)$ contains two parts: (a) $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ is right bounded and (b) $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_d$ is a finite $A_0$-module for $n \gg 0$.

(3) We have no direct analogue of Corollary 3.6(1) for the condition $\chi_i(M)$.

PROPOSITION 3.8. (1) *$\chi_i(M)$ holds if and only if $\chi_i^\circ(M)$ holds, and for every $j \leqslant i$, every $d$, and every $n \gg 0$, $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_d$ is a finite $A_0$-module.*

(2) *If* $\chi_i(M)$ *holds, then for* $j \leqslant i$, $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ *is bounded and independent of* $n \gg 0$.

(3) $\chi_i(M)$ *holds if and only if* $\lim_{n\to\infty} \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ *is a finite* $A$*-module for all* $d$ *and all* $j \leqslant i$.

(4) $\chi_i$ *holds for* $A$ *if and only if* $\chi_i^\circ$ *holds for every finite module* $M$, *every* $j \leqslant i$ *and every* $n \gg 0$, $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_0$ *is a finite* $A$*-module (or a finite* $A_0$*-module).*

*Proof.* (1) First, Corollary 3.6(2) shows that $\chi_i(M)$ implies $\chi_i^\circ(M)$. The "only if" part of (1) follows. Assume conversely that $\chi_i^\circ(M)$ holds and that for every $d$, $\underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M)_d$ is a finite module when $n \gg 0$. By Proposition 3.5(1), $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d}$ is bounded and independent of $n \gg 0$. Therefore its finiteness follows from the finiteness in each degree separately, which shows that $\chi_i(M)$ holds.

(2) This follows from Proposition 3.5(1).

(3) The limit is a torsion module, because each term is torsion. If it is a finite module then it is right bounded. So in either case, if $\chi_i(M)$ holds or if the limit is a finite module, $\chi_i^\circ(M)$ holds. Then by Proposition 3.5(1), $\underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M)_{\geqslant d}$ is bounded and independent of $n \gg 0$. Therefore the limit is taken on for $n \gg 0$. Thus the limit is a finite torsion module if and only if $\chi_i(M)$ holds.

(4) This follows from (1) by shifting $M$. ∎

*Remarks.* (1) We have no example of a right noetherian graded ring which satisfies $\chi_i^\circ$ but not $\chi_i$.

(2) With reference to part (4) of the previous proposition, note that the values of $n$ for which $\underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M)_0$ is a finite module may change when $M$ is shifted. It may happen that $\chi_i(M)$ holds but nevertheless, for every $n$ there is a degree $d$ such that $\underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M)_d$ is infinitely generated. Example 3.10 below illustrates this point.

EXAMPLE 3.9. A *noetherian graded ring* $A$ *which satisfies* $\chi$, *but such that for some finite* $A$*-module* $M$, $\mathrm{Ext}^1(A_0, M)$ *is not a finite right* $A$*-module.* Let $T \subset B$ be two division rings such that ${}_TB$ is finite and $B_T$ is infinitely generated (see [SZ1, Ex. 3.1]), and let $A = A_0 \oplus A_1$ be the graded ring defined by

$$A_0 = \begin{pmatrix} T & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

with $A_n = 0$ if $n \geqslant 2$. Since $B$ is a finite left $T$-module and a finite right $B$-module, $A$ is a left and right noetherian ring. Since $A$ is bounded, it is trivial that $A$ satisfies $\chi$.

We have $\mathrm{Hom}(A_1, A_1) \cong \mathrm{Hom}(B_B, B_B) \cong B$. Left multiplication by $A$ on $A_1$ is through the quotient $T$. So the induced right action on $\mathrm{Hom}(A_1, A_1)$ is through $T$ too, and since $B_T$ is not a finite module, $\mathrm{Hom}(A_1, A_1)$ is not a finite $A$-module (see [SZ1, Ex. 3.1]). We let $M$ be the right $A$-module $A_1$, so that $\mathrm{Hom}(A_1, M)$ is finitely generated. Consider the Ext sequence

$$0 \to \mathrm{Hom}(A_0, M) \to M_0 \to \mathrm{Hom}(A_1, M) \to \mathrm{Ext}^1(A_0, M) \to 0$$

associated to the exact sequence

$$0 \to A_1 \to A \to A_0 \to 0.$$

Since $M_0 = 0$, $\mathrm{Hom}(A_1, M) \cong \mathrm{Ext}^1(A_0, M)$. Thus $\mathrm{Ext}^1(A_0, M)$ is finitely generated too.

EXAMPLE 3.10. *A noetherian graded ring $R$ which satisfies $\chi$, but there is a finite right $R$-module $M$ such that $\underline{\mathrm{Ext}}^1(R/R_{\geq n}, M)$ is not a finite right $R$-module for any $n \geq 1$.* Let $A$ be as in Example 3.9, let $R = A[x]$ with $\deg(x) = 1$, and let $M$ be the module $A_1$, on which $R$ acts via $x = 0$. We will prove later (Corollary 8.12(1)) that $R$ satisfies $\chi$ because $A$ does. We set $N = R$ in (3.1.3), and apply $\underline{\mathrm{Ext}}^j(\cdot, M)$ to this exact sequence, obtaining an exact sequence

$$\cdots \to \underline{\mathrm{Hom}}(R/R_{\geq n+1}, M) \to \underline{\mathrm{Hom}}(R_n, M) \to \underline{\mathrm{Ext}}^1(R/R_{\geq n}, M) \to \cdots.$$

As $A$-module, $R_n \cong A_0[-n] \oplus A_1[-n+1]$. Hence $\underline{\mathrm{Hom}}(R_n, M) \cong \underline{\mathrm{Hom}}(A_0, M)[n] \oplus \underline{\mathrm{Hom}}(A_1, M)[n-1]$, which is infinitely generated by Example 3.9. Since $\underline{\mathrm{Hom}}(R/R_{\geq n+1}, M) = M$ is finitely generated, $\underline{\mathrm{Ext}}^1(R/R_{\geq n}, M)$ is infinitely generated for every $n \geq 1$.

PROPOSITION 3.11. (1) *$\chi_0$ holds for every right noetherian $\mathbb{N}$-graded ring $A$.*

(2) *If $A$ is locally finite, then $\chi_i$ is equivalent with $\chi_i^\circ$.*

(3) *$\chi$ holds if $A$ is commutative.*

*Proof.* (1) In fact, left multiplication by $A$ on $A/A_{\geq n}$ makes $\underline{\mathrm{Hom}}(A/A_{\geq n}, M)$ into a finite right module for every $n$, because the torsion submodule $\tau(M)$ is a finite $A$-module, and $\underline{\mathrm{Hom}}(A/A_{\geq n}, M)$ is isomorphic to a submodule of $\tau(M)$.

(2) We have already seen that $\chi_i(M)$ implies $\chi_i^\circ(M)$. The converse follows from Proposition 3.1(3).

(3) If $A$ is commutative, then the $A$-module structures on $\underline{\mathrm{Ext}}^j(N, M)$ induced by (left = right) multiplication on $N$ and on $M$ are the same, and

by Proposition 3.1(4), $\underline{\mathrm{Ext}}^j(N, M)$ is a finite $A$-module whenever $N$ and $M$ are finite. Thus $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ is a finite $A$-module, as required. ∎

PROPOSITION 3.12. (1) *Let $\{I_n \subset A \mid n \geqslant 0\}$ be a family of right ideals which is cofinal with $A_{\geqslant n}$, i.e., for every n, and every $n' \gg 0$, $I_n \supseteq A_{\geqslant n'}$ and $A_{\geqslant n} \supseteq I_{n'}$. Then $\chi_i^\circ(M)$ holds if and only if for $j \leqslant i$, $\underline{\mathrm{Ext}}^j(A/I_n, M)$ is a bounded module A-module for $n \gg 0$. In this case, there is a k-linear isomorphism*

$$\underline{\mathrm{Ext}}^j(A/I_n, M)_{\geqslant d} \cong \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d} \tag{3.12.1}$$

*for* $j \leqslant i$ and $n \gg 0$. *Hence $\chi_i(M)$ holds if and only if the right A-module structure induced on $\underline{\mathrm{Ext}}^j(A/I_n, M)_{\geqslant d}$ by this isomorphism makes it into a finite module.*

(2) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then $\chi_i(M')$* and *$\chi_i(M'')$ imply $\chi_i(M)$, and $\chi_{i+1}(M')$ and $\chi_i(M)$ imply $\chi_i(M'')$.*

(3) *$\chi_i$ holds for A if and only if every finite nonzero module M has a nonzero submodule M′ such that $\chi_i(M')$ holds.*

(4) *$\chi_i(M)$ holds for every right bounded module M.*

*Proof.* The proofs of assertions (2) and (3) are routine, and (4) follows from Proposition 3.1(5), which shows that for a right bounded module, $M$, $\underline{\mathrm{Ext}}^j(A/A_n, M)_{\geqslant d} = 0$ if $n \gg 0$. The first assertion of (1) follows from Corollary 3.6(2). For the second, we consider the exact sequence

$$0 \to A_{\geqslant n}/I_{n'} \to A/I_{n'} \to A/A_{\geqslant n} \to 0.$$

Lemma 3.4 shows that for $j \leqslant i$, the right bound of $\underline{\mathrm{Ext}}^j(A_{\geqslant n}/I_{n'}, M)$ tends to negative infinity with $n$. Therefore the map $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d} \to \underline{\mathrm{Ext}}^j(A/I_{n'}, M)_{\geqslant d}$ is bijective if $n \gg 0$. Now Proposition 3.8(2) completes the proof. ∎

The next two propositions explain the special nature of the condition $\chi_1$. We use the standard notation which was introduced in Section 2: $\mathscr{A}$, $\mathscr{M}$ denote the images $\pi(A)$, $\pi(M)$ respectively in QGr $A$, $\tilde{M} = \omega(\pi(M))$, and $\tau(M)$ is the torsion submodule of $M$. Also, anticipating the notation to be introduced in the next section, we set $\Gamma(\mathscr{M}) := \underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{M})$. Note that by (2.2.1) and (2.2.2),

$$\lim_{n \to \infty} \underline{\mathrm{Hom}}(A_{\geqslant n}, M) \cong \Gamma(\mathscr{M}) \cong \tilde{M}. \tag{3.12.2}$$

Passing to the limit over the sequence (3.1.4) yields a sequence

$$0 \to \tau(M) \to M \to \tilde{M} \to \lim_{n \to \infty} \underline{\mathrm{Ext}}^1(A/A_{\geqslant n}, M) \to 0. \tag{3.12.3}$$

PROPOSITION 3.13. *Let $M$ be a finite module such that $\chi_1^\circ(M)$ holds. Then*

(1) *For every integer $d$ and every $n \gg 0$,*

$$\underline{\mathrm{Hom}}(A_{\geqslant n}, M)_{\geqslant d} \cong \Gamma(\mathcal{M})_{\geqslant d}.$$

(2) *There is an integer $l$ such that $M_{\geqslant l} \cong \Gamma(\mathcal{M})_{\geqslant l} \cong \tilde{M}_{\geqslant l}$.*

(3) *$\chi_1(M)$ holds if and only if $\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{M})_d$ is a finite $A$-module (and a finite $A_0$-module) for every $d$.*

*Proof.* (1) This follows from (3.12.2) and Proposition 3.5(2).

(2) This follows from (1), Proposition 3.5(3), and (3.12.2).

(3) By Propositions 3.8(2) and 3.11(1), $\chi_1(M)$ holds if and only if $\lim_{n \to \infty} \underline{\mathrm{Ext}}^1(A/A_{\geqslant n}, M)_d$ is a finite module for every $d$. (3.12.2) and (3.12.3) show that this is true if and only if $\Gamma(\mathcal{M})_d = \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{M})_d$ is finite for every $d$. ∎

PROPOSITION 3.14. (1) *A module $M$ is isomorphic to $\tilde{M}$ if and only if $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M) = 0$ for $j = 0, 1$ and for all $n$.*

(2) *Let $M$ be a finite module. Then*

(a) *$\chi_1^\circ(M)$ holds and only if the cokernel of the map $M \to \tilde{M}$ is right bounded.*

(b) *$\chi_1(M)$ holds if and only if for every $d$, $\tilde{M}_{\geqslant d}$ is a finite $A$-module.*

*Proof.* (1) The sequence (3.12.3) shows that the condition is sufficient. Conversely, if $M \to \tilde{M}$ is bijective, then $\tau(M) = 0$, hence $\underline{\mathrm{Hom}}(A/A_{\geqslant n}, M) = 0$ for every $n$. Also, the identity map $M \to M$ is a final object in the category $C_M$ of (2.2.3). Hence $\underline{\mathrm{Ext}}^1(N, M) = 0$ for every torsion module $N$.

(2) (2a) follows from the sequence (3.12.3) and Corollary 3.6(3), and (2b) follows from (3.12.3) and Proposition 3.8(3). ∎

## 4. CHARACTERIZATION OF Proj

Let $(\mathscr{C}, \mathcal{A}, s)$ be a triple of the type considered in Section 2, so that $\mathscr{C}$ is a $k$-linear abelian category, $\mathcal{A}$ is a special object of $\mathscr{C}$ called the *structure sheaf*, and $s$ is an autoequivalence of $\mathscr{C}$ called the *shift operator*. The *global section functor* $\mathrm{H}^0$ on $(\mathscr{C}, \mathcal{A}, s)$ is defined by

$$\mathrm{H}^0(\mathcal{M}) = \mathrm{Hom}(\mathcal{A}, \mathcal{M})$$

for $\mathcal{M} \in \mathscr{C}$. As discussed in the introduction, the purpose of this section is to characterize triples which are isomorphic to (proj $A, s$) for some right noetherian $\mathbb{N}$-graded $k$-algebra $A$. To do this we begin by constructing a graded algebra $A$ and a *representing functor* $\Gamma: \mathscr{C} \to \mathrm{Gr}\, A$.

Suppose first that the shift operator $s$ is an *automorphism* of the category $\mathscr{C}$. In this case, $s$ is invertible, and so its powers $s^d$ are defined for all $d \in \mathbb{Z}$. For $\mathscr{M} \in \mathscr{C}$, we define

$$\Gamma(\mathscr{M}) := \bigoplus_{d=-\infty}^{\infty} \mathrm{H}^0(\mathscr{M}[d]) \tag{4.0.1}$$

where $\mathscr{M}[d] = s^d(\mathscr{M})$ and we set

$$A := \Gamma(\mathscr{A})_{\geqslant 0}. \tag{4.0.2}$$

Multiplication is defined as follows: If $x \in \mathrm{H}^0(\mathscr{M}[i])$, $a \in \mathrm{H}^0(\mathscr{A}[j])$, and $b \in \mathrm{H}^0(\mathscr{A}[k])$, then

$$x \cdot a = s^j(x) \circ a \qquad \text{and} \qquad a \cdot b = s^k(a) \circ b. \tag{4.0.3}$$

With this law of composition, $\Gamma(\mathscr{M})$ becomes a graded right module over the $\mathbb{Z}$-graded algebra $\Gamma(\mathscr{A})$ and over its $\mathbb{N}$-graded subalgebra $A$. The associative law for multiplication, and the fact that $\Gamma$ is a functor from $\mathscr{C}$ to Gr $A$ are easily checked.

In general, the shift operator $s$ may be only an autoequivalence, but it will still have a quasi-inverse, which we denote by $s_{-1}$. This means that $s_{-1}$ is a left and right adjoint to $s$, and hence that there are natural equivalences

$$\eta_{1,-1} : ss_{-1} \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}} \qquad \text{and} \qquad \eta_{-1,1} : s_{-1}s \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$$

such that

$$\eta_{1,-1}s = s\eta_{-1,1} \qquad \text{and} \qquad s_{-1}\eta_{1,-1} = \eta_{-1,1}s_{-1} \tag{4.0.4}$$

(see [Mi, Prop. II.10.1]). In this situation, we set $s_n = s^n$ and $s_{-n} = (s_{-1})^n$ for $n \geqslant 0$, and we define isomorphisms of functors $\eta_{i,j} : s_i s_j \xrightarrow{\sim} s_{i+j}$ as follows: $\eta_{i,j} = \textit{identity}$ if $i$, $j$ have the same sign, and $\eta_{i,j}$ is obtained as the appropriate iteration of the isomorphism $\eta_{1,-1}$ or $\eta_{-1,1}$ if $i, j$ have opposite signs. Then it follows from (4.0.4) that for all $i$, $j$, $k$, the following two isomorphisms of functors which send $s_i s_j s_k \to s_{i+j+k}$ are equal:

$$\eta_{i,j+k} \circ (s_i \eta_{j,k}) = \eta_{i+j,k} \circ (\eta_{i,j} s_k). \tag{4.0.5}$$

We set $\mathscr{M}[d] = s_d(\mathscr{M})$, and we define $\Gamma(\mathscr{M})$, $\Gamma(\mathscr{A})$ and $A$ by the same formulas (4.0.1), (4.0.2) as above, but replacing the multiplication rule (4.0.3) by

$$x \cdot a = \eta_{i,j}(\mathscr{M}) \circ s_j(x) \circ a \qquad \text{and} \qquad a \cdot b = \eta_{j,k}(\mathscr{A}) \circ s_k(a) \circ b.$$

The associative law of multiplication is proved using formula (4.0.5).

The functorial behavior of this definition is explained by the next proposition. We consider a collection of data $\mathscr{D} = \{\mathscr{C}, \mathscr{A}, \{s_i\}, \{\eta_{i,j}\}\}$, where $s_i$ are autoequivalences of $\mathscr{C}$ and $\eta_{i,j}$ are isomorphisms of functors satisfying (4.0.5). A *morphism* $\mathscr{F}: \mathscr{D} \to \mathscr{D}'$ of such data consists of a functor $f: \mathscr{C} \to \mathscr{C}'$, an isomorphism $\theta: f(\mathscr{A}) \xrightarrow{\sim} \mathscr{A}'$, and for each $i$ an isomorphism of functors $\mu_i: fs_i \xrightarrow{\sim} s_i' f$, such that the following diagrams commute:

$$\begin{array}{ccc} fs_is_j & \xrightarrow{s_i'\mu_j \circ \mu_i s_j} & s_i's_j'f \\ {\scriptstyle f\eta_{i,j}}\downarrow & & \downarrow{\scriptstyle \eta_{i,j}'f} \\ fs_{i+j} & \xrightarrow[\mu_{i+j}]{} & s_{i+j}'f. \end{array}$$

**Proposition 4.1.** *Let $\mathscr{F}: \mathscr{D} \to \mathscr{D}'$ be a morphism of data. Define a map $\Phi: \Gamma(\mathscr{M}) \to \Gamma(f(\mathscr{M}))$ as follows: If $x \in \Gamma(\mathscr{M})_i$, then $\Phi(x) = \mu_i(\mathscr{M}) f(x) \theta^{-1}$. This rule defines a homomorphism of graded algebras $\phi = \Gamma(\theta) \circ \Phi: \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}')$, and, for every $\mathscr{M} \in \mathscr{C}$, a $\phi$-linear map $\Phi: \Gamma(\mathscr{M}) \to \Gamma(f(\mathscr{M}))$.* ∎

The isomorphisms $\eta_{i,j}$ clutter up the notation rather badly, so it is more convenient to work with the case that the shift operator $s$ is actually an automorphism. The general case can be referred to this special case by the following method: Given data $\mathscr{D} = \{\mathscr{C}, \mathscr{A}, \{s_i\}, \{\eta_{i,j}\}\}$, we define a new catagory $\mathscr{C}'$ as follows: An object $\mathscr{M}'$ of $\mathscr{C}'$ is a collection $(\{\mathscr{M}_n\}, \{m_{i,j}\})$ consisting of a sequence of objects $\mathscr{M}_n$ of $\mathscr{C}$ indexed by the integers, and for each pair of integers $i, j$, and isomorphism $m_{i,j}: s_i\mathscr{M}_j \xrightarrow{\sim} \mathscr{M}_{i+j}$, such that for all $i, j, k$, the diagram

$$\begin{array}{ccc} s_is_j\mathscr{M}_k & \xrightarrow{s_im_{j,k}} & s_i\mathscr{M}_{j+k} \\ {\scriptstyle n_{i,j}}\downarrow & & \downarrow{\scriptstyle m_{i,j+k}} \\ s_{i+j}\mathscr{M}_k & \xrightarrow[m_{i+j,k}]{} & \mathscr{M}_{i+j+k} \end{array}$$

commutes. A map $(\{\mathscr{M}_n\}, \{m_{i,j}\}) \to (\{\mathscr{N}_n\}, \{n_{i,j}\})$ consists of maps $f_n: \mathscr{M}_n \to \mathscr{N}_n$ for each $n$ which are compatible with the isomorphisms, i.e., such that for all $i, j$, we have a commutative diagram

$$\begin{array}{ccc} s_i\mathscr{M}_j & \xrightarrow{s_if_j} & s_i\mathscr{N}_j \\ {\scriptstyle m_{i,j}}\downarrow & & \downarrow{\scriptstyle n_{i,j}} \\ \mathscr{M}_{i+j} & \xrightarrow[f_{i+j}]{} & \mathscr{N}_{i+j}. \end{array}$$

We define adjoint functors between $\mathscr{C}'$ and $\mathscr{C}$ as follows: An object $(\{\mathscr{M}_n\}, \{m_{i,j}\}) \in \mathscr{C}'$ is sent to $\mathscr{M}_0 \in \mathscr{C}$. In other direction, and object $\mathscr{M} \in \mathscr{C}$ is sent $(\{s_n(\mathscr{M})\}, \{\eta_{i,j}(\mathscr{M})\})$, and (4.0.5) implies that this is an object of $\mathscr{C}'$.

PROPOSITION 4.2. (1) *The category* $\mathscr{C}'$ *defined above has an automorphism* $s'$ *defined by* $s'((\{\mathscr{M}_n\}, \{m_{i,j}\})) = (\{\mathscr{M}'_n\}, \{m'_{i,j}\})$ *where* $\mathscr{M}'_n = \mathscr{M}_{n+1}$ *and* $m'_{i,j} = m_{i,j+1}$.

(2) *Let* $\mathscr{D}'$ *be the data* $\{\mathscr{C}', \mathscr{A}', \{s'^i\}, \{id_{i,j}\}\}$, *where* $\mathscr{A}' = (\{\mathscr{A}_n\}, \{a_{i,j}\})$, *and* $\mathscr{A}_n = s_n(\mathscr{A})$ *and* $a_{i,j} = \eta_{i,j}(\mathscr{A})$. *The functor* $f: \mathscr{C} \to \mathscr{C}'$ *defined above is an equivalence and it extends to a map of data for which the maps described in Proposition* 4.1 *are isomorphisms.* ▮

This proposition justifies restricting to the case that the shift is an automorphism, and we will do this from now on. Thus our functor $\Gamma$ will be defined by (4.0.1) and (4.0.2).

The shift operator $s$ is called *ample* if

$$\begin{array}{ll} \text{(a)} & \text{for every object } \mathscr{M} \text{ of } \mathscr{C}, \text{ there are positive integers} \\ & l_1, ..., l_p \text{ and an epimorphism from } \bigoplus_{i=1}^{p} \mathscr{A}[-l_i] \text{ to } \mathscr{M}. \\ \text{(b)} & \text{for every epimorphism } f: \mathscr{M} \to \mathscr{N}, \text{ there exists an} \\ & \text{integer } n_0 \text{ such that for every } n \geqslant n_0, \text{ the map} \\ & \mathrm{H}^0(\mathscr{M}[n]) \to \mathrm{H}^0(\mathscr{N}[n]) \text{ is surjective.} \end{array} \qquad (4.2.1)$$

The condition (4.2.1)(a) implies that the negative shifts of $\mathscr{A}$ form a set of generators for $\mathscr{C}$. Note that if (4.2.1)(a) holds, then for any integer $r$ and for each object $\mathscr{M}$, there are integers $l_1, ..., l_p > r$ and an epimorphism from $\bigoplus_{i=1}^{p} \mathscr{A}[-l_i]$ to $\mathscr{M}$. To show this, one replaces $\mathscr{M}$ by $\mathscr{M}[r]$.

COROLLARY 4.3. *Let* $F_i = (f_i, \theta_i, \mu_i): (\mathscr{C}, \mathscr{A}, s) \to (\mathscr{C}', \mathscr{A}', s')$ *be right exact morphisms of triples, for* $i = 1, 2$. *Assume that* $s$, $s'$ *are automorphisms and that* $s$ *satisfies* (4.2.1)(a). *Then there is at most one natural equivalence* $\varepsilon: F_1 \to F_2$.

*Proof.* The compatibility of $\varepsilon$ with $\theta$, see (2.3.1), determines $\varepsilon(\mathscr{A})$, and the compatibility with $\mu$, see (2.3.2), determines $\varepsilon(s^n\mathscr{A})$ for every $n \in \mathbb{Z}$. Since the functors $f_i$ are right exact and (4.2.1)(a) holds, $\varepsilon$ is determined on every object $\mathscr{M}$ of $\mathscr{C}$. ▮

PROPOSITION 4.4. (1) *Let* $A$ *be a right noetherian* $\mathbb{N}$*-graded algebra. Then* (4.2.1)(a) *holds for* (qgr $A$, $\mathscr{A}$, $s$).

(2) *Let* $A$ *and* $B$ *be two right noetherian* $\mathbb{N}$*-graded algebras, and let* $F_i = (f_i, \theta_i, \mu_i)$: (qgr $A$, $\mathscr{A}$, $s_A$) $\to$ (qgr $B$, $\mathscr{B}$, $s_B$) *be right exact morphisms of triples, for* $i = 1, 2$. *Then there is at most one natural equivalence* $\varepsilon: F_1 \to F_2$.

*Proof.* (1) For any $\mathscr{M} = \pi(M)$, we have $\mathscr{M} = \pi(M_{\geqslant 1})$ because $M/M_{\geqslant 1}$ is torsion. Since $M_{\geqslant 1}$ is generated by negative shifts of $A$, $\mathscr{M}$ is generated by negative shifts of $\mathscr{A}$, by exactness of $\pi$. This is (4.2.1)(a).

(2) Since $s_A$ and $s_B$ are automorphisms, (2) follows from (1) and Corollary 4.3. ∎

The main theorem of this section is the following noncommutative version of Serre's Theorem. In the statement, $\chi_1$ refers to the condition which was introduced in Section 3.

THEOREM 4.5. (1) *Let* $(\mathscr{C}, \mathscr{A}, s)$ *be a triple as above, such that the following hold*:

(H1) $\mathscr{A}$ *is noetherian,*

(H2) $A_0 := \mathrm{H}^0(\mathscr{A})$ *is a right noetherian ring and* $\mathrm{H}^0(\mathscr{M})$ *is a finite* $A_0$*-module for all* $\mathscr{M}$,

(H3) $s$ *is ample.*

*Let* $A = \Gamma(\mathscr{A})_{\geq 0}$. *Then* $A$ *is right noetherian* $\mathbb{N}$*-graded* $k$*-algebra which satisfies* $\chi_1$, *and* $(\mathscr{C}, \mathscr{A}, s)$ *is isomorphic to* $(\operatorname{proj} A, s)$.

(2) *Conversely, let* $B$ *be a right noetherian* $\mathbb{N}$*-graded algebra satisfying* $\chi_1$. *Then* (H1), (H2), *and* (H3) *hold for the triple* $(\operatorname{qgr} B, \mathscr{B}, s)$. *Moreover, if* $A = \Gamma(\mathscr{B})_{\geq 0}$, *then there is a canonical homomorphism* $B \to A$ *whose kernel and cokernel are right bounded, and which induces an isomorphism* $(\operatorname{proj} B, s) \to (\operatorname{proj} A, s)$.

*Remarks.* (1) The conditions (H1) and (H3) imply that all objects $\mathscr{M}$ of $\mathscr{C}$ are noetherian.

(2) (H1) and (H2) are somewhat related. For example, if $\mathrm{H}^0$ is faithful and $A_0$ is right noetherian, then (H1) holds. If $\mathscr{A}$ is a projective object, then (H1) and (H3) imply (H2) (see, for example, the proof of Proposition 5.6).

(3) The ring $\Gamma(\mathscr{A})$ may not be right noetherian even if (H1), (H2), and (H3) hold for $(\mathscr{C}, \mathscr{A}, s)$. See Example 5.7.

(4) Without the hypotheses (H1), (H2), and (H3), the ring $\Gamma(\mathscr{A})_{\geq 0}$ may not be right noetherian. For example, let $(\mathscr{C}, \mathscr{A}, s)$ be the triple $(\operatorname{Gr} A, A_A, s)$ for some non right noetherian $\mathbb{N}$-graded ring $A$. Then $\Gamma(A_A) = A$ is not right noetherian.

*Proof of Theorem* 4.5. We are going to show that $A$ is right noetherian and that $\Gamma(\mathscr{M})_{\geq 0}$ is a finite $A$-module for every $\mathscr{M} \in \mathscr{C}$. Since $\Gamma(\mathscr{M})$ and $\Gamma(\mathscr{M})_{\geq 0}$ have the same image in $\operatorname{QGr} A$, it will follow that $\Gamma$ actually defines a functor $\bar{\Gamma}: \mathscr{C} \to \operatorname{qgr} A$. This is the functor which induces the isomorphism $(\mathscr{C}, \mathscr{A}, s) \to (\operatorname{proj} A, s)$. But for the proof of these facts, we introduce the following temporary notation. We denote by rtbd the dense subcategory of $\operatorname{Gr} A$ of right bounded modules. Let $\mathscr{D}$ denote the quotient category $\operatorname{Gr} A/\mathrm{rtbd}$, and let $\bar{\Gamma}: \mathscr{C} \to \mathscr{D}$ denote the functor $\pi\Gamma$, where $\pi$ is the canonical functor from $\operatorname{Gr} A$ to the quotient category $\mathscr{D}$.

We break the proof into 10 steps, each of which is relatively easy to prove. We suppose that the triple $(\mathscr{C}, \mathscr{A}, s)$ satisfies (H1), (H2), and (H3).

S1. *$\Gamma$ is left exact and $\bar{\Gamma}$ is exact.*

*Proof of* S1. Being defined in terms of Hom, the functor $\Gamma$ is left exact, and since $\pi$ is exact, $\bar{\Gamma}$ is also left exact. To verify right exactness, let $f: \mathscr{M} \to \mathscr{N}$ be a surjection. Then (4.2.1)(b) implies that $\Gamma(f): \Gamma(\mathscr{M}) \to \Gamma(\mathscr{N})$ is surjective in sufficiently high degree. This means that $\Gamma(\mathscr{N})/\operatorname{im}(\Gamma(\mathscr{M}))$ is right bounded. Thus $\bar{\Gamma}(f)$ is surjective.

S2. *Let $\mathscr{N}$ be a subobject of the object $\mathscr{M}$ of $\mathscr{C}$. Then $\Gamma(\mathscr{M})/\Gamma(\mathscr{N})$ is torsion-free. In particular, $\Gamma(\mathscr{M})$ is torsion-free.*

*Proof of* S2. Let $x$ be a homogeneous element in $\Gamma(\mathscr{M})$ of degree $r$ such that $xA_{\geqslant n} \subseteq \Gamma(\mathscr{N})$. We want to prove $x \in \Gamma(\mathscr{N})$. By definition, $x$ is a map from $\mathscr{A}$ to $\mathscr{M}[r]$, but applying the shift, we may also regard it as a map from $\mathscr{A}[-r]$ to $\mathscr{M}$. We choose an epimorphism $u: \mathscr{P} \to \mathscr{A}[-r]$, where $\mathscr{P}$ denotes a sum of shifts of $\mathscr{A}$, say $\bigoplus_{i=1}^{p} \mathscr{A}[-l_i]$, and where $l_i \geqslant n+r$ for all $i$. The map $u$ is a sum of maps $u_i$: $\mathscr{A}[-l_i] \to \mathscr{A}[-r]$, and the composed map $xu$ is a sum of maps $xu_i$: $\mathscr{A}[-l_i] \to \mathscr{M}$. We shift back, in order to regard $u_i$ as an element of $A_{l_i-r}$ and $xu_i$ as an element of $\mathrm{H}^0(M[l_i]) = \Gamma(\mathscr{M})_{l_i}$. Then $xu_i$ lies in $\Gamma(\mathscr{N})$ because $u_i \in A_{l_i-r}$ and $l_i - r \geqslant n$. By definition of $\Gamma(\mathscr{N})$, this means that the image of $xu_i$: $\mathscr{A}[-l_i] \to \mathscr{M}$ lies in $\mathscr{N}$, and hence the image of $xu$ is also in $\mathscr{N}$. Since $u$ is an epimorphism, $x$ defines a map from $\mathscr{A}[-r]$ to $\mathscr{N}$; i.e., $x$ is in $\Gamma(\mathscr{N})$. The last assertion follows by setting $\mathscr{N} = 0$.

S3. *$\bar{\Gamma}$ is faithful.*

*Proof of* S3. Since $\bar{\Gamma}$ is an exact functor, it is enough to show that $\bar{\Gamma}(\mathscr{M}) \neq 0$ when $\mathscr{M} \neq 0$. Now (4.2.1)(a) provides nonzero maps $\mathscr{A}[-l] \to \mathscr{M}$. Hence $\Gamma(\mathscr{M}) \neq 0$. By S2, $\Gamma(\mathscr{M})$ contains no nonzero, right bounded submodule. Hence $\bar{\Gamma}(\mathscr{M}) \neq 0$.

S4. *$A$ is a right noetherian ring.*

*Proof of* S4. We only need to prove that $A$ is graded right noetherian. Let $N$ be a graded right ideal of $A$. A homogeneous element $x$ of degree $r$ defines a map from $\mathscr{A}[-r]$ to $\mathscr{A}$. Given a finite set $X$ of homogeneous elements of $N$, let $\mathscr{P}_X = \bigoplus \mathscr{A}[-r_x]$, where $r_x$ is the degree of $x$. We obtain a map $f_X: \mathscr{P}_X \to \mathscr{A}$ defined by the elements of $X$, and we denote its image by $\mathscr{N}_X$.

Since $\mathscr{A}$ is noetherian, we may choose a finite set $X$ so that $\mathscr{N}_X$ is maximal among such subobjects. We set $\mathscr{N} = \mathscr{N}_X$, $\mathscr{P} = \mathscr{P}_X$. Also, we set

$N'' := \Gamma(\mathcal{N})_{\geqslant 0}$, $P'' := \bigoplus A[-r_x]$, and we write $N'$ for the image of the map $P'' \to A$ defined by $X$. As above, any element $x \in N_r$ defines a map $\mathcal{A}[-r] \to \mathcal{A}$, and the image of this map is in $\mathcal{N}$ because $\mathcal{N}$ is maximal. This means that $x \in N''$. Hence $N \subset N''$. Also, for $x \in N_r$, the map $A[-r] \to A$ defined by mulplication by $x$ has its image in $N$. Hence $N' \subset N$.

Now $\Gamma(\mathcal{N})/\mathrm{im}(\Gamma(\mathcal{P}))$ is right bounded because $\bar{\Gamma}$ is exact. Furthermore, $\Gamma(\mathcal{P}) = \bigoplus \Gamma(\mathcal{A}[-r_x])$ and hence $\Gamma(\mathcal{P})/P''$ is also right bounded. It follows that $N''/N'$ is bounded. By (H2), $N''/N'$ is a noetherian $A_0$-module. Hence $N/N'$ is noetherian. Since $N'$ is a finitely generated right ideal by its construction, this shows that $N$ is finitely generated.

S5. *For all $\mathcal{M}$ and all $d$, $\Gamma(\mathcal{M})_{\geqslant d}$ is a finite right $A$-module.*

*Proof of* S5. It suffices to prove this for the case $d=0$. Let $M = \Gamma(\mathcal{M})_{\geqslant 0}$. We choose a finite sum $\mathcal{P} = \bigoplus_i \mathcal{A}[-l_i]$ and an epimorphism from $\mathcal{P}$ to $\mathcal{M}$ as in (4.2.1)(a). Let $P = \bigoplus_i A[-l_i]$. Applying $\Gamma$, we obtain a map from $P$ to $M$ whose cokernel is bounded. The cokernel is finitely generated by (H2). Since $P$ is finitely generated, so is $M$.

It follows from S5 that $\Gamma_{\geqslant 0}$ defines a functor $\mathscr{C} \to \mathrm{gr}\, A$. From now on, we let $\pi$ be the canonical functor from gr $A$ to its quotient category qgr $A$. Then $\bar{\Gamma} = \pi\Gamma$ is a functor $\mathscr{C} \to \mathrm{qgr}\, A$. Since the torsion modules in gr $A$ are the right bounded modules, this is just restriction to a dense subcategory of $\mathscr{D}$. Hence the analogues of S1 and S3 still hold.

Next, we are going to prove that $\bar{\Gamma}$ is an equivalence.

S6. *For every finite $A$-module $M$, there is an object $\mathcal{M}$ such that $\bar{\Gamma}(\mathcal{M}) \cong \pi(M)$.*

*Proof of* S6. We may replace $M$ by $M_{\geqslant 1}$. Choose finite sums $P_1$, $P_0$ of negative shifts of $A$ and an exact sequence

$$P_1 \to P_0 \to M \to 0.$$

Let $\mathcal{P}_i$ be the sum of shifts of $\mathcal{A}$ corresponding to $P_i$, and let $\mathcal{M}$ be the cokernel of the induced map $\mathcal{P}_1 \to \mathcal{P}_0$. Then $\bar{\Gamma}(\mathcal{P}_i) \cong \pi(P_i)$. Therefore, applying $\bar{\Gamma}$ to the sequence $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0$ yields an exact sequence $\pi(P_1) \to \pi(P_0) \to \bar{\Gamma}(\mathcal{M}) \to 0$. This shows that $\bar{\Gamma}(\mathcal{M})$ and $\pi(M)$ are isomorphic.

S7. *Let $M = \Gamma(\mathcal{M})$, and let $M \to M'$ be an injective $A$-module homomorphism. If $M'$ is torsion-free, then $M'/M$ is torsion-free.*

Corollary. *$\Gamma(\mathcal{M})$ is isomorphic to $\omega(\pi\Gamma(\mathcal{M})) = \widetilde{\Gamma(\mathcal{M})}$.*

The corollary follows from S7, S2, and Proposition 3.14(1).

*Proof of* S7. Let $x$ be a homogeneous element of degree $r$ of $M'$ such that $xA_{\geqslant n} \subseteq M$. So $x$ defines a map from $A[-r]$ to $M'$. We choose finite sums of negative shifts of $\mathscr{A}$, say $\mathscr{P}_0 = \bigoplus_i \mathscr{A}[-v_i]$ and $\mathscr{P}_1 = \bigoplus_j \mathscr{A}[-w_j]$ and an exact sequence

$$\mathscr{P}_1 \to \mathscr{P}_0 \to \mathscr{A}[-r] \to 0.$$

Denote the maps above by $f = (f_{ij})$ and $g = (g_i)$, respectively, where $f_{i,j} \in A_{w_j - v_i}$ and $g_i \in A_{v_i - r}$, and where $\sum g_i f_{ij} = 0$ for all $j$. We may choose the sequence so that

(i) $v_i - r \geqslant n$ for all $i$, and

(ii) $A/\sum g_i A$ is bounded.

We can achieve (ii) by choosing $\{g_i\}$ to contain a system of generators for the right ideal $A_{\geqslant d}$ for some $d \gg 0$. By (i), $xg_i \in M$ for all $i$. Therefore $xg$ defines a map $\phi: \mathscr{P}_0 \to \mathscr{M}$, and the composed map $\phi f$ is zero. It follows from the exactness of the $\mathrm{Hom}(\cdot, \mathscr{M})$ sequence associated to the above sequence that there is a map $y: \mathscr{A}[-r] \to \mathscr{M}$ such that $yg = \phi = xg$. This map can be interpreted as an element $y \in M_r$ such that $yg_i = xg_i$ for all $i$. Thus $y - x$ is annihilated by the right ideal $\sum g_i A \supseteq A_{\geqslant N}$. Hence $y - x$ is a torsion element. But $M'$ is torsion-free, and therefore $x = y \in M$. This completes the proof.

S8. *The functor* $\bar{\Gamma}: \mathscr{C} \to \operatorname{qgr} A$ *is an equivalence of categories, and it defines an isomorphism of triples* $(\mathscr{C}, \mathscr{A}, s) \to (\operatorname{proj} A, s)$.

*Proof of* S8. Because of S1, S3, and S6, it suffices to show that the functor is full. Given a pair $\mathscr{N}$, $\mathscr{M}$ of objects of $\mathscr{C}$, let $N = \Gamma(\mathscr{N})$ and $M = \Gamma(\mathscr{M})$. To say that $\bar{\Gamma}$ is full means that $\mathrm{Hom}(\pi(N), \pi(M)) \cong \mathrm{Hom}(\mathscr{N}, \mathscr{M})$. By the Corollary of S7 and (2.2.2),

$$\mathrm{Hom}(\pi(N), \pi(M)) = \mathrm{Hom}(\pi(N_{\geqslant n}), \pi(M)) = \mathrm{Hom}(N_{\geqslant n}, M)$$

for all $n$. We do have $\mathrm{Hom}(A, M) = M_0 = \mathrm{Hom}(\mathscr{A}, \mathscr{M})$. So the result is true when $\mathscr{N} = \mathscr{A}$. Similarly, it is true for a shift of $\mathscr{A}$, and for a sum of such shifts. Since every $\mathscr{N}$ has a resolution by such sums, the required equality follows from the left exactness of Hom.

By definition, $\bar{\Gamma}(\mathscr{A}) = \pi(A_A)$ and $\bar{\Gamma}s = s\bar{\Gamma}$. Therefore $\bar{\Gamma}$ defines an isomorphism of triples from $(\mathscr{C}, \mathscr{A}, s)$ to $(\operatorname{proj} A, s)$.

S9. *The algebra A satisfies* $\chi_1$.

*Proof of* S9. Since $A$ is right noetherian, $\chi_0$ is satisfied (Proposition 3.11(1)). S6 tells us that for every finite module $M$, there is an object $\mathscr{M}$

such that $\bar{\Gamma}(\mathcal{M}) \cong \pi(M)$. Applying $\omega$ and S7, we obtain $\tilde{M} \cong \widetilde{\Gamma(\mathcal{M})} = \Gamma(\mathcal{M})$. Then S5 and Proposition 3.14(2b) imply that $A$ satisfies $\chi_1$.

Finally, let us prove the converse statement.

S10. *If $B$ is a right noetherian $\mathbb{N}$-graded algebra satisfying $\chi_1$, then* (H1), (H2), *and* (H3) *hold for the triple* (qgr $B$, $\mathcal{B}$, $s$), *and if $A = \Gamma(\mathcal{B})_{\geq 0}$, there is a canonical homomorphism $B \to A$ whose kernel and cokernel are right bounded, and which induces an isomorphism* (proj $B$, $s$) $\to$ (proj $A$, $s$).

*Proof of* S10. By Proposition 2.3, $\mathcal{B}$ is noetherian, hence (H1) holds. By Proposition 4.4(1), the condition (4.2.1)(a) holds. Next, a surjective map $f\colon \mathcal{M} \to \mathcal{N}$ in qgr $B$, can be represented by a surjective map $M \to N$ of finite $B$-modules. By Proposition 3.13(2), the map $\Gamma(\mathcal{M}) \to \Gamma(\mathcal{N})$ is surjective in large degree. Thus (4.2.1)(b) holds. By Proposition 3.13(3), $\mathrm{H}^0(\mathcal{M}) = \mathrm{Hom}(\mathcal{B}, \mathcal{M})$ is a noetherian right $B_0$-module for all $\mathcal{M}$. Hence $A_0 := \mathrm{H}^0(\mathcal{B})$ is a noetherian right $B_0$-module and so it is a right noetherian ring. This proves (H2).

We have a canonical map $B_n = \mathrm{Hom}(B_B, B_B[n]) \to \mathrm{Hom}(\mathcal{B}, \mathcal{B}[n]) = A_n$ which defines the homomorphism $B \to A$. The fact that its kernel and cokernel are right bounded follows from Proposition 3.13(2), with $B$ replacing $A$ and $M$. Proposition 2.5(2) tells us that this homomorphism induces an isomorphism of projective schemes. ∎

We are specially interested in the case of a locally finite algebra, for example, a connected graded algebra over a field $k$. For such algebras, Theorem 4.5 can be stated as follows.

COROLLARY 4.6. (1) *Suppose that a triple $(\mathcal{C}, \mathcal{A}, s)$ satisfies*

(H1) *$\mathcal{A}$ is noetherian,*

(H2)′ *$\mathrm{H}^0(\mathcal{M})$ is a finite $k$-module for every object of $\mathcal{C}$, and*

(H3) *$s$ is ample.*

*Then $A = \Gamma(\mathcal{A})_{\geq 0}$ is a right noetherian $\mathbb{N}$-graded locally finite $k$-algebra satisfying $\chi_1$ and $(\mathcal{C}, \mathcal{A}, s) \cong$* (proj $A$, $s$).

(2) *Conversely, if $B$ is a right noetherian $\mathbb{N}$-graded locally finite $k$-algebra satisfying $\chi_1$, then* (H1), (H2)′, *and* (H3) *hold for the triple* (qgr $B$, $\mathcal{B}$, $s$). *Moreover, $B$ can be recovered up to a finite $k$-module from the triple by the functor $\Gamma$.* ∎

PROPOSITION 4.7. *Suppose that a triple $(\mathcal{C}, \mathcal{A}, s)$ satisfies* (H1), (H2)′, *and* (H3) *of Corollary* 4.6. *Then for every object $\mathcal{N}$ of $\mathcal{C}$, the following hold:*

(1) *$\mathcal{N}$ is noetherian;*

(2) $\mathrm{Hom}(\mathcal{N}, \mathcal{M})$ *is finite $k$-module for every $\mathcal{M} \in \mathcal{C}$.*

*Proof.* By Corollary 4.6, we may replace $(\mathscr{C}, \mathscr{A}, s)$ by $(\operatorname{proj} A, s)$ for some locally finite ring $A$ satisfying $\chi_1$. (1) follows from Proposition 2.3 and (2) follows from (2.2.1), Proposition 3.5(2), and Proposition 3.1(3). ∎

## 5. Some Examples

In this section we illustrate the results of the last sections with some elementary examples.

Proposition 5.1. *Every noetherian* $\mathbb{N}$*-graded PI algebra* $A$ *satisfies* $\chi$. ∎

This follows from Thm. 3.5 of [SZ1] by setting $T=A$, $R=A$, and $M=A/A_{\geqslant n}$. Another proof is given in Section 8.

Example 5.2. Spec *as a special case of* Proj. Let $R$ be a right noetherian ungraded $k$-algebra and let $s$ be an autoequivalence of the category mod $R$. It is straighforward to check that the triple $(\operatorname{mod} R, R_R, s)$ satisfies (H1), (H2), and (H3) of Theorem 4.5, hence it is isomorphic to a triple $(\operatorname{qgr} A, \mathscr{A}, s)$ where $A$ is the ring $\Gamma(R)_{\geqslant 0}$. If $s$ is the identity functor of mod $R$, then $A \cong R[x]$ is the polynomial extension of $R$ generated by a central element $x$ of degree 1. The shift operator $s$ on $\operatorname{proj} R[x]$ is naturally isomorphic to the identity. Hence $\operatorname{spec} R \cong \operatorname{proj} R[x]$. A $k$-algebra automorphism $\sigma$ of $R$ induces an autoequivalence of mod $R$ which can be used as the shift operator. With this shift, the graded algebra $A$ becomes the Ore extension $R[x, \sigma]$.

Of course, any autoequivalence $s$ can be described as a Morita equivalence by an invertible $(R, R)$-bimodule $P$ [MR, Cor. 3.5.4]. Then $A_n = s^n(R) = P^{\otimes n}$. The following is a simple example:

$$R = \begin{pmatrix} k & 0 & 0 \\ 0 & k & k \\ 0 & k & k \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & k & k \\ k & 0 & 0 \\ k & 0 & 0 \end{pmatrix}.$$

Here $A_n \cong R$ if $n$ is even, and $A_n \cong P$ if $n$ is odd. Denoting $U_0 := k[t^2]$ and $U_1 := tU_0$, we have

$$A \cong \begin{pmatrix} U_0 & U_1 & U_1 \\ U_1 & U_0 & U_0 \\ U_1 & U_0 & U_0 \end{pmatrix}. \tag{5.2.1}$$

Example 5.3. *Changing the structure sheaf.* Though the structure sheaf is part of the structure of Proj, one may ask, given a $k$-linear abelian category $\mathscr{C}$, which objects $\mathscr{A}$ could serve. In other words, for which $\mathscr{A}$ do the conditions (H1), (H2), and (H3) of Theorem 4.5 hold? Since (H3) involves both the structure sheaf and the polarization $s$, the answer may depend on $s$. We propose to illustrate the possibilities by the simple example in which $\mathscr{C} = \text{mod } R$ when $R = k_1 \oplus k_2$, where $k_i = k$ and where $s$ is the symmetry which interchanges the two factors. The objects of $\mathscr{C}$ have the form $V \approx k_1^{r_1} \oplus k_2^{r_2}$, and the only requirement for (H1), (H2), and (H3) is that $r_i$ not both be zero.

Denoting $V' = k_1^{r_2} \oplus k_2^{r_1}$, we have $s^n(V) = V$ if $n$ is even and $s^n(V) = V'$ if $n$ is odd. Thus if we set $\mathscr{A} = V$ and $A = \Gamma(\mathscr{A})$, then $A_n \cong k^{r_1 \times r_1} \oplus k^{r_2 \times r_2}$ if $n$ is even, and $A_n \cong k^{r_1 \times r_2} \oplus k^{r_2 \times r_1}$ if $n$ is odd. For example, if $r_1 = 1$ and $r_2 = 2$, this yields the ring (5.2.1). If $r_1 = 1$ and $r_2 = 0$, then $A = k[y]$, where $y$ is an element of degree 2. Both of the integers $r_i$ would need to be positive if $s$ were the identity functor.

Example 5.4. Proj *of a strongly graded ring.* Recall that a $\mathbb{Z}$-graded algebra $B$ is *strongly graded* if $B_n B_{-n} = B_0$ *for all* $n \in \mathbb{Z}$. By [NV, Thm. I.3.4] the functor $F\colon \text{gr } B \to \text{mod } B_0$ defined by $F(M) = M_0$ is an equivalence of categories. The shift operator on mod $R$ which is induced by the shift operator of gr $B$ is $s(N) = N \otimes_{B_0} B_1$. There are no non-zero torsion modules when $B$ is strongly graded, hence qgr $B \cong$ gr $B$. Setting $A = B_{\geqslant 0}$ and combining with Proposition 2.5, we obtain isomorphisms of triples

$$(\text{mod } B_0, B_0, s) \cong (\text{gr } B, B, s) \cong (\text{qgr } A, \mathscr{A}, s).$$

In this case the structure sheaf $\mathscr{A}$ is projective because $B_0$ is projective in mod $B_0$.

Example 5.5. *Semiprime graded algebras of GK-dimension* 1. Let $k$ be a field, and let $A$ be an $\mathbb{N}$-graded $k$-algebra which is finitely generated by elements of degree 1, and which has GK-dimension 1. According to [SSW], $A$ is a PI ring and a finite module over its center $Z$, and $Z$ is a semiprime, finitely generated, commutative graded $k$-algebra. Hence $A$ contains a homogeneous regular central element $z$. Also $A$ satisfies $\chi$, by Proposition 5.1. Since $A$ is generated in degree 1, the ring $B := A[z^{-1}]$ is strongly graded [NV, Thm. I.3.4], and by Example 5.4, proj $A \cong$ proj $B \cong$ spec $B_0$. Moreover, $A$ and $B$ are locally finite, hence $B_0$ is a finite, semiprime $k$-algebra, and so proj $A$ has the form described in Example 5.2 and 5.3. Note that, by [NV, Cor. I.3.26], $B \cong B_0[x, x^{-1}, \sigma]$ for some $x$ of degree 1 and some algebra automorphism $\sigma$ of $B_0$.

In the above examples, the structure sheaf $\mathscr{A}$ of proj $A$ is a projective object of the category, but in most interesting cases it will not be projective. The next proposition shows that if $\mathscr{A}$ is projective, then $\Gamma(\mathscr{A})$ is closed to being strongly graded.

PROPOSITION 5.6. *Let $A$ be a right noetherian graded ring, let* $(\operatorname{qgr} A, \mathscr{A}, s)$ *be the associated projective scheme, and let* $A' := \Gamma(\mathscr{A})$.

(1) *If $\mathscr{A}$ is a projective object in* $\operatorname{qgr} A$, *then $A'$ is right noetherian and* $(\operatorname{qgr} A, \mathscr{A}, s) \cong (\operatorname{gr} A', A', s)$. *Moreover, $A'$ satisfies the following (equivalent) conditions*:

(i) *there are positive integers $n_1, ..., n_p$ such* $\sum_{j=1}^{p} A'_{n_j} A'_{-n_j} = A'_0$,

(ii) *for any $r$ there are $n_1, ..., n_p > r$ such that for all $s$,* $\sum_{j=1}^{p} A'_{n_j} A'_{s-n_j} = A'_s$.

(2) *Conversely, if $A'$ is a right noetherian graded algebra which satisfies* (i), *then* $\operatorname{gr} A'$ *contains no torsion modules. Hence* $\operatorname{gr} A = \operatorname{qgr} A'$, *and* $A'_{A'}$ *represents the projective structure sheaf $\mathscr{A}'$ of* $\operatorname{qgr} A'$.

*Proof.* (2) If $A'$ satisfies (i), then $xA'_{\geqslant 1} = 0$ implies $xA'_0 = 0$, hence that $x = 0$. This shows that $\operatorname{tors} A = 0$.

(1) Assume that $A$ is right noetherian and that $\mathscr{A}$ is a projective object of $\operatorname{qgr} A$. We verify the conditions of Theorem 4.5: First, condition (H1) of Theorem 4.5 holds, by Proposition 2.3, and condition (4.2.1)(a) holds by Proposition 4.4(1). Since $\mathscr{A}$ is projective, (4.2.1)(b) holds with $n_0 = 0$, and thus (H3) holds. Next, we will prove that $A'$ is right noetherian and that $M' := \Gamma(\mathscr{M})$ is finite $A'$-module for all $\mathscr{M} \in \operatorname{qgr} A$. By Proposition 2.1, this will also prove (H2).

The proof that $A'$ is noetherian is similar to S4 and S5 in the proof of Theorem 4.5. A homogeneous element $x$ of degree $r$ of $M'$ defines a map from $\mathscr{A}[-r]$ to $\mathscr{M}$. Given a finite subset $X$ of homogeneous elements of $M'$, let $\mathscr{P}_X = \bigoplus \mathscr{A}[-r_x]$, where $r_x$ is the degree of $x$. We obtain a map $f_X : \mathscr{P}_X \to \mathscr{M}$ defined by the elements of $X$. Note that $P' := \Gamma(\mathscr{P}_X) = \bigoplus A'[-r_x]$, and that $f'_X := \Gamma(f_X)$ is a map $P' \to M'$. We identify the image $\operatorname{im}(f'_X)$ in two ways: By construction, it is the submodule of $M'$ generated by $X$. On the other hand, since $\mathscr{A}$ is projective, $\Gamma$ is an exact functor. Hence $\operatorname{im}(f'_X)$ is also equal to $\Gamma(im(f_X))$. Since $\mathscr{M}$ is noetherian, the set of images $\operatorname{im}(f_X)$ satisfies the ascending chain condition. Hence the set of finitely generated submodules of $M'$ also satisfies ascending chain condition, which shows that $M'$ is noetherian, as required.

We now apply Theorem 4.5(1) and Proposition 2.5(2), concluding that $(\operatorname{qgr} A, \mathscr{A}, s) \cong (\operatorname{qgr} A', \mathscr{A}', s)$. We still have to show that $\operatorname{qgr} A' \cong \operatorname{gr} A'$, but according to (2) this will follow if we show (i). It is clear that condition (i) is weaker than condition (ii), so we verify (ii). By (4.2.1)(a), for any $r$, there is an epimorphism $\bigoplus_i \mathscr{A}[-l_i] \to \mathscr{A}[r]$ for some finite number $l_i > 0$. Applying $s^{-r}$, we have an epimorphism $\bigoplus_i \mathscr{A}[-n_i] \to \mathscr{A}$ where $n_i = r + l_i > r$. Applying $\Gamma$, we get an epimorphism $\bigoplus_i A'[-n_i] \to A'$. This is the condition (ii) in the proposition. ∎

*Remarks.* (1) Note that $A'$ is $\Gamma(\mathscr{A})$, not $\Gamma(\mathscr{A})_{\geqslant 0}$. In general $A'/A$ is neither left bounded nor right bounded (see Example 5.7 below).

(2) A simple example of an algebra which satisfies the conditions of Proposition 5.6, but is not strongly graded is $A = k[y, y^{-1}]$ where $\deg(y) = 2$. Here $A_2 A_{-2} = A_0$, but $A_1 A_{-1} = 0$.

(3) If $\mathscr{A}$ is not projective, then $A' = \Gamma(\mathscr{A})$ may not be right noetherian even if $(\operatorname{qgr} A, \mathscr{A}, s)$ satisfies (H1), (H2), and (H3) of Theorem 4.5. Example 5.8 illustrates this point.

EXAMPLE 5.7. *A right noetherian $\mathbb{N}$-graded ring $A$ such that $\Gamma(\mathscr{A})/A$ is neither left bounded nor right bounded.* Let

$$A = \begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} k[x, x^{-1}] & k[x, x^{-1}] \\ k[x, x^{-1}] & k[x, x^{-1}] \end{pmatrix}.$$

The algebra $A$ is right noetherian and not left noetherian, $B$ is left and right noetherian and strongly graded, and $A$ is a graded subring of $B$. We claim that $\Gamma(\mathscr{A}) \cong B$, hence $\Gamma(\mathscr{A})/A$ is neither left nor right bounded. This fact can be verified either by direct computation, or as follows, from general principles.

We note that $B_{\geqslant 0} A_{\geqslant 1} \subseteq A_{\geqslant 1}$; i.e., $A_{\geqslant 1}$ is a left ideal of $B_{\geqslant 0}$. Hence $B_{\geqslant 0}/A$ is a torsion right $A$-module, and it follows that $B/A$ is a torsion $A$-module. By Proposition 2.5, $\operatorname{QGr} A \cong \operatorname{Gr} B/\mathbf{T}_A$, where $\mathbf{T}_A$ is the subcategory of $\operatorname{Gr} B$ of modules which are $A$-torsion. Note that $A_{\geqslant n} B = B$. This implies that $\mathbf{T}_A = 0$. So $(\operatorname{QGr} A, \mathscr{A}, s) \cong (\operatorname{Gr} B, B, s)$. Since $A$ and $B$ are right noetherian, $(\operatorname{qgr} A, \mathscr{A}, s) \cong (\operatorname{gr} B, B, s)$. This implies that $\Gamma(\mathscr{A}) \cong B$.

EXAMPLE 5.8. *A right noetherian $\mathbb{N}$-graded ring $A$ which satisfies $\chi_1$ such that $A' = \Gamma(\mathscr{A})$ is not noetherian.* Note that by Theorem 4.5(2), conditions (H1, 2, 3) hold for $(\operatorname{qgr} A, \mathscr{A}, s)$. This example is due to Stafford. Let

$$A = \begin{pmatrix} k[x, y] & k[x] \\ 0 & k[x, y] \end{pmatrix}$$

where $\deg(x) = \deg(y) = 1$ and $k[x]$ denotes the $k[x, y]$-bimodule $k[x, y]/(y)$. Since $A$ is a PI ring, it satisfies $\chi$ (Proposition 5.1). In this case,

$$A' = \Gamma(\mathscr{A}) = \begin{pmatrix} k[x, y] & k[x, x^{-1}] \\ 0 & k[x, y] \end{pmatrix}.$$

Then $A'$ is neither left nor right noetherian, though $A$ is noetherian (see [MR, Prop. 1.1.7]).

EXAMPLE 5.9. *Veronese subrings.* Let $B = \bigoplus_n B_n$ be a graded algebra. For every $r \geqslant 2$, the Veronese subring $A = B^{(r)}$ is defined to be $A = \bigoplus_n A_n$ and $A_n = B_{rn}$. Here are some basic facts about Veronese subrings.

PROPOSITION 5.10. *Let $B$ be an $\mathbb{N}$-graded algebra, and let $A = B^{(r)}$ be a Veronese subring.*

(1) *If $B$ is right noetherian, then so is $A$.*

(2) *If $B$ is generated by elements of degrees* 0 *and* 1, *so is $A$. If in addition $B$ is a finitely generated k-algebra, then $B_A$ and ${}_AB$ are finite modules, and $A$ is right (or left) noetherian if and only $B$ is.*

(3) *If $B$ is generated in degree* 0 *and* 1 *and right noetherian, then* Proj $B$ *is equivalent to* Proj $A$ *via the functor $\pi(M) \to \pi(M^{(r)})$ where $M^{(r)} := \bigoplus_n M_{rn}$. Thus* (Proj $B, s^r_B) \cong$ (Proj $A, s_A$) *and* (proj $B, s^r_B) \cong$ (proj $A, s_A$).

*Proof.* (1) We only need to prove that $A$ is graded right noetherian. If $I$ is a graded right ideal of $A$, then $IB$ is a graded right ideal of $B$ and $I = IB \cap A$. This implies that the noetherian property carries over from $B$ to $A$.

(2) Suppose $B$ is generated by elements of degree 0 and 1. Then $B_n = (B_1)^n$ for all $n > 0$. Hence $A_n = B_{rn} = (B_1)^{rn} = (B_r)^n = A_1^n$, which implies that $A$ is generated in degree 0 and 1. If $B$ is finitely generated, then each $B_i$ is finitely generated right module over $B_0 = A_0$. Thus $B = (\bigoplus_{i=0}^{r-1} B_i)A$ is a finitely generated $A$-bimodule. Hence, if $A$ is right noetherian, then so is $B$.

(3) This follows from [V1, Thm. 4.4]. Note that in that paper, the author works with left modules. ∎

*Remarks.* (1) There are two natural $\mathbb{Z}$-gradings on the Veronese subring $A = B^{(r)}$. One grading is $\deg(A_n) = n$, and the other one is $\deg(A_n) = rn$. With the second grading, $A$ becomes a graded subalgebra of $B$. Let $\hat{A}$ denote the graded algebra graded in this way. So $A$ and $\hat{A}$ have the same underlying ungraded structure, and hence one is noetherian if and only if the other is. However, the categories of graded modules are different because the allowable degrees do not correspond. It is easily see that any $\hat{A}$-module $\hat{M}$ can be decomposed as a direct sum $\bigoplus_i \hat{M}_i$, where $\hat{M}_i$ is the sum of the parts whose degree is congruent $i$ modulo $r$, and that in this way, gr $\hat{A}$ is equivalent with $(\text{gr } A)^r$. The shift is $\hat{s}(\hat{M}_0, \hat{M}_1, ..., \hat{M}_{r-1}) = (\hat{M}_1, \hat{M}_2, ..., \hat{M}_{r-1}, \hat{M}_0[1])$. It also follows that $\chi_1$ (resp. $\chi$) holds for $A$ if and only if it holds for $\hat{A}$.

(2) If $B$ is not generated in degree $\leqslant 1$, then neither Proposition 5.10(3) nor the converse of Proposition 5.10(1) need be true. For example, let $k$ be a field. If $A = k[x]$, where $x$ has degree 1, then as discussed in

Example 5.2, proj $A \cong \operatorname{spec} k$ is a point. On the other hand, if we regrade by assigning degree 2 to $x$, then as above, qgr $A$ is equivalent to two copies of mod $k$. In this case, proj $A$ can be thought of as a pair of points which are interchanged by the shift, while proj $A^{(2)}$ is a single point.

As an example in which the Veronese is noetherian though the original ring is not, let $B = k\{x, y\}/(x^2, xy)$ with $\deg(x) = 1$ and $\deg(y) = 2$. Then $B$ is not right noetherian, but $A = B^{(2)} = k[y]$ is noetherian.

The relationship between $B$ and $B^{(r)}$ about the condition $\chi$ is given in Proposition 8.7.

EXAMPLE 5.11. *Twisted graded rings.* Let $\sigma$ be an automorphism of the graded algebra $A$. We define a new multiplication $*$ on the underlying graded $k$-module $A = \bigoplus_n A_n$ by

$$a * b = a\sigma^n(b),$$

where $a$ and $b$ are homogeneous elements in $A$ and $\deg(a) = n$. The new algebra is called *twist* of $A$ and it is denoted by $A^\sigma$. By [ATV2, Zh], gr $A \cong$ gr $A^\sigma$ and hence proj $A \cong$ proj $A^\sigma$. Moreover, (proj $A, s$) satisfies (H1), (H2), and (H3) if and only if (proj $A^\sigma, s$) does, though these triples are not always isomorphic. Twisted algebras are studied in [Zh].

For example, if $A = k[x, y]$ where $\deg(x) = \deg(y) = 1$, then any linear operator on the space $A_1$ defines an automorphism, and hence a twist, of $A$. If $k$ is an algebraically closed field then, after a suitable linear change of variable, a twist can be brought into one of the forms $k_q[x, y] := k\langle x, y\rangle/(yx - qxy)$ for some $q \in k$, or $k_J[x, y] := k\langle x, y\rangle/(x^2 + xy - yx)$. Hence proj $k[x, y] \cong$ proj $k_q[x, y] \cong$ proj $k_J[x, y]$. The projective scheme associated to any one of these algebras is the projective line $\mathbb{P}^1$.

## 6. CLASSICAL PROJECTIVE SCHEMES

Let $Z$ be a commutative scheme of finite type over a field $k$, and let $\mathcal{A}$ be a finite $\mathcal{O}$-algebra, where $\mathcal{O} = \mathcal{O}_Z$. Let $\mathscr{C} :=$ mod $\mathcal{A}$ denote the category of coherent sheaves with a structure of right $\mathcal{A}$-module. We call the pair (mod $\mathcal{A}, \mathcal{A}$) a *classical scheme*, and in analogy with [EGA II, 1], we denote it by $\underline{\operatorname{Spec}}\, \mathcal{A}$.

The conditions (H1) and (H2) of Theorem 4.5 are satisfied by $\underline{\operatorname{Spec}}\, \mathcal{A}$ whenever $Z$ is proper over $k$. So if $Z$ is proper and if we are also given an ample autoequivalence $s$ of $\mathscr{C}$ to play the role of a shift operator, then the triple $(\mathscr{C}, \mathcal{A}, s)$ becomes a projective scheme in our sense. We call such a triple a *classical projective scheme*. This section describes a autoequivalence of mod $\mathcal{A}$ as a "Morita equivalence" defined by an invertible $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{P}$ (see Corollary 6.9), but we do not discuss the condition that $s$ be ample.

Invertible bimodules were discussed in [AV] in the case $\mathcal{A} = \mathcal{O}$. In that case, an invertible bimodule has the form $\mathcal{L}_\sigma$, where $\sigma$ is an automorphism of the scheme $Z$ and $\mathcal{L}$ is an invertible sheaf on $Z$. This invertible sheaf defines the left module structure of $\mathcal{L}_\sigma$, and its right module structure is obtained by shifting the left structure by $\sigma$ [AV, Prop. 2.15]. If the auto-equivalence $s$ defined by tensoring with $\mathcal{L}_\sigma$ is ample, then the conditions (H1), (H2), and (H3) of Theorem 4.5 are satisfied. The algebra $A = \Gamma(\mathcal{O})_{\geqslant 0}$ has

$$A_n = \mathrm{H}^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}).$$

Note that in [AV, 1.2] left modules are used to define the structure, which reverses the order of multiplication.

We return to the general case. The $\mathcal{A}$-modules of finite length are the finite-dimensional vector spaces over $k$, equipped with an $\mathcal{A}$-module structure. Though $\mathcal{A}$ need not be a projective object of mod $\mathcal{A}$, it behaves like a projective with respect to these modules of finite length, and this fact allows us to recover various concepts from the category of modules alone.

There is a bijective correspondence between maximal (two-sided) ideals of $\mathcal{A}$ and isomorphism classes of simple $\mathcal{A}$-modules. Let us denote the maximal ideal space by $X$, the maximal ideal corresponding to $x \in X$ by $m_x$, and a representative simple module in the corresponding class by $\mathcal{V}_x$. We have a canonical map

$$\phi : X \to Z.$$

PROPOSITION 6.1. *Let $\mathcal{M}$ be a finite $\mathcal{A}$-module. The following subsets of $X$ are equal*:

(1) *the set of $x$ such that $m_x$ contains the right annihilator* ann$(\mathcal{M})$ *of $\mathcal{M}$, and*

(2) *the set of $x$ such that $\mathcal{M}$ has a subquotient isomorphic to $\mathcal{V}_x$.*

We define the *support* of a module $\mathcal{M}$ to be the subset of $X$ defined by either of these conditions, and we denote it by Supp$(\mathcal{M})$.

*Proof.* Denote the sets defined by (1) and (2) by $S(\mathcal{M})$ and $T(\mathcal{M})$, respectively. The assertion being local, we may assume that $Z$ is affine. So it suffices to prove the same assertion when $\mathcal{A}$ is a noetherian PI ring. We first note that if

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

is an exact sequence, then $T(\mathcal{M}_1) \cup T(\mathcal{M}_3) = T(\mathcal{M}_2)$ and $S(\mathcal{M}_1) \cup S(\mathcal{M}_3) = S(\mathcal{M}_2)$. The first equality is clear. For the second, let $\mathrm{Ra}(\mathcal{M})$ denote the radical of the annihilator ideal $\mathrm{ann}(\mathcal{M})$. Then $S(\mathcal{M})$ is also equal to the set of $x \in X$ such that $m_x$ contains $\mathrm{Ra}(\mathcal{M})$, and with $M_i$ as above, we have $\mathrm{Ra}(\mathcal{M}_1) \cap \mathrm{Ra}(\mathcal{M}_3) = \mathrm{Ra}(\mathcal{M}_2)$. The second equality follows.

Taking a suitable composition series reduces the proof of the proposition to the case that the annihilator is a prime ideal. Assume that $\mathrm{ann}(\mathcal{M})$ is a prime ideal. We replace $\mathcal{A}$ by $\mathcal{A}/\mathrm{ann}(\mathcal{M})$, which reduces us to the case that the annihilator is 0 and that $\mathcal{A}$ is a prime noetherian PI ring. In this case we have to show that every simple module $\mathcal{V}_x$ is a subquotient of $\mathcal{M}$. Passing to fractions shows that there is a subquotient of $\bar{\mathcal{M}}$ of $\mathcal{M}$ and an injection of $\mathcal{A}$ to a finite sum of copies of $\bar{\mathcal{M}}$ (see, for example, [SZ1, Lemma 2.1(ii)]). Therefore $T(\mathcal{A}) \subseteq T(\bar{\mathcal{M}})$, which completes the proof because $T(\mathcal{A}) = X$. ▮

PROPOSITION 6.2. *The following concepts are defined in terms of the category* mod $\mathcal{A}$ *alone*:

(1) *the space $X$ of maximal ideals of $\mathcal{A}$*;

(2) *the support of a module $\mathcal{M}$ in $X$*;

(3) *the Zariski topology on $X$.*

*The following concepts are defined in terms of the pair* (mod $\mathcal{A}$, $\mathcal{A}$):

(4) *the anihilator of a module $\mathcal{M}$*;

(5) *two-sided ideals of $\mathcal{A}$.*

*Proof.* Assertions (1) and (2) were treated above.

(3) The Zariski closed sets in $X$ are the supports of finite $\mathcal{A}$-modules.

(4) The annihilator of a module $\mathcal{M}$ of finite length is the intersection of the kernels of homomorphisms $\mathcal{A} \to \mathcal{M}$. Also, the annihilator of any finite module $\mathcal{M}$ is the intersection of the annihilators of the finite-length quotients $\bar{\mathcal{M}}$ of $\mathcal{M}$, hence it is determined by $\mathcal{M}$ and $\mathcal{A}$. For, this intersection obviously contain the annihilator. To check the opposite inclusion, it suffices to do so locally, when $Z$ is affine. Then the data correspond to a finitely generated commutative $k$-algebra $\mathcal{O}$, a finite $\mathcal{O}$-algebra $\mathcal{A}$, and a finite $\mathcal{A}$-module $\mathcal{M}$. Let $a$ be an element of $\mathcal{A}$ which does not annihilate $\mathcal{M}$. So there is an element $m$ such that $ma \neq 0$. Hence there is an ideal $\mathcal{I}$ of $\mathcal{O}$ such that $\mathcal{O}/\mathcal{I}$ has finite length and $ma \not\equiv 0$ (modulo $\mathcal{I}$). Then $a$ does not annihilate the residue of $m$ in the finite length module $\mathcal{M}/\mathcal{M}\mathcal{I}$.

(5) An ideal $\mathcal{I}$ of $\mathcal{A}$ is a subobject which is the annihilator of some module. ▮

COROLLARY 6.3. *Let $\mathcal{A}'$ be a finite $\mathcal{O}_{Z'}$-algebra, where $Z'$ is another commutative scheme of finite type over $k$, and let $\mathcal{C}' = \operatorname{mod} \mathcal{A}$. Let $F_i = (f_i, \theta_i)$, $i = 1, 2$, be right exact morphisms of pairs $(\mathcal{C}, \mathcal{A}) \to (\mathcal{C}', \mathcal{A}')$, where $f_i: \mathcal{C} \to \mathcal{C}'$ and $\theta_i$ is an isomorphism $f_i(\mathcal{A}) \xrightarrow{\sim} \mathcal{A}'$. There is at most one natural equivalence $\varepsilon: f_1 \to f_1$ which is compatible with $\theta$ and $\theta'$.*

*Proof.* This follows because the compatibility with $\theta$ and $\theta'$ determines $\varepsilon$ on $\mathcal{A}$. This determines $\varepsilon$ on quotients of sums of $\mathcal{A}$, hence on the modules of finite length, and from there on all modules. ∎

If $\mathcal{M}$, $\mathcal{N}$ are finite modules whose supports are disjoint, then $\operatorname{Hom}(\mathcal{M}, \mathcal{N}) = 0$. To see this, we may replace $\mathcal{N}$ by a finite length quotient, and apply the definition of $T(\mathcal{M})$. We say that a closed subset $Y$ is *saturated* if it has the following property: For any finite module $\mathcal{M}$ whose support is contained in $Y$, and any $x \notin Y$, we have

$$\operatorname{Ext}^1(\mathcal{M}, \mathcal{V}_x) = 0.$$

It follows that the same condition holds when $\mathcal{V}_x$ is replaced by a finite length module whose support is disjoint from $Y$. For example, if $\phi: X \to Z$ denotes the structure map, and if $Y$ is any closed set in $X$, then $\phi^{-1}(\phi(Y))$ is saturated.

Let $U$ be an open subset of $X$, let $Y$ be its closed complement, and assume that $Y$ is saturated. The finite $\mathcal{A}$-modules whose support is contained in $Y$ form a dense subcategory of $\mathcal{C} = \operatorname{mod} \mathcal{A}$. We denote the quotient category by $\mathcal{C}_U$. For an $\mathcal{A}$-module $\mathcal{M}$, we define $\mathcal{M}_U$ to be the image of $\mathcal{M}$ in $\mathcal{C}_U$, and we set

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) := \operatorname{Hom}(\mathcal{A}_U, \mathcal{M}_U).$$

Then $\mathcal{A}(U)$ is a ring, and $\mathcal{M}(U)$ is a right $\mathcal{A}(U)$-module, multiplication being defined as composition of maps.

LEMMA 6.4. *Let $U$ be an open subset of $X$ whose complement $Y$ is saturated, let $\mathcal{M}$ be a finite module, and let $\mathcal{V}$ be a finite length module whose support contained in $U$. Then with the above notation,*

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{V}) \cong \operatorname{Hom}_{\mathcal{C}_U}(\mathcal{M}_U, \mathcal{V}_U).$$

*Proof.* By construction of the quotient category, $\operatorname{Hom}(\mathcal{M}_U, \mathcal{V}_U) = \varinjlim \operatorname{Hom}(\mathcal{M}', \mathcal{V}'')$, the limit being over pairs of maps $\mathcal{M}' \to \mathcal{M}$ and $\mathcal{V} \to \mathcal{V}''$ whose kernels and cokernels have support in $Y$. The hypothesis that $Y$ is saturated implies that the maps $\mathcal{V} \to \mathcal{V}''$ are split monomorphism, and that every map $\mathcal{M}' \to \mathcal{V}''$ is induced by a unique map $\mathcal{M} \to \mathcal{V}$. ∎

PROPOSITION 6.5. *Let $U$ be an open subset of $X$ whose complement $Y$ is saturated. Suppose furthermore that $\Gamma(U, \cdot): \mathscr{C}_U \to \operatorname{mod} \mathscr{A}(U)$ is an exact functor. Then*

(1) *$\mathscr{A}(U)$ is a right noetherian ring, and for every $\mathscr{M} \in \mathscr{C}$, $\mathscr{M}(U)$ is a finite $\mathscr{A}(U)$-module;*

(2) *$\Gamma(U, \cdot)$ is an equivalence of categories $\mathscr{C}_U \to \operatorname{mod} \mathscr{A}(U)$.*

*Hence the pair $(\mathscr{C}_U, \mathscr{A}_U)$ is isomorphic to* $\operatorname{spec} \mathscr{A}(U)$.

*Proof.* First, note that the functor $\Gamma(U, \cdot)$ is faithful. For, if $\mathscr{M}$ is an $\mathscr{A}$-module whose support is not contained in $Y$, then $\mathscr{M}$ has a nonzero quotient $\mathscr{V}$ of finite length whose support is in $U$. Then $\operatorname{Hom}(\mathscr{A}, \mathscr{V}) = \mathscr{V}(U) \neq 0$. Since $\Gamma(U, \cdot)$ is exact, $\mathscr{M}(U) \neq 0$ as well. Also, the assumption that $\Gamma(U, \cdot)$ is exact means that $\mathscr{A}_U$ is a projective object. Thus $\mathscr{A}_U$ is a progenerator, and Mitchell's theorem (see [Po, Cor. 5.9.4]) applies. ∎

An open subset $U$ of $X$ whose complement $Y$ is saturated and such that $\Gamma(U, \cdot)$ is exact is called an *affine open* subset of $X$, and we identify $U$ with $\operatorname{Spec} \mathscr{A}(U)$.

COROLLARY 6.6. *The notion of affine open set depends only on the category* $\operatorname{mod} \mathscr{A}$. ∎

PROPOSITION 6.7. *Let $\mathscr{P}$ be a finite $\mathscr{A}$-module which has the properties of a progenerator on modules of finite length, i.e., such that*

(i) $\operatorname{Hom}(\mathscr{P}, \cdot)$ *is an exact functor on $\mathscr{A}$-modules of finite length, and*

(ii) *for every finite length module $\mathscr{V}$, there is an integer $r$ and an epimorphism $\mathscr{P}^{\oplus r} \to \mathscr{V}$.*

*Then $\mathscr{P}$ is a* local progenerator, *which means that for every affine open subset $U$ of $X$, $\mathscr{P}(U)$ is a progenerator in the category* $\operatorname{mod} \mathscr{A}(U)$.

*Proof.* The hypotheses are compatible with localization, so it suffices to prove this when $Z$ is affine. Hence it suffices to prove the same assertion in the case that $\mathscr{A}$ is a finite $\mathscr{O}$-algebra, where $\mathscr{O}$ is a commutative $k$-algebra of finite type. We must show that $\mathscr{P}$ is a progenerator.

To show that $\mathscr{P}$ is projective, let $p$ be a closed point of $Z = \operatorname{spec} \mathscr{O}$, and let ^ denote completion at the point $p$. For any finite $\mathscr{A}$-module $\mathscr{M}$, $\operatorname{Hom}_{\mathscr{A}}(\mathscr{P}, \mathscr{M})$ is a finite $\mathscr{O}$-module whose completion $\operatorname{Hom}_{\mathscr{A}}(\mathscr{P}, \mathscr{M})^\wedge$ is isomorphic to $\operatorname{Hom}_{\mathscr{A}^\wedge}(\mathscr{P}^\wedge, \mathscr{M}^\wedge)$. Let $\mathscr{M} \to \mathscr{M}'$ be a surjective map of $\mathscr{A}$-modules. To show that $\operatorname{Hom}_{\mathscr{A}}(\mathscr{P}, \mathscr{M}) \to \operatorname{Hom}_{\mathscr{A}}(\mathscr{P}, \mathscr{M}')$ is surjective, it suffices to show that the induced map of completions is surjective for each closed point $p$. But $\mathscr{M}^\wedge = \varprojlim \bar{\mathscr{M}}$, where $\bar{\mathscr{M}}$ runs over finite length quotients of $\mathcal{M}$, and $\operatorname{Hom}_{\mathcal{A}^\wedge}(\mathcal{P}^\wedge, \mathcal{M}^\wedge) = \varprojlim \operatorname{Hom}_{\mathcal{A}}(\mathcal{P}, \bar{\mathcal{M}})$. The surjectivity follows by inspection of the map of inverse systems $\operatorname{Hom}_{\mathcal{A}}(\mathcal{P}, \bar{\mathcal{M}}) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{P}, \bar{\mathcal{M}}')$. The fact that $\mathcal{P}$ is a generator can also be proved by inspecting the completions. ∎

Now let $\mathcal{A}'$ be a finite $\mathcal{O}_{Z'}$-algebra, where $Z$ is another commutative scheme of finite type over $k$, let $\mathcal{C}' = \operatorname{mod} \mathcal{A}'$, and let $f: \mathcal{C} \to \mathcal{C}'$ be an equivalence of categories. Then $f$ defines a homeomorphisms between the maximal ideal spaces $g: X = \operatorname{Max} \mathcal{A} \to X' = \operatorname{Max} \mathcal{A}'$, and since the notion of affine open set is defined in terms of the module categories, this homeomorphism carries affines opens to affine opens. Let $U$ be an affine open of $X$, and let $U'$ be the corresponding affine open of $X'$. Then $f$ defines an equivalence of categories $\operatorname{mod} \mathcal{A}(U) \to \operatorname{mod} \mathcal{A}'(U')$. To simplify notation, we write $\mathcal{A}$ also for the ring $\mathcal{A}(U)$ and $\mathcal{A}'$ for $\mathcal{A}'(U)$. The equivalence of categories on the open sets is described by Morita theory in terms of an $(\mathcal{A}, \mathcal{A}')$-bimodule $\mathcal{P} = f(\mathcal{A})$, which is a finite left $\mathcal{A}$-module and a right $\mathcal{A}'$-module progenerator. The equivalence can be recovered on the open set as $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{P}$. The next proposition extends this locally defined module to a coherent sheaf over the scheme $Z \times Z'$.

We form the coherent sheaf of algebras $\mathcal{E} := \mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{A}'$ over the scheme $Z \times Z'$. By an $(\mathcal{A}, \mathcal{A}')$*-bimodule*, we mean a coherent sheaf on $Z \times Z'$ with the structure of right $\mathcal{E}$-module.

Denote $\operatorname{Max} \mathcal{E}$ by $Y$. Since the maps $\mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ and $\mathcal{A}' \to \mathcal{E}$ are extensions in the sense of Procesi [Pr, Chap. 2, Defn. 6.3], and since $\operatorname{Max} \mathcal{A}^{\mathrm{op}}$ is canonically homeomorphic to $X$, there is a canonical map $Y \to X \times X'$. This map is bijective if $k$ is algebraically closed, and in any case, it is a surjective map whose fibers are finite sets.

Let $x$ be a point of $X$, let $m = m_x$, and $\mathcal{V} = \mathcal{V}_x$. As we know, $\Lambda = \mathcal{A}/m$ is a matrix algebra over a division ring $D$, and $D = \operatorname{End}(\mathcal{V})$. If $x'$ is the corresponding point of $X'$, then with the obvious notation, $D' = \operatorname{End}(\mathcal{V}')$. Hence the equivalence of categories induces an isomorphism $D \to D'$, and an isomorphism of the centers $K \to K'$ of these division rings. This isomorphism picks out a point $y \in Y$ which lies over $(x, x')$. The following description can be checked by analyzing the finite algebra $\Lambda^{\mathrm{op}} \otimes_k \Lambda'$. The maximal ideals in the finite $k$-algebra $K \otimes_k K'$ correspond bijectively to points of $Y$ which lie over $(x, x')$. The isomorphism $K \to K'$ determines a maximal ideal in the finite $k$-algebra $K \otimes_k K'$, hence a point $y$. Let $\mathcal{V}^*$ denote the simple left $\mathcal{A}$-module which corresponds to $x$. We make $V'$ into a left $D$ module using the isomorphism $D \to D'$. Then $\mathcal{V}^* \otimes_D \mathcal{V}'$ is the simple $\mathcal{E}$-module whose support is $y$.

We denote by $\Gamma$ the set of point $y \in Y$ lying over the graph of $g: X \to X'$ which are determined as above, and by $\mathrm{pr}_i$, $i = 1, 2$, the projections of $Z \times Z'$ onto the two factors.

PROPOSITION 6.8. *With the above notation, there exists an* $(\mathscr{A}, \mathscr{A}')$-*bimodule* $\mathscr{P}$ *with the following properties*:

(1) *If* $U$, $U'$ *are corresponding affine open subsets of* $X$ *and* $X'$, *then* $\mathscr{P}_{U \times U'}$ *is isomorphic to the bimodule* $\mathscr{P} = f(\mathscr{A})$ *described above on* $U \times U'$.

(2) *The support of* $\mathscr{P}$ *as* $\mathscr{E}$-*module is* $\Gamma$, *and as* $\mathcal{O}_{Z \times Z'}$-*module, its support in* $Z \times Z'$ *is finite and proper over* $Z$ *and over* $Z'$.

(3) $\mathrm{pr}_{2*}\mathscr{P}$ *is a local progenerator in* $\mathscr{C}'$, *and* $\mathrm{pr}_{1*}\mathscr{P}$ *is a finite left* $\mathscr{A}$-*module.*

(4) *For* $\mathscr{M} \in \mathscr{C}$, *there is a functorial isomorphism* $f(\mathscr{M}) \cong \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathscr{M} \otimes_{pr_1^* \mathscr{A}} \mathscr{P})$.

*Proof.* Let $\mathscr{Z}$ denote the center of $\mathscr{A}$, which is a finite algebra over $\mathcal{O}_Z$, and set $Z_1 := \underline{\mathrm{Spec}}_{\mathcal{O}}\, \mathscr{Z}$. We may also view $\mathscr{A}$ as a coherent sheaf on $Z_1$, and the two interpretations of the notation $\bmod \mathscr{A}$ yield canonically equivalent categories. So as a technical convenience, we replace $Z$ by $Z_1$, in other words, we assume that $\mathcal{O}$ is the center of $\mathscr{A}$. We do the same with $Z'$ and $\mathscr{A}'$. The reason for doing this is that a Morita equivalence $\bmod A \to \bmod A'$ induces an isomorphism of centers $Z(A) \to Z(A')$ [MR, Thm. 3.5.9(iii)]. This being true on every affine open set, it follows that our equivalence $f: \mathscr{C} \to \mathscr{C}'$ induces an isomorphism of commutative schemes $d: Z \to Z'$.

Let $\Gamma$ denote the graph of the map $g: X \to X'$ induced by $f$, and let $\Delta$ be the graph of $d: Z \to Z'$. Then $\Delta$ is the image of $\Gamma$ in $Z \times Z'$.

For the proof, we will restrict attention to affine open sets of $X$ of the form $U = \phi^{-1}(V)$, where $V$ is an affine open in $Z$, and we will denote by $U'$, $V'$ the corresponding affine opens of $X'$, $Z'$, respectivily. Let $W \subset Z \times Z'$ be the union of the opens $V \times V'$, where $V$ is an affine open in $Z$. This is an open subset which contains $\Delta$.

Item (1) describes the restriction of $\mathscr{P}$ to an open set of the form $U \times U'$, and it is clear that this description is compatible with restriction to a smaller affine open set $U_1 \subset U$. Hence these local descriptions of $\mathscr{P}$ glue together to define a coherent sheaf on the open set $W$.

By construction, $\mathscr{P}$ is an $\mathscr{E}$-module over the open set $W$. We claim that on $W$, the support of $\mathscr{P}$ is $\Gamma$. To show this, one may work locally on an affine open $U \times U'$, where standard Morita theory applies. Suppose that $y \in \Gamma$ lies over $(x, x')$. As above, the simple $\mathscr{E}$-module corresponding to $y$ is $\mathscr{V}^* \otimes_D \mathscr{V}' = \mathscr{V}^* \otimes_D \mathscr{V} \otimes_{\mathscr{A}} \mathscr{P}$. If $e \in \mathscr{E}$ annihilates $\mathscr{P}$, then $e$ also annihilates $\mathscr{V}^* \otimes_D \mathscr{V}'$. Thus $\operatorname{ann} \mathscr{P} \subset m_y$. This shows that $\Gamma \subset \operatorname{Supp}(\mathscr{P})$. Conversely, let $y_1$ be a point of $Y$ which is not on $\Gamma$, let $x$ be its image in $X$, and let $y$ be the point lying over $(x, x')$ which is determined by $f$, as above. There is an element $e \in \mathscr{E}$ which is in $m_{y_1}$ and which is congruent to 1 (modulo $m_y$). Then $b$ does not annihilate the simple module $\mathscr{V}^* \otimes_D \mathscr{V}' = \mathscr{V}' \otimes_D \mathscr{V} \otimes_A \mathscr{P}$. Therefore $b$ is not in $\operatorname{ann} \mathscr{P}$.

Since the support of $\mathscr{P}$ on $W$ lies over $\Delta$ and since $W$ is an open neighborhood of the closed set $\Delta$ in $Z \times Z'$, we may extend $\mathscr{P}$ by zero outside of $W$, thereby obtaining a coherent sheaf over $Z \times Z'$. The $\mathscr{E}$-module structure also extends over $Z \times Z'$, and the support of this sheaf remains $\Gamma$. The assertions (1) and (2) are true, and (3) follows from its local analogue. Also (4) follows from the local analogue, once one has deciphered the notation. Recall that the operation $\mathrm{pr}_1^*$ is the sheaf version of the tensor product $\cdot \otimes_{\mathcal{O}} \mathcal{O}_{Z \times Z'}$, while $\mathrm{pr}_{2*}$ is the sheaf version of restriction of scalars from $\mathrm{mod}\, \mathcal{O}_{Z \times Z'}$ to $\mathrm{mod}\, \mathcal{O}'$. Note that $\mathscr{P}$ has the structures of a left $\mathrm{pr}_1^* \mathscr{A}$-module and a right $\mathrm{pr}_2^* \mathscr{A}'$-module given by the inclusions $\mathrm{pr}_1^* \mathscr{A}^{\mathrm{op}} \to \mathscr{E}$ and $\mathrm{pr}_2^* \mathscr{A}' \to \mathscr{E}$. Also, $\mathrm{pr}_1^* \mathscr{M}$ is a right $\mathrm{pr}_1^* \mathscr{A}$-module. Hence $\mathrm{pr}_1^* \mathscr{M} \otimes_{\mathrm{pr}_1^* \mathscr{A}} \mathscr{P}$ is defined, and it inherits the structure of a right $\mathrm{pr}_2^* \mathscr{A}$-module. So $\mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathscr{M} \otimes_{\mathrm{pr}_1^* \mathscr{A}} \mathscr{P})$ is an $\mathscr{A}'$-module, as required. This is the operation which defines the Morita equivalence. Finally, one must verify that (4) is compatible with the change of scheme $Z_1 \to Z$ which was introduced temporarily at the beginning of the proof. ∎

Now suppose that $Z = Z'$ and $\mathscr{A} = \mathscr{A}'$ in the above discussion. An $(\mathscr{A}, \mathscr{A})$-bimodule $\mathscr{P}$ is called an *invertible* if there is a bimodule $\mathscr{Q}$ such that $\mathscr{P} \otimes_{\mathrm{pr}_1^* \mathscr{A}} \mathscr{Q}$ and $\mathscr{Q} \otimes_{\mathrm{pr}_1^* \mathscr{A}} \mathscr{P}$ are isomorphic to $\mathscr{A}$ as $(\mathscr{A}, \mathscr{A})$-bimodules.

COROLLARY 6.9. *Let $Z$ be a commutative scheme of finite type over a field $k$, let $\mathscr{A}$ be a coherent sheaf of $\mathcal{O}_Z$-algebras, and let $\mathscr{C} = \mathrm{mod}\, \mathscr{A}$.*

(1) *If $\mathscr{P}$ is an invertible $(\mathscr{A}, \mathscr{A})$-bimodule, the functor $s$ defined by*

$$s(\mathscr{M}) = \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathscr{M} \otimes_{\mathrm{pr}_1^* \mathscr{A}} \mathscr{P})$$

*is an equivalence of $\mathscr{C}$.*

(2) *Every autoequivalence of the category $\mathscr{C}$ is naturally isomorphic to one defined in this way.* ∎

EXAMPLE 6.10. *The polarizations and structure sheaves on the projective line.* Let $\mathscr{C}$ denote the category of coherent sheaves on $\mathbb{P}_k^1$, and let $s_1$ denote the standard shift operator $\cdot \otimes \mathcal{O}(1)$. As we know, this category is also equivalent to $\mathrm{qgr}\, A$, where $A = k[x, y]$ is the commutative polynomial ring, but as we saw in Example 5.11, there are other rings $B$ with equivalent module categories. We propose to describe the pairs $(\mathscr{A}, s)$ such that the triple $(\mathscr{C}, \mathscr{A}, s)$ satisfies the conditions (H1), (H2), and (H3) of Theorem 4.5. As we already remarked, or by Proposition 4.7, (H1) and (H2) hold in any case. The problem is to describe the pairs such that $s$ is ample.

First of all, every object $\mathcal{M}$ of $\mathcal{C}$ is isomorphic to a direct sum $\mathcal{M} \cong \mathcal{P} \oplus \mathcal{V}$, where $\mathcal{P}$ is locally free and $\mathcal{V}$ is supported on a finite set of points of $\mathbb{P}^1$. Moreover, a locally free $\mathcal{O}$-module $\mathcal{P}$ on $\mathbb{P}^1$ is isomorphic to a sum $\bigoplus_{i=1}^{r} \mathcal{O}[d_i]$ of shifts of $\mathcal{O}$ [Ha, Exc. V.2.6].

Next, an autoequivalence $s$ is defined by an invertible bimodule which, as mentioned before, has the form $\mathcal{L}_\sigma$. Then $\mathcal{L} \cong \mathcal{O}[m]$ for some $m$. It follows that $s \cong s_1^m \sigma^*$, where $\sigma^*$ denotes the pullback via $\sigma$.

PROPOSITION 6.11. *Let $k$ be a field and let $\mathcal{C}$ denote the category of coherent sheaves on $\mathbb{P}^1$. Let $s$ denote the autoequivalence $s_1^m\sigma^*$, and let $\mathcal{A} = \mathcal{P} \oplus \mathcal{V}$, where $\mathcal{P}$ is locally free and $\mathcal{V}$ is supported on a finite set. The autoequivalence $s$ is ample for the triple $(\mathcal{C}, \mathcal{A}, s)$ if and only if*

(i) $m > 0$,

(ii) $\mathcal{P} \neq 0$, *and*

(iii) *every point in the support of $\mathcal{V}$ has an infinite $\sigma$-orbit.*

*Proof.* For clarity, we replace the notation $\mathcal{M}[n]$ in (4.2.1) by $s^n(\mathcal{M})$. Suppose that (i) holds. We have $s^n(\mathcal{O}[d]) \cong \mathcal{O}[d+mn]$, from which one sees immediately that $s$ is ample if $\mathcal{A} = \mathcal{O}[d]$. Similarly, $s$ is ample when $\mathcal{A}$ is a nonzero direct sum of such sheaves. Also, (i) and (ii) imply that (4.2.1)(a) holds. Conversely, if $m \leqslant 0$ or if $\mathcal{P} = 0$, direct computation shows that $\mathrm{Hom}(s^{-n}(\mathcal{A}), \mathcal{O}[-r]) = 0$ if $n \gg 0$ and $r \gg 0$, hence that (4.2.1)(a) fails.

To explain (iii), we write $\mathrm{H}^0(\mathcal{M}) = \mathrm{Hom}(\mathcal{P}, \mathcal{M}) \oplus \mathrm{Hom}(\mathcal{V}, \mathcal{M})$. Condition (4.2.1)(b) must hold for both summands separately, and, as remarked above, it does hold for $\mathrm{Hom}(\mathcal{P}, \mathcal{M})$ if $m > 0$ and $\mathcal{P} \neq 0$. Consider the term $\mathrm{Hom}(\mathcal{V}, \mathcal{M})$. We substitute the canonical map $\mathcal{O} \to k(p)$, where $p$ is a point of $\mathbb{P}^1$ for $f$ in (4.2.1)(b). We have $\mathrm{Hom}(\mathcal{V}, s^n(\mathcal{O})) = 0$ for every $n$. So in order for (4.2.1)(b) to hold, we must have $\mathrm{Hom}(\mathcal{V}, s^n(k(p))) = 0$ for $n \gg 0$. This will be so if and only if the support of $\mathcal{V}$ does not contain $\sigma^n p$ when $n \gg 0$. Since $p$ is arbitrary, it follows that the orbits of the points in $\mathrm{Supp}(\mathcal{V})$ must be infinite, i.e., that (iii) holds. If (iii) holds, then $\mathrm{Hom}(\mathcal{V}, s^n(\mathcal{N})) = 0$ for every $\mathcal{N}$ and for $n \gg 0$, and this implies (4.2.1)(b). ∎

With the notation of the above proposition, if the triple $(\mathcal{C}, \mathcal{A}, s)$ satisfies the conditions (i)–(iii), then by Theorem 4.5, the algebra $A := \Gamma(\mathcal{A})_{\geqslant 0}$ is right noetherian and satisfies $\chi_1$.

EXAMPLE 6.12. *A particular case.* This ring is given as Example 4.3 in [SZ2]. Consider the case that $\sigma$ is multiplication of a coordinate by $q$, where $q$ is not a root of unity, and that $s = s_1\sigma^*$. If $\mathcal{A} = \mathcal{O}$, then $A$ is isomorphic to the quantum polynomial algebra $R = k_q[x, y]$ (see Example 5.11). But according to the previous proposition, we may also take $\mathscr{A} = \mathcal{O} \oplus k(p)$, where $p$ is any point other than the two fixed points of $\sigma$. In this case,

$$A = \begin{pmatrix} R & 0 \\ M & k \end{pmatrix},$$

where $M$ is the point module corresponding to $p$. This module has the form $R/zR$, where $z = ax + by$ is an element of degree 1, and $a$, $b$ are not zero. Note that the ring $A$ is right but not left noetherian. We omit the proof of the following result.

PROPOSITION 6.13. *With the notation of Proposition* 6.11, *suppose that* (i)–(iii) *hold. The algebra* $A := \Gamma(\mathscr{A})_{\geq 0}$ *is left noetherian if and only if* $\mathscr{V} = 0$, *and if and only if* $A$ *satisfies* $\chi$. ∎

## 7. COHOMOLOGY

Let $A$ be a right noetherian $\mathbb{N}$-graded algebra. The category QGr $A$ has enough injective objects, so the Ext groups can be defined on QGr $A$ by using injective resolutions:

$$\operatorname{Ext}^i_{\mathrm{QGr}}(\mathscr{N}, \mathscr{M}) = \mathrm{h}^i(\operatorname{Hom}_{\mathrm{QGr}}(\mathscr{N}, \mathscr{E}^{\cdot})), \tag{7.0.1}$$

where $\mathrm{h}^i$ denotes cohomology of the complex, and where $\mathscr{E}^{\cdot}$ is an injective resolution of $\mathscr{M}$. In the noetherian category qgr $A$, we define $\operatorname{Ext}^i_{\mathrm{qgr}}(\mathscr{M}, \mathscr{N}) := \operatorname{Ext}^i_{\mathrm{QGr}}(\mathscr{M}, \mathscr{N})$. This Ext group is isomorphic to the "Yoneda Ext," defined in terms of equivalence classes of $i$-fold extensions in qgr $A$ (see [Mi, Chap. VII, Sect. 3]). Thus it is defined in terms of category qgr $A$ alone. Another way to see that Ext is defined in terms of qgr $A$ is to recall that, by Proposition 2.3, qgr $A$ determines QGr $A$ up to equivalence.

The following proposition sums up standard results relating injectives in Gr $A$ and in the quotient category QGr $A$ (see [Ga; Po, 4.5.3, 4.5.4]).

PROPOSITION 7.1. *Let $A$ be a right noetherian $\mathbb{N}$-graded algebra. Then for the category* Gr $A$ *and its quotient category* QGr $A$, *the following hold*:

(1) *The functor $\omega$ carries injective objects of* QGr $A$ *to injective objects of* Gr $A$. *Moreover, $\omega(\mathscr{Q})$ is torsion-free for every injective object* $\mathscr{Q} \in$ QGr $A$.

(2) *Every injective object* $\mathscr{Q} \in$ QGr $A$ *is of form $\pi(Q)$, where $Q$ is an injective, torsion-free right $A$-module.*

(3) *If* $Q \in$ Gr $A$ *is injective and torsion-free, then $Q$ is naturally isomorphic to* $\tilde{Q} := \omega(\pi(Q))$.

(4) *Let $I$ be the injective hull of a module $M$. Then $I$ is right bounded (resp. torsion) if and only if $M$ is right bounded (resp. torsion).*

(5) *Every injective in* Gr $A$ *is a direct sum $Q \oplus I$ where $Q$ is an injective torsion-free right $A$-module and $I$ is an injective torsion right $A$-module, and the modules $Q$ and $I$ are determined up to isomorphism.*

*Proof.* (1) is true because $\pi$ is exact and because $\omega(\mathcal{M})$ is torsion-free for any $\mathcal{M} \in \mathrm{QGr}\, A$. (2) follows from (1) and the fact that $\pi\omega\mathcal{Q} \cong \mathcal{Q}$. (3) follows from Proposition 3.14(1). (4) follows from Proposition 2.2. To prove (5), let $T$ denote the torsion submodule of an injective module $J$, and let $I$ be the injective hull of $T$. Then $I$ is a summand of $J$, and it is torsion. ▮

Let $M$ be a right $A$-module, and let $E^{\cdot}(M)$ denote an injective (resp. a minimal injective) resolution of $M$. Recall that an injective resolution $E^{\cdot}$ of a module $M$ is called *minimal* if $E^0$ is an essential extension of $M$ and for each $j \geqslant 1$, $E^j$ is and essential extension of $dE^{j-1}$. The graded $\underline{\mathrm{Ext}}$ group is

$$\underline{\mathrm{Ext}}^i(N, M) = \mathrm{h}^i(\underline{\mathrm{Hom}}(N, E^{\cdot}(M)),$$

where $\mathrm{h}^i$ denotes the $i$th cohomology group of a complex. We may write the resolution in the form

$$E^{\cdot}(M) := 0 \to M \to Q^0 \oplus I^0 \to Q^1 \oplus I^1 \to \cdots, \tag{7.1.1}$$

where $Q^i$ are injective and torsion-free and $I^i$ are injective torsion modules. Then there is an exact sequence of complexes

$$0 \to I^{\cdot}(M) \to E^{\cdot}(M) \to Q^{\cdot}(M) \to 0, \tag{7.1.2}$$

where $I^j(M) = I^j$ and $Q^j(M) = Q^j$. Since $Q^{\cdot}$ is torsion-free,

$$\underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M) = \mathrm{h}^i(\underline{\mathrm{Hom}}(A/A_{\geqslant n}, I^{\cdot}(M)),$$

we have

$$\mathrm{h}^i(I^{\cdot}(M))) = \mathrm{h}^i(\lim_{n \to \infty} \underline{\mathrm{Hom}}(A/A_{\geqslant n}, I^{\cdot}(M))) = \lim_{n \to \infty} \underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M). \tag{7.1.3}$$

If we apply the exact functor $\pi$ to (7.1.1), we obtain an (resp. a minimal) injective resolution of $\mathcal{M} = \pi(M)$,

$$\mathcal{E}^{\cdot}(\mathcal{M}) := 0 \to \mathcal{M} \to \mathcal{Q}^0 \to \mathcal{Q}^1 \to \cdots, \tag{7.1.4}$$

where $\mathcal{Q}^i = \pi(Q^i)$. Applying $\omega$ to the exact sequence (7.1.4) gives back the complex $Q^{\cdot}(M)$, because $Q^i \cong \tilde{Q}^i$.

If $\mathcal{N} = \pi(N)$ for a graded finite $A$-module $N$, then by definition,

$$\operatorname{Ext}^i(\mathcal{N}, \mathcal{M}) = \mathrm{h}^i(\operatorname{Hom}(\mathcal{N}, \mathcal{E}^{\cdot}(\mathcal{M}))) \cong \mathrm{h}^i(\operatorname{Hom}(N, Q^{\cdot}(M)), \quad (7.1.5)$$

where the last isomorphism is induced by the adjoint isomorphism (2.2.2) $\operatorname{Hom}(N, Q^i) \cong \operatorname{Hom}(\mathcal{N}, \mathcal{Q}^i)$.

Set $N = A$ in (7.1.5). Then $\operatorname{Hom}(A, Q^{\cdot}(M)) = Q^{\cdot}(M)_0$, where $Q^{\cdot}(M)_0$ is the degree 0 part of the complex $Q^{\cdot}(M)$. By definition,

$$\mathrm{H}^i(\mathcal{M}) := \operatorname{Ext}^i(\mathcal{A}, \mathcal{M}) = \mathrm{h}^i(Q^{\cdot}(M))_0. \quad (7.1.6)$$

As usual, we define the graded Ext and graded cohomology by

$$\underline{\operatorname{Ext}}^i(\mathcal{M}, \mathcal{N}) = \bigoplus_{n=-\infty}^{\infty} \operatorname{Ext}^i(\mathcal{N}, \mathcal{M}[n]) \cong \mathrm{h}^i(\underline{\operatorname{Hom}}(N, Q^{\cdot}(M)), \quad (7.1.7)$$

and

$$\underline{\mathrm{H}}^i(\mathcal{M}) = \bigoplus_{n=-\infty}^{\infty} \mathrm{H}^i(\mathcal{M}[n]) \cong \mathrm{h}^i(Q^{\cdot}(M)). \quad (7.1.8)$$

Note that $\underline{\mathrm{H}}^0 = \Gamma$ and that $\underline{\mathrm{H}}^i(\mathcal{M})$ is naturally a right $\Gamma(\mathcal{A})$-module, hence also a right $A$-module.

Proposition 7.2. *Let $A$ be a right noetherian $\mathbb{N}$-graded algebra, let $M$ and $N$ be $A$-modules and let $\mathcal{M} = \pi(M)$ and $\mathcal{N} = \pi(N)$ respectively. Assume that $N$ is a finite module.*

(1) *For $i \geqslant 0$, we have*

$$\underline{\operatorname{Ext}}^i(\mathcal{N}, \mathcal{M}) \cong \lim_{n \to \infty} \underline{\operatorname{Ext}}^i(N_{\geqslant n}, M)$$

and

$$\underline{\mathrm{H}}^i(\mathcal{M}) \cong \lim_{n \to \infty} \underline{\operatorname{Ext}}^i(A_{\geqslant n}, M).$$

(2) *There is an exact sequence*

$$0 \to \tau(M) \to M \to \underline{\mathrm{H}}^0(\mathcal{M}) \to \lim_{n \to \infty} \underline{\operatorname{Ext}}^1(A/A_{\geqslant n}, M) \to 0,$$

*and for $i \geqslant 1$,*

$$\underline{\mathrm{H}}^i(\mathcal{M}) \cong \lim_{n \to \infty} \underline{\operatorname{Ext}}^{i+1}(A/A_{\geqslant n}, M) \cong \mathrm{h}^{i+1}(\Gamma^{\cdot}(M)).$$

(3) *$\underline{\mathrm{H}}^i(\mathcal{M})$ is a torsion $A$-module, if $i \geqslant 1$.*

(4) *$\underline{\mathrm{Ext}}^i(\mathcal{N}, \mathcal{M})$ and $\underline{\mathrm{H}}^i(\mathcal{M})$ are compatible with direct limits of objects $\mathcal{M}$.*

*Proof.* With the above notation, we have (see (2.2.1) and (2.2.2))

$$\begin{aligned}\lim_{n\to\infty} \underline{\mathrm{Ext}}^i(N_{\geq n}, M) &\cong \lim_{n\to\infty} \mathrm{h}^i(\underline{\mathrm{Hom}}(N_{\geq n}, Q^{\cdot}\oplus I^{\cdot}))\\ &\cong \mathrm{h}^i(\lim_{n\to\infty} \underline{\mathrm{Hom}}(N_{\geq n}, Q^{\cdot}\oplus I^{\cdot}))\\ &\cong \mathrm{h}^i(\lim_{n\to\infty} \underline{\mathrm{Hom}}(N_{\geq n}, Q^{\cdot})\oplus \lim_{n\to\infty} \underline{\mathrm{Hom}}(N_{\geq n}, I^{\cdot}))\\ &\cong \mathrm{h}^i(\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{Q}^{\cdot})\oplus 0)\cong \underline{\mathrm{Ext}}^i(\mathcal{N}, \mathcal{M}).\end{aligned}$$

This proves the first part of (1), and the second is the case $N = A$. Next, (2) follows from (1), (3.1.4), (3.1.5), and (7.1.3), and (3) is a consequence of the last assertion of (2). Finally, (4) follows from Proposition 3.1(1b). ∎

COROLLARY 7.3. *Let $A$ be a right noetherian $\mathbb{N}$-graded algebra satisfying $\chi_i^{\circ}$, and let $M$, $N$ be finite $A$-modules. Suppose $j \leq i-1$. Then*

(1) *for all $d$ and all $n \gg 0$, $\underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})_{\geq d} = \underline{\mathrm{Ext}}^j(N_{\geq n}, M)_{\geq d}$, and*

(2) *the natural map from $\underline{\mathrm{Ext}}^j(N, M)$* to *$\underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})$ has right bounded kernel and cokernel.*

(3) If *$A$ is locally finite, then $\underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})$ is also locally finite.*

*Proof.* (1) follows from Proposition 3.5(2) and Proposition 7.2(1). (2) follows from (1) and Proposition 3.5(3). (3) follows from (1) and Proposition 3.1(3). ∎

We are now ready to prove a finiteness theorem for proj $A$.

THEOREM 7.4 (Serre's Finiteness Theorem). (1) *Let $A$ be a right noetherian $\mathbb{N}$-graded algebra satisfying $\chi$, and let $\mathcal{M} \in$ qgr $A$. Then*

(H4) *for every $j \geq 0$, $\mathrm{H}^j(\mathcal{M})$ is a finite right $A_0$-module, and*

(H5) *for every $j \geq 1$, $\underline{\mathrm{H}}^j(\mathcal{M})$ is right bounded; i.e., there is an integer $d_0$ such that, for all $d \geq d_0$, $\mathrm{H}^j(\mathcal{M}[d]) = 0$.*

(2) *Conversely, if $A$ satisfies $\chi_1$ and if* (H4) *and* (H5) *hold for every $\mathcal{M} \in$ qgr $A$, then $A$ satisfies $\chi$.*

*Remarks.* (1) If $\chi_1$ holds, then it follows from Proposition 3.13(3) and Proposition 2.1 that $\mathrm{H}^0(\mathcal{A})$ is a finite right $A_0$-module. Therefore one may replace $A_0$ by $\mathrm{H}^0(\mathcal{A})$ in condition (H4).

(2) There are noetherian algebras $A$ such that (proj $A, s$) satisfies (H1), (H2), and (H3) of Theorem 4.5 and (H4) and (H5) of Theorem 7.4,

but $A$ does not satisfy $\chi_1$. An example is given in [SZ2, Thm. 2.3, Cor. 2.8].

*Proof of Theorem* 7.4 We are going to prove Theorem 7.4 by verifying the following:

($\dagger$) Let $A$ be a right noetherian $\mathbb{N}$-graded algebra and let $M$ be a finite $A$-module. Let $\mathcal{M} = \pi(M)$ and suppose $i \geqslant 2$. If $\chi_i(M)$ holds, then the following two conditions hold:

$(\mathrm{H4})_i(\mathcal{M})$: for every $d$ and every $0 \leqslant j < i$, $\underline{\mathrm{H}}^j(\mathcal{M})_d$ is a finite right $A_0$-module.

$(\mathrm{H5})_i(\mathcal{M})$: for every $1 \leqslant j < i$, $\underline{\mathrm{H}}^j(\mathcal{M})$ is right bounded.
Conversely, if $\chi_1(M)$, $(\mathrm{H4})_i(\mathcal{M})$ and $(\mathrm{H5})_i(\mathcal{M})$ hold, then $\chi_i(M)$ holds.

Since $\chi_1(M)$ holds, by Proposition 3.13(3), $\underline{\mathrm{H}}^0(\mathcal{M})_d$ is a finite $A_0$-module for all $d$. So it suffices to consider $j \geqslant 1$ in $(\mathrm{H4})_i(\mathcal{M})$. Suppose that $\chi_i(M)$ holds. By Proposition 3.5(2) and Proposition 7.2(2), we have $\underline{\mathrm{H}}^j(\mathcal{M})_{\geqslant d} = \underline{\mathrm{Ext}}^{j+1}(A/A_{\geqslant n}, M)_{\geqslant d}$ for $n \gg 0$ and $j < i$. The condition $\chi_i(M)$ implies that $\underline{\mathrm{Ext}}^{j+1}(A/A_{\geqslant n}, M)_d$ is a finite right $A_0$-module and that $\underline{\mathrm{Ext}}^{i+1}(A/A_{\geqslant n}, M)_{\geqslant d}$ is right bounded. Therefore $(\mathrm{H4})_i(\mathcal{M})$ and $(\mathrm{H5})_i(\mathcal{M})$ hold.

For the converse, we assume $\chi_1(M)$, $(\mathrm{H4})_i(\mathcal{M})$ and $(\mathrm{H5})_i(\mathcal{M})$ hold. It suffices to show that $\chi_i^\circ(M)$ holds. If so, then by Proposition 3.5(1) and Proposition 7.2(2), $\underline{\mathrm{H}}^j(\mathcal{M})_{\geqslant d} = \underline{\mathrm{Ext}}^{j+1}(A/A_{\geqslant n}, M)_{\geqslant d}$ for $n \gg 0$ and $0 \leqslant j < i$. Then $(\mathrm{H4})_i(\mathcal{M})$ and $(\mathrm{H5})_i(\mathcal{M})$ imply that $\chi_i(M)$ holds. It remains to prove $\chi_i^\circ$, and the cases $i = 0, 1$ are true by hypothesis. For $i \geqslant 2$ we use induction and assume that $\chi_{i-1}^\circ$ holds. Then $(\mathrm{H5})_i(\mathcal{M})$ and Proposition 7.2(2) imply that $\lim_{n\to\infty} \underline{\mathrm{Ext}}^i(A/A_{\geqslant n}, M)$ is right bounded. By Corollary 3.6(3), $\chi_i^\circ(M)$ holds. ∎

When $A$ is a locally finite $k$-algebra, we have the following.

Corollary 7.5. *Let $A$ be a right noetherian locally finite $\mathbb{N}$-graded $k$-algebra satisfying $\chi$. Let $\mathcal{M}$ be an object in* qgr $A$. *Then*

$(\mathrm{H4})'$ *for each $j \geqslant 0$, $\mathrm{H}^j(\mathcal{M})$ is a finite right $k$-module*;

(H5) *for each $j \geqslant 1$, there is an integer $d_0$ such that, for all $d \geqslant d_0$, $\mathrm{H}^j(\mathcal{M}[d]) = 0$.*

*Conversely, if $A$ satisfies $\chi_1$ and* (proj $A$, $s$) *satisfies* $(\mathrm{H4})'$ *and* (H5), *then $A$ satisfies $\chi$.* ∎

Corollary 7.6. *Suppose the triple $(\mathcal{C}, \mathcal{A}, s)$ satisfies* (H1), (H2), *and* (H3) *of Theorem* 4.5 *and* (H4) *and* (H5) *of Theorem* 7.4. *Then $A := \Gamma(\mathcal{A})_{\geqslant 0}$ is right noetherian graded which satisfies $\chi$. If in addition* $(\mathrm{H4})'$ *of Corollary* 7.5 *holds, then $A$ is locally finite.*

*Proof.* By Theorem 4.5(2), $A$ is right noetherian and satisfies $\chi_1$, and $(\mathscr{C}, \mathscr{A}, s) \cong (\text{proj } A, s)$. So we replace $(\mathscr{C}, \mathscr{A}, s)$ by $(\text{proj } A, s)$, and (H4) and (H5) still hold. By Theorem 7.4, $A$ satisfies $\chi$. To prove the last assertion, note that (H4)′ (letting $j=0$) implies (H2)′. By Corollary 4.6(1), $A$ is locally finite. ▮

We can give a more complete description of the condition $\chi$ in the case that the ground ring $k$ is a field and that $A$ is a connected graded $k$-algebra. Since $A$ is right noetherian, it is also locally finite. For every locally finite $A$-module $M$, the *graded dual* $M^*$ is defined by $M_n^* = \text{Hom}_k(M_{-n}, k)$. If $M$ is a left $A$-module, then $M^*$ is a right $A$-module and vice versa. It is easy to see that $*$ is a contravariant functor between the category of locally finite graded right $A$-modules and the category of locally finite graded left $A$-modules and that $**$ is isomorphic to the identity functor. Hence a locally finite $A$-module $M$ is projective (resp. injective) if and only if $M^*$ is injective (resp. projective).

Now whenever $A$ is right noetherian, a torsion injective $I$ is the injective hull of its *socle* $S = \underline{\text{Hom}}(A_0, I)$. In our case, the socle is a graded vector space over $k$, which is a direct sum of (possibly infinitely many) shifts of $k$. And, the direct sum of injective hulls is an injective hull of the sum because $A$ is right noetherian [AF, Prop. 18.13]. The injective hull of $k$ is the graded dual $A^*$ of the projective cover $A$ of the left module $k$. Thus when $k$ is a field and $A$ is connected, the torsion injectives are sums of shifts of the dual module $A^*$. Dually to $\text{Hom}(A, A[r]) = A_r$, we have

$$\text{Hom}(A^*[-r], A^*) = A_r.$$

Proposition 7.7. *Assume that $k$ is a field and that $A$ is a right noetherian connected graded $k$-algebra. Let $E^\cdot(M)$ be a minimal injective resolution of a finite module $M$, and let $I^\cdot(M)$ be its subcomplex of torsion modules (see (7.1.2)). Then*

(1) *$\underline{\text{Ext}}^j(k, M)$ is isomorphic to the socle $\underline{\text{Hom}}(k, I^j)$ of $I^j$, and*

(2) *$A$ satisfies $\chi$ if and only if each $I^j$ in $I^\cdot(M)$ is a finite sum of shifts of $A^*$.*

*Proof.* Since $A$ is locally finite, $\chi$ is equivalent to $\chi^\circ$, which in this case says that $\underline{\text{Ext}}^j(k, M)$ is finite dimensional for all $j \geq 0$ and all finite $A$-modules $M$. Hence (2) follows from (1), and it suffices to prove (1). Since $E^\cdot$ is a minimal resolution, the socles of $dE^{j-1}$ and $E^j$, are equal if $j \geq 1$, as are the socles of $M$ and $E^0$. This implies that the coboundary map $\underline{\text{Hom}}(k, E^j) \to \underline{\text{Hom}}(k, E^{j+1})$ is zero for all $j \geq 0$. Hence $\underline{\text{Ext}}^j(k, M) = \underline{\text{Hom}}(k, Q^j \oplus I^j) = \underline{\text{Hom}}(k, I^j)$. ▮

For arbitrary right noetherian graded rings, we have the following.

PROPOSITION 7.8. *Let $A$ be a right noetherian $\mathbb{N}$-graded algebra, and let $I^{\cdot}(M)$ denote the torsion subcomplex of a minimal injective resolution of a graded $A$-module $M$, as in (7.1.1). Then $\chi_i^{\circ}(M)$ holds if and only if $I^j(M)$ is right bounded for $j \leqslant i$.*

*Proof.* Set $E^j := E^j(M)$ and $I^j := I^j(M)$. If $I^j$ is right bounded, so is $h^j(I^{\cdot})$. By (7.1.3), $h^j(I^{\cdot}) = \lim_{n \to \infty} \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ is right bounded. Corollary 3.6(3) shows that $\chi_i^{\circ}(M)$ holds. Conversely, suppose that $I^j$ is unbounded for some $j \leqslant i$, and let $j$ be the smallest integer such that $I^j$ is unbounded. Let $Z^j = \mathrm{im}(E^{j-1} \to E^j) = \ker(E^j \to E^{j+1})$, and let $T^j$ denote the torsion submodule of this module. Since $E^{\cdot}$ is a minimal resolution, $E^j$ is an essential extension of $Z^j$ and $I^j$ is an essential extension of $T^j$. Thus $T^j$ is unbounded (see Proposition 7.1(4)). Also, $\mathrm{h}^j(I^{\cdot}) = T^j/\mathrm{im}\, I^{j-1}$. Since $I^{j-1}$ is bounded, this shows that $\mathrm{h}^j(I^{\cdot})$ is unbounded, hence that $\chi_i^{\circ}(M)$ fails.

PROPOSITION 7.9. *If $k$ is a field and $A$ is a locally finite noetherian $\mathbb{N}$-graded $k$-algebra satisfying $\chi$, then for every object $\mathcal{M}$ in* qgr *$A$ and every $i \geqslant 1$, $\underline{\mathrm{H}}^i(\mathcal{M})^*$ is a finite left $A$-module.*

*Proof.* We use a fact which will be proved in Lemma 8.2(5): $A$ satisfies $\chi$ if and only if the connected graded subalgebra $k \oplus A_{\geqslant 1}$ satisfies $\chi$. Replacing $A$ by a connected graded subalgebra of $A$, we may assume $A$ is connected. By Proposition 7.2(2), $\underline{\mathrm{H}}^i(\mathcal{M})$ is a subfactor of $I^{i+1}$. Since $*$ is exact, it suffices to prove that $(I^{i+1})^*$ is a finite left $A$-module. This follows from Proposition 7.7(2). ∎

We define *cohomological dimension* of proj $A$ to be

$$\mathrm{cd}(\mathrm{proj}\, A) = \max\{i \mid \mathrm{H}^i(\mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \mathrm{qgr}\, A\}.$$

Since cohomology commutes with direct limits (see Proposition 7.2(4)), we could replace qgr $A$ by QGr $A$ and proj $A$ by Proj $A$ in the definition. By definition, the cohomological dimension depends on the structure sheaf $\mathcal{A}$ but not on the shift operator $s$. The proof of the following proposition is routine.

PROPOSITION 7.10. (1) *If* $\mathrm{cd}(\mathrm{proj}\, A)$ *is* finite, then it is equal to $\max\{i \mid \underline{\mathrm{H}}^i(\mathcal{A}) \neq 0\}$.

(2) *Suppose that every finite nonzero $A$-module $M$ contains a nonzero submodule $M'$ such that $\max\{i \mid \mathrm{H}^i(\mathcal{M}') \neq 0\} \leqslant d$. Then* $\mathrm{cd}(\mathrm{proj}\, A) \leqslant d$.

(3) *If the global dimension of $A$ is $d < \infty$, then* $\mathrm{cd}(\mathrm{proj}\, A) \leqslant d - 1$.

(4) *If* $\mathrm{cd}(\mathrm{proj}\, A)$ *is finite, then* $\mathrm{cd}(\mathrm{proj}\, A) \leqslant \max\{\mathrm{inj.dim}(A) - 1, 0\}$.

*Proof.* (1) Let $d$ be the cohomological dimension of proj $A$. We pick an object $\mathcal{M}$ in qgr $A$ such that $\underline{\mathrm{H}}^d(\mathcal{M}) \neq 0$. Consider the short exact sequence

$$0 \to \mathcal{N} \to \bigoplus_{i=1}^{p} \mathcal{A}[l_i] \to \mathcal{M} \to 0$$

for some $\mathcal{N}$ in qgr $A$. By the long exact sequence of derived functors $\underline{\mathrm{H}}^i$, we have

$$\to \bigoplus_{i=1}^{p} \underline{\mathrm{H}}^d(\mathcal{A})[l_i] \to \underline{\mathrm{H}}^d(\mathcal{M}) \to \underline{\mathrm{H}}^{d+1}(\mathcal{N}) = 0.$$

Hence $\underline{\mathrm{H}}^d(\mathcal{A}) \neq 0$ and (1) follows.

(2) follows by noetherian induction and the exact cohomology sequence and (3) follows from Proposition 7.2(2).

(4) The injective dimension is the length of an injective resolution of $A$. If inj.dim$(A) = p < \infty$, then $\underline{\mathrm{Ext}}^{p+1}(A/A_{\geqslant n}, A) = 0$ for all $n$. Hence by Proposition 7.2(2), $\underline{\mathrm{H}}^p(\mathcal{A}) = 0$ and by (1), cd(proj $A$) $\leqslant p - 1$. ▮

## 8. Some Graded Algebras Satisfying $\chi$

In this section we describe some graded algebras which satisfy the condition $\chi$, and we bound the cohomological dimension of Proj $A$ for some rings.

One important class of algebras which satisfy $\chi$ is the class of regular algebras. We adopt the terminology of [AS], calling a graded algebra $A$ over a field $k$ *regular* if it is connected graded and if it has the following properties:

(1) $A$ has finite global dimension $d$;

(2) $A$ has finite GK-dimension;

(3) $A$ is Gorenstein, meaning that $\underline{\mathrm{Ext}}^i(k, A) = 0$ if $i \neq d$, and for some $l$, $\underline{\mathrm{Ext}}^d(k, A) \cong k[l]$.

The following generalized Gorenstein condition follows from (3): For every finite length module $T$, $\underline{\mathrm{Ext}}^i(T, A) = 0$ for all $i \neq d$ and

$$\underline{\mathrm{Ext}}^d(T, A) = T^*[l] \tag{8.0.1}$$

where $T^*$ is the graded dual of $T$. It is unknown whether all noetherian connected graded rings having finite global dimension are regular. However, if an algebra of finite global dimension is a PI ring or, more general, if it has enough normalizing elements, then it is regular (see [SZ1]). For more information about regular rings see [AS, Le].

If $A$ is a noetherian connected graded algebra having global dimension 1, then $A$ is isomorphic to $k[x]$, where $\deg(x)=n$ for some $n>0$. There is nothing new in this case. For regular rings of dimension greater than 1, we have the following.

THEOREM 8.1. *Let $A$ be a noetherian regular graded algebra of dimension $d\geqslant 2$ over a field $k$, and let $N$ be a finite $A$-module. Let $\mathscr{A}=\pi(A_A)$ and $\mathscr{N}=\pi(N)$. Then*

(1) *$A$ satisfies the condition $\chi$;*

(2) *$\Gamma(\mathscr{A})=A$;*

(3) *$\underline{\mathrm{H}}^i(\mathscr{A})=0$ for $i\neq 0$, $d-1$, and $\underline{\mathrm{H}}^{d-1}(\mathscr{A})=A^*[l]$, where $l$ is as in* (8.0.1);

(4) $\mathrm{cd}(\mathrm{proj}\ A)=d-1$;

(5) $\underline{\mathrm{Ext}}^i(\mathscr{N},\mathscr{A})=\underline{\mathrm{Ext}}^i(N,A)$ *for all $i\leqslant d-2$.*

*Proof.* (1) Since $A$ is locally finite, we only need to prove that $A$ satisfies $\chi^\circ$ (see Proposition 3.11(2)). We use induction on the projective dimension of $M$. If $\mathrm{pd}(M)=0$, then $M$ is a finite direct sum of shifts of $A$, and $\underline{\mathrm{Ext}}^i(k,M)$ is bounded for all $i$ by the Gorenstein condition. If $\mathrm{pd}(M)>0$, we choose an exact sequence

$$0\to N\to P\to M\to 0$$

where $P$ is a sum of shifts of $A$. Then $\mathrm{pd}(N)=\mathrm{pd}(M)-1$. By induction, $\underline{\mathrm{Ext}}^i(k,N)$ and $\underline{\mathrm{Ext}}^i(k,P)$ are bounded for every $i$, hence so is $\underline{\mathrm{Ext}}^i(k,M)$.

(2) We consider the exact sequence (3.1.4). By the generalized Gorenstein condition (8.0.1), its left and the right terms are zero. Hence

$$A\cong \lim_{n\to\infty}\underline{\mathrm{Hom}}(A_{\geqslant n},A)=\Gamma(\mathscr{A}).$$

(3) Since $\lim_{n\to\infty}(A/A_{\geqslant n})^*=A^*$, this follows from the generalized Gorenstein condition and Proposition 7.2(2).

(4) Since $A$ has global dimension $d$, $\mathrm{cd}(\mathrm{proj}\ A)\leqslant d-1$ (see Corollary 7.8). By (3), $\mathrm{cd}(\mathrm{proj}\ A)=d-1$.

(5) This follows from Proposition 7.2(1), the generalized Gorenstein condition, and the $\underline{\mathrm{Ext}}$ sequence associated to the exact sequence (3.1.1). ∎

Next, we will show that the condition $\chi$ carries over to finite algebras, and to do this we need the following spectral sequence: Let $\phi\colon A\to B$ be a homomorphism of graded algebras. For any right $A$-module $N$ and right $B$-module $M$, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_B(N \underline{\otimes}_A B, M) \cong \underline{\mathrm{Hom}}_A(N, M_A) \tag{8.1.1}$$

where, as before, $\underline{\otimes}$ is the graded tensor product. There is a spectral sequence of graded $k$-modules (see (2.2) in [SZ1]):

$$E_2^{p,q} = \underline{\mathrm{Ext}}_B^p(\underline{\mathrm{Tor}}_q^A(N, B), M) \Rightarrow \underline{\mathrm{Ext}}_A^{p+q}(N, M_A). \tag{8.1.2}$$

The first three terms of the five-term exact sequence from this spectral sequence are

$$0 \to \underline{\mathrm{Ext}}_B^1(N \underline{\otimes}_A B, M) \to \underline{\mathrm{Ext}}_A^1(N, M_A) \to \underline{\mathrm{Hom}}_B(\underline{\mathrm{Tor}}_1^A(N, B), M). \tag{8.1.3}$$

Let $I_n$ denote the right ideal $A_{\geqslant n} B$ of $B$, so that

$$B/I_n = (A/A_{\geqslant n}) \underline{\otimes}_A B. \tag{8.1.4}$$

Set $N = A/A_{\geqslant n}$ in the spectral sequence (8.1.2). For every $p$, $n$ and $d$, the spectral sequence defines a canonical map

$$(\mathrm{E}_2^{p,0})_{\geqslant d} = \underline{\mathrm{Ext}}_B^p(B/I_n, M)_{\geqslant d} \to \underline{\mathrm{Ext}}_A^p(A/A_{\geqslant n}, M_A)_{\geqslant d}. \tag{8.1.5}$$

LEMMA 8.2. *Let $\phi: A \to B$ be a homomorphism of right noetherian $\mathbb{N}$-graded algebras.*

(1) *If ${}_AB$ is a finite module, then the family of right ideals $I_n$ is cofinal with $B_{\geqslant n}$.*

(2) *$\underline{\mathrm{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded for all $q$ and $n$ if either*

(a) *$A$ is noetherian and ${}_AB$ is a finite module, or*

(b) *the kernel and cokernel of $\phi$ are bounded.*

(3) *Assume that $\chi_{i-1}^{\circ}$ holds for $B$, and that $\underline{\mathrm{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded for $n \gg 0$ and $q \leqslant i$. Then the map (8.1.5) is bijective for $n \gg 0$ and $p \leqslant i$.*

(4) *Assume that $B_A$ is a finite module, that the right ideals $I_n$ are cofinal with $B_{\geqslant n}$, and that $\underline{\mathrm{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded for all $n \gg 0$ and $q \leqslant i$. If $A$ satisfies $\chi_i$, then so does $B$.*

(5) *Assume that the kernel and cokernel of $\phi$ are bounded. Then $B$ satisfies $\chi_i^{\circ}$ if and only if $A$ does. If, in addition, $B_0$ is a finite right $A_0$-module, then $B$ satisfies $\chi_i$ if and only if $A$ does.*

*Proof.* (1) If $B$ is a finite left $A$-module, then (8.1.4) shows that $B/I_n$ is bounded, hence that $B_{\geqslant n'} \subset I_n$ for some $n'$. On the other hand, $I_n \subset B_{\geqslant n}$.

(2) If (a) holds, then Proposition 2.4(3), with left and right interchanged, implies that $\underline{\mathrm{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded. Assume that (b) holds. We consider the exact sequence

$$0 \to \ker \to A \to B \to \operatorname{coker} \to 0.$$

Since $A$ is right noetherian and since ker and coker are bounded, Proposition 2.4(3) shows that $\underline{\mathrm{Tor}}_i^A(A/A_{\geqslant n}, \ker)$ and $\underline{\mathrm{Tor}}_i^A(A/A_{\geqslant n}, \operatorname{coker})$ are bounded. Since $A$ is projective, $\underline{\mathrm{Tor}}_i^A(A/A_{\geqslant n}, A)$ is zero if $i \geqslant 1$, and it is bounded if $i = 0$. It follows that $\underline{\mathrm{Tor}}_i^A(A/A_{\geqslant n}, B)$ is bounded for all $n$ and all $i \geqslant 0$.

(3) We set $N = A/A_{\geqslant n}$ in the spectral sequence (8.1.2). Assume that $\underline{\mathrm{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded. Proposition 2.4(6) implies that the left bound tends to infinity with $n$, if $q \geqslant 1$. If $\chi_{i-1}^\circ$ holds for $B$, then by Lemma 3.4, the right bound of $\mathrm{E}_2^{p,q}$ tends to $-\infty$ as $n \to \infty$ for all $p \leqslant i-1$ and all $1 \leqslant q \leqslant i$. Therefore these terms are zero in degree $\geqslant d$, provided that $n \gg 0$, which implies that the abutment is isomorphic to $\mathrm{E}_2^{p,0}$ in degree $\geqslant d$ for $p \leqslant i$ and $n \gg 0$. This is the assertion of the proposition.

(4) Suppose that the hypotheses of this assertion are true. By induction on $i$, we may assume that $\chi_{i-1}$ holds for $B$. Then (3) shows that (8.1.5) is bijective for $p \leqslant i$ and $n \gg 0$. By Proposition 3.12(1), $\chi_i^\circ$ holds for $B$. To prove that $\chi_i$ holds for $B$, it suffices to prove that the left side of (8.1.5) is a finite right $B$-module for $p \leqslant i$ and $n \gg 0$ (Proposition 3.12(1)). Since $B_A$ is a finite module, $M_A$ is a finite $A$-module for all finite $B$-modules $M$. For such an $M$, the right side of (8.1.5) is a finite $A$-module if $p \leqslant i$ and $n \gg 0$, because $\chi_i$ holds for $A$. Hence the left side is a finite $A$-module too, and a fortiori it is a finite $B$-module, as required.

(5) Assume that the kernel and cokernel are bounded. By induction, we may assume that $B$ and $A$ satisfy $\chi_{i-1}^\circ$. In this case, $A_{\geqslant n} \cong B_{\geqslant n}$ and $I_{\geqslant n} = B_{\geqslant n}$ for $n \gg 0$. Assertions (2b) and (3) show that the map (8.1.5) is bijective for $p \leqslant i$ and $n \gg 0$.

Let $M$ be a finite $B$ (or $A$)-module. Then $M_{\geqslant l}$ for some $l$ is a finite $A$ and $B$-module because $A_{\geqslant n} = B_{\geqslant n}$ for all $n \gg 0$ (see Proposition 2.5). By Proposition 3.1(5),

$$\underline{\mathrm{Ext}}^j_B(B/B_{\geqslant n}, M/M_{\geqslant l})_{\geqslant d} = \underline{\mathrm{Ext}}^j_A(A/A_{\geqslant n}, M/M_{\geqslant l})_{\geqslant d} = 0$$

for $n \gg 0$. Hence we may replace $M$ by $M_{\geqslant l}$.

If $B$ satisfies $\chi_i^\circ$ (resp. $\chi_i$ if $B_0$ is a finite right $A_0$-module), the left side of (8.15) is a bounded (resp. finite) $B$-module for every finite $B$-module $M$, and for $p \leqslant i$ and $n \gg 0$ (see Corollary 3.6(1) and Definition 3.7). Hence the right side of (8.1.5) is a bounded (resp. finite) $A$-module and then $A$ satisfies $\chi_i^\circ$ (resp. $\chi_i$). The converse is proved in the same way. ∎

THEOREM 8.3. *Let $\phi: A \to B$ be a homomorphism of right noetherian $\mathbb{N}$-graded algebras such that ${}_AB$ and $B_A$ are finitely generated.*

(1) *If $A$ satisfies $\chi_1$, so does $B$.*

(2) *If $A$ is noetherian and satisfies $\chi_i$, then so does $B$.*

(3) *Assume that $A$ is noetherian and $B$ satisfies $\chi^\circ$. Let $M$ be a finite $B$-module, and denote by $\mathcal{M}$ and $\mathcal{M}_{\mathcal{A}}$ the corresponding objects of* qgr $B$ *and* qgr $A$. *Then for all $q$,*

$$\mathrm{H}^q_{\mathrm{proj}\,B}(\mathcal{M}) \cong \mathrm{H}^q_{\mathrm{proj}\,A}(\mathcal{M}_{\mathcal{A}}).$$

*Proof.* Since ${}_AB$ is finitely generated, the right ideals $I_n$ are cofinal with $B_{\geqslant n}$ (Lemma 8.2(1)).

(1) Let $M$ be a finite $B$-module. Then $M_A$ is a finite $A$-module. We substitute $N = A/A_{\geqslant n}$ into (8.1.3). Then $\chi_1$ implies that the middle term is a finite $A$-module in degree $\geqslant d$. Then $\underline{\mathrm{Ext}}^1_B(B/I_n, M)_{\geqslant d}$ is finite $B$-module and by Proposition 3.12(1), $B$ satisfies $\chi_1$.

(2) Since $A$ is noetherian and ${}_AB$ is finitely generated, $\underline{\mathrm{Tor}}^A_q(A/A_{\geqslant n}, B)$ is bounded for all $n, q$. Now Lemma 8.2(4) applies.

(3) This assertion follows from Lemma 8.2(3), Proposition 7.2(2), and Proposition 3.12(1). ∎

We have an immediate corollary relating the cohomological dimensions of Proj $A$ and Proj $B$. Two basic cases in which Theorem 8.3 applies are (a) $B$ is a factor ring of $A$ and (b) $A$ is a subring of $B$, over which $B$ is a finite module. One example of (b) is that $A$ is a Veronese subring of $B$.

COROLLARY 8.4. (1) *Let $\phi: A \to B$ be a homomorphism of noetherian $\mathbb{N}$-graded algebras. Suppose ${}_AB$ and $B_A$ are finite modules and that $B$ satisfies $\chi$. Then* cd(proj $B$) $\leqslant$ cd(proj $A$).

(2) *Let $B$ be a graded factor ring of a noetherian $\mathbb{N}$-graded algebra $A$ which satisfies $\chi_i$. Then $B$ satisfies $\chi_i$, and* cd(proj $B$) $\leqslant$ cd(proj $A$) *if $A$ or $B$ satisfies $\chi$.*

(3) *If $B$ is graded factor ring of a regular ring of dimension $d$, then $B$ satisfies $\chi$ and* cd(proj $B$) $\leqslant d-1$. ∎

COROLLARY 8.5. *Let $A$ be a noetherian $\mathbb{N}$-graded algebra. Then $A$ satisfies $\chi_i$ if and only if $A/P$ satisfies $\chi_i$ for all minimal prime ideals $P$ of $A$. If $A$ satisfies $\chi$, then* cd(proj $A$) = max{cd(proj $A/P$)}.

*Proof.* If $A$ satisfies $\chi_i$, then by Corollary 8.4(2) every $A/P$ satisfies $\chi_i$. Conversely, if for every minimal prime $P$, $A/P$ satisfies $\chi_i$ then Lemma 8.2 shows that $\chi_i(M_A)$ holds for every finite $A/P$-module $M$. Since every nonzero finite $A$-module has a nonzero submodule which is a right $A/P$-module for some minimal prime $P$, Proposition 3.12(3) implies that $\chi_i$ holds for $A$, and Proposition 7.10(2) shows that $\operatorname{cd}(\operatorname{proj} A) \leqslant \max\{\operatorname{cd}(\operatorname{proj} A/P)\}$. By Corollary 8.4(2), this inequality is an equality. ∎

*Remark.* If $A$ is not left noetherian, then the assertion of Corollary 8.5 for the condition $\chi_i$ may not hold. For example, let $A$ be a right noetherian PI ring which is not left noetherian. By [SZ2, Thm. 4.1], $A$ does not satisfy $\chi_1$, but every prime factor of a right noetherian PI ring $A$ is left and right noetherian [MR, Thm. 13.6.15] and hence it satisfies $\chi$ by Proposition 5.1.

Now let us consider the case that $A$ is a Veronese subring of $B$. We grade $A$ so that it becomes a graded subring of $B$ (see the Remark after Proposition 5.10).

LEMMA 8.6. *Let $B$ be a right noetherian graded algebra and let $A$ be the Veronese subring $B^{(r)}$ for some $r \geqslant 2$. Assume that $B$ is generated in degrees* 0 *and* 1. *Then for all $q \geqslant 1$ and all $n$, $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B)$ is a bounded right $B$-module.*

*Proof.* First, $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B)$ is a finite right $B$-module, by Propositions 5.10(1) and 2.4(4). Next, we have

$$ {}_AB_A = A \oplus B', \tag{8.6.1}$$

where $B'$ is the sum of terms of $B$ of degrees not divisible by $r$. Since $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, A) = 0$, $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B) \cong \underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B')$. Therefore $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B)$ is zero in all degrees divisible by $r$. Since $B$ is generated in degrees $\leqslant 1$, it follows that $\underline{\operatorname{Tor}}_q^A(A/A_{\geqslant n}, B)$ is bounded. ∎

PROPOSITION 8.7. *Let $B$ be a right noetherian graded algebra and let $A$ be the Veronese subring $B^{(r)}$ for some $r \geqslant 2$. Assume that ${}_AB$ and $B_A$ are finite $A$-modules.*

(1) *If $B$ is generated in degree* 0 *and* 1, *then*:

(a) $\operatorname{cd}(\operatorname{proj} A) = \operatorname{cd}(\operatorname{proj} B)$;

(b) *$A$ satisfies $\chi_i$ if and only if $B$ does.*

(2) *If $A$ is noetherian, then $B$ satisfies $\chi$ if and only if $A$ satisfies $\chi$, and in this case* $\operatorname{cd}(\operatorname{proj} A) = \operatorname{cd}(\operatorname{proj} B)$.

*Proof.* (1a) follows from Proposition 5.10(3). For (1b), we note that $A_0 = B_0$. It suffices to show that the appropriate $\underline{\operatorname{Ext}}$ groups are finite $A_0$-modules. By induction, we may assume that $\chi_{i-1}$ holds for $A$ and for $B$. By Proposition 5.10(1), Lemma 8.6 and Lemma 8.2(3), the map (8.1.5) is bijective for $p \leqslant i$ and $n \gg 0$. Hence by Lemma 8.2(1) and Proposition 3.12(1), $\chi_i$ holds for $B$ if it holds for $A$. On the other hand, (8.6.1) shows that $N \otimes_A B = N \oplus (N \otimes_A B')$. Setting $M = N \otimes_A B$ in (8.1.5) shows that $\chi_i$ holds for $A$ if it holds for $B$.

(2) Again, one direction of the first part is Theorem 8.3(2). For the other direction, we assume $\chi_i$ holds for $B$. Since $A$ is left and right noetherian, by Lemma 8.2, (8.1.5) is bijective. Then the same argument as (1b) works. Therefore $A$ satisfies $\chi$ if and only if $B$ does.

For the cohomological dimensions, we have $\mathrm{cd}(\mathrm{proj}\, B) \leqslant \mathrm{cd}(\mathrm{proj}\, A)$ by Corollary 8.4(1). If $\mathrm{cd}(\mathrm{proj}\, B) = d$, then for all $i > d$, $\mathrm{H}^i(\mathcal{M}_A) = 0$. Let $N$ be any finite $A$-module and $M = N \otimes_A B$. Then as before, $M = N \oplus M'$, and $\mathcal{M} = \mathcal{N} \oplus \mathcal{M}'$. Hence $\mathrm{H}^i(\mathcal{N}) = 0$ for all $i > d$ and $\mathrm{cd}(\mathrm{proj}\, A) \leqslant \mathrm{cd}(\mathrm{proj}\, B)$. Therefore $\mathrm{cd}(\mathrm{proj}\, A) = \mathrm{cd}(\mathrm{proj}\, B)$. ∎

*Remark.* The proof of Proposition 8.7 shows the following: If $A$ is a noetherian graded subring of $B$ such that ${}_AB$ and $B_A$ are finite $A$-modules and that (8.6.1) holds, then $A$ satisfies $\chi$ if and only if $B$ does, and in this case $\mathrm{cd}(\mathrm{proj}\, A) = \mathrm{cd}(\mathrm{proj}\, B)$.

THEOREM 8.8. *Let $A$ be a noetherian $\mathbb{N}$-graded algebra with a homogeneous normal element $x$ of positive degree. Then $A$ satisfies $\chi$ if and only if $B := A/(x)$ satisfies $\chi$. In this case,* $\mathrm{cd}(\mathrm{proj}\, B) \leqslant \mathrm{cd}(\mathrm{proj}\, A) \leqslant \mathrm{cd}(\mathrm{proj}\, B) + 1$.

*Proof.* If $A$ satisfies $\chi$, then by Corollary 8.4(2) and Theorem 8.3(2), $B$ satisfies $\chi$ and $\mathrm{cd}(\mathrm{proj}\, B) \leqslant \mathrm{cd}(\mathrm{proj}\, A)$. It remains to treat the case that $B$ satisfies $\chi$.

By Corollary 8.5, we may assume that $A$ is prime. This implies that $x$ is regular and that there is a graded algebra automorphism $\sigma$ of $A$ such that $\sigma(a)x = xa$ for all $a \in A$. Given an $A$-module $M$, we denote by $M^\sigma$ the module obtained by restriction of scalars via the map $\sigma\colon A \to A$. Thus the underlying abelian group of $M^\sigma$ is the same as that of $M$, and scalar multiplication in $M^\sigma$ is given as follows: If $m \in M$ and if $m^\sigma$ denotes the same element, viewed as element of $M^\sigma$, then $m^\sigma \cdot a = m\sigma(a)$.

Clearly, $\sigma$ defines a $k$-linear automorphism of $\mathrm{Gr}\, A$ in this way. So if $N$, $M$ are $A$-modules, then there are induced isomorphisms of graded $k$-modules.

$$\underline{\mathrm{Ext}}^j(N, M) \cong \underline{\mathrm{Ext}}^j(N^\sigma, M^\sigma). \tag{8.8.1}$$

Let $E^\cdot(M)$ be an injective resolution of $M$. Then $E^\cdot(M)^\sigma$ is an injective resolution of $M^\sigma$. The right $A$-module structure of $\underline{\mathrm{Hom}}^j(A/A_{\geqslant n}, M)$ is induced by the right $A$-module structure of $E^\cdot(M)$. Hence we obtain an isomorphism of graded right $A$-modules

$$\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M^\sigma) \cong \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)^\sigma \quad \text{and} \quad \underline{\mathrm{H}}^i(\mathcal{M}^\sigma) \cong \underline{\mathrm{H}}^i(\mathcal{M})^\sigma. \tag{8.8.2}$$

Let $l=\deg(x)$. For any right $A$-module $M$, right multiplication by $x: r_x(m^\sigma)=mx$ defines a homomorphism of graded $A$-modules

$$r_x: M^\sigma[-l] \to M. \tag{8.8.3}$$

Assume that $B$ satisfies $\chi$. Lemma 8.2 applies, and it shows that $\chi_i(N_A)$ holds for all finite $B$-modules $N$ and all $i$. Note that the kernel and cokernel of the map $r_x: M^\sigma[-l] \to M$ are annihilated by $x$, hence they are $B$-modules. By Proposition 3.12(3), it suffices to treat the case that the kernel is zero, so that there is an exact sequence

$$0 \to M^\sigma[-l] \to M \to N_A \to 0, \tag{8.8.4}$$

where $N$ is a $B$-module. We apply $\underline{\mathrm{Ext}}^{\cdot}(A_0, \cdot)$ and Lemma 8.9 (below) to this sequence, obtaining short exact sequences

$$0 \to \underline{\mathrm{Ext}}^i(A_0, M) \to \underline{\mathrm{Ext}}^i(A_0, N_A) \to \underline{\mathrm{Ext}}^{i+1}(A_0, M^\sigma)[-l] \to 0$$

for all $i \geqslant 0$. Since $\chi_i(N_A)$ holds, it follows that $\underline{\mathrm{Ext}}^i(A_0, M)$ is bounded, hence that $\chi_i^\circ(M)$ holds for all $i$. Thus $\chi^\circ$ is true for $A$. Next we need to show that the term of degree $d$ of $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)$ is a finite $A_0$-module for every $d$, $j \leqslant i$ and $n \gg 0$ (Proposition 3.5(1)). By descending induction, starting with the right bound, we may assume that $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_{\geqslant d+1}$ is finite. Applying $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, \cdot)$ to (8.8.4), we obtain he following exact sequence:

$$\underline{\mathrm{Ext}}^{j-1}(A/A_{\geqslant n}, N) \to \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M^\sigma)[-l] \to \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M).$$

Consider the terms of degree $d+l$ of the above sequence, the left end is finite because $\chi(N)$ holds, the right end is finite by induction, hence the middle term is finite. The middle term is $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M^\sigma[-l])_{d+l} \cong \underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_d^\sigma$ (see (8.8.2). Therefore $\underline{\mathrm{Ext}}^j(A/A_{\geqslant n}, M)_d$ is finite and $\chi(M)$ holds for all $M$.

Applying $\underline{\mathrm{H}}^{\cdot}$ to the exact sequence (8.8.4), we obtain that $\underline{\mathrm{H}}^i(\mathcal{M}^\sigma)[-l] \cong \underline{\mathrm{H}}^i(\mathcal{M})$ for all $i \geqslant \mathrm{cd}(\mathrm{proj}\, B)+1$. On the other hand, by (8.8.2), $\underline{\mathrm{H}}^i(\mathcal{M}^\sigma) \cong \underline{\mathrm{H}}^i(\mathcal{M})^\sigma$. Since $\underline{\mathrm{H}}^i(\mathcal{M})$ is right bounded by Theorem 7.4, it follows that $\underline{\mathrm{H}}^i(\mathcal{M})=0$, and Proposition 7.10(2) shows the inequality on cohomological dimensions. ∎

LEMMA 8.9. *Let $r_x$ be as defined in* (8.8.3). *Then for all $A$-modules $M$ and all $j$, the map $\underline{\mathrm{Ext}}^j(A_0, M^\sigma[-l]) \to \underline{\mathrm{Ext}}^j(A_0, M)$ induced by $r_x$ is zero.*

*Proof.* The homomorphism $r_x$ is functorial, so if $E=E^{\cdot}(M)$ is an injective resolution of $M$, then $E^\sigma$ is an injective resolution of $M^\sigma$, and $r_x$ extends to a map $E^\sigma[-l] \to E$. On the other hand, since $x$ has positive degree, it annihilates $A_0$. Therefore the map $\underline{\mathrm{Hom}}(A_0, E^{n}[-l]) \to \underline{\mathrm{Hom}}(A_0, E)$ is zero. ∎

The cohomological dimension of $A$ may be equal to the cohomological dimension of $A/(x)$ even when $x$ is a regular homogeneous element of positive degree.

EXAMPLE 8.10. *A graded algebra A and regular homogeneous normal element x of positive degree such that* $\mathrm{cd}(\mathrm{proj}\, A) = \mathrm{cd}(\mathrm{proj}\, B)$ *where* $B = A/(x)$. Let $R = k[t]$ and $A = R[y, z]$, where $k$ is a field, $\deg(t) = 0$, and $\deg(y) = \deg(z) = 1$. Then $\mathrm{proj}\, A$ is the projective line over $\mathrm{spec}\, R$, which has cohomological dimension 1. Consider the regular homogeneous element $x = yt$ of degree 1 in $A$. Then $B := A/(x)$ has $k[y, z]$ as factor ring, and $\mathrm{proj}\, k[y, z]$ is the projective line over $\mathrm{spec}\, k$, which also has cohomological dimension 1. By Theorem 8.1(4) and Corollary 8.4(2), $\mathrm{cd}(\mathrm{proj}\, B) = 1$, too.

LEMMA 8.11. *Let B be a graded ring and* $B[x]$ *be the polynomial extension of B with* $\deg(x) = 1$. *Let N and M be finite B-modules and let* $i \geqslant 0$. *Then for all* $n \geqslant 1$,

$$\underline{\mathrm{Ext}}^{i+1}_{B[x]}(N[x]/N[x]\,x^n, M[x]) = \underline{\mathrm{Ext}}^{i}_{B}(N, M) \otimes_k k[x]\,x^{-n}/k[x].$$

*Proof.* Let $P^{\cdot}$ be a projective resolution of $N$ and let

$$Q^{\cdot} := 0 \longrightarrow k[x][-n] \xrightarrow{x^n} k[x] \longrightarrow k[x]/k[x]\,x^n \longrightarrow 0 \qquad (8.11.1)$$

be the projective resolution of $k[x]/k[x]\,x^n$. Then the tensor product $P^{\cdot} \otimes_k Q^{\cdot}$ is a projective resolution of $N[x]/N[x]\,x^n$. By using the projective resolution (8.11.1), it is easy to see that $\underline{\mathrm{Ext}}^i(k[x]/k[x]\,x^n, k[x]) = 0$ for all $i \neq 1$ and $\underline{\mathrm{Ext}}^1(k[x]/k[x]\,x^n, k[x]) \cong k[x]\,x^{-n}/k[x]$. Since $k[x]$, $k[x]/k[x]\,x^n$ and $k[x]\,x^{-n}/k[x]$ are free $k$-modules, it is direct to check the following identity of complexes

$$\underline{\mathrm{Hom}}(P^{\cdot}, M) \otimes_k \underline{\mathrm{Hom}}(Q^{\cdot}, k[x]) = \underline{\mathrm{Hom}}(P^{\cdot} \otimes_k Q^{\cdot}, M \otimes_k k[x]),$$

and by the Künneth formula [Ro, Thm. 11.31], we finish the proof. ∎

COROLLARY 8.12. (1) *Let B be a right noetherian* $\mathbb{N}$*-graded algebra with automorphism* $\sigma$, *and let A be the Ore extension* $B[x; \sigma]$, *where* $\deg(x) = 1$. *Then A satisfies* $\chi$ *if and only if B satisfies* $\chi$, *and* $\mathrm{cd}(\mathrm{proj}\, A) = \mathrm{cd}(\mathrm{proj}\, B) + 1$.

(2) *If A is a noetherian* $\mathbb{N}$*-graded algebra such that each torsion-free prime factor of A has a homogeneous regular normal element of positive degree, then A satisfies* $\chi$ *and* $\mathrm{cd}(\mathrm{proj}\, A) \leqslant \mathrm{Kdim}(A) - 1$.

(3) *If $A$ is a noetherian $\mathbb{N}$-graded PI algebra, then $A$ satisfies $\chi$ and* $\operatorname{cd}(\operatorname{proj} A) \leqslant \operatorname{Kdim}(A) - 1$.

*Proof.* (1) By Theorem 8.8, $A$ satisfies $\chi$ if and only if $B$ satisfies $\chi$. Since $B[x, \sigma]$ is a graded twisted algebra of $B[x]$, we may assume $\sigma$ is the identity and $A = B[x]$ (see [Zh]). Let $I_n = B_{\geqslant n}[x] + B[x]\, x^n$, then $I_n$ are cofinal with $A_{\geqslant n}$ and $A/I_n = N_n[x]/N_n[x]\, x^n$ where $N_n = A/A_{\geqslant n}$. By Lemma 8.11, $\lim_{n\to\infty} \underline{\operatorname{Ext}}^{i+1}(A/I_n, A[x]) = \{\lim_{n\to\infty} \underline{\operatorname{Ext}}^i(A/A_{\geqslant n}, A)\}[x^{-1}]$. Then Proposition 3.12, Proposition 7.2(2), and Proposition 7.10(1) imply $\operatorname{cd}(\operatorname{proj} A) = \operatorname{cd}(\operatorname{proj} B) + 1$.

(2) We use induction on $\operatorname{Kdim}(A)$. If $\operatorname{Kdim}(A) = 0$, then $A$ is bounded and there is nothing to prove. Assume now $\operatorname{Kdim}(A) > 0$. By Corollary 8.5, we may assume $A$ is prime, and then $A$ has a positive-degree homogeneous normal element. Theorem 8.8 and induction finish the proof.

(3) Every (torsion-free) prime PI ring $A$ has a central element in $A_{\geqslant 1}$ [MR, Thm. 13.6.4], then there is a homogeneous central element in $A_{\geqslant 1}$. The assertion follows from (2) now. ∎

Using induction, we can also prove that every graded FBN ring satisfies the condition $\chi$ and its cohomological dimension is less than the Krull dimension of the ring. We will only recall the definition and state the theorem, omitting the proof.

An $\mathbb{N}$-graded ring $A$ is called *graded FBN* if following conditions hold: (i) $A$ is noetherian and (ii) for any graded prime factor ring $A' := A/P$, every one-sided essential graded ideal of $A'$ contains a nonzero two-sided graded ideal of $A'$. Definition of an ungraded FBN ring is similar without the word "graded" everywhere. Every noetherian PI ring is an FBN ring [MR, Cor. 13.6.6(iii)], and a graded ring which is FBN as an ungraded ring is a graded FBN ring [NV, Lemma C.I.3.3, Prop. C.I.3.4]. However not every graded FBN ring is FBN as an ungraded ring [NV, p. 241]. Another interesting result is [NV, Thm. C.I.3.12]: A graded ring $A$ is FBN (as an ungraded ring) if and only $A[x]$ is graded FBN. We state the following result without proof.

THEOREM 8.13. *Every $\mathbb{N}$-graded FBN algebra $A$ satisfies $\chi$ and* $\operatorname{cd}(\operatorname{proj} A) \leqslant \operatorname{Kdim}(A) - 1$. ∎

## REFERENCES

[Ar] M. ARTIN, Geometry of quantum planes, *Contemp. Math.* **124** (1992), 1–15.

[AF] F. W. ANDERSON AND K. R. FULLER, "Rings and Categories of Modules," Springer-Verlag, New York, 1992.

[AS] M. ARTIN AND W. SCHELTER, Graded algebras of dimension 3, *Adv. Math.* **66** (1987), 172–216.

[ATV1] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of curves, "The Grothendieck Festschrift," Birkhäuser, Basel, 1990, pp. 33–85.

[ATV2] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), 335–389.

[AV] M. Artin and M. Van den Berg, Twisted homogeneous coordinate rings, *J. Algebra* **188** (1990), 249–271.

[EGAII] A. Grothendieck and J. Dieudonné, Eléments de géométrie algébrique II, *Inst. Hautes Etudes Sci. Publ. Math.* **8** (1961).

[EGAIII] A. Grothendieck and J. Dieudonné, Eléments de géométrie algébrique III, *Inst. Hautes Etudes Sci. Publ. Math.* **11** (1963).

[Ga] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[Ha] R. Hartshorne, "Algebraic Geometry," Springer-Verlag, New York 1977.

[Le] T. Levasseur, Some properties of non-commutative regular graded rings, *Glasgow Math. J.* **34** (1992), 277–300.

[Ma] Yu. I. Manin, "Quantum Groups and Non Commutative Geometry," Pub. Cent. Rech. Math., Univ. de Montréal, 1988.

[MR] J. C. McConnell and J. C. Robson, "Non-commutative Noetherian Rings," Wiley–Interscience, Chichester, 1987.

[Mi] B. Mitchell, "Theory of Categories," Academic Press, New York, 1965.

[NV] C. Năstăcescu and F. Van Oystaeyen, "Graded Ring Theory," North-Holland, Amsterdam, 1982.

[OF1] A. V. Odeskii and B. L. Feigin, Sklyanin Algebras associated with an elliptic curve preprint (1989). [In Russian]

[OF2] A. V. Odeskii and B. L. Feigin, Elliptic Sklyanin Algebras (in Russian), *Funktional Anal. i Prilozhen.* **23** (1989), 45–54.

[Po] N. Popescu, "Abelian Categories with Applications to Rings and Modules," L.M.S. Monographs, Academic Press, New York, 1973.

[Pr] C. Procesi, Rings with Polynomial Identities," Dekker, New York, 1973.

[Ro] J. J. Rotman, "An Introduction to Homological Algebra," Academic Press, New York, 1979.

[Se] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. Math.* **61** (1955), 197–278.

[S1] E. K. Sklyanin, Some algebraic structures connected to the Yang–Baxter equation, *Funct. Anal. Appl.* **16** (1982), 27–34.

[S2] E. K. Sklyanin, Some algebraic structures connected to the Yang–Baxter equation. Representations of quantum algebras, *Funct. Anal. Appl.* **17** (1983), 273–284.

[SSW] L. W. Small, J. T. Stafford and R. B. Warfield, Affine algebras of Gelfand–Kirillov dimension one are PI, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 407–414.

[SZ1] J. T. Stafford and J. J. Zhang, Homological properties of (graded) Noetherian PI rings, *J. Algebra* **168** (1994), 988–1026.

[SZ2] J. T. Stafford and J. J. Zhang, Examples in non-commutative projective geometry, *Math. Proc. Cambridge Phil. Soc.*, to appear.

[St] B. Stenström, "Rings of Quotients," Springer-Verlag, New York, 1975.

[TV] J. Tate and M. Van den Berg, Homological properties of Sklyanin algebras, preprint (1993).

[V1] A. B. Verevkin, On a noncommutative analogue of the category of coherent sheaves on a projective scheme, *Amer. Math. Soc. Transl. (2)* **151** (1992).

[V2] A. B. Verevkin, Serre injective sheaves, *Math. Zametki* **52** (1992), 35–41.

[Zh] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, *Proc. London Math. Soc.*, provisionally accepted.