On Gelfand-Zetlin modules over orthogonal Lie algebras

Volodymyr Mazorchuk

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Abstract

We construct a new family of simple modules over orthogonal complex Lie algebra associated with Gelfand-Zetlin formulae for simple finite dimensional modules and study the corresponding Gelfand-Zetlin subalgebra.

1 Introduction

Explicit formulae which effectively define all simple finite dimensional modules over the groups of unimodular and orthogonal matrices were obtained by Gelfand and Zetlin in their famous papers [GZ1, GZ2]. Using these formulae it is possible to define and investigate big families of modules over the corresponding Lie algebras, as was done (for special real forms on the Lie algebras) in [O1, O2]. The analogous formulae are also known for quantum groups (see [J, NT, GK] and references therein). In the same way, using the formulae for the unimodular group, Drozd, Futorny and Ovsienko constructed a large family of simple modules over the reductive Lie algebra $\mathfrak{G} = \mathfrak{g}l(n, \mathbb{C})$ in [DFO1, DFO2]. Roughly speaking this is an $n(n+1)/2$-parameter family of simple $\mathfrak{G}$-modules and each module is presented in a very convenient base and hence is quite simple for computations. It seems that this is the biggest known family of simple weight modules for $\mathfrak{G}$.

Gelfand-Zetlin modules are defined as $U(\mathfrak{G})$-modules which can be decomposed into a direct sum of finite-dimensional modules with respect to the so-called Gelfand-Zetlin subalgebra, which is a big commutative subalgebra in $U(\mathfrak{G})$. In fact, in [O] it was shown that Gelfand-Zetlin subalgebra is a maximal commutative subalgebra in $U(\mathfrak{G})$. Roughly speaking, Gelfand and Zeltlin show that Gelfand-Zetlin subalgebra has a simple spectrum on all finite-dimensinal modules. Simple modules constructed in [DFO1, DFO2] inherit this property and have a natural base parametrized by the so-called (Gelfand-Zetlin) tableaux (or patterns), associated with the points of the spectrum, making them especially easy to handle.

Further properties of Gelfand-Zetlin modules were obtained in [M1, M2, MO, O]. For example a huge family of Verma and generalized Verma modules were realized as Gelfand-Zetlin modules, which allows one to describe the structure of these modules. It was also proved that any simple module over the Gelfand-Zetlin algebra possesses a non-trivial extension to an $U(\mathfrak{G})$-module. Regretfully, all the above results were obtained only for the Lie algebra $\mathfrak{g}l(n, \mathbb{C})$.

The aim of this paper is to construct the Gelfand-Zetlin modules for orthogonal complex Lie algebras and to investigate their properties. Our technique is based on the Gelfand-Zetlin formulae for orthogonal group, see also [GZ1, BR]. In particular, we construct a new family of simple modules over orthogonal complex Lie algebras, investigate the structure of some modules having a tableaux realization and construct the orthogonal operator algebras associated with Gelfand-Zetlin formulae.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries and notations. In Section 3 we recall the Gelfand-Zetlin formulae for the orthogonal group. In Sections 4,5 we introduce Gelfand-Zetlin subalgebra in an orthogonal Lie algebra and construct the family of simple gelfand-Zetlin modules. In Section 6 we propose a simplification for the Gelfand-Zetlin formulae. In Section 7 using the simplified Gelfand-Zetlin formulae we construct two families of modules having a tableaux realization (in he sense of [M1]). Finally, in Section 8 we construct the corresponding orthogonal algebras.

2 Notations and preliminaries

Throughout the paper we fix a base field $\mathbb{C}$ of complex numbers. As usual, we will denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and positive integers respectively. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$. By module we will mean left module.

Suppose that A is an associative algebra and B is its commutative subalgebra. An A-module V is said to be a B-weight (root) module (or simply a weight or root module, if B is fixed) provided it can be decomposed into a direct sum of one dimensional (finite dimensional) B-modules or, equivalently,

$$V = \bigoplus_{\lambda \in B^*} V_\lambda \quad \left(V = \bigoplus_{\lambda \in B^*} V^\lambda \right),$$

where V_λ (V^λ) is a weight (root) subspace corresponding to the weight (root) $\lambda \in B^*$. The set of all non-zero weights of the given B-weight (root) module V is usually called the B-support of V.

We will also denote by e_{ij} the matrix units, i.e. matrices in $Mat_{n\times n}(\mathbb{C})$ whose ij-component is one and all others are zero.

3 Gelfand-Zetlin formulae for orthogonal Lie algebras

Recall the Gelfand-Zetlin formal construction of simple finite dimensional modules for the Lie algebra $\mathfrak{G} = \mathfrak{O}(n, \mathbb{C})$ ([GZ2] or [BR, cf 10.1.B]). Let $X_{i+1\,i} = e_{i+1\,i} - e_{i\,i+1}$, $1 \leqslant i \leqslant n-1$ denote the standard generators of $\mathfrak{G}$. Let $n = 2k$ or $n = 2k+1$. Fix a vector $m = (m_1, m_2, \dots, m_k)$ with integer or half-integer entries satisfying the following conditions:

1. For $n = 2k$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{k-1} \geqslant |m_k|$.
2. For $n = 2k+1$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{k-1} \geqslant m_k \geqslant 0$.

To proceed we have to introduce the notion of tableaux. For a fixed positive integer d by tableau of size d we will mean the vector $[\,l\,]$ with complex (but often with integer or half-integer) entries, considered as a double indexed family $[\,l_{i\,j}\,]$, where $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant k$ for $i = 2k-1$ or $i = 2k$. By $[\,\delta^{i\,j}\,]$ we will denote the Kronecker tableau, i.e. the one with $\delta^{i\,j}_{k\,l} = 0$ if $k \neq i$ or $l \neq j$ and $\delta^{i\,j}_{i\,j} = 1$. Under the above definition, tableaux of the same size form a vectorspace over $\mathbb{C}$. Now with our fixed vector m one can associate a set $\mathcal{M}(m)$ consisting of all tableaux of size $n-1$ with all integer or all half integer entries satisfying the following conditions:

$$\left.\begin{array}{ll}
l_{n-1\,i} = m_i, & \\
l_{2p+1\,i} \geqslant l_{2p\,i} \geqslant l_{2p+1\,i+1}, & i = 1, 2, \dots, p-1, \\
l_{2p+1\,p} \geqslant l_{2p\,p} \geqslant |l_{2p+1\,p+1}|, & \\
l_{2p\,i} \geqslant l_{2p-1\,i} \geqslant l_{2p\,i+1}, & i = 1, 2, \dots, p-1, \\
l_{2p\,p} \geqslant l_{2p-1\,p} \geqslant -l_{2p\,p} &
\end{array}\right\}.$$

Consider $\mathcal{M}(m)$ as a basis for our module and define an action of the generators $X_{i+1\,i}$ on this basis as follows:

$$X_{2p+1\,2p}[\,l\,] = \sum_{j=1}^{p} A(l_{2p-1\,j})([\,l\,] + [\,\delta^{2p-1\,j}\,]) - \sum_{j=1}^{p} A(l_{2p-1\,j} - 1)([\,l\,] - [\,\delta^{2p-1\,j}\,]),$$
$$X_{2p+2\,2p+1}[\,l\,] = \sum_{j=1}^{p} B(l_{2p\,j})([\,l\,] + [\,\delta^{2p\,j}\,]) - \sum_{j=1}^{p} B(l_{2p\,j} - 1)([\,l\,] - [\,\delta^{2p\,j}\,]) + iC_{2p}[\,l\,].$$

Here the functions A, B and C are defined in the following way: first we substitute $l_{2p-1\,j}$ by $s_{2p-1\,j} = l_{2p-1\,j} + p - j$ and $l_{2p\,j}$ by $s_{2p\,j} = l_{2p\,j} + p - j + 1$ for all possible p, then using these notations we define

$$\begin{aligned}
A(l_{2p-1,j}) = \frac{1}{2}\left(\prod_{r=1}^{p-1}(s_{2p-2\,r} - s_{2p-1\,j} - 1)(s_{2p-2\,r} + s_{2p-1\,j})\right)^{1/2} \times \\
\times \left(\prod_{r=1}^{p}(s_{2p\,r} - s_{2p-1\,j} - 1)(s_{2p\,r} + s_{2p-1\,j})\right)^{1/2} \times \\
\times \left(\prod_{r\neq j}(s^2_{2p-1\,r} - s^2_{2p-1\,j})(s^2_{2p-1\,r} - (s_{2p-1\,j} + 1)^2)\right)^{-1/2},
\end{aligned}$$

$$B(l_{2p,j}) = \left(\prod_{r=1}^{p}(s^2_{2p-1\,r} - s^2_{2p\,j})\prod_{r=1}^{p+1}(s^2_{2p+1\,r} - s^2_{2p\,j})\right)^{1/2} \times$$

$$\times \left(s^2_{2p\,j}(4s^2_{2p\,j} - 1)\prod_{r\neq j}(s^2_{2p\,r} - s^2_{2p\,j})((s_{2p\,r} - 1)^2 - s^2_{2p\,j})\right)^{-1/2},$$

$$C_{2p} = \prod_{r=1}^{p} s_{2p-1\,r}\prod_{r=1}^{p+1} s_{2p+1\,r}\left(\prod_{r=1}^{p} s_{2p\,r}(s_{2p,r} - 1)\right)^{-1}.$$

The classical result by Gelfand and Zetlin ([GZ2]) states that in such a way we indeed obtain a simple finite dimensional $\mathfrak{G}$-module, that all simple finite-dimensional $\mathfrak{G}$-modules can be obtained using this construction and base elements $[\,l\,]$ form an orthonormal basis for operators $X_{ij} = e_{ij} - e_{ji}$ with respect to the standard involution $(X_{ij})^* = -X_{ij}$. In what follows we will call the above formulae Gelfand-Zetlin (or simply GZ) formulae. Originally they are obtained considering the restriction of the given simple $\mathfrak{G}$-module on the smaller orthogonal algebra and continuing this procedure. Thus each basis element $[l]$ belongs to the intersection of the modules over the components of the descending chain of algebras

$$\mathfrak{O}(n,\mathbb{C}) \supset \mathfrak{O}(n-1,\mathbb{C}) \supset \cdots \supset \mathfrak{O}(2,\mathbb{C}),$$

where $\mathfrak{O}(j,\mathbb{C}) \subset \mathfrak{O}(j+1,\mathbb{C})$ is the standard embedding with respect to the left upper corner. Now, by using the Schur and the Quillen lemmas, one obtains that $[\,l\,]$ is an eigenvector for the commutative algebra, generated by all $Z(\mathfrak{O}(j,\mathbb{C}))$ for $2 \leqslant j \leqslant n$. This is the motivation for introducing the notion of Gelfand-Zetlin subalgebra.

4 Gelfand-Zetlin subalgebra

Consider the chain of orthogonal Lie algebras

$$\mathfrak{O}(2,\mathbb{C}) \subset \mathfrak{O}(3,\mathbb{C}) \subset \cdots \subset \mathfrak{O}(n,\mathbb{C}) = \mathfrak{G},$$

embedded in the left upper corner. Then we have the same embeddings for the universal enveloping algebras and thus we can work inside $U = U(\mathfrak{G})$. Fix the standard set of generators $X_{ij}(p) = e_{ij} - e_{ji}$, $1 \leqslant i,j \leqslant p$ in $\mathfrak{O}(p,\mathbb{C})$ and let $X(p)$ be a $p\times p$ matrix, whose ij-component equals $X_{ij}(p)$. Set $C_{2i}(p) = \mathrm{Tr}\ X^{2i}(p)$ and $\hat{C}_s(2s) = \varepsilon^{i_1j_1,i_2j_2,\dots,i_sj_s}X_{i_1j_1}X_{i_2j_2}\dots X_{i_sj_s}$, where $\varepsilon^{i_1j_1,i_2j_2,\dots,i_sj_s}$ is the totally anti-symmetric Levi-Civita tensor ([BR, cf. 9.4.B]). Let Γ denote the subalgebra of U generated by the following elements: X_{21}, $C_{2s}(p)$ for $3 \leqslant p \leqslant n$ and $1 \leqslant i \leqslant k$ if $p = 2k$ or $p = 2k+1$, $\hat{C}_s(2s)$ for $2 \leqslant s \leqslant k$ if $n = 2k$ or $n = 2k+1$. Clearly, Γ is a commutative subalgebra in U and, moreover, any basis element $[\,l\,]$ in the Gelfand-Zetlin model of a simple finite-dimensional $\mathfrak{G}$-module is an eigenvector with respect to Γ. We will call Γ the Gelfand-Zetlin (GZ) subalgebra of U.

A $\mathfrak{G}$-module V will be called Gelfand-Zetlin (GZ) module provided V is a Γ-root module. If a GZ-module V is in fact a Γ-weight module and we want to emphasize this fact, we will call V a GZ-weight module. For example, as was noted in the previos section, any finite dimensional $\mathfrak{G}$ module is a GZ-weight module. In the next section we will construct a large family of simple GZ-weight modules. This also allows us to obtain some interesting results about the structure of GZ subalgebra itself.

5 Generic modules

There is a clear naive way to construct a large family of simple GZ-modules. This can be done without any preliminary preparation and, in some sence, the modules obtained will be opposite to finite dimensional ones.

Fix a tableau $[\,l\,]$ with complex entries $l_{i\,j}$, $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant k$ if $i = 2k-1$ or $i = 2k$ satisfying the following defining conditions:

- all $l_{i\,j}$ are not integers or half-integers for $i < n-1$;
- $l_{i\,j} \pm l_{i\,k}$ is not an integer for all $1 \leqslant i \leqslant n-2$ and all $j \neq k$.

Consider a set $B([\,l\,])$ consisting of all tableaux $[\,t\,]$ such that

- $t_{n-1\,j} = l_{n-1\,j}$ for all j;
- $t_{i\,j} - l_{i\,j}$ is an integer for all $1 \leqslant i \leqslant n-2$ and all j.

Let $V([\,l\,])$ be a vector space with a basis $B([\,l\,])$. For $[\,t\,] \in B([\,l\,])$ set

$$X_{2p+1\,2p}[\,t\,] = \sum_{j=1}^{p} A(t_{2p-1\,j})([\,t\,] + [\,\delta^{2p-1\,j}\,]) - \sum_{j=1}^{p} A(t_{2p-1\,j} - 1)([\,t\,] - [\,\delta^{2p-1\,j}\,]),$$
$$X_{2p+2\,2p+1}[\,t\,] = \sum_{j=1}^{p} B(t_{2p\,j})([\,t\,] + [\,\delta^{2p\,j}\,]) - \sum_{j=1}^{p} B(t_{2p\,j} - 1)([\,t\,] - [\,\delta^{2p\,j}\,]) + iC_{2p}[\,t\,],$$

where the functions A, B and C are taken from GZ formulae (see Section 3). This action can be easily extended to $V([\,l\,])$ by linearity.

Theorem 1. *The formulae above define on $V([\,l\,])$ the structure of a completely reducible $\mathfrak{G}$-module of finite length.*

Proof. Any relation in $U(\mathfrak{G})$ applied to an element $[\,t\,] \in B([\,l\,])$ can be expresed in terms of rational functions in t_{ij}. Since GZ formulae really define the structure of a $\mathfrak{G}$-module in a finite dimensional case one can find sufficiently many zeros for these functions to deduce that they are identically zero. Thus $V([\,l\,])$ is indeed a $\mathfrak{G}$-module.

Clearly, it follows directly from the definition and disscusion in the previous section that $V([\,l\,])$ is a GZ weight module. Now we want to show that under the definition of $B([\,l\,])$ the algebra Γ separates the elements in $B([\,l\,])$, i.e. for $[\,t\,] \neq [\,s\,] \in B([\,l\,])$ there is an element $u \in \Gamma$ such that the eigenvalues of u on $[\,t\,]$ and on $[\,s\,]$ do not coincide. To do this, we first recall that the rows of a tableau indexing a basis element in a finite-dimensional module can be interpreted as a linear combination of the components of the highest weight corresponding to the simple module containing this basis element. By the Harish-Chandra theorem ([D, Theorem 7.4.4]) the central character of a highest weight module can be computed in terms of its highest weight via the Harish-Chandra isomorphism using invariant polynomials with respect to the natural action of the Weyl group on a Cartan subalgebra. But it is easy to find the corresponding symmetry in GZ formulae. From the definition of the action it follows immediately that the eigenvalues of $Z(\mathfrak{O}(i+2,\mathbb{C}))$ can be computed as polynomials in $t_{i\,j}$ (here only j varies). Indeed, this eigenvalue depend only on $t_{k\,j}$ with $k \leqslant i$, but they should not change after application of the generators $X_{k+1\,k}$, $1 \leqslant k \leqslant i+1$. Since these generators change the values of all $t_{k\,j}$ with $k < i$ it follows that the desired polynomials depend only on $t_{i\,j}$. It is easy to see that for odd i, the GZ formulae for $X_{k+1\,k}$, $1 \leqslant k \leqslant i+1$ are invariant with respect to permutations of $t_{i\,j}$ and substitutions $t_{i\,j} \mapsto -t_{i\,j}+1$ and, for even i, the GZ formulae are invariant with respect to permutations of $t_{i\,j}$ and to the sign change on an even number of places. This defines an action of the corresponding Weyl group on the corresponding row of a tableau and we can state that the character of Γ can be computed using the polynomials in $t_{i\,j}$ wich are invariant with respect to the action of the product of Weyl groups (we will denote this algebra of invariants by $\hat{\Gamma}$). Since for the finite-dimensional case, the elements of each row can be interpreted as a linear combination of the components of the highest weight and by virtue of the mentioned Harish-Chandra theorem we conclude that, in fact, this provides a canonical isomorphism between Γ and $\hat{\Gamma}$. Now we note that under our choice of $[\,l\,]$ two different elements $[\,s\,]$ and $[\,t\,]$ in $B([\,l\,])$ can not be contained in one orbit of the action of the described product of Weyl groups on the set of all tableaux. The last two facts together imply that Γ separates the basis elements of $B([\,l\,])$.

The discussion above shows, in fact, that $V([\,l\,])$ is a GZ weight module with one-dimensional weight spaces. Consider a graph, whose verticies are the elements in $B([\,l\,])$ and where two verticies $[\,t\,]$ and $[\,s\,]$ are connected by an edge coming from $[\,t\,]$ to $[\,s\,]$ if there exist $1 \leqslant i \leqslant n-2$ such that $X_{i+1\,i}[\,t\,]$ contains a nonzero coefficient in $[\,s\,]$. It is a simple consiquence of the GZ formulae (which follows from their compatibility with respect to the standard involution of an algebra, see [BR, Chapter 10.1.B]), that any edge from $[\,t\,]$ to $[\,s\,]$ appears together with the reverse edge from $[\,s\,]$ to $[\,t\,]$, so we can view our graph as a non-oriented graph. The last observation together with the fact that of Γ separates the elements in $B([\,l\,])$ implies that $V([\,l\,])$ is completely reducible and its submodules correspond to the connected components of our graph. Further it follows from the GZ formulae that the zero value of A or B function on a tableau $[\,t\,]$ implies that $t_{i\,j} \pm t_{i-1\,k} = 0$ or $t_{i\,j} \pm t_{i-1\,k} - 1 = 0$ for some i, j, k. Since there are only finitely many of these conditions it follows that our graph has finitely many connected components and hence $V([\,l\,])$ is of finite length. This completes the proof. $\square$

For $[\,t\,] \in B([\,l\,])$ let $\varphi_{[\,t\,]}$ denote the corresponding charachter of Γ, i.e. $u[\,t\,] = \varphi_{[\,t\,]}(u)[\,t\,]$ for all $u \in \Gamma$.

Corollary 1. *For any* $[\,t\,] \in B([\,l\,])$ *there is a unique simple GZ-module* $V_{[\,t\,]}$ *such that* $(V_{[\,t\,]})_{\varphi_{[\,t\,]}} \neq 0$. *In fact,* $\dim(V_{[\,t\,]})_{\varphi_{[\,t\,]}} = 1$.

Proof. The existence and the property that $\dim(V_{[\,t\,]})_{\varphi_{[\,t\,]}} = 1$ follow directly from Theorem 1. The proof of uniqueness is analogous to that of [DFO2, Theorem 30]. □

Remark 1. *We have to note that in order to prove the uniqueness, as it was done in [DFO2, Theorem 30], one has to prove that* Γ *is a Harish-Chandra subalgebra in* $U(\mathfrak{G})$ *in the sense of [DFO2, Section 1.3] first. The proof of this fact repeats word by word one in [DFO2, Corollary 26].*

Corollary 2. *Any simple subquotient of* $V([\,l\,])$ *occures with multiplicity one.*

Proof. Follows immediately from Corollary 1 and the construction of $V([\,l\,])$. □

The following corollary is "well-known"; we include a proof having never seen one in the literature. For the case of $gl(n,\mathbb{C})$ the proof (which was also "well-known") is in [O, Corollary 2]; our proof is a slight variation.

Corollary 3. Γ *is a maximal commutative subalgebra in* $U(\mathfrak{G})$.

Proof. Let $u \in U(\mathfrak{G})$ be an element commuting with all elements in Γ. Our aim is to prove that $u \in \Gamma$. Consider a tableau $[\,l\,]$ whose entries $l_{i\,j}$, $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant k$ if $i = 2k-1$ or $i = 2k$ are independent variables. Let $B([\,l\,])$ be the same as above i.e. consisting of all tableaux $[\,t\,]$ such that

- $t_{n-1\,j} = l_{n-1\,j}$ for all j;
- $t_{i\,j} - l_{i\,j}$ is an integer for all $1 \leqslant i \leqslant n-2$ and all j

and let $V([\,l\,])$ be a vector space over the field of rational functions in all $l_{i\,j}$ with a basis $B([\,l\,])$. We can define, on the elements of $B([\,l\,])$, the action of the generators of $\mathfrak{G}$ via GZ formulae, viewing A, B and C as rational functions in $l_{i\,j}$. Clearly, in this way $V([\,l\,])$ becomes a $\mathfrak{G}$-module. Then the action of any element $u \in U(\mathfrak{G})$ on the module $V([\,l\,])$ now can be expresed as a set of rational functions. By virtue of the Harish-Chandra theorem ([D, Theorem 2.5.7]), which states that any non-zero $v \in U(\mathfrak{G})$ acts non-trivially on some finite-dimensional module, and by virtue of GZ fomulae for finite dimensional modules, we conclude that any non-zero $v \in U(\mathfrak{G})$ acts non-trivially on $V([\,l\,])$. Thus u acts non-trivially on $V([\,l\,])$. Moreover, since u commutes with Γ and Γ trivially separates the elements of $B([\,l\,])$, we conclude that u is diagonalizable in the basis $B([\,l\,])$. Thus $u[\,t\,] = f(t_{i\,j})[\,t\,]$ for some rational function f.

Now we have to analyse the GZ formulae once more. For a fixed p it is easy to see that the numerators of a functions of type A are invariant under the actions of the Weyl groups corresponding to the$2p$-th and the $2p-2$-th rows and the set of functions $A(m_{2p-1\,j})$, $A(m_{2p-1\,j}-1)$ where $1 \leqslant j \leqslant p$ is invariant under the action of the Weyl group corresponding to the $2p-1$-th row. Analogously, the numerators of B functions are invariant under the actions of the Weyl groups corresponding to $2p+1$-th and $2p-1$-th rows and the set of functions $B(m_{2p\,j})$, $B(m_{2p\,j}-1)$ where $1 \leqslant j \leqslant p$ is invariant under the action of the Weyl group corresponding to $2p$-th row. From this it follows immediately that f is invariant under the action of the product of the Weyl groups and thus can be expressed as a polynomial in polynomials corresponding to the generators of Γ. Since the annihilator of $V([\,l\,])$ is zero we conclude that u itself is a polynomial in generators of Γ and thus belongs to Γ. This completes the proof. $\square$

6 Simplifying the GZ formulae

The construction described above is a straightforward way to construct a family of simple modules using GZ formulae. But we have to note that in [DFO2] the authors use a simplified version of GZ formulae, which have been found for the $gl(n,\mathbb{C})$ case in [Z, Chapter 10]. The simplification was obtained by a diagonal base change in finite dimensional simple modules. Of course any simplification leads to the loss of some properties. The simplification used in [Z] leeds to the loss of the unitary condition on generating elements, which, in particular, made it possible to construct in a naive way some indecomposable but not simple modules. Regretfully, only the $gl(n,\mathbb{C})$ case was considered in [Z]. Now we are going to simplify the original GZ formulae for orthogonal Lie algebras. This is a straightforward technical procedure which we will present omitting rigorous calculations.

Lemma 1. *Fix a positive integer k and let $N(k)$ denote the set of all vectors $(i_1, i_2, \ldots, i_k)$ with integer (or half-integer) entries satisfying the condition $i_1 > i_2 > \cdots > i_{k-1} > |i_k|$. Set $\delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{ik})$, where δ_{ij} is the Kronecker symbol. Let f be the function from $N(k)$ into positive integers defined as follows:*

$$f(i_1, i_2, \ldots, i_k) = \prod_{a<b} (i_a + i_b)!.$$

Then for $l = (i_1, i_2, \ldots, i_k) \in N(k)$ holds

$$\frac{f(l)}{f(l+\delta_t)} = \left(\prod_{j \neq t} (i_t + i_j + 1) \right)^{-1}$$

and

$$\frac{f(l)}{f(l-\delta_t)} = \prod_{j \neq t} (i_t + i_j).$$

Proof. Direct verification. $\square$

Lemma 2. *Fix the positive integers k, m such that $k = m$ or $k = m+1$ and let $N(k,m)$ denote the set of double indexed vectors $(i_{t\,s})$, $t = 1, 2$, $1 \leqslant s \leqslant k$ (m) if $t = 2$ (1) with integer or half-integer entries satisfying the following conditions: $i_{2\,t} > i_{1\,t} \geqslant i_{2\,t+1}$ for all* *possible t, $i_{2\,m} > i_{1\,m} \geqslant |i_{2\,m+1}|$ if $k = m+1$, $i_{2\,m} > i_{1\,m} \geqslant -i_{2\,m}$ if $k = m$. Let δ_{ij} be the Kronecker delta. Let f be the function from $N(k,m)$ to the positive integers defined as follows:*

$$f(i_1, i_2, \ldots, i_k) = \prod_{a=1}^{k} \prod_{b=1}^{m} (i_{2\,a} + i_{1\,b})!.$$

Then for $l \in N(k,m)$ holds

$$\frac{f(l)}{f(l + \delta_{s\,t})} = \left(\prod_{j \geqslant 1} (i_{s\,t} + i_{3-s\,j} + 1) \right)^{-1}$$

and

$$\frac{f(l)}{f(l - \delta_{s\,t})} = \prod_{j \geqslant 1} (i_{s\,t} + i_{3-s\,j}).$$

Proof. Direct verification. □

Theorem 2. *Let V be a finite-dimensional $\mathfrak{G}$-module and $\mathcal{M}(m)$ be the corresponding GZ basis. There exists a diagonal base change such that in the new base $\tilde{\mathcal{M}}(m)$ the following holds:*

$$X_{2p+1\,2p}[\,l\,] = \sum_{j=1}^{p} A^{+}(l_{2p-1\,j})([\,l\,] + [\,\delta^{2p-1\,j}\,]) - \sum_{j=1}^{p} A^{-}(l_{2p-1\,j})([\,l\,] - [\,\delta^{2p-1\,j}\,]),$$

$$X_{2p+2\,2p+1}[\,l\,] = \sum_{j=1}^{p} B^{+}(l_{2p\,j})([\,l\,] + [\,\delta^{2p\,j}\,]) - \sum_{j=1}^{p} B^{-}(l_{2p\,j})([\,l\,] - [\,\delta^{2p\,j}\,]) + iC_{2p}[\,l\,],$$

where

$$A^{+}(l_{2p-1\,j}) = \frac{\prod_{r=1}^{p} (s_{2p\,r} - s_{2p-1\,j} - 1)(s_{2p\,r} + s_{2p-1\,j})}{2 \prod_{r \neq j} (s_{2p-1\,r}^2 - s_{2p-1\,j}^2)},$$

$$A^{-}(l_{2p-1\,j}) = \frac{\prod_{r=1}^{p-1} (s_{2p-2\,r} - s_{2p-1\,j})(s_{2p-2\,r} + s_{2p-1\,j} - 1)}{2 \prod_{r \neq j} (s_{2p-1\,r}^2 - s_{2p-1\,j}^2)},$$

$$B^{+}(l_{2p,j}) = \frac{\prod_{r=1}^{p} (s_{2p-1\,r} + s_{2p\,j}) \prod_{r=1}^{p+1} (s_{2p+1\,r} - s_{2p\,j})}{\left(s_{2p\,j}(4s_{2p\,j}^2 - 1)\right)^{1/2} \prod_{r \neq j} (s_{2p\,r} - s_{2p\,j})(s_{2p\,r} + s_{2p\,j} - 1)},$$

$$B^-(l_{2p\,j}) = \frac{\prod_{r=1}^{p+1}(s_{2p+1\,r} + s_{2p\,j} - 1)\prod_{r=1}^{p}(s_{2p-1\,r} - s_{2p\,j} + 1)}{\big((s_{2p\,j}-1)(4(s_{2p\,j}-1)^2-1)\big)^{1/2}\prod_{r\neq j}(s_{2p\,r} - s_{2p\,j})(s_{2p\,r} + s_{2p\,j} - 1)}.$$

Proof. The factors of the coefficients in the GZ formulae can be naturally divided into "plus" and "minus" classes. Clearly, using the superposition of the diagonal base changes, one has to prove the statement for these two parts separately. For the "minus" part by [Z, Theorem X.7] the result can be obtained via the base change from [Z, Theorem X.6]. Thus we have to consider only the "plus" factors. Using Lemma 1 and Lemma 2 one easily checks that multiplying any element $[\,l\,] \in \mathcal{M}(m)$ by the scalar

$$\prod_{a\geqslant 1}\left(\left(\prod_{i<j}(s_{2a-1\,i} + s_{2a-1\,j})!\right)^{-1/2}\cdot\left(\prod_{i<j}(s_{2a-1\,i} + s_{2a-1\,j}+1)!\right)^{1/2}\right)\times$$
$$\times\prod_{a\geqslant 1}\left(\left(\prod_{i<j}(s_{2a\,i} + s_{2a\,j}-1)!\right)^{-1/2}\cdot\left(\prod_{i<j}(s_{2a\,i} + s_{2a\,j})!\right)^{1/2}\right)\times$$
$$\times\prod_{a\geqslant 1}\left(\left(\prod_{i=1}^{a+1}\prod_{j=1}^{a}(s_{2a+1\,i} + s_{2a\,j}-1)!\right)^{1/2}\cdot\left(\prod_{i=1}^{a}\prod_{j=1}^{a}(s_{2a\,i} + s_{2a-1\,j}-1)!\right)^{-1/2}\right)$$

one obtains the statement of the theorem. □

Corollary 4. *The formulae from Theorem 2 define on $V([\,l\,])$ (see Section 5) the structure of a $\mathfrak{G}$-module of finite length.*

Proof. Analogous to that of Theorem 1. □

Remark 2. *It is easy to see that the module $V([\,l\,])$ constructed in Corollary 4 is not completely reducible is general.*

7 Modules with tableaux realization

Let V be a GZ module over $\mathfrak{G}$ and $\lambda \in \Gamma^*$ be its nontrivial root subspace. Then there is a canonical way to associate with λ an orbit of the product of the Weyl groups acting on tableaux. Really, the eigenvalues of the generators of Γ can be expressed in terms of the invariants polynomials under the above mentioned action on the entries of a tableau. This defines the desired correspondence. We will call a tableau $[\,l\,]$ of size $n-1$ good provided $l_{i\,j} \neq l_{i\,k}$ for all $j<k$ and all $1 \leqslant i \leqslant n-2$. Following [M1] a $\mathfrak{G}$-module V is said to have a tableaux realization if V is a GZ weight module, all GZ weight spaces of V are one dimensional and the tableaux corresponding to its weights are good. For example, it follows immediately from the GZ formulae that all simple finite dimensional modules have a tableaux realization.

We note that the polynomials expressing the eigenvalues of the generators of Γ do not depend on the choice of original or simplyfied GZ formulae, since their values on the tableaux basis for all finite dimensional simple modules coincide. Hence we can construct modules having a tableaux realization using the simplified GZ formulae. In particular, this allows us to construct non simple indecomposable modules. No we are going to construct two families of $\mathfrak{G}$-modules having a tableaux realization. Using the similarity with the corresponding construction from [M1] we call modules from the first family quasi Verma modules.

Let $n = 2k$ or $n = 2k + 1$. Fix a vector $a \in \mathbb{C}^k$ satisfying the folowing conditions:

- all entries of a are not integers or half-integers;
- $a_i \pm a_j \notin \mathbb{Z}$ for all $1 \leqslant i < j \leqslant k$ if n is odd;
- $a_i \pm a_j \notin \mathbb{Z}$ for all $1 \leqslant i < j < k$ if n is even;
- $a_i \notin \frac{1}{2}\mathbb{N}$ for all i if n is odd;
- $a_i \notin \frac{1}{2}\mathbb{N} \cup \{0\} \cup \{-1/2\}$ for all i if n is even.

Consider a tableau $[\,l\,] = [\,l\,](a)$ of size $n-1$ defined as follows: $l_{n-1\,j} = a_j$ for all j, $l_{2i\,j} = l_{2i+1\,j}$ for all i and j, $l_{2i-1\,j} = l_{2i\,j} - 1$ for all i and j. Clearly, under the above conditions $[\,l\,]$ is uniquely defined. Let $P([\,l\,])$ denote the set of all tableaux $[\,t\,]$ satisfying the following conditions:

1. $t_{n-1\,j} = l_{n-1\,j}$ for all j;
2. $l_{i\,j} - t_{i\,j} \in \mathbb{Z}_+$ for all $i < n-1$ and all j;
3. $t_{i\,j} - t_{i-1\,j} \geqslant 0$ for all j and odd i;
4. $t_{i\,j} - t_{i-1\,j} > 0$ for all j and even i.

Set $a^{\pm}_{i\,j}([s]) = A^{\pm}(l_{i\,j})$ and $b^{\pm}_{i\,j}([s]) = B^{\pm}(l_{i\,j})$ in the notions of Section 6.

Lemma 3. *Let $[\,t\,] \in P([\,l\,])$ and $a^{\pm}_{i\,j}([\,t\,]) \neq 0$ ($b^{\pm}_{i\,j}([\,t\,]) \neq 0$). Then $[\,t\,] \pm [\,\delta^{i\,j}\,] \in P([\,l\,])$.*

Proof. Let $a^{+}_{i\,j} \neq 0$ (the other cases are similar). Then $s = t_{i+1\,j} - t_{i\,j} - 1 \neq 0$ and thus $s > 0$ by the definition of $P([\,l\,])$. The last implies $[\,t\,] + [\,\delta^{i\,j}\,] \in P([\,l\,])$. $\square$

Corollary 5. *Let $V([\,l\,])$ be a vector space with $P([\,l\,])$ as a basis. Then the simplified GZ formulae define on $V([\,l\,])$ a structure of $\mathfrak{G}$-module.*

Proof. Lemma 3 guarantees that the action of the generators on the basis is well-defined. Now the proof is analogous to that of Theorem 1. $\square$

Theorem 3. *1. If n is odd then $V([\,l\,])$ is simple. If n is even then $V([\,l\,])$ is simple if and only if $a_i \pm a_k$ is not a positive integer for all $1 \leqslant i \leqslant k-1$.*

2. *If n is even and $a_i - a_k = x \in \mathbb{N}$ then $V([\,l\,])$ has as a unique non trivial submodule which is isomorphic to $V([\,s\,])$, where $[\,s\,] = [\,s\,](b)$, $b = a^{(ik)}$ (here (ik) is a transposition in the symmetric group acting on $\mathbb{C}^k$ in a natural way).*

3. *If n is even and $a_i + a_k = x \in \mathbb{N}$ then $V([\,l\,])$ has as a unique non trivial submodule M such that $V([\,l\,])/M$ is isomorphic to $V([\,s\,])$, where $[\,s\,] = [\,s\,](b)$, $b_j = a_j^{(ik)}$ for $i \neq j$ and $b_i = -a_k - 1$.*

Remark 3. *The connection between a and b can be easily expressed in terms of the Weyl group acting on $\mathbb{C}^k$ as in the proof of Theorem 1.*

Proof. To prove the first statement we need only note that under our conditions for $[\,t\,], [\,t\,] \pm [\,\delta^{ij}\,] \in P([\,l\,])$ one has $a_{ij}^{\pm} \neq 0$ if i is odd and $b_{ij}^{\pm} \neq 0$ if i is even.

The proofs of the second and the third statements are similar, so we will prove only the second one. Under the conditions of the second statement one has $P([\,s\,]) \subset P([\,l\,])$ up to permutations of elements in the rows of tableaux. From this we easily obtain that $V([\,l\,])$ has a submodule isomorphic to $V([\,s\,])$. By the same arguments as for for the first statement it is easy to see that both this copy of $V([\,s\,])$ and the corresponding quotient are simple. This completes the proof. □

The next family of modules having a tableaux realization can be viewed as an analogue of generalized Verma modules ([M1, MO]).

Fix a tableau $[\,l\,]$ of size $n-1$ satisfying the following conditions:

- all entries of $[\,t\,]$ are not integers or half-integers;
- $l_{i\,1} = l_{i-1\,1}$ for all odd i;
- $l_{i\,1} = l_{i-1\,1} + 1$ for all even i;
- $l_{s\,i} \pm l_{s,j} \notin \mathbb{Z}$ for all i, j and all $s < n-1$;
- $l_{s\,i} \pm l_{s-1\,j} \notin \mathbb{Z}$ for all i, all $j > 1$ and all s;
- $l_{i\,j}$ is not integers or half-integers.

Let $P([\,l\,])$ denote the set of all tableaux $[\,t\,]$ satisfying the following conditions:

1. $t_{n-1\,j} = l_{n-1\,j}$ for all j;
2. $l_{i\,j} - t_{i\,j} \in \mathbb{Z}$ for all $i < n-1$ and all $j > 1$;
3. $l_{i\,1} - t_{i\,1} \in \mathbb{Z}_+$ for all $i < n-1$;
4. $t_{i\,1} - t_{i-1\,1} \geqslant 0$ for all odd i;
5. $t_{i\,1} - t_{i-1\,1} > 0$ for all even i.

Lemma 4. *Let $V([\,l\,])$ be a vector space with $P([\,l\,])$ as a basis. Then the simplified GZ formulae define on $V([\,l\,])$ a structure of a $\mathfrak{G}$-module.*

Proof. Analogous to that of Corollary 5. $\square$

Theorem 4. *1. The module $V([\,l\,])$ is simple if and only if $l_{n-1\,1} - l_{n-1\,j} \notin \mathbb{N}$ and $l_{n-1\,1} + l_{n-1\,j} - 1 \notin \mathbb{N}$ for all $j > 1$.*

2. If $l_{n-1\,1} - l_{n-1\,j} \in \mathbb{N}$ for some j then $V([\,s\,]) \subset V([\,l\,])$ for $[\,s\,]$ defined as follows: $s_{i\,t} = l_{i\,t}$, for $t > 1$ and $i < n-1$; $s_{i\,1} = (l_{i\,1} - l_{n-1\,1}) + l_{n-1\,j}$; $s_{n-1\,j} = l_{n-1\,1}$; $s_{n-1\,t} = l_{n-1\,t}$, $t \neq 1, j$.

3. If n is odd and $l_{n-1\,1} + l_{n-1\,j} - 1 = z \in \mathbb{N}$ for some j then $V([\,s\,]) \subset V([\,l\,])$ for $[\,s\,]$ defined as follows: $s_{i\,t} = l_{i\,t}$, for $t > 1$ and $i < n-1$; $s_{n-1\,1} = l_{n-1\,j}$; $s_{i\,1} = l_{i\,1} - z$, for $i < n-1$; $s_{n-1\,j} = l_{n-1\,1}$; $s_{n-1\,t} = l_{n-1\,t}$, $t \neq 1, j$.

4. If n is even and $l_{n-1\,1} + l_{n-1\,j} - 1 = z \in \mathbb{N}$ for some j then $V([\,l\,])$ has a submodule N such that $V([\,l\,])/N \simeq V([\,s\,])$ for $[\,s\,]$ defined as follows: $s_{i\,t} = l_{i\,t}$, for $t > 1$ and $i < n-1$; $s_{n-1\,1} = l_{n-1\,j}$; $s_{i\,1} = l_{i\,1} - z$, for $i < n-1$; $s_{n-1\,j} = l_{n-1\,1}$; $s_{n-1\,t} = l_{n-1\,t}$, $t \neq 1, j$.

5. Any simple subquotient of $V([\,l\,])$ occures with multiplicity one.

Proof. The last statement follows from an analogue of Corollary 2 for our situation. The rest follows from simplified GZ formulae. $\square$

Remark 4. *We have to note again that the connection between parameters of submodules (quotients) of $V([\,l\,])$ and $[\,l\,]$ can be easily expressed in terms of the natural Weyl group action.*

8 Corresponding orthogonal algebras

Fix $n \in \mathbb{N}$ and $r = (r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ and set $k = |r| = \sum_{i=1}^{n} r_i$. For a field $\mathbb{F}$ consider a vector space $\mathcal{L} = \mathcal{L}(\mathbb{F}, r)$ of dimension k. We will call the elements of $\mathcal{L}$ tableaux and will consider them as doubly indexed families

$$[\,l\,] = \{l_{ij} \,|\, i = 1, \ldots, n; j = 1, \ldots, r_i\}.$$

The element r will be called the signature of $[\,l\,]$. By the rank of $[\,l\,]$ we will mean $\mathrm{rank}([\,l\,]) = n-1$. We will denote by $\delta^{i\,j} = [\,\delta^{ij}\,]$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r_i$, the Kronecker tableau, i.e. $\delta^{ij}_{ij} = 1$ and $\delta^{ij}_{pq} = 0$ for $p \neq i$ or $q \neq j$. Denote by $\mathcal{L}_0$ the subset of $\mathcal{L}$ consisting of all $[\,l\,]$ satisfying the following conditions:

1. $l_{nj} = 0$, $j = 1, \ldots, r_n$;

2. $l_{ij} \in \mathbb{Z}$, $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant r_i$.

Consider a field Λ of rational functions in k variables λ_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r_i$. Let $[\mathfrak{l}] \in \mathcal{L}(\Lambda, r)$ be the tableau defined by $\mathfrak{l}_{ij} = \lambda_{ij}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r_i$.

Consider a vector space M over Λ with the base $v_{[\mathfrak{i}]}$, $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$ (here $[\mathfrak{i}]$ is a formal index and thus M is infinite-dimensional over Λ). For $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$, $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant r_i$ define Λ-linear transformations X_i, $1 \leqslant i \leqslant n$ as follows:

$$X_{2p+1}[\mathfrak{i}] = \sum_{j=1}^{p} a^{+}_{2p-1\,j}([\mathfrak{i}])([\mathfrak{i}] + [\delta^{2p-1\,j}]) - \sum_{j=1}^{p} a^{-}_{2p-1\,j}([\mathfrak{i}])([\mathfrak{i}] - [\delta^{2p-1\,j}]),$$
$$X_{2p+2}[\mathfrak{i}] = \sum_{j=1}^{p} b^{+}_{2p\,j}([\mathfrak{i}])([\mathfrak{i}] + [\delta^{2p\,j}]) - \sum_{j=1}^{p} b^{-}_{2p\,j}([\mathfrak{i}])([\mathfrak{i}] - [\delta^{2p\,j}]) + ic_{2p}([\mathfrak{i}])[\mathfrak{i}],$$

where

$$a^{+}_{2p-1\,j}([\mathfrak{i}]) = \frac{\prod_{r=1}^{p}(\mathfrak{i}_{2p\,r} - \mathfrak{i}_{2p-1\,j} - 1)(\mathfrak{i}_{2p\,r} + \mathfrak{i}_{2p-1\,j})}{2\prod_{r\neq j}(\mathfrak{i}^2_{2p-1\,r} - \mathfrak{i}^2_{2p-1\,j})},$$

$$a^{-}_{2p-1\,j}([\mathfrak{i}]) = \frac{\prod_{r=1}^{p-1}(\mathfrak{i}_{2p-2\,r} - \mathfrak{i}_{2p-1\,j})(\mathfrak{i}_{2p-2\,r} + \mathfrak{i}_{2p-1\,j} - 1)}{2\prod_{r\neq j}(\mathfrak{i}^2_{2p-1\,r} - \mathfrak{i}^2_{2p-1\,j})},$$

$$b^{+}_{2p,j}([\mathfrak{i}]) = \frac{\prod_{r=1}^{p}(\mathfrak{i}_{2p-1\,r} + \mathfrak{i}_{2p\,j}) \prod_{r=1}^{p+1}(\mathfrak{i}_{2p+1\,r} - \mathfrak{i}_{2p\,j})}{\left(\mathfrak{i}_{2p\,j}(4\mathfrak{i}^2_{2p\,j} - 1)\right)^{1/2} \prod_{r\neq j}(\mathfrak{i}_{2p\,r} - \mathfrak{i}_{2p\,j})(\mathfrak{i}_{2p\,r} + \mathfrak{i}_{2p\,j} - 1)},$$

$$b^{-}_{2p\,j}([\mathfrak{i}]) = \frac{\prod_{r=1}^{p+1}(\mathfrak{i}_{2p+1\,r} + \mathfrak{i}_{2p\,j} - 1) \prod_{r=1}^{p}(\mathfrak{i}_{2p-1\,r} - \mathfrak{i}_{2p\,j} + 1)}{\left((\mathfrak{i}_{2p\,j} - 1)(4(\mathfrak{i}_{2p\,j} - 1)^2 - 1)\right)^{1/2} \prod_{r\neq j}(\mathfrak{i}_{2p\,r} - \mathfrak{i}_{2p\,j})(\mathfrak{i}_{2p\,r} + \mathfrak{i}_{2p\,j} - 1)}.$$

$$c_{2p}([\mathfrak{i}]) = \prod_{r=1}^{p} \mathfrak{i}_{2p-1\,r} \prod_{r=1}^{p+1} \mathfrak{i}_{2p+1\,r} \left(\prod_{r=1}^{p} \mathfrak{i}_{2p\,r}(\mathfrak{i}_{2p,r} - 1) \right)^{-1}.$$

For a fixed $1 \leqslant i \leqslant n$ let γ_{ij}, $1 \leqslant j \leqslant r_i$ be a set of independent generators of the algebra of invariant (with respect to the action in the proof of Theorem 1) polynomials in $\lambda_{i1}, \ldots, \lambda_{ir_i}$. Define Λ-linear diagonal transformations $d_{ij} : M \to M$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r_i$ by

$$d_{ij} v_{[\mathfrak{i}]} = \gamma_{ij}(\mathfrak{i}_{i1}, \ldots, \mathfrak{i}_{ir_i}) v_{[\mathfrak{i}]}. \tag{1}$$

Consider an associative operator algebra $\mathcal{U}$ (over $\mathbb{F}$), generated by elements X_i, $i = 1, \ldots, n-1$ and d_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r_i$. We will call it the Orthogonal Gelfand-Zetlin algebra (OGZ algebra) of signature r associated with orthogonal Lie algebras (to differ it from the orthogonal GZ algebra associated with the unimodular Lie algebra in [M2]).

Let $n = 2k-1$ or $n = 2k$.

Theorem 5. *The OGZ algebra of signature* $(1, 1, 2, 2, 3, 3, \ldots, k)$ *(n entries) is isomorphic to* $U(\mathfrak{O}, n+1)$.

Proof. Using Theorem 1 one can prove this fact in the same way as [M2, Theorem 1]. $\square$

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Mechanics and Mathematics Department, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 252033, Kyiv, Ukraine, e-mail: `mazorchu@uni-alg.kiev.ua`

Address until 31 December 1999:

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501, Bielefeld FRG, e-mail: `mazor@mathematik.uni-bielefeld.de`