Koszul Calculus

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Abstract

We present a calculus which is well-adapted to quadratic algebras. This calculus is defined in Koszul cohomology (homology) by cup products (cap products). Koszul homology and cohomology are interpreted in terms of derived categories. If the algebra is not Koszul, Koszul (co)homology provides different information than Hochschild (co)homology. Koszul homology is related to de Rham cohomology. If the algebra is Koszul, Koszul cohomology is related to Calabi-Yau property. The A-$A^!$ duality of Koszul cohomology is proved for any quadratic algebra A, and this duality is extended to Koszul homology. The calculus is made explicit on a non-Koszul example.

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1 Introduction

Quadratic algebras are associative algebras defined by homogeneous quadratic relations. Since their definition by Priddy [15], Koszul algebras form a widely studied class of quadratic algebras [14]. In his monograph [13], Manin brings out a general approach of quadratic algebras (not necessarily Koszul), including the fundamental observation that quadratic algebras form a category which should be a relevant framework for a noncommutative analogue of projective algebraic geometry. According this general approach, non-Koszul quadratic algebras deserve certainly more attention.

The goal of this article is to introduce new general tools for studying quadratic algebras. These tools consist in a (co)homology, called Koszul (co)homology, together with products, called Koszul cup and cap products. They are organized in a calculus, called Koszul calculus. If two quadratic algebras are isomorphic in the sense of the Manin category, their Koszul calculus are isomorphic. If the quadratic algebra is Koszul, the Koszul calculus is isomorphic to Hochschild (co)homology endowed with usual cup and cap products – called Hochschild calculus. In this introduction, we would like to describe the main features of the Koszul calculus and how they are involved in the course of the article.

The Koszul homology $HK_\bullet(A, M)$ of a quadratic algebra A with coefficients in a bimodule M is defined by applying the functor $M \otimes_{A^e} -$ to the Koszul complex of A, analogously

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for the Koszul cohomology $HK^\bullet(A, M)$, see Section 2 for details. If A is Koszul, the Koszul complex is a projective resolution of A, so that $HK_\bullet(A, M)$ (resp. $HK^\bullet(A, M)$) is isomorphic to the Hochschild homology $HH_\bullet(A, M)$ (resp. Hochschild cohomology $HH^\bullet(A, M)$). Restricting the Koszul calculus to $M = A$, we present in Section 9 a non-Koszul quadratic algebra A which is such that $HK_\bullet(A) \ncong HH_\bullet(A)$ and $HK^\bullet(A) \ncong HH^\bullet(A)$. So $HK_\bullet(A)$ and $HK^\bullet(A)$ provide further invariants associated to the Manin category, besides those provided by Hochschild (co)homology. In Section 2, we prove that Koszul homology (cohomology) is isomorphic to a Hochschild hyperhomology (hypercohomology), showing that this new homology (cohomology) becomes natural in terms of derived categories.

In Section 3 and Section 4, the Koszul cup and cap products are introduced by copying the definition of usual cup and cap products on Koszul cochains and chains respectively. It turns out that these definitions make sense and are compatible with the Koszul differentials, providing differential graded algebras and differential graded bimodules. These products pass to (co)homology. The so-obtained Koszul calculus is isomorphic to the Hochschild calculus if A is Koszul.

For any unital associative algebra A, the Hochschild cohomology of A with coefficients in A itself, endowed with the cup product, has a richer structure provided by Gerstenhaber product $\circ$, called Gerstenhaber calculus [5]. When $\circ$ is replaced in the structure by the graded bracket associated to $\circ$, that is, the Gerstenhaber bracket $[-, -]$, the calculus becomes a Gerstenhaber algebra [5]. Next, the Gerstenhaber algebra and the Hochschild homology of A, endowed with cap products, are organized in a Tamarkin-Tsygan calculus [17], see also [10]. In the Tamarkin-Tsygan calculus, the Hochschild differential b is defined from the multiplication μ of A and the Gerstenhaber bracket by

$$b(f) = [\mu, f] \tag{1.1}$$

for any Hochschild cochain f.

The obstruction to see the Koszul calculus as a Tamarkin-Tsygan calculus is the following: the Gerstenhaber product $\circ$ *does not make sense on Koszul cochains*. However, this negative answer can be bypassed by the fundamental formula of the Koszul calculus

$$b_K(f) = -[e_A, f]_{\underset{K}{\smile}} \tag{1.2}$$

where b_K is the Koszul differential, e_A is the fundamental 1-cocycle and f is any Koszul cochain; see Subsection 3.3.

In formula (1.2), $[-, -]_{\underset{K}{\smile}}$ is the graded bracket associated to the Koszul cup product $\underset{K}{\smile}$. In other words, *the Koszul differential may be defined from the Koszul cup product.* Therefore, the Koszul calculus is simpler than the Tamarkin-Tsygan calculus, since no additional product such as $\circ$ is required to express the differential by means of a graded bracket. The Koszul calculus is more flexible since the formula (1.2) is valid for any bimodule M, while the definitions of Gerstenhaber product and bracket are meaningless when considering other bimodules of coefficients [6]; it is also more symmetric since there is an analogue of (1.2) in homology, where the Koszul cup product is replaced by the Koszul cap product. See Section 3 and Section 4.

In the Tamarkin-Tsygan calculus, Connes differential B defined on Hochschild homology is an essential ingredient. Although B does not send Koszul chains to Koszul chains, the question to find such a differential at the level of Koszul homology classes is solved in Subsection 6.3 for some very particular cases, the general case being open. The applications of such a differential to non Koszul quadratic algebras would deserve more attention. In fact, the Rinehart-Goodwillie identity – which holds in the Tamarkin-Tsygan calculus – should be conjecturally replaced by a more sophisticated identity.

In Subsection 6.1, we recall the Rinehart-Goodwillie identity of the Tamarkin-Tsygan calculus, and we emphasize its consequences on Hochschild homology of graded algebras. In particular, we obtain a Poincaré Lemma for graded algebras in characteristic zero in Theorem 6.3. There are two ways to read this Poincaré Lemma, according to the differential one chooses. It turns out that the choice of the Rinehart-Goodwillie operator is better for us, because this differential has an immediate counterpart in Koszul calculus, unlike Connes differential B. In fact, the Koszul Rinehart-Goodwillie operator is given by the operator $e_A \underset{K}{\frown} -$, that is, the left Koszul cap product by the fundamental 1-cocycle e_A. This operator induces a homology theory on $HK_\bullet(A, M)$ called the higher Koszul homology of A with coefficients in the bimodule M. See Section 5 for details. See also Subsections 3.5, 3.6, 3.7 where the higher Koszul cohomology is defined analogously.

In [10], the second author defines the Tamarkin-Tsygan calculi with duality. Specializing this general definition to the Hochschild situation, the Tamarkin-Tsygan calculus of an associative algebra A is said to be "with duality" if there is a class c in a space $HH_n(A)$ for some $n \in \mathbb{N}$, called the fundamental class, such that the k-linear map

$$- \frown c : HH^p(A) \longrightarrow HH_{n-p}(A)$$

is an isomorphism for any p. If the algebra A is n-Calabi-Yau [7], such a calculus exists, and for any bimodule M,

$$- \frown c : HH^p(A, M) \longrightarrow HH_{n-p}(A, M)$$

is an isomorphism coinciding with Van den Bergh duality [19], see also [10] for details.

Assume now that A is a quadratic algebra over a field of characteristic zero, which is Koszul. The Poincaré Lemma for graded algebras holds for A, so that it carries over in Koszul calculus, and A becomes an acyclic object of higher Koszul homology (Theorem 6.4). If furthermore A is n-Calabi-Yau, one applies the Poincaré Lemma and the isomorphisms $- \frown c$ to obtain that the higher Koszul cohomology of A vanishes in any homological degree p, except for $p = n$ where it is one-dimensional (Corollary 7.2). See Subsection 6.2 and Section 7 for details on these two results and for a discussion on their conjectural converses (Conjecture 6.5 and Conjecture 7.3).

The duality between a Koszul algebra A and its Koszul dual $A^!$, called Koszul duality, is an important tool for the study of Koszul algebras. In [7], Ginzburg mentions that the Hochschild cohomology algebras of A and $A^!$ are isomorphic if the quadratic algebra A is Koszul. In Section 8, we obtain such a Koszul duality theorem linking the Koszul cohomology algebras of A and $A^!$ *for any quadratic algebra A, Koszul or not* (Theorem 8.3). The proof of our result lies on a Koszul duality at the level of Koszul cochains and uses standard facts on duality of finite dimensional vector spaces.

Our result reveals two phenomena. Firstly, the homological weight p is changed by the duality into the coefficient weight m. Secondly, the exchange $p \leftrightarrow m$ implies that we have to replace one of both cohomologies by a modified version of Koszul cohomology and Koszul cup product, see Subsection 8.1. The statement of Theorem 8.3 is the following.

Theorem 1.1 *Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ be a quadratic algebra. Let $A^! = T(V^*)/(R^\perp)$ be the Koszul dual of A. There is an isomorphism of $\mathbb{N} \times \mathbb{N}$-graded unital associative algebras*

$$(HK^\bullet(A), \underset{K}{\smile}) \cong (\tilde{HK}^\bullet(A^!), \underset{K}{\tilde{\smile}}). \tag{1.3}$$

In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK^p(A)_m \cong \tilde{HK}^m(A^!)_p. \tag{1.4}$$

This theorem will be illustrated by an example, with direct computations. Both phenomena are shown to be essential in this example. From Theorem 1.1, a higher version is easily deduced (Theorem 8.7). Theorem 1.1 is extended to Koszul homology (Theorem 8.8) as follows (see Subsection 8.3 for details).

Theorem 1.2 *Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ be a quadratic algebra. Let $A^! = T(V^*)/(R^\perp)$ be the Koszul dual of A. There is an isomorphism*

$$HK_\bullet(A) \cong \tilde{HK}^\bullet(A^!, A^{!*}), \tag{1.5}$$

from the $(HK^\bullet(A), \underset{K}{\smile})$-bimodule $HK_\bullet(A)$ for the actions $\underset{K}{\frown}$ to the $(\tilde{HK}^\bullet(A^!), \underset{K}{\tilde{\smile}})$-bimodule $\tilde{HK}^\bullet(A^!, A^{!*})$ for the actions $\underset{K}{\tilde{\smile}}$. In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK_p(A)_m \cong \tilde{HK}^m(A^!, A^{!*})_p. \tag{1.6}$$

The statement of this theorem can be expressed in a categorical way: the duality functor $A \mapsto A^!$ defines a natural isomorphism from the functor $A \mapsto HK_\bullet(A)$ to the functor $A \mapsto \tilde{HK}^\bullet(A^!, A^{!*})$, the three functors being defined on the Manin category. From Theorem 1.2, a higher version is deduced easily (Theorem 8.11), and the same example is used again for direct verifications.

In Section 9, the Koszul calculus is made explicit on an example of non Koszul quadratic algebra A. Moreover, we prove that Koszul homology (cohomology) of A is not isomorphic to Hochschild homology (cohomology) of A. For computing the Hochschild homology and cohomology of A in degrees 2 and 3, we use a projective bimodule resolution of A due to the third author and Chouhy [3]. See Subsection 9.4 for details.

2 Koszul homology and cohomology

Throughout the article, k will denote a field and V a k-vector space. The tensor algebra of V is denoted by $T(V)$. The symbol $\otimes$ will always mean $\otimes_k$. The k-algebra $T(V) = \bigoplus_{m \geq 0} V^{\otimes m}$ is graded by the *weight* m. This grading is inherited by the associative k-algebra $A = T(V)/(R)$ for any subspace R of $V \otimes V$ and it is called *weight grading*. The homogeneous component of weight m of A is denoted by A_m. The graded algebra A is called a *quadratic algebra*.

2.1 Recalling the bimodule complex $K(A)$

Let $A = T(V)/(R)$ be a quadratic algebra. For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by the following equality

$$W_p = \bigcap_{i+2+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

Remark that $W_0 = k$, $W_1 = V$ and $W_2 = R$. It is convenient to use a specific notation for the elements of W_p. This notation will play an important role in Koszul calculus, similar to the role played by Sweedler's notation for Hopf algebras. The notation is the following: an arbitrary element of W_p will be denoted by a product $x_1 \dots x_p$, where $x_1, \dots, x_p$ are in V. This notation should be thought of as a sum of such products. Moreover, regarding W_p as a subspace of $V^{\otimes q} \otimes W_r \otimes V^{\otimes s}$ where $q + r + s = p$, the element $x_1 \dots x_p$ viewed in $V^{\otimes q} \otimes W_r \otimes V^{\otimes s}$ will be denoted by the *same* notation, meaning that the product $x_{q+1} \dots x_{q+r}$ is thought of as an element of W_r and the other x's are thought of as arbitrary in V.

Clearly, V is the component of weight 1 of A, so that $V^{\otimes p}$ is a subspace of $A^{\otimes p}$. As defined by Priddy [15], the *Koszul complex* $K(A)$ of the quadratic algebra A is a weight graded bimodule subcomplex of the bar resolution $B(A)$ of A (see also [14, 12]). Precisely, $K(A)$ is such that $K(A)_p$ is the subspace $A \otimes W_p \otimes A$ of $A \otimes A^{\otimes p} \otimes A$. It is easy to see that $K(A)$ coincides with the following complex

$$\cdots \xrightarrow{d} K_p \xrightarrow{d} K_{p-1} \xrightarrow{d} \cdots \xrightarrow{d} K_1 \xrightarrow{d} K_0 \longrightarrow 0\,, \tag{2.1}$$

where $K_p = A \otimes W_p \otimes A$, and the differential d is defined on K_p as follows

$$d(a \otimes x_1 \dots x_p \otimes a') = ax_1 \otimes x_2 \dots x_p \otimes a' + (-1)^p a \otimes x_1 \dots x_{p-1} \otimes x_p a', \tag{2.2}$$

for a, a' in A and $x_1 \dots x_p$ in W_p. The homology of $K(A)$ is equal to A in degree 0, and to 0 in degree 1.

Definition 2.1 *A quadratic algebra A is said to be Koszul if the homology of $K(A)$ is* 0 *in any positive degree.*

The multiplication $\mu : K_0 = A \otimes A \to A$ defines a morphism from the complex $K(A)$ to the complex A concentrated in degree 0. Whereas $\mu : B(A) \to A$ is always a quasi-isomorphism, A is Koszul if and only if $\mu : K(A) \to A$ is a quasi-isomorphism. So, if the quadratic algebra A is Koszul, the bimodule free resolution $K(A)$ may be used to compute Hochschild homology and cohomology of A instead of $B(A)$. In the next two subsections we apply the same Tor and Ext functors to $K(A)$ even if A is not Koszul. The goal of this article is to show that the so-obtained Koszul (co)homology is of interest for quadratic algebras, providing invariants that are not obtained using Hochschild (co)homology.

2.2 The Koszul homology $HK_\bullet(A, M)$

Let M be an A-bimodule. As usual, M can be considered as a left or right A^e-module, where $A^e = A \otimes A^{op}$. Applying the functor $M \otimes_{A^e} -$ to $K(A)$, we obtain the chain complex

$(M \otimes W_\bullet,\, b_K)$, where $W_\bullet = \bigoplus_{p\geq 0} W_p$. The elements of $M \otimes W_p$ are called the *Koszul p-chains with coefficients in M*. From Equation (2.2), we see that the differential $b_K = M \otimes_{A^e} d$ is given on $M \otimes W_p$ by the formula

$$b_K(m \otimes x_1 \dots x_p) = m.x_1 \otimes x_2 \dots x_p + (-1)^p x_p.m \otimes x_1 \dots x_{p-1}, \tag{2.3}$$

for any m in M and $x_1 \dots x_p$ in W_p. In particular, if $p = 1$, one has $b_K(m \otimes x) = m.x - x.m$ for any x in V.

Definition 2.2 *Let $A = T(V)/(R)$ be a quadratic algebra and M be an A-bimodule. The homology of the complex $(M \otimes W_\bullet,\, b_K)$ is called the Koszul homology of A with coefficients in M, and is denoted by $HK_\bullet(A, M)$. We set $HK_\bullet(A) = HK_\bullet(A, A)$.*

Denote by $\chi : K(A) \to B(A)$ the bimodule morphism of complexes defined by the inclusion of $K(A)$ into $B(A)$. The functor $M \otimes_{A^e} \chi$ defines a morphism of complexes, denoted by $\tilde{\chi}$, from $(M \otimes W_\bullet, b_K)$ to $(M \otimes A^{\otimes \bullet}, b)$, where b is the Hochschild differential. For each degree p, $\tilde{\chi}_p$ coincides with the natural injection of $M \otimes W_p$ into $M \otimes A^{\otimes p}$. Since the complex

$$A \otimes R \otimes A \xrightarrow{d} A \otimes V \otimes A \xrightarrow{d} A \otimes A \xrightarrow{\mu} A \to 0 \tag{2.4}$$

is exact, the k-linear map $H(\tilde{\chi})_p : HK_p(A, M) \to HH_p(A, M)$ is an isomorphism for $p = 0$ and $p = 1$. The next proposition is clear.

Proposition 2.3 *Let $A = T(V)/(R)$ be a quadratic algebra which is Koszul. For any A-bimodule M and any $p \geq 0$, $H(\tilde{\chi})_p : HK_p(A, M) \to HH_p(A, M)$ is an isomorphism.*

Note that there is a converse – see also Subsection 2.4 below. Actually, in order to conclude that A is Koszul, it is sufficient to assume that the isomorphisms hold for $M = A^e$ viewed as right A^e-module, since the functor $M \otimes_{A^e} -$ coincides with the identity functor in this case. In Section 9, a non-Koszul example is presented for which $H(\tilde{\chi})_3$ is not surjective when $M = A$.

It is well-known that quadratic k-algebras and their morphisms form a category, see Manin [13]. In this category, a morphism θ from $A = T(V)/(R)$ to $A' = T(V')/(R')$ is defined by a linear map $\theta : V \to V'$ such that $\theta^{\otimes 2}(R) \subseteq R'$. For each p, $\theta^{\otimes p}$ maps W_p into W'_p, with obvious notation. Moreover, the maps $a \otimes x_1 \dots x_p \mapsto \theta(a) \otimes \theta(x_1) \dots \theta(x_p)$ define a morphism of complexes from $(A \otimes W_\bullet, b_K)$ to $(A' \otimes W'_\bullet, b_K)$. So we obtain a covariant functor $A \mapsto HK_\bullet(A)$. More generally, the same functorial properties of Hochschild homology (see 1.1.4 in Loday's book [11]) stand for Koszul homology.

Let us now show that Koszul homology is isomorphic to a Hochschild hyperhomology, namely

$$HK_\bullet(A, M) \cong \mathbb{HH}_\bullet(A, M \otimes_A K(A)). \tag{2.5}$$

Denote by $\mathcal{A}$ (resp. $\mathcal{E}$) the abelian category of A-bimodules (resp. k-vector spaces). For any A-bimodule M, the left derived functor $M \overset{L}{\otimes}_{A^e} -$ is defined from the triangulated category $\mathcal{D}^-(\mathcal{A})$ to the triangulated category $\mathcal{D}^-(\mathcal{E})$, so that we have

$$HK_p(A, M) \cong H_p(M \overset{L}{\otimes}_{A^e} K(A)). \tag{2.6}$$

The following lemma is standard [19].

Lemma 2.4 *Let M and N be A-bimodules. The k-linear map*

$$\zeta : M \otimes_{A^e} N \to (M \otimes_A N) \otimes_{A^e} A$$

defined by $\zeta(x \otimes_{A^e} y) = (x \otimes_A y) \otimes_{A^e} 1$ is an isomorphism. Moreover, for any complex of A-bimodules C, the map $\zeta : M \otimes_{A^e} C \to (M \otimes_A C) \otimes_{A^e} A$ is an isomorphism of complexes.

In other words, the functor $F : C \mapsto M \otimes_{A^e} C$ coincides with the composite $H \circ G$ where $G : C \mapsto M \otimes_A C$ and $H : C' \mapsto C' \otimes_{A^e} A$. So their left derived functors satisfy $LF \cong L(H) \circ L(G)$, in particular for $C = K(A)$,

$$M \overset{L}{\otimes}_{A^e} K(A) \cong (M \overset{L}{\otimes}_A K(A)) \overset{L}{\otimes}_{A^e} A. \tag{2.7}$$

Passing to homology and using the definition of hypertor [20], we obtain

$$HK_p(A, M) \cong \mathbb{T}or_p^{A^e}(M \otimes_A K(A), A), \tag{2.8}$$

which proves the isomorphism (2.5). If A is Koszul, we recover usual Tor and Proposition 2.3.

2.3 The Koszul cohomology $HK^\bullet(A, M)$

From now on, Hom_k will always be denoted just by Hom. Applying the functor $Hom_{A^e}(-, M)$ to the complex $K(A)$, we obtain the cochain complex $(Hom(W_\bullet, M),\, b_K)$, where

$$Hom(W_\bullet, M) = \bigoplus_{p \geq 0} Hom(W_p, M).$$

The elements of $Hom(W_p, M)$ are called the *Koszul p-cochains with coefficients in M*. If $f : W_p \to M$ is a Koszul p-cochain, its differential $b_K(f)$ is defined with the usual sign convention, that is

$$b_K(f) = -(-1)^p f \circ d. \tag{2.9}$$

From equation (2.2),

$$b_K(f)(x_1 \dots x_{p+1}) = f(x_1 \dots x_p).x_{p+1} - (-1)^p x_1.f(x_2 \dots x_{p+1}), \tag{2.10}$$

for any $x_1 \dots x_{p+1}$ in W_{p+1}. In particular, if $p = 0$ and $f : k \to M$ is identified to $f(1)$, one has $b_K(f)(x) = f(1).x - x.f(1)$ for any x in V.

Definition 2.5 *Let $A = T(V)/(R)$ be a quadratic algebra and M an A-bimodule. The homology of the complex $(Hom(W_\bullet, M),\, b_K)$ is called the Koszul cohomology of A with coefficients in M, and is denoted by $HK^\bullet(A, M)$. We set $HK^\bullet(A) = HK^\bullet(A, A)$.*

Recall that $\chi : K(A) \to B(A)$ denotes the natural injection of $K(A)$ into $B(A)$. The functor $Hom_{A^e}(\chi, M)$ defines a morphism of complexes χ^* from $(Hom(A^{\otimes \bullet}, M), b)$ to $(Hom(W_\bullet, M), b_K)$, where b is the Hochschild differential.

Warning: b is defined with the sign convention as in equality (2.9).

For each degree p, χ_p^* coincides with the natural projection of $Hom(A^{\otimes p}, M)$ onto $Hom(W_p, M)$. Using again the exact complex (2.4), we deduce that the k-linear map $H(\chi^*)_p : HH^p(A, M) \to HK^p(A, M)$ is an isomorphism for $p = 0$ and $p = 1$. The following proposition is clear.

Proposition 2.6 *Let $A = T(V)/(R)$ be a quadratic algebra which is Koszul. For any A-bimodule M and any $p \geq 0$, $H(\chi^*)_p : HH^p(A, M) \to HK^p(A, M)$ is an isomorphism.*

For a converse, see Subsection 2.4. In the non-Koszul example of Section 9, $H(\chi^*)_2$ is not surjective for $M = A$. Here again, it is easily proved that the same functorial properties of Hochschild cohomology (see 1.5.1 and 1.5.5 in [11]) stand for Koszul cohomology. In particular, there is a contravariant functor $A \mapsto HK^\bullet(A, A^*)$, where the A-bimodule $A^* = Hom(A, k)$ is defined by: $(a.f.a')(x) = f(a'xa)$ for any k-linear map $f : A \to k$, and x, a, a' in A.

As we prove now, the Koszul cohomology is isomorphic to the following Hochschild hypercohomology

$$HK^\bullet(A, M) \cong \mathbb{HH}^\bullet(A, Hom_A(K(A), M)). \tag{2.11}$$

For any A-bimodule M, the right derived functor $RHom_{A^e}(-, M)$ is defined from the triangulated category $\mathcal{D}^-(\mathcal{A})$ to the triangulated category $\mathcal{D}^+(\mathcal{E})$, so that we have

$$HK^p(A, M) \cong H^p(RHom_{A^e}(K(A), M)). \tag{2.12}$$

Due to lack of a suitable reference, we include here a proof of the next lemma.

Lemma 2.7 *Let M and N be A-bimodules. The k-linear map*

$$\eta : Hom_{A^e}(N, M) \to Hom_{A^e}(A, Hom_A(N, M))$$

defined by $\eta(f)(a)(x) = f(xa)$ for any A-bimodule map $f : N \to M$, a in A and x in N, is an isomorphism, where $Hom_A(N, M)$ means left A-module morphisms from M to N. Moreover, for any complex of A-bimodules C, $\eta : Hom_{A^e}(C, M) \to Hom_{A^e}(A, Hom_A(C, M))$ is an isomorphism of complexes.

Proof. The A-bimodule $Hom_A(N, M)$ is defined by the following actions: if $f : N \to M$ is left A-linear, then $(a.f)(x) = f(xa)$ and $(f.a)(x) = f(x)a$ for any a in A and x in N. For any A-bimodule map $f : N \to M$, it is easy to check that $\eta(f)$ is an A-bimodule map. Since $\eta(f)(1) = f$, η is k-linear. For any g in $Hom_{A^e}(A, Hom_A(N, M))$, define $\eta'(g) = g(1)$. It is easy to check that $\eta'(g)$ belongs to $Hom_{A^e}(N, M)$ and that the map η' is k-linear. Next, the maps η and η' are inverse each other. The extension of η to A-bimodule complexes C is routine and is left to the reader. Note that the definition of the differentials $Hom_{A^e}(d, M)$ or $Hom_A(d, M)$ follows the sign convention as in equality (2.9). For the A-bimodule cochain complexes (C', d'), the sign convention is reduced to $Hom_{A^e}(A, d')(f) = d' \circ f$. ■

Accordingly, the functor $F : C \mapsto Hom_{A^e}(C, M)$ coincides with the composite $H \circ G$ where $G : C \mapsto Hom_A(C, M)$ and $H : C' \mapsto Hom_{A^e}(A, C')$. As a consequence, their right derived functors RF and $R(H) \circ R(G)$ are isomorphic. In particular for $C = K(A)$

$$RHom_{A^e}(K(A), M) \cong RHom_{A^e}(A, RHom_A(K(A), M)). \tag{2.13}$$

Passing to homology and using the definition of hyperext [20], we obtain

$$HK^p(A, M) \cong \mathbb{E}xt^p_{A^e}(A, Hom_A(K(A), M)), \tag{2.14}$$

which proves the isomorphism (2.11). If A is Koszul, we recover usual Ext and Proposition 2.6.

2.4 Coefficients in k

In this subsection, Koszul homology and cohomology are examined for the trivial bimodule $M = k$. Denote by $\epsilon : A \to k$ the projection of A onto its component of weight 0, so that the A-bimodule k is defined by the following actions: $a.1.a' = \epsilon(aa')$ for any a and a' in A. It is immediate from (2.3) and (2.10) that the differentials b_K vanish in case $M = k$. Denoting $Hom(E,k)$ by E^* for any k-vector space E, we obtain the following.

Proposition 2.8 *Let $A = T(V)/(R)$ be a quadratic algebra. For any $p > 0$, there are isomorphisms $HK_p(A,k) \cong W_p$ and $HK^p(A,k) \cong W_p^*$ for any $p \geq 0$.*

Recall some fundamental facts concerning quadratic algebras viewed as connected graded algebras by the weight [14]. Let $A = T(V)/(R)$ be a quadratic algebra. In the category of graded A-bimodules, A has a minimal projective resolution $P(A)$ whose component of homological degree p has the form $A \otimes E_p \otimes A$, where E_p is a weight graded space. Moreover, the minimal weight in E_p is equal to p and the component of weight p in E_p coincides with W_p. Denote by $\underline{Hom}$ the graded Hom w.r.t. the weight grading of A, and by $\underline{HH}$ the corresponding graded Hochschild cohomology HH. The following fundamental property of $P(A)$ holds for any connected graded algebra A.

Lemma 2.9 *The differentials of the complexes $k \otimes_{A^e} P(A)$ and $\underline{Hom}_{A^e}(P(A),k)$ vanish.*

Consequently, we have $HH_p(A,k) \cong E_p$ and $\underline{HH}^p(A,k) \cong \underline{Hom}(E_p,k)$ for any $p \geq 0$. Since $K(A)$ is a weight graded subcomplex of $P(A)$, $H(\tilde{\chi})_p$ coincides with the natural injection of W_p into E_p and $H(\chi^*)_p$ with the natural projection of $\underline{Hom}(E_p,k)$ onto W_p^*. So we obtain the following converses of Proposition 2.3 and Proposition 2.6.

Proposition 2.10 *Let $A = T(V)/(R)$ be a quadratic algebra. The algebra A is Koszul if one of the following properties holds.*
(i) For any $p \geq 0$, $H(\tilde{\chi})_p : HK_p(A,k) \to HH_p(A,k)$ is an isomorphism.
(ii) For any $p \geq 0$, $H(\chi^*)_p : \underline{HH}^p(A,k) \to HK^p(A,k)$ is an isomorphism.

3 The Koszul cup product

3.1 Definition and first properties

Definition 3.1 *Let $A = T(V)/(R)$ be a quadratic algebra. Let P and Q be A-bimodules. For any Koszul p-cochain $f : W_p \to P$ and any Koszul q-cochain $g : W_q \to Q$, define the Koszul $(p+q)$-cochain $f \underset{K}{\smile} g : W_{p+q} \to P \otimes_A Q$ by the following equality*

$$(f \underset{K}{\smile} g)(x_1 \dots x_{p+q}) = (-1)^{pq} f(x_1 \dots x_p) \otimes_A g(x_{p+1} \dots x_{p+q}), \tag{3.1}$$

for any $x_1 \dots x_{p+q} \in W_{p+q}$.

Clearly, the Koszul cup product $\underset{K}{\smile}$ is k-bilinear and associative. Recall that for any Hochschild cochains $f : A^{\otimes p} \to P$ and $g : A^{\otimes q} \to Q$, their cup product is the Hochschild $(p+q)$-cochain $f \smile g$ given by the same equality where $x_1, \dots, x_{p+q}$ are arbitrary elements in A, see for example [10]. Therefore, using the morphism of complexes χ^* (Subsection 2.3) w.r.t. the bimodules P, Q and $P \otimes_A Q$, we have the formula

$$\chi^*(f \smile g) = \chi^*(f) \underset{K}{\smile} \chi^*(g), \tag{3.2}$$

for any Hochschild cochains $f : A^{\otimes p} \to P$ and $g : A^{\otimes q} \to Q$. For such Hochschild cochains, recall that

$$b(f \smile g) = b(f) \smile g + (-1)^p f \smile b(g).$$

Applying the morphism of complexes χ^* (which is surjective), we obtain the identity

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g), \tag{3.3}$$

for any Koszul p-cochain $f : W_p \to P$ and any Koszul q-cochain $g : W_q \to Q$. In particular, $Hom(W_\bullet, A)$ is a differential graded algebra (dga), and $W_\bullet^* = Hom(W_\bullet, k)$ is a dga as well. For any A-bimodule M, $Hom(W_\bullet, M)$ is a differential graded bimodule over the dga $Hom(W_\bullet, A)$. The following statement is clear.

Proposition 3.2 *Let $A = T(V)/(R)$ be a quadratic algebra. The Koszul cup product $\underset{K}{\smile}$ defines a Koszul cup product, still denoted by $\underset{K}{\smile}$, on Koszul cohomology classes. A formula similar to (3.2) holds for $H(\chi^*)$. Endowed with this product, $HK^\bullet(A)$ and $HK^\bullet(A, k)$ are graded associative algebras. For any A-bimodule M, $HK^\bullet(A, M)$ is a graded $HK^\bullet(A)$-bimodule.*

Since $HK^0(A) = Z(A)$ is the center of the algebra A, $HK^\bullet(A, M)$ is a $Z(A)$-bimodule. From Proposition 2.8, $HK^\bullet(A, k)$ coincides with the graded algebra $W_\bullet^* = \bigoplus_{p\geq 0} W_p^*$ endowed with the graded tensor product of linear forms composed with inclusions $W_{p+q} \hookrightarrow W_p \otimes W_q$. Recall that the graded algebra $(\underline{HH}^\bullet(A, k), \smile)$ is isomorphic to the Yoneda algebra $E(A) = \underline{Ext}_A^*(k, k)$ of the graded algebra A [14]. Formula (3.2) and Proposition 2.10 imply the following proposition.

Proposition 3.3 *Let $A = T(V)/(R)$ be a quadratic algebra. The map $H(\chi^*)$ defines a graded algebra morphism from the Yoneda algebra $E(A)$ of A onto $W_\bullet^*$, which is an isomorphism if and only if A is Koszul.*

3.2 The Koszul cup bracket

Definition 3.4 *Let $A = T(V)/(R)$ be a quadratic algebra. Let P and Q be A-bimodules, at least one of them equal to A. For any Koszul p-cochain $f : W_p \to P$ and any Koszul q-cochain $g : W_q \to Q$, we define the Koszul cup bracket by*

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f. \tag{3.4}$$

The Koszul cup bracket is k-bilinear, graded antisymmetric, and it passes to cohomology. We still use the notation $[\alpha, \beta]_{\underset{K}{\smile}}$ for the cohomology classes α and β of f and g. Clearly, the Koszul cup bracket is a graded biderivation of the graded associative algebras $Hom(W_\bullet, A)$ and $HK^\bullet(A)$. We shall see that the Koszul cup bracket plays in some sense the role of the Gerstenhaber bracket. For this, it is necessary to introduce a Koszul 1-cocycle which plays a fundamental role in Koszul calculus.

3.3 The fundamental 1-cocycle

Lemma 3.5 *Let $A = T(V)/(R)$ be a quadratic algebra. Let $f : V \to V$ be a k-linear map, considered as a Koszul 1-cochain $f : W_1 = V \to A$ with coefficients in A. If f is a coboundary, then $f = 0$. If f is a cocycle, then its cohomology class contains a unique 1-cocycle with image in V and this cocycle is equal to f.*

Proof. If $f = b_K(a)$ for some a in A, then $f(x) = ax - xa$ for any x in V. Since $f(x) \in V$, this implies that $f(x) = a_0 x - x a_0$ with $a_0 \in k$, thus $f = 0$. The second part of the statement is an immediate consequence. ■

If f is equal to the identity map of V, Equation (2.10) shows that for any $x_1x_2 \in W_2 = R$, one has $b_K(f)(x_1x_2) = 2x_1x_2$, so that $b_K(f) = 0$ since x_1x_2 is equal to zero in A. This Koszul 1-cocycle f is denoted by e_A and its cohomology class is denoted by $\overline{e}_A$. By the previous lemma, e_A is not a coboundary if $V \neq 0$. Let us call e_A the *fundamental 1-cocycle* of A, and $\overline{e}_A$ the *fundamental 1-class* of the quadratic algebra A. It is clear that $e_A \underset{K}{\smile} e_A = 0$. The following statement is easily proved, but it is of crucial importance in Koszul calculus.

Theorem 3.6 *Let $A = T(V)/(R)$ be a quadratic algebra. For any Koszul cochain f with coefficients in any A-bimodule M, the following formula holds*

$$[e_A, f]_{\underset{K}{\smile}} = -b_K(f). \tag{3.5}$$

Proof. For any $x_1 \dots x_{p+1}$ in W_{p+1}, one has $(e_A \underset{K}{\smile} f)(x_1 \dots x_{p+1}) = (-1)^p x_1.f(x_2 \dots x_{p+1})$ and $(f \underset{K}{\smile} e_A)(x_1 \dots x_{p+1}) = (-1)^p f(x_1 \dots x_p).x_{p+1}$, so that Formula (3.5) is immediate from (2.10). ■

Corollary 3.7 *The fundamental 1-class $\overline{e}_A$ belongs to the graded center of $HK^\bullet(A)$, that is, for any α in $HK^\bullet(A, M)$, one has $[\overline{e}_A, \alpha]_{\underset{K}{\smile}} = 0$.*

The fundamental Formula (3.5) shows the following remarkable fact: *the Koszul differential b_K may be defined from the Koszul cup product*, and defining b_K by (3.5), we may deduce the identity (3.3) from the biderivation $[-,-]_{\underset{K}{\smile}}$. This remarkable fact does not hold in Hochschild calculus: it is not possible to deduce the Hochschild differential b from the cup product $\smile$ and in that situation it is necessary to add another bracket into the picture, namely the Gerstenhaber bracket $[-,-]$, see [5]. In fact, the simple formula (3.5) has to be replaced in Hochschild calculus by the "more sophisticated" and well-known formula

$$b(f) = [\mu, f], \tag{3.6}$$

where multiplication $\mu = b(Id_A)$ is a 2-coboundary and f is any Hochschild cochain (recall that b is defined with the sign convention as in equality (2.9)).

It is possible to deduce the fundamental formula (3.5) from Gerstenhaber calculus, that is, from Hochschild calculus including the Gerstenhaber product $\circ$. We recall from [5] the Gerstenhaber identity

$$b(f \circ g) = b(f) \circ g - (-1)^p f \circ b(g) - (-1)^p [f, g]_{\smile}, \tag{3.7}$$

for any Hochschild cochains $f : A^{\otimes p} \to A$ and $g : A^{\otimes q} \to A$, where

$$[f, g]_{\smile} = f \smile g - (-1)^{pq} g \smile f.$$

The Gerstenhaber product $f \circ g$ is the $(p + q - 1)$-cochain defined by

$$f \circ g\,(a_1, \ldots, a_{p+q-1}) = \sum_{1 \leq i \leq p} (-1)^{(i-1)(q-1)} f(a_1, \ldots a_{i-1}, g(a_i, \ldots a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1}), \tag{3.8}$$

for any $a_1, \ldots, a_{p+q-1}$ in A.

Definition 3.8 *Let $A = T(V)/(R)$ be a quadratic algebra. The weight map $D_A : A \to A$ of the graded algebra A is defined by $D_A(a) = pa$ for any $p \geq 0$ and any a homogeneous of weight p in A.*

Clearly, D_A is the unique derivation of A extending $e_A : V \to A$. The map D_A is also called the Euler derivation of A. For $g = D_A$, identity (3.7) becomes

$$b(f \circ D_A) - b(f) \circ D_A = -(-1)^p [f, D_A]_{\smile}.$$

Restricting this identity to W_{p+1}, the right-hand side coincides with $[e_A, f']_{\underset{K}{\smile}}$ where f' is the restriction of f to W_p. Since $f \circ D_A = pf'$ on W_p, the restriction of $b(f \circ D_A)$ is equal to $pb_K(f')$. The restriction of $b(f) \circ D_A$ is equal to $(p+1)b_K(f')$. Thus we recover the fundamental formula $[e_A, f']_{\underset{K}{\smile}} = -b_K(f')$.

3.4 Koszul derivations

Definition 3.9 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. Any Koszul 1-cocycle $f : V \to M$ with coefficients in M will be called a Koszul derivation of A with coefficients in M. When $M = A$, we will simply speak about a Koszul derivation of A.*

According to equation (2.10), a k-linear map $f : V \to M$ is a Koszul derivation if and only if

$$f(x_1)x_2 + x_1 f(x_2) = 0, \tag{3.9}$$

for any $x_1 x_2$ in R. If this equality holds, the unique derivation $\tilde{f} : T(V) \to M$ extending f defines a unique derivation $D_f : A \to M$ from the algebra A to the bimodule M. The k-linear map $f \mapsto D_f$ is an isomorphism from the space of Koszul derivations of A with coefficients in M to the space of derivations from A to M. In the non-Koszul example of Section 9, $H(\chi^*)_2$ is not surjective for $M = A$, so that there exists a Koszul 2-cocycle which does not extend to a Hochschild 2-cocycle. Consequently, the identity in the next proposition cannot be proved from Gerstenhaber calculus.

Proposition 3.10 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. For any Koszul derivation $f : V \to M$ and any Koszul q-cocycle $g : W_q \to A$, there is an equality*

$$[f, g]_{\underset{K}{\smile}} = b_K(D_f \circ g). \tag{3.10}$$

Proof. Applying D_f to equation $g(x_1 \dots x_q).x_{q+1} = (-1)^q x_1.g(x_2 \dots x_{q+1})$, we get

$D_f(g(x_1 \dots x_q)).x_{q+1} + g(x_1 \dots x_q).f(x_{q+1}) = (-1)^q(f(x_1).g(x_2 \dots x_{q+1}) + x_1.D_f(g(x_2 \dots x_{q+1})))$,

and equality (3.10) follows from (2.10). ■

Corollary 3.11 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. For any $\alpha \in HK^p(A, M)$ with $p = 0$ or $p = 1$ and $\beta \in HK^q(A)$,*

$$[\alpha, \beta]_{\underset{K}{\smile}} = 0. \tag{3.11}$$

Proof. The case $p = 1$ follows from the proposition. The case $p = 0$ is clear since $HK^0(A, M)$ is the space of the elements of M commuting to any element of A. ■

If A is Koszul, then $[\alpha, \beta]_{\underset{K}{\smile}} = 0$ for any p and q, using Gerstenhaber calculus and the isomorphisms $H(\chi^*)$. We do not know whether $[\alpha, \beta]_{\underset{K}{\smile}} = 0$ holds for any p and q when A is not Koszul (see however Section 9). Note that the equality (3.8) defining the Gerstenhaber product *does not make sense* for $f \circ g : W_{p+q-1} \to A$ when $f : W_p \to A$ and $g : W_q \to A$.

3.5 Higher Koszul cohomology

Let $A = T(V)/(R)$ be a quadratic algebra. Let $f : V \to A$ be a Koszul derivation of A. Denote by $[f]$ the cohomology class of f. Assuming $\mathrm{char}(k) \neq 2$, identity (3.11) shows that $[f] \underset{K}{\smile} [f] = 0$, so that the k-linear map $[f] \underset{K}{\smile} -$ defines a cochain differential on $HK^\bullet(A, M)$ for any A-bimodule M. We obtain therefore a *new cohomology*, called "higher Koszul cohomology" of A with coefficients in M. The Gerstenhaber identity (3.7) implies that $2D_f \smile D_f = b(D_f \circ D_f)$, therefore $[D_f] \smile -$ defines a cochain differential on $HH^\bullet(A, M)$, hence a higher Hochschild cohomology of A with coefficients in M. Moreover $H(\chi^*)$ defines a morphism from the higher Hochschild cohomology to the higher Koszul cohomology, which is an isomorphism if A is Koszul.

Let us limit ourselves to the case $f = e_A$, the fundamental 1-cocycle. In this case, with no assumption on the characteristic of k, the formula $e_A \underset{K}{\smile} e_A = 0$ shows that the k-linear map $e_A \underset{K}{\smile} -$ defines a cochain differential on $Hom(W_\bullet, M)$, and $\overline{e}_A \underset{K}{\smile} -$ defines a cochain differential on $HK^\bullet(A, M)$.

Definition 3.12 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. The differential $\overline{e}_A \underset{K}{\smile} -$ of $HK^\bullet(A, M)$ is denoted by $\partial_\smile$. The homology of $HK^\bullet(A, M)$ endowed with $\partial_\smile$ is called the higher Koszul cohomology of A with coefficients in M and is denoted by $HK^\bullet_{hi}(A, M)$. We set $HK^\bullet_{hi}(A) = HK^\bullet_{hi}(A, A)$.*

If we want to evaluate $\partial_\smile$ on classes, it suffices to go back to the formula

$$(e_A \underset{K}{\smile} f)(x_1 \dots x_{p+1}) = (-1)^p x_1.f(x_2 \dots x_{p+1})$$

for any cocycle $f : W_p \to M$, and any $x_1 \dots x_{p+1}$ in W_{p+1}. Note that the right-hand side of the latter equality is equal to $f(x_1 \dots x_p).x_{p+1}$. Since $HK^0(A, M)$ equals the space $Z(M)$ of the elements of M commuting to any element of A, we obtain the following.

Proposition 3.13 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. $HK^0_{hi}(A, M)$ is the space of the elements u of $Z(M)$ such that there exists $v \in M$ satisfying $u.x = v.x - x.v$ for any x in V. In particular, if the bimodule M is symmetric, then $HK^0_{hi}(A, M)$ is the space of elements of M whose annihilator is equal to $A_+ = \oplus_{p>0} A_p$. If A is a commutative domain and $V \neq 0$, then $HK^0_{hi}(A) = 0$.*

The differential $e_A \underset{K}{\smile} -$ vanishes for $M = k$, so Proposition 2.8 provides the following result.

Proposition 3.14 *Given a quadratic algebra $A = T(V)/(R)$, $HK^p_{hi}(A, k)$ equals W^*_p for any $p \geq 0$.*

3.6 Higher Koszul cohomology with coefficients in A

Lemma 3.15 *Let $A = T(V)/(R)$ be a quadratic algebra. Given α in $HK^p(A)$ and β in $HK^q(A)$,*

$$\partial_{\smile}(\alpha \underset{K}{\smile} \beta) = \partial_{\smile}(\alpha) \underset{K}{\smile} \beta = (-1)^p \alpha \underset{K}{\smile} \partial_{\smile}(\beta).$$

Proof. The first equality comes from the associativity relation

$$\overline{e}_A \underset{K}{\smile} (\alpha \underset{K}{\smile} \beta) = (\overline{e}_A \underset{K}{\smile} \alpha) \underset{K}{\smile} \beta.$$

The second one uses associativity together with the relation $[\overline{e}_A, \alpha]_{\underset{K}{\smile}} = 0$ of Corollary 3.7. ■

Consequently, the Koszul cup product is defined on $HK^\bullet_{hi}(A)$, it is still denoted by $\underset{K}{\smile}$ and $(HK^\bullet_{hi}(A), \underset{K}{\smile})$ is a graded associative algebra. Remark that, if $V \neq 0$, then $\partial_{\smile}(1) = \overline{e}_A \neq 0$ and so 1 and $\overline{e}_A$ *do not survive* in higher Koszul cohomology. To go further in the structure of $HK^\bullet_{hi}(A)$, an assumption is required.

Throughout the remainder of this subsection, assume that V is *finite dimensional*. For any $p \geq 0$, the space $Hom(W_p, A)$ is graded by the weight $m \geq 0$ of the coefficients in A. Precisely, a Koszul p-cochain $f : W_p \to A_m$ is said to be homogeneous of weight m. Therefore, the space of Koszul cochains $Hom(W_\bullet, A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight (p, m), where p is called the *homological weight* and m is called the *coefficient weight*. If $f : W_p \to A_m$ and $g : W_q \to A_n$ are homogeneous of biweights (p, m) and (q, n) respectively, then $f \underset{K}{\smile} g : W_{p+q} \to A_{m+n}$ is homogeneous of biweight $(p + q, m + n)$, and

$$(f \underset{K}{\smile} g)(x_1 \dots x_{p+q}) = (-1)^{pq} f(x_1 \dots x_p) g(x_{p+1} \dots x_{p+q}), \tag{3.12}$$

for any $x_1 \dots x_{p+q} \in W_{p+q}$. Moreover, b_K is homogeneous of biweight $(1, 1)$. Thus the unital associative k-algebras $Hom(W_\bullet, A)$ and $HK^\bullet(A)$ are $\mathbb{N} \times \mathbb{N}$-graded by the biweight. The homogeneous component of biweight (p, m) of $HK^\bullet(A)$ is denoted $HK^p(A)_m$. Clearly, we have

$$\partial_{\smile} : HK^p(A)_m \to HK^{p+1}(A)_{m+1}.$$

Therefore, the associative k-algebra $HK^\bullet_{hi}(A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight, and its (p, m)-component is denoted by $HK^p_{hi}(A)_m$.

From Proposition 3.13 we deduce the following.

Proposition 3.16 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume that V is finite dimensional. If $V \neq 0$, then $HK^0_{hi}(A)_0 = 0$. If A is finite dimensional, $HK^0_{hi}(A)_{\max} = A_{\max}$ where* max *is the highest nonnegative integer m such that $A_m \neq 0$. If the algebra A is commutative, then for any $m \geq 0$, $HK^0_{hi}(A)_m$ equals the space of elements of A_m whose annihilator in A is A_+.*

3.7 Higher Koszul cohomology of symmetric algebras

Throughout this subsection, $A = S(V)$ is the symmetric algebra of the k-vector space V. No assumption is made here on $\dim(V)$ or $\mathrm{char}(k)$. Although A is Koszul and its Hochschild (co)homology is known [11], we prefer to compute the Koszul (co)homology of A using Koszul calculus. Recall the following lemma which describes the spaces W_p.

Lemma 3.17 *Let V be a k-vector space and $A = S(V)$ be the symmetric algebra of V. For any $p \geq 0$, the space W_p is equal to the image of the k-linear map $Ant : V^{\otimes p} \to V^{\otimes p}$ defined by*

$$Ant(v_1, \dots, v_p) = \sum_{\sigma \in \Sigma_p} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \dots v_{\sigma(p)},$$

for any $v_1, \dots, v_p$ in V, where Σ_p is the symmetric group and sgn *is the signature.*

Proposition 3.18 *Let V be a k-vector space and $A = S(V)$ be the symmetric algebra of V. Let M be a symmetric A-bimodule. The differentials b_K of the complexes $M \otimes W_\bullet$ and $Hom(W_\bullet, M)$ vanish. Therefore,*

$$HK_\bullet(A, M) = M \otimes W_\bullet$$

and

$$HK^\bullet(A, M) = Hom(W_\bullet, M).$$

Proof. Equation (2.3) can be written as

$$b_K(m \otimes x_1 \dots x_p) = m.(x_1 \otimes x_2 \dots x_p + (-1)^p x_p \otimes x_1 \dots x_{p-1}),$$

and the right hand side vanishes according to the previous lemma and the relation

$$Ant(v_p, v_1, \dots, v_{p-1}) = (-1)^{p-1} Ant(v_1, \dots, v_p).$$

Similarly, Equation (2.10) can be written as

$$b_K(f)(x_1 \dots x_{p+1}) = f \otimes 1_V (x_1 \dots x_p \otimes x_{p+1} - (-1)^p x_2 \dots x_{p+1} \otimes x_1).$$

So, $b_K(f) = 0$, according to the lemma and the relation

$$Ant(v_2, \dots, v_{p+1}, v_1) = (-1)^p Ant(v_1, \dots, v_{p+1}). \blacksquare$$

In characteristic 0, the next result is a consequence of a general fact concerning Koszul algebras which are Calabi-Yau (see Section 5). For the proof given below, we recall some facts concerning quadratic algebras [14]. Applying the functor $- \otimes_A k$ to the bimodule complex $K(A) = (A \otimes W_\bullet \otimes A, d)$ introduced in Subsection 2.1, one obtains the left Koszul complex $K_\ell(A) = (A \otimes W_\bullet, d_\ell)$ of left A-modules. The algebra A is Koszul if and only if $K_\ell(A)$ is a resolution of k. Note that $\mu \otimes_A k$ coincides with the natural projection $\epsilon : A \to k$. From (2.2), we set – with obvious notation:

$$d_\ell(a \otimes x_1 \dots x_p) = ax_1 \otimes x_2 \dots x_p. \tag{3.13}$$

Theorem 3.19 *Let V be a k-vector space and $A = S(V)$ be the symmetric algebra of V. Assume that* $\dim(V) = n$ *is finite. There are isomorphisms*

$$\begin{aligned} HK^n_{hi}(A) &\cong k, \\ HK^p_{hi}(A) &\cong 0 \text{ if } p \neq n. \end{aligned}$$

If $n \geq 1$, the Koszul cup product of $HK^\bullet_{hi}(A)$ is zero.

Proof. Proposition 3.18 shows that the differential $\partial_\smile$ on $HK^\bullet(A)$ coincides with the differential $e_A \underset{K}{\smile} -$ on $Hom(W_\bullet, A)$. Given $f : W_p \to A$, denote by $F : A \otimes W_p \to A$ the left A-linear extension of f to $A \otimes W_p$. From equation (3.13) applied to $1 \otimes x_1 \dots x_{p+1}$, and from

$$(e_A \underset{K}{\smile} f)(x_1 \dots x_{p+1}) = (-1)^p x_1.f(x_2 \dots x_{p+1}),$$

we deduce that

$$e_A \underset{K}{\smile} f = (-1)^p F \circ d_\ell,$$

where $d_\ell : A \otimes W_{p+1} \to A \otimes W_p$ is restricted to W_{p+1}. Thus the differential $e_A \underset{K}{\smile} -$ coincides with the opposite – according to the convention sign (2.9) – of the differential $Hom_A(d_\ell, A)$, when the functor $Hom_A(-, A)$ is applied to $K_\ell(A)$. Since A is Koszul, we have obtained that

$$HK^\bullet_{hi}(A) \cong Ext^\bullet_A(k, A).$$

Using that A is AS-Gorenstein of global dimension n [14], the theorem is proved. ■

4 The Koszul cap products

4.1 Definition and first properties

Definition 4.1 *Let $A = T(V)/(R)$ be a quadratic algebra. Let M and P be A-bimodules. For any Koszul p-cochain $f : W_p \to P$ and any Koszul q-chain $z = m \otimes x_1 \dots x_q$ in $M \otimes W_q$, we define the Koszul $(q-p)$-chains $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ with coefficients in $P \otimes_A M$ and $M \otimes_A P$ respectively, by the following equalities*

$$f \underset{K}{\frown} z = (-1)^{(q-p)p}(f(x_{q-p+1} \dots x_q) \otimes_A m) \otimes x_1 \dots x_{q-p}, \tag{4.1}$$

$$z \underset{K}{\frown} f = (-1)^{pq}(m \otimes_A f(x_1 \dots x_p)) \otimes x_{p+1} \dots x_q. \tag{4.2}$$

The element $f \underset{K}{\frown} z$ is called the left Koszul cap product of f and z, while $z \underset{K}{\frown} f$ is called their right Koszul cap product.

If $q < p$, both the left and right Koszul cap products $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ annihilate. Both Koszul cap products $\underset{K}{\frown}$ are k-bilinear. It is easy to verify the following *associativity relations* for any pair of Koszul cochains f and g and any Koszul chain z with coefficients in arbitrary bimodules:

$$\begin{aligned} f \underset{K}{\frown} (g \underset{K}{\frown} z) &= (f \underset{K}{\smile} g) \underset{K}{\frown} z, \\ (z \underset{K}{\frown} g) \underset{K}{\frown} f &= z \underset{K}{\frown} (g \underset{K}{\smile} f), \\ f \underset{K}{\frown} (z \underset{K}{\frown} g) &= (f \underset{K}{\frown} z) \underset{K}{\frown} g. \end{aligned}$$

In particular, considering both Koszul cap products $\underset{K}{\frown}$ respectively as left or right action, $M \otimes W_\bullet$ becomes a graded bimodule over the graded algebra $Hom(W_\bullet, A)$.

Recall that for any Hochschild cochain $f : A^{\otimes p} \to P$ and any Hochschild chain $z \in M \otimes A^{\otimes q}$, their left and right cap products are the Hochschild $(q-p)$-chains $f \frown z$ and $z \frown f$ given by the same equalities (4.1) and (4.2) for arbitrary elements $x_1, \ldots, x_{p+q}$ of A [10]. Therefore, using the morphisms of complexes $\tilde{\chi}$ and χ^* of Subsections 2.2 and 2.3, we have the formulas

$$\tilde{\chi}(\chi^*(f) \underset{K}{\frown} z) = f \frown \tilde{\chi}(z), \tag{4.3}$$

$$\tilde{\chi}(z \underset{K}{\frown} \chi^*(f)) = \tilde{\chi}(z) \frown f, \tag{4.4}$$

for any $f : A^{\otimes p} \to P$ and $z \in M \otimes W_q$. For any Hochschild p-cochain f and any Hochschild q-chain z, let us recall the following equalities

$$b(f \frown z) = b(f) \frown z + (-1)^p f \frown b(z),$$

$$b(z \frown z) = b(z) \frown f + (-1)^q z \frown b(f),$$

from which we easily deduce the identities

$$b_K(f \underset{K}{\frown} z) = b_K(f) \underset{K}{\frown} z + (-1)^p f \underset{K}{\frown} b_K(z), \tag{4.5}$$

$$b_K(z \underset{K}{\frown} f) = b_K(z) \underset{K}{\frown} f + (-1)^q z \underset{K}{\frown} b_K(f), \tag{4.6}$$

for any Koszul p-cochain $f : W_p \to P$ and any Koszul q-chain $z \in M \otimes W_q$. In fact, it suffices to apply $\tilde{\chi}$, which is injective, to each side of (4.5) and of (4.6), and next use formulas (4.3) and (4.4) for a Hochschild cochain F such that $\chi^*(F) = f$. So $M \otimes W_\bullet$ becomes a differential graded bimodule over the dga $Hom(W_\bullet, A)$. The next statement is clear.

Proposition 4.2 *Let $A = T(V)/(R)$ be a quadratic algebra. Both Koszul cap products $\underset{K}{\frown}$ at the chain-cochain level define Koszul cap products, still denoted by $\underset{K}{\frown}$, on Koszul (co)homology classes. The associativity relations hold on classes. Formulas (4.3) and (4.4) hold for $H(\tilde{\chi})$ and $H(\chi^*)$ applied to classes. Considering Koszul cap products as actions, for any A-bimodule M, $HK_\bullet(A, M)$ is a graded bimodule on the graded algebra $HK^\bullet(A)$. In particular, $HK_\bullet(A, M)$ is a $Z(A)$-bimodule. Moreover, $HK_\bullet(A, k) = W_\bullet$ is a graded bimodule on the graded algebra $HK^\bullet(A, k) = W_\bullet^*$.*

4.2 The Koszul cap bracket

Definition 4.3 *Let $A = T(V)/(R)$ be a quadratic algebra. Let M and P be A-bimodules such that M or P is equal to A. For any Koszul p-cochain $f : W_p \to P$ and any Koszul q-chain $z \in M \otimes W_q$, we define the Koszul cap bracket $[f, z]_{\underset{K}{\frown}}$ by*

$$[f, z]_{\underset{K}{\frown}} = f \underset{K}{\frown} z - (-1)^{pq} z \underset{K}{\frown} f. \tag{4.7}$$

For $z = m \otimes x_1 \dots x_q$, the explicit expression of the bracket is

$$[f, z]_{\underset{K}{\frown}} = (-1)^{(q-p)p} f(x_{q-p+1} \dots x_q)m \otimes x_1 \dots x_{q-p} - mf(x_1 \dots x_p) \otimes x_{p+1} \dots x_q. \tag{4.8}$$

In particular, if $p = 0$ and identifying as usual $f : k \to P$ with $f(1)$, then $[f, z]_{\underset{K}{\frown}} = [f(1), m]_c \otimes x_1 \dots x_q$ where $[-, -]_c$ denotes the commutator. The Koszul cap bracket is k-bilinear and passes to (co)homology classes. We still use the notation $[\alpha, \gamma]_{\underset{K}{\frown}}$ for classes α and γ corresponding to f and z. When $M = A$, the maps $[f, -]_{\underset{K}{\frown}}$ and $[\alpha, -]_{\underset{K}{\frown}}$ are graded derivations of the graded $Hom(W_\bullet, A)$-bimodule $A \otimes W_\bullet$, and of the graded $HK^\bullet(A)$-bimodule $HK_\bullet(A)$, respectively.

Similarly to what happens in cohomology, *the Koszul differential b_K in homology may be defined from the Koszul cap products*: defining b_K by (4.9) below, we may deduce the identities (4.5) and (4.6) from the derivation $[f, -]_{\underset{K}{\frown}}$. Next theorem is analogue to Theorem 3.6. Recall that $e_A : V \to A$ is the fundamental 1-cocycle defined by the identity map of V.

Theorem 4.4 *Let $A = T(V)/(R)$ be a quadratic algebra. For any Koszul cochain z with coefficients in any A-bimodule M, we have the fundamental formula*

$$[e_A, z]_{\underset{K}{\frown}} = -b_K(z). \tag{4.9}$$

Proof. For any $z = m \otimes x_1 \dots x_q$ in $M \otimes W_q$, equality (4.8) provides

$$[e_A, z]_{\underset{K}{\frown}} = (-1)^{(q-1)} x_q m \otimes x_1 \dots x_{q-1} - m x_1 \otimes x_2 \dots x_q,$$

so that formula (4.9) is immediate from (2.3). ■

Corollary 4.5 *For any γ in $HK_\bullet(A, M)$, the bracket $[\overline{e}_A, \gamma]_{\underset{K}{\frown}}$ vanishes. The fundamental 1-class $\overline{e}_A$ belongs to the graded center of the graded $HK^\bullet(A)$-bimodule $HK_\bullet(A)$, that is: the left action and the right action of the element $\overline{e}_A$ of $HK^\bullet(A)$ on $HK_\bullet(A)$ coincide up to the graded sign.*

4.3 Actions of Koszul derivations

Using Subsection 3.4, we associate to a bimodule M and a Koszul derivation $f : V \to M$ the derivation $D_f : A \to M$. The linear map $D_f \otimes Id_{W_\bullet}$ from $A \otimes W_\bullet$ to $M \otimes W_\bullet$ will still be denoted by D_f.

Proposition 4.6 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. For any Koszul derivation $f : V \to M$ and any Koszul q-cycle $z \in A \otimes W_q$,*

$$[f, z]_{\underset{K}{\frown}} = b_K(D_f(z)). \tag{4.10}$$

Proof. Setting $z = a \otimes x_1 \dots x_q$ and applying D_f to equation

$$ax_1 \otimes x_2 \dots x_q + (-1)^q x_q a \otimes x_1 \dots x_{q-1} = 0,$$

we get

$$(D_f(a)x_1 + af(x_1)) \otimes x_2 \dots x_q + (-1)^q (f(x_q)a + x_q D_f(a)) \otimes x_1 \dots x_{q-1} = 0,$$

and identity (4.10) follows from (4.8) and (2.3). ■

Corollary 4.7 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. For any $p \in \{0, 1, q\}$, $\alpha \in HK^p(A, M)$ and $\gamma \in HK_q(A)$,*

$$[\alpha, \gamma]_{\underset{K}{\frown}} = 0. \tag{4.11}$$

Proof. The case $p = 1$ follows from the proposition. The case $p = 0$ is clear since $HK^0(A, M)$ is the space of the elements m of M commuting to any element of A, so that $[m, z]_{\underset{K}{\frown}} = 0$ for any Koszul chain z. Assume that $p = q$, α is the class of f and γ is the class of $z = a \otimes x_1 \dots x_p$. The equality (4.8) gives

$$[f, z]_{\underset{K}{\frown}} = f(x_1 \dots x_p).a - a.f(x_1 \dots x_p)$$

which is an element of $[M, A]_c$. Since $[\alpha, \gamma]_{\underset{K}{\frown}}$ belongs to $HK_0(A, M)$, we conclude from the isomorphism

$$H(\tilde{\chi})_0 : HK_0(A, M) \to HH_0(A, M) = M/[M, A]_c. \ ■$$

Note that the same proof shows that $[\alpha, \gamma]_{\underset{K}{\frown}}$ is zero if $\alpha \in HK^p(A)$ and $\gamma \in HK_p(A, M)$. The Koszul cap actions when $p = q$ are called *Kronecker actions*, with reference to the Kronecker product used in Hochschild calculus (see 1.5.9 in [11]). We do not know whether the identity $[\alpha, \gamma]_{\underset{K}{\frown}} = 0$ in the previous corollary holds for any p and q – even if A is Koszul! See Section 9 for a non-Koszul example.

5 Higher Koszul homology

5.1 Higher Koszul homology associated to a Koszul derivation

A similar procedure to the one developed in Subsection 3.5 leads to the definition of a higher homology theory in the following situation. Let $A = T(V)/(R)$ be a quadratic algebra, $f : V \to A$ a Koszul derivation of A and M an A-bimodule. Assuming $\mathrm{char}(k) \neq 2$, an associativity relation and the identity $[f] \underset{K}{\smile} [f] = 0$ show that the linear map $[f] \underset{K}{\frown} -$ defines a chain differential on $HK_\bullet(A, M)$. We obtain therefore a *new homology*, called "higher Koszul homology" of A with coefficients in M. Analogously, $[D_f] \frown -$ defines a chain differential on $HH_\bullet(A, M)$, hence a higher Hochschild homology of A with coefficients in M. The map $H(\tilde{\chi})$ defines a morphism from higher Koszul homology to higher Hochschild homology, which is an isomorphism whenever A is Koszul.

It is important to recognize the linear operator $D_f \frown -$ of $M \otimes A^{\otimes \bullet}$ as a Rinehart-Goodwillie operator. This will be essential in the next section, where we will deal with the special case $f = e_A$. For $z = m \otimes a_1 \dots a_p$ in $M \otimes A^p$, we deduce from the Hochschild analogue of equality (4.1) that

$$D_f \frown z = (-1)^{p-1}(D_f(a_p)m) \otimes a_1 \dots a_{p-1}.$$

Thus, $D_f \frown -$ *coincides with the Rinehart-Goodwillie operator associated to the derivation* D_f *of* A*, usually denoted by* e_{D_f}. For $M = A$, see Rinehart [16], Goodwillie [8], 4.1.7 in [11], and 9.9 in [20].

5.2 Higher Koszul homology associated to the fundamental 1-cocycle

Let fix $f = e_A$ for the remainder of the article. In this case, with no assumption on the characteristic of k, the k-linear map $e_A \underset{K}{\frown} -$ defines a chain differential on $M \otimes W_\bullet$, and next $\overline{e}_A \underset{K}{\frown} -$ defines a chain differential on $HK_\bullet(A, M)$.

Definition 5.1 *Let $A = T(V)/(R)$ be a quadratic algebra and let M be an A-bimodule. The differential $\overline{e}_A \underset{K}{\frown} -$ of $HK_\bullet(A, M)$ will be denoted by $\partial_\frown$. The homology of $HK_\bullet(A, M)$ endowed with $\partial_\frown$ will be called the higher Koszul homology of A with coefficients in M, denoted by $HK^{hi}_\bullet(A, M)$. We set $HK^{hi}_\bullet(A) = HK^{hi}_\bullet(A, A)$.*

If we want to evaluate $\partial_\frown$ on classes, it suffices to go back to the formula

$$e_A \underset{K}{\frown} z = (-1)^{(p-1)} x_p m \otimes x_1 \dots x_{p-1}$$

for any cycle $z = m \otimes x_1 \dots x_p$ in $M \otimes W_p$. Note that the right-hand side of the latter equality is equal to $mx_1 \otimes x_2 \dots x_p$. If $M = k$, the differential $e_A \underset{K}{\frown} -$ vanishes, so Proposition 2.8 provides the following.

Proposition 5.2 *Let $A = T(V)/(R)$ be a quadratic algebra. For any $p \geq 0$,*

$$HK^{hi}_p(A, k) = W_p.$$

5.3 Higher Koszul homology with coefficients in A

Lemma 5.3 *Let $A = T(V)/(R)$ be a quadratic algebra. Given α in $HK^p(A)$ and γ in $HK_q(A)$, one has*

$$\partial_\frown(\alpha \underset{K}{\frown} \gamma) = \partial_\smile(\alpha) \underset{K}{\frown} \gamma = (-1)^p \alpha \underset{K}{\frown} \partial_\frown(\gamma),$$

$$\partial_\frown(\gamma \underset{K}{\frown} \alpha) = \partial_\frown(\gamma) \underset{K}{\frown} \alpha = (-1)^q \gamma \underset{K}{\frown} \partial_\smile(\alpha).$$

Proof. Let us only prove the equalities on first line since the proof for the second line is analogue. The first equality in this line comes from the associativity relation

$$\overline{e}_A \underset{K}{\frown} (\alpha \underset{K}{\frown} \gamma) = (\overline{e}_A \underset{K}{\smile} \alpha) \underset{K}{\frown} \gamma.$$

The second equality comes from the same associativity relation after using the relation $[\overline{e}_A, \alpha]_{\underset{K}{\smile}} = 0$ in Corollary 3.7. ■

Consequently, the Koszul cap products are defined in $HK^{\bullet}_{hi}(A)$ acting on $HK^{hi}_{\bullet}(A)$ and are still denoted by $\underset{K}{\frown}$. This makes $HK^{hi}_{\bullet}(A)$ a graded bimodule over the graded algebra $HK^{\bullet}_{hi}(A)$. More generally, $HK^{hi}_{\bullet}(A, M)$ is a graded bimodule over the graded algebra $HK^{\bullet}_{hi}(A)$ for any A-bimodule M.

As we have done in cohomology, but without any assumption on V, we show now that the space $HK^{hi}_{\bullet}(A)$ is bigraded. For any $q \geq 0$, the space $A \otimes W_q$ is graded by the weight $n \geq 0$ of the coefficients in A. Precisely, a Koszul q-chain z in $A_n \otimes W_q$ is said to be homogeneous of weight n. Therefore, the space of Koszul chains $A \otimes W_{\bullet}$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight (q, n), where q is called the *homological weight* and n is called the *coefficient weight*. Moreover, b_K is homogeneous of biweight $(-1, 1)$. Thus the space $HK_{\bullet}(A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight. The homogeneous component of biweight (q, n) of $HK_{\bullet}(A)$ is denoted by $HK_q(A)_n$. Clearly,

$$\partial_{\frown} : HK_q(A)_n \to HK_{q-1}(A)_{n+1}.$$

Therefore, the space $HK^{hi}_{\bullet}(A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight, and its (q, n)-component is denoted by $HK^{hi}_q(A)_n$.

Assume now that V is finite dimensional. If $f : W_p \to A_m$ and $z \in A_n \otimes W_q$ are homogeneous of biweights (p, m) and (q, n) respectively, then $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ are homogeneous of biweight $(q - p, m + n)$, and

$$f \underset{K}{\frown} z = (-1)^{(q-p)p} f(x_{q-p+1} \dots x_q) a \otimes x_1 \dots x_{q-p}, \tag{5.1}$$

$$z \underset{K}{\frown} f = (-1)^{pq} a f(x_1 \dots x_p) \otimes x_{p+1} \dots x_q. \tag{5.2}$$

where $z = a \otimes x_1 \dots x_q$. The $Hom(W_{\bullet}, A)$-bimodule $A \otimes W_{\bullet}$, the $HK^{\bullet}(A)$-bimodule $HK_{\bullet}(A)$ and the $HK^{\bullet}_{hi}(A)$-bimodule $HK^{hi}_{\bullet}(A)$ are thus $\mathbb{N} \times \mathbb{N}$-graded by the biweight.

The proof of the following is straightforward and is left to the reader.

Proposition 5.4 *Let $A = T(V)/(R)$ be a quadratic algebra. We have*

$$HK_0(A)_0 = HK^{hi}_0(A)_0 = k.$$

Moreover $HK_0(A)_1 = HK_1(A)_0 = V$ and $\partial_{\frown} : HK_1(A)_0 \to HK_0(A)_1$ is the identity map of V. As a consequence,

$$HK^{hi}_0(A)_1 = HK^{hi}_1(A)_0 = 0.$$

5.4 Higher Koszul homology of symmetric algebras

Theorem 5.5 *Given a k-vector space V and $A = S(V)$, the symmetric algebra of V, there are isomorphisms*

$$\begin{aligned} HK_0^{hi}(A) &\cong k, \\ HK_p^{hi}(A) &\cong 0 \text{ if } p > 0. \end{aligned}$$

If $V \neq 0$, the Koszul cap products of $HK_{hi}^{\bullet}(A)$ acting on $HK_{\bullet}^{hi}(A)$ are zero.

Proof. Proposition 3.18 shows that the differential $\partial_{\frown}$ on $HK_{\bullet}(A)$ coincides with the differential $e_A \underset{K}{\frown} -$ on $A \otimes W_{\bullet}$. From equation (3.13) applied to any $z = a \otimes x_1 \dots x_p$ in $A \otimes W_p$, and from

$$e_A \underset{K}{\frown} z = ax_1 \otimes x_2 \dots x_p$$

we see that the complex $(HK_{\bullet}(A), \partial_{\frown})$ coincides with the left Koszul complex $K_\ell(A) = (A \otimes W_{\bullet}, d_\ell)$ defined in Subsection 3.7. Since A is Koszul, we deduce $HK_{\bullet}^{hi}(A)$ as stated. The action of $HK_{hi}^p(A)$ on $HK_q^{hi}(A)$ via the Koszul cap product produces elements of $HK_{q-p}^{hi}(A)$. These actions vanish either if $q \neq 0$ or if $q = 0, p \neq 0$. Accordingly, if $V \neq 0$, the second part of the statement follows from $HK_{hi}^0(A) = 0$, since A is a commutative domain (Proposition 3.13). ∎

Our aim is now to generalize this theorem to any Koszul algebra, in characteristic zero. This generalization is presented in the next section. The proof given below uses some standard facts on Hochschild homology of graded algebras including the Rinehart-Goodwillie operator.

6 Higher Koszul homology and de Rham cohomology

6.1 Standard facts on Hochschild homology of graded algebras

For Hochschild homology of graded algebras, we refer to Goodwillie [8], Section 4.1 of Loday's book [11], or Section 9.9 of Weibel's book [20]. In this subsection, A is a unital associative k-algebra which is $\mathbb{N}$-graded by a weight. The homogeneous component of weight p of A is denoted by A_p and we set $|a| = p$ for any a in A_p. Recall that the weight map $D = D_A : A \to A$ of the graded algebra A is defined by $D(a) = pa$ for any $p \geq 0$ and a in A_p. As recalled in Subsection 5.1, the Rinehart-Goodwillie operator $e_D = D \frown -$ of $A \otimes A^{\otimes \bullet}$ is defined by

$$e_D(a \otimes a_1 \dots a_p) = (-1)^{p-1}(|a_p|a_p a) \otimes a_1 \dots a_{p-1},$$

for any a, $a_1, \dots, a_p$ in A with a_p homogeneous. If $p = 0$, note that $e_D(A) = 0$.

Denote by $[D]$ the Hochschild cohomology class of D. Assuming $char(k) \neq 2$ we can make use of Gerstenhaber's identity $2D \smile D = b(D \circ D)$. The map $H(e_D) = [D] \frown -$ defines a chain differential on $HH_{\bullet}(A)$, and $[D] \smile -$ defines a cochain differential on $HH^{\bullet}(A)$. The so-obtained higher Hochschild homology (resp. cohomology) of A with coefficients in A will be denoted by $HH^{hi}_\bullet(A)$ (resp. $HH^\bullet_{hi}(A)$). If A is a quadratic algebra, we recall that $H(\chi^*)$ defines a morphism of graded algebras from $HH^\bullet_{hi}(A)$ to $HK^\bullet_{hi}(A)$ –which is an isomorphism if A is Koszul– inducing thus a structure of $HH^\bullet_{hi}(A)$-bimodule on $HK^{hi}_\bullet(A)$ such that $H(\tilde{\chi})$ defines a morphism of graded $HH^\bullet_{hi}(A)$-bimodules from $HK^{hi}_\bullet(A)$ to $HH^{hi}_\bullet(A)$, which is an isomorphism if A is Koszul.

Let B be the normalized Connes differential of $A\otimes \bar{A}^{\otimes\bullet}$ where $\bar{A}=A/k$ [11, 20]. Recall that

$$B(a\otimes a_1\dots a_p)=\sum_{0\leq i\leq p}(-1)^{pi}1\otimes(a_{p-i+1}\dots a_p\bar{a}a_1\dots a_{p-i}), \tag{6.1}$$

for any $a\in A$, and $a_1,\dots,a_p$ in $\bar{A}$, where $\bar{a}$ denotes the class of a in $\bar{A}$. Note that $B(a)=1\otimes\bar{a}$ for any a in A. The operator B passes to Hochschild homology and defines the cochain differential $H(B)$ on $HH_\bullet(A)$. We follow Van den Bergh [18] for the next definition.

Definition 6.1 *The complex $(HH_\bullet(A),H(B))$ is called the de Rham complex of A. The homology of this complex is called the de Rham cohomology of A and is denoted by $H^\bullet_{dR}(A)$.*

Obviously, this definition stands for any unital associative k-algebra A. The definition is natural because, if A is commutative, the antisymmetrization is a morphism of complexes, which is moreover injective if $char(k)=0$, from the usual de Rham complex of A to the complex $(HH_\bullet(A),H(B))$ – see 1.3.12 and 2.3.3 in [11] –, and it is an isomorphism if A is smooth – this is Hochschild-Kostant-Rosenberg Theorem, see for example 3.4 in [11].

If A is $\mathbb{N}$-graded and $char(k)=0$, it turns out that *one of both differentials $H(B)$ and $H(e_D)$ of $HH_\bullet(A)$ is – up to a normalization – a contracting homotopy of the other one.* This remarkable duality linking $H(B)$ and $H(e_D)$ will be a consequence of the Rinehart-Goodwillie identity (6.2) below. Let us introduce the weight map L_D of $A\otimes\bar{A}^{\otimes\bullet}$. Precisely, L_D is the operator of $A\otimes\bar{A}^{\otimes\bullet}$ defined by

$$L_D(z)=|z|z,$$

for any homogeneous $z=a\otimes a_1\dots a_p$, where $|z|=|a|+|a_1|+\cdots+|a_p|$. Clearly, L_D defines an operator $H(L_D)$ on $HH_\bullet(A)$. Note that $A\otimes A^{\otimes\bullet}$, $HH_\bullet(A)$ and $HH^{hi}_\bullet(A)$ are graded by the *total weight* (called simply the weight), and that the operators $H(e_D)$, $H(B)$ and $H(L_D)$ are weight homogeneous. Next we state the Rinehart-Goodwillie identity; for a proof, see for example Corollary 4.1.9 in [11].

Proposition 6.2 *Let A be a unital associative $\mathbb{N}$-graded k-algebra. The identity*

$$[H(e_D),H(B)]_{gc}=H(L_D), \tag{6.2}$$

holds, where $[-,-]_{gc}$ denotes the graded commutator with respect to the homological degree.

The assumption $char(k)\neq 2$ may be added, just to ensure that $H(e_D)$ is a differential of $HH_\bullet(A)$. The following consequence of the Rinehart-Goodwillie identity is a noncommutative analogue of Poincaré Lemma for graded algebras.

Theorem 6.3 *Let A be a unital associative k-algebra which is $\mathbb{N}$-graded and connected. Assume $char(k)=0$. There are isomorphisms*

$$\begin{aligned}H^0_{dR}(A)&\cong HH^{hi}_0(A)&&\cong k,\\ H^p_{dR}(A)&\cong HH^{hi}_p(A)&&\cong 0 \text{ if } p>0.\end{aligned}$$

Proof. Let $\alpha \neq 0$ be a weight homogeneous element in $HH_p(A)$. Assume firstly that $H(B)(\alpha) = 0$. The identity (6.2) provides

$$H(B) \circ H(e_D)(\alpha) = |\alpha|\alpha. \tag{6.3}$$

If $p > 0$, then $|\alpha| \neq 0$, so that α is an $H(B)$-boundary, showing that $H^p_{dR}(A) = 0$. If $p = 0$, then $|\alpha| = 0$, since $H(e_D)(\alpha) = 0$, but $HH_0(A)_0 = k$ and so $\alpha \in k$. Conversely, $B(\lambda) = 0$ for any λ in k. Thus $H^0_{dR}(A) = k$.

Assume now that $H(e_D)(\alpha) = 0$. Using again identity (6.2) we get

$$H(e_D) \circ H(B)(\alpha) = |\alpha|\alpha. \tag{6.4}$$

If $p > 0$, then $|\alpha| \neq 0$, so that α is a $H(e_D)$-boundary, showing that $HH^{hi}_p(A) = 0$. If $p = 0$, any α in $HH_0(A)$ is a cycle for $H(e_D)$ and if $|\alpha| \neq 0$, it is a boundary by (6.4). If $p = |\alpha| = 0$, α cannot be a boundary since $H(e_D)$ adds 1 to the coefficient weight. Thus $HH^{hi}_0(A) = k$. Note that the assumption $char(k) = 0$ is essential in this proof, except for proving that $H^0_{dR}(A) \cong k$ and that $HH^{hi}_0(A)_0 \cong k$. ■

6.2 Consequences for quadratic algebras

If A is quadratic, then $H(\tilde{\chi}) : HK_p(A) \to HH_p(A)$ is always an isomorphism for $p = 0$ and $p = 1$; moreover, if A is Koszul it is an isomorphism for any p (Subsection 2.2). As a consequence, $H(\tilde{\chi})$ defines an isomorphism from $HK^{hi}_p(A)$ to $HH^{hi}_p(A)$ for $p = 0$, and for any p if A is Koszul. So, generalizing Theorem 5.5 in characteristic zero, we obtain the following consequence of the previous theorem, the last part of the statement being clear by using the biweight.

Theorem 6.4 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume that $char(k) = 0$. We have $HK^{hi}_0(A) \cong k$. If A is Koszul, then for any $p > 0$,*

$$HK^{hi}_p(A) \cong 0.$$

If A is Koszul and $V \neq 0$ is finite dimensional, then the Koszul cap products of $HK^{\bullet}_{hi}(A)$ acting on $HK^{hi}_{\bullet}(A)$ are null.

It would be more satisfactory to find a proof within Koszul calculus, possibly without any assumption on $char(k)$. We would also like to know if the converse of this theorem holds, namely, if the following conjecture is true.

Conjecture 6.5 *Let $A = T(V)/(R)$ be a quadratic algebra. The algebra A is Koszul if and only if there are isomorphisms*

$$\begin{aligned} HK^{hi}_0(A) &\cong k, \\ HK^{hi}_p(A) &\cong 0 \text{ if } p > 0. \end{aligned}$$

Let us comment on this conjecture. In the non-Koszul example of Section 9, we will find that $HK^{hi}_2(A) \neq 0$ – not invalidating the conjecture! Within graded Hochschild calculus, this conjecture is meaningless, since *any* graded algebra has a trivial higher Hochschild homology as stated in Theorem 6.3. Consequently, the higher Koszul homology provides more information on quadratic algebras than the higher Hochschild homology. Moreover, if Conjecture 6.5 is true, then *Koszul algebras would be exactly the acyclic objects for higher Koszul homology.*

In Subsection 3.7, the left Koszul complex $K_\ell(A) = K(A) \otimes_A k$ associated to any quadratic algebra A is recalled. Since A is Koszul if and only if $K_\ell(A)$ is a resolution of k, Conjecture 6.5 is an immediate consequence of the following.

Conjecture 6.6 *Let $A = T(V)/(R)$ be a quadratic algebra. For any $p \geq 0$*

$$HK_p^{hi}(A) \cong H_p(K_\ell(A)). \tag{6.5}$$

A stronger conjecture asserts that there exists a quasi-isomorphism from the complex $(HK_\bullet(A), \partial_\frown)$ to the complex $K_\ell(A)$. The proof of Theorem 5.5 shows that the stronger conjecture holds for symmetric algebras. For any quadratic algebra A, it is well-known that $H_0(K_\ell(A)) \cong k$ and $H_1(K_\ell(A)) \cong 0$, therefore Conjecture 6.6 implies that $HK_0^{hi}(A) \cong k$ and $HK_1^{hi}(A) \cong 0$. What is known about $HK_1^{hi}(A)$ in general is that $HK_1^{hi}(A)_0 \cong 0$ (Proposition 5.4), and $HK_1^{hi}(A)_1 \cong 0$ (next subsection). Note that the non-Koszul example of Section 9 will satisfy Conjecture 6.6.

6.3 Connes differential in Koszul calculus

From the equality (6.1) defining Connes differential B of $A \otimes \bar{A}^{\otimes \bullet}$, observe that $B(A \otimes W_p)$ is not included in $A \otimes W_{p+1}$, so that it seems hard to find an analogue to B at the Koszul chain level. We prefer to search an analogue to $H(B)$ at the Koszul homology level. In this subsection, the notation $H(B)$ is simplified and replaced by B. We are interested in the following question. Let $A = T(V)/(R)$ be a quadratic algebra.

Does there exist a k-linear cochain differential B_K on $HK_\bullet(A)$ such that the diagram

$$\begin{array}{ccc} HK_p(A) & \xrightarrow{B_K} & HK_{p+1}(A) \\ \downarrow H(\tilde{\chi})_p & & \downarrow H(\tilde{\chi})_{p+1} \\ HH_p(A) & \xrightarrow{B} & HH_{p+1}(A) \end{array} \tag{6.6}$$

commutes for any $p \geq 0$?

Since B and $H(\tilde{\chi})$ preserve the total weight, B_K should preserve the total weight too. Therefore, using our notation for coefficient weight, we impose that

$$B_K : HK_p(A)_m \to HK_{p+1}(A)_{m-1}.$$

The answer to the question is affirmative if A is Koszul since the vertical arrows are isomorphisms, and in this case the corresponding Rinehart-Goodwillie identity linking the differentials B_K and $\partial_\frown$ of $HK_\bullet(A)$ holds. If the answer is affirmative for a non-Koszul algebra A, Conjecture 6.5 would imply that this Koszul Rinehart-Goodwillie identity does not hold in characteristic zero, and it would be interesting to measure the defect to be an identity.

Let us begin by examining the diagram (6.6) for $p = 0$. In this case, such a B_K exists since the vertical arrows are isomorphisms. It suffices to pre and post compose the map

$$B : HH_0(A) \to HH_1(A),\ [a] \mapsto [1 \otimes \bar{a}],$$

with the isomorphism and its inverse in order to obtain B_K, however an explicit expression of B_K is not clear. It is easy to obtain it for small coefficient weights. Clearly,

$$B_K : HK_0(A)_0 = k \to HK_1(A)_{-1} = 0$$

is zero, and

$$B_K : HK_0(A)_1 = V \to HK_1(A)_0 = V$$

is the identity of V (see Proposition 5.4). Next, assume $char(k) \neq 2$ and consider the projections ant and sym of $V \otimes V$ defined by

$$ant(x \otimes y) = \frac{1}{2}(x \otimes y - y \otimes x),\ sym(x \otimes y) = \frac{1}{2}(x \otimes y + y \otimes x),$$

for any x and y in V. The proof of the following lemma is straightforward.

Lemma 6.7 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume $char(k) \neq 2$. Using the above notation,*

$$HK_2(A)_0 = R \cap ant(V \otimes V),\ HK_1(A)_1 = \frac{ant^{-1}(R)}{sym(R)},\ HK_0(A)_2 \cong \frac{V \otimes V}{ant(V \otimes V) + R}.$$

The map $B_K : HK_0(A)_2 \to HK_1(A)_1$ is thus defined by $B_K([a]) = [sym(a)]$ for any $[a]$ in $\frac{V \otimes V}{ant(V \otimes V)+R}$, where $a \in V \otimes V$, noting that $sym(V \otimes V)$ is included in $ant^{-1}(R)$.

Let us continue a bit further by defining the map

$$B_K : HK_1(A)_1 \to HK_2(A)_0$$

by $B_K([a]) = 2ant(a)$ for any $[a]$ in $\frac{ant^{-1}(R)}{sym(R)}$. The proof of the following lemma is direct.

Lemma 6.8 *The map $B_K : HK_1(A)_1 \to HK_2(A)_0$ is surjective and together with $B_K : HK_0(A)_2 \to HK_1(A)_1$ it satisfies the Koszul Rinehart-Goodwillie identity*

$$(\partial_{\frown} \circ B_K + B_K \circ \partial_{\frown})([a]) = 2[a],$$

for any $[a]$ in $HK_1(A)_1$.

Note that the total weight is equal to 2. It is also straightforward to check that $H(\tilde{\chi})_2 : HK_2(A)_0 \to HH_2(A)_2$ is an isomorphism.

Warning: $HH_p(A)_t$ denotes the homogeneous component of *total weight* t.

Moreover, using the previous B_K, the diagram (6.6) corresponding to $p = 1$ and total weight 2 commutes. From Lemma 6.8, we obtain immediately the following proposition.

Proposition 6.9 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume $char(k) \neq 2$. There are isomorphisms*

$$HK_2^{hi}(A)_0 \cong HK_1^{hi}(A)_1 \cong 0.$$

Generalizing $B_K : HK_1(A)_1 \to HK_2(A)_0$ as below, we obtain the following.

Proposition 6.10 *Let $A = T(V)/(R)$ be a quadratic algebra. If $p \geq 2$ is not divisible by $char(k)$, then $HK_p^{hi}(A)_0 \cong 0$.*

Proof. Denote $b_{K,p} : W_p \to V \otimes W_{p-1}$ and $b_{K,p-1} : V \otimes W_{p-1} \to A_2 \otimes W_{p-2}$ the differential b_K on p-chains of weight 0 and on $(p-1)$-chains of weight 1. We have

$$HK_p(A)_0 = \ker(b_{K,p}) \subseteq W_p \subseteq V \otimes W_{p-1}, \quad HK_{p-1}(A)_1 = \frac{\ker(b_{K,p-1})}{\mathrm{im}(b_{K,p})},$$

and $e_A \frown z = z$ for any z in $\ker(b_{K,p})$. Considering this z in $V \otimes W_{p-1}$, it is easy to check that $z \in \ker(b_{K,p-1})$. Thus the map

$$\partial_\frown : HK_p(A)_0 \to HK_{p-1}(A)_1$$

is defined by $\partial_\frown(z) = [z]$ for any z in $\ker(b_{K,p})$. In order to show that this map is injective under the hypothesis on the characteristic, it suffices to define

$$B_K : HK_{p-1}(A)_1 \to HK_p(A)_0$$

such that $B_K \circ \partial_\frown = p\,\mathrm{Id}_{HK_p(A)_0}$. For this, restrict the operators t and N of cyclic homology [11] to $V^{\otimes p}$. We get the operators τ and γ of $V^{\otimes p}$ given for any $v_1, \ldots, v_p$ in V and z in $V^{\otimes p}$ by

$$\tau(v_1 \otimes \ldots \otimes v_p) = (-1)^{p-1} v_p \otimes v_1 \otimes \ldots \otimes v_{p-1},$$
$$\gamma(z) = z + \tau(z) + \cdots + \tau^{p-1}(z).$$

Clearly $\tau^p = \mathrm{Id}_{V^{\otimes p}}$ and $(1-\tau) \circ \gamma = \gamma \circ (1-\tau) = 0$.

Lemma 6.11 *If $z \in V \otimes W_{p-1}$ is such that $b_{K,p-1}(z) = 0$, then $\gamma(z) \in W_p$ and $b_{K,p}(\gamma(z)) = 0$.*

Proof. Write $z = x \otimes x_1 \ldots x_{p-1}$ with usual notation. For $1 \leq i \leq p-1$, define

$$\mu_{i,i+1} = \mathrm{Id}_{V^{\otimes i-1}} \otimes \mu \otimes \mathrm{Id}_{V^{\otimes p-i-1}} : V^{\otimes p} \to V^{\otimes i-1} \otimes A_2 \otimes V^{\otimes p-i-1},$$

so that $\mu_{i,i+1}(v_1 \otimes \ldots \otimes v_p) = v_1 \otimes \ldots \otimes v_{i-1} \otimes (v_i v_{i+1}) \otimes \ldots \otimes v_p$. Clearly,

$$\mu_{i+1,i+2} \circ \tau = -\tau \circ \mu_{i,i+1} \tag{6.7}$$

where τ on the right-hand side acts on $A^{\otimes p-1}$ by the same formula, hence with sign $(-1)^{p-2}$. The formula

$$b_{K,p-1}(z) = (xx_1) \otimes x_2 \ldots x_{p-1} + (-1)^{p-1}(x_{p-1}x) \otimes x_1 \ldots x_{p-2}$$

shows that $b_{K,p-1}$ coincides with the restriction of $\mu_{1,2} \circ (1+\tau)$ to $V \otimes W_{p-1}$. Since $\gamma(z)$ is equal to

$$x \otimes x_1 \ldots x_{p-1} + (-1)^{p-1} x_{p-1} \otimes x \otimes x_1 \ldots x_{p-2} + x_{p-2} \otimes x_{p-1} \ldots x_{p-3} + \cdots + (-1)^{p-1} x_1 \otimes x_2 \ldots x$$

we see that

$$\mu_{1,2}(\gamma(z)) = \mu_{1,2}(z + \tau(z)) = b_{K,p-1}(z) = 0$$

by assumption. Therefore, using equation (6.7), we get

$$\mu_{2,3}(\gamma(z)) = \mu_{2,3}(\tau(z) + \tau^2(z)) = -\tau \circ \mu_{1,2}(z + \tau(z)) = 0,$$

and we proceed inductively, up to

$$\mu_{p-1,p}(\gamma(z)) = \mu_{p-1,p}(\tau^{p-2}(z) + \tau^{p-1}(z)) = -\tau \circ \mu_{p-2,p-1}(\tau^{p-3}(z) + \tau^{p-2}(z)) = 0.$$

Thus, we have proved successively that $\gamma(z)$ belongs to $R \otimes V^{\otimes p-2}$, $V \otimes R \otimes V^{\otimes p-3}$, up to $V^{\otimes p-2} \otimes R$, which means that $\gamma(z) \in W_p$. Next, $b_{K,p}(\gamma(z)) = 0$ is clear since $b_{K,p}$ coincides with the restriction of $1 - \tau$ to W_p. ■

So we set $B_K([z]) = \gamma(z)$ for any $[z]$ in $HK_{p-1}(A)_1$ where $z \in \ker(b_{K,p-1})$. It is immediate that $(B_K \circ \partial_{\frown})(z) = \gamma(z) = pz$ for any z in $\ker(b_{K,p})$. Proposition 6.10 is thus proved. ■

Note that the corresponding diagram (6.6) w.r.t. $p-1$ and total weight p commutes, since B is defined by equality (6.1) on $A \otimes \bar{A}^{\otimes p-1}$ and satisfies $B(z) = 1 \otimes \gamma(z)$ for any z in $V \otimes W_{p-1}$. Remark as well that $H_p(K_\ell(A))_0 = 0$, thus Conjecture 6.6 is satisfied in characteristic zero for coefficient weight zero. We leave to the reader the proof of the following consequence of Proposition 6.10 (see Subsection 4.3 for the definition of Kronecker actions).

Proposition 6.12 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume that $V \neq 0$ is finite dimensional and $char(k) = 0$. For any $p \geq 0$, the higher Kronecker actions of $HK^p_{hi}(A)$ on $HK_p^{hi}(A)$ vanish.*

7 Higher Koszul cohomology and Calabi-Yau algebras

For the definition of Calabi-Yau algebras, we refer to Ginzburg [7]. The following is a higher Hochschild cohomology version of Poincaré duality, and it is based on the material recalled in Subsection 6.1.

Theorem 7.1 *Let A be a unital associative k-algebra which is $\mathbb{N}$-graded and connected. Assume that $char(k) = 0$. If A is n-Calabi-Yau, then*

$$\begin{aligned} HH^n_{hi}(A) &\cong k, \\ HH^p_{hi}(A) &\cong 0 \text{ if } p \neq n. \end{aligned}$$

Proof. Let $c \in HH_n(A)$ be the fundamental class of the Calabi-Yau algebra A. As proved by the second author in [10] (Théorème 4.2), Van den Bergh duality [19] can be expressed by saying that the k-linear map

$$- \frown c : HH^p(A, M) \longrightarrow HH_{n-p}(A, M)$$

is an isomorphism for any p and any A-bimodule M. As in Subsection 6.1, D denotes the weight map of A, the map $[D] \frown -$ defines a chain differential on $HH_\bullet(A)$, and $[D] \smile -$ defines a cochain differential on $HH^\bullet(A)$. The associativity relation

$$[D] \frown (\alpha \frown c) = ([D] \smile \alpha) \frown c$$

for any α in $HH^\bullet(A)$ shows that the diagram

$$\begin{array}{ccc} HH^p(A) & \overset{[D]\smile -}{\longrightarrow} & HH^{p+1}(A) \\ \downarrow - \frown c & & \downarrow - \frown c \\ HH_{n-p}(A) & \overset{[D]\frown -}{\longrightarrow} & HH_{n-p-1}(A) \end{array} \tag{7.1}$$

commutes for any $p \geq 0$. Since the vertical arrows are isomorphisms, they induce an isomorphism from $HH^p_{hi}(A)$ to $HH^{hi}_{n-p}(A)$. The result thus follows from Theorem 6.3. ■

Corollary 7.2 *Let $A = T(V)/(R)$ be a quadratic algebra. Assume that char$(k) = 0$. If A is Koszul and n-Calabi-Yau, then*

$$\begin{aligned} HK^n_{hi}(A) &\cong k, \\ HK^p_{hi}(A) &\cong 0 \text{ if } p \neq n. \end{aligned}$$

If $n \geq 1$, then the Koszul cup product of $HK^\bullet_{hi}(A)$ is zero.

Proof. Since A is Koszul, $H(\chi^*)$ defines an isomorphism from $HH^\bullet_{hi}(A)$ to $HK^\bullet_{hi}(A)$. The second part of the statement is obvious. ■

Analogously to Conjecture 6.5, we formulate the following.

Conjecture 7.3 *Let $A = T(V)/(R)$ be a quadratic algebra which is Koszul. The algebra A is n-Calabi-Yau if and only if there are isomorphisms*

$$\begin{aligned} HK^n_{hi}(A) &\cong k, \\ HK^p_{hi}(A) &\cong 0 \text{ if } p \neq n. \end{aligned}$$

It is known that the Koszul algebra $A = S(V)$ is n-Calabi-Yau if $\dim(V) = n$ is finite. In this example, the higher Koszul cohomology is as stated in Corollary 7.2 without any assumption on char(k), see Theorem 3.19.

We will illustrate Conjecture 7.3 by the example $A = T(V)$ when $\dim(V) \geq 2$ – if $\dim(V) \leq 1$, $T(V) = S(V)$. The complex $K_\ell(A)$ is in this case

$$0 \longrightarrow A \otimes V \xrightarrow{\mu} A \longrightarrow 0,$$

so that A is Koszul of global dimension 1, and A is not AS-Gorenstein since $\dim(V) \geq 2$, thus A is not Calabi-Yau. Next proposition shows that Conjecture 7.3 is valid for these algebras.

Proposition 7.4 *Let V be a k-vector space of* $\dim(V) \geq 2$ *(possibly infinite), let $A = T(V)$ be the tensor algebra of V. There are isomorphisms*

$$\begin{array}{lcl} HK^0_{hi}(A) & \cong & 0 \\ HK^1_{hi}(A)_0 & \cong & V^* \\ HK^1_{hi}(A)_1 & \cong & Hom(V,V)/k.\mathrm{Id}_V \\ HK^1_{hi}(A)_m & \cong & Hom(V, V^{\otimes m})/ < v \mapsto av - va; a \in V^{\otimes m-1}) > \quad \text{if } m \geq 2 \\ HK^p_{hi}(A) & \cong & 0 \text{ if } p \geq 2. \end{array} \tag{7.2}$$

The Koszul cup product of $HK^\bullet_{hi}(A)$ is zero.

Proof. The homology of the complex

$$0 \longrightarrow A \xrightarrow{b_K} Hom(V, A) \longrightarrow 0,$$

where $b_K(a)(v) = av - va$ for any a in A and v in V, is $HK^\bullet(A)$. Thus

$$\begin{array}{rcl} HK^0(A) & \cong & Z(A) \ \cong \ k \\ HK^1(A) & \cong & Hom(V, A)/ < v \mapsto av - va; a \in A > \\ HK^p(A) & \cong & 0 \text{ if } p \geq 2. \end{array} \tag{7.3}$$

Next, $\partial_\smile$ is defined from $HK^0(A)_0 \cong k$ to $HK^1(A)_1 \cong Hom(V, V)$ by $\partial_\smile(\lambda) = \lambda.\mathrm{Id}_V$ for any λ in k, hence it is injective. Equations (7.2) follow immediately. The second part of the statement is obvious. ■

Remark 7.5

There is another way to prove Corollary 7.2. Let $A = T(V)/(R)$ be a quadratic algebra with V finite dimensional. Assume that there exist $n \geq 1$ and $c \neq 0$ in W_n such that

$$W_p = \begin{cases} k.c & \text{if } p = n \\ 0 & \text{if } \ p > n. \end{cases} \tag{7.4}$$

See $c = 1 \otimes c$ as a Koszul chain in $A \otimes W_n$. For any A-bimodule M, the diagram

$$\begin{array}{ccc} Hom(W_p, M) & \xrightarrow{e_A \underset{K}{\smile} -} & Hom(W_{p+1}, M) \\ \downarrow - \underset{K}{\frown} c & & \downarrow - \underset{K}{\frown} c \\ M \otimes W_{n-p} & \xrightarrow{e_A \underset{K}{\frown} -} & M \otimes W_{n-p-1} \end{array} \tag{7.5}$$

commutes for any $p \geq 0$. If moreover $b_K(c) = 0$, then the diagram

$$\begin{array}{ccc} HK^p(A, M) & \xrightarrow{\partial_\smile} & HK^{p+1}(A, M) \\ \downarrow - \underset{K}{\frown} [c] & & \downarrow - \underset{K}{\frown} [c] \\ HK_{n-p}(A, M) & \xrightarrow{\partial_\frown} & HK_{n-p-1}(A, M) \end{array} \tag{7.6}$$

commutes as well. Under the assumptions of Corollary 7.2, the vertical arrows are isomorphisms, hence we conclude using Theorem 6.4.

8 Koszul duality of quadratic algebras

Throughout this section, V denotes a finite dimensional k-vector space and $A = T(V)/(R)$ is a quadratic algebra. Let $V^* = Hom(V, k)$ be the dual vector space of V. For any $p \geq 0$, the natural isomorphism from $(V^{\otimes p})^*$ to $V^{*\otimes p}$ is always understood *without sign*. The reason is that in this paper, we are only interested in the *ungraded situation*, meaning that there is no additional $\mathbb{Z}$-grading on V. Let $R^\perp$ be the subspace of $V^* \otimes V^*$ defined as the orthogonal of the subspace R of $V \otimes V$, w.r.t. the natural duality between the space $V \otimes V$ and its dual $(V \otimes V)^* \cong V^* \otimes V^*$.

Definition 8.1 *The quadratic algebra $A^! = T(V^*)/(R^\perp)$ is called the Koszul dual of the quadratic algebra A.*

Recall that A is Koszul if and only if $A^!$ is Koszul [14]. The homogeneous component of weight m of $A^!$ is denoted by $A^!_m$. The subspace of $V^{*\otimes p}$ corresponding to the subspace W_p of $V^{\otimes p}$ is denoted by $W^!_p$. By definition, one has

$$A^!_m = V^{*\otimes m} / \sum_{i+2+j=m} V^{*\otimes i} \otimes R^\perp \otimes V^{*\otimes j}, \tag{8.1}$$

$$W^!_p = \bigcap_{i+2+j=p} V^{*\otimes i} \otimes R^\perp \otimes V^{*\otimes j}. \tag{8.2}$$

8.1 Koszul duality in cohomology

Recall that $HK^\bullet(A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight (p, m), where p is the homological weight and m is the coefficient weight, see Subsection 3.6. The homogeneous component of biweight (p, m) of $HK^\bullet(A)$ is denoted by $HK^p(A)_m$. It will be crucial for the Koszul duality to exchange the weights p and m in the definition of the Koszul cohomology of A. This fact leads to a new version of Koszul cohomology in which the differential b_K and the cup product $\underset{K}{\smile}$ are modified by exchanging homological and coefficient weights in the signs of equalities (2.10) and (3.12). More precisely, for Koszul cochains $f : W_p \to A_m$ and $g : W_q \to A_n$, define $\tilde{b}_K(f)$ and $f \underset{K}{\tilde{\smile}} g$ by

$$\tilde{b}_K(f)(x_1 \dots x_{p+1}) = f(x_1 \dots x_p)x_{p+1} - (-1)^m x_1 f(x_2 \dots x_{p+1}), \tag{8.3}$$

$$(f \underset{K}{\tilde{\smile}} g)(x_1 \dots x_{p+q}) = (-1)^{mn} f(x_1 \dots x_p) g(x_{p+1} \dots x_{p+q}). \tag{8.4}$$

Let us also define the corresponding cup bracket by

$$[f, g]_{\underset{K}{\tilde{\smile}}} = f \underset{K}{\tilde{\smile}} g - (-1)^{mn} g \underset{K}{\tilde{\smile}} f.$$

Lemma 8.2 *The product $\underset{K}{\tilde{\smile}}$ is associative and the following formula holds*

$$\tilde{b}_K(f) = -[e_A, f]_{\underset{K}{\tilde{\smile}}}$$

for any Koszul cochain f with coefficients in A.

Proof. The associativity is obvious. The formula is a consequence of $e_A \underset{K}{\tilde{\smile}} f(x_1 \dots x_{p+1}) = (-1)^m x_1.f(x_2 \dots x_{p+1})$ and $f \underset{K}{\tilde{\smile}} e_A(x_1 \dots x_{p+1}) = (-1)^m f(x_1 \dots x_p).x_{p+1}$. ■

Associativity implies that $[-, -]_{\underset{K}{\tilde{\smile}}}$ is a graded biderivation for the product $\underset{K}{\tilde{\smile}}$. It is straightforward to deduce from this that

$$\tilde{b}_K(f \underset{K}{\tilde{\smile}} g) = \tilde{b}_K(f) \underset{K}{\tilde{\smile}} g + (-1)^m f \underset{K}{\tilde{\smile}} \tilde{b}_K(g),$$

$$\tilde{b}_K(\tilde{b}_K(f)) = 0.$$

Therefore, $(Hom(W_\bullet, A), \underset{K}{\tilde{\smile}}, \tilde{b}_K)$ is a dga w.r.t. the coefficient weight. The following convention is essential for stating the Koszul duality in the next theorem.

Convention: $(Hom(W_\bullet, A), \underset{K}{\tilde{\smile}})$ is considered as $\mathbb{N}\times\mathbb{N}$-graded by the *inverse* biweight (m, p).

The homology of the complex $(Hom(W_\bullet, A), \tilde{b}_K)$ is denoted by $\tilde{HK}^\bullet(A)$, it is a unital associative algebra, $\mathbb{N}\times\mathbb{N}$-graded by the inverse biweight (m, p). The homogeneous component of biweight (m, p) is denoted by $\tilde{HK}^p(A)_m$. Note that $HK^\bullet(A)$ and $\tilde{HK}^\bullet(A)$ are different in general. For example, $HK^0(A) = Z(A)$, while $\tilde{HK}^0(A) = \tilde{Z}(A)$ is the graded center of A, considering A graded by the weight.

Theorem 8.3 *Let V be a finite dimensional k-vector space, $A = T(V)/(R)$ a quadratic algebra and $A^! = T(V^*)/(R^\perp)$ the Koszul dual of A. There is an isomorphism of $\mathbb{N}\times\mathbb{N}$-graded unital associative algebras*

$$(HK^\bullet(A), \underset{K}{\smile}) \cong (\tilde{HK}^\bullet(A^!), \underset{K}{\tilde{\smile}}). \tag{8.5}$$

In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK^p(A)_m \cong \tilde{HK}^m(A^!)_p. \tag{8.6}$$

Proof. Let us first explain the strategy: it suffices to exhibit a morphism of $\mathbb{N}\times\mathbb{N}$-graded unital associative algebras

$$\varphi_A : (Hom(W_\bullet, A), \underset{K}{\smile}) \to (Hom(W^!_\bullet, A^!), \underset{K}{\tilde{\smile}}), \tag{8.7}$$

which is a morphism of complexes w.r.t. b_K and $\tilde{b}_K$, such that $\varphi_{A^!}\circ\varphi_A = \mathrm{id}$ and $\varphi_A\circ\varphi_{A^!} = \mathrm{id}$ –using the natural isomorphisms $W^{!!}_\bullet \cong W_\bullet$ and $A^{!!} \cong A$. In fact, the isomorphism (8.5) will be then given by

$$H(\varphi_A) : (HK^\bullet(A), \underset{K}{\smile}) \to (\tilde{HK}^\bullet(A^!), \underset{K}{\tilde{\smile}}).$$

We begin by the definition of φ_A. Using (8.2) and the natural isomorphism $V^{*\otimes p} \cong (V^{\otimes p})^*$, $W^!_p$ is identified to the orthogonal space of $\sum_{i+2+j=p} V^{\otimes i}\otimes R\otimes V^{\otimes j}$ in $(V^{\otimes p})^*$. The following lemma is standard.

Lemma 8.4 *For any subspace F of a finite dimensional vector space E, denote by $F^\perp$ the subspace of E^* whose elements are the linear forms vanishing on F. The canonical map $(E/F)^* \to E^*$, transpose of $can : E \to E/F$, defines an isomorphism $(E/F)^* \cong F^\perp$, and the canonical map $E^* \to F^*$, transpose of $can : F \to E$, defines an isomorphism $E^*/F^\perp \to F^*$.*

Applying the lemma, we define the k-linear isomorphism $\psi_p : W^!_p \to A^*_p$, where A^*_p denotes the dual vector space of

$$A_p = V^{\otimes p}/\sum_{i+2+j=p} V^{\otimes i}\otimes R\otimes V^{\otimes j}.$$

The transpose $\psi^*_p : A_p \to W^{!*}_p$ is an isomorphism. Replacing A by $A^!$ and using $W^{!!}_p \cong W_p$, $\psi^{!*}_p : A^!_p \to W^*_p$ is an isomorphism as well. According to the lemma, $\psi^{!*}_p$ is induced by the map sending any linear form on $V^{\otimes p}$ to its restriction to W_p.

Definition 8.5 *For any $p \geq 0$, $m \geq 0$ and for any Koszul cochain $f : W_p \to A_m$, we define the Koszul cochain $\varphi_A(f) : W^!_m \to A^!_p$ by the commutative diagram*

$$\begin{array}{ccc} W^!_m & \xrightarrow{\varphi_A(f)} & A^!_p \\ \downarrow \psi_m & & \downarrow \psi^{!*}_p \\ A^*_m & \xrightarrow{f^*} & W^*_p. \end{array} \tag{8.8}$$

The so-defined k-linear map φ_A is homogeneous for the biweight (p, m) of $Hom(W_\bullet, A)$ and the biweight (m, p) of $Hom(W^!_\bullet, A^!)$. The commutative diagram (8.8) applied to $\varphi_A(f)$ instead of f provides, using $W^{!!}_p \cong W_p$ and $A^{!!}_m \cong A_m$,

$$\begin{array}{ccc} W_p & \xrightarrow{\varphi_{A^!}(\varphi_A(f))} & A_m \\ \downarrow \psi^!_p & & \downarrow \psi^*_m \\ A^{!*}_p & \xrightarrow{\varphi_A(f)^*} & W^{!*}_m. \end{array} \tag{8.9}$$

Comparing this diagram to the transpose of Diagram (8.8), we immediately obtain $\varphi_{A^!} \circ \varphi_A(f) = f$. The proof of $\varphi_A \circ \varphi_{A^!}(h) = h$ for any $h : W^!_m \to A^!_p$ is similar. So

$$\varphi_A : Hom(W_\bullet, A) \to Hom(W^!_\bullet, A^!)$$

is a k-linear isomorphism whose inverse isomorphism is

$$\varphi_{A^!} : Hom(W^!_\bullet, A^!) \to Hom(W_\bullet, A).$$

Now we continue the proof of Theorem 8.3 by the following.

Claim 8.6 *The map φ_A is an algebra morphism from $(Hom(W_\bullet, A), \underset{K}{\smile})$ to $(Hom(W^!_\bullet, A^!), \underset{K}{\tilde{\smile}})$.*

Proof. Let $f : W_p \to A_m$ and $g : W_q \to A_n$. For the proof, it is necessary to introduce the cup product *without sign* $\underset{K}{\tilde{\smile}}$ defined on $Hom(W_\bullet, A)$ by

$$(f \underset{K}{\tilde{\smile}} g)(x_1 \ldots x_{p+q}) = f(x_1 \ldots x_p) g(x_{p+1} \ldots x_{p+q}).$$

Conformally to the *ungraded situation* stated in the introduction of this section, the tensor products of linear maps are understood *without sign* in the sequel. In particular, the following diagram, whose transpose is used below, commutes.

$$\begin{array}{ccc} W_p \otimes W_q & \xrightarrow{f \otimes g} & A_m \otimes A_n \\ \uparrow \mathrm{can} & & \downarrow \mu \\ W_{p+q} & \xrightarrow{f \underset{K}{\tilde{\smile}} g} & A_{m+n}. \end{array}$$

Tensoring Diagram (8.8) by its analogue for g, we write down the commutative diagram

$$\begin{array}{ccc} W^!_m \otimes W^!_n & \xrightarrow{\varphi_A(f) \otimes \varphi_A(g)} & A^!_p \otimes A^!_q \\ \downarrow \psi_m \otimes \psi_n & & \downarrow \psi^{!*}_p \otimes \psi^{!*}_q \\ A^*_m \otimes A^*_n & \xrightarrow{f^* \otimes g^*} & W^*_p \otimes W^*_q. \end{array} \tag{8.10}$$

Combining this diagram with the following four commutative diagrams

$$
\begin{array}{ccc}
W^!_{m+n} & \xrightarrow{\text{can}} & W^!_m \otimes W^!_n \\
\downarrow \psi_{m+n} & & \downarrow \psi_m \otimes \psi_n \\
A^*_{m+n} & \xrightarrow{\mu^*} & A^*_m \otimes A^*_n
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A^!_p \otimes A^!_q & \xrightarrow{\mu^!} & A^!_{p+q} \\
\downarrow \psi^{!*}_p \otimes \psi^{!*}_q & & \downarrow \psi^{!*}_{p+q} \\
W^*_p \otimes W^*_q & \xrightarrow{\text{can}} & W^*_{p+q}
\end{array}
$$

$$
\begin{array}{ccc}
W^!_{m+n} & \xrightarrow{\varphi_A(f)\underset{K}{\bar{\smile}}\varphi_A(g)} & A^!_{p+q} \\
\downarrow \text{can} & & \uparrow \mu^! \\
W^!_m \otimes W^!_n & \xrightarrow{\varphi_A(f)\otimes\varphi_A(g)} & A^!_p \otimes A^!_q
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A^*_m \otimes A^*_n & \xrightarrow{f^*\otimes g^*} & W^*_p \otimes W^*_q \\
\uparrow \mu^* & & \downarrow \text{can} \\
A^*_{m+n} & \xrightarrow{(f\underset{K}{\bar{\smile}}g)^*} & W^*_{p+q}
\end{array}
$$

we obtain the commutativity of

$$
\begin{array}{ccc}
W^!_{m+n} & \xrightarrow{\varphi_A(f)\underset{K}{\bar{\smile}}\varphi_A(g)} & A^!_{p+q} \\
\downarrow \psi_{m+n} & & \downarrow \psi^{!*}_{p+q} \\
A^*_{m+n} & \xrightarrow{(f\underset{K}{\bar{\smile}}g)^*} & W^*_{p+q}.
\end{array}
\tag{8.11}
$$

Finally, it is sufficient to compare this diagram to Diagram (8.8) applied to $f\underset{K}{\bar{\smile}}g$ instead of f, for showing that $\varphi_A(f\underset{K}{\bar{\smile}}g) = \varphi_A(f)\underset{K}{\bar{\smile}}\varphi_A(g)$. Multiplying the latter equality by $(-1)^{pq}$, we conclude that $\varphi_A(f \underset{K}{\smile} g) = \varphi_A(f)\underset{K}{\tilde{\smile}}\varphi_A(g)$. ■

Consequently, one has $\varphi_A([f,g]_{\underset{K}{\smile}}) = [\varphi_A(f), \varphi_A(g)]_{\underset{K}{\tilde{\smile}}}$. In particular, $\varphi_A([e_A, f]_{\underset{K}{\smile}}) = [e_{A^!}, \varphi_A(f)]_{\underset{K}{\tilde{\smile}}}$, and therefore $\varphi_A(b_K(f)) = \tilde{b}_K(\varphi_A(f))$ by using the fundamental formulas $b_K(f) = -[e_A, f]_{\underset{K}{\smile}}$ and $\tilde{b}_K(f) = -[e_A, f]_{\underset{K}{\tilde{\smile}}}$. Theorem 8.3 is thus proved. ■

We illustrate Theorem 8.3 by the example $A = k[x]$, that is $V = k.x$ and $R = 0$. The Koszul dual of A is $A^! = k \oplus k.x^*$ with $x^{*2} = 0$. It is straightforward to verify the following isomorphisms for any $m \geq 0$

$$HK^0(A)_m \cong k.(1 \mapsto x^m) \cong k.(x^{*m} \mapsto 1) \cong \tilde{HK}^m(A^!)_0$$

$$HK^1(A)_m \cong k.(x \mapsto x^m) \cong k.(x^{*m} \mapsto x^*) \cong \tilde{HK}^m(A^!)_1$$

$$HK^p(A)_m \cong 0 \cong \tilde{HK}^m(A^!)_p \text{ for any } p \geq 2,$$

and it is also direct to check that the products work well. Remark that $HK^0(A)_m$ is not isomorphic to $\tilde{HK}^0(A^!)_m$ for any $m \geq 2$, so the exchange $p \leftrightarrow m$ is essential in Theorem 8.3. Passing to $\tilde{HK}^m(A^!)_p$ is also essential, since it is easy to check for any $m \geq 0$ that

$$HK^m(A^!)_0 \cong \begin{cases} k.(x^{*m} \mapsto 1) & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

$$HK^m(A^!)_1 \cong \begin{cases} k.(x^{*m} \mapsto x^*) & \text{if } m = 0 \text{ or odd} \\ 0 & \text{otherwise} \end{cases}$$

$$HK^m(A^!)_p \cong 0 \text{ for any } p \geq 2.$$

In particular, $HK^m(A^!)_0$ is not isomorphic to $HK^0(A)_m$ when m is odd.

8.2 Koszul duality in higher cohomology

Now we proceed as in Subsection 3.5 in order to define the tilde version of the Koszul higher cohomology. Clearly, $e_A \underset{K}{\tilde{\smile}} e_A = 0$, so that $e_A \underset{K}{\tilde{\smile}} -$ defines a cochain differential on $Hom(W_\bullet, A)$. Next, $\overline{e}_A \underset{K}{\tilde{\smile}} -$ defines a cochain differential on $\tilde{HK}^\bullet(A)$ denoted by $\tilde{\partial}_\smile$. The homology of $\tilde{HK}^\bullet(A)$ endowed with $\tilde{\partial}_\smile$ is denoted by $\tilde{HK}^\bullet_{hi}(A)$. The vector space $(\tilde{HK}^\bullet_{hi}(A), \underset{K}{\tilde{\smile}})$ is an associative k-algebra which is $\mathbb{N} \times \mathbb{N}$-graded by the inverse biweight. Since

$$H(\varphi_A)(\overline{e}_A \underset{K}{\smile} \alpha) = \overline{e}_{A^!} \underset{K}{\tilde{\smile}} H(\varphi_A)(\alpha),$$

for any α in $HK^\bullet(A)$, Theorem 8.3 implies that the isomorphism $H(\varphi_A) : HK^\bullet(A) \to \tilde{HK}^\bullet(A^!)$ is also an isomorphism of complexes w.r.t. the differentials $\partial_\smile$ and $\tilde{\partial}_\smile$. We have thus proved the following higher Koszul duality theorem.

Theorem 8.7 *Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ a quadratic algebra. Let $A^! = T(V^*)/(R^\perp)$ be the Koszul dual of A. There is an isomorphism of $\mathbb{N} \times \mathbb{N}$-graded associative algebras*

$$(HK^\bullet_{hi}(A), \underset{K}{\smile}) \cong (\tilde{HK}^\bullet_{hi}(A^!), \underset{K}{\tilde{\smile}}). \tag{8.12}$$

In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK^p_{hi}(A)_m \cong \tilde{HK}^m_{hi}(A^!)_p. \tag{8.13}$$

We illustrate this theorem by the same example $A = k[x]$. It is immediate that for a fixed biweight, the nonzero maps $\partial_\smile$ and $\tilde{\partial}_\smile$ are exactly the following

$$\partial_\smile : HK^0(A)_m \to HK^1(A)_{m+1},\ (1 \mapsto x^m) \mapsto (x \mapsto x^{m+1}),$$

$$\tilde{\partial}_\smile : \tilde{HK}^m(A^!)_0 \to \tilde{HK}^{m+1}(A^!)_1,\ (x^{*m} \mapsto 1) \mapsto (x^{*m} \mapsto x^*),$$

for any $m \geq 0$. Thus there are isomorphisms

$$HK^1_{hi}(A)_0 \cong k.(x \mapsto 1) \cong k.(1 \mapsto x^*) \cong \tilde{HK}^0_{hi}(A^!)_1,$$

and the other spaces are 0. The reader can easily compute $HK^\bullet_{hi}(A^!)$. It turns out that in this example $HK^\bullet_{hi}(A^!) \cong \tilde{HK}^\bullet_{hi}(A^!)$.

8.3 Koszul duality in homology

We proceed as we have done for cohomology in Subsection 8.1. We recall that $HK_\bullet(A)$ is $\mathbb{N} \times \mathbb{N}$-graded by the biweight, and its (q, n)-component is denoted by $HK_q(A)_n$ like in Subsection 5.3. We define a new version of Koszul homology in which the differential b_K and the cap products $\underset{K}{\frown}$ are modified by exchanging homological and coefficient weights in the signs of Equalities (2.3), (5.1) and (5.2). Precisely, for $f : W_p \to A_m$ and $z = a \otimes x_1 \dots x_q$ in $A_n \otimes W_q$, we define $\tilde{b}_K(z)$, $f \underset{K}{\tilde{\frown}} z$ and $z \underset{K}{\tilde{\frown}} f$ by

$$\tilde{b}_K(z) = ax_1 \otimes x_2 \dots x_q + (-1)^n x_q a \otimes x_1 \dots x_{q-1}, \tag{8.14}$$

$$f \underset{K}{\tilde{\frown}} z = (-1)^{(n-m)m} f(x_{q-p+1} \dots x_q) a \otimes x_1 \dots x_{q-p}, \tag{8.15}$$

$$z \underset{K}{\tilde{\frown}} f = (-1)^{mn} a f(x_1 \dots x_p) \otimes x_{p+1} \dots x_q. \tag{8.16}$$

The corresponding cap bracket is

$$[f, z]_{\underset{K}{\tilde{\frown}}} = f \underset{K}{\tilde{\frown}} z - (-1)^{mn} z \underset{K}{\tilde{\frown}} f.$$

It is just routine to verify the following associativity relations:

$$\begin{aligned} f \underset{K}{\tilde{\frown}} (g \underset{K}{\tilde{\frown}} z) &= (f \underset{K}{\tilde{\smile}} g) \underset{K}{\tilde{\frown}} z, \\ (z \underset{K}{\tilde{\frown}} g) \underset{K}{\tilde{\frown}} f &= z \underset{K}{\tilde{\frown}} (g \underset{K}{\tilde{\smile}} f), \\ f \underset{K}{\tilde{\frown}} (z \underset{K}{\tilde{\frown}} g) &= (f \underset{K}{\tilde{\frown}} z) \underset{K}{\tilde{\frown}} g, \end{aligned}$$

and the fundamental formula

$$\tilde{b}_K(z) = -[e_A, z]_{\underset{K}{\tilde{\frown}}}.$$

The associativity relations imply that $[-,-]_{\underset{K}{\tilde{\frown}}}$ is a graded biderivation for the product $\underset{K}{\tilde{\smile}}$ in the first argument and the actions $\underset{K}{\tilde{\frown}}$ in the second argument. From that, it is straightforward to deduce

$$\tilde{b}_K(f \underset{K}{\tilde{\frown}} z) = \tilde{b}_K(f) \underset{K}{\tilde{\frown}} z + (-1)^m f \underset{K}{\tilde{\frown}} \tilde{b}_K(z),$$

$$\tilde{b}_K(z \underset{K}{\tilde{\frown}} f) = \tilde{b}_K(z) \underset{K}{\tilde{\frown}} f + (-1)^n z \underset{K}{\tilde{\frown}} \tilde{b}_K(f),$$

$$\tilde{b}_K(\tilde{b}_K(z)) = 0.$$

Therefore, $(A \otimes W_\bullet, \underset{K}{\tilde{\frown}}, \tilde{b}_K)$ is a differential graded bimodule w.r.t. the coefficient weight over the dga $(Hom(W_\bullet, A), \underset{K}{\tilde{\smile}}, \tilde{b}_K)$.

The homology of the complex $(A \otimes W_\bullet, \tilde{b}_K)$ is denoted by $\tilde{HK}_\bullet(A)$. It is a $\tilde{HK}^\bullet(A)$-bimodule, $\mathbb{N} \times \mathbb{N}$-graded by the *inverse* biweight. The homogeneous component of biweight (n, q) is denoted by $\tilde{HK}_q(A)_n$. Note that $HK_0(A)_0 \cong k$, while $\tilde{HK}_0(A)_0 \cong 0$ if $\mathrm{char}(k) \neq 2$.

In order to state the Koszul duality in homology, we need to slightly generalize the formalism described up to now in this section, by replacing the graded space of coefficients, namely A, by an arbitrary $\mathbb{Z}$-graded A-bimodule M, whose degree is still called the *weight*. The formalism described up to now for $M = A$ extends immediately to such a graded M by using *the same* b_K, $\underset{K}{\smile}$, $\underset{K}{\frown}$, $\tilde{b}_K$, $\underset{K}{\tilde{\smile}}$, $\underset{K}{\tilde{\frown}}$. We obtain the following general formalism:

1. $Hom(W_\bullet, M)$ is a $(Hom(W_\bullet, A), \underset{K}{\smile})$-bimodule for $\underset{K}{\smile}$, $\mathbb{N} \times \mathbb{Z}$-graded by the biweight, and $HK^\bullet(A, M)$ is a $\mathbb{N} \times \mathbb{Z}$-graded $(HK^\bullet(A), \underset{K}{\smile})$-bimodule.

2. $Hom(W_\bullet, M)$ is a $(Hom(W_\bullet, A), \underset{K}{\tilde{\smile}})$-bimodule for $\underset{K}{\tilde{\smile}}$, $\mathbb{Z} \times \mathbb{N}$-graded by the inverse biweight, and $\tilde{HK}^\bullet(A, M)$ is a $\mathbb{Z} \times \mathbb{N}$-graded $(\tilde{HK}^\bullet(A), \underset{K}{\tilde{\smile}})$-bimodule.

3. $M \otimes W_\bullet$ is a $(Hom(W_\bullet, A), \underset{K}{\smile})$-bimodule for $\underset{K}{\frown}$, $\mathbb{N} \times \mathbb{Z}$-graded by the biweight, and $HK_\bullet(A, M)$ is a $\mathbb{N} \times \mathbb{Z}$-graded $(HK^\bullet(A), \underset{K}{\smile})$-bimodule.

4. $M \otimes W_\bullet$ is a $(Hom(W_\bullet, A), \underset{K}{\tilde{\smile}})$-bimodule for $\underset{K}{\tilde{\frown}}$, $\mathbb{Z} \times \mathbb{N}$-graded by the inverse biweight, and $\tilde{HK}_\bullet(A, M)$ is a $\mathbb{Z} \times \mathbb{N}$-graded $(\tilde{HK}^\bullet(A), \underset{K}{\tilde{\smile}})$-bimodule.

Apart from the case $M = A$, we will need to consider the graded dual $M = A^* = \bigoplus_{m \geq 0} A^*_m$. It would be more natural to grade A^* by the weight $-m$, but in order to avoid notational complications, we prefer to use the nonnegative weight m. So all the biweights used below will belong to $\mathbb{N} \times \mathbb{N}$. We recall the actions of the graded A-bimodule A^*. For any u in A^*_m and a in A_n, they are defined by $a.u$ and $u.a$ in A^*_{m-n}, where

$$(a.u)(a') = (-1)^n u(a'a), \tag{8.17}$$

$$(u.a)(a') = u(aa'), \tag{8.18}$$

for any a' in A_{m-n}. In (8.17), the sign obeys to the Koszul rule since $(-1)^{nm+n(m-n)}$ is equal to $(-1)^n$. We are now ready to state the Koszul duality theorem in homology.

Theorem 8.8 *Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ a quadratic algebra. Let $A^! = T(V^*)/(R^\perp)$ be the Koszul dual of A. There is an isomorphism*

$$HK_\bullet(A) \cong \tilde{HK}^\bullet(A^!, A^{!*}), \tag{8.19}$$

from the $(HK^\bullet(A), \underset{K}{\smile})$-bimodule $HK_\bullet(A)$ with actions $\underset{K}{\frown}$, $\mathbb{N} \times \mathbb{N}$-graded by the biweight, to the $(\tilde{HK}^\bullet(A^!), \underset{K}{\tilde{\smile}})$-bimodule $\tilde{HK}^\bullet(A^!, A^{!*})$ with actions $\underset{K}{\tilde{\smile}}$, $\mathbb{N} \times \mathbb{N}$-graded by the inverse biweight. In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK_p(A)_m \cong \tilde{HK}^m(A^!, A^{!*})_p. \tag{8.20}$$

Proof. We start by recalling from Theorem 8.3 that there is an isomorphism $HK^\bullet(A) \cong \tilde{HK}^\bullet(A^!)$. Now, it is sufficient to exhibit an isomorphism

$$\theta_A : A \otimes W_\bullet \to Hom(W^!_\bullet, A^{!*}), \tag{8.21}$$

from the $(Hom(W_\bullet, A), \underset{K}{\smile})$-bimodule $A \otimes W_\bullet$ with actions $\underset{K}{\frown}$ to the $(Hom(W^!_\bullet, A^!), \underset{K}{\smile})$-bimodule $Hom(W^!_\bullet, A^{!*})$ with actions $\underset{K}{\tilde{\smile}}$, such that θ_A is homogeneous for the biweights as in the statement and θ_A is a morphism of complexes w.r.t. b_K and $\tilde{b}_K$. After doing so, the isomorphism (8.19) will be given by

$$H(\theta_A) : HK_\bullet(A) \cong \tilde{HK}^\bullet(A^!, A^{!*}).$$

For defining the linear map $\theta_A : A_m \otimes W_p \to Hom(W^!_m, A^{!*}_p)$, we use the linear isomorphisms $\psi^*_m : A_m \to W^{!*}_m$ and $\psi^!_p : W_p \to A^{!*}_p$ defined in the proof of Theorem 8.3. For any $z = a \otimes x_1 \dots x_p$ in $A_m \otimes W_p$, set

$$\theta_A(z)(w) = \psi^*_m(a)(w)\, \psi^!_p(x_1 \dots x_p), \tag{8.22}$$

for any w in $W^!_m$. The so-defined linear map θ_A is homogeneous for the biweight of $A \otimes W_\bullet$ and the inverse biweight of $Hom(W^!_\bullet, A^{!*})$.

We now prove that θ_A is an isomorphism. For this, we define

$$\theta'_A : Hom(W^!_m, A^{!*}_p) \to A_m \otimes W_p, \tag{8.23}$$

by $\theta'_A(f) = \sum_{i\in I} e_i \otimes (\psi_p^{!-1} \circ f \circ \psi_m^{-1}(e_i^*))$ for any linear $f : W^!_m \to A^{!*}_p$, where $(e_i)_{i\in I}$ is a basis of the space A_m and $(e_i^*)_{i\in I}$ is its dual basis. Calculating on one hand

$$\theta'_A \circ \theta_A(z) = \sum_{i\in I} e_i \otimes \psi_p^{!-1}(\psi_m^*(a)(\psi_m^{-1}(e_i^*))\, \psi^!_p(x_1 \dots x_p)),$$

the right-hand side equals z since $< \psi_m^*(a), \psi_m^{-1}(e_i^*) >=< a, e_i^* >$ by the natural pairing $< -, - >$, and $\sum_{i\in I} < a, e_i^* > e_i \otimes x_1 \dots x_p = z$.

On the other hand

$$\begin{aligned}
\theta_A \circ \theta'_A(f)(w) &= \sum_{i\in I} \psi_m^*(e_i)(w)\, \psi^!_p(\psi_p^{!-1} \circ f \circ \psi_m^{-1}(e_i^*)) \\
&= \sum_{i\in I} \psi_m^*(e_i)(w)\, f(\psi_m^{-1}(e_i^*)) \\
&= f(\sum_{i\in I} < w, \psi_m^*(e_i) > \psi_m^{-1}(e_i^*)) \\
&= f(w),
\end{aligned}$$

since $(\psi_m^*(e_i))_{i\in I}$ is the basis of $W^{!*}_m$ dual to $(\psi_m^{-1}(e_i^*))_{i\in I}$. Thus θ_A is an isomorphism whose inverse is θ'_A.

Claim 8.9 *Using φ_A, consider the $Hom(W^!_\bullet, A^!)$-bimodule $Hom(W^!_\bullet, A^{!*})$ as a $Hom(W_\bullet, A)$-bimodule. The map $\theta_A : A \otimes W_\bullet \to Hom(W^!_\bullet, A^{!*})$ is a morphism of $Hom(W_\bullet, A)$-bimodules.*

Proof. This amounts to prove that

$$\theta_A(f \underset{K}{\frown} z) = \varphi_A(f) \underset{K}{\tilde{\smile}} \theta_A(z), \tag{8.24}$$

$$\theta_A(z \underset{K}{\frown} f) = \theta_A(z) \underset{K}{\tilde{\smile}} \varphi_A(f), \tag{8.25}$$

for any $z = a \otimes x_1 \dots x_p$ in $A_m \otimes W_p$ and $f : W_q \to A_n$, with $p \geq q$.

Analogously to $\underset{K}{\bar{\smile}}$, define the cap-products without sign $\underset{K}{\bar{\frown}}$. Firstly we prove

$$\theta_A(f \underset{K}{\bar{\frown}} z) = \varphi_A(f) \underset{K}{\bar{\smile}} \theta_A(z), \tag{8.26}$$

leaving to the reader the proof of

$$\theta_A(z \underset{K}{\bar{\frown}} f) = \theta_A(z) \underset{K}{\bar{\smile}} \varphi_A(f). \tag{8.27}$$

For any $w = y_1 \dots y_{m+n} \in W^!_{m+n}$, we deduce from equality (8.22) that

$$\theta_A(f \underset{K}{\bar{\frown}} z)(w) = \psi^*_{m+n}(f(x_{p-q+1} \dots x_p)a)(w)\, \psi^!_{p-q}(x_1 \dots x_{p-q}).$$

Write $w = w_1 w_2$ where $w_1 = y_1 \dots y_n \in W^!_n$ and $w_2 = y_{n+1} \dots y_{m+n} \in W^!_m$, so that

$$\theta_A(z)(w_2) = \psi^*_m(a)(w_2)\, \psi^!_p(x_1 \dots x_p).$$

Denoting by $\bar{\cdot}$ the left action of an element of $A^!_q$ on an element $A^{!*}_p$ giving an element of $A^{!*}_{p-q}$ as in (8.17) *but without sign*, one has

$$\begin{array}{rcl}(\varphi_A(f) \underset{K}{\smile} \theta_A(z))(w) & = & \varphi_A(f)(w_1) \,\bar{\cdot}\, \theta_A(z)(w_2) \\ & = & \psi^{!*-1}_q \circ f^* \circ \psi_n(w_1) \,\bar{\cdot}\, (\psi^*_m(a)(w_2)\, \psi^!_p(x_1 \dots x_p)) \\ & = & \psi^*_m(a)(w_2)\, (\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1) \,\bar{\cdot}\, \psi^!_p(x_1 \dots x_p)).\end{array}$$

Next, for any $a' \in A^!_{p-q}$, one has

$$(\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1) \,\bar{\cdot}\, \psi^!_p(x_1 \dots x_p))(a') = \psi^!_p(x_1 \dots x_p)(a'(\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1))).$$

The right-hand side is equal to $\psi^!_{p-q}(x_1 \dots x_{p-q})(a')\, \psi^!_q(x_{p-q+1} \dots x_p)(\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1))$, by using the commutative diagram

$$\begin{array}{ccc} W_p & \xrightarrow{\text{can}} & W_{p-q} \otimes W_q \\ \downarrow \psi^!_p & & \downarrow \psi^!_{p-q} \otimes \psi^!_q \\ A^{!*}_p & \xrightarrow{\mu^{!*}} & A^{!*}_{p-q} \otimes A^{!*}_q. \end{array}$$

Therefore, we obtain

$$(\varphi_A(f) \underset{K}{\smile} \theta_A(z))(w) = \psi^*_m(a)(w_2)\, \psi^!_q(x_{p-q+1} \dots x_p)(\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1))\, \psi^!_{p-q}(x_1 \dots x_{p-q}).$$

By duality, $\psi^!_q(x_{p-q+1} \dots x_p)(\psi^{!*-1}_q \circ f^* \circ \psi_n(w_1))$ is equal to $\psi^*_n(f(x_{p-q+1} \dots x_p))(w_1)$. Moreover, the commutative diagram

$$\begin{array}{ccc} A_n \otimes A_m & \xrightarrow{\mu} & A_{m+n} \\ \downarrow \psi^*_n \otimes \psi^*_m & & \downarrow \psi^*_{m+n} \\ W^{!*}_n \otimes W^{!*}_m & \xrightarrow{\text{can}} & W^{!*}_{m+n} \end{array}$$

shows that

$$\psi^*_n(f(x_{p-q+1} \dots x_p))(w_1)\, \psi^*_m(a)(w_2) = \psi^*_{m+n}(f(x_{p-q+1} \dots x_p)a)(w_1 w_2),$$

leading to

$$(\varphi_A(f) \underset{K}{\smile} \theta_A(z))(w) = \psi^*_{m+n}(f(x_{p-q+1} \dots x_p)a)(w)\, \psi^!_{p-q}(x_1 \dots x_{p-q}).$$

Thus equality (8.26) is proved. From this equality and the definition of $f \underset{K}{\frown} z$, we draw the following

$$\theta_A(f \underset{K}{\frown} z) = (-1)^{pq}(-1)^q \varphi_A(f) \underset{K}{\smile} \theta_A(z).$$

Recall that $\varphi_A(f) : W^!_n \to A^!_q$ and $\theta_A(z) : W^!_m \to A^{!*}_p$, so that $(-1)^{pq}$ is equal to the sign defining $\underset{K}{\tilde{\smile}}$ from $\underset{K}{\smile}$, without forgetting the sign $(-1)^q$ defining the left action of $A^!_q$ on $A^{!*}_p$ as in (8.17). Thus $\theta_A(f \underset{K}{\frown} z) = \varphi_A(f) \underset{K}{\tilde{\smile}} \theta_A(z)$.

Similarly, $\theta_A(z \underset{K}{\frown} f) = (-1)^{pq}\, \theta_A(z) \underset{K}{\smile} \varphi_A(f) = \theta_A(z) \underset{K}{\tilde{\smile}} \varphi_A(f)$, since the Koszul sign in the right action (8.18) is equal to $+1$. ■

Consequently, one gets $\theta_A([f, z]_{\underset{K}{\frown}}) = [\varphi_A(f), \theta_A(z)]_{\underset{K}{\tilde{\smile}}}$. In particular, $\theta_A([e_A, z]_{\underset{K}{\frown}}) = [e_{A^!}, \theta_A(z)]_{\underset{K}{\tilde{\smile}}}$, and therefore $\theta_A(b_K(z)) = \tilde{b}_K(\theta_A(z))$ by using the fundamental formulas. Theorem 8.8 is proved. ■

Again, we illustrate Theorem 8.8 by the example $A = k[x]$, $A^! = k \oplus k.x^*$ and $A^{!*} = k \oplus k.x$. It is straightforward to verify the following isomorphisms for any $m \geq 0$

$$HK_0(A)_m \cong k.x^m \cong k.(x^{*m} \mapsto 1) \cong \tilde{HK}^m(A^!, A^{!*})_0$$

$$HK_1(A)_m \cong k.(x^m \otimes x) \cong k.(x^{*m} \mapsto x) \cong \tilde{HK}^m(A^!, A^{!*})_1$$

$$HK_p(A)_m \cong 0 \cong \tilde{HK}^m(A^!, A^{!*})_p \text{ for any } p \geq 2,$$

and that the actions work well. The reader will note that the Koszul signs in the left and right actions of the bimodule $A^{!*}$ are crucial in the calculations. Remark that the exchange $p \leftrightarrow m$ and passing to tilde are also essential in Theorem 8.8.

Remark 8.10

Denote by $\mathcal{C}$ the Manin category of the quadratic k-algebras over finite dimensional vector spaces, and by $\mathcal{E}$ the category of the $\mathbb{N} \times \mathbb{N}$-graded k-vector spaces whose components are finite dimensional. We have pointed out in Subsection 2.2 that $A \mapsto HK_\bullet(A)$ defines a covariant functor F from $\mathcal{C}$ to $\mathcal{E}$, and in Subsection 2.3 we noticed that $A \mapsto HK^\bullet(A, A^*)$ defines a contravariant functor G from $\mathcal{C}$ to $\mathcal{E}$, where A^* is now the graded dual, hence the same holds for $\tilde{G} : A \mapsto \tilde{HK}^\bullet(A, A^*)$. The proof of Theorem 8.8 shows that the duality functor $D : A \mapsto A^!$ defines a natural isomorphism θ from F to $\tilde{G} \circ D$, and we have got another one from $F \circ D$ to $\tilde{G}$.

8.4 Koszul duality in higher homology

Generalizing the tilde version of higher Koszul cohomology to any $\mathbb{Z}$-graded bimodule M, we see that $\overline{e}_A \underset{K}{\tilde{\smile}} -$ defines a cochain differential on $\tilde{HK}^\bullet(A, M)$ still denoted by $\tilde{\partial}_\smile$. The homology of $\tilde{HK}^\bullet(A, M)$ endowed with $\tilde{\partial}_\smile$ is denoted by $\tilde{HK}^\bullet_{hi}(A, M)$. It is a $\tilde{HK}^\bullet_{hi}(A)$-bimodule with action $\underset{K}{\tilde{\smile}}$, which is $\mathbb{N} \times \mathbb{N}$-graded by the inverse biweight. Since

$$H(\theta_A)(\overline{e}_A \underset{K}{\frown} \alpha) = \overline{e}_{A^!} \underset{K}{\tilde{\smile}} H(\theta_A)(\alpha),$$

for any α in $HK_\bullet(A)$, Theorem 8.8 implies that the isomorphism $H(\theta_A) : HK_\bullet(A) \to \tilde{HK}^\bullet(A^!, A^{!*})$ is also an isomorphism of complexes w.r.t. the differentials $\partial_\frown$ and $\tilde{\partial}_\smile$. We have thus proved the following higher Koszul duality theorem in homology.

Theorem 8.11 *Let V be a finite dimensional k-vector space and $A = T(V)/(R)$ a quadratic algebra. Let $A^! = T(V^*)/(R^\perp)$ be the Koszul dual of A. There is an isomorphism of $\mathbb{N}\times\mathbb{N}$-graded $HK_{hi}^\bullet(A)$-bimodules*

$$HK_\bullet^{hi}(A) \cong \tilde{HK}_{hi}^\bullet(A^!, A^{!*}). \tag{8.28}$$

In particular, for any $p \geq 0$ and $m \geq 0$, there is a k-linear isomorphism

$$HK_p^{hi}(A)_m \cong \tilde{HK}_{hi}^m(A^!, A^{!*})_p. \tag{8.29}$$

If $A = k[x]$, one can verify directly that $HK_0^{hi}(A)_0 \cong k \cong \tilde{HK}_{hi}^0(A^!, A^{!*})_0$ and the other spaces are isomorphic to 0.

9 A non-Koszul example

9.1 Koszul algebras with two generators

The Koszul algebras with two generators were explicitly determined by the first author in [1]. The result is recalled below without proof. The paper [1] was devoted to study changes of generators in quadratic algebras and their consequences on confluence. The result was obtained by using Priddy's theorem, which asserts that any weakly confluent quadratic algebra is Koszul, and some lattice techniques for the converse "Koszulity implies strong confluence" in case of two relations.

Assume that $V = k.x \oplus k.y$, R is a subspace of $V \otimes V$ and $A = T(V)/(R)$. If $R = 0$ or $R = V \otimes V$, then A is Koszul. If $\dim(R) = 1$, then A is Koszul according to Gerasimov's theorem [4, 2] (see [1] and [2] for proofs not using Gerasimov's theorem). In case $\dim(R) = 3$, A is Koszul since $\dim(R^\perp) = 1$ and $A^!$ is Koszul. For two relations, the Koszul algebras are given by the next proposition.

We need some preparation. Assume that $x < y$. Order lexicographically the monomials of weight two. We obtain the list $x^2 < xy < yx < y^2$. The two quadratic relations of A are enumerated as follows: pick in the previous list two monomials m_1 and m_2 such that $m_1 > m_2$, and express each m_i as a linear combination of the two other monomials of the list while imposing these monomials are smaller than m_i with respect to the order $<$. Greek letters are parameters belonging to k.

Proposition 9.1 *Under the previous assumptions and identifying A to its quadratic relations, the Koszul algebras with two generators and two relations are the following.*

$$\left\{\begin{array}{lcl} xy & = & 0 \\ x^2 & = & 0 \end{array}\right. \text{ and } \left\{\begin{array}{lcl} yx & = & \alpha xy \\ x^2 & = & 0 \end{array}\right. \text{ are Koszul.} \tag{9.1}$$

$$\left\{\begin{array}{lcl} yx & = & \alpha x^2 \\ xy & = & \beta x^2 \end{array}\right. \text{ is Koszul} \Leftrightarrow \alpha = \beta. \tag{9.2}$$

$$\left\{\begin{array}{lcl} y^2 & = & \alpha xy + \beta yx \\ x^2 & = & 0 \end{array}\right. \text{ is Koszul} \Leftrightarrow \alpha = \beta. \tag{9.3}$$

$$\left\{\begin{array}{lcl} y^2 & = & \alpha x^2 + \beta yx \\ xy & = & \gamma x^2 \end{array}\right. \text{ is Koszul} \Leftrightarrow \alpha = 0 \text{ and } \beta = \gamma. \tag{9.4}$$

$$\left\{\begin{array}{lcl} y^2 & = & \alpha x^2 + \beta xy \\ yx & = & \gamma x^2 + \delta xy \end{array}\right. \quad \textit{is Koszul} \Leftrightarrow \left\{\begin{array}{lcl} \beta(1-\delta) & = & \gamma(1+\delta) \\ \alpha(1-\delta^2) & = & -\beta\gamma\delta \end{array}\right. \tag{9.5}$$

Choosing $\alpha = 1$ and $\beta = 0$ in (9.3), we get a non-Koszul algebra A. Our aim is now to make the Koszul calculus of this algebra A explicit. So, throughout the remainder of this section, A denotes the non-Koszul quadratic algebra

$$A = k\langle x, y\rangle / \langle x^2, y^2 - xy\rangle.$$

It is immediate that the cubic relations $y^3 = xy^2 = yxy = 0$ and $y^2x = xyx$ hold in A. From this, it is easily deduced that A_3 is 1-dimensional generated by xyx and that $A_m = 0$ for any $m \geq 4$. Therefore $\dim(A) = 6$ and 1, x, y, xy, yx, xyx form a basis of A. We will use this basis during the calculations. We will assume that the characteristic of k is zero.

9.2 Koszul homology of A

Lemma 9.2 *For any $p \geq 3$, $W_p = k.x^p$.*

Proof. Given u in W_3, it can be written as

$$u = x^2(\alpha x + \beta y) + (y^2 - xy)(\gamma x + \delta y) = (\alpha' x + \beta' y)x^2 + (\gamma' x + \delta' y)(y^2 - xy).$$

Using the monomial basis of $V^{\otimes 3}$, it is straightforward that all the coefficients vanish, except α and α' which are equal. Hence $W_3 = k.x^3$. Assuming now that $W_p = k.x^p$ for some $p \geq 3$, for any u in $W_{p+1} = (W_p \otimes V) \cap (V^{\otimes p-1} \otimes R)$, u can be written as

$$u = \lambda x^{p+1} + \mu x^p y = u' x^2 + u''(y^2 - xy),$$

where λ, μ are in k, and u', u'' are in $V^{\otimes p-1}$. Necessarily $u'' = 0$, thus $\mu = 0$ and $u = \lambda x^{p+1}$. Conversely, x^{p+1} belongs to W_{p+1}. ■

The homology of the following complex is $HK_\bullet(A)$:

$$\ldots \xrightarrow{b_K} A \otimes x^4 \xrightarrow{b_K} A \otimes x^3 \xrightarrow{b_K} A \otimes R \xrightarrow{b_K} A \otimes V \xrightarrow{b_K} A \longrightarrow 0, \tag{9.6}$$

where the maps b_K are successively given by

$$b_K(a \otimes x) = ax - xa \text{ and } b_K(a' \otimes y) = a'y - ya',$$

$$b_K(a \otimes x^2) = (ax + xa) \otimes x \text{ and } b_K(a' \otimes (y^2 - xy)) = -ya' \otimes x + (a'y + ya' - a'x) \otimes y,$$

$$b_K(a \otimes x^p) = (ax + (-1)^p xa) \otimes x^{p-1},$$

for any a, a' in A, and $p \geq 3$. Decomposing coefficients in the basis of A, the rather long calculations leading to the next proposition are routine. They are left to the reader.

Proposition 9.3 *The Koszul homology of A is given by*

1. *$HK_0(A)$ is 4-dimensional, generated by the classes of 1, x, y and xy,*

2. *$HK_1(A)$ is 3-dimensional, generated by the classes of $1 \otimes x$, $1 \otimes y$ and $y \otimes y$,*

3. *$HK_2(A)$ is 3-dimensional, generated by the classes of $x \otimes x^2$, $yx \otimes x^2 + (xy + yx) \otimes (y^2 - xy)$ and $xyx \otimes (y^2 - xy)$,*

4. *for any $p \geq 3$ odd (resp. even), $HK_p(A)$ is 1-dimensional, generated by the class of $1 \otimes x^p$ (resp. $x \otimes x^p$).*

We recall from Subsection 5.2 that $\partial_\frown$ is defined by the formula

$$\partial_\frown([a \otimes x_1 \dots x_p]) = [ax_1 \otimes x_2 \dots x_p]$$

for any cycle $a \otimes x_1 \dots x_p$ in $A \otimes W_p$. In particular,

$$\partial_\frown([1 \otimes x]) = [x],\ \partial_\frown([1 \otimes y]) = [y],\ \partial_\frown([y \otimes y]) = [xy],$$
$$\partial_\frown([x \otimes x^2]) = 0,\ \partial_\frown([xyx \otimes (y^2 - xy)]) = 0,$$
$$\partial_\frown([yx \otimes x^2 + (xy + yx) \otimes (y^2 - xy)]) = [-xyx \otimes y] = [b_K(xy \otimes (y^2 - xy))] = 0,$$
$$\partial_\frown([1 \otimes x^p]) = [x \otimes x^{p-1}] \text{ if } p \text{ odd} \geq 3,$$
$$\partial_\frown([x \otimes x^p]) = 0 \text{ if } p \text{ even} \geq 4.$$

Therefore, we obtain the following.

Proposition 9.4 *The higher Koszul homology of A is given by*

1. *$HK_0^{hi}(A) \cong k$,*

2. *$HK_1^{hi}(A) \cong 0$,*

3. *$HK_2^{hi}(A)$ is 2-dimensional, generated by the classes of $[yx \otimes x^2 + (xy+yx) \otimes (y^2 - xy)]$ and $[xyx \otimes (y^2 - xy)]$,*

4. *$HK_p^{hi}(A) \cong 0$ for any $p \geq 3$.*

The next proposition shows that A satisfies Conjecture 6.6.

Proposition 9.5 *The homology of the complex $K_\ell(A)$ is given by*

1. *$H_0(K_\ell(A)) \cong k$,*

2. *$H_1(K_\ell(A)) \cong 0$,*

3. *$H_2(K_\ell(A))$ is 2-dimensional, generated by the classes of $yx \otimes (y^2 - xy)$ and $xyx \otimes (y^2 - xy)$,*

4. *$H_p(K_\ell(A)) \cong 0$ for any $p \geq 3$.*

Proof. The complex $K_\ell(A)$ is the following

$$\dots \xrightarrow{d_\ell} A \otimes x^4 \xrightarrow{d_\ell} A \otimes x^3 \xrightarrow{d_\ell} A \otimes R \xrightarrow{d_\ell} A \otimes V \xrightarrow{d_\ell} A \longrightarrow 0, \tag{9.7}$$

where the maps d_ℓ are successively given by

$$d_\ell(a \otimes x) = ax \text{ and } d_\ell(a' \otimes y) = a'y,$$
$$d_\ell(a \otimes x^2) = ax \otimes x \text{ and } d_\ell(a' \otimes (y^2 - xy)) = a'(y - x) \otimes y,$$
$$d_\ell(a \otimes x^p) = ax \otimes x^{p-1},$$

for any a, a' in A, and $p \geq 3$. The homology is then easily computed by writing coefficients in terms of the basis of A. ∎

9.3 Koszul cohomology of A

Recall that for any finite dimensional vector space E, the linear map $can : A \otimes E^* \to Hom(E, A)$ defined by $can(a \otimes u)(x) = u(x)a$ for any a in A, u in E^* and x in E, is an isomorphism. The inverse isomorphism sends any $f : E \to A$ to $\sum_{i \in I} f(e_i) \otimes e_i^*$, where $(e_i)_{i \in I}$ is a basis of E and $(e_i^*)_{i \in I}$ is its dual basis. Using this, define the isomorphism of complexes

$$can : A \otimes W_\bullet^* \cong Hom(W_\bullet, A).$$

The differential of $A \otimes W_\bullet^*$ is obtained by carrying the differential b_K of $Hom(W_\bullet, A)$, and is still denoted by b_K.

The dual basis of V^* corresponding to the basis (x, y) of V is (x^*, y^*). Denote by x^{*2} the restriction to R of the linear form $x^* \otimes x^*$ on $V \otimes V$, and analogously for x^*y^*, y^*x^* and y^{*2}. Clearly x^{*2} and y^{*2} form a basis of R^*, and we have the following relations in R^*:

$$x^*y^* = -y^{*2},\ y^*x^* = 0.$$

For any $p \geq 3$, denote by x^{*p} the restriction to W_p of the linear form $x^{*\otimes p}$ on $V^{\otimes p}$, so that W_p^* is generated by x^{*p}.

The homology of the complex

$$0 \longrightarrow A \xrightarrow{b_K} A \otimes V^* \xrightarrow{b_K} A \otimes R^* \xrightarrow{b_K} A \otimes x^{*3} \xrightarrow{b_K} A \otimes x^{*4} \xrightarrow{b_K} \dots \tag{9.8}$$

is $HK^\bullet(A)$.

Lemma 9.6 *The differential of the complex (9.8) can be written as follows.*

1. $b_k : A \to A \otimes V^*$ *is given by*

$$b_K(a) = (ax - xa) \otimes x^* + (ay - ya) \otimes y^*,$$

2. $b_k : A \otimes V^* \to A \otimes R^*$ *is given by*

$$b_K(a \otimes x^*) = (ax + xa) \otimes x^{*2} - ay \otimes y^{*2},\ b_K(a' \otimes y^*) = (a'y + ya' - xa') \otimes y^{*2},$$

3. $b_k : A \otimes R^* \to A \otimes x^{*3}$ *is given by*

$$b_K(a \otimes x^{*2}) = (ax - xa) \otimes x^{*3},\ b_K(a' \otimes y^{*2}) = 0,$$

4. $b_k : A \otimes x^{*p} \to A \otimes x^{*p+1}$ *is given by*

$$b_K(a \otimes x^{*p}) = (ax - (-1)^p xa) \otimes x^{*p+1},$$

for any a in A, a' in A and $p \geq 3$.

Proof.

1. Since $b_K(a)(v) = av - va$ for any v in V, $can^{-1}(b_K(a)) = (ax - xa) \otimes x^* + (ay - ya) \otimes y^*$.

2. From $can(a \otimes x^*) = f$, where $f(x) = a$ and $f(y) = 0$, draw $b_K(f)(x^2) = ax + xa$ and $b_K(f)(y^2 - xy) = -ay$; next $can^{-1}(b_K(f)) = (ax + xa) \otimes x^{*2} - ay \otimes y^{*2}$. From $can(a' \otimes y^*) = g$, where $g(x) = 0$ and $g(y) = a'$, draw $b_K(g)(x^2) = 0$ and $b_K(g)(y^2 - xy) = a'y + ya' - xa'$; next $can^{-1}(b_K(g)) = (a'y + ya' - xa') \otimes y^{*2}$.

The proof of the other formulas are obtained in the same manner. ■

From this lemma, the routine calculations leading to the next proposition are left to the reader.

Proposition 9.7 *The Koszul cohomology of A is given by*

1. *$HK^0(A)$ is 2-dimensional, generated by 1 and xyx,*
2. *$HK^1(A)$ is 2-dimensional, generated by the classes of $x \otimes x^* + y \otimes y^* \cong e_A$ and $xy \otimes y^*$,*
3. *$HK^2(A)$ is 4-dimensional, generated by the classes of $1 \otimes x^{*2}$, $1 \otimes y^{*2}$, $y \otimes y^{*2}$ and $xyx \otimes y^{*2}$,*
4. *for any $p \geq 3$ odd (resp. even), $HK^p(A)$ is 1-dimensional, generated by the class of $x \otimes x^{*p}$ (resp. $1 \otimes x^{*p}$).*

From Subsection 3.5, recall that

$$\partial_\smile([f]) = [x_1 \dots x_{p+1} \mapsto f(x_1 \dots x_p)x_{p+1}]$$

for any cocycle $f : W_p \to A$, and any $x_1 \dots x_{p+1}$ in W_{p+1}. Using the isomorphism can, we get

$$\partial_\smile(1) = [x \otimes x^* + y \otimes y^*],\ \partial_\smile(xyx) = 0,$$
$$\partial_\smile([x \otimes x^* + y \otimes y^*]) = 0,\ \partial_\smile([xy \otimes y^*]) = 0,$$
$$\partial_\smile([1 \otimes x^{*2}]) = [x \otimes x^{*3}],\ \partial_\smile([1 \otimes y^{*2}]) = \partial_\smile([y \otimes y^{*2}]) = \partial_\smile([xyx \otimes y^{*2}]) = 0,$$
$$\partial_\smile([x \otimes x^{*p}]) = 0 \text{ if } p \text{ odd} \geq 3,$$
$$\partial_\smile([1 \otimes x^{*p}]) = [x \otimes x^{*p+1}] \text{ if } p \text{ even} \geq 4.$$

Proposition 9.8 *The higher Koszul cohomology of A is given by*

1. *$HK^0_{hi}(A)$ is 1-dimensional, generated by the class of xyx,*
2. *$HK^1_{hi}(A)$ is 1-dimensional, generated by the class of $[xy \otimes y^*]$,*
3. *$HK^2_{hi}(A)$ is 3-dimensional, generated by the classes of $[1 \otimes y^{*2}]$, $[y \otimes y^{*2}]$ and $[xyx \otimes y^{*2}]$,*
4. *$HK^p_{hi}(A) \cong 0$ for any $p \geq 3$.*

The next proposition shows that the cup and cap products are graded commutative in the Koszul (co)homology of A. We do not know whether this fact holds for any quadratic algebra.

Proposition 9.9 *The algebra* $(HK^\bullet(A), \underset{K}{\smile})$ *is graded commutative. The* $(HK^\bullet(A), \underset{K}{\smile})$*-bimodule* $HK_\bullet(A)$ *is graded symmetric for the actions* $\underset{K}{\frown}$*. In particular, the same properties hold in higher Koszul (co)homology.*

Proof. We work in cohomology. According to Corollary 3.11, it suffices to multiply a p-class and a q-class when $p \geq 2$ and $q \geq 2$. Carrying $\underset{K}{\smile}$ by the isomorphism *can*, we obtain a cup product $\underset{K}{\smile}$ on $A \otimes W_\bullet^*$. For example,

$$(a \otimes x^{*2} + a' \otimes y^{*2}) \underset{K}{\smile} (b \otimes x^{*2} + b' \otimes y^{*2}) = ab \otimes x^{*4}.$$

Thus all the products of the generators of $HK^2(A)$ given in Proposition 9.7 vanish, except for the square of $1 \otimes x^{*2}$ which is equal to $1 \otimes x^{*4}$. Similarly, the products of $1 \otimes y^{*2}$, $y \otimes y^{*2}$ or $xyx \otimes y^{*2}$ with the generator of $HK^p(A)$ ($p \geq 3$) vanish, while $1 \otimes x^{*2}$ commutes with this generator. The generators of $HK^p(A)$, when $p \geq 3$ is varying, are (graded) commuting.

In homology, it suffices – using Corollary 4.7 – to make a cohomology p-class act on a homology q-class when $p \geq 2$, $q \geq 2$ and $p < q$. Carrying $\underset{K}{\frown}$ by the isomorphism *can*, we obtain actions $\underset{K}{\frown}$ of $A \otimes W_\bullet^*$ on $A \otimes W_\bullet$. For example, for any $q > 2$,

$$(a \otimes x^{*2} + a' \otimes y^{*2}) \underset{K}{\frown} (b \otimes x^q) = ab \otimes x^{q-2} = (a \otimes x^q) \underset{K}{\frown} (b \otimes x^{*2} + b' \otimes y^{*2}).$$

From this, we see that $1 \otimes y^{*2}$, $y \otimes y^{*2}$ and $xyx \otimes y^{*2}$ act on $HK_q(A)$ by zero, and that $1 \otimes x^{*2}$ acts symmetrically. The generator of $HK^p(A)$ acts (graded) symmetrically on the generator of $HK_q(A)$ for any $p, q \geq 3$. ■

In higher Koszul cohomology, the products of two biweight homogeneous classes vanish, except for

$$[xyx] \underset{K}{\smile} [[1 \otimes y^{*2}]] = [[1 \otimes y^{*2}]] \underset{K}{\smile} [xyx] = [[xyx \otimes y^{*2}]].$$

Examining the possible biweights, we see that the higher Koszul cohomology of A acts on the higher Koszul homology of A by zero. We do not know whether the latter fact stands for any quadratic algebra.

9.4 Hochschild (co)homology of A

Apart from standard examples including Koszul algebras, it is difficult to compute the Hochschild (co)homology of an associative algebra given by generators and relations. The bar resolution is too large and, if the algebra is graded, a construction of the minimal projective resolution is too hard to perform in general. Fortunately, in case of monomial relations, Barzell's resolution provides a minimal projective resolution whose calculation is tractable. The differential and the contracting homotopy of Barzell's resolution are simultaneously defined in homological degree p from $(p-1)$-ambiguities. The ambiguities are monomials simply defined from the well-chosen reduction system $\mathcal{R}$ defining the algebra.

The third author and Chouhy have extended Bardzell's resolution to any algebra, not necessarily graded, defined by relations on a finite quiver [3]. The first step consists in well-choosing a reduction system $\mathcal{R}$ of the algebra. Their resolution is in some sense a deformation of Bardzell's one. The bimodules of their resolution are free, and the free bimodule in homological degree p is generated by the $(p-1)$-ambiguities of the associated monomial algebra. The differential and the contracting homotopy are simultaneously defined by induction on p. See [3] for details. Guiraud, Hoffbeck and Malbos [9] have constructed a resolution which may be related to the construction of [3].

If the algebra is quadratic and not Koszul, the bar resolution remains too large and the minimal projective resolution remains too hard to compute in general. It is the case of our favorite non-Koszul algebra A. The resolution defined in [3] is now applied to this particular algebra A in order to compute certain spaces of the Hochschild (co)homology of A. Denote this resolution by $S(A)$ and its differential by d. We do not give the details of the construction of $S(A)$ and d.

The construction of $S(A)$ starts with $x < y$, the corresponding deglex order on the monomials in x and y, and the reduction system

$$\mathcal{R} = \{x^2, y^2 - xy, yxy\}.$$

We obtain that $S(A) = \bigoplus_{p\geq 0} A\otimes k.S_p \otimes A$, where $k.S_p$ denotes the k-vector space generated by the set S_p. Explicitely, $S_0 = \{1\}$, $S_1 = \{x, y\}$ and $S_2 = \{x^2, y^2, yxy\}$ – denoted by S in [3]. For each $p \geq 3$, S_p is the set of the $(p-1)$-ambiguities defined by S_2. The p-ambiguities are the monomials obtained as minimal proper superpositions of p elements of S_2. For example, $S_3 = \{x^3, y^3, yxy^2, y^2xy, yxyxy\}$ and

$$S_4 = \{x^4, y^4, yxy^3, y^3xy, y^2xy^2, yxy^2xy, y^2xyxy, yxyxy^2, yxyxyxy\}.$$

The differential d is defined in S_2 by $d(1\otimes x^2 \otimes 1) = x\otimes x\otimes 1 + 1\otimes x\otimes x$, and

$$d(1\otimes y^2 \otimes 1) = y\otimes y\otimes 1 + 1\otimes y\otimes y - x\otimes y\otimes 1 - 1\otimes x\otimes y,$$

$$d(1\otimes yxy \otimes 1) = yx\otimes y\otimes 1 + y\otimes x\otimes y + 1\otimes y\otimes xy.$$

For any $p \geq 3$, x^p belongs to S_p and $d(1\otimes x^p \otimes 1) = x\otimes x^{p-1}\otimes 1 + (-1)^p 1\otimes x^{p-1}\otimes x$. Therefore, the morphism of graded A-bimodules $\chi : K(A) \to S(A)$ defined by the identity map on the generators of all the spaces W_p, except $y^2 - xy$ which is sent to y^2, is a morphism of complexes, allowing us to view $K(A)$ as a (non-acyclic!) subcomplex of the resolution $S(A)$.

Applying the functors $A\otimes_{A^e} -$ and $Hom_{A^e}(-, A)$ to χ, we get the morphisms of complexes $\tilde{\chi} : A\otimes W_\bullet \to A\otimes k.S_\bullet$ and $\chi^* : Hom(k.S_\bullet, A) \to Hom(W_\bullet, A)$, inducing the graded linear maps

$$H(\tilde{\chi}) : HK_\bullet(A) \to HH_\bullet(A) \text{ and } H(\chi^*) : HH^\bullet(A) \to HK^\bullet(A).$$

We know that $H(\tilde{\chi})_p$ and $H(\chi^*)_p$ are isomorphisms for $p = 0$ and $p = 1$. The proof of the following is omitted; it lies on rather long computations.

Proposition 9.10 *Let A be the algebra considered in this section,*

- *$H(\tilde{\chi})_2 : HK_2(A) \to HH_2(A)$ is an isomorphism.*
- *$HH_3(A)$ is 3-dimensional, generated by the classes of $1\otimes x^3$, $y\otimes y^3 + 1\otimes yxy^2$ and $xy\otimes y^3 + y\otimes yxy^2$. Moreover, $H(\tilde{\chi})_3 : HK_3(A) \to HH_3(A)$ sends $[1\otimes x^3]$ to itself. In particular, $H(\tilde{\chi})_3$ is injective and not surjective.*

- *$HH^2(A)$ is 2-dimensional, generated by the classes of $1 \otimes x^{*2} + y \otimes y^* x^* y^*$ and $x \otimes x^{*2} - y \otimes y^{*2}$. Moreover, $H(\chi^*)_2 : HH^2(A) \to HK^2(A)$ sends the first one to the class of $1 \otimes x^{*2}$, and the second one to the class of $-\frac{1}{2} y \otimes y^{*2}$. In particular, $H(\chi^*)_2$ is injective and not surjective.*

- *$HH^3(A)$ is 1-dimensional, generated by the class of $xyx \otimes y^* x^* y^{*2} + xyx \otimes y^{*2} x^* y^*$. Moreover, $H(\chi^*)_3 = 0$.*

References

[1] R. Berger, Weakly confluent quadratic algebras, *Algebras and Representation Theory* **1** (1998), 189-213.

[2] R. Berger, Gerasimov's theorem and N-Koszul algebras, *J. London Math. Soc.* **79** (2009), 631-648.

[3] S. Chouhy, A. Solotar, Projective resolutions of associative algebras and ambiguities, *J. Algebra* **432** (2015), 22-61.

[4] V. N. Gerasimov, Free associative algebras and inverting homomorphisms of rings, *Amer. Math. Soc. Transl.* **156** (1993), 1-76.

[5] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math.* **78** (1963), 267-288.

[6] M. Gerstenhaber, S. D. Schack, Simplicial cohomology in Hochschild cohomology, *J. of Pure Appl. Algebra* **30** (1983), 143-156.

[7] V. Ginzburg, Calabi-Yau algebras, arXiv:math.AG/0612139v3.

[8] T. G. Goodwillie, Cyclic homology, derivations and the free loop space, *Topology* **24** (1985), 187-215.

[9] Y. Guiraud, E. Hoffbeck, P. Malbos, Linear polygraphs and Koszulity of algebras, arXiv:1406.0815.

[10] T. Lambre, Dualité de Van den Bergh et structure de Batalin-Vilkoviskiĭ sur les algèbres de Calabi-Yau, *J. Noncommut. Geom.* **4** (2010), 441-457.

[11] J-L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften **301**, Second Edition, Springer, 1998.

[12] J-L. Loday, B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften **346**, Springer, 2012.

[13] Yu. I. Manin, *Quantum groups and non-commutative geometry*, CRM, Université de Montréal, 1988.

[14] A. Polishchuk, L. Positselski, *Quadratic algebras*, University Lecture Series 37, AMS, 2005.

[15] S. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.* **152** (1970), 39-60.

[16] G. S. Rinehart, Differential forms on general commutative algebras, *Trans. Amer. Math. Soc.* **108** (1963), 195-222.

[17] D. Tamarkin, B. Tsygan, The ring of differential operators on forms in noncommutative calculus, *Proc. Sympos. Pure Math.* **73**, Amer. Math. Soc. (2005), 105-131.

[18] M. Van den Bergh, Non-commutative homology of some three-dimensional quantum spaces, *K-Theory* **8** (1994), 213-220.

[19] M. Van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, *Proc. Amer. Math. Soc.* **126** (1998), 1345-1348; erratum *ibid.* **130** (2002), 2809-2810.

[20] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.

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