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Twistings and Hopf Galois Extensions[1]

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Let k be a commutative ring, let H be a k-Hopf algebra, and let A be a right H-comodule algebra. A twisting of A is a map τ: $H \otimes A \to A$ such that $(A, *_\tau, \rho_A)$ is also an H-comodule algebra, where the product $*_\tau$ is defined by $a *_\tau b = \Sigma a_0 \tau(a_1 \otimes b)$. In this note, we observe that there is a map of pointed sets from the twistings of A to the H-measurings from $A^{\mathrm{co}\,H}$ to A and study the set of twistings that map to the trivial measuring. If $A/A^{\mathrm{co}\,H}$ is Galois and H is finitely generated projective, then the twistings that map to the trivial measuring can be described as a set of invertible twisted cocycles: φ: $H \otimes H \to A$. An equivalence relation on the set of twisted cocycles corresponds to isomorphism classes of Galois extensions.

INTRODUCTION

If H is a finitely generated projective cocommutative Hopf algebra over a commutative ring k, then it is well known that the isomorphism classes of Galois H-objects A/k with A isomorphic to H as an H-comodule form an Abelian group via the cotensor product, and, furthermore, this group is isomorphic to the second Sweedler cohomology group $H^2(H, k)$

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[12]. In [6] the inclusion of this subgroup in $\mathcal{G}al(H,k)$, the group of isomorphism classes of all Galois H-objects A/k, is shown to be the beginning of an exact sequence. Caenepeel [3] generalized this sequence to the situation when H is cocommutative and faithfully flat over k but not finitely generated.

In this paper, we consider Galois H-objects $A/A^{\mathrm{co}\,H}$ where H is finitely generated projective over k, but we do not assume H is cocommutative, $A^{\mathrm{co}\,H}$ is commutative, or $A \cong A^{\mathrm{co}\,H} \otimes H$ as H-comodules. The possible Galois structures on the H-comodule algebra A correspond to the twistings of A in the sense of [1], and any twisting of A induces an H-measuring of $A^{\mathrm{co}\,H}$ to A. Therefore, we may use the idea of twistings to describe the Galois H-objects $C = (A, \times, \rho_A)$ corresponding to the trivial measuring in terms of a set of "twisted cocycles." If the image of a twisted cocycle lies in $Z(A)$, the centre of A, then the twisted cocycle is a Sweedler cocycle with trivial weak action and satisfying a condition on the coaction.

1. PRELIMINARIES

We work over a commutative ring k and assume that all maps are k-linear. Throughout, H will denote a k-Hopf algebra with bijective antipode S. The composition inverse of S is denoted $\overline{S}$. Until the final section, we do not assume that H is finitely generated.

Let A be a right H-comodule algebra; i.e., A is an algebra in the category $\mathcal{M}^H$ of right H-comodules. We denote by ρ_A (or just ρ if the context makes the meaning clear) the comodule structure map from A to $A \otimes H$.

For C a coalgebra and A an algebra, $\mathrm{Hom}(C, A)$ is an algebra with the convolution product $*$. We write $\mathrm{Reg}(C, A)$ for the convolution invertible elements in $\mathrm{Hom}(C, A)$.

Definition of $\mathcal{T}(A)$, the Set of Twistings of A

Let τ be a map from $H \otimes A$ to A satisfying the normality conditions

$$\tau(1 \otimes a) = a, \qquad \tau(h \otimes 1) = \epsilon(h)1_A \tag{1.1}$$

for all $a \in A$, $h \in H$. Then A^τ is defined to be the H-comodule A with (possibly nonassociative) multiplication $*_\tau$ defined by

$$a *_\tau b = \sum a_0 \tau(a_1 \otimes b)$$

for $a, b \in A$. The normality conditions (1.1) ensure that 1_A is a multiplicative identity for $*_\tau$. Also, if $M \in \mathcal{M}_A^H$, then M^τ is defined to be the

H-comodule M together with the map $M \otimes A \to M$ defined by $m \otimes a \mapsto m *_\tau a = \Sigma m_0 \tau(m_1 \otimes a)$. We often will omit the subscript τ when the meaning is clear. If F_τ defined by $F_\tau(M) = M^\tau$, $F_\tau(f) = f$, is a functor from $\mathscr{M}_A^H$ to $\mathscr{M}_{A^\tau}^H$ then τ is called a twisting map. Then $(A^\tau, *_\tau, \rho_A)$ is an H-comodule algebra and A^τ is called a twisting of A.

There is also a left version of the twistings described above. Suppose ν: $H \otimes A \to A$ satisfies (1.1) and let ${}^\nu A$ denote the H-comodule A with (possibly nonassociative) multiplication $*_\nu$ defined by

$$a *_\nu b = \sum \nu(b_1 \otimes a) b_0.$$

Also for $M \in {}_A\mathscr{M}^H$, let ${}^\nu M$ be the H-comodule M together with the map from $A \otimes M \to M$ given by $a \otimes m \mapsto \Sigma \nu(m_1 \otimes a) m_0 = a *_\nu m$. If $(A, *_\nu, \rho_A)$ is an H-comodule algebra and F_ν: ${}_A\mathscr{M}^H \to {}_{{}^\nu A}\mathscr{M}^H$, $F_\nu(M) = {}^\nu M$, $F(f) = f$, is a functor, then ν is called a left twisting of A.

Proposition 1.1. (i) *A map τ: $H \otimes A \to A$ satisfying* (1.1) *is a twisting if and only if for all $h \in H$, $a, b \in A$;*

$$\sum (1 \otimes h_1) \rho(\tau(h_2 \otimes a)) = \sum \tau(h_1 \otimes a_0) \otimes h_2 a_1; \tag{1.2}$$

$$\tau(h \otimes a *_\tau b) = \sum \tau(h_1 \otimes a_0) \tau(h_2 a_1 \otimes b). \tag{1.3}$$

(ii) *A map ν: $H \otimes A \to A$ satisfying* (1.1) *is a left twisting if and only if for all $h \in H$, $a, b \in A$,*

$$\sum \rho(\nu(h_2 \otimes a))(1 \otimes h_1) = \sum \nu(h_1 \otimes a_0) \otimes a_1 h_2; \tag{1.4}$$

$$\nu(h \otimes a *_\nu b) = \sum \nu(b_1 h_2 \otimes a) \nu(h_1 \otimes b_0). \tag{1.5}$$

Proof. This is proved in [1, Theorem 1.1 and Proposition 2.1]. ∎

It is straightforward to verify that for τ a twisting of A, and for ν a left twisting of A,

$$\rho_A\Big(\sum \tau\big(\bar{S}(a_1) \otimes a_0\big)\Big) = \sum \tau\big(\bar{S}(a_1) \otimes a_0\big) \otimes a_2; \tag{1.6}$$

$$\rho_A\Big(\sum \nu\big(S(a_1) \otimes a_0\big)\Big) = \sum \nu\big(S(a_1) \otimes a_0\big) \otimes a_2. \tag{1.7}$$

These equations will be useful later.

Example 1.2. Let $u(h \otimes a) = \epsilon(h) a$ for all $a \in A$, $h \in H$. This map u is a twisting and $a *_u b = \Sigma a_0 \epsilon(a_1) b = ab$, so that $A^u = A$. Similarly u is a left twisting and ${}^u A = A$. We call $u = \epsilon \otimes \mathrm{id}$ the identity twisting.

In [1], a twisting is viewed as a map from H to $\mathrm{End}(A)$. If τ: $H \otimes A \to A$ is a twisting, we define $\tau' \in \mathrm{Hom}(H, \mathrm{End}(A))$ by $\tau'(h) a = \tau(h \otimes a)$.

DEFINITION 1.3. Let τ be a (left) twisting of A. If $\tau' \in \mathrm{Reg}(H, \mathrm{End}(A))$, then τ is called an invertible (left) twisting.

Note that if τ is an invertible twisting with λ' the convolution inverse to τ', then λ is a twisting of A^τ, not of A. However, (1.2) and (1.6) still hold for λ.

Recall from [1] that if τ is an invertible (left) twisting, then the functor F_τ from $\mathscr{M}_A^H$ to $\mathscr{M}_{A^\tau}^H$ (F_τ from ${}_A\mathscr{M}^H$ to ${}_{{}^\tau A}\mathscr{M}^H$) is an isomorphism of categories.

For a given right H-comodule algebra A, let $\mathscr{T}(A)$ denote the set of twistings on A and $\mathscr{L}(A)$ the set of left twistings on A. Let $\mathscr{U}(\mathscr{T}(A))$ and $\mathscr{U}(\mathscr{L}(A))$ be the sets of invertible twistings and invertible left twistings on A. Since H has a bijective antipode, the following lemma shows that there is a bijection of pointed sets between $\mathscr{U}(\mathscr{T}(A))$ and $\mathscr{U}(\mathscr{L}(A))$.

LEMMA 1.4. *For $\tau \in \mathscr{U}(\mathscr{T}(A))$, with λ' the convolution inverse to τ', define $l(\tau)$: $H \otimes A \to A$ by*

$$l(\tau)(h \otimes a) = \sum \tau'\big(\bar{S}(a_2 h)\big)\lambda'\big(\bar{S}(a_1)\big)(a_0).$$

For $\nu \in \mathscr{U}(\mathscr{L}(A))$, with μ' the convolution inverse to ν', define $r(\nu)$: $H \otimes A \to A$ by

$$r(\nu)(h \otimes a) = \sum \nu'(S(ha_2))(\mu'(S(a_1))(a_0)).$$

Then $l(\tau) \in \mathscr{U}(\mathscr{L}(A))$, $r(\nu) \in \mathscr{U}(\mathscr{T}(A))$, $r(l(\tau)) = \tau$, and $l(r(\nu)) = \nu$. Furthermore, for $\tau \in \mathscr{U}(\mathscr{T}(A))$, the H-comodule algebras A^τ and ${}^{l(\tau)}A$ are isomorphic. For u the identity twisting, $r(u) = l(u) = u$.

Proof. It is shown in the proof of [1, Theorem 2.3] that $l(\tau) \in \mathscr{U}(\mathscr{L}(A))$ with the convolution inverse to $l(\tau)'$ being μ' defined by

$$\mu'(h)(a) = \sum \tau'\big(\bar{S}\big(a_2 h_3 \bar{S}(h_1)\big)\big)\lambda'\big(\bar{S}(a_1 h_2)\big)(a_0).$$

Therefore,

$$\begin{aligned}
&r(l(\tau))(h \otimes a)\\
&\quad = \sum l(\tau)'(S(ha_2))(\mu'(S(a_1))(a_0))\\
&\quad = \sum l(\tau)'(S(ha_6))\tau'\big(\bar{S}\big(a_2 S(a_3)\bar{S}S(a_5)\big)\big)\lambda'\big(\bar{S}(a_1 S(a_4))\big)(a_0)\\
&\quad = \sum l(\tau)'(S(ha_4))\tau'\big(\bar{S}(a_3)\big)\lambda'\big(\bar{S}(a_1 S(a_2))\big)(a_0)\\
&\quad = \sum l(\tau)'(S(ha_2))\big(\tau'\big(\bar{S}(a_1)\big)(a_0)\big)\\
&\quad = \sum \tau'\big(\bar{S}(a_3 S(ha_4))\big)\lambda'\big(\bar{S}(a_2)\big)\tau'\big(\bar{S}(a_1)\big)(a_0) \qquad \text{by (1.6)}\\
&\quad = \sum \tau'\big(ha_2\bar{S}(a_1)\big)(a_0) \qquad \text{since } \lambda', \tau' \text{ are inverse}\\
&\quad = \sum \tau'(h)(a) = \tau(h \otimes a).
\end{aligned}$$

Also, in [1, Theorem 2.3], it is shown that for $\nu \in \mathscr{U}(\mathscr{L}(A))$, with μ' the convolution inverse to ν', $r(\nu)$ is a twisting. It is straightforward to verify that the convolution inverse to $r(\nu)'$ is given by λ',

$$\lambda'(h)(a) = \sum \nu'(S(S(h_1)h_3 a_2))(\mu'(S(h_2 a_1))(a_0)).$$

Then

$$\begin{aligned}
l(r(\nu))(h \otimes a) &\\
&= \sum r(\nu)'(\bar{S}(a_2 h))\lambda'(\bar{S}(a_1))(a_0)\\
&= \sum r(\nu)'(\bar{S}(a_6 h))\nu'\big(S(S(\bar{S}(a_5))\bar{S}(a_3)a_2)\big)\mu'\big(S(\bar{S}(a_4)a_1)\big)(a_0)\\
&= \sum r(\nu)'(\bar{S}(a_2 h))(\nu'(S(a_1))(a_0))\\
&= \sum \nu'\big(S(\bar{S}(a_4 h)a_3)\big)\mu'(S(a_2))(\nu'(S(a_1))(a_0))\\
&= \sum \nu'(S(a_1)a_2 h)(a_0) \qquad \text{since } \mu', \nu' \text{ are inverse}\\
&= \nu(h \otimes a).
\end{aligned}$$

The isomorphism from ${}^{l(\tau)}A$ to A^{τ} is found in [1, Theorem 2.3] and the final statement is clear. ∎

For more detail on twistings of H-comodule algebras, see the definitions and basic results in [1]. The motivating paper for [1] was [13] where H is a group or a semigroup algebra. The literature contains many different definitions of twisted objects; a discussion of these various concepts can be found in [7].

Smash Products $\#(H, A)$ and $\#^{op}(H, A)$

In [8], a twisting is regarded as a map from A to $\mathrm{Hom}(H, A)$. For τ: $H \otimes A \to A$, we define $\tau'' \in \mathrm{Hom}(A, \mathrm{Hom}(H, A))$ by $\tau''(a) = \tau_a$ where $\tau_a(h) = \tau(h \otimes a)$. It will be convenient to think of a twisting as τ, τ', or τ'' depending on the context. Besides the convolution product, $\mathrm{Hom}(H, A)$ is an algebra via a smash product or opposite smash product.

Denote by $\#(H, A)$ the k-module $\mathrm{Hom}(H, A)$ with associative multiplication given by

$$(f \cdot g)(h) = \sum f(g(h_2)_1 h_1) g(h_2)_0 \tag{1.8}$$

for $f, g \in \mathrm{Hom}(H, A)$, $h \in H$. Also $\#^{\mathrm{op}}(H, A)$ is the k-module $\mathrm{Hom}(H, A)$ with associative multiplication

$$(f \cdot g)(h) = \sum f(h_2)_0 g(h_1 f(h_2)_1). \tag{1.9}$$

(We denote multiplication in both $\#(H, A)$ and $\#^{\text{op}}(H, A)$ by $\cdot$; the meaning will be clear from the context.) The map $h \mapsto \epsilon(h)1_A$ is the identity in both $\#^{\text{op}}(H, A)$ and $\#(H, A)$, and A embeds as a subalgebra of either $\#(H, A)$ or $\#^{\text{op}}(H, A)$ by

$$\alpha_A \colon A \to \text{Hom}(H, A), \qquad \alpha_A(a)(h) = \epsilon(h)a. \tag{1.10}$$

Also H^* embeds as an algebra in $\#(H, A)$ and $H^{*\text{op}}$ embeds in $\#^{\text{op}}(H, A)$ by regarding maps from H to k as maps from H to A; i.e.,

$$\gamma \colon \text{Hom}(H, k) \to \text{Hom}(H, A), \qquad \gamma(h^*)(h) = h^*(h)1_A. \tag{1.11}$$

Finally, ev, evaluation at 1_H, maps either $\#(H, A)$ or $\#^{\text{op}}(H, A)$ to A by

$$ev \colon \text{Hom}(H, A) \to A, \qquad ev(f) = f(1). \tag{1.12}$$

Let $\leftharpoonup$ denote the usual right action of H on H^*, namely $(h^* \leftharpoonup h)(l) = h^*(hl)$. Note that the smash product $A\#H^*$ with multiplication $(a\#h^*)(b\#l^*) = \sum ab_0\#(h^* \leftharpoonup b_1)l^*$ is the subalgebra of $\#(H, A)$ generated by $\alpha(A)$ and $\gamma(H^*)$. If H is finitely generated projective over k, then $\#(H, A) = A\#H^*$. For more detail on these maps, see [8].

H-Galois Objects and Crossed Products

Finally, recall that $A/A^{\text{co}\,H}$ is called an H-Galois object if the canonical map can: $A \otimes_{A^{\text{co}\,H}} A \to A \otimes H$ defined by $\text{can}(a \otimes b) = \sum ab_0 \otimes b_1$ is a bijection. Since H has bijective antipode, can is bijective if and only if can$'$: $A \otimes_{A^{\text{co}\,H}} A \to A \otimes H$, defined by $\text{can}'(a \otimes b) = \sum a_0 b \otimes a_1$, is bijective.

Crossed products $A = B\#_\sigma H$ are well-known examples of H-comodule algebras with $\rho_A = 1 \otimes \Delta$ and, if σ is invertible, they are H-Galois objects. Given a weak action $\cdot$ of H on B (i.e., an H-measuring from B to B) and a map $\sigma \in \text{Hom}(H \otimes H, B)$ such that for all $h, k, m \in H$, $b \in B$,

$$\sum h_1 \cdot (k_1 \cdot b)\sigma(h_2, k_2) = \sum \sigma(h_1, k_1)(h_2 k_2) \cdot b \tag{1.13}$$

and

$$\sum h_1 \cdot \sigma(k_1, m_1)\sigma(h_2, k_2 m_2) = \sum \sigma(h_1, k_1)\sigma(h_2 k_2, m), \tag{1.14}$$

then one may form the crossed product $B\#_\sigma H$. We will call the map σ a Sweedler cocycle relative to the given weak action. The set of (convolution invertible) Sweedler cocycles will be denoted $Z^2(H, B)$ (respectively $\mathscr{U}(Z^2(H, B))$). Then $B\#_\sigma H$ is the k-module $B \otimes H$ with associative multiplication given by

$$(b\#h)(c\#l) = \sum b(h_1 \cdot c)\sigma(h_2, l_1)\#h_3 l_2.$$

Crossed products with σ invertible are precisely the cleft extensions of B (see [2, 5, 10]), i.e., the right H-comodule algebras A such that $A^{\mathrm{co}\,H} = B$, and there is a right H-comodule convolution invertible map γ: $H \to A$. Here $\gamma(h) = 1\#h$. Then $\sigma(h,k) = \Sigma\gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2)$ and $h \cdot b = \Sigma\gamma(h_1)b\gamma^{-1}(h_2)$ for all $h, k \in H$, $b \in B$. For more detail on Hopf Galois extensions and crossed products, see [10, Chaps. 7 and 8].

2. THE SET OF TWISTINGS OF AN H-COMODULE ALGEBRA

Now let A be a right H-comodule algebra with ring of coinvariants $B = A^{\mathrm{co}\,H} = \{a \in A \mid \rho(a) = a \otimes 1\}$ and let τ be a twisting of A. Then, since $A = A^\tau$ as H-comodules, $(A^\tau)^{\mathrm{co}\,H}$ is also the k-algebra B. Let $\mathrm{Meas}_H(B, A)$ denote the set of H-measurings from B to A.

LEMMA 2.1. *For A a right H-comodule algebra with twisting τ, the following is a commutative diagram where τ'' is an algebra map from B to the convolution algebra* $\mathrm{Hom}(H, A)$ *and from B to* $\#^{\mathrm{op}}(H, A)$:

$$\begin{array}{ccc} B & \xrightarrow{\tau''} & (\mathrm{Hom}(H,A), *) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ B & \xrightarrow{\tau''} & (\#^{\mathrm{op}}(H,A), \cdot\,) \end{array}$$

Proof. Let $a, b \in B$ and $h \in H$. Then, writing τ_a for $\tau''(a)$, we have

$$\begin{aligned} (\tau_a * \tau_b)(h) &= \sum \tau_a(h_1)\tau_b(h_2) \\ &= \sum \tau(h_1 \otimes a)\tau(h_2 \otimes b) \\ &= \tau(h \otimes a *_\tau b) \qquad \text{by (1.3)} \\ &= \tau(h \otimes ab) = \tau_{ab}(h). \end{aligned}$$

Similarly,

$$\begin{aligned} (\tau_a \cdot \tau_b)(h) &= \sum \tau_a(h_2)_0 \tau_b\big(h_1\tau_a(h_2)_1\big) \qquad \text{by (1.9)} \\ &= \sum \tau(h_3 \otimes a)\tau\big(h_1 S(h_2)h_4 \otimes b\big) \\ &= \sum \tau(h_1 \otimes a)\tau(h_2 \otimes b) = \tau_{ab}(h) \end{aligned}$$

as above. Also $\tau_1(h) = \tau(h \otimes 1) = \epsilon(h)1_A$ by (1.1). Commutativity of the diagram is obvious. ∎

Thus, for every twisting τ of A, $\tau'' \in \mathrm{Meas}_H(B, A) = \mathrm{Alg}(B, \mathrm{Hom}(H, A))$, and $\tau'' \in \mathrm{Alg}(B, \#^{\mathrm{op}}(H, A))$. If u is the identity twisting of Example 1.2, then u'' is the restriction of α_A (see (1.10)) to B. We write α_B to denote this restriction.

For $\mathscr{T}(A)$ the set of twistings of A, let Ω be the map from $\mathscr{T}(A)$ to $\mathrm{Meas}_H(B, A)$ taking τ to $\tau''|_B$. Then $\Omega(\tau) = \alpha_B$ if and only if τ restricted to $B \otimes H$ is the identity twisting. We call the set of such twistings $K(\Omega)$.

LEMMA 2.2. *$K(\Omega) = \{\tau \in \mathscr{T}(A) \mid \Omega(\tau) = \tau''|_B = \alpha_B\} = \{\tau \in \mathscr{T}(A) \mid \tau'(h) \in \mathrm{End}(A_B)$ for all $h \in H\}$.*

Proof. Suppose $\tau \in K(\Omega)$ so that $a *_\tau b = ab$ for $a \in A$, $b \in B$. Then by (1.3),

$$\tau(h \otimes ab) = \sum \tau(h_1 \otimes a_0)\tau(h_2 a_1 \otimes b) = \tau(h \otimes a)b,$$

so that $\tau'(h) \in \mathrm{End}(A_B)$ for all $h \in H$. Conversely if $\tau'\colon H \to \mathrm{End}(A_B)$, then for $b \in B$, $\tau(h \otimes b) = \tau(h \otimes 1)b = \epsilon(h)b$ by (1.1). ∎

We now define some non-identity twistings in $K(\Omega)$.

DEFINITION 2.3. We call $\varphi\colon H \otimes H \to A$ a twisted cocycle if, for all $g, h \in H$, $a \in A$,

(i) $\varphi(1, h) = \varphi(h, 1) = \epsilon(h)1_A$;

(ii) $\rho_A(\varphi(g, h)) = \sum \varphi(g_2, h_2) \otimes S(g_1)g_3 h_3 \bar{S}(h_1)$;

(iii) $\sum \varphi(g_1, a_1)a_0 \varphi(g_2 a_2, h) = \sum \varphi(g, a_2 h_2)a_0 \varphi(a_1, h_1)$.

Remark 2.4. (i) If $b \in B = A^{\mathrm{co}\, H}$ then Definition 2.3(i) and (iii) imply that $b\varphi(g, h) = \varphi(g, h)b$; i.e., $\varphi\colon H \otimes H \to C_A(B)$, the centralizer of B in A.

(ii) If H is cocommutative, then Definition 2.3(ii) is equivalent to saying that φ maps $H \otimes H$ to B, and so by the preceding remark, to $Z(B)$ the centre of B.

(iii) If $\varphi\colon H \otimes H \to Z(A)$, the centre of A, and A/B is H-Galois, then Definition 2.3(iii) is equivalent to

$$\sum \varphi(g_1, t_1)\varphi(g_2 t_2, h) = \sum \varphi(t_1, h_1)\varphi(g, t_2 h_2)$$

for all $t, g, h \in H$. This follows from the fact that since can is onto, $1 \otimes t = \sum c_k b_{k_0} \otimes b_{k_1}$ for some $c_k, b_k \in A$. Thus, here, twisted cocycles satisfy the Sweedler cocycle condition with trivial weak action.

PROPOSITION 2.5. *Let A be a right H-comodule algebra with $B = A^{\mathrm{co}\, H}$. If $\varphi\colon H \otimes H \to A$ is a twisted cocycle, then the map $\tau_\varphi = \tau$, $\tau\colon H \otimes A \to A$ defined by $\tau(h \otimes a) = \sum \varphi(h, a_1)a_0$ is a twisting of A. Furthermore $\tau'(H) \subseteq \mathrm{End}(A_B)$ and so $\Omega(\tau) = \alpha_B$.*

Proof. Since $\tau(1 \otimes a) = \sum \varphi(1, a_1)a_0 = a$ and $\tau(h \otimes 1) = \varphi(h, 1) = \epsilon(h)$, the normality conditions (1.1) are satisfied. To verify (1.2), note that

$$\begin{aligned} &\rho(\tau(h \otimes a)) \\ &\qquad = \rho\left(\sum \varphi(h, a_1)a_0\right) \\ &\qquad = \sum \varphi(h_2, a_3)a_0 \otimes S(h_1)h_3a_4\overline{S}(a_2)a_1 \qquad \text{by Definition 2.3(ii)} \\ &\qquad = \sum \tau(h_2 \otimes a_0) \otimes S(h_1)h_3a_1 \qquad \text{as required.} \end{aligned}$$

Also (1.3) holds because

$$\tau(h \otimes a *_\tau b) = \sum \varphi(h, a_1b_1)a_0 *_\tau b_0 = \sum \varphi(h, a_2b_2)a_0\varphi(a_1, b_1)b_0$$

while

$$\sum \tau(h_1 \otimes a_0)\tau(h_2a_1 \otimes b) = \sum \varphi(h_1, a_1)a_0\varphi(h_2a_2, b_1)b_0,$$

and these expressions are equal to Definition 2.3(iii). The last statement is easy to verify. ∎

Let $Z^2_{\mathrm{tw}}(H, A)$ denote the set of twisted cocycles from $H \otimes H$ to A. Then Γ, the map from $Z^2_{\mathrm{tw}}(H, A)$ to $\mathscr{T}(A)$ defined by $\Gamma(\varphi) = \tau_\varphi$, maps $Z^2_{\mathrm{tw}}(H, A)$ to $K(\Omega)$.

DEFINITION 2.6. We call a twisted cocycle φ: $H \otimes H \to A$ invertible if $\varphi' \in \mathrm{Reg}(H, \#(H, A))$, where $\varphi'(h)(g) = \varphi(h, g)$.

Note that in general $\varphi' \in \mathrm{Reg}(H, \#(H, A))$ is not equivalent to $\varphi \in \mathrm{Reg}(H \otimes H, A)$.

Remark 2.7. Let φ be a map from $H \otimes H$ to $B = A^{\mathrm{co}\,H}$ and define φ' from H to $\#(H, B)$ as above. Then since for all $h, g \in H$, $\lambda \in \mathrm{Hom}(H \otimes H, B)$,

$$\sum \varphi'(h_1) \cdot \lambda'(h_2)(g) = \sum \varphi(h_1 \otimes g_1)\lambda(h_2 \otimes g_2),$$

$\varphi \in \mathrm{Reg}(H \otimes H, B)$ if and only if $\varphi' \in \mathrm{Reg}(H, \#(H, B))$.

The map Γ maps invertible cocycles to invertible twistings.

PROPOSITION 2.8. *Let* $\varphi \in Z^2_{\mathrm{tw}}(H, A)$. *If* $\varphi' \in \mathrm{Reg}(H, \#(H, A))$, *then* $\tau'_\varphi \in \mathrm{Reg}(H, \mathrm{End}(A))$, *so that* τ_φ *is an invertible twisting.*

Proof. Let λ: $H \to \#(H, A)$ be the convolution inverse for φ'. Then, for all $h \in H$, in $\#(H, A)$,

$$\sum \varphi'(h_1) \cdot \lambda(h_2) = \sum \lambda(h_1) \cdot \varphi'(h_2) = \epsilon(h)\epsilon.$$

Define λ': $H \to \mathrm{End}(A)$ by $\lambda'(h)(a) = \sum \lambda(h)(a_1)a_0$; we claim that λ' is the convolution inverse to τ'_φ in $\mathrm{Hom}(H, \mathrm{End}(A))$, i.e., that

$$\sum \tau'_\varphi(h_1)(\lambda'(h_2)(a)) = \sum \lambda'(h_1)\big(\tau'_\varphi(h_2)(a)\big) = \epsilon(h)a$$

for all $a \in A$.

First we check that

$$\begin{aligned}
\sum \lambda'(h_1)\big(\tau'_\varphi(h_2)(a)\big) &= \sum \lambda'(h_1)(\varphi(h_2, a_1)a_0)\\
&= \sum \lambda(h_1)\big(S(h_2)h_4a_4\overline{S}(a_2)a_1\big)\varphi(h_3, a_3)a_0\\
&= \sum (\lambda(h_1) \cdot \varphi'(h_2))(a_1)a_0 \qquad \text{by (1.8)}\\
&= \epsilon(h)a.
\end{aligned}$$

Also

$$\begin{aligned}
\sum \tau'_\varphi(h_1)(\lambda'(h_2)(a)) &= \sum \tau'_\varphi(h_1)(\lambda(h_2)(a_1)a_0)\\
&= \sum \varphi'(h_1)((\lambda(h_2)(a_2))_1a_1)(\lambda(h_2)(a_2))_0a_0\\
&= \sum (\varphi'(h_1) \cdot \lambda(h_2))(a_1)a_0\\
&= \epsilon(h)a.
\end{aligned}$$

∎

Thus we have shown that for A an H-comodule algebra with $B = A^{\mathrm{co}\,H}$, and with $\mathscr{U}(Z^2_{\mathrm{tw}}(H, A))$ and $\mathscr{U}(\mathscr{T}(A))$ the sets of invertible twisted cocycles and invertible twistings, respectively, there are sequences

$$Z^2_{\mathrm{tw}}(H, A) \xrightarrow{\Gamma} K(\Omega) \xrightarrow{\mathrm{Id}} \mathscr{T}(A) \xrightarrow{\Omega} \mathrm{Meas}_H(B, A),$$

$$\mathscr{U}\big(Z^2_{\mathrm{tw}}(H, A)\big) \xrightarrow{\Gamma} K(\Omega) \cap \mathscr{U}(\mathscr{T}(A)) \xrightarrow{\mathrm{Id}} \mathscr{U}(\mathscr{T}(A)) \xrightarrow{\Omega} \mathrm{Meas}_H(B, A).$$

To end this section, we give a sufficient condition for A^τ and A^λ to be isomorphic, where $\tau, \lambda \in \mathscr{T}(A)$.

PROPOSITION 2.9. *Let τ and λ be twistings of A. Let $v \in \mathrm{Hom}(H, A)$ such that for all $h \in H$, $a \in A$,*

(i) $v(1_H) = 1_A$;

(ii) $\rho_A(v(h)) = \sum v(h_2) \otimes S(h_1)h_3$;

(iii) $\sum \lambda(h_1 \otimes a_0)v(h_2a_1) = \sum v(h_1)\tau(h_2 \otimes a_0v(a_1))$.

Then $\psi : A^\lambda \to A^\tau$ defined by $\psi(a) = \sum a_0 v(a_1)$ is a left B-module right H-comodule algebra map which is the identity on B. If $v \in \mathrm{Reg}(H, A)$, then ψ is an isomorphism.

Proof. Clearly ψ is a left B-module map, is the identity on B, and by (ii), is an H-comodule map. We check that ψ preserves multiplication. For $a, b \in A$,

$$\begin{aligned}\psi(a *_\lambda b) &= \sum (a_0 *_\lambda b_0) v(a_1 b_1) \\ &= \sum a_0 \lambda(a_1 \otimes b_0) v(a_2 b_1) \\ &= \sum a_0 v(a_1) \tau(a_2 \otimes b_0 v(b_1)) \qquad \text{by (iii)} \\ &= \sum (a_0 v(a_1)) *_\tau (b_0 v(b_1)) = \psi(a) *_\tau \psi(b).\end{aligned}$$

Note that if $v \in \operatorname{Reg}(H, A)$, then v^{-1}, the convolution inverse to v, also satisfies (i) and (ii) above and thus the map $\psi^{-1} \in \operatorname{End}(A)$ defined by $\psi^{-1}(a) = \sum a_0 v^{-1}(a_1)$ is inverse to ψ. ∎

DEFINITION 2.10. (i) For $\tau, \lambda \in \mathscr{T}(A)$, define $\tau \sim \lambda$ if and only if there exists $v \in \operatorname{Reg}(H, A)$ satisfying the conditions in Proposition 2.9. Then $\sim$ is an equivalence relation on $\mathscr{T}(A)$.

(ii) For $\alpha, \beta \in \operatorname{Meas}_H(B, A)$, define $\alpha \sim \beta$ if there is $v \in \operatorname{Reg}(H, A)$ such that for all $h \in H$, $b \in B$,

$$\alpha(h)(b) = \sum v(h_1) \beta(h_2)(b) v^{-1}(h_3).$$

Then $\sim$ is an equivalence on $\operatorname{Meas}_H(B, A)$.

LEMMA 2.11. *Suppose $\lambda, \tau \in \mathscr{T}(A)$ and $\lambda \sim \tau$.*

(i) *If τ is an invertible twisting, so is λ.*

(ii) *The measurings $\Omega(\lambda)$ and $\Omega(\tau)$ are equivalent.*

Proof. (i) Let $v \in \operatorname{Reg}(H, A)$ satisfy the conditions of Proposition 2.9 and let ψ: $A^\lambda \to A^\tau$ be the algebra isomorphism $\psi(a) = \sum a_0 v(a_1)$. Let τ'^{-1} be the convolution inverse to τ' and define ω: $H \to \operatorname{End}(A)$ by

$$\omega(h)(a) = \psi^{-1}\left\{\sum \tau'^{-1}(h_1)\left(v^{-1}(h_2) a_0 v(h_3 a_1)\right)\right\}.$$

Then for all $h \in H$, $a \in A$, we have

$$\sum \omega(h_1)(\lambda(h_2 \otimes a)) = \psi^{-1} \sum \left\{\tau'^{-1}(h_1)\left(v^{-1}(h_2) c_0 v(h_3 c_1)\right)\right\},$$

where $c = \lambda(h_4 \otimes a)$ so that

$$\sum c_0 \otimes c_1 = \sum v(h_5) \tau(h_6 \otimes \psi(a_0)) v^{-1}(h_7 a_1) \otimes S(h_4) h_8 a_2.$$

Then $\sum \omega(h_1)(\lambda(h_2 \otimes a)) = \psi^{-1} \sum \{\tau'^{-1}(h_1) \tau'(h_2)(\psi(a))\} = \epsilon(h) a$.

Similarly $\sum \lambda'(h_1)(\omega(h_2)(a)) = \sum v(h_1)\tau(h_2 \otimes \psi(c_0))v^{-1}(h_3 c_1)$, where

$$\sum c_0 \otimes c_1 = \rho\big(\omega(h_4)(a)\big) = \sum \psi^{-1}\big\{\tau'^{-1}(h_5)\big(v^{-1}(h_6)a_0 v(h_7 a_1)\big)\big\} \otimes S(h_4)h_8 a_2$$

so that

$$\begin{aligned}
&\sum \lambda'(h_1)\big(\omega(h_2)(a)\big) \\
&\quad = \sum v(h_1)\tau'(h_2)\big\{\tau'^{-1}(h_3)\big[v^{-1}(h_4)a_0 v(h_5 a_1)\big]\big\}v^{-1}(h_6 a_2) \\
&\quad = \epsilon(h)a.
\end{aligned}$$

(ii) Since $\lambda \sim \tau$ implies the existence of $v \in \mathrm{Reg}(H, A)$ satisfying the conditions of Proposition 2.9, this statement is clear. ∎

In general, it is not known whether $A^\tau \cong A^\lambda$ implies that $\tau \sim \lambda$.

3. CROSSED PRODUCTS

In this section, as an example, we study the twistings of the H-comodule algebra $A = B \otimes H$ with $\rho_A = 1 \otimes \Delta_H$. Except for the usual assumption that H has bijective antipode, H is arbitrary. By [1, Theorem 3.4], $\mathscr{T}(A)$, the set of twistings on A, can be identified with the set of crossed products $B\#_\sigma H$ and $\mathscr{U}(\mathscr{T}(A))$ with those crossed products where σ is invertible, i.e., the cleft extensions. We will always assume that $\sigma(1, h) = \sigma(h, 1) = \epsilon(h)$ for all $h \in H$.

If $B\#_\sigma H$ is a crossed product then the corresponding twisting is given by

$$\tau(h \otimes (b \otimes g)) = \sum (h_2 \cdot b)\sigma(h_3, g_1) \otimes S(h_1)h_4 g_2, \tag{3.15}$$

and if τ is a twisting of $B \otimes H$, then $(B \otimes H)^\tau = B\#_\sigma H$ where the weak action from $H \otimes B \to B$ and the cocycle $\sigma\colon H \otimes H \to B$ are defined by

$$\begin{aligned}
h \cdot b &= (1 \otimes \epsilon)\tau(h \otimes (b \otimes 1)) \qquad \text{and} \\
\sigma(h, g) &= (1 \otimes \epsilon)\tau(h \otimes (1 \otimes g)).
\end{aligned} \tag{3.16}$$

PROPOSITION 3.1. *There is a bijection between the set $Z^2_{\mathrm{tw}}(H, B \otimes H)$ of twisted cocycles from $H \otimes H$ to $A = B \otimes H$ and $Z^2(H, B)$, the set of Sweedler cocycles from $H \otimes H$ to B with trivial weak action. Invertible twisted cocycles correspond to invertible Sweedler cocycles under this map.*

Proof. If $\varphi \in Z^2_{\mathrm{tw}}(H, A)$, let $\sigma_\varphi = (1 \otimes \epsilon) \cdot \varphi\colon H \otimes H \to Z(B)$. Then it is easy to see that σ_φ is normal and satisfies (1.14) with trivial weak action.

Conversely, let $\sigma \in Z^2(H, B)$. Define $\varphi_\sigma = \varphi\colon H \otimes H \to B \otimes H$ by $\varphi(h, g) = \sum \sigma(h_2, g_2) \otimes S(h_1)h_3g_3\bar{S}(g_1)$. Since σ is normal, we have $\varphi(h, 1) = \varphi(1, h) = \epsilon(h) \otimes 1$. Also we have

$$\begin{aligned}\rho_A(\varphi(h, g)) &= \sum \sigma(h_3, g_3) \otimes S(h_2)h_4g_4\bar{S}(g_2) \otimes S(h_1)h_5g_5\bar{S}(g_1)\\ &= \sum \varphi(h_2, g_2) \otimes S(h_1)h_3g_3\bar{S}(g_1),\end{aligned}$$

so that Definition 2.3(i) and (ii) hold.

Since σ is associated with the trivial weak action, we have by (1.13) that $\sigma(h, g) \in Z(B)$ for all $h, g \in H$. Then for $b \in B$, $l, g, h \in H$, we have

$$\begin{aligned}&\sum \varphi(g_1, l_2)(b \otimes l_1)\varphi(g_2l_3, h)\\ &\qquad = \sum \sigma(g_2, l_1)\sigma(g_3l_2, h_2)b \otimes S(g_1)g_4l_3h_3\bar{S}(h_1),\\ &\sum \varphi(g, l_3h_2)(b \otimes l_1)\varphi(l_2, h_1)\\ &\qquad = \sum \sigma(l_1, h_2)\sigma(g_2, l_2h_3)b \otimes S(g_1)g_3l_3h_4\bar{S}(h_1).\end{aligned}$$

These two expressions are equal by (1.14), and Definition 2.3(iii) is verified. Thus $\varphi\colon H \otimes H \to A$ is a twisted cocycle. It is straightforward to see that $\varphi_{\sigma_\varphi} = \varphi$ and $\sigma_{\varphi_\sigma} = \sigma$.

Since $\Gamma(\varphi_\sigma) = \tau_{\varphi_\sigma}$ is the twisting associated with σ, then by Proposition 2.8 if φ_σ is invertible, τ_{φ_σ} is an invertible twisting and thus σ is an invertible Sweedler cocycle.

Conversely suppose $\sigma \in Z^2(H, B)$ is invertible with trivial weak action. We must show that $\varphi = \varphi_\sigma$, defined by $\varphi(h, g) = \sum \sigma(h_2, g_2) \otimes S(h_1)h_3g_3\bar{S}(g_1)$, lies in $\mathscr{U}(Z^2_{\mathrm{tw}}(H, A))$; i.e., $\varphi'\colon H \to \#(H, A)$ is convolution invertible. We need a map $\lambda\colon H \to \#(H, A)$ such that $\sum \varphi'(h_1) \cdot \lambda(h_2)(g) = \sum \lambda(h_1) \cdot \varphi'(h_2)(g) = \epsilon(h)\epsilon(g)$ for all $h, g \in H$. Since $B \#_\sigma H$ is a cleft extension, then $\sigma(g, h) = \sum \phi(g_1)\phi(h_1)\phi^{-1}(g_2h_2)$ and $h \cdot b = \sum \phi(h_1)b\phi^{-1}(h_2)$ where $\phi\colon H \to B \#_\sigma H$ is the convolution invertible H-comodule map defined by $\phi(h) = 1 \#_\sigma h$. Note that $\rho_A(\phi^{-1}(h)) = \sum \phi^{-1}(h_2) \otimes S(h_1)$. Now define $\psi\colon H \to \#(H, A)$ by

$$\psi(l)(m) = \sum \phi^{-1}(l_3)\phi(l_4m_2)\phi^{-1}(S(l_2)l_5m_3) \otimes S(l_1)l_6m_4\bar{S}(m_1).$$

By the above observation, the first tensorand does indeed lie in B. Also $\phi(h)$ and $\phi^{-1}(h)$ lie in $C_A(B)$.

Then for $h, g \in H$, in $\#(H, A)$,

$$
\begin{aligned}
\sum (\psi(h_1) \cdot \varphi'(h_2))(g)
&= \sum \psi(h_1)[S(h_2)h_4 g_2]\varphi(h_3, g_1) \\
&= \sum \left[\phi^{-1}(h_3)\phi(h_4 m_2)\phi^{-1}(S(h_2)h_5 m_3) \otimes S(h_1)h_6 m_4 \bar{S}(m_1)\right] \\
&\quad \times \varphi(h_8, g_1) \qquad \text{where } m = S(h_7)h_9 g_2 \\
&= \sum \left[\phi^{-1}(h_3)\phi(h_7 g_3)\phi^{-1}(S(h_2)h_8 g_4)\right. \\
&\qquad\qquad \left.\otimes S(h_1)h_9 g_5 \bar{S}(g_2)\bar{S}(h_6)h_4\right]\varphi(h_5, g_1) \\
&= \sum \phi^{-1}(h_3)\phi(h_{10}g_6)\phi^{-1}(S(h_2)h_{11}g_7)\phi(h_6)\phi(g_2)\phi^{-1}(h_7 g_3) \\
&\quad \otimes S(h_1)h_{12}g_8\bar{S}(g_5)\bar{S}(h_9)h_4 S(h_5)h_8 g_4 \bar{S}(g_1) \\
&= \sum \phi^{-1}(h_3)\phi(h_4)\phi(g_2)\phi^{-1}(h_5 g_3)\phi(h_6 g_4)\phi^{-1}(S(h_2)h_7 g_5) \\
&\quad \otimes S(h_1)h_8 g_6 \bar{S}(g_1) \qquad \text{since } \left[\phi(h_4)\phi(g_2)\phi^{-1}(h_5 g_3)\right] \in B \\
&= \sum \phi(g_2)\phi^{-1}(S(h_2)h_3 g_3) \otimes S(h_1)h_4 g_4 \bar{S}(g_1) = \epsilon(h)\epsilon(g) \otimes 1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum \varphi'(h_1) \cdot \psi(h_2)(g)
&= \sum \left[\varphi'(h_1)(S(h_2)h_9 g_5)\right]\left[\phi^{-1}(h_5)\phi(h_6 g_2)\phi^{-1}(S(h_4)h_7 g_3)\right. \\
&\qquad\qquad\qquad\qquad \left.\otimes S(h_3)h_8 g_4 \bar{S}(g_1)\right] \\
&= \sum \phi(h_2)\phi(m_2)\phi^{-1}(h_3 m_3)\phi^{-1}(h_8)\phi(h_9 g_2)\phi^{-1}(S(h_7)h_{10}g_3) \\
&\quad \otimes S(h_1)h_4 m_4 \bar{S}(m_1)S(h_6)h_{11}g_4\bar{S}(g_1) \qquad \text{where } m = S(h_5)h_{12}g_5 \\
&= \sum \phi(h_2)\phi(S(h_3)h_8 g_4)\phi^{-1}(h_9 g_5)\phi^{-1}(h_5) \\
&\quad \times \phi(h_6 g_2)\phi^{-1}(S(h_4)h_7 g_3) \otimes S(h_1)h_{10}g_6\bar{S}(g_1).
\end{aligned}
$$

Now use the fact that $\phi^{-1}(h_5)\phi(h_6 g_2)\phi^{-1}(S(h_4)h_7 g_3) \in B$ and then straightforward calculation yields that this expression is $\epsilon(h)\epsilon(g) \otimes 1$. ∎

Now let $\Omega' : \mathscr{T}(A) \to \mathrm{Meas}_H(B, B)$ be defined by $\Omega' = (1 \otimes \epsilon)\Omega$. Then Ω' is a map of pointed sets and clearly $K(\Omega) \subseteq K(\Omega') = \{\tau \mid (1 \otimes \epsilon)\tau(h \otimes (b \otimes 1)) = \epsilon(h)b \otimes 1\}$. In fact $K(\Omega) = K(\Omega')$. Suppose $\tau \in K(\Omega')$, and let $(B \otimes H)^\tau = B\#_\sigma H$ as in (3.15) and (3.16). Then, for all $b \in B$, $h \in H$, we have $h \cdot b = \epsilon(h)b$, and thus $\tau(h \otimes (b \otimes 1)) = \sum b\sigma(h_2, 1) \otimes S(h_1)h_3 = b\epsilon(h) \otimes 1$ and $\tau \in K(\Omega)$.

Recall that an H-measuring γ of B is called C-inner if $B \subseteq C$ as algebras and $\gamma(h \otimes b) = \sum u(h_1) b u^{-1}(h_2) \in B$ for some $u \in \operatorname{Reg}(H, C)$. Let $\operatorname{Inn}\operatorname{Meas}_H(B, B)$ denote the set of inner measurings of B which are C-inner for some extension C of B.

THEOREM 3.2. *For $A = B \otimes H$, there are exact sequences of pointed sets*

$$1 \to Z^2_{\mathrm{tw}}(H, A) \xrightarrow{\Gamma} \mathscr{T}(A) \xrightarrow{\Omega'} \operatorname{Meas}_H(B, B)$$

and

$$1 \to \mathscr{U}\big(Z^2_{\mathrm{tw}}(H, A)\big) \xrightarrow{\Gamma} \mathscr{U}(\mathscr{T}(A)) \xrightarrow{\Omega'} \operatorname{Inn}\operatorname{Meas}_H(B, B).$$

Proof. First we note that Γ is injective. Suppose that for $\varphi, \lambda \in Z^2_{\mathrm{tw}}(H, A)$, $\tau_\varphi = \tau_\lambda$. Applying both twistings to $h \otimes (1 \otimes g)$, we obtain $\sum \varphi(h, g_2)(1 \otimes g_1) = \sum \lambda(h, g_2)(1 \otimes g_1)$ for all h, g, so that $\varphi(h, g) = \sum \varphi(h, g_3)(1 \otimes g_2)(1 \otimes \overline{S}(g_1)) = \sum \lambda(h, g_3)(1 \otimes g_2)(1 \otimes \overline{S}(g_1)) = \lambda(h, g)$. The proof now follows from Proposition 3.1. ∎

Finally, we note the correspondence between the equivalence relation $\sim$ on $\mathscr{T}(A)$ of Definition 2.10 and the equivalence of crossed systems in [4]. Recall that crossed systems $(\cdot, \sigma)$ and $(\cdot', \sigma')$ are defined in [4] to be equivalent if there exists $\bar{v} \in \operatorname{Reg}(H, B)$ with inverse $\overline{w}$ such that $\bar{v}(1) = 1$, and for all $h, g \in H$, $b \in B$,

$$h \cdot' b = \sum \bar{v}(h_1)(h_2 \cdot b)\overline{w}(h_3), \tag{3.17}$$

$$\sigma'(h, g) = \sum \bar{v}(h_1)(h_2 \cdot \bar{v}(g_1))\sigma(h_3, g_2)\overline{w}(h_4 g_3). \tag{3.18}$$

PROPOSITION 3.3. *For $A = B \otimes H$, and $\tau, \lambda \in \mathscr{T}(A)$, $\tau \sim \lambda$ if and only if the crossed systems $(\cdot, \sigma)$ and $(\cdot', \sigma')$ corresponding to τ and λ, respectively, are equivalent in the sense of Doi.*

Proof. Suppose first that $\tau \sim \lambda$ and let $\bar{v} = (1 \otimes \epsilon) \cdot v \in \operatorname{Reg}(H, B)$. Applying $1 \otimes \epsilon$ to Proposition 2.9(iii) with $a \in B$ yields (3.17). To obtain (3.18), first note that by Proposition 2.9(iii), for $h, g \in H$, we have

$$\lambda(h \otimes (1 \otimes g)) = \sum v(h_1)\tau\big(h_2 \otimes (1 \otimes g_1)v(g_2)\big)v^{-1}(h_3 g_3).$$

Applying $1 \otimes \epsilon$ to both sides, we obtain

$$\sigma'(h, g) = \sum \bar{v}(h_1)(1 \otimes \epsilon)\big(\tau\big(h_2 \otimes (1 \otimes g_1)v(g_2)\big)\big)\overline{w}(h_3 g_3).$$

But $\sum(1 \otimes g_1)v(g_2) \otimes g_3 = \sum \rho_A((1 \otimes g_1)v(g_2)) = (\mathrm{Id} \otimes \Delta_H)(\sum(1 \otimes g_1)v(g_2))$, and applying $\mathrm{Id} \otimes \epsilon \otimes \mathrm{Id}$ yields $\sum \bar{v}(g_1) \otimes g_2 = \sum(1 \otimes g_1)v(g_2)$, so that by (3.15), Eq. (3.18) holds.

Conversely, let $(\cdot, \sigma)$ and $(\cdot', \sigma')$ be Doi-equivalent. Define $\upsilon \in \operatorname{Reg}(H, A)$ by $\upsilon(h) = \Sigma \bar{\upsilon}(h_2) \otimes S(h_1)h_3$. Clearly (i) and (ii) of Proposition 2.9 hold and we must verify (iii). For $a = b \otimes g$,

$$\begin{aligned}\lambda(h \otimes a) &= \sum (h_2 \cdot' b)\sigma'(h_3, g_1) \otimes S(h_1)h_4 g_2 \qquad \text{by (3.15)} \\ &= \sum \bar{\upsilon}(h_2)(h_3 \cdot b)(h_4 \cdot \bar{\upsilon}(g_1))\sigma(h_5, g_2)\overline{w}(h_6 g_3) \\ &\qquad \otimes S(h_1)h_7 g_4 \qquad \text{by (3.17), (3.18)} \\ &= \sum \upsilon(h_1)\{(h_3 \cdot b)(h_4 \cdot \bar{\upsilon}(g_1))\sigma(h_5, g_2) \\ &\qquad\qquad \otimes S(h_2)h_6 g_3\}\upsilon^{-1}(h_7 g_4).\end{aligned}$$

Here $\Sigma a_0 \upsilon(a_1) = \Sigma(b \otimes g_1)(\bar{\upsilon}(g_3) \otimes S(g_2)g_4) = \Sigma b\bar{\upsilon}(g_1) \otimes g_2$, so now it follows easily that $\lambda(h \otimes a) = \Sigma \upsilon(h_1)\tau(h_2 \otimes a_0 \upsilon(a_1))\upsilon^{-1}(h_3 a_2)$. ∎

COROLLARY 3.4. *For $A = B \otimes H$ and $\tau, \lambda \in \mathscr{U}(\mathscr{T}(A))$, then $\tau \sim \lambda$ if and only if $A^\tau \cong A^\lambda$ as algebras in ${}_B\mathscr{M}^H$.*

Proof. This follows directly from Proposition 2.9, Proposition 3.3, and [4]. ∎

4. TWISTINGS OF AN *H*-GALOIS OBJECT

In this section we study the set of H-Galois structures on a given H-Galois extension $A/A^{\mathrm{co}\,H}$. First we note that if τ is an invertible twisting, then if $A/A^{\mathrm{co}\,H}$ is H-Galois, so is $A^\tau/A^{\mathrm{co}\,H}$. No flatness or finiteness assumptions are needed for this argument. As usual, let $B = A^{\mathrm{co}\,H}$.

PROPOSITION 4.1. *Let A be an H-comodule algebra and let $\tau \in \mathscr{U}(\mathscr{T}(A))$. Then A^τ/B is H-Galois if and only if A/B is H-Galois.*

Proof. Let λ: $H \to \operatorname{End}(A)$ be the convolution inverse to τ'. Denote by can'_τ the canonical map from $A^\tau \otimes_B A^\tau$ to $A^\tau \otimes H$, $a \otimes b \mapsto \Sigma a_0 *_\tau b \otimes a_1$. We show that can'_τ is bijective if and only if can' is.

Note first that f: $A \otimes H \to A \otimes H$, $f(a \otimes h) = \Sigma \tau'(\bar{S}(h_1))(a) \otimes h_2$ is a bijection with inverse f^{-1} defined by $f^{-1}(a \otimes h) = \Sigma \lambda(\bar{S}(h_1))(a) \otimes h_2$. Also g: $A \otimes_B A \to A \otimes_B A$ defined by $g(a \otimes b) = \Sigma \tau'(\bar{S}(a_1))(a_0) \otimes b$ is bijective with inverse g^{-1} defined by $g^{-1}(a \otimes b) = \Sigma \lambda(\bar{S}(a_1))(a_0) \otimes b$. To check that g and g^{-1} are inverses, use (1.6).

Now it is straightforward to check that the diagram

$$\begin{array}{ccc} A \otimes_B A & \xrightarrow{\mathrm{can}'_\tau} & A \otimes H \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A \otimes_B A & \xrightarrow{\mathrm{can}'} & A \otimes H \end{array}$$

commutes, since

$$\begin{aligned} f(\mathrm{can}'_\tau(a \otimes b)) &= f\left(\sum a_0 *_\tau b \otimes a_1\right) \\ &= \sum \tau'\left(\bar{S}(a_1)\right)(a_0 *_\tau b) \otimes a_2 \\ &= \sum \tau'\left(\bar{S}(a_3)\right)(a_0)\tau'\left(\bar{S}(a_2)a_1\right)(b) \otimes a_4 \qquad \text{by (1.3)} \\ &= \sum \tau'\left(\bar{S}(a_1)\right)(a_0)b \otimes a_2 \\ &= \mathrm{can}'\left(\sum \tau'\left(\bar{S}(a_1)\right)(a_0) \otimes b\right) \qquad \text{by (1.6)} \\ &= \mathrm{can}'(g(a \otimes b)). \end{aligned}$$

Thus can'_τ is a bijection if and only if can' is. ∎

COROLLARY 4.2. *If A/B is H-Galois and $\nu \in \mathscr{U}(\mathscr{L}(A))$ then ${}^\nu A/B$ is also H-Galois.*

Proof. This follows immediately from Lemma 1.4 and the proposition. ∎

The next theorem is the left hand version of [8, Theorem 2.3] but we provide most of the details for completeness.

THEOREM 4.3. *Let A be a right H-comodule algebra (with multiplication written as juxtaposition) and suppose H is k-projective. Let $C = (A, \times, \rho_A)$ be the H-comodule A but with a different associative multiplication $\times$. Suppose there is an algebra map ϕ: $\#(H, C) \to \#(H, A)$ such that the diagram*

$$\begin{array}{ccccc} H^* & \xrightarrow{\gamma} & \#(H,C) & \xleftarrow{\alpha_C} & C \\ \| & & \big\downarrow{\scriptstyle\phi} & & \| \\ H^* & \xrightarrow{\gamma} & \#(H,A) & \xrightarrow{ev} & A \end{array}$$

commutes where γ, α, ev are as in (1.10)–(1.12). *Then there is a left twisting ν such that $C = {}^\nu A$.*

Proof. Define ν'': $A \to \#(H, A)$ by $\nu'' = \phi \circ \alpha_C$, so that $\nu(h \otimes a) = \phi(\alpha_C(a))(h)$. We check that ν is a left hand twisting. For the normality conditions (1.1), we note that from the commutativity of the diagram, $1_A = 1_C$, and

$$\nu(1 \otimes a) = \phi(\alpha_C(a))(1) = ev \circ \phi \circ \alpha_C(a) = a,$$

and since ϕ is an algebra map, $\phi(\epsilon) = \epsilon$, and so

$$\nu(h \otimes 1) = \phi(\alpha_C(1))(h) = \phi(\epsilon)(h) = \epsilon(h).$$

Also, in $C\#H^* \subseteq \#(H, C)$, we have that for all $h^* \in H^*$, $a \in C$,

$$\begin{aligned} \alpha_C(a)\gamma_C(h^*) = a\#h^* &= \sum \left(1\#h^* \leftharpoonup \bar{S}(a_1)\right)(a_0\#\epsilon) \\ &= \sum \gamma\left(h^* \leftharpoonup \bar{S}(a_1)\right)\alpha_C(a_0). \end{aligned}$$

Applying ϕ to both sides, we obtain that in $\#(H, A)$

$$(\phi\alpha_C(a)) \cdot (\phi\gamma_C(h^*)) = \sum \left(\phi\gamma\left(h^* \leftharpoonup \bar{S}(a_1)\right)\right) \cdot (\phi\alpha_C(a_0)),$$

and evaluating at $h \in H$, we have

$$\sum \nu(a \otimes h_1)h^*(h_2) = \sum h^*\left(\bar{S}(a_1)(\nu(a_0 \otimes h_2))_1 h_1\right)\nu(a_0 \otimes h_2)_0.$$

Since this equality holds for all $h^* \in H^*$ and H is projective, we have

$$\sum \nu(a \otimes h_1) \otimes h_2 = \sum \nu(a_0 \otimes h_2)_0 \otimes \bar{S}(a_1)\nu(a_0 \otimes h_2)_1 h_1,$$

or equivalently, (1.4); i.e.,

$$\rho(\nu(a \otimes h)) = \sum \nu(a_0 \otimes h_2) \otimes a_1 h_3 \bar{S}(h_1).$$

Also, we see that $(C, \times, \rho_A) = ({}^{\nu}A, *_{\nu}, \rho_A)$ since for $a, b \in C$, we have

$$\begin{aligned} a \times b = \nu''(a \times b)(1) &= (\nu''(a) \cdot \nu''(b))(1) \\ &= \sum \nu''(a)(\nu''(b)(1)_1)\nu''(b)(1)_0 \\ &= \sum \nu(b_1 \otimes a)b_0 = a *_{\nu} b. \end{aligned}$$

Now $\nu''(a *_{\nu} b)(h) = (\nu''(a) \cdot \nu''(b))(h)$ in $\#(H, A)$ together with (1.4) implies (1.5). ∎

THEOREM 4.4. *Suppose A and C are H-Galois objects, and $A = C$ as objects in $\mathscr{M}_B^H$. Also suppose H is finitely generated projective over k. Then $C = {}^{\nu}A$ for an invertible left twisting ν.*

Proof. If $A = C$ as H-comodules, $A^{\mathrm{co}\,H} = C^{\mathrm{co}\,H} = B$ and so $1_A = 1_C$. Since H is finitely generated projective, $\#(H, A) = A\#H^*$ and $\#(H, C) = C\#H^*$. Also since A and C are Galois, the map π_A: $A\#H^* \to \mathrm{End}(A_B)$ is an algebra isomorphism by [9] (or see [10, Chap. 8]) where $\pi_A(a\#h^*)(b) = \Sigma a b_0 h^*(b_1)$. Similarly π_C is an algebra isomorphism and we claim that $\phi = \pi_A^{-1} \circ \pi_C$ makes the diagram in Theorem 4.3 commute.

Since $\pi_A(1\#h^*) = \pi_C(1\#h^*)$ for all $h^* \in H^*$, the left hand side of the diagram commutes. Also, for $a \in C$, $(ev \circ \phi \circ \alpha_C)(a) = [(\pi_A^{-1} \circ \pi_C)(a\#\epsilon)](1)$. Suppose $\pi_C(a\#\epsilon) = \pi_A(\Sigma b_i\#h_i^*)$. Then $\Sigma b_i h_i^*(1) = a$, and the right hand side commutes also.

Thus by Theorem 4.3, $C = {}^{\nu}A$ for some left twisting ν. Similarly $A = {}^{\lambda}C$ for a left twisting λ, and we claim that ν is invertible with λ' the convolution inverse to ν' in $\mathrm{Hom}(H, \mathrm{End}(A))$. For any $a, b \in A$, since $A = {}^{\lambda}({}^{\nu}A)$,

$$\begin{aligned} ab &= \sum \lambda(b_1 \otimes a) *_\nu b_0 = \sum \nu\big(b_1 \otimes \lambda(b_2 \otimes a)\big) b_0 \\ &= \sum \nu'(b_1)\big(\lambda'(b_2)(a)\big) b_0. \end{aligned}$$

For $h \in H$, since A is H-Galois, there exist $b_i, c_i \in A$ such that $\Sigma b_{i_0} c_i \otimes b_{i_1} = 1 \otimes h$. Then for any $a \in A$,

$$\sum \nu'(h_1)\big(\lambda'(h_2)(a)\big) = \sum \nu'\big(b_{i_1}\big)\Big(\lambda'\big(b_{i_2}\big)(a)\Big) b_{i_0} c_i = a \sum b_i c_i = a\epsilon(h).$$

Similarly $\Sigma \lambda'(h_1) \circ \nu'(h_2)(a) = \epsilon(h)a$, and so ν is invertible. ∎

Now, for A/B H-Galois and H finitely generated projective over k, the above implies that there is a bijection between the set $\mathscr{U}(\mathscr{L}(A))$ of invertible left twistings of A and the set $\mathscr{G}al(A)$ of Galois objects C/B with $C = A$ in $\mathscr{M}_B^H$, for ν in $\mathscr{U}(\mathscr{L}(A))$ corresponds to ${}^{\nu}A$ in $\mathscr{G}al(A)$. If ${}^{\nu}A = {}^{\lambda}A$, then ${}^{\mu}({}^{\nu}A) = A$ where μ' is the convolution inverse to λ', and then the proof of Theorem 4.4. shows that μ' and ν' are convolution inverses; i.e., $\lambda = \nu$. Thus we have:

THEOREM 4.5. *For H finitely generated projective over k, the sets $\mathscr{G}al(A)$, $\mathscr{U}(\mathscr{L}(A))$, and $\mathscr{U}(\mathscr{T}(A))$ are in bijective correspondence. The twisting u corresponds to A in $\mathscr{G}al(A)$.*

Now we note that for any H, the map Γ: $Z^2_{\mathrm{tw}}(H, A) \to K(\Omega)$, defined in Section 2, is injective if $A/A^{\mathrm{co}\,H}$ is H-Galois.

Remark 4.6. If A is H-Galois then Γ is injective. For if for all h, a we have $\tau_\varphi(h \otimes a) = \Sigma \varphi(h, a_1) a_0 = \tau_\lambda(h \otimes a)$ and if $1 \otimes g = \Sigma a_{i_0} c_i \otimes a_{i_1}$, then $\varphi(h, g) = \Sigma \varphi(h, a_{i_1}) a_{i_0} c_i = \Sigma \lambda(h, a_{i_1}) a_{i_0} c_i = \lambda(h, g)$, so that $\varphi = \lambda$. Here it is not necessary that H be finite.

THEOREM 4.7. *For A/B H-Galois and H finitely generated projective over k, there is an exact sequence of pointed sets*

$$1 \to \mathscr{U}\big(Z^2_{\mathrm{tw}}(H, A)\big) \xrightarrow{\Gamma} \mathscr{U}(\mathscr{T}(A)) \xrightarrow{\Omega} \mathrm{Meas}_H(B, A)$$

and thus an exact sequence of pointed sets

$$1 \to \mathscr{U}\big(Z^2_{\mathrm{tw}}(H, A)\big) \to \mathscr{G}al(A) \to \mathrm{Meas}_H(B, A).$$

Proof. Since A/B is H-Galois, π: $A \# H^* \to \mathrm{End}(A_B)$, defined by $\pi(a\#h^*)(c) = \Sigma a c_0 h^*(c_1)$, is an algebra isomorphism. Suppose $\tau \in K(\Omega)$; i.e., $\tau'(h) \in \mathrm{End}(A_B)$ for all $h \in H$.

Define φ: $H \otimes H \to A$ by $\varphi(h, g) = \pi^{-1}(\tau'(h))(g)$; i.e., if $\pi^{-1}(\tau'(h)) = \Sigma a_i^h \# f_i^h$, so that $\tau'(h)(a) = \Sigma a_i^h f_i^h(a_1) a_0$, then

$$\varphi(h, g) = \sum a_i^h f_i^h(g).$$

Now $\Gamma(\varphi)(h \otimes a) = \Sigma \varphi(h, a_1) a_0 = \Sigma a_i^h f_i^h(a_1) a_0 = \tau(h \otimes a)$ and it remains to show that φ is an invertible cocycle. Proving normality is straightforward, since $\varphi(h, 1) = \Sigma a_i^h f_i^h(1) = \tau'(h)(1) = \epsilon(h) 1_A$, and since $\Sigma a_i^1 \# f_i^1 = 1 \# \epsilon$, if $1 \otimes g = \Sigma c_{j_0} d_j \otimes c_{j_1}$, $\varphi(1, g) = \Sigma \epsilon(c_{j_1}) c_{j_0} d_j = \epsilon(g) 1_A$.

Now we check the coaction of H on $\varphi(g, h)$. By (1.2), since $\tau(h \otimes a) = \Sigma a_i^h f_i^h(a_1) a_0$, we have for all $a \in A$,

$$\sum a_{i_0}^h f_i^h(a_2) a_0 \otimes a_{i_1}^h a_1 = \sum a_i^{h_2} f_i^{h_2}(a_1) a_0 \otimes S(h_1) h_3 a_2.$$

Now suppose $1 \otimes g = \Sigma c_{j_0} d_j \otimes c_{j_1}$. Then by the equation above,

$$\sum a_{i_0}^h f_i^h(c_{j_2}) c_{j_0} d_j \otimes a_{i_1}^h c_{j_1} = \sum a_i^{h_2} f_i^{h_2}(c_{j_1}) c_{j_0} d_j \otimes S(h_1) h_3 c_{j_2},$$

so that

$$\sum a_{i_0}^h f_i^h(g) \otimes a_{i_1}^h = \sum a_i^{h_2} f_i^{h_2}(g_2) \otimes S(h_1) h_3 g_3 \bar{S}(g_1),$$

in other words,

$$\rho(\varphi(h, g)) = \sum \varphi(h_2, g_2) \otimes S(h_1) h_3 g_3 \bar{S}(g_1).$$

Now we must verify 2.3(iii). From (1.3), since $\rho(a *_\tau b) = \Sigma a_0 *_\tau b_0 \otimes a_1 b_1$, we have

$$\sum a_i^h f_i^h(a_1 b_1)(a_0 *_\tau b_0) = \sum \big(a_i^{h_1} f_i^{h_1}(a_1) a_0\big)\big(a_j^{h_2 a_2} f_j^{h_2 a_2}(b_1) b_0\big);$$

i.e.,

$$\sum a_i^h f_i^h(a_2 b_2) a_0 a_j^{a_1} f_j^{a_1}(b_1) b_0 = \sum a_i^{h_1} f_i^{h_1}(a_1) a_0 a_j^{h_2 a_2} f_j^{h_2 a_2}(b_1) b_0.$$

Suppose $1 \otimes g = \Sigma b_{k_0} c_k \otimes b_{k_1}$. Then, as in previous computations,

$$\sum a_i^h f_i^h (a_2 g_2) a_0 a_j^{a_1} f_j^{a_1}(g_1) = \sum a_i^{h_1} f_i^{h_1}(a_1) a_0 a_j^{h_2 a_2} f_j^{h_2 a_2}(g);$$

i.e.,

$$\sum \varphi(h, a_2 g_2) a_0 \varphi(a_1, g_1) = \sum \varphi(h_1, a_1) a_0 \varphi(h_2 a_2, g).$$

Finally, we show that φ is invertible, i.e., that φ': $H \to \#(H, A)$, $\varphi'(h)(g) = \varphi(h, g)$, is convolution invertible. Let λ' be the convolution inverse to τ' in $\mathrm{Hom}(H, \mathrm{End}(A))$ and let λ: $H \otimes A \to A$ be $\lambda(h \otimes a) = \lambda'(h)(a)$. Then, for $b \in B$, since $\epsilon(h)b = \Sigma\lambda'(h_1)(\tau'(h_2)(b)) = \Sigma\lambda'(h_1)\epsilon(h_2)b = \lambda'(h)(b)$, we have that λ restricted to $H \otimes B$ is the identity twisting. The map λ is a twisting, not of A, but of A^τ, so that (1.3) holds for λ in the algebra A^τ. Then for $a \in A$, $b \in B$, $h \in H$,

$$\begin{aligned}
\lambda'(h)(ab) &= \lambda'(h)(a *_\lambda b) && \text{since } \lambda|_{H\otimes B} \text{ is the identity twisting} \\
&= \sum \lambda'(h_1)(a_0) *_\tau \lambda'(h_2 a_1)(b) && \text{by (1.3) for } \lambda \text{ and } A^\tau \\
&= \sum \lambda'(h)(a) *_\tau b = \lambda'(h)(a) b,
\end{aligned}$$

so that $\lambda'(h) \in \mathrm{End}(A_B)$ for all $h \in H$. Define ω: $H \otimes H \to A$ by $\omega(h, g) = \pi^{-1}(\lambda'(h))(g)$. We are required to show that $\Sigma(\varphi'(h_1) \cdot \omega'(h_2))(g) = \Sigma(\omega'(h_1) \cdot \varphi'(h_2))(g) = \epsilon(h)\epsilon(g)$ for all h, g. If we denote $\pi^{-1}(\lambda'(h)) = \Sigma b_k^h \# l_k^h \in A\#H^*$, then

$$\begin{aligned}
\sum (\varphi'(h_1) \cdot \omega'(h_2))(g) &= \sum \varphi'(h_1)[\omega'(h_2)(g_2)_1 g_1] \omega'(h_2)(g_2)_0 \\
&= \sum \varphi'(h_1)\big(b_{i_1}^{h_2} g_1\big) b_{i_0}^{h_2} l_i^{h_2}(g_2) \\
&= \sum a_j^{h_1} f_j^{h_1}\big(b_{i_1}^{h_2} g_1\big) b_{i_0}^{h_2} l_i^{h_2}(g_2) \\
&= \sum \Big(a_j^{h_1} b_{i_0}^{h_2} \#\big(f_j^{h_1} \leftharpoonup b_{i_1}^{h_2}\big)\big(l_i^{h_2}\big)\Big)(g) \\
&= \sum \big(a_j^{h_1} \# f_j^{h_1}\big)\big(b_i^{h_2} \# l_i^{h_2}\big)(g) \\
&= \sum \pi^{-1}(\tau'(h_1)) \pi^{-1}(\lambda'(h_2))(g) \\
&= (\epsilon(h)\#\epsilon)(g) = \epsilon(h)\epsilon(g).
\end{aligned}$$

Similarly $\Sigma(\omega'(h_1) \cdot \varphi'(h_2))(g) = \epsilon(h)\epsilon(g)$. Thus $\Gamma(\mathscr{U}(Z_{\mathrm{tw}}^2(H, A))) = K(\Omega)$, and the sequence is exact. ∎

EXAMPLE 4.8. (i) Suppose $A^{\mathrm{co}\,H} = B = k$ so that by Lemma 2.2, $K(\Omega) = \mathscr{T}(A)$. If H is finitely generated projective over k and A/k is H-Galois, then Γ is a bijection of pointed sets from $\mathscr{U}(Z_{\mathrm{tw}}^2(H, A))$ to $\mathscr{U}(\mathscr{T}(A))$.

(ii) If $B = k$ and A is commutative, then by Remark 2.4(iii), $\mathcal{G}al(A)$ is in bijective correspondence with the set of invertible Sweedler cocycles $\mathcal{U}(Z^2(H, A))$ with trivial weak action which satisfy 2.3(ii).

(iii) If, as well, H is cocommutative, then $\mathcal{U}(Z^2(H, A))$ is an abelian group under convolution and so $\mathcal{U}(\mathcal{T}(A))$, $\mathcal{U}(\mathcal{L}(A))$, and $\mathcal{G}al(A)$ have induced abelian group structures also. Suppose $\varphi, \lambda \in \mathcal{U}(Z^2(H, A))$. Then

$$\begin{aligned}\tau'_\varphi * \tau'_\lambda(h)(a) &= \sum \tau'_\varphi(h_1)[\lambda(h_2, a_1)a_0] = \sum \varphi(h_1, a_1)\lambda(h_2, a_2)a_0 \\ &= \tau'_{\varphi * \lambda}(h)(a)\end{aligned}$$

and so twistings in $\mathcal{U}(\mathcal{T}(A))$ multiply by $\tau_\varphi \diamond \tau_\lambda(h \otimes a) = \sum \varphi(h_1, a_1)\lambda(h_2, a_2)a_0$. Then in $\mathcal{G}al(A)$, $A^\tau \bigstar A^\mu = A^{\tau \diamond \mu}$. Here A acts as the identity element and the inverse to A^τ is $A^{\tau^{-1}}$.

Finally, we show that for A/B Galois, the equivalence classes of twistings in $\mathcal{U}(\mathcal{T}(A))$ correspond to the isomorphism classes of twisted algebras A^τ. Here no finiteness restriction is imposed on H but if H is not finitely generated projective then we have not proved that every Galois H-object $C = (A, \times, \rho_A)$ is A^τ for some τ.

THEOREM 4.9. *Suppose A/B is H-Galois, $\tau, \lambda \in \mathcal{T}(A)$, and there is a left B-module right H-comodule algebra homomorphism ψ from A^λ to A^τ. Then there is a map v: $H \to A$ satisfying the conditions of Proposition* 2.9. *If ψ is an isomorphism, then* $v \in \operatorname{Reg}(H, A)$.

Proof. We imitate the notation of [11] and denote $\operatorname{can}^{-1}(1 \otimes h)$ by $\sum l_i(h) \otimes_B r_i(h) \in A \otimes_B A$, so that $1 \otimes h = \sum l_i(h) r_i(h)_0 \otimes r_i(h)_1$. Note that juxtaposition denotes multiplication in A and $*_\tau$, $*_\lambda$ denote multiplication in A^τ and A^λ, respectively. For $h \in H$, define

$$v(h) = \sum l_i(h)\psi(r_i(h)).$$

Since ψ is a left B-module map v: $H \to A$ is well defined, and clearly $v(1_H) = 1_A$.

For $a \in A$, from [11] or by just applying can to both sides of this equation, we see that

$$\sum a_0 l_i(a_1) \otimes_B r_i(a_1) = 1 \otimes_B a \in A \otimes_B A.$$

Thus

$$\sum a_0 v(a_1) = \sum a_0 l_i(a_1)\psi(r_i(a_1)) = \psi(a).$$

Since ψ is an H-comodule map, for all $a \in A$, we have $\sum \psi(a_0) \otimes a_1 = \sum \psi(a)_0 \otimes \psi(a)_1$ and thus $\sum a_0 v(a_1) \otimes a_2 = \sum a_0 v(a_2)_0 \otimes a_1 v(a_2)_1$. Then, using the standard argument since can′ is an isomorphism, we have that

$$\sum v(h_1) \otimes h_2 = \sum v(h_2)_0 \otimes h_1 v(h_2)_1.$$

Condition (ii) of Proposition 2.9 follows immediately, and it remains to check condition (iii). For $b, c \in A$, since ψ is an algebra map, we have $\psi(b *_\lambda c) = \psi(b) *_\tau \psi(c)$ which yields

$$\sum b_0 \lambda(b_1 \otimes c_0) v(b_2 c_1) = \sum b_0 v(b_1) \tau\big(b_2 \otimes c_0 v(c_1)\big).$$

Once again using the bijectivity of can′, we obtain (iii).

Now suppose that ψ is an isomorphism so that ψ^{-1} is a left B-module right H-comodule algebra map from A^τ to A^λ. Then there is a map w: $H \to A$ satisfying (i), (ii), and (iii) of Proposition 2.9 such that $\psi^{-1}(a) = \Sigma a_0 w(a_1)$ for all $a \in A$. Then for all $a \in A$ we have that $a = \Sigma a_0 v(a_1) w(a_2) = \Sigma a_0 w(a_1) v(a_2)$ and, again using the fact that A is Galois, we see that w and v are convolution inverses. ∎

EXAMPLE 4.10. Suppose H is cocommutative and A/k is a commutative Galois H-object. Then it is easy to see that cocycles φ and ω are cohomologous; i.e., there is $u \in \operatorname{Reg}(H, A)$ such that for all $h, g \in H$,

$$\varphi(h, g) = \sum u(h_1) u(g_1)\, \omega(h_2, g_2) u^{-1}(h_3 g_3),$$

if and only if $\tau_\varphi \sim \tau_\omega$. Then the group of isomorphism classes of algebras in $\mathcal{G}al(A)$ is isomorphic to the second Sweedler cohomology group $H^2(H, B)$.

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