ALGEBRAIC COMBINATORICS AND QUANTUM GROUPS

edited by

Naihuan Jing

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Naihuan Jing

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Preface

Combinatorial methods have become increasingly important in various research in representation theory, geometry and mathematical physics. Many interesting problems of algebra and geometry are closely related with symmetric groups and Hecke algebras, which have made algebraic combinatorics an indispensable tool in mathematics during the last several decades. For example, the development of Macdonald polynomials provides examples and stimulates interactive research both inside and outside algebraic combinatorics. Recent advances in representations of Hecke algebras and quantum affine algebras have partly relied on corresponding research in algebraic combinatorics.

The current volume consists of papers devoted to algebraic combinatorics, quantum groups and related topics by invited speakers in the CBMS conference "Algebraic Combinatorics" held at North Carolina State University during June 4-8, 2001. The conference's goal was to introduce graduate students and young researchers to this active field and survey some of current researches in related areas. Alain Lascoux gave ten featured lectures on Schur polynomials, Schubert polynomials and their applications. His lecture notes will be published by American Mathematical Society in a separate volume.

Three related areas are emphasized in this volume:

1) The theory of symmetric functions. Schur functions, Schur Q-functions (or P-functions) and Macdonald functions are studied from various angles.
2) Quantum affine algebras, Hecke algebras and quiver varieties.
3) Combinatorial formulas in statistical mechanics.

I would like to thank the support from National Science Foundation and also the mathematics department at NCSU for such an activity, and in particular I am grateful to Alain Lascoux for giving special lectures at the CBMS conference which made the gathering possible. Many thanks are due to anonymous referees who have read the papers and offered constructive suggestions. And last, but not least I would like to thank the authors who have contributed to this volume.

Naihuan Jing
Raleigh, North Carolina
February 2003

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UNO'S CONJECTURE ON REPRESENTATION TYPES OF HECKE ALGEBRAS

SUSUMU ARIKI

ABSTRACT. Based on a recent result of the author and A.Mathas, we prove that Uno's conjecture on representation types of Hecke algebras is true for all Hecke algebras of classical type.

1. INTRODUCTION

Let K be a field of characteristic l, A a finite dimensional K–algebra. We always assume that K is a splitting field of A. We say that A is of finite representation type if there are only finitely many isomorphism classes of indecomposable A–modules. If A is a group algebra, then the following theorem answers when A is of finite representation type.

Theorem 1. [3, 8] *Let G be a finite group, $A = KG$ its group algebra. Then A is of finite representation type if and only if the Sylow l–subgroups of G are cyclic.*

We restrict ourselves to the case where G is a finite Weyl group and see consequences of this result.

Theorem 2. *Let W be a finite Weyl group. Then KW is of finite representation type if and only if l^2 does not divide the order $|W|$.*

For the proof see Appendix. Note that Theorem 2 does not hold if we consider finite Coxeter groups. Dihedral groups $W(I_2(m))$ with odd l and $l^2 \mid m$ are obvious counterexamples.

Uno conjectured a q–analogue of Theorem 2. Let $q \in K$ be an invertible element, (W, S) a finite Weyl group. We denote by $\mathcal{H}_q(W)$ the (one parameter) Hecke algebra associated to W. The quadratic relation we choose is $(T_s + 1)(T_s - q) = 0$ $(s \in S)$. The Poincare polynomial $P_W(x)$ is defined by

$$P_W(x) = \sum_{w \in W} x^{l(w)}.$$

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Key words and phrases. Hecke algebra, finite representation type.

Let e be the smallest positive integer such that $q^{e-1} + \cdots + 1 = 0$. If $q = 1$ then $e = l$, and if $q \neq 1$ then e is the multiplicative order of q. As the condition that l^2 does not divide $|W|$ is the same as $(\frac{x^e-1}{x-1})^2$ evaluated at $x = 1$ does not divide $P_W(1)$, the following is a reasonable guess. We call this Uno's conjecture.

Conjecture 3. Assume that $q \neq 1$ and K a splitting field of $\mathcal{H}_q(W)$. Then $\mathcal{H}_q(W)$ is of finite representation type and not semisimple if and only if q is a simple root of $P_W(x) = 0$, that is, if and only if $(x - q)^2$ does not divide $P_W(x)$.

It is well-known that $\mathcal{H}_q(W)$ is semisimple if and only if $P_W(q) \neq 0$. See [7, Proposition 2.3] for example. This fact will be used later.

Remark 4. In [11], it is proved that the conjecture is true for $\mathcal{H}_q(I_2(m))$. So, unlike the case of $q = 1$, we may ask the same question for finite Coxeter groups instead of finite Weyl groups.

The following theorem is proved in [11].

Theorem 5. [11, Proposition 3.7,Theorem 3.8] *Suppose that $q \neq 1$ and denote its multiplicative order by e. Then $\mathcal{H}_q(A_{n-1})$ is of finite representation type if and only if $n < 2e$.*

As $P_W(x) = \prod_{i=1}^{n} \frac{x^i-1}{x-1}$ in this case, a primitive e^{th} root of unity is a simple root if and only if $n < 2e$. In particular, the conjecture is true if $W = W(A_{n-1})$. The purpose of this article is to prove.

Theorem 6. (Main Theorem) *Assume that W is of classical type and that K is a splitting field of $\mathcal{H}_q(W)$. Then $\mathcal{H}_q(W)$ is of finite representation type and not semisimple if and only if q is a simple root of $P_W(x) = 0$.*

We remark that the exceptional cases are settled recently by Miyachi [10] under the assumption that the characteristic l of the base field K is not too small.

2. Reduction to Hecke algebras associated to irreducible Weyl groups

This is proved by using the complexity of modules. Let A be a self–injective finite dimensional K–algebra, M a finite dimensional A–module, $P^{\cdot} \to M$ be its minimal projective resolution. Then the complexity $c_A(M)$ is the smallest integer $s \geq 0$ such that $\dim_K(P^t)/(t+1)^{s-1}$ $(t = 0, 1, \dots)$ is bounded. The following lemma is fundamental.

Lemma 7. *Suppose that A is self–injective as above. Then*

(1) $c_A(M) = 0$ *if and only if M is a projective A–module.*

(2) *A is semisimple if and only if $c_A(M) = 0$ for all indecomposable A–modules M.*
(3) *If A is of finite representation type and not semisimple then $c_A(M) \leq 1$ for all indecomposable A–modules M and the equality holds for some M.*

Proposition 8. *Let $\mathcal{S}$ be a set of irreducible finite Weyl groups. If Uno's conjecture is true for all $\mathcal{H}_q(W)$ with $W \in \mathcal{S}$ then the conjecture is true for $\mathcal{H}_q(W_1 \times \cdots \times W_r)$ with $W_1, \ldots, W_r \in \mathcal{S}$.*

Proof. Write $W = W_1 \times \cdots \times W_r$. Then $\mathcal{H}_q(W) = \mathcal{H}_q(W_1) \otimes \cdots \otimes \mathcal{H}_q(W_r)$ and $P_W(x) = P_{W_1}(x) \cdots P_{W_r}(x)$.

First assume that $q \neq 1$ is a simple root of $P_W(x) = 0$. Then q is a simple root of $P_{W_i}(x) = 0$ and $P_{W_j}(q) \neq 0$ for all $j \neq i$. Then $\mathcal{H}_q(W_j)$ for $j \neq i$ are all semisimple and $\mathcal{H}_q(W_i)$ is of finite representation type and not semisimple. Thus $\mathcal{H}_q(W)$ is of finite representation type and not semisimple.

Next assume that q is a multiple root of $P_W(x) = 0$. If q is a multiple root of $P_{W_i}(x) = 0$, for some i, then $\mathcal{H}_q(W_i)$ is of infinite representation type by assumption. Thus $\mathcal{H}_q(W)$ is of infinite representation type. If q is a simple root of $P_{W_i}(x) = 0$ and $P_{W_j}(x) = 0$, for some $i \neq j$, then $\mathcal{H}_q(W_i)$ and $\mathcal{H}_q(W_j)$ are of finite representation type and not semisimple. By Lemma 7(3), there exist an indecomposable $\mathcal{H}_q(W_i)$–module M_i and an indecomposable $\mathcal{H}_q(W_j)$–module M_j such that $c_{\mathcal{H}_q(W_i)}(M_i) = 1$ and $c_{\mathcal{H}_q(W_j)}(M_j) = 1$. Write $M = M_i \otimes M_j$. We shall prove that the complexity of M as an indecomposable $\mathcal{H}_q(W_i) \otimes \mathcal{H}_q(W_j)$–module is equal to 2; if we use the fact that $c_{\mathcal{H}_q(W_i \times W_j)}(M)$ is the growth rate of $\mathrm{Ext}^{\bullet}(M, M)$ then the Kunneth formula implies the result. In a more concrete manner, the proof of $c_{\mathcal{H}_q(W_i \times W_j)}(M) = 2$ is as follows.

Let $P_i^{\bullet}$ and $P_j^{\bullet}$ be minimal projective resolutions of M_i and M_j respectively. Then $c_{\mathcal{H}_q(W_i)}(M_i) = 1$ and $c_{\mathcal{H}_q(W_j)}(M_j) = 1$ imply that there exists a constant C such that $1 \leq \dim_K(P_i^t) \leq C$ and $1 \leq \dim_K(P_j^t) \leq C$ for all t. As $P^{\bullet} = P_i^{\bullet} \otimes P_j^{\bullet}$ is a minimal projective resolution of M, we have

$$t + 1 \leq \dim_K(P^t) = \sum_{s=0}^{t} \dim_K(P_i^s)\dim_K(P_j^{t-s}) \leq C^2(t+1).$$

Therefore, the complexity of M is exactly 2. As a result, $\mathcal{H}_q(W_i) \otimes \mathcal{H}_q(W_j)$ is of infinite representation type by Lemma 7(2) and (3). Thus $\mathcal{H}_q(W)$ is also of infinite representation type. □

3. Type B and type D

To prove Theorem 6, it is enough to consider type B and type D by virtue of Theorem 5 and Proposition 8.

Let q and Q be invertible elements of K. The (two parameter) Hecke algebra $\mathcal{H}_{q,Q}(B_n)$ of type B_n is the unital associative K–algebra defined by generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$(T_0+1)(T_0-Q)=0, \quad (T_i+1)(T_i-q)=0\ (1\le i\le n-1),$$
$$T_0T_1T_0T_1=T_1T_0T_1T_0, \quad T_{i+1}T_iT_{i+1}=T_iT_{i+1}T_i\ (1\le i\le n-2),$$
$$T_iT_j=T_jT_i\ (0\le i<j-1\le n-2).$$

The following theorem, together with Theorem 5, will allow us to assume that $-Q$ is a power of q.

Theorem 9. [5, Theorem 4.17] *Suppose that $Q\neq -q^f$ for any $f\in\mathbb{Z}$. Then $\mathcal{H}_{q,Q}(B_n)$ is Morita equivalent to $\bigoplus_{m=0}^n \mathcal{H}_q(A_{m-1})\otimes\mathcal{H}_q(A_{n-m-1})$.*

Corollary 10. *Assume that q is a primitive e^{th} root of unity with $e\ge 2$ as above. If $Q\neq -q^f$ for any $f\in\mathbb{Z}$ then $\mathcal{H}_{q,Q}(B_n)$ is of finite representation type if and only if $n<2e$.*

Proof. If $n\ge 2e$ then $\mathcal{H}_q(A_{n-1})$ is of infinite representation type by Theorem 5. Thus, $\mathcal{H}_{q,Q}(B_n)$ is of inifinite representation type by Theorem 9.

If $n<2e$ then one of m and $n-m$ is smaller than e for each m. Thus, one of $\mathcal{H}_q(A_{m-1})$ and $\mathcal{H}_q(A_{n-m-1})$ is semisimple and the other is of finite representation type for each m. Thus, $\mathcal{H}_{q,Q}(B_n)$ is of finite representation type by Theorem 9. □

Theorem 11. [1, Theorem 1.4] *Suppose that $-Q=q^f$ $(0\le f<e)$ and $e\ge 3$. Then $\mathcal{H}_{q,Q}(B_n)$ is of finite representation type if and only if*

$$n<\min\{e, 2\min\{f,e-f\}+4\}.$$

It is also easy to prove that Theorem 11 is valid for $e=2$; see [2].

Corollary 12. *Uno's conjecture is true if $W=W(B_n)$.*

Proof. See the argument of [1, Introduction]. □

Recall that $\mathcal{H}_q(D_n)$ is the K–algebra defined by generators $T_0^D,\dots,T_{n-1}^D$ and relations

$$(T_i^D+1)(T_i^D-q)=0\ (0\le i\le n-1),$$
$$T_0^DT_2^DT_0^D=T_2^DT_0^DT_2^D, \quad T_0^DT_i^D=T_i^DT_0^D\ (i\neq 2),$$
$$T_{i+1}^DT_i^DT_{i+1}^D=T_i^DT_{i+1}^DT_i^D\ (1\le i\le n-2),$$
$$T_i^DT_j^D=T_j^DT_i^D\ (1\le i<j-1\le n-2).$$

Now assume that $Q=1$ and denote the generators of $\mathcal{H}_{q,1}(B_n)$ by $T_0^B,\dots,T_{n-1}^B$. Then we have an algebra homomorphism

$$\phi:\mathcal{H}_q(D_n)\longrightarrow\mathcal{H}_{q,1}(B_n)$$

defined by $T_0^D\mapsto T_0^BT_1^BT_0^B$ and $T_i^D\mapsto T_i^B$ for $1\le i\le n-1$.

ϕ is injective and we identify $\mathcal{H}_q(D_n)$ with its image. Then we have

$$\mathcal{H}_{q,1}(B_n) = \mathcal{H}_q(D_n) \oplus T_0^B \mathcal{H}_q(D_n) \text{ and } T_0^B \mathcal{H}_q(D_n) = \mathcal{H}_q(D_n) T_0^B.$$

We define an algebra automorphism π of $\mathcal{H}_{q,1}(B_n)$ by

$$\pi(T_1^B) = T_0^B T_1^B T_0^B \text{ and } \pi(T_i^B) = T_i^B \text{ for } i \neq 1.$$

We have $\pi^2 = 1$ and π induces the Dynkin automorphism of $\mathcal{H}_q(D_n)$ defined by $T_i^D \mapsto T_{1-i}^D$ for $i = 0, 1$ and $T_i^D \mapsto T_i^D$ for $i \geq 2$.

Lemma 13. (1) *If e is odd then $\mathcal{H}_{q,1}(B_n)$ is of finite representation type if and only if $n < 2e$.*

(2) *If e is even then $\mathcal{H}_{q,1}(B_n)$ is of finite representation type if and only if $n < e$.*

(3) *If M is a semisimple $\mathcal{H}_{q,1}(B_n)$-module then so is the $\mathcal{H}_q(D_n)$-module* $\mathrm{Res}(M)$.

Proof. (1)(2) These are consequences of Corollary 10 and Theorem 11.
(3) We may assume that M is a simple $\mathcal{H}_{q,1}(B_n)$–submodule without loss of generality. Let N be a simple $\mathcal{H}_q(D_n)$–submodule of $\mathrm{Res}(M)$. Then $T_0^B N$ is also a simple $\mathcal{H}_q(D_n)$–submodule whose action is the twist of the action of N by π. Since $N + T_0^B N$ is $\mathcal{H}_{q,1}(B_n)$–stable, it coincides with M. Hence $\mathrm{Res}(M)$ is semisimple. □

Let D be a $\mathcal{H}_q(D_e)$–module which affords the sign representation $T_i^D \mapsto -1$, for $0 \leq i \leq e-1$. We denote its projective cover by P.

Recall from [1] that simple $\mathcal{H}_{q,1}(B_e)$–modules are indexed by Kleshchev bipartitions. Further, $\lambda = ((0), (1^e))$ is Kleshchev and if $e \geq 3$ is even then the projective cover P^λ of the simple $\mathcal{H}_{q,1}(B_e)$–module D^λ has the property that $\mathrm{Rad}P^\lambda/\mathrm{Rad}^2 P^\lambda$ does not contain D^λ and $\mathrm{Rad}^2 P^\lambda/\mathrm{Rad}^3 P^\lambda$ contains $D^\lambda \oplus D^\lambda$. See the proof of [1, Theorem 4.1] in p.12. D^λ affords the representation $T_i^B \mapsto -1$ $(0 \leq i \leq n-1)$.

Lemma 14. *Assume that e is even. Then $P \simeq \mathrm{Res}(P^\lambda)$.*

Proof. First note that the characteristic l of the base field is odd since e is even. Let D^μ be the simple $\mathcal{H}_{q,1}(B_e)$–module which affords the representation $T_0^B \mapsto 1$ and $T_i^B \mapsto -1$ $(1 \leq i \leq e-1)$. Then $\mathrm{Ind}(D) = D^\lambda \oplus D^\mu$ since the left hand side is given by

$$T_0^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_i^B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ (1 \leq i \leq e-1),$$

and T_0^B is diagonalizable because l is odd. Now the surjection $P \to D$ induces surjective homomorphisms $\mathrm{Ind}(P) \to D^\lambda$ and $\mathrm{Ind}(P) \to D^\mu$. Hence these induce surjective homomorphisms $\mathrm{Ind}(P) \to P^\lambda$ and $\mathrm{Ind}(P) \to P^\mu$. We have that both P^λ and P^μ are direct summands of $\mathrm{Ind}(P)$. On the other hand, Mackey's formula implies that

$$\mathrm{Res}\,(\mathrm{Ind}(P)) \simeq P \oplus {}^\pi P,$$

where ${}^{\pi}P$ is the indecomposable projective $\mathcal{H}_q(D_e)$–module which is the twist of P by π. As the twist of D is D itself, we have $\mathrm{Res}(\mathrm{Ind}(P)) \simeq P \oplus P$. As $\mathrm{Res}(P^{\lambda})$ and $\mathrm{Res}(P^{\mu})$ are direct summands of $\mathrm{Res}(\mathrm{Ind}(P))$, we conclude that $\mathrm{Res}(P^{\lambda})$ and $\mathrm{Res}(P^{\mu})$ are isomorphic to P. □

Next lemma is obvious.

Lemma 15. *Let A be a finite dimensional K–algebra and B a K–subalgebra such that B is a direct summand of A as a (B, B)–bimodule. If A is of finite representation type then so is B.*

Recall that $P_W(x) = (x^n - 1)\prod_{i=1}^{n-1} \frac{x^{2i}-1}{x-1}$ in type D_n. Thus q is a simple root of $P_W(x) = 0$ if and only if either e is odd and $e \le n < 2e$ or e is even and $\frac{e}{2} + 1 \le n < e$.

Proposition 16. *Uno's conjecture is true if $W = W(D_n)$.*

Proof. First assume that e is odd. If $n < 2e$ then Lemma 15 implies that $\mathcal{H}_q(D_n)$ is of finite representation type since $\mathcal{H}_{q,1}(B_n)$ is of finite representation type by Lemma 13(1). If $n \ge 2e$ then it is enough to prove that $\mathcal{H}_q(D_{2e})$ is of infinite representation type by Lemma 15. Using the same lemma again, we further know that it is enough to prove that $\mathcal{H}_q(A_{2e-1})$ is of infinite representation type. However, this is nothing but the result of Theorem 5.

Next assume that e is even. If $n < e$ then Lemma 15 implies that $\mathcal{H}_q(D_n)$ is of finite representation type since $\mathcal{H}_{q,1}(B_n)$ is of finite representation type by Lemma 13(2). If $n \ge e$ then it is enough to prove that $\mathcal{H}_q(D_e)$ is of infinite representation type by Lemma 15. Note that $W(D_2) = W(A_1) \times W(A_1)$ and the conjecture is true in this case by Proposition 8. Thus we may assume that $e \ge 4$. In particular, we have $q \ne -1$, and this implies that $\mathrm{Ext}^1(D, D) = 0$; if we write

$$T_i = \begin{pmatrix} -1 & a_i \\ 0 & -1 \end{pmatrix},$$

then $T_i - q$ is invertible and thus $a_i = 0$.

Let $\overline{P} = \mathrm{Res}(P^{\lambda}/\mathrm{Rad}^3 P^{\lambda})$. Lemma 13(3) and $\mathrm{Ext}^1(D, D) = 0$ imply that $\overline{P}$ has Loewy length 3. Since $\overline{P}$ has unique head D by Lemma 14, there exists a surjective homomorphism $P \to \overline{P}$. Further, as $\overline{P}$ contains $D \oplus D$ as a $\mathcal{H}_q(D_e)$–submodule, $\mathrm{Rad}^2 P/\mathrm{Rad}^3 P$ contains $D \oplus D$. On the other hand, $\mathrm{Ext}^1(D, D) = 0$ implies that $\mathrm{Rad}P/\mathrm{Rad}^2 P$ does not contain D. Hence we conclude that $\mathrm{End}_{\mathcal{H}_q(D_e)}(P/\mathrm{Rad}^3 P)$ is isomorphic to $K[X, Y]/(X^2, XY, Y^2)$, which is not isomorphic to any of the truncated polynomial rings $K[X]/(X^N)$ $(N = 1, 2, \dots)$. As we assume that the base field is a splitting field of $\mathcal{H}_q(D_e)$, this implies that $\mathcal{H}_q(D_e)$ is of infinite representation type. □

4. Appendix

In this section, we prove Theorem 2. If the reader is familiar with the structure of the Sylow subgroups of exceptional Weyl groups then (s)he would not need this proof to know that Theorem 2 is true. In the proof below, we use standard facts about the structure of exceptional Weyl groups; they can be found in [4] or [9, 2.12]. First we consider irreducible Weyl groups.

Type A_{n-1};

W has cyclic Sylow l–subgroups if and only if $n < 2l$, and this is equivalent to the condition that l^2 does not divide $|W| = n!$.

Type B_n;

W has cyclic Sylow l–subgroups if and only if either $l > 2$ and $n < 2l$ or $l = 2$ and $n < 2$, and this is equivalent to the condition that l^2 does not divide $|W| = 2^n n!$.

Type D_n;

W has cyclic Sylow l–subgroups if and only if either $l > 2$ and $n < 2l$ or $l = 2$ and $n < 2$, and this is equivalent to the condition l^2 does not divide $|W| = 2^{n-1}n!$.

Type F_4;

As $|W(F_4)| = 2^7 \cdot 3^2$, we prove that the Sylow l–subgroups for $l = 2, 3$ are not cyclic. Let $\Delta(F_4)$ be the root system of type F_4. The long roots form a root system which is isomorphic to $\Delta(D_4)$. Let Γ be the Dynkin automorphism group of the Dynkin diagram of type D_4. Then it is known that $W(F_4)$ is isomorphic to the semi–direct product of $W(D_4)$ and Γ.

Assume that $l = 2$. Since $W(D_4)$ contains $C_2 \times C_2$, the Sylow 2–subgroup of $W(F_4)$ cannot be cyclic.

Assume that $l = 3$. Since Γ is isomorphic to the symmetric group of degree 3, we can choose $\sigma \in \Gamma$ of order 3. Let P be a Sylow 3–subgroup of $W(F_4)$ containing σ. As the Sylow 3–subgroup of $W(D_4)$ is a cyclic group of order 3, we have $|W(D_4) \cap P| \leq 3$. On the other hand, as $\langle\sigma\rangle$ is a Sylow 3–subgroup of Γ and $|P| = 9$, both $W(D_4) \cap P$ and $P/W(D_4) \cap P$ are isomorphic to the cyclic group of order 3. Let τ be a generator of $W(D_4) \cap P$. Then we have the following split exact sequence.

$$1 \longrightarrow \langle\tau\rangle \simeq C_3 \longrightarrow P \longrightarrow \langle\sigma\rangle \simeq C_3 \longrightarrow 1.$$

As $\mathrm{Aut}(C_3) \simeq C_2$, σ acts on $\langle\tau\rangle$ trivially and $P \simeq C_3 \times C_3$.

Type E_n;

Recall that $|W(E_6)| = 2^7 \cdot 3^4 \cdot 5$, $|W(E_7)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$ and $|W(E_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

Assume that $l = 2$ or $l = 3$. Since $W(F_4) \subset W(E_n)$, The Sylow l–subgroup of $W(E_n)$ contains $C_l \times C_l$. Thus it cannot be a cyclic group.

Assume that $l = 5$. Let Q be the root lattice of type E_8 with scalar product normalized to $(\alpha_i, \alpha_i) = 2$ for simple roots α_i $(1 \le i \le 8)$. Then $q(x) = \frac{(x,x)}{2} \pmod 2$ defines a quadratic form on $Q/2Q \simeq \mathbb{F}_2^8$. Note that if we choose simple roots as a basis, we can write down $q(x)$ explicitly, and the computation of its Witt decomposition shows that its Witt index is 4. Thus, by choosing a different basis, we may assume that $q(x) = \sum_{i=1}^4 x_{2i-1}x_{2i}$. Let $O_8(2)$ be the orthogonal group associated to this form. Then it is known that there is an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow W(E_8) \longrightarrow O_8(2) \longrightarrow 1.$$

Let $q'(x) = x_1x_2 + x_3^2 + x_3x_4 + x_4^2$ be a quadratic form on $\mathbb{F}_2^4$. If we write $O_4^-(2)$ for the orthogonal group associated to this form, we know that its Sylow 5–subgroups are cyclic of order 5. Now explicit computation of Witt decomposition again shows that $q' \oplus q'$ has Witt index 4. Thus $O_8(2)$ contains $C_5 \times C_5$. As a result, the Sylow 5–subgroup of $W(E_8)$ is isomorphic to $C_5 \times C_5$.

Type G_2;

This is the dihedral group of order 12 and its Sylow 2–subgoups are not cyclic.

Now let W be a general finite Weyl group. That is, W is a product of the groups listed above. Then W has a cyclic Sylow l–group if and only if at most one component of the product has a cyclic Sylow l–group and all the other components have trivial Sylow l–groups. This is equivalent to the condition that l^2 does not divide $|W|$.

References

[1] S. Ariki and A. Mathas, The representation type of Hecke algebras of type B, to appear in Adv. Math., **math.RT/0106185**.

[2] S. Ariki and A. Mathas, The Hecke algebras with a finite number of indecomposable modules: a summary of results, preprint.

[3] V.M. Bondarenko and J.A. Drozd, The representation type of finite groups, J. Soviet Math., **20** (1982), 2515–2528.

[4] N. Bourbaki, Groupes et algébres de Lie, (1968), Hermann.

[5] R. Dipper and G. James, Representations of Hecke algebras of type B_n, J. Algebra, **146** (1992), 454–481.

[6] K. Erdmann and D.K. Nakano, Representation type of Hecke algebras of type A, Trans. A.M.S., **354** (2002), 275–285.

[7] M. Geck, Brauer trees of Hecke algebras, Comm. Alg., **20** (1992), 2937–2973.

[8] D. Higman, Indecomposable representations at characteristic p, Duke Math.J., **21** (1954), 377–381.

[9] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics **29** (1990), Cambridge University Press.

[10] H. Miyachi, Uno's conjecture for the exceptional Hecke algebras, in preparation.

[11] K. Uno, On representations of non–semisimple specialized Hecke algebras, J. Algebra, **149** (1992), 287–312.

RIMS, Kyoto University, Kyoto 606–8502, Japan
E-mail address: ariki@kurims.kyoto-u.ac.jp

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QUIVER VARIETIES, AFFINE LIE ALGEBRAS, ALGEBRAS OF BPS STATES, AND SEMICANONICAL BASIS

IGOR FRENKEL, ANTON MALKIN, AND MAXIM VYBORNOV

ABSTRACT. We suggest a (conjectural) construction of a basis in the plus part $\widetilde{\mathfrak{n}}_+$ of the affine Lie algebra of type ADE indexed by irreducible components of certain quiver varieties. This construction is closely related to a string-theoretic construction of a Lie algebra of BPS states. We then study the new combinatorial questions about the (classical) root systems naturally arising from our constructions and Lusztig's semicanonical basis.

1. INTRODUCTION

1.1. Since the appearance of the first manifestation of the connection between theory of representations of quivers and the structure theory of Lie algebras [Gab72], several authors discovered constructions of Lie algebras arising from the quiver theory. C. M. Ringel constructed the plus part of the simple Lie algebra of type ADE, [Rin90], and a (related) construction of the plus part of any Kac-Moody Lie algebra is implicit in Lusztig's construction of (quantized) enveloping algebras of Kac-Moody Lie algebras in terms of functions on a class of remarkable affine varieties Λ_V, [Lus91, Lus92].

Inspired by Ringel and Lusztig, the present authors came up with a construction of the plus part $\widetilde{\mathfrak{n}}_+$ of the affine Lie algebra of type ADE in terms of indecomposable representations of quivers [FMV01].

A similar construction is suggested by string theorists [FM00], who use the *stable* representations of (double) quivers rather than indecomposable representations of oriented quivers (as do Ringel and the present authors). The concept of stability has been very useful in representation theory: H. Nakajima discovered that modules over Kac-Moody Lie algebras may be described using functions on the varieties Λ_V^s/G_V where Λ_V^s are the *stable* points of Lusztig's varieties Λ_V [Nak94, Nak98].

1.2. One of the main goals of this paper is to study the relationship between the Lusztig's construction of $\widetilde{\mathfrak{n}}_+$, the construction by the present authors [FMV01], and the "stable" construction suggested by physicists. Using the notion of *semistable diagonal* in Λ_V suggested in [HM98], we make a conjecture (Conjecture 4.2.5) directly relating the Lusztig's construction and the "stable" construction. Moreover, we (conjecturally) obtain a basis in $\widetilde{\mathfrak{n}}_+$ parameterized by irreducible components of algebraic varieties. One can look at this conjectural basis as "semicanonical" basis for $\widetilde{\mathfrak{n}}_+$.

1.3. In section 5 we use our methods to obtain some simple results describing new aspects (arising from quiver constructions) of combinatorics of the root systems and Weyl groups, and ask many more questions than we can answer at the moment.

One of the results of the Lusztig construction of the (quantized) universal enveloping algebras in terms of functions on the varieties Λ_V is the appearance of the *semicanonical* basis in the non-quantized enveloping algebras of simply laced Kac-Moody Lie algebras [Lus00]. Semicanonical basis is indexed by the irreducible components of Λ_V. It was expected that this basis would coincide with the specialization of the Lusztig's canonical basis to $q = 1$, but a counterexample was found by M. Kashiwara and Y. Saito [KS97]. The relationship between these two bases is quite mysterious at the moment, even though they share their combinatorial properties.

Let $\mathfrak{n}_+ \subset U(\mathfrak{n}_+)$ be the plus part of a simple Lie algebra of type ADE, and $U(\mathfrak{n}_+)$ be its universal enveloping algebra. For every positive root $\alpha \in R_+$ we have a canonically (up to a sign) defined element $E^*_\alpha \in \mathfrak{n}_\alpha \subset U(\mathfrak{n}_+)$, where $\mathfrak{n}_\alpha$ is the one dimensional root subspace of $\mathfrak{n}_+$ corresponding to α. We study the decomposition of E^*_α with respect to the semicanonical basis. More precisely, let us consider Λ_V, $\dim V = \alpha$. Then the irreducible components of Λ_V may be indexed by decompositions of α into the sum of positive roots. Let $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_l$, $\alpha_i \in R_+$ be such a decomposition and let $e_{\alpha_1 + \alpha_2 + \cdots + \alpha_l}$ be the element of the semicanonical basis corresponding to this decomposition (and the corresponding irreducible component of Λ_V). Then

$$E^*_\alpha = \sum_{\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_l} c_{\alpha_1 + \alpha_2 + \cdots + \alpha_l} e_{\alpha_1 + \alpha_2 + \cdots + \alpha_l},$$

where $c_{\alpha_1 + \alpha_2 + \cdots + \alpha_l} \in \mathbb{Z}$ is the coefficient of $e_{\alpha_1 + \alpha_2 + \cdots + \alpha_l}$. In other words, with the help of the semicanonical basis we assign an integer $c_{\alpha_1 + \alpha_2 + \cdots + \alpha_l}$ to every decomposition $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_l$ of the root α into a sum of positive roots. It would be very interesting give a purely combinatorial description of these numbers without appealing to the semicanonical basis. We manage to obtain such a description in the A_n case in terms of the sign character of the Weyl group $W = S_{n+1}$.

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2. Preliminaries

We always assume that our ground field is the field of complex numbers $\mathbb{C}$. Our notation and conventions are mostly lifted from [Lus91, Lus00, FMV01].

2.1. Quivers.

2.1.1. To a graph Q, with no edges joining a vertex with itself, we associate a pair of sets: I (vertices), and H (oriented edges), and two maps from H to I:

(2.1.1.1) a map $H \to I$ denoted $h \to h'$ (initial vertex),
(2.1.1.2) a map $H \to I$ denoted $h \to h''$ (terminal vertex),
(2.1.1.3) an involution $h \to \bar{h}$ on H which maps an oriented edge to the same edge with the opposite orientation.

An *orientation* of (I, H) is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \overline{\Omega} = H$ and $\Omega \cap \overline{\Omega} = \emptyset$. Abusing terminology we will call both (I, H) and (I, Ω) *quivers*.

2.1.2. Let $\mathcal{V}$ be the category of I-graded vector spaces $V = \oplus_{i \in I} V_i$. We define:

$$\mathbf{E}_V = \bigoplus_{h \in H} \operatorname{Hom}(V_{h'}, V_{h''}),$$

$$\mathbf{E}_{V,\Omega} = \bigoplus_{h \in \Omega} \operatorname{Hom}(V_{h'}, V_{h''}).$$

For an element $x \in \mathbf{E}_V$ we denote by x_h, $h \in H$ its component in $\operatorname{Hom}(V_{h'}, V_{h''})$. A pair (V, x), $x \in \mathbf{E}_V$ (resp. $x \in \mathbf{E}_{V,\Omega}$) is called a representation of (I, H) (resp. (I, Ω)). We will also sometimes call (V, x) a module over (I, H), and moreover denote such a module by V if it is clear what x we consider.

Let $\mathbb{Z}[I]$ be the free abelian group generated by the set I. The *dimension* of (V, x) with $x \in \mathbf{E}_V$ or $x \in \mathbf{E}_{V,\Omega}$ is an element of $\mathbb{Z}[I]$ defined as follows:

$$\dim(V, x) = \dim V = \sum_{i \in I} \dim_{\mathbb{C}}(V_i) i \in \mathbb{Z}[I].$$

The algebraic group $G_V = \prod_{i \in I} GL(V_i)$ acts on $\mathbf{E}_V$ in a natural way.

2.1.3. We say that $x \in \mathbf{E}_{V,\Omega}$ is *indecomposable* if (V, x) is indecomposable as a representation of (I, Ω). The subset of indecomposable elements in $\mathbf{E}_{V,\Omega}$ is denoted by $\mathbf{E}^{\mathrm{ind}}_{V,\Omega}$.

2.1.4. We say that $x \in \mathbf{E}_V$ is *nilpotent* if there exists an $N \geq 2$ such that for any sequence $h_1, h_2, \ldots, h_N \in H$, such that $h'_1 = h''_2, h'_2 = h''_3, \ldots, h'_{N-1} = h''_N$, the composition $x_{h_1} x_{h_2} \ldots x_{h_N} = 0$. The subset of nilpotent elements in $\mathbf{E}_V$ is denoted by $\mathbf{E}_V^{nil}$.

2.1.5. Following [Lus91] we consider the moment map attached to the G_V-action on $\mathbf{E}_V$. The i-component m_i of this map is given by

$$m_i(x) = \sum_{h \in H : h'' = i} \epsilon(h) x_h x_{\bar{h}},$$

where $\epsilon : H \to \mathbb{C}^*$ is a function such that $\epsilon(h) + \epsilon(\bar{h}) = 0$ for all $h \in H$. Following Lusztig we introduce the subvariety Λ_V of $\mathbf{E}_V$ as follows

Definition. *Λ_V is the closed subvariety of all nilpotent elements $x \in \mathbf{E}_V$ such that $m_i(x) = 0$ for all $i \in I$.*

2.2. Convolution product.

2.2.1. Let

$$\mathcal{L}(V) = M_{G_V}(\Lambda_V)$$

be the set of constructible $\mathbb{C}$-valued functions on Λ_V which are constant on G_V-orbits in Λ_V. Let $\mathcal{L}_{\mathbb{Q}}(V)$ (resp. $\mathcal{L}_{\mathbb{Z}}(V)$) be the set of all $f \in \mathcal{L}(V)$ with rational (resp. integral) values.

2.2.2. Let $V, V', V'' \in \mathcal{V}$ be such that $\dim V' + \dim V'' = \dim V$. Let $f' \in \mathcal{L}(V')$, $f'' \in \mathcal{L}(V'')$. We lift the convolution construction from [Lus91, 12.10]. Consider the diagram

$$\Lambda_{V'} \times \Lambda_{V''} \xleftarrow{p_1} \mathbf{F}' \xrightarrow{p_2} \mathbf{F}'' \xrightarrow{p_3} \Lambda_V,$$

where $\mathbf{F}''$ is the variety of all pairs (x, W) where $x \in \Lambda_V$ and $W \subset V$ is an x-stable subspace, $\dim W = \dim V''$.

$\mathbf{F}'$ is the variety of all quadruples (x, W, R'', R') where $(x, W) \in \mathbf{F}''$, and R', R'' are $\mathcal{V}$-isomorphisms $R' : V' \simeq V/W$, $R'' : V'' \simeq W$.

We have $p_1(x, W, R'', R') = (x', x'')$ where $x_h R'_{h'} = R'_{h''} x'_h$, and $x_h R''_{h'} = R''_{h''} x''_h$ for all $h \in H$.

We have $p_2(x, W, R'', R') = (x, W)$, $p_3(x, W) = x$.

Let $f_1 \in \mathcal{L}(\Lambda_{V'} \times \Lambda_{V''})$ be given by $f_1(x', x'') = f'(x') f''(x'')$. Then there is a unique function $f_3 \in \mathcal{L}(\mathbf{F}'')$ such that $p_1^* f_1 = p_2^* f_3$. Then by definition

$$f' * f'' = (p_3)_!(f_3).$$

There exists an analogous construction for the oriented quiver [Lus91, 10.19]. We will denote the corresponding convolution product by $*_\Omega$ for an orientation Ω.

2.2.3. If $\dim V = i \in I$, then Λ_V is a point, and we denote by $E_i(\Lambda_V) \equiv 1$ the function which is identically 1 on this point. Let $\mathcal{L}_0$ be the associative algebra with $*$-product generated by $\{E_i\}_{i\in I}$. (The associative algebra of functions on $\mathbf{E}_{V,\Omega}$ with $*_\Omega$-product generated by $\{E_i\}_{i\in I}$ will be denoted by $\mathcal{L}_{0,\Omega}$.) One can consider $\mathcal{L}_0$ as a $\mathbb{Z}_+[I]$-graded Lie algebra over $\mathbb{Q}$, with the following Lie bracket:

$$[f,g] = f * g - g * f.$$

We denote by $\mathfrak{n}^*$ the Lie subalgebra of $\mathcal{L}_0$ generated by $\{E_i\}_{i\in I}$.

2.3. Geometric realization of the enveloping algebra.

2.3.1. Let (I,H) be a quiver. For $i,j \in I$ define $i \cdot j = -\operatorname{Card}\{h \in H \mid h' = i, h'' = j\}$, if $i \neq j$ and $i \cdot j = 2$ if $i = j$. Let U^+ be the $\mathbb{C}$-algebra defined by generators e_i, $i \in I$ and the Serre relations:

$$\sum_{\substack{p,q\in\mathbb{Z}_{\geq 0}\\ p+q=-i\cdot j+1}} (-1)^p \frac{e_i^p}{p!} e_j \frac{e_i^q}{q!} = 0$$

for any $i \neq j$ in I. $U^+ = U(\mathfrak{n})$ is the enveloping algebra of the Lie algebra $\mathfrak{n}$ defined by generators e_i, $i \in I$ and the Serre relations:

$$(\operatorname{ad} e_i)^{-i\cdot j+1}(e_j) = 0$$

for any $i \neq j$ in I.

Let (V,x) be a representation of (I,H). Let $U^+_{\dim V}$ denote the subspace of U^+ generated by $e_{i_1}e_{i_2}\dots e_{i_n}$ for sequences $i_1, i_2, \dots, i_n$ in which i appears $\dim V_i$ times for any $i \in I$.

Let $U^+_{\mathbb{Z}}$ be the subring of U^+ generated by the elements $e_i^p/p!$ for all $i \in I$, $p \in \mathbb{Z}_{\geq 0}$. Then $U^+_{\mathbb{Z},\dim V} = U^+_{\mathbb{Z}} \cap U^+_{\dim V}$, see [Lus00, 1.1].

2.3.2. In [Lus91, 12.12] Lusztig defines a $\mathbb{C}$-linear map $\psi_V : U^+_{\dim V} \to \mathcal{L}(V)$, such that

(2.3.2.1) $\psi = \bigoplus_{\dim V} \psi_V : U^+ \simeq \mathcal{L}_0$ is an isomorphism of algebras,
(2.3.2.2) $\psi(e_i) = E_i$, for any $i \in I$,
(2.3.2.3) $\psi_V(U^+_{\mathbb{Z},\dim V}) \subset \mathcal{L}_{\mathbb{Z}}(V)$.

The isomorphism ψ also restricts to an isomorphism of Lie algebras $\psi : \mathfrak{n} \to \mathfrak{n}^*$ such that $\psi(e_i) = E_i$ for any $i \in I$.

In [Lus00] Lusztig defines a $\mathbb{Q}$ version of the map $\psi_V : U^+_{\dim V} \to \mathcal{L}(V)$, which restricts to a map $\psi_V : U^+_{\mathbb{Z},\dim V} \to \mathcal{L}_{\mathbb{Z}}(V)$.

2.4. Lie algebra based on the Euler cocycle: classical ADE case. Let the graph underlying the quiver (I,H) be the Dynkin diagram of type ADE. In this case we can identify $\mathbb{Z}[I]$ with the root lattice of type ADE. The elements $i \in I$ are considered simple roots, and we have the root system $R \subset \mathbb{Z}[I]$ and its positive part $R_+ \subset \mathbb{Z}_{\geq 0}[I]$.

Let us fix a cocylcle $\epsilon : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}/2\mathbb{Z}$. In the simply laced Dynkin diagram case, the Lie algebra $\mathfrak{n} = \mathfrak{n}^*$ defined above is isomorphic to the Lie algebra n^ϵ spanned by the elements $\tilde{e}_\alpha$, $\alpha \in R_+$ with the bracket defined by ϵ as in [FMV01, 4.2.1.a].

Now let us define the functions $E^*_\alpha \in \mathcal{L}(V)$, $\alpha \in R_+$, $\dim V = \alpha$ as follows:

$$E^*_\alpha = \psi(\tilde{e}_\alpha). \tag{2.4.0.1}$$

Notice that if $\alpha = i \in I$, then Λ_V is a point and $E^*_i = E_i$.

2.5. **Lie algebra based on the Euler cocycle: affine ADE case.** Let the graph underlying the quiver (I, H) be the extended Dynkin diagram of type ADE. In this case we can identify $\mathbb{Z}[I]$ with the affine root lattice of type ADE. The elements $i \in I$ are considered simple roots, and we have the affine root system $R \subset \mathbb{Z}[I]$ and its positive part $R_+ \subset \mathbb{Z}_{\geq 0}[I]$.

Let us choose an extending vertex $p \in I$, and let $I' = I - \{p\}$.

Again, let us fix a cocylcle $\epsilon : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}/2\mathbb{Z}$. Let $\mathfrak{n}_\alpha$ be the root space of the algebra $\mathfrak{n}$ corresponding to $\alpha \in R_+$. Let δ be the indivisible imaginary root. The Lie algebra $\mathfrak{n}$ is isomorphic to the Lie algebra $\mathfrak{n}^\epsilon$ spanned by the elements $\tilde{e}_\alpha \in \mathfrak{n}^\epsilon_\alpha$, $\alpha \in R^{\mathrm{re}}_+$, and $\alpha_k(n) \in \mathfrak{n}^\epsilon_{n\delta}$, $n \geq 1$, where we identify $\mathfrak{n}^\epsilon_{n\delta} = \mathbb{C}[I']$ = the vector space spanned by I', and $\alpha_k = k \in I' \subset \mathbb{Z}[I']$ are simple roots. The bracket is defined by ϵ as in [FMV01, 5.2.1.a]. In particular,

$$[\tilde{e}_\alpha, \tilde{e}_\beta] = \begin{cases} \epsilon(\alpha,\beta)\tilde{e}_{\alpha+\beta} & \text{if } \alpha+\beta \in R^{\mathrm{re}}_+, \\ \epsilon(\alpha,\beta)\alpha(n) & \text{if } \alpha+\beta = n\delta, \\ 0 & \text{if } \alpha+\beta \notin R_+. \end{cases} \tag{2.4.0.2}$$

Now let us define the functions $E^*_\alpha \in \mathcal{L}(V)$, $\dim V = \alpha \in R^{\mathrm{re}}_+$ as follows:

$$E^*_\alpha = \psi(\tilde{e}_\alpha), \tag{2.4.0.3}$$

and the functions $E^*_k(n) \in \mathcal{L}(V)$, $\dim V = n\delta$ as follows:

$$E^*_k(n) = \psi(\alpha_k(n)). \tag{2.4.0.4}$$

Again, if $\alpha = i \in I$, then Λ_V is a point and $E^*_i = E_i$.

3. Stability and simple Lie algebras of type ADE

3.1. Stability after King and Rudakov.

3.1.1. *Stability after King.* Let us consider the abelian category of nilpotent representations of a quiver (I, H) and its Grothendieck group K_0. A *character* on K_0 is an additive function $\theta : K_0 \to \mathbb{R}$.

Definition. [King94] *A point $x \in \mathbf{E}_V$ is called θ-stable (resp. θ-semistable) if $\theta(V) = 0$, and for any x-stable nonzero proper subspace $V' \subset V$ we have $\theta(V') > 0$ (resp. $\theta(V') \geq 0$).*

Definition. *Two θ-semistable points $x', x'' \in \mathbf{E}_V$ are* S-equivalent *(notation: $x' \overset{S}{\sim} x''$) if the orbit closures $\overline{G_V \cdot x'} \cap \overline{G_V \cdot x''}$ intersect in the set of θ-semistable points $\mathbf{E}_V^{ss}$.*

3.1.2. *Stability after Rudakov.* One can also define stable points using a "slope" stability condition: $\mu = c/r$ where c and r are additive functions $K_0 \to \mathbb{R}$ and $r(V) > 0$ for any $V \in \mathcal{V}$, see [Rud97, 3].

Definition. [Rud97] *A point $x \in \mathbf{E}_V$ is called* μ-stable *(resp.* μ-semistable*) if for any x-stable nonzero proper subspace $V' \subset V$ we have $\mu(V') < \mu(V)$ (resp. $\mu(V') \leq \mu(V)$).*

If we fix such slope stability condition μ and $V \in \mathcal{V}$ we can define a character $\theta_\mu : K_0 \to \mathbb{R}$ as follows:

$$\theta_\mu(V') = -c(V') + \frac{c(V)}{r(V)} r(V'). \tag{3.1.2.1}$$

According to [Rud97, Proposition 3.4] a point $x \in \mathbf{E}_V$ is μ-stable if and only if it is θ_μ-stable.

3.1.3. Let us fix an orientation $\Omega \subset H$ until the end of this section. Following [King94] we construct a character $\theta = \Theta_{V,\Omega}$ associated to Ω. Let $V, V' \in \mathcal{V}$. We define

$$\Theta_{V,\Omega}(V') = \sum_{h \in \Omega} (\dim V_{h'} \dim V'_{h''} - \dim V'_{h'} \dim V_{h''}).$$

3.2. **Stability Lemma.** In this section we will consider quivers of finite ADE type, i.e. the underlying non-oriented graph Q is a Dynkin graph of (finite) ADE type. A fact similar to the lemma below was independently conjectured by M. Reineke [Rei02, Conjecture 7.1]

Lemma. *Let $x \in \Lambda_V$, $\dim V = \alpha \in R_+$. Then x is $\Theta_{V,\Omega}$-stable if and only if $x \in \mathbf{E}_{V,\Omega}^{\mathrm{ind}}$.*

Proof. Since in the Dynkin quiver case every root is a Schur root [King94], a generic point in $\mathbf{E}_{V,\Omega}$ is $\Theta_{V,\Omega}$-stable according to [King94, Scho92]. Since $\mathbf{E}_{V,\Omega}^{\mathrm{ind}}$ is an open dense G_V-orbit in $\mathbf{E}_{V,\Omega}$, any $x \in \mathbf{E}_{V,\Omega}^{\mathrm{ind}}$ is $\Theta_{V,\Omega}$-stable.

Let us prove the "only if" part. If $x \in \Lambda_V - \mathbf{E}_{V,\Omega}^{\mathrm{ind}}$, then we have $x = y + z$, where $y \in \mathbf{E}_{V,\Omega}$, $z \in \mathbf{E}_{V,\overline{\Omega}}$. Following [Lus91, 14] we see that there exists a decomposition:

$$(V, y) = \oplus_{p=1}^{\nu} (V^p, y)$$

such that $\mathrm{Hom}(V^p, V^{p'}) = 0$ whenever $p' < p$. There is a direct sum decomposition:

$$\mathbf{E}_{V,\Omega} = \bigoplus_{1 \leq p,p' \leq \nu} \mathbf{E}_{V,\Omega}^{p,p'} \qquad \text{where} \qquad \mathbf{E}_{V,\Omega}^{p,p'} = \mathrm{Hom}_{h \in \Omega}(V_{h'}^p, V_{h''}^{p'}).$$

Now we need the following claim: there exists $y' \in \bigoplus_{p>p'} \mathbf{E}_{V,\Omega}^{p,p'}$ such that $y + y' =: x' \in \mathbf{E}_{V,\Omega}^{\mathrm{ind}}$. It is not very hard to prove this claim using the methods of [Lus90, 4].

Following [Lus91, 14] we denote $V^{(q)} = \oplus_{p>q} V^p$. Then we have the x-invariant filtration:

$$V = V^{(0)} \supset V^{(1)} \supset \cdots \supset V^{(\nu)} = 0.$$

Denote $V_{(q)} = \oplus_{p<q} V^p$. Then we have the x'-invariant filtration:

$$0 = V_{(0)} \subset V_{(1)} \subset \cdots \subset V_{(\nu)} = V.$$

The vector space $V_{(q)}$ is a submodule in (V, x'). Since x' is $\Theta_{V,\Omega}$-stable we have $\Theta_{V,\Omega}(V_{(q)}) > 0$. However $V_{(q)}$ is a quotient module in (V, x). Therefore, (V, x) is unstable. □

3.3. Stable construction.

3.3.1. Let us consider the cocycle ϵ_Ω associated to our fixed orientation Ω, see [FK81], [FMV01, 1.3.4]. By construction, the functions E_α^*, $\alpha \in R_+$ on Λ_V defined as in 2.4 with the cocycle $\epsilon = \epsilon_\Omega$ form a basis of the Lie algebra n^*, and the $*$-bracket is given by:

$$[E_\alpha^*, E_\beta^*] = \begin{cases} \epsilon(\alpha,\beta) E_{\alpha+\beta}^*, & \text{if } \alpha+\beta \in R_+, \\ 0 & \text{if } \alpha+\beta \notin R_+. \end{cases} \tag{3.3.1.1}$$

3.3.2. It is clear that the affine variety $\mathbf{E}_{V,\Omega} \subset \Lambda_V$ is one of the irreducible components of Λ_V. We can define a constructible function E_α, $\alpha \in R_+$ on $\mathbf{E}_{V,\Omega}$, $\dim V = \alpha$ as follows, see [FMV01, 4.3.1.b]:

$$E_\alpha = \begin{cases} 1, & \text{if } x \in \mathbf{E}_{V,\Omega}^{\mathrm{ind}}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.3.2.1}$$

According to [FMV01, 4.3.4], the space spanned by the functions E_α, $\alpha \in R_+$ is a Lie algebra (which we will denote here by $\mathfrak{n}^\Omega$) with the bracket $[f,g]_\Omega = f *_\Omega g - g *_\Omega f$. The map $\mathfrak{n}^\epsilon \to \mathfrak{n}^\Omega$ given by $\tilde{e}_\alpha \to E_\alpha$ is an isomorphism. In other words, the bracket can be explicitly described as follows:

$$[E_\alpha, E_\beta]_\Omega = \begin{cases} \epsilon(\alpha,\beta) E_{\alpha+\beta}, & \text{if } \alpha+\beta \in R_+, \\ 0 & \text{if } \alpha+\beta \notin R_+. \end{cases} \tag{3.3.2.2}$$

3.3.3. Let us take the character $\Theta_{V,\Omega}$, and denote the identity function on Λ_V^s, $\dim V = \alpha$, by $\widetilde{E}_\alpha$, $\widetilde{E}_\alpha(\Lambda_V^s) \equiv 1$, where Λ_V^s is the set of points in Λ_V stable with respect to $\Theta_{V,\Omega}$. Notice that by Lemma 3.2, $\Lambda_V^s = \mathbf{E}_{V,\Omega}^{\mathrm{ind}}$, and so the function $\widetilde{E}_\alpha$ is constant on a single G_V-orbit in Λ_V^s.

Lemma. *Up to a sign:*

(3.3.3.1) $E^*_\alpha|_{\mathbf{E}_{V,\Omega}} = E_\alpha$,

(3.3.3.2) $E^*_\alpha|_{\Lambda^s_V} = \widetilde{E}_\alpha$.

Proof. Follows immediately from definitions, the proof of [Lus91, Theorem 12.13], and the formulas 3.3.1.1, 3.3.2.2. □

3.3.4. Let us denote the space spanned by the functions $\widetilde{E}_\alpha$, $\alpha \in R_+$ by $\mathfrak{n}^{stable}$. Since $\Lambda^s_V = \mathbf{E}^{\mathrm{ind}}_{V,\Omega}$, by lemma 3.2, the restrictions from Λ_V to $\mathbf{E}_{V,\Omega}$ to Λ^s_V provide us with the based isomorphisms of vector spaces (cf. Lemma 3.3.3)

$$\mathfrak{n}^* \xrightarrow[\text{restriction}]{\simeq} \mathfrak{n}^\Omega \xrightarrow[\text{restriction}]{\simeq} \mathfrak{n}^{stable},$$

defined by $E^*_\alpha \mapsto E_\alpha \mapsto \widetilde{E}_\alpha$ for $\alpha \in R_+$. It is clear that the first restriction is an isomorphism of Lie algebras, and the second restriction equips the space $\mathfrak{n}^{stable}$ of functions on Λ^s_V constant on G_V-orbits (or equivalently, functions on $\Lambda^s_V/G_V = \mathrm{pt}_V$) with the structure of a Lie algebra isomorphic to $\mathfrak{n}^* \simeq \mathfrak{n}^\Omega \simeq \mathfrak{n}^\epsilon$.

4. Stability, affine Lie algebras of type ADE, and a conjectural construction of the algebras of BPS states

4.1. Physics.

4.1.1. One of the current models of string theory defines D-branes as objects in the derived category $D^b(Coh(Z))$ of the coherent sheaves on the algebraic (Calabi-Yau) variety Z. Physicists consider the moduli space $\mathcal{M}_\zeta(n)$ (where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_N)$ are the Fayet-Iliopulos terms) of *semi-stable* D-branes of charge n, cf. [HM98, DFR00, FM00]. In the semiclassical approximation, the space of BPS states is given by the cohomology of the moduli space:

$$\mathcal{H}_{\mathrm{BPS}} = H^*(\mathcal{M}_\zeta(n)).$$

4.1.2. *Dictionary.* We will consider the D-branes on the resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of a simple singularity $\mathbb{C}^2/\Gamma$. Let Q be the extended Dynkin diagram corresponding to Γ via the McKay correspondence. We suggest a mathematical model of the physical situation in this case with which we will work. In the next subsection 4.1.3 we will justify our model. For now, we provide the physics/mathematics dictionary:

the charge lattice : the positive part $\mathbb{Z}_{\geq 0}[I]$ of the affine root lattice associated to Q;

places in the the charge lattice occupied by the single particle BPS states : positive roots $R_+ \subset \mathbb{Z}_{\geq 0}[I]$;

a D-brane of charge $\alpha \in \mathbb{Z}[I]$ on $\widetilde{\mathbb{C}^2/\Gamma}$: a representation $x \in \Lambda_V$, $\dim V = \alpha$ of the quiver (I,H) (with the underlying non-oriented graph Q);

Fayet-Iliopulos terms ζ : an additive function c which defines Rudakov's slope stability together with $r(V) = \sum_{i \in I} \dim V_i$;

ζ-stable D-branes : stable elements of Λ_V with respect to a stability condition;

moduli spaces of D-branes of charge $\alpha \in R_+^{\text{re}}$: $\Lambda_V^s / G_V = \text{pt}_\alpha$, $\dim V = \alpha \in R_+^{\text{re}}$;

moduli spaces of D-branes of charge δ = *indivisible imaginary root* : $\Lambda_V^s / G_V = \mathfrak{L} = \mathfrak{L}_1$, where $\mathfrak{L}$ is the exceptional fiber of the resolution of simple singularity;

moduli spaces of D-branes of charge $m\delta$, $m > 1$: $\Lambda_V^{ssd}/(S-\text{equivalence}) = \mathfrak{L} = \mathfrak{L}_m$, where Λ_V^{ssd} is the semistable diagonal (see 4.2.4.1 for the definition, cf. [HM98]), and where $\mathfrak{L}$ is the exceptional fiber of the resolution of simple singularity;

BPS states : certain constructible functions on our moduli spaces, see 4.2.8 for details.

4.1.3. *Justification.* If we consider the D-branes on $\widetilde{\mathbb{C}^2/\Gamma}$ then due to M. Kapranov and E. Vasserot [KV00]:

$$D^b(Coh(\widetilde{\mathbb{C}^2/\Gamma})) \simeq D^b(Rep(Q))$$

where Q is the extended Dynkin diagram associated to the $\Gamma \subset SL(2, \mathbb{C})$ via the McKay correspondence, and $Rep(Q)$ is the category of finite-dimensional *double* representations of Q in the sense of [KV00, 3.4].

We are only interested in D-branes of charge $\alpha \in R_+$ which are 0-complexes on the right hand side, i.e. double representations of Q rather than complexes of such representations. Such representations are identified with elements $x \in \mathbf{E}_V$, $\dim V = \alpha$ satisfying $m_i(x) = 0$ for all $i \in I$ (see section 2.1.5 for the definition of m_i). The notion of stability becomes the usual GIT stability adapted to the quiver situation by King and Rudakov (see section 3). We choose a "non-degenerate" stability condition (cf. 4.2.6).

If $\dim V = \alpha \in R_+^{\text{re}}$, then the moduli space is a point: $\mathcal{M}_\zeta(\alpha) = \text{pt}_\alpha$.

If $\dim V = \delta$, where δ is the indivisible imaginary root, then the moduli space is the resolution of a simple singularity:

$$\mathcal{M}_\zeta(\delta) = \widetilde{\mathbb{C}^2/\Gamma}.$$

In the two cases above one may replace the (middle) cohomology of the moduli spaces with the constructible functions on the exceptional fiber see [Nak94, 10.16]. Following this logic, we suggest the interpretation of BPS states as in the dictionary above and in 4.2.8 below if $\dim V = m\delta$, $m > 1$ where the situation is more complicated and a clear connection between physical and mathematical results was not available before, as far as we know.

4.1.4. Physics implies that the space of BPS states on $\widetilde{\mathbb{C}^2/\Gamma}$ of all possible charges should form a Lie algebra isomorphic to the plus part of the affine Lie algebra corresponding to Γ via the McKay correspondence [FM00]. The main purpose of this section is to offer a conjectural, but mathematically rigorous, validation of this claim in 4.2.8.

4.2. **The conjecture.** In this section we consider quivers of affine ADE type, i.e. the underlying non-oriented graph Q is an extended Dynkin graph of ADE type.

4.2.1. Let us fix an extending vertex p of the graph Q. For every $\alpha = \dim V$ let us fix the Nakajima's character: $\theta(V) = -1$, if $\dim V = k \in I' = I - \{p\}$, and $\theta(V) = \sum_{i \in I'} \dim_{\mathbb{C}}(V_i)$ if $\dim V = p \in I$ (cf. [Nak98]). (Semi)stable points in this section are considered with respect to this stability condition. If $V_p = 0$ we can take a King's character $\Theta_{V,\Omega}$ (see 3.1.3) associated to any orientation that "flows to the extending vertex" i.e. the extending vertex is a sink and one can get to the extending vertex from any other point in the quiver going along the oriented edges. In the DE case there is only one such orientation (once the extending vertex is fixed). In the A_n case there are n such orientations.

4.2.2. For $\dim V = \alpha \in R_+^{\mathrm{re}}$ we have (cf. [CB01]):

$$\Lambda_V^s / G_V = \mathrm{pt}_\alpha . \tag{4.2.2.1}$$

Let us denote by $\widetilde{E}_\alpha$ the function identically 1 on the point pt_α, $\widetilde{E}_\alpha(\mathrm{pt}_\alpha) \equiv 1$.

4.2.3. For $\dim V = \delta =$ indivisible imaginary root we have ([Kron89], [Nak94]):

$$\Lambda_V^s / G_V = \mathfrak{L}, \tag{4.2.3.1}$$

where $\mathfrak{L} = \mathfrak{L}_1$ is the exceptional fiber of the resolution of simple singularity

$$\mathfrak{L} \hookrightarrow \widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma.$$

Here Γ is the finite subgroup of $SL(2,\mathbb{C})$ corresponding to the diagram Q via the McKay correspondence.

It is well known that $\mathfrak{L}$ is a configuration of lines $\mathbb{P}^1$ which are its irreducible components and which may be indexed by the vertices $k \in I' = I - \{p\}$. We will denote the k^{th} irreducible component of $\mathfrak{L}$ by Y_k, $k \in I'$. Let us denote the characteristic function of Y_k by $\widetilde{E}_k(1)$, $k \in I'$, $\widetilde{E}_k(1)(Y_k) \equiv 1$, $\widetilde{E}_k(1)(\mathfrak{L} - Y_k) \equiv 0$.

4.2.4. Finally, for $\dim V = m\delta$, $m > 1$ we introduce the *semistable diagonal* $\Lambda_V^{ssd} \subset \Lambda_V$ as follows (cf. [HM98]): x is S-equivalent to the direct sum of m isomorphic representations in Λ_δ^s. Here $\Lambda_\delta^s = \Lambda_{V'}^s$, $\dim V' = \delta$. Formally:

(4.2.4.1)

$$\Lambda_V^{ssd} = \{x \in \Lambda_V \mid x \overset{S}{\sim} x_1 \oplus x_2 \oplus \cdots \oplus x_m, x_1 \simeq x_2 \simeq \cdots \simeq x_m, x_i \in \Lambda_\delta^s\}$$

It is clear that:

$$\Lambda_V^{ssd}/(S\text{-equivalence}) = \mathfrak{L}$$

where $\mathfrak{L} = \mathfrak{L}_m$ is the same exceptional variety as above. Let us denote the characteristic function of the k^{th} irreducible component Y_k of $\mathfrak{L} = \mathfrak{L}_m$ by $\widetilde{E}_k(m)$, $k \in I'$, $\widetilde{E}_k(m)(Y_k) \equiv 1$, $\widetilde{E}_k(m)(\mathfrak{L} - Y_k) \equiv 0$.

4.2.5. We have defined the functions E_α^* on Λ_V, $\dim V = \alpha \in R_+^{\text{re}}$, and the functions $E_k^*(m)$, $k \in I' = I - \{p\}$ on Λ_V, $\dim V = m\delta$, $m \geq 1$ in section 2.5. We need one more definition (cf. 2.4.0.2):

$$\hat{E}_k^*(m) := \epsilon(\alpha_k, \beta_k)[E_{\alpha_k}^*|_{\Lambda_{V_1}^s}, E_{\beta_k}^*|_{\Lambda_{V_2}^s}],$$

where $\dim V_1 = \alpha_k = k \in I' \subset R_+^{\text{re}}$, $\dim V_2 = \beta_k = (m\delta - \alpha_k) \in R_+^{\text{re}}$ (here we treat elements of I' as simple roots), and the bracket is in the sense of 2.2.3.

Since the functions $E_\alpha^*, \hat{E}_k^*(m)$ are constant on G_V-orbits, we can consider them as functions on Λ_V/G_V. The equalities in the following conjecture are understood in this sense.

Conjecture. (4.2.5.1) *Up to a sign,* $E_\alpha^*|_{\Lambda_V^s} = \widetilde{E}_\alpha$, $\dim V = \alpha \in R_+^{\text{re}}$.

(4.2.5.2) *Up to a sign,* $\hat{E}_k^*(1)|_{\Lambda_V^s} = \widetilde{E}_k(1)$, $\dim V = \delta$.

(4.2.5.3) *Let* $\dim V = m\delta$, $m > 1$. *We conjecture that if* $x', x'' \in \Lambda^{ssd}$ *are two elements in the same S-equivalence class and* $\hat{E}_k^*(m)(x') \neq 0$, $\hat{E}_k^*(m)(x'') \neq 0$, *then* $\hat{E}_k^*(m)(x') = \hat{E}_k^*(m)(x'')$. *Thus we can consider* $\hat{E}_k^*(m)|_{\Lambda_V^{ssd}}$ *as a function on* Λ^{ssd}/S*-equivalence by setting for an S-equivalence class X:*

$$\hat{E}_k^*(m)(X) = \begin{cases} \hat{E}_k^*(m)(x), & \textit{if there exists } x \in X \textit{ with } \hat{E}_k^*(m)(x) \neq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

(4.2.5.4) *Up to a sign,* $\hat{E}_k^*(m)|_{\Lambda_V^{ssd}} = \widetilde{E}_k(m)$, $\dim V = m\delta$.

The conjecture is verified in the $\hat{A}_1$ case.

4.2.6. *Remark.* We would like to explain here that there is nothing special about the Nakajima's choice of character. If $\dim V = m\delta$ is an imaginary root, then the space of all characters defining stability may be naturally identified with a (classical) Cartan subalgebra $\mathfrak{h} = \mathbb{C}$-vector space spanned by $i \in I'$. A character θ is called non-degenerate if it does not lie on a wall of a Weyl chamber. We expect the conjecture to be true for any non-degenerate character in the same Weyl chamber as the Nakajima's character (this is the fundamental chamber). Moreover, we could take a non-degenerate character in any other Weyl chamber, but that would change our choice of simple roots which we identify with the set $I' = I - \{p\}$. One could perhaps reformulate our conjecture in this case modifying the functions $E_k^*(m)$ using the action of the Weyl group.

4.2.7. *Remark.* The functions $\widetilde{E}_k(m)$ parameterize the set $\operatorname{Irr} \mathfrak{L} = \operatorname{Irr} \mathfrak{L}_m$ of the irreducible components of the exceptional variety $\mathfrak{L}$. The components of $\mathfrak{L}$ are a basis in the space $H^2(\widetilde{\mathbb{C}^2/\Gamma})$, see [Nak94].

4.2.8. Denote the Lie algebra spanned by the functions E_α^* on Λ_V, $\dim V = \alpha \in R_+^{\mathrm{re}}$, and the functions $E_k^*(m)$ $k \in I' = I - \{p\}$ on Λ_V, $\dim V = m\delta$, $m \geq 1$ by $n^*(Q)$, cf. section 2.5.

Let us denote the vector space spanned by the functions $\widetilde{E}_\alpha$, $\alpha \in R_+^{\mathrm{re}}$, and the functions $\widetilde{E}_k(m)$ on $\mathfrak{L} = \mathfrak{L}_m = \Lambda_{m\delta}^{ssd}/(S - \text{equivalence})$ by $\mathfrak{n}^{stable}(Q)$. This is the vector space of constructible functions on moduli spaces of (semi)stable points in Λ, which are linear combinations of characteristic functions of irreducible components. The space $\mathfrak{n}^{stable}(Q)$ is our model of the space of BPS states at the orbifold $\mathbb{C}^2/\Gamma$, with Q corresponding to Γ via the McKay correspondence.

The conjecture, if true, would imply that the restriction from Λ to Λ^s (or Λ^{ssd}) gives us the based isomorphism of vector spaces

$$\mathfrak{n}^*(Q) \xrightarrow{\simeq} \mathfrak{n}^{stable}(Q)$$

such that $E_\alpha^* \mapsto \widetilde{E}_\alpha$ and $E_k^*(m) \mapsto \hat{E}_k(m) \mapsto \widetilde{E}_k(m)$ for $\alpha \in R_+^{\mathrm{re}}$, $k \in I'$, $m \geq 1$. This map equips the space of BPS states $\mathfrak{n}^{stable}(Q)$ with the structure of a Lie algebra isomorphic to the plus part of the affine Lie algebra corresponding to Q.

On the other hand the above isomorphism provides the plus part of the affine Lie algebra corresponding to Q, with a basis $\{\widetilde{E}_\alpha, \alpha \in R_+^{\mathrm{re}}; \widetilde{E}_k(m), k \in I', m \in \mathbb{Z}_{>0}\}$ indexed by irreducible components of algebraic varieties $\mathrm{pt}_\alpha = \Lambda_\alpha^s/G_\alpha$, and $\mathfrak{L}_m = \Lambda_{m\delta}^{ssd}/(S - \text{equivalence})$, $m \in \mathbb{Z}_{>0}$.

5. Remarks on semicanonical basis for simple Lie algebras

5.1. **Semicanonical basis.** This subsection is lifted from [Lus00, 2.4-5]. Recall the setup of 2.3. Let $V = \oplus_{i \in I} V_i$ be a I-graded vector space. If $Y \in \operatorname{Irr} \Lambda_V$ is an irreducible component of Λ_V and $f \in \mathcal{L}_{\mathbb{Z}}(V)$ then there is a unique $c \in \mathbb{Z}$ such that $f^{-1}(c) \cap Y$ contains an open dense subset of Y. Note that $f \mapsto c$ is a linear function $\rho_Y : \mathcal{L}_{\mathbb{Z}}(V) \to \mathbb{Z}$.

Lemma. *Let $Y \in \operatorname{Irr} \Lambda_V$. There exists $f = e_Y \in \psi_V(U_{\mathbb{Z},V}^+)$ such that $\rho_Y(f) = 1$ and $\rho_{Y'}(f) = 0$ for any $Y' \in \operatorname{Irr} \Lambda_V - \{Y\}$.*

Lusztig proves that the functions e_Y, $Y \in \operatorname{Irr} \Lambda_V$ form a basis in $\psi_V(U_V^+)$, and therefore, the collection of e_Y, $Y \in \operatorname{Irr} \Lambda_V$ for all possible dimensions of V forms a basis in the algebra $\psi(U^+) = \mathcal{L}_0$ isomorphic to U^+. This basis is called *semicanonical.*

In the remainder of this section we consider quivers of finite ADE type.

5.1.1. Since our quiver is of finite ADE type we know [Lus91, 14.2] that the irreducible components of Λ_V are the closures of conormal bundles of the various G_V-orbits in $\mathbf{E}_{V,\Omega}$, where Ω is some orientation. Such G_V-orbits are indexed by the decomposition of (V, x), $x \in \mathbf{E}_{V,\Omega}$ into a direct sum of indecomposable submodules $(V, x) = (V_1, x_1) \oplus \cdots \oplus (V_l, x_l)$ with $x_i \in \mathbf{E}^{\mathrm{ind}}_{V_i,\Omega}$. (The decompositions are considered up to the order of the summands.) Thus, there is a one-to one correspondence between $\operatorname{Irr} \Lambda_V$ and the decompositions $\dim V = \dim V_1 + \cdots + \dim V_l$ of $\dim V \in \mathbb{Z}[I]$ into a sum of positive roots $\dim V_i \in R_+$. (Recall that $\mathbf{E}^{\mathrm{ind}}_{V,\Omega} \neq \emptyset$ if and only if $\dim V \in R_+$.) Thus the elements of the semicanonical basis may be indexed by decompositions of $\dim V$ into a sum of positive roots.

5.2. **Open problem.** Recall that we have defined the functions E^*_α on Λ_V, $\dim V = \alpha \in R_+$ in 2.4. By construction these functions are defined canonically up to a sign which depends on the choice of the cocycle ϵ. Since $E^*_\alpha \in \psi_V(U^+_{\mathbb{Z},\dim V})$, we can decompose E^*_α with respect to the semicanonical basis:

$$E^*_\alpha = \sum_{Y \in \operatorname{Irr} \Lambda_V} c_Y e_Y. \tag{5.2.0.1}$$

The problem is to calculate the integer coefficients c_Y for all irreducible components Y of Λ_V. These coefficients are independent of any choices up to simultaneous multiplication of all of them by -1. Suppose that an irreducible component $Y \in \operatorname{Irr} \Lambda_V$, $\dim V = \alpha$ corresponds to a decomposition

$$\alpha = \alpha_1 + \cdots + \alpha_l \tag{5.2.0.2}$$

of α into a sum of positive roots $\alpha_1, \ldots, \alpha_l$. Then the formula 5.2.0.1 may be regarded as an assignment $\{\alpha = \alpha_1 + \cdots + \alpha_l\} \mapsto c_Y$ of the integer c_Y to every decomposition 5.2.0.2. It would be very interesting to describe this assignment in terms of combinatorics of the root system without appealing to the semicanonical basis. We obtain such a description in the A_n case using the Weyl group $W = S_{n+1}$. We also succeed in calculating c_Y when $Y = \mathbf{E}_{V,\Omega}$ are irreducible components "arising from orientations", and we calculate the examples of D_4 and D_5.

5.3. **Irreducible components arising from orientations.**

5.3.1. Let us define the subvariety $O_V \subseteq \Lambda_V$ as follows:

$$O_V = \bigcup_{\Omega} \mathbf{E}_{V,\Omega}$$

where the union is taken over all orientations Ω of our quiver. Since each $Y = \mathbf{E}_{V,\Omega}$ is an irreducible component of Λ_V (we will call such Y irreducible components arising from orientations), we have $\operatorname{Irr} O_V \subseteq \operatorname{Irr} \Lambda_V$. We will split the sum 5.2.0.1 into two parts:

$$E^*_\alpha = E'_\alpha + E''_\alpha$$

where

$$E'_\alpha = \sum_{Y \in \operatorname{Irr} O_V} c_Y e_Y \quad \text{and} \quad E''_\alpha = \sum_{Y \in \operatorname{Irr} \Lambda_V - \operatorname{Irr} O_V} c_Y e_Y \tag{5.3.1.1}$$

5.3.2. If a root $\alpha \in R_+$ is not simple, represent it as a sum of simple roots $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_h}$, $k_l \in I$, where $h = \operatorname{ht} \alpha$

$$\alpha = \alpha_{k_1} + \alpha_{k_2} + \cdots + \alpha_{k_h} \tag{5.3.2.1}$$

in such a way that $\alpha_{k_1} + \alpha_{k_2} + \cdots + \alpha_{k_j}$ is a root for $1 \leq j \leq h$.

It follows from our construction and definitions that up to a sign

$$E^*_\alpha = [\ldots [E^*_{\alpha_{k_1}}, E^*_{\alpha_{k_2}}] \ldots E^*_{\alpha_{k_h}}] \tag{5.3.2.2}$$

Now we can calculate the coefficients of E'_α (cf. section 3 for a similar discussion).

Proposition. *Up to a sign,*

$$E'_\alpha = \sum_{Y = \mathbf{E}_{V,\Omega} \in \operatorname{Irr} O_V} (\prod_{i<j} \epsilon_\Omega(\alpha_{k_i}, \alpha_{k_j})) e_Y,$$

where ϵ_Ω is the Frenkel-Kac cocycle corresponding to Ω, [FMV01, 1.3.4]. *This presentation does not depend on the decomposition 5.3.2.1.*

Proof. Let us fix some orientation Ω, and let us assume that $Y = \mathbf{E}_{V,\Omega}$. According to Lusztig [Lus92, 12.13] we have a homomorphism of algebras $\mathcal{L}_0 \to \mathcal{L}_{0,\Omega}$ induced by the restriction on functions from Λ_V to $\mathbf{E}_{V,\Omega}$ (see 2.2.3 for the definitions of $\mathcal{L}_0$ and $\mathcal{L}_{0,\Omega}$). Then

$$E^*_\alpha|_Y = E'_\alpha|_Y = [\ldots [E_{k_1}, E_{k_2}]_\Omega \ldots E_{k_h}]_\Omega = \prod_{i<j} \epsilon_\Omega(\alpha_{k_i}, \alpha_{k_j}) E_\alpha, \tag{5.3.2.4}$$

where E_α is defined by 3.3.2.1, and the last equality follows from [FMV01, 4.3.4.b]. Now it is clear that $\rho_Y(E^*_\alpha) = \rho_Y(E^*_\alpha|_Y)$. Then

$$c_Y = \rho_Y(E^*_\alpha) = \rho_Y(E^*_\alpha|_Y) = \prod_{i<j} \epsilon_\Omega(\alpha_{k_i}, \alpha_{k_j}) \rho_Y(E_{\alpha,\Omega}) = \prod_{i<j} \epsilon_\Omega(\alpha_{k_i}, \alpha_{k_j}).$$

Notice that due to the formula 5.3.2.4 we know explicitly $E^*_\alpha|_O$ i.e., the value $E^*_\alpha(x)$ at any point $x \in O \subseteq \Lambda$. □

5.3.3. Let Q be the Dynkin graph of type A_n. In this case it is clear that for $V \in \mathcal{V}$, $\dim V = \alpha \in R_+$ any irreducible component of Λ_V is of the form $\mathbf{E}_{V,\Omega}$ for some orientation Ω. Thus Proposition 5.3.2 gives a complete answer to our question 5.2 in this case.

5.4. Conjugacy classes of the Weyl group.

5.4.1. Recall the isomorphism $\psi : \mathfrak{n} \to \mathfrak{n}^*$ (see 2.3.2), and let us take $\tilde{e}_\alpha \in \mathfrak{n}^\epsilon \simeq \mathfrak{n}$ such that $\psi(\tilde{e}_\alpha) = E^*_\alpha$, $\alpha \in R_+$, as in 2.4. Let us fix $Y \in \operatorname{Irr}\Lambda_V$ and let $\alpha = \beta_1 + \cdots + \beta_l$ be the corresponding decomposition of α into the sum of positive roots. Now consider the element $\tilde{e}_{\beta_1} + \cdots + \tilde{e}_{\beta_l} \in \mathfrak{n}^\epsilon \simeq \mathfrak{n}$, and consider its adjoint orbit. In such a way we obtain a map from $\operatorname{Irr}\Lambda_V$ to the set $\mathcal{N}$ of nilpotent orbits of the Lie algebra corresponding to the Dynkin diagram Q. Moreover, following Kazhdan-Lusztig [KL88] and Spaltenstein [Spal88], we can construct a map from $\mathcal{N}$ to the set $\mathrm{Cl}(W)$ of conjugacy classes of the Weyl group W. Thus we obtain a map $\mathrm{cl} : \operatorname{Irr}\Lambda_V \to \mathrm{Cl}(W)$ for any $V \in \mathcal{V}$. We will use this map in the following subsection.

5.4.2. *Example:* A_n.

Proposition. *Up to a sign,*

$$E^*_\alpha = \sum_{Y \in \operatorname{Irr}\Lambda_V} \operatorname{sgn}(\mathrm{cl}\, Y)\ e_Y\ ,$$

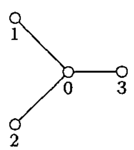
where $\operatorname{sgn}(\mathrm{cl}\, Y)$ *is the value of the one-dimensional sign character of* $W = S_{n+1}$ *on the conjugacy class* $\mathrm{cl}\, Y$.

Proof. The proof is straightforward and is left to the reader. □

5.4.3. *Open question.* We have the map $\mathrm{cl} : \operatorname{Irr}\Lambda_V \to \mathrm{Cl}(W)$ for all ADE quivers, see 5.4.1. Looking at the A_n case (Proposition 5.4.2), one may ask if in the DE case the coefficients c_Y are also governed by characters of the Weyl group. Unfortunately, the authors' naive attempt to replicate the A_n result fails already in the D_4 case (however the explicit calculation for the maximal root of D_4 is given below).

5.5. Example: D_4.

5.5.1. We index the vertices of the Dynkin diagram as follows:



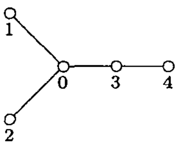
The simple roots are denoted by $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. Let $\alpha = 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ be the maximal root. Indexing the irreducible components of Λ_V, $\dim V =$

α by the decompositions of α we obtain by a straighforward computation using the properties of the semicanonical basis:

$$
\begin{aligned}
E^*_\alpha &= E'_\alpha + E''_\alpha \\
&= E'_\alpha - e_{(\alpha_0+\alpha_1)+(\alpha_0+\alpha_2+\alpha_3)} - e_{(\alpha_0+\alpha_2)+(\alpha_0+\alpha_1+\alpha_3)} \\
&\quad - e_{(\alpha_0+\alpha_3)+(\alpha_0+\alpha_1+\alpha_2)} - e_{\alpha_0+\alpha_1+(\alpha_0+\alpha_2+\alpha_3)} \\
&\quad - e_{\alpha_0+\alpha_2+(\alpha_0+\alpha_1+\alpha_3)} - e_{\alpha_0+\alpha_3+(\alpha_0+\alpha_1+\alpha_2)} \\
&\quad + 2e_{\alpha_0+(\alpha_0+\alpha_1+\alpha_2+\alpha_3)}.
\end{aligned}
$$

5.6. **Example:** D_5.

5.6.1. We index the vertices of the Dynkin diagram as follows:



The simple roots are denoted by $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let $\alpha = 2\alpha_0 + \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ be the maximal root. There are 55 root partitions of α, e.g. $\alpha = \alpha_0 + \alpha_3 + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$. Indexing the irreducible components of Λ_V, $\dim V = \alpha$ by the decompositions of α we obtain by a tedious but straightforward computation using the properties of the semicanonical basis:

$$
\begin{aligned}
&E^*_\alpha \\
&= E'_\alpha + E''_\alpha = E'_\alpha + \\
&+ e_{(2\alpha_0+\alpha_1+\alpha_2+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{\alpha_2+(\alpha_0+\alpha_1)+(\alpha_0+\alpha_3)+(\alpha_3+\alpha_4)} \\
&+ e_{\alpha_1+(\alpha_0+\alpha_2)+(\alpha_0+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{\alpha_1+\alpha_2+(\alpha_0+\alpha_3)+\alpha_0+(\alpha_3+\alpha_4)} \\
&+ e_{(2\alpha_0+\alpha_1+\alpha_2+\alpha_3)+\alpha_3+\alpha_4} && + e_{\alpha_2+(\alpha_0+\alpha_1)+\alpha_3+\alpha_4+(\alpha_0+\alpha_3)} \\
&+ e_{\alpha_1+(\alpha_0+\alpha_2)+(\alpha_0+\alpha_3)+\alpha_3+\alpha_4} && + e_{(\alpha_0+\alpha_3)+\alpha_0+\alpha_2+\alpha_3+\alpha_4} \\
&+ e_{(\alpha_2+\alpha_0+\alpha_3)+(\alpha_1+\alpha_0+\alpha_3+\alpha_4)} && + e_{(\alpha_1+\alpha_2+\alpha_0+\alpha_3)+(\alpha_0+\alpha_3)+\alpha_4} \\
&- e_{(\alpha_2+\alpha_0+\alpha_1)+(\alpha_0+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{(\alpha_2+\alpha_0+\alpha_1)+\alpha_0+\alpha_3+(\alpha_3+\alpha_4)} \\
&+ e_{(\alpha_2+\alpha_0+\alpha_1)+\alpha_0+2\alpha_3+\alpha_4} && - e_{(\alpha_0+\alpha_1+\alpha_2)+(\alpha_0+\alpha_3)+\alpha_4} \\
&- e_{(\alpha_0+\alpha_2)+(\alpha_0+\alpha_1+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{(\alpha_0+\alpha_1+\alpha_3)+(\alpha_0+\alpha_2+\alpha_3+\alpha_4)} \\
&- e_{\alpha_2+(\alpha_0+\alpha_1+\alpha_3)+\alpha_0+(\alpha_3+\alpha_4)} && + e_{\alpha_2+(\alpha_0+\alpha_1+\alpha_3)+(\alpha_0+\alpha_3+\alpha_4)} \\
&- e_{(\alpha_0+\alpha_1)+(\alpha_2+\alpha_0+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{(\alpha_2+\alpha_0+\alpha_3)+(\alpha_0+\alpha_1+\alpha_3+\alpha_4)} \\
&- e_{\alpha_0+\alpha_1+(\alpha_2+\alpha_0+\alpha_3)+(\alpha_3+\alpha_4)} && + e_{\alpha_1+(\alpha_2+\alpha_0+\alpha_3)+(\alpha_0+\alpha_3+\alpha_4)} \\
&+ e_{(\alpha_0+\alpha_2)+\alpha_3+(\alpha_0+\alpha_1+\alpha_3+\alpha_4)} && - e_{(\alpha_0+\alpha_2)+\alpha_3+(\alpha_0+\alpha_1+\alpha_3)+\alpha_4} \\
&+ e_{\alpha_0+\alpha_2+\alpha_3+(\alpha_0+\alpha_1+\alpha_3+\alpha_4)} && - e_{\alpha_0+\alpha_2+\alpha_3+(\alpha_0+\alpha_1+\alpha_3)+\alpha_4}
\end{aligned}
$$

$$
\begin{array}{ll}
+ e_{\alpha_3 + (\alpha_0 + \alpha_1) + (\alpha_0 + \alpha_2 + \alpha_3 + \alpha_4)} & - e_{\alpha_3 + (\alpha_0 + \alpha_1) + (\alpha_0 + \alpha_2 + \alpha_3) + \alpha_4} \\
+ e_{\alpha_3 + \alpha_0 + \alpha_1 + (\alpha_0 + \alpha_2 + \alpha_3 + \alpha_4)} & - e_{\alpha_3 + \alpha_0 + \alpha_1 + (\alpha_0 + \alpha_2 + \alpha_3) + \alpha_4} \\
+ 2e_{\alpha_0 + (\alpha_3 + \alpha_4) + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)} & - 2e_{(\alpha_0 + \alpha_3 + \alpha_4) + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)} \\
- 2e_{\alpha_0 + \alpha_3 + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} & + 2e_{\alpha_0 + \alpha_3 + \alpha_4 + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)} \\
- 2e_{(2\alpha_0 + \alpha_1 + \alpha_2 \alpha_3 + \alpha_4) + \alpha_3)} & - 2e_{\alpha_2 + (\alpha_0 + \alpha_1) + (\alpha_0 + \alpha_3 + \alpha_4) + \alpha_3} \\
- 2e_{\alpha_1 + (\alpha_0 + \alpha_2) + (\alpha_0 + \alpha_3 + \alpha_4) + \alpha_3} & - 2e_{(\alpha_0 + \alpha_3 + \alpha_4) + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3} \\
+ 2e_{(\alpha_0 + \alpha_3 + \alpha_4) + (\alpha_0 + \alpha_1 + \alpha_2) + \alpha_3} &
\end{array}
$$

5.6.2. While in the D_5 case the coefficients in the decomposition for the maximal root are $\pm 1, \pm 2$, in the D_6 case the coefficients are $\pm 1, \pm 2, \pm 4$. We expect that in the D_n case all the coefficients are the powers of 2.

5.7. **More open questions.** The questions of decomposing "canonically" defined basis elements of affine Lie algebra (such as $E^*_\alpha, E^*_k(n)$, see 2.5) with respect to semicanonical basis make sense for any affine Lie algebra of type ADE. It would be very interesting to develop a purely combinatorial approach to these questions in the affine case as well.

References

[CB01] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. 126 (2001), no. 3, 257–293.

[DFR00] M. Douglas, B. Fiol, and C. Römelsberger, *Stability and BPS branes*, preprint 2000, hep-th/0002037.

[FM00] B. Fiol and M. Mariño, *BPS states and algebras from quivers*, J. High Energy Phys. 2000, no. 7, Paper 31, 40 pp.

[FK81] I. Frenkel and V. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. 62 (1980/81), no. 1, 23–66.

[FMV01] I. Frenkel, A. Malkin, and M. Vybornov, *Affine Lie Algebras and Tame Quivers*, Selecta Math. (N.S.) 7 (2001), no. 1, 1–56.

[Gab72] P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. 6 (1972), 309.

[HM98] J. Harvey and G. Moore, *On the algebras of BPS states*, Comm. Math. Phys. 197 (1998), no. 3, 489–519.

[KV00] M. Kapranov and E. Vasserot *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. 316 (2000), no. 3, 565–576.

[KS97] M. Kashiwara and Y. Saito, *Geometric Construction of Crystal bases*, Duke Math. J. 89 (1997), no. 1, 9–36.

[KL88] D. Kazhdan and G. Lusztig, *Fixed point varieties on affine flag manifolds*, Israel J. Math. 62 (1988), no. 2, 129–168.

[King94] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford **(2), 45**, (1994), 515–530.

[Kron89] P. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989), no. 3, 665–683.

[Lus90] G. Lusztig, *Canonocal bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447-498.

[Lus91] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. 4 (1991), no. 2, 365–421.

[Lus92] G. Lusztig, *Affine quivers and canonical bases*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 76, 111–163.

[Lus00] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. 151 (2000), no. 2, 129–139.

[Nak94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76 (1994), no. 2, 365–416.

[Nak98] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. 91 (1998), no. 3, 515–560.

[Rei02] M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, preprint 2002, math.QA/0204059.

[Rin90] C. Ringel, *Hall poynomials for the representation-finite hereditary algebras*, Adv. Math. **84** (1990), no. 2, 137–178.

[Rud97] A. Rudakov, *Stability for an abelian category* J. Algebra 197 (1997), no. 1, 231–245.

[Scho92] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), 46-64.

[Spal88] N. Spaltenstein, *Polynomials over local fields, nilpotent orbits and conjugacy classes in Weyl groups* Orbites unipotentes et représentations, I. Astérisque No. 168 (1988), 10–11, 191–217.

Department of Mathematics, Yale University, 10 Hillhouse Ave, New Haven, CT 06520

E-mail address: `frenkel@math.yale.edu`

Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139

E-mail address: `malkin@math.mit.edu`

Department of Mathematics, University of Massachusetts, Lederle Graduate Research Tower, Amherst, MA 01003

E-mail address: `vybornov@math.umass.edu`

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DIVIDED DIFFERENCES OF TYPE D AND THE GRASSMANNIAN OF COMPLEX STRUCTURES

HAIBAO DUAN AND PIOTR PRAGACZ

Dedicated to Professor Tatsuo Suwa on his 60th birthday

CONTENTS

1. INTRODUCTION

Divided differences are discrete analogues of derivations. They were introduced by Newton in his famous interpolation formula in "Principia Mathematica" (1686). Their importance in geometry was shown in the early 1970's by Bernstein-Gelfand-Gelfand [BGG] and Demazure [De] in the context of Schubert calculus for generalized flag varieties associated with semisimple algebraic groups. More recently, simple divided differences, interpreted as correspondences in flag bundles, were used by Fulton in his study of the classes of degeneracy loci. Divided differences admit still another interpretation as Gysin maps in the cohomology of flag bundles associated with semisimple algebraic groups (cf., e.g., [P2]). We refer to the lecture notes [FP] for a systematic discussion of these issues. The case of $SL(n)$ has been recently developed extensively by Lascoux and Schützenberger (cf., e.g., [LSc]), and serves nowadays as an important and useful tool for multivariate polynomials (cf. [L2]).

The Grassmannian of complex structures parametrizes orthogonal automorphisms of the Euclidean space $\mathbf{R}^{2n}$ whose square is the minus identity. Equivalently, it parametrizes minimal geodesics from the identity to the minus identity in the orthogonal group $SO(2n,\mathbf{R})$ [Mi]. This space is usually denoted by CS_n. It played a significant role in several important achievements in topology:

- in the investigation of orthonormal vector fields on spheres by Hurewicz and Adams;
- in the study of the existence of complex structures on even dimensional spheres by Borel and Serre;
- in the Bott's discovery of the eight-periodicity of homotopy groups of the stable real orthogonal groups.

Also, CS_n serves as the classifying space of all complex bundles whose real reduction is trivial, by a result of the first author [Du1].

The goal of the present paper is to develop in a systematic way a Schubert calculus for CS_n. We hope that it will be useful also for topologists.

The space CS_n has two connected components, each isomorphic to the homogeneous space

$$SO(2n, \mathbf{C})/U(n) \quad \text{or} \quad SO(2n, \mathbf{C})/P,$$

where P is the maximal parabolic corresponding to omitting the "right end root". This space is a connected component of the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. to the orthogonal form induced by the scalar product, and as such, it is also known as the orthogonal Grassmannian.

With the help of the group-theoretic description, we can use the characteristic map of Borel [Bo], and – via the theory of Bernstein-Gelfand-Gelfand [BGG] and Demazure [De] – divided differences of type D to study the intersection theory on the space in question. In order to make the work with the characteristic map efficient, one needs a proper family of "invariant" polynomials that are well suited to divided differences, and also to geometry/topology at the same time.

A result of the second author [P1] identified Schubert classes in the homogeneous space $SO(2n, \mathbf{C})/U(n)$ with suitable Schur P-polynomials. In loc. cit. this identification used a geometric argument, namely an isomorphism $SO(2n, \mathbf{C})/U(n) \simeq SO(2n-1, \mathbf{C})/U(n-1)$, and an identification of the Schubert classes for the latter Grassmannian with Schur P-functions. (This last identification was based on comparison of the Pieri-type formulas from [HB] and [Mo].)

In the present paper we revisit the identification for the Schubert classes for $SO(2n, \mathbf{C})/U(n)$ with Schur P-functions via a direct *group-theoretic* argument based on the calculus of divided differences of type D. More precisely, we give a group-theoretic proof of a Pieri-type formula that is based on some vanishing results for operators composed of divided differences of type D and simple reflections from the Weyl group of type D. These last results form the most technical part of the present work. Our proof of the Pieri-type formula follows a strategy for deriving similar formulas for various homogeneous spaces worked out by Ratajski and the second autor in a series of papers summarized in [P2]. This particular proof was promised in [PR2] – a paper that is now under revision. The proof uses esentially an iteration of the Leibniz-type formula for a simple divided difference applied to the product of two functions.

Combining the Pieri formula with a combinatorial lemma of Schur [S] for the projective characters of the symmetric groups, we get a formula for the degree of Schubert varieties in $SO(2n, \mathbf{C})/U(n)$. (Occasionally, we discuss some alternative derivations of the lemma of Schur with the help of a specialization result from [DP].)

We remark that there exists now a refinement of Schur P-functions that seems to be even better adapted for some aspects of geometry. These are the so called $\widetilde{P}$-functions of [PR1], which are modeled on Schur P-functions. In [LP], Lascoux and the second author worked out a connection of orthogonal divided differences to $\widetilde{P}$-functions using vertex operators. This has led to *orthogonal Schubert polynomials* that are useful in various cohomological computations (cf. a recent work of Kresch and Tamvakis [KT,T2], and Buch [BKT]).

After presenting the Schubert varieties in a group-theoretic way, we also describe them via Schubert-type conditions relative to some flag of linear subspaces, and finally we study them in terms of complex structures. To this end, we are guided by the Mahowald-Vassiljev-type formula ([DV,V]):

$$H_i(CS_n) = \oplus_{k=0}^{n} H_{i-k(k-1)}(G_k(\mathbf{C}^n)) ,$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex k-planes through zero in $\mathbf{C}^n$.

We end the paper by illustrating how the Schubert calculus developed here can be used to solve problems about enumeration of complex structures which satisfy some natural conditions of "partial overlaping" with a certain number of complex structures in general position in $\mathbf{R}^{2n}$. One of the applications leads to an interesting algebraic conjecture about homomorphisms between the cohomology ring of CS_n^+ and that of the Grassmannian $G_k(\mathbf{C}^n)$.

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2. Preliminaries, Notation, and Conventions

We start with some algebraic preliminaries on even orthogonal groups. We fix a positive integer n. Suppose that $H = SO(2n, \mathbf{C})$ is the orthogonal group (of type D_n) over the field of complex numbers. Let us use the following notation:

B - a fixed Borel subgroup of H,

$T \subset B$ - a fixed maximal torus,

$\mathcal{R}$ - the root system of H associated with T,

Σ - a set of simple roots of $\mathcal{R}$ associated with B,

W - the Weyl group of (H, T).

In a standard Bourbaki [Bu] realization, we have:

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \subset \mathbf{R}^n = \oplus_{i=1}^n \mathbf{R} e_i,$$
$$\Sigma = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\},$$
$$W = S_n \ltimes \mathbf{Z}_2^{n-1}.$$

A typical element of W can be written as a pair (τ, ϵ), where $\tau \in S_n$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is a sequence of elements of $\mathbf{Z}_2 = \{-1, 1\}$ such that $\#\{i : \epsilon_i = -1\}$ is even. Multiplication in W is given by

$$(\tau, \epsilon) \cdot (\tau', \epsilon') = (\tau \circ \tau', \delta),$$

where "$\circ$" denotes the composition of permutations and $\delta_i = \epsilon_{\tau'(i)} \cdot \epsilon'_i$. The following lemma can be easily verified (and is pretty well-known). For $w \in W$, let $l(w)$ denote the length of w taken w.r.t. to the above Σ.

Lemma 2.1. *For any $w \in W$, $l(w)$ is*

$$\sum_{i=1}^{n} \#\{j \,:\, j > i \text{ and } w(j) < w(i)\} + \sum_{\epsilon_p = -1} 2\#\{q \,:\, q > p \text{ and } w(q) > w(p)\}.$$

We will use the "barred-permutation" notation, indicating by a bar a place in the permutation $w = [w(1), \dots, w(n)]$ where $\epsilon_i = -1$.

The following lemma, that is easy to prove from Lemma 2.1, gives us the lengths of some barred permutations important for this paper.

Lemma 2.2. *Let $y_1 < \cdots < y_{n-k}$ and $z_k > \cdots > z_1$ be sequences of integers that are complementary in $\{1, \dots, n\}$. Assume that k is even. Then in W we have*

$$l([y_1, y_2, \dots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \dots, \overline{z}_1]) = \sum_{j=1}^{k} (n - z_j).$$

The barred permutations of this type form the poset, denoted W^*, of the minimal length left coset representatives of S_n in W.

Our terminology and all unexplained notation concerning partitions will follow [Ma].

We set $\rho(k) := (k, k-1, \dots, 1)$, a "triangular partition" of length k.

Given a strict partition $\alpha = (\alpha_1 > \cdots > \alpha_l > 0) \subset \rho(n-1)$, we set

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_l + 1)$$

if l is even, and

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_l + 1, 1)$$

if l is odd. Note that α^+ is of even length.

Given a strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0) \subset \rho(n)$ of even length k, we associate with it the following element w_μ of W^*. We set

$$w_\mu := [y_1, y_2, \dots, y_{n-k}, \overline{n+1-\mu_k}, \overline{n+1-\mu_{k-1}}, \dots, \overline{n+1-\mu_1}].$$

Note that $l(w_\mu) = |\mu| - k$ by Lemma 2.2. (Recall that the symbol $|\mu|$ denotes the sum of the parts of μ.)

Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $l(w_\lambda) = |\alpha|$.

Finally, we adopt a convention that all homology or cohomology groups in the present paper are taken with integer coefficients.

3. Combinatorics of Divided Differences of Type D

We define simple divided differences of type D_n which are operators $\partial_i : \mathbf{Z}[X] \to \mathbf{Z}[X]$, $i = 1, \dots, n$, of degree -1 acting on the ring of polynomials $\mathbf{Z}[X]$ where X is a fixed set of indeterminates $X = \{x_1, x_2, \dots, x_n\}$. To this end, we denote by s_i, $1 \le i \le n-1$, the transposition

$$[1, \dots, i-1, i+1, i, i+2, \dots, n] \in S_n \subset W,$$

acting on X by interchanging x_i and x_{i+1}. Moreover, let

$$s_n = [1, \dots, n-2, \overline{n}, \overline{n-1}]$$

be the reflection which transposes x_{n-1} with x_n and changes the signs of both the variables. The remaining variables are invariant under the action of these transpositions. This action is extended multiplicatively to the ring $\mathbf{Z}[X]$. Note that s_n commutes with s_i, $i \neq n-2$, and

$$s_{n-2} \cdot s_n \cdot s_{n-2} = s_n \cdot s_{n-2} \cdot s_n .$$

Simple divided differences of type D are defined as follows:

$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}), \qquad i = 1, \dots, n-1;$$
$$\partial_n(f) = (f - s_n f)/(x_{n-1} + x_n).$$

The algebra generated by the divided diferences ∂_i, $1 \le i \le n-1$ that goes back to Newton, has been recently extensively studied by Lascoux and Schützenberger [LSc]. Here we investigate the effect of adding one extra divided difference ∂_n (which changes the picture in a drastic way).

For every $f, g \in \mathbf{Z}[X]$ and any i, we have:

$$\partial_i(f \cdot g) = f \cdot (\partial_i g) + (\partial_i f) \cdot (s_i g) \tag{1}$$

(a Leibniz-type formula).

For a given $\mathbf{a} = (a_n, a_{n-1}, \dots, a_2, a_1) \in \{-1, 0, 1\}^n$, we define the generating function:

$$E_{\mathbf{a}} = \prod_{i=1}^{n} (1 + a_i x_i). \tag{2}$$

In particular, for $\mathbf{a} = (1, \dots, 1)$, the resulting generating function, denoted by E, is the generating function for the elementary symmetric polynomials $e_i(X) = e_i(x_1, \dots, x_n)$, $i = 1, \dots, n$.

Lemma 3.1. (a) *We have* $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$, *where*

$$\mathbf{a}' = \begin{matrix} (a_n, \dots, a_{i+2}, a_i, a_{i+1}, a_{i-1}, \dots, a_1) & i < n, \\ (-a_{n-1}, -a_n, a_{n-2}, \dots, a_1) & i = n. \end{matrix}$$

(b) *For* $i = 1, 2, \dots, n-1$,

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} if \quad a_i = a_{i+1} + d \qquad (d = -2, -1, 0, 1, 2),$$

where $\mathbf{a}' = (a_n, \dots, 0, 0, \dots, a_1)$ *is the sequence* $\mathbf{a}$ *with* a_{i+1}, a_i *replaced by zeros.*

(c) $\partial_n(E_{\mathbf{a}}) = (a_n + a_{n-1}) \cdot E_{(0,0,a_{n-2},\dots,a_1)}$.

In particular if Δ *is a composition of some* s- *and* ∂-*operations, then for every* $\mathbf{a}$, $\Delta(E_{\mathbf{a}}) = (scalar) \cdot E_{\mathbf{a}'}$, *where* $\mathbf{a}'$ *is uniquely determined if this scalar is not zero.*

Proof. We prove e.g. (c). We have, with $\mathbf{a}' = (0, 0, a_{n-2}, \dots, a_1)$,

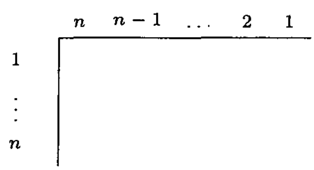
$$\partial_n(E_{\mathbf{a}}) = \frac{(1 + a_{n-1}x_{n-1})(1 + a_n x_n) - (1 - a_{n-1}x_n)(1 - a_n x_{n-1})}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'}$$

$$= \frac{(a_{n-1} + a_n)(x_{n-1} + x_n)}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'} = (a_n + a_{n-1}) \cdot E_{\mathbf{a}'} ,$$

as desired. □

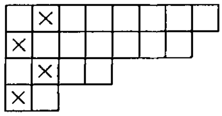
We now recall the following fact from [BGG] and [De]. For any $w \in W$ and any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ one can define $\partial_w = \partial_{i_1} \circ \cdots \circ \partial_{i_l}$ – an operator on $\mathbf{Z}[X]$ of degree $-l(w)$. In fact, since divided differences satisfy braid relations, ∂_w does not depend on the chosen reduced decomposition of w.

Suppose a strict partition μ with even $l(\mu)$ is given. Let us use the following coordinates for boxes in the Ferrers' diagram D_μ of μ:



We associate with μ a certain distinguished reduced decomposition of $w_\mu \in W$. To this end, let us modify the diagram D_μ in the following way. Remove one box from each row of D_μ : from rows with even numbers remove the box in the n-th column, and from rows with odd numbers remove the box in the $(n-1)$-st column.

We display the removed boxes in the picture using the symbol $\times$ and denote the so obtained set of boxes by $\overset{\circ}{D}_\mu$. For example, $\overset{\circ}{D}_{(8,7,4,2)}$ is:



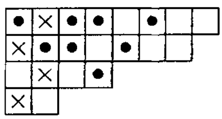
Assume now, that a subset $D \subset \overset{\circ}{D}_\mu$ is given. A box belonging to D will be called *a D-box* and a box from the difference $D_\mu \setminus D$ will be called *a $\sim$$D$-box.*

Definition 3.2. *Read $\overset{\circ}{D}_\mu$ row by row from left to right and from top to bottom. Every D-box (resp. $\sim$ D-box) in the i-th column gives us s_i (resp. ∂_i). Then ∂_μ^D is the composition of the resulting s_i's and ∂_i's (the composition written from right to left).*

Definition 3.3. *Read $\overset{\circ}{D}_\mu$. Every D-box in the i-th column gives us s_i. $\sim$ D-boxes give no contribution. Then, r_D is the word obtained by writing the resulting s_i's from right to left. (In other words, one obtains r_D by erasing all the ∂_i's from ∂_μ^D.)*

For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,

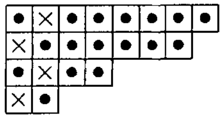
9 8 7 6 5 4 3 2 1



$$\partial_\mu^D = \partial_8 \circ s_6 \circ \partial_7 \circ \partial_9 \circ \partial_3 \circ \partial_4 \circ s_5 \circ \partial_6 \circ s_7 \circ s_8 \circ \partial_2 \circ \partial_3 \circ s_4 \circ \partial_5$$
$$\circ s_6 \circ s_7 \circ s_9$$
$$r_D = s_6 \cdot s_5 \cdot s_7 \cdot s_8 \cdot s_4 \cdot s_6 \cdot s_7 \cdot s_9 .$$

One can easily prove that for $D = \overset{\circ}{D}_\mu$, we have $r_D \in R(w_\mu)$ - the set of reduced decompositions of w_μ. This is our *distinguished reduced decomposition* of w_μ. For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,

9 8 7 6 5 4 3 2 1



$$w_\mu = s_8 \cdot s_6 \cdot s_7 \cdot s_9 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_9 \,.$$

Let now μ be a strict partition with even length. We will examine subsets $D \subset \mathring{D}_\mu$ for which $\partial_\mu^D(E) = 0$ (we say: " D causes the vanishing").

In many computations in this section, we will apply compositions of the operators of boxes of D_μ to the generating functions $E_{\mathbf{a}}$. With the following example we illustrate how such operators act.

Example 3.4. *Let $n = 9$. We apply the operators of boxes from left to right to $E_{\mathbf{a}}$, where $\mathbf{a} = (a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1)$, and obtain $E_{\mathbf{a}'}$; "•" denotes a D-box and empty boxes are $\sim$D-boxes. We give 2 examples of the action of the operators associated with a row in D_μ:*

9	8	7	6	5	4	3	2	1
•	×	•	•	•	•	•	•	

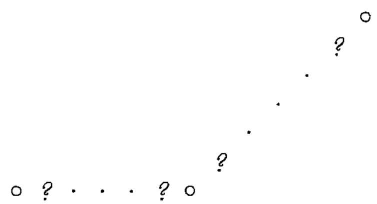
$$\mathbf{a}' = (-a_8, a_7, a_6, a_5, a_4, a_3, a_2, 0, 0)\,,$$

9	8	7	6	5	4	3	2	1
×	•	•	•		•	•		•

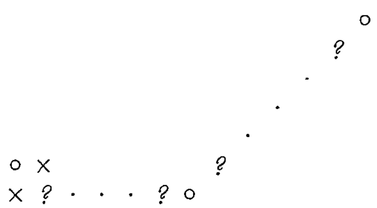
$$\mathbf{a}' = (a_8, a_7, a_6, 0, a_4, a_3, 0, a_1, 0)\,.$$

Lemma 3.5. *The following configurations of $\sim$D-boxes (noted by "○") in D_μ give the vanishing:*

(1) (*A three $\sim$D-box configuration*)



(2) (*A three $\sim$D-box configuration*)



(3) (*A three $\sim$D-box configuration*)

$$\begin{array}{cccccccccccc}
&&&&&&&&&&&\circ\\
&&&&&&&&&&?&\\
&&&&&&&&&\cdot&&\\
&&&&&&&&\cdot&&&\\
&&&&&&&\cdot&&&&\\
\times&\circ&&&&&?&&&&&\\
?&\times&?&\cdots&?&\circ&&&&&&
\end{array}
\qquad \textit{also:} \qquad
\begin{array}{cccccccc}
&&&&&&&\circ\\
&&&&&&?&\\
&&&&&\cdot&&\\
&&&&\cdot&&&\\
&&&\cdot&&&&\\
\times&\circ&?&&&&&\\
\circ&\times&&&&&&
\end{array}$$

(Above, "?" can be $\circ, \bullet$, or $\times$; and the skew directions are all parallel to the antidiagonal.)

Proof. Direct calculation using Lemma 3.1. $\square$

Let us fix an element $w = [y_1, y_2, \dots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \dots, \overline{z}_1] \in W^*$. Recall that k is even. We treat a given reduced decomposition $w = s_{i_1} \cdot s_{i_2} \cdots s_{i_l}$ as a sequence of simple transposition operations, which produces the element in question from the identity permutation:

$$[y_1, y_2, \dots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \dots, \overline{z}_1] = (\cdots([1, 2, \dots, n] \cdot s_{i_1}) \cdots) \cdot s_{i_l}.$$

In the following, the simple transpositions involved will be called the "s_{i_h}-operations" $(h = 1, \dots, l)$.

Proposition 3.6. *We have the following two possibilities for the action of s_{i_h}-operations on the z's:*

(1) *If $i_h = n$ then this operation is:*

$$[\dots, z, z'] \;\to\; [\dots, \overline{z}', \overline{z}],$$

where $z = z_{p-1}$ and $z' = z_p$ for some even p.

(2) *If $i_h < n$, then this operation is:*

$$[\dots, z, x, \dots] \to [\dots, x, z, \dots]$$

where $x \neq z_j$ for $j = 1, \dots, k$.

Proof. We must transpose each pair (z_j, y_i), for $z_j < y_i$, at least once, because the y's preced the (barred) z's in w. Also, we must transpose each pair (z_i, z_j), for $i < j$, at least once, because the (barred) z's appear in w in an descending order. In sum, we need at least

$$\sum \#\{(z_j, y_i) : z_j < y_i\} + \sum \#\{(z_i, z_j) : i < j\}$$

s_{i_h}-operations to reach the sequence w. But by Lemma 2.1 this last number is equal to $l(w)$. This means that the mentioned transpositions exchaust the family of all s_{i_h}-operations under consideration. As a consequence, no s_{i_h}-operation, for $i_h < n$, interchanges two (bar-free) z's (on their way towards the end of the permutation). Moreover, we see that exactly $k/2$ s_{i_h}-operations with $i_h = n$ appear. This implies immediately both assertions of the proposition. □

Now we assume that λ is another strict partition, that $D \subset \overset{\circ}{D}_\mu$, and that $r_D \in R(w_\lambda)$. Suppose that a D-box appears in the i-th column where $i < n$. We define the *mark* of this box to be p, if the corresponding s_{i_h}-operation acts on the i-th and $(i+1)$-st places as follows:

$$[\dots, z_p, x, \dots] \to [\dots, x, z_p, \dots],$$

where $x \neq z_j$, $j = 1, \dots, k$. A $D-box$ in the n-th column has mark $p-1$ if the corresponding $(s_{i_h} = s_n)$-operation acts via

$$[\dots, z_{p-1}, z_p] \to [\dots, \overline{z}_p, \overline{z}_{p-1}].$$

In particular, boxes in the n-th column have only odd marks. In the following lemma, we collect some simple properties of marks.

Lemma 3.7. (1) (*Connectedness*) *The D-boxes with a fixed mark in one row form a connected set. (The boxes marked by $\times$ are treated as empty boxes).*

(2) (*Separation*) *In a fixed row, the two sets of D-boxes equipped with different marks are disconnected (i.e. there is at least one $\sim$$D$-box between them; a box marked by $\times$ is not treated as as $\sim$$D$-box).*

(3) *The sequence of boxes with odd mark p is of the form:*

$$(t_n, n), (t_{n-2}, n-2), \dots, (t_{z_p}, z_p),$$

where $p \le t_n \le t_{n-2} \le \dots \le t_{z_p}$.

(4) *The sequence of boxes with even mark p is of the form:*

$$(t_{n-1}, n-1), (t_{n-2}, n-2), \dots, (t_{z_p}, z_p),$$

where $p \le t_{n-1} \le t_{n-2} \le \dots \le t_{z_p}$.

(5) *The marks of boxes in a fixed column (strictly) increase from top to bottom.*

(6) *The marks of boxes in a fixed row (weakly) decrease from left to right.*

Definition 3.8. *The set of D-boxes with mark p is called the* ribbon of mark p.

We have two basic operations of deforming ribbons.

- ("Push down") Let i be odd and suppose that the boxes

  $$(i, n), (i, n-2), \dots, (i, j)$$

  form an entire ribbon. The operation transforms them to

  $$(i+2, n), (i+2, n-2), \dots, (i+2, j).$$

  Let i be even and suppose that the boxes

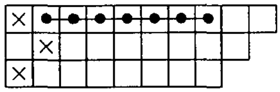
  $$(i, n-1), (i, n-2), \dots, (i, j)$$

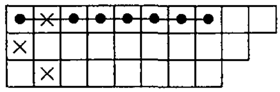
form an entire ribbon. The operation transforms them to

$$(i+2, n-1), (i+2, n-2), \ldots, (i+2, j).$$

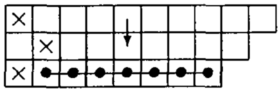
(We assume that the $(i+1)$-st and $(i+2)$-nd row are empty before this operation.)

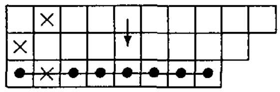
For example, the ribbons





can be pushed down to





- ("Breaking a ribbon") Let $j \le n-2$. The operation transforms a final segment

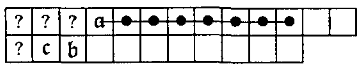
$$(i, j), (i, j-1), \ldots, (i, h)$$

of a ribbon to empty boxes

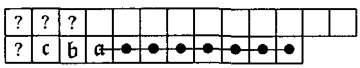
$$(i+1, j), (i+1, j-1), \ldots, (i+1, h),$$

provided $(i+1, j+1)$ is a $\sim D$-box, or it is $\times$ and $(i+1, j+2)$ is a $\sim D$-box. The box (i, j) (before the operation) is called the *breaking box.*

For example, for a breaking box $\mathfrak{a}$, $\mathfrak{b}$ a $\sim D$-box, or $\mathfrak{b} = \times$ and $\mathfrak{c}$ a $\sim D$-box:



can be broken at $\mathfrak{a}$ and transformed to



Using the braid relations in W one easily shows that if $r_D \in R(w_\lambda)$, then, after breaking a ribbon in D, we get D' such that $r_{D'} \in R(w_\lambda)$. In the case of the push down operation, it is clear that we get D' with $r_D = r_{D'}$. Note that any configuration of boxes $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$ can be obtained from $\overset{\circ}{D}_\lambda \subset \overset{\circ}{D}_\mu$ by a sequence of operations of the above described two types.

Definition 3.9. (*"Maximal deformation" of* $\overset{\circ}{D}_\lambda \subset \overset{\circ}{D}_\mu$)

- *Pick the lowest ribbon. Push it down as many times as possible. Then choose the leftmost breaking box on this ribbon (if it exists) and break the ribbon.*
- *Pick a ribbon and suppose that lower ribbons in $\overset{\circ}{D}_\lambda$ have been already deformed. Push down this ribbon as many times as possible. Let $\mathfrak{a}$ be the leftmost breaking box on the ribbon. Break the ribbon at $\mathfrak{a}$ as many times as possible. Then choose the next leftmost breaking box $\mathfrak{b}$ and break the ribbon at $\mathfrak{b}$ as many times as possible etc.*

For some examples of maximal deformations of diagrams, see Example 4.7.

Proposition 3.10. *Let $D \subset \overset{\circ}{D}_\mu$ be such that $r_D \in R(w_\lambda)$. If $\partial_\mu^D(E) \neq 0$ then D is the maximal deformation of $\overset{\circ}{D}_\lambda \subset \overset{\circ}{D}_\mu$.*

Proof. The proof is by descending induction on the mark of a ribbon. Pick the ribbon with mark p. Assume that the ribbons with marks $p+1, \ldots, l(\lambda)$ have been already maximally deformed. Suppose that we have either a possibility of pushing down of a ribbon or breaking a ribbon. In the case of the former operation we will refer to boxes of the three involved rows; in the case of the latter operation, we will refer to the two involved rows. We note that

- any box directly to the right or directly below of the rightmost box of a row of the (deformed) ribbon is a $\sim\!D$-box;
- any box directly to the left or directly above of the leftmost box of a row of the (deformed) ribbon is a $\sim\!D$-box;
- $\sim D$-boxes in the n-th or $(n-1)$-st column cannot be supplied by marks smaller than p.

This implies that if we not perform the operations in a maximal way, then we will either obtain a configuration

$$\begin{array}{ccccccc} & & & & & & \circ \\ \circ & ? & \cdot & \cdot & \cdot & ? & \circ \end{array}$$

or, we will get one of the following two possibilities:

$$\begin{array}{cccccccc} \bullet & \times & & & & & & \\ \times & \mathfrak{b} & \mathfrak{c} & & & & & \\ \mathfrak{a} & \times & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \end{array} \qquad\qquad \begin{array}{cccccccc} \times & & & & & & & \\ \mathfrak{a} & \times & \mathfrak{b} & & & & & \\ \times & \mathfrak{c} & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \end{array}$$

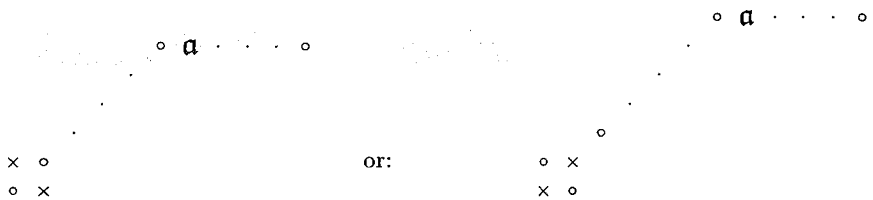
where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are $\sim D$-boxes. By Lemma 3.5 all these three configurations of $\sim D$-boxes cause the vanishing. The obtained contradiction means that the maximal deformation of $\overset{\circ}{D}_\lambda \subset \overset{\circ}{D}_\mu$ is necessary to avoid the vanishing. $\square$

As a corollary of this proposition, we get the following two results bounding the size of μ w.r.t. λ, if one wants to avoid the vanishing. We will need some additional notation. Given a strict partition μ of even length l, we denote by μ^- the strict partition $(\mu_1 - 1, \dots, \mu_l - 1)$. Note that $(\mu^-)^+ = \mu$. We also set $\mathring{l}(\mu) := l(\mu^-)$. Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $\mathring{l}(\lambda) = l(\alpha) = \#D_\alpha = \#\mathring{D}_\lambda$.

Proposition 3.11. *If, for the maximal deformation D of $\mathring{D}_\lambda \subset \mathring{D}_\mu$, one has $\partial_\mu^D(E) \neq 0$, then $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$.*

Proof. Suppose that $\mathring{l}(\mu) \geq \mathring{l}(\lambda) + 2$. Pick the highest, say i-th, row which contains a $\sim D$-box in the n-th or $n-1$-st column. This is the highest row from which some ribbon has been pushed down in the maximal deformation, or, if there was no pushing down, it is the row with number $l(\lambda) + 1$. Since $\mathring{l}(\mu) \geq \mathring{l}(\lambda) + 2$ and by the nature of maximal deformation, we see that the $(i+1)$-st row contains also a $\sim D$-box in its $n-1$-st or n-th row respectively.

After breaking some higher ribbons we get the following configuration of $\sim D$-boxes:



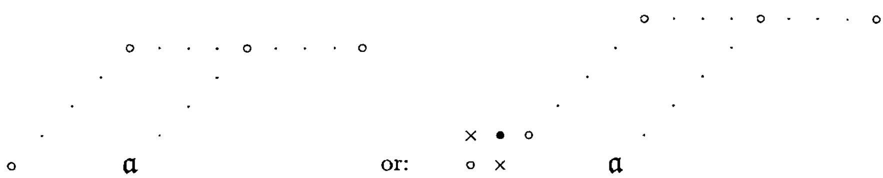
The boxes marked by $\mathfrak{a}$ exist and they are $\sim D$-boxes (because μ is a strict partition). By Lemma 3.5 we get the vanishing – contradiction. $\square$

Proposition 3.12. *Assume that $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$. If for the maximal deformation D of $\mathring{D}_\lambda \subset \mathring{D}_\mu$ one has $\partial_\mu^D(E) \neq 0$, then $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip.*

Proof. Suppose that $\lambda_i < \mu_{i+1}$ for some i. We can assume that for some $j < i$,

$$\mu_i = \lambda_{i-1}, \mu_{i-1} = \lambda_{i-2}, \dots, \mu_{j+2} = \lambda_{j+1} \quad \text{but} \quad \lambda_j > \mu_{j+1}\,.$$

After the maximal deformation, we get in the consecutive rows with numbers $i+1, i, \dots, j+1$ the following configuration of $\sim D$-boxes:



where $\mathfrak{a}$ displays a $\sim D$-box. We get the vanishing by Lemma 3.5 – contradiction. $\square$

The maximal deformation is obtained by breaking each row such that $\lambda_i = \mu_{i+1}$, at one breaking point.

Among the connected components of $\mathring{D}_\mu \setminus \mathring{D}_\lambda$ we have those which do not meet the n-th column: they are ordinary horizontal strips [Ma]. Those which meet the n-th component are of the form:

$$\begin{array}{llllllll}
 & & & & & & & \circ\ \cdot\ \cdot\ \cdot\ \circ \\
 & & & & & & \cdot & \\
 & & & & & \cdot & & \\
 & & & & \cdot & & & \\
 & & & \circ\ \cdot\ \cdot\ \cdot\ \circ & & & & \\
 & & \cdot & & & & & \\
 & \cdot & & & & & & \\
\cdot & & & & & & & \\
\circ\ \times\ \circ\ \cdot\ \cdot\ \cdot\ \circ & & & & & & & \\
\times & & & & & & &
\end{array} \tag{3}$$

Note the following particular case of (3):

$$\begin{array}{l}
\times\ \bullet\ \bullet\ \cdot\ \cdot\ \cdot\ \bullet\ \bullet \\
\circ\ \times\ \circ\ \cdot\ \cdot\ \cdot\ \circ| \\
\times
\end{array} \tag{4}$$

(By "|" we visualize the end of a row.) After the maximal deformation an ordinary horizontal strip becomes

$$\begin{array}{lllll}
 & & & & \circ\ \cdot\ \cdot\ \cdot\ \circ| \\
 & & & \cdot & \\
 & & \cdot & & \\
 & \cdot & & & \\
\circ & & & &
\end{array}$$

In turn, configuration (3) which is not of the form (4), is deformed to:

$$\begin{array}{lllll}
 & & & & \circ\ \cdot\ \cdot\ \cdot\ \circ| \\
 & & & \cdot & \\
 & & \cdot & & \\
 & \cdot & & & \\
\times\ \bullet\ \circ & & & & \\
\circ\ \times & & & & \\
\times & & & &
\end{array}$$

Of course, configuration (4) does not change under the maximal deformation.

Proposition 3.13. *Suppose that $\lambda \subset \mu \subset \rho(n)$ are strict partitions such that $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular, $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$). Let D be the maximal deformation of $\mathring{D}_\mu \setminus \mathring{D}_\lambda$. Then $\partial^D_\mu(E) = 2^m$, where m is the number of connected components of $\mathring{D}_\mu \setminus D$.*

Proof. Different connected components of $\mathring{D}_\mu \setminus D$ lie in separate rows and separate columns. Let us number these components from top to bottom. Pick a connected component of $\mathring{D}_\mu \setminus D$. The part of ∂^D_μ associated with the boxes in the rows preceding the rows of the component, transform E into $2^{m'}E_{\mathbf{a}}$, where m' is the number of components preceding the given one. If the first row of its appearance has odd (resp. even) number, then $\mathbf{a} = (1, 1, \dots, 1, *, \dots, *)$ (resp. $\mathbf{a} = (-1, 1, \dots, 1, *, \dots, *)$) and the cardinality of displayed ±1's is the length of the first row of μ supporting the component, the count including the box $\times$. In turn, the operators of rows supporting the component transform $E_{\mathbf{a}}$ to $2E_{\mathbf{a}'}$ for some $\mathbf{a}'$. The multiplicity 2 comes from the highest leftmost box of the component; the operators of all remaining boxes give the multiplicity 1. If such highest leftmost box lies in the h-th column where $h < n$, then one gets the multiplicity 2 by applying ∂_h to $E_{\mathbf{b}}$ where $\mathbf{b} = (\dots, -1, 1, \dots)$, the displayed entries being in $(h+1)$-st and h-th places. If the component is of the form (4), then we get the multiplicity 2 by applying ∂_n to $E_{\mathbf{c}}$ where $\mathbf{c} = (1, 1, \dots)$. This proves the proposition. □

We summarize the results of this section in the following theorem (observe that the connected components of $\mathring{D}_\mu \setminus D$ are in one-to-one correspondence with the connected components of $D_{\mu^-} \setminus D_{\lambda^-}$):

Theorem 3.14. *Let $\lambda \subset \mu \subset \rho(n)$ be strict partitions. Then for $D \subset \mathring{D}_\mu$, one has $\partial^D_\mu(E) \neq 0$ iff $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular $\mathring{l}(\mu) < \mathring{l}(\lambda) + 1$), and D is obtained by the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$. In this case, $\partial^D_\mu(E) = 2^m$, where m is the number of connected components of $D_{\mu^-} \setminus D_{\lambda^-}$.*

4. A Group-Theoretic Approach to Schubert Calculus for CS_n

We first introduce some notation. Recall that $H = SO(2n, \mathbf{C})$ and $B \subset H$ is a Borel subgroup of H. We denote by P – the maximal parabolic subgroup of H containing B and corresponding to the subset Σ of simple roots minus the right end root, by F – an "isotropic" orthogonal flag manifold H/B, and by G – the orthogonal Grassmannian H/P.

Moreover, the Schubert variety X_w, $w \in W$, is defined as the closure of the Schubert cell B^-wB/B in H/B (B^- is the opposite Borel subgroup to B). We record the following well-known result:

Lemma 4.1. *X_w is a (closed) subvariety of H/B of (complex) codimension $l(w)$.*

Let $X = \{x_1, \dots, x_n\}$ be a sequence of variables. For brevity, we denote also by the symbol X_w the class of the variety X_w in $H^{2l(w)}(F)$. Let $\alpha \subset \rho(n-1)$ be a strict partition and put $\lambda := \alpha^+$; one has $X_{w_\lambda} \in H^{2|\alpha|}(F)$. Since $w_\lambda \in W^*$, it follows from [BGG] that X_{w_λ} belongs already to $H^{2|\alpha|}(G) \subset H^{2|\alpha|}(F)$. Let us denote this element in $H^{2|\alpha|}(G)$ (as well as the representing it Schubert variety in G), by σ_α.

There exists a surjective ring homomorphism $c : \mathbf{Z}[1/2][X] \to H^*(F)$ (called the *Borel characteristic map*) such that for a homogeneous $f \in \mathbf{Z}[X]$ one has

$$c(f) = \sum_{l(w)=\deg f} \partial_w(f)\; X_w \,. \tag{5}$$

(The original Borel's definition [Bo] of the characteristic map was different; the present description comes from [BGG] and [De].)

Note (cf. e.g. [Bo]) that the ring $H^*(G)$ can be identified algebraically as

$$H^*(G) = \mathbf{Z}[X]^{S_n}/(e_i(X^2), i = 1, \dots, n-1; x_1x_2\cdots x_n)\,,$$

where $X^2 = (x_1^2, \dots, x_n^2)$.

We have also another identification of $H^*(G)$ stemming from [Du1] and [P1]:

Lemma 4.2. *Let S be the tautological rank n subbundle on G. The Chern classes $c_i(S)$ are all divisible by 2, and one has the identification $\sigma_i = \frac{1}{2}c_i(S^*)$ for $i = 1, \dots, n-1$. Moreover,*

$$H^*(G) = \mathbf{Z}[\sigma_1, \dots, \sigma_{n-1}]/(R_i, 1 \le i \le n-1)\,,$$

where, with the convention $\sigma_i = 0$ for $k > n-1$, the relations R_i are given by

$$R_i = \sigma_i^2 - 2\sigma_{i-1}\sigma_{i+1} + 2\sigma_{i-2}\sigma_{i+2} - \cdots + 2(-1)^{i-1}\sigma_1\sigma_{2i-1} + (-1)^i\sigma_{2i}\,.$$

It turns out that after restriction to $\mathbf{Z}[X]^{S_n} \otimes \mathbf{Z}[1/2]$, the map c goes onto $H^*(G)$, and we have the following fact. Let, from now on, $e_r = e_r(X)$ denote the r-th elementary symmetric function in $X = \{x_1, \dots, x_n\}$.

Lemma 4.3. *For every $r = 1, \dots, n-1$, one has $c(e_r) = 2\sigma_r$.*

Proof. We have

$$\partial_{n-r}\cdots\partial_{n-2}\partial_n(e_r) = 2\,.$$

Any other divided difference operator of degree r applied to e_r gives 0. This implies the assertion. $\square$

For a strict partition $\alpha \subset \rho(n-1)$, we choose a homogeneous $f_\alpha \in \mathbf{Z}[1/2][X]$ such that $c(f_\alpha) = \sigma_\alpha$. Then, for $w \in W^*$ with $l(w) = l(\alpha)$, one has $\partial_w(f_\alpha) \neq 0$ iff $w = w_\lambda$ and $\partial_{w_\lambda}(f_\alpha) = 1$ for $\lambda = \alpha^+$. We want to find the coefficients d_β in the expansion

$$c(f_\alpha \cdot e_r) = \sum d_\beta \sigma_\beta\,. \tag{6}$$

Proposition 4.4. *In the above notation, setting $\mu := \beta^+$, one has*

$$d_\beta = \sum \partial_\mu^D(e_r),$$

where the sum is over all $D \subset \overset{\circ}{D}_\mu$ such that $r_D \in R(w_\lambda)$ (here, $\lambda = \alpha^+$).

Proof. We have $d_\beta = \partial_{w_\mu}(f_\alpha \cdot e_r)$, and $\partial_{w_\mu} = \partial_\mu^\emptyset$. The integer $d_\beta = \partial_\mu^\emptyset(f_\alpha \cdot e_r)$ is computed by a consecutive application of the Leibniz-type formula (1): we apply only the ∂_i's (and the identity operators) to f_α, and both the s_i's and ∂_i's to the factor e_r. We get

$$d_\beta = \sum \partial_{r_D}(f_\alpha) \cdot \partial_\mu^D(e_r)\,,$$

the sum over all $D \subset \overset{\circ}{D}_\mu$. The summand corresponding to a subset $D \subset \overset{\circ}{D}_\mu$ is not zero only if $\#D = \deg f_\alpha$ and $\#(D_\mu \setminus D) = r$. By the choice of f_α, $\partial_{r_D}(f_\alpha) = 0$ if $r_D \notin R(w_\lambda)$, and equals 1 if $r_D \in R(w_\lambda)$, and thus we get the desired equality. $\square$

Remark 4.5. *This use of an iterated Leibniz-type formula to compute the multiplicities d_β stems from a series of papers of Ratajski and the second author* (*cf.* [P2]). *It was also known to Kostant and Kumar – see* [KK].

Combining this proposition with Theorem 3.14, and taking into account Lemma 4.3, we get a group-theoretic proof of the following result (that is refered to as a "Pieri-type formula"):

Theorem 4.6. *Let $\alpha \subset \rho(n-1)$ be a strict partition. Then for any $1 \leq r \leq n-1$,*

$$\sigma_\alpha \cdot \sigma_r = \sum_\beta 2^{m_\beta}\ \sigma_\beta\,,$$

where the sum is over all strict partitions $\beta \subset \rho(n-1)$ such that $D_\beta \backslash D_\alpha$ is a horizontal strip of length r and m_β is the number of connected components of $D_\beta \setminus D_\alpha$ minus 1.

(Cf. also [P1, Theorem 6.17'].)

Example 4.7. *Let $n = 8$. We examine the product $\sigma_{5,3} \cdot \sigma_4$:*

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \end{array} \qquad \textit{times} \qquad \circ\ \circ\ \circ\ \circ$$

On the LHS we depict the β's; on the RHS we display the unique $D \subset \mathring{D}_\mu$ such that $\partial_\mu^D(E) \neq 0$:

$$\begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & & \end{array} \quad \textit{and} \quad \begin{array}{llllllll} \bullet & \times & \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & \bullet & & \end{array}$$

$$\begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & & & \\ \circ & & & & & & \end{array} \quad \textit{and} \quad \begin{array}{llllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & & & \\ \circ & \times & & & & & & \\ \times & & & & & & & \end{array}$$

$$\begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & & & & \\ \circ & \circ & & & & & \end{array} \quad \textit{and} \quad \begin{array}{llllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & & & & \\ \circ & \times & \circ & & & & & \\ \times & & & & & & & \end{array}$$

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & \\ \circ & & & & & \end{array} \quad \textit{and} \quad \begin{array}{lllllll} \bullet & \times & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & \bullet & \\ \circ & \times & & & & & \\ \times & & & & & & \end{array}$$

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \circ & & \\ \circ & \circ & & & & \end{array} \quad \textit{and} \quad \begin{array}{lllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ \\ \times & \bullet & \bullet & \bullet & \circ & & \\ \circ & \times & \circ & & & & \\ \times & & & & & & \end{array}$$

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \circ & \\ \circ & \circ & \circ & & \end{array} \quad \textit{and} \quad \begin{array}{llllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet \\ \times & \bullet & \circ & \circ & \circ & \\ \circ & \times & \bullet & \bullet & & \\ \times & & & & & \end{array}$$

Thus we get:

$$\sigma_{5,3} \cdot \sigma_4 = \sigma_{7,5} + 4\sigma_{7,4,1} + 2\sigma_{7,3,2} + 2\sigma_{6,5,1} + 4\sigma_{6,4,2} + \sigma_{5,4,3} \,.$$

A fundamental invariant of a projective variety $X \subset \mathbf{P}^N_{\mathbf{C}}$ is its *degree*, defined by

$$\deg X = \int_X {\omega_X}^n \,,$$

where $n = \dim_{\mathbf{C}} X$ and ω_X is the restriction of the standard Kähler form on $\mathbf{P}^N$ to X. The importance of this invariant is seen from its various interpretations [GH, p.171]:

(i) The number $\deg X$ equals to the number of intersection points of X with a general linear subspace in $\mathbf{P}^N$ of complementary dimension.

(ii) The number $n! \deg X$ agrees with the volume of X .

It is well known that σ_1 is the generator of $\mathrm{Pic}(G)$ and also σ_1 is the Kähler class of G.

So by (i),

$$\deg(\sigma_\alpha) = \sigma_\alpha \cdot {\sigma_1}^{n(n-1)/2-|\alpha|}.$$

We invoke Schur P-functions $P_\lambda = P_\lambda(X)$ of [S] whose definition reads:

(1) For a nonnegative integer i, $P_i := \sum s_\lambda$, where the sum is over all hook partitions λ of i, and s_λ denotes the corresponding Schur S-function (cf., e.g., [Ma]).

(2) For integers $i > j > 0$,

$$P_{(i,j)} := P_i P_j + 2 \sum_{1 \le q \le i-1} (-1)^q P_{j+q} P_{i-q} + (-1)^{i+j} P_{i+j}.$$

(3) For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ written with an even k (by putting $\lambda_k = 0$ if necessary),

$$P_{(\lambda_1,\ldots,\lambda_k)} := \mathrm{Pf}[P_{(\lambda_p,\lambda_q)}]_{1\le p<q\le k},$$

where Pf denotes the Pfaffian. See [S], [Ma], [HH], and [P1] for more on Schur P-functions. Sometimes it is more handy to work with Schur Q-functions defined by $Q_\lambda = Q_\lambda(X) = 2^{l(\lambda)} P_\lambda$ for a strict partition λ.

Comparing the Pieri-type formula for P-functions [Ma,III.8.15] (extracted in [P1] from [Mo]) with Theorem 4.6, we get that $\deg\sigma_\alpha$ is the coefficient of $P_{\rho(n-1)}$ in

$$P_\alpha \cdot P_1^{n(n-1)/2-|\alpha|},$$

or the coefficient of $P_{\bar\alpha}$ in

$$P_1^{n(n-1)/2-|\alpha|} = P_1^{|\bar\alpha|}.$$

Here $\bar\alpha$ is the partition whose part complement the parts of α in $\{1, \ldots, n-1\}$.

We define for a partition $\gamma = (\gamma_1, \gamma_2, \ldots)$,

$$g^\gamma = \frac{|\gamma|!}{\gamma!} \prod_{i<j} \frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j}, \tag{7}$$

where $\gamma! = \gamma_1! \gamma_2! \cdots$.

Proposition 4.8. *One has* $\deg(\sigma_\alpha) = g^\gamma$ *for* $\gamma = \bar\alpha$.

Remark 4.9. *Certain special cases of this formula were obtained by Hiller* [Hi]. *Some related computations were performed by Tamvakis* [Ta2] *in the context of heights of homogeneous spaces in arithmetic intersection theory.*

The proposition follows from the following lemma due essentially to Schur [S].

Lemma 4.10. *One has*

$$P_1^k = \sum g^\gamma P_\gamma,$$

the sum over strict partitions γ *of* k.

Proof. We give here a proof using a specialization result from [DP] and the following formula (8). Let $p_i(X) = x_1^i + \cdots + x_n^i$ be the power sum. For a partition $\mu = (\mu_1, \mu_2, \ldots)$ we set $p_\mu(X) = \prod_i p_{\mu_i}(X)$ and $z_\mu = \prod_{i\geq 1} i^{m_i} m_i!$, where $m_i = \#\{j : \mu_j = i\}$. Moreover, by an *odd* partition we understand the one whose all parts are odd. Let $Y = \{y_1, \ldots, y_n\}$ be another set of indeterminates. Then we have [Ma, III.8.13], [HH, Cor.7.15]:

$$\sum_{\lambda \text{ strict}} P_\lambda(X) Q_\lambda(Y) = \sum_{\mu \text{ odd}} 2^{l(\mu)} z_\mu^{-1} p_\mu(X) p_\mu(Y) . \tag{8}$$

We use the following specialization. We set $p_1(Y) = 1/2$ and $p_i(Y) = 0$ for $i \geq 2$. Using

$$Q_i(Y) = \sum_{\nu \text{ odd}} z_\nu^{-1} 2^{l(\nu)} p_\nu(Y)$$

(Ma, p.260], [HH, (7.9)), we see that under this specialization, we have $Q_i(Y) = 1/i!$. The following equality was proved in [DP]: via this specialization, for a strict partition λ,

$$Q_\lambda(Y) = g^\lambda / |\lambda|! . \tag{9}$$

Therefore the specialization under consideration transforms equation (8) into the assertion of the lemma. □

Remark 4.11. *As a matter of fact the key in the original Schur's calculation* [S] (*see also* [Ma, p.267] *and in more detail* [HH]) *is the proof of the following equality: for a strict partition* $\gamma = (\gamma_1 > \cdots > \gamma_l > 0)$,

$$g^\gamma = \sum_{i=1}^{l} g^{\gamma^{(i)}} , \tag{10}$$

where $\gamma^{(i)}$ *is the strict partition obtained from* γ *by subtracting* 1 *from the* i*-th part* γ_i *of* γ. *The original argument rests on the expansion into partial fractions of the function*

$$(2t-1) \prod_i \frac{(t+\gamma_i)(t-\gamma_i-1)}{(t+\gamma_i-1)(t-\gamma_i)} .$$

Here is another way of obtaining (10) *for those who prefer Lagrange interpolation to expansion into partial fractions. Suppose that* $\gamma_1, \ldots, \gamma_l$ *are* l *indeterminates. We start with the equation:*

$$\begin{aligned}(\gamma_1 + \cdots + \gamma_l) \prod_{i<j} (\gamma_i - \gamma_j) \\ = \sum_p (-1)^{p-1} \gamma_p \prod_{\{i,j \neq p;\ i<j\}} (\gamma_i - \gamma_j) \prod_{i \neq p} (\gamma_p + \gamma_i) ,\end{aligned} \tag{11}$$

which, as Lascoux points out, is exactly the content of the Lagrange interpolation for $\gamma_1 + \cdots + \gamma_l$ (cf.[L2]). (*Equation* (11) *is easy to prove, e.g. by showing that its RHS is skew-symmetric.*)

Letting $\overline{Q}_\lambda$ *be a function in the* γ*'s given by the expression for* $Q_\lambda(Y)$ *in* (9), *we rewrite* (11) *as*

$$(\gamma_1 + \cdots + \gamma_l)\overline{Q}_\gamma = \sum_{i=1}^{l} (-1)^{i-1} \overline{Q}_{\gamma_i - 1} \overline{Q}_{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_l} . \tag{12}$$

We now record the following identity for general Q*-functions that follows rather easily from their definition by induction:*

$$\sum_{i=1}^{l} (-1)^{i-1} Q_{\gamma_i - 1} Q_{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_l} = \sum_{j=1}^{l} Q_{\gamma_1, \ldots, \gamma_j - 1, \ldots, \gamma_l} . \tag{13}$$

Comparing (12) *and* (13), *we get*

$$(\gamma_1 + \cdots + \gamma_l)\overline{Q}_\gamma = \sum_{j=1}^{l} \overline{Q}_{\gamma_1, \ldots, \gamma_j - 1, \ldots, \gamma_l} ,$$

which gives (10).

Remark 4.12. *Here is still another derivation of the lemma for the reader knowing Hall-Littlewood functions: combine* [Ma, Ex.III.8.1 p.259; formula III.7.1, p.246; and Ex.III.8.12 p.266].

Up to now, we have considered Schubert varieties as purely group-theoretic objects. We end this section by recalling their interpretation in terms of Schubert-type conditions. This description is a recollection from [LSe] and [P1], and we will need it in the next section.

Let U be a $2n$-dimensional vector space endowed with a nondegenerate orthogonal form $\xi : U \times U \to \mathbf{C}$. Consider

$$Z = \{L \subset U : L \text{ is maximal isotropic subspace in } U\} .$$

This subvariety is canonically embedded in the Grassmannian $G_n(U)$. This last variety has Schubert (sub)varieties which are defined w.r.t. to flag

$$U_1 \subset U_2 \subset \cdots \subset U_{2n} = U$$

(where $\dim(U_i) = i$), in the following way: Given a sequence $1 \le i_1 < \cdots < i_n \le 2n$, we set

$$\Omega(i_1, \ldots, i_n) = \{L \in G_n(U) : \dim(L \cap U_{i_p}) \ge p \ \forall p = 1, \ldots, n\} .$$

One has

$$\dim \Omega(i_1, \ldots, i_n) = i_1 + \cdots + i_n - n(n+1)/2 .$$

It is known that Z has two connected components which are isomorphic to $G = H/P$. Let $v_1, \ldots, v_n, w_1, \ldots, w_n$ be a basis of U such that

$\xi(v_i, v_j) = \xi(w_i, w_j) = 0, \xi(v_i, w_j) = \xi(w_j, v_i) = \delta_{i,j}$. Let V_i be the vector space spanned by the first i vectors of the above basis. Then the Schubert varieties in $G_n(U)$ (determined by the flag $V_1 \subset \cdots \subset V_{2n} = V$) which *give rise* to the Schubert varieties in G (in the sense of [LSe] and [P1]) are indexed by the sequences $(i_1, \ldots, i_n)$ where $i_p \neq 2n+1-i_q$ for $p, q = 1, \ldots, n$, and if k denotes the largest number such that $i_k \leq n$, then $n-k$ is even. Let us denote by $\Omega[i_1, \ldots, i_k]$ the Schubert variety in G determined (via restriction to G) by this Schubert variety in $G_n(U)$, that is:

$$\Omega[i_1, \ldots, i_k] := \{L \in G : \dim(L \cap V_{i_p}) \geq p \ \forall \ p = 1, \ldots, k\}.$$

(Instead refering to the flag $V_1 \subset \cdots \subset V_n$, we will also say that this Schubert variety is defined w.r.t. the ordered basis $\{v_1, \ldots, v_n\}$.) One has

$$\dim \Omega[i_1, \ldots, i_k] = i_1 + \cdots + i_k + n(n-k) - n(n+1)/2.$$

The Schubert classes in $H^*(G)$ determined by these Schubert varieties are related in following way to the Schubert classes σ_α considered earlier in this section. For a strict partition $\alpha = (\alpha_1, \ldots, \alpha_k) \subset \rho(n-1)$, one has $\sigma_\alpha = \Omega[n-\alpha_1, \ldots, n-\alpha_k]$ if $n-k$ is even, and $\sigma_\alpha = \Omega[n-\alpha_1, \ldots, n-\alpha_k, n]$ if $n-k$ is odd.

The corresponding Schubert variety σ_α in G can be defined in the following way w.r.t. the above flag $V_1 \subset \cdots \subset V_n$; it is:

$$\{L \in G : \dim(L \cap V_{n-\alpha_p}) \geq i \ \forall \ p = 1, \ldots, k \text{ and } \operatorname{codim}_{V_n}(L \cap V_n) \text{ is even}\}.$$

5. Schubert Cycles of Complex Structures on $\mathbf{R}^{2n}$

We will adopt the following convention. Let $\mathbf{R}^{2n}$ be the real Euclidean $2n$-space with the standard orthonormal basis $\{e_1, \ldots, e_{2n}\}$.

The $2n$-dimensional complex Euclidean space $\mathbf{C}^{2n}$ will be considered as the complexification of $\mathbf{R}^{2n}$; $\mathbf{C}^{2n} = \mathbf{R}^{2n} \otimes_{\mathbf{R}} \mathbf{C}$. Note the following simple facts:

(a) The set $\{e_1 \otimes 1, \ldots, e_{2n} \otimes 1\}$ is an orthonormal basis for $\mathbf{C}^{2n}$.

(b) If $L, K \subset \mathbf{R}^{2n}$ are two linear subspaces satisfying $\dim_{\mathbf{R}}(L \cap K) \geq i$, then their complexifications $L^{\mathbf{C}}, K^{\mathbf{C}} \subset \mathbf{C}^{2n}$ satisfy $\dim_{\mathbf{C}}(L^{\mathbf{C}} \cap K^{\mathbf{C}}) \geq i$.

(c) Corresponding to an orthogonal decomposition $L = L_1 \oplus L_2$ of a subspace $L \subset \mathbf{R}^{2n}$, one has the orthogonal decomposition $L^{\mathbf{C}} = L_1^{\mathbf{C}} \oplus L_2^{\mathbf{C}}$ of $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

(d) An $\mathbf{R}$-linear endomorphism of a subspace $L \subset \mathbf{R}^{2n}$ induces a $\mathbf{C}$-linear endomorphism of the subspace $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

Let V be an oriented even dimensional real Euclidean space and $\operatorname{Iso}(V)$, the group of orientation preserving isometries of V. Consider

$$CS(V) = \{A \in \operatorname{Iso}(V) \ : \ A^2 = -Id_V\}.$$

It is the space of complex structures on V.

If $V = \mathbf{R}^{2n}$, one has the identification $\mathrm{Iso}(R^{2n}) = SO(2n, \mathbf{R})$, the special orthogonal group of order $2n$, and

$$CS(\mathbf{R}^{2n}) = \{A \in SO(2n, \mathbf{R}) \mid A^2 = -I_{2n}\}.$$

Note that if $A \in CS(\mathbf{R}^n)$ then A is a skew-symmetric matrix. Let us abbreviate $CS(\mathbf{R}^{2n})$ by CS_n as is common. The space CS_n has two connected components which are distinguished by the Pfaffian function

$$\mathrm{Pf} : CS_n \to \{\pm 1\}.$$

We write $CS_n = CS_n^+ \sqcup CS_n^-$ with $\mathrm{Pf}(CS_n^\pm) = \pm 1$.(The symbol $\sqcup$ denotes the disjoint union.) Both manifolds $CS_n^\pm$ are isometric to $SO(2n, \mathbf{C})/U(n)$.

Our goal in this section is to interpret Schubert varieties in $SO(2n, \mathbf{C})/U(n)$ in terms of complex structures. We will give an interpretation, in terms of Schubert varieties, of the following Mahowald-Vassiljev-type formula

$$H_p(CS_n) = \oplus_{k=0}^n H_{p-k(k-1)}(G_k(\mathbf{C}^n)),$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex k-planes through zero in $\mathbf{C}^n$ (see [DV] and also [V]).

We will now define Schubert varieties of complex structures. Let us fix a complex structure $J_0 \in CS_n$. By convention, we will denote by CS_n^+ the connected component of CS_n that contains J_0. We will work here with the component CS_n^+, leaving to the reader details concerning the other component CS_n^-. The results about the component CS_n^- will be summarized in Proposition 5.5.

Let

$$\mathbf{R}^{2n} = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where $\dim_{\mathbf{R}} L_i = 2$, be an invariant subspace decomposition of the orthogonal operator $J_0 : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$. This yields a flag in $\mathbf{R}^{2n}$

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbf{R}^{2n}, \tag{14}$$

where $F_i = L_1 \oplus L_2 \oplus \cdots \oplus L_i$. Furnishing $\mathbf{R}^{2n}$ with the complex structure J_0, we get an n-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$. Since each F_i is an invariant subspace w.r.t. J_0, the flag (14) gives rise to a complex flag

$$W_1 \subset W_2 \subset \cdots \subset W_n = \mathbf{C}^n, \tag{15}$$

where $\dim_{\mathbf{C}} W_i = i$.

Consider the Grassmannian $G_l(\mathbf{C}^n)$ of all complex l-planes through zero in $\mathbf{C}^n$. For a sequence $1 \le j_1 < \cdots < j_l \le n$, one defines a Schubert variety

$$\Omega(j_1, \ldots, j_l) = \{L \in G_l(\mathbf{C}^n) : \dim(L \cap W_{i_p}) \ge p \ \forall p = 1, \ldots, l\}.$$

One has

$$\dim_{\mathbf{C}} \Omega(j_1, \ldots, j_l) = j_1 + \cdots + j_l - l(l+1)/2.$$

For an even l, we define $CS_n^+(j_1, \ldots, j_l)$ as

$$\{A \in CS_n^+ : \exists L \in \Omega(j_1, \ldots, j_l) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}}^\perp = J_0|L_{\mathbf{R}}^\perp\},$$

where $L_{\mathbf{R}}^{\perp}$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of L. This is a closed subvariety in CS_n^+. One has

$$\dim_{\mathbf{C}} CS_n^+(j_1,\dots,j_l) = j_1 + \cdots + j_l - l\,.$$

One verifies easily that the class of the variety $CS_n^+(j_1,\dots,j_l)$ is independent of the choice of J_0. Indeed, a path in CS_n^+ joining J_0 to another $J \in CS_n^+$ yields a one-parameter family of varieties from $CS_n^+(j_1,\dots,j_l)$ attached to J_0, to that attached to J.

Define the complex structure J_n in the initial basis $\{e_1,\dots,e_{2n}\}$ by

$$J_n = J_1 \oplus J_1 \oplus \cdots \oplus J_1 \quad n \text{ times}\,,$$

where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example 5.1. *Suppose that $J_0 = J_n$. For a sequence $1 \le i_1 < \cdots < i_k \le n$, we put*

$$J(i_1,\dots,i_k) = \epsilon_1 J_1 \oplus \epsilon_2 J_1 \oplus \cdots \oplus \epsilon_n J_1$$

where $\epsilon_h = -1$ if $h = i_p$ for $1 \le p \le k$ Then $J(i_1,\dots,i_k) \in CS_n^+$ iff k is even.

We now want to identify the variety $CS_n^+(j_1,\dots,j_l)$ with a suitable Schubert variety $\Omega[i_1,\dots,i_k]$ in G. To this end, we first describe an imbedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$. Every $A \in CS_n$ has two eigenvalues $\pm i$ (here i is the pure imaginary complex number) with equal multiplicities n. Thus, as an endomorphism of $\mathbf{C}^{2n}$, (cf. (d)), A has the eigensubspace decomposition

$$\mathbf{C}^{2n} = L(A,+) \oplus L(A,-) \quad \text{where} \quad \dim L(A,+) = \dim L(A,-) = n\,,$$

with

$$A(v) = iv \text{ for all } v \in L(A,+) \quad \text{and} \quad A(v) = -iv \quad \text{for all } v \in L(A,-)\,.$$

The embedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$ defined by $A \to L(A,+)$ has as its image the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. the orthogonal form induced by the scalar product.

Let us fix the complex structure J_0 to be J_n. By using a simple linear algebra, one shows that w.r.t. the flag (15), associated with this complex structure, the following identification takes place.

Proposition 5.2. *Let l be even. Then the embedding ι restricts to an isomorphism of varieties:*

$$CS_n^+(j_1,\dots,j_l) \quad \text{and} \quad \Omega[n+1-t_{n-l},\dots,n+1-t_1]\,,$$

where $t_1 < \cdots < t_{n-l}$ is the complement of $j_1 < \cdots < j_l$ in $\{1,\dots,n\}$.

Remark 5.3. *The equality in this proposition is a refinement of a result announced by Dynnikov and Veselov in [DV], where some variant of Morse theory was used.*

As a consequence of Proposition 4.8, we get:

Corollary 5.4. *The degree of $CS_n^+(j_1, \dots , j_l)$ is equal to the number g^γ, where $\gamma = (j_l - 1, \dots , j_1 - 1)$.*

Since, for even l, $\Omega[n+1-t_{n-l}, \dots , n+1-t_1]$ is equal to

$$\{A : \exists K \in \Omega(j_1, \dots , j_l) \text{ s.t. } A(K_{\mathbf{R}}) = K_{\mathbf{R}} \ \& \ A|K_{\mathbf{R}}^{\perp} = J_0|K_{\mathbf{R}}^{\perp}\}\,,$$

then, rewriting it for even $n-k$, $\Omega[i_1, \dots , i_k]$ is equal to

$$\{A : \exists K \in \Omega(n+1-s_{n-k}, \dots , n+1-s_1) \text{ s.t.}$$
$$A(K_{\mathbf{R}}) = K_{\mathbf{R}} \ \& \ A|K_{\mathbf{R}}^{\perp} = J_0|K_{\mathbf{R}}^{\perp}\}\,.$$

By taking $L = K^\perp$, we can present this $\Omega[i_1, \dots , i_k]$ as

$$\{A : \exists L \in \Omega(i_1, \dots , i_k) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,. \qquad (16)$$

This last identification (16) seems to be the most handy for applications.

In the following proposition, keeping the notation from this section, we collect properties of Schubert varieties in CS_n^-. For an odd l, we define $CS_n^-(j_1, \dots , j_l)$ as

$$\{A \in CS_n^- : \exists L \in \Omega(j_1, \dots , j_l) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}}^{\perp} = J_0|L_{\mathbf{R}}^{\perp}\}\,,$$

where $L_{\mathbf{R}}^{\perp}$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of L.

Proposition 5.5. (i) *$CS_n^-(j_1, \dots , j_l)$ is a closed subvariety in CS_n^- of dimension $j_1 + \dots + j_l - l$.*
(ii) *$CS_n^-(j_1, \dots , j_l)$ can be identified with the restriction to CS_n^-, properly embedded in $G_n(\mathbf{C}^{2n})$, of the Schubert variety*

$$\Omega(n+1-t_{n-l}, \dots , n+1-t_1, n+j_1, \dots , n+j_l)$$

in this last Grassmannian.
(iii) *$CS_n^-(j_1, \dots , j_l)$ is also identified with*

$$\{A \in CS_n^- : \exists L \in \Omega(i_1, \dots , i_k) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \text{ and } A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

(iv) *The degree of $CS_n^-(j_1, \dots , j_l)$ is equal to g^γ, where $\gamma = (j_l - 1, \dots , j_1 - 1)$.*

Example 5.6. *We describe the Schubert varieties in CS_n^+ which are divisors. We have different description according to the parity of n. If n is odd, then the divisor $\sigma_1 = \Omega[n-1]$ is*

$$\{A : \exists L \subset W_{n-1} \ \textit{s.t.} \ \dim_{\mathbf{C}} L = 1,\ A(L_{\mathbf{R}}) = L_{\mathbf{R}} \text{ and } A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

If n is even, then the divisor $\sigma_1 = \Omega[n-1, n]$ is

$$\{A : \exists L \subset W_n \ \textit{s.t.} \ \dim_{\mathbf{C}} L = 2,\ A(L_{\mathbf{R}}) = L_{\mathbf{R}} \text{ and } A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

We will end this paper with some applications. The identification made in (16) allows us to solve enumerative problems about the number of general complex structures satisfying some constrains. To this end, we need the following definition.

Definition 5.7. *Let A and B be two orthogonal operators on $\mathbf{R}^{2n}$. A linear subspace $L \subset \mathbf{R}^{2n}$ is said to be a common k-space of A and B iff*

$$A(L) = B(L) = L\ ,\ \ A|L = B|L \text{ and } \dim_{\mathbf{R}} L = k\,.$$

We will work in CS_n^+. Let n and $2 \leq k \leq n$ be even integers. Let $1 \leq i_1 < \cdots < i_k \leq n$ be a sequence of integers. Set $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \leq i \leq d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. A has a common $2k$-space from $\Omega(i_1, \ldots, i_k)$ with B_0, and A has a common 4-space with any other B_i from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

As a particular case, we have:

Proposition 5.8. *Let n and $2 \leq k \leq n$ be even integers. Suppose that a list $\{B_i\}$ of general complex structures on $\mathbf{R}^{2n}$ is given, where*

$$0 \leq i \leq 1 + (n-k)(n-k-1)/2\,.$$

Then the number of complex structures $A \in CS_n^+$ having a common fixed $2k$-space $(W_k)_{\mathbf{R}}$ with B_0 and a common 4-space with any other B_i is given by

$$g^{(n-k-1,n-k-2,\ldots,2,1)} = [(n-k)(n-k-1)/2]! \prod_{i=1}^{n-k-1} \frac{(i-1)!}{(2i-1)!}\,.$$

Indeed, this is a restatement of the formula about the degree of

$$\Omega[1,2,\ldots,k-1,k] = CS_n^+(1,2,\ldots,n-k)\,,$$

which is simplified in this case of a triangular partition, cf. [DP].

Example 5.9. *For $n = 8$ and $k = 4$, a list of 7 complex structures $\{B_0, B_1, B_2, B_3, B_4, B_5, B_6\}$ is given. There exist exactly 2 complex structures $A \in CS_n^+$ s.t. A has a common fixed 8-space with B_0 and at least a common 4-space with every B_i, where $1 \leq i \leq 6$.*

If $n = 10$ and $k = 4$, a list of 16 structures $\{B_0, B_1, \ldots, B_{15}\}$ is given. There exist exactly 286 complex structures $A \in CS_n^+$ s.t. A has a common fixed 8-space with B_0 and at least a common 4-space with every B_i, where $1 \leq i \leq 15$.

Let now n and $2 < k \leq n$ be odd integers. Let $1 \leq i_1 < \cdots < i_k \leq n$ be a sequence of integers. Put $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \leq i \leq d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. A has a common $2k$-space from $\Omega(i_1, \ldots, i_k)$ with B_0, and A has a common 2-space from $(W_{n-1})_{\mathbf{R}}$ with any other B_i from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

We leave it to the reader to deduce from it a result analogous to the one in the last proposition.

We will give now another example of enumerating complex structures satisfying some constraints and state some conjecture.

For a complex structure $B_0 \in CS_n$, we have an n-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, B_0)$. Note that for all $L \in G_k(\mathbf{C}^n)$, both $L_{\mathbf{R}}$ and $L_{\mathbf{R}}^{\perp}$ are invariant subspaces of B_0. We have then an embedding $\alpha : G_k(\mathbf{C}^n) \to CS_n$ defined by

$$\alpha(L) = (B_0|L_{\mathbf{R}}) \oplus (-B_0|L_{\mathbf{R}}^{\perp}) .$$

Without loss of generality, we can assume that the image of α lies in CS_n^+.

Let R be the canonical complex k-bundle over $G_k(\mathbf{C}^n)$ and $R^{\perp}$ its orthogonal complement in the trivial complex n-bundle. Let S be the canonical complex n-bundle over CS_n^+. From the definition of α we have

$$\alpha^* S = R \oplus \overline{R^{\perp}} ,$$

where $\overline{R^{\perp}}$ denotes the complex conjugation of $R^{\perp}$. We infer that the pullback of the total Chern class of S, $\alpha^*(1 + c_1(S) + \cdots + c_n(S))$, is equal to

$$(1 + c_1(R) + \cdots + c_k(R))(1 - c_1(R) + c_2(R) - \cdots + (-1)^k c_k(R))^{-1} .$$

From this we get that the induced homomorphism

$$\alpha^* : H^*(CS_n) \to H^*(G_k(\mathbf{C}^n))$$

satisfies $\alpha^*(\frac{1}{2}c_1(S)) = -\alpha^*(\sigma_1) = c_1(R)$. That is, the embedding α preserves the classes of hyperplane sections, or, the Kähler classes of both varieties.

As a consequence, we get results summarized in the following proposition.

Proposition 5.10. (i) *Let n be an even integer. Suppose that $2 \leq k \leq n$ is another integer. Let $\{B_i\}$, $0 \leq i \leq k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. A and B_0 have a common $2k$-space, A and $(-B_0)$ have a common $2(n-k)$-space, and A and each B_i, $i \geq 1$ have at least a common 4-space, is equal to the degree of $G_k(\mathbf{C}^n)$.*
(ii) *Let now n be an odd integer. Suppose that $2 \leq k \leq n$ is another integer. Let $\{B_i\}$, $0 \leq i \leq k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. A and B_0 have a common $2k$-space, A and $(-B_0)$ have a common $2(n-k)$-space, and A and*

each B_i, $i \geq 1$ have at least a common 2-space in $(W_{n-1})_{\mathbf{R}}$, is equal to the degree of $G_k(\mathbf{C}^n)$.

(Recall that

$$\deg G_k(\mathbf{C}^n) = \frac{1!2!\cdots(k-1)![k(n-k)]!}{(n-k)!(n-k+1)!\cdots(n-1)!},$$

a result which goes back to Schubert (1886).)

It is well known that the Grassmannian $G_k(\mathbf{C}^n)$ is an approximation space for the classifying space $BU(k)$ of all complex k-bundles. On the other hand, the space CS_n serves as the classifying space for all complex n-bundles with a trivial real reduction [Du1]. Thus a homotopy classification of continuous maps $G_k(\mathbf{C}^n) \to CS_n$ may suggest possible interesting operators between these two vector bundle theories.

Let $\beta : G_k(\mathbf{C}^n) \to CS_n$ be a continuous map. Since $\sigma_1 = -\frac{1}{2}c_1(S) \in H^*(CS_n^{\pm}) = \mathbf{Z}$ and $c_1(R) \in H^*(G_k(\mathbf{C}^n)) = \mathbf{Z}$ are the only generators in dimension 2, then the induced map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ satisfies

$$\beta^*(\frac{1}{2}c_1(S)) = -\beta^*(\sigma_1) = m \cdot c_1(R)$$

for some $m \in \mathbf{Z}$.

We finish this paper by stating the following conjecture.

Conjecture 5.11 *If $m \neq 0$, then the map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ is given by*

$$\beta^*(x) = m^p \alpha^*(x),$$

for $x \in H^{2p}(CS_n^{\pm})$.

We refer the reader to [Du2] and [Ho] for some background related to this conjecture.

References

[BGG] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and cohomology of the spaces G/P,* Russian Math. Surveys **28:3** (1973), 1–26.

[Bo] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts,* Annals of Math. **57** (1953), 115–207.

[Bu] N. Bourbaki, *Groupes et Algèbrés de Lie,* Chapters 4,5, and 6, Herrmann, Paris, 1968.

[BKT] A.S. Buch, A. Kresch and H. Tamvakis, *Gromov-Witten invariants on Grassmannians,* preprint (April 2002).

[DP] C. De Concini and P. Pragacz, *On the class of Brill-Noether loci for Prym varieties,* Math. Ann. **302** (1995), 687–697.

[De] M. Demazure, *Désingularisation des variétés des Schubert géneralisées,* Ann. E.N.S.**7** (1974), 53–88.

[Du1] H. Duan, *The secondary Chern characteristic classes,* Proc. A.M.S. **128**(8) (2000), 2465-2471.

[Du2] H. Duan, *Self-maps of the Grassmannian of complex structures,* to appear in Compositio Math.

[DV] I.A. Dynnikov and A.P. Veselov, *Integrable gradient flows and Morse Theory,* Algebra i Analiz **8** (1996), 78-103 (Russian); English translation: St. Petersburgh Math. J., **8** (1997), 429-446.

[FP] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci,* Springer LNM **1689** (1998).

[GH] P. Griffith and J. Harris, *Principles of algebraic geometry,* Wiley, New York, 1978.

[HP] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry,* Cambridge University Press, 1954.

[Hi] H. Hiller, *Combinatorics and intersection of Schubert varieties,* Comment. Math. Helvetici **57** (1982), 41–59.

[HB] H. Hiller and B. Boe, *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n,* Adv. in Math. **62** (1986), 49–67.

[Ho] M. Hoffman, *Endomorphisms of the cohomology of complex Grassmannians,* Trans. A.M.S. **281** (1984), 745–760.

[HH] P.N. Hoffman and J.F. Humphreys, *Projective representations of the symmetric groups,* Oxford University Press, 1992.

[KK] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G^*,* Adv. in Math. **62** (1986), 187–237.

[KT] A. Kresch and H. Tamvakis, *Quantum cohomology of orthogonal Grassmannians,* Preprint (2001).

[LSe] V. Lakshmibai and C.S. Seshadri, *Geometry of G/P II,* Proc. Indian Acad. Sci. A **87** (1978), 1–54.

[L1] A. Lascoux, *Notes on interpolation,* Nankai University, Tjanjin (1996).

[L2] A. Lascoux, *Notes for the CBMS Conference on Algebraic Combinatorics,* N.C.S.U., June 2001.

[LP] A. Lascoux and P. Pragacz, *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-functions, and vertex operators,* Michigan Math. J., Fulton's volume **48** (2000), 417–441.

[LSc] A. Lascoux and M.-P. Schützenberger, *Décompositions dans l'algèbre des differences divisées,* Discrete Math. **99** (1992), 165–179.

[Ma] I.G. Macdonald, *Symmetric functions and Hall polynomials,* (Second edition), Clarendon Press, Oxford, 1995.

[Mi] J. Milnor, *Morse Theory,* Annals of Mathematics Studies **51**, Princeton, 1963.

[Mo] A. Morris, *A note on the multiplication of Hall functions,* J. London Math. Soc. **39** (1964), 481–488.

[P1] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials,* Séminaire d'Algèbre Dubreil-Malliavin 1988-89, (M.-P. Malliavin, ed), Lecture Notes in Math., Vol. **1478**, Springer-Verlag, Berlin and New York, 1991, 130–191.

[P2] P. Pragacz, *Symmetric polynomials and divided differences in formulas of intersection theory,* in "Parameter Spaces", Banach Center Publications **36** (1996), 125–177.

[PR1] P. Pragacz and J. Ratajski, *Formulas of Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach,* Compositio Math. **107** (1997), 11–87.

[PR2] P. Pragacz and J. Ratajski, *A Pieri-type theorem for even orthogonal Grassmannians,* MPIM-Preprint 1996-83.

[S] I. Schur, *Uber die Darstellung der symmetrischen und alternierenden Gruppe durch gebrochene lineare Substitutionen,* J. Reine Angew. Math. **139** (1911), 155-250.

[T1] H. Tamvakis, *Height formulas for homogeneous spaces,* Michigan Math. J., Fulton's volume **48** (2000), 593–610.

[T2] H. Tamvakis, *Quantum cohomology of Lagrangian and orthogonal Grassmannians,* "Arbeitstagung 2001", Bonn, MPIM-Preprint 2001-50.

[V] V.A. Vassiljev, *A geometric realization of homology of classical groups* Algebra & Analysis **3(4)** (1991), 113-120 (Russian); English translation: St. Petersburg Math. J. **3** (1992), 809-815.

INSTITUTE OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100080, P.R. OF CHINA

E-mail address: dhb@math.ac.cn

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, ŚNIADECKICH 8, PL 00-956 WARSZAWA 10, P.O. BOX 21, POLAND

E-mail address: pragacz@impan.gov.pl

TABLEAUX STATISTICS FOR TWO PART MACDONALD POLYNOMIALS

L. LAPOINTE AND J. MORSE

ABSTRACT. The Macdonald polynomials expanded in terms of a modified Schur function basis have coefficients called the q,t-Kostka polynomials. We define operators to build standard tableaux and show that they are equivalent to creation operators that recursively build the Macdonald polynomials indexed by two part partitions. We uncover a new basis for these particular Macdonald polynomials and in doing so are able to give an explicit description of their associated q,t-Kostka coefficients by assigning a statistic in q and t to each standard tableau.

1. INTRODUCTION

The Macdonald polynomials, $J_\lambda[X;q,t]$, are a two parameter family of polynomials in N variables, forming a basis for the space of symmetric functions. The polynomials, expanded in a modified Schur function basis $\{S_\lambda[X^t]\}_\lambda$, have coefficients called the q,t-Kostka polynomials, $K_{\lambda\mu}(q,t)$. Macdonald conjectured that $K_{\lambda\mu}(q,t)$ is a polynomial in q and t with positive integer coefficients. This conjecture has been proven [3] based on the representation-theoretic interpretation [2] of the q,t-Kostka polynomials. However, the proof does not reveal a combinatorial interpretation for these coefficients. In this article, we consider the case that $J_\lambda[X;q,t]$ is indexed by partitions with no more than 2 parts. We uncover a new basis for the these particular Macdonald polynomials and in doing so are able to give an explicit description of their associated q,t-Kostka coefficients by assigning a statistic in q and t to each standard tableau.

Our results appeared on the web a few years ago [1]. The recent generalization [6, 7] of the methods introduced here have prompted us to submit this article for publication. Prior to this work, a rigged configuration interpretation for the coefficients associated to Macdonald polynomials indexed by partition with no more than 2 parts was given in [1]. In addition, similar tableaux statistics in the case of partitions whose first part is not larger than 2 were discovered [13] at the same time as ours.

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This paper can be seen as a study of the case $k = 2$ of the k-Schur functions, $s_\lambda^{(k)}[X;t]$, associated to a filtration of the symmetric function space introduced in [7]. There is strong computational evidence to support conjectures asserting that the k-Schur functions obey k-generalizations of many Schur function positivity properties. The results of this paper prove, for $k = 2$, that the k-Schur functions expand positively in terms of Schur functions and that the Macdonald polynomials indexed by partitions whose first part does not exceed 2 can be expanded positively in terms of k-Schur functions. Conjectures generalizing this phenomenon are given in [7].

The connection between the work in this paper and that of the k-Schur functions in [7] can be made using the following notational correspondance:

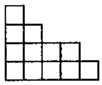
$$\begin{aligned}
F_t^{-1}\, H_\lambda[X;q,t] &\longleftrightarrow q^{n(\lambda')} H_{\lambda'}[X;1/q] \\
F_t^{-1} B_2^{(0)}\, F_t &\longleftrightarrow B_2\, t^{-D}\Big|_{t\to 1/q} \\
F_t^{-1} B_2^{(1)}\, F_t &\longleftrightarrow B_{1,1}\, t^{-2D-1}\Big|_{t\to 1/q} \\
F_t^{-1}\, U_{0^a,1^b}^{(\epsilon)} &\longleftrightarrow q^{n(\lambda)} s_\lambda^{(2)}[X;1/q]\,,
\end{aligned} \tag{1.1}$$

where on an arbitrary symmetric function $P[X]$, we define $F_t\, P[X] = P[X(1-t)]$ and $F_t^{-1}\, P[X] = P[X/(1-t)]$, and where λ is the partition $\lambda = (2^a, 1^{2b}, \epsilon)$. Note that from Property 5, $F_t^{-1}\, U_v^{(\epsilon)}$ is, for any v, equal up to a constant to a 2-Schur function. Also, letting $t \to 1/q$ has the effect of transforming the 2-Schur functions from a basis of the linear span of Macdonald polynomials indexed by partitions whose first part is not larger than 2, to a basis of the linear span of Macdonald polynomials indexed by partitions whith no more than 2 parts.

The paper is divided as follows: section two covers basic definitions used in symmetric function theory. We present in the third section, the creation operator $B_2 = tB_2^{(0)} + B_2^{(1)} q^{-D-1}$, showing several properties that include the action of $B_2^{(0)}$ on the Hall-Littlewood polynomials $H_\lambda[X;q,t]$ and an expansion of $J_\lambda[X;q,t]$ in terms of products of $B_2^{(0)}$ and $B_2^{(1)}$ for $\ell(\lambda) \le 2$. Further, a new basis for these Macdonald polynomials having coefficients in the parameters q and t with positive integer coefficients is uncovered. The fourth section begins with basic definitions used in tableaux theory and then introduces operators on tableaux which correspond, under a morphism $\digamma$, to the operators of section 3. Finally, a statistic on standard tableaux is presented in the fifth section.

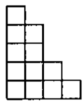
2. Definitions

Partitions are sequences of integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \ge \lambda_2 \ge \cdots > 0$. The order of λ is $|\lambda| = \lambda_1 + \lambda_2 + \dots$, the number of non-zero parts in λ is denoted $\ell(\lambda)$ and $n(\lambda)$ refers to $\sum_i (i-1)\lambda_i$. The *dominance order* on partitions is defined for two partitions with $|\lambda| = |\mu|$, by $\lambda \leq \mu$ when $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. A partition λ may be associated to a Young diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. The Young diagram of $\lambda = (5, 4, 2, 1)$ is



. (2.1)

For each square $s = (i, j)$ in the diagram of λ, let $\ell'(s), \ell(s), a(s)$ and $a'(s)$ be respectively the number of squares in the diagram of λ to the south, north, east and west of the square s. The transposition of a Young diagram associated to λ with respect to the main diagonal gives the conjugate partition λ'. For example, the conjugate of (2.1) is



, (2.2)

which gives $\lambda' = (4, 3, 2, 2, 1)$.

We shall use λ-rings, needing only the formal ring of symmetric functions Sym to act on the ring of rational functions in $x_1, \ldots, x_N, q, t$, with coefficients in $\mathbb{R}$. The ring Sym is generated by power sums Ψ_i, $i = 1, 2, 3 \ldots$. The action of Ψ_i on a rational function $\sum_\alpha c_\alpha u_\alpha / \sum_\beta d_\beta v_\beta$ is by definition

$$\Psi_i \left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta} \right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \tag{2.3}$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in $x_1, \ldots, x_N, q, t$. Since any symmetric function is uniquely expressed in terms of the power sums, (2.3) extends to an action of Sym on rational functions. In particular, a symmetric function $f(x_1, \ldots, x_N)$ can be denoted $f[x_1 + \cdots + x_N]$. We shall use the elements $X := x_1 + \cdots + x_N$, $X^{tq} := X(t-1)/(q-1)$ and $X^t := X(t-1)$.

The Macdonald polynomials can now be defined using a scalar product $\langle\ ,\ \rangle_{q,t}$ defined on $Sym \otimes \mathbb{Q}[q, t]$ by

$$\langle \Psi_\lambda[X], \Psi_\mu[X] \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \tag{2.4}$$

where we associate to a parition λ with $m_i(\lambda)$ parts equal to i the number

$$z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots \tag{2.5}$$

Macdonald polynomials $J_\lambda[X;q,t]$ are thus uniquely specified [11] by

$$\text{(i) } \langle J_\lambda, J_\mu\rangle_{q,t} = 0, \qquad \text{if } \lambda \neq \mu, \tag{2.6}$$

$$\text{(ii) } J_\lambda[X;q,t] = \sum_{\mu\leq\lambda} v_{\lambda\mu}(q,t) S_\mu[X], \tag{2.7}$$

$$\text{(iii) } v_{\lambda\lambda}(q,t) = \prod_{s\in\lambda}(1-q^{a(s)}t^{\ell(s)+1}), \tag{2.8}$$

where $S_\mu[X]$ is the usual Schur function and $v_{\lambda\mu}(q,t) \in \mathbb{Q}[q,t]$.

The expansion coefficients of the Macdonald polynomials when expanded in terms of the basis $\{S_\mu[X^t]\}_\mu$:

$$J_\lambda[X;q,t] = \sum_{\mu} K_{\mu\lambda}(q,t) S_\mu[X^t]\,, \tag{2.9}$$

are studied here in the case that $\ell(\lambda) \leq 2$.

3. Algebraic Side

A Macdonald polynomial indexed by any partition can be constructed by repeated application of creation operators B_k defined [4, 8] such that

$$B_k\, J_\lambda[X;q,t] = J_{\lambda+1^k}[X;q,t]\,, \qquad \ell(\lambda)\leq k\,, \tag{3.1}$$

or more specifically, when $k=2$,

$$B_2\, J_{m,n}[X;q,t] = J_{m+1,n+1}[X;q,t]\,. \tag{3.2}$$

If $\mathcal{V}$ is the $\mathbb{Q}[q,t]$-linear span of $\{J_\lambda[X;q,t]\}_{\ell(\lambda)\leq 2}$, because it is known [11] that

$$J_\lambda[X;q,t] = \sum_{\mu\geq\lambda} c_{\mu\lambda}(q,t)\, S_\mu[X^{tq}]\,, \tag{3.3}$$

for some $c_{\mu\lambda}(q,t) \in \mathbb{Q}[q,t]$, $\mathcal{V}$ must also be the $\mathbb{Q}[q,t]$-linear span of $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\leq 2}$. The action of B_2 on $\mathcal{V}$ can thus be defined by its action on $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\leq 2}$, introduced in [5] as

$$B_2\, S_{m,n}[X^{tq}] = \det \begin{vmatrix} (1-tq^{m+1})S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ (1-tq^{n})S_n[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix}\,. \tag{3.4}$$

It will be convenient to split the operator B_2 into a sum of two operators,

$$B_2 = tB_2^{(0)} + B_2^{(1)} q^{-D-1}, \tag{3.5}$$

where D is the operator such that $Df[X] = d\,f[X]$ on any homogeneous function of degree d and where $B_2^{(0)}$ and $B_2^{(1)}$ are defined on $\mathcal{V}$ by

$$B_2^{(0)}\, S_{m,n}[X^{tq}] = \det \begin{vmatrix} -q^{m+1}S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ -q^{n}S_n[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix}\,, \tag{3.6}$$

and

$$B_2^{(1)}\, S_{m,n}[X^{tq}] = q^{m+n+1} \det \begin{vmatrix} S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ S_n[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix} . \qquad (3.7)$$

These expressions, obtained by expanding (3.4), provide that (3.5) holds on $\mathcal{V}$.

We now introduce a deformation of the Hall-Littlewood polynomials:

$$H_\lambda[X;q,t] = \sum_\mu q^{n(\lambda')} K_{\mu\lambda}(1/q,0) S_{\mu'}[X^t] , \qquad (3.8)$$

which is a basis for the ring of symmetric functions since the $H_\lambda[X;q,t]$'s are linearly independent; i.e., the $K_{\mu\lambda}(1/q,0)$ matrix is triangular with respect to the partial ordering and $K_{\lambda\lambda}(1/q,0) = 1$. In fact, these polynomials are specializations of the Macdonald polynomials. More precisely, let $J_\lambda[X;q,t]^{\{t\}}$ (resp $J_\lambda[X;q,t]^{\{tq\}}$) denote the $\{S_\lambda[X^t]\}$–expansion (resp $\{S_\lambda[X^{tq}]\}$–expansion) of the Macdonald polynomial, $J_\lambda[X;q,t]$. Then

Proposition 1. *$H_\lambda[X;q,t]$ is obtained by taking the coefficient of the maximal t-power in $J_\lambda[X;q,t]^{\{t\}}$ (or $J_\lambda[X;q,t]^{\{tq\}}$). Equivalently,*

$$\begin{aligned} H_\lambda[X;q,t] &= J_\lambda[X;q,t]^{\{t\}}\Big|_{t^{n(\lambda)}} := \sum_\mu K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} S_\mu[X^t] \\ &= J_\lambda[X;q,t]^{\{tq\}}\Big|_{t^{n(\lambda)}} := \sum_{\mu\geq\lambda} c_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} S_\mu[X^{tq}] . \end{aligned} \qquad (3.9)$$

Proof. Given the first identity, the second follows from (3.3) and the relation:

$$S_\lambda[X^t] = \sum_\mu \bar{v}_{\mu\lambda}(q) S_\mu[X^{tq}] , \qquad (3.10)$$

for some $\bar{v}_{\mu\lambda}(q) \in \mathbb{Q}[q]$. Thus, it suffices to prove the first identity in (3.9). We have $K_{\mu\lambda}(q,t) = q^{n(\lambda')} t^{n(\lambda)} K_{\mu'\lambda}(1/q,1/t)$ [11], which gives

$$K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} = q^{n(\lambda')} K_{\mu'\lambda}(1/q,1/t)\Big|_{t^0} . \qquad (3.11)$$

Since $K_{\mu'\lambda}(1/q,1/t)$ is a polynomial in $1/q, 1/t$ (eg. [4],[8]), (3.11) can be rewritten

$$K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} = q^{n(\lambda')} K_{\mu'\lambda}(1/q,0) . \qquad (3.12)$$

Therefore, the first identity in (3.9) is equivalent to (3.8). □

Substituting the known [11] relation, $J_m[X;q,t] = (q;q)_m S_m[X^{tq}]$, where

$$(q;q)_m = (1-q)(1-q^2)\cdots(1-q^m) , \quad m > 0 ; \qquad (q;q)_0 = 1, \qquad (3.13)$$

into $H_m[X;q,t] = J_m[X;q,t]^{\{tq\}}\Big|_{t^0}$, that is into Proposition 1 in the case $\lambda = (m)$, yields

Corollary 2.

$$H_m[X;q,t] = J_m[X;q,t] = (q;q)_m S_m[X^{tq}]\,. \tag{3.14}$$

With H_λ now characterized as a specialization of J_λ, we have

Lemma 3. *$\mathcal{V}$ is the $\mathbb{Q}[q,t]$-linear span of $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$.*

Proof. (3.9) gives that the $\mathbb{Q}[q,t]$-linear span of $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$ is included in the $\mathbb{Q}[q,t]$-linear span of $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\le 2}$ which is equal to $\mathcal{V}$. The lemma then follows because $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$ is a linearly independent set. □

We now establish several properties of the operators $B_2^{(0)}$ and $B_2^{(1)}$ that will later enable us to associate them to tableaux operators. First, by Lemma 3, we define the action of $B_2^{(0)}$ on $\mathcal{V}$ by finding its action on $H_{m,n}[X;q,t]$.

Property 4. *The action of $B_2^{(0)}$ on $H_{m,n}[X;q,t]$ is given by*

$$B_2^{(0)} H_{m,n}[X;q,t] = H_{m+1,n+1}[X;q,t]\,. \tag{3.15}$$

Proof. Definitions (3.6) and (3.7) show that the action of $B_2^{(0)}$ and $B_2^{(1)}$ on $S_{m,n}[X^{tq}]$ gives coefficients involving only the parameter q when expanded in terms of the $\{S_\lambda[X^{qt}]\}_{\ell(\lambda)\le 2}$ basis. This implies that the successive action of $B_2^{(0)}$ and $B_2^{(1)}$ on $J_m[X;q,t] = (q;q)_m S_m[X^{tq}]$ produces coefficients involving only q when expanded in the $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\le 2}$ basis. Therefore, from

$$J_{m+\ell,\ell}[X;q,t] = (B_2)^\ell J_m[X;q,t] = \left(tB_2^{(0)} + B_2^{(1)} q^{-D-1}\right)^\ell J_m[X;q,t]\,, \tag{3.16}$$

the coefficient of the maximal t-power in $J_{m+\ell,\ell}[X;q,t]$ is given by

$$J_{m+\ell,\ell}[X;q,t]^{\{tq\}}\Big|_{t^\ell} = \left(B_2^{(0)}\right)^\ell J_m[X;q,t]. \tag{3.17}$$

The assertion follows by replacing left side of this expression with $H_{m+\ell,\ell}$ yield$[X;q,t]$ by (3.9), and $J_m[X;q,t] = H_m[X;q,t]$. □

Property 5. *On the space $\mathcal{V}$, we have the q-commutation relation*

$$B_2^{(1)} B_2^{(0)} = q\, B_2^{(0)} B_2^{(1)}\,. \tag{3.18}$$

Proof. The action defined in (3.6) and (3.7), and the Pieri rule

$$S_m[X^{tq}] S_n[X^{tq}] = \sum_{\ell=0}^{n} S_{m+n-\ell,\ell}[X^{tq}]\,, \quad \text{where} \quad m \ge n\,, \tag{3.19}$$

yield

$$\begin{aligned}
B_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}] = {} & q^{n+m+2}(q^{m+1}-q^n)\sum_{r=0}^{n}\sum_{\ell=0}^{r}(q^r-q^{m+n+3-r}) \\
& \times S_{m+n+4-\ell,\ell}[X^{tq}] \\
& + q^{2m+2n+5-\ell}(q^{m+1}-q^n)\sum_{\ell=0}^{n}(q^{\ell+1}-1) \\
& \times S_{m+n+3-\ell,\ell+1}[X^{tq}] \\
& + q^{n+m+1}(1-q^{n+1})(q^{n+1}-q^{m+2})\sum_{\ell=0}^{n+1} \\
& \times S_{m+n+4-\ell,\ell}[X^{tq}] \\
& + q^{2m+n+3}(1-q^{n+1})(q^{n+2}-1) \\
& \times S_{m+2,n+2}[X^{tq}]
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
B_2^{(1)}B_2^{(0)}S_{m,n}[X^{tq}] = {} & q^{n+m+4}(q^n-q^{m+1})\sum_{r=0}^{n}\sum_{\ell=0}^{r}(q^{m+n+3-r}-q^r) \\
& \times S_{m+n+4-\ell,\ell}[X^{tq}] \\
& + q^{n+m+3}(q^n-q^{m+1})\sum_{\ell=0}^{n}(1-q^{\ell+1}) \\
& \times S_{m+n+3-\ell,\ell+1}[X^{tq}] \\
& + q^{2m+n+5}(q^{n+1}-1)\sum_{\ell=0}^{n+1}(q^{m+2}-q^{n+1}) \\
& \times S_{m+n+4-\ell,\ell}[X^{tq}] \\
& + q^{2m+n+4}(1-q^{n+1})(q^{n+2}-1) \\
& \times S_{m+2,n+2}[X^{tq}]\,.
\end{aligned} \tag{3.21}$$

First exchange the order of summation in the right side of the top line in (3.20) and (3.21), and then use

$$\sum_{r=\ell}^{n}(q^r-q^{m+n+3-r}) = \frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)}. \tag{3.22}$$

Next, send $\ell \to \ell-1$ in the right side of the second terms in (3.20) and (3.21). By fixing ℓ, we have that $qB_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}] = B_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}]$

if

$$\left(\frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)}+q^{m+n+4-\ell}(q^\ell-1)+(q^{n+1}-1)\right)$$
$$=\left(q\frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)}+(q^\ell-1)+q^{m+3}(q^{n+1}-1)\right), \tag{3.23}$$

which holds by algebraic manipulation. □

Property 6. *Let $\epsilon \in \{0,1\}$. $B_2^{(0)}$ and $B_2^{(1)}$ are such that*

$$(B_2^{(0)}+B_2^{(1)})^m H_\epsilon[X;q,t]=H_{2m+\epsilon}[X;q,t]. \tag{3.24}$$

Proof. Definitions (3.6) and (3.7) give that

$$(B_2^{(0)}+B_2^{(1)})S_k[X^{tq}]=(1-q^{k+1})(1-q^{k+2})S_{k+2}[X^{tq}], \tag{3.25}$$

which, using $H_k[X;q,t]=(q;q)_k S_k[X^{tq}]$, implies that

$$(B_2^{(0)}+B_2^{(1)})H_k[X;q,t]=H_{k+2}[X;q,t]. \tag{3.26}$$

We apply this identity m times, starting with $k=\epsilon$, to complete the proof. □

With relation (3.5), this property shows that any Macdonald polynomial indexed by a partition with no more than 2 parts can be built using $B_2^{(0)}$ and $B_2^{(1)}$ since $H_m[X;q,t]=J_m[X;q,t]$.

Definition 7. *Let $v=(v_1,\ldots,v_k)$ with all $v_i\in\{0,1\}$. For $\epsilon\in\{0,1\}$ we define*

$$U_v^{(\epsilon)}=B_2^{(v_1)}\cdots B_2^{(v_k)}\cdot H_\epsilon[X;q,t]. \tag{3.27}$$

Proposition 8. *For $\epsilon, v_i\in\{0,1\}$ we have*

$$J_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{v=(v_1,\ldots,v_{m+\ell})} q^{(1-d)|v|_\ell+2n(v)_\ell}t^{\ell-|v|_\ell}U_v^{(\epsilon)}, \tag{3.28}$$

where $d=2m+2\ell+\epsilon$, $|v|_\ell=v_1+\cdots+v_\ell$ and $n(v)_\ell=v_2+2v_3+\cdots+(\ell-1)v_\ell$.

Proof. From Property 6, we have that

$$J_{2m+\epsilon}[X;q,t]=\sum_{v=(v_1,\ldots,v_m)} U_v^{(\epsilon)}, \tag{3.29}$$

where $v_i \in \{0,1\}$, proving (3.28) for $\ell = 0$. Proceeding by induction, we assume that (3.28) holds for every ℓ. We thus have, acting with B_2, that

$$\begin{aligned}(tB_2^{(0)} + B_2^{(1)} q^{-D-1}) J_{2m+\ell+\epsilon,\ell} = &\sum_{v'=(0,v)} q^{(1-d)|v'|_{\ell+1}+2(n(v')_{\ell+1}-|v'|_{\ell+1})} \\ &\times t^{\ell+1-|v'|_{\ell+1}} U^{(\epsilon)}_{v'} \\ &+ \sum_{v''=(1,v)} q^{(1-d)(|v''|_{\ell+1}-1)+2(n(v'')_{\ell+1}-|v''|_{\ell+1}+1)-d-1} \\ &\times t^{\ell+1-|v''|_{\ell+1}} U^{(\epsilon)}_{v''}\,. \end{aligned} \tag{3.30}$$

Combining the two sums, we obtain

$$B_2 J_{2m+\ell+\epsilon,\ell} = \sum_{\bar v=(\bar v_1,\dots,\bar v_{m+\ell+1})} q^{\left(1-(d+2)\right)|\bar v|_{\ell+1}+2n(\bar v)_{\ell+1}} t^{\ell+1-|\bar v|_{\ell+1}} U^{(\epsilon)}_{\bar v}, \tag{3.31}$$

which completes the induction argument since $B_2 J_{2m+\ell+\epsilon,\ell} = J_{2m+\ell+1+\epsilon,\ell+1}$. □

Example: We have

$$\begin{aligned} J_{4,2}[X;q,t] = {} & t^2 U^{(0)}_{0,0,0} + t^2 U^{(0)}_{0,0,1} + tq^{-3} U^{(0)}_{0,1,0} + tq^{-3} U^{(0)}_{0,1,1} \\ & + tq^{-5} U^{(0)}_{1,0,0} + tq^{-5} U^{(0)}_{1,0,1} + q^{-8} U^{(0)}_{1,1,0} + q^{-8} U^{(0)}_{1,1,1}\,. \end{aligned} \tag{3.32}$$

Corollary 9. *The maximal t-power in* (3.28) *is*

$$H_{2m+\ell+\epsilon,\ell}[X;q,t] = \sum_{\bar v} U^{(\epsilon)}_{\bar v}\,, \tag{3.33}$$

summing over all $\bar v = (0^\ell, v)$ *where* $v = (v_1, \dots, v_m)$ *such that* $v_i \in \{0,1\}$.

4. Tableaux Side

4.1. Definition and background. Let $\mathcal{A}^*$ be the free monoid generated by the alphabet $\mathcal{A} = \{1,2,3,\dots\}$ and $\mathbb{Q}[\mathcal{A}^*]$ be the free algebra of $\mathcal{A}$. The elements of $\mathcal{A}^*$ are called words. The degree of a word w is denoted $|w|$ and its image in the ring of polynomials $\mathbb{Z}[\mathcal{A}]$ is called the evaluation, denoted $ev(w)$. For example, $w = 131332$ has degree 6 and evaluation (2,1,3). A word w of degree n is said to be standard iff $ev(w) = (1,1,\dots,1)$.

A tableau T will be the pair (λ, w), where λ is a partition and w is a word, such that $|\lambda| = |w|$. We say that λ is the shape of T. A Young diagram associated to λ filled with the letters of w from left to right and top to bottom is a planar representation of T. For example, $\mathsf{T} = ((4,2,2,1), 114356234)$ corresponds to

$$\mathsf{T} = \begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-1} 1 & 4 \\ \cline{1-2} 3 & 5 \\ \hline 6 & 2 & 3 & 4 \\ \hline \end{array}\,. \tag{4.1}$$

A semi-standard tableau is a tableau such that the entries in every row are nondecreasing and such that the entries in every column are increasing. In this case, we do not specify λ in the pair (λ, w), since it can be extracted from w. For instance, $T = 67\ 445\ 11123$ has the representation

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-2} 6 & 7 \\ \cline{1-3} 4 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline \end{array} . \tag{4.2}$$

Notice that a semi-standard tableau is a tableau. Finally, a standard tableau $\mathcal{T}$ is a semi-standard tableau of evaluation $(1, 1, \ldots, 1)$. For example, $\mathcal{T} = 7\ 46\ 1235$ or

$$\mathcal{T} = \begin{array}{|c|c|c|c|} \cline{1-1} 7 \\ \cline{1-2} 4 & 6 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \tag{4.3}$$

is a standard tableau.

As with the Ferrers diagrams, we define T^t to be the transpose of a tableau T. For example, with T as given in (4.1), we have

$$\mathsf{T}^t = \begin{array}{|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-3} 2 & 5 & 4 \\ \hline 6 & 3 & 1 & 1 \\ \hline \end{array} . \tag{4.4}$$

Notice that the transpositon of a semi-standard tableau may not be a semi-standard tableau while the transposition of a standard tableau must be a standard tableau.

Words can be associated to numbers called the charge and cocharge where

$$\text{charge}(w) = n(ev(w)_P) - \text{cocharge}(w) , \tag{4.5}$$

for $ev(w)_P$ the partition obtained by reordering $ev(w)$. The cocharge of a standard word w is defined by the following algorithm;

1. Label the letter 1 in w by $c_1 = 0$
2. If the letter $i + 1$ appears at the left of the letter i in w, then $c_{i+1} = c_i + 1$. Otherwise $c_{i+1} = c_i$.
3. $\text{cocharge}(w) = c_1 + \cdots + c_n$.

For instance, $\text{cocharge}(413265) = 0+0+1+2+2+3 = 8$. Recall that semi-standard tableaux and standard tableaux are simply words, and therefore have an associated charge and cocharge.

Lascoux and Schützenberger defined [9] an action of the symmetric group on $\mathbb{Z}[\mathcal{A}^*]$ that sends a word of evaluation $(ev_1, \ldots, ev_i, ev_{i+1}, \ldots)$ to a word of evaluation $(ev_1, \ldots, ev_{i+1}, ev_i, \ldots)$ under an elementary transposition, σ_i. Their action induces the usual action of the symmetric group on $\mathbb{Z}[\mathcal{A}]$. For our purposes, we define this action only on words such that $(ev_i, ev_{i+1}) \in \{(1,2), (2,1)\}$:

$$aab \overset{\sigma_i}{\leftrightarrows} abb, \quad aba \overset{\sigma_i}{\leftrightarrows} bba, \quad baa \overset{\sigma_i}{\leftrightarrows} bab , \tag{4.6}$$

where a and b stand for i and $i+1$ respectively. This action sends a semi-standard tableau to a semi-standard tableau while preserving its shape. For example, $\sigma_4(215345) = 215344$ and $\sigma_3 \begin{array}{lll} 4 & & \\ 3 & 5 & 5 \\ 1 & 2 & 3 \end{array} = \begin{array}{lll} 4 & & \\ 3 & 5 & 5 \\ 1 & 2 & 4 \end{array}$.

We will use several simple linear operations on words. τ_k is a translation of a to $a+k$ for every letter a in an alphabet. For instance, $\tau_2(231567) = 453789$. The restriction of a word w to the alphabet $a, b, c, \ldots,$ is denoted $w_{\{a,b,c,\ldots\}}$. i.e., $w_{\{3,4\}} = 3343$ for $w = 12334223$. If w is such that $w_{\{a,b\}} = ab$, $r_{(ab\to cd)}$ sends ab to cd. For example, $r_{(23\to 46)}(121543) = 141546$. We further define an operator R_a to remove all letters a in w, and finally, $A_{n+1,n+1}$ is an operator on a tableau T, that adds a horizontal 2-strip of the boxes $n+1$ in all the possible ways to T. For instance

$$A_{5,5} \begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \end{array} = \begin{array}{ll} 5 & \\ 3 & \\ 2 & 5 \\ 1 & 4 \end{array} + \begin{array}{lll} 5 & & \\ 3 & & \\ 2 & & \\ 1 & 4 & 5 \end{array} + \begin{array}{lll} 3 & & \\ 2 & 5 & \\ 1 & 4 & 5 \end{array} + \begin{array}{llll} 3 & & & \\ 2 & & & \\ 1 & 4 & 5 & 5 \end{array}. \tag{4.7}$$

There exists a shape and cocharge preserving standardization [9] of semi-standard tableaux, denoted by VS, that we will use only on semi-standard tableaux with evaluation $(ev_1, \ldots, ev_k)$, where $ev_i \in \{1,2\}$. In the case of a semi-standard tableau T, it is defined as

1. If $ev_1 = 2$ then $T \to \tau_1 r_{(11\to 01)} T$. Proceed to step 2.
2. If $ev(T) = (1,1,\ldots,1)$ then the standardization is complete. Otherwise, $T \to \sigma_1 \cdots \sigma_{i-1} T$ for the smallest i such that $ev_i = 2$. Proceed to step 1.

For example, $T = 45\ 235\ 124$ undergoes the following standardization process:

$$\begin{array}{lll} 4 & 5 & \\ 2 & 3 & 5 \\ 1 & 2 & 4 \end{array} \to \begin{array}{lll} 4 & 5 & \\ 2 & 3 & 5 \\ 1 & 1 & 4 \end{array} \to \begin{array}{lll} 5 & 6 & \\ 3 & 4 & 6 \\ 1 & 2 & 5 \end{array} \to \begin{array}{lll} 5 & 6 & \\ 2 & 3 & 6 \\ 1 & 1 & 4 \end{array} \to \begin{array}{lll} 6 & 7 & \\ 3 & 4 & 7 \\ 1 & 2 & 5 \end{array} \to \begin{array}{lll} 5 & 6 & \\ 2 & 3 & 7 \\ 1 & 1 & 4 \end{array} \to \begin{array}{lll} 6 & 7 & \\ 3 & 4 & 8 \\ 1 & 2 & 5 \end{array}. \tag{4.8}$$

Note that $VS^{(n)}$ will denote the standardization of only the first n letters of a semi-standard tableau of degree $N \geq n$ which sends such a tableau to a semi-standard tableau of evaluation $(1^n, ev_{n+1}, \ldots, ev_N)$.

4.2. **Tableaux operators.** The primary goal of this section is to present tableaux operators that can be related to $B_2^{(0)}$ and $B_2^{(1)}$ using the linear morphism on semi-standard tableaux,

$$F : T \to q^{\text{cocharge}(T)} S_{\text{shape}(T)}[X^t]. \tag{4.9}$$

It is known [10] that the Hall-Littlewood polynomials

$$Q_\lambda[X;t] = \sum_\mu K_{\mu\lambda}(0,t) S_\mu[X^t], \tag{4.10}$$

are equivalently expressed as a sum over semi-standard tableaux T such that

$$Q_\lambda[X;t] = \sum_{T; ev(T)=\lambda} t^{\text{charge}(T)} S_{\text{shape}(T)}[X^t]. \tag{4.11}$$

The substitution of $K_{\mu\lambda}(1/q,0) = K_{\mu'\lambda'}(0,1/q)$ [11] in expression (3.8) thus yields

$$H_\lambda[X;q,t] = \sum_\mu q^{n(\lambda')} K_{\mu'\lambda'}(0,1/q) S_{\mu'}[X^t]$$
$$= \sum_{T; ev(T)=\lambda'} q^{\text{cocharge}(T)} S_{\text{shape}(T)}[X^t]\,, \tag{4.12}$$

using cocharge$(T) = n(ev(T))$ − charge(T). Since standardization VS preserves cocharge and shape, we have

$$H_\lambda[X;q,t] = \sum_{T;ev(T)=\lambda'} q^{\text{cocharge}(VS(T))} S_{\text{shape}\left(VS(T)\right)}[X^t]\,. \tag{4.13}$$

This given, we can equivalently express $H_\lambda[X;q,t]$ as a sum of semi-standard tableaux under the action of $\mathcal{F}$. More precisely,

Definition 10. *For all partitions λ where $\ell(\lambda) \le 2$, let*

$$\mathbb{H}_\lambda = \sum_{T;ev(T)=\lambda'} VS(T)\,. \tag{4.14}$$

Therefore, (4.13) gives that

$$\mathcal{F}(\mathbb{H}_\lambda) = H_\lambda[X;q,t]\,. \tag{4.15}$$

In the spirit of section 3, we define two linear operators on standard tableaux.

Definition 11. *On any standard tableau $\mathcal{T}$ such that $|\mathcal{T}| = n$, let*

$$\mathbb{B}_2^{(0)} : \mathcal{T} \to VS(A_{n+1,n+1}\mathcal{T}) = \tau_1 r_{(11\to 01)} \sigma_1 \cdots \sigma_n A_{n+1,n+1}\mathcal{T} \tag{4.16}$$

and

$$\mathbb{B}_2^{(1)} : \mathcal{T} \to \left(\mathbb{B}_2^{(0)} \mathcal{T}^t\right)^t\,. \tag{4.17}$$

Example: Given a standard tableau, $\mathcal{T} = \begin{array}{|c|cc}\hline 3 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$, we add all possible horizontal 2-strips containing the letter 5 and then standardize:

$$\mathbb{B}_2^{(0)}\left(\begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array}\right) = VS\left(\begin{array}{lll} 5 & & \\ 3 & 5 & \\ 1 & 2 & 4 \end{array} + \begin{array}{llll} 5 & & & \\ 3 & & & \\ 1 & 2 & 4 & 5 \end{array} + \begin{array}{lll} 3 & 5 & 5 \\ 1 & 2 & 4 \end{array} + \begin{array}{llll} 3 & 5 & & \\ 1 & 2 & 4 & 5 \end{array} + \begin{array}{lllll} 3 & & & & \\ 1 & 2 & 4 & 5 & 5 \end{array}\right)$$
$$= \begin{array}{lll} 5 & & \\ 4 & 6 & \\ 1 & 2 & 3 \end{array} + \begin{array}{llll} 6 & & & \\ 4 & & & \\ 1 & 2 & 3 & 5 \end{array} + \begin{array}{lll} 3 & 4 & 6 \\ 1 & 2 & 5 \end{array} + \begin{array}{llll} 4 & 6 & & \\ 1 & 2 & 3 & 5 \end{array} + \begin{array}{lllll} 4 & & & & \\ 1 & 2 & 3 & 5 & 6 \end{array}. \tag{4.18}$$

Note that $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ send $\mathcal{T}$ to a sum of standard tableaux.

We now undertake the task of proving that these operators satisfy properties analogous to Properties 4, 5 and 6 as this will imply that sequences of $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ correspond, under $\mathcal{F}$, to similar sequences of $B_2^{(0)}$ and $B_2^{(1)}$ (see Theorem 26). First we prove the analog of Property 4.

Property 12. *With λ a partition such that $\ell(\lambda) \le 2$,*

$$\mathbb{B}_2^{(0)} \mathbb{H}_\lambda = \mathbb{H}_{\lambda+1^2} . \tag{4.19}$$

Proof. Since $A_{n+1,n+1}$ commutes with $VS^{(n)}$, using the action of $A_{n+1,n+1}$ on a semi-standard tableau of degree $n = |\lambda|$ and (4.14),

$$A_{n+1,n+1} \sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} VS^{(n)}(T) , \tag{4.20}$$

where λ' is a vector of length n. Thus by the definition of $\mathbb{B}_2^{(0)}$,

$$\mathbb{B}_2^{(0)} \sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} \tau_1 r_{(11\to 01)} \sigma_1 \cdots \sigma_n VS^{(n)}(T) . \tag{4.21}$$

and further by the recursion $VS^{(n+2)} = \tau_1 r_{(11\to 01)} \sigma_1 \cdots \sigma_n VS^{(n)}$, we have

$$\mathbb{B}_2^{(0)} \sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} VS^{(n+2)}(T) . \tag{4.22}$$

Because the symmetric group action (4.6) preserves charge and shape, for $\beta(\mu)$ any permutation of the vector μ, we have

$$\sum_{T;ev(T)=\mu} VS(T) = \sum_{\bar{T};ev(\bar{T})=\beta(\mu)} VS(\bar{T}) , \tag{4.23}$$

which then implies

$$\mathbb{B}_2^{(0)} \sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{\bar{T};ev(\bar{T})=(\lambda+1^2)'} VS^{(n+2)}(\bar{T}) . \tag{4.24}$$

Property 12 follows now from definition (4.14). □

The analog of Property 6 in the tableaux world is stated:

Property 13. *For $\epsilon \in \{0,1\}$, we have that*

$$\mathbb{H}_{2m+\epsilon} = (\mathbb{B}_2^{(0)} + \mathbb{B}_2^{(1)})^m \, \mathbb{H}_\epsilon . \tag{4.25}$$

Proof. From Definition 10,

$$\mathbb{H}_n = \sum_{T;ev(T)=(1^n)} VS(T) = \sum_{\mathcal{T}} \mathcal{T} , \tag{4.26}$$

and

$$\mathbb{H}_{n+1,1} = \sum_{T;ev(T)=(2,1^n)} VS(T) = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=12} \mathcal{T}' \tag{4.27}$$

where $\mathcal{T}$ and $\mathcal{T}'$ are standard tableaux of degree n and $n+2$ respectively. Property 12 gives that $\mathbb{B}_2^{(0)} \mathbb{H}_n = \mathbb{H}_{n+1,1}$, implying

$$\mathbb{B}_2^{(0)} \mathbb{H}_n = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=12} \mathcal{T}' . \tag{4.28}$$

On the other hand, since $\mathbb{B}_2^{(1)}\mathcal{T} = (\mathbb{B}_2^{(0)}\mathcal{T}^t)^t$ and $\mathbb{H}_n = \mathbb{H}_n^t$, we have

$$\mathbb{B}_2^{(1)}\mathbb{H}_n = \left(\mathbb{B}_2^{(0)}\mathbb{H}_n\right)^t = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=21} \mathcal{T}' . \tag{4.29}$$

$\mathbb{H}_{n+2}$ is the sum of all standard tableaux of order $n+2$, of which each tableaux contain the subword 12 or 21. Therefore we have

$$\mathbb{H}_{n+2} = (\mathbb{B}_2^{(0)} + \mathbb{B}_2^{(1)})\,\mathbb{H}_n\,, \tag{4.30}$$

from which (4.25) follows. □

It remains to show that $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ satisfy a relation similar to the q-commutation relation of $B_2^{(0)}$ and $B_2^{(1)}$. The proof of this last important property (stated in Property 25) requires a lengthy development of lemmas and identities.

We need two linear operators on standard tableaux, defined by:

Definition 14. *On any standard tableau $\mathcal{T}$ with $\mathcal{T}_{\{1,2\}} = 12$ and $|\mathcal{T}| = n$,*

$$\overset{*}{\mathbb{B}}{}_2^{(0)} : \mathcal{T} \to R_{n-1}\sigma_{n-2}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}\,, \tag{4.31}$$

and on any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}} = 21$ and $|\mathcal{T}| = n$,

$$\overset{*}{\mathbb{B}}{}_2^{(1)} : \mathcal{T} \to (\overset{*}{\mathbb{B}}{}_2^{(0)}\mathcal{T}^t)^t\,. \tag{4.32}$$

Example: Acting with $\overset{*}{\mathbb{B}}{}_2^{(1)}$ on $\begin{matrix}\boxed{5}\\ \boxed{2}\boxed{6}\\ \boxed{1}\boxed{3}\boxed{4}\end{matrix}$, we go through the following steps:

$$\begin{matrix}\boxed{5}\\ \boxed{2}\boxed{6}\\ \boxed{1}\boxed{3}\boxed{4}\end{matrix} \xrightarrow{t} \begin{matrix}\boxed{4}\\ \boxed{3}\boxed{6}\\ \boxed{1}\boxed{2}\boxed{5}\end{matrix} \xrightarrow{(1)} \begin{matrix}\boxed{3}\\ \boxed{2}\boxed{5}\\ \boxed{1}\boxed{1}\boxed{4}\end{matrix} \xrightarrow{(2)} \begin{matrix}\boxed{3}\\ \boxed{2}\boxed{5}\\ \boxed{1}\boxed{2}\boxed{4}\end{matrix} \xrightarrow{(3)} \begin{matrix}\boxed{3}\\ \boxed{2}\boxed{5}\\ \boxed{1}\boxed{3}\boxed{4}\end{matrix} \xrightarrow{(4)} \begin{matrix}\boxed{3}\\ \boxed{2}\boxed{5}\\ \boxed{1}\boxed{4}\boxed{4}\end{matrix} \xrightarrow{(5)} \begin{matrix}\boxed{3}\\ \boxed{2}\boxed{5}\\ \boxed{1}\boxed{4}\boxed{5}\end{matrix} \xrightarrow{(6)} \begin{matrix}\boxed{3}\\ \boxed{2}\\ \boxed{1}\boxed{4}\end{matrix} \xrightarrow{t} \begin{matrix}\boxed{4}\\ \boxed{1}\boxed{2}\boxed{3}\end{matrix}\,. \tag{4.33}$$

For any sum $S = \sum_k \mathsf{T}^{(k)}$ with $\mathsf{T}^{(k)} \neq \mathsf{T}^{(k')}$ for all $k \neq k'$, we shall say that $\mathsf{T} \in S$ if and only if $\mathsf{T} = \mathsf{T}^{(k)}$ for some k. Then, the operators $\overset{*}{\mathbb{B}}{}_2^{(0)}$ and $\overset{*}{\mathbb{B}}{}_2^{(1)}$ can be seen as the inverse of operations $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$. That is,

Property 15. *For $\mathcal{T}_1' \in B_2^{(0)}\mathcal{T}_1$ and $\mathcal{T}_2' \in B_2^{(1)}\mathcal{T}_2$, we have*

$$\overset{*}{\mathbb{B}}{}_2^{(0)}\mathcal{T}_1' = \mathcal{T}_1 \quad \textit{and} \quad \overset{*}{\mathbb{B}}{}_2^{(1)}\mathcal{T}_2' = \mathcal{T}_2\,. \tag{4.34}$$

Proof. These identities follow directly from the definition of the elementary operations comprising the operators and the fact that $R_a\mathsf{T}' = \mathsf{T}$ for any tableau $\mathsf{T}' \in A_{a,a}\mathsf{T}$. □

It is useful to alternatively define the action of $\overset{*}{\mathbb{B}}{}_2^{(1)}$. This is accomplished by introducing a new permutation $\bar{\sigma}_a$, defined by its action on a and $b = a+1$;

$$aab \overset{\bar{\sigma}_a}{\leftrightarrow} bab, \quad aba \overset{\bar{\sigma}_a}{\leftrightarrow} abb, \quad baa \overset{\bar{\sigma}_a}{\leftrightarrow} bba \tag{4.35}$$

Note that $\bar{\sigma}_a$ is such that on a word w with $ev(w_{\{a,b\}}) \in \{(1,2),(2,1)\}$, we have $\sigma_a w^R = (\bar{\sigma}_a w)^R$, where the superscript R is the operation sending a word $w = w_1 \cdots w_n$ to the word $w^R = w_n \cdots w_1$.

Property 16. *The action of* $\overset{*}{\mathbb{B}}{}_2^{(1)}$ *on any standard tableau* $\mathcal{T}$ *where* $\mathcal{T}_{\{1,2\}} =$ 21 *and* $|\mathcal{T}| = n$, *can equivalently be expressed as*

$$\overset{*}{\mathbb{B}}{}_2^{(1)} : \mathcal{T} \to R_{n-1}\bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)^{\mathcal{T}-1}}\mathcal{T}\,. \tag{4.36}$$

Example: This action of $\overset{*}{\mathbb{B}}{}_2^{(1)}$ on $\begin{smallmatrix}5&&\\2&6&\\1&3&4\end{smallmatrix}$ consists of the following steps:

$$\begin{smallmatrix}5&&\\2&6&\\1&3&4\end{smallmatrix} \overset{(1)}{\to} \begin{smallmatrix}4&&\\1&5&\\1&2&3\end{smallmatrix} \overset{(2)}{\to} \begin{smallmatrix}4&&\\2&5&\\1&2&3\end{smallmatrix} \overset{(3)}{\to} \begin{smallmatrix}4&&\\3&5&\\1&2&3\end{smallmatrix} \overset{(4)}{\to} \begin{smallmatrix}4&&\\4&5&\\1&2&3\end{smallmatrix} \overset{(5)}{\to} \begin{smallmatrix}5&&\\4&5&\\1&2&3\end{smallmatrix} \overset{(6)}{\to} \begin{smallmatrix}4&&\\1&2&3\end{smallmatrix}. \tag{4.37}$$

Note that we avoid the transposition steps shown in Example (4.33), and that each tableau here is the transposition of the corresponding tableau from Example (4.33).

Proof. Definition 14 gives that for any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}} = 21$ and $|\mathcal{T}| = n$, we must show

$$\left(\sigma_{n-2}\cdots\sigma_1 r_{(01\to 11)^{\mathcal{T}-1}}\mathcal{T}^t\right)^t = \bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)^{\mathcal{T}-1}}\mathcal{T}\,, \tag{4.38}$$

where we have used that R_{n-1} commutes with transposition. If we let $\bar{T}$ be the semi-standard tableau $\bar{T} = r_{(01\to 11)^{\mathcal{T}-1}}\mathcal{T}^t$, we get that $\bar{T}^t = r_{(10\to 11)^{\mathcal{T}-1}}\mathcal{T}$. Expression (4.38) is then verified since we can obtain $(\sigma_{n-2}\cdots\sigma_1\bar{T})^t = \bar{\sigma}_{n-2}(\sigma_{n-3}\cdots\sigma_1\bar{T})^t = \cdots = \bar{\sigma}_{n-2}\cdots\bar{\sigma}_1\bar{T}^t$ by repeatedly using the identity,

$$\begin{aligned}(\sigma_a T)^t = \bar{\sigma}_a T^t \text{ for any semi-standard } T \text{ with } ev(T_{\{a,b=a+1\}})\\ \in \{(1,2),(2,1)\}\,.\end{aligned} \tag{4.39}$$

This identity can be proven by observing that under such conditions, we have $(T_{\{a,b\}})^R = T^t_{\{a,b\}}$. For example, $T = \begin{smallmatrix}4&&&\\1&2&3&3\end{smallmatrix}$ is such that $T_{\{3,4\}} =$ 433 and $T^t_{\{3,4\}} = 334$. The only possible planar distribution of a and b that could cause this to fail are ${}_b{}^a$ or ${}_a{}^b$, since $T_{\{a,b\}}$ would be the same as $T^t_{\{a,b\}}$. The first case never holds and the second occurs only if we have both another a and another b, which we do not. Thus, using $T^R_{\{a,b\}} = (T^R)_{\{a,b\}} = (T_{\{a,b\}})^R = T^t_{\{a,b\}}$, we have $\bar{\sigma}_a T^t_{\{a,b\}} = \bar{\sigma}_a T^R_{\{a,b\}} = (\sigma_a T_{\{a,b\}})^R = (\sigma_a T)^R_{\{a,b\}} = (\sigma_a T)^t_{\{a,b\}}$, where the second equality follows from the definition of $\bar{\sigma}_a$. This gives that $(\sigma_a T)^t = \bar{\sigma}_a T^t$ on a semi-standard tableau. □

Let us now define an operation Σ_i to act on pairs of words as follows;

$$\Sigma_i : (w_1, w_2) \to (\bar{\sigma}_i\sigma_{i+1}w_1, \sigma_i\bar{\sigma}_{i+1}w_2)\,, \tag{4.40}$$

and consider 6 pairs of words with a and $b = a + 1$;

$$\begin{array}{lll} C_1(a) = (aabb, baab), & C_2(a) = (abab, abab), & C_3(a) = (abba, aabb) \\ C_4(a) = (baab, bbaa), & C_5(a) = (baba, baba), & C_6(a) = (bbaa, abba)\,. \end{array} \tag{4.41}$$

We insert letters into such pairs by defining an insertion operator on words,

$$I_k^{(a)} w_1 \cdots w_n = w_1 \cdots w_{k-1} a w_k \cdots w_n \tag{4.42}$$

with the understanding that $I_1^{(a)} w = aw$ and $I_{n+1}^{(a)} w = wa$. $I_k^{(a)}$ acts on pairs of words by $I_k^{(a)}(w_1, w_2) = (I_k^{(a)} w_1, I_k^{(a)} w_2)$.

Computer experimentation using *ACE* revealed that applying Σ_i to any pair $C_j(i)$ with the letter $(i+2)$ inserted recovered a pair of the same type where $i \to i+1$ with the extra letter i occuring in the same position of both elements of the pair. More exactly,

Lemma 17. *Let $1 \le k, k' \le 5$, $1 \le j, j' \le 6$ and $i > 1$. For any k, j and i,*

$$\Sigma_i \Big(I_k^{(i+2)} C_j(i) \Big) = I_{k'}^{(i)} C_{j'}(i+1), \tag{4.43}$$

for some k' and j'.

Example: We have

$$\begin{aligned} \Sigma_4 \Big(I_2^{(6)} C_3(4) \Big) &= \Sigma_4(46554, 46455) \\ &= (\bar\sigma_4 \sigma_5 46554, \sigma_4 \bar\sigma_5 46455) \\ &= (46565, 46565) = I_1^{(4)} C_5(5)\,. \end{aligned} \tag{4.44}$$

Proof. This is a lemma having 30 possible configurations which are easily verified with a computer. □

Proposition 18. *Let $\mathcal{T}_1$ be any standard tableau of degree n such that $\mathcal{T}_{1\{1,2,3,4\}}$ is 4312 or 3124 and let $\mathcal{T}_2 = \mathcal{T}_1^{2\leftrightarrow 3}$, i.e. $\mathcal{T}_2$ is obtained by permuting 2 and 3 in $\mathcal{T}_1$. If*

$$\mathbb{B} : (\mathcal{T}_1, \mathcal{T}_2) \to \Big(\overset{*}{\mathbb{B}}{}_2^{(1)} \overset{*}{\mathbb{B}}{}_2^{(0)} \mathcal{T}_1, \overset{*}{\mathbb{B}}{}_2^{(0)} \overset{*}{\mathbb{B}}{}_2^{(1)} \mathcal{T}_2 \Big)\,, \tag{4.45}$$

then

$$\mathbb{B}\,(\mathcal{T}_1, \mathcal{T}_2) = (\bar{\mathcal{T}}, \bar{\mathcal{T}})\,, \tag{4.46}$$

for some standard tableau $\bar{\mathcal{T}}$.

Example: Given $\mathcal{T}_1 = \begin{array}{|l} 7 \\ 3\,5 \\ 1\,2\,4\,6\,8 \end{array}$, we have the pair,

$$\mathbb{B}(\mathcal{T}_1, \mathcal{T}_2) = \mathbb{B} \left(\begin{array}{|l} 7 \\ 3\,5 \\ 1\,2\,4\,6\,8 \end{array}, \begin{array}{|l} 7 \\ 2\,5 \\ 1\,3\,4\,6\,8 \end{array} \right) = \left(\begin{array}{|l} 2 \\ 1\,3\,4 \end{array}, \begin{array}{|l} 2 \\ 1\,3\,4 \end{array} \right)\,. \tag{4.47}$$

Proof. $\overset{*}{\mathbb{B}}{}_2^{(0)}$ reduces the degree of $\mathcal{T}_1$ to $n-2$, giving by Definition 14 and Property 16,

$$\begin{aligned}\overset{*}{\mathbb{B}}{}_2^{(1)}\overset{*}{\mathbb{B}}{}_2^{(0)}\mathcal{T}_1 &= R_{n-3}\bar{\sigma}_{n-4}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}R_{n-1}\sigma_{n-2}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1\\ &= R_{n-3}R_{n-2}\big(\bar{\sigma}_{n-4}\sigma_{n-3}\big)\cdots\big(\bar{\sigma}_1\sigma_2\big)r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1\,,\end{aligned} \tag{4.48}$$

where we have considered the relations $\tau_{-1}R_{n-1} = R_{n-2}\tau_{-1}$ and $\tau_{-1}\sigma_i = \sigma_{i-1}\tau_{-1}$. Similarly, acting first with $\overset{*}{\mathbb{B}}{}_2^{(1)}$,

$$\begin{aligned}\overset{*}{\mathbb{B}}{}_2^{(0)}\overset{*}{\mathbb{B}}{}_2^{(1)}\mathcal{T}_2 &= R_{n-3}\sigma_{n-4}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}R_{n-1}\bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T}_2\\ &= R_{n-3}R_{n-2}\big(\sigma_{n-4}\bar{\sigma}_{n-3}\big)\cdots\big(\sigma_1\bar{\sigma}_2\big)r_{(01\to 11)}\tau_{-1}\bar{\sigma}_2\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T}_2\,.\end{aligned} \tag{4.49}$$

We act first with $r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}$ on $\mathcal{T}_1$ in the case that $\mathcal{T}_{1\{1,2,3,4\}}$ is 3124, obtaining a word with subword 1122 and letters $3,\ldots,n{-}2$ occurring exactly once. $\mathcal{T}_2$, defined by permuting 2 and 3 in $\mathcal{T}_1$, thus contains the subword 2134 which is sent to 2112 under $r_{(01\to 11)}\tau_{-1}\bar{\sigma}_2\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}$, while the remaining letters occur exactly as they do in $r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1$. Consequently,

$$\mathbb{B}\left(\mathcal{T}_1,\mathcal{T}_2\right) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_1\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_{k_{n-4}}^{(3)}(1122,2112)\right)\,, \tag{4.50}$$

for some $k_1,\ldots,k_{n-4}$. Since Σ_1 acts only on the letters 1,2 and 3, we may now use Lemma 17, where $i=j=1$, to determine that the action of Σ_1 results in

$$\mathbb{B}\left(\mathcal{T}_1,\mathcal{T}_2\right) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_2\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_r^{(1)}C_j(2)\right)\,, \tag{4.51}$$

for some j and r. Lemma 17, applied repeatedly in this manner, gives

$$\mathbb{B}\left(\mathcal{T}_1,\mathcal{T}_2\right) = R_{n-3}R_{n-2}\left(I_{r_1}^{(1)}I_{r_2}^{(2)}\cdots I_{r_{n-4}}^{(n-4)}C_j'(n-3)\right)\,, \tag{4.52}$$

for some $r_1,\ldots,r_{n-4}$ and some j'. This is to say that the action of $\mathbb{B}$ on such a pair is equivalent to acting with $R_{n-3}R_{n-2}$ on a pair of tableaux that are identical in all letters except $n-3$ and $n-2$. Further, since $R_{n-3}R_{n-2}$ removes these letters, we have proved the identity in the case 3124. A sequence of similar arguments may be used in the case that $\mathcal{T}_{1\{1,2,3,4\}}$ is 4312, and we get

$$\mathbb{B}\left(\mathcal{T}_1,\mathcal{T}_2\right) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_1\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_{k_{n-4}}^{(3)}(2112,2211)\right)\,. \tag{4.53}$$

Again, successive applications of Lemma 17, beginning with the case $i=1$ and $j=4$, prove the identity. □

Lemma 19. *For $\mathcal{T}'$ a standard tableau of degree $n \geq 4$, we have that*

$$
\begin{aligned}
\mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(0)} \sum_{\mathcal{T};|\mathcal{T}|=n-4} \mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}} \in \{1234, 4123, 3412\}\,,\\
\mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)} \sum_{\mathcal{T};|\mathcal{T}|=n-4} \mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}} \in \{4312, 3124\}\,,\\
\mathcal{T}' \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)} \sum_{\mathcal{T};|\mathcal{T}|=n-4} \mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}} \in \{4213, 2134\}\,,\\
\mathcal{T}' \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(1)} \sum_{\mathcal{T};|\mathcal{T}|=n-4} \mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}} \in \{4321, 3214, 2413\}\,.
\end{aligned}
\tag{4.54}
$$

Proof. We begin by simultaneously proving the first two cases of $(\Rightarrow)$ and the others of $(\Rightarrow)$ follow by transposition. Notice first that $B_2^{(0)}\mathcal{T} = \sum_{\mathcal{T}''}\mathcal{T}''$, where $\mathcal{T}''_{\{1,2\}}$ is 12 and $B_2^{(1)}\mathcal{T} = \sum_{\mathcal{T}''}\mathcal{T}''$, where $\mathcal{T}''_{\{1,2\}}$ is 21 (see the proof of Property 13). The following action of $B_2^{(0)}$ begins with $A_{n+1,n+1}$ adding a horizontal 2 strip to $\mathcal{T}''$ resulting in tableaux that are all semi-standard and containing the subword 12 (or 21). We act next with the succession of $\sigma_n, \sigma_{n-1}, \ldots, \sigma_3$ implying that the tableaux remain semi-standard and thus must each contain, in the first case, the subword 1233, 3123 or 3312, and in the second, 3213 or 2133. The remaining operations, aside from τ_1, act exclusively on these subwords as follows;

$$
\begin{aligned}
\tau_1 r_{(11\to01)}\sigma_1\sigma_2 1233 &= 1234\,,\\
\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3123 &= 4123\,,\\
\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3312 &= 3412\,,
\end{aligned}
\tag{4.55}
$$

for the first case and

$$
\begin{aligned}
\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3213 &= 4312\,,\\
\tau_1 r_{(11\to01)}\sigma_1\sigma_2 2133 &= 3124\,,
\end{aligned}
\tag{4.56}
$$

for the second, thus proving $(\Rightarrow)$ for the two first cases. To prove $(\Leftarrow)$, we are given a standard tableau $\mathcal{T}'$ with subword $\mathcal{T}'_{\{1,2,3,4\}}$ in one of the four defined disjoint sets; call this set $S_{\epsilon_1,\epsilon_2}$. Property 13 gives that $\mathcal{T}'$, which is $\in \mathbb{H}_n$, for $n = |\mathcal{T}'|$, is such that $\mathcal{T}' \in \mathbb{B}_2^{(\bar\epsilon_1)}\mathbb{B}_2^{(\bar\epsilon_2)}\mathbb{H}_{n-4} = \mathbb{B}_2^{(\bar\epsilon_1)}\mathbb{B}_2^{(\bar\epsilon_2)}\sum_{\{|\mathcal{T}|=n-4\}}\mathcal{T}$, for some $\bar\epsilon_i \in \{0,1\}$. But since we have just proven that for such $\mathcal{T}'$, $\mathcal{T}'_{\{1,2,3,4\}}$ is contained in the set $S_{\bar\epsilon_1,\bar\epsilon_2}$, we see that $\bar\epsilon_i = \epsilon_i$ and the lemma is proven. □

Proposition 20. *On any standard tableau $\mathcal{T}$, we have*

$$
\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} = \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow 3}\,,
\tag{4.57}
$$

where $2 \leftrightarrow 3$ denotes a permutation of the letters 2 and 3 in each tableaux.

Proof. Suppose there exists $\mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T}$ such that $\mathcal{T}'^{2\leftrightarrow 3} \notin \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}$. Lemma 19 gives that every element in $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T}$, in particular $\mathcal{T}'$, must contain either the subword 4312 or 3124. This implies that $\mathcal{T}'^{2\leftrightarrow 3}$ must contain either 4213 or 2134 and thus by the same lemma we have that

$$\mathcal{T}'^{2\leftrightarrow 3} \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}'' \tag{4.58}$$

for some $\mathcal{T}'' \neq \mathcal{T}$. Observe now that $\overset{*}{\mathbb{B}}{}_2^{(1)}\overset{*}{\mathbb{B}}{}_2^{(0)}\mathcal{T}' = \mathcal{T}$, implies by Proposition 18 that $\overset{*}{\mathbb{B}}{}_2^{(0)}\overset{*}{\mathbb{B}}{}_2^{(1)}\mathcal{T}'^{2\leftrightarrow 3}$ must also be $\mathcal{T}$. Expression (4.58) then yields

$$\overset{*}{\mathbb{B}}{}_2^{(0)}\overset{*}{\mathbb{B}}{}_2^{(1)}\left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}''\right) = \mathcal{T} + \text{other terms} \tag{4.59}$$

which by Property 15 gives that $\mathcal{T}'' = \mathcal{T}$ and we reach a contradiction. We thus have that $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} \subseteq \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow 3}$. We can also show in the same manner that $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} \supseteq \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow 3}$, which proves the proposition. □

We now define four pairs of words on consecutive numbers; $a, b = a+1, c = b+1$ and $d = c+1$.

$$\begin{aligned} D_1(a) &= (bacd, cabd), & D_2(a) &= (dbac, dcab), \\ D_3(a) &= (acdb, abdc), & D_4(a) &= (cdba, bdca)\,. \end{aligned} \tag{4.60}$$

These pairs appear as the only distinct subwords in certain pairs of semi-standard tableaux. More precisely, such a pair of semi-standard tableaux, called $(T_1, T_2)_{D_j(a)}$, satisfies $T_1 = T_2^{b\leftrightarrow c}$ and $(T_{1_{\{a,b,c,d\}}}, T_{2_{\{a,b,c,d\}}}) = D_j(a)$. For example,

$$\left(\begin{array}{|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 7 & 8 \\ \cline{1-3} 1 & 2 & 3 & 6 \\ \hline \end{array}\;,\; \begin{array}{|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 6 & 8 \\ \cline{1-3} 1 & 2 & 3 & 7 \\ \hline \end{array} \right)_{D_3(5)} \tag{4.61}$$

is such a pair. One should note that $(T_1, T_2)_{D_j(a)}$ is a pair of tableaux of the same shape since in any such semi-standard tableaux, b and c never occur in the same row or column. With Ω_i defined such that on pairs of words

$$\Omega_i : (w_1, w_2) \to (\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}w_1, \sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}w_2)\,; \tag{4.62}$$

Lemma 21. *Let* $1 \le k_2, k_2', k_2'' \le 5,\ \ 1 \le k_1, k_1', k_1'' \le 6,\ \ 1 \le j, j', j'' \le 4$ *and* $i > 0$. *For any such* k_1, k_2, j *and* i, *we have*

$$r_{(ii\to ii-1)}\Omega_i I_{k_1}^{(i+4)} I_{k_2}^{(i+4)} D_j(i) \in \left\{ I_{k_1'}^{(i-1)} I_{k_2'}^{(i)} D_{j'}(i+1)\,,\, I_{k_1'}^{(i-1)} I_{k_2'}^{(i+4)} D_{j'}(i) \right\} \tag{4.63}$$

and

$$r_{(ii\to i-1i)}\Omega_i I_{k_1}^{(i+4)} I_{k_2}^{(i+4)} D_j(i) \in \left\{ I_{k_1''}^{(i-1)} I_{k_2''}^{(i)} D_{j''}(i+1)\,,\, I_{k_1''}^{(i-1)} I_{k_2''}^{(i+4)} D_{j''}(i) \right\}, \tag{4.64}$$

for some $k_1', k_1'', k_2', k_2'', j'$ *and* j''.

Example: Starting with $I_4^{(9)} I_2^{(9)} D_4(5) = (798965, 698975)$, we get

$$\Omega_4(798965, 698975) = (597865, 596875)\,. \tag{4.65}$$

Under $r_{(55\to 45)}$, we recover $I_1^{(4)} I_1^{(9)} D_4(5)$ and under $r_{(55\to 54)}$, $I_6^{(4)} I_2^{(9)} D_3(5)$.
Proof. For each i there are 60 cases that have been verified using a computer. $\square$

Lemma 22. *Let $\mathcal{T}$ and $\mathcal{T}'$ be standard tableaux of type $(\mathcal{T}, \mathcal{T}')_{D_j(i)}$ for some i, j. If standard tableaux $\bar{\mathcal{T}} \in \mathbb{B}_2^{(\epsilon)} \mathcal{T}$ and $\bar{\mathcal{T}}' \in \mathbb{B}_2^{(\epsilon)} \mathcal{T}'$ are standard tableaux of the same shape for $\epsilon \in \{0, 1\}$, then $\bar{\mathcal{T}}$ and $\bar{\mathcal{T}}'$ is a pair of type $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+1)}$ or $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+2)}$, for some j'.*

Proof. We start with the case $\epsilon = 0$ and split the action of $B_2^{(0)}$ into a sequence of operations beginning with $A_{n+1,n+1}$. As such, we consider a semi-standard tableau T obtained by adding an arbitrary horizontal 2-strip to $\mathcal{T}$. We denote by T', the semi-standard tableau of the same shape that is obtained by adding this horizontal 2-strip to $\mathcal{T}'$ and thus T and T' are a pair of type $(T, T')_{D_j(i)}$. Next in the sequence of operations defining $B_2^{(0)}$ is $\sigma_{i+4} \cdots \sigma_n$ which, acting only on the letters $i+4, \dots, n$, must preserve the similarity in T and T'. If $i \neq 1$, since acting with σ_{i-1} amounts to applying either $r_{(ii\to ii-1)}$ or $r_{(ii\to i-1i)}$, acting on both elements with $\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}$ using Lemma 21, gives a pair of semi-standard tableaux of type $(\bar{T}, \bar{T}')_{D_{j'}(i+1)}$ or $(\bar{T}, \bar{T}')_{D_{j'}(i)}$. There remains to act with $\tau_1 r_{11\to 01} \sigma_1 \dots \sigma_{i-2}$, which leads to pairs of standard tableaux of type $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+2)}$ or $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+1)}$. In the case where $i = 1$, acting on both elements with $\tau_1 r_{(11\to 01)} \sigma_1\sigma_2\sigma_3\sigma_4$, gives, from Lemma 21, pairs of standard tableaux of type $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}')_{D_{j'}(3)}$ or $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}')_{D_{j'}(2)}$, finally proving the lemma for $\epsilon = 0$. To prove the lemma in the case $\mathbb{B}_2^{(1)}$, we observe that, for $\mathcal{T}_1$ and $\mathcal{T}_2$ standard tableaux,

$$\begin{aligned} (\mathcal{T}_1, \mathcal{T}_2)_{D_1(a)} &\implies (\mathcal{T}_2^t, \mathcal{T}_1^t)_{D_2(a)} \qquad & (\mathcal{T}_1, \mathcal{T}_2)_{D_2(a)} &\implies (\mathcal{T}_2^t, \mathcal{T}_1^t)_{D_1(a)}\,, \\ (\mathcal{T}_1, \mathcal{T}_2)_{D_3(a)} &\implies (\mathcal{T}_2^t, \mathcal{T}_1^t)_{D_4(a)} \qquad & (\mathcal{T}_1, \mathcal{T}_2)_{D_4(a)} &\implies (\mathcal{T}_2^t, \mathcal{T}_1^t)_{D_3(a)}\,. \end{aligned} \tag{4.66}$$

$\mathbb{B}_2^{(1)} \mathcal{T} = (\mathbb{B}_2^{(0)} \mathcal{T}^t)^t$ thus implies that the proof in this case is exactly the proof for $\mathbb{B}_2^{(0)}$ with every pair reversed plus an additional reversal of the pairs at the end, accounting for the last transposition in $\mathbb{B}_2^{(1)}$. $\square$

Lemma 23. *If $\mathcal{T}$ and $\mathcal{T}'$ are a pair of standard tableaux of type $(\mathcal{T}, \mathcal{T}')_{D_j(i)}$ then*

$$F(\mathcal{T}) = q F(\mathcal{T}')\,. \tag{4.67}$$

Proof. We have already noted that $(\mathcal{T}, \mathcal{T}')_{D_j(i)}$ is a pair of tableaux with the same shape. Further, the definition of cocharge gives cocharge$(\mathcal{T})$ =

cocharge($\mathcal{T}'$) + 1; for example, for $D_1(a) = (bacd, cabd)$, we have

$$\begin{aligned}\text{cocharge}(\mathcal{T}) &= c_1 + \cdots + c_a + (c_a+1) + (c_a+1) + (c_a+1)\\ &\quad + c_{a+4} + \cdots + c_n\\ \text{cocharge}(\mathcal{T}') &= c_1 + \cdots + c_a + (c_a) + (c_a+1) + (c_a+1)\\ &\quad + c_{a+4} + \cdots + c_n\,,\end{aligned} \tag{4.68}$$

giving cocharge($\mathcal{T}$) = cocharge($\mathcal{T}'$) + 1 as claimed. □

We now finally have all the ingredients needed to prove the commutation relation.

Definition 24. *We define, for $v = (v_1, \ldots, v_k)$ with $v_i \in \{0,1\}$,*

$$\mathbb{U}_v^{(\epsilon)} = \mathbb{B}_2^{(v_1)} \cdots \mathbb{B}_2^{(v_k)} \mathbb{H}_\epsilon\,, \qquad \epsilon \in \{0,1\}\,. \tag{4.69}$$

Property 25. *For any $v = (v_1, \ldots, v_k)$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_{k'})$, with $v_i, \bar{v}_i \in \{0,1\}$ and $k, k' \geq 0$, we have*

$$\mathcal{F}(\mathbb{U}_{v,1,0,\bar{v}}^{(\epsilon)}) = q\mathcal{F}(\mathbb{U}_{v,0,1,\bar{v}}^{(\epsilon)}) \qquad \epsilon \in \{0,1\}\,. \tag{4.70}$$

Proof. We begin by showing that this identity holds in the case that v is empty. Since $\mathbb{U}_{1,0,\bar{v}}^{(\epsilon)} = \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)} \sum_{\mathcal{T}} \mathcal{T}$ and $\mathbb{U}_{0,1,\bar{v}}^{(\epsilon)} = \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)} \sum_{\mathcal{T}} \mathcal{T}$, each tableau $\mathcal{T}_1 \in \mathbb{U}_{1,0,\bar{v}}^{(\epsilon)}$ can be paired with some $\mathcal{T}_1' \in \mathbb{U}_{0,1,\bar{v}}^{(\epsilon)}$ such that $\mathcal{T}_1' = \mathcal{T}_1^{2\leftrightarrow 3}$ by Proposition 20 and such that $\mathcal{T}_{1\{1,2,3,4\}} \in \{4213, 2134\}$ by Lemma 19. This implies that $\mathcal{T}_1$ and $\mathcal{T}_1'$ are of type $(\mathcal{T}_1, \mathcal{T}_1')_{D_j(1)}$ where $j = 1$ or 2 which, using Lemma 23, proves the identity for $v = ()$. We now proceed to the case when $v = (0)$ or (1) by acting with $\mathbb{B}_2^{(\epsilon)}$ on the pairs obtained when $v = ()$. These pairs $(\mathcal{T}_1, \mathcal{T}_1')_{D_j(1)}$ are thus sent to a pair of sums of standard tableaux that, by Lemma 22, can be paired by types $(\mathcal{T}_2, \mathcal{T}_2')_{D_j(i)}$ for some $j = 1, 2, 3$ or 4 and some $i = 3$ or 4. For any v, we repeat this process and obtain that each $\bar{\mathcal{T}} \in \mathbb{U}_{v,1,0,\bar{v}}^{(\epsilon)}$ can be paired with some standard tableau $\bar{\mathcal{T}}' \in \mathbb{U}_{v,0,1,\bar{v}}^{(\epsilon)}$ where this pair is of type $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_j(i)}$, for $1 \leq j \leq 4$ and $1 \leq i \leq n-3$. Lemma 23 then proves the property. □

Given Properties 12, 13 and 25, analogous to those proven in section 3, we can finally prove the main result of this section.

Theorem 26. *Let $\epsilon, v_i \in \{0,1\}$. For any $v = (v_1, \ldots, v_k)$ we have*

$$\mathcal{F}(\mathbb{U}_v^{(\epsilon)}) = U_v^{(\epsilon)}\,. \tag{4.71}$$

Proof. Recall that the action of $B_2^{(0)}$ and $B_2^{(0)} + B_2^{(1)}$, determined in Properties 4 and 6, led to Corollary 9. That is, to

$$H_{2m+\ell+\epsilon,\ell}[X;q,t] = \sum_{\bar{v}} U_{\bar{v}}^{(\epsilon)}\,, \tag{4.72}$$

where $\bar{v} = (0^\ell, v)$ for some $v = (v_1, \ldots, v_m)$. Observe that we have proved equivalent actions for the operators $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(0)} + \mathbb{B}_2^{(1)}$ in Properties 12 and 13, giving

$$\mathbb{H}_{2m+\ell+\epsilon,\ell} = \sum_{\bar{v}} \mathbb{U}_{\bar{v}}^{(\epsilon)}, \tag{4.73}$$

where $\bar{v}$ is as before. We thus have, since $\mathcal{F}(\mathbb{H}_{m,\ell}) = H_{m,\ell}[X;q,t]$,

$$H_{2m+\ell+\epsilon,\ell}[X;q,t] = \sum_{\bar{v}} \mathcal{F}(\mathbb{U}_{\bar{v}}^{(\epsilon)}) = \sum_{\bar{v}} U_{\bar{v}}^{(\epsilon)}. \tag{4.74}$$

We convert the expression such that we are summing only over dominant vectors $v_d = (0^{m+\ell-k}, 1^k)$ for some k by using the following implication of the q-commutation relations proven in Properties 5 and 25: $U_{\beta(v)}^{(\epsilon)} = q^{\ell(\beta)} U_{v_d}^{(\epsilon)}$ and $\mathcal{F}(\mathbb{U}_{\beta(v)}^{(\epsilon)}) = q^{\ell(\beta)} \mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$, where $\ell(\beta)$ is the length of the permutation β such that $\beta(v) = v_d$. This gives

$$H_{2m+\ell+\epsilon,\ell}[X;q,t] = \sum_{v_d} d_{v_d}^{m,\ell}(q) \mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)}) = \sum_{v_d} d_{v_d}^{m,\ell}(q) U_{v_d}^{(\epsilon)}, \tag{4.75}$$

where $d_{v_d}^{m,\ell}(q) = \sum_{\beta(v)=v_d} q^{\ell(\beta)}$. For $2m+2\ell+\epsilon = n$, the number of possible v_d is $\lfloor n/2 \rfloor + 1$, exactly the number of partitions of n of length ≤ 2. We thus have, from (4.75), that $U_{v_d}^{(\epsilon)}$ and $\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$ are both bases for $\mathcal{V}$, the $\mathbb{Q}[q,t]$-linear span of $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\leq 2}$. We see from expression (4.75) again, that the transition matrices from $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\leq 2}$ to $\{U_{v_d}^{(\epsilon)}\}_{v_d}$ and from $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\leq 2}$ to $\{\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})\}_{v_d}$ are identical. Since these are invertible matrices, we have that $U_{v_d}^{(\epsilon)} = \mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$, which can be extended to $U_v^{(\epsilon)} = \mathcal{F}(\mathbb{U}_v^{(\epsilon)})$ using Properties 5 and 25. □

5. A statistic for Macdonald polynomials in 2 parts

It is now clear from the previous theorem and (3.28) that for $d = 2m + 2\ell + \epsilon$, we have

$$J_{2m+\ell+\epsilon,\ell}[X;q,t] = \sum_{v=(v_1,\ldots,v_{m+\ell})} q^{(1-d)|v|_\ell + 2n(v)_\ell} t^{\ell-|v|_\ell} \mathcal{F}(\mathbb{U}_v^{(\epsilon)}), \tag{5.1}$$

where $|v|_\ell$ and $n(v)_\ell$ are as defined in Proposition 8. To provide an expression for $J_\lambda[X;q,t]$ with coefficients that are determined by statistics, we associate to any standard tableau a vector $\in \{0,1\}^k$ called a "domino" vector. The domino vector is determined by the succession of of operators $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ that build the associated standard tableau. Since any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}} = 12$ (or 21) can be obtained by acting with $\mathbb{B}_2^{(0)}$ (or $\mathbb{B}_2^{(1)}$) on a predecessor $\mathcal{T}'$, such a succession is determined recursively using Property 15.

Theorem 27. *The Macdonald polynomials indexed by partitions with no more than 2 parts are given by*

$$J_{2m+\ell+\epsilon,\ell} = \sum_{|\mathcal{T}|=d} \mathrm{Stat}(\mathcal{T}) S_{\mathrm{shape}(\mathcal{T})}[X^t], \tag{5.2}$$

where $d = 2m + 2\ell + \epsilon$ *and*

$$\mathrm{Stat}(\mathcal{T}) = q^{\mathrm{cocharge}(\mathcal{T})} q^{(1-d)|\mathrm{dv}(\mathcal{T})|_\ell + 2n(\mathrm{dv}(\mathcal{T}))_\ell} t^{\ell - |\mathrm{dv}(\mathcal{T})|_\ell}, \tag{5.3}$$

with the domino vector, $\mathrm{dv}(\mathcal{T}) = (\mathrm{dv}_1, \ldots, \mathrm{dv}_{m+\ell})$, *obtained recursively by*

$$\mathrm{dv}(\mathcal{T}) = \begin{cases} \left(0, \mathrm{dv}(\overset{*}{\mathbb{B}}{}_2^{(0)} \mathcal{T})\right) & \text{if } \mathcal{T}_{\{1,2\}} = 12 \\ \left(1, \mathrm{dv}(\overset{*}{\mathbb{B}}{}_2^{(1)} \mathcal{T})\right) & \text{if } \mathcal{T}_{\{1,2\}} = 21 \\ \emptyset & \text{if } \mathcal{T} \text{ has degree} \leq 1 \end{cases} \tag{5.4}$$

Proof. The theorem follows directly from (5.1) and Property 15. □

Example: The statistic associated to a standard tableau $\mathcal{T} = \begin{array}{|c|c|c|}\hline 4 & 8 \\ \hline 3 & 5 & 7 \\ \hline 1 & 2 & 6 \\ \hline\end{array}$ in the Macdonald polynomial $J_{6,2}[X; q, t]$ is determined by finding the domino vector of $\mathcal{T}$.

$$\begin{aligned} \left(0, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(0)} \begin{array}{|c|c|c|}\hline 4 & 8 \\ \hline 3 & 5 & 7 \\ \hline 1 & 2 & 6 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 4 \\ \hline 2 & 6 \\ \hline 1 & 3 & 5 \\ \hline\end{array}\right)\right) &= \left(0, 1, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(1)} \begin{array}{|c|c|c|}\hline 4 \\ \hline 2 & 6 \\ \hline 1 & 3 & 5 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 4 \\ \hline 3 \\ \hline 1 & 2 \\ \hline\end{array}\right)\right) \\ &= \left(0, 1, 0, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(0)} \begin{array}{|c|c|}\hline 4 \\ \hline 3 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right)\right) \\ &= (0, 1, 0, 1). \end{aligned} \tag{5.5}$$

This gives that $|\mathrm{dv}(\mathcal{T})|_2 = 1$ and $n(\mathrm{dv}(\mathcal{T}))_2 = 1$. The cocharge of $\mathcal{T} = 48\ 357\ 126$ is $0 + 0 + 1 + 2 + 2 + 2 + 3 + 4 = 14$, and we have

$$\mathrm{Stat}(\mathcal{T}) = q^{14} q^{-7+2} t^{2-1} = q^9 t. \tag{5.6}$$

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References

[1] S. Fishel, *Statistics for special Kostka polynomials*, Proc. Amer. Math. Soc. **123** (1995), 2961–2969.

[2] A.M. Garsia and M. Haiman, *A graded representation module for Macdonald's polynomials*, Proc. Natl. Acad. Sci. USA V **90** (1993) 3607–3610.

[3] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math Soc. **14** (2001), 941–1006.

[4] A. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math. J. **93** (1998), 1–39.

[5] L. Lapointe, A. Lascoux and J. Morse, *Determinantal expressions for Macdonald polynomials*, IMRN **18** (1998), 957–978.

[6] L. Lapointe and J. Morse, *Schur function identities, their t-analogs, and k-Schur irreducibility*, Adv. in Math., to appear.

[7] L. Lapointe and J. Morse, *Schur function analogs for a filtration of the symmetric function space*, JCT-A to appear.

[8] L. Lapointe and L. Vinet, *A short proof of the integrality of the Macdonald (q,t)-Kostka coefficients*, Duke Math. J. **91** (1998), 205–214.

[9] A. Lascoux and M.-P. Schützenberger, *Le monoïde plaxique*, Quaderni della Ricerca scientifica **109** (1981), 129–156.

[10] A. Lascoux and M.-P. Schützenberger, *Sur une conjecture de H.O. Foulkes*, C.R. Acad. Sc. Paris. **294** (1978), 323–324.

[11] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Clarendon Press, Oxford, 1995.

[12] S. Veigneau, *ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE, Version 3.0*, Université de Marne-la-Vallée, 1998, http://phalanstere. univ-mlv.fr/~ace/.

[13] M. A. Zabrocki, *A Macdonald vertex operator and standard tableaux statistics for the two-column (q,t)-Kostka coefficients*, Electron. J. Combinat. **5**, R45 (1998), 46pp.

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile

E-mail address: lapointe@inst-mat.utalca.cl

University of Miami, Department of Mathematics, Coral Gables, FL 33124, USA

E-mail address: morsej@math.miami.edu

A CRYSTAL TO RIGGED CONFIGURATION BIJECTION FOR NONEXCEPTIONAL AFFINE ALGEBRAS

MASATO OKADO, ANNE SCHILLING, AND MARK SHIMOZONO

ABSTRACT. Kerov, Kirillov, and Reshetikhin defined a bijection between highest weight vectors in the crystal graph of a tensor power of the vector representation, and combinatorial objects called rigged configurations, for type $A_n^{(1)}$. We define an analogous bijection for all nonexceptional affine types, thereby proving (in this special case) the fermionic formulas conjectured by Hatayama, Kuniba, Takagi, Tsuboi, Yamada, and the first author.

1. INTRODUCTION

The fermionic formula, denoted by M, is a certain polynomial expressed as a sum of products of q-binomial coefficients. It originates in the Bethe Ansatz analysis of solvable lattice models in two dimensional statistical mechanics. The prototypical example is given by the Kostka polynomial $K_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$, which is indexed by a pair of partitions λ, μ. According to Lascoux and Schützenberger [12],

$$K_{\lambda\mu}(q) = \sum_{T \in \mathcal{T}(\lambda,\mu)} q^{c(T)}.$$

Here $\mathcal{T}(\lambda, \mu)$ is the set of semistandard tableaux of shape λ and weight μ, and $c(T)$ is the charge of the tableau T.

We consider the case that μ is a single column (1^L). Kirillov and Reshetikhin [8] gave a fermionic formula for the Kostka polynomial:

$$K_{\lambda,(1^L)}(q) = q^{\binom{L}{2}} M(\lambda, (1^L); q^{-1}) \tag{1.1}$$

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where

$$M(\lambda,(1^L);q)=\sum_{\{m\}} q^{cc(\{m\})} \prod_{\substack{1\le a\le n\\ i\ge 1}} \begin{bmatrix} p_i^{(a)}+m_i^{(a)} \\ m_i^{(a)} \end{bmatrix},$$

$$cc(\{m\})=\frac{1}{2}\sum_{1\le a,b\le n} C_{ab}\sum_{i,j\ge 1}\min(i,j)m_i^{(a)}m_j^{(b)},$$

$$p_i^{(a)}=L\delta_{a1}-\sum_{1\le b\le n} C_{ab}\sum_{j\ge 1}\min(i,j)m_j^{(b)},$$

$\begin{bmatrix} p+m \\ m \end{bmatrix} = (q)_{p+m}/(q)_p(q)_m$ is the q-binomial coefficient, $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$, the sum $\sum_{\{m\}}$ is taken over $\{m_i^{(a)} \in \mathbb{Z}_{\ge 0} \mid 1\le a\le n, i\ge 1\}$, satisfying $p_i^{(a)}\ge 0$ for $1\le a\le n, i\ge 1$ and $\sum_{i\ge 1} im_i^{(a)} = \lambda_{a+1}+\lambda_{a+2}+\cdots+\lambda_{n+1}$ for $1\le a\le n$. Here n is an integer not less than the length of λ minus 1, and $(C_{ab})_{1\le a,b\le n}$ is the Cartan matrix of sl_{n+1}.

To prove that the Kostka polynomial is given by the fermionic formula, Kerov, Kirillov and Reshetikhin (KKR) defined a bijection between $\mathcal{T}(\lambda,(1^L))$ and combinatorial objects called rigged configurations [7]. Expanding the q-binomial coefficients in $M(\lambda,(1^L);q)$, to each term q^c one can associate a rigged configuration having a statistic c. Under the bijection, the charge of a tableau agrees with the statistic on the rigged configuration. This bijection was extended to the larger class of Littlewood-Richardson tableaux and corresponding rigged configurations [10].

The Kostka polynomial is related to the affine Lie algebra of type $A_n^{(1)}$, since the corresponding fermionic formula is derived from the integrable model associated to the quantum affine algebra $U_q(A_n^{(1)})$. The Kostka polynomial $K_{\lambda\mu}(q)$ gives the graded multiplicity of the λ-th irreducible $U_q(A_n)$-module in the restriction of the tensor product of certain finite-dimensional $U_q'(A_n^{(1)})$-modules that have crystal bases. The situation generalizes to the context of any affine Lie algebra. One can define the analogous tensor product modules and graded multiplicities, and a corresponding fermionic formula M [1, 2]. The new combinatorial objects which replace tableaux are called paths. A path is a highest weight element of the aforementioned tensor product crystal base. Paths have a natural statistic called energy. In the case of the Kostka polynomial, paths biject with rigged configurations: one may send the path (which may be viewed as a word) to its Robinson-Schensted recording tableau, which is then sent to a rigged configuration by the KKR bijection. The generating function of paths by energy is called the "one dimensional sum" X. The equality $X = M$ was conjectured in full generality in [1, 2].

The purpose of the paper is to construct the analogue of the KKR bijection and thereby prove the $X = M$ conjecture, for all nonexceptional affine Lie algebras, in the case of the simplest crystal bases. For $A_n^{(1)}$, this case corresponds to the Kostka polynomial $K_{\lambda(1^L)}(q)$ discussed above.

2. Quantum affine algebras and crystals

2.1. Affine algebras. We adopt the notation of [1]. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of nonexceptional affine type $X_N^{(r)}$, that is, one of the types $A_n^{(1)}(n \geq 1)$, $B_n^{(1)}(n \geq 3)$, $C_n^{(1)}(n \geq 2)$, $D_n^{(1)}(n \geq 4)$, $A_{2n}^{(2)}(n \geq 1)$, $A_{2n}^{(2)\dagger}(n \geq 1)$, $A_{2n-1}^{(2)}(n \geq 2)$, $D_{n+1}^{(2)}(n \geq 2)$. Note that $A_{2n}^{(2)\dagger}$ is the same diagram as $A_{2n}^{(2)}$ but with the opposite labeling.

The Dynkin diagram of $\mathfrak{g} = X_N^{(r)}$ is depicted in Table 1 (Table Aff 1-3 in [4]). Its nodes are labeled by the set $I = \{0, 1, 2\dots, n\}$.

Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$. Let δ and c denote the generator of imaginary roots and the canonical central element, respectively. Recall that $\delta = \sum_{i\in I} a_i\alpha_i$ and $c = \sum_{i\in I} a_i^\vee h_i$, where the Kac labels a_i are the unique set of relatively prime positive integers giving the linear dependency of the columns of the Cartan matrix A (that is, $A(a_0, \dots, a_n)^t = 0$). Explicitly,

$$(2.1)\qquad \delta = \begin{cases} \alpha_0 + \cdots + \alpha_n & \text{if } \mathfrak{g} = A_n^{(1)} \\ \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n & \text{if } \mathfrak{g} = B_n^{(1)} \\ \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = C_n^{(1)} \\ \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_n^{(1)} \\ 2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n}^{(2)} \\ \alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + 2\alpha_n & \text{if } \mathfrak{g} = A_{2n}^{(2)\dagger} \\ \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n-1}^{(2)} \\ \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases}$$

The dual Kac label $a_i^\vee$ is the label a_i for the affine Dynkin diagram obtained by "reversing the arrows" of the Dynkin diagram of $\mathfrak{g}$, or equivalently, the coefficients giving the linear dependency of the rows of the Cartan matrix A. Note that $a_0^\vee = 2$ for $\mathfrak{g} = A_{2n}^{(2)\dagger}$ and $a_0^\vee = 1$ otherwise.

Let $(\cdot|\cdot)$ be the normalized invariant form on P [4]. It satisfies

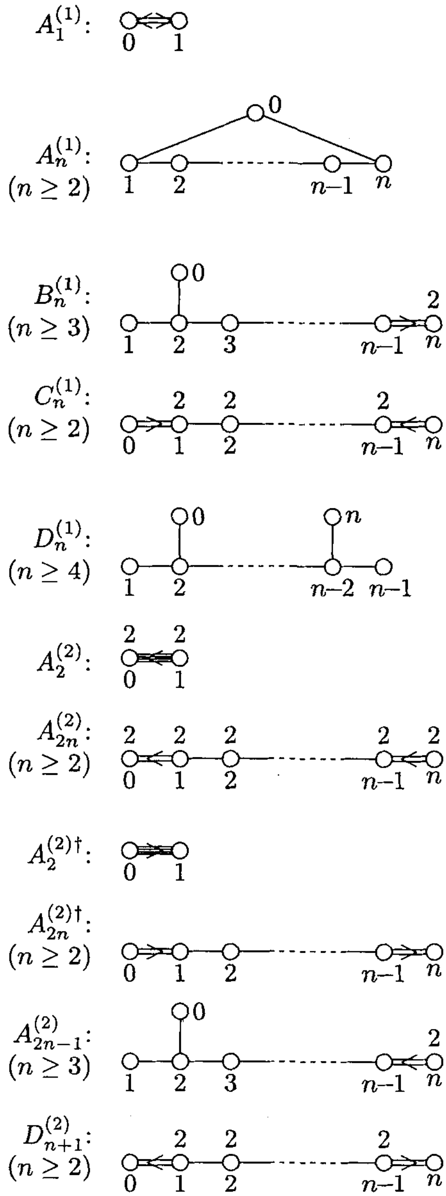
$$(2.2)\qquad (\alpha_i|\alpha_j) = \frac{a_i^\vee}{a_i} A_{ij}$$

for $i, j \in I$. In particular

$$(2.3)\qquad (\alpha_a|\alpha_a) = \frac{2r}{a_0^\vee}$$

if α_a is a long root.

TABLE 1. Dynkin diagrams for $X_N^{(r)}$. The labeling of the nodes (by elements of I) is specified under or the right side of the nodes. The numbers t_i (resp. $t_i^\vee$) defined in (2.4) are attached *above* the nodes for $r = 1$ (resp. $r > 1$) if and only if $t_i \neq 1$ (resp. $t_i^\vee \neq 1$).



For $i \in I$ let

$$t_i = \max(\frac{a_i}{a_i^\vee}, a_0^\vee), \qquad t_i^\vee = \max(\frac{a_i^\vee}{a_i}, a_0). \tag{2.4}$$

The values t_i are given in Table 1. We shall only use $t_i^\vee$ and t_i for $i \in I^* = I\backslash\{0\}$. For $a \in I^*$ we have

$$t_a^\vee = 1 \;\text{ if } r = 1, \qquad t_a = a_0^\vee \;\text{ if } r > 1.$$

We consider two finite-dimensional subalgebras of $\mathfrak{g}$: $\overline{\mathfrak{g}}$, whose Dynkin diagram is obtained from that of $\mathfrak{g}$ by removing the 0 vertex, and $\mathfrak{g}_{\overline{0}}$, the subalgebra of X_N fixed by the automorphism σ given in [4, Section 8.3].

TABLE 2

$\mathfrak{g}$	$X_N^{(1)}$	$A_{2n}^{(2)}$	$A_{2n}^{(2)\dagger}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$
$\overline{\mathfrak{g}}$	X_N	C_n	B_n	C_n	B_n
$\mathfrak{g}_{\overline{0}}$	X_N	B_n	B_n	C_n	B_n

Let $\overline{\mathfrak{g}}$ (resp. $\mathfrak{g}_{\overline{0}}$) have weight lattice $\overline{P}$ (resp. $\tilde{P}$), with simple roots and fundamental weights $\alpha_a, \overline{\Lambda}_a$ (resp. $\tilde{\alpha}_a, \tilde{\Lambda}_a$) for $a \in I^*$. Note that $\overline{\mathfrak{g}} = \mathfrak{g}_{\overline{0}}$ for $\mathfrak{g} \neq A_{2n}^{(2)}$. For $\mathfrak{g} = A_{2n}^{(2)}$, $\overline{\mathfrak{g}} = C_n$ and $\mathfrak{g}_{\overline{0}} = B_n$.

$\tilde{P}$ is endowed with the bilinear form $(\cdot|\cdot)'$, normalized by

$$(\tilde{\alpha}_a|\tilde{\alpha}_a)' = 2r/a_0^\vee \qquad \text{if } \tilde{\alpha}_a \text{ is a long root of } \mathfrak{g}_{\overline{0}}. \tag{2.5}$$

For $A_2^{(2)}$, the unique simple root $\tilde{\alpha}_1$ of $\mathfrak{g}_{\overline{0}} = B_1$ is considered to be short.

Note that $\alpha_a, \overline{\Lambda}_a$ and $(\cdot|\cdot)$ may be identified with $\tilde{\alpha}_a, \tilde{\Lambda}_a$ and $(\cdot|\cdot)'$ if $\mathfrak{g} \neq A_{2n}^{(2)}$.

Define the $\mathbb{Z}$-linear map $\iota : \overline{P} \to \tilde{P}$ by

$$\iota(\overline{\Lambda}_a) = \epsilon_a \tilde{\Lambda}_a \qquad \text{for } a \in I^*, \tag{2.6}$$

where ϵ_a is defined by

$$\epsilon_a = \begin{cases} 2 & \text{if } \mathfrak{g} = A_{2n}^{(2)} \text{ and } a = n \\ 1 & \text{otherwise.} \end{cases} \tag{2.7}$$

In particular $\iota(\alpha_a) = \epsilon_a \tilde{\alpha}_a$ for $a \in I^*$. We have

$$(\iota(\alpha_b)|\iota(\alpha_b))' = a_0(\alpha_b|\alpha_b) \qquad \text{for all } b \in I^*. \tag{2.8}$$

If $\mathfrak{g} = A_{2n}^{(2)}$ both sides of (2.8) are equal to 8 if $b = n$ and 4 otherwise. Especially for $\mathfrak{g} = A_2^{(2)}$ ($n = 1$), we have $(\tilde{\alpha}_1|\tilde{\alpha}_1)' = 2$ and $(\alpha_1|\alpha_1) = 4$. In the rest of the paper we shall write $(\cdot|\cdot)$ in place of $(\cdot|\cdot)'$.

2.2. **Simple subalgebras.** For later use, specific realizations are given for the simple roots and fundamental weights of the simple Lie algebras of types B_n, C_n, and D_n, which appear as the subalgebras $\overline{\mathfrak{g}}$ and $\mathfrak{g}_{\overline{0}}$ of $\mathfrak{g}$. In each case the sublattice of $\overline{P}$ given by the weights appearing in tensor products of the vector representation, is identified with $\mathbb{Z}^n$. Let $\{\epsilon_i \mid 1 \leq i \leq n\}$ be the standard basis of $\mathbb{Z}^n$.

The simple Lie algebra B_n.

$$\begin{aligned} \alpha_a &= \epsilon_a - \epsilon_{a+1} && \text{for } 1 \leq a < n \\ \alpha_n &= \epsilon_n \\ \overline{\Lambda}_a &= \epsilon_1 + \cdots + \epsilon_a && \text{for } 1 \leq a < n \\ \overline{\Lambda}_n &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n). \end{aligned} \tag{2.9}$$

$\lambda \in \mathbb{Z}^n$ is B_n-dominant if and only if

$$\begin{aligned} \lambda_a - \lambda_{a+1} &\geq 0 && \text{for } 1 \leq a < n \\ \lambda_n &\geq 0. \end{aligned} \tag{2.10}$$

The simple Lie algebra C_n.

$$\begin{aligned} \alpha_a &= \epsilon_a - \epsilon_{a+1} && \text{for } 1 \leq a < n \\ \alpha_n &= 2\epsilon_n \\ \overline{\Lambda}_a &= \epsilon_1 + \cdots + \epsilon_a && \text{for } 1 \leq a \leq n. \end{aligned} \tag{2.11}$$

$\lambda \in \mathbb{Z}^n$ is C_n-dominant if and only if it is B_n-dominant (2.10).

The simple Lie algebra D_n.

$$\begin{aligned} \alpha_a &= \epsilon_a - \epsilon_{a+1} && \text{for } 1 \leq a < n \\ \alpha_n &= \epsilon_{n-1} + \epsilon_n \\ \overline{\Lambda}_a &= \epsilon_1 + \cdots + \epsilon_a && \text{for } 1 \leq a \leq n-2 \\ \overline{\Lambda}_{n-1} &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n) \\ \overline{\Lambda}_n &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n) \end{aligned} \tag{2.12}$$

$\lambda \in \mathbb{Z}^n$ is D_n-dominant if and only if

$$\begin{aligned} \lambda_a - \lambda_{a+1} &\geq 0 && \text{for } 1 \leq a < n \\ \lambda_{n-1} + \lambda_n &\geq 0. \end{aligned} \tag{2.13}$$

2.3. **Crystals.** Let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$. Denote the corresponding quantized universal enveloping algebras of $\mathfrak{g} \supset \mathfrak{g}' \supset \overline{\mathfrak{g}}$ by $U_q(\mathfrak{g}) \supset U_q'(\mathfrak{g}) \supset U_q(\overline{\mathfrak{g}})$.

In [2] it is conjectured that there is a family of finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules $\{W_i^{(a)} \mid a \in I^*, i \in \mathbb{Z}_{>0}\}$ which, unlike most finite-dimensional $U_q'(\mathfrak{g})$-modules, have crystal bases $B^{a,i}$. This family is conjecturally characterized in several different ways:

1. Its characters form the unique solutions of a system of quadratic relations (the Q-system) [9].
2. Every crystal graph of an irreducible integrable finite-dimensional $U_q'(\mathfrak{g})$-module, is a tensor product of the $B^{a,i}$.
3. For $\lambda \in P$ let $V(\lambda)$ be the extremal weight module defined in [3, Section 3] and $B(\lambda)$ its crystal base, with unique vector $u_\lambda \in B(\lambda)$ of weight λ. Then the affinization of $B^{a,i}$ (in the sense of [5]) is isomorphic to the connected component of u_λ in $B(\lambda)$, for the weight $\lambda = i\overline{\Lambda}_a$ (except when $\mathfrak{g} = A_{2n}^{(2)\dagger}$ and $a = n$, in which case $\lambda = 2i\overline{\Lambda}_a$).

In light of point (2) above, we consider the category of crystal graphs given by tensor products of the crystals $B^{a,i}$.

We introduce notation for tensor products of $B^{a,i}$. Let $\mu = (L_i^{(a)})_{a\in I^*, i\in\mathbb{Z}_{>0}}$ be a matrix of nonnegative integers, almost all zero. Define

$$B^{(\mu)} = \bigotimes_{(a,i)\in I^*\times\mathbb{Z}_{>0}} (B^{a,i})^{\otimes L_i^{(a)}}. \tag{2.14}$$

In type $A_n^{(1)}$ this is the tensor product of modules, which, when restricted to A_n, are irreducible modules indexed by rectangular partitions. The set of classically restricted paths (or classical highest weight vectors) in $B^{(\mu)}$ of weight $\lambda \in \overline{P}^+ = \bigoplus_{i\in I^*} \mathbb{Z}_{\geq 0}\overline{\Lambda}_i$ is by definition

$$\mathcal{P}(\lambda, \mu) = \{b \in B^{(\mu)} \mid \mathrm{wt}(b) = \lambda \text{ and } \widetilde{e}_i b \text{ undefined for all } i \in I^*\}. \tag{2.15}$$

Here $\widetilde{e}_i$ is given by the crystal graph. For $b, b' \in B^{a,i}$ we have $b' = \widetilde{e}_i(b)$ if there is an arrow $b' \xrightarrow{i} b$ in the crystal graph; if no such arrow exists then $\widetilde{e}_i(b)$ is undefined. Similarly, $b' = \widetilde{f}_i(b)$ if there is an arrow $b \xrightarrow{i} b'$ in the crystal graph; if no such arrow exists then $\widetilde{f}_i(b)$ is undefined. If B_1 and B_2 are crystals, then for $b_1 \otimes b_2 \in B_1 \otimes B_2$ the action of $\widetilde{e}_i$ is defined as

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{else,} \end{cases}$$

where $\varepsilon_i(b) = \max\{k \mid \widetilde{e}_i^k \text{ is defined}\}$ and $\varphi_i(b) = \max\{k \mid \widetilde{f}_i^k \text{ is defined}\}$.

Assumption 2.1. In this paper we shall restrict our attention to the case $B^{(\mu)} = B^{\otimes L}$ where $B = B^{1,1}$. We shall write $B^{(\tilde{\mu})} = B^{\otimes(L-1)}$.

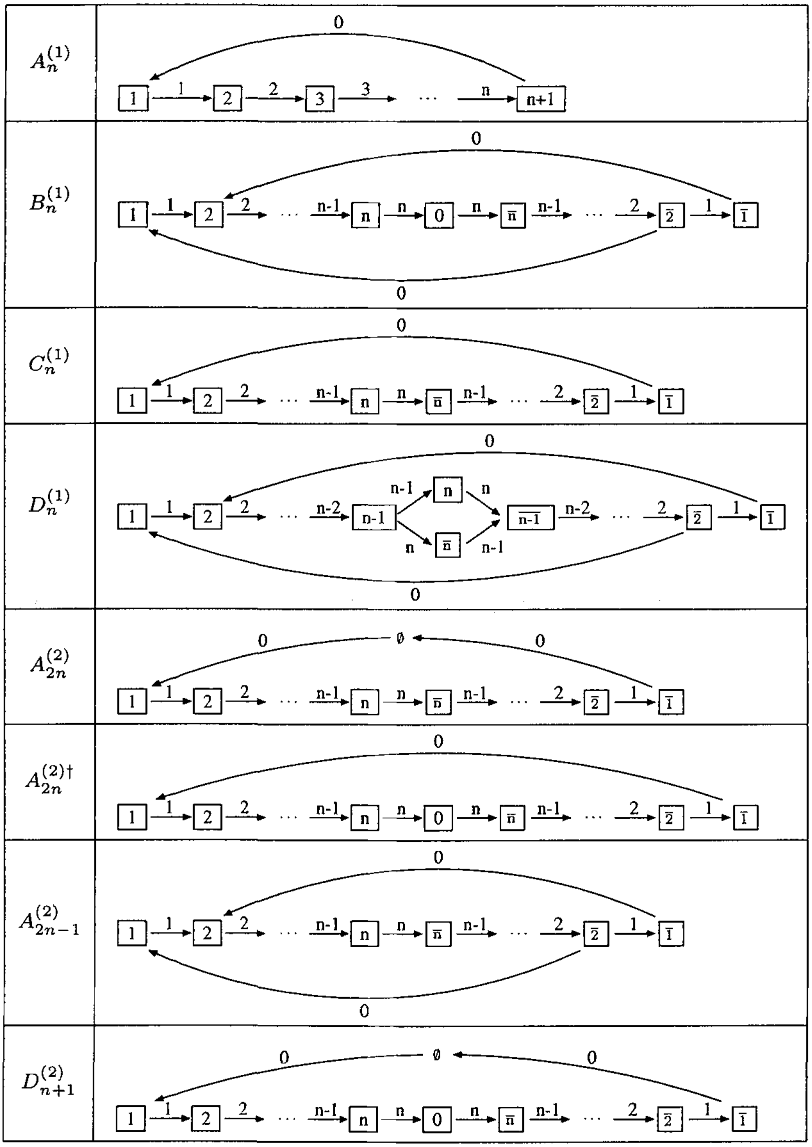


TABLE 3. Crystals $B^{1,1}$

The crystal graphs $B^{1,1}$ are listed in Table 3.

In each case (other than $A_n^{(1)}$) the elements of $B = B^{1,1}$ consist of $\{k, \overline{k} \mid 1 \le k \le n\}$ and possibly elements 0 and ϕ.

Remark 2.2. By glancing at Table 3, one may check that the following are equivalent for $b = b_L \otimes b_{L-1} \otimes \cdots \otimes b_1 \in B^{\otimes L}$ and $\lambda \in \overline{P}^+$.

(1) b is a classically restricted path of weight $\lambda \in \overline{P}^+$.

(2) $\lambda - \mathrm{wt}(b_L) \in \overline{P}^+$, $b_{L-1} \otimes \cdots \otimes b_1$ is a classically restricted path of weight $\lambda - \mathrm{wt}(b_L)$, and if $b_L = 0 \in B$ then $\lambda_n > 0$ (where λ is viewed as an element of $\mathbb{Z}^n$).

The weight function $\mathrm{wt} : B \to \mathbb{Z}^n$ is given by

$$\begin{aligned} \mathrm{wt}(k) &= \epsilon_k && \text{for } 1 \le k \le n \\ \mathrm{wt}(\overline{k}) &= -\epsilon_k && \text{for } 1 \le k \le n \\ \mathrm{wt}(0) &= \mathrm{wt}(\phi) = 0. \end{aligned}$$

The weight function $\mathrm{wt} : B^{\otimes L} \to \mathbb{Z}^n$ is defined by $\mathrm{wt}(b_L \otimes \cdots \otimes b_1) = \sum_{j=1}^{L} \mathrm{wt}(b_j)$. So if $\lambda = \mathrm{wt}(p)$ where $p \in B^{\otimes L}$, then λ_k is the multiplicity of k in p minus the multiplicity of $\overline{k}$ in p.

2.4. **One-dimensional sums.** The energy function $D : B^{(\mu)} \to \mathbb{Z}$ gives the grading on $B^{(\mu)}$. In the case $B^{(\mu)} = B^{\otimes L}$ it takes a simple form. Due to the existence of the universal R-matrix and the fact that $W_1^{(1)}$ is irreducible, by [5] there is a unique (up to global additive constant) function $H : B^{1,1} \otimes B^{1,1} \to \mathbb{Z}$ called the local energy function, such that

(2.16)
$$H(\widetilde{e}_i(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & \text{if } i = 0 \text{ and } \widetilde{e}_0(b \otimes b') = b \otimes \widetilde{e}_0 b' \\ 1 & \text{if } i = 0 \text{ and } \widetilde{e}_0(b \otimes b') = \widetilde{e}_0 b \otimes b' \\ 0 & \text{otherwise.} \end{cases}$$

Let $b^\natural \in B^{1,1}$ be the unique element such that $\varphi(b^\natural) = \Lambda_0$. We normalize H by the condition

$$H(1 \otimes 1) = 0. \tag{2.17}$$

Then

$$\begin{aligned} E(b_L \otimes \cdots \otimes b_1) &= L\ H(b_1 \otimes b^\natural) + \sum_{j=1}^{L-1} (L-j)\ H(b_{j+1} \otimes b_j), \\ D(b_L \otimes \cdots \otimes b_1) &= E(b_L \otimes \cdots \otimes b_1) - E(1 \otimes \cdots \otimes 1). \end{aligned} \tag{2.18}$$

Define the one-dimensional sum $X(\lambda, \mu; q) \in \mathbb{Z}[q, q^{-1}]$ by

$$X(\lambda, \mu; q) = \sum_{b \in \mathcal{P}(\lambda, \mu)} q^{D(b)}. \tag{2.19}$$

Since $B^{(\mu)}$ is completely reducible as a $U_q(\overline{\mathfrak{g}})$-crystal, one has

$$\sum_{b \in B^{(\mu)}} e^{\mathrm{wt}(b)} q^{D(b)} = \sum_{\lambda \in \overline{P}^+} \chi^\lambda X(\lambda, \mu; q)$$

where χ^λ is the character of the irreducible $U_q(\overline{\mathfrak{g}})$-module of highest weight λ. It can be shown that $X(\lambda,\mu;q) \in \mathbb{Z}_{\geq 0}[q^{-1}]$. For convenience we define

$$\overline{H} = -H, \qquad \overline{D} = -D, \qquad \overline{X}(\lambda,\mu;q) = X(\lambda,\mu;q^{-1}). \tag{2.20}$$

3. Rigged configurations and the bijection

3.1. The fermionic formula, $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$. This subsection reviews definitions of [1, 2]. Let $\mathfrak{g}$ be a Kac-Moody algebra of nonexceptional affine type that is not of the form $A_{2n}^{(2)\dagger}$. Fix $\lambda \in \overline{P}^+$ and a matrix $\mu = (L_i^{(a)})$ of nonnegative integers as in subsection 2.4.

Let $\nu = (m_i^{(a)})$ be another such matrix. Say that ν is a λ-configuration if

$$\sum_{\substack{a\in I^* \\ i\in\mathbb{Z}_{>0}}} i\, m_i^{(a)} \tilde{\alpha}_a = \iota\left(\sum_{\substack{a\in I^* \\ i\in\mathbb{Z}_{>0}}} i\, L_i^{(a)} \overline{\Lambda}_a - \lambda\right). \tag{3.1}$$

Say that a configuration ν is μ-admissible if

$$p_i^{(a)} \geq 0 \qquad \text{for all } a \in I^* \text{ and } i \in \mathbb{Z}_{>0}, \tag{3.2}$$

where

$$p_i^{(a)} = \sum_{k\in\mathbb{Z}_{>0}} \left(L_k^{(a)} \min(i,k) - \frac{1}{t_a^\vee} \sum_{b\in I^*} (\tilde{\alpha}_a|\tilde{\alpha}_b) \min(t_b i, t_a k)\, m_k^{(b)} \right). \tag{3.3}$$

Write $C(\lambda,\mu)$ for the set of μ-admissible λ-configurations. Define

$$cc(\nu) = \frac{1}{2} \sum_{a,b\in I^*} \sum_{j,k\in\mathbb{Z}_{>0}} (\tilde{\alpha}_a|\tilde{\alpha}_b) \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)}. \tag{3.4}$$

The fermionic formula is defined by

$$\overline{M}(\lambda,\mu;q) = \sum_{\nu\in C(\lambda,\mu)} q^{cc(\nu)} \prod_{a\in I^*} \prod_{i\in\mathbb{Z}_{>0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}_{q^{t_a^\vee}}. \tag{3.5}$$

The $X = M$ conjecture of [1, 2] states that

$$\overline{X}(\lambda,\mu;q) = \overline{M}(\lambda,\mu;q). \tag{3.6}$$

3.2. Rigged configurations, $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$. The fermionic formula $\overline{M}(\lambda,\mu)$ can be interpreted using combinatorial objects called rigged configurations. These objects are a direct combinatorialization of the fermionic formula $\overline{M}(\lambda,\mu;q)$. Our goal is to prove (3.6) under Assumption 2.1 by defining a statistic-preserving bijection from rigged configurations to paths. For this purpose it is convenient to use an indexing slightly differing from that used above.

For $a \in I^*$, define

$$v_a = \begin{cases} 2 & \text{if } a = n \text{ and } \mathfrak{g} = C_n^{(1)} \\ \frac{1}{2} & \text{if } a = n \text{ and } \mathfrak{g} = B_n^{(1)} \\ 1 & \text{otherwise.} \end{cases} \tag{3.7}$$

v_a is half the square length of α_a for untwisted affine types and is equal to 1 for twisted types.

A quasipartition λ of type $a \in I^*$ is a finite multiset taken from the set $v_a\mathbb{Z}_{>0}$. Denote by $m_i(\lambda)$ the number of times $i \in v_a\mathbb{Z}_{>0}$ occurs in λ. The diagram of such a quasipartition has, for each $i \in v_a\mathbb{Z}_{>0}$, $m_i(\lambda)$ rows consisting of i boxes, where each box has width v_a. Set

$$\mathcal{H} = \{(a,i) \mid a \in I^*, i \in v_a\mathbb{Z}_{>0}\}. \tag{3.8}$$

Denote by $(\nu^\bullet, J^\bullet)$ a pair where $\nu^\bullet = \{\nu^{(a)}\}_{a\in I^*}$ is a sequence of quasipartitions with $\nu^{(a)}$ of type a and $J^\bullet = \{J^{(a,i)}\}_{(a,i)\in\mathcal{H}}$ is a double sequence of partitions. For $(a,i) \in \mathcal{H}$, define

$$\begin{aligned} P_i^{(a)}(\nu^\bullet) &= p_{i/v_a}^{(a)} \\ m_i^{(a)}(\nu^\bullet) &= m_{i/v_a}^{(a)} = m_i(\nu^{(a)}). \end{aligned} \tag{3.9}$$

Then a rigged configuration is a pair $(\nu^\bullet, J^\bullet)$ subject to the restriction (3.1) and the requirement that $J^{(a,i)}$ be a quasipartition contained in a $m_i^{(a)}(\nu^\bullet) \times P_i^{(a)}(\nu^\bullet)$ rectangle. The set of rigged configurations for fixed λ and μ is denoted by $\mathrm{RC}(\lambda,\mu)$. Then (3.5) is equivalent to

$$F(\lambda,\mu) = \sum_{(\nu^\bullet,J^\bullet)\in\mathrm{RC}(\lambda,\mu)} q^{cc(\nu^\bullet,J^\bullet)}$$

where $cc(\nu^\bullet, J^\bullet) = cc(\nu) + |J^\bullet|$ and $|J^\bullet| = \sum_{(a,i)\in\mathcal{H}} t_a^\vee |J^{(a,i)}|$ for ν corresponding to $\nu^\bullet$ under (3.9).

3.3. $A_{2n}^{(2)\dagger}$ **rigged configurations.** In this subsection let $\mathfrak{g} = A_{2n}^{(2)\dagger}$. As this case is not considered in [1] we shall only give the definition in terms of rigged configurations, although it is easy to express the result as a sum of a product of q-binomials (see [13, Section 7.6]). The important feature is that the riggings of odd-sized parts of $\nu^{(n)}$, must have the form $x/2$ where x is an odd integer. So let μ and λ be as in subsection 3.1. Given a matrix $\nu = (m_i^{(a)})$, let $P_i^{(a)}(\nu^\bullet)$ and $m_i^{(a)}(\nu^\bullet)$ be defined as before. Call $\nu^\bullet$ μ-admissible if $P_i^{(a)}(\nu^\bullet) \geq 0$ for all $a \in I^*$ and $i \in \mathbb{Z}_{>0}$, together with the extra condition that

$$P_i^{(n)}(\nu^\bullet) \geq 1 \qquad \text{if } i \text{ is odd and } m_i^{(n)}(\nu^\bullet) > 0. \tag{3.10}$$

A rigging $J^\bullet$ consists of quasipartitions $J^{(a,i)}$ for $a \in I^*$ and $i \in \mathbb{Z}_{>0}$. For $a \neq n$ or i even, $J^{(a,i)}$ is an ordinary partition satisfying the usual properties. For $a = n$ and i odd, $J^{(n,i)}$ is a quasipartition contained in a rectangle with $P_i^{(n)}(\nu^\bullet)$ columns and $m_i^{(n)}(\nu^\bullet)$ rows, but it has cells of width $1/2$ and each part size must be of the form $x/2$ for x an odd integer. This defines the set $RC(\lambda,\mu)$ for $\mathfrak{g} = A_{2n}^{(2)\dagger}$. Then $F(\lambda,\mu)$ is defined as before where $|J^\bullet|$ is the sum of the areas of all the quasipartitions $J^{a,i}$. This definition is compatible with the virtual crystal realization which embeds paths (and rigged configurations) of type $A_{2n}^{(2)\dagger}$ into those of type $A_{2n-1}^{(1)}$ [13].

3.4. **The bijection from RCs to paths.** We now describe the general form of the bijection $\Phi : \mathrm{RC}(\lambda,\mu) \to \mathcal{P}(\lambda,\mu)$ under Assumption 2.1. Let $\mu = (L_i^{(a)})$ be such that $B^{(\mu)} = B^{\otimes L}$, that is, $L_i^{(a)} = L\delta_{a1}\delta_{i1}$. Let $\tilde{\mu}$ be such that $B^{(\tilde{\mu})} = B^{\otimes(L-1)}$.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda,\mu)$. We shall define a map $\mathrm{rk} : \mathrm{RC}(\lambda,\mu) \to B$ which associates to $(\nu^\bullet, J^\bullet)$ an element of B called its rank.

Denote by $\mathrm{RC}_b(\lambda,\mu)$ the elements of $\mathrm{RC}(\lambda,\mu)$ of rank b. We shall define a bijection $\delta : \mathrm{RC}_b(\lambda,\mu) \to \mathrm{RC}(\lambda - \mathrm{wt}(b), \tilde{\mu})$. The disjoint union of these bijections then defines a bijection $\delta : \mathrm{RC}(\lambda,\mu) \to \bigcup_{b\in B} \mathrm{RC}(\lambda - \mathrm{wt}(b), \tilde{\mu})$.

The bijection Φ is defined recursively as follows. For $b \in B$ let $\mathcal{P}_b(\lambda,\mu)$ be the set of paths in $B^{(\mu)} = B^{\otimes L}$ that have b as leftmost tensor factor. For $L = 0$ the bijection Φ sends the empty rigged configuration (the only element of the set $\mathrm{RC}(\lambda,\mu)$) to the empty path (the only element of $\mathcal{P}(\lambda,\mu)$). Otherwise assume that Φ has been defined for $B^{\otimes(L-1)}$ and define it for $B^{\otimes L}$ by the commutative diagram

$$\begin{array}{ccc} \mathrm{RC}_b(\lambda,\mu) & \xrightarrow{\Phi} & \mathcal{P}_b(\lambda,\mu) \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow \\ \mathrm{RC}(\lambda - \mathrm{wt}(b),\tilde{\mu}) & \xrightarrow{\Phi} & \mathcal{P}(\lambda - \mathrm{wt}(b),\tilde{\mu}) \end{array} \tag{3.11}$$

where the right hand vertical map removes the leftmost tensor factor b. In short,

$$\Phi(\nu^\bullet, J^\bullet) = \mathrm{rk}(\nu^\bullet, J^\bullet) \otimes \Phi(\delta(\nu^\bullet, J^\bullet)). \tag{3.12}$$

Remark 3.1. For Φ to be well-defined, by Remark 2.2 it must be shown that if $b = \mathrm{rk}(\nu^\bullet, J^\bullet)$, then $\rho = \lambda - \mathrm{wt}(b)$ is dominant, and if $b = 0$ then $\lambda_n > 0$.

We also require the bijection $\tilde{\Phi} : \mathrm{RC}(\lambda,\mu) \to \mathcal{P}(\lambda,\mu)$ given by $\tilde{\Phi} = \Phi \circ \mathrm{comp}$ where $\mathrm{comp} : \mathrm{RC}(\lambda,\mu) \to \mathrm{RC}(\lambda,\mu)$ with $\mathrm{comp}(\nu^\bullet, J^\bullet) = (\nu^\bullet, \tilde{J}^\bullet)$ is the function which complements the riggings, meaning that $\tilde{J}^\bullet$ is obtained from $J^\bullet$ by complementing all partitions $J^{(a,i)}$ in the $m_i^{(a)} \times P_i^{(a)}(\nu^\bullet)$ rectangle.

Theorem 3.2. *$\Phi : \mathrm{RC}(\lambda,\mu) \to \mathcal{P}(\lambda,\mu)$ is a bijection such that*

$$cc(\nu^\bullet, J^\bullet) = \overline{D}(\tilde{\Phi}(\nu^\bullet, J^\bullet)) \qquad \textit{for all } (\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda,\mu). \tag{3.13}$$

For type $A_n^{(1)}$ a generalization of this theorem for all μ was proven in [10]. For other types Theorem 3.2 is proved in section 5.

4. The bijection for each root system

In this section the maps rk and δ are defined in a case-by-case manner. For each $\mathfrak{g}$, an explicit formula is given for the vacancy numbers $P_i^{(a)}(\nu^\bullet)$ (see (3.9)), obtained by writing (3.3) in terms of the function Q_i (see (4.1)) using the data for the simple Lie algebras given in section 2.2. Then for $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, \mu)$, an algorithm is given which defines $b = \mathrm{rk}(\nu^\bullet, J^\bullet)$, the new smaller rigged configuration $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$ such that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \in \mathrm{RC}(\rho, \tilde{\mu})$ (where $\rho = \lambda - \mathrm{wt}(b)$), and the new vacancy numbers in terms of the old.

For a quasipartition τ with boxes of width v and $i \in v\mathbb{Z}_{\geq 0}$, define

$$Q_i(\tau) = \sum_j \min(\tau_j, i), \tag{4.1}$$

the area of τ in the first i quasicolumns.

The quasipartition $J^{(a,i)}$ is called *singular* (with respect to the configuration $\nu^\bullet$) if it has a part of size $P_i^{(a)}(\nu^\bullet)$. If A is a statement then $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. We also use the Kronecker delta notation $\delta_{a,b} = \chi(a = b)$.

4.1. Bijection algorithm for type $D_n^{(1)}$.

Vacancy numbers.

(4.2)

$$\begin{aligned}
P_i^{(a)}(\nu^\bullet) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq a < n-2 \\
P_i^{(n-2)}(\nu^\bullet) &= Q_i(\nu^{(n-3)}) - 2Q_i(\nu^{(n-2)}) + Q_i(\nu^{(n-1)}) + Q_i(\nu^{(n)}) \\
P_i^{(n-1)}(\nu^\bullet) &= Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) \\
P_i^{(n)}(\nu^\bullet) &= Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n)})
\end{aligned}$$

Constraints.

$$\begin{aligned}
|\nu^{(a)}| &= L - \sum_{b=1}^{a} \lambda_b \qquad\qquad \text{for } 1 \leq a \leq n-2 \\
|\nu^{(n-1)}| &= \frac{1}{2}(L - \sum_{b=1}^{n-1} \lambda_b + \lambda_n) \\
|\nu^{(n)}| &= \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b)
\end{aligned} \tag{4.3}$$

Algorithm δ. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \dots, n-2$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a+1$.

If the process has not stopped at $a = n-2$ continue as follows. Find the minimal indices $i, j \geq \ell^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular. If neither i nor j exist, set $b = n-1$ and stop. If i exists, but not j, set $\ell^{(n-1)} = i$, $b = n$ and stop. If j exists, but not i, set $\ell^{(n)} = j$, $b = \overline{n}$ and stop. If both i and j exist, set $\ell^{(n-1)} = i$, $\ell^{(n)} = j$ and continue with $a = n-2$.

Now continue for $a = n-2, n-3, \dots, 1$ or until stopped. Find the minimal index $i \geq \bar{\ell}^{(a+1)}$ where $\bar{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $P_i^{(a)}(\nu^\bullet)$ in $J^{(a,i)}$). If no such i exists, set $b = \overline{a+1}$ and stop. If the process did not stop, set $b = \overline{1}$.

Set all yet undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ to ∞.

New RC.

$$
m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - 1 \text{ and } 1 \leq a \leq n-2 \\ -1 & \text{if } i = \bar{\ell}^{(a)} \text{ and } 1 \leq a \leq n-2 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}
$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Change in vacancy numbers.

(4.5)

$$
\begin{aligned}
P_i^{(a)}(\tilde{\nu}^\bullet) = P_i^{(a)}(\nu^\bullet) &- \chi(\ell^{(a-1)} \leq i) + 2\chi(\ell^{(a)} \leq i) - \chi(\ell^{(a+1)} \leq i) \\
&- \chi(\bar{\ell}^{(a-1)} \leq i) + 2\chi(\bar{\ell}^{(a)} \leq i) - \chi(\bar{\ell}^{(a+1)} \leq i) \\
&\qquad\qquad\qquad \text{for } 1 \leq a < n-2 \\
P_i^{(n-2)}(\tilde{\nu}^\bullet) = P_i^{(n-2)}(\nu^\bullet) &- \chi(\ell^{(n-3)} \leq i) + 2\chi(\ell^{(n-2)} \leq i) - \chi(\ell^{(n-1)} \leq i) \\
&- \chi(\bar{\ell}^{(n-3)} \leq i) + 2\chi(\bar{\ell}^{(n-2)} \leq i) - \chi(\ell^{(n)} \leq i) \\
P_i^{(n-1)}(\tilde{\nu}^\bullet) = P_i^{(n-1)}(\nu^\bullet) &- \chi(\ell^{(n-2)} \leq i) - \chi(\bar{\ell}^{(n-2)} \leq i) + 2\chi(\ell^{(n-1)} \leq i) \\
P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet) &- \chi(\ell^{(n-2)} \leq i) - \chi(\bar{\ell}^{(n-2)} \leq i) + 2\chi(\ell^{(n)} \leq i).
\end{aligned}
$$

4.2. Bijection algorithm for type $B_n^{(1)}$.

Vacancy numbers.

(4.6)

$$\begin{aligned} P_i^{(a)}(\nu^\bullet) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} && \text{for } i \in \mathbb{Z}_{\geq 0} \\ & && 1 \leq a \leq n-2 \\ P_i^{(n-1)}(\nu^\bullet) &= Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) + 2Q_i(\nu^{(n)}) && \text{for } i \in \mathbb{Z}_{\geq 0} \\ P_i^{(n)}(\nu^\bullet) &= 2Q_i(\nu^{(n-1)}) - 4Q_i(\nu^{(n)}) && \text{for } i \in \frac{1}{2}\mathbb{Z}_{\geq 0}. \end{aligned}$$

Constraints.

$$\begin{aligned} |\nu^{(a)}| &= L - \sum_{b=1}^{a} \lambda_b \qquad \text{for } 1 \leq a \leq n-1 \\ |\nu^{(n)}| &= \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b). \end{aligned} \tag{4.7}$$

Algorithm δ. Call a partition *quasi-singular* if it is not singular and has a part of size $P_i^{(a)}(\nu^\bullet) - 1$.

Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n-1$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If the process has not yet stopped, continue as follows. For brevity let us denote by (S) and (Q) the following conditions:

(S) $i \geq \ell^{(n-1)}$ and $J^{(n,i)}$ is singular.

(Q) $i = \ell^{(n-1)} - \frac{1}{2}$ and $J^{(n,i)}$ is singular; or $i \geq \ell^{(n-1)}$ and $J^{(n,i)}$ is quasi-singular.

Find the minimal index $i \geq \ell^{(n-1)} - \frac{1}{2}$ such that (S) or (Q) holds (note that (S) and (Q) are mutually excluding). If no such i exists, set $b = n$ and stop. If (S) holds set $\bar{\ell}^{(n)} = i$ and $\ell^{(n)} = i - \frac{1}{2}$. Say that case (S) holds. If (Q) holds set $\ell^{(n)} = i$ and find the minimal index $j > i$ such that (S) holds. If no such j exists, set $b = 0$ and stop. Say that case (Q) holds. Otherwise set $\bar{\ell}^{(n)} = j$ and say that case (Q,S) holds.

If the process has not yet stopped continue in the following fashion for $a = n-1, n-2, \ldots, 1$ or until stopped. Find the minimal index $i \geq \bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular (if $\ell^{(a)} = i$ then $J^{(a,i)}$ actually needs to have two parts of size $P_i^{(a)}(\nu^\bullet)$). If no such i exists, set $b = \overline{a+1}$ and stop. Otherwise set $\bar{\ell}^{(a)} = i$ and continue. If the process did not stop for $a \geq 1$ set $b = \bar{1}$.

Set all undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ for $1 \leq a \leq n$ to ∞.

New RC.

$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - v_a \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - v_a \\ -1 & \text{if } i = \bar{\ell}^{(a)} \\ 0 & \text{otherwise}. \end{cases} \tag{4.8}$$

Note that if two or more conditions hold, all of the changes should be performed.

For $1 \le a < n$ the partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$ and leaving it unchanged otherwise. If case (S) occurred $\tilde{J}^{(n,i)}$ is obtained from $J^{(n,i)}$ by removing a part of size $P_i^{(n)}(\nu^\bullet)$ for $i = \bar{\ell}^{(n)}$, adding a part of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ for $i = \bar{\ell}^{(n)} - 1$, and leaving it unchanged otherwise. If case (Q) holds remove the largest part in $J^{(n,i)}$ for $i = \ell^{(n)}$ and add a part of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(n)} - \frac{1}{2}$. If case (Q,S) holds, then apply (S') for $t = \ell^{(n)}$ and (Q') for $t = \bar{\ell}^{(n)}$ where

(S') obtain $\tilde{J}^{(n,i)}$ from $J^{(n,i)}$ by removing the largest part for $i = t$ and adding a part of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ for $i = t - \frac{1}{2}$, leaving all other $J^{(n,i)}$ unchanged;

(Q') obtain $\tilde{J}^{(n,i)}$ from $J^{(n,i)}$ by removing the largest part for $i = t$ and adding a part of size $P_i^{(n)}(\tilde{\nu}^\bullet) - 1$ if $t < \bar{\ell}^{(n-1)}$ and of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ if $t = \bar{\ell}^{(n-1)}$ for $i = t - \frac{1}{2}$, leaving all other $J^{(n,i)}$ unchanged.

Change in vacancy numbers.

(4.9)

$$\begin{aligned} P_i^{(a)}(\tilde{\nu}^\bullet) = P_i^{(a)}(\nu^\bullet) &- \chi(\ell^{(a-1)} \le i) + 2\chi(\ell^{(a)} \le i) - \chi(\ell^{(a+1)} \le i) \\ &- \chi(\bar{\ell}^{(a-1)} \le i) + 2\chi(\bar{\ell}^{(a)} \le i) - \chi(\bar{\ell}^{(a+1)} \le i) \end{aligned}$$

for $1 \le a \le n-1$ and

$$\begin{aligned} P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet) &- \chi(\ell^{(n-1)} - \frac{1}{2} \le i) - \chi(\ell^{(n-1)} \le i) + 2\chi(\ell^{(n)} \le i) \\ &- \chi(\bar{\ell}^{(n-1)} - \frac{1}{2} \le i) - \chi(\bar{\ell}^{(n-1)} \le i) + 2\chi(\bar{\ell}^{(n)} \le i). \end{aligned}$$

4.3. Bijection algorithm for type $C_n^{(1)}$.

Vacancy numbers.

$$P_i^{(a)}(\nu^\bullet) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \qquad \text{for } i \in \mathbb{Z}_{\geq 0} 1 \leq a < n$$

$$P_i^{(n)}(\nu^\bullet) = Q_i(\nu^{(n-1)}) - Q_i(\nu^{(n)}) \qquad \text{for } i \in 2\mathbb{Z}_{\geq 0}\,. \tag{4.10}$$

Constraints.

$$|\nu^{(a)}| = L - \sum_{b=1}^{a} \lambda_b \qquad \text{for } 1 \leq a \leq n\,. \tag{4.11}$$

Algorithm δ. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If the process has not stopped continue as follows for $a = n-1, n-2, \ldots, 1$ or until stopped. Set $\bar{\ell}^{(n)} = \ell^{(n)}$ and reset $\ell^{(n)} = \bar{\ell}^{(n)} - 1$. If $\ell^{(a)} = \bar{\ell}^{(a+1)}$ set $\bar{\ell}^{(a)} = \ell^{(a)}$ and reset $\ell^{(a)} = \bar{\ell}^{(a)} - 1$. Say case (S) holds. Otherwise find the minimal index $i \geq \bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = \overline{a+1}$. Otherwise set $\bar{\ell}^{(a)} = i$ and continue. If the process does not stop for $a \geq 1$ set $b = \bar{1}$.

Set all undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ for $1 \leq a \leq n$ to ∞.

New RC.

$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - 1 \\ -1 & \text{if } i = \bar{\ell}^{(a)} \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

If two or more conditions hold then all changes should be performed.

If $a = n$ or case (S) holds for $1 \leq a < n$ the partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \bar{\ell}^{(a)} - 2$, and leaving it unchanged otherwise. Otherwise $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Change in vacancy numbers.

$$\begin{aligned} P_i^{(a)}(\tilde{\nu}^\bullet) = P_i^{(a)}(\nu^\bullet) &- \chi(\ell^{(a-1)} \leq i) + 2\chi(\ell^{(a)} \leq i) - \chi(\ell^{(a+1)} \leq i) \\ &- \chi(\bar{\ell}^{(a-1)} \leq i) + 2\chi(\bar{\ell}^{(a)} \leq i) - \chi(\bar{\ell}^{(a+1)} \leq i) \end{aligned} \tag{4.13}$$

for $1 \le a \le n-1$ and

$$P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet) - \chi(\ell^{(n-1)} \le i) - \chi(\bar{\ell}^{(n-1)} \le i) \\ + \chi(\ell^{(n)} \le i) + \chi(\bar{\ell}^{(n)} \le i).$$

4.4. **Bijection algorithm for type $A_{2n}^{(2)}$.** Recall here that $\overline{\mathfrak{g}} = C_n$ and $\mathfrak{g}_{\overline{0}} = B_n$.

Vacancy numbers. The vacancy numbers are the same as for type $C_n^{(1)}$ (4.10) with the only exception that now $i \in \mathbb{Z}_{\ge 0}$ even for $a = n$.

Constraints. The constraints are the same as for type $C_n^{(1)}$ (4.11).

Algorithm δ. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \dots, n$ or until stopped. Find the minimal index $i \ge \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If $\ell^{(n)} = 1$ set $b = \phi$ and stop. Otherwise say case (S) holds for $a = n$ and continue.

If the process has not stopped, set $\bar{\ell}^{(n)} = \ell^{(n)}$ and reset $\ell^{(n)} = \bar{\ell}^{(n)} - 1$. Continue as follows for $a = n-1, n-2, \dots, 1$ or until stopped. If $\ell^{(a)} = \bar{\ell}^{(a+1)}$ set $\bar{\ell}^{(a)} = \ell^{(a)}$ and reset $\ell^{(a)} = \bar{\ell}^{(a)} - 1$. Say case (S) holds. Otherwise find the minimal index $i \ge \bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = \overline{a+1}$. Otherwise set $\bar{\ell}^{(a)} = i$ and continue. If the process does not stop for $a \ge 1$ set $b = \overline{1}$.

Set all undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ for $1 \le a \le n$ to ∞.

New RC. The configuration changes in the same way as for type $C_n^{(1)}$ (4.12).

If case (S) holds for $1 \le a \le n$ the partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \bar{\ell}^{(a)} - 2$, and leaving it unchanged otherwise. Otherwise $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Change in vacancy numbers. The change in the vacancy numbers is the same as for type $C_n^{(1)}$ (4.13).

4.5. **Bijection algorithm for type $A_{2n-1}^{(2)}$.**

Vacancy numbers.

(4.14)

$$P_i^{(a)}(\nu^\bullet) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \qquad \text{for } 1 \le a < n-1$$

$$P_i^{(n-1)}(\nu^\bullet) = Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) + 2Q_i(\nu^{(n)})$$

$$P_i^{(n)}(\nu^\bullet) = Q_i(\nu^{(n-1)}) - 2Q_i(\nu^{(n)}).$$

Constraints.

$$(4.15) \qquad \begin{aligned} |\nu^{(a)}| &= L - \sum_{b=1}^{a} \lambda_b \qquad \text{for } 1 \le a < n \\ |\nu^{(n)}| &= \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b). \end{aligned}$$

Algorithm δ. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n$ or until stopped. Find the minimal index $i \ge \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If the process has not stopped set $\bar{\ell}^{(n)} = \ell^{(n)}$ and continue as follows for $a = n-1, n-2, \ldots, 1$ or until stopped. Find the minimal index $i \ge \bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $P_i^{(a)}(\nu^\bullet)$ in $J^{(a,i)}$). If no such i exists, set $b = \overline{a+1}$ and stop. If the process did not stop, set $b = \bar{1}$.

Set all yet undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ to ∞.

New RC.

$$(4.16) \quad m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - 1 \text{ and } 1 \le a \le n-1 \\ -1 & \text{if } i = \bar{\ell}^{(a)} \text{ and } 1 \le a \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ when $1 \le a \le n$ and $i = \bar{\ell}^{(a)}$ when $1 \le a < n$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ when $1 \le a \le n$ and $i = \bar{\ell}^{(a)} - 1$ when $1 \le a < n$, and leaving it unchanged otherwise.

Change in vacancy numbers.

$$(4.17) \quad \begin{aligned} P_i^{(a)}(\tilde{\nu}^\bullet) = P_i^{(a)}(\nu^\bullet) &- \chi(\ell^{(a-1)} \le i) + 2\chi(\ell^{(a)} \le i) - \chi(\ell^{(a+1)} \le i) \\ &- \chi(\bar{\ell}^{(a-1)} \le i) + 2\chi(\bar{\ell}^{(a)} \le i) - \chi(\bar{\ell}^{(a+1)} \le i) \end{aligned}$$

for $1 \leq a \leq n-1$ and

$$P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet) - \chi(\ell^{(n-1)} \leq i) + 2\chi(\ell^{(n)} \leq i) - \chi(\bar{\ell}^{(n-1)} \leq i).$$

4.6. **Bijection algorithm for type $D_{n+1}^{(2)}$.**

Vacancy numbers.

(4.18)

$$P_i^{(a)}(\nu^\bullet) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \text{for } 1 \leq a \leq n-1$$
$$P_i^{(n)}(\nu^\bullet) = 2Q_i(\nu^{(n-1)}) - 2Q_i(\nu^{(n)}).$$

Constraints. The constraints are the same as for type $C_n^{(1)}$ (4.11).

Algorithm δ. Call a partition quasi-singular if it is not singular and has a part of size $P_i^{(a)}(\nu^\bullet) - 1$.

Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n-1$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If the process has not yet stopped, continue as follows. Consider the following conditions:

(S) $J^{(n,i)}$ is singular and $i > 1$;
(P) $J^{(n,i)}$ is singular and $i = 1$;
(Q) $J^{(n,i)}$ is quasi-singular.

Find the minimal index $i \geq \ell^{(n-1)}$ such that one of the mutually exclusive conditions (S), (P) or (Q) holds. If no such i exists, set $b = n$ and stop. If (P) holds set $\ell^{(n)} = i, b = \phi$ and stop. If (S) holds set $\ell^{(n)} = i-1, \bar{\ell}^{(n)} = i$, say case (S) holds for $a = n$ and continue. If (Q) holds set $\ell^{(n)} = i$. Find the minimal $j > i$ such that (S) holds. If no such j exists, set $b = 0$ and stop. Else set $\bar{\ell}^{(n)} = j$, say case (Q,S) holds and continue.

If the process has not stopped continue in the following fashion for $a = n-1, n-2, \ldots, 1$ or until stopped. If $\ell^{(a)} = \bar{\ell}^{(a+1)}$ set $\bar{\ell}^{(a)} = \ell^{(a)}$ and reset $\ell^{(a)} = \bar{\ell}^{(a)} - 1$. Say case (S) holds for a. Otherwise find the minimal index $i \geq \bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = \overline{a+1}$ and stop. Otherwise set $\bar{\ell}^{(a)} = i$ and continue. If the process did not stop for $a \geq 1$ set $b = \bar{1}$.

Set all undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ for $1 \leq a \leq n$ to ∞.

New RC. The new configuration $\tilde{\nu}^\bullet$ is given by (4.12).

If case (S) holds for $1 \leq a \leq n$ the partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \bar{\ell}^{(a)} - 2$, and leaving it unchanged otherwise. If (Q) or (Q,S) holds for $a = n$, then $\tilde{J}^{(n,i)}$ is obtained from $J^{(n,i)}$ by removing a part of size $P_i^{(n)}(\nu^\bullet) - 1$ (resp. $P_i^{(n)}(\nu^\bullet)$) for $i = \ell^{(n)}$ (resp. $i = \bar{\ell}^{(n)}$), adding a part of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ (resp. $P_i^{(n)}(\tilde{\nu}^\bullet)-1$) for $i=\ell^{(n)}-1$ (resp. $i=\bar{\ell}^{(n)}-1$), and leaving it unchanged otherwise. Otherwise $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i=\ell^{(a)}$ and $i=\bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i=\ell^{(a)}-1$ and $i=\bar{\ell}^{(a)}-1$, and leaving it unchanged otherwise.

Change in vacancy numbers.

$$(4.19)\quad \begin{aligned} P_i^{(a)}(\tilde{\nu}^\bullet) = P_i^{(a)}(\nu^\bullet) &- \chi(\ell^{(a-1)}\le i) + 2\chi(\ell^{(a)}\le i) - \chi(\ell^{(a+1)}\le i)\\ &- \chi(\bar{\ell}^{(a-1)}\le i) + 2\chi(\bar{\ell}^{(a)}\le i) - \chi(\bar{\ell}^{(a+1)}\le i)\end{aligned}$$

for $1\le a\le n-1$ and

$$\begin{aligned} P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet) &- 2\chi(\ell^{(n-1)}\le i) + 2\chi(\ell^{(n)}\le i)\\ &- 2\chi(\bar{\ell}^{(n-1)}\le i) + 2\chi(\bar{\ell}^{(n)}\le i).\end{aligned}$$

4.7. Bijection algorithm for type $A_{2n}^{(2)\dagger}$.

Vacancy numbers. The vacancy numbers are given by the same formula as for type $C_n^{(1)}$ (4.10) with the only exception that in this case $i\in\mathbb{Z}_{\ge 0}$ for all $a\in I^*$.

Algorithm δ. If $a=n$ and i is odd, then $J^{(n,i)}$ is never singular. For i odd, call $J^{(n,i)}$ quasi-singular if it has a part of size $P_i^{(n)}(\nu^\bullet)-1/2$.

Set $\ell^{(0)}=0$ and repeat the following process for $a=1,2,\dots,n-1$ or until stopped. Find the minimal index $i\ge\ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b=a$ and stop. Otherwise set $\ell^{(a)}=i$ and continue.

If the process has not yet stopped, continue as follows. Consider the conditions

(S) i is even and $J^{(n,i)}$ is singular;
(Q) i is odd and $J^{(n,i)}$ is quasi-singular.

Find the minimal index $i\ge\ell^{(n-1)}$ such that one of the mutually exclusive conditions (S) or (Q) holds. If no such i exists, set $b=n$ and stop. If (S) holds set $\ell^{(n)}=i-1, \bar{\ell}^{(n)}=i$, say case (S) holds for $a=n$ and continue. If (Q) holds set $\ell^{(n)}=i$. Find the minimal $j>i$ such that (S) holds for j. If no such j exists, set $b=0$ and stop. Else set $\bar{\ell}^{(n)}=j$, say case (Q,S) holds and continue.

If the process has not stopped continue in the following fashion for $a=n-1,n-2,\dots,1$ or until stopped. If $\ell^{(a)}=\bar{\ell}^{(a+1)}$ set $\bar{\ell}^{(a)}=\ell^{(a)}$ and reset $\ell^{(a)}=\bar{\ell}^{(a)}-1$. Say case (S) holds for a. Otherwise find the minimal index $i\ge\bar{\ell}^{(a+1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b=\overline{a+1}$ and stop. Otherwise set $\bar{\ell}^{(a)}=i$ and continue. If the process did not stop for $a\ge 1$ set $b=\bar{1}$.

Set all undefined $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ for $1\le a\le n$ to ∞.

New RC. The new configuration $\tilde{\nu}^\bullet$ is given by (4.12).

If case (S) holds for $1 \le a \le n$ the partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \bar{\ell}^{(a)} - 2$, and leaving it unchanged otherwise.

If (Q) or (Q,S) holds for $a = n$, then $\tilde{J}^{(n,i)}$ is obtained from $J^{(n,i)}$ by removing a part of size $P_i^{(n)}(\nu^\bullet) - 1/2$ for $i = \ell^{(n)}$ (and a part of size $P_i^{(n)}(\nu^\bullet)$ for $i = \bar{\ell}^{(n)} < \infty$), adding a part of size $P_i^{(n)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(n)} - 1$ (and a part of size $P_i^{(n)}(\tilde{\nu}^\bullet) - 1/2$ for $i = \bar{\ell}^{(n)} - 1 < \infty$), and leaving it unchanged otherwise.

Otherwise $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \bar{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Change in vacancy numbers. The vacancy numbers $P_i^{(a)}(\nu^\bullet)$ change as in (4.13).

5. Proof of Theorem 3.2

In the following subsections Theorem 3.2 is proved case-by-case for the various root systems. The following notation is used. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, \mu)$, $b = \mathrm{rk}(\nu^\bullet, J^\bullet) \in B$, $\rho = \lambda - \mathrm{wt}(b)$, and $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$. There are three things that must be verified:

(I) ρ is dominant and b can be appended to any path in $\mathcal{P}(\rho, \tilde{\mu})$ to give an element of $\mathcal{P}(\lambda, \mu)$.

(II) $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \in \mathrm{RC}(\rho, \tilde{\mu})$ where $B^{\tilde{\mu}} = B^{\otimes(L-1)}$.

(III) The conditions of Lemma 5.1 are satisfied.

Parts (I) and (II) show that δ is well-defined. The proof that δ has an inverse, is omitted as it is very similar to the proof of well-definedness. Part (III) suffices to prove that $\tilde{\Phi}$ preserves statistics.

For $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, \mu)$, define $\Delta(cc(\nu^\bullet, J^\bullet)) = cc(\nu^\bullet, J^\bullet) - cc(\delta'(\nu^\bullet, J^\bullet))$ and $\Delta^2(cc(\nu^\bullet, J^\bullet)) = \Delta(cc(\nu^\bullet, J^\bullet)) - \Delta(cc(\delta'(\nu^\bullet, J^\bullet)))$ where $\delta' = \mathrm{comp} \circ \delta \circ \mathrm{comp}$.

Lemma 5.1. *To prove that* (3.13) *holds, it suffices to show that it holds for $L = 1$, and that for $L \ge 2$ with $\tilde{\Phi}(\nu^\bullet, J^\bullet) = b_L \otimes \cdots \otimes b_1$, we have*

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \frac{t_1^\vee}{a_0^\vee} \alpha_1^{(1)} - \chi(b_L = \phi), \tag{5.1}$$

and

$$\overline{H}(b_L \otimes b_{L-1}) = \frac{t_1^\vee}{a_0^\vee} (\alpha_1^{(1)} - \tilde{\alpha}_1^{(1)}) - \chi(b_L = \phi) + \chi(b_{L-1} = \phi) \tag{5.2}$$

where $\alpha_1^{(1)}$ and $\tilde{\alpha}_1^{(1)}$ are the lengths of the first columns in $\nu^{(1)}$ and $\tilde{\nu}^{(1)}$ respectively, and $\delta(\nu^\bullet, J^\bullet) = (\tilde{\nu}^\bullet, \tilde{J}^\bullet)$.

Proof. If $L = 0$, $\mathrm{RC}(\lambda,\mu)$ and $\mathcal{P}(\lambda,\mu)$ are both empty unless $\lambda = 0$, in which case $\mathrm{RC}(\lambda,\mu)$ (resp. $\mathcal{P}(\lambda,\mu)$) is the singleton set containing the empty rigged configuration (resp. the empty path). Both of these objects have statistic zero. The case $L = 1$ is given by hypothesis. For $L \geq 2$, by the definition (2.18) and (2.20) of $\overline{D}$,

(5.3)
$$\overline{D}(b_L \otimes \cdots \otimes b_1) - \overline{D}(b_{L-1} \otimes \cdots \otimes b_1) = \overline{H}(b_1 \otimes b^\natural) + \sum_{j=1}^{L-1} \overline{H}(b_{j+1} \otimes b_j).$$

Therefore by induction on L it suffices to prove that $\Delta(cc(\nu^\bullet, J^\bullet))$ is given by the right hand side of (5.3). By induction and again "taking the difference" it suffices to prove that

$$\Delta^2(cc(\nu^\bullet, J^\bullet)) = \overline{H}(b_L \otimes b_{L-1}).$$

But this follows from (5.1) and (5.2). □

We also need several preliminary lemmas on the convexity and nonnegativity of the vacancy numbers $P_i^{(a)}(\nu^\bullet)$.

Lemma 5.2. *For large i, we have*

$$P_i^{(a)}(\nu^\bullet) = \lambda_a - \lambda_{a+1} \qquad \text{for } 1 \leq a < n$$
$$P_i^{(n)}(\nu^\bullet) = \begin{cases} 2\lambda_n & \text{for } B_n^{(1)},\ D_{n+1}^{(2)} \\ \lambda_n & \text{for } C_n^{(1)},\ A_{2n}^{(2)},\ A_{2n}^{(2)\dagger},\ A_{2n-1}^{(2)} \\ \lambda_{n-1} + \lambda_n & \text{for } D_n^{(1)}. \end{cases}$$

Proof. This follows from the formulas for the vacancy numbers (4.2), (4.6), (4.10), (4.14), (4.18), the constraints (4.3), (4.7), (4.11), (4.15), and the fact that for large i, $Q_i(\nu^{(a)}) = |\nu^{(a)}|$. □

Direct calculations show that

(5.4)

Type $D_n^{(1)}$

$$\begin{aligned} &- P_{i-1}^{(a)}(\nu^\bullet) + 2P_i^{(a)}(\nu^\bullet) - P_{i+1}^{(a)}(\nu^\bullet) \\ &= \begin{cases} m_i^{(a-1)}(\nu^\bullet) - 2m_i^{(a)}(\nu^\bullet) + m_i^{(a+1)}(\nu^\bullet) + L\delta_{a,1}\delta_{i,1} & \text{for } 1 \leq a \leq n-3 \\ m_i^{(n-3)}(\nu^\bullet) - 2m_i^{(n-2)}(\nu^\bullet) + m_i^{(n-1)}(\nu^\bullet) + m_i^{(n)}(\nu^\bullet) & \text{for } a = n-2 \\ m_i^{(n-2)}(\nu^\bullet) - 2m_i^{(n)}(\nu^\bullet) & \text{for } a = n-1, n. \end{cases} \end{aligned}$$

(5.5)

Type $B_n^{(1)}$

$$-P_{i-v_a}^{(a)}(\nu^\bullet)+2P_i^{(a)}(\nu^\bullet)-P_{i+v_a}^{(a)}(\nu^\bullet)$$
$$=\begin{cases} m_i^{(a-1)}(\nu^\bullet)-2m_i^{(a)}(\nu^\bullet)+m_i^{(a+1)}(\nu^\bullet)+L\delta_{a,1}\delta_{i,1} & \text{for } 1\le a\le n-2\\ m_i^{(n-2)}(\nu^\bullet)-2m_i^{(n-1)}(\nu^\bullet) & \\ \quad +2(2m_i^{(n)}(\nu^\bullet)+m_{i+\frac12}^{(n)}(\nu^\bullet)+m_{i-\frac12}^{(n)}(\nu^\bullet)) & \text{for } a=n-1\\ 2m_i^{(n-1)}(\nu^\bullet)-4m_i^{(n)}(\nu^\bullet) & \text{for } a=n. \end{cases}$$

(5.6)

Type $C_n^{(1)}$

$$-P_{i-v_a}^{(a)}(\nu^\bullet)+2P_i^{(a)}(\nu^\bullet)-P_{i+v_a}^{(a)}(\nu^\bullet)$$
$$=\begin{cases} m_i^{(a-1)}(\nu^\bullet)-2m_i^{(a)}(\nu^\bullet)+m_i^{(a+1)}(\nu^\bullet)+L\delta_{a,1}\delta_{i,1} & \\ & \text{for } 1\le a\le n-1\\ m_{i-1}^{(n-1)}(\nu^\bullet)+2m_i^{(n-1)}(\nu^\bullet)+m_{i+1}^{(n-1)}(\nu^\bullet)-2m_i^{(n)}(\nu^\bullet) & \\ & \text{for } a=n. \end{cases}$$

(5.7)

Types $A_{2n}^{(2)}$ and $A_{2n}^{(2)\dagger}$

$$-P_{i-1}^{(a)}(\nu^\bullet)+2P_i^{(a)}(\nu^\bullet)-P_{i+1}^{(a)}(\nu^\bullet)$$
$$=\begin{cases} m_i^{(a-1)}(\nu^\bullet)-2m_i^{(a)}(\nu^\bullet)+m_i^{(a+1)}(\nu^\bullet)+L\delta_{a,1}\delta_{i,1} & \text{for } 1\le a\le n-1\\ m_i^{(n-1)}(\nu^\bullet)-m_i^{(n)}(\nu^\bullet) & \text{for } a=n. \end{cases}$$

(5.8)

Type $A_{2n-1}^{(2)}$

$$-P_{i-1}^{(a)}(\nu^\bullet)+2P_i^{(a)}(\nu^\bullet)-P_{i+1}^{(a)}(\nu^\bullet)$$
$$=\begin{cases} m_i^{(a-1)}(\nu^\bullet)-2m_i^{(a)}(\nu^\bullet)+m_i^{(a+1)}(\nu^\bullet)+L\delta_{a,1}\delta_{i,1} & \text{for } 1\le a< n-1\\ m_i^{(n-2)}(\nu^\bullet)-2m_i^{(n-1)}(\nu^\bullet)+2m_i^{(n)}(\nu^\bullet) & \text{for } a=n-1\\ m_i^{(n-1)}(\nu^\bullet)-2m_i^{(n)}(\nu^\bullet) & \text{for } a=n. \end{cases}$$

(5.9)

Type $D_{n+1}^{(2)}$

$$
\begin{aligned}
&-P_{i-1}^{(a)}(\nu^\bullet)+2P_i^{(a)}(\nu^\bullet)-P_{i+1}^{(a)}(\nu^\bullet)\\
&=\begin{cases} m_i^{(a-1)}(\nu^\bullet)-2m_i^{(a)}(\nu^\bullet)+m_i^{(a+1)}(\nu^\bullet)+L\delta_{a,1}\delta_{i,1} & \text{for } 1\le a\le n-1\\ 2m_i^{(n-1)}(\nu^\bullet)-2m_i^{(n)}(\nu^\bullet) & \text{for } a=n.\end{cases}
\end{aligned}
$$

In particular these equations imply the convexity condition

$$
P_i^{(a)}(\nu^\bullet)\ge\frac{1}{2}(P_{i-v_a}^{(a)}(\nu^\bullet)+P_{i+v_a}^{(a)}(\nu^\bullet))\qquad \text{if } m_i^{(a)}(\nu^\bullet)=0. \tag{5.10}
$$

Lemma 5.3. *Let $\nu^\bullet$ be a configuration in $C(\lambda,\mu)$. The following are equivalent:*

1. $P_i^{(a)}(\nu^\bullet)\ge 0$ *for all* $i\in v_a\mathbb{Z}_{>0}$, $a\in I^*$;
2. $P_i^{(a)}(\nu^\bullet)\ge 0$ *for all* $i\in v_a\mathbb{Z}_{>0}$, $a\in I^*$ *such that* $m_i^{(a)}(\nu^\bullet)>0$.

Proof. This follows immediately from Lemma 5.2 and the convexity condition (5.10). (See also [11, Lemma 10]). □

5.1. Proof for type $D_n^{(1)}$.

Proof of (I) for $D_n^{(1)}$. Here it suffices to show that ρ satisfies (2.13). Suppose not. If $b=k$ with $1\le k\le n$ then

(a) $\lambda_k=\lambda_{k+1}$ if $1\le k\le n-2$
(b) $\lambda_{n-1}=|\lambda_n|$ if $k=n-1$
(c) $\lambda_{n-1}=-\lambda_n$ if $k=n$.

In case (a) we have $P_i^{(k)}(\nu^\bullet)=0$ for large i by Lemma 5.2. Let ℓ be the largest part in $\nu^{(k)}$. By convexity this implies $P_i^{(k)}(\nu^\bullet)=0$ for all $i\ge\ell$. Equation (5.4) in turn yields $m_i^{(k-1)}(\nu^\bullet)=0$ for all $i>\ell$ so that $1\le\ell^{(k-1)}\le\ell$. But this is a contradiction since there is a singular string of length ℓ in $(\nu^\bullet,J^\bullet)^{(k)}$ since $P_\ell^{(k)}(\nu^\bullet)=0$ and $m_\ell^{(k)}(\nu^\bullet)>0$ so that we would have $\mathrm{rk}(\nu^\bullet,J^\bullet)>k$. In case (b) let us first assume that $\lambda_{n-1}=\lambda_n$. Then for large i, $P_i^{(n-1)}(\nu^\bullet)=0$ and by convexity $P_i^{(n-1)}(\nu^\bullet)=0$ for $i\ge\ell$ where ℓ is the largest part in $\nu^{(n-1)}$. By (5.4) we have $m_i^{(n-2)}(\nu^\bullet)=0$ for $i>\ell$. Hence $1\le\ell^{(n-2)}\le\ell$ which yields a contradiction since there is a singular string of length ℓ in $(\nu^\bullet,J^\bullet)^{(n-1)}$ so that $\mathrm{rk}(\nu^\bullet,J^\bullet)\ne n-1$. If $\lambda_{n-1}=-\lambda_n$ the same argument goes through with $n-1$ replaced by n. The case (c) is analogous to the second part of case (b).

Now suppose $b=\overline{k}$ for some $1\le k\le n$. We show again that ρ not dominant will yield a contradiction. If ρ is not dominant one of the following has to be true:

(d) $\lambda_k = \lambda_{k-1}$ if $2 \leq k \leq n-1$

(e) $\lambda_n = \lambda_{n-1}$ if $k = n$.

Case (e) is analogous to case (b). In case (d) some caution is in order. By lemma 5.2 and convexity (5.10) we have $P_i^{(k-1)}(\nu^\bullet) = 0$ for $i \geq \ell$ where ℓ is the largest part in $\nu^{(k-1)}$. By (5.4) it follows that $m_i^{(k)}(\nu^\bullet) = 0$ for $i > \ell$. Hence $\bar{\ell}^{(k)} \leq \ell$. Since $P_\ell^{(k-1)}(\nu^\bullet) = 0$ and $m_\ell^{(k-1)}(\nu^\bullet) > 0$ there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(k-1)}$. Hence $\bar{\ell}^{(k-1)} \leq \ell$ unless $\ell^{(k-1)} = \ell$ and $m_\ell^{(k-1)}(\nu^\bullet) = 1$. We will show that the latter case cannot occur. Equation (5.4) with $a = k-1$ and $i = \ell$ implies that $P_{\ell-1}^{(k-1)} = 0$ and $m_\ell^{(k-2)}(\nu^\bullet) = 0$ since by assumption $\ell^{(k-1)} = \ell^{(k)} = \bar{\ell}^{(k)} = \ell$ and hence $m_\ell^{(k)}(\nu^\bullet) \geq 2$ (or $m_\ell^{(n-1)}(\nu^\bullet) \geq 1$ and $m_\ell^{(n)}(\nu^\bullet) \geq 1$ for $k = n-1$). However this implies that $m_{\ell-1}^{(n-2)}(\nu^\bullet) = 0$ since otherwise $\ell^{(k-1)} \leq \ell - 1$ and not ℓ since there is a singular string of length $\ell - 1$ in $(\nu^\bullet, J^\bullet)^{(k-1)}$. Now by induction on $i = \ell-1, \ell-2, \ldots, 1$ it follows from (5.4) at $a = k-1$ that $P_i^{(k-1)}(\nu^\bullet) = m_i^{(k-2)}(\nu^\bullet) = m_i^{(k-1)}(\nu^\bullet) = 0$. However, this means in particular that $m_i^{(k-2)}(\nu^\bullet) = 0$ for all $1 \leq i \leq \ell$ so that $\ell^{(k-2)} > \ell$ which contradicts $\ell^{(k-1)} = \ell$. □

Proof of (II) for $D_n^{(1)}$. Denote by $J_{\max}^{(a,i)}(\nu^\bullet, J^\bullet)$ the biggest part in $J^{(a,i)}$. To prove admissibility of $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ we need to show for all $i \geq 1, 1 \leq a \leq n$ that

$$0 \leq J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \leq P_i^{(a)}(\tilde{\nu}^\bullet). \tag{5.11}$$

Fix $a \geq 1$. Only one string of size $\ell^{(a)}$ and one string of size $\bar{\ell}^{(a)}$ change in the transformation $(\nu^\bullet, J^\bullet)^{(a)} \to (\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(a)}$. Hence

$$\begin{aligned} J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) &= P_i^{(a)}(\tilde{\nu}^\bullet) && \text{for } i = \ell^{(a)} - 1 \text{ and } i = \bar{\ell}^{(a)} - 1 \\ 0 \leq J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) &\leq J_{\max}^{(a,i)}(\nu^\bullet, J^\bullet) && \text{else.} \end{aligned}$$

Hence by (4.5) the inequality (5.11) can only be violated when $\ell^{(a-1)} \leq i < \ell^{(a)}$ or $\bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)}$ where $\bar{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$. By the construction of $\ell^{(a)}$ and $\bar{\ell}^{(a)}$ there are no singular strings of length i in $(\nu^\bullet, J^\bullet)^{(a)}$ for $\ell^{(a-1)} \leq i < \ell^{(a)}$ or $\bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)}$. This means that $J_{\max}^{(a,i)}(\nu^\bullet, J^\bullet) \leq P_i^{(a)}(\nu^\bullet) - 1$ if i occurs as a part in $\nu^{(a)}$, that is $m_i^{(a)}(\nu^\bullet) > 0$. Hence (5.11) is fulfilled for these i. It remains to prove that $P_i^{(a)}(\tilde{\nu}^\bullet) \geq 0$ for all i such that $m_i^{(a)}(\nu^\bullet) = 0$ and $\ell^{(a-1)} \leq i < \ell^{(a)}$ or $\bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)}$. Note that $m_i^{(a)}(\tilde{\nu}^\bullet) = 0$ if $m_i^{(a)}(\nu^\bullet) = 0$ for $\ell^{(a-1)} \leq i < \ell^{(a)} - 1$ or $\bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)} - 1$. Hence by lemma 5.3 it suffices to prove (5.11) for all a and i such that $m_i^{(a)}(\tilde{\nu}^\bullet) > 0$. Therefore the only remaining case for which (5.11) might be violated occurs when

For $1 \leq a \leq n-2$:

$m_{\ell-1}^{(a)}(\nu^\bullet) = 0$, $P_{\ell-1}^{(a)}(\nu^\bullet) = 0$, $\ell^{(a-1)} < \ell$ (resp. $\bar{\ell}^{(a+1)} < \ell$) and ℓ finite where $\ell = \ell^{(a)}$ (resp. $\ell = \bar{\ell}^{(a)}$)

For $a = n-1, n$:

$m_{\ell-1}^{(a)}(\nu^\bullet) = 0$, $P_{\ell-1}^{(a)}(\nu^\bullet) = 0$, $\ell^{(n-2)} < \ell$ and ℓ finite where $\ell = \ell^{(a)}$.

We show that these conditions cannot be met simultaneously. Let $p < \ell$ be maximal such that $m_p^{(a)}(\nu^\bullet) > 0$; if no such p exists set $p = 0$. By (5.10) $P_{\ell-1}^{(a)}(\nu^\bullet) = 0$ is only possible if $P_i^{(a)}(\nu^\bullet) = 0$ for all $p \leq i \leq \ell$. By (5.4) we find that $m_i^{(a-1)}(\nu^\bullet) = 0$ (resp. $m_i^{(a+1)}(\nu^\bullet) = 0$) for $p < i < \ell$. Since $\ell^{(a-1)} < \ell$ (resp. $\bar{\ell}^{(a+1)} < \ell$) this implies that $\ell^{(a-1)} \leq p$ (resp. $\bar{\ell}^{(a+1)} \leq p$). If $p = 0$ this contradicts the condition $\ell^{(a-1)} \geq 1$ (resp. $\bar{\ell}^{(a+1)} \geq 1$). Hence assume that $p > 0$. Since $P_p^{(a)}(\nu^\bullet) = 0$ and $m_p^{(a)}(\nu^\bullet) > 0$ there is a singular string of length p in $(\nu^\bullet, J^\bullet)^{(a)}$ and therefore $\ell^{(a)} = p$ (resp. $\bar{\ell}^{(a)} = p$). However, this contradicts $p < \ell$. This concludes the proof that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ is well-defined. □

Proof of (III) for $D_n^{(1)}$. Here $b^\natural = 1$, $\overline{H}(b \otimes b') = 0$ if $b \leq b'$, $\overline{H}(b \otimes b') = 1$ if $b \otimes b' = n \otimes \overline{n}, \overline{n} \otimes n$ or $b > b'$ where $b \neq \overline{1}$, $b' \neq 1$, and $H(\overline{1} \otimes 1) = 2$.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \alpha_1^{(1)} \tag{5.12}$$

$$\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) + \chi(\bar{\ell}^{(1)} = 1) \tag{5.13}$$

where $\ell^{(i)}$ and $\bar{\ell}^{(i)}$ are determined by the algorithm δ.

Let $\tilde{\ell}^{(a)}$ and $\tilde{\bar{\ell}}^{(a)}$ be the length of the selected strings defined by the algorithm δ on $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$. To check (5.13) note that if $\ell^{(1)} = 1$ it follows from (4.5) that $\tilde{\ell}^{(a)} \geq \ell^{(a+1)}$ for $1 \leq a \leq n-2$. Hence if $b_L \leq n-1$ then $b_{L-1} < b_L$ and both sides of (5.13) yield 1. If $b_L = n$ then $b_{L-1} \leq n-1$ or $b_{L-1} = \overline{n}$ and both sides of (5.13) are 1. Similarly, if $b_L = \overline{n}$ then $b_{L-1} \leq n$ and both sides if (5.13) are 1. Finally, if $b_L \geq \overline{n-1}$ then $\tilde{\bar{\ell}}^{(a)} \geq \bar{\ell}^{(a-1)}$ and $b_{L-1} < b_L$. If $b_L < \overline{1}$ then both sides of (5.13) are 1. If $b_L = \overline{1}$ and $\bar{\ell}^{(1)} = 1$ then $\tilde{\ell}^{(1)} = \infty$ and hence $b_{L-1} = 1$. In this case both sides of (5.13) are 2. If $b_L = \overline{1}$ and $\bar{\ell}^{(1)} > 1$ then there is a singular string in $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(1)}$ so that $b_{L-1} > 1$. In this case both sides of (5.13) are 1. If $\ell^{(1)} > 1$ then $\tilde{\ell}^{(a)} < \ell^{(a)}$ for $1 \leq a \leq n-2$ and the cases can be checked in a similar fashion as before.

To prove (5.12), by (3.4) and (4.4) we have

$$cc(\tilde{\nu}^\bullet) = \frac{1}{2} \sum_{i,j\geq 1} \sum_{a,b=1}^{n} \min(i,j)(\alpha_a|\alpha_b)$$
$$\times \left(m_i^{(a)} - \delta_{i,\ell^{(a)}} + \delta_{i,\ell^{(a)}-1} - \chi(a \leq n-2)(\delta_{i,\bar{\ell}^{(a)}} - \delta_{i,\bar{\ell}^{(a)}-1})\right)$$
$$\times \left(m_j^{(b)} - \delta_{j,\ell^{(b)}} + \delta_{j,\ell^{(b)}-1} - \chi(b \leq n-2)(\delta_{j,\bar{\ell}^{(b)}} - \delta_{j,\bar{\ell}^{(b)}-1})\right).$$

Applying the data for D_n and using (4.5), a tedious but straightforward calculation yields

$$\Delta cc(\nu^\bullet) = \sum_{a=1}^{n} \sum_{i\geq 1} \left(P_i^{(a)}(\tilde{\nu}^\bullet) - P_i^{(a)}(\nu^\bullet)\right)$$
$$\times \left(m_i^{(a)} - \delta_{i,\ell^{(a)}} - \chi(a \leq n-2)\delta_{i,\bar{\ell}^{(a)}}\right) + \sum_{i\geq 1} m_i^{(1)}.$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\Delta|J^\bullet| = \sum_{a=1}^{n} \sum_{i\geq 1} \left(P_i^{(a)}(\nu^\bullet) - P_i^{(a)}(\tilde{\nu}^\bullet)\right)$$
$$\times \left(m_i^{(a)} - \delta_{i,\ell^{(a)}} - \chi(a \leq n-2)\delta_{i,\bar{\ell}^{(a)}}\right).$$

Hence altogether, using $\sum_{i\geq 1} m_i^{(1)} = \alpha_1^{(1)}$, we obtain (5.12). □

5.2. Proof for type $B_n^{(1)}$.

Proof of (I) for $B_n^{(1)}$. Let us assume that either ρ is not dominant, or that $b = 0$ (so that $\rho = \lambda$) and $\lambda_n = 0$. For $b = k$ with $1 \leq k < n$ the proof that this cannot happen is the same as for type $D_n^{(1)}$. Now assume that $b = n$ and $\lambda_n = 0$. Then $P_i^{(n)}(\nu^\bullet) = 0$ for $i \geq \ell$ where ℓ is the largest part in $\nu^{(n)}$ by Lemma 5.2 and (5.10). By (5.5) with $a = n$ we find that $m_i^{(n-1)}(\nu^\bullet) = 0$ for $i > \ell$, so that $\ell^{(n-1)} \leq \ell$. But there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\ell^{(n)} = \infty$. Next assume that $b = 0$ and $\lambda_n = 0$. By the same arguments as in the previous case $\ell^{(n-1)} \leq \ell$. But there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(n)}$ since $m_\ell^{(n)}(\nu^\bullet) > 0$ and $P_\ell^{(n)}(\nu^\bullet) = 0$. Since (Q) must hold for $b = 0$, there must be a singular string at $\ell^{(n-1)} - \frac{1}{2}$ or a quasisingular string at $\ell^{(n-1)} \leq i < \ell$. But then (S) holds for ℓ which contradicts $b = 0$. The case $b = \bar{k}$ with $1 \leq k \leq n$ is the same as for type $D_n^{(1)}$. □

Proof of (II) for $B_n^{(1)}$. Denote by $J_{\max}^{(a,i)}(\nu^\bullet, J^\bullet)$ the biggest part in $J^{(a,i)}$. To prove admissibility of $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ we need to show for all $i \geq 1, 1 \leq a \leq n$

that

$$0 \le J^{(a,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \le P^{(a)}_i(\tilde{\nu}^\bullet). \tag{5.14}$$

Up to small alterations, the proof of (5.14) for $1 \le a < n$ is the same as for type $D^{(1)}_n$. Let us assume that $a = n$. Only one string of size $\ell^{(n)}$ and one string of size $\bar{\ell}^{(n)}$ change in the transformation $(\nu^\bullet, J^\bullet)^{(n)} \to (\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(n)}$. Hence for the different cases:

$$\begin{array}{rll}
(S) & J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = P^{(n)}_i(\tilde{\nu}^\bullet) & \text{for } i = \bar{\ell}^{(n)} - 1 \\
& 0 \le J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \le J^{(n,i)}_{\max}(\nu^\bullet, J^\bullet)\text{else} & \\
(Q) & J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = P^{(n)}_i(\tilde{\nu}^\bullet) & \text{for } i = \ell^{(n)} - 1/2 \\
& 0 \le J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \le J^{(n,i)}_{\max}(\nu^\bullet, J^\bullet)\text{else} & \\
(Q,S) & J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = P^{(n)}_i(\tilde{\nu}^\bullet) & \text{for } i = \ell^{(n)} - 1/2 \\
& J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = P^{(n)}_i(\tilde{\nu}^\bullet) & \text{for } i = \bar{\ell}^{(n)} - 1/2,\ \bar{\ell}^{(n)} = \bar{\ell}^{(n-1)} \\
& J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = P^{(n)}_i(\tilde{\nu}^\bullet) - 1 & \text{for } i = \bar{\ell}^{(n)} - 1/2,\ \bar{\ell}^{(n)} < \bar{\ell}^{(n-1)} \\
& 0 \le J^{(n,i)}_{\max}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \le J^{(n,i)}_{\max}(\nu^\bullet, J^\bullet)\text{else}\,. &
\end{array}$$

Let us first assume that (S) holds:
By the definition of $\ell^{(n)}$ and $\bar{\ell}^{(n)}$ there is no singular string at $\ell^{(n-1)} - \frac{1}{2}$ and no singular or quasisingular string of length $\ell^{(n-1)} \le i < \bar{\ell}^{(n)} = \ell$. Hence, if $m^{(n)}_i(\nu^\bullet) > 0$, we have $J^{(n,i)}_{\max}(\nu^\bullet, J^\bullet) \le P^{(n)}_i(\nu^\bullet) - 2$ for $\ell^{(n-1)} \le i < \ell$ and $J^{(n,i)}_{\max}(\nu^\bullet, J^\bullet) \le P^{(n)}_i(\nu^\bullet) - 1$ for $i = \ell^{(n-1)} - \frac{1}{2}$. Hence (5.14) holds if $m^{(n)}_i(\nu^\bullet) > 0$. By lemma 5.3, (5.14) can only be violated if

$$m^{(n)}_{\ell-1}(\nu^\bullet) = 0, \quad P^{(n)}_{\ell-1}(\nu^\bullet) = 0 \text{ or } 1, \quad \ell^{(n-1)} \le \ell - 1, \quad \ell \text{ finite}.$$

The case $P^{(n)}_{\ell-1}(\nu^\bullet) = 0$ is the same as before. Hence assume that $P^{(n)}_{\ell-1}(\nu^\bullet) = 1$. If $m_{\ell-\frac{1}{2}}(\nu^\bullet) = 0$, then by (5.5) and (5.10) $P^{(n)}_i(\nu^\bullet) = 1$ for $p \le i < \ell$ where $p < \ell$ is maximal such that $m^{(n)}_p(\nu^\bullet) > 0$. By (5.5) we also have $m^{(n-1)}_i(\nu^\bullet) = 0$ for $p < i < \ell$ so that $\ell^{(n-1)} \le p$. But since $P^{(n)}_p(\nu^\bullet) = 1$ and $m^{(n)}_p(\nu^\bullet) > 0$ there is a (quasi)singular string of length p in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $p < \ell$. If $m^{(n)}_{\ell-\frac{1}{2}}(\nu^\bullet) > 0$, then $P^{(n)}_{\ell-\frac{1}{2}}(\nu^\bullet) \ge 2$ since otherwise there would be a (quasi)singular string of length $\ell - \frac{1}{2}$ in $(\nu^\bullet, J^\bullet)^{(n)}$. By convexity (5.10) and (5.5) this implies $P^{(n)}_{\ell-\frac{3}{2}}(\nu^\bullet) = 0$ and $m_{\ell-\frac{3}{2}}(\nu^\bullet) > 0$. Since $\ell^{(n-1)} \le \ell - 1$, (Q) would hold for $\ell - \frac{3}{2}$ which contradicts our assumptions.

One more problem might occur when $\bar{\ell}^{(n-1)} = \bar{\ell}^{(n)} = \ell$ and $m_{\ell-\frac{1}{2}}(\nu^\bullet) > 0$, $J_{\max}^{(n,\ell-\frac{1}{2})}(\nu^\bullet, J^\bullet) = P_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet)$. But in this case there is a singular string of length $\ell - \frac{1}{2}$ in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\bar{\ell}^{(n)} = \ell$.

Now assume that (Q) holds:
In this case $\bar{\ell}^{(n-1)} = \bar{\ell}^{(n)} = \infty$. By similar arguments as before (5.14) can only be violated if

$$m_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 0, \quad P_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 0, \quad \ell^{(n-1)} - \frac{1}{2} < \ell^{(n)} < \ell^{(n)} = \ell, \quad \ell \text{ finite}$$

$$\text{or} \quad m_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 0, \quad P_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 1, \quad \ell^{(n-1)} < \ell^{(n)} < \ell^{(n)} = \ell, \quad \ell \text{ finite}.$$

Since (Q) holds, we must have $P_\ell^{(n)}(\nu^\bullet) \geq 1$. Hence by convexity (5.10) it follows that $P_i^{(n)}(\nu^\bullet) = 1$ for $p \leq i < \ell$ where $p < \ell$ is maximal such that $m_p^{(n)}(\nu^\bullet) > 0$. Equation (5.5) implies that $m_i^{(n-1)}(\nu^\bullet) = 0$ for $p < i < \ell$ so that $\ell^{(n-1)} \leq p$. But there is a (quasi)singular string of length p in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $p < \ell$.

Finally assume that (Q,S) holds:
For $i < \ell^{(n)}$ the same arguments hold as for case (Q). Since by definition there are no singular strings of length $\ell^{(n-1)} < i < \bar{\ell}^{(n)} = \ell$ in $(\nu^\bullet, J^\bullet)^{(n)}$, case (Q) holds for $i = \ell^{(n-1)}$ and $m_{\ell^{(n)}}^{(n)}(\nu^\bullet) > 0$, the only problem occurs when

$$m_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 0, \quad P_{\ell-\frac{1}{2}}^{(n)}(\nu^\bullet) = 0, \quad \ell^{(n)} + 1 \leq \ell, \quad \ell \text{ finite}.$$

If $p < \ell$ is maximal such that $m_p^{(n)}(\nu^\bullet) > 0$, then by (5.5) and (5.10) $P_i^{(n)}(\nu^\bullet) = 0$ for $p \leq i \leq \ell$. Since $m_{\ell^{(n)}}^{(n)}(\nu^\bullet) > 0$ and $P_{\ell^{(n)}}^{(n)}(\nu^\bullet) > 0$ we must have $\ell^{(n)} < p$. But then there is a singular string of length p in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\bar{\ell}^{(n)} = \ell$. □

Proof of (III) for $B_n^{(1)}$. Here $b^\natural = \bar{1}$. Note that $\overline{H}(b \otimes b') = 0$ if $b \leq b'$ and $b \otimes b' \neq 0 \otimes 0$, $\overline{H}(b \otimes b') = 2$ if $b \otimes b' = \bar{1} \otimes 1$, and $H(b \otimes b') = 1$ otherwise.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \alpha_1^{(1)} \tag{5.15}$$

$$\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) + \chi(\bar{\ell}^{(1)} = 1). \tag{5.16}$$

where $\ell^{(i)}$ and $\bar{\ell}^{(i)}$ are determined by the algorithm δ.

Let $\tilde{\ell}^{(a)}$ and $\tilde{\bar{\ell}}^{(a)}$ be the length of the selected strings defined by the algorithm δ on $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$. To check (5.16) note that if $\ell^{(1)} = 1$ it follows that $\tilde{\ell}^{(a)} \geq \ell^{(a+1)}$ for $1 \leq a \leq n-1$. Hence if $b_L \leq n$ then $b_{L-1} < b_L$ and both sides of (5.16) yield 1. If $b_L = 0$ then $b_{L-1} \leq b_L$ by (4.9) and both sides of (5.13) are 1. If $\bar{n} \leq b_L < \bar{1}$, then $b_{L-1} < b_L$ by (4.9) and both sides of (5.16) are 1. If $b_L = \overline{1}$ and $\bar{\ell}^{(1)} = 1$, then $b_{L-1} = 1$ by (4.9). Hence both sides of (5.16) yield 2. Finally, if $b_L = \overline{1}$ and $\bar{\ell}^{(1)} > 1$, then there exists a singular string of length $\bar{\ell}^{(1)} - 1$ in $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ so that $b_{L-1} \neq 1$. Hence both sides of (5.16) are 1. If $\ell^{(1)} > 1$ then $\tilde{\ell}^{(a)} < \ell^{(a)}$ for $1 \le a \le n-2$ and the cases can be checked in a similar fashion as before.

To prove (5.15), by (3.4) and (4.8) we have

$$\begin{aligned} cc(\tilde{\nu}^\bullet) =& \frac{1}{2} \sum_{i,j\ge 1} \sum_{a,b=1}^{n} \min(t_b i, t_a j)(\alpha_a|\alpha_b) \\ & \times \left(m_i^{(a)} - \delta_{i,\frac{\ell^{(a)}}{v_a}} + \delta_{i,\frac{\ell^{(a)}}{v_a}-1} - \delta_{i,\frac{\bar{\ell}^{(a)}}{v_a}} + \delta_{i,\frac{\bar{\ell}^{(a)}}{v_a}-1} \right) \\ & \times \left(m_j^{(b)} - \delta_{j,\frac{\ell^{(b)}}{v_b}} + \delta_{j,\frac{\ell^{(b)}}{v_b}-1} - \delta_{j,\frac{\bar{\ell}^{(b)}}{v_b}} + \delta_{j,\frac{\bar{\ell}^{(b)}}{v_b}-1} \right). \end{aligned}$$

Applying the data for B_n and using (4.9), a tedious but straightforward calculation yields

$$\begin{aligned} \Delta cc(\nu^\bullet) =& \sum_{a=1}^{n} \sum_{i\ge 1} \left(P_{v_a i}^{(a)}(\tilde{\nu}^\bullet) - P_{v_a i}^{(a)}(\nu^\bullet) \right) \left(m_i^{(a)} - \delta_{v_a i, \ell^{(a)}} - \delta_{v_a i, \bar{\ell}^{(a)}} \right) \\ & + \sum_{i\ge 1} m_i^{(1)} - \chi\left(\ell^{(n)} = \ell^{(n-1)} - \frac{1}{2} \right) + \chi(\bar{\ell}^{(n)} = \bar{\ell}^{(n-1)}) \\ & + \chi(\bar{\ell}^{(n)} = \infty)\chi(\ell^{(n)} < \infty). \end{aligned}$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\begin{aligned} \Delta|J^\bullet| =& \sum_{a=1}^{n} \sum_{i\ge 1} \left(P_{v_a i}^{(a)}(\nu^\bullet) - P_{v_a i}^{(a)}(\tilde{\nu}^\bullet) \right) \left(m_i^{(a)} - \delta_{v_a i, \ell^{(a)}} - \delta_{v_a i, \bar{\ell}^{(a)}} \right) \\ & + \chi(\ell^{(n)} = \ell^{(n-1)} - \frac{1}{2}) - \chi(\bar{\ell}^{(n)} = \bar{\ell}^{(n-1)}) - \chi(\bar{\ell}^{(n)} = \infty)\chi(\ell^{(n)} < \infty), \end{aligned}$$

where the last three terms come from the fact that for n-th rigged partition singular strings can be transformed into quasisingular strings and vice versa. Hence altogether, using $\sum_{i\ge 1} m_i^{(1)} = \alpha_1^{(1)}$, we obtain (5.15). □

5.3. Proof for type $C_n^{(1)}$.

Proof of (I) for $C_n^{(1)}$. If $b = k$ with $1 \le k < n$ the proof that ρ is dominant is analogous to type $D_n^{(1)}$. For $b = n$ a problem occurs if $\lambda_n = 0$. In this case $P_i^{(n)}(\nu^\bullet) = 0$ for $i \ge \ell$ where ℓ is the largest part in $\nu^{(n)}$ by Lemma 5.2 and (5.10). By (5.6) this implies $m_i^{(n-1)}(\nu^\bullet) = 0$ for $i > \ell$. Hence $\ell^{(n-1)} \le \ell$. But there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\ell^{(n)} = \infty$. If $k = \overline{n}$ a problem occurs if $\lambda_n = \lambda_{n-1}$. In this case $P_i^{(n-1)}(\nu^\bullet) = 0$ for $i \ge \ell$ where ℓ is the largest part in $\nu^{(n-1)}$. By (5.6), $m_i^{(n)}(\nu^\bullet) = 0$ for $i > \ell$. Hence $\bar{\ell}^{(n)} \le \ell$. But there is a singular string of size

ℓ in $(\nu^\bullet, J^\bullet)^{(n-1)}$ (this also works if $\ell^{(n-1)} = \bar{\ell}^{(n)} = \ell$) which contradicts $\bar{\ell}^{(n-1)} = \infty$. □

Proof of (II) for $C_n^{(1)}$. We show that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \in \mathrm{RC}(\rho, \tilde{\mu})$. We use the same notation and set-up as in type $D_n^{(1)}$. Then

$$\begin{aligned} J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) &= P_i^{(a)}(\tilde{\nu}^\bullet) && \text{for } i = \ell^{(a)} - 1 \text{ and } i = \bar{\ell}^{(a)} - 1 \\ & && \text{or } i = \bar{\ell}^{(a)} - 2 \text{ if } \ell^{(a)} = \bar{\ell}^{(a+1)} \\ 0 \le J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) &\le J_{\max}^{(a,i)}(\nu^\bullet, J^\bullet) && \text{else.} \end{aligned}$$

The proof that $0 \le J_{\max}^{(a,i)}(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \le P_i^{(a)}(\tilde{\nu}^\bullet)$ for $1 \le a < n$ is the same as usual if $\ell^{(a)} \neq \bar{\ell}^{(a+1)}$. If $\ell^{(a)} = \bar{\ell}^{(a+1)}$, by (4.13) the only problem occurs if

$$m_{\ell-2}^{(a)}(\nu^\bullet) = 0, \quad P_{\ell-2}^{(a)}(\nu^\bullet) = 0, \quad \ell^{(a-1)} < \ell - 1, \quad \ell = \bar{\ell}^{(a)} = \ell^{(a)} + 1 \text{ finite.}$$

We show that these conditions cannot be met simultaneously. Let $p < \ell - 1$ be maximal such that $m_p^{(a)}(\nu^\bullet) > 0$; if no such p exists set $p = 0$. By (5.10), $P_{\ell-2}^{(a)}(\nu^\bullet) = 0$ is only possible if $P_i^{(a)}(\nu^\bullet) = 0$ for $p \le i \le \ell - 1$. By (5.6) this requires $m_i^{(a-1)}(\nu^\bullet) = 0$ for $p < i < \ell - 1$ so that $\ell^{(a-1)} < \ell - 1$ implies $\ell^{(a-1)} \le p$. But there is a singular string of length p in $(\nu^\bullet, J^\bullet)^{(a)}$ which contradicts $\ell^{(a)} = \ell - 1 > p$.

Finally for $a = n$ the only problem occurs if

$$m_{\ell-2}^{(n)}(\nu^\bullet) = 0, \quad P_{\ell-2}^{(n)}(\nu^\bullet) = 0, \quad \ell^{(n-1)} < \ell - 1, \quad \ell = \bar{\ell}^{(n)} \text{ finite.}$$

By convexity (5.10), $P_i^{(n)}(\nu^\bullet) = 0$ for $p \le i \le \ell$ where $p < \ell$ is largest such that $m_p^{(n)}(\nu^\bullet) > 0$. Then by (5.6) we also have $m_i^{(n-1)}(\nu^\bullet) = 0$ for $p < i < \ell$ so that $\ell^{(n-1)} < \ell - 1$ implies $\ell^{(n-1)} \le p$. But there is a singular string of length p in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\bar{\ell}^{(n)} = \ell > p$. □

Proof of (III) for $C_n^{(1)}$. Here $b^\natural = \bar{1}$, $\overline{H}(b \otimes b') = 0$ if $b \le b'$ and $H(b \otimes b') = 1$ otherwise.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \alpha_1^{(1)} \tag{5.17}$$

$$\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) \tag{5.18}$$

where $\ell^{(i)}$ is determined by the algorithm δ. Note that there is no contribution from $\bar{\ell}^{(1)}$ in (5.18) since $\bar{\ell}^{(1)} > 1$.

Let $\tilde{\ell}^{(a)}$ and $\bar{\tilde{\ell}}^{(a)}$ be the length of the selected strings defined by the algorithm δ on $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$. Note that if $\ell^{(1)} = 1$ then (4.13) implies that $b_{L-1} < b_L$ so that both sides of (5.18) are 1. If $\ell^{(1)} > 1$ then $\tilde{\ell}^{(a)} < \ell^{(a)}$ for $1 \le a < n$ so that $b_{L-1} \ge b_L$ and both sides of (5.18) are 0.

For $b_L = \overline{n}$, $\tilde{\overline{\ell}}^{(n)} < \overline{\ell}^{(n)}$ unless $\overline{\ell}^{(n)} = 2$. But note that in this case $\ell^{(n)} = 1$ and hence $\ell^{(1)} = 1$ which contradicts our assumptions. If $\tilde{\overline{\ell}}^{(n)} < \overline{\ell}^{(n)}$ then also $\tilde{\overline{\ell}}^{(a)} < \overline{\ell}^{(a)}$ which implies that $b_{L-1} \geq b_L$. Hence both sides of (5.18) are 0.

To prove (5.17), by (3.4) and (4.12) we have

$$cc(\tilde{\nu}^\bullet) = \frac{1}{2} \sum_{i,j\geq 1} \sum_{a,b=1}^{n} \min(t_b i, t_a j)(\alpha_a|\alpha_b)$$
$$\times \left(m_i^{(a)} - \chi(a<n)(\delta_{i,\ell^{(a)}} - \delta_{i,\ell^{(a)}-1}) - \delta_{i,\frac{\overline{\ell}^{(a)}}{v_a}} + \delta_{i,\frac{\overline{\ell}^{(a)}}{v_a}-1}\right)$$
$$\times \left(m_j^{(b)} - \chi(b<n)(\delta_{j,\ell^{(b)}} - \delta_{j,\ell^{(b)}-1}) - \delta_{j,\frac{\overline{\ell}^{(b)}}{v_b}} + \delta_{j,\frac{\overline{\ell}^{(b)}}{v_b}-1}\right).$$

Applying the data for C_n and using (4.13), a tedious but straightforward calculation yields

$$\Delta cc(\nu^\bullet) = \sum_{a=1}^{n} \sum_{i\geq 1} \left(P_{v_a i}^{(a)}(\tilde{\nu}^\bullet) - P_{v_a i}^{(a)}(\nu^\bullet)\right)$$
$$\times \left(m_i^{(a)} - \chi(a<n)\delta_{i,\ell^{(a)}} - \delta_{v_a i,\overline{\ell}^{(a)}}\right) + \sum_{i\geq 1} m_i^{(1)}.$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\Delta|J^\bullet| = \sum_{a=1}^{n} \sum_{i\geq 1} \left(P_{v_a i}^{(a)}(\nu^\bullet) - P_{v_a i}^{(a)}(\tilde{\nu}^\bullet)\right) \left(m_i^{(a)} - \chi(a<n)\delta_{i,\ell^{(a)}} - \delta_{v_a i,\overline{\ell}^{(a)}}\right).$$

Hence altogether, using $\sum_{i\geq 1} m_i^{(1)} = \alpha_1^{(1)}$, we obtain (5.17). □

5.4. **Proof for type $A_{2n}^{(2)}$.** The proofs of (I) and (II) are analogous to the previous cases. In particular, the proof of (II) is very similar to that for type $C_n^{(1)}$.

Proof of (III) for $A_{2n}^{(2)}$. Here $b^\natural = \phi$, $\overline{H}(b \otimes b') = 0$ if $b \leq b'$, $\overline{H}(b \otimes b') = 2$ if $b > b'$ or $b \otimes b' = \phi \otimes \phi$ and $H(b \otimes b') = 0$ otherwise.

If $L = 1$ then the path is 1 or ϕ. In the former case, the rigged configuration is empty, and both sides of (3.13) are zero. In the other case both sides of (3.13) are 1.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = 2\alpha_1^{(1)} - \chi(\ell^{(n)} = 1) \tag{5.19}$$
$$\overline{H}(b_L \otimes b_{L-1}) = 2\chi(\ell^{(1)} = 1) - \chi(\ell^{(n)} = 1) + \chi(\tilde{\ell}^{(n)} = 1) \tag{5.20}$$

where $\ell^{(i)}$ is determined by the algorithm δ. Note that there is no contribution from $\overline{\ell}^{(1)}$ in (5.20) since $\overline{\ell}^{(1)} > 1$.

Equation (5.20) can be checked in a similar fashion as to the other cases.

To prove (5.19), applying the data for B_n and using (4.13), a tedious but straightforward calculation yields

$$\Delta cc(\nu^\bullet) = 2\sum_{a=1}^{n}\sum_{i\geq 1}\left(P_i^{(a)}(\tilde{\nu}^\bullet) - P_i^{(a)}(\nu^\bullet)\right)$$
$$\times\left(m_i^{(a)} - \chi(a<n)\delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}\right) - \chi(\ell^{(n)}=1) + 2\sum_{i\geq 1} m_i^{(1)}.$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\Delta|J^\bullet| = 2\sum_{a=1}^{n}\sum_{i\geq 1}\left(P_i^{(a)}(\nu^\bullet) - P_i^{(a)}(\tilde{\nu}^\bullet)\right)\left(m_i^{(a)} - \chi(a<n)\delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}\right).$$

Hence altogether, using $\sum_{i\geq 1} m_i^{(1)} = \alpha_1^{(1)}$, we obtain (5.19). □

5.5. Proof for type $A_{2n-1}^{(2)}$.

Proof of (I) for $A_{2n-1}^{(2)}$. The proof that ρ is dominant for $b=k$ with $1\leq k\leq n$ is analogous to the other types. For $b=\overline{k}$ with $1\leq k\leq n$, ρ is not dominant if $\lambda_k=\lambda_{k-1}$. In this case $P_i^{(k-1)}(\nu^\bullet)=0$ for $i\geq\ell$ where ℓ is the largest part of $\nu^{(k-1)}$ by Lemma 5.2 and (5.10). By (5.8), $m_i^{(k)}(\nu^\bullet)=0$ for $i>\ell$ so that $\bar{\ell}^{(k)}\leq\ell$. But since $P_\ell^{(k-1)}(\nu^\bullet)=0$ and $m_\ell^{(k-1)}>0$, there is a singular string of length ℓ in $(\nu^\bullet,J^\bullet)^{(k-1)}$. Hence $\bar{\ell}^{(k-1)}\leq\ell$ (which contradicts $\bar{\ell}^{(k-1)}=\infty$ since $\overline{k}=\mathrm{rk}(\nu^\bullet,J^\bullet)$) unless $\ell^{(k-1)}=\bar{\ell}^{(k)}=\ell$ and $m_\ell^{(k-1)}(\nu^\bullet)=1$. Since $m_\ell^{(k)}(\nu^\bullet)\geq 2$ for $1\leq k<n$ and $m_\ell^{(n)}(\nu^\bullet)\geq 1$, (5.8) for $a=k-1$ and $i=\ell$ implies that $m_\ell^{(k-2)}(\nu^\bullet)=0$ and $P_{\ell-1}^{(k-1)}(\nu^\bullet)=0$. Hence $\ell^{(k-2)}<\ell$ and $m_{\ell-1}^{(k-1)}(\nu^\bullet)=0$ since otherwise $\ell^{(k-1)}\leq\ell-1$. By induction on $i=\ell-1,\ell-2,\dots,1$ (5.8) for $a=k-1$ implies that $m_i^{(k-2)}(\nu^\bullet)=0$ and $P_{i-1}^{(k-1)}(\nu^\bullet)=0$ which in turn requires $m_{i-1}^{(k-1)}(\nu^\bullet)=0$ since else $\ell^{(k-1)}\leq i-1$. But then $m_i^{(k-2)}(\nu^\bullet)=0$ for all $1\leq i\leq\ell$ so that $\ell^{(k-2)}>\ell$ which contradicts our assumptions. □

Proof of (II) for $A_{2n-1}^{(2)}$. To prove that $(\tilde{\nu}^\bullet,\tilde{J}^\bullet)$ is admissible, one finds similarly to the proof of type $D_n^{(1)}$ that the only problem occurs if

$$m_{\ell-1}^{(a)}=0,\quad P_{\ell-1}^{(a)}(\nu^\bullet)=0,\quad \ell^{(a-1)}<\ell \text{ (resp. } \bar{\ell}^{(a+1)}<\ell \text{ for } 1\leq a<n),$$
$$\ell \text{ finite},\quad \ell=\ell^{(a)} \text{ (resp. } \ell=\bar{\ell}^{(a)} \text{ for } 1\leq a<n).$$

Analogous to the case $D_n^{(1)}$ it can be shown that these conditions cannot hold simultaneously. □

Proof of (III) for $A^{(2)}_{2n-1}$. Here $b^\natural = \bar{1}$, $\overline{H}(\bar{1} \otimes 1) = 2$, $\overline{H}(b \otimes b') = 0$ if $b \le b'$ and $H(b \otimes b') = 1$ otherwise.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \alpha^{(1)}_1 \tag{5.21}$$

$$\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) + \chi(\bar{\ell}^{(1)} = 1) \tag{5.22}$$

where $\ell^{(i)}$ is determined by the algorithm δ.

The proof that (5.22) holds is very similar to the previous cases.

To prove (5.21) we apply the data for C_n to (3.4). Using (4.16) and (4.17) a tedious but straightforward calculation yields

$$\begin{aligned}\Delta cc(\nu^\bullet) = \sum_{a=1}^{n} \sum_{i\ge 1} & t^\vee_a \left(P^{(a)}_i(\tilde{\nu}^\bullet) - P^{(a)}_i(\nu^\bullet)\right) \\ &\times \left(m^{(a)}_i - \delta_{i,\ell^{(a)}} - \chi(a<n)\delta_{i,\bar{\ell}^{(a)}}\right) + \sum_{i\ge 1} m^{(1)}_i .\end{aligned}$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\Delta|J^\bullet| = \sum_{a=1}^{n} \sum_{i\ge 1} t^\vee_a \left(P^{(a)}_i(\nu^\bullet) - P^{(a)}_i(\tilde{\nu}^\bullet)\right) \left(m^{(a)}_i - \delta_{i,\ell^{(a)}} - \chi(a<n)\delta_{i,\bar{\ell}^{(a)}}\right).$$

Hence altogether, using $\sum_{i\ge 1} m^{(1)}_i = \alpha^{(1)}_1$, we obtain (5.21). □

5.6. Proof for type $D^{(2)}_{n+1}$.

Proof of (I) for $D^{(2)}_{n+1}$. The proof proceeds as before except in the cases $b = n$ and $b = 0$ (there is nothing to prove for $b = \phi$). Suppose $b = n$ and ρ is not dominant. Since λ is dominant, $\lambda_n = 0$. Then it can be deduced that $P^{(n)}_i(\nu^\bullet) = 0$ for $i \ge \ell$ where ℓ is the largest part in $\nu^{(n)}$ by Lemma 5.2 and (5.10) and the admissibility of $\nu^\bullet$. By (5.9) it follows that $m^{(n-1)}_i(\nu^\bullet) = 0$ for $i > \ell$ so that $\ell^{(n-1)} \le \ell$. But there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(n)}$ since $P^{(n)}_\ell(\nu^\bullet) = 0$ which contradicts $\ell^{(n)} = \infty$. To prove that $b = 0$ cannot occur if $\lambda_n = 0$ we find as for the case $k = n$ that $\ell^{(n-1)} \le \ell$ and that there is a singular string of length ℓ in $(\nu^\bullet, J^\bullet)^{(n)}$ since $P^{(n)}_\ell(\nu^\bullet) = 0$. For the $b = 0$ case (Q) must hold so that there must be a quasisingular string of length $\ell^{(n)} < \ell$ in $(\nu^\bullet, J^\bullet)^{(n)}$. But observe that there is a singular string of length $\ell > \ell^{(n)}$ in $(\nu^\bullet, J^\bullet)^{(n)}$ which contradicts $\bar{\ell}^{(n)} = \infty$. □

Proof of (II) for $D^{(2)}_{n+1}$. Next we need to show that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \in \mathrm{RC}(\rho, \tilde{\mu})$. The case that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(a)}$ is admissible for $1 \le a < n$ works as usual.

Consider $a = n$. First note that there is no problem in case (Q,S) setting the new string of length $\bar{\ell}^{(n)} - 1$ to be quasisingular since the string of length $\bar{\ell}^{(n)} - 1$ is not singular by definition so that $P^{(n)}_{\bar{\ell}^{(n)}-1}(\nu^\bullet) > 0$ and also $P^{(n)}_{\bar{\ell}^{(n)}-1}(\tilde{\nu}^\bullet) > 0$ by (4.19). The only problem occurs if

$$m^{(n)}_{\ell-1}(\nu^\bullet) = 0, \quad P^{(n)}_{\ell-1}(\nu^\bullet) = 0 \text{ or } 1, \quad \ell^{(n-1)} < \ell, \quad \ell = \ell^{(n)} \text{ finite.}$$

Note that $P^{(n)}_i(\nu^\bullet)$ is always even so that $P^{(n)}_{\ell-1} = 1$ is impossible. The proof that these conditions cannot hold simultaneously works as usual. □

Proof of (III) for $D^{(2)}_{n+1}$. Here $b^\natural = \phi$ and $\overline{H}(\phi \otimes \phi) = 2$, $\overline{H}(b \otimes \phi) = \overline{H}(\phi \otimes b) = 1$ if $b \neq \phi$, $\overline{H}(b \otimes b') = 0$ if $b, b' \neq \phi$, $b \leq b'$ and $b \neq b'$ if $b = 0$, and $\overline{H}(b \otimes b') = 2$ if $b > b'$ or $b = b' = 0$.

If $L = 1$ then the path is either 1 or ϕ. In the former case the rigged configuration is empty, and both sides of (3.13) are zero. In the latter case it is also not hard to check that both sides of (3.13) are 1.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = 2\alpha^{(1)}_1 - \chi(\ell^{(n)} = 1) \tag{5.23}$$

$$\overline{H}(b_L \otimes b_{L-1}) = 2\chi(\ell^{(1)} = 1) - \chi(\ell^{(n)} = 1) + \chi(\tilde{\ell}^{(n)} = 1) \tag{5.24}$$

where $\ell^{(i)}$ and $\bar{\ell}^{(i)}$ are determined by the algorithm δ. To obtain (5.24) we used the fact that by definition $\bar{\ell}^{(1)} > 1$. Here $\tilde{\ell}^{(a)}$ is defined by the algorithm δ on $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta'(\nu^\bullet, J^\bullet)$.

It can be checked directly that (5.24) holds. For example, if $\ell^{(1)} = 1$ it follows that $\tilde{\ell}^{(a)} \geq \ell^{(a+1)}$ for $1 \leq a < n$. Hence if $b_L \leq n$ then $b_{L-1} < b_L$ and both sides of (5.24) yield 2. If $b_L = \phi$ then both sides of (5.24) are 1 for $b_{L-1} \neq \phi$ and 2 if $b_{L-1} = \phi$. If $b_L = 0$ then both sides of (5.24) are 2 if $b_{L-1} \leq 0$. Note that $b_{L-1} = \phi$ or $b_{L-1} \geq \bar{n}$ is not possible. Finally, if $b_L \geq \bar{n}$ then $\bar{\tilde{\ell}}^{(a)} \geq \bar{\ell}^{(a)}$ and $b_{L-1} < b_L$. Note that $b_{L-1} = \phi$ is not possible in this case since $\ell^{(n)} > 1$ which implies $\tilde{\ell}^{(n)} > 1$. Both sides of (5.24) yield 2 in this case. If $\ell^{(1)} > 1$ then $\tilde{\ell}^{(a)} < \ell^{(a)}$ for $1 \leq a \leq n$ and the cases can be checked in a similar fashion as before.

To prove (5.23), from (3.4) and (4.12) we obtain

$$\begin{aligned} cc(\tilde{\nu}^\bullet) = & \frac{1}{2} \sum_{i,j \geq 1} \sum_{a,b=1}^{n} \min(i,j)(\alpha_a|\alpha_b) \\ & \times (m^{(a)}_i - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}} + \delta_{i,\ell^{(a)}-1} + \delta_{i,\bar{\ell}^{(a)}-1}) \\ & \times (m^{(b)}_j - \delta_{j,\ell^{(b)}} - \delta_{j,\bar{\ell}^{(b)}} + \delta_{j,\ell^{(b)}-1} + \delta_{j,\bar{\ell}^{(b)}-1}). \end{aligned}$$

Expanding out and using (4.19) a tedious but straightforward calculation yields

$$\Delta cc(\nu^\bullet) = \sum_{a=1}^{n}\sum_{i\geq 1} t_a^\vee \left(P_i^{(a)}(\tilde{\nu}^\bullet) - P_i^{(a)}(\nu^\bullet)\right)\left(m_i^{(a)} - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}\right) + 2\sum_{i\geq 1} m_i^{(1)} - \chi(\bar{\ell}^{(n)} = \infty)\chi(\ell^{(n)} < \infty).$$

For $\Delta|J^\bullet|$ we obtain from the algorithm δ'

$$\Delta|J^\bullet| = \sum_{a=1}^{n}\sum_{i\geq 1} t_a^\vee \left(P_i^{(a)}(\nu^\bullet) - P_i^{(a)}(\tilde{\nu}^\bullet)\right)\left(m_i^{(a)} - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}\right) + \chi(\bar{\ell}^{(n)} = \infty)\chi(1 < \ell^{(n)} < \infty)$$

where the last term comes from the fact that in case (Q) a quasisingular string is changed into a singular string. Hence altogether, using $\sum_{i\geq 1} m_i^{(1)} = \alpha_1^{(1)}$ and the fact that $\bar{\ell}^{(n)} = \infty$ if $\ell^{(n)} = 1$ by the algorithm δ, we obtain (5.23). □

5.7. Proof for type $A_{2n}^{(2)\dagger}$.

Proof of (I) for $A_{2n}^{(2)\dagger}$. The only case that proceeds differently than before is $b = 0$. Suppose $\lambda_n = 0$. Let ℓ be the longest part of $\nu^{(n)}$. As in the proof of the $D_{n+1}^{(2)}$ case, $P_\ell^{(n)}(\nu^\bullet) = 0$ where $\ell \geq \ell^{(n-1)}$. If ℓ is odd then this is a contradiction of the admissibility of $\nu^\bullet$; see (3.10). If ℓ is even then $J^{(n,\ell)}$ is singular and $\ell^{(n)} < \ell$ (as $\ell^{(n)}$ is odd and ℓ is the longest part), contradicting $b = 0$. □

Proof of (II) for $A_{2n}^{(2)\dagger}$. The admissibility of $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ for $1 \leq a < n$ is as before. Let $a = n$. We first observe that in all cases,

$$\ell^{(1)} \leq \ell^{(2)} \leq \cdots \leq \ell^{(n)} \leq \bar{\ell}^{(n)} \leq \bar{\ell}^{(n-1)} \leq \cdots \leq \bar{\ell}^{(1)}, \tag{5.25}$$

with $\ell^{(n)}$ odd and $\bar{\ell}^{(n)}$ even (when they are finite). We also note that by (4.13),

$$P_i^{(n)}(\tilde{\nu}^\bullet) \geq P_i^{(n)}(\nu^\bullet) - 1 \tag{5.26}$$

with equality if and only if $\ell^{(n-1)} \leq i < \ell^{(n)}$.

Let us verify (3.10) for $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$. Let i be odd such that $m_i^{(n)}(\tilde{\nu}^\bullet) > 0$. Suppose first that $m_i^{(n)}(\nu^\bullet) > 0$. By (3.10) for $(\nu^\bullet, J^\bullet)$, $P_i^{(n)}(\nu^\bullet) > 0$. By (5.26) we may assume that $P_i^{(n)}(\nu^\bullet) = 1$ and $\ell^{(n-1)} \leq i < \ell^{(n)}$. But then $J^{(n,i)}$ was quasisingular, which is a contradiction to the definition of δ. So suppose $m_i^{(n)}(\nu^\bullet) = 0$. Since $m_i^{(n)}(\tilde{\nu}^\bullet) > 0$ we are in case (Q, S) with $i = \bar{\ell}^{(n)} - 1$. In case (Q, S) $\ell^{(n)} < \bar{\ell}^{(n)}$, so $\ell^{(n)} \leq i < \bar{\ell}^{(n)}$. Now $i \neq \ell^{(n)}$ since

$m_i^{(n)}(\nu^\bullet) = 0$. So $\ell^{(n)} < i < \bar{\ell}^{(n)}$ with $\ell^{(n)}$ and i odd. By (4.13) $P_i^{(n)}(\tilde{\nu}^\bullet) = P_i^{(n)}(\nu^\bullet)$. There is only a problem if $P_i^{(n)}(\nu^\bullet) = 0$. By (5.10) it follows that $P_{i-1}^{(n)}(\nu^\bullet) = P_{i+1}^{(n)}(\nu^\bullet) = 0$. Since $i-1$ is even, if $m_{i-1}^{(n)}(\nu^\bullet) > 0$ then $J^{(n,i-1)}$ would have been singular with $\ell^{(n)} < i-1 < \bar{\ell}^{(n)}$, contradicting the choice of $\bar{\ell}^{(n)}$. So $m_{i-1}^{(n)}(\nu^\bullet) = 0$. Applying (5.10) again, $P_{i-2}^{(n)}(\nu^\bullet) = 0$. Since $(\nu^\bullet, J^\bullet)$ was admissible and $i-2$ is odd, by (3.10) it follows that $m_{i-2}^{(n)}(\nu^\bullet) = 0$. Continuing in this manner, a contradiction is reached since $P_{\ell^{(n)}}^{(n)}(\nu^\bullet) > 0$.

Now suppose i is even. It must be checked that $P_i^{(n)}(\tilde{\nu}^\bullet) \geq 0$. The only problem is if $P_i^{(n)}(\nu^\bullet) = 0$ and $\ell^{(n-1)} \leq i < \ell^{(n)}$. If $m_i^{(n)}(\nu^\bullet) > 0$ then δ would have chosen the singular partition $J^{(n,i)}$. So $m_i^{(n)}(\nu^\bullet) = 0$. By (5.10) it follows that $P_{i+1}^{(n)}(\nu) = 0$. Arguing as above but with the index increasing from i, a contradiction is reached since $P_{\ell^{(n)}}^{(n)}(\nu^\bullet) > 0$. □

Proof of (III) for $A_{2n}^{(2)\dagger}$. One has $b^\natural = 1$, $\overline{H}(b_2' \otimes b_1') = 0$ if $b_2' \leq b_1'$ (except for $\overline{H}(0 \otimes 0) = 1$), and $\overline{H}(b_2' \otimes b_1') = 1$ for $b_2' > b_1'$.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.2) are given by

$$\Delta(cc(\nu^\bullet, J^\bullet)) = \alpha_1^{(1)} \tag{5.27}$$

$$\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) \tag{5.28}$$

where $\ell^{(i)}$ and $\bar{\ell}^{(i)}$ are determined by the algorithm δ. The term $\chi(\bar{\ell}^{(1)} = 1)$ disappears since the definition of the algorithm forces $\bar{\ell}^{(1)} \geq 2$. The proof of (5.27) is very similar to that in the $D_{n+1}^{(2)}$ case.

Straightforward computations yield

$$\begin{aligned}\Delta cc(\nu^\bullet) = &\sum_{a,b,i} (\alpha_a|\alpha_b)(\chi(i \geq \ell^{(b)}) + \chi(i \geq \bar{\ell}^{(b)}))(m_i^{(a)}(\nu^\bullet) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}) \\ &+ \chi(\ell^{(1)} < \infty) + \chi(\bar{\ell}^{(1)} < \infty) - \frac{1}{2}\chi(\ell^{(n)} < \infty) + \frac{1}{2}\chi(\bar{\ell}^{(n)} < \infty)\end{aligned}$$

and

$$\begin{aligned}&\sum_{a,i}(P_i^{(a)}(\tilde{\nu}^\bullet) - P_i^{(a)}(\nu^\bullet))(m_i^{(a)}(\nu^\bullet) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}) \\ &\quad = \sum_{a,b,i} (\alpha_a|\alpha_b)(\chi(i \geq \ell^{(b)}) + \chi(i \geq \bar{\ell}^{(b)}))(m_i^{(a)}(\nu^\bullet) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}) \\ &\quad\quad - \sum_i m_i^{(a)}(\nu^\bullet) + \chi(\ell^{(1)} < \infty) + \chi(\bar{\ell}^{(1)} < \infty).\end{aligned}$$

Together these yield

$$\begin{aligned}\Delta cc(\nu^\bullet) = &\sum_{a,i}(P_i^{(a)}(\tilde{\nu}^\bullet) - P_i^{(a)}(\nu^\bullet))(m_i^{(a)}(\nu^\bullet) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}) \\ &+ \alpha_1^{(1)} - \frac{1}{2}\chi(\ell^{(n)} < \infty) + \frac{1}{2}\chi(\bar{\ell}^{(n)} < \infty).\end{aligned}$$

One can also show that

$$\begin{aligned}\Delta|J^\bullet| = &\sum_{a,i}(P_i^{(a)}(\nu^\bullet) - P_i^{(a)}(\tilde{\nu}^\bullet))(m_i^{(a)}(\nu^\bullet) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}}) \\ &+ \frac{1}{2}\chi(\ell^{(n)} < \infty) - \frac{1}{2}\chi(\bar{\ell}^{(n)} < \infty).\end{aligned}$$

This proves (5.27). □

References

[1] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Z. Tsuboi, *Paths, crystals, and fermionic formula*, MathPhys odyssey, 2001, 205–272, Prog. Math. Phys., 23, Birkhuser Boston, Boston, MA, 2002.

[2] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, *Remarks on fermionic formula*, Contemporary Math. **248** (1999) 243-291.

[3] M. Kashiwara, *On level zero representations of quantized affine algebras*, Duke Math. J. **112** (2002), no. 1, 117–195.

[4] V. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, 1990.

[5] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, *Affine crystals and vertex models*, Int. J. Mod. Phys. **A7** (suppl. 1A) (1992) 449–484.

[6] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, *Perfect crystals of quantum affine Lie algebras*, Duke Math. J. **68** (1992) 499–607.

[7] S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, *Combinatorics, the Bethe ansatz and representations of the symmetric group*, Zap.Nauchn. Sem. (LOMI) **155** (1986) 50–64. (English translation: J. Sov. Math. **41** (1988) 916–924.)

[8] A. N. Kirillov and N. Y. Reshetikhin, *The Bethe Ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41** (1988) 925-955.

[9] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of Yangians and multiplicity of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras*, J. Sov. Math. **52** (1990) 3156–3164.

[10] A. N. Kirillov, A. Schilling and M. Shimozono, *A bijection between Littlewood-Richardson tableaux and rigged configurations*, Selecta Math. (N.S.) **8** (2002), no. 1, 67–135.

[11] A. N. Kirillov and M. Shimozono, *A generalization of the Kostka-Foulkes polynomials*, J. Algebraic Combin. **15** (2002), no. 1, 27–69.

[12] A. Lascoux and M. P. Schützenberger, *Sur une conjecture de H.O. Foulkes*, CR Acad. Sci. Paris **286A** (1978) 323–324.

[13] M. Okado, A. Schilling, and M. Shimozono, *Virtual crystals and fermionic formulas of types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$*, Representation Theory, to appear (math.QA/0105017).

Department of Informatics and Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
E-mail address: okado@sigmath.es.osaka-u.ac.jp

Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, U.S.A.
E-mail address: anne@math.ucdavis.edu

Department of Mathematics, 460 McBryde Hall, Virginia Tech, Blacksburg, VA 24061-0123, U.S.A
E-mail address: mshimo@math.vt.edu

LITTLEWOOD'S FORMULAS FOR CHARACTERS OF ORTHOGONAL AND SYMPLECTIC GROUPS

ALAIN LASCOUX

ABSTRACT. Littlewood gave expansions of products of the type $\prod 1/(1 - a_i a_j)$. Several authors published generalizations involving a small number of extra factors $\prod 1/(1 - x a_i)$. We show that the method of Littlewood (that we express in terms of λ-rings) covers these extensions. We also express Littlewood's coefficients in terms of orthogonal or symplectic Schur functions.

Given an infinite alphabet $\mathbb{A} = \{a_1, a_2, a_3, \dots\}$, Littlewood [7] gave expansions of products of the type

$$\prod (1 - a_i)^{\pm 1} \prod_{i<j} (1 - a_i a_j)^{\pm 1} \quad \text{and} \quad \prod (1 - a_i)^{\pm 1} \prod_{i \le j} (1 - a_i a_j)^{\pm 1}, \tag{1}$$

the case $\prod (1 - a_i)^{-1} \prod_{i<j} (1 - a_i a_j)^{-1}$ was due to Schur [10].

By functoriality of symmetric functions, Littlewood's formulas imply without further computations expansions of more complicated products like

$$\prod (1 - a_i)^{\pm 1} \prod (1 - b_i)^{\mp 1} \prod_{i \le j} (1 - a_i a_j)^{\pm 1} \prod_{i<j} (1 - b_i b_j)^{\pm 1} \prod (1 - a_i b_j)^{\mp 1} \tag{2}$$

involving a second alphabet $\mathbb{B} = \{b_1, b_2, \dots\}$.

However, Littlewood's method is not limited to expand the above products, and it also allows to expand products of the type

$$\prod_i \frac{1}{(1 - x_1 a_i) \cdots (1 - x_n a_i)} \prod_{i<j} \frac{1}{1 - a_i a_j} \tag{3}$$

involving a finite alphabet $\mathbb{X} = \{x_1, \dots, x_n\}$, the alphabets $\mathbb{A}$ and $\mathbb{X}$ playing now a disymmetrical role.

To obtain his formulas, Littlewood describes in fact the adjoint of the multiplication by certain series of symmetric functions (see the theorem below). We shall state his results in terms of λ-rings.

The case $\mathbb{X} = \{x_1, x_2\}$ has been considered by Ishikawa [2], Ishikawa and Wakayama [3], Bressoud [1], and the case $\mathbb{X} = \{x_1, x_2, x_3\}$ by Jouhet and Zeng [4].

Before reinterpreting their formulas, let us recall some definitions, for which we refer to [8].

Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_j(\mathbb{A}-\mathbb{B})$ are defined by the generating series (with z an extra variable)

$$\sigma_z(\mathbb{A}-\mathbb{B}) := \prod_{b\in\mathbb{B}}(1-zb)/\prod_{a\in\mathbb{A}}(1-za)\,. \tag{4}$$

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{N}^r$, the Schur function $S_\lambda(\mathbb{A}-\mathbb{B})$ is

$$S_\lambda(\mathbb{A}-\mathbb{B}) = \left|S_{\lambda_{i+j-i}}(\mathbb{A}-\mathbb{B})\right|_{1\le i,j\le r}\,.$$

When λ is weakly decreasing, it is a *partition*. In that case, following Frobenius, one uses another description of partitions, by decomposing their diagram into its diagonal hooks, and writing $\lambda = (\alpha\,|\,\beta)$ ([8, p.3]).

On the space $\mathfrak{S}ym(\mathbb{A})$ of symmetric functions in $\mathbb{A}$, there exists a scalar product $(\,,\,)_\mathbb{A}$, and Schur functions indexed by partitions constitute an orthormal basis with respect to it :

$$\big(S_\lambda(\mathbb{A}),\, S_\lambda(\mathbb{A})\big)_\mathbb{A} = 1 \qquad \text{and} \qquad \big(S_\lambda(\mathbb{A}),\, S_\nu(\mathbb{A})\big)_\mathbb{A} = 0 \quad \lambda \neq \nu\,. \tag{5}$$

Another way to state (5) is to assert the existence of a *reproducing kernel*

$$K(\mathbb{A},\,\mathbb{B}) := \sigma_1(\mathbb{A}\mathbb{B}) = \prod_{a\in\mathbb{A},\,b\in\mathbb{B}}(1-ab)^{-1}$$

which satisfies

$$\forall f \in \mathfrak{S}ym\ ,\ \big(K(\mathbb{A},\,\mathbb{B}),\, f(\mathbb{A})\big)_\mathbb{A} = f(\mathbb{B})\ , \tag{6}$$

because the Cauchy formula states that

$$K(\mathbb{A},\,\mathbb{B}) = \sum_\lambda S_\lambda(\mathbb{A})\,S_\lambda(\mathbb{B})\ , \tag{7}$$

where the sum is over all partitions.

Given any symmetric function f, the adjoint $D(f;\bullet)$ of the multiplication by f is defined by

$$\forall f,\,g,\,h \in \mathfrak{S}ym\ ,\ \big(D(f;g)\,,\,h\big) = (g\,,\,fh)\,. \tag{8}$$

The formulas

$$\sigma_1(\Lambda^2(\mathbb{A})) = \sum S_\mu(\mathbb{A})\ ,\ \text{all } \mu \text{ with columns of even lengths} \tag{9}$$

and

$$\sigma_1(\mathbb{A}+\Lambda^2(\mathbb{A})) = \sum S_\lambda(\mathbb{A})\ ,\ \text{all partitions } \lambda \tag{10}$$

are equivalent, because $\Lambda^2(\mathbb{A}+1) = \mathbb{A}+\Lambda^2(\mathbb{A})$. Combinatorially, considering the letter 1 to be bigger than all the letters in $\mathbb{A}$, it is equivalent to enumerate all tableaux in $\mathbb{A}$, or all tableaux in the alphabet $\mathbb{A}+1 := \mathbb{A}\cup\{1\}$

with columns of even length (we write unions of alphabets with a '+'). Indeed, the 1's occupy the horizontal strip such that the inner shape μ has columns of even lengths.

The computations of Littlewood in chapter XI of his book [7] can be rephrased in the λ-rings paradigm as follows :

Theorem 1 (Littlewood). *Given two alphabets* $\mathbb{A}$, $\mathbb{X}$, *then one has the identities*

$$\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A})) = \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(\Lambda^2); S_\mu)\big)(\mathbb{X})\,, \tag{11}$$

$$\sigma_1(\mathbb{A}\mathbb{X} + S^2(\mathbb{A})) = \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(S^2); S_\mu)\big)(\mathbb{X})\,, \tag{12}$$

$$\sigma_1(\mathbb{A}\mathbb{X} - \Lambda^2(\mathbb{A})) = \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(-\Lambda^2); S_\mu)\big)(\mathbb{X})\,, \tag{13}$$

$$\sigma_1(\mathbb{A}\mathbb{X} - S^2(\mathbb{A})) = \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(-S^2); S_\mu)\big)(\mathbb{X})\ , \tag{14}$$

where the sums are over all partitions μ.

Proof. All four generating series factorize : $\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A})) = \sigma_1(\mathbb{A}\mathbb{X})\sigma_1(\Lambda^2(\mathbb{A}))$. Since $\sigma_1(\mathbb{A}\mathbb{X})$ is a reproducing kernel, one has for any μ :

$$\big(\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A}))\,,\, S_\mu(\mathbb{A})\big)_{\mathbb{A}} = \big(\sigma_1(\mathbb{A}\mathbb{X})\,,\, D(\sigma_1(\Lambda^2); S_\mu)(\mathbb{A})\big)_{\mathbb{A}}$$
$$= D(\sigma_1(\Lambda^2); S_\mu)(\mathbb{X})\ .$$

and this proves the first identity.

The same proof would work for $\sigma_1(\mathbb{A}\mathbb{X})\, f(\mathbb{A})$, f being any symmetric function. Indeed property (6) entails

$$\sigma_1(\mathbb{A}\mathbb{X})\, f(\mathbb{A}) = \sum_\mu S_\mu(\mathbb{A})\, D(f; S_\mu)(\mathbb{X})\ , \tag{15}$$

and formulas (12), (13) and (14) are special cases of it. □

Let us mention that other expansions are possible. Indeed,

$$S^2(\mathbb{A} \pm \mathbb{X}) = S^2(\mathbb{A}) \pm S^1(\mathbb{A})S^1(\mathbb{X}) + S^2(\pm\mathbb{X})\,,$$
$$\Lambda^2(\mathbb{A} \pm \mathbb{X}) = \Lambda^2(\mathbb{A}) \pm \Lambda^1(\mathbb{A})\Lambda^1(\mathbb{X}) + \Lambda^2(\pm\mathbb{X})\ .$$

It implies, for example, that

$$\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A})) = \sigma_1(-\Lambda^2(\mathbb{X}))\, \sigma_1(\Lambda^2(\mathbb{A} + \mathbb{X}))$$
$$= \sigma_1(-\Lambda^2(\mathbb{X})) \sum\nolimits_\nu S_\nu(\mathbb{A} + \mathbb{X})\ , \tag{16}$$

sum over all partitions ν with columns of even lengths. To go further, one would decompose $S_\nu(\mathbb{A} + \mathbb{X})$ in the basis of Schur functions in $\mathbb{A}$, and use in general Littlewood-Richardson's rule to collect functions in $\mathbb{X}$. But in the case where $\mathbb{X}$ is of small cardinality, the expansion can be written directly.

For example [1, Theorem IV], if $\mathbb{X} = x_1+x_2$, Equation (16) can be rewritten as

$$\sigma_1\big((x_1+x_2)\mathbb{A} + \Lambda^2(\mathbb{A})\big) = (1-x_1x_2)\sum\nolimits_\nu S_\nu(\mathbb{A}+x_1+x_2) \tag{17}$$

and is easy to expand in terms of the $S_\mu(\mathbb{A})$, interpreting it as the sum of all Young tableaux with even columns, on an alphabet with last two letters x_1, x_2.

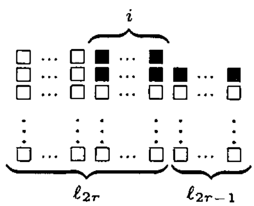
Let us show in more details how to explicitly calculate the coefficients in Littlewood's identities, for $\mathrm{card}(\mathbb{X}) = n = 1, 2, 3$, recovering the related results of [1, 2, 3, 4]. We shall treat only the case of $\sigma_1(\mathbb{X}\mathbb{A} + \Lambda^2\mathbb{A})$, the other being similar.

When $n = 1$, $\mathbb{X} = \{x_1\}$. Then $D(S_\nu; S_\mu)(\mathbb{X})$ is non zero iff the diagram of μ/ν is an horizontal strip. Therefore, given any partition μ, there is one and only one partition ν with even columns such that $D(S_\nu; S_\mu)(\mathbb{X})$ is non zero. Equation (11) specializes to

$$\sigma_1(x_1\mathbb{A} + \Lambda^2(\mathbb{A})) = \sum_\mu x_1^{\mathrm{odd}(\mu)}\, S_\mu(\mathbb{A})\ , \tag{18}$$

where $\mathrm{odd}(\mu)$ is the number of columns of μ of odd length.

Take now $n = 2$. A skew diagram μ/ν gives a non-zero function $S_{\mu/\nu}(x_1+x_2)$ iff it is made of horizontal strips of height 1 and 2. Decompose the partition into blocks $\mathcal{B}_k$, $k = 1, 2, \ldots$ of columns of the same length k (denoting ℓ_k its width). If k is odd, then the restriction of μ/ν to this block must be an horizontal strip of length ℓ_k, giving a factor $S_{\ell_k}(x_1+x_2)$. If k is even, the restriction of μ/ν is a $2\times i$ rectangle, with $0 \le i \le \ell_k$. The figure shows the restriction of a strip to two consecutive blocks :

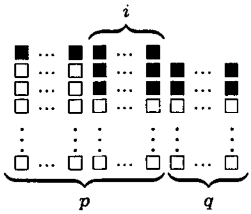


The sum over all possible i gives a factor $S_{\ell_k}(1+x_1x_2)$. Moreover, there is no interference between the different blocks, and therefore, one gets that the coefficient of $S_\mu(\mathbb{A})$ in $\sigma_1\big((x_1+x_2)\mathbb{A} + \Lambda^2(\mathbb{A})\big)$ is

$$D(\sigma_1(\Lambda^2); S_\mu)(x_1+x_2) = \prod_{k=0}^{\infty} S_{\ell_{2k}}(1+x_1x_2)\, S_{\ell_{2k+1}}(x_1+x_2)\ . \tag{19}$$

The case $n = 3$ is treated by Jouhet and Zeng [4]. The independent blocks are now the blocks $\mathcal{B}_k$ of columns of length $2k+1$, $2k$.

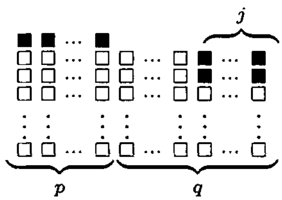
There are two possible configurations for those μ/ν giving a non-zero contribution :



$$, \quad p = \ell_{2k+1},\ q = \ell_{2k}, \quad i \geq 1\,.$$

The sum over all $i = 1, \ldots, p$ gives

$$x_1x_2x_3S_{p-1}(1 + x_1x_2x_3)S_q(\Lambda^2(x_1 + x_2 + x_3))\,,$$

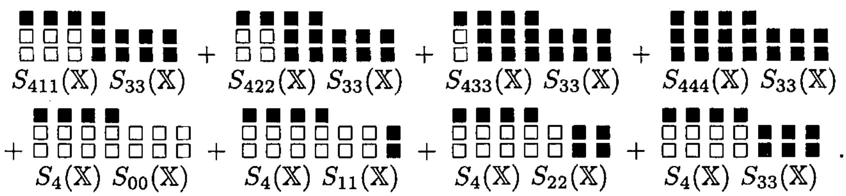


$$, \ i = 0,\ 0 \leq j \leq q\,.$$

The sum over all $j = 0, \ldots, q$ is

$$S_p(x_1 + x_2 + x_3)\, S_q(1 + \Lambda^2(\mathbb{X}))\,.$$

For example, for $\mu = [7, 7, 4]$, the possible configurations, together with their factors, are :



In summary, the contribution of the block $\mathcal{B}_k$ is

$$c(\mathcal{B}_k) = \sum_{i=0}^{p} S_{p-i}(\mathbb{X})S_{qq}(\mathbb{X}) + S_p\mathbb{X})\Big(1 + S_{11}(\mathbb{X}) + \cdots + S_{nn}(\mathbb{X})\Big) \qquad (20)$$

and the coefficient of $S_\mu(\mathbb{A})$ in $\sigma_1\big((x_1 + x_2 + x_3)\mathbb{A} + \Lambda^2(\mathbb{A})\big)$ is the product of all the factors $c(\mathcal{B}_k)$ given by the decomposition of μ into blocks. This expression is equivalent to Formula (8) of Jouhet and Zeng [4].

Littlewood's expansions were motivated by the description of characters of orthogonal and symplectic groups. In turn, these characters allow to expand Littlewood's formulas (13) and (14).

Following [7, p.240], using a second alphabet $\mathbb{X}$, define the *orthogonal Schur function* $\mathcal{O}_\lambda(\mathbb{A})$ by the generating series :

$$\prod_{i\leq j}(1-a_ia_j)\prod_i\prod_j(1-a_ix_j)^{-1}=\sum_\lambda \mathcal{O}_\lambda(\mathbb{X})\,S_\lambda(\mathbb{A})\ , \tag{21}$$

that is,

$$\sigma_1\big(\mathbb{A}\mathbb{X}-S^2(\mathbb{A})\big)=\sum_\lambda \mathcal{O}_\lambda(\mathbb{A})\,S_\lambda(\mathbb{X})\ . \tag{22}$$

Formula (14) can now be rewritten :

$$\mathcal{O}_\lambda = D\big(\sigma_1(-S^2)\,;S_\lambda\big) \tag{23}$$

and the inverse formula is

$$S_\lambda=\mathrm{subs}\Big(S=\mathcal{O}, D(\sigma_1(S^2)\,;S_\lambda)\Big) \tag{24}$$

which means that in that last case, one uses the operator $D\big(\sigma_1(S^2)\,;\bullet\big)$ as a formal operator on a vector space, the basis of which is indexed by partitions (and the action being obtained by identifying this basis with Schur functions).

For example, Littlewood [7, p. 241] gives (writing in the expansion of $\sigma_1(-S^2)$ and $\sigma_1(S^2)$ only the terms which give a non-zero contribution) :

$$\mathcal{O}_{332}=D(S_0-S_2+S_{31}-S_{33}\,;\,S_{332})=S_{332}-S_{332/2}+S_{332/31}-S_2$$

and the identity

$$D(S_0+S_2+S_{22}+S_{222}\,;\,S_{332})=S_{332}+S_{33}+S_{321}+S_{211}+S_{31}+S_{11}$$

implies, by formal change of S into $\mathcal{O}$:

$$S_{332}=\mathcal{O}_{332}+\mathcal{O}_{33}+S_{321}+\mathcal{O}_{211}+\mathcal{O}_{31}+\mathcal{O}_{11}\ .$$

Ishikawa and Wakayama [3] give the expansion of $\sigma_1((x_1+x_2)\mathbb{A}-S^2(\mathbb{A}))$, with $x_1x_2=1$. It means computing the specialization of orthogonal Schur functions for such $\mathbb{X}$. In fact the complete functions (resp. power sums) of $\mathbb{X}$ are Tchebychef polynomials of the first and second kind in the variable $(x_1+x_2)/2$. Ishikawa and Wakayama characterize those partitions for which $\mathcal{O}_\lambda(\mathbb{X})\neq 0$ in terms of their Frobenius decomposition, and obtains that in that case $\pm\mathcal{O}_\lambda(\mathbb{X})$ is equal to a Tchebychef polynomial of the second kind.

Using symmetrizing operators, one easily expands $\sigma_1(-S^2)$ as :

$$\sigma_1(-S^2)=\sum\pm S_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots}=S_{000\ldots}-S_{200\ldots}-S_{040\ldots}+S_{240\ldots}+\cdots \tag{25}$$
$$=S_0-S_2+S_{31}-S_{33}+\cdots\ . \tag{26}$$

In the first summation, Schur functions are indexed by vectors $[\epsilon_1,\epsilon_2,\epsilon_3,\ldots]$ such that $\epsilon_i\in\{0,2i\}$, and not by partitions, and the sign alternates with the parity of the number of non-zero components.

It is remarkable that the operator $D\big(\sigma_1(-S^2);\bullet\big)$ can also be written as an enumeration of vectors :

$$S_\lambda \to \mathcal{O}_\lambda = D\big(\sigma_1(-S^2), S_\lambda\big) = \sum_{\epsilon_1,\epsilon_2,\dots} \pm S_{\lambda/[\epsilon_1,\epsilon_2,\dots]} \tag{27}$$

(recall that, for any two integral vectors λ, ν, $S_{\lambda/\nu}$ stands for the determinant $\big|S_{\lambda_i+j-i-\nu_j}\big|$).

Indeed, the fact that $D(S_\nu; S_\lambda) = S_{\lambda/\nu}$ for two partitions, extends, by reordering the rows of the determinantal expression of S_ν and the columns of $S_{\lambda/\nu}$, to the case where ν is not a partition, i.e. is not decreasing.

Thus $\mathcal{O}_{332}$ can also be written

$$\begin{aligned}\mathcal{O}_{332} &= S_{332/000} - S_{332/200} - S_{332/040} + S_{332/240}\\ &= \begin{vmatrix} S_3 & S_4 & S_5\\ S_2 & S_3 & S_4\\ S_0 & S_1 & S_2\end{vmatrix} - \begin{vmatrix} S_1 & S_4 & S_5\\ S_0 & S_3 & S_4\\ 0 & S_1 & S_2\end{vmatrix} - \begin{vmatrix} S_3 & S_0 & S_5\\ S_2 & 0 & S_4\\ S_0 & 0 & S_2\end{vmatrix} + \begin{vmatrix} S_1 & S_0 & S_5\\ S_0 & 0 & S_4\\ 0 & 0 & S_2\end{vmatrix} .\end{aligned}$$

By mutilinearity of determinants, one can write the above sum as a single determinant, due to Weyl [11, Theorem 7.9.A], that we display for the order 3 :

$$\mathcal{O}_{\lambda_1\lambda_2\lambda_3} = \begin{vmatrix} S_{\lambda_1} - S_{\lambda_1-2} & S_{\lambda_1+1} - S_{\lambda_1+1-4} & S_{\lambda_1+2} - S_{\lambda_1+2-6}\\ S_{\lambda_2-1} - S_{\lambda_2-1-2} & S_{\lambda_2} - S_{\lambda_2-4} & S_{\lambda_2+1} - S_{\lambda_2+1-6}\\ S_{\lambda_3-2} - S_{\lambda_3-2} & S_{\lambda_3-1} - S_{\lambda_3-1-4} & S_{\lambda_3} - S_{\lambda_3-6}\end{vmatrix} . \tag{28}$$

For example, $\mathcal{O}_{332}$, again, is equal to

$$\begin{vmatrix} S_3 - S_1 & S_4 - S_0 & S_5 - 0\\ S_2 - S_0 & S_3 - 0 & S_4 - 0\\ S_0 - 0 & S_1 - 0 & S_2 - 0\end{vmatrix} .$$

We shall refer to [5] for determinantal expressions of orthogonal or symplectic characters. The case of "nearly rectangular shape" leads to an interesting combinatorics obtained by Krattenthaler [6].

Similarly, define the *symplectic Schur function* $\mathcal{S}p_\lambda$ by

$$\mathcal{S}p_\lambda = D(\sigma_1(-\Lambda^2)\,; S_\lambda)\ , \tag{29}$$

but this time, it will be evaluated into the alphabet
$\mathbb{Y} := \{x_1, 1/x_1, x_2, 1/x_2, \dots, x_n, 1/x_n\}$ instead of $\mathbb{X}$.

The identity

$$\sigma_1(-\Lambda^2) = \sum S_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3,\dots}\ , \tag{30}$$

sum over all $\epsilon_i \in \{0, 2i\}$, with $\epsilon_0 = 0$, extends, as before, to

$$\mathcal{S}p_\lambda = D(\sigma_1(-\Lambda^2)\,; S_\lambda) = \sum_{\epsilon_1,\epsilon_2,\dots} S_{\lambda/[0,\epsilon_1,\epsilon_2,\dots]}\ , \tag{31}$$

and this sum can be written as a single determinant due to Weyl [11, Theorem 7.8.E] :

$$\begin{vmatrix} S_{\lambda_1} & S_{\lambda_1+1} - S_{\lambda_1+1-2} & S_{\lambda_1+2} - S_{\lambda_1+2-4} \\ S_{\lambda_2-1} & S_{\lambda_2} - S_{\lambda_2-2} & S_{\lambda_2+1} - S_{\lambda_2+1-4} \\ S_{\lambda_3-2} & S_{\lambda_3-1} - S_{\lambda_3-1-2} & S_{\lambda_3} - S_{\lambda_3-4} \end{vmatrix} . \tag{32}$$

Lovers of λ-rings will prefer the expression

$$Sp_\lambda = S_\lambda(\mathbb{Y}-0, \mathbb{Y}-2, \mathbb{Y}-4, \dots) := \left| S_{\lambda_i+j-i}(\mathbb{Y}-2j+2) \right| , \tag{33}$$

with, for any alphabet $\mathbb{Y}$, any number k,

$$\sigma_z(\mathbb{Y}-k) := \sigma_z(\mathbb{Y}-1-\cdots-1) = (1-z)^k \sigma_z(\mathbb{Y}) .$$

Indeed, because

$$S_i(\mathbb{Y}-k) = S_i(\mathbb{Y}) - \binom{k}{1} S_{i-1}(\mathbb{Y}) + \binom{k}{2} S_{i-2}(\mathbb{Y}) - \cdots \pm S_{i-k}(\mathbb{Y}) ,$$

expanding each term of the determinant $\left|S_{\lambda_i+j-i}(\mathbb{Y}-2j+2)\right|$, one obtains Weyl's determinant by eliminating in each column multiples of the columns on its left.

In particular, when $x_1, \dots, x_n$ are specialized to 1, the determinant specializes to a determinant of binomial coefficients

$$\left| \binom{2n+\lambda_i-i-j+1}{\lambda_i+j-i} \right|_{1\le i,j\le n} \tag{34}$$

which is one of the many expressions of the dimension of a representation of $\mathfrak{S}p_{2n}$.

Let us mention that the interpretation of the orthogonal and symplectic Schur functions as characters depends on the parity of the number of variables. See Proctor [9] for the "'odd symplectic groups".

References

[1] D. Bressoud, *Identities for Schur functions and plane partitions*, Ramanujan J. **4** (2000) 69–80.

[2] M. Ishikawa, *Minor summation formula of Pfaffians and Schur functions identities*, FPSAC 95 (Formal Power Series and Algebraic Combinatorics), Université de Marne La Vallée (1995) 379–386.

[3] M. Ishikawa and M. Wakayama, *Applications of Minor-Summation Formula II. Pfaffians and Schur Polynomials*, J. Combin. Th., Ser. A **88** (1999) 136-157.

[4] F. Jouhet and J. Zeng, *Some new identities for Schur functions*, preprint (2001).

[5] K. Koike and I. Terada, *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, J. Algebra **107** (1987) 466–511.

[6] C. Krattenthaler, *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998) 1–64.

[7] D.E. Littlewood, *The theory of group characters*, Oxford University Press (1950).

[8] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, second edition, Oxford (1995).

[9] R. Proctor, *Interconnections between symplectic and orthogonal characters*, Contemp. Math. **88** (1989) 145–162.
[10] I. Schur, *Gesammelte Abhandlungen*, vol. III, p.456, Springer (1973).
[11] H. Weyl, *The classical groups*, Princeton Univ. Press (1939).

Institut Gaspard Monge, CNRS, Université de Marne-la-Vallée, 77454 Marne-la-Vallée Cedex, France
E-mail address: Alain.Lascoux@univ-mlv.fr

A q-ANALOG OF SCHUR'S Q-FUNCTIONS

GEANINA TUDOSE AND MICHAEL ZABROCKI

ABSTRACT. We present a family of analogs of the Hall-Littlewood symmetric functions in the Q-function algebra. The change of basis coefficients between this family and Schur's Q-functions are q-analogs of numbers of marked shifted tableaux. These coefficients exhibit many parallel properties to the Kostka-Foulkes polynomials.

1. INTRODUCTION

The space of Q-functions, Γ, is defined to be the algebra generated by the odd power sum elements $\{p_1, p_3, p_5, \dots\}$ as a subalgebra of the space of symmetric functions, Λ. Γ is associated to the representation theory of the spin group and is also related to the projective represenation theory of the symmetric and alternating groups. The fundamental basis for this space are Schur's Q-functions, $Q_\lambda[X]$, which are indexed by strict partitions λ. These functions hold the place that the Schur S-functions, $s_\mu[X]$ for μ a partition, represent in the algebra of the symmetric functions.

The space of symmetric functions contains an important basis, $H_\mu[X;q]$, the Hall-Littlewood symmetric functions [6]. Through specializations of the parameter q these functions interpolate several well studied bases of the symmetric functions and generalize features of these bases. They have elegant properties and may be seen in many different contexts of combinatorics, algebra, representation theory, geometry and mathematical physics.

It is natural to ask the question of what the analog of the Hall-Littlewood symmetric functions in the Q-function algebra should be. In this paper we introduce a family of functions $G_\lambda[X;q] \in \Gamma$ that answers this question since we observe that this family shares many of the combinatorial and algebraic properties of the $H_\mu[X;q]$ functions in the space of symmetric functions. We expect that these functions will also be interesting from the perspective of other fields as well.

From the combinatorial standpoint, we note that the coefficient of $s_\lambda[X]$ in the symmetric function $H_\mu[X;q]$ is the well known Kostka-Foulkes polynomial. This family of coefficients are known to be polynomials in the parameter q with non-negative integer coefficients and at $q = 1$ represent the number of column strict tableaux of shape λ and content μ. The combinatorial tools of jeu de taquin and the plactic monoid were in part developed to explain the connections between the Kostka-Foulkes polynomials and the column strict tableaux [15].

By comparison the coefficient of $Q_\lambda[X]$ in the function $G_\mu[X;q]$ is also a polynomial in q and we conjecture (and prove in certain cases) that it also has coefficients that are non-negative integers. At $q=1$ we know that these coefficients are the number of marked shifted tableaux with shape λ and content μ. A version of the RSK-algorithm was developed by Sagan, Worley and others [5], [19], [26], [24], and used to develop the theory of marked shifted tableaux. We hope that this theory can be extended to help answer the question of a combinatorial interpretation for these coefficients.

Our definition for the functions $G_\lambda[X;q]$ is motivated by viewing the symmetric functions $s_\mu[X]$ and $H_\mu[X;q]$ as compositions of operators. In the case of the Schur functions, the Bernstein operator $\mathbf{S}_m \in End(\Lambda)$ ([17] p. 96) has the property for $m \geq \mu_1$,

$$\mathbf{S}_m(s_\mu[X]) = s_{(m,\mu_1,\ldots,\mu_{\ell(\mu)})}[X].$$

That is, this formula is a recursive defintion for the Schur functions of degree $n+m$ as an algebraic relation that raises the degree of a symmetric function by m acting on a Schur function of degree n. For the Hall-Littlewood symmetric functions, the operator $\mathbf{H}_m \in End(\Lambda)$ with

$$\mathbf{H}_m(H_\mu[X;q]) = H_{(m,\mu_1,\ldots,\mu_{\ell(\mu)})}[X;q]$$

for $m \geq \mu_1$ is due to Jing [8]. In [27], it was noticed that these operators (as well as many others) are related by a simple algebraic q-twisting, $\widetilde{\mathbf{S}_m}^q = \mathbf{H}_m$ (the defintion of $\widetilde{\ }^q$ is stated precisely in equation (11) below).

The Schur's Q-functions, $Q_\lambda[X]$ may also be seen from this perspective ([7], [14], [17] p. 262-3). That is, there exists an operator $\mathbf{Q}_m \in End(\Gamma)$ such that for $m > \lambda_1$,

$$\mathbf{Q}_m(Q_\lambda[X]) = Q_{(m,\lambda_1,\lambda_2,\ldots\lambda_{\ell(\lambda)})}[X].$$

Since the defintion of the q-twisting $\widetilde{\ }^q$ extends to $End(\Gamma)$, a natural defintion for an analog to of the Hall-Littlewood symmetric functions in Γ is to define $\mathbf{G}_m := \widetilde{\mathbf{Q}_m}^q$ and then for $m > \lambda_1$, we set

$$G_{(m,\lambda_1,\ldots,\lambda_{\ell(\lambda)})}[X;q] := \mathbf{G}_m(G_\lambda[X;q]).$$

This framework provides us only with a possible definition for the Hall-Littlewood analogs in Γ. It remains to show that these functions share properties similar to those of the Hall-Littlewood functions. In this case we find some striking similarities that say we have indeed found the correct analog. We remark that similar functions were defined in [9] that are deformations of Schur's Q-functions and defined because of Lie algebra considerations.

This work is inspired by the results of the Hall-Littlewood functions and the desire to find analogous structure in the Q-function algebra. In addition, part of the motivation of defining these functions and identifying their properties is to find what features of the Hall-Littlewood symmetric functions are not unique to the symmetric function algebra and should hold in a more general setting. A goal of this reasearch is to possibly identify what the q-twisiting of equation (11) represents on a combinatorial, geometric or representation theoretical level and to show that the $G_\lambda[X;q]$ are another example of a structure that seems to exist in a more general context.

The remainder of this paper is divided into three sections and an appendix. The first section is simply an exposition of definitions and and notation related to the symmetric functions and Q-function algebra. We develop in some detail the perspective that bases of the symmetric functions and Q-functions can be seen as compositions of operators that have simple algebraic definitions. In the next section we introduce the $G_\lambda[X;q]$ functions and derive recurrences and some properties that are analogous to those that exist for the Hall-Littlewood symmetric functions. In a final section we discuss a generalization of the functions $G_\lambda[X;q]$ that are indexed by a sequence of strict partitions $(\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(k)})$ and the motivation for this generalization. These functions correspond to a q-analog of the product $Q_{\mu^{(1)}}[X]Q_{\mu^{(2)}}[X]\cdots Q_{\mu^{(k)}}[X]$ and the coefficients of these functions correspond to analogs of the generalized or parabolic Kostka polynomials of [11], [13], [20], [22], [21] and [23].

Finally, in the appendix we include tables of transition coefficients between the $G_\lambda[X;q]$ basis and the $Q_\lambda[X]$ basis for degrees 3 through 9. These tables are evidence of a very strong conjecture that these coefficients are polynomials in q with non-negative integer coefficients and represent a q analog of the number of marked shifted tableaux. This suggests that the marked shifted tableaux should have a poset structure similar to the charge poset for the column strict tableaux.

2. Notation and Definitions

2.1. Symmetric functions, partitions and columns strict tableaux. Consider Λ^X the ring of series of finite degree in the variables $x_1, x_2, x_3, \dots$ which are invariant under all permutations of the variables. This ring is algebraically generated by the set of elements $\{p_k[X] = \sum_i x_i^k\} \subset \Lambda^X$ and hence Λ^X is isomorphic to the ring $\Lambda = \mathbb{C}[p_1, p_2, p_3, \dots]$ with $deg(p_k) = k$. We will refer to both Λ and Λ^X as the ring of symmetric functions.

Λ is a graded ring and a basis for the component of degree n is given by the monomials $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$ where λ is a non-increasing sequence of non-negative integers such that the values sum to n. Such a sequence is called a partition of n (denoted $\lambda \vdash n$). The entries of λ are called the parts of the partition. The number of parts that are of size i in λ will be represented by $m_i(\lambda)$ and the total number of non-zero parts is represented by $\ell(\lambda) := \sum_i m_i(\lambda)$ and the size by $|\lambda| := \sum_k km_k(\lambda) = \sum_i \lambda_i$. A common statistic associated to partitions is $n(\lambda) := \sum_i (i-1)\lambda_i$.

The partial order on partitions, $\lambda \leq \mu$ if and only if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all $1 \leq k \leq \ell(\lambda)$, is called the dominance order. We call the operators

$$R_{ij}\lambda = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_{\ell(\lambda)})$$

for $1 \leq i \leq j \leq \ell(\lambda)$ 'raising operators' and they have the property that if $R_{ij}\lambda$ is a partition, then $R_{ij}\lambda \geq \lambda$.

We will consider three additional bases of Λ here. Following the notation of [17], we define the homogeneous (complete) symmetric functions as $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$ where $h_n = \sum_{\lambda \vdash n} p_\lambda / z_\lambda$ and $z_\lambda = \prod_{i=1}^{\ell(\lambda)} i^{m_i(\lambda)} m_i(\lambda)!$. The elementary symmetric functions are $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ where $e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} p_\lambda / z_\lambda$. By convention we set $p_0 = h_0 = e_0 = 1$ and $p_{-k} = h_{-k} = e_{-k} = 0$ for $k > 0$. The Schur functions are given by $s_\lambda = det|h_{\lambda_i+i-j}|_{1 \leq i,j \leq \ell(\lambda)}$. The sets $\{p_\lambda\}_{\lambda \vdash n}$, $\{h_\lambda\}_{\lambda \vdash n}$, $\{e_\lambda\}_{\lambda \vdash n}$ and $\{s_\lambda\}_{\lambda \vdash n}$ all form bases for the symmetric functions of degree n.

We will consider the elements of Λ as functors on the space Λ^X. If E is an element of Λ^X then we let $p_k[E] = E$ with x_i replaced with x_i^k, so that $p_k : \Lambda^X \to \Lambda^X$. We then extend this relation algebraically, $p_\lambda[E]$ will represent the expression $p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]$. In particular, if we take $E = X = x_1 + x_2 + x_3 + \cdots$ then $p_k[X] = \sum_i x_i^k$ and the map that sends $p_k \mapsto p_k[X]$ is a ring isomorphism $\Lambda \to \Lambda^X$ since $p_k[X] = x_1^k + x_2^k + x_3^k + \cdots$.

Identities which hold in the ring Λ specialize as well to the ring of symmetric polynomials in a finite number of variables. We will use the notation X_n to represent $x_1 + x_2 + \cdots + x_n$, and the map which sends Λ to Λ^{X_n} by $p_k \mapsto p_k[X_n]$ corresponds to the operation of specializing variables in a symmetric series from an infinite set variables to a symmetric polynomial in a finite set of variables. For notational purposes, capital letters used as variables the end of the alphabet will be used to represent a series of variables (e.g. $X = \sum_i x_i$ or $Y = \sum_i y_i$), while capital letters indexed by a number will represent a polynomial sum of variables (e.g. $X_n = \sum_{i=1}^n x_i$ or $Z_n = \sum_{i=1}^n z_i$).

We will need to adjoin to each of the rings Λ, Λ^X and Λ^{X_n} a special element q (or many special elements, if necessary) which acts much like a variable in this ring, however q will specialize to values in the field. q has the special property that $p_k[qX] = q^k p_k[X]$ and hence is not an element of our base field since for $c \in \mathbb{C}$, we have that $p_k[cX] = cp_k[X]$.

Notice that by definition we have in general $p_k[aX + bY] = ap_k[X] + bp_k[Y]$ for $a, b \in \mathbb{C}$. This implies that $f[-X]$ does not represent the symmetric series $f[X]$ with x_i replaced by $-x_i$ since $p_k[X]\big|_{x_i \to -x_i} = (-1)^k p_k[X]$, while $p_k[-X] = -p_k[X]$. To this end we introduce the notation $f[\epsilon X] =$

$f[qX]\big|_{q=-1}$. In the case of the power sums we have that $p_k[\epsilon X] = (-1)^k p_k[X]$ and hence $f[\epsilon X] = f[X]\big|_{x_i \to -x_i}$.

Consider the series $\Omega = \sum_{n\geq 0} h_n$ which is not an element of the ring Λ, but lies in the completion of this ring. We will use the morphism $p_k \mapsto p_k[X]$ on this element as well and manipulations of this notation allow us to derive the following identities, which we will use repeatedly in our calculations.

$$\Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n\geq 0} h_n[X] \tag{1}$$

$$\Omega[-X] = \prod_i (1-x_i) = 1/\Omega[X] \tag{2}$$

$$\Omega[X+Y] = \Omega[X]\Omega[Y] \tag{3}$$

$$\Omega[-\epsilon X] = \prod_i (1+x_i) = \sum_{n\geq 0} e_n[X]\,. \tag{4}$$

Define a generating function of operators $\mathbf{S}(z) = \sum_m \mathbf{S}_m z^m$ where for an arbitrary symmetric function $P[X]$, $\mathbf{S}(z)P[X] = P[X - 1/z]\Omega[zX]$. In this manner $\mathbf{S}_m$ acts on any symmetric function raising the degree of the function by m and has the action $\mathbf{S}_m P[X] = \mathbf{S}(z)P[X]\Big|_{z^m}$. A composition of the operators $\mathbf{S}(z_i)$ produces the expression

$$\mathbf{S}(z_1)\mathbf{S}(z_2)\cdots\mathbf{S}(z_k)1 = \Omega[Z_k X] \prod_{1\leq i<j\leq n} (1 - z_j/z_i), \tag{5}$$

where Z_k represents the sum $z_1 + z_2 + \cdots + z_k$. Since the coefficient of z^λ in $\Omega[Z_k X]$ then it must be that the coefficient of z^λ in the right hand side of (5) is given by $\prod_{1\leq i<j\leq n}(1 - R_{ij})h_\lambda[X]$, where $R_{ij}h_\lambda[X] = h_{R_{ij}\lambda}[X]$ (considering the h-functions indexed by sequences of numbers). This is an expression for the Schur function $s_\lambda[X]$, hence it follows that $\mathbf{S}_{\lambda_1}\mathbf{S}_{\lambda_2}\cdots\mathbf{S}_{\lambda_k}1 = s_\lambda[X]$. These operators are due to Bernstein [17] p.96. It follows that $\mathbf{S}_m(s_\lambda[X]) = s_{(m,\lambda)}[X]$ where (m,λ) denotes $(m, \lambda_1, \ldots, \lambda_{\ell(\lambda)})$ and $s_{(a_1,\ldots,a_\ell)}[X] = det|h_{a_i+i-j}[X]|$.

Remark 1: We follow [17] in the use of raising operators for our definitions, however we are being imprecise since our raising operators are not associative or commutative as defined. We will consider a symmetric function as a composition of operators (for example in equation 5) and the operators R_{ij} serve to raise or lower the indexing integer of the operator in the i and j^{th} positions respectively.

By acting on an arbitrary symmetric function using these operations and the relations in equations (1) (2) and (3) commutation relations of the operators follow very nicely. By expanding the left and right side of the following expression verifies that

$$\mathbf{S}(z)\mathbf{S}(u)P[X] = -\frac{u}{z}\mathbf{S}(u)\mathbf{S}(z)P[X].$$

By taking the coefficient of $u^m z^n$ in both sides of the equation, we find that $\mathbf{S}_n\mathbf{S}_m = -\mathbf{S}_{m-1}\mathbf{S}_{n+1}$ which also implies that $\mathbf{S}_m\mathbf{S}_{m+1} = 0$. Many of the calculations of commutation relations for other operators are of a similar sort of manipulation.

A Young diagram for a partition will be a collection of cells of the integer grid lying in the first quadrant. For a partition λ, $Y(\lambda) = \{(i,j) : 0 \leq j < \ell(\lambda) \text{ and } 0 \leq i < \lambda_{j+1}\}$. The reason why we consider empty cells rather than say, points, is because we wish to consider fillings of these cells. A tableau is a map from the set $Y(\lambda)$ to $\mathbb{N}$, this may be represented on a Young diagram by writing integers within the cells of a graphical representation of a Young diagram (see figure 1). The shape of the tableau is the partition λ. We say that a tableau T is column strict if $T(i,j) \leq T(i+1,j)$ and $T(i,j) < T(i,j+1)$ whenever the points $(i+1,j)$ or $(i,j+1)$ are in $Y(\lambda)$. Let $m_k(T)$ represent the number of points p in $Y(\lambda)$ such that $T(p) = k$. The vector $(m_1(T), m_2(T), \ldots)$ is the content of the tableau T.

The Pieri rule describes a combinatorial method for computing the product of $h_m[X]$ and $s_\mu[X]$ expanded in the Schur basis. We will use the notation $\lambda/\mu \in \mathcal{H}_m$ to represent that $|\lambda| - |\mu| = m$ and for $1 \leq i \leq \ell(\lambda)$, $\mu_i \leq \lambda_i$ and $\mu_i \geq \lambda_{i+1}$. It may be easily shown that

$$h_m[X]s_\mu[X] = \sum_{\lambda/\mu\in\mathcal{H}_m} s_\lambda[X]. \tag{6}$$

This gives a method for computing the expansion of the $h_\mu[X]$ basis in terms of the Schur functions. Consider the coefficients $K_{\lambda\mu}$ defined by the expression

$$h_\mu[X] = \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}s_\lambda[X]. \tag{7}$$

$K_{\lambda\mu}$ are called the Kostka numbers and are equal to the number of column strict tableaux of shape λ and content μ.

2.2. Kostka polynomials and Hall-Littlewood symmetric functions.

Define the following symmetric functions

$$\begin{aligned} H_\lambda[X;q] &= \prod_{i<j} \frac{1-R_{ij}}{1-qR_{ij}} h_\lambda[X] \\ &= \prod_{1\leq i<j\leq n} (1 + (q-1)R_{ij} + (q^2-q)R_{ij}^2 + \cdots)h_\lambda[X]. \end{aligned} \tag{8}$$

They will be referred to as Hall-Littlewood symmetric functions as they are transformations of the symmetric polynomials defined by Hall [6] (see [17] for a modern account where $Q_\mu(x;q)$ in their notation is $H_\mu[X(1-q);q]$ in ours). The coefficient of $s_\lambda[X]$ in $H_\mu[X;q]$ is known as the Koskta Foulkes polynomial $K_{\lambda\mu}(q)$. That is, we have the expansion

$$H_\mu[X;q] = \sum_\lambda K_{\lambda\mu}(q) s_\lambda[X]. \tag{9}$$

We will present some of the properties of the Kostka-Foulkes polynomials and the Hall-Littlewood symmetric functions below. First, it will be important to establish some identities for manipulating these functions.

Let $\mathbf{H}(z) = \sum_m \mathbf{H}_m z^m$ be defined as the operation $\mathbf{H}(z)P[X] = P[X - (1-q)/z]\Omega[zX]$. Taking the coefficient of z^m defines the operator $\mathbf{H}_m = \mathbf{H}(z)\Big|_{z^m}$ which has the effect of raising the degree of the symmetric function it is acting on by m. A composition of these operators has the expression

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\cdots\mathbf{H}(z_k)1 = \Omega[Z_k X] \prod_{1\le i<j\le k} \frac{1 - z_j/z_i}{1 - qz_j/z_i}. \tag{10}$$

Now since $h_{R_{ij}\lambda} = \frac{z_j}{z_i}\Omega[Z_k X]\Big|_{z^\lambda}$, it is clear that the coefficient of z^λ in the right hand side of (10) is exactly the right hand side of (8) and hence $\mathbf{H}_{\lambda_1}\mathbf{H}_{\lambda_2}\cdots\mathbf{H}_{\lambda_{\ell(\lambda)}}1 = H_\lambda[X;q]$. This operator also satisfies the relations $\mathbf{H}_{m-1}\mathbf{H}_m = q\mathbf{H}_m\mathbf{H}_{m-1}$ and $\mathbf{H}_{m-1}\mathbf{H}_n - q\mathbf{H}_m\mathbf{H}_{n-1} = q\mathbf{H}_n\mathbf{H}_{m-1} - \mathbf{H}_{n-1}\mathbf{H}_m$. This relation can be derived as we did for the Schur function operators by demonstrating $(z-qu)\mathbf{H}(z)\mathbf{H}(u) = (qz-u)\mathbf{H}(u)\mathbf{H}(z)$ on an arbitrary symmetric function $P[X]$.

This family of operators $\mathbf{H}_m$ is due to Jing [8] and they are sometimes referred to as 'vertex operators' for the Hall-Littlewood symmetric functions.

For an element $V \in Hom(\Lambda, \Lambda)$, define

$$\widetilde{V}^q P[X] = V^Y P[qX + (1-q)Y]\Big|_{Y=X}, \tag{11}$$

where V^Y denotes that as an operation on symmetric functions in the Y variables only and $Y = X$ represents setting the Y variables equal to the X variables after the operation is completed. This is a q-analog of the operator V and we remark that $\widetilde{\mathbf{S}(z)}^q = \mathbf{H}(z)$. This follows by calculating

$$\begin{aligned}
\widetilde{\mathbf{S}(z)}^q P[X] &= \mathbf{S}^Y(z)P[qX + (1-q)Y]\Big|_{Y=X} \\
&= P[qX + (1-q)(Y - 1/z)]\Omega[zY]\Big|_{Y=X} \\
&= P[X - (1-q)/z]\Omega[zX] = \mathbf{H}(z)P[X].
\end{aligned}$$

This relationship between $\mathbf{S}(z)$ and $\mathbf{H}(z)$ is the motivation for our definition of the q-analog of Schur's Q-functions.

The functions $H_\lambda[X;q]$ interpolate between the functions $s_\lambda[X] = H_\lambda[X;0]$ and $h_\lambda[X] = H_\lambda[X;1]$. The Kostka-Foulkes polynomials are defined as the q-polynomial coefficient of $s_\lambda[X]$ in $H_\mu[X;q]$ and hence we have the expansion analogous to (7).

$$H_\mu[X;q] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q) s_\lambda[X]. \tag{12}$$

The coefficients $K_{\lambda\mu}(q)$ are clearly polynomials in q, but it is surprising to find that the coefficients of the polynomials are non-negative integers.

A defining recurrence can be derived $K_{\lambda\mu}(q)$ in terms of the Kostka-Foulkes polynomials indexed by partitions of size $|\mu| - \mu_1$ using the formula for $\mathbf{H}_m$. This recurrence is often referred to as the 'Morris recurrence' for the Kostka-Foulkes polynomials [18]. The action of $\mathbf{H}_m$ on the Schur functions is given by

$$\mathbf{H}_m(s_\mu[X]) = \sum_{i \geq 0} \sum_{\mu/\lambda \in \mathcal{H}_i} q^i s_{(m+i,\lambda)}[X]. \tag{13}$$

It is not immediately obvious that at $q = 1$, the previous equation reduces to the Pieri rule and at $q = 0$ the formula is simply $\mathbf{S}_m(s_\mu[X]) = s_{(m,\mu)}[X]$. Using (13) and equating coefficients of s_λ on both sides of the equation $\mathbf{H}_m(H_\mu[X;q]) = \sum_\lambda K_{\lambda\mu}(q)\mathbf{H}_m(s_\lambda[X])$, we arrive at the Morris recurrence

$$K_{\alpha,(m,\mu)}(q) = \sum_{s:\alpha_s \geq m} (-1)^{s-1} q^{\alpha_s - m} \sum_{\lambda:\lambda/\alpha^{(s)} \in \mathcal{H}_{(\alpha_s - m)}} K_{\lambda\mu}(q), \tag{14}$$

where $m > \mu_1$ and $\alpha^{(s)}$ is α with part α_s removed.

The Kostka-Foulkes polynomials and the generating functions $H_\mu[X;q]$ have the following important properties which we simply list here so that we may draw a connection to analogous formulae. For a more detailed reference of these sorts of properties we refer the interested reader to the excellent survey article [1].

(1) the degree in q of $K_{\lambda\mu}(q)$ is $n(\mu) - n(\lambda)$.
(2) $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$ which implies $H_\mu[X;0] = s_\mu[X]$, $K_{\lambda\mu}(1) = K_{\lambda\mu}$, so that $H_\mu[X;1] = h_\mu[X]$, $K_{\lambda\lambda}(q) = 1$ and $K_{(|\mu|)\mu}(q) = q^{n(\mu)}$. We also have that $K_{\lambda\mu}(q) = 0$ if $\lambda < \mu$.
(3) $K_{\lambda\mu}(q) = \sum_T q^{c(T)}$, where the sum is over all column strict tableaux of shape λ and content μ and $c(T)$ denotes the charge of a tableau T (see [15]) and hence is a polynomial with non-negative integer coefficients.
(4) A combinatorial interpretation for these coefficients exists in terms of objects known as rigged configurations [12].
(5) $H_{(1^n)}[X;q] = e_n\left[\frac{X}{1-q}\right](q;q)_n$ where $(q;q)_n = \prod_{i=1}^n (1-q^i)$.
(6) If ζ is k^{th} root of unity, $H_\mu[X;\zeta]$ factors into a product of symmetric functions.
(7) Set $K'_{\mu\lambda}(q) := q^{n(\lambda)-n(\mu)} K_{\mu\lambda}(1/q)$, then $K'_{\mu\lambda}(q) \geq K'_{\mu\nu}(q)$ for $\lambda \leq \nu$.

$$\begin{array}{|c|c|c|c|c|c|}
\hline 4 & 6 & 7 \\
\hline 3 & 5 & 5 \\
\hline 2 & 2 & 3 & 4 & 6 \\
\hline 1 & 1 & 1 & 1 & 2 & 3 \\
\hline
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 5 & & & \\
 & & 3' & 3 & 4 & 5 & \\
 & 2' & 2 & 3' & 4' & 4 & \\
1' & 1 & 2' & 2 & 2 & 3 & 3
\end{array}$$

FIGURE 1. The diagram on the left represents a column strict tableau of shape $(6, 5, 3, 3)$ and content $(4, 3, 3, 3, 2, 2, 2, 1)$. The diagram on the right represents a shifted marked tableau of shape $(7, 5, 4, 1)$ and content $(2, 5, 5, 3, 2)$. This tableau has labels which are marked on the diagonal.

(8) $K_{\lambda+(a),\mu+(a)}(q) \geq K_{\lambda,\mu}(q)$, where $\lambda + (a)$ represents the partition λ with a part of size a inserted into it.

(9) $K_{\lambda\mu}(q) = \sum_{w \in S_n} sign(w)\mathcal{P}_q(w(\lambda + \rho) - (\mu + \rho))$ where $\mathcal{P}_q(\alpha)$ is the coefficient of x^α in $\prod_{1 \leq i < j \leq n}(1 - qx_i/x_j)^{-1}$, a q analog of the Kostant partition function and $\rho = (\ell(\mu) - 1, \ell(\mu) - 2, \ldots , 1, 0)$.

(10) $H_\mu[X; q]H_\lambda[X; q] = \sum_\gamma d^\nu_{\lambda\mu}(q)H_\nu[X; q]$, for some coefficients $d^\nu_{\lambda\mu}(q)$ with the property that if the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu} = 0$ then $d^\nu_{\lambda\mu}(q) = 0$. These coefficients are a transformation of the Hall algebra structure coefficients.

(11) For the scalar product $\langle s_\lambda[X], s_\mu[X]\rangle = \delta_{\lambda\mu}$, the basis $H_\mu[X(1 - q); q]$ is orthogonal with respect to $H_\lambda[X; q]$, that is $\langle H_\lambda[X; q], H_\mu[X(1 - q); q]\rangle = 0$ if $\lambda \neq \mu$.

2.3. **Schur's Q-functions, strict partitions, and marked shifted tableaux.** The Q-function algebra is a sub-algebra of the symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$. A typical monomial in this algebra will be p_λ, where λ is a partition and λ_i is odd. A partition λ is strict if $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq \ell(\lambda) - 1$ and a partition λ is odd if λ_i is odd for $1 \leq i \leq \ell(\lambda)$. We will use the notation $\lambda \vdash_s n$ (respectively $\lambda \vdash_o n$) to denote that λ is a partition of size n that is strict (respectively odd). Note that the number of strict partitions of size n and the number of odd partitions of size n is the same (proof: write out a generating function for each sequence).

The analog of the homogeneous and elementary symmetric functions in Γ are the functions $q_\lambda := q_{\lambda_1}q_{\lambda_2}\cdots q_{\lambda_{\ell(\lambda)}}$, where $q_n = \sum_{\lambda\vdash_o n} 2^{\ell(\lambda)}p_\lambda/z_\lambda$. Define an algebra morphism $\theta : \Lambda \to \Gamma$ by the action on the p_n generators as $\theta(p_n) = (1 - (-1)^n)p_n$. That is $\theta(p_n) = 2p_n$ if n is odd and $\theta(p_n) = 0$ for n even. θ has the property that $\theta(h_n) = \theta(e_n) = q_n$ and may be represented in our notation as $\theta(p_n[X]) = p_n[(1 - \epsilon)X]$. Under this morphism, our Cauchy element may also be considered a generating function for the q_n

elements since

$$\Omega[(1-\epsilon)X] = \sum_{n\geq 0} q_n[X] = \prod_i \frac{1+x_i}{1-x_i}. \tag{15}$$

It follows that $\{p_\lambda\}_{\lambda\vdash_o n}$, $\{q_\lambda\}_{\lambda\vdash_o n}$, $\{q_\lambda\}_{\lambda\vdash_s n}$ are all bases for the subspace of Q-functions of degree n. Another fundamental basis for this space are the Schur's Q-functions $Q_\lambda[X] = \theta(H_\lambda[X;-1])$. These functions hold a similar place in the Q-function algebra that the Schur functions hold in Λ. In particular, $\{Q_\lambda[X]\}_{\lambda\vdash_s n}$ is a basis for the Q-functions of degree n.

In analogy with the Schur functions, $Q_\lambda[X]$ may also be defined with a raising operator formula by setting $q=-1$ and applying the θ homomorphism to equation (8). We arrive at the formula:

$$Q_\lambda[X] = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}} q_\lambda[X] = \prod_{i<j} (1 - 2R_{ij} + 2R_{ij}^2 - \cdots) q_\lambda[X], \tag{16}$$

where the operators now act as $R_{ij} q_\lambda[X] = q_{R_{ij}\lambda}[X]$. Furthermore, they have a formula as the coefficient in a generating function:

$$Q_\lambda[X] = \Omega[(1-\epsilon)Z_n X] \prod_{1\leq i<j\leq n} \frac{1-z_j/z_i}{1+z_j/z_i}\Big|_{z^\lambda}. \tag{17}$$

As with Schur functions and the Hall-Littlewood functions, the raising operator formula leads us to an operator definition. By setting $\mathbf{Q}(z)P[X] = P\left[X - \frac{1}{z}\right]\Omega[(1-\epsilon)zX]$, it is easily shown that

$$\mathbf{Q}(z_1)\mathbf{Q}(z_2)\cdots\mathbf{Q}(z_n)1 = \Omega[(1-\epsilon)Z_n X] \prod_{1\leq i<j\leq n} \frac{1-z_j/z_i}{1+z_j/z_i},$$

and hence if we set $\mathbf{Q}_m P[X] = \mathbf{Q}(z)P[X]\Big|_{z^m}$ then $\mathbf{Q}_m(Q_\lambda[X]) = Q_{(m,\lambda)}[X]$ as long as $m > \lambda_1$. The commutation relations for the $\mathbf{Q}_m$ are

$$\mathbf{Q}_m\mathbf{Q}_n = -\mathbf{Q}_n\mathbf{Q}_m \text{ for } m \neq -n, \tag{18}$$

$$\mathbf{Q}_m\mathbf{Q}_{-m} = 2(-1)^m - \mathbf{Q}_{-m}\mathbf{Q}_m \text{ if } m \neq 0, \tag{19}$$

$$\mathbf{Q}_m^2 = 0 \text{ if } m \neq 0 \text{ and } \mathbf{Q}_0^2 = 1. \tag{20}$$

These formulas allow us to straighten the $Q_\mu[X]$ functions when they are not indexed by a strict partition.

The Q-function algebra is endowed with a natural scalar product. If we set $\langle p_\lambda, p_\mu\rangle_\Gamma = 2^{\ell(\lambda)}\delta_{\lambda\mu} z_\lambda$ for $\lambda, \mu \vdash_o n$, then it may be shown that we also have

$$\langle Q_\lambda[X], Q_\mu[X]\rangle_\Gamma = 2^{\ell(\lambda)}\delta_{\lambda\mu}. \tag{21}$$

A shifted Young diagram for a partition will again be a collection of cells lying in the first quadrant. For a strict partition λ, let $YS(\lambda) = \{(i,j) :$

$0 \le j < \ell(\lambda)$ and $j \le i < \lambda_{j+1} + j\}$. A marked shifted tableau T of shape λ is a map from $YS(\lambda)$ to the set of marked integers $\{1' < 1 < 2' < 2 < \dots\}$ that satisfy the following conditions

- $T(i,j) \le T(i+1,j)$ and $T(i,j) \le T(i,j+1)$
- If $T(i,j) = k$ for some integer k (i.e. has an unmarked label) then $T(i,j+1) \ne k$
- If $T(i,j) = k'$ for some marked label k' then $T(i+1,j) \ne k'$.

We may represent these objects graphically with a Young diagram representing λ and the cells filled with the marked integer alphabet. If T is a marked shifted tableau, then we will set $m_i(T)$ as the number of occurrences of i and i' in T. The sequence $(m_1(T), m_2(T), m_3(T), \dots)$ is the content of T.

The combinatorial definition of the marked shifted tableaux is defined so that it reflects the change of basis coefficients between the q_μ and Q_λ basis. The rule for computing the product of $q_m[X]$ and $Q_\mu[X]$ when expanded in the Schur Q-functions is the analog of the Pieri rule for the Γ space. If $\lambda/\mu \in \mathcal{H}_m$ then $a(\lambda/\mu)$ will represent 1+ the number of $1 < j \le \ell(\lambda)$ such that $\lambda_j > \mu_j$ and $\mu_{j-1} > \lambda_j$. We may show that

$$q_m[X]Q_\mu[X] = \sum_{\lambda/\mu \in \mathcal{H}_m} 2^{a(\lambda/\mu) - \ell(\lambda) + \ell(\mu)} Q_\lambda[X]. \tag{22}$$

Denote by $L_{\lambda\mu}$ the number of marked shifted tableaux T of shape λ and content μ (where λ is a strict partition) such that $T(i,i)$ is not a marked integer. We may expand the function $q_\mu[X]$ in terms of the Q-functions using (22) to show

$$q_\mu[X] = \sum_{\lambda \vdash |\mu|} L_{\lambda\mu} Q_\lambda[X]. \tag{23}$$

3. The Q-Hall-Littlewood basis $G_\lambda[X;q]$ for the algebra Γ

In this section we define a new family of functions $G_\lambda[X;q]$ which seems to play the same role as the Hall-Littlewood functions $H_\lambda[X;q]$ in the Q-functions algebra. These functions are introduced via a raising operator formula similar to (8). This definition permits an equivalent interpretation via a corresponding vertex operator $\mathbf{G}_m$ whose properties are analogues to both the Hall-Littlewood vertex operator $\mathbf{H}_m$ and $\mathbf{Q}_m$.

Note: From here, unless otherwise stated, all partitions are considered strict.

3.1. **Raising operator formula.** We define the following analog of the Hall-Littlewood functions in the subalgebra Γ

$$G_\lambda[X;q] := \prod_{1\le i<j\le n}\left(\frac{1+qR_{ij}}{1-qR_{ij}}\right)\left(\frac{1-R_{ij}}{1+R_{ij}}\right) q_\lambda[X]$$

$$= \prod_{1\le i<j\le n}\left(\frac{1+qR_{ij}}{1-qR_{ij}}\right) Q_\lambda[X]. \tag{24}$$

We call the functions $G_\lambda \in \Gamma \otimes_{\mathbb{C}} \mathbb{C}(q)$ the *Q-Hall-Littlewood functions.*

In $\Gamma \otimes \mathbb{C}(q)$ this family can be expressed in the basis of Q-functions as

$$G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X], \tag{25}$$

which can be viewed as a q-analog of (23). We call the coefficients $L_{\lambda\mu}(q)$ the *Q-Kostka polynomials.* We shall see that this family of polynomials shares many of the same properties with the classical Kostka-Foulkes polynomials. Tables of these coefficients are given in an Appendix. It follows from (24) that $L_{\lambda\mu}(q)$ have integer coefficients and $L_{\lambda\mu}(q) = 0$ if $\lambda < \mu$. This shows

Proposition 1. *The G_λ, λ strict, form a $\mathbb{Z}$-basis for $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[q]$.*

The basis G_λ interpolates between the Schur's Q-functions and the functions q_μ because $G_\lambda[X;0] = Q_\lambda[X]$ and $G_\lambda[X;1] = q_\lambda[X]$ as is clear from (24).

Since the coefficient of z^λ in $\Omega[(1-\epsilon)Z_nX]$ is $q_\lambda[X]$ equation (24) implies

$$G_\lambda[X;q] = \prod_{1\le i<j\le n}\left(\frac{1-z_j/z_i}{1+z_j/z_i}\right)\left(\frac{1+qz_j/z_i}{1-qz_j/z_i}\right)\Omega[(1-\epsilon)Z_nX]\Big|_{z^\lambda}. \tag{26}$$

Define the operator $\mathbf{G}(z)$ acting on an arbitrary symmetric function $P[X]$ as

$$\mathbf{G}(z)P[X] = P\left[X - \frac{1-q}{z}\right]\Omega[(1-\epsilon)zX]. \tag{27}$$

The operator $\mathbf{G}(z)$ defines a family of operators as $\mathbf{G}(z) = \sum_{m\in\mathbb{Z}} \mathbf{G}_m z^m$ and hence $\mathbf{G}_m P[X] = \mathbf{G}(z)P[X]\Big|_{z^m}$.

If we consider a composition of these operators acting on the symmetric function 1 we obtain the following

$$\mathbf{G}(z_1)\mathbf{G}(z_2)\cdots\mathbf{G}(z_n)1 = \prod_{1\le i<j\le n}\left(\frac{1-z_j/z_i}{1+z_j/z_i}\right)\left(\frac{1+qz_j/z_i}{1-qz_j/z_i}\right)$$

$$\times\Omega[(1-\epsilon)Z_nX], \tag{28}$$

which together with relation (26) gives

$$G_\lambda[X;q] = \mathbf{G}_{\lambda_1}\ldots\mathbf{G}_{\ell(\lambda)}(1).$$

Next we investigate some properties of this operator. First, $\mathbf{G}_m$ satisfies the following commutation relation.

Proposition 2. *For all $r, s \in \mathbb{Z}$ we have*

$$(1-q^2)(\mathbf{G}_r\mathbf{G}_s+\mathbf{G}_s\mathbf{G}_r)+q(\mathbf{G}_{r-1}\mathbf{G}_{s+1}-\mathbf{G}_{s+1}\mathbf{G}_{r-1}+\mathbf{G}_{s-1}\mathbf{G}_{r+1}-\mathbf{G}_{r+1}\mathbf{G}_{s-1}) = 2(-1)^r(1-q)^2\delta_{r,-s}.$$

Proof We will prove this relation in a few steps. Consider $\mathbf{G}(u)$ and $\mathbf{G}(z)$ the operator $\mathbf{G}$ defined above on the variable u and z respectively. We are looking at the composition of these two operators.

Step 1. We may write

$$\mathbf{G}(u)\mathbf{G}(z) = G(z,u)F(z/u)F(-qz/u),$$

where $G(z,u)$ is an operator symmetric in z and u defined by

$$G(z,u)P[X] := P\left[X - \frac{1-q}{u} - \frac{1-q}{z}\right]\Omega[(1-\epsilon)(z+u)X]$$

and $F(t) := \dfrac{1-t}{1+t}$. This is easily seen from

$$\text{(29)} \qquad \mathbf{G}(u)\mathbf{G}(z)P[X] = \mathbf{G}(u)P\left[X - \frac{1-q}{z}\right]\Omega[(1-\epsilon)zX] =$$

$$= P\left[X-(1-q)\left(\frac{1}{z}+\frac{1}{u}\right)\right]\Omega[(1-\epsilon)(z+u)X]\Omega\left[(1-\epsilon)(q-1)\frac{z}{u}\right].$$

Note that first two factors are exactly $G(u,z)$ and $\Omega\left[(1-\epsilon)(q-1)\frac{z}{u}\right]$ is equal to $F(z/u)F(-qz/u)$. Denote by $\alpha(t) := F(t) + F(t^{-1}) = \sum_{r\in\mathbb{Z}} 2(-1)^r t^r$.

Step 2. Let $q_i^\perp$ be the adjoint of multiplication by $q_i[X]$ in the algebra Γ under the scalar product $<,>_\Gamma$. Consider $Q^\perp(t)$ its generating function, i.e., $Q^\perp(t) = \sum_i q_i^\perp t^i$.

It is not difficult to see that $P[X+t] = Q^\perp(t)P[X]$. In fact it suffices to show this for a suitable basis element in Γ, namely the power sum $p_\lambda[X]$, where λ is an odd partition. We show that $q_i^\perp p_\lambda[X] = p_\lambda[X+t]\Big|_{t^i}$. We have

$$\text{(30)} \qquad p_\mu[X+t]\Big|_{t^i} = (p_{\mu_1}[X]+t^{\mu_1})\dots(p_{\mu_\ell}[X]+t^{\mu_\ell})\Big|_{t^i}.$$

At the same time we also know that $p_\lambda^\perp p_\mu = 2^{-\ell(\lambda)}\frac{z_\mu}{z_\nu}p_\nu$ if $\nu \uplus \lambda = \mu$ since

$$\text{(31)} \quad p_\lambda^\perp p_\mu\Big|_{p_\nu} = \left\langle p_\lambda^\perp p_\mu, \frac{2^{\ell(\nu)}p_\nu}{z_\nu}\right\rangle_\Gamma = \frac{2^{\ell(\nu)}}{z_\nu}\langle p_\mu, p_{\lambda\uplus\nu}\rangle_\Gamma = \frac{2^{\ell(\nu)}z_\mu}{2^{\ell(\mu)}z_\nu}\delta_{\mu,\lambda\uplus\nu}.$$

Now because $q_i = \sum_{\lambda\vdash_o i} 2^{\ell(\lambda)}p_\lambda/z_\lambda$, then $q_i^\perp p_\mu[X] = \sum_{\lambda\uplus\nu=\mu}\frac{z_\mu}{z_\lambda z_\nu}p_\nu$ where $\lambda\vdash_o i$ and this is exactly the right hand side of (30).

Also recall that $\Omega[(1-\epsilon)tX] = Q(t)$, where Q is the generating functions for q_i. If we replace these in the expression of $G(z,u)$ we obtain

$$G(z,u) = Q(z)Q(u)Q^{\perp}(-u^{-1})Q^{\perp}(-z^{-1})Q^{\perp}(qu^{-1})Q^{\perp}(qz^{-1}).$$

We know that $Q(u)Q(z)\alpha(u/z) = \alpha(u/z)$ ([17] Chap III. 8, p. 263), and hence we have as well $Q^{\perp}(u)Q^{\perp}(z)\alpha(u/z) = \alpha(u/z)$. Therefore

$$G(z,u)\alpha(u/z) = \alpha(u/z).$$

Step 3. Consider the following expression.

$$\begin{aligned}[&\mathbf{G}(u)\mathbf{G}(z)(1-qz/u)(1+qu/z) + \mathbf{G}(z)\mathbf{G}(u)(1-qu/z)(1+qz/u)] \\ &=(1+qz/u)(1+qu/z)G(u,z)\alpha(u/z) \\ &=(1+qz/u+qu/z+q^2)\alpha(u/z),\end{aligned}$$

and so

$$\begin{aligned}\mathbf{G}(u)\mathbf{G}(z)&(1-qz/u+qu/z-q^2) + \mathbf{G}(u)\mathbf{G}(z)(1-qu/z+qz/u-q^2) \\ &= (1+qz/u+qu/z+q^2)\alpha(u/z) = (1-q)^2\alpha(u/z).\end{aligned}$$

since we have as a series $t\alpha(t) = -\alpha(t)$. By taking the coefficient of $u^r z^s$ in the left hand side when expanded as series and equating with the corresponding coefficient in the right hand side we obtain the desired relation. □

For $q = 0$ in the relation above we recover the commutation relations of the operator $\mathbf{Q}$ given in equations (18), (19) and (20). At $q = 1$, $\mathbf{G}_m$ becomes multiplication by q_m and hence is commutative.

Formula (26) may be used to derive the action of the operator $\mathbf{G}_m$ on the basis of Schur's Q-functions.

Proposition 3. *For $m > 0$,*

$$\mathbf{G}_m(Q_\lambda[X]) = \sum_{i\geq 0} q^i \sum_{\mu:\lambda/\mu\in\mathcal{H}_i} 2^{a(\lambda/\mu)}(-1)^{\epsilon(m+i,\mu)}Q_{\mu+(m+i)}[X], \tag{32}$$

where $\mu + (k)$ denotes the partition formed by adding a part of size k to the partition μ, and $\epsilon(k,\mu)+1$ represents which part k becomes in $\mu+(k)$ ($Q_{\mu+(k)}[X] = 0$ if μ contains a part of size k). For $m \leq 0$ a similar statement can be made using the commutation relations (18), (19) and (20).

Proof From (26) the action of $\mathbf{G}_m$ on a function $P[X] \in \Gamma$ can be written as

$$\begin{aligned}\mathbf{G}_m P[X] =& P[X-(1-q)/z]\Omega[(1-\epsilon)zX]\Big|_{z^m} \\ =& \sum_{i\geq 0} q^i z^{-i}(q_i^\perp P)[X-1/z]\Omega[(1-\epsilon)zX]\Big|_{z^m} \\ =& \sum_{i\geq 0} q^i (q_i^\perp P)[X-1/z]\Omega[(1-\epsilon)zX]\Big|_{z^{m+i}},\end{aligned}$$

since $P[X+t] = \sum_{i\geq 0} q_i^\perp P[X]t^i$. Thus

$$\mathbf{G}_m Q_\lambda[X] = \sum_{i\geq 0} q^i \mathbf{Q}_{m+i}(q_i^\perp Q_\lambda[X]),$$

where $q_i^\perp$ applied to Q_λ is

$$q_i^\perp Q_\lambda[X] = \sum_{\mu:\lambda/\mu\in\mathcal{H}_i} 2^{a(\lambda/\mu)} Q_\mu[X],$$

If $m > 0$, equation (32) follows from (18) and (20). If $m \leq 0$ we may need to use the commutation relation (19) for $\mathbf{Q}_n$ to straighten the index to a strict partition. In general we have $\mathbf{Q}_{m+i}(Q_\mu[X]) = Q_{(m+i,\mu)}[X]$, where we prepend the part $(m+i)$ (possibly negative) to the partition μ. □

Example 1. We compute $G_{(3,2,1)}[X;q]$ using the proposition above. We have

$$\begin{aligned}G_{(3,2,1)}[X;q] =& \mathbf{G}_3(\mathbf{G}_2(Q_{(1)}[X])) \\ =& \mathbf{G}_3\left(\sum_{i\geq 0}\sum_{(1)/\mu\in\mathcal{H}_i} 2^{a((1)/\mu)}(-1)^{\epsilon(2+i,\mu)} Q_{\mu+(2+i)}[X]\right) \\ =& \mathbf{G}_3(Q_{(2,1)}[X]) + 2q\mathbf{G}_3(Q_{(3)}[X]) \\ =& \sum_{i\geq 0}\sum_{(2,1)/\mu\in\mathcal{H}_i} 2^{a((2,1)/\mu)}(-1)^{\epsilon(3+i,\mu)} Q_{\mu+(3+i)}[X] + \\ & + 2q\left(\sum_{i\geq 0}\sum_{(3)/\nu\in\mathcal{H}_i} 2^{a((3)/\nu)}(-1)^{\epsilon(3+i,\nu)} Q_{\nu+(3+i)}[X]\right) \\ =& (q^0 2^0 Q_{(3,2,1)} + q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)}) \\ & + 2q(q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)} + q^3 2^1 Q_{(6)}) \\ =& Q_{(3,2,1)} + (2q+4q^2)Q_{(4,2)} + (2q^2+4q^3)Q_{(5,1)} + 4q^4 Q_{(6)}.\end{aligned}$$

3.2. **Properties of the polynomials $L_{\lambda\mu}(q)$.** The Q-Kostka polynomials introduced here have a number of remarkable properties that are very similar to those of Kostka-Foulkes polynomials listed in the previous section. We have already seen the analog of Property 2.2 holds for Q-Kostka polynomials. In what follows we will consider some of the other remaining properties.

An important consequence of equation (32) is a Morris-like recurrence which expresses the Q-Kostka polynomials $L_{\lambda\mu}(q)$ in terms of smaller ones.

Proposition 4. *We have the following recurrence*

$$L_{\alpha,(n,\mu)}(q) = \sum_{s:\alpha_s \geq n} (-1)^{s-1} q^{\alpha_s - n} \sum_{\lambda:\lambda/\alpha^{(s)} \in \mathcal{H}_{(\alpha_s - n)}} 2^{a(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q), \tag{33}$$

where $n > \mu_1$ and $\alpha^{(s)}$ is α with part α_s removed.

Proof If $n > \mu_1$ we have that

$$\mathbf{G}_n G_\mu[X;q] = G_{(n,\mu)}[X;q] = \sum_\alpha L_{\alpha,(n,\mu)}(q) Q_\alpha[X]. \tag{34}$$

On the other hand $G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X]$ and so

$$\mathbf{G}_n \left(\sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X] \right) = \sum_\mu L_{\lambda\mu}(q) \mathbf{G}_n(Q_\lambda[X]).$$

Using the action in (32) we have

(35)

$$\mathbf{G}_n G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) \sum_{i \geq 0} q^i \sum_{\nu:\lambda/\nu \in \mathcal{H}_i} 2^{a(\lambda/\nu)} (-1)^{\epsilon(n+i,\nu)} Q_{\nu+(n+i)}[X].$$

For $\alpha = \nu + (n+i)$, equating the coefficients of Q_α in (34) and (35) we get

$$L_{\alpha,(n,\mu)}(q) = \sum_\lambda \sum_{i \geq 0} q^i 2^{a(\lambda/\alpha-(n+i))} (-1)^{\epsilon(n+i,\alpha-(n+i))} L_{\lambda\mu}(q).$$

By reindexing $i := \alpha_s - n$ for $\alpha_s - n \geq 0$ we obtain the desired recurrence (33). □

Example 2. Let $n = 5$ and $L_{(6,2),(5,2,1)}(q) = 2q + 4q^2$. Using the recurrence we have one s such that $\alpha_s \geq 5$, i.e. $\alpha_1 = 6$. So

$$L_{(6,2),(5,2,1)}(q) = q^{6-5} \sum_{\lambda/(2) \in \mathcal{H}_1} 2^{a(\lambda/(2))} L_{\lambda(2,1)}(q)$$

$$= q\big(2L_{(21),(21)}(q) + 2L_{(3),(21)}(q)\big) = q(2 + 2 \cdot 2q) = 2q + 4q^2.$$

As a consequence of the Morris-like recurrence we have the following

Corollary 5. *Let $\mu \leq \lambda$ in dominance order.*

1. *If $n > \lambda_1$ then $L_{(n,\lambda),(n,\mu)}(q) = L_{\lambda\mu}(q)$.*
2. *$L_{\lambda\lambda}(q) = 1$ and $L_{(|\lambda|)\lambda}(q) = 2^{\ell(\lambda)-1} q^{n(\lambda)}$.*
3. *$2^{\ell(\mu)-\ell(\lambda)}$ divides $L_{\lambda\mu}(q)$.*

Proof 1. There is only one term in the recurrence (33) in this case which is exactly $L_{\lambda\mu}(q)$.

2. The first is a consequence of (1). For the second, we have that the only term on the right hand side is $q^{|\lambda|-\lambda_1}2L_{(|\lambda|-\lambda_1)(\lambda_2,\dots)}(q)$ which by induction is $q^{|\lambda|-\lambda_1+n((\lambda_2,\dots))}2\cdot 2^{\ell(\lambda)-2} = 2^{\ell(\lambda)-1}q^{n(\lambda)}$. This is the analog of Property 2.2 for the Kostka-Foulkes polynomials.

3. We use induction on $\ell(\mu)$ and the Morris recurrence to derive this property.
If $\ell(\mu) = 1$ we know that $G_{(m)}[X;q] = Q_{(m)}[X]$ and thus λ can be only (m) and the assertion holds.

For the induction step we use equation (33). We need to show that $2^{\ell(\mu)-\ell(\alpha)+1}$ divides $L_{\alpha,(n,\mu)}(q)$. From the induction hypothesis we know that $2^{\ell(\mu)-\ell(\lambda)}$ divides $L_{\lambda\mu}(q)$ for every such λ on the left hand side of equation (33).

The partitions λ have $\ell(\lambda)\in\{\ell(\alpha),\,\ell(\alpha)-1\}$. If $\ell(\lambda) = \ell(\alpha)-1$ we are done. If $\ell(\lambda) = \ell(\alpha)$ then $\lambda\neq\alpha^{(s)}$, since $\alpha^{(s)}$ has length one less than α, and thus $a(\lambda/\alpha^{(s)})\geq 1$. This implies that $2^{\ell(\mu)-\ell(\lambda)+1}$ divides each $2^{a(\lambda/\alpha^{(s)})}L_{\lambda,\mu}(q)$ and thus $L_{\alpha,(n,\mu)}(q)$. □

Using the Morris recurrence we are also able to obtain a formula for the degree of $L_{\lambda\mu}(q)$ similar to Property 2.2 for the Kostka-Foulkes polynomials.

Proposition 6. *If $\mu\leq\lambda$ in dominance order, we have*

$$deg_q L_{\lambda\mu}(q) = n(\mu) - n(\lambda).$$

Proof We prove this assertion by induction on $\ell(\mu)$. For $\ell(\mu) = 1$, the equality is obvious.

For the induction step we use the recurrence (33). Fix now an index s on the right hand side of the equation (33). Denote by $\mu^{(s)}_{(0)} = (\alpha_1+\alpha_s - n, \alpha_2,\dots,\alpha_{s-1},\alpha_{s+1},\dots)$.

We claim that $n(\mu^{(s)}_{(0)}) < n(\lambda)$ for any other λ such that $\lambda/\alpha^{(s)}\in\mathcal{H}_{(\alpha_s-n)}$. We have that

$$n(\mu^{(s)}_{(0)}) = \sum_{i=2}^{s-1}(i-1)\alpha_i + \sum_{i\geq 0}(s+i-1)\alpha_{s+i+1}$$

while

$$n(\lambda) = \sum_{i=2}^{s-1}(i-1)(\alpha_i+\epsilon_i) + \sum_{i\geq 0}(s+i-1)(\alpha_{s+i+1}+\epsilon_{s+i}),$$

where $\lambda = (\alpha_1+\epsilon_1,\dots,\alpha_{s-1}+\epsilon_{s-1},\alpha_{s+1}+\epsilon_s,\dots)$ and $\sum_i\epsilon_i = \alpha_s - n$. Moreover if $\lambda\neq\mu^{(s)}_{(0)}$ there exists at least one ϵ_i with $i\geq 2$ such that $\epsilon_i\neq 0$.

Therefore $n(\mu_{(0)}^{(s)}) < n(\lambda)$. We thus have proved that among the polynomials $L_{\lambda\mu}(q)$ in the second sum of (33), the polynomial $L_{\mu_{(0)}^{(s)},\mu}(q)$ has the highest degree, namely $n(\mu) - n(\mu_{(0)}^{(s)})$.

Next we show that in the first sum, the highest degree is obtained for $s = 1$. That is to say $deg_q\left(q^{\alpha_1 - n}L_{\mu_{(0)}^{(1)},\mu}(q)\right) > deg_q\left(q^{\alpha_s - n}L_{\mu_{(0)}^{(s)},\mu}(q)\right)$, hence

$$\alpha_1 - n + n(\mu) - n(\mu_{(0)}^{(1)}) > \alpha_s - n + n(\mu) - n(\mu_{(0)}^{(s)}),$$

$$\alpha_1 + \sum_{i=2}^{s-1}(i-1)\alpha_i + \sum_{j\geq s+1}(j-2)\alpha_j > \alpha_s + \sum_{i=3}^{s-1}(i-2)\alpha_i + \sum_{j\geq s}(j-2)\alpha_j$$

which is

$$\alpha_1 + \alpha_2 + \sum_{i=3}^{s-1}(i-1)\alpha_i > \sum_{i=3}^{s-1}(i-2)\alpha_i + (s-1)\alpha_s$$

$$= (\alpha_3 + \alpha_s) + (2\alpha_4 + \alpha_s) + \cdots + [(s-3)\alpha_{s-1} + \alpha_s] + 2\alpha_s.$$

The last inequality is true as $(i-1)\alpha_i > (i-2)\alpha_i + s$ for $i = 3, \ldots, s-1$ and $\alpha_1 + \alpha_2 > 2\alpha_s$.

Thus we have $deg_q L_{\alpha,(n,\mu)}(q) = (\alpha_1 - n) + n(\mu) - n(\mu_{(0)}^{(1)})$. Finally we need to show this is in fact $n((n,\mu)) - n(\alpha)$. That is

$$\alpha_1 - n + \sum_{i\geq 2}(i-1)\mu_i - \sum_{i\geq 3}(i-2)\alpha_i = \sum_{i\geq 0}\mu_i - \sum_{i\geq 2}(i-1)\alpha_i,$$

and by simplifying we obtain $\sum \alpha_i - n = \sum \mu_i$, which is obviously true.

Hence $deg_q\left(L_{\alpha,(n,\mu))}(q)\right) = n((n,\mu)) - n(\alpha)$ and the proof is complete. □

The property that is most suggestive that these polynomials are analogs of the Kostka-Foulkes polynomials is

Conjecture 7. *The Q-Kostka polynomials $L_{\lambda\mu}(q)$ have non-negative coefficients.*

We will prove this conjecture for some particular cases. In general we believe that there should exist a similar combinatorial interpretation as for the Kostka-Foulkes polynomials. More precisely there should exist a statistic function d on the set of marked shifted tableaux, similar to the charge function on column strict tableaux, such that

$$L_{\lambda\mu}(q) = \sum_T q^{d(T)}$$

summed over marked shifted tableaux of shifted shape λ and content μ with diagonal entries unmarked.

The polynomials $L_{\lambda\mu}(q)$ suggest that many of the tableaux of the same shape and content will have the same statistic. This suggests that the statistic d might satisfy $d(T) = d(S)$ for S and T differing by only marking or unmarking certain cells. This property does not seem to hold for larger sets of tableaux and for large enough examples (for example $\mu = (5,4,3,2,1)$) it is impossible that the statistic is completely independent of the markings.

Another intriguing property of this statistic function d is that the values it takes are not too different than the charge function. It seems that in general we have that for given λ and μ the set of $\{d(T) : T$ in the summation of $L_{\lambda\mu}(q)\}$ is a subset of $\{c(T) : T$ column strict tableaux of shape λ and content $\mu\}$, where c is the usual charge. This suggests that there should be relationship between these two statistics; however, we have so far not been able to establish what that link might be.

Proposition 8. (1) *For μ a two-row partition and $\lambda > \mu$ we have $L_{\lambda\mu}(q) = 2q^{n(\mu)-n(\lambda)}$.*

(2) *If μ has the property $\mu_i \geq \sum_{j \geq i+1} \mu_j$, Conjecture 7 is true.*

Proof (1). Let us consider $G_\mu[X;q]$ and let $\mu = (n,m)$. We have that $G_{(n,m)}[X;q] = \mathbf{G}_n(Q_{(m)}[X])$ which by Proposition 3 is

$$\mathbf{G}_n(Q_{(m)}[X]) = \sum_{i \geq 0} q^i \sum_{\mu:(m)/\mu \in \mathcal{H}_i} 2^{a((m)/\mu)}(-1)^{\epsilon(n+i,\mu)} Q_{\mu+(n+i)}[X].$$

From this we deduce that $i = 0, 1, \ldots, m$ and $\mu = (m-i)$. Thus

$$\mathbf{G}_n(Q_{(m)}[X]) = Q_{(n,m)}[X] + \sum_{i=1}^{m} 2q^i Q_{(n+i,m-i)}[X]$$

and the proof is complete.

(2). In this case we prove it by induction on $\ell(\mu)$ and using the Morris recurrence (33).

The case $\ell(\mu) = 1$ is clear as $L_{\lambda\mu}(q) = \delta_{\lambda\mu}$. For the induction step consider $L_{\alpha(n,\mu)}(q)$ as in the right hand side of (33). Under our assumption there is just one index in the first sum, i.e., only α_1 can be greater than n. This is true since $|\alpha| = n + |\mu|$, $\alpha \geq (n,\mu)$ in dominance order and $n \geq |\mu|$. Thus the right hand side does not contain negative signs and by induction it is non-negative . Hence $L_{\alpha(n,\mu)}(q)$ has non-negative coefficients. □

We also note that monotonicity properties, similar to Property 2.2 and 2.2, hold for the Q-Kostka polynomials.

Conjecture 9. *Let $L'_{\lambda\mu}(q) := q^{n(\mu)-n(\lambda)} L_{\lambda\mu}(q^{-1})$. We have*

$$L'_{\lambda\mu}(q) \geq 2^{\ell(\nu)-\ell(\mu)} L'_{\lambda\nu}(q), \qquad \textit{for } \mu \leq \nu \textit{ in dominance order.}$$

We can prove this fact by using induction and the recurrence (33) for the case $\mu_1 = \nu_1$.

Example 3. Let $\lambda = (6,2)$, $\mu = (4,3,1)$, $\nu = (5,2,1)$. We have $n(\lambda) = 2$, $n(\mu) = 5$, and $n(\nu) = 4$. The L' polynomials are

$$L'_{\lambda\mu} = q^{5-2}(4/q^2 + 4/q^3) = 4 + 4q, \qquad L'_{\lambda\nu} = q^{4-2}(2/q + 4/q^2) = 4 + 2q,$$

and thus $L'_{\lambda\mu}(q) \geq 2^{3-3} L'_{\lambda\nu}(q)$.

Another property of the Kostka-Foulkes polynomials case that seems to hold in our case refers to the growth of the polynomials L. For the Kostka-Foulkes polynomials the conjecture belongs to Gupta (see [1] and references therein).

Conjecture 10. *If r is an integer that is not a part in either partition λ or μ, then*

$$L_{\lambda+(r),\mu+(r)}(q) \geq L_{\lambda\mu}(q).$$

The case where $r > \lambda_1$ (which also ensures that $r > \mu_1$) is obviously true since $L_{(r,\lambda),(r,\mu)}(q) = L_{\lambda\mu}(q)$ (see Corollary 5).

Example 4. Let $\lambda = (5,3), \mu = (4,3,1)$ and $a = 2$. We have

$$L_{(5,3,2),(4,3,2,1)}(q) - L_{(5,3),(4,3,1)}(q) = 2q + 4q^2 + 8q^3 - (2q + 4q^2) = 8q^3.$$

3.3. **Another expression for $L_{\lambda\mu}(q)$.** The polynomials $L_{\lambda\mu}(q)$ have a similar interpretation to property 2.2 using an analog of the q-Kostant partition function. We follow the construction in [1]. In order to write equation (16) as

$$q_\lambda[X] = \prod_{i<j} \left(\frac{1+R_{ij}}{1-R_{ij}}\right)^{-1} Q_\lambda[X] \tag{36}$$

we will use linear maps from the group algebra $\mathbb{Z}[\mathbb{Z}^n]$ to the algebra Γ. A basis of $\mathbb{Z}[\mathbb{Z}^n]$ will consist of formal exponentials $\{e^\alpha\}_{\alpha\in}$ which satisfy relations $e^\alpha e^\beta = e^{\alpha+\beta}$. In fact we identify the ring with the ring of Laurent polynomials in $x_1, \ldots, x_n$ and set $e^\alpha = x^\alpha$. With this in mind we are viewing all our polynomials in Γ_n (or Λ_n) as linear homomorphisms from $\mathbb{Z}[\mathbb{Z}^n]$ to Γ i.e.

$$Q : e^\lambda \to Q(e^\lambda) = Q_\lambda \quad q : e^\lambda \to q(e^\lambda) = q_\lambda.$$

If we now set $\zeta_n := \prod_{1\leq i<j\leq n} \left(\frac{1+x_i/x_j}{1-x_i/x_j}\right)$, we have that $\zeta_n = \sum_{\alpha\in\mathbb{Z}^n} \mathcal{R}(\alpha)e^\alpha$ where $\mathcal{R}(\alpha) = \sum_t a_t 2^t$ and a_t counts the number of ways the vector α can be written as a sum of positive roots of type A_{n-1}, t of which are distinct. The positive roots in the root lattice of A_{n-1} are $\{e_i - e_j\}_{1\leq i<j\leq n}$, where $e_i = (0, \ldots, 1, \ldots 0)$ is the canonical basis of $\mathbb{Z}^n$.

Since $q(e^\lambda) = Q(\zeta_n e^\lambda)$ we have that

$$q_\lambda[X] = q(e^\lambda) = Q(\sum_{\alpha\in\mathbb{Z}^n} \mathcal{R}(\alpha)e^\alpha e^\lambda)$$

$$= \sum_{\alpha \in \mathbb{Z}^n} \mathcal{R}(\alpha) Q_{\lambda+\alpha}[X].$$

If we consider the same argument for $G_\lambda[X;q] = \prod_{1 \le i < j \le n} \left(\frac{1 + qR_{ij}}{1 - qR_{ij}} \right) Q_\lambda[X]$ we need to define the q-analog of ζ_n as

$$\zeta_n(q) := \prod_{i<j} \left(\frac{1 + qx_i/x_j}{1 - qx_i/x_j} \right),$$

and thus $\zeta_n(q) = \sum_{\alpha \in \mathbb{Z}^n} \mathcal{R}_q(\alpha) e^\alpha$, where $\mathcal{R}_q(\alpha) = \sum_{t,k} a_{t,k} 2^t q^k$ and $a_{t,k}$ counts the number of ways the vector α can be written as a sum of k positive roots, t of which are distinct. Hence

$$G_\lambda[X;q] = \sum_{\alpha \in \mathbb{Z}^n} \mathcal{R}_q(\alpha) Q_{\lambda+\alpha}[X].$$

This yields another expression for the Q-Kostka polynomials in terms of $\mathcal{R}_q(\alpha)$ as

$$L_{\lambda\mu}(q) = \sum_{\alpha:\, Q_{\alpha+\mu} = \pm 2^r Q_\lambda} \pm 2^r \mathcal{R}_q(\alpha).$$

The index of the sum reflects the straightening of a Q-function indexed by an integer sequence and it is a consequence of the commutation relations (19). It is possible to express the equation above using the action of the symmetric group on Schur's Q-functions, yielding an alternating sum similar to Property 2.2. Unfortunately the action of the symmetric group on Schur's Q-functions indexed by a general integer vector is not as elegant as for Schur functions.

Example 5. To compute $G_{(3,2,1)}[X;q]$ we compute $\mathcal{R}_q(\alpha)$ for all of the relevant compositions α. There will be a finite number for which the function $Q_{\lambda+\alpha}[X]$ is non-zero and we list only those the expression below. Note that trailing 0s of these composition are removed.

$$\begin{aligned} G_{(3,2,1)}[X;q] =& Q_{(3,2,1)}[X] + (2q + 4q^2) Q_{(4,2)}[X] + (4q^2 + 4q^3) Q_{(5,1)}[X] \\ &+ 2q^2 Q_{(5,0,1)}[X] + (4q^4 + 4q^3) Q_{(6)}[X] + 2q^3 Q_{(6,-1,1)}[X]. \end{aligned}$$

The commutation relations say that $Q_{(5,0,1)}[X] = -Q_{(5,1)}[X]$ and $Q_{(6,-1,1)}[X] = -2Q_{(6)}[X]$ and hence we have used this formula to show

$$\begin{aligned} G_{(3,2,1)}[X;q] = & Q_{(3,2,1)}[X] + (2q + 4q^2) Q_{(4,2)}[X] + (2q^2 + 4q^3) Q_{(5,1)}[X] \\ & + 4q^4 Q_{(6)}[X]. \end{aligned}$$

Remark: Most of the properties of the Q-Kostka polynomials $L_{\lambda\mu}(q)$ are analogous to the Kostka-Foulkes polynomials, but a few properties do not seem to generalize.

(1) The analog of Property 2.2 does not seem to hold since computations of $G_\lambda[X;q]$ where q is set to a root of unity do not factor.
(2) There does not seem to exist an elegant relationship between $G_\lambda[X;q]$ and its dual basis (Property 2.2).
(3) A property similar to that of Property 2.2 does not seem to hold. We do not know if there is a relationship between $G_\lambda[X;q]$ and a Hall-like algebra.
(4) We do not know if an analog of the Macdonald symmetric functions should exist. A family of functions which mimic the formulas for the Macdonald symmetric functions in [27] may easily be defined, but the specializations of the variables indicate that the same sort of positivity and symmetry properties of the coefficients cannot hold through this definition.

4. Generalized (parabolic) Q-Kostka polynomials

There exists in the literature a few generalizations of the Kostka-Foulkes polynomials that correspond to q-analogs of multiplicities of irreducibles in tensor products of irreducible representations (Littlewood-Richardson coefficients). In [23] formulas were introduced for realizing 'generalized' or 'parabolic' Kostka coefficients [22] as coefficients appearing in families of symmetric functions defined as compositions of operators. This construction may also be extended to the Q-function algebra providing a means of defining a generalization of the Q-Kostka polynomials that corresponds to a q-analog of coefficients in products of $Q_\mu[X]$.

Let $\mu^* = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(k)})$ be a sequence of partitions and $\bar{\mu^*}$ the concatenation of all those partitions. Take $\eta = (\ell(\mu^{(1)}), \ell(\mu^{(2)}), \dots, \ell(\mu^{(k)}))$ and $n = \sum_i \eta_i$, then set $Roots_\eta = \{(i,j) : 1 \leq i \leq \eta_1 + \cdots + \eta_r < j \leq n \text{ for some } r\}$.

Now set

$$H_{\mu^*}[X;q] = \prod_{(i,j)\in Roots_\eta} \frac{1}{1-qR_{ij}} s_{\bar{\mu^*}}[X]. \tag{37}$$

The parabolic Kostka polynomials are defined as the coefficients of the Schur basis in these symmetric functions. That is, the polynomials $K_{\lambda;\mu^*}(q)$ are defined by the coefficients in the expansion

$$H_{\mu^*}[X;q] = \sum_{\lambda\vdash|\mu^*|} K_{\lambda;\mu^*}(q) s_\lambda[X]. \tag{38}$$

The functions $H_{\mu^*}[X;q]$ and the parabolic Kostka coefficients have the following properties.

- If $\bar{\mu^*}$ is a partition then it is conjectured that $K_{\lambda;\mu^*}(q)$ has non-negative integer coefficients (in certain cases this is known).
- $H_{\mu^*}[X;0] = s_{\bar{\mu^*}}[X]$. $\bar{\mu^*}$ need not be a partition, but this is consistent with the definition of s_λ in section 2.1.

- $H_{\mu^*}[X;1] = s_{\mu^{(1)}}[X]s_{\mu^{(2)}}[X]\cdots s_{\mu^{(k)}}[X]$ and in this sense the coefficients $K_{\lambda;\mu^*}(q)$ are q-analogs of the Littlewood-Richardson coefficients
- If $\mu^* = ((\gamma_1),(\gamma_2),\ldots,(\gamma_{\ell(\gamma)}))$ where γ is a partition, then $H_{\mu^*}[X;q] = H_\gamma[X;q]$.
- If $\bar{\mu}^*$ is a partition then $H_{\mu^*}[X;q] = s_{\bar{\mu}^*}[X] + \sum_{\lambda > \bar{\mu}^*} K_{\lambda;\mu^*}(q) s_\lambda[X]$.
- There exists an operator $\mathbf{H}_\gamma$ such that $\mathbf{H}_\gamma(H_{\mu^*}[X;q]) = H_{(\gamma,\mu^{(1)},\ldots,\mu^{(k)})}[X;q]$ (see [23]).

In addition, analogs of most properties of the Hall-Littlewood functions and the Kostka-Foulkes polynomials also seem to hold (see for instance [22]).

We should also mention that there is an analog of several other formulas for the Hall-Littlewood functions. It follows from the definition of the functions $H_{\mu^*}[X;q]$ that

$$H_{\mu^*}[X;q] = \Omega[Z_nX] \prod_{1\le i<j\le n} (1 - z_j/z_i) \prod_{(i,j)\in Roots_\eta} \frac{1}{1-qz_j/z_i}\bigg|_{z^{\bar{\mu}^*}}. \tag{39}$$

For $k>0$, if we define the operation,

$$\mathbf{H}(Z^k)P[X] = P[X-(1-q)Z^*]\Omega[ZX] \prod_{1\le i<j\le k} (1-z_j/z_i), \tag{40}$$

where $Z^* = \sum_{i=1}^k \frac{1}{z_i}$, then

$$\begin{aligned} \mathbf{H}(Z^{\eta_1})\mathbf{H}(Z^{\eta_2})\cdots\mathbf{H}(Z^{\eta_{\ell(\eta)}})1 = \Omega[Z_nX] &\prod_{1\le i<j\le n} \\ \times (1-z_j/z_i) &\prod_{(i,j)\in Roots_\eta} \frac{1}{1-qz_j/z_i} \end{aligned} \tag{41}$$

and therefore $\mathbf{H}(Z^k)H_{\mu^*}[X;q]\Big|_{z^\gamma} = H_{(\gamma,\mu^*)}[X;q]$.

This construction exists in complete analogy within the Q-function algebra. We will create a family of functions in Γ which are indexed by a sequence of strict partitions. Let $\mu^* = (\mu^{(1)},\mu^{(2)},\ldots,\mu^{(k)})$ where $\mu^{(i)}$ is a strict partition and set $\eta = (\ell(\mu^{(1)}),\ell(\mu^{(2)}),\ldots,\ell(\mu^{(k)}))$. Define $Roots_\eta$ as before and then define the function

$$G_{\mu^*}[X;q] = \prod_{(i,j)\in Roots_\eta} \frac{1+qR_{ij}}{1-qR_{ij}} Q_{\bar{\mu}^*}[X]. \tag{42}$$

We may also view these elements of Γ as the result of a family of operators acting on 1. Consider the composition of the operators

$$\begin{aligned} \mathbf{Q}_\lambda P[X] &:= \mathbf{Q}_{\lambda_1}\mathbf{Q}_{\lambda_2}\cdots\mathbf{Q}_{\lambda_k}P[X] \\ &= P[X-Z_k^*]\Omega[(1-\epsilon)Z_kX] \prod_{1\le i<j\le k} \frac{1-z_j/z_i}{1+z_j/z_i}\bigg|_{z^\lambda} \end{aligned}$$

and then set $\mathbf{G}_\lambda := \widetilde{\mathbf{Q}_\lambda}^q$, that is

$$\mathbf{G}_\lambda P[X] = P[X + (q-1)Z_k^*]\Omega[(1-\epsilon)Z_k X] \prod_{1\leq i<j\leq k} \frac{1 - z_j/z_i}{1 + z_j/z_i}\Big|_{z^\lambda}. \tag{43}$$

As with the other operators of this sort, it is easily shown that a composition of $G_{\mu^{(i)}}$ acting on 1 is equivalent to the defining relation (42) of the functions $G_{\mu^*}[X;q]$ and hence that

$$\mathbf{G}_\gamma G_{\mu^*}[X;q] = G_{(\gamma,\mu^{(1)},\dots,\mu^{(k)})}[X;q]. \tag{44}$$

The $G_{\mu^*}[X;q]$ functions seem to share many of the same properties of the $H_{\nu^*}[X;q]$ and $G_\gamma[X;q]$ analogs. Define the polynomials $L_{\lambda;\mu^*}(q)$ by the expansion

$$G_{\mu^*}[X;q] = \sum_\lambda L_{\lambda;\mu^*}(q) Q_\lambda[X]. \tag{45}$$

- $G_{\mu^*}[X;0] = Q_{\bar{\mu^*}}[X]$. $\bar{\mu^*}$ need not be a strict partition, however if it is not then the straightening relations (18), (19) and (20) may be applied to reduce the expression.
- $G_{\mu^*}[X;1] = Q_{\mu^{(1)}}[X]Q_{\mu^{(2)}}[X]\cdots Q_{\mu^{(k)}}[X]$ and hence $L_{\lambda;\mu^*}(1)$ is equal to the coefficient of $Q_\lambda[X]$ in the product $Q_{\mu^{(1)}}[X]Q_{\mu^{(2)}}[X]\cdots Q_{\mu^{(k)}}[X]$.
- If $\mu^* = ((\gamma_1),(\gamma_2),\dots,(\gamma_{\ell(\gamma)}))$ where γ is a partition, then $G_{\mu^*}[X;q] = G_\gamma[X;q]$.
- If $\bar{\mu^*}$ is a strict partition then $G_{\mu^*}[X;q] = Q_{\bar{\mu^*}}[X] + \sum_{\lambda>\bar{\mu^*}} L_{\lambda;\mu^*}(q) Q_\lambda[X]$.

Computing these coefficients suggests the following remarkable conjecture and indicates that these coefficients are an important q-analog of the structure coefficients of the $Q_\lambda[X]$ functions in the same way that the $K_{\lambda;\mu^*}(q)$ polynomials are q-analogs of the Littlewood-Richardson coefficients.

Conjecture 11. *For a sequence of partitions μ^*, if $\bar{\mu^*}$ is a partition then $L_{\lambda;\mu^*}(q)$ is a polynomial in q with non-negative integer coefficients.*

If this conjecture is true then the polynomials $L_{\lambda;\mu^*}(q)$ are a q analog of the coefficient of $Q_\lambda[X]$ in the product $Q_{\mu^{(1)}}[X]Q_{\mu^{(2)}}[X]\cdots Q_{\mu^{(k)}}[X]$. A combinatorial description for these coefficients was given in [24] and hence we are looking for an additional statistic on the set of objects counted by them which includes as a special case the coefficients $L_{\lambda\mu}(q)$.

This conjecture suggests that the $L_{\lambda\mu^*}(q)$ should also share many of the properties that are held by the $K_{\lambda;\mu^*}(q)$ and that generalize the case of the Kostka-Foulkes polynomials.

We remark that the parabolic Kostka polynomials indexed by a sequence of partitions μ^* where each $\mu^{(i)}$ is a rectangle (i.e. each $\mu^{(i)} = (a_i, a_i, \dots, a_i)$ for some a_i) is a special subfamily of these polynomials. In this case, explicit combinatorial formulas are known for the coefficients (see for example [20], [21] or [13]) which imply that the coefficients $K_{\lambda;\mu^*}(q)$ are positive. By contrast, for the generalizations of the Q-Kostka polynomials we know that if $\mu^{(i)}$ has two equal parts for any i then $G_{\mu^*}[X;q]=0$, hence this special case is not of interest in this setting.

5. Appendix: Tables of $2^{\ell(\lambda)-\ell(\mu)}L_{\lambda\mu}(q)$ for $n=4,5,6,7,8,9$.

$$\begin{array}{c}\begin{matrix}(3,1) & (4)\end{matrix}\\ \begin{bmatrix}1 & q\\ 0 & 1\end{bmatrix}\end{array}$$

$$\begin{array}{c}\begin{matrix}(3,2) & (4,1) & (5)\end{matrix}\\ \begin{bmatrix}1 & 2q & q^2\\ 0 & 1 & q\\ 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\begin{array}{c}\begin{matrix}(3,2,1) & (4,2) & (5,1) & (6)\end{matrix}\\ \begin{bmatrix}1 & 2q^2+q & 2q^3+q^2 & q^4\\ 0 & 1 & 2q & q^2\\ 0 & 0 & 1 & q\\ 0 & 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\begin{array}{c}\begin{matrix}(4,2,1) & (4,3) & (5,2) & (6,1) & (7)\end{matrix}\\ \begin{bmatrix}1 & q & 2q^2+q & 2q^3+q^2 & q^4\\ 0 & 1 & 2q & 2q^2 & q^3\\ 0 & 0 & 1 & 2q & q^2\\ 0 & 0 & 0 & 1 & q\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\begin{array}{c}\begin{matrix}(4,3,1) & (5,2,1) & (5,3) & (6,2) & (7,1) & (8)\end{matrix}\\ \begin{bmatrix}1 & 2q & 2q^2+q & 2q^2+2q^3 & q^3+2q^4 & q^5\\ 0 & 1 & q & 2q^2+q & 2q^3+q^2 & q^4\\ 0 & 0 & 1 & 2q & 2q^2 & q^3\\ 0 & 0 & 0 & 1 & 2q & q^2\\ 0 & 0 & 0 & 0 & 1 & q\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\begin{array}{cccccccc} (4,3,2) & (5,3,1) & (5,4) & (6,2,1) & (6,3) & (7,2) & (8,1) & (9) \end{array}$$
$$\left[\begin{array}{cccccccc}
1 & 2q+4q^2 & 2q^3+q^2 & 2q^2+4q^3 & q^2+2q^4+4q^3 & 4q^4+q^3+2q^5 & 2q^6+2q^5 & q^7 \\
0 & 1 & q & 2q & 2q^2+q & 2q^2+2q^3 & q^3+2q^4 & q^5 \\
0 & 0 & 1 & 0 & 2q & 2q^2 & 2q^3 & q^4 \\
0 & 0 & 0 & 1 & q & 2q^2+q & 2q^3+q^2 & q^4 \\
0 & 0 & 0 & 0 & 1 & 2q & 2q^2 & q^3 \\
0 & 0 & 0 & 0 & 0 & 1 & 2q & q^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

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REFERENCES

[1] Désarménien, J., Leclerc, B., and Thibon, J.-Y., "Hall-Littlewood Functions and Kostka-Foulkes Polynomials in Representation Theory." *Sèminaire Lotharingien de Combinatoire*, **B32c**(1994), 38 pp.

[2] Foulkes, H. O., "A survey of some combinatorial aspects of symmetric functions." in *Permutations.* (1974) Gauthier-Villars, Paris.

[3] Garsia, A. M., "Orthogonality of Milne's polynomials and raising operators." *Discrete Math*, **99**, 247–64.

[4] Fulton, W., *Young Tableaux.* Cambridge University Press, Cambridge, 1997.

[5] Haiman, M., "On Mixed Insertion, Symmetry, and Shifted Young Tableaux." *J. Comb Theory (A)*, **50** (1989), 196–225.

[6] Hall, P. "The algebra of partitions." *Proc. 4th Canadian Math. Congress, Banff*, (1959).

[7] Jing, N., "Vertex Operators, Symmetric Functions and the Spin Group Γ_n." *J. Algebra* **138**(1991), 340–398.

[8] Jing, N., "Vertex Operators and Hall-Littlewood Symmetric Functions." *Advances in Math.* **87**(1991), 226–248.

[9] Jing, N., "Vertex operators and generalized symmetric functions," in *Proceedings of Quantum Topology, World Sci. Singapore* (1993), 111–128.

[10] Józefiak, T., "Schur Q-functions and applications." Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, (1991) 205–224, .

[11] Kirillov, A., "Ubiquity of Kostka polynomials." *Phys. and Comb., Proceedings of the Nagoya 1999 International Workshop on Physics and Combinatorics, Nagoya University*, August 23-27, (1999), 85–200.

[12] A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe Ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41** (1988), 925–955.

[13] Kirillov, A. N., Shimozono, M., "A generalization of the Kostka-Foulkes polynomials," math.QA/9803062, to appear in *J. Algebraic Combin.*

[14] Lascoux, A., Pragacz, P., "Orthogonal Divided Differences and Schubert Polynomials, $\tilde{P}$-Functions and Vertex Operators." *Mich. Math. J.* **48**(2000), 417–441.

[15] Lascoux, A., Schützenberger M. P., "Sur une conjecture de H. O. Foulkes." *C. R. Acad. Sci. Paris,* **286A**(1978) 323–324.

[16] Lascoux, A. and Schützenberger, M. P., "Le Monoid plaxique." *Noncommutative structures in Algebra and Geometric Combinatorics*, CNR, Rome. **109**(1981), 129–156.

[17] Macdonald, I. G., *Symmetric Functions and Hall Polynomials.* Cambridge University Press, Cambridge, 1995.

[18] Morris, A. O., "The characters of the group $Gl(n;q)$", *Math. Zeitschr.* **81** (1963) 112-123.

[19] Sagan, B. E., "Shifted Tableaux, Schur Q-functions and a Conjecture of R. Stanley." *J. Comb. Theory (A)*, **45**(1987), 62–103.

[20] Schilling, A., Warnaar, S. O. "Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials." *Commun. Math. Phys.* **202** (1999) 359–401.

[21] M. Shimozono, "A cyclage poset structure for Littlewood-Richardson tableaux," European J. Combin. **22** (2001) 365–393.

[22] Shimozono, M., Weyman, J., "Graded characters of modules supported in the closure of a nilpotent conjugacy class." *European J. Combin.* **21** no. 2(2000) 257–288.

[23] Shimozono, M., Zabrocki, M., "Hall-Littlewood Vertex Operators and Generalized Kostka Polynomials." *Adv. Math.* **158**(2001), 66–85.

[24] Stembridge, J. R., "Shifted Tableaux and the Projective Representations of Symmetric Groups." *Advances in Math.* **74**(1989), 87–134.

[25] Stembridge, J. R., "On Schur's Q-functions and the Primitive Idempotents of a Commutative Hecke Algebra." *J. Alg. Combinatorics* **1**(1992), 71–95.

[26] Worley, D. R., "A Theory of Shifted Young Tableaux." Thesis, MIT (1984).

[27] Zabrocki, M., "q-Analogs of symmetric function operators," *Discrete Math.* **256** (2002), 831–853.

School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455

E-mail address: `tudose@math.umn.edu`

Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3

E-mail address: `zabrocki@mathstat.yorku.ca`

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