THE CLASSICAL GROUPS. SPECTRAL ANALYSIS OF THEIR FINITE-DIMENSIONAL REPRESENTATIONS

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THE CLASSICAL GROUPS. SPECTRAL ANALYSIS OF THEIR FINITE-DIMENSIONAL REPRESENTATIONS

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Contents

Introduction

The theory of linear representations is often regarded as an auxiliary powerful device that enables us to clarify the *structure* of one class of abstract groups or another; however, the *study of the various types of linear representations* that a given group G induces in a well-defined system of geometrical objects is also of a certain interest (such objects may be, in particular, functions on a certain space with a given group of motions); the latter point of view has been developed particularly since the creation of quantum mechanics and has found its clearest expression in the well-known monograph of H. Weyl "The classical groups, their invariants and representations". Important connections of the theory of representations with functional analysis appear on this course, in particular, connections with the theory of special functions and the general theory of harmonic analysis on groups and homogeneous manifolds.

In the present paper we shall only deal with the theory of *finite-dimensional* representations of the "classical" linear groups: on the one hand, finite-dimensional representations still play an important role in practical applications; on the other hand, in geometrically clear examples of this theory one can easily see the general rules that also govern, to a certain extent, the theory of infinite-dimensional representations.

In applications of the theory of representations, especially in physics, many problems reduce to spectral analysis, i.e. to the decomposition of complicated representations into irreducible ones. The simplest analogue of this problem is the spectral analysis of linear operators; concerning terminology, we distinguish between *spectral analysis* (description of the points of the spectrum, construction of eigenvectors) and *spectral synthesis* (the determination of projection operators for spectral decomposition); the problems of synthesis of linear representations are only solved in rare cases: typical examples are the well-known decomposition of Clebsch-Gordan for the Lorentz group and the decomposition of affine tensors with respect to symmetry types (Young diagrams). More accurately, by analysis we understand the solution of the following two problems; 1) given a representation to find its spectrum, i.e. to answer the question: what irreducible representations are contained in it and with what multiplicity; 2) to find all the invariant subspaces on which these components act.

Among the known methods of solving problem 1) we must mention the classical *method of characters* of H. Weyl, the *method of dominant weights* of Cartan and the *method of Laplace operators* (Gel'fand-Berezin); the method of dominant weights - or, to put it better, "dominant vectors" of Cartan - is on principle the simplest and enables us to solve both problems at the same time; however, being realized in an *infinitesimal* form, it is not sufficiently flexible, and this is probably the reason for the surprising fact that this method is not, as a rule, used (with the exception, as far as the author is aware, of several simple examples in the account of E. Cartan himself); all the same, many problems of spectral analysis have so far been solved only by either ad hoc or unnecessarily complicated methods.

The object of the present paper is to show that the corresponding *integral* method of dominant vectors is a good key to spectral analysis of linear representations. Physics, in the present state of the theory, naturally prefers the infinitesimal method; however, the integral approach, i.e. the study of the group "as a whole", has a number of advantages of which probably the most fundamental is the possibility of an extremely simple explicit construction of each irreducible representation in the space of functions (polynomials) on a certain special "root" subgroup Z. Such a construction in the form of models plays a fundamental role in the theory of infinite-dimensional representations and was introduced in the well-known papers of I.M. Gel'fand and M.A. Naimark; R. Godemant in the paper [11] remarked that in this way one can obtain with great simplicity a classification of all the irreducible finite-dimensional representations of the classical linear groups. The "universal" manifold Z has a simple algebraic structure and is a natural many-dimensional generalization of the ordinary complex plane.

Brief summary of the paper. The paper consists of two parts. In the first part we give a brief account of the fundamental concepts and solve the problem of a description of the systems of all irreducible representations for each classical linear Lie group (we also develop, in particular, the necessary apparatus for spectral analysis); the second part contains a series of problems to be solved by the method of dominant vectors.

However, we have not made it our object in this paper to give a survey of problems with ready-made answers. Wherever it appears opportune, we have mentioned relations between the various problems studied; it may be hoped that these remarks will help the reader to grasp the contents of representation theory as a whole. A classification of all the irreducible representations of the classical Lie groups was obtained by E. Cartan [3], in particular, the two-valued ("spinor") representations of the orthogonal group were discovered here. However, by the use of the "integral method" we can give an elementary solution of this problem (developing an idea proposed by R. Godemant) and at the same time construct the analytical apparatus of "induced representations". The description of a precise model of the representations of the group Z is a new result; it is remarkable that the spinor representations fall naturally into the general construction.

Some results of Part II are also new; the remaining ones are only of methodical value, since the solution is reached in the simplest - and, probably, the most natural - way; even in such a problem of long standing as the Clebsch-Gordan decomposition of the Lorentz group simplifications are obtained: in this case the analysis and synthesis are reached by means of simple *recurrence* relations (moreover, the decomposition obtained can be extended to a certain class of infinite-dimensional representations).

The study of the classical groups and their representations is made easier by the remarkable fact that in the majority of cases these groups belong to the class of the so-called "semisimple" Lie groups (the full linear group, which contains a non-trivial centre, and the group of motions, which contains the translations, are exceptions); a consequence of semisimplicity is the "complete reducibility" of all linear representations - i.e. the decomposibility of the representation space into a direct sum consisting of irreducible parts. In the general case "Jordan blocks" can occur; a peculiarity of this class of problems will be illustrated (§12, 4) on the example of the group of motions of an n-dimensional complex euclidean space.

A general outline of the method of dominant vectors[1]. Every "semisimple" complex connected Lie group contains two remarkable subgroups: the "root" subgroup Z and the "Cartan" subgroup D (for example, in

[1] A more general outline is given in [12].

the case of the group of all non-singular matrices of order two D consists of all diagonal non-singular matrices, and Z of the matrices of the form $\binom{1\,z}{0\,1}$, where z is an arbitrary complex parameter).

In the space of every irreducible representation of a group G there exists a uniquely defined one-dimensional direction θ whose points are:

1) *fixed under Z;*

2) *eigenvectors with respect to D; the corresponding character $\alpha(\delta)$ of D (the "coefficient of dilatation" in the direction θ) determines the original representation of G uniquely to within equivalence.*

Therefore, if a certain representation of a group G is a direct sum of irreducible representations:

$$E = E_1 + E_2 + \ldots + E_s,$$

where E is the representation space and E_i its irreducible parts, then every E_i contains only one "extremal" direction θ_i and the *spectral analysis in the space E reduces to the search for the invariants of the group Z.*[1]

For a real group G we can formulate a similar, somewhat more complicated principle, however, it is more convenient to go over to the "complex envelope" of the group G (this process is inverse to the device of "unitary constraint" of H. Weyl; from this point of view real subgroups, even when they are compact, are of more complicated structure than their complex envelopes); besides, we shall hardly go at all into the real case.

As a heuristic device, the principle of dominant vectors is useful also for the study of infinite-dimensional representations; a strict proof in this case can perhaps be obtained by the methods of the theory of generalized functions. It ought to be mentioned (having in mind the group Z as a model) that the content of this paper in a certain sense is "the theory of finite-dimensional representations from the infinite-dimensional point of view"; however, the opposite link also exists - from the more intuitive finite-dimensional case to the less explored infinite-dimensional one. Infinite-dimensional representations are remarkable by the fact (and, I suppose, precisely by it) that they contain a series of *unitary* representations for which analysis and synthesis can be realized by means of generalized "Fourier integrals"; but the specific nature of finite-dimensional representations is not less interesting - here the concept of *solubility* plays the principal role. It is an interesting problem to clarify the connections between these two (in a certain sense contrasting) extremes between which the idea of analytic continuation is the connecting link. In the finite-dimensional case the theory is naturally simpler; this comes out by the fact that it is essentially algebraic; topology, in H. Weyl's phrase, here only throws a shadow on the algebraic scene.

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[1] Moreover, every E_i is uniquely determined by its "dominant vector" $x_i \in \theta_i$, in fact, every irreducible space is the "cyclic envelope" of every non-zero element.

Part I

DESCRIPTION OF THE IRREDUCIBLE REPRESENTATION

§1. The elements of the general theory[1]

A *linear representation* of a group G is a homomorphic mapping (preserving the group operation)

$$g \longrightarrow T_g$$

of G into a group of linear operators acting on a vector space E; the space E is here assumed to be finite-dimensional and defined over the field of complex or the field of real numbers; if G is a topological group, then the concept of a homomorphism includes the requirement of continuity, i.e. that the operator function $T(g) = T_g$ must be continuous with respect to $g \in G$. Two linear representations T' and T'' are regarded as *equivalent* if there is a one-to-one linear correspondence between their representation spaces that carries the operations T'_g and T''_g into one another:

$$T'_g = U T''_g U^{-1};$$

here as everywhere, when it is not stated otherwise, g denotes an arbitrary element of G.

1. A representation is called *reducible* if there is in E an invariant subspace ($\neq E$, $\neq (0)$) and *completely reducible* if every invariant subspace $E_1 \subset E$ has an invariant complement $E_2 \subset E$; in the latter case E appears in the form of the direct sum of the two invariant subspaces E_1 and E_2. Continuing this process we arrive at a decomposition of the representation space into a direct sum of a finite number of irreducible subspaces, and in the corresponding bases the matrices of all operators of a representation assume quasi-diagonal form, whereas without the assumption of complete reducibility we can only obtain quasi-triangular matrices by means of successive factorization); the study of a completely reducible representation now comes to that of its elementary cells – the *irreducible representations*. When we combine in a single subspace all the equivalent irreducible components and then write E as a direct sum

$$E = E_1 + E_2 + \ldots + E_s,$$

where the representation acting on each invariant subspace E_i is a *multiple of an irreducible one*, then it is easy to show that the decomposition so obtained is uniquely determined, whereas the decomposition of the multiple representations is essentially non-unique. Symbolically

$$\tau = k_1\tau_1 + k_2\tau_2 + \ldots + k_s\tau_s,$$

where τ_i are the irreducible representations that occur in the decomposition of the representation τ with the multiplicity k_i; the collection of representations τ so obtained will be called the *spectrum* of τ.

2. Before we can tackle the spectral analysis of linear representations we have to know what irreducible representations a given group G

[1] The basic concepts of linear algebra, of the theory of groups and of topology are assumed to be known: see, for example, [5], [19], and also [6], [10], [15], [17], [18].

admits at all; for the "classical groups" to be studied in this paper a detailed classification is well known (we refer to the original paper by E. Cartan [3]) and will be obtained in the next few sections in an entirely elementary way. An auxiliary tool for our solution of this problem is the *embedding of an irreducible representation T_g in the regular representation of the group G*: the (right) regular representations of a topological group G is the representation R_{g_0} defined by the formula

$$R_{g_0}f(g) = f(gg_0)$$

in the infinite-dimensional vector space $C(G)$ consisting of all continuous functions on the group G. The possibility of such an embedding is evident: let the representation T_g act on a space E; in the *dual space* $\hat{E}$ we fix an arbitrary vector $l_0 \neq 0$, i.e. we fix a linear form $l_0(x)$ on the space E and with every vector $x \in E$ we associate the function

$$f_x(g) = l_0(T_g x);$$

the set of functions $f(g)$ so obtained forms a finite-dimensional linear space $E_0 \subset C(G)$, and the mapping $E \to E_0$ is one to one (for the inverse image of the zero of E is an invariant subspace and cannot, therefore, be different from 0, because E is irreducible); it is clear that in this mapping the vector $T_{g_0}x$ corresponds to the function $f_x(gg_0)$, i.e. the operation T_{g_0} goes over into the operation of right translation R_{g_0}.

In a less invariant (matrix) form the process of embedding can be described in the following way: let $T_{ij}(g)$ denote the matrix element of T_g referred to some basis of E; we fix an arbitrary, for example the first, row of T_g and define the space E_0 as the linear envelope of the functions

$$e_i(g) = T_{1i}(g);$$

from the multiplicative property of the operator T_g it follows immediately that the right translation of the argument g induces a transformation of the functions $e_i(g)$ by means of the matrix $(T_{ij}(g_0))$. (Obviously, there exist many distinct methods of embedding, and the irreducible representation T is contained in the regular representation with a multiplicity at least $\dim E$).

3. We mention a few general concepts of the algebra of representations. The representation $\hat{T}$ *contragredient* to T acts on the dual space $\hat{E}$ and is defined by the rule:

$$\hat{T}_g = T'_{g^{-1}},$$

where $'$ denote the operation of transposition in a certain basis of E (the operation T_g carries the linear form $l(x)$ into $T_g l(x) = l(T_{g^{-1}}x)$; in other words, when we reckon the dual space to consist of hyperplanes of E, then the operation $\hat{T}_g$ transforms the hyperplane in such a way that the relations of incidence are preserved: if the vector x_0 belongs to the hyperplane l_0, then the vector $T_g x_0$ belongs to the hyperplane $\hat{T}_g l_0$). It is clear that T is in turn contragredient to $\hat{T}$.

By applying the processes of *raising to a power, alternation and symmetrization* to the linear space T, we can obtain a whole series of other linear representations of G. Let us assume, for example, that G

itself is linear, i.e. that its elements g are matrices in a certain space E; by regarding E as a space of linear forms on $\hat{E}$ and by writing the dual vectors $\xi \in \hat{E}$ in the form of matrices of a single row, we realize the transformations of G in the form

$$T_g l(\xi) = l(\xi g). \tag{1}$$

When we replace in this definition the linear form $l(\xi)$ by a multilinear form of rank m:

$$p(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}),$$

which is *skew-symmetric in all arguments*, we obtain the m-th *alternating power* $\alpha_m(T)$ of the representation T; $\alpha_m(T)$ is a representation that transforms the *multi-vectors* [14] of rank m over the space E (and the matrix elements of $\alpha_m(T)$ are formed as minors of order m from the elements of T_g); in particular, if n is the dimension of E, then $\alpha_{n-1}(T)$ is the dual representation $\hat{T}$, $\alpha_n(T)$ is the one-dimensional representation $g \to \det T_g$, and for higher indices the representation $\alpha_m(T)$ is not defined (because there are no skew-symmetric forms of a rank greater than n other than zero). Similarly, by taking $p(\xi)$ as a *symmetric* form of rank m (this time m can be an arbitrary non-negative integer), we obtain the m-th *symmetric power* $\sigma_m(T)$ of the representation (1.1). Starting out from the simplest representations of a group G and combining the processes of alternation and symmetrization we can construct all the irreducible representations of the classical continuous groups.

Another natural operation on representations is the construction of the tensor product $T = T' \times T''$ of two given representations T' and T'', which acts on the space $E' \times E''$ by the rule

$$T_g(\xi \times \eta) = T'_g\xi \times T''_g\eta, \quad \xi \in E', \quad \eta \in E'';$$

certain physical problems, in particular, lead to the spectral analysis of tensor products.

4. The "classical" groups to be studied in this paper are typical and, for analysis, the most important examples of *Lie groups* (every such group admits a coordinatization, and the group multiplication is given by analytic functions); we shall hardly at all touch on the general theory of Lie groups, however, it is worth mentioning some points of principle that play a role, explicitly or implicitly, in the subsequent account.

The *commutator subgroup* of an arbitrary group G is the subgroup K that is generated by all possible commutators of the elements of G (which means that the elements of K are representable in the form of finite products $k_1 k_2 \ldots k_m$, where each k_i is a commutator of two elements of G); if G is a topological group, then the closure of this commutator subgroup is called the *derived group* G' of G; all the higher derived groups (defined recurrantly) are, obviously, normal subgroups of G. A group G is called *soluble* if the chain of its derived groups

$$G, G', G'', G''', \ldots$$

ends after a finite number of steps in the group $\{e\}$ consisting of the unit element only: a "complementary" class of topological groups are the *semisimple* groups: a group is called semisimple if it does not contain connected soluble normal subgroups. (In an arbitrary group G the maximal connected soluble normal subgroup R is called the *radical*; clearly the factor-group G/R is always semisimple.)

Everyone of these classes of groups has its own characteristic property of linear representations. By a theorem of S. Lie [11] *every irreducible representation of a connected soluble group G in a complex vector space is one-dimensional*, therefore the matrices of every linear representation of G can be reduced in a certain basis simultaneously to *triangular form*; a remarkable property of representations of a semisimple Lie group is the theorem of H. Weyl: *every representation of a semisimple connected Lie group is completely reducible*;[1] by way of contrast a soluble group, if it is not compact, always has substantially "many" not completely reducible representations; as the simplest example we give the Jordan block

$$T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

– a representation of the additive group of real numbers x.

Finally, a remarkable fact is the theorem of E. Levy [21], [16], according to which every connected Lie group can be constructed by means of a so-called "semidirect product" of two subgroups one of which is soluble, the other semisimple (more precisely, from the radical and the maximal connected semisimple subgroup); without going into details we give as an example the group of motion of a euclidean n-dimensional space, where the (commutative) radical is the subgroup of parallel shifts and the maximal semisimple subgroup the subgroup of all rotations around a fixed point. In the paper [12] one can find a generalization of Weyl's theorem: *a representation of a connected Lie group is completely reducible if and only if its restriction to the radical is completely reducible*; the spectral analysis of such representations reduces to the corresponding problem for the semisimple components of G.

The groups that occur in this paper are either semisimple or reductive (the radical is contained in the centre of G); in the latter case the complete reducibility of the elements of the centre is sufficient for the complete reducibility of the representation; we omit the easy verification of this condition and note at once that *all representations that will be considered below are completely reducible, except one special example in* §12.

5. It is important to emphasize the difference between *complex* and *real* Lie groups. Complex groups, as a rule, have a considerably simpler structure, so that it is often convenient to reduce the study of real groups to the complex case, and this happens to be possible owing to a certain general principle of analytic continuation which we shall now briefly describe.

We do not want to reproduce the general definition of a Lie algebra. The Lie algebra corresponding to a group G can be defined as the tangent plane at the unit point of the group manifold, provided we introduce in this plane a "multiplication" (commutation) of vectors that is

1 Various proofs of this theorem can be found in [1], [4], [21], see also [24].

compatible with the multiplication law in G and reflects this law in the "infinitely small". A group G is called *complex* if its Lie algebra can be regarded as an algebra over the field of complex numbers. A complex connected group G_* is called *complexification* of its real connected subgroup G if the Lie algebra of G_* is the complex linear envelope of the Lie algebra of G and if the complex dimension of G_* is equal to the real dimension of G; in this case the subgroup G is called a *real form* of the group G_*; one and the same complex group may have several non-isomorphic real forms.

Let us consider, for example, the group $\mathfrak{U}$, consisting of all unitary matrices of the second order; every one-parameter subgroup of $\mathfrak{U}$ is of the form

$$u(t) = e^{ita},$$

where t is a real parameter and where the matrix a is Hermitian:

$$a = \begin{pmatrix} x & y \\ \bar{y} & x \end{pmatrix}, \quad x = \bar{x};$$

the tangent vector to the subgroup $u(t)$ at the unit point is the skew-Hermitian matrix ia:

$$ia = \lim_{t \to 0} \frac{1}{t} [u(t) - u(0)];$$

therefore, the Lie algebra of $\mathfrak{U}$ consists of all skew-Hermitian matrices of the second order. The complex envelope of these matrices coincides with the set of all complex matrices of the second order; accordingly, the group $\mathfrak{U}_*$ consists of all non-singular complex matrices of the second order. Another real form of $\mathfrak{U}_*$ is the subgroup of real matrices with determinant greater than zero; being non-compact, this subgroup cannot be isomorphic to $\mathfrak{U}$.

Let T_g be a representation of a group G; then the infinitesimal operations

$$A(x) = \lim_{t \to 0} [T_{g(t)} - 1],$$

where $g(t)$ is a one-parameter subgroup with the tangent vector x, form a representation of the Lie algebra of G. Let us assume that this Lie algebra is complex; then the representation $A(x)$ is called *analytic* if it preserves the complex structure:

$A(\lambda x) = \lambda A(x)$ for every complex λ, and *anti-analytic* if it satisfies the condition

$$A(\lambda x) = \bar{\lambda} A(x);$$

it is easy to show that every real representation of a Lie algebra can be put in the form of a direct sum of an analytic and an anti-analytic representation. Similarly, a representation T_g is called *analytic* (*complex-analytic*) if $A(x)$ is analytic and *anti-analytic* if $A(x)$ is anti-analytic.[1]

[1] Strictly speaking, only complex-analytic representations can be regarded as representations of a complex group G. Real representations arise if the complex group G is regarded as real, i.e. if every complex parameter is replaced by two real ones.

We can now, finally, formulate the principle of analytic continuation: *every irreducible complex analytic representation of the group G_* remains irreducible when restricted to the subgroup G; and conversely, every irreducible representation of G can be extended to a complex analytic irreducible representation of G_** (which may, however, turn out to be not unique). The detailed justification of these (rather obvious) statements exceeds the framework of the present paper.

§2. The Gauss decomposition

In this section we shall explain a certain general scheme of constructing irreducible representations; with this aim in mind it is convenient to restrict ourselves to an abstract form of the presentation, and all the concepts to be introduced here will soon find a simple concrete content.

DEFINITION 1. We shall say that a topological group G *admits a Gauss decomposition* if G contains subgroups Z, D and Z with the following properties:

1°. The sets $\mathrm{Z}D$ and DZ are *soluble connected* subgroups of G whose derived groups are Z and Z, respectively.

2°. The intersections $\mathrm{Z} \cap DZ$ and $D \cap Z$ consist only of the unit element, and the set $\mathrm{Z}DZ$ is dense in G:

$$G = \overline{\mathrm{Z}DZ}.$$

From the first condition it follows that D is an abelian connected group, and that Z and Z are soluble and connected; the second condition means that almost every[1] element $g \in G$ has a decomposition

$$g = \zeta\delta z, \quad \zeta \in \mathrm{Z}, \quad \delta \in D, \quad z \in Z,$$

and that this decomposition, if it exists, is unique.

All semisimple complex Lie groups, in particular, admit a Gauss decomposition [12].

Let $\alpha(\delta)$ be a character of D; for the elements of G that have a Gauss decomposition we form the expression $\alpha(g)$ by the rule

$$\alpha(g) = \alpha(\zeta\delta z) = \alpha(\delta),$$

and we formally define the transformation

$$T_g f(z) = \alpha(zg) f(z \cdot g) \tag{2}$$

in the class of functions on the group Z, where $z \cdot g \in Z$ means the "right" component in the Gauss decomposition of the element $zg \in G$; then it is easy to verify [12] that the operations T_g satisfies the relation $T_{g_1} T_{g_2} = T_{g_1} T_{g_2}$, so that they can form a *representation* of G provided there exists a *finite-dimensional* space of functions $f(z)$ in which the mapping $g \to T_g$ can be extended to a *continuous* homomorphism of G.

DEFINITION 2. Let us assume that the formula (2) defines a representation of G in some finite-dimensional space $E(Z)$ of functions

[1] Here and in what follows the term "almost every" is understood in the topological sense (the set of all such elements is everywhere dense in G).

on the group Z; then we shall say that the representation (2) *is induced by the character* $\alpha(\delta)$, and also that the character $\alpha(\delta)$ *is inductive with respect to the group* G.

Note that

$$T_{z_0} f(z) = f(zz_0), \tag{3}$$

$$T_\delta f(z) = \alpha(\delta) f(\delta^{-1} z \delta). \tag{4}$$

THEOREM 1. *Every irreducible representation of a group G is induced by some character $\alpha(\delta)$ of the subgroup D; two irreducible representations of G induced by the characters α_1 and α_2 are equivalent if and only if $\alpha_1 = \alpha_2$.*

PROOF. We shall only give here a brief outline of the argument; the reader can find a more detailed proof in the papers [11] and [12]. Let T_g be an irreducible representation of G; its restriction T_k to the connected soluble subgroup $K = ZD$ can be reduced, by the theorem of Lie, to triangular form:

$$T_k = \begin{pmatrix} \mu_1(k) & & 0 \\ & \mu_2(k) & \\ * & & \\ & & \mu_r(k) \end{pmatrix}, \tag{5}$$

where the $\mu_i(k)$ are characters of K (moreover, obviously, every character $\mu(k)$ of K is trivial on the *derived* subgroup Z and is a character of the abelian group D: $\mu(k) = \mu(\delta)$ for $k = \zeta\delta$). The space E of the representation T_g can be regarded as spanned by the basis elements

$$e_i(g) = T_{1i}(g)$$

("embedding in the right regular representation of G" - see §1); in accordance with (5), all these elements and therefore all the functions $f(g)$ of the space E satisfy the relation

$$f(kg) = \mu_1(k) f(g);$$

in particular, $f(\zeta\delta z) = \mu_1(\delta) f(z)$ and the functions $f(g)$, being continuous, are uniquely determined by their restrictions to the subgroup Z; when we denote by $E(Z)$ the space of the restrictions, we find that the mapping $E \to E(Z)$ is one to one and the representation T_g transferred to the space $E(Z)$ is immediately defined by the formula (2), where the role of $\alpha(\delta)$ is played by the character $\mu_1(\delta)$.

Further, by applying the theorem of Lie to the representation of the connected soluble group $H = DZ$, we conclude that the space E (or $E(Z)$) contains an eigenvector for all operations T_h - this vector is then fixed under the derived subgroup Z; but in the space $E(Z)$ the transformations of the subgroup Z reduce to a right translation, and only a constant vector ($f^0(z) \equiv 1$) can be fixed, moreover

$$T_\delta f^0 = \mu_1(\delta) f^0;$$

hence the inductive character $\mu_1(\delta)$ is uniquely defined and therefore determines the representation T_g to within equivalence. The theorem is now proved.

An eigenvector of all the transformations of the subgroup D:

$$T_\delta x = \mu(\delta)\, x,$$

is usually called a *weight* vector and the corresponding character $\mu(\delta)$ a *weight*; for example, if D is the group of complex diagonal matrices

$$\begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{pmatrix} \qquad (\delta_i \neq 0),$$

then every complex analytical character is of the form

$$\mu(\delta) = \delta_1^{m_1} \delta_2^{m_2} \dots \delta_n^{m_n},$$

where $m_1, m_2, \dots, m_n$ are integers.

COROLLARY 1. *In the space of every irreducible representation of a group G there is one and only one (apart from normalization) invariant of Z that is at the same time a weight:*

$$T_z f^0 = f^0, \quad T_\delta f^0 = \alpha(\delta) f^0.$$

The character $\alpha(\delta)$ is called the *dominant weight*[1] of the representation T_g and the vector f^0 the *dominant vector* of the representation space. The general scheme of spectral analysis can now be formulated as follows.

COROLLARY 2. *Let $g \to T_g$ be a completely reducible representation of a group G in the space E and E^0 the maximal subspace that is fixed under the subgroup Z. Then E^0 is invariant and completely reducible under the abelian subgroup D; the set of irreducible representations contained in E (the "spectrum" of the representation T_g) is uniquely determined by the set of weights of the space E^0 (the "weight spectrum" of the space E^0); finally, every irreducible subspace of E can be obtained as the linear envelope of vectors of the form $T_g x_\alpha$, where x_α is one of the weight vectors of the subspace E^0.*

In other words, the decomposition of the representation T_g into irreducible ones reduces to the task of finding the subspace E^0 and subsequently to the choice of a basis of weight vectors in E^0:

$$T_\delta x_\alpha = \alpha(\delta)\, x_\alpha,$$

1 The origin of the term "dominant weight" can be explained in the following way. Usually in the study of the classical linear groups one selects all the weight vectors (with respect to T) in the space of an irreducible representation T_g; it turns out that a linear order can be introduced in the set of the corresponding weights under which the dominant weight becomes the largest. We note that the subgroups Z and Z occur in the Gauss decomposition symmetrically, in place of the dominant vector we could have taken the "lowest" vector – an invariant of the subgroup Z; in that linear order the lowest vector is then the smallest. If $\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2} \dots \delta_n^{m_2}$, then the system of numbers $(m_1, m_2, \dots, m_n)$ is a complete system of "invariants" that characterizes the given representation to within equivalence.

and every irreducible subspace E_α is spanned by the vectors $T_g x_\alpha$, $g \in G$.

In particular, *in order to prove the irreducibility of a representation T_g it is sufficient to verify*[1] *that the representation space contains only one invariant of the subgroup Z.*

Note that the space of the induced representation (2) is spanned by the vectors

$$f_g(z) = \alpha(zg);$$

it is clear that the functions $\alpha(z, g) = \alpha(zg)$ in this case are *continuous* with respect to $z \in Z$, $g \in G$ (and even analytic if G is a Lie group). The equality $\alpha(z, g) = T_g(1)$ also implies the fundamental algebraic property of the multipliers $\alpha(z, g)$:

$$\alpha(z, g_1, g_2) = \alpha(z, g_1)\alpha(z \cdot g_1, g_2) \tag{6}$$

(composition law for the operators T_g); in particular,

$$\alpha(z, gz_0) = \alpha(z, g),$$
$$\alpha(z, z_0 g) = \alpha(zz_0, g).$$

Turning now to the problem of classification of all irreducible representations of a group G (it is assumed that G has a Gauss decomposition), we see that this problem reduces to a description of the special subset $\mathfrak{A}$ of "*dominant weights*" in the character group X of the abelian group D, i.e. to a description of all characters $\alpha(\delta)$ that are inductive with respect to G. The following proposition is a convenient criterion in this process.

COROLLARY 3. *Let us assume that the Gauss decomposition of a group G induces a Gauss decomposition in a certain subgroup G_0:*

$$G_0 = \overline{Z_0 D_0 Z_0},$$

where Z_0, D_0 *and* Z_0 *are the intersections of* G_0 *with* Z, D *and* Z, *respectively*; *we denote by* $\alpha_0(\delta_0)$ *the restriction of a character* $\alpha(\delta)$ *of the group* D *to the subgroup* D_0. *If the character* α *is inductive with respect to* G, *then the character* α_0 *is inductive with respect to* G_0.

PROOF. The inductiveness of the character α implies that the linear envelope of the functions $f_g(z) = \alpha(z, g)$ is of finite dimension; this is then *a fortiori* true for the functions

$$f_{g_0}(z_0) = \alpha(z_0, g_0), \quad z_0 \in Z_0, \ g_0 \in G_0,$$

which are at the same time continuous functions on Z_0 and depend continuously on the parameter $g_0 \in G_0$ (because they are obtained from $\alpha(z, g)$ by restriction to $Z_0 \times G_0$); consequently a representation of G_0 acts on the linear envelope of these functions; this representation is irreducible, because its space is cyclically spanned by the unique dominant vector $f^0(z_0) \equiv 1$; as a result, the character α_0 is inductive.

[1] Here, as before, we have assumed that T_g is completely reducible.

§3. The Lorentz group and the group of rotations of three-dimensional euclidean space

The Lorentz group has appeared as a touchstone in many papers on representation theory: here is an example exhibiting properties that are characteristic for a wide class of classical Lie groups; the considerations of the present section will also be used as a first (and decisive) inductive step in the solution of the general problem of classification.

1. Fundamental definitions. By R_4^1 we denote the real four-dimensional pseudo-euclidean space in which the metric is given by the quadratic form

$$s^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2. \tag{7}$$

Proper Lorentz group is the name for the maximal connected subgroup in the group of rotations of the space R_4^1; in more detail this means the following: in the group of all non-singular matrices $g = (g_{ij})$ of order four that preserve the form (7) we select the elements satisfying the additional condition

$$\det g = 1, \quad g_{44} \geqslant 1;$$

if the first condition is discarded, then the group so obtained (for whose elements $\det g = \pm 1$) is called the *full Lorentz group*; the second condition ensures the *preservation of the sign* of the fourth coordinate (in the physical interpretation this represents the time coordinate). The Lorentz group contains as a subgroup the rotations of the three-dimensional euclidean space embedded in R_4^1 as the hyperplane $x_4 = 0$.

It is well known ([9], [17], [18]; see also the end of the present section) that the proper Lorentz group can be parametrized by means of the *complex unimodular matrices of the second order*

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1; \tag{8}$$

more precisely: with every proper Lorentz transformation g we can associate a unimodular matrix a, defined to within its sign, such that the relations

$$g_1 g_2 \longrightarrow \pm\, a_1 a_2$$

are satisfied for any two transformations g_1, g_2 and their corresponding matrices a_1, a_2 (if the uniqueness of the function $a(g)$ is required, i.e. if one keeps track of one of its branches, then it is impossible to preserve the continuity of this function; however, in a certain neighbourhood of the unit matrix $g = e$ the parametrization by means of the matrices a is one-to-one and continuous); one says that the group $\mathfrak{A}$ of all matrices of the form (8) is *locally isomorphic* to the proper Lorentz group - more exactly, $\mathfrak{A}$ covers this group twice. To the subgroup of three-dimensional rotations there corresponds the subgroup $\mathfrak{U}$ consisting of all *unitary* matrices (8); every matrix $u \in \mathfrak{U}$ has, of course, the expression

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{8)}$$

Let us agree to call the unimodular group $\mathfrak{A}$ for the sake of brevity the *Lorentz group*.

2. **The Gauss decomposition.** We denote by Z, D and Z the subgroups of the Lorentz group that consist, respectively, of the matrices of the form

$$\zeta=\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix},\quad \delta=\begin{pmatrix}\delta^{-1} & 0\\ 0 & \delta\end{pmatrix},\quad z=\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}$$

(it is customary in this case to denote a matrix and the complex number that is its parameter by one and the same letter); obviously, all these subgroups are commutative, also D is isomorphic to the multiplicative group of all complex numbers and each of the groups Z and Z is isomorphic to the additive group of all complex numbers. By an immediate computation it is easy to verify that every matrix $a \in \mathfrak{A}$ in which the element a_{22} is different from zero has a unique representation in the form

$$a=\zeta\delta z,\quad \zeta\in \mathrm{Z},\quad \delta\in D,\quad z\in Z, \tag{9}$$

and that the parameters ζ, δ, z can be expressed in terms of the elements of the matrix a by the formulae:

$$\zeta=\frac{a_{12}}{a_{22}},\quad \delta=a_{22},\quad z=\frac{a_{21}}{a_{22}}; \tag{10}$$

when we denote by H the group of triangular matrices of the form δz (H is, obviously, connected) and compute the chain of its derived groups:

$$H'=Z,\quad H''=\{e\},$$

we see that H is soluble; the same is true of the group $K = \mathrm{Z}D$; finally, the set of matrices of the form (9) is everywhere dense in $\mathfrak{A}$ (because its complement, which is distinguished by the equation $a_{22} = 0$, is of smaller dimension); consequently, the decomposition (9) is a Gauss decomposition of $\mathfrak{A}$ in the sense of Definition 1.

The subgroup $\mathfrak{U}$, obviously, does not have a Gauss decomposition – from our point of view $\mathfrak{A}$ is of simpler structure than $\mathfrak{U}$ – however, $\mathfrak{A}$ is the "analytic continuation" of $\mathfrak{U}$ and owing to this the properties of the two groups are closely connected.

3. **Classification of all irreducible representations.** Following the general scheme of the preceding paragraph we determine to begin with the explicit form of the transformation $\tilde{z} = z \cdot a$ that is generated in the group Z by the element $a \in \mathfrak{A}$. When we replace in the last formula of (10) the matrix a by the product za, we find that the parameter of the matrix $\tilde{z}$ which can be determined from the relation

$$za=\zeta\delta\tilde{z},$$

is the number

$$\tilde{z}=\frac{\alpha z+\gamma}{\beta z+\delta},$$

where α, β, γ, δ are the elements of the matrix a; consequently, the *transformation $z \to z \cdot a$ is a fractional linear substitution in the complex plane Z.*

Furthermore, every complex analytic character of the group D has the form

$$\alpha(\delta)=\delta^m,$$

where m is an arbitrary integer; when we even allow many-valued characters, we can regard the parameter m as an arbitrary complex number. From Theorem 1 we conclude that every analytic irreducible representation of the Lorentz group must be defined by a formula of the form

$$T_a f(z) = (\beta z + \delta)^m f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right), \tag{11}$$

and it remains to solve the problem what numbers m are here admissible, i.e. what characters $\alpha(\delta)$ are inductive with respect to the group $\mathfrak{A}$.

The problem has a simple solution: we recall that the space E of the representation (11) must contain all the binomials

$$f(z) = (\beta z + \delta)^m,$$

where the parameters β and δ range over all possible values that are admissible for the elements $a \in \mathfrak{A}$; in partucular, the space E must contain all the translations

$$f_i(z) = (z + \delta_i)^m.$$

But E is finite-dimensional, therefore there exists an integer $r \geqslant 1$ such that any $r + 1$ functions $f_i(z)$ are linear dependent, in other words, such that the determinant[1]

$$\Delta(z) = \begin{vmatrix} f_1(z) \; f_2(z) \dots f_{r+1}(z) \\ f_1'(z) \; f_2'(z) \dots f_{r+1}'(z) \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ f_1^{(r)}(z) \; f_2^{(r)}(z) \dots f_{r+1}^{(r)}(z) \end{vmatrix}$$

is identically zero; it is clear that this condition is only satisfied in the case when the parameter m is a *non-negative integer*. For when we set $z = 0$, we find that

$$\Delta(0) = \begin{vmatrix} \delta_1^m & \delta_2^m & \dots \delta_{r+1}^m \\ m\delta_1^{m-1} & m\delta_2^{m-1} & \dots m\delta_{r+1}^{m-1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} = m^r(m-1)^{r-1}\dots(m-r+1)[\delta_1\delta_2\dots\delta_{r+1}]^{m-r}W(\delta),$$

where $W(\delta)$ is the Vandermonde determinant:

$$W(\delta) = \prod_{i<j} (\delta_i - \delta_j);$$

therefore, if m is not equal to one of the numbers 0, 1, ..., $r - 1$, then $\Delta(0) \neq 0$, whenever the numbers δ_i are chosen different from zero and from each other.

Note that for integral $m \geqslant 0$ the space E, which contains all binomials, also contains all the monomials 1, z, z^2, ..., z^m.

Let us summarize the result:

Every complex-analytic irreducible representation of the Lorentz group is given by an integer $m \geqslant 0$ *and can be realized by the formula* (11) *in*

[1] Here we understand by the differentiation operator the "formal" operator

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

the space $E_m(Z)$ that consists of all polynomials of the variable z of a degree not exceeding m.

So far we have had in mind only *complex* (complex-analytic) representations of the Lorentz group; however, by repeating almost word for word all the preceding arguments for the case of a real character $\alpha(\delta) = \delta^p\bar{\delta}^q$, we conclude that every irreducible representation of the Lorentz group is given by a pair of non-negative integers (p, q) and is realized by the formula

$$T_g f(z) = (\beta z + \delta)^p (\overline{\beta z + \delta})^q f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right)$$

in the space $E_{pq}(Z)$ consisting of all the polynomials

$$f(z) = f(z, \bar{z})$$

of a degree not exceeding p in the variable z and not exceeding q in the variable $\bar{z}$; in other words, every real irreducible representation of the Lorentz group has the form of a tensor product

$$\tau_1 \times \bar{\tau}_2,$$

where τ_1 and τ_2 are complex-analytic (irreducible) representations, and $\bar{\tau}$ denotes the representation that is complex-conjugate[1] to τ.

Incidentally we have obtained the statement: *the Lorentz group has no many-valued representations* (this can also be concluded from more general reasonings if it can be shown that the group $\mathfrak{A}$ is *simply connected*; note that the "proper Lorentz group" in its original definition is doubly connected - and the group $\mathfrak{A}$ is a two-valued representation of it).

Finally, turning to the subgroup $\mathfrak{U}$, we can use the general principle of analytic continuation and conclude that every irreducible representation of $\mathfrak{U}$ is a restriction of one of the irreducible *complex* representations of $\mathfrak{A}$ to the elements $u \in \mathfrak{U}$.

4. The matrix elements of irreducible representations. The monomials $1, z, z^2, \ldots, z^m$ form a basis in the space $E_m(Z)$, and this is a "weight" basis with respect to the diagonal group D (in other words, every vector z^l is an eigenvector with respect to all the operators T_δ, $\delta \in D$); by introducing the frequently used numbering $m = 2k$, where k is either an integer or half of an integer depending on the parity of m, we can write the basis vectors in the form

$$e_\mu = z^{k+\mu}, \quad \mu = -k, -k+1, \ldots, k,$$

and find an explicit expression for the matrix elements of the representation T_a defined by the relation

$$T_a e_\mu = \sum_{\nu=-k}^{k} e_\nu T^k_{\nu\mu}(a).$$

From the formula (11), where m is replaced by $2k$, we find

$$T_a e_\mu = (\beta z + \delta)^{k-\mu} (\beta z + \delta)^{k+\mu};$$

---

[1] This means that *for some choice of the basis* the matrices of the representations $\bar{\tau}$ are complex conjugates to those of τ.

hence it follows immediately that

$$T^k_{\nu\mu}(a) = \frac{1}{(k+\nu)!}\frac{\partial^{k+\nu}}{\partial z^{k+\nu}}\{(\beta z+\delta)^{k-\mu}(\alpha z+\gamma)^{k+\mu}\}|_{z=0};$$

we set $x = \beta z + \delta$, $y = \alpha z + \gamma$; then x and y are connected by the relation $x\alpha - y\beta = 1$ which enables us to go over from z to the variable

$$t = \alpha(\beta z + \delta)$$

and obtain as a result the expression

$$T^k_{\nu\mu}(a) = \frac{(-1)^{k+\mu}}{(k+\nu)!}\alpha^{\nu+\mu}\beta^{\nu-\mu}\frac{\partial^{k+\nu}}{\partial t^{k+\nu}}\{t^{k-\mu}(1-t)^{k+\mu}\}|_{t=t_0=\alpha\delta}.$$

So we have illustrated the connection between the matrix elements and the well-known *Jacobi polynomials*; restricting ourselves to the unitary subgroup $\mathfrak{U}$ and applying the parametrization (see (8′)):

$$\alpha = \sqrt{t}e^{i\varphi}, \quad \beta = \sqrt{1-t}\,e^{i\psi}, \quad 0 \leqslant t \leqslant 1, \quad 0 \leqslant \varphi,\ \psi < 2\pi,$$

we can write the matrix elements[1] in the form

$$T^k_{\nu\mu}(u) = \frac{(-1)^{k+\mu}}{(k+\nu)!}e^{i\nu(\varphi+\psi)}e^{i\mu(\varphi-\psi)}P^k_{\nu\mu}(t),$$

where

$$P^k_{\nu\mu}(t) = t^{-\frac{\nu+\mu}{2}}(1-t)^{-\frac{\nu-\mu}{2}}\frac{d^{k+\nu}}{dt^{k+\nu}}[t^{k-\mu}(1-t)^{k+\mu}];$$

further details can be found, for example, in [6], [10]. The result obtained reflects in a primitive form some more general rules that hold for all the other classical Lie groups.

5. Connection of the group $\mathfrak{A}$ with geometry. We confine ourselves to an account of the formal scheme and refer the reader to the rather extensive literature on this problem.

As a starting point we take the group of rotations of a four-dimensional *complex* euclidean space E_4 and denote by G its maximal connected subgroup, i.e. the group of all complex matrices with unit determinant that preserve the quadratic form

$$(x, x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

(where x_0, x_1, x_2, x_3 are complex numbers); with every vector $x \in E_4$ we associate a matrix of order two

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

setting $x_{11} = x_1 + ix_0$, $x_{12} = x_2 + ix_3$, $x_{21} = x_2 - ix_3$, $x_{22} = -x_1 + ix_0$; then the determinant of the matrix X differs only in sign from the scalar square (x, x). Denoting by a_1 and a_2 two arbitrary elements of the group $\mathfrak{A}$, we consider the linear transformation in the space E_4 that is defined by the formula

$$X \longrightarrow a_1 X a_2', \tag{12}$$

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[1] A remarkable property of these elements is their mutual *orthogonality with respect to the invariant measure on the group* $\mathfrak{U}$.

where the matrix a' is obtained from a by transposition; on multiplication by unimodular matrices the value of det X is unchanged; moreover, from the fact that $\mathfrak{A}$ is connected it follows that all the transformations (12) form a *connected* group, consequently, every transformation (12) is orthogonal with respect to the metric (x, x) and belongs to the connected group G; it is not difficult to show that the collection of such transformations *coincides with the whole group* G (making use of the connectedness of G it is sufficient to verify that among the transformations (12) there is contained a complete system of one-parameter subgroups of G; another method of proof will be mentioned in §9); finally, the pairs of matrices (a_1, a_2) and $(-a_1, -a_2)$ determine one and the same transformation g in the space E_4, and this accounts for the ambiguity of the inverse mapping $g \to (a_1, a_2)$.

We conclude that *the orthogonal group G is locally isomorphic to the direct product*

$$\mathfrak{G} = \mathfrak{A} \times \mathfrak{A}$$

(the "square" of the Lorentz group) *and the group G covers the group $\mathfrak{G}$ twice.*

The pseudo-euclidean space R_4^1 can be embedded in E_4 if the coordinates x_1, x_2, x_3 are taken to be real, and the coordinate x_0 purely imaginary: $x_0 = ix_4$; the corresponding matrices

$$X = \begin{pmatrix} x_4 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_4 - x_1 \end{pmatrix} \tag{13}$$

range over the space of all *hermitian* matrices of the second order, and the "proper Lorentz group" is now identified with the subgroup of $\mathfrak{G}$ consisting of all the elements of the form $(a, \bar{a})$, more accurately, with the group of transformations

$$X \longrightarrow aXa^* \tag{14}$$

in the space R_4^1 (here * denotes hermitian conjugacy).[1] By $\mathfrak{M}$ we denote the surface in R_4^1 that consists of all positive definite hermitian matrices X with the determinant equal to 1; it is well known that the group of motions (14) determines on $\mathfrak{M}$ a Lobachevskii geometry; consequently, *the proper Lorentz group is isomorphic to the group of motions of a Lobachevskii space.*

By excluding the "time" coordinate x_4 we can identify the "physical" space of three dimensions with the totality of matrices (14) whose *trace* is equal to zero; the unitary matrices of the group $\mathfrak{A}$ generate a connected group of rotations in this euclidean space:

$$X \longrightarrow uXu^{-1}, \quad u \in \mathfrak{U}.$$

[1] Similarly we can distinguish the subgroup of proper rotations of the pseudo-euclidean space in which the metric is given by a form with two negative squares - for this purpose we have to consider the pairs (a_1, a_2) with *real* matrices a_1, a_2 and the matrix X also is taken as real finally, the transformations $X \to aXa^{-1}$, where X is a complex matrix with *zero trace*, generate the group $SO(3)$ of proper rotations of a three-dimensional complex euclidean space; and so we have proved that the Lorentz group doubly covers the group $SO(3)$.

If we regard $\mathfrak{A}$ as a group of fractional linear transformations of the complex plane Z, then the correspondence between the group of rotations and the unitary group $\mathfrak{U}$ can also be obtained in a well-known way by means of *stereographic projection* of the unit sphere of a three-dimensional space onto the plane Z.

Another remarkable subgroup of $\mathfrak{A}$ consists of all fractional linear transformations that preserve the interior of the unit circle in the Z-plane (it can easily be shown that this subgroup is isomorphic to the group of all real unimodular matrices of the second order); the geometry defined by the corresponding "motions" in the unit circle is a model of the geometry of the Lobachevskii plane (the boundary of the unit circle plays in this realization the role of the Absolute).

§4. The full linear group

Among the square matrices of order n the set of all non-singular[1] matrices forms a group which is called the *full linear group* and is denoted by $GL(n)$ (all our constructions will be carried out over the field of *complex* numbers, with special notes in the various cases when peculiarities arise on transition to the real field; to distinguish the fields occasionally the symbols $GL(n, C)$, $GL(n, R)$ are used).

1. The Gauss decomposition. Having fixed a basis we define the subgroup D as the set of all *diagonal* matrices with a determinant different from zero; it is convenient to take the subgroup Z as consisting of the *upper* triangular matrices

$$z=\begin{pmatrix} 1 & z_{12} & z_{13} & \dots & z_{1n} \\ & 1 & z_{23} & \dots & z_{2n} \\ & & 1 & \dots & z_{3n} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}, \tag{15}$$

with 1's along the main diagonals; the group Z is obtained from Z by transposition; it is easy to verify that the system $\{\mathrm{Z}, D, Z\}$ has all the properties of Definition 1, and the "Gauss decomposition" coincides for the group G with the well known decomposition of matrices into triangular factors (see [7]).

The explicit form of the parameters ζ, δ, z in the decomposition

$$g=\zeta\delta z$$

can easily be found by the rule of multiplication of minors; in fact, among the minors of order p formed from the first p rows of the *lower* triangular matrix $k = \zeta\delta$, only the principal diagonal minor[2]

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1 i.e. with a determinant different from zero.

2 The symbol $\begin{pmatrix} i_1 i_2 \dots i_p \\ j_1 j_2 \dots j_p \end{pmatrix}$ as usual denotes the minor formed from the rows with the numbers $i_1, i_2, \dots, i_p$ and the columns with the numbers $j_1, j_2, \dots, j_p$.

$$\begin{pmatrix} 12\ldots & p \\ 12\ldots & p \end{pmatrix} = k_{11}k_{22}\ldots k_{pp} = \delta_1\delta_2\ldots\delta_p,$$

is different from zero, where the δ_i are the diagonal elements of k; consequently,

$$\begin{pmatrix} 12\ldots & p-1 & p \\ 12\ldots & p-1 & q \end{pmatrix}_g = \delta_1\delta_2\ldots\delta_p \begin{pmatrix} 12\ldots & p-1 & p \\ 12\ldots & p-1 & q \end{pmatrix}_z = \delta_1\delta_2\ldots\delta_p z_{pq},$$

and this relation determines z_{pq}; defining the elements of the matrix in an analogous fashion, we obtain the final formulae

$$\zeta_{pq} = \begin{pmatrix} 12\ldots & q-1 & p \\ 12\ldots & q-1 & q \end{pmatrix} \frac{1}{\Delta_q}, \quad \delta_p = \frac{\Delta_p}{\Delta_{p-1}}, \quad z_{pq} = \frac{1}{\Delta_p} \begin{pmatrix} 12\ldots & p-1 & p \\ 12\ldots & p-1 & q \end{pmatrix}, \tag{16}$$

where all the minors are minors of g, in particular,

$$\Delta_p = \begin{pmatrix} 12\ldots & p \\ 12\ldots & p \end{pmatrix} = \Delta_p(g)$$

is the principal diagonal minor ($\Delta_0 = 1$); from the formulae (16) it follows that the singular points of the Gauss decomposition fill in G the manifold of elements for which at least one of the minors $\Delta_1, \ldots, \Delta_{n-1}$ vanishes; it is clear that this manifold has a smaller dimension in G so that the complementary set ZDZ is everywhere dense in G.

We note that if $\delta = \delta(g)$ is the diagonal component in the decomposition (16), then

$$\Delta_p(g) = \Delta_p(\delta), \tag{17}$$

and the principal minors $\Delta_p(g)$ depend only on the parameters δ.

2. **Classification of irreducible representations.** Every complex-analytic character of the diagonal group D is of the form

$$\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n}, \tag{18}$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are the eigenvalues of the matrix $\delta \in D$.

LEMMA 1. *The character* (18) *is inductive with respect to the group* $GL(n)$ *if and only if all the differences* $m_i - m_j$ *are integers and the order relations*

$$m_1 \geqslant m_2 \geqslant m_3 \geqslant \ldots \geqslant m_n$$

are satisfied.

PROOF. Let us suppose that the character $\alpha(\delta)$ is inductive and denote by α_0 its restriction to the subgroup D_0 consisting of the matrices

$$\begin{pmatrix} \lambda & & & & & & 0 \\ & \lambda^{-1} & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & 1 \end{pmatrix};$$

now we observe that the subgroup G_0 consisting of all the matrices of the form

$$g=\begin{pmatrix} \alpha & \beta & & & & & 0 \\ \gamma & \delta & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ 0 & & & & & & 1 \end{pmatrix}, \quad \alpha\delta-\beta\gamma=1, \tag{19}$$

is isomorphic to the Lorentz group and that the Gauss decomposition in G induces the Gauss decomposition in G_0 (to make the notation consistent with the preceding section we have to bear in mind that the matrices $z \in G$ now are *upper* triangular - this corresponds to a change of the numbering of rows and columns; in particular, in the diagonal subgroup D_0 we must take the element $\delta_{11} = \lambda$ as an independent parameter; from the Corollary 3 of Theorem 1 we infer that the character

$$\alpha_0(\lambda)=\lambda^{m_1-m_2}$$

is inductive with respect to the Lorentz group;[1] therefore $m_1 - m_2$ is a non-negative integer. By shifting the "active" block of the second order along the main diagonal in (19), we obtain a similar statement about the values $m_2 - m_3, \ldots, m_{n-1} - m_n$, and this proves the necessity of the condition of the lemma.

The character (18) can be written in the form

$$\alpha(\delta)=\Delta_1^{r_1}\Delta_2^{r_2}\ldots\Delta_n^{r_n},$$

where $\Delta_p = \delta_1\delta_2 \ldots \delta_p$ and $r_p = m_p - m_{p+1}$ $(m_{n+1} = 0)$; the corresponding function $\alpha(g)$ defined in §2 coincides, by (17), with the product of the principal minors

$$\alpha(g)=\Delta_1^{r_1}(g)\,\Delta_2^{r_2}(g)\ldots\Delta_n^{r_n}(g);$$

assuming that the exponents $r_1, \ldots, r_{n-1}$ are non-negative integers, we see that the functions

$$f_g(z)=\alpha(zg)$$

are *polynomials* in the elements of the matrix z whose degrees are uniformly bounded with respect to $g \in G$; therefore their linear envelope is finite-dimensional, and the character $\alpha(\delta)$ is, in fact, inductive (note that the function $\Delta_n(zg) = \det zg = \det g$ does not depend at all on z, so that the exponent r_n is not subject to any restriction).

This completes the proof of the lemma.

The representation with the *signature* $(m_1, m_2, \ldots, m_n)$ has the usual model

[1] A representation of G_0 with the dominant weight α_0 can also be obtained by the restriction to G_0 of the representation of G with the dominant weight α - for this purpose we have to consider the irreducible subspace containing the "dominant" vector of the representation space of G (the existence of such a subspace follows from the theorem on the complete reducibility *applied to the Lorentz group*).

$$T_g f(z) = \alpha(z, g) f(z \cdot g); \tag{20}$$

here $\alpha(z, g) = \Delta_1^{r_1}(zg) \ldots \Delta_n^{r_n}(zg)$, $r_1 = m_1 - m_2, \ldots, r_{n-1} = m_{n-1} - m_n$, $r_n = m_n$, and the elements of the transformed matrix $\tilde{z} = z \cdot g$ are computed by (16):

$$\tilde{z}_{pq} = \frac{\Delta_{pq}}{\Delta_p},$$

where $\Delta_p = \Delta_p(zg)$ is a diagonal minor of zg and Δ_{pq} the minor obtained from Δ_p is substituting the column with the number q in place of the column with the number p.

All the *real* irreducible representations of the group $GL(n)$ are computed similarly; in particular, every "analytic" representation is induced in a well defined class of polynomials in $\bar{z}_{pq}$ by a character of the form $\alpha(\delta)$, where $\alpha(\delta)$ is an analytic dominant weight, and an arbitrary irreducible representation of $GL(n)$ is the tensor product

$$\tau_1 \times \bar{\tau}_2$$

of an "analytic" and an "anti-analytic" (irreducible) representation.

Note that the full linear group has a series of *one-dimensional* representations – characters of the group G every one of which can be put in the form $\lambda^p \bar{\lambda}^q$, where

$$\lambda \cdot (g) = \det g;$$

the representations of this series may be infinite-valued and may also generate reducible representations that are not completely reducible, for example:

$$\lambda^m \begin{pmatrix} 1 & \log|\lambda| \\ 0 & 1 \end{pmatrix}.$$

(The radical of G is isomorphic to the multiplicative group of complex numbers and consists of the matrices λe, multiples of the unit matrix; all the one-dimensional representations of G are essentially representations of its radical.)

3. The unimodular group $SL(n)$. The *unimodular group* is distinguished in the full linear group by the normalization $\det g = 1$ (all the transformations of this subgroup preserve the volume of the parallelepiped spanned by n arbitrary vectors); it can easily be obtained from Lemma 1 (or independently by means of Schur's Lemma) that every irreducible representation of $GL(n)$, being restricted to the unimodular group, remains irreducible and that all the irreducible representations of $SL(n)$ are obtained in this way. It is clear that two representations with dominant weights of the form (18) generate one and the same representation of $SL(n)$ if and only if they differ from each other by a certain power of the determinant of δ as factors, i.e. if all their exponents differ by a common summand.

When we normalize, for example, the dominant weight (18) by the condition $m_n = 0$, we can say that every irreducible representation of $SL(n)$ is determined by a system of $n - 1$ integers m_i subject to the order relations

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{n-1} \geqslant 0;$$

consequently, *all the representations of the unimodular group are single-valued*;[1] finally, by a theorem by H. Weyl [2], *all the representations of the unimodular group are completely reducible* (these statements can be obtained from the general theory if it is shown that $SL(n)$ is *simply connected and semisimple*; hence it also follows that the full linear group is *reductive*).

EXAMPLE. Every irreducible representation of the unimodular group $SL(3)$ is determined by the signature $m_1 \geqslant m_2 \geqslant 0$ or by the system of non-negative integers $r_1 = m_1 - m_2$, $r_2 = m_2$. In §6 it will be shown that the space of the induced representation with the signature $\{r_1, r_2\}$:

$$T_g f(z) = \Delta_1^{r_1}(z, g)\, \Delta_2^{r_2}(z, g)\, f(z \cdot g)$$

consists of all polynomials of the variables $z = (z_{12}, z_{13}, z_{23})$, satisfying the system of equations

$$D_1^{r_1+1} f(z) = 0, \quad D_2^{r_2+1} f(z) = 0,$$

where D_1, D_2 are the linear differential operators:

$$D_1 = \frac{\partial}{\partial z_{12}} + z_{23} \frac{\partial}{\partial z_{13}}, \quad D_2 = \frac{\partial}{\partial z_{23}},$$

defined in the class of functions on the group Z.

The representations with the signature Δ_1 (vectors) and Δ_2 (bivectors) are in this case dual to one another; with the help of this fact we can write down very simply the explicit form of the multipliers:

$$\Delta_1(z, g) = g_{11} + z_{12} g_{21} + z_{13} g_{31},$$
$$\Delta_2(z, g) = \hat{z}_{31} \hat{g}_{13} + \hat{z}_{32} \hat{g}_{23} + \hat{g}_{33},$$

where the symbol $\hat{g}$ denotes the matrix g'^{-1}, in particular, $\hat{z}_{31} = z_{12} z_{23} - z_{13}$, $\hat{z}_{32} = -z_{23}$; the explicit form of the "fractional-linear substitutions" $z \cdot g$ can be written down similarly.[2]

4. The real forms of the full linear group. Among the real forms of $GL(n)$ (there are four such non-isomorphic forms) we mention the *unitary group* $U(n)$ and the maximal connected subgroup G_+ *in the full linear group over the field of real numbers*. (G_+ is distinguished by the condition $\det g > 0$.)

The unitary matrices in an n-dimensional space preserve the hermitian bilinear form

$$(x, y) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \ldots + x_n \bar{y}_n$$

and are, therefore, distinguished by the condition $u^{*-1} = u$, where $*$ denotes the hermitian conjugate. Let us show that the full linear group is the "complexification" of the group $U(n)$: indeed, every unitary matrix

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[1] In fact, the function $\alpha(z, g)$ is for $m_n = 0$ a polynomial on G and therefore single-valued.

[2] The representation with the signature Δ_1^m (the symmetrized power of a vector) acts on the space of polynomials that depend only on the variable z_{12}, z_{13}; correspondingly, the representation space of Δ_2^m consists of the polynomials in $\hat{z}_{31}$, $\hat{z}_{32}$; note that the operator D_1 becomes the operator of differentiation with respect to z_{12} if the independent variables are taken to be z_{12}, $\hat{z}_{31}$ and $\hat{z}_{32}$.

can be represented in the form e^{ia}, where the matrix a is hermitian ($a^* = a$); therefore the Lie algebra of the group $U(n)$ consists of all "skew-hermitian" matrices whose complex envelope coincides with the set of *all* matrices of order n, i.e. with the Lie algebra of the full linear group.

A remarkable topological property of the group $U(n)$ is its *compactness*.

The elements of the full linear group over the field of real numbers admit a decomposition into triangular factors; however, the diagonal subgroup in this case is *disconnected*, and the conditions of Definition 1 are not satisfied; nevertheless it can be shown that *Theorem* 1 *and all the constructions of the irreducible representations remain valid*. We can also make use of the principle of the "analytic continuation" and conclude (as in the case $U(n)$) that every irreducible representation of the group G_+ is enumerated by a system of numbers[1]

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n$$

and is determined by a formula of the form (20); here (in contrast to the case $U(n)$) the representations can be constructed in terms of polynomials on the "proper" (real) subgroup Z.

5. Conventions concerning notation. For the enumeration of the complex-analytic irreducible representation τ of the full linear group $GL(n)$ we shall write its "signature" in the form

$$\tau = (m_1, m_2, \ldots, m_n)$$

(bearing in mind the definite order of Lemma 1) or in the form

$$\{r_1, r_2, \ldots, r_n\}, \tag{21}$$

where the numbers m_i, r_i are connected by the system of relations

$$\left.\begin{array}{l} r_1 = m_1 - m_2;\ r_2 = m_2 - m_3,\ \ldots,\ r_{n-1} = m_{n-1} - m_n,\ r_n = m_n; \\ m_1 = r_1 + r_2 + r_3 + \ldots + r_n, \\ m_2 = \qquad r_2 + r_3 + \ldots + r_n, \\ \ldots\ldots\ldots\ldots\ldots \\ m_n = \qquad\qquad\qquad r_n; \end{array}\right\} \tag{22}$$

sometimes it is convenient to use a more expressive notation:

$$\tau = \Delta_1^{r_1} \Delta_2^{r_2} \ldots \Delta_n^{r_n};$$

here the number r_n is arbitrary, and the remaining parameters r_i are *non-negative integers*.

A special role is played by the Cartan "dominants" $\Delta_1, \Delta_2, \ldots, \Delta_n$: the representation Δ_p is the p-th *alternating power* of the "primary" representation Δ_1 and corresponds to the transformation law of *multi-vectors* of rank p; in particular, the representation Δ_1 can be identified with the original realization of the (linear) group $GL(n)$; obviously the representation Δ_p^m is the m-th *symmetrized power* of the representation Δ_p.

We preserve the same notation for the group $SL(n)$ and for the various "real" subgroups of the full linear group; note that the representations

[1] All the differences $m_i - m_j$ are integers; when we talk of *single-valued* representations, m_n must be an integer in the case $U(n)$; the remaining parameters m_i are then also integers.

Δ_p and Δ_{n-p} are dual to one another with the respect to the unimodular group:

$$\hat{\Delta}_p = \Delta_{n-p}; \tag{23}$$

in the general case, as is easy to verify, the dual signature has the form

$$(-m_n, \ -m_{n-1}, \dots, \ -m_1).$$

In the set of all signatures (21) we can introduce a linear order in the following way: τ'' is regarded as *higher than* τ' if either $m'_1 < m''_1$, or when these parameters are equal, $m'_2 < m''_2$ and so on; this order is usually called *lexicographic*.

We mention, in passing, one well-known fact (which can also easily be obtained by the method of Z-invariants): in the tensor product

$$W = \Delta_1 \times \Delta_1 \times \dots \times \Delta_2 \times \Delta_2 \times \dots \times \Delta_n,$$

where the factor Δ_p occurs r_p times, the irreducible representation $\tau = \Delta_1^{r_1} \Delta_2^{r_2}, \dots, \Delta_n^{r_n}$ is contained precisely *once*, and all the remaining irreducible representations in W are *lexicographically lower than* τ.

The subgroups Z, D, Z in the full linear group are occasionally denoted by $Z(n)$, $D(n)$, $Z(n)$.

§5. Differential operators on the group Z

The space of every irreducible representation of the group $GL(n)$ is defined as the linear envelope of the corresponding functions $\alpha(z, g)$ – polynomials on Z; however, the problem remains open of an "intrinsic" (functional) definition of this space of polynomials; such a definition turns out to be especially useful for the solution of certain problems of spectral analysis. With this aim we shall construct an uncomplicated formal apparatus that is based on the investigation of certain remarkable differential operators on Z.

For the sake of simplicity we consider first the *real* group $G_+ = G_+L(n)$ (see §4, 4).

Let us consider in Z the one-parameter subgroup

$$z_i(t) = 1 + te_{i,\,i+1} \qquad (i = 1, 2, \dots, n-1),$$

where e_{ij} denotes the matrix whose elements are all zero except the elements in the intersection of the i-th row and the j-th column which is equal to 1. Let $\mathscr{D}_i$ denote the infinitesimal operator of the *left* translation on Z that is generated by the one-parameter subgroup $z_i(t)$:

$$\mathscr{D}_i f(z) = \lim_{t \to 0} \frac{1}{t} \left[f(z_i(t)\, z) - f(z) \right];$$

then in the parameters (15) the differential operators $\mathscr{D}_i$ have the following form:

$$\left.\begin{aligned}\mathscr{D}_1 &= \frac{\partial}{\partial z_{12}} + z_{23}\frac{\partial}{\partial z_{13}} + \dots + z_{2n}\frac{\partial}{\partial z_{1n}},\\ \mathscr{D}_2 &= \qquad\qquad \frac{\partial}{\partial z_{23}} + \dots + z_{3n}\frac{\partial}{\partial z_{3n}},\\ &\dots\dots\dots\dots\dots\dots\dots\\ \mathscr{D}_{n-1} &= \qquad\qquad\qquad\qquad \frac{\partial}{\partial z_{n-1,n}};\end{aligned}\right\} \tag{24}$$

clearly $\mathscr{D}_p$ is the "substitution operator" of the elements of the $(p+1)$-th row in place of the elements of the p-th row of the matrix z, and this leads to the next lemma.

LEMMA 2. *The following identities hold:*

$$\mathscr{D}_p(\Delta_q(z, g)) = 0 \quad \text{for } q \neq p,$$
$$\mathscr{D}_p^2(\Delta_p(z, g)) = 0,$$

where $\Delta_p(z, g)$ denotes the principal diagonal minor formed from the p first rows and the p first columns of the matrix zg and the differentiation is carried out with respect to the parameter z for fixed $g \in G$.

PROOF. The minor

$$\Delta_q(z, g) = \begin{vmatrix} (zg)_{11} & \dots & (zg)_{1q} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ (zg)_{q1} & \dots & (zg)_{qq} \end{vmatrix} = \sum_{i_1 < i_2 < \dots < i_q} z_{i_1 i_2 \dots i_q} g^{i_1 i_2 \dots i_q}$$

is a linear combination of the minors $z_{i_1 i_2 \dots i_q}$ formed from the q first rows of z. If $q < p$, then the function $\Delta_q(z, g)$ does not depend at all on the parameters of the p-th row with respect to which the differentiation $\mathscr{D}_p$ is carried out; if $q > p$, then the action of $\mathscr{D}_p$ leads to the appearance of two identical rows in the minor Δ_q and the result is again equal to zero; finally, if $q = p$, then obviously the function $\Delta_p(z, g)$ is linearly dependent on the elements of the p-th row of z. This completes the proof of the lemma.

Clearly, there is a more general identity

$$\mathscr{D}_p^{N+1}(\Delta_p^N(z, g)) = 0 \qquad (N = 0, 1, 2, \dots);$$

hence it follows that $r_p + 1$ is the least power of $\mathscr{D}_p$ that annihilates the multiplier

$$\alpha(z, g) = \Delta_1^{r_1}(zg) \dots \Delta_n^{r_n}(zg),$$

and we conclude that all the polynomials in the space of the irreducible representation with the signature $\{r_1, \dots, r_n\}$ satisfy the system of differential equations

$$\mathscr{D}_1^{r_1+1} f(z) = 0, \dots, \mathscr{D}_{n-1}^{r_{n-1}+1} f(z) = 0; \tag{25}$$

let us make it our aim to find out, whether the conditions (25) *are also sufficient for the function* $f(z)$ *to be a polynomial and to belong to the space of our irreducible representation.*

Being an operator of left translation, $\mathscr{D}_p$ is permutable with all the operators of *right* translations on Z; this fact can be expressed in the equation

$$\mathscr{D}_p = \widetilde{\mathscr{D}}_p,$$

where $\widetilde{\mathscr{D}}_p$ denotes the operator $\mathscr{D}_p$ rewritten for the transformed variables $\widetilde{z} = zz_0$ with an arbitrary fixed $z_0 \in Z$; it is important for us to clarify the more general problem: how are the $\mathscr{D}_p$ transformed in the change of variables $z \to \widetilde{z}$, where $\widetilde{z} = z \cdot g$, $g \in G$. More precisely, we want to find a correspondence between $\mathscr{D}_p$ and the operator

$$\widetilde{\mathscr{D}}_p = U\mathscr{D}_p U^{-1},$$

where U is the operator of the change of variables: $Uf(z) = f(\widetilde{z})$;[1] in particular, it is easy to show that for $g = \delta$

$$\mathscr{D}_p = \frac{\delta_{p+1}}{\delta_p}\widetilde{\mathscr{D}}_p;$$

in the transition to the general case it is convenient to use a certain determinant identity which can easily be proved by induction: let $[\overset{1}{u}\overset{2}{u}\ldots\overset{p}{u}]$ denote the determinant formed from the coordinates of the vectors $\overset{1}{u}, \overset{2}{u}, \ldots, \overset{p}{u}$ with the numbers $1, 2, \ldots, p$; then

$$[\overset{1}{x}\overset{2}{x}\ldots\overset{p}{x}y][\overset{1}{x}\overset{2}{x}\ldots\overset{p-1}{x}z]-[\overset{1}{x}\overset{2}{x}\ldots\overset{p}{x}z][\overset{1}{x}\overset{2}{x}\ldots\overset{p-1}{x}y]= \\ =[\overset{1}{x}\overset{2}{x}\ldots\overset{p-1}{x}yz][\overset{1}{x}\overset{2}{x}\ldots\overset{p}{x}]; \quad (26)$$

hence we obtain for the minors of an arbitrary matrix g the following relations

$$\begin{pmatrix}1 & 2\ldots p\\ 1 & 2\ldots p\end{pmatrix}\begin{pmatrix}1 & 2\ldots p-1 & \mu\\ 1 & 2\ldots p-1 & \nu\end{pmatrix}- \\ -\begin{pmatrix}1 & 2\ldots p-1 & \mu\\ 1 & 2\ldots p-1 & p\end{pmatrix}\begin{pmatrix}1 & 2\ldots p-1 & p\\ 1 & 2\ldots p-1 & \nu\end{pmatrix}= \\ =\begin{pmatrix}1 & 2\ldots p & \mu\\ 1 & 2\ldots p & \nu\end{pmatrix}\begin{pmatrix}1 & 2\ldots p-1\\ 1 & 2\ldots p-1\end{pmatrix}. \quad (27)$$

For the proof of (26) we carry the right hand side to the left and denote the expression so obtained by A_p. In every summand the first factor is of order $p+1$ and the second of order p, therefore the coordinates with the number $p+1$ are only contained in the first factors, and A_p is linear with respect to these coordinates:

$$A_p = \sum_{i=1}^{p}\overset{i}{x}_{p+1}X_i + y_{p+1}Y + z_{p+1}Z.$$

By writing out the coefficients X_i, Y, Z we find that $Y = Z = 0$; to prove that $X_i = 0$ we make an inductive assumption of the type

$$A_{p-1} = 0$$

[1] In other words, $\widetilde{\mathscr{D}}_p = \sum_{s=p+1}^{n}\widetilde{z}_{p+1,s}\frac{\partial}{\partial\widetilde{z}_{ps}}$, where $\widetilde{z}_{ij}$ are the elements of the matrix $\widetilde{z} = z\cdot g$.

(i.e. we assume the identity (26) to be true with p replaced by $p - 1$); in particular,

$$X_1 = [\overset{2}{x}\overset{3}{x} \dots \overset{p}{x}y]\,[\overset{1}{x}\overset{2}{x} \dots \overset{p-1}{x}\;z] - [\overset{2}{x}\overset{3}{x} \dots \overset{p}{x}z]\,[\overset{1}{x}\overset{2}{x} \dots \overset{p-1}{x}\;y] - [\overset{2}{x}\overset{3}{x} \dots \overset{p-1}{x}\;yz]\,[\overset{1}{x}\overset{2}{x} \dots \overset{p}{x}]$$

depends linearly on the coordinates of the vector $\overset{1}{x}$ and all the coefficients have the type A_{p-1} so that $X_1 = 0$; similarly $X_2 = X_3 = \dots = X_p = 0$. Finally, the induction is justified, because $A_1 = 0$.

The identity (27) is necessarily true when $\mu \leqslant p$ or $\nu \leqslant p$, therefore we shall take it that $p < \mu \leqslant \nu$. We denote by $\overset{1}{x}, \dots, \overset{p-1}{x}$ the first $p-1$ rows of g; for the row with the number p we introduce the notation y and for the column with the number μ the notation z; then when we expand the identity (27) with respect to the elements of the ν-th column of g, we obtain an identity of the type A_{p-1}.

LEMMA 3. *In the change of variables $z \to \tilde{z}$, where $\tilde{z} = z \cdot g$ with an arbitrary matrix $g \in G$, the operators $\mathscr{D}_i$ are transformed by the rule:*

$$\mathscr{D}_p = \frac{\Delta_{p-1}\Delta_{p+1}}{\Delta_p^2}\,\tilde{\mathscr{D}}_p \qquad (p = 1, 2, \dots, n-1); \tag{28}$$

Δ_q in this case denotes the operator of multiplication by $\Delta_q(z, g)$ as a function of z; a more general identity for the powers of the operator $\mathscr{D}_p$ has the form

$$\mathscr{D}_p^{r+1}\Delta_p^r = \frac{(\Delta_{p-1}\Delta_{p+1})^{r+1}}{\Delta_p^{r+2}}\,\tilde{\mathscr{D}}_p^{r+1}. \tag{29}$$

PROOF. The explicit formula for the transformed variables $\tilde{z}$ has already been mentioned above:

$$\tilde{z}_{\mu\nu} = \frac{\Delta_{\mu\nu}(z, g)}{\Delta_\mu(z, g)},$$

where the minor $\Delta_{\mu\nu}(a)$ is obtained by substituting the ν-th column of the diagonal minor $\Delta_\mu(a)$ in place of the μ-th; we have to find the result of applying the operator $\mathscr{D}_p$ to the functions $\tilde{z}_{\mu\nu}$, and afterwards make use of the equation

$$\mathscr{D}_p = \sum_{\mu<\nu} \mathscr{D}_p(\tilde{z}_{\mu\nu})\frac{\partial}{\partial \tilde{z}_{\mu\nu}};$$

here it is only necessary to consider the case $\mu = p$, since in all the remaining cases the result of applying $\mathscr{D}_p$ is equal to zero (see the proof of Lemma 2). Using the identity (27) we obtain:

$$\begin{aligned}
\mathscr{D}_p(\tilde{z}_{p\nu}) &= \frac{1}{\Delta_p^2}\,[\Delta_p\mathscr{D}_p(\Delta_{p\nu}) - \mathscr{D}_p(\Delta_p)\,\Delta_{p\nu}] = \\
&= \frac{1}{\Delta_p^2}\left[\begin{pmatrix} 1 & 2 \dots p \\ 1 & 2 \dots p \end{pmatrix}\begin{pmatrix} 1 & 2 \dots p-1 & p+1 \\ 1 & 2 \dots p-1 & \nu \end{pmatrix} - \right. \\
&\quad \left. - \begin{pmatrix} 1 & 2 \dots p-1 & p+1 \\ 1 & 2 \dots p-1 & p \end{pmatrix}\begin{pmatrix} 1 & 2 \dots p-1 & p \\ 1 & 2 \dots p-1 & \nu \end{pmatrix}\right] = \\
&= \frac{1}{\Delta_p^2}\begin{pmatrix} 1 & 2 \dots p & p+1 \\ 1 & 2 \dots p & \nu \end{pmatrix}\begin{pmatrix} 1 & 2 \dots p-1 \\ 1 & 2 \dots p-1 \end{pmatrix} = \frac{\Delta_{p-1}\Delta_{p+1}}{\Delta_p^2}\,\tilde{z}_{p+1,\,\nu},
\end{aligned}$$

and the relation (28) is proved.

To verify (29) we write this equation in the form

$$\mathscr{D}^{r+1}\mu^{r}=\mu^{-r-2}\widetilde{\mathscr{D}}^{r+1}, \tag{30}$$

where $\mathscr{D}=\mathscr{D}_p$ and μ is the operator of multiplication by $\dfrac{\Delta_p}{\sqrt{\Delta_{p-1}\Delta_{p+1}}}$ (both these expressions are equivalent owing to the fact that Δ_{p-1} and Δ_{p+1} behave like constants with respect to the differentiation $\mathscr{D}_p$). By Leibniz' formula

$$\mathscr{D}^{r+1}\mu=\mu\mathscr{D}^{r+1}+(r+1)\,\mu'\mathscr{D}^{r}=[\mu\mathscr{D}+(r+1)\,\mu']\,\mathscr{D}^{r},$$

where $\mu'=\mathscr{D}(\mu)$ and where we have used the fact that $\mathscr{D}^2(\mu)=0$ (see Lemma 2). When we multiply this equation on the right by μ^{r-1} and apply as induction hypothesis the identity (30) in which r is replaced by $r-1$, we find

$$\mathscr{D}^{r+1}\mu^{r}=[\mu\mathscr{D}+(r+1)\,\mu']\,\mathscr{D}^{r}\mu^{r-1}=[\mu\mathscr{D}+(r+1)\,\mu']\,\mu^{-r-1}\widetilde{\mathscr{D}}^{r};$$

finally, by carrying out the differentiation $\mathscr{D}$ and applying (28) we complete the calculation:

$$\mathscr{D}^{r+1}\mu^{r}=\mu^{-r}\mathscr{D}\widetilde{\mathscr{D}}^{r}+\{(r+1)\,\mu'-(r+1)\,\mu'\}\,\mu^{-r-1}\widetilde{\mathscr{D}}^{r}=\mu^{-r-2}\widetilde{\mathscr{D}}^{r+1}.$$

The lemma is now proved.

This completes the formal part. Note that in all our arguments we have not used the explicit form of the operators $\mathscr{D}_p$ but only their "algebraic" properties of the type of the properties mentioned in Lemma 2.

§6. The space of irreducible representations

The space of the induced representation τ will be denoted by, $\mathfrak{R}_\tau(Z)$; since in the discussion of spaces one-dimensional factors do not play a role, we find it convenient to speak of the *unimodular* group $SL(n)$; we recall that, to begin with, we are studying the group of matrices over the field of *real numbers*. We are now in a position to give a precise characterization of the space $\mathfrak{R}_\tau(Z)$ in terms of the differential operators $\mathscr{D}_p$.

THEOREM 2. *Every irreducible representation of the group $SL(n)$ is determined by a system of non-negative integers $r_1, \ldots, r_{n-1}$ and can be realized by the formula* (20) *in the space $\mathfrak{R}_\tau(Z)$ consisting of the polynomials on Z.*

The space $\mathfrak{R}_\tau(Z)$ is distinguished in the class of all functions on Z as the collection of solutions of the system of differential equations

$$\mathscr{D}_1^{r_1+1}f(z)=0,\ \ldots,\ \mathscr{D}_{n-1}^{r_{n-1}+1}f(z)=0, \tag{24'}$$

where the operators $\mathscr{D}_i$ are defined in accordance with (24).

PROOF. Let us begin by showing the invariance of the space of all solutions under the operation

$$\tau_g f(z) = \alpha(z, g) f(z \cdot g),$$

where $\alpha(z, g) = \Delta_1^{r_1}(zg) \ldots \Delta_n^{r_n}(zg)$. The invariance is dictated by the formula (29); in fact,

$$\mathscr{D}_p^{r_p+1}(\tau_g f(z)) = \mathscr{D}_p^{r_p+1}[\alpha(z, g) f(z \cdot g)] = \beta(z, g)[\mathscr{D}_p^{r_p+1} f(z)]_{z \to z \cdot g},$$

where $\beta(z, g)$ is a certain function whose explicit form is not required; hence it is clear that $\tau_g(f)$ is a solution, no matter what such an f is.

Let us now assume that the *finite dimensionality* of the space of all solutions is already proved, furthermore, that all solutions are *polynomials* (in the variables z_{pq}) on Z; then the operation τ_g is a *rational*[1] representation of G in the space of all solutions; by the theorem on the complete reducibility [2], [24] this representation is completely reducible; moreover, the operations τ_z are continuous, therefore in every irreducible subspace there is a non-zero invariant of Z. But the operations τ_z reduce to a right translation on Z; this implies the *uniqueness* of the dominant vector ($f^0(z) \equiv 1$), and in this case the theorem is proved.[2]

At first sight the finite dimensionality is not obvious, because the number of operators $(n - 1)$ is considerably less than the number of independent variables $(\frac{n(n-1)}{2})$ in Z; an exception is the case $n = 2$, when the system reduces to the single equation

$$\frac{d^{r+1} f(z)}{dz^{r+1}} = 0$$

for the function $f(z)$ of *one* real variable z; we want to apply induction and therefore prove a simple combinatorial lemma.

LEMMA 4. *Let $Y_1, Y_2, \ldots, Y_m$ be a system of permutable linear operators in some vector space F and $p_1(x), \ldots, p_m(x)$ be a system of m linearly independent polynomials in the variable $x = (x_1, \ldots, x_n)$. We denote by $F[X]$ the space of polynomials in x of a certain fixed degree with values in F; let $F_1[X]$ be the subspace of $F[X]$ on which*

$$(p_1 Y_1 + p_2 Y_2 + \ldots + p_m Y_m)^l = 0,$$

where l is an integer $\geqslant 1$; then there exist exponents $N_1, \ldots, N_m$ such that everywhere on $F_l[X]$

$$Y_1^{N_1} = 0, \quad Y_2^{N_2} = 0, \ \ldots, \ Y_m^{N_m} = 0.$$

---

[1] This means that the matrix elements of the representation T_g are rational functions of the elements of the matrix g.

[2] Note that incidentally we have proved the following statement: *in the class of polynomials on Z there exists no finite-dimensional space other than $\mathfrak{R}_T(Z)$ that is invariant under all the transformations τ_g of G.* Hence there follows, in particular, a convenient criterion: *$\mathfrak{R}_T(Z)$ contains all the monomials that occur in the expansion of $\alpha(z, g)$ with respect to minors of z* (in fact, the linear envelope of these monomials is invariant under the transformations τ_g).

PROOF. We set $D = p_1Y_1 + \ldots + p_mY_m$ and for the sake of simplicity consider the case when there is only *one* variable x; then, by expanding it in powers, we can represent the operator D in the form

$$D = D_0 + xD_1 + \ldots + x^pD_p,$$

where the $D_i = \Sigma d_{ij}Y_j$ are operators with "constant" coefficients (independent of x).[1] From the linear independence of the polynomials $p_i(x)$ it follows that the rank of the matrix (d_{ij}) is equal to m, i.e. that the operators Y_j can be expressed as linear combinations (with constant coefficients) of the operators D_i; therefore it is sufficient for the proof of the lemma to verify the existence of a number $N \geqslant 1$ such that

$$D_i^N = 0$$

everywhere on the space $F_l[X]$.

Suppose that the lemma is proved for a certain exponent l, we shall verify that it is true then for $l + 1$; for this purpose we write the vector $f \in F[X]$ in the form

$$f(x) = f_0 + xf_1 + x^2f_2 + \ldots + x^qf_q, \quad f_j \in F,$$

and note that if $f \in F_{l+1}[X]$, then $Df \in F_l[X]$; therefore in the table

$$\begin{matrix} D_0f_0 & D_0f_1 & D_0f_2 \ldots D_0f_q \\ D_1f_0 & D_1f_1 & D_1f_2 \ldots D_1f_q \\ \cdots & \cdots & \cdots \\ D_pf_0 & D_pf_1 & D_pf_2 \ldots D_pf_q \end{matrix} \qquad (T_{pq}),$$

labelled by the numbers $i = 0, 1, \ldots, p$, $j = 0, 1, \ldots, q$, every sum of terms that lie along the line $i + j = \text{const}$ is congruent to zero (mod F_n), where F_N denotes the subspace of F on which

$$D_i^Nf = 0 \qquad (i = 0, 1, 2, \ldots, p);$$

in particular,

$$D_0f_0 \equiv 0 \pmod{F_n}. \tag{31}$$

Let us show that this implies the system of relations

$$f_j \equiv 0 \,(\mathrm{mod}\, F_{n'}) \qquad (j = 0, 1, 2, \ldots, q)$$

for some n' that does not depend on the particular choice of the function $f(x) \in F_{l+1}[X]$ – we shall say briefly that the table T_{pq} is *soluble*.

We apply a double induction with respect to p and q (the number of operators D_i and the number of coefficients f_j of the function $f(x)$); let us assume that the tables $T_{p-1,q}$ and $T_{p,q-1}$ are soluble. Taking the commutativity of the operators D_i into account, we multiply the table T_{pq} by D_0; then it follows from the formulae (2) that in the tables so obtained the whole first column is $\equiv 0$ (mod F_n) and can be discarded; there remains then the table $T_{p,q-1}$, written down for the vectors $f_i' = D_0f_i$ ($i = 1, 2, \ldots, q$); hence, by applying the induction hypothesis we conclude that

[1] The proof in the general case is entirely similar. It is sufficient to remark that the symbol x^mD_m can in the general case be regarded as a *convolution* of two symmetric tensors of rank m of which the first is the m-th symmetrized power of the vector $x = (x_1, \ldots, x_n)$.

$$D_0 f_j \equiv 0 \ (\mathrm{mod}\, F_{n''}) \tag{32}$$

with a certain number n'' $(n'' \geqslant n)$. Combining this statement with (31) we see that the whole first row of the table T_{pq} is congruent to zero (mod $F_{n''}$); therefore this row may be discarded and we obtain for the vector f_j the table $T_{p-1,q}$ which by the induction hypothesis is soluble:

$$D_i^{n'} f_j = 0, \quad i \neq 0;$$

moreover, from (32) it follows that similar conditions are true also in the case $i = 0$. Finally, the basis of the induction follows from the obvious solubility of the tables T_{p0} and T_{0q}.

The lemma is now proved.

In particular, the conditions of the lemma are satisfied for every differential operator of the form

$$D = \frac{\partial}{\partial y_0} + x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + \ldots + x_m \frac{\partial}{\partial y_m},$$

i.e. the fact that the solutions of the equation $D^l f(x, y) = 0$ are polynomials in the variables $x_1, \ldots, x_m$ with a uniformly bounded highest degree implies that these solutions are also polynomials in the variables $y_0, y_1, \ldots, y_m$ (with uniformly bounded highest degree!). When we apply this statement in succession to the operators $\mathscr{D}_{n-1}, \ldots, \mathscr{D}_1$, we find that all the solutions of the system of equations

$$\mathscr{D}_i^{r_i+1} f(z) = 0$$

are polynomials in the elements of the matrix z whose degrees are uniformly bounded.[1] This completes the proof of the theorem.

For $n = 2$ we obtain the familiar case of the Lorentz group; for $n = 3$ the result was presented in the form of an example on p. 24.

Turning now to the case of the full linear group over the field of *complex* numbers it is easy to conclude that Theorem 2 remains valid if the system of operators

$$\mathscr{D}_p = \sum_{s=p+1}^{n} z_{p+1,\,s} \frac{\partial}{\partial z_{ps}}$$

is complemented by the system of complex-conjugate operators

$$\overline{\mathscr{D}}_p = \sum_{s=p+1}^{n} \bar{z}_{p+1,\,s} \frac{\partial}{\partial \bar{z}_{ps}},$$

where the symbols $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial \bar{z}}$ denote, respectively, the expressions

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{for } z = x + iy$$

(it is clear that in the problems to be studied the variables z and $\bar{z}$ can

[1] The remarkable property of the differential operators $\mathscr{D}_i$ that we have established in the process of our proof is a consequence of the simple fact that the one-parameter groups $z_i(t)$ are *a system of generators* of the group $Z = Z(n)$.

be regarded as formally independent); in particular, the condition that the functions $f(z)$ on Z are *complex analytic* comes to this: that the system of equations

$$\overline{\mathcal{D}}_1 f(z) = \overline{\mathcal{D}}_2 f(z) = \ldots = \overline{\mathcal{D}}_{n-1} f(z) = 0$$

– "the Cauchy-Riemann system" – is satisfied for Z.

§7. The Symplectic group

The *symplectic group* $Sp(n)$ consists of all linear transformations of a complex n-dimensional vector space that preserve a non-degenerate skew-symmetric bilinear form; the dimension n must be taken to be even:

$$n = 2\nu,$$

since only in this case a skew-symmetric form can be non-degenerate.

By a special choice of the basis the form reduces to

$$[x, y] = x_1 y_n + x_2 y_{n-1} + \ldots + x_\nu y_{\nu+1} - x_{\nu+1} y_\nu - \ldots - x_n y_1,$$

and the condition that a matrix g belongs to the symplectic group reduces to the relations

$$\sigma^{-1} g \sigma = g'^{-1}, \text{ where } \sigma = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}; \tag{33}$$

for this choice of the basis *the Gauss decomposition of the full linear group $GL(n)$ immediately induces a Gauss decomposition in the symplectic subgroup*

$$Sp(n) = \overline{ZDZ},$$

where Z, D and Z are defined as the intersections of the symplectic group with the groups Z(n), $D(n)$ and $Z(n)$, respectively.[1] Note that the group $Sp(2)$ is isomorphic to $SL(2)$.

1. Classification of the irreducible representations. A diagonal matrix $\delta \in D(n)$ is symplectic if and only if in the sequence of its eigenvalues

$$\delta_1, \delta_2, \ldots, \delta_\nu; \quad \delta_{\nu+1}, \ldots, \delta_{n-1}, \delta_n$$

any two numbers that are symmetrically situated with respect to the centre are opposite in value; by taking as independent parameters the numbers $\delta_1, \ldots, \delta_\nu$, we can write an arbitrary complex analytic character of the group D in the form

$$\alpha(\delta) = \delta_1^{m_1} \delta_2^{m_2} \ldots \delta_\nu^{m_\nu}. \tag{34}$$

---

[1] The "regular" matrices $g \in Sp(n)$ have the decomposition

$$g = \zeta \delta z, \quad \zeta \in Z(n), \quad \delta \in D(n), \quad z \in Z(n),$$

and from the uniqueness of this it is easy to obtain that all these matrices are symplectic; however, it is still necessary to verify (see [12]) that the set ZDZ is everywhere dense in $Sp(n)$.

LEMMA 5. *The character (34) is inductive with respect to $Sp(n)$ if and only if all the parameters $m_1, \ldots, m_\nu$ are integers subject to the order relation*:

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_\nu \geqslant 0.$$

PROOF. The necessity of these conditions follows from the existence in G of a system of subgroups $G_1, G_2, \ldots, G_\nu$ each of which is isomorphic to the Lorentz group: in particular, the subgroup G_1 consists of all linear transformations of the form

$$\begin{pmatrix} \alpha & \beta & & & & & 0 \\ \gamma & \delta & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 0 & & & & & \alpha & -\beta \\ & & & & & -\gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1;$$

the subgroups $G_2, \ldots, G_{\nu-1}$ are similarly defined; G_ν consists of all unimodular transformations that change only the coordinates $x_\nu, x_{\nu+1}$ (arguing as in the proof of Lemma 1 we find that all the values $m_1 - m_2, \ldots, m_{\nu-1} - m_\nu, m_\nu$ are non-negative integers).

The sufficiency becomes obvious when we observe that under the conditions of the lemma the character of the group $D(\hat{n})$ with the signature $(m_1, \ldots, m_\nu, 0, \ldots, 0)$ is inductive with respect to the full linear group, consequently, the character (34) is inductive with respect to the subgroup $Sp(n)$ (Corollary 3 to Theorem 1). The lemma is now proved.

When we define the function $\alpha(z, g)$ by the equation

$$\alpha(z, g) = \Delta_1^{r_1}(zg) \ \ldots \ \Delta_\nu^{r_\nu}(zg),$$

where $r_p = m_p - m_{p+1}$ and $\Delta_i(g)$ is a principal diagonal minor of the matrix $g \in Sp(n)$, we see that all the parameters r_p are non-negative integers and the representation of the symplectic group with the signature $\{r_1, \ldots, r\}$ is defined by the usual formula

$$\tau_g f(z) = \alpha(z, g) f(z \cdot g)$$

in a certain space $\Re_\tau(Z)$ of polynomials on Z for which we shall immediately obtain a precise definition.

2. Parametrization of the group Z. For the choice of independent parameters in the group $Z = ZSp(n)$ we can make use of the fact that Z contains a subgroup Z_0 consisting of the matrices of the form

$$z_0 = \begin{pmatrix} 1 & 0 & 0 \\ & x & 0 \\ & & 1 \end{pmatrix}, \quad x \in ZSp(n-2) \tag{35}$$

(here all the places below the principal diagonal are filled with zeros); and a subgroup Z_1 formed by the matrices

$$z_1 = \begin{pmatrix} 1 & t_2 & t_3 & t_4 \dots t_{n-1} & t_n \\ & 1 & 0 & 0 \dots 0 & \tilde{t}_{n-1} \\ & & 1 & 0 \dots 0 & \tilde{t}_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ & & & 1 & \tilde{t}_2 \\ & & & & 1 \end{pmatrix} \tag{36}$$

(in which the elements above the main diagonal are all zero except in the first row and the last column); the condition that z_1 belongs to $Sp(n)$ reduces to the fact that the parameters $t_2, \dots, t_n$ can be taken as independent and that the column of the elements $\tilde{t}_i$ is computed by the formula

$$\tilde{t}_2 = -t_2, \dots, \tilde{t}_\nu = -t_\nu; \quad \tilde{t}_{\nu+1} = t_{\nu+1}, \dots, \tilde{t}_{n-1} = t_{n-1}.$$

LEMMA 6. *Every matrix $z \in Z$ can be uniquely represented in the form*

$$z = z_0 z_1, \quad z_0 \in Z_0, \quad z_1 \in Z_1.$$

COROLLARY. *As independent parameters in Z we can take the elements of z above and on the second diagonal.*

PROOF. We denote by x the truncation of z that is formed by the elements z_{ij} whose indices take only the values 2, 3, ..., $n - 1$; it is easy to verify (by using block multiplication) that the matrix x is *symplectic*, i.e. belongs to $ZSp(n-2)$; with the help of x we construct by formula (35) the element $z_0 \in Z_0$; computing the element $z_0^{-1}z$ we find that it belongs to the subgroup Z_1. The uniqueness of the decomposition obtained follows from the obvious fact that $Z_0 \cap Z_1 = \{e\}$.

When the choice of the parameters in the group Z_0 isomorphic to $ZSp(n-2)$ has already been made (by the method indicated in the corollary), then we only have to add to these parameters the elements of the first row of Z_1 that coincide with the elements of the first row of $z = z_0 z_1$.

This completes the proof of the theorem.[1]

Let us write down a table containing the independent parameters:

$$\left.\begin{matrix} 1 & z_{12} & z_{13} & z_{14} \dots z_{1,n-2} & z_{1,n-1} & z_{1n} \\ & 1 & z_{23} & z_{24} \dots z_{2,n-2} & z_{2,n-1} & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & 1 & z_{\nu,\nu+1}; & \end{matrix}\right\} \tag{37}$$

from the proof of Lemma 6 it is clear that all the remaining elements of Z can easily be determined and are *polynomials* in these parameters; we only require the elements that are immediately adjacent to and below the second diagonal:

$$w_p = [Z_p, Z_{p-1}] \qquad (p = 2, 3, \dots, \nu), \tag{38}$$

[1] Let us also mention that $ZSp(n)$ has a unique decomposition of the form $Z = Z_1 Z_2 \dots Z_\nu$, where the matrices of Z_ν are determined by the formulae (36) and the product $Z_1 Z_2 \dots Z_{\nu-1} = Z_0$ determines a similar decomposition of the group Z_0 isomorphic to $ZSp(n-2)$, and so on. Here all the groups Z_i are commutative and every set $Z_i Z_{i+1} \dots Z_\nu$ is a normal subgroup of Z.

where Z_p denotes the p-th row of the table (37) complemented at both ends by zeros to a vector of length n and where the square brackets represent the fundamental form $[x, y]$.

When we define the one-parameter subgroup $z_p(t)$ by the formula

$$z_p(t) = 1 + \frac{t}{2}(e_{p,\,p+1} - e_{q-1,\,q}) \qquad (p = 1, 2, \ldots, \nu),$$

where e_{ij} is the usual "matrix unit" and q the complementary index:

$$q = n - p + 1,$$

we can introduce the differential operator $\mathscr{D}_i$ as the infinitessimal operator of left translation on Z corresponding to the one-parameter subgroup $z_i(t)$; obviously, in the parameters (37) these operators have the form

$$\mathscr{D}_1 = \frac{\partial}{\partial z_{12}} + z_{23}\frac{\partial}{\partial z_{13}} + \ldots + z_{2,\,n-1}\frac{\partial}{\partial z_{1,\,n-1}} + z_{2n}\frac{\partial}{\partial z_{1n}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathscr{D}_\nu = \frac{\partial}{\partial z_{\nu,\,\nu+1}};$$

the differential operators $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_\nu$ can be called the *principal translations on* Z.

3. **Description of the space $\mathfrak{R}_\tau(Z)$.** Following the scheme of the preceding section we can easily establish an analogue to Theorem 2 (the symplectic group has a comparatively simple structure and the transfer of the formal apparatus constructed for the full linear group can be accomplished in this case without a specific difficulty).

THEOREM 3. *Every irreducible complex-analytic representation of the group $Sp(n)$ is determined by a system $\{r_1, \ldots, r_\nu\}$ of non-negative integers and has as dominant weight the character*

$$\alpha(\delta) = \Delta_1^{r_1}\Delta_2^{r_2}\ldots\Delta_\nu^{r_\nu}.$$

The space of the induced representation with the signature $\{r_1, \ldots, r_\nu\}$ consists of all solutions of the system of differential equations

$$\mathscr{D}_1^{r_1+1}f(z) = 0, \ldots, \mathscr{D}_\nu^{r_\nu+1}f(z) = 0,$$

where $\mathscr{D}_1, \ldots, \mathscr{D}_\nu$ — are the principal translations on the group $Z = ZSp(n)$.

All the representations of $Sp(n)$ are single-valued.

PROOF. The operators $\mathscr{D}_1, \ldots, \mathscr{D}_\nu$ satisfy the relations

$$\mathscr{D}_p(\Delta_q(z, g)) = 0, \quad q \neq p,$$
$$\mathscr{D}_p^2(\Delta_p(z, g)) = 0 \qquad (p, q = 1, 2, \ldots, \nu),$$

which become obvious when we recall the definition of the operators $\mathscr{D}_i$ as *substitutions* and observe that the minors $\Delta_1, \ldots, \Delta_\nu$ contain only the first ν rows of z; hence there follows the analogue to Lemma 3 and also the invariance of the space of solutions with respect to the operators of the induced representation $\{r_1, \ldots, r_\nu\}$.

The proof of the theorem will be complete (in analogy to the proof of Theorem 2) if it can be established that the space of all solutions is finite-dimensional. Using the relation (38) we write the operator $\mathscr{D}_1$ in the form

$$\mathscr{D}_1 = \frac{\partial}{\partial z_{12}} + z_{23}\frac{\partial}{\partial z_{13}} + \ldots + z_{2,\,n-1}\frac{\partial}{\partial z_{1,\,n-1}} + [1\cdot z_{1,\,n-1} + \ldots - \ldots - z_{2,\,n-1}z_{12}]\frac{\partial}{\partial z_{1n}} =$$
$$= \left(\frac{\partial}{\partial z_{12}} + z_{1,\,n-1}\frac{\partial}{\partial z_{1n}}\right) + \ldots + z_{2,\,n-1}\left(\frac{\partial}{\partial z_{1,\,n-1}} - z_{12}\frac{\partial}{\partial z_{1n}}\right);$$

when we denote the expressions in the round brackets by $Y_1, Y_2, \ldots, Y_{n-2}$, we can easily verify that the conditions of Lemma 4 are satisfied for the operator $\mathscr{D}_1$. Arguing by induction we may assume that it has already been proved that the elements of the second row in table (37) are polynomials (with bounded highest degree); we then deduce from Lemma 4 that all the operators $Y_1, \ldots, Y_{n-2}$ are of degree zero on the solution space. Further we observe that these operators fall into pairs each of which has the form

$$A = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad B = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t};$$

the fact that the operators A, B are of degree zero implies, as is easy to show, the same fact for all the operators $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, i.e. that they are polynomials in all the variables.

The theorem is now proved.

All the real representations of the symplectic group can be written down in a similar way.

Note that the group $Sp(2\nu)$ contains a subgroup isomorphic to $GL(\nu)$, namely the subgroup consisting of the matrices of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix}, \quad \hat{a} = s^{-1}a'^{-1}s,$$

where $a \in GL(\nu)$ and the matrix s is determined in accordance with (33). We can say *that every irreducible representation of* $GL(\nu)$ *with the signature*

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_\nu \geqslant 0, \ m_i \text{ — are integers},$$

is inductive with respect to the symplectic group $Sp(2\nu)$; a more precise meaning of this statement will be indicated in §9; in particular, we see that an irreducible representation of $Sp(2\nu)$ can be realized in a certain space of vector functions assuming values in the space of an irreducible representation of $GL(\nu)$.

§8. The orthogonal group

The *full orthogonal group* $O(n)$ consists of all rotations of a complex n-dimensional space, i.e. of all linear transformations that preserve a non-degenerate symmetric bilinear form (scalar product); the *proper orthogonal group* $SO(n)$ consists of all orthogonal transformations that do not change the orientation (for such transformations $\det g = 1$, whereas the determinant of an arbitrary rotation is ± 1); the group $SO(n)$ is a maximal connected subgroup in $O(n)$. The case $n = 2$ is of no interest, since in this case the group of rotations is commutative; therefore *we shall assume throughout that* $n \geqslant 3$.

When we chose the special basis in which the scalar product is determined by the form

$$(x, y) = x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1,$$

then *the Gauss decomposition of the full linear group induces a Gauss decomposition in* $SO(n)$:

$$SO(n) = \overline{\mathrm{Z}DZ},$$

where Z, D and Z are the intersections of $SO(n)$ with the subgroups $\mathrm{Z}(n)$, $D(n)$ and $Z(n)$, respectively.[1]

A remarkable fact concerning the group $SO(n)$ is that it is doubly connected; this has something to do with the existence of series of *two-valued* representations of the group $SO(n)$ that were first discovered by E. Cartan; for the simplest of these representations, the so-called *spinor* representation, the elegant construction by means of the Clifford algebras is well-known [1], [21], [23]. In our scheme the existence of spinor representations is obtained in the following way (no. 2): in the realization of the irreducible representations of the group $SO(n)$ the principal role is played, as before, by the dominants $\Delta_1, \dots, \Delta_\nu$, $\nu = [\frac{n}{2}]$; however, some of the polynomials $\Delta_i(z, g)$ now turn out to be *reducible*.

In fact, when we are dealing with rotations of a space of odd dimension, $n = 2\nu + 1$, then, as we shall see, there exists on the group $Z = ZO(n)$ a polynomial $\mathfrak{S}_0(z, g)$ (depending on g as on a parameter) such that the last of the minors Δ_i (with respect to a multivector of rank ν) is the square of this polynomial:

$$\Delta_\nu(z, g) = \mathfrak{S}_0^2(z, g); \tag{39}$$

a spinor representation of the group $SO(n)$ can now be constructed in the linear envelope of the functions $f_g(z) = \mathfrak{S}_0(z, g)$, $g \in G$, by means of the usual formula

$$S_g f(z) = \mathfrak{S}_0(z, g) f(z \cdot g);$$

here it follows from the multiplicative relations (6) for the minor Δ_ν that

$$S_{g_1 g_2} = \pm S_{g_1} S_{g_2},$$

and the ambiguity must be avoided if we start out from the requirement of continuity (continuous dependence of S_g on g); linear objects that are transformed according to $\mathfrak{S}_0$ are called *spinors*. A similar model, of course, arises for every single-valued or two-valued representation of the group $SO(2\nu + 1)$.

But if the dimension is even, $n = 2\nu$, then there exist two polynomials $\mathfrak{S}_-$ and $\mathfrak{S}_+$ on Z that satisfy the relations

$$\Delta_{\nu-1}(z, g) = \mathfrak{S}_-(z, g)\, \mathfrak{S}_+(z, g),\ \Delta_\nu(z, g) = \mathfrak{S}_+^2(z, g); \tag{40}$$

---

[1] See footnote on p. 34; as we shall see below, in the case of an even dimension n the *full* orthogonal group $O(n)$ does not have a Gauss decomposition.

the linear objects $\mathfrak{S}_+$ and $\mathfrak{S}_-$ are usually distinguished as *spinors of the first and second kind.*

1. The mirror automorphism of the group $SO(n)$. The full orthogonal group $O(n)$ is the union of two connected components $O^+(n)$ and $O^-(n)$ for whose elements $\det g = \pm 1$, respectively; having fixed an arbitrary element $g_0 \in O^-(n)$ we can obtain all the remaining elements of this subset by multiplying g_0 (on the left or on the right) by the elements $g \in O^+(n)$; for such an element g_0 we choose the matrix $o = -e$, when n is odd; but when n is even, the matrix

$$o = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 \end{pmatrix},$$

which permutes the coordinate axes with the numbers ν and $\nu + 1$; note that in both cases $o^2 = e$.

The proper group $SO(n) = O^+(n)$, obviously, is a normal subgroup of $O(n)$; this means, in particular, that the transition from the matrix g to the matrix

$$\check{g} = ogo^{-1}$$

leaves the subgroup $SO(n)$ invariant. We shall call the transformation o the *mirror mapping* and the automorphism $g \to \check{g}$ of the group $SO(n)$ defined by it the *mirror automorphism* of this group.

The so-defined mirror automorphism has the remarkable property that *it leaves invariant the subgroups* Z, *D and Z* and consequently carries the Gauss decomposition of the element g into the Gauss decomposition of the element $\check{g}$. (This statement is easily verified and follows, in particular, from the explicit formulae §8, 3.)

Since the diagonal matrices $\delta \in D(n)$ are orthogonal with respect to the form (x, y) if and only if in the series $\delta_1, \ldots, \delta_\nu$ the terms are situated symmetrically with respect to their inverses, we have the right in each of the cases $n = 2\nu$ and $n = 2\nu + 1$ to take as independent parameters of the matrix δ the elements $\delta_1, \ldots, \delta_\nu$; therefore, every signature is of the form $(m_1, m_2, \ldots, m_\nu)$ and corresponds to the character

$$\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_\nu^{m_\nu}. \tag{41}$$

LEMMA 7. *If the representation τ of the group $SO(2\nu)$ has the signature $(m_1, m_2, \ldots, m_\nu)$, then the mirror-conjugate representation ($\check{\tau}_g = \tau_{\check{g}}$) has the signature*

$$\check{\tau} = (m_1, \ldots, m_{\nu-1}, -m_\nu);$$

but if the dimension n is odd, $n = 2\nu + 1$, then every representation of *the group $SO(n)$ is the mirror-selfconjugate.*

PROOF. Since the mapping $g \to \check{g}$ preserves the structure of ZDZ, it is sufficient to find the image of the dominant weight $\alpha(\delta)$; in particular, if the dimension is even, $n = 2\nu$, then all the parameters δ_i are preserved, except the parameter δ_ν which is replaced by δ_ν^{-1}, and this proves the lemma.

2. Classification of the irreducible representations. Following the usual scheme we establish this result.

LEMMA 8. *The character (41) is the dominant weight of a single-valued irreducible representation of the group $SO(n)$ if and only if all the numbers m_i are integers and satisfy the system of relations:*

$$1) \text{ for } n=2\nu:\ m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant |m_\nu|,$$
$$2) \text{ for } n=2\nu+1:\ m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu \geqslant 0.$$

Moreover, the group $SO(n)$ has a series of two-valued representations; in this case the signature $(m_1, m_2, \ldots, m_\nu)$ consists of semi-integers[1] *subject to the same order relations.*

PROOF. 1. *The case $n = 2\nu$.* Similarly to the way in which this was done in the proof of Lemma 5, we consider in the group $SO(2\nu)$ subgroups $G_1, G_2, \ldots, G_{\nu-1}$ isomorphic to the Lorentz group or we consider at once the subgroup G_0 isomorphic to $SL(\nu)$ and consisting of the matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix}, \quad a \in SL(\nu),$$

where $\hat{a} = s^{-1}a'^{-1}s$ and the matrix s is determined in accordance with (33).[2] If the character $\alpha(\delta)$ is inductive with respect to the group G, then its restriction to G_0 is inductive with respect to G_0; hence there follow the conditions

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu,$$

which must also hold for the mirror-conjugate representation of G:

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant -m_\nu;$$

by combining the two chains we obtain a system of relations as indicated in the lemma; moreover, any two terms

$$m_i - m_\nu \text{ and } m_i + m_\nu$$

must be integers, consequently all the parameters $m_1, \ldots, m_\nu$ are simultaneously either integers or semi-integers.

On the other hand, let us assume that the conditions of the lemma are satisfied for a certain system of numbers $(m_1, \ldots, m_\nu)$; since mirror-conjugate characters can only be inductive simultaneously, we may assume without loss of generality that $m_\nu \geqslant 0$; but then the character of the diagonal group $D(n)$ with the parameters $(m_1, \ldots, m_\nu, 0, \ldots, 0)$ is

1 A semi-integer is half an odd integer.

2 The invariance of the scalar product (x, y) is equivalent to the conditions

$$g'^{-1} = s_0^{-1} g s_0, \text{ where } \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} = {}^0s$$

inductive with respect to the group $GL(n)$, consequently, its restriction to the orthogonal group is inductive with respect to this group.

2. *The case* $n = 2\nu + 1$. The method of proof is completely analogous; the last condition ($m_\nu \geqslant 0$) follows from considering the subgroup $SO(3)$, whose elements consist of rotations that transform only the coordinates x_ν, $x_{\nu+1}$, $x_{\nu+2}$ (as mentioned in the footnote on p. 19 the group $SO(3)$ is doubly covered by the Lorentz group).

The lemma is now proved.

When we write the character of D by means of principal minors

$$\alpha(\delta) = \Delta_1^{r_1}\Delta_2^{r_2}\dots\Delta_\nu^{r_\nu},$$

we find that in the case of an odd n, $n = 2\nu + 1$, the exponent r_ν may be a semi-integer, whereas all the remaining r_i are integers; similarly, for $n = 2\nu$ the exponents $r_{\nu-1}$, r_ν may be semi-integers; *the formulae* (39) *and* (40), *in particular, follow from this, which indicate the reducibility of certain minors of the matrix zg as polynomials of the variables z.*[1]

We shall now write the signature of an irreducible representation in the form

$$\Delta_1^{r_1}\dots\Delta_{\nu-1}^{r_{\nu-1}}\mathfrak{S}_0^{r_0},$$

if $n = 2\nu + 1$, and in the form

$$\Delta_1^{r_1}\dots\Delta_{\nu-2}^{r_{\nu-2}}\mathfrak{S}_-^{r_-}\mathfrak{S}_+^{r_+},$$

if $n = 2\nu$; in particular, the spinor representation of $SO(2\nu + 1)$ is determined by the signature

$$\mathfrak{S}_0 = (\delta_1\delta_2\dots\delta_\nu)^{1/2},$$

and the spinors of the first and second kind of the group $SO(2\nu)$ are transformed according to the representations

$$\mathfrak{S}_+ = (\delta_1\dots\delta_{\nu-1}\delta_\nu)^{1/2}, \quad \mathfrak{S}_- = (\delta_1\dots\delta_{\nu-1}\delta_\nu^{-1})^{1/2},$$

which are mirror-conjugate to each other.

3. **The formal apparatus.** As in the symplectic case we find that the group $Z = ZO(n)$ has the decomposition

$$Z = Z_0 Z_1,$$

where the group Z_0, isomorphic to $ZO(n - 2)$, consists of the matrices

$$z_0 = \begin{pmatrix} 1 & 0 & 0 \\ & x & 0 \\ & & 1 \end{pmatrix}, \quad x \in ZO(n-2),$$

and the matrices of Z_1 have the form

$$z_1 = \begin{pmatrix} 1 & t & -\frac{v^2}{2} \\ & e_{n-2} & \tilde{t} \\ & & 1 \end{pmatrix},$$

where $t = (t_2, \dots, t_{n-1})$ is an arbitrary row of $n - 2$ complex numbers, and $\tilde{t}$ the column

[1] Two alternative proofs of this statement will be given further on.

$$\begin{pmatrix} -t_{n-1} \\ \cdots \\ -t_2 \end{pmatrix},$$

depending on t, and $v^2 = (t,\ t)$; consequently, for $n = 2\nu + 1$ independent parameters of the group $ZO(n)$ are given by the table

$$\left.\begin{matrix} 1 & z_{12} & z_{13} & z_{14} \cdots z_{1,n-2} & z_{1,n-1} & * \\ & 1 & z_{23} & z_{24} \cdots z_{2,n-2} & * & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 1 & z_{\nu-1,\nu} & * & \\ & & & 1, & & \end{matrix}\right\} \tag{42}$$

and in the case of an even dimension, $n = 2\nu$, by the table

$$\left.\begin{matrix} 1 & z_{12} & z_{13} & z_{14} \cdots z_{1,n-2} & z_{1,n-1} & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & 1 & z_{\nu-1,\nu} & z_{\nu-1,\nu+1} & * & \\ & & 1 & 0; & & \end{matrix}\right\} \tag{42'}$$

here the elements on the second diagonal that are indicated by asterisks in the tables are computed by the formula

$$v_p = (z_p,\ z_p) \quad (p = 1,\ 2,\ \ldots,\ \nu), \tag{43}$$

where z_p denotes the p-th row of one of the tables (42), (42′), extended at both ends by zeros to a vector of length n, and $(x,\ x)$ denotes the scalar square of the vector x.

It is convenient to start the discussion with the case of even dimension. We begin with the remark that among the scalar squares (43) the last one is equal to zero:

$$z_{\nu,\ \nu+1} \equiv 0$$

(the truncation of the matrix z formed from the elements with the indices ν and $\nu + 1$ is an orthogonal matrix of $ZO(2)$, but the group $ZO(2)$ – a Gaussian component of the commutative group $SO(2)$ – consists only of the unit element); this implies the above-mentioned statement that *the mirror automorphism* $\check{g}$ *defined by the matrix* o *(transposition of the indices* ν *and* $\nu + 1$*) preserves the Gauss decomposition of the group* $SO(n)$:

$$\check{Z} = Z,\ \check{D} = D,\ \check{Z} = Z;$$

furthermore, for the principal minors of an arbitrary matrix $g \in SO(n)$ we have the identity

$$\check{\Delta}_p = \Delta_p \qquad (p = 1, 2, \ldots, \nu - 1), \quad \check{\Delta}_\nu = \frac{\Delta_{\nu-1}^2}{\Delta_\nu},$$

where $\Delta_i = \Delta_i(g)$, $\check{\Delta}_i = \Delta_i(\check{g})$ (it is sufficient to verify these identities for *diagonal* matrices δ).

We introduce the "characteristic" differential operators

$$\left.\begin{aligned} \mathscr{D}_p &= \sum_{s=p+1}^{n-p} z_{p+1,\,s}\frac{\partial}{\partial z_{ps}} \qquad (p=1,2,\dots,\nu-2),\\ \mathscr{D}_{\nu-1} &= \mathscr{L}_- = \frac{\partial}{\partial z_{\nu-1,\,\nu}}, \qquad \mathscr{L}_+ = \frac{\partial}{\partial z_{\nu-1,\,\nu+1}}. \end{aligned}\right\} \tag{44}$$

These infinitesimal operators are generated by the one-parameter subgroups

$$z_p(t) = 1 + \frac{t}{2}(e_{p,\,p+1} - e_{q-1,\,q}), \quad q = n-p+1,$$

$$z_+(t) = 1 + \frac{t}{2}(e_{\nu-1,\,\nu+1} - e_{\nu,\,\nu+2}),$$

and consequently have the "algebraic" character of substitutions.

LEMMA 9. *In the change of variables $z \to \tilde{z} = z \cdot g$ the operators $\mathscr{D}_p$, $\mathscr{L}_+$ are transformed by the rule*

$$\mathscr{D}_p^{r+1}\Delta_p^r = \frac{(\Delta_{p-1}\Delta_{p+1})^{r+1}}{\Delta_p^{r+2}}\tilde{\mathscr{D}}_p^{r+1} \qquad (p=1,2,\dots,\nu-1),$$

$$\mathscr{L}_+^{r+1}\Delta_\nu^{r/2} = \frac{\Delta_{\nu-2}^{r+1}}{\Delta_\nu^{r/2+1}}\tilde{\mathscr{L}}_+^{r+1},$$

where the operators Δ_p and $\tilde{\mathscr{D}}$ are defined as in Lemma 3.

PROOF. For the operators $\mathscr{D}_1, \dots, \mathscr{D}_{\nu-2}$ and $\mathscr{L}_- = \mathscr{D}_{\nu-1}$ the arguments of the type used in the proof of Lemma 3 remain valid; for $\mathscr{L}_+$ this method ceases to be valid (in this case Lemma 2 does not hold), however the identity for $\mathscr{L}_+$ is the *mirror-conjugate* with respect to the identity for $\mathscr{L}_-$: it is sufficient to observe that by (43),

$$\Delta_\nu = V^{-1}\left(\frac{\Delta_{\nu-1}^2}{\Delta_\nu}\right)V,$$

where V denotes the operator of the change of variable $Vf(z) = f(\check{z})$; for this purpose we have to use once more the obvious equality $(z \cdot g)^\vee = \check{z} \cdot g$. The lemma is now proved.

In the case of an odd dimension $n = 2\nu + 1$, the differential operator $\mathscr{L}_0 = \frac{\partial}{\partial z_{\nu,\,\nu+1}}$ appears similarly, whose transposition rule takes the form

$$\mathscr{L}_0^{r+1}\Delta_\nu^{r/2} = \frac{\Delta_{\nu-1}^{r+1}}{\Delta_\nu^{r/2+1}}\tilde{\mathscr{L}}_0^{r+1},$$

which is easy to prove by embedding the group $ZO(2\nu+1)$ as the subgroup of $ZO(2\nu+2)$ that leaves the expression $x_{\nu+1} + x_{\nu+2}$ invariant (in the restriction to this subgroup both operators $\mathscr{L}_-$ and $\mathscr{L}_+$ go over into the operator $\mathscr{L}_0$ for $ZO(2\nu+1)$).

4. **The general construction.** When we define for the signature $\tau = \{r_1, \dots, r_{\nu-2};\ r_-, r_+\}$ the space $\mathfrak{R}_\tau(Z)$ as the space of all solutions of the system of differential equations

$$\mathscr{D}_1^{r_1+1}f(z) = \dots = \mathscr{D}_{\nu-2}^{r_{\nu-2}+1}f(z) = \mathscr{L}_-^{r_-+1}f(z) = \mathscr{L}_+^{r_++1}f(z) = 0, \tag{45}$$

we easily obtain from Lemma 4 that $\mathfrak{R}_\tau$ is finite-dimensional and consists of polynomials on Z.

From this, in particular, there follows another proof of the reducibility of the minors $\Delta_{\nu-1}$, Δ_ν; indeed,

$$\mathscr{D}_1(\mathfrak{S}_+)=\ldots=\mathscr{D}_{\nu-2}(\mathfrak{S}_+)=\mathscr{L}_-(\mathfrak{S}_+)=0,$$

where $\mathfrak{S}_+=\mathfrak{S}_+(z,g)$ (the appearance of identical rows in the minor $\Delta_\nu(z,g)$); moreover, by applying both parts of the last identity of Lemma 9 for $r=1$ to the function $f^0(z)\equiv 1$, we conclude that

$$\mathscr{L}_+^2(\mathfrak{S}_+)=0;$$

consequently, the function $f_g(z)\equiv\mathfrak{S}_+(z,g)$ for every $g\in G$ belongs to the null space of one of the systems (45); therefore

$$\mathfrak{S}_+(z,g)=\sqrt{\Delta_\nu(z,g)}$$

is a polynomial in z (whose degree is uniformly bounded with respect to $g\in G$); the polynomial character of $\mathfrak{S}_-(z,g)$ is obtained by mirror-conjugacy, and the polynomial character of $\mathfrak{S}_0(z,g)$ is proved similarly. For odd n the space $\mathfrak{R}_\tau(Z)$ is defined by the system

$$\mathscr{D}_1^{r_1+1}f(z)=\ldots=\mathscr{D}_{\nu-1}^{r_{\nu-1}+1}f(z)=\mathscr{L}_0^{r_0+1}f(z)=0; \tag{45'}$$

as a result an analogue to Theorem 3 is obtained.

THEOREM 4. *Every irreducible representation (single-valued or two-valued) of the orthogonal group $SO(n)$ is induced in one of the spaces $\mathfrak{R}_\tau(Z)$ each of which is defined by the system* (45) *or* (45′) *depending on the parity of n and consists of polynomials on Z.*

The system of dominant weights of $SO(n)$ is described in Lemma 8; *here the parameters m_i and r_i are connected by the relations*:

1) $n=2\nu$: $r_1=m_1-m_2,\ldots,\ r_{\nu-2}=m_{\nu-2}-m_{\nu-1},\ r_\pm=m_{\nu-1}\pm m_\nu$;

2) $n=2\nu+1$: $r_1=m_1-m_2,\ldots,\ r_{\nu-1}=m_{\nu-1}-m_\nu,\ r_0=2m_\nu$;

all the parameters r are arbitrary non-negative integers.[1]

Observe also that (as a consequence of Clifford's theorem [1]) on transition to the full group $O(2\nu)$ the mirror-conjugate representations with the signatures $\{r_1,\ldots,r_{\nu-2};\ r_-,\ r_+\}$ and $\{r_1,\ldots,r_{\nu-2};\ r_+,\ r_-\}$ combine to a single irreducible representation; whereas for odd n irreducibility for $O(n)$ is equivalent to irreducibility for $SO(n)$; and so all the irreducible representations of the *full* orthofonal groups are listed.

§9. The binary decomposition; symmetric powers of spinor representations

By dividing a basis in an n-dimensional vector space into two arbitrary parts: $\{e_1,\ldots,e_p\}$, $\{e_{p+1},\ldots,e_{p+q}\}$, $p+q=n$, we can easily verify that every matrix $g\in GL(n)$ for which $\Delta_p(g)\neq 0$ has a unique representation in the form

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[1] See also the symbolism at the end of §2.

$$g = \begin{pmatrix} e_p & 0 \\ \eta & e_q \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} e_p & y \\ 0 & e_q \end{pmatrix},$$

where e_p, e_q are unit matrices of order p and q, $a \in GL(p)$, $b \in GL(q)$ and η, y are rectangular matrices; therefore

$$G = \overline{\mathscr{H} G^0 Y},$$

where $\mathscr{H}$, G^0 and Y denote the corresponding matrix groups; here G^0 is isomorphic to the direct product $GL(p) \times GL(q)$ and the groups $\mathscr{H}$ and Y are commutative. We call the decomposition so obtained the *binary decomposition* of the group $GL(n)$.

The binary decomposition gives a certain parametrization of the group G which is occasionally useful; for example, the following statement holds.

LEMMA 10. *Let $a \to U_a$ be an irreducible representation of the group $GL(p)$ in the space F with the signature $(m_1, \ldots, m_p)$, where $m_1, \ldots, m_p$ are non-negative integers, and let $f(y)$ denote a vector function on the (commutative) group Y with values in F. Then the formula*

$$T_g f(y) = U_a f(y \cdot g), \tag{46}$$

where the matrices a and $\tilde{y} = y \cdot g$ are computed from the binary decomposition of the matrix $\begin{pmatrix} e_p & y \\ 0 & e_q \end{pmatrix} g$, defines an irreducible representation of $GL(n)$ with the signature $(m_1, \ldots, m_p, 0, \ldots, 0)$.

In particular, the representation Δ_p^m of the group $GL(n)$ is realized in *numerical* functions on the group Y.

Similarly, *every* irreducible representation of $GL(n)$ is induced by some irreducible representation of the subgroup $G^0 \approx GL(p) \times GL(q)$.

Note that the matrix g is expressed in terms of the new parameters by the formula

$$g = \begin{pmatrix} a & ay \\ \eta a & \eta a y + b \end{pmatrix}, \tag{47}$$

hence, in particular, we find the explicit form of the "fractional linear substitutions" $y \cdot g$:

$$y \cdot g = (g_{11} + y g_{21})^{-1} (g_{12} + y g_{22}),$$

where g_{ij} are the *blocks* of the matrix g.

When we set $p = q = \nu$, *then the binary decomposition in $GL(2\nu)$ induces the binary decompositions in the subgroups $Sp(2\nu)$ and $SO(2\nu)$ and in each of these cases the subgroup G^0 consists of the matrices of the form*

$$\begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix}, \quad \hat{a} = s^{-1} a'^{-1} s,$$

i.e. is isomorphic to $GL(\nu)$ (the proof is not difficult: it is sufficient to verify the validity of the decomposition for the subgroups Z and Z); a consequence of this fact was already mentioned in §7 for the symplectic group $Sp(n)$. Similarly we can obtain the decomposition in $SO(2\nu+1)$.[1]

For the groups $ZO(2\nu)$ (similar to $ZSp(2\nu)$) the binary decomposition reduces to the decomposition of the form $Z = XY$, where the subgroup X is isomorphic to $Z(\nu)$ and Y is a normal subgroup of Z; note that the operators $\mathscr{D}_1, \ldots, \mathscr{D}_{\nu-1}$ act only on the parameters X and coincide here with the equally named operators for $GL(\nu)$; whereas the operator $\mathscr{L}_+$ (similar to $\mathscr{D}_\nu$) acts only on the variables Y.

Lemma 10 can be used, in particular, to study the symmetric powers of the spinor representations of $SO(2\nu)$; for the representation with the signature $\mathfrak{S}_+^m$ is induced, according to (46), by the character

$$U_a = (\det a)^{m/2}$$

of the full linear group $GL(\nu)$, and the representation space is spanned by the vectors

$$f_g(y) = (\det(g_{11} + yg_{21}))^m;$$

using the parametrization (47) of g we observe that

$$\det(g_{11} + yg_{21}) = \det(e + y\eta)\det a,$$

therefore the matter reduces to the computation of the value of

$$S_+^2(y, \eta) = \det(e + y\eta). \tag{48}$$

Here the matrices η and y are parameters in the groups $\mathscr{H}$ and Y, respectively; from the orthogonality conditions it follows that they range over the set of all matrices of order ν that are *skew-symmetric with respect to the second diagonal*. In this way one can obtain an elementary proof of the reducibility of the polynomials (48) and also of the minors $\Delta_\nu(z, g)$ on Z; we shall only discuss cases of small dimension: $n = 4$ and $n = 6$; in particular, *the polynomial* $S_+^2(y, \eta) = \det(e + y\eta)$ *in the case* $n = 6$ *has the form*

$$S_+^2(y, \eta) = [1 + y_1\eta_1 + y_2\eta_2 + y_3\eta_3]^2,$$

where y_1, y_2, y_3 *are independent parameters of* y *and* η_1, η_2, η_3 *the corresponding parameters in* η.

PROOF. We can assume that the factors y and η are skew-symmetric with respect to the *principal* diagonal (because one of them can be multiplied on the right, and the other on the left, by the matrix $s = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}$; therefore we choose the parameters in the following way:

[1] Above it was mentioned that the full group $O(2\nu)$ does not have a Gauss decomposition; this follows from the identity $\Delta_\nu(g) \equiv 0$ for all "improper" rotations $g \in O^-(n)$, which is easily verified by means of (47) (multiplication by the matrix o).

$$y=\begin{pmatrix}0 & y_1 & y_2\\ -y_1 & 0 & y_3\\ -y_2 & -y_3 & 0\end{pmatrix},\quad \eta=\begin{pmatrix}0 & -\eta_1 & -\eta_2\\ \eta_1 & 0 & -\eta_3\\ \eta_2 & \eta_3 & 0\end{pmatrix}; \tag{49}$$

then

$$\Delta_\nu=\det(e+y\eta)=\begin{vmatrix}1+y_1\eta_1+y_2\eta_2 & y_2\eta_3 & -y_1\eta_3\\ y_3\eta_2 & 1+y_1\eta_1+y_3\eta_3 & y_1\eta_2\\ -y_3\eta_1 & y_2\eta_1 & 1+y_2\eta_2+y_3\eta_3\end{vmatrix}= \\ =[1+y_1\eta_1+y_2\eta_2+y_3\eta_3]^2 \tag{50}$$

(To verify this we consider Δ_ν as a function of y_1 and find $\Delta_\nu(0)$, $\Delta'_\nu(0)$, $\Delta''_\nu(0)$; note that the parameters y are expressed in terms of the parameters of $z \in XY$ by the rule: $y_1 = z_{14} - z_{12}z_{24}$, $y_2 = z_{15} + z_{13}z_{24}$, $y_3 = z_{24}$.

Thus, in the case $n = 6$ we know a basis in the representation space $\mathfrak{S}_+$ (and also in $\mathfrak{S}_+^m$ for every $m \geqslant 0$); for higher dimensions n the number of independent variables $\left(\frac{\nu(\nu-1)}{2}\right)$ in y remains less than the dimension $(2^{\nu-1})$ of $\mathfrak{S}_+$,[1] correspondingly in the representation space there appear polynomials in y of a degree higher than the first (it can be shown that in this way there arise the relations that are characteristic for the *simple spinors* of Cartan [13]).

In the case $n = 4$ the matrix

$$z=\begin{pmatrix}1 & z_{12} & z_{13} & -z_{12}z_{23}\\ 0 & 1 & 0 & -z_{13}\\ 0 & 0 & 1 & -z_{12}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

determines the general form of an element of $ZO(4)$; the representation $\mathfrak{S}_-^m$ is realized on polynomials in the single variable z_{12}, and the representation $\mathfrak{S}_+^m$ on polynomials in the variable z_{13} and an arbitrary irreducible representation has the signature

$$\mathfrak{S}_-^m\mathfrak{S}_+^n,$$

i.e. is locally isomorphic[2] to the direct product of the two symmetrized powers $\mathfrak{S}_+$ and $\mathfrak{S}_-$.

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1 The dimension of the spinor representation will be computed in §13.

2 The stipulation of being local stems from the fact that if at least one of the exponents m, n is odd, then the corresponding representation is *two-valued*; in particular, for $m = n = 1$ we obtain the local isomorphism of $SO(4)$ and the "square" $\mathfrak{G}\times\mathfrak{G}$ of the Lorentz group; note that the latter statement can be verified in an elementary way for the matrices ζ, δ, z, for example

$$z\approx\begin{pmatrix}1 & z_{12}\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & z_{13}\\ 0 & 1\end{pmatrix},$$

and the correspondence in the large follows from this.

Part II

SOME PROBLEMS OF SPECTRAL ANALYSIS

The general scheme of spectral analysis was explained in the introduction and made more precise in §2; in §§3, 4, 7 and 8 the possible "point spectra" were described for every classical group; the apparatus of differential operators $\mathscr{D}_p$ ("principal translations") will only be used in §§13, 14 and 15.

§10. The product of two irreducible representations of the Lorentz group

We shall give a detailed account of this simple example which admits a complete solution (analysis and synthesis) and is of great importance in physics [15]. As we know, every irreducible (complex) representation of the Lorentz group is a symmetrized power $\mathfrak{D}^m$ of the representation $\mathfrak{D} = \Delta_1$ and can be realized by the formula (11) in the space $E_m(Z)$ consisting of polynomials of degree $\leqslant m$ of one complex variable z.

Correspondingly the tensor product of the representations $\mathfrak{D}^{n_1}$ and $\mathfrak{D}^{n_2}$ can be constructed in the space $\mathscr{E} = E_{n_1}(Z) \times E_{n_2}(Z)$, consisting of all polynomials of two variables z_1 and z_2, of a degree not higher than n_1 in z_1 and n_2 in z_2, and the operators of the representation act according to the formula

$$T_g f(z_1, z_2) = (\beta z_1 + \delta)^{n_1} (\beta z_2 + \delta)^{n_2} f\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right), \tag{51}$$

where α, β, γ, δ are the elements of the matrix $g \in G$; we introduce the notation

$$m = \min(n_1, n_2), \qquad n = n_1 + n_2. \tag{52}$$

We have to decompose the representation T_g into irreducible representations.

1. Analysis. In accordance with the general scheme we restrict T_g to Z:

$$T_z f(z_1, z_2) = f(z_1 + z, z_2 + z);$$

hence we conclude that the invariants of Z can depend only on the difference $z_2 - z_1$, and among such vectors we choose a system of "weight" vectors:

$$1, \; z_2 - z_1, \; (z_2 - z_1)^2, \; \ldots, \; (z_2 - z_1)^m,$$

i.e. of eigenvectors with respect to the transformations of the subgroup D; here the monomial $(z_2 - z_1)^k$ corresponds to the "weight" $\delta^{n_1+n_2-2k}$ (we recall that the independent parameter of $\delta \in D$ is taken to be the element $\delta_{22} = \delta$); from this follows the spectral formula

$$\mathfrak{D}^{n_1} \times \mathfrak{D}^{n_2} = \sum_{k=0}^{m} \mathfrak{D}^{n_1+n_2-2k} = \sum_{l=|n_1-n_2|}^{n_1+n_2}{}' \mathfrak{D}^l, \tag{53}$$

where the dash denotes that the summation index changes each time by two units.

Continuing the analysis let us find invariant subspaces on which the constituants $\mathfrak{D}^l$ act; for this purpose we make use of the following convenient device. According to (53) the space $\mathscr{E}$ can be mapped one-to-one onto the space

$$\mathscr{E}^0 = \sum_{l=|n_1-n_2|}^{n_1+n_2}{}' E_l(Z)$$

(Σ denotes a direct sum) which can be regarded as a space of *vector functions* on the plane Z

$$\varphi(z) = (\varphi_0(z), \varphi_1(z), \dots, \varphi_m(z)), \tag{54}$$

where $\varphi_k(z)$ belongs to $E_{n_1+n_2-2k}(Z)$. We denote by S the operator of the mapping $\mathscr{E} \to \mathscr{E}^0$ it is clear that a knowledge of the operators S^{-1} and S solves the problem of analysis and synthesis (indeed, let P_l be the projection operator from $\mathscr{E}^0$ to the subspace $E_l(Z)$; then $S^{-1}P_lS$ is the projection operator from $\mathscr{E}$ to the corresponding irreducible invariant subspace); we shall not dwell on the description of the projection operators (this can be done quite trivially) and restrict ourselves to a computation of the operators S, S^{-1}. For the solution of the restricted problem of *analysis* it is sufficient to know how the operator S^{-1} acts on the basis vector $z^\nu \in E_l(Z)$ ($l = |n_1 - n_2|, \dots, n_1 + n_2$; $\nu = 0, 1, \dots, l$).

Acting with the operator T_g on the dominant vector of the representation $\mathfrak{D}^{n-2k}$

$$T_g[(z_2-z_1)^k] = (\beta z_1+\delta)^{n_1-k}(\beta z_2+\delta)^{n_2-k}(z_2-z_1)^k =$$
$$= (z_2-z_1)^k \sum C_{n_1-k}^{\nu_1} C_{n_2-k}^{\nu_2} \beta^{\nu_1+\nu_2}\delta^{n_1+n_2-\nu_1-\nu_2} z_1^{\nu_1} z_2^{\nu_2},$$

we bring in for comparison the representation $\mathfrak{D}^{n-2k}$ in its canonical form

$$T_g^0[1] = (\beta z+\delta)^{n-2k} = \sum C_{n-2k}^{\nu}\beta^{\nu}\delta^{n-\nu}z^{\nu};$$

as a result we see that to the basis vector $z \in E_{n-2k}$ there corresponds in $\mathscr{E}$ the vector

$$\varphi_k^\nu = (z_2-z_1)^k \sum_{\nu_1+\nu_2=\nu} p_{\nu_1\nu_2}(k)\, z_1^{\nu_1} z_2^{\nu_2}, \tag{55}$$

where for brevity we have set

$$p_{\nu_1\nu_2}(k) = \frac{C_{n_1-k}^{\nu_1} C_{n_2-k}^{\nu_2}}{C_{n_1+n_2-2k}^{\nu_1+\nu_2}},$$

and C_a^b is the binomial coefficient.* Rewriting the result in the form $\varphi_k^\nu = S^{-1}p_k(z^\nu)$, where p_k is the projection operator from $\mathscr{E}^0$ to the direct summand $\mathscr{E}_k^0 = E_{n-2k}(Z)$, we see that on the subspace $\mathscr{E}_k^0$ the action of the operator S^{-1} reduces to a "partition" or "polarization" of the power z^ν; we can understand this result more clearly after studying in the next section the rules for symmetrization of tensor representations.

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* Note that the symbol C_n^k in the Russian text denotes the binomial coefficient $\binom{n}{k}$. (Ed.)

As a result the space $\mathscr{E}$ is the direct sum

$$\mathscr{E}=\mathscr{E}_0+\mathscr{E}_1+\mathscr{E}_2+\ldots+\mathscr{E}_m,$$

where each $\mathscr{E}_k$ is the linear envelope of the vectors (55) for fixed k; the representation in $\mathscr{E}_k$ has the signature $\mathfrak{D}^{n-2k}$.

In particular, in the irreducible subspace with the highest signature $n = n_1 + n_2$ we have obtained a basis of the vectors:

$$\varphi_0^0=1,$$
$$\varphi_0^1=\frac{1}{n}(n_1z_1+n_2z_2),$$
$$\varphi_0^2=\frac{1}{n(n-1)}(n_1(n_1-1)z_1^2+2n_1n_2z_1z_2+n_2(n_2-1)z_2^2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\varphi_0^n=z_1^{n_1}z_2^{n_2},$$

and every irreducible subspace has a basis of similar form.

The usual Clebsch-Gordan matrices can easily be obtained from (55), when we carry out the multiplication and find the coefficients of the basis quantities $z_1^{s_1}z_2^{s_2}$; however, our system of notation differs in normalization: usually the canonical basis of the representation $\mathfrak{D}^l$ is written in the form

$$e_\mu=\frac{z^{k-\mu}}{\sqrt{(k-\mu)!\,(k+\mu)!}},\quad k=\frac{l}{2},\quad -k\leqslant\mu\leqslant k.$$

All our constructions go over without change to the case of the unitary subgroup $\mathfrak{U}$, since all the representations $\mathfrak{D}^l$ are analytic continuations of the corresponding representations of $\mathfrak{U}$; finally, the investigation of the real representations $\mathfrak{D}^p\overline{\mathfrak{D}}^q$ of the Lorentz group differs only in doubling the number of indices.

2. Synthesis. We introduce into the discussion the differential operators

$$\left.\begin{aligned}
&U_0=1,\\
&U_1=\frac{1}{n}\left(n_1\frac{\partial}{\partial z_2}-n_2\frac{\partial}{\partial z_1}\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&U_k=\frac{1}{C_{n-k+1}^k}\sum_{s_1+s_2=k}(-1)^{s_1}C_{n_1-s_1}^{s_2}C_{n_2-s_2}^{s_1}\frac{\partial^{s_1+s_2}}{\partial z_2^{s_2}\,\partial z_1^{s_1}},\ k=0,1,\ldots,m,
\end{aligned}\right\}\tag{56}$$

and prove the following theorem.

THEOREM 5. *The representation* (1) *is the direct sum of the irreducible representations* $\mathfrak{D}^{n-2k}$ (k = 0, 1, 2, ..., m), *where*

$$n=n_1+n_2,\quad m=\min(n_1,\,n_2).$$

The irreducible representation $\mathfrak{D}^{n-2k}$, *realized in its canonical form in the space* $E_{n-2k}(Z)$, *can be embedded in the representation space* $\mathscr{E}$ (51) *if we associate with every basis vector* $z^\nu \in E_{n-2k}(Z)$ *the corresponding vector* (55); *here the index* k *is fixed and* ν *assumes the values* 0, 1, ..., $n-2k$.

The inverse mapping of the space $\mathscr{E}$ *onto the direct sum of the spaces*

$E_{n-2k}(Z)$ is given by the formula

$$\varphi_k(z) = \frac{1}{k!}[U_k f(z_1, z_2)]_{z_1=z_2=z},$$

where the linear differential operators U_k are defined by (56); in other words, if the functions $f(z_1, z_2)$ are transformed according to the representation (51), then the corresponding functions $\varphi_k(z)$ are transformed according to the irreducible representations $\mathfrak{D}^{n-2k}$.

PROOF. We could verify almost immediately that the following relations hold

$$U_l(\varphi_k^\nu) = \begin{cases} 0, & \text{if } l \neq k, \\ k! z^\nu, & \text{if } l = k, \end{cases}$$

however we prefer to use a simple symbolical device. Being a tensor product the space $\mathscr{E}$ spanned by the vectors

$$f_{g_1 g_2}(z_1, z_2) = \Delta_1^{n_1}\Delta_2^{n_2}, \tag{57}$$

where we have set briefly $\Delta_1 = \beta_1 z_1 + \delta_1$, $\Delta_2 = \beta_2 z_2 + \delta_2$; β, δ are the parameters of the matrix $g \in G$; *it is even sufficient to choose these vectors for $\beta_1 \neq 0$, $\beta_2 \neq 0$, and this we shall do.* On the symbols $\Delta_1^{s_1}\Delta_2^{s_2}$ we define the operators p_1 and p_2 by the rule

$$p_1(\Delta_1^{s_1}\Delta_2^{s_2}) = s_1\Delta_1^{s_1}\Delta_2^{s_2}, \quad p_2(\Delta_1^{s_1}\Delta_2^{s_2}) = s_2\Delta_1^{s_1}\Delta_2^{s_2},$$

and we consider the expression

$$Z = \frac{\partial}{\partial z_2}\frac{1}{p_2} - \frac{\partial}{\partial z_1}\frac{1}{p_1},$$

defined on the same symbols for $s_1 \neq 0$, $s_2 \neq 0$; it is easy to verify[1] that for functions of the form (57) the action of the differential operator U_k coincides to within a numerical factor with the action of the operators Z^k, and the problem now reduces to a verification of the fact that the functions

$$R_k(z) = \{Z^k f_{g_1 g_2}(z_1, z_2)\}_{z_1=z_2=z}$$

are transformed according to $\mathfrak{D}^{n-2k}$, when the original functions f are transformed according to (51).

When we apply Z repeatedly to the functions of the form (57), we find

$$Z(\Delta_1^{n_1}\Delta_2^{n_2}) = \beta_2\Delta_1^{n_1}\Delta_2^{n_2-1} - \beta_1\Delta_1^{n_1-1}\Delta_2^{n_2},$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$Z^k(\Delta_1^{n_1}\Delta_2^{n_2}) = [\beta_2(\beta_1 z_1 + \delta_1) - \beta_1(\beta_2 z_2 + \delta_2)]^k \Delta_1^{n_1-k}\Delta_2^{n_2-k},$$

1 We could also propose another, equivalent symbolism. When we replace the operators p_1 and p_2 in the expressions for Z by the numbers n_1 and n_2 for which we introduce, apart from the usual multiplication, also the "nominal" powers

$$[n_1]^k[n_2]^l = n_1(n_1-1)\dots(n_1-k+1)\,n_2(n_2-1)\dots(n_2-l+1);$$

then we have, for example,

$$[Z]^2 = \frac{\partial^2}{\partial z_2^2}\frac{1}{n_2(n_2-1)} - 2\frac{\partial^2}{\partial z_2\,\partial z_1}\frac{1}{n_2 n_1} + \frac{\partial^2}{\partial z_1^2}\frac{1}{n_1(n_1-1)},$$

which agrees with (56) and, on the other hand, corresponds to the rules of commutation for the operators $\frac{\partial}{\partial z_i}$, p_i that are used in the text.

from which by setting $z_1 = z_2 = z$ we obtain for $R_k(z)$ the expression

$$R_k(z) = [\beta_2\delta_1 - \beta_1\delta_2]^k (\beta_1 z + \delta_1)^{n_1-k} (\beta_2 z + \delta_2)^{n_2-k}. \tag{58}$$

According to the general identity (6), the binomials $\Delta(z, g) = \beta z + \delta$ satisfy the multiplicative relation

$$\Delta(z, g_0 g) = \Delta(z, g_0)\,\Delta(z\cdot g_0, g);$$

consequently, under the influence of the operator T_{g_0} every function (57) goes over into a function of the same form, where the indices g_1, g_2 are replaced by the indices g_0g_1, g_0g_2, respectively. We also observe that under the substitution $g_1 \to g_0g_1$, $g_2 \to g_0g_2$ the expression $\omega(g_1, g_2) = \beta_1\delta_2 - \beta_2\delta_1$ remains *invariant*; now we can easily compute the corresponding transformation for the functions (58):

$$R_k(z) \longrightarrow \omega^k \Delta^{n_1-k}(z, g_0g_1)\,\Delta^{n_2-k}(z, g_0g_2) = \Delta^{n_1+n_2-2k}(z, g_0)\,R_k(z\cdot g_0),$$

and our statement is proved.

The rule of transformation proved for the case of vectors of the special form (57) extends additively to the whole space $\mathscr{E}$. To verify the normalization factor $k!$ it is sufficient to apply U_k to the dominant vector $(z_2 - z_1)^k$ and to use the identity

$$C^k_{n-k+1} = \sum_{s_1+s_2=k} C^{s_2}_{n_1-s_1}\, C^{s_1}_{n_2-s_2},$$

which is easily obtained from relations of the form

$$(1+t)^N[(1+t)^p - (t+\varepsilon)^p] =$$
$$= (1-\varepsilon)[(1+t)^{N+p-1} + (1+t)^{N+p-2}(t+\varepsilon) + \ldots + (1+t)^N(t+\varepsilon)^{p-1}].$$

This completes the proof of the theorem.

3. Another form of the result. In the notation of 1. we have obtained the operator S as a "superposition" of the linear differential operators U_0, U_1, ..., U_m; as regards the inverse mapping S^{-1}, we only know the rule (55) of its application to an element of the canonical basis of $\mathscr{E}^0$; however, it is not difficult to find its expression as a whole; furthermore it turns out that S^{-1} is also a differential operator.

It is most convenient to proceed in the following way. The polynomials of $\mathscr{E}$ are represented in the form

$$f(z_1, z_2) = \sum_{i=0}^{n_2} (z_2 - z_1)^i a_i(z_1),$$

where $a_i(z)$ are polynomials in z of degree not exceeding n_1 (similarly they can be expanded by functions of the single variable z_2); it is convenient here to assume that $n_2 \leqslant n_1$, i.e. $m = n_2$. The correspondence $\mathscr{E} \longrightarrow \mathscr{E}^0$ is now interpreted as a correspondence between the vector functions

$$a(z) = (a_0(z),\ a_1(z),\ \ldots,\ a_m(z))$$

on the complex plane Z and the vector functions (54) of the space $\mathscr{E}^0$ that have the same number of components. From Theorem 5 it now follows that *the correspondence*

$$\varphi = Sa$$

is realized by a triangular matrix S having units along the principal diagonal

$$\varphi_i = \sum_{j=0}^{i} \pi_{ij} D_{i-j} a_j, \tag{59}$$

where the symbol D_l denotes the differential operator

$$D_l = \frac{1}{l!}\left(\frac{d}{dz}\right)^l, \quad D_0 = 1,$$

and the coefficients π_{ij} are computed by the formula

$$\pi_{ij} = (-1)^{i-j} \prod_{\nu=1}^{i-j} \frac{(m-\nu-j+1)}{(n-\nu-i-j+2)}.$$

It is remarkable that we have obtained *recurrence* relations; it is not difficult to invert them and to obtain similar expressions for the transformation $a = S^{-1}\varphi$

$$a_i = \sum_{j=0}^{i} \tilde{\pi}_{ij} D_{i-j} \varphi_j, \tag{60}$$

where

$$\tilde{\pi}_{ij} = \prod_{\nu=1}^{i-j} \frac{(m-\nu-j+1)}{(n-\nu-2j+1)}.$$

COROLLARY. *The decomposition of the representation* (51) *into irreducible representations can be realized for an appropriate choice of the parameters by means of the recurrence relations* (59) *and* (60).

Returning to the previous system of parametrization we can obtain from (60) the general formula

$$f(z_1, z_2) = \sum_{i=0}^{n_2} (z_2 - z_1)^i \sum_{j=0}^{i} \frac{(n-i-j)!}{(n-2j)!} C_{n_1-j}^{i-j} \frac{d^{i-j}}{dz_1^{i-j}} \varphi_j(z_1);$$

note that the coefficients π_{ij}, $\tilde{\pi}_{ij}$ can also be written in the form

$$\pi_{ij} = (-1)^{i-j} \frac{(n-2i+1)!\,(m-j)!}{(n-i-j+1)!\,(m-i)!}, \quad \tilde{\pi}_{ij} = \frac{(n-i-j)!\,(m-j)!}{(n-2j)!\,(m-i)!}, \tag{61}$$

where m and n are defined by (52). Finally, we write down the formulae (59) and (60) for small values of the indices:

$$\begin{aligned}
a_0 &= \varphi_0, \\
a_1 &= \varphi_1 - \frac{m}{n} D_1 \varphi_0, \\
a_2 &= \varphi_2 - \frac{m-1}{n-2} D_1 \varphi_1 + \frac{m(m-1)}{(n-1)(n-2)} D_2 \varphi_0, \\
\varphi_0 &= a_0, \\
\varphi_1 &= a_1 + \frac{m}{n} D_1 a_0, \\
\varphi_2 &= a_2 + \frac{m-1}{n-2} D_1 a_1 + \frac{m(m-1)}{n(n-1)} D_2 a_0.
\end{aligned}$$

It is remarkable that the formulae (60) and (61) enable us to go over immediately to the discussion of the case when *one* of the factors $(\mathfrak{D}^{n_1})$ is replaced by an *infinite-dimensional* representation. This theme goes beyond the framework of our paper, however we mention that every infinite-dimensional irreducible representation of the Lorentz group again has the form $\mathfrak{D}^p\bar{\mathfrak{D}}^q$, where p and q are arbitrary complex numbers whose difference must be an integer, and can be realized[1] in a certain normed space of functions on the plane Z that we denote by $L^2_{pq}(Z)$. The representation space $\mathscr{E}$ (51) contains this time the functions $f(z_1, z_2)$ that are as before polynomials in z_2 and belong to the space $L^2_{pq}(Z)$ as functions of the variable z_1. Then we have the following theorem.

THEOREM 6. *The representation* $\mathfrak{D}^p\bar{\mathfrak{D}}^q \times \mathfrak{D}^{n_2}$ *has the decomposition*

$$\mathfrak{D}^{p+n_2}\bar{\mathfrak{D}}^q + \mathfrak{D}^{p+n_2-2}\bar{\mathfrak{D}}^q + \ldots + \mathfrak{D}^{p-n_2}\bar{\mathfrak{D}}^q,$$

which is realized by the mutually inverse formulae (59) *and* (60), *where the operators* $\mathfrak{D}_l$ *are defined on certain everywhere dense linear manifolds and the coefficients* π_{ij}, $\tilde{\pi}_{ij}$ *are obtained from* (61) *by analytic continuation with respect to the parameter* $\rho = p + n_2$ *(i.e. by replacing the factorial by the gamma function); the space* $\mathscr{E}$ *is mapped here onto the direct sum of the corresponding spaces* $L^2_{p'q}(Z)$, $p' = p + n_2 - 2k$.

M.A. Naimark in a number of papers[2] has studied the product of two *infinite-dimensional unitary* representations of the Lorentz group; the role of the mappings S, S^{-1} is played here by certain *integral* operators; Naimark's integrals have the same relation to our differential operators as Abel's integrals to the usual differential operator $\left(\frac{d}{dx}\right)^m$; the transition to the finite-dimensional case can be effected[3] by way of regularizing Naimark's integral by the methods of the theory of generalized functions.

§11. Decomposition of tensor representations

This problem can be called classical; it consists in studying the transformations that the tensors of all possible ranks over the original n-dimensional vector space undergo, for example relative to the group $GL(n)$. The spectral analysis shows that for every integral signature $\tau = (m_1, \ldots, m_n)$ we can find an irreducibly invariant class of tensors that are transformed according to τ. In fact, this is the way on which the direct algebraic construction of the irreducible representations of classical groups was first obtained.[4] In particular, with a definite convention on the meaning of the prefixes "co" and "contra", representations with positive signature ($m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n \geqslant 0$) correspond to contravariant tensors of rank $m = m_1 + m_2 + \ldots + m_n$ and representations with negative signature ($0 \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n$) to covariant tensors of rank $|m|$. Thus, the tensor laws of transformation are "sufficiently rich"; and this fact, in the expression of H. Weyl, is the real group-theoretical justification of the tensor calculus.

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1 D.P. Zhelobenko, DAN 126, no. 3 (1959).

2 DAN 119, no. 5 (1958); 125, no. 6 (1958); 130, no. 2 (1958), 132, no. 5 (1958); Trudy Mosc. Matem. obšč 8 (1959); 9 (1960); 10 (1961).

3 This was done recently by L.Ya. Oblomskaya who obtained as a result the formulae (59) and (60).

4 In the case of the orthogonal group a new class of objects has to be introduced - spinors and "spin-tensors".

1. **Statement of the problem.** We fix our attention to the case of the full linear group $GL(n)$. Instead of tensors it is convenient to consider *multilinear forms*: as a basis we take a "covariant" vector x – a row of n complex numbers $x_1, \ldots, x_n$, and in the space of linear forms $l(x) = l_1x_1 + \ldots + l_nx_n$ we define the representation

$$T_g\, l(x) = l\,(xg),$$

where the product xg is to be understood as the usual multiplication of matrices, the first of which consists only of a single row. From the equation

$$(x_1, x_2, \ldots, x_n)\begin{pmatrix} 1 & z_{12} & z_{13} & \ldots & z_{1n} \\ & 1 & z_{23} & \ldots & z_{2n} \\ & & 1 & \ldots & z_{3n} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = (x_1,\, x_1z_{12}+x_2,\, \ldots,\, x_1z_{1n}+x_2z_{2n}+\ldots+x_n) \quad (62)$$

it is obvious that the only form invariant under the subgroup Z is the form $\omega_1(x) = x_1$ which has the "weight" δ_1 so that our representation has the signature Δ_1.

The tensor algebra Π is now constructed from all (contravariant) multilinear forms of an arbitrary number of vector arguments $\overset{1}{x}, \overset{2}{x}, \ldots, \overset{m}{x}$:

$$\pi(\overset{1}{x}, \overset{2}{x}, \ldots, \overset{m}{x}) = \sum_{i_1 i_2 \ldots i_m} \pi^{i_1 i_2 \ldots i_m}\, \overset{1}{x}_{i_1} \overset{2}{x}_{i_2} \ldots \overset{m}{x}_{i_m}, \qquad (63)$$

and the representation is extended to multilinear forms by the rule

$$T_g\pi(\overset{1}{x}, \overset{2}{x}, \ldots, \overset{m}{x}) = \pi(\overset{1}{x}g, \overset{2}{x}g, \ldots, \overset{m}{x}g). \qquad (64)$$

The forms of rank m form an invariant subspace Π_m:

$$\Pi_m = \Pi_1 \times \Pi_1 \times \ldots \times \Pi_1 \qquad (m \text{ factors})$$

in which the representation is the m-th tensor power of the *primary* representation Δ_1:

$$\Delta_1^{\otimes m} = \Delta_1 \times \Delta_1 \times \ldots \times \Delta_1. \qquad (65)$$

Our task consists in the decomposition of (65) into irreducible parts; the result is well known (H. Weyl [1]); however, by the method of Z-invariants it becomes ostensibly simpler.

2. **Spectral analysis.** The invariants of the subgroup Z form a subalgebra of Π that we denote by Ω; these so-called "vector" invariants of the triangular group Z are well known.

We form n skew-symmetric multilinear forms of rank $1, 2, \ldots, n$:

$$\omega_1(x) = x_1,$$

$$\omega_2(x, y) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix},$$

$$\dots\dots\dots\dots$$

$$\omega_n(x, y, \dots, w) = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{vmatrix};$$

occasionally we shall use for ω_p the abbreviated notation $[\overset{1}{x}\overset{2}{x} \dots \overset{p}{x}]$. The invariance of $\omega_1, \dots, \omega_n$ under z is visibly demonstrated by the formula (62).

LEMMA 11. *The forms $\omega_1, \omega_2, \dots, \omega_n$ are generators of the algebra Ω.*

In other words, every form of Ω is a linear combination of the monomials

$$\mathbf{s}\omega_1^{r_1}\omega_2^{r_2} \dots \omega_n^{r_n}, \tag{66}$$

where $r_1, \dots, r_n$ are non-negative integers and the symbol $\mathbf{s}$ *denotes an arbitrary substitution of the vector arguments $\overset{i}{x}$ into the various factors $\omega_1, \dots, \omega_n$.*[1]

The proof can be found in the beginning of the book by H. Weyl (p. 74) and is based on the formal apparatus of the theory of invariants (Capelli identities).

The combination of this lemma with the theorem on complete reducibility permits us to state that *we have found the decomposition of the representation T_g into representations that are multiples of irreducible ones.* For every monomial $\mathbf{s}\,\omega_1^{r_1} \dots \omega_n^{r_n}$ has a weight, namely $\delta_1^{r_1+r_2+\dots+r_n}\delta_2^{r_2+\dots+r_n} \dots \delta_n^{r_n}$, and therefore has a dominant vector that generates the representation

$$\Delta_1^{r_1}\Delta_2^{r_2} \dots \Delta_n^{r_n};$$

owing to the complete reducibility we can write

$$\Pi = \sum_{m_1 \geqslant m_2 \geqslant \dots \geqslant m_n \geqslant 0} \Pi_{m_1 m_2 \dots m_n},$$

where $\Pi_\tau = \Pi_{m_1 m_2 \dots m_n}$ is the maximal subspace in which the representations with the given signature τ are concentrated (all m_i are integers).

In order to determine the multiplicity with which τ occurs in the decomposition we have to find the dimension of the subspace $\Omega_\tau = \Omega_{m_1 m_2 \dots m_n}$ consisting of the dominant vectors of given weight,

[1] We recall that multiplication in a tensor algebra is non-commutative: when we multiply a form π_1 of degree m_1 on the left by a form π_2 of degree m_2 we obtain a form $\pi_1(x, y, \dots)\,\pi_2(u, v, \dots)$ of degree $m_1 + m_2$, and the product taken in the opposite order differs from this by the arrangement of the arguments. The introduction of the operator $\mathbf{s}$ makes it unnecessary for us to write down the factors in various orders.

i.e. to find the number of linearly independent monomials among (66). We prefer a roundabout way and prove the recurrence formula

$$k_{m_1 m_2 \dots m_n} = k_{m_1-1,\, m_2 \dots m_n} + k_{m_1,\, m_2-1, \dots,\, m_n} + \dots + k_{m_1 m_2 \dots m_n - 1}, \tag{67}$$

where $k_{m_1 \dots m_n}$ denotes the multiplicity with which the representation $(m_1, \dots, m_n)$ is contained in Π. (Note that owing to the graduation by degrees of homogeneity only those representations coincide in the subspace Π_m for which the "complete signature" $m_1 + \dots + m_n$ is equal to m.) In particular,

$$\Delta_1 = \Delta_1,$$
$$\Delta_1 \times \Delta_1 = \Delta_1^2 + \Delta_2,$$
$$\Delta_1 \times \Delta_1 \times \Delta_1 = \Delta_1^3 + 2\Delta_1\Delta_2 + \Delta_3,$$
$$\Delta_1 \times \Delta_1 \times \Delta_1 \times \Delta_1 = \Delta_1^4 + 3\Delta_1^2\Delta_2 + 2\Delta_2^2 + 3\Delta_1\Delta_3 + \Delta_4,$$

and so on (Δ_p is replaced by zero if p exceeds n); we recall that the parameters m_i, r_i in the signatures $\Delta_1^{r_1} \dots \Delta_n^{r_n}$ are connected by the relations (22). For $n = 2$ (we shall talk of the Lorentz group) our formula is similar to the usual method of computing binomial coefficients:

m \ k	0	1/2	1	3/2	2	5/2
1		1				
2	1		1			
3		2		1		
4	2		3		1	
5		5		4		1

where the number in the intersection of the row numbered m and the column numbered $k = \frac{l}{2}$ determines the multiplicity with which Δ_1^l is contained in the m-th tensor power of the primary representation Δ_1 (if the table is completed by zeros to the left of the vertical boundary, then every number is the sum of the two adjacent numbers in the preceding row).

Let us make a few special remarks.

1°. The individual generator ω_p has the weight $\delta_1\delta_2 \dots \delta_p$ and generates, therefore, the Cartan "dominant" Δ_p: the representation in the space of multivectors of rank p which in this case is realized in the space spanned by the basis elements

$$x_{i_1 i_2 \dots i_p} = \begin{vmatrix} \overset{1}{x}_{i_1} & \overset{1}{x}_{i_2} & \dots & \overset{1}{x}_{i_p} \\ \cdot & \cdot & \cdot & \cdot \\ \overset{p}{x}_{i_1} & \overset{p}{x}_{i_2} & \dots & \overset{p}{x}_{i_p} \end{vmatrix}, \quad i_1 < i_2 < \dots < i_p$$

(for the proof we remark that the vectors $T_g\,\omega_p$ homologous to ω_p split with respect to these minors; on the other hand, in the linear envelope of these minors there is, apart from ω_p, no invariant of Z, skew-symmetric in all the arguments, of rank p); this then gives a description of the subspaces of the form $\Pi_{11\ldots10\ldots0}$.

2°. The decomposition $\Delta_1 \times \Delta_1 = \Delta_1^2 + \Delta_2$ corresponds to the fact that in Π_2 we can choose a basis of symmetrized and alternating products:

$$\frac{1}{2}(x_i y_j + x_j y_i) \text{ and } \frac{1}{2}(x_i y_j - x_j y_i);$$

for the proof we verify that the vectors $T_g\omega_1^2$ $(T_g\omega_2)$ are decomposed only with respect to vectors of the first (second) type. Similarly in the decomposition $\Delta_1^3 + 2\Delta_1\Delta_2 + \Delta_3$ the summands at the end are interpreted as completely symmetrized and completely alternating components, and the tensors that are transformed according to the rule $\Delta_1\Delta_2$ form a certain intermediate type of symmetry.

3. **The symmetric group and its group ring.** Below it will be shown that irreducibly invariant classes of tensors may differ by their symmetry type; here the *symmetric group* $S = S(m)$ comes into action, which consists of the permutations of the abstract elements 1, 2, ...m.

By interpreting the symbol $\begin{pmatrix} i_1 & i_2 & \ldots & i_m \\ j_1 & j_2 & \ldots & j_m \end{pmatrix}$ as the permutation that carries j_α into i_α, we obtain a notation for the composition that recalls the matrix rule of multiplication

$$\begin{pmatrix} i_1 & i_2 & \ldots & i_m \\ j_1 & j_2 & \ldots & j_m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & \ldots & j_m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix} = \begin{pmatrix} i_1 & i_2 & \ldots & i_m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix};$$

the operations defined in Π_m by the formula $\pi \to \mathrm{s}\pi$, where

$$\mathrm{s}\pi(\overset{1}{x}, \overset{2}{x}, \ldots, \overset{m}{x}) = \pi(\overset{i_1}{x}, \overset{i_2}{x}, \ldots, \overset{i_m}{x}) \quad \text{for } \mathrm{s} = \begin{pmatrix} 1 & 2 & \ldots & m \\ i_1 & i_2 & \ldots & i_m \end{pmatrix}$$

form a *representation* of S (obviously the order of composition is preserved). As we wish to have a more flexible apparatus, we go over to the representation of the *group ring* $\mathbf{A}(S)$ of S, i.e. we form the linear combinations

$$\mathbf{a} = \sum_{s \in S} a(s)\,\mathrm{s}, \tag{68}$$

where $a(s)$ is a numerical function on the (finite) group S. (We recall that the group ring consists, by definition, of all such functions $a(s)$ and that the convolution

$$a_1 a_2(s) = \sum_{t \in S} a_1(st^{-1})\, a_2(t)$$

defines the composition in this ring; since it preserves all ring operations, the mapping $a \to \mathbf{a}$ is, in fact, a representation.)

The role of the symmetric group in our immediate problem is explained by the fact that the *operators* T_g, acting on each vector argument independently, *commute with all permutations*; this enables us to construct a subspace invariant with respect to T_g by means of equations of the form

$$\mathbf{a}_0\pi = \lambda_0\pi,$$

where $\mathbf{a}_0$ is a fixed element of $\mathbf{A}(S)$ and λ_0 a fixed number. It is remarkable that on these lines one attains even the detailed decomposition of

Π_m into irreducibly invariant subspaces;[1] we shall base the proof directly on the apparatus of *Young symmetrizers*. We recall their definition.

We start from an arbitrary partition of the number of objects (m) into positive summands $m_1, m_2, \ldots, m_p$ which are arranged in decreasing order:

$$m_1 \geqslant m_p \geqslant \ldots \geqslant m_p;$$

with each such "signature" we associate a *Young diagram* which is a cell-like figure of m_1 cells in the first row, m_2 in the second, and so on. By distributing the objects 1, 2, ..., m on the fields of such a diagram we obtain a table which we shall call *Young tableau* and denote by the symbol

$$Y(1,\ 2,\ \ldots,\ m);$$

every other variant of the distribution on the fields of one and the same diagram can be represented by the symbol sY, i.e. can be obtained from the original by means of a permutation of the objects according to the rule:

$$sY(1,\ 2,\ \ldots,\ m) = Y(i_1,\ i_2,\ \ldots,\ i_m) \quad \text{for} \quad s = \begin{pmatrix} 1 & 2 & \ldots & m \\ i_1 & i_2 & \ldots & i_m \end{pmatrix}$$

(in the tableau sY the objects $i_1, \ldots, i_m$ stand in the fields that were occupied in the previous tableau Y by the objects 1, 2, ..., m, respectively).

Fixing one of the tableaux Y we select in the group S the subgroup $P = P(Y)$ whose elements permute only objects standing in one and the same row of the tableau Y, and similarly the subgroup $Q = Q(Y)$ consisting of the vertical shifts - permutations within the columns of the tableau Y. The element of the group ring

$$d_Y = (\Sigma \pm q)\ (\Sigma p) = \Sigma \pm qp,$$

where the sums are taken over Q and P and the sign $\pm$ is determined by the parity of the permutation q, is called *Young symmetrizer* corresponding to the tableau Y (the corresponding operator $\mathbf{d}_Y$ in Π_m produces symmetrization with respect to the rows and a subsequent alternation with respect to the columns). The tableau sY corresponds to the symmetrizer sd_Ys^{-1}; by averaging over all permutations of each "signature" $\mu = (m_1, m_2, \ldots, m_p)$ we form the *central symmetrizer*

$$\varepsilon_\mu = \frac{1}{l^2} \sum_{s \in S} s d_Y s^{-1}, \tag{69}$$

which, obviously, is permutable with all permutations and therefore belongs to the centre of the group ring $\mathbf{A}(S)$.

An element e_0 of a ring is called *idempotent* if $e_0^2 = e_0$. It is known that the Young symmetrizer is idempotent apart from normalization

$$d_Y d_Y = l d_Y,$$

[1] The principal result is given by the theory of "duality": the problem reduces to a verification of the fact that the ring $\mathbf{A}(S)$ of operators (68) and the so-called enveloping algebra $\mathbf{A}(G)$ of the operators T_g form *mutually commuting* sets; this is the course of explanation taken in the book by H. Weyl.

so that $e_Y = \frac{1}{l} d_Y$ is an idempotent; precisely this constant l is also used in the definition (69).

The central symmetrizers ε are also idempotents, and if the tableau Y is constructed for the signature μ, then

$$\varepsilon_\mu d_Y = d_Y \varepsilon_\mu = d_Y,$$

i.e. d_Y is "subordinate" to ε_μ; a remarkable property of the idempotents ε_μ is their "orthogonality":

$$\varepsilon_\mu \varepsilon_\nu = 0, \quad \text{if} \quad \mu \neq \nu;$$

finally, the collection of all idempotents ε_μ corresponding to the various signatures, i.e. to all possible partitions of the number m, form the "decomposition of the unit element" in the group ring $A(S)$ (we shall not make use of the latter properties). Proofs are given, for example, in the book of H. Weyl and are based on a simple combinatorial lemma that can also be applied directly to advantage.

LEMMA 12. *Suppose that the tableaux Y and Y' are constructed for the diagrams with the signatures μ and μ', respectively.*

If $\mu > \mu'$, then we can find among the objects 1, 2, ..., m at least one pair α, β such that α and β stand in one and the same row of Y and in one and the same column of Y'.

If $\mu = \mu'$, then there is an alternative: either such a pair exists, or we can find permutations $p \in P(Y)$ and $q' \in Q(Y')$ for which the following relation holds

$$q' Y' = pY .$$[1]

We mention one further obvious property of the Young symmetrizers:

$$d_Y p = d_Y, \quad q d_Y = \pm d_Y \tag{70}$$

for arbitrary elements $p \in P$, $q \in Q$, where the sign $\pm$ is determined by the parity of q.

4. Spectral synthesis. In the mapping of the group ring into the ring of operators (68) the image of every idempotent is a projection operator $e_0^2 = e_0$. It remains for us to verify the almost obvious fact that the image of a central symmetrizer ε_μ for $\mu = (m_1, \ldots, m_n)$ is a projection operator on the subspace $\Pi_{m_1 \ldots m_n}$

$$\Pi_{m_1 m_2 \ldots m_n} = \varepsilon_\mu \Pi_m. \tag{71}$$

For the proof *it is sufficient to restrict ourselves to a discussion of the invariants*, i.e. to verify that

$$\Omega_{m_1 m_2 \ldots m_n} = \varepsilon_\mu \Omega_m,$$

where $\Omega_m = \Omega \cap \Pi_m$ is the collection of all Z-invariant forms of rank m; from this (71) follows automatically; for the subspace $\varepsilon_\mu \Pi_m$ consisting of all the elements of Π_m for which $\varepsilon_\mu \pi = \pi$ is *necessarily invariant* under the representation T_g, consequently this subspace is completely determined by the set of its dominant vectors.

[1] The order of signatures is to be understood as their lexicographic order: see end of §4. Note that the statement of the last condition of the lemma appears outwardly to differ from Weyl's; this is explained by different conventions on symbolism.

The formula (71) also yields the spectral synthesis; for every vector $\pi \in \Pi_m$ determines its projection $\varepsilon_\mu \pi$ onto $\Pi_{m_1 m_2 \dots m_n}$; in the further process of detailing the decomposition within $\Pi_{m_1 m_2 \dots m_n}$ the projectors d_Y naturally come into play; however, among the symmetrizers d_Y there are no "orthogonality relations"; hence it follows that the subspaces $d_{sY}\Pi_m$ for various s may be linearly dependent; in other words, the decomposition of a multiple representation into irreducible ones is necessarily non-unique and this accounts for the general difficulty of the process of "detailing".

Among the various "signatures" $(m_1, m_2, \dots, m_p)$ of the abstract scheme under 3. we consider only those for which $p \leqslant n$ and we call such signatures *admissible*. If Y is one of the tableaux with an admissible signature μ, then we associate with it a dominant vector ω_Y in the following way: to every column of the tableau Y of length p containing the indices $i_1, i_2, \dots, i_p$ there corresponds the factor

$$\omega_p = \omega_p(\overset{i_1}{x}, \overset{i_2}{x}, \dots, \overset{i_p}{x});$$

in other words, by writing down the signature in the form $\Delta_1^{r_1} \dots \Delta_n^{r_n}$ we associate with one of the tableaux Y the vector

$$\omega_Y = \omega_1^{r_1}\omega_2^{r_2} \dots \omega_n^{r_n},$$

and with every tableau of the form sY the vector $s\,\omega_Y$. Our Z-invariants are, as it were, "living Young diagrams"; note that they are *skew-symmetric* in the indices of each column. Going over from one signature to another we obtain dominant vectors with distinct "symmetry types", in particular, to the most dominant of all possible signatures $(m, 0, 0, \dots, 0)$ there corresponds the *totally symmetrized* form ω_1^m, and to the least dominant the form with the greatest expression of *skew-symmetry*. This property suggests the proof of the following theorem.

THEOREM 7. *The maximal subspace Π_τ on which the multiple representation $\tau = \Delta_1^{r_1}\Delta_2^{r_2} \dots \Delta_n^{r_n}$ acts can be obtained from the forms $\omega_1^{r_1}\omega_2^{r_2} \dots \omega_n^{r_n}$ by applying the operations T_g, the permutations s, and by taking linear combinations.*

A projection operator on Π_τ in the space of homogeneous forms of degree m is the central symmetrizer ε_τ; the Young symmetrizers corresponding to all possible tableaux Y with the signature τ project onto irreducible subspaces of Π_τ.

The multiplicity of the irreducible representation with the signature $(m_1, m_2, \dots, m_n)$ is computed by the formula (67).

PROOF. (in outline). 1°. To begin with we verify that for every admissible signature $\varepsilon_\mu \neq 0$; for this purpose we choose an arbitrary tableau Y with the signature μ and define a form π_0 on the basis vectors $e_1, e_2, \dots, e_n$ by the rule

$$\pi_0\ (\underbrace{e_1, \dots, e_1}_{r_1};\ \underbrace{e_2, \dots, e_2}_{r_2};\ \dots;\ \underbrace{e_n, \dots, e_n}_{r_n}) = 1;$$

$\pi_0 = \pm 1$ when the arrangement of the vectors e_i changes under the permutation $q \in Q(Y)$; $\pi_0 = 0$ in all remaining cases; then $d_Y \pi_0 = \pi_0$ and therefore also

$$\varepsilon_\mu \pi_0 = \varepsilon_\mu d_Y \pi_0 = d_Y \pi_0 = \pi_0 \neq 0.$$ [1]

2°. If $\tau > \tau'$, then it follows from Lemma 12 that $\varepsilon_\tau \Omega_{\tau'} = (0)$: for if s_0 denotes the transposition of the indices α and β that occur in this lemma, then we have for every tableau $Y = T_\tau$: $d_Y s_0 = d_Y$ (by (70)), on the other hand, $s_0 \omega_{Y'} = -\omega_{Y'}$, for the tableau $Y' = Y_{\tau'}$ ($\overset{\alpha}{x}$ and $\overset{\beta}{x}$ stand in one and the same determinant ω_p); consequently

$$d_Y \omega_{Y'} = (d_Y s_0)\,\omega_{Y'} = d_Y\,(s_0 \omega_{Y'}) = -d_Y \omega_{Y'},$$

and therefore $d_Y \omega_{Y'} = 0$ for arbitrary tableaux $Y = Y_\tau$ and $Y' = Y_{\tau'}$; as a result, $\varepsilon_\tau \omega_{Y'} = 0$, and therefore also $\varepsilon_\tau\ \Omega_{\tau'} = (0)$.

3°. In the chain

$$\Omega_{\tau_1},\ \Omega_{\tau_2},\ \ldots,\ \Omega_{\tau_N},$$

where $\tau_1 > \tau_2 > \ldots > \tau_N$, the first term is the one-dimensional subspace spanned by the form $\omega_1{}^m$; it is clear that ε_{τ_2} – *the operator of complete symmetrization* – annihilates all the terms of this chain except the first; from the "orthogonality relations" for the ε_μ we conclude *successively* that

$$\varepsilon_\tau\ \Omega_{\tau'} = (0) \quad \text{for} \quad \tau \neq \tau'; \quad \varepsilon_\tau\ \Omega_\tau = \Omega_\tau$$

(the latter follows from the fact that ε_τ is non-trivial).

4°. It can be verified that

$$d_Y\,(s\omega_Y) = \begin{cases} 0, \\ \pm\, d_Y \omega_Y \end{cases}$$

according to the alternatives of Lemma 12 for the tableaux Y and $Y' = sY$; consequently, the space $d_Y \Omega_\tau$ is *one-dimensional* and therefore generates an *irreducible* subspace of Π_τ.

This concludes the proof of the theorem.

NOTE 1. The decomposition of a representation of S in Π_m into irreducible ones runs on parallel lines: it can be shown that $\Pi_{m_1 m_2 \ldots m_n}$ is the maximal subspace in which the representation is a multiple of an irreducible one; also the subspace $\Omega_{m_1 m_2 \ldots m_n}$ is *irreducible*, and every other irreducible subspace of $\Pi_{m_1 m_2 \ldots m_n}$ is obtained from it as homologous with respect to the operations T_g and to taking linear combinations.[2] We emphasize the *duality* between the representations of G and S: the irreducible parts with respect to these groups give "mutually perpendicular" decompositions of the space $\Pi_{m_1 m_2 \ldots m_n}$.

NOTE 2. The analogous problem for $SO(n)$ and $Sp(n)$ is solved by means of the operation of "contraction" of tensors; however, it can also be effected directly: *it is sufficient to add to the invariants ω_p the "absolute invariants" – the fundamental bilinear form $(x,\ y)$* or $[x,\ y]$.

5. The universal representation $\mathfrak{T}$. In practice it is hardly convenient to use tensors as models for irreducible representations; however,

[1] In contrast, if the diagram contains more than n rows, then according to the second relation (70) every form of $d_Y \Omega_m$ is skew-symmetric in $p < n$ vectors; but every such form is identically zero, and therefore $d_Y = 0$; consequently, for "inadmissible" diagrams we always have $\varepsilon_\mu = 0$.

[2] In other words, every such subspace has the form $A\,\Omega_{m_1 m_2 \ldots m_n}$, where A is a fixed linear combination $\sum c_i T_{g_i}$.

the geometrical interdependence of the "linear quantities" is clarified in this course, owing to their construction from the basic quantities (vectors or multivectors); hence it is easy to arrive at a more convenient model. The sub-division of a tensor power into irreducible ones is explained essentially by the obvious fact that the transformations of the group $G = GL(n)$ in an n-dimensional space preserve all *incidence relations* among linear subspaces. We automatically take incidence relations into account when we construct the representation $\Delta_1^{r_1} \Delta_2^{r_2} \ldots \Delta_n^{r_n}$ by means of the generating vectors

$$[x]^{r_1} [xy]^{r_2} \ldots [xy \ldots w]^{r_n} \tag{72}$$

in a certain class of polynomials of vector arguments $x, y, \ldots, w$ – and this is in essence the embedding in the right regular representation of G.

A *generalized linear element* over the original n-dimensional space is the name for a chain of embedded linear subspaces whose dimensions differ by one (incident line, plane etc.); parametrically every linear element can be given by a system of n linearly independent vectors $x, y, \ldots, w$ so that the first p vectors determine a p-hyperplane of the element L. Replacing $x, y, \ldots, w$ by $xg, yg, \ldots, wg$, we obtain the transformation $L \to L \cdot g$ of G in the space of linear elements and the representation with dominant vectors (72) can be written by means of the formula

$$T_g p(L) = p(L \cdot g)$$

in a certain class of polynomials $p(L)$. In particular, by choosing the vectors $x, y, \ldots, w$ as mutually orthogonal in a unitary metric we obtain a realization of the representation in a class of polynomials on the unitary subgroup $U(n)$; by choosing them as truncated triangularly (in a certain basis), we obtain a realization on the group $H = DZ$, and after a necessary normalization also on the group Z.

Let us describe the model on H. Denoting by $\mathscr{P}(G)$ the space of all complex-analytic polynomials on G, we define $\mathscr{P}(H)$ by means of restriction to H of all polynomials $p(g)$ that are invariant with respect to the operation $p(g) \to p(\zeta g)$, $\zeta \in Z$, i.e. that are constant on cosets of Z. The representation

$$\mathfrak{T}_g p(h) = p(h \cdot g)$$

in $\mathscr{P}(H)$, where $h \cdot g = \tilde{h}$ is defined by the decomposition $hg = \zeta\tilde{h}$, $\zeta \in Z$, $\tilde{h} \in H$, is in a certain sense *universal*, which will be clear from the following statement.

The representation $\mathfrak{T}$ contains each irreducible representation of integral signature $m_1 \geqslant \ldots \geqslant m_n \geqslant 0$ and every one precisely once.

(The proof follows easily from the general scheme of spectral analysis: dominant vectors are the monomials $h_{11}^{m_1} h_{22}^{m_2} \ldots h_{nn}^{m_n}$, where h_{ii} are the diagonal elements of the triangular matrix $h \in H$.)

Our familiar model of an irreducible representation in polynomials on Z is obtained by means of the mapping $p(h) \to p(z)$ for the polynomials $p(h)$ from a *fixed* irreducible component of $\mathfrak{T}$; the inverse embedding is effected by the formula $p(h) = \alpha(\delta)\, p(z)$ for $h = \delta z$, where $\alpha(\delta) = \delta_1^{m_1} \ldots \delta_n^{m_n}$. It is interesting to compare the two models – the "homogeneous" and the "inhomogeneous": whereas $\mathfrak{T}$ is the collection of all irreducible representations that are realized by one general formula, the transition to the realization on Z causes a "contraction" of all dominant vectors into one single point; the spaces $\mathfrak{R}_T(Z)$ are "threaded" to this point, forming a partially ordered family under inclusion as is clear from Theorem 2.

§12. Symmetric powers

The symmetric powers of a representation T arise under symmetrization of its tensor powers; however, it is more convenient to give a somewhat different definition.

Symmetric algebra over a linear space E is the name for the algebra $\mathcal{P}(A)$ consisting of all polynomials on the dual space $A = \hat{E}$; the collection $E_m = \mathcal{P}_m(A)$ of all *homogeneous* polynomials of degree m forms the m-th *symmetric power* of E; in particular, $E_1 = E$.

Suppose that a representation of a group G acts on the space A; the transformation referring to the element $g^{-1} \in G$ will be denoted by the symbol $a \cdot g$;[1] in E there arises then the contragredient transformation

$$T_g l(a) = l(a \cdot g)$$

(here $l(a)$ is a linear form on A), which can be extended by the formula

$$S_g p(a) = p(a \cdot g) \tag{73}$$

to a representation in the infinite-dimensional space $\mathcal{P}(A)$. The representation S so obtained shall be called the "complete symmetric power" of T in E; correspondingly the m-th *symmetric power* $\sigma_m(T)$ acts on the invariant subspace $\mathcal{P}_m(A)$ consisting of all homogeneous polynomials of degree m.

For example, if $G = GL(n)$ and the space E coincides with the original n-dimensional vector space consisting of column vectors, then the space A consists of row vectors and the operation $a \cdot g$ consists in multiplication of the row a on the right by the matrix g. In this case every representation $\sigma_m(\Delta_1) = \Delta_1^m$ is irreducible. In practice the spectral analysis of symmetric powers is an important task.

For a description of the algebra Ω of all invariants of the subgroup Z it is convenient to begin with selecting a "functional basis" of this algebra; occasionally one succeeds by means of it in finding a polynomial basis, i.e. generators of Ω. In other words, by a suitable replacement of the variables we select (non-linear) parameters of the point a on the two groups: x and y so that every polynomial of Ω is distinguished by an equation $p(x, y) = p(x, y_0)$, i.e. does not depend on the variables y. The simple example to be given below will also outline the practical limitations of this method: luckily the transformations of Z do not differ "too much"

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[1] With this definition the order of composition is preserved, i.e.

$$(a \cdot g_1) \cdot g_2 = a \cdot g_1 g_2.$$

A generalization of our construction is the representation defined by formula (73) on one class of *functions* or another on a manifold A which this time need not be a linear space, and the symbol $a \cdot g$ denotes some "motion" carrying the element g into the manifold A.

from transitive ones.

1. Representations of the group $G=GL(n_1)\times GL(n_2)$. The elements of our group are pairs

$$g=(g_1,\ g_2),\quad g_1\in GL(n_1),\quad g_2\in GL(n_2),$$

with independent multiplication in each component. Obviously, G admits a Gauss decomposition (this is always the case for $G_1\times G_2$ if it is true for each factor); in particular, $D = D(n_1)\times D(n_2)$, and hence it follows that every irreducible representation of G is induced by a dominant weight of the form $\alpha_1(\delta_1)\alpha_2(\delta_2)$, where $\alpha_i(\delta_i)$ is a dominant weight for $GL(n_i)$, i.e. is determined by the double signature

$$\mu=(m_1,m_2,\ldots,m_{n_1}\,|\,l_1,\ l_2,\ \ldots,\ l_{n_2}),$$
$$m_1\geqslant m_2\geqslant\ldots\geqslant m_{n_1},\quad l_1\geqslant l_2\geqslant\ldots\geqslant l_{n_2}.$$

Denoting by A the collection of all rectangular matrices having n_1 rows and n_2 columns, we consider one of the transformations

$$a\cdot g=g_1'ag_2,\quad a\cdot g=g_1{}^*ag_2,\quad a\cdot g=g_1^{-1}ag_2, \tag{74}$$

where $g = (g_1,\ g_2)$; the corresponding dual representation of G consists of transformations of the *coordinates* of a:

$$T_g(a_{ij})=(a\cdot g)_{ij},$$

as elements of a basis in the space $E = \hat{A}$; the symmetric power $\sigma_m(T)$ is now determined by the formula (73) in the space $\mathcal{P}_m(A)$ of homogeneous polynomials of the variables a_{ij} of degree m.

To distinguish one of the three cases (74) we denote the representations T_g by the symbols $T,\ T',\ T''$, respectively; for the sake of definiteness we shall assume that $n_1\leqslant n_2$ and we set $n_1 = n$. The result of the analysis is the following theorem:

THEOREM 8. *In the above notation we have the spectral formulae*:

$$\left.\begin{aligned}
\sigma_m(T)&=\sum(m_1,m_2,\ldots,m_n\,|\,m_1,m_2,\ldots,m_n,0,0,\ldots,0),\\
\sigma_m(T')&=\sum(\overline{m_1,m_2,\ldots,m_n}\,|\,m_1,m_2,\ldots,m_n,0,0,\ldots,0),\\
\sigma_m(T'')&=\sum(-m_n,\ldots,-m_2,-m_1\,|\,m_1,m_2,\ldots,m_n,0,0,\ldots,0),
\end{aligned}\right\} \tag{75}$$

where each sum is taken over all sets of integers $m_1,\ m_2,\ \ldots,\ m_n$ for which

$$m_1\geqslant m_2\geqslant\ldots\geqslant m_n\geqslant 0,$$
$$m_1+m_2+\ldots+m_n=m;$$

a system of dominant vectors will be indicated in the course of the proof.

Consequently, every irreducible representation occurs in the decomposition singly and the spectrum contains only such signatures $(m_1,\ \ldots,\ m_{n_1}\,|\,l_1,\ \ldots,\ l_{n_2})$, *for which* $l_{n_1+1} = l_{n_1+1} = \ldots = l_{n_2} = 0$.

PROOF. 1°. Rectangular matrices "almost always" admit a triangular decomposition of the form $a = \zeta\lambda z$, where $\zeta\in Z(n_1)$, $z\in Z(n_2)$ and λ is a truncated diagonal matrix

$$\lambda=\begin{pmatrix}\lambda_1 & 0 & \ldots & 0 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0 & 0 & \ldots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & \lambda_n & 0 & \ldots & 0\end{pmatrix};$$

a proof can be given, as in §4, by the multiplication law for minors; in particular, $\lambda_p(a) = \frac{\Delta_p}{\Delta_{p-1}}$ and the decomposition is valid when none of the diagonal minors $\Delta_p(a)$ of a is equal to zero.

Hence it follows that the functions $\lambda_1(a), \ldots, \lambda_n(a)$ form a functional basis for the algebra Ω in the case of the representations T and T'. Suppose, for example, that the polynomial $p(a)$ is invariant under T_z; then every matrix a_0 having the form $\zeta_0 \lambda_0 \grave{z}_0$ satisfies the equation

$$p(a_0) = p(\lambda_0) \tag{76}$$

(it is sufficient to consider the element $z = (z_1, z_2)$, for $z_1 = \zeta_0'^{-1}$, $z_2 = z_0^{-1}$) and $p(a)$ is represented in the form of a *rational* function of the principal minors $\Delta_i(a)$ of a; the invariant polynomials can now be represented in the form of a sum of monomials

$$\omega_\nu(a) = \lambda_1^{\nu_1} \lambda_2^{\nu_2} \ldots \lambda_n^{\nu_n}, \tag{77}$$

where $\lambda_i = \lambda_i(a)$ and $\omega_\nu(a)$ is a weight vector with respect to the subgroup D and the corresponding weight has the exponents

$$(\nu_1, \nu_2, \ldots, \nu_n \mid \nu_1, \nu_2, \ldots, \nu_n, 0, 0, \ldots, 0). \tag{78}$$

It is essential that all such weights are *distinct*: this enables us to conclude that every dominant vector is precisely a *monomial* of the form (77) (and not a linear combination of such monomials); finally, the dominant vector $\omega_\nu(a)$ generates a representation of G with the signature (78), and hence it follows that all exponents are necessarily integers satisfying the order relations

$$\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_n \geqslant 0$$

(here the last condition $\nu_n \geqslant 0$ and the fact that ν_n is an integer follow from the polynomial character of (77) or can be proved by restriction to the unimodular subgroup).

2°. In studying T'' we modify the triangular decomposition by carrying it into the form

$$a = z_1 s \lambda z_2,$$

where λ has the previous form, $z_1 \in Z(n_1)$, $z_2 \in Z(n_2)$ and s is the permutation matrix (33): such a decomposition follows from the usual decomposition of the matrix $b = s^{-1}a$, because $sZ(n_1)s^{-1} = Z(n_1)$; the rest of the proof goes similarly.

The dominant vectors in the cases T and T' have the form $\Delta_1^{r_1}(a) \ldots \Delta_n^{r_n}(a)$, where all the r_i are non-negative integers, and in the case T'' Δ_p is replaced by the minor formed from the p first rows and the p last columns of a. This completes the proof of the theorem.

The spectral formula for $\sigma_m(T)$ was obtained by Loo-Keng Hua [22] by the method of characters.

NOTE. Our results have an important interpretation. Let us consider the case of square matrices $(n_1 = n_2 = n)$ and denote by $\mathscr{P}_{m_1 m_2 \ldots m_n}$ the linear subspace of $\mathscr{P}(A)$ spanned by the matrix elements

$$\tau_{ij}(a), \quad 1 \leqslant i, \ j \leqslant \dim \tau$$

of the representation τ of $GL(n)$ with the signature $(m_1, m_2, \ldots, m_n)$; these functions are defined in the first instance only for det $a \neq 0$, but can be extended by continuity to arbitrary values of a. To fix our ideas we shall talk, for example, of the representation T''. The multiplication formula for the matrix elements τ_{ij} indicates that every $\mathscr{P}_{m_1 m_2 \ldots, m_n}$ is invariant with respect to T'', and from Theorem 8 we conclude its irreduciblity. Moreover, Theorem 8 contains a statement on the completeness of the system of all matrix elements.

COROLLARY. *Every polynomial in the variables $a_{pq}(1 \leqslant p, q \leqslant n)$ can be represented in the form of a linear combination of the matrix elements $\tau_{ij}(a)$, corresponding to all possible integral signatures $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n \geqslant 0$.*

From this there follow approximation theorems for wider classes of functions $f(a)$ or for functions $f(g)$ on G; thus, for *continuous* functions $f(a)$ we obtain, by means of Weierstrass' Theorem, the *uniform approximability on every compactum* $Q \subset A$ by means of linear combinations of matrix elements; setting, in particular, $Q = U(n)$ we obtain the *Peter-Weyl Theorem* for the group $U(n)$,[1] and this in turn implies that the functions on the group $U(n)$ can be developed in Fourier series with respect to the elements $\tau_{ij}(u)$ (converging in mean). If a function (for simplicity, a polynomial) is a *class function*, i.e. is invariant under the transformations $a \to h^{-1}ah$, $h \in GL(n)$, then it can be expanded by the *characters* (traces) of all possible representations τ: for by Schur's Lemma [1] the space $\mathscr{P}_{m_1 m_2 \ldots m_n}$ contains only one class function - the trace $\chi_\tau(a)$ of the representation $(m_1 m_2, \ldots, m_n)$.[2]

2. Some related examples. Let A denote the set of all square matrices of order n. We consider in the group $G = GL(n) \times GL(n)$ the subgroup G_0, isomorphic to $GL(n)$, consisting of all pairs of the form (g, g); if we restrict the transformation (74) to G_0, then the representations T, T', T'' go over into representations of $GL(n)$ with the signatures

$$\Delta_1 \times \Delta_1, \ \bar{\Delta}_1 \times \Delta_1, \quad \hat{\Delta}_1 \times \Delta_1$$

respectively, (where $\times$ denotes the tensor product, - transition to the complex-conjugate, and the cap ^ transition to the contragredient representation); in particular, $\Delta_1 \times \Delta_2$ is the "adjoint" representation of $GL(n)$. From Theorem 8 we obtain:

Example 1. $\sigma_m(\Delta_1 \times \Delta_1) = \sum \tau \times \tau$.

Example 2. $\sigma_m(\bar{\Delta}_1 \times \Delta_1) = \sum \bar{\tau} \times \tau$.

Example 3. $\sigma_m(\hat{\Delta}_1 \times \Delta_1) = \sum \hat{\tau} \times \tau$.

1 We remark that by a similar method we can obtain the Peter-Weyl Theorem for *every* compact Lie group; in contrast to the usual "transcendental" proof this proof is purely algebraic.

2 We recall another well-known corollary: every class function can be developed by the elementary symmetric functions $\sigma_i(a)$ - the traces of the representations Δ_i $(i = 1, 2, \ldots, n)$; for the proof it is sufficient to consider the representation $W = W(\tau)$ defined at the end of §4, and to observe that

$$Sp(W) = Sp(\Delta_1 \times \ldots \times \Delta_n) = \sigma_1^{r_1} \sigma_2^{r_2} \ldots \sigma_n^{r_n};$$

besides there are recurrence relations between τ and $W(\tau)$ which enable us to conclude that the trace χ_τ can be expressed recurrently in terms of the monomials $\sigma_1^{r_1}\sigma_2^{r_2}, \ldots, \sigma_n^{r_n}$.

Here the sum is taken each time over all integral signatures $\tau = (m_1, m_2, \ldots, m_n)$ for which $m_1 + \ldots + m_n = m$, $m_n \geqslant 0$; the second of these formulae gives a definite result; the other two reduce the task to the decomposition of representations of the form $\tau \times \tau$ and $\hat{\tau} \times \tau$, respectively, into irreducible representations.

When we replace A by the space of all symmetric or the space of all skew-symmetric matrices of order n, we easily obtain the following spectral decompositions which have applications in "harmonic analysis" of functions of several complex variables.

Example 4. $\sigma_m(\Delta_1^2) = \sum (2m_1, 2m_2, \ldots, 2m_n)$.

Example 5. $\sigma_m(\Delta_2) = \sum (m_1, m_1, m_2, m_2, \ldots)$.

(Here we have the previous summation rule in mind.)

The proof in Example 4 follows (by the scheme of the proof of Theorem 8) from the decomposition $a = z'\lambda z$, $\lambda \in D(n)$, $z \in Z(n)$ which holds for *symmetric* matrices a; similarly, in Example 5, by using the partition into blocks of order two, we reduce "almost every" matrix to diagonal form

$$\begin{pmatrix} a_1 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & \ldots \\ 0 & 0 & a_3 & \\ & & & \ddots \\ \ldots & \ldots & \ldots & \end{pmatrix},$$

where by virtue of the *skew-symmetry* all the blocks a_i have the form $\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$ (if the dimension is odd, then the last block is of order 1 and consists of 0).

3. The representation $\sigma_m(\Delta_1)$ in the case of the orthogonal group. Let x denote a row of n complex numbers $x_1, \ldots, x_n$ and X the space of all such row vectors; then the space $\mathcal{P}_m(X)$ is spanned by the basis vectors

$$x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \quad i_1 + i_2 + \ldots + i_n = m.$$

We introduce in X a scalar product by means of the bilinear form (x, y) of §8 and consider the transformations $x \to xg$, where g is an element of $SO(n)$, and we realize the representation $\sigma_m(\Delta_1)$ by the usual formula

$$T_g\, p(x) = p(xg), \quad g \in SO(n)$$

in the space $\mathcal{P}_m = \mathcal{P}_m(X)$.

When we extend T_g to a representation of the full linear group, then we obtain an *irreducible* representation with the signature Δ_1^m: for $\mathcal{P}_m$ contains only one "vector" invariant $p_m(x) = x_1^m$ of $Z(n)$; the only change in transition to the orthogonal subgroup $Z = ZO(n)$ consists, as was mentioned above, in the appearance of the basis invariant (x, x) which is also invariant in the *full* orthogonal group $SO(n)$.[1]

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[1] An elementary proof of this fact can easily be obtained in our case if we "embed" x by a definite normalization in Z as the first row of the matrix $z \in Z$; see the analogous technique in §13.

Therefore, in $\mathscr{P}_m$ there arises the system of dominant vectors

$$x_1^m,\quad x_1^{m-2}(x,x),\quad x_1^{m-4}(x,x)^4,\ldots,$$

and this corresponds to the spectral formula

$$\sigma_m(\Delta_1)=\Delta_1^m+\Delta_1^{m-2}+\Delta_1^{m-4}+\ldots+\Delta_1^{\varepsilon}, \tag{79}$$

where $\varepsilon = 0$ or 1 depending on the parity of m. (A corresponding fact holds for $ZSp(n)$.)

The invariant subspaces $E_k \in \mathscr{P}_m$ on which the components Δ_1^k act can easily be described recurrently by increasing the exponent m step by step. For when we multiply all the polynomials $p \in E_k$ by the coordinates $x_1, x_2, \ldots, x_n$ and then form linear combinations, we obtain an *invariant* subspace of $\mathscr{P}_{m+1}$ which we denote by $\mathscr{P}_1E_k$; by its definition $\mathscr{P}_{m+1}$ is the direct sum of all these subspaces.

I. *All the spaces $\mathscr{P}_1E_k$ are irreducible with the exception of $\mathscr{P}_1E_1$, which contains two irreducible representations with the signatures Δ_1^2 and Δ_1^0.*

PROOF. If m is even, then the number of irreducible summands in (79) is not increased in the transition to the degree $m + 1$, and our statement is obvious; but if m is odd, then the number of summands increases by one and precisely the space $\mathscr{P}_1E_1$ splits; for it is spanned by the basis vectors $(x, x)^{\frac{m-1}{2}} x_i$ $(i = 1, 2, \ldots, n)$; when these polynomials are multiplied by x_j $(j = 1, 2, \ldots, n)$, there arises a "convolution" – the scalar square (x, x) – and the one-dimensional representation Δ_0 splits off.

Denoting the representation acting on $\mathscr{P}_1E_k$, by the symbol $\Delta_1 \cdot \Delta_1^k$, we can make our statement more precise by the formula

$$\Delta_1\cdot\Delta_1^k=\begin{cases}\Delta_1^{k+1} & \text{for } k\neq 1,\\ \Delta_1^2+\Delta_1^0 & \text{for } k=1.\end{cases}$$

4. Example of a not completely reducible representation. Keeping to the notation in 3. we supplement the row x by a coordinate $x_0 = 1$ to a row $\mathfrak{x} = (1, x)$ of dimension $n + 1$ and consider the transformations of the row vectors so obtained by means of the group of matrices

$$\mathfrak{g}=\begin{pmatrix}1 & t\\ 0 & g\end{pmatrix}, \tag{80}$$

where t is a row of n complex numbers $t_1, \ldots, t_n$ and g a complex orthogonal matrix of order n; it is clear that *this group is isomorphic to the group of all motions of an n-dimensional complex euclidean space X*:

$$x\to xg+t.$$

Restricting ourselves for simplicity to the elements $g \in SO(n)$, i.e. setting $\det g = 1$, we denote the group so obtained by the letter $\mathfrak{G}$. The formula

$$T\mathfrak{g}p(\mathfrak{x})=p(\mathfrak{x}\mathfrak{g})$$

defines a representation of $\mathfrak{G}$ in the space $P_m = \mathscr{P}_0 + \mathscr{P}_1 + \ldots + \mathscr{P}_m$ consisting of all polynomials in the coordinates $x_1, \ldots, x_n$ *of a degree not exceeding* m (the homogeneous polynomials now do not form an invariant subspace). We denote the representation in P_m by Γ_m and proceed to its spectral analysis.

We begin by studying Γ_1. When we restrict ourselves to the subgroup $\mathfrak{G}_0$ consisting of the rotations around the coordinate origin which is distinguished in (80) by the condition $t = 0$, then, as we know from 3., $P_1 = \mathscr{P}_1 + \mathscr{P}_0$ is a decomposition into irreducibly invariant subspaces; coming back to the whole group $\mathfrak{G}$ we see that $\mathscr{P}_0$ as before is invariant, but there exists no invariant complement to it. For let us span on the dominant vectors $p_1(x) = x_1$, $p_0(x) = 1$ a two-dimensional plane N and select in $\mathfrak{G}$ the commutative subgroup R consisting of parallel translations: $x \to x + t$; the transformations of R leave N invariant and the corresponding transformation

$$p_1 \longrightarrow p_1 + t_1 p_0, \qquad p_0 \longrightarrow p_0$$

is a "Jordan block" which is of course indecomposable.
Symbolically we can express the preceding analysis by the formula

$$\Gamma_1 = \Delta_1 \longrightarrow \Delta_0,$$

where the arrow points to the *invariant* representation Δ_0 acting on the subspace $\mathscr{P}_0$; whereas Δ_1 acts on the factor space $P_1/\mathscr{P}_0$.

The general problem now consists in describing the spectra of all invariant subspaces of the representation Γ_m.

We introduce the notation Γ^k_l for the irreducible representation of $\mathfrak{G}_0$ generated by the dominant vector $x_1^k(x, x)^l$. Here the upper index denotes the *signature* Δ_1^k, and the *degree of homogeneity* $k + 2l$ is determined by means of the lower index; we denote the representation space of Γ^k_l by P^k_l.

The principal spectral property of Γ_m consists in the following rule.

II. *The cyclic envelope of* P^k_l *relative to* $\mathfrak{G}$ *consists, to within polynomials of lower degree, of the elements* $P^{k-1}_l + P^{k+1}_{l-1}$, *where for* $k = 0$ $(l = 0)$ *the first (second) vector summand is replaced by zero. Symbolically*:

$$\Gamma^k_l \longrightarrow \Gamma^{k-1}_l + \Gamma^{k+1}_{l-1} \quad (\operatorname{mod} P_{k+2l-2}).$$

From this there follows the complete "linking" scheme for the representation Γ_m, for example Γ_5:

$$\begin{tikzcd}
\underline{s=5} & \underline{s=4} & \underline{s=3} & \underline{s=2} & \underline{s=1} & \underline{s=0} \\
\Gamma^5_0 \arrow[r] & \Gamma^4_0 \arrow[r] & \Gamma^3_0 \arrow[r] & \Gamma^2_0 \arrow[r] & \Gamma^1_0 \arrow[r] & \Gamma^0_0 \\
\Gamma^3_1 \arrow[r] \arrow[ur] & \Gamma^2_1 \arrow[r] \arrow[ur] & \Gamma^1_1 \arrow[r] \arrow[ur] & \Gamma_0 \arrow[ur] & & \\
\Gamma^1_2 \arrow[r] \arrow[ur] & \Gamma^0_2 \arrow[ur] & & & &
\end{tikzcd}$$

(here the number s denotes for every column the degree of homogeneity).

PROOF. In order to compute the required cyclic envelope $\mathscr{E}_0$ it is sufficient to find the *cyclic envelope of a dominant vector* $\xi_0 \in P^k_l$ which coincides with it (we recall that P_l^k is the cyclic envelope of ξ_0 relative to $\mathfrak{G}_0$).

Furthermore, let Ω_0 denote the cyclic envelope of ξ_0 *relative to the group of motions* which we have denoted by R; then it follows from the relation $\mathfrak{G}=\mathfrak{G}_0R$ that $\mathscr{E}_0$ is the cyclic envelope of Ω_0 *relative to the subgroup of rotations*. Hence it follows that if Ω_0 contains some *homogeneous* polynomial ω of degree m', then $\mathscr{E}_0$ contains every subspace $P_{l'}^{k'}$, for which in the decomposition

$$\omega=\omega_{l_1}^{k_1}+\omega_{l_2}^{k_2}+\ldots+\omega_{l_s}^{k_s},\quad k_i+2l_i=m',$$

the corresponding projection $\omega_{l'}^{k'}$ is not zero (here we make use of the fact that every irreducible representation $\Gamma_{l_i}^{k_i}$ is contained *singly* in $\mathscr{P}_{m'}$); conversely, if all the projections of Ω_0 onto $P_{l'}^{k'}$ are zero, then obviously this subspace is not contained in $\mathscr{E}_0$.

When we replace in the expression $\xi_0 = x_1^k(x, x)^l$ the vector argument x by $x + t$ and expand by degrees of homogeneity with respect to the parameter t, then we conclude that among the polynomials of the degree of homogeneity $k + 2l - 1$ the cyclic envelope Ω_0 contains only the polynomials

$$kt_1x_1^{k-1}(x, x)^l+2lx_1^k(t, x)(x, x)^{l-1}$$

and their linear combinations (where t_1 denotes the first coordinate of t). Here the first summand is a dominant vector of P_l^{k-1}, and the second summand belongs to $\mathscr{P}_1P_{l-1}^k$ (in the notation of 3.) which, as we know, either coincides with P_{l-1}^{k+1} or in the case $k = 1$ splits into the direct sum $P_{l-1}^2 + P_l^0$; furthermore, it is clear that the second summand has a non-zero projection onto P_{l-1}^2; hence in every case there arise projections onto P_l^{k-1} and P_{l-1}^{k+1} which can be zero only for $k = 0$ or $l = 0$.

This proves our assertion.

For example, in the scheme for Γ_5 the cyclic envelope of Γ_1^1 contains all the representations of that sector of the scheme which is situated *to the right and higher* than Γ_1^1, provided a shift "higher up" is understood to be in the direction of the slanting arrows; the corresponding invariant subspace can be extended by including, for example, Γ_0^5, and then all the elements of the first row come into the cyclic envelope. Now let us prove the final result.

III. *Every invariant subspace of Γ_m can be obtained as the cyclic envelope of one or several subspaces Γ_l^k.*

PROOF. Let Q be an invariant subspace of P_m and Q_0 an arbitrary subspace of it that is irreducibly invariant under the subgroup of rotations; then a dominant vector $\xi_0 \in Q_0$ necessarily has the form

$$\xi_0=x_1^k(\sigma^l+a_1\sigma^{l-1}+\ldots+a_l\sigma^0)=x_1^kP(\sigma),$$

where $\sigma = (x, x)$ and k is the signature.

For the proof of our statement it is sufficient to verify that Q contains together with every subspace of the type Q_0 all the "canonical" subspaces P_l^k on which the vectors of Q_0 have a non-zero projection; in other words, it is sufficient to verify that Q contains together with the vector ξ_0 all the dominant vectors $x_1^k \sigma^{l-\nu}$ for which the coefficient a_ν is non-zero.

We shall conduct the proof by induction on the highest power l of the polynomial $P(\sigma)$. We shall make use of the obvious fact that *together with every polynomial $p(x)$ its cyclic envelope C with respect to translations* contains all the partial derivatives with respect to $x_1, \ldots, x_n$ - in fact, for example, for $t = (t_1, 0, \ldots, 0)$

$$\frac{\partial p(x)}{\partial x_1} = \lim_{t_1 \to 0} \frac{1}{t_1}[p(x+t) - p(x)] \in C.$$

Consequently, Q contains together with the dominant vector ξ_0 also the dominant vector

$$\frac{1}{2} \frac{\partial \xi_0}{\partial x_n} = x_1^{k+1} P'(\sigma);$$

but then, by the induction hypothesis, Q also contains the dominant vector $x_1^{k+1}\sigma^{l-1}$; it is easy to see (see II) that the sector of cyclicity of the corresponding representation Γ_{l-1}^{k+1} contains all the components Γ_{l-1}^k, Γ_{l-2}^k, Γ_{l-3}^k, ...; as a result, Q contains all the dominant vectors: $\xi_1 = x_1^k\sigma^{l-1}$, $\xi_2 = x_1^k\sigma^{l-2}$, ..., and therefore also the vector $x_1^k\sigma^l$ (which is congruent to the original vector ξ_0 mod$(\xi_1, \xi_2, \ldots)$). This completes the proof.

This example exhibits typical features of the spectral analysis of "semi-reducible" representations.

§13. Reduction of dimension (Restriction of representations from a group to a subgroup)

We have met the first example of a restriction of irreducible representations from a group to a subgroup in §12, 3. We shall now take up the study of certain cases of the same type when a linear group G acting on an n-dimensional vector space E_n contains a subgroup G_0 whose transformations leave a certain fixed vector of E_n invariant or - in a more general setting - when a certain linear subspace E_k of dimension $k < n$ remains invariant. Constructions of this type occur, in particular, in various inductive arguments. Technically this series of problems is distinguished by the fact that the complete algebra of Z-invariants is easily constructed - but the complication consists in selecting those invariants that belong to the space in question; a useful criterion here is the application of an "indicator system" of differential operators on Z.

At the same time we can illustrate how easy the algebraic constructions on the manifold Z are.

1. A subgroup of $GL(n)$ isomorphic to $GL(n-1)$. We form the subgroup G_0 of all matrices of the form

$$g=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in GL(n-1) \tag{81}$$

(where there are zeros at all places of the last row and the last column except for the elements $g_{nn} = 1$). The Gauss decomposition in $GL(n)$ induces a Gauss decomposition in G_0

$$G_0 = \overline{Z_0 D_0 Z_0},$$

where Z_0, D_0 and Z_0 are the intersections of G_0 with Z, D and Z, respectively. This is our problem: *given an irreducible representation of G; into what irreducible representations does it split on restriction to G_0?* We state the result in the form of a theorem.

THEOREM 9. *A representation of the full linear group with the signature $(m_1, m_2, \ldots, m_n)$ on restriction to the subgroup G_0 isomorphic to $GL(n-1)$ contains all irreducible representations of this subgroup with the signatures $(l_1, l_2, \ldots, l_{n-1})$ for which the following conditions are satisfied*

$$m_1 \geqslant l_1 \geqslant m_2 \geqslant l_2 \geqslant m_3 \geqslant \ldots \geqslant m_{n-1} \geqslant l_{n-1} \geqslant m_n; \tag{82}$$

every irreducible component occurs precisely once.

PROOF. 1°. For the representation $\tau = (m_1, m_2, \ldots m_n) = \Delta_1^{r_1} \Delta_2^{r_2} \ldots \Delta_n^{r_n}$ we use its "canonical model" on Z:

$$\tau_g f(z) = \alpha(z, g) f(z \cdot g), \qquad f(z) \in \mathfrak{R}_\tau(Z),$$

where $\mathfrak{R}_\tau(Z)$ is the space of polynomials in the elements of z which, as we know (§ 6), is the null-space of a certain system of differential operators:

$$\mathscr{D}_p = z_{p+1} \frac{\partial}{\partial z_p} \qquad (p = 1, 2, \ldots, n-1).$$

Here z_p is the p-th row of z, and $x\frac{\partial}{\partial y}$ is an abridged notation for the operator of differentiation "in the direction of the vector x", i.e. $x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2} + \ldots + x_n\frac{\partial}{\partial y_n}$. We also recall that the transformation τ_z reduces to a right translation on Z.

2°. When we apply the block notation for the elements $g \in G$ as in (81), we can represent, in particular, the matrices $z \in Z$ in the form

$$z = \begin{pmatrix} x & t \\ 0 & 1 \end{pmatrix} = \{x, t\},$$

where $x \in Z(n-1)$ and t is a column of $n-1$ complex numbers $t_i = z_{in}$; then the elements of Z_0 are distinguished by the condition $t = 0$ and from the relation

$$\{x, t\}\{x, 0\} = \{xx_0, t\}$$

it follows that every numerically given matrix $z = \{x, t\}$ can be brought into the form $\{e, t\}$ by a transformation of Z_0; as a result every polynomial $p(z)$ that is invariant with respect to right translations of Z_0 does not depend on the parameters x and can be expressed as a *polynomial* in the invariant variables t.

3°. We denote by $\Omega(Z)$ the set of polynomials on Z that contain only the variables t_i and observe that among these polynomials only the monomials

$$\omega_{\nu}(t)=t_1^{\nu_1}\, t_2^{\nu_2}\, \dots\, t_{n-1}^{\nu_{n-1}}$$

have weights with respect to D_0, namely

$$T_{\delta}\omega_{\nu}=\delta_1^{m_1-\nu_1}\delta_2^{m_2-\nu_2}\dots\delta_{n-1}^{m_{n-1}-\nu_{n-1}}\omega_{\nu}, \qquad \delta\in D_0;$$

consequently, only they can be dominant vectors with respect to G_0; it remains to explain which of these monomials belong to $\mathfrak{R}_{\tau}(Z)$. The problem has a simple solution, because on $\Omega(Z)$ the action of the operator $\mathscr{D}_i$ reduces to within the factor t_{i+1} to the differentiation $\frac{\partial}{\partial t_i}$; from Theorem 2 we conclude that $\omega_{\nu}\in\mathfrak{R}_{\tau}(Z)$ if and only if the following conditions hold

$$0\leqslant\nu_i\leqslant r_i \qquad (i=1, 2, \dots, n-1);$$

it is easy to verify that these conditions are equivalent to the system of relations (82) for the values $l_i = m_i - \nu_i$ (we recall the connections between the parameters r_i, m_i determined by (22)). This concludes the proof of the theorem.[1]

NOTE 1. Let $z_{i_1 i_2 \dots i_p}$ denote the minor of z formed from the first p rows and from the columns numbered $i_1, i_2, \dots, i_p$; then

$$\Delta_p(z, g)=\sum_{i_1<i_2<\dots<i_p} z_{i_1 i_2\dots i_p} g^{i_1 i_2\dots i_p}, \tag{83}$$

where $g^{i_1 i_2 \dots i_p}$ are the minors of g defined similarly but replacing the word "row" by "column". We denote by $\Delta'_p(z, g)$ the result of substituting in (83) in place of every minor $z_{i_1 i_2\dots i_p}$ of order p the minor $z_{i_1 i_2 \dots i_p n}$ of order $p+1$; then an easy calculation verifies that the vectors $T_g\omega_{\nu}$, $g\in G_0$ (homologous to the dominant vector $\omega_{\nu}(t)$ with respect to G_0) has the form

$$z_{1n}^{\nu_1}\left[\prod_{i=1}^{n-2}\Delta_i^{r_i-\nu_i}\Delta_i'^{\nu_{i+1}}\right](\det g)^{r_{n-1}-\nu_{n-1}-\sum_{s=1}^{n-1}\nu_s}, \quad g\in G_0,$$

where $\Delta_i = \Delta_i(z, g)$, $\Delta'_i = \Delta'_i(z, g)$; the linear envelope of these functions is the corresponding irreducible subspace E_{ν} (in which the representation can be interpreted as "symmetrization" of the tensor product of two irreducible representations with the signatures $\{r_i - \nu_i\}$ and $\{\nu_{i+1}\}$); using these remarks we can, for small dimensions, give an explicit description of the elements of a basis in each space E_{ν} and so obtain a basis in the whole space $\mathfrak{R}_{\tau}(Z)$.

NOTE 2. When we replace in the definition of G_0 the condition $g_{nn} = 1$ by $g_{nn}\neq 0$, we obtain a subgroup G_0 isomorphic to $GL(n-1)\times\Lambda$, where

[1] In the proof we can avoid the rather trivial part of Theorem 2 by using the criterion mentioned in the footnote on p. 31.

Λ is the multiplicative group of complex numbers. It is easy to verify that we now have to ascribe to every dominant vector $\omega_{\nu_1 \ldots \nu_{n-1}}$ the signature of Λ, which is equal to

$$\lambda^{m_1+m_2+\ldots+m_n-\nu_1-\nu_2-\ldots-\nu_{n-1}}.$$

2. A subgroup of $SO^{\sim}(2\nu+1)$ isomorphic to $SO(2\nu)$. The choice of an invariant vector is dictated only by convenience, – we shall assume this that vector is the basis vector $e_{\nu+1}$ (with respect to the basis of §8); then the coordinate $x_{\nu+1}$ (the orthogonal projection $(x, e_{\nu+1})$) also remains invariant. The block notation makes the formal scheme considerably more lucid; in particular, by representing the elements of Z and Z_0, respectively, (in a notation analogous to 1.) in the form

$$z=\begin{pmatrix} z_{11} & \alpha & z_{12} \\ & 1 & \beta \\ & & z_{22} \end{pmatrix}, \quad z_0=\begin{pmatrix} z_{11} & 0 & z_{12} \\ & 1 & 0 \\ & & z_{22} \end{pmatrix} \tag{84}$$

(below the diagonal there are zeros, α is a column, β a row of order ν), we can show the corresponding lemma on the "normal form".

LEMMA 13. *By a right multiplication by an element $z_0 \in Z_0$ every z in* (84) *can be carried into the following form*:

$$\begin{pmatrix} e_\nu & \alpha & \frac{1}{2}\alpha\beta \\ & 1 & \beta \\ & & e_\nu \end{pmatrix},$$

where e_ν is the unit matrix of order ν, the row $\beta = (\beta_\nu, \ldots, \beta_1)$ is expressed in terms of the original column $\alpha = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_\nu \end{pmatrix}$ by $\beta_i = -\alpha_i$, and $\alpha\beta$ is obtained by the multiplication rule for rectangular matrices.

PROOF. The orthogonality relations of Z imply the following relations[1]

$$\left.\begin{array}{l} z_{22}=sz_{11}'^{-1}s, \quad \beta=-z_{11}^{-1}\alpha' s, \\ sz_{12}'s=z_{11}^{-1}(\alpha\beta-z_{12})\,z_{22}^{-1}; \end{array}\right\} \tag{85}$$

hence it follows, in particular, that z_{11} can be taken to be an *arbitrary* matrix of $Z(\nu)$. By carrying out, first of all, a multiplication by the "quasi-diagonal" matrix $z_1 \in Z_0$, we reduce z to the form

$$\begin{pmatrix} e_\nu & \alpha & c \\ & 1 & \beta \\ & & e_\nu \end{pmatrix}, \tag{86}$$

---

[1] We recall, that $s = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ & 1 & \\ 1 & & 0 \end{pmatrix}$.

and then it follows from the orthogonality relations (85) that $\beta = -\alpha' c$, $sc's = \alpha\beta - c$; writing the second relation in the form

$$F(c)=\alpha\beta,$$

where $F(c) = sc's + c$ we easily find that

$$z_2=\begin{pmatrix} e_\nu & 0 & \frac{1}{2}\alpha\beta-c \\ & 1 & 0 \\ & & e_\nu \end{pmatrix}$$

is also orthogonal: in fact, $F(\alpha\beta) = -\alpha\alpha's + \alpha\beta = 2\alpha\beta$, so that $F(\frac{1}{2}\alpha\beta - c) = \alpha\beta - \alpha\beta = 0$. Multiplying (86) on the right by z_2 we obtain the required normal form. This proves the lemma.

THEOREM 10. *Restricting the representation of $SO(2\nu + 1)$ with the signature $\tau = (m_1, m_2, \ldots, m_\nu)$ to the subgroup G_0 isomorphic to $SO(2\nu)$ we obtain a representation*

$$(m_1, m_2, \ldots, m_\nu)|_{G_0} = \sum_q (q_1, q_2, \ldots, q_{\nu-1}),$$

where the parameters q_i assume all integral or all semi-integral values (depending on whether τ is single-valued or two-valued) within the limits

$$m_1 \geqslant q_1 \geqslant m_2 \geqslant q_2 \geqslant \ldots \geqslant m_\nu \geqslant q_\nu \geqslant -m_\nu$$

(and every irreducible representation occurs just once).

Similarly, the representation of $SO(2\nu - 1)$ obtained by restriction from the single-valued (two-valued) representation $(m_1, m_2, \ldots, m)$ of $SO(2\nu)$ contains just once all the signatures $(p_1, \ldots, p_{\nu-1})$ for which

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant \ldots \geqslant p_{\nu-1} \geqslant |m_\nu|,$$

and the indices p_i are simultaneously integers (semi-integers).

In particular, for the spinors of the first and second kind over the space $E_{2\nu}$ we obtain as a consequence

$$\mathfrak{S}_+|_{G_0} = \mathfrak{S}_0, \quad \mathfrak{S}_-|_{G_0} = \mathfrak{S}_0,$$

i.e. under restriction to $SO(2\nu - 1)$ both representations remain irreducible and coincide with the spinor representation $\mathfrak{S}_0$ of $SO(2\nu - 1)$. On the other hand, let $\widetilde{\mathfrak{S}}_0$ be a spinor representation of $SO(2\nu + 1)$; then under restriction to the subgroup G_1 isomorphic to $SO(2\nu)$ it splits:

$$\widetilde{\mathfrak{S}}_0|_{G_1} = \mathfrak{S}_- + \mathfrak{S}_+$$

into the direct sum of two mirror-conjugate representations.[1] Using induction on the number of dimensions we can, in particular, find from these relations the dimension of the linear space on which the spinor representation

$$\dim \mathfrak{S}_+ = \dim \mathfrak{S}_- = 2^{\nu-1}; \ \dim \mathfrak{S}_0 = 2^\nu$$

acts.

PROOF OF THE FIRST PART. Applying Theorem 4 we repeat the outline of

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[1] However, for the *full* orthogonal group $O(2\nu)$ it remains irreducible.

the proof of Theorem 9. The algebra $\Omega(Z)$ contains, by Lemma 13, all the polynomials that depend only on the elements of the $(\nu+1)$-th column of z:

$$\alpha_1 = z_{1,\nu+1}, \quad \alpha_2 = z_{2,\nu+1}, \ldots, \alpha_\nu = z_{\nu,\nu+1},$$

the restrictions of the differential operators $\mathscr{D}_i$ $(i = 1, 2, \ldots, \nu)$, $\mathscr{L}_0$ to $\Omega(Z)$ differ only by inessential factors from the operators of differentiation $\frac{\partial}{\partial\alpha_1}, \ldots, \frac{\partial}{\partial\alpha_\nu}$, respectively; consequently, dominant vectors can only be the monomials

$$\omega_s(\alpha) = \alpha_1^{s_1}\alpha_2^{s_2} \ldots \alpha_\nu^{s_\nu},$$

for which

$$0 \leqslant s_1 \leqslant r_1, \ldots, \quad 0 \leqslant s_{\nu-1} \leqslant r_{\nu-1}, \quad 0 \leqslant s_\nu \leqslant r_0,$$

and for the signatures $q_i = m_i - s_i$ we obtain the required relations (we recall that the parameters $\{r_i, r_0\}$ are expressed in terms of $(m_1, \ldots, m_n)$ by the rule: $r_1 = m_1 - m_2, \ldots, r_{\nu-1} = m_{\nu-1} - m_\nu$, $r_0 = 2m_\nu$; in particular, from the condition $0 \leqslant s_\nu \leqslant r_0 = 2m_\nu$ it follows that $0 \leqslant m_\nu - q_\nu \leqslant 2m_\nu$, hence $m_\nu \geqslant q_\nu \geqslant -m_\nu$).

3. **A subgroup of $SO(2\nu)$ isomorphic to $SO(2\nu-1)$.** The subgroup G_0 is defined as the subgroup of matrices that leave invariant the expression $x_\nu + x_{\nu+1}$.

The analogue to Lemma 13 consists in the fact that the matrix $z \in Z$ can be reduced to the normal form

$$\begin{pmatrix} e_{\nu-1} & \alpha & \alpha & \alpha\beta \\ & 1 & 0 & \beta \\ & & 1 & \beta \\ & & & 1 \end{pmatrix},$$

where $\alpha = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_{\nu-1} \end{pmatrix}$ is a column and $\beta = (\beta_{\nu-1}, \ldots, \beta_1)$ a row with $\beta_i = -\alpha_i$ and the quantities α_i are expressed in terms of the parameters of the original matrix z by the formulae

$$\alpha_p = \frac{1}{2}(z_{p\nu} + z_{p,\nu+1}) \qquad (p = 1, 2, \ldots, \nu-1).$$

The subspace $\Omega(Z)$ consists of the polynomials $p(z)$ that are expanded only in powers of the variables $\alpha_1, \ldots, \alpha_{\nu-1}$.

When we replace the variables $(z_{p\nu}, z_{p,\nu+1})$ by (α_p, θ_p), where

$$\theta_p = \frac{1}{2}(z_{p\nu} - z_{p,\nu+1}) \qquad (p = 1, 2, \ldots, \nu-1),$$

we find that the restrictions of the principal translations to $\Omega(Z)$ have the form

$$\mathscr{D}_1 = \alpha_2 \frac{\partial}{\partial\alpha_1}, \ \ldots, \ \mathscr{D}_{\nu-2} = \alpha_{\nu-1}\frac{\partial}{\partial\alpha_{\nu-2}}; \quad \mathscr{L}_- = \mathscr{L}_+ = \frac{\partial}{\partial\alpha_{\nu-1}};$$

applying Theorem 4 we find that the dominant vectors in the representation $\{r_i;\ r_-,\ r_+\}$ are the monomials

$$\omega_s(\alpha) = \alpha_1^{s_1}\alpha_2^{s_2} \ldots \alpha_{\nu-1}^{s_{\nu-1}},$$

for which $0 \leqslant s_i \leqslant r_i$ $(i = 1, 2, \ldots, \nu - 2)$, $0 \leqslant s_{\nu-1} \leqslant \min(r_-, r_+)$, and this implies the second part of Theorem 10.

4. A subgroup of $Sp(n)$ isomorphic to $Sp(n-1)$. In this case the reduction is non-trivial in that the dimension necessarily decreases at once by two units.

Similarly to the way in which this was done in the Note 2 under 1. it is convenient to consider the subgroup $\tilde{G}_0$ isomorphic to $Sp(n-2) \times \Lambda$; we define this as the subgroup consisting of the symplectic coordinate transformations $x_1, \ldots, x_{\nu-1}$; $x_{\nu+2}, \ldots, x_n$ and the dilatations $x_\nu \to \lambda x_\nu$, $x_{\nu+1} \to \lambda^{-1} x_{\nu+1}$; observe that every signature of $\tilde{G}_0$ has the form

$$(q_1, q_2, \ldots, q_{\nu-1} \mid s),$$

where (q_i) is the signature of $Sp(n-2)$ and s a parameter that determines the character λ^s of Λ.

THEOREM 11. *The restriction of the representation* $(m_1, m_2, \ldots, m_\nu)$ *to* $\tilde{G}_0$ *contains in its spectrum all the signatures of the form* $(q_1, q_2, \ldots, q_{\nu-1} \mid s[p, q])$ *where*

$$s[p, q] = 2\sum_{\alpha=1}^{\nu} p_\alpha - \sum_{\alpha=1}^{\nu-1} q_\alpha - \sum_{\alpha=1}^{\nu} m_\alpha,$$

and the (integral) indices q_i, p_j assume all possible values within the limits

$$\left.\begin{array}{l} m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant m_3 \geqslant \ldots \geqslant p_{\nu-1} \geqslant m_\nu \geqslant p_\nu \geqslant 0, \\ p_1 \geqslant q_1 \geqslant p_2 \geqslant q_2 \geqslant \ldots \geqslant p_{\nu-1} \geqslant q_{\nu-1} \geqslant p_\nu. \end{array}\right\} \qquad (87)$$

PROOF. By the standard arguments we obtain that the required algebra of invariants $\Omega(Z)$ consists of the polynomials that depend only on the elements of the two columns

$$\left.\begin{array}{cc} z_{1\nu} & z_{1,\nu+1} \\ z_{2\nu} & z_{2,\nu+1} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ 1 & z_{\nu,\nu+1} \end{array}\right\} \qquad (88)$$

of Z. To clarify the question of the intersection $\Omega(Z) \cap \mathfrak{R}_\tau(Z)$ we apply *the idea of correspondence in the representations of the full linear group* $G' = GL(\nu+1)$.

Parallel with our problem we consider the representation of G' with the signature $(m_1, \ldots, m, 0)$; we call it τ' and restrict it to the subgroup G'_0 consisting of the matrices of the form

$$g = \begin{pmatrix} a & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad a \in GL(\nu - 1);$$

on this model we can effect a reduction in two consecutive steps, each time applying Theorem 9; as a result we find (see Note 2 under 1.) that the spectrum contains all the signatures of G'_0:

$$(q_1, q_2, \ldots, q_{\nu-1} \mid s_1, s_2), \qquad \textbf{(89)}$$

for which $s_1=\sum_1^{\nu} p_\alpha - \sum_1^{\nu-1} q_\alpha$, $s_2=\sum_1^{\nu} m_\alpha - \sum_1^{\nu} p_\alpha$, and the integers p_i, q_i are contained within the limits (87). The corresponding algebra $\Omega(Z')$ has the elements of the two last columns of $z \in ZL(\nu+1)$ as a system of generators.

Finally, when we compute the restriction of the principal translations to $\Omega(Z)$ and, on the other hand, to $\Omega(Z')$, we easily find that the operators $\mathscr{Z}_i$ so obtained (for the case Z' in the parameters (88)) have *identical expressions*; therefore, we can establish between the spaces $\Omega(Z) \cap \Re_\tau(Z)$ and $\Omega'(Z') \cap \Re_{\tau'}(Z')$ a one-to-one correspondence; in this correspondence weight vectors go over into weight vectors under the formal substitution $\lambda_2 = \lambda_1^{-1}$, i.e. by selecting in the product $\Lambda \times \Lambda$ in a definite way a subgroup isomorphic to Λ; moreover, the signatures (89) go over each time into those points of the spectrum that are given in the statement of the theorem. This completes the proof.

5. Description of the weight spectrum of the space $\Re_\tau(Z)$. For simplicity we consider the case $G = GL(n)$. When we select in G a chain of embedded subgroups: $G \supset G_1 \supset G_2 \supset \ldots \supset G_{n-1}$, where each G_i is isomorphic to $GL(n-i) \times \Lambda^i$ (Λ is the multiplicative group of complex numbers), we reach step by step a subgroup G_{n-1} that coincides with D, i.e. consists of all diagonal matrices $\delta \in G$. From this we conclude that *the restriction T_δ of an arbitrary irreducible representation T_g of G to D is completely reducible*; the set of one-dimensional representations so obtained, i.e. of characters or "weights" shall be called the weight spectrum of $\Re_\tau(Z)$. Using Note 2 under 1. we now prove by induction:

COROLLARY. *In the space $\Re_\tau$ of an irreducible representation of the full linear group with the integral signature $\tau = (m_1, m_2, \ldots, m_n)$ there exists a basis of weight vectors e_μ where the index μ ranges over all the matrices*

$$\mu = \begin{pmatrix} \mu_{12} & \mu_{13} & \cdots & \mu_{1n} \\ & \mu_{23} & \cdots & \mu_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ & & & \mu_{n-1,n} \end{pmatrix}$$

with integral elements μ_{ij} within the limits

$$m_1 \geqslant \mu_{1n} \geqslant m_2 \geqslant \mu_{2n} \geqslant m_3 \geqslant \ldots \geqslant m_{n-1} \geqslant \mu_{n-1,n} \geqslant m_n$$
$$\mu_{1n} \geqslant \mu_{1,n-1} \geqslant \mu_{2n} \geqslant \mu_{2,n-1} \geqslant \ldots \geqslant \mu_{n-2,n-1} \geqslant \mu_{n-1,n}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mu_{13} \geqslant \mu_{12} \geqslant \mu_{23},$$

and each vector e_μ has the weight $\delta_1^{s_2-s_1}\delta_2^{s_3-s_2}\ldots\delta_n^{s_{n+1}-s_n}$, where

$$s_1=\mu_{12}, \quad s_2=\mu_{13}+\mu_{23}, \ \ldots, \ s_n=\sum_{i=1}^{n-1}\mu_{in}, \quad s_{n+1}=\sum_{\alpha=1}^{n} m_\alpha.$$

A similar description can be given for $SO(n)$ and $Sp(n)$. For the cases $GL(n)$ and $SO(n)$ this result is not new: it was published [8], [9] by I.M. Gel'fand and M.L. Tseitlin who defined explicitly infinitesimal operators of the representation $(m_1, m_2, \ldots, m_n)$ in a space obtained formally from symbols $\{\mu\}$, where each $\{\mu\}$ is a collection of integral indices of the type in question.

In the case of the symplectic group the basis consists of vectors of the form e_{pq}, where p and q are triangular matrices:

$$p=\begin{pmatrix} p_{12} & p_{13} & \cdots & p_{1,\nu+1} \\ & p_{23} & \cdots & p_{2,\nu+1} \\ \cdot & \cdot & \cdot & \cdot \\ & & & p_{\nu,\nu+1} \end{pmatrix},\quad q=\begin{pmatrix} q_{12} & q_{13} & \cdots & q_{1\nu} \\ & q_{23} & \cdots & q_{2\nu} \\ \cdot & \cdot & \cdot & \cdot \\ & & & q_{\nu-1,\nu} \end{pmatrix}$$

with integral elements whose limits can easily be written down by means of Theorem 11.

§14. Computation of the spectrum of the tensor product of two irreducible representations

For simplicity we restrict ourselves to the case of the full linear group $GL(n)$; however, from the constructions it will be clear that they are also applicable to every other classical group.

It is very easy to multiply two dominants, for example,

$$\Delta_1\times\Delta_p=\Delta_1\Delta_p\times\Delta_{p+1},$$

as follows from simple arguments concerning Z-invariants (if $p = n$, then Δ_{n+1} has to be replaced by zero). A similar "symbolic" method is obvious, but it becomes rather complicated on transition into the general case; a more natural course of the solution of the problem consists, apparently, in making use of the "canonical model" on Z.

1. Description of the dominant vectors. Let us perform the tensor multiplication of two irreducible representations α and β whose dominant weights shall be denoted by the same letters $(\alpha(\delta),\ \beta(\delta))$. By applying the canonical model we realize the product $\alpha\times\beta$ in the space $\mathscr{E}=\mathfrak{R}_\alpha(Z)\times\mathfrak{R}_\beta(Z)$ by means of the formula

$$T_gf(x,\ y)=\alpha(x,\ g)\,\beta(y,\ g)\,f(x\cdot g,\ y\cdot g);$$

we recall that $\mathscr{E}$ consists of all polynomials $f(x,\ y)$, $x\in Z$, $y\in Z$, that belong to $\mathfrak{R}_\alpha(Z)$ as functions of the first argument for fixed y and to $\mathfrak{R}_\beta(Z)$ with respect to the second argument for fixed x. In particular,

$$T_zf(x,\ y)=f(xz,\ yz),$$
$$T_\delta f(x,\ y)=\alpha(\delta)\,\beta(\delta)\,f(\delta^{-1}x\delta,\ \delta^{-1}y\delta).$$

We denote by $\Omega\subset\mathscr{E}$ the set of all invariants of Z.

LEMMA 14. The formula

$$f(x,\ y)=\varphi(xy^{-1})$$

provides a one-to-one mapping of Ω *onto the space* $\mathscr{P}_{\alpha\beta}(Z)$ *consisting of the polynomials* $\varphi(z)$ *of "one" variable* $z\in Z$, *where*

$$\mathscr{P}_{\alpha\beta}(Z)=\mathfrak{R}_\alpha(Z)\cap\hat{\mathfrak{R}}_\beta(Z),$$

and $\hat{\mathfrak{R}}_\beta$ *denotes the image of* $\mathfrak{R}_\beta$ *in the mapping* $z\to z^{-1}$.

The proof is obvious (we remark only that the elements of z^{-1} are expressed as *polynomials* and by *recurrence* in terms of the elements of z).

It is clear that in this correspondence weight vectors of $\mathfrak{R}_\alpha(Z)$ (or $\mathfrak{R}_\beta(Z)$) go over into weight vectors of Ω, and that in taking intersections their number can only be diminished; hence we obtain by an easy calculation of the multipliers by the following corollary.

COROLLARY 1. *We denote by $\sigma(\alpha\times\beta)$ the spectrum of the representation $\alpha\times\beta$, i.e. the set of signatures in which it is contained and by $S(\tau)$ the weight spectrum of τ, i.e. the set of eigenvalues of D in the space of this representation. Then*

$$\sigma(\alpha\times\beta)\subseteq S(\alpha)\cdot\beta, \quad \sigma(\alpha\times\beta)\subseteq\alpha\cdot S(\beta),$$

where γS denotes the set obtained from S by multiplying all its elements by the weight $\gamma=\gamma(\delta)$.[1]

As an example we recall the case of the Lorentz group where the product $\mathfrak{D}^{n_1}\times\mathfrak{D}^{n_2}$ contains the signatures

$$n_1+n_2, \quad n_1+n_2-2, \ \ldots, \ |n_1-n_2|;$$

thus, under the condition $n_1 \geqslant n_2$ the following relations hold

$$\sigma(\mathfrak{D}^{n_1}\times\mathfrak{D}^{n_2})=\delta^{n_1}S(n_2)\subseteq S(n_1)\,\delta^{n_2}.$$

In the general case the problem reduces to a description of the space $\mathscr{F}_{\alpha\beta}$ of Lemma 14; it is clear that this space is distinguished by the system of equations

$$\mathscr{D}_i^{r_i+1}f(z)=0, \quad \hat{\mathscr{D}}_i^{s_i+1}f(z)=0,$$

where $\{r_1, \ldots, r_n\}=\alpha$, $\{s_1, \ldots, s_n\}=\beta$, and the operators $\hat{\mathscr{D}}_i$ are obtained from the principal translations $\mathscr{D}_i$ by the change of variables $z\to z^{-1}$; more accurately, $\hat{\mathscr{D}}_p$ is the operator

$$\hat{\mathscr{D}}_p=\sum_{s=1}^{p} z_{sp}\frac{\partial}{\partial z_{s,\,p+1}},$$

that is realized by the "substitution" of the elements of the p-th column in place of the elements of the $(p+1)$-th column of z. This differential criterion, although it is somewhat complicated, enables us to solve the problem in various cases of practical importance.

2. The case when one of the signatures is sufficiently "small". Let us agree to interpret the symbol $S(\alpha)$ now not as a set of weights, but of their formal sums, for example,

$$S(\Delta_1)=\delta_1+\delta_2+\ldots+\delta_n,$$
$$S(\Delta_2)=\delta_1\delta_2+\delta_1\delta_3+\ldots+\delta_1\delta_n+\delta_2\delta_3+\ldots+\delta_{n-1}\delta_n;$$

with this definition $S(\alpha)$ coincides with the formal expression for the *trace* function of the representation α, or, as it is sometimes called, the *character* of this representation; more accurately, when numerical values are given to the symbols $\delta_1, \ldots, \delta_n$, then $S(\alpha)$ turns into the ordinary trace of the operator $T_\delta^{(\alpha)}$, $\delta\in D$.

Corollary 1 can now be expressed in the form

$$\alpha\times\beta\subseteq S(\alpha)\cdot\beta, \quad \alpha\times\beta\subseteq\alpha\cdot S(\beta),$$

provided we understand by the symbols on the right hand sides the

[1] The cases of an arbitrary connected complex Lie group or of one of its real forms can be treated in exactly the same way.

corresponding sums of "weights", for example,

$$\delta_1 S(\Delta_1) = \delta_1^2 + \delta_1\delta_2 + \delta_1\delta_3 + \ldots + \delta_1\delta_n;$$

it is obvious that not all the weights on the right hand sides can be "dominant weights" for G, because they do not all satisfy the order relations for the exponents

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n;$$

it can be shown that to obtain the spectrum of $\alpha \times \beta$ it is sufficient to delete the "inadmissible signatures"; however, the actual situation is complicated.

Applying the differential criterion we obtain results for various special cases; however, the general case can, to a certain extent, be reduced to these.

COROLLARY 2. *Let us assume that the following relation holds between the signatures* $\alpha = \{r_i, \ldots, r_n\}$ *and* $\beta = \{s_1, \ldots, s_n\}$:

$$r_1 + r_2 + \ldots + r_{n-1} \leqslant \min(s_1, s_2, \ldots, s_{n-1});$$

then, in the symbolism just given,

$$\alpha \times \beta = S(\alpha)\cdot\beta.$$

The proof consists in an easy verification of the inclusion $\mathfrak{R}_\alpha \subseteq \hat{\mathfrak{R}}_\beta$, whence $\mathscr{P}_{\alpha\beta} = \mathfrak{R}_\alpha$.

COROLLARY 3. *For the product* $\Delta_p \times \beta$ *only those summands have to be rejected from the spectral sum*

$$S(\Delta_p)\beta = \Big(\sum_{i_1 < \ldots < i_p} \delta_{i_1}\delta_{i_2} \ldots \delta_{i_p}\Big)\beta(\delta)$$

whose exponents do not satisfy the order relation.

The proof comes out easily if we observe that $\mathfrak{R}_\alpha(Z)$ for $\alpha = \Delta_p$ is spanned by all possible minors $z_{i_1 i_2 \ldots i_p}$ formed from the p-th first rows of z.

COROLLARY 4. *For the product* $\Delta_1^m \times \beta$ *with* $\beta = \Delta_1^{s_1}\Delta_2^{s_2} \ldots \Delta_n^{s_n}$ *from the spectral sum*

$$S(\Delta_1^m)\beta = \Big(\sum_{\nu_1 + \ldots + \nu_n = m} \delta_1^{\nu_1}\delta_2^{\nu_2} \ldots \delta_n^{\nu_n}\Big)\beta(\delta)$$

those and only those monomials $\delta_1^{\sigma_1}\delta_2^{\sigma_2} \ldots \delta_n^{\sigma_n}$, *have to be rejected for which* $\sigma_1 > s_1, \sigma_2 > s_2, \ldots, \sigma_{n-1} > s_{n-1}$.

The proof reduces to a study of the operators $\dfrac{\partial}{\partial z_{12}}, \dfrac{\partial}{\partial z_{13}}, \ldots, \dfrac{\partial}{\partial z_{1n}}$ which differ only by inessential factors from the operators $\hat{\mathscr{D}}_1, \ldots, \hat{\mathscr{D}}_{n-1}$, restricted to $\mathfrak{R}_\alpha(Z)$.

NOTE 1. Corollary 2 can also be obtained without difficulty by a "symbolic" device which in the general case consists in replacing the complicated linear expression of the signature $\{r_1, \ldots, r_n\}$ by a system consisting of r_1 vectors, r_2 bivectors etc; this is admissible in all problems concerning only the *type of transformations* (we may, of course, go further by dividing the multivectors into systems of ordinary vectors); then we can construct from two such systems all possible "vector" invariants, from which we have to choose those that satisfy the required type of symmetry (according to Young).

In discussing, for example, the product $\Delta_1 \times \beta$, where $\beta = \Delta_1^{s_1} \dots \Delta_n^{s_n}$, we can realize a linear object of the signature β by means of polynomials in the generalized linear element $L = \{\overset{1}{x}, \overset{2}{x}, \dots, \overset{n}{x}\}$ (see §11, 4., which also involves a reduction to a system of multivectors, but where the required symmetry relations are already taken into account); denoting by y a vector to be transformed according to Δ_1, we find a system of invariants linearly dependent on y

$$[y], \ [y\overset{1}{x}], \ [y\overset{1}{x}\overset{2}{x}], \dots, [y\overset{1}{x}\overset{2}{x} \dots \overset{n-1}{x}];$$

to these dominant vectors there corresponds the spectral sum

$$\left(\Delta_1 + \frac{\Delta_2}{\Delta_1} + \frac{\Delta_3}{\Delta_2} + \dots + \frac{\Delta_n}{\Delta_{n-1}}\right)\Delta_1^{s_1}\Delta_2^{s_2} \dots \Delta_n^{s_n},$$

where, however, the "inadmissible" summands have to be thrown out (those that contain at least one factor Δ_i to a negative power). We can also write the result in the form

$$\Delta_1 \times (m_1, m_2, \dots, m_n) =$$
$$= (m_1+1, m_2, \dots, m_n) + (m_1, m_2+1, \dots, m_n) + \dots + (m_1, m_2, \dots, m_n+1),$$

in which the "inadmissible" summands are deleted. This result is well-known and is usually obtained by the method of characters.[1]

NOTE 2. By using the decomposition just found we can now prove the recurrence formula (67) that was promised in §11 and expresses the multiplicity of the points of the spectrum of the tensor power $\pi_m = \Delta_1^{\oplus m}$ in terms of the corresponding multiplicities π_{m-1}; it is sufficient to observe that $\pi_m = \Delta_1 \times \pi_{m-1}$ can contain the signature $(m_1, m_2, \dots, m_n)$ only in virtue of the circumstance that the signatures $(m_1 - 1, m_2, \dots, m_n)$, $(m_1, m_2 - 1, m_3, \dots, m_n)$ etc. are contained in π_{m-1}.

3. The spectral formula in general form. Two determinant formulae are known[2] for the computation of the trace $S(m_1, \dots, m_n)$:

$$S = \frac{1}{\prod\limits_{i<j}(\delta_i - \delta_j)}\begin{vmatrix} \delta_1^{l_1}\delta_2^{l_1} & \dots & \delta_n^{l_1} \\ \delta_1^{l_2}\delta_2^{l_2} & \dots & \delta_n^{l_2} \\ \cdot & \cdot \cdot \cdot & \cdot \\ \delta_1^{l_n}\delta_2^{l_n} & \dots & \delta_n^{l_n} \end{vmatrix}, \quad S = \begin{vmatrix} \sigma_{l_1-(n-1)}\sigma_{l_1-(n-2)} & \dots & \sigma_{l_1} \\ \sigma_{l_2-(n-1)}\sigma_{l_2-(n-2)} & \dots & \sigma_{l_2} \\ \cdot & \cdot \cdot \cdot & \cdot \\ \sigma_{l_n-(n-1)}\sigma_{l_n-(n-2)} & \dots & \sigma_{l_n} \end{vmatrix},$$

$$l_1 = m_1 + (n-1), \quad l_2 = m_2 + (n-2), \dots, l_{n-1} = m_{n-1} + 1, \ l_n = m_n,$$

where σ_m denotes the trace of the symmetric power $\Delta_1^{\overset{m}{\vee}}$:

$$\sigma_1 = \delta_1 + \delta_2 + \dots + \delta_n, \quad \sigma_2 = \delta_1\delta_2 + \delta_1\delta_3 + \dots + \delta_{n-1}\delta_n, \dots;$$

we take the second of these formulae as our basis (the first is complicated for our objective, because it contains the quotient of two polynomials).

We introduce a further refinement in our symbolism. On the monomials $\alpha(\delta) = \delta_1^{m_1} \dots \delta_n^{m_n}$ we define operators $\delta_1, \delta_2, \dots, \delta_n$ by the rule

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1 In cases when the left hand factor has the form Δ_p^{m} one has to use determinant identities of the type (27).

2 H. Weyl [1], p. 200 and 203.

$$\delta_p \alpha(\delta) = \begin{cases} \delta_1^{m_1} \ldots \delta_p^{m_p+1} \ldots \delta_n^{m_n}, & \text{if} \quad m_{p-1} > m_p, \\ 0, & \text{if} \quad m_{p-1} = m_p; \end{cases} \tag{90}$$

when $\alpha(\delta)$ is written in the form $\Delta_1^{r_1} \ldots \Delta_n^{r_n}$, then this rule reduces to multiplication by $\frac{\Delta_1}{\Delta_{p-1}}$ and substitution of zero for the result if negative exponents should arise. Observe that the multipliers δ_i so defined are *non-commutative*; for example, the monomial $\delta_p \ldots \delta_2\delta_1 = \Delta_p$ in which the indices are arranged in decreasing order cannot cause even one negative power to appear, but the same monomial in ascending order of the indices, $\delta_1\delta_2 \ldots \delta_p \sim \Delta_1 \frac{\Delta_2}{\Delta_1} \ldots \frac{\Delta_n}{\Delta_{n-1}}$ has a whole series of such possibilities.

It is easy to verify that the statements of the Corollaries 3 and 4 can now be written in the following form.

COROLLARY 5. *The multipliers $S(\Delta_1^m)$, $S(\Delta_p)$ have the form of "ordered traces":*

$$S(\Delta_1^m) = \sum_{\nu_1+\nu_2+\ldots+\nu_n=m} \delta_1^{\nu_1}\delta_2^{\nu_2} \ldots \delta_n^{\nu_n}, \quad S(\Delta_p) = \sum_{i_1>i_2>\ldots>i_p} \delta_{i_1}\delta_{i_2} \ldots \delta_{i_p}$$

(in the second case the indices come in decreasing order).

THEOREM 12. *In order to compute the spectrum of the representation $\alpha \times \beta$ it is sufficient to apply to the dominant weight $\beta(\delta)$ the multiplier $\Gamma(\alpha)$ defined by the formula*

$$\Gamma_{m_1 m_2 \ldots m_n} = \begin{vmatrix} \Gamma_{m_1}\Gamma_{m_1+1} \ldots \Gamma_{m_1+(n-1)} \\ \Gamma_{m_2-1}\Gamma_{m_2} \ldots \Gamma_{m_2+(n-2)} \\ \ldots\ldots\ldots\ldots\ldots \\ \Gamma_{m_n-(n-1)}\Gamma_{m_n-(n-2)} \ldots \Gamma_{m_n} \end{vmatrix},$$

where Γ_m is the ordered trace of the symmetric representation Δ_1^m:

$$\Gamma_m = \sum_{\nu_1+\nu_2+\ldots+\nu_n=m} \delta_1^{\nu_1}\delta_2^{\nu_2} \ldots \delta_n^{\nu_n} \quad (\Gamma_m = 0 \text{ for } m < 0),$$

and the operators δ_i are defined by (90).[1]

(Corollaries 2-5 furnish very simple criteria for various special cases.)

PROOF. The case $\alpha = \Delta_1^m$ is treated in Corollary 5; to prove the general case we consider the expression of the trace $S(\alpha)$ in terms of the sum of the elementary traces σ_l; this equality of traces signifies an equality of the spectral decompositions

$$\alpha = (m_1, m_2, \ldots, m_n) = \sum_{0 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n-1} \pm \Delta_1^{m_1+i_1} \times \Delta_1^{m_2+i_2-1} \times \ldots \times \Delta_1^{m_n+i_n-(n-1)}$$

(the trace of a tensor product is equal to the sum of the traces of the factors; when the traces are equal, then the representations are equivalent); it remains to replace the product $\alpha \times \beta$ by the alternating

[1] Note that the operators Γ_1, Γ_2, Γ_3, ... are permutable (this follows from the commutativity of the multiplication $\alpha \times \beta$).

sum of the various products of the form

$$\Delta_1^{l_1}\times\Delta_1^{l_2}\times\ldots\times\Delta_1^{l_n}\times\beta.$$

This completes the proof of the theorem.

We mention (as a special case of Corollary 5) the formula for the multiplication of two multivectors Δ_p and Δ_q for $p \geqslant q$:

$$\Delta_p\times\Delta_q=\Delta_p\Delta_q+\Delta_{p+1}\Delta_{q-1}+\ldots+\Delta_{p+q}\Delta_0$$

(the chain breaks off sooner if the ascending index exceeds the dimension n).

Our criterion has a certain similarity with another determinant method the method of R. Brauer and H. Weyl.

§15. Concrete formulae of spectral analysis

In this chapter, in studying reducible representations, we have turned our main attention to the description of their spectra and the search for all dominant vectors; however, we have left open the question of the structure of the projection operators onto invariant subspaces. Now, in conclusion of this paper, we shall give a brief account of a scheme for the solution of this problem. It is curious to observe, how this small amount of information contained in the formula of spectral decomposition becomes more meaningful and intuitive through the introduction of the projection operators (more accurately, in our account, of certain transformation operators); usually these operators are defined on linear groups by means of various substitutions of vector arguments.

The problems to be discussed below are connected by a common construction idea.

1. The tensor product of irreducible representations. We restrict ourselves for simplicity to the special case of the form

$$T=\Delta_1^m\times\alpha,$$

where α is an arbitrary irreducible representation of the group $G = GL(n)$.

We realize the representation T by means of the model indicated at the beginning of §14; we recall that the space $\mathscr{E}$ of T is constructed in this model as a definite class of polynomials

$$f(z',\ z)$$

on the manifold $Z \times Z$.

According to §14 (see Lemma 14 or Corollary 4) the monomials

$$\omega_i=(z',\ z^{-1})_{1,\ i+1},\quad i=1,\ 2,\ \ldots,\ n-1,$$

are generators of the corresponding algebra of Z-invariants; in other words, every dominant vector in $\mathscr{E}$ has the form

$$\omega_\nu=\omega_1^{\nu_1}\omega_2^{\nu_2}\ldots\omega_{n-1}^{\nu_{n-1}} \tag{91}$$

with a definite restriction on the exponents $\nu_1,\ \nu_2,\ \ldots,\ \nu_{n-1}$. We denote by $\mathscr{E}_\nu$ the irreducible subspace with the dominant vector ω_ν and by $\mathscr{E}_\nu^0$ the corresponding irreducible subspace realized in the canonical form (i.e. consisting of polynomials $\varphi(z)$ on Z). Our aim is to find the explicit form of the operator S that maps $\mathscr{E}$ onto the subspace of vector functions $\varphi_\nu(z)$:

$$S : \mathscr{E} \longrightarrow \sum_\nu \mathscr{E}_\nu^0.$$

The result can be explained in the following way: for every index ν we construct a *differential* operator Ω_ν such that if $f(z', z) \in \mathscr{E}$, then the function

$$\varphi_\nu(z) = \Omega_\nu f(z', z)\big|_{z'=z} \tag{92}$$

belongs to $\mathscr{E}_\nu^0$ and is transformed in this space in accordance with the irreducible representation. Furthermore, when a certain symbolical process of multiplication is used, then every operator Ω_ν can be written in the form

$$\Omega_\nu = \Omega_1^{\nu_1}\Omega_2^{\nu_2}\ldots\Omega_{n-1}^{\nu_{n-1}}, \tag{93}$$

where Ω_1, Ω_2, $\ldots$, Ω_{n-1} are definite differential operators *of the first kind* on the manifold $Z \times Z$.

We now proceed to a detailed description of these operators. Suppose that α has the signature $\Delta_1^{r_1}$, $\Delta_2^{r_2}$ $\ldots$ $\Delta_n^{r_n}$. Instead of using the symbolical multiplication it is convenient to introduce *auxiliary variables* λ_1, λ_2, $\ldots$, λ_n and to multiply each function $f(z) \in \mathfrak{R}_\alpha(Z)$ by one and the same factor:[1]

$$m(\lambda) = \lambda_1^{r_1}\lambda_2^{r_2} \ldots \lambda_n^{r_n}.$$

To keep the notation short let us agree to use the previous formula (92) in which it is assumed, however, that the operators Ω_ν mean differentiation with respect to the variables λ, the function f means the factor $m(\lambda)$ and that *after differentiation all auxiliary variables are set equal to one*. We introduce the notation:

$$D_i = \lambda_i \frac{\partial}{\partial\lambda_i}, \qquad i = 1, 2, \ldots, n,$$

and observe that the action of the operators D_1, D_2, $\ldots$, D_n on the space $\mathfrak{R}_\alpha(Z)$ reduces to a multiplication by a number, since

$$D_i[m(\lambda)] = r_i m(\lambda), \qquad i = 1, 2, \ldots, n;$$

consequently, the operators D_i and all the polynomials in these operators have inverses in the space $\mathfrak{R}_\alpha$, except when they are equal to zero on $\mathfrak{R}_\alpha$.

Let D_{ij} be the infinitessimal operator of left translation on Z;

$$D_{ij} = \sum_{\alpha=j}^{n} z_{j\alpha}\frac{\partial}{\partial z_{i\alpha}}, \qquad 1 \leqslant i < j \leqslant n;$$

we arrange these operators in the form of a triangular matrix

[1] Note that this is equivalent to the transition to the homogeneous realization of α (in polynomials on H).

$$D=\begin{pmatrix} 1 & D_{12} & D_{13}\dots D_{1n} \\ & 1 & D_{23}\dots D_{2n} \\ & & 1\dots D_{3n} \\ & & \cdot \\ & 0 & \cdot \\ & & 1 \end{pmatrix}.$$

The central place in our description is occupied by the definition of a certain new matrix formed from the operators D_{ij} which we shall denote by D^{-1}.

The *method of computation of* D^{-1} can conveniently be described by recurrence. For $n = 1$ the matrix D consists of 1 and we obtain

$$D^{-1}=1.$$

Now suppose that D^{-1} is already defined for the dimension $n - 1$; then we represent D of order n in the form of a blocked matrix:

$$D=\begin{pmatrix} 1 & T \\ 0 & X \end{pmatrix},$$

where T is the row of the elements $D_{12}, \dots, D_{1n}$, and we set by definition

$$D^{-1}=\begin{pmatrix} 1 & -T*X^{-1} \\ 0 & X^{-1} \end{pmatrix},$$

where the product $T*X^{-1}$ is computed by the usual method of multiplying matrices, however, with the additional "re-normalization":

$$(T*X^{-1})_{1k}=\mu_{1k}\{T_{12}(X^{-1})_{2k}+\dots+T_{1,n-1}(X^{-1})_{n-1,k}\}.$$

Similarly, in the computation of X^{-1} there arise certain factors μ_{2k} that normalize the second row of D^{-1} etc. We give the values of all these factors μ_{pq}, $1 \leqslant p < q \leqslant n$:

$$\mu_{pq}=\frac{1}{\lambda_{q-1}}\cdot\frac{1}{D_p+D_{p-1}+\dots+D_{q-1}+(q-p-1)} \quad (\lambda_0=1).$$

It is easy to verify that the occurrence of negative powers does not lead to a contradiction and that the elements of D^{-1} are, in fact, defined as operators on $\mathfrak{R}_\alpha(Z)$.

Finally, we denote the matrix D written in the variables z' by D'.

Now it turns out that the elements of the first row of $D'D^{-1}$ are generators of the class of differential operators Ω_ν we are looking for. More accurately, let us multiply all the elements of $\mathscr{E}$ by the auxiliary factor $m(\lambda) = \lambda'^m\lambda_1^{r_1}\lambda_2^{r_2} \dots \lambda_n^{r_n}$ and introduce the notation

$$\Omega_i=\mu(D'D^{-1})_{1,\,i+1}, \qquad i=1,\,2,\,\dots,\,n-1, \tag{94}$$

where $\mu=\frac{1}{\lambda'}\cdot\left(\lambda'\frac{\partial}{\partial\lambda'}\right)^{-1}$ and where the product DD'^{-1} is computed by the usual rule for the multiplication of matrices; then we have the following theorem.

THEOREM 13. *If the functions $f(z', z)$ are transformed according to $\Delta_1^m \times \alpha$, then every function* (92), *where*

$$\Omega_\nu = \Omega_1^{\nu_1}\Omega_2^{\nu_2} \dots \Omega_n^{\nu_n},$$

is transformed according to an irreducible representation of G. The signature of this irreducible representation is equal to the product of the weight of the dominant vector

$$\omega_\nu = \omega_1^{\nu_1}\omega_2^{\nu_2} \dots \omega_n^{\nu_n}$$

and the character $\delta_1^m \alpha(\delta)$.

A proof can easily be given, for example on the model of §10. One considers the special vectors of the form

$$f_{g'g}(z', z) = \Delta_1^m(z'g')\alpha(zg).$$

One verifies that the action of the operator Ω_i reduces to decreasing the exponent m by one and to replacing one of the minors (namely Δ_{i-1}) in the function $\alpha(zg)$ by a similar minor of the next higher order in which one of the columns contains the elements of $z'g'$. Under the substitution $z' = z$ one obtains a function that is transformed according to the required representation.

2. **Reduction to a subgroup.** Let us realize an irreducible representation of the group $G_n = GL(n)$ in the space $\mathfrak{R}_\alpha(Z)$.

We form a subgroup G_{n-1} isomorphic to $GL(n-1)$ from the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \in GL(n-1);$$

we write the elements $f(z) \in \mathfrak{R}_\alpha(Z)$ in the form $f(t, x)$, where the variables t, x are defined as follows:

$$z = \begin{pmatrix} 1 & t \\ 0 & x \end{pmatrix}, \quad x \in Z(n-1).$$

Then the restriction of α to G_{n-1} acts according to the formula

$$T_a f(t, x) = \beta(xa) f(ta, x \cdot a);$$

here the multiplier β has the signature $\Delta_1^{r_2}\ \Delta_2^{r_3} \dots \Delta_{n-1}^{r_n}$ (relative to $GL(n-1)$). The functions $f(t, x)$ are polynomials in t of degree not exceeding $r_1 + r_2 + \dots + r_{n-1}$ and belong to $\mathfrak{R}_\beta(X)$ with respect to the variable x (where $X = Z(n-1)$); hence we conclude that T_α is contained as an invariant part in the tensor product of the irreducible representation β and the representation

$$p(t) \longrightarrow p(ta),$$

acting on the class of polynomials in t of degree not exceeding $r_1 + r_2 + \dots + r_{n-1}$. The representation in the class of polynomials splits into irreducible ones, when every polynomial is decomposed into a sum of homogeneous constituents; and so the solution of our problem reduces to the task discussed under 1.

The result can be formulated in the following way. We represent Z in the form

$$\begin{pmatrix} 1 & \omega \\ 0 & e \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \quad \omega = tx^{-1}; \tag{95}$$

then the elements of the row

$$\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$$

are generators in the algebra of Z-invariants (with respect to the right translations by the elements $z \in G_{n-1} \cap Z$). It is easy to verify that the monomial

$$\omega_\nu = \omega_1^{\nu_1}\omega_2^{\nu_2} \ldots \omega_{n-1}^{\nu_{n-1}} \tag{96}$$

belongs to $\mathfrak{R}_\alpha(Z)$ only under the conditions

$$0 \leqslant \nu_i \leqslant r_i.$$

Multiplying all the elements of $\mathfrak{R}_\alpha(Z)$ by the expression $m(\lambda) = \lambda_1^{r_1}\lambda_2^{r_2} \ldots \lambda_n^{r_n}$, where $\lambda_1, \ldots, \lambda_n$ are auxiliary variables, we consider the matrix D formed from the operators of left translation (see 1). 1). Representing D in block form

$$D = \begin{pmatrix} 1 & T \\ 0 & X \end{pmatrix},$$

we construct, in analogy to (95), the row

$$\Omega = T * X^{-1}, \quad \Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{n-1}), \tag{97}$$

where X^{-1} in T^*X^{-1} is determined in accordance with 1. For every set of exponents $\nu = (\nu_1, \nu_2, \ldots, \nu_{n-1})$ we form the differential operator

$$\Omega_\nu = \Omega_1^{\nu_1}\Omega_2^{\nu_2} \ldots \Omega_{n-1}^{\nu_{n-1}}.$$

THEOREM 14. *With every function $f(z) \in \mathfrak{R}_\alpha(Z)$ we associate the function*

$$\varphi_\nu(x) = \Omega_\nu f(t, x)|_{t=0}.$$

Let σ be the signature of the irreducible representation of G_{n-1} with the dominant vector (96). Then the function $\varphi_\nu(x)$ belongs to $\mathfrak{R}_\sigma(X)$ and is transformed in this space according to the irreducible representation with the signature σ.

The formula so obtained effects the restriction of τ to G_{n-1}.

3. The matrix elements of irreducible representations. We consider a chain of embedded subgroups:

$$G = G_n \supset G_{n-1} \supset G_{n-2} \supset \ldots \supset G_2 = \{e\}, \tag{98}$$

where G_i is isomorphic to $GL(i)$ and is embedded in G_{i+1} by the rule given at the beginning of 2. In other words, if $e_1, e_2, \ldots, e_n$ is a basis in which the matrices of G are expressed, then G_i leaves the linear envelope of $e_1, e_2, \ldots, e_{n-i}$ fixed and the linear envelope of $e_{n-i+1}, \ldots, e_n$ invariant.

The chain (98) enables us to choose in $\mathfrak{R}_\alpha(Z)$ a definite basis consisting of weight vectors with respect to D. This basis is constructed as follows. To begin with the dominant vectors $f_{\nu(1)}$ are chosen with respect to G_{n-1}; next, in every irreducible component with respect to G_{n-1} dominant vectors $f_{\nu(1),\nu(2)}$ are chosen with respect to G_{n-2} etc. As a result every basis vector $e_\nu(z)$ is numbered by a system of indices of the form

$$\nu = (\nu(1),\ \nu(2),\ \dots,\ \nu(n-1)),$$

where $\nu(i)$ is a row of integers ν_{ij}, $1 \leqslant i < j \leqslant n$, with definite restrictions on these numbers (see §13).[1]

Applying Theorem 14 step by step we are led to the construction of operators of the form

$$P_\nu = \Omega^{(n-1)}_{\nu(n-1)} \dots \Omega^{(2)}_{\nu(2)}\Omega^{(1)}_{\nu(1)}, \tag{99}$$

defined on Z that have the following properties:

I. *For every vector $f \in \mathfrak{R}_\alpha(Z)$ the formula*

$$c_\nu = P_\nu f(z)\,|_{z=e}$$

determines the coefficient in the decomposition of f with respect to the basis vectors $e_\nu(z)$.

By carrying out the corresponding construction in the conjugate space, replacing Z by Z, we can easily prove the following statement:

II. *The basis vectors $e_\nu(z)$ are computed by the formula*

$$e_\nu(z) = \Pi_\nu\, \alpha(z\zeta)\,|_{\zeta=e},$$

where Π_ν are the operators corresponding to (99) *constructed for the subgroup of lower triangular matrices ζ (with units along the main diagonal).*

Finally, the propositions I and II together enable us to obtain a general formula for the matrix elements of T_g in the basis $e_\nu(z)$:

$$T_{\mu\nu}(g) = P_\mu \alpha(zg\zeta)\,\Pi_\nu\Big|_{\substack{z=e\\ \zeta=e}}. \tag{100}$$

Here the operators P_μ act on the variable z and the operators Π_ν on the variable ζ, and for the sake of symmetry of notation we have written the operators Π_ν on the *right* of the functions on which they act.

In the special case when $n = 2$, we again obtain the formula of §3 that expresses the matrix elements in terms of Jacobi polynomials.

Now we make a further important remark. In the formula (100) the differential operators P_μ, Π_ν are computed at the unit point, and therefore their explicit form is considerably simplified: indeed, every infinitesimal operator D_{ij} is replaced by the operator $\dfrac{\partial}{\partial z_{ij}}$. As a result we arrive at the following construction.

We denote by $\dfrac{\partial}{\partial z}$ the triangular matrix of differential operators

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[1] We observe that in $\mathfrak{R}_\alpha(Z)$ a scalar product can be introduced with respect to which the basis $\{e_\nu(z)\}$ is orthogonal (this scalar product is invariant for all operators T_u, $u \in \mathfrak{U}$).

$$\frac{\partial}{\partial z}=\begin{pmatrix} 1 & \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{13}} & \cdots & \frac{\partial}{\partial z_{1n}} \\ & 1 & \frac{\partial}{\partial z_{23}} & \cdots & \frac{\partial}{\partial z_{2n}} \\ & & 1 & \cdots & \cdot \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

on Z and by $\frac{\partial}{\partial \zeta}$ the corresponding matrix

$$\frac{\partial}{\partial \zeta}=\begin{pmatrix} 1 & & & \\ \frac{\partial}{\partial \zeta_{21}} & 1 & & \\ \cdots & \cdots & \cdots & \\ \frac{\partial}{\partial \zeta_{n1}} & \frac{\partial}{\partial \zeta_{n2}} & \cdots & 1 \end{pmatrix}$$

on Z. We introduce the modified "inverse" matrix $\left(\frac{\partial}{\partial z}\right)^{-1}$ by the rule under 1. and also the matrix $\left(\frac{\partial}{\partial \zeta}\right)^{-1}$ that is defined similarly, replacing rows by columns. (We recall that auxiliary variables $\lambda_1, \ldots, \lambda_n$ are involved in the definition of the inverse matrices on which the function to be differentiated depends by means of a power product $\lambda_1^{r_1}\lambda_2^{r_2} \ldots \lambda_n^{r_n}$, where $(r_1, r_2, \ldots, r_n)$ is the signature of the irreducible representation in question; after differentiation the auxiliary variables are set equal to one.) As a result we obtain

THEOREM 15. *The matrix elements of the irreducible representation with the dominant weight* $\alpha(\delta)$ *are computed according to the formula*

$$T_{\mu\nu}(g) = Z_\mu \alpha(zg\zeta)Z_\nu|_{z=\zeta=e},$$

where the operators Z_μ, Z_ν *are defined as follows*:

$$Z_\mu=\prod_{i<j}\left[\left(\frac{\partial}{\partial z}\right)^{-1}\right]_{ij}^{\mu_{ij}}, \quad Z_\nu=\prod_{i>j}\left[\left(\frac{\partial}{\partial \zeta}\right)^{-1}\right]_{ij}^{\nu_{ij}}$$

Here the numbers μ_{ij} *determine the index* μ, *the numbers* ν_{ij} *determine the index* ν *and the operators* Z_ν *are written to the right of the functions on which they act.*

The structure of the formula we have obtained has an obvious geometric meaning. Indeed, as was mentioned in §2, the function $\alpha(g)$ can be defined in the following way:

$$\alpha(g)=\frac{(\varepsilon_0, T_g e_0)}{(\varepsilon_0, e_0)},$$

where T_g is an irreducible representation of the signature $\alpha(\delta)$, e_0 is Z-invariant in the representation space ε_0 is Z-invariant in the conjugate linear space. In other words, when the dominant vectors are normalized so that $(\varepsilon_0, e_0) = 1$, then we obtain

$$\alpha(g) = T_{00}(g)$$

- one of the matrix elements of the given representation. Furthermore, every weight vector e_ν in the representation space of T_g can be obtained by applying the operators T_ζ, $\zeta \in Z$, for sufficiently small ζ; similarly the weight vectors ε_μ in the conjugate space can be constructed by means of sufficiently small transformations $\hat{T}_z$, $z \in Z$. Theorem 15 gives the explicit form of corresponding infinitesimal transformation. It is clear that a similar structure also holds for the other classical linear groups.

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References

[1] H. Weyl, *The classical groups, their invariants and representations*, (*Klassicheskie gruppy, ikh invarianty i predstavleniya*, M., IL, 1947). (Translation of English original, 2nd ed., Princeton 1946).

[2] H. Weyl, Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen, Math. Z. 23, 24 (1926). (Russian translation in UMN* 4 (1938), 201-246).

[3] E. Cartan, Les groupes projectivs qui ne laissent invariante aucune multiple plane, Bull. Soc. Math. de France 41 (1913), 53-96.

[4] B.L. Casimir, van der Waerden, Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen, Math. Ann. 111 (1935), 1-12.

[5] I.M. Gel'fand, *Lektsii po lineinoi algebre* (Lectures on linear algebra), M.L., Gostekhizdat, 1948.

[6] I.M. Gel'fand, P.A. Minlos, Z.Ya. Shapiro, *Predstavleniya gruppy vrashcenii i gruppy Lorentsa, ikh primeneniya* (Representations of the rotation group and the Lorentz group, with applications), M. Fizmatgiz, 1958.

[7] I.M. Gel'fand, M.A. Naimark, Unitary representations of the classical groups, Trudy Mat. Inst. Steklov 36 (1950).

[8] I.M. Gel'fand, M.L. Tseitlin, Finite-dimensional representations of the group of unimodular matrices, DAN* 71, No. 5 (1950), 825-828.

[9] I.M. Gel'fand, M.L. Tseitlin, Finite-dimensional representations of the group of orthogonal matrices, DAN 71, No. 6 (1950), 1017-1020.

[10] I.M. Gel'fand, Z.Ya. Shapiro, Representations of the rotation group in three-dimensional space and their applications, UMN 7, 1 (1952), 3-117.

[11] R. Godement, A theory of spherical functions, I, Trans. Amer. Math. Soc. 7, No. 3 (1956), 496-556.

[12] D.P. Zhelobenko, On the theory of linear representations of complex and real Lie groups, Trudy Mockov. Mat. Obshch. 11 (1962).

[13] E. Cartan, *Lecons sur la théorie des spineurs* (*Teoriya spinorov*, M., IL, 1947). (Translation of French original, 2 vols., Paris 1938).

[14] E. Cartan, *Leçons sur la géométrie des espaces de Riemann* (*Geometriya rimanovykh proctranstv*, M.L. ONTI, 1936). (Translation of French original, 2 vols., Paris 1928).

[15] G.Ya. Lyabarskii, *Teoriya grupp i ee primenenie v fisike* (Theory of groups and its applications in physics), M., Fizmatgiz, 1958.

[16] A.I. Mal'tsev, On semisimple subgroups of Lie groups, Izv. Akad. Nauk SSSR. Ser. Mat. 8, No. 4 (1944), 143-174.

[17] M.A. Naimark, *Lineinye predstavleniya gruppy Lorentsa* (Linear representations of the Lorentz group), M., Gostekhizdat, 1958.

[18] M.A. Naimark, Linear representations of the Lorentz group, UMN 9, 4 (1954).

[19] L.S. Pontryagin, *Nepreryvnye gruppy* (Togological groups), Gostekhizdat, 1954.

[20] P.K. Rashevskii, On some fundamental theorems in the theory of Lie groups, UMN 8, 1 (1953).

[21] P.K. Rashevskii, Theory of spinors, UMN 10, 2 (1954).
[22] Loo-keng Hua (Khua Lo-ken), *Garmonicheskii analyz funktsii mnogikh kompleksnykh peremennykh v klassicheskykh oblastyakh* (Harmonic analysis of functions of several complex variables in the classical systems), M., IL 1959.
[23] C. Chevalley, *Theorie des groupes de Lie* (*Teoriya grupp Li*, M., IL 1948). (Translation of French original, Vol. 1, Paris 1951).
[24] C. Chevalley, *Theorie des groupes de Lie* (*Teoriya grupp Li*, M., IL 1958). (Translation of French original, Vol. 2, Paris 1953).

* DAN = Dokl. Akad. Nauk SSSR. UMN = Uspekhi Mat. Nauk

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