THEORY OF GROUP REPRESENTATIONS AND APPLICATIONS

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Dedicated to

OLA and PIERRETTE

Preface

This book is written primarily for physicists, but also for other scientists and mathematicians to acquaint them with the more modern and powerful methods and results of the theory of topological groups and of group representations and to show the remarkable wide scope of applications. In this respect it is markedly different than, and goes much beyond, the standard books on group theory in quantum mechanics. Although we aimed at a mathematically rigorous level, we tried to make the exposition very explicit, the language less abstract, and have illustrated the results by many examples and applications.

During the past two decades many investigations by physicists and mathematicians have brought a certain degree of maturity and completeness to the theory of group representations. We have in mind the new results in the development of the general theory, as well as many explicit constructions of representations of specific groups. At the same time new applications of, in particular, noncompact groups revealed interesting structures in the symmetry, as well as in the dynamics, of quantum theory. The mathematical sophistication and knowledge of the physicists have also markedly increased. For all these reasons it is timely to collect the new results and to present a book on a much higher level than before, in order to facilitate further developments and applications of group representations.

There is no other comparable book on group representations, neither in mathematical nor in physical literature, and we hope that it will prove to be useful in many areas of research.

Many of the results appear, to our knowledge, for the first time in book form. These include, in particular, a systematic exposition of the theory and applications of induced representations, the classification of all finite-dimensional irreducible representations of arbitrary Lie groups, the representation theory of Lie and enveloping algebras by means of unbounded operators, new integrability conditions for representations of Lie algebras and harmonic analysis on homogeneous spaces.

In the domain of applications, we have discussed the general problem of symmetries in quantum theory, in particular, relativistic invariance, group theoretical derivation of relativistic wave equations, as well as various applications of group representations to dynamical problems in quantum theory.

We have tried to achieve a certain amount of completeness so that the book can be used as a textbook for an advanced course in mathematical physics on Lie algebras, Lie groups and their representations. Some of the standard topics can be found scattered in various texts but, so far, not all under a single cover.

A book in the border area of theoretical physics and pure mathematics is always problematic. And so this book may seem to be too difficult, detailed and abstract to some physicists, and not detailed and complete enough for some mathematicians, as we have deliberately omitted a number of proofs. Fortunately the demand for knowledge of modern mathematics among physicists is on the rise. And to give the proofs of all theorems in such a wide area of mathematics is impossible even in a large volume as this one. Where too long technical details would cloud the clarity and when the steps of the proof did not seem to be essential for further development of the subject we have omitted the proofs.

The material collected in this book originated from lectures given by the authors over many years in Warsaw, Trieste, Schladming, Istanbul, Göteborg and Boulder. It has passed several rewritings. We are especially grateful to many friends and colleagues who read, corrected and commented on parts of the manuscript. We would like to thank Dr. S. Woronowicz for his careful and patient reading of the entire manuscript and pointing out numerous improvements and corrections. We have discussed parts of the manuscript with many of our friends and colleagues who made constructive criticism, in particular S. Dymus, M. Flato, B. Kostant, G. Mackey, K. Maurin, L. Michel, I. Segal, D. Sternheimer, S. Ström, A. Sym, I. Szczyrba and A. Wawrzyńczyk.

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Finally, we would like to express our gratitude to Mr J. Panz, editor in the Polish Scientific Publishers, for his great help in preparing this manuscript for printing. We are also obliged to Mrs Z. Osek for her kind help in all phases of preparing the manuscript for publication.

A. O. Barut and R. Rączka

Boulder and Warsaw, August 1976

Contents

Outline of the Book

The book begins with a long chapter on Lie algebras. This is a self-contained detailed exposition of the theory and applications of Lie algebras. The theory of Lie algebras is an independent discipline in its own right and the chapter can be read independently of others. We give, after basic concepts, the structure and theory of arbitrary Lie algebras, a description of nilpotent and solvable algebras and a complete classification of both complex and real simple Lie algebras. Another feature is the detailed discussion of decomposition theorems of Lie algebras, i.e., Gauss, Cartan and Iwasawa decompositions.

Ch. 2 begins with a review of the properties of topological spaces, in order to introduce the concepts of topological groups. The general properties of topological groups such as compactness, connectedness and metric properties are treated. We discuss further integration over the group manifold, i.e., the invariant measure (Haar measure) on the group. The fundamental Mackey decomposition theorem of topological groups is also given.

Ch. 3 begins with a review of differentiable manifolds, their analytic structures and tangent spaces. With these preparations on topological groups and differentiable manifolds we introduce Lie groups as topological groups with an analytic structure and derive the basic relations between Lie groups and Lie algebras. The remaining sections of ch. 3 are devoted to the composition and decomposition properties of groups (i.e., Levi–Malcev, Gauss, Cartan, Iwasawa decompositions), to the classification of Lie groups and to some results on the structure of Lie groups and to the construction of invariant measure and of invariant metric.

In the next chapter, 4, we introduce the concepts of homogeneous and symmetric spaces on which groups act. These concepts play an important role in the modern theory of group representations and in physical applications. We further give a classification of globally symmetric Riemannian spaces associated with the classical simple Lie groups. Also discussed in this chapter is the concept of quasi-invariant measure, because invariant measures do not exist in general on homogeneous spaces.

The theory of group representations, the main theme of the book, begins in ch. 5 where we first give the definitions, the general properties of representations, irreducibility, equivalence, tensor and direct product of representations. We further treat the Mautner and the Gel'fand–Raikov theorems on the decomposition and completeness of group representations.

The detailed group representation theory is then developed in successive steps beginning with the simplest case of commutative groups, in ch. 6, followed by the representations of compact groups, in ch. 7. For completeness we also review here, as a special case, the representations of finite groups. The representation theory of compact groups is complete and we give the general theorems (the Peter–Weyl and Weyl approximation theorems) of this theory. With a view towards applications, we discuss also the projection operators, decomposition of the representations and of tensor products.

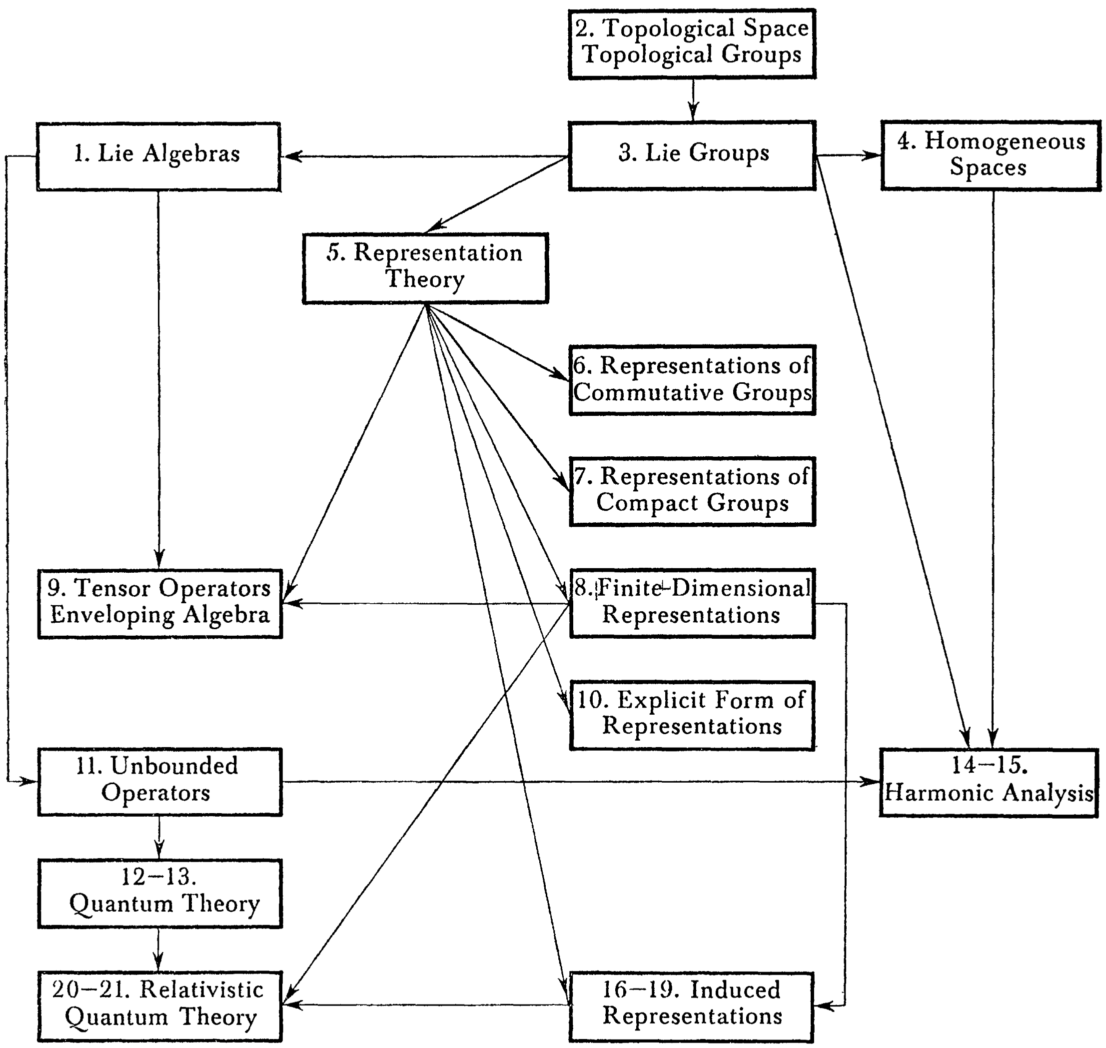
Next comes the description of all finite-dimensional irreducible representations of arbitrary Lie groups (compact or non-compact) (ch. 8). Here we give a more complete treatment of the properties of representations of semisimple groups than is available, to our knowledge, in any other book. The methods for the explicit construction of the finite-dimensional representations are treated in ch. 10, after a necessary discussion of tensor operators, enveloping algebras and invariant or Casimir operators and their spectra in ch. 9. (These concepts are used to specify and label the representations.) Among the methods we give the Gel'fand–Zetlin method, the tensor method, the method of harmonic functions and the method of creation and annihilation operators.

Ch. 11 deals with the representation theory of Lie and enveloping algebras by unbounded operators and the related questions of integrability of Lie algebra representations to the representations of the corresponding Lie groups. This is one of the most important chapters of the book. The theory of unbounded operators is also important for applications because most of the observables in quantum theory are represented by unbounded operators. More specifically, the theory of analytic vectors for Lie groups and Lie algebras is presented.

In chs. 12 and 13 we give a treatment of the role that the theory of group representation plays in all areas of quantum theory and specific applications. The mathematical structure of group representations in the Hilbert space is particularly adapted to quantum theory. In fact, we can base the framework of quantum theory solely on the concept of group representations. Historically, also, the concepts of Hilbert space and representation of groups in the Hilbert space had their origin in quantum theory. We also discuss the concepts of kinematical and dynamical symmetries, a classification of basic symmetries of physics and the use of group representations in solving dynamical problems in quantum mechanics.

The next two chapters (14 and 15) are devoted to harmonic analysis on Lie groups and on homogeneous and symmetric spaces. Here the theory encompasses a generalization of the Fourier expansion for non-commutative groups, the corresponding spectral synthesis and Plancherel formulas. We discuss the general theory as well as specific applications to some simple and semi-direct product groups.

The following four chapters, 16–19, are devoted to the theory of induced representations, one of the most important themes of the book. Already in ch. 8 we have used induced representations to obtain a classification as well as the explicit form of all irreducible finite-dimensional representations of Lie groups. Here the general theory is presented.



Ch. 16 deals with the basic properties of induced representations and the fundamental imprimitivity theorem. In the next chapter, 17, the induced representations of semi-direct product of groups is given, with a derivation of the complete classification of all representations of the Poincaré group. The further properties of induced representations (the induction-reduction theorem, the tensor product theorem and the Frobenius reciprocity theorem) are discussed in ch. 18. In ch. 19 the theory is applied to derive explicitly the induced irreducible unitary, hence infinite-dimensional, representations of principal and supplementary series of complex classical Lie groups.

Finally, in chs. 20–21, we take up applications of the imprimitivity theorem and induced representations of the Poincaré group in quantum physics: first to the concept of relativistic position operator and to the proof of equivalence of Heisenberg and Schrödinger descriptions in non-relativistic quantum mechanics (in ch. 20), next, in ch. 21, to the classification of all finite-dimensional relativistic wave equations, to applications of imaginary mass representations, to Gel'fand–Yaglom type and infinite component relativistic wave equations, and to the problem of group extension of the representations of the relativity group by discrete operations and by other symmetry groups.

A number of mathematical concepts which are not so familiar to physicists and which are essential for the book have been collected in the appendices on functional analysis, and on other results from algebra, topology, integration theory, etc.

Each chapter contains at the end notes on further developments of the subject as well as exercises.

Notations

Our space-time metric $g_{\mu\nu}$ is such that $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The symbol a^* denotes the hermitian conjugation of a matrix or an operator a. The symbols $\bar{a}$ and a^T denote the complex conjugation and the transposition of a matrix a. The symbol ▼ denotes the end of the proof of a theorem or of an example. The direct sum of vector spaces V_i is written as $V_1 \dot{+} V_2 \dot{+} \ldots$ and the direct sum of Lie algebras L_i as $L_1 \oplus L_2 \oplus \ldots$ The semidirect sum of two Lie algebras is denoted by $L_1 \oplus\!\!\!\!\rightarrow L_2$ and the semidirect product of two groups as $G_1 \circledS G_2$, while the direct product of two groups is written as $G_1 \times G_2$. The expression 'th. 8.6.3' means theorem 3 of chapter 8 in section 6. The expression 'exercise 9.7.3.1' means exercise § 3.1 in chapter 9, section 7. The quotation, say 'Lunn 1969', denotes the reference to the paper of the author Lunn from the year 1969: if there are several papers of the same author in a given year we have additional index a, b, ..., etc.

The symbol

$$\left[\frac{1}{2}n\right] = \begin{cases} \frac{1}{2}n & \text{if } n = 2r, \\ \frac{1}{2}(n-1) & \text{if } n = 2r+1; \end{cases}$$

the symbol

$$\left\{\frac{1}{2}n\right\} = \begin{cases} \frac{1}{2}n & \text{if } n = 2r, \\ \frac{1}{2}(n+1) & \text{if } n = 2r+1. \end{cases}$$

We use throughout Einstein summation convention unless stated otherwise.

For the sake of simplicity we use the symbol $\sqrt{}(\ldots)$ instead of $\sqrt{(\ldots)}$ for roots.

Chapter 1

Lie Algebras

For didactic reasons we have found it advantageous to begin with the discussion of Lie algebras first then go over to the topological concepts and to Lie groups. The theory of Lie algebras has become a discipline in its own right.

§ 1. Basic Concepts and General Properties

A. *Lie Algebras*

Let L be a finite-dimensional vector space over the field K of real or complex numbers. The vector space L is called a *Lie algebra over* K if there is a rule of composition $(X, Y) \to [X, Y]$ in L which satisfies the following axioms:

$$[\alpha X+\beta Y, Z] = \alpha[X, Z]+\beta[Y, Z] \quad \text{for } \alpha, \beta \in K \quad \text{(linearity)}, \tag{1}$$

$$[X, Y] = -[Y, X] \quad \text{for all } X, Y \in L \quad \text{(antisymmetry)}, \tag{2}$$

$$[X, [Y, Z]]+[Y, [Z, X]]+[Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in L. \tag{3}$$

The third axiom is the Jacobi identity (or Jacobi associativity). The operation $[\ ,\ ]$ is called *Lie multiplication*. From axiom (3) it follows that this Lie multiplication is, in general, non-associative. If K is the field of real (complex) numbers, then L is called a *real* (*complex*) *Lie algebra*. A Lie algebra is said to be *abelian* or *commutative* if for any $X, Y \in L$ we have $[X, Y] = 0$.

Consider two subsets M and N of vectors of the Lie algebra L and denote by $[M, N]$ the linear hull of all vectors of the form $[X, Y]$, $X \in M$, $Y \in N$. If M and N are linear subspaces of an algebra L, then the following relations hold:

$$[M_1+M_2, N] \subset [M_1, N]+[M_2, N], \tag{4a}$$

$$[M, N] = [N, M], \tag{4b}$$

$$[L, [M, N]] \subset [M, [N, L]]+[N, [L, M]]. \tag{4c}$$

These relations can be readily verified using the axioms (1)–(3). A subspace N of the algebra L is a *subalgebra*, if $[N, N] \subset N$, and an *ideal*, if $[L, N] \subset N$. Clearly, an ideal is automatically a subalgebra. A *maximal ideal* N, which satisfies the condition $[L, N] = 0$ is called the *center* of L, and because $[N, N] = 0$, the center is always commutative.

Let $e_1, \ldots, e_n$ be a basis in our vector space L. Then, because of linearity, the commutator $Z = [X, Y]$, when expressed in terms of coordinates, (i.e. $X = x^i e_i$, etc.) takes the form*

$$z^i = [X, Y]^i = c_{jk}{}^i x^j y^k, \quad i, j, k = 1, 2, \ldots, n, \tag{5}$$

with $[e_j, e_k] = c_{jk}{}^i e_i$. The numbers $c_{jk}{}^i$ are called the *structure constants*, and n the *dimension* of the Lie algebra L. It follows from axioms (2) and (3) that the structure constants $c_{jk}{}^i$ satisfy the conditions:

$$c_{jk}{}^i = -c_{kj}{}^i, \tag{6}$$

$$c_{is}{}^p c_{jk}{}^s + c_{js}{}^p c_{ki}{}^s + c_{ks}{}^p c_{ij}{}^s = 0. \tag{7}$$

The existence of subalgebras or ideals of a Lie algebra L is reflected in certain definite restrictions on the structure constants. If $e_1, e_2, \ldots, e_k$ are the basis elements of a subalgebra, then the structure constants must satisfy the relations

$$c_{ij}{}^s = 0 \quad \text{for } i, j \leqslant k,\ s > k, \tag{8}$$

and, if they are the basis elements of an ideal, then

$$c_{ij}{}^s = 0 \quad \text{for } i \leqslant k,\ s > k \text{ and an arbitrary } j. \tag{9}$$

The structure constants are not constants as their name might imply. In fact, it follows from definition (5) that under a change of basis in the algebra L, the $c_{ij}{}^k$ transform as a third rank tensor with one contravariant and two covariant indices.

EXAMPLE 1. Let L be the set of all skew-hermitian traceless 2×2 matrices. Clearly, L is of (real) dimension three. Let us choose in L the basis

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad e_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \frac{1}{2}\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix},$$

and define the commutator $[X, Y]$ in L as follows:

$$[X, Y] = XY - YX, \quad X, Y \in L. \tag{10}$$

It is readily verified that this commutator satisfies the axioms (1)–(3) for the Lie multiplication. Using (10) we find that e_i satisfy the following commutation relations

$$[e_i, e_k] = \varepsilon_{ikl} e_l, \quad i, k, l = 1, 2, 3,$$

where ε_{ikl} is the totally antisymmetric tensor in R^3. The elements of L are linear combinations of e_i with real coefficients. The matrices $\sigma_k = 2\mathrm{i}\, e_k$ are called Pauli matrices and satisfy $[\sigma_i, \sigma_k] = 2\mathrm{i}\, \varepsilon_{ikl} \sigma_l$.

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* We use the summation convention over repeated indices throughout the book.

Hence L is the three-dimensional, real Lie algebra with the structure constants $c_{ik}{}^{l}$ given by

$$c_{ikl} = \varepsilon_{ikl}.$$

The algebra L so defined is denoted by the symbol su(2) (or o(3)), in anticipation of the classification of Lie algebras discussed in §§ 4 and 5.

Remark: If A is any finite-dimensional associative algebra with the multiplication law $(X, Y) \to X \cdot Y$ one can obtain a Lie algebra by interpreting the Lie composition rule $[X, Y]$ as$(X \cdot Y - Y \cdot X)$. (See also Ado's theorem in § 2.)

EXAMPLE 2. Let L be the vector space of all $n \times n$ real matrices $\{x_{ik}\}$, $i, k = 1, 2, \ldots, n$, over the field R of real numbers. This vector space L with the Lie multiplication (10) is again a real Lie algebra. It is the full real linear Lie algebra and is denoted by the symbol gl(n, R).

The subset M consisting of all skew-symmetric matrices X satisfying $X^T = -X$ is also closed under the Lie multiplication (10). Therefore M is a subalgebra which is denoted by the symbol o(n). The subset N of matrices of the form λI, multiples of identity, obeys

$$[\mathrm{gl}(n, R), N] = 0.$$

Hence, N is a one-dimensional subalgebra contained in the center of gl(n, R).

We can introduce the so-called *Weyl basis* in gl(n, R) by taking as the basis elements e_{ij}, $i, j = 1, 2, \ldots, n$, the $n \times n$ matrices of the form

$$(e_{ij})_{lk} = \delta_{il}\,\delta_{jk}, \tag{11}$$

which satisfy the following commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \quad i, j, k, l = 1, 2, \ldots, n. \tag{12}$$

From (11) and (12) one can immediately read off the structure constants:

$$c_{sm,kr}{}^{ij} = \delta_s{}^i\,\delta_{mk}\,\delta_r{}^j - \delta_k{}^i\,\delta_{rs}\,\delta_m{}^j. \tag{13}$$

The basis vectors $\tilde{e}_{ik}$ for the o(n) subalgebra can be taken to be of the form

$$\tilde{e}_{ik} = e_{ik} - e_{ki}, \quad i, k, = 1, 2, \ldots, n. \;\blacktriangledown \tag{14}$$

The complex extension V_c of a real vector space V is the complex vector space consisting of all elements z of the form $z = x + \mathrm{i}y$, $x, y \in V$. The multiplication of an element $z \in V_c$ by a complex number $\gamma = \alpha + \mathrm{i}\beta \in C$ is defined by

$$\gamma z = \alpha x - \beta y + \mathrm{i}(\alpha y + \beta x).$$

The *complex extension* L^c of a real Lie algebra L is a complex Lie algebra which satisfies the following conditions:

(i) L^c is the complex extension of the real vector space L.

(ii) The Lie multiplication in L^c is

$$\begin{aligned} Z &= [Z_1, Z_2] = [X_1 + \mathrm{i}Y_1, X_2 + \mathrm{i}Y_2] \\ &= [X_1, X_2] - [Y_1, Y_2] + \mathrm{i}[X_1, Y_2] + \mathrm{i}[Y_1, X_2] \\ &\equiv X + \mathrm{i}Y. \end{aligned} \tag{15}$$

A complex Lie algebra L of dimension n with a basis $\{e_i\}_1^n$ can also be considered to be a real Lie algebra of dimension $2n$ with the basis vectors $e_1, ie_1, \dots, e_n, ie_n$. The real Lie algebra so defined will be denoted by the symbol L^R. Conversely, a real form L^r of a complex Lie algebra L^c is a real Lie algebra whose complex extension is L^c.

The Algebras A_n, B_n, C_n and D_n

The complex extension of $\mathrm{gl}(n, R)$ is the set of all complex $n \times n$ matrices with the Lie multiplication (10). It is called the *full complex linear Lie algebra* and is denoted by $\mathrm{gl}(n, C)$. The subset of all $n \times n$ complex matrices with trace zero is a subalgebra of $\mathrm{gl}(n, c)$ and is denoted by $\mathrm{sl}(n, c)$ or A_{n-1}.

Other sequences of complex algebras are associated with various bilinear forms. Let $\Phi(\xi, \eta)$ be a bilinear form defined in an m-dimensional complex vector space V^m. The linear transformations X which act on V^m and satisfy the condition

$$\Phi(X\xi, \eta) + \Phi(\xi, X\eta) = 0, \qquad \xi, \eta \in V^m,$$

generate a linear Lie algebra L. Indeed, if the above relation is true for X and Y, then

$$\begin{aligned} \Phi([X, Y]\xi, \eta) &= \Phi(XY\xi, \eta) - \Phi(YX\xi, \eta) \\ &= -\Phi(\xi, [X, Y]\eta), \end{aligned}$$

where the Lie multiplication of X and Y is defined as $[X, Y] = XY - YX$.

If the bilinear form $\Phi(\xi, \eta)$ is non-singular* and symmetric (e.g. $\Phi \equiv \xi_i \eta_i$), then L is called an *orthogonal Lie algebra*. For $m = 2n+1$, $n = 1, 2, \dots$ the sequence of corresponding algebras is denoted by $\mathrm{o}(2n+1, C)$ or B_n, and for $m = 2n$, by $\mathrm{o}(2n, C)$ or D_n.

The algebras associated with non-singular skew-symmetric bilinear forms are called *symplectic Lie algebras*. It is known from elementary algebra, that skew-symmetric forms in odd-dimensional spaces are always singular (det $= 0$). Therefore the symplectic algebras can only be realized in even-dimensional complex spaces V^{2n} and are denoted by $\mathrm{sp}(n, C)$ or C_n.

The algebras A_n, B_n, C_n and D_n, $n = 1, 2, \dots,$ form the set of the classical complex Lie algebras.

Direct Sums and Quotient Algebras

Let V_i, $i = 1, 2, \dots, k$, be subspaces of a vector space V and let

$$D = \sum_{i=1}^{k} V_i \tag{16}$$

---

* A bilinear form $\Phi(\xi, \eta)$ is non-singular, if for every $\xi_0 \in V^m$, the linear form $\Phi(\xi_0, \eta)$ is not identically 0 in η. In coordinate form, $\Phi(\xi, \eta) = \xi^i a_{ij} \eta^j$ is non-singular if and only if its matrix $a_{ij}]$ is non-singular: $\det[a_{ij}] \neq 0$.

be the collection of all vectors of the form

$$d = \sum_{i=1}^{k} v_i, \qquad v_i \in V_i,\ i = 1, 2, \ldots, k. \tag{17}$$

If each vector $d \in D$ has a unique representation in the form (17), then we say that D is the *direct sum* of subspaces V_i, $i = 1, 2, \ldots, k$, and we write

$$D = V_1 \dot{+} V_2 \dot{+} \ldots \dot{+} V_k = \sum_{i=1}^{k} \dot{+} V_i. \tag{18}$$

If a Lie algebra L, as a vector space, can be written as a direct sum in the form (18), i.e. $L = L_1 \dot{+} L_2 \dot{+} \ldots \dot{+} L_k$, and, if in addition

$$[L_i, L_i] \subset L_i, \qquad [L_i, L_j] = 0, \qquad i, j = 1, 2, \ldots, k, \tag{19}$$

then L is said to be *decomposed* into a direct sum of Lie algebras $L_1, L_2, \ldots, L_k$ and is denoted by $L = L_1 \oplus L_2 \oplus \ldots \oplus L_k$.

Clearly the subalgebras L_i, $i = 1, 2, \ldots, k$, are ideals of L, because

$$[L, L_i] = [L_i, L_i] \subset L_i. \tag{20}$$

Moreover if N is an ideal of a subalgebra L_i, then N is also an ideal of the algebra L.

Let N be a subalgebra of some Lie algebra L. We introduce in the space L the relation

$$X \simeq Y \,(\mathrm{mod}\, N), \tag{21}$$

if $X - Y \in N$, that is a vector in X is a sum of a vector in Y *and* a vector n in N. This relation satisfies

1° $X \simeq X$,

2° if $X \simeq Y$, then $Y \simeq X$,

3° if $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$,

and, therefore, is an equivalence relation. The whole algebra L decomposes into the disjoint classes $K_x = X + N$ of equivalent elements. The set $\{K_x\}$ of all classes does not form in general a Lie algebra: in fact, if

$$X_1 \simeq Y_1 \,(\mathrm{mod}\, N), \quad \text{i.e.,} \quad X_1 = Y_1 + n_1,$$
$$X_2 \simeq Y_2 \,(\mathrm{mod}\, N), \quad \text{i.e.,} \quad X_2 = Y_2 + n_2$$

then

$$[X_1, X_2] = [Y_1, Y_2] + [Y_1, n_2] + [n_1, Y_2] + [n_1, n_2]. \tag{22}$$

Therefore, in general, the relation

$$[X_1, X_2] \simeq [Y_1, Y_2] \,(\mathrm{mod}\, N) \tag{22a}$$

does not hold. However, if the subalgebra N is in addition an ideal, then the last three terms in eq. (22) are contained in N and the condition (22a) is satisfied.

The resulting Lie algebra is called the *quotient Lie algebra* of L with respect to N and is denoted by L/N.

EXAMPLE 3. Let P be the Poincaré algebra with the commutation relations

$$\left.\begin{aligned}[M_{\mu\nu}, M_{\varrho\sigma}] &= g_{\mu\varrho}M_{\nu\sigma}+g_{\nu\sigma}M_{\mu\varrho}-g_{\nu\varrho}M_{\mu\sigma}-g_{\mu\sigma}M_{\nu\varrho},\\ M_{\mu\nu} &= -M_{\nu\mu},\end{aligned}\right\} \tag{23a}$$

$$[M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \tag{23b}$$

$$[P_\mu, P_\nu] = 0, \tag{23c}$$

where $\mu, \nu, \ldots = 0, 1, 2, 3$, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

The set t^4 of linear combinations of P_ν, $\nu = 0, 1, 2, 3$ (the generators of translation), is an ideal of P. If we introduce an equivalence relation

$$X \simeq Y (\mathrm{mod}\, t^4), \quad X, Y \in P,$$

then the set of classes $K_x = X + t^4$, $X \in P$, of equivalent elements forms a six-dimensional (quotient) Lie algebra which is isomorphic to the Lorentz algebra so(3, 1) generated by $M_{\mu\nu}$ of eq. (23a). On the other hand the equivalence relation

$$X \simeq Y \big((\mathrm{mod\, so}(3, 1)\big) \quad X, Y \in P,$$

defines a four-dimensional quotient vector space (but not a quotient Lie algebra). ▼

B. *Operations over Lie Algebras*

We shall now discuss the properties of various operations defined over Lie algebras. Let L and L' be two arbitrary Lie algebras over the set of real or complex numbers and let φ be a map of L into L'. A map φ is called a *homomorphism* if

$$\varphi(\alpha X+\beta Y) = \alpha\varphi(X)+\beta\varphi(Y), \quad X, Y \in L,\ \alpha, \beta \in K, \tag{24a}$$

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)], \quad X, Y \in L. \tag{24b}$$

The set N

$$N = \{X \in L \colon \varphi(X) = 0\}$$

is called the *kernel* of the homomorphism φ. It is an ideal of L. In fact, if $X \in L$ and $Y \in N$ then,

$$\varphi([X, Y]) = [\varphi(X), 0] = 0,$$

i.e., $[X, Y] \in N$. One can readily verify that L/N is isomorphic with $\varphi(L)$.

Let N be an ideal of a Lie algebra L. The map

$$\varphi \colon X \to X+N$$

is called the *natural homomorphism of L onto L/N*. A one-to-one homomorphism of one algebra onto another is called an *isomorphism*, and the corresponding algebras L, and L' are said to be *isomorphic*: in this case we shall write $L \sim L'$

An isomorphic map of L onto itself is called an *automorphism*. An automorphism φ of a Lie algebra L is called *involutive* if $\varphi^2 = I$.

A map σ of a complex Lie algebra into itself which satisfies the conditions

$$\sigma(\lambda X+\mu Y) = \bar{\lambda}\sigma(X)+\bar{\mu}\sigma(Y), \qquad \sigma[X, Y] = [\sigma(X), \sigma(Y)], \qquad \sigma^2 = I, \quad (25)$$

is called a *conjugation*. For instance if L^c is the complex extension of a real Lie algebra L, then the map

$$\sigma\colon X+\mathrm{i}Y \to X-\mathrm{i}Y, \qquad X, Y \in L,$$

defines a conjugation in L^c. Note that a conjugation σ is not an automorphism of L because it is antilinear.

A *derivation** D of a Lie algebra L is a linear mapping of L into itself satisfying

$$D([X, Y]) = [D(X), Y]+[X, D(Y)], \qquad X, Y \in L. \quad (26)$$

It is evident that if D_1 and D_2 are two derivations of L, then $\alpha D_1+\beta D_2$ is also a derivation. Moreover, if D_1 and D_2 are derivations, then

$$\begin{aligned} D_1 D_2([X, Y]) &= D_1\{[D_2X, Y]+[X, D_2Y]\} \\ &= [D_1D_2X, Y]+[D_2X, D_1Y]+[D_1X, D_2Y]+[X, D_1D_2Y]. \end{aligned}$$

Interchanging indices 1 and 2 and subtracting, we get

$$[D_1, D_2]([X, Y]) = [[D_1, D_2]X, Y]+[X, [D_1, D_2]Y], \quad (27)$$

i.e., the commutator of two derivations is again a derivation. Therefore, the set L_A of all derivations forms a Lie algebra itself, the *derivation algebra* L_A. It is interesting to note that the algebra L_A is the Lie algebra of the group of all automorphisms G_A of the original algebra L. In fact, if $\varphi_t = \exp(\mathrm{i}At)$ is a one-parameter group of automorphism of L, that is,

$$\varphi_t([X, Y]) = [\varphi_t(X), \varphi_t(Y)] \in L, \qquad X, Y \in L, \quad (28)$$

then differentiation with respect to t gives for $t = 0$

$$A([X, Y]) = [AX, Y]+[X, AY],$$

i.e., the generator A of the one-parameter subgroup $\varphi(t)$ of automorphisms is a derivation. Conversely, one can show also that if A satisfies eq. (26), then the corresponding one-parameter subgroup satisfies eq. (28) (cf. exercise 1.8).

Let L be a Lie algebra over the real numbers R or the complex numbers C. Consider the linear map $\mathrm{ad}X$ of L into itself defined by

$$\mathrm{ad}X(Y) \equiv [X, Y], \qquad X, Y \in L. \quad (29)$$

Using the Jacobi identity (3), we get

$$\mathrm{ad}X([Y, Z]) = [\mathrm{ad}X(Y), Z]+[Y, \mathrm{ad}X(Z)], \quad (30)$$

* Also called an *infinitesimal automorphism* of L in older publications.

i.e., the map $\operatorname{ad}X$ represents a derivation of L. Furthermore, using (29) and the Jacobi identity we obtain

$$\operatorname{ad}[X, Y](Z) = [\operatorname{ad}X, \operatorname{ad}Y](Z). \tag{31}$$

Hence the set $L_a = \{\operatorname{ad}X, X \in L\}$ is a linear Lie algebra, a subalgebra of the Lie algebra L_A of all derivations and is called the *adjoint algebra*. The map ψ: $X \to \operatorname{ad}X$ is the homomorphism of L onto L_a. Clearly the kernel of the homomorphism ψ is the center of L.

The Lie algebra L_a is moreover an ideal of the Lie algebra L_A of all derivations. In fact, if $D \in L_A$ and $Y \in L$, we have

$$[D, \operatorname{ad}X](Y) = D[X, Y]-[X, DY] = [DX, Y] = \operatorname{ad}DX(Y), \tag{32}$$

i.e.,

$$[D, \operatorname{ad}X] \in L_a.$$

Note finally that if φ is any automorphism of L, then, by (29) and (24b), we have:

$$\operatorname{ad}\varphi(X)(Y) = [\varphi(X), Y] = \varphi([X, \varphi^{-1}Y]) = \varphi\{\operatorname{ad}X[\varphi^{-1}(Y)]\},$$

i.e.,

$$\operatorname{ad}\varphi(X) = \varphi\operatorname{ad}X\varphi^{-1}. \tag{33}$$

We refer further to ch. 3.3 for the groups $G_A\,(G_a)$ of all (inner) automorphisms of L and their Lie algebras.

EXAMPLE 4. Referring to the three-dimensional Lie algebra of example 1, we see that $\operatorname{ad}e_1(e_1) = 0$, $\operatorname{ad}e_1(e_2) = e_3$, $\operatorname{ad}e_1(e_3) = -e_2$, etc. Hence L_a is again a three-dimensional Lie algebra, and can be represented in the basis $\{e_i\}$ by matrices

$$\operatorname{ad}e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \operatorname{ad}e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad}e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., L_a is the set of three-dimensional skew-symmetric matrices, i.e., $\mathrm{o}(3)$.

Similarly, the adjoint algebra for $\mathrm{gl}(n, R)$, example 2, is given by the relations $\operatorname{ad}e_{ij}(e_{kl}) = \delta_{jk}e_{il}-\delta_{il}e_{kj}$ (dimension n^2), and that of $\mathrm{o}(n)$ is given by the set of $n(n-1)/2$-dimensional skew-symmetric matrices. ▼

We introduce now the important concept of a *semidirect sum* of two Lie algebras. Let T and M be two Lie algebras and let D be the homomorphism of M into the set of linear operators in the vector space T such that every operator $D(X)$, $X \in M$, is a derivation of T. We endow the direct sum of vector spaces $T \dotplus M$ with a Lie algebra structure by using the given Lie brackets of T and M in each subspace, and for the Lie brackets between the two subspaces, we set

$$[X, Y] = \big(D(X)\big)(Y) \quad \text{for } X \in M,\ Y \in T. \tag{34}$$

It is evident that all the axioms (1)–(3) for a Lie algebra are satisfied: in particular, by virtue of eq. (34), we have for $X \in M$, $Y_i \in T$

$$[X, [Y_1, Y_2]]+[Y_2, [X, Y_1]]+[Y_1, [Y_2, X]] \\ = D(X)([Y_1, Y_2])-[D(X)Y_1, Y_2]-[Y_1, D(X)Y_2],$$

which is zero since $D(X)$ is a derivation. The Lie algebra so obtained is called the *semidirect sum* of T and M. The subalgebra T, by virtue of eq. (34), is an ideal of the semidirect sum. In other words, a Lie algebra L is a semidirect sum of subalgebras T and M, if $L = T \dot{+} M$, and,

$$[T, T] \subset T, \qquad [M, M] \subset M, \qquad [M, T] \subset T. \tag{35}$$

We shall use the symbol $T \oplus M$ for a semidirect sum, writing the ideal T first, and the subalgebra M second.

EXAMPLE 5. The Poincaré algebra of example 3 is the semidirect sum of the ideal t^4 and the Lorentz algebra so(3, 1) with

$$D(M_{\mu\nu})(P_\sigma) = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$

i.e.

$$P = t^4 \oplus \mathrm{so}(3, 1). \tag{36}$$

C. *Representations of Lie Algebras*

Let L be a Lie algebra over the field K and let H be a linear space. A *representation* of L in H is a homomorphism $X \to T(X)$ of L into the set of linear operators in H, i.e., for X, Y in L and α, β in K, we have

$$\alpha X+\beta Y \to \alpha T(X)+\beta T(Y), \tag{37}$$

$$[X, Y] \to [T(X), T(Y)] \equiv T(X)T(Y)-T(Y)T(X). \tag{38}$$

Notice that by virtue of eq. (38) the Jacobi identity (3) is automatically satisfied.

If the carrier space H is infinite-dimensional, then we assume in addition that the operators $T(X)$ for all $X \in L$ have a common linear invariant domain D which is dense in H. (Cf. ch. 11.1 for an explicit construction of the domain D.)

EXAMPLE 6. Let L be an arbitrary Lie algebra with the commutation relations

$$[X_l, X_k] = c_{lk}{}^j X_j, \qquad l, k = 1, 2, \ldots, n,$$

where the structure constants $c_{lk}{}^j$ are taken to be real.

Then the map

$$X_l \to T(X_l) = C_l \equiv \{-c_{lk}{}^j\} \tag{39}$$

of L into $n \times n$ matrices in R^n provides a finite-dimensional representation of L. Indeed, by virtue of eqs. (6) and (7) we have

$$[T(X_i), T(X_j)] = C_i C_j - C_j C_i = c_{ij}{}^s T(X_s). \tag{40}$$

The representation $X \to T(X)$ of the Lie algebra L given by eq. (39) is called the *adjoint representation* of L.

EXAMPLE 7. Let L be a Lie algebra defined by the following set of commutation relations

$$[P_l, Q_k] = \frac{1}{\mathrm{i}}\delta_{lk}Z, \qquad [P_l, Z] = 0 = [Q_k, Z], \qquad l, k = 1, 2, \ldots, n. \tag{41}$$

Let $H = L^2(R^n)$ and $D = C_0^\infty(R^n)$. Then the map

$$\begin{gathered} Q_l \to x_l, \qquad l = 1, 2, \ldots, n, \\ P_k \to \frac{1}{\mathrm{i}}\frac{\partial}{\partial x_k}, \qquad k = 1, 2, \ldots, n, \\ Z \to I \end{gathered} \tag{42}$$

defines a representation of L in the carrier space H with D as a common linear invariant domain dense in H. ▼

§ 2. Solvable, Nilpotent, Semisimple and Simple Lie Algebras

A. *Theorem of Ado*

One of the central problems in the theory of Lie algebras is the determination and classification of all non-isomorphic Lie algebras. We saw in § 1.A that the matrix algebras A_n, B_n, C_n and D_n provide large classes of Lie algebras. One could therefore conjecture that perhaps the matrix algebras exhaust all the possible Lie algebras. This is indeed true due to the following fundamental result:

THEOREM 1. *Every Lie algebra over the field of complex numbers C is isomorphic to some matrix algebra** (for the proof see Ado 1947).

The theorem is also true for real Lie algebras. Indeed if L is a real Lie algebra then its complex extension L^c is a matrix Lie algebra by th. 1; consequently, the real contraction of L^c to L is also a matrix Lie algebra.

Ado's theorem says in fact that every abstract Lie algebra may be considered to be a subalgebra of the full linear Lie algebra gl(n, C), $n = 1, 2, \ldots$ Therefore the problem of classification of all non-isomorphic abstract Lie algebras may be reduced to the more tractable problem of the enumeration of all non-isomorphic linear Lie subalgebras of gl(n, C).

We now review the most important classes of Lie algebras:

B. *Solvable and Nilpotent Algebras*

If N is an ideal of an algebra L, then $[N, N]$ is also an ideal of L. In fact, by the formula 1(4c), we have

$$[L, [N, N]] \subset [N, [N, L]] + [N, [L, N]] \subset [N, N]. \tag{1}$$

* The corresponding theorem for Lie groups is not true globally, but locally (cf. Birkhoff 1936).

In particular L is an ideal of L, and therefore $[L, L]$ is again an ideal which may be smaller than L, and it may happen that the sequence of ideals

$$L^{(0)} = L, \quad L^{(1)} = [L^0, L^0], \quad \ldots, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}], \quad n = 0, 1, 2, \ldots \quad (2)$$

terminates, i.e. $L^{(n)} = 0$, for some n.

DEFINITION 1. A Lie algebra L is called *solvable* if, for some positive integer n, $L^{(n)} = 0$.

EXAMPLE 1. Consider the Lie algebra $e(2)$ of the group of motions of the two-dimensional real plane consisting of the two-dimensional translations and rotations around an axis perpendicular to the plane. The generators of this group satisfy the following commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = X_1.$$

We see that

$L^{(1)} = t^2$ (the Lie algebra with basis elements X_1 and X_2),

$L^{(2)} = 0.$

Hence $e(2)$ is solvable. ▼

The property of solvability of a Lie algebra is hereditary, i.e. every subalgebra L_s is also solvable. In fact,

$$L_s^{(1)} = [L_s^{(0)}, L_s^{(0)}]$$

is an ideal of L_s which satisfies $L_s^{(1)} \subset L^{(1)}$. Hence $L_s^{(n)} \subset L^{(n)} = 0$, i.e. L_s is a solvable algebra. It is evident also that every homomorphic image of a solvable algebra is solvable. Moreover, if a Lie algebra L contains a solvable ideal N such that the quotient algebra L/N is solvable, then L is also solvable.

Every solvable algebra contains a commutative ideal. In fact, if $L^{(n)} = 0$ and $L^{(n-1)} \neq 0$, then $L^{(n-1)}$ is an ideal of L, and $[L^{(n-1)}, L^{(n-1)}] = 0$.

Next we introduce the following sequence of ideals:

$$L_{(0)} = L, \quad L_{(1)} = [L_{(0)}, L], \quad \ldots, \quad L_{(n+1)} = [L_{(n)}, L]. \quad (3)$$

DEFINITION 2. A Lie algebra is called *nilpotent* if for some positive integer n, $L_{(n)} = 0$.

It is easily verified by induction that $L^{(n)} \subset L_{(n)}$. In fact $L^{(0)} = L_0$ and if $L^{(n)} \subset L_{(n)}$ then

$$L^{(n+1)} = [L^{(n)}, L^{(n)}] \subset [L_{(n)}, L] \subset L_{(n+1)}.$$

Therefore a nilpotent algebra is solvable. The converse is not true: for instance, the two-dimensional non-commutative Lie algebra defined by the commutation relation

$$[X, Y] = X$$

is solvable, but not nilpotent. Similarly, the Lie algebra $e(2)$ of example 1 above is solvable, but not nilpotent.

It is evident from the definition that every subalgebra and every homomorphic image of nilpotent algebra is nilpotent.

Every nilpotent algebra has a nontrivial center. Indeed, if $L_{(n)} = 0$ and $L_{(n-1)} \neq 0$, then by eq. (3), $[L_{(n-1)}, L] = 0$, i.e., $L_{(n-1)}$ is the center of L.

According to th. 1, every Lie algebra is isomorphic to some linear subalgebra of the full linear algebra $\mathrm{gl}(n, C)$. It is instructive to see the form of these linear matrix algebras corresponding to solvable and nilpotent algebras.

Let $T^{(m)}$ denote the vector space of all $m \times m$ upper triangular matrices, and $S^{(m)}$ the vector space of all $m \times m$ upper triangular matrices with equal diagonal elements. Let $S^{(m_1, m_2, \ldots, m_k)}$ denote the set of all linear transformations A acting in the space

$$V = V_1 \dotplus V_2 \dotplus \ldots \dotplus V_k$$

in such a way that

(i) $A \in S^{(m_1, m_2, \ldots, m_k)}$ leaves the subspaces V_i, $i = 1, 2, \ldots, k$, invariant,

(ii) in each subspace V_i with the basis $\xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_{m_i}^{(i)}$, $A \in S^{(m_i)}$ has the form

$$\begin{bmatrix} \lambda_i & & a_{jk}^{(i)} & \\ & \lambda_i & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}.$$

The commutators of triangular matrices are again triangular matrices. Therefore he vector spaces $T^{(m)}$ and $S^{(m_1, m_2, \ldots, m_k)}$ represent Lie algebras. Moreover, the following theorem holds:

THEOREM 2. *An arbitrary, solvable Lie algebra of linear transformations is isomorphic to a subalgebra of some Lie algebra* $T^{(m)}$. *An arbitrary nilpotent linear Lie algebra is isomorphic to a subalgebra of some Lie algebra* $S^{(m_1, m_2, \ldots, m_k)}$.

(For the proof see Dynkin 1947, § 2).

Previous statements about nilpotent and solvable algebras can now easily be verified in terms of triangular matrices.

C. *The Killing Form*

We introduced in 1(29) the homomorphism $X \to \mathrm{ad}X$ by the relation

$$\mathrm{ad}X(Y) = [X, Y].$$

In terms of coordinates we have

$$\big(\mathrm{ad}X(Y)\big)^i = [X, Y]^i = c_{lk}{}^i x^l y^k,$$

i.e.

$$(\mathrm{ad}X)_k{}^i = c_{lk}{}^i x^l. \tag{4}$$

We define now a '*scalar product*' in a Lie algebra by setting

$$(X, Y) = \mathrm{Tr}(\mathrm{ad}X \mathrm{ad}Y). \tag{5}$$

The scalar product (5) has the following properties:

(i) symmetry: $(X, Y) = (Y, X)$, (6a)

(ii) bilinearity: $(\alpha X+\beta Y, Z) = \alpha(X, Z)+\beta(Y, Z)$ for all $X, Y, Z \in L$ and α, β real or complex numbers, (6b)

(iii) $\big(\mathrm{ad}X(Y), Z\big)+\big(Y, \mathrm{ad}X(Z)\big) = 0$, or $([X, Y], Z)+(Y, [X, Z]) = 0$. (6c)

These properties follow directly from the properties of the trace. For example, let

$$\begin{aligned} a = \big(\mathrm{ad}X(Y), Z\big) &= \mathrm{Tr}\,\{\mathrm{ad}([X, Y])\mathrm{ad}\,Z\} \\ &= \mathrm{Tr}\,(\mathrm{ad}X\mathrm{ad}\,Y\mathrm{ad}\,Z)-\mathrm{Tr}(\mathrm{ad}\,Y\mathrm{ad}X\mathrm{ad}\,Z), \\ b = \big(Y, \mathrm{ad}X(Z)\big) &= \mathrm{Tr}\,\{\mathrm{ad}\,Y(\mathrm{ad}[X, Z])\} \\ &= \mathrm{Tr}\,(\mathrm{ad}\,Y\mathrm{ad}X\mathrm{ad}\,Z)-\mathrm{Tr}\,(\mathrm{ad}\,Y\mathrm{ad}\,Z\mathrm{ad}X), \end{aligned}$$

then by equality $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(CAB)$, we have $a+b = 0$, i.e. (6c).

The symmetric bilinear form (5) on $L\times L$ is called the *Killing form.* In terms of the coordinates in some basis, from (4), we have

$$(X, Y) = \mathrm{Tr}\,\big((\mathrm{ad}X)_k{}^i\mathrm{ad}\,(Y)_i{}^s\big) = c_{lk}{}^i x^l c_{si}{}^k y^s = g_{ls}x^l y^s, \tag{7}$$

where the symmetric second rank tensor

$$g_{ls} = c_{lk}{}^i c_{si}{}^k \tag{8}$$

is called the *Cartan metric tensor* of the Lie algebra L. Note that for some algebras (e.g. commutative) the Killing form (5) and consequently the metric tensor (8) can be degenerate, i.e., $\det[g_{kl}] = 0$.

For an arbitrary automorphism ψ of a given Lie algebra L we have by eq. 1(33)

$$\mathrm{ad}\psi(X) = \psi\mathrm{ad}X\psi^{-1}.$$

Therefore,

$$\big(\psi(X), \psi(Y)\big) = (X, Y), \tag{9}$$

i.e. the Killing form is invariant under the action of the group G_A of all automorphisms of the algebra L.

The Killing form (5) and the associated Cartan tensor (8) play a fundamental role in the theory of Lie algebras and their representations.

For example, a simple criterion for the solvability of Lie algebras in terms of the Killing form is given in

THEOREM 3. If $(X, X) = 0$ *for each* $X \in L$ *then* L *is a solvable Lie algebra**
If an algebra L *is nilpotent, then* $(X, X) = 0$ *for all* $X \in L$.

(For the proof see Dynkin 1947, th. V.) ▼

We prove now three useful lemmas.

---

* Note that the converse is not true.

LEMMA 4. *Let* $(\ ,\ )$, $(\ ,\ )^c$ *and* $(\ ,\ )^R$ *denote the Killing forms of the real algebra* L, *its complex extension* L^c *and the real form* $(L^c)^R$ *of the complex algebra* L^c. *Then,*

$$(X, Y) = (X, Y)^c \quad \textit{for } X, Y \in L, \tag{10}$$

$$(X, Y)^R = 2\operatorname{Re}\big((X, Y)^c\big) \quad \textit{for } X, Y \in (L^c)^R. \tag{11}$$

PROOF: We can choose the same basis (i.e. the same set of structure constants) for L and L^c. Then the Cartan metric tensors in L and L^c coincide. This proves eq. (10). In order to prove eq. (11), we consider a linear transformation A and a basis $e_1, \ldots, e_n$ in L^c. Let $A = B + \mathrm{i}C$ be the decomposition of the transformation A on the real and the imaginary parts. Then, in the basis $e_1, \ldots, e_n, \mathrm{i}e_1, \ldots, \mathrm{i}e_n$ of L^R we have

$$A(e_k) = Be_k + C(\mathrm{i}e_k), \qquad k = 1, 2, \ldots, n,$$

$$A(\mathrm{i}e_k) = -Ce_k + B(\mathrm{i}e_k), \qquad k = 1, 2, \ldots, n.$$

Hence, the transformation $\tilde{A}$ in L^R induced by the transformation A in L^c has the form $\tilde{A} = \begin{bmatrix} B & C \\ -C & B \end{bmatrix}$. Putting $A = \operatorname{ad}X \operatorname{ad}Y$ and using the definition (5) we obtain eq. (11). ▼

LEMMA 5. *Let* N *be an ideal of a Lie algebra* L. *If* $X, Y \in N$ *then*

$$(X, Y)_N = (X, Y)_L, \tag{12}$$

i.e. the value of the Killing form on N *taken with respect to* N *is the same as with respect to* L.

PROOF: If $e_{\bar{1}}, e_{\bar{2}}, \ldots, e_{\bar{r}}, e_{r+1}, \ldots, e_n$ is a basis of L such that $e_{\bar{1}}, e_{\bar{2}}, \ldots, e_{\bar{r}}$, $r \leqslant n$, is a basis of N (i.e. the barred indices refer to the ideal N), then for $X, Y \in N$ we have, by 1(9),

$$(X, Y)_L = \operatorname{Tr}_L(\operatorname{ad}X \operatorname{ad}Y) = c_{\bar{l}k}{}^{s} x^{\bar{l}} c_{\bar{t}s}{}^{k} y^{\bar{t}}$$

$$= c_{\bar{l}\,\bar{k}}{}^{\bar{s}} x^{\bar{l}} c_{\bar{t}\,\bar{s}}{}^{\bar{k}} y^{\bar{t}} = \operatorname{Tr}_N(\operatorname{ad}X \operatorname{ad}Y) = (X, Y)_N. \ ▼$$

LEMMA 6. *The orthogonal complement (with respect to the Killing form) of an ideal* $N \subset L$ *is also an ideal.*

PROOF: Let $X \in N^{\perp} \equiv \{X \in L\colon (X, N) = 0\}$. Then, for every $Y \in N$ and $Z \in L$, we have from (6c)

$$\big(\operatorname{ad}Z(X), Y\big) = -\big(X, \operatorname{ad}Z(Y)\big) = 0.$$

Therefore for an arbitrary Z, $\operatorname{ad}Z(X) \in N^{\perp}$, i.e. $N^{\perp}$ is an ideal of L. ▼

EXAMPLE 1. Let us calculate explicitly the Killing form of the Lie algebra $\mathrm{sl}(n, C)$. Using formula 1(13) for the structure constants of $\mathrm{gl}(n, C)$, we first evaluate the Cartan metric tensor (8)

$$g_{sm,s'm'} = c_{sm,kr}{}^{ij} c_{s'm',ij}{}^{kr} = 2n\delta_{sm'}\delta_{ms'} - 2\delta_{sm}\delta_{s'm'}.$$

Therefore the Killing from for $\mathrm{gl}(n, C)$ is

$$(X, Y) = g_{sm,s'm'}x_{sm}y_{s'm'} = 2n\mathrm{Tr}(X\cdot Y) - 2\mathrm{Tr}X\mathrm{Tr}Y. \tag{13}$$

The set $\mathrm{sl}(n, C)$, by definition, consists of elements of $\mathrm{gl}(n, C)$ satisfying the condition $\mathrm{Tr}X = 0$; it is an ideal of $\mathrm{gl}(n, C)$. Therefore by lemma 5 and formula (13) we have

$$(X, Y)_{\mathrm{sl}(n,C)} = 2n\mathrm{Tr}(X\cdot Y), \qquad X, Y \in \mathrm{sl}(n, C). \tag{14}$$

The set $N = \{\lambda I\}$, $\lambda \in C$, is also an ideal of $\mathrm{gl}(n, C)$. The Killing form (13) is zero when X or $Y \in N$. Hence the scalar product (13) for $\mathrm{gl}(n, C)$ is degenerate. We will see in subsection D that the Killing form is always degenerate for a Lie algebra which contains a non-zero commutative ideal.

The subset of $\mathrm{sl}(n, C)$ consisting of all real matrices generates the real subalgebra $\mathrm{sl}(n, R)$ whose complex extension is the algebra $\mathrm{sl}(n, C)$. Therefore, by lemma 4 and eq. (14), the Killing form for $\mathrm{sl}(n, R)$ is

$$(X, Y)_{\mathrm{sl}(n,R)} = 2n\mathrm{Tr}(X\cdot Y), \qquad X, Y \in \mathrm{sl}(n, R). \tag{15}$$

In particular, for $n = 2$ with the basis given in examples 1.1 and 1.4, we find $(e_i, e_j) = -2\delta_{ij}$, $i, j = 1, 2, 3$. ▼

D. *Simple and Semisimple Lie Algebras*

We have separated the class of solvable and nilpotent algebras from the set of all Lie algebras. In this section we define the class of simple and semisimple Lie algebras, which play a fundamental role in the study of the structure and classification of Lie algebras.

DEFINITION 3. A Lie algebra L is *semisimple* if it has no non-zero commutative ideal.

The criterion for semisimplicity is given by the following theorem:

THEOREM 7 (Cartan). *A Lie algebra L is semisimple if and only if its Killing form is non-degenerate.*

PROOF: If the algebra L is not semisimple, then it has a commutative ideal N. If $X_{\bar{1}}, X_{\bar{2}}, \ldots, X_{\bar{r}}$ are basis elements of the ideal N, then the structure constants satisfy the condition 1(9) (the barred indices refer to the ideal N), i.e.

$$c_{i\bar{l}}{}^{s} = 0 \quad \text{for } \bar{l} \leqslant r,\ s > r \text{ and } i = 1, 2, \ldots, n,$$

$$c_{\bar{m}\bar{l}}{}^{\bar{t}} = 0 \quad \text{for } \bar{m}, \bar{l}, \bar{t} \leqslant r.$$

Therefore we obtain

$$g_{i\bar{m}} = c_{is}{}^{t}c_{\bar{m}t}{}^{s} = c_{i\bar{s}}{}^{t}c_{\bar{m}t}{}^{\bar{s}} = c_{i\bar{s}}{}^{\bar{t}}c_{\bar{m}\bar{t}}{}^{\bar{s}} = 0.$$

Due to these vanishing components in the metric tensor, $\det[g_{il}]$ vanishes, i.e., the Killing form (5) is degenerate. In order to prove the second half of the theorem we suppose that the orthogonal complement $L^{\perp}$ of the algebra L is nontrivial. Because L is an ideal of L, the orthogonal complement $L^{\perp}$ is also an ideal of L by lemma 6. If $X \in L^{\perp}$, then $(X, X) = 0$. Hence, by th. 3, $L^{\perp}$ is a solvable ideal. Consequently, $L^{\perp}$ contains a nontrivial, commutative ideal, which is at the same time an ideal of L. Hence we obtain a contradiction because L is semisimple, and it has no commutative ideal. Therefore, $L^{\perp} = 0$, and consequently (X, Y) is non-degenerate. ▼

DEFINITION 4. A Lie algebra L is *simple* if it has no ideals other than $\{0\}$ and L, and if $L^{(1)} = [L, L] \neq 0$. ▼

We shall see in § 4 that the classes of algebras A_n, B_n, C_n and D_n are simple algebras. There are only five more simple algebras.

The condition $L^{(1)} \neq 0$ eliminates the Lie algebras of dimension one, which would be simple but not semisimple. For instance, the algebra of example 1.1 is simple. A solvable algebra L cannot contain a simple subalgebra; this follows from the fact that if L' is any simple subalgebra, then the ideal $[L', L']$ equals L', by def. 4; hence, the sequence of ideals $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$, $L^{(0)} = L$ of algebra L would always contain L' and therefore would never terminate. Thus, if L contains a simple subalgebra, it cannot be solvable.

A Lie algebra L is said to be *compact* if there exists in L a positive definite quadratic form $(\cdot, \cdot)$ satisfying the condition*

$$([X, Y], Z) + (Y, [X, Z]) = 0. \tag{16}$$

All remaining Lie algebras are called *noncompact*. The Killing form (5) satisfies the condition (16). Hence, if a Cartan metric tensor of a semisimple Lie algebra L is positive (or negative) definite, then L is compact.

In a complex Lie algebra any invariant quadratic form is indefinite. Hence, every complex Lie algebra is noncompact; a compact Lie algebra is a certain real form L^r of the complex Lie algebra L (cf. § 5). We now show that for a compact semisimple Lie algebra L the structure constants $c_{rs}{}^{l}$ may be represented by a third-order totally antisymmetric covariant tensor; indeed, if we use the Cartan metric tensor g_{tl} in L for lowering the indices of contravariant tensors, then the tensor

$$c_{rsl} \equiv c_{rs}{}^{t} g_{tl}, \tag{17}$$

by virtue of eq. (8), may be written in the form

$$\begin{aligned} c_{rsl} = c_{rs}{}^{t} c_{tm}{}^{n} c_{ln}{}^{m} &= -c_{sm}{}^{t} c_{tr}{}^{n} c_{ln}{}^{m} - c_{mr}{}^{t} c_{ts}{}^{n} c_{ln}{}^{m}, \qquad \text{by eq. 1(7)},\\ &= c_{sm}{}^{t} c_{rt}{}^{n} c_{ln}{}^{m} + c_{mr}{}^{t} c_{ts}{}^{n} c_{nl}{}^{m}, \qquad \text{by eq. 1(6)}. \end{aligned}$$

The last expression is invariant under cyclic permutations of the indices and is skew in r and s, by eq. 1(6); hence, the tensor (17) is totally antisymmetric.

* We show in ch. 3.8 that a Lie algebra of a compact Lie group is compact. This justifies the extension of the notion of compactness from groups to algebras.

On the other hand in a compact Lie algebra L the Cartan metric tensor may be taken to be in the form $g_{tl} = \delta_{tl}$; hence, by eq. (17),

$$c_{rsl} = c_{rs}{}^{l}, \tag{18}$$

i.e. the structure constant $c_{rs}{}^{l}$ and the components c_{rsl} of tensor (17) coincide.

§ 3. The Structure of Lie Algebras

The class of solvable Lie algebras complements in some sense the class of semi-simple ones: indeed, every solvable Lie algebra contains a commutative ideal, while, on the other hand, a semisimple Lie algebra has no commutative ideal. The following theorems show that in a certain sense the classification of all Lie algebras is reduced to a classification of solvable and semisimple Lie algebras.

We start with the analysis of the structure of compact Lie algebras. We show that an arbitrary compact Lie algebra is a direct sum $N \oplus S$ of two ideals, where N is the center of L and S is semisimple. This fundamental result is obtained in two steps:

PROPOSITION 1. *Let L be a compact Lie algebra. Every ideal N of L is a simple summand, i.e., there exists another ideal S in L such that*

$$N \cap S = 0, \quad N \oplus S = L \quad (\textit{direct sum of ideals}). \tag{1}$$

PROOF: Let $(\cdot, \cdot)$ be a positive definite, quadratic form in L satisfying condition 2(6c). Denote by S the orthogonal complement of the space N in L in the sense of the metric induced by the form $(\cdot, \cdot)$. Clearly, $N \cap S = 0$ and $N \oplus S = L$. It remains to show that S is an ideal of L. Indeed, for an arbitrary $l \in L$, $n \in N$ and $s \in S$, according to eq. 2(6c), we have

$$(n, [l, s]) = -([l, n], s) = 0,$$

because $[l, n] \in N$. Hence, $[l, s] \in S$, i.e. S is an ideal of L. ▼

The main structure theorem for compact Lie algebras is embodied in

THEOREM 2. *A compact Lie algebra L is a direct sum*

$$L = N \oplus S = N \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_n \tag{2}$$

of ideals, where N is the center of L, S is semisimple and S_i are simple algebras.

PROOF: The center N of L is an ideal of L, hence, by proposition 1, L decomposes into a direct sum of its center N and an ideal S without a center. If S is not simple, then, again by proposition 1, S decomposes into a direct sum $S' \oplus S''$ of ideals; every summand must be noncommutative because S has no center. Repeating successively this procedure, we obtain a decomposition of L into a direct sum of its center N and noncommutative simple ideals S_i. ▼

The direct sum of simple ideals is semisimple.

Clearly, the center N of L is commutative. Hence, the problem of classification of all compact Lie algebras is reduced in fact to the problem of classification of all real compact simple Lie algebras. We give a solution of this latter problem in § 5.

We turn now to the structure theorems for arbitrary Lie algebras and prove first the following important property for an arbitrary Lie algebra.

PROPOSITION 3. *Let L be a Lie algebra over R or C. There exists in L a maximal solvable ideal N such that any other solvable ideal of L is contained in N.*

PROOF: Let N be a solvable ideal of L which is not contained in any other solvable ideal, and let M be an arbitrary solvable ideal of L. Let φ be the natural homomorphism of $N+M$ onto $(N+M)/M$. Then, $\varphi(N) = (N+M)/M$ and the kernel of the homomorphism φ restricted to N is $N \cap M$; consequently $(N+M)/M$ and $N/(N \cap M)$ are isomorphic.

Now because $N \cap M$ is solvable, the quotient Lie algebra $N/(N \cap M)$ is also solvable; therefore the isomorphic algebra $(N+M)/M$ is also solvable. Because $(N+M)/M$ and M are solvable, $N+M$ is a solvable ideal of L. Hence, $M \subset N$. ▼

The maximal solvable ideal N, which contains any other solvable ideal of a Lie algebra L is called the *radical*.

For a semisimple Lie algebra L the radical N must be zero; indeed, if $N \neq 0$, then $N^{(k)} = [N^{(k-1)}, N^{(k-1)}]$, $N^{(0)} = N$, are also ideals of L by virtue of eq. 2(2); therefore, if $N^{(n-1)} \neq 0$ and $N^{(n)} = 0$, then, $N^{(n-1)}$ would be a non-zero commutative ideal of L. Consequently, if L is semisimple N must be zero.

Thus, the solvable and semisimple algebras form two disjoint classes of Lie algebras.

It can be guessed at this point that if we separate the radical N from a given Lie algebra L, the resulting Lie algebra is semisimple. Indeed, we have

PROPOSITION 4. *Let L be a Lie algebra over R or C. If N is the radical of L, then the quotient algebra L/N is semisimple.*

PROOF: Let φ be the natural homomorphism of L onto L/N. Suppose that S is a solvable non-zero ideal of L/N and let $\tilde{S} = \varphi^{-1}(S)$. Clearly, since $\varphi(N) = 0$, the ideal $\tilde{S}$ is larger than N and contains N. The algebras $\tilde{S}/N$ and N are solvable. Therefore, $\tilde{S}$ is also solvable and contains N. This, however, contradicts the maximality of the radical N. Thus, $S = \{0\}$. Consequently, L/N does not contain a commutative ideal and therefore is semisimple.

EXAMPLE 1. Let L be the Poincaré Lie algebra P. It follows from the commutation relations 1(23) that the set $\{t^4\}$ of translation generators P_μ, $\mu = 0, 1, 2, 3$, represents the maximal solvable ideal of P. The quotient algebra

$$M = P/t^4$$

is the Lorentz algebra, which is semisimple. ▼

Proposition 4 states in fact that an arbitrary Lie algebra L consists of two pieces: a radical N and a semisimple algebra L/N. The following fundamental theorem gives a fuller description of this decomposition:

THEOREM 5 (Levi–Malcev theorem). *Let L be an arbitrary Lie algebra over R or C with the radical N. Then, there exists a semisimple subalgebra S of L such that*

$$L = N \oplus S. \tag{3}$$

Any two decompositions of L of the form (3) *are related by an automorphism of the algebra L.* ▼

(For the proof cf. Chevalley 1955, vol. III, ch. V, § 4, th. 4.)

The formula (3) is called the *Levi decomposition* of L, and the subalgebra S is called the *Levi factor*.

Theorem 5 implies that

$$[N, N] \subset N, \quad [S, S] \subset S, \quad [N, S] \subset N, \tag{4}$$

i.e. any Lie algebra L is a semidirect sum $N \oplus S$ of the maximal solvable ideal N and a semisimple subalgebra S. For instance, the Poincaré algebra P, given by 1(23) has the following Levi decomposition

$$P = t^4 \oplus M \qquad M = \mathrm{so}(3, 1) \tag{5}$$

with

$$[t^4, t^4] = 0, \quad [M, M] \subset M, \quad [t^4, M] \subset t^4. \tag{6}$$

The Levi–Malcev theorem allows us to reduce the problem of classification of all Lie algebras to the following ones:

(i) Classification of all solvable Lie algebras.
(ii) Classification of all semisimple Lie algebras.
(iii) Classification of all derivations 1(26) of solvable Lie algebras implied by the classification of semisimple Lie algebras.

At the present time a complete solution exists only for the problem (ii), and this is one of the most remarkable and important results in the theory of Lie algebras. For problems (i) and (ii) there are only partial solutions.

In the second step, the problem of classification of all semisimple Lie algebras is reduced to the problem of classification of simple Lie algebras. Indeed, we have

THEOREM 6 (Cartan). *A semisimple complex or real Lie algebra can be decomposed into a direct sum of pairwise orthogonal simple subalgebras. This decomposition is unique.*

PROOF: Suppose N is a non-zero ideal of L. By lemma 2.6 we know that the orthogonal complement $N^\perp$ is again an ideal of L. It is evident that $N \cap N^\perp$ is also an ideal of L; hence, if $X \in N \cap N^\perp$, then $(X, X) = 0$. Consequently, the ideal $N \cap N^\perp$ is solvable by virtue of lemma 2.5 and th. 2.3. The assumed semisimplicity of L implies $N \cap N^\perp = 0$. Therefore, the algebra L has the decomposition

$$L = N \oplus N^\perp, \quad \text{where } [N, N^\perp] = 0 \text{ and } (N, N^\perp) = 0.$$

If N or $N^{\perp}$ is still semisimple, we repeat the procedure until the semisimple algebra L is decomposed onto a direct sum of simple, pairwise orthogonal non-commutative subalgebras:

$$L = N_1 \oplus N_2 \oplus \ldots \oplus N_k, \qquad [N_i, N_j] = 0, \tag{7}$$

$$(N_i, N_j) = 0, \quad i, j = 1, \ldots, k, \quad i \neq j.$$

Let now $M_1 \oplus M_2 \oplus \ldots \oplus M_s$ be another decomposition of L onto simple ideals. Let M_k be a simple ideal which does not occur among the ideals N_i. Then, since M_k and N_i are different simple ideals of L we have

$$[M_k, N_i] \subset M_k \cap N_i = \{0\}.$$

Hence M_k belongs to the center of L which is zero, because L is semisimple. Consequently the decomposition (7) is unique (up to permutation). ▼

The classification of all simple complex and real Lie algebras is treated in the next two sections.

§ 4. Classification of Simple, Complex Lie Algebras

In this section we introduce the important concept of a root system associated with a semisimple complex Lie algebra. Next, we give Dynkin's concept of simple roots, which provide a basis for the classification of all simple complex Lie algebras. Finally, we enumerate the classical and exceptional simple complex Lie algebras.

A. *Root System*

It is well known that the Lie algebra commutation relations of so(3) can be written as $[J_3, J_{\pm}] = \pm J_{\pm}$, $[J_+, J_-] = J_0$ which is often used in physics. In this section we shall give a generalization of this procedure to arbitrary semisimple Lie algebras, which is also of great theoretical importance.

Let V be a vector space. A subspace $W \subset V$ is called *invariant* under a set T of linear transformations of the vector space V if for each $\tau \in T$ we have $\tau W \subset W$. A set T of linear transformations is called *semisimple* if the complement of every invariant subspace of V with respect to T is also an invariant subspace.

DEFINITION 1. A subalgebra H of a semisimple algebra L is called a *Cartan subalgebra* if

1° H is a maximal abelian subalgebra in L.

2° For an arbitrary $X \in H$ the transformation ad$X$ of the space L is semisimple. ▼

Let α be a linear function on a complex vector space $H \subset L$ where H is a Cartan subalgebra of L. Denote by L^{α} the linear subspace of L defined by the condition

$$L^{\alpha} \equiv \{Y \in L\colon [X, Y] = \alpha(X) Y \text{ for all } X \in H\}. \tag{1}$$

If $L^\alpha \neq \{0\}$ then α is called a *root** and L^α the root subspace, actually a *root vector*, as we shall see presently. It follows from the Jacobi identity, that

$$[L^\alpha, L^\beta] \subset L^{\alpha+\beta} \tag{2}$$

for arbitrary complex linear functions α, β on H. The properties of roots and root subspaces are described by the following theorem.

THEOREM 1. *Let L be a semisimple complex Lie algebra and let Δ denote the set of non-zero roots. Then*

1° $L = H \dot{+} \sum_{\alpha\in\Delta} \dot{+} L^\alpha$.

2° *For every* $\alpha \in \Delta$, $\dim L^\alpha = 1$ (*i.e. roots are non-degenerate, except* $\alpha = 0$).

3° *If roots* α, β *satisfy* $\alpha+\beta \neq 0$, *then* $(L^\alpha, L^\beta) = 0$.

4° *The restriction of the Killing form on the Cartan subalgebra, i.e. on* $H\times H$, *is non-degenerate. For every root* $\alpha \in \Delta$ *there exists a unique vector* $H_\alpha \in H$, *such that*

$$(X, H_\alpha) = \alpha(X) \quad \textit{for all } X\in H. \tag{3}$$

5° *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$, *and if* $X_\alpha \in L^\alpha$, $X_{-\alpha} \in L^{-\alpha}$, *then*

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})H_\alpha, \qquad \alpha(H_\alpha) \neq 0.$$

6° *If* α, $\beta \in \Delta$ *and* $\alpha+\beta \neq 0$, *then* $[L^\alpha, L^\beta] = L^{\alpha+\beta}$. ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 4.)

Thus H and the root vectors L^α provide a suitable basis for L. By item 4° of th. 1 we have a one-to-one correspondence between roots α and elements H_α of the Cartan subalgebra H. Clearly a vector $H_\alpha \in H$ corresponds to a root if and only if there exists in L a root vector E_α satisfying the relation

$$[X, E_\alpha] = (X, H_\alpha)E_\alpha \quad \text{for each } X\in H.$$

In what follows we shall for brevity denote the scalar product (H_α, H_β) by (α, β).

We illustrate th. 1 for the A_n-Lie algebra.

EXAMPLE 1. Let $L = \mathrm{sl}(nC)$. This algebra is spanned by the basis vectors e_{ik}, 1(11), satisfying the commutation relations 1(12). Let λ_i, $i = 1, 2, \ldots, n$, be complex numbers such that $\sum_{i=1}^{n} \lambda_i = 0$. Then by virtue of 1(12) the elements

$$A_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \sum_{i=1}^{n} \lambda_i e_{ii} \tag{4}$$

span a maximal commutative subalgebra H of $\mathrm{sl}(n, C)$. Using 1(12) we obtain

$$[A_{\lambda_1, \lambda_2, \ldots, \lambda_n}, e_{ik}] = (\lambda_i - \lambda_k)e_{ik}. \tag{5}$$

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* The name 'root' is due to the fact that $[X, Y] = \alpha Y$ is an eigenvalue equation and α's can be obtained in a coordinate system by the solution of the secular equation $\det[X^i C_{ij}^k - \alpha\delta_j^k] = 0$.

Hence every one-dimensional subspace E_{ik} of $\mathrm{sl}(n, C)$ spanned by the vector e_{ik} is invariant under the operation $\mathrm{ad}X$, for $X \in H$. Therefore the complement of E_{ik} in $\mathrm{sl}(n, C)$ for a given i and k is also invariant under $\mathrm{ad}X$, $X \in H$. Consequently the transformations $\mathrm{ad}X$, $X \in H$, are semisimple; thus, H is the Cartan subalgebra of $\mathrm{sl}(n, C)$, by virtue of def. 1.

Furthermore, by eqs. (1), and (5), the complex linear forms

$$\alpha_{ik}(A_{\lambda_1, \lambda_2, \dots, \lambda_n}) \equiv \lambda_i - \lambda_k, \quad i, k = 1, 2, \dots, n, \ i \neq k$$

are non-zero roots of $\mathrm{sl}(n, C)$. The rays $E_{ik} = ((e_{ik}))$ are one-dimensional root subspaces $L^{\alpha_{ik}}$. The decomposition 1° of th. 1 takes, in the present case, the form

$$\mathrm{sl}(n, C) = H \dotplus \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \dotplus L^{\alpha_{ik}}, \qquad L^{\alpha_{ik}} = ((e_{ik})).$$

The basis of the subalgebra H can be chosen as

$$H_i = e_{ii} - e_{i+1, i+1}, \qquad i = 1, 2, \dots, n-1. \tag{6}$$

Then $\mathrm{Tr}(H_i) = 0$. Next we determine the explicit form of the root system Δ. By eq. (3) we have

$$\alpha_{ik}(X) = (X, H_{\alpha_{ik}})$$

for an arbitrary $X = A_{\lambda_1, \lambda_2, \dots, \lambda_n} \in H$. In order to determine the unknown vector $H_{\alpha_{ik}} \in H$ we represent it in the form (4), i.e.

$$H_{\alpha_{ik}} = \sum_{s=1}^{n} \mu_s e_{ss}, \qquad \sum_{s=1}^{n} \mu_s = 0; \tag{7}$$

we get

$$\alpha_{ik}(A_{\lambda_1, \lambda_2, \dots, \lambda_n}) = (A_{\lambda_1, \lambda_2, \dots, \lambda_n}, A_{\mu_1, \mu_2, \dots, \mu_n}) = \lambda_i - \lambda_k.$$

Using eq. 2(14) and (4), we obtain on the other hand

$$(A_{\lambda_1, \lambda_2, \dots, \lambda_n}, A_{\mu_1, \mu_2, \dots, \mu_n}) = 2n \mathrm{Tr}(A_{\lambda_1, \lambda_2, \dots, \lambda_n} \cdot A_{\mu_1, \mu_2, \dots, \mu_n}) = 2n \sum_{s=1}^{n} \lambda_s \mu_s.$$

Thus the equation

$$2n \sum_{s=1}^{n} \lambda_s \mu_s = \lambda_i - \lambda_k \tag{8}$$

has to be satisfied for arbitrary λ_s, $s = 1, 2, \dots, n$, provided that $\sum_{s=1}^{n} \lambda_s = 0$. It is readily verified that the equation (8) holds if and only if

$$\mu_s = \begin{cases} \dfrac{1}{2n}, & s = i, \\[2ex] -\dfrac{1}{2n}, & s = k, \\[2ex] 0, & s \neq i, k. \end{cases} \tag{9}$$

From (7) and (9) the final form of vectors $H_{\alpha_{ik}} \in H$ corresponding to roots α_{ik} is given by

$$H_{\alpha_{ik}} = \frac{1}{2n}(e_{ii}-e_{kk}), \quad i, k = 1, 2, \ldots, n, \; i \neq k. \tag{10}$$

If we set

$$\tilde{H}_i = \frac{1}{2n} e_{ii}, \tag{11}$$

then the Δ-system of sl(n, C) is finally

$$\Delta(\mathrm{sl}(n, C)) = \{\tilde{H}_i - \tilde{H}_k, \; i, k = 1, 2, \ldots, n, \; i \neq k\}. \tag{12}$$

The reader can easily verify the statements 5° and 6° of the theorem. ▼

The following theorem describes the basic properties of the root system for semisimple complex Lie algebras.

THEOREM 2. 1° *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$, *but for* $k \neq \pm 1$, $k\alpha \notin \Delta$.

2° *Suppose* $\alpha, \beta \in \Delta$, $\alpha \neq \pm\beta$. *If* $\beta_k = \beta + k\alpha$ *and* $\beta_k \in \Delta$ *for integers* k, $p \leqslant k \leqslant q$, *but* $\beta_{p-1} \notin \Delta$, $\beta_{q+1} \notin \Delta$, *then*

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = -(p+q).$$

3° $(\beta, \alpha) = \dfrac{-2\,(p_{\beta,\alpha}+q_{\beta,\alpha})}{\sum\limits_{\varphi\in\Delta}(p_{\varphi,\alpha}+q_{\varphi,\alpha})}$, *where* $p_{\varrho\sigma}(q_{\varrho\sigma})$ *is the smallest (largest) number in the series* $\varrho_k = \varrho + k\sigma$, ϱ, σ, $\varrho_k \in \Delta$ *defined in* 2°.

4° *The Killing form defines on the linear space*

$$H^* = \sum_{\alpha\in\Delta} r_\alpha H_\alpha, \qquad r_\alpha \in R,$$

a real positive definite metric. Moreover $H = H^* \dotplus \mathrm{i}H^*$. ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 4.)

If we choose as the basis of the Lie algebra the Cartan subalgebra and the root vectors, we obtain the so-called *Cartan–Weyl set of commutation relations* for a semisimple complex Lie algebra. This basis is often used by physicists. The properties of the Cartan–Weyl basis are given by the following

THEOREM 3. *For each* $\alpha \in \Delta$ *we can select a vector* $E_\alpha \in L^\alpha$, *such that, for all* $\alpha, \beta \in \Delta$, *we have*

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha \quad \textit{for } H_i \in H,$$

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \textit{if } \alpha+\beta \neq 0 \textit{ and } \alpha+\beta \notin \Delta, \\ H_\alpha, & \textit{if } \alpha+\beta = 0, \\ N_{\alpha,\beta}E_{\alpha+\beta}, & \textit{if } \alpha+\beta \in \Delta, \end{cases} \tag{13}$$

where the constants $N_{\alpha,\beta}$ satisfy

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}. \tag{14}$$

For any such choice

$$N^2_{\alpha,\beta} = \frac{q(1-p)}{2}(\alpha, \alpha), \tag{15}$$

where the numbers p and q are defined by the series $\beta+k\alpha$ of th. 2.2°. ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 5.)

B. *Dynkin Diagrams*

We have seen, th. 1.4°, that to every root there corresponds a unique vector H_α in the Cartan subalgebra H. On the other hand the number of roots is, in general, larger than the dimension of the Cartan subalgebra; this may be clearly seen in example 1, sl(n, C), where n^2-n roots are expressed in terms of $n-1$ basis vectors of H. Hence in general root vectors are linearly dependent. It is therefore natural to introduce a basis in the root space. One might expect that the problem of classification of all root systems Δ might be reduced to a simpler problem of classification of all nonequivalent systems of basis vectors in the root space. This is the main idea of Dynkin which led him to the concept of simple roots and to so-called *Dynkin diagrams.*

Let H^* be a subalgebra of the Cartan subalgebra, defined in th. 2.4°, and let $X_1, X_2, \ldots, X_l$ be a basis in H^*. A vector $X \in H^*$ is said to be *positive* if its first coordinate which is different from zero is positive.

We call a positive root $X \in \Delta$ *simple* if it is impossible to represent it as the sum of two positive roots. The properties of a system $\Pi(L)$ of simple roots of a semisimple Lie algebra L are described in the following theorem:

THEOREM 4. 1° *If $\alpha \in \Pi$, $\varphi \in \Pi$, then $\varphi-\alpha \notin \Pi$.*

2° *If $\alpha \in \Pi$, $\varphi \in \Pi$, $\alpha \neq \varphi$, then $-\dfrac{2(\varphi, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer.*

3° *The Π-system is a linearly independent set and is a basis for the space H^*. An arbitrary root $\varphi \in \Delta$ has a representation in the form*

$$\varphi = \varepsilon \sum_{i=1}^{l} k_i \alpha_i, \tag{16}$$

where $\varepsilon = \pm 1$, k_i are non-negative integers.

4° *If the positive root φ is not simple, then $\varphi = \alpha+\psi$, $\alpha \in \Pi$, $\psi \in \Delta$, $0 < \psi < \varphi$*.*

PROOF: *ad* 1°. Assume $\varphi-\alpha = \psi \in \Delta$. Then by th. 2.1°, $-\psi \in \Delta$ and $\varphi = \alpha+\psi$, $\alpha = \varphi+(-\psi)$. Thus, since either $\psi > 0$ or $-\psi > 0$, then either φ or α is not a simple root. Hence we have a contradiction.

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* $\psi < \varphi$ means that the first non-zero coordinate of $\varphi-\psi$ is positive.

ad 2°. By th. 2.2°

$$\frac{2(\varphi, \alpha)}{(\alpha, \alpha)} = -(p+q),$$

where p, q—integers and $p \leqslant q$. By 1°, $p = 0$. Therefore

$$\frac{2(\varphi, \alpha)}{(\alpha, \alpha)} = -q \leqslant 0.$$

ad 3°. Let λ be a positive root. If λ is simple, then $\lambda = \alpha_i$. If λ is not simple, then $\lambda = \alpha+\beta$, where α and β are positive. If α or β or both are not simple, then we repeat this procedure. Finally we obtain the form (16) with $\varepsilon = +1$. If λ is negative, then we apply our decomposition to the vector $-\lambda$ and get eq. (16) with $\varepsilon = -1$.

A set of positive vectors $x_1, x_2, \dots, x_m$ of R^m obeying the conditions

$$(x_i, x_k) \leqslant 0, \quad i \neq k, \tag{17}$$

is a linearly independent set. In fact, suppose that vectors $x_1, x_2, \dots, x_m$ are linearly dependent and let $y_1, y_2, \dots, y_n$ be a minimal linearly dependent subsystem. Then we would have

$$\sum_{i=1}^{n} a_i y_i = 0, \quad \text{where } a_i \neq 0,\ i = 1, 2, \dots, n. \tag{18}$$

Let u be the sum of all terms in (18) with positive coefficients, and $-v$ the sum of all terms with negative coefficients. Then eq. (18) becomes $u = v$, from which we get $(u, u) = (u, v)$. But $(u, u) > 0$, and (u, v), by (17), is non-positive. Hence we have a contradiction. Therefore simple roots, which are positive, and satisfy the condition (17) are linearly independent. They constitute a basis of the space H^* because of (15) and th. 2.4°.

ad 4°. If we add to the system Π a positive root $\varphi \in \Pi$, then we obtain a linearly dependent system. Therefore at least one of the scalar products of the type (17) is positive, i.e. $(\varphi, \alpha_i) > 0$. By th. 2.2°, this inequality implies, for some simple root α_i, $p \neq 0$ and consequently $\psi = \varphi - \alpha \in \Delta$. The inequality $\psi < 0$ is impossible, because otherwise the simple root α would be represented as the sum of two positive roots. ▼

EXAMPLE 2. We determine now the Π-system for the $\mathrm{sl}(n, C)$-algebra. The Δ-system was given by eq. (12). We choose $\tilde{H}_i$, $i = 1, 2, \dots, n$, as the basis vectors in H^*, and define

$$\sum_{i=1}^{n} \lambda_i \tilde{H}_i > 0,$$

if the first non-zero component of $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is positive. Then the roots

$$H_{\alpha_{ik}} = \tilde{H}_i - \tilde{H}_k, \quad i, k = 1, 2, \dots, n,\ i \neq k,\ i < k,$$

are positive, and the roots

$$H_{\alpha_{i,i+1}} = \tilde{H}_i - \tilde{H}_{i+1} = \frac{1}{2n} H_i, \quad i = 1, 2, \ldots, n-1,$$

are simple where H_i is given by eq. (6). Denoting the root $\alpha_{i,i+1}$ by the symbol α_i and using eq. 2(14) we find

$$(\alpha_i, \alpha_k) = (H_{\alpha_i}, H_{\alpha_k}) = \begin{cases} \dfrac{1}{n} & \text{for } i = k, \\ -\dfrac{1}{2n} & \text{for } |i-k| = 1, \\ 0 & \text{for } |i-k| > 1, \end{cases} \tag{19}$$

i.e. the angles $\langle \alpha_i, \alpha_k \rangle$ between the roots α_i and α_k are

$$\langle \alpha_i, \alpha_k \rangle = \begin{cases} 120°, & \text{if } |i-k| = 1, \\ 90°, & \text{if } |i-k| > 1. \blacktriangledown \end{cases} \tag{20}$$

The metric properties of the Π-system of a semisimple Lie algebra L determine the Δ-system. An inductive method for the construction of the Δ-system is contained in the proof of the next theorem.

THEOREM 5. *The $\Delta(L)$-system of all roots of a given semisimple Lie algebra L can be constructed from its $\Pi(L)$-system of simple roots.*

PROOF: According to th. 2.1°, we can restrict ourselves to the problem of the construction of positive roots only. Let β be a positive root of Δ, and $\beta = \sum_{i=1}^{n} k_i \alpha_i$ be its decomposition in terms of simple roots as in eq. (16). We call a root β a *root of order* s if $\sum_{i=1}^{n} k_i = s$. Clearly the simple roots are all of order one. Suppose now that we have constructed all roots of order less than s. By th. 4.4°, roots of order s have the form $\psi + \alpha$, where ψ is a root of order $s-1$ and $\alpha \in \Pi$. We use the formula

$$q = -p - \frac{2(\psi, \alpha)}{(\alpha, \alpha)} \tag{21}$$

(th. 2.2°), if the vector $\psi + \alpha \in \Delta$. The vectors $\varphi = \psi + k\alpha$, $k = 0, -1, -2, \ldots$, are, by th. 4.3°, positive and of order less than s. Therefore, we can determine, by induction, whether or not they belong to the set Δ, and we can find the smallest value $k_{\min} \equiv p$. Using formula (21), we find a number q. If $q > 0$ then the series $\varphi = \psi + j\alpha$, $j = 1, 2, \ldots, q$, contains the root $\psi + \alpha$. Otherwise the vector $\psi + \alpha$ is not a root. $\blacktriangledown$

The above theorem and th. 3 show that in fact the problem of a classification of all simple complex Lie algebras can be reduced to the problem of classification of all Π-systems of simple roots. According to th. 4 the latter problem can be reduced to a simpler combinatorial problem of the classification of all finite systems Γ of vectors of R^n satisfying the following conditions:

1° Γ is a linearly independent system of vectors.

2° If α, $\beta \in \Gamma$ then $\dfrac{2(\alpha, \beta)}{(\beta, \beta)}$ is a non-negative integer.

Clearly every $\Pi(L)$-system of simple roots is a Γ-system.

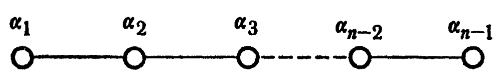
The problem of classification of Π-systems for simple Lie algebras can be simplified by introducing the device of the Dynkin diagrams. Let us observe first that according to th. 4.2°, if $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, the quantity

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \cdot \frac{2(\alpha, \beta)}{(\beta, \beta)} = 4\cos^2\langle\alpha, \beta\rangle \tag{22}$$

is a non-negative integer; hence $4\cos^2\langle\alpha, \beta\rangle$ takes one of the values 0, 1, 2 or 3; consequently the corresponding angles are 90°, 120°, 135° and 150°, respectively.

We can represent graphically a Π-system (or Γ) of vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ as a connected linear complex (or graph). The vertices $(\alpha_1), (\alpha_2), \ldots, (\alpha_n)$ are in one-to-one correspondence with the vectors α_i of the Π-system. Two different vertices of the complex are connected with a single, a double, or a triple line, when the two corresponding vectors span an angle of 120°, 135° or 150°, respectively. If all vectors α_i have the same lengths we denote the vertices by small open circles ○; if vectors α_i have two different lengths we denote the vertices corresponding to vectors with a smaller length by the full dots ●, and by small open circles otherwise.

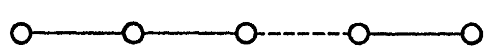
Example 3. Let $L = \mathrm{sl}(n, C)$. We construct the Dynkin diagram for this Lie algebra. According to eq. (19) all simple roots α_i have the same length. Moreover, by virtue of (20), the simple roots α_i and α_{i+1} will be connected with a single line; any other pair α_i, α_k $k \neq i+1$, of simple roots form an angle $\langle\alpha_i, \alpha_k\rangle = 90°$ and therefore is not connected. Consequently the Dynkin diagram for $\mathrm{sl}(n, C)$ ($\sim A_{n-1}$) has the form

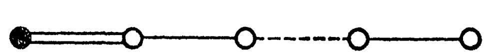
A_{n-1}. 

(23)

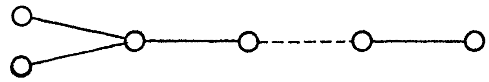
The following fundamental theorem gives a description of the Dynkin diagrams and the Π-systems of simple roots for all simple complex Lie algebras.

Theorem 6. *The four infinite sequences of diagrams*

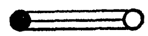
A_n:  (24)

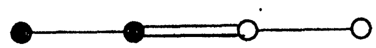
B_n:  (25)

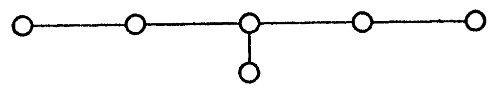
C_n:  (26)

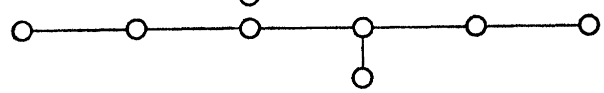
D_n:  (27)

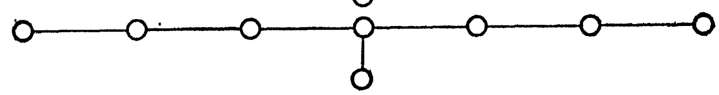
and the five single diagrams

G_2:  (28)

F_4:  (29)

E_6:  (30)

E_7:  (31)

E_8:  (32)

constitute the set of all diagrams which can be associated with Π-systems. The corresponding Π-systems of simple roots are explicitly given by

$$\begin{aligned}
\Pi(A_n) &= \{h_{i+1}-h_i,\ i = 1, \dots, n\},\\
\Pi(B_n) &= \{h_1, h_{i+1}-h_i,\ i = 1, \dots, n-1\},\\
\Pi(C_n) &= \{2h_1, h_{i+1}-h_i,\ i = 1, \dots, n-1\},\\
\Pi(D_n) &= \{h_1+h_2,\ h_{i+1}-h_i,\ i = 1, 2, \dots, n-1\},\\
\Pi(G_2) &= \{h_2-h_1,\ h^{(3)}-3h_2\},\\
\Pi(F_4) &= \{h_3-h_2,\ h_2-h_1,\ h_1,\ \tfrac{1}{2}(h_4-h_1-h_2-h_3)\},\\
\Pi(E_6) &= \{h_{i+1}-h_i,\ i = 1, 2, \dots, 5,\ \tfrac{1}{2}\sqrt{2}h_7+\tfrac{1}{2}h^{(6)}-h_4-h_5-h_6\},\\
\Pi(E_7) &= \{h_{i+1}-h_i,\ i = 1, 2, \dots, 6,\ \tfrac{1}{2}h^{(8)}-h_4-h_5-h_6-h_7\},\\
\Pi(E_8) &= \{h_1+h_2,\ h_{i+1}-h_i,\ i = 1, 2, \dots 6,\ h_8-\tfrac{1}{2}h^{(8)}\}.
\end{aligned}$$

The vectors h_i are orthogonal basis vectors of the corresponding Euclidean space and have the same but arbitrary length and $h^{(r)} = h_1+h_2+ \dots +h_r$.

A simple complex Lie algebra can be associated with each of the diagram (24)–(32).▼

(For the proof cf. Dynkin 1947, § 7.)

We observe that the diagrams A_1, B_1 and C_1 are identical. The same holds for the pair of algebras B_2 and C_2, and for the pair A_3 and D_3. All other diagrams are different. Thus we have the

COROLLARY. *The four infinite sequences of Lie algebras*

$$A_n,\ n \geqslant 1, \quad B_n,\ n \geqslant 2, \quad C_n,\ n \geqslant 3, \quad D_n,\ n \geqslant 4,$$

and the five exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 constitute all the non-isomorphic simple complex Lie algebras.▼

The sequence D_n does not contain the algebra D_2 because it is not simple ($D_2 \sim D_1 \oplus D_1$, direct sum of two ideals). Clearly $A_1 \sim B_1 \sim C_1$, $B_2 \sim C_2$, $A_3 \sim D_3$ on the basis of the identity of their Dynkin diagrams.

A problem of great practical interest is the reconstruction of a simple Lie algebra from its Π-system of simple roots. This problem can be solved along the following steps:

1° Reconstruct the Δ-system from the Π-system (cf. th. 5).

2° Calculate (up to a sign) the structure constants $N_{\alpha,\beta}$ with the help of formula (15).

3° Determine the sign of $N_{\alpha,\beta}$.

The choice of the sign of $N_{\alpha,\beta}$ must be made in such a manner that the axioms 1(2) and 1(3) of a Lie algebra are satisfied.

When these steps are carried out, the commutation relations of an arbitrary simple complex Lie algebra are given by formulas (13). The dimensions of simple complex Lie algebras A_n, B_n, C_n and D_n can be calculated from their defining matrix realization (cf. § 1, A) and are given by

	A_n	B_n	C_n	D_n
Dimension	$n(n+2)$	$n(2n+1)$	$n(2n+1)$	$n(2n-1)$

The dimensions of exceptional Lie algebra are: G_2: 14, F_4: 52, E_6: 78, E_7: 133, E_8: 248. These can be also calculated from their realizations (cf. § 5 and § 9, D).

§ 5. Classification of Simple, Real Lie Algebras

We have given in § 4 the classification of all complex simple Lie algebras. This classification provides also a natural starting point for the classification of all simple real Lie algebras. This is because one can relate to every simple complex Lie algebra a sequence of real simple Lie algebras introduced in § 1. A by means of the following two processes:

A. Selection of all non-isomorphic real forms L^r of a given complex simple Lie algebra L;

B. Construction of a real Lie algebra L^R associated with a given complex simple Lie algebra L.

These two processes give us, as we shall see, all the simple real Lie algebras.

First note that every real form L^r of a complex simple Lie algebra L is simple. In fact, a real form L^r is generated by a special basis of a given complex simple Lie algebra L, in which all structure constants are real. There is, however, no basis in the simple complex Lie algebra L, in which structure constants could satisfy the conditions 1(9). Hence, L^r has no ideals and, therefore, is simple. Using the same arguments, we conclude that the Lie algebra L^R associated with a simple, complex Lie algebra L is also simple.

The converse of the above statement is not true: the complex extension of a real simple Lie algebra may not be simple. For instance, the complex extension of the Lorentz Lie algebra $o(3, 1)$ given by eq. 1(23a) is the complex Lie algebra $o(4, C) \sim D_2$ which is isomorphic to the direct sum $D_1 \oplus D_1$ of two ideals.

THEOREM 1. *All real simple Lie algebras are obtained by applying the processes* A *and* B *to all simple complex Lie algebras.*

PROOF: Let L be an arbitrary simple real Lie algebra and L^c its complex extension. In general, the algebra L^c may not be simple. Accordingly, we distinguish two cases:

(i) L^c is simple. In this case the original real simple Lie algebra L is one of the real forms of the algebra L^c.

(ii) L^c is not simple. By lemma 2.4, L^c is, at any rate, semisimple, and can be decomposed by th. 3.6 into a direct sum of simple ideals. If $L_1 \neq \{0\}$ is a simple direct summand of L^c and σ is the conjugation of L^c with respect to L, then σL_1 is also a simple direct summand of L^c because $[L_1, \sigma L_1] = 0$.

The original real Lie algebra L consists of the elements invariant under σ, i.e., of the elements of the form $X+\sigma X$ with $X \in L_1$. The map $X \to X+\sigma X$ of L_1 into L is a real isomorphism. For

$$X+Y \to (X+\sigma X)+(Y+\sigma Y) = (X+Y)+\sigma(X+Y),$$
$$\alpha X \to \alpha(X+\sigma X) = \alpha X+\sigma(\alpha X), \qquad \alpha \text{ real},$$
$$[X, Y] \to [X, Y]+\sigma[X, Y] = [X+\sigma X, Y+\sigma Y].$$

If L^c would contain more ideals other than L_1 and σL_1, then L would be a direct sum of real ideals. Because L is simple this is impossible, therefore $L^c = L_1 \dot{+} \sigma L_1$.

The last equality follows also from the relation $[L_1, \sigma L_1] = 0$. Consequently, the real simple Lie algebra L is isomorphic to the complex simple Lie algebra L_1 considered as a real Lie algebra L_1^R of twice the dimension. ▼

It follows from the proof of th. 1 that the complex extension $(L^R)^C$ of the real simple Lie algebra L^R obtained by the process B is not simple while the complex extension $(L^r)^c$ of the real simple Lie algebra L^r obtained by the process A is simple. Consequently the processes A and B provide disjoint classes of real simple Lie algebras. The process B associates to every complex simple Lie algebra L a uniquely determined real simple Lie algebra L^R, whose structure constants can be obtained directly from the structure constants of algebra L. Hence the classification of all complex simple Lie algebras given by th. 4.6 provides simultaneously a classification of all real simple Lie algebras obtained by process B. In order to complete a classification of all simple Lie algebras it remains to classify the Lie algebras obtained by the process A. In the solution of this last problem a compact real form of a given complex simple Lie algebra plays an important role. We therefore show first that a compact real form of L exists.

THEOREM 2. *Every semisimple complex Lie algebra has a real form which is compact.*

PROOF: Let H_α, E_α, $\alpha \in \Delta$, be the set of generators, which satisfy the commutation relations of th. 4.3. By items 5° and 3° of th. 4.1, we have $(E_\alpha, E_{-\alpha}) = 1$, and $(E_\alpha, E_\alpha) = 0$. Hence the vectors

$$\begin{aligned} U_\alpha &= \mathrm{i}(E_\alpha + E_{-\alpha}), \\ V_\alpha &= E_\alpha - E_{-\alpha}, \\ \tilde{H}_\alpha &= \mathrm{i}H_\alpha, \quad \alpha \in \Delta \end{aligned}$$

satisfy

$$\begin{aligned} (U_\alpha, U_\alpha) &= -2, \\ (V_\alpha, V_\alpha) &= -2, \\ (U_\alpha, V_\alpha) &= 0, \\ (\tilde{H}_\alpha, \tilde{H}_\alpha) &= -(\alpha, \alpha) < 0. \end{aligned}$$

Because $(E_\alpha, E_\beta) = 0$ for $(\alpha+\beta) \neq 0$, it follows that the Killing form is negative definite on the real linear subspace given by

$$L_k = \sum_{\alpha\in\Delta} R_\alpha \tilde{H}_\alpha + \sum_{\alpha\in\Delta} R_\alpha U_\alpha + \sum_{\alpha\in\Delta} R_\alpha V_\alpha, \qquad R_\alpha \text{ real numbers.}$$

If $X, Y \in L_k$, then by th. 4.3, the commutator $[X, Y]$ is expressed again in terms of elements U_α, V_α and $\tilde{H}_\alpha$ with coefficients proportional to $N_{\alpha,\beta}$ or (α, β). Therefore because both $N_{\alpha,\beta}$ and (α, β) are real, and the Killing form is negative definite, the subspace L_k is the real compact Lie algebra. Moreover we have

$$L = L_k \dotplus \mathrm{i}L_k,$$

i.e. L is the direct sum of the subalgebra L_k and the vector space $\mathrm{i}L_k$. ▼

We give now the explicit construction of all simple real algebras, which admit a given simple complex algebra L as their complex extension.

Let L_k be the compact form of the complex simple algebra L, and let $X_1, X_2, \ldots, X_n$ be a basis of L_k. Clearly, the basis $\{X_i\}$ considered over C provides also a basis in L.

Let P be a linear transformation in L, which transforms the basis X_i, $i = 1, 2, \ldots, n$, into a new basis

$$Y_l = P_{kl} X_k, \qquad l = 1, 2, \ldots, n, \tag{1}$$

in which the structure constants $c_{ij}{}^k$, $i, j, k = 1, 2, \ldots, n$, defined by the commutators

$$[Y_l, Y_s] = c_{ls}{}^k Y_k \tag{2}$$

are real. To each such basis there corresponds a real simple algebra L^r spanned by generators Y_i, $i = 1, 2, \ldots, n$, with the commutation relations (2). The problem of the classification of all non-isomorphic real forms of a given simple complex Lie algebra L is now reduced to the problem of finding all transformations of the form (1) in L_k which lead to non-isomorphic real Lie algebras (2). This problem is solved by the following theorem:

THEOREM 3. *Let L be a complex simple Lie algebra, L_k its compact form, and Σ the set of all involutive automorphisms of L_k. The linear transformations*

$$P = \sqrt{S} \equiv \frac{1-\mathrm{i}}{2} S + \frac{1+\mathrm{i}}{2} I, \qquad S \in \Sigma \tag{3}$$

realize all non-isomorphic real forms of L. Two linear transformations P_1 and P_2 given by eq. (3) *transform L_k into two isomorphic real Lie algebras if and only if*

$$P_1 = AP_2R, \tag{4}$$

where A is an automorphism of L and R is a real transformation in L_k. ▼

(For the proof cf. Gantmacher 1939b, §§ 2 and 3.)

Two involutive automorphisms S_1 and S_2 will be called *equivalent*, if the corresponding transformations P_1 and P_2 satisfy eq. (4).

The following theorem gives a direct method of construction of all non-isomorphic real forms of a complex simple Lie algebra L.

THEOREM 4. *All non-isomorphic real forms of a given complex simple Lie algebra L may be obtained in the following manner:*

1° *Find all nonequivalent involutive automorphisms S of the compact form L_k of L.*

2° *Choose a basis in L_k such that the matrix S is diagonal. Multiply those basis vectors of L_k corresponding to eigenvalue -1 by* i, *and leave the remaining basis vectors unchanged. To the basis so obtained there corresponds a real simple Lie algebra $L^{(S)}$.*

PROOF: An involutive automorphism S of a compact Lie algebra L_k can be brought to the diagonal form with diagonal elements equal to $+1$ or -1. This is because the Killing form is negative definite and consequently the automorphism S is a unitary operator, i.e. $S^* = S^{-1}$. From $S^2 = 1$ it follows that $S^{-1} = S$, or $S^* = S$, hence the automorphism S can be represented as the difference of two projection operators

$$P^+ = \tfrac{1}{2}(1+S) \quad \text{and} \quad P^- = \tfrac{1}{2}(1-S).$$

Putting $L_k^+ = P^+L_k$ and $L_k^- = P^-L_k$, we find that L_k is a direct sum of orthogonal subspaces L_k^+ and L_k^- and each element $X \in L_k^+ (L_k^-)$ is an eigenvector of S with the eigenvalue $+1$ (-1). According to eq. (3), the transformation $P = \sqrt{S}$ has the form

$$P = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & \mathrm{i} & & & \\ & 0 & & & & \mathrm{i} & & \\ & & & & & & \ddots & \\ & & & & & & & \mathrm{i} \end{bmatrix}. \tag{5}$$

Choosing any basis in L_k^+ and L_k^- and applying the transformation (5), we obtain the basis in the real form $L^{(S)}$, specified by th. 4. If we then take all nonequivalent involutive automorphisms S of L_k and apply th. 3, we obtain all non-isomorphic simple real Lie algebras associated with the given simple complex Lie algebra L.

In other words, if

$$L_k = K \dot{+} P \tag{6}$$

is the decomposition of the algebra L_k implied by the involutive automorphism S (i.e., $S(X) = X$ for $X \in K$, and $S(Y) = -Y$ for $Y \in P$), then

$$L^{(S)} = K \dot{+} \mathrm{i}P \tag{7}$$

is the real form of the simple complex Lie algebra L associated with the involutive automorphism S.▼

The Cartan metric tensor $g_{ik}^{(S)}$ in the real Lie algebra $L^{(S)}$ may be put equal to the matrix S. Indeed, because the Killing form for L_k is definite, the Cartan metric tensor in L_k may be taken in the form $g_{ik} = \delta_{ik}$. Hence,

$$g_{ik}^{(S)} = (\sqrt{S})_{il}(\sqrt{S})_{kn} g_{ln} = S_{ik}. \tag{8}$$

It follows, therefore, that two involutive automorphisms S_1 and S_2 of L_k with different signatures lead to non-isomorphic simple real Lie algebras $L^{(S_1)}$ and $L^{(S_2)}$, respectively.

Example 1. Let $L = o(3, C)$. The compact form $L_k = o(3)$ of L is defined by the following commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \tag{9}$$

We have in the present case six involutive transformations in $o(3)$:

$$S_{(1)} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}, \quad S_{(2)} = \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(3)} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix},$$

$$S_{(4)} = \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(5)} = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(6)} = \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{bmatrix}.$$

The transformations $S_{(2)}$ and $S_{(3)}$ are not automorphisms of $o(3)$, because they do not conserve the commutation relations (9). $\sqrt{S_{(1)}} = I$ and transforms $o(3)$ onto $o(3)$. The transformations $S_{(4)}$, $S_{(5)}$ and $S_{(6)}$ are involutive automorphisms of $o(3)$. From eq. (3) we obtain

$$\sqrt{S_{(4)}} = \begin{bmatrix} 1 & & 0 \\ & \mathrm{i} & \\ 0 & & \mathrm{i} \end{bmatrix}, \quad \sqrt{S_{(5)}} = \begin{bmatrix} \mathrm{i} & & 0 \\ & 1 & \\ 0 & & \mathrm{i} \end{bmatrix}, \quad \sqrt{S_{(6)}} = \begin{bmatrix} \mathrm{i} & & 0 \\ & \mathrm{i} & \\ 0 & & 1 \end{bmatrix}. \tag{10}$$

We now show using (4) that authomorphisms $\sqrt{S_{(5)}}$ and $\sqrt{S_{(6)}}$ are equivalent to $\sqrt{S_{(4)}}$. In fact, the automorphisms (10) of o(3) are also automorphisms of o(3, C) in the basis (8). Furthermore the automorphisms of o(3, C) of the form

$$A_{(5)} = \sqrt{S_{(4)}}(\sqrt{S_{(5)}})^3 \quad \text{and} \quad A_{(6)} = \sqrt{S_{(4)}}(\sqrt{S_{(6)}})^3$$

transform both $\sqrt{S_{(5)}}$ and $\sqrt{S_{(6)}}$ into $\sqrt{S_{(4)}}$. Hence, by virtue of eq. (4), they provide the real forms $L^{(S_{(5)})}$ and $L^{(S_{(6)})}$ isomorphic to $L^{(S_{(4)})}$. Using now th. 4, we obtain ($Y_k = (S_{(4)})_{lk}X_l$)

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = -Y_1, \quad [Y_3, Y_1] = Y_2,$$

which is the Lie algebra o(2, 1) of the noncompact Lorentz group in three-dimensional space-time (see also exercice 2.1°). ▼

The problem of classification of all non-equivalent involutive automorphisms of compact simple Lie algebras can be solved by means of geometrical methods (cf. Cartan 1929), or algebraic methods (cf. Gantmacher 1939a,b or Hausner and Schwartz 1968, ch. III). In what follows we restrict ourselves to the enumeration of the concrete forms of the classical simple real Lie algebras implied by these automorphisms.

I. *Real Forms of* sl(n, C) ($\sim A_{n-1}$, $n > 1$)

This Lie algebra has the following real forms

(i) L_k = su(n) — the Lie algebra consisting of all skew-hermitian matrices Z of order n with $\mathrm{Tr}\,Z = 0$.

(ii) sl(n, R) — the Lie algebra consisting of all real matrices X of order n with $\mathrm{Tr}\,X = 0$.

(iii) su(p, q), $p+q = n$, $p \geqslant q$ — the Lie algebra of all matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix},$$

where Z_1, Z_3 are skew-hermitian of order p and q, respectively, $\mathrm{Tr}\,Z_1 + \mathrm{Tr}\,Z_3 = 0$, Z_2 arbitrary.

(iv) su*($2n$) — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix},$$

where Z_1, Z_2 complex matrices of order n, $\mathrm{Tr}\,Z_1 + \mathrm{Tr}\,\bar{Z}_1 = 0$.

II. *Real Forms of* so($2n$, C) ($\sim D_n$, $n \geqslant 1$)

(i) L_k = so($2n$) — the Lie algebra consisting of all real skew-symmetric matrices of order $2n$.

(ii) $\mathrm{so}(p, q)$, $p+q = 2n$, $p \geqslant q$ — the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix},$$

where all X_i real, X_1, X_3 skew-symmetric of order p and q, respectively, and X_2 arbitrary.

(iii) $\mathrm{so}^*(2n)$ — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix},$$

where Z_1, Z_2 complex matrices of order n, Z_1 skew-symmetric and Z_2 hermitian.

III. *Real Forms of* $\mathrm{so}(2n+1, C)$ $(\sim B_n, n \geqslant 1)$

(i) $L_k = \mathrm{so}(2n+1)$ — the Lie algebra of all real skew-symmetric matrices of order $2n+1$.

(ii) $\mathrm{so}(p, q)$, $p+q = 2n+1$, $p \geqslant q$ — the Lie algebra of all real matrices of order $2n+1$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix},$$

where all X_i real, X_1, X_3 skew-symmetric of order p and q, respectively, and X_2 arbitrary.

IV. *Real Forms of* $\mathrm{sp}(n, C)$ $(\sim C_n, n \geqslant 1)$

(i) $L_k = \mathrm{sp}(n)$ — the Lie algebra of all skew-hermitian traceless matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^T \end{bmatrix},$$

where all Z_i complex matrices of order n, Z_2 and Z_3 symmetric, (i.e. $\mathrm{sp}(n) = \mathrm{sp}(n, C) \cap \mathrm{su}(2n)$).

(ii) $\mathrm{sp}(n, R)$ — the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{bmatrix},$$

where X_1, X_2, X_3 real matrices of order n, X_2, X_3 symmetric.

(iii) $\mathrm{sp}(p, q)$, $p+q = n$, $p \geqslant q$ — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{12}^* & Z_{22} & Z_{14}^T & Z_{24} \\ -\bar{Z}_{13} & \bar{Z}_{14} & \bar{Z}_{11} & -\bar{Z}_{12} \\ Z_{14}^* & -\bar{Z}_{24} & -Z_{12}^T & \bar{Z}_{22} \end{bmatrix},$$

where Z_{ij} complex matrices: Z_{11} and Z_{13} of order p, Z_{12} and Z_{14} with p rows and q columns, Z_{11} and Z_{22} skew-hermitian, Z_{13} and Z_{24} symmetric.

The list of global real Lie groups associated with these Lie algebras is given in ch. III, § 7.

There are important isomorphisms among the lowest members of the non-exceptional real simple Lie algebras. They are induced (except in one case) by the isomorphisms of corresponding complex simple algebras.

Table I shows all the known isomorphisms. We include also for convenience the isomorphisms induced by the fact that the complex Lie algebra D_2 is isomorphic to $A_1 \oplus A_1$.

Table I

Isomorphisms of complex algebras	Isomorphisms of real forms
$A_1 \sim B_1 \sim C_1$	$\mathrm{su}(2) \sim \mathrm{so}(3) \sim \mathrm{sp}(1)$
	$\mathrm{sl}(2, R) \sim \mathrm{su}(1, 1) \sim \mathrm{so}(2, 1) \sim \mathrm{sp}(1, R)$
$B_2 \sim C_2$	$\mathrm{so}(5) \sim \mathrm{sp}(2)$
	$\mathrm{so}(3, 2) \sim \mathrm{sp}(2, R)$
	$\mathrm{so}(4, 1) \sim \mathrm{sp}(1, 1)$
$D_2 \sim A_1 \oplus A_1$	$\mathrm{so}(4) \sim \mathrm{so}(3) \oplus \mathrm{so}(3)$
	$\mathrm{so}(2, 2) \sim \mathrm{sl}(2, R) \oplus \mathrm{sl}(2, R)$
	$\mathrm{sl}(2C) \sim \mathrm{so}(3, 1)$
	$\mathrm{so}^*(4) \sim \mathrm{sl}(2, R) \oplus \mathrm{su}(2)$
$A_3 \sim D_3$	$\mathrm{su}(4) \sim \mathrm{so}(6)$
	$\mathrm{sl}(4, R) \sim \mathrm{so}(3, 3)$
	$\mathrm{su}(2, 2) \sim \mathrm{so}(4, 2)$
	$\mathrm{su}(3, 1) \sim \mathrm{so}^*(6)$
	$\mathrm{su}^*(4) \sim \mathrm{so}(5, 1)$

There exists another isomorphism of the real forms which is not induced by the above isomorphisms of the complex simple Lie algebras. Namely

$$\mathrm{so}^*(8) \sim \mathrm{so}(6, 2)$$

(cf. Morita 1956).

For real forms of exceptional Lie algebras see e.g. Helgason 1962, p. 354, or Hausner and Schwartz 1968.

§ 6. The Gauss, Cartan and Iwasawa Decompositions

We have seen that by Levi–Malcev theorem, an arbitrary Lie algebra M admits the decomposition

$$M = N \oplus L,$$

where N is the radical of M and L is a semisimple Lie algebra. In order to better elucidate the structure and the properties of an arbitrary semisimple Lie algebra we give in this section three further decompositions of the Levi factor L. These decompositions play a fundamental role in the representation theory of the Lie algebras as well as of the corresponding Lie groups.

A. *The Gauss Decomposition*

Let L be a semisimple complex Lie algebra and let Δ be its system of non-zero roots. Let Δ^+ denote the set of all positive roots and L^+ the linear hull of eigenvectors E_α defined by the equations

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha, \qquad \alpha \in \Delta^+, \qquad H_i \in H. \tag{1}$$

For α, $\beta \in \Delta^+$ we have from 4(13)

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \text{if } \alpha+\beta \neq 0,\ \alpha+\beta \notin \Delta, \\ N_{\alpha\beta}E_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta. \end{cases} \tag{2}$$

If α, $\beta \in \Delta^+$, then $\alpha+\beta$ is also in Δ^+. Hence, by virtue of (2), the vector space L^+ forms a Lie algebra. Let Δ^- be the set of all negative roots and let L^- be the linear hull of all eigenvectors of eq. (1) for $\alpha \in \Delta^-$. Using eq. (2) we conclude that L^- forms a Lie algebra as well.

From eqs. (1) and (2) and commutativity of the Cartan subalgebra H, it follows that the direct sum $L^+ \dotplus H$ is a subalgebra of L. Moreover, eq. (1) implies that L^+ is an ideal of $L^+ \dotplus H$. Hence, the subalgebra $L^+ \dotplus H$ is, in fact, the semidirect sum $L^+ \oplus H$ of the ideal L^+ and the Cartan subalgebra H. Similarly, $L^- \oplus H$ is the semidirect sum of the ideal L^- and H. Moreover, we have

THEOREM 1. *Let L be a complex semisimple Lie algebra. Then*

1° *The subalgebras L^+ and L^- are nilpotent.*

2° *The subalgebras $L^+ \oplus H$ and $L^- \oplus H$ are solvable.*

3° *L admits the following decomposition*

$$L = L^+ \dotplus H \dotplus L^-. \tag{3}$$

PROOF: *ad* 1°. We first show that L^+ is nilpotent. Because Δ is a finite set, there exists a positive integer N such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_N \notin \Delta$$

for an arbitrary sequence $\alpha_1, \ldots, \alpha_N$ of positive roots. Consequently, by virtue of eq. (2), the multiple commutator

$$[\ldots [[[[E_{\alpha_1}, E_{\alpha_2}], E_{\alpha_3}], E_{\alpha_4}], \ldots, E_{\alpha_{N-1}}], E_{\alpha_N}] \sim E_{\alpha_1+\alpha_2+\ldots+\alpha_N} \tag{4}$$

is zero. Thus, there exists a positive integer N such that the sequence of ideal $(L^+)_{(k)}$, $k = 0, 1, 2, \ldots, N$, given by eq. 2(3) terminates with $(L^+)_{(N)} = 0$. Consequently, L^+ is nilpotent. Similarly, one proves that L^- is nilpotent.

ad 2°. By virtue of eqs. (1) and (2) and commutativity of H we have

$$(L^+ \dotplus H)^{(1)} \equiv [L^+ \dotplus H, L^+ \dotplus H] = L^+.$$

Because L^+ is nilpotent, there exists a positive integer N such that $(L^+ \dotplus H)^{(N)} = 0$. Hence, $L^+ \dotplus H$ is solvable according to def. 2.1. Similarly one shows that $L^- \dotplus H$ is solvable.

ad 3°. According to th. 4.1, we can write

$$L = H \dotplus \sum_{\alpha \in \Delta} \dotplus L^\alpha. \tag{5}$$

Combining, in eq. (5), the root subspaces L^α corresponding to positive and negative roots, respectively, we obtain

$$L = \sum_{\alpha \in \Delta^+} \dotplus L^\alpha \dotplus H \dotplus \sum_{\alpha \in \Delta^-} \dotplus L^\alpha,$$

which gives eq. (3). ▼

The decomposition (3) of a complex semisimple Lie algebra L is called the *Gauss decomposition.* As an illustration we determine the explicit form of the Gauss decomposition for $\mathrm{sl}(n, C)$.

EXAMPLE 1: Let $L = \mathrm{sl}(n, C)$. The Cartan subalgebra H is spanned by the basis vectors $H_i = e_{ii} - e_{i+1,i+1}$, $i = 1, 2, \ldots, n-1$ (cf. example 4.1). The vectors e_{sk}, $s, k = 1, 2, \ldots, n$, $s \neq k$, span the one-dimensional root subspaces $L^{\alpha_{sk}}$. Equation (1) takes in the present case the form

$$[H_i, e_{sk}] = \alpha_{sk}(H_i) e_{sk}.$$

We showed in example 4.2 that the roots α_{sk} for $s < k$ are positive and the roots α_{sk} for $s > k$ are negative. Consequently, for $\mathrm{sl}(n, C)$ we have

$$L^+ = \sum_{s<k} \dotplus L^{\alpha_{sk}}, \quad L^- = \sum_{s>k} \dotplus L^{\alpha_{sk}}. \tag{6}$$

The vectors e_{sk} for $s < k$ have non-vanishing matrix elements above the main diagonal only. Hence, L^+ is the nilpotent subalgebra consisting of all upper triangular matrices. Similarly, L^- is the nilpotent algebra consisting of all lower triangular matrices. Thus, the Gauss decomposition (3) of $\mathrm{sl}(n, C)$ is nothing but the decomposition of an arbitrary traceless matrix into the direct sum of upper triangular, diagonal and lower triangular traceless matrices. ▼

It is useful to extend the concept of the Gauss decomposition also to non-semisimple Lie algebras. We say in general that a Lie algebra L admits a Gauss decomposition (3) if the subalgebras L^+, H and L^- satisfy conditions 1° and 2° of th. 1. In particular the Gauss decomposition of $\mathrm{gl}(n, C)$ is of the form

$$\mathrm{gl}(n, C) = L^+ \dotplus H \dotplus L^-, \tag{7}$$

where L^+ and L^- are nilpotent subalgebras given by eq. (6) and H is the subalgebra of all diagonal complex matrices of order n.

B. *The Cartan Decomposition*

The Cartan decomposition of a semisimple real Lie algebra L is the direct sum of the maximal compact subalgebra and the vector space spanned by the remaining noncompact generators. Consider for example the Lorentz Lie algebra $\mathrm{o}(3, 1)$. Denoting by K_i, $i = 1, 2, 3$, the generators of the compact $\mathrm{so}(3)$ subalgebra and by N_i, $i = 1, 2, 3$, the generators of pure Lorentz transformations, we can write the commutation relations 1(23a) in the form

$$\begin{aligned} [K_i, K_j] &= \varepsilon_{ijl} K_l, \\ [K_i, N_j] &= \varepsilon_{ijl} N_l, \\ [N_i, N_j] &= -\varepsilon_{ijl} K_l. \end{aligned} \tag{8}$$

If K denotes the compact subalgebra $\mathrm{o}(3)$ and N a vector space spanned by the generators N_i, the Lie algebra $L = \mathrm{o}(3, 1)$ can be written in the form

$$L = K \dotplus N.$$

Moreover, the commutation relations (8) become

$$[K, K] \subset K, \qquad [K, N] \subset N, \qquad [N, N] \subset K.$$

Similar decompositions are known for other familiar semisimple real Lie algebras, such as the de Sitter algebra $\mathrm{so}(4, 1)$, the conformal algebra $\mathrm{so}(4, 2)$, etc. This observation is in fact a general property:

THEOREM 2 (Cartan). *A semisimple real Lie algebra L has a decomposition of the form*

$$L = K \dotplus P, \tag{9}$$

satisfying

$$[K, K] \subset K, \qquad [K, P] \subset P, \qquad [P, P] \subset K \tag{10}$$

and

$$\begin{aligned} (X, X) &< 0 \quad \text{for } X \neq 0 \text{ in } K, \\ (Y, Y) &> 0 \quad \text{for } Y \neq 0 \text{ in } P. \end{aligned} \tag{11}$$

If conditions (10) *and* (11) *are satisfied, then K is the maximal compact subalgebra of L.* ▼

(For the proof cf. Helgason 1962, ch. III, § 7.)

The decomposition of a semisimple real algebra specified by th. 2 is called the *Cartan decomposition.*

The form of the commutation relations given by (10) implies that the mapping

$$\theta(X) = X \qquad \text{for } X \in K, \tag{12}$$

$$\theta(Y) = -Y \quad \text{for } Y \in P, \tag{13}$$

is an involutive automorphism of the algebra L (i.e. $\theta^2 = 1$). Conversely, for every involutive automorphism θ of a real semisimple Lie algebra there exists a basis in L such that the formulas (12) and (13), and consequently commutation relations (10), hold. This follows immediately from the fact that every involutive automorphism of a real semisimple Lie algebra can be reduced to the diagonal form with diagonal elements equal to $+1$ or -1 (cf. the proof of th. 5.4).

The vector spaces K and P are orthogonal because

$$(X, Y) = \big(\theta(X), \theta(Y)\big) = -(X, Y) \quad \text{for every } X \in K \text{ and } Y \in P. \tag{14}$$

Hence $(X, Y) = 0$.

EXAMPLE 2. Let $L = \mathrm{sl}(n, R)$. Consider the subalgebra $\mathrm{so}(n)$ spanned by skew-symmetric matrices 1(14). The mapping

$$\theta : X \to -X^T \tag{15}$$

satisfies the condition

$$\theta(X) = X \quad \text{for } X \in \mathrm{so}(n) = K$$

and therefore is an admissible candidate for an automorphism θ of the Cartan decomposition. The vector space P is defined by the condition (13)

$$\theta(Y) = -Y \Rightarrow Y^T = Y,$$

i.e. P is the collection of all symmetric traceless matrices. Because the commutator of a skew-symmetric and a symmetric matrix is a symmetric matrix and the commutator of two symmetric matrices is an anti-symmetric matrix, the commutation relations (10) are satisfied. The Killing form for $\mathrm{sl}(n, R)$ was given in eq. 2(15), i.e.

$$(X, Y) = 2n\mathrm{Tr}(XY).$$

A skew-symmetric matrix, when brought to a diagonal form, has only pure imaginary non-vanishing diagonal elements. Hence if D denotes the matrix, which diagonalizes a given skew-symmetric matrix, then

$$(X, X) = 2n\mathrm{Tr}(DXD^{-1}\, DXD^{-1}) < 0.$$

Similarly one verifies that if Y is a symmetric matrix, then $(Y, Y) > 0$. Therefore conditions (11) are also satisfied. Hence $\mathrm{so}(n)$ is the maximal compact subalgebra of $\mathrm{sl}(n, R)$. Therefore, the Cartan decomposition $L = K \dotplus P$ in the present case is just the well-known decomposition of a matrix into its skew-symmetric and traceless symmetric parts. ▼

The Cartan decomposition takes a particularly simple form for the real Lie algebra L^R associated with a complex simple Lie algebra L. Namely, if U is a compact form of L (which by th. 5.2 always exists), then

$$L^R = U + \mathrm{i}U \tag{16}$$

is the Cartan decomposition of the algebra L^R. In fact, it is evident that commutations relations (10) are satisfied. One can show that conditions (11) are also satisfied: Let $(\cdot,\ \cdot)^R$ and $(\cdot,\ \cdot)$ be the Killing forms on L^R and L, respectively. By lemma 2.4 we have

$$(X, Y)^R = 2\mathrm{Re}(X, Y) \quad \text{for } X, Y \in L^R.$$

Because the Killing form $(\cdot,\ \cdot)$ is negative definite on $U \times U$ and positive definite on $\mathrm{i}U \times \mathrm{i}U$, relations (11) of th. 2 follow. Hence, the decomposition (16) is the Cartan decomposition of the algebra L^R.

C. *The Iwasawa Decomposition*

The third type of decomposition of semisimple real Lie algebras is based on the Cartan decomposition and the decomposition of a complex semisimple algebra on its root subspaces.

Let $L = K \dot{+} P$ be the Cartan decomposition of a semisimple real algebra L and let L^C be the complex extension of L. Let σ and τ be the conjugations of the algebra L^C and of the compact algebra $U = K \dot{+} \mathrm{i}P$, respectively, i.e.,

$$\begin{aligned} \sigma&: X+\mathrm{i}Y \to X-\mathrm{i}Y, \quad X, Y \in L, \\ \tau&: X+\mathrm{i}Y \to X-\mathrm{i}Y, \quad X, Y \in U. \end{aligned} \tag{17}$$

Clearly, by virtue of eq. 1(25), the transformation $\theta = \sigma\tau$ is an automorphism of L^C.

Let α be a root of the algebra L^C. The linear function $\alpha^\theta(X) \equiv \alpha(\theta X)$ on a Cartan subalgebra H is also a root. For if

$$L^\alpha = \{Y \in L:\ [X, Y] = \alpha(X)Y \text{ for all } X \in H\}$$

is the root subspace of L^C corresponding to the root α, then $L^{\alpha^\theta} = \theta^{-1}L^\alpha$ is the root subspace of L^C corresponding to α^θ. Let now

$$\begin{aligned} B_+ &= \{\alpha:\ \alpha \in \Delta^+,\ \alpha \neq \alpha^\theta\}, \quad N = \sum_{\alpha \in B_+} \dot{+} L^\alpha, \\ N_0 &= L \cap N, \quad S_0 = H_P \dot{+} N_0, \end{aligned} \tag{18}$$

where H_P is a maximal abelian subalgebra of P. Then, we have

THEOREM 3. *The spaces N and N_0 are nilpotent Lie algebras, S_0 is a solvable Lie algebra and*

$$L = K \dot{\oplus} H_p \dot{\oplus} N_0. \blacktriangledown \tag{19}$$

(For the proof cf. Helgason 1962, ch. VII, § 3.)

The decomposition (19) of a real semisimple Lie algebra L is called the *Iwasawa decomposition.*

EXAMPLE 3. Consider $L = \mathrm{sl}(n, R)$. Let us find first the form of the nilpotent algebras N and N_0. From example 1 we have that the set Δ^+ of positive roots of $L^C = \mathrm{sl}(n, C)$ consists of

$$\alpha_{ik} = \frac{1}{2n}(e_{ii} - e_{kk}), \qquad i < k,$$

where, from eq. 4(5),

$$\alpha_{ik}(X) = \lambda_i - \lambda_k \quad \text{for } H \ni X = \sum_{s=1}^{n} \lambda_s e_{ss}, \quad \sum_{s=1}^{n} \lambda_s = 0.$$

We verify, using eq. (17), that

$$\sigma(X) = \sum \bar{\lambda}_s e_{ss}, \qquad \tau(X) = \sum -\bar{\lambda}_s e_{ss}.$$

Hence,

$$\theta(X) = \sigma\tau(X) = -X.$$

Consequently,

$$\alpha_{ik}^{\theta}(X) = \lambda_k - \lambda_i \neq \alpha_{ik}(X).$$

This implies that $B_+ = \Delta^+$. Therefore, the nilpotent algebra N has the form

$$N = \sum_{\alpha_{ik} \in \Delta^+} \dot{+} L^{\alpha_{ik}} = \sum_{i<k} \dot{+} ((e_{ik}))_c,$$

where $((e_{ik}))_c$ are one-dimensional complex rays. The algebra $\mathrm{sl}(n, R)$ is spanned by the generators e_{ik}, $i \neq k$, $i, k = 1, 2, \ldots, n$, and the generators of a Cartan subalgebra $H_L = \{e_{ii} - e_{i+1,\, i+1},\ i = 1, 2, \ldots, n-1\}$. Therefore

$$N_0 = L \cap N = \sum_{i<k} \dot{+} ((e_{ik}))_R,$$

where $((e_{ik})_R$ are one-dimensional real rays. This algebra consists of all upper real triangular matrices with zeros on the diagonal. By example 2, the vector space P consists of all symmetric matrices with trace zero. Because all elements of H_L are symmetric matrices, we have $H_P = H_L \cap P = H_L$. Consequently, the solvable algebra S_0 is

$$S_0 = H_P \dot{+} \sum_{i<k} \dot{+} ((e_{ik}))_R,$$

i.e., it consists of all upper real triangular matrices with zero trace.

The Iwasawa decomposition (19) for $\mathrm{sl}(n, R)$ is just the decomposition of an arbitrary real traceless matrix into a sum of a skew-symmetric, a traceless diagonal, and a zero-diagonal upper triangular real matrix. ▼

§ 7. An Application. On Unification of the Poincaré Algebra and Internal Symmetry Algebras

In the physics of elementary particles that interact strongly, it seems to be an empirical fact that these particles and resonances can be grouped into multiplets which correspond to irreducible representations of some so-called *internal symmetry Lie algebras,* like multiplets of isospin algebra su(2), or multiplets of su(3)-algebra. The members of a given multiplet have the same parity and spin, but might have different masses. Thus the overall symmetry algebra cannot be a direct sum $P \oplus S$ of two ideals, the Poincaré algebra P and an internal symmetry algebra S, because otherwise all masses within a multiplet would have to be the same. It is natural therefore to inquire if there exists a larger algebra L which contains P and S as subalgebras in such a manner that at least one of the generators of S does not commute with P, so that the (mass)^2 operator $P_\mu P^\mu$ is no longer an invariant of the larger algebra. Because the eigenvalues of the basis elements H_i of a Cartan subalgebra H of S are used to label the states in multiplets and these so-called quantum numbers (like hypercharge or third component of isospin) are Poincaré invariant, it is natural to demand that H commutes with P. We show here that there are severe restrictions on the form of such combined Lie algebras.

THEOREM 1. *Let L be a Lie algebra which is spanned by the basis elements of the Poincaré algebra P and by those of semisimple Lie algebra S. If H is a Cartan subalgebra of S and*

$$[P, H] = 0 \tag{1}$$

then L is a direct sum of ideals:

$$L = P \oplus S. \tag{2}$$

PROOF: Let X_ϱ, $\varrho = 1, 2, \ldots, 10$, denote the basis elements of the Poincaré algebra satisfying the commutation relations

$$[X_\varrho, X_\sigma] = c_{\varrho\sigma}{}^\tau X_\tau, \tag{3}$$

and let H_i and E_α be the basis elements of S satisfying the Weyl canonical commutation relations 4(13). An arbitrary commutator $[E_\alpha, X_\varrho]$ can be written in the form

$$[E_\alpha, X_\varrho] = x_{\alpha\varrho}{}^\beta E_\beta + y_{\alpha\varrho}{}^j H_j + z_{\alpha\varrho}{}^\tau X_\tau, \tag{4}$$

where $x_{\alpha\varrho}{}^\beta$, $y_{\alpha\varrho}{}^j$ and $z_{\alpha\varrho}{}^\tau$ are expansion coefficients in the basis E_β, H_j, X_τ of the algebra L. The Jacobi identity

$$[[E_\alpha, X_\varrho], H_i] + [[X_\varrho, H_i], E_\alpha] + [[H_i, E_\alpha], X_\varrho] = 0, \tag{5}$$

by virtue of eqs. (1), (3), (4) and 4(13), takes the form:

$$\sum_\beta \left(x_{\alpha\varrho}{}^\beta \beta(H_i) - x_{\alpha\varrho}{}^\beta \alpha(H_i)\right) E_\beta - \sum_j y_{\alpha\varrho}{}^j \alpha(H_i) H_j - \sum_\tau z_{\alpha\varrho}{}^\tau \alpha(H_i) X_\tau = 0. \tag{6}$$

For every root α, $\alpha(H_i) \neq 0$ for at least one basis element H_i and $\alpha(H_i) \neq \beta(H_i)$ for at least one basis element H_i if $\alpha \neq \beta$: hence

$$y_{\alpha\varrho}{}^{j} = 0, \quad z_{\alpha\varrho}{}^{\tau} = 0 \tag{7}$$

and

$$x_{\alpha\varrho}{}^{\beta} = \delta_{\alpha}{}^{\beta} x_{\alpha\varrho} \quad \text{(no summation).} \tag{8}$$

The Jacobi identity

$$[[X_\varrho, X_\sigma], E_\alpha]+[[X_\sigma, E\ ], X_\varrho]+[[E_\alpha, X_\varrho], X_\sigma] = 0 \tag{9}$$

yields then the equation

$$-\sum_{\tau} c_{\varrho\sigma}{}^{\tau} x_{\alpha\tau} E_\alpha - x_{\alpha\sigma} x_{\alpha\varrho} E_\alpha + x_{\alpha\varrho} x_{\alpha\sigma} E_\alpha = -\sum_{\tau} c_{\varrho\sigma}{}^{\tau} x_{\alpha\tau} E_\alpha = 0. \tag{10}$$

The commutation relations 1(23) for P imply that for every τ we can choose ϱ and σ in such a manner that $c_{\varrho\sigma}{}^{\tau} = 0$ and $c_{\varrho\sigma}{}^{\tau'} = 0$ for $\tau' \neq \tau$. Hence $X_{\alpha\beta} = 0$ according to eq. (10). This in turn implies $[E_\alpha, X_\varrho] = 0$ by virtue of eqs. (7), (8) and (4). Consequently, L is the direct sum $P \oplus S$ of two ideals. ▼

The structure th. 3.2 implies the following generalization of th. 1.

THEOREM 2. *Let L be a Lie algebra which is spanned by the basis elements of the Poincaré algebra P, and the basis elements of an arbitrary compact Lie algebra K. Let C be a maximal commutative subalgebra of K. If*

$$[P, C] = 0 \tag{11}$$

then L is the direct sum of ideals:

$$L = P \oplus K. \tag{12}$$

PROOF: According to th. 3.2, a compact Lie algebra is a sum of ideals $N \oplus S$ where N is the center of K and S is semisimple. Clearly $[P, N] = 0$ by virtue of (11). Let H be a Cartan subalgebra of S. Because S is semisimple and $[P, H] = 0$, th. 1 implies that $[P, S] = 0$. Hence $L = P \oplus N \oplus S = P \oplus K$—direct sum of ideals. ▼

These results do not exclude, in principle, a possibility of imbedding part of the Poincaré and other symmetry Lie algebras in some larger symmetry algebras. We shall come back to these problems in ch. 21, § 3.

§ 8. Contraction of Lie Algebras

Let L be a Lie algebra. A contracted Lie algebra L' of L may be abstractly introduced as follows. (We shall later show that this operation has a physical implementation when certain physical parameters tend to zero or infinity.) Let $X_1, \ldots, X_r$ be a basis of L. For a subset $X_1, \ldots, X_\varrho$, $\varrho \leqslant r$, of basis elements, we define

$$Y_i \equiv \lambda^{-1} X_i, \quad i = 1, 2, \ldots, \varrho \leqslant r \tag{1}$$

and express the commutation relations in terms of Y_i:

$$\begin{aligned}[][Y_i, Y_j] &= c_{ij}{}^k \lambda^{-1} Y_k + \lambda^{-2} c_{ij}{}^m X_m, \\ [Y_i, X_m] &= c_{im}{}^k Y_k + c_{im}{}^n \lambda^{-1} X_n, \\ [X_m, X_n] &= c_{mn}{}^i \lambda Y_i + c_{mn}{}^s X_s,\end{aligned} \tag{2}$$

$$i, j, k \leqslant \varrho, \quad \varrho < m, n, s \leqslant r.$$

Now we let $\lambda \to \infty$, and determine when the elements

$$Y_1, \ldots, Y_\varrho, X_{\varrho+1}, \ldots, X_r$$

would form again a Lie algebra, namely the contracted Lie algebra L'_ϱ. This is the case, if the condition

$$c_{mn}{}^i = 0, \quad i \leqslant \varrho, \quad \varrho < m, n \leqslant r \tag{3}$$

is satisfied. Clearly, if $\varrho = r$, L'_r is the abelian Lie algebra

$$[Y_i, Y_j] = 0, \quad i, j = 1, \ldots, r.$$

There are many important and interesting non-trivial cases with $\varrho < r$.

EXAMPLE 1. The contraction of the de Sitter Lie algebras o(3, 2) or o(4, 1) to the Poincaré Lie algebra.

The de Sitter Lie algebras have basis elements $M_{ab} = - M_{ba}$, $a, b = 1, 2, \ldots, 5$, satisfying

$$[M_{ab}, M_{cd}] = -(g_{bc} M_{ad} - g_{ac} M_{bd} + g_{ad} M_{bc} - g_{bd} M_{ac}) \tag{4}$$

where

$$\begin{aligned} g_{ab}&: - - - + + \quad \text{for o}(3, 2), \\ g_{ab}&: - - - + - \quad \text{for o}(4, 1).\end{aligned}$$

Let now $M_{5\mu} = RP_\mu$ and $R \to \infty$. Then $(\mu, \nu, \varrho = 1, 2, 3, 4)$

$$\begin{aligned}[][P_\mu, P_\nu] &= \frac{1}{R^2}[M_{5\mu}, M_{5\nu}] = -\frac{1}{R^2} g_{55} M_{\mu\nu} \to 0, \\ [M_{\mu\nu}, P_\varrho] &= \frac{1}{R}[M_{\mu\nu}, M_{5\varrho}] = -\frac{1}{R}(g_{\mu\varrho} M_{\nu 5} - g_{\nu\varrho} M_{\mu 5}) = -g_{\nu\varrho} P_\mu - g_{\mu\varrho} P_\nu.\end{aligned} \tag{5}$$

Defining now generators $\tilde{M}_{\mu\nu} = -M_{\mu\nu}$ we obtain for $M_{\mu\nu}$ and P_σ the commutation relations of the Poincaré algebra (cf. eq. 1(23)).

EXAMPLE 2. The contraction of the Poincaré Lie algebra to the Galilean Lie algebra.

This problem has one abstract mathematical solution, and another distinct physical solution, as we shall see in ch. 13. For the formal solution, consider the elements of the Poincaré Lie algebra P_μ, $M_{\mu\nu}$ with the commutation relations given by eq. 1(23). Let

$$M_{0i} \equiv cK_i, \quad c \to \infty.$$

Then

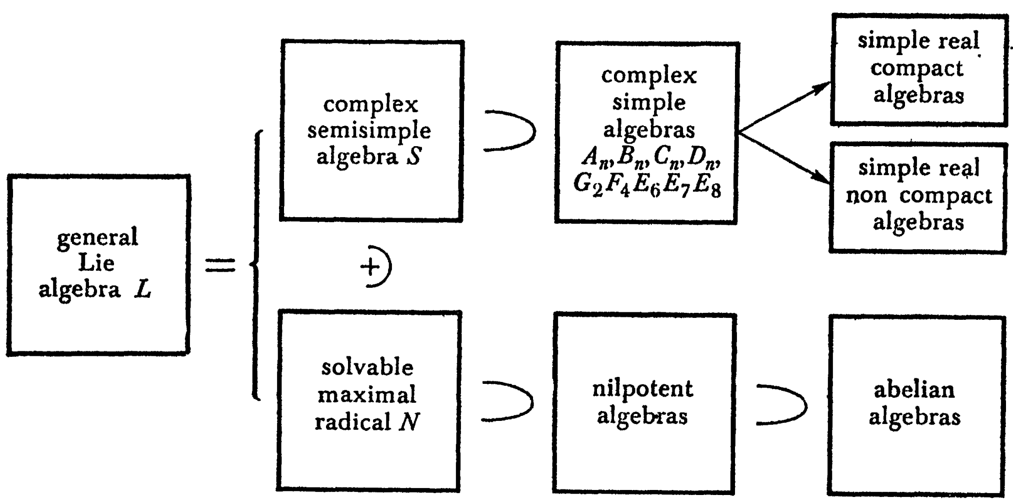
$$
\begin{aligned}
[M_{ij}, K_k] &= \frac{1}{c}[M_{ij}, M_{0k}] = \frac{1}{c}(g_{jk}M_{0i} - g_{ik}M_{0j}) = g_{jk}K_i - g_{ik}K_j, \\
[K_i, K_j] &= \frac{1}{c^2}[M_{0i}, M_{0j}] = \frac{1}{c^2}M_{ij} \to 0, \qquad (6) \\
[K_i, P_\mu] &= \frac{1}{c}[M_{0i}, P_\mu] \to 0.
\end{aligned}
$$

The elements M_{ij}, K_i, P_i, P_0 form a basis of the Lie algebra of the Galilei group.

For the underlying physical motivation and significance of the contraction procedure we refer to ch. 13.

§ 9. Comments and Supplements

A) The following diagram describes the connections between various types of Lie algebras:



B) Figures 1 and 2 describe the set of all *real* simple Lie algebras corresponding to classical simple complex Lie algebras (obtained by both methods A and B of § 5).

C) The first important but incomplete results concerning the classification of complex simple Lie algebras were obtained by Killing 1888–1890. Killing's theory was completed and extended by Cartan in his thesis 1894. He introduced the roots as zeros of the characteristic polynomial $\det[\lambda I - \mathrm{ad}X]$. Later, H. Weyl 1925, 1926, I, II, III and 1935 and B. L. van der Waerden 1933 considerably simplified the theory using the Cartan subalgebra as the main tool in the classification problem. Here we follow the elegant method elaborated by Dynkin 1947; see also the method of H. Freudenthal 1958.

The problem of classification of real simple Lie algebras was also solved by Cartan 1914. Here we followed an algebraic derivation due to Gantmacher 1939a,b.

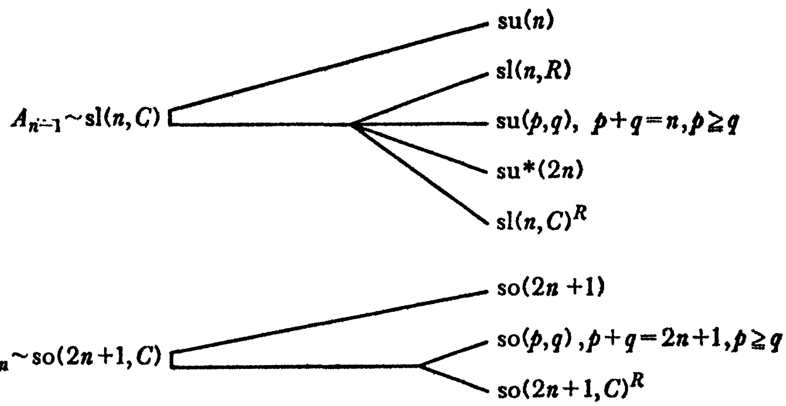


Fig. 1

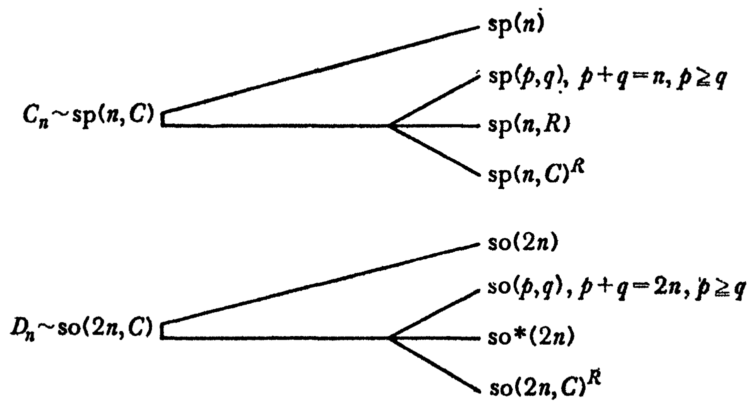


Fig. 2

An excellent readable exposition of the theory of semisimple Lie algebras is given in the monograph of Hausner and Schwartz 1968.

D) The matrix realization of classical Lie algebras A_n, B_n, C_n and D_n is relatively simple; in fact, it is given by the algebra of $n \times n$-matrices satisfying conditions of skew-hermiticity, symmetry, and tracelessness (§ 5). The concrete realization of exceptional Lie algebras is somewhat more complicated; in general, they have the following structure

$$L = S \dot{+} V \dot{+} V',$$

where S is a simple complex Lie algebra, V a complex vector space and V' its conjugate space. The connection between various summands of L is given by the relations

$$[S, V] \subset V, \quad [S, V'] \subset V', \quad [V, V] \subset V', \quad [V', V'] \subset V. \tag{7}$$

For example, for $L = G_2$ we have $S = \mathrm{sl}(3, C)$ and $V = C^3$ (for details cf. Hausner and Schwartz 1968, ch. II.4 and ch. III).

E) Th. 7.1 on space-time and internal symmetry algebras was first proved by McGlinn 1964 following suggestions to couple these two algebras, and in a general form presented here by Coester, Hamermesh and McGlinn 1964. See also the review by Hegerfeldt and Hennig 1968 for a collection of references.

F) The concept of group contractions and their representations was first introduced by Inönü and Wigner 1953.

§ 10. Exercises

§ 1.1. Show that the vector product $a \times b$, $a, b \in R^3$, satisfies the axioms of a Lie algebra.

§ 1.2. Show that the Poisson-brackets of classical mechanics

$$[f, g] \equiv \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right), \quad f, g \in C,$$

where R^{2n} is the phase-space, define a Lie algebra. Show the same for the Jacobi-brackets of vector-valued functions $[f, g] = g \cdot \nabla f - f \cdot \nabla g$.

§ 1.3. The *centralizer* C of a Lie algebra L consists of all elements C such that $[C, X] = 0$ for all $X \in L$. Show that C is a subalgebra.

§ 1.4. Find L^C, $(L^C)^R$ and $((L^C)^R)^C$ for the Lorentz Lie algebra $L = \mathrm{so}(3, 1)$.

§ 1.5. Find all Lie algebras whose basis elements are polynomials of $Q = x$ and $P = -\mathrm{i}d/dx$.

§ 1.6. Find the three-dimensional matrix representation of the nilpotent Heisenberg algebra $[P, Q] = -iI$. (Cf. th. 2.2.)

§ 1.7. Find the second-order differential operators which, together with the Hamiltonian

$$H = \frac{1}{2m} P^2 + \frac{\omega^2}{2} Q^2,$$

form the Lie algebra su(1, 1), where $Q = x$, $P = -\mathrm{i}d/dx$.

§ 1.8. Let L be any non-associative finite-dimensional algebra with a multiplication law $xy \in L$, $x \in L$, $y \in L$, and let D be a derivation of L. Show that

$$\varphi_t \equiv \mathrm{e}^{tD} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu$$

satisfies

$$\varphi_t(xy) = (\varphi_t x)(\varphi_t y)$$

and

$$\varphi_t[x, y] = [\varphi_t x, \varphi_t y].$$

Hint: Use the Leibniz rule

$$D^n(xy)/n! = \sum_{j=0}^{n}\left(\frac{1}{j!}D^j x\right)\left(\frac{1}{(n-j)!}D^{n-j}y\right).$$

§ 1.9. Let L be a Lie algebra with basis elements $X_1, \ldots, X_r$. Let L' be the vector space spanned by $X_1, \ldots, X_r$ and a new element Y. In order that L' be a Lie algebra (extension of L), there must be some restrictions on the coefficients $a^k{}_i$ in the equation

$$[Y, X_i] = a^k{}_i X_k.$$

Show that there is a trivial extension $L' = L \oplus \{Y\}$ (improper extension); and determine the proper nontrivial extension.

§ 1.10. Let H be the Hamiltonian of a physical system in a Hilbert space $\mathscr{H}$ and let $\mathscr{A}$ be the set of all linear operators in $\mathscr{H}$ which commute with H. Show that $\mathscr{A}$ is a Lie algebra (in general, an infinite-dimensional Lie algebra).

§ 1.11. Show that the derivative algebra of the Lie algebra 1(23) of the Poincaré group is the 11-parameter Weyl algebra consisting of the Poincaré Lie algebra plus the generator D of dilatations with

$$[D, P_\mu] = -P_\mu, \quad [D, M_{\mu\nu}] = 0.$$

§ 2.1. Classify all Lie algebras of dimension 2 and 3. (There are two Lie algebras of dimension 2, one abelian and one solvable given by $[e_1, e_2] = e_1$. The Lie algebras of dimension 3 are: (a) an abelian; (b) a nilpotent $[e_1, e_2] = 0$, $[e_2, e_3] = e_1$, $[e_3, e_1] = 0$; (c) $[e_1, e_2] = e_1$, $[e_1, e_3] = 0$, $[e_2, e_3] = 0$; (d) a class of solvable algebras $[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, $\alpha\beta - \beta\gamma \neq 0$ (includes Euclidean algebras e(3) and e(2,1)); (e) two simple algebras so(3) and so(2, 1).)

More specifically, writing $[e_i, e_j] = f_k$ (i, j, k cyclic), we have

	I	II	III	IV	V	VI	VII	VIII	IX
f_1	0	0	e_1	e_2	e_1	e_1	e_1	e_1	e_1
f_2	0	0	0	$-e_1$	e_2	$-e_2$	e_1+pe_2	e_2	$-e_2$
f_3	0	e_1	0	0	0	0	0	e_3	e_3

§ 2.2. Show that the following Lie algebra is nilpotent: $[1, 2] = 5$, $[1, 3] = 6$, $[1, 4] = 7$, $[1, 5] = -8$, $[2, 3] = 8$, $[2, 4] = 6$, $[2, 6] = -7$, $[3, 4] = -5$, $[3, 5] = -7$, $[4, 6] = -8$, all other commutators zero.

§ 2.3. Show that any four-dimensional nilpotent algebra has a three-dimensional ideal.

§ 2.4. Find all solvable subalgebras of the Lie algebras of (a) the Lorentz group and (b) the Poincaré group.

§ 2.5. Evaluate the Killing form for the Poincaré group.

§ 2.6. The commutation relations for so(p, q) Lie algebra are

$$[L_{ab}, L_{cd}] = -g_{bc}L_{ad} - g_{ad}L_{bc} + g_{ac}L_{bd} + g_{bd}L_{ac},$$

where g_{ab} is the metric tensor. Show that the Cartan metric tensor

$$g_{ab,\alpha\beta} = (c_{ab})_{cd}{}^{ef}(c_{\alpha\beta})_{ef}{}^{cd}$$

has the simple form

$$g_{ab,\alpha\beta} = \text{const}(g_{a\beta}g_{b} - g_{a}\, g_{b\beta}).$$

§ 2.7. Let $\psi(x)$ be a nonrelativistic quantum field at fixed t satisfying the canonical commutation relations

$$[\psi(x), \psi^*(y)] = \mathrm{i}\delta^{(3)}(x-y),$$
$$[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0.$$

Define the current

$$J_k(x) = \frac{1}{2\mathrm{i}}\psi^*(x)\overleftrightarrow{\partial}_k\psi(x).$$

Show that

$$[J_k(x), J_l(y)] = -\mathrm{i}\frac{\partial}{\partial x^l}[J_k(x)\delta^{(3)}(x-y)] + \mathrm{i}\frac{\partial}{\partial y^k}[J_l(y)\delta^{(3)}(x-y)].$$

Is $\{J_k(x)\}$ a Lie algebra?

§ 2.8. Set in the previous problem

$$J_k(n) \equiv \int_{-\pi}^{\pi} J_k(x)\mathrm{e}^{-\mathrm{i}n\cdot x}d^3x$$

Show that

$$[J_k(n), J_l(m)] = m_l J_k(m+n) - n_k J_l(m+n).$$

Find a finite-dimensional subalgebra of this Lie algebra.

§ 2.9. Let $\varrho(x) = \psi^*(x)\psi(x)$ be the charge density of the previous problems. Show that

$$[\varrho(x), \varrho(y)] = 0$$

and

$$[\varrho(x), J_k(y)] = \mathrm{i}\frac{\partial}{\partial x^k}\delta^{(3)}(x-y)\varrho(y).$$

Show that $\varrho(x)$ and $J_k(x)$ form an infinite-dimensional Lie algebra.

§ 3.1. Classify all ten-dimensional Lie algebras which contain a four-dimensional abelian algebra (i.e., classification of all space-time groups).

Hint: Use Levi–Malcev theorem.

§ 4.1. Determine the Weyl–Cartan form (cf. th. 4.3) of the Lie algebras su(3), su(2, 2), o(p, q).

§ 4.2. The real Lie algebra so(4, 2) of dimension 15 is given by the following commutation relations of the basis elements:

$$\begin{aligned}
&M_{\mu\nu} \text{ and } P_\mu \text{ exactly as in eqs. 1(23),}\\
&[D, M_{\mu\nu}] = 0,\\
&[D, P_\mu] = -P_\mu,\\
&[K_\lambda, M_{\mu\nu}] = g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu,\\
&[K_\nu, P_\mu] = -2(g_{\mu\nu}D - M_{\mu\nu}),\\
&[D, K_\mu] = K_\mu,\\
&[K_\mu, K_\nu] = 0.
\end{aligned}$$

Show that although the maximal commutative subalgebra is four-dimensional, the Cartan subalgebra is of dimension 3.

§ 4.3. Consider the reflection Σ_α in the l-dimensional root-space by a plane perpendicular to a root α. Show that the reflected root

$$\Sigma_\alpha \beta = \beta - \frac{2(\beta\alpha)}{(\alpha\alpha)}\alpha$$

is also a root and $\Sigma_\alpha \alpha = -\alpha$ (Weyl reflection).

§ 6.1. Find the Iwasawa decomposition for the Lorentz Lie algebra so(3, 1).

Chapter 2

Topological Groups

§ 1. Topological Spaces

In this section we describe the basic properties of topological spaces such as convergence of a sequence, continuity of mappings, compactness, connectedness, etc. We also give the corresponding notions in metric spaces, which are more intuitive.

Notation: The union $A \cup B$ of two sets A and B is the set of all points which belong to A or B, i.e.:

$$A \cup B = \{x: x \in A \text{ or } x \in B\},$$

and the intersection $A \cap B$ is

$$A \cap B = \{x: x \in A \text{ and } x \in B\}.$$

The sets A and B are called *disjoint* if $A \cap B = \emptyset$, where $\emptyset$ is the empty set.

A. *Topological Spaces*

Definition 1. We say that a pair $\{X, \tau\}$, where X is an arbitrary set and τ a collection of subsets $\tau_i \subset X$, is a *topological space* if τ satisfies the following conditions:

1° $\emptyset \in \tau$, $X \in \tau$.

2° If $U_1 \in \tau$ and $U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

3° If $U_s \in \tau$ for each $s \in S$, where S is an arbitrary index set, then $\bigcup_{s \in S} U_s \in \tau$.

Every subset $U \subset X$ belonging to the collection τ is called an *open set* and the collection τ 'a *topology*' in the set X. We also use the expressions: τ defines a topology on X; X is provided with a topology τ. It follows from property 2° that the intersection of an arbitrary finite number of open sets is an open set.

A *neighborhood* of an element $x \in X$ is an arbitrary set which contains an *open neighborhood*, i.e. an open set containing x.

Clearly, a set that has more than one element can be provided with different topologies (cf. exercice 1).

Example 1. Let X be an arbitrary set and τ the family of all of its subsets. Clearly, $\{X, \tau\}$ is a topological space; one says that X has a *discrete topology*, and $\{X, \tau\}$ is a discrete (topological) space. If τ consists of X and the empty subset $\emptyset$ of X, the resultant topology is called the *coarsest topology* on X.

EXAMPLE 2. Consider the real line R and the family τ of all sets $U \subset R$ obeying the condition that for each $x \in U$ there exists an $\varepsilon > 0$ such that the interval $(x-\varepsilon, x+\varepsilon) \subset U$. The family τ satisfies conditions 1°, 2° and 3° of def. 1 for open sets and generates, what is called, the *natural topology* of the real line. The family τ consists of all open intervals with rational end points, their finite intersections and arbitrary unions. ▼

A vector space X over a field K with a topology τ on X is called the *topological vector space* if the maps $(x, y) \to x+y$ of $X \times X$ into X and $(\lambda, x) \to \lambda x$ of $K \times X$ into X are continuous in the topology τ.

B. *Convergence and Continuity*

The usual definitions of a convergent sequence and of a continuous function are reformulated in the language of open sets as follows:

DEFINITION 2. 1° A sequence $\{x_n\}$, $x_n \in X$, *converges to a limit* $x \in X$ if, for each open set U containing x, there is an integer N such that for $n \geqslant N$, $x_n \in U$.

2° A mapping f: $X \to Y$ from a topological space $\{X, \tau\}$ into a topological space $\{Y, \tau'\}$ is *continuous*, if for each $U \in \tau'$, open in Y, the inverse image $f^{-1}(U)$ is open in X. (One can also similarly define the continuity at a point $x \in X$.)

A continuous one-to-one transformation of X onto Y is said to be a *homeomorphism** (also called a *topological mapping*) if f^{-1} is continuous. In other words, X and Y are *homeomorphic* if every open set of X has an open set of Y as image and every open set of Y is the image of an open set of X.

EXAMPLE 3. The transformation $[0, 1) \ni x \to y = \exp(2\pi i x)$ is continuous and one-to-one, but it is not a homeomorphism, because the inverse transformation is not continuous. (See exercise 1.5 for another example.)

C. *Metric Spaces*

We show now that the general defs. 2.1° and 2.2° of convergence and continuity coincide with the usual definitions, if X is a metric space. In this case the concept of distance allows us to give a precise formulation of 'nearness'

A metric space (X, d) is a set X on which a real two-point function $d(\cdot, \cdot)$, the *distance* between the two points, is defined satisfying the following conditions

$$\begin{aligned} &1^\circ\ d(x, y) \geqslant 0, \\ &2^\circ\ d(x, y) = 0, \text{ if and only if } x = y, \\ &3^\circ\ d(x, y) = d(y, x), \\ &4^\circ\ d(x, y) \leqslant d(x, z)+d(z, y) \text{ for all } x,\ y,\ z \text{ in } X. \end{aligned} \tag{1}$$

* Do not confuse this concept with the algebraic concept of *homomorphism*.

EXAMPLE 4. 1° The two-point function

$$d(x,y)=\Big(\sum^{n}|x_i-y_i|^2\Big)^{1/2} \tag{2}$$

in the n-dimensional Euclidean space R^n, and

2° the function

$$d(x,y)=\|x-y\|\equiv[(x-y,x-y)]^{1/2} \tag{3}$$

in a Hilbert space H with scalar product $(\cdot,\cdot)$, define a metric in a finite and infinite-dimensional space, respectively. (See exercise 1.2 for other examples.) ▼

We say that a *metric* is defined on the space, and different metrics can be defined on the same underlying set.

Let $x\in(X,d)$ and let r be a positive number. The sets

$$S(x,r)=\{y\in X:\ d(x,y)<r\}, \tag{4}$$

$$\bar S(x,r)=\{y\in X:\ d(x,y)\leqslant r\} \tag{5}$$

are called the *open* and *closed balls* of radius r with center x. ((4) is also called the *r-neighborhood*).

DEFINITION 3. A set $V\subset X$ of a metric space (X,d) is said to be *open* if it is a union of open balls.

In particular each ball is an open set, and one readily verifies that a collection τ of all open sets in a metric space (X,d) satisfies all the axioms of def. 1 and, therefore, defines a topology in X.

If $x_n\to x$ according to def. 2.1° then, for every open ball $S(x\ \varepsilon)$ we have $d(x_n,x)<\varepsilon$ for $n\geqslant N$. Hence, $\lim x_n=x$ according to the Cauchy definition of convergence. Conversely, if $x_n\to x$ in the sense of Cauchy, then for every open ball $S(x,\varepsilon)$ there exists an N such that for $n\geqslant N$, $x_n\in S(x,\varepsilon)$; this is also true for any open set V in (X,d) containing x according to def. 3. Hence, def. 2.1° and the ordinary Cauchy definition of convergence coincide in metric spaces.

It is also evident that def. 2.2° of continuity in a topological space (X,τ) coincides with the ordinary Cauchy definition if X is a metric space and τ is the topology implied by the metric d.

The defs. 2.1° and 2.2° of convergence and continuity are thus a direct generalization to an aribtrary topological space of ordinary Cauchy definitions in metric spaces. It is in fact necessary to free the notions of neighborhoods, convergence and continuity from the more restrictive concept of distance.

EXAMPLE 5. Let H be a Hilbert space and let formula (3) define a distance in H. Let τ be a collection of open sets which are generated by balls (4) according to def. 3. The topology so obtained is called the *strong topology*, and the corresponding convergence — the *strong convergence*.

Let τ be a collection of all open sets which are unions of 'weak' open spheres of the form

$$S_W(x) = \{u \in H: \ |(g_k, u-x)| < \varepsilon, \ \text{for } k = 1, 2, \ldots, m\} \tag{6}$$

for all possible choices of a positive number ε, a positive integer m, and vectors $g_1, \ldots, g_m$.

The topology so obtained is called the *weak topology* in H and the corresponding convergence—the *weak convergence*.

Notice that a sequence u_n converges weakly to zero if $|(g, u_n)| \to 0$ for all vectors g in H; in particular the sequence $u_n = e_n$, e_n—basis vectors in H, converges weakly to zero, although it does not converge strongly. ▼

The notion of weak and the strong convergence in Hilbert space play a fundamental role in group representation theory.

Although the def. 2 of a convergent sequence and of a continuous mapping are valid in arbitrary topological spaces, they are not sufficiently restrictive in every topological space $\{X, \tau\}$. For example, in a discrete space X every function $f: X \to Y$, where Y is an arbitrary topological space, is continuous, and in the coarsest topology $\tau = \{\emptyset, X\}$ every sequence $\{x_n\}$, $x_n \in X$, is convergent to each point $x \in X$.

In order to define the limiting point of a convergent sequence uniquely, an *axiom of separation* (also called the *Hausdorff separation axiom*) is introduced:

Definition 4. We say that a topological space is a *Hausdorff space* (or *T_2-space*) if for every pair of distinct points x_1 and x_2 there exist neighborhoods U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. ▼

The discrete spaces and the real line with the natural topology are Hausdorff spaces. However, the real line R with the topology $\tau = \{\emptyset, R\}$ is not a Hausdorff space.

Proposition 1. *In Hausdorff spaces every convergent sequence has a unique limit.*

Proof: Let $x_n \to x_1$ and $x_n \to x_2$. Suppose that $x_1 \neq x_2$. On the basis of the separation axiom, we have neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. From the def. 2.1° of a convergent sequence, it follows that there exists an integer N, such that for $n \geqslant N$, $x_n \in U_1$. Therefore, $x_n \notin U_2$ for $n \geqslant N$ and consequently $x_n \nrightarrow x_2$. Thus, $x_1 = x_2$. ▼

The *complement* A' of a set $A \subset X$ is the set

$$A' \equiv X \backslash A \equiv \{x: x \in X \text{ and } x \notin A\}.$$

The set A is called *closed* in X if its complement

$$A' = X \backslash A$$

is open.

It follows directly from the definition of a closed set and a topology τ that in every topological space, the empty set and the whole space are both simultaneously closed and open. In the discrete topology every subset is both open and closed.

Remark: One could use closed sets instead of open sets in the defs. 2.1° and 2.2° with the appropriate changes.

The *closure* $\bar{A}$ of a set $A \subset X$ is the intersection of all closed sets which contain A, i.e. the smallest closed set containing A.

A set $A \subset X$ is called *dense in* X if $\bar{A} = X$. A simple example of a dense set is provided by the set of all rational numbers in R.

A topological space X is *separable*, if there exists a denumerable set $A \subset X$ which is dense in X.

EXAMPLE 6. Let X be the Hilbert space equipped with the weak or strong topology, and let $\{e_n\}^\infty$ be a denumerable basis in X. Because the linear hull $A = \{\sum_{k=1}^{\infty} c_k e_k\}$, c_k rational, $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ is dense in X, X is separable.

D. *Induced Topology, Product Topology and Quotient Topology*

Let X be a topological space and $S \subset X$ a subset. Consider in S a collection τ_s of sets of the form $S \cap U$, where U is an open subset of X. The pair (S, τ_s) satisfies the conditions 1°, 2° and 3° of def. 1. Indeed, condition 1° is satisfied, because $\emptyset = S \cap \emptyset$ and $S = S \cap X$. Using the relations

$$(S \cap U_1) \cap (S \cap U_2) = S \cap (U_1 \cap U_2),$$
$$\bigcup_{k \in K} (S \cap U_k) = S \cap \bigcup_{k \in K} U_k,$$

we see immediately that conditions 2° and 3° of def. 1 are also satisfied. Thus if we consider τ_s as a collection of open sets in S we transform the set S into the topological space (S, τ_s).

DEFINITION 5. The topological space (S, τ_s) is said to be a *subspace* of the space $\{X, \tau\}$ and the topology τ_s is called the *induced* (or *relative*) *topology* by the topology in X. ▼

Let $\{X_1, \tau_1\}$ and $\{X_2, \tau_2\}$ be two topological spaces and let

$$X = X_1 \times X_2 = \{x_1, x_2 \colon x_1 \in X_1 \text{ and } x_2 \in X_2\} \tag{7}$$

be the Cartesian product of X_1 and X_2. We define the *product topology* τ on X by choosing as a basis for the topology τ the class of all sets of the form $V \times U$ where $V \in \tau_1$ and $U \in \tau_2$.

It is evident that one can define in this manner the product topology for any finite number of topological spaces. In particular, the topology on R^1 (or C^1) defines a product topology on R^n (or C^n).

Let $\{X, \tau_X\}$ be a topological space with a topology τ_X and let f be a function on X with the range Y. The inverse function f^{-1} from Y to X defined for $y \in Y$ by $f^{-1}(y) = \{x \in X: f(x) = y\}$ has the properties

$$f^{-1}\Big(\bigcup_i U_i\Big) = \bigcup_i f^{-1}(U_i), \qquad U_i \subset Y,$$

$$f^{-1}\Big(\bigcap_i U_i\Big) = \bigcap_i f^{-1}(U_i).$$

Hence the family τ_Y of *all* sets $U \subset Y$ for which $f^{-1}(U)$ is open in X is the topology on the space Y. The topology τ_Y is the largest topology for Y with the property that the function f is continuous. The topology τ_Y is called the *quotient topology for Y* (the quotient topology relative of f and relative to the topology τ_X of X).

Let (X, τ_X) be a fixed topological vector space and R an equivalence relation on X. Let π be the natural projection of X onto the family X/R of equivalence classes. The quotient space is the family X/R with the quotient topology $\tau_{X/R}$ (relative to the map π and topology τ_X on X).

If $U \subset X/R$, then $\pi^{-1}(U) = \bigcup \{u\colon u \in U\}$. Hence U is open (closed) relative to the quotient topology $\tau_{X/R}$ if and only if $\bigcup \{u\colon u \in U\}$ is open (respectively closed) in X.

E. *Compactness*

The concept of compactness or local compactness of a group will play a fundamental role in representation theory. This notion is purely a topological one and can be defined in terms of open sets only. It is instructive, however, to define this notion first for metric spaces in the language of sequences.

DEFINITION 6. A metric space X is said to be *compact* if from every sequence $p_1, p_2, \ldots$ of points of X we can choose a subsequence, which is convergent to a point $p \in X$, i.e. there exists a sequence of indices

$$k_1 < k_2 < \ldots$$

and a point $p \in X$, such that

$$\lim_{n \to \infty} p_{k_n} = p.$$

EXAMPLE 7. Let X be the closed interval $a \leqslant x \leqslant b$, $a, b < \infty$. The classical Bolzano–Weierstrass theorem states that from every infinite bounded point sequence we can select a convergent subsequence. Hence X is compact. ▼

We can easily prove, using the Bolzano–Weierstrass theorem and the concept of coordinates, that a subset $Y \subset R^n$ is compact if and only if it is closed and bounded. (This is the classical Borel–Lebesgue theorem, also known as the Heine–Borel theorem.)

To give a definition of compactness in an arbitrary topological space we first express it in the language of open sets: in fact, we have*

THEOREM 2 (the Borel–Lebesgue theorem). *Every collection of open sets, whose union covers a compact metric space X, contains a finite subcollection, whose union covers X.* ▼

Consequently in a general topological space we assume

DEFINITION 7. A topological Hausdorff space X is *compact* if every collection of open sets, whose union covers X, contains a finite subcollection, whose union covers X (i.e. every open covering of X contains a finite subcovering).

EXAMPLE 8. 1° A discrete space is compact then and only then, when it is finite. If it is infinite, then the cover $X = \bigcup_{x \in X} \{x_i\}$ does not contain a finite subcovering.

2° A sphere S^n, $n = 1, 2, \ldots < \infty$, is compact. In fact, a sphere S^n is a closed bounded subspace of R^{n+1}. Hence, by the Bolzano–Weierstrass theorem it is compact. ▼

The compactness of topological Hausdorff spaces is invariant not only under homeomorphisms, but also under continuous transformations. In fact, we have

PROPOSITION 3. *Let X be a compact space and Y a Hausdorff space. If there exists a continuous transformation f of the space X onto the space Y, then the space Y is also compact.*

PROOF: Take an arbitrary open cover $\{U_s\}_{s \in S}$ of the space Y. The sets $\{f^{-1}(U_s)\}_{s \in S}$ generate an open cover of the space X. Therefore there exists a finite number of indices $s_1, s_2, \ldots, s_k \in S$ such that

$$f^{-1}(U_{s_1}) \cup f^{-1}(U_{s_2}) \cup \ldots \cup f^{-1}(U_{s_k}) = X.$$

Taking the images of both sides in this equation we obtain

$$U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k} = Y. \blacktriangledown$$

A subset $A \subset X$ is compact if it is compact as a topological space with the induced topology.

DEFINITION 8. A topological space is *locally compact* if each point has a compact neighborhood.

Clearly, every compact space is locally compact. Every discrete space is locally compact. The straight line R is locally compact, because every point $x \in R$ has a neighborhood N (e.g., $N = (x-\varepsilon, x+\varepsilon)$), whose closure, by Bolzano–Weierstrass theorem, is compact.

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* A family $\{U_\lambda\}$ of (open) subsets of X is called an (open) *covering* of a subset $A \subset X$, if $A \subset \bigcup U_\lambda$. A covering is finite (or countable) if it consists of a finite (or countable) number of sets.

The following, fundamental theorem gives a simple characterization of a class of non-locally compact spaces.

THEOREM 4 (Gleason). *A locally convex topological vector space X is locally compact if and only if* $\dim X < \infty$.

In particular every infinite-dimensional Hilbert space is not locally compact. One can show this directly; in fact, for instance, the sequence $x_n = Re_n$, where $\{e_n\}$ is an orthonormal basis in H and R any positive number does not contain a convergent subsequence in the strong topology; hence any ball $S(x, R)$ does not have a compact closure.

F. *Connectedness*

In the theory of topological groups and group representations it is important to know, intuitively speaking, the number of different 'pieces' of a given topological space. We introduce now a mathematical formalism, which describes this feature of topological spaces.

DEFINITION 9. A topological space X is called *connected* if it is *not* the union of two non-empty, disjoint subsets A and B. A subspace $Y \subset X$ is called *connected* if it is connected as a topological subspace with the relative topology.

EXAMPLE 9. Every discrete space X containing more than one point is not connected, because it can be represented as a sum of two sets $A = \{x_0\}$, $x_0 \in X$ and $A' = X \backslash \{x_0\}$, which are non-empty, open and disjoint. ▼

The connectedness is also invariant not only under homeomorphisms, but also under continuous transformations:

PROPOSITION 5. *Let X be a connected space and $f\colon X \to Y$ a continuous transformation onto the space Y. Then Y is connected.*

PROOF: Suppose that Y is not connected. Then, there exist two non-empty open sets A and B in Y such that

$$A \cup B = f(X) = Y$$

and

$$A \cap B = \emptyset.$$

It follows from the definition of the inverse mapping that

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(Y) = X$$

and

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset.$$

The sets $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty. Moreover, the continuity of the mapping f implies that the sets $f^{-1}(A)$ and $f^{-1}(B)$ are open. Hence, X can be represented as the union of two non-empty, disjoint, open sets, which contradicts the connectedness of X. ▼

DEFINITION 10. A *component* of a point x of a topological space X is the union of all connected subspaces of the space X containing the point x.

The closure of a connected space is connected; hence the component is closed.

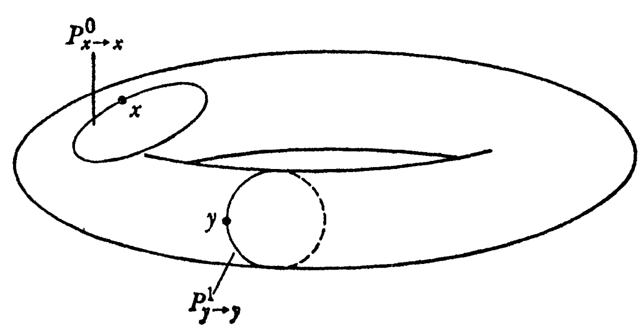
If all components of a space X are one-point sets, then we call the space X *totally disconnected.*

G. *Simply-connected and Multi-connected Spaces*

Two points x and y of a space X are connected by a path $P_{x\to y}$, if one can proceed continuously from x to y passing through the elements of a one-parameter subset P of X. If x coincides with y, one obtains either a closed path $P_{x\to x}$, or the null path P at x. In other words, a *path* in X is a continuous mapping of the closed interval $[0, 1]$ into X.

The two paths $P_{x\to y}$ and $Q_{x\to y}$ are said to be *homotopic* (or *deformable*, or *equivalent*), denoted by $P \simeq Q$, if there exists a continuous deformation of the path $Q_{x\to y}$ into the path $P_{x\to y}$ which leaves the end points unaltered.

EXAMPLE 10. Let X be the surface of a torus. The closed path $P^0_{x\to x}$ which does not wrap the ring is homotopic to the null path. The closed path $P^1_{y\to y}$



which once wraps the ring is neither homotopic to $P^0_{x\to x}$, nor to the path which wraps twice the ring. ▼

A topological space is called *simply-connected* if every closed path is homotopic to the null path. The space in example 10 is not simply-connected.

Similarly, by making the stereographic projection, we conclude that the n-sphere S^n is simply-connected for $n > 1$.

The definition of the homotopy of paths satisfies the following conditions:

$$P \simeq P \quad \text{(reflectivity)},$$
$$P \simeq Q \Rightarrow Q \simeq P \quad \text{(symmetry)},$$
$$P \simeq Q,\ Q \simeq T \Rightarrow P \simeq T \quad \text{(transitivity)}.$$

Hence, it is an equivalence relation: consequently all closed paths (at a point x) are classified into the so-called *homotopy classes.*

A topological space is said to be *n-connected* if it has n homotopy classes at each point.

EXAMPLE 11. Let X be the torus. It is evident that a closed path, which wraps the ring k-times is nonequivalent to a closed path, which wraps the ring l-times for $k \neq l$. Hence, the torus has infinite number of homotopy classes at each point. Consequently it is infinitely connected. ▼

§ 2. Topological Groups

We combine the concepts of an abstract group and a topological space specified on the same set G into that of a topological group G. The consistency of this combination is provided by continuity.

DEFINITION 1. A *topological group* is a set G such that:

1° G is an abstract group,

2° G is a topological space,

3° the function $g(x) = x^{-1}$, $x \in G$, is a continuous map from $G \to G$ and the function of $f(x, y) = x \cdot y$ is a continuous map from $G \times G \to G$.

EXAMPLE 1. Consider the Euclidean space R^n as an abelian algebraic group and as a topological space with the product topology of R^1. The functions

$$g(x) = x^{-1} = -x, \quad \text{and} \quad f(x, y) = x+y$$

are continuous in this topology. Therefore, this abelian group is a topological group. ▼

The condition 3° expresses the compatibility between the algebraical and the topological operations on the set G. The following example illustrates that this compatibility condition is not automatically satisfied on a set, which is both an abstract group and a topological space.

EXAMPLE 2. Consider the cyclic group $C_3 = (1, x, x^2)$.* Let us define a topology τ by means of the following open sets

$$\emptyset,\ \{1\},\ \{x\},\ \{1, x\},\ C_3. \tag{1}$$

The function $g(x) = x^{-1}$ transforms the element x^2 onto $g(x^2) = x$. Therefore the inverse function g^{-1} transforms the open set $\{x\}$ onto the set $\{x^2\}$, which is not open. Therefore the cyclic group C_3 with the topology (1) is not a topological group. ▼

DEFINITION 2. Let G be a topological group. A set $H \subset G$ is called a *topological subgroup* of G if

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* An abstract group G is called *cyclic* if every group element is a power of some group element x, i.e. $g_i = x^{p_i}$, $x \in G$.

1. H is a subgroup of the abstract group G.
2. H is a closed subset of the topological space G. ▼

Thus the fact that G is not only an abstract group, but a topological group, imposes on the subgroup H of G a new condition, namely, that it be closed. The condition 2 could be replaced by other equivalent condition; in fact we have:

PROPOSITION 1. *A subgroup H of a topological group G, that is an open subset of G is also a closed subset.*

PROOF: A coset X is the set of all elements x of G which satisfy the condition $x^{-1}x' \in H$, $x' \in X$ (i.e. $X = x'H$). This definition provides an equivalence relation so that the space G decomposes onto disjoint cosets. Because H is open, each coset is open. The complement of H consists of the union of cosets of H and is therefore open. Consequently, H is closed. ▼

Note that not every abstract subgroup H of the abstract group G is a topological subgroup of G considered as a topological group. For example, the abstract subgroup H of the additive group R of real numbers consisting of rational numbers is not a topological subgroup because it is not closed in R.

A topological subgroup N of a topological group G is called an *invariant subgroup* if for each $n \in N$ and $g \in G$ we have

$$g^{-1}ng \in N, \quad \text{i.e.} \quad g^{-1}Ng \subset N.$$

EXAMPLE 3. Let $\mathrm{GL}(n, R)$ be the group of all real non-singular $n \times n$-matrices under multiplication. We can parametrize an arbitrary group element $x = \{x_{ik}\}$ by the matrix elements $x_{ik} \in R$ so that $\mathrm{GL}(n, R)$ is a subset of R^{n^2}. Let us choose on $\mathrm{GL}(n, R)$ the induced topology of R^{n^2}. Because the matrix elements z_{ik} of $z = xy$ are algebraic functions of x_{il} and y_{lk}, the composition law in G is continuous. Similarly, the matrix elements of x^{-1} are rational, non-singular and, therefore, continuous functions of x. Hence, $\mathrm{GL}(n, R)$ equipped with the induced topology of R^{n^2} is a topological group.

The subgroup $G_1 = \{\lambda I: \lambda \in R,\ \lambda \neq 0\}$ forms a one-parameter invariant topological subgroup of $\mathrm{GL}(n, R)$. Consider the subgroup $\mathrm{O}(n) = \{x \in \mathrm{GL}(n, R), (x^{-1})_{ik} = x_{ki}\}$. The mapping f: $x \to x^T x = e$ of $\mathrm{O}(n)$ is continuous; hence, the inverse image $f^{-1}(e) = \mathrm{O}(n)$ of the closed set e is closed in $\mathrm{GL}(n, R)$; consequently $\mathrm{O}(n)$ is a topological subgroup of $\mathrm{GL}(n, R)$. It is called the *orthogonal group.* ▼

Similarly it follows that with the induced topology of C^{n^2} the group $\mathrm{GL}(n, C)$ of $n \times n$ complex non-singular matrices is a topological group.

Different topological groups might be homeomorphic as topological spaces; for instance, the abelian group G_1 consisting of matrices

$$\begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}, \quad a, b \text{ real},$$

and the noncommutative group G_1 consisting of matrices

$$\begin{bmatrix} e^a & b \\ 0 & e^{-a} \end{bmatrix}, \qquad a, b \ \overline{\text{real}},$$

are both homeomorphic to R^2 as topological spaces.

On the other hand one can obtain different topological groups by taking different topologies on the same abstract group. This leads to the following concept of *isomorphism* of topological groups.

DEFINITION 3. Two topological groups will be called *isomorphic* if there is one-to-one correspondence between their elements, which is a group isomorphism, *and* a space homeomorphism (i.e. preserves open sets).

The topological groups are examples of the so-called homogeneous spaces:

DEFINITION 4. A topological space $\{X, \tau\}$ is *homogeneous* if for any pair $x, y \in X$ there exists a homeomorphism f of the space $\{X, \tau\}$ onto itself such that $f(x) = y$. ▼

Every topological group G is homogeneous, because any two points $x, y \in G$ can be connected by the left translation

$$y = T_a^L x = ax, \qquad a = yx^{-1}, \tag{2}$$

which, by virtue of the uniqueness and the continuity of the group multiplication, is a homeomorphism of G.

The homogeneity of a topological group considerably simplifies the study of their local properties. It is sufficient to investigate the local properties of a topological group in the neighborhood of one point, for example, in the neighborhood of the unit element. Homogeneity will then ensure the validity of these properties at any other point.

We say that a topological group G has a topological property B (e.g., G is compact, connected, separable, quotient group etc.) if G, considered as a topological space, has the property B.

The most important topological properties of G, with fundamental implications in the theory of group representations are compactness and connectedness (cf. § 1). We begin with a simple example to illustrate the concept of compactness of a group.

EXAMPLE 4. 1° Consider the orthogonal group $O(n)$. The real $n \times n$ matrix $x = \{x_{ik}\}$, $i, k = 1, 2, \ldots, n$, that corresponds to a group element $x \in O(n)$ satisfies the condition

$$x^T x = e. \tag{3}$$

It means that columns of a matrix $x \in O(n)$ can be considered as the orthonormal vectors of R^n. Therefore, matrix elements x_{ik}, $i, k = 1, 2, \ldots, n$, obey the condition

$$\sum_{i,k=1}^{n} x_{ik}^2 = n, \tag{4}$$

i.e. group elements of $O(n)$ can be represented as points of the sphere S^{n^2-1} of radius $\sqrt{n}$. The collection of all points on the sphere which correspond to points of $O(n)$ is a closed set. Therefore, the group space of $O(n)$ is a closed, bounded subset of R^{n^2}, in the topology induced by that of R^{n^2}. Hence, by Bolzano–Weierstrass theorem the group $O(n)$ is compact.

2° One can prove in a similar manner that the unitary group $U(n)$ is compact. ▼

Although the underlying, topological space of a group G in R^n or C^n might be bounded, the corresponding topological group might still be noncompact because G is not closed. For instance, the one-parameter Lorentz group can be parametrized by the numbers v/c, where v is the velocity of a reference system and c the velocity of light; however, the Lorentz group is noncompact because the interval $-1 < v/c < 1$ in which the group is defined is not closed.

The group $GL(n, R)$ is noncompact, but it is locally compact; in fact, the map

$$\psi\colon x \to \det x, \qquad x \in GL(n, R) \tag{5}$$

is a continuous map of $GL(n, R)$ into R. Hence, the inverse image $\psi^{-1}(0)$ of the closed set $\{0\}$ in R is closed in R^{n^2}. The complement of $\psi^{-1}(0)$ in R^{n^2} is $GL(n, R)$. Therefore $GL(n, R)$ is an open set in R^{n^2}. Consequently, $GL(n, R)$ is noncompact.

It is, however, locally compact, because each point of the open set in R^{n^2} has a compact neighborhood (cf. def. 1.8).

Similarly it follows that $GL(n, C)$ is locally compact.

Clearly, there exist topological groups which are noncompact and not locally compact. In fact, they play an important role in theoretical physics.

Example 5. 1° Let H be a Hilbert space with the strong topology τ defined by the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. Consider H as an abelian topological group with respect to addition. By virtue of the Gleason theorem (§ 1, th. 4) this group is not locally compact.

2° A physical example of a non-locally compact group is provided by the group of gauge transformations in classical or quantum electrodynamics: Let $A_\mu(x)$ be the vector potential. The gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \varphi(x)$$

leave the field quantities $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ invariant.

Clearly these transformations form a group G with respect to addition. Introducing any topology in G, we can convert the space of gauge functions into a topological vector space: this space is not locally compact by virtue of Gleason theorem. ▼

We shall now derive some properties of connected topological groups. Note first that if G is connected, then the component of the identity coincides with G. On the other hand, if a component of unity contains only unity, then, due to homogeneity of topological groups all components of G are one-point sets, i.e.

G is totally disconnected. The set of rationals considered as an abelian topological group with the relative topology of the reals is an example of such a group.

PROPOSITION 2. *Let G be a topological group, and G_0 the component of the identity* (def. 1.10). *Then G_0 is a closed invariant subgroup of G.*

PROOF: Let C be a connected subset of G; then xC and Cx, $x \in G$, are also connected, because left and right translations are homeomorphisms. Consequently, xG_0x^{-1}, $x \in G$, is a connected component which contains the identity e of G. Thus xG_0x^{-1} coincides with G_0 for every x in G. We know that any component is closed. Hence, G_0 is an invariant topological subgroup of G. ▼

A subgroup H of a topological group G is called *central* if each element of H commutes with every element of the whole group G.

PROPOSITION 3. *If G is a connected topological group and H is an invariant discrete subgroup, then H is central.*

PROOF: Consider the map of G into H

$$G \ni g \to gxg^{-1} \in H, \qquad x \in H.$$

This homeomorphism transforms a connected set onto a connected set. But connected sets in H are only one-point sets $\{x\}$. Therefore the image of G, which contains x, must coincide with $\{x\}$. ▼

The following important example of a simply connected group will be used in this book throughout.

EXAMPLE 6. Let G be the unitary unimodular group SU(2). A group element g can be parametrized by

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6}$$

Because

$$g^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{7}$$

the unitarity condition, $g^* = g^{-1}$, implies

$$d = \bar{a} \quad \text{and} \quad c = -\bar{b}.$$

Consequently, the unimodularity condition $ad-bc = 1$ can be written in the form

$$a\bar{a}+b\bar{b} = 1. \tag{8}$$

Setting $a = x+iy$ and $b = z+it$ we conclude that the group manifold of SU(2) coincides with the surface $x^2+y^2+z^2+t^2 = 1$ of three-dimensional sphere S^3. Since every closed path on S^3 can be contracted to a point, the group SU(2) is simply connected.

The subgroup D

$$D = \{I, -I\} \tag{9}$$

represents the discrete center of SU(2). The quotient group $SU(2)/D$ represents the doubly-connected rotation group SO(3). Thus the group manifold of SO(3) is obtained by identifying the antipodal points of the sphere S^3. ▼

Uniform Continuity

A complex function $\varphi(x)$ on R^1 is said to be *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(x)-\varphi(x+z)| < \varepsilon, \quad \text{whenever } |z| < \delta. \tag{10}$$

This notion has a natural generalization for complex functions defined on a group manifold.

A complex function $\varphi(x)$ defined on a topological group G is said to be *left uniformly continuous* if for arbitrary $\varepsilon > 0$ there exists a neighborhood V of the identity e in G such that for $x^{-1}y \in V$ we have

$$|\varphi(x)-\varphi(y)| < \varepsilon \quad (\text{or } |\varphi(x)-\varphi(xz)| < \varepsilon, \text{ whenever } z \in V). \tag{11}$$

Similarly we say that $\varphi(x)$ is *right uniformly continuous* if for arbitrary $\varepsilon > 0$ there exists a neighborhood U of e such that for $xy^{-1} \in U$ we have

$$|\varphi(x)-\varphi(y)| < \varepsilon \quad (\text{or } |\varphi(x)-\varphi(zx)| < \varepsilon \text{ whenever } z \in U). \tag{12}$$

A function which is both left and right uniformly continuous is said to be uniformly continuous.

PROPOSITION 4. *Let G be a topological group and let S be a compact subset of G. Then a continuous function φ defined on S is uniformly continuous on S.*

PROOF: Because φ is continuous, for every $\varepsilon > 0$ there exists, for each point $y \in S$, a neighborhood V_y of e such that if $x \in S$ and $xy^{-1} \in V_y$, then $|\varphi(x)-\varphi(y)| < \varepsilon/2$. Let W_y be a neighborhood of e such that $W_y^2 \subset V_y$; the collection of open sets $W_y y$ covers S and because S is compact, we can select a finite covering. Let $\{W_{y_i} y_i\}_1^n$ be a finite collection of open sets which cover S, and let

$$V = \bigcap_{i=1}^{n} W_{y_i}.$$

Now, because the sets $W_{y_i} y_i$ cover S for $x, y \in S$, $xy^{-1} \in V$, there exists a number k such that $yy_k^{-1} \in W_{y_k}$ and therefore $|\varphi(y)-\varphi(y_k)| < \varepsilon/2$. Next

$$xy_k^{-1} = xy^{-1}yy_k^{-1} \in VW_{y_k} \subset W_{y_k}^2 \subset V_{y_k}$$

so that $|\varphi(x)-\varphi(y_k)| < \varepsilon/2$; hence we obtain

$$|\varphi(x)-\varphi(y)| < |\varphi(x)-\varphi(y_k)|+|\varphi(y)-\varphi(y_k)| < \varepsilon.$$

Consequently for $x \in S$, $\varphi(x)$ is uniformly right continuous. One shows similarly the uniform left continuity of $\varphi(x)$ on S. ▼

The concept of a left or a right continuity has a natural extension for a function $\varphi(x)$ on G with values in a topological vector space H; for instance, if H is a Hilbert space then $\varphi(x)$ is said to be left uniformly continuous if for arbitrary $\varepsilon > 0$ there exists a neighborhood V of e such that

$$\|\varphi(x)-\varphi(y)\|_H < \varepsilon, \quad \text{whenever } x^{-1}y \in V. \tag{13}$$

§ 3. The Haar Measure

In this section we introduce the important concept of an invariant measure and an invariant integration over a topological group G. Let G be a locally compact group and let $C_0(G)$ and $C_0^+(G)$ denote the space of continuous and continuous non-negative functions on G with a compact support respectively. A positive Radon measure is a positive linear form μ on $C_0(G)$ which is non-negative on $C_0^+(G)$, i.e.,

$$\mu(f) \geqslant 0 \quad \text{for } f \in C_0^+(G). \tag{1}$$

A positive Radon measure μ, which is left-invariant, i.e.,

$$\mu(T_g^L f) = \mu(f), \quad \text{where } T_g^L f(x) = f(g^{-1}x),\ x, g \in G, \tag{2}$$

is called a *left Haar measure* (or a *left Haar integral*).

One defines similarly a right Haar measure λ which satisfies the condition

$$\lambda(T_g^R f) = \lambda(f), \quad \text{where } T_g^R f(x) = f(xg),\ x, g \in G.$$

THEOREM 1. *Every locally compact group has a left Haar measure μ. If ν is any other non-zero left Haar measure, then $\nu = c\mu$ for some positive number c.* ▼

(For the proof cf. Hewitt and Ross 1963, ch. IV, § 15.)

Let G be a locally compact group with the multiplication law xy and let G^* be a new group with the same elements and same topology but with a new group multiplication law $x \times y$ defined by

$$x \times y = yx. \tag{3}$$

If G^* has a left Haar measure μ given by th. 1 then G has a right Haar measure: indeed, for $g^* \in G^*$, we have

$$(T_g^L \times f)(x) = f(g^{*-1} \times x) = f(xg^{-1}) = T_{g^{-1}}^R f(x),$$

hence

$$\lambda(T_g^R f) \equiv \mu(T_g^R f) = \mu(T_{g^{*-1}}^L f) = \mu(f) = \lambda(f), \tag{4}$$

i.e., λ is a right Haar measure. Consequently the existence of a left Haar measure implies the existence of a right Haar measure (see also exercises 4 and 6). Therefore, by th. 1 every locally compact group has also a right Haar measure λ defined up to a positive constant factor. The Haar measure, which is both left- and right-invariant is called the *invariant measure*.

By Riesz theorem we can associate with a measure $\mu(f)$ a set function $\mu(X)$ for a measurable set $X \subset G$, such that

$$\mu(f) = \int_G f(g)\,\mathrm{d}\mu(g). \tag{5}$$

The left-invariance (2) of the Haar measure implies then

$$\mu(gX) = \mu(X), \quad \text{or} \quad \mathrm{d}\mu(gx) = \mathrm{d}\mu(x), \tag{6}$$

for all $X \subset G$, $g, x \in G$.

EXAMPLE 1. Let G be the group of all complex 2×2-matrices with determinant one, i.e., $G = \mathrm{SL}(2, C)$. We shall construct explicitly an invariant Haar measure for G:

Every element of G is a matrix $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and may be identified with a point of C^4: The unimodular matrices form, in C^4, a second order surface $\alpha\delta - \beta\gamma = 1$. We relate with this surface a differential form $\mathrm{d}\omega$ defined by the formula

$$\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta = \mathrm{d}(\alpha\delta-\beta\gamma)\mathrm{d}\omega = J\mathrm{d}(\alpha\delta-\beta\gamma)\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta \tag{7}$$

where J is the Jacobian for the transformation $(\alpha, \beta, \gamma, \delta) \to [(\alpha\delta-\beta\gamma), \beta, \gamma, \delta]$ from which we obtain the following expression for $\mathrm{d}\omega$

$$\mathrm{d}\omega(g) = \frac{1}{\delta}\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta. \tag{8}$$

Under the left translation $g \to g_0 g$ by an element $g_0 \in \mathrm{SL}(2, C)$ the form $\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta$, as well as the determinant $(\alpha\delta-\beta\gamma)$ of the matrix $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ are conserved. Consequently the form $\mathrm{d}\omega$ is also conserved. The same is true for the right translations $g \to gg_0$. Thus the positive form

$$\mathrm{d}\mu(g) = \mathrm{d}\omega\mathrm{d}\bar{\omega} = \frac{1}{|\delta|^2}\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\bar{\beta}\mathrm{d}\bar{\gamma}\mathrm{d}\bar{\delta} \tag{9}$$

satisfies

$$\mathrm{d}\mu(g_0 g) = \mathrm{d}\mu(gg_0) = \mathrm{d}\mu(g). \tag{10}$$

Therefore eq. (9) provides an invariant Haar measure on $\mathrm{SL}(2, C)$. By th. 1 any other Haar measure is then proportional to the measure (9).

Notice that because $g^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$, we have in addition from eq. (9):

$$\mathrm{d}\mu(g^{-1}) = \mathrm{d}\mu(g). \tag{11}$$

which expresses the inversion invariance of the Haar measure. ▼

For other examples see exercises 2 and 5.

Inversion Invariance

Let $\mu(\cdot)$ be a left Haar measure and let $\mu_g(f) \equiv \mu(T_g^R f)$. Because left and right translations commute we have

$$\mu_y(T_g^L f) = \mu(T_y^R T_g^L f) = \mu(T_g^L T_y^R f) = \mu_y(f).$$

Thus a linear positive invariant measure $\mu_y(f)$ is a Haar measure. By th. 1 we conclude that $\mu_y = \Delta(y)\mu$. Hence,

$$\mu(T_y^R f) = \Delta(y)\mu(f). \tag{12}$$

Because the map $G \ni y \to T_y^R f \in C_0(G)$ is continuous, the function $\Delta(y)$ is also continuous. Moreover, it satisfies the functional equation

$$\Delta(xy) = \Delta(x)\Delta(y). \tag{13}$$

Indeed,

$$\Delta(xy)\mu(f) = \mu(T_{xy}^R f) = \mu\big(T_x^R(T_y^R f)\big) = \Delta(x)\mu(T_y^R f) = \Delta(x)\Delta(y)\mu(f).$$

The function $\Delta(x)$ is called the *modular function for the group* G. If $\Delta(x) \equiv 1$, then, by eq. (12) the right and the left Haar measures for the group G coincide. If this is the case the group is called *unimodular*.

Clearly, every abelian locally compact group is unimodular, because $T_{g^{-1}}^R = T_g^L$ in this case. Moreover, we have

PROPOSITION 2. *Every compact group is unimodular.*

PROOF: If G is compact, then the function $f(x) \equiv 1$, $x \in G$, is in $C_0^+(G)$; hence, normalizing the left Haar measure by the condition $\mu(1) = 1$ we obtain

$$\Delta(y) = \Delta(y)\cdot\mu(1) = \mu(T_y^R 1) = \mu(1) = 1. \blacktriangledown$$

The unimodular Lie groups are described in ch. 3, § 10.D.

We now derive the fundamental inversion property of a left Haar measure.

PROPOSITION 3. *Let* $\mu(\cdot)$ *be a left Haar measure and let* $\check{f}(x) \equiv f(x^{-1})$. *Then,*

$$\mu(f) = \mu\left(\check{f}\,\frac{1}{\Delta}\right) \quad \textit{for every } f \in C_0^+(G). \tag{14}$$

PROOF: Let $\check{\mu}(f) \equiv \mu\left(\check{f}\,\dfrac{1}{\Delta}\right)$; then,

$$\check{\mu}(T_y^L f) = \mu\left((T_y^L f)^\vee \frac{1}{\Delta}\right) = \mu\left(T_{y^{-1}}^R \check{f}\,\frac{1}{\Delta}\right) = \Delta(y^{-1})\mu\left(T_{y^{-1}}^R \check{f}\,\frac{1}{T_{y^{-1}}^R \Delta}\right)$$

$$= \Delta(y^{-1})\Delta(y)\mu\left(\check{f}\,\frac{1}{\Delta}\right) = \check{\mu}(f).$$

Therefore, $\check{\mu}(\cdot)$ is a left Haar measure; consequently $\check{\mu}(f) = c\mu(f)$.

Next we show that $c = 1$; let ξ be a positive number and let U be a neighborhood of e in G such that $\left|\dfrac{1}{\Delta(x)} - 1\right| < \varepsilon$ for all $x \in U$. Let h be a non-zero element in $C_0^+(G)$ such that $\check{h} = h$ and h vanishes on the complement U' of U. Then,

$$\left|h(x) - h(x)\frac{1}{\Delta(x)}\right| \leqslant \varepsilon h(x) \qquad \text{for all } x \in G;$$

consequently

$$\left|\mu(h)-\mu\left(h\frac{1}{\Delta}\right)\right| \leqslant \varepsilon\mu(h).$$

This implies $|1-c| < \varepsilon$, i.e., $c = 1$. Thus $\mu(f) = \mu\left(\check{f}\,\frac{1}{\Delta}\right)$ for all $f \in C_0^+(G)$. ▼

Note that for a unimodular group G we have $\mu(f) = \mu(\check{f})$, i.e.,

$$\int_G f(x)\,\mathrm{d}\mu(x) = \int_G f(x^{-1})\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x^{-1}). \tag{15}$$

i.e. $\mathrm{d}\mu(x) = \mathrm{d}\mu(x^{-1})$.

In other words every invariant Haar measure is also invariant under the inversion; eq. (11) expresses explicitly this property of the Haar measure for SL(2, C).

§ 4. Comments and Supplements

A. *Mackey Decomposition Theorem*

The following theorem gives an important decomposition of an arbitrary element of a topological group G.

THEOREM 1. *Let G be a separable locally compact group and let K be a closed subgroup of G. Then there exists a Borel set S in G such that every element $g \in G$ can be uniquely represented in the form*

$$g = ks, \quad k \in K, \ s \in S. \text{▼} \tag{1}$$

(For the proof cf. Mackey 1952, part I, lemma 1.1.)

The decomposition (1) plays a fundamental role in the theory of induced representations of topological groups (cf. ch. 16).

B. *The Universal Covering Group*

The connection between global and local properties of topological groups is described by the following theorem:

THEOREM 2. *Let Γ be the class of all arcwise-connected, locally connected, locally simply-connected topological groups, which are locally isomorphic with a certain topological group G. Then there exists in the class Γ, up to isomorphism, one and only one simply-connected group $\tilde{G}$. Any other group of the class Γ is a quotient group $\tilde{G}/N$, where N is a discrete normal subgroup.* ▼

(For the proof cf. Pontryagin 1966, ch. IX, sec. 51.)

The group $\tilde{G}$ is called the *universal covering group* of all groups in the class Γ.

Th. 2 plays a fundamental role in group representation theory because the connectedness of the group space is directly related to the single-valuedness of the representations of G.

C. *Invariant Metric*

It is interesting that a topological group G possesses not only an invariant measure but also an invariant metric. In fact, we have:

THEOREM 3 (the Birkhoff–Kakutani theorem). *Let G be a topological group whose open sets at the identity e have a countable basis.* Then there exists a distance function $d(\cdot, \cdot)$, which is right-invariant, i.e.*

$$d(xg, yg) = d(x, y) \quad \text{for all } x, y \text{ and } g \text{ in } G, \tag{2}$$

and which induces on G the original topology. ▼

(For the proof cf. Montgomery and Zippin 1955, ch. I, § 22.)

D. *Bibliographical Notes*

The axiomatic definition of a topological group in the form used today was first given by Polish mathematician F. Leja 1927. This subject became very popular in the early 1930's and was investigated by many prominent mathematicians such as D. van Dantzig, A. Haar, J. von Neumann, and others.

The notion of invariant integration on continuous groups had already been introduced in the nineteenth century by Hurwitz 1897. Later on, Weyl 1925–1926, I, II, III computed the invariant integral for $\mathrm{O}(n)$ and $\mathrm{U}(n)$; soon Peter and Weyl 1927 showed the existence of an invariant integral for any compact Lie group. The crucial achievement was Haar's 1933 result, who directly constructed a left-invariant integral for a locally compact group with a countable open basis. This result was surprising even for the best mathematicians like von Neumann, who did not believe in an existence of invariant integrals for such an extensive class of topological groups.

Haar's construction was extended to an arbitrary locally compact group by A. Weil 1936a, b, 1940.

An invariant measure exists also on some non-locally compact groups. In particular, the construction of an invariant measure for the complete metric groups was treated by Oxtoby 1946. For a more detailed discussion, see also the books by Hewitt and Ross 1963 and L. Nachbin 1965. Cf. also Stone 1966.

§ 5. Exercises

§ 1.1. Show that there exist nine topological spaces consisting of 3 elements no two of them homeomorphic.

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* A family $B(x)$, $x \in X$, of neighborhoods of a point x having the property that for every open set V containing x there exists $U \in B(x)$ such that $x \in U \subset V$ is called a *basis* of the topological vector space (X, τ) in the point x.

§ 1.2. Show that the following two-point functions (besides eq. 1(2) and 1(3)) define a metric on the corresponding sets:

(a) On the set of n-tuples of real numbers

$$d(x, y) = \max|x_i - y_i|,$$

(b) X = set of all continuous real functions on the closed interval $[0, 1]$,

$$d(x, y) = \left[\int_0^1 (x(t)-y(t))^2 \mathrm{d}t\right]^{1/2},$$

or

$$d(x, y) = \max_{0\leqslant t\leqslant 1} |x(t)-y(t)|.$$

(c) X = arbitrary set,

$$d(x, y) = 1 \qquad \text{for } x \neq y,$$
$$d(x, x) = 0.$$

§ 1.3. The sequence A_n of operators in a Hilbert space is said to *converge to A in uniform* (or *norm*) *topology* if

$$\lim_{n\to\infty} ||A_n - A|| = 0.$$

Show that the unitary one-parameter group of translations

$$U_t\colon u(x) \to u(x+t)$$

in $H = L^2(R^1)$ is continuous in strong topology but not in the uniform topology. Notice that because for any t, t', $t \neq t'$

$$||U_t - U_{t'}|| = 2,$$

the *curve* $t \to U_t$ is *discrete* in uniform topology. This example illustrates the different nature of continuity in different topologies.

§ 1.4. Let SU(3) be the group of all 3×3 unitary and unimodular matrices. Study the connectedness of $\mathrm{SU}(3)/Z_3$, where

$$Z_3 = \{I, \mathrm{e}^{\mathrm{i}2\pi/3}I, \mathrm{e}^{\mathrm{i}4\pi/3}I\}$$

is the center of SU(3). Generalize it to $\mathrm{SU}(n)/Z_n$.

§ 1.5. Let (R, τ_1) and (R, τ_2) be the real line R equipped with a topology τ_1 and τ_2, respectively. Show that the one-to-one mapping $f\colon x \to y = x$ from (R, τ_1) into (R, τ_2) is continuous, iff τ_1 is stronger than τ_2.

§ 2.1. Let G be the group of all linear transformations in C^n which leaves the quadratic form

$$z_1\bar{z}_1 + \ldots + z_p\bar{z}_p - z_{p+1}\bar{z}_{p+1} - \ldots - z_n\bar{z}_n = 1$$

invariant.

Define a topology τ on G such that G becomes a topological group.

§ 2.2. Let G be the group $O(n, 1)$ of all real transformations in R^{n+1} which conserves the quadratic form:

$$x_0^2 - x_1^2 - x_2^2 - \ldots - x_n^2.$$

Show that $O(2, 1)$ is infinitely many times connected.

§ 2.3. Show that $O(3, 1)$ consists of four components. Verify that the result is also true for the group $O(n, 1)$, $n \geqslant 3$.

§ 2.4. Let $X = R^n$ and let G be the set of all one-to-one C^∞ transformations: $f\colon R^n \to R^n$ such that the inverse transformation is also C^∞. The group G is the group of coordinate transformations (diffeomorphism group). For f and g in G the functions

$$d(f, g)_n = \max_{0 \leqslant |m| \leqslant n} \sup_{x \in R^n} \left|(1+|x|^2)^n \left(f^{(m)}(x) - g(x)^{(m)}\right)\right|$$

where

$$f^{(m)}(x) = (D^m f)(x), \qquad D^m = D^{m_1} \ldots D^{m_l}, \qquad D^{m_j} = \frac{\partial}{\partial x_{m_j}}, \qquad |m| = l$$

define metrics in G. Let τ_d denote the topology in G defined by the metrics d_n. Show that the group operations are continuous relative to τ_d and therefore (G, τ_d) is a topological group.

§ 2.5. Let H be a Hilbert space. Let G be the group of all unitary operators in H. Define a topology τ on G, such that G becomes a topological group.

§ 2.6. Consider the Schwartz space of functions S on R^n as an abelian group. Let $N = S \circledS G$ (G = the diffeomorphism group in R^n, cf. exercise 2.3)) be the group defined by the following composition law

$$(s, g)(s', g') = (s + s' \circ g, g \circ g'),$$

where $s' \circ g$ and $g \circ g'$ denote the composition of the corresponding maps on R^n (e.g. $s' \circ g = s'[g(x)]$).

Show that N equipped with the product topology of Schwartz topology on S and the topology τ_d on G is a topological group. (Note: This is the global group associated with the current algebraic commutation relations; cf. ch. 1.10, exercise 2.9.)

§ 3.1. Let μ be the left Haar measure on G. We define a new measure

$$\check{\mu}(f) = \mu(\check{f}), \quad \text{where } \check{f}(x) = f(x^{-1}).$$

Show that $\check{\mu}$ is the right Haar measure on G.

§ 3.2. Consider the group G of all matrices of the form

$$G \ni g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, \quad x, y \in R,\ x \neq 0.$$

Show that the left Haar measure has the form

$$\mathrm{d}\mu(g) = \frac{\mathrm{d}x\,\mathrm{d}y}{x^2}.$$

§ 3.3. Show that the only translationally, invariant measure on the real line R is proportional to the Lebesgue measure, i.e. $\mathrm{d}\mu(x) = c\,\mathrm{d}x$, $c = \text{const.}$

§ 3.4. Let G be the set of all real non-singular 2×2-matrices of the form

$$G \ni g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad (\text{i.e., } G = \mathrm{GL}(2, R)).$$

Show that the left and the right invariant Haar measures on G have the form

$$\mathrm{d}\mu(g) = \frac{\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\delta}{(\alpha\delta-\beta\gamma)^2}.$$

§ 3.5. Let $G = \mathrm{GL}(n, R)$. Show that the Haar measure has the form

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{|\det X|^n},$$

where

$$\mathrm{d}x = \prod_{i,j=1}^{n} \mathrm{d}x_{ij}, \qquad X = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \cdot & \cdot & \cdot \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \in G.$$

§ 3.6. Let G be the group of all real $n\times n$ triangular matrices

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & x_{22} & \cdots & x_{2n} \\ & & \ddots & \\ & 0 & & x_{nn} \end{bmatrix}.$$

Show that the left Haar measure on G has the form

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x_{11}\,\mathrm{d}x_{12} \dots \mathrm{d}x_{n-1,n}\,\mathrm{d}x_{nn}}{|x_{11}^{n}\,x_{22}^{n-1} \dots x_{n-1,n-1}^{2}\,x_{nn}|},$$

whereas the right Haar measure has the form

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x_{11}\,\mathrm{d}x_{12} \dots \mathrm{d}x_{n-1,n}\,\mathrm{d}x_{nn}}{|x_{11}\,x_{22}^{2} \dots x_{nn}^{n}|}.$$

Chapter 3

Lie Groups

§ 1. Differentiable Manifolds

In this section we introduce the concepts of differentiable (smooth) and analytic manifolds. Let M be a Hausdorff space. A *chart* on M is a pair (U, φ), where U is an open subset of M and φ is a homeomorphism of U onto an open subset of R^n, n-dimensional (real) Euclidean space. The number n is called the *dimension* of the chart, and U the *domain* of the chart.

In other words, a chart is a local coordinate system in M with respect to φ.

A Hausdorff space M is said to be *locally Euclidean* if at each point $p \in M$ there exists a chart (U, φ) on a neighborhood U (called a *coordinate neighborhood*) of p of dimension n. We then say that (U, φ) is a *chart at* p. A Hausdorff space which is locally Euclidean at each point is called a *topological manifold* (of dimension equal to the dimension of the chart).

EXAMPLES. R^n, the sphere S^n, the projective spaces (real or complex), orthogonal group ($n \times n$ orthogonal matrices as a subspace of R^{n^2}) are topological manifolds. ▼

Let S and S' be open subsets of R^n and ψ a map of S into S'. The map ψ is said to be *differentiable* (or *smooth*) if the coordinates $y^j(\psi(p))$, $j = 1, 2, \ldots, n$, are infinitely differentiable functions of the coordinates $x^i(p)$, $i = 1, 2, \ldots, n$, $p \in S$. We shall write in this case $\psi \in$ class $C^\infty(S)$. The map ψ: $S \to S'$ is said to be *analytic* (or *class* C^ω) if for each $p \in S$ there exists a neighborhood U of p, such that for $q \in U$ every one of the coordinates $y^j(\psi(q))$, $j = 1, 2, \ldots, n$, can be expressed as a convergent power series in $x^i(q) - x^i(p)$, $i = 1, 2, \ldots, n$.

DEFINITION 1. A differentiable structure of dimension n (also called an *atlas of class* C^∞) on a Hausdorff space M is a collection of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on M, such that the following conditions are satisfied:

1° $M = \bigcup_{\alpha \in A} U_\alpha$ (i.e., the domains of the charts cover M).

2° For each pair $\alpha, \beta \in A$ the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable map of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.* ▼

* The symbol $\varphi \circ \psi(z)$ means the composition of mappings ψ followed by φ, i.e., $\varphi[\psi(z)]$.

A chart $(U_\alpha, \varphi_\alpha)$, $\alpha \in A$, defines a local coordinate system on the manifold M. The local coordinates of a point $p \in U_\alpha$ are the components of the function $\varphi_\alpha(p) = (x^1(p), \ldots, x^n(p))$.

Condition 2° means, in fact, that the transformation $\varphi_\beta \circ \varphi_\alpha^{-1}$ relating different coordinates introduced on the set $U_\alpha \cap U_\beta$ by the chart $(U_\alpha, \varphi_\alpha)$ and by the chart (U_β, φ_β) is differentiable; it expresses the compatibility of overlapping coordinate systems.

We define an analytic structure on a Hausdorff space M in a similar manner. We just replace the condition of differentiability of $\varphi_\beta \circ \varphi_\alpha^{-1}$ by the condition of analyticity of this map.

A differentiable (analytic) manifold of dimension n is a Hausdorff space M with a differentiable (analytic) structure of dimension n.

The simplest example of an analytic manifold is provided by the Euclidean space R^n. A chart $(U_\alpha, \varphi_\alpha$ lefined by an open set $U_\alpha = R^n$ and the homeomorphism φ_α which assigns to a point $p \in U_\alpha$ its Cartesian coordinates $\varphi_\alpha(p) = (x^1(p), x^2(p), \ldots, x^n(p))$. We shall denote this analytic manifold by R^n.

EXAMPLE 1. Consider the two-dimensional unit sphere S^2 embedded in R^3 with the center at $(0, 0, 0)$. We shall introduce a collection of charts on S^2 by means of stereographic projections. Consider first the stereographic projection from the south pole s with coordinates $s = (0, 0, -1)$ onto the plane through the equator. This is a homeomorphism of the punctured sphere (i.e., without the south pole) into the Euclidean plane R^2. If the Cartesian coordinates of a point $p \in S^2$ are (x, y, z), then the Cartesian coordinates of the projection into the plane are $\left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. Taking an arbitrary point $s' \in S^2$ as a 'new south pole' and utilizing the stereographic projection, we obtain a new local coordinate system. It can be readily verified that the coordinates of a point $q \in S^2 \setminus \{s\} \cap S^2 \setminus \{s'\}$ in the first and the second local coordinate system are related by an analytic transformation. Hence, the collection of local charts constructed by the stereographic projection from each point of S^2 satisfies the conditions 1° and 2° of def. 1, and defines the analytic structure on S^2. ▼

A complex analytic manifold of dimension n is defined in an analogous manner. We replace R^n by C^n in the definition of a chart and we replace the condition 2° of def. 1 by the condition that the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ should be a holomorphic function of coordinates $z^i(p)$, $i = 1, 2, \ldots, n$, of a point $p \in U_\alpha \cap U_\beta$.

In the following, by an analytic manifold, we mean an analytic real manifold.

A real-valued function f on an analytic manifold M is said to be *analytic at* $p \in M$ if there exists a chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$, such that $f \circ \varphi_\alpha^{-1}$ is an analytic function on the set $\varphi_\alpha(U_\alpha)$. The function f is said to be *analytic* if it is analytic at each point $p \in M$. If we restrict the argument p of a function f to a subset

$N \subset M$, we obtain a real-valued function defined on N; we denote the restriction of f to N by $f|N$.

DEFINITION 2. Given two analytic manifolds M and N, we say that N is an *analytic submanifold* of M if:

1° $N \subset M$ (set theoretically).

2° For any chart $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ with $\varphi_\alpha(p) = \big(x^1(p), x^2(p), \ldots, x^n(p)\big)$ the functions $x^i|N$ are analytic functions in N, and at each point $p \in N$ at which they are defined, we can select a subset $(x^{i_1}|N, x^{i_2}|N, \ldots, x^{i_\nu}|N)$ which forms a chart at p. ▼

Simple examples of submanifolds are provided by the set R^m in R^n, $m < n$, by great circles in S^2 (cf. example 1) and by S^2 in R^3.

Let M and N be two analytic manifolds of dimension m and n respectively. We shall now construct the product of these manifolds. Regarding M and N as Hausdorff spaces, we can form their topological product, which consists of all ordered pairs (p, q), $p \in M$, $q \in N$.

The topology in $M \times N$ is defined as the product topology. The analytic structure on the manifold $M \times N$ is defined in a natural manner with the help of analytic structures on M and N as follows: Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ be collections of charts determining the analytic structure on M and N, respectively. Denote by $\varphi_\alpha \times \psi_\beta$, $\alpha \in A$, $\beta \in B$, the mapping $(p, q) \to \big(\varphi_\alpha(p), \psi_\beta(q)\big)$ of the product of open sets $U_\alpha \times V_\beta$ into R^{m+n}. Then the collection $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$ of charts on the product $M \times N$ satisfies conditions 1° and 2° and defines the analytic structure on $M \times N$.

A. *Tangent Spaces and Vector Fields*

Let M be an analytic manifold of dimension n, p a point of M, and $A(p)$ the class of functions analytic at p.

DEFINITION 3. The mapping $L\colon f(p) \to R$, $f \in A(p)$, is said to be a *tangent vector* at p if the following conditions are satisfied:

1° $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, $\alpha, \beta \in R$, $f, g \in A(p)$,

2° $L(fg) = L(f)g(p) + f(p)L(g)$,

i.e., L is a linear functional and a derivation. ▼

If L and L' are tangent vectors, then $\alpha L + \beta L'$ is also a tangent vector. Hence the set of tangent vectors forms a linear vector space over R, the *tangent space*.

Let (U, φ) be any chart at p. If a function f is analytic at p, then the function $f^* = f \circ \varphi^{-1}$ is analytic in a neighborhood of $\big(x^1(p), \ldots, x^n(p)\big)$. We shall write for simplicity $\partial f / \partial x^i$ for

$$\left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)} \tag{1}$$

THEOREM 1. *The mapping $L: f(p) \to R$, $f \in A(p)$ is a tangent vector at p if and only if it is given by the formula*

$$Lf = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} L(x^i). \tag{2}$$

The collection of all tangent vectors at p forms an n-dimensional vector space.
The tangent vectors

$$L_i(p)(f) = \frac{\partial f^*}{\partial x^i}\bigg|_{x^i = x^i(p)}, \qquad i = 1, 2, \ldots, n, \tag{3}$$

form a basis of the tangent space at p. ▼

PROOF: If the action of L is given by eq. (2) then L is evidently a tangent vector. To prove that the action of any tangent vector is given by eq. (2), we note that if $f = \text{const}$, then $Lf = 0$. Expanding f^* around $x(p)$

$$f^* = a_0 + a_1(x^1 - x^1(p)) + \ldots + a_n(x^n - x^n(p)) + \\ + \sum_{i,j=1}^{n} (x^i - x^i(p))(x^j - x^j(p)) g_{ij} + \ldots,$$

where

$$a_i = \frac{\partial f^*}{\partial x^i}\bigg|_{x^i = x^i(p)} \equiv \frac{\partial f}{\partial x^i} \qquad \text{and} \qquad g_{ij} \in A(p),$$

we obtain

$$Lf^* = a_1 L(x^1) + \ldots + a_n L(x^n), \tag{4}$$

which is equivalent to (2).

The set of tangent vectors $L_i(p)$ defined by

$$L_i(p)(f) = \frac{\partial f^*}{\partial x^i}\bigg|_{x^i = x^i(p)}, \qquad i = 1, 2, \ldots, n,$$

forms a basis of the tangent space at p. Indeed,

$$\sum_{i=1}^{n} \lambda_i L_i(p)(x^j) = \lambda_j,$$

hence $L_i(p)$ are linearly independent. Moreover, if $L(p)$ is any tangent vector, then by (2), $L(p)(x^j) = \sum_{i=1}^{n} L(x^i) L_i(p)(x^j)$, $j = 1, 2, \ldots, n$, i.e.,

$$L = \sum_{i=1}^{n} L(x^i) L_i(p). \tag{5}$$

Hence, the tangent space at p is an n-dimensional vector space. ▼

DEFINITION 4. A *vector field* X on an analytic manifold M is a map which assigns to every point $p \in M$ a tangent vector $X(p)$ at p. ▼

A vector field X is sometimes called an *infinitesimal transformation*. The vector field X on M is said to be *analytic at* p if Xf is analytic at p for an arbitrary function f analytic at p and it is said to be *analytic on* M if it is analytic at each $p \in M$. The simplest examples of analytic vector fields are provided by operators L_i given by eq. (3). Indeed, let (U, φ) be a chart at $p \in M$ and let f be an analytic function at p. Then, expressing $f^* = f \circ \varphi^{-1}$ as a function $f^*(x^1, x^2, \ldots, x^n)$ of coordinates

$$\varphi(q) = \{x^1(q), x^2(q), \ldots, x^n(q)\}, \qquad q \in U$$

and setting

$$L_i(p)f = \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)}, \tag{6}$$

we obtain that the map $p \to L_i(p)$ is analytic.

If A is a vector field defined on U, then, by virtue of eq. (3)

$$A(p) = \sum_{i=1}^{n} a^i(p) L_i(p), \tag{7}$$

where a^i, $i = 1, 2, \ldots, n$, are functions defined on U, given by the formula $a^i = Ax^i$. Conversely, if a^i, $i = 1, 2, \ldots, n$, are functions defined and analytic on U, then $A = \sum a^i L_i$ is an analytic vector field on U. At every $p \in U$ we have

$$(Af)(p) = \sum a^i(p) \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)}. \tag{8}$$

Hence, we may represent a vector field A by the symbol $\sum_{i=1}^{n} a^{*i} \frac{\partial}{\partial x^i}$, where the functions $a^{*i} \equiv a^i \circ \varphi^{-1}$ are said to be the components of A with respect to the coordinates $x^1, x^2, \ldots, x^n$.

It follows from def. 4 that if X and Y are vector fields, then $\alpha X + \beta Y$, $\alpha, \beta \in R$, is also a vector field. Hence, the collection of all vector fields forms a real vector space. This vector space is infinite-dimensional. This follows from the fact that we cannot introduce a finite set of basis functions for components $a^i(p)$ which appear in eq. (7).

By def. 4 a vector field X may be viewed as a mapping $X: C^\omega(M) \to C^\omega(M)$. Hence the product XY of two vector fields is well defined. However their product XY is not in general a vector field. For example, if $M = R^n$, $A = \frac{\partial}{\partial x^1}$, $B = \frac{\partial}{\partial x^2}$, then $ABf = \frac{\partial^2}{\partial x^1 \partial x^2} f$, and the map of

$$f \to \left.\frac{\partial^2 f^*}{\partial x^1 \partial x^2}\right|_{x^i = x^i(p)}$$

is not a tangent vector to R^n. It is possible, however, to introduce some type of product in the vector space of analytic vector fields which transforms this space into a Lie algebra; namely, we may associate with analytic vector fields $A = \sum a^i L_i$ and $B = \sum b^i L_i$ an object C

$$C = AB - BA \equiv [A, B], \tag{9}$$

which is called the *Lie product* or the commutator of A and B. In terms of a local coordinate system $\{x^1, x^2, \ldots, x^n\}$ at p, we obtain

$$Af^* = \sum a^{*i}(x) \frac{\partial f^*}{\partial x^i}, \quad Bf^* = \sum b^{*i}(x) \frac{\partial f^*}{\partial x^i},$$

and

$$Cf^* = \sum_{i,j=1}^{n} \left(b^{*i} \frac{\partial a^{*j}}{\partial x^i} - a^{*i} \frac{\partial b^{*j}}{\partial x^i} \right) \frac{\partial f^*}{\partial x^j}. \tag{10}$$

Hence, if A and B are analytic vector fields defined on a manifold M, then their commutator $[A, B]$, by eq. (10), is also an analytic vector field defined on M.

The commutator operation (9) has the following properties

$$\begin{aligned} &1^\circ \; [\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C], \qquad \alpha, \beta \in R, \\ &2^\circ \; [A, B] = -[B, A], \\ &3^\circ \; [A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0. \end{aligned} \tag{11}$$

We see, therefore, that the collection of all analytic vector fields on an analytic manifold is an infinite-dimensional real Lie algebra.

B. *Transformation of Vector Fields*

Let M and N be two differentiable C^∞-manifolds and Ω a mapping of M into N. A mapping Ω is *differentiable at* $p \in M$ if $f \circ \Omega \in C^\infty(p)$ for every $f \in C^\infty(\Omega(p))$. A mapping Ω is *differentiable* if it is differentiable at each $p \in M$. One defines similarly analytical mappings.

Let φ: $q \to (x^1(q), \ldots, x^m(q))$ be a system of coordinates in a neighborhood U of a point $p \in M$ and φ': $r \to (y^1(r), \ldots, y^n(r))$ a system of coordinates in a neighborhood U' of a point $\Omega(p)$ in N and let $\Omega(U) \subset U'$. A mapping $\varphi' \circ \Omega \circ \varphi^{-1}$ of $\varphi(U)$ into $\varphi'(U')$ is given by a system of n functions

$$y^j = \omega^j(x^1, \ldots, x^m), \quad 1 \leqslant j \leqslant n, \tag{12}$$

which represent the mapping Ω in terms of the coordinates.

We now derive the transformation properties of tangent vectors. Note first that if L is a tangent vector at $p \in M$, then the linear mapping L': $C^\infty(N) \to R$ given by $L'(g) = L(g \circ \Omega)$, is a tangent vector in the point $\Omega(p)$. We call the map $d\Omega_p$: $L \to L'$ the differential of the mapping Ω in a point p. By virtue of th. 1 the basis vectors in tangent spaces at a point p and $\Omega(p)$ are given by formulas

$$e_i\colon f \to \left.\frac{\partial f^*}{\partial x^i}\right|_{\varphi(p)}, \quad 1 \leqslant i \leqslant m,\ f^* = f \circ \varphi^{-1}, \tag{13}$$

$$e_j\colon g \to \left.\frac{\partial g^*}{\partial y^j}\right|_{\varphi'(\Omega(p))}, \quad 1 \leqslant j \leqslant m,\ g^* = g \circ \varphi'^{-1}. \tag{14}$$

Hence, we obtain

$$\mathrm{d}\Omega_p(e_i)g = e_i(g \circ \Omega) = \left.\frac{\partial (g \circ \Omega)^*}{\partial x^i}\right|_{\varphi(p)} \tag{15}$$

Because $(g \circ \Omega)^*(x^1, \ldots, x^m) = g^*(y^1, \ldots, y^m)$, where $y^j = \omega^j(x^1, \ldots, x^m)$, then

$$\mathrm{d}\Omega_p(e_i) = \sum_{j=1}^{n} \left.\frac{\partial \omega^j}{\partial x^i}\right|_{\varphi(g)} e_j'. \tag{16}$$

We see, therefore, that if we represent the map $\mathrm{d}\Omega_p$ as a matrix using the basis e_i, $1 \leqslant i \leqslant m$ and e_j', $1 \leqslant j \leqslant n$, we obtain the well-known Jacobi matrix of the system (12).

Vector fields X and Y on manifolds M and N are said to be Ω-related if

$$\mathrm{d}\Omega_p X(p) = Y(\Omega(p)) \quad \text{for all } p \in M. \tag{17}$$

PROPOSITION 2. *Let X_i and Y_i, $i = 1, 2$, be Ω-related. Then,*

$$\mathrm{d}\Omega[X_1, X_2] = [Y_1, Y_2]. \tag{18}$$

PROOF: The formula (17) can be written in the form

$$(Yf) \circ \Omega = X(f \circ \Omega) \qquad \text{for all } f \in C^\infty(N). \tag{19}$$

Hence,

$$Y_1(Y_2 f) \circ \Omega = X_1(Y_2 f \circ \Omega) = X_1\big(X_2(f \circ \Omega)\big). \tag{20}$$

Changing indices 1 and 2 and subtracting expressions (20) we obtain eq. (18). ▼

§ 2. Lie Groups

Having discussed the general properties of analytic manifolds we are now in a position to define Lie groups.

DEFINITION 1. An abstract group G is said to be a *Lie group* if

1. G is an analytic manifold.
2 The mapping $(x, y) \to xy^{-1}$ of the product manifold $G \times G$ into G is analytic. ▼

The condition (2) is equivalent to the following two conditions:

2'. The mapping $x \to x^{-1}$ of G into G is analytic.

2''. The mapping $(x, y) \to x \cdot y$ of $G \times G$ into G is analytic. Indeed, in condition 2, we can set $x = e$, and see that y^{-1} is analytic in y and, hence, $xy = x(y^{-1})^{-1}$

is analytic in both x and y. Conversely, if $2'$ and $2''$ are satisfied, then $(x, y) \to (x, y^{-1})$ is an analytic mapping of $G \times G$ into itself and, therefore, the mapping $(x, y) \to (x, y^{-1}) \to xy^{-1}$ is analytic so that condition 2 holds. Note that by condition $2''$ the left translation $T_x^L y = xy$, and the right translation $T_x^R y = yx$ are both analytic mappings.

Any Lie group is a topological group with respect to the topology induced by its analytic structure. Indeed, a manifold is a Hausdorff space and the analytic mapping $(x, y) \to xy^{-1}$ is continuous. Hence, by def. 2 (2.1) a Lie group is a topological group.

Furthermore any Lie group is locally compact. This follows from the fact that a manifold is locally Euclidean and an Euclidean space R^n is locally compact.

A simple example of a Lie group is the additive group R^n associated with the manifold R^n. The mapping $(x, y) \to xy^{-1} \equiv x-y$, in this case, is evidently analytic. The next example will play an important role in the following considerations.

EXAMPLE 1. Let $G = \mathrm{GL}(n, R)$ (cf. example 2.2.3). Consider the matrix elements x^{ij}, $i, j = 1, 2, \ldots, n$, of an element $x = \{x^{ij}\} \in \mathrm{GL}(n, R)$ as the set of coordinates of a point in R^{n^2}. Because the map

$$\psi \colon x \to \det x,$$

is a continuous map of R^{n^2} into R, the set $\psi^{-1}(0)$ is closed in R^{n^2}. Therefore, its complement $(\psi^{-1}(0))'$ in $\mathrm{GL}(n, R)$ is an open subset of R^{n^2} which is an open analytic submanifold of R^{n^2}. The coordinates z^{ij} of the element $z = xy^{-1}$ can be expressed as rational functions of x^{ls} and y^{tu}, and the denominators of these rational functions are different from zero on $\mathrm{GL}(n, R)$. Hence, the map $(x, y) \to xy^{-1}$ is analytic and, consequently, $\mathrm{GL}(n, R)$ is a Lie group. ▼

Let (U_e, φ) be a chart at the identity e of a Lie group G. We denote by $x^i(p)$ $i = 1, 2, \ldots, n$, the coordinates of a point $p \in U_e$ determined by the homeomorphism $\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p)) \in R^n$. It follows from the condition $2''$ that for every neighborhood U_e of e and for every open set $V \times W$ of $G \times G$ such that* $VW \subset U_e$, the functions f^i, $i = 1, 2, \ldots, n$, defined by

$$(xy)^i = f^i(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n) \equiv f^i(x, y), \qquad x \in V,\ y \in W \tag{1}$$

are analytic functions of their arguments. The functions $f^i(x, y)$ are called the *composition functions of* G. They satisfy the obvious relations:

$$f^i(x, e) = x^i, \quad f^i(e, y) = y^i, \tag{2}$$

$$\left.\frac{\partial f^i}{\partial x^j}\right|_{(e,e)} = \left.\frac{\partial f^i}{\partial y^j}\right|_{(e,e)} = \delta^j_i. \tag{3}$$

* The product VW denotes the subset of G consisting of all elements vw, $v \in V$, $w \in W$.

It follows from the continuity of group multiplication that in locally Euclidean topological groups the composition functions are always continuous.

There arises the natural question as to when a locally Euclidean topological group is a Lie group. This problem was raised by Hilbert in 1900 and is known as Hilbert's Fifth Problem. The following theorem gives the solution of this problem:

THEOREM 1. *A locally Euclidean topological group is isomorphic to a Lie group.* ▼ (For the proof cf. Montgomery and Zippin 1956, ch. IV, § 4. 10.)

Th. 1 asserts in particular that the existence of continuous composition functions in a locally Euclidean topological group implies the existence (in some proper coordinate system) of analytic composition functions.

An interesting class of topological groups, which are not Lie groups, is provided by the class of infinite-dimensional topological groups, which often occur in classical and quantum physics. For example, the abelian group of gauge transformations of classical electrodynamics

$$A_\mu \to A_\mu + \partial_\mu \varphi \tag{4}$$

where φ is a scalar gauge function, is not a Lie group since it is not locally Euclidean (cf. example 2.2.5).

We remark that the def. 1 specifies in fact a real Lie group. In the following, by a Lie group, we shall mean always a real Lie group, unless otherwise stated. The complex Lie group is defined as follows:

DEFINITION 2. An abstract group is said to be a *complex Lie group* if

1° G is a complex analytic manifold.

2° The mapping $(x, y) \to xy^{-1}$ of the product manifold $G \times G$ into G is holomorphic.

EXAMPLE 2. Let $G = \mathrm{GL}(n, C)$. The matrix elements $x^{ij} \in C$, $i, j = 1, 2, \ldots, n$, of a matrix $x = \{x^{ij}\} \in \mathrm{GL}(n, C)$ can be considered as coordinates of a point in C^{n^2}. Because the set $X = \{x\colon \det x = 0\}$ is closed in C^{n^2} (cf. example 1), the group space of $\mathrm{GL}(n, C)$ is an open subset of C^{n^2} and, therefore, is a complex analytic submanifold of $\mathscr{C}^{n^2}$.* Similarly as in example 1, we verify that the coordinates z^{ij}, $i, j = 1, 2, \ldots, n$, of an element $z = xy^{-1}$ are holomorphic functions of the coordinates x^{ls} and y^{tu}, $l, s, t, u = 1, 2, \ldots, n$. ▼

Any complex Lie group of complex dimension n can be considered as a real Lie group of $2n$ real dimensions. In fact, a complex analytic manifold of complex dimension n can be considered as a real one of $2n$ real dimensions and the holomorphic mapping $(x, y) \to xy^{-1}$ becomes an analytic mapping, when considered on this real $4n$-dimensional manifold.

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* $\mathscr{C}^n$ is the complex analytic manifold determined by C^n and the Cartesian coordinates.

Note that there exist genuine real Lie groups which are defined by complex matrices, e.g., the groups SU($2n$), $n = 1, 2, \ldots$, but they cannot be considered as complex groups because they are odd dimensional.

DEFINITION 3. Let G be a Lie group. A subset $H \subset G$ is said to be an *analytic subgroup* of G if

1° H is a subgroup of G.

2° H is an analytic submanifold of G. ▼

One could expect that an analytic subgroup is itself a Lie group. Indeed, we have:

PROPOSITION 2. *Any analytic subgroup H of a Lie group G is a Lie group.*

PROOF: Let $a, b \in H$. Then, $ab \in H$ and there exists a local coordinate system (U, φ), $\varphi(z) = (z^1, z^2, \ldots, z^n)$, at ab in G, such that $z^i|H$, $i = 1, 2, \ldots, \nu$, $\nu = \dim H$, form a local coordinate system at ab in H. The element $xy \in G$ is near to ab, when x and y are near a and b, respectively. This remains true, when x and y are restricted to H. Thus, the map $(x, y) \rightarrow xy$ restricted to $H \times H$ is analytic. We can show similarly that the map $x \rightarrow x^{-1}$ restricted to H is also analytic. Hence, the analytic subgroup H of a Lie group G is a Lie group. ▼

Because analytic subgroups of a Lie group G are themselves Lie groups, they are usually called Lie subgroups of the Lie group G.

An analytic homomorphism $t \rightarrow x(t)$ of R into a Lie group is said to be a *one-parameter subgroup* of G.

EXAMPLE 3. Consider the subgroup GL(m, R), $m < n$, of GL(n, R). The collection of elements of GL(m, R) is a subset of GL(n, R), which, by the results of example 1, is an analytic submanifold of GL(n, R). Therefore, the conditions 1° and 2° of def. 3 are satisfied and GL(m, R), $m < n$, is a Lie subgroup of GL(n, R). ▼

The following theorem gives a convenient criterion for a locally compact topological group to be a Lie group.

THEOREM 3. *A locally compact topological group G is a Lie group if it can be mapped into* GL(n, R) *by a continuous one-to-one homomorphism.* ▼

(For the proof cf. Montgomery and Zippin 1956, ch. II, § 16.)

For example, the groups O(n), U(n), (which is a subgroup of O($2n$)) and Sp(n) (which is a subgroup of O($4n$)) are Lie groups.

A. *The Structure Constants*

Let G be a Lie group and (V_e, φ) a chart at e, with $\varphi(e) = 0$. Consider the Taylor expansion of the composition functions (1) at the point $x^i = y^i = 0$. Using eqs. (1), (2) and (3) we obtain

$$f^i = x^i + y^i + a_{jk}{}^i x^j y^k + b_{jkl}{}^i x^j x^k y^l + d_{jkl}{}^i x^j y^k y^l + r_4, \tag{5}$$

where

$$a_{jk}{}^{i} = \left.\frac{\partial f^{i}}{\partial x^{j}\partial y^{k}}\right|_{00}, \ldots \tag{6}$$

The numbers

$$c_{jk}{}^{i} = a_{jk}{}^{i} - a_{kj}{}^{i} \tag{7}$$

are called the *structure constants.* Under a change of the coordinate system

$$x^{i} \to x^{i'} = x^{i'}(x^{k}).$$

The structure 'constants' are subjected to the following transformation:

$$c_{j'k'}{}^{i'} = \left.\frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{j}}{\partial x^{j'}}\frac{\partial x^{k}}{\partial x^{k'}}\right|_{0} c_{jk}{}^{i}.$$

Therefore, $c_{jk}{}^{i}$ is a tensor with one contravariant and two covariant indices. It also follows from formula (7) that

1. $c_{lk}{}^{i}$ are real numbers.
2. For commutative groups $c_{lk}{}^{i} = 0$.
3. $c_{lk}{}^{i} = -c_{kl}{}^{i}$. (8)
4. $c_{ls}{}^{p} c_{jk}{}^{s} + c_{js}{}^{p} c_{kl}{}^{s} + c_{ks}{}^{p} c_{lj}{}^{s} = 0$.

The last identity is a consequence of the associativity of the group multiplication. To prove it, we first calculate, using expansion (5), the coordinates of the elements $w = x(yz)$, and then $w' = (xy)z$. Comparing third order terms we obtain the last identity.

EXAMPLE 5. As an illustration we shall find the structure constants for GL(n, R). The composition functions in the present case are (cf. example 1):

$$z^{ij} = f^{ij}(x, y) = x^{ik}y^{kj}.$$

Hence, using definitions (5) and (6) we obtain

$$a_{sm,kr}{}^{ij} = \left.\frac{\partial f^{ij}(x, y)}{\partial x^{sm}\partial y^{kr}}\right|_{\substack{x=e\\y=e}} = \delta^{i}_{s}\,\delta_{mk}\,\delta^{j}_{r},$$

and

$$c_{sm,kr}{}^{ij} = \delta^{i}_{s}\,\delta_{mk}\,\delta^{j}_{r} - \delta^{i}_{k}\,\delta_{rk}\,\delta^{j}_{m}. \tag{9}$$

Notice that the structure constants (9) for GL(n, R) coincide with those for the Lie algebra gl(n, R) (cf. 1.1 (13)).

§ 3. The Lie Algebra of a Lie Group

Starting from the concept of Lie groups we shall now establish contact with the theory of Lie algebras discussed in ch. 1 by introducing the concept of the Lie algebra of a Lie group G.

Let $T(e)$ be the algebra of differentiable functions of class C^1 defined in a neighborhood of e, and let $x(t)$, $a \leqslant t \leqslant b$, be a curve representing the homomorphism of class C^1 of $[a, b]$ into G, such that $x(0) = e$. The vector tangent to the curve $x(t)$ at e is the map A: $T(e) \to R$ defined by:

$$Af = \left.\frac{\mathrm{d}f(x(t))}{\mathrm{d}t}\right|_{t=0} \tag{1}$$

In a local coordinate system $\{x^1, x^2, \ldots, x^n\}$ at e we have

$$Af = \left.\frac{\mathrm{d}f(x(t))}{\mathrm{d}t}\right|_{t=0} = \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x^j}\right|_{x^j=x^j(e)} \left.\frac{\mathrm{d}x^j}{\mathrm{d}t}\right|_{t=0} = \sum_{j=1}^{n} a^j L_j(e) f, \tag{2}$$

where

$$L_j(e) f = \left.\frac{\partial f}{\partial x^j}\right|_{x^j=x^j(e)},$$

and the numbers

$$a^j = \left.\frac{dx^j(t)}{dt}\right|_{t=0}, \quad j = 1, 2, \ldots, n, \tag{3}$$

are the components of the vector A (cf. eq. 1(7)). Clearly, a vector tangent to a curve $x(t)$ at e, according to def. 1.3, is a tangent vector at e. Moreover, every tangent vector at e can be considered as a vector tangent to a curve. Indeed, if

$$A = \sum_{j=1}^{n} a^j L_j(e)$$

is any tangent vector at e, the tangent vector to the curve

$$x^i(t) = x^i(e) + a^i t$$

is precisely $\sum a^j L_j(e) = A$.

According to th. 1.1 we can represent the tangent vector (2) by its components, i.e., we set $A = (a^1, a^2, \ldots, a^n)$. We know by th. 1.1 that the tangent space at e is an n-dimensional vector space. We convert this vector space into a Lie algebra by setting

$$c^i = [A, B]^i = c_{jk}{}^i a^j b^k, \tag{4}$$

where the structure constants $c_{jk}{}^i$ are given by eq. 2(7). Indeed, from eq. (4) and 2(8)3 and (8)4 it follows that

$$[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C], \tag{5}$$

$$[A, B] = -[B, A], \tag{6}$$

and

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{7}$$

The Lie algebra so obtained is said to be the *Lie algebra of the Lie group G.*

If we represent an element A of a Lie algebra, in some basis of the vector space L, as $A = a^i X_i$, then, from eqs. (4) and (5), we obtain

$$C = c_{lk}{}^i a^l b^k X_i = [a^l X_l, b^k X_k] = a^l b^k [X_l, X_k],$$

i.e.,

$$[X_l, X_k] = c_{lk}{}^i X_i. \tag{7'}$$

EXAMPLE 1. Consider the group $\mathrm{GL}(n, R)$. The one-parameter subgroups may be written in the form

$$[g^{(ik)}(t)]^{ls} = \delta^{ls} + \delta^{li}\delta^{sk}x^{ik}(t) \quad \text{(no summation over } i, k, l, s = 1, 2, \dots, n) \tag{8}$$

for 'off diagonal' subgroups, and in the form

$$[g^{(ii)}(t)]^{ls} = \begin{cases} \delta^{ls}, & l \neq i \\ x^{ii}(t), & l = i,\ s = i, \end{cases} \tag{9}$$

for diagonal ones. Here the index (i, k), $i, k = 1, 2, \dots, n$, enumerates successive subgroups and the indices l, s; l, $s = 1, 2, \dots, n$, enumerate matrix elements of a group element $g^{(ik)}(t)$. The tangent vector $A^{(ik)}$ to the curve (8)–(9) has the following components

$$(A^{(ik)})^{ls} = a^{ik}\delta^{li}\delta^{sk}, \quad i, k, l = 1, 2, \dots, n, \tag{10}$$

where

$$a^{ik} = \left.\frac{\mathrm{d}x^{ik}(t)}{\mathrm{d}t}\right|_{t=0}.$$

Eq. (10) implies that the basis vectors of $\mathrm{gl}(n, R)$ are given by $n \times n$-matrices e_{ik}, $i, k = 1, 2, \dots, n$, of the form

$$(e_{ik})^{ls} = \delta^{li}\delta^{sk}. \tag{11}$$

The commutation relations for the basis elements e_{ik} follow from eq. (4)

$$[e_{sm}, e_{kr}] = c_{sm,kr}{}^{ij} e_{ij}. \tag{12}$$

Using expression 2(9) for the structure constants of $\mathrm{GL}(n, R)$ we obtain

$$[e_{sm}, e_{kr}] = \delta_{mk} e_{sr} - \delta_{rs} e_{km} \tag{13}$$

(cf. example 1.1.2).

Note that the Lie product (13) for the tangent vectors e_{sm} and e_{kr} coincides with the commutator $[e_{sm}, e_{kr}] = e_{sm}e_{kr} - e_{kr}e_{sm}$ of the corresponding matrices.

A. *Transformation Groups*

The Lie algebras occur in theoretical physics in most cases as the Lie algebras of transformation groups.

DEFINITION 1. A Lie group G is a said to be a (*right*) *Lie transformation group* of a differentiable manifold M if to each pair (p, x), $p \in M$, $x \in G$ there corresponds an element $q \in M$ denoted by px, such that

1. The map $(p, x) \to px$ of $M \times G$ onto M is differentiable.
2. $pe = p$ for all $p \in M$.
3. $(px_1)x_2 = p(x_1 x_2)$ for all $p \in M$ and $x_1, x_2 \in G$.

Similarly one defines a left Lie transformation group $(x, p) \to xp$ of a manifold M.

The group G is said to be *effective* on M; if $x = e$ is the only element of G which satisfies $px = p$ or $xp = p$ respectively for all $p \in M$.

We find now a general expression for the generators of the one-parameter transformation groups.

Let (U, φ) be a chart at e and let (V, ψ) be a chart at a point $p \in M$. Condition 1 states that the coordinates q^i of $q = px$ are analytic functions

$$q^i = \Phi^i(p^s, x^k), \qquad i = 1, 2, \ldots, m, \tag{14}$$

of coordinates p^s, $s = 1, 2, \ldots, m$, of $p \in M$ and x^k, $k = 1, 2, \ldots, n$ of $x \in G$.

Let ψ be an analytic function on M and let $x^i = e^i + \lambda^i \delta t$ be the coordinates of an element x in an infinitesimally small neighborhood of e. Using the Taylor expansion for the function $T^R_x \psi(p) = \psi(px)$ at p we obtain

$$\psi(q) = \psi(p) + \lambda^i \delta t \frac{\partial \psi(q)}{\partial q^k} \frac{\partial q^k}{\partial x^i}\bigg|_{x=e} + \varepsilon[(\delta t)^2]. \tag{15}$$

We denote by $f^k{}_i$ the derivatives $\dfrac{\partial q^k}{\partial x^i}\bigg|_{x=e}$ of the composition function (14) and evaluate the change of a function $\psi(p)$ due to an infinitesimal right translation

$$\delta\psi = \psi(q) - \psi(p) = \lambda^i f^k{}_i \frac{\partial \psi(q)}{\partial q^k}\bigg|_{q=p} \delta t, \tag{16}$$

where we have neglected powers of δt beyond the first. We see, therefore, that the operators

$$X_i = f^k{}_i \frac{\partial}{\partial q_k}, \qquad i = 1, 2, \ldots, n = \dim G, \tag{17}$$

play the role of the generators of one-parameter right translations.

LEMMA 1. *The functions* $f^k{}_i(q) = \dfrac{\partial q^k}{\partial x^i}\bigg|_e$ *satisfy the following equation*

$$\frac{\partial f^i{}_j(q)}{\partial q^a} f^a_k(q) - \frac{\partial f^i{}_k(q)}{\partial q^a} f^a{}_j(q) = c_{jk}{}^b f^i{}_b(q), \tag{18}$$

where $c_{jk}{}^b$ *are the structure constants for* G. ▼

The proof easily follows from proposition 1.1 and the corresponding definitions and we omit it.

By virtue of eq. 1(9) we have

$$[X, Y] = XY - YX. \tag{19}$$

Hence by eq. (18) we obtain

$$[X_k, X_j] = \left(f_k^a \frac{\partial f^i{}_j(q)}{\partial q^a} - f_j^a \frac{\partial f^i{}_k(q)}{\partial q^a}\right)\frac{\partial}{\partial q^i} = c_{kj}{}^b f^i_b \frac{\partial}{\partial q_i} = c_{kj}{}^b X_b, \tag{20}$$

i.e., the set of generators (17) is closed under the Lie multiplication (19).

Notice that if $M = G$, then eq. (17) provides an expression for generators of the right translations

$$T_x^R \psi(y) = \psi(yx) \quad \text{on } G$$

(cf. exercise 3.1).

One can similarly define generators of left translations on M and G.

EXAMPLE 2. The group $\mathrm{GL}(n, R)$ can be considered as an effective transformation group on R^n. We find the Lie algebra of $\mathrm{GL}(n, R)$ corresponding to this realization. Formula (14) takes, in the present case, the form

$$q^i = x^{ik} p^k. \tag{21}$$

Hence,

$$f^i{}_{st}(p) = \left.\frac{\partial x^{ik} p^k}{\partial x^{st}}\right|_{x=e} = \delta^{is} p^t \tag{22}$$

and the generators (17) of one-parameter subgroups are

$$X_{st} = f^i{}_{st}(p)\frac{\partial}{\partial p^i} = \delta^{is} p^t \frac{\partial}{\partial p^i} = p^t \frac{\partial}{\partial p^s}. \tag{23}$$

They satisfy the commutation relations

$$[X_{sm}, X_{kr}] = \delta_{mk} X_{sr} - \delta_{rs} X_{km} \tag{24}$$

(cf. eq. (13)). ▼

B. *Correspondence between Lie Groups and Lie Algebras*

The following theorem establishes a close correspondence between the structures of Lie groups and Lie algebras:

THEOREM 2. *Get G be a Lie group, L its Lie algebra and H a Lie subgroup of G. Denote by N the set of all tangent vectors to differentiable curves at e in H. Then,*

1. *N is a subalgebra of L; it is the Lie algebra of the Lie subgroup H.*
2. *If H is an invariant subgroup, then N is an ideal of L.*
3. *If H is a central invariant subgroup, then N is a central ideal.*

PROOF: By def. 1 a Lie subgroup is at the same time an analytic submanifold. Hence, a subset N is a subspace of L. Let A and B be tangent vectors to the curves $x(t)$ and $y(t)$ in H. The curve $g(t) = x(t)y(t)x^{-1}(t)y^{-1}(t)$ as well as the curve $g(\sqrt{s})$, $t = \sqrt{s}$ lie in H because H is a subgroup. It is easy to verify that the tangent vector C to the curve $q(\sqrt{s})$ is equal to $[A, B]$ given by eq. (4). Hence, the tangent space N is a linear subspace of L which is closed under the Lie multiplication, i.e., it is a Lie subalgebra of L.

Let now H be a normal subgroup of G. Denote by $x(t)$ an arbitrary curve in G with the tangent vector A, and by $y(t)$ a curve in H with the tangent vector B. Then, the curve $x(t)y(t)x(t)^{-1}$ lies in H and, therefore, the curve $q(t) = x(t)y(t)x^{-1}(t)y^{-1}(t)$ as well as the curve $q(\sqrt{s})$, $t = \sqrt{s}$ lie in H. The vector $C = [A, B]$, $A \in L$, $B \in N$ is tangent to the curve $q(\sqrt{s})$. Hence, C belongs to N; consequently, N is an ideal of L.

In the special case, when H is a central invariant subgroup, the curve $q(t)$ reduces to the point e and therefore the tangent vector C is zero, i.e., N is a central ideal of L. ▼

We see that a Lie group determines a Lie algebra up to isomorphism. The following theorem gives an answer to the inverse question: to what extent does a Lie algebra L determine a Lie group?

THEOREM 3. *Every subalgebra of a Lie algebra L of a Lie group G is a Lie algebra of precisely one connected Lie subgroup of G. Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.* ▼

(For the proof cf. Helgason 1962, ch. II, § 1 and § 2.)

C. *Lie Groups with Isomorphic Lie Algebras*

We can also give a relation among global Lie groups which have isomorphic Lie algebras. Indeed, let Γ be the class of all connected Lie groups having isomorphic Lie algebras. Then, by th. 3, any two members of the class Γ are locally isomorphic. Moreover, by virtue of th. 2.4.2 there exists in the class Γ, up to an isomorphism, one and only one simply connected group $\tilde{G}$, the *universal covering group* of the class Γ. Any group of the class Γ is a quotient group $\tilde{G}/N$, where N is a discrete central invariant subgroup.

Let us note that the members of the class Γ, although locally isomorphic, may be totally different globally. The simplest example is the rotation group SO(2) and the translation group T^1. These groups are locally isomorphic, because their Lie algebras are isomorphic. However, as global groups, they are completely different; namely, SO(2) is a compact and infinitely connected group, while T^1 is noncompact and simply connected by th. 2.4.2. There exists between these groups the relation

$$\mathrm{SO}(2) = T^1/N, \tag{25}$$

where the discrete central invariant subgroup N is the subgroup of integers.

Another example: SU(2) is simply connected, SO(3) is doubly connected, they have isomorphic Lie algebras, and $\mathrm{SO}(3) = \mathrm{SU}(2)/Z_2$, where Z_2 is the discrete center $Z_2 = (e, -e)$ of SU(2).

D. *Adjoint Group*

Consider now a Lie group G and denote by L its Lie algebra. The map

$$\psi_x(y) = xyx^{-1}, \tag{26}$$

for a fixed $x \in G$ defines an automorphism of G.

Automorphisms of the form (26) are called *inner automorphisms of* G. Any other automorphism is called an *outer automorphism* of G. To every automorphism of G there corresponds an automorphism of the Lie algebra L. We can calculate the explicit form of the induced automorphism l_x of L by introducing a coordinate system on G. Because

$$y'_x = \psi_x(y) = (xyx^{-1}y^{-1})y$$

then using the coordinates of the product $xyx^{-1}y^{-1}$ we obtain

$$y'^i_x = c_{lk}{}^i x^l y^k + y^i + \varepsilon = (c_{lk}{}^i x^l + \delta^i_k)y^k + \varepsilon^i, \tag{27}$$

where ε^i is of the third order of smallness with respect to coordinates of x and y. The explicit form of the automorphism l_x of L can be calculated from the definition of the tangent vector to the one-parameter subgroup $y^i(t)$. Differentiating both sides of (27) we get, for $t = 0$,

$$a'^i = c_{lk}{}^i x^l a^k, \tag{28}$$

where

$$a^i = \mathrm{d}y^i/\mathrm{d}t|_{t=0}$$

are the coordinates of a vector $A = a_i X^i \in L$ in a basis X_i of L.

The automorphism l_x of the Lie algebra L has then the form

$$(l_x)_k{}^i = c_{lk}{}^i x^l, \tag{29}$$

where x^l, $l = 1, 2, \ldots, n$, are coordinates of an element $x \in G$. The map $h\colon\ x \to l_x$ is the homomorphism of G into the group G_A of all automorphisms of L. Obviously the kernel of this homomorphism is the center of G. The automorphisms (29) of the algebra L induced by the inner automorphisms (26) of G are called *inner automorphisms of* L. All other automorphisms of L are called *outer automorphisms*. The group G_a of all inner automorphisms (29) of L is called the *adjoint group*. We show now that the Lie algebra of the adjoint group G_a is the *adjoint algebra* L_a. In fact, taking a one-parameter subgroup $l_{x(t)}$ we find by eq. (29) that the coordinates of the generator

$$P = \mathrm{d}l_{x(t)}/\mathrm{d}t|_{t=0}$$

are

$$P_k{}^i = c_{lk}{}^i b^l, \qquad \text{where} \qquad b^l = \mathrm{d}x^l/\mathrm{d}t. \tag{30}$$

Putting $B = \sum b^i X_i$, we find

$$P = \operatorname{ad} B, \quad \text{or} \quad P(A) = \operatorname{ad} B(A) = [B, A], \tag{31}$$

i.e., $P \in L_a$. The dimension of the Lie algebra $\tilde{L}_a$ of tangent vectors of inner automorphisms is equal to the dimension of the adjoint algebra L_a. Hence $\tilde{L}_a$ is identical to L_a. For this reason L_a is also called the *algebra of inner derivations.*

E. *Left- and Right- Invariant Lie Algebras*

Let G be a Lie group. Let Ω_{g_0} be the mapping of G onto G given by the left translation Ω_{g_0}: $g \to g_0 g$. It follows from the def. 2.1 of a Lie group that Ω_{g_0} is an analytic isomorphism of G onto G. Let $\mathrm{d}\Omega_{g_0}$ denote the differential of Ω_{g_0} defined in 1.B. It follows from 1.B that $\mathrm{d}\Omega_{g_0}$ transforms the tangent space L_e at the unity e into a tangent space L_{g_0} at g_0.

DEFINITION 2. A vector field $X = \{X_1, g \in G\}$ on G is said to be *left-invariant* if for any $g, g' \in G$ we have

$$\mathrm{d}\Omega_{g'g^{-1}} X_g = X_{g'}. \tag{32}$$

The collection of all left-invariant vector fields on G forms a Lie algebra. Indeed if X and Y are any two left-invariant vector fields, then evidently $\alpha X + \beta Y$ is also left-invariant; moreover, by virtue of proposition 1.2, we have

$$\mathrm{d}\Omega_{g'g^{-1}}([X, Y_g]) = [\mathrm{d}\Omega_{g'g^{-1}}(X), \mathrm{d}\Omega_{g'g^{-1}}(Y)] = [X, Y]_{g'}, \tag{33}$$

i.e., $[X, Y]$ is also left-invariant. The left-invariant Lie algebra is generated by right translations and is realized by virtue of th. 1.1 by first order differential operators. We denote this Lie algebra by L^R.

PROPOSITION 4. *Every left-invariant vector field is analytic.*

PROOF: Let V_1 be a neighborhood of an arbitrary point $g_0 \in G$ and let $\{t_1, \ldots, t_n\}$ be a coordinate system on G at g_0. There exists a neighborhood V_2 of g_0 such that the condition $g, h \in V_2$ implies $g g_0 h^{-1} \in V_1$. For $g \in G$ by virtue of eqs. (1), 1(17), and 1(19), we have

$$X_g t_i = (\mathrm{d}\Omega_{gg_0^{-1}} X_{g_0}) t_i = X_{g_0}(t_i \circ \Omega_{gg_0^{-1}}). \tag{34}$$

Now the coordinates $t_i'(g, h) \equiv t_i(g g_0^{-1} h)$ are analytic on $V_2 \times V_2$; hence $t_i'(g, h) = f_i(t_1(g), \ldots, t_n(g); t_1(h), \ldots, t_n(h))$, where the functions $f_i(y_1, \ldots, y_n; z_1, \ldots, z_n)$ are analytic in all their $2n$ arguments in the neighborhood of the set of values $y_k = t_k(g_0)$, $z_k = t_k(g_0)$, $k = 1, 2, \ldots, n$. We obtain

$$X_g t_i = (X_{g_0} t_j) \left(\frac{\partial f_i}{\partial z_j}\right)\bigg|_{g, g_0}, \tag{35}$$

where the indices g, g_0 mean that the partial derivatives are taken for $y_k = t_k(g)$, $z_k = t_k(g_0)$. Now the quantities $X_{g_0} t_j$ are constant by virtue of th. 1.1 and $(\partial f_i / \partial z_j)|_{g, g_0}$ considered as a function of g is analytic at g_0. Hence, the functions

$X_g t_i$ are analytic at g_0 and consequently the vector field $X = \{X_g, g \in G\}$ is also analytic. ▼

Proposition 4 implies that the left-invariant Lie algebra L^R of G consists of analytic vector fields on G.

Similarly, we can also introduce right-invariant vector fields and show that they are analytic and form a right-invariant Lie algebra L^L.

PROPOSITION 5. *The left and the right-invariant Lie algebras are isomorphic. This isomorphism is analytic.*

PROOF: Let $\mathscr{I}$ be the map $g \to g^{-1}$ of G onto itself. Clearly $\mathscr{I}$ is an analytic isomorphism by def. 2.1. Let X be any left-invariant vector field and let $Y_g \equiv \mathrm{d}\mathscr{I} X_{g^{-1}}$. Then Y is right-invariant. Indeed denoting right infinitesimal translations on G by $d\Sigma$, we have

$$\mathrm{d}\Sigma_g Y_e = \mathrm{d}\Sigma_g(\mathrm{d}\mathscr{I} X_e) = \mathrm{d}(\Sigma_g \circ \mathscr{I}) X_e. \tag{36}$$

Because $(\Sigma_g \circ \mathscr{I})$ maps g_0 into $g_0^{-1} g = (g^{-1} g_0)^{-1}$, we have $\Sigma_g \circ \mathscr{I} = \mathscr{I} \circ \Omega_{g^{-1}}$. Hence

$$\mathrm{d}\Sigma_g Y_e = \mathrm{d}\mathscr{I}(\mathrm{d}\Omega_{g^{-1}} X_e) = \mathrm{d}\mathscr{I}(X_{g^{-1}}) = Y_g. \tag{37}$$

Finally, for arbitrary g_0 we have

$$\mathrm{d}\Sigma_{g_0 g^{-1}} Y_g = \mathrm{d}\Sigma_{g_0 g^{-1}}(\mathrm{d}\Sigma_g Y_e) = \mathrm{d}(\Sigma_{g_0 g^{-1}} \circ \Sigma_g) Y_e = \mathrm{d}\Sigma_{g_0} Y_e = Y_{g_0}, \tag{38}$$

i.e., Y is right-invariant. Because the map $\mathscr{I}$ is analytic the isomorphism $\mathrm{d}\mathscr{I}$ is also analytic. ▼

Th. 1.1 implies that a right-invariant Lie algebra can be represented by means of first-order differential operators; i.e., for $\tilde{X} \in L^L$, we have

$$\tilde{X}_g = a^k(g(t)) \frac{\partial}{\partial t^k}, \tag{39}$$

where $t^k(g)$ are coordinates of an element $g \in G$.

Similarly an element $\tilde{\tilde{Y}}$ in L^R associated with the right translations (that is, in the left-invariant Lie algebra) has the form

$$\tilde{\tilde{Y}}_g = b^k(g(t)) \frac{\partial}{\partial t^k}. \tag{40}$$

By virtue of proposition 1, all functions $a^k(g)$ and $b^k(g)$, $k = 1, 2, \ldots, \dim G$, are analytic on G. Moreover, by virtue of proposition 2, the function $b^k(g)$ can be expressed in terms of $a^k(g)$, or, vice versa, by means of an analytic transformation determined by $\mathrm{d}\mathscr{I}$.

F. *Identities in Lie Algebras*

Let L be a real Lie algebra, and G the corresponding connected and simply connected real Lie group. In this section we derive two identities in the Lie algebra L involving elements, their transforms under inner automorphisms defined by elements of G, and derivatives of local coordinates of the second kind relative to some parameter.

As it is well known, the exponential e^x, for x in some open neighbourhood V of the origin in L, will realize some open neighbourhood of the identity in G. If $x_1, \ldots, x_r$ is a basis of L, V can be chosen small enough so that for any $x \in V$ we shall have $e^x = e^{t_1 x_1} \ldots e^{t_r x_r}$, the coordinates of the second kind, $e^x \to (t_1, \ldots, t_r)$ being a local chart in G over some neighbourhood W of the identity, contained in e^V. Furthermore, we can suppose W to be convex, namely that if e^y and $e^x e^y$ belong to W, then $e^{tx}e^y \in W$, when $0 \leqslant t \leqslant 1$; this follows from the fact that (cf. Helgason 1962, p. 34 and 92–94) the translates $t \to e^{tx}e^y$ of the one-parameter groups are the geodesics of the Cartan–Schouten connection.

Thus if $e^x \in W$ and $0 \leqslant t \leqslant 1$, we have $e^{tx} = e^{t_1 x_1} \ldots e^{t_r x_r}$, where the coordinates t_i are analytic in t.

To simplify the notations, we suppose (using e.g. Ado theorem) that the Lie algebra L is realized faithfully as a matrix algebra. Thus, some neighbourhood of the identity in G, containing W (and all the products of elements of W needed below), will be realized as a matrix group neighborhood.

Therefore, from the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tx} = xe^{tx} = e^{tx}x = \left(\frac{\mathrm{d}t_1}{\mathrm{d}t}x_1 + \ldots + \frac{\mathrm{d}t_r}{\mathrm{d}t}e^{t_1x_1} \ldots e^{t_{r-1}x_{r-1}}x_r e^{-t_{r-1}x_{r-1}} \ldots e^{-t_1x_1}\right)e^{tx}$$
$$= e^{tx}\left(e^{-t_rx_r} \ldots e^{-t_2x_2}x_1\frac{\mathrm{d}t_1}{\mathrm{d}t}e^{t_2x_2} \ldots e^{t_rx_r} + \ldots + x_r\frac{\mathrm{d}t_r}{\mathrm{d}t}\right),$$

we find

$$\begin{aligned} x &= \frac{\mathrm{d}t_1}{\mathrm{d}t}x_1 + \ldots + \frac{\mathrm{d}t_r}{\mathrm{d}t}\operatorname{Int}(t_1x_1) \ldots \operatorname{Int}(t_{r-1}x_{r-1})x_r, \\ x &= \operatorname{Int}(-t_rx_r) \ldots \operatorname{Int}(-t_2x_2)x_1\frac{\mathrm{d}t_1}{\mathrm{d}t} + \ldots + x_r\frac{\mathrm{d}t_r}{\mathrm{d}t}, \end{aligned} \tag{41}$$

where $\operatorname{Int}(tx)$ denotes the inner automorphism $\operatorname{Ad}(e^{tx})$ of L defined by $y \to e^{tx}ye^{-tx}$ in any realization.

Moreover, if $x, y \in L$ and e^y, $e^xe^y \in W$, then, as we have seen, for $0 \leqslant t \leqslant 1$, $e^{tx}e^y \in W$. We can then write:

$$e^{tx}e^y = e^{\alpha_1x_1} \ldots e^{\alpha_rx_r}, \qquad e^y = e^{\beta_1x_1} \ldots e^{\beta_rx_r}$$

and, from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((e^{tx}e^y)e^{-y}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha_1x_1} \ldots e^{\alpha_rx_r}e^{-\beta_rx_r} \ldots e^{-\beta_1x_1})$$
$$= \left(x_1\frac{\mathrm{d}\alpha_1}{\mathrm{d}t} + \ldots + e^{\alpha_1x_1} \ldots e^{\alpha_{r-1}x_{r-1}}x_r\frac{\mathrm{d}\alpha_r}{\mathrm{d}t}e^{-\alpha_{r-1}x_{r-1}} \ldots e^{-\alpha_1x_1}\right)e^{tx}$$

we derive the relation

$$x = \frac{d\alpha_1}{dt} x_1 + \dots + \frac{d\alpha_r}{dt} \mathrm{Int}(\alpha_1 x_1) \dots \mathrm{Int}(\alpha_{r-1} x_{r-1}) x_r. \tag{42}$$

In addition, as is well known, we have for all $x, y \in L$ and $t \in R$,

$$e^{tx} y e^{-tx} = \mathrm{Int}(tx) y = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}(tx))^n y. \tag{43}$$

§ 4. The Direct and Semidirect Products

By means of direct and semidirect products of groups we can construct new groups from given ones and reduce the investigation of some complicated groups to simpler subgroups.

A. *The Direct Product*

DEFINITION 1. Let G_1 and G_2 be abstract groups, then the *direct product* $G_1 \otimes G_2$ is the group of all ordered pairs (g_1, g_2), $g_1 \in G_1$, $g_2 \in G_2$, with the multiplication law

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2'). \tag{1}$$

The unit element of $G_1 \times G_2$ is the element $e = (e_1, e_2)$ and the inverse element of (g_1, g_2) is $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-2})$. ▼

The subgroup $\tilde{G}_1$ of $G_1 \times G_2$ consisting of all pairs of the form (g_1, e_2) is an invariant subgroup of $G_1 \times G_2$ isomorphic to G_1. In fact, for any (g_1, e_2) and $g_1', g_2') \in G_1 \times G_2$ we have

$$(g_1', g_2')^{-1}(g_1, e_2)(g_1', g_2') = (g_1'^{-1} g_1 g_1', e_2) \in \tilde{G}_1.$$

This isomorphism is given by the mapping φ: $g_1 \to (g_1, e_2)$. Similarly $\tilde{G}_2 = \{(e_1, g_2)\}$ is an invariant subgroup of $G_1 \times G_2$ isomorphic to G_2 and the following properties of $\tilde{G}_1$ and $\tilde{G}_2$ hold

$$\tilde{G}_1 \cdot \tilde{G}_2 = G_1 \times G_2, \tag{2}$$

$$\tilde{G}_1 \cap \tilde{G}_2 = (e_1, e_2). \tag{3}$$

Conversely, if a group G contains two invariant subgroups G_1 and G_2 satisfying

$$G_1 \cdot G_2 = G, \tag{4}$$

$$G_1 \cap G_2 = e, \tag{5}$$

then G decomposes onto a direct product of G_1 and G_2. It is easy to verify, using the invariance of G_1 and G_2 in G and eq. (5), that every $g_1 \in G_1$ is commutative with every $g_2 \in G_2$, clearly, by (4) every element g can be represented uniquely as a product $g = g_1 g_2$, $g_1 \in G_1$, $g_2 \in G_2$.

If G_1 and G_2 are topological groups, then the direct product $G_1 \times G_2$ can also be considered as a topological group. The topology on $G_1 \times G_2$ is the product topology. If G_1 and G_2 are Lie groups, then $G_1 \times G_2$ can be considered to be a Lie group, whose analytic manifold is the product of analytic manifolds of G_1 and G_2, respectively.

B. *The Semidirect Product*

Let G be an abstract group, G_A the group of all automorphisms of G, $G_{\tilde{A}}$ a subgroup of G_A and $\Lambda(g)$ the image of $g \in G$ under the analytic automorphism $\Lambda \in G_{\tilde{A}}$ (cf. 1.1.B).

DEFINITION 2. The semidirect product $G \otimes G_{\tilde{A}}$ of G and $G_{\tilde{A}}$ is the group of all ordered pairs (g, Λ) with the group multiplication defined by

$$(g, \Lambda)(g', \Lambda') = \big(g\Lambda(g'), \Lambda\Lambda'\big). \tag{6}$$

The unit element of $G \otimes G_{\tilde{A}}$ is $e = (e, I)$ and the inverse element of the pair (g, Λ) is the pair

$$(g, \Lambda)^{-1} = \big(\Lambda^{-1}(g^{-1}), \Lambda^{-1}\big). \tag{7}$$

The topology in $G \otimes G_{\tilde{A}}$ is the product topology of the product of spaces G and $G_{\tilde{A}}$. ▼

Semidirect products play a fundamental role in the theory of Lie groups and in representation theory. In fact, as we shall show, an arbitrary Lie group is locally isomorphic to a semidirect product of groups (cf. § 5). Moreover all irreducible unitary representations of an important class of semidirect products of groups can be obtained as induced representations (cf. ch. 17).

THEOREM 1. *The semidirect product* $G \otimes G_{\tilde{A}}$ *with* $G = \{(g, I)\}$ *and* $G_{\tilde{A}} = \{(e, \Lambda)\}$ *has the following properties*

1. G *is a normal subgroup of* $G \otimes G_{\tilde{A}}$.
2. $G \otimes G_{\tilde{A}}/G$ *is isomorphic to* $G_{\tilde{A}}$.
3. $G \otimes G_{\tilde{A}} = G \cdot G_{\tilde{A}}$ *and* $G \cap G_{\tilde{A}} = (e, I)$.

PROOF: *ad* 1. Let $(g', I) \in G$ and $(g, \Lambda) \in G \otimes G_{\tilde{A}}$. We have

$$\begin{aligned}(g, \Lambda)(g', I)(g, \Lambda)^{-1} &= \big(g\Lambda(g'), \Lambda\big)\big(\Lambda^{-1}(g^{-1}), \Lambda^{-1}\big)\\ &= \big(g\Lambda(g')g^{-1}, I\big) \in G,\end{aligned}$$

i.e., G is an invariant subgroup in $G \otimes G_{\tilde{A}}$.

ad 2. Note that the set (G, Λ) for a fixed $\Lambda \in G_{\tilde{A}}$ represents a coset of $G \otimes G_{\tilde{A}}/G$. Hence the map φ: $(G, \Lambda) \to (e, \Lambda)$ is the isomorphic map of $G \otimes G_{\tilde{A}}/G$ onto $G_{\tilde{A}}$.

Property 3 follows from the definition of G and $G_{\tilde{A}}$ and eq. (6). ▼

EXAMPLE 1. Let a be a four-vector and Λ a homogeneous Lorentz transformation of the four-dimensional Minkowski space. A Poincaré transformation $L = (a, \Lambda)$ is defined by

$$\tilde{x}_\mu = (Lx)_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu. \tag{8}$$

The product $LL' = (a, \Lambda)(a', \Lambda')$ gives

$$\tilde{\tilde{x}}_\varrho = (L\tilde{x})_\varrho = \Lambda_\varrho{}^\mu \Lambda'_\mu{}^\nu x_\nu + \Lambda^\mu{}_\varrho a'_\mu + a_\varrho. \tag{9}$$

Thus the product $(a, \Lambda)(a', \Lambda')$ can be represented by the transformation

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda'), \tag{10}$$

i.e., the composition law for Poincaré transformations is the same as eq. (6) for semidirect products. Therefore the Poincaré group is the semidirect product $T^4 \rtimes SO(3, 1)$ of the four-dimensional translation grup T^4 and the homogeneous Lorentz group SO(3, 1). The group SO(3, 1) acts on T^4 as a group of automorphisms. Note that T^4 is an invariant subgroup of $T^4 \rtimes SO(3, 1)$ and $T^4 \rtimes SO(3, 1)/T^4$ is isomorphic to SO(3, 1).

Eq. (10) can be written as a matrix product if we let

$$(a, \Lambda) = \begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix}. \tag{11}$$

EXAMPLE 2. Let R^3 be the Euclidean space, R a rotation, $\boldsymbol{v}$ and $\boldsymbol{a}$ 3-vectors, and b a real number. A Galileo transformation $g = (b, \boldsymbol{a}, \boldsymbol{v}, R)$ is defined by

$$\begin{aligned} \boldsymbol{x}' &= R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, \\ t' &= t + b. \end{aligned} \tag{12}$$

This definition implies the composition law for the Galileo group

$$(b', \boldsymbol{a}', \boldsymbol{v}', R')(b, \boldsymbol{a}, \boldsymbol{v}, R) = (b' + b, \boldsymbol{a}' + R'\boldsymbol{a} + b\boldsymbol{v}', \boldsymbol{v}' + R'\boldsymbol{v}, R'R), \tag{13}$$

which is again of the semi-product type and can be written as a matrix product with $(b, \boldsymbol{a}, \boldsymbol{v}, R)$ represented by

$$\begin{bmatrix} R & \boldsymbol{v} & \boldsymbol{a} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

The inverse of element g is $(b, \boldsymbol{a}, \boldsymbol{v}, R)^{-1} = \big(-b, R^{-1}(\boldsymbol{a} - b\boldsymbol{v}), -R^{-1}\boldsymbol{v}, R^{-1}\big)$. These formulas are obtained from the corresponding Poincaré transformations of the previous example by the limiting procedure $(x_0 \equiv ct)$

$$\begin{aligned} &\Lambda_{0i} \to 0,\ c \to \infty \quad \text{such that } \Lambda_{0i}c \to v_i, \\ &\text{but} \\ &a_0 \to \infty,\ c \to \infty \quad \text{such that } a_0/c \to b. \end{aligned} \tag{15}$$

To the limiting procedure (15) there corresponds in the Lie algebra the contraction introduced in ch. 1, § 8, example 2.

§ 5. Levi–Malcev Decomposition

A. *Solvable, Nilpotent, Simple and Semisimple Lie Groups*

Let G be an abstract group. We associate with each pair of elements $x, y \in G$ an element $q = xyx^{-1}y^{-1}$, which is called the *commutator* of x and y. The set Q of all elements of $g \in G$, which can be represented in the form $g = q_1 q_2 \ldots q_m$, where each q_i is a commutator of two elements $x_i, y_i \in G$, is called the *commutant* of G. The commutant Q is an invariant subgroup in G. Indeed, the product of $q = q_1 q_2 \ldots q_n$ and $q' = q'_1 q'_2 \ldots q'_m$ is an element of Q, and the inverse element $q^{-1} = q_n^{-1}, \ldots, q_1^{-1}$ to q is also an element of Q. Moreover if $g \in G$ and $q = q_1 q_2 \ldots q_n \in Q$, then

$$g^{-1}qg = \prod_{i=1}^{n} gq_i g^{-1} = \prod_{i=1}^{n} gx_i g^{-1} gy_i g^{-1} gx_i^{-1} g^{-1} gy_i^{-1} g^{-1} \in Q,$$

i.e., Q is a normal subgroup of G. However in general Q is not a topological subgroup (cf. def. 2.2.2).

Taking the closure of Q in the topology of G, we obtain a normal topological subgroup.

The quotient group G/Q is an abelian group. In fact if $x, y \in G$ and $X = xQ$, $Y = yQ$ are any two elements of G/Q, then

$$XYX^{-1}Y^{-1} = xQyQx^{-1}Qy^{-1}Q = xyx^{-1}y^{-1}Q = qQ = Q \simeq e \in G/Q,$$

i.e., G/Q is commutative.

Consider now the chain of commutants

$$G = Q_0 \supset Q_1 \supset \ldots \supset Q_{n-1} \supset Q_n \supset \ldots, \tag{1}$$

where each Q_n is the commutant of Q_{n-1}. If for some m we have $Q_m = \{e\}$, then the group G is said to be *solvable*. If H is a subgroup of a solvable group G, then the nth commutant of H, $(Q_H)_n \subset Q_n$. Hence every subgroup of a solvable group is solvable. A solvable group always has a commutative invariant subgroup. Indeed, if $Q_m = \{e\}$, but $Q_{m-1} \neq \{e\}$, then for any pair $x, y \in Q_{m-1}$ we have $xyx^{-1}y^{-1} = e$, i.e., $xy = yx$.

EXAMPLE 1. Let G be the group of motions of the Euclidean plane R^2. Every element $g \in G$ can be represented in the form $g = (a, \Lambda)$, where a is an element of the translation group T^2 and Λ is an element of the rotation group SO(2). The group multiplication is (cf. example 4.1).

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda'), \tag{2}$$

i.e., the group of motions of R^2 is the semidirect product $T^2 \circledS \mathrm{SO}(2)$.

The commutator of $x = (a, \Lambda)$ and $y = (a', \Lambda')$ is the element

$$q = (a, \Lambda)(a', \Lambda')(a, \Lambda)^{-1}(a', \Lambda')^{-1} = (a + \Lambda a' - a' - \Lambda' a, I) \in T^2,$$

by eqs. 4(7) and (2). Hence we have $Q_1 = T_2$, $Q_2 = (0, I) = e$, i.e., the group $T^2 \circledS \mathrm{SO}(2)$ is solvable. ▼

THEOREM 1. *Every solvable connected Lie group can be represented as the product*

$$G = T_1 T_2 \ldots T_m$$

of one-parameter subgroups T_i, *where the sets*

$$G_k = T_{k+1} T_{k+2} \ldots T_m$$

for arbitrary k, $1 \leqslant k < m$ *are normal subgroups in* G.

The proof follows directly from th. 1.2.2 and we leave it as an exercise for the reader.

Let K be the set of all elements generated by commutators

$$q = xyx^{-1}y^{-1}, \quad x \in Q, \; y \in G.$$

We readily verify as in the previous case that K is an invariant subgroup of G. Consider the sequence of invariant subgroups

$$G = K_0 \supset K_1 \supset K_2 \supset \ldots \supset K_{n-1} \supset K_n \supset \ldots, \tag{3}$$

where each K_{n+1} is a subgroup of G generated by commutators $q = xyx^{-1}y^{-1}$, $x \in K_n$, $y \in G$. If for some m, $K_m = \{e\}$, then the group G is said to be *nilpotent*. It follows from the definition that any subgroup of a nilpotent group is nilpotent. Moreover since $K_n \supset Q_n$, $n = 1, 2, \ldots$, then every nilpotent group is solvable.

Every nilpotent group has a nontrivial center. In fact, if $K_m = \{e\}$, but $K_{m-1} \neq \{e\}$, then for any $x \in K_{m-1}$ and $y \in G$: $xyx^{-1}y^{-1} = e$, i.e., $xy = yx$.

A Lie group is said to be nilpotent if it is nilpotent as an abstract group.

A Lie group is said to be *simple* if it has no proper, connected invariant Lie subgroup. We emphasize that a simple Lie group, in contrast to a simple finite group, might contain a *discrete* invariant subgroup in G. For example, the group $\mathrm{SU}(n)$ has the discrete cyclic invariant subgroup of order n, Z_n, generated by the element

$$g = \exp\left[\frac{2\pi \mathrm{i}}{n}\right] e, \quad e \in \mathrm{SU}(n). \tag{4}$$

However, all groups $G_i = \mathrm{SU}(n)/Z_i$, where Z_i is a subgroup of Z_n will be considered according to the definition as simple Lie groups.

A Lie group is said to be *semisimple* if it contains no proper invariant connected abelian Lie subgroup.

B. *Levi–Malcev Decomposition*

The solvable and semisimple groups form two disjoint classes. In fact, every solvable Lie group contains an invariant abelian subgroup, whereas a semisimple one does not. The theory of Lie groups might be reduced in a certain sense to an investigation of the properties of solvable and semisimple groups. In fact, we have

THEOREM 1 (Levi–Malcev theorem). *Every connected Lie group G is locally isomorphic to the semidirect product*

$$N \otimes S, \tag{5}$$

where N is a connected maximal solvable invariant subgroup of G and S is a connected semisimple subgroup of G.

PROOF: Let L be the Lie algebra of G. By virtue of Levi–Malcev th. 1.3.5, L is a semidirect sum $\tilde{N} \oplus \tilde{S}$ of the radical $\tilde{N}$ and a semisimple Lie subalgebra $\tilde{S}$. Let $N \otimes S$ denote a connected Lie group, with the Lie algebra $\tilde{N} \oplus \tilde{S}$, where N and S are connected subgroups corresponding to $\tilde{N}$ and $\tilde{S}$, respectively, by virtue of th. 3.3. Then G and $N \otimes S$ are locally isomorphic according to th. 3.3.

§ 6. Gauss, Cartan, Iwasawa and Bruhat Global Decompositions

In ch. 1, § 6 we discussed the Gauss, Cartan and Iwasawa decompositions for Lie algebras. We give now the corresponding global decompositions for Lie groups.

A. *Gauss Decomposition*

DEFINITION 1. A topological group G admits a Gauss decomposition if G contains subgroups $\mathfrak{Z}$, D and Z satisfying the conditions:

1° The sets $\mathfrak{Z}D$ and DZ are solvable connected subgroups in G, whose commutants are $\mathfrak{Z}$ and Z respectively.

2° The intersections $\mathfrak{Z} \cap DZ$ and $D \cap Z$ consist of the unit element only and the set $\mathfrak{Z}DZ$ is dense in G.

It follows from the first condition, that D is an abelian subgroup and $\mathfrak{Z}$ and Z are solvable and connected. The second condition means that almost every element $g \in G$ has the decomposition in the form

$$g = \zeta \delta z, \quad \zeta \in \mathfrak{Z}, \ \delta \in D, \ z \in Z \tag{1}$$

and if such a decomposition exists, then it is unique. An element $g \in G$ is called *regular* if it admits the decomposition (1) and *singular* otherwise.

THEOREM 1. *Every connected semisimple complex Lie group G admits a Gauss decomposition*

$$G = \overline{\mathfrak{Z}DZ}, \tag{2}$$

where the abelian group D is connected, and the groups $\mathfrak{Z}$ and Z are simply connected and nilpotent: $\mathfrak{Z}D$ and DZ are maximal connected solvable subgroups in G. The set of singular points (complementary to $\mathfrak{Z}DZ$) is closed and has a smaller *dimension than G: the components ζ, δ and z of a regular point $g \in \mathfrak{Z}DZ$ are continuous functions of g.*

Two arbitrary decompositions of this type are connected by an automorphism of G. ▼

(For the proof cf. Želobenko 1963, § 5.)

We illustrate this theorem by the example of the group SL(2, C) which is the covering group of the homogeneous Lorentz group.

Example 1. Let $G = \mathrm{SL}(2, C)$. Let $\mathfrak{Z}$, D and Z be the subgroups of SL(2, C), consisting, respectively, of matrices of the type

$$\zeta = \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}. \tag{3}$$

We denoted for simplicity the matrix and the corresponding complex number by the same letter. The group D is isomorphic to the multiplicative group of complex numbers and each one of the groups $\mathfrak{Z}$ and Z is isomorphic to the additive group of complex numbers. We verify, by comparing the product $\zeta\delta z$, and the matrix

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in \mathrm{SL}(2, C), \quad g_{11}g_{22} - g_{12}g_{21} = 1,$$

that every element $g \in \mathrm{SL}(2, C)$ for which $g_{22} \neq 0$ has the unique decomposition in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z}, \; \delta \in D, \; z \in Z, \tag{4}$$

where the components ζ, δ and z are

$$\zeta = \frac{g_{12}}{g_{22}}, \quad \delta = g_{22}, \quad z = \frac{g_{21}}{g_{22}}. \tag{5}$$

Let $S = DZ$; then the commutants are

$$S^{(1)} = Z, \quad \text{and} \quad S^{(2)} = Z^{(1)} = \{e\}.$$

Hence S is a solvable subgroup. Similarly, we verify that $K = \mathfrak{Z}D$ is solvable. Both subgroups are connected, because every element of S or K can be reached continuously from the unit element. The set of matrices of the form (4) is dense in SL(2, C), because its complement defined by the condition $g_{22} = 0$, has a smaller dimension than SL(2, C). Therefore the decomposition (4) represents the Gauss decomposition for SL(2, C). ▼

Remark: If we take the abelian subgroup D to consist of diagonal matrices of the form

$$\delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \tag{6}$$

then $\mathfrak{Z}DZ$ gives a Gauss decomposition of GL(2, C) according to def. 1.

THEOREM 2. *Let $G = \mathrm{SL}(n, C)$ and let $\mathfrak{Z}$, D and Z be the subgroups of G whose elements are of the form*

$$\xi = \begin{bmatrix} 1 & \zeta_{12} & \zeta_{13} & \cdots & \zeta_{1n} \\ & 1 & \zeta_{23} & \cdots & \zeta_{2n} \\ & & & \ddots & \vdots \\ & 0 & & & \zeta_{n-1,n} \\ & & & & 1 \end{bmatrix}, \qquad \delta = \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & 0 \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix},$$

$$z = \begin{bmatrix} 1 & & & \\ z_{21} & 1 & & 0 \\ & & \ddots & \\ z_{n1} & z_{n2} & \cdots & 1 \end{bmatrix}. \tag{7}$$

Then the Gauss decomposition for $\mathrm{SL}(n, C)$ *can be written either in the form*

$$G = \overline{\mathfrak{Z}DZ} \tag{8}$$

or

$$G = \overline{ZD\mathfrak{Z}}. \tag{9}$$

PROOF: We first show that for almost every element $g \in \mathrm{SL}(n, C)$ there exists an element $z \in Z$ such that $gz \in K$ where K is a subgroup of all upper triangular matrices with determinant one. In fact, it follows from the definition of K that all matrix elements of gz under the main diagonal should be equal to zero. Moreover $z_{pq} = 0$ for $p < q$ and $z_{pp} = 1$. Consequently the condition $gz \in K$ is equivalent to the set of linear equations

$$\sum_{s=q}^{n} g_{ps} z_{sq} = 0 \quad \text{for } p > q,$$

i.e.,

$$\sum_{s=q+1}^{n} g_{ps} z_{sq} = -g_{pq}, \quad p = q+1, \ldots, n. \tag{10}$$

For fixed q the determinant of eqs. (10) coincides with the minor g_{p+1} of the form

$$g_s = \begin{vmatrix} g_{ss} & \cdots & g_{sn} \\ \cdots & \cdots & \cdots \\ g_{ns} & \cdots & g_{nn} \end{vmatrix}, \quad s = p+1 = 2, 3, \ldots, n. \tag{11}$$

Hence if g_{p+1} is not equal to zero, eqs. (10) have a solution relative to z_{sq}. Therefore almost all elements $g \in \mathrm{SL}(n, C)$ can be represented in the form $g = kz$. Now every element $k \in K$ can be uniquely represented in the form

$$k = \zeta\delta, \quad \zeta \in \mathfrak{Z} \text{ and } \delta \in D. \tag{12}$$

Indeed the equality (12) means that

$$k_{pq} = \zeta_{pq}\delta_q \quad \text{(no summation)}. \tag{13}$$

In particular for $p = q$, $\zeta_{pp} = 1$; hence

$$\delta_p = k_{pp}, \quad \zeta_{pq} = \frac{k_{pq}}{k_{pp}}. \tag{14}$$

Consequently every $g \in \mathrm{SL}(n, C)$ for which the minors (11) do not vanish has the decomposition in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z}, \ \delta \in D, \ z \in Z. \tag{15}$$

The set of singular points for which at least one of minors (11) does not vanish has a dimension smaller than that of $\mathrm{SL}(n, C)$; therefore its complementary set $\mathfrak{Z}DZ$ is dense in G. Thus the decomposition (8) follows. One verifies easily that the subgroups $\mathfrak{Z}$, D, Z, $\mathfrak{Z}D$ and DZ have all properties stated in th. 1. Hence (8) gives the desired Gauss decomposition for $\mathrm{SL}(n, C)$.

The decomposition (9) is derived in a similar fashion. ▼

Remark: The explicit form of continuous functions $\zeta(g)$, $\delta(g)$ and $z(g)$ is given in exercise 11.6.1 and 6.2. ▼

The compact semisimple Lie groups do not admit a Gauss decomposition, because they do not possess solvable subgroups. However, for the noncompact semisimple real Lie groups there exist some analogues of the Gauss decomposition. Indeed we have

THEOREM 3. *Every connected semisimple real Lie group G admits a decomposition*

$$G = \overline{\mathfrak{Z}DZ}$$

where D is the direct product

$$D = A \otimes K$$

of a simply-connected abelian group A and a connected semisimple compact group K, whereas groups $\mathfrak{Z}$ and Z are nilpotent and simply-connected.

The set of singular points (complementary to $\mathfrak{Z}DZ$) is closed and has a smaller dimension than G: in the decomposition of a regular point $g = \zeta\delta z$ all components ζ, δ and z are continuous functions of g. ▼

(For the proof cf. Želobenko 1963, § 6.)

EXAMPLE 2. Let $G = U(p, q)$. Then $\mathfrak{Z}$ and Z are intersections of $U(p, q)$ with subgroups $\mathfrak{Z}$ and Z of $\mathrm{GL}(n, C)$, respectively. The abelian subgroup A is the product of p-dimenstional toroid with the p-dimensional Euclidean space, whereas $K = U(q-p)$.

B. *The Cartan Decomposition*

Let L be a real semisimple Lie algebra and let

$$L = K \dot{+} P \tag{16}$$

be its Cartan decomposition (cf. th. 1.6.9). There exists a global version of this decomposition which is described by the following

THEOREM 4. *Let G be a connected semisimple Lie group with a finite center. The Lie algebra L of G has the Cartan decomposition* (16). *Let $\mathscr{K}$ be the connected subgroup of G, whose Lie algebra is K, and let $\mathscr{P}$ be the image of the vector space P under the exponential map. Then*

$$G = \overline{\mathscr{P}\mathscr{K}}. \tag{17}$$

(For the proof cf. Cartan 1929.)

EXAMPLE 3. Let $G = \mathrm{SL}(n, R)$. The Cartan decomposition of the Lie algebra L of G is the decomposition of an arbitrary traceless matrix onto skew-symmetric and traceless symmetric parts (cf. example 1.6.2). The connected subgroup $\mathscr{K}$ of G whose Lie algebra is K, consists of orthogonal matrices. On the other hand the set $\mathscr{P}$ is the set of unimodular hermitian matrices. Hence the global decomposition (17) is, in the present case, the well-known polar decomposition of a unimodular matrix onto the product of its hermitian and orthogonal parts.

C. *The Iwasawa Decomposition*

Let L again be a real semisimple Lie algebra and let

$$L = K \dot{+} H_p \dot{+} N_0 \tag{18}$$

be its Iwasawa decomposition (cf. th. 1.6.3). The global version of the decomposition (18) is described in the following theorem.

THEOREM 5. *Let G be a connected group with the Lie algebra L and let $\mathscr{K}$, $\mathscr{A}_p$ and $\mathscr{N}$ be the connected subgroups of G corresponding to the subalgebras K, H_p and N_0 respectively. Then*

$$G = \mathscr{K}\mathscr{A}_p\mathscr{N}, \tag{19}$$

and every element $g \in G$ has a unique decomposition as a product of elements of $\mathscr{K}$, $\mathscr{A}_p$ and $\mathscr{N}$. The groups $\mathscr{A}_p$ and $\mathscr{N}$ are simply connected. ▼

(For the proof cf. Helgason 1962, ch. VI, § 5.)

EXAMPLE 3. Let $L = \mathrm{sl}(n, R)$. The Iwasawa decomposition (18) for $\mathrm{sl}(n, R)$ consists on the decomposition of an arbitrary traceless matrix onto skew-symmetric, diagonal and upper triangular matrices with zeros on the diagonal (cf. example 1.6.3) Thus the group $\mathscr{K}$ is the orthogonal group $\mathrm{SO}(n)$, the group $\mathscr{A}_p$ is the abelian group and the group $\mathscr{N}$ is the nilpotent group consisting of upper triangular matrices with the one's on the main diagonal. Consequently the global Iwasawa decomposition (19) for the group $\mathrm{SL}(n, R)$ consists of the decomposition of an arbitrary unimodular matrix onto the product of orthogonal, diagonal and upper triangular matrices with ones along the diagonal. ▼

D. *The Bruhat Decomposition*

Let $\mathscr{G} = \mathscr{K}\mathscr{A}\mathscr{N}$ be the Iwasawa decomposition of a connected semisimple Lie group $\mathscr{G}$ with a finite center: let $\mathscr{M}$ be the centralizer of the Lie algebra A of $\mathscr{A}$ in $\mathscr{K}$, i.e. $\mathscr{M} = \{k \in \mathscr{K}: \mathrm{Ad}_k X = X$ for each X in $A\}$. Set $\mathscr{P} = \mathscr{M}\mathscr{A}\mathscr{N}$.

Since both $\mathscr{A}$ and $\mathscr{M}$ normalize $\mathscr{N}$, $\mathscr{P}$ is a closed subgroup of $\mathbf{G}$.

The subgroup $\mathscr{P}$ is called *minimal parabolic subgroup* of $\mathscr{G}$.

Let L and A denote the Lie algebras of $\mathscr{G}$ and $\mathscr{A}$, respectively, and let W be the Weyl group of the pair (L, A). Let $\mathscr{M}^*$ be the normalizer of A in $\mathscr{K}$, i.e. $\mathscr{M}^* = \{k \in \mathscr{K}: \mathrm{Ad}_k A \subset A\}$. It is obvious that $\mathscr{M}$ is a normal subgroup of $\mathscr{M}^*$. The Weyl group W can be identified with the quotient $\mathscr{M}^*/\mathscr{M}$.

Let m^*_w be any element of $\mathscr{M}^*$ belonging to the coset associated with w. Denote by $\mathscr{P} w \mathscr{P}$ the double coset $\mathscr{P} m^*_w \mathscr{P}$. The following lemma gives the so-called *Bruhat decomposition* of $\mathscr{G}$.

LEMMA 5. *The mapping*

$$w \to \mathscr{P} w \mathscr{P}, \qquad w \in W$$

is a one-to-one mapping of W onto the set of double cosets $\mathscr{P} x \mathscr{P}$, $x \in \mathscr{G}$, i.e.

$$\mathscr{G} = \bigcup_{w \in W} \mathscr{P} w \mathscr{P} \qquad (\textit{disjoint sum}).$$

(For the proof cf. Bruhat 1956).

EXAMPLE 4. Let $\mathscr{G} = \mathrm{SL}(2, R)$. Then

$$\mathscr{K} = \left\{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \ \varphi \in (0, 2\pi)\right\}, \qquad \mathscr{A} = \left\{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \ a \in R^+\right\},$$

$$\mathscr{N} = \left\{\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \ c \in R\right\}.$$

One readily verifies that the centralizer $\mathscr{M}$ of A in $\mathscr{K}$ consists of two elements $\mathscr{M} = \{e, -e\}$, $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The minimal parabolic subgroups $\mathscr{P}_x$ are

$$P_x = \left\{\pm x \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} x^{-1}, \ x \in \mathrm{SL}(2, R), \ a \in R^+, \ b \in R\right\}.$$

Using the definition of $\mathscr{M}^*$ one readily verifies that the condition $\mathrm{Ad}_k A \subset A$ implies that $\varphi = \frac{n}{2}\pi$. This implies that $\mathscr{M}^*$ is the three-element group

$$\mathscr{M}^* = \left\{e, -e, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\},$$

Consequently,

$$W = \mathscr{M}^*/\mathscr{M} = \left\{e, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}.$$

Therefore, SL(2, R) can be represented as the disjoint sum of two double cosets.

§ 7. Classification of Simple Lie Groups

The Killing–Cartan classification of simple Lie algebras, by virtue of th. 3.3, leads to the classification of the corresponding simple Lie groups. The explicit form of a simple Lie group corresponding to a simple complex or real Lie algebra L can be easily obtained by exponentiating the explicit defining representation of L given in ch. 1.5. For example, the Lie algebra sl(n, C) was realized as the set of all $n \times n$-complex traceless matrices. Hence the group SL(n, C) consists of all elements

$$x = \mathrm{e}^X, \quad X \in \mathrm{sl}(n, C), \tag{1}$$

which by virtue of the identity $\det \mathrm{e}^X = \mathrm{e}^{\mathrm{Tr}\, X}$ is the set of all $n \times n$-unimodular matrices. Similarly, one calculates an explicit realization of all other simple Lie groups corresponding to the explicit realization in ch. 1.5 of associated Lie algebras which we now list:

A. Groups associated with algebras A_{n-1}:

$$\mathrm{SL}(n, C) \begin{array}{l} \nearrow \mathrm{SU}(n) \\ \nearrow \mathrm{SL}(n, R) \\ \rightarrow \mathrm{SU}(p, q), \qquad p+q = n,\ p \geqslant q. \\ \searrow \mathrm{SU}^*(2n) \\ \searrow \mathrm{SL}(n, C)^R \end{array} \tag{2}$$

(i) SU(p, q), $p+q = n$, $p \geqslant q$ is the group of all matrices in SL(n, C) which leaves the quadratic form in C^n

$$z_1 \bar{z}_1 + \ldots + z_p \bar{z}_p - z_{p+1} \bar{z}_{p+1} - \ldots - z_n \bar{z}_n \tag{3}$$

invariant. For $q = 0$, we obtain the unitary group SU(n) of unimodular matrices. The remaining groups SU(p, q), $q \neq 0$, may be called *pseudo-unitary groups.*

(ii) SL(n, R) is the group of all real matrices with determinant one.

(iii) SU*($2n$) (denoted also Q_{2n}) is the group of all matrices in SL($2n$, C) which commute with the transformation σ in C^{2n} given by

$$\sigma \colon (z_1, \ldots, z_{2n}) \to (\bar{z}_{n+1}, \ldots, \bar{z}_{2n}, -\bar{z}_1, \ldots, -\bar{z}_n). \tag{4}$$

(iv) SL(n, C)^R is the group SL(n, C) considered as a real Lie group.

B. Groups associated with algebras B_n:

$$\mathrm{SO}(2n+1, C) \begin{array}{l} \nearrow \mathrm{SO}(2n+1) \\ \rightarrow \mathrm{SO}(p, q), \qquad p+q = 2n+1,\ p \geqslant q. \\ \searrow \mathrm{SO}(2n+1, C)^R \end{array} \tag{5}$$

(i) $SO(2n+1, C)$ is the group of all matrices in $SL(2n+1, C)$ which conserve the quadratic form in C^{2n+1}:

$$z_1^2 + \dots + z_{2n+1}^2. \tag{6}$$

(ii) $SO(p, q), p+q = 2n+1, p \geqslant q$, is the group of all matrices in $SL(2n+1, R)$ which conserve the quadratic form

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{2n+1}^2. \tag{7}$$

For $q = 0$ we obtain the compact orthogonal group $SO(2n+1)$. The remaining groups $SO(p, q)$, $q > 0$, may be called pseudo-orthogonal.

(iii) $SO(2n+1, C)^R$ is the group $SO(2n+1, C)$ considered as a real Lie group.

C. Groups associated with algebras C_n:

$$\mathrm{Sp}(n, C) \left\langle \begin{array}{l} \mathrm{Sp}(n) \\ \mathrm{Sp}(p, q) \\ \mathrm{Sp}(n, R), \quad p+q = n,\ p \geqslant q. \\ \mathrm{Sp}(n, C)^R \end{array} \right. \tag{8}$$

(i) $\mathrm{Sp}(n, C)$ is the group of all matrices in $GL(2n, C)$ which conserve the exterior form in C^{2n}*

$$z_1 z'_{2n} - z_{2n} z'_1 + \dots + z_n z'_{n+1} - z_{n+1} z'_n. \tag{9}$$

(ii) $\mathrm{Sp}(p, q)$ is the group of all matrices in $\mathrm{Sp}(n, C)$ which conserve the hermitian form in C^{2n}

$$z^T \eta_{pq} \bar{z}, \tag{10}$$

where

$$\eta_{pq} = \begin{bmatrix} -I_p & & & 0 \\ & I_q & & \\ & & -I_p & \\ 0 & & & I_q \end{bmatrix}. \tag{11}$$

For $q = 0$ we obtain the compact symplectic group $\mathrm{Sp}(n)$. It is evident from eq. (10) that

$$\mathrm{Sp}(n) = \mathrm{Sp}(n, C) \cap U(2n),$$

and

$$\mathrm{Sp}(p, q) = \mathrm{Sp}(n, C) \cap U(2p, 2q).$$

(iii) $\mathrm{Sp}(n, R)$ is the group of all matrices in $GL(2n, R)$ which conserve the exterior form in R^{2n}:

$$x_1 x'_{2n} - x_{2n} x'_1 + \dots + x_n x'_{n+1} - x_{n+1} x'_n. \tag{12}$$

(iv) $\mathrm{Sp}(n, C)^R$ is the group $\mathrm{Sp}(n, C)$ considered as a real Lie group.

* The exterior form is $(x \wedge y)^{ij} = \frac{1}{2}(x^i y^j - x^j y^i)$.

D. Groups associated with algebras D_n:

$$\mathrm{SO}(2n, C) \begin{cases} \mathrm{SO}(2n) \\ \mathrm{SO}(p, q) \\ \mathrm{SO}^*(2n) \\ \mathrm{SO}(2n, C)^R \end{cases} \qquad p+q = 2n,\ p \geqslant q. \tag{13}$$

(i) The definitions of $\mathrm{SO}(2n, C)$, $\mathrm{SO}(p, q)$, $p+q = 2n$, $p \geqslant q$, $\mathrm{SO}(2n, C)^R$ groups follow from definitions B(i), B(ii) and B(iii) by replacing index $2n+1$ by $2n$ in corresponding formulas.

(ii) The group $\mathrm{SO}^*(2n)$ is the group of all matrices in $\mathrm{SO}(2n, C)$ which conserve in C^{2n} the skew-hermitian form

$$-z_1\bar{z}_{n+1}+z_{n+1}\bar{z}_1-z_2\bar{z}_{n+2}+z_{n+2}\bar{z}_2- \dots -z_n\bar{z}_{2n}+z_{2n}\bar{z}_n. \tag{14}$$

E. *Connectedness of Classical Lie Groups*

We show in ch. 5 that if a Lie group G is n-connected, then there are representations of G which are n-valued. The following theorem gives a description of the connectedness-property of classical Lie groups.

THEOREM 1. (a) *The groups* $\mathrm{GL}(n, C)$, $\mathrm{SL}(n, C)$, $\mathrm{SL}(n, R)$, $\mathrm{SU}(p, q)$, $\mathrm{SU}^*(2n)$, $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n, C)$, $\mathrm{SO}(n)$, $\mathrm{SO}^*(2n)$, $\mathrm{Sp}(n, C)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n, R)$, $\mathrm{Sp}(p, q)$ *are all connected.*

(b) *The groups* $\mathrm{SL}(n, C)$ *and* $\mathrm{SU}(n)$ *are simply-connected.*

(c) *The groups* $\mathrm{GL}(n, R)$ *and* $\mathrm{SO}(p, q)$ $(0 < p < p+q)$ *have two connected components.* ▼

(For the proof cf. Helgason 1962, IX, § 4, and Želobenko 1962.)

The following table gives the description of the center $Z(G)$ of the universal covering group G of the compact simple Lie groups:

Table 1

G	$Z(G)$	$\dim G$
$\mathrm{SU}(n)$	Z_n	n^2-1
$\mathrm{SO}(2n+1)$	Z_2	$n(2n+1)$
$\mathrm{Sp}(n)$	Z_2	$n(2n+1)$
$\mathrm{SO}(2n)$	Z_4 if $n =$ odd $Z_2 \times Z_2$ if $n =$ even	$n(2n-1)$

§ 8. Structure of Compact Lie Groups

We show here the remarkable result that any compact Lie group is the direct product of its center and finite number of compact simple subgroups.

We defined in 1.2.D that a Lie algebra L is compact if there exists in L a positive definite quadratic form $(\cdot, \cdot)$ satisfying the condition

$$([X, Y], Z)+(Y, [X, Z]) = 0. \tag{1}$$

We now show

PROPOSITION 1. *A Lie algebra L of a compact Lie group G is compact.*

PROOF: Let (X, X) be any positive definite quadratic form on L (e.g., $(X, X) = \sum x_i^2$, where x_i are the coordinates of X in a basis).

Set $\varphi_g(X) = (l_g X, l_g X)$, where $l_g X$ denotes the action of the adjoint group in L given by eq. 3.3(29). For fixed $g \in G$, $\varphi_g(X)$ considered as a function of a vector $X \in L$ is a positive definite quadratic form, while for fixed X, $\varphi_g(X)$ is a continuous positive function on G. Because G is compact, the new bilinear form defined by

$$(X, X)' = \int_G \varphi_g(X) \mathrm{d}g$$

is a positive definite quadratic form on L. For an arbitrary $h \in G$ by virtue of invariance of the Haar measure, we have

$$(l_h X, l_h X)' = \int_G (l_{hg} X, l_{hg} X) \mathrm{d}g = \int_G (l_g X, l_g X) \mathrm{d}g = (X, X)', \tag{2}$$

i.e., $(\cdot, \cdot)'$ is invariant relative to the action of the adjoint group. Eq. (1) results by taking in eq. (2) one-parameter subgroups $h(t_i)$, $i = 1, 2, \ldots, \dim G$, of the adjoint group and differentiating. ▼

We now prove the main theorem.

THEOREM 2. *A compact connected Lie group G is a direct product of its connected center G_0 and of its simple compact connected Lie subgroups.*

PROOF: Let L be the Lie algebra of G. By virtue of proposition 1, L is compact. Hence by th. 1.3.2 we conclude that

$$L = N \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_n, \tag{3}$$

where N is the center of L and S_k, $k = 1, 2, \ldots, n$, are simple ideals of L. Consequently, by virtue of th. 3.3 we obtain

$$G = G_0 \times G_1 \times G_2 \times \ldots \times G_n, \tag{4}$$

where G_0 is the connected center of G and G_k, $k = 1, 2, \ldots, n$, are simple connected Lie subgroups of G.

§ 9. Invariant Metric and Invariant Measure on Lie Groups

A. Invariant Metric

We know by the Birkhoff–Kakutani theorem 2.4.3 that every Lie group admits a right invariant metric. We shall now explicitly construct this metric for an arbitrary matrix Lie group.

PROPOSITION 1. *Let G be a matrix Lie group. Let* $\mathrm{d}g$ *be the matrix consisting of differentials of all matrix elements of $g \in G$ and let* $w(g, \mathrm{d}g)$ *be a differential form on G given by the formula* $w(g, \mathrm{d}g) = \mathrm{d}gg^{-1}$. *Then*

$$\mathrm{d}s^2 = \mathrm{Tr}\, ww^T = \sum_{i,j} w_{ij}^2 \tag{1}$$

is a right invariant metric on G.

PROOF: The form $w(g, \mathrm{d}g)$ is right invariant on G. Indeed $w(gh, \mathrm{d}(gh)) = \mathrm{d}ghh^{-1}g^{-1} = w(g, \mathrm{d}g)$. Hence $\mathrm{d}s^2$ is right invariant and positive. ▼

Let now G be a simple Lie group and let $c_{kl}{}^m$ be the structure constants of the Lie algebra L of G. The map $g \to g_0 g g_0^{-1}$ induces the automorphism 3(29) of the Lie algebra L: consequently the structure constants and the Cartan metric tensor

$$g_{ij} = c_{il}{}^m c_{jm}{}^l \tag{2}$$

are two-sided invariant. Hence, the metric

$$\mathrm{d}s^2(t) = g_{ij}\mathrm{d}t^i \mathrm{d}t^j \tag{3}$$

is also two-sided invariant. If G is compact, the metric tensor g_{ij} is positive definite; otherwise, it is indefinite. Hence every simple Lie group is either a Riemannian or pseudo-Riemannian space.

Now if q and h are arbitrary elements of G then we define a distance $d(g, h)$ by the formula

$$d(g, h) \equiv \inf_{\gamma} \int_{\gamma} \mathrm{d}s, \tag{4}$$

where inf is taken with respect to all continuous curves connecting g and h. Clearly the distance (4) has the same invariance properties as the metric $\mathrm{d}s$.

EXAMPLE 1. Let $G = \mathrm{SL}(n, R)$. The invariant Cartan metric tensor for $\mathrm{SL}(n, R)$ has the form 1.2(14)

$$g_{sm, s'm'} = 2n\delta_{sm'}\delta_{ms'}. \tag{5}$$

By virtue of example 2.1 the coordinates t^{ij} of the element

$$x = \{x^{ij}\}_{i,j=1}^n \in \mathrm{GL}(n, R) \tag{6}$$

are matrix elements x^{ij}. Hence by virtue of eqs. (1) and (6) the invariant metric has the form

$$\mathrm{d}s^2 = g_{sm, s'm'}\mathrm{d}x^{sm}\mathrm{d}x^{s'm'} = 2n\mathrm{Tr}(\mathrm{d}x)^2, \tag{7}$$

where $\mathrm{d}x$ is the matrix $[\mathrm{d}x^{ij}]$. This metric is two-sided invariant.

The distance between two arbitrary points is then obtained by inserting (7) into (4).

B. *Invariant Measure*

We have shown in ch. 3, § 3, the existence, on an arbitrary locally compact topological group G of a left- or right-invariant Haar measure $d\mu(x)$. This implies in particular that all Lie groups possess left- or right-invariant Haar measures.

EXAMPLE 2. Let G be the three-dimensional group of triangular matrices

$$g(\alpha, \beta, \gamma) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \equiv (\alpha, \beta, \gamma), \qquad \alpha, \beta, \gamma \in R^1.$$

This group is called the *Weyl group*. The group space of G is isomorphic with R^3. The multiplication law in G is given by the following formula

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha+\alpha', \beta+\beta', \gamma+\gamma'+\alpha\beta').$$

Hence the Euclidean measure on R^3 given by

$$dg(\alpha, \beta, \gamma) = d\alpha d\beta d\gamma$$

is both left- and right-invariant. ▼

Other examples of invariant measures for the specific Lie groups are given in exercises.

§ 10. Comments and Supplements

A. *Exponential Mapping*

We discuss here the properties of one-parameter subgroups $g(t)$ in G obtained by exponentiation of elements X of the Lie algebra L of G. We first elaborate this problem for the matrix Lie algebras.

Let X be an arbitrary $n\times n$-matrix $X = [X_{ij}]$. Let $\mu \equiv \max_{ij} |X_{ij}|$. Then for the matrix elements $(X^k)_{ij}$ of the kth power, X^k $(0 \leqslant k < \infty)$ of X, we have

$$|(X^k)_{ij}| \leqslant (n\mu)^k. \tag{1}$$

Indeed, eq. (1) is true for $k = 0$. Assuming that (1) is true for some integer $k \geqslant 0$, we obtain

$$|(X^{k+1})_{ij}| = |(X^k)_{il}X_{lj}| \leqslant n(n\mu)^k \mu = (n\mu)^{k+1}.$$

Hence by induction, eq. (1) is valid for an arbitrary k. Now set

$$\theta(X) = \exp X \equiv 1 + \frac{X}{1} + \frac{X^2}{2!} + \dots + \frac{X^k}{k!} + \dots \tag{2}$$

By virtue of eq. (1), for fixed i, j, every series

$$\sum \frac{1}{k!}(X^k)_{ij}$$

is majorized by the series $\sum \frac{1}{k!}\mu^k$; hence, it is absolutely convergent. Consequently for a matrix $X = \{X_{ij}\}$ satisfying the condition $|X_{ij}| < \infty$ the exponential $\exp X$ always exists. By virtue of Jacobi equality

$$\det\exp X = \exp\operatorname{Tr} X \tag{3}$$

we conclude that the exponential of any matrix is a regular matrix.

Now every Lie algebra, by virtue of Ado's theorem, has a faithful representation given by a finite-dimensional matrices. Let $X_1, \ldots, X_n$ be a basis in this matrix algebra. Then the map

$$(t_1 X_1 + \ldots + t_n X_n) \to \exp(t_1 X_1 + \ldots + t_n X_n) \tag{4}$$

provides a map of a neighborhood of 0 in L into a neighborhood of the identity e in G.

Clearly, by virtue of eq. (2), we have

$$\exp(t+s)X = \exp tX \exp tY \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp tX = X\exp tX.$$

Hence X is a tangent vector to a curve $\exp tX$ at $t = 0$. Eq. (1) implies that the map $t \to \exp tX$ of R into G is analytic.

In some cases the map $X \to \exp X$, $X \in L$, covers the whole group G (cf. exercises). However, in other cases this is not so. For instance a diagonal matrix in $\mathrm{GL}(n, R)$ with negative matrix elements cannot be represented as an exponential of any real matrix.

In applications it is useful to have an abstract formulation of the concept of exponential maps. This is given by the following theorem.

THEOREM 1. *Let G be a Lie group and L its Lie algebra. Then*

(i) *For every $X \in L$ there exists a unique analytic homomorphism $\theta(t) \equiv \exp tX$ of R into G such that*

$$\exp(t+s)X = \exp tX \exp sX, \tag{6}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\exp tX\right|_{t=0} = X, \tag{7}$$

$$\exp 0X = I. \tag{8}$$

(ii) *For $X, Y \in L$, we have*

$$\exp tX \exp tY = \exp\left\{t(X+Y) + \frac{t^2}{2}[X, Y] + O(t^3)\right\}, \tag{9}$$

$$\exp(-tX)\exp(-tY)\exp tX \exp tY = \exp\{t^2[X, Y] + O(t^3)\}, \tag{10}$$

$$\exp tX \exp tY \exp(-tX) = \exp\{tY + t^2[X, Y] + O(t^3)\}. \tag{11}$$

In each case $O(t^3)$ denotes a vector in L with the following property: there exists an $\varepsilon > 0$ such that $t^{-3}O(t^3)$ is bounded and analytic for $|t| < \varepsilon$.

(iii) *There exists an open neighborhood N_0 of 0 in L and an open neighborhood V_e of identity e in G such that the map* exp *is an analytic diffeomorphism of N_0 onto V_e.* ▼

(For the proof cf. Helgason 1962, ch. II, § 1.)

Let $X_1, \ldots, X_n$ be a basis in L. The mapping

$$\exp(t_1 X_1 + \ldots + t_n X_n) \to (t_1, \ldots, t_n) \tag{12}$$

of V_e onto N_0 is a coordinate system on V_e, the so-called *canonical coordinate system.*

B. *Taylor's Expansion*

Let G be a Lie group and L its Lie algebra. Let $\tilde{X}$ and $\tilde{\tilde{X}}$ be left- and right-invariant vector fields, respectively, given by 3(39) and 3(40), corresponding to an element $X \in L$. Then we have

THEOREM 2. *Let f be an analytic function on G. Then for $0 \leqslant t \leqslant 1$ we have*

$$f(g \exp tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} [\tilde{X}^k f](g), \tag{13}$$

$$f(\exp tXg) = \sum_{k=0}^{\infty} \frac{t^k}{k!} [\tilde{\tilde{X}} f](g). \tag{14}$$

(For the proof cf. Helgason 1962, ch. II, § 1.)

C. *Levi–Malcev Theorem for Groups*

We state an extended version of th. 1 of sec. 5.

THEOREM 3 (the Levi–Malcev theorem). *Let G be a connected Lie group, $L = N \oplus S$, the Levi–Malcev decomposition of its algebra and $\mathcal{N}$ and $\mathcal{S}$ analytic subgroups associated with N and G, respectively. Then*

$$G = \mathcal{N} \rtimes \mathcal{S}, \tag{15}$$

where $\mathcal{N}$ is the invariant subgroup in G and $\mathcal{S}$ is the maximal semisimple, connected subgroup in G.

If G is simply-connected, then the subgroups $\mathcal{N}$ and $\mathcal{S}$ are simply-connected and for any $g \in G$ the decomposition $g = ns$, where $n \in \mathcal{N}$ and $s \in \mathcal{S}$ is unique.

(For the proof cf. Malcev 1942.)

D. *Unimodular Lie Groups*

It is important to know in applications if a given Lie group G is unimodular (cf. 2.3). The following theorem gives the list of known unimodular Lie groups:

THEOREM 4. *The following Lie groups are unimodular:*

1° *Lie groups G for which the set of values of modular functions $\{\Delta(x), x \in G\}$ is compact.*

2° *Semisimple Lie groups.*

3° *Connected nilpotent Lie groups.* ▼

(For the proof cf. Helgason 1962, ch. X, § 1.)

E. *Measures on Semi-Direct Product Lie Groups*

THEOREM 5. *Let $G = T \circledS K$ and let* dt *and* dk *denote left-invariant Haar measures on T and K, respectively. Then the left-invariant Haar measure on G has the form*

$$\mathrm{d}g = \frac{\mathrm{d}t\,\mathrm{d}k}{\delta^T(k)} \tag{16}$$

and the modular function $\Delta^G(g)$ on G has the form

$$\Delta^G(g) = \Delta^T(t)\Delta^K(k)/\delta^T(k), \tag{17}$$

where the function $\delta^T(k)$ is a unique positive function satisfying

$$\int_T f(k^{-1}(t))\mathrm{d}t = \delta^T(k)\int_T f(t)\mathrm{d}t, \quad f \in L(T, \mathrm{d}t). \blacktriangledown \tag{18}$$

(For the proof cf. Nachbin 1965, ch. II, § 7.)

Formula (17) implies that G is unimodular if and only if T is unimodular and $\delta^T(k) = \Delta^K(k)$.

F. *Bibliographical Comments*

The concept of a local Lie group was introduced by Sophus Lie as a tool for an analysis of the properties of partial differential equations (Engel and Lie 1893). The connection between local and global Lie groups was first clarified by E. Cartan 1926, who proved that every Lie algebra over R is a Lie algebra of a Lie group. The first systematic presentation of the theory of Lie groups from a global point of view was given by Chevalley 1946.

§ 11. Exercises

§ 1.1. Let T^1 be the quotient space R/Z where Z is the set of integers. We endow T^1 with the natural topology of the quotient space. Show that if a coset $p \in T^1$

does not contain the number 1/4 or 3/4, the function $\sin 2\pi p$ can be used to define a system of coordinates (i.e., a chart $(U, \sin 2\pi p)$) at p. And if the coset does not contain 0 or 1/2, then the function $\cos 2\pi p$ can be used.

§ 2.1. Show that every element $g \in \mathrm{SU}(2)$ can be written in the form

$$g = u_0\sigma_0 + \mathrm{i}u_k\sigma_k \tag{1}$$

where $u_\mu \in R$ and satisfy the condition

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1, \tag{2}$$

and $\sigma_0 = I$, σ_k were given in ch. 1.1.A.

§ 2.2. Show that SU(1, 1) consists of those elements $g \in \mathrm{SL}(2, C)$ which satisfy the condition

$$g^*\sigma_3 g = \sigma_3. \tag{3}$$

§ 2.3. Show that elements $g \in \mathrm{SU}(1, 1)$ can be written in the form

$$g = v_0\sigma_0 + v_1\sigma_1 + v_2\sigma_2 + \mathrm{i}v_3\sigma_3 \tag{4}$$

where $v_\mu \in R$ and

$$v_0^2 - v_1^2 - v_2^2 + v_3^2 = 1. \tag{5}$$

§ 2.4. Show that the elements of SL(2, R) consist of those elements of SL(2, C) which satisfy the condition

$$g^*\sigma_2 g = \sigma_2 \tag{6}$$

and can be written in the form

$$g = w_0\sigma_0 + w_1\sigma_1 + \mathrm{i}w_2\sigma_2 + w_3\sigma_3 \tag{7}$$

where $w_\mu \in R$.

§ 2.5. Show that the map

$$g' = \varrho^* g \varrho \tag{8}$$

where

$$\varrho = \exp[\mathrm{i}\pi\sigma_1/4] \tag{9}$$

maps SU(1, 1) onto SL(2, R).

§ 2.6. Show that every element $g \in \mathrm{SO}(3)$ may be represented in the form

$$g(\varphi, \vartheta, \psi) = R_z(\varphi)R_y(\vartheta)R_z(\psi) \tag{10}$$

where $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \vartheta \leqslant \pi$ and $0 \leqslant \psi \leqslant 2\pi$ and

$$R_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_y(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix}, \tag{11}$$

are the rotations around the z- and y-axis, respectively. Find the geometric meaning of the Euler angles φ, ϑ and ψ.

§ 2.7. Show that the group SU(2) is the two-fold universal covering group for SO(3).

Hint. Introduce in R^3 the coordinates of the stereographic projection

$$\xi = \frac{x}{\frac{1}{2}-z}, \qquad \eta = \frac{y}{\frac{1}{2}-z}$$

and the complex variable $\zeta = \xi + \mathrm{i}\eta$ and show that the rotation g in R^3 implies the projective mapping

$$\zeta \to \zeta' = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta} \tag{12}$$

in C, where the matrix

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{13}$$

is an element of the group SU(2).

§ 2.8. Show that the elements $g \in \mathrm{SO}(3)$ can be parametrized as 3×3 real orthogonal matrices R with matrix elements

$$R_{ij} = \cos\theta\,\delta_{ij} + (1-\cos\theta)n_i n_j - \sin\theta\,\varepsilon_{ijk}n_k,$$

$$i,j = 1,2,3, \qquad 0 \leqslant \theta \leqslant \pi, \qquad \sum_{i=1}^{3} n_i^2 = 1.$$

Show that the group space is the ball with radius π (the origin and the surface of the ball being identified), or the sphere S^3 in four dimensions with antipodes identified.

§ 2.9. Show that every element of SU(2) can be written in the form

$$u = \exp\left(\mathrm{i}\frac{\mu}{2}\sigma_3\right)\exp\left(\mathrm{i}\frac{\xi}{2}\sigma_2\right)\exp\left(\mathrm{i}\frac{\nu}{2}\sigma_3\right),$$

$$0 \leqslant \mu < 2\pi, \qquad 0 \leqslant \xi \leqslant \pi, \qquad -2\pi < \nu < 2\pi.$$

Show that the Euler angles μ, ξ and ν are coordinates on S^3.

§ 2.10. Show in particular that to a rotation $g(\varphi, \vartheta, \psi)$ there corresponds the unitary matrix of the form

$$u = \pm\begin{bmatrix} \exp(\mathrm{i}\varphi/2) & 0 \\ 0 & \exp(-\mathrm{i}\varphi/2) \end{bmatrix}\begin{bmatrix} \cos\vartheta/2 & \mathrm{i}\sin\vartheta/2 \\ \mathrm{i}\sin\vartheta/2 & \cos\vartheta/2 \end{bmatrix}\begin{bmatrix} \exp(\mathrm{i}\psi/2) & 0 \\ 0 & \exp(-\mathrm{i}\psi/2) \end{bmatrix}$$

$$= \pm\begin{bmatrix} \cos(\vartheta/2)\exp\left(\mathrm{i}\frac{\varphi+\psi}{2}\right) & \mathrm{i}\sin(\vartheta/2)\exp\left(-\mathrm{i}\frac{\psi-\varphi}{2}\right) \\ \mathrm{i}\sin(\vartheta/2)\exp\left(\mathrm{i}\frac{\psi-\varphi}{2}\right) & \cos(\vartheta/2)\exp\left(\quad\frac{\varphi+\psi}{2}\right) \end{bmatrix}. \tag{14}$$

§ 2.11*. The canonical equations of motion of classical mechanics

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k, \qquad \frac{\partial H}{\partial p_k} = \dot{q}_k, \qquad k = 1, 2, \dots, N,$$

where H is the Hamiltonian of the classical system, can be written in the phase-space R^{2n} with coordinates

$$x_i = q_i, \qquad x_{n+i} = p_i, \qquad 1 \leqslant i \leqslant n,$$

in matrix form as $\dfrac{\partial H}{\partial x} = J\dot{x}$, where J is the matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

with $I = (n \times n)$-identity matrix.

(i) Find the maximal symmetry group of canonical equations.

(ii) Show that the evolution of canonical variables is given by the one-parameter group of symplectic transformations.

§ 3.1. Let $x \to T_x$ be a representation of a Lie group G given by right translations on $H = L^2(G)$:

$$T_x^R \psi(y) = \psi(yx), \qquad \psi \in H.$$

Determine the form of the infinitesimal generators as first-order differential operators on H.

§ 3.2. Let $G = \mathrm{SO}(p, q)$. Show that the generators of the Lie algebra $\mathrm{so}(p, q)$ can be represented in $C^3(R^{p+q})$ in the form

$$\begin{aligned} L^i{}_j &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad i, j = 1, 2, \dots, p \text{ or } i, j = p+1, \dots, p+q, \\ B^i{}_j &= x^i \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^i}, \quad i = 1, 2, \dots, p,\ j = p+1, \dots, p+q \end{aligned} \tag{15}$$

and satisfy the following commutation relations

$$\begin{aligned} [L_{ij}, L_{rs}] &= \delta_{is} L_{jr} + \delta_{jr} L_{is} - \delta_{ir} L_{js} - \delta_{js} L_{ir}, \\ [B_{ij}, B_{rs}] &= \delta_{ir} L_{js} + \delta_{is} L_{jr} + \delta_{jr} L_{is} + \delta_{js} L_{ir}, \\ [L_{ij}, B_{rs}] &= \delta_{jr} B_{is} + \delta_{js} B_{ir} - \delta_{ir} B_{js} - \delta_{is} B_{jr}. \end{aligned} \tag{16}$$

Show that the generators L_{ij} form a basis of the maximal compact subalgebra $\mathrm{so}(p) \oplus \mathrm{so}(q)$.

§ 3.3. Let G be a connected Lie group, $\{X_i\}_1^d$—a basis in the left-invariant Lie algebra L^R of G and $\{\tilde{X}_i\}_1^d$—a basis in the right-invariant Lie algebra L^L of G. Let $\alpha = (\alpha_1, \dots, \alpha_p)$, where $\alpha_j = 1, 2, \dots, d$, represent a multi-index, and let

$$X_\alpha = X_{\alpha 1} \dots X_{\alpha p}, \qquad \tilde{X}_\alpha = \tilde{X}_{\alpha 1} \dots \tilde{X}_{\alpha p}.$$

Let $|\alpha|$ denote the order of the multi-index.

Show that

(i)
$$X_\alpha = \sum_{|\beta| \leqslant |\alpha|} a_\alpha{}^\beta \tilde{X}_\beta.$$

(ii) An arbitrary first order differential operator P with $C^\infty(G)$ coefficients can be written in one of the forms

$$\sum p_\alpha X_\alpha \quad \text{or} \quad \sum \tilde{p}_\alpha \tilde{X}_\alpha,$$

where $A = [a_{\alpha\beta}(x)]$ is an analytic matrix on G.

§ 3.4*. Let u be an infinitely differentiable positive definite function on a Lie group G and let K be an element of the right-invariant enveloping algebra of G. Show that $(K^+Ku)(e) \geqslant 0$.

§ 5.1. If $R_1(\alpha)$ is a rotation around x-axis in R^3 by an angle α, $R_2(\beta)$ a rotation around y-axis by an angle β, then the *commutant* $q = R_1(\alpha)R_2(\beta)R_1^{-1}(\alpha)R_2^{-1}(\beta)$ is, for infinitesimal angles, a rotation around z-axis by an angle $\alpha\beta$.

§ 5.2. Show that the Lie group G_n associated with the Canonical Commutation Relations (CCR) in quantum mechanics given by eq. 1.1(41) has the following composition law

$$(\xi, \eta, s)(\xi', \eta', s') = (\xi+\xi', \eta+\eta', \exp(-\mathrm{i}\eta\xi')ss'), \tag{17}$$

where ξ, $\eta \in R^n$ and $s \in S^1$ (one-dimensional sphere). Show that G_n is nilpotent.

Hint. Set

$$G_n \ni g(\xi, \eta, s) = \exp[\mathrm{i}(\xi Q + \eta P + sI)]$$

and use the Baker–Hausdorff formula.

§ 5.3. Show that the infinite-dimensional Lie group G_∞ associated with CCR $[\varphi(x), \pi(y)] = \mathrm{i}\delta(x-y)$ in quantum field theory has the form

$$G_\infty = H \dotplus H \dotplus S^1$$

where H is a real infinite-dimensional Hilbert space.

§ 5.4. Show that the three-dimensional real group defined by the multiplication law

$$(x_1 y_1 z_1)(x_2 y_2 z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1 y_2), \qquad x, y, z \in R^1 \tag{17a}$$

is a nilpolent group. Show that the subgroup $Z = \{(0, 0, z)\}$ is the center of G and the subgroup $N = \{(0, y, z)\}$ is normal in G.

Show that the group G defined by (17a) is the semi-direct product

$$G = N \circledS S, \tag{17b}$$

where $S = \{(x, 0, 0)\}$ (cf. example 9(2)).

§ 6.1. Let $G = \mathrm{SL}(n, C)$ and let $\mathfrak{Z}$, D and Z be the subgroups given by th. 6.2. Show that the matrix elements of Gauss factors 6(7) for the decomposition 6(8) have the form

$$
\begin{aligned}
&\zeta_{pq} = \frac{1}{g_q}\begin{bmatrix} p & q+1 & \dots & n \\ q & q+1 & \dots & n \end{bmatrix}, \quad p < q, \\
&\delta_p = \frac{g_p}{g_{p+1}}, \quad z_{pq} = \frac{1}{g_p}\begin{bmatrix} p & p+1 & \dots & n \\ q & p+1 & \dots & n \end{bmatrix}, \quad p > q,
\end{aligned} \tag{18}
$$

where g_p is the minor given by eq. 6(11) and

$$
\begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ q_1 & q_2 & \cdots & q_m \end{bmatrix}
$$

is the minor of the matrix obtained by deleting from the element $g \in G$ all rows except the rows $p_1, p_2, \dots, p_m$ and all columns except $q_1, q_2, \dots, q_n$.

§ 6.2. Show that the real subgroup of GL(n, C) determined by the condition

$$
s^{-1}gs = g^{*-1}, \quad s = \begin{bmatrix} 0 & 0 & \sigma \\ 0 & e & 0 \\ \sigma & 0 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} 1 & \\ & 1 \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{bmatrix} \tag{19}
$$

where σ—matrix of order p, e—unit matrix of order $n-2p$, is isomorphic to the $U(p, q)$ group. Show that the Gauss factor D of $U(p, q)$ consists of all bloc-diagonal matrices of the form

$$
\delta = \begin{bmatrix} \lambda & 0 & \\ & u & \\ 0 & & \bar{\lambda}^{-1} \end{bmatrix} \tag{20}
$$

where λ—diagonal complex matrix of order p and u—unitary matrix of order $n-2p$. Show that the remaining Gauss factors have the form $\mathfrak{Z}_0 = \mathfrak{Z} \cap U(p, q)$ and $Z_0 = Z \cap U(p, q)$.

§ 6.3. Let $G = \mathrm{GL}(n, C)$ and let $g = \zeta\delta z$ be the Gauss decomposition of an element g. Show that

$$
\det\begin{bmatrix} g_{11} & \cdots & g_{1p} \\ \cdots & \cdots & \cdots \\ g_{p1} & \cdots & g_{pp} \end{bmatrix} = \delta_1 \delta_2 \dots \delta_p \equiv \Delta_p. \tag{21}
$$

§ 6.4. Using Cartan decomposition show that every Lorentz transformation $g \in \mathrm{SO}_0(3, 1)$ can be written as a product of a rotation and a pure Lorentz transformation (boost), i.e.

$$
g = \exp(\mathrm{i}\alpha J)\exp(\mathrm{i}\beta N). \tag{22}
$$

§ 6.5. Find the Iwasawa decomposition for the Lorentz group.

§ 6.6. *Polar decomposition.* Show that an arbitrary element $g \in \mathrm{GL}(n, C)$ admits the following unique decomposition

$$
g = hu \tag{23}
$$

where h is a positive definite hermitian matrix and u is a unitary matrix.

Hint. Take $p = gg^*$ and show that $p = h^2$ where h is positive definite. Show that $u = h^{-1}q$ is unitary.

Note. Every nonsingular operator X in a Hilbert space has the decomposition

$$X = HU, \tag{24}$$

where H is a positive definite self-adjoint operator and U is an isometric operator.

§ 6.7. *Gramm decomposition*. Show that an arbitrary element $g \in \mathrm{GL}(n, C)$ admits the following unique decomposition

$$g = z\varepsilon u \tag{25}$$

where z is the element of lower triangular subgroup of $\mathrm{GL}(n, C)$, ε is an element of the set E of all positive definite diagonal matrices and u is unitary.

§ 6.8. Show that with the same notation as in exercise 6, we have the decomposition

$$\mathrm{GL}(n, C) = \mathrm{U}E\mathrm{U}. \tag{26}$$

Hint. Set $g = hu$ and reduce h to the diagonal form.

Remark: The decomposition (26) for an arbitrary semisimple Lie group G takes the form

$$G = KAK, \tag{27}$$

where K is the maximal compact subgroup of G and A is the factor in the Iwasawa decomposition ($G = NAK$) (cf. Bruhat 1956).

§ 7.1. Show that the group $\mathrm{SL}(2, C)$ is a two-fold universal covering group of $\mathrm{SO}(3, 1)$ given by the formula

$$L_\mu{}^\nu = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \tilde{\sigma}^\nu \Lambda^*) \tag{28}$$

where

$$L \in \mathrm{SO}(3, 1), \quad \Lambda \in \mathrm{SL}(2, C), \quad \sigma_\mu = (I, \sigma) \quad \text{and} \quad \tilde{\sigma}_\mu = (I, -\sigma).$$

§ 7.2. Show the inverse formula

$$\Lambda = \pm N^{-1} L_{\mu\nu} \tilde{\sigma}^\mu \sigma^\nu, \qquad N^2 = L_{\mu\nu} L_{\gamma\delta} \mathrm{Tr}(\sigma^\gamma \tilde{\sigma}^\nu \sigma^\delta \tilde{\sigma}^\gamma). \tag{29}$$

Hence

$$\mathrm{SO}(3, 1) = \mathrm{SL}(2, C)/D,$$

where

$$D = \{I, -I\}.$$

Hint. Use the one-to-one correspondence between the hermitian 2×2-matrices X and the vectors in Minkowski space given by

$$R^4 \ni x \to X = x^0 I + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{bmatrix} x^0+x^3 & x^1-\mathrm{i}x^3 \\ x^1+\mathrm{i}x^3 & x^0-x^3 \end{bmatrix}$$

and the fact that $\mathrm{SL}(2, C)$ transformations $X' = \Lambda X \Lambda^*$ in the matrix space, induce the Lorentz transformations in R^4.

§ 7.3. Show that the element

$$g = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

of the covering group of $\mathrm{SO}_0(3, 1)$ cannot be written in the form (22).

§ 9.1 Let $d(x, y)$ be a left-invariant distance on a connected Lie group G. Set $\tau(x) = d(e, x)$ and $\nabla\tau(x) = \{X_1\tau, \ldots, X_n\tau\}$, where $\{X_i\}_1^n$ is a basis in the left-invariant Lie algebra L of G. Show that

$$|\nabla\tau(x)| \leqslant |\nabla\tau(e)|.$$

§ 9.2. Let $\mu(\cdot)$ be a left-invariant measure on G. Show that there exists a constant λ such that

$$\int_G \exp[-\lambda\tau(x)]\,d\mu(x) < \infty.$$

9.3. Show that the coefficients $a_{\alpha\beta}(x)$ in the formula (i) of exercise 3.3 which connects the elements of the left- and the right-invariant enveloping algebras satisfy the inequality

$$|a_{\alpha\beta}(x)| \leqslant \exp[c + c\tau(x)],$$

where c is a constant.

§ 9.4. Show that the invariant measure for the group SO(3), in terms of the two different parametrizations (exercises to § 2.6 and 2.8) are

$$dg = \frac{1}{8\pi^2}\sin\vartheta\, d\varphi\, d\vartheta\, d\psi, \tag{30}$$

$$dg(\theta, \boldsymbol{n}) = \frac{1}{4\pi^2}\, d\boldsymbol{n}\, \sin^2\frac{\omega}{2}\, d\omega. \tag{31}$$

§ 9.5. Show that the invariant measure for $SU(2)$ is

$$du = \frac{1}{16\pi^2}\sin\xi\, d\xi\, d\mu\, d\nu. \tag{32}$$

Hint. In

$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

set

$$a = \exp[i(\mu+\nu)/2]\cos\xi/2$$

and

$$b = \exp[i/2(\mu-\nu)]\sin\xi/2.$$

§ 9.6. Show that

$$dg(u) = \pi^{-2}\delta\Big(\sum_{j=0}^{3} u_j^2 - 1\Big)\prod_{i=0}^{3} du_i, \tag{33}$$

$$dg(v) = \pi^{-2}\delta(\det v - 1)\prod_{i=0}^{3} dv_i, \tag{34}$$

$$\mathrm{d}g(w) = \pi^{-2}\delta(\det w - 1)\prod_{i=0}^{3}\mathrm{d}w_i \tag{35}$$

are invariant measures on SU(2), SU(1, 1) and SL(2, R) with the parametrizations (1), (4) and (7), respectively.

§ 9.7. Let $G = T^2 \otimes \mathrm{SO}(2)$. Show that in terms of the parameters x_1, x_2, α where $x = (x_1, x_2) \in T^2$ and $\alpha \in \mathrm{SO}(2)$, $0 \leqslant \alpha < 2\pi$ the invariant measure has the form

$$\mathrm{d}\mu(x, \alpha) = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}\alpha. \tag{36}$$

§ 9.8. Show that the invariant measure on the Poincaré group $G = T^4 \otimes \mathrm{SL}(2, C)$ has the form

$$\mathrm{d}g = \mathrm{d}^4 a\,\mathrm{d}\hat{g}, \tag{37}$$

where $\mathrm{d}^4 a$ a is the Lebesgue measure on R^4 and $\mathrm{d}\hat{g}$ is the invariant measure on SL(2, C) given by 2.3(9).

§ 9.9. Let $G = Z$, the lower triangular complex subgroup of GL(n, C), whose elements are matrices of the form

$$z = \begin{bmatrix} 1 & & & & \\ z_{21} & 1 & & 0 & \\ z_{31} & z_{32} & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \\ z_{n1} & \cdots & \cdots & z_{n,n-1} & 1 \end{bmatrix}, \quad z_{kj} = x_{kj} + \mathrm{i}y_{kj},\ x_{kj}, y_{kj} \in R.$$

Show that the invariant Haar measure on Z is the Euclidean measure in $C^{n(n-1)/2}$ given by

$$\mathrm{d}\mu(z) = \prod_{\substack{k,j=1\\ k<j}}^{N} \mathrm{d}x_{kj}\,\mathrm{d}y_{kj}. \tag{38}$$

Hint. Find a composition law in Z as in example 9.1.

§ 10.1. Let $G = \mathrm{GL}(n, C)$. Show that the exponential mapping $\exp X, X \in L$, covers the whole group G.

Hint. Use Jordan form of an arbitrary element $g \in \mathrm{GL}(n, C)$.

§ 10.2. Show that $G = \mathrm{GL}(n, R)$ cannot be covered by an exponential mapping.

Hint. Consider the diagonal matrix with all negative matrix elements.

Chapter 4

Homogeneous and Symmetric Spaces

§ 1. Homogeneous Spaces

Let Γ be a topological space and G a topological group. We say that G is a *topological* (left) *transformation group* on Γ if the following conditions are satisfied:

1° With each $g \in G$ there is associated a homeomorphism $\gamma \to g\gamma$ of Γ onto Γ.

2° The identity element e of G is the identity homeomorphism of Γ.

3° The mapping $(g, \gamma) \to g\gamma$ of $G \times \Gamma$ into Γ is continuous.

4° $(g_1 g_2)\gamma = g_1(g_2\gamma)$ for $g_1, g_2 \in G$ and $\gamma \in \Gamma$.

The topological space Γ on which G acts is called a *G-space*.

We say that G acts *transitively* on Γ if for every pair of points $\gamma_1, \gamma_2 \in \Gamma$ there exists an element $g \in G$ such that $\gamma_2 = g\gamma_1$. If e is the only element of G which leaves each $\gamma \in \Gamma$ fixed, then it is said that G acts *effectively* on Γ and G is called *effective*.

It follows from def. 2.2.4 that a Hausdorff space Γ is homogeneous if G acts on Γ transtitively.

The subgroup of G which leaves a point $\gamma \in \Gamma$ fixed is called the *stability* (stationary, isotropy, little) *group* of γ. If H_γ is the stability group of γ, and $\gamma' = g\gamma$, then the stability group of the point γ' is the group $H_{\gamma'} = gH_\gamma g^{-1}$. Hence, the stability groups of any two points of a homogeneous space Γ are isomorphic.

An important realization of homogeneous spaces is provided by the quotient spaces G/H as follows. Let G be a topological group, H a closed subgroup of G, and G/H the collection of left cosets xH, $x \in G$. We define the topology on the space G/H by means of the canonical projection π: $G \ni x \to xH \in G/H$; namely, we say that a set $X \subset G/H$ is open in G/H if $\pi^{-1}(X)$ is open in G. It is easily verified that such a collection of open sets defines a Hausdorff topology on G/H. If we assign to each $g \in G$ the map g: $xH \to gxH$, then G becomes a transitive topological transformation group acting on G/H and consequently G/H is a homogeneous space. We verify these statements by using the continuity of the group multiplication of G.

The group G acts effectively on G/H if and only if H does not contain a normal subgroup N of G. In fact, if $N \subset H$ is a normal subgroup of G, $n \in N$, and $x \in G$, then $x^{-1}nx = n' \in N$ and $nxH = xn'H = xH$, i.e., to every $n \neq e$ there corresponds the identity transformation; to prove the second part of the statement, note that by condition 2°, a set N of elements $n \in G$, which satisfy the condition $nxH = xH$ for each $x \in G$, generates a subgroup of G. For each x, $g \in G$, $n \in N$ and $h \in H$ we have $(gng^{-1})xH = xH$, and $(hnh^{-1})H = H$. Hence, N is a normal subgroup of G contained in H.

This construction shows that every quotient space G/H of a topological group G over a closed subgroup H is a homogeneous space. In particular, if $H = \{e\}$ we obtain that G itself is a homogeneous space.

Analogously we denote the homogeneous space $\{Hg\}$ of *right* cosets by the symbol $H \backslash G$.

There arises an interesting question as to whether every homogeneous G-space can be represented in this form. This question is answered positively by the following

Theorem 1. *Let G be a locally compact topological group with a countable basis acting transitively on a locally compact Hausdorff space Γ. Let γ be any point of Γ and H the subgroup of G which leaves γ unchanged. Then,*

1° *H is closed.*

2° *The map*

$$gH \to g\gamma$$

is a homeomorphism of G/H onto Γ. ▼

(For the proof cf. Helgason 1962, ch. II, th. 3.2.)

The same is true for the homogeneous spaces of the right cosets $H \backslash G$.

Homogeneous spaces play an important role in the representation theory. We shall use them for the construction of induced representations of various groups (ch. 16 ff).

§ 2. Symmetric Spaces

In this section we consider a special class of homogeneous spaces whose fundamental group G is a Lie group.

Let G be a connected Lie group and let σ be an involutive automorphism of G (i.e., $\sigma^2 = 1$, $\sigma \neq 1$). Let G_σ be a closed subgroup of G consisting of all fixed points of G under σ and G_σ^I the identity component of G_σ. Let H be a closed subgroup such that $G_\sigma \supset H \supset G_\sigma^I$. We shall then say that G/H is a *symmetric homogeneous space* (defined by σ). If we denote by σ the involutive automorphism of the Lie algebra L of G induced by σ, then, by virtue of the considerations given in ch. 1, § 6, eqs. (9)–(13), we obtain

$$L = K \dotplus P, \tag{1}$$

where

$$K = \{X \in L\colon\ \sigma(X) = X\} \tag{2}$$

and coincides with the subalgebra corresponding to H, and

$$P = \{X \in L\colon\ \sigma(X) = -X\}. \tag{3}$$

We have obviously (cf. th. 1.6.2)

$$[K, K] \subset K, \quad [K, P] \subset P \quad \text{and} \quad [P, P] \subset K. \tag{4}$$

We call a two-point function $f(x, y)$ which satisfies the condition

$$f(gx, gy) = f(x, y), \quad x, y \in \Gamma,\ g \in G, \tag{5}$$

the *invariant* of the symmetric space.

The rank of a symmetric space G/H is defined as the dimension of the maximal abelian subalgebra of P in the decomposition (1). This notion is of great importance in representation theory, since it gives the number of algebraically independent invariant differential operators in the space $L^2(G/H)$ (cf. th. 15.1.1).

EXAMPLE 1. Let $G = \mathrm{SO}(n+1)$, and σ be defined by the formula

$$\sigma(g) = SgS^{-1}, \quad g \in G, \tag{6}$$

where

$$S = \begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix} \tag{7}$$

and I_n is the unit matrix in R^n. We find that $G_\sigma = \mathrm{SO}(n)$ and is equal to its identity component, hence $H = \mathrm{SO}(n)$. The symmetric space $\Gamma = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is homeomorphic with the n-dimensional sphere S^n. In fact, the group $G = \mathrm{SO}(n+1)$ acts transitively on the manifold S^n given by the equation

$$(x^1)^2 + (x^2)^2 + \ldots + (x^{n+1})^2 = 1. \tag{8}$$

The transitivity of the sphere S^n with respect to the group $\mathrm{SO}(n+1)$ follows from the fact that any real vector $x = (x^1, x^2, \ldots, x^{n+1})$ satisfying eq. (8) can be attained from the vector $e^1 = (1, 0, 0, \ldots, 0)$ by a rotation matrix $g(x)$, whose first column $g^i{}_1(x) = x^i$. Therefore, any two vectors $x', x'' \in S^n$ can be related to each other by the rotation matrix $g = g(x')g^{-1}(x'')$. The subgroup of $\mathrm{SO}(n+1)$ which leaves the point $x = e^1 \in S^n$ invariant is isomorphic to $H_\sigma = \mathrm{SO}(n)$. Hence, by th. 1.1 the map

$$gH \to ge^1 \tag{9}$$

is a homeomorphism of $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ onto S^n.

The Cartan decomposition of the Lie algebra of $\mathrm{SO}(n+1)$ is given by

$$\mathrm{so}(n+1) = \mathrm{so}(n) \dotplus P.$$

Because P is spanned by the elements $M_{1,n+1}, \ldots, M_{n,n+1}$ it can be seen from the commutation relations that the maximal abelian subalgebra of P is one-dimensional. Hence, the rank of S^n is equal to one. ▼

THEOREM 1. *Every homogeneous space G/H, where G is a Lie group and H is a compact subgroup, admits an invariant metric.*

PROOF: Let o be the point of G/H represented by the coset H and $\tilde{H}$ a group of linear transformations of the tangent space $T_0(G/H)$, induced by the elements of H. Because H is compact, so is $\tilde{H}$ and, by a procedure similar to that in 3.8., eq. (2), there exists a positive definite inner product, say g_0, in $T_0(G/H)$ which is invariant under $\tilde{H}$. For each $\gamma \in G/H$ we take an element $x \in G$ such that $x(o) = \gamma$ and define an inner product g_γ in $T_\gamma(G/H)$ by $g_\gamma(X, Y) = g_0(x^{-1}X, x^{-1}Y)$, $X, Y \in T_\gamma(G/H)$. The set X of points $x \in G$ which transform $o \to \gamma$ generates a left coset of G with respect to H. The metric g_γ is independent of the choice of an element $x \in \tilde{X}$. In fact, if $x, y \in \tilde{X}$ (i.e., $x^{-1}y = h \in H$), then

$$g_0(y^{-1}X, y^{-1}Y) = g_0(h^{-1}x^{-1}X, h^{-1}x^{-1}X) = g_0(x^{-1}X, x^{-1}Y).$$

It is also readily verified that the Riemannian metric so obtained is invariant with respect to G. ▼

Let G be a connected Lie group and G/H a symmetric homogeneous space with a compact H. The space G/H equipped with the G-invariant Riemannian metric given by th. 1 is called a *globally symmetric Riemannian space*. According to eqs. (1)–(4) we can associate, with every globally symmetric Riemannian space G/H, a pair (L, σ) with the following properties:

(i) L is a Lie algebra of G.

(ii) σ is an involutive automorphism of L.

(iii) The set $K = \{X \in L\colon \sigma(X) = X\}$ is a compact subalgebra of L, and K, P satisfy the commutation relation (4).

The pair (L, σ) is called the *orthogonal symmetric Lie algebra*.

The association of an orthogonal symmetric algebra with a globally symmetric Riemannian space Γ allows us to reduce the problem of the classification of these spaces to the problem of the classification of the orthogonal symmetric algebras.

A globally symmetric Riemannian space is said to be *irreducible* if the associated orthogonal symmetric Lie algebra satisfies the following conditions:

(i) L is semisimple and K contains no ideals $\neq \{o\}$ of L.

(ii) K is a maximal proper subalgebra of L.

EXAMPLE 2. Let H_n be the set of all positive definite hermitian matrices of order n with determinant one. Consider motions in H_n given by the formula

$$h \to h_g = ghg^* \in H_n, \tag{10}$$

where $g \in \mathrm{SL}(n, C)$. We verify that $\mathrm{SL}(n, C)$ acts transitively on H_n. The stability subgroup H of the point $I_n \in H_n$ is, by eq. (10), the set of all matrices satisfying the condition $I_n = gg^*$, i.e., the subgroup $\mathrm{SU}(n)$. An involutive automorphism σ of $\mathrm{SL}(n, C)$, which leaves every point of $\mathrm{SU}(n)$ fixed, is given by formula $\sigma(g)$

$= g^{*-1}$. Hence, G/H is a globally symmetric Riemannian space which, by th. 1.1, is homeomorphic with H_n. We also verify that the orthogonal symmetric algebra $(\mathrm{sl}(n, C), \sigma)$ is irreducible. Consequently, the globally symmetric Riemannian space $SL(n, C)/SU(n)$ is also irreducible. ▼

Cartan showed that the problem of classification of globally symmetric Riemannian spaces can be reduced to the problem of the classification of irreducible ones and solved the latter problem (1926a, b; 1927a, b, c). In table I we list the compact and noncompact *irreducible* globally symmetric Riemannian spaces (type I and III in Cartan's classification), whose transformation group is a simple real connected classical Lie group.

Table 1

Irreducible Globally Symmetric Riemannian Spaces Whose Transformation Group Is a Simple Real Connected Lie Group

Compact	Noncompact	Rank	Dimension
$SU(n)/SO(n)$	$SL(n, R)/SO(n)$	$n-1$	$(n-1)(n+2)/2$
$SU(2n)/Sp(n)$	$SU^*(2n)/Sp(n)$	$n-1$	$(n-1)(2n+1)$
$SU(p+q)/S(U(p)\times U(q))$	$SU(p, q)/S(U(p)\times U(q))$	$\min(p, q)$	$2pq$
$SO(p+q)/SO(p)\times SO(q)$	$SO_0(p, q)/SO(p)\times SO(q)$	$\min(p, q)$	pq
$SO(2n)/U(n)$	$SO^*(2n)/U(n)$	$[n/2]$	$n(n-1)$
$Sp(n)/U(n)$	$Sp(n, R)/U(n)$	n	$n(n+1)$
$Sp(p+q)/Sp(p)\times Sp(q)$	$Sp(p, q)/Sp(p)\times Sp(q)$	$\min(p, q)$	$4pq$

All spaces in table I are simply connected. We did not include in table I symmetric spaces associated with the exceptional groups, because we shall not use them in the following.

In addition to the irreducible symmetric spaces listed, there exists still two other classes. The first one consists of irreducible globally symmetric Riemannian spaces which are the simple compact connected Lie groups (type II). The last class (type IV) contains irreducible globally symmetric Riemannian spaces which are cosets spaces G/H, where G is a connected Lie group whose Lie algebra is $(L)^R$, the real form of a simple complex Lie algebra L, and H is the maximal compact subgroup of G.

It should be also noted that isomorphisms of lower-dimensional complex and real simple Lie algebras (cf. ch. 1, § 5, table I) imply a series of coincidences of lower dimensional symmetric spaces; e.g.,

$$SU(2, 2)/S(U(2)\times U(2)) \sim SO_0(4, 2)/SO(4)\times SO(2)$$

(corresponding to the isomorphism $\mathrm{su}(2, 2) \sim \mathrm{so}(4, 2)$). For the full list of coincidences cf. Helgason 1962, ch. IX.

In the next table we list symmetric spaces with noncompact stability groups

Table II. Symmetric Spaces G/H with Noncompact Stability Group

$H \diagdown G$	$SL(n, C)$	$SL(n, R)$	$SU(p, q)$
	$SO(n, C)$	$SL(p, R)\times SL(q, R)\times R^1$	$SU(k, k+h)\times SU(p-k, n-k-h)\times U(1)$
	$SL(n, R)$	$SO(p, q)$	$SO(p, q)$
	$SL(p, C)\times SL(q, C)\times C^1$	$Sp(n/2, R)$	$Sp(p/2, q/2)$
	$SU(p, q)$	$Sp(n/2, C)\times R^1$	$\left.\begin{array}{l} SO^*(n) \\ Sp(n, R) \\ SL(n, C)\times R^1 \end{array}\right\} p=q=n/2$
	$Sp(n/2, C)$		
	$SU^*(n)$		
$H \diagdown G$	$SU^*(n)$	$SO(n, C)$	$SO(p, q)$
	$SU^*(p)\times SU^*(q)\times R^1$	$SO(p, C)\times SO(q, C)$, $p=1$ or $p>2, q>2$	$SO(p, k+h)\times SO(p-k, n-k-h)$, $k+h>2, n-k-h>2$
	$Sp(p/2, q/2)$		
	$SO^*(n)$		
	$SL(n, C)\times U(1)$	$SO(n-2)\times C^1$	$SO(p-2, q)\times U(1)$
		$SO(p, q)$	$SO(p-1, q-1)\times R^1$
		$SL(n/2, C)\times C^1$	$SU(p/2, q/2)\times U(1)$
		$SO^*(n)$	$\left.\begin{array}{l} SL(n/2, R)\times R^1 \\ SO(n/2, C) \end{array}\right\} p=q=n/2$
		$SO^*(n/2, C)$	
		$SU(p, q)\times U(1)$	$SO(n-2)\times U(1) \left\} \begin{array}{l} p=2 \\ q=n-2 \end{array}\right.$
$H \diagdown G$	$Sp(n, C)$	$Sp(n, R)$	$Sp(p, q)$
	$SL(n, C)\times C^1$	$Sp(p, R)\times Sp(q, R)$	$Sp(k, k+h)\times Sp(p-k, n-k-h)$
	$Sp(n, R)$	$SU(p, q)\times U(1)$	$SU(p, q)\times U(1)$
	$Sp(p, C)\times Sp(q, C)$	$SL(n, R)\times R^1$	
	$Sp(p, q)$	$Sp(n/2, C)$	
		$\left.\begin{array}{l} SU^*(n)\times R^1 \\ Sp(n/2, C) \end{array}\right\} p=q=n/2$	

Note: $p+q=n$, $R^1(C^1)$ additive group of real (complex) numbers. When $n/2, p/2, \ldots$ occur, n and p are even.

The full classification of symmetric spaces, including symmetric spaces associated with exceptional simple Lie groups was elaborated by Berger 1957.

§ 3. Invariant and Quasi-Invariant Measures on Homogeneous Spaces

Let X be a homogeneous space with a transformation group which is a locally compact separable group G. We know from th. 1.1 that X is isomorphic to the coset space $H\backslash G$ or G/H, where H is the stability subgroup of a point $x_0 \in X$.

Let S be a subset of $X = H\backslash G$. By the 'translate' of a subset S by an element $g \in G$ we mean the set $Sg = \{xg\colon x \in S\}$. Let $d\mu(x)$ be a positive measure on X; a measure $d\mu_g(x) \equiv d\mu(xg)$ will be, by definition, the measure given by the following formula

$$\mu_g(f) \equiv \int_X f(x)\,d\mu(xg) = \int_X f(xg^{-1})\,d\mu(x) \quad \text{for every } f \in C_0(X). \tag{1}$$

In other words, $\mu_g(S) \equiv \mu(Sg)$ for every Borel set S in the space X.

We have seen that on every locally compact topological group there exists an invariant measure (cf. th. 2.3.1). The following example shows that on a homogeneous space an invariant measure might not exist.

Example 1. Let G be the group of triangular real matrices of the form

$$g = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix}, \quad \alpha > 0 \tag{2}$$

and let

$$H = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\right\}.$$

Every element $g \in G$ may be represented in the form (Mackey decomposition, cf. th. 2.4.1)

$$g = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \gamma\alpha & 1 \end{bmatrix}. \tag{3}$$

Hence, every element of a right coset Hg may be uniquely represented by a point $x = \gamma\alpha$ of the real line R. Consequently, $X = H\backslash G = R$. Because an element $x \in X$ corresponds to the group element $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$, we obtain the action of G in X by the formula

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \alpha x+\gamma & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \alpha^2 x+\alpha\gamma & 1 \end{bmatrix}, \tag{4}$$

or,

$$g\colon x \to \alpha^2 x+\alpha\gamma. \tag{5}$$

The invariant measure on X relative to G should be, in particular, invariant relative to translations $x \to x+\gamma$; hence, it should be proportional to the Lebesgue measure on R (cf. exercise 2.3.3). Such a measure cannot, however, be invariant relative to homothetic transformations $x \to \alpha^2 x$; consequently, there is no measure $d\mu(x)$ on X invariant relative to G. ▼

The lack of invariant measures on homogeneous spaces leads to the concept of quasi-invariant measures.

DEFINITION 1. A positive measure $\mathrm{d}\mu(x)$ on X is said to be *quasi-invariant* if the measure $\mathrm{d}\mu_g(x) \equiv \mathrm{d}\mu(xg)$ and $\mathrm{d}\mu(x)$ are equivalent for every $g \in G$.

Remark: Two positive measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$ are said to be *equivalent* if they have the same sets of measure zero. According to the Radon–Nikodym theorem (app. A.5) there exists then a function $\varrho(x) \geqslant 0$ such that

$$\mathrm{d}\mu_1(x) = \varrho(x)\mathrm{d}\mu_2(x). \tag{6}$$

The function $\varrho(x) = \mathrm{d}\mu_1(x)/\mathrm{d}\mu_2(x)$ is called the *Radon–Nikodym derivative.* ▼

The following theorem describes the main properties of quasi-invariant measures on homogeneous spaces.

THEOREM 1. *Let G be a locally compact separable group, H a closed subgroup of G and $X = H\backslash G$. Then*

1° *There exists a quasi-invariant measure on X such that the Radon–Nikodym derivative $\mathrm{d}\mu_g(x)/\mathrm{d}\mu(x)$ is a continuous function on $G \times X$.*

2° *Any two quasi-invariant measures on X are equivalent.*

3° *All quasi-invariant measures may be obtained in the following manner: Let $\varrho(g)$ be a strictly positive locally integrable Borel function satisfying*

$$\varrho(hg) = \frac{\Delta_H(h)}{\Delta_G(h)}\varrho(g) \quad \text{for all } h \in H, \tag{7}$$

where Δ_H and Δ_G are modular functions for H and G, respectively.

Then ϱ is related to the quasi-invariant measure μ on X by the formula

$$\int_G f(g)\varrho(g)\mathrm{d}g = \int_X \mathrm{d}\mu(\mathring{g}) \int_H f(hg)\mathrm{d}h, \quad \mathring{g} \equiv Hg, \tag{8}$$

for $f \in C_0(G)$. The measure μ satisfies the condition

$$\mathrm{d}\mu(\mathring{g}g') = \frac{\varrho(\mathring{g}g')}{\varrho(g')}\mathrm{d}\mu(g'), \tag{9}$$

and for a given ϱ, is determined uniquely up to a multiplicative constant. ▼

(For the proof cf. Mackey 1952 and Loomis 1960.)

The following corollary provides a convenient criterion for the existence of an invariant measure on the space $X = H\backslash G$.

COROLLARY 1. *An invariant measure μ exists on a homogeneous space $X = H\backslash G$ if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. This measure is unique up to a multiplicative constant and satisfies*

$$\int_G f(g)\mathrm{d}g = \int_X \mathrm{d}\mu(\mathring{g}) \int_H f(hg)\mathrm{d}h, \quad \mathring{g} = Hg. \tag{10}$$

PROOF: It is sufficient to take $\varrho(g) = 1$ for every g in (8). ▼

In particular, if G is unimodular and H is also unimodular, then $\Delta_G(h) = \Delta_H(h) = 1$ and $X = H\backslash G$ possesses an invariant measure.

EXAMPLE 2. Let $G = \mathrm{SO}(3, 1)$ and $H = \mathrm{SO}(3)$. The homogeneous space $X = H\backslash G$ may be represented as a hyperboloid

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = \varrho^2 > 0 \tag{11}$$

(cf. exercise 1.2). Because G and H are unimodular, we have $\Delta_G(h) = \Delta_H(h) = 1$ for all $h \in H$. Consequently, X possesses an invariant measure by virtue of corollary 1.

We now calculate the explicit form of the invariant measure $\mathrm{d}\mu$ on $X = H\backslash G$. We first relate with the second order surface (11) a differential form $\mathrm{d}\mu$ defined by the formula

$$\mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \mathrm{d}(x_\varrho x^\varrho)\,\mathrm{d}\mu = \mathscr{I}\mathrm{d}(x_\varrho x^\varrho)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{12}$$

The Jacobian $\mathscr{I}$ for the transformation $(x_0, x_1, x_2, x_3) \to (x_\varrho x^\varrho, x_1, x_2, x_3)$ takes the form $\mathscr{I} = 1/2x_0$, where $x_0 = (\varrho^2 + x^2)^{1/2}$ (for the upper hyperboloid (11)). Hence, by virtue of eq. (12),

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}\,x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}{2x_0}. \tag{13}$$

The right translations $x \to xg$ does conserve the differential forms $\mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$ and $\mathrm{d}(x_\varrho x^\varrho)$. Hence, the differential form (13) is invariant and gives an invariant measure on $X = H\backslash G$. This measure is unique up to a multiplicative constant by virtue of corollary 1.

In relativistic kinematics, the energy momentum four-vector of a particle of positive mass satisfies $p_0^2 - p^2 = m^2$, hence it is a point on the homogeneous space $H\backslash G$, and the invariant measure is therefore $\mathrm{d}^3p/2p_0$. ▼

We now give the so-called measure disintegration theorem. Let X be a locally compact space, countable at infinity, r an equivalence relation in X, Y the quotient space X/r. Denote by π the canonical map of X onto Y. Let μ be a finite measure on X. We say that a set $E \subset Y$ is a Borel set iff $\pi^{-1}(Y)$ is a Borel set in X. This gives a Borel structure in Y induced by the Borel structure in X. We define a measure $\tilde{\mu}$ on Y by the formula

$$\tilde{\mu}(E) = \mu(\pi^{-1}E), \qquad E\text{—a Borel set in } Y. \tag{14}$$

If the Borel structure on Y is separable (i.e., there exists a sequence of Borel sets at Y which separates points of Y) then we have the following theorem:

THEOREM 2. *For every $y \in Y$ there exists a measure μ_y on X, with the support $\pi^{-1}(y)$ (i.e. $\mu_y\ (X - \pi^{-1}(y)) = 0$), such that for every function $f \in L^1(X, \mu)$ we have*

$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_Y \mathrm{d}\tilde{\mu}(y) \int_X f(x)\,\mathrm{d}\mu_y(x). \tag{15}$$

(For the proof cf. Mackey 1952, § 11.)

§ 4. Comments and Supplements

A. The following theorem gives a useful measure disintegration theorem implied by the Iwasawa decomposition.

THEOREM 1. *Let G be a connected semisimple Lie group and let $\mathscr{K}\mathscr{A}_p\mathscr{N}$ be its Iwasawa decomposition. Let* dk, da *and* dn *be the left-invariant measures on* $\mathscr{K}$, $\mathscr{A}_p$ *and* $\mathscr{N}$, *respectively. Then the left-invariant measure* dg *on G can be normalized such that*

$$\int_G f(g)\,\mathrm{d}g = \int_{\mathscr{K}\times\mathscr{A}_p\times\mathscr{N}} f(kan)\exp[2\varrho(\log a)]\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$$
$$= \int_{\mathscr{K}\times\mathscr{A}_p\times\mathscr{N}} f(kna)\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}a, \tag{1}$$

where $\log a$ *denotes the unique element X in the Lie algebra* H_p *for which* $\exp X = a$ *and* $\varrho = \frac{1}{2}\sum_{\alpha\in B_+}\alpha$. ▼

(Cf. eq. 1.6(18).)

(For the proof cf. Helgason 1962, ch. X, § 1.)

B. *Historical Notes*

É. Cartan after completing the classification of complex simple Lie groups (1894), real simple Lie groups (1914) began in 1925 an analysis of the properties of homogeneous spaces associated with simple Lie groups. In a series of impressive papers (1926–27), he succeeded in completing the classification of global irreducible symmetric Riemannian spaces. He also gave a geometric description of all of these spaces.

The classification of symmetric spaces with non-compact stability groups was given by Berger 1957.

The description of quasi-invariant measures on homogeneous spaces in the version presented by th. 3.1 was given by Loomis 1960.

§ 5. Exercises

§ 2.1. Let $G = \mathrm{U}(n)$ and $H = \mathrm{U}(n-1)$. Show that the space $X = G/H$ is symmetric and can be represented as the manifold in C^n given by

$$z_1\bar{z}_1 + \dots + z_n\bar{z}_n = 1. \tag{1}$$

§ 2.2. Let $G = \mathrm{SO}(p, q)$ and $H = \mathrm{SO}(p-1, q)$, show that the space $X = G/H$ is symmetric and can be represented as a hyperboloid given by

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 = 1. \tag{2}$$

§ 2.3. Show that the stability group of the cone

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \tag{3}$$

is the group $H = T^2 \circledS \mathrm{S\mathring{O}}(2)$.

Hint. Use 2×2-matrix description of the Minkowski space M given by the correspondence

$$M \ni x \to X = x^\mu \sigma_\mu$$

(cf. exercise 3.7.3) and find the stability subgroup of the point $x = (1, 0, 0, 1)$ in this realization.

§ 2.4. Show that the symmetric space $X = \mathrm{SU}(1,1)/\mathrm{U}(1)$ may be realized as the unit disc $D = \{z \in C: |z| < 1\}$. Show that the action of $\mathrm{SU}(1,1)$ on D is given by the projective transformations $\left(g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \ \alpha, \beta \in C, \ |\alpha|^2 - |\beta|^2 = 1\right)$

$$g: z \to \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}. \tag{4}$$

§ 2.5. Show that the Riemannian metric tensor on the space D of the previous problem has the form

$$\begin{aligned} g_{ij} &= (1-|z|^2)^{-2}\delta_{ij}, \\ g^{ij} &= (1-|z|^2)^2 \delta_{ij}, \end{aligned} \tag{5}$$

and that the volume element $\mathrm{d}\mu$ on D is given by

$$\mathrm{d}\mu(z) = \sqrt{(\det g)}\,\mathrm{d}x\mathrm{d}y = [1-(x^2+y^2)]^{-2}\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

§ 3.1. Show that the invariant metric tensor $g_{\alpha\beta}$ on the hyperboloid (2) has the form

$$g_{\alpha\beta}(t) = g_{ik}\frac{\partial x^i}{\partial t^\alpha}\frac{\partial x^k}{\partial t^\beta}, \tag{7}$$

where $\{x^i\}_{i=1}^{p+q}$ are the Cartesian coordinates on the Minkowski space $M^{p,q}$ in which the hyperboloid (2) is embedded and $\{t^\alpha\}_{\alpha=1}^{p+q-1}$ are any 'internal' coordinates on hyperboloid (e.g., spherical).

§ 3.2. Find a measure on the cone (3) invariant relative to $\mathrm{SO}(3, 1)$.

§ 3.3. Show that the $\mathrm{SO}(p, q)$-invariant measure on the hyperboloid (2) has the form

$$\mathrm{d}\mu(t) = (\det g)^{1/2} \prod_{\alpha=1}^{p+q-1} \mathrm{d}t^\alpha$$

where $g_{\alpha\beta}(t)$ is given by eq. (7).

Chapter 5

Group Representations

§ 1. Basic Concepts

Let G be a locally compact, separable, unimodular topological group and let H be a separable complex Hilbert space.

DEFINITION 1. A map $x \to T_x$ of G into the set $L(H)$ of linear bounded operators in H is said to be a *representation* of G in H if the following conditions are satisfied:

$$T_{xy} = T_x T_y, \quad T_e = I. \blacktriangledown \tag{1}$$

The condition $T_{xy} = T_x T_y$ means that the map $x \to T_x$ is a homomorphism of G into a set of linear operators in H. The condition $T_e = I$ guarantees that the representation $x \to T_x$ is in terms of invertible operators; indeed

$$T_x T_{x^{-1}} = T_{x^{-1}} T_x = T_e = I.$$

Hence,

$$T_x^{-1} = T_{x^{-1}}.$$

In addition certain continuity conditions are also imposed on a representation $x \to T_x$ of G in H. A representation is said to be *strongly continuous* if for all $u \in H$ the map $x \to T_x u$ is a continuous map of G into H. This means that for any $x_0 \in G$

$$\|T_x u - T_{x_0} u\| \to 0, \quad \text{as } x \to x_0 \tag{2}$$

for all vectors $u \in H$.

The condition (2) is equivalent to the following, apparently stronger, condition which is usually given in the definition of a representation of topological groups:

The map $(u, x) \to T_x u$ of $H \times G$ into H is continuous.

PROOF: It is necessary to show that for $x \to x_0 \in G$ and $u \to u_0 \in H$

$$T_x u \to T_{x_0} u_0.$$

Let K be a compact neighborhood of the point x_0 in G. For any $u \in H$ the set $S = \{T_x u : x \in K\}$, by condition (2), is a compact (and therefore bounded) subset of H. The Principle of Uniform Boundedness (cf. Appendix A.4) assures that there exists a constant C_k such that

$$\|T_x\| \leqslant C_k \quad \text{for all } x \in K. \tag{3}$$

We can assume, without loss of generality, that $x \in K$. Then,

$$||T_x u - T_{x_0} u_0|| \leqslant ||T_x u - T_x u_0|| + ||T_x u_0 - T_{x_0} u_0|| \leqslant C_k ||u - u_0|| + ||T_x u_0 - T_{x_0} u_0||.$$

The first term tends to zero since $u \to u_0$ and the second term tends to zero due to condition (2). Therefore for $x \to x_0 \in G$ and $u \to u_0 \in H$, $T_x u \to T_{x_0} u_0$, and the map $(u, x) \to T_x u$ of $H \times G$ into H is continuous. ▼

A representation is said to be *bounded*, if $\sup||T_x|| < \infty$. It follows from eq. (3) that a representation of any locally compact separable, topological group is bounded for any compact subset $K \subset G$. In particular a representation of a compact group is bounded.

A representation is said to be *unitary* if each T_x, $x \in G$, is a unitary operator in H, and *trivial* if $T_x = I$ for all x in G.

We denote for brevity a representation $x \to T_x$ of G by the symbol T. The space H in which a representation T acts is called the *carrier space* of T.

The def. 1 specifies in fact linear representations of G. In the following, by a representation of a topological, locally compact, separable group G we shall mean a 'linear strongly continuous representation in a separable complex Hilbert space H', unless explicitly stated otherwise (e.g., nonlinear, weakly-continuous or discontinuous).

EXAMPLE 1. Let $G = R^1$, and H be an arbitrary Hilbert space. Consider the maps

(i) $R^1 \ni x \to T_x = \exp[ipx] \cdot I$, $p \in R^1$,

and

(ii) $R^1 \ni x \to T'_x = \exp[px] \cdot I$, $p \in R^1$.

The conditions (1) and (2) for the maps (i) and (ii) are obviously satisfied. Hence, T and T' are representations of R^1. The representation T' is not bounded on R^1, although it is bounded on every bounded subset of R^1.

EXAMPLE 2. Let G be a topological transformation group which acts continuously on a locally compact measure space S and leaves the measure invariant. Set $H = L^2(S, \mu)$. Let the map $x \to T_x$ be defined by means of the left translation, i.e.,

$$(T_x u)(s) = u(x^{-1}s), \quad u \in H, \ s \in S, \ x \in G. \tag{4}$$

Clearly, every T_x is a linear operator. Moreover

$$[T_x(T_y u)](s) = (T_y u)(x^{-1}s) = u(y^{-1}x^{-1}s) = (T_{xy}u)(s),$$

i.e.,

$$T_x T_y = T_{xy} \quad \text{and} \quad T_e = 1.$$

Hence, the map (4) defines a representation of G in H.

The invariance of the measure implies

$$(T_x u, T_x v) = \int u(x^{-1}s)\overline{v(x^{-1}s)}\,d\mu(s) = (u, v),$$

i.e. T_x is isometric, and because the domain D_{T_x} of T_x is equal to H, every T_x is unitary.

If $u \in C_0(S)$,

$$\sup|u(x^{-1}s)-u(s)| \to 0,$$

as $x \to e$ by uniform continuity (cf. proposition 2.2.4). Moreover, there exists a fixed compact set $K \subset S$, supporting u and $T_x u$, for x sufficiently near e, and we have

$$\begin{aligned}||T_x u-u|| &= \Big[\int_S |u(x^{-1}s)-u(s)|^2 \mathrm{d}\mu(s)\Big]^{1/2}\\ &\leqslant \max_{s\in K}|u(x^{-1}s)-u(s)|\sqrt{\mu(K)} \to 0, \quad \text{as } x \to e.\end{aligned}$$

The continuity property as $x \to y$ follows if one replaces u by $T_y u$.

Now if $u \in L^2(S, \mu)$ and $\varepsilon > 0$, there exists a $v \in C_0(S)$ such that $||u- v|| < \varepsilon$ Then

$$||T_x u-u|| \leqslant ||T_x(u-v)||+||T_x v-v||+||v-u||$$

and the invariance of μ under G implies

$$||T_x u-u|| \leqslant 2\varepsilon+||T_x v-v||.$$

Hence $||T_x u-u|| \leqslant 3\varepsilon$, if x is sufficiently close to e. Consequently, the map (4) defines a strongly continuous unitary representation of G in $L^2(S, \mu)$. ▼

If $S = G$ the representation (4) is called the *left regular representation.* If $S = G/K$, where K is a closed subgroup of G, the representation (4) is called the *quasi-regular representation.* Clearly by the right translations

$$T_x u(y) = u(yx) \tag{5}$$

one can define similarly the *right regular representation* of G in $L^2(G, \mu)$.

A representation $x \to T_x$ of G in H is said to be *weakly continuous,* if for arbitrary u, $v \in H$,

$$(T_x u, v) \to (T_{x_0}u, v), \quad \text{as } x \to x_0. \tag{6}$$

The weak and the strong continuity for unitary representations are equivalent. Indeed, we have

PROPOSITION 1. *Let T be a unitary representation of a group G in a Hilbert space H. Then the following statements are equivalent.*

1° *T is strongly continuous.*

2° *T is weakly continuous.*

3° *The function $x \to (T_x u, u)$ is continuous at e for all $u \in H$.*

PROOF: Clearly, 1° ⇒ 2° and 2° ⇒ 3°. Hence, it suffices to show that 3° ⇒ 1°. Indeed, for any $u \in H$ and x, $y \in G$ we have

$$\begin{aligned}||T_x u-T_y u||^2 &= (T_x u, T_x u)-(T_x u, T_y u)-(T_y u, T_x u)+(T_y u, T_y u)\\ &= 2(u, u)-2\,\mathrm{Re}(T_y u, T_x u) \leqslant 2|(u, u)-(T_y u, T_x u)|\\ &= 2|(u, u)-(T_{x^{-1}y}u, u)|.\end{aligned} \tag{7}$$

Consequently, if 3° is satisfied, $||T_x u - T_y u|| \to 0$, for $x \to y$, by virtue of eq. (7). Thus, 3° ⇒ 1°. ▼

Remark: For unitary representations the weak continuity also implies the strong left uniform continuity (cf. eq. 2.2.(13)). Indeed

$$||T_x u - T_y u|| = ||u - T_{x^{-1}y} u||.$$

Hence, for arbitrary $\varepsilon > 0$ there exists a neighborhood V_ε of e such that

$$||T_x u - T_y u|| < \varepsilon, \quad \text{whenever } x^{-1}y \in V_\varepsilon. ▼$$

It is interesting that a unitary representation $x \to T_x$ of a group G might not be continuous. This important fact is illustrated by the following

EXAMPLE 3. Let $G = R$ and let $\{\xi^\alpha\}$ be a Hamel basis* for R. Take $\xi^1 = 1$ and let ξ^2 be any other basis element, which from r-independence of basis elements, must be irrational. Clearly, any element $x \in G$ can be written as $x = \sum r_\alpha \xi^\alpha$. Consider the map

$$\varphi\colon\ x \to T_x = \exp(ir_2)I, \tag{8}$$

where $x \in G$ and I is the unit operator in the carrier Hilbert space H. We have $T_{xy} = T_x T_y$ and $T_e = I$. Moreover, $T_x = I$, when x is rational (because in this case $x = x\xi^1$) and $T_{\xi^2} = \exp \mathrm{i} \cdot I$. Hence, the map (8) provides a discontinuous, unitary representation of G in H. ▼

Remark: The representations of a topological group G given by def. 1 do not exhaust all possible representations of G which one encounters in theoretical physics or geometry. For instance, if a carrier space of a representation is a group manifold itself, then, the map $x \to T_x$ defined by

$$T_x^R y = yx, \quad \text{or} \quad T_x^L y = x^{-1}y, \tag{9}$$

satisfies the conditions (1), and due to continuity of a group multiplication, also satisfies continuity conditions. However, the map $x \to T_x$ is nonlinear even if G is commutative, and therefore, there appears a new type of representation of G. A similar situation occurs if, for instance, G acts on a curved homogeneous space G/K, or on a nonlinear differential manifold M. To distinguish these from linear representations they are often called *realizations*. ▼

Let φ be the homomorphism $x \to T_x$ given in def. 1. The set K of all elements of G, which satisfies the condition $\varphi(x) = I$, $x \in G$, is said to be the *kernel* of the homomorphism φ. If $x, y \in K$, then $xy \in K$ and $x^{-1} \in K$. Moreover, if $x \in K$ and $y \in G$, then $\varphi(yxy^{-1}) = T_y T_{y^{-1}} = I$, i.e., $yxy^{-1} \in K$. Therefore, K is an invariant subgroup of G.

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* The Hamel base in R is an (uncountable) base of R, considered as a vector space over the field Q of rational numbers.

A representation $x \to T_x$ of G is said to be *faithful*, if the map $x \to T_x$ is one-to-one. In this case, $K = \{e\}$. If $K \neq \{e\}$, then all elements of a coset xK for a fixed $x \in G$ are represented by the same operator and two different cosets by different operators. Hence, the homomorphism φ: $x \to T_x$, which provides a non-faithful representation of G, can be considered as a faithful representation of the quotient group G/K given by the isomorphism φ: $xK \to T_x$.

Note that simple Lie groups without nontrivial discrete centers, e.g., $SU(n)/Z_n$, where

$$Z_n = \left\{\exp\left[\frac{2\pi i}{k}\right]e, \quad k = 1, 2, \ldots, n-1\right\},$$

have no invariant subgroups. Hence, all representations of simple Lie groups without nontrivial discrete centers are either faithful or trivial ones.

The Matrix Form of Representations

Let $\{e_i\}_1^N$, $N \leqslant \infty$, be an orthonormal basis in the carrier space H. An operator T_x, $x \in G$, transforms a basis element e_j into $T_x e_j \in H$. The latter can be represented in the form

$$T_x e_j = D_{ij}(x) e_i, \quad j = 1, 2, \ldots, N. \tag{10}$$

Hence,

$$D_{ij}(x) = (T_x e_j, e_i), \quad i, j = 1, 2, \ldots, N. \tag{11}$$

Therefore, an operator T_x can be represented in the basis $\{e_i\}_1^N$ by the finite, or infinite matrix $[D_{ij}(x)]$. It follows from proposition 1 that every matrix element (11) is a continuous function on G. The matrix of the operator T_{xy} is the product of matrices (11), i.e.,

$$D_{ij}(xy) = D_{ik}(x) D_{kj}(y). \tag{12}$$

Indeed,

$$\begin{aligned} D_{ij}(xy) &= (T_{xy} e_j, e_i) = (T_x T_y e_j, e_i) = (T_y e_j, T_x^* e_i) \\ &= (T_y e_j, e_k)(e_k, T_x^* e_i) = (T_y e_j, e_k)(T_x e_k, e_i) \\ &= D_{ik}(x) D_{kj}(y). \end{aligned}$$

Clearly, the matrix form (11) of the operator T_x depends upon the choice of a basis in the carrier space H. If U is a unitary operator which maps H onto itself, and if $h_i = Ue_i$, $i = 1, 2, \ldots, N$, then the basis $\{h_i\}_1^N$ is orthonormal and the matrix form of T_x in the new basis is

$$U^{-1} D(x) U, \tag{13}$$

where the matrix elements U_{ij} of the operator $U = \{U_{ij}\}$ are

$$U_{ij} = (Ue_j, e_i) = (h_j, e_i).$$

Indeed,

$$\begin{aligned} D_{ij}^{(h)}(x) &= (T_x Ue_j, Ue_i) = (T_x Ue_j, e_s)(e_s, Ue_i) \\ &= (Ue_j, T_x^* e_s)(e_s, Ue_i) = (Ue_j, e_p)(e_p, T_x^* e_s)(e_s, Ue_i) \\ &= U_{is}^{-1} D_{sp}(x) U_{pj}. \end{aligned}$$

A unitary operator T_x is represented by a unitary matrix $[D_{ij}(x)]$ only if the basis $\{e_i\}_1^N$ is orthonormal.

Let T be a matrix representation of a topological group G in a Hilbert space H. Consider the maps

$$\begin{aligned} &1^\circ\ x \to \check{T}_x \equiv T^T_{x^{-1}}, \\ &2^\circ\ x \to \hat{T}_x = T_x^{*T} = \overline{T}_x, \\ &3^\circ\ x \to \tilde{T}_x = T^*_{x^{-1}}. \end{aligned} \tag{14}$$

It is easy to verify that any of the maps (14) defines a representation of G, for instance,

$$\tilde{T}_e = I, \quad \text{and} \quad \tilde{T}_{xy} = T^*_{(xy)^{-1}} = (T_{y^{-1}} T_{x^{-1}})^* = \tilde{T}_x \tilde{T}_y.$$

The continuity of the maps (14) follows from the fact that the operations 'T' and '*' are continuous.

The representations (14)1°, (14)2° and (14)3° are said to be *contragradient*, *conjugate* and *conjugate-contragradient* to the representation T, respectively. For unitary representations contragradient and conjugate representations coincide.

§ 2. Equivalence of Representations

Let $x \to T_x$ be a representation of a topological group G in a Hilbert space H. Let S be a bounded isomorphism of H onto a Hilbert space H'. Then the map φ: $x \to T'_x = ST_xS^{-1}$ defines a representation of G in H'. Indeed

$$T'_{xy} = ST_xS^{-1}ST_yS^{-1} = T'_xT'_y, \quad T'_e = I, \tag{1}$$

and

$$\begin{aligned} ||T'_xu' - T'_yu'|| &= ||S(T_xS^{-1}u' - T_yS^{-1}u')|| \\ &\leqslant ||S||\,||T_xu - T_yu|| \to 0 \quad \text{as } x \to y. \end{aligned}$$

In this manner a whole class of new representations may be constructed, starting from a given representation $x \to T_x$ of G, acting in the same or isomorphic carrier spaces. These representations, however, are not essentially different. Hence, we collect them into one class of representations by means of the notion of equivalence of the representations.

DEFINITION 1. A representation $x \to T_x$ of a topological group G in a Hilbert space H is said to be *equivalent* to a representation $x \to T'_x$ in H', if there exists a bounded isomorphism S of H onto H' such that

$$ST_x = T'_xS \quad \text{for all } x \in G. \tag{2}$$

We shall write in this case $T_x \simeq T'_x$. This operation '$\simeq$' is reflexive, symmetric and transitive. It means

$$\begin{gathered} T \simeq T, \\ T \simeq T' \Rightarrow T' \simeq T, \\ T \simeq T' \text{ and } T' \simeq T'' \Rightarrow T \simeq T''. \end{gathered} \tag{3}$$

Hence, it is an equivalence relation. Therefore it partitions the set of all representations of G into disjoint classes of equivalent representations.

Next we introduce the narrower concept of unitary equivalence of two representations in the Hilbert spaces H and H'.

DEFINITION 2. Two representations: $x \to T_x$ in H, and $x \to T'_x$ in H' are *unitarily equivalent* if there exists a unitary isomorphism U: $H \to H'$ such that $UT_x = T'_xU$ for every $x \in G$.

EXAMPLE 1. Let T^L and T^R be the left and the right regular representation of a group G. The involution I: $u(x) \to u(x^{-1})$ defines a unitary map of H onto itself. We have

$$(IT^R_x u)(y) = (T^R_x u)(y^{-1}) = u(x^{-1}y) = (Iu)(y^{-1}x) = (T^L_y Iu)(x).$$

Hence

$$IT^R_x = T^L_x I.$$

PROPOSITION 1. *Two equivalent unitary representations are unitarily equivalent.*

PROOF: Taking the adjoint of both sides of eq. (2) we obtain

$$T_x S^* = S^* T'_x. \tag{4}$$

Hence, by eqs. (4) and (2) we have

$$SS^*T'_x = ST_xS^* = T'_xSS^*,$$

i.e., every T'_x commutes with the positive hermitian operator SS^* and, thereby, also with $A = \sqrt{(SS^*)}$. The operator $A^{-1}S$ is a unitary operator, which satisfies the condition (2). Indeed,

$$A^{-1}ST = A^{-1}T'S = T'A^{-1}S.$$

Hence, T and T' are unitarily equivalent. ▼

Two equivalent unitary representations: $x \to T_x$ in H, and $x \to T'_x$ in H', can be described by the same matrices by a proper choice of bases in H and H'. Indeed, let S be an isomorphism of H onto H', such that $ST_x = T'_xS$, and let $\{e_i\}_1^N$, $N \leqslant \infty$, be a basis in H. Then, taking as a basis in H' the set $\{e'_i = Se_i\}_1^N$, we obtain

$$T_x e_j = D_{ij}(x) e_i \tag{5}$$

and

$$D'_{ij}(x)e'_i = T'_x e'_j = T'_x Se_j = ST_x e_j = SD_{ij}(x)e_i = D_{ij}(x)e'_i,$$

i.e., both matrices coincide.

Let T and T' be representations of G in H and H', respectively. A bounded operator S from H into H' is said to be an *intertwining operator* for T and T', if $ST_x = T'_x S$ for every $x \in G$. The set of all intertwining operators forms a linear space, which we denote by $R(T, T')$. The proposition 1 can now be restated in the following form: Two unitary representations T and T' are equivalent if and only if there exists in $R(T, T')$ a unitary operator from H onto H'

Note that for $T = T'$, $R(T, T)$ is an algebra.

§ 3. Irreducibility and Reducibility

Let $x \to T_x$ be a representation of a topological group G in a Hilbert space H. A subspace or subset H_1 of H is said to be *invariant* (with respect to T) if $u \in H_1$ implies $T_x u \in H_1$ for every $x \in G$.

Every representation has at least two invariant subspaces: the null-space $\{0\}$ and the whole space H. These invariant subspaces are said to be trivial. The non-trivial invariant subspaces or subsets will be called *proper*. We introduce now the concept of irreducibility, which plays a fundamental role in the representation theory.

Definition 1 (Algebraic irreducibility). A representation $x \to T_x$ of a group G in H is said to be *algebraically irreducible*, if it has no proper invariant subsets in H.

Definition 2 (Topological irreducibility). A representation $x \to T_x$ of a topological group G in H is said to be *topologically irreducible* if it has no proper *closed* invariant subspace.

Clearly, algebraic irreducibility implies topological irreducibility. A representation, which has proper invariant subspaces is said to be *reducible*. Unless otherwise stated, we use in the following the name irreducible (reducible) for topologically irreducible (reducible) representations.

At least two new representations can be associated with every topologically reducible representation. The first one is obtained by the restriction of every T_x to the closed subspace H_1. This representation is called the *subrepresentation of* T and is denoted by ${}^{H_1}T$. The second one can be realized in the quotient space H/H_1. Indeed, because H_1 is invariant, a coset $u+H_1$ is transformed by T_x into $T_x u+H_1 \in H/H_1$, i.e., H/H_1 is also an invariant space.

Let H be a Hilbert space, and H_1 a proper invariant subspace. The orthogonal complement $H_1^\perp$ of H_1 may not, in general, be an invariant subspace of H. However, if a representation $x \to T_x$ in H is unitary, then this is true. Indeed, we have

Proposition 1. *Let $x \to T_x$ be a unitary representation of any group G in a Hilbert space H. Let H_1 be a subspace of H and let P_1 be the projection operator in H, whose range is H_1. Then,* 1° *The orthogonal complement* $H_1^\perp$ *of* H_1 *is invariant if and only if* H_1 *is invariant.*

2° H_1 *is invariant if and only if* $P_1T_x = T_xP_1$, *for every* $x \in G$.

PROOF: *ad* 1°. Let H_1 be invariant. Then, for $u \in H_1$, $v \in H_1^\perp$ and any T_x we have

$$(T_xv, u) = (v, T_x^*u) = (v, T_{x^{-1}}u) = 0$$

because $T_{x^{-1}}u \in H_1$. Hence $H_1^\perp$ is also invariant. The converse statement follows by exchanging the roles of H_1 and $H_1^\perp$.

ad 2°. Let H_1 be invariant and $u \in H$. Then, $T_xP_1u \in H_1$ for every $x \in G$ and $P_1T_xP_1u = T_xP_1u$; because u is arbitrary, we have $P_1T_xP_1 = T_xP_1$. Taking the adjoint of both sides, we obtain $P_1T_x^*P_1 = P_1T_x^*$, or $P_1T_{x^{-1}}P_1 = P_1T_{x^{-1}}$. Setting $y = x^{-1}$, we have $P_1T_y = P_1T_yP_1 = T_yP_1$ for every $y \in G$. Conversely, if $P_1T_x = T_xP_1$ for every $x \in G$, then, for $u_1 \in H_1$, we get $T_xu_1 = T_xP_1u_1 = P_1T_xu_1 \in H_1$. Hence, H_1 is invariant.▼

The following example shows that the assumption of unitarity in the proposition 1 is essential, i.e. for nonunitary representations the orthogonal complement of an invariant subspace is, in general, not invariant. ▼

EXAMPLE 1. Let $G = R^1$, and let H be the two-dimensional real Hilbert space with the scalar product $(u, v) = u_1v_1 + u_2v_2$. We represent R^1 in H by the triangular nonunitary matrices

$$R \ni x \to T_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \text{i.e.,} \quad T_x\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + xu_2 \\ u_2 \end{bmatrix}. \tag{1}$$

It follows from eq. (1) that the subspace H_1 consisting of vectors $u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$ is invariant with respect to T, while the orthogonal complement $H_1^\perp$ consisting of vectors $u = \begin{bmatrix} 0 \\ u_1 \end{bmatrix}$ is not invariant. ▼

A Hilbert space H is said to be the *direct sum* of its subspaces $H_1, H_2, \ldots$, i.e.,

$$H = H_1 \oplus H_2 \oplus \ldots = \sum_i \oplus H_i, \tag{2}$$

if the following conditions are satisfied:

1° $H_i \perp H_j$ for $i \neq j$.

2° Every element $u \in H$ decomposes into the convergent series

$$u = \sum_i u_i, \quad \text{where } u_i \in H_i.$$

DEFINITION 3. A representation T of G in a Hilbert space H is said to be the *direct sum* of representations T_i of G in H_i if H_i are invariant subspaces of H, such that $H = \sum_i \oplus H_i$ and if each T_i is a subrepresentation of T. ▼

We write in this case

$$T = \sum_i \oplus T_i. \tag{3}$$

A representation T of G in H is said to be *fully* or *completely reducible* (or *discretely decomposable*) if it can be expressed as a direct sum of irreducible subrepresentations. Finite-dimensional representations which are reducible but not fully reducible are called *indecomposable* representations (e.g. example 1). If we take as basis vectors in $\sum_i \oplus H_i$, the basis vectors of orthogonal subspaces H_i, we see that the matrix representation of a completely reducible representation is of the form

$$D(x) = \begin{bmatrix} D^1(x) & 0 & \dots & 0 & \dots \\ 0 & D^2(x) & \dots & 0 & \dots \\ 0 & 0 & \dots & D^i(x) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \tag{4}$$

where each $D^i(x)$ is irreducible. Example 1 shows that a reducible representation need not be completely reducible. However, for unitary *finite-dimensional* representations, we have

COROLLARY 2. *A finite-dimensional unitary representation of any group is completely reducible.*

PROOF: If H_1 is a proper, invariant subspace of H, then, by proposition 1, $H_1^\perp$ is also invariant and $H = H_1 \oplus H_1^\perp$. If H_1 or $H_1^\perp$ contains a proper invariant subspace, then we use again proposition 1 until we obtain a decomposition into irreducible invariant subspaces of H_1, provided this procedure converges (hence finite-dimensional case). ▼

The following proposition is fundamental in the group representation theory:

PROPOSITION 3 (Schur's lemma). *Let T and T' be unitary, irreducible representations of G in H and H', respectively. If S is a bounded linear map of $H \to H'$ such that*

$$ST_x = T'_x S \quad \textit{for every } x \in G, \tag{5}$$

then, either S is an isomorphism of the Hilbert spaces H and H' (i.e., $T \simeq T'$), or $S = 0$.

PROOF: The adjoint of eq. (5) gives $TS^* = S^*T'$. Hence, the positive definite, hermitian operator $V = S^*S$ commutes with T.

If $V = \int \lambda dE(\lambda)$ is the spectral decomposition of V, then $TE(\lambda) = E(\lambda)T$. Therefore every closed subspace $H(\lambda) = E(\lambda)H$ is invariant. Since H is irreducible, $H(\lambda)$ coincides with H or with the null-space $\{0\}$. This implies $V = \lambda I$. Similarly, one obtains $V' = SS^* = \lambda' I'$. Because $\lambda S = SS^*S = \lambda' S$, either $\lambda = \lambda'$ if $S \neq 0$, or $S = 0$ otherwise. In the first case setting $U = \lambda^{-1/2}S$ we obtain

$U^*U = I$ and $UU^* = I'$; hence S is an isomorphism of H and H' and according to def. 2.1, we have $T \simeq T'$. ▼

The Schur's Lemma 3 implies the following criterion of irreducibility:

PROPOSITION 4 (Schur's lemma—unitary case). *A unitary representation T of G in H is irreducible if and only if the only operators commuting with all the T_x are scalar multiples of the identity.*

PROOF: If $ST_x = T_xS$, then $S^*T_x = T_xS^*$. Hence, the self-adjoint operators $S_1 = \frac{1}{2}(S+S^*)$ and $S_2 = \frac{1}{2i}(S-S^*)$ also commute with all the T_x. Therefore, $S = \lambda_1 I + \lambda_2 I = \lambda I$, by proposition 3. Conversely, if every operator S commuting with T has the form λI, then the projection operator P commuting with T is either I or 0. Hence, by proposition 1.2°, the only closed invariant subspaces are the null-space or the whole carrier space H. Consequently T is irreducible. ▼

The result of the proposition 4 allows us to give a new definition of irreducibility:

DEFINITION 2′ A unitary representation T of G in H is said to be *irreducible* if the only operators, which commute with all the T_x, are scalar multiples of the identity. This formulation of irreducibility is called *operator irreducibility* of T. ▼

The proposition 4 has the following analogon for finite-dimensional (unitary or not) representations.

PROPOSITION 5 (Schur's lemma—finite-dimensional case). *Let T be an irreducible representation of G in H,* $\dim H < \infty$. *The only operators which commute with all T_x are scalar multiples of the identity.*

PROOF: Let

$$ST_x = T_xS \qquad \text{for all } x \in G, \tag{6}$$

and let $N = \{u \in H\colon Su = 0\}$. By virtue of (6) we have

$$\{0\} = T_xSN = ST_xN.$$

Therefore, $T_xN \subset N$, i.e., N is invariant subspace of H. Because T is irreducible $N = \{0\}$ or H. Hence, S is either an isomorphism or $S = 0$. Now, let S be any isomorphism which commutes with all T_x and let $\lambda \neq 0$ be an eigenvalue of S· clearly, $(S-\lambda I)$ is not an isomorphism of H; hence $(S-\lambda I) = 0$. ▼

Reducible Representations

There exists a useful classification of reducible representations according to properties of the center $CR(T, T)$ of the algebra $R(T, T)$ of intertwining operators. We start with the case when $CR(T, T)$ is minimal.

DEFINITION 4. A representation T of G is said to be a *factor representation* if the center of $R(T, T)$ contains only multiples of the identity. Representations of this type are called the *primary representations.*

Clearly, an irreducible representation is a factor representation. An interesting feature of factor representations is given by the following

PROPOSITION 6. *Let T be a factor representation which contains an irreducible subrepresentation V. Then, there exists an integer n, $n = 1, 2, 3, \ldots$, such that $T \simeq nV \equiv V \oplus V \oplus V \oplus \ldots + \oplus V$ (n terms).* ▼

(For the proof cf., e.g., Pozzi 1966, proposition 6.14.)

This result implies the following definition of the so-called type I factor representations.

DEFINITION 5. If a factor representation contains an irreducible subrepresentation, it is said to be of *type I*. A group G is said to be of *type I* if it has only type I factor representations.

In this book we shall deal exclusively with factor representations of type I. The factor representations of type I appear most often in applications in the problem of the reduction of the tensor products of representations. We need, then, additional invariant operators (quantum numbers) to split out the factor nT^i onto its irreducible parts. We deal with this problem in sec. 6 and ch. 18, § 2.

Another interesting class of representations is obtained if the center of $R(T, T)$ is as large as possible, i.e., if it coincides with the whole $R(T, T)$.

DEFINITION 6. A unitary representation T is said to be *multiplicity-free*, if $R(T, T)$ is commutative.

Note that if T is both a factor and multiplicity free, then $R(T, T) = \{\lambda I\}$, i.e., T is irreducible by virtue of Schur's lemma.

If T is multiplicity free and discretely decomposable, then

$$T = \sum_i \oplus T^i,$$

where all T^i are mutually inequivalent and irreducible. This follows directly from the fact that $R(T, T)$ contains in this case the maximal number of mutually commuting invariant operators. Thus, in contradistinction to factor representations, in the case of the multiplicity-free representations the generators of $R(T, T)$ provide a labelling of representations.

§ 4. Cyclic Representations

It is useful, in the analysis of the properties of representations of a given group G, to decompose any representation T into more elementary constituents. The cyclic representations may be used for this purpose.

A representation T of G in H is said to be *cyclic* if there is a vector $v \in H$ (called a *cyclic vector* for T), such that the closure of the linear span of all $T_x v$ is H itself. The following theorem allows us to restrict our attention, in the case of unitary representations, to cyclic representations only.

THEOREM 1. *Every unitary representation T of G in H is a direct sum of cyclic subrepresentations.*

PROOF: Let $v_1 \in H$ be any non-zero vector and let H_{v_1} be the closure of the linear span of all vectors $T_x v_1$, $x \in G$. The space H_{v_1} is invariant relative to T. Indeed, let $\tilde{H}_{v_1}$ be the linear span of all vectors $T_x v_1$; then, for each $u \in H_{v_1}$ there exists a sequence $\{u_n\}$ of vectors $u_n \in \tilde{H}_{v_1}$, which converges to u. Clearly $T_x u_n \in \tilde{H}_{v_1}$. The continuity of each T_x implies $T_x u_n \to T_x u$. Hence, the vector $T_x u \in H_{v_1}$ and consequently H_{v_1} is invariant. Thus, the subrepresentation ${}^{H_{v_1}}T$ is cyclic and v_1 is the cyclic vector for it. If $H_{v_1} = H$, the proof is completed. Otherwise, choose any non-zero vector v_2 in $H_{v_1}^{\perp} = H - H_{v_1}$ and consider the closed linear span H_{v_2}, which is invariant relative to T and orthogonal to H_{v_1}, and so forth.

Let τ denote the family of all collections $\{H_{v_i}\}$, each composed of a sequence of mutually orthogonal, invariant and cyclic subspaces and order the family τ by means of the inclusion relation $\subset$. Then τ is an ordered set to which Zorn's lemma (cf. app. A.1) applies, which assures the existence of a maximal collection $\{H_{v_i}\}_{\max}$. By the separability of H, there can be at most a countable number of subspaces in $\{H_{v_i}\}_{\max}$, and their direct sum, by the maximality of $\{H_{v_i}\}_{\max}$, must coincide with H. ▼

Using the th. 1, we can give now a convenient criterion for the irreducibility of a unitary representation.

PROPOSITION 2. *A unitary representation T of G in H is irreducible if and only if every non-zero vector $u \in H$ is cyclic for T.*

PROOF: If T is irreducible, then by the proof of th. 1, every non-zero vector is cyclic for T. In order to prove the converse statement, suppose that H_1 is a non-trivial invariant subspace of H and choose a vector $0 \neq v_1 \in H_1$. Due to the invariance of H_1 we have $T_x v_1 \in H_1$, moreover the closure of the linear span of all $T_x v_1$, which by assumption is the whole H, is contained in H_1. Hence, we have a contradiction. Therefore H does not contain a nontrivial invariant subspace and consequently T is irreducible. ▼

We have, moreover, the following convenient criterion for the unitary equivalence of cyclic representations.

PROPOSITION 3. *Let T and T' be unitary cyclic representations of G in H and H' with cyclic vectors $v \in H$ and $v' \in H'$, respectively.*

If

$$(T_x v, v)_H = (T'_x v', v')_{H'} \quad \textit{for every } x \in G, \tag{1}$$

then T is unitarily equivalent to T'.

PROOF: Let $\tilde{H}$ (resp. $\tilde{H}'$) be the linear span of all vectors $T_x v$ (resp. $T_x v'$), which is dense in H (resp. H'). Then any $u \in \tilde{H}$ is of the form

$$u = \sum_{i=1}^{n} \alpha_i T_{x_i} v. \tag{2}$$

We define a map S by the formula

$$Su = \sum_{i=1}^{n} \alpha_i T'_{x_i} v'. \tag{3}$$

Then, by eqs. (3) and (1):

$$\begin{aligned}
||Su||^2_{H'} &= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T'_{x_i} v', T'_{x_j} v')_{\tilde{H}'} \\
&= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T'_{x_j^{-1} x_i} v', v')_{\tilde{H}'} \\
&= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T_{x_j^{-1} x_i} v, v)_{\tilde{H}}. \\
&= ||u||^2_{\tilde{H}}.
\end{aligned}$$

Therefore, S is a linear isometric (hence, continuous) map, such that $ST_x u = T'_x Su$ for all $u \in H$. Thus, S can be uniquely extended to a unitary map $\bar{S}$ from H onto H', such that $\bar{S}T = T'\bar{S}$. Consequently T is unitarily equivalent to T'.▼

§ 5. Tensor Product of Representations

A. *Tensor Product of Spaces and Operators*

Definition 1. Let $\overset{1}{E}$ end $\overset{2}{E}$ be two vector spaces. Let $\overset{1}{E}\Box\overset{2}{E}$ be a vector space whose elements are formal linear combinations

$$\sum c_{x,y}(x, y), \quad x \in \overset{1}{E},\ y \in \overset{2}{E},$$

with a finite number of coefficients $c_{x,y} \in C$ different from zero. Let N denote the subspace of $\overset{1}{E}\Box\overset{2}{E}$ spanned by all vectors of the form

$$(x, y_1+y_2)-(x, y_1)-(x, y_2), \quad (x_1+x_2, y)-(x_1, y)-(x_2, y),$$
$$(\lambda x, y)-\lambda(x, y), \quad (x, \lambda y)-\lambda(x, y).$$

The tensor product is then defined as the quotient space.

$$\overset{1}{E}\otimes\overset{2}{E} = \overset{1}{E}\Box\overset{2}{E}/N. \blacktriangledown$$

Let φ_2 be the restriction of the canonical map $\varphi: \overset{1}{E}\Box\overset{2}{E} \to \overset{1}{E}\otimes\overset{2}{E}$ to the Cartesian product space $\overset{1}{E}\times\overset{2}{E}$: then we set $\varphi[(x, y)] \equiv x\otimes y$. We have

$$\begin{aligned}&\text{(i)}\ x\otimes(y_1+y_2) = x\otimes y_1 + x\otimes y_2,\\ &\text{(ii)}\ (x_1+x_2)\otimes y = x_1\otimes y + x_2\otimes y,\\ &\text{(iii)}\ (\lambda x)\otimes y = x\otimes(\lambda y) = \lambda(x\otimes y).\end{aligned} \tag{1}$$

Let $\{\overset{k}{e}_i\}_{i=1}^{\dim \overset{k}{E}}$, $k = 1, 2$, be bases in $\overset{1}{E}$ and $\overset{2}{E}$ respectively. Then the map φ associates with each pair $(\overset{1}{e}_i, \overset{2}{e}_k)$ in $\overset{1}{E}\times\overset{2}{E}$ an element $\overset{1}{e}_i\otimes\overset{2}{e}_k$; for $x = x^i \overset{1}{e}_i$ and $y = y^k \overset{2}{e}_k$ where only finite number of coordinates is different from zero, we have

$$x\otimes y = x^i y^k \overset{1}{e}_i\otimes\overset{2}{e}_k. \tag{2}$$

If $\overset{1}{E}$ and $\overset{2}{E}$ are Hilbert spaces with scalar products $(\cdot\,,\cdot)_i$, $i = 1, 2$, then the scalar product in $\overset{1}{E}\otimes\overset{2}{E}$ can be defined by the formula

$$(x_1\otimes y_1, x_2\otimes y_2) \equiv (x_1, x_2)_1 (y_1, y_2)_2. \tag{3}$$

If either $\overset{1}{E}$ or $\overset{2}{E}$ is finite-dimensional, then the space $\overset{1}{E}\otimes\overset{2}{E}$ equipped with the scalar product (3) is complete. If both $\overset{1}{E}$ and $\overset{2}{E}$ are infinite-dimensional, we complete $\overset{1}{E}\otimes\overset{2}{E}$ in the norm defined by (3) and denote it by the symbol $\overset{1}{E}\dot{\otimes}\overset{2}{E}$.

EXAMPLE 1. Let $\overset{i}{H} = L^2(\overset{i}{\Omega}, \overset{i}{\mu})$, where $\overset{i}{\Omega}$ are open subsets of R^n and $\overset{i}{\mu}$ are measures on Ω_i. Let $\overset{1}{\mu}\otimes\overset{2}{\mu}$ denote the product measure on $\overset{1}{\Omega}\times\overset{2}{\Omega}$. Then

$$L^2(\overset{1}{\Omega}, \overset{1}{\mu})\dot{\otimes}L^2(\overset{2}{\Omega}, \overset{2}{\mu}) = L^2(\overset{1}{\Omega}\times\overset{2}{\Omega}, \overset{1}{\mu}\otimes\overset{2}{\mu}).\blacktriangledown$$

If A and B are bounded operators in $\overset{1}{E}$ and $\overset{2}{E}$, respectively, then the tensor product $A\otimes B$ of A and B is defined in $\overset{1}{E}\dot{\otimes}\overset{2}{E}$ by the formula

$$(A\otimes B)(x\otimes y) \equiv Ax\otimes By. \tag{4}$$

It follows from eq. (4) that

$$(A\otimes B)(A'\otimes B') = AA'\otimes BB'. \tag{5}$$

B. *Tensor Product of Representations*

Let $\overset{1}{G}$ and $\overset{2}{G}$ be topological groups. Let $\overset{1}{g} \to \overset{1}{T}_g$ and $\overset{2}{g} \to \overset{2}{T}_g$ be representations of $\overset{1}{G}$ and $\overset{2}{G}$ acting in $\overset{1}{E}$ and $\overset{2}{E}$, respectively. We define in $\overset{1}{E}\otimes\overset{2}{E}$ the operator function

$$\overset{1}{G}\times\overset{2}{G} \ni (\overset{1}{g}, \overset{2}{g}) \to \overset{1}{T}_{\overset{1}{g}}\otimes\overset{2}{T}_{\overset{2}{g}}. \tag{6}$$

We have

$$\overset{1}{T}{}^{1}_{e}\otimes\overset{2}{T}{}^{2}_{e} = \overset{1}{I}\otimes\overset{2}{I}, \tag{7}$$

and by virtue of (5)

$$(\overset{1}{T}{}^{1}_{g}\otimes\overset{2}{T}{}^{2}_{g})(\overset{1}{T}{}^{1}_{g_0}\otimes\overset{2}{T}{}^{2}_{g_0}) = (\overset{1}{T}{}^{1}_{g}\overset{1}{T}{}^{1}_{g_0}\otimes\overset{2}{T}{}^{2}_{g}\overset{2}{T}{}^{2}_{g_0}) = \overset{1}{T}{}^{11}_{gg_0}\otimes\overset{2}{T}{}^{22}_{gg_0}. \tag{8}$$

Hence the map (6) provides a representation of the direct product group $G_1\times G_2$ in the tensor product space $\overset{1}{E}\otimes\overset{2}{E}$.

Definition 2. The representation (6) of $\overset{1}{G}\times\overset{2}{G}$ in the space $\overset{1}{E}\otimes\overset{2}{E}$ is called the *outer tensor product representation*. If $\overset{1}{G}=\overset{2}{G}$ and $\overset{1}{g}=\overset{2}{g}$ then the prepresentation (6) is called the *inner tensor product* (or *Kronecker product*) *representation.*

Let $\overset{1}{E}$ and $\overset{2}{E}$ be Hilbert spaces and let $H=\overset{1}{E}\dot{\otimes}\overset{2}{E}$ be the Hilbert space tensor product. It follows then from eqs. (6) and (4) that if the representations $\overset{1}{T}$ and $\overset{2}{T}$ of $\overset{1}{G}$ and $\overset{2}{G}$, respectively, are continuous, then $\overset{1}{T}\otimes\overset{2}{T}$ is also continuous.

Let $\overset{1}{e}_i\otimes\overset{2}{e}_k$ be the basis in the Hilbert space $\overset{1}{E}\dot{\otimes}\overset{2}{E}$. Then by virtue of eqs. (4) and (3), the matrix elements of the tensor product representation $\overset{1}{T}\otimes\overset{2}{T}$ have the form

$$\begin{aligned} D_{ik,jm}(\overset{1}{g},\overset{2}{g}) &= \big((\overset{1}{T}{}^{1}_{g}\otimes\overset{2}{T}{}^{2}_{g})(\overset{1}{e}_j\otimes\overset{2}{e}_m),\overset{1}{e}_i\otimes\overset{2}{e}_k\big)\\ &= (\overset{1}{T}{}^{1}_{g}\overset{1}{e}_j,\overset{1}{e}_i)_1(\overset{2}{T}{}^{2}_{g}\overset{2}{e}_m,\overset{2}{e}_k)_2 = \overset{1}{D}_{ij}(\overset{1}{g})\overset{2}{D}_{km}(\overset{2}{g}). \end{aligned} \tag{9}$$

Example 1. Let $\overset{1}{Q}$ be a relativistic particle with mass $\overset{1}{m}$. Its wave function $\overset{1}{\psi}(\overset{1}{p})$ in the momentum space is an element of the space $\overset{1}{E}=L^2(\overset{1}{\Omega},\overset{1}{\mu})$, where $\overset{1}{\Omega}$ is the mass hyperboloid $\overset{1}{p}{}^2=\overset{1}{m}{}^2$, and $\mathrm{d}\overset{1}{\mu}=\dfrac{\mathrm{d}^3\overset{1}{p}}{\overset{1}{p}_0}$. The Poincaré group $\Pi=T^4 \rtimes \mathrm{SO}(3,1)$ has the unitary continuous representation in $\overset{1}{E}$ given by the formula

$$\overset{1}{T}_{\{a,\Lambda\}}\overset{1}{\psi}(\overset{1}{p}) = \exp(ipa)\psi(\Lambda^{-1}\overset{1}{p}). \tag{10}$$

Let $\overset{2}{Q}$ be a second relativistic particle with mass $\overset{2}{m}$. The wave function $\psi(\overset{1}{p},\overset{2}{p})$ of two-particle systems is an element of the tensor product space $L^2(\overset{1}{\Omega},\overset{1}{\mu})\dot{\otimes}L^2(\overset{2}{\Omega},\overset{2}{\mu})$. One readily verifies using the def. 1 that

$$L^2(\overset{1}{\Omega},\overset{1}{\mu})\dot{\otimes}L^2(\overset{2}{\Omega},\overset{2}{\mu}) = L^2(\overset{1}{\Omega}\times\overset{2}{\Omega},\overset{1}{\mu}\otimes\overset{2}{\mu}), \tag{11}$$

where $\overset{1}{\mu}\otimes\overset{2}{\mu}$ denotes the product of measures $\overset{1}{\mu}$ and $\overset{2}{\mu}$. The tensor product representation $\overset{1}{T}_g\otimes\overset{2}{T}_g$ of the Poincaré group in the tensor product space (11), by virtue of eq. (4), is given by the formula

$$(\overset{1}{T}_{\{a,\Lambda\}}\otimes\overset{2}{T}_{\{a,\Lambda\}})\psi(\overset{1}{p},\overset{2}{p}) = \exp[i(\overset{1}{p}+\overset{2}{p})a]\psi(\Lambda^{-1}\overset{1}{p},\Lambda^{-1}\overset{2}{p}).$$

Evidently this representation is unitary and continuous in the tensor product space (11). ▼

Let us note that even if the representations $\overset{1}{T}$ and $\overset{2}{T}$ of G are irreducible, the inner tensor product $\overset{1}{T}\otimes\overset{2}{T}$ is in general highly reducible, i.e.,

$$\overset{1}{T}_g\otimes\overset{2}{T}_g \cong \oplus m_\lambda \overset{\lambda}{T}_g \tag{12}$$

where $\overset{\lambda}{T}$ are irreducible representations of G and m_λ is the multiplicity of $\overset{\lambda}{T}$ in the tensor product $\overset{1}{T}\otimes\overset{2}{T}$.

The determination of the multiplicities m_λ of $\overset{\lambda}{T}$ in the tensor product $\overset{1}{T}\otimes\overset{2}{T}$ is the 'Clebsch–Gordan series' problem. It is one of the most difficult problems in group representation theory, whose solution is known only for several groups and certain types of representations. Even for such important groups like the Lorentz group, this problem is not yet completely solved.

Let C be an operator in $\overset{1}{E}\otimes\overset{2}{E}$ which reduces the inner tensor product $\overset{1}{T}\otimes\overset{2}{T}$ of G to a block diagonal form, i.e.,

$$C(\overset{1}{T}\otimes\overset{2}{T})C^{-1} = \oplus m_\lambda \overset{\lambda}{T}, \quad C(\overset{1}{E}\otimes\overset{2}{E}) = \oplus m_\lambda \overset{\lambda}{E}. \tag{13}$$

The matrix elements of the operator C are called the 'Clebsch–Gordan coefficients'. They allow one to express the basis elements $\overset{\lambda}{e}_i$ of the carrier space $\overset{\lambda}{E}$ of an irreducible representation $\overset{\lambda}{T}$ in terms of the tensor basis $\overset{1}{e}_i\otimes\overset{2}{e}_k$. The Clebsch–Gordan coefficients play for physical symmetry groups (like rotation, Lorentz or Poincaré groups) a fundamental role in particle physics.

§ 6. Direct Integral Decomposition of Unitary Representations

Let $g \to T_g$ be a unitary representation of a physical symmetry group in a Hilbert space H. In applications in most cases the representation T is reducible. However, only the irreducible components $T(\lambda)$ of T have a more direct physical meaning. Hence it is of fundamental importance to have a formalism which provides the description of T in terms of its irreducible components.

In general the decomposition of a given reducible unitary representation onto a direct *sum* of irreducible representations is impossible, and one must use the concept of direct integral of representations and the direct integral of the corresponding carrier spaces. We illustrate this on a simple example.

Let G be the translation group of the real line R and let $g \to T_g$ be a unitary representation of G in the Hilbert space $H = L^2(R)$ given by

$$T_g u(x) = u(x+g). \tag{1}$$

By virtue of Schur's lemma every irreducible representation of G is one-dimensional (cf. proposition 6.1). Hence representation (1) is reducible. Suppose that H_1 is a one-dimensional invariant subspace in H. Then for every u_1 in H_1 we have

$$T_g u_1(x) = u_1(x+g) = \lambda_1(g) u_1(x).$$

Hence $u_1(x)$ must be an exponential function. But the only exponential function which is in $L^2(R^1)$ is $u_1 = 0$. Consequently, $H_1 = \{0\}$. Thus H does not contain one-dimensional invariant subspaces. However, if we pass to the direct integral of Hilbert spaces we find explicit one-dimensional spaces in which irreducible representations of G are realized. Indeed, let $A = \mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x}$ be a self-adjoint operator in H. Using the spectral theorem we know that A induces a decomposition of H onto the direct integral (cf. app. B.3)

$$H \leftrightarrow \hat{H} = \int_{\Lambda} H(\lambda)\mathrm{d}\mu(\lambda), \tag{2}$$

where Λ is the spectrum of A, $H(\lambda)$ are the one-dimensional Hilbert spaces and $\mathrm{d}\mu(\lambda)$ is the spectral measure associated with A. Each element u of $\hat{H}$ is a vector-function $u = \{u(\lambda),\ u(\lambda) \in H(\lambda)\}$. In the present case the connection between the elements $u(\lambda)$ of $H(\lambda)$ and $u(x) \in H$ are given by the ordinary Fourier transform:

$$u(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\mathrm{i}\lambda x) u(x)\mathrm{d}x.$$

We obtain the transformation law of the elements $u(\lambda)$ in $H(\lambda)$ by taking the Fourier transform of $T_g u(x)$. We have

$$(T_g u)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\mathrm{i}\lambda x)(T_g u)(x)\mathrm{d}x = \exp(-\mathrm{i}\lambda g) u(\lambda).$$

Hence every Hilbert space $H(\lambda)$ in the direct integral (2) is an invariant space of $T_g(\lambda)$. Thus the decomposition (2) of the carrier space H induced by the operator A implies the decomposition

$$T_g \leftrightarrow \hat{T}_g = \int T_g(\lambda)\mathrm{d}\mu(\lambda) \tag{3}$$

of T_g onto the direct integral of irreducible representations. Notice that the operator A (or more precisely, its spectral projections $E(\lambda)$) is in the commutant

T' of the representation T. Thus the decompositions (2) and (3) may be considered to be the direct integral decomposition implied by the abelian '*'-algebra T'.

We now give a general definition of a *direct integral of representations.* Let (Λ, μ) be a Borel space with a measure μ and let

$$\hat{H} = \int_\Lambda H(\lambda)\mathrm{d}\mu(\lambda)$$

be a direct integral of Hilbert space (cf. app. B, § 3). Suppose that for each $\lambda \in \Lambda$ an operator $T(\lambda)$ on $H(\lambda)$ is defined. We say that the operator field $\lambda \to T(\lambda)$ is *integrable* iff the following conditions are satisfied:

(i) The $\{T(\lambda)\}$ are uniformly bounded, i.e. there exists a number M such that

$$\|T(\lambda)\|_{H(\lambda)} \leqslant M \quad \text{for any } \lambda \in \Lambda.$$

(ii) Given any $u, v \in H$, the complex valued function $\lambda \to \big(T(\lambda)u(\lambda), v(\lambda)\big)_\lambda$ is μ-measurable.

Then it is possible to define an operator T on $\hat{H}$ by setting

$$Tu \equiv \int_\Lambda T(\lambda)u(\lambda)\mathrm{d}\mu(\lambda) \quad \text{whenever } u = \int_\Lambda u(\lambda)\mathrm{d}\mu(\lambda). \tag{4}$$

Conditions (i) and (ii) assure that T is a bounded operator on $\hat{H}$, i.e. $Tu \in \hat{H}$ for $u \in \hat{H}$ and $\|T\|_{\hat{H}} \leqslant M$.

Let G be a group and Λ a Borel space with a measure μ. Suppose that for every $\lambda \in \Lambda$ a unitary representation $T(\lambda)$ of G is given in $H(\lambda)$: we say that the representation field $\lambda \to T(\lambda)$ is *integrable* iff for any $g \in G$ the operator field $\lambda \to T_g(\lambda)$ is integrable. Since $\|T_g(\lambda)\| = 1$ a representation field $T_g(\lambda)$ is integrable iff the function $\big(T_g(\lambda)u(\lambda), v(\lambda)\big)_\lambda$ is μ-integrable.

For any integrable representation field we can define an operator

$$T_g = \int_\Lambda T_g(\lambda)\mathrm{d}\mu(\lambda) \tag{5}$$

in $\hat{H}$. It is evident from definition that

1° $T_e = I$,

2° $T_{g_1 g_2} = T_{g_1}T_{g_2}$,

3° $T_g^* = T_g^{-1}$.

Hence the map $g \to T_g$ in H provides a unitary representation of G in H.

The importance of the concept of a direct integral of representations follows from the following

THEOREM 0. *Every representation T of a separable locally compact group G is a direct integral of irreducible representations*

$$T = \int_\Lambda T(\lambda)\mathrm{d}\mu(\lambda), \tag{6}$$

where (Λ, μ) is some measure space and the $T(\lambda)$ are irreducible.

If G is of type I then the decomposition in (6) *is essentially unique.* ▼

(For the proof cf. Mackey (1955), ch. 1, § 4.)

We shall now elaborate the general formalism of the decomposition of a unitary representation $g \to T_g$ of G onto irreducible components. This formalism is based on a theorem of von Neumann on diagonal and decomposable operators (cf. app. B.3).

The basic steps of the decomposition of a reducible unitary representation T_g onto irreducible components are the following:

(i) Consider a '$*$'-algebra T of operators generated by T_g

$$T = \left\{ \sum_{i=1}^{n} c_i T_{x_i} :\ c_i \in C \right\}.$$

(ii) Find an abelian '$*$'-subalgebra $\mathscr{A}$ in the commutant T' of T.

(iii) Apply the theorem of von Neumann and obtain a decomposition of H onto a direct integral of Hilbert spaces:

$$H \leftrightarrow \hat{H} = \int H(\lambda) \mathrm{d}\mu(\lambda) \tag{7}$$

implied by the algebra $\mathscr{A}$.

Because T_g is in $\mathscr{A}'$, T_g is a decomposable operator. Hence

$$T_g \leftrightarrow \hat{T}_g = \int T_g(\lambda) \mathrm{d}\mu(\lambda). \tag{8}$$

Clearly if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two abelian '$*$'-algebras in T', and $\mathscr{A}_1 \subset \mathscr{A}_2$, then the decomposition $\int_{\Lambda_2} T(\lambda_2) \mathrm{d}\overset{2}{\mu}(\lambda_2)$ implied by $\mathscr{A}_2$ is a refinement of the decomposition $\int_{\Lambda_1} T(\lambda_1) \mathrm{d}\overset{1}{\mu}(\lambda_1)$ implied by $\mathscr{A}_1$. One may expect a most effective decomposition in the case when $\mathscr{A}$ is an abelian maximal '$*$'-algebra in T'. This is the content of the following fundamental theorem.

THEOREM 1 (Mautner). *Let G be a separable locally compact group. Let $g \to T_g$ be a continuous unitary representation of G in a Hilbert space H. Let $\mathscr{A}$ be an abelian '$*$'-algebra in the commutant T' of T. Then*

(i) *there exists a direct integral decomposition of H and T given by eqs.* (4) *and* (5), *respectively,*

(ii) *$T(\lambda)$ are μ-a.a. irreducible in $H(\lambda)$ if and only if $\mathscr{A}$ is maximal in T'.*

SKETCH OF A PROOF: The decomposition (7) and (8) of H and T onto direct integral (5) is a direct consequence of von Neumann theorem (cf. app. B.3). We now show that maximality of $\mathscr{A}$ implies essentially the irreducibility of $T(\lambda)$.

Let $\Lambda_0 \subset \Lambda$ be a set which has a positive measure $\mu(\Lambda_0) > 0$. For each $\lambda_0 \in \Lambda_0$, let $B(\lambda_0) \neq I(\lambda_0)$ be a bounded operator in $H(\lambda)$ which commutes with each $T_g(\lambda_0)$, $g \in G$, and put $B(\lambda) = 0$ for $\lambda \in \Lambda - \Lambda_0$. One may show that there exists a measurable operator field $B(\lambda)$. Set $B \equiv \int_{\Lambda_0} B(\lambda)\,\mathrm{d}\mu(\lambda)$. Because B is not diagonal in $H = \int_{\Lambda} \hat{H}(\lambda)\,\mathrm{d}\mu(\lambda)$, $B \notin \mathscr{A}$, but since B is decomposable $B \in \mathscr{A}'$. Thus the algebra $\mathscr{A} \cup \mathscr{B}$ is a commutative subalgebra in T' and $\mathscr{A} \cup \mathscr{B}$ contains properly $\mathscr{A}$. Thus $\mathscr{A}$ is not maximal in T'. Consequently, the maximality of $\mathscr{A}$ implies the irreducibility of $T(\lambda)$ for μ-a.a. λ.

In order to show the converse, let

$$T = \int_{\Lambda} T(\lambda)\,\mathrm{d}\mu(\lambda)$$

where for μ-.a.a. λ the representations $T(\lambda)$ are irreducible in $\tilde{H}(\lambda)$. If the commutative algebra $\mathscr{A}$ is not maximal, then there exists an orthogonal nontrivial projection $E \in (T' - \mathscr{A})$ commuting with $\mathscr{A}$. Consequently,

$$E = \int_{\Lambda_0} E(\lambda)\,\mathrm{d}\mu(\lambda),$$

where $E(\lambda)$ is a non-zero projection in $H(\lambda)$ for λ in a certain set Λ_0 with $\mu(\Lambda_0) > 0$. Now, because $E \in T'$, we have

$$E(\lambda)\,T(\lambda) = T(\lambda)E(\lambda).$$

Because for μ-a.a. λ, $T(\lambda)$ is irreducible, $E(\lambda) = I$ and we obtain a contradiction. Hence $\mathscr{A}$ must be maximal. ▼

(For the complete proof cf., e.g., Mackey 1955.)

The Mautner theorem plays an important role in group representation theory, and in its applications.

The following fundamental theorem which is a direct consequence of the theorem of Mautner shows that a topological group has always nontrivial irreducible representations.

THEOREM 2 (the Gel'fand–Raikov theorem). *Let G be a separable topological group. Then for every two elements $g_1, g_2 \in G$, $g_1 \neq g_2$, there exists an irreducible representation $g \to T_g$ of G such that $T_{g_1} \neq T_{g_2}$.*

PROOF: Let T^L be a left-regular representation of G in $L^2(G)$. Because T^L is faithful, $T^L_{g_1} \neq T^L_{g_2}$. Let

$$T^L_g = \int_{\Lambda} T_g(\lambda)\,\mathrm{d}\mu(\lambda)$$

be a direct integral decomposition of T^L. If $T_{g_1}(\lambda) = T_{g_2}(\lambda)$ for μ-a.a. λ, $\lambda \in \Lambda$, then $T^L_{g_1}$ would be equal to $T^L_{g_2}$ which is a contradiction. ▼

The Gel'fand–Raikov theorem was proved (by technique of positive definite functions) at the beginnings of the development of group representation theory in 1943. It presents one of the most important results in representation theory. In case of abelian groups, th. 2 gives:

COROLLARY. *Let G be abelian separable topological group. Then for each $g_1 \neq g_2$ there exists a character $\chi(g)$ such that $\chi(g_1) \neq \chi(g_2)$.*

It should be stressed that the selection of a maximal commuting algebra $\mathscr{A}$ in T' is nonunique. An explicit example of a selection of different, unitarily non-equivalent, sets of commuting operators in T' is given in eq. 9.6(11).

There is a general feeling among physicists that a set of invariant operators of a group G (and consequently also T') is commutative; the following is a counterexample.

EXAMPLE 1. Let K be a closed subgroup of a Lie group G such that $X = G/K = \{gK, g \in G\}$ possesses an invariant measure μ. Let $N(K)$ be the normalizer of K in G, i.e., the set of all $n \in G$ such that $nKn^{-1} \subset K$. Let $H = L^2(X, \mu)$ and let $g \to T_g$ be a unitary representation of G in H given by

$$T_g u(x) = u(g^{-1}x). \tag{9}$$

Let T_n^R, $n \in N(K)$, be an operator in H defined by

$$T_n^R u(gK) = u(gKn) = u(gnK).$$

Then

$$\begin{aligned}(T_n^R T_{g_0})u(gK) &= (T_{g_0})u(gKn) = u(g_0^{-1}gKn) \\ &= (T_{g_0}T_n^R u)(gK).\end{aligned} \tag{10}$$

Hence the right translations by elements $n \in N(K)$ are well defined in X and commute with all T_{g_0}, $g_0 \in G$. Thus every right translation T_n^R, $n \in N(K)$, is in the commutant T'. Hence if $N(K)/K$ is a noncommutative subgroup, then T is non-abelian. ▼

One may expect that the decomposition (8) would be simplest in the case T' is abelian. The representations with this property are called *multiplicity-free*. In this case, we can take $\mathscr{A} = T'$ and obtain an essentially unique decomposition of T onto irreducible components. The following theorem shows that for type I groups we have essentially a unique decomposition of the representation T onto its irreducible components:

THEOREM 3. *Let G be a separable, topological type I group. Let $g \to T_g$ be a unitary representation of G in a Hilbert space H, and let $\hat{G}$ be the set of equivalence classes of irreducible representations. Then there exists a standard Borel measure $\hat{\mu}$ on $\hat{G}$ and a function $\hat{n}(\lambda)$ on $\hat{G}$ such that*

$$H \leftrightarrow H = \int_{\hat{G}} H(\lambda)\hat{n}(\lambda)\,d\hat{\mu}(\lambda)$$

and

$$T \leftrightarrow \hat{T} = \int_{\hat{G}} T(\lambda)\hat{n}(\lambda)\,d\hat{\mu}(\lambda). ▼$$

(For the proof cf. Maurin 1968, ch. V, § 2.)

§ 7. Comments and Supplements

A. *Some Generalizations of Continuous Unitary Representations*

So far we have discussed the properties of unitary, continuous linear representations of a topological group G. One could ask about the extensions of the theory if some of these conditions imposed on the representatives T_x of G are relaxed. Firstly, one can construct homomorphisms of G into $L(H)$ for which $T_e \neq I$; indeed, if $G = R^1$, for example, then the map

$$R^1 \ni x \to T_x = \begin{bmatrix} \exp(ix) & 0 \\ 0 & 0 \end{bmatrix}, \quad 0 = \text{zero operator in a subspace of } H,$$

satisfies the condition $T_{xy} = T_x T_y$ in any Hilbert space, but $T_e \neq I$. The following proposition shows, however, that we practically lose nothing by imposing the condition $T_e = I$. Indeed, we have

PROPOSITION 1. *Let $x \to T_x$ be a homomorphism of G into $L(H)$. Then H is the direct sum $H_1 \oplus H_0$ of invariant subspaces and T has the form*

$$T_x = \begin{bmatrix} \overset{1}{T}_x & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where $\overset{1}{T}$ is the representation of G in H_1. ▼

PROOF: Set $H_1 = \{u \in H\colon T_e u = u\}$ and $H_0 = \{u \in H\colon T_e u = 0\}$. Clearly, $H_1 \cap H_0 = \{0\}$. For arbitrary u in H we have $u = T_e u + (u - T_e u)$, where $T_e(T_e u) = T_e u$ and $T_e(u - T_e u) = 0$. Hence, H is the direct sum $H_1 \oplus H_0$. It is evident that H_1 and H_0 are closed invariant subspaces of H. For u in H_0 implies $T_x u = T_x T_e u = 0$. Thus, $T_x H_0 = \{0\}$, and the map $x \to T_x$ takes the form (1). ▼

Secondly, we might drop the continuity condition. In order to see what happens in this case we introduce the notion of the so-called *measurable representations*. Let μ be the Haar measure on G and let T be a unitary representation of G in a separable Hilbert space H. T is said to be *μ-measurable* if the function $x \to (T_x u, v)$ is μ-measurable for all u, v in H. We have

PROPOSITION 2. *A unitary representation T is continuous iff it is μ-measurable.* ▼ (For the proof cf. Hewitt and Ross 1963, I (22.20b)).

This result shows that discontinuous representations must be nonmeasurable. Because for nonmeasurable functions we have only existence theorems, we do not expect explicit constructive realizations of discontinuous representations (cf. example 1.3). Hence, their physical meaning is doubtful.* One could obtain, however, measurable discontinuous representations if one would admit nonseparable Hilbert spaces; for example, the von Neumann infinite tensor product of Hilbert spaces $\prod_i \otimes H_i$.

An interesting characterization of unitary representations in nonseparable Hilbert spaces is given by the following

PROPOSITION 3. *Let T be a unitary, μ-measurable representation of a locally compact group G on a nonseparable Hilbert space H. Let H_s be the subspace of all vectors u in H such that for all φ in $L^1(G)$ and all v in H we have*

$$\int \varphi(x)(T_x u, v)\,\mathrm{d}\mu(x) = 0.$$

Then,

1° *H is the direct sum of two invariant subspaces: $H = H_c \oplus H_s$.*

2° *The representation ${}^{H_c}T$ is continuous. The representation ${}^{H_s}T$ is singular in the sense that the map $x \to (T_x u, u)$ is equivalent to zero for all u in H_s.***

3° *If H is separable, then the subspace H_s is absent.* ▼

(For the prof cf. Segal and Kunze 1968.)

B. *Comments*

(i) In mathematics equivalent representations T and T' of a group G are indistinguishable. However, in physics representations which are unitarily equivalent are not necessarily physically equivalent. Indeed, let H be a Hamiltonian of a system of two interacting non-relativistic particles, or of a particle in a potential field. Suppose that for such an interaction, H has a continuous spectrum only (i.e., absence of bound states). Then, there exists a unitary scattering operator S such that

$$SH_0 = HS,$$

where H_0 is the free Hamiltonian. Hence, the time displacement operators $U_t = \exp(itH)$ and $U_t^0 = \exp(itH_0)$ are unitarily equivalent. However, they are not physically equivalent as they describe the time evolution of essentially different physical systems. The same conclusion holds if one considers equivalent representations of the Galilei group or the Poincaré group. The reason for this distinction lies in the fact that in physics we do not use abstract groups but groups whose generators are identified with physical observables. The same group can be used to describe different physical situations.

(ii) In § 3 we described type I factor representations. Other types of factor representations can be described by using the notion of *finite representations.* If $\infty T \simeq T$ we say that T is *infinite.* If no one subrepresentation of T is infinite we say that T is *finite.* A factor representation T, which has a finite subrepresentation but no irreducible subrepresentation is said to be of type II.

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* A noncontinuous three-dimensional representation of the rotation group SO(3) is obtained if we replace in the matrix elements the functions $\cos\varphi$ and $\sin\varphi$ by $\cos f(\varphi)$, $\sin f(\varphi)$ where $f(\varphi_1+\varphi_2) = f(\varphi_1)+f(\varphi_2)$, and $f(\varphi)$ discontinuous.

** A function $f(x)$ on G (in our case $f(x) = (T_x u, u)$) is *equivalent* to zero, if for μ-almost all $x \in G$. $f(x) = 0$

Finally if T is not irreducible, but every proper subrepresentation of T is equivalent to T, then T is said to be the *factor representation* of type III. Such a T is necessarily a factor representation and infinite.

A general representation T of G need not belong to any of the above three types. We have, however,

THEOREM 1. *Let T be any unitary representation of G. Then, there exists uniquely determined projections P_1, P_2 and P_3 in the center of $R(T, T)$ such that*

(i) $P_1+P_2+P_3 = I$.

(ii) ${}^{P_1H}T$, ${}^{P_2H}T$ *and* ${}^{P_3H}T$ *are of type I, II and III, respectively.*

(iii) *For $i \neq j$ no subrepresentation of ${}^{P_iH}T$ is equivalent to a subrepresentation of ${}^{P_jH}T$.* ▼

This theorem shows that we can restrict our analysis to type I, II, and III factor representations.

In almost all applications we encounter type I representations only. The properties of type II and III factor representations are less intuitive. A relatively simple construction of factor II representations is given in Naimark's book (1970, § 38, p. 484). Explicit examples of factor representations of type III were recently also constructed (cf. Dixmier 1969).

It is interesting that in relativistic quantum field theory we probably cannot avoid the use of type III factors. In fact, Araki and Woods have recently shown that the representation of the canonical commutation relations of a scalar relativistic quantum field leads to type III factors (cf. Araki and Woods 1966).

The beautiful exposition of the theory of factors and its applications to group representation theory is presented in Dixmier's book (1969).

§ 8. Exercises

§ 1.1. Show that the matrix elements of irreducible representations of SO(3) group have the form

$$D^J_{MM'}(\varphi, \vartheta, \psi) = \exp(-\mathrm{i}M\varphi)d^J_{MM'}(\vartheta)\exp(-\mathrm{i}M\psi), \tag{1}$$

where $\varphi \in [0, 2\pi)$, $\vartheta \in [0, \pi)$, $\psi \in [0, 2\pi]$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, M, M' = -J, -J+1, \ldots, J-1, J$ and

$$d^J_{MM'}(\vartheta) = \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0,2M}_{J-M}(\cos\vartheta). \tag{2}$$

Here $P^{\alpha,\beta}_\gamma(x)$ it the Jacobi polynomial.

§ 5.1. Show that the 'Clebsch–Gordan series' for SO(3) has the following form:

$$T^{J_1}\otimes T^{J_2} = \sum_{J=|J_1-J_2|}^{|J_1+J_2|} T^J. \tag{3}$$

Chapter 6

Representations of Commutative Groups

We begin the analysis of the representation theory of locally compact groups with the commutative groups. The commutativity of the group multiplication implies a considerable simplification of representation theory. This does not make, however, the theory trivial, because in most cases we have to use direct integrals to describe the properties of the representations.

§ 1. Irreducible Representations and Characters

We first show the simple but fundamental property of irreducible representations of abelian groups. Unless stated otherwise we consider locally compact abelian group.

PROPOSITION 1. *Any irreducible unitary representation of an abelian group G in complex space is one-dimensional.*

PROOF: For every $x \in G$ and a fixed $y \in G$ we have

$$T_x T_y = T_{xy} = T_{yx} = T_y T_x. \tag{1}$$

Hence, by proposition 5.3.4

$$T_y = \alpha(y) I, \quad \alpha(y) \in C. \tag{2}$$

Consequently any one-dimensional subspace H_1 of the carrier space H is invariant. But T is irreducible; so H_1 must coincide with H. ▼

A *character* of an abelian locally compact group G is any continuous function $\hat{x}\colon G \to C$, which satisfies

$$|\hat{x}(x)| = 1, \tag{3}$$

$$\hat{x}(x_1 x_2) = \hat{x}(x_1)\hat{x}(x_2). \tag{4}$$

It follows from eqs. (3) and (4) that $\hat{x}(e) = 1$ and $\hat{x}(x^{-1}) = \overline{\hat{x}(x)} = \hat{x}(x)^{-1}$.

Therefore, a character is a one-dimensional continuous unitary representation of G.

The *dual space* $\hat{G}$ of an arbitrary group G, is the set of equivalent classes of all continuous, irreducible unitary representations of G. According to proposition 1 for abelian groups $\hat{G}$ consists then of all characters of G. If $\hat{x}_1$ and $\hat{x}_2$ are in $\hat{G}$, then the function $x \to (\hat{x}_1 \hat{x}_2)(x) = \hat{x}_1(x)\hat{x}_2(x)$ satisfies the conditions

(i) $|(\hat{x}_1 \hat{x}_2)(x)| = 1,$ (5)

(ii) $\hat{x}_1\hat{x}_2(xy) = \hat{x}_1(xy)\hat{x}_2(xy) = (\hat{x}_1\hat{x}_2)(x)(\hat{x}_1\hat{x}_2)(y)$. (6)

Moreover, because $\hat{x}^{-1}(x) = \overline{\hat{x}(x)}$, $\hat{G}$ is also an abelian group.

EXAMPLE 1. 1° Consider $G = R^n$ as an additive vector group. Then, every character $\hat{x}(\cdot)$ has the form

$$\hat{x}(x) = \exp i(\hat{x}_1 x_1 + \dots + \hat{x}_n x_n) = \exp[i(\hat{x}\cdot x)], \quad \hat{x} \in R^n . \tag{7}$$

Thus, the character group $\hat{G}$ is isomorphic with G.

2° If G is the multiplicative group of complex numbers of modulus one, $x = \exp(i\theta)$, then, every character has the form

$$\hat{x}(x) = \exp(in\theta), \quad n = \text{integers}, \tag{8}$$

Thus, in this case $\hat{G}$ is isomorphic with the additive group of integers. If n is an arbitrary real number, then the character (8) is a multi-valued function on G. It is however a single-valued representation of the covering group $\tilde{G} = R^1$ of G. ▼

In these examples the character group G is also a locally compact abelian topological group. One can show that this property holds for an arbitrary abelian locally compact group if we endow $\hat{G}$ with the topology of uniform convergence on compact sets (cf., e.g., Weil 1940, § 2g).

In order to take advantage of the symmetry between G and $\hat{G}$ we introduce a more symmetric notation for characters by setting $\hat{x}(x) = \langle x, \hat{x}\rangle$. Then, eqs. (3)–(6) take the form:

$$|\langle x, \hat{x}\rangle| = 1, \tag{3'}$$

$$\langle x_1 x_2, \hat{x}\rangle = \langle x_1, \hat{x}\rangle\langle x_2, \hat{x}\rangle, \tag{4'}$$

$$\langle x, \hat{x}_1\hat{x}_2\rangle = \langle x, \hat{x}_1\rangle\langle x, \hat{x}_2\rangle. \tag{6'}$$

A complex character of an abelian locally compact group G is a representation of G in C.

§ 2. Stone and SNAG Theorems

We now derive a fundamental decomposition theorem for an arbitrary unitary representation of an abelian group.

THEOREM 1 (Stone, Naimark, Ambrose, Godement theorem). *Let T be an unitary continuous representation of an abelian locally compact group G in a Hilbert space H. Then, there exists on the character group $\hat{G}$ a spectral measure $E(\cdot)$ such that**

$$T_x = \int_{\hat{G}} \langle x, \hat{x}\rangle \, dE(\hat{x}). \tag{1}$$

* See app. B.3, for properties of spectral measure dE and von Neumann spectral theory.

PROOF: Let $u \in H$. Then the function $x \to (T_x, u, u)$ is positive definite; hence, by the Bôchner theorem, there exists a finite regular Borel measure $\mu_{u,u}$ on $\hat{G}$ such that

$$(T_x u, u) = \int_{\hat{G}} \langle x, \hat{x} \rangle \, \mathrm{d}\mu_{u,u}(\hat{x}),$$

and, in particular,

$$\int_{\hat{G}} \mathrm{d}\mu_{u,u} = \mu_{u,u}(\hat{G}) = (u, u).$$

Using the polar decomposition, one can write $(T_x u, v)$ as a linear combination of terms like $(T_x u', u')$. Hence there exists a unique complex measure $\mu_{u,v}$ such that

$$(T_x u, v) = \int_{\hat{G}} \langle x, \hat{x} \rangle \, \mathrm{d}\mu_{u,v}(\hat{x}).$$

Now fix any Borel set $\hat{B} \subset \hat{G}$; then $\mu_{u,v}(\hat{B})$ is a bilinear functional $F_{\hat{B}}(u, v)$ on H, which is hermitian since

$$F_{\hat{B}}(u, v) = \mu_{u,v}(\hat{B}) = \mu_{v,u}(\hat{B}) = F_{\hat{B}}(v, u)$$

and is bounded since

$$|F_{\hat{B}}(u, v)|^2 \leqslant \mu_{u,u}(\hat{B})\mu_{v,v}(\hat{B}) \leqslant ||u||^2 ||v||^2. \tag{2}$$

Indeed, set, for any real λ,

$$u' \equiv u + \lambda \mu_{v,u}(\hat{B}) v$$

and then it follows that

$$0 \leqslant \mu_{u',u'}(\hat{B}) = \mu_{u,u}(\hat{B}) + 2\lambda |\mu_{u,v}(\hat{B})|^2 + \lambda^2 |\mu_{u,v}(\hat{B})|^2 \mu_{v,v}(\hat{B}),$$

which implies that

$$|\mu_{u,v}(\hat{B})|^4 - \mu_{u,u}(\hat{B})\mu_{v,v}(\hat{B})|\mu_{u,v}(\hat{B})|^2 \leqslant 0$$

from which eq. (2) follows.

Hence, by the Riesz theorem, for each Borel set $\hat{B} \subset \hat{G}$ there exists an operator $E(\hat{B})$ on H such that for any $u, v \in H$ one has:

$$\big(E(\hat{B})u, v\big) = F_{\hat{B}}(u, v) = \mu_{u,v}(\hat{B}). \tag{3}$$

It is obvious that $[E(\hat{B})]^* = E(\hat{B})$; moreover, a simple calculation shows that

$$\big(E(\hat{B}) T_x u, v\big) = \int_{\hat{B}} \langle x, \hat{x} \rangle \big(E(\mathrm{d}\hat{x})u, v\big).$$

Therefore, if one sets, for every Borel set $\hat{B}_1$, $Q(\hat{B}_1) \equiv E(\hat{B} \cap \hat{B}_1)$ one has

$$\int_{\hat{B}} \langle x, \hat{x}\rangle (E(\mathrm{d}\hat{x})u, v) = \int_{\hat{G}} \langle x, \hat{x}\rangle (Q(\mathrm{d}\hat{x})u, v)$$

$$= (E(\hat{B})T_x u, v) = \int_{\hat{G}} \langle x, \hat{x}\rangle (E(\hat{B})E(\mathrm{d}\hat{x})u, v)$$

from which it is easy to conclude that

$$E(\hat{B} \cap \hat{B}_1) = E(\hat{B})E(\hat{B}_1).$$

One readily verifies that the operator function $\hat{B} \to E(\hat{B})$ satisfies all conditions imposed on the spectral measure (cf. § 3 of app. B). Hence by eq. (3) we have

$$(T_x u, v) = \int_{\hat{G}} \langle x, \hat{x}\rangle \mathrm{d}\mu_{u,v}(\hat{x}) = \int_{\hat{G}} \langle x, \hat{x}\rangle (E(\mathrm{d}\hat{x})u, v).$$

This equality implies the assertion of th. 1. ▼

In the special, but very important, case of the abelian vector groups we obtain

THEOREM 2 (Stone's theorem). *Consider $G = R^n$ as an additive vector group, and let T be a unitary continuous representation of G in a Hilbert space H. Then, there exists a unique set of mutually strongly commuting self-adjoint operators $Y_1, \ldots, Y_n$ such that*

$$T_x = \prod_{k=1}^{n} \exp(\mathrm{i}x_k Y_k). \tag{4}$$

PROOF: By virtue of example 1.1° the character group $\hat{G} = R^n$ and $\langle x, \hat{x}\rangle = \exp[\mathrm{i}(x\hat{x} + \ldots + x_n\hat{x}_n)]$. Hence by virtue of eq. (1) we have:

$$T_x = \int_{R^n} \exp[\mathrm{i}(x_1\hat{x}_1 + \ldots + x_n\hat{x}_n)]\mathrm{d}E(\hat{x}). \tag{5}$$

Using now ths. 4.3, item 3°, and 4.2 of app. B one obtains

$$T_x = \prod_{k=1}^{n} \int_{R^n} \exp[\mathrm{i}x_k\hat{x}_k]\mathrm{d}E(\hat{x}) = \prod \int_{R^1} \exp[\mathrm{i}x_k\hat{x}_k]\mathrm{d}E(\hat{x}_k) = \prod \exp[\mathrm{i}x_k Y_k], \tag{6}$$

where

$$\mathrm{d}E(\hat{x}_k) = \int_{R^{n-1}} \mathrm{d}E(\hat{x}) \quad \text{and} \quad Y_k = \int \hat{x}_k \mathrm{d}E(\hat{x}_k). \tag{7}$$ ▼

EXAMPLE 1. Let $G = T^{3,1}$ be the translation group of the Minkowski space M^4 and let $x \to T_x$ be a unitary representation of G in a Hilbert space H. The dual space $\hat{G}$ is identified in physics with the momentum space P which is isomorphic to M^4. Hence, formula (4) can be written in the form

$$T_x = \int_P \exp(\mathrm{i}xp)\mathrm{d}E(p), \quad xp = x^\mu p_\mu, \tag{8}$$

where $E(\cdot)$ is a spectral measure on momentum space. The commutative set of self-adjoint operators defined by eq. (7) are in this case

$$P_\mu = \int_P p_\mu \mathrm{d}E(p), \quad \mu = 0, 1, 2, 3, \tag{9}$$

and represent the energy-momentum four-vector.

§ 3. Comments and Supplements

A. *Duality Theorem of Pontryagin*

We describe in this section a fundamental property of representations of abelian, locally compact groups.

Note first that the map $\hat{G} \ni \hat{x} \to \langle x, \hat{x} \rangle$ defines a continuous function on $\hat{G}$, which satisfies eqs. 1(3′) and 1(6′). Hence, every $x \in G$ defines a character $\hat{\hat{x}}$ of the group $\hat{G}$; consequently $G \subset \hat{\hat{G}}$, the set of all $\hat{\hat{x}}$. The following theorem asserts that there are no other characters on $\hat{G}$ besides those induced by the elements of G.

THEOREM 1. *The map* $G \ni x \to \hat{\hat{x}} \in \hat{\hat{G}}$ *is a topological isomorphism*

$$G \cong \hat{\hat{G}}. \blacktriangledown$$

(For the proof cf. Hewitt and Ross 1963, § 24.)

Example 1 provides two simplest illustrations of the Pontryagin duality.

B. *Comments*

The SNAG theorem is usually presented in the operator form given by eq. 2(1). The following interesting form of this theorem based on Bochner theorem was recently given by Hewitt and Ross.

THEOREM 2. *Let T be a continuous cyclic unitary representation of an abelian locally compact group G. There is a positive measure $\nu(\cdot)$ on G such that T is unitarily equivalent to the following representation*

$$U_x v(\hat{x}) = \hat{x}(x) v(\hat{x}) = \hat{\hat{x}}(\hat{x}) v(\hat{x}), \quad x \in G, \; v \in L^2(\hat{G}, \nu). \blacktriangledown \tag{1}$$

(For the proof cf. Hewitt and Ross 1970, § 33.8.)

The SNAG Theorem was originally proved by Stone for $G = R^n$ (1930, 1932). Later Naimark 1943, Ambrose 1944 and Godement 1944 gave various extensions of this theorem for an arbitrary abelian locally compact group. We follow here the presentation given by Maurin 1963, 1968, ch. VI.

C. The harmonic analysis on locally compact commutative groups will be considered in ch. 14, § 1

D. *Indecomposable Representations*

We give the construction of indecomposable representations of vector groups, which are most important in applications. Let $G = R^n$. The simplest example of an indecomposable representation is given by the formula

$$R^n \ni x \to T_x = \exp(\mathrm{i}px)\begin{bmatrix} 1 & \gamma(px) \\ 0 & 1 \end{bmatrix},$$

where $px = p_k x^k$ is the scalar product in R^n and $\gamma \in C$. Using the induction method one may find that an n-dimensional indecomposable representation of R^n may be taken to be in the form:

$$R^n \ni x \to T_x = \exp(\mathrm{i}px) \times$$

$$\times \begin{bmatrix} 1 & \gamma_{n-1}(px) & \gamma_{n-2}\gamma_{n-1}(px)^2/2 \ldots & \dfrac{\gamma_2 \cdots \gamma_{n-1}}{(n-2)!}(px)^{n-2} & \dfrac{\gamma_1 \cdots \gamma_{n-1}}{(n-1)!}(px)^{n-1} \\ & 1 & \gamma_{n-2}(px) \qquad \ldots & \dfrac{\gamma_2 \cdots \gamma_{n-2}}{(n-3)!}(px)^{n-3} & \dfrac{\gamma_1 \cdots \gamma_{n-2}}{(n-2)!}(px)^{n-2} \\ & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ & 0 & 1 \qquad \ldots & \gamma_2(px) & \dfrac{\gamma_1\gamma_2}{2}(px)^2 \\ & & & 1 & \gamma_1(px) \\ & & & & 1 \end{bmatrix}$$

These representations are nonunitary. They are used for a group-theoretical description of unstable particles (see ch. 17, § 4).

The indecomposable representations of other commutative groups may be constructed similarly with the help of triangular matrices: for instance the indecomposable representation of the multiplicative group of complex numbers may be taken to be in the form

$$C \ni z \to T_z = \begin{bmatrix} 1 & \ln z \\ 0 & 1 \end{bmatrix}$$

§ 4. Exercises

§ 1.1. Let $D_n = \{\delta = (\delta_1, \ldots, \delta_n),\ \delta_k \in C\}$ be the multiplicative group of complex numbers. Show that the map

$$\delta \to \chi_\delta = \prod_{s=1}^{n} |\delta_{ss}|^{m_s + i\varrho_s} \delta_{ss}^{-m_s} \tag{1}$$

where m_s are integers and ϱ_s are real numbers, is the character of D_n.

§ 1.2. Set $\varrho_s \in C$ in eq. (1). Show that in this case the map (1) gives the complex character of D_n.

§ 1.3. Construct a discontinuous irreducible unitary representation of $G = R^1$.

Hint: Use the Hamel basis.

§ 1.4. Let $G = N \otimes K$ where N is commutative. Let $n \to U_n$ and $k \to V_k$ be representations of N and K, respectively, in a carrier space H. What conditions U must satisfy in order that the map $(n, k) \to U_n V_k$ be a representation of G?

§ 3.1.** Classify all finite-dimensional indecomposable representations of $G = R^1$

§ 3.2.*** Classify all indecomposable representations of R^1 in Hilbert space.

§ 3.3.** Classify all finite-dimensional indecomposable representations of $G = R^n$

Chapter 7

Representations of Compact Groups

§ 1. Basic Properties of Representations of Compact Groups

The representation theory of compact groups forms a bridge between the relatively simple representation theory of finite groups and that of noncompact groups. Most of the theorems for the representations of finite groups have direct analogues for compact groups and these results in turn serve as the starting point for the representation theory of noncompact groups.

Everywhere in this section G will denote a compact topological group and dx an invariant measure on G, normalized to unity. And by a representation of G we shall mean a strongly continuous representation in a Hilbert space H. Recall that by eq. 5.1(3), any representation of a compact, topological group is bounded.

We first show that in the case of representations of compact topological groups we can restrict ourselves without loss of generality to an analysis of unitary representations only.

THEOREM 1. *Let T be an arbitrary representation of a compact group G in H. There exists in H a new scalar product defining a norm equivalent to the initial one, relative to which the map $x \to T_x$ defines a unitary representation of G.*

PROOF: Let $(\cdot, \cdot)$ be the initial scalar product in H. We define the new scalar product by

$$(u, v)' \equiv \int_G (T_x u, T_x v) dx. \tag{1}$$

It is easily verified that $(\cdot, \cdot)'$ is a scalar product in H. In particular

$$(u, u)' = \int_G (T_x u, T_x u) dx = 0 \Rightarrow u = 0.$$

Indeed, $(T_x u, T_x u)$ is zero almost everywhere; if $x \in G$ is such that $T_x u = 0$, then $T_x^{-1} T_x u = u = 0$.

For every T_y, we have then

$$(T_y u, T_y v)' = \int_G (T_{yx} u, T_{yx} v) dx = \int_G (T_z u, T_z v) dz = (u, v)'. \tag{2}$$

Hence, every T_y, $y \in G$, is isometric and $D_{T_y} = H$. Thus, every T_y, $y \in G$, is unitary.

To show the continuity of the representation in the topology induced by the scalar product (1) we first prove the equivalence of norms; note that

$$\begin{aligned}||u||'^2 = (u, u)' = \int_G (T_x u, T_x u)\mathrm{d}x \leqslant (\sup_{x\in G}||T_x||)^2 \int_G (u, u)\mathrm{d}x \\ = N^2(u, u) = N^2||u||^2,\end{aligned}$$

where we set $N = \sup_{x\in G}||T_x||$. Conversely, from the inequality

$$||u||^2 = (T_{x^{-1}}T_x u, T_{x^{-1}}T_x u) \leqslant (\sup_{x\in G}||T_x||)^2(T_x u, T_x u) = N^2||T_x u||^2$$

it follows that

$$||u||^2 = \int_G (u, u)\mathrm{d}x \leqslant N^2 \int (T_x u, T_x u)\mathrm{d}x = N^2(u, u)' = N^2||u||'^2.$$

Hence,

$$N^{-1}||u|| \leqslant ||u||' \leqslant N||u||, \tag{3}$$

i.e., the norms $||\cdot||$ and $||\cdot||'$ are equivalent. The equivalent norms $||\cdot||$ and $||\cdot||'$ define equivalent strong topologies τ and τ', respectively, on H. Hence the map $x \to T_x u$ of G into H is continuous relative to τ' and, consequently, the map $x \to T_x$ is the unitary continuous representation of G in H. ▼

The next important result for compact groups shows that every irreducible unitary representation is finite-dimensional. We first prove the following useful lemma.

LEMMA 2. *Let T be a unitary representation of G and let u be any fixed vector in the carrier space H. Then, the Weyl operator K_u defined for all $v \in H$ by the formula*

$$K_u v = \int_G (v, T_x u)T_x u\mathrm{d}x \tag{4}$$

has the following properties

1° *K_u is bounded.*

2° *$K_u T_x = T_x K_u$ for every $x \in G$ and $u \in H$.* ▼

PROOF: *ad* 1°.

$$||K_u v|| \leqslant \int_G |(v, T_x u)|\,||T_x u||\mathrm{d}x \leqslant \int_G ||v||\,||T_x u||\,||u||\mathrm{d}x = ||u||^2||v||.$$

ad 2°. For every $y \in G$ and $v \in H$ we have

$$\begin{aligned}T_y K_u v = \int_G (v, T_x u)T_{yx}u\mathrm{d}x = \int_G (T_y v, T_{yx}u)T_{yx}u\mathrm{d}x \\ = \int_G (T_y v, T_x u)T_x u\mathrm{d}x = K_u T_y v,\end{aligned}$$

i.e., $T_y K_u = K_u T_y$. ▼

We come now to the fundamental theorem.

THEOREM 3. *Every irreducible unitary representation T of G is finite-dimensional.*

PROOF: We have by lemma 2: $K_uT_y = T_yK_u$ for every $y \in G$ and $u \in H$. Hence by proposition 5.3.4, $K_u = \alpha(u)I$ and consequently

$$(K_u v, v) = \int_G (v, T_x u)(T_x u, v)\,dx = \alpha(u)(v, v).$$

Hence

$$\int_G |(T_x u, v)|^2 dx = \alpha(u)\|v\|^2. \tag{5}$$

By interchanging the roles of u and v in (5) and using the equality

$$\int_G f(x^{-1})\,dx = \int_G f(x)\,dx,$$

we get

$$\begin{aligned}\alpha(v)\|u\|^2 &= \int_G |(T_x v, u)|^2 dx = \int_G |(u, T_x v)|^2 dx \\ &= \int_G |(T_{x^{-1}} u, v)|^2 dx = \int_G |(T_x u, v)|^2 dx = \alpha(u)\|v\|^2.\end{aligned}$$

Hence, $\alpha(u) = c\|u\|^2$ for all $u \in H$, where c is a constant. Setting $v = u$ and $\|u\| = 1$ in eq. (5) we obtain in particular that

$$\int_G |(T_x u, u)|^2 dx = \alpha(u)\|u\|^2 = c\|u\|^4 = c. \tag{6}$$

Hence, $c > 0$, because the non-negative continuous function $x \to |(T_x u, u)|$ assumes the value $\|u\| = 1$ at $x = e$. Now we prove the essential part of the theorem. Let $\{e_i\}_1^n$ be any set of orthonormal vectors in H. Setting $u = e_k$ and $v = e_1$ in eq. (5) we obtain

$$\int_G |(T_x e_k, e_1)|^2 dx = \alpha(e_k)\|e_1\|^2 = c, \qquad k = 1, 2, \ldots, n.$$

Hence, using orthonormality of the vectors $T_x e_k$, $k = 1, 2, \ldots, n$ and the Parseval inequality, we obtain

$$nc = \sum_{k=1}^n \int_G |(T_x e_k, e_1)|^2 dx = \int_G \sum_{k=1}^n |(T_x e_k, e_1)|^2 dx \leqslant \int_G \|e_1\|^2 dx = 1. \tag{7}$$

Eq. (7) shows that the dimension of the carrier space H cannot exceed $1/c$ and hence it is finite. ▼

We have seen, by corollary 5.3.2, that a finite-dimensional unitary representation of any group is completely reducible. This result, in the case of compact topological groups, can be sharpened to the following one:

THEOREM 4. *Every unitary representation T of G is a direct sum of irreducible finite-dimensional unitary subrepresentations.*

PROOF: We first show that the operator K_u, which is defined by eq. (4) has the following properties:

1° $K_u^* = K_u$.

2° K_u is a Hilbert–Schmidt operator.

3° Every eigenspace H_i of K_u is T-invariant.

4° $H = H_0 + \sum_i \oplus H_i$, $\dim H_i < \infty$, where H_0 is the 0-eigenspace of K_u, which may be infinite-dimensional.

ad 1°. For every $v, w \in H$ we have

$$\begin{aligned}(K_u v, w) &= \int (v, T_x u)(T_x u, w)\mathrm{d}x = \int \overline{(w, T_x u)}(v, T_x u)\mathrm{d}x \\ &= \left(v, \int (w, T_x u) T_x u \,\mathrm{d}x\right) = (v, K_u w).\end{aligned}$$

Hence, $K_u^* = K_u$.

ad 2°. We recall that an operator A is Hilbert–Schmidt if and only if for arbitrary basis $\{e_i\}$ in H we have $\sum_i ||Ae_i||^2 < \infty$. In our case we have

$$\begin{aligned}\sum_i ||K_u e_i||^2 &= \sum_i \int_G \int_G (e_i, T_x u)(T_x u, T_y u)(T_y u, e_i)\mathrm{d}x\mathrm{d}y \\ &= \int_G \int_G \sum_i (e_i, T_x u)(T_x u, T_y u)(T_y u, e_i)\mathrm{d}x\mathrm{d}y \\ &= \int_G \int_G (T_x u, T_y u)(T_y u, T_x u)\mathrm{d}x\mathrm{d}y < \infty.\end{aligned}$$

The change of the order of summation and integration is justified by Lebesgue theorem and that

$$\sum_i |(e_i, T_x u)(T_y u, e_i)|^2 \leqslant \left(\sum_i |(e_i, T_x u)|^2 \sum_k |(e_k, T_y u)|^2\right)^{1/2} = ||u||^2.$$

ad 4°. Follows from Rellich–Hilbert–Schmidt spectral theorem for compact operators (cf. app. B.3).

We now come to the proof of the main theorem. For all vectors $v \in H$, which are not orthogonal to u, we have $(K_u v, v) > 0$ (cf. eq. (6) and below). Therefore, the operator K_u has at least one eigenvalue different from zero, i.e., the space $H \ominus H_0$ is not empty.

We have shown, therefore, that the space $H \ominus H_0 = \sum_i \oplus H_i$ is the direct orthogonal sum of finite-dimensional, invariant subspaces H_i. Using corollary 5.3.2 we can split each H_i into the direct, orthogonal sum of irreducible, invariant subspaces. In order to split the space H_0 we consider the subrepresentation $T_x^0 \equiv {}^{H_0}T_x$ and construct for it the operator (4). Therefore, according to the properties 3° and 4° of K_u and corollary 5.3.2, we conclude that H_0 contains also a non-trivial, finite-dimensional, minimal invariant subspace. It is evident from these considerations that the conclusion is true for any invariant space.

To complete the proof of th. 4 we show that the smallest subspace M of H containing all the minimal, mutually orthogonal, invariant subspaces is H itself. Indeed, because T is unitary and M is invariant, then $M^\perp$ is also invariant. Hence, by the above conclusion $M^\perp$ contains a nontrivial, invariant, finite-dimensional subspace, which contradicts the definition of M. Therefore $M^\perp = 0$ and $M = H$. ▼

Next we derive the useful orthogonality relations for matrix elements of irreducible, unitary representations.

THEOREM 5. *Let T^s and $T^{s'}$ be any two irreducible unitary representations of G labelled by indices s and s', respectively. Then, their matrix elements satisfy the relations*

$$\int_G D^s_{ij}(x)\bar{D}^{s'}_{mn}(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } T^s \text{ and } T^{s'} \text{ are not equivalent,} \\ \dfrac{1}{d_s}\delta_{im}\delta_{jn}, & \text{if } T^s \cong T^{s'}, \end{cases} \tag{9}$$

where d_s is the dimension of T^s.

PROOF: Consider the operators

$$E_{ij} = \int_G T^s_x e_{ij} T^{s'}_{x^{-1}}\,\mathrm{d}x, \tag{10}$$

where $(e_{ij})^{mn} = \delta^m_i \delta^n_j$, $i, m = 1, 2, \ldots, d_s$, $j, n = 1, 2, \ldots, d_s$. For every $y \in G$ the operators (10) satisfy the relation

$$T^s_y E_{ij} = E_{ij} T^{s'}_y .$$

Indeed,

$$T^s_y E_{ij} = \int_G T^s_{yx} e_{ij} T^{s'}_{x^{-1}}\,\mathrm{d}x = \int_G T^s_{x'} e_{ij} T^{s'}_{x'^{-1}y}\,\mathrm{d}x' = E_{ij} T^{s'}_y . \tag{11}$$

Hence, if T^s is not equivalent to $T^{s'}$, then, by Schur's lemma, we have $E_{ij} = 0$, or in matrix form,

$$\int_G D^s_{li}(x) D^{s'}_{jk}(x^{-1})\,\mathrm{d}x = \int_G D^s_{li}(x)\overline{D^{s'}_{kj}(x)}\,\mathrm{d}x = 0. \tag{12}$$

If $s = s'$, the operator given by (10) also satisfies the condition (11) and, therefore, by proposition 5.4.3, $E_{ij} = \lambda_{ij} I$. Hence, for $(l, i) \neq (k, j)$ the orthogonality relations (12) are still satisfied. If, however, $(l, i) = (k, i)$, then, by eq. (10) and $E_{ii} = \lambda_{ii} I$ (no summation) we obtain

$$(E_{ii})_{ll} = \int_G D^s_{li}(x) D^s_{il}(x^{-1})\,\mathrm{d}x = \int_G |D^s_{li}(x)|^2\,\mathrm{d}x = \lambda_{ii} \tag{13}$$

(no summation).

In order to calculate the constant λ_{ii} we set $i = j$ in eq. (10) and take the trace of both sides. We obtain

$$\mathrm{Tr} E_{ii} = d_s \lambda_{ii} = \int_G \mathrm{Tr}(T^s_x e_{ii} T^s_{x^{-1}}) \mathrm{d}x = \mathrm{Tr} e_{ii} = 1$$

or, $\lambda_{ii} = 1/d_s$. This completes the proof of eq. (9). ▼

We now show that the right regular representation contains all irreducible representations. Indeed, we have

PROPOSITION 6. *Every irreducible unitary representation T^s of G is equivalent to a subrepresentation of the right regular representation.*

PROOF: Let $\{D^s_{jk}(x)\}$, $j, k = 1, 2, \ldots, d_s$, be a matrix form of T^s and let H^s be a subspace of $L^2(G)$ spanned by the orthonormal vectors $e^s_k = \sqrt{(d_s)} D^s_{ik}(x)$. The subrepresentation ${}^{H^s}T^R$ of the right regular representation T^R is irreducible and is equivalent to T^s. In fact,

$$T^R_{x_0} e^s_k(x) = D^s_{1k}(xx_0) = D^s_{1l}(x) D^s_{lk}(x_0) = D^s_{lk}(x_0) e^s_l(x). \,▼$$

A *character* $\chi(x)$ of a finite-dimensional representation T of G is the trace of the operator T_x, i.e.,

$$\chi(x) = \mathrm{Tr}\, T_x = (T_x e_i, e_i) = D_{ii}(x). \tag{14}$$

The properties of the characters of irreducible unitary representations of G are summarized as follows:

PROPOSITION 7.

$$\begin{aligned}
&1^\circ\ \chi(y^{-1}xy) = \chi(x),\\
&2^\circ\ \chi(x^{-1}) = \overline{\chi}(x),\\
&3^\circ\ \textit{If } T^s \simeq T^{s'}, \textit{ then } \chi^s = \chi^{s'},\\
&4^\circ \int_G \chi^s(x)\overline{\chi}^{s'}(x)\,\mathrm{d}x = \begin{cases} 0 & \textit{if } T^s \not\simeq T^{s'},\\ 1 & \textit{if } T^s \simeq T^{s'}.\end{cases}
\end{aligned} \tag{15}$$

PROOF: *ad* 1° $\chi(y_y^{-1}xy) = \mathrm{Tr}(T_{y^{-1}xy}) = \mathrm{Tr}(T_{y^{-1}} T_x T_y) = \mathrm{Tr}\, T_x = \chi(x)$.

ad 2° $\chi(x^{-1}) = \mathrm{Tr}\, T_{x^{-1}} = \mathrm{Tr}\, T^*_x = \overline{D}_{ii}(x) = \overline{\chi}(x)$.

ad 3° If $T \simeq T'$, then $T = S^{-1}T'S$ and $\mathrm{Tr}\, T_x = \mathrm{Tr}\, S^{-1} T'_x S = \mathrm{Tr}\, T'_s$.

ad 4° From eq. (9),

$$\int_G \chi^s(x)\overline{\chi}^{s'}(x)\,\mathrm{d}x = \int_G D^s_{ii}(x)\overline{D}^{s'}_{jj}(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } T^s \not\simeq T^{s'},\\ 1 & \text{if } T^s \simeq T^{s'}. \,▼\end{cases}$$

Let T be a finite dimensional representation of G. Then, by corollary 5.3.2 and eq. (14) we have

$$\chi(x) = m_i \chi_i(x), \tag{16}$$

where m_i is the multiplicity with which an irreducible representation T^i, $i = 1, 2, \ldots, n$, of G appears in the decomposition of T. Using (15)4° we obtain

$$m_i = \int_G \chi(x)\overline{\chi}_i(x)\,\mathrm{d}x \tag{17}$$

and

$$\sum_{i=1}^{n} m_i^2 = \int_G \chi(x)\bar{\chi}(x)\,\mathrm{d}x. \tag{18}$$

Eq. (17) shows that a character χ defines a finite-dimensional representation T of G up to an equivalence. Formula (18) can be used as a criterion for irreducibility of T; namely a representation T is irreducible if and only if $\int_G \chi(x)\bar{\chi}(x)\,\mathrm{d}x = 1$. If T is reducible, then $\int_G \chi(x)\bar{\chi}(x)\,\mathrm{d}x > 1$.

§ 2. Peter–Weyl and Weyl Approximation Theorems

In this section we prove several useful theorems which will allow us to extend the ordinary Fourier analysis on the real line to the harmonic analysis on compact groups. We begin with the celebrated Peter–Weyl theorem.

THEOREM 1 (Peter–Weyl). *Let $\hat{G} = \{T^s\}$ be the set of all irreducible non-equivalent unitary representations of G. The functions*

$$\sqrt{(d_s)}\, D^s_{jk}(x), \quad s \in \hat{G},\ 1 \leqslant j,\ k \leqslant d_s, \tag{1}$$

where d_s is the dimension of T^s and $D^s_{jk}(x)$ are the matrix elements of T^s, form a complete orthonormal system in $L^2(G)$.

PROOF: Let L be the linear closed subspace of $L^2(G)$ spanned by all functions (1) and let $L^\perp$ be the orthogonal complement of L. The space L is invariant under the right translations T^R (cf. the proof of proposition 1.6). Hence, by proposition 5.3.1, $L^\perp$ is also an invariant subspace relative to T^R. Let $0 \neq v \in L^\perp$ and set

$$u(x) = \int (T^R_x v(y))\bar{v}(y)\,\mathrm{d}y. \tag{2}$$

This is a continuous function on G belonging to $L^\perp$: indeed by (2) we have

$$\int u(x)\bar{D}^s_{jk}(x)\,\mathrm{d}x = \sum_l \int v(x)\bar{D}^s_{jl}(x)\,\mathrm{d}x \overline{\int v(y)\bar{D}^s_{kl}(y)\,\mathrm{d}y} = 0.$$

Moreover $u(e) = ||v||^2 > 0$. Set now

$$w(x) = u(x) + \overline{u(x^{-1})} \tag{3}$$

and consider the operator

$$A\psi(x) = \int w(xy^{-1})\psi(y)\,\mathrm{d}y. \tag{4}$$

By virtue of (3) and because invariant measure on G is finite, A is a self-adjoint compact operator. Because $w \neq 0$ there exists an eigenvalue λ, $\lambda \neq 0$ and a finite-dimensional eigensubspace $H(\lambda)$ of A. Let $\psi_\lambda(x)$ be an eigenfunction of A. By virtue of (4) we have

$$\int \psi_\lambda(x)\overline{D^s_{ij}}(x)\,\mathrm{d}x = \frac{1}{\lambda}\int (A\psi_\lambda)(x)\overline{D^s_{ij}}(x)\,\mathrm{d}x$$
$$= \frac{1}{\lambda}\sum_k \int w(x')\overline{D^s_{ik}}(x')\,\mathrm{d}x' \int \psi_\lambda(x)\overline{D^s_{kj}}(x)\,\mathrm{d}x = 0.$$

Hence $\psi_\lambda(x) \in L^\perp$. The operator A is T^R-invariant; indeed $(T^R_z A\psi)(x) = \int w(xzy^{-1})\psi(y)\mathrm{d}y = \int w(xy^{-1})\psi(zy)\mathrm{d}y = (AT^R_z\psi)(x)$. Consequently $H(\lambda)$ is also invariant with respect to T^R and the formula $T_x\psi_\lambda(y) = \psi_\lambda(yx)$ defines the representation of G in $H(\lambda)$. This representation is fully reducible. Let $\{e^s_k\}_1^{d_s}$ be a basis of an irreducible subspace $H^s(\lambda)$ of $H(\lambda)$, given by eigenfunction of A. We have

$$T^R_x e^s_k(y) = e^s_k(yx) = D^s_{jk}(x)e^s_j(y).$$

Since all eigenfunctions of A are continuous, we have (no summation over s)

$$e^s_k(x) = D^s_{jk}(x)e^s_j(e),$$

i.e., $e^s_k \in L$. Hence $H(\lambda) = \{0\}$ contrary to the previous conclusion: consequently $L^\perp = 0$ by (3) and (2). ▼

COROLLARY. *Let $u(x) \in L^2(G)$. Then*

$$u(x) = \sum_{s\in\hat{G}} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x), \tag{4}$$

and

$$\int_G |u(x)|^2\mathrm{d}x = \sum_{s\in\hat{G}} d_s \sum_{j,k=1}^{d_s} |c^s_{jk}|^2, \tag{5}$$

where

$$c^s_{jk} = d_s\int_G u(x)\overline{D^s_{jk}(x)}\,\mathrm{d}x \tag{6}$$

and the convergence in eq. (4) is understood in the sense of norm on $L^2(G)$.

PROOF: Formula (4) follows from the completeness of functions (1). To show (5), set $u(x) = u_N(x)+\varepsilon_N(x)$, where

$$u_N(x) = \sum_{s=1}^{N} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x).$$

Clearly, $\|\varepsilon_N(x)\| \to 0$ for $N \to \infty$. Then, by orthogonality relations 1(9), one obtains

$$\int_G |u(x)|^2\mathrm{d}x = \sum_{s=1}^{N} d_s^{-1} \sum_{j,k=1}^{d_s} |c^s_{jk}|^2 + (u_N, \varepsilon_N) + (\varepsilon_N, u_N) + (\varepsilon_N, \varepsilon_N).$$

Using the Schwartz inequality, one obtains

$$\Big|\int_G |u(x)|^2 \mathrm{d}x - \sum_{s=1}^{N} \sum_{j,k=1}^{d_s} |c^s_{jk}|^2\Big| \leqslant (2\|u_N\|+\|\varepsilon_N\|)\|\varepsilon_N\| \underset{N\to\infty}{\to} 0. \blacktriangledown$$

The equality (5) is called the *Parseval equality*.

Due to proposition 1.6 we know that every irreducible, unitary representation of G is a subrepresentation of the right regular representation. Moreover, we also know from th. 1.4 that the regular representation is a direct sum of irreducible, unitary (hence, finite-dimensional) representations. The following theorem completes the description of the structure of the regular representation in terms of its irreducible components.

THEOREM 2. *Every irreducible unitary representation T^s of G occurs in the decomposition of the regular representation with a multiplicity equal to the dimension of T^s. The orthonormal vectors*

$$Y^s_{(j)k}(x) = \sqrt{d_s}\, D^s_{jk}(x), \qquad s \in \hat{G} \text{ and } j,k = 1,2,\dots,d_s \tag{7}$$

for fixed j and fixed s span the invariant irreducible subspaces of the right regular representation and the orthonormal vectors:

$$\tilde{Y}^s_{k(j)}(x) = \sqrt{d_s}\, \overline{D^s_{kj}}(x), \qquad s \in \hat{G} \text{ and } j,k = 1,2,\dots,d_s \tag{8}$$

for fixed j and fixed s span the invariant, irreducible subspaces of the left regular representation.

PROOF: Denote by $H^s_{(j)}$, $s \in \hat{G}$ and j fixed, the invariant irreducible subspaces of $H = L^2(G)$ spanned by the orthonormal vectors (7). Let $u(x) \in L^2(G)$. According to formula (4), we have

$$u(x) = \sum_{s\in\hat{G}} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x) = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} u^s_{(j)}(x), \tag{9}$$

where $u^s_{(j)} \in H^s_{(j)}$. Because $H^s_{(j)} \perp H^{s'}_{(j')}$, if $(s,j) \neq (s',j')$, the decomposition (9) is unique. Hence,

$$H = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} \oplus\, H^s_{(j)}. \tag{10}$$

Moreover,

$$T^R_{x_0} u(x) = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} {}^{H^s_{(j)}}T^R_{x_0} u^s_{(j)}(x), \tag{11}$$

where ${}^{H^s_{(j)}}T^R$ are irreducible unitary subrepresentations of T^R in $H^s_{(j)}$ given by formula

$$^{H^s_{(j)}}T^R_{x_0} Y^s_{(j)k}(x) = D^s_{lk}(x_0)\, Y^s_{(j)l}(x). \tag{12}$$

Thus, for every $j = 1, 2, \ldots, d_s$, ${}^{H^s_{(j)}}T^R \simeq T^s$ and, therefore, by def. 5.3.3.,

$$T^R = \sum_{s\in\hat{G}} \oplus\, d_s T^s. \tag{13}$$

The same result can be proved for the left regular representation if we take the functions (8) as basis vectors of irreducible subspaces. ▼

The Peter–Weyl theorem can be considerably sharpened. For continuous functions on G instead of the approximation in the norm of L^2-space, we can obtain a uniform approximation. This is the content of the following Weyl approximation theorem.

THEOREM 3. *Let f be a continuous function on G. For every $\varepsilon > 0$ there exists a linear combination*

$$\sum_{s=1}^{N_\varepsilon} \sum_{j,k} c^s_{jk} D^s_{jk}(x)$$

of the matrix elements of irreducible unitary representations such that

$$\Big| f(x) - \sum_{s=1}^{N_\varepsilon} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x) \Big| \leqslant \varepsilon \quad \textit{for all } x \in G. \tag{14}$$

PROOF: We shall prove th. 3 using the method of the so-called 'smeared-out' operators. This method is very useful in the solution of many problems in the representation theory. Let $\varphi \in C(G)$, $f \in L^2(G)$ and let L_φ be the operator

$$L_\varphi f(x) = \int_G \varphi(y) T^L_y f(x)\,\mathrm{d}y = \int_G \varphi(y) f(y^{-1}x)\,\mathrm{d}y. \tag{15}$$

The operation (15) is also denoted by $\varphi * f$ and called the *convolution* of φ and f. The operator L_φ has the following properties

1° It is a continuous map from $L^2(G)$ into $C(G)$.

2° If H_N is the set of finite linear combinations

$$\sum_{s=1}^{N} \sum_{j,k} c^s_{jk} D^s_{jk}(x),$$

where $D^s_{jk}(x)$ are matrix elements of irreducible, unitary representation of G, then,

$$L_\varphi(H_N) \subset H_N. \tag{16}$$

Indeed, using the Cauchy inequality, we have

$$|L_\varphi f(x)|^2 = \Big|\int_G \varphi(y) f(y^{-1}x)\,\mathrm{d}y\Big|^2 \leqslant \int_G |\varphi(y)|^2\,\mathrm{d}y \int_G |f(y^{-1}x)|^2\,\mathrm{d}y.$$

Let $\|f\|_{C(G)} \equiv \sup_{x\in G} |f(x)|$, then

$$\|L_\varphi f\|_{C(G)} \leqslant c\|f\|_{L^2}, \quad \text{where } c = \|\varphi\|_{L^2}. \tag{17}$$

Moreover, if $f_N(x) \in H_N$, we find

$$\begin{aligned} L_\varphi f_N(x) &= \sum_{s=1}^{N} \sum_{jk} c_{jk}^s \int_G \varphi(y) D_{jk}^s(y^{-1}x) \mathrm{d}y \\ &= \sum_{s=1}^{N} \sum_{jkp} c_{jk}^s D_{pk}^s(x) \int_G \varphi(y) D_{jp}^s(y^{-1}) \mathrm{d}y \\ &= \sum_{s=1}^{N} \sum_{kp} \tilde{c}_{pk}^s D_{pk}^s(x) \in H_N, \quad \text{where } \tilde{c}_{pk}^s = \sum_j c_{pj}^s(\varphi) c_{jk}^s(f). \end{aligned}$$

To prove the main theorem let f be any element of $C(G)$. Because any continuous function on G, by proposition 2.2.4., is uniformly continuous, there exists a neighborhood V_ε of unity such that

$$|f(x_1)-f(x_2)| < \varepsilon, \quad \text{whenever } x_1 x_2^{-1} \in V_\varepsilon. \tag{18}$$

Let $\varphi_\varepsilon \in C(G)$ be a non-negative function, which differs from zero only on V_ε and satisfies $\int_G \varphi_\varepsilon(x)\mathrm{d}x = 1$. Then, for the smeared out operator L_{φ_ε} we obtain

$$||L_{\varphi_\varepsilon} f - f||_{C(G)} < \varepsilon. \tag{19}$$

Indeed, by eq. (18),

$$\begin{aligned} ||L_{\varphi_\varepsilon} f - f||_{C(G)} &= \sup_{x\in G} \left| \int_G \varphi_\varepsilon(y)[f(y^{-1}x) - f(x)] \mathrm{d}y \right. \\ &= \sup_{x\in G} \int_{V_\varepsilon} \varphi_\varepsilon(x)|f(x^{-1}y) - f(y)| \mathrm{d}x < \varepsilon. \end{aligned}$$

We know by th. 1 that every element $f \in C(G)$ can be arbitrarily approximated in the norm of $L^2(G)$ by elements of H_N, i.e., in particular,

$$||f-f_N||_{L^2} \leqslant ||\varphi_\varepsilon||_{L^2}^{-1} \cdot \varepsilon.$$

Thus, according to eqs. (19) and (17), we have

$$\begin{aligned} |f(x) - L_{\varphi_\varepsilon} f_N(x)| &\leqslant |f(x) - L_{\varphi_\varepsilon} f(x)| + |L_{\varphi_\varepsilon}(f(x) - f_N(x))| \\ &< \varepsilon + ||f-f_N||_{L^2} ||\varphi_\varepsilon||_{L^2} < 2\varepsilon. \end{aligned}$$

Because the function $L_{\varphi_\varepsilon} f_N(x)$, $f_N \in H_N$, is an element of H_N, the proof of th. 3 is completed. ▼

Note that from eqs. (4) and (6) we obtain the relation

$$\sum_{s=1}^{\infty} \sum_{j,k=1}^{d_s} d_s \bar{D}_{jk}^s(x') D_{jk}^s(x) = \delta(x-x'). \tag{20}$$

This is an alternate form of the completeness relation of D_{jk}^s-functions, which is very useful in calculations.

§ 3. Projection Operators and Irreducible Representations

We consider in this section the properties of the projection operators associated with the irreducible representations of compact groups. The technique of projection operators is extremely useful, elegant and effective in the solution of various practical problems in representation theory and quantum physics.

Let $D^s_{pq}(x)$ be the matrix elements of an irreducible representation T^s and define the operators

$$P^s_{pq} \equiv d_s \int_G \bar{D}^s_{pq}(x)\, T_x \,\mathrm{d}x, \tag{1}$$

where

d_s: the dimension of an irreducible representation T^s,

dx: the invariant Haar measure on G,

$x \to T_x$: a unitary representation of G in the carrier space H.

Because $D^s_{pq}(x)$ and T_x are continuous functions on G, and because G is compact, the operator integral (1) is well defined, (cf. app. B.2). In particular, all operators P^s_{pq} are bounded. Indeed,

$$\|P^s_{pq}u\| \leqslant d_s \int |\bar{D}^s_{pq}(x)|\, \|T_x u\| \,\mathrm{d}x \leqslant d_s \sup_{x\in G} |D^s_{pq}(x)|\, \|u\|.$$

Hence,

$$\|P^s_{pq}\| \leqslant d_s \sup_{x\in G} |D^s_{pq}(x)|.$$

PROPOSITION 1. *The operators P^s_{pq} have the following properties*

$1°$ $(P^s_{pq})^* = P^s_{qp}$, (2)

$2°$ $P^s_{pq}P^{s'}_{p'q'} = \delta^{ss'}\delta_{qp'}P_{pq'}$. (3)

PROOF: *ad* 1°. Because for every bounded operator A we have $\|A^*\| = \|A\|$, the map $A \to A^*$ is continuous in weak operator topology. Hence, in eq. (1) we can interchange the adjoint operation and integration, i.e.,

$$(P^s_{pq})^* = d_s \int_G D^s_{pq}(x)\, T^*_x \,\mathrm{d}x = d_s \int_G D^s_{pq}(x^{-1})\, T_x \,\mathrm{d}(x^{-1}) = d_s \int \bar{D}^s_{qp}(x)\, T_x \,\mathrm{d}x = P^s_{qp},$$

where we used $T^*_x = T^{-1}_x = T_{x^{-1}}$, as well as the invariance of the Haar measure.

ad 2°. From (1)

$$P^s_{pq}P^{s'}_{p'q'} = d_s d_{s'} \int \bar{D}^s_{pq}(x)\,\bar{D}^{s'}_{p'q'}(x')\, T_x T_{x'} \,\mathrm{d}x\,\mathrm{d}x'.$$

Using the group property $T_x T_{x'} = T_{xx'}$ and the relation

$$D^s_{pq}(x) = D^s_{pq}(\tilde{x}x'^{-1}) = D^s_{pr}(\tilde{x})\,D^s_{rq}(x'^{-1}) = D^s_{pr}(\tilde{x})\,\bar{D}^s_{qr}(x'),$$

where $xx' = \tilde{x}$, as well as the orthonormality relations 1(9), we obtain

$$P^s_{pq}P^{s'}_{p'q'} = d_s d_{s'} \int D^s_{qr}(x')\,\bar{D}^{s'}_{p'q'}(x')\,\mathrm{d}x' \int \bar{D}^s_{pr}(\tilde{x})\, T_{\tilde{x}} \,\mathrm{d}\tilde{x} = \delta^{ss'}\delta_{qp'}P^s_{pq'}. \blacktriangledown$$

COROLLARY. *The operators $P^s_p \equiv P^s_{pp}$ are projection operators, i.e.,*

$$(P^s_p)^* = P^s_p, \qquad P^s_p P^{s'}_{p'} = \delta^{ss'}\delta_{pp'}P^s_p. \blacktriangledown \tag{4}$$

The operators P^s_{pq} have simple transformation properties with respect to the action of the group G. Indeed, we have

PROPOSITION 2. *Let P^s_{pq} be given by formula* (1). *Then* (*no summation over s*)

$$T_x P^s_{pq} = D^s_{rp}(x) P^s_{rq}, \tag{5}$$

$$P^s_{pq} T_x = D^s_{qr}(x) P^s_{pr}. \tag{6}$$

PROOF: Because T_x is continuous, it can be brought under the integral sign in (1). Using the group properties of D^s_{pq} functions, one obtains

$$\begin{aligned} T_x P^s_{pq} &= d_s \int \bar{D}^s_{pq}(x') T_{xx'} \mathrm{d}x' = d_s \int \bar{D}^s_{pq}(x^{-1}\tilde{x}) T_{\tilde{x}} \mathrm{d}\tilde{x} \\ &= d_s D^s_{rp}(x) \int \bar{D}^s_{rq}(\tilde{x}) T_{\tilde{x}} \mathrm{d}\tilde{x} = D^s_{rp}(x) P^s_{rq}. \end{aligned}$$

Formula (6) is proved in a similar manner. $\blacktriangledown$

Note that the vectors

$$|s;p\rangle = P^s_{pq} u, \quad q \,\text{fixed}, \; u \in H,$$

by virtue of eq. (5), transform as the basis vectors of the carrier space H^s of the irreducible representation T^s. This fact is the starting point of most applications of the projection operators P^s_{pq} (cf. § 4.A and § 4.B).

Note also that by eqs. (5) and (6) we have

$$T_x P^s_{pq} T^{-1}_x = D^s_{rp}(x) \bar{D}^s_{tq}(x) P^s_{rt}. \tag{7}$$

Formula (7) means that P^s_{pq} transforms as a tensor operator corresponding to the tensor product of a basis vector e^s_p and an adjoint vector to e^s_q (i.e., as the product $|s;p\rangle\langle s;q|$ in Dirac's notation).

There are also useful projection operators associated with the characters

$$\chi^s(x) = \sum_{p=1}^{d_s} D^s_{pp}(x).$$

They are defined in the following manner:

$$P^s = d_s \int_G \chi^s(x) T_x \mathrm{d}x. \tag{8}$$

PROPOSITION 3. *The operators P^s have the following properties*

$$(P^s)^* = P^s, \tag{9}$$

$$P^s P^{s'} = \delta_{ss'} P^s, \tag{10}$$

$$T_x P^s = P^s T_x. \tag{11}$$

PROOF: Because $P^s = \sum P^s_p$, eqs. (9)–(10) follow directly from propositions 1 and 2, and eq. (10) follows from (8) and the fact that $\chi(x) = \chi(yxy^{-1})$.

We prove still another useful result

PROPOSITION 4. *Let T be a unitary representation of G in H. Then*

$$\sum_s P^s = I, \tag{12}$$

and

$$\sum_{s,p} P^s_p = I. \tag{13}$$

PROOF: Let $\{e^s_r\}$ be a basis in H. Then by virtue of (8) we have

$$\sum_{s'} P^{s'} e^s_r = \sum_{s'} d_{s'} \int \mathrm{d}x \sum_p \bar{D}^{s'}_{pp}(x) D^s_{mr}(x) e^s_m = e^s_r. \tag{14}$$

This implies eq. (12). Eq. (13) follows from the definition of P^s and eq. (12).

EXAMPLE. Let $G = \mathrm{SO}(3)$. If we describe the rotations in terms of Euler angles φ, ϑ and ψ,

$$0 \leqslant \varphi < 2\pi, \quad 0 \leqslant \vartheta < \pi, \quad 0 \leqslant \psi < 2\pi \tag{15}$$

then, by virtue of exercise 5.8.1.1 and 3.11(30) we have

$$\begin{aligned} d_J &= 2J+1, \quad \mathrm{d}x = (8\pi^2)^{-1} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi, \\ D^J_{M,M}(\varphi, \vartheta, \psi) &= \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0,2M}_{J-M}(\cos\vartheta) \exp[-\mathrm{i}M(\varphi+\psi)], \\ \chi(\varphi, \vartheta, \psi) &= \sum_{M=-J}^{J} D^J_{MM}(\varphi, \vartheta, \psi), \\ T_{x(\varphi,\vartheta,\psi)} &= \exp(-\mathrm{i}\varphi J_z)\exp(-\mathrm{i}\vartheta J_y)\exp(-\mathrm{i}\psi J_z). \end{aligned} \tag{16}$$

Therefore, the projection operators P^J_M and P^J are given by

$$P^J_M = \frac{2J+1}{8\pi^2} \int \bar{D}^J_{M,M}(\varphi, \vartheta, \psi) T_{x(\varphi,\vartheta,\psi)} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi, \tag{17}$$

$$P^J = \frac{2J+1}{8\pi^2} \int \bar{\chi}^J(\varphi, \vartheta, \psi) T_{x(\varphi,\vartheta,\psi)} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi. \;\blacktriangledown \tag{18}$$

§ 4. Applications

A. *Decomposition of a Factor Representation onto Irreducible Representations*

In many problems an irreducible representation T^s of G appears several times in the carrier space H. It is then required, in many applications, to decompose a factor representation $n_s T^s$ (no summation) onto its irreducible components and to construct explicitly the corresponding orthogonal carrier spaces. We solve these problems using the projection operators P^s_{pq}. As we noted the vectors

$$|s;p\rangle \equiv P^s_{pq} u, \quad q = \text{fixed}, \; u \in H, \tag{1}$$

transform as basis vectors e^s_p of the carrier space H^s of the irreducible representation T^s, i.e.

$$T_x|s;p\rangle = D^s_{rp}(x)P^s_{rq}u = D^s_{rp}(x)|s;r\rangle. \tag{2}$$

This relation provides us with a simple and elegant method of an explicit construction of the orthogonal basis vectors $|s;p\rangle$ of the irreducible carrier space H^s, from the vectors of the Hilbert space H, in which the reducible unitary representation T of G is realized. Consider first the case, when every irreducible representation T^s appears in the decomposition of T only once. Let $P^s_{pq}u \neq 0$, $p = 1, 2, \ldots, d_s$, d_s = dimension of the irreducible representation T^s, for some $u \in H$ and for fixed s and q. Then, the vectors

$$|s;p\rangle \equiv \frac{1}{N} P^s_{pq}u, \quad p = 1, 2, \ldots, d_s,\ s \text{ and } q \text{ fixed}, \tag{3}$$

where $N^2 = (u, P^s_{qq}u)$, form an orthonormal set of vectors. In fact,

$$\begin{aligned}\langle s;p'|s;p\rangle &= \frac{1}{N^2}(P^s_{p'q}u, P^s_{pq}u) = \frac{1}{N^2}(u, (P^s_{p'q})^* P^s_{pq}u) \\ &= \frac{1}{N^2}(u, P^s_{qp'}P^s_{pq}u) = \delta_{p'p}\frac{1}{N^2}(u, P^s_{qq}u) = \delta_{p'p}.\end{aligned} \tag{4}$$

According to eq. (2), the closed linear hull H^s of orthonormal vectors (3) forms the carrier space in which the irreducible unitary representation $T^s = \{D^s_{ij}\}$ is realized.

It is remarkable that this method works even when a reducible unitary representation T contains an irreducible representation T^s several, say (n_s), times. Indeed, let $T = \sum_s \oplus n_s T^s$ be a decomposition of T onto factor representations $n_s T^s$, and let $H = \sum_s \oplus n_s H^s$ be the corresponding decomposition of the carrier space H. Moreover, let $H^s_q = P^s_q H$, where s and q are arbitrary, but fixed. Then, we have

PROPOSITION 1. *If $u \neq 0$ is an arbitrary vector from H^s_q, s and q fixed, then there exists only one subspace $H^s_{(u)}$ containing this vector, which is the carrier space of the irreducible representation $T^s = \{D^s_{ij}\}$.*

If $u \neq 0$ and $v \neq 0$ are orthogonal vectors from H^s_q then the spaces $H^s_{(u)}$ and $H^s_{(v)}$ are the orthogonal carrier spaces of the irreducible representation $T^s = \{D^s_{ij}\}$.

PROOF: Let $0 \neq u \in H^s_q$ and let $||u|| = 1$. The vectors

$$|s;p\rangle = P^s_{pq}u, \quad p = 1, 2, \ldots, d_s,\ s \text{ and } q \text{ fixed}, \tag{5}$$

constitute the orthonormal set of vectors. According to eq. (2), the closed linear hull $H^s_{(u)}$ of all vectors (5) constitutes the carrier space of the irreducible representation $T^s = \{D^s_{ij}\}$. The vector $|s;q\rangle = P^s_{qq}u = u$ cannot belong to two different irreducible subspaces $H^s_{(u)}$ and $\hat{H}^s_{(u)}$ because the intersection $L = H^s_{(u)} \cap \hat{H}^s_{(u)}$ is an invariant subspace, which satisfies the following conditions:

$$L \subset H^s_{(u)}, \quad L \subset \hat{H}^s_{(u)}, \quad L \neq H^s_{(u)}, \quad L \neq \hat{H}^s_{(u)}.$$

Consequently, because of the irreducibility of $H^s_{(u)}$ and $\hat{H}^s_{(u)}$, this invariant subspace is empty, i.e., $L = 0$.

If $u \neq 0$ and $v \neq 0$ are orthogonal vectors from H^s_q, then the spaces $H^s_{(u)}$ and $H^s_{(v)}$ are the orthogonal carrier spaces of the same irreducible representation $T^s = \{D^s_{ij}\}$. In fact,

$$\begin{aligned}(P^s_{p'q}v, P^s_{p''q}u) &= \big(v, (P^s_{p'q})^* P^s_{p''q}u\big) = (v, P^s_{qp'}P^s_{p''q}u) \\ &= \delta_{p'p''}(v, P^\lambda_{qq}u) = \delta_{p'p''}(v, u) = 0. \blacktriangledown\end{aligned}$$

Thus, by taking successive orthogonal vectors from the space H^s_q, we obtain as many orthogonal carrier spaces of the same irreducible representation $T^s = \{D^s_{ij}\}$ as the dimension of the subspace H^s_q. Clearly, $\dim H^s_q = n_s$. Thus, proposition 1 provides, in the general case, a systematic method of separation of irreducible carrier subspaces $H^s_{(u_i)}$, $i = 1, 2, \ldots, n_s$, from the reducible space H along the following steps:

1° Find the subspace $H^s_q = P^s_q H$ (s fixed, q arbitrary but fixed),

2° Select in an arbitrary manner in the subspace H^s_q an orthogonal base $u_1, u_2, \ldots, u_{n_s}$, $n_s = \dim H^s_q$,

3° Apply successively the formula (5) to each of the vectors u_i, $i = 1, 2, \ldots, n_s$, to find irreducible subspaces $H^s_{(u_i)}$ containing these vectors.

According to proposition 1 the corresponding irreducible subspaces $H^s_{(u_1)}, H^s_{(u_2)}, \ldots, H^s_{(u_{n_s})}$, will be mutually orthogonal. The collection of all of them provides the effective decomposition of the reducible subspace $P^sH = n_s H^s$ into irreducible components $H^s_{(u_i)}$, $i = 1, 2, \ldots, n_s$ in which the same irreducible unitary representation $T^s = \{D^s_{ij}\}$ is realized. Applying successively this method for all s we obtain the effective decomposition of T into irreducible components T^s.

B. *The Coupling Coefficients* (*'Clebsch–Gordan Coefficients'*)

Let T^{s_1} and T^{s_2} be two irreducible representations of G in Hilbert spaces H^{s_1} and H^{s_2}, respectively. Let $|s_i p_i\rangle$ be the orthonormal basis vectors in H^{s_i}, $i = 1, 2$. Suppose first that G is *simply reducible*, i.e., that in the *tensor product* definition of the two irreducible representations the multiplicity of a given representation is at most one. In the tensor product space $H = H^{s_1} \otimes H^{s_2}$, we can construct two sets of orthogonal basis vectors. The first consists of the Kroenecker product of the original basis vectors

$$|s_1 p_1, s_2 p_2\rangle = |s_1 p_1\rangle |s_2 p_2\rangle, \quad p_1 = 1, 2, \ldots, d_{s_1},\ p_2 = 1, 2, \ldots, d_{s_2}, \tag{6}$$

while the second one contains the basis vectors

$$|sps_1 s_2\rangle \tag{7}$$

which span an irreducible carrier space H^s contained in the tensor product space $H^{s_1}\otimes H^{s_2} = \sum_s \oplus H^s$. According to eq. (3) the basis vectors (7) can be obtained from the basis vectors (6) by means of the formula

$$|s_1 s_2 sp\rangle = (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} d_s \int_G \bar{D}^s_{pp'}(x) T_x |s_1 p_1' s_2 p_2'\rangle \mathrm{d}x, \quad p', p_1', p_2' \text{ fixed}, \tag{8}$$

where $N^{ss_1s_2}_{p'p_1'p_2'}$ is a normalization constant. The operator T_x acts on the basis vectors (6) by means of the formula

$$T_x|s_1 p_1 s_2 p_2\rangle = D^{s_1}_{p_1'p_1}(x) D^{s_2}_{p_2'p_2}(x) |s_1 p_1' s_2 p_2'\rangle. \tag{9}$$

The so-called 'Clebsch–Gordan coefficients' are the matrix elements of the unitary operator (called the *transition matrix*) connecting the basis vectors (6) and (7) and are given by (from (8) and (9))

$$\langle s_1 p_1 s_2 p_2 | sps_1 s_2\rangle = (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} d_s \int_G \bar{D}^s_{pp'}(x) D^{s_1}_{p_1 p_1'}(x) D^{s_2}_{p_2 p_2'}(x) \mathrm{d}x. \tag{10}$$

In order to find the normalization constant $N^{ss_1s_2}_{p'p_1'p_2'}$ we calculate the square of basis vector (8)

$$\begin{aligned}
1 &= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2} \langle s_1 p_1' s_2 p_2' | P^{s*}_{pp'} P^s_{pp'} | s_1 p_1' s_2 p_2'\rangle \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2} \langle s_1 p_1' s_2 p_2' | P^s_{p'p'} | sp_1' s_2 p_2'\rangle \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2} d_s \int_G \mathrm{d}x\, D^s_{p'p'}(x) \bar{D}^{s_1}_{p_1'p_2'}(x) \bar{D}^{s_2}_{p_1'p_2'}(x) \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} \overline{\langle s_1 p_1' s_2 p_2' | s_1 s_2 sp'\rangle}.
\end{aligned}$$

Hence

$$N^{ss_1s_2}_{pp_1'p_2'} = \overline{\langle s_1 p_1' s_2 p_2' | s_1 s_2 sp'\rangle} = \langle s_1 s_2 sp' | s_1 p_1' s_2 p_2'\rangle. \tag{11}$$

We see, therefore, that the normalization constant is itself another Clebsch–Gordan coefficient. Because the indices p', p_1' and p_2' are arbitrary, we can select C–G coefficient (11) as simple as possible.

The C–G coefficient (10) are not determined uniquely. Indeed, if we multiply the basis vectors (6) by constant phase factors φ_s, $|\varphi_s| = 1$, we obtain again a complete orthonormal system. Hence, the C–G coefficients are determined up to a phase factor. We can use this arbitrariness to set one of the C–G coefficients to be non-negative. Then the normalization constant in (10) is determined uniquely. Indeed, setting in eq. (10) $p = p'$, $p_1 = p_1'$, and $p_2 = p_2'$ one obtains, by virtue of eq. (11),

$$N^{ss_1s_2}_{p'p_1p_2'} = \left[\int_G \bar{D}^s_{p'p'}(x) D^{s_1}_{p_1'p_1'}(x) D^{s_2}_{p_2p_2}(x)\, \mathrm{d}x\right]^{1/2}. \tag{12}$$

Thus, the knowledge of the matrix elements D^s_{pq} of irreducible representations allows us to determine completely the C–G coefficients for simple reducible groups.

If the given group G is not simply reducible we first split out a factor representation $n_s T^s$ in the tensor product space onto irreducible representations using the technique of the subsection A. Having constructed the basis vectors (7) in the carrier spaces $H^s_{(u_1)}, H^s_{(u_2)}, \ldots, H^s_{(u_{n_s})}$, we proceed as above.

The formulas (10) and (12) constitute the basis for the explicit determination of the C–G coefficients for compact groups of physical interest, such as SO(3), SO(4), SU(3), etc. Because the matrix elements $D^s_{pq}(x)$ can be expressed in terms of the products of special functions (cf. ch. 14) the problem of the determination of C–G coefficients reduces to the problem of integration of the product of three special functions over a finite region.

C. *A Physical Application*: *Distribution of Isotopic-Spin States*

The strongly interacting particles can be assigned certain internal quantum numbers, in addition to mass, total angular momentum (spin) and parity. The internal quantum numbers distinguish particles with the same spin and parity, and they are conserved in strong interactions. In particular we assign to each such particle an isotopic spin t, associated with an (internal) SU(2)-symmetry group. Let I_1, I_2 and I_3 be the generators of this SU(2)-group. The state of the particle is then characterized by $|t, \theta, x\rangle$, where $I^2 = I_1^2+I_2^2+I_3^2$ and I_3 have the eigenvalues $t(t+1)$ and θ, respectively, and x stands for the remaining set of quantum numbers. For a collection of free particles we take, as far as the isotopic spin quantum number is concerned, the tensor product space $|t_1\theta_1, t_2\theta_2, \ldots\rangle$. The total isotopic spin of the systems is defined by the vector sum $I = \sum_i t_i$. In a strong collision process the total isotopic spin of the initial particles is equal to the total isotopic spin of the final particles. In other words the S-matrix expressing the transition probability amplitude from the initial state to the final state commutes with the representation $g \to T_g$ of the isotopic spin group SU(2) in the Hilbert space of state vectors. (See ch. 13 for more details on quantum mechanical invariance properties.)

Consider the process of scattering of two particles in the reaction

$$A+B \to C_1+C_2+ \ldots +C_n,$$

where the particle C_i, $i = 1, 2, \ldots, n$, has the isospin t_i and the third isospin component θ_i. The probability that a certain final state with values $\theta_1, \theta_2, \ldots, \theta_n$ will be found for a given total isospin I and its third component I_3 is

$$P^{I}_{\theta_1,\ldots,\theta_n} = \Big[\sum_{\alpha}|\langle t_1\theta_1,\ldots,t_n\theta_n|II_3\alpha\rangle|^2\Big]\Big[\sum_{(\theta_i)}\sum_{\alpha}|\langle t_1\theta_1,\ldots,t_n\theta_n|II_3\alpha\rangle|^2\Big]^{-1}. \quad (13)$$

Here α stands for all additional isotopic quantum numbers required to describe the n-particle state completely in the isospin space. We have assumed that each state α for fixed I, I_3 occurs with equal probability. Using the technique of projection operators we can easily find the final expression for the probability (13). In fact the quantity

$$\sum_{\alpha}|\langle t_1\theta_1,\ldots,t_n\theta_n|II_3\alpha\rangle|^2 \quad (14)$$

is nothing but the square of the projection of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ onto the subspace in the tensor product space, spanned by the basis vectors $|I, I_3, \alpha\rangle$ with given I, I_3. Therefore, if one could compute directly the length of this projection one could dispense with the lengthy computation of the components and the determination of the complete set of quantum numbers α. In general, more than one $(2I+1)$-dimensional subspace of a representation T^I may occur in the tensor product of one-particle representations $\prod_{i=1}^{n}\otimes T^{t_i}$. Let us denote the direct sum of these subspaces by $\Gamma^{(I)}$. Then the projection on $\Gamma^{(I)}$ is given by the formula 3(18) where now

$$P^{\Gamma^{(I)}} = \frac{2I+1}{8\pi^2}\int\bar{\chi}^{(I)}(\varphi,\vartheta,\psi)T_{g(\varphi,\vartheta,\psi)}\sin\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,\mathrm{d}\psi, \quad (15)$$

$$T_g = \prod_{k=1}^{n} T_g^{(k)}, \qquad T_g^{(k)} = \exp[-\mathrm{i}\varphi(t_k)_z]\exp[-\mathrm{i}\vartheta(t_k)_y]\exp[-\mathrm{i}\psi(t_k)_z]$$

or in the matrix form

$$[T_{g(\varphi,\vartheta,\psi)}]_{\theta'_1\theta_1,\ldots,\theta'_n\theta_n} = \prod_{k=1}^{n} D^{t_k}_{\theta'_k\theta_k}(\varphi,\vartheta,\psi).$$

The square of the projection of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ onto $\Gamma^{(I)}$ is

$$|P^{\Gamma^{(I)}}|t_1\theta_1,\ldots,t_n\theta_n\rangle|^2 = \langle t_1\theta_1,\ldots,t_n\theta_n|P^{\Gamma^{(I)}}|t_1\theta_1,\ldots,t_n\theta_n\rangle,$$

where we made use of formula 3(10). Consequently

$$P^{I}_{\theta_1,\ldots,\theta_n} = \frac{2I+1}{8\pi^2}\int\bar{\chi}^{(I)}(\varphi,\vartheta,\psi)\prod_{k=1}^{n} D^{(t_k)}_{\theta_k\theta_k}(\varphi,\vartheta,\psi)\sin\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,\mathrm{d}\psi. \quad (16)$$

Using the representation of $D^J_{MM}(g)$ in the form (cf. exercise 5.8.1.1)

$$D^J_{M,M}(\varphi,\vartheta,\psi) = \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0,2M}_{J-M}(\cos\vartheta)\exp[-\mathrm{i}M(\varphi+\psi)],$$

where $P^{0,2M}_{J-M}(x)$ is the Jacobi polynomial, and using the expression 3(16) for $\chi^{(I)}$ we get, after an integration over φ and ψ,

$$P^{(I)}_{\theta_1,\dots,\theta_n} = (2I+1)2^{-2I_3-1}\int_{-1}^{+1} dx(1+x)^{2I_3}P^{0,2I_3}_{I-I_3}(x)\prod_{i=1}^{n} P^{0,2\theta_i}_{t_i-\theta_i}(x). \tag{17}$$

In the derivation of the last formula the condition $I_3 \geqslant 0$ was assumed. This is no restriction, however, because one can prove easily that

$$P^{I}_{\theta_1,\dots,\theta_n} = P^{I}_{-\theta_1,\dots,-\theta_n}.$$

The integrand in eq. (17) is a polynomial of degree $N = I+\sum_{i=1}^{n} t_i-2I_3$. If we represent this polynomial by

$$\sum_{j=0}^{N} a_j x^j$$

then

$$P^{I}_{\theta_1,\dots,\theta_n} = (2I+1)2^{-2I_3}\sum_{j=0}^{[N/2]'}\frac{a_{2j}}{2j+1}. \tag{18}$$

From eqs. (17) or (18) we obtain the final expression for $P^{I}_{\theta_1,\dots,\theta_n}$ in various special cases:

(i) Case of n pions (e.g., $p+\bar{p} \to n_+\pi^+ + n_0\pi^0 + n_-\pi^-$)

$$P^{I}_{n_+,n_0,n_-} = (2I+1)2^{-2n_+-I+I_3}(-1)^{I-I_3}\times$$
$$\times\sum_{\nu=0}^{I-I_3}\binom{I-I_3}{\nu}\binom{I+I_3}{I-I_3-\nu}(-1)^{\nu}[A(2n_+ +\nu, n_0, I-I_3-\nu)+$$
$$+(-1)^{n_0}A(I-I_3-\nu, n_0, 2n_+ +\nu)],$$

where

$$A(a,b,c) \equiv \sum_{p=0}^{a}\binom{a}{p}\frac{(p+b)!\,c!}{(p+b+c+1)!}.$$

(ii) Case of one nucleon and n pions

$$P^{I}_{m_1;n_+,n_0,n_-} = (J+1)2^{-(n_++n_-+J+1)}(-1)^{J-M}\times$$
$$\times\sum_{\nu=0}^{J-M}(-1)^{\nu}\binom{J-M}{\nu}\binom{J+M+2}{J-M-\nu}[A(2n_+ +1+\nu, n_0, J-M-\nu)+$$
$$+(-1)^{n_0}A(J-M-\nu, n_0, 2n_+ +1+\nu)],$$

where $J = I-1/2$, $M = I_3-1/2$, and m_1 is the third component of isospin of nucleon.

Note that the permutations which shuffle particles of the same kind and charge among themselves only do not lead to distinguishable isospin states. Therefore, in order to find the probability of a certain charge distribution, one has to multiply the coefficient $P^I_{\theta_1, \dots, \theta_n}$ with the number of permutations of the $\theta_1, \dots, \theta_n$, which result in a reordering of the numbers $\theta_1, \dots, \theta_n$ only. For example the weight of the charge distribution of n pions, without consideration of their momenta is given by

$$\tilde{P}^I_{n_+, n_0, n_-} = \frac{n!}{n_+! n_0! n_-!} P^I_{n_+, n_0, n_-}.$$

§ 5. Representations of Finite Groups

In this section we treat the properties of finite groups and their representations. Finite groups have many important applications in quantum physics, especially in atomic, molecular and solid state physics. For this reason, and in order to be able to discuss the representations of the symmetric group S_N, we give a concise discussion of the representations of finite groups.

Every finite group is compact. Hence, all theorems of this chapter remain true for finite groups. It is only necessary to replace in all formulas the integral over the group manifolds $\int dx$ by the sum over group elements.

A. *The Symmetric Group S_N*

The symmetric group S_N, the group of permutations of N objects, of order $N!$, is fundamental in the study of finite groups, in the applications as well as in the representation theory of continuous groups. We have namely:

THEOREM 1 (Cayley). *Every finite group G of order N is isomorphic to a subgroup of S_N.*

PROOF: Consider the permutation of the elements of G defined by left-multiplication by an element x

$$_x\pi = \downarrow \begin{pmatrix} x_1 & x_2 & \dots x_N \\ xx_1 & xx_2 & \dots xx_N \end{pmatrix}. \tag{1}$$

We denote the permutation of a set $(x_1 \dots x_N)$ into the set $(xx_1 \dots xx_N)$, i.e., an element of S_N, by an arrow as above. All permutations $\{_x\pi, x \in G\}$ form a group. The map f: $x \to {_x\pi}$ is one-to-one, and

$$f\colon xy \to {_x\pi}\, {_y\pi} = {_{xy}\pi}. \blacktriangledown$$

Clearly, one can also define a permutation by a right-multiplication.

The elements of S_N can be generated from the simpler elements called cycles and transpositions. A *cycle* is a permutation in which some $r \leqslant N$ objects are permuted among themselves in a cyclic way. For example,

$$\downarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 & 8 & 9 & 10 \end{pmatrix} = (1\ 2\ 3\ 4)(5\ 6\ 7)(8)(9)(10), \tag{2}$$

where we have introduced the standard notation for cycles. A cycle can be written in any order: $(1\ 2\ 3\ 4) = (2\ 3\ 4\ 1) = (3\ 4\ 1\ 2) = (4\ 1\ 2\ 3)$; the product of disjoint cycles is commutative: $(1\ 2\ 3\ 4)(5\ 6\ 7) = (5\ 6\ 7)(1\ 2\ 3\ 4)$; and one-cycles may be omitted in the decomposition (2). A cycle of two symbols is called a *transposition.* Any cycle can be written as a product of transpositions: $(1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$ (operations from left to right). Continuing this process we have

THEOREM 2. 1° *Every permutation may be represented by a product of disjoint cycles (unique up to an ordering of factors).*

2° *Every permutation may be represented by a product of transpositions of adjacent symbols; the number of transposition in any decomposition is either always even or odd for a given $x \in S_N$.* ▼

Two group elements x_1 and x_2 are *conjugate* if there exists another group element y such that $x_1 = yx_2y^{-1}$. This relationship is 1° reflexive: $x_1 = ex_1e^{-1}$, 2° symmetric: $x_2 = y^{-1}x_1y$, and 3° transitive: if $x_1 = yx_2y^{-1}$ and $x_2 = zx_3z^{-1}$, then $x_1 = (yz)x_3(yz)^{-1}$. Hence the group can be divided into *classes of conjugate group elements.* The identity element is a class itself. Elements in a class have a lot of things in common.

In S_N, if $x_1 = (1\ 5\ 3\ 6\ 7\ 4\ 2)(8\ 10)$, for example, and $y = \downarrow\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 \ldots \\ i_1 & i_2 & i_3 & i_4 & \ldots & \end{pmatrix}$, then $x_2 = yx_1y^{-1} = (i_1\ i_5\ i_3\ i_6\ i_7\ i_4\ i_2)(i_8\ i_{10})$. Thus, all elements in a class have the same cycle structure. The cycle lengths themselves are characterized by the partitions of N, hence the number of classes in S_N is equal to the number of partitions of N.

Because the cycles commute, we can order them from large to small. Thus a cycle structure (a partition) is given by the set of numbers λ_i satisfying

$$N = \lambda_1 + \lambda_2 + \ldots + \lambda_k, \quad \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \ldots \geqslant \lambda_k \geqslant 0, \tag{3}$$

where k is arbitrary. Alternatively, the partitions may be characterized by the set of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ such that

$$N = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + N\alpha_N, \quad \alpha_i \geqslant 0. \tag{4}$$

Here α_k is the number of cycles of length k. Clearly $\alpha_1 \leqslant N$, $\alpha_N \leqslant 1$, etc.

There is no general formula for the number of partitions, excepting infinite series (e.g. Rademacher 1937) but there are tables (cf. Gupta 1958). We shall, however, answer the question as to the number of elements in each class of S_N.

Let U be a subgroup of G. The elements of the form $\{xu|\ u \in U\} = xU$ form a coset. Clearly, two cosets xU and yU are either identical or have no elements in common. Every element x of G is in some coset, namely in the coset xU. All cosets have the same order equal to the order of U. Thus the group is divided into ν disjoint cosets, where

$$[G:U] = \nu \equiv \text{index of } U \text{ in } G = \frac{\text{order of } G}{\text{order of } U}. \tag{5}$$

Thus the order of a subgroup U (or of a coset xU) is a factor of the order of G (Lagrange's theorem).

The divisibility of the order of G also holds with respect to classes. We have namely

THEOREM 3. *The order of a class of conjugate elements is a factor of the order of the group G.*

PROOF: We define a subgroup U_x called *centralizer of* x in G

$$U_x = \{y \mid yxy^{-1} = x\}. \tag{6}$$

We want to know the number of distinct conjugate elements to x. Two elements uxu^{-1} and vxv^{-1} are identical if and only if u and v belong to the same left coset of U_x. Hence the number of distinct elements conjugate to x is equal to the number of cosets of U_x, or to the index of U_x which is a factor of the order of G by the previous theorem.

EXAMPLE 1. As an example we compute the order h_α of the class of S_N defined by eq. (4). By eq. (5) it is sufficient to evaluate the order of U_x defined by eq. (6). A permutation x of cycle structure $\alpha = (\alpha_1, \ldots, \alpha_N)$ is left-invariant in the form $yxy^{-1} = x$ by $\prod_j \alpha_j! j^{\alpha_j}$ permutations, because a cycle of length j remains unchanged by j cyclic permutations and furthermore α_j-cycles can be permuted among themselves. Thus

$$h_\alpha = N!/[\alpha_1! 1^{\alpha_1} \alpha_2! 2^{\alpha_2} \ldots \alpha_N! N^{\alpha_N}] \tag{7}$$

and $\sum_\alpha h_\alpha = N!$. ▼

A normal subgroup N of G consists entirely of classes.

EXAMPLE 2. For $N \neq 4$, the *alternating* group A_N, the subgroup of S_N consisting of even permutations, is the only proper normal subgroup of S_N; its index is 2. For $N \neq 4$, A_N is *simple*, i.e., contains no proper normal subgroup. For $N = 4$, A_4 contains the proper normal subgroup V_4 (the Klein four-group). (Note: A_4 is also known as T, the *tetrahedral group*, the group of rotations and reflections which leave a regular tetrahedron invariant.)

B. *Properties of Representations of Finite Groups*

The general definitions of ch. 5 naturally apply here, except that the concept of continuity is not needed. The criteria in recognizing equivalent and irreducible representations are embedded again in two Schur's lemmas. We now list some useful results in the language of finite groups:

1° Every representation of a finite group G is equivalent to a unitary representation (th. 1.1).

2° Every irreducible representation is finite-dimensional (th. 1.3).

3° Theorem of Maschke: Every reducible representation of finite groups is completely reducible, i.e., is a direct sum of irreducible representations (th. 1.4).

4° Let T^s and $T^{s'}$ be two *irreducible* representations of G. Then the corresponding matrix elements in an orthonormal basis satisfy

$$\sum_{x\in G} D_{ji}^{(s)-1}(x) D_{mn}^{(s')}(x) = \delta^{ss'}\delta_{im}\delta_{jn}\frac{h}{d_s}, \tag{8}$$

where d_s is the dimension of D^s, h is the order of the group and s labels different irreducible representations. If D^s is unitary, then $\overline{D_{ji}^{s-1}(x)} = D_{ij}^s(x)$ (th. 1.5).

5° If $d_1, d_2, \ldots, d_k$ are the dimensions of the irreducible representations, then $h = d_1^2+d_2^2+\ldots+d_k^2$.

6° The number of distinct irreducible representations is equal to the number of conjugate classes.

Thus, the symmetric group S_N can have as many irreducible representations as there are partitions of N.

Because an arbitrary representation is a direct sum of irreducible finite-dimensional representations, an arbitrary character is given by

$$\chi(x) = \sum_s \lambda_s \chi^{(s)}(x), \qquad x \in G, \tag{9}$$

where $\chi^{(s)}$, $s = 1, 2, \ldots$, are the so-called *primitive* or *simple characters* of irreducible representations satisfying (8). Hence the general or *compound character* (9) satisfies

$$\sum_{x\in G} \bar{\chi}(x)\chi(x) = h\sum_s \lambda_s^2 \geqslant h. \tag{10}$$

Eq. (10) is a criterion of reducibility; a criterion of irreducibility is

$$(\chi, \chi) = h. \tag{11}$$

Consider now the right regular representation of G defined in th. 1. The character of this representation is

$$\chi^{\text{reg}}(x) = \begin{cases} h, & x = e, \\ 0, & x \neq e. \end{cases} \tag{12}$$

Thus

$$\sum_{x\in G} \bar{\chi}^{\text{reg}}_{(x)} \chi^{\text{reg}}_{(x)} = h^2. \tag{13}$$

Consequently the regular representation is reducible in the form (9) and the coefficients λ_s satisfy

$$\sum_s (\lambda_s^{\text{reg}})^2 = h. \tag{14}$$

On the other hand, from

$$\chi^{\mathrm{reg}}(x) = \sum_s \lambda_s^{\mathrm{reg}} \chi^{(s)}(x)$$

for $x = e$, we directly obtain

$$h = \sum_s \lambda_s^{\mathrm{reg}} l_s. \tag{15}$$

From (14) and (15) we have $l_s = \lambda_s^{\mathrm{reg}}$ and $h = \sum_s l_s^2$, thus

7° The dimension of an irreducible representation is equal to the number of times it is contained in the regular representation (th. 2.2). Every irreducible representation occurs in the regular representation (th. 1.6) and

$$h = \sum_{s=1}^{f} l_s^2. \tag{16}$$

C. *Representations of* S_N

We begin with the regular representation of S_N which is particularly suited to many physical situations. Consider functions of N objects, $f(1\ 2 \dots N)$, for example, the wave function of a system of N particles; each argument $1, 2, \dots, i, \dots$ stands for the set of quantum numbers of the ith particle. Let $x(1\ 2 \dots N)$ be a permutation of the N objects, $x \in S_N$. The $N!$ functions

$$f\{x(1\ 2 \dots N)\} = f_x(1\ 2 \dots N), \qquad x \in S_N, \tag{17}$$

form a basis of the regular representation of dimension $h = N!$ In this basis a group element x is represented by

$$x f_{x_i} = \sum_{j=1}^{N!} D(x)_{ji} f_{x_j}, \tag{18}$$

where

$$D(x)_{ji} = \delta_{jk}, \qquad \text{if } x x_i = x_k. \tag{19}$$

The regular representation is reducible. For the completely symmetric and the completely antisymmetric functions, denoted by $f(1\ 2 \dots N)$ and $f\{1\ 2 \dots N\}$ respectively, for example, form each a one-dimensional invariant subspace under the transformations (1). In fact, we know from the general theorems of the previous section that the regular representation must contain every irreducible representation of dimension l_s, l_s times. Because the number of irreducible representations (equal to the number of conjugate classes) is given by the number of partitions of N, an irreducible representation corresponding to a partition

$$(\lambda_1, \dots, \lambda_k), \quad \sum \lambda_i = N, \quad \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k \geqslant 0,$$

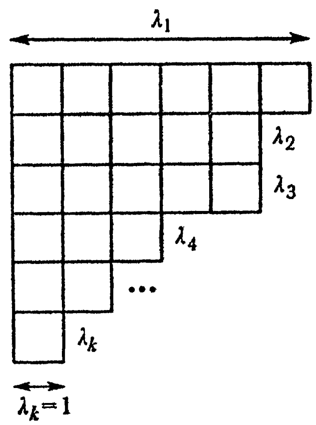
defines a set of functions of a definite symmetry type

$$f\{[1\,2\,\ldots\,\lambda_1][\lambda_1+1,\,\ldots,\,\lambda_1+\lambda_2],\,\ldots\} \tag{20}$$

which are completely symmetric in the first λ_1 variables, completely symmetric with respect to the next λ_2 variables, etc. The number of linearly independent functions of a definite symmetry type is equal to the dimension of the corresponding irreducible representation. Functions of a given symmetry type transform among themselves.

The above discussion characterizes in an intuitive way all the irreducible representations of S_N, but it is necessary to make the matters much more explicit and precise.

First of all, a partition of N can be shown diagrammatically by the following 'Young frame', for example $(\lambda_1, \ldots, \lambda_\kappa)$ is



These are N boxes and $N!$ ways of distributing the N numbers $1, 2, \ldots, N$ into these boxes. A Young frame with the numbers written in it is called a *'Young tableau'* (Frobenius–Young tableau). Thus there are $N!$ tableaux corresponding to any frame.* In terms of our functions (4), the $N!$ functions corresponding to $N!$ tableaux are not linearly independent. The number of linearly independent functions are given by the number of standard tableaux.

A *standard tableaux* is one in which the integers $1, 2, \ldots, N$ are distributed in increasing order in every row from left to right, and in every column from top to bottom in the Young frame of the figure above.

It is then a combinatorial problem to evaluate the number of standard tableaux $l_{(\lambda)}$ corresponding to partition (λ). We give three such formulas:

1)
$$l_{(\lambda)} = \frac{N!}{\prod_{i=1}^{k} n_i!} \prod_{i<k} (n_i - n_k), \tag{21}$$

* 'Frame' and 'tableau' are also called 'table' and 'diagrams', respectively.

where (see figure)

$$\begin{aligned} n_1 &= \lambda_1 + (k-1), \\ n_2 &= \lambda_2 + (k-2), \\ &\cdots\cdots\cdots\cdots \\ n_k &= \lambda_k. \end{aligned}$$

It can be verified that

$$\sum_{(\lambda)} l_{(\lambda)}^2 = N!.$$

2) Formula of W. Feit 1953

$$l_{(\lambda)} = N! \det[1/(\lambda_i - i + j)!], \qquad i, j = 1, 2, \ldots, N. \tag{22}$$

The determinant in (22) is that of an $N \times N$-matrix A with matrix elements $a_{ij} = 1/(\lambda_i - i + j)!$. Note that

$$\frac{1}{x!} = 0 \quad \text{if } x < 0, \quad \text{and} \quad 1/0! = 1.$$

3) Formula of J. S. Frame, G. de B. Robinson and R. M. Thrall 1954

$$l_{(\lambda)} = N! \Big/ \prod_{i,j} h_{ij}, \tag{23}$$

$$h_{ij} = 1 + \lambda_i + \bar{\lambda}_j - (i + j).$$

Here $\bar{\lambda}_j$ = number of boxes in column j.

We shall see that to each frame corresponds an irreducible representation of S_N of dimension $l_{(\lambda)}$.

In order to discuss the irreducible representations of the group S_N, we consider the group algebra $\mathscr{A}$ as the regular representation and decompose $\mathscr{A}$ into its irreducible parts by projection operators (idempotents).

Consider a tableau T. Let

$H(T)$ = the set of horizontal permutations, that is, the set of permutations $p \in S_N$ which permute the numbers in each row of T but do not move any number from one row to another,

$V(T)$ = the set of vertical permutations, that is, permutations $q \in S_N$ which permute the numbers in each column of T but do not move any number from one column to another.

$H(T)$ and $V(T)$ are subgroups of S_N, and clearly, they have only the identity element in common: $H(T) \cap V(T) = I$.

PROPOSITION. *The quantities defined, for each tableau, by*

$$e(T) = \sum_{\substack{p \in H(T) \\ q \in V(T)}} \varepsilon_q pq \in \mathscr{A},$$

$$\varepsilon_q = \begin{cases} +1 & \textit{for even permutations,} \\ -1 & \textit{for odd permutations} \end{cases}$$

are essentially idempotent (that is a scalar multiple of an idempotent element in it; $e(T)^2 = \frac{N!}{l_{(\lambda)}} e(T)$*) and provide us with the decomposition of* $\mathscr{A}$ *into irreducible subspaces.*

PROOF: Clearly $e(T) \neq 0$ in $\mathscr{A}$. Furthermore, we have the relations

$$p_1 e(T) = \sum \varepsilon_q p_1 pq = \sum \varepsilon_q p' q = e(T),$$

for $p_1 \in H(T)$, and similarly $e(T)q_1 = \varepsilon_{q_1} e(T)$ for $q_1 \in V(T)$. We form the ideal $\mathscr{A}e(T)$. We shall show that all these left ideals are minimal, that ideals coming from different tableaux but the same frame, are equivalent (isomorphic), and that ideals coming from different frames are nonequivalent.

Consider two tableaux T and T' of a given frame. We write $T' = gT$, where $g \in S_N$ is the permutation which changes the digits of T into those of T', that is, if the number α is in the position (i,j) of T, then $g\alpha$ is in the position (i,j) of $T' = gT$. For these two tableaux, $e(T)$, $e(T')$, $H(T)$, $H(T')$, are related by

$$e(T') = ge(T)g^{-1} = e(gT),$$
$$H(T') = gH(T)g^{-1} = H(gT),$$
$$V(T') = gV(T)g^{-1} = V(gT).$$

For, if $p \in H(T)$, then p permutes the rows of T, and gpg^{-1} permutes the rows of T', and so on. Let us look now at the corresponding ideals $\mathscr{A}e(T)$ and $\mathscr{A}e(T')$:

$$\mathscr{A}e(T') = \mathscr{A}ge(T)g^{-1} = \mathscr{A}e(T)g^{-1}.$$

The two ideals are related to each other by a right multiplication, hence they are equivalent.

Consider now two different frames associated with the partitions $(n_1 n_2 \dots n_\kappa)$ and $(n_1' \, n_2' \dots n_n')$, respectively We write $(n_1 \, n_2 \cdots n_k) > (n_1' \, n_2' \dots n_n')$ meaning that at the first position where the arrays differ, $n_i > n_i'$. For two tableaux from two such frames we have

$$e(T_1)e'(T_2) = 0.$$

To show this we notice that there must exist two symbols α and β which are collinear somewhere in T and co-columnar somewhere in T' otherwise one can show by suitable permutations that $n_1 = n_1'$, $n_2 = n_2'$, ... Let h be the permutation changing α and β. Then $h \in H(T_1)$ and $h \in V(T_2)$. Hence

$$e(T_2)e(T_1) = e(T_2')hhe(T_1) = -e(T_2')e(T_1),$$

where we used the property $e(T)q = \varepsilon_q q$. Hence

$$e(T_1)e'(T_2) = 0.$$

Next we show the idempotent character of $e(T)$. For any number α, we get $p\alpha e(T)q = \varepsilon_q \alpha e(T)$. Conversely, let $a \in \mathscr{A}$ be such that $paq = \varepsilon_q a$, for all $p \in H(T)$ and $q \in V(T)$. Then there exists a number α such that $a = \alpha e(T)$. To see this, let $a = \sum \alpha(x)x$, $x \in S_N$. Then

$$a = \varepsilon_q p^{-1} a q^{-1} = \varepsilon_q \sum_x \alpha(x)(p^{-1}xq^{-1}) = \varepsilon_q \sum_y \alpha(pyq)y.$$

Thus

$$\alpha(y) = \varepsilon_q \alpha(pyq) \quad \text{for } p \in H(T),\ q \in V(T).$$

Setting $y = 1$ we have $\alpha(1) = \varepsilon_q \alpha(pq)$. To complete the proof we must show that $\alpha(x) = 0$, if x is not of the form pq, $p \in H(T)$, $q \in V(T)$. This is indeed the case: if x is not of the form pq, there must exist symbols collinear in T and co-columnar in $T' = xT$. Let h be the permutation of α and β, then $h \in H(T)$ and $h \in V(xT)$, and so $h = xqx^{-1}$ for some $q \in V(T)$ and

$$\alpha(x) = \varepsilon_{q^{-1}} \alpha(hyq^{-1}) = \varepsilon_{q^{-1}} \alpha(x) = -\alpha(x).$$

Therefore $\alpha(x) = 0$, if x is not of the form pq. Now consider

$$pe(T)^2 q = pe(T)e(T)q = \varepsilon_q e(T)^2.$$

By the preceding result, we have then $e(T)^2 = \alpha e(T)$. To evaluate α we consider the map $T(a) = ae(T)$, $a \in \mathscr{A}$, and the matrix form of T in the basis consisting of group elements $x_1 = 1, x_2, x_3, \ldots, x_N$. Then if

$$e(T) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots$$

we have

$$x_1 e(T) = \alpha_1 x_1 + \alpha_2(x_1 x_2) + \ldots,$$

$$x_2 e(T) = \alpha_1(x_2 x_1) + \alpha_2 x_2 + \ldots,$$

so that $\mathrm{Tr}(T) = \alpha_1 N!$. Furthermore $\alpha_1 = 1$, since $x_1 = 1$ occurs with coefficient 1 in $e(D)$. Consider now a second basis $(y_1, \ldots, y_l, \ldots, y_N)$, such that $(y_1, \ldots, y_l)$ is a basis for the ideal $J = \mathscr{A}e(T)$. Now $a_1 e(T) = \alpha a_1$ for $a_1 \in J$, and so

$y_1 e(T) = \alpha y_1,$

$y_l e(T) = \alpha y_l,$

$y_{l+i} e(T) =$ the first l elements non-zero, the remaining elements zero since $y_{l+i} e(T) \in \mathscr{A}e(T)$.

Now trace $(T) = \alpha l$. Since trace is an invariant

$$\alpha l = N!, \qquad \alpha = N!/l.$$

Thus, the quantity $u = \dfrac{l}{N!} e(T)$ is truly idempotent.

Finally we show that the ideal $\mathscr{A}e(T)$ is minimal. It suffices to show that $e(T)\mathscr{A}e(T)$ is a numerical multiple of $e(T)$. Now

$$pe(T)\mathscr{A}e(T)q = e(T)\mathscr{A}e(T)\varepsilon_q, \quad p \in H(T),\ q \in V(T),$$

and by a previous lemma $e(T)\mathscr{A}e(T)$ is a multiple of identity.

If two tableaux T_1 and T_2 belong to different frames, we know that $e(T_1)e(T_2) = 0$. Hence for $x \in S_N$, $e(T_2)xe(T_1) = e(T_2)e(xT_1)x = 0$. Hence the two ideals $\mathscr{A}e(T_1)$ and $\mathscr{A}e(T_2)$ are inequivalent. ▼

To summarize we have shown that each tableau T corresponding to a Young frame $(n_1\ n_2\ \dots\ n_k)$ defines an essentially idempotent element $e(T) = \sum_{p,q} \varepsilon_q pq$, such that $\mathscr{A}e(T)$ is a minimal left ideal of the group algebra $\mathscr{A}$ of S_N and thus an irreducible component of the regular representation. Further, ideals coming from different tableaux with the same frame are isomorphic, but ideals from different frames are not.

Unfortunately, there is no general algorithm which gives the minimal ideals of any finite group algebra as was the case for S_N. This seems to be an unsolved problem.

§ 6. Comments and Supplements

(i) In the proofs of most of the theorems in this chapter, we used explicitly the finiteness of the group volume $V = \int_G dx$. Hence, we cannot expect that these theorems can be directly extended to noncompact groups for which $V = \infty$. However, a generalization of some of the theorems (in particular the Peter–Weyl theorem) to noncompact groups is possible (cf. ch. 14, § 2).

(ii) The functions $D^s_{pq}(x)$ play a special role in the theory of representations of compact groups and in its applications. Unfortunately these functions are explicitly known only in few cases: SO(3) (cf. exercise 5.8.1.1), SO(4) (cf. exercise 7.1.2), U(3) (cf. Chacón and Moshinski 1966). Gel'fand and Graev derived recursive formulas for D functions for $U(n)$. See also the work of Leznov and Fedoseev 1971.

(iii) One can show that for simple Lie groups the functions $D^s_{pq}(x)$ are eigenfunctions of a maximal set of commuting operators in the enveloping algebra. (Cf. ch. 14, § 2 for a general proof of this statement for compact and noncompact groups.) This property is the starting point in the explicit calculation of $D^s_{pq}(x)$ for specific groups.

(iv) The th. 1.3 was first proved by Gurevich 1943. Here we followed the elegant proof given by Nachbin 1961 (cf. also Koosis 1956). In the proof of th. 1.4 we followed Auslander 1961. The projection operators for finite and compact groups were extensively used by Wigner 1959. He first showed the effectiveness of this technique in the solution of a number of problems in quantum mechanics.

It is interesting that the theory of projection operators can also be extended to noncompact groups (cf. ch. 14, § 5).

The first explicit calculation of C–G coefficients is due to Wigner, who derived formula 4(10) for SO(3) group.

In the calculation of the weights of isospin states we followed the work of Cerulus 1961. He also calculated the other special cases of eq. 4(18). More general formulas summing over all possible final states have been discussed recently (cf. P. Rotelli and L. G. Suttorp 1972).

(v) In table 7.I we give all finite groups up to order 15. Referring to this table:

1) A_n = cyclic group of order n. If n is a prime number there is only one group, namely the cyclic group.

2) If p and q are relatively prime to each other, then $Z_{pq} \sim Z_p \times Z_q$ (isomorphic to the direct product).

3) D_n = dihedral groups of order $2n$ (group of transformations which map a regular n-polygon into itself, consisting of n rotations by an angle $2\pi r/n$, $r = 0, 1, 2, \ldots, n-1$, and the reflection of the plane plus a rotation by $2\pi r/n$). D_n may be generated by two elements x and y satisfying

$$x^2 = e, \quad y^n = e, \quad (xy)^2 = e.$$

4) $\langle 2, 2, m\rangle$ = dicyclic group of order $4m$. It is generated by two elements x, y satisfying*

$$x^4 = e, \quad x^2 = y^m \quad \text{and} \quad yx = xy^{-1}.$$

Table I. All Finite Groups up to Order 15

Order	Groups
1	Z_1
2	$Z_2 \sim S_2$
3	Z_3
4	Z_4, $Z_2 \times Z_2 \sim D_2$
5	Z_5
6	$Z_6 \sim Z_2 \times Z_3$, $S_3 \sim D_3$
7	Z_7
8	Z_8, D_4, $Z_4 \times Z_2$, Q, $Z_2 \times Z_2 \times Z_2$
9	Z_9, $Z_3 \times Z_3$
10	$Z_{10} \sim Z_2 \times Z_5$, D_5
11	Z_{11}
12	$Z_{12} \sim Z_3 \times Z_4$, $Z_2 \times Z_6 \sim Z_2 \times Z_2 \times Z_3$, $D_6 \sim Z_2 \times D_3$
	A_4, $\langle 2, 2, 3\rangle$
13	Z_{13}
14	$Z_{14} \sim Z_2 \times Z_6$, D_7
15	$Z_{15} \sim Z_3 \times Z_5$

* H. S. M. Coxeter and W. O. J. Moser 1965 give large classes of groups generated by relations of this type.

(vi) We discussed in this chapter the strongly continuous representations only. It turns out however that compact groups have interesting non-continuous representations. We give two theorems which nicely illustrate this problem.

THEOREM 1. *A unitary representation of a connected compact semisimple Lie group in a finite-dimensional Hilbert space is neccessarily continuous.*

(For the proof cf. Van der Waerden 1933).

Clearly, by virtue of the structure of compact groups discussed in ch. 3.8 this theorem is false for nonsemisimple groups.

THEOREM 2. *Let G be a locally compact topological group whose every irreducible unitary representation in a Hilbert space is continuous. Then G is discrete.*

(For the proof see Bichteler 1968).

Theorem 2 implies that every connected compact semisimple Lie group must admit a noncontinuous infinite-dimensional irreducible unitary representation in a Hilbert space. It is interesting that in case of the rotation group such representations have an important physical meaning.

§ 7. Exercises

§ 1.1. Show that the matrix elements of irreducible representations of SO(4) have the form

$$D^{J_1,J_2}_{M_1M_1',M_2M_2'}(\varphi,\vartheta,\psi,\alpha,\beta,\gamma) = D^{J_1}_{M_1M_1'}(\varphi,\vartheta,\psi)D^{J_2}_{M_2M_2'}(\alpha,\beta,\gamma)$$

where the functions $D^J_{MM'}$ are given by eq. 5.8 (1).

Hint. Use isomorphism so(4) $\sim$ so(3) $\oplus$ so(3) given in table 1.5.1.

§ 3.1. Let T^{λ_1} and T^{λ_2} be irreducible representations of SO(3). Derive the following relation for matrix elements:

$$D^{\lambda_1}_{m_1m_1'}(g)\,D^{\lambda_2}_{m_2m_2'}(g) = \sum_{\lambda=|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \langle\lambda_1 m_1\lambda_2 m_2|\lambda_1\lambda_2\lambda m\rangle D^{\lambda}_{mm'}(g)\times$$
$$\times\langle\lambda_1\lambda_2\lambda m'|\lambda_1 m_1'\lambda_2 m_2'\rangle.$$

Hint. Find the matrix elements of operator equality 5.8(3).

§ 4.1. Show that the Clebsch-Gordan series (5.8 (3) for SO(3)) implies the following relation for the matrix elements

$$D^{J_1}_{M_1M_1'}(g)D^{J_2}_{M_2M_2'}(g) = \sum_{J=|J_1-J_2|}^{J_1+J_2}\langle J_1M_1J_2M_2|J_1J_2JM\rangle\times$$
$$\times D^J_{M_1+M_2,M_1'+M_2'}(g)\langle J_1J_2JM'|J_1M_2'J_2M_2'\rangle.$$

Hint. Use eq. 5.8(3) and the completeness relation for $|J_1J_2JM\rangle$ states.

§ 5.1. Show that S_N can be generated from two elements $x = (1\,2)$ and $y = (1\,2\ldots N)$.

Hint. Any permutation can be written as products of cycles. Any cycle is of the form

$$(i_1 i_2 \dots i_p) = (i_1 i_2)(i_2 i_3) \dots (i_{p-1} i_p).$$

Any transposition is of the form

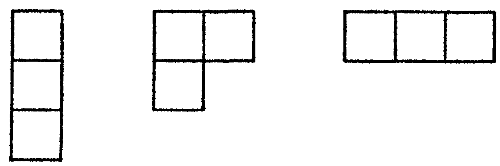
$$(i, j+1) = (j, j+1)(ij)(j, j+1)^{-1}.$$

Then $y^n x (y^n)^{-1}$ gives all transpositions of the form $(j, j+1)$.

§ 5.2. Find the multiplication table and a 2-dimensional and a 3-dimensional representation of $D_3 \sim S_3$. Find the normal subgroups and the conjugate classes.

§ 5.3. Show that the Pauli matrices $\pm I$, $\pm\sigma_1$, $\pm\sigma_2$, $\pm\sigma_3$ constitute a realization of the quaternion group $Q \sim \langle 2, 2, 2\rangle$ (see table 7.I).

§ 5.4. Consider S_3. There are three Young frames



The corresponding essential idempotents are

$$e_1 = \sum_{x \in G} x,$$

$$e_2^{(1)} = I + (1\,2) - (1\,3) - (1\,2),$$

$$e_2^{(2)} = I + (1\,3) - (1\,2) - (1\,3\,2),$$

$$e_3 = \sum_x \varepsilon(x) x.$$

Discuss the properties of the idempotents, the minimal left ideals and two-sided ideals that they generate, and the corresponding irreducible representations.

§ 5.5. If x, y, z are three identical objects show that $\frac{1}{\sqrt{6}}(x+y+z)$ and $\begin{pmatrix} \frac{1}{\sqrt{6}}(2x-y-z) \\ \frac{1}{\sqrt{6}}(y-z) \end{pmatrix}$ transform according to the one and 2-dimensional representation, respectively, of the permutation group S_3.

§ 5.6. Show that the Dirac matrices defined by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3.$$

generate (with respect to matrix multiplication) a finite group of order $h = 32$; that it has 17 classes, hence 17 irreducible representations, 16 of them of dimension 1 and one of dimension 4.

Chapter 8

Finite-Dimensional Representations of Lie Groups

In this chapter we present the theory of finite-dimensional, irreducible representations of an arbitrary, connected Lie group in global form. The global approach brings considerable simplifications to the theory as compared to the infinitesimal, Cartan–Weyl approach; it provides the classification of finite-dimensional, irreducible representations in terms of the highest weights, and at the same time, a simple canonical realization of the carrier space in terms of polynomials in some complex variables. This in turn makes the solution of various practical problems possible, such as the reduction of a representation of the given group to a subgroup, the decomposition of the tensor product, the explicit calculation of the 'Clebsch–Gordan coefficients', and so on.

In § 1 we discuss the general properties of the representations of solvable and semisimple Lie groups. In particular, we derive the global forms of the celebrated Lie and Weyl theorems.

In § 2 we elaborate the techniques of induced, finite-dimensional representations of Lie groups and prove the main theorem that every finite-dimensional irreducible representation of a Lie group G, which admits a Gauss decomposition, is the representation induced by a one-dimensional representation of a certain subgroup.

In §§ 3–6 we develop the global representation theory of complex and real classical Lie groups.

Finally, in § 7 the classification of finite-dimensional irreducible representations of arbitrary connected Lie groups is discussed. The method is based on the use of the Levi–Malcev decomposition of G and the properties of irreducible representations of solvable and semisimple groups.

In this chapter, because we are dealing exclusively with the finite-dimensional representations, we shall often omit, for simplicity, the term 'finite-dimensional'.

§ 1. General Properties of Representations of Solvable and Semisimple Lie Groups

The representation theory of Lie groups is based on the existence, for every complex or real Lie group, of a characteristic solvable connected subgroup. The explicit form of this subgroup is determined by the Levi–Malcev, Gauss or Iwasawa decompositions. We first prove the fundamental theorem of Lie on the representations of solvable groups. This theorem is the key for the classification of irreducible, finite-dimensional representations of arbitrary Lie groups. We give the global version of Lie's theorem which is convenient in the theory of induced representations.

A. *Representation Theory of Solvable Groups*

Let N be a solvable topological group. Let $Q(N)$ be its commutator subgroup i.e. the closure in the topology of G of the set generated by the elements of the form $xyx^{-1}y^{-1}$. Set $Q_i(N) = Q(Q_{i-1}(N))$. Because N is solvable, we have $Q_p(N) = \{e\}$, for some p; the smallest p is called the *height* of N. One easily sees that if N is a connected solvable group then $Q(N)$ is also connected and solvable.

THEOREM 1 (Lie). *Every finite-dimensional irreducible representation of a connected topological, solvable group N in a complex carrier space is one-dimensional.*

PROOF: We prove the theorem by induction. If N has the height one (i.e. N is abelian), the theorem follows from Schur's lemma 5.3.5. Assume that N has height p and the theorem is proved for groups of height $p-1$. Let $n \to T_n$ be a finite-dimensional, irreducible representation of N in a vector space H. The subgroup $Z = Q(N)$ is of height $p-1$. Hence the representation $z \to T_z$ of the subgroup $Z = Q(N)$ contains a one-dimensional representation of Z. Consequently, we can find a complex character $z \to \chi(z)$ and a non-zero vector u_χ in H such that

$$T_z u_\chi = \chi(z) u_\chi, \tag{1}$$

for every z in Z. Denote by Φ the set of characters χ of Z such that (1) has a non-zero solution u_χ in H. Clearly, Φ is a finite set. Because Z is invariant in N, we can define, for every character χ of Z and every $n \in N$, a new character χ_n given by the formula

$$\chi_n(z) \equiv \chi(n^{-1}zn). \tag{2}$$

Equation (1) implies

$$T_z T_n u_\chi = T_n T_n^{-1} T_z T_n u_\chi = \chi_n(z) T_n u_\chi. \tag{3}$$

Consequently, $\chi \in \Phi$ implies $\chi_n \in \Phi$ for every $n \in N$. Let us now introduce the following topology in Φ: $\chi \to \chi'$ if $\chi(z) \to \chi'(z)$ for every z. In this topology Φ is a discrete space. The continuity of group multiplication implies that for a given χ the character χ_n depends continuously on $n \in N$. On the other hand, the connectedness of N implies that for every χ in Φ the set of all χ_n is connected. Because this set is also finite we obtain $\chi_n = \chi$ for every $\chi \in \Phi$ and every $n \in N$. Thus we conclude that if $\chi \in \Phi$, the vectors u_χ, which satisfy eq. (1), span an invariant subspace of H under T_n. Because H is irreducible under T_n we conclude that for every $z \in Z$,

$$T_z = \chi(z)\cdot I. \tag{4}$$

Incidentally, Φ contains one character of Z only.

Let n_0 be an arbitrary element of N and let c be any root of the polynomial $x \to \det(T_{n_0} - xI)$. Because T_{n_0} is nonsingular, c cannot be zero. Hence, there exists a non-zero u_0 in H such that

$$T_{n_0} u_0 = c u_0. \tag{5}$$

Because $n_0 n\, n_0^{-1} n^{-1} \in Z$, eq. (4) implies

$$T_{n_0} T_n = \chi(n_0 n n_0^{-1} n^{-1}) T_n T_{n_0}.$$

From this and eq. (5), we obtain

$$T_{n_0} T_n u_0 = c\chi(n_0 n n_0^{-1} n^{-1}) T_n u_0.$$

Hence, for every $n \in N$, $T_n u_0$ is an eigenvector of T_{n_0}. The corresponding eigenvalue $c\cdot\chi(n_0 n n_0^{-1} n^{-1})$ depends continuously on n and has only finite number of possible values. Hence, the connectedness of N implies that χ does not depend on n. Setting $n = n_0$ we obtain $\chi(n_0 n n_0^{-1} n^{-1}) = 1$. Thus

$$T_{n_0} T_n u_0 = c T_n u_0.$$

Therefore the linear subspace $\{u_0 \in H;\ T_{n_0} u_0 = c u_0\}$ is invariant under all T_n. Consequently it must coincide with H. This, in turn, implies by Schur's lemma that T_{n_0} reduces to the scalar cI for every $n_0 \in N$. Consequently, because of irreducibility of T_{n_0}, we obtain the assertion of Lie's theorem. ▼

Corollary 1. *In any representation space of a connected solvable group N, there is a non-zero vector and a non-zero continuous multiplicative function $\chi(n)$ such that*

$$T_n u_\chi = \chi(n) u_\chi \quad \textit{for all } n \in N.$$

Proof: It is sufficient to pick up an irreducible subspace and apply Lie's theorem. ▼

Corollary 2. *Every representation of a connected, solvable group N can be reduced to the triangular form*

$$T_n = \begin{bmatrix} \chi^1(n) & & & 0 \\ & \chi^2(n) & & \\ & & \ddots & \\ & * & & \chi^N(n) \end{bmatrix}. \tag{6}$$

PROOF: The existence of an eigenvector for all T_n is equivalent to reducibility of the matrices T_n, i.e.,

$$T_n = \left[\begin{array}{ccc:c} & & & 0 \\ & \tilde{T}_n & & \vdots \\ & & & 0 \\ \hdashline * & \dots & * & \chi^N(n) \end{array}\right].$$

The matrices $\tilde{T}_n$ again form a representation of N. Hence, successive application of this form yields (6). ▼

Let us note that many important groups considered in physics are solvable: for instance, the Poincaré group $\Pi_2 = T^{(1,1)} \circledS \mathrm{SO}(1, 1)$ in two-dimensional space-time and the Heisenberg group associated with commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0, \quad (\text{or } [a, a^*] = 1),$$

are solvable groups of height 2. Indeed, for instance, the group multiplication in Π_2 implies $Q_1(\Pi_2) = T^{(1,1)}$, $Q_2(\Pi_2) = \{e\}$. Because these groups are connected we have by Lie's theorem

COROLLARY 3. *Every finite-dimensional, irreducible representation of the Poincaré group Π_2 and the Heisenberg group is one-dimensional.* ▼

Thus the groups of motion of one or two-dimensional Minkowskian and Euclidean space-times must be solvable.

B. *Representation Theory of Semisimple Lie Groups*

A connected simple Lie group has only the trivial one-dimensional representation, $g \to I$. Indeed, a connected simple group can have only two kinds of invariant subgroups: G_1 = discrete center of G (or a subgroup of it), and $G_2 = G$. If the homomorphism $g \to T_g$ has G_1 as a kernel, then T_g is a faithful representation of G/G_1. Consequently, it cannot be one-dimensional, because G/G_1 is noncommutative: if the kernel is G_2, then T_g is the identity.

Because semisimple connected Lie groups are direct products of invariant, simple, connected subgroups, these groups also have only the trivial one-dimensional representation.

Next we prove an interesting property of the representations of simple Lie groups:

THEOREM 2. *A connected, simple, noncompact Lie group G admits no finite-dimensional, unitary representations beside the trivial one.*

PROOF: Suppose first that the kernel of the mapping $g \to T_g$ consists of the identity only. Then, the homomorphism $g \to T_g$ is faithful. Consequently, the group $\{T_g\}$ is isomorphic to G. Moreover, $\{T_g\}$ is connected by continuity of the map $g \to T_g$. Hence, if G admits a finite, say, n-dimensional, unitary representation, then $\{T_g\}$ is a connected and simple subgroup of $U(n)$ which by th. 3.10 is closed. Consequently, G is compact.

Let, now, Z_G be the center of G. Clearly, Z_G is discrete. Let $\tilde{Z}_G$ be a subgroup of Z_G which is the kernel of the homomorphism $g \to T_g$. G is locally isomorphic to G/Z_G. By noncompactness of G the Killing forms of both G and $G/\tilde{Z}_G$ are not definite. Hence $G/\tilde{Z}_G$ is also noncompact and satisfies the assumption of the first part of the proof. ▼

This theorem has important consequences in quantum theory. Because a representation $g \to T_g$ of a physical symmetry group G must conserve the probability (scalar product), T must be unitary. (See ch. 14). On the other hand, many physical symmetry groups such as the Lorentz group SO(3, 1), or de Sitter group SO(4, 1), are simple and noncompact. Hence, we have to use infinite-dimensional representations for the description of states of the underlying physical objects.

Remark: Th. 2 is in general not true for *semisimple*, noncompact Lie groups. For instance, the semisimple, connected, noncompact Lie group

$$G = \mathrm{SO}(3,1) \times \mathrm{SU}(3)$$

has a unitary finite-dimensional representation:

$$(g_1, g_2) \to I \cdot T_{g_2}, \quad g_1 \in \mathrm{SO}(3,1), \quad g_2 \in \mathrm{SU}(3). ▼$$

However, th. 2 implies the following corollary for semisimple groups.

COROLLARY 4. *A connected, semisimple, noncompact Lie group cannot admit faithful unitary finite-dimensional representations.*

PROOF follows from the decomposition of G onto simple factors and from th. 2. ▼

Let now G be a complex Lie group and let $t_1, t_2, \ldots, t_n$ be local (complex) coordinates in G. We distinguish the following classes of representations of G.

DEFINITION 1. A representation $g \to T_g$ of a complex group G is said to be *complex-analytic* if it depends analytically on the parameters $t_1, \ldots, t_n$, *complex-antianalytic* if it depends analytically on $\bar{t}_1, \ldots, \bar{t}_n$ and *real-analytic*, if it depends analytically on parameters $\mathrm{Re}\, t_1, \mathrm{Im}\, t_1, \ldots, \mathrm{Re}\, t_n, \mathrm{Im}\, t_n$ (or $t_1, \ldots, t_n, \bar{t}_1, \ldots, \bar{t}_n$).

EXAMPLE 1. Let G be a complex matrix group. Then, the representation $g \to g$ is analytic, $g \to \bar{g}$ is antianalytic and $g \to g \otimes \bar{g}$ is real.

If $g \to T_g$ is the complex-analytic irreducible representation of G in H, its restriction to a subgroup N might in general be reducible. However, if N is a real form of G (i.e., complex extension of N coincides with G), then we have

THEOREM 3. *Let T_G be a complex-analytic representation of G and T_N the restriction of T_G to a real form N of G. Then, T_G is irreducible (fully reducible), if and only if T_N is irreducible (fully reducible).*

PROOF: By assumption, every matrix element is an analytic function of the complex parameters $t_1, \ldots, t_n$ in G. If some matrix element is zero on G, then it is in particular zero on N: conversely by virtue of uniqueness of analytic continuation if a matrix element is zero on N then it is zero also on G. This implies the assertion of th. 3. ▼

Using the 'Weyl unitary trick' (the construction of representations of real forms G_R of a given complex Lie group G_C by the restriction of the representations of G_C to G_R) we obtain the global representations of real semisimple groups such as $SL(n, R)$, $SU(n)$, $SU(p, q)$ and so on, from the representations of the complex Lie group $GL(n, C)$. Because the structure of complex groups is simpler than the real groups, one obtains in this fashion a considerable simplification in the representation theory of semisimple Lie groups.

We now prove the fundamental Weyl theorem about full reducibility of representations of semisimple Lie groups

Theorem 4 (Weyl). *Let G be a connected semisimple Lie group and let $g \to T_g$ be any finite-dimensional representation of G in a carrier space H. Then,*

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_n, \tag{7}$$

where each H_i is invariant.

PROOF: The proof consists of the reduction of the problem to the complete reducibility in the case of compact groups. Let L be a Lie algebra of G, L^c its complex extension and $L^c{}_k$ a maximal compact subalgebra of L^c. We know by th. 1.5.2 that $L^c{}_k$ is also a real form of L^c, i.e., complex extension of $L^c{}_k$ coincides with L^c. The representation T of G induces the representation $L \ni X \to T(X)$ of L in H by means of linear (matrix) transformations. Because $L^c = L + iL$, the representation $T(X)$ of L provides a representation T^c of L^c and also a representation $T^c{}_k$ of $L^c{}_k$. By th. 3, a Lie algebra of linear transformations is completely reducible if and only if its complexification is completely reducible. Consequently, we can reduce the problem of proving the complete reducibility of $T(X)$ to that (via T^c) of $T^c{}_k$. Let G_k be a compact Lie group associated with $L^c{}_k$. Then, by th. 7.1.4 we know that every representation of G_k is completely reducible. Thus $T^c{}_k$ of $L^c{}_k$ and consequently also $T(X)$ must be completely reducible. Now exponentiating the representation $T(X)$ of L to the global representation T of G we obtain the desired complete reducibility of T. ▼

The Weyl theorem states in fact that every representation of a semisimple Lie group G is built out of irreducible ones. Hence, the problem of classification of finite-dimensional representations of semisimple Lie groups reduces to the problem of classification of all irreducible representations. This problem we solve in §§ 3, 4 and 5.

The generalization of the Weyl theorem to arbitrary connected Lie groups is given in § 7 of this chapter.

§ 2. Induced Representations of Lie Groups

We have seen in proposition 7.6 that every finite-dimensional irreducible representation of a compact group occurs in the regular representation. We now show that every continuous irreducible representation $g \to T_g$ of an arbitrary topological group G can be imbedded in the regular representation realized in the space $C(G)$. Indeed, let H be the carrier space of T and let $\hat{H}$ be the dual space. Take a fixed $0 \neq v \in \hat{H}$ and set

$$f_u(g) = \langle T_g u, v \rangle, \qquad u \in H.$$

The set of functions so obtained forms a linear subspace $\tilde{H} \subset C(G)$. The mapping $V: H \to \tilde{H}$ is one-to-one because the inverse image of the zero of H is an invariant subspace which cannot be different from 0 because H is irreducible. Under V, the function $f_u(gg_0)$ corresponds to the vector $T_{g_0}u$, i.e., G is represented in $\tilde{H}$ by the right translations $T^R_{g_0}$. We can select in the space $\tilde{H}$ a basis consisting of the functions

$$e_i(g) = D_{li}(g), \qquad i = 1, 2, \ldots, \dim H, \tag{1}$$

where $D_{ij}(g)$ are the matrix elements of the representation T_g. Then

$$T^R_{g_0} e_i(g) = e_i(gg_0) = D_{li}(gg_0) = D_{lk}(g) D_{ki}(g_0) = D_{ki}(g_0) e_k(g).$$

Thus the space $\tilde{H}$ spanned by the continuous functions $e_i(g)$, $i = 1, 2, \ldots, n = \dim H$, in G can be taken to be the carrier space of the given irreducible representation $g \to T_g$ of G.

We now give a method of construction of induced finite-dimensional representations of complex, classical Lie groups based on a generalization of this idea. We shall start with the construction of the carrier space.

Representations of G Induced by a Representation L of a Subgroup K*

Let K be a closed subgroup of G and let $k \to L_k$ be a finite-dimensional representation of K in a Hilbert space H. Consider a linear space $\tilde{H}^L$ of functions u with domain in G, range in H, satisfying the following conditions:

$$\begin{aligned} &1^\circ \text{ The scalar product } \big(u(g), v\big)_H \text{ is continuous in } G \text{ for arbitrary } v \in H,\\ &2^\circ\ u(kg) = L_k u(g) \text{ for all } k \in K. \end{aligned} \tag{2}$$

Define

$$T^L_{g_0} u(g) = u(gg_0). \tag{3}$$

* Induced representations, specially infinite-dimensional ones, will be discussed in more detail in ch. 16 ff.

The function $u(gg_0)$ satisfies conditions 1° and 2° and therefore belongs to $\tilde{H}^L$ We have, moreover,

$$(T^L_{g_1}T^L_{g_2}u)(g) = u(gg_1g_2) = T^L_{g_1g_2}u(g).$$

Consequently,

$$T^L_{g_1}T^L_{g_2} = T^L_{g_1g_2} \quad \text{and} \quad T^L_e = I.$$

By condition (2)1° the map $g \to T^L_g$ is continuous. Hence, the map $g \to T^L_g$ defines a continuous representation of G, in general infinite-dimensional.

The map $g \to T^L_g$ is called the *representation of G induced by the representation L of K.*

The realization of an induced representation $g \to T^L_g$ of G by means of the right regular representation obliterates the individuality of a given representation. Therefore, we give another realization of T^L in the linear space $H^L(Z)$ of functions in $Z = K\backslash G$.

Let G be a classical Lie group, which admits the Gauss decomposition of the form

$$G = \overline{\mathfrak{Z}DZ}. \tag{4}$$

where D is the abelian closed subgroup of G, $\mathfrak{Z}D$ and DZ are solvable, connected subgroups in G, whose commutator subgroups are $\mathfrak{Z}$ and Z, respectively, and

$$\mathfrak{Z} \cap DZ = \{e\}, \quad D \cap Z = \{e\}.$$

Let $K = \mathfrak{Z}D$ and let $k \to L_k$ be a one-dimensional representation of K. Because $\mathfrak{Z}$ is the commutator subgroup of K the representation $k \to L_k$ is trivial on $\mathfrak{Z}$, i.e., $L_\zeta = I$. Consequently, the map

$$\mathfrak{Z}D \ni k = \zeta\delta \to IL_\delta \tag{5}$$

defines, in fact, the one-dimensional representation of D. If $\tilde{H}^L$ is the space of functions satisfying conditions (2) with L_δ given by (5), then it follows from eqs. (4) and (5) that for $u(g) \in \tilde{H}^L$, we have

$$u(g) = u(\zeta\delta z) = L_{\zeta\delta}u(z) = L_\delta u(z), \quad z \in Z. \tag{6}$$

Because L_δ is fixed, we can replace each function $u(g) \in \tilde{H}^L$ by its contraction $u(z)$ defined on the domain Z and consider instead of the linear space $\tilde{H}^L$ on G the corresponding linear space $\tilde{H}^L(Z)$ of functions with the domain Z. The map $\tilde{H}^L \to \tilde{H}^L(Z)$ is one-to-one. In fact, the inverse image of zero in $\tilde{H}^L(Z)$ is zero in $\tilde{H}^L$. And, if $u(z) \equiv 0$, then $u(g) = 0$ for a regular point g in G, which admits the decomposition $g = kz$. On the other hand $u(g)$ is continuous and $\mathfrak{Z}DZ$ is dense in G. Consequently, $u(g) = 0$. We now find the representation $g \to T^L_g$ in this realization of the carrier space:

LEMMA 1. *The action of operators T^L_g in the space $\tilde{H}^L(Z)$ is given by the formula*

$$T^L_{g_0}u(z) = L_{\tilde{\delta}}\, u(z\tilde{g}), \tag{7}$$

where $\tilde{\delta}$ and $z_{\tilde{g}}$ are determined from the Gauss decomposition of the element $\tilde{g} = zg_0 = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

PROOF: Writing

$$gg_0 = kzg_0 = k\tilde{\zeta}\tilde{\delta}z_{\tilde{g}} \tag{8}$$

we obtain from eqs. (6) and (5) that the vector $T^L_{g_0}u(z)$ in $\tilde{H}^L(Z)$ corresponding to the vector $(T^L_{g_0}u)(g) = u(gg_0)$ in $\tilde{H}^L$ has the form

$$L_\delta^{-1}u(gg_0) = L_\delta^{-1}L_kL_{\tilde{\zeta}}\,_{\tilde{\delta}}\,u(z_{\tilde{g}}) = L_{\tilde{\delta}}\,u(z_{\tilde{g}}) \tag{9}$$

by (5) and (6). Hence eq. (7) follows. ▼

Remark: Strictly speaking the vectors $u(z)$ as well as the representation in $\tilde{H}^L(Z)$ should be denoted by different symbols, say $\tilde{u}(z)$ and $\tilde{T}^L_g$. For simplicity we have used the same symbol, as there will be no confusion.

It should be also stressed that the representation T^L in $\tilde{H}^L(Z)$ might be reducible and infinite-dimensional. An irreducible subspace of $\tilde{H}^L(Z)$ which contains the function $u_0(z) \equiv 1$ we shall denote by $H^L(Z)$. The restriction of the representation T^L to $H^L(Z)$ we shall denote, for the sake of simplicity, also by the symbol T^L.

Equation (7) implies further

$$T^L_{z_0}u(z) = u(zz_0) \quad \text{for all } z_0 \in Z. \tag{10}$$

$$T^L_\delta u(z) = L_\delta u(\delta^{-1}z\delta) \quad \text{for all } \delta \in D. \tag{11}$$

Clearly, the one-dimensional representations L of D (and therefore also of K) are given by the characters. If

$$D \ni \delta = \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix}, \quad \delta_i \neq 0, \tag{12}$$

then, the most general complex-analytic character $\delta \to L_\delta$ has the form

$$L^\delta = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n}, \tag{13}$$

where m_i, $i = 1, \ldots, n$, are integers. The most general complex antianalytic character has the form:

$$L_\delta = \bar{\delta}_1^{m_1}\bar{\delta}_2^{m_2}\ldots\bar{\delta}_n^{m_n}$$

where m_i are integers.

A character L which determines an induced irreducible finite-dimensional representation T^L of G is said to be *inductive relative* to the group G. We shall prove later that only certain characters L of D can be inductive relative to G.

The following theorem constitutes the main result in the theory of finite-dimensional representations of Lie groups.

THEOREM 2. *Let G be a Lie group which admits a Gauss decomposition $G = \overline{\mathfrak{Z}DZ}$. Then, every irreducible, finite-dimensional representation of G is a representation*

T^L induced in the space H^L by a uniquely defined character $\delta \to L_\delta$ of the subgroup D. Two irreducible representations T^{L_1} and T^{L_2} are equivalent if and only if $L^1 = L^2$.

PROOF: Let T be an irreducible representation of G and let T_K denote its restriction on the solvable connected subgroup $K = \mathfrak{Z}D$. We know by corollary 2 to Lie's theorem that all operators T_k, $k \in K$, can be simultaneously reduced to the triangular form, i.e.,

$$T_k = \begin{bmatrix} L_k^1 & & & \\ & L_k^2 & & 0 \\ & * & \ddots & \\ & & & L_k^r \end{bmatrix}, \tag{14}$$

where $k \to L_k^i$ are characters of the group K. Now, every character of K is trivial on the commutator group $\mathfrak{Z}$ of K. Hence

$$L_k^i = L_{\zeta\delta}^i = L^i{}_\delta .$$

L_k^i is in fact a character of D. The carrier space H of the irreducible representation T can be spanned by vectors

$$e_i(g) = D_{1i}(g), \tag{15}$$

where $D_{ji}(g)$ are matrix elements of T_g (cf. eq. (1)). By virtue of eq. (14) we have

$$e_i(kg) = D_{1s}(k) D_{si}(g) = L_k^1 e_i(g).$$

Therefore, every element $u(g)$ in $\tilde{H}^L$ is a continuous function on G, which satisfies the condition

$$u(kg) = L_k^1 u(g).$$

Thus, the conditions 1° and 2° of eq. (2) are satisfied. Consequently, T can be realized as the representation T^{L^1} of G induced by the one-dimensional representation $k \to L_k^1$ of the subgroup K. The action of $T_g^{L^1}$ in the carrier space $\tilde{H}^{L^1}$ is given by the right translation (3) and in the carrier space $\tilde{H}^L(Z)$ by the formula (7).

Applying corollary 1 to Lie's theorem for the solvable subgroup $N = DZ$ we conclude that the space $\tilde{H}^L(Z)$ contains a common eigenvector $u_0(z)$ for all operators T_n, $n \in N$. Clearly, this vector is invariant under the action of the commutator group Z of N, i.e., $T_{z_0} u_0(z) = u_0(z)$ for all z_0 in Z. Because, by eq. (10), the subgroup Z acts in $\tilde{H}^L(Z)$ by the right translation, the fixed eigenvector of all T_z can be only a constant, e.g., $u_0(z) \equiv 1$. Hence $\tilde{H}^L$ coincides with $H^L(Z)$ and, by virtue of eq. (11), we obtain

$$T_\delta^{L^1} u_0(z) = L_\delta^1 u_0(z). \tag{16}$$

Consequently, the inductive character L^1 is uniquely defined.

If $L^1 = L^2$, then, obviously T^{L^1} and T^{L^2} are equivalent. Conversely, if T^{L^1} and T^{L^2} are equivalent, then there exists an operator V such that $VT^{L^1}V^{-1} = T^{L^2}$ and $H_2 = VH_1$. Hence, by virtue of (16), we obtain $VT_\delta^{L^1}V^{-1}(Vu_0)(z) = L^1{}_\delta(Vu_0)(z)$: this implies $L^1 = L^2$. ▼

The final part of the proof gives the following important result:

COROLLARY 1. *There is one and only one (apart from normalization) invariant vector $u_0(z)$ of the subgroup Z in the carrier space H^L of every irreducible, finite-dimensional representation of G, i.e.,*

$$T_z^L u_0 = u_0 \quad \textit{for all } z \in Z. \tag{17}$$

This invariant vector satisfies, moreover, the condition

$$T_\delta^L u_0 = L_\delta u_0 \quad \textit{for all } \delta \in D, \tag{18}$$

and can be properly normalized such that

$$u_0(z) = 1. \blacktriangledown \tag{19}$$

The character $\delta \to L_\delta$ is said to be the *integral highest weight* of the irreducible representation T^L. Because $T^L = \exp(\sum_k H_k \alpha_k)$, where H_k are the generators of the representation of the subgroup D, and α_n are the parameters in the Lie algebra of D, we can pass to the infinitesimal transformations and obtain by eqs. (18) and (13)

$$H_k u_0 = m_k u_0, \quad k = 1, 2, \ldots, n. \tag{20}$$

The vector $m = (m_1, m_2, \ldots, m_n)$ is called the *highest weight* of the representation T^L and the vector u_0 the *highest vector.* By th. 2, m is uniquely determined and in turn determines the irreducible representation T^L. The highest vector u_0 corresponding to m will be also denoted by u_m.

COROLLARY 2. *If the carrier space H of a representation T contains only one invariant vector of the subgroup Z, then T is irreducible.*

PROOF: Every T of G is completely reducible by the Weyl theorem. Hence, it can be reduced to a bloc diagonal form 5.3(4) of irreducible representations D^i, $i = 1, 2, \ldots, N$. Repeating the construction of th. 2 for each bloc D^i we find N invariant vectors of the subgroup Z. Hence, if $N = 1$, T must be irreducible. $\blacktriangledown$

Eq. (19), the Gauss decomposition and the composition law for the operators T_g^L imply the following useful result:

COROLLARY 3. *The carrier space $H^L(Z)$ of the irreducible representation T^L is spanned by vectors*

$$u_g(z) = L_\delta u_0 = L_{\tilde{\delta}}, \tag{21}$$

where g ranges over G. The Gauss factor $\tilde{\delta}$ of the element zg is a continuous function of z and g. These functions satisfy the relation

$$L_{\tilde{\delta}(z, g_1 g_2)} = L_{\tilde{\delta}(z, g_1)} L_{\tilde{\delta}(z g_1, g_2)}. \blacktriangledown \tag{22}$$

Eqs. (7) and (21) imply that if L is an analytic (antianalytic) representation then the representation T_g^L is also analytic (antianalytic).

The corollaries 2 and 3 give the following procedure for the decomposition of a reducible representation T of G:

1° Find in the carrier space H a maximal subspace H_0 which is fixed under the subgroup Z.

2° Select a normalized basis $u_0^{(i)}$ in H_0. Then, the carrier space $H^{(i)}$ of the irreducible representation T^i is spanned by the vectors

$$u_g^{(i)}(z) = T_g u_0^{(i)} = L^{(i)}{}_{\tilde{z}} u_0^{(i)} = L^i{}_{\tilde{z}} .$$

The action of $T^{(i)}$ in the carrier space $H^{(i)}$ is given by (7). ▼

The following proposition describes the structure of the carrier space $H^L(Z)$ of the irreducible representation T^L. For simplicity we assume that Z is a connected nilpotent group (it is just this case that we shall need later).

PROPOSITION 3. *The carrier space $H^L(Z)$ of an irreducible representation $T^L(Z)$ consists of functions $u(z)$, which are polynomials in the matrix elements z_{pq} of an element $z \in Z$.*

PROOF: By virtue of corollary 2 to th. 1.1 every representation $n \to T_n$ of the solvable connected group $N = DZ$ can be written in the triangular form. Hence, the representations T_z of the commutator subgroup Z can be written in the triangular form with L's on the main diagonal. This means that the Lie algebra A of Z generated by matrices X_{pq}, $p > q$, is mapped into an algebra of nilpotent matrices (i.e., for $X \in A$, $X^m = 0$ for some integer m). Therefore, the matrix elements of matrices T_z $(= \exp \sum_{p>q} z_{pq} X_{pq})$ are polynomials in the matrix elements z_{pq} of element $z \in Z$. On the other hand by eqs. (15) and (14) we have

$$e_i(g) = D_{ii}(g) = D_{ii}(kz) = D_{is}(k) D_{si}(z) = L_k^1 D_{ii}(z) = L_\delta^1 e_i(z).$$

Because the map $H^L(G) \to H^L(Z)$ is one-to-one the matrix elements $e_i(z)$ of T_z span the space $H(Z)$. ▼

The following proposition is useful in the determination of all characters $\delta \to L_\delta$ of D, which are inductive relative to G.

PROPOSITION 4. *Suppose that the Gauss decomposition of G induces a Gauss decomposition of a subgroup G_0 of G:*

$$G_0 = \mathfrak{Z}_0 D_0 Z_0 ,$$

where $\mathfrak{Z}_0$, D_0 and Z_0 are intersections of G_0 with the subgroups $\mathfrak{Z}$, D and Z of G, respectively: Let $L^0{}_{\delta_0}$ be the restriction of the character L_δ of D to the subgroup D_0. If the character L is inductive relative to G, then the character L^0 is inductive relative to the subgroup G_0.

PROOF: Because the character L of D is inductive, the linear hull of functions $u_g(z) = L_{\tilde{z}}$, $\tilde{\delta} = \tilde{\delta}(z, g)$, consists of polynomials in z_{pq} and has a finite dimension by proposition 3. This is also true for functions

$$u_{g_0}(z_0) = L_{\tilde{z}_0}, \quad \tilde{\delta}_0 = \tilde{\delta}(z_0, g_0), \quad z_0 \in Z_0, \; g_0 \in G_0 .$$

Clearly, the functions $u_{g_0}(z_0)$ are continuous on $Z_0 \times G_0$. Consequently, some representation of G_0 is realized in the linear envelope H_0 of these functions.

The vector $u_0(z_0) = 1$ is the unique vector in H_0 which is fixed for the subgroup Z_0. Hence, the representation of G_0 in H_0 is irreducible by corollary 3. Consequently, the character L^0 of D_0 is inductive. ▼

The method of induced representations has a number of advantages when compared with the infinitesimal Cartan–Weyl method. It provides the classification of irreducible representations in terms of the highest weights and at the same time, it gives a natural realization of the carrier space as the linear space $H^L(Z)$ of polynomials over the standard subgroup Z. This is very useful in the solution of various practical problems.

We now consider the explicit construction of the operators T_g^L and the carrier space $H^L(Z)$ for the group SL(2, C), the covering group of the Lorentz group SO(3, 1).

EXAMPLE 1. Let $G = \mathrm{SL}(2, C)$. The Gauss factors $\mathfrak{Z}$, D and Z in this case are given by (cf. eq. 3.6 (3)):

$$\mathfrak{Z} = \left\{\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}\right\}, \quad D = \left\{\begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}\right\}, \quad Z = \left\{\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}\right\},$$

where ζ, δ and z are in C^1. If $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, C)$, then the factors $\tilde{\delta}$ and $z_{\tilde{g}}$ of the element $\tilde{g} = zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$ have the form

$$\tilde{\delta} = \beta z + \delta, \qquad z_{\tilde{g}} = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

An arbitrary complex analytic character of D, by eq. (13), is given by

$$\delta \to L_\delta = \delta^m, \tag{23}$$

where m is an integer, which we determine below. According to th. 2 and eq. (7), every irreducible representation $g \to T_g^L$ induced by the one-dimensional representation (23) of D is given by the formula

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}) = (\beta z + \delta)^m u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{24}$$

It remains only to determine the number m. By corollary 3, the carrier space $H^L(Z)$ is spanned by the vectors

$$u_g(z) = L_{\delta_{\tilde{g}}} = (\beta z + \delta)^m \tag{25}$$

where β and δ take all admissible values for the elements g in SL(2, C). Thus, in particular the space $H^L(Z)$ contains all translations

$$u_i(z) = (z + \delta_i)^m.$$

Because $H^L(Z)$ is finite-dimensional, there exists a number $r \geqslant 1$ such that an arbitrary set of $r+1$ functions is linearly dependent. Consequently the determinant

$$\Delta(z) = \begin{vmatrix} u_1(z) & u_2(z) & \dots & u_{r+1}(z) \\ u_1'(z) & u_2'(z) & \dots & u_{r+1}'(z) \\ \dots & \dots & \dots & \dots \\ u_1^{(r)}(z) & u_2^{(r)}(z) & \dots & u_{r+1}^{(r)}(z) \end{vmatrix}$$

is identically zero. Here

$$u_1^{(s)}(z) \equiv \frac{\partial^s}{\partial z^s} u_1(z)$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right),$$

Setting $z = 0$, we obtain

$$\begin{aligned} \Delta(0) &= \begin{vmatrix} \delta_1^m & \delta_2^m & \dots & \delta_{r+1}^m \\ m\delta_1^{m-1} & m\delta_2^{m-1} & \dots & m\delta_{r+1}^{m-1} \\ \dots & \dots & \dots & \dots \end{vmatrix} \\ &= m^r(m-1)^{r-1} \dots (m-r+1)[\delta_1 \delta_2 \dots \delta_{r+1}]^{m-r} \cdot W(\delta), \end{aligned}$$

where

$$W(\delta) = \prod_{i<j} (\delta_i - \delta_j).$$

One can always choose the numbers $\delta_i \neq 0$ and different between themselves. Hence if m is not equal to one of the integers $0, 1, \dots, r-1$, then $\Delta(0) \neq 0$. Consequently, only non-negative integers $m = 0, 1, 2, \dots$ give the inductive complex analytic characters L of the form (23). For integer $m \geqslant 0$, according to eq. (25), the carrier space H^L contains all monomials $1, z, z^2, \dots, z^m$ and is spanned by them. To summarize:

THEOREM 5. *Every complex analytic irreducible representation of* SL(2, *C*) *determines and is determined by an integer $m \geqslant 0$. It is realized by formula* (24) *in the space $H^m(Z)$ of all polynomials of degree not greater than m.* ▼

We have also complex 'antianalytic' irreducible representations $T^{\bar{L}}$ induced by the character

$$\bar{L}_\delta = \bar{\delta}^n. \tag{26}$$

By the same considerations n must again be a non-negative integer. The representation $g \to T_g^{\bar{L}}$ is given in the space $H^n(\bar{Z})$ of polynomials in the variables $1, \bar{z}, \bar{z}^2, \dots, \bar{z}^n$ by the formula

$$T_g^{\bar{L}} u(z) = \overline{(\beta z + \delta)}^n u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{27}$$

Finally, if we take a real analytic character $\delta \to L_\delta = \delta^m \bar{\delta}^n$ we obtain the real analytic representations of SL(2, C). Thus, every irreducible, finite-dimensional representation of SL(2, C) is determined by a pair (m, n) of non-negative integers. It is given in the space of all polynomials $u(z, \bar{z})$ of degree not greater than m relative to z and not greater than n relative to $\bar{z}$, by the formula

$$T_g^L u(z,\bar{z}) = (\beta z + \delta)^m \overline{(\beta z + \delta)}^n u\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha}\bar{z} + \bar{\gamma}}{\bar{\beta}\bar{z} + \bar{\delta}}\right). \tag{28}$$

In other words, every real, analytic, irreducible representation of SL(2, C) is a tensor product of the form

$$T^{L_1} \otimes \overline{T^{L_2}}, \tag{29}$$

where T^{L_1} and T^{L_2} are complex-analytic, irreducible representations of G and $\overline{T^L}$ denotes the representation conjugate to T^L.

The formula (24) restricted to the subgroup SU(2) provides an irreducible, unitary representation of SU(2) (cf. exercise 9.2.1). In fact, using the 'Weyl unitary trick' we conclude that every irreducible representation of SU(2) is the restriction of an irreducible, complex-analytic representation of SL(2, C) to the subgroup SU(2).

§ 3. The Representations of GL(n, C), GL(n, R), $U(p, q)$, $U(n)$, SL(n, C), SL(n, R), SU(p, q), and SU(n)

Th. 2.2 reduces the problem of the classification of all irreducible representations of a group G to the problem of the enumeration of all inductive highest weights. In this section we solve this problem for the full linear group GL(n, C), for SL(n, C) and for their real forms.

A. *The Representation of* GL(n, C)

The Gauss decomposition of GL(n, C), $G = \overline{\mathfrak{Z}DZ}$ is given by

$$\zeta = \begin{bmatrix} 1 & \zeta_{12} & \cdots & \zeta_{1n} \\ & 1 & \cdot & \vdots \\ & & \ddots & \vdots \\ 0 & & \cdot & \zeta_{n-1,n} \\ & & & 1 \end{bmatrix}, \qquad \delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix},$$

$$z = \begin{bmatrix} 1 & & & & \\ z_2 & 1 & & 0 & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ z_{n1}, & z_{n2} & \cdots & z_{n,n-1} & 1 \end{bmatrix}. \tag{1}$$

A complex, analytic character L of D has the form

$$\delta \to L_\delta = \delta_1^{m_1}\delta_2^{m_2} \dots \delta_n^{m_n}, \tag{2}$$

where m_i are integers.

THEOREM 1. *Every complex analytic irreducible representation of* GL(n, C) *determines and is in turn determined by the highest weight* $m = (m_1, m_2, \dots, m_n)$, *whose components are integers satisfying the conditions*

$$m_1 \geqslant m_2 \geqslant m_3 \dots \geqslant m_n. \tag{3}$$

The carrier space $H^L(Z)$ *consists of polynomials of the matrix elements* z_{pq} *of the elements* $z \in Z$.

The representation T^L *is realized in* $H^L(Z)$ *by means of the formula*

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{4}$$

where $\tilde{\delta}$ *and* $z_{\tilde{g}}$ *are factors of the Gauss decomposition of the element* $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

PROOF: According to th. 2.2 every complex analytic irreducible representation of GL(n, C) is the representation T^L induced by the complex analytic character $\delta \to L_\delta$ of D. Hence m_i are integers. Proposition 2.3 assures that the representation T^L can be realized in the space $H^L(Z)$ of polynomials $u(z)$ of matrix elements z_{pq} of $z \in Z$. To classify all irreducible representations, it is necessary to find all admissible highest weights.

Let L^0 be the restriction of L to the subgroup D_0 consisting of all matrices of the form

$$\delta_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda^{-1} & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}. \tag{5}$$

The subgroup G_0 of all matrices of the form

$$g = \begin{bmatrix} \alpha & \beta & & & \\ \gamma & \delta & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}, \qquad \alpha\delta - \beta\gamma = 1 \tag{6}$$

is isomorphic to SL$(2, C)$. Hence, by proposition 2.4 we conclude that the character

$$L^0{}_{\delta_0} = \lambda^{m_1 - m_2}$$

is inductive relative to SL$(2, C)$. Consequently, $m_1 - m_2$ is a non-negative integer by th. 2.5.

Moving along the main diagonal in (6), we conclude similarly that $m_2-m_3, \ldots, m_{n-1}-m_n$ are also non-negative integers. This proves the necessity of condition (3).

Conversely, when conditions (3) are satisfied the functions

$$u_g(z) = L_{\tilde{\delta}}, \qquad \tilde{\delta} = \tilde{\delta}(z, g) \tag{7}$$

are polynomials of the matrix elements z_{pq} of z, whose degrees are uniformly bounded with respect to $g \in G$. Indeed, the character (2) can be written in the form

$$L_\delta = \Delta_1^{f_1}\Delta_2^{f_2} \ldots \Delta_n^{f_n}, \tag{8}$$

where

$$\Delta_p = \delta_1\delta_2 \ldots \delta_p$$

and

$$f_p = m_p - m_{p+1}, \quad p = 1, 2, \ldots, n, \; m_{n+1} = 0.$$

According to exercise 3.11.6.2, Δ_p is equal to the main diagonal minor of g of the order p. Consequently, $L_{\tilde{\delta}}$ is the polynomial in the matrix elements of $z_{\tilde{g}}$, where $\tilde{\delta}$ and $z_{\tilde{g}}$ are factors in the Gauss decomposition of the element $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$. Therefore, the linear hull H^L of vectors (8) is finite-dimensional, and the representation T^L of G in H^L is given by eq. (4). The vector $u_0(z) \equiv 1$ is the only element of H^L, which is fixed relative to the subgroup Z. Hence, by corollary 2 to th. 2.2 the representation T^L is irreducible. This proves that the condition (3) is also sufficient.

These considerations and th. 2.2 imply that an irreducible representation T^L can be realized in the space $H^L(Z)$ of polynomials on Z by means of the formula (4). ▼

Remark 1: We can write the formula (4) in a more explicit form convenient for further calculations. Indeed, by virtue of eqs. (4) and (8)

$$\begin{aligned} T_g^L u(z) &= L_{\tilde{\delta}} u(z_{\tilde{g}}) = \Delta_1^{f_1}(\tilde{\delta})\Delta_2^{f_2}(\tilde{\delta}) \ldots \Delta_n^{f_n}(\tilde{\delta}) u(z_{\tilde{g}}) \\ &= (zg)_{11}^{m_1-m_2} \begin{vmatrix} (zg)_{11} & (zg)_{12} \\ (zg)_{21} & (zg)_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det g)^{m_n} u(z_{\tilde{g}}). \end{aligned} \tag{9}$$

Here we used the fact that $\det(zg) = \det g$ for the last minor.

Remark 2: One often uses the symbol $f = [f_1, f_2, \ldots, f_n]$, $f_k = m_k - m_{k+1}$, $m_{n+1} = 0$, for labelling an irreducible representation associated with the highest weight $m = (m_1, m_2, \ldots, m_n)$. In the following we use the curly bracket for the highest weight, and the square bracket for the symbol f. ▼

The antianalytic representations of GL(n, C) are realized in the space of polynomials of variables $\bar{z}_{pq}$ of $\bar{z} \in Z$. These representations are induced by the characters $\delta \to \overline{L_\delta}$, where L_δ is the complex-analytic integral highest weight.

An arbitrary real analytic irreducible finite-dimensional representation of

GL(n, C) is induced by a character $L_1\bar{L}_2$. According to eq. (4) and eq. 2(21) it can be written as the tensor product

$$T^{L_1} \otimes T^{\bar{L}_2} \tag{10}$$

of complex-analytic and complex-antianalytic irreducible representations. (Cf. example 2.1, eq. 2(29).)

The group GL(n, C) has in particular a series of one-dimensional representations. These representations are of the form $\eta^p \bar{\eta}^q$, where

$$\eta(g) = \det g.$$

One can associate with these representations the infinitely many valued and indecomposable representations

$$g \rightarrow \eta^r(g) \begin{bmatrix} 1 & \log|\eta| \\ 0 & 1 \end{bmatrix}, \tag{11}$$

where r is a complex number.

B. *Representations of* SL(n, C)

Let T^L be an irreducible representation of GL(n, C) in the space $H^L(Z)$. The formula (9) implies that every irreducible representation T^L is of the form

$$T_g{}^L = (\det g)^{m_n} \tilde{T}_g{}^L \tag{12}$$

where $g \rightarrow (\det g)^{m_n}$ is the one-dimensional representation of GL(n, C) and $\tilde{T}_g^L$ is again a representation of GL(n, C) which is also irreducible. The representation $g \rightarrow \tilde{T}_g^L$ restricted to SL(n, C) provides an irreducible representation of SL(n, C) in the carrier space $H^L(Z)$. Equation (12) implies that two irreducible representations of GL(n, C) give the same representation of SL(n, C) if and only if they differ by some power of $\det g$. This fact, by virtue of (9) allows us to normalize the integral highest weight $m = (m_1, \ldots, m_n)$ in such a fashion that $m_n = 0$. By a direct application of th. 2.2 to SL(n, C) we see that every complex analytic irreducible representation of SL(n, C) is a restriction of a representation of GL(n, C). Consequently, we have

THEOREM 2. *Every irreducible, finite-dimensional representation* $\tilde{T}^L$ *of* SL(n, C) *determines and is in turn determined by the highest weight* $m = (m_1, m_2, \ldots, m_{n-1})$, *whose components are integers satisfying the condition*

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{n-1} \geqslant 0. \tag{13}$$

All of these representations are realized in a space $H^L(Z)$ *of polynomials by means of formula* (4). ▼

Clearly, one has also the corresponding complex-antianalytic and real analytic representations of SL(n, C). Because SL(n, C) is simply-connected, every irreducible representation is single-valued.

C. *The Representations of* $\mathrm{GL}_+(n, R)$ *and* $\mathrm{SL}(n, R)$

The group $\mathrm{GL}(n, R)$ has two connected components by virtue of th. 3.7.1. The complex extension of the group $\mathrm{GL}_+(n, R)$, whose elements satisfy the condition $\det g > 0$, coincides with $\mathrm{GL}(n, C)$. Hence we restrict our attention to $\mathrm{GL}_+(n, R)$. Using the th. 1.3 and th. 1 we conclude that every analytic irreducible, finite-dimensional representation of $\mathrm{GL}_+(n, R)$ determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_n)$, whose components are integers satisfyings the conditions

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n. \tag{14}$$

Let $\mathfrak{Z}_R D_R Z_R$ be the Gauss decomposition for $\mathrm{GL}_+(n, R)$. Then, the irreducible representation T^L induced by the one-dimensional representation L of D_R is realized by the formula

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{15}$$

where $\tilde{\delta} \in D_R$ and $z_{\tilde{g}} \in Z_R$ are determined from the decomposition $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

Using similar arguments as in case of $\mathrm{SL}(n, C)$ we obtain that every irreducible representation of $\mathrm{SL}(n, R)$ determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_{n-1})$, whose components are integers satisfying the condition (13). The action of T^L in the carrier space $H^L(Z_R)$ is given by eq. (4).

D. *The Representations of* $U(p, q)$, $p+q = n$, $U(n)$, $\mathrm{SU}(p, q)$, $p+q = n$, $\mathrm{SU}(n)$, *and* Q_{2n}

These groups are also real forms of $\mathrm{GL}(n, C)$ and $\mathrm{SL}(n, C)$, respectively. Hence, by the same arguments as in subsection C, every irreducible representation of $U(p, q)$, $p+q = n$, and of $U(n)$ is characterized by the highest weight $m = (m_1, m_2, \ldots, m_n)$ whose components satisfy the conditions (3). Similarly, irreducible representations of $\mathrm{SU}(p, q)$, $p+q = n$, $\mathrm{SU}(n)$ and Q_{2n} are characterized by the highest weight $m = (m_1, m_2, \ldots, m_{n-1})$, whose components are integers satisfying the condition (13).

§ 4. The Representations of the Symplectic Groups $\mathrm{Sp}(n, C)$, $\mathrm{Sp}(n, R)$ and $\mathrm{Sp}(n)$

The symplectic group $\mathrm{Sp}(n, C)$ can be realized as the set of all linear transformations of the n-dimensional complex vector space (n even $= 2\nu$), which conserve the skew-symmetric form

$$[x, y] = x_1 y_n + x_2 y_{n-1} + \ldots + x_\nu y_{\nu+1} - x_{\nu+1} y_\nu - \ldots - x_n y_1. \tag{1}$$

Thus, $g \in \mathrm{Sp}(n, C)$, if and only if

$$\sigma^{-1} g \sigma = (g^T)^{-1}, \quad \text{where } \sigma = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix} \tag{2}$$

and S is the ν-by-ν matrix given by

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

In particular, Sp(2, C) is isomorphic to SL(2, C).

The Gauss decomposition of GL(n, C) induces the Gauss decomposition of Sp(n, C):

$$\mathrm{Sp}(n, C) = \overline{\mathfrak{Z}_S D_S Z_S}, \tag{3}$$

where $\mathfrak{Z}_S$, D_S and Z_S are intersections of Sp(n, C) and the corresponding subgroups of GL(n, C).

It follows from eq. (2) that

$$\delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}$$

is an element of D_S if any two of the numbers

$$\delta_1, \delta_2, \dots, \delta_\nu, \delta_{\nu+1}, \dots, \delta_{n-1}, \delta_n \tag{4}$$

in symmetric positions with respect to the center are mutually inverse; i.e., $\delta_n = \delta_1^{-1}$, $\delta_{n-1} = \delta_2^{-1}$, etc. Taking $\delta_1, \dots, \delta_\nu$ as independent parameters of $\delta \in D_S$ we see that every complex analytic character of D_S has the form

$$\delta \to L_\delta = \delta_1^{m_1} \delta_2^{m_2} \dots \delta_\nu^{m_\nu}. \tag{5}$$

We have then

THEOREM 1. *Every complex-analytic, irreducible representation T^L of* Sp(n, C) *determines and is in turn determined by the highest weight $m = (m_1, m_2, \dots, m_\nu)$, whose components are integers satisfying the condition*

$$m_1 \geqslant m_2 \geqslant \dots \geqslant m_\nu \geqslant 0. \tag{6}$$

The carrier space $H^L(Z_S)$ of T^L consists of polynomials of the matrix elements z_{pq} of elements $z \in Z_S$. The representation T^L is realized in $H^L(Z_S)$ by means of the formula

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{7}$$

where $\tilde{\delta}$ and $z_{\tilde{g}}$ are factors of the Gauss decomposition (3) *of the element $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.*

PROOF: Let T^L be an irreducible representation of Sp(n, C) induced by the character $\delta \to L_\delta$ of D_S given by eq. (5). Let $G_1, G_2, \dots, G_\nu$ be a sequence of subgroups of Sp(n, C) isomorphic to SL(2, C). In particular, G_1 consists of all linear transformations of the form

$$\begin{bmatrix} \alpha & \beta & & & & & \\ \gamma & \delta & & & 0 & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & 0 & & & \alpha & -\beta \\ & & & & & -\gamma & \delta \end{bmatrix}, \qquad \alpha\delta-\beta\gamma = 1. \tag{8}$$

The subgroups $G_2, \ldots, G_{\nu-1}$ are obtained similarly by moving along the main diagonal. The subgroup G_ν consists of all unimodular transformations, which change the coordinates x_ν and $x_{\nu+1}$ only. Repeating now the arguments in the proof of th. 3.1, we conclude that all numbers $m_1-m_2, \ldots, m_{\nu-1}-m_\nu, m_\nu$ are non-negative integers. This proves the necessity of condition (6).

Suppose now that $m = (m_1, \ldots, m_\nu)$ satisfies condition (6). Then the highest weight $m = (m_1, \ldots, m_\nu, 0, \ldots, 0)$ of the subgroup D of GL(n, C) is inductive relative to GL(n, C) by th. 3.1. Consequently the character (5) is inductive relative to Sp(n, C) by proposition 2.4. This proves that condition (6) is also sufficient.

It follows from the proposition 2.3 and eq. 3.(7) that the carrier space $H^L(Z_S)$ consists of polynomials in Z_S. The formula (7) results then from lemma 2.1. ▼

The properties of the complex-antianalytic and real analytic irreducible representations of Sp(n, C) are analogous to the corresponding representations of SL(n, C). Because Sp(n, C) is simply connected, all irreducible representations T^L of Sp(n, C) are single-valued.

Using the th. 1.3 and the th. 1 we conclude that every analytic irreducible representations of the real symplectic group Sp(n, R) determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_\nu)$, whose components are integers satisfying the condition (6). The same is true for the compact, symplectic groups Sp(n) = Sp$(n, C) \cap U(2n)$.

§ 5. The Representations of Orthogonal Groups SO(n, C), SO(p, q), SO*(n), and SO(n)

The defining representation of the orthogonal group SO(n, C) is the set of all linear unimodular transformations, which preserve the quadratic form

$$z_1^2+z_2^2+ \ldots +z_n^2.$$

However, for our purposes it is more convenient to realize SO(n, C) as the group of unimodular, linear transformations which conserve the form

$$z_1 z_n+z_2 z_{n-1}+ \ldots +z_n z_1. \tag{1}$$

Over the field of complex numbers both forms coincide. Every element $g \in$ SO(n, C) satisfies the condition

$$\sigma^{-1} g \sigma = (g^T)^{-1}, \quad \text{where } \sigma = \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix} \tag{2}$$

and

$$S = \begin{bmatrix} 0 & \dots & & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \dots & & 0 \\ 1 & 0 & \dots & & 0 \end{bmatrix}.$$

In the realization (1) the Gauss decomposition of GL(n, C) induces the corresponding Gauss decomposition of SO(n, C), i.e.,

$$SO(n, C) = \overline{\mathfrak{Z}_0 D_0 Z_0} \tag{3}$$

where $\mathfrak{Z}_0$, D_0 and Z_0 are intersections of SO(n, C) with the subgroups $\mathfrak{Z}$, D and Z respectively of GL(n, C).

The group SO$(2n+1, C)$ is doubly connected and SO$(2n, C)$ is fourfold connected (cf. ch. 3.7.E). Hence, one might expect that SO(n, C) has also multivalued irreducible representations. This is the special feature of the representation theory of orthogonal groups. For example, the four-dimensional Dirac equation can be viewed as a direct consequence of the existence of additional spinor representations of the orthogonal group SO$(4, C)$.

The full orthogonal group $O(n, C)$ consists of two connected components $O^+(n, C)$ and $O^-(n, C)$, whose elements satisfy the condition $\det g = \pm 1$, respectively. Hence, starting from an arbitrary element o in O^-, one can obtain all other elements of O^- by applying the left or the right translation by $g \in O^+$. For the element o one can take the matrix $o = -e$ in case when n is odd, or the $2\nu \times 2\nu$ matrix

$$o = \left[\begin{array}{ccccc:cc:ccccc} 1 & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & 1 & & & & & & & 0 & & \\ & & & \ddots & & & & & & & & \\ & & & & 1 & & & & & & & \\ \hdashline & & & & & 0 & 1 & & & & & \\ & & & & & 1 & 0 & & & & & \\ \hdashline & & & & & & & 1 & & & & \\ & & & & & & & & \ddots & & & \\ & & 0 & & & & & & & 1 & & \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & & 1 \end{array}\right] \tag{4}$$

in case when n is even, $n = 2\nu$. The matrix (4) corresponds to the transposition of the coordinates z_ν and $z_{\nu+1}$. In both cases we have $o^2 = e$.

The group $\mathrm{SO}(n, C) = O^+(n, C)$ is clearly the normal subgroup in $O(n, C)$. This means, in particular, that the map

$$g \to \check{g} \equiv ogo^{-1} \tag{5}$$

leaves the subgroup $\mathrm{SO}(n, C)$ invariant. We shall call the outer automorphism (5) of $\mathrm{SO}(n, C)$ the mirror automorphism, and the corresponding transformation o the mirror reflection. One readily verifies, using the explicit form of the subgroups $\mathfrak{Z}_0$, D_0 and Z_0 that the mirror automorphism leaves the subgroups $\mathfrak{Z}_0$, D_0 and Z_0 invariant. Consequently the Gauss decomposition $g = \zeta\delta z$ goes over into the corresponding Gauss decomposition of the element $\check{g}$.

The matrix $\delta \in D$ conserves the form (1) if in the series $\delta_1, \delta_2, \ldots, \delta_\nu, \delta_{\nu+1}, \ldots, \delta_{n-1}, \delta_n$ the symmetrically situated elements are mutually inverse (i.e., $\delta_1 = \delta_n^{-1}$, $\delta_2 = \delta_{n-1}^{-1}$, etc.). Hence, in case of even n ($n = 2\nu$) as well as odd ($n = 2\nu+1$) we have only the numbers $\delta_1, \delta_2, \ldots, \delta_\nu$ as the independent matrix elements of $\delta \in D_0$. Consequently, every one-dimensional complex analytic representation L of D_0 has the form

$$\delta \to L_\delta = \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_\nu^{m_\nu}. \tag{6}$$

LEMMA 1. *If T is an irreducible representation of* $\mathrm{SO}(2\nu, C)$ *corresponding to the highest weight $m = (m_1, m_2, \ldots, m_\nu)$, then, the mirror-conjugated representation $\check{T}_g = T_{\check{g}}$ corresponds to the highest weight*

$$\check{m} = (m_1, \ldots, m_{\nu-1}, -m_\nu). \tag{7}$$

If $n = 2\nu+1$, then every irreducible representation is mirror-self-conjugated.

PROOF: Because the map $g \to \check{g}$ conserves the structure of $\mathfrak{Z}_0$, D_0, Z_0 it is sufficient to find the image of the character L_δ. If n is even, $n = 2\nu$, then all parameters δ_i remain unchanged except the parameter δ_ν, which goes into δ_ν^{-1}. If n is odd, then $o = -e$ and δ is unchanged. ▼

The following theorem gives the classification of all complex analytic, irreducible representations of $\mathrm{SO}(n, C)$.

THEOREM 2. *The group* $\mathrm{SO}(n, C)$ *has two series of complex analytic irreducible representations. Every representation of the first series determines and is in turn determined by a highest weight $m = (m_1, m_2, \ldots, m_\nu)$ whose components m_i are integers and satisfy the conditions*

$$\begin{aligned} &1° \text{ for } n = 2\nu\colon m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant |m_\nu|, \\ &2° \text{ for } n = 2\nu+1\colon m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu \geqslant 0. \end{aligned} \tag{8}$$

Every representation of the second series determines and is determined by a highest weight $m = (m_1, m_2, \ldots, m_\nu)$ whose component m_i are half-odd integers and also satisfy the conditions (8).

PROOF: *The case* $n = 2\nu$. Let G_0 be the subgroup of $SO(2\nu, C)$ consisting of all matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix}, \quad g \in SL(\nu, C), \tag{9}$$

where $\hat{g} = S^{-1}(g^T)^{-1}S$. Clearly, G_0 is isomorphic with $SL(\nu, C)$. If the character L^m of D_0 is inductive relative to $SO(2\nu, C)$, then its restriction to G_0 is inductive relative to G_0. Hence, by virtue of eq. 3(13) we get

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu, \tag{10}$$

where m_i are integers. Repeating these arguments for mirror-conjugated representation we find

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant -m_\nu. \tag{11}$$

Moreover, by eq. 3(13), we see that both of the two numbers

$$m_i - m_\nu \quad \text{and} \quad m_i + m_\nu$$

are integers. Consequently, all components $m_1, \ldots, m_\nu$ should be simultaneously either all integers, or all half-odd integers. This proves the necessity of condition (8) 1°.

Suppose now that the condition (8) 1° is satisfied for some weight $m = (m_1, m_2, \ldots, m_\nu)$, where all m_i are simultaneously integers or half-odd integers. Because a miror-conjugated character can be inductive only simultaneously with a given character, we can suppose that $m_\nu \geqslant 0$. Then, the character of the subgroup $D(2\nu)$ of $GL(2\nu, C)$ defined by the weight $m = (m_1, \ldots, m_\nu, 0, \ldots, 0)$ is inductive relative to $GL(2\nu, C)$ by th. 3.1. Consequently, by virtue of proposition 2.4, its restriction on $SO(2\nu, C)$ is inductive relative to this group.

The case $n = 2\nu+1$. The proof runs similar to the case of even n. The last condition in eq. (8) 2° ($m_\nu \geqslant 0$) follows from the consideration of the subgroup $SO(3, C)$, whose elements consist of rotations transforming coordinates x_ν, $x_{\nu+1}$ and $x_{\nu+2}$ only. Indeed, since $SO(3, C)$ is locally isomorphic to $SL(2, C)$ we obtain $m_\nu \geqslant 0$ by th. 3.2. ▼

The irreducible representations of $SO(n, C)$ associated with the highest weights with integer components are the tensor representations. The remaining representations are called spinor representations.

The lowest spinor representations play an essential role in physics. In the case of $SO(2\nu+1, C)$ the lowest spinor representation is determined by the highest weight

$$m_+ = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}). \tag{12}$$

In the case of $SO(2\nu, C)$ there are two lowest spinor representations, namely m_+ (eq. (12)), and

$$m_- = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{13}$$

These representations are mirror-conjugated. The linear objects transforming according to representations $T^{L^{m}+}$ and $T^{L^{m}-}$ are called *spinors of the first* and the *second kind*, respectively.

The group $\mathrm{SO}(n, C)$ has also complex-antianalytic and real-analytic irreducible representations. Their properties are analogous to those of the corresponding representations of $\mathrm{SL}(n, C)$.

Using the th. 3.1 we conclude that th. 2 is also true for the connected components of the real forms of $\mathrm{SO}(n, C)$, i.e., $\mathrm{SO}(p, q)$, $p+q = n$, $\mathrm{SO}^*(n)$, $n = 2\nu$, and $\mathrm{SO}(n)$.

§ 6. The Fundamental Representations

Let G be a Lie group which admits a Gauss decomposition $G = \overline{\mathfrak{Z}DZ.}$ As we showed in § 2 every irreducible representation T^{L^m} of G is an induced representation, namely induced by the one-dimensional representation

$$\delta \rightarrow L_\delta^m = \delta_1^{m_1}\delta_2^{m_2}\dots\delta_n^{m_n} \tag{1}$$

of the subgroup D of G. The components of the highest weight $m = (m_1, m_2, \dots, m_n)$ satisfy certain restrictions which we have stated for each class of the classical groups.

Definition 1. An irreducible representation T^{L^m} is called the *Young product* of irreducible representations $T^{L^{m'}}$ and $T^{L^{m''}}$ if

$$L_\delta^m = L_\delta^{m'}L_\delta^{m''}. \blacktriangledown \tag{2}$$

Eq. (2) implies by virtue of eq. 2 (21) that the carrier space $H^{L^m}(Z)$ is the linear envelope of the products of polynomials $p'(z)p''(z)$, where $p'(z) \in H^{L^{m'}}(Z)$ and $p''(z) \in H^{L^{m''}}(Z)$. Note the difference between this Young product and the tensor product $T^{L^{m'}} \otimes T^{L^{m''}}$, whose carrier space is spanned by the products $p'(z')p''(z'')$.

Using the concept of Young product, we can express an arbitrary irreducible representation T^{L^m} in terms of a set of simplest representations, called the *fundamental representations*. In the case of $\mathrm{GL}(n, C)$ we can take the following representations to be the fundamental ones (we assume for simplicity that m_n is integer):

$$\begin{aligned} \overset{1}{m} &= (1, 0, \dots, 0), \\ \overset{2}{m} &= (1, 1, 0, \dots, 0), \\ &\dots\dots\dots \\ \overset{k}{m} &= (\underbrace{1, 1, \dots, 1}_{k}, 0, \dots, 0), \\ &\dots\dots\dots \\ \overset{n}{m} &= (\underbrace{1, 1, \dots, 1}_{n}). \end{aligned} \tag{3}$$

It is evident from eq. (1) that every other irreducible representation of GL(n, C) is the Young product of representations of the type (3).

In the case of SL(n, C) and Sp$(2n, C)$ the fundamental weights coincide with the first $n-1$ fundamental weights of GL(n, C). In the case of SO$(2\nu+1)$ the fundamental highest weights are:

$$\begin{aligned} \overset{1}{m} &= (1, 0, 0, \ldots, 0), \\ \overset{2}{m} &= (1, 1, 0, \ldots, 0), \\ &\cdots\cdots\cdots \\ \overset{\nu-1}{m} &= (1, 1, \ldots, 1, 0), \\ \overset{\nu}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}). \end{aligned} \tag{4}$$

The last weight corresponds to ~~the~~ spinor representation. Finally, in the case of SO(2ν) we have two spinor representations and the fundamental weights have the form

$$\begin{aligned} \overset{1}{m} &= (1, 0, \ldots, 0), \\ \overset{2}{m} &= (1, 1, 0, \ldots, 0), \\ &\cdots\cdots\cdots \\ \overset{\nu-2}{m} &= (1, 1, \ldots, 1, 0, 0), \\ \overset{\nu-1}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}), \\ \overset{\nu}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}). \end{aligned} \tag{5}$$

If we use the conditions imposed on the components of highest weights corresponding to an inductive character, we obtain

THEOREM 1 (the Cartan theorem). *For every simple, classical Lie group G of rank n there exists n fundamental weights $\overset{i}{m}$, $i = 1, 2, \ldots, n$, such that every highest weight $m = (m_1, m_2, \ldots, m_n)$ corresponding to the irreducible representation T^{L^m} of G is given by the linear combination*

$$m = \sum_{i=1}^{n} f_i \overset{i}{m}$$

with non-negative integral coefficients

$$f_i = m_i - m_{i+1}, \quad i = 1, 2, \ldots, n, \ m_{n+1} = 0. \blacktriangledown \tag{6}$$

In the current literature the symbol $D^N[f_1, f_2, \ldots, f_n]$ is used for labelling an irreducible representation of dimension N with the highest weight $m = f_1\overset{1}{m} + f_2\overset{2}{m} + \ldots + f_n\overset{n}{m}$ expressed in terms of the fundamental weights. This symbol should be distinguished from the symbol $D^N(m_1, m_2, \ldots, m_n)$, where

m_i are the components of the highest weight, which define the inductive character $L^m = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n}$.

Clearly, by virtue of formula (1) and def. 1, the representation T^{L^m} corresponding to a highest weight $m = (m_1, m_2, \ldots, m_n)$ is the Young product of the fundamental representations $T^{\overset{i}{L^m}}$ each taken f_i times.

§ 7. Representations of Arbitrary Lie Groups

We know, by Levi–Malcev theorem, that an arbitrary simply-connected Lie group G can be represented as a semidirect product

$$G = R \otimes G_S, \tag{1}$$

where R is the maximal, simply-connected, solvable, normal subgroup in G and G_S is a semisimple, simply-connected subgroup of G. The subgroup R is called the *radical* of G. This property of Lie groups allows us to develop a representation theory for arbitrary Lie groups analogous to that of semisimple groups. We first show an interesting property of the restriction T_R of an irreducible representation T of G to the radical R.

THEOREM 1. *Let $g \to T_g$ be an irreducible representation of a simply-connected Lie group G and let N be an arbitrary connected solvable normal subgroup of G. Then, for all $n \in N$, we have*

$$T_n = \chi(n)I, \tag{2}$$

where χ is the character of N satisfying the condition

$$\chi(g^{-1}ng) = \chi(n) \quad \textit{for all } g \in G. \tag{3}$$

PROOF: By Lie's theorem there exists in the carrier space H of T a non-zero vector u_χ and a character χ of N such that

$$T_n u_\chi = \chi(n)u_\chi \quad \text{for all } n \in N.$$

Because N is a normal subgroup of G, we have

$$T_n T_g u_\chi = T_g T_{g^{-1}} T_n T_g u_\chi = \chi(g^{-1}ng) T_g u_\chi, \tag{4}$$

i.e., $T_g u_\chi$ is also an eigenvector of T_n, for all $n \in N$. This implies that the restriction T_N of T_G contains together with a character $\chi(n)$ all characters

$$\chi_g(n) \equiv \chi(g^{-1}ng).$$

Repeating now the arguments of the proof of Lie's Theorem (below eq. 1 (3)) we obtain that

$$\chi(g^{-1}ng) = \chi(n) \quad \text{for all } g \in G. \tag{5}$$

Because T_G is irreducible, the invariant linear hull of all vectors $T_g u_\chi$ coincides with the whole space H. Hence, $T_n = \chi(n)I$, by eqs. (4) and (5). ▼

As a corollary to th. 1, we obtain

THEOREM 2. *Every irreducible representation T of a simply-connected Lie group G is of the form*

$$T = \chi \otimes \tilde{T},$$

where $\chi(g)$ is a character of G which is equal to identity for $g \in G_S$, and satisfies the condition (3), *and $\tilde{T}$ is an irreducible representation of the semisimple group G_S.*

PROOF: The restriction T_R of T_G to R, by th. 1, is given by the character $\chi(r)$ $r \in R$. By virtue of Levi–Malcev theorem every $g \in G$ can be represented in the form $g = \{r, g_s\}$, $r \in R$, $g_s \in G_s$. Setting $\chi(g) = \chi(\{r, g_s\}) = \chi(r)$ and using eq. (3), we can extend the character χ onto G. The representation of G defined by $\tilde{T} \equiv \chi^{-1} \otimes T$ of G is trivial on R. Hence, $\tilde{T}$ provides a representation of the factor group $G/R \simeq G_S$. Consequently, $T = \chi \otimes \tilde{T}$. Clearly, the irreducibility of T implies the irreducibility of $\tilde{T}$. ▼

Thus, the problem of the classification of irreducible representations of an arbitrary, simply-connected Lie group is reduced to the problem of the classification of characters of solvable groups and the classification of irreducible representations of semisimple Lie groups. Furthermore, the representation theory of semisimple Lie groups allows us to give the explicit realization of the operators T_g and the carrier space H of an irreducible representation T of G. In fact, if the Levi factor G_S of G is complex, then every irreducible representation of a simply-connected Lie group G is realized by means of the formula

$$U_{\{r, g_s\}} u(z) = \chi(r) L_{\tilde{\delta}} u(z_{\tilde{g}}), \quad g = \{r, g_s\} \in G = R \rtimes G_S, \tag{6}$$

where χ is a character of the radical R satisfying eq. (3), $\delta \to L_\delta$ is the character of the Gauss factor D in the Gauss decomposition $G_S = \overline{\mathfrak{Z}DZ}$ and $\tilde{\delta}$ and $z_{\tilde{g}}$ are determined by the decomposition $\tilde{g} = zg_s = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$ of the element zg_s. The elements $u(z)$ of the carrier space H are polynomials in the matrix elements z_{pq} of elements $z \in Z$.

If the Levi factor G_S of G is a real semisimple Lie group, then the representation $T_{\{r, g_s\}}$ of G can also be realized by eq. (6); the subgroup Z represents in this case the Gauss factor of the complex extension $(G_S)^c$ of G_S.

The following example shows that the condition (3) imposes severe restrictions on the class of admissible characters, and consequently on the class of finite-dimensional irreducible representations of a simply-connected Lie group G.

EXAMPLE 1. Let $G = T^4 \rtimes \mathrm{SL}(2, C)$, i.e., the Poincaré group. An arbitrary character of T^4 has the form

$$\chi(a) = \exp i pa, \quad pa = p_\mu a^\mu, \quad p_\mu \in C^1. \tag{7}$$

From the group multiplication law, we find for $(g = \{b, \Lambda\}, a = \{a, I\})$

$$g^{-1} a g = \{-\Lambda^{-1} b, \Lambda^{-1}\}\{a, I\}\{b, \Lambda\} = \{\Lambda^{-1} a, I\}.$$

Hence, eq. (3) implies

$$\chi(\Lambda^{-1}a) = \chi(a) \quad \text{for all } \Lambda \text{ in } SL(2, C).$$

Hence $\Lambda p = p$ for all Λ, by virtue of eq. (7). This is only possible if p is the zero vector. Hence, $\chi(a) = I$.

Consequently, all irreducible, finite-dimensional representations of the Poincaré group are the irreducible finite-dimensional representations of SL(2, C) lifted to the Poincaré group by means of eq. (6). ▼

The th. 2 has various interesting consequences. In particular, we have

PROPOSITION 3. *A simply-connected Lie group G admits a non-trivial irreducible unitary representation T with $1 < \dim T < \infty$, if and only if the Levi factor G_S contains a non-trivial compact normal subgroup.*

PROOF: By th. 2, $T = \chi \otimes \tilde{T}$, where $\tilde{T}$ is an irreducible representation of a connected, semisimple subgroup G_S of G. Hence, if G_S contains a non-trivial, compact normal subgroup K, then choosing χ to be unitary and $\tilde{T}$ to be an irreducible unitary representation of K, we can extend $\tilde{T}$ to G_S and we obtain a unitary irreducible representation T of G with $1 < \dim T < \infty$. If, however, G_S does not contain a compact normal subgroup, then it is a direct product of simple, connected, noncompact groups. Hence, by th. 1.2, G_S does not admit a non-trivial, finite-dimensional unitary representation. ▼

It is interesting that the Weyl theorem can be generalized to arbitrary simply-connected Lie groups:

THEOREM 4. *A representation T_G of a simply-connected Lie group G is fully reducible if and only if its restriction T_R to the radical R is fully reducible.*

PROOF: It is sufficient to consider the case when an invariant subspace H_1 of the carrier space H and the factor space H/H_1 are irreducible. If the restriction T_R of T_G to the radical R is fully reducible, then there exists a space $H_2 \subset H$, the complement of H_1, which is invariant relative to R. It follows from th. 1 that the action of the radical R in H_1 and H_2 reduces to the multiplication by characters χ_1 and χ_2 of R, respectively. If $\chi_1 \neq \chi_2$, then H_2 is the maximal subspace in H, on which the action of the radical R is given by the multiplication by χ_2; the considerations used in the proof of th. 1 show that H_2 is invariant also relative to the whole G. Hence, T is fully reducible in H.

If $\chi_1 = \chi_2 = \chi$ then, setting $\chi(g) = \chi(\{r, g_s\}) = \chi(r)$ and using eq. (3), we can extend the character χ to the whole group G. Taking now the tensor product $\chi^{-1}(g) \otimes T_g$, we obtain the representation of G, which is trivial on the radical. Thus, it gives the representation of the semisimple group $G_S \simeq G/R$, which is fully reducible by Weyl's theorem. ▼

§ 8. Further Results and Comments

We discuss now briefly a series of further results which are important for applications.

A. *Reduction of a Representation to a Subgroup*

In many physical problems the following question arises: what irreducible representations of the subgroup G_0 of a group G occur if an irreducible representation T of G is restricted to the subgroup G_0?

We first give the main result for GL(n, C).

THEOREM 1. *An irreducible representatiom of* GL(n, C) *determined by the highest weight* $m = (m_1, m_2, \ldots, m_n)$, *restricted to the subgroup* $G_0 \simeq$ GL($n-1$, C) *contains all irreducible representations of* G_0 *with highest weights* $l = (l_1, \ldots, l_{n-1})$ *for which the following conditions are satisfied*

$$m_1 \geqslant l_1 \geqslant m_2 \geqslant l_2 \geqslant m_3 \geqslant \ldots \geqslant m_{n-1} \geqslant l_{n-1} \geqslant m_n. \tag{1}$$

Every irreducible component occurs with multiplicity one. ▼

(For the proof cf. e.g. Želobenko 1962, § 13.)

By virtue of th. 3.1 this theorem is also true for all real forms of GL(n, C) and in particular for the reduction of irreducible representations of the unitary group $U(n)$ with respect to $U(n-1)$.

For orthogonal groups we have a similar result.

THEOREM 2. *An irreducible representation of* SO($2\nu+1$, C) *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ *with integer (half-integer) components restricted to the subgroup* $G_0 \simeq$ SO(2ν, C) *contains all irreducible representations of* G_0 *with highest weights* $q = (q_1, q_2, \ldots, q_\nu)$ *for which the following conditions are satisfied*

$$m_1 \geqslant q_1 \geqslant m_2 \geqslant q_2 \geqslant \ldots \geqslant m_\nu \geqslant q_\nu \geqslant -m_\nu. \tag{2}$$

The components q_i *are simultaneously all integers (if* m_i *are integers) or all half-odd integers (if* m_i *are half-odd integers). Every irreducible representation occurs with multiplicity one.*

Similarly, the restriction of the irreducible representations of SO(2ν) *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ *with integral (or half-odd integral) components contains all irreducible representations of the subgroup* $G_0 \simeq$ SO($2\nu-1$) *with the highest weights* $p = (p_1, p_2, \ldots, p_{\nu-1})$ *for which*

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \ldots \geqslant m_{\nu-1} \geqslant p_{\nu-1} \geqslant |m_\nu|.$$

The components p_i *are simultaneously all integers (all half-odd integers) together with* m_i. *Every irreducible representation occurs with multiplicity one.* ▼

(For the proof cf. Želobenko 1962, § 13.)

Clearly, all statements of th. 2 hold for real forms of SO(n, C) and in particular for orthogonal real groups SO(n) and SO(p, q), $p+q = n$.

Analogous, but more complicated results, hold for symplectic groups.

The proofs of these theorems can be given by various methods. In particular, an elementary proof can be given by means of Young diagrams (cf. Hamermesh 1962, ch. 10).

The technique of induced representations used by Želobenko 1962 allows one not only to prove ths. 1 and 2 in an elegant way, but also to construct the carrier spaces in which irreducible representations of the corresponding subgroups are realized.

The problem of reduction of representations of $SU(m+n)$ with respect to $SU(m)\times SU(n)$ was considered by Hagen and Macfarlane 1966. Some special cases of reduction of $SU(m+n)$ with respect to $SU(m)\times SU(n)$ and $SU(n)$ with respect to $SO(n)$ were considered by Želobenko 1970, ch. XVIII. These problems were also treated by Whippman 1965.

B. *Weight Diagrams*

Let T^{L^m} be an irreducible representation of a semisimple Lie group G corresponding to the highest weight m and let $u_m(z) \equiv 1$ be the highest vector in the carrier space $H^m(Z)$ of T^{L^m}. Denoting the generators of the subgroup $Z(3)$ by $E_\alpha(E_{-\alpha})$ and the generators of the subgroup D by H_i and using the Cartan–Weyl commutation relations, one obtains

$$H_i E_{\pm\alpha} u_m = (E_{\pm\alpha} H_i \pm \alpha(H_i) E_{\pm\alpha}) u_m = (m_i \pm \alpha(H_i)) E_{\pm\alpha} u_m. \tag{3}$$

The eigenvectors of H_i are called the *weight vectors* and the eigenvalues are the components of a *weight*. Hence, the vectors $E_{\pm\alpha}u_m$ are, together with u_m, formally also the weight vectors in the carrier space H^m. However, the action of the subgroup Z in H^m implies (cf. eq. 2(17))

$$T^{L^m}_{z_0} u_m(z) = u_m(zz_0) = u_m, \tag{4}$$

or, infinitesimally,

$$E_\alpha u_m = 0. \tag{5}$$

Hence by eq. (3), there cannot be a weight vector in H^m with the weight $m' = (m_1 + \alpha(H_1), \ldots, m_n + \alpha(H_n))$.* This explains the names 'highest weight' and 'highest vector' for m and u_m, respectively.

Let

$$v = E_{-\alpha^{(1)}} E_{-\alpha^{(2)}} \ldots E_{-\alpha^{(s-1)}} E_{-\alpha^{(s)}} u_m, \qquad s = 1, 2, \ldots, \tag{6}$$

where the generators $E_{-\alpha}$ and E_α are in an arbitrary order. Then, eq. (3) implies

$$H_i v = (m_i - \alpha^{(1)}(H_i) - \alpha^{(2)}(H_i) + \ldots + \alpha^{(s-1)}(H_i) - \alpha^{(s)}(H_i)) v, \tag{7}$$

i.e., every non-zero vector (6) is a weight vector. Now, the highest vector u_m is cyclic for the representation T^{L^m}, by corollary 3 to th. 2.2. Hence, the weight vectors (6) span the carrier space H^m of T^{L^m}.

* We say that a weight m' is *higher* than m if the first non-vanishing component of the vector $m' - m$ is positive.

By virtue of eq. (7), an arbitrary weight has the form

$$m - k_1 \alpha_1 - k_2 \alpha_2 - \dots - k_n \alpha_n, \tag{8}$$

where k_i are non-negative integers and $\alpha_1, \alpha_2, \dots, \alpha_n$ are simple roots and n is the dimension of the Cartan subalgebra. We can restrict ourselves, in eq. (8), to simple roots only, due to the fact that every positive root is a sum of simple roots with non-negative coefficients. Because the dimension of H^m is finite, the number of different weights is finite.

It is convenient to associate with every weight (8) a point of an n-dimensional vector space R^n. The diagram in R^n corresponding to the collection of all weights is called the *weight diagram of a given representation.*

Eq. (7) implies, in particular, that all generators H_i of the Cartan subgroup D are diagonal in the space H^m. In physical applications, if G is a symmetry group of some physical system, the generators H_i are simultaneously diagonalizable, hence observables. For instance, in the case of the SU(3)-symmetry in particle physics, the generator H_1 can be taken to be the third component of isospin, and the generator H_2 to be the hypercharge. Hence, the weights give the values of measurable quantities.

It is crucial for applications to determine all weights associated with a given highest weight and their multiplicities. This problem was solved by Freudenthal (cf. Freudenthal and De Vries 1969) and Kostant 1959.

In order to state the Freudenthal theorem, it is necessary to introduce a scalar product for roots and weights. Notice first, that roots and weights are elements of the dual space H^* of the Cartan algebra H. On the other hand, by formula 1.4(3), for every $\lambda \in H^*$, there is a uniquely determined element $H_\lambda \in H$ such that

$$\lambda(X) = (H_\lambda, X) \quad \text{for all } X \in H, \tag{9}$$

where $(\cdot, \cdot)$ is the Killing form of the Lie algebra L of G. When $\lambda, \mu \in H^*$, the scalar product (λ, μ) can be defined by

$$(\lambda, \mu) = (H_\lambda, H_\mu). \tag{10}$$

Because G is semisimple, by th. 1.4.1, the restriction of the Killing form on H, and consequently, the scalar product (10) are non-degenerate. The Freudenthal formula expresses the multiplicity n_M of a weight M in terms of multiplicities of weights $M+k\alpha$, $\alpha > 0$. Explicitly, we have

THEOREM 3. *The multiplicity n_M of a weight M in the weight diagram associated with a highest weight m is given by the recursion formula*

$$[(m+r, m+r) - (M+r, M+r)] n_M = 2 \sum_{k=1}^{\infty} \sum_{\alpha>0} n_{M+k\alpha} (M+k\alpha, \alpha), \tag{11}$$

where

$$r = \frac{1}{2}\sum_{\alpha>0}\alpha. \blacktriangledown$$

(For the proof cf. Freudenthal and De Vries 1969.)

The formula (11) gives an effective method of calculating the multiplicity n_M of a weight M starting with $n_m = 1$. Clearly, by virtue of eq. (8) the summation over k in eq. (11) is finite.

We now introduce the Weyl group in order to give the Kostant formula. Let μ be a vector from the weight (or root) space and let α be a root. Set

$$\mu' \equiv S_\alpha(\mu) \equiv \mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)}\alpha. \tag{12}$$

Because $S_\alpha(\alpha) = -\alpha$, and for $\mu \perp \alpha$, $S_\alpha(\mu) = \mu$ the map $\alpha \to S_\alpha$ is a reflection with respect to the hyperplane perpendicular to the vector α. Clearly, $S_\alpha^2 = I$ and S_α is an orthogonal transformation, i.e.,

$$\big(S_\alpha(\mu_1), S_\alpha(\mu_2)\big) = (\mu_1, \mu_2). \tag{13}$$

The linear group generated by the transformations S_{α_i}, α_i = simple roots, is called the *Weyl group W*. If $s \in W$ is represented by an even product of reflections relative to hyperplanes perpendicular to roots, then $\det S = 1$. Otherwise $\det S = -1$.

THEOREM 4. *Let $P(R)$, $R \in H^*$, be a partition function, which is equal to the number of solutions $(k_\alpha, k_\beta, \ldots, k_\zeta)$ of the equation*

$$R = \sum_{\alpha>0} k_\alpha \alpha, \tag{14}$$

where α's are positive roots and k_α's are non-negative integers. Then, the multiplicity n_M of a weight M associated with a highest weight m is given by the formula

$$n_M = \sum_{S\in W} (\det S) P[S(m+r)-(M+r)]. \blacktriangledown \tag{15}$$

(For the proof cf. Kostant 1959a.)

The Kostant formula is useful rather in theoretical considerations. It is tedious in practical calculations because there is no effective method of computing the partition function $P(R)$.

In the case of the algebras of rank 2 and the algebra A_3, the explicit expressions for the function $P(R)$ were found by Janski 1963.

The other formulas for multiplicities of weights, sometimes more convenient for applications, were found by Klimyk 1966 and 1967a, b.

C. *Decomposition of the Tensor Product*

The problem of the decomposition of a tensor product $T\otimes T'$ of irreducible representations T and T' of a topological group G is called the *'Clebsch–Gordan series' problem.*

First, we consider the case of $\mathrm{GL}(n, C)$. We define, on representations

$$L^m = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n},$$

of the subgroup D, the following operators $\hat{\delta}_k$, $k = 1, 2, \ldots, n$,

$$\hat{\delta}_k L^m \equiv \begin{cases} \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_k^{m_k+1}\ldots\delta_n^{m_n}, & \text{if } m_{k-1} > m_k, \\ 0, & \text{if } m_{k-1} = m_k. \end{cases} \tag{16}$$

Note that the operators $\hat{\delta}_k$ are noncommutative. The following theorem gives the general formula for the decomposition of the tensor product $T^{L^m}\otimes T^{L^{m'}}$ of irreducible representations in terms of the operators $\hat{\delta}_k$:

THEOREM 5. *Let T^{L^m} and $T^{L^{m'}}$ be irreducible representations of* $\mathrm{GL}(n, C)$ *induced by the representations L^m and $L^{m'}$ of the subgroup D, respectively. Then the tensor product $T^{L^m}\otimes T^{L^{m'}}$ reduces to*

$$\sum_{m''}\oplus T^{L^{m''}}$$

where $L^{m''}$ are summands in the expansion of the following determinant

$$\Gamma_{m_1 m_2\ldots m_n}L^{m'} \equiv \begin{vmatrix} \Gamma_{m_1} & \Gamma_{m_1+1} & \ldots & \Gamma_{m_1+(n-1)} \\ \Gamma_{m_2-1} & \Gamma_{m_2} & \ldots & \Gamma_{m_2+(n-2)} \\ \ldots & \ldots & \ldots & \ldots \\ \Gamma_{m_n-(n-1)} & \Gamma_{m_n-(n-2)} & \ldots & \Gamma_{m_n} \end{vmatrix} \tag{17}$$

where

$$\Gamma_m = \sum_{\nu_1+\nu_2+\ldots+\nu_n=m}\hat{\delta}_1^{\nu_1}\hat{\delta}_2^{\nu_2}\ldots\hat{\delta}_n^{\nu_n}, \qquad \Gamma_m = 0 \quad \textit{for } m < 0. \blacktriangledown$$

(For the proof cf. Želobenko 1963, th. 12.)

In the following, for simplicity, we shall use the symbol $m\otimes m'$ for the tensor product $T^{L^m}\otimes T^{L^{m'}}$.

As examples, we consider two important classes of tensor products of representations of $\mathrm{GL}(n, C)$.

1° Multiplication of the vector $\overset{(1)}{m}$ by a tensor

$$m' = (m_1', m_2', \ldots, m_n'), \qquad \min(m_i' - m_j') \geqslant 1.$$

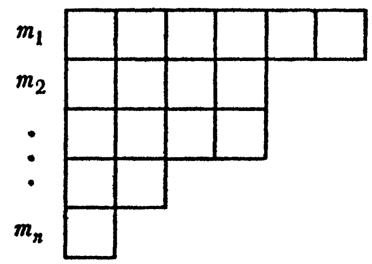
The formula (17) gives immediately

$$\Gamma_{1,0,\ldots,0}L^{m'} = \hat{\delta}_1 L^{m'} + \hat{\delta}_2 L^{m'} + \ldots + \hat{\delta}_n L^{m'}. \tag{18}$$

We will now give a useful graphical representation of these results. First note that to each highest weight

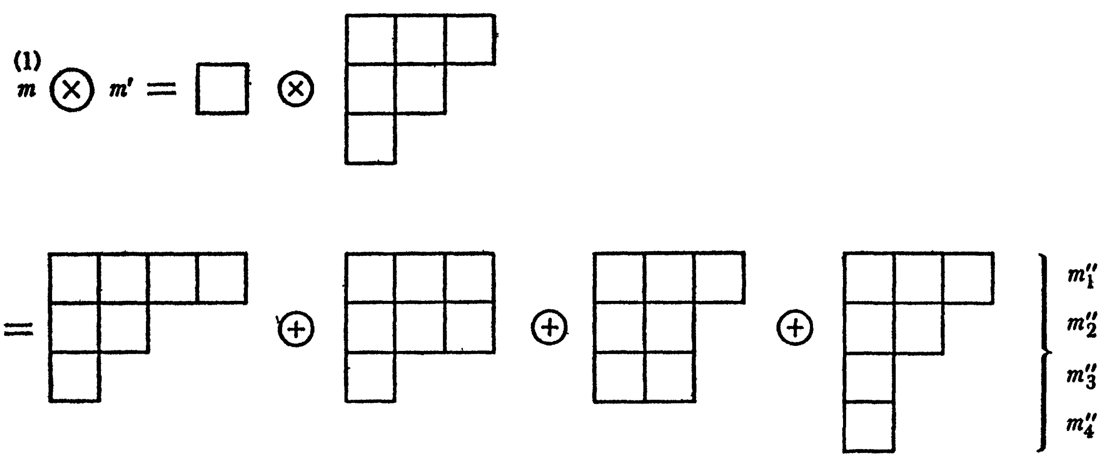
$$m = (m_1, \ldots, m_n), \qquad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n, \tag{19}$$

we can associate the following diagram



where the first row contains m_1 boxes, the second m_2 boxes, etc. It follows from (19) that the lengths of successive rows are non-increasing and the number of rows is at most n. Such diagrams are called admissible. They are in one-to-one correspondence with the highest weights of irreducible representations. These diagrams are precisely the Young frames characterizing the irreducible representations of the permutation group (cf. ch. 7.5.C).

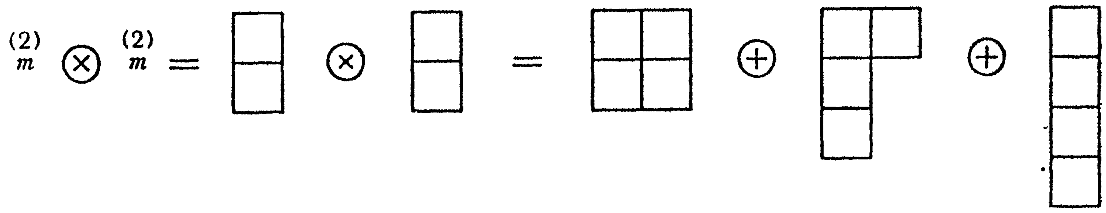
Using the Young diagrams, we can illustrate eq. (18) graphically as follows, e.g., for $m' = (3, 2, 1, 0)$, one obtains



2° Tensor product of two polyvectors: let $\overset{(i)}{m}$ and $\overset{(k)}{m}$ denote the highest weight associated with polyvectors. Then, th. 5 gives immediately

$$\Gamma_{(\underbrace{111\ldots1}_{i}\,000\ldots0)} \overset{(k)}{L^m} = \overset{(i)}{L^m}\overset{(k)}{L^m} + \overset{(i+1)}{L^m}\overset{(k-1)}{L^m} + \ldots + \overset{(i+k)}{L^m} L^0. \tag{20}$$

This result can also be illustrated graphically by means of Young frames, e.g.,



These two examples suggest that there should exist a graphical method of the decomposition of an arbitrary tensor product $m \otimes m'$. Indeed, by virtue of eq. (16), and the fact that a component m_i of m is represented as the length of the ith row in the Young frame, the formula (17) yields the admissible Young frames of the irreducible representations occurring in the decomposition. The general rule can be stated in the following manner:

We take one Young frame corresponding to the highest weight m as fixed, and label the rows of the second frame as follows, for example

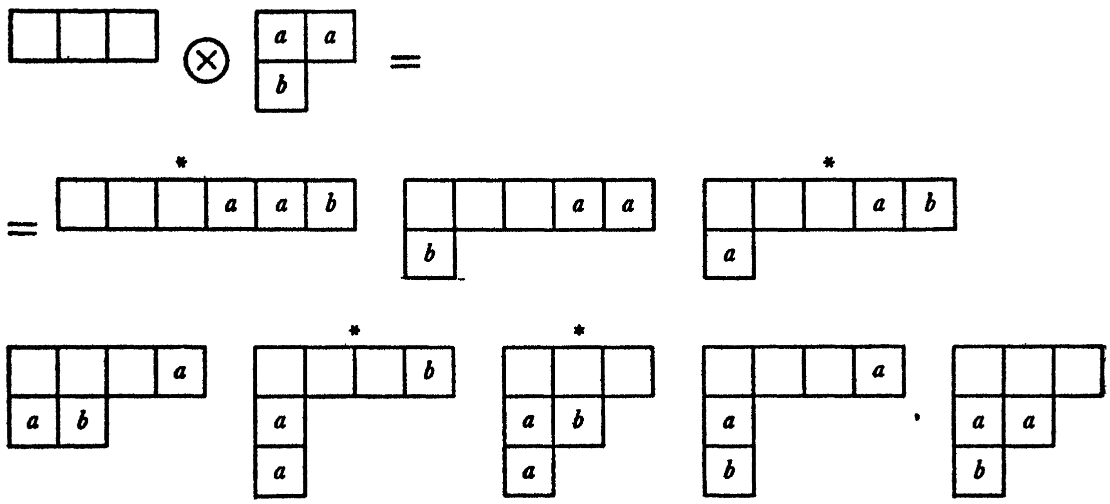
a	a	a	a
b	b	b	
c	c		

We add now the boxes of the first row labelled 'a' to the first frame in all possible ways so as to obtain again admissible frames. To all the frames thus obtained we add next the boxes of the second row, then the third row, etc., in each step requiring that the resulting frames are admissible (i.e., $m_k'' \geqslant m_{k-1}''$).

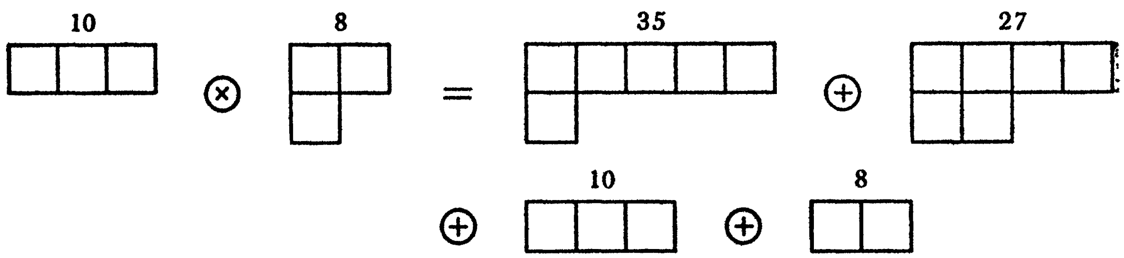
From this set of frames we eliminate those frames which contain equal labels appearing in a column. Next we drop the columns of length n from the tableau. Finally, we order the boxes of the frames: We start with the first row and take the boxes in the order from right to left. Then we run through the second row from right to left and so on. This ordered sequence contains boxes with labels and empty boxes. If we cut this sequence at any point, the number of labels b must not exceed the number of a's, the number of c's must not exceed the number of b's, etc., counted from the start until the cut.

The resulting frames, which differ in form or in the places of the labels, correspond to distinct irreducible representations contained in the tensor product.

Example 1. Let $G = SU(3)$. Consider the tensor product of the representations $m = (3, 0)$ and $m' = (2, 1)$. The rule of adding of boxes of the second frame corresponding to m' to the first frame corresponding to m gives the following set of frames:



The frames with '*' are forbidden by our rules. Hence, we obtain



We have put over the frames the dimension of the resulting irreducible representations obtained from the Weyl formula (29) below (for the detailed proof of these rules of decomposition of tensor product $m \otimes m'$, see Itzykson and Nauenberg 1966 or Boerner 1963). Clearly, th. 1, by virtue of the th. 3.1, is valid for $GL(n, R)$, $U(n)$, $SL(n, C)$, $SL(n, R)$ and $SU(n)$.

A general formula for the decomposition of the tensor product of irreducible representations of an arbitrary semisimple Lie group G was obtained by Kostant and Steinberg.

THEOREM 6. *Let m and m' be two irreducible representations of a semisimple Lie group. Then, the multiplicity $n_{m''}$ of the irreducible representation m'' in the tensor product $m \otimes m'$ is given by the formula*

$$n_{m''} = \sum_{S,T \in W} \det(ST) \cdot P\{S(m+r)+T(m'+r)-(m''+2r)\}, \tag{21}$$

where W is the Weyl group of G, $r = \frac{1}{2}\sum_{\alpha>0} \alpha$ and the partition function $P(R)$ is the same as defined in th. 4. ▼

(For the proof cf. Steinberg 1961.)

The use of formula (21) is tedious even for low-dimensional Lie algebras. Fortunately, special computer programs has been elaborated by Pajas and tables of multiplicities for most important groups are published (Pajas 1967).

Some variants of formulae (21) for multiplicities were derived by Straumann 1965 and Klimyk 1966. There exists also an interesting graphical method elaborated by Speiser 1964. For more recent work we refer to Gruber 1968.

The problem of the decomposition of the tensor product $m \otimes m'$ onto irreducibles is complete if we can give a method of separation of the carrier space $H^{m''}$ in which the irreducible representation m'' is realized. To solve this problem, it is sufficient to express the basis vectors e_k'', $k = 1, 2, \ldots, \dim H^{m''}$, of $H^{m''}$ in terms of the basis vectors $e_i e_j'$ of the tensor product space $H^m \otimes H^{m'}$, i.e.,

$$e_k'' = c_k{}^{ij} e_i e_j'. \tag{22}$$

The coefficients $c_k{}^{ij}$ are called the *Clebsch–Gordan coefficients*. Clearly, they depend on the bases in the space H^m, $H^{m'}$ and $H^{m''}$, respectively. Unfortunately, we know the explicit form of the Clebsch–Gordan coefficients only in few cases: for SU(2) (cf., e.g., Edmonds 1957), SL(2, C) (cf., e.g., Gel'fand *et al.* 1958) and SU(n), e.g., De Swart 1963 for SU(3) and Shelepin 1967 for SU(n).

Ths. 5 and 6 were derived by means of algebraic methods. They can also be obtained by means of global methods. Indeed, one could use the tensor product theorem for global induced unitary irreducible representations of compact groups (cf. ch. 18, § 2). Then, using th. 3.1, one would extend these results for all noncompact complex and real semisimple groups associated with a given compact group. In all known cases this method is very elegant and effective.

D. *Characters and Dimensions of Representations*

The *character of a representation* T of G is defined by the formula

$$\chi(\delta) = \operatorname{Tr} T_\delta, \qquad \delta = \text{Gauss factor of } g \text{ in } g = \zeta \delta z. \tag{23}$$

This fundamental concept was introduced by Weyl. The character (23) does not have the multiplicative property $\chi(\delta + \delta') = \chi(\delta)\chi(\delta')$, valid for the characters of abelian groups. Weyl has derived a general formula for the characters of irreducible representations of all simple Lie groups.

THEOREM 7. *Let T^{L^m} be an irreducible representation of G determined by the highest weight $m = \sum_i^i f_i m$. Let $k = r + m$, where $r = \frac{1}{2} \sum_{\alpha > 0} \alpha$ and the sum is taken over positive roots. Set*

$$\xi(m) = \sum_{\ldots} \det S \exp[\mathrm{i}(Sk)\,\delta], \tag{24}$$

where W is the Weyl group (defined by eqs. (12) *and* (13)) *and* $a\delta = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$.* *Then,*

$$\chi^m(\delta) = \frac{\xi(m)}{\xi(0)}. \blacktriangledown \tag{25}$$

(For the proof cf. Weyl 1925.)

The formula (25) gives simpler expressions in the case of classical groups. In particular, for GL(n, C), we have (cf. Weyl 1939, ch. VII.6)

$$\chi^m(\delta) = \frac{d(l_1, l_2, \ldots, l_n)}{d(n-1, n-2, \ldots, 0)}, \tag{26}$$

where $l_i = m_i + n - i$ and $d(l_1, l_2, \ldots, l_n)$ is the determinant

$$\begin{vmatrix} \delta_1^{l_1} & \delta_1^{l_2} & \ldots & \delta_1^{l_n} \\ \delta_2^{l_1} & \delta_2^{l_2} & \ldots & \delta_2^{l_n} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_n^{l_1} & \delta_n^{l_2} & \ldots & \delta_n^{l_n} \end{vmatrix}. \tag{27}$$

Clearly, the formula (26) provides also a character for all real forms of GL(n, C) and in particular for $U(n)$. In these cases $T_\delta = \exp(i\delta_j H_j)$ where H_j are generators of Cartan subalgebra.

The explicit forms of the characters for the symplectic and orthogonal groups were also given by Weyl 1939, ch. VI, 8 and 9 respectively.

The formula (25) for characters can be used to calculate the dimension N^m of an irreducible representation T^{L^m} of G.

Indeed, because

$$N^m = \chi^m(e), \tag{28}$$

one obtains the dimension of T^{L^m} by the limiting procedure $\delta \to e$. This gives

THEOREM 8.

$$N^m = \frac{\prod_{\alpha>0} (\alpha, r+m)}{\prod_{\alpha>0} (\alpha, r)}, \tag{29}$$

where the multiplication is taken over all positive roots.

(For the proof cf. Weyl 1934.)

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* $Sm = m - \dfrac{2m\cdot r}{r\cdot r}r$ is the reflection of the weight vector m through a hyperplane with the normal r.

The formula (29) in case of GL(n, C) (and in particular for $U(n)$) becomes

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$$N^m = \frac{\prod\limits_{i<j} (l_i - l_j)}{\prod\limits_{i<j} (l_i^0 - l_j^0)}, \tag{30}$$

where $l_j = m_j + n - j$ and $l_j^0 = m_j - j$.

E. *Comments*

The proof of Lie's theorem in the form given here was elaborated by Godement 1956, appendix. The properties of the representations of semisimple Lie groups were investigated by Cartan 1914 and Weyl (cf., e.g., Weyl 1934–35 or 1939). The possibility of using Lie's theorem as a tool for a global classification of irreducible representations of semisimple Lie groups, was demonstrated by Godement 1956, appendix. This approach was used and extended to arbitrary Lie groups by Želobenko 1962, 1963. Here we followed the approach of Želobenko.

Note that every irreducible representation of a semisimple Lie group G is induced by the character of the subgroup D. We show in ch. 19 that infinite-dimensional representations of simple complex Lie groups are also induced by a complex character L of D.

§ 9. Exercises

§ 1.1. Show that the Heisenberg algebra

$$[Q_i, P_j] = \delta_{ij} I, \quad i,j = 1, 2, \ldots, n, \tag{1}$$

has no finite-dimensional irreducible representations.

Hint: Use the fact that $\operatorname{Tr}[A, B] = 0$ for matrices.

§ 1.2. Construct the three-dimensional representation of the Lie algebra:

$$[Q, P] = Z, \quad [Q, Z] = 0, \quad [P, Z] = 0 \tag{2}$$

§ 2.1. Let $G = \mathrm{SU}(2)$ and let H^J, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, be the space of all homogeneous monomials of degree $2J$

$$f(z_1, z_2) = \sum_{M=-J}^{J} a_M z_1^{J-M} z_2^{J+M}, \quad a_M \in C. \tag{1}$$

Set

$$u(z) = f(z, 1), \quad g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha = \bar{\delta}, \quad \gamma = -\bar{\beta} \tag{2}$$

and show that the transformation

$$T_g^J f(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2) \tag{3}$$

implies the transformation

$$(T_g^J u)(z) = (\beta z + \delta)^{2J} u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{4}$$

which are spinor representations of SU(2) of weight J.

§ 2.2. Show that there exist a scalar product in H^J with respect to which the vectors

$$\psi_M^J(z) = [(J-M)!(J+M)!]^{-\frac{1}{2}} z^{J-M} \equiv |JM\rangle \tag{5}$$

are orthonormal

Hint. Use th. 7.1.1.

§ 2.3. Show that the representation (4) is irreducible in H^J, and that every irreducible representation of SU(2) is equivalent to T^J.

§ 2.4. Show that on the space of functions $u(z)$ of problem 1.1 the generators of SU(2) are given by the differential operators

$$\begin{aligned} J_+ u &= -\frac{\mathrm{d}}{\mathrm{d}z} u, \\ J_- u &= z^2 \frac{\mathrm{d}}{\mathrm{d}z} u - 2Jzu, \\ J_3 u &= -z\frac{\mathrm{d}}{\mathrm{d}z} u + Ju. \end{aligned} \tag{6}$$

§ 2.5. Let $G = \mathrm{SL}(2, C)$ and H be the space of polynomials $p(z, \bar{z})$ of degree $\leqslant m$ in z and of degree $\leqslant n$ in $\bar{z}$. The irreducible spinor and tensor representations of G of dimension $(m+1)(n+1)$ in H can be written as

$$\begin{gathered} T(g)p(z, \bar{z}) = (\beta z + \gamma)^m (\bar{\beta}\bar{z} + \bar{\delta})^n p\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha}\bar{z} + \bar{\gamma}}{\bar{\beta}\bar{z} + \bar{\delta}}\right), \\ g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1. \end{gathered} \tag{7}$$

The equivalence of this form to two-component spinors with dotted and undotted indices is obtained if we represent $p(z, \bar{z})$ as follows:

$$p(z, z) = \sum_{A_i, \dot{B}_i = 0}^{1} \psi_{A_1 \dots A_m \dot{B} \dots \dot{B}_n} z^{(A_1 + \dots + A_m)} \bar{z}^{(\dot{B}_1 + \dots + \dot{B}_n)} \tag{8}$$

where the $(m+1)(n+1)$-dimensional 2-component spinor ψ is symmetric in the indices A_k, and symmetric in the indices $\dot{B}_k$. Show that under $T(g)$ the ψ's transform as

$$\psi'_{A_1 \dots A_m \dot{B}_1 \dots \dot{B}_n} = g_{A_1 C_1} \cdots g_{A_m C_m} \bar{g}_{\dot{B}_1 \dot{D}_1} \cdots \bar{g}_{\dot{B}_n \dot{D}_n} \psi_{C_1 \dots C_m \dot{D}_1 \dots \dot{D}_n} \tag{9}$$

where

$$g_{11} = \alpha, \quad g_{12} = \beta, \quad g_{21} = \gamma, \quad g_{22} = \delta.$$

Remark: The irreducible representations of SL(2, C) given by (7) are denoted by $D^{(m,n)}$.

§ 2.6. Let σ_k be the Pauli matrices (cf. example 1.1.1). Show that the matrices

$$J_k = \tfrac{1}{2}\sigma_k, \quad N_k = \tfrac{1}{2}\mathrm{i}\sigma_k, \quad k = 1, 2, 3 \tag{10}$$

are generators of the representation $D^{(1/2,0)}$ of the Lorentz group. Show that the matrices

$$J_k = \tfrac{1}{2}\sigma_k, \quad N_k = -\tfrac{1}{2}\mathrm{i}\sigma_k \tag{11}$$

are generators of the representation $D^{(0,1/2)}$ of the Lorentz group.

§ 2.7. Let $\{\gamma_\mu\}_0^3$ be the set of Dirac matrices given by

$$\gamma_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}. \tag{12}$$

Show that the matrices

$$M_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{13}$$

are generators of the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of the Lorentz group in block-diagonal form

$$D(\Lambda) = \begin{bmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{bmatrix}. \tag{14}$$

§ 2.8. Show that in the so-called Dirac representation of γ matrices given by

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \tag{15}$$

the generators (13) have the form

$$J_k = \varepsilon_{klm} M_{lm} = \frac{\mathrm{i}}{2}\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \quad N_k = M_{0k} = \frac{1}{2}\begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}.$$

§ 3.1. Show that SU(3)/Z_3 has self-conjugate representations of dimensions $1, 8, 27, 64, \ldots, n^3, \ldots$ only. Show in particular that the fundamental representations of dimension 3 of SU(3) (so-called *quark representations*) are not representations of SU(3)/Z_3.

§ 7.1. Classify the finite-dimensional representations of the Euclidean group $T^3 \otimes$SO(3)

§ 8.1. Let H^m be the carrier space of the irreducible representation of $u(n)$ characterized by the highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$. Show that every vector in H^m may be represented by a pattern given by

$$m = \begin{vmatrix} m_{1n} \cdots\cdots\cdots\cdots m_{nn} \\ m_{1,n-1} \cdots m_{n-1,n-1} \\ \cdots\cdots\cdots \\ m_{12} \ m_{22} \\ m_{11} \end{vmatrix},$$

where m_{ij} satisfy the condition

$$m_{ij} \geqslant m_{i,j-1} \geqslant m_{i+1,j}, \quad i = 1, 2, \ldots, n-1,\ j = 2, 3, \ldots, n.$$

Hint: Use the th. 8.8.1.

§ 8.2. Let H^2 be the carrier space of the irreducible representation of $u(2)$ characterized by the highest weight $m = (m_{12}\, m_{22})$. Show that the representation of the generators A^j_i, $i, j = 1, 2$, is given by the formula

$$A_{kk}m = (r_k - r_{k-1})m \ (k = 1, 2), \quad A_{21}m = a^1_1(m)^1_1, \quad A_{12}m = b^1_1(m)m^{\hat{1}}_1,$$

where $r_0 = 0$, $r_k = \sum_{j=1}^{k} m_{jk}$ $(k = 1, 2)$, and

$$a^1_1(m) = \left[\frac{\prod_{i=1}^{2} (l_{i2} - l_{11} + 1)}{(l_{21} - l_{11} + 1)(l_{21} - l_{11})} \right]^{1/2},$$

$$b^1_1(m) = \left[- \frac{\prod_{i=1}^{2} (l_{i2} - l_{11})}{(l_{21} - l_{11})(l_{21} - l_{11} - 1)} \right]^{1/2},$$

with

$$l_{ik} = m_{ik} - i, \quad m^1_1 = \begin{vmatrix} m_{12} & m_{22} \\ & m_{11}+1 \end{vmatrix}, \quad m^{\hat{1}}_1 = \begin{vmatrix} m_{12} & m_{22} \\ & m_{11}-1 \end{vmatrix}.$$

§ 8.3. Using the graphical method show that the tensor product of a fundamental representation $\overset{p}{T}{}^{m}$, $\overset{p}{m} = (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0)$, with an arbitrary representation T^m, $m = (m_1, \ldots, m_n)$ of $U(n)$ has the following decomposition:

$$\overset{p}{T}{}^{m} \otimes T^m = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} \oplus T^{m' = (m_1, \ldots, m_{i_1}+1, \ldots, m_{i_p}+1, \ldots, m_n)}.$$

§ 8.4. Show that the irreducible representations of a semisimple Lie group G form a semigroup with respect to multiplication of highest weights.

§ 8.5. Find unitary irreducible infinite-dimensional discontinuous representations of SU(n).

Hint. Take the discontinuous representations of the abelian subgroup D of SL(n, C) given e.g. by example 5.1.3, induce them to SL(n, C) and then restrict them to obtain representations of SU(n).

Chapter 9

Tensor Operators, Enveloping Algebras and Enveloping Fields

Tensor operators $\{T_a\}$ and tensor-field operators $\{T_a(x)\}$ associated with group representations play a fundamental role in quantum theory. Physical quantities like the angular momentum J, the energy-momentum four-vector P_μ, spin, fields and currents are identified with objects of this kind. The description of most physical phenomena in quantum theory reduces therefore to an analysis of the properties of certain tensor operators.

In sec. 1 we describe the basic properties of tensor operators and derive the Wigner–Eckart theorem.

In sec. 2 we discuss the basic properties of the enveloping algebra, which is in principle an algebra of the tensor operators. The properties of the invariant operators which are, in fact, the simplest tensor operators are given in sec. 3.

If a given group G is a symmetry group of some physical system, then the spectra of the invariant operators associated with G determine the observable quantum numbers of the physical system. Therefore, from the point of view of physical applications we are deeply interested to find explicitly:

(i) The set $\{C_p\}$ of independent invariant operators which generates the ring of the invariant operators in the enveloping algebra E of the Lie algebra L of G.

(ii) The spectra of these independent invariant operators C_p.

In sec. 4 we give an explicit solution to problems (i) and (ii) for all classical simple Lie algebras.

Finally, in sec. 5 we discuss the important concept of the enveloping field of a Lie algebra and in particular the famous Gel'fand–Kirillov theorem on the generators of the enveloping field.

§ 1. The Tensor Operators

In the quantum theory of atomic and nuclear spectroscopy a set of operators $\{T^J_m\}$, $m = -J, -J+1, \ldots, J-1, J$, appear which transform under the rotation group SO(3) as the spherical harmonics $Y^J_m(\vartheta, \varphi)$ (or, as the state vectors), i.e.,*

* We use the definition of tensor operators due to Wigner 1959.

$$U_g^{-1} T_m^J U_g = D_{mm'}^J(g) T_{m'}^J. \tag{1}$$

This has lead to the introduction, in mathematical physics, of the concept of tensor operators. We now give a general definition. Because we shall deal with non-compact groups, we shall distinguish between contravariant $\{T^a\}$ and covariant $\{T_a\}$ tensor operators.

DEFINITION 1. Let $g \to D(g)$ be a finite-dimensional representation of a group G in a vector space V and let $\{D^a{}_b\}$ be its matrix form in a basis $\{e_a\}_1^{\dim V}$ of V. Let $g \to U_g$ be a unitary representation of G in a Hilbert space H.

A set $\{T^a\}$, $a = 1, 2, \ldots, \dim D$, of operators is said to be a *contravariant tensor operator* if

$$U_g^{-1} T^a U_g = D^a{}_b(g) T^b. \blacktriangledown \tag{2}$$

Thus a contravariant tensor operator $\{T^a\}$ in H transforms as a contravariant vector with respect to the representation $g \to D(g)$ in V.

The corresponding definition of the tensor operator on the level of Lie algebra is obtained if we insert the representation of the generators

$$D(X) \equiv \frac{\mathrm{d}}{\mathrm{d}\Theta} D(\exp \Theta X)_{\Theta=0}, \quad \text{and} \quad \mathrm{i}U(X) \equiv \frac{\mathrm{d}}{\mathrm{d}\Theta} U_{\exp \Theta X}|_{\Theta=0} \tag{3}$$

into formula (2). We find

$$[U(X), T^a] = \mathrm{i} D^a{}_b(X) T^b, \quad X \in L. \tag{4}$$

Remark 1: We shall assume that the generators $U(X)$, $X \in L$, have a common dense invariant domain $D \subset H$. The construction of such domains is given in 11.1, as well as the precise definitions of generators (3). We shall assume that D is a domain for operators T^a. We shall concentrate in this chapter on algebraic properties of tensor operators so the concrete form of D is irrelevant.

In general, the definitions (2) and (4) of a tensor operator are equivalent if the representation of the Lie algebra L of G in (4) can be integrated to a global representation U_g of G. And it is instructive to show also that (4) implies (2). The method used here is useful in many practical calculations. Let the global representation $g \to U_g$ be given by

$$U_g = \exp\left(\mathrm{i} s_\varrho U(X_\varrho)\right),$$

where we have denoted the representations of the generators by $U(X_\varrho)$. Consider a 'one-parameter' subgroup of the global representation $U_{g(\lambda)} = \exp\left(\mathrm{i}\lambda s_\varrho U(X_\varrho)\right)$. We set

$$T'^a(\lambda) = U_{g(\lambda)}^{-1} T^a U_{g(\lambda)} \tag{5}$$

and differentiate this equality with respect to λ

$$\frac{\mathrm{d}T'^a(\lambda)}{\mathrm{d}\lambda} = -\mathrm{i} s_\varrho \exp\left(-\mathrm{i}\lambda s_\varrho U(X_\varrho)\right)[U(X_\varrho), T^a] \exp\left(\mathrm{i}\lambda s_\varrho U(X_\varrho)\right).$$

Using eqs. (4) and (5), we obtain

$$\frac{\mathrm{d}T'^a}{\mathrm{d}\lambda} = s_\varrho D^a{}_b(X_\varrho)T'^a(\lambda).$$

This differential equation with the initial conditions $T'^a(0) = T^a$ has the following integral

$$T'^a(\lambda) = [\exp(\lambda s_\varrho D(X))]^a{}_b T'^b.$$

For $\lambda = 1$ we obtain the formula (2), by virtue of eq. (5).

The tensor operator $\{T^{(s)a}\}$ is said to be *irreducible*, if $g \to D^{(s)}(g)$ is irreducible. The simplest example of an irreducible tensor operator is provided by any invariant operator C of G. In this case

$$U_g^{-1}CU_g = C,$$

i.e., $g \to D(g) \equiv 1$. A less trivial example is the following:

EXAMPLE 1. Let G be the rotation group in R^n, $H = L^2(R^n)$ and $(U_g\psi)(x) = \psi(g^{-1}x)$. Let $T^\mu = \hat{x}^\mu$ be the coordinate operator $(\hat{x}^\mu\psi)(x) = x^\mu\psi(x)$. Then

$$\begin{aligned}(U_g^{-1}\hat{x}^\mu U_g\psi)(x) &= (\hat{x}^\mu U_g\psi)(gx)\\ &= g^\mu{}_\nu x^\nu(U_g\psi)(gx) = g^\mu{}_\nu x^\nu\psi(x)\\ &= g^\mu{}_\nu\hat{x}^\nu\psi(x).\end{aligned}$$

Hence,

$$U_g^{-1}\hat{x}^\mu U_g = g^\mu{}_\nu\hat{x}^\nu, \tag{6}$$

i.e., the set $\{\hat{x}^\mu\}$ is a contravariant tensor operator. ▼

DEFINITION 2. A set $\{T_a\}$, $a = 1, 2, \ldots, \dim D$, of operators is said to be a *covariant tensor operator* if it transforms according to the representation $\check{D}(g) = D^T(g^{-1})$ contragradient relative to $D(g)$, i.e.,

$$U_g^{-1}T_aU_g = D_a^{Tb}(g^{-1})T_b \equiv D^b{}_a(g^{-1})T_b. \text{▼} \tag{7}$$

On the level of Lie algebra, eqs. (7) and (3) give

$$[U(X), T_a] = -\mathrm{i}D^b{}_a(X)T_b. \tag{8}$$

Remark 1 also applies to def. 2.

If $g \to D(g)$ is a unitary representation of G, then the space V has the metric tensor $g^{ab} = \delta^{ab}$. Consequently, we may set $D^a{}_b(g) = D_{ab}(g)$ in all formulas. We see then that the standard definition of tensor operator T^j_m for SO(3) given by eq. (1) correspond to def. 1 of a contravariant tensor operator.

Let L be an arbitrary Lie algebra with a basis X_a and let $X_a \to U(X_a)$ be a representation of L by self-adjoint operators in H. Then, the set $\{T_a\} = \{U(X_a)\}$, $a = 1, 2, \ldots, \dim L$, represents a covariant tensor operator for L. Indeed, by virtue of commutation relations in L we have for this special tensor operator

$$[U(X_b), T_a] = \mathrm{i}c_{ba}{}^cT_c. \tag{9}$$

The set of matrices $D(X_a) = -C_a \equiv ||-c_{ab}{}^c||$ provides a representation of L by virtue of Jacobi identity 1.1(7). Hence, the condition (8) is satisfied. Thus, the set $\{U(X_a)\}$ is a covariant tensor operator.

For arbitrary tensor operators $\{Q_a\}$ and $\{T^a\}$ the operator $C = Q_a T^a$ is invariant: indeed

$$U_g^{-1} C U_g = D^b{}_a(g^{-1}) D^a{}_{b'}(g) Q_b T^{b'} = D^b{}_{b'}(g^{-1}g) Q_b T^{b'} = \delta^b{}_{b'} Q_b T^{b'} = C, \quad (10)$$

or,

$$[C, U(X_a)] = 0. \quad (10')$$

This provides a convenient method of construction of invariants of a Lie algebra, which we shall frequently use in what follows.

DEFINITION 3. A set $\{T^{\mu_1\mu_2\cdots\mu_r}\}$ is said to be *contravariant tensor operator of the rank r if*

$$U_g^{-1} T^{\mu_1\mu_2\cdots\mu_r} U_g = D^{\mu_1}{}_{\nu_1}(g) D^{\mu_2}{}_{\nu_2}(g) \dots D^{\mu_r}{}_{\nu_r}(g) T^{\nu_1\nu_2\cdots\nu_r}. \quad (11)$$

One defines analogously the covariant tensor operators $\{T_{\mu_1\mu_2\dots\mu_r}\}$ and *mixed tensor operators* $\{T^{\nu_1\nu_2\dots\nu_s}_{\mu_1\mu_2\dots\mu_r}\}$.

Remark 2: Not every set of operators $\{T_a\}$, which carries a tensor index a represents a tensor operator. An important counter-example is provided by generators of the Poincaré group; indeed, under the Poincaré group the generators P_μ transform in the following manner

$$\begin{aligned}[U^{-1}{}_{(a\Lambda)} P_\mu U_{(a\Lambda)} \psi](p) &= (P_\mu U_{(a\Lambda)}) \exp(-\mathrm{i}pa) \psi(\Lambda p) \\ &= (\Lambda)_\mu{}^\nu p_\nu \exp(\mathrm{i}pa)(U_{(a\Lambda)}) \psi(\Lambda p) \\ &= \Lambda_\mu{}^\nu P_\nu \psi(p) = (\Lambda^{-1})^\nu{}_\mu P_\nu \psi(p),\end{aligned}$$

i.e.,

$$U^{-1}{}_{(a\Lambda)} P_\mu U_{(a\Lambda)} = (\Lambda^{-1})^\nu{}_\mu P_\nu. \quad (11')$$

Thus, P_μ transform according to the contragradient representation of the Poincaré group given by

$$(a, \Lambda) \rightarrow (0, \Lambda)^{-1^T} = (0, \Lambda^{-1^T}),$$

i.e., P_μ is covariant tensor operator, according to def. 2. On the other hand the commutation relations of generators $M_{\mu\nu}$ with, for instance, P_μ gives

$$[P_\sigma, M_{\mu\nu}] = \mathrm{i}(g_{\mu\sigma} P_\nu - g_{\nu\sigma} P_\mu).$$

Therefore, the set $\{M_{\mu\nu}\}$ alone cannot form a tensor operator for the Poincaré group Π according to defs. (4) and (8). However the set $\{P_\mu, M_{\mu\nu}\}$ forms a tensor operator according to eq. (9).

DEFINITION 4. A covariant tensor $\{g_{\mu_1\dots\mu_p}\}$ in V is said to be *invariant* if it satisfies

$$D^{\nu_1}{}_{\mu_1}(g) \dots D^{\nu_p}{}_{\mu_p}(g) g_{\nu_1\dots\nu_p} = g_{\mu_1\dots\mu_p}. \quad (12)$$

The Kronecker symbol δ_{ij} and the Levi–Civita symbol ε_{ijk} are the only invariant tensors in R^3 with respect to SO(3).

The following theorem describes some important properties of tensor operators.

THEOREM 1. 1° *The contraction* $\{T^{\mu_1\ldots\mu_s\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_s\beta_1\ldots\beta_q}\}$ *of a tensor operator* $\{T^{\nu_1\ldots\nu_s\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_s\beta_1\ldots\beta_q}\}$ *is again a tensor operator.*

2° *If* $\{T^{\mu_1\cdots\mu_p}\}$ *is a tensor operator which transforms according to the tensor product* $D\otimes\ldots\otimes D$ *of representations* $g\to D(g)$, *and* $g_{\mu_1\ldots\mu_p}$ *is a covariant invariant tensor relative to contragradient representation* $\hat{D}(g)=D^T(g^{-1})$, *then the operator*

$$T = g_{\mu_1\ldots\mu_p}T^{\mu_1\cdots\mu_p}$$

is an invariant of G, i.e.,

$$U_g^{-1}TU_g = T.$$

PROOF: *ad* 1°. This assertion follows from eq. (10).

ad 2°. Using eq. (11) and eq. (12) we obtain

$$U_g^{-1}TU_g = g_{\mu_1\ldots\mu_p}D^{\mu_1}{}_{\nu_1}(g)\ldots D^{\mu_p}{}_{\nu_p}(g)\,T^{\nu_1\cdots\nu_p} = g_{\nu_1\ldots\nu_p}T^{\nu_1\cdots\nu_p} = T. \blacktriangledown$$

If $\{\overset{1}{T}{}^a\}$ and $\{\overset{2}{T}{}^a\}$, $a=1,2,\ldots,\dim D$, are two contravariant tensor operators, which satisfy eq. (2), then,

$$T^a = \overset{1}{T}{}^a+\overset{2}{T}{}^a$$

is also a contravariant tensor operator of the same kind according to def. 1.

If $\{\overset{1}{T}{}^a\}$ and $\{\overset{2}{T}{}^a\}$ are two contravariant tensor operators which transform according to representations $\overset{1}{D}$ and $\overset{2}{D}$, respectively, then the set $\{T^{ab}=\overset{1}{T}{}^a\overset{2}{T}{}^b\}$ defines a contravariant tensor operator, which transforms as

$$U_g^{-1}T^{ab}U_g = D^{ab}{}_{a'b'}T^{a'b'}, \tag{11''}$$

where

$$D^{ab}{}_{a'b'}(g) = D^a{}_{a'}(g)D^b{}_{b'}(g).$$

The tensor operator $\{T^{ab}\}$ is called the *tensor product* of the tensor operators $\{\overset{1}{T}_a\}$ and $\{\overset{2}{T}_b\}$.

One can form a new irreducible tensor operator from the tensor product of two irreducible tensor operators $\{T^{\lambda_1}_{m_1}\}$ and $\{T^{\lambda_2}_{m_2}\}$. We show this construction for the case when G is a *simply reducible compact group.** Because every representation of a compact group is equivalent to a unitary representation we shall use only lower indices.

* A compact group is said to be *simply reducible* if in the decomposition of the tensor product of any two irreducible representations every irreducible component appears at most once.

Let $\{|\lambda_1; m_1\rangle\}$ be a basis in an irreducible space H^{λ_1}, $\{|\lambda_2; m_2\rangle\}$ a basis in an irreducible space H^{λ_2} and $\{|\lambda_1\,\lambda_2\,\lambda m\rangle\}$ an orthonormal basis in the irreducible subspace H^{λ} of $H^{\lambda_1}\otimes H^{\lambda_2}$. Set

$$T^{\lambda}_m = \sum_{m_1 m_2} \langle\lambda_1\,\lambda_2\,\lambda m|\lambda_1\,m_1\,\lambda_2 m_2\rangle T^{\lambda_1}_{m_1} T^{\lambda_2}_{m_2}, \tag{13}$$

where

$$|\lambda_1\,m_1\,\lambda_2 m_2\rangle \equiv |\lambda_1\,m_1\rangle|\lambda_2 m_2\rangle. \tag{14}$$

Using eq. (1) for $T^{\lambda_1}_{m_1}$ and $T^{\lambda_2}_{m_2}$, we have

$$U_g^{-1}\,T^{\lambda}_m\,U_g = \sum \langle\lambda_1\,\lambda_2\,\lambda m|\,\lambda_1\,m_1\,\lambda_2 m_2\rangle D^{\lambda_1}_{m_1 m_1'}(g)\cdot D^{\lambda_2}_{m_2 m_2'}(g)\,T^{\lambda_1}_{m_1'}T^{\lambda_2}_{m_2'}. \tag{15}$$

According to exercise 7.7.3.1.

$$D^{\lambda_1}_{m_1 m_1'}(g)\,D^{\lambda_2}_{m_2 m_2'}(g)$$

$$= \sum_{\tilde{\lambda}=|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \langle\lambda_1\,m_1\,\lambda_2 m_2|\,\lambda_1\,\lambda_2\,\tilde{\lambda}\tilde{m}\rangle D^{\tilde{\lambda}}_{\tilde{m}\tilde{m}_1}(g)\langle\lambda_1\,\lambda_2\,\tilde{\lambda}\tilde{m}_1|\lambda_1\,m_1'\,\lambda_2 m_2'\rangle \tag{16}$$

Inserting this expression in eq. (15), and using completeness and orthogonality relations for vectors $|\lambda_1\,m_1\,\lambda_2 m_2\rangle$ and $|\lambda_1\,\lambda_2\,\lambda m\rangle$ respectively one obtains

$$U_g^{-1}\,T^{\lambda}_m\,U_g = D^{\lambda}_{mm'}(g)\,T^{\lambda}_{m'}.$$

Hence, the object (13) is an irreducible tensor operator.

The following theorem describes the fundamental property of an irreducible tensor operator, very useful in applications:

THEOREM 2 (the Wigner–Eckart theorem). *Let $U^{\lambda_1}_g$ and $U^{\lambda_2}_g$ be irreducible unitary representations of a simple reducible compact group G in the Hilbert spaces H^{λ_1} and H^{λ_2}, respectively. Let $\{|\lambda_1 m_1\rangle\}$ and $\{|\lambda_2 m_2\rangle\}$ be orthogonal sets of basis vectors in H^{λ_1} and H^{λ_2}. Let $\{T^{\lambda}_m\}$ be an irreducible tensor operator. Then,*

$$\langle\lambda_2 m_2|T^{\lambda}_m|\lambda_1 m_1\rangle = \langle\lambda\lambda_1\,\lambda_2 m_2|\lambda m\lambda_1\,m_1\rangle\,T(\lambda,\,\lambda_1,\,\lambda_2), \tag{17}$$

where $\langle\lambda\lambda_1\,\lambda_2 m_2|\lambda m\lambda_1\,m_1\rangle$ is the Clebsch–Gordan coefficient. $T(\lambda,\,\lambda_1,\,\lambda_2)$ is the so-called reduced matrix element of the tensor operator $\{T^{\lambda}_m\}$ given by

$$T(\lambda,\lambda_1,\lambda_2) = \frac{1}{d_{\lambda_2}}\sum_{n_1,n_2}\langle\lambda n\,\lambda_1 n_1|\lambda\lambda_1\lambda_2 n_2\rangle\langle\lambda_2 n_2|T^{\lambda}_n|\lambda_1 n_1\rangle \tag{18}$$

and d_{λ_2} is the dimension of T^{λ_2}.

PROOF: By virtue of eq. (7) we have:

$$\langle\lambda_2 m_2|T^{\lambda}_m|\lambda_1 m_1\rangle = \sum_n D^{\lambda}_{nm}(g)\langle\lambda_2 m_2|U_g^{-1}T^{\lambda}_n U_g|\,\lambda_1 m_1\rangle, \tag{19}$$

using the equality $U_g|\,\lambda m\rangle = D^{\lambda}_{nm}(g)|\,\lambda n\rangle$ one obtains

$$\langle\lambda_2 m_2|\,T^{\lambda}_m|\,\lambda_1 m_1\rangle = \sum_{n,n_1,n_2} \bar{D}^{\lambda_2}_{n_2 m_2}(g)\,D^{\lambda}_{nm}(g)\,D^{\lambda_1}_{n_1 m_1}(g)\langle\lambda_2 n_2|\,T^{\lambda}_n|\,\lambda_1 n_1\rangle. \tag{20}$$

Integrating now over the group space G and using the relation 7.4(10) and 7.4(11) one obtains

$$\langle \lambda_2 m_2 | T^\lambda_m | \lambda_1 m_1 \rangle$$
$$= \langle \lambda\lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle d_{\lambda_2}^{-1} \sum_{n, n_1, n_2} \langle \lambda n \lambda_1 n_1 | \lambda\lambda_1 \lambda_2 n_2 \rangle \langle \lambda_2 n_2 | T^\lambda_n | \lambda_1 n_1 \rangle. \tag{21}$$

This gives the assertion of the theorem. ▼

Remark: If G is not simply reducible there is a complication due to the fact that in the tensor product $U^{\lambda_1} \otimes U^{\lambda_2}$ the irreducible representation U^λ may occur more than once, i.e.,

$$U^{\lambda_1} \otimes U^{\lambda_2} = \sum_\lambda \oplus c_{\lambda_2} U^{\lambda_2}, \quad c_{\lambda_2} \geqslant 1.$$

In this case one should split out the factor representation $c_\lambda U_\lambda$ onto irreducible components using formalism of sec. 7.4.A and proceed as above. However contrary to the common belief even in case of $U(n)$ groups, $n > 3$, one meets considerable difficulties with a derivation of Wigner–Eckart theorem. (Cf. Holman and Biedenharn 1971.)

In many applications we are interested in the ratios of the matrix elements (17) of a tensor operator $\{T^\lambda_m\}$: in such cases for fixed invariant numbers λ, λ_1 and λ_2 eq. (17) gives

$$\frac{\langle \lambda_2 m_2 | T^\lambda_m | \lambda_1 m_1 \rangle}{\langle \lambda_2 m_2' | T^\lambda_{m'} | \lambda_1 m_1' \rangle} = \frac{\langle \lambda\lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle}{\langle \lambda\lambda_1 \lambda_2 m_2' | \lambda m' \lambda_1 m_1' \rangle}, \tag{22}$$

i.e., the problem is reduced to a calculation of the ratios of C–G coefficients only.

Notice also that for fixed invariant numbers λ, λ_1, λ_2 eq. (17) allows us to calculate an arbitrary matrix element of T^λ_m from the knowledge of any particular matrix element.

More general objects are defined when a tensor operator $\{T_a\}$ depends on coordinates x; such objects are frequently encountered in quantum field theory and are called the *tensor field operators*. The transformation properties of the tensor field operators are defined by the formula

$$U_g^{-1} T^\mu(x) U_g = D^\mu{}_\nu(g) T^\nu(g^{-1}x). \tag{23}$$

In particular, if G is the Poincaré group, $\{x\} = M$ is the Minkowski space and $T(x)$ is a scalar field operator, the formula (23) gives

$$U_{(a\Lambda)}^{-1} T(x) U_{(a\Lambda)} = T\big(\Lambda^{-1}(x-a)\big). \tag{24}$$

An example of tensor field operators is provided by the currents $\{j^\mu{}_k(x)\}$ which transform according to the direct product $G \otimes \Pi$ of an internal symmetry group G (like SU(2) or SU(3)) and the Poincaré group Π. The transformation properties of $\{j^\mu{}_k(x)\}$ are

$$U_g^{-1} j^\mu{}_k(x) U_g = D_{k'k}(g^{-1}) j^\mu{}_{k'}(x), \quad g \in G, \tag{25}$$

and

$$U_{(a\Lambda)}^{-1} j^\mu{}_k(x) U_{(a\Lambda)} = \big((\Lambda)^\mu{}_\nu j^\nu{}_k\big(\Lambda^{-1}(x-a)\big)\big), \quad (a, \Lambda) \in \Pi \tag{26}$$

§ 2. The Enveloping Algebra

Let L be a Lie algebra over $K = R$ or C. Let τ be the (free) tensor algebra over L considered as a vector space, i.e.,

$$\tau = \bigoplus_{r=0}^{\infty} \tau^r = K \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \dots \tag{1}$$

The vector space τ is an associative algebra with the abstract multiplication law given by the tensor product $\otimes$. Let J be the two-sided ideal in τ generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y], \quad \text{where } X, Y \in L. \tag{2}$$

Then, the quotient algebra $E = \tau/J$ is called the *universal enveloping algebra* of the Lie algebra L. Clearly, the enveloping algebra is associative.

Let $\pi\colon Z \to Z + J \equiv \tilde{Z}$, $\tilde{Z} \in E$, denote the canonical map (i.e. natural homomorphism) of τ onto E. Clearly, $\pi(Z_1 \otimes Z_2) = \pi(Z_1)\pi(Z_2) = \tilde{Z}_1 \tilde{Z}_2$.* The vector subspace of E spanned by all elements $X_{i_1} X_{i_2} \dots X_{i_r}$, $X_{i_k} \in L$, of order r will be denoted by E^r. The element

$$([\pi(X), \pi(Y)] - \pi[X, Y]), \quad X, Y \in L \tag{3}$$

is the image of an element (2) under canonical map and is, therefore, zero. Clearly the canonical map π of the algebra τ onto E induces a linear map of L into E.

Let $X \to T(X)$ be a representation of a Lie algebra in a vector space H. The formula

$$T(X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_r}) = T(X_{i_1}) T(X_{i_2}) \dots T(X_{i_r}) \tag{4}$$

defines uniquely a representation $\tilde{T}$ of the associative algebra E in H.** Clearly, $\tilde{T}(X) = T(X)$ for X in L. Moreover,

$$\tilde{T}(X \otimes Y - Y \otimes X - [X, Y]) = T(X)T(Y) - T(Y)T(X) - T([X, Y]) = 0.$$

Thus, every representation T of L can be extended to the representation $\tilde{T}$ of the universal enveloping algebra E of L.

Bases in the Enveloping Algebra

We shall construct two convenient bases in E. Notice first that if $X_1, \dots, X_n$ is a basis in L then monomials

$$\tilde{X}_{i_1} \tilde{X}_{i_2} \dots \tilde{X}_{i_r}, \quad \tilde{X}_{i_k} = \pi(X_{i_k}), \tag{5}$$

---

* For simplicity we shall omit the symbol of multiplication in E.

** The construction of the dense, invariant domain for operators (4) is given in 11.2.

span the space E^r. Using the relation $[\tilde{X}_1, \tilde{X}_2] = c_{ik}{}^l \tilde{X}_l$ for basis elements of L in E we can reduce (5) to the standard monomial form

$$e_{j_1 j_2 \ldots j_r} = \tilde{X}_{j_1} \tilde{X}_{j_2} \ldots \tilde{X}_{j_r}, \quad \text{where } j_1 \leqslant j_2 \leqslant \ldots \leqslant j_r, \tag{6}$$

at the expense of introducing elements (5) of E^{r-1}. Then, the elements (5) of E^{r-1} in turn can be reduced to the standard form (6). Thus, instead of n^r elements, which span E^r, we obtain

$$\frac{(n+r-1)!}{(n-1)!\,r!} \tag{7}$$

elements which span E^r. For example, the space R^3 associated with the Lie algebra su(2) contains 10 basis elements of the form (6) instead of 27 of the form (5). Taking the collection of vectors (6) in $E^0, E^1, E^2, \ldots$ we obtain a set $\{e_{i_1 i_2 \ldots i_r};\ i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r,\ r = 0, 1, 2, \ldots\}$ which span E. We leave as an exercise for the reader the demonstration that elements (6) are linearly independent. The basis (6) in E is called the *Poincaré–Birkhoff–Witt basis.* The elements (6) can also be written in the form $e_{i_1 i_2 \ldots i_r} = \tilde{X}_1^{k_1} \tilde{X}_2^{k_2} \ldots \tilde{X}_n^{k_n}$ where $k_1 + k_2 + \ldots + k_n = r$.

In the applications it is convenient to use the following symmetric basis in E.

PROPOSITION 1. *Let L be a Lie algebra with a basis $X_1, \ldots, X_r$. The elements*

$$e_{\{i_1 i_2 \ldots i_r\}} \equiv \frac{1}{r!} \sum_\sigma \tilde{X}_{i_{\sigma(1)}} \ldots \tilde{X}_{i_{\sigma(r)}}, \quad r = 0, 1, \ldots, \tag{8}$$

where $i_k = 1, 2, \ldots, \dim L$, and σ runs over all permutations of the set $(1, 2, \ldots, r)$, form a basis in the universal enveloping algebra E of L.

PROOF: An element

$$\tilde{X}_{i_1} \ldots \tilde{X}_{i_r} - \tilde{X}_{i_{\sigma(1)}} \ldots \tilde{X}_{i_{\sigma(r)}} \tag{9}$$

of E^r may be expressed in terms of the elements of E^{r-1} if use is made of the commutation relations $\tilde{X}_i \tilde{X}_k = \tilde{X}_k \tilde{X}_i + c_{ik}{}^l \tilde{X}_l$. If we sum eq. (9) over all permutations σ, we obtain

$$\tilde{X}_{i_1} \ldots \tilde{X}_{i_r} = e_{\{i_1 \ldots i_r\}} + \text{terms in } E^{r-1}.$$

Repeating now this procedure for terms in E^{r-1}, E^{r-2}, etc., we find

$$\tilde{X}_{i_1} \ldots \tilde{X}_{i_r} = \sum_{\substack{i_1, i_2, \ldots, i_k = 1 \\ 0 \leqslant k \leqslant r}}^{\dim L} c^{i_1 \ldots i_k} e_{\{i_1 \ldots i_k\}}. \tag{10}$$

The map σ: $e_{i_1 i_2 \ldots i_r} \leftrightarrow e_{\{i_1 i_2 \ldots i_r\}}$ establishes a one to one correspondence of basis elements (6) and symmetric elements (8). Hence elements (8) form also a basis in E. ▼

The *center* Z of the universal enveloping algebra E is the set of all elements C in E which satisfy

$$[C, \tilde{X}] = 0 \quad \text{for all } \tilde{X} \text{ in } L. \tag{11}$$

One of the main problems in the representation theory is to find the center Z and to determine the spectrum of the elements of Z in irreducible carrier spaces. For semisimple Lie algebras this problem is solved explicitly in secs. 4 and 6A.

§ 3. The Invariant Operators

We describe in this section the general properties of invariant operators of arbitrary Lie algebras.

According to eq. 2(6) any element of the enveloping algebra E of a given Lie algebra L can be expressed as the sum of elements of the form*

$$g^{i_1 \dots i_s} X_{i_1} \dots X_{i_s}, \qquad s = 0, 1, \dots \tag{1}$$

The elements $X_{i_1} \dots X_{i_s}$ are tensor operators relative to the adjoint representation (or adjoint group). Therefore, the problem of construction of invariants of E reduces, by th. 1.1.2, to the problem of finding appropriate invariant tensors $g^{i_1 \dots i_s}$. Indeed, we have

THEOREM 1 (Gel'fand). *In order that an element P*

$$P = cI + \sum_i g^i X_i + \sum_{i,k} g^{ik} X_i X_k + \sum_{i,k,j} g^{ikj} X_i X_k X_j + \dots \tag{2}$$

of the enveloping algebra E belong to the centre Z of E it is sufficient that the coefficients

$$g^i, g^{ik}, g^{ikj}, \dots \tag{3}$$

are invariant tensors for the adjoint group G_A. If, in addition, P is written in the form in which the coefficients g^{ik}, g^{ikj}, ... are symmetric, then this condition is also necessary.

PROOF: Denote by P_r an element $g^{i_1 \dots i_r} X_{i_1} \dots X_{i_r}$. Then by virtue of the invariance of the tensor $g^{i_1 \dots i_r}$ we obtain

$$\operatorname{Ad} g P_r = g P_r g^{-1} = g^{i_1 \dots i_r} \prod_{k=1}^{r} \operatorname{Ad} g X_{i_k} = g^{i_1 \dots i_r} \prod_{k=1}^{r} (\operatorname{Ad} g)^{j_l}{}_{i_k} X_{j_l} = P_r .$$

Taking $g = \exp(tX_l)$ and passing to the infinitesimal form with the above equality we obtain

$$[P_r, X_l] = 0 .$$

This implies the first part of th. 1. Now let tensors (3) be symmetric; then by virtue of eq. (2.8) every element P_r in (2) can be written in the form

$$P_r = \sum_{i_1, \dots, i_l} g^{i_1 \dots i_r} e_{\{i_1 \dots i_r\}} .$$

* From now on, we denote for simplicity the product $\tilde{X}_{i_1} \dots \tilde{X}_{i_s}$ in E by $X_{i_1} \dots X_{i_s}$.

If $P \in Z$ then $P_r \in Z$ and $[P_r, X_l] = 0$. This, for arbitrary $g = \exp(tX_l)$, implies

$$gP_rg^{-1} = \operatorname{Ad} gP_r = P_r.$$

Hence

$$g^{i_1 \dots i_r} \prod_{k=1}^{r} (\operatorname{Ad} g)^{j_k}{}_{i_k} e_{\{j_1 \dots j_r\}} = g^{i_1 \dots i_r} e_{\{i \dots i_r\}}.$$

Consequently every tensor (3) must be invariant of G_A. ▼

The problem of the explicit construction of invariant operators for semisimple Lie algebras was first considered by Casimir 1931. Using the Cartan metric tensor g_{ik} in L he constructed the second order operator

$$C_2 = g^{ik} X_i X_k \tag{4}$$

which, due to 1.2(18) satisfies

$$[C_2, X_l] = g^{ik}[X_1, X_l]X_k + g^{ik}X_i[X_k, X_l] = (c_{kls} + c_{slk})X^s X^k = 0.$$

In order to understand the invariance property of the operator (4) from the point of view of th. 1, note that according to eq. 1.2(8) the tensor g_{ik} can be written in the form

$$g_{ik} = \operatorname{Tr} \hat{X}_i \hat{X}_k, \tag{5}$$

where $\hat{X}_i = -C_i \equiv \{-c_{il}{}^s\}$ is the adjoint representation of the Lie algebra L determined by the structure constants. The elements of the adjoint group transform each $\hat{X}_i$ into $g\hat{X}_i g^{-1}$ and therefore leave the tensor (5) invariant. This implies that the operator (4) is an invariant for L by virtue of th. 1.

It becomes straightforward now to construct higher order invariant operators; in fact, the tensor

$$g_{i_1 i_2 \dots i_p} = \operatorname{Tr} \hat{X}_{i_1} \hat{X}_{i_2} \dots \hat{X}_{i_p} = c_{i_1 l_1}{}^{l_2} c_{i_2 l_2}{}^{l_3} \dots c_{i_{p-1} l_{p-1}}{}^{l_p} c_{i_p l_p}{}^{l_1} \tag{6}$$

is an invariant of the adjoint group by the same arguments. Hence, the operators

$$C_p = g_{i_1 i_2 \dots i_p} X^{i_1} X^{i_2} \dots X^{i_p}, \qquad p = 2, 3, \dots \tag{7}$$

are invariants of the enveloping algebra E by virtue of th. 1.

For the construction of invariant tensors (6) we may take finite-dimensional representations $X \to V(X)$ of a given Lie algebra. Indeed. if

$$g_{i_1 i_2 \dots i_p} = \operatorname{Tr}\big(V(X_{i_1}) V(X_{i_2}) \dots V(X_{i_p})\big), \tag{8}$$

then, the adjoint transformation $X = gXg^{-1}$ implies that

$$\begin{aligned} g'_{i_1 i_2 \dots i_p} &= \operatorname{Tr}\big(V(gX_{i_1}g^{-1}) \dots V(gX_{i_p}g^{-1})\big) \\ &= \operatorname{Tr}\big(V_g V(X_{i_1}) V_g^{-1} \dots V_g V(X_{i_p}) V_g^{-1}\big) \\ &= \operatorname{Tr}\big(V(X_{i_1}) \dots V(X_{i_p})\big) = g_{i_1 i_2 \dots i_p}. \end{aligned} \tag{9}$$

i.e., the tensor (8) is invariant. We shall use this fact for the construction of independent invariant operators for arbitrary semisimple Lie groups. In the applications it is crucial to know the minimal number of invariant operators which generate the center Z of the enveloping algebra E. The following theorem gives the solution of this problem for semisimple Lie algebras.

THEOREM 2. *For every semisimple Lie algebra L of rank n there exists a set of n invariant polynomials of generators X_i, whose eigenvalues characterize the finite-dimensional irreducible representations.*

(For the proof cf., e.g., Chevalley 1955.)

The problem of finding the explicit form of the spectra of the invariant operators is of fundamental importance, in particular, in applications. This problem can be solved easily for the second order invariant operators of semisimple Lie algebras. In the standard Cartan–Weyl basis of L the operator C_2, eq. (4), has the form:

$$C_2 = \sum_{i=1}^{l} g^{ik} H_i H_k + \sum_{\alpha} E_\alpha E_{-\alpha}. \tag{10}$$

When this operator acts on the highest weight vector u_m of an irreducible representation, one obtains, because of the condition $E_\alpha u_m = 0$ for positive roots (eq. 8.8(5)),

$$C_2 u_m = \left\{ g^{ik} m_i m_k + \sum_{\alpha>0} [E_\alpha, E_{-\alpha}] \right\} u_m = \left[m^2 + \sum_{\alpha>0} (\alpha, m) \right] u_m. \tag{11}$$

We know that every irreducible representation is characterized by the components of the highest weight vector $m = (m_1, m_2, \ldots, m_n)$. By virtue of Schur's lemma every invariant operator in the carrier space of an irreducible representation is proportional to the identity, i.e., $C_i = \lambda_i I$; the number λ_i is a function of the components of the highest weight $m = (m_1, m_2, \ldots, m_n)$ and represents a spectrum of the Casimir operator C_i, i.e.,

$$C_2(m) = m^2 + 2rm, \tag{12}$$

where

$$r = \frac{1}{2} \sum_{\alpha>0} \alpha \tag{13}$$

and summation runs over the positive roots only. With $k = m + r$ we have then

$$C_2(m) = k^2 - r^2. \tag{14}$$

In this form the eigenvalue of the Casimir operator C_2 is invariant relative to the action of the Weyl group $k \to Sk$. This fact holds for arbitrary invariant operators. Indeed, we have

THEOREM 3 (S-theorem). *Let C be any invariant operator and H^m the carrier space of an irreducible representation of L^m determined by the highest weight m. Then,* *the eigenvalue $C(m)$ of C expressed in terms of $k = m+r$ is invariant under the transformation of the Weyl group, i.e.,*

$$C'(Sk) = C'(k) \quad \textit{for all } S \in W, \tag{15}$$

where

$$C'(k) = C(k-r). \tag{16}$$

PROOF: The character $\chi^m(g) = D^m_{pp}(g)$ is an eigenfunction for an arbitrary invariant operator. Because the trace of the matrix D^m is invariant under the similarity transformation $D^m(g) \to D^m(g')D^m(g)D^m(g'^{-1})$, the character $\chi^m(g)$ of a semisimple Lie group is a function of the classes of conjugate elements and takes the Weyl form 8.8(25). It is evident from the Weyl formula that a character $\chi^m(\delta)$ is left-invariant under the transformation S: $k \to Sk$ except for a possible change of sign. Consequently the eigenvalue

$$C'\chi^m(\delta) = C'(k)\chi^m(\delta)$$

is invariant under the transformation of the Weyl group. ▼

This S-theorem is useful in the determination of the explicit form of the spectra of Casimir operators for semisimple Lie groups, and will be used in the following sections.

§ 4. Casimir Operators for Classical Lie Group

A. *Casimir Operators and Their Spectra for $U(n)$*

Let us consider first the group $U(n)$. The $n \times n$ matrices $u \in U(n)$ obey the condition $u^*u = 1$. Therefore the n^2 generators $M_i{}^k$, $i, k = 1, 2, \ldots, n$ of one-parameter subgroups satisfy

$$(M_i{}^k)^* = M_i{}^k. \tag{1}$$

However, because the commutation relations of the generators $M_i{}^k$ are not in a symmetric form we usually pass to the Lie algebra $\mathrm{gl}(n, R)$, whose commutation relations are simply

$$[A_j{}^i, A_l{}^k] = \delta_l{}^i A_j{}^k - \delta_j{}^k A_l{}^i. \tag{2}$$

If the generators $A_i{}^k$ satisfy the condition $(A_i{}^k)^* = A_k{}^i$, then the n^2 independent hermitian generators obeying (1) are given by

$$\begin{aligned} M_k{}^k &= A_k{}^k, \quad k = 1, 2, \ldots, n, \\ M_k{}^l &= A_k{}^l + A_l{}^k, \quad k < l \leqslant n \\ M_l{}^k &= \mathrm{i}(A_k{}^l - A_l{}^k), \quad k < l \leqslant n. \end{aligned} \tag{3}$$

If an element F of the enveloping algebra E satisfies

$$[F, A_i{}^k] = 0 \quad \text{for all } i,\ k,$$

then, due to (3), it also satisfies

$$[F, M_i{}^k] = 0, \quad i, k = 1, 2, \ldots, n.$$

Therefore, the problem of invariant operators of $u(n)$ is reduced to that of $\mathrm{gl}(n, R)$. The latter problem can be solved easily using th. 3.1. Indeed, using 3(8) and the adjoint representation of $\mathrm{gl}(n, R)$

$$V(A_i^j)_l^s = \delta_{il}\delta^{js}, \tag{4}$$

we obtain

$$\begin{aligned} g_{i_1}{}^{j_1}{}_{i_2}{}^{j_2}{}_{\dots i_p}{}^{j_p} &= \mathrm{Tr}\left(V(A_{i_1}{}^{j_1}) \dots V(A_{i_p}{}^{j_p})\right) \\ &= \delta_{i_1 l_1}\delta^{j_1 s_1}\delta_{i_2 s_1}\delta^{j_2 s_2} \dots \delta_{i_p s_{p-1}}\delta^{j_p l_1} \\ &= \delta_{i_1}{}^{j_p}\delta_{i_2}{}^{j_1}\delta_{i_3}{}^{j_2} \dots \delta_{i_p}{}^{j_{p-1}}. \end{aligned} \tag{5}$$

Therefore, by eq. 3(7) the invariant operators have the form

$$\begin{aligned} C_p &= g_{i_1}{}^{j_1}{}_{i_2}{}^{j_2}{}_{\dots i_p}{}^{j_p} A_{j_1}{}^{i_1} A_{j_2}{}^{i_2} \dots A_{j_p}{}^{i_p} \\ &= A_{i_2}{}^{i_1} A_{i_3}{}^{i_2} \dots A_{i_p}{}^{i_{p-1}} A_{i_1}{}^{i_p}, \qquad p = 1, 2, \dots \end{aligned} \tag{6}$$

The next two theorems give the explicit form of the spectra of the invariant operators (6) in carrier spaces of irreducible representations. Note that the use of tensor operators considerably simplifies the proof of th. 1.

THEOREM 1. *Let H^m be the carrier space of an irreducible representation of the group $U(n)$ determined by the highest weight $m = (m_1, \dots, m_n)$. Then, the spectra of the invariant operators* (6) *in H^m have the form*

$$C_p(m_1, \dots, m_n) = \mathrm{Tr}(a^p E), \tag{7}$$

where the matrix $a = \{a_{ij}\}$, $i, j = 1, 2, \dots, n$, is

$$\begin{aligned} a_{ij} &= (m_i + n - i)\delta_{ij} - Q_{ij}, \\ Q_{ij} &= \begin{cases} 1 & \text{for } i < j, \\ 0 & \text{for } i \geqslant j; \end{cases} \end{aligned} \tag{8}$$

a^p is the p-th power of the matrix a and E is the matrix with all elements $E_{ij} = 1$.

PROOF: In order to calculate the spectra of the invariant operators C_p, $p = 1, 2, \dots, n$, we use an idea which Racah has used in the calculation of the spectrum of the Casimir operator C_2 of an arbitrary simple Lie group (cf. eqs. 3 (10)–3 (12)). Let us recall first the connection between the generators $A_i{}^k$ and the Cartan–Weyl generators H_i, E_α (cf. 1.4 (10) and 1.4 (11))

$$\begin{aligned} H_i &= A_i{}^i, \quad i = 1, 2, \dots, n, \\ E_{(e_i - e_k)} &= A_i{}^k, \quad i \neq k, \end{aligned} \tag{9}$$

where $e_i = (0, 0, \dots, \underset{(i)}{1}, \dots, 0, 0)$, $i = 1, 2, \dots, n$, are the orthonormal vectors of R^n. Therefore, the generators $A_i{}^k$, $i > k$, are associated with positive roots of the algebra $u(n)$ and they play the role of the raising operators.

Consider now an irreducible representation of $u(n)$ which is determined by the highest weight $m = (m_1, \dots, m_n)$ and denote by ψ_m the highest weight vector.

Because for $i > j$ the operators A_i^j are the raising operators, we obtain

$$A_i^j \psi_m = 0, \quad i > j. \tag{10}$$

Let us rewrite now eq. (6) in the form

$$C_p = (T_{p-1})_j^i A_i^j, \quad \text{where } (T_q)_j^i \equiv A_{i_1}{}^i A_{i_2}{}^{i_1} \dots A_j{}^{i_{q-1}}. \tag{11}$$

The operator $(T_{p-1})_j{}^i$ has the same transformation property with respect to $U(n)$ as $A_j{}^i$, so it represents a tensor operator. Consequently, by virtue of eqs. 1(9) and (2)

$$[A_j{}^i, (T_{p-1})_l{}^k] = \delta_j{}^k (T_{p-1})_l{}^i - \delta_l{}^i (T_{p-1})_j{}^k \tag{12}$$

and

$$(T_{p-1})_j{}^i \psi_m = 0 \quad \text{for } i < j. \tag{13}$$

From (10), (11) and (13) it follows that

$$\begin{aligned} C_p \psi_m &= \sum_{i=1}^{n} (T_{p-1})_i{}^i A_i{}^i \psi_m + \sum_{\substack{i,j \\ i>j}} [(T_{p-1})_i{}^j, A_j{}^i] \psi_m \\ &= \left\{ \sum_{i=1}^{n} (T_{p-1})_i{}^i A_i{}^i + \sum_{\substack{i,j \\ i>j}} (T_{p-1})_j{}^j - (T_{p-1})_i{}^i \right\} \psi_m \end{aligned} \tag{14}$$

(no summation convention in eq. (14)).

Using $A_i{}^i \psi_m = m_i \psi_m$ we find that

$$C_p \psi_m = \sum_{i=1}^{n} (m_i + n + 1 - 2i)(T_{p-1})_i{}^i \psi_m. \tag{15}$$

The quantity $(T_{p-1})_i{}^i \psi_m$ can be calculated recursively. Namely, using (11), (13) and (12) we get

$$(T_q)_i{}^i \psi_m = \sum_{j=1}^{n} (T_{q-1})_j{}^i A_i{}^j \psi_m = \sum_{j=1}^{n} a_{ij} (T_{q-1})_j{}^j \psi_m, \tag{16}$$

where the matrix a_{ij} is

$$a_{ij} = (m_i + n - i)\,\delta_{ij} - Q_{ij}, \tag{17}$$

$$Q_{ij} = \begin{cases} 1 & \text{for } i < j, \\ 0 & \text{for } i \geqslant j. \end{cases}$$

Successively lowering the degree of T_q by using (16) and the identity

$$\sum_{i=1}^{n} a_{ij} = m_j + n + 1 - 2j, \quad \sum_{j=1}^{n} a_{ij} = m_i \tag{18}$$

we get the result:

$$C_p(m_1, m_2, \dots, m_n) = \sum_{i,j=1}^{n} (a^p)_{ij}. \tag{19}$$

Introducing the matrix E with the matrix elements $E_{ij} = 1$ $i, j = 1, 2, \ldots, n$, we may write the formula (19) in the form

$$C_p(m_1, \ldots, m_n) = \operatorname{Tr}(a^p E). \blacktriangledown \tag{20}$$

The next theorem gives a convenient generating function for the spectrum of successive Casimir operators C_p, $p = 1, 2, \ldots$

THEOREM 2. *The function*

$$G(z) = z^{-1}(1 - \Pi(z)), \quad z \in C^1, \tag{21}$$

where

$$\Pi(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{1 - \lambda_i z}\right) \tag{22}$$

and

$$\lambda_i = m_i + n - 1, \tag{23}$$

is a generating function for the spectrum of the Casimir operators, i.e.,

$$G(z) = \sum_{p=0}^{\infty} C_p(m_1, \ldots, m_n) z^p. \tag{24}$$

PROOF: By elementary methods the triangular matrix $a = \{a_{ij}\}$ can be reduced to the diagonal form. Hence we can express $C_p(m_1, \ldots, m_n)$ by the eigenvalues λ_i of the matrix a in the form

$$C_p(m_1, \ldots, m_n) = \sum_{i=1}^{n} \lambda_i^p \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{\lambda_i - \lambda_j - 1}{\lambda_i - \lambda_j}, \tag{25}$$

$$\lambda_i = m_i + n - i.$$

We can further simplify eq. (25) using the following integral representation

$$C_p(m_1, \ldots, m_n) = \frac{1}{2\pi i} \oint \lambda^p \prod_{i=1}^{n} \left(1 - \frac{1}{\lambda - \lambda_i}\right) d\lambda, \tag{26}$$

where the contour of integration encloses (in the positive direction) all poles $\lambda = \lambda_i$. With $\lambda = z^{-1}$ we have

$$C_p(m_1, \ldots, m_n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{p+2}} \prod_{i=1}^{n} \left(1 - \frac{z}{1 - \lambda_i z}\right). \tag{27}$$

From (27) it follows that the function

$$\Pi(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{1 - \lambda_i z}\right) = 1 - C_0 z - C_1 z^2 - \ldots, \quad C_0 = n \tag{28}$$

has, at $|z| < 1/\lambda_i$, the eigenvalues of successive Casimir operators as its expansion coefficients. Therefore the function

$$G(z) = z^{-1}[1-\Pi(z)] \tag{29}$$

is the generating function for the Casimir operators, i.e.,

$$G(z) = \sum_{p=0}^{\infty} C_p z^p. \blacktriangledown \tag{30}$$

The generating function (21) is very convenient for the calculation of the eigenvalues of an arbitrary Casimir operator C_p. In order to illustrate its power, let us consider the following example.

EXAMPLE 1. Let us calculate the eigenvalues of the Casimir operators for the totally symmetric $(f, 0, 0, \ldots\ 0)$, or the totally antisymmetric $\{1^k\}$ representation of $u(n)$. Actually, both of these representations are special cases of the more general representation characterized by the highest weight $(\underbrace{f, f, \ldots, f}_{k\ \text{times}}, 0, \ldots, 0)$, $k \leqslant n$. For this more general case, we have

$$\lambda_i = \begin{cases} f+n-i & \text{for } 1 \leqslant i \leqslant k, \\ n-i & \text{for } i > k. \end{cases} \tag{31}$$

From (22) and (31) it follows that

$$\Pi(z) = \frac{[1-(f-n)z][1-(n-k)z]}{1-(f+n-k)z}, \tag{32}$$

hence

$$G(z) = n + \frac{kfz}{1-(f+n-k)z}. \tag{33}$$

Using (30) we find

$$C_p(\underbrace{f, \ldots, f}_{k\ \text{times}}, 0, \ldots, 0) = kf(f+n-k)^{p-1}. \tag{34}$$

If we put in this expression $k = 1$ we get the spectra of the operators C_p for the totally symmetric representation $(f, 0, \ldots, 0)$ and if we put $f = 1$, k arbitrary, $k \leqslant n$, we get the spectra of C_p for the totally antisymmetric representations $\{1^k\}$, respectively. ▼

The formula (6) gives an infinite number of invariant operators. It is not evident however that it provides all the generators of the center Z of the enveloping algebra E of $u(n)$.

The algebra $u(n)$ has n independent Casimir operators, by virtue of th. 3.2. One would expect that the first n Casimir operators $C_1, C_2, \ldots, C_n$ given by eq. (6) generate the center Z. Indeed, by an elementary computation one can show that:

$$\frac{\partial(C_1, C_2, \ldots, C_n)}{\partial(m_1, m_2, \ldots, m_n)} = n! \prod_{i<j} (\lambda_i - \lambda_j). \tag{35}$$

Therefore, for $i < j$ we have $(\lambda_i - \lambda_j) > 0$; the Jacobian (35) is positive. Hence, the invariant operators $C_1, C_2, \ldots, C_p$ are independent and their eigenvalues determine uniquely the irreducible representations of $U(n)$.

B. *Casimir Operators of* SU(n)

The Lie algebra su(n) is generated by the operators A_i^j, $i \neq j$, and by $\tilde{A}_i^i$ of the form

$$\tilde{A}_i^i = A_i^i - \frac{1}{n}\sum_{l=1}^{n} A_l^l. \tag{36}$$

The action of $\tilde{A}_i^i$ on the highest vector ψ_m is given by eq. (10)

$$\tilde{A}_i^i \psi_m = \left(m_i - \frac{1}{n}\sum_{l=1}^{n} m_l\right)\psi_m = \tilde{m}_i \psi_m.$$

Thus, applying the same argument used in the derivation of (19) we obtain

$$C_p^{(su)}(\tilde{m}_1, \ldots, \tilde{m}_n) = \sum_{i,j}^{n} (\tilde{a}^p)_{ij}, \tag{37}$$

where

$$\tilde{a} = a - \frac{m}{n}\cdot I, \qquad m = \sum_{i=1}^{n} m_i$$

and I denotes the unit matrix.

Therefore we obtain the corresponding expressions for the spectrum of the invariant operators $C_p^{(su)}$ of SU(n) by simply replacing in all formulas for $C_p^{(u)}$ the numbers m_i by

$$\tilde{m}_i = m_i - \frac{1}{n}\sum_{s=1}^{n} m_s.$$

In particular, for the representations which are determined by the highest weight $(\underbrace{f, f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0)$, we get

$$C_p^{(su)}(\underbrace{f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0) = \frac{kf(n+f)(n-k)}{n(n+f-k)} \cdot \left\{\left[\frac{(f+n)(n-k)}{n}\right]^{p-1} - \left[-\frac{kf}{n}\right]^{p-1}\right\}. \tag{38}$$

C. *Casimir Operators and Their Spectra for $O(n)$ and* Sp(n)

1. The orthogonal group $O(n)$ consists of all linear transformations of the n-dimensional Euclidean space E^n which conserve the quadratic form

$$(\xi^1)^2 + (\xi^2)^2 + \ldots + (\xi^n)^2 = 1. \tag{39}$$

The symplectic group $\mathrm{Sp}(n, C)$ is formed by all the transformations of $2n$-dimensional complex space C^{2n}, that preserve the bilinear form

$$[x, y] = \sum_{i,j=-n}^{n} h_{ij} x^i y^j = \sum_{i=1}^{n} (x^i y^{-i} - x^{-i} y^i), \tag{40}$$

where the metric tensor h_{ij} is

$$h_{ij} = \varepsilon_i \delta_{i,-j}, \qquad \varepsilon_i = \begin{cases} 0 & \text{for } i = 0, \\ 1 & \text{for } i > 0, \\ -1 & \text{for } i < 0. \end{cases} \tag{41}$$

In what follows it is convenient to consider both groups together. Therefore we go over in the case of orthogonal group from the Cartesian coordinates ξ^i, $i = 1, 2, \ldots, n$ to the 'spherical coordinates' x^i, $i = \pm 1, \pm 2, \ldots, \pm \left[\frac{n}{2}\right]$ (and x^0 if n is odd)

$$x_1 = \frac{\xi^1 + \mathrm{i}\xi^2}{\sqrt{2}}, \quad x^{-1} = \frac{\xi^1 - \mathrm{i}\xi^2}{\sqrt{2}}, \ldots, \quad \text{and } x^0 = \xi^n \text{ if } n \text{ is odd.}$$

The quadratic form (39) takes now the form analogous to (40)

$$(x, y) = \sum_{i,j=-n}^{n} g_{ij} x^i y^j, \qquad g_{ij} = \delta_{i,-j}.$$

The generators X_j^i of one parameter subgroups obey the following commutation relations:*

$$\begin{aligned} [X_j^i, X_l^k] = {} & \delta_j^k X_l^i - \delta_l^i X_j^k + \\ & + \begin{cases} \delta_j^{-l} X_{-i}^{k} - \delta_{-i}^{k} X_l^{-j} & \text{for } O(n), \\ \varepsilon_i \varepsilon_j \delta_j^{-l} X_{-i}^{k} - \varepsilon_j \varepsilon_k \delta_{-i}^{k} X_l^{-j} & \text{for } \mathrm{Sp}(2n). \end{cases} \end{aligned} \tag{42}$$

The equality

$$X_{ij} = -X_{ji}, \quad i, j = 1, 2, \ldots, n, \tag{43}$$

for generators of $O(n)$ in cartesian coordinates corresponds now the following equality

$$X_j^i = -X_{-i}^{-j}, \quad i, j = -n, \ldots, +n. \tag{44}$$

For generators of $\mathrm{Sp}(2n)$ we have correspondingly

$$X_j^i = -\varepsilon_i \varepsilon_j X_{-i}^{-j}, \quad i, j = -n, \ldots, +n. \tag{45}$$

It follows from the commutation relations (42) that the operators X_i^i commute with one another and correspond to the generators H_i of Cartan basis, while the

* We conserve here notation of Perelomov and Popov 1966 for generators X_j^i. However in other works these generators are denoted usually by the symbol $X^i{}_j$.

the generators X_j^i with $i > j$ correspond to the generators E_α associated with the positive roots of the algebra. The commutation relations for a tensor operator T_l^k, which has the same transformation properties as X_j^i are given by the formula:

$$[X_j^i, T_l^k] = \delta_j^k T_l^i - \delta_l^i T_j^k + \\ + \begin{cases} \delta_j^{-l} T_{-i}^{\ k} - \delta_{-i}^{\ k} T_l^{-j} & \text{for } O(n), \\ \varepsilon_i \varepsilon_j \delta_j^{-l} T_{-i}^{\ k} + \varepsilon_j \varepsilon_k \delta_{-i}^{\ k} T_l^{-j} & \text{for } \mathrm{Sp}(2n). \end{cases} \tag{46}$$

In particular

$$[X_j^i, T_i^j] = (1 \mp \delta_{-j}^{\ i})(T_i^i - T_j^j), \tag{47}$$

where the sign $-(+)$ refers to the orthogonal (symplectic) group.

Utilizing Gel'fand theorem 3.1 we easily verify that the operators

$$C_p = \sum_{i_1, \ldots, i_p} X_{i_2}^{\ i_1} X_{i_3}^{\ i_2} \ldots X_{i_1}^{\ i_p}, \tag{48}$$

are the invariant operators for $O(n)$ or $\mathrm{Sp}(2n)$ since the corresponding polynomials 3(6) in a dual space are invariants of the adjoint group. We shall calculate the spectra of invariant operators by the same method as that of $U(n)$. We shall take advantage of the fact that in an irreducible representation space H^m, which is determined by the highest weight $m = (m_1, \ldots, m_n)$ (n—rank of a group) the highest weight vector ψ_m has the following properties

$$X_j^i \psi_m = 0 \quad \text{for } i > j$$

and

$$X_i^i \psi_m = m_i \psi_m. \tag{49}$$

On the basis of eqs. (44) and (45) we get $m_{-i} = -m_i$. Representing now the invariant operator (48) in the form

$$C_p = \sum_{i,j} (T^{(p-1)})_j^{\ i} X_i^j,$$

where

$$(T^{(p-1)})_j^{\ i} = \sum_{i_1, \ldots, i_{p-2}} X_{i_1}^{\ i} X_{i_2}^{\ i_1} \ldots X_j^{\ i_{p-2}}, \tag{50}$$

and using (49) and (47), we get:

$$C_p \psi_m = \Big(\sum_i (T^{(p-1)})_i^{\ i} X_i^i + \sum_{i>j} [(T^{(p-1)})_j^{\ i}, X_i^j] \Big) \psi_m = \sum_{i=-n}^{n} (m_i + 2r_i)(T^{(p-1)})_i^{\ i} \psi_m, \tag{51}$$

where

$$r_i = \tfrac{1}{2} \sum_{j<i} (1 \mp \delta_{-j}^{\ i}) \tag{52}$$

The expression $(T^{(p-1)})_i{}^i$ can be calculated recursively

$$(T^{(q)})_i{}^i\psi_m = \Big\{(T^{(q-1)})_i{}^i X_i{}^i + \sum_{i>j} [(T^{(q-1)})_j{}^i, X_i{}^j]\Big\}\psi_m$$

$$= \Big\{m_i(T^{(q-1)})_i{}^i + \sum_{j<i}(1 \mp \delta_{-j}{}^i)[(T^{(q-1)})_i{}^i - (T^{(q-1)})_j{}^j]\Big\}\psi_m$$

$$= \sum_{j=-n}^{n} a_{ij}(T^{(q-1)})_j{}^j\psi_m$$

where

$$a_{ij} = (l_i+\alpha)\,\delta_{ij} - \theta_{ji} + \tfrac{1}{2}\beta(1+\varepsilon_i)\,\delta_{i,-j},$$
$$l_i = m_i + r_i, \qquad \theta_{ji} = \begin{cases} 1 & \text{for } j < i, \\ 0 & \text{for } j \geqslant i. \end{cases} \tag{53}$$

The constants α and β for various groups as well as the explicit expressions for r_i are given in the table.

Table I

Algebra	Group	Invariant form	α	β	r_i	Index 'i' runs over values
A_{n-1}*	SU(n)	$\sum_{i=1}^{n} \bar{x}_i\,\bar{y}_i$	$\frac{n-1}{2}$	0	$\frac{n+1}{2}-i$	$1, 2, \ldots, n$
B_n	$O(2n+1)$	$\sum_{i=-n}^{n} x^i y^{-i}$	$n-\frac{1}{2}$	1	$(n+\frac{1}{2})\varepsilon_i - i$	$1, 2, \ldots, n, 0, -n, \ldots$ $\ldots, -2, -1$
C_n	Sp($2n$)	$\sum_{i=1}^{n}(x^i y^{-i} - x^{-i}y^i)$	n	-1	$(n+1)\varepsilon_i - i$	$1, 2, \ldots, n, -n, \ldots$ $\ldots, -2, -1$
D_n	$O(2n)$	$\sum_{i=1}^{n}(x^i y^{-i} + x^{-i}y^i)$	$n-1$	1	$n\varepsilon_i - i$	$1, 2, \ldots, n, -n, \ldots$ $\ldots, -2, -1$

Using (51) and (53) we obtain finally

$$C_p(m_1, m_2, \ldots, m_n) = \mathrm{Tr}(a^p E). \tag{54}$$

Therefore the problem of finding of spectra of invariant operators (48) is reduced to the elementary problem of finding of pth power of the known matrix $[a_{ij}]$.

Using this formula (54) we easily calculate that the spectra of the lowest order invariant operators of $O(2n)$, $O(2n+1)$ and Sp($2n$) are of the form

$$C_2 = 2S_2, \quad C_3 = 2[\alpha - \tfrac{1}{2}(\beta-1)]C_2,$$
$$C_4 = 2S_4 - (2\alpha\beta + \beta - 1)S_2, \tag{55}$$

* For $U(n)$ the values of α, β and r_i are the same as for SU(n).

where

$$S_k = \sum_{i=1}^{n} (l_i^k - r_i^k). \tag{56}$$

From (55) and (56) it follows that the spectrum of C_2 is

$$C_2 = 2(m^2 + 2rm). \tag{57}$$

This is (up to factor 2) Racah's result, given in eq. 3(12).

The important problem of finding the independent invariant operators which generate the centre of enveloping algebra, can be solved with the help of Racah's S-theorem, considered in subsec. A. This theorem asserts, in the language of the variables $l_1, \ldots, l_n$, that the spectra of invariant operators are invariant under the action of Weyl's S-group. In terms of the variables $l_1, \ldots, l_n$ any element $s \in S$ for $O(2n+1)$ or $\mathrm{Sp}(2n)$ can be represented as a permutation of the numbers $l_1, \ldots, l_n$ and as an arbitrary number of inversions

$$l_i \to -l_i, \quad l_j \to l_j, \quad j \neq i.$$

Therefore the spectra of invariant operators can be expressed in terms of symmetric polynomials of even order of the variables

$$\tilde{S}_k = \sum_{i=1}^{n} l_i^k$$

or the variables

$$S_k = \sum_{i=1}^{n} (l_i^k - r_i^k) \tag{58}$$

which are more convenient for practical calculations (e.g. for the identity representation $m = (0, \ldots, 0)$ from $S_k = 0$ it follows directly $C_p = 0$).

Consequently the invariant operators C_p with odd p are not independent and can be expressed in terms of C_{2q} operators with $2q < p$. It can be shown by direct calculation that for $O(2n+1)$ and $\mathrm{Sp}(2n)$ the Jacobian

$$\frac{\partial(C_2, C_4, \ldots, C_{2n})}{\partial(m_1, m_2, \ldots, m_n)}$$

does not vanish.* Thus the set of operators $C_2, C_4, \ldots, C_{2n}$ generates the ring of invariant operators of $O(2n+1)$ and $\mathrm{Sp}(2n)$ groups. A somewhat different situation occurs for $O(2n)$ group. The spectra of invariant operators C_{2i}, $i = 1, 2, \ldots, n$, are furthermore invariant under the action of Weyl's group, which in the case of $O(2n)$ reduces to the permutations of numbers l_i, $i = 1, 2, \ldots, n$, and pair inversions

$$l_i \to -l_i, \quad l_j \to -l_j, \quad l_k \to l_k, \quad k \neq i, j. \tag{59}$$

However, as we have shown in 8.5 for this group there exist two nonequivalent

* For analogous calculation see : M. Micu, 1964, Construction of Invariants for Simple Lie Groups, *Nuclear Physics* **60**, 353–362.

fundamental spinor representations Δ_+ and Δ_-, whose highest weights are of the form

$$m_+ = (\tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}), \qquad m_- = (\tfrac{1}{2}, \dots, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{60}$$

The spectra of invariant operators C_{2i}, $i = 1, \dots, n$, are expressed in terms of sums S_k with even k and therefore they are not affected by the substitution $m_n \to -m_n$ (which induces $l_n \to l_n$ since $r_n = 0$). The same holds also for the pair of representations $\Delta_+^p \Delta_-^q$ and $\Delta_+^q \Delta_-^p$. Therefore the set of invariant operators C_{2i} cannot characterize the various nonequivalent representations. Because for the spinor representations $m_i \geqslant 0$, $i = 2, 3, \dots, n-1$, and only m_n can take both positive and negative values, in order to establish a one-to-one correspondence between the components of the highest weight m_i and spectra of invariant operators, it is sufficient to replace the operator, say, C_{2n} by a new invariant operator which is affected by the substitution $m_n \to -m_n$. Such an invariant operator can be constructed with the help of the totally antisymmetric tensor $\varepsilon_{i_1 j_1 \dots i_n j_n}$, all of whose non-zero components in the spherical coordinates, are defined by the condition $\varepsilon_{n, n-1, \dots, -n+1, -n} = -1$. The invariant operator constructed with the help of the totally antisymmetric tensor has the form:

$$C'_n = \sum \varepsilon_{i_1 j_1 \dots i_n j_n} X^{i_1 j_1} \dots X^{i_n j_n} = \sum_{i_1 j_1 \dots i_n j_n} \varepsilon^{i_1 \dots i_n}{}_{j_1 \dots j_n} X_{i_1}{}^{j_1} \dots X_{i_n}{}^{j_n}. \tag{61}$$

Acting by C'_n on the highest weight vector ψ_m and using (49) we can express the eigenvalue of C_n in terms of the eigenvalues m_i of diagonal operators $X_i{}^i$. The leading term has the form

$$\sum_{i_1, \dots, i_n} \varepsilon^{i_1 \dots i_n}{}_{i_1 \dots i_n} m_{i_1} \dots m_{i_n} = (-1)^{n(n-1)/2} 2^n n! m_1 \dots m_n.$$

Therefore passing to the variables l_i we find that C'_n is a polynomial of degree n in the variables $l_1, \dots, l_n$ with the leading term $(-1)^{n(n-1)/2} 2^n n!\, l_1 \cdot l_2 \dots l_n$. The symmetry with respect to the Weyl S-group expressed in terms of l_i of $O(2n)$ asserts that the spectrum of invariant operators should remain unchanged under any permutation of the variables $l_1, \dots, l_n$ and under any 'pair inversions' (59). These conditions are satisfied by the following symmetric polynomials of degree less than or equal to $nl_1 \cdot l_2 \dots l_n$ and $l_1^\alpha + l_2^\alpha + \dots + l_n^\alpha$ for even $\alpha < n$. In order to find the final form of the spectrum of C_n we shall utilize the fact that the operator (61) is a pseudoscalar operator in the extended $O(2n)$ group containing space reflections. We have shown that if a given irreducible representation T_g of SO$(2n)$ is characterized by the highest weight $m = (m_1, \dots, m_{n-1}, m_n)$ then the mirror-conjugate representation

$$\check{T}_g = T_{ogo^{-1}}, \qquad g \in \mathrm{SO}(2n), \qquad o \text{ — reflection},$$

has the highest weight $m = (m_1, \ldots, m_{n-1}, -m_n)$ (see lemma 8.5.1). Since for $O(2n)$ group $r_n = 0$ we get $m_n = l_n$ and

$$C'_n(l_1, \ldots, l_{n-1}, -l_n) = -C'_n(l_1, \ldots, l_{n-1}, l_n). \tag{62}$$

Due to Racah's S-theorem C'_n is a symmetric function of the numbers $l_1, \ldots, l_n$ and therefore eq. (62) is satisfied for any l_i, $i = 1, 2, \ldots, n$. Therefore the expression for the spectrum of C'_n can contain only the term proportional to $l_1, l_2, \ldots, l_n$, i.e.,

$$C'_n(m_1, \ldots, m_n) = (-1)^{n(n-1)/2} \cdot 2^n n!\, l_1 \cdot l_2 \ldots l_n.$$

It can be verified by direct calculation that

$$\frac{\partial(C_2, C_4, \ldots, C_{2(n-1)}, C'_n)}{\partial(m_1, m_2, \ldots, m_n)} \neq 0$$

(for analogous calculations see footnote on p. 363).

2. Special Cases

Consider first the case of totally symmetric representations of $O(2n)$, $O(2n+1)$ or $\mathrm{Sp}(2n)$, which are characterized by the highest weight vector $m = (f, 0, \ldots, 0)$. Utilizing formula (54) we get:

$$\begin{aligned} C_p(f, 0, \ldots, 0) = {} & (f+2\alpha)^p + (-f)^p + (2\alpha+\beta-1) + \\ & + (2\alpha-1)\left(1 + \frac{\beta+1}{2(\alpha-1)}\right)\left[\frac{(-f)^p - 1}{f+1} - \frac{(f+2\alpha)^p - 1}{f+2\alpha-1}\right] + \\ & + \frac{\alpha(\beta+1)}{2(\alpha-1)} \frac{(f+2\alpha)^p - (-f)^p}{f+\alpha}. \end{aligned} \tag{63}$$

For lowest values of p this formula simplifies

$$C_2 = 2f(f+2\alpha), \tag{64}$$

$$C_4 = 2f(f+2\alpha)[f^2 + 2\alpha f + 2\alpha^2 - \alpha\beta - \tfrac{1}{2}(\beta-1)]. \tag{65}$$

In the case of totally antisymmetric fundamental representations characterized by the highest weights $m = (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, \ldots, 0)$, the spectrum of C_p operators is of the form:

$$\begin{aligned} C_p(\{1^k\}) = {} & -(2\alpha+2-k)^p - k^p + (-1)^p(2\alpha+\beta+3) + \\ & + (2\alpha+3)\left(1 + \frac{\beta-1}{2(\alpha+2)}\right)\left[\frac{k^p - (-1)^p}{k+1} + \frac{(2\alpha+2-k)^p - (-1)^p}{2\alpha+3-k}\right] + \\ & + \frac{(\alpha+1)(\beta-1)}{2(\alpha+2)} \cdot \frac{k^p - (2\alpha+2-k)^p}{\alpha+1-k}. \end{aligned} \tag{66}$$

For $p = 2, 4$ we obtain

$$C_2 = 2k(2\alpha+2-k), \tag{67}$$

$$C_4 = 2k(2\alpha+2-k)[k^2-2(\alpha+1)k+(\alpha+1)(2\alpha+2-\beta)+\tfrac{1}{2}(\beta+1)]. \tag{68}$$

As we have mentioned already, for $O(2n+1)$ and $O(2n)$ we have besides the fundamental tensor representations $\{1^k\}$ still the fundamental spinor representations, whose highest weights are:

$$m = \begin{cases} (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) & \text{for } O(2n+1), \\ (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm\tfrac{1}{2}) & \text{for } O(2n). \end{cases}$$

The spectra of C_p for these representations are:

$$C_p = \begin{cases} n[n^{p-1}-(-\tfrac{1}{2})^{p-1}] & \text{for } O(2n+1), \\ (n-\tfrac{1}{2})[(n-\tfrac{1}{2}]^{p-1}-(-\tfrac{1}{2})^{p-1}] & \text{for } O(2n). \end{cases}$$

§ 5. The Enveloping Field

Certain physical observables are described by operators that are quotients of polynomials of the generators of a Lie algebra; for instance, the square of the relativistic spin operators has the form

$$S^2 = \frac{W_\mu W^\mu}{P_\mu P^\mu}, \quad \text{where } W_\mu = \tfrac{1}{2}\varepsilon_{\mu\alpha\beta\gamma} M^{\alpha\beta} P^\gamma, \tag{1}$$

where P^μ, $M^{\alpha\beta}$ are the generators of the Poincaré group Π. The quantities of type (1) are not in the enveloping algebra of Π. The latter consists of polynomials in the generators only. Therefore, we introduce the concept of the *enveloping field* of a Lie algebra which incorporates in a natural way, quotients of polynomials of the generators of the Lie algebra. The enveloping field has other interesting properties: it depends only weakly on the original Lie algebra and can be generated by a Heisenberg algebra $p_1, q_1, \ldots, p_n, q_n$, and a certain number of commutative operators $C_1, \ldots, C_k$. This result is of importance for the theory of dynamical groups in particle physics.

Rings and Quotient Fields

We begin with an analysis of certain properties of rings and fields. We recall that a *ring* R is an abelian group with respect to addition and a (in general noncommutative) multiplicative semigroup with or without the unit element. An element b in R is said to be a *non-zero divisor* if there is no $c \in R$, $c \neq 0$, such that $cb = 0$ or $bc = 0$.

A ring R is called a *left Noether ring* if every chain of (left) ideals of R

$$R_1 \Subset R_2 \Subset \ldots \tag{2}$$

terminates* (i.e., there exists an index n such that $R_n = R_{n+1} = \ldots$).

* $R_1 \Subset R_2$ means 'R is a proper ideal of R_2'.

A ring R is called a (*left*) *Ore ring* if for every a, $b \in R$, where b is a non-zero divisor, there exist a', $b' \in R$, where b' is a non-zero divisor, such that $b'a = a'b$.

We now introduce the important concept of quotients. Let (a, b) and (c, d) be two ordered pairs from $R \times R$ and let a and c be a non-zero divisors; we say that (a, b) is equivalent to (c, d) if there exist $x, y \in R$ such that

$$(xa, xb) = (yc, yd). \tag{3}$$

We can easily verify that all the axioms of equivalence are satisfied.

A *quotient* associated with an Ore ring R is defined as the ordered pair (a, b), a, $b \in R$, a a non-zero divisor, equipped with the above equivalence relation. We shall denote the (a, b)-quotient by the symbol $a^{-1}b$. Similarly one defines the right quotient, denoted by ab^{-1}.

Further, we introduce in the set of all quotients the operations of addition, subtraction, division and multiplication in the following natural manner

$$a^{-1}b_1 \pm a^{-1}b_2 \equiv a^{-1}(b_1 \pm b_2), \tag{4}$$

$$(a^{-1}b_1)^{-1}(a^{-1}b_2) \equiv b_1{}^{-1}b_2, \tag{5}$$

$$(a_1{}^{-1}b_1)(a_2{}^{-1}b_2) \equiv (b_1{}^{-1}a_1)^{-1}(a_2{}^{-1}b_2). \tag{6}$$

We recall that a ring R with the unit element I is called a *field*, if for any $a \in R$, $a \neq 0$, there exists an a^{-1} such that $aa^{-1} = a^{-1}a = I$. We see that the set of all quotients associated with an Ore ring without zero divisor and equipped with operations (4)–(6) forms a field which we shall call the *quotient field*.

EXAMPLE 1. Let R be the ring of all even integers $R = \{\pm 2n,\ n = 0, 1, \ldots\}$. Every element $a \in R$, $a \neq 0$ is a non-zero divisor of R. If $a = 2n$, $b = 2m$, $m \neq 0$, then there exist a' (equal, e.g., $2n$) and $b' \neq 0$ (equal, e.g., $2m$) such that $b'a = a'b$. Hence, R is an Ore ring.

Let $(a, b) \equiv (2n, 2m)$, $a \neq 0$. By virtue of the condition (3) any pair $(c, d) \equiv (2n', 2m')$ equivalent to (a, b) satisfies the condition

$$\frac{m'}{n'} = \frac{m}{n}. \tag{7}$$

Thus, the abstract quotient $a^{-1}b$ corresponds to the class of all pairs $(2n', 2m')$ of the form (xa, xb), $x \in R$ which have the constant ratios $\dfrac{m'}{n'} = \dfrac{b}{a}$. Consequently, the abstract field of quotients defined by relations (4)–(6) corresponds in the present case to the field of all rational numbers. ▼

Heisenberg Ring and Field

We now introduce the field associated with the Heisenberg algebra generated by p, q and a Noether ring A.

Let A be an arbitrary Noether ring over R or C without zero divisor. We denote by $R_n(A)$ an algebra over A with $2n$ generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ satisfying the commutation relations

$$[p_i, q_j] = \delta_{ij} I, \quad [p_i, p_j] = 0, \quad [q_i, q_j] = 0. \tag{8}$$

We first introduce a basis in $R_n(A)$.

PROPOSITION 1. *The algebra $R_n(A)$ is a free A-module and has a basis consisting of all monomials of the form*

$$(p_1)^{k_1} \ldots (p_n)^{k_n} (q_1)^{l_1} \ldots (q_n)^{l_n} \equiv p^{(k)} q^{(l)}, \tag{9}$$

where $p_i^{(0)} = I_R$ and $q_i^{(0)} = I_R$.

PROOF: We show first that monomials (9) generate the A-module $R_n(A)$. Let $[R_n(A)]_k$ denote the set of all elements in $R_n(A)$ which can be written as polynomials in $\{p, q\}$ with coefficients in A and of degree $\leqslant k$. Clearly,

$$R_n(A) = \bigcup_k [R_n(A)]_k. \tag{10}$$

Assume that our assertion is true for $k < k_0$; by virtue of commutation relations (8) we can reduce any monomial of degree k_0 to a monomial of the form (9) (i.e., with all p's on the left side) modulo $[R_n(A)]_{k_0-2}$. Because the statement is obviously true for $k = 0$ and $k = 1$ our assertion follows by the method of induction. Next we show that the monomials (9) are linearly independent over A. Let $x = \Sigma\, a_{kl} p^{(k)} q^{(l)} = 0$, where $a_{kl} \in A$, and at least one of the a_{kl} be non-zero. Let us introduce a lexicographic ordering of $(k, l) = (k_1, \ldots, k_n, l_1, \ldots, l_n)$ and et (k^0, l^0) be the greatest set (according to the lexicographic ordering) for which $a_{kl} \neq 0$. By a simple calculation one verifies that

$$\prod_{i=1}^{n} ad^{k_i^0}(-q_i)\, ad^{l_i^0} p_i x = \prod_{i=1}^{n} k_i!\, l_i!\, a_{k^0 l^0} \neq 0 \tag{11}$$

which contradicts the equality $x = 0$. Hence, $p^{(k)} q^{(l)}$ cannot be linearly dependent. ▼

PROPOSITION 2. *$R_n(A)$ is an Ore ring.*

PROOF: We introduce a filtration of $R_n(A)$ given by

$$[R_n(A)]_0 \subset [R_n(A)]_1 \subset [R_n(A)]_2 \subset \ldots, \tag{12}$$

where $R_n(A)_k$ are the same objects as those used in the proof of the proposition 1. Consider the *graded ring structure* given by

$$\mathrm{gr}^{(k)} R_n(A) \equiv [R_n(A)]_k / [R_n(A)]_{k-1} \tag{13}$$

and

$$\mathrm{gr}\, R_n(A) \equiv \sum_{k=0}^{\infty} \mathrm{gr}^{(k)} R_n(A).$$

By virtue of proposition 1 one obtains the one-to-one correspondence

$$\operatorname{gr} R_n(A) \leftrightarrow R[p, q], \tag{14}$$

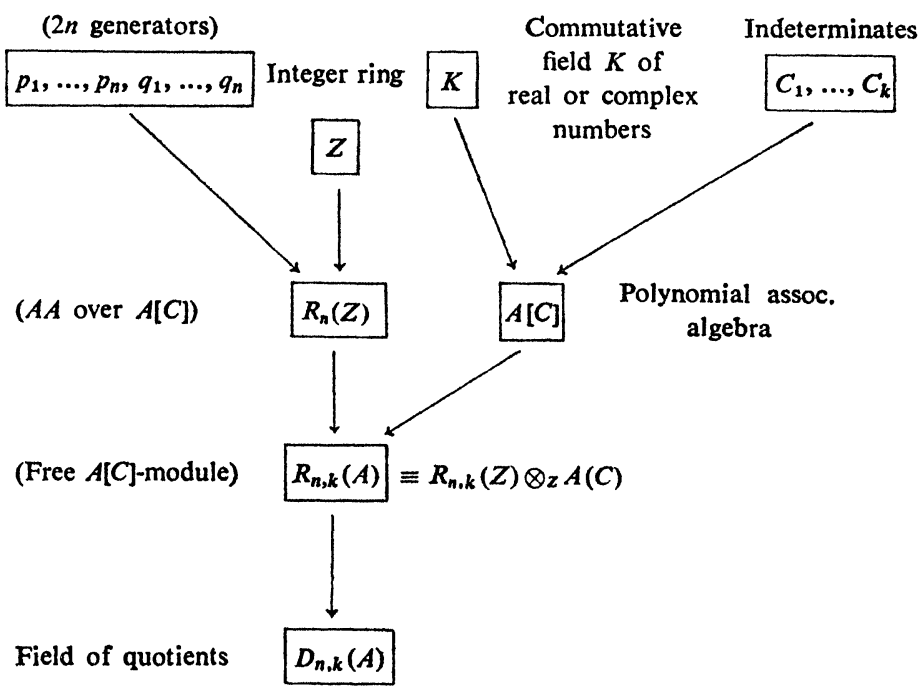
where $R[p, q]$ is the polynomial ring of (p, q). Clearly, $R[p, q]$ is a Noether ring without zero divisors (because it is a polynomial ring). Consequently, $\operatorname{gr} R_n(A)$ is also a Noether ring. Hence, by virtue of propositions 1 and 2 of app. A.2 we conclude that $R_n(A)$ is an Ore ring without zero divisors. ▼

We now associate, by virtue of proposition 2, with the ring $R_n(A)$ a quotient field $D_n(A)$. In what follows the ground ring A is a polynomial ring over a set of k indeterminates $\{C_1, C_2, \ldots, C_k\} \equiv \hat{C}$ with coefficients in the set K of real or complex numbers. Set

$$R_{n,k}(K) = R_n(A[\hat{C}]), \tag{15}$$

$$D_{n,k}(K) \equiv \text{field of quotients associated with } R_{n,k}(K). \tag{16}$$

The following picture illustrates the connection between various objects.



Enveloping Field of a Lie Algebra

Let L be a Lie algebra over the set K of real or complex numbers and let $E(L)$ be the enveloping algebra of L. We have

PROPOSITION 3. *$E(L)$ is an Ore ring.*

PROOF: Let us introduce the *filtration* of $E(L)$ given by

$$E^{(0)} \subset E^{(1)} \subset E^{(2)} \subset \ldots \tag{17}$$

where $E^{(k)}$ denotes the set of all elements in E which can be written as polynomials in the generators of L with coefficients in K and of degree $\leqslant k$. Consider the corresponding *graded ring structure* defined by

$$\mathrm{gr}^k E \equiv E^{(k)}/E^{(k-1)} \tag{18}$$

and

$$\mathrm{gr}\, E \equiv \sum_{k=0}^{\infty} \mathrm{gr}^k E. \tag{19}$$

By virtue of proposition 1 and definitions (18) and (19) we have the one-to-one corespondence

$$\mathrm{gr}\, E \leftrightarrow R(X_1, \ldots, X_m), \tag{20}$$

where $R(X_1, \ldots, X_m)$ is the ring of all polynomials of m variables, $m = \dim L$. Therefore, $\mathrm{gr}\, E$ is a Noether ring without zero-divisors (because it is a polynomial ring). By virtue of propositions 1 and 2 of app A.2 one concludes that E is an Ore ring without zero divisors. ▼

The above property permits us to construct the field of quotients $D(L)$ from $E(L)$. This field we shall call the enveloping field of L or simply the *Lie field.*

We see that the concept of the field of quotients allows us to consider in a natural manner the quotients of arbitrary elements of an enveloping algebra $E(L)$ of a given Lie algebra L. If L is a Poincaré algebra the elements of the form (1) will be in the field $D(L)$ of L.

This is not the only advantage of the concept of a field $D(L)$ associated with $E(L)$. It is interesting that the properties of the field $D(L)$, in contradistinction to the properties of $E(L)$, depend only weakly on the original Lie algebra. We may roughly state this in the form:

The field $D(L)$ of a Lie algebra L is isomorphic to a Heisenberg field $D_{n,k}(K)$.

We show this result first for the field $D(L)$ of the full linear algebra $\mathrm{gl}(n, C)$.

THEOREM 4. *The enveloping field $D(\mathrm{gl}(n, C))$ of the Lie algebra $\mathrm{gl}(n, C)$ is isomorphic to the Heisenberg field $D_{\frac{1}{2}n(n-1),n}(C)$.*

PROOF: We first show that if L_n is the Lie algebra of all $n \times n$ matrices which have zeros in the last row, then

$$D(L_n) = D_{(n/2)(n-1),0}(C). \tag{21}$$

We show this by induction over n. Let e_{ik}, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n+1$, be a natural basis in L_{n+1} given by 1.1(11). Set $q_i = e_{i,n+1}$, $p_i = e_{i,i} q_i^{-1}$, $\tilde{e}_{ik} = e_{ik} q_i^{-1} q_k$ (no summation).

Because $\det q_i = 0$, the quantity q_i^{-1} is not defined as a matrix, but it is defined as a formal quantity whose rules of manipulation are conformed to the equivalence relation of quotients. By direct computation we verify that

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [p_i, q_j] = \delta_{ij} I, \tag{22}$$

$$[\tilde{e}_{ik}, q_j] = \delta_{kj} q_j, \quad [\tilde{e}_{ik}, p_j] = -\delta_{kj} p_j \quad \text{(no summation)} \tag{23}$$

Indeed, for instance

$$\begin{aligned}[q_i, q_j] = q_i q_j - q_j q_i &= e_{i,n+1} e_{j,n+1} - e_{j,n+1} e_{i,n+1} \\ &= \delta_{j,n+1} e_{i,n+1} - \delta_{i,n+1} e_{j,n+1} = 0.\end{aligned} \tag{24}$$

This relation implies also, by the definition of quotients, that

$$q_j q_i^{-1} = q_i^{-1} q_j, \tag{25}$$

i.e., q_i^{-1} and q_j commute.

Similarly we compute

$$\begin{aligned}[p_i, q_j] &= e_{ii} q_i^{-1} q_j - q_j e_{ii} q_i^{-1} = e_{ii} q_j q_i^{-1} - e_{j,n+1} e_{ii} q_i^{-1} \quad \text{(no summation)} \\ &= e_{ii} e_{j,m+1} q_i^{-1} - \delta_{i,n+1} e_{ji} q_i^{-1} = \delta_{ij} e_{i,m+1} q_i^{-1} = \delta_{ij} q_i q_i^{-1} = \delta_{ij} I.\end{aligned}$$

Now, if the coefficients c_{ik} of a matrix c obey the condition

$$\sum_k c_{ik} = 0, \quad i = 1, 2, \ldots, n, \tag{26}$$

then we see, by virtue of eq. (23), that the operator $\alpha(c) = c_{ik} \tilde{e}_{ik}$ commutes with all the operators p_i and q_j. Let $\tilde{L}$ denote the set of all matrices c of order n whose entries satisfy the conditions (26). The set $\tilde{L}$ is isomorphic with L_n; indeed, if l_{ij} are the matrix entries of an element $l \in L_n$, then the correspondence given by

$$c_{ij} = l_{ij}, \quad i = 1, 2, \ldots, n-1, \; j = 1, 2, \ldots, n, \tag{27}$$

and

$$c_{nj} = -\sum_{i=1}^{n-1} l_{ij}, \quad j = 1, 2, \ldots, n, \tag{28}$$

gives a one-to-one mapping of $\tilde{L}$ onto L_n.

With every matrix $c \in \tilde{L}$ we now associate an element $\alpha(c) \equiv c_{ik} \tilde{e}_{ik} \in D(L_{n+1})$ One readily verifies that

$$\alpha([c_1, c_2]) = [\alpha(c_1), \alpha(c_2)]. \tag{29}$$

It is clear that the Lie field $D(L_{n+1})$ is generated by elements of the form $\alpha(c)$, $c \in \tilde{L}$, and by the elements $p_1, \ldots, p_n, q_1, \ldots, q_n$. Hence, in order to conclude our assertion it is sufficient to show that the field generated by the elements $\alpha(c)$ is isomorphic to the field $D_{\frac{n}{2}(n-1),0}(L)$. Now $\tilde{L}$ is isomorphic to L_n; hence, eq. (21) for L_{n+1} follows by induction.

The center of $E(\mathrm{gl}(n, C))$ is generated by n elements of the form $C_1, C_2, \ldots, C_n$, $C_s = \mathrm{Tr}\, e^s$, $e \equiv \{e_{ik}\}$.

The elements of the last row enter linearly into the monomials C_i; consequently, $D(\mathrm{gl}(n, C))$ is generated by $D(L_n)$ and by the elements $C_1, C_2, \ldots, C_n$. Thus,

$$D(\mathrm{gl}(n, C)) = D_{(n/2)(n-1),n}(C). \blacktriangledown \tag{30}$$

COROLLARY. $D(\mathrm{sl}(n, C)) = D_{\frac{1}{2}n(n-1),\, n-1}(C)$.

PROOF: $D(\mathrm{sl}(n, C))$ is generated by $C_2, C_3, \ldots, C_n$ and $D(L_n)$. Hence eq. (30) follows from eq. (21). ▼

In order to generalize the basic result to an arbitrary semisimple Lie algebra L it is convenient to use the concrete realization of L as differential operators on the manifold $X = N\backslash G$, where N is a nilpotent subgroup from the Iwasawa decomposition $G = KAN$. In this case there might exist invariant operators in the carrier space $H = L^2(X)$ which are not elements of the center of $E(L)$ of the Lie algebra L. Let $\tilde{E}(L)$ denote the extension of $E(L)$ by the ring of the invariant operators which are not generated by the elements of the center* Z of $E(L)$. Then, we have

THEOREM 5. *Let L be a semisimple algebra and let $D(L)$ be the enveloping field of the extended enveloping algebra $\tilde{E}(L)$ realized in the space $H = L^2(X)$, $X = N\backslash G$. Let (t_α, τ_i), $\alpha = 1, 2, \ldots, n$, $n = \dim N$ and $i = 1, 2, \ldots, k = \operatorname{rank} G$, be coordinates in the space X. Then, the field $D(L)$ is isomorphic to the Heisenberg field $D_{n,k}(C)$ generated by operators*

$$p_\alpha = \frac{\partial}{\partial t_\alpha}, \quad q_\alpha = t_\alpha, \quad \textit{and} \quad C_i = \tau_i \frac{\partial}{\partial \tau_i} \quad (\textit{no summation}). ▼ \tag{31}$$

(For the proof cf. Gel'fand and Kirillov 1969.)

These authors also proved an analogous theorem for a class of nilpotent Lie algebras (cf. Gel'fand and Kirillov 1966).

These theorems show that there is a common underlying structure of the enveloping fields of all semisimple Lie algebras and this structure is isomorphic to the structure of Heisenberg fields.

The numbers n and k which appear in the Heisenberg field $D_{n,k}(C)$ have a definite meaning in the representation theory: namely, in the case of finite-dimensional representations k represents the number of components of the highest weight $m = (m_1, \ldots, m_k)$ which characterizes the irreducible representation T^{L^m} of G (cf. ch. 8). The number n represents the dimension of the domain $G_0\backslash G$ of functions of the carrier space $H(G_0\backslash G)$ of an irreducible representation T^{L^m} of G (cf. ch. 8). This interpretation of the numbers k and n is also valid in the theory of infinite-dimensional representations (cf. ch. 19).

The fact that an enveloping field of a Lie algebra is generated by the Heisenberg algebra p_i, q_i and a set $C_1, \ldots, C_k$ of invariant operators is useful in quantum mechanics and in particle physics; in particular, it serves as a tool in the interpretation and analysis of the so-called dynamical groups (cf. ch. 13).

* Note that the center Z of $E(L)$ contains invariant operators which are polynomials in the generators; $\tilde{E}(L)$ contains invariant operators which are rational functions of the generators. However, in general, there might be invariant operators which are more general functions of generators, e.g. pseudo-differential operators.

§ 6. Further Results and Comments

A. *Casimir Operators and Their Spectra for Semisimple Lie Algebras*

We now extend th. 4.2 to other semisimple Lie algebras. There is no complete tensor calculus for arbitrary semisimple Lie algebras. Therefore we use eq. 3(7) for the Casimir operators

$$C_p = g_{\mu_1\mu_2\ldots\mu_p} X^{\mu_1}X^{\mu_2}\ldots X^{\mu_p}, \qquad p = 2, 3, \ldots, \tag{1}$$

where

$$g_{\mu_1\mu_2\ldots\mu_p} = \mathrm{Tr}(\hat{X}_{\mu_1}\hat{X}_{\mu_2}\ldots\hat{X}_{\mu_p}) \tag{2}$$

and $\hat{X}_\mu$ is the representation of X_μ in an arbitrary irreducible representation of the given Lie algebra. For classical Lie algebras A_n, B_n, C_n and D_n, we take for the representation $X_\mu \to \hat{X}_\mu$ the simplest fundamental representation $m = (1, 0, \ldots, 0)$. The following theorem gives the direct generalization of results of 4.2 for semisimple Lie algebras.

THEOREM 1. *Let $m = (m_1, \ldots, m_n)$ be the highest weight of an irreducible representation of anyone of the classical Lie algebras A_n, B_n, C_n, D_n, $n = 1, 2, \ldots$, and let $X_\mu \to \hat{X}_\mu$ be the simplest fundamental representation $m = (1, 0, \ldots, 0)$. The function*

$$G(z) = z^{-1}\left(1 + \frac{\beta z}{2-(2\alpha+1)z}\right)(1 - \Pi(z)), \tag{3}$$

where

$$\Pi(z) = \prod_{(i)}\left(1 - \frac{z}{1-\lambda_i z}\right), \quad \lambda_i = l_i + \alpha,\ l_i = m_i + r_i,\ i > 0,\ l_{-i} = -l_i, l_0 = 0, \tag{4}$$

is a generating function for the spectrum of the Casimir operators (1), *i.e.*,

$$G(z) = \sum_{p=0}^{\infty} C_p(m_1, \ldots, m_n) z^p. \tag{5}$$

The parameters α, β and r_i are given in table I for various Lie algebras. ▼

(For the proof cf. Perelomov and Popov 1968.)

These authors have also calculated the form of the spectrum of Casimir operators for some exceptional Lie algebras.

B. *Comments*

(i) The concept of a tensor operator was first introduced by Wigner 1931. The technique of tensor operators was successfully used by Racah, Eliott, Jahn and others in the theory of atomic and nuclear spectra (cf. the collection by Biedenharn and van Dam 1965 for references). More recently this technique has been used also in various problems of elementary particle theory and in particular in the current algebra approach (cf., e.g., Adler and Dashen 1968).

(ii) The various generalizations of the Wigner–Eckárt theorem were given by Sharp 1960, Stone 1961, Biedenharn 1963, Din 1963, Ginibre 1963, Moshinsky 1963, and Klimyk 1971.

(iii) The selection of the generators of the center Z of the enveloping algebra E is not unique; for instance, in the case of $u(n)$ one could take instead of the generators 4(6) the operators

$$C_p' = A_{i_1}{}^{i_2} A_{i_2}{}^{i_3} \dots A_{i_p}{}^{i_1} = \mathrm{Tr}\, A^p. \tag{6}$$

These are invariant operators of $u(n)$ by virtue of th. 1.1. Using the commutation relations 4(2) one may express the Casimir operators C_p' in terms of $C_p, C_{p-1}, \dots, C_1$. For instance, in the case $p = 3$, one obtains

$$C_3' = C_3 - nC_2 + (C_1)^2. \tag{7}$$

One often uses also the symmetrized Casimir operators of the form

$$C_p'' = \frac{1}{p!} P(X_{i_2}{}^{i_1} X_{i_3}{}^{i_2} \dots X_{i_1}{}^{i_p}), \tag{8}$$

where the symbol P denotes summation over $p!$ permutations of the generators in the bracket (cf., e.g., Gel'fand 1950, Berezin 1957). It seems, however, that the formula (1) for the Casimir operators is the most convenient one in the calculation of the spectra of invariant operators.

(iv) The invariant operators C and C' in E always commute. This property is, however, not true for arbitrary invariant operators; for instance, the operators r and $\mathrm{d}/\mathrm{d}r$, $r = |\vec{x}|$, are invariant operators of the rotation group acting in $H = L^2(R^3)$, but they do not commute.

(v) There exists two important results concerning the structure of the ring of invariant operators in the enveloping algebra in the space $H = L^2(X)$, where X is a symmetric space.

Gel'fand–Chevalley Theorem. *The number of generators of the ring of invariant operators in $L^2(X)$ is equal to the rank of symmetric space X.*

(For the proof see Helgason 1962, ch. 10.)

Thus in particular on symmetric spaces of rank one the center of enveloping algebra is generated by a single element. In this case we have

Theorem 2. *The ring of invariant operators in $L^2(X)$, where X is a symmetric space of rank one, is generated by the Laplace–Beltrami operator*

$$\Delta(x) = |\bar{g}|^{-1/2} \partial_\alpha g^{\alpha\beta}(x) |\bar{g}|^{1/2} \partial_\beta, \tag{9}$$

where $g^{\alpha\beta}(x)$ is the left-invariant metric tensor on X and

$$\bar{g}(x) = \det[g_{\alpha\beta}(x)].$$

(For the proof cf. Helgason 1962, ch. 9.)

(vi) We now give the important so-called Commutativity Theorem of Segal concerning the structure of the algebra of invariant operators in $L^2(G, \mu)$. Let G be a unimodular locally compact group with a Haar measure μ and let $g \to T_g^{\mathrm{L}}$ and $g \to T_g^{\mathrm{R}}$ be the left and the right regular representations of G in $L^2(G, \mu)$.

Denote by $\mathscr{R}_{\mathrm{L}}$ (or $\mathscr{R}_{\mathrm{R}}$) the closure in the weak operator topology of the set of all linear combinations of the T_g^{L} or T_g^{R}. Then we have

THEOREM 3. *If G is unimodular then we have*

$$\mathscr{R}'_{\mathrm{L}} = \mathscr{R}_{\mathrm{R}}, \qquad \mathscr{R}'_{\mathrm{R}} = \mathscr{R}_{\mathrm{L}}, \tag{10}$$

$$(\mathscr{R}_{\mathrm{L}} \cup \mathscr{R}_{\mathrm{R}})' = \mathscr{R}'_{\mathrm{L}} \cap \mathscr{R}'_{\mathrm{R}} = \mathscr{R}_{\mathrm{L}} \cap \mathscr{R}'_{\mathrm{L}} = \mathscr{R}_{\mathrm{R}} \cap \mathscr{R}'_{\mathrm{R}}. \tag{11}$$

(For the proof cf. Segal 1950 or Maurin 1968, ch. 6, § 7.)

§ 7. Exercises

§ 1.1. Let $G = \mathrm{SO}(3)$. Show that the hermitean adjoint of the tensor operator Y_M^J given by

$$(Y_M^J)^* = (-1)^M Y_{-M}^J$$

is also a tensor operator.

§ 1.2. Prove the Wigner–Eckart theorem for arbitrary compact groups.

Hint: Decompose onto irreducible components that representation of G with respect to which the product $T_m^{\lambda} U_{m_1}^{\lambda_1}$ transforms and use the method of proof of th. 2.

§ 2.1. Show that the enveloping algebra of the Heisenberg algebra $[P, Q] = I$ has the trivial center.

§ 3.1. Show that the center of the enveloping algebra of the Euclidean Lie algebra $T^n \mathbin{\otimes\!\!\!\supset} \mathrm{so}(n)$ is generated by $\{n/2\}$ elements.

§ 3.2. Find the generators of the center of the enveloping algebra of the semi-direct product $C^3 \mathbin{\otimes\!\!\!\supset} \mathrm{SU}(3)$.

§ 3.3. Show that the following operators

$$C_2 = \tfrac{1}{2} M_{\mu\nu} M^{\mu\nu} = J^2 - N^2,$$

$$C'_2 = -\tfrac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta} = J \cdot N,$$

where $J = (M_{23}, M_{31}, M_{12})$ and $N = (M_{01}, M_{02}, M_{03})$ generate the center of the enveloping algebra of the Lorentz group.

§ 5.1. Is the operator of helicity $J \cdot p/|p|$ an element of the enveloping field of the Poincaré group?

§ 5.2.** Elaborate an extension of the concept of enveloping field of Euclidean Lie algebra which would contain invariant operators of the form $|r|$, $\dfrac{1}{|r|}$, $\dfrac{\mathrm{d}}{\mathrm{d}|r|}$, etc.

§ 5.3.* Let $g \to T_g$ be a quasi-regular representation of SO(3) in the space $H = L^2(X), X = \mathrm{SO}(3)/\mathrm{SO}(2)$. Show that there are no pseudodifferential invariant operators in H.

§ 5.4.* Find the generators of the center of the enveloping field $D(L)$ for semisimple algebras.

§ 5.5. Let $\Pi = T^4 \otimes \mathrm{SO}(3,1)$, N be the Iwasawa factor of SO(3, 1) and $G_0 = T^4 \otimes N$. Show that in the space $H = L^2(X)$, $X = \Pi/G_0$ there are more than two invariant differential operators of Π.

§ 6.1.** Find the spectra of the Casimir operators for the irreducible representations of the exceptional Lie algebras.

§ 6.2.** Find the generating function 6 (5) for the spectra of the Casimir operators for exceptional Lie algebras.

Chapter 10

The Explicit Construction of Finite-Dimensional Irreducible Representations

The method of induced representations presented in ch. 8 solves the problem of classification of all finite-dimensional irreducible representations of all simple Lie groups. However, in order to examine all the consequences for physical applications, we need to determine explicitly:

1° The set of independent invariant operators.

2° The complete set of commuting operators (CSCO) which we interpret as physical observables; the nature and the range of their spectra (cf. ch. 13).

3° The properties of the basis functions and the dimension of the carrier space that is identified with the space of physical states.

4° The properties of the decomposition of a representation of G with respect to a subgroup G_0 of G.

5° The matrix elements of the operators T_g or $T(X)$ of the representation of G or Lie algebra L.

In this chapter we discuss four general methods of construction of irreducible representations for which some or all of the problems 1°–5° are explicitly solved.

§ 1. The Gel'fand–Zetlin Method

The first method of construction of irreducible representations which also provides solutions to the problems 1°–5° listed above is the so-called Gel'fand–Zetlin formalism. It can be applied to both compact and noncompact groups and seems to be especially suitable for applications in quantum physics. We describe this formalism in detail using the examples of the algebras $u(n)$ and $so(n)$.

A. *The Representations of* $u(n)$

The group $U(n)$ is defined as the transformation group in C^n which conserves the hermitian form

$$z_1\bar{z}_1+z_2\bar{z}_2+\ldots+z_n\bar{z}_n = \text{const}. \tag{1}$$

Thus, for $u \in U(n)$, $u^*u = uu^* = 1$. Consequently the generators of the one-parameter subgroups obey the hermiticity condition*

$$M^*_{ik} = M_{ik}. \tag{2}$$

The set of n^2 generators (2) span the Lie algebra of $U(n)$. However, as we have noted in ch. 9, because the commutation relations of the M_{ik} cannot be written in a symmetric way we start from the Lie algebra of the group $\mathrm{GL}(n, R)$ whose elements satisfy the commutation relations

$$[A_{ij}, A_{kl}] = \delta_{jk} A_{il} - \delta_{il} A_{kj}, \qquad i, j = 1, 2, \ldots, n. \tag{3}$$

The Lie algebra (3) has an n^2-dimensional representation given by the Cartan–Weyl matrices

$$A_{ij} \to (e_{ij})_{lk} = \delta_{il}\delta_{jk}. \tag{4}$$

These matrices obey the condition

$$e^*_{ij} = e_{ji}. \tag{5}$$

Let us introduce the following n^2 operators

$$M_{kk} = A_{kk},$$
$$M_{kl} = A_{kl} + A_{lk}, \qquad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}), \qquad k < l \leqslant n. \tag{6}$$

When $A_{ij} = e_{ij}$ the operators (6) are $n \times n$-matrices obeying the hermiticity condition (2). Thus they are generators of $U(n)$ and automatically fulfil the commutation relations of the algebra $u(n)$. Therefore, an arbitrary representation of the Lie algebra of $\mathrm{GL}(n, R)$ for which the condition (5) is fulfilled, induces a hermitian representation of $u(n)$ determined by eq. (6). But clearly this fact does not imply that the Lie algebras of $U(n)$ and $\mathrm{GL}(n, R)$ are isomorphic since both algebras are real and the transformation given by (6) is a complex linear substitution.

Next we construct the canonical basis of an arbitrary irreducible representation space of $u(n)$. The construction is based on the following two properties of irreducible representations of $u(n)$:

(i) A finite-dimensional irreducible representation of the Lie algebra $u(n)$ is uniquely determined by an n-dimensional vector $m_n = (m_{1n}, m_{2n}, \ldots, m_{nn})$ with integer components m_{in} obeying the condition:

$$m_{1n} \geqslant m_{2n} \geqslant \ldots \geqslant m_{nn}. \tag{7}$$

This vector represents the highest weight of the representation. The space in which the irreducible representation, determined by the vector m_n, is realized is denoted by H^{m_n}. We shall assume that the Lie algebra $u(n-1)$ is imbedded in a natural manner in $u(n)$, i.e., it is spanned by the generators A_{ij} $i, j = 1, 2, \ldots, n-1$ (cf. sec. 8.3.D).

* In ch. 9 we used tensors of the form $M_i{}^j$ or $A_i{}^j$. This was convenient for the calculation of the invariants. In this section we shall use covariant tensors of the form A_{ij}, A_{kl}, etc.

(ii) In the decomposition of the irreducible finite-dimensional representations of $u(n)$ only those irreducible representations of $u(n-1)$ appear for which the components $m_{i,n-1}$ of the highest weight obey the condition

$$m_{in} \geqslant m_{i,n-1} \geqslant m_{i+1,n}, \qquad i = 1, 2, \dots, n-1. \tag{8}$$

The multiplicity of every irreducible representation of $u(n-1)$ which appears in the decomposition of $u(n)$ is equal to one (cf. th. 8.8.1).

Let us consider the decreasing chain of algebras

$$u(n) \supset u(n-1) \supset \dots \supset u(2) \supset u(1) \tag{9}$$

and decompose the irreducible space H^{m_n} into subspaces $H^{m_{n-1}}$ which are irreducible with respect to $u(n-1)$. Each of these subspaces decomposes further with respect to irreducible representations of $u(n-2)$, and so on till $u(1)$. Because the irreducible representations of $u(1)$ are one-dimensional, the intersection of the decreasing chain of subspaces

$$H^{m_n} \supset H^{m_{n-1}} \supset \dots \supset H^{m_2} \supset H^{m_1} \tag{10}$$

determines uniquely this one-dimensional subspace. The uniqueness follows from the result (ii). We denote the unit vector which spans this one-dimensional subspace by the so-called *Gel'fand-Zetlin pattern* m:

$$m = \left| \begin{array}{ccccccc} m_{1n} & m_{2n} & \dots & m_{n-1,n} & & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{array} \right|. \tag{11}$$

The first row of the pattern is determined by the components of the highest weight of an irreducible representation of $u(n)$. This row is fixed for a given irreducible representation of $u(n)$. In the following rows there are arbitrary integers which obey the following inequalities

$$m_{ij} \geqslant m_{i,j-1} \geqslant m_{i+1,j}, \qquad \begin{array}{l} j = 2, 3, \dots, n, \\ i = 1, 2, \dots, n-1. \end{array} \tag{12}$$

These inequalities are reflected in the Gel'fand–Zetlin pattern by the fact that the numbers $m_{i,j-1}$ at the $(j-1)$st row are placed between the numbers m_{ij} and $m_{i+1,j}$ at the jth row. For a definite k, $k = 1, 2, \dots, n-1$, the numbers m_{ik}, $1 \leqslant i \leqslant k$, represent the components of the highest weight m_k of an irreducible representation of $u(k)$ which will appear in the decomposition of an irreducible representation of $u(n)$.

In order to determine a hermitian representation of the Lie algebra $u(n)$, eq. (6), it is sufficient to define the action of the operators A_{ij}, $i, j = 1, 2, \dots, n$, on the pattern m and check that the commutation relations (3) and eq. (5) are fulfilled. We can restrict ourselves to the generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ because the action of the other generators can be obtained from the commutators of these generators: e.g., from eq. (3) we have

$$A_{k-2,k} = [A_{k-2,k-1}, A_{k-1,k}],$$

and generally,

$$\begin{aligned} A_{k,k-h} &= [A_{k,k-1}, A_{k-1,k-h}], \\ A_{k-h,k} &= [A_{k-h,k-1}, A_{k-1,k}], \end{aligned} \qquad h > 1. \tag{13}$$

In case of Lie algebra $u(2)$ the action of generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ is well known (cf. exercise 8.9.8.2). Guided by this case we define the action of the operators A_{kk}, $A_{k,k-1}$ and $A_{k-1,k}$ for an arbitrary $u(n)$ as follows:

$$A_{kk}m = (r_k - r_{k-1})m, \tag{14}$$

$$A_{k,k-1}m = \sum_{j=1}^{k-1} a_{k-1}^j(m) m_{k-1}^j, \tag{15}$$

$$A_{k-1,k}m = \sum_{j=1}^{k-1} b_{k-1}^j(m) \hat{m}_{k-1}^j, \tag{16}$$

where

$$r_0 = 0, \quad r_k = \sum_{j=1}^{k} m_{jk}, \quad k = 1, 2, \ldots, n, \tag{17}$$

$$a_{k-1}^j(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2}, \tag{18}$$

$$b_{k-1}^j(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2} \tag{19}$$

and

$$l_{i,k} = m_{i,k} - i.$$

Here $m_{k-1}^j(\hat{m}_{k-1}^j)$ represents the pattern obtained from m by replacing the number $m_{j,k-1}$ in the $(k-1)$st row of m by the number $m_{j,k-1}-1$ $(m_{j,k-1}+1)$. Note that formally there are patterns m_{k-1}^j and $\hat{m}_{k-1}^j$ which do not obey the inequalities (12). However, such patterns do not arise in (15) or (16) because the coefficients a_{k-1}^j (b_{k-1}^j) are different from zero only for the patterns obeying inequalities (12). Moreover, for admissible patterns the denominators of the coefficients a_{k-1}^j and b_{k-1}^j are not equal to zero and that the expressions under the square roots are non-negative. Therefore,

$$\bar{a}_{k-1}^j = a_{k-1}^j \quad \text{and} \quad \bar{b}_{k-1}^j = b_{k-1}^j. \tag{20}$$

We also have

$$a^j_{k-1}(m) = b^j_{k-1}(m^j_{k-1}), \quad b^j_{k-1}(m) = a^j_{k-1}(m^{\hat{j}}_{k-1}). \tag{21}$$

All these statements follow directly from eqs. (18), (19) and (12).

It should be remarked that for some non-admissible patterns the denominators of a^j_{k-1} or b^j_{k-1} can be equal to zero. However, in such cases the numerators are also zero and that the ratios, by definition, are equal to zero.

We show first that the operators A_{kk} and $A_{k,k-1}$ satisfy the correct commutation relations, i.e.,

$$[A_{kk}, A_{k,k-1}] = A_{k,k-1}, \quad k = 2, 3, \ldots, n. \tag{22}$$

Indeed, from (15) and (14) it follows that

$$X \equiv A_{kk}A_{k,k-1}m = \sum_{j=1}^{k-1} a^j_{k-1}(m)[r_k(m^j_{k-1}) - r_{k-1}(m^j_{k-1})]m^j_{k-1}.$$

Using the definition of the pattern m^j_k and (17), we find

$$\begin{aligned} X &= [r_k(m) - r_{k-1}(m)]\sum_{j=1}^{k-1} a^j_{k-1}(m)m^j_{k-1} + \sum_{j=1}^{k-1} a^j_{k-1}(m)m^j_{k-1} \\ &= A_{k,k-1}A_{kk}m + A_{k,k-1}m \end{aligned}$$

so that

$$[A_{kk}, A_{kk-1}]m = A_{kk-1}m \tag{23}$$

which is precisely eq. (22), because of the arbitrariness of the pattern m. Similarly one shows that operators A_{kk}, $A_{k-1,k}$ and $A_{k,k-1}$ satisfy the commutation relations (3). Hence by virtue of (13) the operators A_{ij}, $i, j = 1, 2, \ldots, n$, satisfy the commutation relations (3). Next we verify the hermiticity condition (5) imposed on the generators A_{ij}, which in turn insures that the generators (6) of $u(n)$ are hermitian, namely,

$$(n, A_{ij}m) = \overline{(n, A^*_{ji}m)}, \quad i, j = 1, 2, \ldots, n, \tag{24}$$

for arbitrary patterns n and m.

The generators A_{kk}, $k = 1, 2, \ldots, n$, are hermitian, because they are diagonal in the representation space and their eigenvalues are real by eqs. (14)–(17). For the generators $A_{k,k-1}$ we have:

$$(n, A^*_{k,k-1}m) = \overline{(A^*_{k,k-1}m, n)} = \overline{(m, A_{k,k-1}n)} = \sum_{j=1}^{k-1} \overline{a^j_{k-1}}(n)\,\delta_{m,n^j_{k-1}}. \tag{25}$$

Because the $a^j_{k-1}(n)$ are real and obey (21) we obtain

$$\begin{aligned} (n, A^*_{k,k-1}m) &= \sum_{j=1}^{k-1} b^j_{k-1}(n^j_{k-1})\,\delta_{m,n^j_{k-1}} \\ &= \sum_{j=1}^{k-1} b^j_{k-1}(m)\,\delta_{n,m^{\hat{j}}_{k-1}} = (n, A_{k-1,k}m). \end{aligned} \tag{26}$$

The equality

$$\delta_{m,n^j_{k-1}} = \delta_{n,m^{\hat{j}}_{k-1}}$$

used here follows from the fact that n coincides with m^j_{k-1} if and only if m coincides with $n^{\hat{j}}_{k-1}$.

Due to the arbitrariness of n and m in (25) and (26) we have

$$A^*_{k,k-1} = A_{k-1,k}, \quad A^*_{k-1,k} = A_{k,k-1}. \tag{27}$$

Applying the method of induction and utilizing the recurrence formulas (13), we obtain

$$\begin{aligned} A^*_{k,k-h} &= [A_{k,k-1}, A_{k-1,k-h}]^* = [A^*_{k-1,k-h}, A^*_{k,k-1}] \\ &= [A_{k-h,k-1}, A_{k-1,k}] = A_{k-h,k}. \end{aligned} \tag{28}$$

Therefore, the hermiticity condition (5) is fulfilled for an arbitrary A_{ij}, $i, j = 1, 2, \dots, n$, in the representation space H^{m_n}. Consequently the generators (6) of $u(n)$ are represented in H^{m_n} by hermitian operators.

The space H^{m_n} in which the representation (14)–(19) is realized is by definition irreducible. An independent formal proof of irreducibility follows from the fact that the following pattern

$$m = \left| \begin{array}{ccccccccccc} m_{1n} & & m_{2n} & & m_{3n} & \cdots & m_{n-2,n} & & m_{n-1,n} & & m_{nn} \\ & m_{2n} & & m_{3n} & & \cdots\cdots\cdots & & m_{n-1,n} & & m_{nn} & \\ & & m_{3n} & & & \cdots\cdots\cdots\cdots & & & m_{nn} & & \\ & & & \cdot & & & & \cdot & & & \\ & & & & \cdot & & \cdot & & & & \\ & & & & & \cdot & & & & & \\ & & & & m_{n-1,n} & & m_{nn} & & & & \\ & & & & & m_{nn} & & & & & \end{array} \right|$$

is the only invariant vector of the subgroup Z (i.e., $A_p{}^q m = 0$ for $p > q$) and from corollary 2 to th. 8.2.2.

EXAMPLE 1. The simplest Gel'fand–Zetlin pattern is obtained when all components of the highest weight are equal to each other:

$$m_{1m} = m_{2n} = \dots = m_{nn} = m. \tag{29}$$

In this case, due to inequalities (12) all other entries in the pattern are also equal to m. Therefore, we get the one-dimensional representation of $U(n)$ which is of the type

$$U(n) \ni g \to T_g{}^{L^m} = (\det g)^m. \tag{30}$$

EXAMPLE 2. The Gel'fand–Zetlin pattern for the group $U(3)$. (This is an important internal higher symmetry group for fundamental particles; see ch. 13.) States spanning an irreducible representation space of $U(3)$ may be labelled by the eigenvalues of four commuting operators, namely I^2, I_3 which are associated with SU(2) subgroup and Y and B. It turns out that Gel'fand–Zetlin patterns correspond exactly to the physical states labelled by these quantum numbers. Expressing the operators I^2, I_3 and Y in terms of $A_i^{\ k}$ of the subgroup $U(2)$ and utilizing formulas (14), (15) and (16), we obtain

$$m = \begin{vmatrix} m_{13} & & m_{23} & & m_{33} \\ & I+\frac{1}{2}Y+B & & -I+\frac{1}{2}Y+B & \\ & & I_3+\frac{1}{2}Y+B & & \end{vmatrix},$$

where

$$m_{13}+m_{23}+m_{33} = 3B.$$

The quantity B may be interpreted as the baryon number. We see that for a definite I and Y there are $2I+1$ patterns with different I_3. We may also express the components of the highest weight of $u(3)$ in physical terms: indeed let Y_h be the highest possible values of Y and I_h be the corresponding (unique) value of I: then from eq. (8) we have

$$m_{13} = B+\tfrac{1}{2}Y_h+I_h, \quad m_{23} = B+\tfrac{1}{2}Y_h-I_h, \quad m_{33} = B-Y_h. \tag{31}$$

Also it follows from the inequalities (12) and formula (31) that irreducible representations with the same Y_h and I_h but different values of the baryon number B have the same dimensions. ▼

We are now in a position to give the solutions of the problems listed in the introduction.

(1) *The dimension* of the carrier space H^{m_n} of an irreducible representation is given by the Weyl formula (cf. eq. 8.8(30))

$$N = \frac{\prod\limits_{i<j}(l_i-l_j)}{\prod\limits_{i<j}(l_i^0-l_j^0)}, \tag{32}$$

? n-variable

where

$$l_j = m_{jn}+n-j, \quad l_j^0 = n-j.$$

(2) *The maximal set of commuting operators* consists of the invariant operators of the following chain of subalgebras

$$u(n) \supset u(n-1) \supset \ldots \supset u(2) \supset u(1), \tag{33}$$

i.e., it contains $(n^2+n)/2$ operators:

$$\begin{matrix} C_{1n}, & C_{2n}, & \ldots, & C_{n-1,n}, & C_{nn} \\ & C_{1,n-1}, & \ldots, & C_{n-1,n-1} & \\ & & \cdots\cdots\cdots & & \\ & & C_{12}, \ C_{22} & & \\ & & C_{11} & & \end{matrix} \tag{34}$$

Here

$$C_{pk} = A_{i_2}^{\ i_1} A_{i_3}^{\ i_2} \ldots A_{i_p}^{\ i_{p-1}} A_{i_1}^{\ i_p}, \quad p = 1, 2, \ldots, p \leqslant k,$$

where A_i^j are generators of $\mathrm{gl}(k,R)$ and summation over repeated indices runs from 1 to k.

(3) *The eigenvalues* of any of these operators were explicitly expressed in terms of the highest weights in ch. 9, § 4. For example,

$$C_{1,k} = \sum_{i=1}^{k} m_{i,k}, \quad C_{2,k} = \sum_{i=1}^{k} m_{i,k}(m_{i,k}+n+1-2i), \tag{36}$$

and so forth.

(4) *The matrix elements of the generators* of the Lie algebra $u(n)$ follow from eqs. (14), (15), (16) and (6). For example, the matrix elements of the generators $A_{k,k-1}$ are

$$(m', A_{k,k-1}m) = \sum_{j=1}^{k=1} a_{k-1}^j(m)\,\delta_{m_{k-1}^j, m'}, \quad k = 2, 3, \ldots, n-1. \tag{37}$$

We see that $A_{k,k-1}$ has non-vanishing matrix elements only between neighboring patterns m and m'. Using the recursion formula (13) we can determine the action of any generator A_{ik} on any pattern m. The explicit formulas are given in ch. 11, § 8, eqs. (15)–(24) for both the compact and the noncompact generators of the Lie algebra $u(p, q)$.

(5) Let $g \to \hat{T}_g$ be the conjugate representation to a group representation $g \to T_g$ of $u(n)$. Then by virtue of 5.1(14)2° we obtain for hermitian generators $\hat{M}_{ik}$

$$\hat{M}_{ik} = (-M_{ik})^T. \tag{38}$$

Thus for $U(n)$, we see using (6) that a representation is conjugate to a given representation if the generators $A_{ik}(= M_{ik}-i\tilde{M}_{ik})$ obey the condition (38). It can be verified, using (37), and eqs. (14) and (19), that the irreducible representations determined by $\hat{m}_n = (\hat{m}_{1n}, \hat{m}_{2n}, \ldots, \hat{m}_{nn})$ and $m_n = (m_{1n}, m_{2n}, \ldots, m_{nn})$ are conjugate to each other if and only if

$$\hat{m}_{in} = -m_{n+1-i,n}, \quad i = 1, 2, \ldots, n, \tag{39}$$

and a representation m is self-conjugate if

$$m_{in} = -m_{n+1-i,n}, \quad i = 1, 2, \ldots, n.$$

(6) The set of all representations determined by a highest weight $m_n = (m_{1n}, \ldots, m_{nn})$ can be divided into equivalence classes of projectively equivalent representations, obtained as follows: We associate with a given representation determined by the highest weight m_n the subclass of all irreducible representations determined by $\tilde{m}_n$ for which the components of the highest weights $\tilde{m}_{in}$ obey the condition

$$\tilde{m}_{in} - m_{in} = s, \tag{40}$$

where s is an arbitrary integer. We can easily verify, on the basis of the Weyl formula (32) that the representations associated with any highest weight $\tilde{m}_n$

obeying condition (40) have the same dimension as the original representation associated with m_n. Moreover, on the basis of eqs. (14), (15) and (16), we verify that the matrix elements of any generator A_i^j are related by

$$(\tilde{m}', A_i^j \tilde{m}) = (m', A_i^j m) + s\delta_i^j \delta_{m,m'}.$$

Therefore, the global representations $T^{L^{\tilde{m}_n}}$ and $T^{L^{m_n}}$ of $U(n)$ are related to each other. Indeed, from (42),

$$T^{L^{\tilde{m}_n}} = \exp is(\varphi_1 + \varphi_2 + \ldots + \varphi_n) T^{L^{m_n}} = (\det \delta)^s T^{L^{m_n}} = (\det g)^s T^{L^{m_n}}$$

where $\exp(\varphi_k) = \delta_k$ and $\det \delta = \det g$ (cf. exercise 3.11.6.3). Consequently the representations $T^{L^{\tilde{m}_n}}$ and $T^{L^{m_n}}$ are projectively equivalent, and the set of irreducible representations of $U(n)$ can be divided into subclasses of projectively equivalent representation. Each subclass of projectively equivalent representations contains the infinite set of irreducible representations whose highest weights obey the condition (40). Any subclass of projectively equivalent irreducible representations of $U(n)$ becomes one irreducible representation of the group SU(n).

B. *The Representations of* $o(n)$

The orthogonal group $O(n)$ is the set of all linear transformations g of the n-dimensional Euclidean space R^n,

$$x_l' = g_{ls} x_s, \qquad s, l = 1, 2, \ldots, n,$$

which conserve the quadratic form

$$x_1^2 + x_2^2 + \ldots + x_n^2.$$

The group $O(n)$ contains $\frac{1}{2}n(n-1)$ different one-parameter subgroups, namely rotations in the planes (x_i, x_k)

$$g_{ik}(\vartheta) = \begin{array}{c@{}l} \begin{array}{cccccccccc} & & & & (i) & & (k) & & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & & 0 & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & & 0 & & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \cos\vartheta & \cdots & \sin\vartheta & & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -\sin\vartheta & \cdots & \cos\vartheta & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \cdots (i) \\ \\ \\ \cdots (k) \\ \\ \\ \end{array} \end{array}. \tag{41}$$

In the representation $g \to T_g = g$, the generator X_{ik} of the one parameter subgroup g_{ik} is represented by a skew-symmetric $n \times n$-matrix with elements $(X_{ik})_{ik} = -(X_{ik})_{ki} = 1$ and zero for others. Therefore, the generators of $O(n)$

can be expressed in terms of the generators e_{ik} (eq. (4)) of the group $\mathrm{GL}(n, R)$ as follows:

$$X_{ik} = e_{ik} - e_{ki}. \tag{42}$$

The commutation relations for the generators X_{ik} can be obtained from those of e_{ik}:

$$[X_{ik}, X_{lm}] = \delta_{kl} X_{im} + \delta_{im} X_{kl} - \delta_{km} X_{il} - \delta_{il} X_{km}. \tag{43}$$

As in the case of $u(n)$ the construction of irreducible representations of $o(n)$ is based on the following two results:

(i) A finite-dimensional irreducible representation of the Lie algebra $o(n)$, $n = 2\nu$ or $n = 2\nu+1$, is uniquely determined by the highest weight $m = (m_1, m_2, \ldots, m_\nu)$ with integral or half-integral components obeying the condition

1° for $n = 2\nu$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant |m_\nu|$,

2° for $n = 2\nu+1$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu \geqslant 0$

(cf. th. 8.5.2).

(ii) In the decomposition of an irreducible finite-dimensional representation of $o(n)$ every irreducible representation of $o(n-1)$ appear with multiplicity one: the components p_i of the highest weight of these representations obey the conditions:

1° for $n = 2\nu$:

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant p_{\nu-1} \geqslant |m_\nu|, \tag{44}$$

2° for $n = 2\nu+1$:

$$m_1 \geqslant q_1 \geqslant m_2 \geqslant q_2 \geqslant \ldots \geqslant m_\nu \geqslant q_\nu \geqslant -m_\nu$$

(cf. th. 8.8.2).

The construction of the irreducible representations of the Lie algebra (43) will be accomplished by the following steps:

(i) Construction of the set of orthonormal states associated with a given highest weight.

(ii) Determination of the action of the generators X_{ik} on the basis states and verification of the commutation relations (43).

We denote the components of the highest weight by

$$m = (m_{1,2k+1}, m_{2,2k+1}, \ldots, m_{k+1,2k+1}), \tag{45}$$

when n is even ($n = 2k+2$), and by

$$m = (m_{1,2k}, m_{2,2k}, \ldots, m_{k,2k}), \tag{46}$$

when n is odd ($n = 2k+1$).

With a given highest weight we associate a Gel'fand–Zetlin pattern m. Repeating the arguments given for $U(n)$ and using the result (44), we conclude that the patterns are given by:

for n even ($n = 2k+2$):

$$m = \begin{vmatrix} m_{1,2k+1} & m_{2,2k+1} & \cdots & m_{k,2k+1} & m_{k+1,2k+1} \\ m_{1,2k} & \cdots & & m_{k,2k} & \\ m_{1,2k-1} & \cdots & & m_{k,2k-1} & \\ & m_{1,2k-2} & \cdots & m_{k-1,2k-2} & \\ & m_{1,2k-3} & \cdots & m_{k-1,2k-3} & \\ & \cdots & \cdots & \cdots & \\ & & m_{14} \; m_{24} & & \\ & & m_{13} \; m_{23} & & \\ & & m_{12} & & \\ & & m_{11} & & \end{vmatrix}, \tag{47}$$

for n odd ($n = 2k+1$):

$$m = \begin{vmatrix} m_{1,2k} & m_{2,2k} & \cdots & m_{k-1,2k} & m_{k,2k} \\ m_{1,2k-1} & m_{2,2k-1} & \cdots & m_{k-1,2k-1} & m_{k,2k-1} \\ & m_{1,2k-2} & & m_{k-1,2k-2} & \\ & m_{1,2k-3} & . & m_{k-1,2k-3} & \\ & \cdots & \cdots & \cdots & \\ & & m_{14} \; m_{24} & & \\ & & m_{13} \; m_{23} & & \\ & & m_{12} & & \\ & & m_{11} & & \end{vmatrix}. \tag{48}$$

The patterns (47) or (48) are determined by the upper row, which contains the fixed components of the highest weight of an irreducible representation. For $n = 2k+2$ the numbers m_{ij} in the other rows obey the inequalities (cf. eq. (44))

$$\begin{gathered} m_{1,2k+1} \geqslant m_{1,2k} \geqslant m_{2,2k+1} \geqslant m_{2,2k} \geqslant \ldots m_{k,2k+1} \geqslant m_{k,2k} \geqslant |m_{k+1,2k+1}|, \\ m_{1,2k} \geqslant m_{1,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \ldots \geqslant m_{k,2k-1} \geqslant -m_{k,2k}, \\ m_{1,2k-1} \geqslant m_{1,2k-2} \geqslant m_{2,2k-1} \geqslant \ldots \geqslant m_{k-1,2k-2} \geqslant |m_{k,2k-1}|, \end{gathered} \tag{49}$$

and so on, for an arbitrary row

$$\begin{gathered} m_{i,2p+1} \geqslant m_{i,2p} \geqslant m_{i+1,2p+1}, \quad i = 1, 2, \ldots, p-1, \\ m_{p,2p+1} \geqslant m_{p,2p} \geqslant |m_{p+1,2p+1}|, \\ m_{i,2p} \geqslant m_{i,2p-1} \geqslant m_{i+1,2p}, \quad i = 1, 2, \ldots, p-1, \\ m_{p,2p} \geqslant m_{p,2p-1} \geqslant -m_{p,2p}. \end{gathered}$$

The numbers m_{ij} in the jth row represent the components of the highest weight of $o(j+1)$. The numbers m_{ij} associated with a given pattern are simultaneously all integers or all half-integers. This is in contrast to the highest weight of the group $U(n)$ where all m_{ij} were integers.

For $n = 2k+1$ we have

$$\begin{gathered} m_{1,2k} \geqslant m_{1,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \cdots \geqslant m_{k,2k-1} \geqslant m_{k,2k}. \\ m_{1,2k-1} \geqslant m_{1,2k+2} \geqslant m_{2,2k-1} \geqslant \cdots \geqslant m_{k-1,2k-2} \geqslant |m_{2k-1,k}| \end{gathered}$$

and so on.

It follows from the commutation relations (43) that the action of the entire

Lie algebra $o(n)$ can be reproduced, if the action of generators $X_{2p+1,2p}$, $p = 1, 2, \ldots, [(n-1)/2]$ and $X_{2p+2,2p+1}$, $p = 0, 1, 2, \ldots, [(n-2)/2]$, is explicitly known. Now for $n = 3$ and 4 one may easily calculate directly the action of these generators: using these expressions one can deduce the action of generators $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$ for an arbitrary n. Because these are straightforward algebraic calculations, we restrict ourselves for giving the final formulas only (for the complete derivation see Ottoson 1968).

Let $m_k^j(\hat{m}_k^j)$ be the pattern obtained from m by replacing m_{jk} by $m_{jk}-1(m_{jk}+1)$. Then the operators $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$ are defined by the following relations:

$$X_{2p+1,2p}m = \sum_{j=1}^{p} A_{2p-1,j}(m)\hat{m}^j_{2p-1} - \sum_{j=1}^{p} A_{2p-1,j}(m^j_{2p-1})m^j_{2p-1},$$

$$p = 1, 2, \ldots, \left[\frac{n-1}{2}\right], \tag{50a}$$

and

$$X_{2p+2,2p+1}m = \sum_{j=1}^{p} B_{2p,j}(m)\hat{m}^j_{2p} - \sum_{j=1}^{p} B_{2p,j}(m^j_{2p})m^j_{2p} + \mathrm{i}C_{2p}(m)m. \tag{50b}$$

Using the notation

$$\begin{gathered} m_{p,2p-1} = l_{p,2p-1}, \quad m_{p,2p}+1 = l_{p,2p}, \\ m_{p-1,2p-1}+1 = l_{p-1,2p-1}, \quad m_{p-1,2p}+2 = l_{p-1,2p} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m_{1,2p-1}+p-1 = l_{1,2p-1}, \quad m_{1,2p}+p = l_{1,2p}, \end{gathered} \tag{51}$$

we define the coefficients A, B and C by the following formulas

$$\begin{aligned} A_{2p-1,j}(m) = \frac{1}{2}\Big[\prod_{r=1}^{p-1} (l_{r,2p-2}-l_{j,2p-1}-1)(l_{r,2p-2}+l_{j,2p-1})\Big]^{1/2} \times \\ \times\Big[\prod_{r=1}^{p} (l_{r,2p}-l_{j,2p-1}-1)(l_{r,2p}+l_{j,2p-1})\Big]^{1/2} \times \\ \times\Big\{\prod_{r\neq j} (l^2_{r,2p-1}-l^2_{j,2p-1})[l^2_{r,2p-1}-(l_{j,2p-1}+1)^2]\Big\}^{-1/2}, \end{aligned}$$

$$B_{2p,j}(m) = \left[\frac{\prod_{r=1}^{p} (l^2_{r,2p-1}-l^2_{j,2p}) \prod_{r=1}^{p+1} (l^2_{r,2p+1}-l^2_{j,2p})}{l^2_{j,2p}(4l^2_{j,2p}-1) \prod_{r\neq j} (l^2_{r,2p}-l^2_{j,2p})[(l_{r,2p}-1)^2-l^2_{j,2p}]}\right]^{1/2}, \tag{52}$$

$$C_{2p} = \frac{\prod_{r=1}^{p} l_{r,2p-1} \prod_{r=1}^{p+1} l_{r,2p+1}}{\prod_{r=1}^{p} l_{r,2p}(l_{r,2p}-1)}.$$

From the commutation relations (43) and the expressions (50) for $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$, we can then obtain the explicit form of any generator X_{ik} of the Lie algebra $o(n)$. By straightforward calculation, as in the case of $u(n)$, we can check that the commutation relations for the generators X_{ik} are fulfilled. Furthermore, if the Gel'fand–Zetlin patterns associated with a given weight m are assumed to be orthonormal, then the generators X_{ij} obey the hermiticity condition

$$X_{ij} = -X_{ji}. \tag{53}$$

EXAMPLE 1. The irreducible representations of the Lie algebra $o(4)$. The Gel'fand–Zetlin pattern in this case has the form

$$m = \begin{vmatrix} m_{13} \ m_{23} \\ m_{12} \\ m_{11} \end{vmatrix} \equiv \begin{vmatrix} m_1 \ m_2 \\ J \\ M \end{vmatrix}, \tag{54}$$

where the numbers m_{13} and m_{23} are the fixed components of the highest weight. It follows from the commutation relations (43) that in order to determine the action of any generator it is sufficient to define the action of X_{21}, X_{32} and X_{43}. Using (50) we get

$$X_{21} m = \mathrm{i} M m,$$

$$X_{43} \begin{vmatrix} m_1 \ m_2 \\ J \\ M \end{vmatrix}$$
$$= \left[\frac{(J+M+1)(J-M+1)(m_1-J)(J-m_2+1)(J+m_2+1)(m_1+J+2)}{(2J+1)(2J+3)(J+1)^2}\right]^{1/2} \times$$
$$\times \begin{vmatrix} m_1 \ m_2 \\ J+1 \\ M \end{vmatrix} + \mathrm{i} M \frac{(m_1+1)m_2}{J(J+1)} \begin{vmatrix} m_1 \ m_2 \\ J \\ M \end{vmatrix} -$$
$$- \left[\frac{(J+M)(J-M)(m_1-J+1)(m_1+J+1)(J-m_2)(J+m_2)}{(2J+1)(2J-1)J^2}\right]^{1/2} \begin{vmatrix} m_1 \ m_2 \\ J-1 \\ M \end{vmatrix},$$

$$X_{32} \begin{vmatrix} m_1 \ m_2 \\ J \\ M \end{vmatrix} = \frac{1}{2}[(J-M)(J+M+1)]^{1/2} \begin{vmatrix} m_1 \ m_2 \\ J \\ M+1 \end{vmatrix} -$$
$$- \frac{1}{2}[(J-M+1)(J+M)]^{1/2} \begin{vmatrix} m_1 \ m_2 \\ J \\ M-1 \end{vmatrix},$$

where according to (49)

$$m_1 \geqslant J \geqslant |m_2|, \quad J \geqslant M \geqslant -J$$

and m_1, m_2, J and M are simultaneously all integers or all half-integers. ▼

The carrier space H^m of the representation (50) or $o(n)$ is spanned by the basis vectors (47) or (48), and is, by definition, irreducible. An independent formal proof using the method of Z-invariants can again be given, as in the case of $u(n)$.

The dimension of an irreducible representation determined by the components of the highest weight is given by Weyl's formula (see eq. 8.8(29)).

The maximal set of commuting operators in the representation space contains the following operators:

(i) $O(2k+2)$:

$$\begin{array}{ccccc} C_2(2k+2) & C_4(2k+2) & \cdots & C_{2k}(2k+2) & C'_k(2k+2) \\ C_2(2k+1) & C_4(2k+1) & \cdots & C_{2(k-1)}(2k+1) & C_{2k}(2k+1) \\ C_2(2k) & C_4(2k) & & C_{2(k-1)}(2k) & C'_k(2k) \\ & & \cdots\cdots\cdots\cdots & & \\ & & C_2(4)\ \ C'_2(4) & & \\ & & C_2(3) & & \\ & & X_{21} & & \end{array} \tag{55}$$

(ii) $O(2k+1)$:

$$\begin{array}{ccccc} C_2(2k+1) & C_4(2k+1) & \cdots & C_{2(k-1)}(2k+1) & C_{2k}(2k+1) \\ C_2(2k) & C_4(2k) & \cdots & C_{2(k-1)}(2k) & C'_k(2k) \\ & C_2(2k-1) & \cdots & C_{2(k-1)}(2k-1) & \\ & & \cdots\cdots\cdots\cdots & & \\ & & C_2(4)\ \ C'_2(4) & & \\ & & C_2(3) & & \\ & & X_{21} & & \end{array} \tag{56}$$

Here

$$C_{2i}(p) = \operatorname{Tr} X^{2i}(p) \tag{57}$$

and

$$C'_l(2l) = \varepsilon^{i_1j_1, i_2j_2, \ldots, i_lj_l} X_{i_1j_1} X_{i_2j_2} \ldots X_{i_lj_l}, \tag{58}$$

where $X^{2i}(p)$ denotes $2i$th power of the matrix $X(p) \equiv (X_{ik}(p))$ which is composed of the generators $X_{ik}(p)$ of the group $O(p)$ and $\varepsilon^{i_1j_1, i_2j_2, \ldots, i_lj_l}$ is the totally anti-symmetric Levi–Civita tensor (cf. 9.4.B). It should be noted that for the group $O(2k)$, in contradistinction to group $O(2k+1)$, the set of Casimir operators (57) does not provide a set of independent invariant operators and therefore the pseudoscalar operator (58) has to be included. The spectra of the operators (57) and (58) were given in ch. 9.4.B.

In general, if the components of the highest weight of $o(n)$ are not restricted by relations other than (44), we have in the representation space spanned by Gel'fand–Zetlin patterns

$$N = \frac{1}{2}\left[\frac{n(n-1)}{2} + \left\{\frac{n-1}{2}\right\}\right] \tag{59}$$

independent commuting operators. where $\frac{1}{2}n(n-1)$ and $\left\{\frac{n-1}{2}\right\}$ are the dimension and the rank of $O(n)$, respectively. However, if the components of the highest weight are not independent then some of the operators (57) or (58) become functions of others and the number of independent commuting operators is smaller. For example, if the highest weight is of the form

$$m(f, 0, 0, \ldots, 0), \tag{60}$$

then by virtue of the structure of the patterns (47) and eq. (56) only $n-1$ commuting operators

$$C_2(n), C_2(n-1), \ldots, C_2(3), X_{21}, \tag{61}$$

generate the ring of the commuting operators in the irreducible representation space of $o(n)$, which is defined by the highest weight (60).

Notice finally that because any highest weight (44) with all integer or all half-integer components gives rise, in the Gel'fand–Zetlin approach, to an irreducible representation, this formalism provides a description of all irreducible finite-dimensional representations of $o(n)$.

§ 2. The Tensor Method

Many physical laws such as the Maxwell or Einstein equations are most concisely stated in the language of tensors, objects transforming according to the finite-dimensional tensor representations of a physical symmetry group G.

In this section we elaborate the connection between the theory of tensor representations and the theory of induced representations. In particular, we give the classification of all irreducible tensor representations of groups $GL(n, C)$ and $SO(n, C)$. Clearly, using the th. 8.3.1 we also obtain a description of all irreducible tensor representations of the real compact and noncompact forms of these groups, e.g., $U(p, q)$ and $SO(p, q)$, $p+q = n$.

The tensor method does not solve directly the practical problems 1°–5° presented in the introduction. It gives, however, a beautiful connection between the representation theory of a group G and the representation theory of the permutation group S_n. This connection cannot be seen by means of other methods.

A. *Tensors*

Let a group G be realized as a matrix group, i.e. $G \ni g \leftrightarrow \{g_i^k = D_i^k\} D_i^k$, $i, k = 1, 2, \ldots, N$. A quantity $T = \{T_{i_1 i_2 \ldots i_r}\}$, $T_{i_1 i_2 i_r} \in C$, $i_k = 1, 2, \ldots, N$, $k = 1, 2, \ldots, r$, is called a *tensor of rank r relative to G* if it has the following transformation law

$$T_{i_1 i_2 \ldots i_r} = D_{i_1}{}^{k_1} D_{i_2}{}^{k_2} \ldots D_{i_r}{}^{k_r} T_{k_1 k_2 \ldots k_r}. \tag{1}$$

If $\overset{1}{T}$ and $\overset{2}{T}$ are tensors of rank r then $c_1\overset{1}{T}+c_2\overset{2}{T}$ is also a tensor of the same rank; hence the set of all tensors of a given rank forms a vector space $\overset{r}{H^T}$. It follows from (1) that $\overset{r}{H^T}$ is the carrier space of the tensor product $T_g \otimes T_g \otimes \ldots \otimes T_g$ (r times). This representation is called *tensor representation.*

The following proposition gives a description of the space $\overset{r}{H^T}$.

PROPOSITION 1. *The components*

$$T_{i_1 i_2 \ldots i_r} = \overset{k_1}{x}_{i_1}\overset{k_2}{x}_{i_2} \ldots \overset{k_r}{x}_{i_r}, \qquad k_s = 1, 2, \ldots, N \tag{2}$$

where $\overset{k}{x}$, $k = 1, 2, \ldots, N$, *are linearly independent complex N-dimensional vectors, span the tensor space* $\overset{r}{H^T}$ *of all tensors T of rank r.*

PROOF: The components $T_{i_1 i_2 \ldots i_r}$ are points of C^{N^r}. Setting $\overset{k}{x} = e_k$, where e_k are basis vectors of C^N, we obtain a basis in C^{N^r}. Hence the vectors (2) form a basis in $\overset{r}{H^T}$. ▼

The basic property of tensors, observed first by Weyl, is that the operation of permutations of indices $i_1, i_2, \ldots, i_r$ commutes with the action of G in H^T. Indeed, if s is an element of the symmetric group S_r, and

$$(sT)_{i_1 i_2 \ldots i_r} \equiv T_{s(i_1 i_2 \ldots i_r)} = T_{s(i_1)s(i_2)\ldots s(i_r)}, \tag{3}$$

then

$$\begin{aligned}(sT')_{i_1 i_2 \ldots i_r} &= T'_{s(i_1)s(i_2)\ldots s(i_r)} \\ &= D_{s(i_1)}{}^{s(k_1)} D_{s(i_2)}{}^{s(k_2)} \ldots D_{s(i_r)}{}^{s(k_r)} T_{s(k_1)s(k_2)\ldots s(k_r)} \\ &= D_{s(i_1)}{}^{s(k_1)} D_{s(i_2)}{}^{s(k_2)} \ldots D_{s(i_r)}{}^{s(k_r)} (sT)_{k_1 k_2 \ldots k_r} \\ &= D_{i_1}{}^{k_1} D_{i_2}{}^{k_2} \ldots D_{i_2}{}^{k_2} \ldots D_{i_r}{}^{k_r} (sT)_{k_1 k_2 \ldots k_r} = (sT)'_{i_1 i_2 \ldots i_r}.\end{aligned}$$

Consequently, those tensors $\{T_{i_1 i_2 \ldots i_r}\}$ whose components have a symmetry property corresponding to a given Young frame form an invariant subspace in the space H^T. Thus tensor representations are in general reducible.

A tensor $T = \{T_{i_1 i_2 \ldots i_r}\}$ is said to be *symmetric* if

$$(sT)_{i_1 i_2 \ldots i_r} \equiv T_{s(i_1 i_2 \ldots i_r)} = T_{i_1 i_2 \ldots i_r}$$

for every permutation $s \in S_r$ of indices.

A tensor is *skew-symmetric* if

$$sT = (-1)^{\delta_s} T, \qquad s \in S_r, \tag{4}$$

where δ_s is the parity of the permutation s. The skew-symmetric tensors with respect to all indices are also called *polyvectors.* Using the Young idempotent operators and eq. (2), we can obtain in some cases explicit compact formulas for components of a tensor with a given symmetry property corresponding to a Young frame. Indeed, for instance, the components of a symmetric tensor correspond to the Young frame

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\ \hline\end{array}$$

and can be represented as elements of a tensor product of the vector x, i.e.,

$$T_{i_1 i_2 \ldots i_r} = x_{i_1} x_{i_2} \ldots x_{i_r} \equiv T_{\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_r \\ \hline\end{array}} \tag{5}$$

Antisymmetric tensors can be represented in terms of determinants; for instance, the quantities

$$e_{ij} = \begin{vmatrix} \overset{1}{x}_i & \overset{1}{x}_j \\ \overset{2}{x}_i & \overset{2}{x}_j \end{vmatrix} \equiv e_{\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}}, \qquad \overset{i}{x} \in C^n,\ i, j = 1, \ldots, n \tag{6}$$

represent the components of a skew-symmetric tensor of rank two associated with the Young frame $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$. In general, the quantities

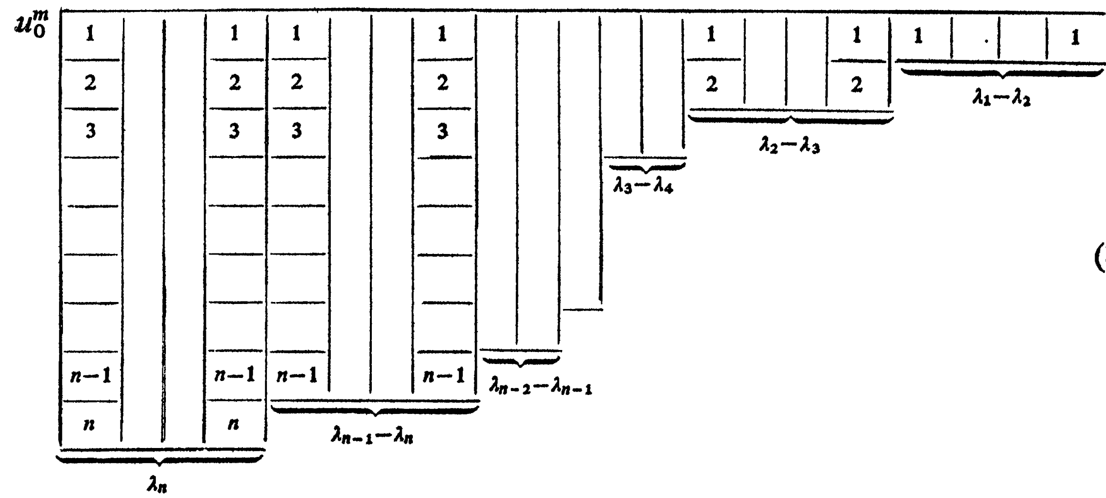
$$e_{i_1 i_2 \ldots i_r} = \begin{vmatrix} \overset{1}{x}_{i_1} & \overset{1}{x}_{i_2} & \ldots & \overset{1}{x}_{i_r} \\ \overset{2}{x}_{i_1} & \overset{2}{x}_{i_2} & \ldots & \overset{2}{x}_{i_r} \\ \cdots & \cdots & \cdots & \cdots \\ \overset{r}{x}_{i_1} & \overset{r}{x}_{i_2} & \ldots & \overset{r}{x}_{i_r} \end{vmatrix} e_{\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_r \\ \hline\end{array}}, \qquad i_1, i_2, \ldots, i_r = 1, \ldots, n \tag{7}$$

represent the components of a skew-symmetric tensor of rank r determined by the Young partition $\overset{r}{\lambda} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$, $r = 1, 2, \ldots, n$. The tensors (7) are *polyvectors*.

One can realize tensors (5) and (7) as functions on the Z factor of the Gauss decomposition $G = \mathfrak{Z}DZ$. Indeed, let G be e.g. $GL(n,C)$. Then Z consists of all lower triangular complex matrices given by eq. 3.6 (7). The polyvectors (7) can be expressed in terms of minors of the elements $z \in Z$ in the form

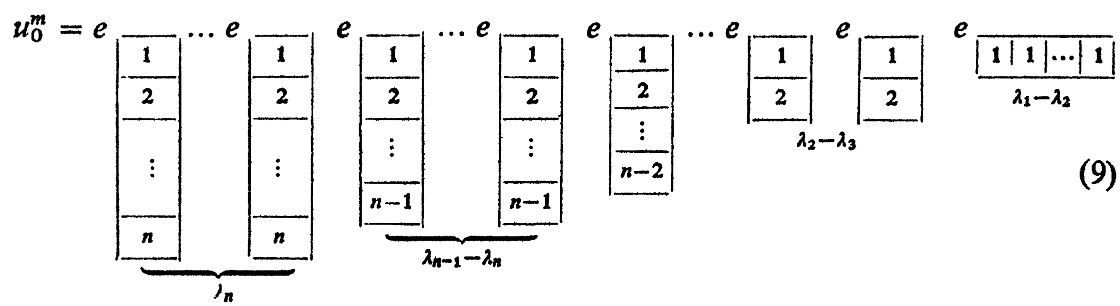
$$e_{\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_r \\ \hline\end{array}} = \begin{vmatrix} z_{i_1 1} & z_{i_1 2} & \ldots & z_{i_1 r} \\ z_{i_2 1} & z_{i_2 2} & \ldots & z_{i_2 r} \\ \cdots & \cdots & \cdots & \cdots \\ z_{i_r 1} & z_{i_r 2} & \ldots & z_{i_r r} \end{vmatrix}, \qquad 1 < r \leqslant n.$$

Similarly one can show that symmetric tensors $T_{\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_r \\ \hline\end{array}}$ can be realized in the class of polynomials of elements of the first colum of the matrices $z \in Z$: $1, z_{21}, z_{31}, \ldots, z_{n1}$ which are homogeneous of the degree r. Among the components of a tensor T with a given Young symmetry λ there exists a distinguished component given by the following Young tableau

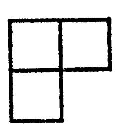
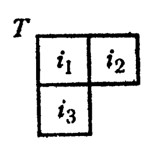
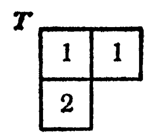
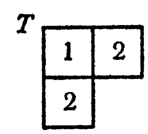


(8)

associated wih a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. This component can be represented by the following product of polyvectors and a symmetric tensor



(9)

We arrange the indices of a tensor with a given Young symmetry $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1+\lambda_2+ \ldots +\lambda_n = r$, in such a way that they correspond to the so-called *standard Young tableaux*. Accordingly, independent components of a tensor T correspond to the standard tableaux which have non-decreasing indices from the left to the right and with increasing indices from the top to the bottom. For instance, for the tensor of order three, associated with the Young frame , we write , where, according to the Young rule $i_1, i_2, i_3 = 1, 2, \ldots, n$ and $i_1 \leqslant i_2$, $i_1 < i_3$; for $n = 2$ we have therefore the following independent components  

In particular, for the components of totally symmetric and totally antisymmetric tensors, according to this convention, we have then, respectively

$$T_{\boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_r}} \qquad \begin{array}{c} i_1, i_2, \ldots, i_r = 1, 2, \ldots, n \\ \text{and } i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r \end{array} \qquad T_{\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_r \\ \hline \end{array}} \qquad \begin{array}{c} i_1, i_2, \ldots, i_r = 1, 2, \ldots, n \\ \text{and } i_1 < i_2 < \ldots < i_r. \end{array} \tag{10}$$

B. *Tensor Representations of* GL(n, C), SL(n, C), GL(n, R), SL(n, R) $U(p, q)$, SU(p, q), $U(n)$, SU(n) *and* SU*(n)

In this section we shall identify an irreducible representation T^{L^m} of GL(n, C) with a certain tensor representation. We first associate a tensor representation to the representation $T^{L^{\overset{i}{m}}}$ with the fundamental weight $\overset{i}{m} = (\underbrace{1, 1, \ldots, 1}_{(i)}, 0, \ldots, 0)$ of GL(n, C).

Let $x = (x_1, \ldots, x_n)$ be an element of the linear space $H^{\overset{1}{T}} = C^n$, in which GL(n, C) is realized as $g \to T_g = D(g) = g$. Let Z be the factor of the Gauss decomposition $G = \overline{\mathfrak{Z}DZ}$. Then the action (1) in $H^{\overset{1}{T}}$ of an element $z \in Z$ is given by the formula

$$zx \equiv \begin{Vmatrix} 1 & & & & \\ z_{21} & 1 & & 0 & \\ z_{31} & z_{32} & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} & z_{n3} & \cdot\;\cdot & 1 \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Vmatrix} = \begin{Vmatrix} x_1 \\ z_{21}x_1 + x_2 \\ z_{31}x_1 + z_{32}x_2 + x_3 \\ \cdots\cdots\cdots \\ z_{n1}x_1 + z_{n2}x_2 + \ldots + x_n \end{Vmatrix}. \tag{11}$$

We see, therefore, that element $T_{\boxed{1}\boxed{\cdots}\boxed{1}} = x_1 \ldots x_1$, in space spanned by components $T_{\boxed{i_1}\boxed{\cdots}\boxed{i_r}}$, is the only invariant vector of Z defined up to a normalization constant. Hence, $T_{\boxed{1}\boxed{\cdots}\boxed{1}}$ is the highest vector by virtue of corollary 1 to th. 8.2.2. For $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in D$ we obtain

$$T_\delta u_0^m = \delta^r u_0^m. \tag{12}$$

Hence $m = (r, 0, \ldots, 0)$. This implies in particular that the vector representation T: $g \to T_g = g$ corresponds to the representation T^{L^m} associated with the fundamental highest weight $m = (0, 0, \ldots, 1)$.

The following theorem shows that all fundamental representations of GL(n, C) can be realized as polyvector representations.

THEOREM 2. *The linear space $H^{\overset{r}{T}}$ of all polyvectors of rank r is the carrier space of the irreducible representation $T^{L^{\overset{r}{m}}}$ of* GL(n, C) *associated with the fundamental highest weight $\overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$.*

PROOF: We can take as the basis vectors in $\overset{r}{H^T}$ the polyvectors (7). Because for $\delta \in D$, $\delta x = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_n x_n)$, we have

$$(T_\delta e)_{i_1 i_2 \ldots i_r} = \delta_{i_1} \delta_{i_2} \ldots \delta_{i_r} e_{i_1 i_2 \ldots i_r}, \tag{13}$$

i.e., every basis vector $e_{i_1 i_2 \ldots i_r}$ is a weight vector. Using eqs. (11) and (7) we verify that the vector $e_{12\ldots r}$ is the only invariant of the subgroup Z. Hence by virtue of corollary 2 to th. 8.2.2, it is the highest vector of an irreducible representation. By virtue of (13) the corresponding integral highest weight is $\delta_1 \delta_2 \ldots \delta_r$; hence the irreducible representation $T^{L^{\overset{r}{m}}}$ in $\overset{r}{H^T}$ is determined by the highest weight $m = \overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$. ▼

Consequently, the representation (1) of $\mathrm{GL}(n, C)$ in the space of all polyvectors $\{e_{i_1 i_2 \ldots i_r}\}$ of rank r corresponds to the representation $T^{L^{\overset{r}{m}}}$ determined by the fundamental weight $\overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$.

Furthermore, by def. 8.6.1, and th. 8.6.1 we have

COROLLARY 1. *Every irreducible representation of* $\mathrm{GL}(n, C)$ *is a Young product of polyvector representations.* ▼

According to eq. (4), the linear space $\overset{r}{H^T}$ spanned by the components of tensor of a rank r with a given Young symmetry, forms an invariant subspace relative to G. Thus, we expect a close connection between the induced representations T^{L^m} and the tensor representations of G realized in the space $\overset{r}{H^T}$. In fact, we have

THEOREM 3. *Let $\overset{r}{H^T}$ be the space of tensors of rank r with respect to* $\mathrm{GL}(n, C)$ *which have the symmetry defined by the Young partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $r = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Then, the representation of* $\mathrm{GL}(n, C)$ *realized in $\overset{r}{H^T}$ is equivalent to the representation T^{L^m} with the highest weight*

$$m = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

PROOF: Let $\overset{1}{x}, \overset{2}{x}, \ldots, \overset{r}{x}$ be a set of r independent vectors from C^n. The tensor $\tilde{T}$ with components $\tilde{T}_{i_1 i_2 \ldots i_r} = \overset{1}{x}_{i_1} \overset{2}{x}_{i_2} \ldots \overset{r}{x}_{i_r}$ has no symmetry with respect to the symmetric group S_r. Hence, acting on $\tilde{T}_{i_1 i_2 \ldots i_r}$ by the Young idempotent operator Y_λ which corresponds to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = r$, we obtain a non-zero tensor T with this Young symmetry: hence $Y_\lambda \tilde{T} \in \overset{r}{H^T}$. We claim that the element u_0^m of $\overset{r}{H^T}$ whose only non vanishing component is given by eq. (8) is invariant relative to the action of the subgroup Z of $\mathrm{GL}(n, C)$.

Indeed, consider the action of the one-parameter subgroup of Z in $\overset{r}{H^T}$ given for instance by matrices

$$z = \begin{bmatrix} 1 & & & 0 \\ z & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}. \tag{15}$$

By virtue of eq. (1), $z\overset{k}{x} = (\overset{k}{x}_1, z\overset{k}{x}_1+\overset{k}{x}_2, \overset{k}{x}_3, \ldots, \overset{k}{x}_n)$, $k = 1, 2, \ldots, r$. This gives

$$(T_z e)_{123\ldots r} = e_{123\ldots r} + z e_{113\ldots r} = e_{123\ldots r}. \tag{16}$$

Using the representation (9) for the component (8) and eq. (16) we conclude that the component (8) is invariant under the action of the subgroup (15). Using eq. (11) one immediately verifies that the action of any other one-parameter subgroup of Z also leaves the component $e_{12\ldots r}$ as well as the component $T_{\boxed{1}\cdots\boxed{1}}$ unchanged. Consequently the component (8) is Z-invariant and therefore the element u_0^m represents the highest vector in $\overset{r}{H^T}$. The inspection of the action of the projector Y_λ shows that for any element T in $\overset{r}{H^T}$ other than u_0^m one can easily find a one-parameter subgroup of Z, which changes this element. Hence, the linear space $\overset{r}{H^T}$ has only one Z-invariant element and therefore is irreducible according to corollary 2 to th. 8.2.2.

For $\overset{k}{x} \in C^n$, $k = 1, 2, \ldots, r$, and $\delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} \in D$, we have

$$\delta\overset{k}{x} = (\delta_1 \overset{k}{x}_1, \delta_2 \overset{k}{x}_2, \ldots, \delta_n \overset{k}{x}_n).$$

Thus, by virtue of eqs. (8) and (14) the action of T_δ on the component (8) of the tensor T is given by

$$T_\delta u_0^m = \delta_1^{\lambda_1} \delta_2^{\lambda_2} \ldots \delta_n^{\lambda_n} u_0^m. \tag{17}$$

Consequently, by eq. 8.2(18) the highest weight m associated with the irreducible representation $g \to T_g$ in $\overset{r}{H^T}$ has the form $m = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. ▼

Clearly, if $m = \overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$, then th. 3 reduces to th. 2. If $m = (f, 0, \ldots, 0)$, the representation T^{L^m} can be realized in the space of totally symmetric tensors of order f.

It is interesting that in quantum physics we use totally symmetric representations for a collection of bosons, and totally antisymmetric ones for a collection of fermions. Representations with other symmetry types are used in the so-called *parastatistics*, or in the description of systems with 'hidden' variables.

Remark: All irreducible tensor representations of $\mathrm{GL}(n, C)$ remain irreducible, when restricted to the subgroups $\mathrm{GL}(n, R)$; $U(p, q)$, $p+q = n$, $U(n)$, $\mathrm{SL}(n, C)$, $\mathrm{SL}(n, R)$, $\mathrm{SU}^*(2n)$, $\mathrm{SU}(p, q), p+q = n$, or $\mathrm{SU}(n)$ as a result of the Weyl unitary trick.

C. *Tensor Representations of* SO(n, C), SO(n), SO(p, q) *and* SO*(n)

The elements of the orthogonal group $\mathrm{SO}(n, C)$ are matrices that satisfy the additional condition

$$gg^T = g^T g = I, \quad \text{or} \quad g_{ij}g_{ik} = \delta_{jk}. \tag{18}$$

The condition (18) implies that there is a new operation which commutes with the group action in the linear space $\overset{r}{H^T}$ spanned by the components $T_{i_1 i_2 \dots i_r}$ of a tensor T. Indeed, consider a contraction (trace) of the tensor given by the formula

$$T^{(12)}_{i_3 i_4 \dots i_r} \equiv T_{iii_3 i_4 \dots i_r} = \delta_{i_1 i_2} T_{i_1 i_2 i_3 \dots i_r}. \tag{19}$$

This operation of contraction and the group transformation by an element $g \in \mathrm{SO}(n, C)$ commute:

$$\begin{aligned}(T_g T)^{(12)}_{i_3 i_4 \dots i_r} &= g_{ik_1} g_{ik_2} g_{i_3 k_3} \cdots g_{i_r k_r} T_{k_1 k_2 \dots k_r} = \delta_{k_1 k_2} g_{i_3 k_3} \cdots g_{i_r k_r} T_{k_1 k_2 k_3 \dots k_r} \\ &= g_{i_3 k_3} \cdots g_{i_r k_r} T^{(12)}_{k_3 k_4 \dots k_r} = (T_g T^{(12)})_{i_3 i_4 \dots i_r}.\end{aligned} \tag{20}$$

The operation of contraction can be applied to any pair of indices of a tensor $\{T_{i_1 i_2 \dots i_r}\}$. A tensor $\{T_{i_1 i_2 \dots i_r}\}$ is said to be *traceless* if the contraction of any pair of indices vanishes. By virtue of eq. (20) the traceless tensors transform among themselves under $\mathrm{SO}(n, C)$, hence form an invariant subspace. Moreover, we have

PROPOSITION 4. *Every tensor* $\{T_{i_1 i_2 \dots i_r}\}$ *can be decomposed uniquely into a traceless tensor* $\overset{\circ}{T}$ *and a tensor* Q *with components*

$$\begin{aligned}Q_{i_1 i_2 \dots i_r} = \delta_{i_1 i_2} R^{(12)}_{i_3 i_4 \dots i_r} + \dots + \delta_{i_p i_q} R^{(pq)}_{i_1 \dots i_{p-1} i_{p+1} \dots i_{q-1} i_{q+1} \dots i_r} + \dots + \\ + \delta_{i_{r-1} i_r} R^{(r-1, r)}_{i_1 i_2 \dots i_{r-2}} \left(\frac{r(r-1)}{2} \text{ terms}\right),\end{aligned} \tag{21}$$

i.e.,

$$T = \overset{\circ}{T} + Q.$$

This decomposition is invariant under $\mathrm{SO}(n, C)$. ▼

PROOF: We introduce in the space $\overset{r}{H^T}$ of all tensors of rank r a scalar product $(\cdot, \cdot)$ given by the formula

$$(T, T') = T_{i_1 i_2 \dots i_r} \overline{T'}_{i_1 i_2 \dots i_r}. \tag{22}$$

Let $\mathscr{K}$ be the subspace of $\overset{r}{H^T}$ consisting of all tensors of the form (21). A tensor T is orthogonal to the subspace $\mathscr{K}$, i.e.,

$$(T, Q) = T_{i_1 i_2 \dots i_r} \bar{Q}_{i_1 i_2 \dots i_r} = 0, \tag{23}$$

if all traces of T vanish. Indeed, for a particular $Q \in \mathscr{K}$ such that only $R^{(12)} \neq 0$, eq. (23) implies that $T^{(12)}$ must be zero. By taking successive non-vanishing terms in eq. (21) we see that all traces $T^{(pq)}$ of T must be zero. Consequently, a linear set of all traceless tensors $\mathring{T} \in \overset{r}{H^T}$ forms a subspace orthogonal to $\mathscr{K}$. Because the whole space $\overset{r}{H^T}$ is the sum of $\mathscr{K}$ and $\mathscr{K}^\perp$, an arbitrary tensor T can be uniquely represented in the form

$$T = \mathring{T} + Q, \qquad \mathring{T} \in \mathscr{K}^\perp,\ Q \in \mathscr{K}. \tag{24}$$

Because $\mathring{T}$ form an invariant subspace, the decomposition (24) is invariant under $\mathrm{SO}(n, C)$ by the Weyl theorem on full reducibility. ▼

Applying successively the proposition 4 to the tensors $R^{(pq)}$, etc., we obtain the invariant decomposition of any tensor T onto traceless tensors of rank r, $r-2$, $r-4$, and so on.

Under the action of the symmetric group S_r a traceless tensor is transformed into another traceless tensor. Hence, we expect that irreducible representations of $\mathrm{SO}(n, C)$ are realized in the linear subspace of $\overset{r}{H^T}$ spanned by the components of a traceless tensor $\{T_{i_1 i_2 \dots i_r}\}$ with a given Young symmetry defined by the partition $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of the number $r = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

We first find the induced representations corresponding to the polyvector representations.

THEOREM 5. *The tensor representation of* $\mathrm{SO}(n, C)$, $n = 2\nu$ *or* $n = 2\nu+1$, *realized in the linear space* $\overset{r}{H^T}$ *of all polyvectors of rank r associated with the Young frame* $\overset{r}{\lambda} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0)$ *is equivalent to the representation* $T^{L^{\overset{r}{m}}}$ *associated with the highest weight* $\overset{r}{m} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0)$. *The tensor representations* $\overset{r}{\lambda}$ *and* $\overset{n-r}{\lambda}$ *are equivalent.*

PROOF: The space $\overset{r}{H^T}$ is spanned by the basis vectors $\hat{e}_{i_1 i_2 \dots i_r}$ defined in the proof of th. 2. Because $\delta x = (\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$, we get

$$(T_\delta \hat{e})_{i_1 i_2 \dots i_r} = \delta_{i_1} \delta_{i_2} \dots \delta_{i_r} \hat{e}_{i_1 i_2 \dots i_r}, \tag{25}$$

i.e., every vector $\hat{e}_{i_1 i_2 \dots i_r}$ is a weight vector. The group $\mathrm{SO}(n, C)$ conserves the form

$$x_1 x_n + x_2 x_{n-1} + \dots + x_n x_1 \tag{26}$$

so that the one-parameter subgroups of Z have now the form, e.g.,

$$z = \begin{bmatrix} 1 & 0 & & & & & 0 \\ z & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 1 & 0 \\ 0 & & & & & -z & 1 \end{bmatrix} \tag{27}$$

and all other one-parameter subgroups of Z are obtained by shifting properly the 'active blocks' $\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix}$. The action of (27) on a vector $x \in C^n$ follows from (11). This implies, as in (16),

$$(T_z e)_{123\ldots r} = e_{123\ldots r} + ze_{113\ldots r} - ze_{123\ldots n-1,n-1} = e_{123\ldots r} \tag{28}$$

i.e., T_z reproduces $e_{123\ldots r}$. An analogous result holds for all other one-parameter subgroups of Z acting of the vector $e_{12\ldots r}$. Thus $e_{12\ldots r}$ is the invariant of the subgroup Z. One can verify that any vector in $\overset{r}{H^T}$ other than $e_{12\ldots r}$ is not invariant under Z. Hence $e_{12\ldots r}$ is the highest vector. By virtue of eq. (25) the corresponding highest weight has the form

$$m = \overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0).$$

Now let $\{e_{i_1 i_2 \ldots i_{n-r}}\}$, $r < \nu$, be the polyvector associated with the Young frame $\overset{n-r}{\lambda} = (\underbrace{1, 1, \ldots, 1}_{(n-r)}, 0, \ldots, 0)$. Then $e_{12\ldots n-r}$ is the highest vector, by the previous argument. The corresponding integral highest weight, according to eq. (25), has the form $L^m = \delta_1 \delta_2 \ldots \delta_{n-r}$. But for SO$(n, C)$ the element δ of the subgroup D has to conserve the form (26); this implies the restriction $\delta_i = \delta_{n-i+1}$ for arbitrary n, and the additional restriction $\delta_{\nu+1} = 1$, for $n = 2\nu+1$. Hence $L^{\overset{n-r}{m}} = \delta_1 \delta_2 \ldots \delta_r = L^{\overset{r}{m}}$, $r < \nu$. Consequently the representations determined by the Young frames $\overset{r}{\lambda}$ and $\overset{n-r}{\lambda}$, $r < \nu$, are equivalent, by virtue of th. 8.2.2. ▼

Remark 1: Notice that for SO$(2\nu+1)$ the highest weight $\overset{r}{m}$, $r < \nu$, corresponding to the polyvector representation $\overset{r}{\lambda}$, coincides with the fundamental highest weight $\overset{r}{m}$ given by eq. 8.6(4). However, the polyvector representation $\overset{(\nu)}{\lambda} = (1, 1, \ldots, 1) \equiv \overset{(\nu)}{m}$ is given by the Young product of spinor representations—indeed, from eq. 8.6 (4), we have

$$L^{\overset{(\nu)}{m}} = L^{\overset{\nu}{m}} \cdot L^{\overset{\nu}{m}}. \tag{29}$$

In the case of the group SO(2ν) the polyvector representation $\overset{r}{\lambda}$ coincides with the fundamental representation 8.6(5) for $r < \nu-1$. The representations $\overset{(\nu-1)}{\lambda}$ and $\overset{(\nu)}{\lambda}$ are Young products of spinor representations: indeed, according to 8.6(5) we have

$$\overset{(\nu-1)}{L^m} = \overset{\nu-1}{L^m}\overset{\nu}{L^m} \quad \text{and} \quad \overset{(\nu)}{L^m} = \overset{\nu-1}{L^m}\overset{\nu-1}{L^m}. \tag{30}$$

Remark 2: The equivalence of the polyvector representations $\overset{r}{\lambda}$ and $\overset{n-r}{\lambda}$, $r < \nu$, is a generalization of the fact, well-known in physics, of equivalence under rotation between a three-dimensional vector $\{T_i\}$ ($\sim \overset{1}{\lambda} = (1\ 0\ 0)$), and the skew-symmetric tensor $\{T_{ij}\}$($\sim \overset{2}{\lambda} = (1\ 1\ 0)$). ▼

The following theorem gives the connection between the tensor representations and the induced irreducible representations:

THEOREM 6. *The representation* T^{L^m} *of* SO(n, C), $n = 2\nu$ *or* $n = 2\nu+1$, *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ (*with integer components*) *is equivalent to the tensor representation realized in the space of traceless tensors*:

$$\{T_{i_1 i_2 \ldots i_r}\}, \qquad r = \sum_{i=1}^{\nu} m_i,$$

with the Young symmetry defined by the partition $\lambda = (m_1, m_2, \ldots, m_\nu)$.

PROOF: The representation T^{L^m} for which m_i are positive integers can be realized as the Young product of representations $T^{L^{\overset{k}{m}}}$ with $\overset{k}{m} = (\underbrace{1, 1, \ldots, 1}_{(k)}, 0, \ldots, 0)$, $k = 1, 2, \ldots, \nu$. The carrier space of a Young product of representations is spanned by the product of basis vectors of $T^{L^{\overset{(k)}{m}}}$ (cf. remark below eq. 8.6(2)). Hence the component

$$\underbrace{e_{12\ldots\nu} \cdots e_{12\ldots\nu}}_{m_\nu \text{ factors}} \cdots \underbrace{e_{12} \cdots e_{12}}_{m_2-m_3} \cdots T_{\boxed{1\,|\,2\,|\ldots|\,m_1-m_2}} \tag{31}$$

where $e_{12\ldots s}$ are given by eq. (7), is the highest vector u_0^m with $m = (m_1, m_2, \ldots, m_r)$. This vector has the Young symmetry defined by the partition $\lambda = (m_1, m_2, \ldots, m_r)$. The tensor space associated with the representation T^{L^m} is obtained from the highest vector (31) by the formula 8.2(21). Since the group action commutes with permutations, the obtained tensor has the symmetry defined by partition $\lambda = (m_1, m_2, \ldots, m_r)$. Finally, since the carrier space is irreducible, the obtained tensor must be traceless. ▼

The representations of SO(n, C) determined by the highest weights with half-integer coefficients are spinor representations and cannot be described in the language of Young diagrams. The remaining highest weights have integer components. Hence th. 6 gives the description of all tensor representations of SO(n, C). This provides also, by the th. 8.3.1, the description of all irreducible tensor representations of SO(n), SO(p, q), $p+q = n$ and SO*(n).

§ 3. The Method of Harmonic Functions

We have shown that for the rotation group SO(3) the basis vectors e^J_M of the carrier space H^J of an irreducible representation can be realized in terms of spherical harmonic functions $Y^J_M(\vartheta, \varphi)$ defined over a symmetric space $X = \mathrm{SO}(3)/\mathrm{SO}(2)$. This realization is extremely useful in the solution of various problems of representation theory and in physical applications. We expect therefore that a realization of the basis vectors of the carrier space H as general harmonic functions* which are the eigenfunctions of a maximal set of commuting operators in H, will also be useful for other groups.

A. *Harmonic Functions for* SO(p).

We consider in detail in this subsection the representation theory of the group SO(p) in the framework of harmonic functions. The approach is in fact a direct extension of the harmonic functions method for SO(3).

Symmetric Spaces and Representation Theory

It follows from Table I in ch. 4.2 that there are two series of symmetric spaces X which can be related to the group SO(n):

X	rank	dimension of X	
$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(p)$	$\min(p, q)$	pq	(1)
$\mathrm{SO}(2n)/U(n)$	$[\frac{1}{2}n]$	$n(n-1)$	

According to the Gel'fand–Chevalley theorem (cf. 9.6.B.(v)) the number of generators of the ring of invariant operators in the enveloping algebra in the space $H = L^2(X, \mu)$ is equal to the rank of the symmetric space X. Thus construction of the representations will be simplest in the symmetric spaces of rank one. The inspection of Table I shows that we obtain symmetric spaces of rank one for SO(p) only in the following cases:

$$X = \mathrm{SO}(p)/\mathrm{SO}(p-1), \quad X_1 = \mathrm{SO}(4)/U(2) \quad \text{and} \quad X_2 = \mathrm{SO}(6)/U(3). \tag{2}$$

The th. 9.6.2 states that in symmetric spaces of rank one the ring of invariant operators in the enveloping algebra is generated by the *Laplace–Beltrami operator* only. This operator has the form

$$\varDelta(x) = |\bar{g}|^{-1/2}\partial_\alpha g^{\alpha\beta}(x)|\bar{g}|^{1/2}\partial_\beta, \tag{3}$$

where $g_{\alpha\beta}(x)$ is the left-invariant metric tensor on X and $\bar{g}(x) = \det[g_{\alpha\beta}(x)]$.

Let $\psi_\lambda(x)$ be the eigenfunctions of $\varDelta(x)$, i.e.,

$$\varDelta(x)\psi_\lambda(x) = \lambda\psi_\lambda(x). \tag{4}$$

* The term 'harmonic functions' is sometimes used (in mathematical literature) as eigenfunctions of the invariant operators $C_1, \ldots, C_N$ with zero eigenvalue, i.e. as solutions of the equation $C_p u = 0$, $p = 1, 2, \ldots, N$ [e.g. $\varDelta u = 0$]. We shall use this term for all eigenfunctions of C_p.

Then because any generator Y of SO(p) commutes with $\Delta(x)$.

$$[\Delta(x), Y] = 0,$$

the linear hull H^{λ} of all functions $\psi_{\lambda}(x)$ forms an invariant subspace of $H(X)$. Thus the construction of irreducible representations in $H(X)$ may be performed along the following steps.

(i) Construction of a convenient model of the abstract quotient space (2), and the selection of a proper coordinate system on X such that the metric tensor $g_{\alpha\beta}(x)$ is diagonal.

(ii) Solution of the eigenfunction problem (4) for the Laplace–Beltrami operator $\Delta(x)$.

(iii) Proof of the irreducibility and unitarity of the representations $T_g\psi_{\lambda}(x) = \psi_{\lambda}(g^{-1}x)$ associated to with the set of harmonic functions $\{\psi_{\lambda}(x)\}$, λ fixed.

The Construction of Harmonic Functions

First we shall introduce a convenient model of the abstract quotient space $X = \mathrm{SO}(p)/\mathrm{SO}(p-1)$. A model must be a transitive manifold with respect to the group and have the same dimension and stability group as the given symmetric space X. We choose the sphere S^{p-1} embedded in the p-dimensional Euclidean space R^p, given by the equation

$$(x^1)^2+(x^2)^2+ \dots +(x^p)^2 = 1. \tag{5}$$

The transitivity of the sphere S^{p-1} with respect to the group SO(p) follows from the fact that any real vector $x = (x^1, x^2, \dots, x^p)$ fulfilling eq. (5) can be attained from the vector $e^1 = [1, 0, 0, \dots, 0]$ by the rotation matrix $g(x)$ for which the first column $g^i{}_1(x) = x^i$. Therefore any two vectors x' and x'' obeying eq. (5) can be related by the rotation matrix $g = g(x')g(x'')^{-1}$. The stability group of the point $e^1 = [1, 0, 0, \dots, 0]$ is the group SO($p-1$). The stability group of any other point x of the hypersurface S^{p-1} is the group

$$G_0 \simeq g(x)\mathrm{SO}(p-1)g^{-1}(x)$$

which, for fixed $g(x)$, is isomorphic to SO($p-1$). Consequently the stability group of the sphere S^{p-1} is the same as the stability group of the abstract quotient space (2). The dimension of X defined by (2) is

$$\dim X = \dim \mathrm{SO}(p) - \dim \mathrm{SO}(p-1) = p-1$$

and equals to the dimension of the sphere S^{p-1}.

Generally there exists a large number of different coordinate systems on the sphere S^{p-1}. It turns out, however, that the most convenient one is the *biharmonic coordinate system* because in this system, not only can the Laplace–Beltrami operator be separated but also the Cartan subalgebra is diagonal and therefore the harmonic functions can be expressed solely in terms of well-known exponentials and $d^J_{N,M}(\cos\vartheta)$-functions.

We shall construct the biharmonic coordinate system on the sphere S^{p-1} by means of a recursion formula. Suppose that p is even ($p = 2n$) and that we have constructed a coordinate system for $x'^1, \ldots, x'^{2k-2}$ ($k \leqslant n$). Then the expression for the variables $x^1, \ldots, x^{2k}$ obeying eq. (5) is given by

$$\begin{aligned} x^i &= x'^i \sin\vartheta^k, \quad & i &= 1, 2, \ldots, 2k-2, \\ x^{2k-1} &= \cos\varphi^k \cos\vartheta^k, \quad & \varphi^k &\in [0, 2\pi),\ k = 2, 3, \ldots, n, \\ x^{2k} &= \sin\varphi^k \cos\vartheta^k, \quad & \vartheta^k &\in \left[0, \frac{\pi}{2}\right),\ k = 2, 3, \ldots, n. \end{aligned} \tag{5a}$$

Therefore, putting

$$\begin{aligned} x'^1 &= \cos\varphi^1, \\ x'^2 &= \sin\varphi^1, \quad \varphi^1 \in [0, 2\pi), \end{aligned}$$

and applying successively the procedure (5a), we obtain the parametrization of all coordinates for the sphere S^{p-1} for an arbitrary even dimension p.

If p is odd ($p = 2n+1$) we first construct the coordinates x'^i ($i = 1, 2, \ldots, 2n$) by means of the method described above for $p = 2n$; we then obtain the corresponding x^k, $k = 1, 2, \ldots, 2n+1$, by

$$\begin{aligned} x &= x'^i \sin\vartheta^{n+1}, \quad & i &= 1, 2, \ldots, 2n, \\ x^{2n+1} &= \cos\vartheta^{n+1}, \quad & \vartheta^{n+1} &\in [0, 2\pi). \end{aligned} \tag{6}$$

In what follows we denote the set of angles $(\varphi^1, \varphi^2, \ldots, \varphi^{[p/2]}, \vartheta^2, \vartheta^3, \ldots, \vartheta^{\{p/2\}})$ by ω, where the brackets around the indices are defined as follows:

$$\left[\frac{p}{2}\right] = \begin{cases} \dfrac{p}{2} \text{ if } p = 2n, \\[2mm] \dfrac{p-1}{2} \text{ if } p = 2n+1, \end{cases} \quad \text{and} \quad \left\{\frac{p}{2}\right\} = \begin{cases} \dfrac{p}{2} \text{ if } p = 2n, \\[2mm] \dfrac{p+1}{2} \text{ if } p = 2n+1, \end{cases} \tag{7}$$

$$n = 1, 2, \ldots$$

The metric tensor $g_{\alpha\beta}(S^{p-1})$ on the sphere S^{p-1} is induced by the metric tensor $g_{ab}(R^p)$ on the Euclidean space R^p in which the sphere S^{p-1} is embedded and is given by

$$g_{\alpha\beta}(S^{p-1}) = g_{ik}(R^p)\, \partial_\alpha x^i(\omega)\, \partial_\beta x^k(\omega), \quad \alpha, \beta = 1, 2, \ldots, p-1, \tag{8}$$

where

$$g_{ik}(R^p) = \delta_{ik}, \quad i, k = 1, 2, \ldots, p,$$

and ∂_α denotes partial differentiation with respect to an angle φ^α, for $\alpha = 1, 2, \ldots, \left[\frac{p}{2}\right]$, and with respect to $\vartheta^2, \ldots, \vartheta^{\{p/2\}}$ for $\alpha = \left[\frac{p}{2}\right], \left[\frac{p}{2}\right]+1, \ldots$ $\ldots, \left[\frac{p}{2}\right] + \left\{\frac{p}{2}\right\} - 1$ respectively.

Using formulae (8), we find that the metric tensor $g_{\alpha\beta}(S^{p-1})$ is diagonal when expressed in the biharmonic coordinate system and has the following form:

(i) SO($2n$)

$$g_{\varkappa\lambda}(S^{2n-1}) = \sin^2\vartheta^n g_{\varkappa\lambda}(S^{2n-3}), \qquad \varkappa, \lambda = 1, 2, \ldots, 2n-3,$$

and

$$\begin{aligned} g_{\alpha, 2n-1}(S^{2n-1}) &= \cos^2\vartheta^n \delta_{\alpha, 2n-1}, \qquad \alpha = 1, 2, \ldots, 2n, \\ g_{\varepsilon, 2n-2}(S^{2n-1}) &= \delta_{\varepsilon, 2n-2}, \qquad \varepsilon = 1, 2, \ldots, 2n-2. \end{aligned} \tag{9}$$

(ii) SO($2n+1$)

$$g_{\varkappa\lambda}(S^{2n}) = \sin^2\vartheta^{n+1} g_{\varkappa\lambda}(S^{2n-1}), \qquad \varkappa, \lambda = 1, 2, \ldots, 2n-1,$$

and

$$g_{\alpha, 2n}(S^{2n}) = \delta_{\alpha, 2n}, \qquad \alpha = 1, 2, \ldots, 2n. \tag{10}$$

Because the metric tensor $g_{\alpha\beta}(S^n)$ is diagonal in this coordinate system, the Laplace–Beltrami operator (3) can be decomposed into two or three parts:

(i) SO($2n$):

for $n = 1$:

$$\Delta(S^1) = \partial^2/\partial(\varphi^1)^2,$$

and for $n = 2, 3, \ldots$

$$\Delta(S^{2n-1}) = (\cos^2\vartheta^n)^{-1}\frac{\partial^2}{\partial\varphi^{n2}} + (\sin^{2n-3}\vartheta^n\cos\vartheta^n)^{-1}\frac{\partial}{\partial\vartheta^n}(\sin^{2n-3}\vartheta^n\cos\vartheta^n)\frac{\partial}{\partial\vartheta^n} + (\sin^2\vartheta^n)^{-1}\Delta(S^{2n-3}).$$

(ii) SO($2n+1$), $n = 1, 2, \ldots$

$$\Delta(S^{2n}) = (\sin^{2n-1}\vartheta^{n+1})^{-1}\frac{\partial}{\partial\vartheta^{n+1}}(\sin^{2n-1}\vartheta^{n+1})\frac{\partial}{\partial\vartheta^{n+1}} + \frac{\Delta(S^{2n-1})}{\sin^2\vartheta^{n+1}}.$$

Here, $\Delta(S^{2n-3})$ and $\Delta(S^{2n-1})$ are again the invariant Laplace–Beltrami operators for the groups SO($2n-2$) and SO($2n$), respectively, which can be decomposed further in a similar manner. Therefore, by the method of separation of variables we can express any eigenfunction of the invariant operator $\Delta(S^p)$ as a product of functions of one variable only. Because the Laplace–Beltrami operator $\Delta(S^{p-1})$ is equal to the second-order Casimir operator 9.6(48) of SO(p), up to a factor -2 its eigenvalues $\lambda_{\{\frac{p}{2}\}}$ are given by eq. 9.4(64). Using this formula one obtains*

$$\lambda_{\{\frac{p}{2}\}} = -l_{\{\frac{p}{2}\}}(l_{\{\frac{p}{2}\}} + p - 2), \qquad \begin{aligned} &p = 3, 4, 5, \ldots, \\ &l_{\{\frac{p}{2}\}} = 0, 1, 2, \ldots \end{aligned} \tag{11}$$

Due to the inductive construction of the Laplace–Beltrami operator we can separate variables in the eigenvalue problem for the operator $\Delta(S^{p-1})$ and finally

* For the proof that this representation is associated with the highest weight $m = (l_{\{\frac{p}{2}\}}, 0, \ldots, 0)$ see subsec. B.

obtain the second-order ordinary differential equation:

$$\left[\frac{1}{\sin^{(2n-3)}\vartheta^n\cos\vartheta^n}\frac{d}{d\vartheta^n}\sin^{(2n-3)}\vartheta^n\cos\vartheta^n\frac{d}{d\vartheta^n}-\frac{m_n^2}{\cos^2\vartheta^n}-\right.$$

$$\left.-\frac{l_{n-1}(l_{n-1}+2n-4)}{\sin^2\vartheta^n}+l_n(l_n+2n-2)\right]\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=0, \qquad (12)$$

if $p=2n$, or

$$\left[\frac{1}{\sin^{(2n-1)}\vartheta^{n+1}}\frac{d}{d\vartheta^{n+1}}\sin^{(2n-1)}\vartheta^{n+1}\frac{d}{d\vartheta^{n+1}}-\frac{l_n(l_n+2n-2)}{\sin^2\vartheta^{n+1}}+\right.$$

$$\left.+l_{n+1}(l_{n+1}+2n-1)\right]\psi^{l_{n+1}}_{l_n}(\vartheta^{n+1})=0, \qquad (13)$$

if $p=2n+1$.

Because the spectrum (11) of $\Delta(S^{p-1})$ is purely discrete solutions of eqs. (12) or (13) belong to the Hilbert space of square integrable functions with respect to the measure

$$d\mu(S^{p-1})=|\bar{g}(S^{p-1})|^{1/2}d\omega=\begin{cases}\prod\limits_{k=2}^{n}\cos\vartheta^k\sin^{(2k-3)}\vartheta^k\,d\vartheta^k\prod\limits_{l=1}^{n}d\varphi^l, & p=2n,\\ \sin^{2n-1}\vartheta^{n+1}d\vartheta^{n+1}\prod\limits_{k=2}^{n}\cos\vartheta^k\sin^{(2k-3)}\vartheta^k d\vartheta^k\times & \\ \qquad\times\prod\limits_{l=1}^{n}d\varphi^l, & p=2n+1.\end{cases} \qquad (14)$$

Setting in eq. (12)

$$\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=\tan^{|l_{n-1}|}\vartheta^n\cos^{l_n}\vartheta^n u(\vartheta^n)$$

and introducing the new variable $y=-\tan^2\vartheta^n$ we reduce eq. (12) to the standard hypergeometric equation and we obtain

$$\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=\tan^{|l_{n-1}|}\vartheta^n\cos^{l_n}\vartheta^n\times$$

$$\times{}_2F_1\left[\frac{1}{2}(|l_{n-1}|-l_n+m_n),\frac{1}{2}(|l_{n-1}|-l_n-m_n),l_{n-1}+n-1;-\tan^2\vartheta^n\right], \qquad (15)$$

for $p=2$

$$\psi_{m_1}(\varphi^1)=(2\pi)^{-1/2}\exp(im_1\varphi^1),$$

where l_n, l_{n-1}, m_n are restricted by the condition that $\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)$ is a square integrable function with respect to the measure (14), i.e.,

$$|m_n|+|l_{n-1}|=l_n-2s, \quad s=0,1,\ldots,[\tfrac{1}{2}l_n]. \qquad (16)$$

Similarly setting in eq. (13) $\psi_{l_n}^{l_{n+1}}(\vartheta^{n+1}) = \tan^{l_n}(\vartheta^{n+1})\cos l_{n+1}(\vartheta^{n+1})u(\vartheta^{n+1})$ we reduce it to the hypergeometric equation and we obtain:

$$\psi_{l_n}^{l_{n+1}}(\vartheta^{n+1}) = \tan^{l_n}\vartheta^{n+1}\cos^{l_{n+1}}\vartheta^{n+1} \times \\ \times {}_2F_1[\tfrac{1}{2}(l_n-l_{n+1}), \tfrac{1}{2}(l_n-l_{n+1}+1), l_n+n; -\tan^2\vartheta^{n+1}] \tag{17}$$

with the restriction

$$l_n = l_{n+1}-k, \quad k = 0, 1, \dots, l_{n+1}. \tag{18}$$

Both solutions (15) and (17) can be expressed in terms of $\mathrm{d}^J_{\Lambda M}$ — functions of the ordinary rotation group SO(3) (cf. eq. 5.8(1)). The orthonormal basis of the corresponding Hilbert spaces $H^{l_n}(S^{2n-1})$ and $H^{l_{n+1}}(S^{2n})$ are then given by the expressions:

(i) SO($2n$):

$$Y^{l_2,\dots,l_n}_{m_1,\dots,m_n}(\omega) = N_n^{-1/2}\prod_{k=2}^{n}\sin^{2-k}\vartheta^k \mathrm{d}^{J_k}_{M_k M_k'}(2\vartheta^k)\prod_{s=1}^{n}\exp(\mathrm{i}m_s\varphi^s), \tag{19}$$

(ii) SO($2n+1$)

$$Y^{l_2,\dots,l_{n+1}}_{m_1,\dots,m_n}(\omega) = N_{n+1}^{-1/2}\sin^{1-n}\vartheta^{n+1}\mathrm{d}^{J_{n+1}}_{M_{n+1},0}(\vartheta^{n+1}) \times \\ \times\prod_{k=2}^{n}\sin^{2-k}(\vartheta^k)\mathrm{d}^{J_k}_{M_k M_k'}(2\vartheta^k)\prod_{s=1}^{n}\exp(\mathrm{i}m_s\varphi^s), \tag{20}$$

where N_n, N_{n+1} are normalization factors given by

$$\begin{aligned} N_n &= 2\pi^n\prod_{k=2}^{n}(l_k+k-1)^{-1}, \\ N_{n+1} &= 4\pi^n[2(l_{n+1}+n)-1]^{-1}\prod_{k=2}^{n}(l_k+k-1)^{-1}, \end{aligned} \tag{21}$$

and the indices $J_{k'}$, $M_{k'}$, M_k' are defined as

$$\begin{aligned} J_k &= \frac{1}{2}(l_k+k-2), \\ M_k &= \frac{1}{2}(m_k+l_{k-1}+k-2), \quad l_1 \equiv m_1, \\ M_k' &= \frac{1}{2}(m_k-l_{k-1}-k+2), \quad k = 2, 3, \dots, n, \\ J_{n+1} &= l_{n+1}+n-1, \quad M_{n+1} = l_n+n-1. \end{aligned} \tag{22}$$

Here l_k, $k = 2, 3, \dots, n+1$, are non-negative integers, m_k, $k = 1, 2, \dots, n$ are integers restricted by the conditions (16) and (18).

B. *Irreducibility and Unitarity*

Let $H = L^2(S^{p-1}, \mu)$. The global transformation of SO(p) in H are defined by means of the left translations, i.e.,

$$T_g u(x) = u(g^{-1}x). \tag{23}$$

The generators of one-parameter subgroups $g_{(ik)}(\vartheta)$, defined by eq. 1(41), have, according to eq. (23), the form

$$L_{ik} = x_i\partial_k - x_k\partial_i, \qquad i, k = 1, 2, \ldots, p. \tag{24}$$

The set of all generators L_{ik} span the Lie algebra of SO(p) with the following commutation relations:

$$[L_{ij}, L_{rs}] = \delta_{is}L_{jr} + \delta_{jr}L_{is} - \delta_{ir}L_{js} - \delta_{js}L_{ir}.$$

Clearly,

$$[\varDelta(S^p), L_{ij}] = 0, \qquad i, j = 1, 2, \ldots, p, \tag{25}$$

so that the harmonic functions associated with a definite eigenvalue of $\varDelta(S^p)$ span an invariant subspace of H. We shall show now that this subspace is irreducible; the set $\left\{L_{2k,2k-1},\ k = 1, 2, \ldots, \left[\frac{1}{2}p\right]\right\}$ of commutative generators forms a Cartan subalgebra of $o(p)$. Using eq. (24) and introducing biharmonic coordinates, one obtains

$$\mathrm{i}L_{2k,2k-1} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi^k}, \qquad k = 1, 2, \ldots, \left[\frac{1}{2}p\right]. \tag{26}$$

Consider first the group SO($2n$) and denote the space spanned by vectors (19) by H^{l_n}. By virtue of eqs. (19) and (26), every basis vector in H^{l_n} is the weight vector with the weight $m = (m_n, m_{n-1}, \ldots, m_1)$. Using eq. (16) we find that m_n is maximal (and then $m_n = l_n$) if $l_{n-1} = 0$; this in turn implies by successive applications of eq. (16) that $m_{n-1} = m_{n-2} = \ldots = m_1 = 0$. Hence the weight

$$m = (l_n, 0, \ldots, 0) \tag{27}$$

represents the highest weight in H^{l_n}. Any other admissible highest weight in H^{l_n}, by eq. (16), would have the form

$$m' = (m_n, l_n - m_n, 0, 0, \ldots, 0). \tag{28}$$

However, the inspection of eq. 9.4 (57) shows that only the weight (27) gives the eigenvalue (11) for the second-order Casimir operator $\varDelta(S^{p-1})$. Thus the highest weight (27) is unique. Consequently the carrier space H^{l_n} is irreducible.

Using similar arguments and eqs. (16) and (18), one can show that the space $H^{l_{n+1}}$ spanned by basis vectors (20) carries the irreducible representation of SO($2n+1$), which is defined by the unique highest weight m of the form

$$m = (l_{n+1}, 0, \ldots, 0). \tag{29}$$

The global representations of $SO(p)$ defined in H^{l_n} or $H^{l_{n+1}}$ by eq. (23) are unitary because of the invariance of the measure μ on S^{p-1}.

Remark 1: The irreducible representations of $SO(p)$, $p = 2n$ or $p = 2n+1$, determined by the highest weight (27) or (29) correspond by virtue of th. 2.6 to the tensor representations determined by the Young partition $\lambda = (\lambda_1, 0, \dots, 0)$ with $\lambda_1 = l_n$ or l_{n+1}. Hence the representations in H^{l_n} or in $H^{l_{n+1}}$ are equivalent to tensor representations realized in the space of symmetric tensors of rank l_n or l_{n+1}, respectively.

Remark 2: In general the maximal set of commuting operators for $SO(p)$, by virtue of eqs. 1(55) and (56), contains $\frac{p^2}{4}$ (p even), or $\frac{p^2-1}{4}$ (p odd) operators.

For these special representations, the maximal set of commuting operators is smallest and consists of

$$Q_p \equiv \begin{cases} \Delta[SO(p)],\ \Delta[SO(p-2)], \dots, \Delta[SO(4)] & \text{for } p \text{ even},\\ \Delta[SO(p)], \Delta[SO(p-1)], \Delta[SO(p-3)], \dots, \Delta[SO(4)] & \text{for } p \text{ odd},\\ & p > 3,\\ \Delta[SO(3)] \quad \text{for } p = 3, \end{cases}$$

and

$$H = \left\{L_{2k,2k-1} = -\frac{\partial}{\partial \varphi^k},\ k = 1, 2, \dots, \left[\frac{1}{2}p\right]\right\}.$$

The set H contains the operators of the Cartan subalgebra. The spectra of the eigenvalues $l_2, \dots, l_{\{\frac{p}{2}\}}$, $m_1, \dots, m_{\{\frac{p}{2}\}}$ are determined by eqs. (16) and (18).

The biharmonic coordinate system seems to be most convenient from the point of view of physical applications: If we associate with each operator appearing in the maximal set of commuting operators a physical observable then, because every operator of the Cartan subalgebra is diagonal in the biharmonic coordinate system, we have in fact thus exhibited the maximal number of linear conservation laws.

§ 4. The Method of Creation and Annihilation Operators

It is well known in the elementary quantum mechanics of a single particle with conjugate dynamical variables p and q, $[p, q] = -i$, that the operators defined by

$$a = \frac{1}{\sqrt{2}}(q+ip), \qquad a^* = \frac{1}{\sqrt{2}}(q-ip), \tag{1}$$

called *annihilation and creation operators*, respectively, satisfy the commutation relations

$$[a, a^*] = 1. \tag{2}$$

Their names are due to the fact that the Hamiltonian $H = \frac{p^2}{2} + \frac{q^2}{2}$ of the linear harmonic oscillator takes the form (in units $\hbar\omega = 1$, $m = 1$)

$$H = a^* a + \tfrac{1}{2} \tag{3}$$

so that, if $|0\rangle$ $(=\pi^{1/4} \exp[-\frac{1}{2}x^2])$ is the ground eigenstate of H with energy 1/2, $a^*|0\rangle$ is another eigenstate of H with energy $(1+1/2)$, $a^*a^*|0\rangle$ another with energy $(2+1/2)$, etc. Moreover, $a(a^{*n})|0\rangle$ is proportional to $a^{*n-1}|0\rangle$ and $a|0\rangle = 0$. Thus a^* creates one unit of 'excitation' (i.e., increases the energy), and a annihilates one unit of 'excitation' (decreases the energy).

The Lie algebra (2) is clearly equivalent to the Heisenberg algebra $[p, q] = -\mathrm{i}$.

We easily generalize (1), (2) and (3) to the set of n independent creation and annihilation operators

$$[a_i, a_j] = [a_i^*, a_j^*] = 0, \qquad [a_i, a_j^*] = \delta_{ij},\ i, j = 1, 2, \ldots, N \tag{4}$$

and

$$H = \sum_{i=1}^{N} a_i^* a_i + N/2 \tag{5}$$

so that the eigenstates of H contain n_1 excitation of type 1, n_2 excitations of type 2, etc., i.e.,

$$a_1^{*n_1} a_2^{*n_2} \ldots a_N^{*n_N} |0\rangle. \tag{6}$$

The operators a_i, a_i^* are also called *boson operators.* In order to explain this nomenclature, we reinterpret the states (6) as follows. Consider the quantum mechanics of N identical particles. Let i be an index counting the set of quantum numbers characterizing the states of a single particle (these quantum numbers may be discrete or continuous with appropriate ranges); in other words, the complete set of one-particle states are φ_i, $i = 1, 2, \ldots$ According to a general postulate of quantum theory for indistinguishable particles, the distinct states of the system of N identical particles are only those characterized by the number of particles n_i in the state φ_i, $i = 1, 2, \ldots$ Thus the state (6) can be interpreted as n_1 particles in the state φ_1, n_2 particles in the state φ_2, etc. Hence a_k^* creates a particle in the state k, a_k annihilates one.

Let H be the space of states of a system of bosons with the basis vectors $|n_1, n_2, n_3, \ldots\rangle$ labelled by the occupation numbers, and V the space of one-particle states.

In H the action of a_i and a_j^* are expressed as follows:

$$\begin{aligned} a_i|n_1 n_2 \ldots n_n\rangle &= \sqrt{n_i}\,|n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_n\rangle, \\ a_i^*|n_1 n_2 \ldots\rangle &= \sqrt{n_i + 1}\,|n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots\rangle. \end{aligned} \tag{7}$$

Then, by direct computation, we indeed obtain from these equations the commutation relations (4). We will show in 20.2 that within equivalence there exists only one irreducible (integrable to the group) representation of the canonical commutation relations (4).

The following example shows another interesting realization of canonical commutation relations (4).

EXAMPLE 1. Let H be a Hilbert space of functions of complex variables $z_1, \ldots, z_n$ with the scalar product

$$(u, v) = \int u(z)\bar{v}(z)\exp(-\bar{z}z)\mathrm{d}\bar{z}\mathrm{d}z, \tag{8}$$

where

$$\mathrm{d}\bar{z}\mathrm{d}z = \pi^{-n}\prod_{k=1}^{n}\mathrm{d}x_k\mathrm{d}y_k, \qquad z_k = x_k+\mathrm{i}y_k. \tag{9}$$

Then the map

$$a_l \to \frac{\partial}{\partial z_l}, \qquad a_l^* \to z_l, \tag{10}$$

defined on analytic functions $u(z)$ in H, gives the representation of the Lie algebra (4). ▼

Construction of Lie Algebras from Bilinear Combinations of Creation and Annihilation Operators

The physical quantities like energy (3) angular momentum $M_{ij} = q_ip_j-p_jq_i$, etc. are bilinear forms of the form

$$c_{ij}a_i^*a_j.$$

This suggests to construct all basis elements X_i of a Lie algebra L in terms of such bilinear combinations. Indeed, let a_i, $i = 1, 2, \ldots, n$, be a set of boson operators in a carrier Hilbert space H. Set $A_{ij} = a_i^*a_j$. Then using eq. (4) we obtain

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il}-\delta_{il}A_{jk}. \tag{11}$$

Hence, by virtue of 9.4 (2) the set $\{A_{ij}\}_{i,j=1}^n$ forms the set of generators of the Lie algebra gl(n, c). Because any Lie algebra is a subalgebra of gl(n, C) by Ado's Theorem any other complex or real Lie algebra is generated by a subset of $\{A_{ij}\}_{i,j=1}^n$. In particular using eq. 1 (6) we find that operators

$$\begin{aligned} M_{kk} &= a_k^*a_k, \qquad k = 1, \ldots, n, \\ M_{kl} &= a_k^*a_l+a_l^*a_k, \\ \tilde{M}_{kl} &= \mathrm{i}(a_k^*a_l-a_l^*a_k), \end{aligned} \quad k < l \leqslant n \tag{12}$$

generate the Lie algebra $u(n)$.

Similarly using eq. 1 (42) we find that the operators

$$X_{ik} = a_i^*a_k-a_k^*a_i \tag{13}$$

generate the Lie algebra so(n). The explicit construction of generators of sp(n)-algebra is given in exercise 6.4.5.

One may construct also with the help of annihilation and creation operators non-compact Lie algebras like $u(p, q)$, so(p, q), sp(p, q) etc. We give for an illustration the construction of generators for $u(p, q)$ Lie algebras, which are often used in particle physics. Let a_i, a_j^*, $i, j = 1, 2, \ldots, p$ and $b_{\hat{i}}$, $b_{\hat{j}}^*$, $\hat{i}, \hat{j} = p+1, \ldots$ $\ldots, p+q$, be the sets of annihilation and creation operators satisfying the relations:

$$[a_i, a_j^*] = \delta_{ij}, \qquad [b_{\hat{i}}, b_{\hat{j}}^*] = \delta_{\hat{i}\hat{j}} \tag{14}$$

and all other commutators equal to zero. Define the set A of operators by the following array of bilinear products

$$A = \begin{bmatrix} A_{ij} & A_{i\hat{j}} \\ A_{\hat{j}i}^* & A_{\hat{i}\hat{j}} \end{bmatrix} = \begin{bmatrix} -a_i^* a_j + r\delta_{ij} & a_i^* b_{\hat{j}}^* \\ -b_{\hat{i}} a_j & b_{\hat{i}} b_{\hat{j}}^* + r\delta_{\hat{i}\hat{j}} \end{bmatrix}, \tag{15}$$

where r is any real number. One readily verifies that elements of the set A satisfy the commutation relations for the Lie algebra gl$(p+q, R)$, whereas the operators

$$\begin{gathered} M_{kk} = A_{kk}, \qquad k = 1, 2, \ldots, n, \\ M_{kl} = A_{kl} + A_{lk}, \quad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}), \quad k \leqslant l \leqslant p \text{ or } p < k < l, \\ N_{k\hat{l}} = A_{k\hat{l}} - A_{\hat{l}k}, \quad \tilde{N}_{k\hat{l}} = \mathrm{i}(A_{k\hat{l}} + A_{\hat{l}k}), \quad k \leqslant p < l, \end{gathered} \tag{16}$$

generate the Lie algebra $u(p, q)$.

Using (7) we can then construct representations of all these Lie algebras.

§ 5. Comments and Supplements

(a) The algebraic method of construction of irreducible representations considered in sec. 1 was elaborated by Gel'fand and Zetlin in 1950a, b. They only gave the final formulas such as 1(14)–(19). An interesting derivation of these formulas, based on the Weyl theory of tensor representations, was given by Baird and Biedenharn in 1963; they also corrected some formulas in the original work of Gel'fand–Zetlin. In 1965 Gel'fand and Graev extended and improved this formalism and succeeded in calculating the matrix elements of the global finite-dimensional representations of GL(n, C); they also extended the algebraic approach to the representation theory of non-compact Lie algebras. We present this theory in ch. 11. The detailed analysis of representations of $u(n)$, so(n), $u(n, 1)$, and so$(n, 1)$ algebras in terms of Gel'fand–Zetlin patterns was given by Ottoson 1967. Holman and Biedenharn gave the alternative derivation of various results for representations of $u(n)$ using Gel'fand–Zetlin technique 1971.

There is a one-to-one correspondence between the Gel'fand–Zetlin basis vectors and the tensor components. In fact, consider the Gel'fand–Zetlin patterns 1(11) associated with a highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and form the Young tableau which contains:

in the first row: m_{11} entries 1 followed by

$$(m_{12}-m_{11}) \text{ entries } 2, \ldots, (m_{1n}-m_{1,n-1}) \text{ entries } n,$$

in the second row m_{22} entries 2 followed by

$$(m_{23}-m_{22}) \text{ entries } 3, \ldots, (m_{2n}-m_{2n-1}) \text{ entries } n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

in the kth row m_{kk} entries 1 followed by

$$(m_{k,k+1}-m_{kk}) \text{ entries } (k+1), \ldots, (m_{kn}-m_{kn-1}) \text{ entries } n.$$

It is evident that this prescription gives a one-to-one correspondence between the Gel'fand–Zetlin basis vectors and the tensor components; the tensor component 2(14), corresponding to the highest vector m obtained in this manner has the highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$ because of eq. 2(17). The correspondence between Gel'fand–Zetlin patterns and basic vectors $|n_1, \ldots, n_n\rangle$ defined by eq. 4 (6) was given by Holman and Biedenharn 1971.

(b) The theory of tensor representations of simple Lie groups is due mainly to Weyl 1939. The Weyl theory was based on the connection between the permutation group and the linear groups. Here we presented an approach based on the concept of induced representations. This approach was originated by Godement 1956, app. and finally elaborated by Želobenko 1962.

(c) The theory of harmonic functions on compact symmetric spaces was also originated by Weyl in 1934. Later on Godement 1952, 1956 and Harish-Chandra 1958 made this one of the main tools in the analysis of irreducible representations of semisimple Lie groups. The case of SO(n) harmonic functions presented in § 3 was elaborated by Rączka, Limić and Niederle 1966a. They also extended this approach to non-compact groups SO(p, q) (cf. 15, § 3).

One can construct the harmonic functions and the corresponding irreducible representations also for other classical Lie groups. The case of $U(n)$ was studied by Rączka and Fischer 1966b. They constructed a class of harmonic functions determined by one and two invariant numbers. The case of symplectic groups Sp(n) was treated by Pajas and Rączka 1968.

The harmonic functions for non-compact unitary groups $U(p, q)$ were constructed by Fischer and Rączka 1965c and for Sp(p, q) by Pajas 1969.

Wigner showed in 1955 that a representation theory of Euclidean group provides a basis for the theory of Bessel functions; in particular, he demonstrated that the composition law of group elements imply various functional relations for Bessel functions. Later on Vilenkin in a series of papers extended this approach to other special functions. The results of Vilenkin are collected in his monograph 1965. Recently other monographs appeared devoted to the study of the special functions of mathematical physics from the point of view of group representation: Miller 1968, and Talman 1968 based on Wigner's lectures of 1955.

(d) The origin of the creation and annihilation operators goes back to the quantum theory of oscillators and of radiation (cf. Dirac 1928). The Fock-space was introduced in 1932 (Fock). The representations of the infinite-dimensional Fermi and Bose operators were first given by Gårding and Wightman 1954 a, b and there has been considerable work on this case since then. (See the reviews by Berezin and Golodets 1969.)

The Hilbert space of entire analytic functions goes back apparently to London, but was introduced in a complete form by Bargmann 1961 and extended to the infinitely many creation and annihilation operators by Segal 1965.

The construction of the representations of the compact Lie algebra su(2) by creation and annihilation operators is due to Schwinger 1952. The non-compact case was discussed by Barut and Fronsdal 1965 for su(1, 1) and by Anderson, Fischer and Rączka 1968 for $u(p, q)$.

§ 6. Exercises

§ 1.1.** Construct the irreducible representations of the Lie algebra sp(n) using Gel'fand–Zetlin method

Hint: Elaborate the structure of Gel'fand–Zetlin patterns using the decomposition of irreducible representations of sp(n) with respect to a sequence of successive maximal subalgebras of sp(n).

§ 2.1. Show that the defining representation L of the Lorentz group in R^4:

$$x \to x' = Lx$$

is equivalent to the $D^{(1/2,1/2)}$ representation, i.e.

$$D^{(1/2,\,1/2)} = TLT^{-1}.$$

§ 2.2. Show that the generators of the Lorentz group for $D^{(1/2,0)}$ and $D^{(0,1/2)}$ representation can be taken in the form ($k = 1, 2, 3$):

$$\begin{aligned} J_k^{(1/2,\,0)} &= -\frac{1}{2}\mathrm{i}\sigma_k, \quad & N_k^{(1/2,0)} &= \frac{1}{2}\sigma_k, \\ J_k^{(0,1/2)} &= -\frac{1}{2}\mathrm{i}\sigma_k, \quad & N_k^{(0,1/2)} &= -\frac{1}{2}\sigma_k. \end{aligned} \tag{1}$$

§ 2.3. Show that the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of SL(2, C) can be written in the form

$$(D^{(1/2,0)} \oplus D^{(0,1/2)})(\Lambda) = \begin{cases} \exp\left(\dfrac{\mathrm{i}}{2}\, \boldsymbol{w}\cdot\boldsymbol{\Sigma}\right) \equiv \cos\dfrac{w}{2} + \mathrm{i}\hat{\boldsymbol{w}}\cdot\boldsymbol{\Sigma}\sin\dfrac{w}{2} \\ \qquad\qquad \text{for special rotations,} \\ \exp\left(-\dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{\alpha}\right) \equiv \cosh\dfrac{u}{2} - \hat{\boldsymbol{u}}\cdot\boldsymbol{\alpha}\sinh\dfrac{u}{2} \\ \qquad\qquad \text{for special Lorentz transformations,} \end{cases} \tag{2}$$

where $\hat{w}$ and $\hat{u}$ are unit vectors in the direction of w and u, $w = |w|$, $u = |u|$ and

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \alpha = \begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix}. \tag{3}$$

§ 2.4. Show that the irreducible representation $D^{(i,j)}$ of SL(2, C) can be realized in the space of symmetric traceless tensors $T_{\mu_1 \mu_{2j}}$.

§ 2.5. Show that the representations $D^{(j,0)}$ and $D^{(0,j)}$ of SL(2, C) satisfy the following equalities:

$$D^{(j,0)}\left(\frac{\sigma p}{m}\right) = m^{-2j}(\sigma p)\otimes(\sigma p)\otimes \dots \otimes(\sigma p) \quad (2j \text{ times}),$$

$$D^{(0,j)}\left(\frac{\sigma p}{m}\right) = m^{-2j}(\tilde{\sigma} p)\otimes(\tilde{\sigma} p)\otimes \dots \otimes(\tilde{\sigma} p) \quad (2j \text{ times}),$$

where $\tilde{\sigma} = -\sigma$.

§ 4.1. Let

$$a_k = 2^{-1/2}(x_k + i\partial_k), \quad a_k^* = 2^{-1/2}(x_k - i\partial_k), \quad k = 1, 2, 3$$

and

$$X_{kl} = a_k^* a_l + \tfrac{1}{2}\delta_{ij}. \tag{4}$$

Show that the operators (4) span the Lie algebra $u(3)$.

Show that in the carrier space $H = L^2(R^3)$ only the most degenerate representations of $u(3)$ characterized by highest weights $m = (m, 0, 0)$ are realized. Show that lowest dimensions of these representations are 1, 6, 10 and 15.

Hint: Show that higher order Casimir operators $C_2, C_3, C_4, \dots$, are functions of $C_1 = \sum_1^3 X_{ll}$.

§ 4.2. Let $H = L^2(R^n)$ and let a_l and a_l^*, $l = 1, 2, \dots, n$, be the creation and annihilation operators given by

$$a_l = 2^{-1/2}(q_l + ip_l), \quad a_l^* = 2^{-1/2}(q_l - ip_l). \tag{5}$$

Show that the vector

$$|0\rangle = \pi^{-n/4}\exp\left(-\frac{x^2}{2}\right) \tag{6}$$

satisfies the condition

$$a_l|0\rangle = 0$$

and

$$(a_l^*)^p|0\rangle = H_p(x_l)|0\rangle, \quad p = 1, 2, \dots, \tag{7}$$

where $H_p(x_l)$ are the normalized Hermite polynomials.

§ 4.3. Set

$$e_a(z) = \exp(\bar{a}z). \tag{8}$$

Show that functions (8) satisfy the conditions (with respect to the scalar product $(\cdot,\cdot)$ given by 4(8))

$$(e_a, e_b) = e_b(a), \qquad (e_a, u) = u(a) \tag{9}$$

so they play the role of Dirac δ-functions.

§ 4.4. Let M, N and R be $n\times n$ matrices. Show that the following identities are satisfied

$$\begin{aligned}
[a^*Ma, a^*Na] &= a^*[M, N]a,\\
[aNa, a^*Ra^*] &= a^*[R^TN^T+R^TN+RN^T+RN)a+\\
&\quad +\mathrm{Tr}(NR^T)+\mathrm{Tr}(NR),\\
[a^*Na, aRa] &= -a[R^TN+RN)a,\\
[a^*Ma, a^*Ra^*] &= a^*(R^TM^T+RM^T)a^*,
\end{aligned} \tag{10}$$

where $a^*Ma = a_i^* M_{ik} a_k$, etc.

§ 4.5. Let a_i, a_i^*, $i = 1, 2, \ldots, n$, be a set of boson operators. Show that the bilinear combinations

$$F_{ij} = a_i a_j, \qquad G_{ij} = a_i^* a_j + \tfrac{1}{2}\delta_{ij}, \qquad H_{ij} = a_i^* a_j^* \tag{11}$$

satisfy the following commutation relations

$$[G_{ij}, G_{kl}] = \delta_{jk} G_{il} - \delta_{il} G_{jk}, \tag{12}$$

$$[F_{ij}, G_{kl}] = \delta_{jk} F_{il} + \delta_{ik} F_{il} \tag{13}$$

and generate the Lie algebra sp(n, C).

§ 4.6. Find the set M^{nm} of matrices in order that the bilinear combinations $a^*M^{nm}a$ form

(i) Lie algebra $u(n)$,

(ii) Lie algebra so(n).

§ 4.7. Let the operators b_k satisfy the following anticommutation relations

$$[b_l, b_k]_+ = 0 = [b_l^*, b_k^*]_+, \qquad [b_l, b_k^*]_+ = \delta_{lk} I, \qquad l, k = 1, 2, \ldots, n \tag{14}$$

Show that the representations of the algebra (14) is equivalent to those of a finite group of order 2^n.

§ 4.8. Denote by Z^+ the set of positive integers and by A_k, $k \in Z^+$ a copy of the algebra of all matrices of rank two with complex entries. Let $A = \bigotimes_{k\in Z} A_k$ and denote by σ_k^μ, $\mu = 0, 1, 2, 3$ the canonical imbedding of the identity σ^0 and Pauli matrices σ^l, $l = 1, 2, 3$ from A_k into A. Show that

$$\begin{aligned}
b_k &= \tfrac{1}{2}\sigma_1^3\sigma_2^3 \ldots \sigma_{k-1}^3(\sigma_k^1 + \mathrm{i}\sigma_k^2),\\
b_k^* &= \tfrac{1}{2}\sigma_1^3\sigma_2^3 \ldots \sigma_{k-1}^3(\sigma_k^1 - \mathrm{i}\sigma_k^2),
\end{aligned} \tag{15}$$

$k = 1, 2, \ldots$, satisfy canonical anticommutation relations (14) with $n = \infty$.

Chapter 11

Representation Theory of Lie and Enveloping Algebras by Unbounded Operators: Analytic Vectors and Integrability

We present in this chapter the general theory of representations of Lie and enveloping algebras by linear unbounded operators in a Hilbert space. This is one of the most interesting and at the same time difficult branches of modern mathematics. It requires a knowledge of algebra, topology, functional analysis and differential manifolds. The theory provides also a rigorous framework for various problems in quantum theory and particle physics.

Even in non-relativistic quantum mechanics the observables such as position, momenta and angular momenta are represented by partial differential operators which are unbounded in the space of physical states.

In sec. 1 we discuss Gårding's representation theory of Lie algebras by unbounded operators. In sec. 2 we extend this theory to the enveloping algebra E of a Lie algebra. In particular, we derive the fundamental theorem which determines when an element Y of E is essentially self-adjoint.

The basic concept of analytic vectors and analytic dominance of operators is introduced in sec. 3. We derive here a series of important theorems on analytic vectors for self-adjoint operators and analytic dominance in Lie and enveloping algebras.

In sec. 4 we introduce the concept of analytic vectors for a representation T of a group G and show that analytic vectors for Lie algebras and for group representations coincide. We show also that analytic vectors for the Nelson operator $\Delta = X_1^2 + \dots + X_d^2$, $d = \dim L$, are also analytic vectors for the group representations.

Sec. 5 contains the Nelson's criterion for a skew-symmetric representation of a Lie algebra L to be integrable to a global unitary representation of the corresponding simply-connected Lie group G.

In sec. 6 the beautiful integrability theory of Lie algebras representations of Flato, Simon, Snellman and Sternheimer is presented. This theory is based on the concept of weak analyticity and allows to express the integrability conditions in term of properties of Lie generators of the Lie algebra. In contrast to Nelson's theory it reduces in most practical cases the problem of integrability to the problem of verification of simple properties of first order differential operators. The derived criteria of integrability can be easily verified in applications and are rather important in quantum physics.

We present in sec. 7 an elegant method of an explicit construction of a dense set of analytic vectors for a representaton T of G using the solutions of the heat equation on G. This set represents a common dense invariant domain for the Lie and enveloping algebras of G.

Finally in sec. 8 we present the Gel'fand–Zetlin technique for the construction of irreducible representations of $u(p, q)$ using diagrammatic method.

The applications to quantum theory will be considered in chs. 12, 13, 17, 20, and 21.

§ 1. Representations of Lie Algebras by Unbounded Operators

A. *General Properties of Representations of Lie Algebras*

In the finite-dimensional case, a representation $X \to T(X)$ of a Lie algebra L was defined as a homomorphism of L into gl(n, C), i.e., for X, Y in L and α, β in C^1, we have

$$\alpha X+\beta Y \to \alpha T(X)+\beta T(Y), \tag{1}$$

$$[X, Y] \to [T(X), T(Y)] = T(X)T(Y)-T(Y)T(X), \tag{2}$$

where $T(\cdot)$ is an element of gl(n, C) (cf. ch. 1.1.C). The equality $[T(X), T(Y)] = T(X)T(Y)-T(Y)T(X)$ follows from the fact that every Lie algebra L is a subalgebra of gL(n, C) in which the commutation relations are $[X, Y] = XY-YX$.

One of the principal difficulties in the general representation theory of Lie algebras is due to the fact that, in many important cases, representatives $T(X)$ of elements of a Lie algebra are given by unbounded operators (cf. example 1). Hence, we have to consider the problem of the selection of a proper common domain D for a set of unbounded operators. This is a fundamental problem in functional analysis. A review of basic results in functional analysis is given in appendix B. We refer readers not familiar with these results to consult this appendix.

Because we want to consider the adjoint $T(X)^*$ together with a representative $T(X)$, a common domain D has to be dense in the carrier space H. The domain D cannot, however, be the whole space, because on such a domain only bounded operators would be defined (cf. appendix B, lemma 1.2) Moreover, because we want to define the commutator $T(X)T(Y)-T(Y)T(X)$ for representatives $T(X)$ and $T(Y)$, the range $R(T(X))$ has to be in D for any X in L. Hence, D must be invariant. We, therefore, arrive at the following general definition of a representation of an abstract Lie algebra L:

Definition 1. *A representation T of L in a Hilbert space H* is any homomorphism $X \to T(X)$, $X \in L$, of L into a set of linear operators having a common linear dense invariant domain D. ▼

Definition 1 means that for arbitrary X, Y in L, α, β in C^1 and u in D, we have

$$T(\alpha X+\beta Y)u = \alpha T(X)u+\beta T(Y)u, \tag{3}$$

$$T([X, Y])u = [T(X), T(Y)]u = \big(T(X)T(Y)-T(Y)T(X)\big)u. \tag{4}$$

Note that by eq. (4)

$$[T(X), [T(Y), T(Z)]]u+[T(Y), [T(Z), T(X)]]u+[T(Z), [T(X), T(Y)]]u = 0,$$

i.e., the Jacobi identity is automatically satisfied.

Because the domain D is invariant, any representation T of a Lie algebra L can be extended to a representation of the enveloping algebra of L.

The set $N = T^{-1}(0) \subset L$ is an ideal of L. Indeed, if $X \in N$ and $Y \in L$, then $T([X, Y]) = [T(X), T(Y)] = 0$, i.e., $[X, Y] \in N$. Hence, in particular, non-trivial representations of simple Lie algebras are faithful, i.e., the map $X \to T(X)$ is one-to-one.

A representation T of L is said to be *skew-adjoint* (*skew-symmetric*), if the homomorphism $X \to T(X)$ maps L into a set of skew-adjoint (skew-symmetric) operators. Clearly, in a skew-adjoint representation T, the operators $iT(X)$ are self-adjoint (hermitian) and satisfy the commutation relations with purely imaginary structure constants.

A representation T is said to be *topologically irreducible* if there is no proper closed subspace $H' \subset H$ containing a common, linear invariant domain $D' \subset H'$ of L which is dense in H'.

EXAMPLE 1. Let L be the Poincaré–Lie algebra, and let $H = L^2(\Omega)$, where Ω is the four-dimensional Minkowski space. The commutation relations for L are given by eqs. 1.1 (23a–c). We verify that these commutation relations are satisfied by the following formal differential operators

$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad P_\mu = \partial_\mu, \quad \nu, \mu = 0, 1, 2, 3, \tag{5}$$

where $\partial_\mu = \partial/\partial x^\mu$. In order to obtain a representation T of L, we have to determine a common, dense, linear invariant domain D for operators (5). We can take either one of the following two dense subspaces of $L^2(\Omega)$

1° $D = C_0^\infty(\Omega)$, (6)

2° $D = S(\Omega)$, (7)

where $S(\Omega)$ is the Schwartz space of $C^\infty(\Omega)$-functions $\varphi(x)$ with

$$\sup_x |x^\alpha D^\beta \varphi(x)| < \infty, \tag{8}$$

where

$$x^\alpha \equiv x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \tag{9}$$

$$D^\beta \equiv \partial^{|\beta|}/\partial x_0^{\beta_0}\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}, \quad |\beta| = \beta_0+\beta_1+\beta_2+\beta_3, \tag{10}$$

$$\alpha_\mu, \beta_\mu = 0, 1, 2, \ldots, \quad \mu = 0, 1, 2, 3.$$

It is easy to verify that all generators (5) are skew-symmetric on these domains, and that these domains are invariant under the operators (5).

B. *Gårding's Theory*

We now discuss the standard method for the construction of a common linear invariant dense domain for a representation $X \to T(X)$ of the algebra L, given a representation $x \to T_x$ of the associated Lie group G. Let $x(t) = \exp(tX)$, $X \in L$, be a one-parameter subgroup of G and $T_{x(t)}$ the corresponding one-parameter subgroup of operators. If for $u \in H$, $\lim_{t\to 0} t^{-1}(T_{x(t)} - I)u$ exists, then the action of the generator $T(X)$ of the subgroup $T_{x(t)}$ is defined by the formula

$$T(X)u = \lim_{t\to 0} t^{-1}(T_{x(t)} - I)u, \qquad x(t) = \exp(tX). \tag{11}$$

The set of all $u \in H$, for which the right hand side of eq. (11) is defined, is said to be the *domain* of $T(X)$.

Let $C_0^\infty(G)$ be the set of all infinitely differentiable functions with compact support in the group space G, and let $T(\varphi)$, $\varphi \in C_0^\infty(G)$, be a new 'smeared out' operator defined by (cf. appendix B.2 for the theory of integration of operator functions)

$$T(\varphi)u = \int_G \varphi(x) T_x u \mathrm{d}x, \qquad u \in H. \tag{12}$$

Denote by D_G the linear subspace spanned by all vectors $u(\varphi) \equiv T(\varphi)u$, $u \in H$. For $u(\varphi) \in D_G$ we have therefore

$$T_y u(\varphi) = u(L_y \varphi).$$

Indeed,

$$T_y u(\varphi) = \int_G \varphi(x) T_{yx} u \mathrm{d}x = \int_G \varphi(y^{-1}z) T_z u \mathrm{d}z = \int (L_y\varphi)(z) T_z u \mathrm{d}z. \tag{13}$$

THEOREM 1. *Let T be a representation of a Lie group G in a Hilbert space H. Then*

1° *The subspace D_G is dense in H.*

2° *The subspace D_G is a common linear invariant domain for the generators of one-parameter subgroups of G.*

PROOF: *ad* 1°. Let $\varphi \in C_0^\infty(G)$ with support K be such that

$$\varphi \geqslant 0, \qquad \int_K \varphi(x) \mathrm{d}x = 1.$$

Then, for any u in H, by eq. (12), we have

$$u(\varphi) - u = \int_G \varphi(x)(T_x - I)u \mathrm{d}x.$$

Hence,

$$\|u(\varphi) - u\| \leqslant \max_{x\in K} \|T_x u - u\|,$$

Consequently, if K shrinks to the identity e in G, the vector $u(\varphi) \to u$ by continuity of T_x. Because u is an arbitrary vector in H, this shows that the set D_G of vectors (2) is dense in H.

ad 2°. Let $y(t) = \exp(tY)$ be a one-parameter subgroup of G. Due to the invariance of the Haar measure dx on G, we obtain

$$\int_G \varphi(y^{-1}(t)x)T_x u\,dx = \int_G \varphi(x)T_{y(t)x}u\,dx = T_{y(t)}\int \varphi(x)T_x u\,dx.$$

Hence,

$$t^{-1}(T_{y(t)}-I)u(\varphi) = \int t^{-1}[\varphi(y^{-1}(t)x)-\varphi(x)]T_x u\,dx. \tag{14}$$

For every v in H the function $|t^{-1}[\varphi(y^{-1}(t)x)-\varphi(x)](T_x u, v)|$, is integrable on G and the limit $t \to 0$ is in $C_0^\infty(G)$; hence, using Lebesgue theorem (app. A.6) we can interchange $\lim\limits_{t\to 0}$ with the integral sign. Thus for $t \to 0$ one obtains

$$T(Y)u(\varphi) = u(\tilde{Y}\varphi), \tag{15}$$

where

$$(\tilde{Y}\varphi)(x) = \lim_{t\to 0}\frac{\varphi(y^{-1}(t)x)-\varphi(x)}{t} \in C_0^\infty(G) \tag{16}$$

gives the action of a left regular representation of L in the space $C_0^\infty(G)$. Hence, for any $u(\varphi)$ in D_G and any generator $T(Y)$, $T(Y)u(\varphi)$ is also in D_G. This means that D_G is the common invariant dense domain for all elements of the Lie algebra L of the Lie group G. It is evident that this domain is linear. ▼

The domain D_G in question is called the *Gårding subspace*.

Remark 1: The map $Y \to \tilde{Y}$ given by eq. (16) is a representation of the Lie algebra L by means of right invariant first order differential operators acting on $C_0^\infty(G)$.

COROLLARY 1. *Let L be the Lie algebra of G and let $x \to T_x$ be a representation of G. Then, the map $X \to T(X)$, $X \in L$, given by eq.* (15), *is the representation of L.*

PROOF: Because D_G is the common, linear, dense invariant subspace of H, and the condition (3) is obviously satisfied, it suffices to verify the condition (4). Indeed, we have by eq. (15)

$$\begin{aligned} T[X, Y]u(\varphi) &= u([\tilde{X}, \tilde{Y}]\varphi) = u[(\tilde{X}\tilde{Y}-\tilde{Y}\tilde{X})\varphi] \\ &= T(X)u(\tilde{Y}\varphi) - T(Y)u(\tilde{X}\varphi) \\ &= \big(T(X)T(Y)-T(Y)T(X)\big)u(\varphi) = [T(X), T(Y)]u(\varphi). \end{aligned}$$

Remark 2: Due to the invariance of D_G, we can define by means of eq. (15) the action of any element

$$M = \sum_{\substack{n \\ i_1 \dots i_n}} a_{i_1\dots i_n}X_{i_1}\dots X_{i_n}, \qquad X_{i_r} \in L,$$

of the enveloping algebra E of the Lie algebra L by the formula

$$T(M)u(\varphi) = u(\tilde{M}\varphi), \qquad M \in E, \tag{17}$$

where $\tilde{M}$ is the 'left' differential operator on $C_0^\infty(G)$ corresponding to the element M in E. Hence, the representation $X \to T(X)$ of L can be extended to a representation $M \to T(M)$ of the enveloping algebra E of L.

PROPOSITION 2. *Let $x \to T_x$ be a unitary representation of a Lie group G. Then, the operators* $\mathrm{i}T(X)$, $X \in L$, *are symmetric.*

PROOF: Let $u, v \in D_G$. Then,

$$\big(\mathrm{i}T(X)u, v\big) = \lim_{t\to 0} t^{-1}\big((\mathrm{i}T_{x(t)}-I)u, v\big) = \lim_{t\to 0} t^{-1}\big(u, -\mathrm{i}(T^*_{x(t)}-I)v\big).$$

Because for a unitary representation $T^*_{x(t)} = T^{-1}_{x(t)} = T_{x(t)^{-1}} = T_{x(-t)}$, we obtain

$$\begin{aligned}\big(\mathrm{i}T(X)u, v\big) &= \lim_{t\to 0} t^{-1}\big(u, -\mathrm{i}(T_{x(-t)}-I)v\big) = \lim_{s\to 0} s^{-1}\big(u, \mathrm{i}(T_{x(s)}-I)v\big) \\ &= \big(u, \mathrm{i}T(X)v\big).\end{aligned}$$

Let T be a representation of G in H. A vector u in H is said to be an *infinitely differentiable* or *regular* vector for T if the mapping $x \to T_x u$ of G into H is of class C^∞. A vector u in H is said to be *analytic* for T if the mapping $x \to T_x u$ of G into H is analytic. Every element $u(\varphi)$ in D_G is a regular vector for T. In fact, using the same arguments as in the proof ad 2° of th. 1, for every $n = 1, 2, \ldots$, we obtain

$$\partial_i^{(n)} T_x u(\varphi) = \partial_i^{(n)} \int_G \varphi(y) T_{xy} u \, \mathrm{d}y = \partial_i^{(n)} \int_G \varphi(x^{-1}y) T_y u \, \mathrm{d}y = \int_G \partial_i^{(n)} \varphi(x^{-1}y) T_y u \, \mathrm{d}y. \tag{18}$$

Because $\partial_i^{(n)}\varphi(x^{-1}y) \in C_0^\infty$, partial mixed derivatives of all orders are well defined and therefore D_G is a dense set of regular vectors for T. In secs. 4 and 6, we describe the construction, due to Nelson and Gårding, of a dense set of analytic vectors for T.

Sometimes it is convenient to take as the domain of the operators representing a given Lie algebra L, a subspace D in H other than the Gårding subspace D_G. For example

1° If T is a quasi-regular representation of G on a homogeneous space G/H, then $D = C_0^\infty(G/H)$ is the natural domain.

2° If T is the restriction to G of a representation of a larger group, then the Gårding subspace of the larger representation might be taken as the domain. We use this domain in the derivation of the Nelson–Stinespring results (cf. sec. 2, corollaries 1–5).

3° If $H = L^2(\Omega)$ and L is given by formal differential operators, then the subspace $C_0^\infty(\Omega)$ or Schwartz's space S can be taken as the domain for a representation T of L.

4° The space of analytic vectors for T of G, associated with the operator $T(\Delta) = T(X_1)^2 + \ldots + T(X_d^2)$, $d = \dim L$, can be used as the domain (cf. sec. 4). We use this domain for the solution of the problem of integrability of a given skew-symmetric representation of a Lie algebra to a global unitary representation of the corresponding Lie group (cf. sec. 5).

5° The space of analytic vectors for T of G associated with Lie generators of $T(L)$. This space is most convenient in applications (cf. sec. 6).

6° The space of analytic vectors for T of G, associated with the solutions of the so-called heat equation on the Lie group, can be taken as the domain D (cf. sec. 7).

§ 2. Representations of Enveloping Algebras by Unbounded Operators

We have stated that most of the observables in quantum theory and in particle physics are elements of enveloping algebras. In order to insure a proper interpretation of measurements, we require that these observables be represented by at least essentially self-adjoint operators. There is a widespread belief among physicists that in a unitary representation of a Lie group the elements of an enveloping algebra are always essentially self-adjoint operators. The following counter-example, due to von Neumann (unpublished), shows that this is not true.

EXAMPLE 1. Let G be the three-dimensional nilpotent group of all real matrices of the form

$$\begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}, \qquad \alpha, \beta, \gamma \in R^1. \tag{1}$$

The composition law in G is given by the formula

$$[\alpha, \beta, \gamma][\alpha', \beta', \gamma'] = [\alpha+\alpha', \beta+\beta', \gamma+\alpha\beta'+\gamma']. \tag{2}$$

The subgroup $[0, 0, \gamma]$, $\gamma \in R^1$, is the center of G.

Let $H = L^2(-\infty, \infty)$. The map

$$T_{[\alpha,\beta,\gamma]}u(x) = \exp[\mathrm{i}\lambda(\gamma+x\beta)]u(x+\alpha), \qquad u \in H, \tag{3}$$

defines a unitary representation of G in H. By definition 1 (11) we find that the generators of the one-parameter subgroups of G corresponding to the parameters α, β and γ have the form

$$T(X) = \mathrm{d}/\mathrm{d}x, \qquad T(Y) = \mathrm{i}\lambda x, \qquad T(Z) = \mathrm{i}\lambda. \tag{4}$$

For example, for the subgroup $[0, \beta, 0]$, we have by eq. 1 (11) and eq. (3),

$$T(Y)u = \lim_{\beta\to 0} \frac{\exp(\mathrm{i}\lambda x\beta)u-u}{\beta} = \mathrm{i}\lambda xu. \tag{5}$$

The generators (4) satisfy $[T(X), T(Y)] = T(Z)$ which for $\lambda = -1$ is equivalent to the Heisenberg commutation relations $[p, q] = -\mathrm{i}$. Therefore, the enveloping algebra of G is mapped onto all ordinary differential operators with polynomial coefficients. It is well known that many of these operators are symmetric but not essentially self-adjoint. ▼

This example shows that the unitarity of a group representation does not guarantee that images of elements of the enveloping algebra E are represented by essentially self-adjoint operators. Hence, we have to find some additional criteria which will allow us to determine when an element M of E is represented by an essentially self-adjoint operator $T(M)$.

We have seen that representatives $iT(X)$, $X \in L(G)$, associated with a unitary representation T of G are symmetric operators on the Gårding domain D_G (cf. sec. 1.B., proposition 2). We now extend this result to certain elements of the enveloping algebra E of L. In the following we deal with the universal enveloping algebra of the right invariant real Lie algebra L (cf. ch. 3.3.F).

We define in E a $+$-operation by

$$M = \sum a_{i_1 \dots i_n} X_{i_1} \dots X_{i_n} \to M^+ \equiv \sum_{\substack{n \\ i_1 \dots i_n}} \bar{a}_{i_1 \dots i_n} X_{i_n}^+ \dots X_{i_1}^+, \tag{6}$$

where for every X in L

$$X^+ \equiv -X. \tag{7}$$

The map $M \to M^+$ defines an involution in E. An element M is said to be *symmetric* in E if $M^+ = M$.

PROPOSITION 1. *Let*

$$M = \sum_{\substack{n \\ i_1 \dots i_n}} a_{i_1 \dots i_n} X_{i_1} \dots X_{i_n} \in E. \tag{8}$$

The operator $T(M)$ defined by eq. 1 (17) *satisfies*

$$(T(M)u, v) = (u, T(M^+)v), \qquad u, v \in D_G. \tag{9}$$

In particular if $M = M^+$ in E then $T(M)$ is a symmetric operator in H.

PROOF: We repeat the derivation of proposition 1.2 for a product $X_{i_1}^+ \dots X_{i_n}^+$, and obtain

$$\begin{aligned}(T(M^+)u, v) &= \Big(\sum_{\substack{n \\ i_1 \dots i_n}} \bar{a}_{i_1 \dots i_n} T(X_{i_n}^+) \dots T(X_{i_1}^+)u, v\Big) \\ &= \Big(u, \sum_{\substack{n \\ i_1 \dots i_n}} a_{i_1 \dots i_n} T(X_{i_1}) \dots T(X_{i_n})v\Big) = (u, T(M)v). \end{aligned} \tag{10}$$

Hence, if $M^+ = M$, then $T(M)^* = T(M)$, i.e., $T(M)$ is symmetric on D_G. ▼

We now derive the Nelson–Stinespring criteria which will determine when a symmetric representative $T(M)$ of an element M of the enveloping algebra E is given by an essentially self-adjoint operator.

In these considerations an important role is played by the so-called elliptic elements of the enveloping algebra. An element L in E is said to be *elliptic* if it is elliptic as a partial differential operator on G. We recall that a formal differential operator

$$L(x, D) = \sum_{0 \leqslant \alpha \leqslant \sigma} a_\alpha(x) D^\alpha, \qquad x = \{x_1, \dots, x_n\} \in G, \tag{11}$$

where

$$D^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \dots + \alpha_n \tag{12}$$

is elliptic if for any vector $\xi = (\xi_1, \dots, \xi_n) \in R^n$ the σ-linear form

$$L(x, \xi) = \sum_{|\alpha| = \sigma} a_\alpha(x) \xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}, \ \xi \neq 0, \tag{13}$$

is different from zero.

THEOREM 2. *Let G be a Lie group and $x \to T_x$ be a unitary representation of G. If L is an elliptic element of the right invariant enveloping algebra of G, then*

$$\overline{T(L^+)} = \big(T(L)\big)^*. \tag{14}$$

In particular, if L is elliptic and symmetric, then $T(L)$ is essentially self-adjoint (e.s.a.).

PROOF: Consider first the special case of an elliptic element L represented in the form $L = K^+K$ for some $K \in E$. We combine the following two observations to prove that $A \equiv T(L)$ is e.s.a.

(i) $(A+I)^{-1}$ is bounded. To show this notice that $(Au, u) \geqslant 0$ for all u in Gårding subspace D_G. In fact, because the map $X \to T(X)$ defines a representation of E, we have $A = T(K^+K) = T(K^+)T(K)$. Thus it follows from proposition 1 that

$$(Au, u) = \big(T(K^+)T(K)u, u\big) = \big(T(K)u, T(K)u\big) \geqslant 0,$$

for all u in D_G. Consequently,

$$\big((A+I)u, (A+I)u\big) = (Au, Au) + 2(Au, u) + (u, u) \geqslant (u, u) > 0$$

hence $(A+I)$ is positive definite, and on the domain on the operator $(A+I)^{-1}$ consisting of vectors $v = (A+I)u$,

$$||(A+I)^{-1}v||^2 \leqslant ||v||^2,$$

i.e., $(A+I)^{-1}$ is bounded.

(ii) $A+I$ has a dense range. To show this, suppose that $u \in H$ is orthogonal to the range of $A+I$. Then,

$$\big((A+I)T(\varphi)u, u\big) = 0 \qquad \text{for all } \varphi \in C_0^\infty(G). \tag{15}$$

It follows from eq. 1 (17) that for $M \in E$, $T(M)T(\varphi)u = T(\tilde{M}\varphi)u$. Hence, eq. (15) can be written in the form

$$\int_G [(\tilde{L}+1)\varphi](x)(T_x u, u)\,\mathrm{d}x = 0. \tag{16}$$

This means that the function $f(x) = (T_x u, u)$ is a weak solution of the partial differential equation $(\tilde{L}+1)f = 0$ on G. Because $\tilde{L}+1$ is elliptic, $(T_x u, u)$ is analytic (cf. F. John 1951) and

$$(\tilde{L}+1)(T_x u, u) = 0 \tag{17}$$

in the ordinary pointwise sense. Because the function $x \to (T_x u, u)$ is positive definite from exercise 3.11.3.4* we have, for $u \neq 0$,

$$((\tilde{L}+1)f)(e) = (\tilde{L}f)(e) + (u, u) > 0$$

which contradicts eq. (17). Hence, $u = 0$ and thus $(A+I)^{-1}$ is bounded and densely defined. These two observations imply that A is e.s.a. by lemma 5.3 of app. B.

Now let L be a general elliptic element. From the previous case, we deduce that $T(L^+L) = T(L^+)T(L)$ is e.s.a. This implies

$$\overline{T(L^+)} = T(L)^*$$

according to lemma 5.4 of app. B. Consequently, if an elliptic element $L \in E$ is symmetric (i.e., $L^+ = L$), then

$$\overline{T(L)} = (T(L))^*,$$

i.e., $T(L)$ is essentially self-adjoint. ▼

A more general criterion for the essential self-adjointness of elements of the enveloping algebra is provided by the following important theorem.

THEOREM 3. *Let G be a Lie group and let T be a unitary representation of G. Let L be an elliptic element of the right invariant enveloping algebra E of G, such that $L^+ = L$. If M is any arbitrary element of E such that $T(M^+M)$ commutes with $T(L)$, then*

$$\overline{T(M^+)} = T(M)^*. \tag{18}$$

In particular, if M is in addition symmetric, then $T(M)$ is essentially self-adjoint.

PROOF: Let r be a positive integer greater than the order of the differential operator $\tilde{M}$ corresponding to an element M in E, and let $A = T(L^{2r})$, $B = T(\tilde{M}^+\tilde{M})(= T(\tilde{M}^+)T(\tilde{M}))$ and $C = A+B$. Then, L^{2r} is elliptic because L is elliptic; and $L^{2r}+\tilde{M}^+\tilde{M}$ is elliptic, because the order of L^{2r} is greater than the order of $\tilde{M}^+\tilde{M}$. Consequently, A and C are representatives of elliptic symmetric operators in E and consequently are e.s.a. by th. 2. Moreover, these operators commute on the Gårding domain.

We now show that the closures $\bar{A}$ and $\bar{C}$ also commute, i.e., these operators have mutually commuting spectral resolutions. To see this, notice that the bounded operators $(1+A)^{-1}(1+C)^{-1}$ and $(1+C)^{-1}(1+A)^{-1}$ coincide on their common domain, i.e. on the range of $(1+A)(1+C) = (1+C)(1+A)$. Moreover, the operator $(1+A)(1+C)$ is the representative of a symmetric elliptic operator and so is e.s.a., by th. 2. It has also a dense range. This follows from the fact that the operator $D \equiv A+C+AC$ is positive definite. Hence, the operator $I+\overline{D}$ is positive definite and self-adjoint; and, therefore $(I+\overline{D})^{-1}$ is bounded. This implies that the range $R(I+\overline{D}) = R(\overline{I+D}) = H$, i.e., $R(I+D)$ is dense in H. Consequently, $\overline{A}$ and $\overline{C}$ commute.

We now prove the main part of the theorem, namely, $\overline{T(M^+)} = T(M)^*$. Notice first that $T(M^+) \subset (T(M))^*$ by proposition 1. Hence, if we can show that the operator $B = T(M^+)T(M)$ is e.s.a., then the assertion of the theorem follows from lemma 5.4, app. B. Set $B_1 = \overline{C}-\overline{A}$. To show that B is e.s.a., we first prove that $B_1 \subset \overline{B}$. Let $v \in D(B_1) = D(\overline{A}) \cap D(\overline{C})$. We can select a sequence $\{v_n\}_1^\infty$ such that

$$D_G \in v_n \to v \quad \text{and} \quad Cv_n \to \overline{C}v, \tag{19}$$

because $\overline{C}$ is the closure of C. For an arbitrary u in D_G, we have $||Au|| \leqslant ||Cu||$. In fact, because $A = T(L^r)T(L^r)$; and the fact that $T(L)$ and $T(M^+M)$ commute, we have

$$(Cu, Cu) = (Au, Au)+(Bu, Bu)+2\big(BT(L^r)u, T(L^r)u\big).$$

Here the second and the third terms are positive (because B is positive definite) and consequently $||Cu|| \geqslant ||Au||$. Setting $u = v_n - v_m$, we obtain

$$||Av_n - Av_m|| \leqslant ||Cv_n - Cv_m|| \to 0 \quad \text{for} \quad m > n \to \infty.$$

Hence, the sequence $\{Av_n\}$ is convergent and

$$Av_n \to \overline{A}v, \tag{20}$$

because $\overline{A}$ is closed. Using eqs. (19) and (20), we obtain

$$Bv_n = Cv_n - Av_n \to \overline{C}v - \overline{A}v = B_1 v.$$

Therefore, $B_1 v = \overline{B}v$ for $v \in D(B_1)$, i.e., $B_1 \subset \overline{B}$. The operator B_1 is e.s.a., by lemma 5.4 of app. B

$$\overline{B}_1 = B_1^* \supset \overline{B}^* \supset \overline{B} \supset \overline{B}_1.$$

Consequently, we have $\overline{B}^* = \overline{B}$, i.e. B is e.s.a. ▼

Th. 3 implies a series of corollaries to determine when an element M in the enveloping algebra E is represented by an essentially self-adjoint operator.

In the following, an important role is played by the so-called *Nelson operator* Δ,

$$\Delta = X_1^2 + \ldots + X_d^2, \quad d = \dim L, \tag{21}$$

where X_i, $i = 1, 2, \ldots, d$, are generators of G. This operator is elliptic. Indeed, because $X_i = a_{ik}(x)\partial^k$, then by eq. (13) we have

$$\Delta(x, \xi) = a_{ik}(x)\, a_{ik'}(x)\, \xi^k \xi^{k'} = b^2 > 0, \tag{22}$$

where $b_i(x) = a_{ik}(x)\xi^k$.

COROLLARY 1. *Let G be an abelian or a compact Lie group. Then the representative T(M) of an arbitrary element M of the enveloping algebra E satisfies the condition* (18), *i.e.*,

$$\overline{T(M^+)} = (T(M))^*.$$

In particular, if $M^+ = M$, then $T(M)$ is essentially self-adjoint.

PROOF: The enveloping algebra of every abelian Lie algebra contains the elliptic, symmetric element of the form

$$\Delta_0 = X_1^2 + \dots + X_d^2,$$

which obviously is the center of E. Hence, every M in E is e.s.a. by th. 3.

Every compact Lie group is a direct product of its center G_0 and of invariant simple subgroups G_i, $i = 1, 2, \dots, N$ (cf. th. 3.8.2). The operator

$$\Delta^{(i)} = (X_1^{(i)})^2 + (X_2^{(i)})^2 + \dots + (X_{d_i}^{(i)})^2, \quad i = 0, 1, 2, \dots, N, \quad d_i = \dim G_i,$$

in the basis in which the Cartan metric tensor of the Lie algebra of G_i is diagonal, is the central symmetric elliptic element of the enveloping algebra E_i of G_i, $i = 0, 1, \dots, N$. Thus, the operator $\Delta = \sum_{i=0}^{N} \Delta^{(i)}$ is the central symmetric elliptic element of the enveloping algebra E of G. Consequently, for every M in E, we have

$$\overline{T(M^+)} = (T(M))^*,$$

by th. 3. Then, if $M^+ = M$, $T(M)$ is e.s.a. ▼

Next we consider the case of noncompact, semisimple Lie groups. In this case, we have

COROLLARY 2. *Let G be a noncompact semisimple Lie group, K a maximal compact subgroup of G and let*

$$\Delta_k = \sum_{i=1}^{\dim K} X_i^2$$

be the second-order Casimir operator of K. Then, the representative T(M) of any element M of the enveloping algebra E of G, which commutes with Δ_k, satisfies the condition $\overline{T(M^+)} = (T(M))^*$.

Remark: In particular, all symmetric Casimir operators of G or of a subgroup $G_i \supset K$ are essentially self-adjoint.

PROOF OF COROLLARY 2: It follows from th. 1.2.7 that the Cartan metric tensor of the Lie algebra of G can be diagonalized. Hence the second-order Casimir operator of G has the form $C_2 = -\Delta_K + \Delta_P$, where $\Delta_p = \sum_{\dim K+1}^{\dim L} X_i^2$. The corresponding Nelson operator $\Delta = (\Delta_K + \Delta_P)$ is elliptic and symmetric on G (cf. eq. (22)). Because $-\Delta_K + \Delta_P$ is central in E, any M in the enveloping algebra E

of G, which commutes with Δ_K, also commutes with $(\Delta_K+\Delta_P) = C_2+2\Delta_K$. Consequently,

$$[M^+,(\Delta_K+\Delta_P)] = [M, (\Delta_K+\Delta_P)]^+ = 0.$$

Hence,

$$\overline{T(M^+)} = (T(M))$$

by th. 3. ▼

In the general case we have

COROLLARY 3. *Let G be an arbitrary Lie group and let M be a central element of E. Then, $\overline{T(M^+)} = (T(M))^*$. In particular if $M^+ = M$, then $T(M)$ is essentially self-adjoint.*

PROOF: The element M in E commutes with the symmetric elliptic element Δ given by eq. (21). Consequently, $[M^+, \Delta] = [\Delta, M]^+ = 0$. Hence, the assertion follows by th. 3. ▼

Remark: If M and N are central and symmetric elements of E, then the operators $T(M)$ and $T(N)$ are e.s.a. by corollary 3 and by virtue of the formula 1(17).

However, it is not *a priori* evident that the self-adjoint operators $\overline{T(M)}$ and $\overline{T(N)}$ strongly commute and whether $\overline{T(M)}$ and T_x, $x \in G$, commute. These important problems are solved in sec. 5 after the elaboration of a criterion of commutativity (cf. th. 5.3). ▼

The next corollary shows that representatives of all generators associated with a unitary representation of G are essentially self-adjoint, i.e., they have proper spectral properties. The fact is of importance in physical applications.

COROLLARY 4. *Let $x \to T_x$ be a unitary representation of an arbitrary Lie group G. Let X be an arbitrary element of the Lie algebra of G and $p(X)$ any real polynomial. Then the operator $T(p(\mathrm{i}X))$ is essentially self-adjoint on D_G. In particular, $T(\mathrm{i}X)$ is essentially self-adjoint.*

PROOF: The element $p(\mathrm{i}X)$ is the symmetric element, $((p(\mathrm{i}X))^+ = p(\mathrm{i}X)$, of the enveloping algebra of the one-parameter (and therefore abelian) subgroup G_1 generated by X; hence, the representative $T(p(\mathrm{i}X))$ is e.s.a. on the Gårding domain D_G, by corollary 1. In particular, the representative of $p(\mathrm{i}X) = \mathrm{i}X$ is e.s.a. ▼

It should be emphasized that in a unitary representation of a Lie group the symmetric elements of the enveloping algebra are not necessarily represented by essentially self-adjoint operators on the Gårding subspace. To illustrate this important fact, we consider the following simple counter-example.

EXAMPLE 2. Let G be the two-parameter group of transformations $x' = ax+b$, $a > 0$ of the real line R and let $H = L^2(-\infty, +\infty)$. The unitary representation of G in H is given by the formula

$$(T_g u)(x) = a^{-1/2} u\left(\frac{x+b}{a}\right), \quad u \in H, \tag{23}$$

The generators of one-parameter subgroups are given by

$$T(X) = \mathrm{d}/\mathrm{d}x, \qquad T(Y) = \mathrm{i}\exp x. \tag{24}$$

The symmetric element $M = XY+YX$ in the enveloping algebra E of G has the representative $T(XY+YX) = 2\mathrm{i}e^x\mathrm{d}/\mathrm{d}x+\mathrm{i}e^x$, which is a symmetric operator by proposition 2.1. In order to verify that M is e.s.a. or may be extended to an e.s.a. operator we have find its deficiency index according to th. 1.4 of app. B. Solving the first-order differential equation $M^*u_\pm = \pm iu_\pm$ we find

$$\begin{aligned} D_+ &= \left\{C\exp\left[-\frac{1}{2}(x+\exp(-x))\right]\right\}. \\ D_- &= \{0\}. \end{aligned} \tag{25}$$

The set D_- is trivial because the second solution

$$u_- = C\exp\left[-\frac{1}{2}(x-\exp(-x))\right]$$

does not belong even to $L^2(-\infty, +\infty)$. Hence, the deficiency indices are $(1, 0)$ and consequently the operator $T(XY+YX)$ has no self-adjoint extension by th. of app. B. 1.4. ▼

It should also be emphasized that all the above results are true under the assumption that the representation of the Lie algebra was derived from a given unitary representation of the corresponding Lie group according to eq. 1 (11). These results might not be true if we have a representation of a Lie algebra which cannot be integrated to a global-unitary representation of the corresponding Lie group. An example is considered in ch. 21, § 5.

We now give an interesting application of group representation theory and of Nelson operator to quantum mechanics. It is well known that one of the basic problems of quantum mechanics is the construction of a domain $D \subset H$ on which the energy operator is self-adjoint. We give an example which provides a solution of this problem using group theoretical technique.

EXAMPLE 3. Let $X_1 = \partial/\partial x, X_2 = \mathrm{i}x, X_3 = \mathrm{i}\lambda x^2$ and $X_4 = \mathrm{i}I$ be skew-symmetric bases of a Lie algebra L on $L^2(R^1)$ with all $\lambda \in R^1$. It is evident that L is nilpotent with obvious commutation relations and that the subalgebra $\tilde{L}$ generated by elements X_2, X_3 and X_4 is invariant. By virtue of th. 3.5.1 we know that every group element of nilpotent Lie group G associated with L can be written as the product of one-parameter subgroups. Since all one-parameter subgroups in $L^2(R^1)$ generated by X_k are unitary the map

$$g(\alpha) \to T_{g(\alpha)} = \mathrm{e}^{\alpha_1 X_1}\mathrm{e}^{\alpha_2 X_2}\mathrm{e}^{\alpha_3 X_3}\mathrm{e}^{\alpha_4 X_4} \tag{26}$$

provides a unitary representation of G in $L^2(R^1)$.

By virtue of eqs. (22) and (6) the Nelson operator Δ is elliptic and symmetric. Hence by virtue of th. 2 the operator

$$T(\Delta) = \sum_{k=1}^{4} T(X_k)^2 = \frac{d^2}{dx^2} - x^2 - \lambda x^4 - I$$

is essentially self-adjoint on the Gårding domain D_G for the representation (26). This implies that the energy operator for the anharmonic oscillator which coincides with $-T(\Delta)-I$ is also essentially self-adjoint on D_G. ▼

It is evident that the above method can be generalized to a large class of energy operators $H = H_0+V$ with the potential $V(x) = \sum_{k=1}^{n} c_k x^{2k}$.

§ 3. Analytic Vectors and Analytic Dominance

In this section we introduce two fundamental concepts: the concept of analytic vectors and the concept of analytic dominance.

In subsec. A, we discuss the basic properties of the analytic vectors for an unbounded operator and, in particular, prove that every self-adjoint operator has a dense set of analytic vectors. In subsec. B we develop a calculus of absolute values of operators, which provides the basis for the theory of analytic dominance of operators.

In subsec. C we prove the main theorem of the calculus of absolute values and we introduce, by means of it, the concept of analytic dominance.

Finally, we develop the theory of analytic dominance for a Lie algebra of operators.

A. *Analytic Vectors*

Let A be an operator in a Hilbert space H. An element $u \in H$ is said to be an *analytic vector for A* if the series expansion of $\exp(As)u$ has a positive radius of absolute convergence, i.e.,

$$\sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n < \infty, \tag{1}$$

for some real $s > 0$. If A is bounded, i.e., $||Au|| \leqslant C||u||$ for every $u \in H$, then

$$\sum_{n=0}^{\infty} \frac{1}{n!} ||A^n u|| s^n \leqslant \sum_{n=0}^{\infty} \frac{1}{n!} C^n s^n = \exp(Cs) < \infty.$$

Hence, for a bounded operator A, every vector u in H is an analytic vector for A. Thus, only analytic vectors for unbounded operators will be of interest.

If u and v in H are analytic vectors for A, then $\alpha u+\beta v$, $\alpha, \beta \in C^1$, is also an analytic vector. Indeed, using the inequality

$$||A(\alpha u+\beta v)|| \leqslant |\alpha|\,||Au|| + |\beta|\,||Av||,$$

we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} ||A^n(\alpha u+\beta v)||s^n \leqslant |\alpha| \sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n + |\beta| \sum_{n=0}^{\infty} \frac{||A^n v||}{n!} s^n < \infty .$$

Hence, analytic vectors for an operator A form a linear subspace in H.

The concept of an analytic vector is useful because for some class of unbounded operators they form a dense set in the Hilbert space H. Indeed, we have:

LEMMA 1. *If an operator A is self-adjoint, then A has a dense set of analytic vectors.*

PROOF: This lemma is a direct consequence of the spectral theorem. Indeed, let $E(\lambda)$ be the spectral resolution of the identity, associated with A, and let $\delta = [a, b]$ be a bounded interval of the real line. Then, any vector in the range of $E(\delta)$ is an analytic vector for A. In fact, if $A = \int_{\mathrm{Sp}A} \lambda \, \mathrm{d}E(\lambda)$, then by eq. 3(16) of app. B we obtain

$$||A^n E(\delta)u|| \leqslant |c|^n ||E(\delta)u|| \leqslant |c|^n ||u||,$$

where $c = \max(|a|, |b|)$. Hence,

$$\sum_{n=0}^{\infty} \frac{||A^n E(\delta)u||s^n}{n!} \leqslant ||u|| \sum_{n=0}^{\infty} \frac{|c|^n s^n}{n!} = ||u|| \exp(|c|s) < \infty . \tag{2}$$

Consequently, $E(\delta)u$ is an analytic vector for A. According to the corollary 2 to the spectral theorem (cf. app. B.3), the set of all vectors of the form $v = E(\delta)u$, where $\delta = [a, b]$ runs over all finite intervals of R and u runs over H is dense in H. Hence, A has a dense set of analytic vectors. ▼

Remark 1: The inverse of lemma 1 is also true: If a closed symmetric operator has a dense set of analytic vectors, then it is self-adjoint (cf. Nelson 1959, lemma 5.1).

Remark 2: Clearly, iX has the same analytic vectors as X. Hence, lemma 1 and remark 1 remain true if 'symmetric' is replaced by 'skew-symmetric' and 'self-adjoint' by 'skew-adjoint'.

EXAMPLE 1. Let A be a self-adjoint unbounded operator in H. If A has a discrete spectrum (as for instance for

$$A = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\varphi} \quad \text{in } L^2(0, 2\pi)),$$

then every eigenvector u_n of A with eigenvalue λ_n is an analytic vector. Indeed,

$$\sum_{k=0}^{\infty} \frac{||A^k u_n||s^k}{k!} = \sum_{k=0}^{\infty} \frac{||u_n||(\lambda_n s)^k}{k!} = ||u_n|| \exp(\lambda_n s) < \infty .$$

The linear envelope of $\{u_n\}$ forms the dense set of analytic vectors for A.

If A has only continuous spectrum (e.g.,

$$A = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{in } L^2(-\infty, \infty)) ,$$

then, by the proof of lemma 1, the dense set of analytic vectors consists of all vectors of the form

$$v = E(\delta)u, \qquad E(\delta) = E(\lambda)-E(\mu), \tag{3}$$

where $E(\lambda)$ is the spectral resolution of the identity associated with A, δ runs over all bounded subsets of R and u runs over the space H. ▼

B. *The Absolute Value of an Operator*

We now develop the calculus of the so-called absolute values of operators. Let $O(H)$ be the set of all linear operators in H. We recall that for A, B in $O(H)$ the sum $A+B$ has as domain $D(A)\cap D(B)$ and the product AB has as domain the set of all vectors $u \in D(B)$ such that Bu is in $D(A)$. If $u \in D(A)$, then $||Au||$ is well defined. If $u \notin D(A)$, then we set $||Au|| = \infty$.

Let A, B, $C \in O(H)$. If the relation

$$||Cu|| \leqslant ||Au||+||Bu|| \tag{4}$$

is satisfied for all $u \in H$, we *symbolically* represent it in the form

$$|C| \leqslant |A|+|B|. \tag{5}$$

The following relations are true for all operators A, B, C in $O(H)$:

1° $$|A+B| \leqslant |A|+|B|. \tag{6}$$

2° $$\text{If } |A| \leqslant |B|, \quad \text{then } |AC| \leqslant |BC|. \tag{7}$$

Indeed, if $u \in D(A)\cap D(B)$, then $||(A+B)u|| \leqslant ||Au||+||Bu||$. On the other hand if $u \notin D(A)\cap D(B)$, then $||Au||+||Bu|| = \infty$ by our convention, i.e., eq. (3) is still valid. If $||Au|| \leqslant ||Bu||$ for all u, then in particular for a vector $v = Cu$, we have $||ACu|| \leqslant ||BCu||$, i.e., inequality (7) is true.

By analogy with the absolute values of ordinary numbers, the symbol $|A|$ is called the *absolute value of A*. Formally, we can define the absolute value $|A|$ of A as the set consisting of A alone. Let $|O(H)|$ be the free abelian semigroup with the set of all $|A|$, with A in $O(H)$ as generators (cf. app. A.3). By definition of a free abelian semigroup, an element $\alpha \in |O(H)|$ is a finite formal sum of the form

$$\alpha = |A_1|+ \dots +|A_l|. \tag{8}$$

We used the concept of a free semigroup in the definition of $|O(H)|$ because we do not consider an inverse element to an element α in $|O(H)|$.

An element $\beta \in |O(H)|$ of the form

$$\beta = |B_1|+ \dots +|B_m| \tag{9}$$

is equal to α if the summands are identical, except possibly for their order. If a is a positive number, we shall identify a with $|aI|$, where I is the identity operator on H. We define the product $\alpha\beta$ of α and β given by eqs. (8) and (9) by the formula

$$\alpha\beta \equiv \sum_{i=1}^{l}\sum_{j=1}^{m} |A_i B_j|. \tag{10}$$

The set $|O(H)|$ with operation (10) is a semiring, and because of the identification $a \equiv |aI|$, it is a semialgebra. For $\alpha \in |O(H)|$ given by eq. (8), we define $||\alpha u||$ for all u in H by the formula

$$||\alpha u|| \equiv ||A_1 u|| + \ldots + ||A_l u||. \tag{11}$$

We set $\alpha \leqslant \beta$ if $||\alpha u|| \leqslant ||\beta u||$ for all u in H. In the following, we shall also use the elements

$$\varphi = \sum_{n=0}^{\infty} \alpha_n s^n \tag{12}$$

of the semialgebra consisting of all power series in some variable s with coefficients in $|O(H)|$. If $\psi = \sum_{n=0}^{\infty} \beta_n s^n$, we define $\varphi \leqslant \psi$ if $\alpha_n \leqslant \beta_n$, $n = 0, 1, 2, \ldots$, and we define $||\varphi u||$ for u in H by the formula

$$||\varphi u|| \equiv \sum_{u=0}^{\infty} ||\alpha_n u|| s^n. \tag{13}$$

We further define $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\varphi(s)$ and $\int_0^s \varphi(t)\mathrm{d}t$ by formal differentiation and integration

With these conventions we have

$$||\exp(|A|s)u|| = \sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n. \tag{14}$$

Thus, with the concept $|A|$ we are able to give a closed symbol for the sum in eq. (1). Note the difference between the symbols $\exp(As)$ and $\exp(|A|s) = \sum \frac{|A|^n}{n!} s^n$ (implied, e.g., by eq. (11)).

For α given by eq. (8), we obtain by eqs. (13), (12) and (10)

$$\begin{aligned} ||\exp(\alpha s)u|| &= \sum_{n=0}^{\infty} \frac{1}{n!} ||(|A_1| + \ldots + |A_l|)^n u|| s^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant l} ||A_{i_1} \ldots A_{i_n} u|| s^n. \end{aligned} \tag{15}$$

Comparing the last expression with the expression (1), we see that $\|\exp(\alpha s)u\| < \infty$ if, and only if, the series expansion of $\exp(A_1 s_1 + A_2 s_2 + \ldots + A_l s_l)u$ is absolutely convergent for any $(s_1, \ldots, s_l)$ sufficiently small. We see, therefore, that by means of the notion of absolute value we can conveniently describe the properties of absolute convergence of the series expansion of $\exp(A_1 s_1 + \ldots + A_l s_l)u$. A vector u in H is said to be an *analytic vector for* α in $|O(H)|$ if $\|\exp(\alpha s)u\| < \infty$ for some $s > 0$. Notice that by virtue of eq. (15), if u is an analytic vector for α, then it is also an analytic vector for any A_i, $i = 1, 2, \ldots, l$.

Let us remark that the above definition of analytic vectors in the case of Lie algebras of operators gives the following

DEFINITION 1. A vector $u \in H$ is said to be *analytic vector* for the whole Lie algebra L if for some $s > 0$ and some linear basis $\{X_1, \ldots, X_d\}$ of the Lie algebra, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant d} \|X_{i_1} \ldots X_{i_n} u\| s^n$$

is absolutely convergent. ▼

The last condition is equivalent to $\|X_{i_1}, \ldots, X_{i_n} u\| \leqslant C^n n!$ for some constant $C > 0$.

We had defined in a Lie algebra the operation $(\mathrm{ad}X)Y = XY - YX$. Clearly, this operation can be defined also in $O(H)$. We now extend this operation to elements $\xi = |X_1| + \ldots + |X_d|$ and $\alpha = |A_1| + \ldots + |A_m|$ in $|O(H)|$ by the formula

$$(\mathrm{ad}\,\xi)\,\alpha = \sum_{i=1}^{d} \sum_{j=1}^{m} |X_i A_j - A_j X_i| = \sum_{i=1}^{d} \sum_{j=1}^{m} |\mathrm{ad}X_i(A_j)|. \tag{16}$$

Hence,

$$(\mathrm{ad}\,\xi)^n \alpha = \sum_{1 \leqslant i_1, \ldots, i_n \leqslant d} \sum_{j=1}^{m} |\mathrm{ad}X_{i_n} \ldots \mathrm{ad}X_{i_1} A_j|. \tag{17}$$

We shall need the notion of commutators for absolute values. For this purpose we discuss first a commutator calculus for operators, namely we evaluate the commutator of an operator A with a product of n other operators $X_n \ldots X_1$. The following lemma provides a convenient algorithm for the solution of this problem.

LEMMA 2. *If $A, X_1, \ldots, X_n$ are operators on H, then**

$$AX_n \ldots X_1 \supset X_n \ldots X_1 A - Q_n, \tag{18}$$

where

$$Q_n = \sum_{k=1}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)} A)\, X_{\sigma(n)} \ldots X_{\sigma(k+1)}, \tag{19}$$

---

* Recall that $A \supset B$, means that $D(A) \supset D(B)$, and $Au = Bu$ for all $u \in D(B)$.

and (n, k) *denotes the set of all* $\binom{n}{k}$ *permutations* σ *of* $1, 2, \ldots, n$ *such that* $\sigma(n) > \sigma(n-1) > \ldots > \sigma(k+1)$ *and* $\sigma(k) > \sigma(k-1) > \ldots > \sigma(1)$.

If $X_1 \ldots, X_n, A$ *have a common invariant domain, then equality holds in eq.* (18) *and we have*

$$[X_n \ldots X_1, A] = Q_n. \tag{20}$$

Remark: The equality (20) can be written in another form which is often used in quantum theory. In fact, for $n = 2$, and for operators having a common invariant domain, we obtain from (20)

$$\begin{aligned}[X_2 X_1, A] = Q_2 &= \mathrm{ad}X_1(A)X_2 + \mathrm{ad}X_2(A)X_1 + \mathrm{ad}X_2\,\mathrm{ad}X_1(A) \\ &= \mathrm{ad}X_2(A)X_1 + X_2\,\mathrm{ad}X_1(A) = [X_2, A]X_1 + X_2[X_1, A],\end{aligned} \tag{21}$$

and in general

$$\begin{aligned}[X_n \ldots X_1, A] = Q_n &= [X_n, A]X_{n-1} \ldots X_1 + \\ &+ X_n[X_{n-1}, A]X_{n-2} \ldots X_1 + \ldots + X_n \ldots X_2[X_1, A].\end{aligned} \tag{22}$$

This equality can be easily proved by the method of induction using eq. (21).

Proof of lemma 2: We prove formula (18) by the method of induction. We first show that

$$X_n \ldots X_1 A \supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)} A) X_{\sigma(n)} \ldots X_{\sigma(k+1)}. \tag{23}$$

For $n = 0$, eq. (23) states that $A \supset A$ and for $n = 1$ it states that

$$X_1 A \supset AX_1 + (\mathrm{ad}X_1)A, \tag{24}$$

which is also true.

Suppose now that eq. (23) holds for n and let X_{n+1} be an operator in H. Then, by eq. (24), we obtain

$$\begin{aligned}X_{n+1}X_n \ldots X_1 A &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} X_{n+1}(\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma_1} A) X_{\sigma(n)} \ldots X_{\sigma(k+1)} \\ &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} \{(\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)} A) X_{\sigma(n)} \ldots X_{\sigma(k+1)} X_{n+1} + \\ &\quad + (\mathrm{ad}X_{n+1}\,\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)} A) X_{\sigma(n)} \ldots X_{\sigma(k+1)}\}.\end{aligned} \tag{25}$$

We now show that a term

$$(\mathrm{ad}X_{\tau(k)} \ldots \mathrm{ad}X_{\tau(1)} A) X_{\tau(n+1)} \ldots X_{\tau(k+1)}, \tag{26}$$

corresponding to a permutation τ in $(n+1, k)$, occurs either before the + sign in the curly brackets of eq. (25) or after the + sign. In fact, it occurs before the + sign in the brackets (corresponding to a σ in (n, k)) if $\tau(n+1) = n+1$, and as a term after the + sign (corresponding to σ in $(n, k-1)$), if $\tau(n+1) \neq n+1$, since either $\tau(n+1)$ or $\tau(k)$ must be equal to $n+1$, by the definition of (n, k). Because $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, the correspondence is one-to-one. Consequently, eq. (23) holds for $n+1$.

In order to prove eq. (18), we note that it is true for $n = 0$ and $n = 1$. Assume that it is true for n and let X_{n+1} be an operator in H. Then eq. (18) holds with $X_n, \ldots, X_1$ replaced by $X_{n+1}, \ldots, X_2$ and with Q_n modified accordingly. Setting Q'_n for the expression which replaces Q_n and multiplying both sides by X_1, we obtain

$$AX_{n+1} \ldots X_2X_1 \supset X_{n+1} \ldots X_2AX_1 - Q'_nX_1 \supset X_{n+1} \ldots$$
$$\ldots X_2X_1A - X_{n+1} \ldots X_2(\mathrm{ad}X_1)A - Q'_nX_1.$$

Applying eq. (23) to $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A$ {with $(\mathrm{ad}X_1)A$ playing the role of A in eq. (23)}, we see that $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A + Q'_nX_1 \supset Q_{n+1}$, where each permutation τ in $(n+1, k)$ with $\tau(1) = 1$ corresponds to a term in $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A$ and each τ with $\tau(1) \neq 1$ corresponds to a term in Q'_nX_1.

Finally, if $X_1, \ldots, X_n, A$ have a common invariant domain, then both sides of eq. (15) have the same domain and, therefore, they are equal. ▼

We now prove the analog of lemma 2 for elements in $|O(H)|$.

LEMMA 3. *Let ξ and α be in $|O(H)|$. Then*

$$\alpha\xi^n \leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} [(\mathrm{ad}\,\xi)^k\alpha]\,\xi^{n-k}. \tag{27}$$

PROOF: Let $\xi = |X_1| + \ldots + |X_d|$, $\alpha = |A_1| + \ldots + |A_l|$ and let Σ^* denote the sum over all $1 \leqslant i \leqslant l$, $1 \leqslant g_1 \leqslant d, \ldots, 1 \leqslant g_n \leqslant d$. Then, using eqs. (10) and (18), we obtain

$$\begin{aligned}\alpha\xi^n &= \Sigma^*|A_iX_{g_n} \ldots X_{g_1}| \leqslant \Sigma^*|X_{g_n} \ldots X_{g_1}A_i| + \\ &\quad + \Sigma^*\left|\sum_{k=1}^{n}\sum_{\sigma\in(n,k)} (\mathrm{ad}X_{g_{\sigma(k)}} \ldots \mathrm{ad}X_{g_{\sigma(1)}}A_i)\,X_{g_{\sigma(n)}} \ldots X_{g_{\sigma(k+1)}}\right| \\ &= \xi^n\alpha + \Sigma^*\left|\sum_{k=1}^{n}\binom{n}{k}(\mathrm{ad}X_{g_k} \ldots \mathrm{ad}X_{g_1}A_i)\,X_{g_n} \ldots X_{g_{k+1}}\right|.\end{aligned} \tag{28}$$

In the last step, we took advantage of the fact that there are $\binom{n}{k}$ permutations in (n, k). Hence, by virtue of the summation over g_i we find that each term $(\mathrm{ad}X_{g_n} \ldots \mathrm{ad}X_{g_1}A_i)\,X_{g_n} \ldots X_{g_{k+1}}$ occurs $\binom{n}{k}$ times. The second term of the last equality is equal to the second term of eq. (27). ▼

C. *Analytic Dominance*

We now introduce the fundamental concept of analytic dominance for elements α and ξ in $|O(H)|$.

THEOREM 4. *Let* ξ and α *be in* $|O(H)|$. *Let* $\xi \leqslant c\alpha$, $(\operatorname{ad}\xi)^n\alpha \leqslant c_n\alpha$ *and*

$$\nu(s) = \sum_{n=1}^{\infty} \frac{c_n s^n}{n!}, \tag{29}$$

$$\varkappa(s) = \int_0^s \frac{\mathrm{d}t}{1-\nu(s)}. \tag{30}$$

Then $\exp(\xi s) \leqslant \exp[c\alpha\varkappa(s)]$. ▼

If c and c_n are such that $c < \infty$ and $\nu(s) < \infty$ *for some* $s > 0$, we shall say that α *analytically dominates* ξ.

In order to clarify the content of the theorem we first prove a corollary.

COROLLARY 1. *Let* ξ *and* α *be in* $|O(H)|$. *If* α *analytically dominates* ξ, *then every analytic vector for* α *is an analytic for* ξ.

PROOF: If α analytically dominates ξ, then ν has a positive radius of convergence and so does $\varkappa$. Consequently, by definition every analytic vector for α is an analytic vector for ξ. ▼

PROOF OF TH. 4: Let us define the elements $\pi_n \in |O(H)|$ by the recursion formulae

$$\pi_0 = |I|, \quad \pi_{n+1} = c\pi_n\alpha + \sum_{k=1}^{n} \binom{n}{k} c_k \pi_{n+1-k}. \tag{31}$$

Clearly, $\pi_1 = c\alpha$, $\pi_2 = c^2\alpha^2 + c_1 c\alpha$ and each π_n is a polynomial in α. We first show by the method of induction that

$$\alpha\xi^{n-1} \leqslant \frac{1}{c}\pi_n. \tag{32}$$

Notice that $\xi \leqslant c\alpha$ implies $\xi^n \leqslant c\alpha\xi^{n-1}$. Hence, eq. (32) implies

$$\xi^n \leqslant \pi_n. \tag{33}$$

For $n = 1$, eq. (32) says that $\alpha \leqslant \alpha$. Suppose that eq. (32) is satisfied for all $k \leqslant n$. Then, using lemma 3 the assumption of the theorem, eqs. (32) and (33), we obtain

$$\begin{aligned}
\alpha\xi^n &\leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} [(\operatorname{ad}\xi^k)\alpha]\xi^{n-k} \\
&\leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} c_k \alpha\xi^{n-k} \\
&\leqslant \pi_n\alpha + \sum_{k=1}^{n} \binom{n}{k} c_k \frac{1}{c}\pi_{n+1-k} \leqslant \frac{1}{c}\pi_{n+1}.
\end{aligned}$$

Thus, eq. (32) and consequently eq. (33) hold for any n. Let $\pi(s)$ be the power series

$$\pi(s) = \sum_{n=0}^{\infty} \frac{\pi_n}{n!} s^n. \tag{34}$$

By eq. (31) and the relation $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ we have

$$(n+1)\frac{\pi_{n+1}}{(n+1)!} = c\frac{\pi_n \alpha}{n!} + \sum_{k=1}^{n} \frac{c_k}{k!}(n+1-k)\frac{\pi_{n+1-k}}{(n+1-k)!}. \tag{35}$$

Using the definitions (29) and (30) of $\pi(s)$ and $\nu(s)$ and eq. (35), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\pi(s) = c\pi(s)\alpha + \nu(s)\frac{\mathrm{d}}{\mathrm{d}s}\pi(s), \tag{36}$$

i.e.,

$$\frac{\mathrm{d}\pi(s)}{\mathrm{d}s} = c\alpha\pi(s)/\big(1-\nu(s)\big). \tag{37}$$

Hence, from the definition of $\varkappa(s)$, we obtain

$$\pi(s) = \exp\big(c\alpha\varkappa(s)\big). \tag{38}$$

In fact, differentiating formula (38), we obtain eq. (37), and setting $s = 0$, we get $\pi(0) = |I|$, in agreement with the definition (34). Using eq. (32), we finally obtain

$$\exp(\xi s) \leqslant \pi(s) = \exp\big(c\alpha\varkappa(s)\big). \;\blacktriangledown$$

Notice that nowhere have we used the fact that the operators $A_1, \ldots, A_l$ are linear or that the carrier space H is complete.

The next useful corollary of th. 4 can be stated without using the terminology of the theory of absolute values.

COROLLARY 2. *Let $X_1, \ldots, X_d$, and A be operators on a Hilbert space H. Let k and k_n, $n = 1, 2, \ldots$, be positive numbers such that for all u in the domain of A*

$$\|X_i u\| \leqslant k(\|Au\| + \|u\|), \quad 1 \leqslant i \leqslant d, \tag{39}$$

and

$$\|\mathrm{ad}X_{i_1} \ldots \mathrm{ad}X_{i_n} Au\| \leqslant k_n(\|Au\| + \|u\|) \quad \text{for } 1 \leqslant i_1, \ldots, i_n \leqslant d. \tag{40}$$

Suppose that $k < \infty$ and that

$$\sum_{n=1}^{\infty} (k_n/n!) s^n < \infty$$

for some $s > 0$. If there is an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\|A^n u\|}{n!} s^n < \infty, \tag{41}$$

then for $(s_1, \dots, s_d)$ *sufficiently close to* $(0, \dots, 0)$, *we have*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \dots, i_n \leqslant d} \|X_{i_1} \dots X_{i_n} u\| s_{i_1} \dots s_{i_n} < \infty . \blacktriangledown \tag{42}$$

PROOF: Let $\xi = |X_1| + \dots + |X_d|$ and $\alpha = |A| + |I|$. By eq. (16), $(\operatorname{ad}\xi)^n \alpha = (\operatorname{ad}\xi)^n |A|$. By virtue of eq. (39), $\xi \leqslant dk\alpha$ and by eq. (40), $(\operatorname{ad}\xi)^n \alpha \leqslant \mathrm{d}^n k_n \alpha$. Because $\nu(s) = \sum \frac{\mathrm{d}^n k_n}{n!} s^n < \infty$ for some $s > 0$, we obtain, using th. 4, that α analytically dominates ξ. Consequently, by corollary 1, every analytic vector for α is an analytic vector for ξ, i.e., eq. (41) implies eq. (42). $\blacktriangledown$

LEMMA 5. *Let* $X_1, \dots, X_d$, *and* A *be symmetric operators on a Hilbert space* H *with a common invariant domain* D, *and let* A *be essentially self-adjoint. Let* $\xi = |X_1| + \dots + |X_d|$, $\alpha = |A| + |I|$, $\xi \leqslant c\alpha$ and $(\operatorname{ad}\xi)^n \alpha \leqslant c_n \alpha$ *with* $c < \infty$ *and* $c_n < \infty$ *for all* $n \geqslant 1$. *Then, for all finite sequences* $i_1, \dots, i_n$, *we have*

$$D(\bar{A}^n) \subset D(\bar{X}_{i_1} \dots \bar{X}_{i_n}). \tag{43}$$

Let $\tilde{D} = \bigcap_{n=1}^{\infty} D(\bar{A}^n)$ *and let* $\tilde{X}_1, \dots, \tilde{X}_d$ *and* $\tilde{A}$ *be the restriction of* $\bar{X}_1, \dots, \bar{X}_d$ *and* $\bar{A}$ *respectively, to* $\tilde{D}$. *Let* $\tilde{\xi} = |\tilde{X}_1| + \dots + |\tilde{X}_d|$, $\tilde{\alpha} = |\tilde{A}| + |I|$. *Then*

$$\tilde{\xi} \leqslant c\tilde{\alpha}, \quad (\operatorname{ad}\tilde{\xi})^n \tilde{\alpha} \leqslant c_n \tilde{\alpha} \quad \textit{for all } n \geqslant 1. \tag{44}$$

If α *analytically dominates* ξ, *then there is an* $s > 0$ *such that the set of* $u \in \tilde{D}$ *for which* $\|\exp(\tilde{\xi}s)u\| < \infty$ *is dense in* H, *and each* X_i *is essentially self-adjoint.*

PROOF: We prove the relation (43) by the method of induction. Let $n = 1$, If $u \in D(\bar{A})$, then by definition of the closure $\bar{A}$ there is a sequence u_j in $D(A) = D$ with $u_j \to u$ and $Au_j \to v = \bar{A}u$. Because $\xi \leqslant c\alpha$, we have $|X_i| \leqslant c\alpha$, $i = 1, 2, \dots, d$. Consequently, for all $i = 1, 2, \dots, d$,

$$\|X_i(u_j - u_k)\| \leqslant c(\|A(u_j - u_k)\| + \|u_j - u_k\| \to 0,$$

as $j, k \to \infty$. Hence, u is in $D(\bar{X}_i)$. By the same argument, we obtain that if $C = \operatorname{ad}X_{i_k} \dots \operatorname{ad}X_{i_1} A$, then u is in $D(\bar{C})$, i.e.,

$$D(\bar{A}) \subset D(\overline{\operatorname{ad}X_{i_k} \dots \operatorname{ad}X_{i_1} A}). \tag{45}$$

We now show that

$$D(\bar{A}^n) \subset D(\bar{A}\bar{X}_{i_1} \dots \bar{X}_{i_{n-1}}). \tag{46}$$

Notice that eq. (46) implies eq. (43) for the case $n = 1$ considered above. Suppose that relation (46) is satisfied for some n and let u be in $D(\bar{A}^{n+1})$. Because $\bar{A} = A^*$, it will suffice to show that $\bar{X}_{i_1} \dots \bar{X}_{i_n} u$ is in $D(A^*)$, i.e., that

$$(Av, \bar{X}_{i_1} \dots \bar{X}_{i_n} u) = (X_{i_n} \dots X_{i_1} Av, u)$$

is a continuous linear functional of v in D. By virtue of eq. (20), we have $X_{i_n} \dots X_{i_1} A = A X_{i_n} \dots X_{i_1} + Q_n$. By the induction hypothesis, and lemma 2, $Q_n(v, u)$ is a linear continuous functional of v in D. Moreover, because u is in $D(\bar{A}^{n+1})$. $A^* u = (\bar{A} u)$ is in $D(\bar{A}^n)$; and, therefore, $(A X_{i_n} \dots X_{i_1} v, u) = (X_{i_n} \dots \dots X_{i_1} v, A^* u)$ is, by the induction hypothesis, a continuous linear functional of v. Hence, relation (46) is satisfied.

Furthermore, eq. (46) implies that the operator $\tilde{X}_1, \dots, \tilde{X}_d$ and $\tilde{A}$ leave $\tilde{D}$ invariant.

We now show that $\tilde{\xi} \leqslant c\tilde{\alpha}$. If u is in $\tilde{D}$, then u is in $D(\bar{A})$, and there is a sequence u_j in D with $u_j \to u$, $A u_j \to \bar{A} u$. According to the relation $\xi \leqslant c\alpha$, we have

$$\sum_{i=1}^{d} \|\tilde{X}_i u\| = \sum_{i=1}^{d} \lim_{j\to\infty} \|X_i u_j\| \leqslant \lim_{j\to\infty} c(\|A u_j\| + \|u_j\|) = c(\|\tilde{A} u\| + \|u\|).$$

Hence, $\tilde{\xi} \leqslant c\tilde{\alpha}$. Similarly, it can be shown using $(\operatorname{ad} \xi)^n \alpha \leqslant c_n \alpha$ that $(\operatorname{ad} \tilde{\xi})^n \tilde{\alpha} \leqslant c_n \tilde{\alpha}$.

Now let α analytically dominate ξ, so that the power series $\nu(s)$ and $\varkappa(s)$ of th. 4 have positive radii of convergence. Let $E(\lambda)$ be the resolution of the identity for the self-adjoint operator $\bar{A}$ and let B be the set of all vectors u, such that for the same bounded set Δ, $E(\Delta) u = u$. We know by corollary 2 of the spectral theorem (app. B.3) that B is dense in H. Moreover, $B \subset \tilde{D}$ and $\|\exp(\tilde{\alpha} t) u\| < \infty$, for all u in B and $0 \leqslant t \leqslant \infty$ by eq. (2). Taking s such that $\varkappa(s) < \infty$, we obtain, by th. 4, that $\|\exp(\tilde{\xi} s) u\| \leqslant \|\exp[c\tilde{\alpha} \varkappa(s)] u\| < \infty$ for all u in B. Any analytic vector for $\tilde{\xi}$ is an analytic vector for each $\tilde{X}_i$. Hence, each $\tilde{X}_i$ is essentially self-adjoint, by remark 1 to lemma 1. Because $X_i \subset \tilde{X}_i \subset X_i^*$, we see that each $\bar{X}_i (= \bar{\tilde{X}}_i)$ is self-adjoint, i.e., X_i is essentially self-adjoint. ▼

Let L be a Lie algebra of skew-symmetric operators having a linear subspace $D \subset H$ as a common dense invariant domain and let E be the enveloping algebra of L. An element $Y \in E$ is said to be of order $\leqslant n$ if it is a real linear combination of operators of the form $Y_1 Y_2 \dots Y_k$ with $k \leqslant n$ and each Y_i in L. The set of all elements of E of order $\leqslant n$ will be denoted by $E^{(n)}$. If $X_1, \dots, X_d$, $d = \dim L$, are generators of L, then the operators

$$X_i \quad \text{and} \quad H_{ij} \equiv X_i X_j + X_j X_i \tag{47}$$

constitute a set of linear generators for $E^{(2)}$ (cf. proposition 9.2.1).

The next two lemmas and the remark show that the elliptic operator $\Delta = X_1^2 + \dots + X_d^2$ (cf. eq. 2. (21)) plays a special role in the representation theory of Lie algebras.

Lemma 6. *Let $\Delta = X_1^2 + \dots + X_d^2$. If A is an arbitrary operator in $E^{(2)}$, then for some $k < \infty$*

$$|A| \leqslant k|\Delta - I|. \tag{48}$$

Proof: For any $B = a_i X_i + a_{ij} H_{ij} \in E^{(2)}$, we have $|B| \leqslant |a_i||X_i| + |a_{ij}||H_{ij}|$, hence it suffices to prove lemma 6 for generators (47) of $E^{(2)}$. For the generators X_i, $i = 1, \ldots, d$, and all u in D, we get

$$\sum_{i=1}^{d} ||X_i u||^2 = \sum_{i=1}^{d} (X_i u, X_i u) = (-\Delta u, u)$$
$$\leqslant \left(\left(\frac{1}{2}\Delta^2 - \Delta + \frac{1}{2}\right)u, u\right) = \left(\frac{1}{2}(\Delta - I)^2 u, u\right) = \frac{1}{2}||(\Delta - I)u||^2. \quad (49)$$

Consequently, we can write

$$||X_i u|| \leqslant \left(\frac{d}{2}\right)^{1/2} ||(\Delta - I)u||. \quad (50)$$

If $u \notin D$ both sides of eq. (50) are infinite. Thus

$$|X_i| \leqslant (d/2)^{1/2} |\Delta - I|. \quad (51)$$

We now prove eq. (48) for $A = H_{ij}$. Let B^+ denote the restriction of B^* to D if B is in E. Thus, $(X_{i_1} \ldots X_{i_n})^+ = (-1)^n X_{i_n} \ldots X_{i_1}$. Let P be the set of elements in E consisting of finite sums of the form $\sum_r Y_r^+ Y_r$. The operator $-\Delta$ is clearly in P. Moreover,

$$-\Delta + H_{ij} = (X_i - X_j)^+ (X_i - X_j) + \sum_{k \neq i,j} X_k^+ X_k, \quad i \neq j,$$
$$-2\Delta + H_{ii} = 2 \sum_{k \neq i} X_k^+ X_k, \quad (52)$$
$$-\Delta - H_{ij} = (X_i + X_j)^+ (X_i + X_j) + \sum_{k \neq i,j} X_k^+ X_k.$$

Hence, if $A = \sum a_{ij} H_{ij}$, a_{ij} real, then there is an $a \geqslant 0$ such that

$$-a\Delta + A \in P. \quad (53)$$

Consider now the operator $4\Delta^2 - H_{ij}^2$. We have

$$4\Delta^2 - H_{ij}^2 = (2\Delta - H_{ij})(2\Delta + H_{ij}) + A_1,$$

where $A_1 = 2[H_{ij}, \Delta]$ is in $E^{(3)}$ by eq. (21). The operator $-(2\Delta - H_{ij}) = \sum_k Y_k^+ Y_k$ with Y_k in L by eq. (52). Similarly $-(2\Delta + H_{ij}) = \sum_l Z_l^+ Z_l$ with Z_l in L. Therefore, using the commutation relations of the Lie algebra L, we obtain

$$4\Delta^2 - H_{ij}^2 = \sum_{k,l} (Y_k Z_l)^+ (Y_k Z_l) + A_2, \quad (54)$$

where A_2 is in $E^{(3)}$.

By proposition 9.2.1, $E^{(3)}$ is spanned by operators (47), and by those of the form

$$H_{ijk} = X_i X_j X_k + X_i X_k X_j + X_j X_i X_k + X_j X_k X_i + X_k X_i X_j + X_k X_j X_i. \quad (55)$$

Consequently, we may write $A_2 = \sum a_{ij}H_{ij}+S$, where S is a real linear combination of the X_i and H_{ijk}. Because the other terms in eq. (54) are symmetric, S must be symmetric. But the X_i and H_{ijk} are skew-symmetric so that S is also skew-symmetric. Consequently, $S = 0$. Therefore, by eq. (53) there is an $a \geqslant 0$ such that $-a\Delta + A_2$ is in P. Hence, by eq. (54), $4\Delta^2 - H_{ij}^2 - a\Delta$ is also in P. Using the fact that $(\Delta^2 u, u) \geqslant 0$, $(\Delta u, u) \leqslant 0$ and setting $k = \max(2, a/4)$, we obtain for all u in D

$$\begin{aligned}||H_{ij}u||^2 &\leqslant \big((4\Delta^2 - a\Delta)u, u\big) \leqslant 4(\Delta^2 u, u) - a(\Delta u, u) + \frac{a^2}{16}(u, u)\\ &\leqslant k^2(\Delta^2 u, u) - 2k^2(\Delta u, u) + k^2(u, u) = k^2||(\Delta - I)u||^2.\end{aligned}$$

Hence, $||H_{ij}u|| \leqslant k||(\Delta - I)u||$. If $u \notin D$, then both sides of this inequality are infinite. Hence, $|H_{ij}| \leqslant k|\Delta - I|$. ▼

LEMMA 7. *Let* $\xi = |X_1| + \ldots + |X_d|$ *and let* $\alpha = |\Delta - I|$. *Then* α *analytically dominates* ξ. *In fact,* $\xi \leqslant \left(\frac{d}{2}\right)^{1/2} \alpha$ *and there is a* $c < \infty$ *such that for all* $n \geqslant 1$, $(\operatorname{ad}\xi)^n \alpha \leqslant c^n \alpha$. *Also,* $|\Delta| + |I|$ *analytically dominates* ξ.

PROOF: The inequality $\xi \leqslant \left(\frac{d}{2}\right)^{1/2} \alpha$ directly follows from eq. (49). To prove $(\operatorname{ad}\xi)^n \alpha \leqslant c^n \alpha$, we first introduce the norm $|||\cdot|||$ in $E^{(2)}$. Notice that $E^{(2)}$ is in fact a finite-dimensional vector space because it is spanned by elements (47). If A is in $E^{(2)}$, we define $|||A|||$ to be the smallest number, such that $|A| \leqslant k\alpha$. By lemma 6 this is always finite and provides a norm in $E^{(2)}$. Moreover, if $|||A||| = 0$, then $A = 0$. Hence, $E^{(2)}$ with this norm is a finite-dimensional Banach space. For any A in $E^{(2)}$ $(\operatorname{ad}X_i)A$ is in $E^{(2)}$ by eq. (21). Because $(\operatorname{ad}X_i)$ is a linear map in the finite-dimensional space $E^{(2)}$, it is continuous in the norm $|||\cdot|||$, hence there is a $c_i < \infty$ such that $|||(\operatorname{ad}X_i)A||| \leqslant c_i|||A|||$. Set $c = d\max c_i$. By eq. (17) $(\operatorname{ad}\xi)^n\alpha$ is the sum of d^n terms of the form $|\operatorname{ad}X_{i_n} \ldots \operatorname{ad}X_{i_1}\Delta|$ which is $\leqslant c_{i_n} \ldots c_{i_1}\alpha$. Therefore, $(\operatorname{ad}\xi)^n\alpha \leqslant c^n\alpha$. Because $c < \infty$ and the quantity $\nu(s)$ of th. 4 is finite ($\nu(s) = \exp(cs)$), α analytically dominates ξ. If we set $\alpha' = |\Delta| + |I|$, then $(\operatorname{ad}\xi)^n\alpha' < c^n\alpha \leqslant c^n\alpha'$, by eq. (17) and inequality $||(\Delta - I)u|| \leqslant ||\Delta u|| + ||u||$. Hence, $|\Delta| + |I|$ also analytically dominates ξ. ▼

Remark: Lemma 6 can be generalized. One can show that if B is in $E^{(2m)}$ ($m = 1, 2, \ldots$), then for some $k < \infty$

$$|B| \leqslant k\alpha^m,$$

where $\alpha^m = |(\Delta - I)^m|$. Moreover, if the operator coresponding to $\eta = |Y_1| + \ldots + |Y_l|$ is in $E^{(2m)}$ and $\operatorname{ad}Y_j$, $j = 1, 2, \ldots, l$, maps $E^{(2m)}$ onto itself, then α^m analytically dominates η (cf. Nelson 1959, lemma 6.3). However, we shall not need these results.

§ 4. Analytic Vectors for Unitary Representations of Lie Groups

We have shown in § 1 that every representation T of a Lie group G gives rise in a natural manner to a representation of its Lie algebra L, defined on the Gårding subspace D_G. However, this correspondence as it stands is not very satisfactory. In fact, it may occur that a subspace $D \subset D_G$ or D_G itself which is invariant under the Lie algebra L is not invariant under G. For instance, if G is the one-parameter translation group represented in the Hilbert space $L^2(-\infty, +\infty)$ by the formula $T_x f(y) = f(x+y)$, then every subspace $C_0^\infty(0, n)$, $n = 1, 2, \ldots$, is invariant under the operator $X = \dfrac{\mathrm{d}}{\mathrm{d}x}$ of the Lie algebra. However, it is obviously not invariant under the group of translations.

In general, the problem stems from the fact that a Taylor series of regular functions does not necessarily converge to a regular function. Indeed on the Gårding subspace for a generator X_i in L, φ in $C_0^\infty(G)$ and $u(\varphi)$ in D_G, the vector

$$T(X_i)^n u(\varphi) = u(\tilde{X}_i^n \varphi) \tag{1}$$

is a regular vector for T of G, because $T_x u(\tilde{X}_i^n \varphi) = \int (\tilde{X}_i^n \varphi)(x^{-1}y)\, T_y u \mathrm{d}y$, but the expansion

$$\sum_{n=0}^{N} \frac{t^n}{n!} T(X_i)^n u(\varphi) = \int \sum_{n=0}^{N} \frac{t^n}{n!} \tilde{X}_i^n \varphi(l)\, T_x u \mathrm{d}x, \tag{2}$$

in general, does not even converge as $N \to \infty$. Those vectors u in H, for which the expansion $\sum \dfrac{t^n}{n!} T(X_i)^n u$, $i = 1, 2, \ldots, \dim L$, converges, are of special interest and, according to sec. 3, are called *analytic vectors* for the representatives $T(X_i)$. The properly chosen analytic vectors assure the satisfactory connection between representations of L and G in H. We shall show in particular that the invariant subspaces of analytic vectors relative to $T(L)$ are also invariant subspaces of analytic vectors relative to $T(G)$ and conversely.

This section is devoted to the analysis of properties of analytic vectors for unitary representations of Lie groups. We first establish the connection between analytic vectors for a group representation $x \to T_x$ and analytic vectors for operators $T(X), X \in L$, in the sense of eq. 3 (1).

LEMMA 1. *Let $x \to T_x$ be a representation of a Lie group G in a Hilbert space H. Let $X_1, \ldots, X_d$, $d = \dim L$, be a basis for the Lie algebra L of G and let $\xi = |T(X_1)| + \ldots + |T(X_d)|$. Then, if $u \in H$ is an analytic vector for ξ, then u is an analytic vector for the representation T of G.*

PROOF: Notice that if $T_x u$ is analytic in a neighborhood of e in G, then $T_x u$ is analytic everywhere. Indeed,

$$T_x u = T_y T_{y^{-1}x} u$$

is analytic if x is near to a fixed element $y \in G$. We know, moreover, that the exponential map is an analytical isomorphism of a neighborhood of 0 in L with a neighborhood of e in G (cf. th. 3.10.1). Hence, it is sufficient to prove that if $u \in H$ is an analytic vector for ξ, then $X \to T_{\exp X} u$ is analytic in some neighborhood of 0 in L. Let $X = X_1 t_1 + \ldots + X_d t_d$. Then, $T_{\exp X} u$ is analytic in some neighborhood of 0 in L, if and only if

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha_1 + \ldots + \alpha_d = n} \|\psi_{\alpha_1 \ldots \alpha_d}\| t_1^{\alpha_1} \ldots t_d^{\alpha_d} < \infty \tag{3}$$

for $t_1, \ldots, t_d$ sufficiently small, where $\psi_{\alpha_1 \ldots \alpha_n}$ is the coefficient of $t_1^{\alpha_1} \ldots t_d^{\alpha_d}$ in the expansion of $\exp[T(X_1)t_1 + \ldots + T(X_d)t_d]u$. Using eq. 3 (15), we see that $\|\psi_{\alpha_1 \ldots \alpha_d}\|$ is the norm of the sum of several term which also occur in the expansion of $\|\exp(\xi s)u\|$, if we take $t_1 = \ldots = t_d = s$. Therefore, the left-hand side of eq. (3) is $\leqslant \|\exp(\xi s)u\|$, when $s = \max(|t_1|, \ldots, |t_d|)$ and consequently it is $< \infty$, by our assumption. ▼

Remark 1: The above result can be sharpened. One can show that a vector $u \in H$ is an analytic vector if and only if u is analytic for the representation T of G (cf. Nelson 1959, lemma 7.1). ▼

We now show two basic properties of the set of analytic vectors for a unitary representation T of a Lie group G.

THEOREM 2. *Let $x \to T_x$ be a unitary representation of a Lie group G on the Hilbert space H. Then the set A_T of analytic vectors for T is dense in H. If $X_1, \ldots, X_d$ is a basis for the Lie algebra L of G, and $\Delta = X_1^2 + \ldots + X_d^2$, then any analytic vector for $\overline{T(\Delta)}$ is an analytic vector for T of G, and the set $A_{T(\Delta)}$ of such vectors is dense in H.*

PROOF: Let $T(X_i)$, $i = 1, 2, \ldots, d$, denote the images of the generators X_i in L, and let $T(\Delta) = T(X_1)^2 + \ldots + T(X_d^2)$. By proposition 1.2, the $T(X_i)$ are skew-symmetric on the Gårding domain D_G, and by th. 2.3 $T(\Delta)$ is essentially self-adjoint.

Set $\tilde{D} = \bigcap_{n=1}^{\infty} D(\overline{T(\Delta)}^n)$. By eq. 3 (43), every vector in $\tilde{D}$ is in $D\big(\overline{T(X_{i_1})} \ldots \overline{T(X_{i_n})}\big)$ for all finite sequences $i_1, \ldots, i_n$. By Stone's theorem the domain $D\big(\overline{T(X)}\big)$ for X in L is the set of all vectors u in H, for which the limit (i.e., derivative) in eq. 1(11) exists. Therefore, if u is in $\tilde{D}$, then u is in $D\big(\overline{T(X)}\big)$ and, consequently, $T_x u$ has all partial derivatives at $x = e$. Now the inner automorphism $y \to xyx^{-1}$ in G implies the inner automorphism

$$X \to \mathrm{Ad}_x(X) = xXx^{-1} \tag{4}$$

in the Lie algebra L of G (cf. eq. 3.3 (28)). Hence, for images (relative to T), we have

$$\overline{T(X)}\, T_x u = T_x \big(T_{x^{-1}} \overline{T(X)}\, T_x\big) u = T_x \overline{T(Y)} u, \tag{5}$$

where $Y = \mathrm{Ad}_{x^{-1}}(X)$. For X', X'' in L, we have

$$\overline{T(X'')}\,\overline{T(X')}\,T_x u = T_x \overline{T(Y'')}\,\overline{T(Y')}u$$

and similar expression for arbitrary products $\overline{T(X_{i_1})} \dots \overline{T(X_{i_n})}$. Thus, $T_x u$ has partial derivatives of all orders for all x in G and u in $\tilde{D}$. Consequently, $\tilde{D}$ is contained in the space $\tilde{D}_G$ of all infinitely differentiable vectors. Clearly $\tilde{D}_G \supset D_G$. If u is in $\tilde{D}_G$, then u is in $D(\overline{T(\Delta)})^n$ for any n and, therefore, $u \in \tilde{D}$. Hence, $\tilde{D} \supset \tilde{D}_G$; consequently, $\tilde{D} = \tilde{D}_G$. Any analytic vector for $\overline{T(\Delta)}$ is in $\tilde{D}$ and, consequently, it is also an analytic vector for $T(\Delta)$. The operator $\overline{T(\Delta)}$ is self-adjoint and so it has a dense set of analytic vectors by lemma 3.1. Because, by lemma 3.7, $\alpha = |T(\Delta) - I|$ analytically dominates ξ, these are all analytic vectors for ξ and by lemma 1 they are analytic vectors for the representation T of G. ▼

EXAMPLE 1. Let G be the three-dimensional nilpotent group of example 2.1, and let

$$T_{[\alpha\beta\gamma]}u(x) = \exp[-\mathrm{i}(\gamma + x\beta)]u(x+\alpha) \tag{6}$$

be the unitary representation of G in $L^2(-\infty, +\infty)$. Then the generators of one-parameter subgroups are

$$P = \frac{\mathrm{d}}{\mathrm{d}x}, \quad Q = -\mathrm{i}x, \; C = -\mathrm{i}I. \tag{7}$$

They satisfy the Heisenberg commutation relation $[P, Q] = -\mathrm{i}I$. We take as a common dense invariant domain D for the Lie algebra (7) the Schwartz's space S of C^∞-functions $u(x)$ with

$$\sup\left|x^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\beta u(x)\right| < \infty, \qquad \alpha, \beta = 0, 1, \dots$$

Now we calculate the dense set of analytic vectors for the operator $T(\Delta) = P^2 + Q^2 + C^2$. The unit operator $-C^2$ in $T(\Delta)$ can be dropped since every analytic vector for $T(\Delta)$ is analytic for

$$T(\Delta') = P^2 + Q^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2. \tag{8}$$

The operator (8) has the same form as the Hamiltonian for one-dimensional harmonic oscillator. It is well known that eigenfunctions $u_n(x)$ normalized to one of $T(\Delta')$ can be expressed in terms of Hermite polynomials, i.e.,

$$u_n(x) = (\sqrt{(\pi)}2^n n!)^{-1/2}\exp(-x^2/2)H_n(x) \tag{9}$$

and correspond to the eigenvalues $\lambda_n = 2n+1$ (cf. e.g. Messiah 1961, vol. I, p. 492). These eigenfunctions are analytic vectors for $T(\Delta')$. Indeed, for $s < \infty$,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|T(\Delta')^k u_n\| s^k = \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda_n s)^k = \exp(\lambda_n s) < \infty.$$

The set $\{u_n(x)\}$ forms a complete orthonormal set in $L^2(-\infty, +\infty)$. Hence by virtue of th. 2, the finite linear hull $A_{T(\Delta)}$ of $\{u_n(x)\}$ provides a dense set of analytic vectors for the representation (6) of G.

Every eigenfunction $u_n(x)$ lies in the Schwartz space S, which is the domain D for $T(L)$. Moreover, from elementary properties of Hermite polynomials, it follows that

$$\begin{aligned} Pu_n &= -\left(\frac{n+1}{2}\right)^{1/2} u_{n+1} + \left(\frac{n}{2}\right)^{1/2} u_{n-1}, \\ Qu_n &= -\mathrm{i}\left(\frac{n+1}{2}\right)^{1/2} u_{n+1} - \mathrm{i}\left(\frac{n}{2}\right)^{1/2} u_{n-1}. \end{aligned} \tag{10}$$

Hence, the dense subspace $A_{T(\Delta)}$ of analytic vectors of $T(\Delta)$ is the common invariant dense domain for the representation (7) of the Lie algebra of Heisenberg commutation relations. ▼

The proof of lemma 3.1 provides a method for the explicit construction of a dense set of analytic vectors for $\overline{T(\Delta)}$. Namely, let $E(\lambda)$ be the spectral resolution of the identity for $\overline{T(\Delta)}$ and $\delta = [a, b]$ run over bounded intervals of R^1, then the vectors of the form $E(\delta)u$, $u \in H$, constitute the dense set of analytic vectors for $\overline{T(\Delta)}$.

We shall see that with a proper choice, the dense set A_{T_G} of analytic vectors for a representation $x \to T_x$ of G forms a common dense invariant domain for a representation $X \to T(X)$ of the Lie algebra L of G and its enveloping algebra E and every element of A_{T_G} is in $A_{T(\Delta)}$ (cf. eq. 7(32)).

If we take as a common dense invariant domain for a representation $X \to T(X)$ of a Lie algebra L, the dense set A_{T_G} of analytic vectors for a representation $x \to T_x$ of G, then we have a satisfactory connection between invariant closed subspaces of the Lie algebra and those of the Lie group. Indeed, if $D \subset A_{\overline{T(\Delta)}}$ is an invariant closed subspace for Lie algebra, then for any $T(X)$ in L and u in D, $T(X)^n u$ is in D and, consequently,

$$T_{\exp(hX)}u = \exp[hT(x)]u = \sum_{k=0}^{\infty} \frac{h^n}{n!} T(x)^n u \tag{11}$$

converges and gives an element in D. Consequently, D is also invariant relative to the representation T of G. Conversely if D is a closed subspace invariant relative to G containing a dense invariant set A of analytic vectors then A is also invariant relative to the Lie algebra L. In fact, by eq. (11) for any $h > 0$ and u in A, we have

$$\frac{T_{\exp(hX)}u - u}{h} = T(X)u + hT(X)^2 u + \ldots \in D. \tag{12}$$

Setting $h \to 0$, we obtain that $T(X)u \in D$.

§ 5. Integrability of Representations of Lie Algebras

In quantum theory and in particle physics, we work in general directly with the representations of the Lie algebras. However, in many problems, the global group transformations themselves have a direct physical significance. For instance, if the group G contains the physical Poincaré group Π as a subgroup, then the (unitary) representatives T_x of $x \in \Pi$ will describe changes of the given physical system associated with the changes of the reference frames. Thus, in many cases, we are interested in those representations of the Lie algebra L, which can be integrated to a global (unitary) representation of the group G. There are many examples where the global representations bring new physical relationships showing that Nature makes use of the representations of groups rather than of Lie algebras.

In this section we give Nelson's fundamental theorem which establishes when a representation of a Lie algebra L in terms of skew-symmetric operators can be associated with a unitary representation of the simply connected Lie group G having L as its Lie algebra.

We first discuss the connection of this problem with analytic vectors.

LEMMA 1. *Let a Lie algebra L be represented by skew-symmetric operators on a Hilbert space H having a common, invariant, dense domain D. Let $X_1, \ldots, X_d$ be an operator basis for L, $\xi = |X_1| + \ldots + |X_d|$. If for some $s > 0$ the set of vectors u in D for which $\|\exp(\xi s)u\| < \infty$ is dense in H, then there is on H a unique unitary representation T of the simply connected Lie group G, having L as its Lie algebra such that for all X in L, $\overline{T(X)} = \bar{X}$.*

PROOF: The condition $\|\exp(\xi s)u\| < \infty$ for some $s > 0$ means that u is an analytic vector for ξ. Because any analytic vector for ξ is an analytic vector for any element $X \in L$, we conclude that any $X \in L$ has a dense set of analytic vectors. Consequently, by remark 1 to lemma 3.1, the operator $i\bar{X}$ is self-adjoint.

Let exp be the exponential mapping in the sense it is defined for Lie groups (cf. ch. 3.10.A), and let N be a neighborhood of e in G such that the exp is a one-to-one mapping from a neighborhood of 0 in L to N. For $x = \exp X$ in N, we define T_x to be the unitary operator $\exp\bar{X}$.

Let X, Y and Z be in L and suppose that $\exp X \exp Y = \exp Z$ in G. Then the two power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} Z^n \quad \text{and} \quad \sum_{k,l=0}^{\infty} \frac{1}{k!}\frac{1}{l!} X^k Y^l$$

are formally equal. Consequently, if u is a vector such that $\|\exp(|X|+|Y|)u\| < \infty$, $\|\exp(|Z|)u\| < \infty$, then $\exp(\bar{X})\exp(\bar{Y})u = \exp(\bar{Z})u$, i.e., $T_{\exp Z}u = T_{\exp X}T_{\exp Y}u$. Now, for X, Y, Z sufficiently close to 0 in L (such that the absolute values of coordinates are less than $\frac{1}{2}s$), there is by our supposition a dense set of vectors

$u \in D$, such that $\|\exp(|X|+|Y|)u\| < \infty$ and $\|\exp(|Z|)u\| < \infty$. Therefore, if x and y are sufficiently close to e in G, $T_x T_y = T_{xy}$. It means that in a neighborhood of e the map $x \to T_x$ defines a unitary homomorphism of the local group. This homomorphism is strongly continuous. Indeed, as $X \to 0$ in L, $\|\exp(X)u - u\| \to 0$ on the dense set, because $\|\exp(\xi s)u\| < \infty$ on the dense set by our supposition. Moreover, the set $\{T_x\}$ being uniformly bounded, $\|T_x\| = 1$, strong continuity on the dense set can be extended to strong continuity in H by continuity of all operators T_x. Consequently, the map $T:(x, u) \to T_x u$ of $N \times H$ into H defines a local unitary representation of G. Because G is simply connected, there is a unique extension of T to a unitary representation of G on H. ▼

This lemma, in fact, provides the criterion for the integrability of a skew-symmetric representation of a Lie algebra. However, its criterion may not be easily applicable in concrete cases. The next theorem provides a simpler criterion for integrability.

THEOREM 2. *Let L be a Lie algebra of skew-symmetric operators on a Hilbert space H which have a common invariant dense domain D. Let $X_1, \ldots, X_d$ be a operator, basis for L and $\Delta = X_1^2 + \ldots + X_d^2$. If Δ is essentially self-adjoint, then there is on H a unique unitary representation T of the simply connected Lie group G which has L as its Lie algebra such that for all X in L, $\overline{T(X)} = \bar{X}$.*

PROOF: Let $\xi = |X_1| + \ldots + |X_d|$ and $\beta = |\Delta| + |I|$. By lemma 3.7, β analytically dominates ξ, and by the last assertion of lemma 3.5 the set of vectors for which $\|\exp(\xi s)u\| < \infty$ for some $s > 0$ is dense in H. Hence, by lemma 1 we obtain the assertion of the theorem. ▼

Th. 2 implies the following corollary which provides a useful criterion for the *non*-integrability of a given skew-symmetric representation of a Lie algebra L.

COROLLARY 1. *Let L be a Lie algebra as in th. 2. If even a single element $\mathrm{i}X$ for X in L is not essentially self-adjoint, then no unitary representation $T(G)$ of the simply connected group G can be associated with L. Consequently, Δ cannot be essentially self-adjoint.*

PROOF: If Δ is e.s.a., then by th. 2, there is a unique unitary representation $T(G)$ of the simply-connected group G and $\overline{T(X)} = \bar{X}$ for all X in L. By corollary 4 to th. 2.3, $\overline{T(\mathrm{i}X)}$ is self-adjoint and consequently every $\mathrm{i}X$ must be essentially self-adjoint. Hence if some $\mathrm{i}X$ is not essentially self-adjoint, then there cannot exist a global unitary representation T of G such that $\overline{T(X)} = \bar{X}$. Consequently, Δ cannot be essentially self-adjoint. ▼

EXAMPLE 1. Let L be the Heisenberg Lie algebra, defined by the following commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0. \tag{1}$$

Let $H = L^2(S)$ where S is the open interval $(0, 2\pi)$ and let

$$X = \frac{\mathrm{d}}{\mathrm{d}\varphi}, \quad Y = \mathrm{i}\varphi, \quad Z = \mathrm{i}I \tag{2}$$

be the skew-symmetric representation of L defined on $C_0^\infty(S)$ as the common invariant dense domain. For u in $C_0^\infty(S)$ and any v in $C^1(S)$ we have

$$(Xu, v) = \int_0^{2\pi} Xu(\varphi)v(\varphi)\,\mathrm{d}\varphi = (u, -Xv)+u(2\pi)v(2\pi)-u(0)v(0)$$
$$= (u, -Xv) = (u, X^*v). \tag{3}$$

Hence $D(X^*) \supset D(X)$, and $X^* = -\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$ on $C^1(S)$. Now the operator X would be essentially self-adjoint if it would have the deficiency indices $(n_+, n_-) = (0, 0)$. From app. B.1(11) it follows that the deficiency indices n_+ and n_- of a symmetric operator $\mathrm{i}X$ are equal to the number of solutions of the equation

$$X^*v_\pm = \pm v_\pm. \tag{4}$$

For $X^* = -\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$ on $C^1(S)$, we obtain $v_\pm = \exp(\mp\varphi)$. Hence deficiency indices n_+ and n_- of the operator X are (1,1); thus the operator X is not essentially self-adjoint. Consequently the representation (2) of the Heisenberg algebra is not integrable to a global representation of the corresponding nilpotent group, by virtue of corollary 1. ▼

The following useful corollary provides the main result of th. 2 but with a weaker assumption on the domain of the representation. Roughly speaking, th. 2 concerns the passage from a domain D of the type C^∞ to C^k, $k < \infty$, and the corollary below the passage from C^k to C^2.

COROLLARY 2. *Let L be a real Lie algebra and H a Hilbert space. For each X in L let $\varrho(X)$ be a skew-symmetric operator on H. Let D be a dense, linear subspace of H, such that for all X, Y in L, D is contained in the domain of $\varrho(X)\varrho(Y)$. Suppose that for all X, Y in L, u in D and real numbers a and b, we have*

$$\varrho(aX+bY)u = a\varrho(X)u+b\varrho(Y)u, \tag{5}$$
$$\varrho([X, Y])u = \big(\varrho(X)\varrho(Y)-\varrho(Y)\varrho(X)\big)u. \tag{6}$$

Let $X_1, \ldots, X_d$ be a basis for L. If the restriction A of $\varrho(X_1)^2+ \ldots +\varrho(X_d)^2$ to D is essentially self-adjoint, then there is on H a unique unitary representation T of the simply-connected Lie group G, which has L as its Lie algebra such that for all X in L, $\overline{T(X)} = \overline{\varrho(X)}$.

PROOF: In the present case, as in lemma 3.5, we have for each n

$$D(\bar{A}^n) \subset D(\overline{\varrho(X_{i_1})} \ldots \overline{\varrho(X_{i_n})}) \tag{7}$$

The proof of this relation is identical with the proof of eq. 3 (43). Only instead of considering $\mathrm{ad}\,\varrho(X)A$, which might have only 0 in its domain, we consider $\varrho\big((\mathrm{ad}X)\varDelta\big)$ where $\varDelta = X_1^2+ \ldots +X_d^2$. Because $(\mathrm{ad}X)\varDelta \in E^2$ by eq. 3 (21), the operator $\varrho\big((\mathrm{ad}X)\varDelta\big)$ is well defined on D. Set $\tilde{D} = \bigcap_{n=1}^{\infty} D(\bar{A}^n)$, $\tilde{A}$ the restriction of $\bar{A}$ to $\tilde{D}$ and $\tilde{X}_i$ the restriction of $\overline{\varrho(X_i)}$ to $\tilde{D}$. By eq. (7), $\tilde{D}$ is invariant relative to $\tilde{X}_i$. Moreover, since $A \subset \tilde{A} \subset \bar{A}$ we find that $\bar{\tilde{A}}(=\bar{A})$ is self-adjoint, i.e., $\tilde{A}$ is essentially self-adjoint. Thus, all assumptions of th. 2 are satisfied and, consequently, the corollary follows. ▼

As a byproduct of th. 2 and corollary 2 we obtain a convenient criterion for the strong commutativity of unbounded operators. We recall that two unbounded self-adjoint operators are called *strongly commuting* if their spectral resolutions commute.

COROLLARY 3. *Let A and B be symmetric operators on a Hilbert space H and let D be a dense linear subspace of H, such that D is contained in the domain of A, B, A^2, AB, BA and B^2, and such that $ABu = BAu$ for all u in D. If the restriction of A^2+B^2 to D is essentially self-adjoint, then*

1° *A and B are essentially self-adjoint,*

2° *$\bar{A}$ and $\bar{B}$ strongly commute.*

PROOF: ad 1°. Let L be a two-dimensional abelian Lie algebra with a basis X, Y and let $\varrho\,(aX+bY) = \mathrm{i}aA+\mathrm{i}bB$. Then all assumptions of corollary 2 are satisfied. Consequently there exists on H a unique unitary representation T of a simply-connected abelian group (i.e. isomorphic to R^2) G, such that $\overline{T(Z)} = \overline{\varrho(Z)}$ for all Z in L. By corollary 4 to th. 2.3. $\mathrm{i}\overline{T(Z)}$ is self-adjoint. Consequently, $A = -\mathrm{i}T(X)$ and $B = -\mathrm{i}T(Y)$ are e.s.a.

ad 2°. Because G is abelian, the unitary operators $T_{\exp Z} = \exp[\overline{T(Z)}]$, $x = \exp Z \in G$, $Z \in L$ commute. Hence the self-adjoint operators $\mathrm{i}\overline{T(X)} = \bar{A}$ and $\mathrm{i}\overline{T(Y)} = \bar{B}$ are strongly commuting. ▼

We know by corollary 3 to th. 2.3 that symmetric elements L, M of the center Z of an enveloping algebra E are mapped onto essentially self-adjoint operators $T(L)$ and $T(M)$. However, if G is a physical symmetry group, we require that the corresponding self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$, as well as T_x and $T(L)$, $x \in G$, $L = L^+ \in Z$ are strongly commuting. The following theorem shows that this is indeed the case.

THEOREM 3. *Let T be a unitary representation of a connected Lie group G in a Hilbert space H and let Z be the center of the left invariant enveloping algebra E of G. Then*

1° *For any symmetric L, M in Z the self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$ are strongly commuting.*

2° *For any symmetric N in Z and x in G, the operators $\overline{T(N)}$ and T_x are strongly commuting.*

PROOF: ad 1°. For any

$$u(\varphi) = \int_G \varphi(x)\, T_x u \,\mathrm{d}x \in D_G$$

(D_G-Gårding domain), we have by formula 1(17)

$$T(L)T(M)u(\varphi) = u(\tilde{L}\tilde{M}\varphi) = u(\tilde{M}\tilde{L}\varphi) = T(M)T(L)u(\varphi).$$

Hence, taking $D = D_G$, we obtain that all assumptions of corollary 3 are satisfied. Consequently, the self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$ are strongly commuting.

ad 2°. Let $u(\varphi) \in D_G$. Then using the fact that the left translations $L_y\varphi(x) = \varphi(y^{-1}x)$ commute with any element $N \in Z$ and by eqs. 1(17) and 1(13) we obtain for any $u(\varphi) \in D_G$ the equality

$$\begin{aligned} T_y T(N)u(\varphi) &= T_y u(\tilde{N}\varphi) = u(L_y\tilde{N}\varphi) = u(\tilde{N}L_y\varphi) = T(N)u(L_y\varphi) \\ &= T(N)T_y u(\varphi). \end{aligned}$$

Hence, the operators $T(N)$ and T_x for any N in Z and x in G commute on the Gårding domain. The operators $\overline{T(N)}$ and T_x also commute on $D(\overline{T(N)})$. Indeed, if $u \in D(\overline{T(N)})$, then from the definition of the closure $\overline{T(N)}$ of $T(N)$ it follows that there exists a sequence $D_G \ni u_n \to u$ such that $T(N)u_n \to v$ and $T(N)u_n \to \overline{T(N)}u = v$. Because every operator T_x, $x \in G$, is continuous, then for every $u \in D(\overline{T(N)})$ we have

$$\begin{aligned} T_x\overline{T(N)}u &= T_x \lim_{n\to\infty} T(N)u_n = \lim_{n\to\infty} T_x T(N)u_n \\ &= \lim_{n\to\infty} T(N)T_x u_n = \overline{T(N)}T_x \lim u_n = \overline{T(N)}T_x u. \end{aligned}$$

The assertion of item 2° follows now from the fact that every bounded operator commuting with a self-adjoint operator is strongly commuting. ▼

§ 6. FS^3-Theory of Integrability of Lie Algebra Representations

We shall present in this section a beautiful theory of integrability of Lie algebra representations elaborated by Flato, Simon, Snellman and Sternheimer ($\equiv FS^3$-theory). In contradistinction to Nelson's theory presented in sec. 5 it gives integrability criteria directly in terms of the properties of the the generators of the Lie algebras: therefore it is generally more effective in practical applications, especially for higher dimensional Lie algebras.

We begin with the observation that in the case of finite-dimensional real Lie algebras one may introduce several other definitions of analytic vectors, which are inequivalent (and weaker) than that used in secs. 3–4 (compare definition 3.1):

(i) A vector $u \in H$ is analytic for every element in the Lie algebra (in the sense of analyticity for a single operator as defined in sec. 3).

(ii) A vector $u \in H$ is analytic for every element X_i, $i = 1, 2, \ldots, \dim L$, in a given linear basis of the Lie algebra.

(iii) A vector $u \in H$ is analytic for every element X_k of Lie generators of the Lie algebra.*

It is evident that the notions of analyticity according (i)–(iii) are weaker than the notion introduced in sec. 4.

We shall now prove the basic result which says that it is enough to have a common invariant dense set of analytic vectors only for a basis of the Lie algebra (in the sense (ii)) to ensure the integrability of the *a priori* given Lie algebra representation. We begin with some preliminary results.

Let $x \to T(x)$ be a representation of a Lie algebra L on a complex Hilbert space H by skew-symmetric operators defined over a common dense invariant domain D. It is evident that such a representation is strongly continuous in H. In what follows we shall write $X = T(x)$, $Y = T(y)$, etc., and we shall denote

$$A(tX, Y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathrm{ad}X)^n Y.$$

LEMMA 1. *For any $x, y \in L$ and $u \in D$, the series*

$$A(tX, Y)u = \sum_{n=0}^{\infty} \frac{t^n}{n!} ((\mathrm{ad}X)^n Y)u \tag{1}$$

is convergent for all $t \in R$ and we have

$$T(e^x y e^{-x} y)u = A(X, Y)u. \tag{2}$$

PROOF: Eq. (2) follows from eq. 3.3(43) and from the above-mentioned strong continuity. Changing x into tx, we obtain the convergence of the series $A(tX; Y)u$ for all $t \in R$. ▼

One readily verifies that for any $x, y \in L$ and $u \in D$ we have

$$YX^m u = \sum_{p=0}^{m} \binom{m}{p} X^p((-\mathrm{ad}X)^{m-p} Y)u. \tag{3}$$

PROPOSITION 2. *Let T and T' be two representations of the real Lie algebra L by skew-symmetric operators over common invariant domains D and D' (respectively), dense in H, with $D \subset D'$, and such that for any $y \in L$, $Y = T(y)$ is the restriction to D of $Y' = T'(y)$. Then if D is a domain of analytic vectors for some $X = T(x)$ $\in T(L)$ we have, denoting by $\langle , \rangle$ the scalar product in H, for any $t \in R$, $u \in D$ and $v \in D'$:*

$$\langle -e^{t\bar{X}} Yu, v\rangle = \langle e^{t\bar{X}} u, A(tX', Y')v\rangle. \tag{4}$$

---

* We recall that a set $\{X_k\}_{k=1}^n$, $n \leqslant \dim L$ is called a set of Lie generators of L if L is generated by $\{X_k\}_{k=1}^n$ by linear combinations of repeated commutators.

PROOF: By virtue of remark 1 to lemma 3.1 the closure $\overline{X}$ of X (and of X') is skew-adjoint and therefore generates a unique one-parameter unitary group which we denote by $e^{t\overline{X}}$. For all $u \in D$, the function $t \to e^{t\overline{X}} u$ and $t \to e^{t\overline{X}} Yu$ are analytic in R. By lemma 1, the function $t \to A(tX', Y')v$, with $v \in D'$, is also analytic in R. The functions $a(t) = \langle -e^{t\overline{X}} Yu, v\rangle$ and $b(t) = \langle e^{t\overline{X}} u, A(tX', Y')v\rangle$ are therefore also analytic for all real t. Now we have

$$\frac{d^n a}{dt^n}(0) = \langle -X^n Yu, v\rangle, \tag{5}$$

$$\frac{d^n b}{dt^n}(0) = \sum_{p=0}^{n} \binom{n}{p} \langle X^p u, (\mathrm{ad}X')^{n-p} Y'v\rangle. \tag{6}$$

Since $Y \subset Y' \subset -Y'^*$ (and the same for X), we obtain from (3) that $\frac{d^n a}{dt^n}(0) = \frac{d^n b}{dt^n}(0)$, hence $a(t) = b(t)$ for all $t \in R$. ▼

COROLLARY 1. *Under the conditions of proposition* 1, $e^{-t\overline{X}}v$ *belongs to the domain* $D(Y^*)$ *of the adjoint* Y^* *of* Y *for all* $v \in D'$ *and* $t \in R$, *and*

$$T'(e^{tx} y e^{-tx})v = -e^{t\overline{X}} Y^* e^{-t\overline{X}} v. \tag{7}$$

Indeed, due to the continuity in u of the right-hand side of (4), $e^{-t\overline{X}}v \in D(Y^*)$, whence the formula (since D is dense in H).

HYPOTHESIS (A): *T is a representation of a Lie algebra L on a dense invariant domain D of vectors that are analytic for all skew-symmetric representatives $X_i = T(x_i)$ of a basis $x_1, \ldots, x_r$ of L.*

LEMMA 3. *Hypothesis* (A) *being satisfied, define H_∞ as the intersection of the domains of all monomials $\overline{X}_{i_1} \ldots \overline{X}_{i_n}$, for all $1 \leqslant i_1, \ldots, i_n \leqslant r$, $n \in N$. Let X_i' be the restriction of $\overline{X}_i$ to H_∞ and define, for all* $y = \sum_{i=1}^{r} \lambda_i x_i \in L$,

$$Y' \equiv T'(y) \equiv \sum_{i=1}^{r} \lambda_i X_i'$$

(with invariant domain H_∞). Then T' is a representation of L by skew-symmetric operators (on H_∞) and we have, for any two elements x_i and x_j in the basis and $v \in H_\infty$:

$$A(tX_i', X_j')v = e^{t\overline{X}_i} \overline{X}_j e^{-t\overline{X}_i} v. \tag{8}$$

PROOF: By definition, H_∞ contains D and is invariant under all

$$\overline{X}_i = \overline{X}_i' = -X_i^* = -X_i'^*,$$

hence under all Y' which, due to the hypothesis made, are skew-symmetric. By definition also, T' is linear. Now, if

$$[x_i, x_j] = \sum_{k=1}^{r} c_{ijk} x_k$$

we have, for all $v \in H_\infty$ and $u \in D$:

$$\begin{aligned}\langle (X'_i X' - X'_j X'_i) v, u \rangle &= \langle v, (X_j X_i - X_i X_j) u \rangle \\ &= \langle v, -\sum_k c_{ijk} X_k u \rangle = \langle \sum_k c_{ijk} X'_k v, u \rangle.\end{aligned}$$

Therefore T' is a representation and we can apply formula (7) with $D' = H_\infty$ to $y = x_j$, whence (8). ▼

LEMMA 4. *Under hypothesis* (A), *the above-defined domain H_∞ is invariant under $T'(L)$ and under all one-parameter groups $e^{t\bar{X}_i}$, and if $t_1, \ldots, t_r$ are differentiable functions of some parameter t, then for all $u \in H_\infty$ the vector-valued function*

$$t \to e^{t_1 \bar{X}_1} \ldots e^{t_r \bar{X}_r} u$$

has a first derivative in t.

PROOF: From (8) and the invariance of H_∞ under $T'(L)$ we obtain

$$A(tX'_i, X'_{j_1}) \ldots A(tX'_i, X'_{j_n}) v = e^{t\bar{X}_i} \bar{X}_{j_1} \ldots \bar{X}_{j_n} e^{-t\bar{X}_i} v$$

for all base elements $x_i, x_{j_1}, \ldots, x_{j_n}$ and all $v \in H_\infty$, and $e^{-t\bar{X}_i} v$ belongs to the domain of all operators $\bar{X}_{j_1} \ldots \bar{X}_{j_n}$, whence the invariance of H_∞ under the $e^{-t\bar{X}_i}$.

The differentiability property follows by induction from the differentiability of the vector-valued function $t \to U(t)u(t)$ where $t \to u(t) \in H_\infty$ is strongly differentiable and $t \to U(t)$ is a unitary operator-valued function strongly differentiable on H_∞ (such that the map $(t, u) \to U(t)u$ is continuous $R \times H \to H$). ▼

We now give the main result.

THEOREM 5. *Let T be a Lie algebra representation in a complex Hilbert space satisfying hypothesis* (A). *Then T is integrable to a unique unitary group representation.*

PROOF: Let x, y be any elements of L close enough to 0 so that e^x, e^y, $e^x e^y$, and therefore $e^{tx} e^y$ for $0 \leqslant t \leqslant 1$, belong to the neighbourhood W of the identity of G introduced in 3.3.G. We shall write

$$\begin{aligned} e^{tx} e^y &= e^{\alpha_1 x_1} \ldots e^{\alpha_r x_r}, \\ e^{tx} &= e^{t_1 x_1} \ldots e^{t_r x_r}, \\ e^y &= e^{\beta_1 x_1} \ldots e^{\beta_r x_r}. \end{aligned} \tag{9}$$

For any $z \in L$ such that $e^z \in W$, we write (in a unique way, once the basis is chosen) $e^z = e^{z_1 x_1} \ldots e^{z_r x_r}$ and define

$$T(e^z) = e^{z_1 \bar{X}_1} \ldots e^{z_r \bar{X}_r}. \tag{10}$$

Since G is generated by finite products of elements of W, we only have to show that the group law holds in W, i.e. that for any $e^x, e^y \in W$ such that $e^x e^y \in W$, we have $T(e^x e^y) = T(e^x) T(e^y)$, and that $T(L)$ is on D the differential of $T(G)$. The uniqueness is obvious since relation (10) is a necessary condition.

From lemma 4, for $u \in H_\infty$, $T(e^{tx})u$ and $T(e^{tx}e^y)T(e^y)^{-1}u$ are differentiable functions of t. Since

$$\frac{d}{dt_i} e^{t_i \bar{X}_i} u = \bar{X}_i e^{t_i \bar{X}_i} u = e^{t_i \bar{X}_i} \bar{X}_i u,$$

we have by direct computation

$$\frac{d}{dt} T(e^{tx})u = \left(\frac{dt_1}{dt} \bar{X}_1 + \ldots + \frac{dt_r}{dt} e^{t_1 \bar{X}_1} \ldots e^{t_{r-1}\bar{X}_{r-1}} \bar{X}_r e^{-t_{r-1}\bar{X}_{r-1}} \ldots e^{-t_1 \bar{X}_1} \right) T(e^{tx})u,$$

and similarly

$$\frac{d}{dt} T(e^{tx})u = T(e^{tx}) \left(\frac{dt_1}{dt} e^{-t_r \bar{X}_r} \ldots e^{-t_2 \bar{X}_2} \bar{X}_1 e^{t_2 \bar{X}_2} \ldots e^{t_r \bar{X}_r} + \ldots + \frac{dt_r}{dt} \bar{X}_r \right) u.$$

From relations 3.3(41), (2), (8), (9), and (10) we then get:

$$\frac{d}{dt} T(e^{tx})u = X' T(e^{tx})u = T(e^{tx}) X' u. \tag{11}$$

On the other hand we have by direct computation, for all $u \in H_\infty$

$$\frac{d}{dt} T(e^{tx}e^y) T(e^y)^{-1} u$$

$$= \left(\frac{d\alpha_1}{dt} \bar{X}_1 + \ldots + \frac{d\alpha_r}{dt} e^{\alpha_1 \bar{X}_1} \ldots e^{\alpha_{r-1}\bar{X}_{r-1}} \bar{X}_r e^{-\alpha_{r-1}\bar{X}_{r-1}} \ldots e^{-\alpha_1 \bar{X}_1} \right) T(e^{tx}e^y) T(e^y)^{-1} u.$$

Hence, from relations 3.3(42), (2), (8), (9) and (10) we obtain that for all $u \in H_\infty$, $T(e^{tx}e^y)T(e^y)^{-1}u$, which belongs to H_∞, is also a differentiable solution of the vector-valued differential equation (with values in H_∞ and derivation in the H-topology)

$$\frac{d}{dt} u(t) = X' u(t), \qquad u(t) \in H_\infty. \tag{12}$$

Such an equation has a unique solution (cf. e.g. Kato 1966, p. 481). Indeed one checks easily that for any solution $u(s) \in H_\infty$, and $0 \leqslant s \leqslant t \leqslant 1$, $T(e^{(t-s)x})u(s)$ is differentiable in s and that

$$\frac{d}{ds} \left(T(e^{(t-s)x}) u(s) \right) = -T(e^{(t-s)x}) X' u(s) + T(e^{(t-s)x}) X' u(s) = 0.$$

Therefore $T(e^{(t-s)x})u(s)$ does not depend on s. Equating its values for $s = 0$ and $s = t$ we obtain $u(t) = T(e^{tx})u(0)$, whence the uniqueness of the solution of (12) in H_∞ and the group law (which we can extend from H_∞ to H by continuity)

$$T(e^{tx}e^y) = T(e^{tx}) T(e^y).$$

Moreover, relation (11) shows that $T(L)$ is the restriction to D of the differential of $T(G)$. ▼

Let us note that while according to Nelson's result, an analytic vector for Δ was necessarily analytic for the whole algebra the situation in th. 5 is not similar: we only know from what was said up till now that the existence of a dense invariant set of analytic vectors for the basis implies integrability and therefore, by the global theorem, the existence of (another) dense invariant set of analytic vectors for the whole algebra (in the Nelson sense), therefore analytic for the group.

We now give a stronger version of th. 5 which utilizes the weakest concept of analyticity as defined by (iii).

THEOREM 6. *Let hypothesis* (A) *be satisfied. Let* $\{x_1, \ldots, x_n\}$ *be a set of Lie generators of the Lie algebra L, and let A denote a set of analytic vectors for* $T(x_1), \ldots, T(x_n)$ *separately. Then*

(i) *The set of all analytic vector for a given arbitrary element of L is invariant under* $T(L)$.

(ii) *There exists a unique unitary representation of the corresponding connected and simply-connected Lie group (having L as its Lie algebra) on the closure of the smallest set A′ containing A and invariant under* $T(L)$, *the differential of which on A′ is equal to T.* ▼

(For the proof cf. Flato and Simon 1973.)

It is noteworthy that the invariance of the set of analytic vectors under $T(L)$ was obtained automatically.

The following theorem clarifies completely the connection between the weak analyticity, the strong analyticity of Nelson and the analyticity for group representation.

THEOREM 7. *Let G be a real finite-dimensional Lie group. Then there exists a basis* $\{x_1, \ldots, x_n\}$ *of the corresponding Lie algebra L such that given any representation of G on a Hilbert space, any vector analytic separately for the closures of the representatives of the basis* $\{x_1, \ldots, x_n\}$ *will be analytic for the group representation (which means analytic for the whole Lie algebra namely jointly analytic).* ▼

(For the proof cf. Flato and Simon 1973.)

We now come to another question: Are analytic vectors really necessary in order to ensure integrability?

The first example showing the necessity was constructed by Nelson 1959. The following theorem well illustrates this problem.

THEOREM 8. *Every compact Lie algebra of dimension* $n > 1$, *has at least one representation in Hilbert space on an invariant domain such that every element of the algebra is represented by an essentially skew-adjoint operator on this domain, every element of some linear basis of the algebra is integrable to a one-parameter compact group, but the representation is not integrable.*

(For the proof cf. Flato, Simon and Sternheimer 1973.)

The FS^3-theory provides a very convenient framework for applications. In particular Niederle and Mickelsson 1973 and Niederle and Kotecký 1975 utilizing FS^3-integrability criteria have shown that the representations of su(p, q) and so(p, q) obtained by Gel'fand–Zetlin method (see sec. 8) are integrable.

The FS^3-theory has also found an interesting applications in Wightman quantum field theory. In particular Snellman 1972 has proved that the action of the polynomial ring in the field operators on the vacuum contains a dense set of analytic vectors for the Poincaré group. This analysis was extended later on by Nagel and Snellman 1974.

The FS^3-theory admits generalizations to other spaces from the Hilbert space. In fact, the theory of integrability of representations of Lie algebras in quasi-complete locally convex spaces was given by FS^3 (see Flato *et al.* 1972, sec. 5) The case of Banach space representations was treated in details by Kisyński 1973.

§ 7. The 'Heat Equation' on a Lie Group and Analytic Vectors

We describe in this section an interesting global method for the construction of analytic vectors for a representation T of a Lie group G. This is a generalization of the Gårding's method for the construction of regular vectors. We replace in fact the function $\varphi \in C_0^\infty(G)$ in the integral 1 (12), i.e.

$$u(\varphi) = \int_G \varphi(x) T_x u \, dx, \tag{1}$$

by a certain analytic function which decreases rapidly enough at infinity together with all its mixed partial derivatives. The class of all such functions is provided by the solutions of the heat equation on a Lie group. The case of nonunitary representations requires a slight extension of the proof of lemma 3 (below) only. Hence we treat here both the cases of unitary and nonunitary representations simultaneously. For simplicity of exposition we restrict ourselves to unimodular Lie groups.

Let X_i, $i = 1, 2, \ldots, d = \dim L$, be the generators of the right translations in $H = L^2(G)$ and let $\Delta = X_1^2 + \ldots + X_d^2$. Then by the heat equation on the Lie group G, we mean the equation of the form

$$\left(\frac{\partial}{\partial t} - \Delta\right) \varphi(t, x) = 0, \tag{2}$$

where t is a real parameter and $x \in G$.

If G is the translation group of the real line, then eq. (2) has the form

$$\left(\frac{\partial}{\partial t} - \frac{d^2}{dx^2}\right) \varphi(t, x) = 0, \tag{3}$$

i.e., it is the ordinary one-dimensional heat equation.

We know by th. 2.2 that the symmetric elliptic operator Δ in $L^2(G)$ is essentially self-adjoint. We denote by $\bar{\Delta}$ its self-adjoint extension in H and set

$$\varphi(t, x) = \big(\exp(t\bar{\Delta})f\big)(x), \tag{4}$$

where $f \in C_0^\infty(G)$ and $t > 0$. By the spectral theorem (app. B.3, th. 1) we have $\partial_t\varphi = \bar{\Delta}\varphi = \Delta\varphi$. Hence, φ is a solution of the heat equation (2). Moreover, as $t \to 0$, $\varphi(t, x) \to f(x)$.

EXAMPLE 1. Let G be the translation group of the real line. Then $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ and the general solutions (4) of the corresponding heat equation (3) have the form

$$\varphi(t, x) = \exp\left(t\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)f(x) = \int_{-\infty}^{+\infty} \exp\left(t\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\delta(x-y)f(y)\,\mathrm{d}y.$$

The kernel function $\exp(t\mathrm{d}^2/\mathrm{d}x^2)\,\delta(x-y)$ can be written in the form

$$\exp(t\mathrm{d}^2/\mathrm{d}x^2)\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mathrm{i}p(x-y)]\,\mathrm{d}p = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-tp^2)\exp[\mathrm{i}p(x-y)]\,\mathrm{d}p$$

$$= \frac{\exp\left(-\dfrac{(x-y)^2}{4t}\right)}{\sqrt{4\pi t}}. \tag{5}$$

Hence

$$\varphi(t, x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4t}\right]f(y)\,\mathrm{d}y, \quad f \in C_0^\infty(G). \tag{6}$$

The kernel (5) is the well-known fundamental solution (Green's function) of the heat equation.

Note that if $t \to 0$,

$$\frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{(x-y)^2}{4t}\right] \to \delta(x-y),$$

and consequently, $\varphi(t, x) \to f(x)$. ▼

We observe that the functions (5) and (6), as well as all their derivatives, decrease more rapidly at infinity than any exponential function. This turns out to be a general feature of the solution (4) of the heat equation (2) for an arbitrary Lie group. To state this property precisely, we make use of the concept of distance on an arbitrary Lie group (cf. ch. 3.9).

Let $\tau(x) \equiv \tau(x, e)$ be the distance between the unity e and x in a connected Lie group G. We shall show that the function $\varphi(t, x) = \exp(t\bar{\Delta})f$ with f in $C_0^\infty(G)$ decreases more rapidly than any exponent.

LEMMA 1. *Let* $\varphi(t, x) = \exp(t\bar{\Delta})f(x)$, $f \in C_0^\infty(G)$ *and let* s *be a real number. Then*

$$(\exp(s\tau)\varphi, \varphi) \leqslant \exp(\tfrac{1}{2}s^2 t)(\exp(s\tau)f, f), \tag{7}$$

where $(\cdot, \cdot)$ *denotes the scalar product in* $H = L^2(G)$. ▼

Remark: Clearly the right hand side of eq. (7) is finite because f has a compact support. We multiply both sides of eq. (7) by $\exp[-\frac{1}{2}s^2(t+\varepsilon)]$, where $\varepsilon > 0$,

$$(\exp[-\tfrac{1}{2}s^2(t+\varepsilon)+s\tau]\varphi, \varphi) \leqslant (\exp[-\tfrac{1}{2}\varepsilon s^2+s\tau]f, f), \tag{8}$$

then integrate both sides over s in the interval $(-\infty, +\infty)$ and get

$$(t+\varepsilon)^{-1/2}\left(\exp\left[-\frac{\tau^2}{2(t+\varepsilon)}\right]\varphi, \varphi\right) \leqslant \varepsilon^{-1/2}(\exp[-\tau^2/2\varepsilon]f, f). \tag{9}$$

Thus due to the factor $\exp\left[-\dfrac{\tau^2}{2(t+\varepsilon)}\right]$ on the left hand side of eq. (9), the decrease of the function φ at fixed t and for $x \to \infty$ must be at least as fast as

$$\exp\left(-\frac{\tau^2(x)}{4t}\right). \tag{10}$$

This is a direct generalization to an arbitrary connected Lie group of the result which we derived for the translation group on the straight line (cf. eq. (6)).

PROOF OF LEMMA 1: Let $X_1, \ldots, X_d$ be a basis in the left invariant Lie algebra L and let

$$\nabla u = \{X_1 u, \ldots, X_d u\} \tag{11}$$

be the gradient of a function u in H. The scalar product of ∇u and ∇v is

$$(\nabla u, \nabla v) = \sum_{i=1}^{d}(X_i u, X_i v). \tag{12}$$

Clearly,

$$(\nabla u, \nabla v) = (-\Delta u, v) = (u, -\Delta v),$$

where $\Delta = X_1^2 + \ldots + X_d^2$ is the left-invariant Laplacian on G. Let H^1 be the linear subspace of H consisting of elements u in H satisfying the condition

$$||u||+||\nabla u|| < \infty. \tag{13}$$

For $v \in H^1$, differentiable relative to t and for a real function h which is bounded together with its derivatives, $h_t = \partial_t h$ and $X_i h$, the following identities are satisfied ($v_t \equiv \partial_t v$)

$$\begin{aligned}(v_t, h^2 v)+(h^2 v, v_t) &= \partial_t(hv, hv)-2(h_t hv, v),\\ (\nabla v, \nabla(h^2 v))+(\nabla(h^2 v), \nabla v) &= 2(\nabla(hv), \nabla(hv))-2(v\nabla h, v\nabla h).\end{aligned} \tag{14}$$

Summing up these identities and dividing by 2, we obtain

$$\operatorname{Re}\{(v_t, h^2v)+(\nabla v, \nabla(h^2v))\} = \tfrac{1}{2}\partial_t\|hv\|^2+\|\nabla(hv)\|^2-\|v\nabla h\|^2-(h_thv, v). \quad (15)$$

Let $N > 0$. Set

$$\psi(x) = \psi_N(x) \equiv \min\{\exp[\tfrac{1}{2}s\tau(x)], N\}.$$

Then ψ is bounded. By virtue of exercise 3.11.9.1 we obtain

$$|\nabla\exp[\tfrac{1}{2}s\tau(x)]|^2 = \sum_{n=1}^{d}|X_n\exp[\tfrac{1}{2}s\tau(x)]|^2 = \frac{s^2}{4}\exp[s\tau(x)]\sum_{n=1}^{d}|X_n\tau(x)|^2$$

$$\leqslant \frac{s^2}{4}\psi^2(x)|\nabla\tau(e)|^2 = \frac{s^2}{4}\psi^2(x). \quad (16)$$

In eq. (15) we put $v(t, x) = \varphi(t, x) = \exp[t\Delta f(x)]$ and $h(t, x) = \psi(x)$. Then because

$$(\varphi_t, \omega)+(\nabla\varphi, \nabla\omega) = (\varphi_t-\Delta\varphi, \omega) = 0,$$

for any ω in H^1, the left hand side of eq. (15) is zero. Hence

$$\partial_t\|\psi\varphi\|^2 \leqslant 2\|\varphi\nabla\psi\|^2. \quad (17)$$

From inequality (16) we obtain

$$\partial_t\|\psi\varphi\| \leqslant \tfrac{1}{2}s^2\|\psi\varphi\|, \quad \text{i.e.} \quad \partial_t\exp(-\tfrac{1}{2}s^2t)\|\psi\varphi\| \leqslant 0.$$

Consequently we can put

$$\|\psi\varphi\| \leqslant \exp(\tfrac{1}{2}s^2t)\|\psi f\|.$$

For $N \to \infty$ we obtain the inequality (7).▼

Next we derive asymptotic properties analogous to (7) and (10) for arbitrary finite mixed derivatives of $\varphi(t, x)$. Let $\alpha = \alpha_1, \ldots, \alpha_p$ represent a multi-index, where $\alpha_j = 1, 2, \ldots, d$, and let X_α be the product

$$X_\alpha = X_{\alpha_1}\ldots X_{\alpha_p}, \quad (18)$$

where X_{α_j} are left-invariant generators of L.

Set

$$P_\alpha = \partial_t^{(k)}X_\alpha, \quad \tilde{P}_\alpha = \partial_t^{(k)}\tilde{X}_\alpha, \quad (19)$$

where $\tilde{X}_\alpha \equiv \tilde{X}_{\alpha_1}\ldots\tilde{X}_{\alpha_p}$ are the corresponding elements of the right-invariant enveloping algebra (of eq. 1(16)):

LEMMA 2. *Let*

$$\psi(t, x) = (P_\alpha\varphi)(t, x), \quad \textit{or} \quad \psi(t, x) = (\tilde{P}_\alpha\varphi)(t, x), \quad (20)$$

and let $\chi(x)$ be a numerical continuous function on G, which does not increase more rapidly than some exponent. Then $(\chi\psi, \psi)$ is bounded for bounded t.

PROOF: Consider first $\psi = \tilde{P}_\alpha \varphi$. Then, because left- and right-invariant generators commute, we have

$$\tilde{P}_\alpha \varphi = \exp(t\bar{\Delta})\,(\Delta)^k \tilde{X}_\alpha f = \exp(t\bar{\Delta})\tilde{f} = \tilde{\varphi}, \tag{21}$$

where $\tilde{f} \in C_0^\infty(G)$. Consequently, $\tilde{\varphi}$ is the solution of the heat equation. Hence, in this case Lemma 2 is a corollary of lemma 1.

The case $\psi = P_\alpha \varphi$ can be reduced to the previous one if use is made of the relation

$$X_\alpha = \sum_\beta a_{\alpha\beta}(x)\tilde{X}_\beta$$

between the elements of the left- and the right-invariant enveloping algebra.

We have then

$$P_\alpha \varphi = \partial_t^k X_\alpha \varphi = \sum_\beta a_{\alpha\beta}(x)\,\partial_t^k \tilde{X}_\beta \varphi = \sum_\beta a_{\alpha\beta}(x)\tilde{P}_\beta \varphi.$$

By exercise 3.11.9.3 the coefficients $a_{\alpha\beta}(x)$ do not increase more rapidly than some exponential. Hence, the last sum and, consequently, $P_\alpha \varphi$ satisfies the assertion of the lemma.

We shall now construct a new set of vector solutions of the heat equation and show that this set is a dense set of analytic vectors for a representation T of G.

LEMMA 3. *Let $f \in C_0^\infty(G)$ and $\varphi(t, x) = \exp(t\bar{\Delta})\,f(x)$. Then*

$$\Phi(t, x) \equiv \int_G \varphi(t, y^{-1}x)\,T_y u\,\mathrm{d}y, \qquad u \in H, \tag{22}$$

is also a solution of the heat equation (2), *regular for $t \geqslant 0$ with the initial condition*

$$\Phi(0, x) = \int f(y^{-1}x)\,T_y u\,\mathrm{d}y. \tag{23}$$

Moreover, for $t > 0$ the function $\Phi(t, x)$ is analytic relative to x in G.

PROOF: Let L_y be a left translation on G, i.e., $L_y\varphi(x) = \varphi(y^{-1}x)$. Because Δ is left-invariant, we have $[L_y, \Delta] = 0$ for every y in G. Hence

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\Phi(t, x) &= \int_G \left(\frac{\partial}{\partial t} - \Delta\right)\varphi(t, y^{-1}x)\,T_y u\,\mathrm{d}y \\ &= \int_G L_y\left(\frac{\partial}{\partial t} - \Delta\right)\varphi(t, x)\,T_y u\,\mathrm{d}y = 0,\end{aligned} \tag{24}$$

i.e., $\Phi(t, x)$ is the solution of the heat equation (2). For $t \to 0$, $\varphi(t, y^{-1}x) \to f(y^{-1}x)$ and, consequently, $\Phi(t, x) \to \int f(y^{-1}x)\,T_y u\,\mathrm{d}y$. Let ψ be the derivative (20) of the function φ. Consider the integral

$$\Psi(t, x) \equiv \int \psi(t, y^{-1}x)\,T_y u\,\mathrm{d}y. \tag{25}$$

Setting $y \to xy^{-1}$, we obtain

$$\Psi(t, x) = \int \psi(t, y) H(x, y) \mathrm{d}y, \tag{26}$$

where

$$H(x, y) = T_{xy^{-1}} u.$$

We now show that the integral (26) is absolutely convergent. Indeed, by Schwartz inequality

$$\begin{aligned}
&\Big\{\int_G |\psi(t, y)| \|H(x, y)\| \mathrm{d}y\Big\}^2 \\
&= \Big\{\int |\psi(t, y)| \exp\Big[\frac{\lambda}{2}\tau(y)\Big] \|H(x, y)\| \exp\Big[-\frac{\lambda}{2}\tau(y)\Big] \mathrm{d}y\Big\}^2 \\
&\leqslant \int_G |\psi(t, y)|^2 \exp[\lambda\tau(y)] \| H(x, y)\|^2 \mathrm{d}y \int_G \exp[-\lambda\tau(y)] \mathrm{d}y,
\end{aligned} \tag{27}$$

where λ is real. Taking λ sufficiently large and using exercise 3.11.9.2, we obtain that the last integral in (27) is convergent: for the first integral we use lemma 2. Since

$$\|H(x, y)\| \leqslant \exp[c + c\tau(y)],$$

where c is finite for finite x, then the next to the last integral in (27) is also finite by lemma 2. Consequently, the integral (26) is absolutely convergent for bounded t and x.

Let $G_N = \{x \in G: \tau(x) \geqslant N\}$. Then if we replace G by G_N in eq. (27), the right-hand side of (27) provides a convenient majorant for

$$\Big\{\int_{G_N} |\psi(t, y)| \, \|H(x, y)\| \mathrm{d}y\Big\}^2. \tag{28}$$

Indeed due to the last integral in eq. (27) the expression (28) approaches zero as $N \to \infty$, uniformly for bounded t and x. Consequently, the function

$$\Psi(t, x) = \lim_{N \to \infty} \int_{G - G_N} \psi(t, y) H(x, y) \mathrm{d}y \tag{29}$$

is continuous because it is the uniform limit of continuous functions. Set now $\psi = P\varphi$ and take $\chi \in C_0^\infty(Z \times G)$, where Z is semi-straight line $t > 0$. Because P is left-invariant, we have

$$\begin{aligned}
\int \varphi(t, y^{-1}x) P\chi(t, x) \mathrm{d}t \mathrm{d}x &= \int L_y P\varphi(t, x) \chi(t, x) \mathrm{d}t \mathrm{d}x \\
&= \int \psi(t, y^{-1}x) \chi(t, x) \mathrm{d}t \mathrm{d}x.
\end{aligned}$$

Multiplying both sides by $T_y u$ and integrating over y, we can, according to the Fubini theorem change the order of integration. From the definitions (22) and (25), we obtain

$$\int \Phi(t, x) P^* \chi(t, x) \, \mathrm{d}t \, \mathrm{d}x = \int \Psi(t, x) \chi(t, x) \, \mathrm{d}t \, \mathrm{d}x. \tag{30}$$

Because this equality is satisfied for an arbitrary $\chi \in C_0^\infty(Z\times G)$, all derivatives $P\Phi$ are continuous in the sense of distribution theory. Consequently, Φ is regular. Strictly speaking, this was proved only for numerical functions. However, the elementary derivation of this fact (Schwartz 1957) can be easily extended to functions with values in a Hilbert space. Now the generators

$$X_i = \sum_{k=1}^{d} a_{ki}(x)\frac{\partial}{\partial x_k}, \qquad i = 1, 2, \ldots, d = \dim L,$$

are analytic vector fields on G. Hence, the coefficients $a_k(x)$ are analytic. Therefore, the operator $\varDelta$ is an elliptic operator with analytic coefficients. It is well known that any solution of a parabolic equation with analytic coefficients is analytic (cf. Gårding 1960, lemma 10.1). Consequently, the function $\Phi(t, x)$ is analytic relative to x in G.

We now give the main theorem of this section.

THEOREM 4. *Let $x \to T_x$ be a representation of a connected Lie group G in a Hilbert space H. Let $f \in C_0^\infty(G)$ and $\varphi(t, x) = \exp(t\bar{\varDelta})f(x)$ be a solution of the heat equation. Then for any fixed finite $t > 0$ (and all f in $C_0^\infty(G)$ and u in H), the set of all vectors*

$$u(t) = \int_G \varphi(t, y^{-1})T_y u\,\mathrm{d}y \tag{31}$$

forms a dense set of analytic vectors for T.

PROOF: The function

$$T_x u(t) = \int \varphi(t, y^{-1})T_{xy}u\,\mathrm{d}y = \int \varphi(t, y^{-1}x)T_y u\,\mathrm{d}y$$

satisfies all the assumptions of lemma 3. Consequently, the function $T_x u(t) \equiv \Phi(t, x)$ is regular for $t \geqslant 0$ and analytic relative to x in G for $t > 0$. For $t \to 0$ the vector $u(t)$ tends to the regular vector

$$v = u(0) = \int f(y^{-1})T_y u\,\mathrm{d}y.$$

The set of all such vectors is dense in H by th. 1.1. Consequently, due to the continuity relative to t, the set of analytic vectors (31) is also dense in H. ▼

The dense set of analytic vectors (31) for T of G forms the common dense invariant domain for the representation $T(X)$ of the Lie algebra L of G. In fact, for any $h > 0$ and X in L we have

$$\frac{1}{h}[T_{\exp(hX)}u(t) - u(t)] = \frac{1}{h}\int \varphi(t, y^{-1})\,(T_{\exp(hX)} - I)T_y u\,\mathrm{d}y$$

$$= \int \left[\frac{1}{h}(\varphi(t, y^{-1}\exp(hX)) - \varphi(t, y^{-1})\right]T_y u\,\mathrm{d}y.$$

The function in the bracket has the limit

$$\frac{1}{h}[\varphi(t,y^{-1}\exp(hX))-\varphi(t,y^{-1})] = \frac{\exp(t\bar{\Delta})}{h}[f(y^{-1}\exp(hX))-f(y^{-1})]$$
$$\underset{h\to 0}{\to} \exp(t\bar{\Delta})[(\tilde{X}f)(y^{-1})] = \exp(t\bar{\Delta})\tilde{f}(y^{-1}) \equiv \tilde{\varphi}(t,y^{-1}),$$

where

$$\tilde{f}(x) \equiv (\tilde{X}f)(x) = \sum_{i=1}^{d} a_i(x)\frac{\partial f}{\partial x_i}.$$

Because $\tilde{f}(y^{-1}) \in C_0^\infty(G)$, the function $\tilde{\varphi}(t,y^{-1})$ is a solution of the heat equation. Consequently, the vector

$$[T(X)u](t) \equiv \int \tilde{\varphi}(t,y^{-1})T_y u\,dy \tag{32}$$

is analytic by th. 4. Clearly, in the same manner, the action of any product of generators can be defined.

Consequently, the linear hull of all analytic vectors (31) provides a common dense invariant domain for the representation $X_{i_1}\dots X_{i_n} \to T(X_{i_1})\dots T(X_{i_n})$ of the enveloping algebra E.

§ 8. Algebraic Construction of Irreducible Representations

In the previous sections we discussed the general representation theory of Lie algebras by unbounded operators. Here we present some alternative explicit construction of irreducible representations for non-compact Lie algebras based on diagrammatical technique. This technique works for any simple noncompact classical Lie algebra. We shall illustrate it in the case of $u(p,q)$-algebras, since $u(1,1)$, $u(2,1)$, $u(2,2)$, $u(6,6)$, etc. occur in particle physics.

We consider the algebraic description of irreducible self-adjoint representations of Lie algebras of the class $u(p,q)$. The approach is a direct generalization of the Gel'fand–Zetlin approach for compact Lie algebras $u(n)$ considered in ch. 10, § 1. The discrete series of irreducible representations of $u(p,q)$ which is constructed here can be considered as 'branches' of the discrete series of irreducible representations of the compact Lie algebra $u(p+q)$.

We recall that the pseudo-unitary group $U(p,q)$ is defined as the group of linear transformations of $(p+q)$-dimensional complex space C^{p+q}, which conserve the hermitian form

$$\bar{z}^1z^1+\dots+\bar{z}^pz^p-\bar{z}^{p+1}z^{p+1}-\dots-\bar{z}^{p+q}z^{p+q}. \tag{1}$$

Because $U(p,q)$ and $U(q,p)$ are isomorphic, we can restrict ourselves to the case of $U(p,q)$ with $p \geqslant q$.

The hermitian form (1) can be written as

$$z^*\sigma z, \tag{2}$$

where z represents the column consisting of z_k, $k = 1, 2, \ldots, p+q$, $z^* = (\bar{z}^1, \ldots, \bar{z}^{1+q}) = \bar{z}^T$ and the matrix σ is of the form

$$\sigma = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad \text{where } I_p = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}. \tag{3}$$

Using the definition of $U(p, q)$ we get

$$z^*u^*\sigma uz = z^*\sigma z, \quad u \in U(p, q).$$

Therefore due to the arbitrariness of z, we have

$$u^*\sigma u = \sigma, \quad \text{for} \quad u \in U(p, q). \tag{4}$$

We can now give an equivalent definition of $U(p, q)$; namely, $U(p, q)$ is the group of all linear transformations of C^{p+q} which obey the condition (4). Setting $u(t) = \exp(\mathrm{i}tM)$ in eq. (4), differentiating with respect to t and putting $t = 0$, we find that the generators M obey the condition

$$M^* = \sigma M\sigma, \tag{5}$$

with the commutation relations

$$[M, M'] = MM' - M'M. \tag{6}$$

Consider now the matrix A_{ij} given by

$$(A_{ij})_{lk} = \delta_{il}\delta_{jk}, \quad i, j, l, k = 1, 2, \ldots, p+q. \tag{7}$$

These matrices obey the commutation relations of the Lie algebra $\mathrm{gl}(p+q, R)$

$$[A_{ij}, A_{kl}] = \delta_{kj}A_{il} - \delta_{il}A_{kj}. \tag{8}$$

Clearly every $n \times n$ complex matrix can be expressed as a linear combination of the A_{ij}-matrices. In particular, every generator $M \in u(p, q)$ obeying the condition (5) can be expressed in terms of A_{ij}. In fact, the $(p+q)^2$ independent matrices

$$\begin{gathered} M_{kk} = A_{kk}, \quad k = 1, 2, \ldots, n, \\ M_{kl} = A_{kl} + A_{lk}, \quad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}); \quad k < l \leqslant p, \text{ or } p < k < l, \\ N_{kl} = A_{kl} - A_{lk}, \quad \tilde{N}_{kl} = \mathrm{i}(A_{kl} + A_{lk}), \quad k \leqslant p < l, \end{gathered} \tag{9}$$

obey the condition (5) and therefore represent the generators of $U(p, q)$. The commutation relations for the generators (9) of $U(p, q)$ can be obtained with the help of that of $\mathrm{gl}(p+q, R)$ given in (8). The hermiticity condition for generators M_{ik} and N_{ik}

$$M^*_{ik} = M_{ik}, \quad N^*_{ik} = N_{ik} \tag{10}$$

impose the following conditions on the generators A_{kl}

$$A_{kl}^* = \varepsilon_{kl} A_{lk}, \quad \varepsilon_{kl} = \begin{cases} +1, & k, l \leqslant p \text{ or } p < k, l, \\ -1, & k \leqslant p < l \text{ or } l \leqslant p < k. \end{cases} \tag{11}$$

Therefore, the problem of finding the irreducible hermitian representations of $u(p, q)$ is reduced to that of finding those irreducible representations of $\mathrm{gl}(p+q, R)$ whose generators obey the condition (11). From the practical point of view this is a considerable simplification of the problem, because the generators of $\mathrm{gl}(n, R)$, in contradistinction to those of $u(p, q)$, obey the simple and symmetric commutation relations of the form (8) (see also chs. 10.2 and 9.4).

The construction of hermitian irreducible representations of $u(p, q)$ can be carried out along the following steps:

(i) The construction of the carrier space.

(ii) The definition of the action of the generators A_{ij} obeying the commutation relations (8) and the 'hermiticity' condition (11).

(iii) The proof of irreducibility and inequivalence of the representations obtained.

In the case of the compact algebra $u(n)$ the carrier space H_{m_n} was specified by the highest weight $m_n = (m_{1n}, \dots, m_{nn})$, which was placed in the top row of the Gel'fand–Zetlin pattern (see ch. 10, § 1). We specify the carrier space of the non-compact Lie algebra $u(p, q)$ also by the top row ('highest weight') $m_n = (m_{1n}, \dots, m_{nn})$, $n = p+q$, of patterns and by the so-called '*type*' of representation. The type of the representation is determined by the decomposition of the number p into two non-negative integers α and β:

$$p = \alpha + \beta. \tag{12}$$

If the top row $m_n = (m_{1n}, \dots, m_{nn})$, and the type (α, β) are given, then we define the generalized Gel'fand–Zetlin patterns by the following set of inequalities

$$\begin{gathered} m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}, \quad k = 1, 2, \dots, p-1, \\ m_{1k} \geqslant m_{1,k+1}+1 \geqslant m_{2k} \geqslant m_{2,k+1}+1 \geqslant \dots \geqslant m_{\alpha k} \geqslant m_{\alpha,k+1}+1, \\ \hfill k = p, \dots, n-1, \\ m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}, \quad j = \alpha+1, \dots, k-\beta, k = p, \dots, n-1, \\ m_{k-\beta+2,k+1}-1 \geqslant m_{k-\beta+1,k} \geqslant m_{k-\beta+3,k+1}-1 \geqslant \dots \geqslant m_{k+1,k+1}-1 \geqslant m_{kk}, \\ \hfill k = p, \dots, n-1. \end{gathered} \tag{13}$$

In the present non-compact case the numbers m_{1k} run over the interval $[m_{1,k+1}+ +1, \infty)$, and the numbers m_{kk} over the interval $(-\infty, m_{k+1,k+1}-1]$. Therefore, the carrier space H_{m_n} is always infinite-dimensional.

The structure of a pattern m is defined as follows: place all numbers m_{jk} of the kth row between the numbers $m_{j,k+1}$ and $m_{j+1,k+1}$ of the $(k+1)$th row, as in the case of the compact $u(n)$ patterns. Then shift each of the first α elements

$m_{1,n-1}, \ldots, m_{\alpha,n-1}$ of the $(n-1)$th row one place to the left. Similarly, shift each of the last β elements of the $(n-1)$th row one place to the right. Then shift the elements $m_{i,n-2}$ of the $(n-2)$nd row with respect to the $(n-1)$st row and so on, till the elements m_{ip} of the pth row, inclusive. The position of the elements of m_{ij} of the jth row, $j < p$ with respect to the neighbouring higher row, remains unchanged. In this manner the structure of the pattern reflects the properties of the inequalities (13) imposed on the numbers m_{ij}. It is assumed that the unit vectors corresponding to different patterns are orthonormal and they span the carrier space H_{m_n}.

EXAMPLE. *The Lie algebra* $u(2,1)$. In this case we have three types:

$$(\alpha, \beta) \sim (2,0),\ (1,1) \text{ and } (0,2).$$

The corresponding patterns are:

$$\left|\begin{matrix} & m_{13} & m_{23} & m_{33} \\ m_{12} & & m_{22} & \\ & m_{11} & & \end{matrix}\right|, \quad \left|\begin{matrix} & m_{13} & m_{23} & m_{33} & \\ m_{12} & & & & m_{22} \\ & & m_{11} & & \end{matrix}\right|, \quad \left|\begin{matrix} m_{13} & m_{23} & m_{33} & \\ & m_{12} & & m_{22} \\ & & m_{11} & \end{matrix}\right|.$$

The numbers m_{ij} of the first pattern obey the following inequalities:

$$m_{12} \geqslant m_{13}+1, \quad m_{13} \geqslant m_{22} \geqslant m_{23}, \quad m_{12} \geqslant m_{11} \geqslant m_{22}$$

and of the second one the inequalities

$$m_{12} \geqslant m_{13}+1, \quad m_{33}-1 \geqslant m_{22}, \quad m_{12} \geqslant m_{11} \geqslant m_{22}.$$

Thus the infinite-dimensional carrier space H_{m_n} is determined by a set of integers $m_n = (m_{1n}, \ldots, m_{nn})$ ('highest weight') which obey the inequalities

$$m_{1n} \geqslant m_{2n} \geqslant \ldots \geqslant m_{nn}, \quad n = p+q,$$

and by a type (α, β). ▼

As in the case of $u(n)$, let the symbol $m^{\hat{j}}_{k-1}(m^{j}_{k-1})$ denote the pattern obtained from a pattern m by replacing the number $m_{j,k-1}$ in the $(k-1)$st row of m by the number $m_{j,k-1}-1$ $(m_{j,k-1}+1)$.

In order to obtain a hermitian representation of $u(p,q)$ in H_{m_n}, we shall define the action od A_{ij}, $i,j = 1, 2, \ldots, n$, on the basis vectors of H_{m_n} in such a way that the commutation relations (8) and the 'hermiticity' conditions (11) will be fulfilled. It was shown in ch. 10, § 1 that any generator $A_{k-h,k}$, $h > 0$, can be expressed as the commutator of the generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ by

$$A_{k,k-h} = [A_{k,k-1}, A_{k-1,k-h}], \quad A_{k-h,k} = [A_{k-h,k-1}, A_{k-1,k}], \quad h > 1. \tag{14}$$

Therefore, in order to define the action of an arbitrary generator A_{ij}, $i,j = 1, 2, \ldots, p+q$, in the carrier space H_{m_n} it is sufficient to define the action of the generators $A_{kk}, A_{k-1,k}$, and $A_{k,k-1}$, $k = 1, 2, \ldots, p+q-1$. The action of these generators on the pattern $m \in H_{m_n}$ may be specified in a similar manner as in the case of the algebra $u(n)$. We denote by $m^{j}_{k-1}(m^{\hat{j}}_{k-1})$ the pattern obtained from m by replacement of $m_{j,k-1}$ by $m_{j,k-1}(m_{j,k-1}+1)$.

THEOREM 1. *Let H_{m_n} be a linear space spanned by patterns defined by eqs.* (12) *and* (13). *Let the action of operators A_{kk}, $A_{k-1,k}$ and $A_{k,k-1}$, $k = 1, 2, \ldots, p+q$, be defined by*

$$A_{kk} m = (r_k - r_{k-1}) m, \tag{15}$$

$$A_{k,k-1} m = \sum_{j=1}^{k-1} a_{k-1}^{j}(m) m_{k-1}^{j}, \tag{16}$$

$$A_{k-1,k} m = \sum_{j=1}^{k-1} b_{k-1}^{j}(m) m_{k-1}^{\hat{j}}, \tag{17}$$

where

$$r_0 = 0, \quad r_k = \sum_{j=1}^{k} m_{jk}, \quad k = 1, 2, \ldots, n, \tag{18}$$

and

$$a_{k-1}^{j}(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq k} (l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2}, \tag{19}$$

$$b_{k-1}^{j}(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2}, \tag{20}$$

$$l_{ik} = m_{ik} - i, \tag{21}$$

$$\arg a_{k-1}^{j} = \arg b_{k-1}^{j} = \begin{cases} 0 & \textit{for } k \neq p+1, \\ \dfrac{\pi}{2} & \textit{for } k = p+1. \end{cases} \tag{22}$$

Then the action of an arbitrary operator A_{kl} is given by the following formula

$$A_{kl} m = \sum_{i_{k-1} \ldots i_l} a_{i_{k-1} \ldots i_l}(m) m_{i_{k-1} \ldots i_l}, \quad k > l \tag{23}$$

and

$$A_{kl} m = \sum_{i_k \ldots i_{l-1}} b_{i_k \ldots i_{l-1}}(m) m_{\hat{i}_k \ldots \hat{i}_{l-1}}, \quad k < l, \tag{24}$$

where

$$a_{i_{k-1} \ldots i_l}(m) = \prod_{s=l+1}^{k} a_{i_{s-1}}(m) \prod_{s=l+2}^{k} \frac{\varepsilon_{i_{s-1} i_{s-2}}(m)}{c_{i_{s-1} i_{s-2}}(m)}, \quad k > l^{*}, \tag{25}$$

---

* The product $\prod_{s=l+2}^{k}$ is, by definition, equal to 1 when $l = k-1$.

$$b_{i_k\ldots i_{l-1}}(m) = \prod_{s=k+1}^{l} b_{i_{s-1}}(m) \prod_{s=k+2}^{l} \frac{\varepsilon_{i_{s-1}i_{s-2}}(m)}{c_{i_{s-1}i_{s-2}}(m)}, \quad k < l, \tag{26}$$

$$a_{i_s}(m) = a_s^{i_s}(m), \quad b_{i_s}(m) = b_s^{i_s}(m), \tag{27}$$

$$c_{i_s i_t}(m) = [(l_{i_s s} - l_{i_t t} + 1)(l_{i_s s} - l_{i_t t})]^{1/2} \geqslant 0, \tag{28}$$

$$\varepsilon_{i_s i_t}(m) = \operatorname{sign}(l_{i_s s} - l_{i_t t}), \tag{29}$$

and i_s runs over the set of values $1, 2, \ldots, s$. The patterns $m_{i_{k-1},\ldots,i_l}$ and $m_{\hat{i}_k,\ldots,\hat{i}_{l-1}}$ are defiined inductively as follows:

$$m_{i_{k-1}\ldots i_l} = (m_{i_{k-1}\ldots i_{l-1}})_l^{i_l}, \quad k > l, \quad m_{\hat{i}_k\ldots\hat{i}_{l-1}} = (m_{\hat{i}_k\ldots\hat{i}_{l-2}})_{l-1}^{\hat{i}_{l-1}}, \quad k < l, \tag{30}$$

where

$$m_{i_s} = m_s^{i_s}, \quad m_{\hat{i}_s} = m_s^{\hat{i}_s}. \tag{31}$$

The operators A_{kl} satisfy the hermiticity conditions (11) *and commutation relations* (8).

Every self-adjoint representation of $u(p, q)$, $p+q = n$, determined by the highest weight $m_n = (m_{1n}, \ldots, m_{nn})$ and the type (α, β) is irreducible. Two representations are unitarily equivalent if and only if their highest weights and types coincide. ▼

The proof of this theorem involves algebraic manipulations only, and because it is very lengthy we omit it. (For details cf. Nikolov and Rerich 1966.)

The set of self-adjoint representations of $u(p, q)$ given by th. 1 is called the *discrete series*. It is evident from the formulas (15), (23) and (24) that a common, dense, linear invariant domain D in H_{m_n} for $u(p, q)$ consists of all finite linear combinations of basis elements m determined by the type (12) and inequalities (13). Clearly the space D is also the invariant domain for the enveloping algebra E of $u(p, q)$.

Decomposition with Respect to Subalgebras. The set of generators A_{ij}, $i, j = 1, 2, \ldots, p+q-1$, determines a subalgebra $u(p, q-1)$ of $u(p, q)$. A subalgebra $u(p, k)$ can be selected in a similar manner from the algebra $u(p, k+1)$. Therefore, we finally obtain the decreasing chain of algebras

$$u(p, q) \supset u(p, q-1) \supset \ldots u(p, 0) \supset \ldots \supset u(2, 0) \supset u(1, 0). \tag{32}$$

We end the section with some comments:

(a) The above construction of representations follows the work of Gel'fand and Graev 1965. They simply guessed the form of patterns for $u(p, q)$ and the action of generators A_{kk}, $A_{k,k-1}$ and $A_{k-1,k}$ in $H_{m,n}^{(\alpha,\beta)}$ on the basis of the compact case $u(n)$.

There is so far no systematic derivation of the structure of admissible patterns of irreducible representations for $u(p, q)$, $q > 1$.

(b) *Representations of* $u(p, 1)$. The case of $u(p, 1)$ was treated in more detail by Gel'fand and Graev 1965 and Ottoson 1967. This case is considerably simpler than the general case with $q > 1$ because $u(p, 1)$ has the compact subalgebra $u(p)$ as a maximal subalgebra; hence the well-elaborated Gel'fand–Zetlin technique for $u(p)$ can be used for an analysis of irreducible representations of $u(p, 1)$. The very detailed derivation of the structure of patterns for $u(p, 1)$, the action of generators $A_{k,k}$, $A_{k-1,k}$ and $A_{k-1,k}$ as well as the classification of all irreducible representations of $u(p, 1)$ were given by Ottoson 1967. It is interesting that in the case of $u(p, 1)$ one obtains—besides a discrete series of representations—semidiscrete ones which are defined by $p-1$ discrete parameters and a complex number.

(c) *Degenerate Representations of* $u(p, q)$. It was remarked by Todorov 1966 that there exists a discrete series of so-called degenerate representations which are not contained in the set of discrete representations considered up to now. In fact, if we impose the restriction

$$m_{k,n} = m_{k+1,n} = \dots = m_{k+r,n}, \qquad r > q \tag{33}$$

upon r neighboring components of a 'weight' m_n then we have

$$n-r \leqslant \alpha+\beta < p \tag{34}$$

and therefore we get the new discrete series of representations of $u(p, q)$. Moreover, the inequalities $m_{\alpha,n} \geqslant m_{\alpha+1,n}$ and $m_{n-\beta,n} \geqslant m_{n-\beta+1,n}$ might be violated. The resulting $(a_{jk})^2$ and $(b_{jk})^2$ which would have in general the wrong sign for such α and β here either vanish (due to the large number of inequalities between m_{ik}) or may be redefined in such a way as to give a hermitian representation of $u(p, q)$. There are two interesting cases. In the first one, we have,

$$m_{2,n} = \dots = m_{n,n} = 0, \qquad \alpha = 1,\ \beta = 0. \tag{35}$$

The basis vectors for such a representation are of the form

$$\left|\begin{array}{ccccccccc}
 & & & m_{1n} & 0 & 0 & 0\ \dots & 0 & 0 \\
 & & m_{1,n-1} & & 0 & 0 & & 0 & \\
 & \cdot & & & \cdot & & & \cdot & \\
m_{1p} & & & & 0 & & \dots\ 0 & & \\
 & \cdot & & & & \cdot & & & \\
 & & \cdot & & & & \cdot & & \\
 & & & m_{12} & & 0 & & & \\
 & & & & m_{11} & & & &
\end{array}\right| \tag{36}$$

These degenerate representations correspond to the so-called 'ladder' representations, which were used for the description of the properties of elementary particles multiplets (see, e.g., Anderson and Rączka 1967 a, b). The other discrete series of so-called *maximally degenerate self-conjugate representations* is obtained when the components m_{in} obey:

$$m_{1n} = -m_{n,n}, \quad m_{2n} = \ldots = m_{n-1,n} = 0, \quad \alpha = \beta = 1. \tag{37}$$

The corresponding patterns are

$$\left|\begin{array}{ccccccccc}
 & & m_{1n} & & 0\ 0 \ldots 0\ 0 & & -m_{1n} & & \\
 & m_{1,n-1} & & 0 & & 0 & & m_{1,n-1} & \\
 \cdot^{\cdot} & & & \vdots & & \vdots & & & \ddots \\
m_{p1} & & & & 0 \ldots 0 & & & & m_{p,p} \\
 \ddots & & & & & & & & \cdot^{\cdot} \\
 & m_{13} & & & & & & m_{33} & \\
 & & & m_{12} & 0 & m_{22} & & & \\
 & & & & m_{11} & & & &
\end{array}\right|. \tag{38}$$

These representations were also used for the description of the properties of elementary particle multiplets (Todorov 1966). The full classification of discrete degenerate series and semidiscrete series for the general $u(p, q)$-algebra has not yet been worked out.

Representations of $U(p, q)$. The problem of integrability of representations of discrete series was solved by Mickelsson and Niederle 1973 and Kotecký and Niederle 1975. Using J. Simon's integrability condition for real Lie algebras they proved that every representation of the discrete series of $u(p, q)$ constructed by Gel'fand–Graev method is differential of a unitary one-valued representation of group $U(p, q)$.

The global form of one-parameter subgroups of $u(p, q)$ was explicitly calculated by Gel'fand and Graev. Then using the representation of a group element $g \in U(p, q)$ in terms of one-parameter subgroups, one may calculate the global form of the representation. Gel'fand and Graev expressed the matrix elements for global representations in terms of generalized β-functions. They were not able, however, to obtain the final formulas in a closed form: an effective solution of this problem would be very useful for many applications.

Representations of $\mathrm{so}(p, q)$. The discrete series of irreducible hermitian representation of $\mathrm{so}(p, q)$ was constructed by Nikolov 1967. The very detailed analysis of all principal supplementary and exceptional series of representations of $\mathrm{so}(n, 1)$ in the framework of Gel'fand patterns was given by Ottoson 1968.

§ 9. Comments and Supplements

A. *Comments*

The representation theory of Lie algebras discussed in sec. 1 is based on the work of Gårding 1947. It is interesting that the th. 1.1 giving a common dense invariant domain for the generators of a Lie algebra can be easily extended to semigroups and even to arbitrary differentiable manifolds.

The representation theory of enveloping algebras in sec. 2 is based on the work of Nelson and Stinespring 1959. Some scattered results were known before; in particular corollary 4 to th. 2.3 was proved previously by Segal 1951.

The theory of analytic vectors was originated by Harish–Chandra 1953. He showed, in particular, that for certain representations of semisimple Lie groups, the set of analytic vectors (well-behaved vectors in his terminology) is dense in H. Next Cartier and Dixmier 1958 showed that if T is either bounded or scalar-valued on a certain discrete central subgroup Z of G then the set of analytic vectors for T is dense.

The content of secs. 3, 4 and 5 on analytic vectors and their applications is based on the fundamental work of Nelson 1959. A more detailed proof of theorem of Nelson 8.5.2 and various extensions of the concept of analytic vectors were given by Goodman 1969. The most important th. 5.2 which gives a convenient criterion for integrability of a representation of a Lie algebra found many applications in particle physics and quantum field theory (cf. ch. 21).

The example 5.1 of nonintegrable representations has the shortcoming that the operator $X = \frac{\mathrm{d}}{\mathrm{d}\varphi}$ is not essentially self-adjoint. Nelson 1959 has found an example which was surprising to many physicists: namely, he has constructed two essentially self-adjoint operators A and B commuting on the common dense invariant domain and showed that the global transformations $\exp(\mathrm{i}tA)$ and $\exp(\mathrm{i}tB)$ do not commute. This demonstrates that an abelian Lie algebra might not be integrable to a global abelian group if the Nelson operator $\Delta = X_1^2 + \ldots + X_n^2$ is not essentially self-adjoint.

The th. 5.3 was derived by K. Maurin and L. Maurin 1964.

The theory of integrability of Lie algebras representation using the concept of weak analyticity was elaborated by Flato, Simon, Snellman and Sternheimer 1972. The simplified version of this theory, using the conditions for Lie generators of Lie algebra only was elaborated by Simon 1972 and Flato and Simon 1973.

The idea of using the solution of the heat equation on a Lie group G for the construction of analytic vectors for a representation T of G was first suggested by Nelson 1959, § 8. In sec. 6 we followed the simplified version of this theory elaborated by Gårding 1960.

One could ask whether it is not possible to develop a representation theory of Lie algebras dealing with skew-symmetric bounded operators only. One could then get rid of almost all difficulties encountered in this chapter. This interesting problem was considered by Doebner and Melsheimer who proved

THEOREM 1. *A nontrivial representation of a noncompact Lie algebra by skew-symmetric operators contains at least one unbounded operator.*

(For the proof cf. Doebner and Melsheimer 1967, th. 1.)

§ 10. Exercises

§ 1.1. Show that every skew-symmetric representation $X \to T(X)$ of a Lie algebra L in a complex Hilbert space having a common dense invariant domain D is strongly continuous on D.

Hint: Endow with the Euclidean topology and use the linearity of the representation.

§ 1.2. Let $G = \mathrm{SO}(3)$ and $H = L^2(S^2, \mu)$. Show that the self-adjoint generators of the left quasi-regular representation $T_x u(s) = u(x^{-1}s)$ have the form

$$\begin{aligned} L_x &= \mathrm{i}\left(\sin\varphi\frac{\partial}{\partial\vartheta}+\cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}\right),\\ L_y &= \mathrm{i}\left(\cos\varphi\frac{\partial}{\partial\vartheta}+\cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi}\right),\\ L_z &= -\mathrm{i}\frac{\partial}{\partial\varphi}. \end{aligned} \tag{1}$$

§ 1.3. Let $G = \mathrm{SO}(3)$, φ, ϑ, ψ the Euler angles for G and $H = L^2(G, \mu)$. Let T be the right regular representation of G in H. Show that the generators of the Lie algebra so (3) have the form

$$\begin{aligned} L_x &= -\cot\vartheta\sin\psi\frac{\partial}{\partial\psi}+\frac{\sin\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi}+\cos\psi\frac{\partial}{\partial\vartheta},\\ L_y &= -\cot\vartheta\cos\psi\frac{\partial}{\partial\psi}+\frac{\cos\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi}-\sin\psi\frac{\partial}{\partial\vartheta},\\ L_z &= \frac{\partial}{\partial\psi} \end{aligned} \tag{2}$$

Find the Gårding domain D_G in H. Find generators of the left regular representation T^L in H.

§ 1.4. Let $G = T^4 \circledS \mathrm{SO}(3, 1)$ and let $H = L^2(R^4)$. Let the T be the left quasi-regular representation of G in H. Find the form of generators of the Lie algebra $t^4 \circledS \mathrm{so}(3, 1)$ in H.

§ 1.5.* Let $E(R^{2n})$ be the Schwartz space of test functions of R^{2n}. Let $F(q,p)$ be a function on the phase space R^{2n} which is in $E(R^{2n})$. Show that the operator $\hat{F}$ associated with the function F by the formula

$$F \to \hat{F} = F-\frac{1}{2}\sum_l\left(q_l\frac{\partial F}{\partial q_l}+p_l\frac{\partial F}{\partial p_l}\right)-\mathrm{i}\sum_l\left(\frac{\partial F}{\partial q_l}\frac{\partial}{\partial p_l}-\frac{\partial F}{\partial p_l}\frac{\partial}{\partial q_l}\right) \tag{3}$$

provides in the space $E(R^{2n})$ a representation of the Lie algebra of classical Poisson brackets, i.e.

$$\widehat{\{F, G\}} = -\mathrm{i}[\hat{F}, \hat{G}]. \tag{4}$$

§ 1.6.* Let W be any function in $E(R^{2n})$. Show that the map

$$\hat{F}' = \{W, F\}+\hat{F} \tag{5}$$

still provides a representation of the Lie algebra of Poisson brackets in space $E(R^{2n})$.

§ 1.7.* Let σ be a canonical transformation in R^{2n}, $\sigma: (p, q) \to (p^\sigma, q^\sigma)$. Let for an arbitrary F in $E(R^{2n})$ the quantity $\hat{F}$ and $\hat{F}^\sigma$ denote a representation (3) in variables (p, q) and (p^σ, q^σ) respectively. Show that there exists a unitary transformation U_σ in $L^2(R^{2n})$ such that

$$U_\sigma \hat{F} U_\sigma^{-1} = \hat{F}^\sigma \quad \text{for all } F. \tag{6}$$

§ 1.8.** Let $\mathscr{F}$ be the space of initial conditions (φ, π) for a classical scalar field $\Phi(x)$ satisfying the nonlinear relativistic wave equation

$$(\Box + m^2)\Phi(x) = \lambda \Phi^3(x), \qquad \lambda < 0, \tag{7}$$

$$\Phi(0, x) = \varphi(x), \qquad (\partial_t \Phi)(0, x) = \Pi(0, x) = \pi(x). \tag{8}$$

Let $F(\varphi, \pi)$ be a smooth functional over the inital data. Show that the operator $\hat{F}$ associated with F by the formula

$$F \to \hat{F} = F - \frac{1}{2} DF\,[(\varphi, \pi)]\,(\varphi, \pi) - \mathrm{i} \int \mathrm{d}^3 z \left(\frac{\delta F}{\delta \varphi(z)} \frac{\delta}{\delta \pi(z)} - \frac{\delta F}{\delta \pi(z)} \frac{\partial}{\partial \varphi(z)} \right) \tag{9}$$

where DF is the Frechet's differential in the point (φ, π) and $\dfrac{\delta}{\delta \varphi}$ and $\dfrac{\delta}{\delta \pi}$ are Frechet's derivatives, provides an algebraic representation of the classical Lie algebra of Poisson brackets.

§ 1.9.** Introduce a topology in the space $E(\mathscr{F})$ of smooth functionals over the space $\mathscr{F}$ of initial data and find a class of smooth functionals for which the map $F \to \hat{F}$ given by eq. (9) will provide an operator representation (cf. Bałaban, Jezuita and Rączka 1976 for a particular solution of this problem).

§ 1.10. Let $\Phi[x|\varphi, \pi]$ be a solution of eq. (7) defined by the initial conditions (φ, π). Show that the operators $\hat{\Phi}$ and $\hat{\Pi}$ associated with Φ and Π respectively by the formula (9) satisfy the following equal time commutation relations

$$[(\hat{\Phi}(t, x), \hat{\Pi}(t, y)] = i\delta^{(3)}(x - y), \tag{10}$$

$$[\hat{\Phi}(t, x), \hat{\Phi}(t, y)] = [\hat{\Pi}(t, x), \hat{\Pi}(t, y)] = 0. \tag{11}$$

§ 1.11. Let $\{h_l\}_1^\infty$ be an orthonormal basis in $L^2(R^3)$. Let $\varphi(x) = \Sigma q_l h_l(x)$ and $\pi(x) = \Sigma p_l h_l(x)$ be the expansion of the canonical variables (φ, π) in the basis $\{h_l\}$. Show that the quantization formula (9) written in terms of variables $\{q_l\}_1^\infty$ and $\{p_l\}_1^\infty$ coincides with the formula (3).

§ 2.1.* Let $G = \mathrm{SO}(3)$. Show that the angular momentum operator $\boldsymbol{J}^2$ associated in the space $L^2(R^3)$ with the quasi-regular representation $T_x u(\gamma) = u(x^{-1}\gamma)$ has only non-negative integer eigenvalues, i.e.

$$J^2 \Psi_\lambda(r) = \lambda \Psi_\lambda(r) \quad \text{with} \quad \lambda = J(J+1),\ J = 0, 1, 2, \ldots, \tag{12}$$

whereas in the space $L^2(\mathrm{SO}(3))$ it has integer and half-integer nonnegative eigenvalues.

§ 2.2. Let T be the representation of $T^4 \circledS \mathrm{SO}(3, 1)$ as in exercise 1.4. Find the spectrum of the Casimir operator $M^2 = P_\mu P^\mu$. Show that the second Casimir operator $S^2 = W_\mu W^\mu$ is identically zero in H.

§ 4.1.* Let $x \to T_x$ be the left regular representation of the translation group R on $H = L^2(R)$. Show that the function

$$u(x) = \sum_{n=1}^{\infty} 2^{-n}[(x+n)^2+n^{-1}]^{-1}, \quad x \in R \tag{13}$$

is analytic in H but is not an analytic vector for T.

§ 4.2.* Show that $u(x)$ is the analytic vector for the representation of 4.1 if its Fourier transform $\hat{u}(p)$ satisfies the condition

$$\exp(\lambda|p|)\hat{u}(p) \in L^2(R) \tag{14}$$

for some $\lambda > 0$.

§ 5.1. Show that the operators

$$\begin{aligned} J_+ &= J_x + \mathrm{i} J_y = z^2 \frac{\mathrm{d}}{\mathrm{d}z} - 2\lambda z, \\ J_- &= J_x - \mathrm{i} J_y = -\frac{\mathrm{d}}{\mathrm{d}z}, \\ J_3 &= z\frac{\mathrm{d}}{\mathrm{d}z} - \lambda, \end{aligned} \tag{15}$$

with $\lambda \in C$, form the infinite-dimensional representation of the Lie algebra so(3) in the space of analytic functions integrable with respect to the Gaussian measure. Show that the Lie algebra cannot be integrated to a global continuous representation of the group SO(3).

§ 5.2.*** Give the classification of non integrable representations of the canonical commutation relations

$$[q, p] = \mathrm{i}I.$$

§ 6.1. Show that the functions

$$\psi(t, \varphi, \vartheta, \psi) = \exp[-tJ(J+1)]D^J_{M\lambda}(\varphi, \vartheta, \psi) \tag{16}$$

are solutions of the heat equation on the rotation group SO (3)

§ 6.2.*** Find the solutions of the heat equation on the Poincaré group.

§ 7.1.*** Give the full classification of discrete degenerate and semidiscrete degenerate series of $u(p, q)$.

§ 7.2.*** Let X be a linear operator in a Hilbert space H. Find a necessary condition for the existence of a dense set of analytic vectors of X in H.

§ 7.3.*** Show that every irreducible essentially skew-adjoint representation of a real finite-dimensional Lie algebra defined on an invariant dense domain in a Hilbert space is integrable.

§ 7.4.*** Give a Lie algebraic formulation of axiomatic quantum field theory.

§ 7.5.*** Find integrability conditions for representations of infinite-dimensional (Hilbertian) Lie algebras in a Hilbert space.

§ 7.6. Show that three series of selt-adjoint irreducible representations of the Lie algebra su(1, 1) $\sim$ so(2, 1) $\sim$ sp(2R) $\sim$ sl(2, R) can be constructed in terms of creation and annihilation operators as follows: Let the Lie algebra be represented by

$$X_1 = \frac{\mathrm{i}}{2}a^*\sigma_1 a, \quad X_2 = \frac{\mathrm{i}}{2}a^*\sigma_2 a, \quad X_3 = \frac{1}{2}a^*\sigma_3 a \quad \text{with } \tilde{C}_2 = X_3^2 - X_1^2 - X_2^2,$$

where σ_k, $k = 1, 2, 3$, are the Pauli matrices and $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a^* = (a_1^*, a_2^*)$; $[a_i, a_j^*] = \delta_{ij}$; $i, j = 1, 2$. Consider the vectors

$$|\varphi, m\rangle = N_{\varphi m} a_1^{*(\varphi+m)} a_2^{*(\varphi-m)}|0\rangle, \quad a_i|0\rangle = 0, \ i = 1, 2.$$

Then $X_3|\varphi, m\rangle = m|\varphi, m\rangle$ and $C_2|\varphi, m\rangle = \varphi(\varphi+1)|\varphi, m\rangle$ and the unitary representations ($\varphi(\varphi+1)$ real, X_i hermitian) are

(i) discrete series $D^\pm$ for $\varphi < 0$, real. For D^+:

$$m = -\varphi, -\varphi+1, -\varphi+2, \ldots \quad \text{and for} \quad D^-: m = \varphi, \varphi-1, \varphi-2, \ldots;$$

(ii) supplementary series: U^{φ, E_0} for $-1+|E_0| < \varphi < -|E_0|$, $-1 < |E_0| < \frac{1}{2}$

$$m = E_0, E_0 \pm 1, E_0 \pm 2, \ldots;$$

(iii) principal series $D^{\varphi,\sigma}$ for $\varphi = -\frac{1}{2}+i\sigma$, σ real; for global forms of these representations see ch. 16.

§ 7.7. For the Lie algebra so(3, 1) $\sim$ sl(2, C), with generators

$$J_k = \frac{1}{2}a^*\sigma_k a, \quad N_k = \frac{1}{2}\mathrm{i}(a^*\sigma_k C a^* + aC\sigma_k a), \quad k = 1, 2, 3, \ C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

acting on the Hilbert space of states spanned by

$$|jm\rangle = [(j+m)!(j-m)!]^{-1/2} a_1^{*(j+m)} a_2^{*(j-m)}|0\rangle$$

show that $C_2 = J^2 - N^2 = 0$ and $C_2' = J \cdot N = -\frac{3}{4}$. If the lowest state is $|0\rangle$ or $a^*|0\rangle$, respectively, one obtains two self-adjoint representations with $j_0 = j_{\min} = 0$ or $\frac{1}{2}$, respectively. The algebra sl(2, C) has two series of unitary representations labelled by two numbers:

(i) principal series: $j_0 = \frac{1}{2}, 1, \frac{3}{2}, \ldots, -\infty < a < +\infty$,

(ii) supplementary series: $j_0 = 0$, $0 \leqslant \mathrm{i}a \leqslant 1$, where

$$C_2 = j_0 a, \quad C_2' = 1 + a^2 - j_0^2.$$

Next use the two pairs of boson operators $a_i, b_i, i = 1, 2$, and

$$J_k = \tfrac{1}{2}(a^*\sigma_k a + b^*\sigma_k b), \quad N_k = -\tfrac{1}{2}(a^*\sigma_k C b^* - aC\sigma_k b),$$

construct other representations (cf. also ch. 19).

Chapter 12

Quantum Dynamical Applications of Lie Algebra Representations

In this chapter we discuss the direct applications of Lie algebra representations in solving kinematical and dynamical problems in quantum theory. It is designed to show how these representations occur and how they are used without first reading about all the subtleties of the formalism of quantum theory. Only the knowledge of the Schrödinger equation is assumed. The more detailed discussion of quantum mechanical framework, the concept of symmetry and group representations is relegated to ch. 13 and the relativistic problems to ch. 20 and 21.

§ 1. Symmetry Algebras in Hamiltonian Formulation

We begin with the theory of non-relativistic quantum systems described by the Schrödinger equation ($\hbar = 1$)

$$i\partial_t \psi(q, t) = H\psi(q, t), \tag{1}$$

where ψ is the wave function of the system, q the set of dynamical coordinates and H the Hamiltonian operator of the system, a differential operator, function of q and $\partial/\partial q$. Alternatively, we could consider, the Heisenberg algebra $[p_i, q_j] = -i\delta_{ij}I$ and H would be a function of p's and q's. The equivalence of Heisenberg and Schrödinger pictures is treated in ch. 20. As we noted at the beginning of ch. 11 most quantum mechanical operators are unbounded operators and for a rigorous treatment the theory developed in ch. 11 must be used.

For stationary solutions of eq. (1) of the form $\psi(t, q) = \exp(-iEt)u(q)$ we obtain the eigenvalue equation

$$Hu = Eu. \tag{2}$$

The symmetry of the Hamiltonian H of a quantum system is generated by those operators which commute with H and possess together with H a common dense invariant domain D in the carrier space. If

$$[H, X_1] = 0 \quad \text{and} \quad [H, X_2] = 0$$

holds on D, then

$$[H, [X_1, X_2]] = 0$$

also on D, by virtue of the fact that for linear operators the Jacobi identity holds. Hence the subset of operators satisfying the above condition forms a Lie algebra. Usually, the representations of the Lie algebras occurring in simple problem are integrable, and symmetry Lie algebras and symmetry Lie groups are used in physics literature generally indiscriminatly.

This notion of symmetry group in the narrow sense should be really called 'group of degeneracy of the energy', because an eigenspace of H for a fixed value E of the energy, is a representation space of the Lie algebra commuting with H. We first discuss in terms of examples this simpler notion of symmetry, in § 2 we shall generalize it considerably. It should be remarked that in non-relativistic quantum mechanics we are led directly to the representation of Lie algebras, the representation of the corresponding symmetry groups enter eventually, because certain important global concepts belong to the level of groups, not algebras.

The largest Lie group (or algebra) whose elements commute with H will be called the *maximal symmetry group*. In many cases it is much larger than the kinematical symmetry group.

EXAMPLE 1. The simple quantum rotator with the Hamiltonian $H = \dfrac{1}{2I}J^2$ where I is the moment of inertia and $J^2 = J_1^2+J_2^2+J_3^2$ is the Casimir operator of the Lie algebra of SO (3) provides a simple example of a quantum system with the symmetry group, in this case the group of rotations. In the carrier space of quantum mechanics $H = L^2(R^3)$ this Hamiltonian has eigenfunctions $u_\lambda(q)$ given by spherical harmonics $Y_M^J(\vartheta, \varphi)$ with eigenvalues $\lambda = (2I)^{-1}J(J+1)$, $J = 0, 1, 2, \ldots$ Hence the eigenspace $H^\lambda \subset H$ is the carrier space of $(2J+1)$-dimensional irreducible representation of SO (3).

According to the basic postulates of quantum mechanics, we must realize H and J as self-adjoint operators, hence by virtue of Nelson theorem we are restricted ourselves to integrable representations of the Lie algebra, i.e. to quantum mechanical representations of the rotation group SO(3). Without this restriction, the Lie algebra of SO (3) admits a large class of irreducible representations which are however not all integrable (see 11.10.5.1).

We also note that the Hamiltonian does not tell us the multiplicity of each representation of the symmetry group, i.e. how many times a given value J occur. This question can be answered, as we shall see, within the framework of larger groups, namely by the existence of other operators distinguishing states with the same J.

EXAMPLE 2. *Quantum mechanical rigid rotator.* Here we have two sets of commuting angular momentum operators $\boldsymbol{J}_s$ and $\boldsymbol{J}_b$, namely angular momenta with respect to the space-fixed and body-fixed axes. For a completely symmetric rigid rotator the Hamiltonian is $H = \dfrac{1}{2I}(J_s^2+J_b^2)$. The Casimirs of the two algebras are however equal by definition of the rigid rotator $J_s^2 = J_b^2 = J^2$ and the Hamiltonian is again proportional to J^2.

Because we have two SO(3) algebras, the symmetry Lie algebra of H now is $SO(3)\times SO(3) \sim SO(4)$. The relation $J_s^2 = J_b^2$ indicates that only special representations of SO(4) of dimension $(2J+1)^2 = n^2$ occur. If we define the generators of SO(4), $\boldsymbol{L} = \boldsymbol{J}_s + \boldsymbol{J}_b$, $\boldsymbol{K} = \boldsymbol{J}_s - \boldsymbol{J}_b$, we have $\boldsymbol{K}^2 = 0$, $\boldsymbol{K}\cdot\boldsymbol{L} = 0$ and for each J, L ranges from 0 to $2J$. Notice that in the present case the symmetry Lie algebra is larger than the geometrical symmetry SO(3).

EXAMPLE 3. *Three-dimensional harmonic oscillator.* The dynamical conjugate variables are the three position operators q_i and the three momenta p_i (represented by $\frac{\partial}{i\partial q_i}$ in the Schrödinger picture). The Hamiltonian is given by (in suitable units)

$$H = \tfrac{1}{2}(p^2+q^2). \tag{3}$$

We consider again the stationary states given by (2).

One way to exhibit the group of degeneracy of the Hamiltonian (3) is to introduce the creation and annihilation operators a_l^*, a_l, $l = 1, 2, 3$ (cf. eqs. 10.4 (1), (2)) by $a_l = \frac{1}{\sqrt{2}}(q_l + ip_l)$, $a_l^* = \frac{1}{\sqrt{2}}(q_l - ip_l)$ and write (3) as

$$H = \sum_{l=1}^{3} a_l^* a_l + \tfrac{3}{2}. \tag{4}$$

It is easy to see that the operators $X_{ij} = a_i^* a_j + \frac{1}{2}\delta_{ij}$ commute with H. By virtue of eq. 10.4(12) X_{ij} form the basis elements of the Lie algebra $u(n)$. In fact, $H = \sum_{i=1}^{3} X_{ii} = \mathrm{Tr}(X) = C_1$. Hence, the remaining elements of $u(3)$ form the Lie algebra su(3). The calculation of the invariant operators of su(3) shows that higher order invariant operators C_2, C_3, etc. are functions of the Hamiltonian. Hence only most degenerate representations of $u(3)$ of dimension 1, 6, 10, 15, ..., are realized in the present example.

An important feature of this example is the fact that the Hamiltonian (3) possesses a larger symmetry group than the immediate geometric symmetry of the problem, namely the rotational invariance. The generators of the rotation group $J_1 = \frac{1}{2}(a_1^* a_2 + a_2^* a_1)$, $J_2 = -\frac{i}{2}(a_1^* a_2 - a_2^* a_1)$, $J_3 = \frac{1}{2}(a_1^* a_1 - a_2^* a_2)$ (cf. 10.4 (16)) are among the elements of the symmetry algebra su(3), but the latter has additional elements. For this reason it is called a *dynamical symmetry of H* (in contrast to the geometric symmetry of H). In an irreducible representation of su(3), the representations of the rotation group occur more than once, but the multiplicity of the representations of the dynamical symmetry group is one. This is because the energy values and the quantum numbers of the representations of the maximal symmetry group constitute a complete set of labellings of all states.

EXAMPLE 4. *Non-relativistic Kepler problem.* Here the dynamical variables are as in example 3: p_i, q_i. The Hamiltonian is given by (again in suitable units)

$$H = \tfrac{1}{2}p^2 - \frac{\alpha}{r}, \quad r \equiv |q|, \quad \alpha > 0. \tag{5}$$

The obvious geometrical symmetry of the problem is still the rotational symmetry: $[H, L] = 0$, $L = r \times p$. However, it is easy to verify that one other vector operator

$$A = \frac{1}{\sqrt{-2E}}\left\{\tfrac{1}{2}[J\times p - p\times J] + \frac{\alpha r}{r}\right\} \tag{6}$$

commutes with H on the space of eigenfunctions of H corresponding to a fixed eigenvalue E. One readily verifies that the operators A (called the *Runge–Lenz vector* for historical reasons) and L form the basis element of the Lie algebra so(4). For $E < 0$, bound states, the dynamical symmetry group of (5) is SO(4). For the representation (6) one of the Casimir operators of SO(4) vanishes:

$$L\cdot A = 0 \tag{7}$$

Hence only special representations of SO(4) are realized. With $J_1 = \frac{1}{2}(L+A)$, $J_2 = \frac{1}{2}(L-A)$, $J_1^2 = J_2^2 = J^2$, the degeneracy of the discrete levels is $(2J+1)^2 = n^2$, $J = 0, \frac{1}{2}, 1, \ldots$ (same as in example 2). The states can be labelled by $|n, J, M\rangle$ where n is equivalent to the energy label or to the remaining Casimir operator of SO(4). Hence the representations of the dynamical symmetry algebra have multiplicities one.

For $E > 0$, because of the occurrence of the factor i in eq. (6), we see that the real dynamical symmetry Lie algebra is now so(3, 1) and eq. (7) still holds. Physically these states correspond to the scattering states of the particle in the Kepler potential. The unitary representations of SO(3, 1) are infinite-dimensional: this implies that in a scattering experiment with a fixed energy, there are infinitely many partial waves of angular momentum, each with multiplicity one, equal to the multiplicity of irreducible representation T^L of SO(3) in the representation of SO(3,1).

Remark: The concept of symmetry can be applied to any other observable beside the energy, e.g. angular momentum, spin, etc. If A is an observable, and we consider the eigenvalue problem

$$Af = af$$

then the operators commuting with A generate a Lie algebra whose representations determine the degeneracy of the states with the same value a.

§ 2. Dynamical Lie Algebras

In § 1 we discussed the group of degeneracy of the Hamiltonians. The representations of the maximal symmetry group gives us the dimensionality of the eigenspace of H for a given energy E. In order to solve the quantum mechanical problem 1(2) completely, we have still to determine the spectrum of H. We shall solve this problem in the framework of the following general formalism.

Let W be a differential operator and consider the (wave) equation

$$W\psi = 0. \tag{$*$}$$

If there are operators L_i, $i = 1, \ldots, r$, forming a Lie algebra L and satisfying, on the space of solutions of $(*)$,

$$[W, L_i]\psi = 0, \quad i = 1, \ldots, r,$$

then all solutions of the wave equation $(*)$ span a representation space for the Lie algebra L. Clearly if ψ is a solution of $(*)$, so is $L_i\psi$ and we have $[W, L_i] = f(W)$ where f is an arbitrary polynomial with coefficients depending on coordinates and satisfying $f(0) = 0$. In particular, if $W = \mathrm{i}\partial_t - H$, or $W = (\partial_\mu - A_\mu)^2$, then the Lie algebra L is called the *dynamical Lie algebra* of the quantum system. It contains in general time-dependent operators $L(t)$ which on the space of solutions ψ of eq. $W\psi = 0$ satisfy the Heisenberg equation

$$[\mathrm{i}\partial_t, L_k(t)] = [H, L_k(t)].$$

The subalgebra $L' \subset L$ consisting of operators commuting with W is a more narrow definition of symmetry; both L' and L are represented on the same Hilbert space. Finally, the subalgebra $L'' \subset L'$ of time-independent operators satisfies $[H, L] = 0$, and is the *symmetry algebra of* H discussed in § 1.

The Heisenberg equation has the solution $L_k(t)$ given by the formula

$$L_k(t) = \exp[\mathrm{i}tH]L_k(0)\exp[-\mathrm{i}tH].$$

Because the energy operator H commutes with the evolution operator $\exp[\mathrm{i}tH]$ the time dependent dynamical Lie algebra $\{H, L_k(t)\}$ and the time independent dynamical Lie algebra $\{H, L_k(0)\}$ are unitarily equivalent. This allows us to restrict ourselves in concrete problems to the analysis of time-independent dynamical Lie algebras.

We now solve explicitly some important quantum-dynamical problems by the method of Lie algebra representations. We begin with a presentation of some supplementary results in the form of lemmas. The proof of these lemmas is straightforward and is left as an exercise for the reader.

LEMMA 1. *The following three operators* ($[p, q] = -\mathrm{i}$)

$$\begin{aligned} \Gamma_0 &= \tfrac{1}{4}(p^2+q^2), \\ T &= \tfrac{1}{4}(pq+qp), \\ \Gamma_4 &= \Gamma_0 - \tfrac{1}{2}q^2 \end{aligned} \tag{1}$$

satisfy the commutation relation of the Lie algebra $o(2, 1)$ (su(1, 1))

$$[\Gamma_0, \Gamma_4] = \mathrm{i}T, \quad [\Gamma_4, T] = -\mathrm{i}\Gamma_0, \quad [T, \Gamma_0] = \mathrm{i}\Gamma_4. \tag{2}$$

The Casimir operator

$$C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 \tag{3}$$

is calculated to be

$$C_2 = -\tfrac{3}{16} = \varphi(\varphi+1), \quad \varphi = -\tfrac{3}{4}, -\tfrac{1}{4}. \blacktriangledown$$

Thus eq. (1) is a realization of a representation D^+ of su(1, 1) (see 11.10.7.6 (i)). Equations (1) and (3) immediately solve the dynamical equation for the linear oscillator with the Hamiltonian equation

$$Hu = \frac{\hbar\omega}{2}\left(p^2 + \frac{\omega m}{\hbar}x^2\right)u = Eu, \tag{4}$$

for with the substitutions $q = \left(\dfrac{m\omega}{\hbar}\right)^{1/2} x$ and $2\Gamma_0 = \dfrac{1}{\hbar\omega} H$, eq. (4) can be written as

$$(2\Gamma_0 - E/\hbar\omega)u = 0.$$

Thus, the eigenstates $|n\rangle$ of Γ_0 by 11.10.7.6(i) have discrete eigenvalues $n+\frac{1}{2}$, $n = 0, 1, 2, \dots$ and provide the space of solutions

$$H|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle.$$

As a second typical example, we use

LEMMA 2. *The operators*

$$\begin{aligned} \Gamma_0 &= \tfrac{1}{2}(rp^2+r), \\ \Gamma_4 &= \tfrac{1}{2}(rp^2-r), \\ T &= \boldsymbol{r}\cdot\boldsymbol{p} - \mathrm{i}, \end{aligned} \tag{5}$$

where $r = \sqrt{\boldsymbol{r}^2}$, $p = \sqrt{\boldsymbol{p}^2}$, *again satisfy the commutation relations* (2) *of the Lie algebra* $o(2, 1)$. *The Casimir operator* (3) *has the value*

$$C_2 = \boldsymbol{J}^2 = (\boldsymbol{r}\times\boldsymbol{p})^2 = j(j+1), \quad \textit{i.e.} \quad \varphi = -j-1 \text{ or } \varphi = j. \blacktriangledown$$

Consider now the Hamiltonian equation

$$Hu = \left(\frac{p^2}{2m} - \frac{\alpha}{r}\right)u = Eu \tag{6}$$

for the stationary solutions of the motion of a particle in a Coulomb field.

We introduce the related equation

$$\Theta\psi = [r(H-E)]\psi = 0. \tag{7}$$

The operator (6) can be expressed as a linear combination of the generators (4):

$$\Theta = \left(\frac{1}{2m} - E\right)\Gamma_0 + \left(\frac{1}{2m} + E\right)\Gamma_4 - \alpha. \tag{8}$$

To solve eq. (7), we first diagonalize Γ_0. Defining

$$\tilde{\psi} \equiv \exp(-\mathrm{i}\theta T)\psi, \tag{9}$$

where T is given in (5), and using the commutation relations (2), we have

$$\left[\left(-E + \frac{1}{2m}\right)(\Gamma_0 \cosh\theta + \Gamma_4 \sinh\theta) + \left(E + \frac{1}{2m}\right)(\Gamma_4 \cosh\theta + \Gamma_0 \sinh\theta) - \alpha\right]\tilde{\psi} = 0. \tag{10}$$

Hence if we choose

$$\tanh\theta = (E + 1/2m)/(E - 1/2m), \tag{11}$$

we obtain immediately from (10) the simple equation

$$[(-2E/m)^{1/2}\Gamma_0 - \alpha]\tilde{\psi} = 0. \tag{12}$$

The spectrum of Γ_0 in the discrete representation of $o(2, 1)$ determined by 11.10.7.6(i) are the values: $n = s+j+1$, $s = 0, 1, 2, \ldots$ Hence eq. (12) yields

$$E_n = -\frac{\alpha^2 m}{2n^2}, \tag{13}$$

which is the well-known H-atom spectrum. The solutions $\tilde{\psi}$ are now normalized as follows:

$$\int \overline{\tilde{\psi}}\tilde{\psi} r^{-1} \mathrm{d}^3x = 1. \tag{14}$$

In order to determine the continuous spectrum of our problem, we choose in eq. (10) the parameter θ differently, namely,

$$\tanh\theta = (E - 1/2m)/(E + 1/2m) \tag{15}$$

and obtain, instead of (12),

$$[(2E/m)^{1/2}\Gamma_4 - \alpha]\psi = 0. \tag{16}$$

Denoting the generalized (non-normalizable) eigenvectors of Γ_4 by

$$\Gamma_4|\varrho, \lambda\rangle = \lambda|\varrho, \lambda\rangle \tag{17}$$

we obtain

$$E_\lambda = \frac{\alpha^2 m}{2\lambda^2}, \quad \lambda \in R. \tag{18}$$

The dynamical algebra (5) does not solve the complete degeneracy of the levels of the Hamiltonian (6), because we have not yet studied the angular momenta of the levels. The following lemma solves this problem.

LEMMA 3. *The operators* (5) *together with*

$$J = r \times p \tag{19}$$

and

$$\begin{aligned} A &= \tfrac{1}{2} r p^2 - p(r, p) - \tfrac{1}{2} r, \\ M &= \tfrac{1}{2} r p^2 - p(r, p) + \tfrac{1}{2} r, \\ \Gamma &= r p \end{aligned} \tag{20}$$

satisfy the commutation relations of the Lie algebra of so(4, 2). *The second, third and fourth order invariant operators have the values*

$$C_2 = -3, \quad C_3 = 0, \quad C_4 = -12, \tag{21}$$

where $C_2 = \frac{1}{2} L_{ab} L^{ab}$, $C_3 = \varepsilon_{abcdef} L^{ab} L^{cd} L^{ef}$, $C_4 = L_{ab} L^{bc} L_{cd} L^{de}$, $a, b = 0, 1, 2, 3, 4, 5$.

[Note: $L_{ij} = \varepsilon_{ijk} J_k$, $L_{i4} = A_i$, $L_{i0} = M_i$, $L_{i5} = \Gamma_i$, $L_{05} = \Gamma_0$, $L_{45} = \Gamma_4$, $L_{04} = T$.]

The next lemma gives the degeneracy of states of Hamiltonian (6).

LEMMA 4. *The representation given by* (5), (19), (20), *and* (21) *of* so(4, 2) *in the basis where* Γ_0, J^2 *and* J_3 *are diagonalized with eigenvalues* n, $j(j+1)$ *and* m, *respectively, has the weight diagram, i.e., the states* $|njm\rangle$, *given by Fig.* 1.

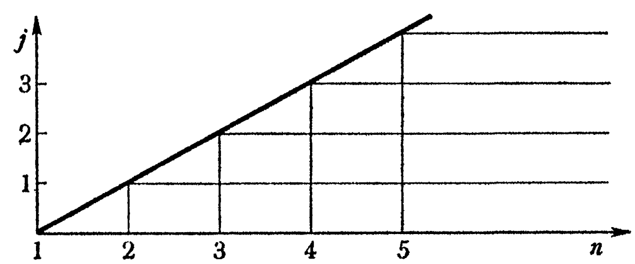


Fig. 1

PROOF: The elements Γ_0, Γ_4, T form the algebra su(1, 1). The Casimir operator $C_2(o(2, 1)) = \Gamma_0^2 - \Gamma_4^2 - T^2 = J^2$. Let the eigenvalues of the operator Γ_0 be denoted by n. Because J^2 has eigenvalues $j(j+1)$ for fixed j, the range of n is $j+1 \leqslant n < \infty$. This gives the horizontal lines in Fig. 1. ▼

For more general problems, we can use the following generalization of lemma 3:

LEMMA 5. *The operators*

$$\begin{aligned} J &= r \times \pi - \mu \hat{r}, \\ A &= \tfrac{1}{2} r \pi^2 - \pi(r \cdot \pi) + \frac{\mu}{r} J + \frac{\mu^2}{2r^2} r - \tfrac{1}{2} r, \\ M &= \tfrac{1}{2} r \pi^2 - \pi(r \cdot \pi) + \frac{\mu}{r} J + \frac{\mu^2}{2r^2} r + \tfrac{1}{2} r, \\ \Gamma &= r \pi, \end{aligned} \tag{22}$$

$$\begin{aligned}\Gamma_0 &= \tfrac{1}{2}(r\pi^2+r+\mu^2/r),\\ \Gamma_4 &= \tfrac{1}{2}(r\pi^2-r+\mu^2/r),\\ T &= \boldsymbol{r}\cdot\boldsymbol{\pi}-\mathrm{i}\end{aligned}$$

with

$$\hat{\boldsymbol{r}} = \boldsymbol{r}/r, \quad \boldsymbol{\pi} = \boldsymbol{p}-\mu\boldsymbol{D}(\boldsymbol{r}), \quad \boldsymbol{D}(\boldsymbol{r}) = \frac{\boldsymbol{r}\times\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{n})}{r[r^2-(\boldsymbol{r}\cdot\boldsymbol{n})^2]}$$

and $\boldsymbol{n}$ *an arbitrary constant unit vector also have the commutation relations of* so(4, 2) *where, instead of* (21), *we have now*

$$C_2 = -3(1-\mu^2), \quad C_3 = 0, \quad C_4 = 0. \tag{23}$$

For each value of $\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots$ *eq.* (22) *gives an irreducible representation of* so(4, 2) *in the discrete series. The representations* (22) *can be characterized by a so-called representation relation*

$$\{L_{AB}, L^A_C\} = -2ag_{BC}, \tag{24}$$

where L_{AB} *are the generators of* so(4, 2) *and* $a = 1-\mu^2$. ▼

A more general theory of symmetries in quantum mechanics is presented in chs. 13 and 21.

§ 3. Exercises

§ 2.1. Consider the angular momentum of a charge-monopole system given in eq. (22), i.e.

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{\pi}-\mu\hat{\boldsymbol{r}},$$

$$\boldsymbol{\pi} = \boldsymbol{p}-\mu\boldsymbol{D}(\boldsymbol{r}), \quad \boldsymbol{D}(\boldsymbol{r}) = \frac{\boldsymbol{r}\times\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{n})}{r\big(r^2-(\vec{r}\cdot\boldsymbol{n})^2\big)}.$$

Show that

(a) For fixed $\boldsymbol{n}$, $\boldsymbol{D}$ is singular; $\boldsymbol{J}$ can be represented on the space of functions which rapidly go to zero along the singularity line $\hat{\boldsymbol{r}} = \boldsymbol{n}$; it has a deficiency index (1, 1), hence can be extended to a self-adjoint operator. The integrability conditions on the representations lead to the so-called charge quantization condition, $\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots$

(b) If $\boldsymbol{n}$ is rotated together with $\boldsymbol{r}$, $\boldsymbol{D}$ is rotationally invariant and we have a configuration space $R^3\otimes S^2$. However, two different choices $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ can be connected by a gauge transformation provided we use the group space S^3 of SU(2). Hence we can represent $\boldsymbol{J}$ on $L^2(S^3)$, and consequently both integer and half integer values of spin are allowed.

(Cf. C. A. Hurst 1968 and A. O. Barut 1974.)

§ 2.2. Consider the differential equation

$$(H-W)u = \left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+q^2+\frac{K}{q^2}-W\right)u = 0, \quad 0 < q < \infty.$$

Let

$$\Gamma_0 = \frac{1}{4}\left(p^2+q^2+\frac{K}{q^2}\right), \quad T = \tfrac{1}{4}(pq+qp) \text{ and } \Gamma_4 = \tfrac{1}{4}(H-2q^2)$$

and calculate the spectrum of H by the representations of $o(2, 1)$ (cf. § 2. eq. (1)).

Hint: $C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = \frac{1}{4}(K - \frac{3}{4})$.

§ 2.3.* Show that the radial Schrödinger wave equation for an N-dimensional oscillator, with an addition potential a/r^2, can be brought to the form given in the previous exercise.

§ 2.4. Show that the radial wave equation for the non-relativistic Kepler problem can also be brought to the form given in exercise 2.2 by suitable substitutions, and obtain the Balmer formula.

Hint: $\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2me^2}{\hbar r} - \frac{2mE}{\hbar^2}\right] u_l(r) = 0.$ Set $\varepsilon = 2mE/\hbar^2$, $q = (2r\sqrt{-\varepsilon})^{1/2}$, $u_l(r) = (2r\sqrt{-\varepsilon})^{1/4}\tilde{u}_l(q)$, $K = \frac{3}{4} + 4l(l+1)$, $W = \frac{4me^2}{\sqrt{-2E}}$.

§ 2.5. Consider the equation

$$[(1+b)\Gamma_0 + (1-b)\Gamma_4 + c]\tilde{\psi} = 0,$$

where Γ_0, Γ_4 and T are generators of SO(2, 1) as before. Many equations of quantum theory can be written in this form. Give a complete classification of solutions of this equation as a function of b, c and the eigenvalues of the Casimir operator $C_2 = \Gamma_0^2 - \Gamma_4^2 - T = \varphi(\varphi+1)$ (cf. 11.10.7.6).

(a) Let $\tilde{\psi} = e^{i\theta T}\psi$, $\tanh\theta = \left(\frac{b-1}{b+1}\right)$ then $\left[b^{1/2}\Gamma_4 + \frac{c}{2}\right]\psi = 0$. Find the range of the spectrum for discrete, principal and supplementary series of representations of so(2, 1).

(b) Let $\tilde{\psi} = e^{i\theta T}\psi$, $\tanh\theta = \frac{b+1}{b-1}$, then $\left[(-b)^{1/2}\Gamma_4 + \frac{c}{2}\right]\psi = 0$. Find the nature and range of the spectra. (Note that Γ_0 has discrete but Γ_4 continuous spectrum.)

(Cf. Barut 1973.)

§ 2.6.** *Occurrence of principal and supplementary series of* $o(2,1)$ *and self-adjoint extension of Hamiltonians.* Consider the $o(2,1)$-algebra representation given by

$$\Gamma_0 = \frac{1}{2}\left(r\pi^2 - \frac{a}{r} + r\right), \quad \Gamma_4 = \frac{1}{2}\left(r\pi^2 - \frac{a}{r} - r\right), \quad T = r \cdot \pi - i$$

(π was defined in Lemma 15, § 2).

Show that in this representation the Casimir operator C_2 has the value $J^2 - \mu^2 - a$. Show that as in lemmas 3, 4 we can solve the Hamiltonian $H = \pi^2 + a/r^2 + b + c/r$. For large positive a, H is no longer self-adjoint (Kato 1966). Let $C_2 = \varphi(\varphi+1)$ and for simplicity, $\mu = 0$. Show that for $a < j(j+1)$, $C_2 > 0$ we have to use the discrete series of representation of $o(2, 1)$. For $j(j+1) < a < j(j+1)+\frac{1}{4}$, $-\frac{1}{4} < C_2 < 0$ and we must use the supplementary series of representations, and for $a > j(j+1)+\frac{1}{4}$, $C_2 < -\frac{1}{4}$, i.e. $\varphi = -\frac{1}{2} + i\lambda$

and we must use the principal series of unitary representations of $O(2, 1)$. It is remarkable that all *three* series of representations of $O(2, 1)$ occur in physical problems, and that the dynamical group provides a method of self-adjoint extension for a class of Hamiltonians which includes relativistic Dirac Hamiltonian for Coulomb problem (cf. Barut 1973).

2.7.* *Tensor method for* so(4, 2)*-algebra*; so(4), su(2), su(1, 1)*-subalgebras, Clebsch–Gordan coefficients.* Consider two su(2) algebras with generators

$$(J_1)_{ij} = \tfrac{1}{2}\varepsilon_{ijk}(a^*\sigma_k a), \qquad (J_2)_{ij} = \tfrac{1}{2}\varepsilon_{ijk}(b^*\sigma_k b) \tag{1}$$

where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are two boson annihilation operators $[a, a^*] = 1$, $[b, b^*] = 1$, and σ_k are the Pauli matrices.

Show that in an su(2)⊗su(2)-basis with vectors

$$\begin{aligned} |j_1 m_1 j_2 m_2\rangle &= N^{j_1 j_2}_{m_1 m_2} a_1^{*j_1+m_1} a_2^{*j_1-m_1} b_1^{*j_2+m_2} b_2^{*j_2-m_2}|0\rangle, \\ (N^{j_1 j_2}_{m_1 m_2})^{-2} &= [(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!] \end{aligned} \tag{2}$$

the basis elements of the SU(2, 2)-algebra are represented by

$$\begin{aligned} L_{ij} &= \frac{1}{2}\varepsilon_{ijk}[a^*\sigma_k a + b^*\sigma_k b] = (J_1)_{ij} + (J_2)_{ij} \equiv J_{ij}, \\ L_{i4} &= -\frac{1}{2}(a^*\sigma_i a - b^*\sigma_i b) = A_i, \\ L_{i5} &= -\frac{1}{2}(a^*\sigma_i Cb^* - aC\sigma_i b) = M_i, \\ L_{i6} &= \frac{1}{2i}(a^*\sigma_i Cb^* + aC\sigma_i b) = \Gamma_i, \\ L_{46} &= \frac{1}{2}(a^*Cb^* + aCb) = T, \\ L_{45} &= \frac{1}{2i}(a^*Cb^* - aCb) = \Gamma_4, \\ L_{56} &= \frac{1}{2}(a^*a + b^*b + 2) = \Gamma_0, \qquad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{3}$$

Prove that the most degenerate discrete self-adjoint irreducible representations are labelled by the eigenvalue $\mu = (j_1 - j_2)$ of

$$K = \frac{1}{2}(a^*a - b^*b), \qquad [K, L_{ab}] = 0 \quad \text{for all } a, b, \tag{4}$$

and that this corresponds precisely to the representation given in eq. 2(22) by differential operators with the Casimir operators $C_2 = -3(1-\mu^2)$, $C_3 = 0$, $C_4 = 0$. The basis $|\mu, m, n, \alpha\rangle$ where the labels are the eigenvalues of K, L_{12}, L_{56}

and L_{34}, respectively, we call parabolic states. Find by direct calculation

$$\mu = j_1 - j_2, \quad m = m_1 + m_2, \quad n = j_1 + j_2 + 1, \quad \alpha = m_2 - m_1,$$

and show that the series of representations satisfy the representation relation

$$\{L_{AB}, L^A_C\} = L_{AB}L^A_C + L^A_C L_{AB} = 2(\mu^2 - 1)g_{BC}, \quad A, B, C = 1, 2, \ldots, 6.$$

Consider the reduction ($J_k \equiv \varepsilon_{klm}J_{lm}$)

$$\mathrm{su}(2)\otimes\mathrm{su}(2) \supset \mathrm{su}(2) \supset u(1), \quad \text{with } \boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2.$$

We define the spherical basis $|\mu, m; n; j(j+1)\rangle$, eigenvectors of $(K, L_{12}, L_{56}, \boldsymbol{J}^2)$ by

$$|(j_1 j_2)jm\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | (j_1 j_2) jm\rangle | j_1 m_1 j_2 m_2\rangle \delta_{m_1+m_2, m}. \tag{5}$$

This expansion can be viewed as the expression of the spherical states in terms of the parabolic states.

Next consider the subgroup $O(2, 1)\times O(2, 1) \subset O(4, 2)$ generated by

$$N_1^{(r)} = \frac{1}{2}(L_{46} + (3-2r)L_{35}),$$

$$N_2^{(r)} = \frac{1}{2}(L_{45} - (3-2r)L_{36}),$$

$$N_3^{(r)} = \frac{1}{2}(L_{56} + (3-2r)L_{34}), \quad r = 1, 2,$$

and the states

$$\begin{aligned} |\varphi_1 n_1 \varphi_2 n_2\rangle &= N b_1^{*n_1+\varphi_1-1} a_2^{*n_1-\varphi_1} a_1^{*n_2+\varphi_2-1} b_2^{*n_2-\varphi_2}|0\rangle, \\ N^{-2} &= [(n_1+\varphi_1-1)!(n_1-\varphi_1)!(n_2+\varphi_2-1)!(n_2-\varphi_2)!], \end{aligned} \tag{7}$$

where

$$\begin{aligned} N_3^{(r)}|\varphi_1 n_1 \varphi_2 n_2\rangle &= n_r|\varphi_1 n_1 \varphi_2 n_2\rangle, \quad r = 1, 2, \\ N^{(r)2}|\varphi_1 n_1 \varphi_2 n_2\rangle &= \varphi_r(\varphi_r - 1)|\varphi_1 n_1 \varphi_2 n_2\rangle, \quad r = 1, 2, \\ N^{(r)2} &= N_3^{(r)2} - N_1^{(r)2} - N_2^{(r)2}, \quad r = 1, 2. \end{aligned} \tag{8}$$

Now in the reduction

$$\mathrm{su}(1, 1)\otimes\mathrm{su}(1, 1) \supset \mathrm{su}(1, 1) \supset u(1)$$

with $\boldsymbol{N} = \boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}$, show that

$$|(\varphi_1\varphi_2)\varphi n\rangle = \sum_{n_1 n_2} \langle \varphi_1 n_1 \varphi n_2 | (\varphi_1 \varphi_2)\varphi n\rangle |\varphi_1 n_1 \varphi_2 n_2\rangle \delta_{n_1+n_2, n}. \tag{9}$$

Prove that a property of the discrete, degenerate series of representations of $O(4, 2)$ is that

$$\boldsymbol{J}^2 = \boldsymbol{N}^2 \tag{10}$$

Hence in the basis $|(\varphi_1\varphi_2)\varphi n\rangle$ the set K, J^2, L_{56}, L_{12} is again diagonal with $\mu = \varphi_2-\varphi_1$, $m = \varphi_1+\varphi_2-1$, $n = n_1+n_2$. On the other hand in the states $|\varphi_1 n_1 \varphi_2 n_2\rangle$ the set $K, L_{12}, L_{56}, L_{34}$ is diagonal with $\alpha = n_1-n_2$.

Hence (5) and (9) represent expansions of the $O(4,2)$-states into the same states, consequently with proper identification of indices show that the Clebsch–Gordan coefficients of SU(2) are identical with those of SU(1, 1) for coupling of discrete series of representations, namely

$$\langle j_1 m_1 j_2 m_2|(j_1 j_2)jm\rangle = \langle \varphi_1 n_1 \varphi_2 n_2|(\varphi_1\varphi_2)\varphi n\rangle$$

with

$$j_1 = \frac{1}{2}(n_1+n_2+\varphi_2-\varphi_1-1),$$

$$j_2 = \frac{1}{2}(n_1+n_2+\varphi_1-\varphi_2-1),$$

$$m_1 = \frac{1}{2}(n_2-n_1+\varphi_2+\varphi_1-1),$$

$$m_2 = \frac{1}{2}(n_1-n_2+\varphi_2+\varphi_1-1),$$

$$j = \varphi-1, \quad m = \varphi_1+\varphi_2-1$$

and

$$\mu = j_1-j_2 = \varphi_2-\varphi_1.$$

§ 2.8. *Coherent States for* SU(1, 1). For the Heisenberg algebra $\{1, a, a^*\}$ with

$$[a, a^*] = 1$$

the coherent states $|z\rangle$ are the eigenstates of a:

$$a|z\rangle = z|z\rangle$$

and can be written as

$$|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = e^{|z|^2/2}e^{za^*}|0\rangle$$

in terms of the eigenstates $|n\rangle$ of the number operator a^*a. They satisfy $\langle z|z\rangle = 1$, $\langle z'|z\rangle = \exp\left[-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2+\bar{z}'z\right]$,

$$\frac{1}{\pi}\int |z\rangle\langle z|\mathrm{d}^2z = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1.$$

For the su(1, 1)-algebra $\{L^+, L^-, L_3\}$ with

$$[L^+, L^-] = -L_3, \quad [L_3, L^\pm] = \pm L^\pm$$

define similar coherent states by

$$L^-|z\rangle = z|z\rangle.$$

Show that for the discrete-representation $D^+(\varphi)$:

$$|z\rangle = [\Gamma(-2\varphi)]^{1/2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}z)^n}{[n!\Gamma(-2\varphi+n)]^{1/2}} |\varphi, n\rangle,$$

and the resolution of identity becomes

$$\frac{4}{\pi\Gamma(-2\varphi)} \int r\mathrm{d}r\mathrm{d}\theta(\sqrt{2}r)^{-2\varphi-1} K_{1/2+\varphi}(2\sqrt{2}r)|z\rangle\langle z| = 1, \quad z = re^{i\theta}$$

and we obtain a family of Hilbert spaces of entire functions of growth (1, 1) (cf. A. O. Barut and L. Girardello 1971).

Chapter 13

Group Theory and Group Representations in Quantum Theory

§ 1. Group Representations in Physics

Discrete and continuous groups occur in classical physics as transformation groups and express generally the symmetry of dynamical equations of particles or fields: crystal symmetry, the Galilei, Poincaré or Einstein group of transformations of space-time, the group of canonical transformations acting on the phase-space, the group of gauge transformation, or conformal transformations of the electromagnetic potentials, symplectic transformation of thermodynamic functions, etc. We gave even a Lie algebra structure in the Poisson-brackets of classical mechanics and classical field theory, for which the canonical transformations act as the group of automorphisms. However, in all these only a defining representation of the groups occurs. The theory of group representations in linear spaces which is the subject matter of this book, is really the domain of quantum physics. The particular adaptability of the theory of group representations to quantum physics stems from the following basic kinematical difference between classical and quantum theories: In both classical and quantum theory a physical system can be described by the notion of 'state', and in both cases the system has continuously infinitely many states. In quantum theory, due to the linearity of the equations of motions, the states can be represented as linear combinations of a selected orthogonal sets of states, often a set of countable many elements; i.e., the states form a linear vector space. In contrast, in the classical case the dynamical equations in terms of coordinates and momenta are non-linear, hence there is no such set of basis states.

The group-structure of physical theories may be studied along the following two different approaches:

A. *Theories Based on Dynamical Equations*

Most physical theories postulate a certain set of dynamical equations, e.g., equations of particle mechanics, fluid dynamics, electrodynamics, statistical and quantum mechanics. These equations govern the behavior of functions ψ of particle or field coordinates, and may be written symbolically as

$$L\psi = 0. \tag{1}$$

These equations may be linear or non-linear, differential or integral equations, or more general operator equations. The role of group theory in this line of approach is to facilitate the solutions of eq. (1) and to better recognize the structure of the underlying dynamics. If we start from this point of view, we are led to study the space Φ of solutions of eq. (1). In order to characterize the space we may search first for set of the operators $\{X_i\}$ for which

$$X_i\psi \in \Phi, \quad \text{if} \quad \psi \in \Omega \subset \Phi,\ i = 1, 2, \ldots$$

For a class of operators X_i we have then the property that $[L, X_i]\psi = 0$ and $[L, [X_i, X_j]]\psi = 0$: we may then give the set a Lie algebra structure. Note that the X's are general operators transforming in general both the arguments of ψ and the form of ψ itself. More generally, we may have $[L, X_i]\psi = \lambda L\psi$, i.e., a multiple of the eq. (1), which leads again to a Lie algebra structure on the space of solutions.

If such a formulation is possible, then the set X_i contains not only the usual symmetry operations, but also the more general dynamical symmetry transformations.

In particular let Y be an operator satisfying the eigenvalue equation

$$Y\psi_n = y_n \psi_n, \quad \psi_n \in \Omega.$$

For a maximal subset C of operators such that

$$[C_i, Y] = 0, \quad C_i \in C,$$

we have

$$Y(C_i\psi_n) = y_n(C_i\psi_n),$$

i.e., the 'value' of Y is 'conserved' under the operation C_i. The set C is a symmetry algebra with respect to the quantity Y. In particular, if Y is the Hamiltonian of the system in classical or quantum mechanics, C is called the *symmetry algebra* or the *algebra of degeneracy of the energy.*

This approach is formally the same for classical and quantum theories, and leads directly to Lie algebras rather than Lie groups, provided we can define the product of operators. This discussion can also be based on a variational principle instead of eq. (1).

In contrast, in the next approach the concept of groups plays a fundamental role.

B. *Theories Based on Prescribed Symmetries*

If the dynamical equations of the system are unknown, we can be guided by general symmetry principles to discover or guess the equations, or we can investigate general properties of the system which results from prescribed symmetries. The general covariance principle of Einstein in general relativity, or the properties of the S-matrix of the interaction of elementary particles are typical examples.

C. *The Idea of Relativity*

In Euclidean geometry points are all indistinguishable and have an objective existence whereas coordinates are man-made. All Cartesian coordinates are equally admissible. Thus the objective properties of points must be independent of the choice of coordinate frame. The coordinates of a point in different frames are related by a group of transformations; or a transformation of the group maps one point into another (passive and active views, respectively). Hence we have a group of automorphisms of our geometry. Conversely, any group of transformations may serve as the group of automorphism of some geometry. The characterization of the geometry by its group of authomorphisms is the basic idea in Klein's Erlangen Program (1872). This idea of the relativity of coordinate frames extends from points, to lines, areas,... and to physical quantities, like forces, velocities, fields... More generally, physical processes taking place at some time at some place are independent of the observers (frames). All observers in a class are equally admissible. Observers are related to each other by a group of transformations, or a transformation takes one observer into another. This group of transformations is the group of authomorphisms of a physical theory, or of a physical law. Physicists call the class of observers related by this group of transformations also as inertial frames. This idea is basic to much of the kinematical applications of group theory to physics.

We begin with the general kinematical framework of quantum theory which is essential for the formulation of symmetry principles and for the use of symmetry and dynamical group.

§ 2. Kinematical Postulates of Quantum Theory

Superposition Principle and Probability Interpretation

Quantum theory originated from the wave properties of matter. The most important property of wave phenomena is the property of superposition or interference: linear combinations of solutions are again solutions of the wave equation because the equation is linear. Furthermore, the matter waves describe probability amplitudes rather than the amplitudes for certain densities of particles or fields. These two fundamental physical ideas put together give us a general framework in which physical states $\psi, \varphi, \ldots$ are described by elements of a linear space, and the observed positive definite conditional probabilities are represented by the square of the sesquilinear forms $|(\psi, \varphi)|^2$. Historically, in simple cases, the space of solutions of the wave equations (e.g., Schrödinger equation) can be imbedded in a Hilbert space, and hence the theory has been formulated traditionally within a Hilbert space framework since its axiomatization by von Neumann. However, the two basic physical ideas allow more general spaces of physical states. The requirement that limiting points of sequences of states ψ_n should be included gives a general topological vector space V and its dual V' so that the sesquilinear form (probability amplitudes) can be taken to be

$$(\psi, \varphi), \quad \varphi \in V \text{ and } \psi \in V'.$$

Even within the Hilbert space formalism it is often convenient to take a more general formalism and use a Gel'fand triplet

$$\Phi \subset H \subset \Phi',$$

where Φ is a dense nuclear subspace in the Hilbert space H and Φ' is dual to Φ, in order to accommodate operators with continuous spectra and their generalized eigenvectors in Φ' which are not normalizable (cf. app. B).

Besides these mathematical generalizations of the framework of quantum theory in the use of more general spaces, physical generalizations of the theory may also be considered in the future, for example, one may weaken the universal validity of the linearity of states, i.e., the superposition principle. Keeping these points in mind, we describe in this section the standard form of quantum postulates and the role of group representations.

States and Rays

The basic framework of the quantum theoretical description of a physical system is a linear space H whose unit rays are in one-to-one correspondence with the states of the system which are called *pure states*. A *unit ray* Ψ is the set of vectors $\{\lambda\psi\}$, $\|\psi\| = 1$, $\lambda = \exp(\mathrm{i}\alpha)$, $\psi \in H$. The reason for the introduction of rays rather than vectors themselves lies a) in the use of a space over the complex numbers, and b) in the basic probability interpretation of quantum theory. The quantities related to observable effects are the absolute values of a sesquilinear form $|(\psi, \varphi)|^2$ which are independent of the parameters λ, λ' characterizing a ray. Consequently the space of rays is the quotient space $H = H/S^1$, i.e. the projective space of one dimensional subspaces of H.

This basic correspondence between physical states and the elements of H incorporates the *superposition principle* of quantum theory—namely, that there is a set of basis states out of which arbitrary states can be constructed by linear superpositions. Thus if the rays $\{\lambda\psi_n\}$, $n = 1, 2, \ldots$, describe physical states, then $\psi'_\alpha = \sum_n \alpha_n \psi_n$ is another vector of H, so that the ray $\{\lambda\psi'_\alpha\}$ corresponds to another possible state of the system. Note that $\psi'_\alpha = \sum_n \alpha_n \psi_n$ and $\lambda\psi'_\alpha$ represent the same state, but $\sum_n \alpha_n(\lambda_n \psi_n)$ is in general a different state, although ψ_n and $(\lambda_n \psi_n)$ represent the same state. Herein lies the problem of relative phases of quantum theory.

States that can be obtained from each other by linear superpositions are called pure 'coherent' states.

Superselection Rules

In general, there are physical limitations on the validity of the superposition principle. One cannot realize pure states out of superposition of certain states; for example, one cannot form a pure state consisting of a positively and a negatively charged particle, or a pure state consisting of a fermion and a boson. This does not mean that two such states cannot interact; it only means that their formal linear combination is not a physically realizable pure state (superselection rule). The existence of superselection rules is connected with the measurability of the relative phase of such a superposition and depends on further properties of the system, like charge, baryon number, etc. The superselection rule on fermions (i.e., separation of states of integral and half integral fermions) follows from the rotational invariance (see below). In all such cases we divide the linear space H into subsets, such that the superposition principle holds within each subset. These subsets are called *coherent subspaces*. In each subspace, $\sum \alpha_n \psi_n$ and $\lambda \sum \alpha_n \psi_n$ correspond to the same state, but $\sum \alpha_n \psi_n$ and $\sum \alpha'_n \psi_n$, in general, correspond to distinct states. We shall come back to this problem at the end of this section.

Probability Interpretation

Physical experiments consist in preparing definite states, in letting them interact and in observing the rate of occurrence of other well-defined states. The transition probability between two states ψ and φ is defined by the square of a sesquilinear form $|(\psi, \varphi)|^2$. We can also say the transition probability between two rays Ψ and Φ because this quantity is the same for all vectors of the rays; overall phases are unimportant. However, if ψ and φ are themselves linear combinations of some basis vectors, then the transition probability depends on the relative phases of their components. The quantity $|(\psi, \varphi)|^2$ can be related, by multiplying it with certain kinematical factors, to the experimentally observed quantities like cross sections of reactions, and lifetimes of unstable states.

The Dynamical Problem

Now in order to evaluate quantities like $|(\psi, \varphi)|^2$ we must have a definite realization of the linear space H, and must obtain a *number* that can be compared with experiment. Thus we need a definite labelling of the states $\psi, \varphi, \ldots$ and a definite expression for the sesquilinear product. We shall refer to this realization as the *concrete linear space* (CLS). This is the more important and the more difficult part of the theory. Although all Hilbert spaces of the same dimension are isomorphic and one can transform one realization into another, some definite explicit realization with a physical correspondence is necessary.

If the linear or the Hilbert space framework provides the *kinematical principle* of quantum theory, the explicit calculation of states $\psi, \varphi, \ldots$ or, of sesquilinear (or scalar) products (ψ, φ), is the *dynamical* part of quantum theory.

In simple cases, the dynamical problems is solved by postulating a differential equation for the states $\psi, \varphi, \ldots$ represented, for example, as elements of $H = L^2(R^3)$, and identifying all solutions of the equation with all the states of the physical system. This is the case in Schrödinger theory. For more complicated systems, or for unknown new systems, this is not possible. Even if we know all the states of an isolated system, measurements on the system are carried out by additional external interactions which change the system.

Short of the complete calculation of the sesquilinear products (ψ, φ) some very general principles allow one to derive a number of important symmetry properties of these quantities. It is along these lines that the traditional use of group representations in quantum theory has been developed. More recently the quantum theoretical Hilbert space has been identified with the explicit carrier space of the representations of more general groups and algebras. In this second sense the group representations solve also the dynamical problem. We shall elaborate both of these aspects. In order to be specific, we consider from now on the space of states to be a Hilbert space.

Equivalent Description or Symmetry Operations

As in any correspondence, one should first consider equivalent mappings between the physical states and the rays in the Hilbert space. For the knowledge of physically equivalent descriptions of a system reflects already, as we shall see, important properties of the system itself.

If the same physical system can be described in two different ways in the same coherent subspace of the Hilbert space H, once by rays $\Psi_1, \Phi_1, \ldots$ and once by rays $\Psi_2, \Phi_2, \ldots$ (for example, by two different observers), such that the same physical state is once described by Ψ_1, in the other case by Ψ_2—equivalently we can speak of symmetry operation of the system—then the transition probabilities must be the same by the definition of physical equivalence. We have then a norm preserving mapping $\hat{T}$ between the rays Ψ_1 and Ψ_2. Mathematically, it is more convenient to find out the corresponding map $H \to H$ between the vectors $\psi, \varphi, \ldots$ in the Hilbert space. Because only the absolute values are invariant, the transformation in the Hilbert space can be unitary or anti-unitary. In fact, one can prove that given two descriptions of a system in the space of rays, *one can choose* unit vectors, $\psi_1, \varphi_1, \ldots$ from the rays $\Psi_1, \Phi_1, \ldots$ in the first description, and unit vectors $\psi_2, \varphi_2, \ldots$ from the rays $\Psi_2, \Phi_2, \ldots$ in the second description, such that the correspondence $\psi_1 \leftrightarrow \psi_2$, $\varphi_1 \leftrightarrow \varphi_2, \ldots$ is either *unitary* or *anti-unitary*. That is, one can *construct* a unitary or anti-unitary correspondence $H \to H$. More precisely, we have

THEOREM 1 (Wigner). *Let $\Psi_2 = \hat{T}\Psi_1$ be a mapping of the rays of a Hilbert space H which preserves the inner product of rays, then there exists a mapping $\psi_2 = T\psi_1$ of all vectors of H such that $T\psi$ belongs to the ray $\hat{T}\Psi$* if *ψ belongs to the ray Ψ and, in addition* $1°T(\psi+\varphi) = T\psi+T\varphi$, $2°T(\lambda\psi) = \chi(\lambda)T(\psi)$, $3°(T\psi, T\varphi) = \chi[(\psi, \varphi)]$, *where either $\chi(\lambda) = \lambda$ (unitary case), or $\chi(\lambda) = \bar{\lambda}$ (anti-unitary case) for all λ.*

PROOF: Let $\psi_1, \varphi_1, \ldots$ and $\psi_2, \varphi_2, \ldots$ be two sets of orthonormal bases chosen from the first and second set of rays, respectively. We are given a transformation which preserves the absolute values. The problem is to *construct* the corresponding transformation T on the vectors by suitable choices of the phases. It could be *a priori* that $T\psi_1 = c_1 \psi_2, T\varphi_1 = c_2 \varphi_2, \ldots$ and that no relations between the c's can be established, in which case T is not even a linear operator. We want, in fact, to show that T can be so defined that it is a unitary or an anti-unitary operator.

We single out the unit vector ψ_1 and choose ψ_2, and define $T\psi_1 = \psi_2$. This is the only arbitrary choice, and shall show that all other phases are uniquely determined. Thus T is determined up to an overall phase factor.

Next consider the vector $\psi_1+\varphi_1$, where φ_1 is orthogonal to ψ_1. It is easy to show that a representative vector of the corresponding ray in the second description is $a\psi_2+b\varphi_2$. We have then

$$T(\psi_1+\varphi_1) = c(a\psi_2+b\varphi_2) = \psi_2+b'\varphi_2,$$

where we must have $c = 1/a$ by the previous choice, and put $cb = b' = b/a$. We now *define* $T\varphi_1$ by $T(\psi_1+\varphi_1)-\psi_2$ or simply by $b'\varphi_2$. Hence we can set

$$T(\psi_1+\varphi_1) = T\psi_1+T\varphi_1.$$

Similarly for a general $f_1 = a_\psi\psi_1+a_\varphi\varphi_1+ \ldots$ we choose a representative $f_2 = \hat{a}_\psi\psi_2+\hat{a}_\varphi\varphi_2+ \ldots$, write $Tf_1 = cf_2$ with $c\hat{a}_\psi = a_\psi$, so that

$$\begin{aligned} T(a_\psi\psi_1+a_\varphi\varphi_1+ \ldots) &= a_\psi\psi_2+c\hat{a}_\varphi\varphi_2+ \ldots \\ &= a_\psi T\psi_1+a'_\varphi T\varphi_1+ \ldots \end{aligned}$$

Now we form the absolute values of the scalar products

$$|(\psi_1+\varphi_1, f_1)| = |a_\psi+a_\varphi|$$

and

$$|(T\psi_1+T\varphi_1, Tf_1)| = |a_\psi+a'_\varphi|.$$

These two numbers must be equal. This plus the fact that $|a'_\varphi| = |a_\varphi|$ allows us to calculate a'_φ in terms of a_φ and a_ψ. One obtains two solutions:

$$a'_\varphi = a_\varphi \quad \text{and} \quad a'_\varphi = \bar{a}_\varphi\frac{a_\psi}{\bar{a}_\psi}.$$

Clearly for the first solution T is linear *and* unitary. For the second solution we find $Tf_1 = (a_\psi/\bar{a}_\psi)[\bar{a}_\psi T\psi_1 + \bar{a}_\varphi T\varphi_1 + \ldots]$. An overall phase factor is unimportant and by a new normalization of T—which by the way does not change the choice $T\psi_1 = \psi_2$—we obtain an *anti-unitary* operator. ▼

Remarks: 1. The two possibilities in th. 1 come from the fact that the complex field has two (and only two) automorphisms that preserve the absolute values: the identity automorphism and the complex conjugation. In the case of the Hilbert space over a real field Wigner's theorem yields only unitary transformations (up to a phase), because the only automorphism of the real field is the identity automorphism. In fact, Wigner's theorem is closely related to the fundamental theorem of projective geometry.

2. One or the other case occurs for a given situation. Whether the transformation is unitary or anti-unitary depends on further properties of the two equivalent descriptions of the system. It does *not* depend, however, on the choice of vectors $\psi, \varphi, \ldots$ from the rays; if the transformation is, for example, unitary for a choice $\psi_1, \varphi_1, \ldots$ there is no other choice $\lambda\psi_1, \lambda'\varphi_1, \ldots$ such that it becomes anti-unitary and vice versa. Furthermore, once a vector ψ_2 is chosen, the others, $\varphi_2, \chi_2, \ldots$ are uniquely determined from the requirement that the correspondence is unitary (or anti-unitary).

Symmetry Transformations

The description of the *symmetry properties* of the system in the standard sense belongs to the situation characterized by the above theorem. For, if under a symmetry transformation measured probabilities are unchanged, we obtain automatically two equivalent descriptions in H, one corresponding to the original and the other to the transformed frame; and these two descriptions must be related to each other by unitary (or anti-unitary) transformations. Conversely, and this is more important from our point of view, *the Hilbert space of states must be isomorphic to the carrier space of unitary (or anti-unitary) representations of the symmetry transformations* (they may form a group or an algebra, etc.). Note that we wish to obtain a concrete Hilbert space to calculate transition probabilities. Thus, if we know the symmetry transformations of the system we can start from an arbitrary *collection* of irreducible unitary (or anti-unitary) representation spaces of the symmetry transformations to build up the Hilbert space H. This solves the problem partly, but not completely because we do not know *what* collection of irreducible representations we have. The explicit forms of the symmetry operations are discussed in sec. 3.

Uniqueness of Operators

We have said that representative vectors from the rays of two equivalent descriptions can be so chosen that the mapping of vectors $\psi_1 \leftrightarrow \psi_2$ is either unitary, or anti-unitary. There is one other important phase problem in quantum theory and this concerns the *uniqueness* of the unitary (or anti-unitary) correspondence $\psi_1 \leftrightarrow \psi_2$. It follows from the proof of Wigner's theorem that this correspondence is unique *up to an overall phase factor.*

Ray Representations or Projective Representations

If there are two equivalent descriptions with rays $\Psi_1, \Phi_1, \ldots$ and $\Psi_2, \Phi_2, \ldots$ respectively, corresponding to the same physical states as seen by the two different observers (passive view), or with rays $\Psi_1, \Phi_1, \ldots$ corresponding to states $\{s\}$ in the first description and to the transformed states $\{gs\}$ in the second description (active view), then we know that we can choose vectors $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2, \ldots$ such that

$$\psi_2 = T_g\psi_1, \quad \varphi_2 = T_g\varphi_1, \ldots \tag{1}$$

That is, if ψ_1 is a vector associated with Ψ_1, then $T_g\psi_1$ is a vector associated with the ray Ψ_2. Now if there are two operators T_g and $T_{g'}$ with the property (1), they can differ only in a constant factor of modulus 1. This result has an implication on the group law of transformations. For the product of two transformations $T_g T_{g'}$ gives the same results as the transformation $T_{gg'}$. Consequently,

$$T_{gg'} = \omega(g, g') T_g T_{g'}, \tag{2}$$

where $\omega(g, g')$ is a phase factor. Because T_g is a representation of the symmetry group, the group law for the representations is more general than the group law itself $g(g's) = (gg')s$. Representation of the type (2) are called 'ray representations' or 'representations up to a factor', or 'projective representations'. This is again the result of the fact that we have a correspondence between physical states and the rays in Hilbert space, not vectors.

DEFINITION. A *ray* (or projective) *representation T of* a topological group G is a continuos homomorphism $T: G \to L(\hat{H})$, the set of linear operators in the projective space $\hat{H}$ with the quotient topology relative to map $H \to \hat{H}$, i.e., $\psi \to \Psi$. ▼

Although the representation T is determined up to an arbitrary factor, the phase $\omega(g, g')$ in eq. (2) is not arbitrary. First of all, two phase systems $\omega(g, g')$ and $\omega'(g, g')$ may be defined to be equivalent if

$$\omega'(g, g') = \omega(g, g')\frac{c(gg')}{c(g)c(g')}, \quad g, g' \in G, \tag{2a}$$

where $c(g)$ is an arbitrary continuous function, because then the corresponding T_g and $T'_g = c(g) T_g$ have the same phase $\omega'(g, g')$. Furthermore, the associativity law of the group multiplication puts another restriction on the phase system $\omega(g, g')$, namely,

$$\omega(g, g')\omega(gg', g'') = \omega(g', g'')\omega(g, g'g''). \tag{2b}$$

Note that $\omega'(g, g')$ defined in (2a) satisfies (2b) if $\omega(g, g')$ does.

It is easy to see by taking very simple examples that eq. (2b), even up to equivalence given by (2a), does not uniquely determine the phases $\omega(g, g')$, so that we have, in general, a number of new inequivalent ray representations for a given group G, besides the usual representations with $\omega = 1$.

Let $g \to T_g$ be a projective representation of G, $g \in G$, in H. Let v_i be the components of some $v \in H$ in some basis. A ray can be represented by the quantities, $\bar{v}_i \equiv v_i/v_l$ where v_l is any component of v. For clearly all vectors in the ray $\{\lambda v\}$ induce the same $\bar{v}$. By choosing a special vector v with $v_1 = 1$ we see that the transformations induced on $\bar{v}$ by T are

$$\bar{v}_1' = v_1,$$

$$\bar{v}_i' = \Big[\sum_{k=2}^{\infty} D_{ik}(g)\bar{v}_k + D_{i1}(g)\Big] \Big/ \Big(\sum_{k=2}^{\infty} D_{1k}(g)\bar{v}_k + D_{11}(g)\Big),$$

$$i = 2, 3, \ldots$$

These transformations are nonlinear and are called *projective transformations*. It is easy to check that representations T_g and $c(g)T_g$ as well as inequivalent ray representations $\omega(g, g') \neq 1$ induce the same projective representation. The phase ambiguity has completely disappeared in this formulation, but it is only hidden, because the inverse problem of finding all inequivalent projective representations is equivalent to finding all inequivalent phases.

Projective Representations and the Central Extension

The remaining phase ambiguity precludes the application of the mathematical theory of ordinary representations when $\omega(g, g') \neq 1$. In this case we can try to construct a larger (or extended) group $\mathscr{E}$ whose ordinary representations give *all* the inequivalent ray representations (2) of G. This is the problem of *lifting* the projective representations of G into the ordinary representations of $\mathscr{E}$ and can be done simply as follows: Let K be the abelian group generated by multiplying the inequivalent phases $\omega(x, y)$ satisfying (2b). Consider the pairs (ω, x), $\omega \in K$, $x \in G$. In particular, $K = \{(\omega, e)\}$ and $G = \{(e, x)\}$. The pairs (ω, x) form a group with the multiplication law of a semidirect product

$$(\omega_1, x_1)(\omega_2, x_2) = \big(\omega_1 \omega(x_1, x_2)\omega_2, x_1 x_2\big).$$

In the present case we can think of (ω, x) as ωx. The group $\mathscr{E} = \{(\omega, x)\}$ is called a *central extension* of G by K (cf. 21, § 4) and we see that the vector representations of $\mathscr{E}$ contain all ray representations of G. Thus the extended group $\mathscr{E}$ may be considered as the proper *quantum mechanical group*. The theory and applications of group extensions will be discussed later in ch. 21. Here we give only

PROPOSITION 2. *Finite-dimensional projective representations of simply connected continuous groups are equivalent to ordinary representations.*

PROOF: First, quite generally, we take the determinant of eq. (2): $\det T(x)\det T(y) = \omega^n(x, y)\det T(x, y)$, where n is the dimension of the representations. The new representation $T'(x) = T(x)/[\det T(x)]^{1/n}$ formally satisfies $T'(x)T'(y) = T'(xy)$: There are different values of $[\det T]^{1/n}$ and we can pass to an equivalent phase system such that $T'(x)T'(y) = \omega'(x, y)T'(xy)$ with $\omega'^n = 1$. Now if the group space is simply connected $[x']^{1/n}$ can be uniquely defined and is the same for all x by continuity. Hence we arrive at an equivalent ordinary representation. ▼

Similarly the ray representations of the one-parameter subgroups of Lie groups are always equivalent to ordinary representations (cf. ch. 21, § 4).

In many physical examples, such as the cases of rotation, Lorentz and Poincaré groups, the projective representations of the group can be reduced to true unitary representations of its universal covering group (Bargmann 1954). A notable exception is the Galilei group, where the 'quantum mechanical group' is truly an eleven-parameter group, a central extension of the universal covering group of the Galilei group. The cases where we do not need a central extension are covered by

LIFTING CRITERION. *Let G be a connected and simply connected Lie group with Lie algebra L. Assume that for each skew-symmetric real valued bilinear form $\theta(x, y)$ on L satisfying*

$$\theta([x, y]z)+\theta([y, z]x)+\theta([z, x], y) = 0$$

there exists a linear form f on L such that

$$\theta(x, y) = f([x, y])$$

for all x, y in L. Then each strongly continuous projective representation of G is induced by a strongly continuous unitary representation on the corresponding Hilbert space. ▼

(For proof see Bargmann 1954, or Simms 1971.)

Remark: This condition is often expressed by the statement that the second cohomology group $H^2(G, R)$ is trivial (cf. ch. 21, § 4).

Summarizing, we have: A symmetry group G of the physical system *induces* a representation T of invertible mappings of H onto itself, which is unitary or anti-unitary and is a representation of a central extension $\mathscr{E}$ of G or of the covering group of G. In the unitary case we proved that H is a certain direct integral of irreducible carrier spaces of T by virtue of Mautner theorem (5.6.1).

Continuity

Mathematical theory of representations of topological groups requires the assumption of continuity of representations. Physically this means the continuity of the probabilities $|(\varphi, T_g\psi)|^2$ as a function of g, i.e., when we compare the probabilities of finding two states $T_g\psi$ and $T_{g'}\psi$ in a fixed state φ.

Unitary and Anti-Unitary Operators

The group property of the transformations, eq. (2), and the continuity allow us to determine the unitary or the anti-unitary character of the representation T of the symmetry group G.

If for every element g of G we have

$$g = h^2, \tag{3}$$

where h is also a group element, we have

$$T_g = \omega(g) T_h^2, \tag{4}$$

where $\omega(g)$ is a phase factor.

The square of an anti-unitary or unitary operator is unitary. Thus, T is unitary. For the identity component of any Lie group G, eq. (3) is satisfied. Indeed let $g(t)$ be a one parameter subgroup of G such that $g(t_0) = g$. Then for $h = g(t_0/2)$, eq. (3) holds. Consequently connected Lie symmetry groups will be represented by unitary operators. For the anti-unitary case, eq. (3), must break down. If eq. (3) does not hold, as for the extended Poincaré group with space and time reflections, further physical considerations are necessary to decide the unitary or anti-unitary character of T.

One can also see that the invariance of a state, which is a superposition of two stationary states with different energies, at time t under a symmetry transformation also eliminates the anti-unitary representation. For then the operator corresponding to the second solution in Wigner's theorem denoted by A would give

$$A(\psi_1 \exp(-iE_1 t) + \varphi_1 \exp(-iE_2 t)) = \exp(iE_1 t) A\psi_1 + \exp(iE_2 t) A\varphi_1,$$

whereas the correct evolution of the state obtained from the Schrödinger equation is

$$\exp(-iE_1 t) A\psi_1 + \exp(-iE_2 t) A\varphi_1 .$$

PROPOSITION 3. *The symmetry corresponding to the 'reversal of the direction of motion' (time reversal) must be represented by anti-unitary operators A.*

PROOF: An arbitrary state $\psi(t)$ can be represented as a superposition of stationary states ψ_n. The state $\psi(t)$ and the time reversed state $(A\psi)(t)$ evolve as

$$\psi(t) = \sum_n \exp(-iE_n t)\psi_n, \quad (A\psi)(t) = \sum_n \exp(-iE_n t) A\psi_n.$$

This last state must be, by time reversal invariance, also the transform of the state $\psi(-t)$, i.e.,

$$A\psi(-t) = A \sum_n \exp(iE_n t)\psi_n.$$

Thus, A must be anti-unitary in order for both states to be the same. ▼

In the unitary case, one can define a normalized operator T_g such that $T_{g^{-1}}$

$= T_g^{-1}$. Then $T_{gg^{-1}} = \omega(g, g^{-1})I$ by eq. (2). For two commuting transformations we have from (2)

$$T_g T_{g'} = c(g, g') T_{g'} T_g, \quad c(g, g') = \frac{\omega(g', g)}{\omega(g, g')}$$

and we find $c(g, g') = +1$ only if T_g and $T_{g'}$ also commute. In general, if the *commutator* $T_g T_{g'} T_g^{-1} T_{g'}^{-1}$ (which is independent of the normalizations of T_g and $T_{g'}$ and which is uniquely determined from T_g and $T_{g'}$) is a multiple of I, i.e., $C = gg'g^{-1}g'^{-1} = I$, then $T_C = c(g, g')I$. The factor $c(g, g')$ is a characteristic of the coherent subspace only, i.e., it has a unique value in each coherent subspace. If in particular $T_{g'}$ and T_g are members of the same one-parameter subgroup, then $c(g, g') = 1$.

Superselection Rules and Symmetry

In section 1.2, we have seen that the vectors $\sum_n \alpha_n \psi_n$ and $\sum_n \alpha_n(\lambda_n \psi_n)$ belong to different rays (states) although ψ_n and $\lambda_n \psi_n$ belong to the same ray. Now if a physical symmetry transformation of the system changes ψ_n into $\lambda_n \psi_n$ then, because the state of the system has not changed, a superposition of the form $\sum_n \alpha_n \psi_n$ is not possible, unless $\lambda_n = 1$. The relative phase λ_n between vectors in different coherent sectors is not an observable because the physics has not changed under the symmetry transformation. If this is the case, no physical measurement can distinguish the state $\sum_n \alpha_n \psi_n$ from the state $\sum_n \alpha_n(\lambda_n \psi_n)$. Thus, to show the existence of a superselection rule, we need a symmetry transformation (a physical postulate) and the existence of vectors ψ_n which go into $\lambda_n \psi_n$ under this transformation, and which represent eigenstates of a measurable physical quantity, e.g. charge.

EXAMPLE 1. *Rotational Invariance and Fermion Superselection Rule.* Consider, for concreteness, a state $\psi_1 = \left|\frac{j}{2}, m\right\rangle$ belonging to the representation $D^{j/2}$ of the rotation group SO(3) and a state $\psi_2 = |j, m'\rangle$ belonging to the representation $D^j, j =$ integer. Consider a rotation $\hat{n}\omega$ by an angle ω in a direction $\hat{n}$. The states transform by the formula

$$|JM\rangle' = D^J_{M',M}(\hat{n}\omega)|JM'\rangle.$$

It follows from exercise 5.8.3.1 that matrices D^J satisfy the condition

$$D^J(\hat{n}\omega + 2\pi n) = (-1)^{2nJ} D^J(\hat{n}\omega).$$

Thus for the rotation $\hat{n}_y 2\pi$, $\hat{n}_y = (0, 1, 0)$ we have in particular

$$|JM\rangle' = D^J_{M',M}(\hat{n}_y 2\pi)|JM'\rangle = (-1)^{2J}|JM\rangle.$$

Thus, we get an extra relative phase of $(-1)^{2J}$ in the linear combination of the two states ψ_1 and ψ_2; hence, if rotational invariance holds, according to our previous discussion, there is a superselection rule between the states with integer J and those with half-odd integer J-values; they cannot mix in physically realizable states.

EXAMPLE 2. SU(2) *Group for Isospin and Superselection Rules.* We give here an example for an approximate symmetry group. It was mentioned in ch. 7, § 4.C that particles of same spin and parity and of roughly equal masses with strong interactions may be grouped into multiplets according to irreducible representations of the group SU(2) (just like spin). The corresponding new quantum numbers are I and I_3, called *isospin.*

If the group $\mathrm{SU}(2)_I$ describing the isotopic spin multiplets of particles were an exact symmetry group of nature in the same way as the group SU(2) for spin, then by the result of example 1 there would be a superselection rule between the integer and half-odd integer I-spin states. Now for strong interactions which are independent of the electric charge, $\mathrm{SU}(2)_I$ is a good symmetry group. This means that there are no *pure* states of the form $|I = 1\rangle + |I = 1/2\rangle$, {e.g., $|\Sigma\rangle + |\Lambda\rangle$}.* There are, however, pure states like $|I = 1/2\rangle + |I = 1/2\rangle$, for example $|n\rangle + |p\rangle$ for strong interactions alone. These superpositions violate, however, the superselection rule on charge (see example 3 below); consequently, there is no superselection rule for charge *for strong interactions alone.* In the presence of electromagnetic and weak interactions, $\mathrm{SU}(2)_I$ is not a symmetry group, but then charge superselection rule holds exactly; a pure state $|\Sigma^0\rangle + |\Lambda^0\rangle$ now exists, but not a state $|n\rangle + |p\rangle$. In fact, an SU(2) -rotation taking n into p does not leave the system unchanged but corresponds to the weak interaction process: $n \to p + e + \bar{\nu}$.

Similarly, if a hypothetical 'superweak interaction' would violate the rotational invariance, then we could have pure states of the form $|j = 1/2\rangle + |j = 0)$, e.g., $|N\rangle + |\pi\rangle$.

EXAMPLE 3. *Superselection Rules for Gauge Groups.* Two equivalent descriptions obtained from each other by a commutative one-parameter continuous group (not obviously related to space-time transformations) implies the existence of an additive quantum number a, and the eigenstates transform as

$$|q\rangle' = \exp(\mathrm{i}\lambda q)|q\rangle.$$

* $|n\rangle$, $|p\rangle$, $|\Lambda\rangle$, $|\Sigma\rangle$, $|\pi\rangle$, ... denote particle states with definite values of isotopic spin I, i.e. the neutron, the proton, the Λ-particle, the sigma particle, the pion, etc.

For two-states with different values of q, e.g. $+1$ and -1, we obtain two different phases $\exp(\mathrm{i}\lambda)$ and $\exp(-\mathrm{i}\lambda)$, hence a superselection rule for q. The basic *physical* assumption underlying all such superselection rules, such as electric charge, baryon number, lepton number, we repeat, is the requirement that the multiplication of all states by $\exp(\mathrm{i}\lambda q)$ produces no observable change in the system, hence equivalent descriptions and gauge groups.

One can form, instead of pure states, *mixed* states out of vectors from different coherent subspace. But this will not interest us here any further.

Implications of the Superselection Rules on Parity and Other Group Extensions

Within a coherent subspace the parity of each state (relative to one of them) is well determined. In fact, we use the ray representations of the full orthogonal group $O(3)$, or the full Lorentz group, including reflections. In this case the parity is defined either in the same representation space as SO(3) (or proper homogeneous Lorentz group) or in a doubled Hilbert space. Thus relative parities are well determined, e.g., for the levels of H-atom and for particle-antiparticle pair in Dirac theory. However, for states in different coherent subspaces, the relative parity is not determined because we cannot take a linear combination of two such states and see how it transforms under parity.

Very similar considerations apply to other group extensions, e.g., by charge conjugation.

The extension of the isotopic spin group SU(2) by a reflection operator implies a doubling of $I = n/2$ states, but not necessarily of $I = n$, n = integer, states. Now this extension is carried out by C = charge conjugation, or by isospin parity $G = Ce^{i\pi}$ also called G-parity. (The use of C or G, respectively, corresponds in the rotation group for spin to the use of reflection operator Σ or the parity P; G commutes with all isospin rotations as P commutes with all space rotations.) G tells us whether we have polar or axial vectors in I-space (e.g., π meson is a polar vector). Therefore, the doubling with G takes us into antiparticles. Consequently, we have among others the result that $I = n/2$ boson-multiplets cannot contain antiparticles; they must lie in the other half of the doubled space. In the limit of an exact $\mathrm{SU}(2)_I$ the relative G-parity (isospin-parity) between $I = n/2$ and $I = n$ multiplets is not defined; nor is it defined between states with different charges or baryon numbers. It is defined, however, between, e.g., $I = 1$ multiplet (π) and two $I = \frac{1}{2}$-multiplets with $N = 0$ (e.g., $N\bar{N}$) (cf. ch. 21.4).

See § 4 for another example of the superselection rule, the mass superselection rule in non-relativistic quantum mechanics.

§ 3. Symmetries of Physical Systems

The concept of symmetry is associated with the following statements, which are all different expressions of the same fundamental phenomenon:

(i) It is impossible to know or measure certain quantities e.g., the absolute positions, directions, the absolute left or right, ...

(ii) It is impossible to distinguish between a class of situations, e.g., two identical particles.

(iii) The physical equations (or laws) are independent of some coordinates, e.g., the equations may contain only relative coordinates, and are independent of absolute coordinates.

(iv) The invariance of the equations of physics under a certain group of transformations, e.g., rotational invariance of Newton equations for the Kepler problem.

(v) The existence of certain permanencies pertaining to a system in spite of the constant change of its motion, or state,

(vi) Equivalent descriptions of the same physical system by two observers which are in different states.

If we have an established theory, we can explicitly study its symmetry properties. On the other hand, because symmetries and permanencies are easier to recognize, we can use these properties as requirements in establishing new theories of physical phenomena. How do the group representations enter into the discussion of the symmetry of physical systems?

The symmetry operation takes one state into another possible state which are in the same equivalence class with respect to the symmetry group G. Having recognized an equivalence relation between the phenomena we can classify them into equivalence classes. Thus the dual object $\hat{G}$ to G, i.e. the set of all irreducible representations of G, really enumerates distinct physical objects. For example, from the point of view of relativistic invariance distinct objects are the possible mass and spin values.

A. *Geometric Symmetry Principles*

The geometric invariant principles refer to the description of physical phenomena in space and time.

An *event* is a point P in space-time manifold M with a scale of units λ at P, (M, λ), i.e. we make a *measurement* at space-point x at time t and with a scale $\lambda(t, x)$, e.g. a position measurement relative to an origin with some choice of scale.

The manifold can be given the structure of the Minkowski space, in relativistic theories, or the structure of $R^1(t) \times R^3(x)$ in non-relativistic theories, for a fixed scale.

An *observer* is a local coordinate system (a *chart*) on M. A space-time *interval* is the dimensionless distance between two events measured in some units $dI = (dx, dx)^{1/2}\lambda$. In the case of the Minkowski structure, the distance is given by the indefinite scalar product $(x, x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$, relative to an observer.

An equivalent description by another observer corresponds by the principle of relativity to a map of the space-time-scale manifold into itself which preserves the interval dI. For fixed scale, in the Minkowski space, the group of transformations which preserve the distance $d(x, y) = (x-y, x-y)^{1/2}$ is isomorphic with the semidirect product $T^{3,1} \circledS O(3, 1)$ of the group of translations and full homogeneous Lorentz transformations on the Minkowski space M relative to the natural action of $O(3, 1)$ on M.

A partial ordering of events can be introduced by the relation

$$x > y, \text{ if and only if } x^0 > y^0 \text{ and } (x-y, x-y) > 0, \tag{1}$$

i.e., the event x is 'later' than the event y, and the relative vector $(x-y)$ is time-like. A transformation on the space-time manifold $\varphi : M \to M$ for which eq. (1) implies

$$\varphi(x) > \varphi(y) \tag{2}$$

and vice versa, is called a causal *automorphism* of space-time with respect to a local coordinate system. The causal automorphisms form a group (*causality group*). We have then

THEOREM 1 (Zeeman). *For fixed scale of units, the complete group of causal automorphisms of the Minkowski space is the semidirect product* $T^{3,1} \circledS (\Lambda^{\uparrow} \times D)$, *where* $\Lambda^{\uparrow}$ *is the group of orthochronous Lorentz transformations, and* D *the group of dilatations* $x \to \varrho x$, $x \in M$, $\varrho \in$ *the multiplicative group of non-zero real numbers.* ▼

We indicate the main steps of the proof. Given a causal automorphism $\varphi: M \to M$ which keeps the origin in M fixed (without loss of generality), we choose four linearly independent light-like vectors l_i, $i = 1, 2, \ldots, 4$, as a basis in M, i.e. $x = x^i l_i$ for all $x \in M$. Let g be the *linear* map in M given by $gx = x^i \varphi(l_i)$. Then it can be proved that φ is linear by showing that $\varphi = g$ by induction on the subspaces M_i spanned by vector l_j, $1 \leqslant j \leqslant i$, for each i. Then because φ preserves the light cone, it follows then that φ belongs to $T^{3,1} \circledS (\Lambda^{\uparrow} \otimes D)$. ▼

Remark: The group of causal automorphisms of M is thus isomorphic to the group of automophisms of the Lie algebra of the Poincaré group. The latter is $T^{3,1} \circledS (\Lambda^{\uparrow} \times D)$ (cf. exercise 1.10.1.11). Another proof of Zeeman's theorem can be given by this isomorphism.

The complete relativistic invariance implies the *full* inhomogeneous Lorentz group, thus includes the discrete transformations of space-reflection, as well as time reversal. The connected component of the identity in homogeneous Lorentz group is the restricted Lorentz group $\Lambda_{+}^{\uparrow}$, where $+$ refers to the condition $\det \Lambda = +1$.

According to sec. 1, we are interested in the projective representations of the group of relativity, or in the representations of the extended groups. The simply connected covering group of $T^{3,1} \circledS SO(3, 1)$ is the group $T^{3,1} \circledS SL(2, C)$, also called the *Poincaré group*.

In non-relativistic theories we replace the full inhomogeneous Lorentz group by the full *Galilei group*. The latter is a *contraction* of the Poincaré group as we showed in example 3.4.2.

If we allow the change of units in measuring the interval of events, and further change the units from point to point in space-time, we arrive at the *group of conformal transformations* of the Minkowski space, which thus contains in addition to the inhomogeneous Lorentz transformations, the dilatations

$$D_1 : x'^{\mu} = \varrho x^{\mu} \tag{3}$$

and the special conformal transformations denoted by

$$\begin{aligned} C_4 :\ x'^{\mu} &= (x^{\mu} + c^{\mu} x^2)/\sigma(x), \\ \sigma(x) &= 1 + 2c^{\nu} x_{\nu} + c^2 x^2. \end{aligned} \tag{4}$$

Equations (4) are *nonlinear* transformations. The 15-parameter conformal group is realized in a nonlinear manner as a group of transformations in the Minkowski space although the Poincaré subgroup is linear. It is possible to introduce a six-dimensional space, and a linear realization in it, because of the isomorphism of the conformal group with the group O (4, 2). It is reasonable to use the six-dimensional manifold as the basic space-time-scale to describe physical systems, the additional two coordinates may be viewed as representing the scale and the change of scale from point to point. As such, this space is the largest possible geometric framework, and its group of motion will be the inhomogeneous conformal group with a covering group

$$T^{4,2} \circledS \mathrm{SU}(2, 2). \tag{5}$$

The group (5) is perhaps the largest *kinematical* group (excluding the curved space of general relativity). For $m = 0$ the usual wave equations are exactly invariant under (5) (cf. § 4). We shall see, however, that much larger groups occur in quantum theory, but not to represent geometric symmetries, but to represent dynamics.

Representations of the Geometric Symmetry Groups

According to the results of sec. 1, we need the unitary representations of the covering groups of geometric symmetry groups, also unitary representations of most of the discrete symmetry groups, except those which contain the time reversal operation T. The discrete operations are best introduced, after the representations of the connected part of the symmetry Lie groups (e.g. Poincaré group) have been determined (cf. ch. 21, § 2).

Because the geometric symmetry groups are in general non-compact, the space of states quantum theory, even for a single particle, is infinite-dimensional; in general, we will have a direct integral of infinite-dimensional representations.

We have noted that the elements of the Lie algebra of the symmetry group have a direct physical meaning, in the sense that we prepare quantum systems initially to be in the eigenstates of a complete set of commuting operators, including the elements of the Cartan subalgebra and those from the enveloping algebra; and these operators have physical names, as linear and angular momentum, spin, helicity, etc. In fact, we begin, in quantum theory, with the representations of the Lie algebra. However, many results show that physics is really using the global representations of the group; i.e., those representations of the Lie algebra, which are integrable.

We know that some of the generators of non-compact symmetry groups are unbounded, and have continuous spectra. Because we do want to label the physical states by these continuous eigenvalues (rather than use always wave packets) and because such states are not in Hilbert space, we see that it is more convenient to base quantum theory on a more general fremework than that given by a Hilbert space. The use of such generalized eigenvectors, their normalization with Dirac δ-functions, etc. is now made mathematically rigorous by the use of distributions and nuclear spectral theory (cf. app. B, § 3).

The representations of the discrete geometric symmetry operations, like parity and time reversal, lead to the concept of central group extensions and the representations of the extended groups (cf. ch. 21, § 4).

Conservation laws or the constants of motion are initimately related with the notion of symmetry. Physically important quantities like energy, momentum, angular momentum, ... are the generators of symmetry transformations. In quantum theory their eigenvalues label the representations of the symmetry groups, hence the physical states.

B. *Symmetry Transformations Which are not of Geometric Origin*

As in geometrical symmetry transformations, the non-geometric symmetry transformations are associated with the impossibility of making certain absolute measurements:

(a) The impossibility of knowing the absolute difference between identical particles leads to symmetry properties of the total wave function of a system of N identical particles. The symmetry group here is the permutation group S_N and the physical requirement is that two states differing in their description by the exchange of identical particles represent the same physics and hence belong to the carrier space of a representation of S_N. As a result of this symmetry, one deduces that, for example, two electrons cannot be simultaneously in the same quantum state. Results following from the permutation symmetry have the quality of dynamical laws. For if we did not know the postulate of indistinguishability of identical particles, we would have been forced to invent fictitious forces between the electrons to prevent them from occupying the same state. This is an instructive example in which certain dynamics is best expressed in terms of a symmetry.

(b) The impossibility of knowing the absolute *sign* of the electric charge in the interactions of a number of charged particles. The physical phenomena depend only on the relative signs of the charges; they do not change if we replace all charges by their negatives. This symmetry is called *charge conjugation.* In the interactions of elementary particles, we have to interpret this symmetry more generally as *particle-antiparticle conjugation*, everywhere each particle is replaced by its antiparticle which has opposite values of *all* additive quantum numbers.

(c) The impossibility of knowing the relative phases between certain states. We know from sec. 1, that absolute value of phases of a state $\psi \in H$ is immaterial. Here we say that even the *relative* phase is sometimes not measurable. We already discussed this symmetry in detail under the name of superselection rules in section 1. As noted there, the superselection rules are associated with the absolute additive quantum numbers, like charge Q, baryon number B, lepton numbers L (and L—the muonic lepton number). These in turn can be represented as generators of one-parameter groups of transformations (abelian gauge groups).

Table I shows the list of symmetry principles in physics. The approximate symmetry groups and dynamical groups are discussed in the next chapter.

Table I
Physical Symmetries and Dynamical Groups

	Conserved Generators or Physical Implications
A. Geometrical Symmetry Groups	
Translations	P_μ
Rotations	J_i
Lorentz or Galileo Transformations	N_i
Parity P	P
Time Reversal T	anti-unitary
B. Group of Scale Transformations	
Dilatations	D
Special Conformal Transformations	K_μ
C. Nongeometrical Symmetries	
Identical Particles	Symmetry types of the wave functions
Gauge Groups (Non-Measurability of Relative Phases)	Charge Q Baryon number B Lepton numbers L
Particle-Antiparticle Conjugation	C
D. General Covariance	Equivalence principle Equations of motion of sources and fields
E. Approximate Dynamical Symmetries, e.g., US(2), SU(3), SU(6), $O(4)$	Multiplets
F. Dynamical Groups	
$O(3, 1)$, $O(4, 2)$	Infinite multiplets
G. Groups of Diffeomorphisms	
Infinite Parameter Groups	Geometrization of dynamics

§ 4. Dynamical Symmetries of Relativistic and Non-Relativistic Systems

According to our discussion in § 2, the dynamical problem in quantum theory amounts to an explicit construction of the concrete linear space (CLS), to identify the states, and to label them with physical observables in order to calculate the transition probabilities. The kinematical symmetries such as the Poincaré or the conformal invariance allow us to label these states by global quantum numbers, e.g. total linear and angular momenta. Consequntly to assure relativistic invariance on the one hand, and to partly determine the concrete linear space on the other hand, it is always appropriate to determine which class of representations of the symmetry group are realized for a given quantum system. For non-relativistic problems the Poincaré invariance goes over into the Galilean invariance (cf. ch. 1, § 8 for the contraction of the Poincaré Lie algebra to the Galilean Lie algebra). It is more convenient however, even for non-relativistic problems to start from the conformal invariance, as we shall see.

We start from the 15 parameter conformal group of the Minkowski space isomorphic to SO(4, 2), introduced in § 3.

PROPOSITION 1. *The Lie algebra basis elements of the conformal group on the space of scalar functions over the Minkowski space are represented by the following differential operators*

$$\begin{aligned} M_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu, \\ P_\mu &= \partial_\mu, \\ K_\mu &= 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, \\ D &= x^\nu \partial_\nu, \end{aligned} \qquad \mu, \nu = 0, 1, 2, 3 \tag{1}$$

where $M_{\mu\nu}$, P_μ are the basis elements of the Poincaré subalgebra, K_μ the generators of the so-called special conformal transformations and D generates the dilatations.

The proof follows from the definition of the conformal group given by eqs. 3(3), 3(4).

Modifications in the case of spinor or vector valued functions can easily be found (see exercise 4.1).

Remark: Strictly speaking the conformal group (4) is not well defined globally in the Minkowski space. A proper definition needs a compactified form of the Minkowski space. For the properties of the Lie algebra however the Minkowski space can be used without difficulties. ▼

The commutation relations of the conformal Lie algebra are easily derived from eq. (1) to be those of the Poincaré Lie algebra plus the following additional ones:

$$[M_{\mu\nu}, K_\lambda] = g_{\nu\lambda} K_\mu - g_{\mu\lambda} K_\nu,$$

$$[M_{\mu\nu}, D] = 0,$$

$$
\begin{aligned}
[P_\mu, K_\nu] &= 2(g_{\mu\nu} D - M_{\mu\nu}), \\
[P_\mu, D] &= P_\mu, \\
[K_\mu, K_\nu] &= 0, \\
[K_\mu, D] &= -K_\mu.
\end{aligned}
\tag{2}
$$

The important property of the conformal group from the physical point of view is embedded in the following observation:

PROPOSITION 2. *Relativistic wave equations for massless particles of spin* 0 *and spin* $\frac{1}{2}$ *are invariant under the conformal group of the Minkowski space.*

The statement means that if

$$W\varphi = 0 \tag{3}$$

is the wave equation, one can realize the generators L_i *of the conformal Lie algebra on the space of solutions such that in general*

$$[W, L_i] = \lambda W. \tag{4}$$

Hence L_i *acting on the space of solutions of the wave equation takes one solution into another.*

PROOF: Consider first the equation for a spinless and massless particle, i.e. the ordinary wave equation

$$\Box\varphi \equiv (\partial_t^2 - \partial_i \partial_i)\varphi = 0. \tag{5}$$

It is easy to verify that K_μ and D given by eq. (1) result in

$$
\begin{aligned}
[\Box, K_\mu] &= 4x_\mu \Box, \\
[\Box, D] &= 2\Box.
\end{aligned}
\tag{6}
$$

The wave equation for spin $\frac{1}{2}$ can be treated in a similar fashion (cf. exercise 6.4.2).

PROPOSITION 3. *Wave equations for massive particles, e.g.*

$$(\Box - m^2c^2/\hbar^2)\varphi = 0, \tag{7}$$

are formally invariant under the conformal group provided the mass m *is transformed as follows:*

under dilatations:

$$m^2 \to e^{2\alpha} m^2, \tag{8}$$

under special conformal transformations:

$$m^2 \to \sigma(x)^2 m^2, \qquad \sigma(x) = 1 + 2c^\mu x_\mu + c^2 x^2,$$

or, alternatively, m^2 *is viewed as an operator having the same commutation relations as* $\Box$ *in eq.* (6). ▼

The proof follows by direct calculation and is straightforward.

Remark: The invariance in proposition 3 is not a symmetry, in the usual sense, for a definite single particle of mass m, because it connects states of particles of mass m with those of another particle of mass m'. However, the transformations (8) have some important applications in physics. Furthermore, if the conformal symmetry is interpreted as a change of scale (cf. § 3), eq. (8) can be interpreted as the change due to the dimension of mass.

We shall now discuss the contraction of the wave equation (7) to the nonrelativistic limit. In this limit the energy is measured after the rest mass is taken out. Hence we set

$$\partial_0 \to mc + \frac{1}{c}\partial_t. \tag{9}$$

Then the generators of the conformal group become

$$\begin{aligned}
P_0 &= mc + \frac{1}{c}\partial_t, \\
P_i &= \partial_i, \\
M_{ij} &= (x_i \partial_j - x_j \partial_i), \\
M_{0i} &= c(t\partial_i - mx_i) - \frac{1}{c} x_i \partial_t, \\
D &= c^2 mt + (t\partial_t - x_k \partial_k), \\
K_0 &= c^3 mt^2 + c(t^2 \partial_t - 2tx_k \partial_k + mx^2) + \frac{1}{c} x^2 \partial_t, \\
K_i &= c^2(2x_i mt - t^2 \partial_i) + (2x_i t \partial_t - 2x_i x_k \partial_k + x^2 \partial_i).
\end{aligned} \tag{10}$$

At the same time the wave operator becomes

$$\Box - m^2 c^2 \to (2m\partial_t - \partial_i \partial_i) + \frac{1}{c^2}\partial_t \partial_t, \tag{11}$$

i.e. the Schrödinger operator plus an additional term of order $1/c^2$.

PROPOSITION 4. *The operators*

$$\begin{aligned}
\bar{P}_0 &= \partial_t, \\
P_i &= \partial_i, \\
M_{ij} &= x_i \partial_j - x_j \partial_i, \\
\bar{M}_{0i} &= (t\partial_i - mx_i)
\end{aligned} \tag{12}$$

commute with the Schrödinger operator $S = (2m\partial_t - \partial_i \partial_i)$ *and generate the Lie algebra of the Galilean group* (cf. ch. 1.8), *except for the following commutator*:

$$[P_i, \bar{M}_{0j}] = -m\delta_{ij} \tag{13}$$

for which the right-hand side was zero in the purely geometric definition of the Galilei group. ▼

The proof of this and the following propositions is again by direct computation and we leave it to the reader.

The difference noted in proposition 4 is due to the replacement (9) and expresses the fact that the mass m is really an operator commuting with all the ten generators of the Galilei group. The solutions of the Schrödinger equation thus realizes a slightly different representation of the Galilei group, namely the projective representation given by eq. (12). Equivalently we can say that eq. (12) is an extension of the Galilean Lie algebra, or its quantum mechanical representation (cf. ch. 21, § 4).

In fact, when we go over to the global form of the representation (12) we obtain an example of a representation up to a factor discussed in § 2 and an example of the superselection rule, namely the

MASS SUPERSELECTION RULE (Bargmann). *For two elements of the Galilei group*

$$g = (b, \boldsymbol{a}, \boldsymbol{v}, R), \qquad g' = (b', \boldsymbol{a}', \boldsymbol{v}', R')$$

we have the global representation

$$U(g')\,U(g) = \omega(g', g)\,U(g'g), \tag{14}$$

where the phase $\omega(g', g)$ *is given by*

$$\omega(g', g) = \exp\left[\mathrm{i}\frac{m}{2}(\boldsymbol{a}'\cdot R'\boldsymbol{v} - \boldsymbol{v}'\cdot R\boldsymbol{a} + b\boldsymbol{v}'\cdot R'\boldsymbol{v})\right].\blacktriangledown \tag{14'}$$

The superselection rule arises because the following *identity* group element

$$(0, 0, -\boldsymbol{v}, 1)(0, -\boldsymbol{a}, 0, 1)(0, 0, \boldsymbol{v}, 1)(0, \boldsymbol{a}, 0, I) = (0, 0, 0, I)$$

is not represented by $U = I$, but by the phase factor

$$\exp(-\mathrm{i}m\boldsymbol{a}\cdot\boldsymbol{v}). \tag{15}$$

Hence the superposition of two states with different masses m_1 and m_2 transforms into

$$\psi(m_1) + \psi(m_2) \to \exp(-\mathrm{i}m_1\boldsymbol{a}\cdot\boldsymbol{v})\psi(m_1) + \exp(-\mathrm{i}m_2\boldsymbol{a}\cdot\boldsymbol{v})\psi(m_2). \tag{16}$$

This implies that the relative phase is not observable, hence according to our general discussion, the superposition $|m_1\rangle + |m_2\rangle$ is not a realizable state.

PROPOSITION 5. *The Schrödinger operator* $S = (2m\partial_t - \partial_i\partial_i)$ *is also invariant under a modified dilatation operator* $\tilde{D}$ *and a modified special conformal transformation with generator* $\tilde{K}_0$ (17) *which together with the Galilei Lie algebra form a* 12*-parameter Lie algebra, called the Schrödinger Lie algebra.* $\blacktriangledown$

The modification comes because, as we have seen, the operators D and K_0 as invariant operators of the wave operator $(\Box - m^2c^2)$ also transform the mass m. Now in the Schrödinger operator $(2m\partial_t - \partial_i\partial_i)$ the mass m occurs as a factor of ∂_t. Hence if we wish to determine the invariance of the Schrödinger operator for fixed m, we can transfer the transformation property of m to ∂_t or to $\partial_i\partial_i$.

This process gives immediately the following expressions for the modified generators

$$\begin{aligned}\tilde{D} &= 2t\,\partial_t + x_k\,\partial_k + 3/2,\\ \tilde{K}_0 &= t^2\,\partial_t + tx_k\,\partial_k + \frac{3}{2}t - \frac{m}{2}x^2.\end{aligned} \tag{17}$$

PROPOSITION 6. *The Schrödinger group generators are represented in the momentum space (in Schrödinger picture of quantum mechanics) by*

$$\begin{aligned}H_0 &= \frac{1}{2m}p^2,\\ P &= p,\\ J &= q\times p,\\ M &= -tp+mq,\\ \tilde{D} &= \frac{t}{m}p^2 + \frac{1}{2}(p\cdot q + q\cdot p),\\ \tilde{K}_0 &= -\frac{t^2}{2m}p^2 + \frac{1}{2}t(p\cdot q+q\cdot p) - \frac{m}{2}q^2.\end{aligned} \tag{18}$$

PROPOSITION 7. *All the generators in the previous proposition including the time dependent ones, satisfy*

$$[H_0, L_A] + \frac{\partial L_A}{\partial t} \equiv \dot{L}_A = 0 \tag{19}$$

and their explicit time dependence is given by

$$L_A(t) = \exp(-\mathrm{i}tH_0)L_A(0)\exp(\mathrm{i}tH_0). \;\blacktriangledown \tag{20}$$

In particular, the time independent generators commute with the Hamiltonian, and generate, what is commonly called a degeneracy group of the Hamiltonian. All operators satisfying (19) generate a symmetry group in the large sense, not of Hamiltonian, but of the time dependent operator $(\mathrm{i}\partial_t - H)$, hence of the quantum mechanical system *per se*.

PROPOSITION 8. *The Schrödinger group has as a subgroup, a dynamical group* SU (1, 1), *generated by* $\tilde{D}$, $\tilde{K}_0$, H, *or*

$$\begin{aligned}L_1(t) &= \frac{1}{2}(\tilde{K}_0 + H_0),\\ L_2(t) &= -\frac{1}{2}D,\\ L_3(t) &= -\frac{1}{2}(\tilde{K}_0 - H_0).\end{aligned} \tag{21}$$

Here L_3 is the compact generator of SU(1, 1) *with a discrete spectrum.*

In particular, the algebra at $t = 0$ is

$$L_1(0) = \frac{1}{2}\left(\frac{p^2}{2m} - \frac{m}{2}q^2\right),$$
$$L_2(0) = -\frac{1}{2}(p\cdot q + q\cdot p), \tag{22}$$
$$L_3(0) = -\frac{1}{2}\left(\frac{p^2}{2m} + \frac{m}{2}q^2\right). \blacktriangledown$$

PROPOSITION 9. *The Lie algebra* (22) *of proposition* 8 *solves the dynamical problem for the* 3*-dimensional quantum oscillator with the Hamiltonian*

$$H = \frac{p^2}{2m} + \lambda q^2 \tag{23}$$

and the free particle with Hamiltonian $H = L_1 - L_3 = \dfrac{p^2}{2m}$. ▼

The proof consist in the observation that in suitable units, the Hamiltonian (23) is identical with the compact generator of the su(1, 1) algebra (22). The Casimir operator

$$C_2 = L_3^2 - L_1^2 - L_2^2$$

can be evaluated in the representation (22):

$$C_2 = \frac{1}{4}(q\times p) - \frac{3}{16}.$$

Thus we know the representation of the Lie algebra, hence the spectrum of H, as well as the states of the oscillator.

Remark: Note that a free particle and the quantum mechanical oscillator are realized on the same representation space of the group SU(1,1). The difference is that the energy eigenstates are obtained by diagonalizing the compact generator L_3 in the oscillator case, but the non-compact generator $(L_1 - L_3)$ in the case of the free particle. The latter operator has of course a continuous spectrum. But even for a free particle there exists an operator with a discrete spectrum. This fact illustrates an important point in quantum theory: The physical identification of the Lie algebra elements is essential in quantum theory; we do not use abstract groups rather groups with definite identifications.

§ 5. Comments and Supplements

Historical Remarks

Quantum theory, developed by Heisenberg, Schrödinger, Dirac, Pauli, Born, Jordan, and others, found its first mathematical formulation by von Neumann, who gave an axiomatic Hilbert space formulation and proved the uniqueness and equivalence of the Heisenberg and Schrödinger formulations. This equivalence was also proved by Pauli and Lanczos. The applications of group representations to quantum theory were originated by Wigner. The formulation of relativistic invariance in quantum theory is due to Dirac and Wigner; the latter gave the first complete discussion of the representations of the Poincaré group (1939). The group theoretical discussion of wave equations is due to Bargmann and Wigner 1948 (cf. chs. 16-21).

The concept of superselection rules was introduced by Wick, Wightman and Wigner 1952. The representations of symmetry groups by unitary or anti-unitary operators in Hilbert space (Wigner theorem) has been elaborated by Wigner, Bargmann 1964 and, more generally, by Emch and Piron 1963 and Uhlhorn 1963.

The invariance of classical Maxwell's equations under the conformal group goes back to Bateman 1910 and Cunningham 1910. The theory of conformally invariant wave equations goes back to Dirac 1936. Dynamical groups were introduced by Barut 1964.

§ 6. Exercises

§ 3.1. Show that the dilatations and the non-linear special conformal transformations eqs. 3 (3), 3 (4) can be written as linear transformations in the six-dimensional space of the form

$$\begin{aligned} D_1: \quad & \eta'^\mu = \eta^\mu, \ k' = \varrho^{-1} k, \ \lambda' = \varrho\lambda, \\ C_4: \quad & \eta'^\mu = \eta^\mu + c^\mu \lambda, \\ & k' = -2c_\nu \eta^\nu + k + c^2 \lambda, \\ & \lambda' = \lambda, \end{aligned}$$

where $\eta^\mu = kx^\mu$, k and $\lambda = kx^2$ are taken to be the six new coordinates.

§ 3.2. Show that the Maxwell equations

$$\begin{gathered} \operatorname{div} \boldsymbol{E}(x) = \varrho(x), \\ \operatorname{div} \boldsymbol{H}(x) = 0, \\ \operatorname{curl} \boldsymbol{H}(x) - \frac{1}{c}\frac{\partial \boldsymbol{E}(x)}{\partial t} = \frac{1}{c}\varrho(x)\boldsymbol{u}(x) = \frac{1}{c}\boldsymbol{j}(x), \\ \operatorname{curl} \boldsymbol{E}(x) + \frac{1}{c}\frac{\partial \boldsymbol{H}(x)}{\partial t} = 0, \end{gathered}$$

where $x = (t, \boldsymbol{x})$, are not invariant under the Galilei transformations

$$\begin{aligned} t' &= t, \\ \boldsymbol{x}' &= \boldsymbol{x} - \boldsymbol{v}t. \end{aligned}$$

§ 3.3. Show that in the Minkowski space M^2 (with one space and one time dimensions) the causality group is much larger than $T^{1,1} \otimes(\mathrm{SO}(1,1)\otimes D)$. In particular non-linear transformations which map space and time lines into curved lines would be allowed.

§ 3.4. On the Minkowski space M^4 define the relation xLy, if $(x-y)$ is an oriented light-like vector: $x^0 > y^0$, $(x-y)^2 = 0$. Let $\varphi: M \to M$ be a one-to-one mapping. Show that φ preserve the partial ordering $x > y$ if and only if it preserves the relation xLy. Note that the relation xLy is not a partial ordering because it is not transitive. Show also that a causal automorphism maps light rays into light rays.

§ 4.1. Show that for a Dirac particle the generators of the conformal group (corresponding to eq. (4.1)) are

$$M_{\mu\nu} = \mathring{M}_{\mu\nu} + \frac{i}{2}\gamma_\mu\gamma_\nu, \quad \mu < \nu,$$

$$P_\mu = \mathring{P}_\mu - \frac{1}{2}\gamma_\mu(1+\gamma_5),$$

$$K_\mu = \mathring{K}_\mu - \frac{1}{2}\gamma_\mu(1-\gamma_5),$$

$$D = \mathring{D} - \frac{1}{2}\gamma_5,$$

where $\mathring{M}_{\mu\nu}$, $\mathring{P}$, ... are given in eq. (4.1).

§ 4.2. The wave operator for a massless Dirac particle is $\gamma^\mu \partial_\mu$. Show that the wave equation $\gamma^\mu \partial_\mu \psi = 0$ is invariant under conformal transformations in the sense of eq. (4.4), using the representation given in the previous exercise.

§ 4.3. Discuss the conformal group in M^2, two-dimensional space-time.

§ 4.4.* Show the conformal invariance of massless spin 1 wave equations in fact of all massless wave equations.

§ 4.5.* Let M^n, $n = 2, 3, \ldots$ be the Minkowski space. Show that the conformal invariance restricts the interaction term $F(\varphi)$ of the nonlinear relativistic equation

$$\Box\varphi = F(\varphi)$$

to the form $F(\varphi) = \lambda\varphi^{(n+2)/(n-2)}$. Note that the resulting interaction is, on the second quantized level, renormalizable, but not superrenormalizable.

§ 4.6.*** Derive the similar result for the nonlinear Dirac equation

$$\partial_\mu\gamma^\mu\psi = F(\bar{\psi}, \psi).$$

Hint: Use only invariance under dilatation $x \to \varrho x$, $\varrho > 0$.

Chapter 14

Harmonic Analysis on Lie Groups. Special Functions and Group Representations

Besides the fundamental role of the theory of group representations in formulating the basic equations of physics, we must also mention the important method of *harmonic analysis* in the solution of dynamical problems.

In many physical problems we deal with functions over the homogeneous or symmetric spaces, in particular, on group spaces. For example, functions over the so-called mass-hyperboloid in momentum space: $\varphi(p_\mu), p_\mu^2 = m^2$. The argument in this case is an element of the homogeneous space SO(3, 1)/SO(3). These functions can be decomposed over the set of eigenfunctions of Casimir operators. Such decompositions are extremely powerful and have a physical interpretation. They also form a basis for approximations when in suitable cases only few terms of the expansion are important. The expansions in terms of the special functions of mathematical physics can be reformulated in terms of harmonic analysis on homogeneous spaces. These problems will be treated in detail in chs. 14 and 15.

Let G be a unimodular Lie group with a Haar measure μ and let $H = L^2(G, \mu)$. We restrict our analysis to type I groups only. The main purpose of harmonic analysis in H is to solve the following problems:

(i) To determine a basis* $\{e_k(\lambda, g)\}$ in H and a dense subspace $\Phi \subset H$ such that if the generalized Fourier transform of $\varphi \in \Phi$ is given by the formula

$$\hat{\varphi}_k(\lambda) = (\varphi, e_k(\lambda)), \tag{1}$$

then the spectral synthesis formula for φ is given by

$$\varphi(g) = \int_\Lambda d\varrho(\lambda) \sum_k \hat{\varphi}_k(\lambda) e_k(\lambda, g). \tag{2}$$

* We assume that the index λ corresponds to the set of eigenvalues of invariant operators of G and k corresponds to the set of eigenvalues of the remaining operators, which together with the invariant operators form a maximal set of commuting operators in H. For convenience we use this notation as though $\{\lambda\}$ would be a continuous set and $\{k\}$ would be a discrete one. However in general both sets might be discrete, continuous or mixed.

(ii) To establish the Plancherel equality*

$$(\varphi, \psi) = \int_{\Lambda} d\varrho(\lambda) \sum_{k} \hat{\varphi}_k(\lambda)\overline{\hat{\psi}_k(\lambda)}. \tag{3}$$

(iii) To construct explicitly the measure $d\varrho(\lambda)$ (Plancherel measure).

The main difficulty in the harmonic analysis on Lie groups is associated with the fact that in most cases a maximal set of commuting operators in H, which determines the basis $e_k(\lambda, g)$ contains unbounded operators with continuous spectra and therefore the eigenvectors $e_k(\lambda, g)$ are distributions. Consequently, in order to give a proper interpretation to the functions $e_k(\lambda, g)$ and to the eigenfunction expansion (2), we have to deal with the so-called *Gel'fand triplet* $\Phi \subset H \subset \Phi'$ rather than with a simple Hilbert space H. In this triplet Φ is a certain nuclear space of smooth functions dense in H, and Φ' is the dual space to Φ. It is evident, therefore, that a natural framework for harmonic analysis on Lie groups is via the nuclear spectral theory. This theory allows a clear and elegant formulation of harmonic analysis on groups. At the same time, the theory provides a generalization of the classical Fourier analysis, and is useful for applications in quantum physics, where the concepts of eigenfunctions and eigenfunction expansions play the central role.

§ 1. Harmonic Analysis on Abelian and Compact Lie Groups

We shall first give the extension of ordinary Fourier analysis on R^n to an arbitrary abelian Lie group.

THEOREM 1. *Let G be an arbitrary abelian Lie group and let $H = L^2(G, \mu)$, where μ is the Haar measure on G. Let $g \to T_g$ be the regular representation of G in H given by*

$$T_g u(\tilde{g}) = u(\tilde{g}+g). \tag{1}$$

Then

(i) *There exists a generalized Fourier transform F such that*

$$\begin{aligned} F\colon H \to FH \equiv \hat{H} = \int_{\Lambda} \hat{H}(\lambda) d\varrho(\lambda), \\ F\colon T_g \to FT_gF^{-1} \equiv \hat{T}_g = \int_{\Lambda} \hat{T}_g(\lambda) d\varrho(\lambda), \end{aligned} \tag{2}$$

where $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$ are ϱ-a.a. irreducible and $\dim \hat{H}(\lambda) = 1$. *The spectrum Λ coincides with the character group $\hat{G}$ of G.*

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e(\lambda, g)$ in $H(\lambda)$ such that for every element X in the enveloping algebra E for ϱ-a.a. λ, we have***

* We follow the convention adopted in mathematical literature and take the scalar product in H which is linear with respect to the first factor and antilinear with respect to the second one.

** For the definition of the Gel'fand triplet, the formulation of nuclear spectral theorem and notation see app. B, § 3.

$$\overline{\langle \overline{T(X)}\varphi, e(\lambda)\rangle} = \hat{X}(\lambda)\langle\varphi, e(\lambda)\rangle, \tag{3}$$

where $\hat{X}(\lambda)$ is a real number. The basis elements $e(\lambda, g)$ are regular functions on G.

(iii) *The spectral synthesis formula has the form*

$$\varphi(g) = \int \mathrm{d}\varrho(\lambda)\hat{\varphi}(\lambda)e(\lambda g), \qquad \varphi \in \Phi, \tag{4}$$

where

$$\hat{\varphi}(\lambda) = \int \varphi(g)\overline{e(\lambda, g)}\mathrm{d}\mu(g), \qquad \hat{\varphi}(\lambda) \in \hat{H}(\lambda). \tag{5}$$

(iv) *For $\varphi, \psi \in \Phi$ the Plancherel equality has the form*

$$\int_G \varphi(g)\overline{\psi(g)}\mathrm{d}\mu(g) = \int_\Lambda \hat{\varphi}(\lambda)\overline{\hat{\psi}}(\lambda)\mathrm{d}\varrho(\lambda). \tag{6}$$

PROOF: ad (i). An abelian Lie group is of type I: hence th. 5.6.3 implies the decomposition (2). By virtue of proposition 6.1.1 every irreducible component $\hat{T}_g(\lambda)$ is one-dimensional.

ad (ii)–(iv). Let $D_G \subset H$ be the Gårding domain for the enveloping algebra E of G. Because the elliptic operator $T(\Delta) = \sum_{i=1}^{\dim G} T(X_i)^2$ commutes with all the elements $T(X), X \in E$ of G, by virtue of th. 11.2.3 the closure $\overline{T(X)}$ of a symmetric element $X^+ = X \in E$ of $T(X)$ is a self-adjoint operator. By virtue of th. 11.5.3, all operators $\overline{T(X)}$, $X \in E$, are mutually commuting and commute also with all T_g, $g \in G$. Let $\{X_i\}_{i=1}^{\dim G}$ be a basis in the Lie algebra L of G. Then the self-adjoint operators $\overline{T(X_i)}$ provide a maximal set of commuting operators in H and the elliptic Nelson operator $\overline{T(\Delta)}$ is also diagonal. Consequently all assertions ad (ii)–ad (iv) follow from the nuclear spectral theorem. ▼

The measure $\mathrm{d}\varrho(\lambda)$ on the spetral set $\Lambda = G$ in eq. (2), is called the *Plancherel measure*.

Remark 1: By virtue of eq. app. B.3(27), eq. (3) may be written in the form

$$\overline{T(X)}'\, e(\lambda, g) = \hat{X}(\lambda)e(\lambda, g), X \in L, \tag{7}$$

where $\overline{T(X)}'$ is the extension of $\overline{T(X)}$ obtained by extending the domain $D(\overline{T(X)})$ by those elements φ' in Φ' for which the equality

$$\langle\overline{T(X)}\varphi, \varphi'\rangle = \langle\varphi, \overline{T(X)}'\varphi'\rangle, \qquad \varphi \in \Phi,\ \varphi' \in \Phi', \tag{8}$$

is satisfied.

Remark 2: Since the Plancherel measure on the spectral set $\Lambda = \hat{G}$ is absolutely continuous relative to the Lebesgue measure $\mathrm{d}\lambda$ (i.e., $\mathrm{d}\varrho(\lambda) = \varrho(\lambda)\mathrm{d}\lambda$, $\varrho(\lambda)$ continuous on Λ), by virtue of eq. app. B.3(29) we obtain the following orthogonality relation for generalized eigenvectors

$$\int_G e(\lambda, g)\overline{e(\lambda', g)}\mathrm{d}\mu(g) = \varrho^{-1}(\lambda)\delta(\lambda - \lambda'). \tag{9}$$

This is a generalization of the well-known orthogonality relation in ordinary Fourier analysis on R:

$$\int_{R^n} \exp(i\lambda x)\overline{\exp(i\lambda' x)}\,\mathrm{d}^n x = (2\pi)^n \delta^{(n)}(\lambda-\lambda'). \tag{10}$$

In both cases these integrals are understood as weak integrals of the regular distributions $e(\lambda, g)\,\overline{e(\lambda', g)}$ on G.

We see therefore that the nuclear spectral theory provides a direct extension of harmonic analysis on R^n to arbitrary abelian Lie groups.

In the case of compact groups the harmonic analysis in the Hilbert space $H = L^2(G, \mu)$, μ – normalized Haar measure on G, is essentially given by the Peter–Weyl theorem 7.2.1, which states that an arbitrary function $u(g) \in H$ may be represented in the form

$$u(g) = \sum_{\lambda, p, q} \hat{u}_{pq}(\lambda) D^{\lambda}_{pq}(g), \tag{11}$$

where $\Lambda = \{\lambda\}$ is the dual object $\hat{G}$ of G and $D^{\lambda}_{pq}(g)$ are the matrix elements of the irreducible representation T^{λ} of G. The generalized Fourier transform $\hat{u}_{pq}(\lambda)$ of $u \in H$ is given by the formula 7.2(6)

$$\hat{u}_{pq}(\lambda) = d^{\lambda} \int_G u(g)\overline{D^{\lambda}_{pq}(g)}\,\mathrm{d}\mu(g), \tag{12}$$

where d^{λ} = dimension of the representation T^{λ} of G. The matrix elements $D^{\lambda}_{pq}(g)$ satisfy the following orthogonality and the completeness relations:

$$\int_G D^{\lambda}_{pq}(g)\overline{D^{\lambda'}_{p'q'}}(g)\,\mathrm{d}\mu(g) = \frac{1}{d^{\lambda}} \delta^{\lambda\lambda'} \delta_{pp'} \delta_{qq'}, \tag{13}$$

$$\sum_{\lambda, p, q} d^{\lambda} D^{\lambda}_{pq}(g)\overline{D^{\lambda}_{pq}}(g') = \delta(g - gg'^{-1}) \tag{14}$$

(cf. eq. 7.1 (9) and 7.2 (20)).

§ 2. Harmonic Analysis on Unimodular Lie Groups

The simplicity of harmonic analysis on compact groups was associated with the fact that the commutant T' of an arbitrary representation T of G was generated by a compact self-adjoint operator K_u given by eq. 7.1(4). Because every compact operator has only discrete spectrum the decomposition of an arbitrary function was given in the form of a discrete sum 1(11). In addition, the basic functions $D^{\lambda}_{pq}(g)$ which provide an expansion of an arbitrary function $u \in L^2(G, \mu)$ were matrix elements of irreducible representations T^{λ} of G and satisfied the orthogonality and completeness relations (1(13) and 1(14)).

In the case of an arbitrary Lie group the commutant T' of the regular representation T of G might contain operators with continuous spectra. Hence, in general, one will obtain a direct integral decomposition of both the representations in $H = L^2(G, \mu)$ and functions $u \in H$. This is a typical feature of non-compact groups. In addition, the orthogonality and the completeness relations 1(13) and 1(14) will hold only in special cases and will need an additional interpretation as products of distributions. Because the eigenfunctions of operators with continuous spectra are not elements of $L^2(G, \mu)$, but only linear functionals over a dense space $\Phi \subset H$ of smooth functions, we have to deal with the triplet $\Phi \subset H \subset \Phi'$ rather than with a single space $H = L^2(G, \mu)$. The elegant and effective formalism for dealing with continuous spectra of self-adjoint operators is provided by the nuclear spectral theory presented in app. B, § 3. This theory provides a satisfactory framework for an extension of harmonic analysis from abelian and compact groups to the case of noncompact Lie groups.

Let G be a unimodular Lie group. We first give a description of invariant operators which generate the center of the commutant T' of the regular representation T in the Hilbert space $H = L^2(G, \mu)$.

Let $g \to T^L_g$ and $g \to T^R_g$ be the left and the right regular representations of G in H, i.e.,

$$T^L_g u(\tilde{g}) = u(g^{-1}\tilde{g}), \quad T^R_g u(\tilde{g}) = u(\tilde{g}g), \quad u \in H. \tag{1}$$

Denote by $\mathscr{R}_L$ (or $\mathscr{R}_R$) the closure, in the weak operator topology of $L(H)$, of the set of all linear combinations of the T^L_g (or T^R_g). Then by virtue of Segal's theorem 9.6.3 we have

$$\mathscr{R}'_L = \mathscr{R}_R, \quad \mathscr{R}'_R = \mathscr{R}_L \tag{2}$$

and

$$(\mathscr{R}_L \cup \mathscr{R}_R)' = \mathscr{R}'_L \cap \mathscr{R}'_R, \tag{3}$$

i.e. the commutant of the algebra $\mathscr{R}_L \cup \mathscr{R}_R$ is the intersection of the center of the algebra $\mathscr{R}'_L$ and the center of $\mathscr{R}'_R$. Segal's theorem shows the important fact that in the space $L^2(G, \mu)$ we have no other invariant operators besides those associated with the algebra of spectral resolutions of two-sided invariant operators.

In order to determine the generalized Fourier expansion for non-compact groups, in analogy to the compact case, we introduce an additional set of non-invariant operators in the carrier space H. To carry out this, we first construct the Gårding domain D_G for the elements of the enveloping algebras E^L and E^R of G.

Let $\{g_1, g_2\} \to T_{\{g_1, g_2\}}$ be a unitary representation of $G \times G$ in $H = L^2(G, \mu)$ given by

$$T_{\{g_1, g_2\}} u(g) = u(g_1^{-1} g g_2). \tag{4}$$

It is evident that $T^L_g = T_{\{g, e\}}$ and $T^R_g = T_{\{e, g\}}$. (5)

The Gårding subspace D_G associated with the representation $T_{\{g_1, g_2\}}$ consists of the elements $u(\varphi)$, $\varphi \in C_0^\infty(G\times G)$, of the form

$$u(\varphi) = \int\limits_{G\times G} \varphi(g_1, g_2) T_{\{g_1, g_2\}} u \, \mathrm{d}\mu(g_1) \mathrm{d}\mu(g_2), \; u \in H. \tag{6}$$

It provides a common, dense, linear, invariant domain for all operators in the enveloping algebra of $G\times G$. Thus by virtue of eq. (4) it provides also a common, dense, linear invariant domain for the left- and right-invariant enveloping algebras E^L and E^R of G respectively.

Let $\{A_j\}_{j=1}^m$ be a maximal set of self-adjoint mutually commuting operators in the right-invariant enveloping algebra E^L of G. Clearly all operators A_j commute with two-sided invariant operators C_i, $i = 1, 2, \ldots, n$. Let $\{B_k\}_{k=1}^m$ be the corresponding maximal set of self-adjoint mutually commuting operators in the left-invariant enveloping algebra E^R of G. The operators B_k commute with all C_i as well as with all A_j. Clearly we can select operators B_k in such a manner that the algebraic form of B_k is the same as that of A_k, $k = 1, 2, \ldots, m$. Then B_k and A_k are related by a unitary transformation. In fact, let J denote the involution operator in H defined by

$$(Ju)(g) \equiv u(g^{-1}). \tag{7}$$

This operator is unitary by virtue of the invariance of the Haar measure. Moreover,

$$JT_g^L J^{-1} = T_g^R. \tag{8}$$

Equation (8) implies

$$JX_i J^{-1} = Y_i, \tag{9}$$

where X_i and Y_i are generators of one-parameter subgroups $T^L_{g(t_i)}$ and $T^R_{g(t_i)}$, respectively. Next, if

$$M^R \equiv \sum a_{\alpha_1 \ldots \alpha_k} X_{\alpha_1} \ldots X_{\alpha_k} \in E^R$$

and

$$M^L \equiv \sum a_{\alpha_1 \ldots \alpha_k} Y_{\alpha_1} \ldots Y_{\alpha_k} \in E^L,$$

then we have also

$$JM^L J^{-1} = M^R. \tag{10}$$

This equation implies that the spectrum p_i of an operator A_i and the spectrum q_i of the corresponding operator B_i coincide. Consequently, the range of the set $p = \{p_1, \ldots, p_{l(s)}\}$ and $q = \{q_1, \ldots, q_{l(s)}\}$ is the same. In addition, if operators A_j are related with compact subgroups of G, then all indices p and q are discrete.

The sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ may often be associated with a specific sequence of successive maximal subgroups of G. For instance, if $G = \mathrm{SO}(p, 1)$, then the sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ may be associated with the set of Casimir operators of successive maximal subgroups $\mathrm{SO}(p) \supset \mathrm{SO}(p-1) \supset \ldots \supset \mathrm{SO}(2)$. If we take

$C_2(\mathrm{SO}(p, 1))$ and $C_2(\mathrm{SO}(p))$ to our system of commuting operators, then by virtue of corollary 2 to th. 11.2.3, the closures of all operators $\{C_i\}_1^{[p/2]}$, $\{A_j\}_1^m$ and $\{B_k\}_1^m$ are self-adjoint.

Because we have two sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ of operators, in addition to the set $\{C_i\}_1^n$ of two-sided invariant operators, we shall denote the eigenvectors by the symbol $e_{pq}(\lambda, g)$, where the multiindex $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is associated with the spectrum of two-sided-invariant operators, the multiindex $p = \{p_1, \ldots, p_m\}$ corresponds to the set of eigenvalues of operators $A_1, \ldots, A_m$, and $q = \{q_1, \ldots, q_m\}$ is the multiindex associated with a set of eigenvalues of operators $B_1, \ldots, B_m$. For instance, in the case of $G = \mathrm{SL}(2, C)$ one could take $G_1 = \mathrm{SU}(2)$ $G_2 = U(1)$; in this case $\lambda = (\varrho, m)$ are eigenvalues of two-sided-invariant operators C_2 and C_2' of $\mathrm{SL}(2, C)$, $p = \{J, M\}$ and $q = \{J', M'\}$.

We use in the following for the sake of simplicity a continuous direct sum notation for direct integral decomposition associated with the invariant operators $\bar{C}_i$ and the direct sum notation for decompositions associated with operators A_i and B_i, $i = 1, 2, \ldots, m$ although in general these operators might have continuous, discrete or mixed spectra.

The following theorem represents a generalization of the Peter–Weyl theorem for non-compact groups.

THEOREM 1. *Let G be a unimodular Lie group, $H = L^2(G, \mu)$, and let $g \to T_g$ be the regular representation of G in H given by eq. (4). Let $\{C_i\}_1^n$ be the maximal set of algebraically independent '+'-symmetric two-sided-invariant operators in the center Z of the enveloping algebra E, and $\{A_i\}_1^m$ and $\{B_i\}_1^m$ be the maximal set of right- and left-invariant, respectively, self-adjoint commuting operators in $E^L \oplus E^R$. Then*

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\,\mathrm{d}\varrho(\lambda), \qquad T_g \to \hat{T}_g = \int_\Lambda \hat{T}_g(\lambda)\,\mathrm{d}\varrho(\lambda) \tag{11}$$

of H and T_g such that $(\hat{H}(\lambda), \hat{T}_g(\lambda))$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e_{pq}(\lambda, g)$ in $H(\lambda)$ such that for $\varphi \in \Phi$ we have*

$$\langle C_i\varphi, e_{pq}(\lambda)\rangle = \hat{C}_i(\lambda_i)\langle\varphi, e_{pq}(\lambda)\rangle, \tag{12}$$

$$\langle A_j\varphi, e_{pq}(\lambda)\rangle = \hat{A}_j(p_j)\langle\varphi, e_{pq}(\lambda)\rangle, \tag{13}$$

$$\langle B_k\varphi, e_{pq}(\lambda)\rangle = \hat{B}_k(q_k)\langle\varphi, e_{pq}(\lambda)\rangle. \tag{14}$$

(iii) *For $\varphi \in \Phi$ the spectral synthesis formula has the form*

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{p,q=1}^{\dim\hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda) e_{pq}(\lambda, g), \tag{15}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \langle\varphi, e_{pq}(\lambda)\rangle. \tag{16}$$

(iv) *For $\varphi, \psi \in \Phi$ the Plancherel equality has the form*

$$\int_G \varphi(g)\overline{\psi(g)}\,\mathrm{d}\mu(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda)\overline{\hat{\psi}_{pq}(\lambda)}. \tag{17}$$

PROOF: Let $D_G \subset H$ be the Gårding domain for operators C_i, A_j and B_k, whose elements are given by eq. (6).

By virtue of th. 11.2.3 we conclude that the closures $\bar{C}_i$ of C_i are self-adjoint operators on D_G. Using then th. 11.5.3, we conclude that all C_i, $i = 1, 2, \ldots, n$, are mutually strongly commuting and also commute with all T_g, $g \in G$.

Now let A be a maximal '*'-algebra in the commutant T' of T which contains all spectral resolutions of $\bar{C}_i$, $i = 1, 2, \ldots, n$. Then using the Mautner theorem 5.6.1 we obtain the decomposition (11).

The assertions (ii)–(iv) follow from the nuclear spectral theorem.▼

The measure $\mathrm{d}\varrho(\lambda)$ on the spectral set Λ in eq. (11) is called the *Plancherel measure* and is determined by the spectral measure of the Casimir operators C_i.

Remark 1: By virtue of eq. app. B.3(27) eqs. (12)–(14) may be written in the form

$$\bar{C}_i' e_{pq}(\lambda, g) = \hat{C}_i(\lambda_i) e_{pq}(\lambda, g), \tag{18}$$
$$A_j' e_{pq}(\lambda, g) = \hat{A}_j(p_j) e_{pq}(\lambda, g), \tag{19}$$
$$B_k' e_{pq}(\lambda, g) = \hat{B}_k(q_k) e_{pq}(\lambda, g), \tag{20}$$

where, e.g., $\bar{C}_i'$ is the extension of $\bar{C}_i$ obtained by extending the domain $D(\bar{C}_i)$ by those elements φ' in Φ' for which the equality

$$\langle \bar{C}_i \varphi, \varphi' \rangle = \langle \varphi, \bar{C}_i' \varphi' \rangle, \quad \varphi \in \Phi$$

is satisfied.

Remark 2: In many cases the set Λ coincides with R^n or is a regular subset of it on which Lebesgue measure is well defined.

If the spectral measure $\mathrm{d}\varrho(\lambda)$ in the decomposition (11) is absolutely continuous relative to the Lebesgue measure $\mathrm{d}\lambda$ (i.e., $\mathrm{d}\varrho(\lambda) = \varrho(\lambda)\mathrm{d}\lambda$, $\varrho(\lambda)$ continuous on Λ), then eq. app. B.3(29) provides the following orthogonality relation for the generalized eigenvectors $e_{pq}(\lambda, g)$

$$\int_G e_{pq}(\lambda, g)\overline{e_{p'q'}(\lambda', g)}\,\mathrm{d}\mu(g) = \varrho^{-1}(\lambda)\,\delta(\lambda - \lambda')\,\delta_{pp'}\,\delta_{qq'}. \tag{21}$$

These formulas represent an alternative way of writing the Plancherel equality (17) and are understood as weak integrals of the distributions

$$e_{pq}(\lambda, g)\overline{e_{p'q'}(\lambda', g)} \quad \text{defined on } G.$$

Notice that a generalized Fourier component $\hat{\varphi}_{pq}(\lambda)$ of an element $\varphi \in \Phi$ given by eq. (16) cannot be represented in general as an integral of the form 1(5). However, we have

PROPOSITION 2. *Let the set $\{C_i\}_1^n$, $\{A_j\}_1^m$, $\{B_k\}_1^m$ of differential operators contain an elliptic operator. Then all eigenvectors are regular functions on G. For $\varphi \in \Phi$ we have*

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g)\overline{e_{pq}(\lambda, g)}\, \mathrm{d}\mu(g). \tag{22}$$

We have also the following completeness relation

$$\int \Big(\sum_{pq} e_{pq}(\lambda, g)\overline{e_{pq}(\lambda, g')}\Big)\, \mathrm{d}\mu(\lambda) = \delta(gg'^{-1}). \tag{23}$$

PROOF: Equation (22) and eq. (23) follow from assertion 3(22) and 3(24) of app. B respectively, of nuclear spectral theorem. The integral (23) is understood in the sense of the weak integral of the regular distributions.

Remark 1: If G is a semisimple Lie group which contains among its maximal subgroups a maximal compact subgroup K (like, e.g., $\mathrm{SO}(n, 1)$), $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$), then the assumption of proposition 2 can be easily satisfied. Indeed, it is sufficient to take in this case the sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ to be the operators associated with Casimir operators of the successive maximal subgroups $G_1 \supset G_2 \supset \ldots \supset G_s$, where $G_1 \doteq K$. Then, because the elliptic Nelson operator Δ satisfies the equality

$$\Delta(G) = \sum_{i=1}^{\dim G} X_i^2 = C_2(G) + 2C_2(K),$$

it is simultaneously diagonalized together with operators $C_2(G)$ and $C_2(K)$ which enter into the maximal set of commuting operators in $L^2(G, \mu)$. Hence for this class of groups the Fourier transform $\hat{\varphi}_{pq}(\lambda)$ and completeness relation have the explicit forms (22) and (23), respectively.▼

We assume in the following that the eigenfunctions $e_{pq}(\lambda, g)$ are regular functions (cf. proposition 2) and we normalize them in the following manner:

$$e_{pq}(\lambda, e) = \delta_{pq}, \tag{24}$$

where e is the unity of G. The following proposition shows that the eigenfunctions $e_{pq}(\lambda, g)$ are in fact matrix elements of irreducible representations. Indeed, we have

PROPOSITION 3. *The eigenfunctions $e_{pq}(\lambda, g)$ satisfy the following relations*:

$$e_{pq}(\lambda, g^{-1}) = \overline{e_{qp}(\lambda, g)}, \tag{25}$$

$$e_{pq}(\lambda, g_1 g_2) = e_{pr}(\lambda, g_1) e_{rq}(\lambda, g_2). \tag{26}$$

PROOF: Let us perform a 'rotation' in the space $H \doteq L^2(G, \mu)$ by means of the operator T_g^R. Then, because the operators $\overline{T(C_i)}$ are two-sided invariants and $\overline{T(A_j)}$ are right-invariant, we have

$$T_g^R \overline{C}_i T_{g^{-1}}^R = \overline{C}_i, \quad T_g^R A_j T_{g^{-1}}^R = A_j, \\ T_g^R B_k T_{g^{-1}}^R = \check{B}_k. \tag{27}$$

The new eigenfunctions in the 'rotated' system are

$$e_{pq}^{(g)}(\lambda, \tilde{g}) \equiv (T_g^R e)_{pq}(\lambda, \tilde{g}) = e_{pq}(\lambda, \tilde{g}g). \tag{28}$$

On the other hand, because the operators A_j are unchanged, we have

$$(T_g^R e)_{pq}(\lambda, \tilde{g}) = D_{q'q}^{\lambda}(g) e_{pq'}(\lambda, \tilde{g}), \tag{29}$$

where $D_{q'q}^{\lambda}(g)$ are the matrix elements of the operator T_g^R in the subspace $\hat{H}(\lambda)$. Clearly because $T_{g_1g_2}^R = T_{g_1}^R T_{g_2}^R$ and $T_{g^{-1}}^R = (T_g^R)^*$, the functions $D_{q'q}^{\lambda}(g)$ satisfy the conditions

$$D_{q'q}^{\lambda}(g_1g_2) = D_{q'r}^{\lambda}(g_1) D_{rq}^{\lambda}(g_2), \tag{30}$$

and

$$D_{q'q}^{\lambda}(g^{-1}) = \overline{D_{qq'}^{\lambda}(g)}. \tag{31}$$

Equations (28) and (29) imply

$$e_{pq}(\lambda, \tilde{g}g) = D_{q'q}^{\lambda}(g) e_{pq'}(\lambda, \tilde{g}). \tag{32}$$

Setting $\tilde{g} = e$ and utilizing the normalization condition (24), one obtains

$$e_{pq}(\lambda, g) = D_{pq}^{\lambda}(g). \tag{33}$$

The assertion of proposition 3 follows now from eqs. (30) and (31).▼

Following the current convention we shall use the symbol $D_{pq}^{\lambda}(g)$ for the generalized eigenvectors $e_{pq}(\lambda, q)$. If the set $\{\overline{T}(C_i)\}$, $\{\overline{T}(A_j)\}$ and $\{\overline{T}(B_k)\}$ of commuting operators in $H = L^2(G, \mu)$ satisfies the assumptions of remark 2 and proposition 2, then we have

$$\int_G D_{pq}^{\lambda}(g) D_{p'q'}^{\lambda'}(g) \mathrm{d}\mu(g) = \varrho^{-1}(\lambda)\, \delta(\lambda - \lambda')\, \delta_{pp'}\, \delta_{qq'}, \tag{34}$$

$$\int_{\Lambda} \mathrm{d}\varrho(\lambda) \sum_{pq} D_{pq}^{\lambda}(g) \overline{D_{pq}^{\lambda}(g')} = \delta(gg'^{-1}) \tag{35}$$

and

$$\varphi(g) = \int_{\Lambda} \sum_{pq} \hat{\varphi}_{pq}(\lambda) D_{pq}^{\lambda}(g) \mathrm{d}\varrho(\lambda), \tag{36}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g) \overline{D_{pq}^{\lambda}}(g) \mathrm{d}\mu(g). \tag{37}$$

Formulas (34)–(37) provide a generalization of the corresponding formulas for compact groups given in § 1, to the case of the unimodular Lie groups (satisfying the assumption of remark 2 and proposition 2).

It is interesting and very useful in applications that the spectral synthesis (15) and the Plancherel equality (17) can be put in an operator form. In fact, let $\varphi \in \Phi(G)$ and set

$$F(\lambda) \equiv \int_G \varphi(g) T_{g^{-1}}(\lambda) \mathrm{d}\mu(g), \tag{38}$$

where $\hat{T}_g(\lambda)$ is an irreducible component of $\hat{T}^R_g$ on the carrier space $\hat{H}(\lambda)$. Since $\hat{T}^R$ is a factor representation in $\hat{H}(\lambda)$ all irreducible components of it are equivalent. We have

PROPOSITION 4. *Let $D^\lambda_{pq}(g)$ be the eigenfunctions in $L^2(G)$, which satisfy assertions of proposition* 2. *Then we have*

(i) *The operator $F(\lambda)$ is a Hilbert–Schmidt operator in $\hat{H}(\lambda)$ for $\varrho-aa\lambda$.*

(ii) *The spectral synthesis formula* (15) *has the form*

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \mathrm{Tr}[F(\lambda) T_g(\lambda)]. \tag{39}$$

(iii) *The Plancherel equality* (17) *has the form*

$$(\varphi, \psi)_H = \int_\Lambda \mathrm{d}\varrho(\lambda) \mathrm{Tr}\big(F(\lambda) G^*(\lambda)\big), \qquad \varphi, \psi \in \Phi,$$

where $G(\lambda)$ is the $\hat{T}(\lambda)$-transform of ψ given by eq. (38).

PROOF: Let $\{\hat{e}_s(\lambda)\}$ be an orthonormal basis in $\hat{H}(\lambda)$; then by eq. (38), we have

$$\begin{aligned} F(\lambda)\hat{e}_s(\lambda) &= \int_G \varphi(g) D^\lambda_{qs}(g^{-1}) \hat{e}_q(\lambda) \mathrm{d}\mu(g) \\ &= \int_G \varphi(g) \overline{D}^\lambda_{sq}(g) \hat{e}_q(\lambda) \mathrm{d}\mu(g) = \hat{\varphi}_{sq}(\lambda) \hat{e}_q(\lambda). \end{aligned}$$

Hence, the square of the Hilbert–Schmidt norm $|F(\lambda)|$ of the operator $F(\lambda)$ is

$$\begin{aligned} |F(\lambda)|^2 &= \sum_{p=1}^{\dim \hat{H}(\lambda)} \|F(\lambda)\, \hat{e}_p(\lambda)\|^2_{\hat{H}(\lambda)} \\ &= \sum_{p,q,q'=1}^{\dim H(\lambda)} \hat{\varphi}_{pq}(\lambda) \overline{\hat{\varphi}_{pq'}(\lambda)} \big(\hat{e}_q(\lambda), \hat{e}_{q'}(\lambda)\big)_{\hat{H}(\lambda)} \\ &= \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda) \overline{\hat{\varphi}_{pq}(\lambda)} = \|\hat{\varphi}(\lambda)\|^2_{\hat{H}(\lambda)}. \end{aligned}$$

Since $\{\hat{\varphi}(\lambda)\} \in L^2(\Lambda)$, the Hilbert–Schmidt norm $|F(\lambda)|$ is ϱ-almost everywhere bounded and consequently $F(\lambda)$ is a ϱ-almost everywhere Hilbert–Schmidt operator.

The matrix elements of the operator $F(\lambda)$ are

$$F_{pq}(\lambda) = \big((F(\lambda)\hat{e}_q(\lambda), \hat{e}_p(\lambda)\big)_{\hat{H}(\lambda)}$$

$$= \int_G \varphi(g)\left((T_g(\lambda))^{-1}\hat{e}_q(\lambda),\, \hat{e}_p(\lambda)\right)_{\hat{H}(\lambda)} \mathrm{d}\mu(g)$$

$$= \int_G \varphi(g)\overline{\left(T_{g(\lambda)}\hat{e}_p(\lambda),\, \hat{e}_q(\lambda)\right)}_{\hat{H}(\lambda)} \mathrm{d}\mu(g)$$

$$= \int_G \varphi(g)\overline{D^{\lambda}_{pq}(g)}\,\mathrm{d}\mu(g) \tag{40}$$

Hence, $F_{pq}(\lambda) = \hat{\varphi}_{qp}(\lambda)$ and thus, the spectral synthesis (15) of φ in $\Phi(G)$ can be written in the form

$$\varphi(g) = \int_{\Lambda} \sum_{pq=1}^{\dim H(\lambda)} F_{qp}(\lambda) D^{\lambda}_{pq}(g)\,\mathrm{d}\varrho(\lambda)$$

$$= \int_{\Lambda} \mathrm{Tr}\left(F(\lambda) T_g(\lambda)\right) \mathrm{d}\varrho(\lambda). \tag{41}$$

Here, the integration runs over the set of unitary irreducible representations on which the Plancherel measure $\varrho(\cdot)$ does not vanish.

The Plancherel equality (17) can now be put in the form:

$$(\varphi, \psi)_H = (\hat{\varphi}, \hat{\psi})_{\hat{H}} = \int_{\Lambda} \sum_{pq} F_{qp}(\lambda)\overline{G_{qp}(\lambda)}\,\mathrm{d}\varrho(\lambda)$$

$$= \int_{\Lambda} \mathrm{Tr}\{F(\lambda) G^*(\lambda)\}\,\mathrm{d}\varrho(\lambda). \blacktriangledown \tag{42}$$

§ 3. Harmonic Analysis on Semidirect Product of Groups

The general theory of harmonic analysis on unimodular Lie groups embraces also the case of unimodular semidirect products $G = N \rtimes G_A$, where G_A is the group of automorphisms of N. We restrict ourselves in this section to the presentations of general theory for the two most important semidirect products, namely the Euclidean groups $E_n = T^n \rtimes \mathrm{SO}(n)$ and the generalized Poincaré groups $\Pi_n = T^n \rtimes \mathrm{SO}(n-1, 1)$, $n = 2, 3, ..$ One readily verifies using th. 3.10.5 that all groups E_n and Π_n are unimodular.

We shall first describe explicitly the maximal set of commuting differential operators in the space $H = L^2(G, \mu)$. We know that the set $\{C_i\}$ of algebraically independent invariant operators of E_n or Π_n consists of $\left\{\frac{n}{2}\right\}$ operators (see 9.7.3.1). We shall now determine the additional operators which together with the set $\{C_i\}$ will provide a maximal set of algebraically independent commuting operators.

Let

$$\mathrm{SO}(n) \supset \mathrm{SO}(n-1) \supset \ldots \supset \mathrm{SO}(2), \tag{1}$$

$$\mathrm{SO}(n-1, 1) \supset \mathrm{SO}(n-1) \supset \ldots \supset \mathrm{SO}(2) \tag{2}$$

be a sequence of successive maximal subgroups of SO(n) and SO($n-1$,1), respectively. Let $\{A_j\}_1^m$ be the maximal set of '+'-symmetric, algebraically independent Casimir operators in the enveloping algebra E^L of G, associated with successive subgroups in the sequence (1) or (2). Let $\{B_k\}_1^m$ be the corresponding maximal set of '+'-symmetric, algebraically independent operators in the enveloping algebra E^R of G.

One readily verifies that $m = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \dim G_A\right)$. Because $\left[\frac{n+1}{2}\right] + 2m = \dim G$, the sets $\{C_i\}_1^{\left[\frac{n+1}{2}\right]}$, $\{A_j\}_1^m$ and $\{B_k\}_1^m$ provide the maximal set of independent commuting operators in the carrier space $H = L^2(G, \mu)$.

The basic features of harmonic analysis on E_n or Π_n are described by the following theorem.

THEOREM 1. *Let G_n be the group E_n or Π_n, $n = 2, 3, \ldots$, and let $g \to T_g$ be the regular representation of G_n in the Hilbert space $H = L^2(G, \mu)$, given by eq.* 2(4).

Let $\{C_i\}_1^{[(n+1)/2]}$, be the sequence of two-sided invariant operators of G_n in H and let $\{A_j\}_1^m$ and $\{B_k\}_1^m$ be the maximal sets of independent '+'-symmetric Casimir operators associated with the sequence of subgroups (1) *or* (2), *respectively. Then*

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\mathrm{d}\varrho(\lambda), \qquad T_g \to \hat{T}_g = \int \hat{T}_g(\lambda)\mathrm{d}\varrho(\lambda) \tag{3}$$

of H and T_g such that $(\hat{H}(\lambda), \hat{T}_g(\lambda))$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e_{pq}(\lambda, g)$ such that the relations* 2(12)–2(14) *hold.*

(iii) *For $\varphi \in \Phi$ the spectral synthesis formula has the form*

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda) e_{pq}(\lambda, g), \tag{4}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g)\overline{e_{pq}(\lambda, g)}\mathrm{d}\mu(g). \tag{5}$$

(iv) *For $\varphi, \psi \in \Phi$ the Plancherel equality has the form* 2(17).

PROOF: The proofs of all assertions of Theorem 1, except eq. (5), are parallel to those of 2.1, and we omit them. The relation (5) is implied by the fact that, e.g., for E_n, together with the operators $C_2(\mathrm{SO}(n))$ and $M^2 = P_\mu P^\mu$, the Nelson operator $\Delta = P_\mu P_\mu + C_2(\mathrm{SO}(n))$ is also diagonalized. Because Δ is elliptic, the formula (5) follows from proposition 2.2.▼

Clearly propositions 2.3 and 2.4 are also valid for the groups E_n and Π_n.

We shall now consider an example which clearly illustrates the main features of harmonic analysis on non-compact, non-commutative Lie groups.

EXAMPLE 1. Let $G = E_2 = T^2 \rtimes SO(2)$. If $x = (x_1, x_2) \in T^2$ and $\alpha \in SO(2)$, $0 \leqslant \alpha < 2\pi$ then the composition law in G is given by the formula

$$(x, \alpha)(x', \alpha') = (x+x'_\alpha, \alpha+\alpha'), \tag{6}$$

where

$$x'_\alpha = (x'_1 \cos\alpha - x'_2 \sin\alpha,\ x'_1 \sin\alpha + x'_2 \cos\alpha).$$

The composition law (6) implies

$$(x, \alpha)^{-1} = (-x_{-\alpha}, 2\pi-\alpha). \tag{7}$$

One readily verifies that the invariant measure on G has the form

$$d\mu[(x, \alpha)] = dx_1 dx_2 d\alpha. \tag{8}$$

Let $H = L^2(G, \mu)$. The right and left regular representations T^R and T^L of G in H are given by the formulas

$$(T^R_{(x',\alpha')}u)[(x, \alpha)] = u[(x, \alpha)(x', \alpha')] = u[(x+x'_\alpha, \alpha+\alpha')] \tag{9}$$

and

$$(T^L_{(x',\alpha')}u)[(x, \alpha)] = u[(x', \alpha')^{-1}(x, \alpha)] = u[(-x'_{-\alpha'}+x_{2\pi-\alpha'}, 2\pi-\alpha'+\alpha)]. \tag{10}$$

The generators X_i, $i = 1, 2, 3$, of the one-parameter subgroups $g(t_i)$ associated with the left regular representation (i.e., belonging to the right-invariant Lie algebra) are given by the formula

$$X_i u = \lim_{t\to 0} \left(\frac{T^L_{g(t_i)} - I}{t_i} u\right), \tag{11}$$

where u is an element of the Gårding domain. Using eqs. (10) and (11), one obtains

$$X_1 = -\frac{\partial}{\partial x_1}, \quad X_2 = -\frac{\partial}{\partial x_2}, \quad X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial \alpha}. \tag{12}$$

Similarly, using the formula

$$Y_i u = \lim_{t_i\to 0} \left(\frac{T^R_{g(t_i)} - I}{t} u\right) \tag{13}$$

and eq. (9), one obtains the following expressions for the generators of the left-invariant Lie algebra of G

$$\begin{aligned} Y_1 &= \cos\alpha \frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_2}, \\ Y_2 &= -\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2}, \\ Y_3 &= \frac{\partial}{\partial \alpha}. \end{aligned} \tag{14}$$

Clearly, each operator X_i commutes with all operators Y_k, $i, k = 1, 2, 3$. The left- and the right-invariant Lie algebras satisfy the commutation relations of the form

$$[Z_1, Z_2] = 0, \quad [Z_2, Z_3] = Z_1, \quad [Z_3, Z_1] = Z_2. \tag{15}$$

One readily verifies that the invariant operator has the form

$$C_1 = Z_1^2 + Z_2^2. \tag{16}$$

Hence,

$$C_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{17}$$

or, in the spherical coordinates,

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi,$$
$$C_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}. \tag{18}$$

Using the general method prescribed in th. 1 for the selection of the maximal set of commuting operators, one then obtains

$$C_1, \quad A_1 = X_3 = -\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\alpha}, \quad B_1 = Y_3 = \frac{\partial}{\partial\alpha}. \tag{19}$$

This is the maximal set of independent, commuting, differential operators, because the dimension of G equals three.

Now, we shall find an explicit form of the eigenfunctions $e_{pq}(\lambda, g) \equiv D^\lambda_{pq}(g)$. The expressions (18) and (19) suggest that we should look for common eigenfunctions of the operators C_1, A_1 and B_1 of the form

$$D^\lambda_{pq}(\varphi, r, \alpha) = \exp(\mathrm{i}p\varphi)\mathrm{d}^\lambda_{pq}(r)\exp[-\mathrm{i}q(\alpha+\varphi)]. \tag{20}$$

These are the eigenfunctions of the operators X_3 and Y_3, while $C_1 D^\lambda_{pq} = \hat{C}_1(\lambda) D^\lambda_{pq}$ provides the following equation for $d^\lambda_{pq}(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(p-q)^2}{r^2}\right) d^\lambda_{pq}(r) = \hat{C}_1(\lambda) d^\lambda_{pq}(r). \tag{21}$$

This is the classical form of the Bessel equation, whose (regular) solution is

$$d^\lambda_{pq}(r) = i^{p-q} J_{p-q}\left(-i\sqrt{\hat{C}_1(\lambda)}\,r\right). \tag{22}$$

The operator C_1 is skew-adjoint (by the Nelson–Stinespring theorem) and negative definite by virtue of eq. (17). Hence, the eigenvalues $\hat{C}_1(\lambda)$ of C_1 are negative. Setting $\hat{C}_1(\lambda) = -\lambda^2$, $\lambda \in R$, one obtains by eqs. (22) and (20)

$$D^\lambda_{pq}(\varphi, r, \alpha) = i^{p-q}\exp[\mathrm{i}(p-q)\varphi] J_{p-q}(\lambda r)\exp[\mathrm{i}q\alpha]. \tag{23}$$

It is well known that the spectral measures $\mathrm{d}\varrho(\lambda)$ for the Bessel equation has the form $d\varrho(\lambda) = \lambda d\lambda$. Hence, the orthogonality and completeness relations for the functions $D^{\lambda}_{pq}(\varphi, r, \alpha)$ take the form

$$\int D^{\lambda}_{pq}(\varphi, r, \alpha)\overline{D^{\lambda'}_{p'q'}(\varphi, r, \alpha)}\, r dr d\varphi d\alpha = \delta_{p'p}\delta_{q'q}\frac{\delta(\lambda-\lambda')}{\lambda}, \tag{24}$$

$$\int_{-\infty}^{+\infty} d\lambda \sum_{pq} D^{\lambda}_{pq}(\varphi r \alpha)\overline{D^{\lambda}_{pq}(\varphi' r' \alpha')} = \delta(\varphi-\varphi')\frac{1}{r}\delta(r-r')\delta(\alpha-\alpha'). \tag{25}$$

Because $J_{p-q}(0) = \delta_{pq}$, the functions $D^{\lambda}_{pq}(e)$ satisfy the normalization condition 2 (24), i.e., $D_{pq}(e) = \delta_{pq}$, where $e = (0, 0, 0)$ is the unity of G. Therefore, the functions $D^{\lambda}_{pq}(g)$ satisfy the unitarity condition

$$D^{\lambda}_{pq}(g^{-1}) = \overline{D^{\lambda}_{qp}(g)} \tag{26}$$

and the composition law

$$\sum_{s} D^{\lambda}_{ps}(g_1)D^{\lambda}_{sq}(g_2) = D^{\lambda}_{pq}(g_1 g_2). \tag{27}$$

Notice that the last formula allows us to derive various composition laws for Bessel functions. One readily verifies that if $g_1 = (0, r_1, 0)$ and $g_2 = (\varphi_2, r_2, 0)$, then, $g_1 \cdot g_2 = (\varphi, r, \alpha)$ is given by the relations

$$r = [r_1^2+r_2^2+2r_1 r_2 \cos\varphi_2]^{1/2}, \quad \exp(i\varphi) = \frac{r_1+r_2\exp(i\varphi_2)}{r}, \quad \alpha = 0. \tag{28}$$

Then, for $\lambda = 1$, eq. (27) gives

$$\sum_{k=-\infty}^{\infty} \exp(ik\varphi_2) J_{n-k}(r_1) J_k(r_2) = \exp(in\varphi) J_n(r). \tag{29}$$

In particular, for $\varphi_2 = 0$, we have $r = r_1 + r_2$ and eq. (29) gives

$$\sum_{k=-\infty}^{\infty} J_{n-k}(r_1) J_k(r_2) = J_n(r_1+r_2) \tag{30}$$

and, for $\varphi_2 = \pi$, $r_1 \geqslant r_2$ we have $r = r_1 - r_2$, and $\varphi = 0$. Hence eq. (29) gives

$$\sum_{k=-\infty}^{\infty} (-1)^k J_{n-k}(r_1) J_k(r_2) = J_n(r_1-r_2). \tag{31}$$

§ 4. Comments and Supplements

A. *Plancherel Measure*

The first significant general result of harmonic analysis on separable unimodular groups was obtained by Segal 1950. With each $\varphi \in H = L^2(G, \mu)$, he associated an operator

$$T(\varphi) = \int \varphi(g) T_g \mathrm{d}\mu(g), \tag{1}$$

where T_g is a unitary representation of G.

The Mautner theorem implies that $T(\varphi)$ has the following direct integral decomposition

$$T(\varphi) \to \hat{T}(\varphi) = \int \hat{F}(\lambda) \mathrm{d}\varrho(\lambda).$$

Segal calls $\{\hat{F}(\lambda)\}$ a Fourier transform of φ and proves the following Plancherel Theorem

$$\int_G |\varphi(g)|^2 \mathrm{d}\mu(g) = \int_\Lambda \langle \hat{F}(\lambda), \hat{F}(\lambda) \rangle_\lambda a(\lambda) \mathrm{d}\varrho(\lambda), \tag{2}$$

where $\langle \cdot, \cdot \rangle_\lambda$ denotes the norm square in the Banach space of bounded operators in the space $\hat{H}(\lambda)$ and $a(\lambda)$ is a positive, ϱ-measurable function.

(For the proof see Segal 1950, th. 3.)

It is crucial to know for applications the explicit form of the Plancherel measure. This problem was solved for classical complex Lie groups by Gel'fand and Naimark 1950 (see also Gel'fand 1963 for simplified derivation). For instance in the case of SL(n, C) group the irreducible unitary representations $\hat{T}(\lambda)$ are labelled by the multiindex $\lambda = (m_2, \ldots, m_n, \varrho_2, \ldots, \varrho_n)$ where m_i are integers and ϱ_i—real numbers. (cf. ch. 19, § 3).

The explicit form of Plancherel measure in these variables is

$$\mathrm{d}\varrho(\lambda) = c \prod_{1 \leqslant p \leqslant q \leqslant n} [(\varrho_p - \varrho_q)^2 + (m_p - m_q)^2], \quad \varrho_1 = m_1 = 0, \tag{3}$$

where $c = 2^{n/2(n-1)}[n!(2\pi)^{(n-1)(n+2)}]^{-1}$ and m_i, ϱ_i, $i = 1, 2, \ldots, n$, are invariant numbers which characterize an irreducible unitary representation of SL(n, C) The Plancherel equality is

$$\int \varphi(g) \overline{\psi}(g) \mathrm{d}\mu(g) = \int \mathrm{Tr}\left(\hat{F}(\lambda) \hat{G}^*(\lambda)\right) \mathrm{d}\varrho(\lambda),$$

where

$$\hat{F}(\lambda) = \int \varphi(g) \hat{T}_g(\lambda) \mathrm{d}\mu(g).$$

The Plancherel measure is zero on the supplementary series of SL(n, C). The generalization of Gel'fand–Naimark results to arbitrary connected semisimple Lie groups was given by Harish–Chandra in a series of papers (cf. 1954, and 1970).

The explicit form of the Plancherel measure for some real groups was found recently. In particular, Hirai found the explicit form of the Plancherel measure for Lorentz type groups SO$(n, 1)$ 1966 and for SU(p, q) groups 1970. Romm has found Plancherel measure for group SL(n, R) 1965. Recently Leznov and Savelev gave an explicit form of the Plancherel measure for all real forms of complex classical groups 1970 for principal nondegenerate series. (See also 1971 for SU(p, q) groups). However, their derivation contains some assumptions whose validity was not verified.

B. *Comments*

The harmonic analysis on unimodular groups presented in secs. 2 and 3 is based on lecture notes given by Rączka 1969. This presentation based on the nuclear spectral theory provides the most natural extension to non-compact noncommutative groups of the classical Fourier theory for abelian groups and of Peter–Weyl theory for compact groups. It provides also a convenient framework for applications in quantum theory, where the concept of generalized eigenfunctions plays a central role. A series of fundamental problems of harmonic analysis for semisimple Lie groups were solved by Harish–Chandra 1954, 1965, 1966 and review article 1970. The monumental works of Harish–Chandra are presented in two-volume monograph elaborated by Warner 1972. An interesting treatment of various problems of harmonic analysis on semisimple complex Lie groups is presented in the recent monograph of Želobenko 1974. It contains in particular an excellent reviev of achievements of Russian mathematicians in this domain.

Example 3.1 demonstrates the usefulness of harmonic analysis on groups for the analysis of properties of special functions. This subject is extensively treated in monographs of Vilenkin 1968, Miller 1968 and lecture notes of Wigner elaborated by Talman 1968.

The harmonic analysis on the Lorentz group SL(2, C) is extensively treated in the monograph of Gel'fand and Vilenkin, v. 5, 1966, and in the recent book by Rühl 1970 which also contains interesting applications in particle physics. The harmonic analysis on Poincaré group was treated by Rideau 1966 and by Nghiem Zuan Hai in the doctoral dissertation (Orsay 1969).

A very elegant and general treatment of harmonic analysis on locally compact groups based on the C^*-algebra theory is presented in the monograph of Dixmier 1969 (see also Naimark 1970).

Chapter 15

Harmonic Analysis on Homogeneous Spaces

The harmonic analysis on homogeneous spaces is another one of the most important but difficult parts of group representation theory. The degree of difficulty is well illustrated by the classical treatise *Abstract Harmonic Analysis* by Hewitt and Ross, where the concept of harmonic analysis appears only after some 1065 pages of 'introductory material'.

We shall first state the basic problems in harmonic analysis.

Let X be a homogeneous space, G a locally compact transformation group on X, and K the stability subgroup of G. Let $d\mu(x)$ be a quasi-invariant measure on X provided by the Mackey Theorem (ch. 4.3.1) and let $H = L^2(X, \mu)$. The map $g \to T_g$ given by

$$T_g u(x) = \sqrt{\frac{d\mu(xg)}{d\mu(x)}}\, u(xg) \tag{1}$$

provides a unitary representation T of G in H.

The two basic problems of harmonic analysis are the following:

(i) *Spectral analysis*: The decomposition of the representation (1) and of the carrier space H onto the direct integrals

$$T_g \to \hat{T}_g = \int_\Lambda \hat{T}_g(\lambda)\, d\varrho(\lambda), \qquad H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\, d\varrho(\lambda) \tag{2}$$

of irreducible representations $\hat{T}_g(\lambda)$ of G in $\hat{H}(\lambda)$ and the determination of the spectrum.

(ii) *Spectral synthesis*: The determination of a dense subspace $\Phi \subset H$ such that for every $\varphi \in \Phi$ we have:

$$\varphi(x) = \int_\Lambda d\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x)\, d\varrho(\lambda), \tag{3}$$

where $\{e_k(\lambda, x\}_{k=1}^{\dim H(\lambda)}$ is a basis in $H'(\lambda)$ and

$$\hat{\varphi}_k(\lambda) = \langle \varphi, e_k(\lambda) \rangle \tag{4}$$

is a component of $\varphi \in H$ in $\hat{H}(\lambda)$.

* Note that $\hat{H}$ (λ) and H' $(\lambda) \subset \Phi'$ are isomorphic but different; cf. app. B.3(30).

§ 1. Invariant Operators on Homogeneous Spaces

Associated with the problem of spectral analysis is the problem of finding the maximal set $\{C_i\}_1^n$ of independent invariant commuting operators. Contrary to a common belief among physicists, the set $\{C_i\}_1^n$ might contain more invariant operators than those obtained from the center Z of the enveloping algebra E of G. In fact, let $N(K)$ be a nontrivial normalizer in G of the stability subgroup K of X, i.e., the set of all $n \in G$ such that $nKn^{-1} \subset K$. Then using the correspondence $x_g \to Kg$ between elements of the space X and the cosets Kg we obtain

$$nx_g \sim nKg = Kng = x_{ng}.$$

This implies that the left translations of X by elements of N and the right translations of X by elements of G commute. Hence

$$(T_n^L T_g u)(x) = (T_g T_n^L u)(x). \tag{5}$$

Consequently, if the quotient group $N(K)/K$ is nontrivial, then the maximal set of operators associated with the group $N(K)/K$ provides an additional set of invariant operators besides those from the center Z of the enveloping algebra E of G. If $N(K)/K$ is non-commutative, the additional set of invariant differential operators associated with $N(K)/K$ is also non-commutative.

Let us note that a Lie group might have invariant operators which are not elements of the enveloping algebra nor even differential operators; for instance, in the case of the Poincaré group in addition to the mass square operator $P_\mu P^\mu$ $(\sim m^2 = p_\mu p^\mu)$ and the square of the spin operator $W_\mu W^\mu$ $(\sim m^2 J(J+1))$ which are differential operators in the center $Z(E)$, we have the invariant operator $Q = \operatorname{sign} p_0$, where p_0 is the eigenvalue of the generator P_0 in the carrier space. The operator Q is neither an element of $Z(E)$ nor it is a differential operator.

EXAMPLE 1. Let G be the Poincaré group $T^4 \otimes \mathrm{SL}(2, C)$ and let

$$K = T^4 \otimes Z, \tag{6}$$

where Z is the two-dimensional nilpotent group consisting of all complex matrices of the form

$$z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, \quad z \in C^1. \tag{7}$$

Hence $X = G/K$ is a four-dimensional homogeneous space. The subgroup $S = ZD$, where

$$D = \left\{ \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}, \ \delta \in C^1 \right\},$$

has Z as its normal subgroup by virtue of Gauss decomposition 3.6(3) for $\mathrm{SL}(2, C)$. Hence the normalizer $N(K)$ for K is the group

$$N(K) = T^4 \otimes S. \tag{8}$$

Thus

$$N(K)/K = D. \tag{9}$$

Consequently, in the space $H = L^2(X, \mu)$, $X = G/K$, there will be two additional invariant operators associated with the generators of D, in addition to mass square $P_\mu P^\mu$ and spin-square $W_\mu W^\mu$ operators which are generators of the center Z of the enveloping algebra E of the Poincaré group. ▼

In the case of arbitrary Lie groups G and $K \subset G$ the problem of finding the normalizer $N(K)$ of K in G is unsolved. Hence, we do not have a general characterization of the set $\{C_i\}_1^n$ of independent invariant operators in the space $H = L^2(X, \mu)$, $X = G/K$. However, for more special groups G and K, there is a more specific characterization of the set $\{C_i\}_1^n$. For intance, if G is a semisimple connected Lie group and K is the maximal compact subgroup of G then it follows from the Cartan decomposition of G: $G = KP$ that there is no subgroup G_0 between G and K, of dimension greater than the dimension of K. Hence $N(K)/K$ is at most discrete. This implies that there are no additional invariant differential operators in $\{C_i\}_1^n$ which come from $N(K)/K$.

One should also remark the following connection between the properties of homogeneous spaces and the properties of the set $\{C_i\}_1^n$ of invariant operators: if the stability group K of X is small, then the number of invariant operators in $\{C_i\}$ is large and may contain invariant operators even from outside of the enveloping algebra; conversely, if the stability subgroup K becomes bigger, there are more constraints in X and the number of invariant operators decreases; in this case even those invariant operators which were algebraically independent in the center Z of E become dependent as the differential operators on the space $H = L^2(X, \mu)$. The following example will illustrate this point.

EXAMPLE 2. Let G be the Euclidean group $T^n \circledS \mathrm{SO}(n)$. In general G has $[(n+1)/2]$ algebraically independent operators in the center Z of E and $X = G/K \sim R^n$. We shall show that the center Z of E in $L^2(X, dx)$ is generated by a single operator. Indeed, let $C_i(x_k, \partial/\partial x_l\}$ be differential operators in Z. Because C_i must be invariant under all translations $t \in T^n$, the differential operators $C_i(x_k, \partial/\partial x_l)$ have constant coefficients. Consequently, $C_i = P_i(\partial_1, \ldots, \partial_n)$, where P_i is a polynomial. Now the only rotational invariant quantity in R^n is the radius $r = |x| = \sqrt{\sum x_i^2}$. Hence the polynomials P_i must be functions of $\partial_1^2 + \ldots + \partial_n^2 \equiv \Delta$. Hence, an arbitrary invariant differential operator in $L^2(X, dx)$ from the enveloping algebra E of G has the form

$$C = \sum_{l=1}^{s} C_l \Delta^l. \tag{10}$$

Clearly, if $G = T^n \circledS \mathrm{SO}(p, q)$, $p+q = n$, then an arbitrary invariant operator in $L^2(X, dx)$ from the enveloping algebra E has also the form (10) with

$$\Delta = \partial_1^2 + \ldots + \partial_p^2 - \partial_{p+1}^2 - \ldots - \partial_{p+q}^2 . \blacktriangledown \tag{11}$$

The following theorem gives the description of the set of invariant operators for the symmetric space $X = G/K$.

THEOREM 1. *Let $X = G/K$ be a symmetric space of rank l. Then the algebra of all G-invariant differential operators in the space $H = L^2(X, \mu)$ is a commutative algebra with l algebraically independent generators.*

If X is of rank one, then every invariant differential operator C is a polynomial in the second order Casimir operator of G which in a proper coordinate system on X is equal to the Laplace–Beltrami operator

$$\Delta = \bar{g}^{-1/2}\,\partial_\alpha\, g^{\alpha\beta}\sqrt{\bar{g}}\,\partial_\beta \tag{12}$$

where $g^{\alpha\beta}(x)$ is the invariant metric tensor on the space X and $\bar{g} \equiv |\det g|$.

(For the proof cf. Helgason 1962, ch. 10).

§ 2. Harmonic Analysis on Homogeneous Spaces

We shall now elaborate on the harmonic analysis on general homogeneous spaces $X = G/K$ where G is a connected Lie group and K a closed subgroup of G. The following theorem provides a general solution of the basic problems of harmonic analysis on homogeneous spaces.

THEOREM 1. *Let $H = L^2(X, \mu)$ where μ is a quasi-invariant measure on X and let $g \to T_g$ be a unitary representation of G in $H = L^2(X, \mu)$ given by eq.* 1(1).

Let $\{T(C_i)\}_1^n$ be the maximal set of independent G-invariant operators in the representation $T(E)$ of the enveloping algebra E of G. Let $Q = \{A_j\}_1^m$ be the maximal set of commuting self-adjoint (non-invariant) operators in $T(E)$. Then

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\mathrm{d}\varrho(\lambda), \qquad T_g \to \hat{T}_g = \int \hat{T}_g(\lambda)\mathrm{d}\varrho(\lambda) \tag{1}$$

of H and T_g such that $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $\{e_k(\lambda, x)\}$ in $H(\lambda)$ such that*

$$\langle \overline{T(C_i)}\varphi, e_k(\lambda)\rangle = \hat{C}_i(\lambda)\langle\varphi, e_k(\lambda)\rangle, \quad i = 1, \ldots, n, \tag{2}$$

$$\langle A_j\varphi, e_k(\lambda)\rangle = \hat{A}_j(k)\langle\varphi, e_k(\lambda)\rangle, \quad j = 1, \ldots, m. \tag{3}$$

(iii) *For $\varphi \in \Phi(X)$ the spectral synthesis formula is*

$$\varphi(x) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x), \tag{4}$$

where

$$\hat{\varphi}_k(\lambda) = \langle\varphi, e_k(\lambda)\rangle.$$

(iv) *For $\varphi, \psi \in \Phi(X)$ the Parseval equality is given by*

$$\int \varphi(x)\overline{\psi}(x)\mathrm{d}\mu(x) = \int_{A} \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)\overline{\hat{\psi}_k(\lambda)}. \tag{6}$$

PROOF: We shall first construct the Gel'fand triplet $\Phi(X) \subset H \subset \Phi'(X)$. Let $u_1 \in H$ and let $A_{u_1} \equiv \{\psi \in D(G)\colon T(\psi)u_1 = 0\}$, where $T(\psi) = \int_G \psi(g)T_g \mathrm{d}g$. If $\psi_n \to \psi$ in $D(G)$, then $T(\psi_n)u_1 \to T(\psi)u_1$. Thus, if $\psi_n \in A_{u_1}$, then $\lim \psi_n = \psi \in A_{u_1}$, i.e., A_{u_1} is closed in $D(G)$. Hence the quotient space $\tilde{D}(G) \equiv D(G)/A_{u_1}$ is nuclear. Take $u_1 \in H$ and set $H_{u_1} \equiv \{T(\psi)u_1, \psi \in D(G)\}$ and equip H_{u_1} with the nuclear topology of the space $\tilde{D}(G)$. If the space H_{u_1} is not dense in H, we take $u_2 \perp H_{u_1}$ and form the topological direct sum $H_{u_1} \oplus H_{u_2}$, and so on until we reach a dense space $\bigoplus_k H_{u_k}$ in H. Because H is separable, this is always possible. The space $\Phi \equiv \bigoplus_k H_{u_k}$ is a countable sum of nuclear spaces and it is therefore itself nuclear. The natural embedding H_{u_k} in H is continuous; in fact,

$$||T(\psi)u_k|| = ||\int \psi(g)T_g u_k \mathrm{d}g|| \leqslant |[\psi]_p|||u_k||,$$

where $|[\psi]_p|$ is a seminorm in $\tilde{D}(G)$ induced by the Schwarz norm p in $D(G)$. Consequently, the imbedding $I\colon \Phi \to H$ is also continuous.

For an element M in the enveloping algebra E of G, we obtain by virtue of eq. 11.1 (17), $T(M)\Phi \subset \Phi$. By the same equation, for every $M \in E$, $T(M)$ is a continuous operator of Φ.

We shall now prove the direct integral decomposition (1). Let C_i, $i = 1, 2, \ldots, n$, be the set of algebraically independent symmetric elements in the center $Z(E)$ of E (i.e., $C_i^+ = C_i$). By virtue of corollary 3 to th. 11.2.3, we conclude that the closures $\overline{T(C_i)}$ of $T(C_i)$ are self-adjoint operators. Using then th. 11.5.3, we conclude that all $\overline{T(C_g)}$, $i = 1, 2, \ldots, n$, are mutually strongly commuting and also commute with all $T_g, g \in G$.

Let A be a maximal '*'-algebra in the commutant T' of T, which contains all spectral resolutions of $\overline{T(C_i)}$, $i = 1, 2, \ldots, n$. Then using the Mautner Theorem 5.6.1 we obtain the decomposition (1). The spectral set Λ contains as a subset the spectra Λ_{C_i} of the self-adjoint operators $\overline{T(C_i)}$, $i = 1, 2, \ldots, n$.

Let $\{e_k(\lambda, x\}$ be the common generalized eigenvectors of the operators $\overline{T(C_i)}$ and $\{A_j\}_1^m$ provided by the nuclear spectral theorem. Then eq. (2) and (3), spectral synthesis formula (4), and Parseval equality (5) follow from assertions 3(23), 3(25) and 3(24) of app. B.▼

Remark 1. By virtue of eq. app. B3(27), eqs. (2) and (3) can be written in the form

$$\overline{T(C_i)}' e_k(\lambda, x) = \hat{C}_i(\lambda)e_k(\lambda, x) \tag{7}$$

and

$$A_j' e_k(\lambda, x) = \hat{A}_j(k) e_k(\lambda, x), \tag{8}$$

where, e.g., A_j' denotes the extension of A_j obtained by extending the domain $D(A_j)$ by those elements $\varphi' \in \Phi'$ for which the equality

$$\langle A_j \varphi, \varphi' \rangle = \langle \varphi, A_j' \varphi' \rangle, \quad \varphi \in \Phi, \quad \varphi' \in \Phi'$$

is satisfied.▼

For the sake of simplicity we shall use a notation as though all Casimir operators would have purely continuous spectrum and all noninvariant operators A_j would have purely discrete spectrum.

PROPOSITION 2. *Let the set $\{T(C_i)\}_1^n$ or $\{A_j\}_1^m$ contain an elliptic differential operator. Then all eigenvectors $e_k(\lambda, x)$ are regular functions on X. For $\varphi \in \Phi$ we have*

$$\hat{\varphi}_k(\lambda) = \int_X \varphi(x) \overline{e_k(\lambda, x)} \mathrm{d}\mu(x). \tag{9}$$

We have also the following completeness relation

$$\int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} e_k(\lambda, x) \overline{e_k(\lambda, x')} = \delta(x - x'). \tag{10}$$

The proof of proposition 2 is similar to that of proposition 14.2.2 and we omit it. The integral (10) is understood as weak integral of regular distributions $e_k(\lambda, x) e_k(\lambda, x')$ on $X \times X$ and is essentially equivalent to Parseval equality.

Finally, if, in addition, all operators in $\{T(C_i)\}_1^n$ have spectra which are absolutely continuous with respect to the Lebesgue measure (i.e. $\mathrm{d}\varrho(\lambda) = \varrho(\lambda)\mathrm{d}\lambda$), then we have the following orthogonality relation

$$\int_X e_k(\lambda, x) \overline{e_{k'}(\lambda', x)} \mathrm{d}\mu(x) = \varrho(\lambda)^{-1} \delta(\lambda - \lambda') \delta_{kk'}. \tag{11}$$

The condition of proposition 2 that the maximal set of commuting operators in $L^2(X, \mu)$ contains an elliptic operator, and the condition that all operators in $\{T(C_i)\}_1^n$ and $\{A_j\}_1^m$ have absolutely continuous spectra and purely discrete spectra, respectively, are often satisfied in applications. In particular, these conditions are satisfied in the case considered in the next section (sec. 3), where X is a symmetric space of rank one, with respect to group $SO(p, q)$.

It is interesting that the eigenvectors $e_k(\lambda, x)$ are linear combinations of matrix elements $D^\lambda_{pq}(g)$ of the representation $\hat{T}_g(\lambda)$. In fact, let $\{e_k(\lambda, x)\}$ be a basis in $H(\lambda)$, whose elements satisfy assertions of proposition 2: then

$$\hat{T}_g(\lambda) e_k(\lambda, x) = e_k(\lambda, xg) = D^\lambda_{lk}(g) e_l(\lambda, x). \tag{11'}$$

Now by virtue of the homogeneity of X, any point in X can be written in the form og_x, where o is the origin of X (o corresponds to the coset K). Hence, formula (11) implies

$$e_k(\lambda, x) = e_k(\lambda, og_x) = D^\lambda_{lk}(g_x)e_l(\lambda, o), \tag{12}$$

where g_x is an element of the Borel set $S \subset G$ determined by the Mackey decomposition $G = KS$ of G. Formula (12) reveals the important fact that the eigenvectors $e_k(\lambda, x)$ in the space $L^2(X, \mu)$, $X = K^{\cdot} G$, may be obtained by the reduction of the matrix elements $D^\lambda_{pq}(g)$ on G to the Borel set S. This procedure is well known to physicists in the case of the group SO(3) where the eigenvectors $Y^J_M(\vartheta, \varphi)$ on the sphere S^2 may be obtained from the matrix elements $D^J_{MN}(\varphi, \vartheta, \psi)$ by the formula $Y^J_M(\vartheta, \varphi) = (-1)^M \left(\frac{4\pi}{(2J+1)}\right)^{1/2} D^J_{-M,O}(\varphi, \vartheta, 0)$.

EXAMPLE 1. Let G be the Lorentz group SO(3, 1) and let

$$X = \mathrm{SO}(3, 1)/\mathrm{SO}(3).$$

According to ch. 4, table 1, X is a Cartan symmetric space of rank one. It can be realized as the three-dimensional hyperboloid

$$(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2 = 1. \tag{13}$$

Because SO(3, 1) and SO(3) are unimodular groups, it follows from corollary 1 to th. 4.3.1 that there exists an invariant measure $\mathrm{d}\mu(x)$ on X. This measure has the form $\mathrm{d}\mu(x) = \mathrm{d}^3x/x^0$ (cf. example 4.3.2). We shall find the explicit form of harmonic functions on X which provide the spectral synthesis formula (4), completeness (10) and orthogonality relation (11), as well as the direct integral decomposition (1) of the space $H = L^2(X, \mathrm{d}\mu)$ and of the representation T_g: $u(x) \to u(g^{-1}x)$ onto irreducible components.

We know by th. 1.1 that in the space $L^2(X, \mu)$ the ring of invariant differential operators is generated by the Laplace-Beltrami operator Δ, 1(12). We shall find first the explicit form of Δ. Introducing the polar coordinates

$$\begin{aligned} x^1 &= \sin\vartheta\cos\varphi\,\mathrm{sh}\,\theta, && \varphi \in [0, 2\pi), \\ x^2 &= \sin\vartheta\sin\varphi\,\mathrm{sh}\,\theta, && \vartheta \in [0, \pi/2], \\ x^3 &= \cos\vartheta\,\mathrm{sh}\theta, && \\ x^0 &= \mathrm{ch}\theta, && \theta \in [0, \infty) \end{aligned} \tag{14}$$

and utilizing expression 1(12) for the Laplace–Beltrami operator, one finds

$$\Delta(X) = -\frac{1}{\mathrm{sh}^2\theta}\frac{\partial}{\partial\theta}\,\mathrm{sh}^2\theta\frac{\partial}{\partial\theta}+\frac{\boldsymbol{J}^2}{\mathrm{sh}^2\theta}, \tag{15}$$

where $\boldsymbol{J}^2$ is the invariant operator of SO(3). Setting

$$e_k(\lambda, x) = V^\lambda_J(\theta)\,Y^J_M(\vartheta, \varphi), \tag{16}$$

where $Y_M^J(\vartheta, \varphi)$ are the harmonic functions on the sphere S^2, one reduces the eigenvalue equation

$$\Delta(X) e_k(\lambda, x) = \hat{\Delta}(\lambda) e_k(\lambda, x) \tag{17}$$

to the solution of a second-order ordinary differential equation of the form

$$\left[-\frac{1}{\sinh^2\theta}\frac{d}{d\theta}\sinh^2\theta\frac{d}{d\theta}+\frac{J(J+1)}{\sinh^2\theta}\right]V_J^\lambda(\theta) = \hat{\Delta}(\lambda)V_J^\lambda(\theta). \tag{18}$$

The spectrum Λ of this operator and the spectral measure $d\varrho(\lambda)$ may be found either by casting this equation into a Schrödinger equation, or by using the standard Titchmarsh–Kodaira technique. One finds that

$$\hat{\Delta}(\lambda) = -\Lambda^2-1, \quad \lambda \in \Lambda = [0, \infty), \tag{19}$$

and the spectral measure $d\varrho(\lambda) = d\lambda$ (cf. Limić, Niederle and Rączka 1967). Equation (17) is a hypergeometric equation whose solution, regular at $\xi = 0$, is

$$V_J^\lambda(\theta) = N^{-1/2}\tanh^J\theta\cosh^{[i\lambda-1]}\theta\, {}_2F_1\left\{\frac{1}{2}(J-i\lambda+1),\right.$$
$$\left.\frac{1}{2}\left(J-i\lambda+2;\ J+\frac{3}{2},\ \tanh\theta\right)\right\}, \tag{20}$$

where

$$N = \left|\frac{(2\pi)^{1/2}\Gamma(i\lambda)\Gamma(J+{}^3/_2)}{\Gamma[\frac{1}{2}(i\lambda+1+J)]\Gamma(\frac{1}{2}(i\lambda+2+J)]}\right|^2. \tag{21}$$

Consequently, the direct integral 2(1) for $H = L^2(X, \mu)$ implied by Δ has the form

$$\hat{H} = \int_0^\infty \hat{H}(\lambda)d\lambda.$$

Because the second-order Casimir operators of SO(3, 1) and of the maximal compact subgroup $K = \mathrm{SO}(3)$ are diagonal, the elliptic Nelson operator

$$\Delta_N = \sum_1^{\dim G} X_i^2 = C_2(\mathrm{SO}(3,1))+2C_2(\mathrm{SO}(3))$$

is also diagonal. This implies by virtue of proposition 2 that we have:

(i) *Spectral synthesis*: The nuclear space $\Phi(X)$ can be taken to be the Schwartz space $S(X)$. For $\varphi \in S(X)$, the spectral synthesis (3) is

$$\varphi(\theta, \vartheta, \varphi) = \int_0^\infty d\lambda \sum_{J=0}^\infty \sum_{M=-J}^J \hat{\varphi}_{JM}(\lambda)V_J^\lambda(\theta)Y_M^J(\vartheta, \varphi), \tag{22}$$

where

$$\hat{\varphi}_{JM}(\lambda) = \int_X d\mu(x)\varphi(\theta, \vartheta, \varphi)V_J^\lambda(\theta)Y_M^J(\vartheta, \varphi). \tag{23}$$

(ii) Parseval equality

$$\int_X \varphi(x)\overline{\psi}(x)\mathrm{d}\mu(x) = \int_0^\infty \mathrm{d}\lambda \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \hat{\varphi}_{JM}(\lambda)\overline{\hat{\psi}_{JM}(\lambda)}. \tag{24}$$

In eqs. (23) and (24):

$$\mathrm{d}\mu(x) = \cosh^2\theta\,\mathrm{d}\theta\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi. \tag{25}$$

(iii) Completeness relation (8)

$$\int_0^\infty \mathrm{d}\lambda \sum_{J=0}^{\infty} \sum_{M=-J}^{J} V_J^\lambda(\theta)\overline{Y_M^J(\theta,\varphi)}\,V_J^\lambda(\theta')\overline{Y_M^J(\theta',\varphi')}$$
$$= \mathrm{ch}^{-2}\theta\,\delta(\theta-\theta')\,\delta(\cos\vartheta-\cos\vartheta')\,\delta(\varphi-\varphi'). \tag{26}$$

(iv) The orthogonality relation (9) takes the form

$$\int_X V_{J'}^{\lambda'}(\theta)\,Y_{M'}^{J'}(\vartheta,\varphi)\,V_J^\lambda(\theta)\,Y_M^J(\vartheta,\varphi)\,\mathrm{d}\mu(x) = \delta(\lambda-\lambda')\,\delta_{JJ'}\,\delta_{MM'} \tag{27}$$

Every generator Y of SO(3, 1) commutes with the operator Δ. Hence, each Hilbert space $H(\lambda)$ is invariant relative to the action of the representation T_g of SO(3, 1). Using the Bruhat criterion given by th. 19.1.2 one may verify that almost every representation $\hat{T}_g(\lambda)$ in $\hat{H}(\lambda)$ obtained by the restriction of $\hat{T}_g$ to $\hat{H}(\lambda)$ is irreducible. Hence the decomposition (1) of H implies the decomposition

$$T_g \to \hat{T}_g = \int_0^\infty \hat{T}_g(\lambda)\,\mathrm{d}\lambda \tag{28}$$

of T_g onto irreducible components.▼

§ 3. Harmonic Analysis on Symmetric Spaces Associated with Pseudo-Orthogonal Groups SO(p, q)

The pseudo-orthogonal groups play an important role in theoretical physics. The most important one is the Lorentz group SO(3, 1) of special theory of relativity.

The group SO(4, 1), known as the de Sitter group, appears in general relativity, as the dynamical group in the theory of hydrogen atom in the periodic table of elements, as well as in hadron models. The conformal group SO(4, 2) is the symmetry group of Maxwell equations and also appears as a symmetry or dynamical group in the theory of elementary particles. Other groups like SO(2, 1) or SO(4, 3) also often appear in applications. Hence it is appropriate to present a detailed study of the properties of harmonic analysis on symmetric spaces associated with the groups SO(p, q) and the associated representations of SO(p, q) on symmetric spaces.

The full classification of symmetric spaces associated with the groups SO(p, q) was given in ch. 4, tables I and II. The most important for applications are the following symmetric spaces:

(i) Cartan symmetric spaces

$$X = \mathrm{SO}_0(p, q)/\mathrm{SO}(p)\otimes \mathrm{SO}(q). \tag{1}$$

The rank k of these spaces is $k = \min(p, q)$ and the dimension equals pq.

(ii) The symmetric spaces with noncompact semisimple stability groups

$$X_{r,s} = \mathrm{SO}_0(p, q)/\mathrm{SO}_0(r, s)\otimes \mathrm{SO}_0(p-r, q-s), \quad 0 \leqslant r \leqslant p, \quad 0 \leqslant s \leqslant q. \tag{2}$$

(iii) The symmetric spaces with noncompact nonsemisimple stability group

$$X_0 = \mathrm{SO}_0(p, q)/T^{p-1,q-1} \otimes\!\!\!\!\!\!\;\, \mathrm{SO}_0(p-1, q-1). \tag{3}$$

Here $T^{n,m}$ is the group of translations of the Minkowski space $M^{n,m}$.

We shall consider in this section the harmonic analysis on symmetric spaces (1)–(3) of rank one; these are

$$\begin{aligned} X_+^{p+q-1} &\equiv \mathrm{SO}_0(p, q)/\mathrm{SO}_0(p-1, q), \\ X_-^{p+q-1} &\equiv \mathrm{SO}_0(p, q)/\mathrm{SO}_0(p, q-1) \end{aligned} \tag{4}$$

and X_0 given in (3).

The groups $\mathrm{SO}_0(p, q)$ and the stability subgroups of the spaces X_+, X_- and X_0 are all unimodular. Hence by corollary 1 to th. 4.3.1, there exists on these spaces X_+, X_- and X_0 an invariant measure $\mathrm{d}\mu(x)$. Hence a unitary representation $g \to T_g$ of $\mathrm{SO}_0(p, q)$ is given in the space $H = L^2(X, \mu)$ by the formula

$$T_g u(x) = u(g^{-1}x). \tag{5}$$

We know by virtue of th. 1.1 that in the spaces $H(X_+)$, $H(X_-)$ and $H(X_0)$ the ring of invariant differential operators is generated by the second order Casimir operator C_2, which on the spaces $H(X_+)$ and $H(X_-)$ is equal to the Laplace–Beltrami operator.

We may always select a coordinate system on X_+, X_- or X_0 such that the second-order Casimir operators of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ will be diagonal. Hence the elliptic Nelson operator

$$\Delta_N = \sum_{i=1}^{\dim G} X_i^2 = C_2(\mathrm{SO}(p, q)) - 2C_2(\mathrm{SO}(p)) - 2C_2(\mathrm{SO}(q))$$

is also diagonal. Consequently, all assumptions of th. 2.1 are satisfied and we obtain:

(i) The space H and the representation T_g given by eq. (5) can be represented as a direct integral

$$H \to \hat{H} = \int_{\Lambda} \hat{H}(\lambda)\mathrm{d}\mu(\lambda), \qquad T_g \to \hat{T}_g = \int_{\Lambda} T_g(\lambda)\mathrm{d}\mu(\lambda)$$

of components $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$, respectively.

(ii) There exists a Gel'fand-triplet $\Phi \subset H \subset \Phi'$ such that all elements $e(\lambda)$ in $H'(\lambda)$ satisfy the eigenvalue eq. 2(7).

(iii) The spectral synthesis 2(4) for functions $\varphi(x) \in \Phi$ and Parseval equality 2(6) hold.

To carry out the decomposition of H and T_g onto irreducible components explicitly and to get the spectral synthesis for the functions $\varphi(x) \in \Phi$ in explicit form, we have to find the maximal abelian algebra A in the commutant T' of the representation T and find its spectrum Λ. We shall solve these problems along the following steps:

(i) Construct a convenient coordinate system on X for which the metric tensor $g_{\alpha\beta}(x)$ is diagonal.

(ii) Solve the eigenvalue problem for the Laplace–Beltrami operator

$$\Delta(X)e(\lambda, x) = \hat{\Delta}(\lambda)e(\lambda, x).$$

(Because $[\Delta, Y] = 0$ for all Y in the Lie algebra L of G, the space $H'(\lambda)$ for fixed λ, spanned by all $e(\lambda, x)$ is invariant.)

(iii) Find additional invariant operators which decompose the space $\hat{H}(\lambda)$ onto irreducible subspaces.

We stress that according to th. 1.1 the ring of invariant differential operators is generated by the Laplace–Beltrami operator. It, however, says nothing about the other invariant operators in H. We shall find that in the case of symmetric spaces of rank one these additional invariant operators are certain reflection operators in $H(X)$.

A. *Harmonic Analysis on Symmetric Spaces Associated with Groups* $SO_0(p, q)$, $p \geqslant q > 2$

To choose a suitable coordinate system, we have to introduce some convenient model for the space $X_{\pm}^{p+q-1}$ in (4). This means that we have to introduce a manifold, with the same dimension and the same stability group as $X_{\pm}^{p+q-1}$ on which the group $SO_0(p, q)$ acts transitively.

It can be shown, by means of considerations similar as in the case of the group $SO(p)$ (cf. 4.2, example 1) that a model for the space X_{+}^{p+q-1} can be realized by the hyperboloid $H^{p,q}$ determined by the equation

$$(x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^{p+q})^2 = 1. \tag{6}$$

And as a model appropriate for the space X_{-}^{p+q-1}, we take the hyperboloid $H^{q,p}$ defined by the equation

$$(x^1)^2 + \dots + (x^q)^2 - (x^{q+1})^2 - \dots - (x^{p+q})^2 = 1. \tag{7}$$

If we introduce internal coordinates $\Omega = \{\eta^1, \dots, \eta^{p+q-1}\}$ on the space $H^{p,q}$ (which is imbedded in the flat Minkowski space $M^{p,q}$), then the metric tensor

$g_{\alpha\beta}(H^{p,q})$ on the hyperboloid $H^{p,q}$ induced by the metric tensor $g_{ab}(M^{p,q})$ on the Minkowski space $M^{p,q}$, is given by

$$g_{\alpha\beta}(H^{p,q}) = g_{ab}(M^{p,q})\,\partial_\alpha x^a(\Omega)\,\partial_\beta x^b(\Omega), \tag{8}$$

where $a, b = 1, 2, \ldots, p+q$, $\alpha, \beta = 1, 2, \ldots, p+q-1$ and x^a are Cartesian coordinates of the hyperboloid Ω.

Generally, we may choose a large number of different coordinate systems on the hyperboloid $H^{p,q}$ such that the Laplace–Beltrami operator admits separation of variables. However, the most convenient coordinate system is the biharmonic one, because in this system the generators of the maximal abelian compact subgroup of the group $SO_0(p, q)$ are automatically contained in the maximal set of commuting operators.

The biharmonic coordinate system on the sphere S^n was introduced in eq.10.3(5a).

The biharmonic coordinate system on the hyperboloid $H^{p,q}$, (6), is constructed as follows:

$$\begin{aligned} x^k &= x'^k\cosh\theta, & k &= 1, 2, \ldots, p, \\ x^{p+l} &= \tilde{x}^l\sinh\theta, & l &= 1, 2, \ldots, q, \end{aligned} \qquad \theta \in [0, \infty), \tag{9}$$

where the form of the x'^k and $\tilde{x}^l$ depends on whether p and q are even or odd. We must distinguish four cases:

$$\begin{aligned} &\text{(i) } p = 2r, & q &= 2s, \\ &\text{(ii) } p = 2r, & q &= 2s+1, \\ &\text{(iii) } p = 2r+1, & q &= 2s, \\ &\text{(iv) } p = 2r+1, & q &= 2s+1, \end{aligned} \qquad r, s = 1, 2, \ldots \tag{10}$$

If p is even ($p = 2r$), the corresponding x'^k ($k = 1, 2, \ldots, 2r$) are given by the recursion formulas

$$\begin{aligned} &\text{for } r = 1 & x'^1 &= \cos\varphi^1, & & \\ & & x'^2 &= \sin\varphi^1, & & \varphi^1 \in [0, 2\pi), \\ &\text{for } r > 1 & x'^i &= x^{*i}\sin\vartheta^r, & & i = 1, 2, \ldots, 2r-2, \\ & & x'^{2r-1} &= \cos\varphi^r\cos\vartheta^r, & & \varphi^j \in [0, 2\pi),\ j = 1, 2, \ldots, r, \\ & & x'^{2r} &= \sin\varphi^r\cos\vartheta^r, & & \vartheta^k \in [0, \tfrac{1}{2}\pi],\ k = 2, 3, \ldots, r, \end{aligned} \tag{11}$$

where x^{*i} are the coordinates for $p = 2(r-1)$.

If p is odd ($p = 2r+1$), we first construct the x^{*i}, $i = 1, 2, \ldots, 2r$, by using the above-mentioned method for $p = 2r$; we then obtain the corresponding x'^k, $k = 1, 2, \ldots, 2r+1$, as

$$\begin{aligned} x'^i &= x^{*i}\sin\vartheta^{r+1}, & & i = 1, 2, \ldots, 2r, \\ x'^{2r+1} &= \cos\vartheta^{r+1}, & & \vartheta^{r+1} \in [0, \pi). \end{aligned} \tag{12}$$

The recursion formulas for $\tilde{x}^l$, q even or odd, are the same as those for x'^k, p even or odd, respectively, except that angles φ^i, ϑ^j in x'^k are replaced by $\tilde{\varphi}^i$, $\tilde{\vartheta}^j$.

Choosing the parametrization $\Omega \equiv \{\omega, \tilde{\omega}, \theta\}$ on the hyperboloid $H^{p,q}$ in the form

$$\begin{aligned} \omega &\equiv \{\varphi^1, \dots, \varphi^{[p/2]}, \vartheta^2, \dots, \vartheta^{\{p/2\}}\}, \\ \tilde{\omega} &\equiv \{\tilde{\varphi}^1, \dots, \tilde{\varphi}^{[q/2]}, \tilde{\vartheta}^2, \dots, \tilde{\vartheta}^{\{q/2\}}\}, \end{aligned} \tag{13}$$

and denoting

$$\{\partial_\gamma\} \equiv \left\{\frac{\partial}{\partial\varphi^1}, \frac{\partial}{\partial\vartheta^2}, \dots; \frac{\partial}{\partial\varphi^{[p/2]}}, \frac{\partial}{\partial\varphi^{\{p/2\}}}, \frac{\partial}{\partial\tilde{\varphi}^1}, \frac{\partial}{\partial\tilde{\vartheta}^2}, \dots, \frac{\partial}{\partial\tilde{\varphi}^{[q/2]}}, \frac{\partial}{\partial\tilde{\vartheta}^{\{q/2\}}}, \frac{\partial}{\partial\theta}\right\}, \gamma = 1, 2, \dots, p+q-1, \tag{14}$$

we can calculate the metric tensor $g_{\alpha\beta}(H^{p,q})$ as well as the Laplace–Beltrami operator $\Delta(H^{p,q})$.

Since in all four cases shown in (10) the variables in the Laplace–Beltrami operator are separated in the same way due to the properties of the metric tensor (8), we can write the operator $\Delta(H^{pq})$ in the form

$$\Delta(H^{pq}) = -(\cosh^{p-1}\theta\sinh^{q-1}\theta)^{-1}\frac{\partial}{\partial\theta}\cosh^{p-1}\theta\sinh^{q-1}\theta\frac{\partial}{\partial\theta} + \\ + \frac{\Delta(S^{p-1})}{\cosh^2\theta} - \frac{\Delta(S^{q-1})}{\sinh^2\theta}, \tag{15}$$

where $\Delta(S^{p-1})[\Delta S^{q-1})]$ is the Laplace–Beltrami operator of the rotation group SO(p) [SO(q)]. (See ch. 10, sec. 3.) If we represent the eigenfunctions of $\Delta(H^{p,q})$ as a product of the eigenfunctions of $\Delta(S^{p-1})$, $\Delta(S^{q-1})$, and a function $\psi^\lambda_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}}(\theta)$ we obtain the following equation:

$$\left[-(\cosh^{p-1}\theta\sinh^{q-1}\theta)^{-1}\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh^{p-1}\theta\sinh^{q-1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{l_{\{p/2\}}(l_{\{q/2\}}+p-2)}{\cosh^2\theta} + \\ + \frac{\tilde{l}_{\{q/2\}}(\tilde{l}_{\{q/2\}}+q-2)}{\sinh^2\theta} - \hat{\Delta}(\lambda)\right] \cdot \psi^\lambda_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}}(\theta) = 0, \tag{16}$$

where $l_{\{p/2\}}(l_{\{p/2\}}+p-2)$, $[\tilde{l}_{\{q/2\}}(\tilde{l}_{\{q/2\}}+q-2)]$ are eigenvalues of the operator Δ $(S^{p-1}), [\Delta(S^{q-1})]$ with $l_{\{p/2\}}[l_{\{q/2\}}]$ certain non-negative integers for $p > 2$ ($q > 2$]).

A discrete series of representations exists if there exist solutions of (16), which are square integrable functions $\psi^\lambda_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}}(\theta)$, $\theta \in [0, \infty)$, with respect to the measure

$$\mathrm{d}\mu(\theta) = \cosh^{p-1}\theta\sinh^{q-1}\theta\mathrm{d}\theta, \tag{17}$$

which is induced by the measure $\mathrm{d}\mu(x)$ on the hyperboloid $H^{p,q}$, which in the biharmonic coordinates has the form*

$$\mathrm{d}\mu(\Omega) = \bar{g}(H^{p,q})^{1/2}\mathrm{d}\Omega = \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tilde{\omega})\cosh^{p-1}\theta\sinh^{q-1}\theta\mathrm{d}\theta. \tag{18}$$

* The measure $\mathrm{d}\mu(x) = [\bar{g}(H^{p,q})]^{1/2}\mathrm{d}\Omega$ is the Riemannian measure, which is left-invariant under the action of $SO_0(p, q)$. See ch. 4, § 3.

The left-invariant measure $\mathrm{d}\mu(\omega)$ (with respect to SO(p)) is defined in eq. 10.3 (14). Since the differential eq. (16) has meromorphic coefficients regular in the interval $(0, \infty)$, any two linearly independent solutions are also regular analytic in this interval (Ince, 1956). Since at the origin and at infinity the coefficients are singular, the solutions are not generally regular there, and we can easily find two essentially distinct behaviours of the solutions at the origin:

$$\psi_1^0 \sim \theta^{\tilde{l}_{\{q/2\}}}, \qquad \psi_2^0 \sim \theta^{-\tilde{l}_{\{q/2\}}-q+2},$$

and at infinity:

$$\psi_{1,2}^{\infty} \sim \exp\left\{-\frac{1}{2}(p+q-2)\pm\left[\frac{1}{2}(p+q-2)^2-\hat{\Delta}(\lambda)\right]^{1/2}\right\}\theta.$$

The only satisfactory solution, i.e., the solution which is square-integrable with respect to our measure $\mathrm{d}\mu(\theta)$ (17), is one that behaves like $\psi_1^0(\theta)$ at the origin and like $\psi_2^{\infty}(\theta)$ at infinity. We obtain the solution of (16) with these properties by converting (16) into the hypergeometric equation whose solution is

$$\psi^{\lambda}_{l_{\{p/2\}}\tilde{l}_{\{q/2\}}}(\theta) = \tanh^{\tilde{l}_{\{q/2\}}}\theta\cdot\cosh^{-\{(p+q-2)/2+[(p+q-2)^2/2-\hat{\Delta}(\lambda)]^{1/2}\}}\theta\times$$
$$\times\, {}_2F_1\left(-n+l_{\{p/2\}}+\frac{p-2}{2},-n;\ \tilde{l}_{\{q/2\}}+\frac{q}{2};\ \tanh^2\theta\right),$$

where the non-negative integer n is connected with $l_{\{p/2\}}$, $l_{\{q/2\}}$ and $\hat{\Delta}(\lambda)$ by the condition that ${}_2F_1$ be a polynomial, i.e.,

$$l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-2n = \tfrac{1}{2}(p+q-2)+\{[\tfrac{1}{2}(p+q-2)]^2-\Delta(\lambda)\}^{1/2}-p+2, \qquad (19)$$
$$n = 0,1,2,\dots$$

From this restrictive condition we can find that the discrete spectrum of the operator $\Delta(H^{p,q})$ is of the form

$$\hat{\Delta}(\lambda) = -L(L+p+q-2), \quad L = -\{\tfrac{1}{2}(p+q-4)\},-\{\tfrac{1}{2}(p+q-4)\}+1,\dots \qquad (20)$$

and

$$L = l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-q-2n. \qquad (21)$$

Thus the orthonormal eigenfunctions of the invariant operator $\Delta(H^{p,q})$ are:

$$Y^{L,l_2,\dots,l_{\{p/2\}}\tilde{l}_2,\dots,\tilde{l}_{\{q/2\}}}_{m_1,\dots,m_{[p/2]},\tilde{m}_1,\dots,\tilde{m}_{[q/2]}}(\omega,\tilde{\omega},\theta)$$
$$= Y^{l_2,\dots,l_{\{p/2\}}}_{m_1,\dots,m_{[p/2]}}(\omega)\cdot Y^{\tilde{l}_2,\dots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\dots,\tilde{m}_{[q/2]}}(\tilde{\omega})\cdot V^L_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta), \qquad (22)$$

where

$$Y^{l_2,\dots,l_{\{p/2\}}}_{m_1,\dots,m_{[p/2]}}(\omega)$$
$$\equiv\begin{cases} Y^{l_2,\dots,l_r}_{m_1,\dots,m_r}(\omega) = (N_r^{-1/2})\prod_{k=2}^{r}\sin^{2-k}(\vartheta^k)\cdot \mathrm{d}^{J_k}_{M_k,M_k'}(2\vartheta^k)\cdot\prod_{k=1}^{r}\exp \mathrm{i}m_k\varphi^k, \ \text{if } p=2r,\\ Y^{l_2,\dots,l_{r+1}}_{m_2,\dots,m_r}(\omega) = (N_{r+1}^{-1/2})\sin^{1-r}(\vartheta^{r+1})\cdot \mathrm{d}^{J_{r+1}}_{M_{r+1},0}(\vartheta^{r+1})\times \\ \quad\times\prod_{k=2}^{r}\sin^{2-k}(\vartheta^k)\cdot \mathrm{d}^{J_k}_{M_k,M_k'}(2\vartheta^k)\cdot\prod_{k=1}^{r}\exp \mathrm{i}m_k\varphi^k, \quad \text{if } p=2r+1,\end{cases} \qquad (23)$$

are eigenfunctions of $\Delta(S^{p-1})$ derived in ch. 10, § 3, eq. (19), eq. (20); and $Y^{\tilde{l}_2,\ldots,\tilde{l}_{[q/2]}}_{\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\tilde{\omega})$ are eigenfunctions of $\Delta(S^{q-1})$ expressed as the product of the usual d-functions of angular momenta and exponential functions exactly as in (23), but of the variables $\tilde{\varphi}^i$, $\tilde{\vartheta}^j$ and $\tilde{l}_k$, $\tilde{m}_l$ instead of φ^i, ϑ^j and l_k, m_l. The function $V^L{}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$ is the solution of (16) given by

$$V^L_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta) = (N^{-1/2})\tanh^{\tilde{l}_{\{2/q\}}}(\theta)\cdot\cosh^{-(L+p+q-2)}(\theta)\times$$
$$\times {}_2F_1[\tfrac{1}{2}(p+q-2+l_{\{p/2\}}+\tilde{l}_{\{q/2\}}+L), \tfrac{1}{2}(L+q+\tilde{l}_{\{q/2\}}-l_{\{p/2\}});\ \tilde{l}_{\{q/2\}}+\tfrac{1}{2}q;\ \tanh^2\theta], \quad (24)$$

where, for a definite representation, L is fixed and $l_{\{p/2\}}$, $\tilde{l}_{\{q/2\}}$ are restricted by the condition that ${}_2F_1$ be a polynomial, i.e.,

$$l_{\{p/2\}}-\tilde{l}_{\{q/2\}} = L+q+2n, \qquad n = 0, 1, 2, \ldots \quad (25)$$

In eq. (24) N is a normalization factor given by

$$N = \frac{\Gamma[\frac{1}{2}(l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-L-q+2)]\Gamma(\tilde{l}_{\{q/2\}}+q/2)\Gamma[\frac{1}{2}(L-\tilde{l}_{\{q/2\}}+l_{\{p/2\}}+p)]}{2[L+\frac{1}{2}(p+q-2)]\Gamma[\frac{1}{2}(l_{\{p/2\}}+\tilde{l}_{\{q/2\}}+L+p+q-2)]\Gamma[\frac{1}{2}(l_{\{p/2\}}+\tilde{l}_{\{q/2\}}-L)]}. \quad (26)$$

Let $\hat{H}(L)$, L fixed, denote the subspace of $H = L^2(H^{p,q}, \mu)$ spanned by the harmonic functions (23). Because

$$[\Delta(H^{p,q}), Z_{ij}] = 0, \quad i,j = 1, 2, \ldots, p+q$$

for any generator $Z_{ij} \in \mathrm{so}(p, q)$, the space $\hat{H}(L)$ is an invariant space for the quasi-regular representation (1.1). We denote the unitary representation by (1.1) restricted to $\hat{H}(L)$ by $\hat{T}(L)$.

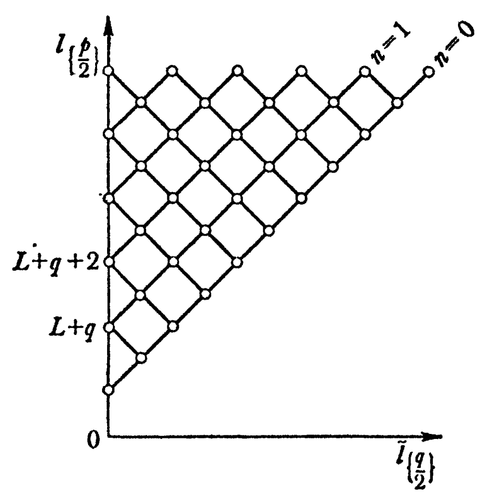


Fig. 1. Representations $\hat{T}(L)$ in $L^2(H^{p,q}, \mu)$ for $p > q > 2$.

The structure of the representation space $\hat{H}(L)$ can be illustrated graphically by a net in the plane $(l_{\{p/2\}}\, l_{\{q/2\}})$. Namely, utilizing relation (25), we get the diagram shown in Fig. 1. Every node of the net in the figure represents a finite-dimensional subspace $\hat{H}^{(L)}_{l_{\{p/2\}}, l_{\{q/2\}}}$ of an irreducible representation of the maximal compact subgroup $SO(p)\times SO(q)$ which is determined by the pair of integers $l_{\{p/2\}}$ and $\tilde{l}_{\{q/2\}}$. The generators $L_{ij}\in SO(p)\times SO(q)$ act inside $\hat{H}^{(L)}_{l\{p/2\},\tilde{l}_{\{q/2\}}}$. On the other hand, the generators B_{ij} of the non-compact type map the subspace $\hat{H}^{(L)}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}$ into four neighboring subspaces $\hat{H}^{(L)}_{l_{\{p/2\}}\pm 1,\tilde{l}_{\{q/2\}}\pm 1}$. It is interesting that the structure of the whole representation space is determined by the lowest value of $l_{\{p/2\}}$.

The set of all representations

$$\hat{T}(L),\quad L=-\{\tfrac{1}{2}(p+q)-4\},-\{\tfrac{1}{2}(p+q)-4\}+1,\ldots,$$

constitutes the discrete series of most degenerate irreducible unitary representations of the group $SO_0(p,q)$ which are realized in the Hilbert spaces $L^2(H^{p,q},\mu)$. Let $\tilde{H}=L^2(H^{p,q},\mu)$. There also exists a discrete series of representations $\tilde{T}(L)$ on the Hilbert space $\tilde{H}(L)\subset\tilde{H}$ spanned by the harmonic functions obtained by exchanging p, $l_{\{p/2\}}$ with q, $\tilde{l}_{\{q/2\}}$, respectively, in eq. (22). The representations $\hat{\tilde{T}}(L)$ of $SO_0(p,q)$ realized $\tilde{H}(L)$ are not unitarily equivalent to $\hat{T}(L)$ on $\hat{H}(L)$ except in the case $p=q$, which is the case when both Hilbert spaces coincide. The structure of the representation space $\hat{\tilde{T}}(L)$ is illustrated graphically in Fig. 2.

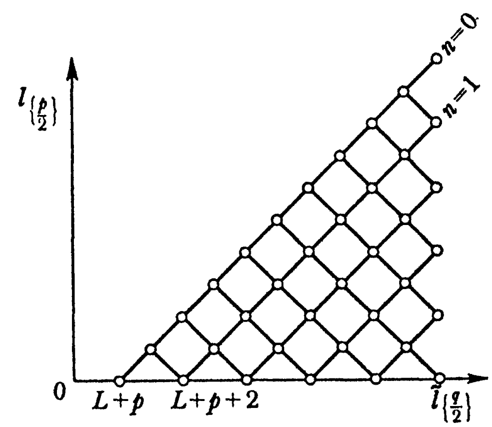


Fig. 2. Representation $\hat{\tilde{T}}(L)$ in $L^2(H^{p,q},\mu)$ for $p\geqslant q>2$.

The differential operator (16) and therefore the Laplace–Beltrami operator (1), has, in the case of the group $SO_0(p,q)$, a discrete spectrum given by eq. (20) as well as a continuous spectrum. The latter is of the form (see Limić, Niederle and Rączka 1967)

$$\hat{\Delta}(\lambda)=+\lambda^2+\left(\frac{p+q-2}{2}\right)^2,\quad \lambda\in[0,\infty). \tag{27}$$

For the continuous spectrum the solution of eq. (16), regular at the origin, is given by the function

$$V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{\tilde{q}/2\}}}(\theta)=(N^{-1/2})\cdot\tanh^{|\tilde{l}_{\{q/2\}}|}\theta\cdot\cosh^{[-(p+q-2)/2+i\lambda]}\theta\times$$
$$\times{}_2F_1\{\tfrac{1}{2}[|\tilde{l}_{\{q/2\}}|+|l_{\{p/2\}}|-i\lambda+\tfrac{1}{2}(p+q-2)],$$

$$\cdot\ \tfrac{1}{2}[|\tilde{l}_{\{q/2\}}|-|l_{\{p/2\}}|-\mathrm{i}\lambda+\tfrac{1}{2}(q-p+2)];\ |\tilde{l}_{\{q/2\}}|+\tfrac{1}{2}q;\ \tanh^2\theta\}, \tag{28}$$

with

$$N = \left|\frac{\sqrt{2\pi}\,\Gamma(|\tilde{l}_{\{q/2\}}+\tfrac{1}{2}q|)\cdot\Gamma(\mathrm{i}\lambda)}{\Gamma\{\tfrac{1}{2}[\mathrm{i}\lambda+|l_{\{p/2\}}|+|\tilde{l}_{\{q/2\}}|+\tfrac{1}{2}(p+q-2)]\}\cdot\Gamma\{\tfrac{1}{2}[\mathrm{i}\lambda+|\tilde{l}_{\{q/2\}}|-|l_{\{p/2\}}|+\tfrac{1}{2}(q-p+2)]\}}\right|^2.$$

These functions obey the following orthogonality condition

$$\int V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\overline{V}^{\lambda'}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\,\mathrm{d}\mu(\theta) = \delta(\lambda-\lambda'), \tag{29}$$

where $\mathrm{d}\mu(\theta)$ is given by formula (17).

The orthogonal eigenfunctions of the Laplace–Beltrami operator $\Delta(H^{p,q})$ are then harmonic functions of the form

$$Y^{\lambda,l_2,\ldots,l_{\{p/2\}},\tilde{l}_2,\ldots,\tilde{l}_{\{q/2\}}}_{m_1,\ldots,m_{[p/2]},\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\theta,\omega,\tilde{\omega}) = V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\cdot Y^{l_2,\ldots,l_{\{p/2\}}}_{m_1,\ldots,m_{[p/2]}}(\omega)\cdot Y^{\tilde{l}_2,\ldots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}} \tag{30}$$

where $V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$ is given in (28), and

$$Y^{l_2,\ldots,l_{\{p/2\}}}_{m_1,\ldots,m_{[p/2]}}(\omega) \quad \text{and} \quad Y^{\tilde{l}_2,\ldots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\tilde{\omega})$$

are the eigenfunctions of $\Delta(S^{p-1})$ and $\Delta(S^{q-1})$ respectively. Because of the orthogonality property (29), the set of eigenfunctions (30) associated with a definite point λ of the continuous part of the spectrum does not span a Hilbert subspace* $\hat{H}(\lambda)$ of $\hat{H}$, but an isomorphic space $H'(\lambda)\subset\Phi'(H^{p,q})$.

It turns out that the set of invariant operators of the so(p, q) algebra in H contains an invariant operator which is outside the enveloping algebra of so(p, q). Namely, the reflection operator R defined by

$$Rf(x) = f(-x), \quad f(x)\in H \tag{31}$$

commutes with all the generators of so(p, q) and therefore represents an invariant operator.** In the case of the discrete series of representations $\hat{T}^L$ the eigenvalue p of the reflection operator R is determined by the invariant number L, i.e.,***

$$RY^{L,l,\tilde{l}}_{m,\tilde{m}} = (-1)^{L+q}Y^{L,l,\tilde{l}}_{m,\tilde{m}} \quad \text{for } Y^{L,l,\tilde{l}}_{m,\tilde{m}}\in H(H^{p,q},\mu)$$

and

$$RY^{L,l,\tilde{l}}_{m,\tilde{m}} = (-1)^{L+p}Y^{L,l,\tilde{l}}_{m,\tilde{m}} \quad \text{for } Y^{L,l,\tilde{l}}_{m,\tilde{m}}\in H(H^{p,q},\mu). \tag{32}$$

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* $\hat{H}(\lambda)$ represents in our case l^2—Hilbert space of sequences. The vectors $\hat{e}_k(\lambda)$ are the sequences of type $(0, 0, \ldots, 0, 1, 0, \ldots)$ with a unit on the k-th place and zero elsewhere.

** The existence of the invariant operator R is not in contradiction with Helgason's theorem which states that the ring of invariant operators in the enveloping algebra of so(p, q) for a symmetric space of rank one is generated by the Laplace–Beltrami operator.

*** In what follows we shall use for the harmonic function (22) and (30) the abbreviations

$$Y^{L,l,\tilde{l}}_{m,\tilde{m}} \quad \text{and} \quad Y^{\lambda,l,\tilde{l}}_{m,\tilde{m}}.$$

However, in the case of the continuous series of representations, the eigenfunctions (30) of $\Delta(H^{p,q})$ associated with a definite eigenvalue λ are eigenvectors of R with eigenvalues

$$r = (-1)^{(l_{\{p/2\}} + \tilde{l}_{\{q/2\}}}. \tag{33}$$

Therefore, the set of functions $\{Y^{\lambda, l, \tilde{l}}_{m, \tilde{m}}\}$ will be split into two subsets of functions

$$\{Y^{\lambda, +, l, \tilde{l}}_{m, \tilde{m}}\} \quad \text{and} \quad \{Y^{\lambda, -, l, \tilde{l}}_{m, \tilde{m}}\}$$

spanning invariant linear subspaces of $\mathrm{so}(p, q)$.

We shall now give the general form of harmonic analysis on homogeneous spaces for the present special case (cf. th. 2.1). In our case the decomposition 2(1) of $H = L^2(H^{p,q}, \mu)$ into direct integral $\hat{H}$ is determined by the set of invariant operators $\{\Delta, R\}$ and has the form

$$H \xrightarrow{F} \hat{H} = \sum_{L=-\{\frac{1}{2}(p+q-4)\}}^{\infty} \hat{H}(L, (-1)^{L+q}) + \sum_{\pm} \int_0^{\infty} \hat{H}(\lambda, \pm)\, d\lambda. \tag{34}$$

As the nuclear space $\Phi \subset H$ we can take the Schwartz S-space on X. The space Φ is a dense invariant domain of the invariant operators $\Delta[\mathrm{SO}_0(p, q)]$ and R, as well as of all the generators of both compact and non-compact types.

The isomorphism F in (34) is given by means of the generalized Fourier transform with respect to the eigenfunctions (22) and (30) associated with the discrete and continuous part of the spectrum of the invariant operators $\Delta[\mathrm{SO}_0(p, q)]$ and R

$$H \supset \Phi \ni \psi(x) \xrightarrow{F} \{(F\psi)(\lambda, r)\} = \{\hat{\psi}(\lambda, r)\}, \tag{35}$$

where $\lambda = L$, or λ and $r = \pm 1$. For a definite λ the vector $\hat{\psi}(\lambda, r)$ is an element of l^2-Hilbert space, whose components $\psi^{\lambda, r, l, \tilde{l}}_{m, \tilde{m}}$ are given by the formula

$$\hat{\psi}^{\lambda, r, l, \tilde{l}}_{m, \tilde{m}} = \langle \psi, Y^{\lambda, r, l, \tilde{l}}_{m, \tilde{m}} \rangle = \int_{H^{p,q}} \psi(\theta, \omega, \tilde{\omega}) \overline{Y^{\lambda, r, l, \tilde{l}}_{m, \tilde{m}}}\, d\mu(\theta, \omega, \tilde{\omega}). \tag{36}$$

Using eqs. 2(4)–2(6) one readily writes in the present case the spectral synthesis formula and the Parseval equality.

The action of any generator $Z_{ij} \in \mathrm{so}(p, q)$ in the Hilbert space $H(\lambda, r)$ is given by

$$Z_{ij} \hat{\psi}^{\lambda, r} \equiv \{\langle Z_{ij} \psi, Y^{\lambda, r}_{m, \tilde{m}} \rangle\}, \qquad \psi \in \Phi. \tag{37}$$

The proof of irreducibility of the space $H(L, \pm)$ and $\hat{H}(\lambda, \pm)$ with respect to the action (38) of the Lie algebra $\mathrm{so}(p, q)$ is straightforward and we omit it. (See Limić, Niederle and Rączka 1966, for details.)

The harmonic functions on the hyperboloid $H^{q,p}$ can be obtained by exchanging p, $l_{\{p/2\}}$ with q, $\tilde{l}_{\{q/2\}}$, respectively, in eqs. (22) and (30).

The continuous series $\hat{T}(\lambda)$ of the irreducible unitary representations of $\mathrm{SO}(p, q)$, $p \geqslant q > 1$, on $H = L^2(H^{q,p}, \mu)$ can be constructed by the same procedure as described above.

B. *Decomposition with Respect to the Maximal Compact and Maximal Noncompact Subgroups.*

The decomposition of an irreducible representation of the group $SO_0(p, q)$ with respect to the irreducible representations of the maximal compact subgroup $SO(p)\times SO(q)$ can be easily obtained from Figures 1 and 2. For example, for the discrete representations $\hat{T}(L)$ the carrier space $\hat{H}(L)$ can be represented as the direct sum

$$\hat{H}(L) = \sum_{l_{\{p/2\}},\,\tilde{l}_{\{q/2\}}} \oplus \hat{H}^{(L)}_{l_{\{p/2\}},\,\tilde{l}_{\{q/2\}}}, \tag{38}$$

where the sum runs over all $l_{\{p/2\}}$ and $\tilde{l}_{\{q/2\}}$ obeying the condition

$$l_{\{p/2\}}-\tilde{l}_{\{q/2\}} = L+2n+q, \qquad n = 0, 1, \ldots,$$

and every finite-dimensional space $H^{(L)}_{l_{\{p/2\}},\,\tilde{l}_{\{q/2\}}}$ enters with multiplicity one. Thus for $g \in SO(p)\times SO(q)$ we have

$$\hat{T}_g(L) = \sum_{l_{\{p/2\}},\,\tilde{l}_{\{q/2\}}} \oplus \hat{T}^{\,l_{\{p/2\}}}\otimes\hat{T}^{\tilde{l}_{\{q/2\}}}, \tag{39}$$

where $T_g^{\,l_{\{p/2\}}}$ ($T_g^{\,\tilde{l}_{\{q/2\}}}$ are symmetric finite-dimensional representations of $SO(p)$ [$SO(q)$] determined by the highest weight m of the type (see eq. 10.3(29))

$$m = (l_{\{p/2\}}, 0, 0, \ldots, 0). \tag{40}$$

Every representation $\hat{T}_g^{\,l_{\{p/2\}}} \otimes \hat{T}_g^{\,\tilde{l}_{\{q/2\}}}$ enters in the decomposition (40) with multiplicity one.

The decomposition of a representation $\hat{T}(\Lambda, +)$ and $\hat{T}(\Lambda, -)$ of the continuous series has the same form as that of the discrete series $\hat{T}(L)$; only the summation in (39) and (40) is taken over all $l_{\{p/2\}}+l_{\{q/2\}}$, even or odd, respectively.

In physical problems for which a noncompact higher symmetry group exists, we are also interested in the decomposition of a given irreducible representation of the noncompact group with respect to irreducible representations of a maximal noncompact subgroup. We can do this in the case of the group $SO_0(p, q)$ if, on the manifold $H^{p,q}$ we introduce the biharmonic coordinate system in which the Laplace–Beltrami operator $\Delta[SO(p, q-1)]$ will be diagonal. Such a biharmonic coordinate system is given by

$$\begin{aligned} x^i &= x'^i\cosh\eta, \qquad i = 1, 2, \ldots, p+q-1, \qquad p \geqslant q > 2, \\ x^{p+q} &= \sinh\eta, \qquad \eta \in (-\infty, \infty), \end{aligned} \tag{41}$$

where the x'^i are the biharmonic coordinates on the hyperboloid $H^{p,q-1}$ given by eq. (9). Using the formula 1(12), we find that in the coordinate system (41) the Laplace–Beltrami operator has the following form

$$\Delta(H^{p,q}) = \frac{-1}{\cosh^{p+q-2}\eta}\,\frac{\partial}{\partial\eta}\cosh^{p+q-2}\eta\frac{\partial}{\partial\eta} + \frac{\Delta(H^{p,q-1})}{\cosh^2\eta}. \tag{42}$$

Here $\Delta(H^{p,q-1})$ is the Laplace–Beltrami operator of the group $SO_0(p, q-1)$ related to the hyperboloid $H^{p,q-1}$. The discrete and continuous parts of the spectra of $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ are clearly the same as those found in subsec. A. If we represent the eigenfunctions of $\Delta(H^{p,q})$ as a product of an eigenfunction of

$\Delta(H^{p,q-1})$ and a function $V_{\tilde{L}}^{L}(\eta)$, we obtain the following differential equation for the latter function

$$\left[\frac{-1}{\cosh^{p+q-2}\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\cosh^{p+q-2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}-\frac{\tilde{L}(\tilde{L}+p+q-3)}{\cosh^2\eta}+L(L+p+q-2)\right]V_{\tilde{L}}^{L}(\eta)=0. \tag{43}$$

Using the transformation $V_{\tilde{L}}^{L}(\eta)=\cosh^{(2-p+q)/2}\eta\cdot\psi_{\tilde{L}}^{L}(\eta)$, we obtain for $\psi_{\tilde{L}}^{L}(\eta)$ the differential equation of the type which has been treated by Titchmarsch (1962, part I, § 4, 19). Therefore, we immediately know that both independent solutions ${}_{(\alpha)}V_{\tilde{L}}^{L}(\eta)$, $\alpha=1,2$, enter into the eigenfunction expansion associated with the differential operator (43). The final solution of eq. (42) for the case when spectra of $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ are discrete can be written in the form

$$ {}_{(\alpha)}Y_{m\tilde{m}\Omega}^{L\tilde{L}l\tilde{l}}={}_{\alpha}V_{\tilde{L}}^{L}(\eta)\,Y_{m\tilde{m}}^{\tilde{L}l\tilde{l}}(\theta,\omega,\tilde{\omega}),\qquad \alpha=1,2, \tag{44}$$

with

$$\begin{aligned}\tilde{L}&=L+(2n+3-\alpha),\qquad n=0,1,2,\ldots,\\ L&=-\{\tfrac{1}{2}(p+q-4)\}+k,\qquad k=1,2,\ldots\end{aligned} \tag{45}$$

The functions ${}_{\alpha}V_{\tilde{L}}^{L}(\eta)$ can be expressed in terms of Gegenbauer polynomials

$$ {}_{\alpha}V_{\tilde{L}}^{L}(\eta)=\frac{(-1)^{(\tilde{L}-L+1-\alpha)}}{\sqrt{{}_{\alpha}M}}\cosh^{-(L+p-1)}\eta\; C_{\tilde{L}-L-1}^{L+p/2}(\eta),\qquad \alpha=1,2, \tag{46}$$

where

$$ {}_{(1)}M=\frac{{}_{(1)}N\cdot\Gamma^2[\frac{1}{2}(L+\tilde{L}+p)]}{\Gamma^2(L+\frac{1}{2}p)\Gamma^2[\frac{1}{2}p(\tilde{L}-L)]},$$

$$ {}_{(\alpha)}N=\frac{2\pi\Gamma[\frac{1}{2}(\tilde{L}-L+\alpha-1)]\Gamma[\frac{1}{2}(\tilde{L}+L+p+q+\alpha-3)]}{(2L+p+q-2)\Gamma[\frac{1}{2}(\tilde{L}+L+p+q-\alpha)]\Gamma[\frac{1}{2}(\tilde{L}-L-\alpha+2)]}$$

and

$$ {}_{(2)}M=\frac{{}_{(2)}N\cdot 4\cdot\Gamma^2[\frac{1}{2}(L+\tilde{L}+p+1)]}{(L+\tilde{L}+p-1)^2\cdot\Gamma^2(L+\frac{1}{2}p)\cdot\Gamma^2[\frac{1}{2}(\tilde{L}-L+1)]}.$$

The solutions corresponding to the continuous part of the spectra of the invariant operators $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ can be found in a similar manner (Niederle and Limić 1968).

The form (44) of the harmonic functions for the group $SO_0(p,q)$ implies that the carrier space $\hat{H}(L)$ has the following structure

$$H(L)=\sum_{\tilde{L}=L+1}^{\infty}\oplus H(L,\tilde{L}), \tag{47}$$

where $\hat{H}(L,\tilde{L})$ is the infinite dimensional space on which the irreducible representation $\hat{T}(L,\tilde{L})$ of $SO(p,q-1)$ is realized. The space $\hat{H}(L,\tilde{L})$ is spanned by the harmonic functions (22) with L and $\tilde{L}$ fixed. The decomposition of the representation $\hat{T}(L)$ is

$$g\in SO_0(p,q-1),\qquad \hat{T}_g(L)=\sum_{\tilde{L}=L+1}^{\infty}\oplus\hat{T}_g(L,\tilde{L}).$$

C. *Maximal Set of Commuting Operators and Their Spectra*

For applications to physical problems of the group $SO_0(p, q)$ the discrete and continuous series of most degenerate representations $\hat{T}(L)$, $\tilde{T}(L)$ and $\hat{T}(\Lambda, \pm)$, $\tilde{T}(\Lambda, \pm)$ are especially convenient due to the following facts:

(i) The maximal set of commuting operators is maximally reduced in these representations of groups $SO_0(p, q)$. That is, for the discrete most degenerate representations of $SO_0(p, q)$, the maximal set of independent commuting operators in the enveloping algebra consists of

$$\Delta[SO(p, q)],\ R,$$
$$C_p \equiv \begin{Bmatrix} \Delta[SO(p)], \Delta[SO(p-2)], \ldots, \Delta[SO(4)] & \text{for } p \text{ even} \\ \Delta[SO(p)], \Delta[SO(p-1)], \Delta[SO(p-3)], \ldots, \Delta[SO(4)] & \text{for } p \text{ odd} \end{Bmatrix}, \tag{48}$$
$$\tilde{C}_q \equiv \begin{Bmatrix} \Delta[SO(q)], \Delta[SO(q-2)], \ldots, \Delta[SO(4)] & \text{for } p \text{ even} \\ \Delta[SO(q)], \Delta[SO(q-1)], \Delta[SO(q-3)], \ldots, \Delta[SO(4)] & \text{for } p \text{ odd} \end{Bmatrix},$$
$$H \equiv \begin{Bmatrix} -\dfrac{\partial}{\partial\varphi^k}, -\dfrac{\partial}{\partial\varphi^l}, & k = 1, 2, \ldots, [\tfrac{1}{2}p] \\ & l = 1, 2, \ldots, [\tfrac{1}{2}q] \end{Bmatrix},$$

where $\Delta[SO(p, q)]$ represents the second-order Casimir operator of $SO(p, q)$, and C_p and C_q the sequence of corresponding Casimir operators of the maximal compact subgroup $SO(p) \times SO(q)$. The set H contains operators of the Cartan subalgebra, except when p and q are odd, in which case H represents the maximal abelian compact subgroup of $SO_0(p, q)$. It should be noted that the reflection operator R, which is outside the enveloping algebra of $SO_0(p, q)$ is necessary for the characterization of an irreducible representation of the continuous series.

The number of operators contained in the maximal set of commuting operators in the enveloping algebra for the discrete most degenerate representatons of $SO_0(p, q)$ is equal to

$$N = p+q-1, \tag{49}$$

while the corresponding number for principal non-degenerate representations is

$$N' = \tfrac{1}{2}(r+l) = \tfrac{1}{4}[N(N+1)+2l],$$

where r and l are the dimension and the rank of $SO_0(p, q)$, respectively.

(ii) The additive quantum numbers may be related to the eigenvalues of the set H. It turns out that the set H is largest in the biharmonic coordinate system, which we have used.

(iii) The eigenfunctions of the maximal commuting set of operators are given in explicit form by formulas (22) and (30) ; the range of the numbers $L, l_2, \ldots, \ldots, l_{\{p/2\}} \tilde{l}_2, \ldots, \tilde{l}_{\{q/2\}}, m_1, \ldots, m_{[p/2]}, \tilde{m}_1, \ldots, m_{[q/2]}$ which may play the role of quantum numbers, is determined by (19), (21) and 10.3(22), respectively.

The main achievement of the present method consists in recasting some of the difficult problems of the representation theory of locally compact Lie groups into the language of the relatively simple theory of second-order differential equations. This method may also be applied to the explicit construction of the less degenerate representations which are determined by two, three, ..., k ($k \leqslant n$— rank of $SO_0(p, q)$) invariant numbers.

§ 4. Generalized Projection Operators

We developed in 7.3 the formalism of projection operators P^{λ}_{pq} which we used for an effective solution of various problems in the representation theory of compact groups and in particle physics. The operators P^{λ}_{pq} were defined by the formula

$$P^{\lambda}_{pq} = d_{\lambda} \int_G \overline{D^{\lambda}_{pq}}(x) T_x \mathrm{d}\mu(x). \tag{1}$$

If one tries to extend the formula (1) for noncompact groups one encounters the following difficulties:

(i) the matrix elements $D^{\lambda}_{pq}(x)$ are distributions from $\Phi'(G)$,

(ii) the volume $\int_G \mathrm{d}x$ is infinite.

Hence, the proper meaning of the integral (1) should be clarified. Consider first as an illustration the case of an abelian vector group G. In this case $D^{\lambda}_{pq}(X)$ reduces to $\exp(ipx)$ and the integral (1) takes the form

$$P^{\lambda} = (2\pi)^{-n/2} \int_G \exp(-\mathrm{i}\lambda x) T_x \mathrm{d}x, \tag{2}$$

where $x \to T_x$ is the regular representation of G. For $\varphi(x) \in \Phi(G)$ we have

$$\begin{aligned}(P^{\lambda}\varphi)(x) &= (2\pi)^{-n/2} \int_G \exp(-\mathrm{i}\lambda x')\varphi(x+x')\mathrm{d}x' \\ &= (2\pi)^{-n/2} \exp(\mathrm{i}\lambda x) \int_G \exp(-i\lambda y)\varphi(y)\mathrm{d}y \\ &= \exp(\mathrm{i}\lambda x)\hat{\varphi}(\lambda) \in H(\lambda) \subset \Phi'.\end{aligned}$$

Thus P^{λ} represents a map from $\Phi(G)$ into $\Phi'(G)$, i.e. it is an operator-valued distribution. To be precise, one should consider first the quantity

$$P^{\lambda}_N = (2\pi)^{-n/2} \int_{G_N} \exp(-\mathrm{i}\lambda x) T_x \mathrm{d}x, \tag{3}$$

where G_N is a compact subset of G and $\lim_{N\to\infty} G_N = G$. Since $\exp(-\mathrm{i}\lambda x) T_x$ is a continuous function on G and G_N has a finite Haar measure, the integral (3) is well defined. We have, moreover,

$$(P^{\lambda}\varphi)(x) \stackrel{\text{def}}{=} \lim_{N\to\infty} (P^{\lambda}_N\varphi)(x) = \exp(\mathrm{i}\lambda x)\hat{\varphi}(\lambda),$$

where the limit is taken in the topology of $\Phi'(G)$. We see therefore that the quantity P^λ given by eq. (2) is well defined for noncompact abelian vector groups as a weak limit of the operators given by (3), i.e.,

$$P^\lambda = \lim_{N\to\infty} (2\pi)^{-n/2} \int_{G_N} \exp(-i\lambda x) T_x dx.$$

If G is a vector group, then, the space Λ of indices λ which determines the irreducible representations of G is also a vector group. Consider the Schwartz's space $D(\Lambda)$ of functions with support in Λ. Then for $f \in D(\Lambda)$, one can define a 'smeared out' operator $P(f)$ of the form

$$P(f) = \int_\Lambda f(\lambda) P^\lambda d\lambda. \tag{4}$$

For $u \in \Phi(G)$ we have

$$\|P(f)u\| \leqslant \max_{\lambda\in\hat{G}} |f(\lambda)|\, \|u\|_H.$$

This means that $P(f)$ is a densely defined bounded operator in $H(G)$.

One readily verifies that

$$P(f_1)P(f_2) = P(f_1 f_2).$$

This equality can be written in the form of product of operator-valued distributions

$$P^\lambda P^{\lambda'} = \delta(\lambda-\lambda')P^\lambda.$$

This is the generalization for noncompact groups of the orthogonality relations 7.3(3).

We see therefore that for a proper description of the projection operators for noncompact groups we have to use the technique of operator-valued distributions. Hence we begin with a review of the basic notions concerning operator-valued distributions in a Hilbert space.

Let $H = L^2(\Lambda)$, where Λ is a subset of an n-dimensional Euclidean or Minkowski space R^n. Let $K(\Lambda)$ be a space of test functions (e.g. Schwartz $D(\Lambda)$ or $S(\Lambda)$ space).

DEFINITION 1. An *operator-valued distribution* P is a mapping $f \to P(f)$, $f \in K(\Lambda)$, with values in a set of linear operators in H satisfying

1° The operators $P(f)$ and $P(f)^*$, $f \in K(\Lambda)$, have a common dense domain D, which is a linear subset of H satisfying

$$P(f)D \subset D, \quad P(f)^*D \subset D. \tag{5}$$

2° On the domain D, $P(f)$ fulfils the conditions

$$P(\alpha f) = \alpha P(f),$$
$$P(f_1+f_2) = P(f_1)+P(f_2),$$

where $\alpha \in C^1$ and $f_1, f_2 \in K(\Lambda)$.

3° $P(f)$ is weakly continuous in $K(\Lambda)$, i.e. if $u, v \in D$ and $f \to 0$ then

$$(P(f)u, v) \to 0. \tag{6}$$

The operator-valued distribution $f \to P(f)$ can sometimes be written in the form of an integral

$$P(f) = \int_{\Lambda} f(\lambda) P(\lambda) \mathrm{d}\lambda. \tag{7}$$

The symbol $P(\lambda)$ has often a direct meaning and is currently also called an *operator-valued distribution.*

DEFINITION 2. The *adjoint* $P^*(\lambda)$ *of an operator distribution* $P(\lambda)$, is an operator distribution such that to a test function $f(\lambda)$ in $K(\Lambda)$ it assigns the operator $[P(\bar{f})]^*$, i.e.,

$$P^*(f) \equiv \int_{\Lambda} f(\lambda) P^*(\lambda) \mathrm{d}\lambda \equiv [P(\bar{f})]^* = \Big[\int_{\Lambda} \overline{f(\lambda)} P(\lambda) \mathrm{d}\lambda\Big]^*. \tag{8}$$

An operator-valued distribution is said to be *real* if $P(\lambda)^* = P(\lambda)$.

Let G be a locally compact Lie group, for which the assertions of th. 14.2.1 are valid, e.g. G is a semisimple Lie group, a group of motions of Euclidean or Minkowski space, etc. Let $C_1, \ldots, C_n$ be '+'-symmetric generators in the center $Z(E)$ of the enveloping algebra E of G and let $\bar{C}_1, \ldots, \bar{C}_n$ be the corresponding self-adjoint invariant operators. Let $H = L^2(G)$ and let $\Phi(G) \subset H \subset \Phi'(G)$ be a Gel'fand triplet, such that all $\bar{C}_i$ map Φ continuously into Φ. Let $\{D^{\lambda}_{pq}(x)\}$ be a set of generalized eigenfunctions of the operators $\bar{C}_1, \ldots, \bar{C}_n$ provided by th. 14.2.1 which satisfy the completeness 14.2(35) and orthogonality 14.2(34) relations and give the decomposition 14.2(36) of any element φ in $\Phi(G)$.

Let us now introduce the quantity

$$P^{\lambda}_{pq} = \varrho(\lambda) \int_{G} \overline{D^{\lambda}_{pq}(x)}\, T_x \mathrm{d}x, \tag{9}$$

where T_x is the right regular representation, i.e.,

$$(T_x\varphi)(y) = \varphi(yx), \tag{10}$$

and $\varrho(\lambda)$ is a spectral measure associated with the invariant operators $\bar{C}_1, \ldots, \bar{C}_n$ of G. We have

PROPOSITION 1. *The quantity* P^{λ}_{pq} *given by* eq. (9) *represents an operator-valued distribution in the space* $H = L^2(G)$ *and maps* $\Phi(G)$ *into* $\Phi'(G)$.

PROOF: Set

$$P^{\lambda}_{pqN} = \varrho(\lambda) \int_{G_N} \overline{D^{\lambda}_{pq}(x)}\, T_x \mathrm{d}x, \tag{11}$$

where G_N is a compact subset of G and $\lim_{N\to\infty} G_N = G$. Since $\overline{D^{\lambda}_{pq}(x)}\, T_x$ is continuous on G, the integral (11) gives a well defined operator in H. For $\varphi \in \Phi(G)$ we have

$$(P^{\lambda}_{pqN}\varphi)(y) = \varrho(\lambda) \int_{G_N} dx\overline{D^{\lambda}_{pq}(x)}(T_x\varphi)(y) = \varrho(\lambda) \int_{yG_N} dz\overline{D^{\lambda}_{pq}(y^{-1}z)}\varphi(z)$$
$$= \varrho(\lambda) \sum_r \overline{D^{\lambda}_{pr}(y^{-1})} \int_{yG_N} dz\overline{D^{\lambda}_{rq}(z)}\varphi(z)$$
$$\xrightarrow[N\to\infty]{} \varrho(\lambda) \sum_r D^{\lambda}_{rp}(y)\hat{\varphi}_{rq}(\lambda). \tag{12}$$

The interchange of integration and summation in eq. (12) is justified by Fubini–Tonelli theorem if $\varphi \in L^1(G)$. Indeed,

$$\Big|\sum_r \overline{D^{\lambda}_{pr}(y^{-1})}\,\overline{D^{\lambda}_{rq}(z)}\,\varphi(z)\Big|^2 \leqslant |\varphi(z)|^2 \sum_r |\overline{D^{\lambda}_{pr}(y^{-1})}|^2 \sum_{r'} |\overline{D^{\lambda}_{r'q}(z)}|^2 = |\varphi(z)|^2,$$

since e.g.

$$\sum_r |\overline{D^{\lambda}_{pr}(z)}|^2 = \sum_r \langle e_p, T^{\lambda}_z e_r\rangle_{\hat{H}(\lambda)}\langle T^{\lambda}_z e_r, e_p\rangle_{\hat{H}(\lambda)} = 1.$$

We see that, as for abelian vector groups, we have

$$P^{\lambda}_{pq} = \lim_{N\to\infty} \varrho(\lambda) \int_{G_N} dx\overline{D^{\lambda}_{pq}(x)}\, T_x, \tag{13}$$

in the sense of the weak limit of operators $(P^{\lambda}_{pq})_N$. The expression (4.12) shows that for any $\varphi(y)$ in $\Phi(G)$ the quantity $(P^{\lambda}_{pq}\varphi)(y)$ is an element of $\varphi(\lambda, y) \in H(\lambda) \subset \Phi'(G)$. Therefore, by eq. (3) of app. B, $P^{\lambda}_{pq}\varphi$ is a generalized eigenvector of the invariant operators $C_1, \ldots, C_n$. Thus, the domain $D(P^{\lambda}_{pq})$ of P^{λ}_{pq} in $H = L^2(G)$ consists of the null-vector only and consequently, P^{λ}_{pq} cannot be considered as an operator in H. However, if $f(\lambda)$ is an element of the space $C_0(\Lambda)$ of continuous functions on Λ then, the smeared-out operator

$$P_{pq}(f) = \int_{\Lambda} f(\Lambda) P^{\lambda}_{pq} d\lambda, \tag{14}$$

represents a bounded linear operator. Indeed, for $\varphi \in \Phi(G)$, one obtains from eq. (12)

$$P_{pq}(f)\,\varphi(y) = \int_{\Lambda} d\lambda f(\lambda)\varrho(\lambda) \sum_r D^{\lambda}_{rp}(y)\hat{\varphi}_{rq}(\lambda).$$

Using the Plancherel equality 14.2(17) we have

$$||P_{pq}(f)\varphi||^2 = \int_{\Lambda} |f(\lambda)|^2 \sum_r \hat{\varphi}_{rq}(\lambda)\overline{\hat{\varphi}_{rq}(\lambda)}\varrho(\lambda)\, d\lambda \leqslant \max_{\lambda\in\Lambda}|f(\lambda)|^2||\varphi||^2,$$

i.e.

$$||P_{pq}(f)\varphi|| \leqslant \max_{\lambda\in\Lambda} |f(\lambda)|\, ||\varphi||. \tag{15}$$

Since the operators $P_{pq}(f)$ are bounded for any $f \in C_0(\Lambda)$ one can take the whole space H as a common dense domain D in def. 1. It is then evident that $P_{pq}(f)$ satisfies the conditions 1° and 2° of def. 1. Condition 3° follows from eq. (15). In fact,

$$|(P_{pq}(f)\varphi, \psi)|^2 \leqslant \|P(f)\varphi\|^2\|\psi\|^2 \leqslant \max_{\lambda\in\Lambda}|f(\lambda)|^2\|\varphi\|^2\|\psi\|^2.$$

Hence, $(P_{pq}(f)\varphi, \psi) \to 0$ whenever $f \to 0$ in $C_0(\Lambda)$. We see, therefore, that the mappings $P^\lambda_{pq}\colon \Phi(G) \to H(\lambda) \subset \Phi'(G)$ given by eq. (13) represent operator-valued distributions in the Hilbert space $H = L^2(G)$. ▼

Remark: If T_x represents the operator of a left translation in $L^2(G)$ then the operator distribution P^λ_{pq} has to be taken in the form

$$P^\lambda_{pq} = \varrho(\lambda)\int \overline{D^\lambda_{pq}(x)}\, T_{x^{-1}}\mathrm{d}x. \tag{16}$$

It is only with such a definition that $(P^\lambda_{pq}\varphi)(y)$ represents an element $\varphi(\lambda, y)$ in $H(\lambda)$ (cf. eq. (12)).

The operator-valued distribution P^λ_{pq} satisfies certain hermicity and orthogonality relations which are very useful in applications. In fact, we have:

PROPOSITION 2. *Let P^λ_{pq} be an operator-valued distribution given by formula* (9). *Then,*

$$(P^\lambda_{pq})^* = P^\lambda_{qp} \tag{17}$$

and

$$P^\lambda_{pq}P^{\lambda'}_{p'q'} = \delta(\lambda-\lambda')\,\delta_{qp'}P^\lambda_{pq'}. \tag{18}$$

PROOF: The operator $P^\lambda_{pq}(f)$ is bounded. Hence, for any φ, ψ in $\Phi(G)$, by eqs. (7), (8) and (12) and the Plancherel equality, 14.2(17) one obtains

$$(\varphi, P^{*\lambda}_{pq}(f)\psi) = (P^\lambda_{pq}(\bar f)\varphi, \psi) = \Big(\int d\lambda\overline{f(\lambda)}P^\lambda_{pq}\varphi, \psi\Big) = \int \mathrm{d}\lambda\overline{f(\lambda)}\varrho(\lambda)\hat\varphi_{rq}(\lambda)\overline{\hat\psi_{rp}(\lambda)}.$$

Using the same formulae one verifies that the last expression is equal to $(\varphi, P^\lambda_{pq}(f)\psi)$. Hence,

$$P^{*\lambda}_{pq}(f) = P^\lambda_{qp}(f), \tag{19}$$

and eq. (17) follows by def. 1 and eq. (7). We now prove eq. (18). In fact, for any f, g in $C_0(\Lambda)$ and φ, ψ in $L^2(G)$, eqs. (19) and (12) and the Plancherel equality 14.2 (17) yield

$$\begin{aligned}(P^\lambda_{pq}(f)P^\lambda_{p'q'}(g)\varphi, \psi) &= (P^\lambda_{p'q'}(g)\varphi, P^\lambda_{qp}(\bar f)\psi)\\ &= \delta_{p'q}\int \mathrm{d}\lambda f(\lambda)g(\lambda)\varrho(\lambda)\hat\varphi_{rq'}(\lambda)\overline{\hat\psi_{rp}(\lambda)}.\end{aligned} \tag{20}$$

One readily verifies that the last expression can be written in the form

$$\Big(\Big\{\int_\Lambda f(\lambda')d\lambda'\int_\Lambda g(\lambda)d\lambda\,\delta(\lambda-\lambda')\,\delta_{p'q}P^\lambda_{pq'}\Big\}\varphi, \psi\Big).$$

Comparing this with the first expression of eq. (20), one obtains eq. (18) by proposition 1 and eq. (7). ▼

The operator-valued distributions P^{λ}_{pq} have simple transformation properties with respect to the action of the group G. Indeed, we have

PROPOSITION 3. *Let P^{λ}_{pq} be an operator-valued distribution given by formula (9) and let $x \in G$. Then*

$$T_x P^{\lambda}_{pq} = \sum_r D^{\lambda}_{rp}(x) P^{\lambda}_{rq}, \tag{21}$$

$$P^{\lambda}_{pq} T_x = \sum_r D^{\lambda}_{rq}(x) P^{\lambda}_{pr}. \tag{22}$$

PROOF: Since T_x is continuous, the product $T_x P^{\lambda}_{pq}$ is an operator-valued distribution and $T_x P_{pq}(f)$ is a bounded operator in H by proposition 1. Utilizing formula (9) and the Plancherel equality one obtains

$$\begin{aligned}(T_x P^{\lambda}_{pq}(f)\varphi, \psi) &= \int_G \mathrm{d}y (P^{\lambda}_{pq}(f)\varphi)(yx)\overline{\psi(y)} \\ &= \int_\Lambda \mathrm{d}\lambda f(\lambda)\varrho(\lambda) D^{\lambda}_{q'p}(x)\hat{\varphi}_{p'q}(\lambda)\overline{\hat{\psi}_{p'q'}(\lambda)} \\ &= \left(\int_\Lambda \mathrm{d}\lambda f(\lambda) D^{\lambda}_{q'p}(x') P^{\lambda}_{q'q}\varphi, \psi\right).\end{aligned}$$

Comparing the first and last terms of this equality and utilizing def. 1 and eq. (7), one obtains eq. (21). Similarly, one proves eq. (22). ▼

By eqs. (21) and (22) one also obtains

$$T_x P^{\lambda}_{pq} T_x^{-1} = D^{\lambda}_{rp}(x)\overline{D^{\lambda}_{sq}(x)} P^{\lambda}_{rs}. \tag{23}$$

Formula (23) means that P^{λ}_{pq} transforms as a tensor operator corresponding to the tensor product of a basis vector $e_p(\lambda)$ and an adjoint vector to $e_q(\lambda)$ (i.e., as the product $|\lambda{:}p\rangle\langle\lambda{:}q|$ in Dirac's notation).

In some cases, like e.g. in the case of semisimple groups or the Poincaré group, the character $\chi^{\lambda}(x) = \operatorname{Tr} T_x(\lambda)$ is a well defined distribution on G. In this case one may define the following operator-valued distributions in $H = L^2(G)$

$$P^{\lambda} = \varrho(\lambda) \int_G \mathrm{d}x \overline{\chi^{\lambda}(x)} T_x. \tag{24}$$

One verifies similarly as in propositions 2 and 3 that

$$(P^{\lambda})^* = P^{\lambda}, \tag{25}$$

$$P^{\lambda} P^{\lambda'} = \delta(\lambda - \lambda') P^{\lambda}, \tag{26}$$

$$T_x P^{\lambda} = P^{\lambda} T_x. \tag{27}$$

The operator-valued distributions P^{λ} are useful in applications. If $H(X)$ is a carrier Hilbert space of a unitary representation T of G and $\Phi \subset H \subset \Phi'$ is the Gel'fand triplet, then P^{λ} projects Φ onto the generalized eigenspace $H(\lambda) \subset \Phi'$ The space $P^\lambda \Phi$ isomorphic with $H(\lambda)$ is invariant under T and it is isomorphic to the Hilbert space $\hat{H}(\lambda)$ by formula 3(30) of app. B.

We have considered so far the formalism of operator-valued distributions in the Hilbert space $H = L^2(G)$. However, all of our results can be extended to the space $H = L^2(X)$, where $X = G/G_0$ is the homogeneous space of right G-cosets $\{G_0 g\}$, G is a connected Lie group and G_0 is a closed subgroup of G. The group G acts on the elements $\varphi \in \Phi(x) \subset L^2(x)$ by means of the right translation

$$(T_g \varphi)(x) = \varphi(xg),$$

i.e., the map $g \to T_g$ gives the unitary quasi-regular representation of G in $L^2(X)$.

Suppose that all assumptions of th. 2.1 are satisfied. Then the generalized Fourier expansion of an element $\varphi \in \Phi(X) \subset L^2(X)$ is given by the formula 1(4). Using formula (9) for P^λ_{pq} and eqs. 1(4) and 2(12) one obtains

$$\begin{aligned}(P^\lambda_{pq}\varphi)(x) &= \varrho(\lambda)\int \mathrm{d}g\,\overline{D^\lambda_{pq}(g)}\,\varphi(xg)\\ &= \varrho(\lambda)\int_G \mathrm{d}g\,\overline{D^\lambda_{pq}(g)}\int_{\tilde{\Lambda}}\sum_{r,s} D^{\lambda'}_{s,r}(g)\hat{\varphi}_r(\lambda')e_s(\lambda' x)\,\mathrm{d}\tilde{\varrho}(\lambda')\\ &= \tilde{\varrho}(\lambda)\hat{\varphi}_g(\lambda)e_p(\lambda, x) \in H(\lambda) \subset \Phi'(X),\end{aligned} \tag{28}$$

provided $\hat{\varphi}_r(\lambda)$ and $e_s(\lambda, x)$ are continuous functions of λ; this condition is satisfied in most cases of practical interest. Hence, as in case of $L^2(G)$, the quantity P^λ_{pq} represents a map from $\Phi(X)$ into $H(\lambda) \subset \Phi'(x)$, i.e., it defines an operator-valued distribution. The proof of propositions 2 and 3 for operator-valued distributions P^λ_{pq} in $L^2(X)$ runs similarly and we omit them.

As we have shown, the spaces $H(\lambda)$ spanned by the generalized eigenvectors $e_k(\lambda)$ play an important role in applications. The operator-valued distributions P^λ_{pq} and P^λ provide a natural tool for the separation of these spaces from the space $H(G)$ or $H(X)$.

Operator-valued distributions P^λ_{pq} provide a convenient method for the solution of various practical problems encountered in group representation theory and quantum physics. For instance, one can derive a general formula for the Clebsch–Gordan coefficients for non-compact Lie groups. Indeed, let $\tilde{H} = H(\lambda_1) \otimes H(\lambda_2)$ be the carrier space of the tensor product $T = T^{\lambda_1} \otimes T^{\lambda_2}$ of irreducible representations T^{λ_1} and T^{λ_2} of G and let $\{e_k(\lambda_i)\}_{k=1}^{\infty}$ be a basis in $H(\lambda_i)$, $i = 1, 2$. Then, for constant q, r and s the element

$$e_p(\lambda) = P^\lambda_{pq} e_r(\lambda_1) e_s(\lambda_2), \tag{29}$$

by virtue of eq. (21), has the following transformation properties

$$T_x e_p(\lambda) = \sum_r D^\lambda_{rp}(x) e_r(\lambda),$$

i.e., it transforms according to an irreducible unitary representation T^λ of G. Consequently the expression

$$\begin{aligned}\langle\lambda, p|\lambda_1, p_1;\, \lambda_2 p_2\rangle &\equiv N^{-1}\langle e_p(\lambda),\, e_{p_1}(\lambda_1)e_{p_2}(\lambda_2)\rangle \\ &\equiv N^{-1}\langle P^{\lambda}_{pq}e_r(\lambda_1)e_s(\lambda_2),\, e_{p_1}(\lambda_1)e_{p_2}(\lambda_2)\rangle \end{aligned}\tag{30}$$

is the projection of a basis vector $e_p(\lambda)$ upon the basis vector $e_{p1}(\lambda_1)e_{p2}(\lambda_2)$ and represents the Clebsch–Gordan coefficient. The constant N represents a normalization constant of the vector (29). We see that the Clebsch–Gordan coefficient (30) is, in fact, the matrix element of the operator-valued distribution P^{λ}_{pq} in the tensor product basis of the space $\tilde{H}$. Using eq. (9) for P^{λ}_{pq} and the relation

$$\begin{aligned}T_x e_r(\lambda_1)e_s(\lambda_2) &= T^{\lambda_1}_x e_r(\lambda_1)\cdot T^{\lambda_2}_x e_s(\lambda_2) \\ &= \sum_{r_1} D^{\lambda}_{r_1 r}(x)e_{r_1}(\lambda_1)\sum_{s_2} D^{\lambda_2}_{s_2 s}e_{s_2}(\lambda_2),\end{aligned}$$

one obtains

$$\langle\lambda, p|\lambda_1, p_1;\, \lambda_2 p_2\rangle = N^{-1}d^{\lambda}\int_G \overline{\mathrm{d}x\, D^{\lambda}_{pq}(x)\, D^{\lambda_1}_{p_1 r}(x)\, D^{\lambda_2}_{p_2 s}(x)}. \tag{31}$$

§ 5. Comments and Supplements

A. *Harish–Chandra and Helgason Theory*

We shall now describe a very interesting approach to harmonic analysis on symmetric spaces G/K based on geometric ideas. This theory was originated by Gel'fand and Harish–Chandra and finally completed by Helgason 1967, 1972, 1973, 1974.

In the case of ordinary Fourier analysis, we have

$$\hat{\varphi}(p) = \int_{R^n} \varphi(x)\exp\overline{[\mathrm{i}(x, p)]}\,\mathrm{d}x,$$

where $(x, p) \equiv x_\mu p^\mu$, and

$$\varphi(x) = (2\pi)^{-n}\int_{R^n} \hat{\varphi}(p)\exp[\mathrm{i}(x, p)]\,\mathrm{d}p,$$

or, in polar coordinates $p = \lambda\omega$, $\lambda \geqslant 0$ and ω a unit vector,

$$\hat{\varphi}(\lambda\omega) = \int_{R^n} \varphi(x)\exp[\mathrm{i}\lambda(x, \omega)]\,\mathrm{d}x, \tag{1}$$

$$\varphi(x) = (2\pi)^{-n}\int_{R^+}\int_{S^{n-1}} \hat{\varphi}(\lambda\omega)\exp[\mathrm{i}\lambda(x, \omega)]\,\lambda^{n-1}\mathrm{d}\lambda\,\mathrm{d}\omega, \tag{2}$$

where $R^+ = \{\lambda \in R\colon \lambda \geqslant 0\}$ and $\mathrm{d}\omega$ is the volume element on the unit sphere S^{n-1}. The function $e_p\colon x \to \exp[\mathrm{i}(x, p)]$ has the following properties;

(i) e_p is an eigenfunction of the Laplace operator on R^n.

(ii) e_p is constant on each hyperplane perpendicular to p (i.e., e_p is a plane wave with the normal p).

In order to extend the harmonic analysis from R^n to the symmetric spaces $X = G/K$, one needs a generalized 'plane wave' e_p on X which would satisfy properties (i) and (ii) and provide an expansion of functions from $L^2(X, \mu)$.

The simplest nontrivial case where this extension can be done is the symmetric space $X = \mathrm{SU}(1, 1)/U(1)$, which is isomorphic to a disc $D = \{z \in C\colon |z| < 1\}$. (See ch. 4, exercise 5.2.4.) We shall now generalize the geometric properties of plane waves to the curved space D. Let B be the boundary of D, i.e., $B = \{z \in C\colon |z| = 1\}$. The parallel geodesics in D are by definition geodesics originating from the same point b on the boundary B of D (see Fig. 1).

A horocycle with normal $b \in B$ is by definition an orthogonal trajectory to the family of all parallel geodesics corresponding to b (see Fig. 1.) Hence a horo-

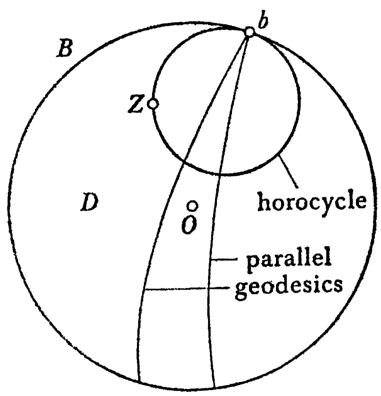

Fig. 1

cycle in D is the non-euclidean analog of a hyperplane in R^n. Now the inner product (x, ω) in eq. (1) is the distance from the origin to the hyperplane with normal ω passing through x. By analogy, we define $\langle z, b\rangle$ for $z \in D$, $b \in B$ to be the Riemannian distance from 0 to the horocycle $\xi(z, b)$ with normal b, passing through z. Now the function

$$e_{p,b}\colon z \to \exp(p\langle z, b\rangle), \qquad p \in C,\ b \in B,\ z \in D \tag{3}$$

has the properties of plane waves on R^n. Indeed,

(i) $e_{p,b}$ is the eigenfunction of the Laplace–Beltrami operator Δ on D ($\Delta = [1-(x^2+y^2)]^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, see exercise 4.5.2.5).

(ii) $e_{p,b}$ is constant on each horocycle $\xi(z, b)$ with normal b.

The following theorem shows that the eigenfunctions $e_{p,b}(z)$ on D are, in fact, the natural extension of the plane waves e_p on R^n.

THEOREM 1. *Let $\varphi \in C_0(D)$. Set*

$$\hat{\varphi}(p, b) = \int_D \varphi(z)\exp[(-ip+1)\langle z, b\rangle]\mathrm{d}z, \qquad p \in R,\ b \in B, \tag{4}$$

where $\mathrm{d}z = (1-|z|^2)^{-2}\mathrm{d}x\mathrm{d}y$ *is the volume element on* D. *Then*

$$\varphi(z) = (2\pi)^{-2} \int_R \int_B \hat{\varphi}(p, b)\exp[(ip+1)\langle z, b\rangle]p\tanh(\tfrac{1}{2}\pi p)\,\mathrm{d}p\,\mathrm{d}b, \tag{5}$$

where $\mathrm{d}b$ *is the ordinary angular measure on* B $(= S^1)$. ▼

(For the proof see Helgason 1967, th. 3.1.)

To extend this theorem to an arbitrary symmetric space, we need a generalization of the concepts of the boundary B, the horocycle ξ and the complex distance $\langle z, b\rangle$.

Let K be the maximal compact subgroup of a semisimple Lie group G and let L be the Lie algebra of G. Let $L = K \oplus H_p \oplus N_0$ be the Iwasawa decomposition of L. Denote by M the centralizer of H_p in K; that is,

$$M = \{k \in K : (\mathrm{Ad}k)Y = Y, \text{ for all } Y \in H_p\}. \tag{6}$$

It turns out that a generalization of the boundary B for an arbitrary symmetric space is given by the coset space $B = K/M$.

The group-theoretical analysis of the horocycle on the disc D reveals that horocycles are the orbits in D of all groups of the form $g\mathcal{N}g^{-1}$, where $\mathcal{N}$ is the nilpotent group associated with the algebra $\mathcal{N}_0$ of SU(1,1). This suggests that for an arbitrary symmetric space $X = G/K$ we may define a horocycle to be an orbit in X of a subgroup of G of the form $g\mathcal{N}g^{-1}$, where $\mathcal{N}$ is the nilpotent group in the Iwasawa decomposition $G = K\mathscr{A}_p\mathcal{N}$. One readily verifies the following properties of horocycles:

LEMMA 2.

(i) *The group* G *permutes the horocycles transitively.*

(ii) *Each horocycle* ξ *can be written in the form*

$$\xi = ka\xi_0, \tag{7}$$

where a *is the unique element of the subgroup* $\mathscr{A}_p$.

(iii) *Given* $x \in X$, $b \in B$, *there exists exactly one horocycle passing through* x *with normal* b. ▼

(For the proof see Helgason 1967.)

The element $a \in \mathscr{A}_p$ in eq. (7) is called the *complex distance* from the coset $K = 0$ to ξ. We denote the complex distance $a \in \mathscr{A}_p$ from 0 to the horocycle determined by lemma 2, (iii), by the symbol $\exp A(x, b)$, $A(x, b) \in H_p$.

Now we give a generalization of plane waves on arbitrary symmetric spaces. Let $b \in B$ and let p be a complex linear functional on H_p and set

$$e_{p,b}: x \to \exp[p(A(x, b))], \quad x \in X.$$

It is evident that $\exp[p(A(x, b))]$ is constant on horocycles. We have

THEOREM 3. *Let* $X = G/K$ *be a Cartan symmetric space. Then*

(i) *The functions* $e_{p,b}$ *are eigenfunctions of all the invariant operators* C *from* *the center Z of the enveloping algebra E of G, represented by differential operators in $L^2(X, \mu)$.*

(ii) *Define for $\varphi \in C_0^\infty(X)$, a generalized Fourier transform by the formula*

$$\hat{\varphi}(p, b) = \int_X \varphi(x) \exp[(-\mathrm{i}p + \varrho) A(x, b)] \,\mathrm{d}\mu(x), \tag{8}$$

where p is an element of the real dual H_p^* of H_p, $b \in B$ and $\varrho = \sum_{\alpha > 0} \alpha$. Then the spectral synthesis formula for $\varphi(x)$ has the form

$$\varphi(x) = \int_{H_p^*} \int_B \hat{\varphi}(p, b) \exp[(\mathrm{i}p + \varrho) A(x, b)] |c(p)|^{-2} \mathrm{d}p \,\mathrm{d}b, \tag{9}$$

if the Euclidean measure $\mathrm{d}p$ on H_p^* is suitably normalized. The spectral density $|c(p)|^{-2}$ is defined by the formula

$$c(p) = \int_{\overline{\mathcal{N}}} \exp[(-\mathrm{i}p - \varrho) Y(\bar{n})] \mathrm{d}\bar{n}, \tag{10}$$

where $\overline{\mathcal{N}}$ is the analytic subgroup associated with the Lie algebra $\bar{N} = \sum_{\alpha < 0} L_\alpha$ and $Y(\bar{n}), \bar{n} \in \overline{\mathcal{N}}$ is determined by the Iwasawa decomposition $\bar{n} = k(\bar{n}) \exp[Y(\bar{n})] n(\bar{n})$.

Furthermore, we have the Parseval formula

$$\int |\varphi(x)|^2 \mathrm{d}\mu(x) = \int_{H_p^*} \int_B \hat{\varphi}(p, b) |c(p)|^{-2} \mathrm{d}p \,\mathrm{d}b, \tag{11}$$

and the direct integral representation of the space $H = L^2(X, \mu)$ and of the representation T_g: $\varphi(x) \to \varphi(g^{-1}x)$ are given by

$$H \to \hat{H} = \int \hat{H}(p) |c(p)|^{-2} \mathrm{d}p, \quad \hat{T}_g = \int \hat{T}_g(p) |c(p)|^{-2} \mathrm{d}p, \tag{12}$$

where p runs over H_p^* modulo the Weyl groups. All the functions $u_p(x) \in H(p)$ given by

$$u_p(x) = \int_B \exp[\mathrm{i}(p + \varrho) A(x, b)] u(b) \mathrm{d}b, \quad u(b) \in L^2(B, \mathrm{d}b), \tag{13}$$

are eigenfunctions of all invariant differential operators of the center Z of the enveloping algebra E of G. ▼

(For the proof see Helgason 1967.)

Th. 3 represents one of the most remarkable results in the theory of harmonic analysis on homogeneous spaces. We emphasize that the proof of the theorem is essentially geometric and does not use the spectral analysis of self-adjoint operators and all the machinery of functional analysis, yet it still provides the spectral measure in explicit form.

B. *Comments*

(i) The first work dealing with harmonic analysis on homogeneous spaces was done by Hecke in 1918, where finite-dimensional spaces of continuous functions on the sphere S^2, invariant under rotations were classified. Later on E. Cartan in 1929 extended Peter–Weyl harmonic analysis on compact groups to harmonic analysis on compact Riemannian spaces with transitive compact Lie group of isometries. However, the full scale activity in this field of research began after 1950. The most important contributions were done in the works of Gel'fand 1950, Godement 1952, Berezin and Gel'fand 1956, Gel'fand and Graev 1959, Harish-Chandra 1958, I and II, Berezin 1957, Gindikin and Karpelevic 1962, Helgason 1959, 1962, 1965, 1967, 1970, 1972, 1973, 1974, and Vilenkin 1956, 1963 and 1968.

The harmonic analysis on arbitrary homogeneous spaces, presented in sec. 2 was elaborated by K. Maurin and L. Maurin 1964 (see also K. Maurin 1969, ch. VII). It provides an elegant solution of basic problems of harmonic analysis on homogeneous spaces, based on general nuclear spectral theorem. The harmonic analysis on symmetric spaces of rank one with pseudoorthogonal transformation group was elaborated by Limić, Niederle and Rączka 1966 a and b and 1967. The harmonic analysis on homogeneous spaces of rank one of $SO_0(p, q)$ groups with noncompact stability group was elaboratet by Niederle 1967 and Limić and Niederle 1968. The extension of this theory for symmetric spaces associated with pseudo-unitary groups $U(p, q)$ and symplectic groups $Sp(n)$ was elaborated by Fischer and Rączka 1966, 1967 and Pajas and Rączka 1969, respectively.

The geometric approach to harmonic analysis on symmetric spaces presented in sec. 4.A was originated by Harish-Chandra 1958, I and II and completed by Helgason 1967, 1970, 1972, 1973, 1974.

(ii) We considered in sec. 3 the harmonic analysis on symmetric spaces of rank one associated with pseudo-orthogonal groups $SO_0(p, q)$, $p \geqslant q > 2$. The same analysis can be done for conformal type groups $SO_0(p, 2)$ and Lorentz type groups $SO_0(p, 1)$. The detailed analysis may be found in a series of papers by Limić, Niederle and Rączka 1966 a, b, 1967. In the papers a further case where the stability group is not simple is also treated, e.g., for symmetric spaces X_0 given by eq. 3(3).

(iii) The generalized projection operators P^{λ}_{pq} for noncompact groups were introduced by Rączka 1969. The application of these operators for the explicit calculation of Clebsch–Gordan coefficients for Lorentz group was given by Anderson, Rączka, Rashid and Winternitz 1970 a, b.

§ 6. Exercises

§ 1.1. Show that the Laplace–Beltrami operator does not exist on the cone

$$(x^1)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - \ldots - (x^{p+q})^2 = 0. \tag{1}$$

Hint: Show that the metric tensor 3(8) is singular.

§ 1.2. Show that on symmetric spaces $X^{p,q}_{+} = U(p,q)/U(p-1,q)$ given by

$$|z^1|^2 + \ldots + |z^p|^2 - |z^{p+1}|^2 - \ldots - |z^{p+q}|^2 = 1 \tag{2}$$

the ring of invariant operators is generated by the Laplace-Beltrami operator and the operator $C_1 = \sum_{i=1}^{p+q} X_i$, where X_i are generators of $U(p,q)$.

§ 2.1. Find the matrix elements of irreducible most degenerate representations of the group $\mathrm{Sp}(n)$.

Hint: Represent every element x of $\mathrm{Sp}(n)$ as a product of one-parameter subgroups and use the method of Pajas and Rączka 1968 for the explicit construction of the carrier space.

§ 3.1. Let $G = \mathrm{SO}(2,2)$ and let $H = L^2(X, \mu)$, where $X = \mathrm{SO}(2,2)/\mathrm{SO}(1,2)$, is realized by the hyperboloid

$$(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 = 1. \tag{3}$$

Show that there exists in H the discrete series of representations of G characterized by the eigenvalues

$$\lambda = -L(L+2), \qquad L = 0, 1, 2, \ldots \tag{4}$$

of the Laplace–Beltrami operator and by the eigenvalues of the reflection operator $Ru(x) = u(-x)$.

§ 3.2. Show that the carrier spaces $H_{\pm}(L)$ of irreducible representations of the discrete series of the previous exercise have the structure given on p. 472, Fig. 3, where m and $\tilde{m}$ are invariant numbers characterizing the representation of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ subgroup, satisfying the condition $|m| - |\tilde{m}| = \pm(L+2+2n)$, $n = 0, 1, 2, \ldots$

The representations obtained after changing m and $\tilde{m}$ are equivalent to a pair of previous representations (and are denoted by the dotted lines in the figure).

Hint: Use the reduction of the Laplace–Beltrami operator for $\mathrm{SO}(2,2)$ with respect to $\mathrm{SO}(2) \times \mathrm{SO}(2)$-subgroup.

§ 3.3.* Let $G = U(2,2)$ and let $H = L^2(X, \mu)$, where $X = U(2,2)/U(1,2)$ is represented by the following hypersurface in C^4

$$|z^1|^2 + |z^2|^2 - |z^3|^2 - |z^4|^2 = 1. \tag{5}$$

Show that there exists in H the discrete series of representations of G, which is characterized by the eigenvalues

$$\lambda = -L(L+6), \qquad L = -2, -1, 0, 1, 2, \ldots,$$

of the Laplace–Beltrami operator and by the eigenvalues of the operator $C_1 = \sum_{i=1}^{4} X_i$, where X_i are generators of the Cartan subgroup of G.

Hint: Introduce the biharmonic coordinate system on X and reduce $\Delta(X)$ to a one-dimensional Schrödinger operator.

§ 3.4.*** Let $G = \mathrm{Sp}(p, q)$. Construct the degenerate series of irreducible representations of G in the spaces $H = L^2(X^+, \mu)$, where $X^+ = \mathrm{Sp}(p, q)/\mathrm{Sp}(p-1, q)$ and $X^- = \mathrm{Sp}(p, q)/\mathrm{Sp}(p, q-1)$.

Hint: Use the method of Pajas and Rączka 1968 which they applied for the construction of degenerate series of irreducible representations of $\mathrm{Sp}(n)$.

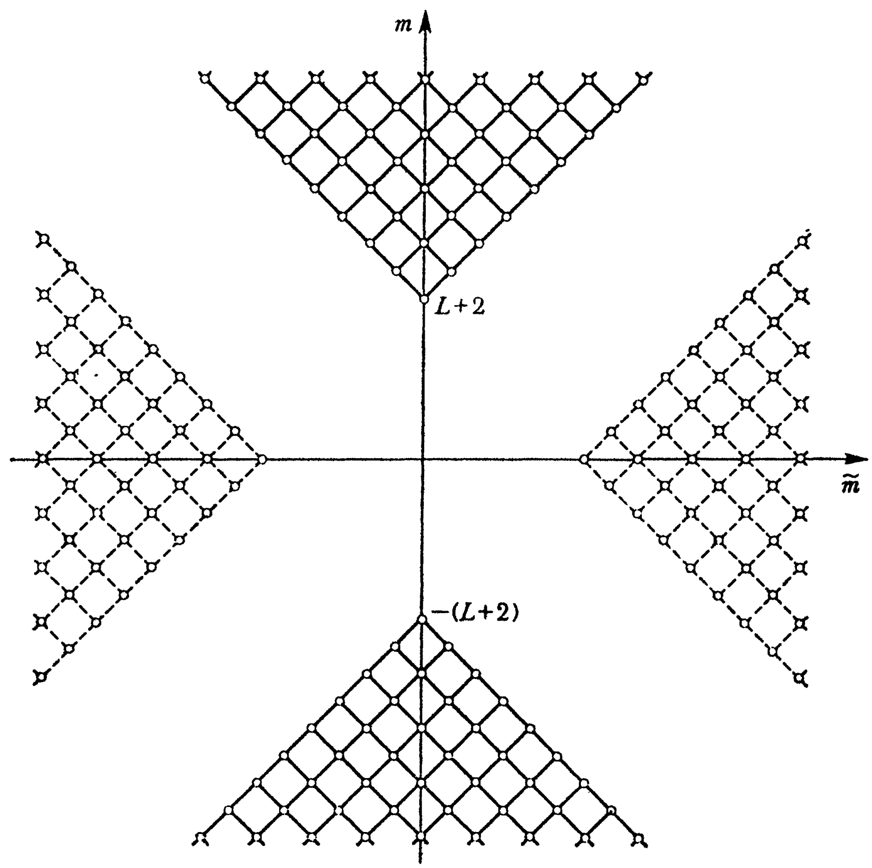

Fig. 3

Chapter 16

Induced Representations

§ 1. The Concept of Induced Representations

We constructed in ch. 7 all irreducible finite-dimensional representations of an arbitrary connected Lie group, using the technique of induced representations. In this chapter we give a method of construction of induced unitary representations of *any* topological, locally compact, separable group G. We begin with the construction of the carrier space, which is a generalization of the construction given in ch. 8, § 2, for the finite-dimensional case.

A. *The Construction*

Let K be a closed subgroup of G and let $k \to L_k$ be a unitary representation of K in a separable Hilbert space H. Let μ be any quasi-invariant measure in the homogeneous space $X = K\backslash G = \{Kg, g \in G\}$ of the right K-cosets. Consider the set H^L of all functions u with domain in G and range in H, satisfying the following conditions:

1° $\big(u(g), v\big)$ is measurable (relative to $\mathrm{d}g$) for all $v \in H$.

2° $u(kg) = L_k u(g)$, for all $k \in K$ and all $g \in G$. (1)

3° $\int_X ||u(g)||^2 \mathrm{d}\mu(\dot{g}) < \infty$, $\dot{g} = Kg$, where $||u(g)||$ is the norm in the space H.

Condition 3° requires some further explanation. We first note that

$$||u(kg)|| = ||L_k u(g)|| = ||u(g)||,$$

because L_k is unitary. Hence, $||u(g)||$ defines a function on the right K-cosets (i.e., on the space $X = K\backslash G$).

Remark: Each point $x = Kg$ of $K\backslash G$ remains fixed under the action of the subgroup $g^{-1}Kg \cong K$ from right. Hence the subgroup K is called the *stability* (stationary, isotropy, little) *group* of the space X (cf. ch. 4, § 1).

LEMMA 1. *The space H^L defined by eq.* (1) *is isomorphic to the Hilbert space $L^2(X, \mu, H)$ of square integrable vector functions with domain in $X = K\backslash G$ and values in the Hilbert space H. This isomorphism is given by the formula*

$$u(g) = L_{k_g}\tilde{u}(\dot{g}),$$

where k_g is the factor of g in the Mackey decomposition $g = k_g s_g$ (2.4(1)).

PROOF: The vector $u(g)$ plays, by condition 1°(1), the role of a linear, continuous functional in H. The norm $||u(g)||$ of this functional is, by definition, given by the formula

$$||u(g)|| = \sup_{||v||\leqslant 1} |(u(g), v)|.$$

Because H is separable, there exists a sequence v_n of elements in H with $||v_n|| \leqslant 1$ and dense in the unit ball in H. Hence, $||u(g)|| = \sup_n |(u(g), v_n)|$. Because, by condition 1°(1) all of the functions $(u(g), v_n)$ are measurable, and because the upper limit of a sequence of measurable functions is itself measurable, it follows that $||u(g)||$ is a measurable function of g, and consequently also of $\dot{g}$ (cf. app. A.5). Hence, the integral (3° of eq. (1)) is meaningful. If $u \in H^L$, $\lambda \in C^1$ then $\lambda u \in H^L$. Moreover if u_1 and u_2 satisfy conditions 1°, 2° and 3° of eq. (1), then u_1+u_2 evidently satisfy 1° and 2°. Due to the inequality

$$||u_1(g)+u_2(g)||^2 \leqslant 2\left(||u_1(g)||^2+||u_2(g)||^2\right), \tag{2}$$

u_1+u_2 also satisfies the condition 3°. Thus the functions, which satisfy conditions 1°–3° form a vector-space. Using the identity

$$\begin{aligned} 4\,(u_1(g), u_2(g)) = {} & ||u_1(g)+u_2(g)||^2-||u_1(g)-u_2(g)||^2+ \\ & +\mathrm{i}||u_1(g)+\mathrm{i}u_2(g)||^2-\mathrm{i}||u_1(g)-\mathrm{i}u_2(g)||, \end{aligned} \tag{3}$$

one deduces that the function $g \to (u_1(g), u_2(g))$ is measurable. From condition 2° it follows that $(u_1(kg), u_2(kg)) = (L_k u_1(g), L_k u_2(g)) = (u_1(g), u_2(g))$. Hence, the function $(u_1(g), u_2(g))$ is a measurable function of $\dot{g}$. This function is also summable relative to the measure $\mu(\cdot)$ in X. In fact, by Schwarz's inequality, we have

$$|(u_1(g), u_2(g))| \leqslant ||u_1(g)||\,||u_2(g)||.$$

By condition 3° the functions $||u_i(g)||^2$, $i = 1, 2$, are summable. Furthermore, by Hölder's inequality the product $||u_1(g)||\,||u_2(g)||$ is also summable.* Consequently, the function $(u_1(g), u_2(g))$ is summable with respect to $\mathrm{d}\mu(\dot{g})$.

The above properties of the function $g \to (u_1(g), u_2(g))$ allows us to introduce a positive, hermitian form in the space H^L by the formula

$$[(u_1, u_2)]_{H^L} \equiv \int_X (u_1(g), u_2(g))_H \,\mathrm{d}\mu(\dot{g}). \tag{4}$$

Identifying in H^L two functions which are equal almost everywhere, and utilizing the last equation for the scalar product in H^L, we can convert the space H^L into

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* If $f(x) \in L^p(\Omega)$ and $g(x) \in L^q(\Omega)$, where

$$1 < p < \infty, \quad 1 < q < \infty, \quad 1/p+1/q = 1,$$

then the product $f(x)g(x)$ is integrable and

$$\left|\int f(x)g(x)\mathrm{d}x\right| \leqslant ||f||_p\,||g||_q.$$

a scalar product space. In order to show the isomorphy of H^L to $L^2(X, \mu; H)$ let $\tilde{u}(g) \in L^2(X, \mu; H)$ and set

$$u(g) = L_{k_g}\tilde{u}(\dot{g}).^* \tag{5a}$$

The functions (5a) are in the space H^L. In fact, the measurability of the map $g \to kg$ and continuity of L_k imply that for any $v \in H$, the function $g \to (u(g), v)$ is measurable. Because $kg = kk_g s_g$ implies $k_{kg} = kk_g$, we have $u(kg) = L_{kk_g}u(g) = L_k u(g)$. Finally, the unitartiy of L_{k_g} implies

$$\int\limits_X \|u(g)\|^2 \mathrm{d}\mu(\dot{g}) = \int\limits_X \|\tilde{u}(\dot{g})\|^2 \mathrm{d}\mu(\dot{g}) < \infty.$$

The map $\tilde{u}(\dot{g}) \to u(g)$ given by eq. (5a) is thus the isometry of $L^2(X, \mu, H)$ into H^L. Conversely, if $u(g) \in H^L$, then the function

$$\tilde{u}(\dot{g}) = L_{k_g}^{-1}u(g) \tag{5b}$$

satisfies $\tilde{u}(k\dot{g}) = L_{kk_g}^{-1}u(kg) = L_{kk_g}^{-1}L_k u(g) = \tilde{u}(\dot{g})$ and belongs to $L^2(X, \mu; H)$. Consequently, the map (5a) represents the isomorphism of $L^2(X, \mu; H)$ onto H^L. ▼

Next we construct a unitary representation of G in the Hilbert space H^L.

LEMMA 2. *The map $g_0 \to U_{g_0}^L$ given by*

$$U_{g_0}^L u(g) \equiv (\varrho_{g_0}(g))^{1/2} u(gg_0), \tag{6}$$

where $\varrho_{g_0}(g) = \mathrm{d}\mu(\dot{g}g_0)/\mathrm{d}\mu(\dot{g})$ is the Radon–Nikodym derivative of the quasi-invariant measure $\mathrm{d}\mu$ in X, defines a unitary representation of G in H^L.

PROOF: The function $v(g) = [\varrho_{g_0}(g)]^{1/2}u(gg_0)$ clearly satisfies the condition 1°(1). Furthermore, because

$$\int\limits_X \|v(g)\|^2 \mathrm{d}\mu(\dot{g}) = \int\limits_X \varrho_{g_0}(g)\|u(gg_0)\|^2 \mathrm{d}\mu(\dot{g}) = \int\limits_X \|u(g)\|^2 \mathrm{d}\mu(\dot{g}) < \infty, \tag{7}$$

the function $v(g) \in H^L$. The operator U_g^L is isometric and possesses an inverse $(U_g^L)^{-1}$. Consequently, the map $g \to U_g^L$ is unitary. Using eqs. (6) and the composition law for $\varrho_{g_0}(g)$ we get

$$\begin{aligned}[U_{g_1}^L U_{g_2}^L u](g) &= (\varrho_{g_1}(g))^{1/2} (\varrho_{g_2}(gg_1))^{1/2} u(gg_1g_2) \\ &= (\varrho_{g_1g_2}(g))^{1/2} u(gg_1g_2) = U_{g_1g_2}^L u(g).\end{aligned}$$

Consequently,

$$U_{g_1}^L U_{g_2}^L = U_{g_1g_2}^L. \tag{8}$$

It remains to show that the map $g \to U_g^L$ is strongly continuous. To see this, note that in the formula

$$(U_g^L u, v) = \int\limits_X (\varrho_g(g_1))^{1/2} (u(g_1 g), v(g_1))\, \mathrm{d}\mu(\dot{g}_1) \tag{9}$$

the integrand is a measurable function of both variables by th. 4.3.1 and eq. (3). Hence, $((U_g^L u, v))$ is a measurable function of g. By proposition 5.7.2.a (weakly)

---

* Here and in some next formulas we omit, for the sake of simplicity, the standard verification that the equality is independent on the choice of element u from the equivalence class.

measurable unitary representation is strongly continuous. Thus, the map $g \to U_g^L$, given by eq. (6), defines a strongly continuous unitary representation of G in the Hilbert space H^L, called the *representation of G induced by L*, or simply, the *induced representation.* ▼

If L is a one-dimensional representation of the closed subgroup $K \subset G$, U^L is called a *monomial representation.* If L is the one-dimensional identity representation of K, then U^L is called the *quasi-regular representation* of G. In this case $H^L = L^2(X, \mu)$. If K is the identity subgroup, then U^L is the right regular representation of G. The monomial induced representations play a fundamental role in representation theory of complex, classical Lie groups (cf. ch. 19) and of nilpotent groups.

EXAMPLE 1. Let $G = N \rtimes M$ be the two-dimensional Poincaré group. The action of G in the two-dimensional space-time is given by the formula

$$\begin{bmatrix} x \\ t \end{bmatrix} \to [n, \Lambda] \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} n_x \\ n_t \end{bmatrix}, \qquad -\infty < \alpha < +\infty .$$

This implies the following composition law

$$(n, \Lambda)(n_0, \Lambda_0) = (n + \Lambda n_0, \Lambda\Lambda_0).$$

We consider representations of G induced by the one-dimensional characters $n \to L_n = \langle n, \hat{n} \rangle = \exp[\mathrm{i}(n, \hat{n})]$ of N where $(n, \hat{n}) = n_\mu \hat{n}^\mu$ Because N is normal in G, the space $X = N \backslash G$ is isomorphic with the group space of the subgroup M. The Haar measure $\mathrm{d}\mu$ on M considered as the measure on X is G-invariant. In fact, from the composition law for G it follows that

$$xg \simeq (0, \Lambda_x)(n, \Lambda) = (\Lambda_x n, \Lambda_x \Lambda) \simeq (0, \Lambda_x \Lambda) \in X.$$

Hence $\mathrm{d}\mu(xg) = \mathrm{d}\mu(\Lambda_x \Lambda) = \mathrm{d}\mu(x)$. This implies

$$\mathrm{d}\mu(xg)/\mathrm{d}\mu(x) \equiv 1.$$

The carrier space H^L of U^L consists of functions $u(g)$ on G satisfying the condition 2°(1), i.e.,

$$u\big((n', I)(n, \Lambda)\big) = \langle n', \hat{n} \rangle u\big((n, \Lambda)\big).$$

The action of $U_{g_0}^L$ in H^L is given by the formula (6) with $\varrho \equiv 1$:

$$U_{g_0}^L u(g) = u(gg_0).$$

By lemma 1 we can represent U^L in the carrier space $L^2(X, \mathrm{d}\mu(x))$. Eq.(5b), mapping the space H^L unitarily onto $L^2(X, \mathrm{d}\mu(x))$ becomes the resctriction operator

$$H^L \ni u \to u|_M M \in L^2(M, \mathrm{d}\mu) \cong L^2(X, \mathrm{d}\mu).$$

The action of G in the last space is now given by ($g_0 = (n_0, \Lambda_0)$) (cf. eq. (15))

$$U_{g_0}^L u(\Lambda) = \langle \Lambda n_0, \hat{n} \rangle u(\Lambda\Lambda_0). ▼$$

We show in ch. 17, § 1, that every such induced representation is irreducible.

There arises the following natural questions:

(i) Are there functions, not identically zero, which satisfy the conditions 1°, 2° and 3° of eq. (1)?

(ii) How are the representations ${}^{\mu}U^{L}$ and ${}^{\nu}U^{L}$ of G corresponding to two different quasi-invariant measures μ and ν on $K\backslash G$ related to each other?

(iii) Can one define a representation $g \to U_g^L$ induced by a representation L of K directly on the space $L^2(X, \mu, H)$, where $X = K\backslash G$?

The answer to the first question, as well as a clear description of the structure of the space H^L is given by the following

PROPOSITION 3. *Let $w(g)$ be an arbitrary, continuous function with domain in G, range in H and with a compact support. Set*

$$\hat{w}(g) \equiv \int_K L_k^{-1} w(kg)\mathrm{d}k, \tag{10}$$

where $\mathrm{d}k$ *is right-invariant Haar measure in K. Then,*

1° *$\hat{w}(g)$ is a continuous function on G with compact support* on $K\backslash G$.*

2° *$\hat{w}(g)$ belongs to H^L.*

3° *The set $C_0^L = \{\hat{w}(g);\ w(g) = \lambda(g)v,\ \lambda(g) \in C_0(G), v \in H\}$ forms a dense set in H^L.*

PROOF: Ad 1°. Every continuous function $w(g)$ on G with a compact support D is uniformly continuous (cf. proposition 2.2.4) i.e., for any $\varepsilon > 0$ there exists a compact neighborhood V of the identity, such that

$$g_1^{-1}g_2 \in V \Rightarrow \|w(g_1) - w(g_2)\| < \varepsilon.$$

Let $g_0 \in G$ and let $g \in g_0 V$. Then,

$$\begin{aligned}\hat{w}(g) - \hat{w}(g_0) &= \int_K (L_k^{-1} w(kg) - L_k^{-1} w(kg_0))\mathrm{d}k \\ &= \int_{K \cap DV^{-1}g_0^{-1}} L_k^{-1}(w(kg) - w(kg_0))\mathrm{d}k.\end{aligned}$$

Hence,

$$\|\hat{w}(g) - \hat{w}(g_0)\| \leqslant \int_{K\cap DV^{-1}g_0^{-1}} \|w(kg) - w(kg_0)\|\mathrm{d}k < \varepsilon \operatorname*{mes}_K (K \cap DV^{-1}g_0^{-1}),$$

i.e., $\hat{w}(g)$ is a continuous function on G.

Ad 2°. For any $k_0 \in K$ we have

$$\hat{w}(k_0 g) = \int_K L_k^{-1} w(kk_0 g)\mathrm{d}k = \int_K L_{\tilde{k}k_0^{-1}}^{-1} w(\tilde{k}g)\mathrm{d}\tilde{k} = L_{k_0}\hat{w}(g).$$

* I.e., $\pi(\operatorname{supp} \hat{w})$ is compact in $X = K\backslash G$, where π is the canonical projection $\pi: G \to X$.

Furthermore, it follows from eq. (10) that $\hat{w}(g) = 0$ if $g \notin KD$. Thus, $\|w(\dot{g})\|$ is a continuous function of $\dot{g}$ with a support $\pi(D)$. Therefore, all conditions 1°–3°(1) are satisfied and consequently $\hat{w} \in H^L$.

Ad 3°. Let u be an element of H^L such that $(u, \hat{w})_{H^L} = 0$ for an arbitrary continuous function w with domain in G, range in H and a compact support. By virtue of 2°(1) we have

$$(u(g), \hat{w}(g))_H = \int_K (u(g), L_k^{-1} w(kg))_H \, \mathrm{d}k = \int_K (u(kg), w(kg))_H \, \mathrm{d}k.$$

Set now $w(g) = \lambda(g)v$, where $\lambda \in C_0(G)$. Then,

$$(u(g), \hat{w}(g))_H = \int_K \lambda(kg)\big(u(kg), v\big)_H \mathrm{d}k,$$

and by eq. (4)

$$(u, \hat{w})_{H^L} = \int_{K\backslash G} \Big[\int_K \lambda(kg)\big(u(kg), v\big)_H \mathrm{d}k\Big] \mathrm{d}\mu(\dot{g}).$$

Then, using the equality 4.3(8) one obtains

$$(u, \hat{w})_{H^L} = \int_G \lambda(g)\big(u(g), v\big)\varrho(g)\mathrm{d}g = 0.$$

This equation implies, because λ is arbitrary, $\big(u(g), v\big) = 0$ up to a set $N_v \subset G$ of measure zero. Let now $\{v_n\}$ be an everywhere dense sequence in H and let $N = \bigcup_n N_{v_n}$. Because N is of measure zero, for $g \notin N$ we have $\big(u(g), v_n\big)_H = 0$ for every n. Consequently $u(g) = 0$. ▼

We shall now solve the problem (ii).

Let μ and ν be two quasi-invariant measures in $X = K\backslash G$. Denote by ${}^{\mu}H^L$ and ${}^{\nu}H^L$ the corresponding carrier spaces and by ${}^{\mu}U^L$ and ${}^{\nu}U^L$ unitary representations of G in ${}^{\mu}H^L$ and ${}^{\nu}H^L$, respectively. Then one has

Proposition 4. *There exists a unitary transformation V from ${}^{\mu}H^L$ onto ${}^{\nu}H^L$ such that*

$$V({}^{\mu}U_g^L)V^{-1} = {}^{\nu}U_g^L \tag{11}$$

for all g in G.

Proof: The measures μ and ν in X are equivalent by th. 4.3.1. Let ψ denote a Radon–Nikodym derivative of μ with respect to ν. It is a measurable function by th. 4.3.1. Let π denote the canonical projection of G onto $K\backslash G$ given by $\pi: g \to Kg$. Then, for each u in ${}^{\mu}H^L$ the function $\sqrt{(\psi \circ \pi)}u$ is in ${}^{\nu}H^L$ and the norm of u in ${}^{\mu}H^L$ is equal to that of $\sqrt{(\psi \circ \pi)}u$ in ${}^{\nu}H^L$. It is evident that every v in ${}^{\nu}H^L$ is of the form $\sqrt{(\psi \circ \pi)}u$ for some u in ${}^{\mu}H^L$. Denote by V the operator of the multiplication by $\sqrt{\psi \circ \pi}$. Then V defines a unitary map of ${}^{\mu}H^L$ onto ${}^{\nu}H^L$. In fact, let $v = \sqrt{(\psi \circ \pi)}u \in {}^{\nu}H^L$. Then, because $\mathrm{d}\mu = \psi \mathrm{d}\nu$ one obtains

$$\varrho^{\mu}_{g_0}(g) \equiv \frac{\mathrm{d}\mu(\dot{g}g_0)}{\mathrm{d}\mu(\dot{g})} = \frac{\mathrm{d}\nu(\dot{g}g_0)}{\mathrm{d}\nu(\dot{g})}\frac{\psi(\dot{g}g_0)}{\psi(\dot{g})} = \varrho^{\nu}_{g_0}(g)\frac{\psi(\dot{g}g_0)}{\psi(\dot{g})},$$

and consequently, by eq. (6),

$$V^{\mu}U^{L}_{g_0}V^{-1}(u) = V\big(\varrho^{\mu}_{g_0}(g)\big)^{1/2}u(gg_0) = (\psi\circ\pi)^{1/2}(g)\,\big(\varrho^{\mu}_{g_0}(g)\big)^{1/2}u(gg_0)$$
$$= (\psi\circ\pi)^{1/2}(gg_0)\,\big(\varrho^{\nu}_{g_0}(g)\big)^{1/2}u(gg_0) = {}^{\nu}U^{L}_{g}u(g),$$

i.e. eq. (11) follows. ▼

We give now the solution of problem (iii). The knowledge of the representation $g \to U^L_g$ directly on the space $L^2(X, \mu; H)$ is essential in many applications; for instance, in particle physics one wants to know properties of representations of the Poincaré group Π in the space of functions with domain on the mass hyperboloid $p_0^2-p_1^2-p_2^2-p_3^2 = m^2 > 0$ which is the quotient space Π/K, where K is the semi-direct product $T^4 \rtimes \mathrm{SO}(3)$.

Let $g \to B_g$ be an operator function from G into the set of unitary operators in H which satisfies

1° $B_{kg} = L_k B_g$, for all $k \in K$ and all $g \in G$. (12)

2° The map $g \to B_g$ is weakly measurable.

The last condition means that for each pair u, v in H the function

$$g \to (B_g u, v) \tag{13}$$

is dg-measurable.

Denote by s_g the unique element of G characterizing the coset $Kg = Kk_g s_g = Ks_g$ and by $k_{s_g g_0} \in K$ the factor of the Mackey decomposition 2.4 (1) of the element $s_g g_0$. We have:

PROPOSITION 5. *Let K be a closed subgroup of G and let $k \mapsto L_k$ be a unitary representation of K in H. If $g \to B_g$ is an operator function satisfying the conditions* (12), *then the map $g_0 \to U^L_{g_0}$ given by**

$$U^L_{g_0}u(\dot{g}) = \varrho^{1/2}_{g_0}(g)B_g^{-1}B_{gg_0}u(\dot{g}g_0) \tag{14}$$

provides a unitary representation of G in $L^2(X, \mu, H)$. If we put $B_g = L_{k_g}$ and $\dot{g} \equiv x$ then

$$U^L_{g_0}u(x) = [\mathrm{d}\mu(xg_0)/\mathrm{d}\mu(x)]^{1/2}L_{k_{s_g g_0}}u(xg_0). \tag{15}$$

PROOF: Let $\tilde{u}(\dot{g}) \in L^2(X, \mu; H)$ and set

$$u(g) = B_g\tilde{u}(\dot{g}). \tag{16a}$$

The functions (16a) are in the space H^L. In fact, by condition 2°(12), for any $v \in H$, the function $g \to \big(u(g), v\big)$ is measurable. By condition 1° (12) and $\dot{\widehat{kg}} = Kkg = Kg = \dot{g}$ we have $u(kg) = L_k u(g)$. Finally, the unitarity of B_g implies

$$\int_X \|u(g)\|^2\mathrm{d}\mu(\dot{g}) = \int_X \|\tilde{u}(\dot{g})\|^2\mathrm{d}\mu(\dot{g}) < \infty.$$

* We use for simplicity the same symbol U^L_g for representations (6) and (15), and we omit the tilda over $u(\dot{g})$.

The map $u(g) \to \tilde{u}(\dot{g})$ given by eq. (16a) is thus the isometry of $L^2(\mu, X, H)$ into H^L. Conversely, if $u \in H^L$ then the function

$$\tilde{u}(\dot{g}) = B_g^{-1}u(g) \tag{16b}$$

satisfies $\tilde{u}(k\dot{g}) = B_{k_g}^{-1}L_k u(g) = B_g^{-1}u(g) = \tilde{u}(\dot{g})$ and belongs to $L^2(X, \mu, H)$. Consequently, the map (16a) represents the isomorphism of $L^2(X, \mu; H)$ onto H^L.

The following diagram illustrates the transformations of the operators $\tilde{U}_g^L$ by isomorphisms (16)

$$\begin{array}{ccc}
\tilde{u}(\dot{g}) & \xrightarrow{\quad B_g \quad} & u(g) = B_g\tilde{u}(\dot{g}) \in H^L \\
\Big\downarrow \tilde{U}^L_{g_0} & & \Big\downarrow U^L_{g_0} \\
\varrho_{g_0}^{1/2}(g)B_g^{-1}B_{gg_0}\tilde{u}(\dot{g}g_0) & \xleftarrow{\quad B_g^{-1} \quad} & \varrho_{g_0}^{1/2}(g)B_{gg_0}\tilde{u}(\dot{g}g_0)
\end{array}$$

which proves the assertion (14).

In order to complete the solution of problem (iii), one should show that there always exists an operator function $g \to B_g$ which satisfies conditions (12). This can be done by means of th. 2.4:1. In fact, let S be a Borel set such that any $g \in G$ can be uniquely written in the form $g = k_g s_g$, $k_g \in K$ and $s_g \in S$. Then, setting

$$B_g = L_{k_g}, \tag{17}$$

where $k \to L_k$ is a unitary representation of the subgroup K, one obtains an operator function $g \to B_g$ which satisfies conditions 1°(12) and 2°(12).

Because $gg_0 = k_g s_g g_0 = k_g k_{s_g g_0} s_{s_g g_0}$ we obtain $k_{gg_0} = k_g k_{s_g g_0}$. Hence

$$B_g^{-1}B_{gg_0} = L_{k_{s_g g_0}} \tag{18}$$

which proves the formula (15). ▼

In the following we denote the space $L^2(X, \mu; H)$ also by the symbol H^L, whenever their distinction is unimportant.

EXAMPLE 2. Let G be the set of all 2×2, real, unimodular matrices

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1,$$

i.e., $G = \mathrm{SL}(2, R)$.

Take as the subgroup K the set of all elements k in G of the form

$$k = \begin{bmatrix} \lambda & \nu \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \neq 0. \tag{19}$$

In fact $K = R \circledS \Lambda$, where

$$R = \left\{\begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix}\right\}$$

is the invariant subgroup in K, and

$$\Lambda = \left\{\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right\}.$$

We want to find unitary representations U^L of G induced by the one-dimensional representations (characters) of K which are of the form

$$k \to L_k \equiv L_{(\nu,\lambda)} = I \cdot L_\lambda = |\lambda|^{i\sigma}\left(\frac{\lambda}{|\lambda|}\right)^{\varepsilon} \tag{20}$$

where $\sigma \in (-\infty, \infty)$ and $\varepsilon = 0$ or 1. Because the carrier space H of L is C^1, the carrier space of U^L is $H^L = L^2(X, \mu)$, where $X = K\backslash G$ and μ is a quasi-invariant measure on X. By virtue of eq. (15) the action of $U^L_{g_0}$ in H^L is given by formula

$$U^L_{g_0} u(x) = (\mathrm{d}\mu(xg_0)/\mathrm{d}\mu(x))^{1/2} L_{k_{s_g g_0}} u(xg_0). \tag{21}$$

In order to obtain the explicit form of the operator $U^L_{g_0}$ we have to find the explicit realization of

(i) the homogeneous space X and the action of G on X,

(ii) the measure $\mathrm{d}\mu(x)$, and

(iii) the function $L_{k_{s_g g_0}}$.

We now give the solutions of these problems.

(i) To find an explicit realization of the space X we use the Mackey decomposition 2.4(1), i.e., $g = k_g s_g$. Notice first that every g for which $\delta \neq 0$ can be represented in the form

$$g = \begin{bmatrix}\delta^{-1} & \beta\\ 0 & \delta\end{bmatrix}\begin{bmatrix}1 & 0\\ \gamma/\delta & 1\end{bmatrix}. \tag{22}$$

The remaining elements in G of the form $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$, $\gamma = -\beta^{-1}$, can be represented as

$$g = \begin{bmatrix}\beta & -\alpha\\ 0 & \beta^{-1}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{23}$$

Thus any element $g \in G$ can be represented as

$$g = k_g s_g, \tag{24}$$

where $k_g \in K$, and

$$s_g = \begin{bmatrix}1 & 0\\ x & 1\end{bmatrix}, \quad x \in (-\infty, \infty), \quad \text{or} \quad s_g = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \equiv S_0, \tag{25}$$

respectively.

The set S of all points (25) provides the Mackey set of the th. 2.4(1). Since

KS_0 is of measure zero in $X = K\backslash G$ we can disregard it in the following considerations. Then

$$X \in x \equiv \dot{g} = Kg = Kk_g s_g = Ks_g, \tag{26}$$

thus every element of the homogeneous space $X = K\backslash G$ can be uniquely determined by the parameter x of s_g; hence by virtue of eq. (25) we have a one-to-one correspondence between the ponts of X and the points of the real line R^1.

To find the point x corresponding to a coset Kg we decompose g by the formula (22) and set $x = \gamma/\delta$.

We obtain the action of $g_0 \in G$ in X on a characteristic element $s_g = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ of a given coset Kg as follows:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} &= \begin{bmatrix} \alpha_0 & \beta_0 \\ \alpha_0 x+\gamma_0 & \beta_0 x+\delta_0 \end{bmatrix} \\
&= \begin{bmatrix} (\beta_0 x+\delta_0)^{-1} & \beta_0 \\ 0 & \beta_0 x+\delta_0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \dfrac{\alpha_0 x+\gamma_0}{\beta_0 x+\delta_0} & 1 \end{bmatrix}.
\end{aligned} \tag{27}$$

This implies

$$x \to xg_0 = \frac{\alpha_0 x+\gamma_0}{\beta_0 x+\delta_0}, \tag{28}$$

i.e., the action of G in X is given by the fractional transformation (28).

(ii) We now find the quasi-invariant measure $\mathrm{d}\mu(x)$ on X.

The differential of eq. (28), gives

$$\mathrm{d}(xg_0) = (\beta_0 x+\delta_0)^{-2}\mathrm{d}x. \tag{29}$$

This shows that one can take the Lebesgue measure $\mathrm{d}x$ as a quasi-invariant measure on X. By virtue of (29), the Radon–Nikodym derivative $\varrho_{g_0}(g)$ is

$$\varrho_{g_0}(g) = \frac{\mathrm{d}(xg_0)}{\mathrm{d}x} = (\beta_0 x+\delta_0)^{-2}. \tag{30}$$

(iii) From eq. (27), we have immediately

$$k_{s_g g_0} = \begin{bmatrix} (\beta_0 x+\delta_0)^{-1} & \beta_0 \\ 0 & \beta_0 x+\delta_0 \end{bmatrix}. \tag{31}$$

Thus by virtue of eq. (20)

$$L_{k_{s_g g_0}} = |\beta_0 x+\delta_0|^{-i\sigma}\left(\frac{\beta_0 x+\delta_0}{|\beta_0 x+\delta_0|}\right)^{\varepsilon} \tag{32}$$

Using eqs. (21), (30) and (32), we obtain the explicit form of induced representations

$$(U_g^L u)(x) = |\beta x+\delta|^{-i\sigma-1}\left(\frac{\beta x+\delta}{|\beta x+\delta|}\right)^{\varepsilon} u\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right). \tag{33}$$

It will be proven in ch. 19 that all these representations are irreducible except the one corresponding to $\varepsilon = 1$, $\sigma = 0$.

Remark on the Choice of the Quasi-Invariant Measure

One can use for the construction of the representation U^L of G the carrier space $H^L = L^2(X, \nu; H)$ with any quasi-invariant measure $\nu(\cdot)$ on X. In general, the requirement that the measure is quasi-invariant implies only the condition $\mathrm{d}\nu(x) = \varphi(x)\mathrm{d}x$, $\varphi \geqslant 0$. The function $\varphi(x)$ can be uniquely determined, for instance, by the requirement that $\nu(\cdot)$ is invariant under the subgroup of rotations $\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}$, for which

$$x \to \frac{x\cos\vartheta-\sin\vartheta}{x\sin\vartheta+\cos\vartheta}. \tag{34}$$

This condition implies $\varphi(x) = (1+x^2)^{-1}$. It is evident that the measure so obtained, $\mathrm{d}\nu(x) = \varphi(x)\mathrm{d}x$, is also quasi-invariant relative to the remaining subgroup consisting of elements of the form $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, or $g = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$. In the present case the Radon–Nikodym derivative has the form

$$\tilde{\varrho}_{g_0}(g) = \frac{\mathrm{d}\nu(xg_0)}{\mathrm{d}\nu(x)} = \frac{1+x^2}{(\alpha_0 x+\gamma_0)^2+(\beta_0 x+\delta_0)^2}. \tag{35}$$

The carrier space $H^L = L^2\,(X, \nu)$ consists now of all functions satisfying

$$\int_X |\tilde{u}(x)|^2 \frac{\mathrm{d}x}{1+x^2} < \infty. \tag{36}$$

The representation U^L induced by the one-dimensional representation (20) of the subgroup K has now the form

$$(U_g^L\tilde{u})(x) = (1+x^2)^{1/2}[(\alpha x+\beta)^2+(\beta x+\delta)^2]^{-1/2} \times$$
$$\times |\beta x+\delta|^{-i\sigma}\cdot\left(\frac{\beta x+\delta}{|\beta x+\delta|}\right)^{\varepsilon}\tilde{u}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right). \tag{37}$$

This representation, by proposition 4, is unitarily equivalent to the representation (33). The operator V which realizes the equivalence has the form

$$\tilde{u} \xrightarrow{V} u(x) = \tilde{u}(x)/(1+x^2)^{1/2}.$$

Although the representations with different measures are mathematically equivalent, they may not be equivalent in physical applications.

B. *Induced and Multiplier Representations*

Historically most of the irreducible representations of various groups were first derived in the form of the so-called multiplier representations defined below. The proposition following the definition describes the one-to-one correspondence between induced and multiplier representations.

DEFINITION 1. The representation $g \to U_g$ of G in $L^2(X, \mu, H)$ defined by

$$U_g u(x) = [\mathrm{d}\mu(xg)/\mathrm{d}\mu(x)]^{1/2}\sigma(x, g)u(xg), \tag{38}$$

where $\sigma(x, g)$ is a measurable operator function on $X \times G$ and satisfies the following functional equation

$$\sigma(x, g_0 g_1) = \sigma(x, g_0)\sigma(xg_0, g_1) \tag{39}$$

is called a *multiplier representation.* The function $\sigma(x, g)$ is called the *multiplier.*

PROPOSITION 6. *The representation $g \to U_g^L$ of G induced by the representation $k \to L_k$ of K provides a multiplier* (18), *i.e.,*

$$\sigma(\dot{g}, g_0) = B_g^{-1} B_{gg_0}, \quad \dot{g} \equiv gK. \tag{40}$$

Conversely, for every multiplier $\sigma(x, g_0)$ with values in the set of unitary operators in the Hilbert space H, the corresponding multiplier representation (38) *is a unitary representation in $L^2(X, \mu, H)$ of G which is equivalent to a representation U^L induced by the representation $k \to L_k$ of K defined by the formula*

$$K \in k \to L_k = \sigma(K, k). \tag{41}$$

PROOF: The operator-valued function

$$G \times G \ni (g, g_0) \to B_g^{-1} B_{gg_0} \in L(H)$$

is left K-invariant with respect to the first variable:

$$(kg, g_0) \to B_{kg}^{-1} B_{kgg_0} = (L_k B_g)^{-1} L_k B_{gg_0} = B_g^{-1} B_{gg_0}.$$

It follows that the map on $X \times G$:

$$(\dot{g}, g_0) \to \sigma(\dot{g}, g_0) \equiv B_g^{-1} B_{gg_0}$$

is well defined.

The map σ satisfies the functional equation (39) (also be referred to as multiplier equation). Indeed,

$$\sigma(\dot{g}, g_0 g_1) = B_g^{-1} B_{gg_0g_1} = B_g^{-1} B_{gg_0} B_{gg_0}^{-1} B_{gg_0g_1} = \sigma(\dot{g}, g_0)\sigma(\dot{g}g_0, g_1).$$

It is easily seen that the function $\varrho_{g_0}^{1/2}(g)$ also satisfies this equation.

We have then obtained the correspondence between induced representations U^L given by equation (14) and the multipliers. It is our aim to prove that this correspondence is, in fact, one-to-one. Indeed, it follows from the multiplier equation that $k \to L_k \equiv \sigma(K, k)$ defines the unitary representation of the subgroup K. The function

$$B_g = \sigma(K, g)$$

satisfies condition (12); hence the corresponding multiplier representation (38) is unitarily equivalent to the representation U^L defined by the formula (14). Indeed,

$$B_g^{-1} B_{gg_0} = \sigma(K, g)^{-1} \sigma(K, gg_0) = \sigma(\dot{g}, g_0)$$

and the proof is complete. ▼

C. *Realization of Induced Representations in Terms of Left Cosets*

So far we have discussed the properties of induced representations realized in a space $H^L = L^2(X, \mu; H)$, where X was a quotient space $K\backslash G$ of right cosets Kg, $g \in G$. However, in many applications the space $\hat{X} = G/K$ of left cosets gK, $g \in G$ is customary. Hence, we give an alternative formulation of the results in the space of left cosets.

The carrier space $\hat{H}^L$ of induced representations $\hat{U}^L$ is now defined by the conditions 1° (1), 3° (1), but instead of 2° (1), we have

$$u(gk) = L_k^{-1} u(g). \tag{42}$$

The action of $\hat{U}^L$ in the carrier space $\hat{H}^L$ is now given by the formula (cf. eq. (6))

$$\hat{U}_{g_0}^L u(g) = \hat{\varrho}_{g_0}^{1/2}(g) u(g_0^{-1} g)$$

where, in the present case, $\hat{\varrho}_{g_0}(g) = \mathrm{d}\mu(g_0^{-1}\dot{g})/\mathrm{d}\mu(\dot{g})$. We introduce, as previously, an operator function $g \to \hat{B}_g$ from G into a set of unitary operators in H which satisfies the following conditions (cf. eq. (12)):

$$\left.\begin{array}{l} 1^\circ\ \hat{B}_{gk} = L_{k^{-1}} \hat{B}_g. \\ 2^\circ\ \text{The map } g \to \hat{B}_g \text{ is weakly measurable.} \end{array}\right\} \tag{43}$$

We readily verify as previously that the map

$$u(g) = \hat{B}_g u(\dot{g}) \tag{44}$$

defines a one-to-one isomorphism of the spaces $\hat{H}^L$ and $L^2(\hat{X}, \mu; H)$ (cf. eqs. (16a) and (16b)). This implies that the action of $\hat{U}_{g_0}^L$ in the space $L^2(\hat{X}, \mu; H)$ is given by the formula (cf. proposition 5)

$$\hat{U}_{g_0}^L u(\dot{g}) = \hat{\varrho}_{g_0}^{1/2}(g) \hat{B}_g^{-1} \hat{B}_{g_0^{-1} g} u(g_0^{-1} \dot{g}). \tag{45}$$

Setting

$$\hat{B}_g = L_{k_g}^{-1}, \tag{46}$$

one verifies that both of the conditions of eq. (43) are satisfied.

The element g in the formula (45) can be taken to be any element of the left coset $\dot{g} = gK = s_g k_g K = s_g K$; hence we can set s_g instead of g in eq. (45). Because $\hat{B}_{s_g} = I$, the formula (45) can be written in the form

$$\hat{U}_{g_0}^L u(x) = (\mathrm{d}\mu(g_0^{-1} x)/\mathrm{d}\mu(x))^{1/2} L_{k_{g_0^{-1} s_g}}^{-1} u(g_0^{-1} x), \tag{47}$$

where $g_0, g \in G$, $x \equiv \dot{g} = gK$, $s_g \in S$ (i.e. $g = s_g k_g$) and $\hat{\varrho}_{g_0}(g)$ is the Radon–Nikodym derivative $\mathrm{d}\mu(g_0^{-1}x)/\mathrm{d}x$.

This completes the construction of induced, unitary representations of a group G associated with a space $\hat{X}$ of left cosets $\dot{g} = gK$. ▼

PROPOSITION 7. *If G is unimodular, then the representations U^L and $\hat{U}^L$ of G given by* (15) *and* (47), *respectively, both induced by the same unitary representation L of K are unitarily equivalent, i.e.,*

$$J\hat{U}^L J^{-1} = U^L, \tag{48}$$

where J denotes the involution given by the formula

$$(Ju)(g) = u(g^{-1}). \tag{49}$$

PROOF: If $u(g) \in H^L$, (i.e., $u(kg) = L_k u(g)$) then

$$Ju(gk) = u(k^{-1}g^{-1}) = L_{k^{-1}}u(g^{-1}) = L_k^{-1}(Ju)(g).$$

Hence, $Ju(g) \in H^{\hat{L}}$.

Given the quasi-invariant measure $\mathrm{d}\mu$ on $K\backslash G$, let us define a measure $\mathrm{d}\hat{\mu}$ on G/K by the formula:

$$\int_{G/K} f(\dot{g})\mathrm{d}\hat{\mu}(\dot{g}) = \int_{K\backslash G} I(f\circ\pi)(\dot{g})\mathrm{d}\mu(\dot{g}).$$

Because $J(f\circ\pi)$ is the left K-invariant function on G, the right hand side of this formula is meaningful.

The measure $\mathrm{d}\hat{\mu}$ is quasi-invariant on G, and by a simple computation we find

$$\hat{\varrho}_{g_0}(g) \equiv \frac{\mathrm{d}\hat{\mu}(g_0^{-1}\dot{g})}{\mathrm{d}\hat{\mu}(\dot{g})} = \frac{\mathrm{d}\hat{\mu}(g_0^{-1}gK)}{\mathrm{d}\hat{\mu}(gK)} = \frac{\mathrm{d}\mu(Kg^{-1}g_0)}{\mathrm{d}\mu(Kg^{-1})} = \varrho_{g_0}(g^{-1}).$$

Then

$$J\hat{U}^L_{g_0}u(g) = (\hat{U}^L_{g_0}u)(g^{-1}) = \hat{\varrho}_{g_0}(g^{-1})u(g_0^{-1}g^{-1}) = \varrho_{g_0}(g)\,(Ju)(gg_0) = (U^L_{g_0}Ju)(g).$$

Because $J = J^{-1}$ one obtains

$$J\hat{U}^L J^{-1} = U^L$$

which is precisely eq. (48). ▼

The following example shows the beauty and the effectiveness of the theory of induced representations.

EXAMPLE 3. Let G be the Poincaré group $T^4 \circledS \mathrm{SO}(3,1)$ and $K = T^4 \circledS \mathrm{SO}(3)$. Let $k \to L_k$, $k = (a, r)$, $a \in T^4$, $r \in \mathrm{SO}(3)$ be given by the formula

$$L_k = L_{(a,r)} = \exp[\mathrm{i}(\overset{0}{p}, a)]\,D^J(r), \tag{50}$$

where $\overset{0}{p} = (m, 0, 0, 0)$ and $D^J(r)$ is an irreducible representation of SO(3). We shall construct the representation U^L of G using eq. (47). We first find the explicit realization of the space $X = G/K$. By virtue of the Cartan decomposition of SO(3, 1) we have

$$\mathrm{SO}(3,1) = \mathscr{P}\mathrm{SO}(3) \tag{51}$$

where $\mathscr{P}$ is the set of pure Lorentz transformations, which may be parametrized by velocities $v = \{v_\mu\}$, $v_\mu v^\mu = 1$, or momenta $p = \{p_\mu\}$, $p_\mu = mv_\mu$. Hence the space X

$$X = T^4 \rtimes \mathrm{SO}(3,1)/T^4 \rtimes \mathrm{SO}(3) \sim \mathscr{P}$$

and may be realized as the mass hyperboloid

$$\{p_\mu : p_\mu p^\mu = m^2\}.$$

Now, by virtue of example 4.3.2 the measure $d\mu$ on the mass hyperboloid has the form $d\mu(p) = d^3p/p_0$; this measure is invariant under the action of G on X, hence the Radon–Nikodym derivative in (47) equals to one.

To complete the construction of U^L we have to find the operator $L^{-1}_{k_{g_0^{-1}s_g}}$. Note first that by virtue of (51) an element g in SO(3, 1) may be written in the form: $g = \Lambda_p r$, where $\Lambda_p \in \mathscr{P}$ and $r \in \mathrm{SO}(3)$. Observing that $\Lambda_p = s_g$ we obtain

$$\begin{aligned} g_0^{-1}s_g &= (a_0, \Lambda_0)^{-1}(0, \Lambda_p) = (-\Lambda_0^{-1}a_0, \Lambda_0^{-1})(0, \Lambda_p) \\ &= (-\Lambda_0^{-1}a_0, \Lambda_0^{-1}\Lambda_p) = (0, \Lambda_{\Lambda_0^{-1}p})(-\Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}a_0, \Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}\Lambda_p). \end{aligned} \tag{52}$$

Hence

$$k_{g_0^{-1}s_g} = (-\Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}a_0, \Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}\Lambda_p). \tag{53}$$

Taking into account that $\Lambda_p \overset{0}{p} = p$ and $(\Lambda p, \Lambda a) = (p, a)$ we obtain by eq. (50)

$$\begin{aligned} \bar{L}^{-1}_{k_{g_0^{-1}s_g}} &= \exp[i(\overset{0}{p}, \Lambda_p^{-1}a][D^J(\Lambda^{-1}_{\Lambda_p^{-1}}\Lambda_0^{-1}\Lambda_p]^{-1} \\ &= \exp[i(p, a)]D^J(\Lambda_p^{-1}\Lambda_0\Lambda_{\Lambda_p^{-1}}). \end{aligned}$$

Setting $r_\Lambda \equiv \Lambda_p^{-1}\Lambda\Lambda_{\Lambda_p^{-1}}$ we obtain: $((p, a) \equiv pa)$

$$U^L_{(a,\Lambda)}u(p) = \exp[ipa]D^J(r_\Lambda)u(\Lambda^{-1}p). \tag{54}$$

We show in ch. 17 that for an arbitrary $m > 0$ and J the representations (54) are irreducible. ▼

§ 2. Basic Properties of Induced Representation

We shall now derive a series of important properties of induced representations.

A. *Conjugate Representations**

We first show the equivalence of a representation $\overline{U^L}$ conjugate to U^L and a representation $U^{\bar{L}}$ induced by a representation $\bar{L}$ that can be written in the form

$$\bar{L}_k = CL_kC,$$

---

* Note that for unitary representations the conjugate and contragradient representations coincide (cf. eq. 5.1(14)).

where C is a conjugation (5.1 (14)) in the carrier space H of L. If we form a representation $U^{\bar{L}}$ induced by $\bar{L}$, then every vector function $\bar{u}(g)$ in $H^{\bar{L}}$ satisfies the condition

$$\bar{u}(kg) = \bar{L}_k \bar{u}(g) = CL_k Cu(g). \tag{1}$$

By virtue of proposition 3.1.3°, the conjugation C can be lifted to the space $H^{\bar{L}}$ ie. $\bar{u}(g) = Cu(g)$. Thus eq. (1) and 1(6) implies

$$U^{\bar{L}}_{g_0}\bar{u}(g) = \left(\varrho_{g_0}(g)\right)^{1/2}\bar{u}(gg_0) = C\,U^L_{g_0}u(g) = \bar{U}^L_{g_0}\bar{u}(g),$$

i.e.,

$$U^{\bar{L}}_g = \bar{U}^L. \tag{2}$$

B. *Representations Induced by the Direct Sum of Representations*

Let $\overset{1}{L}$ and $\overset{2}{L}$ be two unitary representations of a closed subgroup K of a group G in Hilbert spaces $\overset{1}{H}$ and $\overset{2}{H}$, respectively. The direct sum $\overset{1}{L}\oplus\overset{2}{L}$ of these representations is given in the Hilbert space $\overset{1}{H}\oplus\overset{2}{H}$ by the formula

$$\Big(\bigoplus_{i=1}^{2} L^i_g\Big)\{\overset{1}{u}, \overset{2}{u}\} = \{L^1_g\overset{1}{u}, L^2_g\overset{2}{u}\}$$

(cf. def. 5.3.3). The carrier space $H^{\overset{1}{L}\oplus\overset{2}{L}}$ of induced representation $U^{\overset{1}{L}\oplus\overset{2}{L}}$ will consist of all vector functions $\{\overset{1}{u}(g), \overset{2}{u}(g)\}$, with values in $\overset{1}{H}\oplus\overset{2}{H}$, each $\overset{i}{u}(g)$ satisfying conditions 1°, 2° and 3° of eq. 1(1).

This implies

$$H^{\overset{1}{L}\oplus\overset{2}{L}} = H^{\overset{1}{L}}\oplus H^{\overset{2}{L}} \tag{3}$$

and consequently

$$U^{\overset{1}{L}\oplus\overset{2}{L}} \doteq U^{\overset{1}{L}}\oplus U^{\overset{2}{L}}.$$

Hence the operations of inducing and taking direct sum are interchangable. This result can be extended for any discrete sum $\oplus\overset{i}{L}$. In general we have

THEOREM 1. *Let K be a closed subgroup of a locally compact separable group G. Let L be a unitary representation of K, which decomposes into a direct integral of unitary representations $\overset{s}{L}$ of K, i.e.,*

$$L = \int L^s \mathrm{d}\mu(s). \tag{4}$$

Let H be the carrier space of L

$$H = \int \overset{s}{H}\mathrm{d}\mu(s)$$

where every $\overset{s}{H}$ is a separable Hilbert space.

Then, the representation U^L of G induced by L is unitarily equivalent to $\int U^{\overset{s}{L}}\mathrm{d}\mu(s)$.

SKETCH OF THE PROOF: The representation U^L of G is realized in the space H^L of vector functions on G with values in H. The assertion of the theorem follows essentially from the fact that, as in the case of the finite sum (3), to the decomposition $H = \int \overset{s}{H}\mathrm{d}\mu(s)$ there corresponds a decomposition

$$H^L = \int \overset{s}{H^L}\mathrm{d}\mu(s),$$

where $\overset{s}{H^L}$ is a Hilbert space of vector functions on G with values in $\overset{s}{H}$. For measure-theoretical details of the proof cf. Mackey 1952, th. 10.1. ▼

COROLLARY. *If the representation U^L of G, induced by a representation L of a subgroup K is irreducible, then the representation L is also irreducible.*

PROOF: In fact, if L is reducible it could be written as a direct sum $L = \overset{1}{L} \oplus \overset{2}{L}$. This would imply by th. 1 that $U^L = U^{\overset{1}{L}} \oplus U^{\overset{2}{L}}$, which contradicts its irreducibility. ▼

Remark: The converse of this statement is false: L might be irreducible, while U^L is reducible. For instance, let G be any group, $K = \{e\}$ and let L be the one-dimensional identity representation of K in $H = C^1$. Then U^L is the regular representation which is reducible.

C. *'Inducing in Stages'*

Let N and K be two closed subgroups of G, such that $N \subset K$ and let L be a representation of N. One may form an induced representation of G either directly, by forming ${}_GU^L$, or in stages, i.e., by forming first the induced representation ${}_KU^L \equiv V$, and then, constructing ${}_GU^V$. The following theorem states that both methods lead to an equivalent result.

THEOREM 2. *Let $N, K, N \subset K$, be closed subgroups of the separable, locally compact group G. Let L be a representation of N and let $V \equiv {}_KU^L$. Then, ${}_GU^L$ and ${}_GU^V$ are unitarily equivalent representations of G.* ▼

The proof of this important theorem is long and difficult (cf. Mackey 1952, th. 4.1). In order to give, however, the idea of the proof we consider a special case, when homogeneous spaces $N\backslash K$ and $K\backslash G$ both possess invariant measures.

PROOF: We start with the construction of carrier spaces ${}_GH^L$ and ${}_GH^V$ for induced representations ${}_GU^L$ and ${}_GU^V$, respectively. Let H be the carrier space of L. Then, the representation $V = {}_KU^L$ of K is realized in the Hilbert space ${}_KH^L$ of functions on K with values in H, satisfying the condition (cf. eq. 1(1))

$$u(nk) = L_n u(k), \quad n \in N,\ k \in K. \tag{5}$$

The representation $k \to {}_KU^L = V_k$ of K in ${}_KH^L$ is given by the formula

$$V_{k_0}u(k) = u(kk_0). \tag{6}$$

On the other hand, the space ${}_GH^V$ in which the representation ${}_GU^V$ is realized consists of functions on G with values in ${}_KH^L$ satisfying the condition

$$v(kg) = V_k v(g), \quad k \in K,\ g \in G. \tag{7}$$

Because values of the function $v(g) \in {}_GH^V$ belong to ${}_KH^L$, one can consider elements of ${}_GH^V$ as vector functions $F(g, k)$ on $G \times K$ with values in H. Equations (5) and (7) with this convention take the form

$$F(g, nk) = L_n F(g,k), \tag{5'}$$

$$F(k_0 g, k) = V_{k_0} F(g, k) = F(g, kk_0). \tag{7'}$$

Setting in eq. (7′) $k = e$, one obtains

$$F(g, k_0) = F(k_0 g, e) \equiv \Phi(k_0 g). \tag{8}$$

We now show that the map $S: F \to \Phi$ provides an isomorphism between the spaces ${}_GH^L$ and ${}_GH^V$ and an isomorphism of the operators ${}_GU_g^V$ and ${}_GU_g^L$. In fact, eq. (5′) implies

$$\Phi(ng) = L_n \Phi(g), \quad n \in N, \tag{5''}$$

i.e., $\Phi(g) \in {}_GH^L$. Conversely, to each function $\Phi \in {}_GH^L$ there corresponds, by eq. (8), the function $F(g, k)$ satisfying (5′) and (7′). Moreover, ${}_GU_{g_0}^V F(g, k) = F(gg_0, k)$ implies $\Phi(gg_0) = {}_GU_{g_0}^L \Phi(g)$. Consequently,

$$S_G U^V S^{-1} = {}_GU^L.$$

Thus, to conclude the theorem it is now sufficient to show that the map $S: F \to \Phi$ is unitary. Denote by X, Y and Z the homogeneous spaces $N \backslash G$, $K \backslash G$ and $N \backslash K$ respectively. The norm of a function Φ in ${}_GH^L$ is given by the formula

$$\|\Phi\|^2_{{}_GH^L} = \int_X \|\Phi(g_x)\|^2_H \mathrm{d}\mu(x), \tag{9}$$

where g_x is any element of the right coset Ng, corresponding to a point $x \in X$ and $\mathrm{d}\mu(x)$ is an invariant measure on X. Similarly

$$\|F\|^2_{{}_GH^V} = \int_Y \|F(g_y, k)\|^2_{{}_KH^L} \mathrm{d}\sigma(y), \tag{10}$$

$$\|v\|^2_{{}_KH^L} = \int_Z \|v(k_z)\|^2_H \mathrm{d}\varrho(z). \tag{11}$$

In eq. (10), g_y is any element of the right coset Kg corresponding to a point y in Y (cf. th. 2.4.1) and $\mathrm{d}\sigma(y)$ is an invariant measure in Y. The notation in eq. (11) is analogous. Notice that after a selection of g_y and k_z corresponding to a point $y \in Y$ and $z \in Z$, an element g_x corresponding to $x \in X$ can be taken as $k_z g_y$. Hence substituting eq. (11) into eq. (10), one obtains

$$\begin{aligned} \|F\|^2_{{}_GH^V} &= \int_Y \Big(\int_Z (\|F(g_y, k_z)\|^2_H \mathrm{d}\varrho(z) \Big) \mathrm{d}\sigma(y) \\ &= \int_{Y \times Z} \|\Phi(k_z g_y)\|^2_H \mathrm{d}\varrho(z) \mathrm{d}\sigma(y) = \int_X \|\Phi(g_x)\|^2_H \mathrm{d}\tilde{\mu}(x), \end{aligned} \tag{12}$$

where $d\tilde{\mu}(x) = d\sigma(y)d\varrho(z)$. We now show that $d\tilde{\mu}(x)$ is an invariant measure on X equal to $d\mu(x)$. In fact, if $g_x = k_z g_y$ then, by th. 2.4.1, for any $g \in G$, one obtains

$$g_x g = k_z g_y g = k_z k_{(y,g)} g_{yg} = n_{(z,y,g)} k_{zk(y,g)} g_{yg}, \tag{13}$$

where $n_{(z,y,g)} \in N$.

This implies, because we assumed $d\varrho$ and $d\sigma$ to be invariant measures,

$$d\tilde{\mu}(xg) = d\varrho(zk_{(y,g)})d\sigma(yg) = d\varrho(z)d\sigma(y) = d\tilde{\mu}(x). \tag{14}$$

Because the invariant measures $d\mu$, $d\sigma$ and $d\varrho$ are defined up to a constant factor one can normalize them in such a manner that $d\mu = d\varrho\, d\sigma$. Thus the invertible and isometric map $S: F \to \Phi$ is unitary. ▼

D. *Representations Induced by the Tensor Product of Representations*

Let $\overset{1}{T}_{g_1}$ and $\overset{2}{T}_{g_2}$ be representations of G_1 and G_2 in Hilbert spaces H_1 and H_2, respectively. According to def. 5.5.2 the *outer tensor product representation* $\overset{1}{T}_{g_1} \otimes \overset{2}{T}_{g_2}$ of the direct product $G_1 \times G_2$ in the tensor product space $H_1 \otimes H_2$ is given by means of the formula:

$$\begin{aligned}(\overset{1}{T} \otimes \overset{2}{T})_{(g_1,g_2)}(u_1 \otimes u_2) &= (\overset{1}{T}_{g_1} \otimes \overset{2}{T}_{g_2})(u_1 \otimes u_2) \\ &= \overset{1}{T}_{g_1} u_1 \otimes \overset{2}{T}_{g_2} u_2. \end{aligned} \tag{15}$$

The next theorem describes the connection between a representation of $G_1 \times G_2$ induced by an outer tensor product of the representation $L_1 \otimes L_2$ of closed subgroups K_1 and K_2 and outer tensor product of induced representations.

THEOREM 3. *Let L_1 and L_2 be unitary representations of the closed subgroups K_1 and K_2 of the separable, locally compact groups G_1 and G_2, respectively. Then, the two representations*

$$_{G_1 \times G_2}U^{L_1 \otimes L_2} \quad \textit{and} \quad {}_{G_1}U^{L_1} \otimes {}_{G_2}U^{L_2}$$

of $G_1 \times G_2$ are unitarily equivalent.

PROOF: Let H_1 (respectively H_2) denote the carrier space of the representation $L_1(L_2)$ of $K_1(K_2)$. The proof consists in showing the existence of a unitary map S of $H_1^{L_1} \otimes H_2^{L_2}$ onto $(H_1 \otimes H_2)^{L_1 \otimes L_2}$ such that

$$S({}_{G_1}U_{g_1}^{L_1} \otimes {}_{G_2}U_{g_2}^{L_2}) = {}_{(G_1 \times G_2)}U_{(g_1,g_2)}^{L_1 \otimes L_2} S. \tag{16}$$

In order to show this, let us associate with each pair

$$\{u_1(g_1), u_2(g_2)\}; \quad u_1(g_1) \in H_1^{L_1}, \quad u_2(g_2) \in H_2^{L_2},$$

a function

$$G_1 \times G_2 \ni (g_1, g_2) \to u_1(g_1) \otimes u_2(g_2). \tag{17}$$

One readily verifies that in this manner a linear map S of $H_1^{L_1} \otimes H_2^{L_2}$ into $(H_1 \otimes H_2)^{L_1 \otimes L_2}$ is defined. In fact, the conditions 1° and 2° of eq. 1(1) are satisfied. For instance, if $(k_1, k_2) \in K_1 \times K_2$, then,

$$\begin{aligned} u_1(k_1 g_1) \otimes u_2(k_2 g_2) &= L_{1k_1} \otimes L_{2k_2}(u_1(g_1) \otimes u_2(g_2)) \\ &= (L_1 \otimes L_2)_{(k_1, k_2)}(u_1(g_1) \otimes u_2(g_2)). \end{aligned}$$

The condition 3°1(1) is also satisfied. In fact, if $d\mu_1$ (resp. $d\mu_2$) is a quasi-invariant measure on $X_1 = K_1 \backslash G_1$ ($X_2 = K_2 \backslash G_2$) and $\varrho_{g_1}(x_1)\,[\varrho_{g_2}(x_2)]$ is the corresponding Radon–Nikodym derivative, then $d\mu_1 d\mu_2$ is a quasi-invariant measure on $X_1 \times X_2 \cong K_1 \times K_2 \backslash G_1 \times G_2$ with $\varrho_{g_1}(x_1)\varrho_{g_2}(x_2)$ being the corresponding Radon–Nikodym derivative. We have then

$$\begin{aligned} &\int_{X_1 \times X_2} \|u_1(g_1) \otimes u_2(g_2)\|^2_{H_1 \otimes H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \int_{X_1 \times X_2} \|u_1(g_1)\|^2_{H_1} \|u_2(g_2)\|^2_{H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \|u_1\|^2_{H_1 L_1} \|u_2\|^2_{H L} < \infty . \end{aligned}$$

Consequently, $u_1(g_1) \otimes u_2(g_2) \in (H_1 \otimes H_2)^{L_1 \times L_2}$.

Thus, there exists a densely defined linear map S of the tensor product space $H_1^{L_1} \otimes H_2^{L_1}$ into $(H_1 \otimes H_2)^{L_1 \otimes L_2}$. This map is isometric. In fact, if

$$w = \sum_{i=1}^{n} u_i(g_1) \otimes v_i(g_2),$$

where

$$u_i(g_1) \in H_1^{L_1}, \quad v_i(g_2) \in H_2^{L_2},$$

then

$$\begin{aligned} \|S(w)\|^2_{(H_1 \otimes H_2)^{L_1 \times L_2}} &= \int_{X_1 \times X_2} \Big\|\sum_{i=1}^{n} u_i(g_1) \otimes v_i(g_2)\Big\|^2_{H_1 \otimes H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \int_{X_1 \times X_2} \sum_{i,j=1}^{n} (u_i(g_1), u_j(g_1))_{H_1} (v_i(g_2), v_j(g_2))_{H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \sum_{i,j=1}^{n} (u_i, u_j)_{H_1 L_1} (v_i, v_j)_{H_2 L_2} \\ &= \Big\|\sum_{i=1}^{n} u_i \otimes v_i\Big\|^2_{H_1 L_1 \otimes H_2 L_2} \\ &= \|w\|^2_{H_1 L_1 \otimes H_2 L_2}. \end{aligned}$$

Therefore, the map S is extendible to a unitary operator of the whole space $H_1^{L_1} \otimes H_2^{L_2}$ into $(H_1 \times H_2)^{L_1 \otimes L_2}$. In order to conclude the proof, it will be sufficient to show that the image of S forms a total set* in $(H_1 \otimes H_2)^{L_1 \otimes L_2}$.

In fact, by proposition 1.3, elements $\hat{\Phi}(g_1, g_2)$ given by eq. 1(10), where

$$\Phi(g_1, g_2) = \varphi(g_1, g_2)(u \otimes v),$$

$\varphi \in C_0(G_1 \times G_2)$, $u \in H_1$ and $v \in H_2$, form a total set in the space $(H_1 \otimes H_2)^{L_1 \times L_2}$. Because the functions

$$\varphi(g_1, g_2) = \psi(g_1)\eta(g_2), \quad \psi \in C_0(G_1), \ \eta \in C_0(G_2),$$

form a total set in $C_0(G_1 \times G_2)$, the images $\hat{\Phi}$ of the functions $\Phi = \psi(g_1)u \otimes \eta(g_2)v = u_\psi \otimes v_\eta$ still form a total set in $(H_1 \otimes H_2)^{L_1 \times L_2}$. Using formula 1(10), one readily verifies that

$$\hat{\Phi} = S(\hat{u}_\psi \otimes \hat{u}_\eta).$$

Consequently, the assertion of th. 3 follows. ▼

§ 3. Systems of Imprimitivity

A. *The Imprimitivity Theorem*

Let K be a closed subgroup of a locally compact, separable group G, L a unitary representation of K in H, and U^L the induced representation of G in H^L. Let Z be a Borel subset of $X = K\backslash G$ and χ_Z the characteristic function of the set Z. For any $u \in H^L$ let

$$(E(Z)u)(g) \equiv \chi_Z(\dot{g})u(g), \quad \dot{g} = Kg. \tag{1}$$

This function is weakly measurable (cf. eq. 1.1°(1)). Moreover, for $k \in K$, we have

$$E(Z)u(kg) = \chi_Z(\dot{g})u(kg) = L_k(\chi_Z(\dot{g})u(g)) = L_k E(Z)u(g).$$

We have also

$$\int_X ||\chi_Z(\dot{g})u(g)||^2 d\mu(\dot{g}) = \int_X \chi_Z(\dot{g})||u(g)||^2 d\mu(\dot{g})$$

$$= \int_Z ||u(g)||^2 d\mu(\dot{g}) < \infty.$$

Hence, the function (1) satisfies the conditions 1°, 2° and 3° of eq. 1(1) and, consequently, it belongs to H^L. It follows, moreover, from eq. (1) that the operator function $Z \to E(Z)$ has the properties

$$\begin{gathered} E(X) = I, \quad E(\emptyset) = 0, \\ E(Z_1 \cap Z_2) = E(Z_1)E(Z_2), \\ E^*(Z) = E(Z), \end{gathered} \tag{2}$$

* A subset Q in Hilbert space H is called a *total set* if the linear envelope of elements of Q is a dense set in H.

and $E(\cdot)$ is countably additive in the strong operator topology of $\mathscr{L}(H^L)$. Hence the map

$$X \supset Z \to E(Z) \in \mathscr{L}(H^L)$$

defines on the space X a spectral measure (cf. app. B. 3). This measure has the definite transformation properties under the representations U^L of G. In fact,

$$\begin{aligned}(U^L_{g_0} E(Z) U^L_{g_0^{-1}} u)(g) &= [\varrho_{g_0}(g)]^{1/2} \big(E(Z) U^L_{g_0^{-1}} u\big)(gg_0) \\ &= [\varrho_{g_0}(g)]^{1/2} \chi_Z(\dot{\widehat{gg_0}}) \, (U^L_{g_0^{-1}} u)(gg_0) \\ &= [\varrho_{g_0}(g)]^{1/2} [\varrho_{g_0^{-1}}(gg_0)]^{1/2} \chi_Z(\dot{\widehat{gg_0}}) u(g) = E(Z_{g_0^{-1}}) u(g). \qquad (3)\end{aligned}$$

The last step follows from the obvious equality $\chi_Z(\dot{\widehat{gg_0}}) = \chi_{Z_{g_0^{-1}}}(\dot{g})$ and the composition law for Radon–Nikodym derivatives. Thus

$$U^L_g E(Z) U^L_{g^{-1}} = E(Zg^{-1}). \qquad (4)$$

We see, therefore, that with every induced representation U^L of G, one can associate a spectral measure $E(Z)$ having the transformation property (4).

In general, let X be a G-space and let U be a unitary representation of G in a Hilbert space H. If $E(Z)$, $Z \subset X$, is a spectral measure with values in $\mathscr{L}(H)$, which transforms under U_g according to (4), then $E(Z)$ is called a *system of imprimitivity* for U based on X. If the base X is transitive under G, $E(Z)$ is called a *transitive system of imprimitivity.* A representation which has at least one system of imprimitivity is said to be *imprimitive.* The system of imprimitivity given by a spectral function (1) is called the *canonical system of imprimitivity.*

EXAMPLE 1. Let P_μ be the momentum operator of a single relativistic particle and let $\Lambda \to U_\Lambda$ be a unitary representation of the Lorentz group in the Hilbert space H of wave functions. The spectral decomposition of P_μ has the form (cf. example 6.2.1)

$$P_\mu = \int_{p^2 = m^2} p_\mu \mathrm{d}E(p),$$

where $E(p)$ is the spectral measure associated with momenta P_μ. Because P_μ is a tensor operator, we have (cf. 9.1(11))

$$U_\Lambda^{-1} P_\mu U_\Lambda = \Lambda_\mu^{-1\nu} P_\nu.$$

Thus

$$U_\Lambda^{-1} P_\mu U_\Lambda = \int_{p^2 = m^2} \Lambda_\mu^{-1\nu} p_\nu \mathrm{d}E(p) = \int_{p'^2 = m^2} p'_\mu \mathrm{d}E(\Lambda p').$$

On the other hand, we have

$$U_\Lambda^{-1} P_\mu U_\Lambda = \int p_\mu \mathrm{d}\big(U_\Lambda^{-1} E(p) U_\Lambda\big).$$

Consequently, for a Borel subset Z on the mass hyperboloid we have*

$$U_\Lambda^{-1} E(Z) U_\Lambda = E(\Lambda Z).$$

Thus, the spectral measure of the momentum operator is a system of imprimitivity for U based on the momentum space. Because the action of the translations in momentum space is trivial, $E(Z)$ is also a system of imprimitivity for the Poincaré group.

In a similar manner one may show that every self-adjoint tensor operator acting in a carrier space of physical states provides an imprimitivity system for the representation U based on momentum space. ▼

We denote $E(\psi) = \int \psi(z) \mathrm{d}E(z)$.

Equations (1)–(4) show that with every induced representation U^L one can associate the transitive canonical system of imprimitivity. It turns out that conversely a unitary representation possessing a transitive system of imprimitivity is unitarily equivalent to an induced representation. This fundamental result is described by the following theorem.

THEOREM. *Let U be a unitary continuous representation of G in a Hilbert space $H, X = K\backslash G$ and E: $C_0(X) \to L(H)$ a $*$-holomorphism with $E[C_0(X)]H$ dense in H and*

$$U_g E(\psi) U_{g^{-1}} = E(T_g^R \psi), \quad g \in G, \ \psi \in C_0(X). \tag{5}$$

Then there exists a unique (up to unitary equivalence) continuous unitary representation L of K in a Hilbert space H^L such that the pair (U, E) is unitarily equivalent to the pair (U^L, E^L), i.e. there exists a unitary operator V: $H \to H^L$ such that

$$VU_g = U_g^L V, \quad g \in G \tag{6}$$

and

$$VE(\psi) = E^L(\psi) V, \quad \psi \in C_0(X). \tag{7}$$

PROOF: Since the kernel of E is a translation invariant ideal we see that $\|E(\psi)\| = \|\psi\|_\infty$, the supremum norm of ψ; now consider the Gårding domain D_G

$$D_G = \operatorname{span} \{u(\varphi) = \int \varphi(g) U_g u \mathrm{d}g, \ u \in H, \ \varphi \in C_0(G)\} \tag{8}$$

and the Radon measure $\varphi \to (E(\tau\varphi)u, v)$, $\tau\varphi(x) = \int_K \varphi(kx)\mathrm{d}k$ for $u, v \in H$, denoted $\mathrm{d}\mu_{u,v}$, so

$$(E(\tau\varphi)u, v) = \int_G \varphi(g) \mathrm{d}\mu_{u,v} g, \quad \varphi \in C_0(G).$$

We claim that for $u, v \in D_G$, $\mathrm{d}\mu_{u,v}$ is a continuous function. To see this, let $u, v \in H$ and $\psi_1, \varphi \in C_0(G)$ then

$$\begin{aligned} |(E(\tau\varphi)u(\psi_1), v)| &\leqslant \|\tau\varphi\|_\infty \|\psi_1\|_\infty \operatorname{vol}(\operatorname{supp}\psi_1) \|u\| \|v\| \\ &\leqslant C \operatorname{vol}(\operatorname{supp}\psi_1) \|u\| \|v\| \|\varphi\|_\infty \|\psi_1\|_\infty \end{aligned}$$

* Here G acts on the left in momentum space, because in physics we realize momentum space by means of left cosets, $p \sim \Lambda_p \mathrm{SU}(2)$.

where C is a constant depending on the support of φ, so $(E(\tau, \varphi)u(\psi_1), v)$ defines a Radon measure $d\lambda(g_1, g_2)$ on $G\times G$.

By Fubini theorem we have:

$$\int_G \varphi(g)\, d\mu_{u(\psi_1),v(\psi_2)}(g) = \big(E(\tau\varphi)u(\psi_1), v(\psi_2)\big)$$

$$= \int_G \bar{\psi}_2(g)\big(E(\tau(\varphi))u(\psi_1), U_g v\big)\, dg = \int_G \bar{\psi}_2(g)\big(E\tau(T^R_{g^{-1}}\varphi)u(T^R_{g^{-1}}\psi_1), v\big)\, dg$$

$$= \int_G \bar{\psi}_2(g) \iint_{G\times G} \varphi(g_1 g^{-1})\psi_1(g_2 g^{-1})\, d\lambda(g_1, g_2)\, dg$$

$$= \int_G \varphi(g) \iint_{G\times G} \bar{\psi}_2(g_1^{-1}g^{-1})\psi_1(g_2 g_1^{-1}g^{-1})\, d\lambda(g_1, g_2)\, dg,$$

where we have made a change of variables in the g-integration. Hence

$$d\mu_{u(\psi_1),v(\psi_2)}(g) = h_{u(\psi_1),v(\psi_2)}(g)$$

where

$$h_{u(\psi_1),v(\psi_2)}(g) = \iint_{G\times G} \bar{\psi}_2(g_1^{-1}g^{-1})\psi_1(g_2 g_1^{-1}g^{-1})\, d\lambda(g_1, g_2)$$

is a continuous function on G by standard arguments. Note also that $h_{U_g u(\psi_1),v(\psi_2)}(g_1)$ is a continuous function in (g, g_1) and more generally for $u, v \in D_G$ that $h_{U_{g_1}u,U_{g_2}v}$ is a continuous function in (g_1, g_2, g).

We define a sesquilinear form on $D_G\times D_G$ by $\beta(u, v) = h_{u,v}(e)$ and check that

(i) $\beta(u, u) \geqslant 0$, $u \in D_G$,

(ii) $\beta(U_k u, U_k v) = \beta(u, v)$,

(iii) $\big(E(\tau\varphi)u, v\big) = \int_G \varphi(g)\beta(U_g u, U_g v)\, dg$.

Now the rest of the proof is standard: we let

$$H^L = \{[u \in D_G|\beta(u, u) < \infty]/[u \in D_G|\beta(u, u) = 0]\}$$

(Hilbert space completion) and $L_k[u] = [U_k u]$ (the class defined by $U_k u$). Then $(L_k[u], [v]) = \beta(U_k u, v)$ is continuous and L is a unitary continuous representation of K in H^L. For $u \in D_G$, $w_u(g) = [U_g u]$ is a continuous H^L-valued function on G satisfying $w_u(kg) = L_k w_u(g)$, $k \in K$, $g \in G$; in fact $V: u \to w_u$ extends to an isometry from H onto H^L intertwining (U, E) and (U^L, E^L) as required. The uniqueness of L follows from $(U^L, E^L) \simeq (U^{L'}, E^{L'})$ (unitary equivalence) if and only if $L = L'$. ▼

This fundamental theorem is a starting point for many applications in physics; in particular, the classification of irreducible representations of Euclidean, Galilei and Poincaré groups, the construction of relativistic position operators as well as the proof of the equivalence of Schrödinger and Heisenberg formulation of quantum mechanics can be achieved with the help of the Imprimitivity Theorem (cf. ch. 17.2 and 20.1 and 2).

B. *Irreducibility and Equivalence of Induced Representations*

The remaining part of this section is devoted to an elaboration of convenient criteria for the irreducibility of an induced representation U^L and for the equivalence of two induced representations in terms of a system of imprimitivity.

Let $g \to U_g^L$ be an induced unitary representation of a topological locally compact group G in a Hilbert space H^L. A linear hull of the set of vectors of the form

$$\int_G \varphi(g)\, U_g^L v\, dg \quad \text{for all } \varphi \in C_0(G) \text{ for all } v \in H^L,$$

is called the *Gårding space* D_G of the representation U^L. One readily verifies, similarly as in the case of Lie groups, that D_G is a linear and dense subset of H^L invariant with respect to U_g (cf. th. 11.1.1).

A vector $v \in H^L$ is said to be *continuous* if it can be represented as a continuous vector function on G.

LEMMA 1. *Let $\mu(\cdot)$ be a quasi-invariant measure on $K\backslash G$ and choose its continuous Radon–Nikodym derivative. Then every vector $v \in D_G$ is a continuous vector function on G.*

We leave the proof as an exercise for the reader.

In the sequel we shall represent elements $v \in D_G$ by continuous vector functions on G. We now construct a dense set of elements in the space H.

LEMMA 2. *The set*

$$\{v(e) \colon v \in D_G\} \tag{9}$$

is dense in the carrier space H of a representation L of the subgroup K.

The proof follows almost directly from proposition 16.1.3.

Let T and T' be representations of G in $H(T)$ and $H(T')$, respectively. Let $R(T, T')$ be the set of all intertwining operators (cf. ch. 5, § 2). $R(T, T')$ is a vector space. In the case $T = T'$, $R(T, T)$ is a closed subalgebra of $\mathscr{L}(H)$. In fact, if for a sequence $R_n \in R(T, T)$ and $\lim_{n\to\infty} R_n = R \in \mathscr{L}(H(T))$, then for an arbitrary u in $H(T)$

$$\begin{aligned} T_g R u &= T_g \lim_n R_n u = \lim_n T_g R_n u \\ &= \lim_n R_n T_g u = R T_g u. \end{aligned} \tag{10}$$

The algebra $R(T, T)$ is called a *commuting algebra* of the representation T. If $R(T, T')$ contains a unitary operator $\tilde{V}$, then $\tilde{V} T_g \tilde{V}^{-1} = T'_g$ for all $g \in G$, and consequently T and T' are unitarily equivalent.

THEOREM 3. *Let U^L and $U^{L'}$ be representations of G in Hilbert spaces H^L and $H^{L'}$, respectively, induced by the representations L and L' of the closed subgroup $K \subset G$. Let $E(Z)$ (resp. $E'(Z)$) denote the corresponding canonical system of im-*

primitivity, where Z is a Borel set in $K\backslash G$. Then, the set $R(L, L')$ is isomorphic with the set S of all operators $V \in \mathscr{L}(H^L \to H^{L'})$ such that

1° $U_g^{L'} V = V U_g^L$ *for all g in G, i.e. $V \in R(U^L, U^{L'})$,*

2° $E'(Z)V = VE(Z)$ *for all Borel sets $Z \subset K\backslash G$.* (11)

Proof: The proof consists in constructing the isomorphism φ of the vector space $R(L, L')$ onto the vector space S of operators which satisfy conditions (11). Let $R \in R(L, L')$ and $v \in H^L$. Set

$$(\tilde{R}v)(g) \underset{g \in G}{\equiv} Rv(g). \tag{12}$$

We first show that $\tilde{R}v$ is in $H^{L'}$. In fact, it is evident that $\tilde{R}v(g)$ is weakly measurable (cf. 1(1)1°). Moreover, for any $k \in K$

$$\begin{aligned}(\tilde{R}v)(kg) &= Rv(kg) = RL_k v(g)\\ &= L'_k Rv(g) = L'_k(\tilde{R}v)(g).\end{aligned}$$

Finally, from the inequality

$$\begin{aligned}\int_X (\tilde{R}v(g), \tilde{R}v(g))\mathrm{d}\mu(\dot{g}) &= \int_X (Rv(g), Rv(g))\mathrm{d}\mu(\dot{g})\\ &\leqslant \|R\|^2 \int_X (v(g), v(g))\mathrm{d}\mu(g) = \|R\|^2(v, v)\end{aligned} \tag{13}$$

it follows that $\tilde{R}v$ satisfies condition 1(1)3°. Hence $\tilde{R}v \in H^{L'}$. It is evident from the definition (12) that the operator $\tilde{R}$ is linear. Moreover, eq. (13) implies

$$\|\tilde{R}v\| \leqslant \|R\|\,\|v\|, \quad \text{i.e.} \quad \|\tilde{R}\| \leqslant \|R\|.$$

Thus $\tilde{R} \in \mathscr{L}(H^L \to H^{L'})$. We now show that $\tilde{R} \in S$. In fact,

$$\begin{aligned}(U_{g_0}^{L'} \tilde{R}v)(g) &= \varrho_{g_0}^{1/2}(g)\,(\tilde{R}v)(gg_0) = \varrho_{g_0}^{1/2}(g) Rv(gg_0)\\ &= R(U_{g_0}^L v)(g) = (\tilde{R}U_{g_0}^L v)(g).\end{aligned}$$

Moreover, for any Borel set Z in $K\backslash G$, one obtains

$$\begin{aligned}(E'(Z)\tilde{R}v)(g) &= \chi_Z(\dot{g})(\tilde{R}v)(g) = \chi_Z(\dot{g}) Rv(g)\\ &= R\big(E(Z)v\big)(g) = \big(\tilde{R}E(Z)v\big)(g).\end{aligned}$$

Denote by φ the map $R(L, L') \ni R \to \tilde{R} \in S$ given by eq. (12). In order to conclude the proof of th. 3, it is now sufficient to show that φ is the map of $R(L, L')$ onto S, i.e., that for every V in S there exists R in $R(L, L')$ such that $\varphi(R) = V$.

Let D_G and D'_G be the Gårding subspaces of U^L and $U^{L'}$, respectively, and let $V \in S$. Then, by virtue of (11)1° $VD_G \subset D'_G$. Let $v \in D_G$ and let Z be an arbitrary Borel subset of X. Then, utilizing (11)2°, one obtains

$$\begin{aligned}\int_Z \big((Vv)(g), Vv(g)\big)\mathrm{d}\mu(\dot{g}) &= \|E'(Z)Vv\|^2\\ &= \|VE(Z)v\|^2 \leqslant \|V\|^2\|E(Z)v\|^2 = \|V\|^2 \int_Z \big(v(g), v(g)\big)\mathrm{d}\mu(\dot{g}).\end{aligned}$$

Because the functions under the integral are continuous by lemma 1, then

$$((Vv)(g), Vv(g)) \leqslant \|V\|^2 (v(g), v(g))$$

for all g in G. In particular

$$\|(Vv)(e)\| \leqslant \|V\| \, \|v(e)\|.$$

It follows therefore from lemma 2 that there exists a linear map $\tilde{R} \in \mathscr{L}(H \to H')$ such that

$$Vv(e) = \tilde{R}v(e).$$

We now show that $\tilde{R} \in R(L, L')$. In fact, for any $g \in G$ we have

$$\begin{aligned}(Vv)(g) &= \varrho_g^{-1/2}(k)[U_g^{L'} Vv](e) = \varrho_g^{-1/2}(k)[VU_g^L v](e) \\ &= \varrho_g^{-1/2}(k)\tilde{R}[U_g^L v](e) = \tilde{R}v(g).\end{aligned} \tag{14}$$

Now, let $k \in K$. Because v (resp. Vv) satisfies the condition 1(1)2° relative to the representation L_k (resp. L'_k), then, by eq. (14),

$$L'_k \tilde{R} v(e) = L'_k(Vv)(e) = (Vv)(k) = \tilde{R}v(k) = \tilde{R}L_k v(e).$$

Thus, by lemma 1,

$$L'_k \tilde{R} = \tilde{R} L_k \quad \text{for all } k \in K,$$

i.e., $\tilde{R} \in R(L, L')$. Consequently, by eq. (14) and the definition of the map φ for an arbitrary v in D_G, one obtains

$$Vv = \varphi(\tilde{R})v.$$

Because D_G is dense in H^L, then, $V = \varphi(\tilde{R})$. ▼

Remark 1: The irreducibility of L does not guarantee, in general, the irreducibility of U^L (cf. the remark and the counter-example after corollary to th. 2.1). However, in many important cases, for instance in the case of semi-direct products, the irreducibility of L implies the irreducibility of U^L (cf. th. 17.1.5).

We now prove a theorem, which provides in many important cases a convenient criterion for the irreducibility of an induced representation U^L of G.

Let U^L be an induced representation in H^L and let $E(Z)$ be the corresponding canonical system of imprimivity. We call a pair (U^L, E) *irreducible*, if for an arbitrary $V \in \mathscr{L}(H^L)$, $g \in G$, and a Borel subset $Z \subset X$,

$$\begin{pmatrix} VU_g^L = U_g^L V \\ VE(Z) = E(Z)V \end{pmatrix} \to (V = \lambda I). \tag{15}$$

The following theorem gives a simple criterion for the irreducibility of the pair (U^L, E).

THEOREM 4. *Let U^L be a representation of a locally compact, separable group G induced by the representation L of the closed subgroup $K \subset G$. Then*

$$\begin{pmatrix} \textit{A pair } (U^L, E) \textit{ is} \\ \textit{irreducible} \end{pmatrix} \leftrightarrow \begin{pmatrix} \textit{The representation } L \\ \textit{is irreducible} \end{pmatrix}.$$

PROOF: We apply th. 3 in the special case in which $L = L'$. It then follows from the definition of irreducibility, that

$$\begin{pmatrix}\text{A pair } (U^L, E) \text{ is}\\ \text{irreducible}\end{pmatrix} \leftrightarrow (S = \{\lambda I, \lambda \in C^1\}),$$

$$\begin{pmatrix}\text{A representation } L\\ \text{is irreducible}\end{pmatrix} \leftrightarrow \big(R(L, L) = \{\lambda I, \lambda \in C^1\}\big).$$

The assertion of the theorem follows from the fact that the map φ defined by eq. (12) establishes one-to-one correspondence of $R(L, L)$ and S. ▼

Now let U^L (resp. $U^{L'}$) be a representation of G in H^L (resp. $H^{L'}$) induced by a representation L (resp. L') of the closed subgroup $K \subset G$ and let $E(Z)$ (resp. $E'(Z)$) be the corresponding system of imprimitivity. A pair (U^L, E) is said to be *equivalent* to a pair $(U^{L'}, E')$ (which we write as $(U^L, E) \simeq (U^{L'}, E')$), if there exists a unitary operator $V: H^L \to H^{L'}$ such that

$$VU_g^L V^{-1} = U_g^{L'} \quad \text{for all } g \text{ in } G,$$
$$VE(Z)V^{-1} = E'(Z) \quad \text{for all Borel subsets } Z \subset X. \tag{16}$$

The following theorem provides a criterion of equivalence of the pairs (U^L, E) and $(U^{L'}, E')$

THEOREM 5.

$$[(U^L, E) \simeq (U^{L'}, E')] \Leftrightarrow (L \simeq L').$$

PROOF: It follows from the definition of equivalence that $(U^L, E) \simeq (U^{L'}, E')$ if and only if there exists a unitary operator V satisfying conditions (16). On the other hand, $L \simeq L'$ if and only if there exists a unitary, intertwining operator $R \in R(L, L')$. By virtue of eq. (13), R is unitary if and only of the corresponding $V = \varphi(R)$ is unitary. Thus the assertion of the theorem follows from th. 3 which, in fact, states that the map φ is one to one. ▼

Remark 2: It follows from th. 5 and eq. (16) that $L \simeq L'$ implies that the induced representation U^L is equivalent to the induced representation $U^{L'}$.

Theorems 3–5 do not provide in general a direct criterion for the irreducibility of a representation U^L of G induced by an irreducible representation L of a subgroup $K \subset G$. However, in cases in which the system of imprimitivity $E(Z)$ is associated with a spectral measure of a representation of an invariant commutative subgroup N of G (as it is in case of a semidirect product $G = N \rtimes M$) then th. 4 provides the irreducibility of U^L (cf. th. 17.1.5).

The direct criteria of irreducibility for induced representations of semisimple Lie groups are given in ch. 19, sec. 1.

§ 4. Comments and Supplements

The theory of induced representations was originated by Frobenius in 1898. He gave the basic construction of inducing presented in § 1 in the case of finite groups. It is interesting that this simple method which could be extended immediately to many groups was used in the case of continuous groups only forty years later. This was done by Wigner in 1939 in his classic paper on the classification of irreducible unitary representations of the Poincaré group. Later on this method was applied by Bargmann 1947 and Gel'fand and Naimark 1947 for the construction of representations of the Lorentz group. Soon Gel'fand and Naimark recognized the generality and the power of the technique of induced representations and in their fundamental paper in 1950 gave the construction of 'almost all' irreducible unitary representations of all complex classical simple Lie groups.

The systematic analysis of the general properties of induced representations was done by Mackey. He gave a general construction of induced representations for an arbitrary locally compact topological group, proved with the generality the Imprimitivity Theorem, the Induction in Stages Theorem, the Frobenius Reciprocity Theorem, the Tensor Product Theorem and others. The work of Mackey and Gel'fand and Naimark stimulated the development of group representation theory and its applications in quantum physics. In particular the method of induced representations was applied for various concrete groups like the groups of motion of the n-dimensional Minkowski or Euclidean space, $SL(n, R)$, $SU(p, q)$ and other groups. The systematic analysis of the properties of induced representations of real semisimple Lie groups was carried out by Bruhat (1956). In particular he derived the important criteria for irreducibility of induced representations.

The generalization of the technique of induced representation to the construction of the so-called holomorphic and partially holomorphic induced representations as well as the simplified derivation of some of Mackey's results was done by Blattner 1961a, b.

The idea of the proof of Imprimitivity Theorem given in sec. 3 is due to N. S. Poulsen (unpublished) and in the present form was communicated to us by B. Ørsted.

§ 5. Exercises

§ 1.1. Let G be the three-dimensional real nilpotent group with the composition law

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2), \qquad x, y, z \in R^1.$$

The group G is the semidirect product $G = N \circledS S$ of the abelian normal subgroup $N = \{(0, y, z)\}$ and $S = \{(x, 0, 0)\}$. Let $(0, y, z) \to L_{(0, y, z)} = \exp(i\hat{z}z)$, $\hat{z} \in R$ be the one-dimensional representation of N. Show that the representation U^L_g has the form

$$U^L_{(x, y, z)}u(\xi) = \exp i[\hat{z}(z+\xi y)]u(\xi+x)]. \tag{1}$$

§ 1.2. Let $G = SO(3)$. Show that any representation of G induced by an irreducible representation of any subgroup K of G is reducible.

§ 1.3. Let G be the Euclidean group $T^n \rtimes SO(n)$ and let $K = T^n \rtimes SO(n-1)$. Take

$$k \to L_k = L_{(a,r)} = \exp(i\overset{0}{p}a)D^m(r), \tag{2}$$

where $\overset{0}{p} = (M, 0, 0, \ldots, 0)$ and $D^m(r)$ is an irreducible representation of $SO(n-1)$ characterized by highest weight m. Show that

1° The space $X = G/K$ is isomorphic with the sphere $p_\mu p_\mu = M^2$.

2° The action of the induced representation U^L of G in $H = L^2(X, \mu)$ where $d\mu(p)$ is the invariant measure on the sphere is given by the formula

$$(U^L_{(a,R)}u)(p) = \exp(ipa)\, D^J(r_R)\, u(R^{-1}p), \tag{3}$$

where $r_R = R_p^{-1} R R_{R^{-1}p}$ and R_p is the rotation defined by the formula

$$p = R_p \overset{0}{p}.$$

Hint: Use the formula 1 (47) and the method of example 1.3.

§ 2.1. Let $k \to L_k$ be an indecomposable representation of a closed subgroup K of a topological group G. Show that the induced representation U^L of G is also indecomposable.

§ 1.4.* Let $G = SO(3)$ and $K = SO(2)$. Take irreducible representation L of K (given by a character) and find the induced representation U^L of G in $H = L^2(X, \mu)$, $\mu = K/G \simeq S^2$. Show that the obtained representation is reducible. Show that every induced representation of G from any subgroup K is reducible.

§ 1.5.* Show that the group $SL(2, R)$ has also a supplementary series of representations given by the formula

$$T_g u(x) = |\beta x + \delta|^{\varrho - 1} u\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right), \quad g = \alpha \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{4}$$

which is realized in a Hilbert space with $(\cdot, \cdot)$ given by

$$(u, v) = \Gamma^{-1}(-\varrho) \int |x_1 - x_2|^{-1-\varrho} u(x_1)\, \overline{u}(x_2)\, dx_1\, dx_2$$

with $-1 < \varrho < 1$, $\varrho \neq 0$. Find the little group for the representation (4).

Chapter 17

Induced Representations of Semidirect Products

The method of induced representations is most effective in the case of regular semidirect products $G = N \circledS S$ where the subgroup N is abelian. We show that in this case *every* irreducible unitary representation of G is induced from a nontrivial subgroup of G. Further we give the complete classification of all induced irreducible unitary representations of G. The classification is given in terms of the parameters which characterize an orbit $\hat{O}$ of S in the dual $\hat{N}$ of the group $\overline{N}$, and additional parameters which characterize the representation of the stability subgroup $K_{\hat{O}} \subset S$ of the orbit $\hat{O}$. If G is a physical symmetry group, these parameters have a direct physical meaning.

Once the general theory is established, the construction of all irreducible unitary representations of any specific semidirect product becomes an easy exercise. We present the details of the derivation for the Euclidean and the Poincaré group. In the case of the Poincaré group the parameter which characterizes an orbit $\hat{O}$ has the meaning of mass and the parameter which characterizes the representation of the stability group has the meaning of the spin of the particle. The remaining quantum numbers for a particle, like the components of momentum and the projection of the spin also appear in a natural manner. These examples clearly show the power, elegance and usefulness in physics of the method of induced representations.

§ 1. Representation Theory of Semidirect Products

Let G be a semidirect product $N \circledS S$ of separable, locally compact groups N and S, and let N be abelian. We recall that the composition law in G is given by the formula (cf. ch. 3, § 4, def. 2)

$$(n_1, s_1)(n_2, s_2) = \big(n_1\, s_1(n_2),\, s_1 s_2\big), \quad s_1(n_2) \in N. \tag{1}$$

Because N is abelian one can also write $n_1 s_1(n_2) = n_1 + s_1(n_2)$. Let o and e be the identity elements in N and S, respectively. The sets

$$\{(n, e),\ n \in N\} \quad \text{and} \quad \{(o, s), s \in S\} \tag{2}$$

are closed subgroups of G isomorphic, in a natural manner, to the subgroups N and S, respectively. We shall identify N and S with these subgroups. The subgroup S acts as a group of automorphisms of N by the formula

$$sns^{-1} = (o, s)(n, e)(o, s^{-1}) = (s(n), e).$$

We denote this automorphism by the symbol α_s, i.e.,

$$\alpha_s(n) \equiv s(n). \tag{3}$$

Let T be a unitary representation of G and let U and V be its restriction to N and S, respectively, i.e.

$$U_n \equiv T_{(n, e)}, \quad V_s \equiv T_{(o, s)}.$$

Because $(n, s) = (n, e)\,(o, s)$, for all $n \in N$ and $s \in S$, then

$$T_{(n, s)} = U_n V_s. \tag{4}$$

Thus a representation T of G is completely determined by its restriction U and V to the subgroups N and S, respectively. The representations U of N and V of S cannot be chosen arbitrarily. In fact, the composition law in G

$$(n_1, s_1)(n_2, s_2) = (n_1 + s_1(n_2), s_1 s_2) \tag{5}$$

implies that the representations U and V must satisfy the following operator equation

$$U_{n_1} V_{s_1} U_{n_2} V_{s_2} = U_{n_1} U_{\alpha_{s_1}(n_2)} V_{s_2} V_{s_2} \tag{6}$$

which reduces to the following equation

$$V_s U_n V_s^{-1} = U_{\alpha_s(n)} = U_{s(n)}. \tag{7}$$

We now show that if T is a unitary representation then this equation is equivalent to a system of imprimitivity for T. In fact, let $\hat{N}$ be the dual space (of characters) to N (cf. ch. 6). If $\hat{n}(n) = \langle n, \hat{n}\rangle$ is a character and α is an automorphism of N, then $\hat{n}\,(\alpha(n))$ is also a character, which we denote $\hat{n}\alpha$. Using the relation 6.1(6′)

$$\langle n, \hat{n}_1\rangle\langle n, \hat{n}_2\rangle = \langle n, \hat{n}_1\hat{n}_2\rangle \tag{8}$$

one readily verifies that the map $\hat{n} \to \hat{n}\alpha$ is an automorphism of the character group $\hat{N}$, i.e.,

$$(\hat{n}_1\hat{n}_2)\alpha = (\hat{n}_1\alpha)(\hat{n}_2\alpha), \qquad \hat{n}_i\alpha \in \hat{N}.$$

In particular, the automorphism (3) of N implies the automorphism $\hat{n} \to \hat{n}s$ of $\hat{N}$.

Now, using the SNAG theorem (cf. ch. 6, § 2) and the equality

$$\langle s(n), \hat{n}\rangle = \langle n, \hat{n}s\rangle$$

one obtains

$$U_n = \int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}E(\hat{n})$$

and, by (7),

$$V_s U_n V_{s^{-1}} = U_{s(n)} = \int_{\hat{N}} \langle s(n), \hat{n}\rangle \mathrm{d}E(\hat{n})$$
$$= \int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}E(\hat{n}s^{-1}). \tag{9}$$

On the other hand, the left-hand side of eq. (7) is

$$\int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}(V_s E(\hat{n}) V_{s^{-1}}). \tag{9'}$$

Because the characters separate the points of G, equations (9) and (9′) imply

$$V_s E(Z) V_s^{-1} = E(Zs^{-1}), \tag{10}$$

for any Borel set $Z \subset \hat{N}$. Thus, $E(Z)$ is a system of imprimitivity for a representation V based on the dual space $\hat{N}$. Because,

$$U_n E(Z) U_n^{-1} = E(Z),$$

the projection $E(Z)$ form also a system of imprimitivity for a representation T of G. Consequently, every unitary representation of a semidirect product is imprimitive.

The set of all $\hat{n}s$ for a given $\hat{n} \in N$ and all s in S is called the *orbit of the character* $\hat{n}$, and is denoted by the symbol $\hat{O}_{\hat{n}}$. We assume that topology on the orbit $\hat{O}_{\hat{n}}$ was chosen in such a manner that it is a locally compact space. It is evident that two orbits $\hat{O}_{\hat{n}_1}$ and $\hat{O}_{\hat{n}_2}$ either coincide or are disjoint. Consequently, the dual space $\hat{N}$ decomposes onto nonintersecting sets $\hat{O}_{\hat{n}}$.

Any two points $\hat{n}s_1$ and $\hat{n}s_2$ of $\hat{O}_{\hat{n}}$ may be connected by the transformation $s_1^{-1}s_2 \in S$. Hence an orbit $\hat{O}_{\hat{n}}$ is a homogeneous G-space. By virtue of th. 4.1.1, this space is homeomorphic with the homogeneous space $X = K_{\hat{n}} \backslash S$, where $K_{\hat{n}}$ is the closed subgroup consisting of all elements s in S for which $\hat{n}s = \hat{n}$.

By th. 4.3.1, one knows that there exists a measure $\mathrm{d}\mu(x)$ in the homogeneous space $X = K_{\hat{n}} \backslash S$, quasi-invariant with respect to S. It might occur, however, that there exist quasi-invariant measures in $\hat{N}$ which are not concentrated on any orbit $\hat{O}_{\hat{n}}$ of $\hat{N}$. To see this, consider the following example:

EXAMPLE 1. Let G be the semidirect product consisting of all pairs (z, m), $z \in C^1$, m an integer, in which the composition law is given by the following formula

$$(z_1, m_1)(z_2, m_2) = (z_1 + \exp(\mathrm{i} m_1 \pi \alpha) z_2, m_1 + m_2), \tag{11}$$

where α is an irrational number.

Here, the abelian, invariant subgroup N consists of all elements of the form $u = (z, 0)$ and $S = \{(0, m)\}$. Because N is a non-compact vector group, the dual group $\hat{N}$ is isomorphic to N (cf. example 6.1.1). The group S acts on $\hat{N}$ by the formula

$$\hat{N} \ni \hat{n} \to (\hat{n})s = \exp(\mathrm{i}m\pi\alpha)\hat{n}, \tag{12}$$

where $s = (0, m) \in S$.

Thus every orbit $\hat{O}_{\hat{n}}$ consists of a countable number of points lying on the circle of radius $r = |\hat{n}|$.

For every $r > 0$, let $\mu_r(Z)$ denote a linear Lebesgue's measure of the intersection of a Borel set $Z \subset N$ and the circle $|\hat{n}| = r$. Clearly, every $\mu_r(Z)$ is invariant because the action of S, by eq. (12), corresponds to a rotation. However, since every orbit $\hat{O}_{\hat{n}}$ is a countable set, $\mu_r(\hat{O}_{\hat{n}}) = 0$ for every $\hat{O}_{\hat{n}} \subset \hat{N}$. Thus none of the measures $\mu_r(\cdot)$ is concentrated on an orbit $\hat{O}_{\hat{n}}$. ▼

In order to avoid such pathological cases we impose some regularity conditions on the semidirect product $G = N \circledS S$. We say that G is a *regular semidirect product* of N and S if $\hat{N}$ contains a countable family $Z_1, Z_2, \ldots$ of Borel subsets, each a union of G orbits, such that every orbit in $\hat{N}$ is the intersection of the members of a subfamily $Z_{n_1}, Z_{n_2}, \ldots$ containing that orbit. Without loss of generality, we can suppose that the intersection of a finite number of Z_i is an element of the family $\{Z_i\}_1^\infty$. This is equivalent to the assumption that any orbit is the limit of a decreasing sequence $\{Z^{\hat{n}}_{n_l}\} \subset \{Z_i\}_1^\infty$ i.e.: $Z^{\hat{n}}_{n_l} \searrow O_{\hat{n}}$.

We shall see that most of the interesting semidirect products occurring in physical applications are regular. One readily verifies however that the semidirect product of example 1 is not regular.

PROPOSITION 1. *Let T be a unitary representation of a regular semidirect product $G = N \circledS S$, and let $E(\cdot)$ be the projection-valued measure associated with the restriction U of the representation T to N. Then, if T is irreducible there exists an orbit $\hat{O}_{\hat{n}}$ such that $E(\hat{O}_{\hat{n}}) = 1$ and $E(\hat{N} - \hat{O}_{\hat{n}}) = 0$.*

PROOF: Because every set Z_i is a union of orbits then, $Z_i g = Z_i$, for all $g \in G$. The irreducibility of T implies then, by eq. (10), and by Schur's Lemma, that $E(Z_i) = 0$ or 1. Similarly, $E(\hat{O}_{\hat{n}}) = 0$ or 1 for every orbit $\hat{O}_{\hat{n}} \subset \hat{N}$. Suppose that $E(\hat{O}_{\hat{n}}) = 0$ for all orbits $\hat{O}_{\hat{n}}$. Because the orbit is the limit

$$Z^{\hat{n}}_{n_l} \searrow \hat{O}_{\hat{n}}$$

and because the countably additive measure $E(\cdot)$ satisfies

$$E\Big(\bigcap_l Z^{\hat{n}}_{n_l}\Big) = \prod_l E(Z^{\hat{n}}_{n_l}) = \lim_{l\to\infty} E(Z^{\hat{n}}_{n_l})$$

we obtain

$$\lim_{l\to\infty} E(Z^{\hat{n}}_{n_l}) = E(\hat{O}_{\hat{n}}) = 0.$$

Thus for any orbit $O_{\hat{n}}$ there exists an element $Z^{\hat{n}}_{n_l} \supset \hat{O}_{\hat{n}}$ of measure zero. Then because the set of such $Z^{\hat{n}}_{n_l}$ covers $\hat{N}$, we obtain $E(\hat{N}) = 0$ which is a contradiction.

Hence, there exists at least one orbit $\hat{O}_{\hat{n}}$ such that $E(\hat{O}_{\hat{n}}) = 1$. If there were two orbits $\hat{O}_{\hat{n}_1}$ and $\hat{O}_{\hat{n}_2}$ satisfying the condition $E(\hat{O}_{\hat{n}_i}) = I$, $i = 1, 2$, then

$$E(\hat{O}_{\hat{n}_1} \cup \hat{O}_{\hat{n}_2}) = E(\hat{O}_{\hat{n}_1}) + E(\hat{O}_{\hat{n}_2}) = 2I.$$

But $E(\hat{N}) = I$. Hence the spectral measure $E(\cdot)$ is concentrated only on one orbit $\hat{O}_{\hat{n}}$. ▼

Let $\hat{O}$ denote an orbit, which is the support of the spectral measure $E(\cdot)$. We showed that the orbit $\hat{O}$ is homeomorphic with a transitive space $S_{\hat{O}}\backslash S$, where $S_{\hat{O}}$ is the stability subgroup of a point $\hat{n}_0$ of the orbit $\hat{O}$. Because for every $n \in N$ and $\hat{n} \in \hat{O}$, $\hat{n}n = \hat{n}$, the orbit $\hat{O}$ can also be considered as homeomorphic to a transitive manifold $N \otimes S_{\hat{O}} \backslash G$. Then, eq. (10) implies

$$T_{(n,s)} E(Z) T_{(n,s)}^{-1} = E\big(Z(n,s)^{-1}\big), \tag{13}$$

where Z is a Borel subset of $\hat{N}$.

Thus, by the Imprimitivity Theorem (ch. 16, § 3), every irreducible representation T of a regular, semidirect product is unitarily equivalent to a representation $U^{\hat{L}}$ of G induced by a representation $\hat{L}$ of the subgroup $N \otimes S_{\hat{O}}$. By the corollary to th. 16.2.1, the representation $\hat{L}$ is irreducible. The representation $U^{\hat{L}}$ is realized in a Hilbert space $H^{\hat{L}} = L^2(\hat{O}, \mu: H)$, where μ is a quasi-invariant measure in $\hat{O}$ and H is the carrier space of the representation $\hat{L}$ of the subgroup $N \otimes S_{\hat{O}}$.

We now prove the important property of the representation $\hat{L}$ of the stability group $N \otimes S_{\hat{O}}$ of the orbit $\hat{O}$:

Lemma 2. *The restriction $\hat{L}_n$ of an irreducible inducing representation $\hat{L}$ of $N \otimes S_{\hat{O}}$ to the subgroup N is a one-dimensional representation, i.e.,*

$$\hat{L}_n u = \langle n, \hat{n}_0 \rangle u, \tag{14}$$

where $u \in H$ and $\hat{n}_0$ is that element of $\hat{O}$ which has the stability subgroup $N \otimes S_{\hat{O}}$.

Proof: As we have just proved, the irreducible representation U of $N \otimes S$ is unitarily equivalent to an induced representation $U^{\hat{L}}$. By virtue of eq. 16.1(15) the action of $U_g^{\hat{L}}|_N$ is given in $H^{\hat{L}} = L^2(\hat{O}, \mu; H)$ by the formula

$$U_n^{\hat{L}} u(\hat{n}) = \hat{L}_{sns^{-1}} u(\hat{n}), \tag{15}$$

where $\hat{n}_0 s = \hat{n}$ and $\hat{n}_0$ is the element of $\hat{O}$ having the stability subgroup $N \otimes S_{\hat{O}}$. On the other hand, by virtue of SNAG's Theorem for $u, v \in H^{\hat{L}}$ we have

$$(U_n^{\hat{L}} u, v) = \int_{\hat{N}} \langle n, \hat{n} \rangle \, d\mu_{u,v}(\hat{n}) = \int_{\hat{O}} \langle n, \hat{n} \rangle \big(u(\hat{n}), v(\hat{n})\big)_H \, d\mu(\hat{n}). \tag{16}$$

By proposition 16.1.3 we can set $u(\hat{n}) = \alpha(\hat{n})u$, $v(\hat{n}) = \beta(\hat{n})v$, $\alpha, \beta \in L^2(\hat{O}; \mu)$, $u, v \in H$. Then eqs. (15) and (16) imply

$$\int_{\hat{O}} \langle n, \hat{n} \rangle \alpha(\hat{n}) \bar{\beta}(\hat{n}) (u, v)_H \, d\mu(\hat{n}) = \int_{\hat{O}} \alpha(\hat{n}) \bar{\beta}(\hat{n}) (\hat{L}_{sns^{-1}} u, v)_H \, d\mu(\hat{n}). \tag{17}$$

Consequently

$$\hat{L}_{sns^{-1}}u = \langle n, \hat{n}\rangle u \tag{18}$$

for all $n \in N$, $u \in H$ and almost all $\hat{n} \in \hat{O}$. If $\hat{n} = \hat{n}_0 s$ is such that the last formula is true and recalling that $\langle n, \hat{n}_0 s\rangle = \langle sns^{-1}, \hat{n}_0\rangle$ we obtain

$$\hat{L}_n u = \langle n, \hat{n}_0\rangle u. \blacktriangledown$$

The next lemma provides a convenient characterization of irreducible representations $\hat{L}$ of the stability group $N \circledS S_{\hat{O}}$ in terms of irreducible representations L of $S_{\hat{O}}$.

LEMMA 3. *Every irreducible unitary representation L of $N \circledS S_{\hat{O}}$ is determined and determines an irreducible unitary representation L of $S_{\hat{O}}$.*

PROOF: Let L be a unitary representation of $S_{\hat{O}}$. For

$$g_{\hat{O}} = (n, s_{\hat{O}}), \quad g_{\hat{O}} \in N \circledS S_{\hat{O}}, \quad n \in N, \; s_{\hat{O}} \in S_{\hat{O}}$$

set

$$\hat{L}_{(n, s_{\hat{O}})} = \langle n, \hat{n}_0\rangle L_{s_{\hat{O}}}, \tag{19}$$

where

$$\hat{n}_0 s_{\hat{O}} = \hat{n}_0 \quad \text{for all } s_{\hat{O}} \in S_{\hat{O}}.$$

The map $(n, s_{\hat{O}}) \to \hat{L}_{(n, s_{\hat{O}})}$ defines a unitary representation of $N \circledS S_{\hat{O}}$. Indeed,

$$\hat{L}_{(n, s_{\hat{O}})(n', s'_{\hat{O}})} = \hat{L}_{(n(s_{\hat{O}}n'), s_{\hat{O}}s'_{\hat{O}})} = \langle n(s_{\hat{O}}n'), \hat{n}_0\rangle L_{s_{\hat{O}}s'_{\hat{O}}} = \langle n, \hat{n}_0\rangle \langle s_{\hat{O}}n', \hat{n}_0\rangle L_{s_{\hat{O}}} L_{s'_{\hat{O}}}$$
$$= \hat{L}_{(n, s_{\hat{O}})} \hat{L}_{(n', s'_{\hat{O}})}$$

and

$$\hat{L}^*_{(n, s_{\hat{O}})} = \overline{\langle n, \hat{n}_0\rangle} L^*_{s_{\hat{O}}} = \langle n^{-1}, \hat{n}_0\rangle L^{-1}_{s_{\hat{O}}} = \hat{L}^{-1}_{(n, s_{\hat{O}})}.$$

If L is irreducible, then $\hat{L}$ is also irreducible. Conversely, by eqs. (4) and (19), every irreducible representation $\hat{L}$ of $N \circledS S_{\hat{O}}$ defines an irreducible unitary representation of $S_{\hat{O}}$. $\blacktriangledown$

In the following we shall denote the representation $\hat{L}$ of $N \circledS S_{\hat{O}}$ described in lemma 3 by the symbol $\hat{n}L$ where $\hat{n} \in \hat{O}$ and L is a representation of $S_{\hat{O}}$. The following theorem gives a convenient characterization of irreducible, unitary representations of semidirect products.

THEOREM 4. *Let G be a regular, semidirect product $N \circledS S$ of separable, locally compact groups N and S, and let N be abelian. Let T be an irreducible unitary representation of G. Then*

1° *One can associate with T an orbit $\hat{O}$ in $\hat{N}$.*

2° *The representation T is unitarily equivalent to an induced representation $U^{\hat{n}L}$,* *where L is an irreducible unitary representation of $S_{\hat{O}}$ in a Hilbert space H. The representation $U^{\hat{n}L}$ is realized in the Hilbert space $H^{\hat{n}L} = L^2(\hat{O}, \mu; H)$.*

PROOF: By proposition 1 the spectral measure $E(\cdot)$ associated with the restriction of T to N is concentrated on an orbit $\hat{O}$. Further, by eq. (13) and the Imprimitivity Theorem 16.3.1, T is unitarily equivalent to a representation $U^{\hat{L}}$ induced by a representation $\hat{L}$ of the stability subgroup $N \rtimes S_{\hat{O}}$. Finally, by lemmas 2 and 3, $\hat{L}$ is of the form $\hat{n}L$, where L is an irreducible, unitary representation of the subgroup $S_{\hat{O}}$. ▼

The next theorem states that, conversely, a representation $U^{\hat{L}}$ induced by an irreducible unitary representation $\hat{L}$ of the stability subgroup $N \rtimes S_{\hat{O}}$ of an orbit $\hat{O}$ is irreducible. In fact, we have

THEOREM 5. *Let G be as in th. 4. Then:*

1° *With each orbit $\hat{O}$ in $\hat{N}$ and with each irreducible unitary representation $\hat{n}L$ of the stability subgroup $N \rtimes S_{\hat{O}}$ one can associate the induced representation $U^{\hat{n}L}$ which is irreducible.*

2° *The spectral measure $E(\cdot)$, which is defined by the restriction of $U^{\hat{n}L}$ to N, is concentrated on the orbit $\hat{O}$.*

3° *The representation $U^{\hat{n}L}$ is realized in the Hilbert space $H^{\hat{n}L} = L^2(\hat{O}, \mu; H)$, where H is the carrier space of the representation L and μ is a quasi-invariant measure in $\hat{O}$. We have*

$$U^{\hat{n}L}_{(n,s)}u(\hat{n}) = \langle n, \hat{n} \rangle_S U^L_s u(\hat{n}), \tag{20}$$

where ${}_SU^L$ is a representation of S given by eq. 16.1(15) *which is induced by the representation L of the stability subgroup $S_{\hat{O}} \subset S$.*

PROOF: Let $\hat{O}$ be an orbit in $\hat{N}$ and let $S_{\hat{O}} \subset S$ be the stability subgroup of a point $\hat{n}_0 \in \hat{O}$. Because N acts in $\hat{N}$ as the identity, we have:

$$\hat{O} \cong S_{\hat{O}} \backslash S \cong N \rtimes S_{\hat{O}} \backslash N \rtimes S = N \rtimes S_{\hat{O}} \backslash G. \tag{21}$$

Moreover a measure μ on $\hat{O}$, quasi-invariant relative to S, remains quasi-invariant for G.

We now find an explicit form of the induced representation $U^{\hat{n}L}$. Let ϱ denote the Radon–Nikodym derivative of a measure μ on $\hat{O}$ and B_g an operator function on S satisfying conditions 16.1(12) relative to $S_{\hat{O}}$. Let L be an irreducible unitary representation of the subgroup $S_{\hat{O}}$ and let H be the carrier space of L. Then, the representation ${}_SU^L$ of the group S induced by the representation L has the form

$${}_SU^L_s u(\hat{n}) = \varrho_s(\gamma) B_\gamma^{-1} B_{\gamma s} u(\hat{n}s),$$

where $u \in L^2(\hat{O}, \mu, H)$ and $\gamma \in S$ is such that $\hat{n} = n_0\gamma$.

Let us now find functions $\tilde{\varrho}$, $\tilde{B}$, which allow to form the representation ${}_GU^{\hat{n}L}$ induced by the representation $\hat{n}L$ of the subgroup $N \rtimes S_{\hat{O}}$. Because N acts as the identity in $\hat{N}$, we have $(g = (n, s))$

$$\tilde{\varrho}_g(\gamma) = \frac{\mathrm{d}\mu(\hat{n}g)}{\mathrm{d}\mu(\hat{n})} = \frac{\mathrm{d}\mu(\hat{n}s)}{\mathrm{d}\mu(\hat{n})} = \varrho_s(\gamma), \quad \hat{n} = \hat{n}_0\gamma.$$

Further, setting

$$\hat{B}_g = \langle n, \hat{n}_0\rangle B_s \tag{22}$$

we readily verify that both conditions of eq. 16.1(12) relative to the subgroup $N \circledS S_{\hat{O}}$ are satisfied. Hence, by eqs. 16.1(15) and (22) we obtain

$$\begin{aligned} {}_GU_g^L u(\hat{n}) &= \tilde{\varrho}_g^{1/2}(\gamma)\tilde{B}_\gamma^{-1}\tilde{B}_{\gamma g}u(\hat{n}g) \\ &= \varrho_s^{1/2}(\gamma)\tilde{B}_\gamma^{-1}\tilde{B}_{\gamma n\gamma^{-1}s}u(\hat{n}(n, s)) \\ &= \varrho_s^{1/2}(\gamma)B_\gamma^{-1}\langle \gamma n\gamma^{-1}, \hat{n}_0\rangle B_{\gamma s}u(\hat{n}s) \\ &= \langle n, \hat{n}\rangle\, {}_SU_s^L u(\hat{n}). \end{aligned}$$

The restriction of $U^{\hat{n}L}$ to the subgroup N by virtue of eq. (18) gives

$${}_GU_n^{\hat{n}L}u(\hat{n}) = \langle n, \hat{n}\rangle u(\hat{n}).$$

Thus,

$$\begin{aligned} ({}_GU_n^{\hat{n}L}u, v) &= \int_{\hat{O}} \langle n, \hat{n}\rangle \big(u(\hat{n}), v(\hat{n})\big)\mathrm{d}\mu(\hat{n}) \\ &= \int_{\hat{O}} \langle n, \hat{n}\rangle \mathrm{d}\mu_{u,v}(\hat{n}). \end{aligned}$$

Comparing this with the expression resulting from SNAG's theorem:

$$({}_GU_n^{\hat{n}L}u, v) = \int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}\mu_{u,v}(\hat{n}),$$

one sees that the spectral measure $E(\cdot)$ associated with the unitary operators $n \to U_n^{\hat{n}L}$ is concentrated on the orbit $\hat{O}$.

If an operator A commutes with all operators ${}_GU_g^{\hat{n}L}$ (i.e., $A \in R(U^{\hat{n}L}, U^{\hat{n}L})$) then it also commutes with all ${}_GU_n^{\hat{n}L}$, $n \in N$, and consequently with the spectral measure $E(\cdot)$. Thus, by th. 16.3.3, we conclude that the dimension of algebra of operators commuting with all ${}_GU_g^{\hat{n}L}$ is equal to $\dim R(\hat{n}L, \hat{n}L)$. If L is irreducible, then $\hat{n}L$ is also irreducible by proposition 3; consequently $\dim R(\hat{n}L, \hat{n}L) = 1$. Hence, ${}_GU^{\hat{n}L}$ is irreducible. ▼

According to ths. 4 and 5, the classification of all irreducible unitary representations of a regular, semidirect product $N \circledS S$ can be performed along the following steps:

1° Determine the set $\hat{N}$ of all characters of N.

2° Classify all orbits $\hat{O}$ in $\hat{N}$ under the subgroup S.

3° Select an element $\hat{n}_0$ in a given orbit $\hat{O}$ and determine the stability subgroup $S_{\hat{O}} \subset S$.

4° Take an irreducible representation L of $S_{\hat{O}}$ and form the induced irreducible unitary representation ${}_SU^L$ and finally form ${}_GU^{\hat{n}L}$ by formula (20).

EXAMPLE 2. Let G be the semidirect product $N \circledS S$ of the translation group N in the Euclidean space R^3 and the rotation group $S = \mathrm{SO}(3)$. Because N is a noncompact vector group, the dual space $\hat{N}$ is isomorphic to N, i.e., $\hat{N} \simeq R^3$. If $n = (n_1, n_2, n_3,) \in N$ and $\hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{N}$, then an arbitrary unitary character has the form

$$\langle n, \hat{n}\rangle = \exp \mathrm{i}(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3).$$

The set of all $\hat{n}_0 s$ for a given $\hat{n}_0 \in \hat{N}$ and all $s \in \mathrm{SO}(3)$ forms an orbit $\hat{O}$ associated with a character $\hat{n}_0$. It is evident that in the present case the orbits are spheres with the center at $(0, 0, 0) \in R^3$ and the radius $r \geqslant 0$. We now verify that the Euclidean group $R^3 \circledS \mathrm{SO}(3)$ is a regular semidirect product. In fact, let Z be a countable family of Borel subsets of the dual space $\hat{N}$ consisting of the following sets

(i) Z_{00} = the orbit $r = 0$,

(ii) $Z_{r_1r_2}$ = the union of all orbits with $r > 0$ such that $r_1 < r < r_2$, where $r_1, r_2, r_1 < r_2$, are any two positive rational numbers.

We see that $Z_{r_1r_2}g = Z_{r_1r_2}$ for all $Z_{r_1r_2} \in Z$ and all g in $R^3 \circledS \mathrm{SO}(3)$. Moreover, each orbit is the intersection of the members of a subfamily of Z which contain the orbit. Thus, $R^3 \circledS \mathrm{SO}(3)$ is a regular semidirect product. Consequently, according to ths. 4 and 5, every irreducible, unitary representation of this group is a representation induced by an irreducible unitary representation of the stability subgroup associated with orbits $r = 0$ or $r > 0$. We shall consider the cases $r > 0$ and $r = 0$ separately.

1° $r > 0$. Take $\hat{n}_0 = (0, 0, r)$. The stability subgroup $S_{\hat{O}} \subset S$ of the point $\hat{n}_0$ is isomorphic to SO(2). The irreducible representations L of $S_{\hat{O}}$ have the form

$$\varphi \to \exp(\mathrm{i}l\varphi), \quad \varphi \in [0, 2\pi], \quad l = 0, \pm 1, \pm 2, \ldots$$

The carrier space H of L is C^1. The measure μ on the orbit $\hat{O} = S_{\hat{O}}\backslash S = \mathrm{SO}(2)\backslash \mathrm{SO}(3) \cong S^2$ is the ordinary invariant measure relative to rotations on the sphere S^2. Hence, the Hilbert space $H^{\hat{n}L}(\hat{O}, \mu; H)$ consists now of all complex functions on the sphere, square integrable relative to the measure $\mu(\cdot)$. Every irreducible representation L gives rise to an irreducible representation $U^{\hat{n}L}$ of G. According to eq. (20), the action of $U_g^{\hat{n}L}$ in the space $H^{\hat{n}L}(\hat{O}, \mu, H)$ is

$$U_{(n,s)}^{\hat{n}L}u(\hat{n}) = \exp[\mathrm{i}(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3)]_S U_s^L u(\hat{n}), \tag{23}$$

$$n \in N, \ s \in S, \ \hat{n} \in S^2.$$

Here, ${}_SU^L$ is the representation of SO(3) induced by the representation L of SO(2). They are constructed in exercise 16.4.1.4.

2° $r = 0$. Taking $\hat{n}_0 = (0, 0, 0)$ we see that the stability subgroup $S_{\hat{O}} = S = \mathrm{SO}(3)$. The irreducible induced representations of G associated with this orbit are all finite-dimensional, irreducible, unitary representations of SO(3) 'lifted' to the group G. By virtue of ths. 4 and 5, these are all irreducible unitary representations of $R^3 \circledS SO(3)$. ▼

Th. 5 gives the method of construction of irreducible representations $U^{\hat{n}L}$ of $N \circledS S$ using the special properties of the semidirect products. One may also construct the unitary representation $U^{\hat{n}L}$ induced by the representation $\hat{n}L$ of the subgroup $N \circledS S_{\hat{o}}$ directly, using the general method described in ch. 16, § 1. We recall that in that method the representation $U^{\hat{n}L}$ was constructed in the Hilbert space of functions $u(g)$ on G satisfying the condition

$$u(kg) = \hat{L}_k u(g), \quad k \in K \equiv N \circledS S_{\hat{o}}.$$

In our case this condition has the form

$$u\big((n, s_{\hat{o}})g\big) = (\hat{n}L)_{(n,s_{\hat{o}})} u(g) = \langle n, \hat{n}_0 \rangle L_{s_{\hat{o}}} u(g). \tag{24}$$

The set of functions $u(g)$ on G satisfying eq. (24) can easily be found. In fact, let $u(s)$ be the functions on S satisfying, for all $s_{\hat{o}} \in S_{\hat{o}}$, the condition $u(s_{\hat{o}}s) = L_{s_{\hat{o}}} u(s)$, i.e., $u(s) \in H^L$ which is the carrier space of the representation ${}_S U^L$. Then setting

$$u(g) = \langle n, \hat{n}_0 \rangle u(s), \quad g = (n, s), \quad n \in N,\ s \in S \tag{25}$$

and using eqs. (1), (25), (8) and (24), one obtains

$$\begin{aligned} u\big((n', s_{\hat{o}})g\big) &= u\big((n' s_{\hat{o}} n, s_{\hat{o}} s)\big) = \langle n' s_{\hat{o}} n, \hat{n}_0 \rangle u(s_{\hat{o}} s) \\ &= \langle n', \hat{n}_0 \rangle \langle s_{\hat{o}} n, \hat{n}_0 \rangle L_{s_{\hat{o}}} u(s) = \langle n', \hat{n}_0 \rangle L_{s_{\hat{o}}} u(g) \\ &= \hat{n}L_{(n', s_{\hat{o}})} u(g). \end{aligned}$$

Using eq. (25) one verifies that the representation $U^{\hat{n}L}$ of G induced by the representation $\hat{n}L$ of $N \circledS S_{\hat{o}}$ has the form

$$U^{\hat{n}L}_{g'} u(g) = \varrho^{1/2}_{g'}(g) u\big(g(n's')\big) = \varrho^{1/2}_{g'}(g) \langle n, \hat{n}_0 \rangle \langle n', \hat{n}_0 s \rangle u(ss'). \tag{26}$$

Furthermore, if we use the equality

$$\varrho_{g'}(g) = \frac{d\mu(\hat{n}g')}{d\mu(\hat{n})} = \frac{d\mu(\hat{n}s')}{d\mu(\hat{n})} = \varrho_{s'}(s)$$

and cancel on both sides of eq. (26) the factor $\langle n, \hat{n}_0 \rangle$ we obtain

$$U^{\hat{n}L}_{(n', s')} u(s) = \varrho^{1/2}_{s'}(s) \langle n', \hat{n}_0 s \rangle u(ss') = \langle n', \hat{n} \rangle {}_S U^L_s u(s). \tag{27}$$

Thus we see that the representation $U^{\hat{n}L}$ of G is realized in the carrier space of functions over the group S of the representations ${}_S U^L$, induced by the representation L of $S_{\hat{o}}$. This method of construction of the induced representations $U^{\hat{n}L}$ of $N \circledS S$ is often convenient in applications. The representation ${}_S U^L_{s'}$ in (27) is an application of eq. 16.1(6). If we realize it on the space $H^L(\hat{O}, \mu; H)$ using eq. 16.1(15), we obtain precisely eq. (20).

§ 2. Induced Unitary Representations of the Poincaré Group

A. *The Lorentz and the Poincaré Groups*

In this section we give a complete classification of irreducible unitary representations of the Poincaré group, using the general formalism of induced representations developed in sec. 1.

We start with a discussion of some of the properties of the Lorentz and the Poincaré groups. Let $g^{\mu\nu} = g_{\mu\nu}$ be the diagonal metric tensor in the four-dimensional Minkowski space M, with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. For

$$x = \{x^0, x^1, x^2, x^3\} = \{x^\mu\}, \quad \text{and} \quad y = \{y^\mu\}$$

let

$$x \cdot y = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y} = x^\mu y_\mu, \quad y_\mu = g_{\mu\nu} y^\nu \tag{1}$$

be the (indefinite) scalar product in M.

The Lorentz group is the set of all linear transformations L of M into M, which preserve the scalar product (1), i.e.,

$$(Lx) \cdot (Ly) = xy. \tag{2}$$

From eq. (2) one obtains

$$L^\alpha{}_\mu L_{\alpha\nu} = g_{\mu\nu}, \text{ or } L^T g L = g, \tag{3}$$

where

$$L_{\alpha\nu} = g_{\alpha\beta} L^\beta{}_\nu = (gL)_{\alpha\nu} \quad \text{and} \quad (L^T)_\mu{}^\alpha = L^\alpha{}_\mu. \tag{4}$$

Clearly from eq. (3) we have

$$L^T = gL^{-1}g \quad \text{or} \quad L^T{}_\mu{}^\alpha = g_{\mu\tau}(L^{-1})^\tau{}_\varrho g^{\varrho\alpha} = L^{-1}{}_\mu{}^\alpha. \tag{5}$$

Equation (3) also implies $\det L = \pm 1$. Furthermore, from eq. (3) for $\mu = 0$, $\nu = 0$ we have

$$(L^0{}_0)^2 - \sum_{k=1}^{3} (L^k{}_0)^2 = 1.$$

Thus, $|L^0{}_0| \geqslant 1$; consequently $\det L$ and $\operatorname{sign} L^0{}_0$ are both continuous functions of the variables $L^\mu{}_\nu$, and therefore they must be constant on every component of the Lorentz group. Thus every Lorentz transformation falls into one of the four pieces:

$$\begin{aligned} &\text{I. } L^\uparrow_+\colon \det L = +1, \ \operatorname{sign} L^0{}_0 = +1,\\ &\text{II. } L^\uparrow_-\colon \det L = -1, \ \operatorname{sign} L^0{}_0 = +1,\\ &\text{III. } L^\downarrow_+\colon \det L = +1, \ \operatorname{sign} L^0{}_0 = -1,\\ &\text{IV. } L^\downarrow_-\colon \det L = -1, \ \operatorname{sign} L^0{}_0 = -1. \end{aligned} \tag{6}$$

The transformations $L \in L^\uparrow_+$ form a subgroup, which is called the *proper orthochronous Lorentz* group. It is the connected component of the identity (i.e., it consists of all Lorentz transformations, which can be reached from the identity in a continuous manner). The remaining components do not form subgroups of the Lorentz group.

We now establish a connection between $L_+^\uparrow$ and the group SL(2, C) of all 2×2 complex matrices of determinant one. Let $\sigma = \{\sigma^\mu\}$ be the set of four hermitian matrices of the form:

$$\sigma_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \sigma_2 = \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}, \quad \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{7}$$

With each four-vector $x = (x^\mu) \in M$ one associates a 2×2-hermitian matrix X by the formula

$$X = \sum_{\mu=0}^{3} x^\mu \sigma_\mu = \begin{bmatrix}x^0+x^3 & x^1-\mathrm{i}x^2\\ x_1+\mathrm{i}x^2 & x^0-x^3\end{bmatrix}. \tag{8}$$

The map $x \to X$ is linear and one-to-one. In fact, using the formula ($\tilde{\sigma}^\mu \equiv \sigma_\mu$)

$$\mathrm{Tr}(\tilde{\sigma}^\mu \sigma_\nu) = 2g^\mu{}_\nu$$

one can associate to every hermitian matrix X a real four-vector

$$x^\mu = \tfrac{1}{2}\mathrm{Tr}\,(X\sigma^\mu)$$

such that $X = x^\mu \sigma_\mu$.

Using eq. (8) we obtain

$$\det X = x^\mu x_\mu, \qquad \tfrac{1}{2}[\det(X+Y)-\det X-\det Y] = x^\mu y_\mu. \tag{9}$$

Set now

$$\hat{X} = \Lambda X \Lambda^*, \qquad \Lambda \in SL(2, C). \tag{10}$$

The matrix $\hat{X}$ is hermitian. Therefore the corresponding vector $\hat{x}$ belongs to M. Consequently, eq. (10) defines a real linear map $\Lambda \to L_\Lambda$ of M into itself. By eq. (10), we have $\Lambda_1\Lambda_2 \to L_{\Lambda_1\Lambda_2} = L_{\Lambda_1}L_{\Lambda_2}$. Moreover, because $\det\Lambda = 1$, we obtain by eqs. (8) and (10)

$$\hat{x}_\mu \hat{x}_\mu = \det\hat{X} = \det X = x^\mu x_\mu.$$

Hence, by eq. (9), the transformations L_Λ conserve the scalar product (1) and, consequently, represent Lorentz transformations in M. By virtue of eq. (3), $\det L_\Lambda$ is either $+1$ or -1. If there would exist elements L_Λ with determinant $+1$ as well as -1, then, the set of all elements L_Λ would be disconnected. This is, however, impossible because SL(2, C) is connected and the map $\Lambda \to L_\Lambda$ is continuous. Consequently, the map λ: $\Lambda \to L_\Lambda$ is a homomorphism of SL(2C) into $L_+^\uparrow$.

We now show that the homomorphism λ is 'two-to-one'. To see this, we find the kernel Z of the homomorphism λ. This is the set of all Λ in SL(2, C), which for any hermitian matrix X satisfy the equality

$$X = \Lambda X \Lambda^*. \tag{11}$$

Taking, in particular, $X = e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we see that the elements $\Lambda \in Z$ satisfy the condition $\Lambda\Lambda^* = e$, i.e., $\Lambda = \Lambda^{-1}$. Thus, eq. (11) reduces to

$$X\Lambda - \Lambda X = 0$$

which must be satisfied for any hermitian X. This implies $\Lambda = \lambda I$. By virtue of the condition $\det \Lambda = 1$, we obtain $\Lambda = \pm I$. Consequently, $L_{\Lambda_1} = L_{\Lambda_2}$ if and only if $\Lambda_1 = \pm\Lambda_2$.

The group SL(2, C) is simply connected (th. 3.7.1). Hence, it is the universal covering group of the proper Lorentz group $L_+^\uparrow$. Denoting the invariant subgroup of SL(2, C) consisting of elements I and $-I$ by Z, we have

$$L_+^\uparrow = \mathrm{SL}(2, C)/Z. \tag{12}$$

There exists two automorphisms of SL(2, C), which are important in applications:

$$\Lambda \to (\Lambda^T)^{-1} \quad \text{and} \quad \Lambda = \bar{\Lambda}. \tag{13}$$

It is interesting that the first automorphism has an explicit realization. In fact, if $\Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ then the matrix $\sigma_2 \Lambda (\sigma_2)^{-1} = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$, is inverse to $\Lambda^T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Hence,

$$(\Lambda^T)^{-1} = \sigma_2 \Lambda (\sigma_2)^{-1}. \tag{14}$$

The Poincaré Group

Notice that the Lorentz group leaves the interval $(x-y)^2$ in the Minkowski space invariant. On the other hand, the translations $x^\mu \to x^\mu + a^\mu$ where a^μ is a constant four-vector also leave the length $(x-y)^2$ invariant. This leads to the definition of the Poincaré group Π as the group of all real transformations in the Minkowski space M,

$$x^\mu \to L^\mu{}_\nu x^\nu + a^\mu \tag{15}$$

leaving the length $(x-y)^2$ invariant.

Definition (15) gives the following composition law for the elements of the Poincaré group

$$\{n_1, L_1\}\{n_2, L_2\} = \{n_1 + L_1 n_2, L_1 L_2\}. \tag{16}$$

Thus Π is the semidirect product $N \otimes L$ of the translation group N and the Lorentz group L. Similarly, as for the Lorentz group, Π has four pieces distinguished by $\det L$ and sign $L^0{}_0$, namely $\Pi_+^\uparrow, \Pi_-^\uparrow, \Pi_+^\downarrow, \Pi_-^\downarrow$.

In the following we shall consider the inhomogeneous group $\tilde{\Pi}$ corresponding to SL(2, C) group. It is the semidirect product $N \otimes \mathrm{SL}(2, C)$ defined by the following composition law

$$\{n_1, \Lambda_1\}\{n_2, \Lambda_2\} = \{n_1 + L_{\Lambda_1} n_2, \Lambda_1 \Lambda_2\}. \tag{17}$$

We recall that in a semidirect product $N \otimes S$, the topology is defined by the topology of the Cartesian product $N \times S$ of the group spaces N and S (cf. ch. 3, § 4). Thus, because N and SL(2, C) are simply connected, the semidirect product $\tilde{\Pi} = N \otimes \mathrm{SL}(2, C)$ is also simply connected. By virtue of the connection between SL(2, C) and the proper orthochronous Lorentz group $L_+^\uparrow$ we see that $\tilde{\Pi}$ is the universal covering group of the group $\Pi_+^\uparrow$. Moreover, because N and SL(2,C) act in N by unimodular transformations, the group $\tilde{\Pi}$ is also unimodular by th. 3.10.5. In addition, by virtue of eq. 3.10(16) the product of the invariant measures in N and SL(2, C) provides an invariant measure $\mu(\cdot)$ on $\tilde{\Pi}$, i.e.,

$$\begin{aligned} \mathrm{d}\mu(\{n, \Lambda\}) &= \mathrm{d}\sigma(n)\mathrm{d}\nu(\Lambda) \\ &= \mathrm{d}^4 n \frac{\mathrm{d}\beta \mathrm{d}\gamma \mathrm{d}\delta \mathrm{d}\bar{\beta} \mathrm{d}\bar{\gamma} \mathrm{d}\bar{\delta}}{|\delta|^2}, \qquad n \in N,\ \Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, C) \end{aligned} \tag{18}$$

(cf. eq. 2.3 (9)).

B. *Classification of Orbits*

The translation subgroup $N = \{(n, I)\}$ is a non-compact vector group. Hence, the corresponding dual group $\hat{N}$ can be identified with N. To every $\hat{n} = (\hat{n}_0, \hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{N}$ there correspond a character given by the formula:

$$\langle n, \hat{n} \rangle = \exp[\mathrm{i}(n_0 \hat{n}_0 - n_1 \hat{n}_1 - n_2 \hat{n}_2 - n_3 \hat{n}_3)] = \exp(\mathrm{i} n^\mu \hat{n}_\mu). \tag{19}$$

The action of SL(2, C) in $\hat{N}$ by eqs. (17) and (5) follows from the equality

$$\langle L_\Lambda n, \hat{n} \rangle = \exp(\mathrm{i} L_\Lambda{}^\mu{}_\nu n^\nu \hat{n}_\mu) = \exp[\mathrm{i} n^\nu (L_\Lambda^T)_\nu{}^\mu \hat{n}_\mu] = \langle n, L_\Lambda^{-1} \hat{n} \rangle, \tag{20}$$

i.e.,

$$\hat{n} \to L_\Lambda^{-1} \hat{n}, \tag{21}$$

where $\Lambda \in \mathrm{SL}(2, C)$ and $L_\Lambda^{-1} \in L_+^\uparrow$. Thus, the group SL(2, C) acts in the dual space $\hat{N}$ in the same manner as in N. Consequently, the set of all $L_\Lambda \hat{n}_0$ for a given $\hat{n}_0 \in \hat{N}$ and all $L_\Lambda \in L_+^\uparrow$ forms an orbit $\hat{O}$ associated with the character $\hat{n}_0$. This implies that every orbit is contained in one of the hyperboloides

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \tag{22}$$

where m^2 is any real number.

If $m^2 > 0$, then eq. (22) describes a two-sheeted hyperboloid (see Fig. 1a). The upper sheet $\hat{O}_m^+$ and the lower sheet $\hat{O}_m^-$ represent, separately, orbits relative to $L_+^\uparrow$. If $m^2 < 0$, eq. (22) defines a one-sheeted hyperboloid in $\hat{N}$ (see Fig. 1b).

Finally, if $m^2 = 0$, eq. (22) describes a cone, which consists of three orbits: $\hat{O}_0^+$—the upper cone, $\hat{O}_0^0$ consisting of the point (0, 0, 0, 0) only, and $\hat{O}_0^-$—the lower cone (see Fig. 1c).

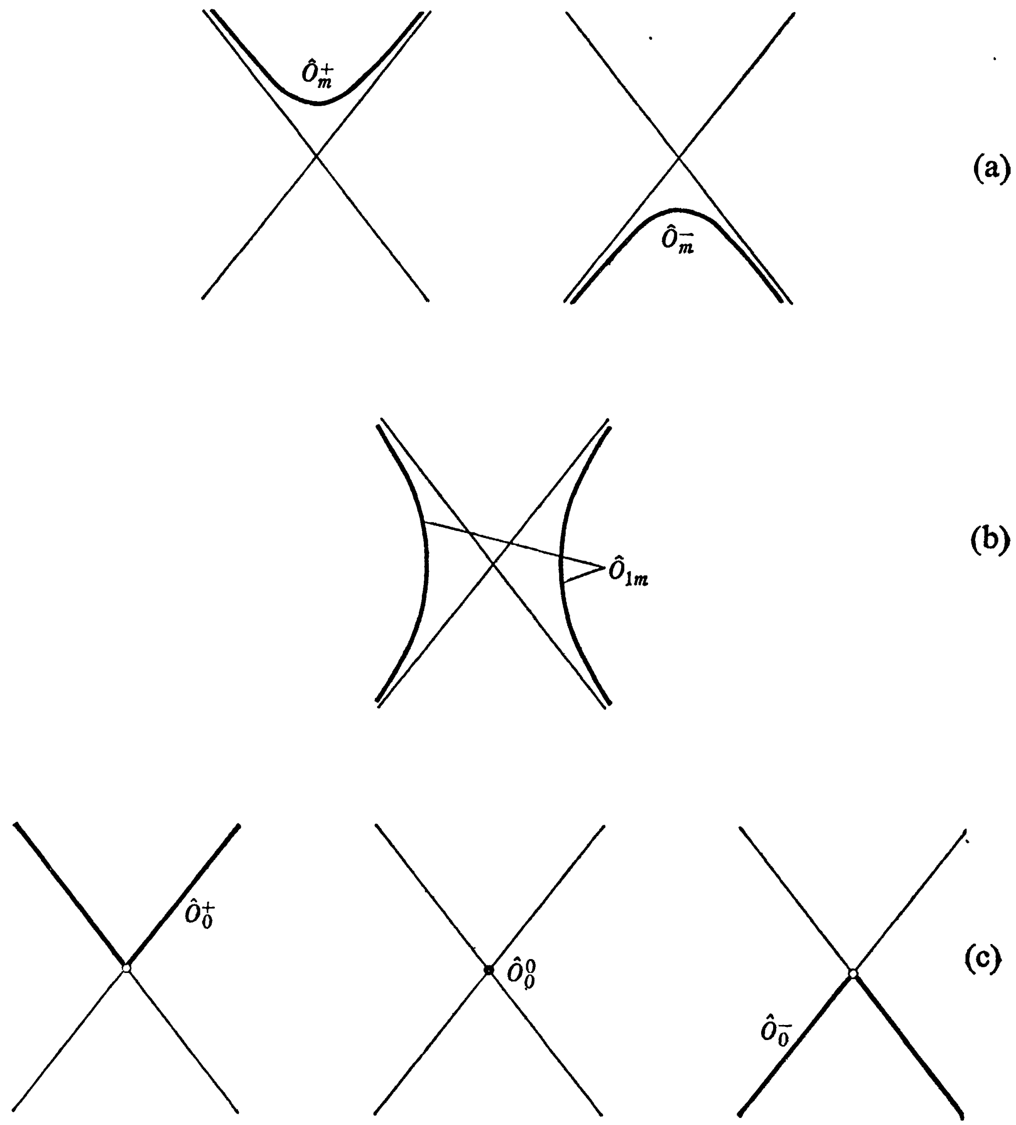

Fig. 1

Summarizing, we have six types of orbits:

1° $\hat{O}_m^+$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = m^2$, $m > 0$, $\hat{n}_0 > 0$,
2° $\hat{O}_m^-$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = m^2$, $m > 0$, $\hat{n}_0 < 0$,
3° $\hat{O}_{im}$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = -m^2$, $m > 0$,
4° $\hat{O}_0^+$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = 0$, $m = 0$, $\hat{n}_0 > 0$,
5° $\hat{O}_0^-$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = 0$, $m = 0$, $\hat{n}_0 < 0$,
6° $\hat{O}_0^0$: the point $0 = (0, 0, 0, 0)$, $m = 0$.

C. *The Classification of Irreducible Unitary Representations*

We verify firstly that the Poincaré group $\tilde{\Pi} = N \circledS \mathrm{SL}(2, C)$ is indeed a regular semidirect product as defined earlier, i.e. there exists a countable family Z of

Borel sets $Z_1, Z_2, \ldots$ of $\hat{N}$, each a union of orbits, such that every orbit in $\hat{N}$ is the intersection of a subfamily $Z_{n_1}, Z_{n_2}, \ldots$ of sets containing the orbit. Using the classification of orbits given in subsec. B, we can easily construct the family Z. Indeed, consider the family of subsets of the dual space $\hat{N}$ consisting of the following sets

(i) $\hat{O}_0^0$, $\hat{O}_0^+$ and $\hat{O}_0^-$;

(ii) $Z_m^+(r_1, r_2)$ = the union of all orbits $\hat{O}_m^+$ with $r_1 < m < r_2$, where r_1, r_2, $r_1 < r_2$, are any two positive rational numbers;

(iii) The sets $Z_m^-(r_1, r_2)$ and $Z_{im}(r_1, r_2)$ defined in the same way as $Z_m^+(r_1, r_2)$.

Then, the countable family of sets (i), (ii) and (iii) satisfies all the conditions imposed on the family Z. Thus, the Poincaré group $\tilde{\Pi}$ is a regular semidirect product. Consequently, we can directly apply for the classification of irreducible unitary representations of $\tilde{\Pi}$ the general formalism of induced representations developed in § 1 for the regular semidirect products.

Next we enumerate the classes of irreducible unitary representations of the Poincaré group $\tilde{\Pi}$ associated with each type of orbit:

1° $\hat{O}_m^+$: A representative of this orbit is the character $\hat{n}_0 = (m, 0, 0, 0)$, $m > 0$. The stability subgroup $S_{\hat{O}_m^+}$ of the point $\hat{n}_0$ is the unitary group SU(2). The group SU(2) has irreducible unitary representations $L^j \equiv D^j$ of dimension $2j+1$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Hence, the corresponding representations $U^{\hat{n}L^j}$ of $\tilde{\Pi}$ induced by these irreducible representations D^j of SU(2) will be labelled by two parameters: m and j. These parameters can be identified in particle physics with the total mass and the total spin of a stable free system (cf. subsec. D). We denote the irreducible induced representations $U^{\hat{n}L}$ by the symbol $U^{m,+,j}$

2° $\hat{O}_m^-$: Take $\hat{n}_0 = (-m, 0, 0, 0)$. Then the stability subgroup of $\hat{n}_0$ is again SU(2). Thus we obtain again a series of induced irreducible unitary representations $U^{m,-,j}$ of $\tilde{\Pi}$ labelled by the mass m and the spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ In the physical identification $U^{m,\pm,j}$ the sign $\pm$ refers to the sign of the energy.

3° $\hat{O}_{im}$: We can choose the character $\hat{n}_0 = (0, m, 0, 0)$. The (2×2)-hermitian matrix (8) corresponding to $\hat{n}_0$ has the form $\hat{n}_0 = m\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = mi\sigma_2$. Thus the stability subgroup $S_{\hat{O}}$ consists of all matrices $g \in \mathrm{SL}(2, C)$ satisfying the condition

$$\sigma_2 = \Lambda\sigma_2\Lambda^*, \quad \text{or} \quad (\sigma_2)^{-1}\Lambda\sigma_2 = (\Lambda^*)^{-1}. \tag{23}$$

Because $(\sigma_2)^{-1} = \sigma_2$, eqs. (14) and (23) imply that $(\Lambda^T)^{-1} = (\Lambda^*)^{-1}$, i.e., $\Lambda = \bar{\Lambda}$. Consequently, the stability group of any orbit $\hat{O}_{im}$ is the group of (2×2)-real unimodular matrices, i.e., SL(2, R).

The group SL(2, R) has three series of unitary irreducible representations.

(i) The principal series $D^{i\sigma,\varepsilon}$, σ real, $\varepsilon = 0$ or 1. We have explicitly constructed these series of irreducible representations in example 16.1.2. The representations $D^{i\sigma,\varepsilon}$ and $D^{-i\sigma,\varepsilon}$ are equivalent.

(ii) The discrete series D^n, $n = 0, 1, 2, \ldots$ These representations can be realized in the Hilbert space H^n of complex functions with the domain in the upper half-plane, $\operatorname{Im} z > 0$. The scalar product in H^n is given by the formula*

$$(u, v) = \frac{\mathrm{i}}{2\pi\Gamma(n)} \int_{\operatorname{Im} z>0} u(z)\overline{v(z)}\,(\operatorname{Im} z)^{n-1}\mathrm{d}z\,\mathrm{d}\bar{z}, \qquad n = 0, 1, 2, \ldots \tag{24}$$

If $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, R)$, then,

$$(D^n_g u)(z) = (\beta z + \delta)^{-n-1} u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{25}$$

These discrete series are of two types $D^{n,+}$ and $D^{n,-}$ in which the spectrum of the compact generator is bounded below and above respectively (cf. exercise 11.10.7.6).

(iii) The supplementary series of representations D^ϱ, $-1 < \varrho < 1$, $\varrho \neq 0$. These can be realized in the Hilbert space H^ϱ of functions with domain on the real line, and with the scalar product

$$(u, v) = \frac{1}{\Gamma(-\varrho)} \int_{R^1 \times R^1} |x_1 - x_2|^{-1-\varrho} u(x_1)\overline{v(x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{26}$$

The action of D^ϱ in H^ϱ is given by the formula

$$(D^\varrho_g u)(x) = |\beta x + \delta|^{\varrho-1} u\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right). \tag{27}$$

(cf. exercise 16.4.1.5).

Corresponding to each series of these irreducible representations $D^{i\sigma,\varepsilon}$, $D^{n,\pm}$ and L^ϱ of $\mathrm{SL}(2, R)$ and to a given orbit $\hat{O}_{im}$ we have the irreducible, unitary representations $U^{im,i\sigma,\varepsilon}$, $U^{im,n,\pm}$ and $U^{im,\varrho}$ of the Poincaré group $\tilde{\Pi}$. Isolated free quantum systems with imaginary masses are not known, but two particle states with a total imaginary mass can be constructed if one of the particles has a space-like momentum. These representations have found some applications also in the harmonic analysis of scattering amplitudes (cf. ch. 21.6).

4° $\hat{O}^+_0$. We can choose the character $\hat{n}_0 = (\frac{1}{2}, 0, 0, \frac{1}{2})$. Then, the corresponding 2×2-matrix is $\hat{n}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The condition $\Lambda \hat{n}_0 \Lambda^* = \hat{n}_0$ implies

$$\begin{bmatrix} \alpha\bar{\alpha} & \alpha\bar{\gamma} \\ \bar{\alpha}\gamma & \gamma\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

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* For $n = 0$, we have $\lim_{\varepsilon\to 0} y_+^{\varepsilon-1}/\Gamma(\varepsilon) = \delta(y)$. Hence the scalar product (24) takes the form

$$(u, v) = \int_{-\infty}^{\infty} u(x)\overline{v(x)}\,\mathrm{d}x.$$

Note that D^0 is reducible.

which gives $\alpha = \exp(i\theta)$, $\gamma = 0$ and β an arbitrary complex number. Thus, the stability subgroup $S_{\hat{O}_0}$ of the point $\hat{n}_0$ consists of all 2×2 matrices of the form

$$\begin{bmatrix} \exp(i\theta) & z \\ 0 & \exp(-i\theta) \end{bmatrix}, \qquad \theta \in [0,2\pi],\ z \in C^1. \tag{28}$$

The group $S_{\hat{O}_0}^+$ is actually isomorphic to a semidirect product. To see this we write an arbitrary element $k \in S_{\hat{O}_0}^+$ in the form

$$k = \begin{bmatrix} \exp\left(i\frac{\theta}{2}\right) & \exp\left(-\frac{i\theta}{2}\right)z \\ 0 & \exp\left(-\frac{i\theta}{2}\right) \end{bmatrix}, \qquad \theta \in [0, 4\pi]. \tag{29}$$

Then the product $k_1 \cdot k_2$, $k_1, k_2 \in S_{\hat{O}_0}^+$ is given by the formula

$$k_1 k_2 = \begin{bmatrix} \exp\left[\frac{i(\theta_1+\theta_2)}{2}\right] & \exp\left[-\frac{i(\theta_1+\theta_2)}{2}\right](z_1+\exp(i\theta_1)z_2) \\ 0 & \exp\left[-\frac{i(\theta_1+\theta_2)}{2}\right] \end{bmatrix}.$$

If we set $k = (z, \theta)$, then we have the following composition law

$$(z_1, \theta_1)(z_2, \theta_2) = (z_1+\exp(i\theta_1)z_2,\ \theta_1+\theta_2). \tag{30}$$

This shows that $S_{\hat{O}}^+$ itself is indeed the semidirect product $T^2 \circledS S^1$ of the two-dimensional translation group $T^2 \doteq \{(z, 0)\}$ and the rotation group $S^1 = \{(0, \theta)\}$. Notice that the composition law (30) in the semidirect product $T^2 \circledS S^1$ is the same as that of the group of the motion of the two-dimensional Euclidean space, i.e., $T^2 \circledS O(2)$. However, in the present case $(0, 2\pi) \neq (0, 0)$, but only $(0, 4\pi) = 0$ (i.e., $T^2 \circledS S^1$ covers twofold the Euclidean group).

Because T^2 is a noncompact vector group, the dual group $\hat{T}^2$ can be identified with T^2. Every character $\hat{z} \in \hat{T}^2$ is given by the formula

$$\langle z, \hat{z}\rangle = \exp[i(x\hat{x}+y\hat{y})], \qquad z = (x, y),\ z = (\hat{x}, \hat{y}). \tag{31}$$

The action of S^1 in T^2 is

$$(0, \theta)(z, 0)(0, -\theta) = (\exp(i\theta)z, 0),$$

i.e., it produces a rotation in T^2 by an angle θ. Hence, by eq. (31) the action of S^1 in the dual space is of the form $\hat{z} \to \exp(i\theta)\hat{z}$. The orbits are circles in the complex plane $\hat{T}^2$, with the center at $0 = (0, 0)$ and radius $r \geqslant 0$. Consequently, we can distinguish two kinds of orbits:

(i) $r = 0$: In this case the stability subgroup is S^1. The irreducible representations of S^1 are one-dimensional: $\theta \to \exp(ij\theta)$ $j = 0, \pm 1/2, \pm 1, \ldots$ The representations $\theta \to \exp(ij\theta)$ and $\theta \to \exp(-ij\theta)$ are equivalent but not unitarily equivalent. The equivalence is given by the anti-unitarity transformation V: $u \to \bar{u}$ of the carrier space $H = C^1$. We denote the irreducible representations of $S_{\hat{O}_0}^+$

$= T^2 \rtimes S^1$ induced by these one-dimensional representations of S^1 by the symbol L^j or L^{-j}, respectively. The representations L^j, $j = 0, \pm\frac{1}{2}, \pm 1, \ldots$ give rise to a series of irreducible unitary induced representations of $\tilde{\Pi}$. We denote these by the symbol $U^{0,+;j}$.

(ii) $r > 0$: Choose $\hat{z}_0 = (r, 0)$. The stability subgroup of the point $\hat{z}_0$ is the center $Z = \{I, -I\}$ of SL(2, C). This group has two irreducible, unitary representations. Hence, we obtain two series $L^{r,\varepsilon}$, $r > 0$, $\varepsilon = 0$ or 1, of irreducible, unitary representations of $S\hat{o}_0^+$ induced by one-dimensional irreducible representations of Z. Consequently, we obtain two series of irreducible representations of $\tilde{\Pi}$ induced by the representations $L^{r,\varepsilon}$ of $S\hat{o}_0^+$. We denote them by the symbol $U^{0,+;r,\varepsilon}$, $r > 0$, $\varepsilon = 0, 1$.

5° $\hat{O}_0^-$: The classification of irreducible, unitary representations of $\tilde{\Pi}$ associated with this orbit runs parallel to the case 4°. We obtain three new series of irreducible, unitary representations of $\tilde{\Pi}$, which we denote by the symbols $U^{0,-;j}$, $U^{0,-;r,\varepsilon}$ $r > 0$, $\varepsilon = 0, 1$, respectively.

6° $\hat{O}_0^0$: In this case the stability subgroup of the orbit $\hat{O}_0^0 = (0, 0, 0, 0)$ is the whole group SL(2, C) itself. The irreducible unitary representations of $\tilde{\Pi}$ associated with this orbit, are all the irreducible unitary representations of SL(2, C), '*lifted*' to the group $\tilde{\Pi}$. The full classification of these representations is given in ch. 19, § 1 and 2. The group SL(2, C) has a principal series and a supplementary series of irreducible unitary representations. Representations of the principal series are labelled by two numbers $i\varrho$ and j, $\varrho \geqslant 0$, $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$. The representations of the supplementary series are labelled by a single real number ϱ, $-1 < \varrho < 1$, $\varrho \neq 0$. The corresponding representations of $\tilde{\Pi}$ induced by the irreducible representations of the principal and the supplementary series are denoted by the symbols $U^{0,i\varrho,j}$ and $U^{0,\varrho}$, respectively.

To summarize, we have the following series of irreducible, unitary representations of the Poincaré group $\tilde{\Pi}$, associated with orbits 1°–6°; the first upper index characterizes the orbit $\hat{O}$ (i.e., mass and the sign of energy), and the second one characterizes the irreducible, unitary representations of the stability subgroup of $\hat{O}$.

1° $U^{m,+;j}$, $m > 0$, $\hat{n}_0 > 0$, $j = 0, \frac{1}{2}, 1 \ldots$,

2° $U^{m,-;j}$, $m > 0$, $\hat{n}_0 < 0$, $j = 0, \frac{1}{2}, 1, \ldots$,

3° $U^{im;i\sigma,\varepsilon}$, $m > 0$, $\sigma \geqslant 0$, $\varepsilon = 0, 1$,

$U^{im;n,\pm}$, $m > 0$, $n = 0, 1, 2, \ldots$,

$U^{im;\varrho}$, $m > 0$, $-1 < \varrho < 1$, $\varrho \neq 0$,

4° $U^{0,+;j}$, $m = 0$, $\hat{n}_0 > 0$, $j = 0, \pm\frac{1}{2}, \pm 1 \ldots$,

$U^{0,+;r,\varepsilon}$, $m = 0$, $\hat{n}_0 > 0$, $r > 0$, $\varepsilon = 0, 1$,

5° $U^{0,-;j}$, $m = 0$, $\hat{n}_0 < 0$, $j = 0, \pm\frac{1}{2}, \pm 1, \ldots$,

$U^{0,-;r,\varepsilon}$, $m = 0$, $\hat{n}_0 < 0$, $r > 0$, $\varepsilon = 0, 1$,

6° $U^{0,0;i\varrho,j}$, $m = 0$, $\hat{n}_0 = 0$, $\varrho \geqslant 0$, $j = 0, \frac{1}{2}, 1 \ldots$,

$U^{0,0;\varrho}$, $m = 0$, $\hat{n}_0 = 0$, $-1 < \varrho < 1$, $\varrho \neq 0$.

D. *The Explicit Realization of Irreducible Representations* ($m > 0$)

We now derive explicit formulas for the unitary operators $U_g^{m,+;j}$, $g \in \tilde{\Pi}$, and discuss the physical identification of these representations.

The stability subgroup of the orbit $\hat{O}_m^+$ is the group $K = T^4 \circledS \mathrm{SU}(2)$. By lemmas 1.2 and 3, the irreducible, unitary representations of $T^4 \otimes \mathrm{SU}(2)$ are of the form

$$k = (a, r) \to L_k^j = L_{(a,r)}^j = \exp(i\overset{0}{p}a) D^j(r), \tag{31}$$

where $\overset{0}{p} = (m, 0, 0, 0)$ and D^j is a unitary irreducible representation of SU(2) derived in exercise 5.8.1. The action of $U_g^{m,+j}$ in the space $H^{m,+;j}$ is given by eq. 16.1(47). Hence to complete the construction of $U_g^{m,+;j}$ we have to find the operator $L^{-1}_{k g_0^{-1} s_g}$. Note first that by virtue of the Cartan decomposition 3.6 (17) every element of $\mathrm{SL}(2, C)$ has the following decomposition

$$\Lambda = \Lambda_p r, \tag{32}$$

where

$$\Lambda_p = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R,\ z \in C \text{ and } r \in \mathrm{SU}(2) \tag{33}$$

(cf. exercise 3). By virtue of eq. (10) we have

$$\hat{p} = \begin{bmatrix} p_0 - p_3 & p_2 + ip_1 \\ p_2 - ip_1 & p_0 + p_3 \end{bmatrix} = \Lambda \hat{\overset{\circ}{p}} \Lambda^* = \Lambda_p \hat{\overset{\circ}{p}} \Lambda_p^*. \tag{34}$$

Hence, Λ_p has the meaning of a pure Lorentz transformations which transform $\overset{\circ}{p}$ onto p. Using the map $\Lambda \to L_\Lambda$ of $\mathrm{SL}(2, C)$ onto the Lorentz group SO(3,1) we can write eq. (34) in the form

$$p = L_\Lambda \overset{\circ}{p} = L_{\Lambda_p} L_r \overset{\circ}{p} = L_{\Lambda_p} \overset{\circ}{p}. \tag{35}$$

Using eqs. (34) and (33) we find that the, explicit correspondence between the elements Λ_p and p is given by the formula

$$\begin{aligned} p_0 &= \frac{m}{2}(\lambda^{-2} + \lambda^2 + |z|^2), \\ p_3 &= \frac{m}{2}(\lambda^{-2} - \lambda^2 - |z|^2), \\ p_2 - ip_1 &= \frac{m}{2} \lambda^{-1} \bar{z}. \end{aligned} \tag{36}$$

These equations give the explicit correspondence between the elements of the set S of Mackey decomposition $G = SK$ and the homogeneous space O_m^+. Observing that $\Lambda_p = s_g$ we obtain

$$\begin{aligned} g_0^{-1} s_g &= (a_0, \Lambda_0)^{-1}(0, \Lambda_p) = (-L_{\Lambda_0}^{-1} a_0, \Lambda_0^{-1})(0, \Lambda_p) \\ &= (-L_{\Lambda_0}^{-1} a_0, \Lambda_0^{-1} \Lambda_p) = (0, \Lambda_{L_{\Lambda_0 p}^{-1}})(-L^{-1}_{\Lambda_{L_{\Lambda_0 p}^{-1}}} L_{\Lambda_0}^{-1} a_0, \Lambda^{-1}_{L_{\Lambda_0 p}^{-1}} \Lambda_0^{-1} \Lambda_p). \end{aligned}$$

Hence,

$$k_{g_0^{-1}s_g} = (-L^{-1}_{L^{-1}_{A_0}p} L^{-1}_{A_0} a_0,\ A^{-1}_{L^{-1}_{A_0}p} A_0^{-1} A_p). \tag{37}$$

Taking into account that $L_{A_p}\mathring{p} = p$ and $(L_A p,\ L_A a) = (p, a)$ we obtain for the representation (31)

$$L^{-1}_{k_{g_0^{-1}s_g}} = \exp[ipa] D^j(A_p^{-1} A_0 A_{L^{-1}_{A_0}p}). \tag{38}$$

Setting $r_{A_0} = A_p^{-1} A_0 A_{L^{-1}_{A_0}p}$ and noticing that the Radon–Nikodym derivative equals to one we finally obtain by virtue of eq. 16.1(47), the formula

$$U^{m,+;j}_{(a,A)} u(p) = \exp[ipa] D^j(r_A) u(L_A^{-1} p). \tag{39}$$

We recall that $u(p)$ is a vector-valued function from $\hat{O}_m^+$ into the $(2j+1)$-dimensional vector space H of the representation D^j of SU(2): $u(p) = \{u_n(p)\}$, $n = -j, -j+1, \ldots, j-1, j$. In terms of the components, eq. (39) can be written as

$$(U^{m,+;j}_{(a,A)} u)_n(p) = \exp(ipa) D^j_{nn'}(A_p^{-1} A A_{L_A^{-1}p}) u_{n'}(L_A^{-1} p). \tag{40}$$

The representation D^j of SU(2) may be extended to $D^{(j,0)}$ of SL(2, C). Utilizing the multiplicative properties of $D^{(j,0)}$ matrices and introducing the so-called spinor basis $v_l(p) \equiv D^{(j,0)}_{ll'}(A_p) u_{l'}(p)$ one may also write eq. (40) in the simpler form:

$$(U^{m,+;j}_{(a,A)} v)_l(p) = \exp(ipa) D^{(j,0)}_{ll'}(A) v_{l'}(L_A^{-1} p). \tag{41}$$

The set of functions $\{u_n(p)\}_{n=-j}^{j} \in H^{m,+;j}$ may be identified with the wave functions of a free physical system with spin j and mass m. Indeed, in the rest system $(p = \overset{0}{p})$, and under rotations, $(0, r) \in \mathrm{SU}(2)$, the formula (40) gives

$$(U^{m,+;j}_{(0,r)} u_n)(\mathring{p}) = D^j_{nn'}(r) u_{n'}(\mathring{p}), \tag{42}$$

i.e., the set $\{u_n(\overset{0}{p})\}$ transforms in the rest frame according to the spinor representations D^j. This is, of course, the property of a free physical system which has a total spin j. The number n, $n = -j, -j+1, \ldots, -1, j$, in the rest system represents the projection of the spin on a given axis of quantization.

Next we consider the generators of the translations by taking the one-parameter subgroups of the form $a_\mu(t) = (a_\mu(t), I)$, $\mu = 0, 1, 2, 3$ and by using eq. (40). We find

$$(P_\mu u)_n(p) = p_\mu u_n(p). \tag{43}$$

Thus, for the mass operator $M = \sqrt{P_\mu P^\mu}$ we obtain

$$M u_n(p) = \sqrt{(p_0^2 - \mathbf{p}^2)}\, u_n(p) = m u_n(p). \tag{44}$$

Equations (42) and (44) show that the set $\{u_n(p)\}$ describes a physical system with the rest mass m, in addition to the spin j.

If j is an integer, the representation $U^{m,+;j}$ of $\tilde{\Pi}$ are also representations of the proper Poincaré group $\Pi_+^\uparrow$. However, if j is half-odd-integer, then the representation (40) becomes a two-valued representation of $\Pi_+^\uparrow$. This follows from the fact that for half-odd-integer j the representation D^j of SO(3) becomes two-valued. (Cf. group extension by parity, ch. 21, § 4.)

We note, for completeness, that the proposition 16.1.3 provides a full description of the structure of elements $u(p) = \{u_n(p)\}$ of the carrier space $H^{m,+;j}$. In fact, the space $H^{m,+;j}$ is spanned by vectors of the form

$$u(p) = \sum_{n=-j}^{j} u_n(p) Y_n^j, \qquad u_n(p) \in C_0(\hat{O}_m^+), \tag{45}$$

where Y_n^j are the basis vectors of the carrier space H^j of the representation D^j of the stability subgroup SU(2). We have shown that Y_n^j can be represented as homogeneous polynomials of order $2j$ in the form

$$Y_n^j(\xi_1, \xi_2) = \frac{\xi_1^{j+n}\xi_2^{j-n}}{\sqrt{[(j+n)!(j-n)!]}}, \tag{46}$$

where $\xi_1, \xi_2 \in C^1$ (cf. exercise 8.9.2.1). The functions $\{Y_n^j\}, n = -j, -j+1, \ldots, j-1, j$, are called spinors of order $2j$. The action of an irreducible representation D^j of SU(2) in the space H^j is given by the formula

$$(D^j(r) Y^j)_n(\xi_1, \xi_2) = D^j_{n'n}(r) Y^j_{n'}(\xi_1, \xi_2). \tag{47}$$

Elementary Systems

We consider the largest symmetry groups associated with the geometrical transformations of space-time (with fixed scales): the Galilei group or the Poincaré group. All the transformations of these groups have a physical, geometrical interpretation:

a) space and time translations (displacements of the coordinate frame);
b) rotations and reflections;
c) transformations which give to the system a velocity ('boost' transformations).

An isolated system must allow equivalent descriptions under the Poincaré group. Consequently we can define *elementary systems* whose concrete Hilbert space (CHS) is the carrier space of a single irreducible representation of the full Poincaré group Π. An elementary system is characterized by the invariants of Π, mass (m^2) and spin $(j(j+1))$, or helicity. The eigenstates of the displacements are labelled by $|p_\mu; \sigma\rangle$, $\sigma \equiv \{j, n\}$; the rest frame states $|p_0, \boldsymbol{p} = 0, \sigma\rangle$ are rotational invariant and a velocity imparting is given by the transformation

$$|p_\mu; \sigma\rangle = \exp(\mathrm{i}\xi \boldsymbol{M})|p_0, \boldsymbol{p} = 0, \sigma\rangle$$

for every fixed ξ. Here $\boldsymbol{M}$ are the generators of pure Lorentz transformations for the rest states and

$$\xi = \hat{p}\cosh^{-1}\frac{p_0}{m} = \hat{p}\sinh^{-1}\frac{p}{m}, \quad p = \sqrt{\boldsymbol{p}^2}, \quad \hat{p} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$$

for massive particles.

An elementary system may reveal under external probing a more complex internal structure. We can give an operational definition of an elementary particle. To do this we first say that two elementary systems are *connected* if physical interactions can connect the Hilbert spaces $H^{m_1,+;j_1}$ and $H^{m_2,+;j_2}$ of the two systems. For example, the connection of $1s$ and $2p$ states of the H-atom by photo-absorption or the connection of the neutron and proton states by β-decay. We have then:

DEFINITION. An *elementary particle* (EP) is an elementary system whose states in no way can be physically connected to the states of other systems. Its Hilbert space is isolated, i.e., the states of one elementary particle $|1\rangle$ do not form a linear space with those of other systems $|2\rangle$, that is, the superposition $|1\rangle+|2\rangle$ is physically meaningless. The only effect that outside interactions can have on an EP is to change the state *within* the irreducible representation, i.e., to change its momentum and spin projection. ▼

It follows that an EP can have only those internal quantum numbers for which there are absolute superselection rules.

This operational definition of an elementary particle reflects the dependence of the concept of elementarity on the nature of interactions, as it should be. Clearly, in the kinetic theory of gases, for example, the molecules are elementary particles for, under the processes considered, the internal structure of the molecule is not excited and there is no connection to other parts of the Hilbert space. Similarly, nuclei are elementary particles in atomic phenomena, and so on.

As we did in the previous section, a 'measurement' on an elementary particle will be described by an interaction vertex with a coupling constant λ

$$M = \lambda\langle p', \sigma'|T|p, \sigma\rangle = \lambda\langle \overset{0}{p}\sigma'|\exp(-i\xi'\cdot M)T\exp(i\xi\cdot M)|\overset{0}{p}\sigma\rangle.$$

Here m and j are fixed on both sides, and T is a general tensor operator and represents the external agent.

The use of the Poincaré group in the definition of elementary systems presupposes a flat Minkowski space as the physical space. If we change the topology of the space, or its metric, or both, the group of motions of the space will change too. For example, the asymptotically flat space of general relativity leads to the Bondi–Meltzner–Sachs group which is the semidirect product of an infinite abelian group with SL(2, C) and one could base the concept of elementary particles on the representations of this group.

§ 3. Representation of the Extended Poincaré Group

We shall now analyse the properties of representations of the Poincaré group including space and time reflections. Let I denote the space or the time inversion operators P. T in the carrier Hilbert space H of a unitary representation U of the

Poincaré group and let $\hat{I}$, $\hat{P}$ and $\hat{T}$ denote the corresponding operators in space-time or momentum space. By definition we have

$$\hat{P}a \equiv a_{\hat{P}} = (a_0, -a), \quad \hat{T}a \equiv a_{\hat{T}} = (-a_0, a). \tag{1}$$

We assume that the action of the transformation P and T on $U_{(a,\Lambda)}$ is given by the formula

$$I^{-1}U_{(a,\Lambda)}I = U_{(a_{\hat{I}},\Lambda_{\hat{I}})}. \tag{2}$$

The momentum vector $\hat{p}$ must transform like $\hat{a}$ under Lorentz transformations and must be linear in p: hence it must be of the form

$$p_{\hat{I}} = \lambda(p_0, -p),$$

where $\lambda \in C^1$. Because $\hat{I}\hat{I}p = p$ we have $\lambda^2 = 1$, i.e., $\lambda = \pm 1$. In order to determine the sign of λ we impose an additional condition, namely the definiteness of the energy. This gives $\lambda = +1$ and

$$p_{\hat{I}} = (p_0, -p).$$

Consequently

$$(\hat{P}a, \hat{P}p) = (a, p), \quad (\hat{T}a, \hat{T}p) = -(a, p). \tag{3}$$

By virtue of eq. (2) we have

$$(I^{-1}U_{(a,e)}I)\psi(p) = U_{(\hat{I}a,e)}\psi(p) = \exp[\mathrm{i}(\hat{I}ap)]\psi(p). \tag{4}$$

This implies by virtue of eq. (3) that P must be linear and T antilinear transformation in the carrier Hilbert space H i.e.

$$P\psi(p) = \eta\psi(p), \quad T\psi(p) = C\psi^*(p), \tag{5}$$

where η and C are matrices.

We now show that

$$\Lambda_{\hat{I}} = \hat{I}^{-1}\Lambda\hat{I} = \Lambda^{*-1}. \tag{6}$$

Indeed it follows, e.g. from the form of generators of the Lorentz group ($M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$), that space and time reflections commute with the rotation group and anticommute with pure Lorentz transformations. Because by virtue of eq. 3.11.6.8 we have

$$\Lambda = u_1\varepsilon u_2, \quad u_1, u_2 \in \mathrm{SU}(2), \tag{7}$$

where $\varepsilon = \left[\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right]$ is a pure Lorentz transformation we obtain

$$\Lambda_{\hat{I}} = u_1\varepsilon^{-1}u_2 = \Lambda^{*-1}. \tag{8}$$

LEMMA 1. *The matrix η in eq.* (5) *must satisfy the condition*

$$D^*(\Lambda)\eta D(\Lambda) = \eta, \tag{9}$$

whereas the matrix C is given by the formula

$$C = \lambda D(\mathrm{i}\sigma_2), \quad |\lambda| = 1, \quad \sigma_2 = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}. \tag{10}$$

If η exists, then it represents the parity operator for the representation $D(\Lambda)$.

PROOF: Because

$$U_{(0,\Lambda)}\psi(p) = D(\Lambda)\psi(L_\Lambda^{-1}p) \tag{11}$$

by virtue of eq. (5) we obtain

$$D(\Lambda_{\hat{P}}) = \eta D(\Lambda)\eta^{-1}, \quad D(\Lambda_{\hat{T}}) = C\bar{D}(\Lambda)C^{-1}. \tag{12}$$

Now, eq. (8) implies

$$D(\Lambda_{\hat{T}}) = D(\Lambda^{*-1}) = D^{*-1}(\Lambda). \tag{13}$$

Using eqs. 2 (14) we see that the condition (12) for T can always be satisfied if $C = \lambda D(\sigma_2)$, $|\lambda| = 1$. To prove the last assertion we note that every representation $D(\Lambda)$ of SL(2, C) is the direct sum of irreducible representations $D^{(j_1, j_2)}$ by th. 8.1.4 Now in the SU(2) × SU(2) basis of representation $D^{(j_1, j_2)}$ all generators of SU(2) are hermitian whereas all generators of pure Lorentz transformation are antihermitian. Hence for $r \in$ SU(2), by virtue of eq. (9), we have

$$D^*(r)\eta \cdot D(r) = D^{-1}(r)\eta D(r) = \eta \tag{14}$$

and for pure Lorentz transformations Λ_p we have

$$D^*(\Lambda_P)\eta D(\Lambda_P) = \exp[\mathrm{i}\vartheta_k N_k]\eta\exp[\mathrm{i}\vartheta_k N_k] = \eta. \tag{15}$$

Eqs. (14) and (15) show that η must commute with J_k and anticommute with N_k, i.e. η is the parity operator for $D(\Lambda)$.▼

We now determine which representations of SL(2, C) satisfy the condition (9). It follows from eq. (1) that P commutes with the generators J of SL(2, C) and anticommutes with generators $N = (M_{01}, M_{02}, M_{03})$ of pure Lorentz transformations. The irreducible representation $D^{(j_1, j_2)}$ of SL(2, C) refer to SU(2) × SU(2) basis associated with generators $J_1 = \frac{1}{2}(J+\mathrm{i}N)$ and $J_2 = \frac{1}{2}\mathrm{i}(J-\mathrm{i}N)$ Consequently j_1 and j_2 are interchanged under parity. Hence among the irreducible representations of SL(2, C) only $D^{(j_1, j_2)}$ admit the definition of parity in the carrier space. If the wave function $\psi(p)$ transform according to the representation $D^{(j_1, j_2)}$, $j_1 \neq j_2$ of SL(2, C), then we have at least to double the carrier space by considering the representation

$$D = D^{(j_1, j_2)} \oplus D^{(j_2, j_1)}$$

in order to be able to define the parity operator P. The specific examples of this procedure as well as the general theory of the extentions of the Poincaré group are discussed in ch. 21.

§ 4. Indecomposable Representations of Poincaré Group

We present in this section the construction of indecomposable representations of the Poincaré group induced from the indecomposable representations of the stability subgroup $T^4 \otimes SU(2)$. We give also an application of these representations to a description of unstable particles of arbitrary spin.

A. *SLS Representations of Topological Groups*

A *sesquilinear system*, SLS, is a pair $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle \equiv \Phi$ of complex linear spaces $\overset{1}{\Phi}$ and $\overset{2}{\Phi}$ together with a sesquilinear (linear-antilinear) form $(\cdot, \cdot)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$, i.e.

$$(\alpha_i u_i, \beta_k w_k) = \alpha_i \overline{\beta}_k (u_i, w_k)$$

and

$$\begin{cases} (u, \overset{2}{\Phi}) = 0 & \text{iff } u = 0, \\ (\overset{1}{\Phi}, w) = 0 & \text{iff } w = 0. \end{cases}$$

An isomorphism F between two SLS, Φ and $\tilde{\Phi}$ is a pair $\langle F_1, F_2 \rangle$, where F_i is a linear isomorphism of $\overset{i}{\Phi}$ onto $\overset{i}{\tilde{\Phi}}$, $i = 1, 2$ and $(F_1 u, F_2 w) = (u, w)$ for all $u \in \overset{1}{\Phi}$, $w \in \overset{2}{\Phi}$.

Using the sesquilinear form $(\cdot, \cdot)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$ one can define a locally convex topology $\tau(\overset{1}{\Phi})$ on $\overset{1}{\Phi}$, generated by functionals $u \to (u, w)$, $u \in \overset{1}{\Phi}$, where w runs over $\overset{2}{\Phi}$: one may define similarly a topology on $\overset{2}{\Phi}$.

A SLS representation T of a locally compact group G on a SLS, $\Phi(T) = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ is a pair $\langle \overset{1}{T}, \overset{2}{T} \rangle$, where

1. $\overset{1}{T}$ (resp. $\overset{2}{T}$) is a homomorphism of G into the group of invertible linear endomorphisms of $\overset{1}{\Phi}$ (resp. $\overset{2}{\Phi}$).
2. $(\overset{1}{T}_g u, \overset{2}{T}_g w) = (u, w)$ for all $g \in G$, $u \in \overset{1}{\Phi}$, $w \in \overset{2}{\Phi}$.
3. The map $g \to (\overset{1}{T}_g u, w)$ is continuous on G for each $u \in \overset{1}{\Phi}$, and $w \in \overset{2}{\Phi}$.

If X is a bounded linear operator in $\overset{1}{\Phi}$ then the adjoint X^* is defined by the equality

$$(X^* u, w) = (u, Xw).$$

The condition 2 means that

$$\overset{1}{T}_g = (\overset{2}{T}{}_g^{-1})^*,$$

i.e. a representation $g \to \overset{1}{T}_g$ in $\overset{1}{\Phi}$ is contragradient to $g \to \overset{2}{T}_g$. Clearly $\overset{1}{T} = \overset{2}{T}$ if $\overset{2}{T}$ is unitary and sesquilinear form is the scalar product.

A representation T is (topologically) irreducible if $\overset{1}{\Phi}$ has no non-trivial $\tau(\overset{1}{\Phi})$-closed $\overset{1}{T}$-stable subspaces. (Clearly this implies that $\overset{2}{\Phi}$ has no non-trivial $\tau(\overset{2}{\Phi})$-closed $\overset{2}{T}$-stable subspaces.)

B. *SLS Induced Representations of the Poincaré Group*

We shall now construct a class of nonunitary representations of the Poincaré group which might correspond to unstable particles. Let G be the Poincaré group $G = \Pi = T^4 \otimes \mathrm{SL}(2, C)$, and let K be the closed subgroup of P such that G/K has an invariant measure. Let $k \to L_k = \langle \overset{1}{L}_k, \overset{2}{L}_k \rangle$ be a finite-dimensional SLS representation of K in the vector space $\tilde{\Phi} = \langle \overset{1}{\tilde{\Phi}}, \overset{2}{\tilde{\Phi}} \rangle$.

If $\langle u, w \rangle \in \tilde{\Phi}$ then the sesquilinear form $(\cdot, \cdot)$ can be written as

$$(u, w) = u_s \bar{w}_s, \quad s = 1, 2, \dots, \dim \overset{1}{L} \tag{1}$$

and the SLS representation L_k, by definition, satisfies

$$(\overset{1}{L}_k u, \overset{2}{L}_k w) = (u, w) \quad \text{for all } \langle u, w \rangle \in \tilde{\Phi} \text{ and } k \in K. \tag{2}$$

Now let $D(G)$ be the vector space of functions $\langle u(g), w(g) \rangle$ on Π with values in $\tilde{\Phi}$ such that each component $u_i(g)$ or $w_s(g)$, $i, s = 1, 2, \dots, \dim \overset{1}{L}$, is an element of the Schwartz space of infinitely differentiable functions with compact support. Denote by $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ the vector space of all functions $\langle u(g), w(g) \rangle \in D(G)$ such that

$$w(gk) = \overset{2}{L}{}_k^{-1} w(g) \tag{3}$$

and

$$u(gk) = (\overset{1}{L}{}_k^{-1}) u(g). \tag{4}$$

The symbol 'C' denotes the operation of taking the contragradient (i.e. for a bounded operator X in $\overset{2}{\Phi}$: $X^C \equiv (X^*)^{-1}$). The vector space of functions satisfying the conditions (3) and (4) can be easily constructed. Indeed, if $\langle u(g), w(g) \rangle \in D(G)$ then

$$\hat{w}(g) = \int_K L_k w(gk) \mathrm{d}k \tag{5}$$

and

$$\hat{u}(g) = \int_K L_k^C u(gk) \mathrm{d}k \tag{6}$$

satisfy the conditions (3) and (4), respectively. It is evident from eq. (5) (resp. (6)) that $\hat{w}(g) = 0$ (resp. $\hat{u}(g) = 0$) if $g \notin SK$ where S is the compact support of the function $w(g)$ (resp. $u(g)$). Hence if

$$\langle u(g), w(g) \rangle \in D(G) \quad \text{then} \quad \langle \hat{u}(g), \hat{w}(g) \rangle$$

has a compact support on G/K. Consequently the sesquilinear form

$$(\hat{u}, \hat{w}) \equiv \int_{G/K} \bar{\hat{u}}_s(\dot{g})\hat{w}_s(\dot{g})\mathrm{d}\mu(\dot{g}), \qquad \dot{g} \equiv gK, \tag{7}$$

is well defined.

The action of the SLS representation $T^L = \langle \overset{1}{T^L}, \overset{2}{T^L}\rangle$ of Π in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi}\rangle$ is given by the left translation

$$\overset{2}{T}{}^L_{g_0} w(g) = w(g_0^{-1}g), \tag{8}$$

$$\overset{1}{T}{}^L_{g_0} u(g) = u(g_0^{-1}g). \tag{9}$$

The sesquilinear form (7) is conserved by the representation $g \to T^L_g$. Indeed using Mackey decomposition $g = s_g k$, where s_g belongs to the Borel set $S \subset G (S \sim G/K)$ and $k \in K$ one obtains ($\dot{g} \equiv x_g$, $G/K \equiv X$):

$$\begin{aligned}
(\overset{1}{T}_{g_0} u, \overset{2}{T}_{g_0} w) &= \int_X u_s(g_0^{-1}g)\overline{w}_s(g_0^{-1}g)\mathrm{d}\mu(\dot{g}) \\
&= \int_X u_s(s_{g_0^{-1}g}k)\overline{w}_s(s_{g_0^{-1}g}k)\mathrm{d}\mu(x_g) \text{ by virtue of (3) and (4)} \\
&= \int_X u_s(g_0^{-1}x_g)\overline{w}_s(g_0^{-1}x_g)\mathrm{d}\mu(x_g) \\
&= \int_X u(x')\overline{w}_s(x')\mathrm{d}\mu(x') = (u, w).
\end{aligned} \tag{10}$$

Because $u(g), w(g) \in C_0(G/K)$ the map $g \to (u, T^L_g w)$ is continuous. Consequently the map $g \to T^L_g = \langle \overset{1}{T}{}^L_g, \overset{2}{T}{}^L_g\rangle$ is an SLS representation of Π in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi}\rangle$ induced by the SLS representation $k \to L_k = \langle \overset{2}{L}_k, \overset{2}{L}_k\rangle$.

The formulas (8) and (9) give the action of induced representation $\langle \overset{1}{T}, \overset{2}{T}\rangle$ in the space $\langle \overset{1}{\Phi}, \overset{2}{\Phi}\rangle$ of functions defined on the group manifold. In many applications it is more convenient to have a realization directly on the function space on the homogeneous space $X = G/K$. This can be easily calculated; in fact, using Mackey decomposition $g = s_g k_g$ and the condition (3) one obtains a map $w(g) = L_{k_g} w(g)$ from the space of functions defined on the group manifold to the space of functions defined on the coset space $X = G/K$. The transformed function $(\overset{2}{T}{}^L_{g_0} w)(g)$ is mapped onto

$$L_{k_g}(\overset{2}{T}_{g_0} w)(g) = L_{k_g} L^{-1}_{k_{g_0^{-1}g}}\, w(x_{g_0^{-1}g}) = L^{-1}_{k_{g^{-1}s_g}} w(g_0^{-1}x_g).$$

Hence,

$$\overset{2}{T}{}^L_{g_0} w(x_g) = L^{-1}_{k_{g_0^{-1}s_g}} w(g_0^{-1}x_g). \tag{11}$$

Consequently, selecting a definite stability subgroup K of G and its arbitrary representation $k \to L_k$ one obtains an explicit realization of the induced representation $\overset{2}{T}{}^L$ of G by formula (11). Similarly we have

$$\overset{1}{T}{}^L_{g_0} u(x_g) = (L^{-1}_{k_{g_0^{-1}g}}) u(g_0^{-1}x), \qquad u \in \overset{1}{\Phi}(X). \tag{12}$$

Consequently, an SLS representation $g \to T^L_g = \langle \overset{1}{T}{}^L_g, \overset{2}{T}{}^L_g \rangle$ of G induced by a representation $k \to L_k$ of a closed subgroup K of Π is realized in the space $\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$ by formula (11) and (12). The sesquilinear form $(\cdot, \cdot)$ on $\overset{1}{\Phi}(X) \times \overset{2}{\Phi}(X)$ is given by the formula

$$(u, w) = \int_X u_s(x) \overline{w}_s(x) \mathrm{d}\mu(x), \qquad X = \Pi/K, \tag{13}$$

where $u \in \overset{1}{\Phi}(X)$, $w \in \overset{2}{\Phi}(X)$, and $\mathrm{d}\mu(x)$ is an invariant measure on $X = \Pi/K$.

The whole formalism can be directly applied to an arbitrary locally compact topological group G. In the general case it is only necessary to put in front of formulas (11) and (12) the factor $\left(\dfrac{\mathrm{d}\mu(g_0^{-1}x)}{\mathrm{d}\mu(x)}\right)^{1/2}$ representing the square root of the Radon–Nikodym derivative of $\mathrm{d}\mu$ on X.

C. *Applications of SLS Indecomposable Representations of the Poincaré Group for a Description of Unstable Particles*

There is so far no completely satisfactory definition of an unstable particle. Hence it seems most reasonable to use as a guide a phenomenological description. An unstable particle is experimentally determined as an object with the following properties:

(i) It has a definite spin J and a definite space parity P.

(ii) It has a mass distribution or equivalently it has a definite decay law.

The decay law is for most particles exponential, i.e. $p(t) \sim e^{-\Gamma t}$. However, it was suggested in some cases, as for instance for the A_2 meson, that a decay law might be an algebraic-exponential of the form $p(t) = (a+bt+ct^2)e^{-\Gamma t}$. In what follows we mean by an isolated unstable particle one which is under the influence of forces causing the decay only.

We shall construct in this section a class of indecomposable representations of the Poincaré group Π by means of which we can reproduce all properties possessed by the phenomenological unstable particle.

We begin with the determination of the stability subgroup K of the Poincaré group Π.

It is generally accepted that an unstable particle has a complex mass M. A complex mass determines a complex orbit $\mathcal{O}$ in the space of complex momenta $p = k - iq$ for which $p^2 = M^2$. The stability subgroup G_p of a vector $p \in \mathcal{O}$ is the subgroup $T^4 \rtimes G_k \cap G_q$. Putting k in the rest system and setting $q = (q_0, 0, 0, q_3)$

by a proper rotation, we conclude that in general $G_p = T^4 \otimes U(1)$. Since we want to have a definite spin J as a quantum number characterizing an unstable particle we must have $G_k \cap G_q = \mathrm{SU}(2)$: this is only possible if $q = \lambda k$. Hence $p = \lambda q + \mathrm{i}q$. It is convenient to write $p = Mv$ where $M = M_0 - \mathrm{i}(\Gamma/2)$ and $v = (v_0, \boldsymbol{v})$ is the relativistic four-velocity ($v_\mu v^\mu = 1$).

We usually consider in particle physics the irreducible representations of Π. It seems however that for the description of an unstable particle or a composite system a reducible representation of Π is more appropriate. Hence we now give a general construction of nonunitary representations T^L of Π induced by an arbitrary nonunitary reducible representation L of $K = T^4 \otimes \mathrm{SU}(2)$.

Let $k \to L_k = \langle \overset{1}{L}_k, \overset{2}{L}_k \rangle$ be an SLS representation of K in $\tilde{\Phi} = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$:

$$k = (a, r) \to \overset{2}{L}_k = N_a D^J(r), \qquad \overset{1}{L} = \overset{2}{L}{}^c, \; a \in T^4, \; r \in \mathrm{SU}(2) \tag{14}$$

where $a \to N_a$ is a reducible representation of the translation group T^4 and $r \to D^J(r)$ is an irreducible representation of SU(2), characterized by an integer or half-integer number J. The composition law in K:

$$(a, r)(a', r') = (a + ra', rr') \tag{15}$$

implies that N_a must be of the form $N_{(a, \mathring{v})}$ where $\mathring{v} = (v_0, 0, 0, 0)$ is a timelike vector and $(a, \mathring{v})$ is the Minkowski scalar product. The sesquilinear form $(u, w)_L$ in $\tilde{\Phi} = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ has now the form

$$(u, w)_L = u_{i\mu} \overline{w}_{i\mu} \tag{16}$$

where $i = 1, 2, \ldots, \dim N_{(a, \mathring{v})}$, and $\mu = -J, -J, -J+1, \ldots, J-1, J$, is the spin index.

In this case the indecomposable representations $a \to N_{(av)}$ of T^4 play an important role. The simplest example of such a representation is given by the formula

$$T^4 \ni a \to N_{(a,v)} = \mathrm{e}^{-\mathrm{i}M(a,v)} \begin{bmatrix} 1 & \gamma\,(a, v) \\ 0 & 1 \end{bmatrix}, \qquad \gamma \in C^1. \tag{17}$$

Using the induction method one may find that an n-dimensional indecomposable representation of T^4 may be taken to be in the form

$$T^4 \ni a \to N_{(a,v)}$$

$$= \mathrm{e}^{-\mathrm{i}M(a,v)} \begin{bmatrix} 1 & \gamma_{n-1}(a, v) & \gamma_{n-2}\gamma_{n-1}(a, v)^2/2 \ldots & \dfrac{\gamma_1 \cdots \gamma_{n-1}}{(n-1)!}(a, v)^{n-1} \\ 0 & 1 & \gamma_{n-2}(a, v) \quad \ldots & \dfrac{\gamma_1 \cdots \gamma_{n-2}}{(n-2)!}(a, v)^{n-2} \\ & \ddots & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \\ & & 1 \quad \gamma_2(a, v) & \dfrac{\gamma_1\gamma_2}{2}(a, v)^2 \\ 0 & & 1 & \gamma_1(a, v) \\ & & 0 & 1 \end{bmatrix}. \tag{18}$$

One may construct also other classes of indecomposable representations of T^4. However, the representations (18) are most important for us, since they provide an algebraic-exponential decay law (cf. eq. (32)).

We now give the explicit form of the representation T_g^L of Π induced by a generally reducible representation $k \to L_k$ of the subgroup $K = T^4 \circledS \mathrm{SU}(2)$.

PROPOSITION 1. *Let* $k \to L_k$ *be a representation of the subgroup* $K = T^4 \circledS \mathrm{SU}(2)$ *given by* eq. (14) *and let* $\overset{i}{\Phi}(X)$, $i = 1, 2$, *be the Schwartz space* $D(X)$, *where* $X = \Pi/K$. *Then the SLS representation*

$$g \to T_g^L = \langle \overset{1}{T_g^L}, \overset{2}{T_g^L} \rangle$$

of the Poincaré group Π *is given in the space*

$$\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$$

by the formulas

$$\overset{2}{T}{}^L_{\{a,\Lambda\}} w(v) = N_{(a,v)} D^J(r_\Lambda) w(L_\Lambda^{-1} v), \quad w \in \overset{2}{\Phi}(X), \tag{19}$$

and

$$\overset{1}{T}{}^L_{\{a,\Lambda\}} u(v) = N^C_{(a,v)} (D^J)^C(r_\Lambda) u(L_\Lambda^{-1} v), \quad u \in \overset{1}{\Phi}(X), \tag{20}$$

where $X = \Pi/K$ *is the velocity hyperboloid* ($X \ni x \sim \{v_\mu\}$, $v_\mu v^\mu = 1$, $v_0 > 0$), *and* $r_\Lambda = \Lambda_v \Lambda \Lambda_{L_\Lambda^{-1} v}$ *is the Wigner rotation,* Λ_v *is the Lorentz transformation implied by the Mackey decomposition*

$$\Lambda = \Lambda_v r, \quad \Lambda_v = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R^1,\ z \in C^1,\ r \in \mathrm{SU}(2) \tag{21}$$

of $\mathrm{SL}(2, C)$ *and* $L_v \in \mathrm{SO}(3, 1)$ *is the pure Lorentz transformation in* T^4 *implied by the element* $\Lambda_v \in \mathrm{SL}(2, C)$.

The sesquilinear form $(\cdot, \cdot)$ *in* $\Phi(X)$ *is given now by the formula*

$$(u, w) = \int \frac{d_3 v}{v_0} u_{i\mu}(v) \overline{w}_{i\mu}(v) \tag{22}$$

where $i = 1, 2, \ldots, \dim N_{(a,v)}$ *and* $\mu = -J, -J+1, \ldots, J-1, J$.

PROOF: Using eq. 2 (36) we find that the homogeneous space $X = \Pi/K$, can be realized as the velocity hyperboloid. The correspondence $\Lambda_v \to v$ is given by the formula

$$\Lambda_v = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix} \to v = \begin{bmatrix} v_0 - v_3 & v_2 + iv_1 \\ v_2 - iv_1 & v_0 + v_3 \end{bmatrix} = \Lambda_v \mathring{v} \Lambda_v^*, \quad \mathring{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{23}$$

The explicit action of $\overset{2}{T_g^L}$ in $\overset{2}{\Phi}(X)$ can be calculated in the following manner:

$$\begin{aligned}
(\overset{2}{T}_{\{a\Lambda\}} w)(0, \Lambda_v) &= w((a, \Lambda)^{-1}(0, \Lambda_v)) \\
&= w((-L_\Lambda^{-1} a, \Lambda^{-1}))(0, \Lambda_v) = w((-L_\Lambda^{-1} a, \Lambda^{-1}\Lambda_v)) \\
&= w((0, \Lambda_{L_{\Lambda_v}^{-1}})(-L^{-1}_{\Lambda_{L_{\Lambda_v}^{-1}}} L_\Lambda^{-1} a, \Lambda^{-1}_{L^{-1}_{\Lambda_v}} \Lambda^{-1} \Lambda_v)) \\
&= N_{(a, v)} D^J(\Lambda_v^{-1} \Lambda \Lambda_{L_{\Lambda_v}^{-1}}) w(0, L_\Lambda^{-1} v).
\end{aligned} \tag{24}$$

The element $\Lambda_v^{-1}\Lambda\Lambda_{L_{\Lambda_v}^{-1}}$ transforms $\mathring{v}$ into $\mathring{v}$: consequently it represents a rotation $r_\Lambda \in \mathrm{SU}(2)$ (Wigner's rotation). If we use the correspondence $\Lambda_v \sim v$ given by eq. (23), then the formula (24) can be written in the form

$$(T_{\{a,\Lambda\}}\, w)(v) = N_{(a,\,v)} D^J(r_\Lambda) w(L_\Lambda^{-1} v).$$

Similarly one obtains formula (20). ▼

Let us find now a physical interpretation for the pair of spaces $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$. According to the basic concept of quantum mechanics a measurement is an operation which prescribes a number to every wave function w; consequently a measurement is in a fact a functional on the space of wave functions. This suggests to consider in our case the space $\overset{2}{\Phi}$ as the space of wave functions and the space $\overset{1}{\Phi}$ as the space of measuring devices. The probability amplitude in the measurement of a state $w \in \overset{2}{\Phi}$ by a measuring device being in the state $u \in \overset{1}{\Phi}$ is then given by the sesquilinear form (22). Clearly by virtue of eq. (10) this probability amplitude is invariant with respect to simultaneous transformations of the state w and the measuring device u.

Consider now various special cases:

C$_1$. *Scalar Unstable Particle*

Consider first the case of a one-dimensional nonunitary representation of the translation group

$$a \to N_{(a,v)} = \mathrm{e}^{-\mathrm{i}M(a,v)} \quad \text{where} \quad M = M_0 - \frac{\mathrm{i}\Gamma}{2}, \quad v = (v_0 \vec{v}).$$

Let $u(v) \in \overset{1}{\Phi}$ and $w(v) \in \overset{2}{\Phi}$ be the states of the measuring device and of the unstable particle, respectively, at $t = 0$. The time evolution of the wave function is given by the formula (19), i.e. $w(t, v) = N_{tv_0} w(v)$. By virtue of eq. (22) the probability of measuring the state $w(t, v)$ by a measuring device in a state u is given by the formula

$$p(t) = |(u(t = 0), w(t))|^2. \tag{25}$$

This is the probability that an unstable particle has not decayed at time t. To obtain an expression for $p(t)$ in the rest frame of the unstable particle we assume a measuring device in the state $u(t = 0, v)$ in the form $u(t = 0; v) = \delta_\varepsilon(v)$, where $\delta_\varepsilon(v)$ is an ε-model with compact support of the Dirac δ-function. By virtue of eq. (22) we obtain the following formula for $p(t)$:

$$p(t) = |(u(t = 0), w(t))|^2 = \left| \int \frac{\mathrm{d}_3 v}{v_0} \delta_\varepsilon(v) w(v) \mathrm{e}^{-\mathrm{i}Mv_0 t} \right|^2 \underset{\varepsilon \to 0}{\to} |w(0)|^2 \mathrm{e}^{-\Gamma t}. \tag{26}$$

This formula agrees with the conventional expression for the time-dependence of probability obtained in the Weisskopf–Wigner formalism.

Consider now the case of a scalar particle ($J = 0$) whose wave function $w_i(v)$, $i = 1, 2, \ldots, \dim N_{(a,\hat{v})}$, transforms according to a reducible representation of the translation group $a \to N_{(a,v)}$, $\dim N_{(a,\hat{v})} > 1$. Assuming now the state of the measuring device to be in the form $u_i(v) = \delta_\varepsilon(v)\alpha_i$ where $\{\alpha_i\}$ is a vector in $\overset{1}{\tilde{\Phi}}$ one obtains the following expression for the probability amplitude

$$\big(u(t = 0), w(t)\big) = \int \frac{d_3 v}{v_0} \delta_\varepsilon(v)\alpha_i \overline{(N_{tv_0})_{ik} w_k(v)}. \tag{27}$$

In the limit $\varepsilon \to 0$ this gives

$$p(t) = |\big(u(t = 0), w(t)\big)|^2 \to |(\alpha, N_{tv_0}\beta)|^2, \quad \beta \equiv w \quad (t = 0; v = 0). \tag{28}$$

The explicit form of the time-dependence of $p(t)$ depends now on the form of the representation of the translation group. In particular if we take the two-dimensional representation (17) then one obtains the following expression:

$$p(t) = (a + bt + ct^2)e^{-\Gamma t} \tag{29}$$

where

$$\begin{aligned} a &= |(\alpha, \beta)|^2, \\ b &= 2\mathrm{Re}[(\alpha, \beta)\gamma\alpha_1\overline{\beta}_2], \\ c &= |\gamma\overline{\alpha}_1\beta_2|^2. \end{aligned} \tag{30}$$

It is interesting that this type of time-dependence was sometimes suggested on the basis of experimental results for the decay of A_2 meson.

Notice that the states of the form

$$w(v) = \begin{bmatrix} w_1(v) \\ 0 \end{bmatrix} \tag{31}$$

form an invariant subspace $\overset{2}{\check{\Phi}}$ in $\overset{2}{\Phi}$ for the representation $\overset{2}{T^L}$; for $w \in \overset{2}{\check{\Phi}}$ by virtue of eq. (30) the probability $p(t)$ has the form $p(t) = ae^{-\Gamma t}$. This provides an illustration of a phenomenon that the decay law $p(t)$ depends on production and on detection arrangements (cf. eqs. (28) and (30)).

In general taking an n-dimensional representation $a \to N_{(a,v)}$ of the translation group T^4 given by eq. (18) one obtains the decay law $p(t)$ in the form

$$p(t) = e^{-\Gamma t} \sum_{k=0}^{2(n-1)} a_k t^k \tag{32}$$

where a_k, $k = 0, 1, \ldots, 2(n-1)$, in formula (32) depend on detection and production arrangements (cf. eq. (28)).

C_2. *Unstable Particle with Spin*

The wave function of an unstable particle with spin J is a vector function $w_{i\mu}(v)$, $i = 1, 2, \ldots, \dim N_{(a\hat{v})}$, $\mu = -J, -J+1, \ldots, J-1, J$, on the velocity hyperboloid. Using the same arguments as above we obtain the following expression for the probability amplitude

$$\big(u(t=0), N_{tv_0} w(t=0)\big) = \int \frac{d_3 v}{v_0}\, \delta_\varepsilon(v) \alpha_{l\mu} \overline{(N_{tv_0})_{ik} w_{k\mu}(v)}. \tag{33}$$

In the limit $\varepsilon \to 0$ this gives

$$p(t) = |\big(u, w(t)\big)|^2 = |\alpha_\mu \overline{N_{tv_0} \beta_\mu}|^2, \quad \beta_{l\mu} \equiv w_{l\mu} \quad (t = 0; v = 0). \tag{34}$$

This shows that the decay law for an unstable particle with an arbitrary spin J is determined in fact by the representation $a \to N_{(a,v)}$ of the translation group T^4.

For physical interpretation of the present theory and further results cf. Rączka 1973.

§ 5. Comments and Supplements

A. *Irregular Semidirect Products*

The theory presented in sec. 1 provides a complete description of all irreducible unitary representations in the case of regular semidirect products $N \otimes S$. If $N \otimes S$ is *not* regular the formalism of sec. 1 can still be applied and one obtains an extensive class of irreducible induced unitary representations. The difference is that we are not able to prove, in this case, that every irreducible unitary representation is induced. Hence the class of irreducible unitary representations so obtained might not be complete.

B. *Semidirect Products of Type I*

It is important to know, in applications, whether a given semidirect product is a group of type I. The following theorem provides a convenient criterion for the solution of this problem.

THEOREM 1 (Mackey). *A regular, semidirect product $N \otimes S$ is a group of type I, if and only if, for every $\hat{n} \in \hat{N}$ the stability subgroup $K_{\hat{o}_{\hat{n}}}$ is of type I.*

The Euclidean and the Poincaré groups are of type I. Indeed, let $G = E^3 \otimes SO(3)$ be the Euclidean group. We have shown in example 1.2 that the stability subgroups are isomorphic to SO(2) in the case $r > 0$ and to SO(3) in the case $r = 0$. Because every compact group is of type I, th. 1 implies that the Euclidean group is also of type I.

For Poincaré group $\tilde{\Pi}$, stability subgroups of any character $\hat{n} \in \hat{N}$ are either simple Lie groups (i.e., SU(2), SL(2, R), SL(2, C)), or the semidirect product $T^2 \otimes S^1$. Simple groups are of type I. The semidirect product $T^2 \otimes S^1$ has in turn only compact stability subgroups. Hence, it is also of type I. Consequently, $\tilde{\Pi}$ has stability subgroups all of type I, and, therefore, by virtue of th. 1, is itself of type I.

C. *Comments*

The representations $U^{m,j}$, $m \geqslant 0, j = 0, \frac{1}{2}, 1, \ldots$ are usually used for the description of elementary relativistic free particles with mass m and spin j. The representations with imaginary mass ($m^2 < 0$) have no direct interpretation. They appear however in the description of interactions of relativistic two-particle systems: in particular these representations were extensively used in the harmonic analysis of scattering amplitudes (cf. ch. 21.6).

§ 6. Exercises

§ 1.1. Let G be a topological group and $g \to T_g$ a representation of G in a topological vector space Φ. Show that the set $\{\Phi, G\}$ forms a semidirect product $\Phi \circledS G$ with the composition law

$$(\varphi, g)(\varphi', g') = (\varphi + T_g \varphi', gg'). \tag{1}$$

§ 1.2. Show that $G = T^n \circledS \mathrm{SO}(n)$ has only two distinct classes of irreducible representations.

§ 1.3. Give the classification of irreducible representations for Lorentz type groups $T^{n+1} \circledS \mathrm{SO}_0(n, 1)$.

§ 1.4. Let G be the affine group of the real line

$$x \to ax+b, \quad a > 0, \; b \in R,$$

i.e. $G = N \circledS K$ where $N = \{(b, 1)\}$. Show that G has only three distinct irreducible representations U^+, U^- and U^s given by the formula

$$U^{\pm}_{(a,b)} u(x) = \exp(\pm \mathrm{i}\, e^{*} b) u(x+\log a), \quad u \in L^2(R) \tag{2}$$

and U^s being the character given by the formula

$$U^s_{(a,b)} = \exp(\mathrm{i} s \log a) I, \quad s \in R. \tag{3}$$

§ 1.5. Show that the group K(2) consisting of all upper triangular matrices of the form

$$\begin{bmatrix} \delta & \zeta \\ 0 & \delta^{-1} \end{bmatrix}, \quad \delta, \zeta \in C,$$

is the semidirect product $\mathfrak{Z} \circledS D$ of two Abelian subgroups $\mathfrak{Z}$ and D. Show that K(2) has two distinct irreducible infinite-dimensional representations.

§ 2.1. Let $g \to U_g$ be a unitary representation of the Poincaré group in a carrier space H. Show that the following operator

$$Z = P_0 (P_0^2)^{-1/2}, \tag{4}$$

where P_0 is the generator of time translation, belongs to the intertwining algebra $R(U, U)$.

Is Z an element of the enveloping field of the Lie algebra of the Poincaré group?

§ 2.2. Show that the operators for two-particle system

$$\lambda_i,\ M,\ S_\mu S^\mu,\ P_\mu \quad (i=1,2), \quad P_\mu = P_{(1)\mu}+P_{(2)\mu}, \tag{5}$$

where

$$M=(P_\mu P^\mu)^{1/2}, \quad S_\mu = \varepsilon_{\mu\nu\varrho\lambda}J^{\nu\varrho}P^\lambda, \quad J^{\nu\varrho}=J^{\nu\varrho}_{(1)}+J^{\nu\varrho}_{(2)},$$
$$\lambda_i = [(P_i^\mu P_\mu)^2-m_i^2M^2]^{-1/2}S_i^\mu P_\mu, \tag{6}$$

form a maximal set of commuting operators in the tensor product space $H = H^{m_1J_1}\otimes H^{m_2J_2}$.

§ 2.3. Let $|p_i\lambda_i[m_iJ_i]\rangle$ be the basis vectors in the spaces H^{m_i,J_i}, $i=1,2$, normalized in the following manner

$$\langle p_i\lambda_i[m_iJ_i]|p_i'\lambda_i[m_iJ_i]\rangle = \varepsilon_i\delta^{(3)}(p_i-p_i'), \quad \varepsilon_i=\sqrt{p_i^2+m_i^2}.$$

Set $p=\frac{1}{2}(p_1-p_2)$ and $P=p_1+p_2$. Because $P^2=(4M^2)^{-1}(M^4+m_1^4+m_2^4-2M^2m_1^2-2M^2m_2^2-2m_1^2m_2^2)$, we can represent P by means of M, and the angles φ and ϑ. Show that the vectors

$$|P,\Lambda[M,J,\lambda_{(1)},\lambda_{(2)}]\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2}\int d(\cos\vartheta)\,d\varphi\,\bar{D}^J_{\Lambda\lambda}(\varphi,\vartheta,0)\,|PM\varphi\vartheta\lambda_{(1)}\lambda_{(2)}\rangle \tag{7}$$

where $\lambda=\lambda_1-\lambda_2$, are the common eigenvectors of operators (6) and are normalized in the following manner

$$\langle P'\Lambda'[M'J'\lambda_1'\lambda_2']|P\Lambda[MJ\lambda_1\lambda_2]\rangle = \delta^{(4)}(p-p')\delta_{JJ'}\,\delta_{\lambda_1\lambda_1'},\,\delta_{\lambda_2\lambda_2'}. \tag{8}$$

§ 2.4. Let $g\to U_g^{(m,j)}$ be an irreducible representation of the Poincaré group characterized by a positive mass m and a spin j. Find the set of analytic vectors for $U^{(m,j)}$.

§ 2.5. Find the form and spectrum of invariant operators of the Poincaré group in n-dimensional space-time.

Hint: Use the fact that irreducible representations are characterized by an orbit and the invariant numbers of an irreducible representation of the stability group.

§ 2.6. Give the explicit realizations of the representations $U^{im,n,\pm}$, $U^{im,\rho}$, $U^{0,\pm,j}$, $U^{0,\pm,r,\varepsilon}$ of the Poincaré group in the same way as that of $U^{m,j}$ given in the text.

§ 2.7. *Representations of the Poincaré Group in Different Basis.* In some physical applications it is convenient not to use the explicit form of the induced representations but realizations in which other operators are diagonalized. Discuss: (i) Total angular momentum basis: $P_0, P^2, P\cdot J, J_3$, (ii) Lorentz subgroup basis: $J^2-N^2, J-N, J^2, J_3$. This realization also solves the problem of decomposition of the representations of the Poincaré group with respect to Lorentz group.

§ 3.1. Let $G=\mathrm{SL}(2,C)$ and $\Lambda\to D(\Lambda)$ be a finite-dimensional representation of G. Show that the parity conjugate of the representation $D(\Lambda_p)$ is $D^{*-1}(\Lambda_p)$.

§ 4.1. Let $G = T^2 \circledS \mathrm{SO}(2)$. Show that the representation U^L induced by a character χ of T^2 is irreducible.

Hint: Use th. 3.4.

§ 4.2. Discuss and classify the induced representations of inhomogeneous conformal group $T^6 \circledS \mathrm{SO}_0(4, 2)$.

Chapter 18

Fundamental Theorems on Induced Representations

Let U^L be the representation of G induced by a representation L of a subgroup K. Let U_N^L denote the restriction of U^L to another subgroup N of G. We derive in sec. 1 the so-called Induction-Reduction Theorem (I-R Theorem) which provides an elegant and effective method of the decomposition of U_N^L into irreducible representations of N. The solution of this problem is crucial for many applications of group representation theory in particle physics.

We derive in sec. 2 the Tensor-Product Theorem which is, in fact, a direct consequence of I-R Theorem. The Tensor-Product Theorem allows us to decompose effectively the tensor product $U^{L_1} \otimes U^{L_2}$ of two arbitrary induced representations of G. In particular, we give the special form of both theorems in the case of semidirect products, for which we have a complete solution of the problem of the decomposition into irreducible components.

In sec. 3 we present a 'continuous' version of the classical Frobenius-Reciprocity Theorem. This theorem has many applications especially in representation theory of complex classical Lie groups.

§ 1. The Induction-Reduction Theorem

Let G be a locally compact group and K and N any two closed subgroups of G. Let U^L be a representation of G induced by a unitary representation L of K, and let U_N^L be its restriction to a subgroup N. It is crucial to know, in many applications, the multiplicity of an irreducible representation M of N in the representation U_N^L. We show that the theory of induced representations provides a satisfactory solution to this problem.

Let $X = K\backslash G$ be the space of right K-cosets. The group G acts transitively in X. The subgroup N, however, does not in general act transitively in X. Let X_1 and X_2 be any two N-invariant Borel subsets of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = 0$. Let $H(X_1)$ and $H(X_2)$ be the closed subspaces of the carrier space H^L of U^L consisting of functions $u(x) \in H^L$, which vanish outside X_1 and X_2, respectively. The spaces $H(X_1)$ and $H(X_2)$ are invariant under the action of representation U_N^L and are orthogonal complements of each other. Thus, we have

$$H^L = H(X_1) \oplus H(X_2) \quad \text{and} \quad U_N^L(H^L) = U^1(H(X_1)) \oplus U^2(H(X_2)). \tag{1}$$

The space X_1 (or X_2) is invariant relative to N, but in general not transitive. Thus, one can split them further until one obtains orbits relative to N. If X_i, $i = 1, 2, \ldots$, is a countable set of orbits of X relative to N, then one obtains, by previous procedure, the following decompositions

$$H^L = \sum_i \oplus H(X_i), \qquad U_N^L(H^L) = \sum_i \oplus U_N^i(H(X_i)). \tag{2}$$

Notice that each orbit X_i of X relative to N represents a double coset of G, i.e., the set of elements of the form Kg_iN, $g_i \in G$. Clearly, two double cosets are either disjoint, or they coincide. Thus, the summation in eqs. (2) extends over different double cosets K: N of G relative to subgroups K and N. This is, in fact, the essence of Induction-Reduction Theorem. The only problem, which is left, is to determine what representations U^i of N are associated with each double coset X_i. We show that it is again a certain induced representation of N.

We shall formulate the Induction-Reduction Theorem in the most general case, which includes the case, when all the orbits are sets of measure zero. In such a case, the direct sum (2) becomes a direct integral over double cosets.

The measure ν_D on the set $\mathscr{D}$ of double cosets is defined in the following way: let $s(g)$, $g \in G$, denote the double coset KgN associated with g, and let $\tilde{\nu}$ be any finite measure in G with the same sets of measure zero as the Haar measure. Then, if $E \subset \mathscr{D}$ we define $\nu = \tilde{\nu}(s^{-1}(E))$. Such a measure is said to be an *admissible measure*. Finally, we assume, as in the case of semidirect products, some regularity conditions for the subgroups K and N: Namely, we say that K and N act regularly in G (by $g(k, n) = k^{-1}gn$) if there exists a sequence of Borel sets Z_i in G such that

(i) $\tilde{\nu}(Z_0) = 0$, $Z_i(k, n) = Z_i$, for each $(k, n) \in K \times N$ and all i.

(ii) Every orbit $\hat{O}$ not contained in Z_0 relative to the action of $K \otimes N$, is an intersection of sets Z_i containing the orbit $\hat{O}$.

We are now in the position to formulate the Induction-Reduction Theorem.

THE INDUCTION-REDUCTION THEOREM. *Let G be a separable, locally compact group and let K and N be closed subgroups of G acting regularly in G. Let U^L be a representation of G induced by the representation L of K and let U_N^L denote its restriction to the subgroup N. Then,*

1° $$U_N^L \simeq \int_{\mathscr{D}} U_N(D) \mathrm{d}\nu(D), \tag{3}$$

where $\mathscr{D}$ is the set of double cosets $K \backslash G/N$, $U_N(D)$ is a unitary representation of N and $\nu(\cdot)$ is any admissible measure on $\mathscr{D}$.

2° *A representation $N \ni n \to U_n(D)$ in the decomposition* (3) *is determined to within equivalence by a double coset D. For every $g \in D$ the subgroup $N \cap g^{-1}Kg$ depends on the double coset D only: representations of it defined as $\gamma \to L_{g\gamma g^{-1}}$,* *are equivalent for all $g \in D$: hence representations of N induced by them are also equivalent, and $U_N(D)$ can be taken to be any one of them.* ▼

PROOF: Ad 1°. Let ν be an admissible measure in $\mathscr{D}$ and ν the corresponding finite measure in G. We define the finite quasi-invariant measure μ in $X = K\backslash G$ by the formula $\mu(E) = \tilde{\nu}(\pi^{-1}(E))$, where π is the canonical projection from G onto X. Let r be the equivalence relation in X defined by the formula $x_1 \approx x_2$ if and only if $x_1 = x_2 n$, $n \in N$. Then, because K and N are acting regularly in G, r is a regular equivalence relation. Hence, we can use the measure disintegration th. 4.3.2 for the measure μ on X. Thus, for every $D \ni K\backslash G/N \equiv \mathscr{D}$ we obtain the measure μ_D in X, such that $\mathrm{d}\mu(x) = \mathrm{d}\nu(D)\mathrm{d}\mu_D(x)$. Each measure μ_D in X is concentrated on the orbit of X relative to N, i.e., $\mu_D(X - r^{-1}(D)) = 0$ and this measure is quasi-invariant with respect to the action of N in X, for almost all D relative to measure $\mathrm{d}\nu(D)$.

The representation U^L of G induced by a unitary representation L of K is realized in the Hilbert space $H^L = L^2(X, \mu, H(L))$ by the standard formula (cf. eq. 16.1(14))

$$U^L_{g_0}u(\dot{g}) = \varrho^{1/2}_{g_0}(g)B_g^{-1}B_{gg_0}u(\dot{g}g_0), \tag{4}$$

where by virtue of eq. 4.3 (9)

$$\varrho_{g_0}(g) = \varrho^{-1}(g)\varrho(gg_0), \tag{5}$$

$\dot{g} = x = x_0 g$, $x_0 = \dot{e} = K$, and $u(\dot{g})$ is a function in X with values in the carrier space $H(L)$ of the representation L of the stability subgroup K.

Let $H(\mathscr{D})$ be the Hilbert space $L^2(X, \mu, H(L))$. Then, the disintegration of the measure $\mathrm{d}\mu(x) = \mathrm{d}\nu(D)\mathrm{d}\mu_D(x)$ and lemma app. B.3.3 imply that the Hilbert space $H^L = L^2(X, \mu, H(L))$ can be written in the form

$$H^L = \int_{\mathscr{D}} H(D)\mathrm{d}\nu(D). \tag{6}$$

This implies in particular

$$||u||_{H^L} = \int_{\mathscr{D}} \mathrm{d}\nu(D) \int_X \mathrm{d}\mu_D(x)||u(x)||_{H(L)} = \int_{\mathscr{D}} \mathrm{d}\nu(D)||u||_{H(\mathscr{D})}. \tag{7}$$

The inspection of formula (4) shows that U^L_N is decomposable with respect to the decomposition (6). Hence

$$U^L_N = \int_{\mathscr{D}} U_N(D)\mathrm{d}\nu(D). \tag{8}$$

Ad 2°. We now determine the representations $U(D)$ occurring in the decomposition (8).

Let $x_D \in D$. Then, $x_D = x_0 g_D$, where $g_D \in G$ and $x_0 = e = K$. The stabilizer of x_D in N is the subgroup $N \cap g_D^{-1}Kg_D$. We know, by virtue of the Measure Disintegration Theorem, that if the measure μ is quasi-invariant relative to the group G, then the measure μ_D is also quasi-invariant relative to N. More precisely, if $n, n' \in N$, and $x = x_D n = x_0 g_D n$, we have

$$\mathrm{d}\mu_D(xn') = \frac{\varrho(g_D n n')}{\varrho(g_D n)} \mathrm{d}\mu_D(x). \tag{9}$$

Consequently, the function $\varrho_D(\cdot)$ corresponding to the measure μ_D is, for dν—almost all D

$$\varrho_D(\dot{n}) = \varrho(g_D n).$$

Set $B_n(D) \equiv B_{g_D n}$. Then, eqs. (4) and (5) imply ($g = g_D n$, $x = x_D n$)

$$U_{n'}(D)u(x) = \varrho_D^{-1/2}(n)\varrho_D(\dot{n}n')B_n^{-1}(D)B_{nn'}(D)u(xn').$$

Now, if $\dot{n} \in N$, and $y \in N \cap g_D^{-1} K g_D$ then,

$$B_{yn}(D) \equiv B_{g_D yn} = B_{g_D y g_D^{-1} g_D n}.$$

Because $g_D y g_D^{-1} \in K$ and $B_{kg} = L_k B_g$ for $k \in K$ and $g \in G$ by virtue of eq. 16.1(12) one obtains

$$B_{yn}(D) = L_{g_D y g_D^{-1}} B_{g_D n} = L_{g_D y g_D^{-1}} B_n(D). \tag{10}$$

This shows that a representation $n \to U_n(D)$ in the Hilbert space $H(D)$ is induced by a representation $y \to L_{g_D y g_D^{-1}}$ of the subgroup $N \cap g_D^{-1} K g_D$. ▼

EXAMPLE 1. Let G be the Poincaré group (i.e., $G = T^4 \circledS \mathrm{SO}(3, 1)$) and let $K = T^4 \circledS \mathrm{SO}(3)$. Let L be an irreducible representation of K defined by the formula $k = (a, R) \to L_k = \exp(ipa)I$, where $p^2 = m^2 > 0$. The representation U^L of G induced by this representation L of K is irreducible (cf. ch. 17, sec. 2.C) and corresponds to a particle with a positive mass m and a spin zero.

Consider first the restriction of U^L to the subgroup $N = T^4 \circledS \{e\}$. The space $X = K/G \cong \mathrm{SO}(3)/\mathrm{SO}(3, 1)$ is isomorphic to the hyperboloid $H^{(1,3)}$ given by the equation $p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2$. The subgroup N, by virtue of eq. 17.1.20, leaves each point of the hyperboloid invariant. Hence, each point of the hyperboloid represents an orbit for N, and, consequently, the space $\mathscr{D}$ of double cosets $K\backslash G/N$ is isomorphic with the set of orbits of the hyperboloid $H^{(1,3)}$. Applying now the Induction-Reduction Theorem, one obtains

$$U_{T^4}^L = \int_{H^{(1,3)}} U_{T^4}(p) \mathrm{d}\nu(p), \qquad H^{(1,3)} = \{p\colon p^2 = m^2\}. \tag{11}$$

The representations $U(p)$ of T^4 occurring in (11) are induced by a representation $y \to L_{gyg^{-1}}$ of the subgroup $T^4 \cap g_{-1}(T^4 \circledS \mathrm{SO}(3))g = T^4$. Hence, $T^4 \ni a \to U_a(p) = \exp(ipa)I$, $p^2 = m^2$. Consequently, $U_{T^4}^L$ is a direct integral of irreducible, one-dimensional representations $U(p)$ of T^4.

Take next $N = \mathrm{SO}(3, 1)$. This subgroup acts transitively on the hyperboloid $H^{(1,3)}$ isomorphic to $X = K \backslash G$. Consequently, we have in this case only one double coset $K:N$. Thus, by virtue of the Induction-Reduction Theorem, we obtain

$$U^L_{\mathrm{SO}(3,1)} = U(D) = U(H^{(1,3)}). \tag{12}$$

The representation $U(H^{(1,3)})$ occurring in (12) is a representation of SO(3, 1) induced by the representation $y \to L_{gyg^{-1}}$ of the subgroup $\mathrm{SO}(3,1) \cap g_{-1} T^4 \otimes \mathrm{SO}(3) g \cong \mathrm{SO}(3)$. Setting $g = e$, one obtains $\mathrm{SO}(3) \ni y \to L_y = I$. Thus $U(H^{(1,3)})$ is the representation of SO(3, 1) induced by the identity representation, of SO(3). It may be realized as the quasi-regular representation

$$U_n u(p) = u(pn), \quad p \in H^{(1,3)}, \quad n \in \mathrm{SO}(3,1). \tag{13}$$

We showed (cf. ch. 15, sec. 3(28)) that $U_N(H^{(1,3)})$ has the following decomposition in terms of the so-called degenerate representations U^Λ_N, $\Lambda \in [0, \infty)$:

$$U_N(H^{(1,3)}) = \int_0^\infty U^\Lambda_N(H^{(1,3)})\,\mathrm{d}\Lambda, \tag{14}$$

where $U^\Lambda(H^{(1,3)})$ is irreducible.

Consequently, the irreducible representation U^L of the Poincaré group restricted to the Lorentz group is the direct integral (14) of irreducible representations U^Λ. ▼

We now derive a special case of the I-R Theorem for the case when G is a semidirect product $N \otimes M$, where N and M are separable and locally compact and N is commutative. This class of groups contains the Euclidean, the Galilean and the Poincaré groups, which play a fundamental role in physics. We show that in these cases one can obtain an effective decomposition of the representations $U(D)$, in eq. (3), associated with a double coset D into its irreducible components.

Let $G = N \otimes M$ and let $W = N_0 \otimes M_0$, where N_0 and M_0 are any closed subgroups of N and M, respectively. Let $\hat{n}$ be any member of the dual group $\hat{N}$ (of characters) of N, and let $M_{\hat{n}}$ be the subgroup of all $m \in M$ such that $\hat{n}m = \hat{n}$. Let L be any irreducible representation of $M_{\hat{n}}$ and let $U^{\hat{n}L}$ denote the (irreducible) representation of G induced by the representation $(n, m) \to \hat{n}(n) L_m$ of the subgroup $K = N \otimes M_{\hat{n}}$. We have

THEOREM 2. *Let $G = N \otimes M$ be a semidirect product and let M_0 and $M_{\hat{n}}$ be regularly related subgroups of M. Then,*

1° $$U^{\hat{n}L}_W \cong \int_{\mathscr{D}} U_W(D)\,\mathrm{d}\nu(D), \tag{15}$$

where $\mathscr{D}$ is the space of double cosets $K: W$ and $U_W(D)$ is a unitary representation of W.

2° *The integrand $U(D)$ corresponding to the double coset D containing an element $m \in M$ can be computed as follows: Let $L^{(m)}$ denote the representation of $M_{\hat{n}m} \equiv m^{-1} M_{\hat{n}} m$ which takes $m^{-1} y m$ into Ly. Restrict $L^{(m)}$ to $M_{\hat{n}m} \cap M_0$ and then induce to $M_{0,\chi}$, where χ is the restriction of $\hat{n}m$ to N_0 and $M_{0,\chi}$ is the subgroup* *of all* $m \in M_0$ *with* $\chi m = \chi$. *Let* $\int L^\lambda d\varrho(\lambda)$ *denote the decomposition of this induced representation into irreducible components. Then*

$$U_W(D) \cong \int U_W^{\chi L^\lambda} d\varrho(\lambda), \tag{16}$$

where $U^{\chi L^\lambda}$ *are irreducible representations of* W *induced by the representation* $(n, m) \to \chi(n) L_m^\lambda$ *of* $N_0 \rtimes M_{0,\chi}$.

PROOF: 1° and the first part of 2° follow directly from the I-R Theorem and we leave elaboration of details for the reader. Eq. (16) follows from the theorem about the decomposition of a representation induced by a direct integral of representations (cf. th. 16.2.1). Irreducibility of $U_W^{\chi L^\lambda}$ results from th. 17.1.5. ▼

Notice that the th. 2 does not provide an explicit solution of the problem of decomposition of the representation $U^{\hat{n}L}$ into irreducible components. It reduces, however, this problem to a problem of the decomposition of a representation of the subgroup $M_{0,\chi} \subset M_0$. In all known cases this is sufficient for an effective solution of the problem of decomposition of the representation $U_W^{\hat{n}L}$ into irreducible components.

EXAMPLE 2. Let $G = N \rtimes M$ be the Euclidean group in E^3, i.e., $N = T^3$, $M = SO(3)$. Let $U^{\hat{n}L}$, $\hat{n} = (1, 0, 0)$, be an (irreducible) representation of G induced by the representation $(n, m) \to \hat{n}(n) L_m$ of the subgroup $K = T^3 \rtimes SO(2)_z$, where $SO(2)_z$ is the rotation group around z-axis. We find the restriction of $U^{\hat{n}L}$ to the subgroup $W = T^2 \rtimes SO(2)_x$. In the present case, $K \backslash G$ is the two-dimensional sphere S^2. Thus, the space of double cosets $K \backslash G / W$ coincides with the set of one-dimensional circles on S^2. Let D be a double coset containing an element $m \in SO(3)$; $m \notin SO(2)_z$. Then, by th. 2, the character $\chi \neq 0$ and consequently $M_{0,\chi} = \{e\}$. Thus, the induced representation of $M_{0,\chi}$ is the identity representation. Consequently, by eq. (16), we have

$$U_W(D) \cong U^{\chi L}, \tag{17}$$

where $U^{\chi L}$ is the representation of W induced by the representation $n \to \chi(n) I$ of $T^2 \rtimes M_{0,\chi}$.

If $m \in SO(2)_z$, then,

$$D = KmW = KW$$

has zero Haar measure (relative to G). Hence, it does not give a contribution to the decomposition (15). Consequently, the representation $U^{\hat{n}L}$ of the Euclidean group $T^3 \rtimes SO(3)$ restricted to the subgroup $T^2 \rtimes SO(2)_x$ is the direct integral (15) over the irreducible representations (17). ▼

The Induction-Reduction Theorem is very useful in an explicit solution of various problems encountered in group representation theory. We shall use it extensively in the next sections of this chapter and in the representation theory of classical Lie groups.

§ 2. Tensor-Product Theorem

Let U^1 and U^2 be irreducible unitary representations of a group G. One of the central problems in the theory of group representations and in the applications is the problem of the reduction of the tensor product $U^1 \otimes U^2$ into its irreducible constituents. We show in this section that the Induction-Reduction Theorem provides an effective method for the decomposition of the tensor product of any two induced representations of a separable, locally compact group G. In fact, let K_1 and K_2 be two closed subgroups of G and let L and M be representations of K_1 and K_2, respectively. Let, further, U^L and U^M be representations of G induced by the representations L of K_1 and M of K_2, respectively. Let

$$\mathscr{G} \equiv G \times G = \{(g_1, g_2); g_1, g_2 \in G\}, \qquad \tilde{G} \equiv \{(g, g); g \in G\}$$

and $K = K_1 \times K_2$. Clearly, G is isomorphic with $\tilde{G}$. Let $U^L \otimes U^M$ be a representation of $\mathscr{G}$ given by the outer tensor product. This representation is equivalent to the representation $U^{L \otimes M}$ of $\mathscr{G}$ by th. 16.2.3. On the other hand, $U^L \otimes U^M$ restricted to $\tilde{G}$ is equivalent to the (inner) tensor product $U^L \otimes U^M$. We have, therefore,

$$U^L \otimes U^M \cong U^{L \otimes M}_{\tilde{G}}. \tag{1}$$

Thus, the problem of the reduction of the (inner) tensor product $U^L \otimes U^M$ of G is, in fact, the problem of the decomposition of induced representations $U^{L \otimes M}$ of $G \otimes G$ restricted to the subgroup $\tilde{G} \simeq G$. This problem, in turn, is solved by the Induction-Reduction Theorem. We have

TENSOR-PRODUCT THEOREM. *Let G be a separable, locally compact group, and let K_1 and K_2 be two regularly related, closed subgroups of G. Let L and M be representations of K_1 and K_2, respectively, and let U^L and U^M be representations of G induced by the representations L and M, respectively. Then,*

1° $$U^L \otimes U^M \cong \int_{\mathscr{D}} U(D)\, \mathrm{d}\nu(D), \tag{2}$$

where D is the set of double cosets $K_1 \backslash G / K_2$, $U(D)$ is a unitary representation of G und ν is any admissible measure in $\mathscr{D}$.

2° *A representation $G \ni g \to U_g(D)$ in the decomposition (2) is determined to within an equivalence by the double coset D. If L: $y \to \tilde{L}_{gyg^{-1}}$ and $M: y \to \tilde{M}_{\gamma y \gamma^{-1}}$, $g, \gamma \in G$, $g\gamma^{-1} \in D$ are representations of the subgroup $g^{-1}K_1 g \cap \gamma^{-1}K_2\gamma$ and $\tilde{L} \otimes \tilde{M}$ denote their tensor product, then $U(D)$ is unitarily equivalent to the representation $U^{\tilde{L} \otimes \tilde{M}}$.*

PROOF: The proof is reduced by eq. (1) to the I-R Theorem. We first determine the set $K \backslash \mathscr{G} / \tilde{G}$ of double cosets.

Two elements (g, γ) and (g_1, γ_1) of $\mathscr{G}$ belong to the same double coset in $K{:}\tilde{G}$ if and only if the formula

$$(k_1, k_2)(g, \gamma)(\tilde{g}, \tilde{g}) = (g_1, \gamma_1)$$

holds for some $k_i \in K_i$ and $\tilde{g} \in G$. This condition is equivalent to

$$k_1 g\gamma^{-1}k_2^{-1} = g_1\gamma_1^{-1}$$

for some $k_i \in K_i$, that is, to

$$\pi(g\gamma^{-1}) = \pi(g_1\gamma_1^{-1}),$$

where π denotes the canonical projection of G into $K_1\backslash G/K_2$. Hence the map $\theta(g, \gamma) \equiv \pi(g\gamma^{-1})$ defines a one-to-one mapping between the (Borel) spaces $K\backslash\mathscr{G}/\tilde{G}$ and $K_1\backslash G/K_2$ in which $K(g, \gamma)\tilde{G}$ corresponds to $K_1 g\gamma^{-1}K_2$. One verifies, using the fact that K_1 and K_2 are regularly related, that θ is a Borel isomorphism. Thus, K and $\tilde{G}$ are regularly related in $G \times G$ and all hypotheses of the I-R Theorem are satisfied. This theorem states that $U^{L\otimes M}$ restricted to $\tilde{G}$ is a direct integral over the set of double cosets $\mathscr{D} = K(g, \gamma)\tilde{G}$ of the form

$$U_{\tilde{G}}^{L\times M} \simeq \int_{\mathscr{D}} U_{\tilde{G}}(D)\,\mathrm{d}\nu(D).$$

Each term $U_{\tilde{G}}(D)$ in the integrand is a representation of $\tilde{G}$ induced by the representation $(y, y) \to (L\times M)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ of the subgroup

$$\tilde{G} \cap (g, \gamma)^{-1}(K_1 \times K_2)(g, \gamma).$$

But the subgroup $\tilde{G} \cap (g, \gamma)^{-1}(K_1 \times K_2)(g, \gamma)$ lifted to G by the isomorphism $(g, g) \to g$ is the subgroup $g^{-1}K_1 g \cap \gamma K_2\gamma^{-1}$. Finally, the representation $(y, y) \to (L\otimes M)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ becomes the representation $\tilde{L} \otimes \tilde{M}$, where $L\colon y \to L_{gyg^{-1}}$ and $M\colon y \to M_{\gamma y\gamma^{-1}}$ are representations of the subgroup $g^{-1}K_1 g \cap \gamma^{-1}K_2\gamma$. ▼

The Tensor-Product Theorem provides an elegant method of the reduction of tensor products of unitary representations of various physical symmetry groups, such as the Lorentz group, the Euclidean group, the Galilean group or the Poincaré group.

The last three groups are of the form of a semidirect product $N \circledS M$, where N and M are separable and locally compact and N is commutative. Hence, we now derive a special case of the T-P Theorem for this class of groups.

Let $\hat{n}_1$ and $\hat{n}_2$ be two characters of N. Let $M_{\hat{n}_i}$ be the closed subgroup of M consisting of all $m \in M$ with $\hat{n}_i m = \hat{n}_i$, $i = 1, 2$. Let L^i be an irreducible representation of $M_{\hat{n}_i}$ and let $\hat{n}_i L_i$ be the representation $(n, m) \to \hat{n}_i(n)L^i_m$ of $N \circledS M_{\hat{n}_i}$, $i = 1, 2$. Then, by virtue of th. 17.1.5, the representations $U^{\hat{n}_i L_i}$ induced by $\hat{n}_i L_i$, $i = 1, 2$, are irreducible. We have

THEOREM 2. *Let $G = N \circledS M$ be a semidirect product of separable, locally compact groups N and M, N commutative, and let $M_{\hat{n}_1}$ and $M_{\hat{n}_2}$ be regularly related subgroups of M. Then,*

1° $$U^{\hat{n}_1 L_1} \otimes U^{\hat{n}_2 L_2} \simeq \int_{\mathscr{D}} U_G(D)\,\mathrm{d}\nu(D), \tag{3}$$

where $\mathscr{D}$ is the space of double cosets $M_{\hat{n}_1}\backslash M/M_{\hat{n}_2}$ in M and $U(D)$ is a unitary representation of G.

2° *The representations $U(D)$ of G occurring in* (3) *corresponding to the double coset D containing an element $m \in M$ can be computed as follows: let $\chi_1 = \hat{n}_1 m$ and let $\chi = \hat{n}_2 \chi_1$. Let $\tilde{L}^i$ be the restriction of L^i to $M_{\chi_1} \cap M_{\hat{n}_2} \subseteq M_\chi$. Form the inner tensor product $\tilde{L}^1 \otimes \tilde{L}^2$, then form the representation $U^{\tilde{L}^1 \otimes \tilde{L}^2}$ of M_χ induced by $\tilde{L}^1 \otimes \tilde{L}^2$. Let $\int L^\lambda \mathrm{d}\varrho(\lambda)$ be the decomposition of $U^{\tilde{L}^1\otimes\tilde{L}^2}$ as a direct integral of irreducible representations. Then,*

$$U_G(D) \simeq \int U_G^{\chi L^\lambda} \mathrm{d}\varrho(\lambda), \tag{4}$$

where $U^{\chi L^\lambda}$ are irreducible representations of $N \otimes\!\!\!\!\!\!\supset M$ induced by the representation $(n, m) \to \chi(n) L_m^\lambda$ of $N \otimes\!\!\!\!\!\!\supset M_\chi$.

PROOF: 1° and the first part of 2° follow from th. 1.2. It remains only to observe the obvious one-to-one correspondence of the space $N \otimes\!\!\!\!\!\!\supset M_{\hat{n}_1}\backslash \backslash G/N \otimes\!\!\!\!\!\!\supset M_{\hat{n}_2}$ to the space $D = M_{\hat{n}_1}\backslash M/M_{\hat{n}_2}$, because N is normal in G: we leave an elaboration of details for the reader. Eq. (4) results from th. 16.2.1. The irreducibility of $U^{\chi L^\lambda}$ follows from th. 17.1.5. ▼

Th. 2 provides then a method of the decomposition of the tensor product of irreducible representations into irreducible constituents. We illustrate now the power of th. 2 in the case of the reduction of the tensor product for the Poincaré group in two-dimensions.

EXAMPLE 1. Let $G = N \otimes\!\!\!\!\!\!\supset M$ be the two-dimensional Poincaré group. The action of G in the two-dimensional space-time is given by the formula

$$\begin{bmatrix} x \\ t \end{bmatrix} \to \begin{bmatrix} \operatorname{ch}\alpha, & \operatorname{sh}\alpha \\ \operatorname{sh}\alpha, & \operatorname{ch}\alpha \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} n_x \\ n_t \end{bmatrix}. \tag{5}$$

The dual $\hat{N}$ of N consists of 'momentum' vectors $\hat{n} = (\hat{n}_x, \hat{n}_t) = p$, with 'mass' square $p^2 = \hat{n}_t^2 - \hat{n}_x^2$. Every non-zero $\hat{n} \in \hat{N}$ has the stability group $M_{\hat{n}} = \{e\}$. For $\hat{n} = (0, 0)$, $M_{\hat{o}} = M$. Hence, every irreducible representation of G is the representation $U^{\hat{n}I}$ induced by a representation $n \to \hat{n}(n) \cdot I$ of the stability subgroup $N \otimes\!\!\!\!\!\!\supset \{e\}$ for $\hat{n} \neq \hat{o}$, or the representation of the Lorentz group M lifted to G for $\hat{n} = \hat{o}$. We want to decompose the tensor product $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ into irreducible components. Because the stability group $M_{\hat{n}}$ of every non-zero character $\hat{n}$ is the identity, one obtains $M_\chi = \{e\}$. Consequently, $U^{\tilde{L}^1\otimes\tilde{L}^2}$ is the identity representation. Thus,

$$\int U^{\chi L^\lambda} \mathrm{d}\varrho(\lambda) = U^{\chi I} \tag{6}$$

is the contribution to $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ of the double coset containing $m \in M$ (i.e., $U(D) = U^{\chi I}$). The decomposition (3) of $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ will be the direct integral of representations (6) over admissible characters χ. Because by th. 2.2°

$$\chi(n) = \hat{n}_2(n)\chi_1(n) = \langle n, \hat{n}_2 + \chi_1 \rangle = \exp[\mathrm{i}(\hat{n}_2 + \chi_1) n],$$

where $\chi_1 = \hat{n}_1 m$, we obtain

$$\chi^2 = \chi_1^2 + 2\chi_1 \hat{n}_2 + \hat{n}_2^2 = p_1^2 + 2|p_1||p_2|\operatorname{ch}\alpha + p_2^2, \qquad |p_i| = \sqrt{p_i^2}\,, \tag{7}$$

i.e.,

$$|p_1| + |p_2| \leqslant |\chi| < \infty .$$

This implies

$$U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I} \cong \int_{|p_1|+|p_2|}^{\infty} U^{\chi I} \mathrm{d}|\chi| . \tag{8}$$

The formula (8) shows that the abstract decomposition (3) into double cosets has a nice physical interpretation. In fact, eq. (8) represents the decomposition of a two-particle system 'with masses' $|p_1|$ and $|p_2|$ into subsystems with invariant mass $|p| = |\chi|$. ▼

Th. 2 can equally be easily applied in the case of the Poincaré group in four dimensions. In this case, the subgroup M_χ is nontrivial and one obtains additional invariant numbers λ_1, λ_2 which remove the degeneracy of the representation $U(D)$. These additional invariant numbers correspond to the helicities of particles one and two. This again shows that the abstract decomposition (3) applied to physical problems provides the results, which have an interesting physical interpretation (cf. exercises 5.2.2*).

§ 3. The Frobenius Reciprocity Theorem

First, we consider the case of finite groups. The classical Frobenius Reciprocity Theorem states:

THEOREM 1. *Let G be a finite group and K a subgroup of G. Let U^{L^i} be a representation of G induced by an irreducible representation L^i of K. Then, the multiplicity of an irreducible representation U^j of G in U^{L^i} is equal to the multiplicity of the representation L^i in the restriction of U^j to K.*

This theorem plays an important role in the representation theory of finite groups and in their applications. We now give Mackey's generalization of the Reciprocity Theorem to induced representations of locally compact, topological groups. We begin with a reformulation of th. 1.

Let us consider the array

$$\begin{bmatrix} n(1,1) & \dots & n(1,s) \\ n(2,1) & \dots & n(2,s) \\ \cdots & \cdots & \cdots \\ n(r,1) & \dots & n(r,s) \end{bmatrix},$$

where rows are indexed by a number 'i' labelling the irreducible representations L^i of K and columns are indexed by a number 'j' labelling irreducible representations U^j of G and in the position (i,j), we put the multiplicity $n(i,j)$ of

U^j in U^{L^i}. The multiplicity $n(i,j)$ represents a function from the group $\hat{K}\times\hat{G}$ dual to the non-negative integers. The classical Frobenius Reciprocity Theorem can now be restated in the following form.

THEOREM 1′. *There exists a function $n(\cdot,\cdot)$ from the group $\hat{K}\times\hat{G}$ dual to non-negative integers such that*:

$$U_G^{L^i} = \sum_{j\in\hat{G}} n(i,j)\,U^j, \quad \textit{and} \quad U_K^j = \sum_{h\in\hat{K}} n(h,j)\,L^h. \blacktriangledown \tag{1}$$

It turns out that the Reciprocity Theorem in this formulation can be generalized for the case when both G and K are not necessarily compact.

We first discuss some properties of measures on $\hat{K}\times\hat{G}$. Let Z_1 and Z_2 be Borel spaces and let α be a finite measure on $Z_1\times Z_2$. Let α_1 and α_2 be the projections of α on Z_1 and Z_2, respectively, i.e., for Borel sets $E_1\subset Z_1$ and $E_2\subset Z_2$, we have

$$\alpha_1(E_1) = \alpha(E_1\times Z_2), \qquad \alpha_2(E_2) = \alpha(Z_1\times E_2). \tag{2}$$

The Measure Disintegration Theorem (cf. th. 4.3.2) assures that there exists a finite Borel measure β_x in Z_2 such that

$$\alpha = \int_{Z_1} \beta_x \, d\alpha_1(x).$$

This means that for all Borel sets $E\subset Z_1\times Z_2$ we have

$$\alpha(E) = \int_{Z_1} \beta_x\{y:(x,y)\in E\}\, d\alpha_1(x).$$

The measure β_x in Z_2 is said to be *x-slice* of the measure α. Similarly, one introduces a finite Borel measure γ_y on Z_1 such that

$$\alpha = \int_{Z_2} \gamma_y \, d\alpha_2(y).$$

The measure γ_y is called *y-slice* of the measure α. Now we are in a position to state Mackey's generalization of the Frobenius Reciprocity Theorem.

THEOREM 2. *Let K be a closed subgroup of a separable, locally compact group G. Let both G and K be of Type I and have smooth duals $\hat{G}$ and $\hat{K}$, respectively. Let U^{L^x} be a representation of G induced by a unitary, irreducible representation L^x of K. Then, there exists a finite Borel measure α in $\hat{K}\times\hat{G}$ and a measurable function $n(\cdot,\cdot)$ from $\hat{K}\times\hat{G}$ to the non-negative integers including $+\infty$ such that*

1° *The projections α_1 and α_2 of α onto $\hat{K}$ and $\hat{G}$ are equivalent to measures defined by the regular representations of K and G, respectively.*

2° *For α_1-almost all x in $\hat{K}$*

$$U^{L^x} \simeq \int_{\hat{G}} n(x,y)\,U^y \, d\beta_x(y), \tag{3}$$

where U^y are irreducible representations of G and β_x is the x-slice of α.

3° *For α_2-almost all y in $\hat{G}$*

$$U^y_K \simeq \int_{\hat{K}} n(x, y) L^x \mathrm{d}\gamma_y(x), \tag{4}$$

where γ_y is y-slice of α. ▼

The proof is given in Mackey's paper, 1952, th. 5.1. A simplified proof is presented in Mackey's Chicago Lecture Notes, 1955.

Remark 1: Mackey's Theorem provides a two-fold generalization of the Reciprocity Theorem. In fact, it gives a single function $n(x, y)$, from which the multiplicities of both U^y in U^{L^x} and L^x in U^y_K, respectively, are obtained as well as a single measure α from which the families of measures $\beta_x(\cdot)$ and $\gamma_y(\cdot)$ are obtained. ▼

If G is compact, then the direct integrals (3) and (4) reduce to direct sums and we have

$$U^{L^i}_G \cong \sum_{j\in\hat{G}} n(i,j)\, U^j_G \quad \text{and} \quad U^j_K = \sum_{h\in\hat{K}} n(h,j) L^h_K, \tag{5}$$

where $n(i,j)$ is a function from $\hat{K}\times\hat{G}$ to the non-negative integers. This result is an extension to compact groups of th. 1′ for finite groups.

We now give two interesting applications of the Reciprocity Theorem.

Example 1. Let G be a compact group and let $g \to U_g$ be the regular representation of G in the Hilbert space $H = L^2(G)$. This representation can be considered as the representation U^L of G induced by the identity representation $L = I$ of the subgroup $K = \{e\}$, where e is the unit element of G. Because the multiplicity of L in an irreducible representation U^j of G restricted to K is equal to $\dim U^j$, then, by virtue of th. 2, we have

$$\text{multiplicity } n_j \text{ of } U^j \text{ in } U^L = \dim U^j.$$

Let now G be a noncompact, simple Lie group. We know that every non-trivial, unitary representation of G is infinite-dimensional. Hence, we conclude by the same arguments that every non-trivial unitary, irreducible representation of G which enters in the regular representation is contained an infinite number of times. ▼

Next we consider an example, when K is noncompact.

Example 2. Let $G = T^4 \rtimes \mathrm{SO}(3, 1)$ be the Poincaré group and let $K = T^4 \rtimes \mathrm{SO}(3)$. We ask for the multiplicity of the representation $\hat{n}^1 L^1$: $(a, r) \to \hat{n}^1(a) L^1_r$ of K in the representation $U^{\hat{n}L}$ of G induced by the representation $\hat{n}L$ of K. We know that $U^{\hat{n}L}$ is irreducible (cf. th. 17.1.5). Hence, the representation $U^{\hat{n}L}$ restricted to K contains $\hat{n}^1 L^1$ at most once. ▼

§ 4. Comments and Supplements

A. The assumption in th. 2 that G and K are both of Type I is essential if K is noncompact. Indeed, Mackey has shown that if G is the discrete group of transformations of the real line

$$x \to ax+b,$$

$a > 0$, a, b rationals, and K corresponds, e.g., to the noncompact subgroup $(\{1, b\})$, then, Frobenius Reciprocity Theorem does not hold (cf. Mackey 1951, p. 216).* However, Mautner has proved that if K is compact, then the Reciprocity Theorem does hold even for groups that are not of Type I. (Cf. also Mackey 1962, § 6.)

B. It is interesting that the classical Frobenius Reciprocity Theorem can be reformulated still in another more symmetric form. In fact, let K_1 and K_2 be subgroups of the finite group G and let U^{L^i} and U^{M^j} be representations of G induced by irreducible representations L^i and M^j of K_1 and K_2, respectively. Then, the Generalized Reciprocity Theorem can be stated in the following form (cf. Osima 1952).

THEOREM 3. *There exists a function $n(\cdot\,, \cdot)$ from the dual group $\hat{K}_1 \times \hat{K}_2$ to non-negative integers such that*

$$U^{L^i}_{K_2} = \sum_{l \in \hat{K}_2} n(i, l) M^l$$

and

$$U^{M^j}_{K_1} = \sum_{h \in \hat{K}_1} n(h, j) L^h. \blacktriangledown \tag{6}$$

Notice that, if $K_2 = G$, this theorem coincides with th. 1′. Similarly, th. 2 can also be stated for non-compact groups in the form analogous to th. 3 (cf. Mackey 1952, § 7).

C. The technique of induced representations provides the complete classification of irreducible unitary representations of regular semidirect products (see ch. 17). It was also shown by Dixmier 1957 that every irreducible unitary representation of a connected nilpotent group G is induced by a one-dimensional representation of some subgroup of G. These two examples illustrate the power of the theory of induced representations.

Kirillov presented a certain variant of the theory of induced representations for nilpotent groups based on the methods of orbits in the dual space to the vector space of the Lie algebra (1962). This method was extended to other classes of groups by Bernat 1965, Kostant 1965, Pukansky 1968 and others. In particular Auslander and Moore 1966 gave the classiffication of induced representations of certain solvable Lie groups.

There is an interesting connection of the method of orbits with the quantiza-

* Let us note that this is the classical example of von Neumann of a group whose factor representations are of type II (cf. e.g. Naimark 1968, p. 558).

tion problem of classical mechanics. This problem was analyzed by Kostant 1965, 1970, Kirillov 1970 and Simms 1973.

D. The generalization of the theory of induced representations to group extensions was also elaborated by Mackey 1958.

§ 5. Exercises

§ 1.1.*** Find the formulation of induction-reduction theorem for nonunitary, e.g. indecomposable induced representations.

§ 2.1. Let $G = T^n \rtimes SO(3,1)$ and $K = T^n \rtimes SO(2)$. Show that the representation U^L induced by the irreducible representation L_k of K of the form

$$k = (a, \varphi) \to L_k = \exp[i\mathring{p}a]\exp[iM\varphi], \tag{1}$$

where $\mathring{p} = (m, 0, 0, 0)$ and $M = 0, \pm 1, \pm 2$ is irreducible and has the following decomposition

$$U^L = \sum_{J \geqslant |M|}^{\infty} \oplus U^{m,J}. \tag{2}$$

§ 2.2.* Let U^{M_1,J_1} and U^{M_2,J_2}, $M_1, M_2 \in (0, \infty)$, $J_1, J_2 = 0, \frac{1}{2}, 1, \frac{1}{2}, \ldots$ be two irreducible representations of the Poincaré group. Show that

(a) for $M_1, M_2 > 0$ we have

$$U^{M_1,J_1} \otimes U^{M_2,J_2} \cong \int_{M_1+M_2}^{\infty} dM \sum_{l=0}^{\infty} \sum_{s=|J_1-J_2|}^{J_1+J_2} \sum_{J=|s-l|}^{s+l} \oplus U^{M,J}, \tag{3}$$

(b) $$U^{M_1,J_1} \otimes U^{0,J} \cong \int_{M_1}^{\infty} dM \sum_{l=|J_2|}^{\infty} \sum_{J=|l-J_1|}^{l+J_1} \oplus U^{M,J}, \tag{4}$$

(c) $$U^{0,J_1} \otimes U^{0,J_2} \cong \int_{0}^{\infty} dM \sum_{J=|J_1-J_2|}^{\infty} \oplus U^{M,J}. \tag{5}$$

Hint: Use the th. 2.2.

§ 2.3. Let $U^{(J_1,J_2)}$ be the finite-dimensional representation of the Poincaré group obtained by lifting the representation $D^{(J_1,J_2)}$ of SL(2, C) to the Poincaré group. Is the tensor product

$$U^{(J_1,J_2)} \otimes U^{M,J}$$

reducible?

§ 2.4. Let $H = H^{M_1,J_1} \otimes H^{M_2,J_2}$ be the carrier space of the tensor product $U^{M_1,J_1} \otimes U^{M_2,J_2}$. Show that the operators

$$\lambda_i = [(P_i^{\mu}P_{\mu})^2 - m_i^2 M^2]^{-1/2} \zeta_i^{\mu} P_{\mu}, \quad i = 1, 2,$$

where

$$P^{\mu} = P_1^{\mu} + P_2^{\mu}, \quad M^2 = P_{\mu}P^{\mu}, \quad m_i^2 = P_i^{\mu}P_{i\mu}$$

and

$$\zeta_i^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\varrho\lambda} M_{i\nu\varrho} P_{i\lambda}$$

are invariant operators in H. Show that in the center of mass system we have

$$\lambda_i = \boldsymbol{P}_i \boldsymbol{J}_i / |\boldsymbol{P}_i|, \qquad i = 1, 2,$$

i.e., operators λ_i are helicity operators.

Chapter 19

Induced Representations of Semisimple Lie Groups

We shall now construct several series of unitary irreducible representations of semisimple classical Lie groups using the method of induced representations. The set of unitary representations of semisimple noncompact groups, in contrast to the set of finite-dimensional representations, is very rich. We usually distinguish four series of unitary representations: the principal nondegenerate, principal degenerate, supplementary nondegenerate and supplementary degenerate series. The corresponding series are determined by various sets of invariant numbers: for instance, in the case of the group $SL(n, C)$ the principal nondegenerate series is determined by $2n-2$ invariant numbers and successive degenerate ones by $2n-2k$, $k = 2, 3, \ldots, n-1$, invariant numbers.

In sec. 1 we present a general theory of induced representations of semisimple Lie groups. The construction of induced representations is based on the induction of representation from irreducible representations of a minimal parabolic subgroup of G. We analyze also problems of irreducibility and nonequivalence of the resulting induced representations. Next we analyze in details the construction of induced representations of principal, supplementary and degenerate series for the groups $SL(n, C)$ and $GL(n, C)$, which are frequently considered as symmetry groups of various physical systems. Finally in sec. 6 we give a brief discussion of properties of induced representations of other classical Lie groups.

It is remarkable that the same technique of induced representations which we used in ch. 8 for construction of all irreducible finite-dimensional representations of classical Lie groups will allow us in the present case the construction of irreducible infinite-dimensional unitary representations of classical groups. The corresponding formulas which give the realization of finite and infinite-dimensional representations are almost identical (cf. eq. 8.3(4) and 3(16)).

§ 1. Induced Representations of Semisimple Lie Groups.

In this section we present the construction of induced representations for semisimple classical Lie groups and include some recent results concerning irreducibility.

Let G be a connected semisimple Lie group with a finite centre, $G = KAN$

the Iwasawa decomposition for G, $P = MAN$ the corresponding minimal parabolic subgroup of G (cf. 3.6.D for definitions). Let L be a finite-dimensional representation of P. The following lemma describes the structure of L.

LEMMA 1. *A finite-dimensional continuous irreducible representation L of P in a space H has the form*

$$L_{man} = \chi(a)L_m, \quad m \in M, \; a \in A, \; n \in N, \tag{1}$$

where χ is a character of A and $m \to L_m$ is a continuous irreducible representation of M in H.

PROOF: By Iwasawa Theorem AN is connected and solvable: consequently by Lie's Theorem there exists a non-zero vector $u_0 \in H$ such that $L_{an}u_0 = \chi(an)u_0$, where χ is a one-dimensional continuous representation of AN: Because N is the derived group of AN $\chi(n) = 1$ for all $n \in N$. Since $L_{an}L_m u_0 = \chi(a)L_m u_0$ the span $L_m u_0$ is L-stable; hence irreducibility of L implies that this span must coincide with H. Therefore $L_{an}u = \chi(a)u$ for all $a \in A$, $n \in N$ and $u \in H$. ▼

Let χL be a finite-dimensional irreducible unitary representation of P, $X = G/P$ or $X = P\backslash G$ and $\mu(\cdot)$ a quasi-invariant measure on X. The action of the induced representation $U^{\chi L}$ of G in the space $H^{\chi L} = L^2(X, \mu)$ is given by the formula 16.1(15) if we use the right translation $x \to xg$, or by the formula 16.1(47) if we use the left translation $x \to g^{-1}x$.

The unitary representations $U^{\chi L}$ of G induced by finite-dimensional irreducible representations χL of P are called the *principal P-series* of unitary representations of G.

Notice that the above construction of induced representation may be used for complex as well as real semisimple Lie groups. If G is complex then, by 3.6.C, K is the compact form of G, M is a maximal torus in K and MA is a Cartan subgroup of G: thus the members of the principal P-series of G are unitary representations of G induced from a Cartan subgroup of G: consequently the invariant numbers which characterize the representations of the principal P-series of G are pairs consisting of an integer and real which label characters of MA.

EXAMPLE 1. Let $G = \mathrm{SL}(2, R)$. The subgroups K, A, N and M were given in example 3.6.4. Since $M = \{e, -e\}$ it has only two irreducible nonequivalent representations given by

$$L_m^+ = 1, \; m \in M, \quad \text{and} \quad L_m^- = m_1, \; m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in M. \tag{2}$$

Consequently, by lemma 1 the irreducible finite-dimensional unitary representations of $P = MAN$ are one-dimensional and have the form:

$$man = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \to \chi(a)L_m^{\pm}, \tag{3}$$

where $\chi(a) = |a|^{ir}$, $r \in R$. Therefore the principal P-series of unitary representations $U^{\chi L}$ of SL(2, R) consist of two series $U^{r,+}$ and $U^{r,-}$. The explicit realization of these representations was given in example 16.1.2.▼

One of the basic problems of representation theory is the determination of irreducibility and equivalence of representations $U^{\chi L}$ of principal P-series. To formulate the corresponding theorems we have to introduce the action of the Weyl group on representations χL of MA and the concept of extendible representations. Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras of G and A respectively and let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$; let χL be an irreducible finite-dimensional representation of MA and $w \in W$; then $w\chi L$ denote the representation of MA given by

$$ma \to \chi(m_w^{*-1} a m_w^*) L_{m_w^{*-1} m m_w^*}, \qquad m \in M, \ a \in A, \tag{4}$$

where m_w^* is any element of the normalizer M^* of A in K associated with w (cf. sec. 3.6.D for properties of M^*).

Let us assume that the nilpotent subgroup N in the Iwasawa decomposition corresponds to the positive restricted root spaces of the Lie algebra $\mathfrak{a}$ and let N^- be the nilpotent subgroup of G corresponding to the negative restricted root spaces. Let L be a unitary irreducible representation of M in a Hilbert space H. We say that the pair (L, H) is extendible if there is an irreducible finite-dimensional complex G module, V, so that the M-module,

$$V^{N-} = \{u \in V;\ nu = u \text{ for all } n \in N^-\} \tag{5}$$

is equivalent with (L, H). We call V an extension of L.

THEOREM 2. *Let χL be a finite-dimensional representation of P. Then*

(i) *If $\chi L \not\sim w\chi L$ for every $w \neq I$ of W then $U^{\chi L}$ is irreducible.*

(ii) *$U^{\chi' L'} \sim U^{\chi'' L''}$ if and only if there exists a $w \in W$ such that $\chi' L' \sim w\chi'' L''$*

(iii) *If $U^{\chi L}$ is reducible then*

$$U^{\chi L} = \sum_{i=1}^{r} U^i$$

where U^i is irreducible unitary subrepresentation of $U^{\chi L}$, $U^i \not\sim U^{\chi' L'}$ for any representation $\chi' L'$ of P and $r \leqslant m$—the multiplicity of L in V as an M-module.

(The proof of these results was essentially given by Bruhat 1956. The upper bound on r given in (iii) was derived by Wallach 1971.)

Th. 2 provides an effective tool in the verification of irreducibility of unitary representations of the principal P-series. We shall illustrate its power for two classes of groups.

THEOREM 3. *Let G be a connected complex semisimple Lie group. Then every member of the principal P-series is irreducible.*

PROOF: In this case MA is a Cartan subgroup of G and W acts on MA as the Weyl group of G relative to MA. Let χ be a character of A and L—a character of M. Let V be the finite-dimensional irreducible (induced) representation of G with the lowest integral weight L constructed in ch. 8, § 2. The subgroup

N^- can be identified with the subgroup Z of the Gauss decomposition. Hence the M-module V^{N^-} defined by eq. (5), by virtue of corollary 1 to th. 8.2.2 is one-dimensional and by virtue of eq. 8.2(18) is equivalent with (L, H). Hence V is an extension of L. Now the corollary 1 to th. 8.2.2 implies that the multiplicity of L in V as an M-module is one. Consequently by assertion (iii) of th. 2 the representation $U^{\chi L}$ is irreducible. ▼

The second application concerns $\mathrm{SL}(n, R)$ groups. In this case $K = \mathrm{SO}(n)$ and A can be taken as the set of all diagonal matrices of G with positive entries. N is the group of all upper triangular matrices with ones on the diagonal. The centralizer M is the group of all diagonal elements of G with entries ± 1 on the diagonal. Let $m = \mathrm{diag}(m_1, \ldots, m_n)$ be an element of M: set $\varepsilon_0(m) = 1$ and $\varepsilon_i(m) = m_i$, $i = 1, \ldots, n-1$. Then every nontrivial unitary character of M is of the form $\varepsilon_{i_1} \ldots \varepsilon_{i_r}$, $1 \leqslant i_1 < \ldots < i_r \leqslant n-1$. Set $\varepsilon_n = \varepsilon_1 \ldots \varepsilon_{n-1}$. Then we have:

THEOREM 4. *Let $G = \mathrm{SL}(n, R)$.* (i) *If n is odd, every element of the principal P-series is irreducible.* (ii) *If n is even, χ—the character of A and*

$$L = \varepsilon_{i_1} \ldots \varepsilon_{i_j}, \qquad 1 \leqslant i_1 < \ldots < n-1 \ \text{and}\ j \neq \frac{n}{2}, \tag{6}$$

then the representation $U^{\chi L}$ is irreducible. If $j = n/2$ and if $U^{\chi L}$ is reducible then

$$U^{\chi L} = U^1 \oplus U^2$$

is the direct sum of irreducible unitary representations. ▼

PROOF: The Weyl group W acts on M by permuting the entries along the diagonal: hence W also permutes $\varepsilon_1, \ldots, \varepsilon_n$. Consider now the representations V^i, $i = 0, \ldots, n-1$, where V^0 is the trivial representation of G, V^1 is the standard (matrix) action of G on C^n and $V^i = V^1 \wedge V^1 \wedge \ldots \wedge V^1$ (i times) is the polyvector representation of G which is determined by the highest weight $(1, \ldots, 1, 0, \ldots, 0)$ (cf. ch. 8.3). Let $e_1, \ldots, e_n$ be the standard basis of C^n. Then if $m \in M$, $me_i = \varepsilon_i(m) e_i$; hence in general as an M-module V^k splits into a direct sum

$$V^0 = 1, \quad V^k = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n-1} \oplus\, \varepsilon_{i_1} \ldots \varepsilon_{i_k} + \sum_{1 \leqslant j_1 < \ldots < j_{n-k} \leqslant n-1} \oplus\, \varepsilon_{j_1} \ldots \varepsilon_{j_{n-k}} \quad \text{for } n-1 \geqslant k > 0. \tag{7}$$

Let χ be a character of A and L be the character of M which can be taken in the form $L = \varepsilon_0 \varepsilon_1 \ldots \varepsilon_r$, $r = 0, \ldots, n-1$. If $r = 0$ then ε_0 is the action of M on $(V^0)^{N^-}$: hence V^0 is the extension of L. Similarly, if $L = \varepsilon_0 \varepsilon_1 \ldots \varepsilon_r$ then V^{n-r} is the extension of L; indeed the M-module $(V^{n-r})^{N^-}$ by virtue of corollary 1 to th. 8.2, is one-dimensional with the lowest integral weight L. Now if n is odd by virtue of eq. (7) every representation of M appears exactly once: by virtue of th. 2(iii) this implies assertion (i). Similarly using (7) and th. 2 one verifies the assertion (ii).

EXAMPLE 2. Let $G = \mathrm{SL}(2, R)$. The Weyl group $W = M^*/M$ for SL(2, R) was calculated in example 3.6.4 and $W = \left\{w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$.

Since the action of W on the subgroup A has the form

$$w_1^{-1} a w_1 = a, \quad w_2^{-1} a w_2 = a^{-1}, \tag{8}$$

and the action of W on M is trivial the representations $L^{r,\pm}$ and $L^{-r,\pm}$ *of* MA satisfy

$$w_2 L^{r,\pm} = L^{-r,\pm}.$$

Consequently, by th. 2(ii) the induced representations $U^{r_1,\pm}$ and $U^{r_2,\pm}$ are equivalent if $r_2 = -r_1$. Now if $r = 0$ the representation $L^- = \varepsilon_1$ of M is contained twice in the representation V^1 of SL(2, R) considered as an M-module. Consequently, by virtue of th. 2(iii) the representation $U^{0,-}$ is a direct sum of two irreducible representations. Similarly using ths. 2 and 3 one verifies that $U^{r,+}$, for $r \neq 0$ and $U^{r,-}$ are irreducible. ▼

§ 2. Properties of the Group SL(n, C) and Its Subgroups

In the case $G = \mathrm{SL}(n, C)$, K is a compact form of G, M is a maximal torus in K and MA is the Cartan subgroup D of G consisting of all unimodular diagonal $n \times n$ matrices. The parabolic subgroup $P = MAN$ consists of all upper triangular unimodular matrices. Comparison with the Gauss decomposition of SL(n, C)

$$\mathrm{SL}(n, C) = \overline{\mathfrak{Z} D Z}$$

given in 3.6.A shows that the parabolic subgroup P coincides with the subgroup $\mathfrak{Z}D$ of SL(n, C). This fact implies that the Gauss decomposition plays an important role in the explicit construction of unitary representations induced from the subgroup P.

We now recall the basic properties of the Gauss decomposition for SL(n, C). The subgroup $P = \mathfrak{Z}D$ consists of upper triangular matrices whose diagonal elements satisfy the condition

$$\prod_{i=1}^{n} k_{ii} = 1 \tag{1}$$

and is solvable. The commutative (Cartan) subgroup D consists of diagonal matrices also satisfying the condition (1). The subgroup $\mathfrak{Z}$ consists of upper triangular matrices with diagonal elements equal to unity. The subgroup Z consists of lower triangular matrices with diagonal elements equal to unity and is also nilpotent. The subgroup $\mathfrak{Z}$ is the normal subgroup of P, and P is the semidirect product $\mathfrak{Z} \otimes D$.

We recall that the Gauss decomposition $\mathrm{SL}(n, C) = \overline{\mathfrak{Z} D Z}$ means that almost every element $g \in \mathrm{SL}(n, C)$ can be uniquely decomposed in the form

$$g = \zeta \delta z, \quad \zeta \in \mathfrak{Z}, \ \delta \in D, \ z \in Z, \tag{2a}$$

or

$$g = kz, \tag{2b}$$

where

$$k = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ 0 & k_{22} & \dots & k_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & k_{nn} \end{bmatrix} \in P = \mathfrak{Z}D \tag{3}$$

and

$$z = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{n,n-1} & 1 \end{bmatrix} \in Z. \tag{4}$$

The matrix elements z_{pq}, $n \geqslant p > q > 0$ of a group element $z \in Z$ are arbitrary complex numbers. Hence, we can represent them in the form $z_{pq} = x_{pq} + iy_{pq}$. We know, due to the results of ch. 3, § 11, eq. (38) that the group of triangular matrices with diagonal elements one has invariant measure which can be taken to be of the form

$$d\mu(z) = \prod_{p,q=1} \mathrm{d}x_{pq}\mathrm{d}y_{pq}, \qquad n \geqslant p > q > 0. \tag{5}$$

In the following we shall utilize this measure on the group Z for the construction of the space $L^2(Z, \mu)$ in which the irreducible, unitary representations of $\mathrm{SL}(n, C)$ are realized.

§ 3. The Principal Nondegenerate Series of Unitary Representations of SL(*n*, *C*)

We now apply the general formalism of induced representations in order to construct a class of irreducible unitary representations of SL(n, C). Let us recall the basic steps of construction of induced representations. Let P be a closed subgroup of G, $X = P\backslash G$, and μ a quasi-invariant measure on X. Let $k \to L_k$ be a unitary representation of P in the Hilbert space H. Then, the carrier space H^L of the representation U^L induced by the representation L of P consists of functions satisfying

$$u(kg) = L_k u(g), \qquad k \in P \tag{1}$$

and

$$\int_X \|u(g)\|_H^2 \mathrm{d}\mu(x) < \infty. \tag{2}$$

In the realization of the representation U^L in $H^L = L^2(X, \mu)$ the action of $U^L_{g_0}$ is given by the formula 16.1 (15), i.e.,

$$U^L_{g_0} u(x) = \sqrt{\left(\frac{d\mu(xg_0)}{d\mu(x)}\right)} L_{k_{x_g g_0}} u(xg_0), \tag{3}$$

where $\frac{d\mu(xg_0)}{d\mu(x)}$ is the Radon–Nikodym derivative, $x = \dot{g} = Pg = Pk_g x_g = Px_g$ and $k_{x_g g_0}$ is determined by the Mackey decomposition (2.4.1) of $x_g g_0$, i.e. $x_g g_0 = k_{x_g g_0} x'$ with $x' \in S$, $k_{x_g g_0} \in P$.

In the case of SL(n, C) we have $P = \mathfrak{Z}D$. Since D is commutative it has only one-dimensional irreducible representations, and because $\mathfrak{Z}$ is normal in P, this representation extends to P, by virtue of (3). Thus the construction of induced representations of SL(n, C) is reduced to a simple computation of the Radon–Nikodym derivative $d\mu(xg)/d\mu(x)$ and of the element $k_{x_g g_0} \in P$ corresponding to $x_g g_0 \in G$.

A. *The Determination of the Factor* $k_{x_g g_0}$

Let D be the diagonal (Cartan) subgroup of SL(n, C) and let

$$\delta \to L_\delta = \chi(\delta) = \prod_{s=2}^{n} |\delta_{ss}|^{m_s + i\varrho_s} \delta_{ss}{}^{-m_s}, \tag{4}$$

where m_s are integers and ϱ_s are real numbers, $s = 2, 3, \dots, n$, be the one-dimensional representation of D determined by the character χ. The subgroup P is the semidirect product $\mathfrak{Z} \otimes D$ and $\mathfrak{Z}$ is normal in P. Hence, the map L: $(\zeta, \delta) \to I \cdot \chi(\delta)$ is the most general unitary one-dimensional representation L of P:

$$k = (\zeta, \delta) \to L_k = I\chi(\delta) = \prod_{s=2}^{n} |k_{ss}|^{m_s + i\varrho_s} k_{ss}{}^{-m_s}, \qquad k_{ss} = \delta_{ss}. \tag{5}$$

We shall use L to construct the induced representation U^L of SL(n, C).

A function $u(x)$ in the carrier space H^L is defined in the space $X = P \backslash G$. It is, however, more convenient to consider it as a function over the group space of the subgroup Z. This is possible due to the following lemma:

LEMMA 1. *The set $X = P \backslash G$ coincides with group space of the subgroup Z up to a subset of measure zero with respect to any quasi-invariant measure μ on X.*

PROOF: By the Gauss decomposition if follows that the set $G - KZ$ is a set of measure zero with respect to the Haar measure dg on G. The canonical projection π: $G \to P \backslash G$ maps dg-null sets into $d\mu$-null sets in X. Hence $X - Z$ is $d\mu$-null set and the lemma is proved. ▼

This lemma allows us to associate uniquely with almost every coset $Pg = x \in X$ the element $z_g \in Z$ defined by the equality $g = k_g z_g$. Comparing the decomposition $g = k_g z_g$ with the Màckey decomposition 2.4 (1), $g = k_g x_g$, we conclude that elements $z_g \in Z$ play the role of elements $x_g \in S$.

Let now the decomposition of the element $z_g g_0$ be defined by

$$\tilde{g} \equiv z_g g_0 = k_{\tilde{g}} z_{\tilde{g}} . \tag{6}$$

For the sake of simplicity of notation we set $z_{\tilde{g}} = \tilde{z}$, $k_{\tilde{g}} = \tilde{k}$. The element $\tilde{z}$ corresponds to the transformed point xg_0 in eq. (3). The explicit formulas for the matrix elements $\tilde{k}$ and $\tilde{z}$ can be obtained with the help of the Gauss decomposition. In fact, applying formula 3.11(18) for the group element

$$\tilde{g} = z_g g_0 = k\tilde{\tilde{z}} \qquad \left(\text{i.e., } \tilde{g}_{pq} = \sum_{s=1}^{p-1} z_{ps} g_{sq} + g_{pq}\right)$$

we obtain

$$(\tilde{z})_{pq} = \frac{\begin{vmatrix} \tilde{g}_{pq} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{p,n} \\ \tilde{g}_{p+1,q} & \tilde{g}_{p+1,p+1} & \cdots & \tilde{g}_{p+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{nq} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}}{\begin{vmatrix} \tilde{g}_{pp} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{np} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}},$$

$$\tilde{k}_{pp} = \frac{\begin{vmatrix} \tilde{g}_{pp} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{np} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}}{\begin{vmatrix} \tilde{g}_{p+1,p+1} & \tilde{g}_{p+1,p+2} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{n,p+1} & \tilde{g}_{n,p+2} & \cdots & \tilde{g}_{nn} \end{vmatrix}}. \tag{7}$$

In particular, in the case of SL(2, C), we obtain

$$(\tilde{z})_{21} = \frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}, \qquad z \in G, \tag{8}$$

$$\tilde{k}_{11} = (g_{12}z + g_{22})^{-1}, \qquad \tilde{k}_{12} = g_{12}, \qquad \tilde{k}_{22} = g_{12}z + g_{22}. \tag{9}$$

We see that the action of SL(2, C) on Z, which in this case is isomorphic to the additive group C of complex numbers, is given by a fractional linear transformation (8). In the case of SL(n, C), $n > 2$, the mapping $z \to \tilde{z}$ is a natural generalization of the fractional linear transformation (8).

B. *Determination of the Radon–Nikodym Derivative* $\dfrac{d\mu(\tilde{z})}{d\mu(z)}$

The invariant measure $d\mu(z)$ on the nilpotent subgroup Z is given by formula 2(5). Consequently, the Radon–Nikodym derivative $d\mu(\tilde{z})/d\mu(z)$ represents, in fact, the Jacobian of the transformation $z \to \tilde{z}$. It compensates the factor, which results from the non-invariance of the measure $d\mu(z)$ on Z with respect to the action of the group $SL(n, C)$ on Z. We have

LEMMA 2. *The Radon–Nikodym derivative is given by the formula*

$$\frac{d\mu(\tilde{z})}{d\mu(z)} = \prod_{s=2}^{n} |\tilde{k}_{ss}|^{-4(s-1)}, \tag{10}$$

where $\tilde{k}_{ss}$ is given by equation (7).

PROOF: Let $(\tilde{z})_{pq}$ and z_{pq} be the matrix elements of $\tilde{z}$ and z, respectively, $(\tilde{g} = zg_0 = \tilde{k}\tilde{z})$. We introduce for simplicity some ordering in the variables $(\tilde{z})_{pq}$ and $(z)_{pq}$, $n \geqslant p > q \geqslant 1$, by setting

$$(\tilde{z})_{pq} = w_l = u_l + iv_l, \tag{11}$$

$$l = 1, 2, \ldots, \frac{n(n-1)}{2} = N.$$

$$(z)_{pq} = z_l = x_l + iy_l, \tag{12}$$

Note that by eq. (7) the variables (11) are analytic functions of the variables (12).

Eq. 2(5) implies that the Radon–Nikodym derivative is given by the following Jacobian

$$\frac{D(u_1, v_1, \ldots, u_n, v_n)}{D(x_1, y_1, \ldots, x_n, y_n)}.$$

In the evaluation of this Jacobian we use the following well-known fact: if $w_l = u_l + iv_l$, $l = 1, 2, \ldots, r$, are analytic functions of variables $z_k = x_k + iy_k$, $k = 1, 2, \ldots, r$, then

$$\frac{D(u_1, v_1, \ldots, u_r, v_r)}{D(x_1, y_1, \ldots, x_r, y_r)} = \left|\frac{D(w_1, w_2, \ldots, w_r)}{D(z_1, z_2, \ldots, z_r)}\right|^2. \tag{13}$$

We prove this by the method of induction. For $r = 1$, using Cauchy–Riemann equations and eqs. (8) and (9), we obtain

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left|\frac{d\tilde{z}}{dz}\right|^2.$$

Assuming that eq. (13) is true for $s = m-1$, we find for $s = m$:

$$\frac{D(u_1, v_1, u_2, v_2, \ldots, u_m, v_m}{D(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)} = \frac{D(u_1, v_1, u_2, v_2, \ldots, u_m, v_m)}{D(x_1, y_1, u_2, v_2, \ldots, u_m, v_m)} \times$$

$$\times \frac{D(x_1, y_1, u_2, v_2, \ldots, u_m, v_m)}{D(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)} = \frac{D(u_1, v_1)}{D(x_1, y_1)} \cdot \frac{D(u_2, v_2, \ldots, u_m, v_m)}{D(x_2, y_2, \ldots, x_m, y_m)}$$

$$= \left|\frac{D(w_1)}{D(z_1)}\right|^2 \cdot \left|\frac{D(w_2, \ldots, w_m)}{D(z_2, \ldots, z_m)}\right|^2 = \left|\frac{D(w_1, w_2, \ldots, w_m)}{D(z_1, z_2, \ldots, z_m)}\right|^2$$

Using now eqs. (11) and (7) we obtain

$$\frac{D(w_1, w_2, \ldots, w_n)}{D(z_1, z_2, \ldots, z_n)} = \prod_{s=2}^{n} (\tilde{k}_{ss})^{-2(s-1)}, \tag{14}$$

where $\tilde{k} \in K$ is defined by the decomposition $\tilde{g} = zg_0 = \tilde{k}\tilde{z}$ and the explicit expressions for the matrix elements $\tilde{k}_{ss}$ are given in eq. (7). ▼

C. *Principal Series of Representations*

By virtue of eqs. (1)–(3), the carrier space H^L of the representation U^L induced by the one-dimensional representation L of K consists of all measurable functions $\varphi(z) = \varphi(\ldots, z_{pq}, \ldots)$, $0 < q < p \leqslant n$, satisfying the condition

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\varphi(\ldots, z_{pq}, \ldots)|^2 \mathrm{d}x_{pq} \mathrm{d}y_{pq} < \infty, \qquad z_{pq} = x_{pq} + \mathrm{i}y_{pq}. \tag{15}$$

The operator U_g^L is, by virtue of eqs. (3), (5) and (10), given by

$$U_{g_0}^L \varphi(z) = \sqrt{\frac{\mathrm{d}\mu(\tilde{z})}{\mathrm{d}\mu(z)}} L_{\tilde{k}} \varphi(\tilde{z}) = \prod_{s=2}^{n} |\tilde{k}_{ss}|^{m_s + \mathrm{i}\varrho - 2(s-1)} \tilde{k}_{ss}^{-m_s} \varphi(\tilde{z}), \tag{16}$$

where

$$\tilde{g} = zg_0 = \tilde{k}\tilde{z}$$

and the parameters $\tilde{k}_{ss}$ and the components $(\tilde{z})_{pq}$ of $\tilde{z}$ are given by eq. (7).

This series of representations is called the *principal non-degenerate series*. It is characterized by two sets of invariant numbers: the integers $m_2, m_3, \ldots, m_n$, and the real numbers $\varrho_2, \varrho_3, \ldots, \varrho_n$, determining the characters 2(4) of D.

EXAMPLE 1. Let $G = \mathrm{SL}(2, C)$, i.e. G is the universal covering group of the Lorentz group SO(3, 1). The subgroup Z, by example 3.6.1, consists of the matrices z of the form

$$z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, \qquad z = x + \mathrm{i}y \text{ and } x, y \in R^1, \tag{17}$$

where we denoted the matrix element z_{21} by the same letter z as the matrix $z \in Z$.

The subgroup P consists of matrices of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix}, \qquad \text{with} \qquad k_{11}k_{22} = 1. \tag{18}$$

The carrier space $H^L \doteqdot L^2(Z, \mu)$ consists of the equivalence classes of measurable functions for which

$$\int |\varphi(z)|^2 \mathrm{d}\mu(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\varphi(z)|^2 \mathrm{d}x\mathrm{d}y < \infty. \tag{19}$$

Let

$$\mathrm{SL}(2, \mathrm{C}) \ni g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Then, by eqs. (5), (8), (9) and (10), one obtains

$$L_{\tilde{k}} = |\tilde{k}_{22}|^{m_2 + i\varrho_2} \tilde{k}_{22}^{-m_2}, \quad \tilde{z} = \frac{\alpha z + \gamma}{\beta z + \delta}, \quad \tilde{k}_{22} = \beta z + \delta \tag{20}$$

and

$$\frac{d\mu(\tilde{z})}{d\mu(z)} = |\tilde{k}_{22}|^{-4}.$$

Hence, the representation U^L given in eq. (16) takes the form

$$U_g^L \varphi(z) = \sqrt{\frac{d\mu(\tilde{z})}{d\mu(z)}} \cdot L_{\tilde{k}} \varphi(\tilde{z}) = |\beta z + \delta|^{m_2 + i\varrho_2 - 2} (\beta z + \delta)^{-m_2} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{21}$$

Note that by analytic continuation of the multiplier, eq. (21) can be made to coincide with the finite-dimensional induced representations of SL(2, C), 8.2(24), (cf. exercise 8.2.1).

Th. 1.3 shows that every representation U^L of the principal series is irreducible.

In order to state the equivalence properties of irreducible representations of SL(n, C), it is convenient to introduce another normalization of the invariant numbers m_i and ϱ_i, $i = 1, 2, \ldots, n$. Because $\prod_{s=1}^{n} k_{ss} = 1$ we can either set $m_1 = \varrho_1 = 0$ (as we did), or we can keep the element k_{11} in

$$\chi(\delta) = \prod_{s=1}^{n} (k_{ss})^{m_s + i\varrho_s} k_{ss}^{-m_s}$$

and introduce a more 'symmetric' normalization

$$\begin{aligned} m_1 + m_2 + \ldots + m_n &= 0, \\ \varrho_1 + \varrho_2 + \ldots + \varrho_n &= 0. \end{aligned} \tag{22}$$

The problem of the equivalence of irreducible representations is solved by the following.

THEOREM 3. *Two irreducible representations of the principal non-degenerate series are equivalent if and only if their collection of pairs of invariant numbers satisfying* (22)

$$(m_1, \varrho_1), \ (m_2, \varrho_2), \ \ldots, \ (m_n, \varrho_n)$$

and

$$(m_1', \varrho_1'), \ (m_2', \varrho_2), \ldots, \ (m_n, \varrho_n'),$$

which determine these irreducible representations can be obtained from each other by a permutation. ▼

PROOF: The Weyl group W permute characters; hence theorem follows from th. 1.2(iii).

D. *Reduction of the Principal Series of Representations to the Subgroup* SU(n)

We now consider the problem of determining the irreducible representations of SU(n) which occur in the restriction of a representation U^L of the principal series of SL(n, C) to the subgroup SU(n). This problem appears in many applications of group theory in particle physics. The following theorem gives the complete answer to this question. The proof of the theorem demonstrates again the power of the I-R Theorem.

THEOREM 4. *Let U^L be an irreducible representation of* SL(n, C) *induced by a one-dimensional representation L of the minimal parabolic subgroup P, and let T be an irreducible representation of* SU(n). *Let M be the subgroup M = SU(n) $\cap$ $\cap$ P. Then, the multiplicity of T in $U^L_{\mathrm{SU}(n)}$ is equal to the multiplicity of the one-dimensional representation* L_M in T_M.*

PROOF: The theorem represents a special case of the I-R Theorem. Because every element of $g \in \mathrm{SL}(n, C)$ can be written in the form $g = ku$ (cf. eq. 2(2a)), there is only one double coset P: SU(n). Applying the I-R Theorem, we find that U^L restricted to SU(n) is the representation of SU(n) induced by the representation L restricted to the subgroup SU(n) $\cap P = M$. Applying now the Frobenius Reciprocity Theorem we obtain the assertion of the th. 4. ▼

Remark 1: The representation L of P restricted to M has the form

$$M \ni \gamma = \begin{bmatrix} \exp(i\varphi_1) & \cdots & 0 \\ & \ddots & \vdots \\ 0 & & \exp(i\varphi_n) \end{bmatrix} \to L_\gamma = \exp[i(m_2\varphi_2 + \dots + m_n\varphi_n)]. \quad (23)$$

Every irreducible representation T of SU(n) is uniquely determinined by its highest weight m and the weight diagram is associated with the highest weight m. The assertion of th. 4 states, in fact, that the multiplicity of T in the representation $U^L_{\mathrm{SU}(n)}$ is equal to the multiplicity of the weight $m_L = (m_2, m_3, \dots, m_n)$ in the weight diagram associated with the highest weight m.

Remark 2: The restriction of the representation L of P to the subgroup M does not depend on the invariant numbers $\varrho_2, \dots, \varrho_n$ because L is, in fact, the character of D (cf. eq. (4)). Consequently, the non-equivalent members of the principal

---

* L_M and T_M denote the restriction of the representation L of P and T of SU(n) to the subgroup M, respectively.

series with the same invariant numbers $m_2, \ldots, m_n$, but with different $\varrho_2, \ldots, \varrho_n$ have the same content with respect to SU(n). ▼

We can derive from th. 4 the following useful corollary:

COROLLARY. *The irreducible representation T of* SU(n) *corresponding to the lowest possible highest weight is contained in $U^L_{\mathrm{SU}(n)}$ only once. This lowest highest weight is equal to the weight m_L.*

PROOF: Indeed, the representation of T, whose highest weight m is equal m_L. satisfies the condition of th. 5 and therefore is contained in $U^L_{\mathrm{SU}(n)}$. It is contained only once because the highest weight in any irreducible representation is nondegenerate. Any other representation T', in which m_L is not a highest weight, is determined by a highest weight m', which is higher than m_L. ▼

The following example illustrates the content of th. 4.

EXAMPLE 2. Let $G = \mathrm{SL}(2, C)$ and let U^L be a representation of the principal series induced by a representation L of P. The representation L is defined by the character $\chi(\delta) = |\delta|^{m_2+i\varrho_2}\delta^{-m_2}$, $\delta = \begin{bmatrix}\delta^{-1} & 0\\ 0 & \delta\end{bmatrix} \in D$. An irreducible representation T^J of SU(2) is determined by J, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Let Y^J_m, $m = -J$, $-J+1, \ldots, J-1, J$, be a basis of the carrier space of T^J. It is well known that an element $\gamma \in M$,

$$\gamma = \begin{bmatrix}\exp(-i\varphi) & 0\\ 0 & \exp(i\varphi)\end{bmatrix}$$

corresponds to the rotation around z-axis by an angle 2φ. Hence, $T^J_\gamma Y^J_m = \exp(2im\varphi) Y^J_m$, and every representation $\varphi \to \exp(2im\varphi)$ of M appears with multiplicity one. Consequently, T^J restricted to M (i.e., T^J_M) contains L_M if and only if $m_2/2$ is one of the numbers $J, J-1, \ldots, -J$. Thus if $J \geqslant m_2/2$, the representation T^J enters in $U^L_{\mathrm{SU}(2)}$ with the multiplicity one, i.e.,

$$U^L_{\mathrm{SU}(2)} = \sum_{J=\left|\frac{m_2}{2}\right|}^{\infty} \oplus T^J(\mathrm{SU}(2)). \tag{24}$$

Note that two non-equivalent representations U^L and $U^{L'}$, for which $\varrho_2 \neq \varrho_2'$ but $m_2 = m_2'$ have the same decomposition (24).

§ 4. Principal Degenerate Series of SL(n, C).

We give now a description of so-called *principal degenerate series*. These series have various degrees of degeneracy and are described by $2n-2k$, $k = 2, 3, \ldots, n-1$, invariant numbers respectively.

Let

$$n = n_1+n_2+ \ldots + n_r, \quad r \geqslant 2, \; r \neq n, \tag{1}$$

be a partition of the integer n into positive integers and let

$$g = \begin{bmatrix} g_{11} & \cdots & g_{1r} \\ \cdots & \cdots & \cdots \\ g_{r1} & \cdots & g_{rr} \end{bmatrix} \tag{2}$$

be a decomposition of $g \in \mathrm{SL}(n, C)$ into matrices g_{pq}, $p, q = 1, 2, \ldots, r$, with n_p rows and n_q columns. We choose the matrix blocks g_{pq} in such a way that when inserted into the eq. (2) they give exactly the matrix $g \in \mathrm{SL}(n, C)$. We introduce, moreover, the matrices k and z, $k, z \in \mathrm{SL}(n, C)$, of the form

$$k = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1r} \\ 0 & k_{22} & \cdots & k_{2r} \\ & & & \vdots \\ 0 & \cdots & \cdots & k_{rr} \end{bmatrix}, \quad z = \begin{bmatrix} I_{n_1} & 0 & \cdots & & 0 \\ z_{21} & I_{n_2} & 0 & \cdots & 0 \\ \vdots & & & & \\ z_{r1}, & z_{r2} & \cdots & & I_{n_r} \end{bmatrix}, \tag{3}$$

where k_{pq} and z_{pq} are arbitrary matrices of dimension $n_p \times n_q$, and I_{n_k}, $k = 1, 2, \ldots, r$, are the square unit matrices of order n_k. The set of all matrices k and z given by eq. (3) are subgroups of $\mathrm{SL}(n, C)$, which we denote by $P_{n_1 n_2, \ldots, n_r}$ and $Z_{n_1 n_2, \ldots, n_r}$, respectively.

The following lemma provides a decomposition of $g \in \mathrm{SL}(n, C)$, which is analogous to the decomposition 1(2b).

LEMMA 1. *Almost every element $g \in \mathrm{SL}(n, C)$ can be uniquely represented in the form*

$$g = k_g z_g, \tag{4}$$

where $k_g \in P_{n_1 n_2, \ldots, n_r}$ and $z_g \in Z_{n_1 n_2, \ldots, n_r}$.

The proof is straightforward and we omit it.

The unitary degenerate representations of $\mathrm{SL}(n, C)$ are constructed in a similar manner as the nondegenerate ones. Therefore, we restrict ourselves to a discussion of the main steps only.

We shall construct unitary representations U^L of $\mathrm{SL}(n, C)$ induced by the one-dimensional representations $k \to L_k$ of the subgroup $P_{n_1 n_2, \ldots, n_r}$. Let $\Lambda_j = \det k_{jj}$, where k_{jj} are $n_j \times n_j$ matrices given in eq. (3). Then the map

$$L\colon k \to \chi(k) = \prod_{s=2}^{r} |\Lambda_s|^{m_s + i\varrho_s} \Lambda_s^{-m_s}, \tag{5}$$

where $\varrho_2, \ldots, \varrho_r$ are arbitrary real numbers and $m_2, \ldots, m_r$ are arbitrary integers, gives the one-dimensional representation of the subgroup $D_{n_1, \ldots, n_r}$ consisting of all matrices of the form $\begin{bmatrix} k_{11} & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & k_{rr} \end{bmatrix}$. The subgroup $\mathfrak{Z}_{n_1 n_2, \ldots, n_r}$ consisting of all matrices of the form

$$\begin{bmatrix} I_{n_1} & k_{12} & \cdots\cdots & k_{1r} \\ & I_{n_2} & k_{23} \ \cdots & k_{2r} \\ 0 & & \ddots & \vdots \\ & & & I_{n_k} \end{bmatrix} \tag{6}$$

is a normal subgroup in $P_{n_1,\ldots,n_r}$, and $P_{n_1,\ldots,n_r} = \mathfrak{Z}_{n_1,\ldots,n_r} \otimes D_{n_1,\ldots,n_r}$. Hence, the representation (5) of $D_{n_1,\ldots,n_r}$ can be lifted to the one-dimensional representation L of the subgroup $P_{n_1,\ldots,n_r}$. The representation U^L of SL(n, C) induced by the representation L of $P_{n_1,\ldots,n_r}$ is realized in the Hilbert space $L^2(X, \mu)$, where $X = P_{n_1,\ldots,n_r}\backslash G$ and μ is a quasi-invariant measure in X. However, using the decomposition (4), one can show that the group space of $Z_{n_1,\ldots,n_r}$ coincides with X up to a subset of a smaller dimension in X (cf. lemma 3.1). Consequently, we can take the measure μ as the invariant measure on the subgroup $Z_{n_1,\ldots,n_r}$. This measure is induced by the measure 2(5) on Z and is given by the formula

$$\mathrm{d}\tilde{\mu}(z) = \Pi \mathrm{d}x_{pq}\mathrm{d}y_{pq}, \tag{7}$$

where only those factors $\mathrm{d}x_{pq}\mathrm{d}y_{pq}$ occur, which correspond to the matrix elements z_{ij}, $i > j$, of the matrix $z \in Z_{n_1,\ldots,n_r}$.

A carrier space $L^2(Z_{n_1,\ldots,n_r}, \tilde{\mu})$ of a degenerate representation U^L consists of all functions $\varphi(z)$ measurable in $Z_{n_1,\ldots,n_r}$ which satisfy the condition

$$\int |\varphi(z)|^2 \mathrm{d}\tilde{\mu}(z) < \infty. \tag{8}$$

The representation U^L of SL(n, C) is given explicitly in the space $L^2(Z_{n_1,\ldots,n_r}, \tilde{\mu})$ by the formula 3(3), i.e.,

$$U_g^L\varphi(z) = \sqrt{\left(\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)}\right)} L_{\tilde{k}}\, \varphi(\tilde{z}), \tag{9}$$

where $\tilde{k}$ and $\tilde{z}$ are factors of the decomposition (4) of an element $\tilde{g} \equiv zg$, i.e., $zg = \tilde{k}\tilde{z}$. To complete the construction of U^L it is still necessary to compute the Radon–Nikodym derivative $\mathrm{d}\tilde{\mu}(\tilde{z})/\mathrm{d}\tilde{\mu}(z)$. The computations analogous to that in Lemma 3.2 give

$$\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)} = |\Delta_2|^{-2(n_1+n_2)}|\Delta_3|^{-2(n_1+2n_2+n_3)}\ldots|\Delta_r|^{-2(n_1+2n_2+\ldots+2n_{r-1}+2n_r)} \tag{10}$$

where Δ_s, $s = 2, \ldots, r$, denote the determinants of the corresponding block-diagonal elements of the element $\tilde{k}$.

Notice that the number of the invariant labels ϱ_s, m_s depends on the partition of n into n_i given by eq. (1). In the case $r = 2$ we obtain the *so-called most degenerate series* of SL(n, C).

EXAMPLE 1. Consider the most degenerate representations of SL(n, C), which are defined by the following partition of n

$$n \equiv n_1 + n_2 = (n-1)+1. \tag{11}$$

In this case the subgroups $Z_{n-1,1}$ and $P_{n-1,1}$ consists of matrices of the form

$$z = \begin{bmatrix} I_{n_1} & 0 \\ z & 1 \end{bmatrix}, \quad k = \begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix} \tag{12}$$

Here, I_{n_1} is the $(n-1)\times(n-1)$ unit matrix,

$$z = (z_{n,1}, z_{n,2}, \ldots, z_{n,n-1}) \equiv (z_1, z_2, \ldots, z_{n-1})$$

is the $1\times(n-1)$ matrix, k_{11} is the $(n-1)\times(n-1)$ matrix, k_{12} is the $(n-1)\times 1$ matrix, and k_{22} is a complex number. Note that in the present case, $\Lambda_2 = \det k_{22} = k_{22}$.

Hence, by eqs. (10) and (5), we have

$$\sqrt{\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)}} = |\Lambda_2|^{m_2+i\varrho_2-(n_1+n_2)}\Lambda_2^{-m_2}$$
$$= |\tilde{k}_{22}|^{m_2+i\varrho_2-n}\tilde{k}_{22}^{-m_2}.$$

Therefore, in order to define explicitly the action of an operator U_g^L, we have to find the form of $\tilde{k}_{22}$ and $\tilde{z}$. Comparing the matrix element of the matrix $\tilde{g} = zg$ with those of the product $\tilde{k}\tilde{z}$, we obtain

$$(\tilde{z})_{np} \equiv (\tilde{z})_p = \Big(\sum_{j=1}^{n-1} g_{jp}z_j + g_{np}\Big)\Big/\Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big) \tag{13}$$

and

$$\tilde{k}_{22} = \Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big) \tag{14}$$

Therefore, the explicit form of the action of an operator U_g^L in the space $L^2(Z_{n-1,1}, \tilde{\mu})$ is given by the formula

$$U_g^L\varphi(\ldots, z_p, \ldots) = \Big|\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big|^{m_2+i\varrho_2-n} \times$$
$$\times \Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big)^{-m_2} \varphi\Big(\ldots, \Big(\sum_{j=1}^{n-1} g_{jp}z_j + g_{np}\Big)\Big/\Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big), \ldots\Big). \tag{15}$$

Eq. (13) shows that SL(n, C) acts on the manifold $Z_{n-1,1}$ as a group of projective transformations, i.e., $Z_{n-1,1}$ is an $(n-1)$-dimensional projective space.

One can readily show, applying the I-R theorem, that every representation of the principal degenerate series is also irreducible. One can also derive the analogs of ths. 3.3 and 3.4 for degenerate series (cf. Gel'fand and Naimark 1950, ch. 3.4.)

§ 5. Supplementary Nondegenerate and Degenerate Series

We considered so far unitary representations U^L of a group G induced by *unitary* representations $k \to L_k$ of a subgroup $P \subset G$. These representations were realized by functions on G satisfying the condition

$$u(kg) = L_k u(g). \tag{1}$$

The action of U_g^L in the carrier space H^L was given by formula 3 (3) and the scalar product was defined by 3(2).

One could also try to obtain a unitary induced representation U^L of G from a *nonunitary* representation L of P. We now show the explicit construction of such representations. The new class of representations of G induced by nonunitary representations of P is the class of supplementary series of representations.

Clearly, the scalar product 3(2) for a nonunitary representation L of P cannot be invariant under U^L. It turns out, however, that it is sufficient to replace 3(2) by the scalar product $(\cdot, \cdot)$ of the form

$$(\varphi, \psi)_{H^L} = \int\limits_{X\times X} K(x_1, x_2)\,(\varphi(x_1), \psi(x_2))_H \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2), \tag{2}$$
$$x \in X = P\backslash G.$$

The kernel $K(x_1, x_2)$ is selected in such a manner that it compensates the additional factor resulting from the nonunitarity of the representation L of P.

A. *Supplementary Series for* SL(2, C)

First, we shall construct the supplementary series of representations for SL(2, C) (cf. example 3.1). In this case the subgroup P consists of matrices of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix} \quad \text{with } k_{11}k_{22} = 1.$$

We find unitary representations U^L of SL(2, C) induced by one-dimensional nonunitary representations of P given by formula

$$k \to L_k = |k_{22}|^{m+i\varrho} k_{22}^{-m}, \tag{3}$$

where now ϱ is not real.

Using 3(21) one obtains

$$U_g^L \varphi(z) = \sqrt{\frac{\mathrm{d}\mu(\tilde{z}}{\mathrm{d}\mu(z)}} L_{\tilde{k}}\,\varphi(\tilde{z}) = |\beta z+\delta|^{m+i\varrho-2}(\beta z+\delta)^{-m}\varphi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right), \tag{4}$$

i.e., the action of U^L in the carrier space H^L is in fact the same as in case of the principal series. The scalar product in H^L will, however, be different. We find it using the invariance and positive definiteness requirements

$$(U_g^L\varphi, U_g^L\psi) = (\varphi, \psi) \equiv \int K(z_1', z_2')\varphi(z_1')\overline{\psi}(z_2')\mathrm{d}z_1'\mathrm{d}z_2' \tag{5}$$

where we set $\mathrm{d}\mu(z) = \mathrm{d}z \equiv \mathrm{d}x\mathrm{d}y$.

LEMMA 1. *The kernel $K(z_1, z_2)$ has the form*

$$K(z_1, z_2) = |z_1 - z_2|^{-2+\sigma}, \tag{6}$$

where $\sigma = -i\varrho$ and $0 < \sigma < 2$.

PROOF: Set $z_1' = \tilde{z}_1 = \dfrac{\alpha z_1 + \gamma}{\beta z_1 + \delta}$ and $z_2' = \tilde{z}_2$ in the right-hand side of eq. (5). Then, using the formula $\mathrm{d}\tilde{z}/\mathrm{d}z = |\beta z + \delta|^{-4}$ one obtains

$$(\varphi, \psi) = \int K(\tilde{z}_1, \tilde{z}_2)\varphi(\tilde{z}_1)\overline{\psi(\tilde{z}_2)}|\beta z_1 + \delta|^{-4}|\beta z_2 + \delta|^{-4}\mathrm{d}z_1\,\mathrm{d}z_2.$$

Further, putting expression (4) in the left-hand side of eq. (5), and by virtue of the arbitrariness of $\varphi(\tilde{z}_1)$ and $\psi(\tilde{z}_2)$, one obtains

$$\begin{aligned} K(z_1, z_2)|\beta z_1 + \delta|^{m+i\varrho-2}(\beta z_1 + \delta)^{-m}|\beta z_2 + \delta|^{m-i\bar{\varrho}-2}\overline{(\beta z_2 + \delta)}^{-m} \\ = K\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right)|\beta z_1 + \delta|^{-4}|\beta z_2 + \delta|^{-4}. \end{aligned}$$

Consequently,

$$\begin{aligned} K\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right) \\ = K(z_1, z_2)|\beta z_1 + \delta|^{m+i\varrho+2}(\beta z_1 + \delta)^{-m}|\beta z_2 + \delta|^{m-i\bar{\varrho}+2}\overline{(\beta z_2 + \delta)}^{-m}. \end{aligned} \tag{7}$$

For the particular value $g = z_0 = \begin{bmatrix} 1 & 0 \\ z_0 & 1 \end{bmatrix}$ of g we obtain

$$K(z_1 + z_0, z_2 + z_0) = K(z_1, z_2).$$

And for $z_0 = -z_2$

$$K(z_1, z_2) = K(z_1 - z_2, 0) \equiv K_1(z_1 - z_2). \tag{8}$$

Using eqs. (8) and (7), one obtains

$$\begin{aligned} K_1\left(\frac{z_1 - z_2}{(\beta z_1 + \delta)(\beta z_2 + \delta)}\right) = K_1(z_1 - z_2)|\beta z_1 + \delta|^{m+i\varrho+2} \times \\ \times (\beta z_1 + \delta)^{-m}|\beta z_2 + \delta|^{m-i\bar{\varrho}+2}\overline{(\beta z_2 + \delta)}^{-m}. \end{aligned} \tag{9}$$

Setting here $z_2 = 0$ and $\beta = \dfrac{1-\delta}{z_1}$, we have

$$K_1\left(\frac{z_1}{\delta}\right) = K_1(z_1)|\delta|^{m-i\bar{\varrho}+2}\bar{\delta}^{-m}. \tag{10}$$

Now, putting in eq. (9), $z_1 = 0$ and $\beta = \dfrac{1-\delta}{z_2}$, we obtain

$$K_1\left(-\frac{z_2}{\delta}\right) = K_1(-z_1)|\delta|^{m+i\varrho+2}\delta^{m}. \tag{11}$$

Eqs. (10) and (11), by virtue of arbitrariness of z_1 and z_2, imply

$$|\delta|^{-i\bar{\varrho}}\bar{\delta}^{-m} = |\delta|^{i\varrho}\delta^{-m}. \tag{12}$$

Setting here $\delta = \exp(i\theta)$, θ real, we obtain

$$\exp(im\theta) = \exp(-im\theta), \quad \text{i.e.,} \quad m = 0. \tag{13}$$

Therefore, by eq. (12), we have $\varrho = -\bar{\varrho}$, i.e., ϱ is a purely imaginary number, $\varrho = i\sigma$, σ real. Putting in eq. (10), $\delta = z_1 \equiv z$ we have

$$K_1(z) = C(z)^{-2+\sigma}$$

and

$$K(z_1, z_2) = C(z_1 - z_2)^{-2+\sigma},$$

where $C = K_1(1)$ is an arbitrary constant.

The application of the standard Fourier analysis of functions of one complex variable shows that the scalar product (5) with the kernel (6) is positive definite only for $0 < \sigma < 2$ (cf. Naimark 1964, ch. III, § 12). ▼

Equations (13) and (4) imply the following expression for U_g^L

$$U_g^L \varphi(z) = |\beta z + \delta|^{-2-\sigma} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{14}$$

This formula defines unitary representations of SL(2, C), when $0 < \sigma < 2$. It is instructive to investigate, what representations we obtain, when $\sigma \geqslant 2$. In the case $\sigma = 2$ the scalar product (5) takes the form

$$(\varphi, \psi) = \iint \varphi(z_1)\bar{\psi}(z_2) dz_1 dz_2. \tag{15}$$

In particular

$$(\varphi, \varphi) = \left|\int \varphi(z) dz\right|^2 \geqslant 0$$

and

$$(\varphi, \varphi) = 0, \quad \text{if} \quad \int \varphi(z) dz = 0. \tag{16}$$

It is natural to consider the set of functions $\varphi(z)$ on the manifold $Z = C^1$ as elements of a Hilbert space H', obtained by identifying functions with the same value of integral $\int \varphi(z) dz$. The scalar product in H_1 is induced by the form (15); in fact, H' is one-dimensional. Indeed, if $\int \varphi_2(z) dz \neq 0$, then setting

$$c = \frac{\int \varphi_1(z) dz}{\int \varphi_2(z) dz}, \quad \varphi = \varphi_1 - c\varphi_2,$$

we obtain

$$\int \varphi(z) dz = 0 \Rightarrow \varphi_1 = c\varphi_2,$$

i.e., any two elements of $H'(z)$ are linearly dependent. Using eq. (14) and setting $\sigma = 2$, we find

$$\int U_g^L \varphi(z) dz = \int |\beta z + \delta|^{-4} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) dz.$$

Passing to the variable $\tilde{z} = \dfrac{\alpha z+\gamma}{\beta z+\delta}$ and utilizing the Jacobian of the transformation $z \to \tilde{z}$ (eq. 2(20) and below), we obtain

$$\int U_g^L \varphi(z)\,\mathrm{d}z = \int \varphi(z)\,\mathrm{d}\tilde{z} = \int \varphi(z)\,\mathrm{d}z,$$

i.e.,

$$U_g^L \varphi(z) = |\beta z+\delta|^{-4}\varphi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) = \varphi(z) \quad \text{for every } \varphi \in H'(Z) \tag{17}$$

or

$$U_g^L = 1.$$

It should be noted that this is the only representation of the Lorentz group which is unitary and finite-dimensional.

When we set $\sigma > 2$, we obtain a Hilbert space with an indefinite metric, which contains a finite number of negative squares. These are the so-called *Pontryagin spaces.* The representation theory in Pontryagin spaces was originated by Gel'fand and Naimark 1947 and recently systematically developed by Naimark and collaborators (cf. excellent review by Naimark and Ismagilov 1968).

B. *Supplementary Series for* SL(n, C)

The derivation of the explicit form of representations of supplementary series for SL(n, C) is similar to the one of SL$(2, C)$. We again start with the class of functions on SL(n, C) satisfying the condition $\tilde{\varphi}(kg) = L_k \varphi(g)$, where the representation $k \to L_k$ of K is now not unitary. We write an element $k \in K$ in the form

$$k = \begin{bmatrix} k_{11} & k_{12} & \cdots & & & & & & k_{1n} \\ & k_{22} & & & & & & & k_{2n} \\ & & \ddots & & & & & & \vdots \\ & & & k_{n-2\tau,n-2\tau} & & & & & \vdots \\ & & & & \lambda_1 & & & & \vdots \\ & & & & & \mu_1 & & & \vdots \\ & & & & & & \lambda_2 & & \vdots \\ & 0 & & & & & & \mu_2 & \vdots \\ & & & & & & & \ddots & \vdots \\ & & & & & & & \lambda_\tau & k_{n-1,n} \\ & & & & & & & & \mu_\tau \end{bmatrix} \in P, \qquad \tau = 1, 2, \ldots, \left[\frac{n}{2}\right] \tag{18}$$

and take a nonunitary one-dimensional representation $k \to L_k$ of K of the form

$$L_k = \prod_{p=2}^{n-2\tau} |k_{pp}|^{m_p+i\varrho_p} k_{pp}^{-m_p} \prod_{q=1}^{\tau} |\lambda_q|^{m'_q+i\sigma'_q+\sigma''_q} \lambda^{-m'_q} |\mu_q|^{m'_q+i\sigma'_q-\sigma''_q} \mu_q^{-m'_q}, \tag{19}$$

$$\varrho_p,\ \sigma'_q,\ \sigma''_q \ \text{real}.$$

Clearly, if $\sigma''_q = 0$, for $q = 1, 2, \ldots, \tau$, then L_k becomes a unitary representation. It is also evident that the number τ represents 'the degree of nonunitarity' of the representation $k \to L_k$; for $\tau = 0$ we obtain a unitary representation of P; for $\tau = \left[\frac{n}{2}\right]$ we have a 'maximally' nonunitary one.

The action of the representation $g \to U^L_q$ for supplementary series is given by the standard formula 3(3). It remains only the derivation of the form of the invariant scalar product (2). It is similar to the derivation in the case of SL(2, C), but fairly long (cf. Gel'fand and Naimark 1950, ch. IV). Therefore, we restrict ourselves to a presentation of the final results.

THEOREM 2. *The representation $g \to U^L_q$ of the supplementary, nondegenerate series of* SL(n, C) *induced by a representation $k \to L_k$ of P given by eq.* (19) *is defined by the formula*

$$U^L_g \varphi(z) = \sqrt{\frac{d\mu(\tilde{z})}{d\mu(z)}}\, L_{\tilde{k}}\, \varphi(\tilde{z}), \tag{20}$$

where $\tilde{k} \in P$ and $\tilde{z} \in Z$ are given by eq. 3(7) *and* R-N *derivative by eq.* 3(10).

The invariant scalar product for the representation (20) *is given by the formula*

$$(\varphi, \psi) = \int K(\dot{z})\varphi(z)\overline{\psi(\dot{z}z)}\, d\mu(\dot{z})\, d\mu(z), \tag{21}$$

where $\dot{z}$ is an element of Z of the form

$$\dot{z} = \left[\begin{array}{c|cccc} \underbrace{\begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & & \\ 0 & \ldots & \ldots & 1 \end{matrix}}_{n-2\tau} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n-2\tau & & 0 & & \\ \hline & \begin{matrix} 1 & 0 \\ z_1 & 1 \end{matrix} & & & 0 \\ & & \begin{matrix} 1 & 0 \\ z_2 & 1 \end{matrix} & & 0 \\ 0 & & & \ddots & \\ & & & & \begin{matrix} 1 & 0 \\ z_\tau & 1 \end{matrix} \end{array}\right] \tag{22}$$

and

$$K(\dot{z}) = \prod_{j=1}^{\tau} |z_j|^{2(\sigma_j''-1)},$$
$$0 < \sigma_j'' < 1, \quad \mathrm{d}\mu(\dot{z}) = \prod_{p=1}^{\tau} \mathrm{d}x_p \mathrm{d}y_p. \tag{23}$$

The invariant numbers, which define a given representation of the supplementary nondegenerate series are the integers $m_1, m_2, \ldots, m_{n-2\tau}, m_1', m_2', \ldots, m_\tau'$ and the real numbers $\varrho_1, \varrho_2 \ldots, \varrho_{n-2\tau}, \sigma_1', \sigma_2', \ldots, \sigma_\tau', \sigma_1'', \sigma_2'', \ldots, \sigma_\tau''$, $0 < \sigma_p'' < 1$, $p = 1, 2, \ldots, \tau$. One readily verifies that for $n = 2$ the formula (20) coincides with (14) and the invariant scalar product (21), after changing variables coincides with (5).

C. *The Supplementary Degenerate Series of* SL(n, C)

The supplementary degenerate series of representations are constructed in a similar manner. We start with a class of functions on SL(n, C) satisfying the condition

$$\varphi(kg) = L_k \varphi(g), \qquad k \in P_{n_1, n_2, \ldots, n_r}, \tag{24}$$

where a representation $k \to L_k$ of $P_{n_1, n_2, \ldots, n_r}$ is again a nonunitary one. We write it in the form (cf. eq. 4(5))

$$L_k \doteq \prod_{p=2}^{r-2\tau} |\Lambda_p|^{m_p + \mathrm{i}\varrho_p} \Lambda_p^{-m_p} \prod_{q=1}^{\tau} |\lambda_q|^{m_q' + \mathrm{i}\sigma_q' + \sigma_q''} \lambda_q^{-m_q'} |\mu_q|^{m_q' + \mathrm{i}\sigma_q' - \sigma_q''} \mu^{-m_q'}, \tag{25}$$

where $p = \det k_{pp}$, $p = 2, 3, \ldots, r-2\tau$, and λ_q, μ_q, $q = 1, 2, \ldots, \tau$, are complex numbers which represent the 2τ last diagonal elements of $k \in P_{n_1, n_2, \ldots, n_r}$ (cf. eq. 3(3)). The number τ represents 'the degree of nonunitarity' of representation L_k, as before.

THEOREM 3. *The representation $g \to U_g^L$ of the supplementary degenerate series of* SL(n, C) *induced by the representation $k \to L_k$ of $K_{n_1, n_2, \ldots, n_r}$ given by eq.* (25) *is defined by the formula*

$$U_g^L \varphi(z) = \sqrt{\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)}} L_{\tilde{k}} \varphi(\tilde{z}), \tag{26}$$

where $\tilde{k} \in P_{n_1, n_2, \ldots, n_r}$ and $\tilde{z} \in Z_{n_1, n_2, \ldots, n_r}$ are factors of the canonical decomposition 4(4) *of the element $\tilde{g} = zg$, i.e., $\tilde{g} = zg = \tilde{k}\tilde{z}$ and* $\mathrm{d}\tilde{\mu}(\tilde{z})/\mathrm{d}\mu(z)$ *is given by eq.* 4(10). ▼

The expressions for the invariant scalar product for the representation (26) and for the variable $\dot{z}$ are the same as for the supplementary nondegenerate series, but the element z should be taken from $Z_{n_1, \ldots, n_r}$.

§ 6. Comments and Supplements

1. The construction of irreducible unitary representations of other semisimple Lie groups can be carried out as in case of SL(n, C) using the general construction presented in sec. 1. In particular, the construction of principal supplementary and degenerate series for SO(n, C) and Sp(n, C) was given by Gel'fand and Naimark 1950. The properties of induced irreducible representations of SO(n, 1) groups were considered by Hirai 1962. The problem of construction of all irreducible representations of SO(p, q), SU(p, q) and Sp(p, q) groups is still not completed so far: partial results were given by Graev 1954, Leznov and Fedoseev 1971 and Leznov and Savelev 1976. The construction of various series of representations of semisimple Lie groups, e.g. principal induced from cuspidal parabolic subgroup is discussed by Lipsman 1974.

2. The problem of a classification of irreducible unitary representations of semisimple Lie groups is in general open. Naimark 1954 published two papers in which he claimed that he would give a complete description of all unitary irreducible representations of complex classical groups: however, the next papers have not appeared so far. In the meantime Stein 1967 has shown that by the method of analytic continuation one can construct new irreducible unitary representations which were not contained in Gel'fand and Naimark classification of (1950) paper.

The full classification of all irreducible unitary representation is known only for several low-dimensional groups like SL(2, R) and SL(2, C) (cf. Gel'fand, Graev and Vilenkin 1966). Dixmier in 1961 published a complete classification in case of De Sitter group SO(4, 1).

3. The existence and properties of irreducible representations of so-called discrete series has been attracting considerable attention in recent years: an irreducible representation $g \to U_g$ of G in H is called *discrete* if there exists a non-zero vector u in H such that the matrix element $(u, U_g u)$ is square integrable on G. The set G_d of all discrete inequivalent irreducible representation of G is called the *discrete series*. Harish-Chandra in two long papers in 1965 and 1966 gave a description of discrete series for semisimple Lie groups (cf. also Warner 1972, I and II). He showed in particular that G has a discrete series if and only if $\operatorname{rank} G = \operatorname{rank} K$; this implies in particular that $U(p, q)$ groups have a discrete series. Special class of discrete series representations for $U(p, q)$ groups was constructed by Graev 1954. The most degenerate discrete series representations for $U(p, q)$ and $SO(p, q)$ were constructed by Rączka and Fischer 1966 and by Rączka, Limić and Niederle 1966 respectively.

4. The properties of irreducibility of representations of semisimple Lie groups induced from parabolic subgroup were discussed in the fundamental paper of Bruhat 1956. The extension of these results were obtained by Wallach in a series of papers 1969, 1971.

5. We now give the important results of Scull 1973, 1976, concerning spectra

of generators of the Lie algebra L of a simple Lie group G. Let $\int \lambda \mathrm{d}E(\lambda)$ be the spectral resolution of $-\mathrm{i}U(X)$, $X \in L$ and let the projection-valued measure $E(\cdot)$ be absolutely continuous with respect to Lebesgue measure $\mu(\cdot)$ on R. Then we say that $-\mathrm{i}U(X)$ has *two-sided spectrum* if $\operatorname{supp}\mu$ is $(-\infty, \infty)$ and *one-sided spectrum* if $\operatorname{supp}\mu$ is either $(-\infty, 0)$ or $(0, \infty)$. It turns out that spectrum of a noncompact generator $-\mathrm{i}U(X)$ depends on G but not on X or U. Indeed we have

THEOREM 1. *Let U be a continuous irreducible unitary representation of a connected simple Lie group G, such that G is not a group of automorphisms of an irreducible hermitian symmetric space. Then if $X \in L$ generates a noncompact one-parameter subgroup* $\exp(tX)$, $-\mathrm{i}U(X)$ *has two-sided spectrum.* ▼

We recall that an irreducible symmetric space X is hermitian if it is of the form $X = K\backslash G$ where G is a noncompact simple Lie group with trivial centre and K is a maximal compact subgroup with nondiscrete nontrivial center. The inspection of Table 4.2.I shows that among the classical Lie groups the following ones are groups of automorphisms of irreducible hermitian symmetric spaces: $SL(2, R)$, $SU(p, q)$, $SO_0(p, 2)$, $Sp(n, R)$ and $SO^*(2n)$. Hence the corresponding noncompact generators have two-sided spectrum.

In some, rather exceptional cases, a noncompact generator may have a one-sided spectrum. Indeed we have:

THEOREM 2. *Let G be a connected simple Lie group of automorphisms of an irreducible hermitian symmetric space. There exists an $X \in L$ such that $-\mathrm{i}U(X)$ has one-sided spectrum for any representation U of the holomorphic discrete series.* ▼

For various generalizations of these results cf. Scull 1973 and 1976. The detailed analysis of holomorphic discrete series of representations is given in the work of Rossi and Vergne 1973.

§ 7. Exercises

§ 1.1.*** Analyze irreducibility properties of representations $U^{\chi L}$ of $SO(p, q)$ and $SU(p, q)$ induced from the minimal parabolic subgroups.

Hint: Use th. 1.4.

§ 1.2.*** Classify irreducible unitary representations of the conformal group $SO(4, 2)$.

Hint: Use results of sec. 1 and extend technique of Dixmier 1961.

§ 3.1. Show that the Casimir operators of $SL(2, C)$ in the carrier space of irreducible representation $[m, \varrho]$ have the eigenvalues

$$C_2\psi = -\tfrac{1}{2}(m^2-\varrho^2-4)\psi,$$
$$C_2'\psi = m\varrho\psi,$$

where

$$C_2 = \tfrac{1}{2} M_{\mu\nu} M^{\mu\nu} = \boldsymbol{J}^2 - \boldsymbol{N}^2,$$
$$C_2' = -\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta} = \boldsymbol{J}\cdot\boldsymbol{N}$$

and

$$\boldsymbol{J} = (M_{32}, M_{13}, M_{21}), \qquad \boldsymbol{N} = (M_{01}, M_{02}, M_{03}).$$

§ 3.2. Show that the two irreducible representations $[m, \varrho]$ and $[-m, -\varrho]$ of SL(2, C) are equivalent.

§ 3.3. Show that the representation $U^*_{g^{-1}}$ conjugate-contragradient to an irreducible representation $U_g^{[m,\varrho]}$ of SL(2, C) is irreducible and is defined by the parameters $[m, -\varrho]$.

Note: Because the representations $[m, \varrho]$ and $[-m, -\varrho]$ are equivalent, the conjugate-contragradient representation $(U_g^{[m,\varrho]})^*$ is equivalent to $U^{[m,\varrho]}$ if and only if either $m = 0$ or $\varrho = 0$.

§ 3.4.* Let $U^{(j_0,j_1)}$ be an irreducible representation of SL(2, C) and let $|j_0, j_1: JM\rangle \equiv e_{JM}$ be the canonical basis associated with the set C_2, C_2', $\boldsymbol{J}^2$ and J_3 of commuting operators of SL(2, C).* Show that the matrix elements of generators $J_\pm = J_1 \pm iJ_2$, J_3, $N_\pm = N_1 \pm iN_2$ and N_3 have the form

$$\begin{aligned}
J_3 e_{JM} &= M e_{JM},\\
J_- e_{JM} &= \sqrt{[(J+M)(J-M+1)]}\, e_{J,M-1},\\
J_+ e_{JM} &= \sqrt{[(J+M+1)(J-M)]}\, e_{J,M+1},\\
M &= -J, -J+1, \dots, J-1, J.
\end{aligned}$$

$$\begin{aligned}
N_3 e_{JM} &= C_J \sqrt{(J^2-M^2)}\, e_{J-1,M} - A_J M e_{JM} - C_{J+1}\sqrt{[(J+1)^2-M^2]}\, e_{J+1,M},\\
N_+ e_{JM} &= C_J \sqrt{[(J-M)(J-M-1)]}\, e_{J-1,M+1} -\\
&\quad - A_J \sqrt{[(J-M)(J+M+1)]}\, e_{J,M+1} +\\
&\quad + C_{J+1}\sqrt{[(J+M+1)(J+M+2)]}\, e_{J+1,M+1},\\
N_- e_{JM} &= -C_J \sqrt{[(J+M)(J+M-1)]}\, e_{J-1,M-1} -\\
&\quad - A_J \sqrt{[(J+M)(J-M+1)]}\, e_{J,M-1} -\\
&\quad - C_{J+1}\sqrt{[(J-M+1)(J-M+2)]}\, e_{J+1,M-1},
\end{aligned}$$

where

$$A_J = \frac{i j_0 j_1}{J(J+1)}, \qquad C_J = \frac{i}{J}\sqrt{\left[\frac{(J^2-j_0^2)(J^2-j_1^2)}{4J^2-1}\right]}, \qquad J = j_0, j_0+1, \dots,$$

$$M = -J, -J+1, \dots, J-1, J, \qquad J = j_0, j_0+1, \dots,$$

$$e_{JM} \in H^{(J)}, \qquad H = \bigoplus_{J=j_0}^{\infty} H^{(J)}.$$

Show further that (a) for j_1 pure imaginary, j_0 non-negative half integers we have the principal series, (b) for j_1 real, $0 \leqslant j_1 \leqslant 1$, $j_0 = 0$ supplementary series,

* The parameters j_0 and j_1 which characterize irreducible unitary representations of SL(2, C) are connected with m, ϱ by the formulas:

$$j_0 = \left|\frac{m}{2}\right|, \qquad j_1 = -i\,(\operatorname{sign} m)\frac{\varrho}{2} \quad \text{for } m \neq 0,$$

$$j_0 = 0, \qquad j_1 = \pm i\frac{\varrho}{2} \quad \text{for } m = 0.$$

(c) for $j_1^2 = (j_0+n)^2$ for some integer n, we have finite-dimensional representations.

§ 5.1.*** Let $U^{(0,j_1)} \otimes U^{(0,j_1')}$ be the tensor product of irreducible representations of the supplementary series of SL(2, C). Find the Clebsch–Gordan coefficients

Hint: Use the technique of generalized projection operators developed in 15.4, cf. also Anderson, Rączka, Rashid and Winternitz 1970 a, b for a solution of a similar problem for representations of the principal series.

Chapter 20

Applications of Induced Representations

We present here two interesting applications of the general theory of induced representations. In sec. 1 we discuss the concept of localizability in relativistic quantum mechanics. We derive also the explicit form of the relativistic position operator. In sec. 2 we discuss the problem of the representations of the Heisenberg canonical commutation relations for finite number of degrees of freedom. We show here the uniqueness of the Schrödinger representation of the canonical commutation relations in the global Weyl form. We discuss also the problem of the equivalence of the Heisenberg and the Schrödinger formulations of quantum mechanics.

§ 1. The Relativistic Position Operator

We shall discuss in this section two basic concepts of relativistic quantum mechanics: the localizability and the position operators of relativistic systems. In subsec. A we introduce, using the concept of the imprimitivity system for the Euclidean group, the notion of localizability. In subsec. B we derive the explicit form of the position operators for a relativistic system, which transforms according to an irreducible representation of the Poincaré group.

A. *Localizable Relativistic Systems*

We begin with a review of the properties of the non-relativistic position operator. In non-relativistic quantum mechanics the position observables are defined by the formula

$$(q_k\psi)(x) = x_k\psi(x), \qquad k = 1, 2, 3, \tag{1}$$

where ψ is the wave function of the particle in the Hilbert space $H = L^2(R^3, \mathrm{d}^3\boldsymbol{x})$. The Fourier transform of the operator q_k is

$$Fq_kF^{-1} = \mathrm{i}\frac{\partial}{\partial p_k}. \tag{2}$$

This operator is hermitian with respect to the scalar product

$$(\varphi, \psi) = \int \varphi(p)\overline{\psi}(p)\,\mathrm{d}^3\boldsymbol{p}.$$

On the other hand, for a relativistic particle of mass m the invariant scalar product in the Hilbert space $L^2(h^m, \mu)$, $\mathrm{d}\mu^{(p)} = \mathrm{d}^3p/p_0$, h^m = mass hyperboloid, $p^2 = m^2$ is given by

$$(\varphi, \psi) = \int_{h^m} \varphi(p)\bar{\psi}(p)\frac{\mathrm{d}^3\boldsymbol{p}}{p_0}. \tag{3}$$

Hence q_k of eq. (1) is not hermitian with respect to this scalar product, for

$$\begin{aligned}(q_k\varphi, \psi) &= \mathrm{i}\int\left(\frac{\partial}{\partial p^k}\varphi\right)(p)\bar{\psi}(p)\frac{\mathrm{d}^3p}{p_0}\\ &= \int\varphi(p)\left[\left(-\mathrm{i}\frac{\partial}{\partial p^k}+\mathrm{i}\frac{p_k}{p^2+m^2}\right)\bar{\psi}\right](p)\frac{\mathrm{d}^3\boldsymbol{p}}{p_0}\\ &\neq (\varphi, q_k\psi).\end{aligned}$$

Consequently, the operator $q_k = \mathrm{i}\dfrac{\partial}{\partial p^k}$ cannot represent the position operator for a relativistic particle.

Thus, for wave functions defined on the hyperboloid, the position operator cannot be obtained by a simple Fourier transform characteristic of R^3.

We shall extend now the concept of the position operator to relativistic systems using the Imprimitivity Theorem. In fact: suppose that we have found the set of three commuting, self-adjoint operators Q_1, Q_2, Q_3, which represent the position operators of a relativistic particle of mass m and spin J. Then, by the Spectral Theorem, there exists a common spectral measure $E(S)$, $S \subset R^3$, such that every operator Q_i has the representation

$$Q_i = \int_{R^3} x_i \mathrm{d}E(x). \tag{4}$$

If $S \subset R^3$ and $\psi(x)$ represents the state of the particle in the Hilbert space H, then the expression

$$p(S) = \frac{||E(S)\psi||^2}{||\psi||^2}$$

represents the probability of measuring the position of the particle in the state ψ to be inside the set S.

The spectral measure $E(S)$, which defines the operators Q_i, is strongly limited by Euclidean invariance. In fact, let $\mathscr{E}^3$ denote the Euclidean group in R^3 and let $\mathscr{E}^3 \ni g \to U_g$ be a unitary representation of $\mathscr{E}^3$ in the Hilbert space H. Then, the Euclidean invariance of probability $p(S) \to p(gS)$ implies that

$$U_g E(S) U_g^{-1} = E(gS), \quad g \in \mathscr{E}^3. \tag{5}$$

Because the space R^3 is transitive relative to the group $\mathscr{E}$, eq. (5) means that the spectral function $E(\cdot)$ represents a transitive system of imprimitivity based on the space R^3.

Thus the existence of a position operator Q implies the existence of a transitive system of imprimitivity for the unitary representation of the Euclidean subgroup. This suggests the following definition of the localizability of a relativistic system.

Definition 1. A representation U of the Poincaré group Π defines a *localizable system*, if and only if the restriction $U_{\mathscr{E}}$ of U to the Euclidean group $\mathscr{E}^3$ possesses a transitive system of imprimitivity $E(S)$ based on the space R^3. ▼

Let us note that the condition (5) will be satisfied if the representation $U_{\mathscr{E}}$ of $\mathscr{E}^3$ is induced from SO(3): in this case $X = \mathscr{E}^3/\mathrm{SO}(3) \cong R^3$ and $E(S)$ can be taken as the canonical spectral measure 16.3(1). It remains therefore to verify if the reduction of a representation U of the Poincaré group to $\mathscr{E}^3$ gives the representation $U_{\mathscr{E}^3}$ of the Euclidean group, which is induced from a certain representation L of SO(3). The following proposition shows that this is indeed the case for irreducible representations of Π associated with massive particles of arbitrary spin.

Proposition 1. *Let $(a, \Lambda) \to U^{m,J}_{(a,\Lambda)}$ be a unitary irreducible representation of the Poincaré group corresponding to a particle of mass m with arbitrary spin J. Let χD^J denote the irreducible representation of $K = T^4 \otimes \mathrm{SO}(3)$ used for the induction of $U^{m,J}$. The restriction $U^{m,J}_{\mathscr{E}^3}$ of $U^{m,J}$ to $\mathscr{E}^3$ is unitarily equivalent to a representation induced by the representation of* SO(3) *given by the direct integral of irreducible representations L^J of* SO(3) *with $L^J \cong D^J$.* ▼

Proof: In th. 18.1.2 take $G = N \otimes M$ to be the Poincaré group and $W = N_0 \otimes M_0$ to be $\mathscr{E}^3 = T^3 \otimes \mathrm{SO}(3)$.

Then $K = T^4 \otimes \mathrm{SO}(3)$ and by 18.1(15) the restriction $U^{m,J}_{\mathscr{E}^3}$ of $U^{m,J}$ is the direct integral

$$U^{m,J}_{\mathscr{E}^3} \cong \int_{\mathscr{D}} U^J_{\mathscr{E}^3}(D)\,\mathrm{d}\nu(D) \tag{9}$$

where $\mathscr{D}$ is the space of double cosets K: $\mathscr{E}^3$.

To determine the content of the representation $U^{m,J}_{\mathscr{E}^3}$ we use the prescription given by item 2° of th. 18.1.2. In our case $M_{\hat{n}} = \mathrm{SO}(3)$, $M_{\hat{n}m} = m^{-1}M_{\hat{n}}m \sim \mathrm{SO}(3)$ and $M_0 = \mathrm{SO}(3)$: hence $M_{\hat{n}m} \cap M_0 \sim \mathrm{SO}(3)$. It follows from the definition that the representation $L_r^{(m)} = D^J(m)D^J(r)D^J(m^{-1})$. The character being the restriction of $\hat{n} = (m, 0, 0, 0)$ to N_0 is identically zero. Consequently $\chi L^{(m)}$ is equivalent to D^J representation: therefore the representation $\int L^\lambda d\varrho(\lambda)$ in th. 18.1.2.2° is irreducible and equivalent to D^J; this implies that the representation $U_{\mathscr{E}^3}(D)$ given by eq. 18.1(16) is equivalent to the representation of $\mathscr{E}^3$ induced by the representation D^J of SO(3). By virtue of 18.1(15) the representation $U^{m,J}_{\mathscr{E}^3}$ is the direct integral over the space of double cosets K: W of above (equivalent) representations of $U_{\mathscr{E}^3}(D)$ of $\mathscr{E}^3$. ▼

Finally, theorem 16.2.1 implies that the representation (9) of $\mathscr{E}^3$ is unitary equivalent to a representation of $\mathscr{E}^3$ induced by a representation of SO(3) given by the direct integral over the set of double cosets K: W of representations equivalent to D^J. ▼

Consequently, massive particles are localizable in the sense of the def. 1.

Theorem 18.1.2 also allows us to see the precise content of the reduction $U^{m;J}_{\mathscr{E}^3}$ of other representations $U^{m,J}$ of the Poincaré group ($m^2 \leqslant 0$, or $m^2 < 0$) when restricted to $\mathscr{E}^3$. In these cases $U^{m;J}_{\mathscr{E}^3}$ is not an induced representation from SO(3), hence they are not localizable in the sense of the definition 1, except for the case $m = 0$, $J = 0$ in which case the representation of SO(3) is trivial.

We can and perhaps we should change the def. 1 of localizability in the cases $m^2 = 0$, $m^2 < 0$. Physically these systems cannot be localized in $\mathscr{E}^3/\mathrm{SO}(3) \sim R^3$ but in $\mathscr{E}^3/\mathscr{E}(2)$ or $\mathscr{E}^3/\mathrm{SO}(2, 1)$, hence the corresponding imprimitivity system must be based on these spaces. (See exercise 1.3).

Finally the non-uniqueness of E, is determined by the existence of unitary operators V which commute with $U^{m;J}_{\mathscr{E}^3}$ but not with $E(S)$. Then all other systems of imprimitivity are given by $F(S) = VE(S)V^{-1}$. The position operator is unique only under further assumptions of time-reversal invariance and of regularity of $E(S)$.

It follows immediately from Euclidean covariance, eq. (6), and eq. (4), that the transformation property of the position operators Q_i is given by

$$U_{\{a,R\}} Q_i U^{-1}_{\{a,R\}} = D^1_{ij}(R)[Q_j - a_j]. \tag{7}$$

Infinitesimally, eq. (7) means

$$[Q_i, P_j] = \mathrm{i}\delta_{ij}, \tag{8}$$

and

$$[Q_j, J_k] = \mathrm{i}\varepsilon_{jkl} Q_l, \tag{9}$$

which are physically desired relations. Thus the Heisenberg commutation relations (7) may be viewed as one of the implications of the existence of a transitive imprimitivity system for $\mathscr{E}^3$.

B. *Construction of Relativistic Position Operators*

We now derive the explicit form of position operators Q_i, $i = 1, 2, 3$, for a relativistic particle with mass $m > 0$ and spin J. We shall work in momentum representation. The action of the operator $U^{m,J}_{\{a,\Lambda\}}$, $a \in T^4$, $\Lambda \in \mathrm{SL}(2, C)$, in the carrier space $H^{m,J}$ of a particle $[m, J]$ is given by the formula (cf. eq. 17.2 (39))

$$(U^{m,J}_{\{a,\Lambda\}}\psi)(p) = \exp(\mathrm{i}pa) D^J(\Lambda_p^{-1}\Lambda\Lambda_{L_\Lambda^{-1}p})\psi(L_\Lambda^{-1}p). \tag{10}$$

Here, $\psi(p) = \{\psi_k(p)\}_1^{2J+1}$ is a $(2J+1)$-component vector function on the positive mass hyperboloid h^m, $p^2 = m^2$, square integrable with respect to the invariant measure $d\mu(p) = d^3\boldsymbol{p}/p_0$. The element $\Lambda_p \in SL(2, C)$ is defined by the Mackey decomposition of $SL(2, C)$ (cf. eq. 17.2 (33))

$$\Lambda = \Lambda_p r, \quad \Lambda_p = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R^1,\ z \in C^1,\ r \in SU(2). \tag{11}$$

Hence, if $\Lambda \in SU(2)$, then $\Lambda_p = I$. Consequently, the restriction $U^{m,J}_{\mathscr{E}^3}$ of the representation $U^{m,J}_{(a,\Lambda)}$ of the Poincaré group Π to the (covering of the) Euclidean group $\mathscr{E}^3$ gives

$$(U^{m;J}_{(a,r)}\psi)(p) = \exp(ip\cdot a) D^J(r)\psi(R_r^{-1}p), \tag{12}$$

where $R_r \in SO(3)$ is the rotation corresponding to an element $r \in SU(2)$.

We shall now introduce the position coordinates x_k as follows.

Consider the operator V defined by

$$(V\psi)(x) \equiv (2\pi)^{-3/2} \int p_0^{1/2} \exp(ipx)\psi(p)\frac{d^3p}{p_0}. \tag{13}$$

Note that the integral (13) is not the ordinary three-dimensional Fourier transform. The operator V is a unitary operator. We show this explicitly for $J = 0$. The inverse transformation V^{-1} is given by

$$(V^{-1}\varphi)(p) = (2\pi)^{-3/2} \int p_0^{1/2} \exp(-ipx)\varphi(x) d^3x. \tag{14}$$

Let F denote the ordinary three-dimensional Fourier transform. Then

$$\|V\psi\|^2 = \int |V\psi|^2 d^3x = \int |F(p_0^{-1/2}\psi)|^2 d^3x = \int |p_0^{-1/2}\psi|^2 d^3p = \int |\psi|^2 \frac{d^3p}{p_0} = \|\psi\|^2.$$

Thus, V is the isometric transformation from $L^2(h^m, d\mu(p))$ into $L^2(R^3, d^3\boldsymbol{x})$. The set of functions $\{p_0^{1/2} H_l(p_1) H_k(p_2) H_n(p_3)\}$ forms a basis for the space $L^2(h^m, d\mu)$, where H_i is the ith Hermite function of a single variable. Using eq. (13) one obtains

$$\begin{aligned}(Vp_0^{1/2}H_l H_k H_n)(x) &= (2\pi)^{-3/2}\int \exp(ipx) p_0 H_l H_k H_n d^3p/p_0 \\ &= i^{l+k+n} H_l(x_1) H_k(x_2) H_n(x_3).\end{aligned} \tag{15}$$

Hence, V has a dense range in $L^2(R^3, d^3x)$. Consequently, V is a unitary mapping of $L^2(h^m, d\mu)$ onto $L^2(R^3, d^3\boldsymbol{x})$.

For an arbitrary $J > 0$ the proof runs similarly. Thus, the scalar product for $(V\psi)(x)$ is as in the non-relativistic quantum mechanics.

Set now

$$U^L_{(a,r)} \equiv VU_{(a,r)}V^{-1}. \tag{16}$$

Using eqs. (13), (12) and (14), one obtains

$$(U^L_{(a,r)}\psi)(x) = D^J(r)\psi(R_r^{-1}(x-a)). \tag{17}$$

We now construct explicitly a transitive system of imprimitivity $E^L(S)$ based on R^3. For this purpose let us define the canonical spectral measure $R^3 \supset S \to \to E^L(S)$ by the formula

$$\big(E^L(S)\psi\big)(x) \equiv \chi_S(x)\psi(x), \tag{18}$$

where χ_S is the characteristic function of the set S.

Using eqs. (17) and (18) we obtain

$$U_g^L E^L(S)\, U_g^{L^{-1}} = E^L(gS), \quad g = (a, r). \tag{19}$$

Indeed:

$$\begin{aligned}\big(U_g^L E^L(S)\, U_{g^{-1}}^L\psi\big)(x) &= \big(D^J(r) E^L(S)\, U_{g^{-1}}^L\psi\big)\big(R_r^{-1}(x-a)\big)\\ &= D^J(r)\chi_S\big(R_r^{-1}(x-a)\big)U_{g^{-1}}^L\psi\big(R_r^{-1}(x-a)\big) = E^L(gS)\psi(x).\end{aligned}$$

The position operators Q_i are now defined by eq. (4) with the spectral measure (18), and consequently satisfy

$$(VQ_k\psi)(x) = x_k(V\psi)(x). \tag{20}$$

Hence, in the momentum space,

$$\begin{aligned}(Q_k\psi)(p) = (V^{-1}x_kV\psi)(p) &= (2\pi)^{-3}\int \exp(-\mathrm{i}px)x_k p_0^{1/2}\mathrm{d}^3x\exp(\mathrm{i}xp')\psi(p')\,\frac{\mathrm{d}^3p'}{p_0'^{1/2}}\\ &= \mathrm{i}\left(\frac{\partial}{\partial p^k} - \frac{p_k}{2p_0^2}\right)\psi(p).\end{aligned} \tag{21}$$

This operator is self-adjoint relative to the scalar product (3).

It follows from eqs. (12) and (21) that the transformation properties of the position operators Q_k relative to Euclidean group $\mathscr{E}^3$ are

$$\begin{aligned}U_{(a,0)}^L Q_k\, U_{(a,0)}^{L^{-1}} &= Q_k - a_k,\\ U_{(0,r)}^L Q_k U_{(0,r)}^{L^{-1}} &= D_{k'k}^1(R_r)Q_{k'}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}[Q_i, Q_k] &= 0,\\ [Q_k, P_j] &= \mathrm{i}\delta_{kj}.\end{aligned}$$

The time derivative of the position operators (21) in the Heisenberg representation is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_k = \mathrm{i}[H, Q_k] = \mathrm{i}[p_0, Q_k] = \frac{p_k}{p_0}, \tag{22}$$

i.e., it represents the operator of the velocity of the particle. We see, therefore, that the operators Q_k given by formula (21) satisfy all the *physical requirements* that could be imposed on position operators.

We now consider in more detail the position operator for a scalar particle. The eigenfunctions of the operators Q_k in the momentum representation localized at a point $x \in R^3$ have in this case the form

$$\psi_x(p) = p_0^{1/2}\exp(-\mathrm{i}px). \tag{23}$$

Indeed,

$$Q_k \psi_x(p) = \mathrm{i}\left(\frac{\partial}{\partial p^k} - \frac{p_k}{2p_0^2}\right)\psi_x(p) = x_k \psi_x(p).$$

Let us also remark that by virtue of eq. (23) the formula (13) can be interpreted as the probability amplitude of finding a scalar particle in a state $\psi(p)$ at the position x at $t = 0$.

Now we perform another 'Fourier' transformation of the eigenfunction (23) of the position operators Q_i. Note that the coordinates obtained by this Fourier transformation are different than the x-coordinate in V-transformation of eq. (13).

$$\begin{aligned}\psi_x(\xi, t = 0) &= (2\pi)^{-3/2}\frac{1}{\sqrt{2}}\int p_0^{1/2}\exp[-\mathrm{i}p(x-\xi)]\frac{\mathrm{d}^3 p}{p_0} \\ &= \mathrm{const}\left(\frac{m}{r}\right)^{5/4} H^{(1)}_{5/4}(\mathrm{i}mr),\end{aligned} \tag{24}$$

where $r = |x-\xi|$ and $H^{(1)}_{5/4}$ denotes the Hankel function of the first kind of order 5/4. The space extension of this function is of the order $1/m$ and for large r it falls off as $\exp(-mr)/r$. Notice, however, that by eq. (13)

$$(V\psi_x)(y) = (2\pi)^{-3/2}\int p_0^{1/2}\exp(-\mathrm{i}p\cdot x)p_0^{1/2}\exp(\mathrm{i}p\cdot y)\frac{\mathrm{d}^3 p}{p_0} = \delta^3(x-y),$$

again as in the non-relativistic case.

The relativistic position operators can also be written in the space determined by this Fourier transform. Indeed, taking the inverse Fourier transform of (21) one obtains

$$(Q_k\psi)(\xi) = \xi_k\psi(\xi) + \frac{1}{8\pi}\int \frac{\exp(-m|\xi-\eta|)}{|\xi-\eta|}\frac{\partial\psi(\eta)}{\partial\eta^k}\mathrm{d}^3\eta. \tag{25}$$

We see that the operators Q_i in this space are represented by non-local operators.

Let us note that position operators Q_k given by eq. (4) are localized at a moment t in the plane determined by the normal vector $n = (1,0,0,0)$; the stability group of this vector is just $\mathscr{E}^3$. This explains why we took $\mathscr{E}^3$ as a covariance group for position operators. Now in case of massless particles $p_\mu p^\mu = 0$, $p^\mu \neq 0$ and the stability subgroup of any point on the cone is $T^4 \circledS \mathscr{E}^2$: the subgroup $\mathscr{E}^2$ acting on M^4 leaves invariant a null hyperplane $\overset{\circ}{H}$ determined by a normal vector $n = (1,0,0,1)$: it is on this null hyperplane alone that we must be able to localize any massless particle (think on a localization of a photon in a photographic plate). Consequently, we must take a subgroup $G_0 = T^3 \circledS \mathscr{E}^2$ instead of $\mathscr{E}^3$ as a covariance group of a position operator of a massless particle. If a representation $U^{0,J}$ of Poincaré group restricted to $T^3 \circledS \mathscr{E}^2$ is a representation induced by a representation L of $\mathscr{E}^2$ then there will exist an imprimitivity system $(E(S), U^{0,J}_{G_0})$, $S \subset T^3 \circledS \mathscr{E}^2/\mathscr{E}^2 \sim R^3$ based on R^3 and by formula (4) we shall have three position operators for photon. Surprisingly enough we have:

PROPOSITION 2. *Let $U^{0,J}$ be an irreducible representation of the Poincaré group, corresponding to a massless particle with spin J. The restriction $U^{0,J}_{T^3 \otimes \mathscr{E}}$ is unitarily equivalent to a representation of $T^3 \otimes \mathscr{E}^2$ induced by a reducible representation of $R^1 \otimes \mathscr{E}^2$.* ▼

The proof can be carried out using Induction-Reduction Theorem 18.2.1 as in proposition 1. The alternative proof was given by Angelopoulos, Bayen and Flato 1975.

Proposition 2 implies that on purely group theoretical basis a massless particle has only two position operator and may be localizable in the plane perpendicular to a direction of motion. This mathematical result coincides with an intuitive idea of a photon hitting a photographic plate and reacting to an ion.

§ 2. The Representations of the Heisenberg Commutation Relations

We consider in this section the problem of representations of Heisenberg (canonical) commutation relation

$$[q_j, p_k] = \mathrm{i}\delta_{jk} I, \quad j, k = 1, 2, \ldots, n. \tag{1}$$

The entity q_j has in non-relativistic quantum mechanics the meaning of the position operator and p_k the meaning of the momentum operator of a particle. Hence, it is of great importance to know the number of non-equivalent, irreducible representations of the algebra (1). The best known representation is the Schrödinger representation

$$\begin{aligned} q_j &: u(x) \to x_j u(x), \\ p_k &: u(x) \to \frac{1}{\mathrm{i}} \frac{\partial u}{\partial x^k} \end{aligned} \tag{2}$$

or, in the global form,

$$\begin{aligned} \exp(\mathrm{i}\beta_j q_j) &: u(x) \to \exp(\mathrm{i}\beta_j x_j) u(x), \\ \exp(\mathrm{i}\alpha_k p_k) &: u(x) \to u(x+\alpha), \end{aligned} \tag{3}$$

which is realized in the Hilbert space $H = L^2(R^n)$. We show that any other representation of the canonical commutation relations (1), integrable to a global representation of the corresponding group is equivalent to the Schrödinger representation. To show this, we first bring the Heisenberg relations (1) into the so-called *Weyl form*.

We perform this using the Baker–Hausdorff formula

$$\exp A \exp B = \exp(A+B+\tfrac{1}{2}[A, B]) = \exp([A, B]) \exp B \exp A, \tag{4}$$

valid for operators A and B whose commutator is a c-number. Because we assume integrability of the representation of (1), this formula holds on the invariant dense set of Nelson–Gårding analytic vectors for the global representation (3) (cf. ch. 11, § 7).

Setting $A = i\alpha_k p_k$ and $B = i\beta_k q_k$ where α_k and β_k are real numbers, we obtain

$$\exp(i\alpha p)\exp(i\beta q) = \exp(i\alpha\beta)\exp(i\beta q)\exp(i\alpha p), \tag{5}$$

where

$$\alpha\beta = \alpha_k\beta_k, \quad k = 1, 2, \ldots, n.$$

The Baker–Hausdorff formula allows us also to find a composition law for the group associated with the Lie algebra (1). Associating with the generators p_k, q_k and I the group parameters α_k β_k and γ_k, respectively, we obtain

$$\begin{aligned}&\exp[i(\alpha p+\beta q+\gamma I)]\exp[i(\alpha' p+\beta' q+\gamma' I)]\\ &\qquad = \exp[i\{(\alpha+\alpha')p+(\beta+\beta')q+(\tfrac{1}{2}(\alpha\beta'-\alpha'\beta)+\gamma+\gamma')I\}].\end{aligned} \tag{6}$$

This gives the following composition law for group elements

$$(\alpha, \beta, \exp(i\gamma))(\alpha', \beta', \exp(i\gamma')) = (\alpha+\alpha', \beta+\beta', \exp[i(\tfrac{1}{2}(\alpha\beta'-\alpha'\beta)+\gamma+\gamma')]). \tag{7}$$

Notice that the Lie algebra L determined by eq. (1) is nilpotent. Indeed, we have

$$L_{(1)} = [L, L] = \{I\}, \quad L_{(2)} = [L_{(1)}, L_{(1)}] = \{0\}.$$

Consequently the global group (7) is also nilpotent. One may easily verify this property on the group level using the definition of nilpotent groups given in 3.5.

Note that the Weyl relations (5) imply the Heisenberg relations (1). In fact, taking derivatives $\partial^2/\partial\alpha_k\partial\beta_j$ on both sides of eq. (5), one recovers eq. (1). The converse statement is, however, not true. The derivation of the Weyl relation (5) was based on the fact that the Schrödinger representation (3) is integrable. If the representation of the Lie algebra (1) is not integrable, then one cannot associate with it a corresponding Weyl formula (5). Consequently, the Heisenberg and the Weyl relations are in fact not equivalent.

Now we show, using the imprimitivity theorem, that every integrable representation of (1) is unitarily equivalent to the Schrödinger representation (3). Set

$$V_\alpha \equiv \exp(i\alpha p), \quad U_\beta \equiv \exp(i\beta q), \quad \langle\alpha, \beta\rangle \equiv \exp(i\alpha\beta). \tag{8}$$

Then, eq. (5) takes the form

$$V_\alpha U_\beta = \langle\alpha, \beta\rangle U_\beta V_\alpha. \tag{9}$$

Here $\langle\alpha, \beta\rangle$ plays the role of the character $\beta(\alpha)$ of an abelian group $G = R^n$. We have, moreover,

$$V_\alpha V_{\alpha'} = V_{\alpha+\alpha'}, \quad U_\beta U_{\beta'} = U_{\beta+\beta'},$$

i.e., the maps $\alpha \to V_\alpha$ and $\beta \to U_\beta$ give representations of the abelian groups isomorphic to R^n. This observation is the starting point of the following theorem.

THEOREM 1. *Let G be a separable, locally compact, abelian group and $\hat{G}$ its dual group of characters. Let $\alpha \to V_\alpha$ and $\beta \to U_\beta$ be unitary representations of G and $\hat{G}$, respectively, in the same Hilbert space H, and satisfy the conditions*

(i) $$V_\alpha U_\beta = \langle\alpha, \beta\rangle U_\beta V_\alpha \quad \textit{for all } \alpha \in G, \ \beta \in \hat{G}. \tag{10}$$

(ii) *The set* $\{V_\alpha, \alpha \in G, U_\beta, \beta \in \hat{G}\}$ *is irreducible.*
Then, there exists a unitary isomorphism $S\colon H \to L^2(G)$, *such that*

$$SV_\alpha S^{-1}\varphi(x) = \varphi(x+a), \quad SU_\beta S^{-1}\varphi(x) = \langle x, \beta\rangle\varphi(x), \quad x \in G. \tag{11}$$

PROOF: By SNAG's theorem we have

$$(U_\beta\varphi, \psi) = \int_G \langle\alpha, \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big). \tag{12}$$

Here the symbol $\mathrm{d}\big(E(\alpha)\varphi, \psi\big)$ means that the character $\langle\alpha, \beta\rangle$ is to be integrated as a function of α with respect to the set function $G \supset A \to \big(E(A)\varphi, \psi\big)$. We replace in eq. (12) U_β by $V_{\alpha'} U_\beta V_{\alpha'}^{-1}$. Then, using the fact that

$$V_{\alpha'} U_\beta V_{\alpha'}^{-1} = \langle\alpha', \beta\rangle U_\beta$$

(by supposition (10)), we conclude that

$$\int_G \langle\alpha, \beta\rangle\langle\alpha', \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big) = \int_G \langle\alpha+\alpha', \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big)$$

$$= \int_G \langle\alpha'', \beta\rangle \mathrm{d}\big(E(\alpha''-\alpha')\varphi, \psi\big) = \int_G \langle\alpha'', \beta\rangle \mathrm{d}\big(V_{\alpha'}E(\alpha'')V_{\alpha'}^{-1}\varphi, \psi\big). \tag{13}$$

The characters separate points of G. Hence, the measures in eq. (13) are equal. This implies

$$E(A\alpha^{-1}) = V_\alpha E(A) V_\alpha^{-1} \tag{14}$$

for all $A \subset G$ and $\alpha \in G$. Thus, $E(A)$ represents a system of imprimitivity based on G.

The supposition (ii) implies that the pair (V, E) is irreducible. In turn, the Imprimitivity Theorem implies that there exists a unitary map S, such that

$$\begin{aligned} SV_\alpha S^{-1} &= V_\alpha^L \quad \text{for all } \alpha \in G, \\ SE(A)S^{-1} &= E^L(A) \quad \text{for all Borel sets } A \subset G, \end{aligned} \tag{15}$$

where V^L is a representation induced by the stability subgroup K and $E^L(A)$ is the canonical set of projections based on G, i.e.,

$$E^L(A)\varphi(\alpha) = \chi_A(\alpha)\varphi(\alpha). \tag{16}$$

Because $K = \{e\}$ and L are irreducible, the representation V^L is the right regular representation, i.e.,

$$(V_{\alpha_0}^L\varphi)(\alpha) = \varphi(\alpha\alpha_0) = \varphi(\alpha+\alpha_0). \tag{17}$$

Finally, eqs. (12) and (16) imply

$$(SU_\beta S^{-1}\varphi)(\alpha) = \int_G \langle\alpha', \beta\rangle \mathrm{d}\big(SE(\alpha')S^{-1}\varphi\big)(\alpha)$$

$$= \int_G \langle\alpha', \beta\rangle \mathrm{d}\big(E^L(\alpha')\varphi\big)(\alpha) = \langle\alpha, \beta\rangle\varphi(\alpha).$$

This concludes the proof of th. 1. ▼

If we put $G = R^n$ in th. 1, then the composition law (10) is the same as the composition law (9) for the Weyl group. Hence, the formula (11) provides an irreducible unitary representation of the Weyl group. It is evident that the generators of U_β have the same form as q_j given by eq. (2), and the generators of V_α have the same form as p_k. Thus every irreducible integrable representation of the canonical commutation relations is equal to the Schrödinger representation.

The set $\{V, U\}$, eq. (10), might be in general reducible. In this case one can show, however, that the carrier Hilbert space H can be decomposed into the orthogonal direct sum $\oplus H_s$ of subspaces, each invariant and irreducible relative to the set $\{V, U\}$ (cf. Mackey 1949, th. 1). Consequently, one concludes that any integrable representation of the canonical commutation relations (1) is unitarily equivalent to at most a countable sum of replicas of the Schrödinger representation. This shows, in fact, the equivalence of the Heisenberg and the Schrödinger formulations of quantum mechanics in the case of integrable representations.

§ 3. Comments and Supplements

A. The construction of the relativistic position operator and the proof of equivalence of the Schrödinger and the Heisenberg formulations in quantum mechanics given in this chapter were based in fact on the Imprimitivity Theorem. This once more demonstrates the importance and the power of this theorem. In fact, the exposition of quantum mechanics—nonrelativistic as well as relativistic—could be based on this theorem.

Historically, the equivalence of the Heisenberg and Schrödinger representations were proved by Lanczos 1925, Schrödinger 1926 and Pauli.

The concept of a relativistic position operator was first introduced in a fundamental paper by Newton and Wigner 1949. The derivation based on the Imprimitivity Theorem was given by Wightman 1962 and Mackey 1963.

The construction of position operators given in 1.B was elaborated by Lunn 1969. Position operators for massless particles were discussed by Bertrand 1972. The new and physically satisfactory theory of position operator for massless particles was presented in the beautiful paper by Angelopoulos, Bayen and Flato 1975.

The problem of representations of the canonical commutation relations was extensively investigated by Stone 1930 and v. Neumann 1931. The derivation of the equivalence of any irreducible representation to the Schrödinger representation based on the Imprimitivity Theorem was given by Mackey 1949. Here we follow essentially the Mackey derivation.

The Weyl group has apparently no direct meaning like the Galilei or the Poincaré group. Hence non-integrable or partially integrable representations of canonical commutation relations 2(1) might be also of some interest. An example of such representation (which is non-equivalent to the Schrödinger representation!) was constructed by Doebner and Melsheimer 1967. The physical meaning of this representation is, however, so far unclear. It would be very interesting to construct an explicit example of partially integrable (with respect to p_j) representation of canonical commutation relations and look for their physical meaning.

B. *Algebraic Definition of Position Operators*

We give now, for completeness two other definitions of position operators, which are frequently considered as more physical than the definition given in § 1. These definitions in general are not equivalent with definition 1.1. Given an irreducible representation of the generators P_μ and $M_{\mu\nu}$ of the Poincaré group Π for a massive particle we wish to define the position operators Q_j in the envelopping field of the Lie algebra of Π.

(1) From the physical assumptions of translational and rotational invariance, we have, as in eqs. 1 (8)–1 (9):

$$[Q^j, P^k] = i\delta^{jk},\qquad [Q^j, J^k] = i\varepsilon^{jkl}Q_l. \tag{1}$$

Further, time translations and pure Lorentz transformations (boosts) give the conditions

$$[Q^j, P^0] = iP^jP_0^{-1},\qquad [Q^j, M^{0k}] = -i\delta^{jk}Q_0 + iQ^jP^kP_0^{-1}, \tag{2}$$

where $Q_0 = t$ is assumed to be a number in the carrier space. Notice that because of the spectral condition P_0^{-1} is well defined.

A form of Q_μ satisfying these requirements is

$$Q_\mu = tP_\mu P_0^{-1} + \frac{1}{m^2}M_{\mu\nu}P^\nu + \frac{1}{m^2}M^{\nu 0}P_\mu P_\nu P_0^{-1}. \tag{3}$$

Indeed, we have $Q_0 = t$, and in the rest frame $(P_0, \mathbf{0})$:

$$\mathring{Q}_i = \frac{1}{m}M_{i0},$$

which is the correct relation. We remark that Q_μ in (3) is not a four-vector.

However, contrary to the condition used in § 1, we have

$$[Q^j, Q^k] = \frac{1}{m^2}i\varepsilon^{jkl}S_l, \tag{5}$$

where S_l is the spin operator. For spinless particles the position operators commutes. But particles with spin cannot be localized better than their Compton wave-lengths. Note that for spin $J = 0$, the position operator (3) in momentum space is

$$Q_j = \mathrm{i}\frac{\partial}{\partial p^j} - \mathrm{i}\frac{p_i}{p_0^2} \tag{6}$$

whereas the Newton–Wigner hermitian position operator is

$$Q_j^{(NW)} = \mathrm{i}\partial/\partial p^j - \tfrac{1}{2}\mathrm{i}p_j/p_0^2. \tag{7}$$

(2) Often another position operator, namely a four-vector X_μ is introduced by the commutation relations

$$[X^\mu, P^\nu] = -\mathrm{i}g^{\mu\nu}. \tag{8}$$

In terms of X^μ and P^ν we can represent

$$M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + S_{\mu\nu}, \tag{9}$$

where $S_{\mu\nu}$ is the spin part of $M_{\mu\nu}$: $S_{\mu\nu}P^\nu = 0$. Clearly, X_0 plays a different role than Q_0 in (3).

If we introduce $D = Q_\mu P^\mu$, then we can easily derive from (9).

$$X_\mu = [\{D, P_\mu\} + \{M_{\mu\nu}, P^\nu\}]/2P^2. \tag{10}$$

In the 11-parameter Lie algebra P_μ, $M_{\mu\nu}$, D these operators have the commutation relations, in addition to (8)

$$\begin{aligned} [M^{\mu\nu}, X^\lambda] &= -\mathrm{i}(g^{\nu\lambda}X^\mu - g^{\mu\lambda}X^\nu), \\ [X^\mu, D] &= -\mathrm{i}X^\mu, \\ [X^\mu, X^\nu] &= -\mathrm{i}\varepsilon^{\mu\nu\lambda\sigma}P_\lambda W_\sigma/P^4, \end{aligned} \tag{11}$$

where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}M^{\nu\lambda}P^\sigma$. This position operator is formally covariant but does not satisfy the physical requirements (2).

§ 4. Exercises

§ 1.1. Show that the imprimitivity system $U_{\mathscr{E}^3}$, $E(\cdot)$ given by eq. 17.1(10) is not proper for the construction of position operators Q_i, $i = 1, 2, 3$.

§ 1.2.* Show that the position operator Q_k, $k = 1, 2, 3$, in the space H of positive energy solutions of the Dirac equation $(\gamma_\mu p^\mu - m)\psi(p) = 0$ has the following form

$$Q_k = \mathrm{i}\frac{\partial}{\partial p_k} + \mathrm{i}\frac{\gamma_k}{2p_0} - \frac{\mathrm{i}(\gamma p)p_k + (\Sigma \times p)_k p_0}{2p_0^2(p_0+m)} - \frac{\mathrm{i}p_k}{p_0^2}. \tag{1}$$

Hint: Use the Induction-Reduction Theorem 18.1.1.

§ 1.3.*** Construct position operators for massless particle with arbitrary spin J.

Hint: Use Induction-Reduction Theorem 18.1.1 for the representation $U^{0,J}$ of Poincaré group and find the base $X = \Pi/K$ of spectral measure $E(S)$, $S \subset X$. Deduce from this what physical quantities of massless particle can be localized.

§ 1.4.*** Construct position operators for tachyons $m^2 < 0$ of arbitrary spin.

Hint: Use the same method as in the previous exercise.

§ 1.5. Show that for position operator (1) of a Dirac particle

$$\frac{\mathrm{d}Q_k}{\mathrm{d}t} = \mathrm{i}[H, Q_k] = \frac{p_k}{p_0}\,\frac{\gamma_0 m + \gamma_0 \gamma p}{p_0}, \tag{2}$$

which is equal to p_k/p_0 in H, whereas

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{i}\frac{\partial}{\partial p^k}\right) = \mathrm{i}\left[H, \mathrm{i}\frac{\partial}{\partial p^k}\right] = \gamma_0 \gamma_k, \tag{3}$$

which by virtue of the fact that $(\gamma_0 \gamma_k)^2 = I$ is equal to the velocity of the light.

§ 1.6.* Show that every Galilei invariant system with mass $m > 0$ is localizable.

§ 1.7.* Show that an elementary Galilean system with $m = 0$ described by an irreducible representation of the Galilei group is not localizable.

§ 1.8. Construct a position operator for a massive particle of arbitrary spin.

Hint: Use Induction-Reduction Theorem 18.1.1.

§ 1.9. Complete position operators Q_k, $k = 1, 2, 3$, given by 1(21) to a covariant position operator $Q = (Q_0, Q_1, Q_2, Q_3)$, $Q_0 = t$. Show that this operator associated with hyperplane H defined by the normal vector $n = (1, 0, 0, 0)$ transform covariantly in the following manner

$$Q_\mu(H) \xrightarrow[(a,\Lambda)]{} Q'_\mu(H') = \Lambda^\nu_\mu Q_\nu(H) + a_\mu$$

where

$$H = (n, \tau), \quad Q_\mu n^\mu = \tau$$

and

$$H' = (n', \tau'), \quad n'_\mu = \Lambda^\nu_\mu n_\nu, \quad \tau' = \tau + n'_\mu a^\mu.$$

§ 1.10.* Show that the operator $Q'_\mu(H')$ is defined by the system of imprimitivity $(E'(\cdot), U'^{m,J})$, where

$$U'^{m,J}_{(a,\Lambda)}\psi(p) = \mathrm{e}^{\mathrm{i}pa} D^J(r')\psi(\Lambda^{-1}p)$$

$r' = \Lambda_\omega^{-1} r \Lambda_\omega$, Λ_ω^{-1} is the Lorentz transformation which takes H into H' and $E'(\cdot)$ is the spectral measure associated with U' as in proposition 1 based on $\mathrm{SO}(3)'\backslash T^3 \otimes \mathrm{SO}(3)' \sim H'$ where $\mathrm{SO}(3)' = \Lambda_\omega^{-1}\mathrm{SO}(3)\Lambda_\omega$. ▼

§ 1.11. Show that the classical Poyting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ and the energy density $U = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2)$ are invariants of $E(2)$.

Hint: Introduce a complex vector $\boldsymbol{E} + \mathrm{i}\boldsymbol{H}$ and show that for $A \in \mathscr{E}^2$, $A = \begin{bmatrix} \alpha & \bar{\alpha} z \\ 0 & \bar{\alpha} \end{bmatrix}$, $|\alpha| = 1$, $z \in C$ one has $\boldsymbol{E} + \mathrm{i}\boldsymbol{H} \xrightarrow{A} \alpha^2(\boldsymbol{E} + \mathrm{i}\boldsymbol{H})$.

§ 2.1.** Construct simplest indecomposable representations of canonical commutation relations (2.1).

Hint: Use the indecomposable representations 6.3.D for an abelian subgroup of the Weyl group and induce them to the whole group.

§ 2.2.*** Find all nonintegrable representations of canonical commutation relations (2.1). Can they have a physical interpretation?

§ 2.3.*** Find all partially integrable representations (with respect to momenta P_j) of canonical commutation relations (2.1). Give a physical interpretation for these representations.

Chapter 21

Group Representations in Relativistic Quantum Theory

In this chapter we discuss a number of selected basic applications of group representations which are at the foundation of many approaches to relativistic quantum theory. We do not have a closed complete relativistic quantum theory in the same level of development as the nonrelativistic quantum mechanics. For this reason the group theoretical framework of relativistic theory plays a basic and guiding role in the establishment of models and theories for relativistic processes.

§ 1. Relativistic Wave Equations and Induced Representations

Wave equations for quantum systems are modelled after the wave equations of classical physics: electromagnetic waves, sound or water waves, etc., however, with a different interpretation of the wave function. In quantum physics the wave function represents a probability amplitude.

Relativistic wave equations provide an effective and practical way of implementing the induced representations of the Poincaré group. The solutions of the wave equations, the wave functions of the system, carries all the information about the spin and momenta of the system provided by the Poincaré group. In addition, the wave equation provides a conserved current density for the quantum system. And via the so-called minimal coupling to the electromagnetic field it also gives a very simple and natural covariant description of the interaction of the quantum system with the external electromagnetic field or radiation. These two last properties of the wave equations go much beyond the theory of induced representations. The real importance of wave equations lie in the covariant description of interactions.

The Klein–Gordon and Dirac equations are the best known examples of relativistic wave equations. But there are infinitely many other possible relativistic wave equations. In fact, we show in this section not only the relation of the wave equations to induced representations, but also give in a unified manner the wave equations corresponding to all induced representations of the Poincaré group.

We go even further and discuss the so-called infinite-component wave equations, which use representations of the Poincaré group induced from infinite-dimensional representations of the homogeneous Lorentz-group, or even of more general groups. These wave equations, we shall see, describe composite quantum systems with internal degrees of freedom.

A. *From Induced Representations to Wave Equations*

We start generally from a group G and its unitary representation U^L induced by a representation $k \to L_k$ of a closed subgroup K of G. We assume for simplicity that the space $X = K\backslash G$ has an invariant measure.

As we know from chs. 8 and 16, the Mackey 'wave functions' $f(g)$, $g \in G$, transform in the following manner

$$[U^L_{g_0} f](g) = f(gg_0), \quad g_0, g \in G, \tag{1}$$

and satisfy the subsidiary condition

$$f(kg) = L_k f(g), \quad k \in K, \tag{2}$$

where K is the closed inducing subgroup of G and $K \to L_k$ is a continuous unitary representation of K. The scalar product in the representation space H of L determines the scalar product for U^L (cf. 16.1(1.3°))

$$(f_1, f_2) = \int_X \mathrm{d}\mu(\dot{g})(f_1, f_2)_H, \quad \dot{g} \in K\backslash G = X, \tag{3}$$

where

$$\mathrm{d}\mu(\dot{g}g) = \mathrm{d}\mu(\dot{g}), \quad \dot{g} \in X,\ g \in G.$$

The subsidiary condition (2) shows that the integrand in (3) depends only on $\dot{g} \in X$. In order not to carry along the subsidiary condition it is convenient to have it automatically built-in into the formalism. One method for this has already been discussed in detail, namely to write eq. (1) on the coset space $X = K\backslash G$ by introducing wave functions over the coset space. These are called the *Wigner-states* in the case of the Poincaré group.

A second method is to write covariant wave functions: Instead of inducing from the representation L of K, we start from a representation $\tilde{L}$ of G containing L as its restriction, reduce it with respect to K and then induce it to get another representation U^L of G. Let $f(g)$ satisfy eqs. (1) and (2) and define

$$h(g) \equiv \tilde{L}_g^{-1} f(g). \tag{4}$$

role k_g

Then it follows that

$$h(kg) = h(g), \quad k \in K,\ g \in G, \tag{5}$$

that is $h(g)$ depends only on cosets $\dot{g} = Kg$:

$$h(g) \equiv \psi(\dot{g}), \quad \dot{g} \in X = K\backslash G. \tag{6}$$

We have then by virtue of eqs. 16.1(14) and (12) ($\dot{g} \equiv x$, $\tilde{L}_g \equiv B_g$)

$$[U^L_{g_0} \psi](x) = \tilde{L}_{g_0} \psi(xg_0) \tag{7}$$

which is a simple 'covariant' transformation law, without subsidiary conditions; $\psi(x)$ is called, in the case of the Poincaré group, the *spinor wave functions.*

The scalar product (3) becomes

$$(\psi_1, \psi_2) = \int \mathrm{d}\mu(x)\,(\tilde{L}_g\psi_1(x), \tilde{L}_g\psi_2(x))_H, \qquad x = \dot{g}. \tag{8}$$

If the restriction L of $\tilde{L}$ to K is unitary, then the induced representation U^L of G is also unitary.

If we wish to obtain the particular representation U^{L^j} only, a complication of this method arises if the restriction of $\tilde{L}_g$ to K contains many representations of K other than L^j. Let $L = \sum \oplus L^j$ be the decomposition of the representation $\tilde{L}$ of G when restricted to K. Then by virtue of th. 16.2.1 the induced representation U^L of G have the form

$$U^L = \sum_j \oplus U^{L^j} \quad \text{and} \quad H^L = \sum_j \oplus H^{L^j}. \tag{9}$$

If we are interested in the specific representation U^{L^j} and the corresponding carrier space H^{L^j} we can eliminate the unwanted representations by imposing the subsidiary condition

$$\pi f(g) = f(g), \quad \text{or} \quad \pi f(e) = f(e), \quad f \in H^{L^j}, \tag{10}$$

where π is the projector for the representation $L^j \subset L$. The equivalence of the above relations follows from the fact that for arbitrary f in H^L, $f(g) = (U_g^L)f(e)$ by virtue of eq. (1). Because, by virtue of eqs. (6) and (4) $f(g) = \tilde{L}_g\psi(x)$ the eq. (10) gives

$$\tilde{L}_g^{-1}\pi\tilde{L}_g\psi(x) = \psi(x). \tag{11}$$

Note that the projector $\tilde{L}_g^{-1}\pi\tilde{L}_g$ depends only on the coset x. Indeed by definition of π we have

$$\tilde{L}_{kg}^{-1}\pi\tilde{L}_{kg} = \tilde{L}_g^{-1}L_k^{-1}\pi L_k\tilde{L}_g = \tilde{L}_g^{-1}\pi\tilde{L}_g.$$

Hence setting $\pi(x) = \tilde{L}_g^{-1}\pi\tilde{L}_g$, $x = \dot{g}$ we obtain

$$\pi(x)\psi(x) = \psi(x). \tag{12}$$

This is the general 'wave equation' for a function in the reducible carrier space H^L, which transforms according to the representations U^{L^j}. If only one representation of K occurs in L then $\pi = I$; hence $\pi(x) = I$ and we have no wave equation.

Note that if one uses left translations in eq. (1), we have to change $\tilde{L}$ into $\tilde{L}^{-1}$ in equations (4), (8) and (11) (cf. eq. 16.1(44) and (46)). Using eq. 16.1(45) we find that the transformation law for the wave function is

$$[\hat{U}_g\psi](x) = \tilde{L}_g^{-1}\psi(g^{-1}x)$$

(cf. eq. 17.2(41)).

B. *General Wave Equations for the Poincaré Group*

We shall now derive a general wave equation for a massive particle of arbitrary spin. In this case the stability subgroup is $K = T^4 \rtimes SU(2)$ (ch. 17, § 2) and it is convenient to realize the quotient space $X = G/K$ as the mass hyperboloid $p^2 = m^2, p = (p_0, p_1, p_2, p_3)$.

The irreducible representations of the Poincaré group were constructed in ch. 17, § 2. The wave function $\psi(p)$ corresponding to a massive particle with a spin j transforms in the spinor basis 17.2 (41) in the following manner

$$U^{mj}_{(a\Lambda)}\psi(p) = \exp[\mathrm{i}pa]D^{(j,0)}(\Lambda)\psi(L_\Lambda^{-1}p). \tag{13}$$

The wave function $\psi(p)$ does not satisfy any wave equation besides the trivial one

$$(p^2-m^2)\psi(p) = 0 \tag{14}$$

which expresses the mass irreducibility condition.

However by virtue of the fact that the parity operator transforms $D^{(j,0)}$ into $D^{(0,j)}$-representation (cf. 17, § 3) the wave function $P\psi$ will transform according to $D^{(0,j)}$-representation and therefore it will not be an element of the carrier space H^{mj}. Hence, if we want to construct a carrier space for a massive spin j particle which admits the parity operator we must start with a reducible representation $D(\Lambda)$ of SL(2, C), e.g. $D(\Lambda) = (D^{(j,0)}+D^{(0,j)})(\Lambda)$, restrict it to SU(2) and then induce to the Poincaré group. The elements $\psi(p)$ in this space will transform in the following manner

$$U^m_{(a,\Lambda)}\psi(p) = \exp(\mathrm{i}pa)D(\Lambda)\psi(L_\Lambda^{-1}p). \tag{15}$$

However we have now twice as many components of the wave function $\psi(p)$ than we need for a description of the particle with the spin j. Hence we have to get rid of the redundant components. We now show that the condition which removes redundant components is just the wave equation.

It follows from the theory developed in ch. 17, § 2 that the wave function $\psi(p)$ defined on the single orbit $p^2 = m^2$ will transform with respect to the irreducible representation U^{mj} if and only if the wave function $\psi(\mathring{p}), \mathring{p} = (m, 0, 0, 0)$, in the rest frame will transform according to the single representation D^j of SU(2). The wave function $\psi(\mathring{p})$ which satisfies this condition is selected by the requirement

$$\pi\psi(\mathring{p}) = \psi(\mathring{p}), \tag{16}$$

where π is the projector for the representation $D^j \subset D$ of SU(2). Using this equation for the function $\varphi(\mathring{p}) = (U_{(0,\Lambda)}\psi)(\mathring{p})$ and utilizing the transformation law (15) we obtain

$$\pi D(\Lambda)\psi(p) = D(\Lambda)\psi(p), \quad p = L_\Lambda^{-1}\mathring{p},$$

or

$$\pi(p)\psi(p) = \psi(p), \tag{17}$$

where

$$\pi(p) = D^{-1}(\Lambda)\pi D(\Lambda). \tag{18}$$

Using the Mackey decomposition $\Lambda = \Lambda_p r$ of SL(2, C) (cf. 17.2 (32)), and the fact that the projection operator π commutes with the transformations $D(r)$, $r \in \mathrm{SU}(2)$, we obtain

$$\pi(p) = D^{-1}(\Lambda_p)\pi D(\Lambda_p). \tag{19}$$

Using again the Mackey decomposition of SL(2, C) we find

$$D^{-1}(\Lambda')\pi(p)D(\Lambda') = D^{-1}(\Lambda_{\Lambda'^{-1}p})\pi D(\Lambda_{\Lambda'^{-1}p}) = \pi(L_{\Lambda'}^{-1}p). \tag{20}$$

Hence $\pi(p)$ is a covariant matrix operator. Because $D(\Lambda)$ is finite-dimensional the rank of tensor coefficients in the powers of p in $\pi(p)$ is bounded. Consequently $\pi(p)$ is a covariant polynomial in p. If one makes the identification $p_\mu = \mathrm{i}\partial_\mu$ then eq. (17) becomes a covariant differential equation of finite order.

Let us note that every covariant relativistic wave equation for a massive particle with an arbitrary spin is a special case of eq. (17). Indeed to each pair $\{D(\Lambda), \pi\}$ there corresponds the unique covariant wave equation for the massive particle with spin j, and conversely to every covariant wave equation there corresponds a unique pair $\{D(\Lambda), \pi\}$. Hence eq. (17) represents the most general covariant relativistic wave equation.

We shall now find the form of the scalar product in the spinor basis. Using eq. (8) and the note at the end of subsec. A we find

$$(\psi_1, \psi_2) = \int (D^{-1}(\Lambda_p)\psi_1(p), D^{-1}(\Lambda_p)\psi_2(p))_H \frac{\mathrm{d}^3 \boldsymbol{p}}{p_0}. \tag{21}$$

We utilized in the last formula the fact that by 17.2(32) $\Lambda = \Lambda_p r$ and $\big(D^{-1}(r) \times \psi_1(p), D^{-1}(r)\psi_2(p)\big)_H = \big(\psi_1(p), \psi_2(p)\big)_H$.

Using now the relation $\Lambda_p \sigma \cdot \mathring{p} \Lambda_p^* = \sigma \cdot p$, $\sigma\mathring{p} = m$, we obtain

$$D^{-1*}(\Lambda_p)D^{-1}(\Lambda_p) = D\big((\Lambda_p\Lambda_p^*)^{-1}\big) = D\left(\left(\frac{\sigma \cdot p}{m}\right)^{-1}\right) = D\left(\frac{\sigma \cdot p}{m}\right). \tag{22}$$

Hence

$$(\psi_1, \psi_2) = \int \left(\psi_1(p), D\left(\frac{\sigma \cdot p}{m}\right)\psi_2(p)\right)_H \frac{\mathrm{d}^3 \boldsymbol{p}}{p_0}. \tag{23}$$

The formula (23) simplifies if the rest frame states $\psi(\mathring{p})$ are eigenstates of the parity operator η for $D(\Lambda)$, e.g.

$$\eta\psi(\mathring{p}) = \psi(\mathring{p}). \tag{24a}$$

Using this formula for $\varphi(\mathring{p}) = (U_{(0,\Lambda)}\psi)(\mathring{p}) = D(\Lambda)\psi(p)$ we obtain

$$\eta D(\Lambda)\psi(p) = D(\Lambda)\psi(p) \tag{24b}$$

for arbitrary $\psi(p)$ in the carrier space and arbitrary $\Lambda \in \mathrm{SL}(2, C)$. Replacing now $D(\Lambda_p)\psi_2(p)$ in eq. (21) by $\eta D(\Lambda_p)\psi_2(p)$ and using the formula 17.3(9) we obtain finally

$$(\psi_1, \psi_2) = \int (\psi_1(p), \eta\psi_2(p))_H \frac{d^3p}{p_0}. \tag{26}$$

This is the most convenient form of the scalar product which will be frequently used.

§ 2. Finite Component Relativistic Wave Equations*

We now derive all conventional relativistic wave equations for a massive particle with spin j by putting the particular forms of the representation $D(\Lambda)$ and projector π in the general wave equation 1 (17)

A. *Dirac Equation*

We want to derive a wave equation for a particle with a positive mass m and spin 1/2. The wave functions $\psi(p)$ from the carrier space $H^{m,1/2}$ of irreducible representations $U^{m,1/2}$ by virtue of eq. 17.2 (41) transform in the spinor basis in the following manner

$$U^{m,1/2}_{(a,\Lambda)}\psi(p) = \exp[ipa]D^{(1/2,0)}(\Lambda)\psi(L_\Lambda^{-1}p). \tag{1}$$

However the parity operator P cannot be defined in $H^{m,1/2}$ because $PD^{(1/2,0)}P^{-1} = D^{(0,1/2)}$ The minimal extension H^m of the space $H^{m,1/2}$ in which the parity operator is defined is the space of wave functions which transform according to $D^{(1/2,0)}\oplus D^{(0,1/2)}$ representation, i.e.

$$U^m_{(a,\Lambda)}\psi(p) = \exp[ipa]\,(D^{(1/2,0)}\oplus D^{(0,1/2)})\,(\Lambda)\psi(L_\Lambda^{-1}p). \tag{2}$$

However, we have now four components of the wave function, instead of two needed for the description of the particle with two spin projections. The projection operator π which removes the unwanted components has in the present case the form

$$\pi = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} = \frac{1}{2}(\gamma_0 + I) \quad \text{with } \gamma_0 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}. \tag{3}$$

Using the formulas 8.9(14)–(15) for the representation $D^{(1/2,0)}\oplus D^{(0,1/2)}$ and γ_μ-matrices we readily verify that

$$(D^{(1/2,0)}\oplus D^{(0,1/2)})^{-1}(\Lambda)\gamma_\mu(D^{(1/2,0)}\oplus D^{(0,1/2)})(\Lambda) = (L_\Lambda^{-1})_\mu{}^\nu\gamma_\nu. \tag{4}$$

Hence by virtue of eq. 1(18) we obtain

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* In order to have a correspondence with notation used in the physical literature we have used exceptionally in this section a scalar product which is antilinear with respect to the first factor and linear with respect to the second one.

$$(D^{(1/2,0)}\oplus D^{(0,1/2)})^{-1}(\Lambda_p)\pi(D^{(1/2,0)}\oplus D^{(0,1/2)})(\Lambda_p) = \frac{1}{2m}(\gamma_\mu p^\mu+m), \tag{5}$$

where $p = L_\Lambda \mathring{p}$, $\mathring{p} = (m, 0, 0, 0)$.

Consequently, using the general wave equation 1(17), we obtain the equation

$$(\gamma_\mu p^\mu - m)\psi(p) = 0 \tag{6}$$

which is the Dirac equation. This example clearly shows that the subsidiary condition for the spin irreducibility represents the wave equation.

Using the explicit form of $D^{(1/2,0)}\oplus D^{(0,1/2)}$-representation given by eqs. 8.9(14) and formula 17.3(9) we conclude that in the case of $D^{(1/2,0)}\oplus D^{(0,1/2)}$-representation the parity operator η must satisfy the following conditions

$$\eta\gamma_k\eta = -\gamma_k \quad \text{and} \quad \eta\gamma_0\eta = \gamma_0. \tag{7}$$

This conditions are satisfied by $\eta = \gamma_0$. Hence the scalar product 1(26) takes the form

$$(\psi_1, \psi_2) = \int \frac{d_3 p}{p_0}(\psi_1(p), \gamma_0\psi_2(p))_H \doteq \int \bar{\psi}_{1\alpha}(p)\psi_\alpha(p)\frac{d^3p}{p_0}, \tag{8}$$

where

$$\bar{\psi}(p) = \psi^*(p)\gamma_0.$$

B. *Proca Equations*

We now want to derive the wave equation for a particle with a positive mass and spin 1. For reasons of relativistic covariance such a particle could be described by the three-component wave function $\tilde{\Phi}(p) = \{\tilde{\Phi}_k(p)\}_{k=1}^3$ (each component corresponding to a spin projection) which transforms in the following manner (cf. eq. 17.2(41))

$$U^{m,1}_{(a,\Lambda)}\tilde{\Phi}(p) = \exp(ipa)D^{(1,0)}(\Lambda)\tilde{\Phi}(L_\Lambda^{-1}p). \tag{9}$$

However the representation $D^{(1,0)}$ does not admit the parity operator p in the carrier space $H^{m,1}$. Hence we have either to double the space and consider the representation $D^{(1,0)}\oplus D^{(0,1)}$, or to start with the four-vector function $\Phi(p) = \{\Phi_\mu(p)\}_{\mu=0}^3$ which transforms with respect to the representation $D^{(1/2,1/2)}$. Because $D^{(1/2,1/2)}|_{SU(2)} \simeq D^1+D^0$ we have in the latter case in addition to the spin 1 particle, another scalar particle. Because the $D^{(1/2,1/2)}$-representation gives the wave function with lesser number of components than $D^{(1,0)}\oplus D^{(0,1)}$ we shall use it for the description of spin 1 particles. The projector π onto the three-vector space can be written in the form

$$\pi = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \tfrac{1}{2}[\delta_{\mu\nu} - g_{\mu\nu}]. \tag{10}$$

The $D^{(1/2,1/2)}$-representation is the regular four-dimensional representation L of the Lorentz group given by eq. 17.2(2). Thus by virtue of eqs. (10) we obtain

$$\pi(p) = (D^{(1/2,\,1/2)})^{-1}(\Lambda)\,\pi D^{(1/2,\,1/2)}(\Lambda) = \frac{1}{2}\left[\frac{p_\mu p_\nu}{m^2} - g_{\mu\nu}\right].$$

Hence using eq. 1(17) we obtain

$$\frac{1}{2}\left(\frac{p_\nu p_\mu}{m^2} - g_{\mu\nu}\right)\Phi^\mu(p) = \Phi_\nu(p).$$

Multiplying both sides by p^ν we have

$$p^\nu \Phi_\nu(p) = 0 \tag{11}$$

which is the Proca equation. Clearly, since $p^2 = m^2$ each component $\Phi_\mu(p)$ satisfies also the Klein–Gordon equation

$$(p^2 - m^2)\Phi_\mu(p) = 0. \tag{12}$$

Because $D^{(1/2,1/2)}(\Lambda) = L_\Lambda$ the parity operator η for $D^{(1/2,1/2)}(\Lambda)$ satisfying eq. 17.3(9) is given by virtue of 17.2(5) by the metric tensor $g = ||g_{\mu\nu}||$. Hence by virtue of eq. 1(26) the scalar product has the form

$$(\Phi, \Phi) = \int \Phi^*(p) g \Phi(p) \frac{d^3p}{p_0} = \int \Phi_\mu^*(p)\Phi^\mu(p)\frac{d^3p}{p_0}. \tag{13}$$

The Proca equations (11) and (12) in coordinate space have the following form

$$\partial^\mu \Phi_\mu(x) = 0 \quad \text{and} \quad (\Box - m^2)\Phi_\mu(x) = 0. \tag{14}$$

These equations can be cast into the form of a set of first order equations. Indeed, setting

$$B_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu \tag{15}$$

we obtain

$$\partial^\mu B_{\mu\nu} - m^2 \Phi_\nu = 0. \tag{16}$$

One readily verifies by differentiation that original Proca equations (11) and (12) are equivalent to eqs. (15) and (16).

C. *Massive Tensor Fields Equations*

We start with the massive spin-2 field. For the description of this field we may use either $D^{(1,1)}$-representation or $D^{(2,0)} \oplus D^{(0,2)}$. We know that $D^{(1,1)}$-representation may be realized in the space of traceless symmetric tensor $\Phi_{\mu_1\mu_2}$ of order two. Because $D^{(1,1)}|_{SU(2)} \simeq D^2 \oplus D^1 \oplus D^0$ we have to cut down unwanted spin-1 and spin-0 components. Following the vector case we take the projector π in the form

$$\pi = \bigotimes_{r=1}^{2} \tfrac{1}{2}||\overset{r}{\delta} - \overset{r}{g}||. \tag{17}$$

Because the tensor $\Phi_{\mu_1\mu_2}$ transforms according to $L\otimes L$ representation, the projector $\pi(p)$ by virtue of 1(18) has the form

$$\pi(p) = \bigotimes_{r=1}^{2} \tfrac{1}{2}\left\| g^{\mu_r}{}_{\nu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right\|. \tag{18}$$

The wave function $\Phi_{\mu_1\mu_2}(p)$ in momentum space, by virtue of eq. 1(17) satisfies the condition

$$\pi(p)\Phi = \Phi. \tag{19}$$

Multiplying both sides by p^μ we obtain

$$p^\mu \Phi_{\mu_1\mu_2}(p) = 0. \tag{20}$$

Solving this equation with respect to $\Phi_{0,\mu}$ we obtain

$$\Phi_{0,\mu}(p) = \frac{p_k \Phi_{k\mu}}{p_0}. \tag{21}$$

Hence the wave equations (20) allows us to express the component $\Phi_{0,0}(p)$ (spin-0 particle) and the components $\Phi_{0,k}(p)$ (spin-1 particle) in terms of five independent components $\Phi_{k,l}(p)$ corresponding to a spin-2 particle. Clearly by virtue of the mass condition we have also

$$(p^2 - m^2)\Phi_{\mu_1,\mu_2}(p) = 0. \tag{22}$$

The procedure of this example can be applied directly to the representations $D^{(j,j)}$ of symmetric traceless tensors $\Phi_{\mu_1,\dots,\mu_{2j}}$. The resulting wave equations have the form

$$p^\mu \Phi_{\mu_1,\mu_2,\dots,\mu_{2j}}(p) = 0, \quad (p^2 - m^2)\Phi_{\mu_1,\mu_2,\dots,\mu_{2j}} = 0. \tag{23}$$

D. *Rarita–Schwinger Equations*

The example of the Dirac and tensor particles suggest to use for the description of massive particles with arbitrary half-integer spin the tensor product of the Dirac and the tensor representations of SL(2, C), i.e.,

$$D = (D^{(1/2,0)} \oplus D^{(0,1/2)}) \otimes D^{(j,j)}. \tag{24}$$

In this case the wave function $\psi(p)$ carries both a spinor index and symmetric traceless tensor indices: $\psi_{\alpha;\mu_1,\dots,\mu_{2j}}(p)$. The projector π onto the highest spin $2j+\frac{1}{2}$ of the representation (24) is the tensor product of the Dirac (3) and the symmetric tensor (18) projectors, i.e.

$$\pi = \tfrac{1}{2}(\gamma_0 + I)^2 \bigotimes_{r=1}^{2j} \tfrac{1}{2}(\overset{r}{I} - \overset{r}{g}). \tag{25}$$

Hence by virtue of eq. 1 (18), we obtain

$$\pi(p) = \frac{1}{2m}(\gamma p + m) \bigotimes_{r=1}^{2j} \left\| g^{\mu_r}{}_{\nu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right\|. \tag{26}$$

Multiplying both sides of eq. (26) by $(\gamma p - m)$ and p^μ respectively we obtain

$$(\gamma p - m)\psi_{\mu_1, \dots, \mu_{2j}}(p) = 0, \tag{27}$$

$$p^\mu \psi_{\mu, \mu_2, \dots, \mu_{2j}}(p) = 0. \tag{28}$$

We have also

$$(p^2 - m^2)\psi_{\mu_1, \dots, \mu_{2j}}\, p) = 0. \tag{29}$$

Eqs. (27)–(29) are called *Rarita–Schwinger equations*. We leave as an exercise for the reader to verify that the spin-tensor wave function $\psi_{\alpha;\mu_1, \dots, \mu_{2j}}(p)$ satisfying eqs. (27)–(29) has $4j+1$ independent components.

E. *Bargmann–Wigner Equations*

We now want to derive the wave equation for a massive particle with arbitrary integer spin j. Clearly we can construct a particle with spin j from the tensor product of Dirac particles: the corresponding representation $D(\Lambda)$ of SL(2, C) has the form

$$D = \bigotimes_{r=1}^{j} (\overset{r}{D}{}^{(1/2,0)} \oplus \overset{r}{D}{}^{(0,\,1/2)}). \tag{30}$$

By virtue of eq. (3) the projector onto the highest spin has the form

$$\pi = \bigotimes_{r=1}^{j} \tfrac{1}{2}(\overset{r}{\gamma}_0 + \overset{r}{I}). \tag{31}$$

Using eqs. 1(18) and (5) we find that the projector $\pi(p)$ has now the form

$$\pi(p) = \bigotimes_{r=1}^{2j} \frac{1}{2m}(\overset{r}{\gamma} p + m). \tag{32}$$

Multiplying the equation 1(17), i.e.,

$$\pi(p)\psi(p) = \psi(p) \tag{33}$$

by $\overset{i}{\gamma} p - m$, $i = 1, 2, \dots, j$, and using the mass condition $p^2 = m^2$ one obtains a series of spinor equations

$$(\overset{i}{\gamma} p - m)_{\alpha_i \beta_i} \psi_{\beta_1, \dots, \beta_i, \dots, \beta_{2j}}(p) = 0, \quad i = 1, 2, \dots, 2j. \tag{34}$$

These equations are the Bargmann–Wigner wave equations. They represent a direct generalization of the Dirac equation to particles with arbitrary integer spin.

F. 2(2j+1)-*Component Wave Equations*

In order to describe a massive particle with an arbitrary spin one may also use the representation $D = D^{(j,0)} \oplus D^{(0,j)}$ which is another direct generalization of the Dirac representation. We take the projector π in the form

$$\pi = \tfrac{1}{2}(\eta + I), \tag{35}$$

where similarly as in the Dirac case the operator η is the parity operator for the representation D, $\eta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $\eta D^{(j,0)}\eta = D^{(0,j)}$. Using eqs. 1(17), 1(18) and (22), we obtain the following wave equation

$$\begin{aligned} \psi(p) &= \pi(p)\psi(p) = \tfrac{1}{2}D^{-1}(\Lambda)(\eta+I)D(\Lambda)\psi(p) \\ &= \tfrac{1}{2}(\eta D^*(\Lambda)D(\Lambda)+I)\psi(p) = \tfrac{1}{2}\left[\eta D\left(\frac{\sigma p}{m}\right)+I\right]\psi(p) \\ &= \frac{1}{2}\begin{bmatrix} I & D^{(j,0)}\left(\dfrac{\sigma p}{m}\right) \\ D^{(0,j)}\left(\dfrac{\sigma p}{m}\right) & I \end{bmatrix}\psi(p). \end{aligned} \tag{36}$$

The spinor $\psi(p)$ has the form

$$\psi(p) = \begin{bmatrix} \psi_{\alpha_1,\dots,\alpha_j}(p) \\ \psi_{\dot\beta_1,\dots,\dot\beta_j}(p) \end{bmatrix},$$

where the first row transforms according to $D^{(j,0)}$ representation and the second row according to $D^{(0,j)}$ representation. By convention the components of the spinor transforming according to $D^{(0,j)}$ representation are denoted by dotted indices. Using the equality

$$D^{(j,0)}\left(\frac{\sigma p}{m}\right) = \left(\frac{\sigma p}{m}\right)\otimes\dots\otimes\left(\frac{\sigma p}{m}\right) \quad (2j\text{-times}) \tag{37}$$

and

$$D^{(0,j)}\left(\frac{\sigma p}{m}\right) = \left(\frac{\tilde\sigma p}{m}\right)\otimes\dots\otimes\left(\frac{\tilde\sigma p}{m}\right) \quad (2j\text{-times}) \tag{38}$$

we can write eq. (36) in the spinorial form

$$\begin{aligned} (\sigma\cdot p)_{\alpha_1\dot\beta_1}(\sigma\cdot p)_{\alpha_2\dot\beta_2}\cdots(\sigma p)_{\alpha_j\dot\beta_j}\psi^{\dot\beta_1\dots\dot\beta_j} &= m^{2j}\psi_{\alpha_1\alpha_2\dots\alpha_j}, \\ (\tilde\sigma\cdot p)^{\dot\beta_1\alpha_1}(\tilde\sigma\cdot p)^{\dot\beta_2\alpha_2}\dots(\tilde\sigma p)^{\dot\beta_j\alpha_j}\psi_{\alpha_1,\dots\alpha_j} &= m^{2j}\psi^{\dot\beta_1\dots\dot\beta_j}. \end{aligned} \tag{39}$$

Projection operators of this type have been discussed by Joos 1962, Barut, Muzinich and Williams 1963, and Weinberg 1964.

G. *Wave Equations for Massless Particles*

If we use wave functions for a massless particle which transform covariantly under representations $D(\Lambda)$ of the Lorentz group according to eq. (15) we must project it onto the irreducible representations of the subgroup $\tilde{E}_2$ of SL(2, C)

which is the little group for massless particles. This different little group, as compared to massive particles, which is non-compact and non-semisimple has important consequences for the physics of massless particles. The finite dimensional representations of SL(2, C) reduced with respect to the subgroup $\tilde{E}(2)$ are indecomposable, hence nonunitary representations of the latter, unless it is one-dimensional. The structure of $\tilde{E}^2$ is $\tilde{E}^2 \simeq T^2 \rtimes U(1)$ and the restriction of the representation D of SL(2, C) to $\tilde{E}(2)$ is of the form $D(T^2)\exp(ij\Phi)$ where $D(T^2)$ is in general indecomposable. In ch. 17, § 2.C, we have discussed the orbits of the representations of $\tilde{E}_2$. Thus, for a wave equation with a positive scalar product we must make sure that the representations $D(T^2)$ of T^2 are trivial, i.e.

$$D(T^2)\psi(\mathring{p}) = \psi(\mathring{p}), \quad \mathring{p} = (1, 0, 0, 1).$$

Let the generators of SL(2, C) be J_k and N_k, $k = 1, 2, 3$, then $J_1 - N_2$ and $J_2 + N_1$ are the commuting generators of T_2. Then for the trivial representation of T_2 the wave function must satisfy

$$(J_1 - N_2)\psi(\mathring{p}) = 0, \quad \text{and} \quad (J_2 + N_1)\psi(\mathring{p}) = 0. \tag{40}$$

Take the representation $D^{(j_1, j_2)}(A)$ of SL(2, C) in the SU(2)×SU(2)-basis with $\boldsymbol{J}_1 = \boldsymbol{J} + \mathrm{i}\boldsymbol{N}$, $\boldsymbol{J}_2 = \boldsymbol{J} - \mathrm{i}\boldsymbol{N}$. The elementary calculation shows that equations (40) imply that $\psi(\mathring{p})$ is a highest weight vector for $\boldsymbol{J}_1$ and a lowest weight vector for $\boldsymbol{J}_2$, i.e.

$$J_1^{(+)}\psi(\mathring{p}) = 0, \quad J_2^{(-)}\psi(\mathring{p}) = 0, \tag{41}$$

where

$$J_k^{(\pm)} = (J_k)_1 + \mathrm{i}(J_k)_2, \quad k = 1, 2$$

or,

$$(J_1)_3\psi(\mathring{p}) = j_1\psi(\mathring{p}) \quad \text{and} \quad (J_2)_3\psi(\mathring{p}) = -j_2\psi(\mathring{p}). \tag{42}$$

In an arbitrary Lorentz frame, these equations take the form

$$(\boldsymbol{J}\cdot\boldsymbol{p})\psi(p) = p_0(j_1 - j_2)\psi(p) \tag{43a}$$

or, equivalently,

$$(\boldsymbol{N}\cdot\boldsymbol{p})\psi(p) = \mathrm{i}p_0(j_1 + j_2)\psi(p). \tag{43b}$$

The wave equations (43) can also be written as

$$W_0\psi(p) = p_0(j_1 - j_2)\psi(p), \tag{44}$$

where $W_\mu = \varepsilon_{\mu\lambda\sigma\nu}M^{\lambda\sigma}P^\nu$ is the spin operator. Thus eq. (44) indicates that $\psi(p)$ has only one component, namely, with helicity $\lambda = j_1 - j_2$, as we know from the representation theory generally. Parity invariance necessitates again the doubling of the space, so that massless particles, when parity is defined, have two states of polarization. The scalar product 1(23) reduces in this case to

$$(\psi_1, \psi_2) = \int \frac{\mathrm{d}^3 p}{p_0} \psi_1^*(p) p_0^{-2(j_1+j_2)} \psi_2(p), \quad p_0 = |\boldsymbol{p}|. \tag{45}$$

EXAMPLE 1. $D(\Lambda) = D^{(1/2,0)}(\Lambda)$. Then $\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma}$, hence either one of eq. (43) gives

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi(p) = p_0\psi(p) \tag{46}$$

which is the Weyl-equation for the neutrino.

EXAMPLE 2. Take $D(\Lambda) = (D^{(1,0)}\oplus D^{(0,1)})(\Lambda)$. Then $\boldsymbol{\psi} = \boldsymbol{\psi}_1\oplus\boldsymbol{\psi}_2$, $\boldsymbol{J}^{(1,0)} = \boldsymbol{J}^{(0,1)}$ with $(J_a^{(1,0)})_{bc} = \mathrm{i}\varepsilon_{abc}$; hence $(\boldsymbol{J}^{(1,0)}\boldsymbol{p})_{bc} = \mathrm{i}\varepsilon_{abc}p_a = (\boldsymbol{J}^{(0,1)}\boldsymbol{p})_{bc}$ and from eq. (43a) we get

$$\boldsymbol{p}\times\boldsymbol{\psi}_1(p) = -\mathrm{i}\omega\boldsymbol{\psi}_1(p), \quad \boldsymbol{p}\times\boldsymbol{\psi}_2(p) = \mathrm{i}\omega\boldsymbol{\psi}_2(p).$$

Setting $\boldsymbol{\psi}_1 = \boldsymbol{B}+\mathrm{i}\boldsymbol{E}$ and $\boldsymbol{\psi}_2 = \boldsymbol{B}-\mathrm{i}\boldsymbol{E}$ we obtain finally the following equations in momentum space

$$\begin{aligned} \boldsymbol{p}\times\boldsymbol{E} &= \omega\boldsymbol{B}, & \boldsymbol{p}\cdot\boldsymbol{B} &= 0, \\ \boldsymbol{p}\cdot\boldsymbol{E} &= 0, & \boldsymbol{p}\times\boldsymbol{B} &= -\omega\boldsymbol{E} \end{aligned} \tag{47}$$

which are the free Maxwell equations.

H. *General Remarks*

(i) It is remarkable that all existing finite-dimensional relativistic wave equations are special cases of the general wave equation 1(17)

$$\pi(p)\psi(p) = \psi(p)$$

derived on the basis of the theory of induced representation. We have obtained particular wave equations by taking specific representations $D(\Lambda)$ of SL(2, C) and calculating the corresponding projection operators $\pi(p)$. These results show again the effectiveness and the elegance of the theory of induced representations.

(ii) In general the wave equation 1(17) represents the irreducibility condition for spin. The Klein–Gordon equation

$$(p^2-m^2)\psi(p) = 0 \tag{48}$$

satisfied by all wave functions represents the irreducibility condition for mass. In the case of the Dirac equation the mass irreducibility follows from the spin irreducibility. Indeed multiplying the Dirac equation

$$(\gamma_\mu p^\mu - m)\psi(p) = 0$$

by the operator $(\gamma_\mu p^\mu + m)$ we obtain eq. (48). However in the case of the Proca equation one cannot derive the mass irreducibility (48) from the spin irreducibility condition $p^\nu\Phi_\nu(p) = 0$. One can write however the wave equation in a form from which the mass and spin irreducibility follow: for instance the corresponding equation for a massive spin 1 particle has the form

$$(m^2 g_\nu{}^\mu + p_\nu p^\mu)\Phi_\mu(p) = p^2\Phi_\nu(p);$$

conversely one can write the Dirac equation in the form

$$(\gamma_\mu p^\mu - (p^2)^{\frac{1}{2}})\psi(p) = 0$$

from which the mass irreducibility does not follow. These two examples illustrate the fact that the mass and the spin irreducibility conditions are on the same footing and it is only a question of convenience if we represent them by a single or by two separate equations.

§ 3. Infinite Component Wave Equations

A. *Gel'fand–Yaglom Equations*

We have shown in sec. 1 that any representation of the Lorentz group whose restriction to SU(2) was reducible could be used for a construction of manifestly covariant wave equation. In sec. 2 we used finite-dimensional reducible representations of SL(2, C) to produce conventional finite component relativistic wave equations. One observes however in experiments that the elementary particles and resonances may be grouped into possible infinite families of particles. Fig. 1 shows an example of mass-spin relations for a multiplet of mesons (so-called ϱ and π Regge trajectories).

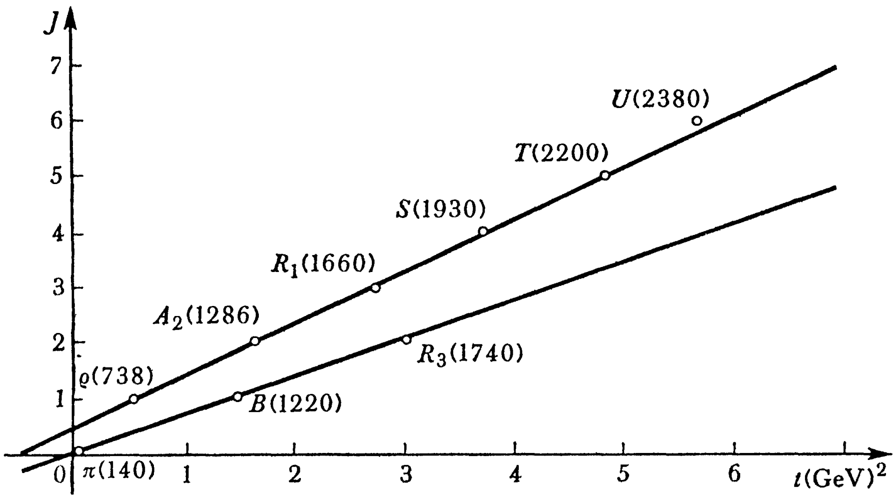


Fig. 1

It is therefore a highly attractive idea to consider infinite-component wave equations describing the properties of a whole family of particles. The simplest equation of this type would be a generalization of the Dirac equation

$$(\Gamma_\mu p^\mu - \varkappa)\psi(p) = 0, \tag{1}$$

where Γ_μ is a vector operator in the carrier space and $\varkappa$ is a scalar. However there arises the problem if and when in any infinite-dimensional space a vector operator exists which must satisfy the covariance condition

$$U_g^{-1}\Gamma_\mu U_g = \Lambda_\nu^{\ \mu}\Gamma_\nu \tag{2}$$

exists. This problem was studied by Gel'fand and Yaglom who found

THEOREM 1. *Let the irreducible representation of* SL(2, C) *be labelled by a pair of numbers* $[j_0, j_1]$, *where* j_0 *is the lowest spin in the representation and takes in-*

tegral or half-odd-integral values and j_1 *is an arbitrary complex number* (cf. ch. 19). *A four-vector operator* Γ^μ *exists in the direct sum* $H = \bigoplus_s H^{[j_0^s, j_1^s]}$ *of irreducible carrier spaces if for every irreducible component* $H^{[j_0, j_1]}$ *in H there exists an irreducible component* $H^{[j_0', j_1']}$ *in H whose invariant numbers are related to each other by*

$$\begin{aligned}[j_0', j_1'] &= [j_0+1, j_1] \\ &= [j_0-1, j_1] \\ &= [j_0, j_1+1] \\ &= [j_0, j_1-1]. \blacktriangledown \end{aligned} \tag{3}$$

PROOF: By virtue of eq. (2) we obtain

$$[\Gamma_\mu, M_{\lambda\varrho}] = \mathrm{i}(g_{\mu\lambda}\Gamma_\varrho - g_{\mu\varrho}\Gamma_\lambda), \tag{4}$$

where $M_{\lambda\varrho}$ are generators of SL(2, C). Setting $\boldsymbol{J} = (M_{32}, M_{13}, M_{21})$ and $\boldsymbol{N} = (M_{01}, M_{02}, M_{03})$, we obtain in particular

$$\mathrm{i}\Gamma_k = [\Gamma_0, N_k], \tag{5}$$

$$[\Gamma_0, J_k] = 0, \quad [\Gamma_3, N_3] = \mathrm{i}\Gamma_0 = -\mathrm{i}[[\Gamma_0, N_3], N_3]. \tag{6}$$

By virtue of eq. (5) it is sufficient to find Γ_0 in order to obtain $\{\Gamma_\mu\}_{\mu \neq 0}^3$. Hence it is sufficient to verify when the operator Γ_0 which is determined by eq. (6) exists. Let H be a reducible carrier space of representation $g \to U_g$ of SL(2, C) and let $\sum_\tau \oplus H^\tau$ be its decomposition onto irreducible subspaces H^τ, $\tau \equiv [j_0, j_1]$. Let $|\tau, JM\rangle$ be the canonical basis in H^τ. Let $[c_{JM, J'M'}^{\tau\tau'}]$ be the matrix elements of Γ_0 in this basis in the carrier space H. Then by virtue of eq. (6) we have

$$c_{JM,J'M'}^{\tau\tau'} = c_J^{\tau\tau'} \delta_{JJ'} \delta_{MM'}. \tag{7}$$

The action of the generator N_3 on the elements of the canonical basis is given in exercise 19.7.3.4. Taking now the matrix element of the equation $\Gamma_0 = [[\Gamma_0, N_3], N_3]$ in the canonical basis between basis elements $|\tau JM\rangle$ and $|\tau(J \pm 1)M\rangle$ we obtain six linear equations for three unknowns $c_J^{\tau\tau'}$, $c_{J-1}^{\tau\tau'}$ and $c_{J+1}^{\tau\tau'}$. We leave as an exercise for the reader to write down explicitly these equations. Solving the first three equations with respect to these unknowns and inserting the obtained expressions to the remaining three equations we readily verify that $c_J^{\tau\tau'}$ can be different from zero only then if $\tau(j_0 j_1)$ and $\tau'(j_1' j_0')$ are such that

$$[j_0', j_1'] = [j_0 \pm 1, j_1] \tag{8}$$

or

$$[j_0', j_1'] = [j_0, j_1 \pm 1]. \blacktriangledown \tag{8'}$$

Remark 1: Let us note that the condition (3) is satisfied also by finite dimensional representations. Indeed using the correspondence between indices $[j_0 j_1]$ and

(J_1, J_2) which characterize the finite-dimensional representation $D^{(J_1,J_2)}$ given by the formula

$$J_1 = \frac{j_0+j_1-1}{2}, \quad J_2 = \frac{j_1-j_0-1}{2}$$

we find that for instance the direct sums

$$D(\Lambda) = D^{(0,0)} \oplus D^{(1/2,1/2)},$$
$$D(\Lambda) = D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)}$$

satisfy condition (3). Hence there are in these cases finite-dimensional Gel'fand–Yaglom equation of the type (1).

Remark 2: The eigenvalues of Casimir operators $C_2 = \boldsymbol{J}^2 - \boldsymbol{N}^2$ and $C_2' = \boldsymbol{JN}$ for representation $[j_0, j_1]$ are $j_0^2 + j_1^2 - 1$ and $2_{ij_0j_1}$ (see exercise 19.7.3.1). The action of the parity operator is: $\boldsymbol{P}$: $\boldsymbol{J} \to \boldsymbol{J}$ and $N \to -N$. Hence the parity transform of $[j_0 j_1]$ is $[j_0, -j_1]$. Consequently by eq. (3) Γ_μ exists on the direct sum of spaces of

(a) $\qquad [0, j_1]$ with $[1, j_1]$, $[0, j_1+1]$, $[0, j_1-1]$ $\qquad$ (9)

and

$[j_0, 0]$ with $[j_0+1, 0]$, $[j_0-1, 0]$, $[j_0, 1]$, $[j_0, -1]$, $\qquad$ (10)

(b) $\qquad [j_0, j_1]$ $(j_0 \neq 0,\ j_1 \neq 0)$ with the set $[j_0', j_1']$, $\qquad$ (11)

as in eq. (3) and with the same set corresponding to $[j_0', -j_1']$.

B. *Majorana Wave Equation*

The Gel'fand–Yaglom Theorem shows that in general one needs at least a pair of irreducible representations in order to be able to determine a vector operator Γ_μ in the carrier space H. However if we take the unitary irreducible representation of SL(2, C) in the form $[j_0, j_1] = [0, \frac{1}{2}]$ then by virtue of th. 1, $[j_0', j_1'] = [0, -\frac{1}{2}]$ is the second representation which together with $[0, \frac{1}{2}]$ defines the operator Γ_μ. However $[0, -\frac{1}{2}]$ is equivalent to $[0, \frac{1}{2}]$ by virtue of exercise 19.7.3.2. Hence, in this case the vector operator can be defined in the carrier space $H^{[0,1/2]}$ of the irreducible representation $[0, \frac{1}{2}]$. A similar situation holds for the representation $[\frac{1}{2}, 0]$. The corresponding wave equation

$$(\Gamma_\mu p^\mu - \varkappa)\psi(p) = 0 \tag{12}$$

associated with $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 0]$ representation is called the Majorana equation. These equations were introduced by Majorana in 1932 as a possibility for avoiding 'negative energy' states in Dirac theory, which caused a serious embarrassment at that time.

We find now the spectrum of the Majorana equation associated with $[\frac{1}{2}, 0]$ representation. Assuming that p is a time-like momentum and going to the rest frame $\mathring{p} = (m, 0, 0, 0)$ we obtain

$$(\Gamma_0 m - \varkappa)\psi(\mathring{p}) = 0. \tag{13}$$

To get the spectrum of Γ_0 we take a special realization of $[\frac{1}{2}, 0]$ representation. Let a_i and a_i^*, $i = 1, 2$, be the creation and annihilation operators. Then the vectors $|[\frac{1}{2}, 0]; JM\rangle$ in the space $H^{[1/2, 0]}$ can be realized by means of the formula

$$|[\tfrac{1}{2}, 0];\ JM\rangle = N a_1^{*J+M} a_2^{*J-M}|0\rangle, \tag{14}$$

where N is a normalization factor. The generators $\boldsymbol{J}$ and $\boldsymbol{N}$ of SL(2, C) have in the realization (3) the following form

$$\begin{aligned} \boldsymbol{J} &= \tfrac{1}{2} a^* \boldsymbol{\sigma} a, \\ \boldsymbol{N} &= \frac{i}{4}(a^* \boldsymbol{\sigma} C a^* + a C \boldsymbol{\sigma} a), \end{aligned} \tag{15}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and C is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. By virtue of eq. (6) the operator Γ_0 must commute with $\boldsymbol{J}$ and satisfy the equality $[[\Gamma_0, N_3], N_3] = -\Gamma_0$. One readily verifies that a second order operator in a and a^* which satisfies these conditions has the form of the particle number operator

$$\Gamma_0 = \tfrac{1}{2}(a^* a + 1). \tag{16}$$

Using the formula (5) one obtains

$$\boldsymbol{\Gamma} = -\frac{i}{4}(a^* \boldsymbol{\sigma} C a - a C \boldsymbol{\sigma} a). \tag{17}$$

The action of Γ_0 on the states (14) gives

$$\Gamma_0 |[\tfrac{1}{2}, 0];\ JM\rangle = (J + \tfrac{1}{2}) |[\tfrac{1}{2}, 0];\ JM\rangle. \tag{18}$$

Using then eq. (13) we obtain the mass formula for the Majorana equation

$$m_J = \frac{\varkappa}{J + \frac{1}{2}}, \quad J = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \tag{19}$$

One obtains the same mass formula for Majorana equation associated with $[0, \frac{1}{2}]$ representation (with $J = 0, 1, 2, \ldots$, in this case).

It is interesting that the set of operators $\boldsymbol{J}$, $\boldsymbol{N}$ and Γ_μ closes to the Lie algebra so(2, 3). Consequently the carrier spaces $H^{[1/2, 0]}$ or $H^{[0, 1/2]}$ are at the same time the carrier spaces of irreducible representations of SO(2, 3).

The Majorana equations (12) have also solutions for space-like momenta $p^2 = m^2 < 0$. In this case in the rest frame $\mathring{p} = (0, 0, 0, m)$ the resulting equation has the form

$$(\Gamma_3 m - \varkappa)\psi(\mathring{p}) = 0. \tag{20}$$

Diagonalizing now Γ_3 we find a continuous mass spectrum in this case. This follows also directly from the observation that Γ_3 is a generator of a noncompact subgroup of SO(3, 2) and the fact that all such generators have continuous spectra.

Minimal Coupling

The importance of linear equations of the type of Dirac equation, 2(6), or Majorana eq. (12) lies in the fact that the behavior of the particles in an external electromagnetic field with potential $A_\mu(x)$ is described by the substitution

$$P_\mu \to P_\mu - eA_\mu(x), \tag{21}$$

where e is the electric charge. Hence we have the equation

$$(\Gamma_\mu p^\mu - e\Gamma_\mu A^\mu(x) - \varkappa)\psi = 0. \tag{22}$$

Compared with the free particle equation the second term $e\Gamma_\mu A^\mu(x)$ appears as the interaction term. This process obtained by the rule (21) is called the *minimal coupling*. Thus the operator $e\Gamma_\mu$ is the current operator of the quantum system. The potentials $A_\mu(x)$ themselves are in turn produced by currents. Consequently, in general, we have a system of coupled equations, (22) and

$$\Box A_\mu(x) = j_\mu(x). \tag{23}$$

For given external fields $A_\mu(x)$ one can however confine oneselve to eq. (22).

C. *Generalizations of Gel'fand–Yaglom Equations*

The mass spectrum (19) is rather unphysical: a decreasing mass with increasing spin. This form of the spectrum is typical for the general Gel'fand–Yaglom equation (1). Also the magnetic moment derived from eq. (23) turns out to have the wrong sign. Hence it is natural to look for a more general relativistic wave equation of the form

$$(\Gamma_\mu p^\mu - K)\psi(p) = 0, \tag{24}$$

where K is an invariant operator of the Lorentz group. For $K = \alpha p_\mu p^\mu + \varkappa$ one obtains

$$(\Gamma_\mu p^\mu - \alpha p_\mu p^\mu - \varkappa)\psi = 0. \tag{25}$$

This equation can also be solved exactly by going to the rest frame. Proceeding as in the case of the simple Majorana equation one obtains:

$$m_J = \frac{J+\frac{1}{2}}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\varkappa\alpha}{(J+\frac{1}{2})^2}}\right). \tag{26}$$

In particular, for $\varkappa = 0$, we have a linear spectrum in spin

$$m_J = \frac{1}{2\alpha}(J+\tfrac{1}{2}).$$

One obtains another interesting model of generalized equation (24) by taking the wave function $\psi(p)$ which transforms according to $(D^{(1/2,0)} \oplus D^{(0,1/2)}) \otimes U^{[j_0,0]}$ representation. Setting in this case $K = -(M_0 + M_1 \sigma_{\mu\nu} M^{\mu\nu})$ where $M^{\mu\nu}$ are generators of $U^{[j_0,0]}$ representation and M_0 and M_1 are scalars we obtain the following wave equation

$$(\gamma_\mu p^\mu + M_0 + M_1 \sigma_{\mu\nu} M^{\mu\nu})\psi(p) = 0. \tag{27}$$

Passing to the rest frame and performing the similar analysis as in case of Majorana equations one obtains the following mass formula

$$\pm m_J = M_1(J+\tfrac{1}{2}) \pm \{(M_0 - M_1)^2 + M_1^2[J(J+1) - j_0(j_0+1) - \tfrac{3}{4}]\}. \tag{28}$$

Eq. (27) is called the Abers, Grodsky and Norton equation (1967).

D. *Applications of Infinite Component Wave Equations*

In sec. 1, we have embedded the inducing representation of the subgroup K of G into a representation $D(G)$ of G in order to have manifestly covariant wave equations. We can also embed $D(K)$ into a representation $D(\check{G})$ of a larger group $\check{G}$ containing G (hence K). Now the multiplicity of $D(K)$ in $D(\check{G})$ will be in general much larger. Physically these multiplicities will be identified with additional internal degrees of freedom of the system.

We now give an important example of this method. Let $\check{G} = \mathrm{SO}(4,2)$. The Lie algebra of SO(4, 2) has a basis $L_{ab} = -L_{ba}$, $a, b = 0, 1, 2, \ldots 5$, containing the generators $M_{\mu\nu} = L_{\mu\nu}$; $\mu, \nu = 0, 1, 2, 3$, of SO(3, 1). With respect to $M_{\mu\nu}$, the elements $L_{\mu 5} = \Gamma_\mu$ are components of a four-vector operator, and $L_{45} = S$ is a scalar operator. Hence the most general Lorentz covariant wave equation linear in the elements of the Lie algebra is

$$(\Gamma_\mu p^\mu + \beta S + \gamma)\psi(p) = 0. \tag{29}$$

Eq. (29) can be solved again by transforming it into the rest frame $\mathring{p} = (p_0, 0, 0, 0)$:

$$(\Gamma_0 p^0 + \beta S + \gamma)\psi(0) = 0. \tag{30}$$

Here we can either diagonalize Γ_0 which has a discrete spectrum as the generator of the compact subgroup, or S with a continuous spectrum as the generator of the noncompact subgroup.

We choose for example the most degenerate discrete class of representations of SO(4, 2) discussed in ch. 15 and choose as a basis the eigenvectors of Γ_0, J^2, J_3 labelled by $|n, J, J_3\rangle$. In this representation the invariant operators of SO(4, 2) have the values

$$\begin{aligned} C_2 &= \tfrac{1}{2} L_{ab} L^{ab} = -3, \\ C_3 &= \varepsilon_{abcdef} L^{cd} L^{ef} L^{ab} = 0, \\ C_4 &= L_{ab} L^{bc} L_{cd} L^{da} = -12. \end{aligned} \tag{31}$$

Due to the fact that Γ_0, S and $L_{04} \equiv T$ generate an SU(1, 1) subgroup, we can solve (30) by defining $\tilde{\psi}(\mathring{p})$ by

$$\psi(\mathring{p}) \equiv \exp(\mathrm{i}\theta_n L_{04})\tilde{\psi}(\mathring{p}), \qquad \Gamma_0|\tilde{\psi}_n(\mathring{p})\rangle = n|\tilde{\psi}_n(\mathring{p})\rangle \tag{32}$$

and choosing θ_n appropriately. Then

$$[(m^2-\beta^2)^{1/2}\Gamma_0+\gamma)\tilde{\psi}(\mathring{p}) = 0 \tag{33}$$

or

$$m^2 = (\beta^2-\gamma^2/n^2). \tag{34}$$

Consider the physically interesting case when β and γ are functions of the total mass m of the system. Clearly, in this case, the mass spectrum might be different from that given by eq. (34). Set

$$\beta = m\frac{\omega^2-m_1^2+(m-m_2)^2}{\omega^2+m_1^2-(m-m_2)^2}, \quad \gamma = \frac{-2\alpha\omega m(m+m_1-m_2)}{\omega^2+m_1^2-(m-m_2)^2}. \tag{35}$$

Solving (34) with respect to m one obtains

$$m = m_2+m_1\frac{1-\alpha^2/n^2}{1+\alpha^2/n^2}. \tag{36}$$

Expansion with respect to α^2/n^2 gives:

$$m = m_1+m_2-2m_1\alpha^2/n^2,$$

which is the mass spectrum of nonrelativistic hydrogen atom. We now show that choosing properly the representation of operators Γ_0 and S we obtain that eq. (30) with β and γ given by eq. (35) can be interpreted as the Klein–Gordon equation with scalar and vector potential of the type $1/r$ and equal coupling constants at both interactions. Indeed, take a representation of so(4, 2) algebra in terms of differential operators on $L^2(R^3)$ (cf. ch. 12, § 2). Generators of su(1, 1) subalgebra have in this case the form (cf. eq. 12.2(5))

$$\begin{aligned} \Gamma_0 &= \frac{1}{2\omega}(-r\nabla^2+\omega^2 r), \\ S &= \frac{1}{2\omega}(-r\nabla^2-\omega^2 r), \\ T &= -\mathrm{i}r\nabla-\mathrm{i}. \end{aligned} \tag{37}$$

Inserting this generators into eq. (30), defining $E_1 \equiv m-m_2$ and using expressions (35) for β and γ one obtains:

$$\left[\frac{1}{2\omega}(\omega^2+m_1^2-E_1^2)(-r\nabla^2+\omega^2 r)+\frac{1}{2\omega}(\omega^2-m_1^2+E_1^2)(-r\nabla^2-\omega^2 r)\right]\psi = 2\alpha\omega(E_1+m_1)\psi.$$

After elementary calculations one obtains

$$\left[-\nabla^2-\left(E_1+\frac{\alpha}{r}\right)^2+\left(m_1-\frac{\alpha}{r}\right)^2\right]\psi = 0. \tag{38}$$

This is the Klein–Gordon equation with the scalar potential $\varphi(r) = -\alpha/r$ and the vector potential $V(r) = -\alpha/r$. From eq. (36) we get the following spectrum for E_1:

$$E_1 = m_1\frac{1-\alpha^2/n^2}{1+\alpha^2/n^2}.$$

Expanding with respect to α^2/n^2 we get:

$$E_1 = m_1 - 2m_1 \alpha^2/n^2.$$

This corresponds to the spectrum of the Schrödinger equation with the potential $U(r) = -2\alpha/r$: this fact is not surprising since in nonrelativistic approximation the scalar and the vector potential add together.

It is thus remarkable, that equations of the type (29) describe relativistically composite quantum systems, such as the H-atom. Notice that the labeling of the states $|nJJ_3\rangle$ agrees with the quantum numbers of the H-atom. With the covariant equations (29) we thus established contact to the dynamical group formalism of quantum mechanics discussed in ch. 12.

Another representation of the su(1, 1) algebra (53), namely

$$\begin{aligned} \Gamma_0' &= \tfrac{1}{2}\left[-r\nabla^2 + r + \frac{1}{r}(-\alpha^2 - \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right], \\ S' &= \tfrac{1}{2}\left[-r\nabla^2 - r + \frac{1}{r}(-\alpha^2 - \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right], \\ T' &= T = -\mathrm{i}r\nabla - \mathrm{i}, \qquad \hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|, \end{aligned} \tag{40}$$

where $\boldsymbol{\alpha}$'s are the Dirac matrices, leads with a suitable choice of the parameters to

$$[\Gamma_0' + S' - (E^2 - m^2)(\Gamma_0' - S') - 2\alpha E]\psi = 0$$

or

$$\left[p^2 - (E^2 - m^2) - \frac{2\alpha E}{r} - \frac{1}{r^2}(\alpha^2 + \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right]\psi = 0 \tag{41}$$

which is the second order Dirac equation for the Coulomb potential.

Finally using the representation given in eq. 12.2 (22) we obtain the equation for dyonium which is an atom of two dyons, particles having both electric and magnetic charges. The Lie algebra solving the dyonium is the same as in eqs. (35) or (40), except for the replacements

$$\begin{aligned} \boldsymbol{p} &\to \boldsymbol{\pi} = \boldsymbol{p} - \mu\boldsymbol{D}(\boldsymbol{r}), \\ \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}} &\to (\mu\boldsymbol{\sigma} + \mathrm{i}\alpha\boldsymbol{\alpha})\cdot\hat{\boldsymbol{r}}, \end{aligned} \tag{42}$$

where $\boldsymbol{D}(\boldsymbol{r})$ and μ are defined in lemma 12.2.5.

E. *The Dynamical Group Interpretation of Wave Equations*

In the previous sections, covariant wave equations have been interpreted as projections on certain subspaces of the reducible representations of the Poincaré group induced from those of SL(2, C) (cf. eq. 1(12)).

A second interpretation (which ties up with the discussion of ch. 13, § 2) is to specify the space of states in the rest frame of a system by an irreducible representation U of a dynamical group $\mathscr{G} \supset$ SL(2, C). Then states of momentum P_μ are obtained by a Lorentz transformation (i.e. a boost). This method is particularly useful for infinite-component wave equations, but is also applicable to finite-component equations.

EXAMPLE 1. Let $\mathscr{G} = O(4,2)$. Take U to be the 4-dimensional non-unitary representation in which the generators of $\mathscr{G}$ are given in terms of the 16 elements of the algebra of Dirac matrices as in exercise 13.6.4.1.

Because $\frac{1}{2}L_{56} = \gamma_0$ has eigenvalues $n = \pm 1$, taking the simplest mass relation $mn = \varkappa$, we can write

$$(m\gamma_0 - \varkappa)\psi(\mathring{p}) = 0, \tag{43}$$

where $\varkappa$ is a fixed constant.

Transforming this equation with the Lorentz transformation of parameter ξ

$$\begin{aligned} \psi(p) &= \exp(\mathrm{i}\xi N)\psi(p), \\ N &= \tfrac{1}{2}\gamma_0\gamma \end{aligned} \tag{44}$$

gives

$$(\gamma^\mu p_\mu - \varkappa)\psi(p) = 0$$

which is the Dirac Equation 2(6).

EXAMPLE 2. Let $\mathscr{G} = O(4,2)$. Let U be an infinite-dimensional unitary representation of $\mathscr{G}$ of the most degenerate series. Now $\frac{1}{2}L_{56} = \Gamma_0$ has the spectrum $n = \mu+1, \mu+2, \ldots$ The simplest mass relation linear in the group generators is

$$(M\Gamma_0 - \varkappa)\psi(\mathring{p}) = 0.$$

Corresponding to masses $M_n = \varkappa/n, n = \mu+1, \ldots$ Transforming this equation by $\exp(\mathrm{i}\xi N)$ (Lorentz transformations) we obtain

$$(\Gamma_\mu p_\mu - \varkappa)\psi(p) = 0$$

which is the Majorana equation 3(12). Or, taking the more general mass relation

$$(M\Gamma_0 + \alpha M^2 - \varkappa)\psi(\mathring{p}) = 0$$

we obtain the generalized Majorana Equation 3(25). ▼

As these examples show we have again an inducing process. The rest frame states $\psi(\mathring{p})$ transform according to a representation L of K in H: $\psi_k(\mathring{p}) = L_k\psi(\mathring{p})$, $k \in K$, and then we pass to the induced representations of the Poincaré group Π by (44).

The reduction of U with respect to SL(2, C) gives a reducible representation $\tilde{L}$ of the latter. Hence the spinorial wave functions $\psi(p)$ transform, according to a, in general reducible, representation $\tilde{L}$ of SL(2, C):

$$(U_{(a,\Lambda)}\psi)(p) = \exp(\mathrm{i}pa)\tilde{L}(\Lambda^{-1}p).$$

The general procedure is as follows:

(1) The choice of the dynamical group $\mathscr{G}$ and its representation U depends on the internal dynamics and degrees of freedom of the system (i.e. spin, or orbital and radial excitations). This is reflected in the number of quantum numbers and of states. Figs. 2 and 3 show two weight diagrams for $\mathscr{G} = O(4, 2)$ where j and n are the eigenvalues of L^2 and $L_{56} = \Gamma_0$, respectively (cf. also ch. 12, Fig. 1).

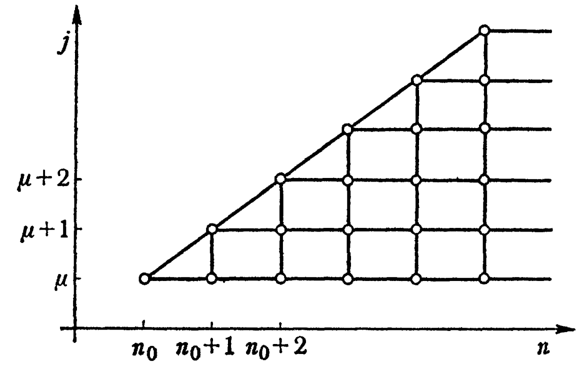


Fig. 2

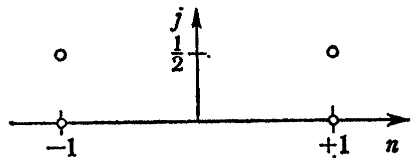


Fig. 3

(2) Let Γ_μ be a vector operator in U and S a scalar operator. Take any relation involving the quantities $P_0\Gamma^0$, $P_\mu P^\mu$, S, ... on the carrier space H of U:

$$f(\Gamma_0 P^0, P^2, S)\psi(\mathring{p}) = 0, \quad \psi(\mathring{p}) \in H \tag{45}$$

and transform this equation by the Lorentz transformation of the type (44)

$$f(\Gamma_\mu P^\mu, P^2, S)\psi(p) = 0, \tag{46}$$

which is a covariant equation incorporating the postulated mass relation (45).

The role of the dynamical group $\mathscr{G}$ in these considerations lies in the fact that it carries information about the internal dynamics of the system and dictates therefore which representation of SL(2, C) has to be chosen to construct a covariant wave equation. This latter choice was so far arbitrary in our discussion in § 1. Eqs. (45) and (46) can be generalized to spin dependent mass formulas.

F. *Physical Applications of Matrix Elements of Representations of Semisimple Non-Compact Lie Groups*

Let G be a Lie group and U_g a representation in H. Certain group elements can be parametrized as

$$\exp(\mathrm{i}\theta^k X_k),$$

where θ^k are the group parameters and X_k the generators of the Lie algebra of G. Let $u_m \in H$ be a basis in the carrier space of U_g. The matrix elements will be denoted by

$$D_{mn}(\theta) = \langle m|\exp(\mathrm{i}\theta\cdot X)|n\rangle \tag{47}$$

with respect to the scalar product in H. Clearly these functions generalize the functions D^j_{mn} of the group SU(2).

For the compact semisimple groups SO(n), the expressions for $D(\theta)$ has been given by Vilenkin 1968. For the noncompact case SO(2,1) these functions were first given by Bargmann 1947. Many other results are known for SO(3,1) and certain representations of SO(4,1), SO(4,2), ... in the physical literature.

Consider the wave equation of the type (12), (25) or (29).

The electromagnetic coupling of the system is described by the process of minimal coupling (cf. (21)).

The physical transition probability amplitude is given by the matrix elements of the current operator Γ_μ between the two states of momenta p_1 and p_2 (cf. 13).

$$A_\mu = \langle \psi_m(p_1) | \Gamma_\mu | \psi_n(p_2) \rangle .$$

Using Lorentz transformations we have

$$A_\mu = \langle \psi_m(\mathring{p}) | \exp(-\mathrm{i}\xi \cdot N) \Gamma_\mu \exp(\mathrm{i}\xi \cdot N) | \psi_n(\mathring{p}) \rangle .$$

These matrix elements are to be evaluated in general in an infinite-dimensional representation U of the dynamical group $\mathscr{G}$, SO(4, 2), for example. Thus transition amplitudes can be reduced to the matrix elements $D_{mn}(\theta)$ of the type (47). More general matrix elements appear in the physical applications (cf. Barut and Wilson 1976).

§ 4. Group Extensions and Applications

A. *Group Extension*

Group extension plays an important role in many parts of quantum theory. We discuss here the relation of group extension to cohomology.

Consider a following sequence of groups and homomorphisms

$$\to G_n \xrightarrow{f_n} G_{n+1} \xrightarrow{f_{n+1}} G_{n+2} \to . \tag{1}$$

Recall that if the kernel $\mathrm{Ker} f_n = I$, f_n is called injective (then Image $f_n \cong G_n$), if Image $f_n = G_{n+1}$, then f_n is called surjective.

DEFINITION 1. The sequence (1) is an *exact sequence* if Image $f_n = \mathrm{Ker} f_{n+1}$, for all n.

EXAMPLES:

(1°) $1 \to \mathrm{Ker} f \to G \xrightarrow{f} \mathrm{Im} f \to 1,$ (2)

(2°) $1 \to C(G) \to G \xrightarrow{f} J(G) \to 1,$ (3)

where $C(G)$ is the center of G and $J(G)$ is the group of non-trivial inner automorphisms. Clearly, $G/\mathrm{Ker} f \cong \mathrm{Im} f$ and $G/C(G) \cong J(G)$.

(3°) $1 \to J(G) \to \mathrm{Aut}\, G \to A(G) \to 1,$ (4)

where $A(G)$ represents the automorphism classes of G modulo inner automorphisms, $A(G) \cong \mathrm{Aut}\; G/J(G)$.

DEFINITION 2. A group E is called an *extension of G by K* if it satisfies an exact sequence

$$1 \to K \to E \to G \to 1. \blacktriangledown \tag{5}$$

We have then that K is an invariant (normal) subgroup of E and $E/K = G$; thus E can be thought of as consisting of the cosets kG, $k \in K$ (written multiplicatively). An important mathematical problem is to find all groups E such that $E/K = G$, and K is an invariant subgroup of E.

$$\begin{tikzcd} & 1 \arrow[d] & & & \\ & C(K) \arrow[d] & & & \\ 1 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & J(K) \arrow[r] \arrow[d] & \mathrm{Aut}K \arrow[r] & A(K) \arrow[r] & 1 \\ & 1 & & & \end{tikzcd} \tag{6}$$

Noting that elements from E induce automorphisms of K by $k \to x^{-1}kx$ for any $x \in E$ we get homomorphisms $E \to \mathrm{Aut}K$ and $G \to A(K)$. In general however the latter homomorphism does not come from a homomorphism $G \to \mathrm{Aut}K$.

B. *Extensions of the Poincaré Group*

The full relativistic invariance includes also the discrete symmetry operations of space-reflection Σ and time reflection Θ.

We introduce the two reflection operations Σ and Θ. Their identification with parity P and time reflection T, or their combinations, eg. PC, PT, ... depends on superselection rules. The full Lorentz group then consists of the cosets

$$L = (\Lambda, \Lambda\Sigma, \Lambda\Theta, \Lambda\Sigma\Theta), \tag{7}$$

where Λ is the restricted Lorenz group. L is the extension of Λ by the group of reflections. We again use the covering group SL(2, C) of Λ and write the group elements of the Poincaré group as

$$(a, \Lambda) \to (a \cdot \sigma, A), \quad A \in \mathrm{SL}(2, C),$$

where $a \cdot \sigma = a^\mu \sigma_\mu$ (cf. ch. 17.2). We denote by

$U(a \cdot \sigma, A)$ = unitary representations of the restricted Poincaré group,

$U(\Sigma)$ = operators assigned to Σ, assumed to be unitary. (8)

$U(\Theta)$, $U(\Theta\Sigma)$ = operators assigned to Θ and $\Theta\Sigma$, assumed to be anti-unitary

In ch. 13 we discussed the quantum mechanical unitary and antiunitary operators. The relation between the restricted groups, the full groups and the covering groups can be arranged into following sequences:

(i) The group of translations T^4, the restricted Poincaré group Π_0, the restricted Lorentz group Λ, the full Poincaré group Π, and the full Lorentz group L satisfy

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & T_4 & \rightarrow & \Pi_0 & \rightarrow & \Lambda & \rightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & T_4 & \rightarrow & \Pi & \rightarrow & L & \rightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C_2\oplus C_2 & & C_2\oplus C_2 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \tag{9}$$

(ii) The quantum mechanical covering groups of (9) are

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Z_2 & & Z_2 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & T_4 & \rightarrow & \overline{\Pi}_0 & \rightarrow & \mathrm{SL}(2, C) & \rightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & T_4 & \rightarrow & \Pi & \rightarrow & \Lambda & \rightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \tag{10}$$

The group Z_2 here is the first homotopy group of Λ, i.e. $\mathrm{SL}(2, C)/Z_2 = \Lambda$.

Now we shall determine the representations of the extended group. By a choice of equivalent phase factors we can set $U(\Sigma) = I$ (cf. ch. 13). Let $\overset{*}{U}(\Theta)^2 = \varepsilon_\theta$ and $\overset{*}{U}(\Theta)^2 = \varepsilon_{\theta\Sigma}$. It follows from the associative law $\overset{*}{U}\overset{*}{U}{}^2 = \overset{*}{U}{}^2\overset{*}{U}$ that both ε's satisfy $\varepsilon^2 = \pm 1$. Correspondingly, we have four types of representations depending on $\varepsilon_\theta = \pm 1$, $\varepsilon_{\theta\Sigma} = +1$. $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$ are then determined up to factor $e^{i\alpha}$. Next, we determine the phases between the products of $U(\Sigma)$, $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$, and obtain from the group law the following multiplication table:

	$U(\Sigma)$	$\overset{*}{U}(\Theta)$	$\overset{*}{U}(\Theta\Sigma)$
$U(\Sigma)$	1	$\overset{*}{U}(\Theta\Sigma)$	$\overset{*}{U}(\Theta)$
$\overset{*}{U}(\Theta)$	$\varepsilon_\theta\varepsilon_{\theta\Sigma}\overset{*}{U}(\Theta\Sigma)$	ε_θ	$\varepsilon_\theta U(\Sigma)$
$\overset{*}{U}(\Theta\Sigma)$	$\varepsilon_\theta\varepsilon_{\theta\Sigma}\overset{*}{U}(\Theta)$	$\varepsilon_\theta U(\Sigma)$	$\varepsilon_{\theta\Sigma}$

(11)

This is a multiplication table of a finite group (cf. ch. 7).

The next problem would be to determine the phases between $U(\Sigma)$, $\overset{*}{U}(\Theta)$, $\overset{*}{U}(\Theta\Sigma)$ and the representations $U(a, \Lambda)$ of the restricted group. We write the group law as

$$U(\Sigma)\, U(a, \Lambda)\, U(\Sigma^{-1}) = \omega(a, \Lambda)\, U(\Sigma a, \Sigma\Lambda\Sigma^{-1}).$$

If one considers the product of two such transformations one obtains for the phase the equation

$$\omega\big(a + \Lambda(a)b, AB\big) = \omega(a, A)\omega(a, B). \tag{12}$$

Thus, ω is a one-dimensional representation of the Poincaré group Π. But Π has no invariant subgroup with an abelian factor group, hence $\omega = 1$.

Similarly, the phase ω' between $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$ must be one.

Consequently no new possibilities are introduced, and we have the four possible quantum mechanical full Poincaré groups corresponding to eq. (8) depending on the factors in the multiplication table,

$$U(a, L) = \big(U(a, \pm A),\ U(a, \pm A)\,U(\Sigma),\ U(a, \pm A)\overset{*}{U}(\Theta),\ U(a, \pm A)\overset{*}{U}(\Theta\Sigma)\big). \tag{13}$$

Note that, of course, the use of $\pm A \in \mathrm{SL}(2, C)$ in the quantum mechanical representation of $A \in \mathrm{SO}(3, 1)$ is already a group extension from the mathematical point of view. The Poincaré group can be further extended by other reflection operators, such as C = charge conjugation.

Let $\{|p, \sigma\rangle\}$ be the carrier-space vectors of the representation $U(a, A)$ of the restricted Poincaré group, where p^2 defines an orbit (cf. 17.2). Consider now the vectors $U(\Sigma)|\tilde{p}, \sigma\rangle$ where $\tilde{p}_\mu = (p_0, -\boldsymbol{p})$, and where we have identified Σ with the parity operator. Both vectors $|p, \xi\rangle$ and $U(\Sigma)|\tilde{p}, \sigma\rangle$ transform in the same way under the restricted Poincaré group. Hence we can define eigenstates of parity by

$$|p, \sigma, \pm\rangle = |p, \sigma\rangle \pm U(\Sigma)|p, \sigma\rangle.$$

If neither of these states vanishes, we have a new quantum number parity, with eigenvalues ± 1, hence a doubling of the representation space. Whether both of these spaces are observable depends on the existence of superselection rules (cf. ch. 13).

C. *Classification of Extensions*

Definition 3. If K is abelian and $K \subset C$ the center of E, then E is called a *central extension*. We have then $C(K) = K$, $J(K) = 1$, $\mathrm{Aut}\,K = A(K)$.

Suppose we are given a homomorphism $\sigma\colon G \to \mathrm{Aut}\,K$; this is frequently described by saying that G acts via σ on K, or that K is a G-module (with respect to σ). We shall investigate the problem of describing group extensions

$$1 \to K \xrightarrow{f} E \xrightarrow{h} G \to 1$$

such that inner automorphisms of E restricted to K coincide with $\sigma \circ h$. Note that there always exists such an extension, namely the semidirect product $E = K \otimes G$ with the multiplication defined by

$$(\alpha, a)(\beta, b) = (\alpha + \sigma_a \beta, ab), \quad \alpha, \beta \in K,\ a, b \in G, \tag{14}$$

is such extension; it is called a *trivial extension*. Observe that for trivial extensions

$$1 \to K \xrightarrow{f} E \xrightarrow{h} G \to 1$$

there exists a group homomorphism $D\colon G \to E$ such that $h \circ D$ = identity (D is called a *section* or *splitting*). We shall characterize nontrivial extensions by means of non necessarily homomorphic sections, i.e. extensions

$$1 \to K \xrightarrow{f} E \underset{D}{\overset{h}{\rightleftarrows}} G \to 1,$$

where D: $G \to E$ satisfies $h \circ D =$ identity. We shall deal exclusively with the case of an abelian K. Consider $f(\alpha)D(a)$, $\alpha \in K$, $a \in G$. For fixed a, varying α we get fibers of the bundle E with base G (i.e. cosets of $f(K)$). For fixed α, varying a, we have sections. In particular, $f(1)D(G) \cong G$. $D(a)D(b)$ and $D(ab)$ are in the same coset (fiber) of $f(K)$, since they are both mapped under the homomorphism h into ab. Hence, they can only differ by a 'phase'

$$D(a)D(b) = \omega(a, b)D(ab). \tag{15}$$

The associativity of the group multiplication law implies

$$\omega(a, b)+\omega(ab, c)-\sigma_a\omega(b, c)-\omega(a, bc) = 0. \tag{16}$$

Such a map ω: $G \times G$ into K (identifying K with $f(K)$) is called a *factor system.* However we wish to disregard differences coming from replacing a map D by another D' such that $D'(a) = \varphi(a)D(a)$, where $\varphi(a) \in K$. D' leads to another factor system ω', such that

$$\omega'(a, b) = \omega(a, b)+\theta(a, b), \tag{17}$$

where $\theta(a, b)$ is given by

$$\theta(a, b) = \varphi(a)+\sigma_a\varphi(b)-\varphi(ab)$$

and is said to be a trivial factor system.

A normalized factor system by the definition satisfies

$$D(1) = 1, \quad \text{or} \quad \omega(a, 1) = \omega(1, b) = \omega(1, 1) = 0. \tag{18}$$

Conversely, given a homomorphism σ: $G \to \mathrm{Aut}K$ and a normalized factor system $\omega(a, b) \in K$, we can construct an extension E of G by K with elements (α, a), $\alpha \in K$, $a \in G$, with the composition law

$$(\alpha, a)(\beta, b) = \big(\alpha+\sigma_a\beta+\omega(a, b), ab\big) \tag{19}$$

where $\omega(a, b)$ satisfies eq. (18). The unit element of E is $(0, 1)$ and the inverse element of (α, a) is

$$(\alpha, a)^{-1} = \big(\sigma_{a^{-1}}\alpha\omega(a^{-1}, a), a^{-1}\big). \tag{20}$$

Two factor systems differing by a trivial factor system provide isomorphic or equivalent extensions. Furthermore, the inner automorphisms on E induces on K the automorphism σ because

$$(\alpha, a)(\alpha', 1)(\alpha, a)^{-1} = (\sigma_a\alpha', 1). \tag{21}$$

The mathematical problem now is to find all normalized factor systems up to a trivial one. The result can be expressed as

PROPOSITION 1. *The two-dimensional cohomology group $H^2(G, K)$ is exactly the group of all (equivalent classes of) those extensions E of G by K which realize the given group action σ of G on K.*

In order to explain this proposition we discuss now the relationship of group extensions to homology and cohomology.

'Homology' of a Lie algebra, or of a group (or of an associative algebra) goes back to the homology of topological spaces (i.e. the connectivity, cf. ch. 2). The homology groups quite generally are described in terms of chains and their boundaries. An n-dimensional chain is a formal linear combination of n-simplices in the space. All n-chains form a free abelian group C_n. The chain complex C is the sequence

$$C_0 \leftarrow C_1 \leftarrow \dots \leftarrow C_{n-1} \overset{\partial_n}{\leftarrow} C_n \overset{\partial_{n+1}}{\leftarrow} C_{n+1} \leftarrow \dots$$

where the boundary homomorphisms ∂ satisfy $\partial\partial = 0$. Its homology group in n-dimension is

$$H_n(C) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}. \tag{22}$$

Now we describe the corresponding cohomology groups. An n-dimensional cochain is a homomorphism f: $C_n \to K$ (abelian group), coboundary operator δ_n is the map δ: $\operatorname{Hom}(C_n, K) \to \operatorname{Hom}(C_{n+1}, K)$ defined (uniquely) by the condition

$$\delta_n f = f\partial.$$

Hence we get the cochain complex

$$\to C^n \equiv \operatorname{Hom}(C_n, K) \overset{\delta_n}{\to} C^{n+1} = \operatorname{Hom}(C_{n+1}, K) \to .$$

Kernels of δ are cocycles (those of ∂, cycles) and Image δ are coboundaries (those of ∂, are boundaries). Consequently, the n-dimensional cohomology of the complex C (or of the underlying space) with coefficients in K is

$$H^n(C, K) = \frac{\operatorname{Ker} \delta_n}{\operatorname{Im} \delta_{n-1}}. \tag{23}$$

Cohomology of groups deals with the action of a group G on an abelian group K:

$$\alpha \to \sigma_a \alpha, \quad \alpha \in K, \quad a \in G, \ \sigma: G \to \operatorname{Aut} K$$

and is defined, dimension by dimension, in terms of the following cocycles.

In dimension one, the cocycles are 'crossed homomorphisms' ϕ: $G \to K$ such that $\phi(ab) = \sigma_a \phi(b) + \phi(a)$, for all $a, b \in G$. All such homomorphisms form a group. The 'principal crossed homomorphisms', i.e. coboundaries are $k_\alpha a = \sigma_a \alpha - \alpha$, $a \in G$, for each α. Then we define

$$H^1(G, K) = \{\text{crossed homomorphisms}\}/\{\text{principal crossed homomorphisms}\}. \tag{24}$$

In dimension two: cocycles are defined to be the factor systems introduced in eq. (15), i.e. maps $\theta: G \times G \to K$ such that

$$\sigma_a \theta(b, c) + \theta(a, bc) = \theta(ab, c) + \theta(a, b). \tag{25}$$

All solutions of this equation form a group. The trivial factor systems which satisfy $\theta_\phi(a, b) = \sigma_a \phi(b) - \phi(ab) + \phi(a)$ are declared coboundaries and the two-dimensional cohomology group is defined by

$$H^2(G, K) = \frac{\{\theta: G \times G \to A\}}{\{\theta = \theta_\phi\}},$$

i.e. all solutions of the factor set equation (25) modulo trivial solutions. This proves proposition 1.

Generally we define $\alpha_n: G\times G\times \ldots \times G \to K$ and the sequence of abelian groups

$$C^n(G, K)\colon \{\alpha_n(a_1 \ldots a_n),\ a_i \in G,\ \alpha_n \in K | \alpha_n = 0 \text{ if at least one } \alpha_i = 1\}$$

with the homomorphisms (boundary operator)

$$\begin{aligned}\delta_n[\alpha_n(a_1, \ldots, a_n)] &\equiv (\delta_n \alpha_n)(a_1, \ldots, a_{n+1})\\ &\overset{\text{Def}}{\equiv} a_1\alpha_{n}(a_2, \ldots, a_{n+1})+\sum_{k=1}^{n}(-1)^k\alpha_n(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1})+\\ &\quad +(n-1)^{n+1}\alpha_n(a_1, \ldots, a_n),\end{aligned}$$

so that n-dimensional cohomology group $H^n(G, K)$ can be defined as above.

Indeed, we have immediately $C^0(G, K) = K$, $(\delta_0\alpha_0)a = a\alpha_0-\alpha_0$. Let $\sigma_a K = aK = aC^0\cdot\operatorname{Ker}\delta_0 = K^G$ = fixed point of K under G. In this case we have $B^0 = 0$, by definition, consequently $H^0(G, K) = K^G$.

If we look at δ_1 and δ_2 we find that

$$\delta_1[\alpha_1(a)] \equiv (\delta_1\alpha_1)(a, b) = a\alpha_1(b)-\alpha_1(ab)+\alpha_1(a), \tag{26}$$

$$\delta_2[\alpha_2(a, b)] \equiv (\delta_2\alpha_2)(a, b, c) = a\alpha_2(b, c)-\alpha_2(ab, c)+\alpha_2(a, bc)-\alpha_2(a, b), \tag{27}$$

so it coincides with the definitions given above.

As a special case the problem of phase representations of a Lie group G becomes simply a group extension of G by a one-dimensional abelian group K (cf. ch. 12), and we have

THEOREM 2 (Bargmann). *Let G be a connected and simply connected Lie group with a trivial second cohomology group $H^2(G, K)$, where K is a one-dimensional abelian group. Then any projective representation of G admits a lifting to a representation of G.*

(For the proof see V. Bargmann 1954, D. T. Simms 1971.)

D. *Examples: Some Further Applications of Group Extensions in Physics*

1) Extensions of U_1: $\{e^{i\theta}, -\pi \leqslant \theta \leqslant \pi\}$.

The only inner automorphism is the identity automorphism; and the only outer automorphism is $e^{i\theta} \to e^{-i\theta}$. If we represent U_1 by $e^{i\theta Q}$ (for example, Q is the charge operator) and the automorphisms by $\varrho e^{i\theta Q}\varrho^{-1} = e^{\pm i\theta Q}$, then

$$[\varrho, Q] = 0,\ \varrho^2 = 1 \text{ for inner automorphism,}$$

$$[\varrho, Q]_+ = 0,\ \varrho^2 = 1,$$

or

$$[\varrho, Q]_+ = 0,\ \varrho^2 = e^{i\pi Q} = (-1)^Q\} \text{ for outer automorphisms.}$$

2) Extension of U_2.

Writing $U_2 = (U_1\otimes SU(2))/C_2$, where C_2 has two elements $(1, 1)$ and $(-1, -1)$, we interpret in strong interactions SU(2) with the isospin group, and U_1 with hypercharge Y. The commutation relations of isospin with the ordinary electric charge Q are

$$[I_3, Q] = 0, \quad [I_\pm, Q] = \pm I_\pm.$$

Hence defining $Y = Q - I_3$, we have

$$[I, Y] = 0,$$

thus the direct product $U_1 \otimes \mathrm{SU}(2)$. The representations of U_2 are $D^I(u)\mathrm{e}^{\mathrm{i}\alpha Y}$ with both elements of C_2 represented by identity. Thus $D^I(-1)\mathrm{e}^{\mathrm{i}\pi Y} = 1$, or

$$(-1)^{2I} = (-1)^Y.$$

This relation is satisfied empirically for all known particles.

Let now C be the charge conjugation operator which reverses the eigenvalues of Q. We can extend U_2 by C, $C^2 = I$, or more conveniently by $G = C\mathrm{e}^{\mathrm{i}\pi I}$ $= G$-parity. We obtain $[G, I] = 0$. This extension corresponds to the automorphism $\mathrm{e}^{\mathrm{i}\alpha} \to \mathrm{e}^{-\mathrm{i}\alpha}$, hence we obtain (as in example 1)

$$GY + YG = 0.$$

The operator G is just the parity operator in isospin space, distinguishing axial or polar vectors, or tensor in the isospin space.

3) The physically meaningful representations of the Galilei group (in the space of solutions of the Schrödinger equation) are the projective unitary representations of its universal covering group (cf. ch. 13). Let $g = (b, \boldsymbol{a}, \boldsymbol{v}, R)$ be an element of the Galilei group. The projective representations satisfy

$$U(g')\,U(g) = \omega(g', g)\,U(g'g)$$

and the factor system $\omega(g', g)$ is explicitly given by

$$\omega(g', g) = \exp\left[\mathrm{i}\frac{m}{2}\left(\boldsymbol{a}'(R'\boldsymbol{v}) - \boldsymbol{v}'(R\boldsymbol{a}) + b\boldsymbol{v}'(R'\boldsymbol{v})\right)\right].$$

This procedure is equivalent to finding a central extension E of the Galilei group G by a one-dimensional abelian group $K = R$. This 11-parameter group E has the mass operator m as one of its invariants and leads to a superselection rule on mass, as we have noted in ch. 13.4. The exact sequence in this case is

$$1 \to R_1 \to E \to G \to 1.$$

Galilei group G does not satisfy Bargmann's criterion. The projective unitary representations of G are not induced by the unitary representations of G but of its extension E.

§ 5. Space-Time and Internal Symmetries

In ch. 1, § 7, we discussed possible unification of the space-time symmetries and internal symmetries of fundamental particles of physics on the algebraic level. Theorems 1 and 2, there give the limitations for attempts to combine these two types of symmetries within a larger finite Lie algebra. In this section we shall elaborate on the same problem on the group level. The following theorem, corollary and counterexamples describe the scope and results obtained. Finally we shall state how this problem is solved in practice from the physical point of view.

THEOREM (Jost). *Let G be a finite connected Lie group containing the Poincaré group as an analytic subgroup. Let U be a continuous unitary representation of G and P_μ the energy-momentum vector. Let the spectrum of P_μ be contained in $\{0\} \cup \cup V_+$, V_+ = forward cone in the Minkowski space M^4. If the mass operator $M = (P_\mu P^\mu)^{1/2}$ has an isolated eigenvalue $m_1 > 0$, then the corresponding eigenspace H_1 is invariant under G.*

PROOF: Let the spectral resolution of continuous abelian group of translations be

$$e^{-ia^\mu P_\mu} = \int e^{-ia^\mu p_\mu} dE(p). \tag{1}$$

If the mass operator $M = \int \sqrt{p^2}\, dE(p)$ has isolated eigenvalues, e.g. $m_1 > 0$ and the rest, its spectrum consists of the mass hyperboloid $M_1 = \{p : p^0 > 0; p^2 = m_1^2\}$ and the rest M_2, which are O_M-separated: Two closed sets $M_\alpha \subset M^4$, $\alpha = 1, 2$, are said to be *O_M-separated*, if there are functions $h_\alpha \in O_M$ such that

(i) $0 \leqslant h_\alpha(p) \leqslant 1$,

(ii) $h_\alpha(p) = 1$ if $p \in M_\alpha$,

(iii) $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \varnothing$.

Let

$$E_\alpha = \int_{M_\alpha} dE(p) = \int_{M_\alpha} h_\alpha(p) dE(p), \qquad \alpha = 1, 2. \tag{2}$$

The theorem is proved if we can show that

$$E_\alpha U_g = U_g E_\alpha \tag{3}$$

for every $g \in G$, for then $U_g \varphi_1 \in H_1$ together with $\varphi_1 \in H_1$. We must show that, because G is connected,

$$E_\alpha e^{itX} = e^{itX} E_\alpha, \qquad t \in R^1, \tag{4}$$

$$X = U(x), \quad x \in L, \text{ the Lie algebra of } G,$$

or,

$$E_1 X \psi = X E_1 \psi,$$

$$\psi \in \Delta(X) = \text{the domain of definition of } X. \tag{5}$$

This last assertion follows from

LEMMA. *Let $h_\alpha \in O_M$ and bounded, $\alpha = 1, 2$, and let $\varphi \in D$ (Gårding domain). Then $h_\alpha(p)\varphi \in \Delta(X)$ for every X. Further if $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \varnothing$ then*

$$h_1(p) X h_2(p) \varphi = h_2(p) X h_1(p) \varphi = 0. \blacktriangledown \tag{6}$$

For then using (2) we get

$$E_1 X E_2 \varphi = E_2 X E_1 \varphi = 0 \tag{7}$$

and, because $E_1+E_2 = I$, $E_1X\varphi = XE_1\varphi$. And if we have a sequence φ_k, $\varphi_k \in D$, for which $\varphi_k \to \psi$, then $X\varphi_k \to X\psi$, also $E_1X\varphi_k \to E_1X\psi$, hence $XE_1\varphi_k \to E_1X\psi$, Consequently, $E_1\psi \in \Delta(X)$ and $E_1X\psi = XE_1\psi$, or $E_1X \subset XE_1$.

Only the proof of the Lemma remains: For $\varphi \in D$ we have from $Adg: x \to gxg^{-1}$

$$U_g X U_{g^{-1}}\varphi = AdgX\varphi. \tag{8}$$

In particular

$$e^{ia^\mu P_\mu} X e^{-ia^\mu P_\mu}\varphi = C(a)X\varphi \tag{9}$$

where $C(a)$ is a real polynomial in a_μ. (The last statement follows from the fact that $ad(a^\mu P_\mu)$ is nilpotent, i.e. there exists an N such that $[ad(a^\mu P_\mu)]^N = 0$.) Hence

$$X e^{-ia^\mu P_\mu}\varphi = e^{-ia^\mu P_\mu} C(a) X\varphi. \tag{10}$$

We can apply this equation to finite sums $c_k e^{-ia_k^\mu P_\mu}$ and finally to the limits

$$X\int \tilde{f}(a) e^{-ia^\mu P_\mu} d^4a\varphi = \int \tilde{f}(a) C(a) e^{-ia^\mu P_\mu} d^4a X\varphi,$$

or, using (1) and $f(p) = \int \tilde{f}(a) e^{-ia^\mu P_\mu} d^4a$, $f \in S(R^4)$

$$X\int f(p)dE(p)\varphi = \int (C(\nabla)f)(p)dE(p)X\varphi,$$

where $\nabla = i\dfrac{\partial}{\partial p}$. Or in the operator form

$$Xf(p)\varphi = [C(\nabla)f](p)X\varphi, \quad f \in S(R^4),\ \varphi \in D.$$

If $g \in O_M$ and g is bounded with all its derivatives up to order $N-1$, then

$$Xg(p)\varphi = [C(\nabla)g](p)X\varphi. \tag{11}$$

The Gårding domain D is invariant under U_g: $XD \subset D$. Let

$$||P||^2 = P_0^2+P_1^2+P_2^2+P_3^2 = \int ||p||^2 dE(p).$$

For every $\varphi_1 \in D$ and $n \in Z_+$, $(1+||P||^2)^n\varphi_1 \in D$, hence

$$Xg(p)(1+||p||^2)^n \varphi_1 = [C(\nabla)g](p)X(1+||p||^2)^n\varphi_1.$$

Finally let $h \in O_M$. There exists $n \in Z_+$ such that

$$g = \frac{h}{(1+||p||^2)^n}$$

is bounded with all its derivatives up to order $N-1$. Hence

$$Xh(p)\varphi_1 = [C(\nabla)g](p)X(1+||p||^2)^n\varphi_1$$

and $\operatorname{supp} h = \operatorname{supp} g$.

Now for $h_\alpha \in O_M$, $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \emptyset$, clearly $\operatorname{supp} h_1 \cap \operatorname{supp} C(\nabla)g_2 = \emptyset$. Then $h_1 C(\nabla)g_2 = 0$, or

$$h_1(p)Xh_2(p)\varphi = (h_1 C(\nabla)g_2)(p)X(1+||p||^2)^n\varphi = 0$$

and the lemma is proved. ▼

COROLLARY. *In an irreducible representation of G there can be no mass-splitting.*

In other words, in an irreducible representation an isolated mass value is the whole spectrum; or, the spectrum must be a connected set.

This still leaves reducible representations, or, non-integrable irreducible representations of the Lie algebra L of G, in which cases a discrete mass spectrum is possible.

EXAMPLE 1 (Flato, Sternheimer). This example is based on the simple observation that the Hamiltonian $H = p^2/2m$ of a non-relativistic free particle has a continuous spectrum when represented as a self-adjoint operator in $L^2(-\infty, \infty)$, but a discrete spectrum when represented as a self-adjoint operator in $L^2(a, b)$, in a box with suitable boundary conditions.

Consider now the su$(2, 2)$-algebra containing the Poincaré algebra Π (cf. ch. 13, 4(2)). Consider $H = L^2(Q)$, where Q is a 'box' in the Minkowski space, Q: $\{0 \leqslant x_\mu \leqslant a\}$, $x_\mu \in M^4$. The SU$(2, 2)$ generators define hermitian operators in the domain of absolutely continuous functions, with L^2 derivatives, vanishing on the boundary ∂Q; and $-\partial_\mu^2$ is self-adjoint on $D(P_\mu^2)$ of $f \in L^2(Q)$ and $\partial_\mu f$ absolutely continuous with respect to x_μ, $\partial_\mu^2 f \in L^2(Q)$ and with periodic boundary conditions on f and $\partial_\mu f$. The mass operator $\square = -\partial_\mu \partial^\mu$ is defined on the common domain $\bigcap_{\mu=1}^{3} D(P_\mu^2)$, is essentially self-adjoint and has a spectral resolution

$$M^2 = \left(\frac{2\pi}{a}\right)^2 \sum_{m=-\infty}^{m=\infty} mE(m),$$

where $E(m)$ is the projection operator on the subspace H_m by $\exp\left[\left(i\frac{2\pi}{a}\right) n^\mu x_\mu\right]$. This representation is partially integrable on its translation subalgebra, but this domain of integrability does not coincide with the invariant domain of the entire su$(2, 2)$. ▼

Finally, we reformulate th. 1 and corollary 2 of ch. 1, § 7 on the group level in a somewhat stronger form.

LEMMA 2. *Let G be any group and S and $\Pi T^4 \otimes$ SO $(3, 1) = T \oplus \Lambda$ (Poincaré group) subgroups of G such that any $g \in G$ has a unique decomposition into a product $g = s\pi$, $s \in S$, $\pi \in \Pi$. If there is one $\varrho \in \Pi$, $g \notin T$, such that $s^{-1}gs \in \Pi$ for all $s \in S$, then $G = S \otimes \Pi$.*

LEMMA 3. *Let S and Π be subgroups of G such that $G = S\Pi = \Pi S$ and $S \cap \Pi = \{1\}$. If there is one element $s \in S$ such that $\pi s \pi^{-1} \in S$ for all $\pi \in \Pi$, and if no proper invariant subgroup of S contain s, then $G = S \otimes \Pi$.*

(Proofs are straightforward, cf. Michel 1964.)

Solution of the Mass Spectrum Problem

From a physical point of view there are no compelling reasons to consider irreducible representations of finite Lie groups G containing Π. The origin of the idea of larger dynamical groups G goes back to the description of mass spectrum of composite systems, like H-atom, *at rest* by non-compact groups (cf. ch. 12). The relativistic generalization of this idea to *moving systems*, i.e. the inclusion of the momenta P_μ is also possible. An elegant solution of this problem is via the infinite-component wave equations (§ 3) which describe covariantly systems with internal degrees of freedom and provides a *discrete* mass spectrum. Here the irreducible representations of the dynamical group G is used again to generate the states of the system at rest, and the wave equation defines boosted states to a momentum p_μ. This turns out to be the proper relativistic generalization of the dynamical groups, and not a finite Lie group containing G. The algebraic structure of the infinite-component wave equations involves an infinite-parameter Lie algebra hence there is no contradiction with the above theorems.

§ 6. Comments and Supplements

Let us stress again that the general formalism presented in sec. 1 may be applied to the derivation of wave equations covariant with respect to an arbitrary topological group. In particular one may derive wave equations covariant with respect to SO(3), SO(4), SO(4, 2), $T^3 \circledS \mathrm{SO}(3)$ or generalized Poincaré group $T^n \circledS \mathrm{SO}(n-1, 1)$; in each case wave equations will represent irreducibility conditions with respect to invariant quantities characterizing irreducible representations of a given group like mass and spin irreducibility conditions in the case of the Poincaré group.

We now give for completeness a brief characterization of other conventional finite-component wave equations which were not discussed in sec. 2. They are all special cases of eq. 1 (17) or of Gel'fand–Yaglom equation 3 (1).

A. *Fierz–Pauli Equations*

These equations were postulated by Fierz 1939 and Fierz and Pauli 1939 in the form ($\partial_{\alpha\dot\beta} \equiv (\sigma_\mu \partial^\mu)_{\alpha\beta}$)

$$\partial_{\alpha\dot\beta}\varphi^{\alpha,\alpha_1,\dots,\alpha_l}_{\dot\beta_1,\dots,\dot\beta_k}(x) = \mathrm{i}\varkappa\chi^{\alpha_1,\dots,\alpha_l}_{\dot\beta,\dot\beta_1,\dots,\dot\beta_k}(x) \tag{1}$$

and

$$\partial^{\alpha\dot\beta}\chi^{\alpha_1,\dots,\alpha_l}_{\dot\beta,\dot\beta_1,\dots,\dot\beta_k}(x) = \mathrm{i}\varkappa\varphi^{\alpha,\alpha_1,\dots,\alpha_l}_{\dot\beta_1,\dots,\dot\beta_k}(x). \tag{2}$$

Here both spinors are symmetric: hence they transform according to the following representation of SL(2, C)

$$D(\Lambda) = D^{[(l+1)/2,\, k/2]} \oplus D^{[l/2,\, (k+1)/2]}. \tag{3}$$

Consequently the highest spin is $j = \frac{1}{2}(l+k+1)$. Thus the Fierz–Pauli equation is characterized by the pair $(D(\Lambda), \pi)$ where $D(\Lambda)$ is given by eq. (3) and π is a projector onto the highest spin. The Fierz–Pauli equation does not admit a parity operator unless $l = k$. Cleary for $l = k = 0$ we obtain the Dirac equation.

B. *Duffin–Kemmer–Petiau Equations*

We noticed that the Gel'fand–Yaglom condition given by th. 3.1 may be also satisfied by a direct sum of finite-dimensional representations. In particular the representations

$$D(\Lambda) = D^{(0,0)} \oplus D^{(1/2,1/2)} \tag{4}$$

or

$$D(\Lambda) = D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)} \tag{5}$$

satisfy the condition 3 (3).

The corresponding wave equation are

$$(\beta_\mu p^\mu + m)\psi(p) = 0 \tag{6}$$

where in the present case $\{\beta_\mu\}_{\mu=0}^3$ is a set of 5×5 or 10×10 matrices, respectively. The explicit form of these matrices can be found using eqs. 3 (6) and 3 (5). Eq. (6) represents the wave equation for spin zero and spin one particles, respectively and exhibit the spin irreducibility condition 1 (17). Eq. (6) with $D(\Lambda)$ given by eqs. (4) or (5) are called the *Duffin–Kemmer–Petiau equations.* They are equivalent to Proca equation: indeed setting

$$\begin{aligned} \psi &= (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10}) \\ &= \left(-\frac{1}{m}B_{14}, -\frac{1}{m}B_{24}, -\frac{1}{m}B_{34}, -\frac{1}{m}B_{23}, -\frac{1}{m}B_{31}, -\frac{1}{m}B_{12}, \Phi_1, \Phi_2, \Phi_3, \Phi_4\right) \end{aligned} \tag{7}$$

we find that Proca equations 2(15) and 2(16)

$$p_\mu \Phi_\nu - p_\nu \Phi_\mu = B_{\mu\nu} \quad \text{and} \quad p^\mu B_{\mu\nu} = m^2 \Phi_\nu \tag{8}$$

can be written in the form of eq. (6).

C. *Bhabba Equations*

Covariant wave equations like (6) in the case when $\psi(p)$ transforms according to a finite-dimensional representation $D(\Lambda)$ of SL(2, C) are often called *Bhabba type equations.* The Gel'fand–Yaglom Theorem gives a general method for construction of such equations. A non-trivial example is provided by the following representation $D(\Lambda)$ of SL(2, C)

$$D(\Lambda) = (D^{(1/2,0)} \oplus D^{(0,1/2)} \oplus D^{(1,1/2)} \oplus D^{(1/2,1)})(\Lambda) \tag{9}$$

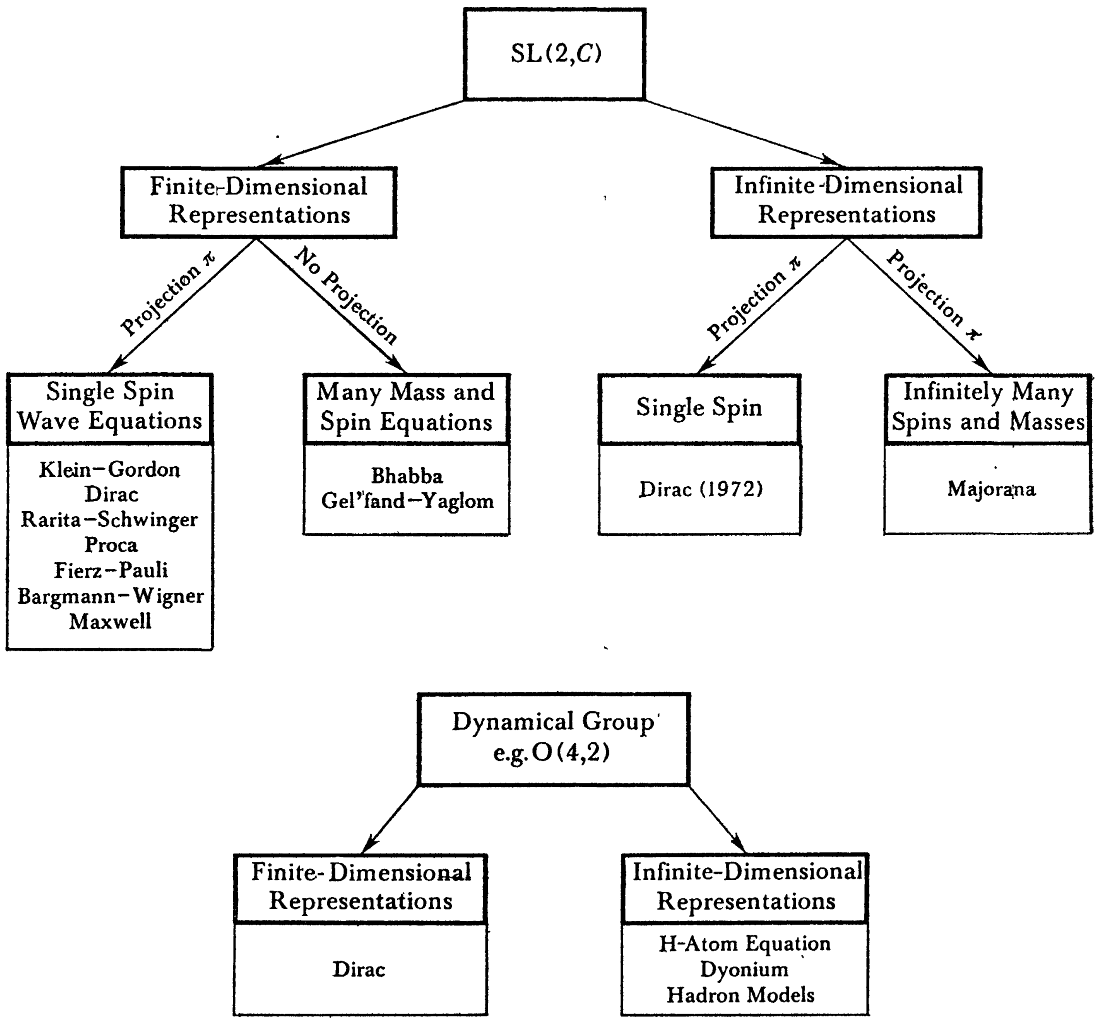
which satisfies conditions 3 (3). Clearly this equation describes particles with spin $\frac{3}{2}$ and $\frac{1}{2}$. Solving eq. 3 (6) for the representation (9) we obtain the matrix β_0. Passing to the rest frame and solving the eigenvalue equation $\beta_0 p_0 \psi = -m\psi$ one finds that the masses of corresponding particles are

$$m_{3/2} = m, \; m_{1/2}^{(1)} = \lambda_1^{-1}, \; m_{1/2}^{(2)} = \lambda_2^{-1}, \tag{10}$$

where λ_1 and λ_2 are characteristic roots of the matrix β_0. Hence we have a mass spectrum as in the case of general Gel'fand–Yaglom equations 3 (1). This is a typical situation for general Bhabba-like equations, they contain a finite number of masses and spins.

Table 1 shows the group structure of different relativistic wave equations.

Table 1



D. *Applications of the Imaginary Mass* ($m^2 < 0$) *Representations of the Poincaré Group*

Although the representations of the Poincaré group with $m^2 < 0$ cannot be interpreted physically as isolated quantum systems with measurable mass, they occur in relativistic two-particle systems as follows.

Consider the scattering process of two particles a and b producing two same or other particles c and d:

$$a+b \to c+d. \tag{11}$$

The energy-momentum for vectors satisfy the conservation law

$$p_a+p_b = p_c+p_d. \tag{12}$$

For spinless particles the amplitudes for the scattering process depends only on the invariant products of the momenta which can be taken as, e.g.

$$s = (p_a+p_b)^2 \quad \text{and} \quad t = (p_a-p_c)^2. \tag{13}$$

For a fixed value of s the states of the two-particle system (a, b) belong to irreducible representations of the Poincaré group, the generators of which being $P^\mu = P^\mu_a + P^\mu_b$ and $J^{\mu\nu} = J^{\mu\nu}_a + J^{\mu\nu}_b$.

In the physical region of the process (11),

$$s > (m_a+m_b)^2 > 0, \tag{14}$$

and this is the usual representation of the Poincaré group in the tensor product space of the two particles.

On the other hand for each fixed value of t, we may consider another Poincaré group with generators

$$P'^\mu = P^\mu_a - P^\mu_c \quad \text{and} \quad J^{\mu\nu} = J^{\mu\nu}_a + J^{\mu\nu}_c, \tag{15}$$

so that we have, again in the physical region of the process (1) representations of the Poincaré group, but with $p'_\mu p'^\mu = (p^\mu_a - p^\mu_c)^2 < 0$. The significance of these remarks can be seen from the partial-wave expansions (i.e. harmonic analysis) of the scattering amplitude.

For the reaction (11) the scattering amplitude can be written as a matrix element

$$T^{S_c S_d S_a S_b}_{\lambda_c \lambda_d \lambda_a \lambda_b}(p) = \langle m_c S_c p_c \lambda_c ; m_d S_d p_d \lambda_d | T | m_a S_a p_a \lambda_a, m_b S_b p_b \lambda_b \rangle, \tag{16}$$

where m, S, p, λ denote mass, spin, momentum and helicity of each particle. The relativistic invariance implies that these matrix elements transform according to the tensor products of representations of the Poincaré group to which the particles belong, denoted by $U(c)^*$, $U(d)^*$, $U(a)$ and $U(b)$ symbolically.

We can decompose the product $U(a) \otimes U(b)$ and similarly the product $U(c)^* \otimes \otimes U(d)^*$, into the direct integral of representations of the Poincaré group. We then obtain an expansion of the T-matrix (16) in terms of the two-particle states (a, b), or (c, d). Physically these are the states formed by the particles a and b, for example, resonances or compound states of a and b. The relevant Poincaré algebra then is (14). In particular, if we use in the expansion, a basis consisting of P^2, J^2, J_3 for (14), we obtain the so-called partial wave (or direct channel) expansion of the T-matrix, a sum over all possible total angular momenta of the intermediate states formed by a and b which then decay into the particles c and d.

We can also decompose the direct products

$$U(a)\, U(c)^* \quad \text{or} \quad U(a)\, U(d)^* \tag{17}$$

into the direct integrals of the representations of the Poincaré group. However, in this case, we see immediately that we must use the Poincaré group generated by (15). The little group in this case is $O(2, 1)$, so we are led to $m^2 = t < 0$— representations of the Poincaré group. Physically, the momenta $p_a - p_c$ in (15) correspond to the momentum of the 'exchange particle' (Fig. 1.B), and the expansion is called a crossed channel expansion of the T-matrix in terms of the possible two-particles states in the crossed channel.

These expansions gain a further significance due to the analytic continuation properties of the T-matrix elements in momenta, or, analytic properties of the representations of the Poincaré group. If we continue analytically p_c into $-p'_c$ and p_b into $-p'_b$, eq. (12) goes into

$$p_a+p'_c \to p'_b+p_d \tag{18}$$

and describes the physical momentum balance in the reaction

$$a+c \to b+d. \tag{19}$$

Looking at Fig. 1.B, we see that the 'exchanged' particles for the process (1) are just the two particle-states of the reaction (19). It is assumed in particle physics that a single analytic function describes both processes (11) and (19)—and others—in their respective physical region of momenta satisfying (12) or (18).

There exist a great many other forms of expansion of the T-matrix depending on which homogeneous or symmetric spaces we represent T.

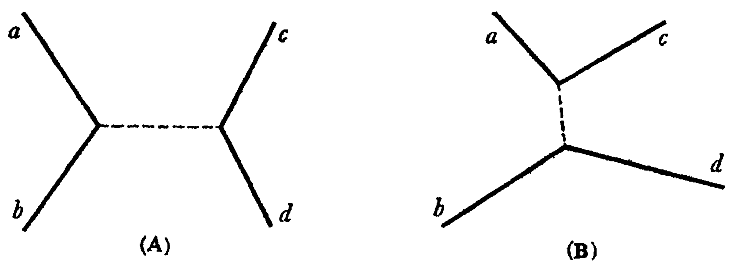


Fig. 1

E. *Bibliographical Remarks*

The relativistic wave equation for a massive spin $\frac{1}{2}$ particle was proposed by Dirac in 1928. At that time only tensor representations of the Lorentz group were known and the question arose if the Dirac equation can be covariant at all, as it does not transform according to any tensor representations. A clarification of this problem gave rise to a new class of representations of the Lorentz group, i.e., spinor representations. Next it was observed that Dirac equation admits negative energy solutions which represented a difficulty in the physical interpretation of the equation. In order to remove this difficulty Majorana proposed in 1932 the infinite-component wave equation which does not contain the negative energy states. It was however realized soon that the Majorana equation has also solutions $\psi(p)$ with $p^2 = m^2 \leqslant 0$.

It was later recognized that negative energy states of the Dirac equation have a natural interpretation in quantum field theory as antiparticles and in 1931 antiparticles were discovered experimentally. The wave equation for massless spin $\frac{1}{2}$ particle was first proposed by Weyl in 1932. This equation was discussed by

Pauli in 1933, but it was rejected because of its noninvariance under space inversions. When it was verified experimentally that parity is violated in reactions involving neutrinos, Landau 1957, Salam 1957 and Lee and Yang proposed the Weyl equation for the description of neutrino states.

The wave equation for a massive spin 1 particle was proposed by Proca in 1936 and by Petiau in his thesis also in 1936. The general aspects of these equations were discussed by Duffin 1938 and Kemmer 1939.

The general wave equations (1) and (2) for particles with arbitrary spin were discussed first by Fierz 1939 and by Fierz and Pauli 1939 in the language of spinors. The alternative formulation in the language of spin-tensors was given by Rarita and Schwinger 1941. The interesting description of tensor particles in terms of symmetric tensor wave functions was given by Bargmann and Wigner in 1946.

A relatively simple form of covariant wave equations for massive and massless particles with arbitrary spin was proposed by Joos 1962, Barut, Muzinich and Williams 1963, and Weinberg 1964.

The derivation of the finite-component relativistic wave equations presented in this chapter is based on the theory of induced representations and enables us to give a unified derivation of all finite-component covariant wave equations starting from the single equation 1(17) (cf. also Niederer and O'Raifeartaigh 1974).

All these equations concern massive particles with a specific mass. Bhabba proposed in 1945 an analogue of the Dirac equation for the description of a system with several masses and spins (see also 1949). A general theory of such equations was developed by Gel'fand and Yaglom in 1947, which included also infinite-component wave equations of Majorana type. Gel'fand–Yaglom equations provide in general a mass spectrum which is decreasing with increasing spin. Various other equations have been recently given and discussed by Nambu 1967, Fronsdal 1968, Abers, Grodsky and Norton 1967, Barut *et al.* 1967 and others. These equations modify the Gel'fand–Yaglom form in order to obtain an increasing mass spectrum.

A detailed discussion of Bhabba like equations and some other covariant wave equations is given in Umezawa's book 1956.

The application of infinite-component wave equations for the description of properties of realistic quantum systems was developed mainly by Barut and collaborators. Using this approach combined with SO(4, 2)-symmetry of quantum systems they succeeded to obtain not only the spectrum of hydrogen-like atom, but also they were able to take into account the recoil correction, to find forms factors and to describe other characteristic features of quantum systems (see Barut and Kleinert 1967 and Barut *et al.* 1968–75). The application of this approach for a description of the internal structure of hadrons was also successful (cf. Barut 1970).

§ 7. Exercises

§ 1.1. Can one derive the general Poincaré invariant wave equation 1(17) from the general wave equation 1(12).

Hint: The Poincaré group has no other finite-dimensional representations beside those of SL (2, C) lifted to Π (cf. example 8.7.1).

§ 2.1. Derive the wave equation for a spin 1 massive particle using the representation $D^{(1,0)} \oplus D^{(0,1)}$ of SL(2, C).

Hint: Follow the derivation of Dirac equation.

§ 2.2. 'Maxwell Group': consider the wave equation of a particle in a fixed given external field. For example the Schrödinger equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}(p - \mathrm{e}A)^2 + \mathrm{e}A_0\right]\psi$$

or the Dirac equation

$$[\gamma^\mu(p_\mu - eA_\mu) - m]\psi = 0.$$

The symmetry groups of these equations are now only subgroups of the Galilei group, or Poincaré group, respectively. (Or a subgroup of the conformal group if either m is transformed, or $m = 0$.) Because of gauge invariance, this subgroup is given by those transformations which leave $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ invariant. In particular, determine the symmetry group in the case of constant $F_{\mu\nu}$ and study the induced projective representations of this group. (cf. H. Bacry, Ph. Combe and J. L. Richard 1970; J. L. Richard 1972; R. Schrader 1972). If the fields are also transformed, we regain the full invariance. Cf. also the following problem.

§ 3.1 Discuss the light-like solutions of the Majorana equation.

§ 4.1. Study the extension of the space-time group Π (Poincaré group or Galilei group) by the electromagnetic 'gauge group' K, i.e. abelian group of all real-valued functions $\in C^\nu$ or C^∞, on the Minkowski space M^4 (cf. U. Cattaneo and A. Tanner 1974 and references therein).

§ 4.2. Show that the group of automorphisms of the Poincaré group Π is

$$\mathrm{Aut}\Pi = \Pi \circledS (C_2 \times C_2 \times D),$$

where D is the multiplicative group of real numbers $\varrho < 0$ occurring in $(a, \Lambda) \to (\varrho a, \Lambda)$, and $C_2 \times C_2$ is the group generated by space and time reflecting. What is the group of automorphisms of the Galilei group (compare exercise 1.10.1.11).

Appendix A

Algebra, Topology, Measure and Integration Theory

1. *Zorn's Lemma*

A relation R in a set X is called *reflexive* if xRx for $\forall x \in X$, *symmetric* if xRy implies yRx, *antisymmetric* if xRy and yRx implies $x = y$ for $\forall x, y \in R$, and *transitive* if xRy and yRz implies xRz for $\forall x, y, z \in X$.

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive.

A relation R on a set X is called a *partial ordering* if it is reflexive, transitive and antisymmetric. If R is a partial ordering we write $x \prec y$ instead of xRy.

If for all x, y in X either $x \prec y$ or $y \prec x$, X is said to be *linearly ordered.*

An element $u \in X$ is called an *upper bound* for a subset $Y \subset X$ if $y \prec u$ for all $y \in Y$. An element $m \in X$ is called a maximal element of X if $m \prec x$ implies $x = m$.

ZORN'S LEMMA. *Every nonempty partially ordered set X with the property that every linearly ordered subset has an upper bound in X possesses a maximal element.* ▼

2. *Rings*

PROPOSITION 1. *Let R be a ring. If grR is a (left) Noether ring without zero-divisors then R is also (left) Noether ring without zero-divisors.*

(For the proof cf. Chow (1969), proposition 13.)

PROPOSITION 2. *A (left) Noether ring without zero-divisors satisfies the (left) Ore condition.*

(For the proof cf. Chow 1969, proposition 14.)

3. *Semigroups*

A semigroup G is a nonempty set S and a mapping $(x, y) \to xy$ of $G \times G$ into G such that

$$x(yz) = (xy)z \quad \text{for all } x, y, z \text{ in } G.$$

Let X be any nonempty set. A *free* semigroup S is the set of all finite formal products

$$x_1 \cdot x_2 \dots x_n,$$

where $x_1, x_2, \ldots, x_n \in X$, $n \geqslant 1$ with the multiplication law

$$(x_1 \cdot x_2 \ldots x_a)(y_1 y_2 \ldots y_a) = x_1 \cdot x_2 \ldots x_n \cdot y_1 \cdot y_2 \ldots y_n.$$

4. *Principle of Uniform Boundedness*

A complete metric vector space over a field K is called an *F-space* if

(i) $d(x, y) = d(x-y, 0)$ for all $x, y \in X$,

(ii) the map $(\lambda, x) \to \lambda x$ of $K \times X$ into X is continuous with respect to λ for every x and continuous with respect to x for every λ.

PRINCIPLE OF UNIFORM BOUNDEDNESS. *Let, for every element a of a set A, T_a be a continuous linear map of an F-space X into an F-space Y. If for every $x \in X$ the set $\{T_a x | a \in A\}$ is bounded then $\lim_{x \to 0} T_a x = 0$ uniformly relative to $a \in A$.* ▼

(For the proof see Dunford and Schwartz 1958, ch. I.)

5. *Measures and Integration*

A Borel structure on a set X is a family B of subsets of X such that

(i) $\emptyset, X \in B$.

(ii) If $X_j \in B$, then $X - X_j$, $\bigcap_{j=1}^{\infty} X_j$ and $\bigcup_{j=1}^{\infty} X_j$ are in B.

The pair (X, B) is called a *Borel space*. Elements of B are called *Borel subsets* of X.

Every set X has at least one Borel structure. In fact if F is any family of subsets of a set X then there exists a smallest Borel structure for X which contains F.

If X and Y are Borel spaces then the map $F\colon X \to Y$ is called a *Borel map* if $f^{-1}(Y_i)$ is a Borel subset of X for every Borel subset Y_i of Y. A Borel map is called a *Borel isomorphism* if f is one-to-one and onto and if f^{-1} is a Borel map.

A Borel space is called *standard* if it is Borel isomorphic to the Borel space associated with Borel subsets of a complete separable metric space.

A Borel measure is a map $\mu\colon B \to [0, \infty)$ such that

(i) $\mu(X_i) \geqslant 0$ for $X_i \in B$, $\mu(\emptyset) = 0$.

(ii) If $X_i \in X$ are disjoint elements of B then

$$\mu(\bigcup_i X_i) = \sum_i \mu(X_i).$$

(iii) There exists $X_k \in B$, $k = 1, 2, \ldots$ such that

$$\mu(X_j) < \infty \quad \text{and} \quad X = \bigcup_j X_j.$$

A Borel measure on X is called *standard* if there exists a Borel subset N such that $\mu(N) = 0$ and $X - N$ is a standard Borel space.

A real-valued nonnegative function on a Borel space X is called *measurable* if for any $a > 0$ the set $\{x\colon f(x) > a\}$ is measurable. A complex-valued function is measurable if it may be written in the form $f = f_1 - f_2 + i(f_3 - f_4)$ with f_i real, nonnegative and measurable. A function obtained from measurable functions by means of algebraic operations or passage to the limit is measurable.

The limit of a Lebesgue sum relative to the measure μ is called the *integral* $\int_X f(x)\,d\mu(x)$ of f. If $\int_X |f(x)|\,d\mu(x)$ is finite then f is called *integrable.*

Given a measure μ on a measure space X we can define a set of integrable functions on X and we can form $L^p(X, \mu)$ spaces as in the case when $X = R$.

A measure ν is said to be *absolutely continuous with* respect to the measure μ if and only if $\mu(A) = 0$ implies $\nu(A) = 0$. We have

RADON–NIKODYM THEOREM. *A measure ν is absolutely continuous with respect to μ if and only if there is a measurable function ϱ such that*

$$\nu(A) = \int \varrho(x)\,\chi_A\,d\mu(x)$$

for any measurable set A, where χ_A is the characteristic function of the set A. ▼

One often uses this theorem in the differential form

$$\frac{d\nu}{d\mu}(x) = \varrho(x).$$

6. *Lebesgue Theorems*

DOMINATED CONVERGENCE THOREM. *If $\{u_n\}_1^\infty$ is a sequence of integrable functions, convergent almost everywhere to a function u and if $|u_n(x)| \leqslant v(x)$ almost everywhere, where v is integrable, then the function u is also integrable and*

$$\int_X u(x)\,d\mu(x) = \lim_{n\to\infty} \int_X u_n(x)\,d\mu(x). \quad ▼$$

The assumptions of the last theorem are satisfied in particular, when $\mu(X) < \infty$ and $|u_n(x)| \leqslant M$ almost everywhere, and M is a constant. In this case we obtain

BOUNDED CONVERGENCE THEOREM. *If $\{u_n\}_\infty^1$ is a nondecreasing sequence of integrable functions, convergent to a function u almost everwyhere then*

$$\lim_{n\to\infty} \int_X u_n(x)\,d\mu(x) = \int_X u(x)\,d\mu(x). \quad ▼$$

Let X_1 and X_2 be two locally compact spaces with measures μ_1 and μ_2, respectively. Then there exists on the space $X = X_1 \times X_2$ a measure ν such that

$$\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$

for every measurable sets $A_i \subset X_i$. The measure ν is called the *tensor product* of measures μ_1 and μ_2 and is denoted by $\mu_1 \otimes \mu_2$.

7. FUBINI THEOREM

Version I. *If f: $X_1 \times X_2 \to R$ (or C) is $\mu_1 \otimes \mu_2$ integrable then the integrals*

$$I_{12} = \int_{X_1} d\mu_1(x_1) \int_{X_2} f(x_1, x_2) d\mu_2(x_2)$$

and

$$I_{21} = \int_{X_2} d\mu_2(x_2) \int_{X_1} f(x_1, x_2) d\mu_1(x_1)$$

exist and

$$\int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \otimes \mu_2) = I_{12} = I_{21}.$$

Version II. *If f is $\mu_1 \otimes \mu_2$ measurable and $f \geqslant 0$ then all three above integral exists (finite or infinite) and are equal.* ▼

The Fubini Theorem implies the following

THEOREM ON INTEGRATION OF SERIES. *Let X be a Borel space with a Borel measure μ. Let $\{f_n(x)\}_1^\infty$ be a sequence of μ-measurable functions. If*

$$\int_X \sum_{n=1}^{\infty} |f_n| d\mu(x) \quad \text{or} \quad \sum_{n=1}^{\infty} \int |f_n(x)| d\mu(x)$$

exists, then

$$\int_X \sum_{n=1}^{\infty} f_n(x) d\mu(x) = \sum_{n=1}^{\infty} \int f_n(x) d\mu(x). \ ▼$$

8. *Various Results*

FRECHET AND RIESZ THEOREM. *To every linear continuous functional L in a Hilbert space H there correspond a vector $l \in H$ such that*

$$L(u) = (u, l) \quad \text{for all } u \in H. \ ▼$$

(For the proof see Maurin 1969, ch. I.)

Appendix B

Functional Analysis

§ 1. Closed, Symmetric and Self-Adjoint Operators in Hilbert Space

We begin with some basic properties of operators in the Hilbert space.

An operator A with the domain $D(A) \subset H$ is said to be *continuous* at a point $u_0 \big(u_0 \in D(A)\big)$, if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$||u-u_0|| \leqslant \delta, \ u \in D(A) \Rightarrow ||Au-Au_0|| < \varepsilon.$$

An operator A is said to be *bounded* if there exists a constant C such that

$$||Au|| \leqslant C||u|| \qquad \text{for all } u \in D(A).$$

A bounded, linear operator is (uniformly) continuous. Indeed,

$$||Au-Au_0|| = ||A(u-u_0)|| \leqslant C||u-u_0||. \tag{1a}$$

Hence, if $||u-u_0|| \to 0$, then $||Au-Au_0|| \to 0$, i.e., $Au \to Au_0$. Conversely, if a linear operator A is continuous at a point u_0 (e.g., $u_0 = 0$), then A is bounded. In fact, let $||Au-Au_0|| < \varepsilon$ for $||u-u_0|| < \delta(\varepsilon)$. Then $||Av|| < \varepsilon$ for $||v|| \leqslant \delta(\varepsilon)$ by linearity of A. Because for every $w \in H$ we have $\left\|\frac{w}{||w||}\,\delta(\varepsilon)\right\| = \delta(\varepsilon)$, then

$$||Aw|| = \left\|A\frac{w}{||w||}\,\delta(\varepsilon)\right\|\frac{||w||}{\delta(\varepsilon)} < \varepsilon\frac{||w||}{\delta(\varepsilon)}. \tag{1b}$$

Setting $C = \varepsilon/\delta(\varepsilon)$, we obtain

$$||Aw|| < C||w||$$

for every $w \in D(A)$. Hence A is bounded. We see therefore that for linear operators in Hilbert space the continuity and boundedness are equivalent properties.

An operator A is said to be *positive definite*, if

$$(Au, u) \geqslant m||u||^2$$

for some $m > 0$ and all $u \in D(A)$, and is said to be *positive* if $(Au, u) \geqslant 0$ for all $u \in D(A)$. An operator A is said to be *closed* if from

$$u_m \in D(A), \quad \lim u_m = u, \quad \lim Au_m = v$$

it follows that

$$u \in D(A) \quad \text{and} \quad v = Au.$$

Notice the essential difference between continuous and closed operators: if A is continuous, then the existence of a limit

$$\lim_{n\to\infty} u_n \to u \in D(A)$$

implies the existence of a limit $\lim_{n\to\infty} Au_n$; on the other hand, if A is only closed, then from the convergence of a sequence

$$u_1, u_2, \ldots, u_m \in D(A), \tag{2}$$

it does not follow that the sequence

$$Au_1, Au_2, \ldots \tag{3}$$

is also convergent.

If A is not closed we define the *closure* $\bar{A}$ of A to be an operator with the following properties.

1° The domain $D(\bar{A})$ of $\bar{A}$ consists of all vectors $u \in H$ for which there exists at least one sequence (2) generating a convergent sequence (3).

2° The action of $\bar{A}$ is defined by the equality

$$\bar{A}u = \lim_{n\to\infty} Au_n, \quad u \in D(\bar{A}).$$

It follows from the definition of the closure that an operator A admits a closure $\bar{A}$, if and only if, from the relations

$$u_n \to 0, \quad Au_n \to v$$

it follows that $v = 0$.

Let A be a linear operator in H (bounded or not). There exist pairs $v, v' \in H$ such that the equality

$$(Au, v) = (u, v') \tag{4}$$

is satisfied for every $u \in D(A)$. In fact, equality (4) is satisfied at least for $v = v' = 0$. It seems natural to set $v' = A^*v$ and call the operator A^* the *adjoint* of A. However, this definition of the adjoint operator A^* will be meaningful only if an element v' is uniquely determined by v. This condition will be satisfied if and only if $D(A)$ is dense in H. Indeed, if $D(A)$ is not dense in H and $w \neq 0$ is orthogonal to $D(A)$, then for every $u \in D(A)$ together with (4), we have

$$(Au, v) = (u, v'+w),$$

i.e., the symbol A^*v is meaningless. Conversely, if $D(A)$ is dense in H and if for any $u \in D(A)$

$$(Au, v) = (u, v_1'),$$

$$(Au, v) = (u, v_2'),$$

then for an arbitrary $u \in D(A)$ we have

$$(u, v_1' - v_2') = 0,$$

which is impossible if $v_1' \neq v_2'$. Therefore, if $D(A)$ is dense in H, then an operator A has the adjoint operator A^*: the domain $D(A^*)$ of A^* is the set of all $v \in H$, for which there exists $v' \in H$ satisfying eq. (4) for an arbitrary $u \in D(A)$. The action of A^* on $D(A^*)$ is given by the formula

$$A^*v = v'. \tag{5}$$

Notice that $v \in D(A^*)$ if and only if (Au, v) is a linear continuous functional on $D(A)$. The adjoint operator A^* has a series of interesting properties:

LEMMA 1. 1° *The operator A^* is linear.*

2° *The operator A^* is closed even if A is not closed.*

3° *If the operator A has a closure $\bar{A}$, then $(\bar{A})^* = A^*$.*

All of these properties directly follow from the definition of the adjoint operator A^*. Moreover, it can be proved that

$$A^{**} = \bar{A} \tag{6}$$

(cf. e.g. Stone 1932b, th. 2.9).

Let A, B be operators in H with $D(B) \supset D(A)$ and let

$$Bu = Au \quad \text{for } u \in D(A),$$

then the operator B is said to be an *extension* of A, and A is said to be a *reduction* of B. We shall write $B \supset A$. In particular the closure $\bar{A}$ is an extension of A. Note that

$$A \subset B \Rightarrow A^* \supset B^*. \tag{7}$$

Indeed, if $v \in D(B^*)$, then there exists $v' \in H$ such that $(Bu, v) = (u, v')$ for all $u \in D(B)$. If $A \subset B$, then also $(Au, v) = (u, v')$ for all $u \in D(A)$, because $Au = Bu$. Therefore $v' \in D(A^*)$ and $A^*v = v' = B^*v$. Hence $A^* \supset B^*$.

An operator A is said to be *symmetric* if $A^* \supset A$. In other words, a densely defined operator A is symmetric if

$$(Au, v) = (u, Av) \quad \text{for all } u, v \in D(A). \tag{8}$$

An operator A is said to be *self-adjoint* if $A^* = A$. Because A^* is closed (cf. lemma 1), then a self-adjoint operator is closed.

LEMMA 2. *Let A be a linear symmetric operator and $D(A) = H$. Then*

1° $A^* = A$.

2° *A is a bounded operator.*

PROOF: Ad 1°. Because $A^* \supset A$, then $D(A^*) \supset D(A) = H$. Hence $D(A^*) = H$ and consequently $A^* = A$.

Ad 2°. Because A is symmetric we have

$$(Au, v) = (u, Av), \quad u, v \in H.$$

Let $u_n \to u_0$ and $Au_n \to v_0$. For any $v \in H$, we have

$$(v_0, v) = \lim(Au_n, v) = \lim(u_n, Av)$$
$$= (u_0, Av) = (Au_0, v).$$

Hence $Au_0 = v_0$, i.e., A is continuous and therefore bounded. ▼

This lemma indicates that an unbounded self-adjoint operator cannot be defined on the whole Hilbert space H. Hence we might define them only on some dense domain in H. The selection of a proper dense domain for an unbounded operator is one of the most difficult problems of functional analysis and consequently also of quantum physics.

We shall now discuss the properties of symmetric operators and self-adjoint extensions of operators. We have

LEMMA 3. *An operator A having a symmetric extension $\tilde{A}$ is itself symmetric. Every symmetric extension $\tilde{A}$ of an operator A is a reduction of the operator A^*.*

PROOF: By assumption $\tilde{A} \supset A$ and $\tilde{A} \subset \tilde{A}^*$. Then $\tilde{A}^* \subset A^*$, by eq. (7), and consequently

$$A \subset \tilde{A} \subset \tilde{A}^* \subset A^*. \tag{9}$$

We also readily verify, using the definition of closed and symmetric operators and the properties of their domains that the closure $\bar{A}$ of a symmetric operator A is a symmetric operator, i.e.

$$(\bar{A})^* \supset \bar{A}. \;\blacktriangledown \tag{10}$$

We know that only self-adjoint operators are proper candidates for physical observables. On the other hand, symmetric operators most often occur in quantum physics. Hence, the important question: when does a symmetric operator admit a self-adjoint extension? This problem can be solved with the help of the concept of *deficiency indices.*

Let A be a linear densely defined operator in a Hilbert space H, and let A^* be the adjoint of A. Set

$$D_+ = \{u \in D(A^*) : A^*u = iu\}, \quad D_- = \{u \in D(A^*) : A^*u = -iu\}. \tag{11}$$

The spaces D_+ and D_- are said to be the spaces of positive and negative deficiency of the operator A, respectively. Their dimensions (finite of infinite numbers) denoted by n_+ and n_-, respectively, are called *deficiency indices* of the operator A.

THEOREM 4. *Let A be a symmetric operator. Then*

1° *A has a self-adjoint extension if and only if $n_+ = n_-$.*

2° *If $n_+ = n_- = 0$, then the unique self-adjoint extension of A is its closure $\bar{A} = A^*$.*

(For the proof of the theorem cf. Dunford and Schwartz, vol. II, ch. 12, sec. 4.)

This theorem solves completely the problem of the existence of a self-adjoint extensions of a symmetric operator.

Let us note that with a given physical symmetric operator one can associate many self-adjoint extensions, and consequently many physical observables. It is evident that the selection of the proper representative for a physical observable must be a physical problem. In fact, the different self-adjoint extensions might have (in the same space H) different eigenvalues and different complete sets of orthogonal eigenfunctions; for example, consider the operator $d_\vartheta = i^{-1}\dfrac{d}{d\varphi}$ in $L^2(0, 2\pi)$ whose eigenfunctions u

$$d_\vartheta u = \lambda u$$

satisfy the boundary condition

$$u(2\pi) = \vartheta u(0), \quad \vartheta = \exp(i\omega), \quad \omega = \text{const} \in R.$$

The solutions have the form

$$u_n(\varphi) = \exp\left[i\left(n+\frac{\omega}{2\pi}\right)\varphi\right], \quad n = 0, \pm 1, \pm 2, \dots \tag{12}$$

Hence, the eigenvalues of the operator d_ϑ are numbers $\lambda_n = n+\dfrac{\omega}{2\pi}$. It follows from the spectral theorem that every set of functions (12) for a fixed $\vartheta\{u_n\}$ forms a complete orthonormal set of functions in $L^2(0, 2\pi)$. Note that in quantum mechanics we select the extension d_ϑ with $\vartheta = 1$ by the requirement of the uniqueness of the wave function. This requirement is not however universal, because, for example, for half-integer spins we admit two-valued wave functions and some authors even consider infinite valued wave functions.

§ 2. Integration of Vector and Operator Functions

Let $u(t)$ be a function defined on the interval $[a, b] \subset R$ with values in a Hilbert space H. The Riemann integral of the function $u(t)$ is defined in the same manner as the Riemann integral of ordinary functions.

A subdivision Δ^i of the interval $[a, b]$ is a system of n_i points

$$a = t_0^{(i)} < t_{n_1}^{(i)} < \dots < t_{n_i}^{(i)} = b. \tag{1}$$

A sequence $\{\Delta^i\}$ of subdivisions is said to be normal if

$$\lim_{i\to\infty} \sup_{1\leqslant k\leqslant n_i} |t_k^{(i)} - t_{k+1}^{(i)}| = 0. \tag{2}$$

Set

$$S(u(t), \Delta^i, t_k^i) = \sum_{k=1}^{n} u(t_k)(t_k^{(i)} - t_{k-1}^{(i)}), \tag{3}$$

where t_k is an arbitrary point satisfying the inequalities $t_{k-1}^{(i)} \leqslant t_k \leqslant t_k^{(i)}$. If the limit

$$\lim_{i\to\infty} S(u(t), \Delta^i, t_k^i) \tag{4}$$

exists for an arbitrary normal sequence of subdivisions and for an arbitrary choice of points t_k, then this limit is called the *Riemann integral* of the function $u(t)$ and is denoted by

$$\int_{[a,b]} u(t)\mathrm{d}t. \tag{5}$$

One can show, as in the case of ordinary functions, that this limit does not depend on the choice of the normal sequences of subdivisions or on the choice of the properties of t_k.

The Riemann integral (5) of a vector function $u(t)$ has all the properties of Riemann integrals of ordinary functions. In particular, we have

LEMMA 1. *A vector function $u(t)$ in H, continuous on the interval $[a, b] \subset R$, is integrable on $[a, b]$.*

PROOF: A bounded interval $[a, b]$ of R is compact. We know that a continuous map $t \to u(t)$ of compact set is uniformly continuous. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$||u(t_1| - u(t_2)|| < \varepsilon \quad \text{if } |t_1 - t_2| < \delta. \tag{6}$$

Take a subdivision Δ^i of the interval $[a, b]$ with a diameter smaller than δ. For every Δ^l, $l > i$, such that Δ^i is a subdivision of Δ^l, we have by eq. (6)

$$\begin{aligned}
&||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^i, t_k^i)|| \\
&= \left|\left|\sum_{k=1}^{n_l} u(\tau_k)(\tau_k^{(l)} - \tau_{k-1}^{(l)}) - \sum_{k=1}^{n_l} u(t_k)(t_k^{(i)} - t_{k-1}^{(i)})\right|\right| \\
&= \left|\left|\sum_{k=1}^{n_l} (u(\tau_k) - u(t_k'))(\tau_k^{(l)} - \tau_{k-1}^{(l)}))\right|\right| \leqslant |b-a|\varepsilon,
\end{aligned}$$

where $|\tau_k - t_k'| < \delta$. Consequently, for any subdivisions $\Delta^l, \Delta^s, l, s > i$, such that Δ^i is a subdivision of Δ^s and Δ^l, we have

$$\begin{aligned}
&||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^s, \tau_k^s)|| \\
&\quad \leqslant ||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^i, t_k^i)|| + ||S(u(t), \Delta^s, \tau_k^s) - S(u(t), \Delta^i, t_k^i)|| \\
&\quad \leqslant 2|b-a|\varepsilon.
\end{aligned}$$

Hence, a sequence $\Delta^i \to S(u(t), \Delta^i, \tau_k^i)$ is a Cauchy sequence. The completeness of H implies the existence of an element v in H such that

$$\lim_{i \to \infty} S(u(t), \Delta^i, \tau_k^i) = v. \blacktriangledown$$

LEMMA 2. *The Riemann integral of a vector function $u(t)$, $t \in [a, b] \subset R$, has the following properties*:

1° *Linearity*: *if $u_1(t)$ and $u_2(t)$ are integrable on $[a, b]$, then for $\alpha, \beta \in R$ the vector function $\alpha u_1(t) + \beta u_2(t)$ is integrable and*

$$\int_{[a,b]} (\alpha u_1(t)+\beta u_2(t))\mathrm{d}t = \alpha \int_{[a,b]} u_1(t)\mathrm{d}t+\beta \int_{[a,b]} u_2(t)\mathrm{d}t. \tag{7}$$

2° *If $u(t)$ is continuous on $[a, b]$, then $||u(t)||$ is integrable and*

$$\left\| \int_{[a,b]} u(t)\mathrm{d}t \right\| \leqslant \int_{[a,b]} ||u(t)||\mathrm{d}t \leqslant C|a-b|, \tag{8}$$

where

$$C = \sup_{t\in[a,b]} ||u(t)||.$$

3° *If A is a bounded linear operator in H and $u(t)$ is integrable on $[a, b] \subset R^1$, then $Au(t)$ is also integrable and*

$$\int_{[a,b]} Au(t)\mathrm{d}t = A \int_{[a,b]} u(t)\mathrm{d}t. \tag{9}$$

PROOF: The property 1° follows directly from the definition of the Riemann integral of a vector function. The first step in inequalities (8) follows also from the definition of the integral. The second step follows from the fact that the map $u \to ||u||$ is continuous. Hence, the numerical continuous function $||u(t)||$ is bounded in the bounded interval $[a, b]$.

Ad 3°. A bounded linear operator is continuous by eq. 1 (1a). Hence, by eqs. (4) and (3) we have

$$\begin{aligned} A \int_{[a,b]} u(t)\mathrm{d}t &= A\lim_{i\to\infty} S(u(t), \Delta^{[i]}, t_k) \\ &= \lim_{i\to\infty} S(Au(t), \Delta^{(i)}, t_k) = \int_{[a,b]} Au(t)\mathrm{d}t. \blacktriangledown \end{aligned} \tag{10}$$

Remark: Let D be a closed bounded domain of R^n and $u(t), t \in D$, a vector function on D with values in H. Then, in a similar manner as for ordinary functions one readily shows that all previous results remain valid for $u(t), t \in D$. In particular,

$$\left\| \int_D u(t)\mathrm{d}t \right\| \leqslant \int_D ||u(t)||\mathrm{d}t \leqslant CV_D, \tag{11}$$

where $C = \sup_{t\in D} ||u(t)||$ and V_D is the volume of D.

Let A_t be a strongly continuous operator function in a closed bounded domain $D \subset R^n$. By definition, the vector function $u(t) = A_t u$ is continuous for any u in H. Set

$$\tilde{A}u = \int_D A_t u\mathrm{d}t. \tag{12}$$

Clearly, the operator $\tilde{A}$ is linear. It is also bounded. In fact, because the map

$\Phi: A \to ||A||$ is continuous, the numerical function $||A_t||$ is continuous in the closed, bounded domain D. Therefore, $\sup_{t\in D}||A_t|| = C < \infty$. Hence,

$$||\tilde{A}u|| \leqslant \int ||A_t u|| \mathrm{d}t \leqslant ||u|| \int_D ||A_t|| \mathrm{d}t \leqslant CV_D ||u||. \tag{13}$$

A bounded, linear operator $\tilde{A}$ is said to be the *integral* of the operator function A_t over the domain D. We denote it as

$$\tilde{A} = \int_D A_t \mathrm{d}t. \tag{14}$$

We have by formula (12)

$$\left(\int_D A_t \mathrm{d}t\right) u = \int_D A_t u \mathrm{d}t \tag{15}$$

for any u in H. Moreover, by eq. (13), we have

$$\left\|\int_D A_t \mathrm{d}t\right\| \leqslant CV_D. \tag{16}$$

If B is a bounded operator, then by eq. (9) we have

$$B\int A_t \mathrm{d}t = \int BA_t \mathrm{d}t. \tag{17}$$

Similarly, one can define improper integrals over unbounded domains of strongly continuous operator function.

Example. Let $G = \mathrm{SO}(2)$ and $H = L^2(G)$. Let T be the right regular representation of G in H, i.e., $T_x u(\varphi) = u(\varphi+x)$. The operator function

$$A_t = \frac{1}{2\pi}\exp(-\mathrm{i}mt)T_t, \qquad t \in [0, 2\pi], \qquad m \in Z \tag{18}$$

is continuous because T is continuous. Hence, the integral

$$\tilde{A} = \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)T_t \mathrm{d}t \tag{19}$$

is well-defined. The action of $\tilde{A}$ on any element $u(\varphi)$ in H gives the mth Fourier component of u. In fact,

$$\begin{aligned}\tilde{A}u &= \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)T_t u(\varphi)\mathrm{d}t = \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)u(\varphi+t)\mathrm{d}t \\ &= \frac{1}{2\pi}\exp(\mathrm{i}m\varphi)\int_G \exp(-\mathrm{i}m\psi)u(\psi)\mathrm{d}\psi = \exp(\mathrm{i}m\varphi)\hat{u}(m).\end{aligned}$$

Similarly, all projection operators

$$P_{ij}^{\lambda} = \frac{\dim T^{\lambda}}{\mathrm{vol}\, G}\int_G \bar{D}_{ij}^{\lambda}(x)\, T_x \mathrm{d}x$$

considered in ch. 7, § 3, are integrals over G of continuous operator functions of the form

$$A_x = \bar{D}^{\lambda}_{ij}(x)T_x,$$

where $x \in G$ and $D^{\lambda}_{ij}(x)$ are the matrix elements of the irreducible representation T^{λ} of G which are continuous on G. If $G = R^1$, then the operator function

$$A_t = \exp(-\mathrm{i}\,\lambda t)T_t \tag{20}$$

is still continuous in G. However, the integral

$$\tilde{A} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\mathrm{i}\,\lambda t)T_t\,\mathrm{d}t \tag{21}$$

does not provide an operator in H because the numerical function $\alpha(t) = ||A_t|| = 1$ is not integrable on the real line. In fact the quantity (21) represents an operator-valued distribution (cf. ch. 15.4).

Notice, however, that the unboundedness of the path of integration is not an obstacle in the construction of the integrable operator functions. Indeed, the operator function

$$A_t = \exp(-t^2)T_t$$

is continuous and $\alpha(t) = ||A_t|| = \exp(-t^2)$. Hence, the integral

$$\tilde{A} = \int_{-\infty}^{\infty}\exp(-t^2)T_t\,\mathrm{d}t \tag{22}$$

is well defined. The analytic vectors for a group representation are constructed by means of operators of the form (22) (cf. ch. 11, § 4).

§ 3. Spectral Theory of Operators

A. *Spectral Theorem*

The theory of the spectral decomposition of self-adjoint operators was developed mainly by Hilbert and v. Neumann. It provides extremely useful tools for the elaboration of representations of Lie groups and Lie algebras.

Let $[a, b]$ be a finite or infinite interval of the real line R. An operator function $E(\lambda)$, $\lambda \in [a, b]$, is said to be a *resolution of the identity* (or a spectral function), if it satisfies the following conditions:

$$\begin{aligned}
&1^\circ\ E(\lambda)^* = E(\lambda).\\
&2^\circ\ E(\lambda)E(\mu) = E(\min(\lambda, \mu)).\\
&3^\circ\ E(\lambda+0) = E(\lambda).\\
&4^\circ\ E(-\infty) = 0,\ E(\infty) = I.
\end{aligned} \tag{1}$$

Conditions 1° and 2° mean that $E(\lambda)$, $\lambda \in [a, b]$, are bounded hermitian operators of orthogonal projections (cf. ch. 7, § 3). For an interval $\Delta = [\lambda_1, \lambda_2] \subset [a, b]$

we shall denote the difference $E(\lambda_2)-E(\lambda_1)$ by $E(\Delta)$. If Δ_1 and Δ_2 are two such intervals, then, by condition 2°, we have

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2). \tag{2}$$

In particular, if Δ_1 and Δ_2 have no common points, then

$$E(\Delta_1)E(\Delta_2) = 0, \tag{3}$$

i.e., the subspaces $H_1 = E(\Delta_1)H$ and $H_2 = E(\Delta_2)H$ are orthogonal.

Condition 3° means that the operator function $E(\lambda)$ is strongly right continuous. The convergence in 4° is meant in the strong sense, i.e.,

$$\lim_{\lambda\to-\infty} E(\lambda)u = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} E(\lambda)u = u \tag{4}$$

for every u in H.

EXAMPLE 1. Let G be the translation group of the real line R and $H = L^2(R^1)$. Let T be the right regular representation of G, i.e., $T_a u(x) = u(x+a)$. The operator function

$$E(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda a)\, T_a \mathrm{d}a \tag{5}$$

represents the resolution of the identity. We now directly verify the properties 1°–4° of eq. (1). For any $u(x)$ in H we have

$$E(\lambda)u(x) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda a)u(x+a)\mathrm{d}a$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp[-\mathrm{i}\lambda(y-x)]u(y)\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} \exp(\mathrm{i}\lambda x)\hat{u}(\lambda)\mathrm{d}\lambda. \tag{6}$$

Here

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda y)u(y)\mathrm{d}y \tag{7}$$

is the Fourier transform of the element u in H. Thus,

$$E(-\infty)u = 0 \quad \text{and} \quad E(\infty)u = u$$

for any u in H. Consequently, $E_{-\infty} = 0$ and $E_\infty = I$.

Ad 3°. By eq. (6) for any u in H we have

$$\lim_{\Delta\lambda\to 0} E(\lambda+\Delta\lambda)u = E(\lambda)u + \lim_{\Delta\lambda\to 0}\frac{1}{\sqrt{2\pi}}\int\limits_{\lambda}^{\lambda+\Delta\lambda}\exp(\mathrm{i}\lambda x)\hat{u}(\lambda)\,\mathrm{d}\lambda = E(\lambda)u.$$

Hence, $E(\lambda+0) = E(\lambda)$.

Ad 2°. For any u in $L^2(R)$ we have

$$E(\lambda)E(\mu)u = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\lambda}\mathrm{d}\lambda\int\limits_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda a)T_a\mathrm{d}a\int\limits_{-\infty}^{\mu}\mathrm{d}\mu\int\limits_{-\infty}^{\infty}\exp(-\mathrm{i}\mu b)\,T_b u(x)\mathrm{d}b.$$

We can take the bounded operator T_a under the last integral sign. Then, $T_aT_bu(x) = T_{a+b}u(x) = u(x+a+b)$. Setting $y = x+a+b$, we obtain

$$\begin{aligned}
E(\lambda)E(\mu)u & \\
&= \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\lambda}\mathrm{d}\lambda\int\limits_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda a)\mathrm{d}a\int\limits_{-\infty}^{\mu}\exp[\mathrm{i}\mu(x+a)]\mathrm{d}\mu\int\limits_{-\infty}^{\infty}\exp(-\mathrm{i}\mu y)u(y)\mathrm{d}y\\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\lambda}\mathrm{d}\lambda\int\limits_{-\infty}^{\mu}\exp(\mathrm{i}\mu x)\,\delta(\lambda-\mu)\hat{u}(\mu)\mathrm{d}\mu\\
&= \int\limits_{-\infty}^{\min(\lambda,\mu)}\exp(\mathrm{i}\mu x)\hat{u}(\mu)\mathrm{d}\mu = E\big(\min(\lambda,\mu)\big)u. \qquad (8)
\end{aligned}$$

The interchange of integrals relative to 'a' and 'μ' is justified by Fubini theorem (cf. app. A). Utilizing eq. (5) we have

$$E^*(\lambda) = \frac{1}{2\pi}\int\limits_{-\infty}^{\lambda}\mathrm{d}\lambda\int\limits_{-\infty}^{\infty}\exp(\mathrm{i}\lambda a)T_a^*\mathrm{d}a = \frac{1}{2\pi}\int\limits_{-\infty}^{\lambda}\mathrm{d}\lambda\int\limits_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda b)T_b\mathrm{d}b = E(\lambda).$$

Here we have used the fact that T is unitary, i.e., $T_a^* = T_{a^{-1}} = T_{-a}$, and put $b = -a$. ▼

It follows from the formula (1) that, for any u in H, the function

$$\sigma_u(\lambda) \equiv \big(E(\lambda)u, u\big) \qquad (9)$$

is a right continuous, non-decreasing function of bounded variation for which

$$\sigma_u(-\infty) = 0, \quad \sigma_u(\infty) = ||u||^2.$$

In fact, for $\mu < \lambda$,

$$\big(E(\mu)u, u\big) = ||E(\mu)u||^2 = ||E(\mu)E(\lambda)u||^2 \leqslant ||E(\lambda)u||^2 = \big(E(\lambda)u, u\big). \qquad (10)$$

The function $\sigma_u(\lambda)$ defines another function $\sigma_u(\Delta) = \big(E(\Delta)u, u\big)$ which is determined for any interval $\Delta \subset [a, b]$ and can be extended to all Borel subsets of $[a, b]$. The function $\sigma_u(\Delta)$ is positive by eq. (10). Moreover, it is denumerable additive. In fact, if

$$\Delta = \bigcup_{n=1}^{\infty} \Delta_n \quad \text{and} \quad \Delta_n \cap \Delta_m = 0 \quad \text{for } n \neq m,$$

then

$$E(\Delta) = \sum_{n=1}^{\infty} E(\Delta_n) \quad \text{and} \quad \sigma_u(\Delta) = \sum_{n=1}^{\infty} \sigma_u(\Delta_n). \tag{11}$$

The function $\sigma_u(\Delta)$ is called the *spectral measure.*

Let A be a self-adjoint operator in a Hilbert space H. We now state the fundamental theorem in spectral decomposition theory:

THEOREM 1. *Every self-adjoint operator in H has the representation*

$$A = \int_{-\infty}^{\infty} \lambda \mathrm{d}E(\lambda), \tag{12}$$

where $E(\lambda)$ is a spectral family, which is uniquely determined by the operator A. ▼

Because this is a classical result of functional analysis, we shall not give the proof here (cf., e.g., Maurin 1967, ch. 6). In (12) neither the domain of integration nor the operator function $\lambda \mathrm{d}E(\lambda)$ are bounded; it is therefore necessary to make precise the meaning of an integral of this type. The domain $D(A)$ of the operator A consists of all vectors u for which

$$(Au, Au) = \int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\sigma_u(\lambda) < \infty. \tag{13}$$

For u in $D(A)$ the operator (12) is defined by the formula

$$Au = \int_{-\infty}^{\infty} \lambda \mathrm{d}\big(E(\lambda)u\big), \tag{14}$$

where this equality is understood in the weak sense, i.e.,

$$(Au, v) = \int_{-\infty}^{\infty} \lambda \mathrm{d}\big(E(\lambda)u, v\big)$$

for any v in H.

COROLLARY 1. *If Δ is any interval in $[a, b]$, then*

$$E(\Delta)A = AE(\Delta) = \int_{\Delta} \lambda \mathrm{d}E(\lambda), \tag{15}$$

i.e., the spectral function $E(\Delta)$ is permutable with A.

PROOF: By virtue of eq. (12)

$$E(\Delta)A = \int_{-\infty}^{\infty} \lambda E(\Delta) \mathrm{d}E(\lambda).$$

By eq. (2), $E(\Delta)\mathrm{d}E(\lambda) = 0$, if $\lambda \notin \Delta$ and $E(\Delta)\mathrm{d}E(\lambda) = \mathrm{d}E(\lambda)$ if $\lambda \in \Delta$. Therefore

$$E(\Delta)A = \int_{\Delta} \lambda \mathrm{d}E_{\lambda}$$

Similarly,

$$AE(\Delta) = \int_{\Delta} \lambda \mathrm{d}E(\lambda)$$

and consequently eq. (15) follows. ▼

Notice, that if $u \in H_{\Delta} \equiv E(\Delta)H$, where $\Delta = [\lambda, \mu]$ then, by eq. (15),

$$||Au - \lambda u|| \leqslant (\mu - \lambda)||u||. \tag{16}$$

Hence, if $\mu - \lambda$ is small, then u is 'almost an eigenvector' of the operator A. If

$$\bigcup_{i=1}^{\infty} \Delta_i = [a, b]$$

and $\Delta_i \cap \Delta_k = 0$ for $i \neq k$, then the space H can be represented as the orthogonal direct sum of subspaces $H_{\Delta_n} = E(\Delta_n)H$ in which the operator A 'almost' reduces to the multiplication operator.

Clearly, if A has only continuous spectrum, then all eigenvectors are outside the Hilbert space.

COROLLARY 2. *The set* $D = \{E(\Delta_i)u\}$, *where* Δ_i *runs over all finite intervals of* $[a, b]$ *and* u *runs over* H, *is dense in* H. *All powers* A^n, $n = 1, 2, \ldots,$ *are defined on* D.

PROOF: Let $\{\tilde{\Delta}_i\}_{i=1}^{\infty}$ be a collection of finite intervals such that

$$\bigcup_{i=1}^{\infty} \tilde{\Delta}_i = [a, b] \quad \text{and} \quad \tilde{\Delta}_i \cap \tilde{\Delta}_k = 0 \quad \text{for } i \neq k. \tag{17}$$

Then, for any v in H we have:

$$v = \lim_{N \to \infty} \sum_{i=1}^{N} E(\tilde{\Delta}_i)v.$$

Hence, the set $\{E(\tilde{\Delta}_i u)\}$, $u \in H$, is dense in H. Now, for any u in H, by eq. (15) we have

$$v = A^n E(\Delta)u = A^{n-1} \int_{\Delta} \lambda \mathrm{d}E(\lambda)u = A^{n-2} \int_{\Delta} \lambda^2 \mathrm{d}E(\lambda)u = \int_{\Delta} \lambda^n \mathrm{d}E(\lambda)u. \tag{18}$$

Since

$$||v||^2 \leqslant \int_{\Delta} |\lambda|^{2n} \, ||\mathrm{d}E(\lambda)u||^2 \leqslant \max_{\lambda \in \Delta} |\lambda|^{2n} |\Delta| \, ||u||^2, \tag{19}$$

we obtain that any vector $E(\Delta_i)u$ lies in the domain of A^n. ▼

In a given Hilbert space we have many resolutions of the identity associated with various self-adjoint operators. In applications, it is useful to know when a resolution of the identity $E(\lambda)$ can be associated with a self-adjoint operator A in H. This problem is solved by the following theorem.

THEOREM 2. *A resolution of the identity $E(\lambda)$ represents the spectral function of an operator A if and only if*

1° *$E(\Delta)$ reduces A for any interval $\Delta \subset [-\infty, \infty]$.*

2° *The condition $u \in (E(\lambda)-E(\mu))H$, $-\infty \leqslant \mu < \lambda \leqslant \infty$ implies the inequality*

$$\mu\|u\|^2 \leqslant (Au, u) \leqslant \lambda\|u\|^2. \tag{20}$$

(For the proof cf., e.g., Akhiezer and Glatzman 1966, § 75.)

EXAMPLE 2. Let $H = L^2(R)$; let $A = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}$ and let $D(A)$ consist of all functions $u(x)$ in $L^2(R)$ such that

1° $u(x)$ is absolutely continuous in every finite interval

$$\Delta \subset [-\infty, \infty],$$

2° $u'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(R^1)$.

The functions u in $D(A)$ automatically satisfy the boundary condition i.e.,

$$\lim_{x\to-\infty} u(x) = \lim_{x\to\infty} u(x) = 0.$$

We readily verify that A is self-adjoint on $D(A)$.

We now show that the resolution of the identity

$$E(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda a) T_a \mathrm{d}a$$

given by eq. (5) is the spectral function for A. In fact, since $T_a u(x) = u(x+a)$ $= \exp[a(\mathrm{d}/\mathrm{d}x)]u(x)$, the operators A and T_a commute. Hence, $E(\Delta)$ reduces A for any interval $\Delta \subset [-\infty, \infty]$. Moreover, by eq. (6)

$$\begin{aligned}(AE(\Delta)u, u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \lambda\exp(\mathrm{i}\lambda x)\hat{u}(\lambda)u(x)\mathrm{d}\lambda\mathrm{d}x \\ &= \int_{\Delta} \lambda u(\lambda)\overline{u(\lambda)}\mathrm{d}\lambda \leqslant \max_{\lambda\in\Delta} \lambda\|u\|^2.\end{aligned}$$

Hence, the condition 2° of th. 2 is also satisfied. Consequently $E(\lambda)$ is the spectral function of A. ▼

B. *Spectral Theory of Compact and Hilbert–Schmidt Operators*

A linear operator X: $H_1 \to H_2$ from a Hilbert space H_1 into a Hilbert space H_2 is said to be compact if it maps the unit ball in H_1 into a precompact set in H_2.

RELLICH–HILBERT–SCHMIDT THEOREM. *Let X be a linear bounded self-adjoint (or normal) operator in a Hilbert space H. Then X is compact if and only if the following conditions hold:*

(i) $X = \sum_{k=1}^{\infty} \lambda_k E_k$, *where E_k are mutually orthogonal projections on finite-dimensional subspaces $H_k = E_k H$ and $|\lambda_k| \to 0$ as $k \to \infty$,*

(ii) *for every $u_k \in H_k$ we have $Xu_k = \lambda_k u_k$,*

(iii) $H = \oplus H_k + H_0$ *where* $H_0 = X^{-1}(0)$. ▼

(For the proof cf. Maurin 1968, ch. I.)

We now introduce the important class of the so called Hilbert–Schmidt operators. Let H be a Hilbert space and let $\{e_i\}_1^\infty$ be an orthonormal basis in H. A linear bounded operator X in H is said to be a *Hilbert–Schmidt operator* if

$$|X| \equiv \sqrt{\sum_i \|Xe_i\|^2} < \infty .$$

HILBERT–SCHMIDT THEOREM. *A linear bounded self-adjoint operator X is Hilbert–Schmidt if and only if*

$$X = \sum_{k=1}^{\infty} \lambda_k E_k ,$$

where the projections E_k are of finite dimensions and

$$\Sigma_k (|\lambda_k|^2 \dim E_k) < \infty . \blacktriangledown$$

(For the proof cf. Maurin 1968, ch. I.)

C. *Nuclear Variant of the Spectral Theorem*

In order to obtain a convenient interpretation of the eigenvectors in the case of continuous spectra we present in this section the so-called nuclear variant of the spectral theorem (cf. Maurin 1968, ch. II). We begin with the basic facts concerning nuclear spectral theory. We first recall the concept of direct integral of Hilbert spaces.

Let (Λ, μ) be a space with a measure. Consider a family $\{H(\lambda)\}_{\lambda \in \Lambda}$ of Hilbert spaces $H(\lambda)$, each equipped with a scalar product $(\cdot, \cdot)_\lambda$, and form the Cartesian product

$$\bigtimes_{\lambda \in \Lambda} H(\lambda) .$$

We shall call elements $u, v \in \bigtimes_{\lambda \in \Lambda} H(\lambda)$ *vector fields*. We shall call $\Gamma = \{\overset{i}{e}\}_{i \in J}$ a *fundamental family of μ-measurable vector fields* if

1° for arbitrary $i, j \in J$ the function $\Lambda \to \lambda \to (\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda \in C$ is μ-measurable,

2° for every $\lambda \in \Lambda$, vectors $\{\overset{i}{e}(\lambda)\}_{i \in J}$ span the space $H(\lambda)$.

A field $u \in \bigtimes_{\lambda \in \Lambda} H(\lambda)$ is said to be *measurable* if all functions $\lambda \to (u(\lambda), \overset{i}{e}(\lambda))_\lambda$ are μ-measurable. It is evident that μ-measurable fields form a vector subspace of the space $\bigtimes_{\lambda \in \Lambda} H(\lambda)$.

A measurable field u is called *square integrable* if

$$\int \|u(\lambda)\|_\lambda^2 \, \mathrm{d}\mu(\lambda) < \infty.$$

Two measurable fields are said to be equivalent if they are equal μ-almost everywhere on Λ.

Definition 1. The *direct integral* H of Hilbert spaces $H(\lambda)$ is the space of equivalence classes of measurable and integrable vector fields $\{u(\lambda)\}$ equipped with the scalar product

$$(u, v) = \int (u(\lambda), v(\lambda))_{H(\lambda)} \mathrm{d}\mu(\lambda). \blacktriangledown$$

Using the same arguments as in the proof of the classical Riesz–Fischer theorem one shows that H is complete. Consequently H is a Hilbert space: we shall denote it by the symbol

$$H = \int_\Lambda H(\lambda) \mathrm{d}\mu(\lambda).$$

Examples. 3′. Let $\Lambda = N$—the set of natural numbers and let $\mu(n) = 1$; then

$$H = \int_\Lambda H(\lambda) \mathrm{d}\mu(\lambda) = \bigoplus_{\lambda \in N} H(\lambda),$$

i.e. in this case the direct integral reduces to a direct sum.

3″. Let $H(\lambda) = \hat{H}$, for all $\lambda \in \Lambda$, where $\hat{H}$ is a Hilbert space; then

$$H = \int \hat{H}(\lambda) \mathrm{d}\mu(\lambda) = L^2(\Lambda, \mu, \hat{H}),$$

in particular if $\hat{H} = C$, then $H = L^2(\mu)$. $\blacktriangledown$

We give now a useful lemma implied by the measure desintegration theorem 4.3.2 and the concept of direct integral of Hilbert spaces. Let $X, Y, r, \mu, \tilde{\mu}$ and μ_y be such as in the formulation of th. 4.3.2. Then we have:

Lemma 3. *Set* $H = L^2(X, \mu; \tilde{H})$, $H(y) = L^2(X, \mu_y, \tilde{H})$, $\tilde{H}$ *a Hilbert space. Then we can equip the field* $y \to H(y)$ *with the structure of* μ*-measurable fields of Hilbert spaces such that*

$$H = \int_Y H(y) \mathrm{d}\tilde{\mu}(y). \blacktriangledown$$

(For the proof cf. Mackey 1952, § 12.)

DIAGONAL AND DECOMPOSABLE OPERATORS

Let t be a measurable essentially bounded function, $t \in L^\infty(\Lambda, \mu)$ and let $I(\lambda)$ be the identity operator in $H(\lambda)$. The operator field

$$\lambda \to t(\lambda)I(\lambda) \in L(H(\lambda), H(\lambda)),$$

is called the *diagonal operator in the Hilbert space* $H = \int H(\lambda)\,\mathrm{d}\mu(\lambda)$. The diagonal operator field $\{t(\lambda)I(\lambda)\}$ defines an operator T in H by the formula

$$(Tu)(\lambda) = t(\lambda)u(\lambda), \qquad u \in H.$$

One readily verifies that $||T|| = ||t||_\infty$.

An operator field $T(\cdot)$: $\Lambda \ni \lambda \to T(\lambda) \in L(H(\lambda), H(\lambda))$ is said to be *measurable* if all functions $\lambda \to (T(\lambda)\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda$, where $\{\overset{i}{e}\}$ is the fundamental family of vector fields, are measurable.

If $u(\cdot)$ and $T(\cdot)$ are measurable, then $\lambda \to T(\lambda)u(\lambda) \in H(\lambda)$ is a measurable vector field. Indeed since $(\overset{i}{e}(\lambda), T(\lambda)\overset{j}{e}(\lambda))_\lambda = (T^*(\lambda)\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda$, the vector field $\lambda \to T^*(\lambda)\overset{i}{e}(\lambda)$ is measurable. Hence $(T(\lambda)u(\lambda), \overset{i}{e}(\lambda))_\lambda = (u(\lambda), T^*\overset{i}{e}(\lambda))_\lambda$ is measurable. This implies that $\lambda \to T(\lambda)u(\lambda)$ is measurable.

We now introduce an important concept of a decomposable operator. Take $T(\lambda) \in L(H(\lambda), H(\lambda))$ such that the numerical function $||T(\cdot)|| = (\lambda \to ||T(\lambda)||_\lambda) \in L^\infty(\Lambda, \mu)$. Set $N = \operatorname*{ess\,sup}_{\lambda\in\Lambda} ||T(\lambda)||_\lambda$. The vector field $\lambda \to T(\lambda)u(\lambda)$, for every $u(\cdot) \in H$ is measurable and we have $||T(\lambda)u(\lambda)||_\lambda = N||u(\lambda)||_\lambda$, almost everywhere.

Consequently

$$\int ||T(\lambda)u(\lambda)||_\lambda^2\,\mathrm{d}\mu(\lambda) \leqslant N^2 \int ||u(\lambda)||_\lambda^2\,\mathrm{d}\mu(\lambda) = N^2||u||^2.$$

Denoting the vector field $\{(Tu)(\lambda)\}$ by Tu we have $Tu \in H$ and $||Tu|| \leqslant N||u||$. Thus $T = \{T(\lambda)\}$ represents a bounded operator in H. This operator is called a *decomposable operator* and is denoted by the symbol

$$T = \int_\Lambda T(\lambda)\,\mathrm{d}\mu(\lambda).$$

Clearly $||T|| = \operatorname*{ess\,sup}_{\lambda\in\Lambda} ||T(\lambda)||_\lambda = N.$

One readily verifies that the decomposable operators have the following properties:

$$\int (T(\lambda)+U(\lambda))\,\mathrm{d}\mu(\lambda) = \int T(\lambda)\,\mathrm{d}\mu(\lambda) + \int U(\lambda)\,\mathrm{d}\mu(\lambda),$$

$$\int T^*(\lambda)\,\mathrm{d}\mu(\lambda) = \left(\int T(\lambda)\,\mathrm{d}\mu(\lambda)\right)^*,$$

$$\int T(\lambda)\,\mathrm{d}\mu(\lambda) \int U(\lambda)\,\mathrm{d}\mu(\lambda) = \int T(\lambda)U(\lambda)\,\mathrm{d}\mu(\lambda).$$

If almost all $U(\lambda)$ are unitary then $\int U(\lambda)\mathrm{d}\mu(\lambda)$ is an unitary operator in H.

It is evident from the above properties that the set of decomposable operators forms a $*$-algebra in H.

A $*$-subalgebra $\mathscr{A}$ of $L(H, H)$ is said to be a *von Neumann algebra* if it satisfies one of the following (equivalent) conditions:

(i) $\mathscr{A}$ is closed in the weak operator topology of H.

(ii) $\mathscr{A}$ is closed in the strong operator topology of H.

(iii) $\mathscr{A}$ coincides with its bicommutant $\mathscr{A}''$.

THE VON NEUMANN THEOREM.

(i) *The algebra $\mathscr{D}$ of diagonal operators is a commutative von Neumann algebra.*

(ii) *The commutant $\mathscr{D}'$ of $\mathscr{D}$ is the von Neumann algebra $\mathscr{R}$ of decomposable operators in H, i.e.*

$$\mathscr{D}' = \mathscr{R}, \quad \mathscr{R}' = \mathscr{D}.$$

(For the proof cf. Maurin 1969, ch.I, § 6.)

Let $H(X)$ be a Hilbert space of functions with the domain X and let $A_1, A_2, \ldots, A_n$ be a set of self-adjoint, strongly commuting operators in $H(X)$ which contains an elliptic differential operator.

Let Φ be a dense, linear subset of H endowed with a nuclear topology, which is, stronger than the relative topology induced by H. This means, in particular that the natural embedding i: $\Phi \to H$ is continuous. Suppose that the space Φ is so chosen that the map A_i: $\Phi \to \Phi$ is a continuous one. Let Φ' be the conjugate space of linear continuous functionals over the space Φ. Then the triplet (Φ, H, Φ') is called the *Gel'fand triplet*. Let Λ denote a subset of E^n. Then we have

THE NUCLEAR SPECTRAL THEOREM.

1° *There exists a direct integral* $\hat{H} = \int_\Lambda \hat{H}(\lambda) \mathrm{d}\varrho(\lambda)$ *and a generalized Fourier transform F* :

$$F: H \to FH \equiv \hat{H} = \int_\Lambda \hat{H}(\lambda) \mathrm{d}\varrho(\lambda), \tag{21}$$

which is given by the formula

$$\Phi \ni \varphi(x) \to \hat{\varphi}_k(\lambda) \doteq (F\hat{\varphi})_k(\lambda) = \langle \varphi, e_k(\lambda) \rangle = \int_X \varphi(x) \overline{e_k(\lambda, x)} \mathrm{d}x, \tag{22}$$

where $k = 1, 2, \ldots, \dim \hat{H}(\lambda)$, *and* $e_k(\lambda) \in \Phi'$ *are generalized eigenvectors. If* $A_i \varphi \in \Phi$, $i = 1, 2, \ldots, n$, *then*

$$\big(A_j \varphi, e_k(\lambda)\big) = \hat{A}_j(\lambda) \langle \varphi, e_k(\lambda) \rangle \tag{23}$$

for ϱ-almost all $\lambda \in \Lambda$, where $\hat{A}_i(\lambda)$ is the spectrum of A_i. The eigenvectors $e_k(\lambda, x)$ are regular functions.*

2° *For every φ, ψ in Φ, the so-called Parseval equality is satisfied*:

$$(\varphi, \psi)_H = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) \overline{\hat{\psi}_k(\lambda)} = (\varphi, \psi)_{\hat{H}}. \tag{24}$$

3° *The generalized Fourier inversion formula* (*the spectral synthesis*) *of an element* $\varphi(x)$ *in* Φ *is*

$$\varphi(x) = \int_{\Lambda} \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x) \mathrm{d}\varrho(\lambda). \tag{25}$$

(For the proof. cf. Maurin 1968, ch. II.)

Remark 1: If we drop the condition that the set $\{A_i\}_1^n$ contains an elliptic operator, then the nuclear spectral theorem still holds. However, the generalized eigenvectors, $e_k(\lambda, x)$, are in general no longer regular functions and we do not have a representation of $\hat{\varphi}_k(\lambda)$ as an integral over the x-space as given by eq. (22).

Remark 2: The so-called 'complete von Neumann spectral' theorem for a set $\{A_i\}_1^n$ of self-adjoint, strongly commuting operators states that the map $F: H \to \hat{H} = \int \hat{H}(\lambda) \mathrm{d}\varrho(\lambda)$ is unitary. Therefore, the Parseval equality (24) holds for any φ, ψ in H. However, we have a representation $\hat{\varphi}_k(\lambda) = \int \varphi(x) \overline{e_k(\lambda, x)} \mathrm{d}x$ only for elements φ in $\Phi(X)$.

The formula (23) requires some additional comments. Let A_i' be the natural extension of A given by the identity

$$\langle A_i \varphi, \psi' \rangle = \langle \varphi, A_i' \psi' \rangle, \tag{26}$$

where φ runs over Φ and ψ' is an element in Φ'. Eq. (26) means that the domain $D(A_i')$ is the extension of the domain $D(A_i)$ of A_i by those elements ψ' in Φ' for which identity (26) is satisfied, i.e. $A_i' \supset A_i^* = A_i$. Equation (23) can now be written concisely in the form

$$A_i' e_k(\lambda, x) = \hat{A}_i(\lambda) e_k(\lambda, x), \qquad k = 1, 2, \ldots, \dim \hat{H}(\lambda), \tag{27}$$

where $\hat{A}_i(\lambda)$ is the spectrum of A_i, $i = 1, 2, \ldots, n$. Formula (25) for the spectral synthesis allows us to write the following completeness relation, which is often used by physicists:

$$\int \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} e_k(\lambda, x) \overline{e_k(\lambda, y)} = \delta(x-y). \tag{28}$$

This integral is understood in the sense of the weak integral of the regular distributions $e_k(\lambda, x)\overline{e_k(\lambda, y)}$ on $X \times X$. The weak integral (28) applied to a function $\varphi(y) \in \Phi(X)$ gives the spectral synthesis (25) of φ, and applied to a function $\varphi(y)\overline{\psi(x)} \in \Phi(X) \times \Phi(X)$, it gives the Parseval equality (24).

Notice that the spectral theorem says nothing about the orthogonality of the generalized eigenvectors. However, in many cases the spectral function $\mathrm{d}\varrho(\lambda)$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\lambda$ on the set Λ.

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* We recall that a function $f(x)$ in $H(X)$ is said to be regular if $\varphi f \in \mathscr{D}(X)$ ($\mathscr{D}(X)$ is the Schwartz space) for every $\varphi \in \mathscr{D}(X)$.

This implies $d\varrho(\lambda) = \varrho(\lambda)d\lambda$ by the Radon–Nikodym theorem. The inverse formula (22) allows us to write the following orthogonality relation:

$$\int_X e_k(\lambda, x)\overline{e_{k'}(\lambda', x)}dx = \varrho^{-1}(\lambda)\,\delta(\lambda-\lambda')\,\delta_{k,k'}. \tag{29}$$

This is a generalization of the well-known orthogonality relation in ordinary Fourier analysis

$$\int_{-\infty}^{\infty} \exp(i\lambda x)\overline{\exp(i\lambda' x)}dx = 2\pi\delta(\lambda-\lambda').$$

In both cases these integrals are understood as weak integrals of distributions $e_k(\lambda, x), e_k(\lambda', x)$ defined in the subspace $\hat{\Phi}(\Lambda) = F[\Phi(X)]$.

The spectral synthesis (25) of an element $\varphi(X)$ in Φ suggests that it would be useful to introduce a space $H'(\lambda) \subset \Phi'$ isomorphic with $\hat{H}(\lambda)$, i.e.

$$\hat{H}(\lambda) \ni \{\hat{\varphi}_k(\lambda)\} \to \hat{\varphi}(\lambda) = \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)\hat{e}_k(\lambda) \to$$

$$\to \varphi(\lambda, x) = \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)e_k(\lambda, x) \in H'(\lambda) \subset \Phi'. \tag{30}$$

Here $\hat{e}_k(\lambda)$ are orthonormal basis vectors in $\hat{H}(\lambda)$, i.e., $(\hat{e}_k(\lambda), \hat{e}_{k'}(\lambda))_{\hat{H}(\lambda)} = \delta_{kk'}$, whereas $e_k(\lambda, x) \in \Phi'(X)$. Formula (30) shows that each $H'(\lambda)$ is a linear subset of Φ' and, by eqs. (27) and (30),

$$A_i'\varphi(\lambda, x) = \hat{A}_i(\lambda)\varphi(\lambda, x), \tag{31}$$

i.e. any element $\varphi(\lambda, x)$ in $H'(\lambda)$ is the generalized eigenvector of all A_i', $i = 1, 2, \ldots, n$. Formula (25) may now be concisely written in the form

$$\varphi(x) = \int_\Lambda \varphi(\lambda, x)d\varrho(\lambda). \tag{32}$$

It is interesting that the isomorphism given by (30) between the spaces $\hat{H}(\lambda)$ and $H'(\lambda)$ induces a finite scalar product in the spaces $H(\lambda) \subset \Phi'$. We shall illustrate this with two examples:

EXAMPLE 4. Let $H = L^2(R^4)$, where R^4 is the four-dimensional Minkowski space. Let $A = \Box = -\nabla^2 + \dfrac{\partial^2}{\partial t^2}$ be the wave operator in H. Let Φ be the Schwartz S-space and $\Phi' = S'(R^4)$. Then, all assumptions of the nuclear spectral theorem are satisfied. The generalized eigenvectors

$$\Box e(\lambda, x) = \hat{\Box}(\lambda)e(\lambda, x) \quad (k = 1) \tag{33}$$

are plane waves

$$e(\lambda, x) = (2\pi)^{-2}\exp(ipx), \quad p^2 = \lambda^2 = \hat{\Box}(\lambda), \tag{34}$$

where the parameter λ plays the role of mass of a scalar particle.

The space $\hat{H}(\lambda)$, where

$$\lambda^2 = p_0^2 - \boldsymbol{p}^2, \tag{35}$$

is the momentum space in which the scalar product (for positive masses) is

$$(\hat{\varphi}, \hat{\psi})_{\hat{H}(\lambda)} = \int_{p_0=[\boldsymbol{p}^2+\lambda^2]^{1/2}} p_0^{-1} \mathrm{d}^3\boldsymbol{p}\hat{\varphi}(p)\overline{\psi(p)}. \tag{36}$$

This scalar product induces the following formula in $H'(\lambda)$:

$$(\varphi, \psi)_{H'(\lambda)} = \mathrm{i} \int_t \mathrm{d}^3 x [\varphi(\lambda, x)\partial_t \overline{\psi(\lambda, x)} - (\partial_t \varphi(\lambda, x))\overline{\psi(\lambda, x)}]$$
$$\equiv \mathrm{i} \int_t \mathrm{d}^3 x \varphi(\lambda, x) \overleftrightarrow{\partial_t} \overline{\psi(\lambda, x)}, \tag{37}$$

where the symbol t affixed to the integral means that all functions under the integral sign are taken at the same time t. One readily verifies that the formula (37) possesses all the properties required from a scalar product and is conserved in time. The generalized Fourier expansion (25) in this case is just the ordinary Fourier expansion

$$\varphi(\lambda, x) = 2^{1/2}[2\pi]^{-3/2} \int \mathrm{d}^4 p \exp(-\mathrm{i}px)\delta(p^2-\lambda^2)\theta(p_0)\varphi(p), \tag{38}$$

EXAMPLE 5. We have a similar result for the Dirac equation in the space $(L^2)^4(R^4)$. In fact, let

$$\gamma^\mu \partial_\mu \psi(\lambda, x) = \lambda\psi(\lambda, x), \tag{39}$$

where $\lambda = m$ and

$$\psi(\lambda, x) = \exp(-\mathrm{i}px) \begin{bmatrix} \psi_1(p) \\ \psi_2(p) \\ \psi_3(p) \\ \psi_4(p) \end{bmatrix}, \quad p_\mu p^\mu = m^2 = \lambda^2.$$

The scalar product in $\hat{H}(\lambda)$*

$$(\varphi, \psi)_{\hat{H}(\lambda)} = \int_{p_0=[\boldsymbol{p}^2+\lambda]^{1/2}} p_0^{-1} \mathrm{d}^3\boldsymbol{p} \sum_{\alpha=1}^{4} \hat{\varphi}_\alpha(p)\overline{\hat{\psi}_\alpha(p)}, \tag{40}$$

induces the following finite time invariant scalar product in $H'(\lambda)$

$$(\varphi, \psi)_{H'(\lambda)} = \int_t \mathrm{d}_3 x \sum_{\alpha=1}^{4} \varphi_\alpha(\lambda, x)\overline{\psi_\alpha(\lambda, x)} = \int_{\sigma(t)} \mathrm{d}\sigma^\mu \varphi(x)\gamma^\mu\overline{\psi(x)}. \blacktriangledown \tag{41}$$

We would like to stress that in quantum physics we are more often interested n the spaces $H'(\lambda) \subset \Phi'$, than in the Hilbert spaces $\hat{H}(\lambda)$.

* In physical literature one uses the scalar product for the Dirac wave functions which is antilinear with respect to the first factor.

§ 4. Functions of Self-Adjoint Operators

Let A be a self-adjoint operator and $E(\lambda)$ the resolution of the identity associated with A. We want to define a function $f(A)$ of the operator A and elaborate an operational calculus for the collection of functions of A.

Let $f(\lambda)$ be a complex, continuous function defined on the real line. Then, for any u in H, the set of vectors $v(\lambda) = f(\lambda)E(\lambda)u$ represents a one-parameter (strongly) continuous curve in H. Thus, for any α, β satisfying the condition $-\infty < \alpha < \beta < \infty$, the integral

$$\int_\alpha^\beta f(\lambda)\mathrm{d}E(\lambda)u$$

can be defined as the Riemann integral considered in § 2, i.e.,

$$\int_\alpha^\beta f(\lambda)\mathrm{d}E(\lambda)u = \lim_{i\to\infty}\sum_i f(\lambda_i')[E(\lambda_{i+1})-E(\lambda_i)]u, \tag{1}$$

where

$$\alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_n = \beta, \qquad \lambda_i' \in (\lambda_i, \lambda_{i+1}],$$

and

$$\max|\lambda_{i+1}-\lambda_i| \to 0 \quad \text{as } i\to\infty.$$

By the results of § 2, the integral (1) always exists.

We can also define an improper integral by the following formulae:

$$\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}E(\lambda)u = \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\int_\alpha^\beta f(\lambda)\mathrm{d}E(\lambda)u, \tag{2}$$

if such a limit of Riemann integrals exists. The problem of existence of the integral (2) is solved by the following theorem.

THEOREM 1. *Let $E(\lambda)$ be a resolution of the identity. Then, for a given u in H, the following conditions are equivalent*:

(i) $\displaystyle\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}E(\lambda)u$ *exists*, (3)

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(\lambda)|^2\mathrm{d}\|E(\lambda)u\|^2 < \infty$, (4)

(iii) $F(v) \equiv \displaystyle\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}\big(E(\lambda)v, u\big)$ *is a bounded linear functional.*▼ (5)

(For the proof cf. Yoshida 1965, ch. 11.)

We now show that a self-adjoint operator $f(A)$ can be associated with every real, continuous function $f(\lambda)$, $\lambda \in (-\infty, \infty)$. In fact, we have:

THEOREM 2. *Let $E(\lambda)$ be the resolution of the identity associated with a self-adjoint operator A, and let $f(\lambda)$ be a real, continuous function. The equality*

$$(f(A)u, v) = \int_{-\infty}^{\infty} f(\lambda)\mathrm{d}(E(\lambda)u, v), \tag{6}$$

where

$$u \in D = \left\{u \in H: \int_{-\infty}^{\infty} |f(\lambda)|^2 \mathrm{d}||E(\lambda)u||^2 < \infty\right\} \tag{7}$$

and v is any element in H, defines a self-adjoint operator $f(A)$ in H with $D(f(A)) = D$. Moreover, $f(A)E(\lambda) \supset E(\lambda)$, i.e., the operators $f(A)$ and $E(\lambda)$ commute. ▼

(For the proof cf. Yoshida 1965, ch. 11.)

§ 5. Essentially Self-Adjoint Operators

We know that the spectrum of a self-adjoint operator is real and that the corresponding eigenfunctions form a complete orthogonal set of functions. Hence, the self-adjoint operators defined in the Hilbert space of physical state vectors are proper candidates for physical observables. However, in the representation theory of Lie algebras self-adjoint operators are not the most convenient objects for the formulation of a series of interesting theorems. Hence, it is necessary to introduce a wider class of operators, which on the one hand still have good spectral properties and on the other hand allow us to express certain fundamental mathematical results of representation theory.

Consider first symmetric operators. The spectrum of a symmetric operator is also real and any two eigenvectors corresponding to different eigenvalues are orthogonal (cf., e.g., Akhiezer and Glazman 1966, p. 136). However, a symmetric operator A defined in the Hilbert space H of physical state vectors is not the proper candidate to be an observable. This is because the set of all eigenvectors of a symmetric operator A does not form a complete set, i.e., there exists state vectors in H which will be orthogonal to all eigenvectors of the operator A. On such vectors the observable A does not yield an eigenvalue.

If we want to replace a symmetric operator by its self-adjoint extension, we find that in many cases a given symmetric operator has many, or even infinitely many, self-adjoint extensions. Hence, in general, symmetric operators are not the proper candidates for observables. However, special symmetric operators, which have a unique self-adjoint extension, could be used in physics. Such operators are called essentially self-adjoint.

A symmetric operator A is said to be *essentially self-adjoint* (e.s.a.) if its closure $\bar{A}$ is self-adjoint, i.e., $(\bar{A})^* = \bar{A}$. These operators, on the one hand, admit through their closure a physical interpretation as observables and, on the other hand, they are convenient objects for stating a number of interesting theorems in the representation theory of Lie algebras

Let us now derive the simplest properties of essentially self-adjoint operators.

LEMMA 1. *An operator A is essentially self-adjoint if and only if* $\bar{A} = A^*$.

PROOF: Let A be e.s.a. Then, $\bar{A} = (\bar{A})^* = A^*$ by lemma 1 of § 1. Conversely, if $\bar{A} = A^*$, then $(\bar{A})^* = A^{**} = \bar{A}$ by eq. (6).▼

Note that by th. 4 in § 1 a symmetric operator is essentially self-adjoint if it has both deficiency indices equal zero.

LEMMA 2. *Let* $A_k^* = A_k$, $k = 1, 2$, *and let the spectral families of these operators mutually commute. Then,* $A_1 \pm A_2$ *is essentially self-adjoint.*

PROOF: The proof follows directly from the spectral theorem.▼

A useful criterion for essential self-adjointness is given in:

LEMMA 3. *Let A be symmetric operator and let* $(A+I)^{-1}$ *be bounded and densely defined. Then, A is essentially self-adjoint.*

PROOF: A bounded linear operator is continuous by eq. 1(1). Hence, the closed operator $(A+I)^{-1} = (\bar{A}+I)^{-1}$ is defined on the whole space. Consequently, $R(\bar{A}+I) = H$. The operator $\bar{A}+I$ is symmetric by eq. 1(16). Therefore, $\bar{A}+I$ is self-adjoint by lemma 1.2. Hence, $\bar{A} = (\bar{A}+I)-I$ is essentially self-adjoint by lemma 2.▼

We prove another useful result

LEMMA 4. *Let D be a dense linear manifold in a Hilbert space H. Let A and A be linear transformations whose domains are D and whose ranges are contained in D such that A' is contained in the adjoint of A. If $A'A$ is essentially self-adjoint, then the closure of A' is the adjoint of A.*

PROOF: We must show that the graph of A^* contains no non-zero element, orthogonal to the graph of A'. Suppose $\{a, b\}$ is an element of the graph of A^* that is orthogonal to the graph of A'. In other words, $b = A^*a$, but $(y, a)+ +(A'y, b) = 0$ for all y in D. If x is in D then Ax is in D and hence $(Ax, a)+ +(A'Ax, b) = 0$; that is, $(x, b)+(A'Ax, b) = 0$. But $1+A'A$ has a dense range. Consequently $b = 0$. So $(y, a) = 0$ for all y in D; therefore $a = 0$.▼

Let us note finally that properties of a given operator are, to some extent, under our control, as the following two striking examples show:

1° Th. 1.4 states that a symmetric operator with different deficiency indices has no self-adjoint extension. However, if we embed the original Hilbert space H in a larger space $\tilde{H}$, then we have

THEOREM 5. *Every symmetric operator A in H with arbitrary deficiency indices* (n_+, n_-) *can be extended to a self-adjoint operator A acting in a larger Hilbert space* $\tilde{H} \supseteq H$.▼

(For the proof of the theorem cf. B. Sz. Nagy 1955.)

Hence, if elements of a larger Hilbert space $\tilde{H}$ admit a physical interpretation as state vectors, we can, in principle, take any symmetric operator in H to be a physical observable.

2° The differential operator $d = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\varphi}$ is unbounded in $H = L^2(0, 2\pi)$ and therefore discontinuous. This is, however, true only if we consider the continuity implied by the strong topology in the Hilbert space. However, in the space of distribution the differential operator d is continuous. To show this, we recall that a sequence $\{F_n\}$ of distributions is said to be *convergent* to a distribution F, if for an arbitrary test function ψ we have

$$\lim_{n\to\infty} (F_n, \psi) = (F, \psi).$$

Let $\psi \in C^\infty(0, 2\pi)$ and let $F_n \to F$. Then,

$$\left(\frac{\partial F_n}{\partial\varphi}, \psi\right) = \left(F_n, -\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right) \to \left(F, -\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right) = \left(\frac{\partial F}{\partial\varphi}, \psi\right).$$

Hence, if $F_n \to 0$, then $\frac{1}{\mathrm{i}} \frac{\mathrm{d}F_n}{\mathrm{d}\varphi} \to 0$, i.e., the operator d is continuous. ▼

These two examples indicate that we can considerably improve properties of operators, if we introduce a properly chosen structure in the carrier space.

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List of Important Symbols

$g_{\mu\nu}$ — space-time metric XIX
a^* — hermitian conjugation of a matrix or operator a XIX
$\bar{a}$ — complex conjugation of a matrix a XIX
$[\frac{1}{2}n]$ — XIX, 304
$\{\frac{1}{2}n\}$ — XIX, 304
$[X, Y]$ — Lie multiplication 1
$[M, N]$ — linear hull of vectors of the form $[X, Y]$, $X \in M$, $Y \in N$, $M \subset L$, $N \subset L$ 1
C^l_{jk} — structure constants of a Lie algebra or Lie group 2, 85
L^c — complex extension of a Lie algebra L 3
$\Phi(\xi, \eta)$ — bilinear form 4
$V_1 \dot{+} V_2 \dot{+} \ldots$ — direct sum of vector spaces V_i 5
$L_1 \oplus L_2 \oplus \ldots$ — direct sum of Lie algebras L_i 5
L/N — quotient Lie algebra of L with respect to N 6
$\mathrm{ad}X(Y) \equiv [X, Y]$ — 7
L_A — adjoint algebra of L 8
$L_1 \oplus\!\!\!\!+ L_2$ — semidirect sum of two Lie algebras L_1 and L_2 9
(X, Y) — Killing form in a Lie algebra 12
L^α — subspace of the root α 21
$\Delta(L)$ — system of nonzero roots of a semisimple Lie algebra 21
$\Pi(L)$ — system of simple roots of a semisimple Lie algebra 24
$\mathrm{gl}(n, R)$ — 3
$\mathrm{gl}(n, C)$ — 4
$\mathrm{sl}(n, C)$ — 4
$O(2n+1, C)$ — 4
$O(2n, C)$ — 4
A_n, B_n, C_n, D_n — classical Lie algebras 4
$\mathrm{su}(n)$ — 34
$\mathrm{sl}(n, R)$ — 34
$\mathrm{su}(p, q)$ — 34
$\mathrm{su}^*(2n)$ — 34
$\mathrm{so}(2n)$ — 35
$\mathrm{so}(2n+1)$ — 35
$\mathrm{sp}(n)$ — 35
$\mathrm{sp}(n, R)$ — 35
$\mathrm{sp}(p, q)$ — 36
$A \cup B$ — union of two sets A and B 52
$A \cap B$ — intersection of two sets A and B 52
$\{X, \tau\}$ — topological space 52
$d(x, y)$ — distance in metric space 53
$S(x, r) = \{y \in X: d(x, y) < r\}$ — neighbourhood of a point x 54
$A' \equiv X \backslash A \equiv \{x: x \in X \text{ and } x \notin A\}$ — complement A' of a set $A \subset X$ 55

$P \simeq Q$ — two homotopic paths P and Q 60
$\|\cdot\|$ — norm 64
$\mu(X)$ — measure of a set X 68
$f|N$ — restriction of function f to subset N 77
$\Delta(x) \equiv \Delta^G(x)$ — modular function for group G 69
$d\Omega_p$ — differential of mapping Ω in a point p 80
$G_1 \times G_2$ — direct product of groups G_1 and G_2 95
$G_1 \circledS G_2$ — semidirect product of groups G_1 and G_2 96
GL(n, R) — 62
O(n) — 62, 63
GL(n, C) — 62
SU(p, q) — 106
SL(n, R) — 106
SL$(n, C)^R$ — 106
SO$(2n+1, C)$ — 107
SO$(2n+1, C)^R$ — 107
SO(p, q) — 107
Sp(n, C) — 107
Sp(n) — 107
Sp(p, q) — 107
Sp(n, R) — 107
Sp$(n, C)^R$ — 107
SO$(2n, C)$ — 108
SO$(2n, C)^R$ — 108
SO*$(2n)$ — 108
$U(n)$ — 277
T_x — representation of a group G in Hilbert space 134
$D_{ij}(x)$ — matrix elements of the operator T_x 138
$H_1 \oplus H_2 \oplus \ldots$ — direct sum of subspaces H_i 142
$\overset{1}{E} \otimes \overset{2}{E}$ — tensor product of vector spaces $\overset{1}{E}$ and $\overset{2}{E}$ 147
$A \otimes B$ — tensor product of operators A and B 148
$\hat{x}(x) \equiv \langle x, \hat{x} \rangle$ — character of an abelian group 159, 160
(u, v) — scalar product in Hilbert space 166
K_n — Weyl operator 167
$X(x)$ — character of a representation T of group G 171
T^L_g or U^L_g — representation of group G induced by the representation L of subgroup K 206
$\{D^a_1(g)\}$ — matrix form of the representation $g \to D(g)$ 243
$\{T^a\}$ — contravariant tensor operator 243
$\{T_a\}$ — covariant tensor operator 244
$E(L)$ — enveloping algebra of a Lie algebra L 269
$D(L)$ — enveloping field of a Lie algebra L 270
$\dot{g} = Kg$ — 273
$T = \{T_{i_1 i_2 \ldots i_r}\}$ — tensor of rank r relative to group G 241
$T = \{e_{i_1 i_2 \ldots i_r}\}$ — polyvector 293
$\Delta = X_1^2 + \ldots X_r^2$ — Nelson operator 327
$|A|$ — absolute value of an operator A 333
$P^*(\lambda)$ — adjoint of an operator distribution $P(\lambda)$ 461
$K \backslash G$ — quotient space of right cosets 485
G/K — quotient space of left cosets 485

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