CANONICAL GELFAND-ZEITLIN MODULES OVER ORTHOGONAL GELFAND-ZEITLIN ALGEBRAS

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Abstract. We prove that every orthogonal Gelfand-Zeitlin algebra U acts on its Gelfand-Zeitlin subalgebra Γ. Considering the dual module, we show that every Gelfand-Zeitlin character of Γ is realizable in a U-module. We observe that the Gelfand-Zeitlin formulae can be rewritten using divided difference operators. It turns out that the action of the latter operators on Γ gives rise to an explicit basis in a certain Gelfand-Zeitlin submodule of the dual module mentioned above. This gives, generically, both in the case of regular and singular Gelfand-Zeitlin characters, an explicit construction of simple modules which realize given Gelfand-Zeitlin characters.

1. Introduction and description of the results

In this paper we work over the field $\mathbb{C}$ of complex numbers. For a positive integer $n \geq 1$, consider the flag

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

of general linear Lie algebras where each $\mathfrak{gl}_i$ is embedded into $\mathfrak{gl}_{i+1}$ in the obvious way with respect to the top left corner. This flag induces the following flag

$$U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U_n$$

of embeddings of the corresponding universal enveloping algebras, where U_i denotes the universal enveloping algebra of $\mathfrak{gl}_i$. Let Z_i be the center of U_i. Then the subalgebra Γ_n of U_n, generated by all Z_i, where $i = 1, 2, \dots, n$, is called the *Gelfand-Zeitlin*[1] *subalgebra*. It is a maximal commutative subalgebra of U_n, see [Ov1, Corollary 1].

A U_n-module M is called a *Gelfand-Zeitlin module* provided that the action of Γ_n on U_n is locally finite. The theory of Gelfand-Zeitlin modules originates in the papers [DOF1, DOF2, DOF3, DOF4], inspired by the description, due to I. Gelfand and M. Zeitlin, of a basis in finite dimensional U_n-modules consisting of Γ_n-eigenvectors in [GZ1], see also [GZ2] for a similar result for orthogonal Lie algebras. The main value of the original general theory of Gelfand-Zeitlin modules was that it produced a family of simple U_n-module which depends on $n(n+1)/2$ complex parameters, the largest known family of simple U_n-modules to date. This theory was generalized to orthogonal Lie algebras in [Ma3] and to quantum algebras in [MT]. It was also very useful for the study of various categories of U_n-modules, see [Kh, Ma1, Ma4, Ma5, MO, CM, KM, MS]

A major step in the development of the theory of Gelfand-Zeitlin modules was made in [Ov1, Ov2] (it was also put into a more general setup in [FO], see also [FMO, FOS]) where it was, in particular, shown that all Gelfand-Zeitlin characters

[1]The surname Zeitlin appears in the literature in different spellings, that is in different latinizations of the Cyrillic version of the original Latin (German) surname, in particular, it was spelled as Cetlin, Zetlin, Tzetlin and Tsetlin. Here we use the original Latin spelling.

lift to U_n-modules and that the number of such non-isomorphic simple lifts is finite. This naturally motivated the question of classification and explicit construction of simple Gelfand-Zeitlin modules. The main difficulty is to construct and classify so-called *singular* Gelfand-Zeitlin modules, that is modules on which the (rational) coefficients of the classical Gelfand-Zeitlin formulae have potential singularities. A lot of progress in this direction was made recently in [FGR1, FGR2, FGR3, FGR4, FGR5, FRZ1, FRZ2, GR, RZ, Vi1, Vi2, Za] using a variety of different methods.

In the present paper we observe that the approach to construct singular simple Gelfand-Zeitlin proposed in [Vi1, Vi2] works, with minimal adjustment, for a much larger class of algebras, called *orthogonal Gelfand-Zeitlin algebras* which were introduced in [Ma2] and studied in [MPT].

Let us now describe the results and the structure of the paper. Let U be an orthogonal Gelfand-Zeitlin algebra and Γ its Gelfand-Zeitlin subalgebra. Our first interesting observation, presented in Proposition 1, is that the regular action of Γ on itself extends, via the Gelfand-Zeitlin formulae, to an action of U on Γ. Using the standard adjunction argument it follows that the dual module Γ^* contains each simple Γ-module as a submodule. This gives a very short and non-technical proof of the original statement [Ov2, Theorem 2] on existence of Gelfand-Zeitlin U-modules for arbitrary Gelfand-Zeitlin characters. Our arguments and results also work for an arbitrary orthogonal Gelfand-Zeitlin algebra, while [Ov2, Theorem 2] is proved just for U_n.

The module Γ^* can be used to define, for every fixed Gelfand-Zeitlin character, what we call a *canonical* simple Gelfand-Zeitlin module for this character, see Subsection 3.4. In full generality, simple Gelfand-Zeitlin modules are not classified. So, it is a very unexpected feature that, for a fixed Gelfand-Zeitlin character, one can define the simple Gelfand-Zeitlin module in a way which does not involve any choices.

We study Γ^* and its Gelfand-Zeitlin submodules more closely in Section 4. There we first observe that the Gelfand-Zeitlin formulae can be rewritten using divided difference operators, see [BGG, De]. As an immediate byproduct, we obtain that the action of U on Γ extends to the action of U on a certain Galois extension of Γ, see Subsection 4.3. Further, we use divided difference operators to write down an explicit U-submodule of Γ^* and show, in Subsection 4.5, that this module is generically simple and hence also canonical.

Section 2 below contains all necessary preliminaries.

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2. Orthogonal Gelfand-Zeitlin algebras

2.1. **Setup.** Let m be a fixed positive integer and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ a composition of m with k non-zero parts. This means that all λ_i are positive integers and, moreover, $\lambda_1 + \lambda_2 + \cdots + \lambda_k = m$. Denote by $I = I_\lambda$ the set of all pairs (i, j) such that $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, \lambda_i\}$. Let $\Omega = \Omega_\lambda$ be the field of rational functions in m variables $x_{\mathbf{a}}$, where $\mathbf{a} \in I$. For $\mathbf{a} = (i, j) \in I$, we will use the notation

$i_{\mathbf{a}} := i$ and $j_{\mathbf{a}} := j$. For $i = 1, 2, \dots, k$, we also denote by $I^{(i)}$ the set of all $\mathbf{a} \in I$ for which $i_{\mathbf{a}} = i$.

For each $\mathbf{a} \in I$, let $\varphi_{\mathbf{a}}$ denote the automorphism of the field extension $\mathbb{C} \subset \Omega$ which is uniquely defined via

$$\varphi_{\mathbf{a}}(x_{\mathbf{b}}) = x_{\mathbf{b}} + \delta_{\mathbf{a},\mathbf{b}}, \quad \text{for all} \quad \mathbf{b} \in I,$$

where $\delta_{\mathbf{a},\mathbf{b}}$ is the Kronecker symbol. We denote by $\beth$ the (abelian) group generated by all $\varphi_{\mathbf{a}}$ with $i_{\mathbf{a}} < k$.

For each $f \in \Omega$, we have the $\mathbf{C}$-linear transformation of Ω given by multiplication with f.

2.2. **Definition.** For $i = 1, 2, \dots, k-1$, define the $\mathbb{C}$-linear operators E_i and F_i on Ω by the following *Gelfand-Zeitlin formulae*:

$$E_i := \sum_{j=1}^{\lambda_i} \frac{\prod\limits_{\mathbf{a}\in I^{(i+1)}} (x_{(i,j)} - x_{\mathbf{a}})}{\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{(i,j)\}} (x_{(i,j)} - x_{\mathbf{b}})} \varphi_{(i,j)} \quad F_i := \sum_{j=1}^{\lambda_i} \frac{\prod\limits_{\mathbf{a}\in I^{(i-1)}} (x_{(i,j)} - x_{\mathbf{a}})}{\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{(i,j)\}} (x_{(i,j)} - x_{\mathbf{b}})} \varphi_{(i,j)}^{-1},$$

where the set $I^{(0)}$ is, by convention, empty and hence the product over this set equals 1. For $\mathbf{a} \in I$, we also define the $\mathbb{C}$-linear operators $\gamma_{\mathbf{a}}$ on Ω as multiplication with the $j_{\mathbf{a}}$-th elementary symmetric polynomial in $\{x_{\mathbf{b}} : \mathbf{b} \in I^{(i_{\mathbf{a}})}\}$.

The *orthogonal Gelfand-Zeitlin* (OGZ)-algebra U_λ associated to λ is the subalgebra of the algebra of all $\mathbb{C}$-linear transformations of Ω, generated by all E_i, F_i, where $i = 1, 2, \dots, k-1$, and all $\gamma_{\mathbf{a}}$, where $\mathbf{a} \in I$, see [Ma2, Section 3].

2.3. **Gelfand-Zeitlin subalgebra and Gelfand-Zeitlin modules.** The commutative subalgebra Γ_λ of U_λ, generated by all $\gamma_{\mathbf{a}}$, where $\mathbf{a} \in I$, is called the *Gelfand-Zeitlin subalgebra*. A U_λ-module M is called a *Gelfand-Zeitlin* module provided that the action of Γ_λ on this module is locally finite. By [Ma2, Corollary 1], the algebra Γ_λ is a *Harish-Chandra* subalgebra of U_λ in the sense of [DOF4].

For a character $\chi : \Gamma_\lambda \to \mathbb{C}$ and a Gelfand-Zeitlin module M, we denote by $M(\chi)$ the set of all vectors in M which are annihilated by some power of the kernel of χ. Then we have

$$M = \bigoplus_{\chi} M(\chi).$$

We denote by pr_χ the projection map $M \twoheadrightarrow M(\chi)$ with respect to this decomposition.

2.4. U_n **as an OGZ algebra.** If we take $m = n(n+1)/2$ and $\lambda = (1, 2, \dots, n)$, then $U_n \cong U_\lambda$ such that this isomorphism identifies Γ_n with Γ_λ, see [Ma2, Section 4] for details.

2.5. **Group action.** Let $G = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$. Then, for $i = 1, 2, \dots, k$, the permutation group S_{λ_i} permutes $\{x_{\mathbf{a}} : \mathbf{a} \in I^{(i)}\}$ by acting on the second component of the pair $\mathbf{a}$. This defines an action of G on Ω.

Let R denote the polynomial ring in all $x_{\mathbf{a}}$, where $\mathbf{a} \in I$. Then R is a subring of Ω and Ω is the field of fractions of the domain R. The algebra Γ_λ is canonically isomorphic to the algebra $\mathbf{R} := R^G$ of all polynomials in $\{x_{\mathbf{a}} : \mathbf{a} \in I\}$ that are invariant with respect to the action of G. In what follows we will identify the algebras Γ_λ and $\mathbf{R}$.

We also set $\underline{G} := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{k-1}}$, which is a subgroup of G in the obvious way.

3. Action of U_λ on $\mathbf{R}$ and its consequences

3.1. **Dual spaces.** Let V be the $\mathbb{C}$-vector space spanned by all $x_{\mathbf{a}}$, where $\mathbf{a} \in I$. Consider the dual vector space V^* of V. Each $\mathtt{v} \in V^*$ gives rise to the *evaluation* ring homomorphism $\mathrm{ev}_{\mathtt{v}} : R \twoheadrightarrow \mathbb{C}$ by sending each $x_{\mathbf{a}}$, where $\mathbf{a} \in I$, to $\mathtt{v}(x_{\mathbf{a}})$.

As usual, R^* denotes the vector space of all $\mathbb{C}$-linear maps from R to $\mathbb{C}$. Then the elements $\mathrm{ev}_{\mathtt{v}}$, where $\mathtt{v} \in V^*$, are linearly independent elements in R^*. The action of $\daleth$ on Ω induces an action of $\daleth$ on both V^* and $\{\mathrm{ev}_{\mathtt{v}} : \mathtt{v} \in V^*\}$.

For $\mathtt{v} \in V^*$, we have the algebra homomorphism $\mathbf{ev}_{\mathtt{v}} : \mathbf{R} \to \mathbb{C}$ given by

$$\mathbf{ev}_{\mathtt{v}} : \mathbf{R} \hookrightarrow R \xrightarrow{\mathrm{ev}_{\mathtt{v}}} \mathbb{C}.$$

The corresponding 1-dimensional $\mathbf{R}$-module is denoted $\mathbb{C}_{\mathtt{v}}$. Note that, for $\mathtt{v}, \mathtt{w} \in V^*$, we have $\mathbb{C}_{\mathtt{v}} \cong \mathbb{C}_{\mathtt{w}}$ if and only if $\mathbf{ev}_{\mathtt{v}} = \mathbf{ev}_{\mathtt{w}}$ if and only if $\mathtt{v} \in G \cdot \mathtt{w}$. In particular, $\mathbf{R}$ distinguishes G-orbits on V^*. Furthermore, each character of $\mathbf{R}$ is of the form $\mathbf{ev}_{\mathtt{v}}$, for some $\mathtt{v} \in V^*$. We denote by $\mathbf{m}_{\mathtt{v}}$ the kernel of $\mathbf{ev}_{\mathtt{v}}$ and by $\chi_{\mathtt{v}}$ the quotient map $\mathbf{R} \twoheadrightarrow \mathbf{R}/\mathbf{m}_{\mathtt{v}} \cong \mathbb{C}$.

3.2. **Action on invariant polynomials.**

Proposition 1. *The algebra $\mathbf{R}$ is invariant under the natural (left) action of U_λ.*

Proof. For $f \in \mathbf{R}$, clearly $\gamma_{\mathbf{a}} \cdot f \in \mathbf{R}$, for any $\mathbf{a} \in I$. If $i \in \{1, 2, \dots, k-1\}$, then both $E_i \cdot f$ and $F_i \cdot f$ are G-invariant rational function whose denominators consist of products of $x_{\mathbf{a}} - x_{\mathbf{b}}$, where $\mathbf{a}$ and $\mathbf{b}$ are different elements of $I^{(i)}$.

Let g denote the element $E_i \cdot f$ (or $F_i \cdot f$). Let $\mathbf{a}$ and $\mathbf{b}$ be different elements of $I^{(i)}$. We want to prove that $x_{\mathbf{a}} - x_{\mathbf{b}}$ disappears from the denominator of g. Note that each denominator in the formulae for both E_i and F_i contains the factor $x_{\mathbf{a}} - x_{\mathbf{b}}$ at most once. Therefore, we can write $g = \frac{h}{x_{\mathbf{a}} - x_{\mathbf{b}}}$ such that h is a rational function in which $x_{\mathbf{a}} - x_{\mathbf{b}}$ does not appear in the denominator anymore and which maps to $-h$ after swapping $x_{\mathbf{a}}$ and $x_{\mathbf{b}}$. The classical description of alternating polynomials then implies that $x_{\mathbf{a}} - x_{\mathbf{b}}$ is a factor of h. Hence this factor cancels in the numerator and in the denominator of g and the claim follows. $\square$

Proposition 1 says that the vector space Γ_λ has the natural structure of a left U_λ-module extending the left regular action of Γ_λ on itself. We refer to [Ni1, Ni2] for similar phenomena for some Lie algebras. Comparing with the main result of [Ni1], it would be interesting to classify all possible U_λ-module structures on Γ_λ extending the left regular action of Γ_λ on itself.

We note that the U_λ-module Γ_λ is not simple as Γ_λ does not have a central character. Indeed, the S_{λ_k}-invariant polynomials in $x_{(k,1)}, x_{(k,2)}, \dots, x_{(k,\lambda_k)}$ clearly belong to the center of U_λ and the algebra of such polynomials acts freely on Γ_λ. However, it would be interesting to know, for which maximal ideals $\mathbf{m}$ in the center of U_λ, the U_λ-module $\Gamma_\lambda/\mathbf{m}\Gamma_\lambda$ is simple, alternatively, has finite length.

3.3. **Generic regular modules.** We denote by $\mathbf{R}^*$ the set of all $\mathbb{C}$-linear maps from $\mathbf{R}$ to $\mathbb{C}$. From Proposition 1 we have that the space $\mathbf{R}^*$ has the natural structure of a right U_λ-module.

Fix some $\mathtt{v} \in V^*$ such that $\mathtt{v}(x_{\mathbf{a}}) - \mathtt{v}(x_{\mathbf{b}}) \notin \mathbb{Z}$, for all $i \in \{1, 2, \dots, k-1\}$ and all different $\mathbf{a}, \mathbf{b} \in I^{(i)}$. Denote by $M_{\mathtt{v}}$ the subspace of $\mathbf{R}^*$ generated by all elements of the form $\mathbf{ev}_{\mathtt{w}}$, where $\mathtt{w} \in \gimel \cdot \mathtt{v}$.

Proposition 2. *The space $M_{\mathtt{v}}$ is invariant under the right U_λ-action.*

Proof. We note that our choice of $\mathtt{v}$ ensures that each $\mathbf{ev}_{\mathtt{w}}$, where $\mathtt{w} \in \gimel \cdot \mathtt{v}$, evaluates denominators in the formulae for all E_i and F_i to non-zero elements. Therefore, directly from the definitions, it follows that the precomposition of any $\mathbf{ev}_{\mathtt{w}}$ as above with any E_i, F_i or $\gamma_{\mathbf{a}}$ stays inside $M_{\mathtt{v}}$. The claim follows. $\square$

The modules described in Proposition 2 are the so-called *generic regular Gelfand-Zeitlin* modules over U_λ, cf. [DOF4, Theorem 24] and [Ma2, Section 8]. These U_λ-modules are indeed Gelfand-Zeitlin modules as Γ_λ acts via scalars on each $\mathbf{ev}_{\mathtt{w}}$ by construction. It is easy to see that each $M_{\mathtt{v}}$ as above has finite length and its simple subquotients are easy to describe in terms of connected components of $\gimel \cdot \mathtt{v}$ obtained by cutting $\gimel \cdot \mathtt{v}$ using the hyperplanes derived from the conditions that some numerator in the Gelfand-Zeitlin formulae is zero.

We also note that the above construction does not work if $\mathtt{v}(x_{\mathbf{a}}) - \mathtt{v}(x_{\mathbf{b}}) \in \mathbb{Z}$, for some $\mathbf{a}$ and $\mathbf{b}$ in the same $I^{(i)}$ with $i \in \{1, 2, \dots, k-1\}$. Indeed, in this case $\mathbf{ev}_{\mathtt{v}}$ evaluates to zero the denominators of some of the coefficients in the the Gelfand-Zeitlin formulae and the whole construction collapses.

3.4. **Existence of singular modules.** The following result generalizes [Ov2, Theorem 2] to our setup.

Corollary 3. *Any character of Γ_λ extends to a non-zero right U_λ-module.*

Proof. For any $\mathtt{v} \in V^*$, the U_λ-submodule of the right U_λ module $\mathbf{R}^*$ generated by $\mathbf{ev}_{\mathtt{v}}$ contains the non-zero element $\mathbf{ev}_{\mathtt{v}}$. By construction, the algebra Γ_λ acts on $\mathbf{ev}_{\mathtt{v}}$ via scalars prescribed by $\mathtt{v}$. As Γ_λ is identified with $\mathbf{R}$, varying $\mathtt{v}$ will exhaust all characters of Γ_λ. The claim follows. $\square$

From [Ma2, Corollary 1] and [DOF4, Subsection 1.4] it follows that the module constructed in the proof of Corollary 3 is, in fact, a Gelfand-Zeitlin module.

The assertion of Corollary 3 can be strengthened as follows.

Proposition 4. *For each $\mathtt{v} \in V^*$, the element $\mathbf{ev}_{\mathtt{v}}$ is, up to scalar, the unique element of $\mathbf{R}^*$ annihilated by $\mathbf{m}_{\mathtt{v}}$.*

Proof. The fact that Γ_λ acts on $\mathbf{ev}_{\mathtt{v}}$ via scalars prescribed by $\mathtt{v}$ follows directly from the definitions. So, we just need to prove the uniqueness. Our proof follows the argument from [Ni2, Proposition 2]. Using adjunction, we compute:

$$\begin{aligned} \mathrm{Hom}_{\Gamma_\lambda}\big(\mathbb{C}_{\mathtt{v}}, \mathbf{R}^*\big) &= \mathrm{Hom}_{\Gamma_\lambda}\big(\mathbb{C}_{\mathtt{v}}, \mathrm{Hom}_{\mathbb{C}}(\mathbf{R}, \mathbb{C})\big) \\ &\cong \mathrm{Hom}_{\mathbb{C}}\big(\mathbb{C}_{\mathtt{v}} \otimes_{\Gamma_\lambda} \mathbf{R}, \mathbb{C}\big) \end{aligned}$$

and the claim of the proposition follows from the observation that $\mathbb{C}_{\mathtt{v}} \otimes_{\Gamma_\lambda} \mathbf{R} \cong \mathbb{C}_{\mathtt{v}}$ as $\mathbf{R}$ is a free Γ_λ-module of rank one. $\square$

A similar adjunction argument shows that a simple right U_λ-module M is a submodule of $\mathbf{R}^*$ if and only if $M \otimes_{U_\lambda} \mathbf{R} \neq 0$.

For each $\mathtt{v} \in V^*$, we denote by $N_{\mathtt{v}}$ the U_λ-submodule of $\mathbf{R}^*$ generated by $\mathbf{ev}_{\mathtt{v}}$, that is $N_{\mathtt{v}} := \mathbf{ev}_{\mathtt{v}} \cdot U_\lambda$.

Proposition 5. *The module $N_{\mathtt{v}}$ has a unique maximal submodule.*

Proof. The unique maximal submodule of $N_{\mathtt{v}}$ is the sum of all submodules M of $N_{\mathtt{v}}$ with the property $\mathbf{ev}_{\mathtt{v}} \notin M$. The detailed proof is similar to the proof of [Di, Proposition 7.1.11]. □

The quotient of $N_{\mathtt{v}}$ by its unique maximal submodule will be denoted $L_{\mathtt{v}}$ and called the *canonical* simple Gelfand-Zeitlin module associated to $\mathtt{v}$. Our terminology is motivated by the fact that the construction of $L_{\mathtt{v}}$ does not use any choice and is given purely in terms of the original Gelfand-Zeitlin formulae and dualization. Clearly, $L_{\mathtt{v}} \cong L_{\mathtt{w}}$ if $\mathtt{v} \in G \cdot \mathtt{w}$. However, we do not know general sufficient and necessary condition, in terms of $\mathtt{v}$ and $\mathtt{w}$, for $L_{\mathtt{v}}$ and $L_{\mathtt{w}}$ to be isomorphic.

Generically, for $\mathtt{v} \in V^*$, there exists a unique simple Gelfand-Zeitlin module containing a non-zero element annihilated by $\mathbf{m}_{\mathtt{v}}$. This module is then automatically canonical. This is the case, for example, in the generic regular situation.

4. Singular Gelfand-Zeitlin modules

4.1. **Divided difference operators.** Let $\mathbf{a}, \mathbf{b} \in I^{(i)}$ be an ordered pair of different elements, for some $i < k$, and $(\mathbf{a}, \mathbf{b})$ denote the transposition swapping $\mathbf{a}$ and $\mathbf{b}$. Then we have, see [BGG, De], the corresponding *divided difference operator* $\partial_{\mathbf{a},\mathbf{b}} : \Omega \to \Omega$ given by

$$\partial_{\mathbf{a},\mathbf{b}} = \frac{\mathrm{id} - (\mathbf{a}, \mathbf{b})}{(x_{\mathbf{a}} - x_{\mathbf{b}})}.$$

The operators $\partial_{\mathbf{a},\mathbf{b}}$ satisfy the following *Leibniz rule*:

$$\partial_{\mathbf{a},\mathbf{b}}(fg) = \partial_{\mathbf{a},\mathbf{b}}(f)g + f^{(\mathbf{a},\mathbf{b})}\partial_{\mathbf{a},\mathbf{b}}(g),$$

where $f^{(\mathbf{a},\mathbf{b})}$ denotes the outcome of the action of $(\mathbf{a}, \mathbf{b})$ on f. This Leibniz rule implies the following variation which should be understood as an equality of operators acting on Ω, where $\gamma \in \daleth$,

$$\partial_{\mathbf{a},\mathbf{b}} \circ f \circ \gamma = \partial_{\mathbf{a},\mathbf{b}}(f) \circ \gamma + f^{(\mathbf{a},\mathbf{b})} \circ \partial_{\mathbf{a},\mathbf{b}} \circ \gamma.$$

We also have $\partial_{\mathbf{a},\mathbf{b}} = -\partial_{\mathbf{b},\mathbf{a}}$.

For a fixed linear order $\prec$ on $I^{(i)}$, we set

$$\partial_{(\mathbf{a},\mathbf{b})} = \begin{cases} \partial_{\mathbf{a},\mathbf{b}}, & \mathbf{a} \prec \mathbf{b}; \\ \partial_{\mathbf{b},\mathbf{a}}, & \mathbf{b} \prec \mathbf{a}. \end{cases}$$

The order $\prec$ allows us to view G as a Coxeter group such that simple reflections are given by those transpositions which swap neighboring elements with respect to $\prec$. Then the divided difference operators which correspond to simple reflections satisfy the defining relations of the nil-Coxeter algebra. In particular, the dimension of the algebra which such operators generate coincides with the cardinality of G. Furthermore, to each $w \in G$ with a fixed reduced expression $w = s_1 s_2 \cdots s_l$, we can associate the element $\partial_w = \partial_{s_1}\partial_{s_2} \cdots \partial_{s_l}$ and this element will not depend on the reduced expression, see [BGG, Theorem 3.4 b)]. If the expression is not reduced, then $\partial_w = 0$, see [BGG, Theorem 3.4 a)]. The elements ∂_w, for $w \in G$, are linearly independent as operators on Ω. A similar construction works for any parabolic subgroup of G viewed as a Coxeter group in the natural way.

For $u \in G$ and simple reflection s, we set $\partial_s^{(u)} = u\partial_s u^{-1}$ and, for $w \in G$, define $\partial_w^{(u)} = \partial_{s_1}^{(u)}\partial_{s_2}^{(u)}\cdots\partial_{s_l}^{(u)}$, where $w = s_1 s_2 \cdots s_l$ is a reduced expression.

4.2. **Realization of U_λ via divided difference operators.** The results of this subsection are partially inspired by [Er].

For a fixed element $i \in \{1, 2, \dots, k-1\}$, set, for simplicity, $m = \lambda_i$. Let $\prec$ be a fixed linear order on $\{1, 2, \dots, m\} = \{a_1, a_2, \dots, a_m\}$, where $a_1 \prec a_2 \prec \cdots \prec a_m$. Let μ be a composition of m with non-zero parts. We identify μ with the following decomposition of $\{1, 2, \dots, m\}$ into subsets:

$$\{a_1, a_2, \dots, a_{\mu_1}\}, \quad \{a_{\mu_1+1}, a_{\mu_1+2}, \dots, a_{\mu_1+\mu_2}\}, \quad \dots \quad .$$

Let

$$\underline{\mu_j} := \{a_{\mu_1+\mu_2+\dots+\mu_{j-1}+1}, a_{\mu_1+\mu_2+\dots+\mu_{j-1}+2}, \dots, a_{\mu_1+\mu_2+\dots+\mu_j}\}$$

denote the subset corresponding to the part μ_j (here $\mu_0 = 0$ by convention).

For a part μ_j, we set $\min(\mu_j) = \mu_1 + \mu_2 + \cdots + \mu_{j-1} + 1$. This is the index of the minimal element in $\underline{\mu_j}$ with respect to $\prec$. If $\mu_j = 1$, we define $\partial(\mu, j)$ to be the identity operator on $\overline{\mathbf{R}}$. If $\mu_j > 1$, we define $\partial(\mu, j)$ as the operator

$$\partial_{(a_{\min(\mu_j)+\mu_j-2}, a_{\min(\mu_j)+\mu_j-1})} \circ \cdots \circ \partial_{(a_{\min(\mu_j)+1}, a_{\min(\mu_j)+2})} \circ \partial_{(a_{\min(\mu_j)}, a_{\min(\mu_j)+1})}.$$

Finally, we denote by $f(\mu, j)^{\pm}$ the following rational function:

$$f(\mu, j)^{\pm} := \frac{\displaystyle\prod_{\mathbf{a} \in I^{(i\pm1)}} (x_{(i, a_{\min(\mu_j)})} - x_{\mathbf{a}})}{\displaystyle\prod_{\mathbf{b} \in I^{(i)} \setminus \underline{\mu_j}} (x_{(i, a_{\min(\mu_j)})} - x_{\mathbf{b}})}.$$

Proposition 6. *The generators E_i and F_i of U_λ can be written as follows:*

$$E_i = \sum_j \partial(\mu, j) f(\mu, j)^{+} \varphi_{(i, \min(\mu_j))},$$
$$F_i = \sum_j \partial(\mu, j) f(\mu, j)^{-} \varphi_{(i, \min(\mu_j))}.$$

Proof. We prove the statement for E_i, the proof for F_i is similar. Let s be some element in $\{1, 2, \dots, m\}$. We need to check that the coefficient at $\varphi_{(i,s)}$ on the right hand side of the formula in the formulation is the correct one. We use induction on λ_i via the size of the part μ_j of μ containing s. If $\mu_j = 1$, then the claim follows directly from the Gelfand-Zeitlin formulae. In the inductive procedure below we will prove the result at the same time for all $s \in \underline{\mu_j}$.

To prove the induction step, we assume that $\mu_j > 1$ and set

$$\mathbf{c} := (i, a_{\mu_1+\mu_2+\dots+\mu_j-1}), \quad \mathbf{d} := (i, a_{\mu_1+\mu_2+\dots+\mu_j}).$$

Note that $a_{\mu_1+\mu_2+\dots+\mu_j}$ is the maximum element of $\underline{\mu_j}$ with respect to $\prec$ and $a_{\mu_1+\mu_2+\dots+\mu_j-1}$ is the maximum element of $\underline{\mu_j} \setminus \{a_{\mu_1+\mu_2+\dots+\mu_j}\}$ with respect to $\prec$.

We now use the induction hypothesis and compute:

$$\frac{\mathrm{id}-(\mathbf{c},\mathbf{d})}{x_{\mathbf{c}}-x_{\mathbf{d}}}\left(\sum_{\mathbf{u}\in\underline{\mu_j}\setminus\{\mathbf{d}\}}\frac{\prod\limits_{\mathbf{a}\in I^{(i+1)}}(x_{\mathbf{u}}-x_{\mathbf{a}})}{\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{\mathbf{u},\mathbf{d}\}}(x_{\mathbf{u}}-x_{\mathbf{b}})}\varphi_{\mathbf{u}}\right).$$

The coefficients at $\varphi_{\mathbf{c}}$ and $\varphi_{\mathbf{d}}$ are, clearly, correct. For $\mathbf{u}\in I^{(i)}\setminus\{\mathbf{c},\mathbf{d}\}$, the coefficient at $\varphi_{\mathbf{u}}$ is

$$\frac{1}{(x_{\mathbf{c}}-x_{\mathbf{d}})}\left(\frac{\prod\limits_{\mathbf{a}\in I^{(i+1)}}(x_{\mathbf{u}}-x_{\mathbf{a}})}{\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{\mathbf{u},\mathbf{d}\}}(x_{\mathbf{u}}-x_{\mathbf{b}})}-\frac{\prod\limits_{\mathbf{a}\in I^{(i+1)}}(x_{\mathbf{u}}-x_{\mathbf{a}})}{(x_{\mathbf{u}}-x_{\mathbf{d}})\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{\mathbf{u},\mathbf{c},\mathbf{d}\}}(x_{\mathbf{u}}-x_{\mathbf{b}})}\right).$$

Setting f to be the common numerator and g to be the product expression in the denominator of the last fraction, we have

$$\begin{aligned}\frac{1}{(x_{\mathbf{c}}-x_{\mathbf{d}})}&\left(\frac{f}{(x_{\mathbf{u}}-x_{\mathbf{c}})g}-\frac{f}{(x_{\mathbf{u}}-x_{\mathbf{d}})g}\right)=\\&=\frac{f}{g}\left(\frac{1}{(x_{\mathbf{c}}-x_{\mathbf{d}})(x_{\mathbf{u}}-x_{\mathbf{c}})}-\frac{1}{(x_{\mathbf{c}}-x_{\mathbf{d}})(x_{\mathbf{u}}-x_{\mathbf{d}})}\right)=\\&\qquad=\frac{f}{g}\cdot\frac{1}{(x_{\mathbf{u}}-x_{\mathbf{c}})(x_{\mathbf{u}}-x_{\mathbf{d}})}=\frac{\prod\limits_{\mathbf{a}\in I^{(i+1)}}(x_{\mathbf{u}}-x_{\mathbf{a}})}{\prod\limits_{\mathbf{b}\in I^{(i)}\setminus\{\mathbf{u}\}}(x_{\mathbf{u}}-x_{\mathbf{b}})}.\end{aligned}$$

This shows that the coefficient at $\varphi_{\mathbf{u}}$ is also correct and completes the proof. $\square$

Note that, varying $\prec$, we obtain different ways to write down the action of E_i and F_i on Ω. The assertion of Proposition 6 seems very surprising as it looks as a manipulation, which is not G-invariant, with some elements, which are not G-invariant either, that results in a G-invariant element.

4.3. **Action on polynomials.** An immediate corollary from Proposition 6 is the following statement.

Corollary 7. *The vector space R has the natural structure of a U_λ-module, where each $\gamma_{\mathbf{a}}$ acts by multiplication and each E_i and F_i acts by the corresponding operator described in Proposition 6, for μ having just one part.*

Proof. The only thing to note here is that divided differences act on R, which is noticed already in [BGG]. $\square$

Note that the U_λ-module $\mathbf{R}$ is a submodule of the U_λ-module R.

4.4. **Construction of singular Gelfand-Zeitlin modules.** For $\mathtt{v}\in V^*$, we denote by $G_{\mathtt{v}}$ the stabilizer of $\mathtt{v}$ in $\underline{G}$. For $\rho\in G_{\mathtt{v}}$ and Q an element of a set on which $G_{\mathtt{v}}$ acts, we will denote by Q^ρ the outcome of the action of ρ on Q.

Fix $\mathtt{v}\in V^*$ with the following properties:

- The group $G_{\mathtt{v}}$ is the maximum element, with respect to inclusions, in the set $\{G_{\mathtt{w}}:\mathtt{w}\in\daleth\cdot\mathtt{v}\}$.

- Every orbit of $G_{\mathtt{v}}$ on each $I^{(i)}$ has the form
$$\{(i,p),(i,p+1),(i,p+2),\dots,(i,p+q)\},$$
for some i,p,q.

The set $\gimel\cdot\mathtt{v}$ is invariant under the action of $G_{\mathtt{v}}$ and we can write it as a disjoint union of orbits:
$$\gimel\cdot\mathtt{v}=\coprod_{j\in J}\mathcal{O}_j,$$
where J is just some indexing set. For each element $j\in J$, we fix the unique representative $\mathtt{u}_j=\xi_j^{-1}(\mathtt{v})$ of the orbit $\mathcal{O}_j$, where $\xi_j\in\gimel$, which has the following property: If (i,p) and $(i,p+1)$ belong to the same orbit of $G_{\mathtt{v}}$, for some i and p, then

$$\mathbb{Z}\ni(\mathtt{u}_j-\mathtt{v})(x_{(i,p)})\geq(\mathtt{u}_j-\mathtt{v})(x_{(i,p+1)})\in\mathbb{Z}. \tag{1}$$

Let X_j denote the set of shortest representatives of cosets in $G_{\mathtt{v}}/G_{\mathtt{u}_j}$.

We fix the natural Coxeter presentation of G, i.e. the one where all simple reflections have the form $((i,p),(i,p+1))$, for some i and p. Consider the following two subsets in $\mathbf{R}^*$:

$$\mathbf{B}:=\bigcup_{j\in J}\bigcup_{w\in X_j}\{\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j\}\qquad\text{and}\qquad\underline{\mathbf{B}}:=\bigcup_{j\in J}\bigcup_{w\in G_{\mathtt{v}}}\{\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j\}. \tag{2}$$

We note the use of $\mathrm{ev}_{\mathtt{v}}$ instead of $\mathbf{ev}_{\mathtt{v}}$ here due to the presence of the shifts ξ_j.

Lemma 8. *For any $\rho\in G_{\mathtt{v}}$ and any $\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j\in\underline{\mathbf{B}}$, we have*
$$\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j=\mathrm{ev}_{\mathtt{v}}\circ\partial_w^{(\rho)}\circ\xi_j^{\rho},$$
as elements of $\mathbf{R}^*$.

Proof. As $\rho\in G_{\mathtt{v}}$, we have $\mathrm{ev}_{\mathtt{v}}=\mathrm{ev}_{\mathtt{v}}\circ\rho$. We have $\rho\circ\partial_w\circ\rho^{-1}=\partial_w^{(\rho)}$ by [BGG, Lemma 3.3] and hence
$$\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j=\mathrm{ev}_{\mathtt{v}}\circ\partial_w^{(\rho)}\circ\xi_j^{\rho}\circ\rho^{-1}.$$
As the set $\mathbf{R}$ consists of G-invariant polynomials, the claim follows. □

Lemma 9. *The element $\mathrm{ev}_{\mathtt{v}}\circ\partial_w\circ\xi_j$ of $\underline{\mathbf{B}}$ is zero (as an element of $\mathbf{R}^*$) provided that $w\notin X_j$.*

Proof. If $w\notin X_j$, we may write $\partial_w=\partial_{w'}\partial_\tau$ for some simple reflection $\tau\in G_{\mathtt{u}_j}$. For any $f\in\mathbf{R}$, we then have that $\tau(\xi_j(f))=\xi_j(f)$. Therefore $\partial_\tau\xi_j(f)=0$. The claim follows. □

From Lemma 9 we get that either $\underline{\mathbf{B}}=\mathbf{B}$ or $\underline{\mathbf{B}}=\mathbf{B}\cup\{0\}$. Let $M_{\mathtt{v}}$ be the linear span, in $\mathbf{R}^*$, of $\underline{\mathbf{B}}$. Then $M_{\mathtt{v}}$ coincides with the linear span, in $\mathbf{R}^*$, of $\mathbf{B}$.

Theorem 10. *The space $M_{\mathtt{v}}$ is closed under the action of U_λ, moreover, $\mathbf{B}$ is a basis in $M_{\mathtt{v}}$.*

Proof. To prove that $M_{\mathtt{v}}$ is closed under the action of U_λ, let us check that it is closed under the right multiplication with the generators. Multiplying on the right with an element $h\in\mathbf{R}$, we can move this element past ξ_j by acting on it, that is
$$\xi_j\circ h=h^{\xi_j}\circ\xi_j,$$

and then we can move the new element h^{ξ_j} past ∂_w using the Leibniz rule. When we reach $\mathrm{ev}_{\mathtt{v}}$, we evaluate at the point $\mathtt{v}$. This means that $M_{\mathtt{v}}$ is closed under the action of all $\gamma_{\mathbf{a}}$, where $\mathbf{a} \in I$.

Let us prove that $M_{\mathtt{v}}$ is closed under the action of E_i and F_i, where $i < k$. We prove this for E_i and in case of F_i the proof is similar. Consider the element

$$\mathrm{ev}_{\mathtt{v}} \circ \partial_w \circ \xi_j \circ E_i, \tag{3}$$

for some $\mathrm{ev}_{\mathtt{v}} \circ \partial_w \circ \xi_j \in \mathbf{B}$. We want to prove that (3) can be written as a linear combination of elements in $\underline{\mathbf{B}}$.

Let μ be the composition of $\{1, 2, \dots, \lambda_i\}$ corresponding to the orbits of $G_{\mathtt{u}_j}$ on $\{1, 2, \dots, \lambda_i\}$. We write E_i in the form given by Proposition 6, for the composition μ. Then every divided difference operator $\partial_{\mathbf{a},\mathbf{b}}$ appearing in this expression has the property that ξ_j is invariant under $(\mathbf{a}, \mathbf{b})$. In this case the application of ξ_j to $x_{\mathbf{a}} - x_{\mathbf{b}}$ gives $x_{\mathbf{a}} - x_{\mathbf{b}}$. This implies the relation

$$\xi_j \circ \partial_{\mathbf{a},\mathbf{b}} = \partial_{\mathbf{a},\mathbf{b}} \circ \xi_j$$

which allows us to move all operators $\partial_{\mathbf{a},\mathbf{b}}$ to the left of ξ_j.

For each part μ_s of μ, the corresponding rational function $f(\mu, s)^+$ has, by construction, the property that, for any $w \in G_{\mathtt{u}_j}$, the evaluation of $(\xi_j(f(\mu, s)^+))^w$ at $\mathtt{v}$ is well-defined as none of the denominators of $(\xi_j(f(\mu, s)^+))^w$ evaluates to zero. Therefore we may use the Leibniz rule to move $(\xi_j(f(\mu, s)^+))^w$ to the left past all the divided difference operators which appear in our expression and then evaluate the resulting function at $\mathtt{v}$. Note that the composition $\xi_j \varphi_{(i,s)}$ may be an element in $\gimel$ which does not coincide with any $\xi_{j'}$. In the latter case we may apply Lemma 8 and, finally, get a linear combination of elements in $\underline{\mathbf{B}}$. This completes the proof of the first statement.

It remains to prove that $\mathbf{B}$ is a basis. Recall that the divided difference operators annihilate all symmetric polynomials. By [So, Endomorphismensatz 3(i)] the shift of symmetric polynomials corresponding to going from $\mathtt{u}_j$ hits, after factoring all symmetric polynomials out, exactly the algebra of G_{ξ_j}-invariants in the coinvariant algebra for $G_{\mathtt{v}}$. By [BGG, Theorems 5.5], the action of the divided difference operators from X_j on the socle of the algebra of G_{ξ_j}-invariants in the coinvariant algebra for $G_{\mathtt{v}}$ gives a linearly independent system of elements of the coinvariant algebra. This implies that elements in $\mathbf{B}$ are linearly independent and completes the proof. □

The map $w \mapsto w(\xi_j)$ is a bijection from X_j to $\mathcal{O}_j$. Therefore the elements of the basis $\mathbf{B}$ of $M_{\mathtt{v}}$ are in a natural bijection with the elements in $\gimel \cdot \mathtt{v}$.

From [So, Endomorphismensatz 3(i)] it even follows that the space $M_{\mathtt{v}}(\chi_{\mathtt{w}})$, considered as a module over $\Gamma_\lambda / \mathrm{Ann}_{\Gamma_\lambda}(M_{\mathtt{v}}(\chi_{\mathtt{w}}))$, is isomorphic to the regular representation of the algebra of $G_{\mathtt{w}}$-invariants in the coinvariant algebra of $G_{\mathtt{v}}$.

We expect that each simple Gelfand-Zeitlin U_λ-module is a subquotient of $M_{\mathtt{v}}$, for some $\mathtt{v}$.

4.5. **Sufficient conditions for simplicity.**

Theorem 11. *Let $\mathtt{v} \in V^*$ be as in Subsection 4.4. Further, assume that, for any $i = 1, 2, \dots, k-1$ and for any $\mathbf{a} \in I^{(i)}$ and $\mathbf{b} \in I^{(i+1)}$, we have $\mathtt{v}(x_{\mathbf{a}}) - \mathtt{v}(x_{\mathbf{b}}) \notin \mathbb{Z}$. Then the module $M_{\mathtt{v}}$ is simple. Consequently, $M_{\mathtt{v}} \cong L_{\mathtt{v}}$ in this case.*

Proof. We start by prove the following two facts:

- For any $\mathtt{w} \in \daleth \cdot \mathtt{v}$, any generator E_i and any $j \in \{1, 2, \ldots, \lambda_i\}$, the vector $\mathrm{pr}_{\chi_{\varphi_{(i,j)}(\mathtt{w})}}(\mathbf{ev}_{\mathtt{w}} \circ E_i)$ is non-zero.
- For any $\mathtt{w} \in \daleth \cdot \mathtt{v}$, any generator F_i and any $j \in \{1, 2, \ldots, \lambda_i\}$, the vector $\mathrm{pr}_{\chi_{\varphi^{-1}_{(i,j)}(\mathtt{w})}}(\mathbf{ev}_{\mathtt{w}} \circ F_i)$ is non-zero.

We will prove the first statement and the second one is proved similarly. Without loss of generality we may assume that $\mathtt{w} = \mathtt{u}_s$, for some $s \in J$ as in Subsection 4.4.

Let μ be the partition of λ_i corresponding to orbits of $G_{\mathtt{w}}$ on $I^{(i)}$. Write E_i in the form given by Proposition 6 and consider the element $\mathbf{ev}_{\mathtt{w}} \circ E_i$. Note that, if we vary j along some orbit of $G_{\mathtt{w}}$ on $I^{(i)}$, this does not affect the Gelfand-Zeitlin character $\chi_{\varphi_{(i,j)}(\mathtt{w})}$. This, in fact, defines a bijection between orbits of $G_{\mathtt{w}}$ on $I^{(i)}$ and different Gelfand-Zeitlin characters of the form $\chi_{\varphi_{(i,j)}(\mathtt{w})}$. Therefore in what follows we may assume that $j = \min(\mu_s)$, for some part μ_s of μ.

The summands of E_i which correspond to the parts different from μ_s do not effect $\mathrm{pr}_{\chi_{\varphi_{(i,j)}(\mathtt{w})}}(\mathbf{ev}_{\mathtt{w}} \circ E_i)$ and thus we only need to consider the summand corresponding to μ_s.

Let $a_1 < a_2 < \cdots < a_{\mu_s}$ be elements of $\underline{\mu_s}$. For $t = 1, 2, \ldots, \mu_s - 1$, set

$$w_t = (a_t, a_{t+1}) \ldots (a_2, a_3)(a_1, a_2).$$

Using the Leibniz rule, we can write

$$\begin{aligned}\partial(\mu, s) \circ f(\mu, s)^+ = w_{\mu_s - 1}(f(\mu, s)^+) \circ \partial_{a_{\mu_s-1}, a_{\mu_s}} \circ \cdots \circ \partial_{a_2,a_3} \circ \partial_{a_1,a_2} + \\ + \partial_{a_{\mu_s-1}, a_{\mu_s}}(w_{\mu_s-2}(f(\mu, s)^+)) \circ \partial_{a_{\mu_s-2}, a_{\mu_s-1}} \circ \cdots \circ \partial_{a_1,a_2} + \ldots \\ \cdots + \partial_{a_{\mu_s-1}, a_{\mu_s}} \circ \cdots \circ \partial_{a_2,a_3} \circ \partial_{a_1,a_2}(f(\mu, s)^+).\end{aligned}$$

As $\mathbf{ev}_{\mathtt{w}}(w_{\mu_s-1}(f(\mu, s)^+)) \neq 0$, due to our assumptions on $\mathtt{v}$, we see that the element $\mathbf{ev}_{\mathtt{w}} \circ E_i$ has a non-zero coefficient at the element $\mathbf{ev}_{\mathtt{w}} \circ \partial(\mu, s) \circ \varphi_{(i, \min(\mu_s))}$ and the rest is a linear combination of divided difference operators of strictly smaller degree. This implies that $\mathrm{pr}_{\chi_{\varphi_{(i,j)}(\mathtt{w})}}(\mathbf{ev}_{\mathtt{w}} \circ E_i)$ is non-zero.

The above implies that every submodule of $M_{\mathtt{v}}$ contains $\mathbf{ev}_{\mathtt{v}}$. Morover, a similar argument to the above implies that $\mathbf{ev}_{\mathtt{v}} \circ U_\lambda$ contains all $\mathrm{ev}_{\mathtt{v}} \circ \partial_w \circ \xi_j$, where ∂_w has maximal possible degree. It follows that $\mathbf{ev}_{\mathtt{v}} \circ U_\lambda = M_{\mathtt{v}}$ and hence $M_{\mathtt{v}}$ is simple, completing the proof. $\square$

If $\lambda = (1, 2, \ldots, n)$, then, under the assumptions of Theorem 11 and for any $\mathtt{w} \in \daleth \cdot \mathtt{v}$, the module $M_{\mathtt{v}}$ is the unique simple Gelfand-Zeitlin module extending $\mathbb{C}_{\mathtt{w}}$ as it has the same Gelfand-Zeitlin character as the module $U_\lambda / U_\lambda(\mathbf{m}_{\mathtt{w}})$.

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