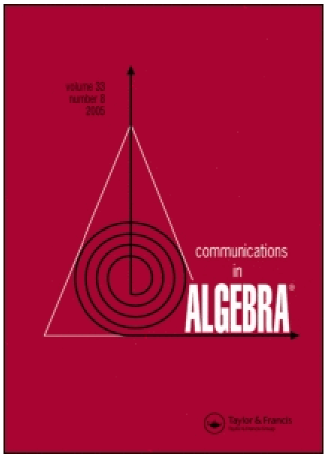

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

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Online Publication Date: 19 June 2002
To cite this Article: Dumas, François and Rigal, Laurent (2002) 'PRIME SPECTRUM AND AUTOMORPHISMS FOR 2×2 JORDANIAN MATRICES', Communications in Algebra, 30:6, 2805 - 2828
To link to this article: DOI: 10.1081/AGB-120003990
URL: http://dx.doi.org/10.1081/AGB-120003990

COMMUNICATIONS IN ALGEBRA, 30(6), 2805–2828 (2002)

PRIME SPECTRUM AND AUTOMORPHISMS FOR 2×2 JORDANIAN MATRICES

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ABSTRACT

This paper is devoted to some ring theoretic properties of the jordanian deformation $\mathcal{O}^J(M_2)$ of the algebra of regular functions on the 2×2 matrices with coefficients in an algebraically closed field of characteristic zero, and of the associated factor algebra $\mathcal{O}^J(SL_2)$. We prove in particular that the prime spectrum of $\mathcal{O}^J(M_2)$ is the disjoint union of five components, each of which being homeomorphic to the spectrum of a commutative (possibly localised) polynomial ring. So we can give an explicit description of the prime spectrum of $\mathcal{O}^J(SL_2)$, and check that any prime factor of $\mathcal{O}^J(M_2)$ satisfies the Gelfand-Kirillov property. Then we study the automorphism groups of the algebras $\mathcal{O}^J(M_2)$ and $\mathcal{O}^J(SL_2)$ and prove that they are generated by linear automorphisms and exponentials of locally nilpotent derivations.

INTRODUCTION

Up to isomorphism, the search of Hopf algebras with the same representation theory as SL_2 leads to two noncommutative deformations of the Hopf algebra $\mathcal{O}(SL_2)$ of regular functions on SL_2: the quantum deformation $\mathcal{O}_q(SL_2)$ and the jordanian deformation $\mathcal{O}^J(SL_2)$ (see for instance [1,2]). The first one has been extensively studied; this paper is devoted to the study of the algebra structure of the second one (when the base field is algebraically closed of characteristic zero). More precisely, we are interested in the birational equivalence of Gelfand and Kirillov, the structure of the prime spectrum and the automorphism group. It turns out that, from the ring theoretic point of view, the behavior of the jordanian deformation is closer to the classical framework of Weyl algebras and enveloping algebras than to the framework of quantum groups.

In the first section we recall the construction of the bialgebra $\mathcal{O}^J(M_2)$ of 2×2 jordanian matrices (applying the Faddeev-Reshetikhin-Takhtajan procedure to a convenient Hecke symmetry). The Hopf algebra $\mathcal{O}^J(SL_2)$ is then the factor algebra of $\mathcal{O}^J(M_2)$ with respect to the two-sided ideal generated by $z - 1$, where z is a suitably defined "jordanian determinant" (see 1.2 for more references). The FRT construction leads to a description of $\mathcal{O}^J(M_2)$ by generators and relations from which it follows that this algebra is an iterated Ore extension. We establish the properties of $\mathcal{O}^J(M_2)$ useful for our purposes. In particular, we describe its center and classify its normal elements. Moreover, we show that $\mathcal{O}^J(M_2)$ satisfies the Gelfand-Kirillov property: its division ring of fractions is isomorphic to that of a Weyl algebra.

The second part is devoted to the prime spectrum of the algebras $\mathcal{O}^J(M_2)$ and $\mathcal{O}^J(SL_2)$. We first check, using classical results, that prime ideals are completely prime. By means of a noncommutative analogue of Nagata's lemma, we show that $\mathcal{O}^J(M_2)$ is a unique factorisation domain in the sense of Chatters and Jordan (see [3]). The prime spectrum of $\mathcal{O}^J(M_2)$ is then described as the disjoint union of five components, each of which being homeomorphic to the spectrum of a commutative (possibly localised) polynomial ring. This description allows us to distinguish primitive ideals among prime ideals as those which are maximal in their component. It also allows us to check that, for a prime ideal, it is equivalent to be primitive, rational or locally closed. We prove that all the prime factors of $\mathcal{O}^J(M_2)$ satisfy the Gelfand-Kirillov property (the quantum analogue of this property has been established in any dimension by G. Cauchon in [4]). Finally, we give an explicit description of the prime spectrum of $\mathcal{O}^J(SL_2)$.

In the third part, we study the automorphism group of the algebras $\mathcal{O}^J(M_2)$ and $\mathcal{O}^J(SL_2)$. Let us recall that the structure of the automorphism group of a commutative algebra of polynomials in n indeterminates is known only for $n \leq 2$. For $n \geq 3$, it is not known (see for example [5]) whether an automorphism has to be tame (i.e., product of linear automorphisms and triangular automorphisms) or semi-tame (i.e., product of linear automorphisms and exponentials of locally nilpotent derivations). As far as we know, the same questions remain open for $\mathcal{O}(SL_2)$. In the case of the quantum algebras $\mathcal{O}_q(M_2)$ and $\mathcal{O}_q(SL_2)$, J. Alev and M. Chamarie, and A. Braverman, have shown in [6] and [7] that, due to the quantisation, the equivalent problem turns out to be trivial: the automorphism group is reduced to the product of a subgroup of order two by the natural action of a torus and the zero derivation is the only locally nilpotent derivation. The fact that the quantisation leads to a more rigid situation is a well known phenomenon, observed in many other cases (see for instance [8] and [9]). The jordanian deformation gives rise to a very different picture as we show in this paper; in some sense, it is intermediate between the extremely rich commutative situation and the very rigid quantum case. Indeed, the main theorem of Sec. 3 (Theorem 3.3) describes the automorphism group of $\mathcal{O}^J(M_2)$ as the semi-direct product of four subgroups: the two first are composed of linear automorphisms while the elements of the two others are exponentials of locally nilpotent derivations. This result shows that any automorphism of $\mathcal{O}^J(M_2)$ is semi-tame in a non-trivial way. At this point, it is worth mentioning the results obtained for some classical noncommutative algebras by J. Dixmier and M. Smith: the automorphisms of the Weyl algebra A_1 and of certain enveloping algebras of solvable Lie algebras in small dimensions are tame (see [10] and [11]), and by A. Joseph and J. Alev: in the enveloping algebras $U(sl_2)$ and $U(sl_3^+)$ there exist wild automorphisms (see [12] and [13]). We also describe the locally nilpotent derivations of $\mathcal{O}^J(M_2)$ as well as the automorphisms of the factor algebra $\mathcal{O}^J(SL_2)$. Finally, we use the description of the linear automorphisms of $\mathcal{O}^J(M_2)$ (resp. $\mathcal{O}^J(SL_2)$) to describe its spectrum in terms of stratification.

In this paper, k denotes an algebraically closed field of characteristic zero.

If A is a prime noetherian ring and x a non zero normal element of A, we denote by A_x the localisation of A with respect to the powers of x.

1 THE ALGEBRAS $\mathcal{O}^J(\mathrm{M}_2)$ AND $\mathcal{O}^J(\mathrm{SL}_2)$

Definition 1.1. *The jordanian deformation of the ring of regular functions on the plane (or to simplify the jordanian plane), denoted P^J, is the algebra* *generated over k by two generators x and y subject to the relation: $xy - yx = y^2$.*

The jordanian deformation of the ring of regular functions on the 2×2 matrices with coefficients in k, denoted $R = \mathcal{O}^J(M_2)$, is the algebra generated over k by four generators a, b, c, d subject to the relations:

$$\begin{cases} [a,c] = c^2, & [a,b] = ad - bc + ac - a^2, & [d,b] = ad - bc + ac - d^2, \\ [d,c] = c^2, & [a,d] = dc - ac, & [b,c] = dc + ac - c^2. \end{cases} \tag{1}$$

The jordanian determinant in $\mathcal{O}^J(M_2)$ is the element:

$$z = ad - bc + ac, \text{ which is central in } R.$$

The jordanian deformation of the ring of regular functions on the group SL_2 is the factor algebra $\overline{R} = \mathcal{O}^J(SL_2)$ of $R = \mathcal{O}^J(M_2)$ by the two-sided ideal $(z-1)R$. Its canonical generators are denoted $\overline{a}, \overline{b}, \overline{c}, \overline{d}$.

Remark 1.2. (i) The first references concerning the algebras $\mathcal{O}^J(M_2)$ and $\mathcal{O}^J(SL_2)$ are [14,15,16,17,1] and [18]. Further studies can be found in [19,20,21,22,23] and [24]. The papers [25] and [26] deal with the corresponding forms of de Rham complexes. A two-parameter version of the construction recalled below is studied in [27] and [28]. The paper [2] concerns the possible extensions to SL_3.

(ii) Recall that $\mathcal{O}^J(M_2)$ is obtained applying the method of Faddeev, Reshetikhin and Takhtajan (see [29]) to the R-matrix:

$$\mathcal{R}_h^J = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with parameter } h \in k.$$

For all $h \in k$, by identification of coefficients, the equality:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{R}_h^J = \mathcal{R}_h^J \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is equivalent to the relations:

$$\begin{cases} [a,c]=hc^2, & [a,b]=h(ad-bc+hac-a^2), & [d,b]=h(ad-bc+hac-d^2), \\ [d,c]=hc^2, & [a,d]=h(dc-ac), & [b,c]=h(dc+ac-hc^2). \end{cases}$$

Let R_h denote the algebra generated over k by a, b, c, d subject to the above relations, and let $\overline{R}_h$ be the factor algebra $R_h/(z_h-1)R_h$, where z_h is the central element of R_h defined by $z_h = ad - bc + hac$. The algebra R_h can be equipped with the structure of a bialgebra with coproduct and counit defined by:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix},$$

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $\Delta(z_h) = z_h \otimes z_h$, the ideal $(z_h-1)R_h$ is a coideal of R_h and the factor bialgebra is a Hopf algebra with antipode defined by:

$$S\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} = \begin{pmatrix} \overline{d} + h\overline{c} & -\overline{b} + h\overline{d} - h\overline{a} + h^2\overline{c} \\ -\overline{c} & \overline{a} - h\overline{c} \end{pmatrix}.$$

(iii) When $h = 0$, $\mathcal{R}_0^J$ is the usual flip and we recover the commutative algebras $R_0 = k[a, b, c, d]$ and $\overline{R}_0 = \mathcal{O}(SL_2)$. When $h \neq 0$, the assignment $a \mapsto a$, $b \mapsto hb$, $c \mapsto h^{-1}c$, $d \mapsto d$ defines a bialgebra isomorphism from R_h to $R_1 = R = \mathcal{O}^J(M_2)$ that induces a Hopf algebra isomorphism from $\overline{R}_h$ to $\overline{R}_1 = \overline{R} = \mathcal{O}^J(SL_2)$. Thus, it suffices to study the case $h = 1$, that is the algebras defined in 1.1.

(iv) Finally recall that there is a coaction of $R = \mathcal{O}^J(M_2)$ on the jordanian plane P^J via the algebra homomorphism $\Delta' : P^J \to R \otimes P^J$ defined by $\Delta'(x) = a \otimes x + b \otimes y$ and $\Delta'(y) = c \otimes x + d \otimes y$.

In Lemma 1.3 we define a new set of generators of R that clarifies the commutation relations. We also introduce certain subalgebras of R useful for our purposes. Many proofs in this work are based on the use of the localisation $S = R_c$ of R with respect to the powers of the normal element c and of the factor algebra $T = R/cR$. Technical results concerning S and T are proved in 1.4, 1.5 and 1.6.

Lemma 1.3. *Let $u = a - d \in R$.*

(i) The elements c and u are normal elements in R. They induce the same inner automorphism γ of R which is defined by $\gamma(a) = a - c$, $\gamma(c) = c$, $\gamma(b) = -a + b + c - d$, $\gamma(d) = d - c$.

(ii) *The subalgebra* R_0 *of* R *generated by* c, u, a *is the Ore extension* $R_0 = k[c, u][a\,;\,D]$ *where* D *is the* k*-derivation* $c(c\partial_c + u\partial_u)$ *of* $k[c, u]$.

(iii) *Let* σ *denote the* k*-automorphism of* R_0 *such that* $\sigma(a) = a + c$ *and whose restriction to* $k[c, u]$ *is the identity and* δ *be the* σ*-derivation of* R_0 *defined by* $\delta(c) = c(c - u + 2a)$, $\delta(u) = u(c - u + 2a)$ *and* $\delta(a) = (c + u)a$. *Then,* R *is the iterated Ore extension* $R = R_0[b\,;\,\sigma, \delta] = k[c, u][a\,;\,D][b\,;\,\sigma, \delta]$.

(iv) *We have* $z = -cb + a^2 - (c + u)a = a^2 - ua - c(b + a)$ *and the subalgebra of* R *generated by* z, u, c *is the commutative polynomial ring* $k[z, u, c]$.

(v) *The subalgebra* R' *generated by* z, u, c, a *is the Ore extension* $R' = k[z, u, c][a\,;\,D] = R_0[z]$ *where* D *is extended to* $k[z, u, c]$ *by* $D(z) = 0$.

Proof. From the defining relations of R (see 1.1) it follows that:

$$
\begin{array}{llll}
ca = (a - c)c, & cd = (d - c)c, & cb = (b + u - 2a + c)c, & cu = uc, \\
ua = (a - c)u, & ud = (d - c)u, & ub = (b + u - 2a + c)u, & uc = cu,
\end{array}
$$

which proves (i). From the same relations we deduce that R coincides with the algebra generated over k by a, b, c, u subject to the relations:

$$
cu = uc, \qquad \begin{cases} ac - ca = c^2, \\ au - ua = cu, \end{cases} \qquad \begin{cases} bc - cb = c(c - u + 2a), \\ bu - ub = u(c - u + 2a), \\ ba - (a + c)b = (c + u)a, \end{cases} \tag{2}
$$

which proves (ii) and (iii). Using the relations above, z may be written in $R = k[c, u][a\,;\,D][b\,;\,\sigma, \delta]$ as $z = -cb + a^2 - (c + u)a$. The fact that z, u, c commute with each other has already been proved and using their degree in b it is easily checked that they are algebraically independent over k. From this, (iv) and (v) follow. ■

Lemma 1.4. *Let* $S = R_c$ *be the localisation of* R *with respect to the multiplicative set generated by the normal element* c.

(i) S *is the iterated Ore extension* $S = k[c^{\pm 1}, u][a\,;\,D][b\,;\,\sigma, \delta]$ *where* D, σ *and* δ *are extended in the natural way.*

(ii) *The element* $v = c^{-1}u$ *is central in* S *and, if* D *is the* k*-derivation of the commutative ring* $k[z, u, c^{\pm 1}] = k[z, v, c^{\pm 1}]$ *defined by* $D(z) = D(v) = 0$, $D(c^{-1}) = -1$, *we have* $S = k[z, u, c^{\pm 1}][a\,;\,D] = k[z, v, c^{\pm 1}][a\,;\,-\partial_{c^{-1}}]$.

Proof. Point (i) is clear from 1.3 (iii). The centrality of v follows from 1.3 (i). By 1.3 (iv) we have $b = -c^{-1}z + (c^{-1}a^2 - a - c^{-1}ua)$, so $S = k[u, c^{\pm 1}]\,[a\,;\,D][z]$. This completes the proof. ∎

Lemma 1.5. *Let $T = R/cR$ be the factor algebra of R by the two-sided ideal cR. Then, T is isomorphic to the iterated Ore extension $k[a', u'][b'\,;\,\delta']$ where δ' is the k-derivation of the commutative polynomial ring $k[a', u']$ defined by $\delta'(a') = u'a'$ and $\delta'(u') = u'(2a' - u')$.*

Proof. This is clear from 1.3 (iii). ∎

Lemma 1.6.

(i) For $j \in \mathbb{N}^*$, $z^j \equiv \sum_{i=0}^{j}(-1)^i\binom{j}{i}u^i a^{2j-i}$ modulo cR.
(ii) The following equalities hold: $k[u, z, c^{\pm 1}] \cap R = k[u, z, c]$ and $k[z, v] \cap R = k[z]$.

Proof. By 1.3 (iv), we have $z^j \equiv (a^2 - ua)^j$ modulo cR from which point (i) follows using relations (2). We now prove (ii). Assume there exists $r \in k[z, u, c^{\pm 1}] \cap R$ such that $r \notin k[z, u, c]$. We can write $r = p + s$ with $p \in k[z, u, c]$ and $s = \sum_{i=1}^{n} s_i c^{-i}$, where $s_i \in K[z, u]$ for all $1 \leq i \leq n$ and $s_n \neq 0$. We thus have $s \in R$ from which it follows that $s_n \in kcR$ by right multiplication of s by c^n. Now, let us write $s_n = \sum_{j=0}^{m} e_j z^j$ with $e_j \in k[u]$ and $e_m \neq 0$. Applying point (i) and if $t = \sum_{j=0}^{m} e_j \sum_{i=0}^{j}(-1)^i\binom{j}{i}u^i a^{2j-i}$, we get $t \in cR$. Since t is a non zero linear combination of the elements of the free family $\{u^i a^j\}_{i,j \geq 0}$, we get a contradiction with 1.3 (iii). Thus $k[u, z, c^{\pm 1}] \cap R = k[u, z, c]$. Since $v = uc^{-1}$, the equality $k[z, v] \cap R = k[z]$ follows at once. ∎

Remark 1.7. In 1.8 and 1.9, we shall establish the first properties of $\mathcal{O}^J(M_2)$. Let us recall useful notations. Let B be a (not necessarily commutative) noetherian k-algebra which is an integral domain with division ring of fractions $K = \text{Frac}\,B$. We denote by $A_1(B)$ the first Weyl algebra over B (generated over B by two generators x and y that commute with the elements of B and such that $xy - yx = 1$). Its division ring of fractions is the first Weyl skew-field $D_1(K) = \text{Frac}\,A_1(B) = \text{Frac}\,A_1(K)$.

Proposition 1.8. *The center of $R = \mathcal{O}^J(M_2)$ is $k[z]$ and the center of $S = R_c$ is $k[z, v]$. Moreover* $\text{Frac}\,R = \text{Frac}\,S \simeq D_1(K)$ *where $K = k(z, v)$ is a field of rational fractions.*

Proof. By 1.4 (ii), the relation $[c^{-1}, a] = 1$ holds in $S = k[z, v][c^{\pm 1}][a\,;\,D]$. Thus, for all $s = \sum_{i=0}^{n} s_i a^i \in S$ with $s_i \in k[z, v, c^{\pm 1}]$, we have $[c^{-1}, s] = \sum_{i=0}^{n} s_i[c^{-1}, a^i] = \sum_{i=0}^{n} i s_i a^{i-1}$. If s is in the center $Z(S)$ of S, it follows that $s = s_0 \in k[z, v, c^{\pm 1}]$. Since moreover $D(s) = [a, s] = 0$, we conclude that $s \in k[z, v]$. The center $Z(R)$ of R is then $Z(R) = k[z, v] \cap R = k[z]$ by 1.6 (ii). Finally, the subalgebra S^- of S generated by z, v, c^{-1} and a is isomorphic to $A_1(B)$ where $B = Z(S) = k[z, v]$. Hence $\text{Frac}\, R = \text{Frac}\, S = \text{Frac}\, S^- \simeq D_1(K)$ where $K = \text{Frac}\, B = k(z, v)$. ■

We shall prove in 2.3.2 that $\text{Frac}\, \overline{R} \simeq D_1(k(\overline{v}))$. Moreover, it can be shown that the center of $\overline{R} = \mathcal{O}^J(SL_2)$ is k; we leave the details to the reader. We now describe the set of normal elements of $\mathcal{O}^J(M_2)$. It will be used in 3.3 to calculate the automorphism group of $\mathcal{O}^J(M_2)$.

Proposition 1.9. *For $m \in \mathbb{N}$, let E_m stand for the $k[z]$-module of elements of the commutative ring $k[z, u, c] = k[z][u, c]$ which are homogeneous of degree m in u and c. The set $N(R)$ of normal elements of $R = \mathcal{O}^J(M_2)$ is the disjoint union $N(R) = \cup_{m \in \mathbb{N}} E_m$.*

Proof. For $m \in \mathbb{N}$, the inclusion $E_m \subseteq N(R)$ is clear from 1.3 (i). For the converse, we start proving that $N(R) \subseteq k[u, z, c]$. For this, let $\deg_a$ denote the degree in a defined in the obvious way (recall that S is the Ore extension described in 1.4 (i)). Let $s \in R \setminus \{0\}$. Write s as $s = \sum_{i=0}^n s_i a^i$ with $s_i \in k[v, z, c^{\pm 1}]$ and $s_n \neq 0$ and assume $s \notin k[v, z, c^{\pm 1}]$, i.e., $n = \deg_a s \geq 1$. An inductive use of the relation $ac = ca - c^2$ shows that $\deg_a(cs - sc) = n - 1$. If s was normal in R we would have $cs \in sR$ and thus $cs - sc \in sR \subseteq sS$; since $cs - sc \neq 0$, it would then follow that $\deg_a(cs - sc) \geq n$, a contradiction. Thus, any normal element in R must belong to $k[v, z, c^{\pm 1}]$. Using Lemma 1.6 (ii) it follows that $N(R) \subseteq k[u, z, c]$. Now, let $s \in N(R) \setminus \{0\}$. Since $s \in k[u, z, c] = \oplus_{m \geq 0} E_m$ we can write $s = s_p + \cdots + s_q$, where $s_m \in E_m$ for all $p \leq m \leq q$, $s_p \neq 0$, $s_q \neq 0$. Let us assume that $q > p \geq 0$. Using the automorphism γ of R defined in 1.3 (i), we get $sa = \sum_{m=p}^q \gamma^m(a) s_m = \sum_{m=p}^q (a - mc) s_m$, from which it follows that $sa - (a - pc)s = -\sum_{i=1}^{q-p} ics_{p+i}$. Since $s \in N(R)$, $sa - (a - pc)s$ is in the two-sided ideal $sR = Rs$. Thus, there exists $r \in R \setminus \{0\}$ such that $c(s_{p+1} + 2s_{p+2} + \cdots + (q-p)s_q) = (s_p + s_{p+1} + \cdots + s_q)r$. Since $u = vc$, each term s_m can be written $s_m = t_m c^m$ with $t_m \in k[z, v]$. We get:

$$\begin{aligned} & t_{p+1}c^{p+2} + 2t_{p+2}c^{p+3} + \cdots + (q-p)t_q c^{q+1} \\ & \quad = (t_p c^p + t_{p+1}c^{p+1} + \cdots + t_q c^q) r. \end{aligned} \tag{3}$$

Considering this identity in the Ore extension S (see 1.4 (ii)), it follows that $r \in k[v, z][c^{\pm 1}]$. Letting $r = r_e c^e + \cdots + r_f c^f$ with $r_i \in k[v, z]$ for all $e \leq i \leq f$ and $r_e \neq 0 \neq r_f$, an identification in $k[v, z][c^{\pm 1}]$ of the two terms of (3) leads to $f = 1$ and $e \geq 2$. This is a contradiction. ■

2 SPECTRUM OF $\mathcal{O}^J(\mathrm{M}_2)$ AND $\mathcal{O}^J(\mathrm{SL}_2)$

2.1 Prime Factors Are Integral Domains

Theorem 2.1.1. *In the ring $R = \mathcal{O}^J(M_2)$, prime ideals are completely prime.*

Proof. Because of Lemmas 1.4 and 1.5 and since we assume that k is of characteristic zero, Corollary 2.6 of [30] shows that in $S = R_c$ and $T = R/(c)$ prime ideals are completely prime. Since c is normal, the result follows. ■

2.2 The Ring R Is a Unique Factorisation Ring

In this subsection we prove that R is a unique factorisation ring in the sense of Chatters and Jordan (see [3]). Details concerning unique factorisation rings in the noncommutative setting can be found in [3]. Recall that if A is a ring, an element x in A is called prime if it is normal and generates a proper prime ideal. An ideal I in A is called principal if there exists a normal element x in A such that $I = (x)$.

Definition 2.2.1. *A ring A is called a noetherian unique factorisation ring (UFR for short) if A is a prime noetherian ring in which any non zero prime ideal contains a non zero principal prime ideal.*

Remark 2.2.2. If A is a prime noetherian ring that satisfies the descending chain condition on prime ideals, then A is a noetherian UFR if and only if height one primes are principal (see [3]). Using standard results (see [31] 6.4.5 and 6.5.4), it is clear that the classical Krull dimension of R is at most four. In particular, R satisfies the descending chain condition on prime ideals.

To prove that R is a UFR, we will use Lemma 2.2.3 which is a noncommutative analogue of Nagata's Lemma (in the commutative case, see [32] 19.20). We leave its proof to the reader since it is similar to that of the commutative case.

Lemma 2.2.3. *Let A be a prime noetherian ring and x a non zero normal element in A such that (x) is a completely prime ideal of A. If P is a prime ideal of A not containing x and such that the prime ideal PA_x of A_x is principal, then P is principal. In particular, if A_x is a noetherian UFR, then so is A.*

We now prove that the ring R is a noetherian UFR. For this, we use [3, Theo. 5.5] which provides a criterion allowing to prove that certain Ore extensions are noetherian UFR. Recall that, if A is a ring and δ a derivation of A, a δ-ideal is an ideal I such that $\delta(I) \subseteq I$. Moreover, an ideal P is called

δ-prime if it is a δ-ideal such that for each pair I, J of δ-ideals, $IJ \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$.

Proposition 2.2.4. *The ring* $R = \mathcal{O}^J(M_2)$ *is a noetherian UFR.*

Proof. Lemma 1.5 shows that $R/(c)$ is an integral domain thus, by Lemma 2.2.3, it is enough to show that $S = R_c$ is a noetherian UFR. Recall from 1.4 (ii) that $S = k[v, z, c^{\pm 1}][a; -\partial_{c^{-1}}]$. Since $k[v, z, c^{\pm 1}]$ is a noetherian UFR (in the usual commutative sense and thus also in the noncommutative sense) [3, Theorem 5.5] reduces the problem to showing that any non zero ideal of $k[v, z, c^{\pm 1}]$ which is $(\partial_{c^{-1}})$-prime contains a non zero principal $(\partial_{c^{-1}})$-ideal.

An element f in $k[v, z, c^{\pm 1}]$ may be written in a unique way as $f = \sum a_i c^i$, with $i \in \mathbb{Z}$ and the $a_i \in k[v, z]$ being almost all zero. We call length of f the number of integers i such that $a_i \neq 0$. Let P be a non zero $(\partial_{c^{-1}})$-ideal of $k[v, z, c^{\pm 1}]$ and let f be an element of P with minimal length among non zero elements of P. One may assume (multiplying by a convenient power of c if necessary) that $f = \sum_{i=0}^{d} a_i c^{-i}$ with $a_i \in k[v, z]$ for $0 \leq i \leq d$, $a_0 \neq 0$. We then have $\partial_{c^{-1}}(\sum_{i=0}^{d} a_i c^{-i}) = \sum_{i=1}^{d} i a_i c^{1-i} \in P$. The minimality of the length of f implies that $a_i = 0$ for $1 \leq i \leq d$, from which it follows that $f \in k[v, z]$. Thus, any non zero $(\partial_{c^{-1}})$-ideal P of $k[v, z, c^{\pm 1}]$ contains a non zero principal $(\partial_{c^{-1}})$-ideal; the proof is complete. ∎

Corollary 2.2.5.

(i) The (completely) prime ideals of R *of height one are principal.*
(ii) The ring $R/(z-1)$ *is an integral domain.*
(iii) For any $\lambda \in k$, *the ring* $R/(z-1, u-\lambda c)$ *is an integral domain.*

Proof. Point (i) follows from Remark 2.2.2 and Proposition 2.2.4. Moreover, since R is a noetherian UFR, any normal element x in R may be written as $x = up_1 \dots p_s$ where u is a unit of R and the p_i are prime elements (see [3]). Since it is clear (for degree reasons) that $z-1$ is irreducible in R, the centrality of $z-1$ shows that it is prime. Thus $(z-1)$ is (completely) prime. The arguments to prove point (iii) are similar to those used for (ii). We give a sketch of proof and leave the details to the reader. Let $P = (u - \lambda c)$; in the notations of 1.3 (ii) and (iii), the ring R/P is isomorphic to $k[\bar{c}][\bar{a}; \bar{D}][\bar{b}; \bar{\sigma}, \bar{\delta}]$, where $\bar{r}$ means the residue class of $r \in R$ modulo P. In this ring, $\bar{c}$ is normal and generates a (completely) prime ideal. By the same arguments as those used for R, it can be shown that $(R/P)_{\bar{c}}$ is a noetherian UFR, and then so is R/P. Here again, $\bar{z} - 1$ turns out to be a central irreducible element of R/P. It follows that $(z-1, u-\lambda c)$ is completely prime. ∎

2.3 Spectrum and Prime Factors of $\mathcal{O}^J(M_2)$ and $\mathcal{O}^J(SL_2)$

The prime spectrum of R is equipped with the Jacobson topology. In this subsection, we show that Spec R is the disjoint union of five *components* which, with respect to the induced Jacobson topology, are homeomorphic to spectra of commutative rings and that an ideal in Spec R is primitive if and only if it is maximal in its component. Finally, we show that for any prime $P \in$ Spec R, the division ring of fractions of R/P is isomorphic to a (commutative) field or to a Weyl skew-field.

We will call components of Spec R the following subsets of Spec R:

– $\mathcal{S}^c$: the set of prime ideals of R not containing c;
– $\mathcal{S}^a_{c,u}$: the set of prime ideals of R containing c and u and not containing a;
– $\mathcal{S}^{a,u}_c$: the set of prime ideals of R containing c and containing neither a nor u;
– $\mathcal{S}^u_{c,a}$: the set of prime ideals of R containing c and a and not containing u;
– $\mathcal{S}_{c,a,u}$: the set of prime ideals of R containing c, a and u.

Each of the components of Spec R is endowed with the topology induced from the Jacobson topology of Spec R. Moreover, it is clear that

$$\text{Spec } R = \mathcal{S}^c \cup \mathcal{S}^a_{c,u} \cup \mathcal{S}^{a,u}_c \cup \mathcal{S}^u_{c,a} \cup \mathcal{S}_{c,a,u}. \quad \text{(Disjoint union.)} \tag{4}$$

Recall that, if B is a commutative k-algebra which is an integral domain, the Weyl algebra over B is the iterated Ore extension $B[x][y; -\partial_x]$, where ∂_x is the derivation of the polynomial ring $B[x]$ such that $\partial_x(B) = 0$ and $\partial_x(x) = 1$. It is well known that x generates a denominator set in $A_1(B)$ (we use notations of 1.7). We write $A_1(B)_x$ for the corresponding localised ring. As one can easily check, there exist maps

$$\begin{array}{ccc} \text{Spec } A_1(B)_x \longrightarrow \text{Spec } B & \text{and} & \text{Spec } B \longrightarrow \text{Spec } A_1(B)_x \\ P \mapsto P \cap B & & \mathfrak{p} \mapsto \mathfrak{p}A_1(B)_x \end{array} \tag{5}$$

which are inclusion preserving homeomorphisms, inverse to each other. We start describing $\mathcal{S}^c$. Recall from 1.4 that $S = R_c = k[v,z][c^{\pm 1}][a; -\partial_{c^{-1}}]$.

Proposition 2.3.1. *The maps*

$$\begin{array}{ccc} \mathcal{S}^c \to \text{Spec } S \to \text{Spec } k[v,z] & \text{and} & \text{Spec } k[v,z] \to \text{Spec } S \to \mathcal{S}^c \\ P \mapsto PS \mapsto PS \cap k[v,z] & & \mathfrak{p} \mapsto \mathfrak{p}S \mapsto \mathfrak{p}S \cap R \end{array}$$

are inclusion preserving homeomorphisms which are inverse to each other.

Proof. Clearly, extension-contraction defines an inclusion preserving homeomorphism between $\mathcal{S}^c$ and Spec S (see [31] 2.1.16). But S is isomorphic to $A_1(B)_x$ with $B = k[v, z]$ and $x = c^{-1}$, thus the remark above completes the proof. ∎

Proposition 2.3.2. *If P is a (completely) prime ideal of $\mathcal{S}^c$, then* Frac $R/P \cong D_1(K)$, *where K is the field of fractions of $k[v, z]/\mathfrak{p}$, with $\mathfrak{p} = PS \cap k[v, z]$.*

Proof. If P is a (completely) prime ideal of $\mathcal{S}^c$, we have Frac $R/P \cong$ Frac S/PS. Moreover, if $\mathfrak{p} = PS \cap k[v, z]$, then $PS = \mathfrak{p}S$ and $S/PS \cong (k[v, z]/\mathfrak{p})[c^{\pm 1}][a\,; -\partial_{c^{-1}}])$ from which the claim follows. ∎

It remains to consider the ideals of Spec R containing c, that is the spectrum of T (recall from 1.5 that $T = R/(c)$). For the remainder of this paragraph, to simplify, we will still denote a, u, ... the images of a, u, ... under the canonical projection of R onto T. The following lemma describes $\mathcal{S}_{c,a,u}$ and $\mathcal{S}^u_{c,a}$.

Lemma 2.3.3.

(i) The ideals (a) and (u, a) of T are (completely) prime and if P is a prime ideal of T properly containing (a), then P contains u. In particular, $\mathcal{S}^u_{c,a} = \{(c, a)\}$.

(ii) The prime ideals of T containing (a, u) are (a, u) and the ideals $(a, u, b - \lambda)$ where $\lambda \in k$. In particular, $\mathcal{S}_{c,a,u}$ is homeomorphic to the spectrum of a polynomial ring in one indeterminate.

(iii) If P is a prime ideal in $\mathcal{S}^u_{c,a} \cup \mathcal{S}_{c,a,u}$, the division ring of fractions of R/P is a (commutative) field extension of k or a Weyl skew-field $D_1(K)$ where K is a (commutative) field extension of k.

Proof. We use 1.5 and its notations. The ring T is isomorphic to the Ore extension $k[a', u'][b'; \delta']$. Moreover, (a') and (a', u') are δ'-stable ideals of $k[a', u']$. Thus, using obvious notations, $T/(a) \cong (k[a', u']/(a'))[b'; \bar{\delta}'] \cong k[u'][b'; \bar{\delta}']$ and $T/(a, u) \cong (k[a', u']/(a', u'))[b'; \bar{\delta}'] \cong k[b']$. The complete primeness of the ideals (a) and (a, u) of T follows at once. Thus, $\mathcal{S}_{c,a,u}$ is clearly homeomorphic to the spectrum of a polynomial ring in one indeterminate. Moreover, $u + (a)$ is a normal element in $T' = T/(a)$ and $T'_{u+(a)} \cong k[u^{\pm 1}][b; \bar{\delta}']$ is a localisation of its subring generated by u^{-1} and b which is isomorphic to the Weyl algebra $A_1(k)$. Since $A_1(k)$ is a simple ring, it follows that $\mathcal{S}^u_{c,a} = \{(c, a)\}$ (and is homeomorphic to the spectrum of k). Points (i), (ii) and (iii) follow. ∎

It remains to consider the case of $\mathcal{S}^{a}_{c,u}$ and $\mathcal{S}^{a,u}_{c}$. Let us remark that by 1.5 (and its proof) a is normal in T. Moreover, by 1.3 (iv) (and its proof), $a^2 - z = au$ in T and so $a^2 - z$ is normal in T. Let $U = T_a$; if D' is the derivation of the commutative ring $k[a^{\pm 1}, z]$ such that $D'(z) = 0$ and $D'(a) = a^2 - z$, U can be described in the following way:

$$U = k[a^{\pm 1}, z][b; D']. \tag{6}$$

Clearly $\mathcal{S}^{a}_{c,u} \cup \mathcal{S}^{a,u}_{c}$ is homeomorphic to Spec U. More precisely, we have the following proposition which describes $\mathcal{S}^{a}_{c,u}$ and $\mathcal{S}^{a,u}_{c}$.

Proposition 2.3.4. *We use the notations introduced above.*

(i) The normal element $a^2 - z = au$ of U generates a completely prime ideal of U and $U/(a^2 - z) = U/(u) \cong (T/(u))_{a+(u)}$ is isomorphic to a localisation $B := k[X^{\pm 1}, Y]$ of a polynomial ring in two indeterminates X and Y. In particular, $\mathcal{S}^{a}_{c,u}$ is homeomorphic to Spec B *(in an inclusion preserving homeomorphism).*

(ii) Extension and contraction define a one to one correspondence between the spectrum of U_{a^2-z} and the spectrum of its commutative subalgebra $k[z]$. In particular, $\mathcal{S}^{a,u}_{c}$ is homeomorphic to the spectrum of a polynomial ring in one indeterminate (in an inclusion preserving homeomorphism).

(iii) If P is a (completely) prime ideal of $\mathcal{S}^{a}_{c,u} \cup \mathcal{S}^{a,u}_{c}$, the division ring of fractions of R/P is a (commutative) field extension of k or a Weyl skew-field $D_1(K)$ where K is a (commutative) field extension of k.

Proof. (i) The ideal $(a^2 - z)$ of $k[a^{\pm 1}, z]$ is prime and D'-stable (see notations of (6)). Thus $U/(a^2 - z) \cong (k[a^{\pm 1}, z]/(a^2 - z))[b; D']$ is an integral domain obviously isomorphic to a localisation $k[X^{\pm 1}, Y]$ of a polynomial ring in two indeterminates.

(ii) Since $au = a^2 - z$ is a normal element of U, one may consider the localisation U_{a^2-z}. Let $\xi = b(a^2 - z)^{-1} \in U_{a^2-z}$. Clearly, U_{a^2-z} is the algebra generated by $a^{\pm 1}$, z and ξ subject to the relations $az = za$, $\xi z = z\xi$ and $\xi a - a\xi = 1$. In other words, $V := U_{a^2-z} = k[z, a^{\pm 1}][\xi; \partial_a]$. Point (ii) then follows from the remark before 2.3.1 since V is isomorphic to a convenient localisation of the Weyl algebra $A_1(k[z])$.

(iii) If $P \in \mathcal{S}^{a}_{c,u}$, the claim follows at once from (i). Now, let $P \in \mathcal{S}^{a,u}_{c}$ (that is $a^2 - z \notin P$). Let P' stand for the image of P in T. If we denote by $\overline{a^2 - z}$ the residue class of $a^2 - z \in U$ modulo $P'U$, then $\overline{a^2 - z}$ is a (regular) normal element of $U/P'U$ which thus generates a denominator set. Moreover, $(U/P'U)_{\overline{a^2-z}} \cong V/P'V$. If $\mathfrak{q} = P'V \cap k[z]$, point (ii) shows that $V/P'V \cong (k[z]/\mathfrak{q})[a^{\pm 1}][\xi, \partial_a]$. Thus, the division ring of fractions of $U/P'U$ is isomorphic to the Weyl skew-field $D_1(K)$, where K is the field of fractions of the (commutative) ring $k[z]/\mathfrak{q}$. Since Frac $R/P \cong$ Frac $U/P'U$, the proof is complete. ■

Now, summing up the equality (4) and claims 2.3.1 to 2.3.4 (and their proofs), one obtains immediately the two following theorems.

Theorem 2.3.5. *The spectrum of $R = \mathcal{O}^J(M_2)$ is the disjoint union of five components each of which is homeomorphic (in an inclusion preserving homeomorphism) to the spectrum of a commutative ring.*

Theorem 2.3.6. *If P is a (completely) prime ideal of $R = \mathcal{O}^J(M_2)$, the division ring of fractions of R/P is a (commutative) field extension of k or a Weyl skew-field $D_1(K)$ where K is a (commutative) field extension of k.*

To complete the picture, we now show how the structure of Spec R described above allows us to distinguish (left) primitive ideals of R among prime ideals. In passing, we check that R satisfies the Dixmier-Moeglin equivalence.

Recall that if A is a noetherian (not necessarily commutative) k-algebra, we say that a prime ideal in A is rational if the center of the total ring of fractions of A/P (which is a field extension of k) equals k (recall we assume that k is algebraically closed). Moreover, a prime ideal of A is locally closed (with respect to the Jacobson topology of Spec A) if and only if it is strictly included in the intersection of those prime ideals that properly contains it (here, we adopt the convention that the intersection of an empty family of prime ideals of A is A). As it is well known, if in addition the k-algebra A satisfies the Nullstellensatz (see [31] 9.1.2, 9.1.4), any locally closed prime ideal is primitive and any primitive ideal is rational.

Theorem 2.3.7. *Let P be an ideal of* Spec R. *The following are equivalent:*

(i) P is maximal in its component,
(ii) P is locally closed,
(iii) P is primitive,
(iv) P is rational.

Proof. Note first that, since R is an iterated Ore extension of a field, it satisfies the Nullstellensatz (see [31] 9.4.21). Thus, it remains to show that if P is maximal in its component then it is locally closed and that if P is not maximal in its component, it is not rational. We proceed component by component. It is clear from 2.3.3 (i) that (c, a) is locally closed. The claim is thus true for $\mathcal{S}^u_{c,a}$ since $\mathcal{S}^u_{c,a} = \{(c, a)\}$. By 2.3.3 (ii), $\mathcal{S}_{c,a,u} = \{(c, a, u)\} \cup \{(c, a, u, b - \lambda),\ \lambda \in k\}$. So, the maximal ideals of $\mathcal{S}_{c,a,u}$ are maximal in Spec R and thus are locally closed. Moreover, the proof of

2.3.3 shows that (c, a, u) is not rational. We have shown the claim for $\mathcal{S}_{c,a,u}$. The proof of 2.3.2 shows that if P is not a maximal ideal of $\mathcal{S}^c$, then it is not rational. Now, let P be a maximal ideal of $\mathcal{S}^c$. If it is maximal in Spec R, it is locally closed. Otherwise, there exist prime ideals properly containing P and any such ideal must contain c. It follows that P is locally closed and the claim is true for $\mathcal{S}^c$. The same arguments show it is true also for $\mathcal{S}_c^{a,u}$ and $\mathcal{S}_{c,u}^a$. ■

Remark 2.3.8. Spectrum of $\mathcal{O}^J(SL_2)$.

From the above results, it is easy to deduce an explicit description of the lattice of prime ideals containing $(z-1)$, that is of the spectrum of $\mathcal{O}^J(SL_2) = R/(z-1)$. The exhaustive list of (completely) prime ideals of R containing $(c, z-1)$ is: $(c, z-1)$, $(c, z-1, a+1)$, $(c, z-1, a-1)$, $(c, z-1, a+1, b-\mu)$ and $(c, z-1, a-1, b-\mu)$, where $\mu \in k$. The exhaustive list of (completely) prime ideals of R containing $(z-1)$ and not containing c is: $(z-1)$ and $(z-1, u-\lambda c)$, $\lambda \in k$. Remark that the list of generators given there for each ideal is a normalising sequence. Moreover, among these ideals, the primitive ideals are $(z-1, c, a-1, b-\mu)$, $(z-1, c, a+1, b-\mu')$, $(z-1, c)$ and $(z-1, a-d-\lambda c)$. Again, for an element of Spec $\mathcal{O}^J(SL_2)$, it is equivalent to be locally closed, primitive and rational. Thus, we have the following scheme which is the lattice of prime ideals of R containing $(z-1)$ $(\mu, \mu', \lambda \in k)$.

$$\begin{tikzcd}
(z-1, c, a-1, b-\mu) \arrow[d, dash] & & (z-1, c, a+1, b-\mu') \arrow[d, dash] \\
(z-1, c, a-1) \arrow[d, dash] \arrow[drr, dash] & & (z-1, c, a+1) \arrow[d, dash] \arrow[dll, dash] \\
(z-1, c) \arrow[dr, dash] & & (z-1, a-d-\lambda c) \arrow[dl, dash] \\
& (z-1) &
\end{tikzcd}$$

3 AUTOMORPHISMS OF $\mathcal{O}^J(\mathrm{M}_2)$ AND $\mathcal{O}^J(SL_2)$

Our main aim in this section is the description of the group Aut R of k-algebra automorphisms of $R = \mathcal{O}^J(M_2)$. A first natural subgroup of Aut$R$ is the subgroup $\mathrm{Aut}_L R$ of linear automorphisms, i.e., those who restrict to an automorphism of the k-vector space $V = ka \oplus kb \oplus kc \oplus kd = ka \oplus kb \oplus kc \oplus ku$. It will be described in Corollary 3.4. Another standard process to construct automorphisms is to consider the exponential of locally nilpotent derivations. Recall that a k-derivation Δ of R is called locally nilpotent if, for any $x \in R$, there exists some integer $n \geq 1$ such that $\Delta^n(x) = 0$. It is well known that in this case, the locally finite sum $\sigma = \sum_{n \geq 0} \frac{1}{n!} \Delta^n$ defines an automorphism of R. Following the terminology in use for polynomial automorphisms of affine spaces, an automorphism of R is called *semi-tame* if it is a product in $\operatorname{Aut} R$ of linear automorphisms and exponentials of locally nilpotent derivations. So we will obtain, as a consequence of the main theorem (Theorem 3.3), that any k-automorphism in $\operatorname{Aut} R$ is semi-tame.

We start introducing certain particular automorphisms and their restriction to the subalgebra $R' = k[z, u, c][a; D] = R_0[z]$ defined in 1.3 (v).

Proposition 3.1.

(i) *There exists* $\tau \in \operatorname{Aut} R$ *such that:* $\tau(a) = d$, $\tau(b) = b$, $\tau(c) = c$, $\tau(d) = a$, *and* $T = \{\mathrm{id}_R, \tau\}$ *is a subgroup of order 2 of* $\operatorname{Aut} R$. *Moreover,* τ *restricts to an automorphism of* R' *such that* $\tau(z) = z$ *and* $\tau(u) = -u$.

(ii) *For all* $\alpha \in k^*$, *there exists* $\sigma_\alpha \in \operatorname{Aut} R$ *such that:*

$$\sigma_\alpha(a) = \alpha a, \quad \sigma_\alpha(b) = \alpha b, \quad \sigma_\alpha(c) = \alpha c, \quad \sigma_\alpha(d) = \alpha d,$$

and $H = \{\sigma_\alpha \,;\, \alpha \in k^*\}$ *is a subgroup of* $\operatorname{Aut} R$ *isomorphic to* k^*. *Moreover,* σ_α *restricts to an automorphism of* R' *such that* $\sigma_\alpha(z) = \alpha^2 z$ *and* $\sigma_\alpha(u) = \alpha u$.

(iii) *For all* $q(z) \in k[z]$, *there exists* $\eta_q \in \operatorname{Aut} R$ *such that*:

$$\eta_q(a) = a, \quad \eta_q(b) = b + q(z)a, \quad \eta_q(c) = c, \quad \eta_q(d) = d + q(z)c,$$

and $G_1 = \{\eta_q \,;\, q(z) \in k[z]\}$ *is a subgroup of* $\operatorname{Aut} R$ *isomorphic to the additive group* $k[z]$. *Moreover,* η_q *restricts to an automorphism of* R' *such that* $\eta_q(z) = z$ *and* $\eta_q(u) = u - q(z)c$.

(iv) *For all* $p(z, u, c) \in k[z, u, c]$, *there exists* $\xi_p \in \operatorname{Aut} R$ *such that:*

$$\xi_p(a) = a + p(z, u, c)c, \quad \xi_p(c) = c, \quad \xi_p(d) = d + p(z, u, c)c,$$
$$\xi_p(b) = b + ap(z, u, c) + p(z, u, c)d + p(z, u, c)^2 c,$$

and $G_2 = \{\xi_p \,;\, p(z, u, c) \in k[z, u, c]\}$ *is a subgroup of* $\operatorname{Aut} R$ *isomorphic to the additive group* $k[z, u, c]$. *Moreover,* ξ_p *restricts to an automorphism of* R' *such that* $\xi_p(z) = z$ *and* $\xi_p(u) = u$.

Proof. Points (i) and (ii) are clear. For (iii), remark that there exists a triangular derivation Δ of R' such that: $\Delta(a) = \Delta(z) = \Delta(c) = 0$ and $\Delta(u) = -c$ which extends to a derivation of $S = R'_c$ by $\Delta(c^{-1}) = 0$. An easy computation shows that $\Delta(b) = \Delta(-c^{-1}z + c^{-1}a^2 - a - c^{-1}ua) = -c^{-1}\Delta(u)a = a$. This derivation of S induces a locally nilpotent derivation of R still denoted by Δ. Any polynomial $q(z) \in k[z]$ is central and belongs to $\ker \Delta$. Thus, $q(z)\Delta$ is again a locally nilpotent derivation of R and $\eta_q = \exp q(z)\Delta$ is the automorphism mentioned in point (iii).

For point (iv) we proceed in a similar fashion. For all $p(z, u, c) \in k[z, u, c]$, there is a triangular derivation Δ_p of R' defined by: $\Delta_p(u) = \Delta_p(z) = \Delta_p(c) = 0$ and $\Delta_p(a) = p(z, u, c)c$ which extends to a derivation of S such that $\Delta_p(c^{-1}) = 0$ and $\Delta_p(b) = \Delta_p(-c^{-1}z + c^{-1}a^2 - a - c^{-1}ua) = c^{-1}\big(a\Delta_p(a) + \Delta_p(a)a\big) - \Delta_p(a) - c^{-1}u\Delta_p(a) = ap + pa - up = dp + pa$. The restriction to R of this derivation of S is a locally nilpotent derivation of R still denoted by Δ_p. Let ξ_p denote the automorphism $\exp \Delta_p$. It is the automorphism mentioned in point (iv). ∎

Lemma 3.2. *The subgroup* $\mathrm{Aut}^+ R$ *of* $\mathrm{Aut}\, R$ *generated by* H, G_1 *and* G_2 *can be described as the semi-direct product* $\mathrm{Aut}^+ R = (G_2 \rtimes G_1) \rtimes H$. *The subgroup of* $\mathrm{Aut} R$ *generated by* $\mathrm{Aut}^+ R$ *and* T *equals the semi-direct product* $\mathrm{Aut}^+ R \rtimes T$.

Proof. One easily checks that $\sigma_\alpha \eta_q \sigma_\alpha^{-1} = \eta_{\sigma_\alpha(q)}$, $\sigma_\alpha \xi_p \sigma_\alpha^{-1} = \xi_{\sigma_\alpha(p)}$ and $\eta_q \xi_p \eta_q^{-1} = \xi_{\eta_q(p)}$ in $\mathrm{Aut}^+ R$. The action of τ by conjugation on $\mathrm{Aut}^+ R$ follows from the relations $\tau \xi_p \tau = \xi_{\tau(p)}$, $\tau \eta_q \tau = \xi_q \eta_{-q} = \eta_{-q} \xi_q$ and $\tau \sigma_\alpha \tau = \sigma_\alpha$. ∎

Theorem 3.3. *For the algebra* $R = \mathcal{O}^J(M_2)$, *one has:*

$$\mathrm{Aut} R = \mathrm{Aut}^+ R \rtimes T = [(G_2 \rtimes G_1) \rtimes H] \rtimes T.$$

Proof. Let $\theta \in \mathrm{Aut} R$. The set $N(R)$ of normal elements of R is stable under θ, as well as the subspace $E_0 = k[z] = Z(R)$ (see 1.8 and 1.9). An easy argument based on the total degree with respect to c and u shows that both $\theta(c)$ and $\theta(u)$ must belong to the subspace E_1. Consequently there exist $\alpha(z), \beta(z), \gamma(z), \delta(z) \in k[z]$ such that $\alpha(z)\delta(z) - \beta(z)\gamma(z) \in k^*$ and

$$\begin{cases} \theta(c) = \alpha(z)c + \beta(z)u \\ \theta(u) = \gamma(z)c + \delta(z)u. \end{cases}$$

Let us first consider the relation $[\theta(a), \theta(c)] = \theta(c)^2$ in R. Expanding $\theta(a)$ in S as $\theta(a) = \sum_{i=0}^n s_i a^i$ with $s_i \in k[z, u, c^{\pm 1}]$, $s_n \neq 0$, we get $\sum_{i=0}^n s_i [a^i, \theta(c)] = (\alpha(z)c + \beta(z)u)^2$. Thus we have $n \geq 1$ and, since the degree of $[a^i, \theta(c)]$ in S with respect to a is $i-1$, we conclude that $n = 1$. Moreover, $[a, \theta(c)] = \alpha(z)[a, c] + \beta(z)[a, u] = \theta(c)c$, so that the above equality reduces to $s_1\theta(c)c = \theta(c)^2$, from which it follows that $s_1 = \alpha(z) + \beta(z)uc^{-1}$. But $\theta(a) = s_0 + s_1 a$ belongs to R, thus by routine computations using Lemma 1.6, we obtain $s_0 \in k[z, u, c]$ and $s_1 \in k[z, u, c]$. In particular $\beta(z) = 0$. It follows that $\alpha(z)\delta(z) \in k^*$, thus $\alpha(z) = \alpha \in k^*$ and $\delta(z) = \delta \in k^*$. For consistence of notations, rename $q(z)$ the polynomial $\gamma(z)$ in $k[z]$. We then have:

$$\begin{cases} \theta(c) = \alpha c & \\ \theta(u) = \delta u + q(z)c & \text{with } \alpha, \delta \in k^*, \quad q(z) \in k[z], \\ & s_0(z, u, c) \in k[z, u, c] \\ \theta(a) = \alpha a + s_0(z, u, c) & \end{cases}$$

Let us now consider the automorphisms $\sigma = \sigma_{\alpha^{-1}} \in H$ and $\eta = \eta_{\delta^{-1}q(z)} \in G_1$ and let us put $\theta' = \sigma\eta\theta$. For $p_0 = \sigma\eta(s_0) \in k[z, u, c]$, the automorphism θ' satisfies:

$$\begin{cases} \theta'(c) = c & \\ \theta'(u) = \delta\alpha^{-1}u & \text{with } \alpha, \delta \in k^*, p_0(z, u, c) \in k[z, u, c] \\ \theta'(a) = a + p_0(z, u, c). & \end{cases}$$

We write p_0 in $k[z, u, c]$ as $p_0(z, u, c) = t_0(z, u) + cr(z, u, c)$ where $r(z, u, c) \in k[z, u, c]$ and $t_0(z, u) \in k[z, u]$. For $\xi = \xi_{-r(z,u,c)} \in G_2$, the automorphism $\theta'' = \xi\theta'$ satisfies:

$$\begin{cases} \theta''(c) = c & \\ \theta''(u) = \delta\alpha^{-1}u & \text{with } \alpha, \delta \in k^*, \ t_0(z, u) \in k[z, u]. \\ \theta''(a) = a + t_0(z, u) & \end{cases}$$

Moreover, θ'' restricts to an automorphism of the center $k[z]$ of R; it follows that there exist $\lambda \in k^*, \mu \in k$ such that $\theta''(z) = \lambda z + \mu$. By definition of the jordanian determinant z, we get $c(a + b) = a^2 - ua - z$, and so $\theta''(a^2 - ua - z) \in cR$, that is $(a + t_0)^2 - \delta\alpha^{-1}u(a + t_0) - \lambda z - \mu \in cR$. But, on one hand $z \equiv a^2 - ua$ modulo cR, on the other hand $at_0 \equiv t_0 a$ modulo cR because $D(t_0) \in D(R) \subseteq cR$. It follows:

$$(1 - \lambda)a^2 + \big(2t_0 + (\lambda - \delta\alpha^{-1})u\big)a + (t_0^2 - \delta\alpha^{-1}t_0 u - \mu) \in cR. \tag{7}$$

When $t_0 = 0$, (7) shows that $\lambda = 1$, $\delta = \alpha$ and $\mu = 0$ (see 1.5). Thus, the restriction of θ'' to R' is $\mathrm{id}_{R'}$. Clearly, it follows that $\theta'' = \mathrm{id}_R$ from which we deduce that $\theta = \eta^{-1}\sigma^{-1}\xi^{-1}\theta'' \in \mathrm{Aut}^+ R$. Suppose now that t_0 is a non zero element of $k[z,u]$ and let m be its degree in z. By Lemma 1.6 (i), there exist $w_0(u), \ldots, w_{2m}(u) \in k[u]$ such that $t_0 \equiv \sum_{j=0}^{2m} w_j(u)a^j$ modulo cR, with $w_{2m}(u) \neq 0$. It then follows from (7) that $m = 0$ and so $t_0(z,u) = w_0(u) \in k[u]$. Let us rewrite (7) as

$$(1-\lambda)a^2 + \big(2w_0(u) + (\lambda - \delta\alpha^{-1})u\big)a + (w_0(u)^2 - \delta\alpha^{-1}w_0(u)u - \mu)cR.$$

We get $\lambda = 1$, $w_0(u) = \frac{1}{2}(\delta\alpha^{-1} - 1)u$ and $\frac{1}{4}(\delta\alpha^{-1} - 1)(\delta\alpha^{-1} + 1)u^2 = \mu = 0$. Since $w_0(u)$ is non zero, we must have $\delta = -\alpha$. To sum up: $\theta''(c) = c$, $\theta''(u) = -u$, $\theta''(a) = a - u$, $\theta''(z) = z$. We conclude that θ'' and τ coincide on the subring R'. We then have $\tau(b) = b$ so finally $\theta'' = \tau$ and $\theta = \eta^{-1}\sigma^{-1}\xi^{-1}\theta''(\mathrm{Aut}^+ R \rtimes T)$. ∎

Corollary 3.4. *For the algebra $R = \mathcal{O}^J(M_2)$, let $\mathrm{Aut}_L R$ be the subgroup of linear automorphisms of R (see the introduction of this section) and $\mathrm{Aut}_L^+ R = \mathrm{Aut}^+ R \cap \mathrm{Aut}_L R$. Then the following holds:*

(i) $\mathrm{Aut}_L^+ R$ *is the set of automorphisms of the form:*

$$\phi_{\alpha,\lambda,\mu} : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & 0 & \lambda & 0 \\ \mu & \alpha & \lambda\mu\alpha^{-1} & \lambda \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & \mu & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad \text{with } \alpha \in k^*, \lambda, \mu \in k.$$

(ii) $\mathrm{Aut}_L^+ R$ *is abelian, isomorphic to the group $k^* \times k^2$ equipped with the product defined by $\phi_{\alpha,\lambda,\mu}\phi_{\alpha',\lambda',\mu'} = \phi_{\alpha\alpha',\alpha\lambda'+\alpha'\lambda,\alpha\mu'+\alpha'\mu}$.*

(iii) $\mathrm{Aut}_L R = \mathrm{Aut}_L^+ R \rtimes T$, *with $\phi_{\alpha,\lambda,\mu}\tau = \tau\phi_{\alpha,\mu,\lambda}$.*

Proof. By 3.3, any $\theta \in \mathrm{Aut}^+ R$ satisfies $\theta(u) = \alpha u + q(z)c$ and $\theta(a) = \alpha a + r(z,u,c)c$, with $\alpha \in k^*, q(z) \in k[z], r(z,u,c) \in k[z,u,c]$. It is linear if and only if there exist $\lambda, \mu \in k$ such that $r(z,u,c) = \lambda$ et $q(z) = \lambda - \mu$. The corollary follows easily. ∎

Remark 3.5. Locally nilpotent derivations of $R = \mathcal{O}^J(M_2)$.

Let Der R be the k-Lie algebra of k-derivations of R, $\mathrm{Der}_Z R$ the Lie subalgebra of Der R of the derivations vanishing on the center $k[z]$ of R and $\mathrm{Der}_{LN} R$ the subset of Der$R$ of locally nilpotent derivations. In the proof of Proposition 3.1, we introduced the derivation $\Delta \in \mathrm{Der}_Z R \cap \mathrm{Der}_{LN} R$ defined by:

$$\Delta(a) = \Delta(c) = 0, \quad \Delta(u) = -c, \quad \Delta(b) = a.$$

Moreover, for $i, j \in \mathbb{N}$, consider the derivation $\Delta_{i,j} \in \mathrm{Der}_Z R \cap \mathrm{Der}_{LN} R$ defined by:

$$\Delta_{i,j}(u) = \Delta_{i,j}(c) = 0, \quad \Delta_{i,j}(a) = u^i c^{j+1}, \quad \Delta_{i,j}(b) = au^i c^j + u^i c^j a - u^{i+1} c^j$$

(this is a special case of the derivation Δ_p of the proof of 3.1 with $p(z, u, c) = u^i c^j$); for $i, j, i', j' \in \mathbb{N}$, we have:

$$[\Delta_{i,j}, \Delta] = i\Delta_{i-1,j+1}, \quad [\Delta_{i,j}, \Delta_{i',j'}] = 0.$$

Applying Theorem 3.3 to the exponential of any locally nilpotent derivation, one may check that $\mathrm{Der}_{LN} R \subseteq \mathrm{Der}_Z R$, that $\{\Delta\} \cup \{\Delta_{i,j},\ i, j \in \mathbb{N}\}$ is a basis of $\mathrm{Der}_{LN} R$ as a $k[z]$-module and that $\mathrm{Der}_{LN} R$ is a Lie subalgebra of $\mathrm{Der}\, R$. The details are left to the interested reader. Recall in contrast that the quantum deformation $\mathcal{O}_q(M_2)$ has no non zero locally nilpotent derivation (see [6]) and that the description of the locally nilpotent derivations in commutative polynomial algebras in more than two indeterminates remains a widely open problem (see [33]).

Remark 3.6. Automorphisms of $\overline{R} = \mathcal{O}^J(SL_2)$.

We consider now the factor algebra $\mathcal{O}^J(SL_2) = \bar{R} = R/(z-1)$. Let $\mathrm{Aut}^z R = \{\theta \in \mathrm{Aut} R\,;\, \theta(z) = z\}$. It is clear that any automorphism $\theta \in \mathrm{Aut}^z R$ induces an automorphism $\overline{\theta}$ of $\overline{R}$. The canonical map $\theta \mapsto \overline{\theta}$ is a group homomorphism $\Gamma : \mathrm{Aut}^z R \to \mathrm{Aut}\overline{R}$. In $\mathrm{Aut}^z R$, we consider the subgroups T and $H' = \{\sigma_\alpha \in H\,;\, \alpha = \pm 1\}$ of order 2, $G'_1 = \{\eta_q \in G_1\,;\, q \in k\} \simeq k$ and $G'_2 = \{\xi_p \in G_2\,;\, p(u, c) \in k[u, c]\} \simeq k[u, c]$ (see 3.1 for the notations). It can be shown that the restriction Γ_1 of Γ to $[(G'_2 \rtimes G'_1) \rtimes H'] \rtimes T$ is an isomorphism, and so $\mathrm{Aut}\overline{R} \simeq [(G'_2 \rtimes G'_1) \rtimes H'] \rtimes T$. The proof is left to the reader. Let us indicate only the starting argument: as any $\theta \in \mathrm{Aut}\, \bar{R}$ leaves the set of height one primes of $\overline{R}$ invariant, the results of Remark 2.3.8 show that there exist $\alpha, \beta, \gamma, \delta \in k$ satisfying $\alpha\delta - \beta\gamma \neq 0$ and such that $\theta(\overline{c}) = \alpha\overline{c} + \beta\overline{u}$ and $\theta(\overline{u}) = \gamma\overline{c} + \delta\overline{u}$. The rest of the proof is an adaptation to $\overline{R}$ of the calculations detailed for R in the proof of 3.3. It can also be checked that the subgroup of Hopf algebra automorphisms of $\overline{R}$ is isomorphic via Γ_1 to the subgroup $\{\phi_{1,\lambda,-\lambda}\,;\, \lambda \in k\} \cong k$ of linear automorphisms of R (see 3.4 for the notations). Recall that the corresponding results are proved in [7] for the quantum group $\mathcal{O}_q(SL_2)$ and remain unknown (as far as we know) for the commutative coordinate ring $\mathcal{O}(SL_2)$.

Remark 3.7. We conclude this analysis by showing that Spec $\mathcal{O}^J(M(2))$ (resp. Spec $\mathcal{O}^J(SL(2))$) may be described as the union of finitely many $\mathcal{H}$-strata for a suitable algebraic group $\mathcal{H}$ acting rationally by algebra automorphisms on $\mathcal{O}^J(M(2))$ (resp. $\mathcal{O}^J(SL(2))$). For details about stratifications of spectra, the reader is referred to [34], where we also took our notations.

1. Stratification of Spec $\mathcal{O}^J(M(2))$. Let $\mathcal{H}$ be the subgroup of $\mathrm{Aut}R$ generated by H and G_1' (see 3.1 and 3.6 for the notation). Clearly, $\mathcal{H}$ is the subgroup $\{\phi_{\alpha,0,\mu},\ \alpha \in k^*,\ \mu \in k\}$ of $\mathrm{Aut}_L^+ R$ (see Cor. 3.4). A careful use of the previous results allows to describe Spec $\mathcal{O}^J(M(2))$ as a union of $\mathcal{H}$-strata as follows.

(i) The component $\mathcal{S}^c$, homeomorphic to $k[v,z]$ by 2.3.1, is the union of two strata.
- The first one is composed of the ideals in one-to-one correspondence with the ideals of $k[v,z]$ of the set $\{(z, v-\lambda),\ \lambda \in k\} \cup \{(z)\}$. It is the stratum of those $P \in$ Spec R such that $(\mathcal{H} : P) = (z)$.
- The second one is the complement of the first one in $\mathcal{S}^c$ and its ideals are those $P \in \text{Spec}\, R$ such that $(\mathcal{H} : P) = (0)$.

(ii) The component $\mathcal{S}^a_{c,u}$ is a stratum: $\mathcal{S}^a_{c,u} = \{P \in \text{Spec}\, R \mid (\mathcal{H} : P) = (c,u)\}$.

(iii) The component $\mathcal{S}_{c,a,u}$ is the union of two strata.
- The first one is $\{(c,a,u)\} \cup \{(c,a,u,b-\lambda),\ \lambda \in k^*\} = \{P \in \text{Spec}\, R \mid (\mathcal{H} : P) = (c,a,u)\}$.
- The second one is $\{(c,a,u,b)\} = \{P \in \text{Spec}\, R \mid (\mathcal{H} : P) = (c,a,u,b)\}$.

(iv) The component $\mathcal{S}^u_{c,a} = \{(c,a)\}$ is a stratum: $\mathcal{S}^u_{c,a} = \{P \in \text{Spec}\, R \mid (\mathcal{H} : P) = (c,a)\}$.

(v) The component $\mathcal{S}^{a,u}_c$, homeomorphic to Spec $k[z]$ by 2.3.4, is the union of two strata.
- The first one is composed of the ideals of $\mathcal{S}^{a,u}_c$ in one-to-one correspondence with the ideals of Spec $k[z]$ in $\{(0)\} \cup \{(z-\lambda),\ \lambda \in k^*\}$. It is composed of those $P \in \text{Spec}\, R$ such that $(\mathcal{H} : P) = (c)$.
- The second one is composed of the ideal of $\mathcal{S}^{a,u}_c$ in one-to-one correspondence with $(z) \in \text{Spec}\, k[z]$, that is (c,d). It is the stratum of the ideals $P \in \text{Spec}\, R$ such that $(\mathcal{H} : P) = (c,d)$.

2. Stratification of Spec $\mathcal{O}^J(SL(2))$. Recall that $\overline{R} = \mathcal{O}^J(SL(2))$ and let now $\mathcal{H}$ be the group of automorphisms of $\overline{R}$ induced by the automorphisms of G_1'. The description of Spec $\overline{R}$ given in 2.3.8 allows to decompose it as a union of the four following $\mathcal{H}$-stata:

(i) $\{P \in \text{Spec}\, \overline{R} \mid (\mathcal{H} : P) = (0)\} = \{(0)\} \cup \{(u-\lambda c), \lambda \in k\}$,

(ii) $\{P \in \operatorname{Spec} \overline{R} \mid (\mathcal{H} : P) = (c)\} = \{(c)\}$,

(iii) $\{P \in \operatorname{Spec} \overline{R} \mid (\mathcal{H} : P) = (c, a-1)\} = \{(c, a-1)\} \cup \{(c, a-1, b-\mu), \mu \in k\}$,

(iv) $\{P \in \operatorname{Spec} \overline{R} \mid (\mathcal{H} : P) = (c, a+1)\} = \{(c, a+1)\} \cup \{(c, a+1, b-\mu), \mu \in k\}$.

It is now clear from 2.3.7 and 2.3.8 that, both in R and $\overline{R}$, a prime ideal is primitive if and only if it is maximal in its stratum.

ACKNOWLEDGMENTS

We thank the referee for raising the question which led to Remark 3.7.

REFERENCES

1. Woronowicz, S.L. New Quantum Deformation of $\mathrm{SL}(2, C)$. Hopf Algebra Level. Rep. Math. Phys. **1991**, *30* (2), 259–269.
2. Ohn, C. Quantum $\mathrm{SL}(3, C)$'s with Classical Representation Theory. J. Algebra **1999**, *213* (2), 721–756.
3. Chatters, A.W.; Jordan, D.A. Noncommutative Unique Factorisation Rings. J. London Math. Soc. **1986**, *33* (1), 22–32.
4. Cauchon, G. Quotients Premiers de $\mathcal{O}_q(\mathfrak{m}_n(k))$. J. Algebra **1996**, *180* (2), 530–545.
5. Van Den Essen, A. *Polynomial Automorphisms and the Jacobian Conjecture*; Progress in Math. 190; Birkhäuser: Basel, 2000.
6. Alev, J.; Chamarie, M. Dérivations et Automorphismes de Quelques algèbres Quantiques. Comm. Algebra **1992**, *20* (6), 1787–1802.
7. Braverman, A. On Embeddings of Quantum Groups. C. R. Acad. Sci. Paris Sér. I Math. **1994**, *319* (2), 111–115.
8. Alev, J.; Dumas, F. Rigidité des Plongements des Quotients Primitifs Minimaux de $U_q(\mathrm{sl}(2))$ dans l'algèbre Quantique de Weyl-Hayashi. Nagoya Math. J. **1996**, *143*, 119–146.
9. Fleury, O. Automorphismes de $\check{U}_q(\mathfrak{b}^+)$. Beiträge Algebra Geom. **1997**, *38* (2), 343–356.
10. Dixmier, J. Sur les algèbres de Weyl. Bull. Soc. Math. France **1968**, *96*, 209–242.
11. Smith, M.K. Automorphisms of Enveloping Algebras. Comm. Algebra **1983**, *11* (16), 1769–1802.
12. Joseph, A. A Wild Automorphism of $\mathrm{U}(sl_2)$. Math. Proc. Cambridge Philos. Soc. **1976**, *80* (1), 61–65.

13. Alev, J. Un Automorphisme Non-modéré de $U(\mathfrak{g}_3)$. Comm. Algebra **1986**, *14* (8), 1365–1378.
14. Demidov, E.E. Some Aspects of the Theory of Quantum groups. Russian Math. Surveys **1993**, *48* (6), 41–79.
15. Demidov, E.E.; Manin, Yu.I.; Mukhin, E.E.; Zhdanovich, D.V. Nonstandard Quantum Deformations of $GL(n)$ and Constant Solutions of the Yang-Baxter Equation. Progr. Theoret. Phys. Suppl. **1990**, *102*, 203–218.
16. Dubois-Violette, M.; Launer, G. The Quantum Group of a Non-degenerate Bilinear Form. Phys. Lett. B **1990**, *245* (2), 175–177.
17. Lazarev, A.Yu.; Movshev, M.V. On the Quantization of Certain Lie Groups and Algebras. Russian Math. Surveys **1991**, *46* (6), 225–226.
18. Zakrzewski, S. A Hopf Star-Algebra of Polynomials on the Quantum $SL(2, R)$ for a "unitary" R-matrix. Lett. Math. Phys. **1991**, *22* (4), 287–289.
19. Kupershmidt, B.A. The Quantum Group $GL_h(2)$. J. Phys. A **1992**, *25* (22), L1239–L1244.
20. Aghamohammadi, A.; Shariati, A.; Khorrami, M. The Universal R-matrix for the Jordanian Deformation of sl(2), and the Contracted Forms of so(4). Modern Phys. Lett. A **1996**, *11* (3), 187–197.
21. Aghamohammadi, A.; Shariati, A.; Khorrami, M. Jordanian Deformation of SL(2) as a Contraction of its Drinfeld-Jimbo Deformation. J. Phys. A **1995**, *28* (8), L225–L231.
22. Korenskii, S.V. Representations of the Quantum Group $SL_J(2)$. Russian Math. Surveys **1991**, *46* (6), 221–222.
23. Ohn, C. A $*$-product on SL(2) and the Corresponding Nonstandard Quantum $U(\mathfrak{sl}(2))$. Lett. Math. Phys. **1992**, *25* (2), 85–88.
24. Ohn, C. The "Jordanian" Quantum $SL(2, C)$ has no Compact Form. C. R. Acad. Sci. Paris Sér. I Math. **1995**, *320* (5), 529–532.
25. Karimipour, V. The Quantum de Rham Complexes Associated with $SL_h(2)$. Lett. Math. Phys. **1994**, *30* (2), 87–98.
26. Karimipour, V. Bicovariant Differential Geometry of the Quantum Group $SL_h(2)$. Lett. Math. Phys. **1995**, *35* (4), 303–311.
27. Aghamohammadi, A. The Two Parametric Extension of h Deformation of GL(2), and the Differential Calculus on its Quantum Plane. Modern Phys. Lett. A **1993**, *8* (27), 2607–2613.
28. Aneva, B.L.; Dobrev, V.K.; Mihov, S.G. Duality for the Jordanian Matrix Quantum Group $GL_{g,h}(2)$. J. Phys. A **1997**, *30* (19), 6769–6781.
29. Kassel, C. *Quantum Groups*; Graduate Texts in Mathematics 155; Springer-Verlag: New-York, 1995.
30. Sigurdsson, G. Differential Operator Rings whose Prime Factors have Bounded Goldie Dimension. Arch. Math. **1984**, *42*, 348–353.

31. McConnell, J.C.; Robson, J.C. *Noncommutative Noetherian Rings*; Wiley: Chichester, 1987.
32. Eisenbud, D. *Commutative Algebra*; Graduate Texts in Mathematics 150; Springer-Verlag: New-York, 1995.
33. Daigle, D.; Freudenburg, G. Locally Nilpotent Derivations over a U.F.D. and an Application to Rank Two Derivations of $k[X_1, \ldots, X_n]$. J. Algebra **1998**, *204*, 353–371.
34. Goodearl, K.R.; Letzter, E.S. The Dixmier-Moeglin Equivalence in Quantum Coordinate Rings and Quantized Weyl Algebras. Trans. Amer. Math. Soc. **2000**, *352* (3), 1381–1403.

Received December 2000