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Gourmet's Guide to Gorensteinness

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0. INTRODUCTION

If we were very clever, we might be able to solve the following problem: Let A be a noetherian $\mathbb{N}$-graded connected k-algebra given by generators and relations. Now determine, by looking at the relations, if A is AS-Gorenstein.

It must, however, be admitted that the problem as stated seems rather hard. A more modest goal is hence the following: Find some criteria by which it can be decided whether some naturally occurring algebras are AS-Gorenstein. That is what we shall do in this paper.

We prove generalizations of several well-known theorems from commutative algebra, notably Watanabe's theorem [2, Theorem 5.3.2] and Stanley's theorem [10, Theorem 4.4]. Throughout this paper, "algebra" means connected $\mathbb{N}$-graded algebra over a base field k. The main results are the following:

THEOREM 4.7 (Watanabe). *Let A be a noetherian AS-regular algebra with dual $A^{\#}$, let G be a group, and let*

$$\phi\colon G \to \mathrm{GrAut}(A) \qquad \textit{resp.}$$

$$\theta\colon G \to \mathrm{GrAut}(A)$$

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be a homomorphism resp. an anti-homomorphism satisfying hdet $\phi(g) =$ hdet $\theta(g)$ *for each* $g \in G$ (*we define homological determinant* hdet *in Section* 3). *Then* G *acts on* $A \otimes A^{\#}$ *through the induced homomorphism*

$$g \mapsto \phi(g) \otimes \overline{\theta(g)}$$

for all $g \in G$.

Suppose that its image is of order coprime to char(k). *Then the fixed ring* $(A \otimes A^{\#})^G$ *is AS-Gorenstein.*

THEOREM 6.2 (Stanley). *Let* A *be a noetherian AS-Cohen–Macaulay domain which satisfies one of the following conditions*:

- *A has enough normal elements.*
- *A is quotient of an Auslander-regular algebra.*

Then A *is AS-Gorenstein if and only if the Hilbert series of* A *satisfies the functional equation*

$$H_A(t) = \pm t^{-m} H_A(t^{-1})$$

as rational functions over $\mathbb{Q}$ *for some* m.

There are also several corollaries and related results, some of which solve problems posed in [3], to which this paper can be viewed as a sequel. For instance,

THEOREM 3.6. *Let* A *be noetherian and AS-Gorenstein and* ℓ *be as in Definition* 0.2(1).

(1°) *If* h *is a positive integer such that* $h^{-1} \in k$ *and that* $h \mid \ell$, *then the Veronese subalgebra* $A^{(h)}$ *is AS-Gorenstein.*

(2°) *If* $h \nmid \ell$, *and* $\dim A_d > 1$ *for all* $d > 0$, *then* $A^{(h)}$ *has infinite injective dimension over itself, and so is not AS-Gorenstein.*

This generalizes [3, Proposition 6.5].

Theorem 4.7 builds on a theory of traces and determinants which extends the methods described in [3]. We thus continue the line laid out in [3] of using traces for invariant theory, even though the underlying rings are non-commutative (in the commutative case, traces are a classical tool for invariant theory). Theorem 6.2 builds on methods for working with dualizing modules akin to those of [7], and on a generalization of the Similar Submodule Condition from [16].

The paper is structured thus: Sections 1 to 3 study traces and homological determinants. Section 4 uses these methods to prove (among other things) the Watanabe theorem. Section 5 introduces the generalization of the Similar Submodule Condition and studies its properties. Section 6 uses it to prove (among other things) the Stanley theorem.

To close the Introduction, let us run through some notation and conventions. Throughout k is a fixed ground field, and the word "algebra" means $\mathbb{N}$-graded, connected, locally finite k-algebra. We denote the group of graded k-algebra automorphisms of A by $\mathrm{GrAut}(A)$. The word "A-module" means graded left A-module, and obtain right A-modules as left A°-modules where A° is the opposite ring of A. The category of (graded left) A-modules and homomorphisms of degree zero is denoted $\mathbf{GrMod}(A)$, while the subcategory of finitely generated modules is denoted $\mathbf{grmod}(A)$. The degree of the lowest non-vanishing piece of the module M is denoted by $\mathrm{i}(M)$; if M is not left bounded, then $\mathrm{i}(M) = -\infty$.

A module M is called *locally finite* if each graded piece M_i is a finite dimensional vector space over k. For a locally finite module, we introduce the Hilbert function $H(M, -)$ defined by

$$H(M, i) = \dim_k M_i.$$

If M is also right resp. left bounded, we introduce the Hilbert series

$$H_M(t) = \sum_i H(M, i)\, t^i;$$

it is a Laurent series in t^{-1} resp. t.

Shifting of modules is defined by $M(i)_j = M_{i+j}$. The *Matlis dual* is defined by $(M')_i = (M_{-i})'$, where the right hand prime denotes taking the dual vector space. Matlis dual sends left modules to right modules and vice versa.

If $M \in \mathbf{GrMod}(A)$ and $\sigma \in \mathrm{GrAut}(A)$, then the module ${}^{\sigma}M$ is determined by having the same underlying vector space as M, but multiplication defined by $a \cdot m := \sigma(a)\, m$.

Since A is connected, the graded bimodule $A/A_{\geqslant 1}$ is one-dimensional over k. We usually denote it simply by k, and call it the *trivial module*.

We denote "graded Ext" by $\underline{\mathrm{Ext}}$, so for $M, N \in \mathbf{GrMod}(A)$, we have

$$\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_m \mathrm{Ext}^i_{\mathbf{GrMod}(A)}(M, N(m)),$$

where $\mathrm{Ext}^*_{\mathbf{GrMod}(A)}$ is the Ext^* of the category $\mathbf{GrMod}(A)$. If $\mathfrak{m} = A_{\geqslant 1}$ is the graded maximal ideal of A, we use the notation $H^*_{\mathfrak{m}}$ for the local cohomology functors, so

$$H^i_{\mathfrak{m}}(M) = \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geqslant m}, M).$$

There are certain classes of particularly interesting algebras:

DEFINITION 0.1. Let A be an algebra. We say that A is *AS-Cohen–Macaulay* if there exists an integer n such that

$$i \neq n \Rightarrow H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}^\circ}(A) = 0.$$

AS stands for Artin and Schelter. Next two definitions were in fact introduced by Artin and Schelter.

DEFINITION 0.2. Let A be an algebra. We say that A is *AS-Gorenstein* if the following conditions are satisfied:

- We have

$$\mathrm{id}_A(A) = \mathrm{id}_{A^\circ}(A) = n < \infty,$$

where id is the graded injective dimension.

- There is an integer ℓ such that

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^\circ}(k, A) \cong \begin{cases} 0 & \text{for} \quad i \neq n, \\ k(\ell) & \text{for} \quad i = n. \end{cases} \tag{1}$$

DEFINITION 0.3. Let A be an algebra. We say that A is *AS-regular* if it is AS-Gorenstein and it has finite left and rigth global dimension.

If A is AS-regular, then the trivial module k has a minimal free resolution of the form

$$0 \to F_n \to \cdots \to F_0 \to k \to 0,$$

where each F_i is finitely generated, and where $F_n = A(-\ell)$. The integers n and ℓ appearing here agree with the n and ℓ appearing in Definition 0.2. Note also the following:

Important Convention. Throughout the paper, when a theorem/lemma/proposition begins, "Let A be an AS-Gorenstein (resp. AS-regular) algebra," the proof of the result uses n and ℓ to mean the integers appearing in Definition 0.2.

1. THE TRACE OF A σ-LINEAR MAP

A classical tool in invariant theory is trace functions. The study of noncommutative fixed rings can be viewed as a noncommutative generalization of invariant theory. So it is natural to try applying trace functions to fixed rings. This was done in [3]. The present paper continues the application of traces; this section sets up a few definitions, including the various forms of rationality that we will use a great deal.

DEFINITION 1.1. Let A be an algebra, and let $\sigma \in \mathrm{GrAut}(A)$. Let $M, N \in \mathbf{GrMod}(A)$. A k-linear graded map $f: M \to N$ will be called a *σ-linear map* if it gives a homomorphism of graded A-modules, $f: M \to {}^{\sigma}N$, i.e. if $f(am) = \sigma(a) f(m)$.

Note that $\sigma: A \to A$ is itself σ-linear. If $f: M \to M$ is σ-linear, then $f: M \to {}^{\sigma}M$ is A-linear. So if $M \to E^{\cdot}$ is an injective resolution in the category $\mathbf{GrMod}(A)$, then f lifts to an A-linear chain map $E^{\cdot} \to {}^{\sigma}E^{\cdot}$, that is, to a σ-linear map $E^{\cdot} \to E^{\cdot}$. In other words, σ-linear maps lift to σ-linear chain maps, and hence the σ-linear map f induces σ-linear maps

$$H^i_{\mathfrak{m}}(f): H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(M).$$

DEFINITION 1.2. Let A be an algebra. A module $M \in \mathbf{GrMod}(A)$ is called *Γ-finite* if it satisfies the following two conditions:

- We have $H^i_{\mathfrak{m}}(M) = 0$ for $i \gg 0$.
- Each $H^i_{\mathfrak{m}}(M)$ is locally finite and right bounded.

DEFINITION 1.3. Let A be an algebra, and let $\sigma \in \mathrm{GrAut}(A)$. Let $M \in \mathbf{GrMod}(A)$, and let $f: M \to M$ be σ-linear. We write

$$\mathrm{Tr}_M(f, t) = \sum_d \mathrm{tr}(f|_{M_d})\, t^d,$$

whenever this is well defined.

When M is left bounded and locally finite, we interpret $\mathrm{Tr}_M(f, t)$ as an element of $k((t))$, the fraction field of the power series ring $k[[t]]$.

When M is right bounded and locally finite, we interpret $\mathrm{Tr}_M(f, t)$ as an element of $k((t^{-1}))$, the fraction field of the power series ring $k[[t^{-1}]]$.

When M is Γ-finite, we also write

$$\mathrm{Br}_M(f, t) = \sum_i (-1)^i\, \mathrm{Tr}_{H^i_{\mathfrak{m}}(M)}(H^i_{\mathfrak{m}}(f), t),$$

interpreting this as an element of $k((t^{-1}))$.

When M is left bounded, locally finite, and Γ-finite, we say that f is *rational over k* if it satisfies the conditions:

- $\mathrm{Tr}_M(f, t)$ resp. $\mathrm{Br}_M(f, t)$ are rational functions over k (inside $k((t))$ resp. $k((t^{-1}))$).

- As rational functions over k, we have

$$\mathrm{Tr}_M(f, t) = \mathrm{Br}_M(f, t).$$

DEFINITION 1.4. Let A be an algebra, and let $M \in \mathbf{GrMod}(A)$.

If M is left bounded and locally finite, we can interpret the Hilbert series $H_M(t)$ as an element of $k((t))$.

When M is Γ-finite, we introduce

$$B_M(t) = \sum_i (-1)^i H_{H^i_{\mathfrak{m}}(M)}(t),$$

and we can interpret this as an element of $k((t^{-1}))$.

When M is left bounded, locally finite, and Γ-finite, we say that M is *rational over* k if it satisfies the conditions:

- $H_M(t)$ and $B_M(t)$ are rational functions over k (inside $k((t))$ resp. $k((t^{-1}))$).
- As rational functions over k, we have

$$H_M(t) = B_M(t).$$

Similarly, when M is left bounded, locally finite, and Γ-finite, we can interpret $H_M(t)$ resp. $B_M(t)$ as elements of $\mathbb{Q}((t))$ resp. $\mathbb{Q}((t^{-1}))$. We say that M is *rational over* $\mathbb{Q}$ of it satisfies the conditions:

- $H_M(t)$ and $B_M(t)$ are rational functions over $\mathbb{Q}$ (inside $\mathbb{Q}((t))$ resp. $\mathbb{Q}((t^{-1}))$).
- As rational functions over $\mathbb{Q}$, we have

$$H_M(t) = B_M(t).$$

2. HOMOLOGICAL DETERMINANT

When A is an AS-Gorenstein algebra, it turns out that one can define a group homomorphism, called homological determinant,

$$\mathrm{hdet}\colon \mathrm{GrAut}(A) \to k^*,$$

where k^* is the multiplicative group of the field k. This generalizes the case $A = k[x_1, \ldots, x_n]$, where graded automorphisms are given by elements of the group $\mathrm{GL}_n(k)$, on which the usual (matrix) determinant is defined.

If G is a finite subgroup of $\mathrm{GrAut}(A)$, then the restriction $\mathrm{hdet}|_G$ is important for questions concerning properties of the fixed ring A^G. The present section sets up the homomorphism hdet.

First a lemma about (not necessarily noetherian) AS-Gorenstein algebras.

LEMMA 2.1. *Let A be AS-Gorenstein. Then*

$$H^i_{\mathfrak{m}}(A) \cong \begin{cases} 0 & \text{for} \quad i \neq n, \\ {}_A A'(\ell) & \text{for} \quad i = n. \end{cases}$$

Proof. Let

$$0 \to A \to E^0 \to \cdots \to E^n \to 0$$

be a minimal injective resolution of A as an object of the category $\mathbf{GrMod}(A)$. Let $j \leqslant n$. If E^j has a nonzero torsion element e, then we can also find a nonzero element $e_1 \in E^j$ such that $A_{\geqslant 1} e_1 = 0$. This yields $k(-\deg(e_1)) \hookrightarrow E^j$ by $1 \mapsto e_1$. But k is a simple module, so the condition of minimality on E shows that the composition $k \to E^j \to E^{j+1}$ is zero, so $\underline{\mathrm{Ext}}^j_A(k, A) \neq 0$, so $j = n$. In other words, $E^0, ..., E^{n-1}$ are torsionfree. So since $\underline{\mathrm{Ext}}^n_A(k, A) \cong k(\ell)$, we must have $k(\ell) \hookrightarrow E^n$, and hence $k(\ell)$'s injective envelope is a direct summand in E^n,

$$E^n \cong A'(\ell) \oplus I,$$

and here I must be torsionfree (otherwise $\dim_k \underline{\mathrm{Ext}}^n_A(k, A)$ would be larger than 1). This implies the statement on $H^*_{\mathfrak{m}}(A)$. ∎

Let A be AS-Gorenstein, let $\sigma \in \mathrm{GrAut}(A)$, and consider the map of right modules

$$\sigma^{-1} \colon A_A(-\ell) \to A_A(-\ell),$$

which is σ^{-1}-linear (in the equation, we really should write $\sigma^{-1}(-\ell)$ rather than just σ^{-1}, but we will be sloppy on this point). The Matlis dual map

$$(\sigma^{-1})' \colon {}_A A'(\ell) \to {}_A A'(\ell)$$

is σ-linear.

LEMMA 2.2. *Let A be AS-Gorenstein, let $\sigma \in \mathrm{GrAut}(A)$. There exists a scalar $c \in k^*$ such that the σ-linear map*

$$H^n_{\mathfrak{m}}(\sigma) \colon H^n_{\mathfrak{m}}(A) \to H^n_{\mathfrak{m}}(A)$$

is equal to

$$c(\sigma^{-1})' \colon {}_A A'(\ell) \to {}_A A'(\ell).$$

Proof. First note that by Lemma 2.1, the module $H^n_{\mathfrak{m}}(A)$ is equal to ${}_AA'(\ell)$, so the statement of the lemma makes sense.

Now, the lemma really tells us to consider the A-linear map

$$\sigma\colon {}_AA \to {}^{\sigma}_{A}A,$$

and apply $H^n_{\mathfrak{m}}$, getting the A-linear map

$$H^n_{\mathfrak{m}}(\sigma)\colon H^n_{\mathfrak{m}}(A) \to H^n_{\mathfrak{m}}({}^{\sigma}A) = {}^{\sigma}H^n_{\mathfrak{m}}(A).$$

Using Lemma 2.1, we may write this as

$$H^n_{\mathfrak{m}}(\sigma)\colon {}_AA'(\ell) \to {}^{\sigma}_{A}A'(\ell).$$

But using Matlis duality, it is trivial to see that the only A-linear maps from ${}_AA'(\ell)$ to ${}^{\sigma}_{A}A'(\ell)$ are of the form $(\sigma^{-1})'$, where $c \in k$. So

$$H^n_{\mathfrak{m}}(\sigma) = c(\sigma^{-1})'$$

for some $c \in k$.

Finally we want to see $c \neq 0$. But that is easy, since

$$H^n_{\mathfrak{m}}(\sigma)\, H^n_{\mathfrak{m}}(\sigma^{-1}) = H^n_{\mathfrak{m}}(\sigma^{-1}\sigma) = \mathrm{id}_{H^n_{\mathfrak{m}}(A)},$$

whence the map $H^n_{\mathfrak{m}}(\sigma) = c(\sigma^{-1})'$ cannot be zero. ▮

This enables us to give the definition of the homological determinant:

DEFINITION 2.3. Let A be AS-Gorenstein. For each $\sigma \in \mathrm{GrAut}(A)$, we have by Lemma 2.2 that

$$H^n_{\mathfrak{m}}(\sigma) = c(\sigma^{-1})',$$

and write

$$\mathrm{hdet}\,\sigma := c^{-1},$$

calling this the *homological determinant of* σ. In this way, we have defined a map,

$$\mathrm{hdet}\colon \mathrm{GrAut}(A) \to k^*.$$

Remark 2.4. Note that in Definition 2.3, the map σ is equal to the identity in graded degree zero, that is, in its lowest non-vanishing degree. The map

$$(\sigma^{-1})'\colon A'(\ell) \to A'(\ell)$$

is therefore equal to the identity in graded degree $-\ell$, that is, in its highest non-vanishing degree. The map

$$H^n_{\mathfrak{m}}(\sigma) = (\mathrm{hdet}\,\sigma)^{-1}\,(\sigma^{-1})'$$

is therefore, in its highest non-vanishing degree (that is, degree ℓ), just multiplication with the scalar $(\mathrm{hdet}\,\sigma)^{-1}$. ∎

As claimed in the beginning of this section, the map hdet has the crucial property of being a group homomorphism:

Proposition 2.5. *Let A be AS-Gorenstein. Then the determinant map of Definition 2.3 satisfies $\mathrm{hdet}(\tau\sigma) = \mathrm{hdet}(\tau)\,\mathrm{hdet}(\sigma)$ for any $\tau, \sigma \in \mathrm{GrAut}(A)$, and is thus a group homomorphism*

$$\mathrm{hdet}\colon \mathrm{GrAut}(A) \to k^*.$$

Proof. Let $\tau, \sigma \in \mathrm{GrAut}(A)$. We have a commutative triangle of A-linear maps,

$$\begin{tikzcd} {}_A A \arrow[r, "\sigma"] \arrow[dr, "\tau\sigma"'] & {}^{\sigma}_{A}A \arrow[d, "\tau"] \\ & {}^{\tau\sigma}_{A}A, \end{tikzcd}$$

so we get

$$\begin{tikzcd} H^n_{\mathfrak{m}}(A) \arrow[r, "H^n_{\mathfrak{m}}(\sigma)"] \arrow[dr, "H^n_{\mathfrak{m}}"'] & H^n_{\mathfrak{m}}({}^{\sigma}A) \arrow[d, "H^n_{\mathfrak{m}}(\tau)"] \\ & H^n_{\mathfrak{m}}({}^{\tau\sigma}A). \end{tikzcd}$$

Now we use Remark 2.4: It states that in graded degree $-\ell$, the map $H^n_{\mathfrak{m}}(\sigma)$ is multiplication with $(\mathrm{hdet}\,\sigma)^{-1}$, the map $H^n_{\mathfrak{m}}(\tau)$ is multiplication with $(\mathrm{hdet}\,\tau)^{-1}$, whereas the map $H^n_{\mathfrak{m}}(\tau\sigma)$ is multiplication with $(\mathrm{hdet}\,\tau\sigma)^{-1}$. But then the identity $H^n_{\mathfrak{m}}(\tau)\,H^n_{\mathfrak{m}}(\sigma) = H^n_{\mathfrak{m}}(\tau\sigma)$ clearly means that

$$(\mathrm{hdet}\,\tau\sigma)^{-1} = (\mathrm{hdet}\,\tau)^{-1}\,(\mathrm{hdet}\,\sigma)^{-1},$$

which implies the proposition. ∎

Let us finish the section with a proof of another key property of the homological determinant, which links it to the trace of graded automorphisms.

LEMMA 2.6. *Let A be AS-Gorenstein, and let $\sigma \in \mathrm{GrAut}(A)$.*
If σ is k-rational in the sense of Definition 1.3, then the rational function $\mathrm{Tr}_A(\sigma, t)$ has the form

$$\mathrm{Tr}_A(\sigma, t) = (-1)^n\,(\mathrm{hdet}\,\sigma)^{-1}\,t^{-\ell} + \textit{lower terms},$$

when we write it out as a Laurent series in t^{-1}.

Proof. When σ is k-rational, we have

$$\begin{aligned}\mathrm{Tr}_A(\sigma, t) &\overset{(a)}{=} \mathrm{Br}_A(\sigma, t)\\ &\overset{(b)}{=} (-1)^n\,\mathrm{Tr}_{H^n_{\mathfrak{m}}(A)}(H_{\mathfrak{m}}(\sigma), t)\\ &\overset{(c)}{=} (-1)^n\,(\mathrm{hdet}\,\sigma)^{-1}\,t^{-\ell} + \text{lower terms},\end{aligned}$$

where (a) is by definition of rationality, (b) is by definition of Br, and (c) is by Remark 2.4. ∎

Among other things, this lemma enables us to motivate the name "homological determinant" for the map hdet by proving the claim made at the beginning of the section: If $A = k[x_1, ..., x_n]$ is a commutative polynomial algebra, and the automorphism $\sigma \in \mathrm{GrAut}(A)$ is given by the invertible matrix $C \in \mathrm{GL}_n(k)$, then our $\mathrm{hdet}\,\sigma$ coincides with the usual matrix determinant of C, $\det C$. To see this, note that by [3, (1-1)], we have

$$\mathrm{Tr}_A(\sigma, t) = \frac{1}{\det(I - Ct)} = \frac{1}{(-1)^n \det(C)\,t^n + \text{lower terms}} = (*),$$

and writing this out as a Laurent series in t^{-1},

$$(*) = (-1)^n\,(\det C)^{-1}\,t^{-n} + \text{lower terms}$$

(for the polynomial algebra, $n = \ell$). Comparison with the formula of Lemma 2.6 then gives

$$\mathrm{hdet}\,\sigma = \det C.$$

Note that our definition of homological determinant works also for algebras not generated in degree 1.

3. THE INVARIANT RING OF A FINITE GROUP

Suppose that B is an AS-Gorenstein algebra, and that G is a finite subgroup of $\mathrm{GrAut}(B)$. It turns out that the restriction $\mathrm{hdet}|_G$ is connected to the structure of the dualizing module of the fixed ring B^G. The connection results in one of our two principal tools for proving Gorenstein property, Theorem 3.3, which says that if B if noetherian and $\mathrm{hdet}|_G$ trivial, then B^G is AS-Gorenstein.

In this section, fix the following notation: B is an algebra, and G is a finite subgroup of $\mathrm{GrAut}(B)$ for which $|G|^{-1} \in k$. We write $A = B^G$ for the invariant ring of G; it is again an algebra. We write $\mathfrak{n} = B_{\geqslant 1}$ and $\mathfrak{m} = A_{\geqslant 1}$. It is standard that B is a graded A-bimodule, and that as an A-bimodule, B splits as $B = A \oplus C$. The map

$$F: B \to B, \qquad F(b) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(b)$$

is a projection of B onto the summand A. If B is (left) noetherian, so is A, and in this case, B is (left) finitely generated over A. For proofs of these statements, see [9, Corollaries 1.12 and 5.9].

If B is noetherian and satisfies condition χ, then A also satisfies condition χ by the remark following [1, Proposition 8.7]. If B is noetherian and has a balanced dualizing complex, then A also has a balanced dualizing complex by [15, Proposition 4.17]. We begin by computing the dualizing module of the fixed ring, A.

LEMMA 3.1. *Suppose that B is noetherian and AS-Gorenstein. Then*

(1°) *When $i \neq n$, we have $H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}^\circ}(A) = 0$. Consequently, A is AS-Cohen–Macaulay.*

(2°) *We have as left A-modules*

$$H^n_{\mathfrak{m}}(A) = \{x \in B'(\ell) \mid (\mathrm{hdet}\,\sigma)^{-1}\,(\sigma^{-1})'\,(x) = x \text{ for all } \sigma \in G\}$$

(*and of course, there is a similar formula of right A-modules*).

Proof. First note that B satisfies condition χ, by [14, Corollies 4.3(1)]. As observed, the same is the case for A, and since B is a module finite over A, [15, Lemma 4.13] implies that $H^i_{\mathfrak{m}} = H^i_{\mathfrak{n}}$ and $H^i_{\mathfrak{m}^\circ} = H^i_{\mathfrak{n}}$ for all i.

(1°) We have

$$H^i_{\mathfrak{n}}(B) = H^i_{\mathfrak{m}}(B) = H^i_{\mathfrak{m}}(A \oplus C) = H^i_{\mathfrak{m}}(A) \oplus H^i_{\mathfrak{m}}(C).$$

By Lemma 2.1, we have that $H^i_{\mathfrak{n}}(B) = 0$ for $i \neq n$, so we must have $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq n$. Of course, one can do the same computation for the opposite rings.

(2°) The A-bimodule homomorphism F introduced at the beginning of this section is a splitting of the inclusion of A into B. So the A-bimodule homomorphism

$$H^n_{\mathfrak{m}}(F)\colon H^n_{\mathfrak{m}}(B) \to H^n_{\mathfrak{m}}(B)$$

is a splitting of the inclusion of $H^n_{\mathfrak{m}}(A)$ into $H^n_{\mathfrak{m}}(B)$. So the module $H^n_{\mathfrak{m}}(A)$ can be obtained as the set of fixed points inside $H^n_{\mathfrak{m}}(B)$ of the map $H^n_{\mathfrak{m}}(F)$,

$$H^n_{\mathfrak{m}}(A) = \{z \in H^n_{\mathfrak{m}}(B) \mid H^n_{\mathfrak{m}}(F)\, z = z\}.$$

But the group G acts on $H^n_{\mathfrak{m}}(B)$ through the maps $H^n_{\mathfrak{m}}(\sigma)$, and

$$H^n_{\mathfrak{m}}(F) = \frac{1}{|G|} \sum_{\sigma \in G} H^n_{\mathfrak{m}}(\sigma).$$

Now let $z \in H^n_{\mathfrak{m}}(B)$. On one hand, suppose that z is invariant under all maps $H^n_{\mathfrak{m}}(\sigma)$ for $\sigma \in G$. Then it is clear that $H^n_{\mathfrak{m}}(F)\, z = z$, and hence $z \in H^n_{\mathfrak{m}}(A)$. On the other hand, suppose that $z \in H^n_{\mathfrak{m}}(A)$. Then $H^n_{\mathfrak{m}}(F)\, z = z$, and hence for each $\sigma \in G$, we get

$$\begin{aligned} H^n_{\mathfrak{m}}(\sigma)(z) &= H^n_{\mathfrak{m}}(\sigma)(H^n_{\mathfrak{m}}(F)\, z) \\ &= H^n_{\mathfrak{m}}(\sigma) \sum_{\tau \in G} H^n_{\mathfrak{m}}(\tau)(z) \\ &= \sum_{\tau \in G} H^n_{\mathfrak{m}}(\sigma\tau)(z) \\ &= H^n_{\mathfrak{m}}(F)\, z \\ &= z. \end{aligned}$$

So inside $H^n_{\mathfrak{m}}(B)$, one gets $H^n_{\mathfrak{m}}(A)$ by taking the module of elements which are fixed under all the maps $H^n_{\mathfrak{m}}(\sigma)$:

$$H^n_{\mathfrak{m}}(A) = \{x \in H^n_{\mathfrak{m}}(B) \mid H^n_{\mathfrak{m}}(\sigma)(x) = x \text{ for all } \sigma \in G\}.$$

Moreover, we know that $H^n_{\mathfrak{m}} = H^n_{\mathfrak{n}}$, so actually

$$H^n_{\mathfrak{m}}(A) = \{x \in H^n_{\mathfrak{n}}(B) \mid H^n_{\mathfrak{n}}(\sigma)(x) = x \text{ for all } \sigma \in G\}.$$

Replacing $H^n_{\mathfrak{n}}(B)$ with $B'(\ell)$, and using $H^n_{\mathfrak{n}}(\sigma) = (\operatorname{hdet} \sigma)^{-1}\, (\sigma^{-1})'$, we get the lemma's statement.

Gorenstein property of an AS-Cohen–Macaulay algebra can be read off from its dualizing module:

LEMMA 3.2. *Let C be a noetherian AS-Cohen–Macaulay algebra with a balanced dualizing complex. Write $\mathfrak{o} = C_{\geqslant 1}$. If C satisfies*

$$H^n_{\mathfrak{o}}(C) = C'(\ell)$$

as left C-modules, then C is AS-Gorenstein.

Proof. There is a convergent spectral sequence,

$$E_2^{pq} = \mathrm{Ext}^p_C(k, H^q_{\mathfrak{o}}(C)) \Rightarrow \mathrm{Ext}^{p+q}_C(k, C),$$

see [5, Proposition 1.1]. It implies that for any algebra satisfying the lemma's assumptions,

$$\mathrm{Ext}^i_C(k, C) = \begin{cases} 0 & \text{for} \quad i \neq n, \\ k(\ell) & \text{for} \quad i = n, \end{cases}$$

and using [6, Theorem 4.5] this implies that $\mathrm{id}_C(C) < \infty$ (here we used the existence of a balanced dualizing complex for C, which is enough to ensure the validity of the proof of [6, Theorem 4.5]). So C is left AS-Gorenstein.
And then by [4, Corrolary 4.6], we know that C is also right AS-Gorenstein, so all in all, C is AS-Gorenstein. ∎

The following theorem generalizes a result by Watanabe, [2, Theorem 4.6.2]. It will serve as one of our two key tools for proving Gorenstein property (the other one being Stanley's theorem, Theorem 6.1).

THEOREM 3.3. *Suppose that B is noetherian and AS-Gorenstein, and that G is a finite subgroup of* $\mathrm{GrAut}(A)$. *If* $\mathrm{hdet}\,\sigma = 1$ *for each $\sigma \in G$, then the fixed ring $A = B^G$ is AS-Gorenstein.*

Proof. First note that B has a balanced dualizing complex, by [13, Corollary 4.14] (the proof of which works for AS-Gorenstein algebras). Then note that by the remarks at the beginning of this section, A is noetherian, and has a balanced dualizing complex. Note also that by Lemma 3.1, part (1°), the algebra A is AS-Cohen–Macaulay.
However, by Lemma 3.1, part (2°), we may obtain the left A-module $H^n_{\mathfrak{m}}(A)$ as the submodule of $B'(\ell)$ which is fixed under all maps $(\mathrm{hdet}\,\sigma)^{-1}(\sigma^{-1})'$, for $\sigma \in G$.
Under the present assumption of $\mathrm{hdet}(\sigma) = 1$ for all $\sigma \in G$, we therefore get $H^n_{\mathfrak{m}}(A)$ as the submodule of $B'(\ell)$ fixed under all homomorphisms

$(\sigma^{-1})'$, for $\sigma \in G$. And since A is the fixed ring of G, it is clear that this submodule is $A'(\ell)$, that is,

$$H^n_{\mathfrak{m}}(A) = A'(\ell).$$

By Lemma 3.2, A is then AS-Gorenstein. ∎

As an illustration of how Theorem 3.3 might be used, let us write down an immediate corollary.

COROLLARY 3.4. *Suppose that B is noetherian and AS-Gorenstein. Under any one of the following conditions, the fixed ring $A = B^G$ is AS-Gorenstein*:

(1°) $G = \langle \sigma_1, ..., \sigma_r \rangle$, *and* $\operatorname{hdet} \sigma_i = 1$ *for each* i.

(2°) $G = [G, G]$.

(3°) *G is non-abelian simple.*

(4°) *For any $\sigma \in G$ we have $\sigma^n = e$, but k contains no non-trivial nth root of unity.*

(5°) *$|G|$ is odd, and $k = \mathbb{Q}$.*

Proof. Under any of the assumptions, it is clear that the homomorphism hdet: $G \to k^*$ is trivial, in other words that $\operatorname{hdet} \sigma = 1$ for any $\sigma \in G$. Hence A is AS-Gorenstein by Theorem 3.3. ∎

The following corollary is less striking than Corollary 3.4, but will serve us in the next section, when we prove our generalization of Watanabe's theorem.

COROLLARY 3.5. *Let B be noetherian and AS-Gorenstein, and G be a finite subgroup of* $\operatorname{GrAut}(B)$ *generated by the elements $\sigma_1, ..., \sigma_r$. Suppose that the following two conditions are satisfied*:

- *If M is a finitely generated B-module, and $1 \leqslant i \leqslant r$, and $f: M \to M$ is a σ_i-linear map, then f is rational over k.*
- *If $1 \leqslant i \leqslant r$, then*

$$\operatorname{Tr}_B(\sigma_i, t) = (-1)^n \, t^{-\ell} + \textit{lower terms}.$$

Then the fixed ring $A = B^G$ is AS-Gorenstein.

Proof. Let $1 \leqslant i \leqslant r$. By the first assumption in the corollary, the map σ_i is k-rational. Using Lemma 2.6, we have that

$$\operatorname{Tr}_B(\sigma_i, t) = (-1)^n \, (\operatorname{hdet} \sigma_i)^{-1} \, t^{-\ell} + \text{lower terms}.$$

Combining this with the second assumption in the corollary, we see that

$$\operatorname{hdet} \sigma_i = 1.$$

But by Corollary 3.4, part (1°), we then get that $A = B^G$ is AS-Gorenstein. ∎

Let us round off with an application of Corollary 3.4, proving a generalization of [3, Proposition 6.5].

THEOREM 3.6. *Let C be noetherian and AS-Gorenstein.*

(1°) *If h is a positive integer such that $h^{-1} \in k$ and that $h \mid \ell$, then the Veronese subalgebra $C^{(h)}$ is AS-Gorenstein.*

(2°) *If $h \nmid \ell$, and $\dim C_d > 1$ for all $d > 0$, then $C^{(h)}$ has infinite injective dimension over itself, and so is not AS-Gorenstein.*

Proof. Part (2°) follows from Lemma 3.1 and the last part of [3, proof of Proposition 6.5]. We now prove part (1°), dividing the proof into three steps.

Step 1. First we assume that the field k contains a primitive hth root of unity, ζ. We can define a graded automorphism of C by $\tau(x) = \zeta^{\deg(x)}x$. We write $G = \langle \tau \rangle$ for the (finite cyclic) subgroup of $\operatorname{GrAut}(C)$ generated by τ. It is clear that

$$C^G = C^{(h)},$$

when we think of the Veronese algebra as graded by

$$C_i^{(h)} = \begin{cases} 0 & \text{for} \quad h \nmid i, \\ C_i & \text{for} \quad h \mid i. \end{cases}$$

It is easy to see that $\operatorname{hdet} \tau = \zeta^{-\ell}$. If $h \mid \ell$, then $\operatorname{hdet} \tau = 1$. But then $C^{(h)} = C^G$ is AS-Gorenstein by Corollary 3.4, part (1°).

Step 2. Now we suppose that k is arbitrary. We write $F = k(\zeta)$ for the extension field obtained by adjoining a primitive hth root of unity.

The F-algebra $C \otimes_k F$ is noetherian, since $k \subset F$ is a finite field extension. There is an obvious "change of scalars" functor,

$$- \otimes F\colon \mathbf{GrMod}(C) \to \mathbf{GrMod}(C \otimes F),$$

which is exact (we omit the subscript k on $\otimes$ from here on), and for $M, N \in \mathbf{GrMod}(C)$, there are natural isomorphisms,

$$\operatorname{Ext}^i_{C \otimes F}(M \otimes F, N \otimes F) = \operatorname{Ext}^i_C(M, N) \otimes F.$$

This easily implies that $C \otimes F$ is AS-Gorenstein with the same values of n and ℓ as C. So Step 1 above applies to $C \otimes F$, and tells us that the algebra

$$(C \otimes F)^{(h)} = C^{(h)} \otimes F$$

is AS-Gorenstein.

Step 3. We repeat the trick from Step 2, observing that for $M, N \in \mathbf{GrMod}(C^{(h)})$, we have natural isomorphisms

$$\mathrm{Ext}^i_{C^{(h)} \otimes F}(M \otimes F, N \otimes F) = \mathrm{Ext}^i_{C^{(h)}}(M, N) \otimes F.$$

When $C^{(h)} \otimes F$ is AS-Gorenstein, this equation easily yields that $C^{(h)}$ itself is AS-Gorenstein. ∎

4. WATANABE'S THEOREM

This section proves our generalizations of Watanabe's theorem (Theorems 4.7 and 4.8). These results generalize [3, Theorem 6.6]. Most of the section consists of technicalities related to traces and rationality; these will bring us in a position to use Corollary 3.5 in the cases we are interested in.

LEMMA 4.1. *Let A be an algebra, let $\sigma \in \mathrm{GrAut}(A)$, let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\mathbf{GrMod}(A)$, and let $f_i: M_i \to M_i$ be σ-linear maps which are compatible with the maps in the exact sequence. Then

(1°) *If two of M_1, M_2, M_3 are Γ-finite, so is the third.*

(2°) *We have*

$$\sum_i (-1)^i \operatorname{Tr}_{M_i}(f_i, t) = 0,$$

$$\sum_i (-1)^i \operatorname{Br}_{M_i}(f_i, t) = 0.$$

Proof. This is elementary. ∎

PROPOSITION 4.2. *Let A be AS-regular. Let M be a finitely generated A-module which has a finite free resolution* (*that is, a free resolution terminating after finitely many steps, and consisting of finitely generated* *modules). Let* $\sigma \in \mathrm{GrAut}(A)$, *and suppose that* $f\colon M \to M$ *is* σ*-linear. Then* f *is rational over* k.

Proof. Observe that since A itself in Γ-finite (see Definition 1.2), by working our way along the finite free resolution of M, we can see that M is Γ-finite. We prove the proposition in four steps.

Step 1. Suppose that $M = A(s)$ and $f = \sigma$. Note that $\mathrm{Tr}_M(f, t) = \mathrm{Tr}_{A(s)}(\sigma, t)$ is a rational function over k, by [3, Theorem 2.3(3)]. We have $H^n_{\mathfrak{m}}(A) = A'(\ell)$, and can compute as

$$\begin{aligned}
\mathrm{Br}_M(f, t) &= \mathrm{Br}_{A(s)}(\sigma, t) \\
&= (-1)^n \, \mathrm{Tr}_{H^n_{\mathfrak{m}}(A(s))}(H^n_{\mathfrak{m}}(\sigma), t) \\
&\overset{(a)}{=} (-1)^n \, \mathrm{Tr}_{A'(s+\ell)}((\mathrm{hdet}\, \sigma)^{-1} (\sigma^{-1})', t) \\
&= (-1)^n \, (\mathrm{hdet}\, \sigma)^{-1} \, t^{-s-\ell} \, \mathrm{Tr}_{A'}((\sigma^{-1})', t) \\
&= (-1)^n \, (\mathrm{hdet}\, \sigma)^{-1} \, t^{-s-\ell} \, \mathrm{Tr}_A(\sigma^{-1}, t^{-1}) \\
&\overset{(b)}{=} (-1)^n \, (\mathrm{hdet}\, \sigma)^{-1} \, t^{-s-\ell} a t^m \, \mathrm{Tr}_A(\sigma, t) \\
&= b t^{m-\ell} t^{-s} \, \mathrm{Tr}_A(\sigma, t) \\
&= b t^{m-\ell} \, \mathrm{Tr}_{A(s)}(\sigma, t) \\
&= b t^{m-\ell} \, \mathrm{Tr}_M(f, t),
\end{aligned}$$

where $b = (-1)^n \, (\mathrm{hdet}\, \sigma)^{-1} \, a$ is a scalar. Here (a) is by Lemma 2.2 and Definition 2.3, and (b) is by [3, Theorem 3.1]. The integer m which appears in (b) is defined by $m = \deg(p_\sigma(t))$, where $p_\sigma(t)$ is the polynomial from [3, Theorem 3.1].

So we get the equation

$$\mathrm{Br}_M(f, t) = b t^{m-\ell} \, \mathrm{Tr}_M(f, t). \tag{2}$$

Note that b and m do not depend on s.

Step 2. Suppose that M is finitely generated free, f an arbitrary σ-linear map. We will prove by induction on the rank of M that $\mathrm{Tr}_M(f, t)$ is rational over k, and that Eq. (2) is satisfied.

To start the induction, assume $\mathrm{rank}(M) = 1$. Then M equals $A(s)$ for some integer s, and f is σ itself, so the assertion follows from Step 1.

Now assume $\mathrm{rank}(M) \geqslant 2$. We can use an argument from [3, Proof of Lemma 2.2] to get a short exact sequence

$$0 \to A(s) \to M \to M'' \to 0,$$

where $A(s)$ is a direct summand in M, where f preserves $A(s)$, and where $\operatorname{rank}(M'') = \operatorname{rank}(M) - 1$. Using the inductional assumption, and Lemma 4.1, we learn that $\operatorname{Tr}_M(f, t)$ is rational over k. We also have

$$\begin{aligned}\operatorname{Br}_M(f, t) &= \operatorname{Br}_{M''}(f, t) + \operatorname{Br}_{A(s)}(f, t)\\ &\overset{(a)}{=} bt^{m-\ell}\operatorname{Tr}_{M''}(f, t) + bt^{m-\ell}\operatorname{Tr}_{A(s)}(f, t)\\ &= bt^{m-\ell}\operatorname{Tr}_M(f, t),\end{aligned}$$

where (a) is by the inductional assumption on M'' and Step 1. So M and f satisfy Eq. (5.3).

Step 3. Now look at the general case where M has a finite free resolution, and where f is an arbitrary σ-linear map. We want to prove that f satisfies Eq. (2). We can lift $f: M \to M$ to a chain map defined on the finite free resolution of M, and by Step 2, each homomorphism which is a component of this chain map has rational trace, and satisfies Eq. (2). Therefore f has the same properties.

Step 4. To finish the proof, we just need to check that the constant b from Eq. (2) equals 1, and that the integer $m - \ell$ from Eq. (2) equals 0. But that follows from setting $M = k$ and $f = \mathrm{id}_k$. ∎

COROLLARY 4.3. *Let A be noetherian and AS-regular, and let $\sigma \in \operatorname{GrAut}(A)$. Let $M \in \mathbf{grmod}(A)$, and let $f: M \to M$ be σ-linear. Then f is rational over k.*

Proof. Since A is noetherian and AS-regular, M has a finite free resolution. Now use Proposition 4.2. ∎

LEMMA 4.4. *Let A and B be noetherian algebras with balanced dualizing complexes. Suppose that A is a graded subalgebra of B, in a way such that B_A and ${}_AB$ are finitely generated.*

Let $\sigma \in \operatorname{GrAut}(B)$ be such that $\sigma(A) = A$, let $M \in \mathbf{grmod}(B)$, and let $f: M \to M$ be σ-linear. Then

(1°) $H^i_{\mathfrak{n}}(M) = H^i_{\mathfrak{m}}(M)$.

(2°) $\operatorname{Br}_{{}_BM}(f, t) = \operatorname{Br}_{{}_AM}(f, t)$.

(3°) *If each $\sigma|_A$-linear map on a finitely generated A-module is rational over k, then each σ-linear map on a finitely generated B-module is rational over k.*

Proof. (1°) Since A has a balanced dualizing complex, it satisfies condition χ by [12, Theorem 6.3]. And B is module finite over A, so the lemma's part (1°) can be read in [15, Lemma 4.13].

(2°) The isomorphism in (1°) is functorial, so f induces the same maps on $H^*_{\mathfrak{m}}(M)$ respectively $H^*_{\mathfrak{n}}(M)$. This implies (2°).

(3°) Any finitely generated B-module can be viewed as a finitely generated A-module. And the trace of f does not depend on whether M is considered to be an A- or a B-module. So this follows from (2°). ∎

For any algebra A, we can equip the dual

$$A^{\#} = \underline{\mathrm{Ext}}^*_A(k, k)$$

with the Yoneda product. With this product, $A^{\#}$ is a priori an ungraded algebra. We need to view it both as a $\mathbb{Z}^2$-graded algebra, with

$$A^{\#}_{ij} = \underline{\mathrm{Ext}}^j_A(k, k)_i,$$

and as an $\mathbb{N}$-graded algebra, with

$$A^{\#}_m = \underline{\mathrm{Ext}}^*_A(k, k)_{-m}.$$

Note the change of sign between the two gradings which makes

$$A^{\#}_i = A^{\#}_{-i,*}.$$

The context will make it clear which grading we think of.

Note that if A is a Koszul algebra, then $A^{\#}$ is equal to the Koszul dual $A^!$ of A, if we equip $A^{\#}$ with the $\mathbb{N}$-grading described above (cf. [3, p. 367]). One might have expected a generalization of the Watanabe theorem to deal with a Koszul algebra A, and to involve $A^!$, but as we shall see, the Koszul condition is superfluous if we use $A^{\#}$ rather than $A^!$.

If A is AS-regular, then [3, Lemma 4.2] tells us that any $\sigma \in \mathrm{GrAut}(A)$ induces an automorphism $\bar{\sigma}$ of $A^{\#}$. The definition of $\bar{\sigma}$ makes it clear that it respects both the above gradings on $A^{\#}$. The map $\sigma \mapsto \bar{\sigma}$ is an anti-homomorphism

$$\mathrm{GrAut}(A) \to \mathrm{GrAut}(A^{\#})$$

by [3, Lemma 4.2], that is, it satisfies

$$\overline{\tau \circ \sigma} = \bar{\sigma} \circ \bar{\tau}.$$

If we use the $\mathbb{N}$-grading on $A^{\#}$, the tensor algebra $A \otimes A^{\#}$ also becomes $\mathbb{N}$-graded, and σ induces an element,

$$\sigma \otimes \bar{\sigma} \in \mathrm{GrAut}(A \otimes A^{\#}).$$

But note that the map $\sigma \mapsto \sigma \otimes \bar{\sigma}$ is not a group homomorphism (nor an anti-homomorphism), so a priori, even if G acts on A, it does not act on $A \otimes A^{\#}$.

COROLLARY 4.5. *Let A be noetherian and AS-regular. Let $\sigma \in \mathrm{GrAut}(A)$, and $\tau \in \mathrm{GrAut}(A^{\#})$, and consider the induced automorphism*

$$\sigma \otimes \tau \in \mathrm{GrAut}(A \otimes A^{\#}).$$

Let $M \in \mathbf{grmod}(A \otimes A^{\#})$, and let $f: M \to M$ be $\sigma \otimes \tau$-linear. Then f is rational over k.

Proof. We have

$$A = A \otimes k \hookrightarrow A \otimes A^{\#},$$

and $A \otimes A^{\#}$ is a module finite over A, since $\dim_k A^{\#} < \infty$. Clearly, when we view A as a subalgebra of $A \otimes A^{\#}$ in this way, it is preserved by any $\sigma \otimes \tau$. Also, both A and $A \otimes A^{\#}$ have balanced dualizing complexes, so we are in the situation of Lemma 4.4.

By part (3°) of that lemma, the present corollary will follow if we can see that any $(\sigma \otimes \tau)|_A$-linear map is rational over k, for $\sigma \in G$. But $(\sigma \otimes \tau)|_A = \sigma$, so the statement we need follows from Corollary 4.3. ∎

Note that the following lemma is in a sense the "dual" of Lemma 2.6.

LEMMA 4.6. *Let A be noetherian and AS-regular, and let $\sigma \in \mathrm{GrAut}(A)$. Then*

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t) = \mathrm{hdet}(\sigma)\, t^{\ell} + \textit{lower terms},$$

when we write the trace as Laurent series in t^{-1}.

Proof. First observe that as $A^{\#}_{ij} = \underline{\mathrm{Ext}}^{j}_{A}(k, k)_i$, we have by [11, Proposition 3.1(3)] that

$$A^{\#}_{\leqslant -\ell, *} = A^{\#}_{-\ell, n} = k.$$

So

$$\mathrm{tr}(\bar{\sigma}_{< -\ell, *}) = 0, \tag{3}$$

and

$$\mathrm{tr}(\bar{\sigma}_{-\ell, q}) = 0 \qquad \text{for} \quad q \neq n. \tag{4}$$

Now we have equations of Laurent series in t^{-1},

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, -1)^{-1} \overset{(a)}{=} \mathrm{Tr}_A(\sigma, t)$$
$$\overset{(b)}{=} (-1)^n\,(\mathrm{hdet}\,\sigma)^{-1}\, t^{-\ell} + \text{lower terms},$$

where (a) is by [3, (4-2)], while (b) is by Lemma 2.6, whence

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, -1) = (-1)^n\,\mathrm{hdet}(\sigma)\, t^{\ell} + \text{lower terms}.$$

On the other hand, by definition

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, -1) = \sum_{p,\,q} \mathrm{tr}(\bar{\sigma}\,|_{A^{\#}_{pq}})\, t^{-p}(-1)^q,$$

see [3, p. 366].

Comparing the two formulae for $\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, -1)$ and using the knowledge of Eqs. (3) and (4) shows that

$$\mathrm{tr}(\bar{\sigma}_{-\ell,\,n}) = \mathrm{hdet}\,\sigma, \tag{5}$$

and combining (3), (4), and (5) yields that as Laurent series in t^{-1},

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, 1) = \mathrm{hdet}(\sigma)\, t^{\ell} + \text{lower terms}.$$

But then as Laurent series in t^{-1},

$$\mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t) = \mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t^{-1}, 1) = \mathrm{hdet}(\sigma)\, t^{\ell} + \text{lower terms},$$

as claimed. Note how the first "=" here involves the sign inversion between the two gradings of $A^{\#}$, changing a t into a t^{-1}. ∎

With all the above machinery in place, we can finally prove the main results of this section, the noncommutative generalizations of the Watanabe theorem. Since G does not a priori act on $A \otimes A^{\#}$, even if G acts on A, one has to think of some recipe to get a group which does; there are (at least) two ways of doing so, hence two different generalizations of Watanabe (Theorems 4.7 and 4.8).

THEOREM 4.7 (Watanabe I). *Let A be noetherian and AS-regular, let G be a group, and let*

$$\phi\colon G \to \mathrm{GrAut}(A) \qquad \textit{resp.}$$

$$\theta\colon G \to \mathrm{GrAut}(A)$$

be a homomorphism resp. an anti-homomorphism satisfying $\operatorname{hdet}\phi(g) = \operatorname{hdet}\theta(g)$ *for each* $g \in G$. *Then* G *acts on* $A \otimes A^{\#}$ *through the induced homomorphism*

$$g \mapsto \phi(g) \otimes \overline{\theta(g)}$$

for all $g \in G$.

Suppose that its image is of order coprime to $\operatorname{char}(k)$. *Then the fixed ring* $(A \otimes A^{\#})^G$ *is AS-Gorenstein.*

Proof. $A \otimes A^{\#}$ is AS-Gorenstein of injective dimension n, by [3, Corollary 6.3(1)]. Denoting the image of G in $\operatorname{GrAut}(A \otimes A^{\#})$ by G', we want to use Corollary 3.5 on $A \otimes A^{\#}$, so need to check its two conditions on $A \otimes A^{\#}$ and each $\phi(g) \otimes \overline{\theta(g)}$.

(1°) This follows immediately from Corollary 4.5.

(2°) For each $g \in G$, we can compute

$$\begin{aligned}
\operatorname{Tr}_{A \otimes A^{\#}}(\phi(g) \otimes \overline{\theta(g)}, t) &\overset{(a)}{=} \operatorname{Tr}_A(\phi(g), t) \operatorname{Tr}_{A^{\#}}(\overline{\theta(g)}, t) \\
&\overset{(b)}{=} ((-1)^n (\operatorname{hdet}\phi(g))^{-1} t^{-\ell} + \text{lower terms}) \\
&\quad (\operatorname{hdet}\theta(g)\, t^{\ell} + \text{lower terms}) \\
&\overset{(c)}{=} (-1)^n t^0 + \text{lower terms},
\end{aligned}$$

where (a) is by [3, (2-4)], (b) is by Lemmas 2.6 and 4.6, and (c) is because $\operatorname{hdet}\phi(g) = \operatorname{hdet}\theta(g)$ for each g. ∎

This theorem contains the commutative Watanabe theorem [2, Theorem 5.3.2], as a special case: Let V be a finite dimensional vector space over k, and let $A = k[V]$. Let G be a finite subgroup of

$$\operatorname{GL}_n(k) = \operatorname{GrAut}(A)$$

for which $|G|^{-1} \in k$. We set ϕ equal to the inclusion of G in $\operatorname{GrAut}(A)$, and θ equal to the anti-homomorphism obtained by composing the inclusion with transposition of matrices.

The dual of A is the exterior algebra $A^{\#} = \bigwedge(V')$, and one can check that the action of G on $A \otimes A^{\#} = k[V] \otimes \bigwedge(V')$ given by the recipe of Theorem 4.7,

$$g \mapsto \phi(g) \otimes \overline{\theta(g)},$$

is the action appearing in [2, Theorem 5.3.2]. But now Theorem 4.7 says that the fixed algebra

$$(A \otimes A^{\#})^{G} = \left(k[V] \otimes \bigwedge (V')\right)^{G}$$

is AS-Gorenstein, and that is exactly the statement of [2, Theorem 5.3.2].

THEOREM 4.8 (Watanabe II). *Let A be noetherian and AS-regular, let G be a finite subgroup of* $\mathrm{GrAut}(A)$, *and consider the elements*

$$\sigma \otimes \bar{\sigma} \in \mathrm{GrAut}(A \otimes A^{\#})$$

for $\sigma \in G$. They generate a subgroup of $\mathrm{GrAut}(A \otimes A^{\#})$ *which we call G'. Assume that the order of G is coprime to* $\mathrm{char}(k)$. *Then* $(A \otimes A^{\#})^{G'}$ *is AS-Gorenstein.*

Proof. Since G' is a subgroup of $G \times G^{opp}$, it is finite. If the order of G is coprime to $\mathrm{char}(k)$, then the order of G' is coprime to $\mathrm{char}(k)$.

This proof is almost the same as the proof of Theorem 4.7: $A \otimes A^{\#}$ is AS-Gorenstein of injective dimension n, by [3, Corollary 6.3(1)]. We want to use Corollary 3.5 on $A \otimes A^{\#}$, so need to check its two conditions on $A \otimes A^{\#}$ and a chosen set of generators of G'. Naturally, we choose the generators $\sigma \otimes \bar{\sigma}$.

(1°) This follows immediately from Corollary 4.5.

(2°) For each $\sigma \in G$, we can compute

$$\begin{aligned}
\mathrm{Tr}_{A \otimes A^{\#}}(\sigma \otimes \bar{\sigma}, t) &\overset{(a)}{=} \mathrm{Tr}_{A}(\sigma, t)\, \mathrm{Tr}_{A^{\#}}(\bar{\sigma}, t) \\
&\overset{(b)}{=} ((-1)^{n}\, (\mathrm{hdet}\,\sigma)^{-1} t^{-\ell} + \text{lower terms}) \\
&\quad\ (\mathrm{hdet}(\sigma)\, t^{\ell} + \text{lower terms}) \\
&= (-1)^{n}\, t^{0} + \text{lower terms},
\end{aligned}$$

where (a) is by [3, (2-4)], and (b) is by Lemmas 2.6 and 4.6. ∎

5. THE SIMILAR SUBMODULE CONDITION

The present section introduces a condition, the Similar Submodule Condition, and shows that an algebra satisfying this condition has a number of beneficial properties. In the next section, they will help us to check that various algebras satisfy the conditions which will appear in the core form of our generalized Stanley theorem, Theorem 6.1.

First we generalize [16, p. 397]:

DEFINITION 5.1. Let A be an algebra, and let $M, N \in \mathsf{GrMod}(A)$. We say that M and N are *similar*, abbreviated to $M \sim N$, if they satisfy the following conditions:

(S1) $M \cong N$ as graded k-vector spaces, and

(S2) $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N)$ as graded k-vector spaces, for each integer i.

If A is noetherian and AS-Gorenstein, the local duality theorem [13, Theorem 4.18] implies that this definition coincides with the definition of similarity given in [16, p. 397].

DEFINITION 5.2. Let A be an algebra, and let $M \in \mathsf{GrMod}(A)$. We say that M has a *proper similar submodule* if: There exists an $M' \in \mathsf{GrMod}(A)$ such that M' is similar to M, and an integer $\ell > 0$ such that $M'(-\ell)$ is isomorphic to a proper graded submodule of M.

If A is an algebra such that any torsionfree graded module M has a graded submodule N, such that N has a proper similar submodule, we say that A satisfies the *Similar Submodule Condition*, abbreviated to (SSC).

If A is noetherian and AS-Gorenstein, the present condition (SSC) coincides with condition (SSC) introduced on AS-Gorenstein algebras in [16, p. 398].

The following two results exhibit the basic good behaviour which results from condition (SSC). They generalize [16, Lemma 2.2 and Theorem 3.1]. The notion of multi-polynomial functions is used in the first of the results; see [16, p. 399] for the definition.

LEMMA 5.3. *Let A be a noetherian algebra satisfying condition* (*SSC*), *and let $M \in \mathsf{grmod}(A)$. Then*

(1°) *The Hilbert function $H(M, -)$ is multi-polynomial for large values of the argument, and*

$$\mathrm{GKdim}(M) = \deg(H(M, -)) + 1 < \infty.$$

(2°) $\mathrm{Kdim}(M) \geqslant \mathrm{GKdim}(M)$.

(3°) *If the graded module N is isomorphic to M as graded k-vector spaces, and there is an integer $m > 0$, and an exact sequence*

$$0 \to K_1 \to N(-m) \to M \to L \to K_2 \to 0,$$

then we have $\mathrm{GKdim}(M) \leqslant \mathrm{GKdim}(L) + 1$. *And if* $K_1 = K_2 = 0$, *we even have* $\mathrm{GKdim}(M) = \mathrm{GKdim}(L) + 1$.

Proof. Use the proof of [16, Lemma 2.2]. ∎

The following is [15, Theorem 5.14], and see [15] for definitions of Auslander and GKdim-Macaulay conditions.

THEOREM 5.4. *Let A be a noetherian algebra satisfying conditions* (*SSC*) *and* χ. *Then*

(1°) *A has a balanced dualizing complex, and satisfies the Auslander and* GKdim-*Macaulay conditions. Moreover*, $\mathrm{GKdim}(A) = \mathrm{lcd}(A) = \mathrm{lcd}(A^{\circ})$.

(2°) *For any finitely generated graded module, M, we have*

$$\mathrm{GKdim}(M) = \mathrm{Kdim}(M) < \infty.$$

We go on to show some properties of algebras with (SSC) which will be handy for the next section. The key results are Propositions 5.5 and 5.7.

PROPOSITION 5.5. *Let A be a noetherian algebra. Suppose that A satisfies one of the following conditions*:

- *A satisfies* (*SSC*) *and* χ.
- *A is quotient of a noetherian AS-regular algebra.*

Then any $M \in \mathbf{grmod}(A)$ is rational over $\mathbb{Q}$, in the sense of Definition 1.4.

Proof. The Case Where A Satisfies (*SSC*) *and* χ. First note that we are in the situation of Theorem 5.4. In particular, $\mathrm{lcd}(A) < \infty$ and A satisfies χ, so $B_M(t)$ is well-defined. We also know $\mathrm{GKdim}(M) < \infty$.

We use induction on $d = \mathrm{GKdim}(M)$. For $d = 0$, rationality of M over $\mathbb{Q}$ is obvious, since M has finite length. So assume $d > 0$, and suppose that rationality over $\mathbb{Q}$ holds for any $L \in \mathbf{grmod}(A)$ which has $\mathrm{GKdim}(L) < d$.

Since $H_M(t)$ and $B_M(t)$ are both additive with respect to short exact sequences in M, a standard noetherian argument shows that we just need to produce a nonzero graded submodule N in M such that $H_N(t)$ and $B_N(t)$ are rational over $\mathbb{Q}$, and such that the equation $H_N(t) = B_N(t)$ holds for N.

If M has torsion, we can use $N = \tau(M)$.

If M is torsionfree, we use condition (SSC) to choose $N \subseteq M$ such that N has a proper similar submodule: There is $\tilde{N}$ such that $N \sim \tilde{N}$, and such that $\tilde{N}(-m) \subset N$ for some $m > 0$. So there is a short exact sequence,

$$0 \to \tilde{N}(-m) \to N \to N'' \to 0. \tag{6}$$

Lemma 5.3, part (3°), tells us that

$$\mathrm{GKdim}(N'') = \mathrm{GKdim}(N) - 1 \leqslant \mathrm{GKdim}(M) - 1 = d - 1,$$

so by induction, we have that N'' is rational over $\mathbb{Q}$. In other words, $H_{N''}$ and $B_{N''}$ are rational functions, and

$$H_{N''}(t) = B_{N''}(t). \tag{7}$$

However, the sequence (6) clearly gives us the equation

$$(1 - t^m)\, H_N(t) = H_{N''}(t), \tag{8}$$

and using that $N \sim \tilde{N}$, whence $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(\tilde{N})$ for all i, the sequence (6) also gives the equation

$$(1 - t^m)\, B_N(t) = B_{N''}(t). \tag{9}$$

By Eqs. (8) and (9), $H_N(t)$ and $B_N(t)$ are rational over $\mathbb{Q}$. And combining Eqs. (7), (8), and (9), we have the desired equality of rational functions,

$$H_N(t) = B_N(t).$$

The Case Where A is Quotient of an AS-Regular Algebra. Let B be the AS-regular algebra of which A is quotient. By [15, Lemma 4.13], local cohomology of A-modules can be computed equally well over A and B. It is therefore enough to show that any finitely generated B-module is rational over $\mathbb{Q}$.

First note that both $H_M(t)$ and $B_M(t)$ are additive with respect to short exact sequences in M. Since each finitely generated module has a finite free resolution consisting of direct sums of shifts of B, it is enough to see that $H_B(t)$ and $B_B(t)$ are rational, and that $H_B(t) = B_B(t)$.

But this follows easily from [11, Theorem 2.4(2)], and the remark preceding that theorem. ∎

LEMMA 5.6. *Let A be a noetherian Auslander–Gorenstein algebra. Then A has an Artinian ring of quotients.*

Proof. This follows from [15, Theorem 6.23]. ∎

PROPOSITION 5.7. *Let A be a noetherian AS-Cohen–Macaulay algebra which has a balanced dualizing complex given by* $K[n]$, *where K is the dualizing module. Suppose also that A satisfies one (or both) of the following conditions:*

- *A satisfies (SSC) and* χ.
- *A is quotient of a noetherian Auslander-regular algebra.*

If $s > -\mathrm{i}(K)$, *then the algebra* $A \times K(-s)$ *has an Artinian ring of quotients (the product in this algebra is given by* $(a,k)(a_1,k_1) = (aa_1, ak_1 + ka_1)$).

Proof. Generally speaking, according to [7, Proposition 1.5], the algebra $A \times K(-s)$ is AS-Gorenstein when $s > -\mathrm{i}(K)$.

Now, in the case where A satisfies (SSC) and χ, according to Theorem 5.4, part (1°), the algebra A satisfies the Auslander condition.

In the case where A is quotient of a noetherian Auslander-regular algebra, A also satisfies the Auslander condition, by [15, Corollary 4.15(2)].

And $A \times K(-s)$ is module-finite over A, so again by [15, Corollary 4.15(2)], the algebra $A \times K(-s)$ satisfies the Auslander condition in either of the two cases we are considering.

Combining the properties of $A \times K(-s)$, we see that $A \times K(-s)$ is Auslander–Gorenstein when $s > -\mathrm{i}(K)$. But then Lemma 5.6 says that it has an Artinian ring of quotients. ∎

Finally, let us note that (SSC) is satisfied for many practically occurring algebras.

Lemma 5.8. *Let A be an algebra which has enough normal elements, in the sense of* [16, *p.* 392]. *Then*

(1°) *A satisfies condition* (*SSC*) *of Definition* 5.2.

(2°) *A satisfies condition χ.*

Proof. Part (2°) is [1, Corollary 8.12(2)].

(1°) Let $M \in \mathbf{grmod}(A)$ be torsionfree. As in [16, proof of Proposition 2.3(2)], we may choose $N \neq 0$ as a graded M-submodule, such that $\mathfrak{p} = \mathrm{Ann}(N)$ is a prime ideal in A, and such that N is fully faithful over $A/\mathfrak{p}$. Since M is torsionfree, we have $\mathfrak{p} \neq \mathfrak{m}$, so choose a homogeneous regular normal element $x \in (A/\mathfrak{p})_{\geqslant 1}$. Write $m = \deg(x)$, and let σ be the graded automorphism of $A/\mathfrak{p}$ such that $ax = x\sigma(a)$ for all $a \in A/\mathfrak{p}$.

Look at N as an $A/\mathfrak{p}$-module. It contains the graded submodule xN, and in fact as $A/\mathfrak{p}$-modules we have

$$xN \cong {}^{\sigma}N(-m).$$

So in particular, we have

$$xN \cong N(-m) \qquad \text{as graded vector spaces,}$$

that is, condition (S1) of Definition 5.1 is satisfied for xN and $N(-m)$.

We also get, for each integer i, an isomorphism over $A/\mathfrak{p}$,

$$H^i_{\mathfrak{m}/\mathfrak{p}}(xN) \cong H^i_{\mathfrak{m}/\mathfrak{p}}({}^{\sigma}N(-m)) \cong {}^{\sigma}(H^i_{\mathfrak{m}/\mathfrak{p}}(N(-m))),$$

so in particular, as graded vector spaces, $H^i_{\mathfrak{m}/\mathfrak{p}}(xN) \cong H^i_{\mathfrak{m}/\mathfrak{p}}(N(-m))$. But A satisfies χ by part (2°), and by [15, Lemma 4.13] we then have $H^i_{\mathfrak{m}/\mathfrak{p}}(L) \cong H^i_{\mathfrak{m}}(L)$ for any $L \in \mathbf{grmod}(A/\mathfrak{p})$. All in all,

$$H^i_{\mathfrak{m}}(xN) \cong H^i_{\mathfrak{m}}(N(-m)) \qquad \text{as graded vector spaces,}$$

that is, condition (S2) of Definition 5.1 is satisfied for xN and $N(-m)$.

So $(xN)(m) \sim N$, and N has a submodule isomorphic to xN, that is, to the $-m$th shift of $(xN)(m)$. ∎

6. STANLEY'S THEOREM

This section proves our generalizations of Stanley's theorem. There are both an "abstract" version, Theorem 6.1, which contains some rather complicated conditions on the algebra in question, a more "concrete" version with more comprehensible conditions, Theorem 6.2, and finally a version for fixed rings, Theorem 6.4.

THEOREM 6.1 (Stanley). *Let A be a noetherian AS-Cohen–Macaulay algebra which is a domain, and satisfies the following conditions.*

(1°) *A has a balanced dualizing complex. By the AS-Cohen–Macaulay condition, the dualizing complex is equal to $K[n]$ for some bimodule K; we call K the dualizing module.*

(2°) *There is an integer $s > -\mathrm{i}(K)$ such that the algebra $A \times K(-s)$ has an Artinian ring of quotients.*

(3°) *The Hilbert series $H_A(t)$ and $H_K(t)$ are both rational functions over $\mathbb{Q}$, and as rational functions they satisfy*

$$H_K(t) = \pm H_A(t^{-1}).$$

Then A is AS-Gorenstein if and only if the equation of rational functions

$$H_A(t) = \pm t^{-m} H_A(t^{-1}) \tag{10}$$

is satisfied for some integer m.

A comment on the conditions (1°) to (3°) in the theorem: We shall see below, using the results from Section 5, that they are all satisfied whenever A is AS-Cohen–Macaulay, and satisfies conditions (SSC) and χ.

Proof of Theorem 6.1. Suppose on one hand that A is AS-Gorenstein. Then the dualizing module K is just $A(-m)$, so we have the equation

$$H_A(t) = t^{-m} H_K(t) \overset{(a)}{=} t^{-m}(\pm H_A(t^{-1})) = \pm t^{-m} H_A(t^{-1}),$$

where (a) is by condition (3°). This shows that Eq. (10) holds.

Suppose on the other hand that A satisfies Eq. (10). We write $B = A \times K(-s)$; by condition (2°), there is a choice of s such that B has an Artinian ring of quotients. We can then argue in the same way as the proof of [7, Theorem 2.1], and obtain the same statement as in [7, Theorem 2.1(4)]: If $x \in A$ is a non-zero-divisor of A, then x is regular on K, both from the left and from the right. Since A is a domain by assumption, this means that any nonzero $x \in A$ is regular on K, both from the left and from the right.

Combining Eq. (10) and condition (3°), we have the equation

$$H_K(t) = \pm H_A(t^{-1}) = \pm(\pm t^m H_A(t)) = \pm t^m H_A(t),$$

which is valid when viewed as an equation of power series with rational coefficients, and clearly the plus sign must apply, so

$$H_K(t) = t^m H_A(t),$$

hence the Hilbert series of K is just a shift of the Hilbert series of A.

Let y be a nonzero homogeneous element of minimal degree in K; we have $\deg(y) = m$. Look at the left submodule $Ay \subseteq K$. It was established above that any nonzero $x \in A$ is regular on K, and so $H_{Ay}(t) = t^m H_A(t)$. But then

$$H_{Ay}(t) = t^m H_A(t) = H_K(t),$$

and we conclude that $Ay = K$, that is, K is a free module with one generator, y. Thus we see that

$$\mathrm{id}_A(A) = \mathrm{id}_A(K) < \infty.$$

The same device, from the right, can be employed to see that

$$\mathrm{id}_{A^\circ}(A) = \mathrm{id}_{A^\circ}(K) < \infty.$$

And A and A° satisfy χ by condition (1°) and [12, Theorem 6.3], so by [14, Theorem 4.2(3)] the algebra A is AS-Gorenstein. ∎

As an application, we will show that Stanley's theorem is valid in certain practically occurring cases.

THEOREM 6.2 (Stanley's Theorem for Rings with Enough Normal Elements and Quotients of Auslander-Regular Algebras). *Let A be an AS-Cohen–Macaulay algebra which is a domain. Suppose that A satisfies one of the following conditions*:

- *A has enough normal elements in the sence of* [16, *p.* 392].
- *A is quotient of an Auslander-regular algebra.*

Then A is AS-Gorenstein if and only if the equation of rational functions

$$H_A(t) = \pm t^{-m} H_A(t^{-1})$$

is satisfied for some integer m.

Proof. It is clear that we can prove this by checking that the algebra A satisfies conditions $(1°)$ to $(3°)$ from Theorem 6.1.

The Case Where A has Enough Normal Elements. First note that by Lemma 5.8, A satisfies both (SSC) and χ.

Condition $(1°)$. A satisfies condition χ, and by Theorem 5.4, part $(1°)$, we also have $\operatorname{lcd}(A) < \infty$. The same statements hold for $A°$, since the assumption of enough normal elements is left/right symmetric. By [12, Theorem 6.3], A has a balanced dualizing complex, so condition $(1°)$ holds.

Condition $(2°)$. If $s > -\mathrm{i}(K)$, we know from Proposition 5.7 that the algebra $A \times K(-s)$ has an Artinian ring of quotients.

Condition $(3°)$. Every finitely generated A-module is rational over $\mathbb{Q}$ by Proposition 5.5.

The Case Where A is Quotient of an Auslander-Regular Algebra. Note again that A satisfies condition χ, by [14, Corollary 4.3(1)].

Condition $(1°)$. A has a balanced dualizing complex by [12, Theorem 6.3].

Condition $(2°)$. This works just as when A has enough normal elements, by appealing to Proposition 5.7.

Condition $(3°)$. Note that an Auslander-regular algebra is AS-regular by [8, Theorem 6.3]. Now use Proposition 5.5 to see that every finitely generated A-module is rational over $\mathbb{Q}$. ∎

This theorem contains Stanley's original result, [10, Theorem 4.4], as a special case, since any commutative noetherian graded k-algebra is quotient of a polynomial algebra $k[x_1, ..., x_n]$ (where the generators are not necessarily of degree one).

As another application of Stanley's theorem in the form of Theorem 6.1, we will give a necessary and sufficient condition for a fixed subring of an Auslander-regular ring to be AS-Gorenstein. First a lemma.

LEMMA 6.3. *Let B be a noetherian algebra with a balanced dualizing complex. let G be a finite subgroup of* $\mathrm{GrAut}(B)$ *with $|G|^{-1} \in k$, and set $A = B^G$.*

If every σ-linear map on every finitely generated left B-module is rational over k, then every finitely generated left A-module is rational over k.

Proof. We recall from Section 3 that as A-bimodules, $B = A \oplus C$, and that the projection onto the A-summand is the map

$$F = \frac{1}{|G|} \sum_{\sigma \in G} \sigma.$$

Let $M \in \mathbf{grmod}(A)$, and set $M' = B \otimes_A M$. We have

$$M' = (A \oplus C) \otimes_A M = M \oplus (C \otimes_A M).$$

Clearly, the projection onto the M-summand is the map $F' = F \otimes \mathrm{id}_M$. It is also clear that we have the fact

$$\begin{gathered}\text{when } G \text{ acts on } M' \text{ by } m \otimes b \mapsto m \otimes \sigma(b),\\ \text{the fixed module is precisely } M.\end{gathered} \tag{11}$$

Using the same observation as in the proof of Lemma 3.1, we now see that

$$H^i_{\mathfrak{m}}({}_AM') = H^i_{\mathfrak{m}}(M) \oplus H^i_{\mathfrak{m}}(C \otimes_A M),$$

and that we have the fact

$$\begin{gathered}\text{The projection onto the } H^{\mathrm{i}}_{\mathfrak{m}}(M)\text{-summand is}\\ H^{\mathrm{i}}_{\mathfrak{m}}(F') = \frac{1}{|G|} \sum_{\sigma \in \mathrm{G}} H^{\mathrm{i}}_{\mathfrak{m}}(\sigma).\end{gathered} \tag{12}$$

We can now compute with Laurent series over k:

$$\begin{aligned} B_{{}_AM}(t) &\overset{(a)}{=} \mathrm{Br}_{{}_AM'}(F', t) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Br}_{{}_AM'}(\sigma \otimes \mathrm{id}, t) \end{aligned}$$

$$\overset{(b)}{=} \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Br}_{{}_B M'}(\sigma \otimes \mathrm{id}, t)$$
$$\overset{(c)}{=} \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Tr}_{M'}(\sigma \otimes \mathrm{id}, t)$$
$$\overset{(d)}{=} \mathrm{Tr}_{(M')^G}(\mathrm{id}, t)$$
$$\overset{(e)}{=} \mathrm{Tr}_M(\mathrm{id}, t)$$
$$= H_M(t).$$

The (a) simply expresses that the dimension of a space is equal to the trace of a projection mapping to that space, cf. (12). The (b) is by Lemma 4.4, part 3°. The (c) is by our assumption on B. The (d) is by an obvious generalization of the formula in [3, Lemma 5.2]. And (e) is by (11).

The computation shows that $H_M(t)$ and $B_{{}_A M}(t)$ are both rational functions over k, since $\frac{1}{|G|}\sum_{\sigma \in G} \mathrm{Tr}_{M'}(\sigma \otimes \mathrm{id}, t)$ is rational by assumption. And the computation also shows $B_{{}_A M}(t) = H_M(t)$ as rational functions over k. So all in all, M is rational over k. ∎

THEOREM 6.4 (Stanley's Theorem for the Fixed Ring of a Finite Group). *Suppose* $\mathrm{char}(k) = 0$. *Let B be a noetherian Auslander-regular algebra, and let G be a finite subgroup of* $\mathrm{GrAut}(B)$. *Then the fixed ring $A = B^G$ is AS-Gorenstein if and only if the Hilbert series satisfies the functional equation*

$$H_A(t) = \pm t^{-m} H_A(t^{-1}),$$

for some integer m.

Proof. By [8, Theorem 4.8], B is a domain, so the same is true for A. And A is AS-Cohen–Macaulay by Lemma 3.1, part (1°).

So the theorem follows from Theorem 6.1 if we can prove that A satisfies conditions (1°) to (3°) of that theorem.

Condition (1°). Since B has a balanced dualizing complex, so does A by [15, Proposition 4.17].

Condition (2°). B is Auslander-regular, and by [15, Proposition 4.18], A therefore satisfies the Auslander condition. By [15, Corollary 4.15(2)], this again means that $A \times K_A(-s)$ satisfies the Auslander condition, since this algebra is a module finite over A.

On the other hand, when $s > -\mathrm{i}(K_A)$, the algebra $A \times K_A(-s)$ is AS-Gorenstein by [7, Proposition 1.5]. Putting together the results, for $s > -\mathrm{i}(K_A)$ we have that $A \times K_A(-s)$ is in fact graded Auslander–Gorenstein.

By Lemma 5.6, $A \times K_A(s)$ therefore has an Artinian ring of quotients, when $s > -\mathrm{i}(K_A)$.

Condition (3°). By Corollary 4.3, each σ-linear map of a finitely generated B-module is rational over k. So by Lemma 6.3, each finitely generated left A-module is rational over k. And k has characteristic zero, so since $k(t) \cap \mathbb{Q}((t)) = \mathbb{Q}(t)$ inside $k((t))$, Condition (3°) is satisfied. ▮

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