SEMI-SIMPLE ALGEBRAS AND GENERALIZED DERIVATIONS.*

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Introduction. Roughly speaking, a derivation of an algebra is the infinitesimal operation corresponding to an automorphism. This notion plays an important part in the study of the structure of Lie algebras and associative algebras, and its structural significance will be understood best by first recalling the connections between derivations and automorphisms in general.

The term "algebra" will be used to denote a linear set (not necessarily of finite dimension) over a field, in which an operation of "multiplication" is defined as usual, except that we do not require associativity. By a derivation of such an algebra we shall mean a linear mapping of the algebra into itself which possesses the formal properties of ordinary differentiation, i. e. a linear mapping $a \rightarrow D(a)$ of an algebra $\mathfrak{A}$ into itself such that for any $x, y \in \mathfrak{A}$ we have $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$, where the dot indicates the operation of "multiplication" in $\mathfrak{A}$.[1]

A simple example is given by the ring of all polynomials (with real coefficients) in one variable. We may regard this ring as an algebra (in the above general sense) over the field of real numbers. The mapping which sends every polynomial $P(x)$ into its derivative $D_x(P)$ is clearly a derivation in the above sense. Consider now the operator T_t, depending on the real parameter t, which is defined by the formula

$$T_t = 1 + tD_x + (1/2!)t^2D_x{}^2 + \cdots = \exp(tD_x).$$

Then, if $P(x)$ is any polynomial, we clearly have $T_t(P(x)) = P(x+t)$, and the family of mappings $\{T_t\}$ constitutes a continuous one parameter group of automorphisms.

Let us now consider a more general situation: Let $\mathfrak{A}$ be an arbitrary algebra of finite dimension over the field of the real numbers. The set of all non singular linear transformations of $\mathfrak{A}$ into itself can be given the structure of a Lie group G. The corresponding Lie algebra, $\mathfrak{L}$, is the set of all linear transformations of $\mathfrak{A}$, commutation being defined by the formula $T_1 \circ T_2 = T_2T_1 - T_1T_2$. If T is any element of $\mathfrak{L}$, we obtain a one parameter sub-

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[1] This definition and the elementary theory of derivations will be found in N. Jacobson, "Abstract derivation and Lie algebras," *Transactions of the American Mathematical Society*, vol. 42 (1937), p. 206.

group $g = \{g_t\}$ of G, whose tangent at the identity E of G is T, if we define $g_t = E + t \cdot T + (1/2!)t^2 \cdot T^2 + \cdots = \exp(t \cdot T)$. Since the condition that a transformation $g \in G$ be an automorphism can be expressed by a finite number of analytic equations in the coefficients of the matrix representing the element g, it is clear that the automorphisms of $\mathfrak{A}$ constitute a Lie-subgroup (i. e. a closed subgroup) H of G.

On the other hand, the derivations of $\mathfrak{A}$ constitute a subalgebra $\mathfrak{D}$ of $\mathfrak{L}$; for, if D_1, D_2 are any two derivations of $\mathfrak{A}$, so is $D_1 \circ D_2$. We shall show that $\mathfrak{D}$ is the subalgebra of $\mathfrak{L}$ which corresponds to the subgroup H of G.

Let $h = (h_t)$ be a one parameter subgroup of H which passes through the identify E for $t = 0$, and let $L \in \mathfrak{L}$ be its tangent at E, i. e. $L = \lim_{t \to 0} (h_t - E)/t$. If x, y are arbitrary elements of $\mathfrak{A}$, we have $h_t(x \cdot y) = h_t(x) \cdot h_t(y)$, which is easily seen to imply that $L(x \cdot y) = L(x) \cdot y + x \cdot L(y)$, i. e. that L is a derivation, or $L \in \mathfrak{D}$.

Conversely, let $D \in \mathfrak{D}$; the corresponding one parameter subgroup of G is given by $g_t = \exp(t \cdot D)$, and a direct computation shows that $g_t(x \cdot y) = g_t(x) \cdot g(y)$, i. e. that $g_t \in H$. This proves our assertion.

Let $\mathfrak{A}$ be an associative algebra with a unity element. If c is any regular element of $\mathfrak{A}$, the automorphism $a \to c.\, a.\, c^{-1}$ is called an inner automorphism. Algebraically, the inner automorphisms constitute an invariant subgroup K of the group H of all automorphisms of $\mathfrak{A}$. Although we cannot say in general that K is also a topological (Lie-) subgroup of H, it is evident that we can define a topology in K such that it becomes a Lie group K^* which is locally isomorphic with the group K, regarded as a subspace of H. It follows that there corresponds to K a certain ideal $\mathfrak{J}$ of the Lie algebra $\mathfrak{D}$ of H. In fact, the elements of $\mathfrak{J}$ are the tangents at E of those one parameter subgroups of H which are contained in K, and if k_t is such a subgroup the corresponding element $I \in \mathfrak{J}$ is given by the equation $I = \lim_{t \to 0} (k_t - E)/t$, so that we have $k_t(a) = a + t \cdot I(a) + o(t)$, for every element $a \in \mathfrak{A}$.

The group constituted by the regular elements of $\mathfrak{A}$ can be given the structure of a Lie group, and it can be shown that the one parameter subgroups, in a sufficiently small neighborhood of the identity e, are of the form $a_t = \exp(ta_o)$, where a_o is an element of $\mathfrak{A}$. Furthermore, it is easily seen that we can find an analytic mapping $k_t \to a_t$ of any given one parameter subgroup of K onto a one parameter subgroup through e of the group of regular elements of $\mathfrak{A}$ such that $k_t(a) = a_t \cdot a \cdot a_t^{-1}$ for every $a \in \mathfrak{A}$. For sufficiently small t we have therefore $k_t(a) = \exp(ta_o) \cdot a \cdot \exp(-ta_o) = a + t(a_o \cdot a - a \cdot a_o) + o(t)$, and a comparison of this with the expression above shows that we have $I(a) = a_o \cdot a - a \cdot a_o$. Derivations of this form are called inner derivations.

Suppose that I is any given inner derivation, i. e. $I(a) = a_0 \cdot a - a \cdot a_0$; the corresponding one parameter group of automorphisms is given by $k_t = \exp(tI)$. A straightforward computation shows that $k_t(a) = a_t \cdot a \cdot a_t^{-1}$, where $a_t = \exp(ta_0)$, i. e. the automorphisms corresponding to the inner derivations are the inner automorphisms. We have shown, therefore, that $\mathfrak{J}$ is the set of inner derivations of $\mathfrak{A}$.

We have a similar situation in the case of a Lie algebra. Let $\mathfrak{L}$ be any Lie algebra over the field of real numbers and let Γ be a connected and simply connected Lie group whose Lie algebra is $\mathfrak{L}$. An automorphism of Γ will induce an automorphism of $\mathfrak{L}$, and conversely. An automorphism of $\mathfrak{L}$ will be called an inner automorphism if it corresponds to an inner automorphism of Γ. As in the case of an associative algebra, it can be shown that to the invariant subgroup of the group of automorphisms which consists of the inner automorphisms there corresponds an ideal of the Lie algebra of the derivations of $\mathfrak{L}$, whose elements will be called inner derivations. A derivation D of the Lie algebra $\mathfrak{L}$ is an inner derivation if and only if there exists an element $l_0 \in \mathfrak{L}$ such that $D(l) = l \circ l_0$, for every $l \in \mathfrak{L}$. In the case of a general groundfield this is taken as the definition of an inner derivation.

We shall be concerned with the study of the behavior of Lie algebras and associative algebras with regard to linear mappings of the derivational type which map a given algebra $\mathfrak{A}$, not necessarily into itself, but, more generally, into some algebra $\mathfrak{B}$ containing $\mathfrak{A}$ as a subalgebra. These "generalized derivations," as we should expect from the above considerations, are found to be significant for the structure of an algebra. In fact, we shall obtain a characterization of semi-simple Lie algebras and semi-simple associative algebras (over a non-modular field) in terms of their generalized derivations. In this respect there is a very close analogy between Lie algebras and associative algebras, which we shall exhibit by treating the two cases side by side.

I have found it necessary to include proofs of a number of known results on representations of semi-simple Lie algebras, some of which (like the proofs of Whitehead's lemmas) cannot be found in the literature. For these proofs, which served me as a guide in the present investigation, I am indebted to Professor C. Chevalley of Princeton University.

1. Representations. For the sake of future reference we briefly review some fundamental results of the theory of representations of associative and Lie algebras.

Definition 1. 1. A representation is said to be completely decomposable if the corresponding representation space is the direct sum of irreducible invariant subspaces.

Proposition 1.1. A representation, with the representation space $\mathfrak{P}$, is completely decomposable if and only if for every invariant subspace $\mathfrak{Q}$ of $\mathfrak{P}$ there exists another invariant subspace $\mathfrak{Q}'$ such that $\mathfrak{P}$ is the direct sum of $\mathfrak{Q}$ and $\mathfrak{Q}'$.

The proof of this proposition is very easy and may be omitted here.

THEOREM 1.1. *The necessary and sufficient condition for an associative algebra to be semi-simple is that every representation be completely decomposable.*

This result is classical, and no proof need be given here.

THEOREM 1.2. *A sufficient condition for a Lie algebra to be semi-simple is that every representation be completely decomposable.*

Proof. We must show that a Lie algebra $\mathfrak{L}$ satisfying this condition cannot possess a (non zero) solvable ideal.

Let $\mathfrak{H}$ be any solvable ideal of $\mathfrak{L}$. Since in particular the adjoint representation of $\mathfrak{L}$ is completely decomposable we can find another ideal $\mathfrak{H}^*$ of $\mathfrak{L}$ such that $\mathfrak{L}$ is the direct sum of $\mathfrak{H}$ and $\mathfrak{H}^*$. Hence it is clear that every representation of $\mathfrak{H}$ can be extended to a representation of $\mathfrak{L}$ by representing the elements of $\mathfrak{H}^*$ by zero matrices. It follows that $\mathfrak{H}$ still has the property that all representations are completely decomposable. Hence it will suffice to show that every solvable Lie algebra $\mathfrak{H}$ of dimension $d > 0$ possesses a representation which is not completely decomposable.

Now, if $\mathfrak{H}$ is solvable it contains an ideal $\mathfrak{H}^*$ of dimension $d - 1$. Let h_1 be an element of $\mathfrak{H}$ which does not belong to $\mathfrak{H}^*$; then every element $h \in \mathfrak{H}$ can be written in the form $h = \alpha(h)h_1 + h^*$, where $\alpha(h)$ is a linear function on $\mathfrak{H}$ (with values in the groundfield) and $h^* \in \mathfrak{H}^*$. Let $\mathfrak{P}$ be a linear space of dimension 2 over the groundfield, and let the vectors $\vec{e}_1$, $\vec{e}_2$ constitute a basis for $\mathfrak{P}$.

If we write $P_h(\vec{e}_1) = \alpha(h)\vec{e}_1$; $P_h(\vec{e}_2) = \alpha(h)(\vec{e}_1 + \vec{e}_2)$, the mapping $h \rightarrow P_h$ is clearly a representation of $\mathfrak{H}$ in which $\mathfrak{H}^*$ is annulled. The subspace spanned by $\vec{e}_1$ is invariant, but it is clearly the only invariant subspace of dimension 1. It follows that the representation P is not completely decomposable. This completes the proof.

THEOREM 1.3. (Cartan). *Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic zero. If there exists a faithful representation $l \rightarrow P_l$ of $\mathfrak{L}$, such that* $\mathrm{Sp}(P_l^2) = 0$ *for every $l \in \mathfrak{L}$, (where* $\mathrm{Sp}(A)$ *denotes the trace of the matrix* A*), then $\mathfrak{L}$ is solvable.*

THEOREM 1.4. (Cartan). *Let $\mathfrak{K}$ be a semi-simple Lie algebra over a field of characteristic zero. Let $k \rightarrow P_k$ be a faithful representation of $\mathfrak{K}$, and suppose that $k_1, k_2, \cdots, k_n$ constitute a basis for $\mathfrak{K}$. Then the determinant* $|\operatorname{Sp}(P_{k_i}P_{k_j})| \neq 0$.

Both these results are well known.

THEOREM 1.5. *If $\mathfrak{K}$ is a semi-simple Lie algebra over a field of characteristic 0, the adjoint representation A of $\mathfrak{K}$ is completely decomposable.*

Proof. Let $\mathfrak{H}$ be any ideal of $\mathfrak{K}$. We have to show that there exists another ideal $\mathfrak{H}^*$ of $\mathfrak{K}$ such that $\mathfrak{K}$ is the direct sum of $\mathfrak{H}$ and $\mathfrak{H}^*$. Let $\mathfrak{H}^*$ be the set of all elements h^* of $\mathfrak{K}$ such that $\operatorname{Sp}(A_h A_{h^*}) = 0$ for all $h \in \mathfrak{H}$. $\mathfrak{H}^*$ is an ideal of $\mathfrak{K}$, and since the adjoint representation of a semi-simple Lie algebra is faithful we may conclude from Theorem 1.3 that $\mathfrak{H}^* \cap \mathfrak{H}$ is a solvable ideal. Hence we must have $\mathfrak{H}^* \cap \mathfrak{H} = \{0\}$. A dimension argument now shows that $\mathfrak{K}$ is the direct sum of $\mathfrak{H}$ and $\mathfrak{H}^*$.

THEOREM 1.6. *Let $\mathfrak{A}$ be a semi-simple associative algebra over a field of characteristic* 0. *Let $(a_1, a_2, \cdots, a_n)$ be a basis of $\mathfrak{A}$, and suppose that $P_{a_1}, P_{a_2}, \cdots, P_{a_n}$ are the matrices corresponding to the elements a_i of this basis in a faithful representation P of $\mathfrak{A}$. Then* $|\operatorname{Sp}(P_{a_i}P_{a_j})| \neq 0$.

Proof. Let us write $\mathfrak{A}$ as a direct sum of two sided simple ideals, say $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \cdots + \mathfrak{A}_k$. We have then k sets of indices, $I_1, I_2, \cdots, I_k$ such that $a_i \in \mathfrak{A}_\nu$ for $i \in I_\nu$. Now $P_{a_i}P_{a_j} = 0$ if i and j do not belong to the same set I_ν, whence it is clear that

$$|\operatorname{Sp}(P_{a_i}P_{a_j})|_{(i,j=1,2\ldots n)} = \prod_{\nu=1}^{k} |\operatorname{Sp}(P_{a_i}P_{a_j})|_{(i,j\in I_\nu)}.$$

Every irreducible representation of an algebra is equivalent to an irreducible representation contained in the adjoint representation, and a simple algebra possesses only one irreducible representation (to within equivalence). Hence, if we denote the matrices in the adjoint representation by A, we have, for $i, j \in I_\nu$, $\operatorname{Sp}(P_{a_i}P_{a_j}) = r_\nu \cdot \operatorname{Sp}(A_{a_i}A_{a_j})$, where r_ν is a positive rational number depending only on ν. It follows that $|\operatorname{Sp}(P_{a_i}P_{a_j})|$ is a (non zero) multiple of $|\operatorname{Sp}(A_{a_i}A_{a_j})|$, which is known to be different from zero for a semi-simple algebra. (Cf. Deuring, "Algebren," *Ergebnisse der Mathematik*, vol. 4, part 1, pp. 33-34).

THEOREM 1.7. (Casimir). *Let $\mathfrak{L}$ be a semi-simple Lie algebra over a field of characteristic zero. Let P be any* (*non null*) *representation of $\mathfrak{L}$, and let $\mathfrak{H}^*$ be the kernel of this representation* (i. e. *the ideal of $\mathfrak{L}$ which is annulled* *in the representation P). Let $\mathfrak{H}$ be another ideal such that $\mathfrak{L}$ is the direct sum of $\mathfrak{H}$ and $\mathfrak{H}^*$; ($\mathfrak{H}$ exists, by Theorem 1.5). If $(h_1, h_2, \cdots, h_k)$ is any basis of $\mathfrak{H}$ there exists a dual basis $(\bar{h}_1 \cdot h_2, \cdots, \bar{h}_k)$ of $\mathfrak{H}$ such that $\mathrm{Sp}(P_{h_i} P_{\bar{h}_j}) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol ($= 0$ for $i \neq j$, and $= 1$ for $i = j$).*

Then, if l is any element of $\mathfrak{L}$, and $h_i \circ l = \sum_{j=1}^{k} \gamma_{ij} h_j$, we have $l \circ \bar{h}_i = \sum_{j=1}^{k} \gamma_{ji} \bar{h}_j$. If we define a transformation Γ by the equation $\Gamma = \sum_{i=1}^{k} P_{h_i} P_{\bar{h}_i}$, we have $\Gamma P_l = P_l \Gamma$ for every $l \in \mathfrak{L}$.

Proof. The existence of the dual basis $(\bar{h}_i)$ follows immediately from Theorem 1.4 since P gives a faithful representation of $\mathfrak{H}$. If $h_i \circ l = \sum_{j=1}^{k} \gamma_{ij} h_j$ and

$l \circ \bar{h}_j = \sum_{i=1}^{k} \epsilon_{ij} \bar{h}_i$, we have $\gamma_{ij} = \mathrm{Sp}(P_{h_i \circ l} P_{\bar{h}_j}) = \mathrm{Sp}(P_{h_i} P_{l \circ \bar{h}_j}) = \epsilon_{ij}$. Finally,

$$\Gamma P_l - P_l \Gamma = \sum_{i=1}^{k} (P_{h_i} P_{\bar{h}_i} P_l - P_l P_{h_i} P_{\bar{h}_i}) = \sum_{i=1}^{k} (P_{h_i} P_{l \circ \bar{h}_i} - P_{h_i \circ l} P_{\bar{h}_i})$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} (\gamma_{ji} P_{h_i} P_{\bar{h}_j} - \gamma_{ij} P_{h_j} P_{\bar{h}_i}) = 0.$$

THEOREM 1.8. *Let $\mathfrak{A}$ be a semi-simple associative algebra over a field of characteristic zero. Let P be any (non null) representation of $\mathfrak{A}$, and let $\mathfrak{H}^*$ be the kernel of this representation. $\mathfrak{H}^*$ is a two sided ideal of $\mathfrak{A}$, and there exists another two sided ideal $\mathfrak{H}$ such that $\mathfrak{A}$ is the direct sum of $\mathfrak{H}$ and $\mathfrak{H}^*$. To any basis $(h_1, h_2, \cdots, h_k)$ of $\mathfrak{H}$ we can find a dual basis $(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_k)$ such that $\mathrm{Sp}(P_{h_i} P_{\bar{h}_j}) = \delta_{ij}$. If a is any element of $\mathfrak{A}$, and $h_i \cdot a = \sum_{j=1}^{k} \gamma_{ij} h_j$, we have $a \cdot \bar{h}_i = \sum_{j=1}^{k} \gamma_{ji} \bar{h}_j$. The transformation $\Gamma = P_h$, where $h = \sum \bar{h}_i h_i$, commutes with all the transformations of the representation P.*

Proof. The existence of the dual basis follows from Theorem 1.6. The remaining relations are easily verified as in the proof of Theorem 1.7.

2. Whitehead's lemmas and their analogues for associative algebras. It is known that every representation of a semi-simple Lie algebra over a field of characteristic zero is completely decomposable. This result follows from a lemma of Whitehead's which can be stated as follows:

THEOREM 2.1. *Let $\mathfrak{L}$ be a semi-simple Lie algebra over a field of characteristic zero. Let $\mathfrak{P}$ be a representation module of $\mathfrak{L}$, and denote the transform* *of a vector* $\vec{e} \in \mathfrak{P}$ *by the transformation corresponding to an element* $l \in \mathfrak{L}$ *by* $l \cdot \vec{e}$. *Suppose we have a linear mapping* $l \rightarrow \vec{e}(l)$ *of* $\mathfrak{L}$ *into* $\mathfrak{P}$ *such that* $\vec{e}(l \circ m) = m \cdot \vec{e}(l) - l \cdot \vec{e}(m)$ *for all* $l, m \in \mathfrak{L}$. *Then there exists a fixed vector* $\vec{e}_0$ *in* $\mathfrak{P}$ *such that* $\vec{e}(l) = l \cdot \vec{e}_0$ *for every* $l \in \mathfrak{L}$.

Proof. If $\mathfrak{P}$ is annulled by $\mathfrak{L}$ we may take $\vec{e}_0 = 0$, and there is nothing to prove. If not, we employ the notations and results of Theorem 1. 7 and define a vector $\vec{f}_0 = \sum_{i=1}^{k} h_i \cdot \vec{e}(\bar{h}_i)$. We have then

$$l \cdot \vec{f}_0 = \sum_{i=1}^{k} l \cdot (h_i \cdot \vec{e}(\bar{h}_i)) = \sum_{i=1}^{k} [(h_i \circ l) \cdot \vec{e}(\bar{h}_i) + h_i \cdot (l \cdot \vec{e}(\bar{h}_i))]$$
$$= \sum_{i=1}^{k} [h_i \cdot \vec{e}(l \circ \bar{h}_i) + h_i \cdot (l \cdot \vec{e}(\bar{h}_i))] = \sum_{i=1}^{k} h_i \cdot (\bar{h}_i \cdot \vec{e}(l)) = \Gamma\{\vec{e}(l)\}.$$

We are now in a position to prove our theorem by an induction on the dimension d of the representation space $\mathfrak{P}$. By the remark made at the outset we may suppose that $d > 0$, and that the theorem holds for every representation space of dimension $< d$. If our representation is not the null representation (which we may suppose), we have $\mathrm{Sp}(\Gamma) = k \neq 0$, since the groundfield is of characteristic 0. Hence $\Gamma \neq 0$; therefore, if $\mathfrak{Q}$ is the linear subspace formed by the vectors $\vec{f} \in \mathfrak{P}$ such that $\Gamma(\vec{f}) = 0$, we have $\mathfrak{Q} \neq \mathfrak{P}$. If $\mathfrak{Q} = \{0\}$, Γ is a linear isomorphism of $\mathfrak{P}$ and therefore possesses an inverse Γ^{-1} which also commutes with all the transformations of our representation. Hence, if we put $\vec{e}_0 = \Gamma^{-1}(\vec{f}_0)$, we have $l \cdot \vec{e}_0 = l \cdot \{\Gamma^{-1}(\vec{f}_0)\} = \Gamma^{-1}(l \cdot \vec{f}_0) = \vec{e}(l)$, as required by the theorem. We may therefore suppose that $\mathfrak{Q} \neq \{0\}$. Now $\mathfrak{Q}$ is an invariant subspace of $\mathfrak{P}$ since $\Gamma(\vec{f}) = 0$ implies that $\Gamma(l \cdot \vec{f}) = l \cdot \Gamma(\vec{f}) = 0$. Hence our representation in $\mathfrak{P}$ induces a representation in $\mathfrak{P}/\mathfrak{Q}$, and, if $\vec{E}(l)$ denotes the coset, modulo $\mathfrak{Q}$, of $\vec{e}(l)$, the mapping $l \rightarrow \vec{E}(l)$ of $\mathfrak{L}$ into $\mathfrak{Q}/\mathfrak{P}$ still satisfies the conditions of our theorem. Since $\mathfrak{P}/\mathfrak{Q}$ is of dimension $< d$, it follows from our inductive hypothesis that there exists an element $\vec{E}_1 \in \mathfrak{P}/\mathfrak{Q}$ such that $\vec{E}(l) = l \cdot \vec{E}_1$ for every $l \in \mathfrak{L}$. Let $\vec{e}_1$ be any vector in this coset $\vec{E}_1$; if we put $\vec{f}(l) = \vec{e}(l) - l \cdot \vec{e}_1$, the mapping $l \rightarrow \vec{f}(l)$ is a linear mapping of $\mathfrak{L}$ into $\mathfrak{Q}$ such that $\vec{f}(l \circ m) = m \cdot \vec{f}(l) - l \cdot \vec{f}(m)$. Since $\mathfrak{Q}$ is also of dimension $< d$ there exists a vector $\vec{e}_2 \in \mathfrak{Q}$ such that $\vec{f}(l) = l \cdot \vec{e}_2$ for all $l \in \mathfrak{L}$. The vector $\vec{e}_0 = \vec{e}_1 + \vec{e}_2$ will be the required vector, and our theorem is proved.

Before stating the analogue of Theorem 2.1 for associative algebras we make the following definition:

Definition 2.1. Let $\mathfrak{A}$ be an associative algebra over a field $\mathfrak{F}$. Let $\mathfrak{P}$ be a linear set over $\mathfrak{F}$ which is at the same time an $\mathfrak{A}$-right and an $\mathfrak{A}$-left module, the operations being such that (apart from the usual associativity requirements for one sided modules) we have $a \cdot (\vec{e} \cdot b) = (a \cdot \vec{e}) \cdot b$ for all $a, b \in \mathfrak{A}$ and every $\vec{e} \in \mathfrak{P}$. Then $\mathfrak{P}$ is called a two sided $\mathfrak{A}$-module.

THEOREM 2.2. *Let $\mathfrak{A}$ be a semi-simple associative algebra over a field $\mathfrak{F}$ of characteristic* 0. *Let $\mathfrak{P}$ be a two sided $\mathfrak{A}$-module over $\mathfrak{F}$, and suppose we have a linear mapping $a \rightarrow \vec{e}(a)$ of $\mathfrak{A}$ into $\mathfrak{P}$ such that $\vec{e}(a \cdot b) = a \cdot \vec{e}(b) + \vec{e}(a) \cdot b$. Then there exists $\vec{e}_0 \in \mathfrak{P}$ such that $\vec{e}(a) = a \cdot \vec{e}_0 - \vec{e}_0 \cdot a$.*

Proof. In the notation of Theorem 1.8 we have:

$$\begin{aligned} h \cdot \vec{e}(a) &= \sum_{i=1}^{k} \bar{h}_i h_i \cdot \vec{e}(a) = \sum_{i=1}^{k} \bar{h}_i \cdot [\vec{e}(h_i a) - \vec{e}(h_i) \cdot a] \\ &= \sum_{i=1}^{k} a\bar{h}_i \cdot \vec{e}(h_i) - \sum_{i=1}^{k} \bar{h}_i \cdot \vec{e}(h_i) \cdot a = a \cdot \vec{f}_0 - \vec{f}_0 \cdot a, \end{aligned}$$

where $\vec{f}_0 = \sum_{i=1}^{k} \bar{h}_i \cdot \vec{e}(h_i)$.

The proof now runs like that of Theorem 2.1; we may assume that the left representation in $\mathfrak{P}$ is not null, for otherwise we could operate in the same manner from the right, while—if both the left and the right representations were null—we could take $\vec{e}_0 = 0$. The subspace $\mathfrak{Q}$, defined as before, is invariant for both the right and the left representation, and the induction goes exactly as before.

An interesting application of Theorem 2.2 is the following: Let P_n be the algebra of all matrices of order n over a field P of characteristic 0. Let $\mathfrak{A}$ be a semi-simple subalgebra and U an arbitrary matrix of P_n. The elements of the form $UA - AU$, where $A \in \mathfrak{A}$, together with all products of such elements by elements of $\mathfrak{A}$, span a certain linear subspace of P_n, which we may regard as a two sided $\mathfrak{A}$-module; $\mathfrak{M}$, say. It is easily verified that the mapping $A \rightarrow UA - AU$ of $\mathfrak{A}$ into $\mathfrak{M}$ satisfies the conditions of Theorem 2.2. It follows that there exists a matrix $M \in \mathfrak{M}$ such that $U - M$ commutes with all the matrices of $\mathfrak{A}$.

THEOREM 2.3. *If an associative algebra $\mathfrak{A}$ possesses the property of Theorem 2.2 then every representation of $\mathfrak{A}$ is completely decomposable.*

Proof. Let $\mathfrak{P}$ be the representation space of a representation P of $\mathfrak{A}$. If $\mathfrak{Q}$ is any invariant subspace of $\mathfrak{P}$ we have to show that there exists another invariant subspace $\mathfrak{R}$ such that $\mathfrak{P} = \mathfrak{Q} + \mathfrak{R}$. Let us choose a linear subspace $\mathfrak{T}$ (not necessarily invariant, of course) such that $\mathfrak{P} = \mathfrak{Q} + \mathfrak{T}$. Let P_a be the transformation of $\mathfrak{P}$ which corresponds to the element a of $\mathfrak{A}$. Let $\vec{t}$ denote an arbitrary vector in $\mathfrak{T}$. According to the decomposition $\mathfrak{P} = \mathfrak{Q} + \mathfrak{T}$ we write: $P_a(\vec{t}) = Q_a(\vec{t}) + T_a(\vec{t})$, where $Q_a(\vec{t}) \in \mathfrak{Q}$, $T_a(\vec{t}) \in \mathfrak{T}$. It is clear that the mapping $\vec{t} \to Q_a(\vec{t})$ is a linear mapping of $\mathfrak{T}$ into $\mathfrak{Q}$ and the mapping $\vec{t} \to T_a(\vec{t})$ a linear mapping of $\mathfrak{T}$ into itself. From the fact that P is a representation we obtain the following relations:

$$\begin{aligned} P_{ab}(\vec{t}) &= Q_{ab}(\vec{t}) + T_{ab}(\vec{t}) = P_aP_b(\vec{t}) = P_a(Q_b(\vec{t}) + T_b(\vec{t})) \\ &= (P_aQ_b(\vec{t}) + Q_aT_b(\vec{t})) + T_aT_b(\vec{t}). \end{aligned}$$

Since the sum $\mathfrak{Q} + \mathfrak{T}$ is direct it follows that we must have

$$Q_{ab} = P_aQ_b + Q_aT_b; \qquad T_{ab} = T_aT_b.$$

The set of all linear mappings Q of $\mathfrak{T}$ into $\mathfrak{Q}$ forms a linear space over our groundfield: we denote this linear space by $\mathfrak{S}$. Now we may regard $\mathfrak{S}$ as a two-sided $\mathfrak{A}$-module if we define the operations by elements $a \in \mathfrak{A}$ on elements $Q \in \mathfrak{S}$ by the formulae: $a \cdot Q = P_aQ$, $Q \cdot a = QT_a$. Consider the linear mapping $a \to Q_a$ of $\mathfrak{A}$ into $\mathfrak{S}$. We have $Q_{ab} = P_aQ_b + Q_aT_b = a \cdot Q_b + Q_a \cdot b$. Now, since $\mathfrak{A}$ possesses the property of Theorem 2. 2, there exists an element $Q_0 \in \mathfrak{S}$ such that $Q_a = Q_0T_a - P_aQ_0$.

Consider now the mapping $\vec{t} \to \vec{t} + Q_0(\vec{t})$ of $\mathfrak{T}$ into $\mathfrak{P}$. This maps $\mathfrak{T}$ onto some linear subspace $\mathfrak{R}$ of $\mathfrak{P}$. We have $P_a(\vec{t} + Q_0(\vec{t})) = P_a(\vec{t}) + P_aQ_0(\vec{t})$ $= P_a(\vec{t}) + Q_0T_a(\vec{t}) - Q_a(\vec{t}) = T_a(\vec{t}) + Q_0T_a(\vec{t})$, which shows that $\mathfrak{R}$ is an invariant subspace of $\mathfrak{P}$. Furthermore, $\vec{t} + Q_0(\vec{t}) = 0$ only if $\vec{t} = \theta$, so that our mapping $\vec{t} \to \vec{t} + Q_0(\vec{t})$ is a linear isomorphism, whence we have that dim. $\mathfrak{R} =$ dim. $\mathfrak{T}$. Finally, if $\vec{e} \in \mathfrak{Q} \cap \mathfrak{R}$, i. e. if $\vec{e} \in \mathfrak{Q}$, and $\vec{e} = \vec{t} + Q_0(\vec{t})$ for some $\vec{t} \in \mathfrak{T}$, we must have $\vec{t} = 0$, and hence also $\vec{e} = 0$, whence $\mathfrak{Q} \cap \mathfrak{R} = \{0\}$. It follows that $\mathfrak{P}$ is the direct sum of $\mathfrak{Q}$ and $\mathfrak{R}$, and our theorem is proved.

The proof of Theorem 2. 3 we have just given is the precise analogue of a known proof for Theorem 2. 4 below:

THEOREM 2. 4. *If a Lie algebra $\mathfrak{L}$ possesses the property of Theorem 2. 1 then every representation of $\mathfrak{L}$ is completely decomposable.*

By Theorems 2.1, 2.4, and 1.2 we have now:

THEOREM 2.5. *The necessary and sufficient condition for a Lie algebra $\mathfrak{L}$ over a field of characteristic zero to be semi-simple is that every representation of $\mathfrak{L}$ be completely decomposable.*

We shall pursue the analogy between Lie algebras and associative algebras a little further: another lemma of Whitehead, which is used in the proof of Levi's theorem on the decomposition of a Lie algebra into the direct sum of its maximal solvable ideal and a semi-simple subalgebra, can be stated as follows:

THEOREM 2.6. *Let $\mathfrak{P}$ be a representation space of a semi-simple Lie algebra $\mathfrak{L}$ over a field of characteristic 0. Suppose we have a bilinear mapping $(x,y) \to \vec{f}(x,y)$ of $\mathfrak{L} \times \mathfrak{L}$ into $\mathfrak{P}$ such that*

1) $\vec{f}(x,y) + \vec{f}(y,x) = 0,$

2) $x \cdot \vec{f}(y,z) + y \cdot \vec{f}(z,x) + z \cdot \vec{f}(x,y) = \vec{f}(y \circ x, z) + \vec{f}(z \circ y, x) + \vec{f}(x \circ z, y).$

Then there exists a linear mapping $x \to \vec{e}(x)$ of $\mathfrak{L}$ into $\mathfrak{P}$ such that $\vec{f}(x,y) = x \cdot \vec{e}(y) - y \cdot \vec{e}(x) + \vec{e}(x \circ y)$.

We shall not write out the proof of this theorem. It is very similar to that of Theorem 2.7 below and requires the same methods we have employed already in the proof of Theorem 2.1.

There is an interesting analogue of this theorem for associative algebras; it is as follows:

THEOREM 2.7. *Let $\mathfrak{A}$ be a semi-simple associative algebra over a field of characteristic 0. Let $\mathfrak{P}$ be a two-sided $\mathfrak{A}$-module and suppose we have a bilinear mapping $(a,b) \to \vec{f}(a,b)$ of $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{P}$ such that $a \cdot \vec{f}(b,c) + \vec{f}(a,bc) = \vec{f}(a,b) \cdot c + \vec{f}(ab,c)$. Then there exists a linear mapping $a \to \vec{e}(a)$ of $\mathfrak{A}$ into $\mathfrak{P}$ such that $\vec{f}(a,b) = a \cdot \vec{e}(b) + \vec{e}(a) \cdot b - \vec{e}(ab)$.*

Proof. In the notation of Theorem 1.8 we have: $h \cdot \vec{f}(a,b) = \sum_{i=1}^{k} \bar{h}_i h_i \cdot \vec{f}(a,b) = \sum_{i=1}^{k} \bar{h}_i \cdot (\vec{f}(h_i,a) \cdot b + \vec{f}(h_i a, b) - \vec{f}(h_i, ab)) = \vec{e}_1(a) \cdot b - \vec{e}_1(ab) + \sum_{i=1}^{k} \bar{h}_i \cdot \vec{f}(h_i a, b)$, where we have put $\vec{e}_1(a) = \sum_{i=1}^{k} \bar{h}_i \cdot \vec{f}(h_i, a)$. Now, as a consequence of the properties of the bases (h_i) and $(\bar{h}_i)$, we have $\sum_{i=1}^{k} \bar{h}_i \cdot \vec{f}(h_i a, b) = \sum_{i=1}^{k} a\bar{h}_i \cdot \vec{f}(h_i, b) = a \cdot \vec{e}_1(b)$. Hence $h \cdot \vec{f}(a, b) = a \cdot \vec{e}_1(b) + \vec{e}_1(a) \cdot b - \vec{e}_1(ab)$.

Let now $\mathfrak{Q}$ be the linear subspace of $\mathfrak{P}$ formed by those elements $\vec{g}$ for which $h \cdot \vec{g} = 0$. Suppose first that $\mathfrak{Q} = \{0\}$. Then the mapping $\vec{e} \rightarrow h \cdot \vec{e} = \Gamma(\vec{e})$ possesses an inverse Γ^{-1}, which commutes with all the elements of $\mathfrak{A}$, regarded as operators on $\mathfrak{P}$, and we may then take $\vec{e}(a) = \Gamma^{-1}\{\vec{e}_1(a)\}$.

Next, let us suppose that both the left representation and the right representation are null. Our condition on the mapping $\vec{f}$ then becomes: $\vec{f}(a, bc) = \vec{f}(ab, c)$. Let $(\mathfrak{A}, \mathfrak{P})$ denote the direct sum of the linear spaces $\mathfrak{A}$ and $\mathfrak{P}$. To every element a of $\mathfrak{A}$ we have a transformation of $(\mathfrak{A}, \mathfrak{P})$ which is defined by the formula $(b, \vec{e}) \rightarrow a \cdot (b, \vec{e}) = (ab, \vec{f}(a, b))$; it is easily verified that this defines a representation of $\mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{P})$ as representation space. The subspace $(0, \mathfrak{P})$ is clearly an invariant subspace. Since every representation of is completely decomposable there must exist another invariant subspace $\mathfrak{Q}^*$ of $(\mathfrak{A}, \mathfrak{P})$ such that $(\mathfrak{A}, \mathfrak{P})$ is the direct sum of $(0, \mathfrak{P})$ and $\mathfrak{Q}^*$. Let us write, according to this decomposition, $(a, 0) = (0, \vec{e}(a)) + q^*(a)$. We have then

$$b \cdot (a, 0) = b \cdot (0, \vec{e}(a)) + b \cdot q^*(a), \text{ i.e. } (ba, \vec{f}(b, a)) = b \cdot q^*(a) \in \mathfrak{Q}^*.$$

On the other hand, we have

$$(ba, \vec{f}(b, a)) = (ba, 0) + (0, \vec{f}(b, a)) = (0, \vec{e}(ba)) + q^*(ba) + (0, \vec{f}(b, a)).$$

It follows that $(0, \vec{e}(ba)) + (0, \vec{f}(b, a)) = 0$; the mapping $a \rightarrow -\vec{e}(a)$ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{P}$ such that $\vec{f}(a, b) = -\vec{e}(ab)$, and our theorem is verified for this case. The theorem is trivial if $\mathfrak{P}$ is of dimension 0, and we can now proceed by induction as in the proof of Theorem 2. 1, since we shall have $\mathfrak{Q} \neq \mathfrak{P}$ for either the left representation or for the right representation, unless we have the case just disposed of. As an application of Theorem 2. 4 we give a proof of the well-known result that every associative algebra over a field of characteristic zero can be represented as the direct sum of its radical and a semi-simple subalgebra.[2]

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[2] J. H. C. Whitehead has communicated a proof of this result to N. Jacobson, which utilizes the same ideas as the proof given here.

Let $\mathfrak{A}$ be any associative algebra over a field of characteristic zero, and let $\mathfrak{N}$ denote the radical of $\mathfrak{A}$. We may clearly suppose that $\mathfrak{N} \neq \{0\}$. If we have $\mathfrak{N}^2 \neq \{0\}$, $\mathfrak{A}/\mathfrak{N}^2$ is of smaller dimension than $\mathfrak{A}$. Hence, if we suppose that the result has already been proved for all algebras of smaller dimension than $\mathfrak{A}$ (it is obvious for dimension one), we have that $\mathfrak{A}/\mathfrak{N}^2 = \mathfrak{S}^* + \mathfrak{A}^*_1$, where $\mathfrak{S}^*$ is the radical of $\mathfrak{A}/\mathfrak{N}^2$ and $\mathfrak{A}^*_1$ is a semi-simple subalgebra of $\mathfrak{A}/\mathfrak{N}^2$ which is isomorphic with $(\mathfrak{A}/\mathfrak{N}^2)/\mathfrak{S}^*$. But it is clear that $\mathfrak{S}^* = \mathfrak{N}/\mathfrak{N}^2$, and hence that $\mathfrak{A}^*_1$ is isomorphic with $\mathfrak{A}/\mathfrak{N}$. Now, there clearly exists a subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ such that $\mathfrak{A}^*_1 = \mathfrak{A}_1/\mathfrak{N}^2$, and we have then $\mathfrak{A}/\mathfrak{N}^2 = \mathfrak{N}/\mathfrak{N}^2 + \mathfrak{A}_1/\mathfrak{N}^2$. Since $\mathfrak{N}/\mathfrak{N}^2 \neq \{0\}$, we have dim. $\mathfrak{A}_1 <$ dim. $\mathfrak{A}$. Hence there exists a subalgebra $\mathfrak{A}_2$ of $\mathfrak{A}_1$, i. e. also of $\mathfrak{A}$, which is isomorphic with $\mathfrak{A}_1/\mathfrak{N}^2$, and hence also with $\mathfrak{A}/\mathfrak{N}$, such that $\mathfrak{A}_1 = \mathfrak{N}^2 + \mathfrak{A}_2$, and we have $\mathfrak{A} = \mathfrak{N} + \mathfrak{A}_2$.

The only difficulty arises if $\mathfrak{N}^2 = \{0\}$. In this case we note that $\mathfrak{N}$ may be regarded as a two-sided $\mathfrak{A}/\mathfrak{N}$-module, the operation with elements of $\mathfrak{A}/\mathfrak{N}$ on $\mathfrak{N}$ being that of multiplication of the elements of $\mathfrak{N}$ by an arbitrary representant in $\mathfrak{A}$ of an element of $\mathfrak{A}/\mathfrak{N}$. (Since $\mathfrak{N}^2 = \{0\}$, all representants of a given coset, modulo $\mathfrak{N}$, will induce the same transformation of $\mathfrak{N}$.)

Let us choose a linear subspace $\mathfrak{S}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{N} + \mathfrak{S}$. We have then a one to one correspondence $a \leftrightarrow a^*$ between the elements a of $\mathfrak{A}/\mathfrak{N}$ and the elements a^* of $\mathfrak{S}$. Set $\tilde{f}(a, b) = (ab)^* - a^*b^*$. Then $\tilde{f}(a, b) \in \mathfrak{N}$ for any $a, b \in \mathfrak{A}/\mathfrak{N}$ and gives a bi-linear mapping of $\mathfrak{A}/\mathfrak{N} \times \mathfrak{A}/\mathfrak{N}$ into $\mathfrak{N}$. Now, we have $\tilde{f}(a, b) \cdot c + \tilde{f}(ab, c) = ((ab)^* - a^*b^*) \cdot c + (abc)^* - (ab)^*c^*$, $a \cdot \tilde{f}(b, c) + \tilde{f}(ab, c) = a \cdot ((bc)^* - b^*c^*) + (abc)^* - a^*(bc)^*$. But if $a \in \mathfrak{A}/\mathfrak{N}$, $\tilde{r} \in \mathfrak{N}$, we have $a \cdot \tilde{r} = a^*\tilde{r}$. Hence we have $\tilde{f}(a, b) \cdot c + \tilde{f}(ab, c) = (abc)^* - a^*b^*c^* = a \cdot \tilde{f}(b, c) + \tilde{f}(a, bc)$, i. e. $\tilde{f}$ satisfies the conditions of Theorem 2. 4. It follows that there exists a mapping $a \rightarrow \tilde{a}$ of $\mathfrak{A}/\mathfrak{N}$ **into** $\mathfrak{N}$, such that $\tilde{f}(a, b) = a^*\tilde{b} + \tilde{a}b^* - \widetilde{ab}$, i. e. $(ab)^* - a^*b^* = a^*\tilde{b} + \tilde{a}b^* - \widetilde{ab}$. If we set $\bar{a} = a^* + \tilde{a}$ we have, therefore, $\bar{a}\bar{b} = a^*b^* + a^*\tilde{b} + \tilde{a}b^* = (ab)^* + \widetilde{ab} = \overline{ab}$, whence our result follows immediately.

3. Derivations.

Definition 3. 1. Let $\mathfrak{A}$ be any algebra (not necessarily associative); let $\mathfrak{B}$ be another algebra (of the same kind) which contains $\mathfrak{A}$ as a subalgebra. A linear mapping $a \rightarrow D(a)$ of $\mathfrak{A}$ into $\mathfrak{B}$ such that, for any elements $x, y \in \mathfrak{A}$, we have $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$ is called a derivation of $\mathfrak{A}$ into $\mathfrak{B}$. A derivation of $\mathfrak{A}$ into itself will sometimes be simply called a derivation of $\mathfrak{A}$.

If $\mathfrak{A}$ is an associative algebra contained in another associative algebra $\mathfrak{B}$, it is easy to verify that the mapping $a \rightarrow ba - ab$, where b is any fixed element of $\mathfrak{B}$, is a derivation of $\mathfrak{A}$ into $\mathfrak{B}$. This derivation can trivially be extended to a derivation of $\mathfrak{B}$ into itself.

Similarly, if $\mathfrak{L}$ is a Lie algebra contained in another Lie algebra $\mathfrak{K}$, we see from Jacobi's identity that the mapping $l \to l \circ k$, where k is any fixed element of $\mathfrak{K}$, is a derivation of $\mathfrak{L}$ into $\mathfrak{K}$ which can be extended to a derivation of $\mathfrak{K}$ into itself.

Definition 3.2. If $\mathfrak{A}$ is an associative algebra, the derivation $a \to a_0 a - aa_0$, where a_0 is any fixed element of $\mathfrak{A}$, is called an inner derivation of $\mathfrak{A}$.

If $\mathfrak{L}$ is a Lie algebra, the derivation $l \to l \circ l_0$, where l_0 is any fixed element of $\mathfrak{L}$, is called an inner derivation of $\mathfrak{L}$.

It is known that a semi-simple Lie algebra possesses only inner derivations.[3]

For associative algebras it has been shown [4] that, if $\mathfrak{A}$ is a semi-simple subalgebra of a normal simple algebra $\mathfrak{R}$, any derivation of $\mathfrak{A}$ into itself can be extended to an inner derivation of $\mathfrak{R}$.

Before we generalize these results we make the following definition:

Definition 3.3. Let $\mathfrak{A}$ be an associative algebra, or a Lie algebra. $\mathfrak{A}$ is said to be reflexive if every derivation of $\mathfrak{A}$ into an arbitrary algebra $\mathfrak{B} \supseteqq \mathfrak{A}$ can be extended to an inner derivation of $\mathfrak{B}$.

THEOREM 3.1. *A Lie algebra is reflexive if and only if it possesses the property of Theorem* 2.1.

Proof. 1). Suppose that a Lie algebra $\mathfrak{L}$ satisfies the condition of Theorem 2.1. If $\mathfrak{K}$ is any Lie algebra containing $\mathfrak{L}$, we may regard it as a representation space of $\mathfrak{L}$, the transformation of $\mathfrak{K}$ corresponding to an element $l \in \mathfrak{L}$ being the mapping $k \to k \circ l$. If now D is any derivation of $\mathfrak{L}$ into $\mathfrak{K}$ we have $D(l \circ m) = D(l) \circ m + l \circ D(m) = D(l) \circ m - D(m) \circ l$, which shows that the mapping $l \to D(l)$ satisfies the conditions of Theorem 2.1. Hence there exists an element $k_0 \in \mathfrak{K}$ such that $D(l) = k_0 \circ l$, whence we see that $\mathfrak{L}$ is reflexive. 2). Suppose that $\mathfrak{L}$ is reflexive. Let $\mathfrak{P}$ be a representation space of $\mathfrak{L}$. We define a Lie algebra $\mathfrak{K}$ whose underlying space is $\mathfrak{L} \times \mathfrak{P}$ by the commutation rule $(l, \vec{e}) \circ (m, \vec{f}) = (l \circ m,\ m \cdot \vec{e} - l \cdot \vec{f})$. (It is easy to verify that Jacobi's identity is satisfied). $\mathfrak{K}$ contains $(\mathfrak{L}, 0)$ as a subalgebra isomorphic with $\mathfrak{L}$.

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[3] K. Yosida, "A characterization of the adjoint representations of the semi-simple Lie rings," *Japanese Journal of Mathematics*, vol. 14 (1938), p. 170.

[4] Cf. N. Jacobson, "Abstract derivation and Lie algebras," *Transactions of the American Mathematical Society*, vol. 42 (1937), p. 214.

Suppose now we have a mapping $l \to \vec{e}(l)$ of $\mathfrak{L}$ into $\mathfrak{P}$, as is considered in Theorem 2.1. This gives a mapping $(l, 0) \to (0, \vec{e}(l))$ of $(\mathfrak{L}, 0)$ into $\mathfrak{K}$ which is a derivation since $(0, \vec{e}(l \circ m)) = (0, m \cdot \vec{e}(l) - l \cdot \vec{e}(m)) = (0, \vec{e}(l)) \circ (m, 0) + (l, 0) \circ (0, \vec{e}(m))$. But $(\mathfrak{L}, 0)$, being isomorphic with $\mathfrak{L}$, is reflexive, and hence there exists an element $(l_0, \vec{e}_0) \in \mathfrak{K}$ such that $(0, \vec{e}(l)) = (l, 0) \circ (l_0, \vec{e}_0)$, which implies that $\vec{e}(l) = l \cdot (-\vec{e}_0)$, as required in Theorem 2.1. This completes the proof.

THEOREM 3.2. *An associative algebra is reflexive if and only if it possesses the property of Theorem 2.2.*

Proof. 1). If $\mathfrak{B}$ is any associative algebra containing $\mathfrak{A}$ we may regard $\mathfrak{B}$ as a two-sided $\mathfrak{A}$-module. A derivation D of $\mathfrak{A}$ into $\mathfrak{B}$ is a mapping of the type considered in Theorem 2.2. The conclusion of Theorem 2.2 becomes the statement that D is an inner derivation of $\mathfrak{L}$. Hence if $\mathfrak{A}$ has the property of Theorem 2.2 then it is reflexive. 2). Suppose that $\mathfrak{A}$ is reflexive and let $\mathfrak{P}$ be a two-sided $\mathfrak{A}$-module. We construct an algebra $\mathfrak{B}$ over the linear space $\mathfrak{A} \times \mathfrak{P}$ by defining multiplication as follows: $(a, e) \cdot (b, f) = (ab, a \cdot \vec{f} + \vec{e} \cdot b)$. The rest of the proof is the same as the corresponding part of the proof of Theorem 3.1.

From Theorems 3.1, 2.4, 2.1, 1.2 and 3.2, 2.3, 2.2, we have now:

THEOREM 3.3. *If an algebra (Lie or associative) is reflexive then every representation of it is completely decomposable.*

THEOREM 3.4. *An algebra over a field of characteristic zero is semi-simple if and only if it is reflexive.*

Remarks. 1). We have shown without any assumption on the groundfield that if an algebra is reflexive then it is semi-simple. The converse was proved with the restriction that the groundfield be of characteristic zero. That this restriction is necessary for associative algebras is seen from results of N. Jacobson who considered the derivation algebras of certain fields of characteristic p. (Clearly, if a field were reflexive, its only derivation would be the mapping which maps every element on 0; but this is not the case for characteristic p). We refer to: N. Jacobson, "Abstract Derivation and Lie Algebras," *Transactions of the American Mathematical Society,* vol. 42 (1937), p. 217, for these results.

For Lie algebras, the necessity of our restriction is seen from the following example:

Let P be the field of the integers mod. 2 and define a Lie algebra of dimension 3 over P by the following equations of structure (x, y, z is a basis of the Lie algebra): $x \circ y = z$, $y \circ z = x$, $z \circ x = y$. The Lie algebra thus defined is clearly a simple Lie algebra. But the linear mapping D defined by the equations $D(x) = x$, $D(y) = y$, $D(z) = 0$ is a derivation which is not inner.

2). It may be useful to point out that the condition that an algebra $\mathfrak{A}$ be reflexive is genuinely stronger than the condition that all derivations of $\mathfrak{A}$ *into itself* be inner.

In fact, consider the Lie algebra $\mathfrak{L}$ of dimension 2 over (e. g.) the field of the rational numbers, with generators x, y, where $x \circ y = x$. It is easily verified that every derivation D of $\mathfrak{L}$ into itself is of the form $D(x) = \alpha x$, $D(y) = \beta x$. If we put $z = \alpha y - \beta x$ we have $D(x) = x \circ z$, $D(y) = y \circ z$, i. e. D is an inner derivation. But $\mathfrak{L}$ cannot be reflexive, since it is solvable.

Similarly, the associative algebra of all matrices of the form $\left(\begin{smallmatrix}\lambda & 0\\ \mu & \nu\end{smallmatrix}\right)$, with coefficients in the field of the rational numbers, is clearly not semi-simple and hence cannot be reflexive; but it is easily seen that every derivation D of this algebra into itself is of the form $D(E_{11}) = \alpha E_{21}$, $D(E_{21}) = \beta E_{21}$, $D(E_{22}) = -\alpha E_{21}$, where we have written E_{11} for the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, etc. Now, if $a_0 = \alpha E_{21} + \beta E_{22}$, we $D(a) = a_0 a - a a_0$ for every element a of this algebra, i. e. D is an inner derivation.

4. Derivation algebras. If $\mathfrak{A}$ is any algebra, the derivations $D_1, D_2, \cdots$ of $\mathfrak{A}$ constitute a Lie algebra of linear transformations, commutation being defined by the formula $D_1 \circ D_2 = D_2 D_1 - D_1 D_2$. The Lie algebra thus obtained is called the derivation algebra of $\mathfrak{A}$. We propose to study the structure of derivation algebras.

Definition 4. 1. A subset $\mathfrak{J}$ of an algebra (Lie or associative) is called characteristic if $D(\mathfrak{J}) \subseteq \mathfrak{J}$ for every derivation D of the algebra.

For example, it is easy to verify that the center of a Lie algebra is always a characteristic ideal. Similarly, the center of an associative algebra is always a characteristic subalgebra.

We shall require the following:

THEOREM 4. 1.[5] *Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic* 0; *let $\mathfrak{H}$ be its radical (the maximal solvable ideal). Then $\mathfrak{H}$ is a characteristic ideal.*

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[5] This was first proved by H. Zassenhaus, *Hamb. Abhandlungen*, Band. 13 (1940), p. 79.

Proof. Let $\mathfrak{H} \supset \mathfrak{H}^{(1)} \supset \mathfrak{H}^{(2)} \supset \cdots \mathfrak{H}^{(p+1)} = \{0\}$ be the sequence of the successive derived algebras of $\mathfrak{H}$. Let D be any derivation of $\mathfrak{L}$. Suppose that $D^i(\mathfrak{H}^{(k+1)}) \subseteq \mathfrak{H}$ for all $i = 1, 2, \cdots$; (this holds trivially for $k = p$). Then we shall show that $D^i(\mathfrak{H}^{(k)}) \subseteq \mathfrak{H}$; $i = 1, 2, \cdots$.

Since $\mathfrak{H}^{(k)}$ is an ideal of $\mathfrak{L}$ it is easy to see that the set $[\mathfrak{H} + D(\mathfrak{H}^{(k)})]$ of all elements of the form $h + D(h')$, where $h \in \mathfrak{H}$ and $h' \in \mathfrak{H}^{(k)}$ constitutes an ideal in $\mathfrak{L}$. If $h_1, h_2 \in \mathfrak{H}^{(k)}$ we have $2\,D(h_1) \circ D(h_2) \equiv D^2(h_1 \circ h_2)$ mod. $\mathfrak{H}$, and it follows from our inductive hypothesis that $D(h_1) \circ D(h_2) \in \mathfrak{H}$.

Hence the derived algebra $[\mathfrak{H} + D(\mathfrak{H}^{(k)})]'$ is contained in $\mathfrak{H}$: $[\mathfrak{H} + D(\mathfrak{H}^{(k)}]$ is therefore a solvable ideal of $\mathfrak{L}$, and since $\mathfrak{H}$ is maximal we must have $D(\mathfrak{H}^{(k)}) \subseteq \mathfrak{H}$.

Suppose we have already proved that $D^i(\mathfrak{H}^{(k)}) \subseteq \mathfrak{H}$ for all $i < n$. Then $[\mathfrak{H} + D^n(\mathfrak{H}^{(k)})]$ is an ideal in $\mathfrak{L}$, for $D^n(h) \circ x \equiv D^n(h \circ x)$ mod. $\mathfrak{H}$ for all $h \in \mathfrak{H}^{(k)}$ and $x \in \mathfrak{L}$. The derived algebra $[\mathfrak{H} + D^n(\mathfrak{H}^{(k)})]'$ consists of sums of elements of $\mathfrak{H}$ and elements of the form $D^n(h_1) \circ D^n(h_2)$, with $h_1, h_2 \in \mathfrak{H}^{(k)}$. But $D^n(h_1) \circ D^n(h_2) \equiv 1/{}_{2n}C_n \cdot D^{2n}(h_1 \circ h_2)$ mod. $\mathfrak{H}$, whence we may conclude, as before, that we must have $D^n(\mathfrak{H}^{(k)}) \subseteq \mathfrak{H}$. Thus we have shown that, if $D^i(\mathfrak{H}^{(k+1)}) \subseteq \mathfrak{H}$ for all i, then also $D^i(\mathfrak{H}^{(k)}) \subseteq \mathfrak{H}$ for all i, and our theorem follows by induction on k.

In exactly the same manner we can prove:

THEOREM 4.2. *The radical of an associative algebra over a field of characteristic* 0 *is a characteristic ideal.*

Let now $\mathfrak{A}$ be a Lie algebra or an associative algebra over a field of characteristic 0. In either case we may write $\mathfrak{A} = \mathfrak{H} + \mathfrak{K}$, where $\mathfrak{H}$ is the radical of $\mathfrak{A}$ and $\mathfrak{K}$ is a semi-simple subalgebra. If D is any derivation of $\mathfrak{A}$ we have $D(\mathfrak{H}) \subseteq \mathfrak{H}$. On the other hand we know that there exists an element $a_0 \in \mathfrak{A}$ such that $D(k) = k \circ a_0$ for every $k \in \mathfrak{K}$. (In the associative case, $k \circ a_0$ will stand for the expression $a_0 k - k a_0$). Let D_{a_0} denote the inner derivation effected by a_0. If we put $D' = D - D_{a_0}$ we have $D'(\mathfrak{K}) = \{0\}$. We have, then, $D'(h \circ k) = D'(h) \circ k$ (for Lie algebras) or

$$\left\{\begin{array}{l} D'(kh) = kD'(h) \\ D'(hk) = D'(h)k \end{array}\right\} \text{(for associative algebras).} \qquad \cdots\cdots\cdots\cdot (A)$$

Conversely, any derivation of $\mathfrak{H}$ satisfying the condition (A) gives a derivation of $\mathfrak{A}$ if we define $D(k) = 0$ for every $k \in \mathfrak{K}$. Hence we have:

THEOREM 4.3. *Every derivation of* $\mathfrak{A}$ *is the sum of an inner derivation and a derivation which annuls* $\mathfrak{K}$.

A derivation of $\mathfrak{H}$ can be extended to a derivation of $\mathfrak{A}$ if and only if it is the sum of an inner derivation (by means of an element of $\mathfrak{A}$, not necessarily of $\mathfrak{H}$) and a derivation with the property (A).

We are now in a position to prove the following result:

THEOREM 4. 4. *The derivation algebra $\mathfrak{D}$ of a Lie algebra $\mathfrak{L}$ over a field of characteristic* 0 *is semi-simple if and only if $\mathfrak{L}$ is semi-simple.*

Proof. If $\mathfrak{L}$ is semi-simple, $\mathfrak{D}$ consists only of inner derivations. Moreover, since the center of $\mathfrak{L}$ reduces to $\{0\}$, $\mathfrak{D}$ is isomorphic with $\mathfrak{L}$ and hence is semi-simple.

Suppose now that $\mathfrak{D}$ is semi-simple. Let us write $\mathfrak{L} = \mathfrak{H} + \mathfrak{K}$, as before. Denote by $\mathfrak{D}_{\mathfrak{H}}$ the set of the inner derivations of $\mathfrak{L}$ which are effected by the elements of $\mathfrak{H}$. If D_h is such a derivation, and D is any other derivation of $\mathfrak{L}$ we have $D_h \circ D = D_{D(h)}$, whence it is clear that $\mathfrak{D}_{\mathfrak{H}}$ is an ideal in $\mathfrak{D}$. Since $\mathfrak{H}$ is solvable, it follows moreover that $\mathfrak{D}_{\mathfrak{H}}$ is solvable, and hence must reduce to $\{0\}$. This means that $\mathfrak{H}$ is the center of $\mathfrak{L}$. Since every derivation is inner as far as $\mathfrak{K}$ is concerned, this implies that every derivation maps $\mathfrak{K}$ into itself. Finally, every derivation of the center $\mathfrak{H}$ can be extended to a derivation of $\mathfrak{L}$ which annuls $\mathfrak{K}$, and the same statement is true if $\mathfrak{H}$ and $\mathfrak{K}$ are interchanged. Every derivation is the sum of derivations obtained in this manner; in fact, $\mathfrak{D}$ is isomorphic with the direct sum of the derivation algebra of $\mathfrak{H}$ and the derivation algebra of $\mathfrak{K}$. Since $\mathfrak{H}$ is abelian, its derivation algebra is the Lie algebra of all linear transformations of $\mathfrak{H}$, and this is clearly not semi-simple unless $\mathfrak{H} = \{0\}$. This completes the proof.

With regard to associative algebras it is known that the derivation algebra of a normal simple algebra is simple (N. Jacobson, "Abstract Derivation and Lie Algebras," *Transactions of the American Mathematical Society,* vol. 42). We shall prove:

THEOREM 4. 5. *Let $\mathfrak{A}$ be an associative algebra over a field of characteristic* 0, *and let $\mathfrak{D}$ be the derivation algebra of $\mathfrak{A}$. Then $\mathfrak{D}$ is semi-simple or* $\{0\}$ *if and only if $\mathfrak{A}$ is semi-simple.*

Proof. If $\mathfrak{A}$ is semi-simple its derivation algebra consists only of inner derivations. Let us denote by $\mathfrak{A}_l$ the Lie algebra obtained from $\mathfrak{A}$ by defining the commutator of two elements as $a \circ b = ba - ab$. Then it is clear that the derivation algebra of $\mathfrak{A}$ is isomorphic with $\mathfrak{A}_l/\mathfrak{Z}$, where $\mathfrak{Z}$ is the center of $\mathfrak{A}_l$. Now, it is known that the derived algebra $\mathfrak{A}'_l$ of $\mathfrak{A}_l$ is semi-simple or $\{0\}$.[6]

[6] W. Landherr, "Uber einfache Lie-sche Ringe," *Hamb. Abhandlungen,* Band 11 (1935), pp. 41-64. N. Jacobson, "Simple Lie algebras over a field of characteristic 0," *Duke Mathematical Journal,* vol. 4 (1938), p. 536.

Let $\mathfrak{H}$ be the radical of $\mathfrak{A}_l$. Then $\mathfrak{H} \circ \mathfrak{A}_l$ is a solvable ideal in $\mathfrak{A}'_l$ and hence $\mathfrak{H} \circ \mathfrak{A}_l = \{0\}$. But this means that $\mathfrak{H} \subseteq \mathfrak{Z}$ (in fact $\mathfrak{H} = \mathfrak{Z}$). Hence $\mathfrak{A}_l/\mathfrak{Z}$ is semi-simple or $\{0\}$, which proves the first part of our theorem.

Suppose now that $\mathfrak{D}$ is semi-simple or $\{0\}$. Write $\mathfrak{A} = \mathfrak{R} + \mathfrak{K}$, where $\mathfrak{K}$ is a semi-simple subalgebra and $\mathfrak{R}$ the radical of $\mathfrak{A}$. It follows, then, as in the proof of Theorem 4. 4, that $\mathfrak{R}$ is contained in the center of $\mathfrak{A}$ and hence that $\mathfrak{K}$ is mapped into itself by every derivation of $\mathfrak{A}$. Let $\mathfrak{R} \supset \mathfrak{R}^2 \supset \cdots \supset \mathfrak{R}^k = \{0\}$ be the series of the successive powers of $\mathfrak{R}$. It is easily seen by an induction on the exponent i that every $\mathfrak{R}^i$ is a characteristic ideal of $\mathfrak{A}$. Suppose that $\mathfrak{R} \neq \{0\}$.

If $\mathfrak{R}^2 = \{0\}$ we can define a derivation D of $\mathfrak{A}$ as follows:

$$D(r) = r \quad \text{if } r \in \mathfrak{R}; \qquad D(k) = 0 \quad \text{if } k \in \mathfrak{K}.$$

It is clear that D is really a derivation of $\mathfrak{A}$. If D^* is any other derivation of $\mathfrak{A}$ we have

$$(D \circ D^*)(r) = (D^*D - DD^*)(r) = D^*(r) - D(D^*(r)) = 0 \text{ if } r \in \mathfrak{R}$$
$$\text{(since } D^*(r) \in \mathfrak{R})$$

and $(D \circ D^*)(k) = \ldots\ldots\ldots\ldots\ldots\ldots = -D(D^*(k)) = 0$ if $k \in \mathfrak{K}$
$$\text{(since } D^*(k) \in \mathfrak{K}).$$

Hence $D \circ D^* = 0$ for every $D^* \in \mathfrak{D}$, which contradicts the hypothesis that $\mathfrak{D}$ is semi-simple or $\{0\}$. Hence $\mathfrak{R}^2 \neq \{0\}$, and so, in the series above, $k > 2$. If $r_0 \in \mathfrak{R}^{k-2}$ we can define a derivation D_{r_0} of $\mathfrak{A}$ as follows:

$$D_{r_0}(r) = r_0 \cdot r \quad \text{if } r \in \mathfrak{R}; \qquad D_r(k) = 0 \quad \text{if } k \in \mathfrak{K}.$$

(Again, the verification that this is really a derivation is trivial.)

If D is any other derivation of $\mathfrak{A}$ we have

$$(D_{r_0} \circ D)(r) = D(r_0 r) - r_0 \cdot D(r) = D(r_0) \cdot r \qquad \text{if } r \in \mathfrak{R},$$
$$\text{(since } D(r) \in \mathfrak{R})$$

and $(D_{r_0} \circ D)(k) = -D_{r_0}(D(k)) = 0$ if $k \in \mathfrak{K}$, (since $D(k) \in \mathfrak{K}$).

Hence $D_{r_0} \circ D = D_{D(r_0)}$, which shows that the derivations of the form D_{r_0}, $r_0 \in \mathfrak{R}^{k-2}$, constitute an ideal in $\mathfrak{D}$. Since there is at least one non-zero derivation of this type (otherwise we should have $\mathfrak{R}^{k-1} = \{0\}$), and since the ideal just defined is clearly abelian, we have again reached a contradiction. Hence $\mathfrak{R} = \{0\}$ and $\mathfrak{A}$ is semi-simple.

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