A q-IDENTITY RELATED TO A COMODULE

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Dedicated to Mia Cohen, coauthor and friend, on the occasion of her retirement

1. Introduction

In this paper we show that a certain algebra being a comodule algebra over the Taft Hopf algebra of dimension n^2 is equivalent to a set of identities related to the q-binomial coefficient, when q is a primitive n^{th} root of 1. We then give a direct combinatorial proof of these identities. To be consistent with the usual notation for the Taft algebra, we will write $q = \omega$ for our n^{th} root of 1.

Let $\Bbbk$ be a field of characteristic 0 which contains a primitive n^{th} root of 1, ω. Consider the algebra $A = A_n(\omega) = \Bbbk[z]/(z^n - \omega)$. It was proposed by Cohen, Fischman, and the second author [CFM] that A is a right H-comodule for the Taft Hopf algebra $H = T_{n^2}(\omega)$ of dimension n^2, for a particular map $\rho : A \to A \otimes H$. [CFM, Proposition 2.2(d)] proved that ρ is a comodule map when $n \leq 4$. However the question for general n was left open, since the general case seemed to lead to some rather complicated identities.

The comodule problem was later solved for arbitrary n in [MS] by indirect means: it was shown there that A is a module for the Drinfel'd double $D(H)$, giving an action of the dual $(H^*)^{cop}$ on A. This action dualizes exactly to the [CFM] coaction of H. Moreover [MS] show that A is always a Yetter-Drinfeld module for H; this had been proved in [CFM] for $n \leq 4$.

The question was raised as to whether a direct proof of the comodule property for A via ρ could be given, by determining precisely the identities involved (see [MS, p. 357]). In Theorem 3.9 we determine exactly the identities needed, using the q-binomial coefficient with $q = \omega$. In Theorem 4.2 we then give a combinatorial proof of the identities. This gives an alternative to the methods of [MS].

In Section 5 we also show directly that our algebra $A = A_n(\omega)$ is in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfel'd modules for H, for any n, using the form of the comodule map ρ. As a consequence A is always a commutative algebra in the category ${}^H_H\mathcal{YD}$, answering another question of [CFM].

Finally in Section 6 we discuss in more detail the dual approach of [MS].

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2. Preliminaries

We let H denote the Taft Hopf algebra of dimension n^2, that is

$$H = T_{n^2}(\omega) = \Bbbk\langle x, g | x^n = 0,\ g^n = 1,\ xg = \omega gx\rangle,$$

where ω is a fixed primitive n^{th} root of 1, with Hopf structure given by

$$\Delta(g) = g \otimes g,\ \ \Delta(x) = x \otimes 1 + g \otimes x$$

$$\epsilon(g) = 1,\ \epsilon(x) = 0,\ S(g) = g^{-1} \text{ and } S(x) = -g^{-1}x.$$

We also need some well-known facts about q-binomial coefficients [K]. Recall that

$$\binom{b}{k}_q := \frac{(b)!_q}{(k)!_q\ (b-k)!_q}, \text{ where } (b)!_q := \frac{(q-1)(q^2-1)\cdots(q^b-1)}{(q-1)^b}.$$

So for $k, s \in \mathbb{N}$,

$$\binom{k+s}{k}_q = \frac{(1-q)\cdots(1-q^s)(1-q^{s+1})\cdots(1-q^{s+k})}{(1-q)\cdots(1-q^s)(1-q)\cdots(1-q^k)} = \frac{(1-q^{s+1})\cdots(1-q^{s+k})}{(1-q)\cdots(1-q^k)}.$$

Lemma 2.1. *Given $x, g \in H$ as above and $b \in \mathbb{N}$,*

$$\Delta(x^b) = \sum_{k=0}^{b} \binom{b}{k}_\omega g^k x^{b-k} \otimes x^k.$$

Proof. Since $\Delta(x^b) = (\Delta(x))^b = (x \otimes 1 + g \otimes x)^b$, the lemma follows from the q-binomial theorem [K, IV.2.2], as follows: in [K], the theorem is stated for $(x+y)^b$, where $yx = qxy$. Here we replace x by $g \otimes x$, y by $x \otimes 1$ and q by ω. □

Corollary 2.2. *For any $a, b \in \mathbb{N}$,*

$$\Delta(x^b g^a) = \sum_{k=0}^{b} w^{-k(b-k)} \binom{b}{k}_\omega x^{b-k} g^{k+a} \otimes x^k g^a.$$

Proof.

$$\begin{aligned}\Delta(x^b g^a) &= \Delta(x^b)\Delta(g^a) = \sum_{k=0}^{b} \binom{b}{k}_\omega g^k x^{b-k} g^a \otimes x^k g^a \\ &= \sum_{k=0}^{b} w^{-k(b-k)} \binom{b}{k}_\omega x^{b-k} g^{k+a} \otimes x^k g^a.\end{aligned}$$

□

3. The comodule algebra for H

As noted in the introduction, [CFM] proposed that A will be an H-comodule. We let u denote the coset $z + I$, where $I = (z^n - \omega)$, and thus $\{1, u, u^2, \ldots, u^{n-1}\}$ will be a basis for A.

For our given root of unity ω, we define

$$a_i := (\omega - 1)^i \omega^{\frac{i(i+1)}{2}}. \tag{3.1}$$

The explicit coaction $\rho : A \to A \otimes H$ is now defined by

$$\rho(u) = \sum_{i=0}^{n-1} a_i x^i g^{-(i+1)} \otimes u^{i+1}. \tag{3.2}$$

We must prove that

$$(id \otimes \rho)\rho = (\Delta \otimes id)\rho. \tag{3.3}$$

Now [CFM] showed that $\rho(u)^n = \omega 1$, and thus ρ is a homomorphism since $u^n = \omega 1$. Since Δ is also a homomorphism and the powers of u are a basis for A, it will suffice to check that Equation (3.3) holds when applied to the element u.

Evaluating Equation (3.3) on u, we obtain the new equation:

$$\sum_{s=0}^{n-1} a_s x^s g^{-(s+1)} \otimes \rho(u)^{s+1} = \sum_{m=0}^{n-1} a_m \Delta(x^m g^{-(m+1)}) \otimes u^{m+1}, \tag{3.4}$$

where by Corollary 2.2, the right hand side is

$$\sum_{m=0}^{n-1} a_m \left(\sum_{k=0}^{m} \omega^{-k(m-k)} \binom{m}{k}_\omega x^{m-k} g^{k-(m+1)} \otimes x^k g^{-(m+1)} \right) \otimes u^{m+1}.$$

In order to compute the left hand side of (3.4), we need to find an explicit formula for $\rho(u)^s$ for any $1 \leq s \leq n$. We start with an auxiliary lemma:

Lemma 3.5. *(i) Given $r, s \in \mathbb{N}$, $a_r a_s = a_{r+s}\omega^{-rs}$ and more generally*

$$\Pi_{i=1}^{t} a_{r_i} = a_{(\sum_{i=1}^{t} r_i)} \omega^{-\sum_{j<i} r_i r_j}$$

(ii) For all $1 \leq i \leq n-1$,

$$(\sum_{j=0}^{i} \omega^{j-i}) a_i + a_{i-1} = \omega^{i+1} a_{i-1}.$$

Proposition 3.6. *For any $1 \leq s \leq n$,*

$$\rho(u)^s = \sum_{k=0}^{n-1} a_k \left(\sum_{\{0 \leq i_1, \ldots, i_s \leq k \ \mid \ \sum_{j=1}^{s} i_j = k\}} \omega^{\sum_{j=2}^{s} i_j (j-1)} \right) x^k g^{-(k+s)} \otimes u^{k+s}.$$

Proof. Let $\rho(u)_j = \sum_{i_j=0}^{n-1} a_{i_j} x^{i_j} g^{-(i_j+1)} \otimes u^{i_j+1}$ denote the j-th copy of $\rho(u)$ in $\rho(u)^s$. As we multiply one term from each of the s factors $\rho(u)_j$ in $\rho(u)^s$, we obtain a sum of terms of the form

$$a_{i_1} \cdots a_{i_j} \cdots a_{i_s} x^{i_1} g^{-(i_1+1)} \cdots x^{i_j} g^{-(i_j+1)} \cdots x^{i_s} g^{-(i_s+1)} \otimes u^{i_1+1} \cdots u^{i_j+1} \cdots u^{i_s+1}. \tag{3.7}$$

Let $k = \sum_{j=1}^{s} i_j$. Using Lemma 3.5 (i) and the fact that $g^r x^s = \omega^{-rs} x^s g^r$, (3.7) becomes

$$a_k \omega^{-(\sum_{t<r} i_r i_t)} \Pi_{j=2}^{s} \omega^{\sum_{l=1}^{j-1} i_j (i_l+1)} x^k g^{-(k+s)} \otimes u^{k+s}.$$

Simplifying, we have

$$\begin{aligned}
(3.8)\qquad \omega^{-(\sum_{t<r} i_r i_t)}\prod_{j=2}^{s}\omega^{\sum_{l=1}^{j-1} i_j(i_l+1)} &= \omega^{-(\sum_{t<r} i_r i_t)}\omega^{\sum_{j=2}^{s}\sum_{l=1}^{j-1}(i_j i_l+i_j)} \\
&= \omega^{-(\sum_{t<r} i_r i_t)}\omega^{\sum_{j=2}^{s}(\sum_{l=1}^{j-1} i_j i_l)}\omega^{\sum_{j=2}^{s} i_j(j-1)} \\
&= \omega^{\sum_{j=2}^{s} i_j(j-1)},
\end{aligned}$$

since the first two powers of ω which appear have opposite exponents.

Finally, since such a term arises whenever $i_1+\cdots+i_s=k$, by ordering the terms according to powers of x we have that

$$\rho(u)^s=\sum_{k=0}^{n-1} a_k\left(\sum_{\{0\le i_1,\ldots,i_s\le k\ \mid\ \sum_{j=1}^{s} i_j=k\}}\omega^{\sum_{j=2}^{s} i_j(j-1)}\right)x^k g^{-(k+s)}\otimes u^{k+s}.$$

□

Using Proposition 3.6 with $s+1$ instead of s, we have all the components of our desired equation (3.4). Substituting them in (3.4), we may compare the coefficients on both sides:

$$\begin{aligned}
\sum_{s=0}^{n-1}\sum_{k=0}^{n-1} a_s a_k\left(\sum_{\{0\le i_1,\ldots,i_{s+1}\le k\ \mid\ \sum_{j=1}^{s+1} i_j=k\}}\omega^{\sum_{j=2}^{s+1} i_j(j-1)}\right)x^s g^{-(s+1)}\otimes x^k g^{-(k+s+1)}\otimes u^{k+s+1} \\
=\sum_{m=0}^{n-1}\sum_{l=0}^{m} a_m\omega^{l(m-l)}\binom{m}{l}_{\omega} x^{m-l}g^{l-(m+1)}\otimes x^l g^{-(m+1)}\otimes u^{m+1}
\end{aligned}$$

By linear independence, the coefficients of each term on both sides should agree. Thus we have:

Theorem 3.9. *Fix a primitive nth root of unity ω in $\Bbbk$, and let $A=A_n(\omega)$ and $H=T_{n^2}(\omega)$ be as above. Then A is a right H-comodule algebra via the coaction ρ in Equation (3.2) $\iff$ for all pairs of natural numbers $0\le k,s\le n-1$,*

$$\sum_{\{0\le i_1,\ldots,i_{s+1}\le k\ \mid\ \sum_{j=1}^{s+1} i_j=k\}}\omega^{\sum_{j=2}^{s+1} i_j(j-1)}=\begin{cases}\binom{k+s}{k}_{\omega} & \text{if } k+s<n\\ 0 & \text{if } k+s\ge n.\end{cases}$$

4. A proof of the identities

In this section we give a direct combinatorial proof of the identities in Theorem 3.9. We thank Jason Fulman for pointing it out to us.

We consider the expansion of $\dfrac{1}{(1-z)(1-z\omega)\cdots(1-z\omega^s)}$ as a formal power series in the ring $\Bbbk[[z]]$. Write

$$\frac{1}{(1-z)(1-z\omega)\cdots(1-z\omega^s)}=\sum_{k\ge 0}\beta_k z^k.$$

Lemma 4.1. *For each $k \geq 0$,*

$$\beta_k = \sum_{\{0 \leq i_1, \ldots, i_{s+1} \leq k \ \mid \ \sum_{j=1}^{s+1} i_j = k\}} \omega^{\sum_{j=2}^{s+1} i_j (j-1)}.$$

Proof. We know that

$$\sum_{k \geq 0} \beta_k z^k = \Pi_{l=1}^{s+1} \left(\frac{1}{1 - z\omega^{l-1}} \right) = \Pi_{l=1}^{s+1} \left(\sum_{i_l \geq 0} (z\omega^{l-1})^{i_l} \right).$$

Whenever $\sum_{l=1}^{s+1} i_l = k$, the last product gives a term $z^k \omega^{\sum_{l=2}^{s+1} i_l (l-1)}$, where the sum in the exponent starts at $l = 2$ because $l - 1 = 0$ for $l = 1$. Thus

$$\beta_k = \sum_{\{0 \leq i_1, \ldots, i_{s+1} \leq k \ \mid \ \sum_{l=1}^{s+1} i_l = k\}} \omega^{\sum_{l=2}^{s+1} i_l (l-1)}$$

and thus the left hand side of the identity in Theorem 3.9 is the coefficient of z^k in the power series. □

Theorem 4.2. *The identities in Theorem 3.9 hold, for all $n > 1$, any given primitive n^{th} root of unity ω in $\mathbb{k}$, and all pairs of natural numbers $0 \leq k, s \leq n - 1$.*

Proof. We evaluate the coefficient β_k in a different way, using Theorem 349 in [HW] which states that given $\omega \in \mathbb{k}$,

$$\frac{1}{(1 - z\omega)(1 - z\omega^2) \cdots (1 - z\omega^j)} = 1 + z\omega \frac{1 - \omega^j}{1 - \omega} + z^2 \omega^2 \frac{(1 - \omega^j)(1 - \omega^{j+1})}{(1 - \omega)(1 - \omega^2)} + \cdots .$$

Replacing $z\omega$ by z we get

$$\frac{1}{(1 - z)(1 - z\omega) \cdots (1 - z\omega^{j-1})} = 1 + z \frac{1 - \omega^j}{1 - \omega} + z^2 \frac{(1 - \omega^j)(1 - \omega^{j+1})}{(1 - \omega)(1 - \omega^2)} + \cdots$$

and if we choose $j = s + 1$ then

$$\frac{1}{(1 - z)(1 - z\omega) \cdots (1 - z\omega^s)} = 1 + z \frac{1 - \omega^{s+1}}{1 - \omega} + z^2 \frac{(1 - \omega^{s+1})(1 - \omega^{s+2})}{(1 - \omega)(1 - \omega^2)} + \cdots .$$

In particular, the coefficient β_k of z^k turns out to be

$$\frac{(1 - \omega^{s+1}) \cdots (1 - \omega^{s+k})}{(1 - \omega) \cdots (1 - \omega^k)} = \binom{k + s}{k}_\omega .$$

Since β_k is unique, both forms must agree and

$$\sum_{\{0 \leq i_1, \ldots, i_{s+1} \leq k \ \mid \ \sum_{j=1}^{s+1} i_j = k\}} \omega^{\sum_{j=2}^{s+1} i_j (j-1)} = \binom{k + s}{k}_\omega .$$

When $k + s \geq n$ with $0 \leq k \leq n - 1$, one of the factors in the numerator $(1 - \omega^{s+1}) \cdots (1 - \omega^{s+k})$ is $1 - \omega^n = 0$ while the denominator $(1 - \omega) \cdots (1 - \omega^k) \neq 0$, making $\binom{k+s}{k}_\omega = 0$ as required. □

Corollary 4.3. *The algebra A is an H-comodule algebra, via the coaction in Equation (3.2).*

5. YD-module algebras and H-commutativity

In this section we consider the (left, left) Yetter-Drinfel'd category ${}^H_H\mathcal{YD}$. Recall that a module M is in ${}^H_H\mathcal{YD}$ if it is both a left H-module, a left H-comodule (via ρ), and

$$h \cdot \rho(m) = \sum \rho(h_1 \cdot m)(h_2 \otimes 1). \tag{5.1}$$

[CFM, Prop 2.2(e)] prove that our algebra $A = A_n$ is in ${}^H_H\mathcal{YD}$ for $H = T_{n^2}(\omega)$, for all $n \leq 4$. Here we show this for all n. We use a result from [CFM] which holds for any H and any A:

Lemma 5.2. [CFM, Lemma 2.10] *Let A be a left H-module and a left H-comodule.*

(a) Let M be an H-submodule of A. If the Yetter-Drinfel'd condition is satisfied for all $m \in M$ and all algebra generators of H (from some chosen generating set), then it is satisfied for all $m \in M$ and all $H \in H$.

(b) If A is also an H-module algebra and an H-module coalgebra, and if the Yetter-Drinfeld condition holds for all $h \in H$ and all algebra generators of A (from some generating set), then $A \in {}^H_H\mathcal{YD}$.

Proposition 5.3. *The algebra $A = A_n(\omega)$ is in ${}^H_H\mathcal{YD}$ for the Taft algebra $H = T_{n^2}(\omega)$, for all n.*

Proof. By Corollary 4.3, A is a left H-comodule, and so Lemma 5.2 will apply. We use that A is generated as an algebra by the H-submodule $M = k\{1, u\}$ and H is generated as an algebra by the set $\{g, x\}$. Thus A will be in ${}^H_H\mathcal{YD}$ provided we show the Yetter-Drinfeld condition (5.1) when $a = u$ and either $h = g$ or $h = x$.

First assume $h = g$. Then, using $\rho(u)$ as in (3.2),

$$\begin{aligned} g \cdot \rho(u) &= \sum_{i=0}^{n-1} a_i g x^i g^{-(i+1)} \otimes g \cdot u^{i+1} \\ &= \sum_{i=0}^{n-1} a_i \omega^{-i} x^i g^{-i} \otimes w^{i+1} u^{i+1} \\ &= \sum_{i=0}^{n-1} \omega a_i x^i g^{-i} \otimes u^{i+1}. \end{aligned}$$

On the other hand, since $g \cdot u = \omega u$

$$\begin{aligned} \rho(g \cdot u)(g \otimes 1) &= \omega \left(\sum_{i=0}^{n-1} a_i x^i g^{-(i+1)} \otimes u^{i+1} \right) (g \otimes 1) \\ &= \sum_{i=0}^{n-1} \omega a_i x^i g^{-i} \otimes u^{i+1}. \end{aligned}$$

Thus the Yetter-Drinfel'd condition holds for g and u.

Now assume that $h = x$. First, since $\Delta(x) = x \otimes 1 + g \otimes x$ and $x \cdot u = 1$, it is easy to see that $x \cdot u^{i+1} = (\sum_{j=0}^{i} \omega^j)u^i$. Thus in (5.1),

$$\begin{aligned}
x \cdot \rho(u) &= \sum_{i=0}^{n-1} a_i x x^i g^{-(i+1)} \otimes u^{i+1} + \sum_{i=0}^{n-1} a_i g x^i g^{-(i+1)} \otimes x \cdot u^{i+1} \\
&= \sum_{i=0}^{n-1} a_i x^{i+1} g^{-(i+1)} \otimes u^{i+1} + \sum_{i=0}^{n-1} \omega^{-i} a_i x^i g^{-i} \otimes (\sum_{j=0}^{i} \omega^j) u^i \\
&= 1 \otimes 1 + \sum_{i=1}^{n-1} \left((\sum_{j=0}^{i} \omega^{j-i}) a_i + a_{i-1} \right) x^i g^{-i} \otimes u^i.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum \rho(x_1 \cdot m)(x_2 \otimes 1) &= \rho(x \cdot u)(1 \otimes 1) + \rho(g \cdot u)(x \otimes 1) \\
&= \rho(1)(1 \otimes 1) + \omega \left(\sum_{i=0}^{n-1} a_i x^i g^{-(i+1)} \otimes u^{i+1} \right) (x \otimes 1) \\
&= 1 \otimes 1 + \omega \sum_{i=0}^{n-1} \omega^{i+1} a_i x^{i+1} g^{-(i+1)} \otimes u^{i+1} \\
&= 1 \otimes 1 + \sum_{i=1}^{n-1} \omega^{i+1} a_{i-1} x^i g^{-i} \otimes u^i,
\end{aligned}$$

where in both cases the term corresponding to $i = n$ vanishes because $x^n = 0$. Thus for A to be a Yetter-Drinfel'd module algebra, we need that

$$(\sum_{j=0}^{i} \omega^{j-i}) a_i + a_{i-1} = \omega^{i+1} a_{i-1}, \text{ for all } 1 \leq i \leq n-1.$$

However this holds by Lemma 3.5 (ii). □

[CFM] also study when A is commutative as an algebra in the category ${}_H^H\mathcal{YD}$.

Recall that for any braided monoidal category $\mathcal{C}$, with braiding $\tau : V \otimes W \to W \otimes V$ for $V, W \in \mathcal{C}$, an algebra A in $\mathcal{C}$ is *commutative in* $\mathcal{C}$ if for all $a, b \in A$,

$$m_A(a \otimes b) = m_A \circ \tau(a \otimes b). \tag{5.4}$$

Several authors have considered this generalized commutativity. In particular Cohen and Westreich considered the case when $\mathcal{C}$ is the module category of a quasi-triangular Hopf algebra in [CW].

In our situation ${}_H^H\mathcal{YD}$ has the structure of a braided monoidal category, as follows: for two modules $M, N \in {}_H^H\mathcal{YD}$, the braiding is given as follows [Y]:

$$\tau : M \otimes N \to N \otimes M \quad \text{via} \quad m \otimes n \mapsto \rho(m)(n \otimes 1) = \sum (m_{-1} \cdot n) \otimes m_0.$$

Thus an algebra A in ${}^H_H\mathcal{YD}$ is commutative in ${}^H_H\mathcal{YD}$ if

$$ab = \sum (a_{-1} \cdot b) a_0. \tag{5.5}$$

Corollary 5.6. *For the given algebra $A_n = k[u]$ and $H = T_{n^2}(\omega)$, A is commutative in ${}^H_H\mathcal{YD}$, for any n.*

Proof. It is shown in [CFM, Prop 2.2(e)] that if A_n is in ${}^H_H\mathcal{YD}$, then it is commutative in ${}^H_H\mathcal{YD}$. In fact their argument uses only that A_n is an H-module H-comodule algebra; again it suffices to check on generators of A and of H. □

6. The dual action

In this section, for the sake of completeness, we sketch the approach of [MS] for the action of H^* on A. As noted in the introduction, it is shown there that A is a $D(H)$-module algebra (and thus a Yetter-Drinfeld module algebra).

The Taft Hopf algebras $H = T_{n^2}(\omega)$ are known to be self-dual; thus we may write

$$H^* = \Bbbk\langle G, X \mid G^n = \varepsilon, X^n = 0, XG = \omega GX\rangle, \tag{6.1}$$

where $\Delta(G) = G \otimes G$, $\Delta(X) = X \otimes \varepsilon + G \otimes X$, $\langle G, 1\rangle = 1$, and $\langle X, 1\rangle = \varepsilon_{H^*}(X) = 0$.

The dual pairing between H and H^* is determined by

$$\langle G, g\rangle = \omega^{-1},\ \langle G, x\rangle = 0,\ \langle X, g\rangle = 0,\ \text{and}\ \langle X, x\rangle = 1. \tag{6.2}$$

Lemma 6.3. *As an algebra, $D(H)$ is generated by $\{x, g, X, G\}$. The relations among these generators, in addition to the relations in H and H^*, are as follows:*

$$gG = Gg, \quad xG = \omega^{-1} Gx, \quad Xg = \omega^{-1} gX, \quad \textit{and} \quad xX - Xx = G - g.$$

One may check that $(H^*)^{cop} = \Bbbk\langle G^{-1}, XG^{-1}\rangle \subset D(H)$, and that these generators give the usual relations in $D(H)$. The generators given in Lemma 6.3 are used since X and x behave similarly when acting as skew derivations. [MS] then use properties of higher skew derivations and the relations in Lemma 6.3 to prove:

Theorem 6.4. [MS, Theorem 4.5] *Let $H = T_{n^2}(\omega)$ be the Taft Hopf algebra and H^* its dual as above. Then $A = A_n$ becomes a $D(H)$-module algebra via the following:*
(a) $g \cdot u = \omega u$ and $G \cdot u = \omega^{-1} u$, and
(b) $x \cdot u = 1$ and $X \cdot u = (\omega^{-1} - 1)u^2$.

To see that A is an algebra in the category ${}^H_H\mathcal{YD}$ of left, left Yetter-Drinfeld modules, one may use a theorem of Majid [Mj] that $D(H)$-modules maybe be identified with ${}^H_H\mathcal{YD}$-modules. The only difficulty remains in showing that dualizing the left $(H^*)^{cop}$-action in Theorem 6.4 to a left H-comodule action gives the desired coaction.

Theorem 6.5. [MS, Theorem 5.7] *Let $H = T_{n^2}(\omega)$, $A = A_n$, and the H-action on A be as described in Theorem 6.4. Then there is a unique left H-comodule algebra structure ρ on A such that A is in ${}^H_H\mathcal{YD}$, given by*

$$\rho(u) = \sum_{m=0}^{n-1} a_m x^m g^{-(m+1)} \otimes u^{m+1},$$

where the coefficient a_m is given by

$$a_m = ((1-\omega^{-1})\omega)^m \omega^{\frac{m(m+1)}{2}} = (\omega - 1)^m \omega^{\frac{m(m+1)}{2}}.$$

This coefficient a_m is exactly our coefficient in Definition (3.1), and so the coaction in (6.5) is exactly our coaction in Equation (3.2). Thus [MS, Theorem 5.7] gives an alternate proof of Corollary 4.3 and Proposition 5.3.

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