ON THE COHOMOLOGY OF RELATIVE HOPF MODULES

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Abstract. Let H be a Hopf algebra over a field k, and A an H-comodule algebra. The categories of comodules and relative Hopf modules are then Grothendieck categories with enough injectives. We study the derived functors of the associated Hom functors, and of the coinvariants functor, and discuss spectral sequences that connect them. We also discuss when the coinvariants functor preserves injectives.

Introduction

Let k be a field, and H a Hopf algebra with bijective antipode, and A an H-module algebra. We can then consider the smash product $A\#H$ and the subring of invariants A^H. A left H-module M is called locally finite if $\dim_k(Hm)$ is finite, for every $m \in M$. In [12], the second author studied homological algebra for H-locally finite $A\#H$-modules, with emphasis to injective modules, minimal injective resolutions and cohomology. He also calculated the Picard group of A^H in terms of the Picard group of A and various subgroups of the group $Z(H, A)$ consisting of linear maps from $H \to A$ satisfying the cocycle condition. In the particular situation where H is the enveloping algebra of a finite dimensional Lie algebra, we refer to [10, 11]. The methods in [10, 11, 12] are based on Magid's papers [15, 16] on rational algebraic group actions.

The aim of this paper is to discuss the homological algebra for relative Hopf modules. If H is a Hopf algebra, and A is an H-comodule algebra, then a relative Hopf module is a vector space with an A-action and an H-coaction with a certain compatibility relation. In the case where H is finite dimensional, the category of relative Hopf modules is isomorphic to the category of modules over the smash product $A\#H^*$, providing the connection to the theory developed in [12]. However, the situation is more interesting in the case where H is infinite dimensional. Given two H-comodules M and N, we can consider the space $\mathrm{Hom}^H(M, N)$ of H-colinear morphisms between M and N, and also the H-comodule $\mathrm{HOM}(M, N)$, consisting of rational k-linear maps $M \to N$. We can consider the right derived functors of these two Hom functors, given rise to two different versions of the Ext functors.

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Hom^H can be viewed as the composition of HOM and the coinvariants functor, and the results in a spectral sequence connecting the two versions of Ext. This is discussed in Section 1. In Section 2, we look at relative Hopf modules. Again, we have two versions of the Hom functor, and, with some addtional conditions, the corresponding right derived functors are connected by a spectral sequence, see Propositions 2.11 and 2.15. More specific results can be obtained in the case where H is cosemisimple, this is discussed in Section 3.

1. The right derived functors of the coinvariant functor and the HOM functor

Throughout this paper, k is a field, and H is a Hopf algebra with bijective antipode. We recall that $\mathcal{M}^H$, the category of H-comodules and H-colinear maps, is a Grothendieck category with enough injectives (see for example [8]). We say that H has the symmetry property if $M \otimes N$ and $N \otimes M$ are isomorphic as comodules, for any $M, N \in \mathcal{M}^H$. If H is an almost commutative Hopf algebra, then its antipode is bijective and it has the symmetry property (see [17, 10.2.11, 10.2.12]).

Let A be an H-comodule algebra. A relative left-right (A, H)-Hopf module is a vector space with a left A-action and a right H-coaction ρ such that $\rho(am) = \rho(a)\rho(m)$, for all $a \in A$ and $m \in M$. The category of relative (A, H)-Hopf modules ${}_A\mathcal{M}^H$ has direct sums, and is a Grothendieck category with enough injective objects. If A is noetherian, then direct sums of injectives are injective, see [21, 3.1, 3.2].

We will use the Sweedler-Heyneman notation for comultiplications and coactions; if Δ is the comultiplication on H, then we write

$$\Delta(h) = h_1 \otimes h_2,$$

where the summation is implicitly understood. In a similar way, if M is a right H-comodule, with right H-coaction ρ, then we write, for all $m \in M$:

$$\rho(m) = m_0 \otimes m_1.$$

$M^{\mathrm{co}H} = \{m \in M \mid \rho(m) = m \otimes 1\}$ is called the k-submodule of coinvariants of M. $\mathcal{M}^H$ is a monoidal category: if $M, N \in \mathcal{M}^H$, then $M \otimes N \in \mathcal{M}^H$, with H-coaction

$$\rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1.$$

The unit object is k, with coaction $\rho(x) = x \otimes 1_H$.

Take $f \in \mathrm{Hom}(M, N)$, and consider $\rho(f) \in \mathrm{Hom}(M, N \otimes H)$ given by

$$\rho(f)(m) = f(m_0)_0 \otimes S^{-1}(m_1) f(m_0)_1.$$

As k is a field, $\mathrm{Hom}(M, N) \otimes H \subset \mathrm{Hom}(M, N \otimes H)$, and we introduce

$$\mathrm{HOM}(M, N) = \{f \in \mathrm{Hom}(M, N) \mid \rho(f) \in \mathrm{Hom}(M, N) \otimes H\}.$$

A morphism $f \in \mathrm{HOM}(M,N)$ is called a rational morphism. If H is finite dimensional, then all morphisms are rational. It is well-known (see for example [18, 20]) that $\mathrm{HOM}(M,N)$ is an H-comodule, and that it is the largest H-comodule contained in $\mathrm{Hom}(M,N)$. Also recall that $\rho(f) = f_0 \otimes f_1$ if and only if

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes S^{-1}(m_1) f(m_0)_1. \tag{1}$$

Lemma 1.1. *For any $M, N \in \mathcal{M}^H$, we have that $\mathrm{HOM}(M,N)^{\mathrm{co}H} = \mathrm{Hom}^H(M,N)$.*

Proof. If $\rho(f) = f \otimes 1$, then it follows from (1) that

$$f(m_0) \otimes m_1 = f(m_0)_0 \otimes m_2 S^{-1}(m_1) f(m_0)_1 = f(m)_0 \otimes f(m)_1$$

and f is H-colinear. Conversely, if f is H-colinear, then

$$f(m_0)_0 \otimes S^{-1}(m_1) f(m_0)_1 = f(m_0) \otimes S^{-1}(m_2) m_1 = f(m) \otimes 1,$$

and it follows from (1) that $\rho(f) = f \otimes 1$. □

Proposition 1.2. *Let $M, N, P \in \mathcal{M}^H$, and consider the natural isomorphism of vector spaces*

$$\phi : \ \mathrm{Hom}(N \otimes M, P) \to Hom(M, Hom(N,P)), \ \ \phi(f)(m)(n) = f(n \otimes m).$$

(1) *If $f \in \mathrm{Hom}(N \otimes M, P)$ is H-colinear, then $\phi(f)(m) \in \mathrm{HOM}(N,P)$, for every $m \in M$; furthermore $\phi(f)$ is H-colinear.*
(2) *ϕ induces a k-isomorphism*

$$\phi : \ \mathrm{Hom}^H(N \otimes M, P) \to \mathrm{Hom}^H(M, \mathrm{HOM}(N,P)).$$

(3) *If H has the symmetry property, then ϕ induces a k-isomorphism*

$$\psi : \ \mathrm{Hom}^H(M \otimes N, P) \to \mathrm{Hom}^H(M, \mathrm{HOM}(N,P)).$$

Proof. (1) Let f be H-colinear. We claim that

$$\rho(\phi(f)(m)) = f(- \otimes m_0) \otimes m_1.$$

Indeed, we show easily that (1) is satisfied:

$$\begin{aligned}
&(\phi(f)(m)(n_0))_0 \otimes S^{-1}(n_1)(\phi(f)(m)(n_0))_1 \\
&\quad = \ f(n_0 \otimes m)_0 \otimes S^{-1}(n_1) f(n_0 \otimes m)_1 \\
&\quad = \ f(n_{00} \otimes m_0) \otimes S^{-1}(n_1) n_{01} m_1 \\
&\quad = \ f(n_0 \otimes m_0) \otimes S^{-1}(n_2) n_1 m_1 = f(n \otimes m_0) \otimes m_1,
\end{aligned}$$

as needed. $\phi(f)$ is H-colinear

$$\phi(f)(m_0) \otimes m_1 = \rho(\phi(f)(m)) = f(- \otimes m_0) \otimes m_1,$$

for all $m \in M$. This is equivalent to

$$\phi(f)(m_0)(n) \otimes m_1 = f(n \otimes m_0) \otimes m_1,$$

for all $m \in M$ and $n \in N$. This is obvious.

(2) Take $f : N \otimes M \to P$, and assume that $\phi(f) \in \mathrm{Hom}^H(M, \mathrm{HOM}(N,P))$. Then we compute that

$$\begin{aligned}\rho(f(n\otimes m)) = \rho((\phi(f)(m))(n) &= \rho\Big((\phi(f)(m))(n_0)\Big)\varepsilon(n_1)\\ &= \Big((\phi(f)(m))(n_0)\Big)_0 \otimes n_2 S^{-1}(n_1)\Big((\phi(f)(m))(n_0)\Big)_1\\ &= f(n_0\otimes m_0)\otimes n_1 m_1,\end{aligned}$$

and it follows that f is right H-colinear.

By the symmetry property, there is an H-colinear isomorphism $\tau:\ N\otimes M \to M\otimes N$. The map

$$\mathrm{Hom}^H(\tau, P):\ \mathrm{Hom}^H(M\otimes N, P) \to \mathrm{Hom}^H(N\otimes M, P)$$

is an isomorphism of vector spaces, and $\psi = \phi \circ \mathrm{Hom}^H(\tau, P)$ is the required isomorphism. □

Corollary 1.3. *Let $M, V \in \mathcal{M}^H$, with V finite dimensional, and M projective in $\mathcal{M}^H$. Then $V \otimes M \in \mathcal{M}^H$ is also projective.*

Proof. As V is finite dimensional, the H-comodules $\mathrm{HOM}(V,P) = \mathrm{Hom}(V,P)$ and $V^*\otimes P$ are isomorphic, for all $P \in \mathcal{M}^H$. Therefore $\mathrm{Hom}(V,-) : \mathcal{M}^H \to \mathcal{M}^H$ is exact. Also $\mathrm{Hom}^H(M,-) :\ \mathcal{M}^H \to \mathcal{M}$ is exact, since $M \in \mathcal{M}^H$ is projective. It then follows from Proposition 1.2 (2) that $\mathrm{Hom}^H(V\otimes M,-) :\ \mathcal{M}^H \to \mathcal{M}^H$ is exact, and $V\otimes M$ is a projective object in $\mathcal{M}^H$. □

Recall that $M \in \mathcal{M}^H$ is called simple if it has no proper subobjects. M is semisimple or completely reducible if it is isomorphic to the direct sum of simple objects. $\mathcal{M}^H$ is called semisimple or completely reducible if every object is semisimple. It is well-known that $\mathcal{M}^H$ is semisimple if and only if H is cosemisimple, see for example [17, Lemma 2.4.3]. We present another criterion in the Lemma 1.4.

Lemma 1.4. *$\mathcal{M}^H$ is semisimple if and only if k is a projective object in $\mathcal{M}^{\mathrm{fd}H}$, the category of finite dimensional H-comodules.*

Proof. Take an exact sequence

$$0 \to W_1 \to W_2 \to W_3 \to 0$$

in $\mathcal{M}^{\mathrm{fd}H}$, and let V be a finite dimensional right H-comodule. Then we have the following exact sequence in $\mathcal{M}^{\mathrm{fd}H}$:

$$0 \to \mathrm{Hom}(V,W_1) \to \mathrm{Hom}(V,W_2) \to \mathrm{Hom}(V,W_3) \to 0.$$

If k is a projective object in $\mathcal{M}^{\mathrm{fd}H}$, then we have the following exact sequence of vector spaces

$$\begin{array}{rcccc} 0 & \to & \mathrm{Hom}^H(k,\mathrm{Hom}(V,W_1)) & \to & \mathrm{Hom}^H(k,\mathrm{Hom}(V,W_2))\\ & \to & \mathrm{Hom}^H(k,\mathrm{Hom}(V,W_3)) & \to & 0\end{array}$$

It follows from (2) in Proposition 1.2 that the sequence

$$0 \to \mathrm{Hom}^H(V, W_1) \to \mathrm{Hom}^H(V, W_2) \to \mathrm{Hom}^H(V, W_3)) \to 0$$

is exact, so V is a projective object in $\mathcal{M}^{\mathrm{fd}H}$, and therefore any subcomodule of V is a direct summand of V in $\mathcal{M}^H$. It follows that V is semisimple in $\mathcal{M}^H$. Let M be in $\mathcal{M}^H$. By the Fundamental Theorem of comodules [8, Theorem 2.1.7], each element $m \in M$ is contained in a finite-dimensional subcomodule V_m of M. In particular, every $m \in M$ is contained in a sum of simple subcomodules of M, this implies that M is the sum of a family of simple subojects. Using Zorn's Lemma we can show that this sum is direct. □

Using Proposition 1.2, we now give necessary and sufficient conditions for the rationality of $f \in \mathrm{Hom}(M, N)$.

Proposition 1.5. *Take two H-comodules M and N. For $f \in \mathrm{Hom}(M, N)$, the following assertions are equivalent.*

(1) $f \in \mathrm{HOM}(M, N)$*;*
(2) *there exists an H-comodule V, an element v in V and an H-colinear map $F : M \otimes V \to N$ such that $F(m \otimes v) = f(m)$ for all m in M;*

If H has the symmetry property, then (1) and (2) are equivalent to

(3) *there exists an H-comodule V, an element v in V and an H-colinear map $F' : M \to \mathrm{Hom}(V, N)$ such that $F'(m)(v) = f(m)$ for all m in M.*

In (2) and (3), we can choose V to be finite dimensional.

Proof. (2)$\Longleftrightarrow$(3) follows from Proposition 1.2.

(2)$\Longrightarrow$(1). We claim that $\rho(f) = F(- \otimes v_0) \otimes v_1$. Using the H-colinearity of F, we obtain

$$\begin{aligned} f(m_0)_0 \otimes S^{-1}(m_1) f(m_0)_1 &= F(m_0 \otimes v)_0 \otimes S^{-1}(m_1) F(m_0 \otimes v)_1 \\ &= F(m_0 \otimes v_0) \otimes S^{-1}(m_2) m_1 v_1 = F(m \otimes v_0) \otimes v_1 \end{aligned}$$

and (1) holds, as needed.

(1)$\Longrightarrow$(2). Take a finite dimensional H-subcomodule of $\mathrm{HOM}(M, N)$ containing f. Such a V exists by the Fundamental Theorem [8, Theorem 2.1.7]. Then define $F : M \otimes V \to N$ by

$$F(m \otimes v) = v(m)$$

Clearly $F(m \otimes f) = f(m)$, so we are done if we can show that F is H-colinear. Using the fact that $v \in V$ is rational, we find

$$\begin{aligned} F(m_0 \otimes v_0) \otimes m_1 v_1 &= v_0(m_0) \otimes m_1 v_1 \\ &= v(m_0)_0 \otimes m_2 S^{-1}(m_1) v(m_0)_1 = v(m)_0 \otimes v(m)_1 \\ &= F(m \otimes v)_0 \otimes F(m \otimes v)_1. \end{aligned}$$

□

Corollary 1.6. *Take $M,N,P\in\mathcal{M}^H$ be H-comodules. If $g\in\mathrm{HOM}(M,N)$ and $f\in\mathrm{HOM}(N,P)$, then $f\circ g\in\mathrm{HOM}(M,P)$.*

Proof. By Proposition 1.5, there exist finite dimensional H-comodules V and W, $v\in V$, $w\in W$ and H-colinear maps $G:\ M\otimes V\to N$, $F:\ N\otimes W\to P$ such that $G(m\otimes v)=g(m)$, $F(n\otimes w)=f(n)$ for all $m\in M$ and $n\in N$. The map

$$K:\ M\otimes V\otimes W\to P,\ K(m\otimes s\otimes t)=F(G(m\otimes s)\otimes t)$$

is H-colinear, and $K(m\otimes(v\otimes w))=(f\circ g)(m)$. □

Corollary 1.7. *For any $T\in\mathcal{M}^H$, $\mathrm{HOM}(-,T)$ and $\mathrm{HOM}(T,-)$ are left exact endofunctors of $\mathcal{M}^H$.*

Proof. Let $0\to M\xrightarrow{i}N\xrightarrow{\pi}P\to 0$ be an exact sequence in $\mathcal{M}^H$. Then

$$0\to\mathrm{Hom}(P,T)\to\mathrm{Hom}(N,T)\to\mathrm{Hom}(M,T)\to 0$$

is an exact sequence of vectorspaces. π is H-colinear, hence $\pi\in\mathrm{HOM}(N,P)$, by Lemma 1.1. Consequently, $f\circ\pi\in\mathrm{HOM}(N,T)$, for all $f\in\mathrm{HOM}(P,T)$. In a similar way, $f\circ i\in\mathrm{HOM}(M,T)$, for all $f\in\mathrm{HOM}(N,T)$, and it follows that $\mathrm{HOM}(-,T)$ is left exact. □

Proposition 1.8. *Let I be an injective object of $\mathcal{M}^H$. Then*

(1) *$\mathrm{HOM}(N,I)$ is an injective object of $\mathcal{M}^H$, for any $N\in\mathcal{M}^H$;*
(2) *$\mathrm{HOM}(-,I)$ is an exact endofunctor of $\mathcal{M}^H$.*

Proof. (1) follows from Proposition 1.2 and the fact that $N\otimes -:\ \mathcal{M}^H\to\mathcal{M}^H$ is exact.
(2) Let $0\to M\xrightarrow{i}N\xrightarrow{\pi}P\to 0$ be an exact sequence in $\mathcal{M}^H$. We know from Corollary 1.7 that

$$0\to\mathrm{HOM}(P,I)\to\mathrm{HOM}(N,I)\to\mathrm{HOM}(M,I)$$

is exact in $\mathcal{M}^H$. Take $f\in\mathrm{HOM}(M,I)$, and let V be a finite dimensional H-subcomodule of $\mathrm{HOM}(M,I)$ containing f. Clearly $i\otimes V:\ M\otimes V\to N\otimes V$ is an H-colinear monomorphism. As in the proof of (1) $\Rightarrow$ (2) in Proposition 1.5, we can show that

$$F:\ M\otimes V\to I,\ F(m\otimes v)=v(m)$$

is rational. Since $I\in\mathcal{M}^H$ is injective, there exists an H-colinear map $G:\ N\otimes V\to I$ such that $G\circ(i\otimes V)=F$. It follows from Proposition 1.5 that

$$g:\ N\to I,\ g(n)=G(n\otimes f)$$

is rational. On the other hand

$$f(m)=F(m\otimes f)=G((i\otimes V)(m\otimes f))=G(i(m)\otimes f)=(g\circ i)(m),$$

and it follows that $\mathrm{HOM}(N,I)\to\mathrm{HOM}(M,I)$ is surjective. □

We will use the following notation.

- $R^p a(\mathrm{co}H, -)$ are the right derived functors of the covariant left exact functor $(-)^{\mathrm{co}H} :\ \mathcal{M}^H \to \mathcal{M}$;
- $\mathrm{EXT}^p(M, -)$ are the right derived functors of $\mathrm{HOM}(M, -) :\ \mathcal{M}^H \to \mathcal{M}^H$;
- $\mathrm{Ext}^{H^p}(-, -)$ are the right derived functors of $\mathrm{Hom}^H(-, -) :\ \mathcal{M}^H \times \mathcal{M}^H \to \mathcal{M}$.

In particular, if M and N are H-comodules, then $\mathrm{EXT}^p(M, N)$ is also an H-comodule. If $V \in \mathcal{M}^H$ is finite dimensional, then $\mathrm{Hom}(V, -) \cong V^* \otimes M$, hence $\mathrm{HOM}(V, -)$ is exact, and $\mathrm{EXT}^q(V, M) = 0$ for all $q \geq 1$.

Proposition 1.9. *Let $M, N \in \mathcal{M}^H$.*

(1) *We have a spectral sequence*

$$R^p a(\mathrm{co}H, \mathrm{EXT}^q(M, N)) \ \Rightarrow\ \mathrm{Ext}^{H^{p+q}}(M, N)$$

with $p, q \geq 0$;

(2) *if M is finite dimensional, then*

$$R^p a(\mathrm{co}H, M^* \otimes N) = \mathrm{Ext}^{H^p}(M, N),$$

for all $p \geq 0$.

Proof. By Lemma 1.1, $\mathrm{HOM}(M, N)^{\mathrm{co}H} = \mathrm{Hom}^H(M, N)$, and the result follows from Proposition 1.8 (1) and Grothendieck's spectral sequence for composite functors. □

In order to be able to compute right derived functors, we describe injective resolutions of $M \in \mathcal{M}^H$.
Let V be a vector space. Then $V \otimes H$ is a right H-comodule, with coaction induced by the comultiplication, and we call $V \otimes H$ a free H-comodule. Recall [8, Prop. 2.4.7] that a right H-comodule M is an injective object in $\mathcal{M}^C$ if and only if it is a direct summand in a free H-comodule. In particular H is injective. Lemma 1.10 is the analog of [13, Prop. 3.10 (c)] for the category of H-comodules.

Lemma 1.10. *Take $M, N \in \mathcal{M}^H$. If $N \in \mathcal{M}^H$ is injective, then $M \otimes N$ is also injective. In particular, $M \otimes H$ is an injective object of $\mathcal{M}^H$.*

Proof. As we have seen above, N is a direct summand of $V \otimes H$, with V a vector space. Then $M \otimes N$ is a direct summand of $M \otimes V \otimes H$. Let M_{tr} be the vector space M with trivial H-coaction. We have an isomorphism of H-comodules

$$f :\ M \otimes V \otimes H \to M_{\mathrm{tr}} \otimes V \otimes H,\ f(m \otimes v \otimes h) = m_0 \otimes v \otimes m_1 h,$$

with inverse given by $f^{-1}(m \otimes v \otimes h) = m_0 \otimes v \otimes S(m_1)h$. So $M \otimes N$ is a direct summand of the free comodule $M_{\mathrm{tr}} \otimes V \otimes H$, and is an injective object of $\mathcal{M}^H$. □

For $M \in \mathcal{M}^H$, we define $C^q(M)$ and $\varphi_q:\ C^q(M) \to C^{q+1}(M)$ recursively by

$$C^{-1}(M) = M \text{ and } C^{q+1}(M) = C^q(M) \otimes H;$$
$$\varphi_{-1}:\ M \to M \otimes H,\ \varphi_{-1}(m) = m \otimes 1;$$
$$\varphi_{q+1}(u \otimes h) = u \otimes h \otimes 1 - \varphi_q(u) \otimes h.$$

It is clear that φ_q is H-colinear. Using induction on q, we easily show that $\varphi_{q+1} \circ \varphi_q = 0$, hence $\{C^q(M)\}_{q \geq 0}$ is a complex in $\mathcal{M}^H$. Now consider

$$\psi_q:\ C^q(M) \to C^{q-1}(M),\ \psi_q(u \otimes h) = \varepsilon(h)u.$$

Then a straightforward computation shows that

$$\varphi_{q-1} \circ \psi_q + \psi_{q+1} \circ \varphi_q = C^q(M),$$

the identity map on $C^q(M)$, for all $q \geq 0$. Hence $\mathrm{Im}\,(\varphi_q) \supset \mathrm{Ker}\,(\varphi_{q+1})$, and $C^*(M)$ is an acyclic complex. It follows from Lemma 1.10 that $C^q(M)$ is an injective object in $\mathcal{M}^H$, for all $q \geq 0$, hence $C^*(M)$ is an injective resolution of $M \in \mathcal{M}^H$. It follows that $R^p a(\mathrm{co}H, M)$ is the cohomology group of the complex $C^*(M)^{\mathrm{co}H}$, and $\mathrm{EXT}^p(M, N)$ is the cohomology group of the complex $\mathrm{HOM}(M, C^*(N))$.

2. The right derived functors of ${}_A\mathrm{HOM}(-,-)$ and ${}_A\mathrm{Hom}^H(-,-)$

Let A be a right H-comodule algebra. Recall that this is an algebra with a right H-coaction ρ_A such that the unit and the multiplication are right H-colinear, that is,

$$\rho_A(ab) = a_0b_0 \otimes a_1b_1 \quad \text{and} \quad \rho_A(1_A) = 1_A \otimes 1_H.$$

A vector space M with a left A-action and a right H-coaction ρ_M is called a relative (A, H)-Hopf module if

$$\rho_M(am) = a_0m_0 \otimes a_1m_1,$$

for all $a \in A$ and $m \in M$. ${}_A\mathcal{M}^H$ is the category of relative Hopf module and A-linear H-colinear maps. For two relative Hopf modules M and N, we let ${}_A\mathrm{Hom}^H(M, N)$ be the space of A-linear H-colinear maps, and

$${}_A\mathrm{HOM}(M, N) = {}_A\mathrm{Hom}(M, N) \cap \mathrm{HOM}(M, N).$$

The aim of this Section is to relate the right derived functors of ${}_A\mathrm{HOM}(-,-)$ and ${}_A\mathrm{Hom}^H(-,-)$ by a spectral sequence. The sequence collapses if H is cosemisimple. We can improve the results if A is left noetherian.

Lemma 2.1. *Let M and N be relative (A, H)-Hopf modules, and take $f \in {}_A\mathrm{Hom}(M, N)$.*

(1) *The k-linear map $\rho(f):\ M \to N \otimes H$ defined by*

$$\rho(f)(m) = f(m_0)_0 \otimes S^{-1}(m_1)f(m_0)_1$$

is A-linear, hence $\rho(f) \in {}_A\mathrm{Hom}(M, N \otimes H)$;

(2) *$f \in {}_A\mathrm{HOM}(M,N)$ if and only if there exists $f_0 \otimes f_1 \in {}_A\mathrm{Hom}(M,N) \otimes H$ such that*

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes S^{-1}(m_1)f(m_0)_1,$$

for all $m \in M$.

Proof. For all $a \in A$ and $m \in M$, we have

$$\begin{aligned}\rho(f)(am) &= f(a_0m_0)_0 \otimes S^{-1}(a_1m_1)f(a_0m_0)_1 \\ &= a_0 f(m_0)_0 \otimes S^{-1}(m_1)S^{-1}(a_2)a_1 f(m_0)_1 \\ &= a(\rho(f)(m)).\end{aligned}$$

This shows that $\rho(f) \in {}_A\mathrm{Hom}(M, N \otimes H)$. The second part then follows easily. □

Lemma 2.2. *Let M and N be relative (A,H)-Hopf modules.*

(1) *${}_A\mathrm{HOM}(M,N)$ is an H-subcomodule of* $\mathrm{HOM}(M,N)$*; it is the largest H-comodule contained in ${}_A\mathrm{Hom}(M,N)$;*
(2) *${}_A\mathrm{HOM}(M,N)^{\mathrm{co}H} = {}_A\mathrm{Hom}^H(M,N)$;*
(3) *if $M \in {}_A\mathcal{M}$ is finitely generated, then ${}_A\mathrm{HOM}(M,N) = {}_A\mathrm{Hom}(M,N)$.*

Proof. (1) follows from Lemma 2.1 and the fact that $\mathrm{HOM}(M,N)$ is the largest comodule contained in $\mathrm{Hom}(M,N)$ (see Section 1).
(2) similar to the proof of Lemma 1.1.
(3) follows from [5, Prop. 4.2]. □

Now let $M \in \mathcal{M}^H$, and $N \in {}_A\mathcal{M}^H$. We have seen in [5, Lemma 1.1] that $N \otimes M \in {}_A\mathcal{M}^H$, with A-action $a(n \otimes m) = an \otimes m$.
If A is commutative and $M, N \in {}_A\mathcal{M}^H$, then $M \otimes_A N \in {}_A\mathcal{M}^H$, with H-coaction

$$\rho_{M \otimes_A N}(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1.$$

Lemma 2.3. *Let $M \in \mathcal{M}^H$, and $N, P \in {}_A\mathcal{M}^H$. The isomorphism of vector spaces*

$$\phi : \ \mathrm{Hom}^H(N \otimes M, P) \to \mathrm{Hom}^H(M, \mathrm{HOM}(N,P)),\ \phi(f)(m)(n) = f(n \otimes m),$$

as discussed in Proposition 1.2, induces an isomorphism

$$\phi : \ {}_A\mathrm{Hom}^H(N \otimes M, P) \to \mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N,P)).$$

Proof. If $f \in {}_A\mathrm{Hom}^H(N \otimes M, P)$, then $\phi(f)(m)$ is A-linear, for all $m \in M$. If $g \in \mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N,P))$, then $\phi^{-1}(g)$ is A-linear. □

Corollary 2.4. *Let I be an injective object of ${}_A\mathcal{M}^H$.*

(1) *For every $N \in {}_A\mathcal{M}^H$, ${}_A\mathrm{HOM}(N,I)$ is an injective object of $\mathcal{M}^H$;*
(2) *I is an injective object of $\mathcal{M}^H$.*

Proof. (1) follows from Lemma 2.3 and the exactness of the functor $N \otimes (-) : \mathcal{M}^H \to {}_A\mathcal{M}^H$.
(2) By (1), ${}_A\mathrm{Hom}(A,I)$ is an injective object of $\mathcal{M}^H$. By [5, Lemma 1.1], ${}_A\mathrm{Hom}(A,I) \cong I$ in ${}_A\mathcal{M}^H$. □

Let $M \in {}_A\mathcal{M}^H$. We will use the following notation.

- ${}_A\mathrm{EXT}^q(M,-)$ are the right derived functors of
$$ {}_A\mathrm{HOM}(M,-):\ {}_A\mathcal{M}^H \to \mathcal{M}^H; $$
- ${}_A\mathrm{Ext}^{H^q}(-,-)$ are the right derived functors of
$$ {}_A\mathrm{Hom}^H:\ {}_A\mathcal{M}^H \times {}_A\mathcal{M}^H \to \mathcal{M}. $$

In particular, if $M, N \in {}_A\mathcal{M}^H$, then ${}_A\mathrm{EXT}^p(M,N) \in \mathcal{M}^H$.

Lemma 2.5. *For any $N \in {}_A\mathcal{M}^H$, we have $R^p a(\mathrm{co}H, N) = {}_A\mathrm{Ext}^{H^p}(A,N)$.*

Proof. By Lemma 2.2, ${}_A\mathrm{Hom}^H(A,N) = {}_A\mathrm{Hom}(A,N)^{\mathrm{co}H}$. By [5, Lemma 1.1], ${}_A\mathrm{Hom}(A,N)^{\mathrm{co}H} = N^{\mathrm{co}H}$. By Corollary 2.4 (2), an injective resolution of $N \in {}_A\mathcal{M}^H$ is also an injective resolution of $N \in \mathcal{M}^H$. □

Proposition 2.6. *Let $M, N \in {}_A\mathcal{M}^H$. Then we have a spectral sequence*

$$ R^p a(\mathrm{co}H, {}_A\mathrm{EXT}^q(M,N)) \ \Rightarrow\ {}_A\mathrm{Ext}^{H^{p+q}}(M,N). \tag{2} $$

Proof. We have that
$$ {}_A\mathrm{Hom}^H(M,N) = {}_A\mathrm{HOM}(M,N)^{\mathrm{co}H}. $$
By Corollary 2.4, the functor ${}_A\mathrm{HOM}(M,-)$ takes injective objects of ${}_A\mathcal{M}^H$ to injective objects of $\mathcal{M}^H$. The result then follows from Grothendieck's spectral sequence for composite functors. □

Corollary 2.7. *Assume that H is cosemisimple, and take $M, N \in {}_A\mathcal{M}^H$. Then*
$$ {}_A\mathrm{EXT}^q(M,N)^{\mathrm{co}H} = {}_A\mathrm{Ext}^{H^q}(M,N). $$

Proof. We know that $\mathcal{M}^H$ is a semisimple category. The result follows from Proposition 2.6. □

Proposition 2.8. (1) *For any $I \in {}_A\mathcal{M}^H$, the functors ${}_A\mathrm{HOM}(I,-)$ and ${}_A\mathrm{HOM}(-,I)$, from ${}_A\mathcal{M}^H$ to $\mathcal{M}^H$, are left exact.*
(2) *If $I \in {}_A\mathcal{M}^H$ is injective, then ${}_A\mathrm{HOM}(-,I)$ is exact.*

Proof. (1) Let $0 \to M \xrightarrow{i} N \xrightarrow{\pi} P \to 0$ be an exact sequence in ${}_A\mathcal{M}^H$. By Corollary 1.7,
$$ 0 \to \mathrm{HOM}(P,I) \to \mathrm{HOM}(N,I) \to \mathrm{HOM}(M,I) $$
is an exact sequence in $\mathcal{M}^H$. It is clear that $f \circ \pi \in {}_A\mathrm{HOM}(N,I)$ for all $f \in {}_A\mathrm{HOM}(P,I)$, and $f \circ i \in {}_A\mathrm{HOM}(M,I)$ for all $f \in {}_A\mathrm{HOM}(N,I)$. It follows that
$$ 0 \to {}_A\mathrm{HOM}(P,I) \to {}_A\mathrm{HOM}(N,I) \to {}_A\mathrm{HOM}(M,I) $$
is an exact sequence in $\mathcal{M}^H$. So the functor ${}_A\mathrm{HOM}(-,I)$ is left exact. In a similar way, we show that the functor ${}_A\mathrm{HOM}(I,-)$ is left exact.

(2) Let $f \in {}_A\mathrm{HOM}(M, I)$ and let V be a finite-dimensional H-subcomodule of $\mathrm{HOM}(M, I)$ containing f. The k-linear map

$$i \otimes V : M \otimes V \to N \otimes V$$

is a monomorphism of relative (A, H)-Hopf modules. The map

$$F : \ M \otimes V \to I, \ F(m \otimes v) = v(m)$$

is A-linear. As in the proof of (1) $\Rightarrow$ (2) in Proposition 1.5, we can show that F is H-colinear, hence F is a morphism in ${}_A\mathcal{M}^H$. Since I is injective, there exists a morphism $G : \ N \otimes V \to I$ in ${}_A\mathcal{M}^H$ such that $G \circ (i \otimes V) = F$. The map

$$g : \ N \to I, \ g(n) = G(n \otimes f)$$

is right A-linear, and it follows from Proposition 1.5 that g is rational, hence $g \in {}_A\mathrm{HOM}(N, I)$. Finally

$$f(m) = F(m \otimes f) = G \circ (i \otimes V)(m \otimes f) = G(i(m) \otimes f) = (g \circ i)(m),$$

and it follows that ${}_A\mathrm{HOM}(N, I) \to {}_A\mathrm{HOM}(M, I)$ is surjective. $\square$

Proposition 2.9. *Let A be left noetherian and $M \in {}_A\mathcal{M}^H$ finitely generated as a left A-module. If $I \in {}_A\mathcal{M}^H$ is injective, then ${}_A\mathrm{Ext}^p(M, I) = 0$, for all $p > 0$.*

Proof. Since M is finitely generated, there exist a finite dimensional H-comodule V_0 and an epimorphism $p_0 : \ P_0 = A \otimes V_0 \to M$ in ${}_A\mathcal{M}^H$, by [5, Prop. 4.1]. Then $K = \mathrm{Ker}\,(p_0)$ is a subobject of P_0 in ${}_A\mathcal{M}^H$. $P_0 \in {}_A\mathcal{M}$ is finitely generated, and A is left noetherian, so $K \in {}_A\mathcal{M}$ is also finitely generated. So we can find a finite dimensional H-comodule V_1 and an epimorphism $p_1 : \ P_1 = A \otimes V_1 \to K$, and we have that $\mathrm{Im}\,(p_1) = K = \mathrm{Ker}\,(p_0)$. Repeating this construction, we find an A-free resolution $P_\star$ of M in ${}_A\mathcal{M}^H$,

$$\cdots \to P_i = A \otimes V_i \to \ldots \to P_1 = A \otimes V_1 \to P_0 = A \otimes V_0 \to M \to 0,$$

with each V_i a finite dimensional H-comodule. For each $p > 0$, we have

$${}_A\mathrm{Ext}^p(M, I) = H^p({}_A\mathrm{Hom}(P_*, I)).$$

From Lemma 2.2, we know that

$${}_A\mathrm{HOM}(M, I) = {}_A\mathrm{Hom}(M, I) \text{ and } {}_A\mathrm{HOM}(P_i, I) = {}_A\mathrm{Hom}(P_i, I),$$

for all $i \geq 0$. On the other hand, $P_\star$ is an acyclic complex in ${}_A\mathcal{M}^H$. We deduce from Corollary 2.4 and Proposition 2.8 that ${}_A\mathrm{HOM}(P_\star, I)$ is an injective resolution of ${}_A\mathrm{HOM}(M, I)$ in $\mathcal{M}^H$, and it follows that $H^p({}_A\mathrm{HOM}(P_\star, I)) = 0$ for all $p > 0$. $\square$

Corollary 2.10. *Let A be left noetherian. Take $M, N \in {}_A\mathcal{M}^H$, with M finitely generated as an A-module and $E^* = \{E^i\}$ an injective resolution of N in ${}_A\mathcal{M}^H$. Then for all $p \geq 0$*

$$\text{(3)} \qquad {}_A\mathrm{Ext}^p(M, N) = {}_A\mathrm{EXT}^p(M, N) = H^p({}_A\mathrm{Hom}(M, E^*)).$$

Proof. For all $p \geq 0$, we have that

$$ {}_A\mathrm{Ext}^p(M,N) = H^p({}_A\mathrm{Hom}(P_\star,N)) = H^p({}_A\mathrm{HOM}(P_\star,N)), $$

where $P_\star$ is the A-free resolution of M constructed in Proposition 2.9. ${}_A\mathrm{HOM}(P_\star,N)$ is a complex in $\mathcal{M}^H$ which induces on each $H^p({}_A\mathrm{HOM}(P_\star,N))$ $= {}_A\mathrm{Ext}^p(M,N)$ a structure of H-comodule, so ${}_A\mathrm{Ext}^p(M,-)$ is a cohomological functor from ${}_A\mathcal{M}^H$ to $\mathcal{M}^H$ and, by Proposition 2.9, ${}_A\mathrm{Ext}^p(M,I)=0$ for all $p>0$ if $I \in {}_A\mathcal{M}^H$ is injective. Clearly the same property holds for ${}_A\mathrm{EXT}^p(M,-)$. By Lemma 2.2, we have that

$$ {}_A\mathrm{Ext}^0(M,N) = {}_A\mathrm{Hom}(M,N) = {}_A\mathrm{HOM}(M,N) = R^0({}_A\mathrm{HOM}(M,-))(N) $$

in ${}_A\mathcal{M}^H$. It follows that ${}_A\mathrm{Ext}^p(M,-)$ and ${}_A\mathrm{EXT}^p(M,-)$ coincide on ${}_A\mathcal{M}^H$ for $p \geq 0$, and we obtain the first equality of (3). The second one follows after we observe that ${}_A\mathrm{EXT}^p(M,N) = H^p({}_A\mathrm{HOM}(M,E^*)$ and, by Lemma 2.2, ${}_A\mathrm{HOM}(M,E^p) = {}_A\mathrm{Hom}(M,E^p)$, for all $p \geq 0$. □

Proposition 2.11. *Let A be left noetherian. Take $M,N \in {}_A\mathcal{M}^H$, with M finitely generated as a left A-module. Then we have a spectral sequence*

$$ R^p a(\mathrm{co}H, {}_A\mathrm{Ext}^q(M,N)) \;\Rightarrow\; {}_A\mathrm{Ext}^{H\,p+q}(M,N). $$

Proof. By Lemma 2.2, we know that ${}_A\mathrm{Hom}(M,N)^{\mathrm{co}H} = {}_A\mathrm{Hom}^H(M,N)$. By Proposition 2.8, the functor ${}_A\mathrm{Hom}(M,-)$ takes injective objects of ${}_A\mathcal{M}^H$ to injective objects of $\mathcal{M}^H$. Now $R^q({}_A\mathrm{Hom}(M,-))(N) = {}_A\mathrm{Ext}^q(M,N)$ for every $q \geq 0$, by Corollary 2.10. The result then follows from the Grothendieck spectral sequence for composite functors. □

Corollary 2.12. *Assume that H is cosemisimple, and that A is left noetherian. Take $M,N \in {}_A\mathcal{M}^H$, with M finitely generated as a left A-module. Then*

$$ {}_A\mathrm{Ext}^q(M,N)^{\mathrm{co}H} = {}_A\mathrm{Ext}^{H\,q}(M,N). $$

Proof. We know that $\mathcal{M}^H$ is a semisimple category, so the result follows from Corollary 2.10. □

With notation and assumptions as in Corollary 2.12, it follows that if $M \in {}_A\mathcal{M}^H$ is finitely generated and projective in ${}_A\mathcal{M}$, then M is also projective in ${}_A\mathcal{M}^H$.

Lemma 2.13. *Let A and H be commutative. Let take $M,N \in {}_A\mathcal{M}^H$. Then ${}_A\mathrm{HOM}(M,N) \in {}_A\mathcal{M}^H$. A fortiori ${}_A\mathrm{EXT}^p(M,N) \in {}_A\mathcal{M}^H$.*

Proof. By Lemma 2.2, ${}_A\mathrm{HOM}(M,N)$ is a a right H-comodule. For $a \in A$, we consider the k-linear map

$$ L(a):\; M \to M,\; L(a)(m) = am. $$

Then for all $m \in M$, we have that

$$ \begin{aligned} &(L(a)(m_0))_0 \otimes (L(a)(m_0))_1 S(m_1) = (am_0)_0 \otimes (am_0)_1 S(m_1) \\ &\quad = \; a_0 m_0 \otimes a_1 m_1 S(m_2) = a_0 m \otimes a_1 = L(a_0)(m) \otimes a_1, \end{aligned} $$

so $L(a)_0\otimes L(a)_1 = L(a_0)\otimes a_1$, and $L \in {}_A\mathrm{HOM}(M,M)$. For $f \in {}_A\mathrm{HOM}(M,N)$, we now set $af = f \circ L(a)$. It follows from Proposition 1.5 that $af \in \mathrm{HOM}(M,N)$, and it is clear that af is left A-linear. Hence ${}_A\mathrm{HOM}(M,N)$ is a left A-module. Let us finally check the compatibility relation between the action and coaction on ${}_A\mathrm{HOM}(M,N)$. For all $f \in {}_A\mathrm{HOM}(M,N)$, $m \in M$ and $a \in A$, we have

$$\begin{aligned}((af)_0 \otimes (af)_1)(m) &= ((af)(m_0))_0 \otimes ((af)(m_0))_1 S(m_1)\\ &= a_0(f(m_0)_0) \otimes a_1(f(m_0)_1)S(m_1)\\ &= a_0(f(m_0)_0) \otimes a_1(f(m_0)_1 S(m_1))\\ &= a_0(f_0(m)) \otimes a_1 f_1\\ &= (a_0 f_0)(m) \otimes a_1 f_1 = (a_0 f_0 \otimes a_1 f_1)(m).\end{aligned}$$

□

Let A be commutative, and take $M, N \in {}_A\mathcal{M}^H$. By [5, Lemma 1.1], $M \otimes_A N \in {}_A\mathcal{M}^H$. The action and coaction are given by the formulas

$$a(m \otimes n) = am \otimes n = m \otimes an;$$

$$\rho_{M\otimes_A N}(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1.$$

Proposition 2.14. *Let A and H be commutative, take $M, N, P \in {}_A\mathcal{M}^H$, and consider the natural k-isomorphism*

$$\phi :\ {}_A\mathrm{Hom}(M\otimes_A N, P) \to {}_A\mathrm{Hom}(M, {}_A\mathrm{Hom}(N,P)),\ \phi(f)(m)(n) = f(m\otimes n).$$

(1) *If $f \in {}_A\mathrm{Hom}(M\otimes_A N, P)$ is H-colinear, then $\phi(f)(m) \in {}_A\mathrm{HOM}(N,P)$, for every $m \in M$; furthermore $\phi(f)$ is H-colinear;*
(2) *ϕ induces a k-isomorphism*

$$\phi :\ {}_A\mathrm{Hom}^H(M \otimes_A N, P) \to {}_A\mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N,P));$$

(3) *If N is flat as a left A-module, then ${}_A\mathrm{HOM}(N,-)$ preserves the injective objects of ${}_A\mathcal{M}^H$.*

Proof. (1) and (2): an easy adaptation of the proof of (1) and (2) in Proposition 1.2.
(3) If $I \in {}_A\mathcal{M}^H$ is injective, then the functor ${}_A\mathrm{Hom}^H(-,I)$ is exact. N is flat as a left A-module, so $-\otimes_A N$ is an exact endofunctor of ${}_A\mathcal{M}^H$. It then follows from (2) that the functor ${}_A\mathrm{Hom}^H(-, {}_A\mathrm{HOM}(N,I))$ is exact. □

Proposition 2.15. *Let A and H be commutative, and take $M, N, P \in {}_A\mathcal{M}^H$. If N is flat as a left A-module, then we have a spectral sequence*

$${}_A\mathrm{Ext}^{Hp}(M, {}_A\mathrm{EXT}^q(N,P)) \ \Rightarrow\ {}_A\mathrm{Ext}^{Hp+q}(M \otimes_A N, P).$$

Proof. The functors ${}_A\mathrm{Hom}(M \otimes_A N, -)$ and ${}_A\mathrm{Hom}(M, {}_A\mathrm{HOM}(N,-))$ coincide on ${}_A\mathcal{M}^H$, by Proposition 2.14 (2). ${}_A\mathrm{HOM}(M,-)$ preserves the injectives of ${}_A\mathcal{M}^H$, by Proposition 2.14 (3). □

3. The functor ${}_B\mathrm{HOM}(A,-)$

Recall that $\varsigma \in H^*$ is called a left integral on H if $h^*\phi = h^*(1)\phi$ for all $h^* \in H^*$. Throughout this Section, we assume that H is cosemisimple, which is equivalent to the existence of a left integral ϕ on H^* such that $\phi(1) = 1$ (see e.g. [19]). For every $M \in \mathcal{M}^H$, we then have an H-colinear epimorphism (see [22, Prop. 1.5])

$$p_M:\ M \to M^{\mathrm{co}H},\ p_M(m) = \phi(m_1)m_0.$$

$M \in \mathcal{M}^H$ is called ergodic if $M^{\mathrm{co}H} = 0$. A subcomodule of an ergodic comodule is ergodic, and, for every $M \in \mathcal{M}^H$, $M/M^{\mathrm{co}H}$ is ergodic. Let $M_{\mathrm{co}H}$ be the maximal ergodic subcomodule of M. It is obvious that $M^{\mathrm{co}H} \cap M_{\mathrm{co}H} = 0$, and we have

Lemma 3.1. *Let H be a cosemisimple Hopf algebra. Then for all $M \in \mathcal{M}^H$,*

$$M = M^{\mathrm{co}H} \oplus M_{\mathrm{co}H}$$

as H-comodules.

The decomposition of Lemma 3.1 is functorial in the following sense. If $f:\ M \to M'$ is H-colinear, then $f(M^{\mathrm{co}H}) \subseteq M'^{\mathrm{co}H}$ and $f(M_{\mathrm{co}H}) \subseteq M'_{\mathrm{co}H}$. In particular, the projection $p_M:\ M \to M^{\mathrm{co}H}$ is H-colinear, and $f \circ p_M = p_{M'} \circ f$.
Let M be a B-module, and let H coact trivially on M. In particular, H coacts trivially on B, B is an H-comodule algebra, and M is a relative (B,H)-Hopf module.
Take $M \in {}_A\mathcal{M}^H$. For $b \in B = A^{\mathrm{co}H}$, the map $f_b \in \mathrm{End}(M)$ given by $f_b(m) = bm$ is H-colinear, so $f_b \circ p_M = p_M \circ f_b$. It follows that $f_b(M^{\mathrm{co}H}) \subseteq M^{\mathrm{co}H}$ and $f_b(M_{\mathrm{co}H}) \subseteq M_{\mathrm{co}H}$, that is, $M^{\mathrm{co}H}$ and $M_{\mathrm{co}H}$ are B-submodules (hence (B,H)-Hopf submodules) of M and p_M is B-linear (hence a morphism of (B,H)-Hopf modules).
Recall from [22, Lemmas 2.1 and 2.2] that ${}_B\mathrm{HOM}(A,M) \in {}_A\mathcal{M}^H$. The left A-action is given by the formula

$$(af)(a') = f(a'a).$$

For every M (resp. N) in ${}_B\mathcal{M}$ (resp. in ${}_A\mathcal{M}^H$), ${}_BE(M)$ (resp. ${}_AE^H(N)$) will be the injective hull of M in ${}_B\mathcal{M}$ (resp. of N in ${}_A\mathcal{M}^H$).

Lemma 3.2. *Let A be an H-comodule algebra.*

(1) *If $M \in {}_B\mathcal{M}$, then $A \otimes_B M \in {}_A\mathcal{M}^H$.*
(2) (a) *For $M \in {}_B\mathcal{M}$ and $N \in {}_A\mathcal{M}^H$, we have an isomorphism of k-vector spaces*

$${}_A\mathrm{Hom}^H(A \otimes_B M, N) \cong {}_B\mathrm{Hom}(M, N^{\mathrm{co}H});$$

(b) *For $M \in {}_B\mathcal{M}$ and $N \in {}_B\mathcal{M}^H$, we have an isomorphism of k-vector spaces*

$${}_B\mathrm{Hom}^H(M,N) \cong {}_B\mathrm{Hom}(M, N^{\mathrm{co}H}).$$

Proof. (1) is obvious. (2a) follows from the fact that we have a pair of adjoint functors $(A \otimes_B -, (-)^{\mathrm{co}H})$ between ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}$. (2b) follows after we take $A = B$ in (2a), with trivial coaction on A. □

Also recall the following results from [22, Theorem 2.3, Cor. 2.4 and 2.5].

Proposition 3.3. *Let A be an H-comodule algebra, and assume that H is cosemisimple. Take $N \in {}_B\mathcal{M}$ and $M \in {}_A\mathcal{M}^H$.*

(1) *The map*

$$\phi:\ {}_A\mathrm{Hom}^H(M, {}_B\mathrm{HOM}(A, N)) \to {}_B\mathrm{Hom}(M^{\mathrm{co}H}, N),\ \phi(f)(p_M(m)) = f(m)(1)$$

is an isomorphism of k-vector spaces;

(2) *the map*

$$F : {}_B\mathrm{HOM}(A, N)^{\mathrm{co}H} \to N,\ F(f) = f(1)$$

is an isomorphism of B-modules;

(3) *if $I \in {}_B\mathcal{M}$ is injective, then ${}_B\mathrm{HOM}(A, I) \in {}_A\mathcal{M}^H$ is injective.*

Theorem 3.4. *Let A be an H-comodule algebra, and assume that H is cosemisimple.*

(1) *If $N \in {}_B\mathcal{M}$ and M is an (A, H)-Hopf submodule of ${}_B\mathrm{HOM}(A, N)$, then $M^{\mathrm{co}H} = 0$ implies $M = 0$;*
(2) *if $M \to N$ is an essential monomorphism in ${}_B\mathcal{M}$, then ${}_B\mathrm{HOM}(A, M) \to {}_B\mathrm{HOM}(A, N)$ is an essential monomorphism in ${}_A\mathcal{M}^H$;*
(3) *if $N \in {}_B\mathcal{M}$, then ${}_AE^H({}_B\mathrm{HOM}(A, N)) \cong {}_B\mathrm{HOM}(A, {}_BE(N))$;*
(4) *if $N \in {}_B\mathcal{M}$, then $({}_AE^H({}_B\mathrm{HOM}(A, N)))^{\mathrm{co}H} \cong {}_BE(N)$.*

Proof. For any subset T of ${}_B\mathrm{HOM}(A, M)$, set $T(1) = \{f(1) \mid f \in T\}$.
(1) If $M^{\mathrm{co}H} = 0$, then ${}_A\mathrm{Hom}(M^{\mathrm{co}H}, N) = 0$, and, by Proposition 3.3(1), ${}_A\mathrm{Hom}^H(M, {}_B\mathrm{HOM}(A, N)) = 0$. Hence the inclusion map $M \to {}_B\mathrm{HOM}(A, N)$ is the zero map, hence $M = 0$.
(2) If L is a nonzero (A, H)-Hopf submodule of ${}_B\mathrm{HOM}(A, N)$, then by (1), $L^{\mathrm{co}H}$ is a nonzero B-submodule of ${}_B\mathrm{HOM}(A, N)$. By Proposition 3.3(2), this means that $L(1)$ is a nonzero B-submodule of N, so $L(1) \cap M \neq 0$. But $L(1) \cap M = ({}_B\mathrm{HOM}(A, N) \cap L)(1)$; so L meets ${}_B\mathrm{HOM}(A, N)$ nontrivially.
(3) By (2), ${}_B\mathrm{HOM}(A, N) \to {}_B\mathrm{HOM}(A, {}_BE(N))$ is an essential monomorphism in ${}_A\mathcal{M}^H$. But, by (2), ${}_B\mathrm{HOM}(A, {}_BE(N))$ is an injective object of ${}_A\mathcal{M}^H$.
(4) It follows from (3) that ${}_AE^H({}_B\mathrm{HOM}(A, N)) = {}_B\mathrm{HOM}(A, {}_BE(N))$, and from Proposition 3.3(3) that ${}_B\mathrm{HOM}(A, {}_BE(N))^{\mathrm{co}H} = {}_BE(N)$. □

By (1), the nonzero subobjects of ${}_B\mathrm{HOM}(A, N)$ in ${}_A\mathcal{M}^H$ contain nonzero coinvariants. We will see below that this - rather strong - property implies that M is an essential extension of $AM^{\mathrm{co}H}$.

Let H^* be the linear dual of H, and consider the smash product $A\#H^*$ (see e.g. [9]). Then we have a functor ${}_A\mathcal{M}^H \to {}_{A\#H^*}\mathcal{M}$, and, conversely, a left $A\#H^*$-module which is rational as an H^*-module can be regarded as a relative (A,H)-Hopf module.

Corollary 3.5. *Take $N \in {}_B\mathcal{M}$ and let $M \neq 0$ be a subobject of ${}_B\mathrm{HOM}(A,N)$ in ${}_A\mathcal{M}^H$. Take $m \in M$.*

1. *M is an essential extension of $AM^{\mathrm{co}H}$ in ${}_A\mathcal{M}^H$.*
2. *If $p_M(am)=0$ for all $a \in A$, then $m = 0$.*

Proof. (1) Let $L \neq 0$ be a subobject of M in ${}_A\mathcal{M}^H$. By Theorem 3.4(1), $L^{\mathrm{co}H} \neq 0$ and $L^{\mathrm{co}H} \subseteq M^{\mathrm{co}H}$, so $L \cap AM^{\mathrm{co}H} \neq 0$.
(2) By [8, p. 247], $A\#H^*$ is isomorphic as a left H^*-module to $H^* \otimes A$. So each element of $A\#H^*$ can be written as a finite sum $\sum_i h_i^* a_i$, with $h_i^* \in H^*$ and $a_i \in A$. Consider the $A\#H^*$-submodule P of M generated by m. If $m \neq 0$ then, by Theorem 3.4(1), P contains a nonzero coinvariant element $y = \sum h_i^* a_i m$. But $p_M(y) = y$ while $p_M(\sum h_i^* a_i m) = \sum h_i^* p_M(a_i m) = 0$, since p_M is H^*-linear. So $y = 0$, which is a contradiction. We conclude that $m = 0$. □

For $M \in {}_A\mathcal{M}^H$, we set

$$ {}^\bullet M = \{m \in M \mid p_M(am) = 0 \quad \text{for all} \quad a \in A\}. $$

Note that if $M \in {}_A\mathcal{M}^H$ is simple, and $M^{\mathrm{co}H} \neq 0$, then ${}^\bullet M = 0$. Indeed, if ${}^\bullet M = M$, then $p_M(m) = 0$ for every $m \in M$, so $M = M_{\mathrm{co}H}$, hence $M^{\mathrm{co}H} = 0$ which is a contradiction.

Lemma 3.6. *Let $M \in {}_A\mathcal{M}^H$ and consider the natural transformation (see* [22, Prop. 2.7]*)*

$$ \nu_M :\ M \to {}_B\mathrm{HOM}(A, M^{\mathrm{co}H}),\ \nu_M(m)(a) = p_M(am). $$

1. *${}^\bullet M = \mathrm{Ker}\,\nu_M$;*
2. *if $f : M \to M'$ is a morphism in ${}_A\mathcal{M}^H$ then $f({}^\bullet M) \subset {}^\bullet M'$;*
3. *${}^\bullet(M/{}^\bullet M) = 0$;*
4. *if M is a subobject of N in ${}_A\mathcal{M}^H$, then ${}^\bullet N \cap M = {}^\bullet M$;*
5. *if ${}^\bullet M = 0$, then ν_M is an essential monomorphism in ${}_A\mathcal{M}^H$.*

Proof. (1) is obvious, and (2) follows from the fact that $f \circ p_M = p_{M'} \circ f$.
(3) Observe that $({}^\bullet M)^{\mathrm{co}H} = 0$. Hence $(M/{}^\bullet M)^{\mathrm{co}H} = M^{\mathrm{co}H}$. As $\mathrm{Ker}\,(\nu_M) = {}^\bullet M$, the map ν_M factorizes through

$$ \overline{\nu}_M :\ M/{}^\bullet M \to {}_B\mathrm{HOM}(A, M^{\mathrm{co}H}). $$

Now ${}_B\mathrm{HOM}(A, M^{\mathrm{co}H}) \cong {}_B\mathrm{HOM}(A, (M/{}^\bullet M)^{\mathrm{co}H})$, so it follows that $\overline{\nu}_M = \nu_{M/{}^\bullet M}$. Take $m \in M$ such that the corresponding $[m] \in M/{}^\bullet M$ is in ${}^\bullet(M/{}^\bullet M)$. It follows from (1) that $\nu_{M/{}^\bullet M}([m]) = 0$, hence $m \in {}^\bullet M$, and it follows that ${}^\bullet(M/{}^\bullet M) = 0$.
(4) follows from the definition of ${}^\bullet M$, and the fact that the restriction of p_N to M is p_M.
(5) Assume that ${}^\bullet M = 0$, and identify M with $\nu_M(M)$. If $L \neq 0$ is a

subobject of ${}_B\mathrm{HOM}(A, M^{\mathrm{co}H})$, then, by Theorem 3.4, $L^{\mathrm{co}H} \neq 0$, and, by Proposition 3.3(2), $M^{\mathrm{co}H} = {}_B\mathrm{HOM}(A, M^{\mathrm{co}H})^{\mathrm{co}H}$, so $L \cap M \neq 0$. □

Lemma 3.6 can be used to characterise injective objects in ${}_A\mathcal{M}^H$ of the form ${}_B\mathrm{HOM}(A, I)$, with $I \in {}_B\mathcal{M}$ injective.

Theorem 3.7. (1) *If $E \in {}_A\mathcal{M}^H$ is injective and ${}^\bullet E = 0$, then $E^{\mathrm{co}H} \in {}_B\mathcal{M}$ is injective and $E \cong {}_B\mathrm{HOM}(A, E^{coH})$ in ${}_A\mathcal{M}^H$.*

(2) *If $M \in {}_A\mathcal{M}^H$ with ${}^\bullet M = 0$, then ${}_AE^H(M) \cong {}_B\mathrm{HOM}(A, {}_BE(M^{\mathrm{co}H}))$ in ${}_A\mathcal{M}^H$.*

Proof. (1) Set $E' = {}_BE(E^{\mathrm{co}H})$. Then $E^{\mathrm{co}H} \to E'$ is an essential monomorphism of B-modules, so, by Theorem 3.4(2), ${}_B\mathrm{HOM}(A, E^{\mathrm{co}H}) \to {}_B\mathrm{HOM}(A, E')$ is an essential monomorphism in ${}_A\mathcal{M}^H$. Since E is injective in ${}_A\mathcal{M}^H$, we have that

$$E \cong {}_B\mathrm{HOM}(A, E^{\mathrm{co}H}) \cong {}_B\mathrm{HOM}(A, E').$$

By Proposition 3.3(2), ${}_B\mathrm{HOM}(A, E')^{\mathrm{co}H} \cong E'$ is an injective B-module, so $E^{\mathrm{co}H} \cong E'$ is an injective B-module.
(2) Set $E = {}_BE(M^{\mathrm{co}H})$. By Proposition 3.3(3), ${}_B\mathrm{HOM}(A, E) \in {}_A\mathcal{M}^H$ is injective, and by Lemma 3.6(5) and Theorem 3.4(2), we have essential monomorphisms

$$M \to {}_B\mathrm{HOM}(A, M^{\mathrm{co}H}) \to {}_B\mathrm{HOM}(A, E)$$

in ${}_A\mathcal{M}^H$, and it follows that ${}_AE^H(M) \cong {}_B\mathrm{HOM}(A, E)$ in ${}_A\mathcal{M}^H$. □

Remark 3.8. It follows from Corollary 3.5(2) that we have the following converse of Theorem 3.7(1): if an injective object $E \in {}_A\mathcal{M}^H$ is isomorphic to ${}_B\mathrm{HOM}(A, E^{coH})$ in ${}_A\mathcal{M}^H$, then ${}^\bullet E = 0$.

It is well-known that ${}_A\mathcal{M}$ has an injective cogenerator I, and it follows from [14, Prop. 1, Theorem 3] that $I \otimes H$ is an injective cogenerator of ${}_A\mathcal{M}^H$. If ${}_A\mathcal{M}^H$ has an injective cogenerator C with ${}^\bullet C = 0$, then it follows that ${}^\bullet M = 0$, for every $M \in {}_A\mathcal{M}^H$, by Lemma 3.6(4). In this case, we will say that ${}_A\mathcal{M}^H$ satisfies the condition (α).

Proposition 3.9. *Assume that ${}_A\mathcal{M}^H$ satisfies condition (α).*

(1) *Every injective object of ${}_A\mathcal{M}^H$ is isomorphic to ${}_B\mathrm{HOM}(A, I)$, for some injective left A-module I;*
(2) *For $M \in {}_B\mathcal{M}$ and $N \in {}_A\mathcal{M}^H$, we have*

$$_A\mathrm{Ext}^{Hp}(A \otimes_B M, N) \cong {}_B\mathrm{Ext}^p(M, N^{\mathrm{co}H}),$$

for all $p \geq 0$.

Proof. (1) follows immediately from Theorem 3.7 and Remark 3.8.
(2) Let $E^* = \{E^i\}$ be an injective resolution of N in ${}_A\mathcal{M}^H$. By Lemma 3.2(2),

$${}_A\mathrm{Hom}^H(A \otimes_B M, E^i) \cong {}_B\mathrm{Hom}(M, E^{i\,\mathrm{co}H}),$$

for every i, so we have that

$$ {}_A\mathrm{Ext}^{H^p}(A\otimes_B M, N) = H^p({}_B\mathrm{Hom}(M, E^{*\mathrm{co}H})). \tag{4} $$

for all $p \geq 0$. But the functor $(-)^{\mathrm{co}H}$ is exact and by Theorem 3.7(1), each $E^{i^{\mathrm{co}H}}$ is injective in ${}_B\mathcal{M}$. Hence $\{E^{i^{\mathrm{co}H}}\}$ is an injective resolution of $N^{\mathrm{co}H}$ in ${}_B\mathcal{M}$, and the right hand side of (4) is ${}_B\mathrm{Ext}^p(M, N^{\mathrm{co}H})$. □

Lemma 3.10. *Let A be noetherian, and $\{E^i \mid i \in I\}$ be a set of injective B-modules. We have the following isomorphism in ${}_A\mathcal{M}^H$:*

$$ E = \bigoplus_{i\in I} {}_B\mathrm{HOM}(A, E^i) \cong {}_B\mathrm{HOM}\Big(A, \bigoplus_{i\in I} E_i\Big). $$

Proof. $E \in {}_A\mathcal{M}^H$ is injective, by Proposition 3.3(3), and ${}^\bullet_B\mathrm{HOM}(A, E^i) = 0$ for all $i \in I$, by Corollary 3.5(2), hence ${}^\bullet E = 0$. $E^{\mathrm{co}H} = \bigoplus_{i\in I} E^i$, by Theorem 3.4. We have seen in Lemma 3.6(5) that ν_E is an essential monomorphism in ${}_A\mathcal{M}^H$ and, since $E \in {}_A\mathcal{M}^H$ is injective, ν_E is an isomorphism in ${}_A\mathcal{M}^H$. □

Lemma 3.11. *Let $I \in {}_B\mathcal{M}$ be injective, and take $M \in {}_A\mathcal{M}^H$. Assume that ${}^\bullet M = 0$ and that $f:\ M \to {}_B\mathrm{HOM}(A, I)$ is an essential monomorphism in ${}_A\mathcal{M}^H$. Then*

$$ M^{\mathrm{co}H} \to {}_B HOM(A, I)^{\mathrm{co}H} = I $$

is an essential monomorphism in ${}_B\mathcal{M}$.

Proof. By Lemma 3.6(5), ν_M is an essential monomorphism in ${}_A\mathcal{M}^H$ and, by Proposition 3.3(3), ${}_B\mathrm{HOM}(A, I)$ is injective in ${}_A\mathcal{M}^H$. So there exists a morphism $h:\ {}_B\mathrm{HOM}(A, M^{\mathrm{co}H}) \to {}_B\mathrm{HOM}(A, I)$ in ${}_A\mathcal{M}^H$ such that $f = h \circ \nu_M$. Let L be a B-submodule of I such that $L \cap M^{\mathrm{co}H} = 0$. Then

$$ {}_B\mathrm{HOM}(A, M^{\mathrm{co}H}) \cap {}_B\mathrm{HOM}(A, L) = 0 $$

and ${}_B\mathrm{HOM}(A, L)$ is a relative (A, H)-Hopf submodule of ${}_B\mathrm{HOM}(A, I)$. If ${}_B\mathrm{HOM}(A, L) \neq 0$ then ${}_B\mathrm{HOM}(A, L)$ meets $M = \mu_M(M)$ nontrivially because $\nu_M(M) \subseteq {}_B\mathrm{HOM}(A, M^{\mathrm{co}H})$. This is impossible, so ${}_B\mathrm{HOM}(A, L) = 0$. We deduce from Proposition 3.3(2) that $0 = {}_B\mathrm{HOM}(A, L)^{\mathrm{co}H} = L$. □

Now we are ready to show that the functor $(-)^{\mathrm{co}H}$ takes minimal injective resolutions of ${}_A\mathcal{M}^H$ to minimal injective resolutions of ${}_B\mathcal{M}$.

Proposition 3.12. *Assume that ${}_A\mathcal{M}^H$ satisfies condition (α), and that A and B are noetherian. Take $M \in {}_A\mathcal{M}^H$, and let $\{{}_A E^{H^i}(M)\}$ be the minimal injective resolution of M in ${}_A\mathcal{M}^H$ and $\{{}_B E^i(M^{\mathrm{co}H})\}$ the minimal injective resolution of $M^{\mathrm{co}H}$ in ${}_B\mathcal{M}$. Then $({}_A E^{H^i}(M))^{\mathrm{co}H} = {}_B E^i(M^{\mathrm{co}H})$, for all i.*

Proof. Set $E^i = {}_A E^{H^i}(M)$ and $K^i = \mathrm{Ker}\,(E^i \to E^{i+1})$, for all $i \geq 0$. It follows from Theorem 3.7(1) that $I^i = E^{i^{\mathrm{co}H}}$ is an injective B-module and $E^i = {}_B\mathrm{HOM}(A, I^i)$. Since H is cosemisimple, the sequence

$$ I^0 \to I^1 \to \cdots \to I^i \to \cdots $$

is exact in ${}_B\mathcal{M}$ and $(K^i)^{\mathrm{co}H} = \mathrm{Ker}\,(I^i \to I^{i+1})$. Since ${}^\bullet(K^i) = 0$ and $K^i \to E^i = {}_B\mathrm{HOM}(A, I^i)$ is an essential monomorphism in ${}_A\mathcal{M}^H$, ${K^i}^{\mathrm{co}H} \to I^i = {}_B\mathrm{HOM}(A, I^i)^{\mathrm{co}H}$ is an essential monomorphism of B-modules, by Lemma 3.11, so $\{I^i\}$ is a minimal injective resolution of ${K^0}^{\mathrm{co}H} = M^{\mathrm{co}H}$ in ${}_B\mathcal{M}$. □

Theorem 3.13. *Assume that ${}_A\mathcal{M}^H$ satisfies condition (α), and that A and B are noetherian, with B commutative. Take $M \in {}_A\mathcal{M}^H$. For every $P \in \mathrm{Spec}\,(B)$, let $\mu_i(P, M^{\mathrm{co}H})$ be the number of times that ${}_BE(B/P)$ occurs in ${}_BE^i(M^{\mathrm{co}H})$. Then*

$$ {}_A{E^H}^i(M) = \bigoplus_{P \in \mathrm{Spec}\,(B)} {}_AE^H(A/PA)^{\mu_i(P, M^{\mathrm{co}H})}. $$

Proof. By Theorem 3.7(1) and Proposition 3.12, $({}_A{E^H}^i(M))^{\mathrm{co}H}$ is B-injective and ${}_A{E^H}^i(M) \cong {}_B\mathrm{HOM}(A, ({}_A{E^H}^i(M))^{\mathrm{co}H}) = {}_B\mathrm{HOM}(A, {}_BE^i(M^{\mathrm{co}H}))$ in ${}_A\mathcal{M}^H$. So by the definition of μ_i and Lemma 3.10, ${}_A{E^H}^i(M)$ is the direct sum over $P \in spec(B)$ of $\mu_i(P, M^{\mathrm{co}H})$ copies of ${}_B\mathrm{HOM}(A, {}_BE(B/P))$. But $(A/PA)^{\mathrm{co}H} = B/P$, so, by Theorem 3.7(2), ${}_B\mathrm{HOM}(A, {}_BE(B/P)) \cong {}_AE^H(A/PA)$ in ${}_A\mathcal{M}^H$. □

Lemma 3.14. *Take $M \in {}_B\mathcal{M}$, $N \in {}_A\mathcal{M}^H$ and $V \in \mathcal{M}^H$ finite dimensional.*

1. *Assume that H has the symmetry property. We have the following isomorphisms in $\mathcal{M}^H$:*
$$ {}_A\mathrm{Hom}(A \otimes V, N) \cong \mathrm{Hom}(V, N) \cong V^* \otimes N \cong N \otimes V^*. $$
Consequently, $(N \otimes V^*)^{\mathrm{co}H}$ and $\mathrm{Hom}^H(V, N)$ are isomorphic as vector spaces.
2. *If B be commutative, then the map*
$$ \phi : \ {}_B\mathrm{Hom}^H(V \otimes A, M) \to \mathrm{Hom}^H(V, {}_B\mathrm{HOM}(A, M)), \ \ \phi(f)(v)(a) = f(v \otimes a) $$
is an isomorphism of k-vectorspaces.

Proof. (1) The first two isomorphisms are well-known; the third one is a consequence of the symmetry property.
(2) is a direct consequence of Proposition 1.2(3). □

Remark 3.15. If A and H are commutative, then the isomorphisms in Lemma 3.14(1) are left A-linear, and therefore $(N \otimes V^*)^{\mathrm{co}H}$ and $\mathrm{Hom}^H(V, N)$ are isomorphic left B-modules.

The condition that ${}_A\mathcal{M}^H$ satisfies the condition (α) is quite restrictive; it implies that the coinvariants functor ${}_A\mathcal{M}^H \to {}_B\mathcal{M}$ preserves injectivity (this follows from Proposition 3.9(2)). We will see that - given some finiteness condition of the ring morphism $B \to A$ - this comes down to A being flat as a left B-module.

Let V be a simple subcomodule of a right H-comodule V. The sum M_V of all the subcomodules of M isomorphic to V will be called the *H-isotypic component* of M. This sum is a direct sum, and M_V is a semisimple subcomodule of M.

We want to describe the H-isotypic components of ${}_B\mathrm{HOM}(A,M)$. First we recall the following Lemma (see [13, 2.14] in the case where H is cocommutative). Also recall from [8, Prop. 2.4.13] that a simple H-subcomodule of an H-comodule is finite dimensional.

Lemma 3.16. *Let k be algebraically closed. Take $N \in \mathcal{M}^H$ and $V \in \mathcal{M}^H$ a simple H-subcomodule. Then*

$$\mathrm{Hom}^H(V,N)\otimes V \cong N_V$$

as H-comodules, and

$$N = \bigoplus \{N_V \mid V \subset N \text{ is a simple subcomodule}\}.$$

Remark 3.17. If H is commutative, and $N \in {}_A\mathcal{M}^H$, then $\mathrm{Hom}(V,N) \cong V^*\otimes N \in {}_A\mathcal{M}^H$, so $\mathrm{Hom}^H(V,N) \in {}_B\mathcal{M}^H$, and $\mathrm{Hom}^H(V,N)\otimes V \cong N_V$ is an isomorphism in ${}_B\mathcal{M}^H$.

Lemma 3.18. *Let k be algebraically closed. Take $M \in {}_A\mathcal{M}$ and $V \in \mathcal{M}^H$ simple. Then*

$${}_B\mathrm{HOM}(A,M)_V \cong {}_B\mathrm{HOM}(A_{V^*},M)$$

in $\mathcal{M}^H$.

Proof. Consider the canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}^H(V,{}_B\mathrm{HOM}(A,M)) &\cong {}_B\mathrm{Hom}^H(V\otimes A,M)\\ &\cong {}_B\mathrm{Hom}^H(V\otimes A_{V^*},M) \cong \mathrm{Hom}^H(V,{}_B\mathrm{HOM}(A_{V^*},M)).\end{aligned}$$

The first and third isomorphism follow from Lemma 3.14(2); the second follows from the fact that M is a trivial H-comodule and from the definition of A_{V^*}, namely, if W is another simple H-comodule, then $\mathrm{Hom}^H(V\otimes W^*,k) = \mathrm{Hom}^H(V,W^*) = 0$ if $W^* \neq V$. Now it follows from Lemma 3.16 that ${}_B\mathrm{HOM}(A,M)_V \cong {}_B\mathrm{HOM}(A_{V^*},M)$ as H-comodules. If W is another simple H-comodule, not isomorphic to V, then $\mathrm{Hom}^H(V\otimes A_{V^*},M) = 0$, since $\mathrm{Hom}^H(W\otimes V^*,k) = 0$. So we find that

$${}_B\mathrm{HOM}(A_{V^*},M) = \bigoplus_W {}_B\mathrm{HOM}(A_{V^*},M)_W = {}_B\mathrm{HOM}(A_{V^*},M)_V.$$

□

Remark 3.19. If H and A are commutative, then the isomorphism of Lemma 3.18 is an isomorphism in ${}_B\mathcal{M}^H$.

Corollary 3.20. *Assume that k is algebraically closed and that A and H are commutative. If the functor $(-)^{\mathrm{co}H}: {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ preserves injectives, then for every injective left B-module I, and for every simple H-comodule V, ${}_B\mathrm{HOM}(A_{V^*}, I)$ is an injective left B-module.*

Proof. Set $W = V^*$ and $E = {}_B\mathrm{HOM}(A_{V^*}, I)$. Then by Proposition 3.3(2), E is an injective object of ${}_A\mathcal{M}^H$. Then $M = A \otimes V$ is finitely generated in ${}_A\mathcal{M}^H$ and A-free. By Remark 3.15, $E \otimes W \cong {}_A\mathrm{Hom}(M, E)$ in ${}_A\mathcal{M}^H$. By Lemma 2.2, ${}_A\mathrm{HOM}(M, E) = {}_A\mathrm{Hom}(M, E)$, so it follows from Proposition 2.14 that ${}_A\mathrm{Hom}^H(-, E \otimes W) = {}_A\mathrm{Hom}^H(- \otimes_A M, E)$. Now ${}_A\mathrm{Hom}^H(-, E): {}_A\mathcal{M}^H \to \mathcal{M}$ is exact, so $E \otimes W$ is an injective object of ${}_A\mathcal{M}^H$, and it follows from the hypotheses that $(E \otimes W)^{\mathrm{co}H}$ is injective in ${}_B\mathcal{M}$. By Lemma 3.14 and Remark 3.15, $(E \otimes W)^{\mathrm{co}H} \cong \mathrm{Hom}^H(V, E)$ as a left B-module. Now, $E_V \cong \mathrm{Hom}^H(V, E) \otimes V$ in ${}_B\mathcal{M}^H$, so E_V is injective in ${}_B\mathcal{M}$. By Lemma 3.16, $E_V = {}_B\mathrm{HOM}(A, I)_V = {}_B\mathrm{HOM}(A_{V^*}, I)$. □

Under the assumptions of Corollary 3.20, if A_{V^*} is finitely generated as a left B-module, then ${}_B\mathrm{Hom}(A_{V^*}, I) = {}_B\mathrm{HOM}(A_{V^*}, I)$ is an injective left B-module. We will apply this result in Theorem 3.21.

Theorem 3.21. *Let k be an algebraically closed field and take A and H commutative. Assume that the functor $(-)^{\mathrm{co}H}: {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ preserves injectives, and let V be a simple H-comodule. If A_V is finitely generated as a B-module, then A_V is flat as a left B-module. If A_W is finitely generated as a B-module, for every simple H-comodule W, then A is B-flat.*

Proof. Let $I \in {}_B\mathcal{M}$ be injective. Then by [7, Prop. 6.5.1], we have the duality isomorphism

$$ {}_B\mathrm{Hom}(\mathrm{Tor}_1^B(M, A_V), I) = {}_B\mathrm{Ext}^1(M, {}_B\mathrm{Hom}(A_V, I)). $$

Lemma 2.2 and Corollary 3.20 show that ${}_B Hom(\mathrm{Tor}_1^B(M, A_V), I) = 0$ for all $M \in {}_B\mathcal{M}$, and it follows that $\mathrm{Tor}_1^B(M, A_V) = 0$: it suffices to take $I = {}_BE(\mathrm{Tor}_1^B(M, A_V))$. □

Lemma 3.22. *Let A be flat as a right B-module. Then the functor $(-)^{\mathrm{co}H}: {}_A\mathcal{M}^H \to {}_B\mathcal{M}$ preserves injectives.*

Proof. We have that ${}_A\mathrm{Hom}^H(A \otimes_B (-), I) = {}_B\mathrm{Hom}(-, I^{\mathrm{co}H})$ in ${}_B\mathcal{M}$, by Lemma 3.2(2). The functor $A \otimes_B (-): {}_B\mathcal{M} \to {}_A\mathcal{M}^H$ is exact. If $I \in {}_A\mathcal{M}^H$ is injective, then the functor ${}_A\mathrm{Hom}^H(-, I)$ is exact. □

Lemma 3.23. *Let M be a finitely generated left B-module, and $N \in {}_A\mathcal{M}^H$. Then for every i, ${}_B\mathrm{Ext}^i(M, N)$ is an H-comodule and*

$$ {}_B\mathrm{Ext}^i(M, N)^{\mathrm{co}H} = {}_B\mathrm{Ext}^i(M, N^{\mathrm{co}H}). $$

Proof. Let $\{F_i\}$ be a finitely generated free resolution of M. We can regard each F_i and M as objects of ${}_B\mathcal{M}^H$, with trivial H-coaction. It follows from [5, Lemma 1.1] that each ${}_B\mathrm{Hom}(F_i, N)$ is an H-comodule. Therefore, each

${}_B\mathrm{Ext}^i(M,N)$ is an H-comodule. Applying [5, Lemma 1.1] again, we find that ${}_B\mathrm{Hom}(F_i,N)^{\mathrm{co}H}={}_B\mathrm{Hom}(F_i,N^{\mathrm{co}H})$ for all i. The last assertion follows from the fact that the functor $(-)^{\mathrm{co}H}$ commutes with homology. □

Proposition 3.24. *Let A be finitely generated as a left B-module, $M\in {}_B\mathcal{M}$, and $N\in {}_A\mathcal{M}^H$. Then we have a spectral sequence*

$$ {}_A\mathrm{Ext}^{H^i}(N,{}_B\mathrm{Ext}^j(A,M)) \;\Rightarrow\; {}_B\mathrm{Ext}^{i+j}(N^{\mathrm{co}H},M);\; i,j\geq 0. $$

If A is left noetherian and N is finitely generated as a left A-module, then

$$ {}_A\mathrm{Ext}^i(N,{}_B\mathrm{Ext}^j(A,M))^{\mathrm{co}H} \;\Rightarrow\; {}_B\mathrm{Ext}^{i+j}(N^{\mathrm{co}H},M);\; i,j\geq 0. $$

Proof. Since A is a finitely generated B-module, we have ${}_B\mathrm{Hom}(A,M) = {}_B\mathrm{HOM}(A,M)$ and the first assertion follows from Proposition 3.3 and the Grothendieck spectral sequence for composite functors. The second assertion then follows from Corollary 2.12. □

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