To Bertram Kostant

A CONSTRUCTION OF GENERALIZED HARISH-CHANDRA MODULES FOR LOCALLY REDUCTIVE LIE ALGEBRAS

Ivan Penkov and Gregg Zuckerman

Abstract

We study cohomological induction for a pair $(\mathfrak{g}, \mathfrak{k})$, $\mathfrak{g}$ being an infinite dimensional locally reductive Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ being of the form $\mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$, where $\mathfrak{k}_0 \subset \mathfrak{g}$ is a finite dimensional reductive in $\mathfrak{g}$ subalgebra and $C_{\mathfrak{g}}(\mathfrak{k}_0)$ is the centralizer of $\mathfrak{k}_0$ in $\mathfrak{g}$. We prove a general non-vanishing and $\mathfrak{k}$-finiteness theorem for the output. This yields in particular simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$ which are analogs of the fundamental series of generalized Harish-Chandra modules constructed in [PZ1] and [PZ2]. We study explicit versions of the construction when $\mathfrak{g}$ is a root-reductive or diagonal locally simple Lie algebra.

(2000 MSC): Primary 17B10, Secondary 17B55

1 Introduction

A *locally reductive Lie algebra* is defined as a union $\cup_{n\in\mathbb{Z}_{>0}}\mathfrak{g}_n$ of nested finite dimensional reductive Lie algebras $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ such that each $\mathfrak{g}_n$ is reductive in $\mathfrak{g}_{n+1}$. The class of locally reductive Lie algebras is a very natural and interesting class of infinite dimensional Lie algebras, and no classification is known. There are two (intersecting) subclasses of locally reductive Lie algebras which are relatively well-understood, see Subsection 2.3: the root-reductive Lie algebras, [DP], [B], and the locally simple diagonal Lie algebras, [BZh]. For instance, the Lie algebra $g\ell(\infty)$ of infinite matrices with only finitely many non-zero entries is root-reductive, and the Lie algebra $g\ell(2^\infty)$, defined as the union $\cup_{n\in\mathbb{Z}_{>0}}g\ell(2^n)$ via the injections

$$\begin{array}{ccc} g\ell(2^n) & \subset & g\ell(2^{n+1}) \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \end{array}$$

is diagonal. Both of the above classes of Lie algebras yield explicit examples of the general construction of this paper.

Representations of direct limit Lie groups have been studied for quite a considerable time now, [Ha], [Ne], [O1], [O2], [NO], [W], [NRW], however the theory of direct limit group representations has not been related in a systematic way to modules over the direct limit Lie algebra. In our opinion, this problem deserves further investigation.

In this paper we restrict ourselves to representations of locally reductive Lie algebras $\mathfrak{g}$ and we initiate the study of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$. More specifically, we provide a construction of such modules when $\mathfrak{k}$ is the form $\mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ for a finite-dimensional reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}_0$ ($C_{\mathfrak{g}}(\cdot)$ denotes centralizer in $\mathfrak{g}$). If $\mathfrak{g}$ is root-reductive, such subalgebras $\mathfrak{k}$ may equal the fixed vectors of an involution on $\mathfrak{g}$, hence $(\mathfrak{g}, \mathfrak{k})$-modules of finite type generalize Harish-Chandra modules. Our main construction is a generalization of the fundamental series for subalgebras $\mathfrak{k} \subset \mathfrak{g}$ of the form $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$, cf. [PZ2]. We use the derived functor of the functor of locally finite $\mathfrak{k}_0$-vectors. Its output is automatically endowed with a $(\mathfrak{g}, \mathfrak{k})$-module structure. Our finiteness result is based on a general finiteness theorem for cohomological induction which asserts $\mathfrak{k}$-finiteness of the output provided the input is $\mathfrak{k} \cap \mathfrak{m}$-finite, $\mathfrak{m}$ being the reductive part of the compatible parabolic subalgebra. A main technical observation of this paper is that one can construct reasonably large classes of parabolically induced modules which are $\mathfrak{k} \cap \mathfrak{m}$-finite, both when $\mathfrak{g}$ is root-reductive and when $\mathfrak{g}$ is a diagonal. This is based on the stabilization of the branching multiplicities of certain tensor representations of classical Lie algebras of increasing rank.

Our main interest is in constructing simple $(\mathfrak{g}, \mathfrak{k})$-modules M which in addition to being of finite type are also strict, i.e. for which $\mathfrak{k}$ coincides with the subalgebra of $\mathfrak{g}$ consisting of all elements $g \in \mathfrak{g}$ which act locally finitely on M (the Fernando-Kac subalgebra of M). In particular, we provide sufficient conditions for strictness of the modules constructed.

The theory of $(\mathfrak{g}, \mathfrak{k})$-modules for locally reductive Lie algebras $\mathfrak{g}$ is still in its infancy and many questions remain off limits for this paper. This concerns for instance the problem of unitarizability of the $(\mathfrak{g}, \mathfrak{k})$-modules we construct. Another very interesting problem is to describe the locally reductive subalgebras $\mathfrak{k} \subset \mathfrak{g}$ which admit strict simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type. Our paper deals with subalgebras of the form $\mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$, and hence not with the case when $\mathfrak{k} = \mathfrak{h}$ is a splitting Cartan subalgebra of $s\ell(\infty), so(\infty)$ and $sp(\infty)$. In fact, using a theorem of S. Fernando, one can show that strict simple $(\mathfrak{g}, \mathfrak{h})$-modules of finite type exist only for $s\ell(\infty)$ and $sp(\infty)$ (I. Dimitrov, unpublished). Finally, we would like to point out that the idea of studying direct limits of cohomologically induced modules was first suggested by A. Habib in [Ha] and that this idea has been an inspiration for us.

Acknowledgement. We thank G. Olshanskii for helpful comments and J. Willenbring for a detailed discussion of Proposition 2.1. G. Zuckerman acknowledges the hospitality and support of the Jacobs University Bremen.

2 Preliminaries

2.1 Conventions

All vector spaces and Lie algebras are defined over $\mathbb{C}$. If p is a positive integer and W is a vector space or a Lie algebra, we set $W^p := \underbrace{W \oplus \ldots \oplus W}_{p \text{ times}}$. $T^{\cdot}(W) = \bigoplus_{k \geq 0} T^k(W)$ is the tensor algebra of W. The superscript * indicates dual space, and $\otimes = \otimes_{\mathbb{C}}$. If $\mathfrak{g}$ is a Lie algebra, $Z_{\mathfrak{g}}$ stands for the center of $\mathfrak{g}$, $C_{\mathfrak{g}}(\alpha)$ stands for the centralizer in $\mathfrak{g}$ of a subset $\alpha \subset \mathfrak{g}$, $U(\mathfrak{g})$ stands for the enveloping algebra and $Z_{U(\mathfrak{g})}$ stands for the center of $U(\mathfrak{g})$.

The sign $\in\!\!\!\!+$ denotes semidirect sum of Lie algebras. A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is *reductive in* $\mathfrak{g}$ if under the adjoint action of $\mathfrak{k}, \mathfrak{g}$ is a semisimple $\mathfrak{k}$-module. If $\mathfrak{l}$ is any subalgebra of $\mathfrak{g}$ and M is an $\mathfrak{l}$-module, we denote the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M$ by $\mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}} M$. If $\mathfrak{l}'$ is a finite dimensional Lie algebra, by $V_{\mathfrak{l}'}(\lambda)$ we denote the simple finite dimensional $\mathfrak{l}'$-module with highest weight λ. When we write a vector space W as $\cup_{n\in\mathbb{Z}_{>0}} W_n$ we automatically assume that $W_n \subset W_{n+1}$ for $n \in \mathbb{Z}_{>0}$.

2.2 A stabilization result

Proposition 2.1 *Let $\mathfrak{s}_n$ be a sequence of classical finite dimensional simple Lie algebras of rank n and of fixed type A, B, C or D. Denote by V_n the natural $\mathfrak{s}_n$-module. Then, for any fixed $a, b, c, k \in \mathbb{Z}_{>0}$ the length of the $\mathfrak{s}_n$-module $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ stabilizes when $n \to \infty$ (here $\mathbb{C}$ stands for the trivial 1-dimensional $\mathfrak{s}_n$-module).*

Proof. This result is a relatively straightforward corollary of the results in [HTW], and we describe the argument only very briefly. Assume that $\mathfrak{s}_n = s\ell(n+1)$, let $\mathfrak{h}_n$ be the diagonal subalgebra, $\mathfrak{b}_n$ be the upper-triangular subalgebra, and $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n+1}$ be the standard basis in $\mathfrak{h}_n^*$. We will view any $\mathfrak{b}_n$-dominant weight $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i$ of $\mathfrak{s}_n$, $\lambda_1 \geq \dots \geq \lambda_n$, $\lambda_i \in \mathbb{Z}$ as a $\mathfrak{b}_{n+k}$-dominant weight of $\mathfrak{s}_{n+k}$ by inserting k zeroes in the non-increasing sequence $\lambda_1 \geq \dots \geq \lambda_{n+1}$ so that the remaining sequence remains non-increasing. Therefore, for a fixed n_0 and a $\mathfrak{b}_{n_0}$-dominant weight λ as above, the $\mathfrak{s}_n$-module $V_{\mathfrak{s}_n}(\lambda)$ is well defined for $n \geq n_0$. The first fact needed in the proof of Proposition 2.1 is that for fixed a, b, c, k, there is an integer n_0 such that all simple constituents of $X_n := T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ are of the form $V_{\mathfrak{s}_n}(\lambda)$ for $n \geq n_0$, where λ runs over a finite set of $\mathfrak{b}_{n_0}$-dominant weights of $\mathfrak{s}_{n_0}$. This is proved by a straightforward induction on k.

All that remains to show now is that for each $V_{\mathfrak{s}_n}(\lambda)$ with λ as above, $\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n)$ stabilizes when $n \to \infty$. This can also be done by induction on k. The case $k = 1$ is obvious, so we can assume that the statement is true for $1, 2, \dots, k$. Then, in order to prove the Proposition for $k+1$, it suffices to show that $\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n)$ and $\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n^*)$ stabilize for $n \to \infty$. Note that

$$\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n) = \dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*, X_n),$$

$$\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda), X_n \otimes V_n^*) = \dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda) \otimes V_n, X_n).$$

The statement follows now from the induction assumption and from the key formula 1.2.1 in [HTW] which implies that

$$\mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda'), V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*) \neq 0$$

for an independent on n finite set of weights λ' only (respectively,

$$\mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda''), V_{\mathfrak{s}_n}(\lambda) \otimes V_n) \neq 0$$

for an independent on n finite set λ'' only), and that $\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda'), V_{\mathfrak{s}_n}(\lambda) \otimes V_n^*)$ (resp., $\dim \mathrm{Hom}_{\mathfrak{s}_n}(V_{\mathfrak{s}_n}(\lambda''), V_{\mathfrak{s}_n}(\lambda) \otimes V_n)$) stabilizes for $n \to \infty$. The reader will easily fill in the details.

For $\mathfrak{s}_n$ of types B, C, D the argument is essentially the same and uses formulas 1.2.2 and 1.2.3 in [HTW]. $\square$

2.3 Locally reductive Lie algebras

We defined locally reductive Lie algebras in the Introduction. In the rest of this paper, when writing $\mathfrak{g} = \cup_{n\in\mathbb{Z}_{>0}}\mathfrak{g}_n$ for a locally reductive Lie algebra $\mathfrak{g}$, we will always assume that the $\mathfrak{g}_n$'s form a chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n \subset \mathfrak{g}_{n+1} \subset \ldots \tag{1}$$

of finite dimensional reductive Lie algebras such that each $\mathfrak{g}_n$ is reductive in $\mathfrak{g}_{n+1}$.

An important but quite restrictive class of locally reductive Lie algebras are the *root-reductive* Lie algebras. They have the form $\cup_{n\in\mathbb{Z}_{>0}}\mathfrak{g}_n$, where the chain (1) satisfies the requirement that each inclusion $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ is a root homomorphism, i.e. maps a Cartan subalgebra of $\mathfrak{g}_n$ into a Cartan subalgebra of $\mathfrak{g}_{n+1}$ and any root space of $\mathfrak{g}_n$ into a root space of $\mathfrak{g}_{n+1}$. A most natural example of a root-reductive Lie algebra is the Lie algebra $g\ell(\infty)$, defined via the chain $g\ell(i) \subset g\ell(i+1)$ of upper left-hand corner embeddings.

A Lie algebra $\mathfrak{s}$ is *locally simple* if $\mathfrak{s} = \cup_{n\in\mathbb{Z}_{>0}}\mathfrak{s}_n$ where $\mathfrak{s}_n$ are simple Lie algebras (in this case $\mathfrak{s}_n$ is automatically reductive in $\mathfrak{s}_{n+1}$), in particular a locally simple Lie algebra is locally reductive. Up to isomorphism, there are three simple infinite dimensional locally simple root-reductive Lie algebras: $s\ell(\infty), so(\infty)$ and $sp(\infty)$. They are defined by obvious chains of inclusions which are root-homomorphisms (in the case of $so(\infty)$ there are two natural choices: $\ldots \subset so(2i) \subset so(2i+2) \subset \ldots$ and $\ldots \subset so(2i+1) \subset so(2i+3) \subset \ldots$, however these yield isomorphic locally simple Lie algebras). The following structure theorem has been proved in [DP].

Theorem 2.2 *Let $\mathfrak{g}$ be a root-reductive Lie algebra.*

(a) The exact sequence

$$0 \to [\mathfrak{g},\mathfrak{g}] \to \mathfrak{g} \to \mathfrak{a} := \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to 0$$

splits, hence $\mathfrak{g}$ is isomorphic to the semidirect sum $[\mathfrak{g},\mathfrak{g}] \ni \mathfrak{a}$ ($\mathfrak{a}$ being an abelian Lie algebra).

(b) $[\mathfrak{g},\mathfrak{g}]$ is isomorphic to a direct sum of at most countably many copies of $s\ell(\infty), so(\infty), sp(\infty)$, as well as of simple finite dimensional Lie algebras.

A more general and very interesting class of locally reductive Lie algebras which are not necessarily root-reductive are the *diagonal* Lie algebras. By definition, a chain (1) of classical finite dimensional Lie algebras is *diagonal*, if for any n, the natural representation of $\mathfrak{g}_{n+1}$ is isomorphic to a direct sum of copies of the natural representation of $\mathfrak{g}_n$, of its dual and of the trivial representation. Locally simple diagonal Lie algebras have been classified up to isomorphism in [BZh]. In the present paper, we will restrict ourselves to the simplest subclass of diagonal Lie algebras $g\ell(p\Theta)$ defined below, however our results should extend without significant difficulty to general diagonal Lie algebras. Let $\theta_1, \theta_2, \ldots$ be an infinite sequence of integers greater than 1. We denote by Θ the formal product $\theta_1\theta_2\ldots$ and, for each $p \in \mathbb{Z}_{\geq 1}$, we define the Lie algebra $g\ell(p\Theta)$ (for $p = 1$ we write simply $g\ell(\Theta)$) as the union of the following diagonal chain

$$g\ell(p) \subset g\ell(p\theta_1) \subset g\ell(p\theta_1\theta_2) \subset \ldots$$

where, for $n \in \mathbb{Z}_{\geq 0}$, $g\ell(p\theta_1\theta_2\ldots\theta_{n-1})$ is embedded into $g\ell(p\theta_1\ldots\theta_n)$ by repeating a matrix $A \in g\ell(p\theta_1\ldots\theta_{n-1})$ θ_n times along the main diagonal in $g\ell(p\theta_1\ldots\theta_n)$. The locally simple diagonal Lie algebra $s\ell(p\Theta)$ is defined in the same way with $g\ell(p\theta_1\ldots\theta_n)$ replaced by

$s\ell(p\theta_1 \dots \theta_n)$. The reader will check immediately that $g\ell(p\Theta) = Z_{g\ell(p\Theta)} \oplus s\ell(p\Theta)$, the center $Z_{g\ell(p\Theta)}$ being 1-dimensional. The Lie algebra $g\ell(2^\infty)$ (see the Introduction) is the simplest example of a Lie algebra of the form $g\ell(p\Theta)$ (here $p = 2 = \theta_n, n \in \mathbb{Z}_{>0}$).

2.4 $(\mathfrak{g}, \mathfrak{k})$-modules

If $\mathfrak{g}$ is a locally reductive Lie algebra and M is a $\mathfrak{g}$-module, the *Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ consists of all elements $g \in \mathfrak{g}$ which act locally finitely on M, see [F], [DMP] and the references therein.

If $\mathfrak{g}$ is locally reductive and $\mathfrak{k} \subset \mathfrak{g}$ is a Lie subalgebra, we call a $\mathfrak{g}$-module M a *$(\mathfrak{g}, \mathfrak{k})$-module* if $\mathfrak{k} \subset \mathfrak{g}[M]$. In other words, M is a $(\mathfrak{g}, \mathfrak{k})$-module if for any $m \in M$ and any $n \in \mathbb{Z}_{>0}$ the $\mathfrak{k}_n$-submodule of M generated by m is finite-dimensional. We call a $(\mathfrak{g}, \mathfrak{k})$-module M *strict* if $\mathfrak{k} = \mathfrak{g}[M]$. Sometimes we use the term *$\mathfrak{k}$-integrable $\mathfrak{g}$-module* as an equivalent to $(\mathfrak{g}, \mathfrak{k})$-module.

Furthermore, we define a $(\mathfrak{g}, \mathfrak{k})$-module M to be of *finite type* if the following two conditions hold:

- every finitely generated $\mathfrak{k}$-submodule M' of M has finite length as a $\mathfrak{k}$-module;
- for every fixed simple integrable $\mathfrak{k}$-module L, the multiplicity of L as a subquotient of M' is bounded when M' runs over all finitely generated $\mathfrak{k}$-submodules of M.

If a $(\mathfrak{g}, \mathfrak{k})$-module M is not of *finite type*, we say that M is of infinite type. A *generalized Harish-Chandra module* is a finitely generated $\mathfrak{g}$-module M such that M is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type for some Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$.

Note that given any integrable $\mathfrak{k}$-module E, the induced $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}} E$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module, however in general (and more specifically, for $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ as in Section 3 below) $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}} E$ has infinite type[1]. Therefore for the construction of strict simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite t ype, one needs more sophisticated techniques than ordinary induction. As we show below, cohomological induction is an ideal tool for this purpose.

Here are two examples illustrating the notions of a $(\mathfrak{g}, \mathfrak{k})$-module of finite and of infinite type in the extreme case of an integrable $\mathfrak{g}$-module.

Proposition 2.3 *Let $\mathfrak{s} = \cup_{n \in \mathbb{Z}_{>0}} \mathfrak{s}_n$ be any infinite dimensional locally simple Lie algebra and $\mathfrak{k}_0 \subset \mathfrak{s}_1$ be a finite dimensional subalgebra of $\mathfrak{s}_1$. Let M be any non-trivial integrable $\mathfrak{s}$-module. Then M is an $(\mathfrak{s}, \mathfrak{k}_0)$-module of infinite type.*

Proof. Note first that $\dim M = \infty$. This follows from the fact that all $\mathfrak{s}_n$ have no non-trivial common finite dimensional module since $\dim \mathfrak{s}_n$ tends to ∞ when $n \to \infty$. Now, assume to the contrary that M is an $(\mathfrak{s}, \mathfrak{k}_0)$-module of finite type. Then M is a $(\mathfrak{s}, \mathfrak{s}_n)$-module of finite type for any $\mathfrak{s}_n$. We claim that this contradicts a result of Willenbring and Zuckerman. Indeed, Theorem 4.0.11 in [WZ] implies that if the difference of dimensions $\dim \mathfrak{s}_n - \dim \mathfrak{s}_1$ is sufficiently large, then there is a finite number of simple finite dimensional $\mathfrak{s}_1$-modules $W_1, \dots, W_x$ such that any simple finite dimensional $\mathfrak{s}_n$ module contains some W_j as a $\mathfrak{s}_1$-submodule. It is an immediate consequence of this fact that any infinite dimensional $(\mathfrak{s}, \mathfrak{s}_n)$-module of finite type is an $(\mathfrak{s}, \mathfrak{s}_1)$-module of infinite type as some W_j

[1] An interesting case when $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}} E$ has finite $\mathfrak{k}$-type is as follows. Using results of [NP] it is easy to construct an embedding $g\ell(\infty) \simeq \mathfrak{k} \subset \mathfrak{g} \simeq g\ell(\infty)$, so that $\mathfrak{g}/\mathfrak{k}$ is isomorphic as a $\mathfrak{k}$-module to natural $\mathfrak{k}$-module V (i.e. to the union of natural $\mathfrak{k}_n$-modules V_n, where $\mathfrak{k}_n \simeq g\ell(n)$). Then $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}} \mathbb{C} \simeq S^{\cdot}(\mathfrak{g}/\mathfrak{k}) \simeq S^{\cdot}(V)$, and it is easy to see that the symmetric algebra is a multiplicity free $\mathfrak{k}$-module, i.e., in particular, $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}}$ has finite type as a $(\mathfrak{g}, \mathfrak{k})$-module.

will appear with infinite multiplicity. This contradiction shows that our assumption was false, i.e. M is an $(\mathfrak{s}, \mathfrak{k}_0)$-module of infinite type. $\square$

Let now $\mathfrak{g} = g\ell(p\Theta)$ where $\Theta = \theta_1\theta_2\dots$ with $\theta_n > 1$ for all $n \in \mathbb{Z}_{>0}$, and let $\mathfrak{k}_0 := \mathfrak{g}_1 = g\ell(p)$. Set $\mathfrak{k}_n := \mathfrak{k}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$ for $\mathfrak{g}_n = g\ell(p\theta_1\dots\theta_{n-1})$, and $\mathfrak{k} := \cup_{n\in\mathbb{Z}_{>0}}\mathfrak{k}_n$. Then, as it is easy to check, $C_{\mathfrak{g}_n}(\mathfrak{k}_0) = g\ell(\theta_1\dots\theta_{n-1})$, and the inclusion $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \subset C_{\mathfrak{g}_{n+1}}(\mathfrak{k}_0)$ is nothing but the θ_n-diagonal inclusion. Hence $\mathfrak{k} \simeq g\ell(p) + g\ell(\Theta)$.

Proposition 2.4 *The adjoint representation of $g\ell(p\Theta)$ is a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module of finite length and thus, in particular, a $(g\ell(p\Theta), \mathfrak{k})$-module of finite type.*

Proof. The statement follows from the observation that for each n, the adjoint representation of $g\ell(p\theta_1\dots\theta_{n-1})$ considered as a $C_{\mathfrak{g}_n}(\mathfrak{k}_0) = g\ell(\theta_1\dots\theta_{n-1})$-module is a submodule of $T^2(V_n^p \oplus (V_n^*)^p)$, where V_n is the natural $g\ell(\theta_1\dots\theta_{n-1})$-module. By Proposition 2.1, the length of $T^2(V_n^p \oplus (V_n^*)^p)$ as an $s\ell(\theta_1\dots\theta_{n-1})$-module stabilizes for $n \to \infty$, hence the length of $g\ell(p\theta_1\dots\theta_{n-1})$ considered as a $C_{\mathfrak{g}_n}(\mathfrak{k}_0)$-module is bounded for $n \to \infty$. The reader will check immediately that this implies that the adjoint module of $g\ell(p\Theta)$ has finite length as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. $\square$

2.5 The Zuckerman functor

In this subsection $\mathfrak{g}$ is any Lie algebra and $\mathfrak{k}' \subset \mathfrak{g}$ is a finite dimensional subalgebra which acts locally finitely and semisimply on $\mathfrak{g}$. For instance, if $\mathfrak{g} = \cup_n\mathfrak{g}_n$ is locally reductive and $\mathfrak{k}' \subset \mathfrak{g}_n$ is a reductive in $\mathfrak{g}_n$ subalgebra for some n, the above condition is satisfied.

By $\mathcal{C}(\mathfrak{g}, \mathfrak{k}')$ we denote the category of all $(\mathfrak{g}, \mathfrak{k}')$-modules which are semisimple over $\mathfrak{k}'$. For any reductive in $\mathfrak{k}'$ subalgebra $\mathfrak{m}' \subset \mathfrak{k}'$, we consider the left exact functor

$$\begin{aligned} \Gamma_{\mathfrak{k}',\mathfrak{m}'} : \mathcal{C}(\mathfrak{g}, \mathfrak{m}') &\to \mathcal{C}(\mathfrak{g}, \mathfrak{k}') \\ M &\mapsto \Gamma_{\mathfrak{k}',\mathfrak{m}'}(M) := \sum_{X \subset M, X \in Ob(\mathcal{C}(\mathfrak{g},\mathfrak{k}'))} X \ . \end{aligned}$$

The category $\mathcal{C}(\mathfrak{g}, \mathfrak{m}')$ has sufficiently many injectives and hence one can introduce the right derived functor $R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}$. This functor is known as *the Zuckerman functor.*

A well known property of the Zuckerman functor which we use below is that if $Z_{U(\mathfrak{g})}$ acts via a fixed character on M, then $Z_{U(\mathfrak{g})}$ acts via the same character on $R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M)$. The following two propositions discuss some further fundamental properties of the functor $R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}$.

Proposition 2.5

(a) (restriction principle). Let $\mathfrak{g}' \subset \mathfrak{g}$ be an arbitrary Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{k}' \subset \mathfrak{g}'$. Then the diagram of functors

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{g}, \mathfrak{m}') & \xrightarrow{R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}} & \mathcal{C}(\mathfrak{g}, \mathfrak{k}') \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathfrak{g}', \mathfrak{m}') & \xrightarrow{R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}} & \mathcal{C}(\mathfrak{g}', \mathfrak{k}'), \end{array}$$

whose vertical arrows are restriction functors, is commutative.

(b) Let $U^0(\mathfrak{k}') := \Gamma_{\mathfrak{k}',\mathfrak{m}'}(\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{k}'),\mathbb{C}))$*. Then* $U^0(\mathfrak{k}')$ *is a* $U(\mathfrak{k}')$*-bimodule, and for any* M *in* $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$ *there is a natural isomorphism of* $\mathfrak{k}'$*-modules*

$$R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M) \cong H^{\cdot}(\mathfrak{k}',\mathfrak{m}', M \otimes U^0(\mathfrak{k}'))$$

(here we apply $R^{\cdot}\Gamma_{\mathfrak{k}'\mathfrak{m}'}$ *to objects of* $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$ *by setting* $\mathfrak{g}' = \mathfrak{k}'$*, see (a)).*

(c) Let M *be an inductive limit* $\varinjlim M_i$ *of modules* M_i *in* $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$*. Then*

$$R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M) \cong \varinjlim R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M_i).$$

Proof.

(a) It suffices to show that an injective object I in $\mathcal{C}(\mathfrak{g},\mathfrak{m}')$ is also injective in $\mathcal{C}(\mathfrak{g}',\mathfrak{m}')$. If Q is an arbitrary object in $\mathcal{C}(\mathfrak{g}',\mathfrak{m}')$, then $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q$ is an object in $\mathcal{C}(\mathfrak{g},\mathfrak{m}')$, and the functor

$$Q \mapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q$$

is exact. The natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} Q, I) = \mathrm{Hom}_{\mathfrak{g}'}(Q,I)$ shows that I represents an exact functor in $\mathcal{C}(\mathfrak{g}',\mathfrak{m}')$. Therefore I is injective in $\mathcal{C}(\mathfrak{g}',\mathfrak{m}')$, and (a) follows.

(b) This statement is a rephrasing of the isomorphism (4.5) in [EW].

(c) For any M in $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$, we use the standard complex for relative Lie algebra cohomology:

$$C^{\cdot}(\mathfrak{k}',\mathfrak{m}', M \otimes U^0(\mathfrak{k}')) = \mathrm{Hom}_{\mathfrak{m}'}(\Lambda^{\cdot}(\mathfrak{k}'/\mathfrak{m}'), M \otimes U^0(\mathfrak{k}')).$$

As $\mathfrak{k}'$ is finite-dimensional, we have an isomorphism

$$C^{\cdot}(\mathfrak{k}',\mathfrak{m}', M \otimes U^0(\mathfrak{k}')) \simeq \varinjlim C^{\cdot}(\mathfrak{k}',\mathfrak{m}', M_i \otimes U^0(\mathfrak{k}')),$$

and the fact that cohomology commutes with inductive limits implies (c). □

Proposition 2.6 *(comparison principle). Suppose* $\mathfrak{k}' = \mathfrak{k}'' \oplus \mathfrak{k}'''$ *is a decomposition into two ideals, and let* $\mathfrak{m}''$ *be a reductive in* $\mathfrak{k}''$ *subalgebra. Set* $\mathfrak{m}' := \mathfrak{m}'' \oplus \mathfrak{k}'''$*. Then for any* $(\mathfrak{g},\mathfrak{m}')$*-module* M*, there is a natural isomorphism of* $\mathfrak{g}$*-modules*

$$R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M) \simeq R^{\cdot}\Gamma_{\mathfrak{k}'',\mathfrak{m}''}(M). \tag{2}$$

Lemma 2.7 *Under the assumptions of Proposition 2.6, let* I *be an injective object in* $\mathcal{C}(\mathfrak{g},\mathfrak{m}')$*. Then*

$$R^t\Gamma_{\mathfrak{k}'',\mathfrak{m}''}(I) = 0 \text{ for } t > 0.$$

Proof of Lemma 2.7. As a $\mathfrak{k}'$-module I can be decomposed as $\oplus_\lambda (J_\lambda \boxtimes V_{\mathfrak{k}'''}(\lambda))$, where λ runs over all dominant integral weights of $\mathfrak{k}'''$ and where the J_λ's are $(\mathfrak{k}'',\mathfrak{m}'')$-modules. We claim that each J_λ is injective in $\mathcal{C}(\mathfrak{k}'',\mathfrak{m}'')$. Indeed, by the proof of the restriction principle (Proposition 2.5(a)) I is injective in $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$, hence for each λ, $J_\lambda \boxtimes V_{\mathfrak{k}'''}(\lambda)$ is injective in $\mathcal{C}(\mathfrak{k}',\mathfrak{m}')$. Therefore J_λ is injective in $C(\mathfrak{k}'',\mathfrak{m}'')$.

By Proposition 2.5(b)

$$R^{\cdot}\Gamma_{\mathfrak{k}'',\mathfrak{m}''}(I) \cong H^{\cdot}(\mathfrak{k}'',\mathfrak{m}'', I \otimes U^0(\mathfrak{k}'')),$$

and thus (since relative Lie algebra cohomology commutes with direct sums), it suffices to show that

$$H^t(\mathfrak{k}'',\mathfrak{m}'', (J_\lambda \boxtimes V_{\mathfrak{k}'''}(\lambda)) \otimes U^0(\mathfrak{k}'')) = 0 \tag{3}$$

for $t > 0$. However,

$$\begin{aligned}&H^t(\mathfrak{k}'', \mathfrak{m}'', (J_\lambda \boxtimes V_{\mathfrak{k}'''}(\lambda)) \otimes U^0(\mathfrak{k}'')) = \\ &= H^t(\mathfrak{k}'', \mathfrak{m}'', J_\lambda \boxtimes U^0(\mathfrak{k}'')) \boxtimes V_{\mathfrak{k}'''}(\lambda) = \\ &= R^t\Gamma_{\mathfrak{k}'',\mathfrak{m}''}(J_\lambda) \boxtimes V_{\mathfrak{k}'''}(\lambda) = 0\end{aligned}$$

since J_λ is injective in $C(\mathfrak{k}'', \mathfrak{m}'')$, and the Lemma follows. $\square$

Proof of Proposition 2.6 By Lemma 2.7, any $\mathcal{C}(\mathfrak{g}, \mathfrak{m}')$-injective resolution of M is $\Gamma_{\mathfrak{k}'',\mathfrak{m}''}$-acyclic hence it can be used both for the computation of $R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M)$ and of $R^{\cdot}\Gamma_{\mathfrak{k}'',\mathfrak{m}''}(M)$. This yields the natural isomorphism (2). $\square$

3 The Construction

Let $\mathfrak{g} = \cup_n \mathfrak{g}_n$ be a locally reductive Lie algebra and $\mathfrak{k}_0 \subset \mathfrak{g}_1$ be a finite dimensional subalgebra reductive in $\mathfrak{g}$ (equivalently, in $\mathfrak{g}_1$). Fix a Cartan subalgebra $\mathfrak{t}_0$ in $\mathfrak{k}_0$. For any $\mathfrak{g}_n$ we have the notion of a $\mathfrak{t}_0$-*compatible parabolic subalgebra* of $\mathfrak{g}_n$: by definition this is a parabolic subalgebra $\mathfrak{p}_n \subset \mathfrak{g}_n$ of the form $\bigoplus_{\sigma, \mathrm{Re}\sigma \geq 0} (\mathfrak{g}_n)^\sigma_{h_n}$, where h_n is a semisimple element of $\mathfrak{t}_0, \sigma$ runs over the eigenvalues of h_n in $\mathfrak{g}_n$, and $(\mathfrak{g}_n)^\sigma_{h_n}$ are the corresponding eigenspaces. We call a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ a $\mathfrak{t}_0$-*compatible parabolic subalgebra* if, for all n, $\mathfrak{p} \cap \mathfrak{g}_n$ is a $\mathfrak{t}_0$-compatible parabolic subalgebra of $\mathfrak{g}_n$ and $\mathfrak{n}_n = \mathfrak{n}_{n+1} \cap \mathfrak{g}_n$, where $\mathfrak{n}_n$ is the nilradical of $\mathfrak{p}_n$. It is possible (but not required) that there is a semisimple element $h \in \mathfrak{t}_0$ such that $\mathfrak{p} = \bigoplus_{\sigma, \mathrm{Re}\sigma \geq 0} \mathfrak{g}^\sigma_h$.

One can always choose decompositions $\mathfrak{p}_n = \mathfrak{m}_n \supsetplus \mathfrak{n}_n$ where, for each n, $\mathfrak{m}_n$ is a reductive in $\mathfrak{g}_n$ subalgebra such that $\mathfrak{m}_{n+1} \cap \mathfrak{g}_n = \mathfrak{m}_n$. This yields a decomposition $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$, where $\mathfrak{m} = \cup_n \mathfrak{m}_n$ and $\cup_n \mathfrak{n}_n$. By definition, $\mathfrak{n}$ is the *nilradical* of $\mathfrak{p}$ and $\mathfrak{m}$ is a locally reductive subalgebra of $\mathfrak{g}$. In what follows, we consider the decomposition $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ fixed and define $\bar{\mathfrak{n}}$ as the union $\cup_n \bar{\mathfrak{n}}_n$, where for each n, $\mathfrak{g}_n = \bar{\mathfrak{n}}_n \oplus \mathfrak{m}_n \oplus \mathfrak{n}_n$ is the canonical $\mathfrak{m}_n$-module decomposition. In this way, $\bar{\mathfrak{n}}$ is of course an integrable $\mathfrak{m}$-module.

Let $\mathfrak{k} := \mathfrak{k}_0 + C_\mathfrak{g}(\mathfrak{k}_0)$. Then $\mathfrak{k}_n = \mathfrak{k}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$ is reductive in $\mathfrak{g}$ for each n. Note that $\mathfrak{k} \cap \mathfrak{m} = \mathfrak{m}_0 + C_\mathfrak{g}(\mathfrak{k}_0)$, where $\mathfrak{m}_0 := \mathfrak{k}_0 \cap \mathfrak{m}$. Our goal is to construct nontrivial $(\mathfrak{g}, \mathfrak{k})$-modules by starting with a nontrivial $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module E and then applying a functor of cohomological induction type. We first extend E to a $\mathfrak{p}$-module by setting $\mathfrak{n} \cdot E = 0$. We then consider the induced module $M(\mathfrak{p}, E) := \mathrm{ind}^\mathfrak{g}_\mathfrak{p} E$. This is an integrable $\mathfrak{m} \cap \mathfrak{k}$-module. Indeed, the equality of $\mathfrak{m}$-modules $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$ implies via the Poincaré-Birkhoff-Witt theorem that $M(\mathfrak{p}, E)$ has an $\mathfrak{m}$-module filtration with associated graded equal to $S^{\cdot}(\bar{\mathfrak{n}}) \otimes E$. Both $S^{\cdot}(\bar{\mathfrak{n}})$ and E are integrable $\mathfrak{m} \cap \mathfrak{k}$-modules, thus $M(\mathfrak{p}, E)$ is also $\mathfrak{m} \cap \mathfrak{k}$-integrable.

We now set $A(\mathfrak{p}, E) := R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(M(\mathfrak{p}, E))$, where $s := \frac{1}{2}\dim(\mathfrak{k}_0/\mathfrak{m}_0)$. By definition $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k}_0)$-module, but as we show below $A(\mathfrak{p}, E)$ is in fact a $(\mathfrak{g}, \mathfrak{k})$-module. We also set $A(\mathfrak{p}_0, E) := R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{ind}^{\mathfrak{k}_0}_{\mathfrak{p}_0} E)$, where $\mathfrak{p}_0 := \mathfrak{k}_0 \cap \mathfrak{p}$ and we regard E as a module over $\mathfrak{m}_0 + C_\mathfrak{g}(\mathfrak{k}_0)$ and $\mathrm{ind}^{\mathfrak{k}_0}_{\mathfrak{p}_0} E$ as a $\mathfrak{k}_0 + C_\mathfrak{g}(\mathfrak{k}_0)$ module. By Proposition 2.5(a) there is a functorial morphism of $\mathfrak{k}_0$-modules

$$\Psi_E : A(\mathfrak{p}_0, E) \to A(\mathfrak{p}, E).$$

Knapp and Vogan [KV] call Ψ_E the *bottom layer map*. In the present paper, we call any $\mathfrak{g}$-subquotient of $A(\mathfrak{p}, E)$ generated by vectors in $\mathrm{im}\Psi_E$ *a bottom layer subquotient* of $A(\mathfrak{p}, E)$.

Note that $\mathfrak{m}_0 \cap C_{\mathfrak{g}}(\mathfrak{k}_0) = Z_{\mathfrak{k}_0}$. Therefore, if $\mathfrak{b}_{\mathfrak{m}_0}$ is a fixed Borel subalgebra of $\mathfrak{m}_0$, we can decompose E as

$$\bigoplus_{\nu} V_{\mathfrak{m}_0}(\nu) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''_{\nu},$$

where we consider $E''_{\nu} := \mathrm{Hom}_{\mathfrak{m}_0}(V_{\mathfrak{m}_0}(\nu), E)$ as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module and ν runs over all $\mathfrak{b}_{\mathfrak{m}_0}$-dominant integral weights of $\mathfrak{m}_0$.

Fix now a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{k}_0$ such that $\mathfrak{b}_0 \cap \mathfrak{m}_0 = \mathfrak{b}_{\mathfrak{m}_0}$. This defines two Weyl group elements: the element $w_{\mathfrak{k}_0} \in W_{\mathfrak{k}_0}$ of maximal length with respect to $\mathfrak{b}_0$, and the element $w_{\mathfrak{m}_0} \in W_{\mathfrak{m}_0}$ of maximal length with respect to $\mathfrak{b}_0 \cap \mathfrak{m}_0$. For any $\mathfrak{b}_{\mathfrak{m}_0}$-dominant $\mathfrak{k}_0$-integral weight ν, we set

$$\nu^{\vee} := w_{\mathfrak{k}_0} \circ w_{\mathfrak{m}_0}^{-1}(\nu + \rho_{\mathfrak{b}_0}) - \rho_{\mathfrak{b}_0},$$

where $\rho_{\mathfrak{b}_0}$ is the half-sum of the $\mathfrak{b}_0$-positive roots of $\mathfrak{k}_0$.

Lemma 3.1 *The $\mathfrak{k}$-module $A(\mathfrak{p}_0, E)$ is $\mathfrak{k}$-integrable and is isomorphic to $\bigoplus_{\nu} V_{\mathfrak{k}_0}(\nu^{\vee}) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''_{\nu}$, where as above ν runs over all dominant integral weights of $\mathfrak{m}_0$, and where $V_{\mathfrak{k}_0}(\nu^{\vee}) := 0$ whenever $\nu^{\vee}$ is not $\mathfrak{b}_0$-dominant and integral for $\mathfrak{k}_0$.*

Proof. This statement is a direct corollary of the Bott-Borel-Weil theorem proved in [EW], see [EW, Proposition 6.3]. $\square$

The following theorem is our main result.

Theorem 3.2

(a) $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module.

(b) If $M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type, then $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type.

(c) Assume $E = \cup_n E_n$ where each E_n is an $(\mathfrak{m}_n, \mathfrak{k} \cap \mathfrak{m}_n)$-module on which $Z_{\mathfrak{m}_n}$ acts via a 1-dimensional representation. Then the bottom layer map Ψ_E is an injection. Assume that for some ν, $E''_{\nu} \neq 0$ and $\nu^{\vee}$ is dominant integral for $\mathfrak{k}_0$. Then $\mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\nu^{\vee}), A(\mathfrak{p}, E)) = E''_{\nu}$. Hence $A(\mathfrak{p}, E)$ has a simple bottom layer subquotient.

(d) Assume $E = \cup_n E_n$ where each E_n is an $(\mathfrak{m}_n, \mathfrak{k} \cap \mathfrak{m}_n)$-module with $Z_{U(\mathfrak{m}_n)}$-character, that $A(\mathfrak{p}, E) \neq 0$, and that for some N the $Z_{U(\mathfrak{g}_N)}$ -character of $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ is not regular integral. Then some bottom layer subquotient of $A(\mathfrak{p}, E)$ is not an integrable $\mathfrak{g}$-module. If in addition, $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$, then some simple bottom layer subquotient of $A(\mathfrak{p}, E)$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module.

(e) Under the assumptions of (c) assume further that $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0)$ and that E is simple. Then $\mathfrak{t}_0$ acts via weight $\mu \in \mathfrak{t}_0^*$ on E, $\mu^{\vee}$ is dominant integral for $\mathfrak{k}_0$, and there is an isomorphism of $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$-modules

$$A(\mathfrak{p}_0, E) \simeq V_{\mathfrak{k}_0}(\mu^{\vee}) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''.$$

where E'' equals E considered as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Furthermore, Ψ_E yields an isomorphism between the $\mathfrak{k}$-modules $A(\mathfrak{p}_0, E)$ and $V_{\mathfrak{k}_0}(\mu^{\vee}) \otimes \mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\mu^{\vee}), A(\mathfrak{p}, E))$.

(f) If, under the assumptions of e), $\mathrm{im}\Psi_E$ is a simple $\mathfrak{k}$-submodule of $A(\mathfrak{p}, E)$, then $A(\mathfrak{p}, E)$ has a unique simple bottom layer subquotient. A sufficient condition for the simplicity of $\mathrm{im}\Psi_E$ is the inclusion $\mathfrak{m} \subset \mathfrak{k}$.

Proof.

(a) By construction, $M(\mathfrak{p},E)$ is a $(\mathfrak{g},\mathfrak{k}\cap\mathfrak{m})$-module. Since $\mathfrak{m}\cap\mathfrak{k}\supset C_{\mathfrak{g}}(\mathfrak{k}_0)$, $M(\mathfrak{p},E)$ is an integrable $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Let $\tilde{M}$ denote the restriction of $M(\mathfrak{p},E)$ to $\mathfrak{k}$: by Proposition 2.5(a) $A(\mathfrak{p},E)$ is isomorphic as a $\mathfrak{k}$-module to $R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\tilde{M})$. By the Poincaré-Birkhoff-Witt Theorem, the $\mathfrak{k}$-module $\tilde{M}$ has an increasing filtration with associated graded

$$\mathrm{Gr}\ \tilde{M}=\bigoplus_{t\in\mathbb{Z}_{\geq 0}}\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}(S^t(\mathfrak{k}_0^c\cap\bar{\mathfrak{n}})\otimes E),\tag{4}$$

where $\mathfrak{k}_0^c$ is a fixed $\mathfrak{k}_0$-module complement of $\mathfrak{k}_0$ in $\mathfrak{g}$. □

Lemma 3.3 *$R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\ \tilde{M})$ is a graded integrable $\mathfrak{k}$-module.*

Proof of Lemma 3.3. Decompose the $\mathfrak{m}_0+C_{\mathfrak{g}}(\mathfrak{k}_0)$-module $S^t(\mathfrak{k}_0^c\cap\bar{\mathfrak{n}})\otimes E$ as

$$\bigoplus_{\nu}V_{\mathfrak{m}_0}(\nu)\boxtimes_{U(Z_{\mathfrak{k}_0})}X_{\nu,t}$$

for some $C_{\mathfrak{g}}(\mathfrak{k}_0)$-modules $X_{\nu,t}$. Observe that each $X_{\nu,t}$ is an integrable $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. We obtain a $\mathfrak{k}$-module isomorphism

$$\mathrm{Gr}\ \tilde{M}\cong\bigoplus_{\nu,t}\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}\left(V_{\mathfrak{m}_0}(\nu)\boxtimes_{U(Z_{\mathfrak{k}_0})}X_{\nu,t}\right).$$

For each ν, let $G_{\nu}^{\cdot}$ be a resolution of $\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}V_{\mathfrak{m}_0}(\nu)$ by $\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}$-acyclic $(\mathfrak{k}_0,\mathfrak{m}_0)$-modules. We can compute $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\ \tilde{M})$ as

$$H^{\cdot}(\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\bigoplus_{\nu,t}G_{\nu}^{\cdot}\boxtimes_{U(Z_{\mathfrak{k}_0})}X_{\nu,t})),$$

which is isomorphic as a $\mathfrak{k}$-module to

$$\bigoplus_{\nu,t}H^{\cdot}(\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(G_{\nu,t}^{\cdot}))\boxtimes_{U(Z_{\mathfrak{k}_0})}X_{\nu,t}$$

and hence to

$$\bigoplus_{\nu,t}R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(V_{\mathfrak{m}_0}(\nu))\boxtimes_{U(Z_{\mathfrak{k}_0})}X_{\nu,t}.\tag{5}$$

Therefore $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\tilde{M})$ is an integrable $\mathfrak{k}$-module. This proves the Lemma. □

To complete the proof of (a) note that, by Proposition 2.5(c), $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}$ commutes with inductive limits. Since furthermore, $\mathcal{C}_{\mathfrak{g}}(\mathfrak{k}_0)$ acts by $\mathfrak{k}_0$-endomorphisms on $\tilde{M}$, $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\tilde{M})$ has an increasing filtration of $\mathfrak{k}_0+\mathcal{C}_{\mathfrak{g}}(\mathfrak{k}_0)$-modules induced by the filtration on $\tilde{M}$. An obvious induction argument using the fact that $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\tilde{M})$ is a $\mathfrak{k}$-integrable module (Lemma 3.3) implies that $R^{\cdot}\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\tilde{M})$ is filtered by $\mathfrak{k}$-integrable modules, and hence is itself $\mathfrak{k}$-integrable. This proves (a).

(b) Suppose $M(\mathfrak{p},E)$ is of finite type over $\mathfrak{k}\cap\mathfrak{m}=\mathfrak{m}_0+C_{\mathfrak{g}}(\mathfrak{k}_0)$. We can rewrite (4) as

$$\mathrm{Gr}\ \tilde{M}=\bigoplus_{\nu}(\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}V_{\mathfrak{m}_0}(\nu))\boxtimes_{U(Z_{\mathfrak{k}_0})}Y_{\nu}$$

with each $Y_\nu = \oplus_t X_{\nu,t}$ an integrable $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Since $\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} V_{\mathfrak{m}_0}(\nu)$ is a $(\mathfrak{k}_0, \mathfrak{m}_0)$-module, we conclude that every Y_ν is of finite type over $C_{\mathfrak{g}}(\mathfrak{k}_0)$. Combining (5) with Lemma 3.3, we obtain

$$R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\tilde{M}) \cong \bigoplus_\nu V_{\mathfrak{k}_0}(\nu^\vee) \boxtimes_{U(Z_{\mathfrak{k}_0})} Y_\nu. \tag{6}$$

The right hand side of (6) is of finite type over $\mathfrak{k}$ as each Y_ν is of finite type over $C_{\mathfrak{g}}(\mathfrak{k}_0)$ and $V_{\mathfrak{k}_0}(\nu'^\vee) \ncong V_{\mathfrak{k}_0}(\nu''^\vee)$ for $\nu' \neq \nu''$. Finally, the fact that $R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\tilde{M})$ is of finite type over $\mathfrak{k}$ implies that $R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(M)$ is of finite type over $\mathfrak{k}$. Indeed, this follows from the observation, that since $R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}$ commutes with inductive limits,

$$\mathrm{Gr}(R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\tilde{M})) \cong R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{Gr}\tilde{M}), \tag{7}$$

where the left hand side of (7) refers to the filtration of $R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\tilde{M})$ induced by the filtration on $\tilde{M}$. This proves (b).

(c) The theory of the bottom layer map in the finite dimensional case is elaborated by Knapp and Vogan in [KV, Ch.$\overline{\underline{\mathrm{V}}}$, Sec.6]. There the authors assume that they are working with a symmetric pair. However, a careful examination of Theorem 5.80 in [KV] reveals that the assumption that $\mathfrak{k}_0$ is symmetric in $\mathfrak{g}_n$ is not needed; hence our hypothesis on E_n implies that Ψ_{E_n} is an injection from $A(\mathfrak{p}_0, E_n)$ to $A(\mathfrak{p}_n, E_n) = R^s\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n)$ for each n. Furthermore, we have an injection of $\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n$ to $\mathrm{ind}_{\mathfrak{p}_{n+1}}^{\mathfrak{g}_{n+1}} E_{n+1}$ which induces a $\mathfrak{g}_n$-module homomorphism $\varphi_n : A(\mathfrak{p}_n, E_n) \to A(\mathfrak{p}_{n+1}, E_{n+1})$.

On the other hand, we have a canonical $\mathfrak{k}_0$-module homomorphism $\chi_n : A(\mathfrak{p}_0, E_n) \to A(\mathfrak{p}_0, E_{n+1})$ induced by the inclusion of E_n into E_{n+1}. Moreover, the diagram

$$\begin{tikzcd} A(\mathfrak{p}_0, E_{n+1}) \arrow[r, "\Psi_{E_{n+1}}"] & A(\mathfrak{p}_{n+1}, E_{n+1}) \\ A(\mathfrak{p}_0, E_n) \arrow[u, "\chi_n"] \arrow[r, "\Psi_{E_n}"] & A(\mathfrak{p}_n, E_n) \arrow[u, "\varphi_n"'] \end{tikzcd} \tag{8}$$

is commutative, and Ψ_{E_n} and $\Psi_{E_{n+1}}$ are injections. Consider the inductive limit homomorphism

$$\varinjlim \Psi_{E_n} : \varinjlim A(\mathfrak{p}_0, E_n) \to \varinjlim A(\mathfrak{p}_n, E_n).$$

By Proposition 2.5(c) $\Psi_E = \varinjlim \Psi_{E_n}$ is an injection.

Assume now that for some ν, $E''_\nu \neq 0$ and $\nu^\vee$ is dominant integral for $\mathfrak{k}_0$. For sufficiently large n, $E''_{n,\nu} := \mathrm{Hom}_{\mathfrak{m}_0}(V_{\mathfrak{m}_0}(\nu), E_n)$ is always nonzero. The fact that $\mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\nu^\vee), A(\mathfrak{p}_n, E_n)) \cong \mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\nu^\vee), A(\mathfrak{p}_0, E_n))$ ([KV, Theorem 5.80]), together with the fact that $\Psi_E = \varinjlim \Psi_{E_n}$, implies

$$\mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\nu^\vee), A(\mathfrak{p}, E)) = E''_\nu$$

as required. In particular, the bottom layer $\mathrm{im}\Psi_E \subset A(\mathfrak{p}, E)$ is non-zero. Finally, to construct a simple bottom layer quotient of $A(\mathfrak{p}, E)$ it suffices to consider a simple quotient of a cyclic module $U(\mathfrak{g}) \cdot v$, where $v \in \mathrm{im}\Psi_E$. This proves (c).

For the proof of (d) we need the following lemma.

Lemma 3.4 *Suppose F is an integrable $\mathfrak{m}_0$-module. Extend F to a $\mathfrak{p}_0$-module so that $\mathfrak{n}_0 \cdot F = 0$. Then if $i < s$, $R^i\Gamma_{\mathfrak{k}_0,\mathfrak{m}_0}(\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} F) = 0$.*

Proof of Lemma 3.4. According to Proposition 2.5(b) we need to show that

$$H^i(\mathfrak{k}_0, \mathfrak{m}_0, (\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} F) \otimes U^0(\mathfrak{k}_0)) = 0$$

for $i < s$. Since $U^0(\mathfrak{k}_0)$ is a semisimple integrable $\mathfrak{k}_0$-module, it is enough to show that $H^i(\mathfrak{k}_0, \mathfrak{m}_0, V \otimes \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} F) = 0$ for $i < s$ and for any simple finite-dimensional $\mathfrak{k}_0$-module V. By Poincaré duality for relative Lie algebra cohomology we must show that

$$H_{2s-i}(\mathfrak{k}_0, \mathfrak{m}_0, V \otimes \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} F) = 0$$

for $i < s$. It is well known that

$$V \otimes \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} F \cong \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}(V \otimes F).$$

So we must show that

$$H_{2s-i}(\mathfrak{k}_0, \mathfrak{m}_0, \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}(V \otimes F)) = 0$$

for $i < s$. But Shapiro's Lemma implies that the above homology is isomorphic to $H_{2s-i}(\mathfrak{p}_0, \mathfrak{m}_0, V \otimes F)$, and the latter vanishes for $i < s$ because $\dim(\mathfrak{p}_0/\mathfrak{m}_0) = s$. The Lemma follows. □

(d) Consider the short exact sequence

$$0 \to \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} E_n \to \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} E_{n+1} \to \mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0}(E_{n+1}/E_n) \to 0.$$

It yields a long exact sequence for $R^{\cdot}\Gamma_{\mathfrak{k}_0, \mathfrak{m}_0}$. Lemma 3.4 implies that each χ_n is an injection. Therefore, by the commutativity of diagram (8), $\varphi_n \circ \Psi_{E_n}$ is an injection for each n, and hence the maps $\varphi_n \circ \Psi_{E_n}$ induce an injection

$$i_n : A(\mathfrak{p}_0, E_n) \to A(\mathfrak{p}, E)$$

for each n.

Fix a value of N so that $A(\mathfrak{p}_0, E_N) \neq 0$, and so that the $Z_{U(\mathfrak{g}_N)}$-character of $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ is not regular integral. Fix a nonzero vector $v \in A(\mathfrak{p}_0, E_N)$, let A_v be the $\mathfrak{g}$-submodule generated by $\tilde{v} := \Psi_E(i_n(v))$ (note that $\tilde{v} \neq 0$), and let A'_v be a simple quotient of A_v. We claim that A'_v is not $\mathfrak{g}$-integrable. To see this consider the image $A'_{v,N}$ in A'_v of the $\mathfrak{g}_N$-submodule $U(\mathfrak{g}_N) \cdot \tilde{v} \subset A(\mathfrak{p}, E)$. The commutativity of the diagram

$$\begin{tikzcd} A(\mathfrak{p}_0, E) \arrow[r, "\Psi_E"] & A(\mathfrak{p}, E) \\ A(\mathfrak{p}_0, E_N) \arrow[u, "i_N"] \arrow[r, "\Psi_{E_N}"] & A(\mathfrak{p}_N, E_N) \arrow[u] \end{tikzcd}$$

implies that $A'_{v,N}$ is isomorphic to a subquotient of $A(\mathfrak{p}_N, E_n)$. Since $Z_{U(\mathfrak{g}_N)}$ acts by one and the same character on $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ and on $A(\mathfrak{p}_N, E_N)$, $A'_{v,N}$ is a $\mathfrak{g}_N$-module with a central character which is not regular integral, and is thus not an integrable $\mathfrak{g}_N$-module. This implies that A'_v itself is not an integrable $\mathfrak{g}$-module.

(e) Note that, under our assumptions, $\mathfrak{m}_0 = \mathfrak{t}_0$. As $\mathfrak{t}_0 \subset Z_{\mathfrak{m}}$, $\mathfrak{t}_0$ acts via weight μ on E, and moreover, $E = \mathbb{C}_\mu \boxtimes_{U(Z_{\mathfrak{k}_0})} E''$ where $\mathbb{C}_\mu$ is the 1-dimensional $\mathfrak{t}_0$-module corresponding to μ. Lemma 3.1 yields now (3), and (c) implies that Ψ_E is an isomorphism between $A(\mathfrak{p}_0, E)$ and $V_{\mathfrak{k}_0}(\mu^\vee) \otimes \mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\mu^\vee), A(\mathfrak{p}, E))$.

(f) Assume in addition that $\mathrm{im}\Psi_E$ is a simple $\mathfrak{k}$-module. Let $A^{\#}$ denote the $\mathfrak{g}$-submodule of $A(\mathfrak{p}, E)$ generated by $\mathrm{im}\Psi_E$, and let $A^{\$}$ be the sum of all $\mathfrak{g}$-submodules X of $A^{\#}$ with $\mathrm{Hom}_{\mathfrak{k}_0}(V_{\mathfrak{k}_0}(\mu^\vee), X) = 0$. Then (e) together with the $\mathfrak{k}_0$-semisimplicity of $A(\mathfrak{p}, E)$ imply that $A^{\$}$ is a maximal proper $\mathfrak{g}$-submodule of $A^{\#}$, and hence $A^{\#}/A^{\$}$ is the unique bottom layer subquotient of $A(\mathfrak{p}, E)$.

Finally, the inclusion $\mathfrak{m} \subset \mathfrak{k}$ yields $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0) \subset \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ which implies that $\mathfrak{m} = \mathfrak{t}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$. As $\mathfrak{t}_0$ is abelian, E'' is a simple $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module, and the isomorphism (3) of (e) implies that $A(\mathfrak{p}_0, E)$ is a simple $\mathfrak{k}$-module. Therefore (by (c)) $\mathrm{im}\Psi_E$ is isomorphic to $A(\mathfrak{p}_0, E)$, and is thus a simple $\mathfrak{k}$-module. □

In the spirit of [PSZ] we call a locally reductive subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of a locally reductive Lie algebra $\mathfrak{g}$ *primal*, if there exists a simple strict $(\mathfrak{g}, \mathfrak{l})$-module M of finite type, such that $\mathfrak{l}$ is a maximal locally reductive subalgebra of $\mathfrak{g}[M]$. Using Theorem 3.2, one can prove that certain subalgebras $\mathfrak{l}$ are primal, for instance a subalgebra $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ is primal whenever there exists an $\mathfrak{m}$-module E satisfying the assumption of Theorem 3.2(d). Below we show the primality of $\mathfrak{k}$ in some special cases.

4 The case $\mathfrak{g} = g\ell(p\Theta)$

To illustrate our main result in the specific case of $\mathfrak{g} = g\ell(p\Theta)$, fix the exhaustion $\mathfrak{g} = \cup_n g\ell(p\theta_1 \dots \theta_{n-1})$ as in Subsection 2.3. Let $\mathfrak{k}_0 \subset \mathfrak{g}_1 = g\ell(p)$ be any reductive in $\mathfrak{g}_1$ subalgebra which contains a $\mathfrak{g}_1$-regular element h, and such that the p-dimensional natural $g\ell(p)$-module $\mathbb{C}^p$ is simple as a $\mathfrak{k}_0$-module. For instance, $\mathfrak{k}_0$ may equal $g\ell(p)$, $s\ell(p)$ or a principal $s\ell(2)$-subalgebra of $s\ell(p)$. Let $\mathfrak{t}_0 := C_{\mathfrak{k}_0}(h)$. We define $\mathfrak{p}$ as the $\mathfrak{t}_0$-compatible parabolic subalgebra $\bigoplus_{\sigma, \mathrm{Re}\sigma \geq 0} \mathfrak{g}_h^\sigma$.

Lemma 4.1

(a) $\mathfrak{m} \cap \mathfrak{g}_n \simeq g\ell(\theta_1 \dots \theta_{n-1})^p$*.*

(b) $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \simeq g\ell(\theta_1 \dots \theta_{n-1})$ *is the diagonal subalgebra in* $g\ell(\theta_1 \dots \theta_{n-1})^p$*.*

Proof. As an $C_{\mathfrak{g}_n}(h)$-module, the natural representation V_n of $g\ell(p\theta_1 \dots \theta_{n-1})$ decomposes as a direct sum of p isotypic components each of dimension $\theta_1 \dots \theta_{n-1}$. This yields (a).

As a $\mathfrak{k}_0$-module V_n decomposes as a direct sum of $\theta_1 \dots \theta_{n-1}$ copies of the simple $\mathfrak{k}_0$-module $\mathbb{C}^p$. This implies (b). □

Corollary 4.2

(a) $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0) = g\ell(\Theta)^p$*;*

(b) $\mathfrak{k} \simeq \mathfrak{k}_0 + g\ell(\Theta)$*,* $\mathfrak{k}_0 \cap g\ell(\Theta) \subset Z_{g\ell(\Theta)}$*;*

(c) if $\mathfrak{k}_0 = g\ell(p)$*, then* $\mathfrak{k} \simeq g\ell(p) + g\ell(\Theta)$ *is a maximal proper subalgebra of* $g\ell(p\Theta)$*.*

We now construct a class of simple $g\ell(\Theta)$-modules. Let V_n denote the natural representation of $g\ell(\theta_1 \dots \theta_{n-1})$. Fix $n_0 > 1$ and let $V(\lambda_{n_0})$ be the simple finite dimensional $g\ell(\theta_1 \dots \theta_{n_0-1})$-module with highest weight $\lambda_{n_0} = (\lambda^1, \dots, \lambda^{\theta_1 \dots \theta_{n_0-1}}), \lambda^i \geq \lambda^{i+1}$. Define $n' = n'(\lambda_{n_0-1})$ as the largest index for which the entry $\lambda^{n'}$ is non-negative; if $\lambda^1 < 0$, we put $n' = 0$. To λ_{n_0} we assign the following highest weight of $g\ell(\theta_1 \dots \theta_{n_0})$:

$$\lambda_{n_0+1} := (\lambda^1, \dots, \lambda^{n'}, \quad \underbrace{0, 0, \dots, 0,}_{\theta_1 \dots \theta_{n_0}(\theta_{n_0+1}-1)\text{times}} \quad \lambda^{n'+1}, \dots, \lambda^{\theta_1 \dots \theta_{n_0}}).$$

Lemma 4.3 *There is a natural injection of $g\ell(\theta_1 \dots \theta_{n_0-1})^{\theta_{n_0}}$-modules*

$$V(\lambda_{n_0})^{\theta_{n_0}} \to V(\lambda_{n_0+1}),$$

and hence a diagonal injection of $g\ell(\theta_1 \dots \theta_{n-1})$-modules

$$V(\lambda_n) \to V(\lambda_{n+1})$$

for any $n > n_0$.

Proof. The natural injection $V_{n_0}^{\theta_{n_0}} \to V_{n_0+1}$ induces a natural injection of $g\ell(\theta_1 \dots \theta_{n_0-1})^{\theta_{n_0}}$-modules

$$T^{\cdot}(V_{n_0} \oplus V_{n_0}^*)^{\theta_{n_0}} \to T^{\cdot}(V_{n_0+1} \oplus V_{n_0+1}^*)$$

which in turn induces an injection

$$V(\lambda_{n_0})^{\theta_{n_0}} \to V(\lambda_{n_0+1})$$

as required. $\square$

Corollary 4.4 *For every n_0 and any dominant integral weight λ_{n_0} of $g\ell(\theta_1 \dots \theta_{n_0-1})$, $\tilde{V}(\lambda_{n_0})$ is a simple $g\ell(\Theta)$-module defined as the direct limit $\varinjlim_{n \geq n_0} V(\lambda_n)$, where $V(\lambda_n)$ is embedded diagonally into $V(\lambda_{n+1})$ according to Lemma 4.3.*

Let now $\lambda_{n_0^1} \dots \lambda_{n_0^p}$ be p dominant weights as in Corollary 4.4. Assume that the ordering of the weights is compatible with $\mathfrak{n}$, i.e. that the h value of any root $\varepsilon_i - \varepsilon_j, i < j$, of $\mathfrak{g}_1 = g\ell(p)$ has non-negative real part. Define E as $V(\lambda_{\mathfrak{n}_0^1}) \boxtimes \dots \boxtimes \tilde{V}(\lambda_{n_0^p})$ with trivial action of $\mathfrak{n}$.

Proposition 4.5 $M(\mathfrak{p}, E) = \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} E$ *is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type.*

Proof. It suffices to show that $\mathrm{Gr}\, M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type. As a $\mathfrak{m}$-module $\mathrm{Gr}\, M(\mathfrak{p}, E)$ is isomorphic to $S^{\cdot}(\bar{\mathfrak{n}}) \otimes E$, and is in particular a weight module over the Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. This subalgebra acts via a single weight on E and via arbitrary sums of $\mathfrak{p}$-negative $\mathfrak{t}_0$-weights on $S^{\cdot}(\bar{\mathfrak{n}})$. Since each $\mathfrak{t}_0$-weight of $S^{\cdot}(\bar{\mathfrak{n}})$ occurs only in finitely many symmetric powers of $\bar{\mathfrak{n}}$, it suffices to show that each fixed tensor product $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $\mathfrak{k} \cap \mathfrak{m}$-module of finite length. Notice that E is a direct limit $\varinjlim_{\mathfrak{n} \geq \max(n_0^1, \dots \mathfrak{n}_0^p)} E_n$ such that each E_n is a $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \simeq g\ell(\theta_1 \dots \theta_{n-1})$-submodule of a fixed tensor power $T^k(V_n^p \oplus (V_n^*)^p)$. Hence $S^t(\bar{\mathfrak{n}}_n) \otimes E_n$ is also contained in a fixed tensor power $T^k(V_n^p \oplus (V_n^*)^p)$. Proposition 2.1 now implies that, for each n, $S^t(\bar{\mathfrak{n}}_n) \otimes E_n$ is a $C_{\mathfrak{g}}(\mathfrak{k}_0) \cap \mathfrak{g}_n$-module of finite length, hence $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $\mathfrak{k} \cap \mathfrak{m}$-module of finite length. The Proposition follows. $\square$

Note now that the assumptions of Theorem 3.2(e) apply to the case we consider. Therefore, to ensure that $A(\mathfrak{p}, E)$ is non-zero, it suffices to ensure that the weight $\mu^{\vee}$ is integral $\mathfrak{k}_0$-dominant. An easy computation shows that the weight μ is nothing but the weight $(\sum_i \lambda_{n_0^1}^i, \sum_i \lambda_{n_0^2}^i, \dots, \sum_i \lambda_{n_0^p}^i)$ of $\mathfrak{g}_1$, restricted to $\mathfrak{t}_0$. Let $\mathfrak{k}_0 = g\ell(p)$. Then the regularity and $\mathfrak{k}_0$-dominancy condition on $\mu^{\vee}$ are equivalent to the condition

$$\sum_i \lambda_{n_0^1}^i \leq \sum_i \lambda_{n_0^2}^i \leq \dots \leq \sum_i \lambda_{n_0^p}^i.$$

Note furthermore, that our choice of weights $\lambda_{n_0^1}, \dots, \lambda_{n_0^p}$ allows for the possibility the $Z_{U(\mathfrak{g}_N)}$-character of $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ to be non-regular for some N, and hence in the latter case, no irreducible bottom layer quotient of $A(\mathfrak{p}, E)$ is $\mathfrak{g}$-integrable. Since $\mathfrak{k}_0 = g\ell(p)$, $\mathfrak{k}$ is a maximal proper subalgebra of $g\ell(p\Theta)$. This implies (via Theorem 3.2(d)) that whenever $A(\mathfrak{p}, E)$ is not integrable, any irreducible bottom layer quotient of $A(\mathfrak{p}, E)$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module. In particular, $\mathfrak{k} = g\ell(p) + g\ell(\Theta)$ is a primal subalgebra of $g\ell(p\Theta)$.

Finally, Lemma 4.1 (a) and (b) imply that the condition $\mathfrak{m} \subset \mathfrak{k}$ from Theorem 3.2(f) holds only when $p = 1$. However, in this case $s = 0$, hence the claim of (f) is trivial. Nevertheless, there is an interesting non-trivial case in which Theorem 3.2 (f) applies: this is when $\lambda_{\mathfrak{n}_0^1} = \ldots = \lambda_{n_0^{p-1}} = 0$ and $\lambda_{n_0^p} \neq 0$. In this latter case E'' is clearly a simple $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Furthermore, as it is easy to see, for large n the $Z_{U(\mathfrak{g}_n)}$-character of $\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n$ is integral but not regular, hence the $(\mathfrak{g}, \mathfrak{k})$-module $A(\mathfrak{p}, E)$ has a unique strict simple subquotient.

5 The root-reductive case

Let now $\mathfrak{g}$ be a simple infinite dimensional root-reductive Lie algebra, i.e. $\mathfrak{g} \cong s\ell(\infty), so(\infty), sp(\infty)$. Fix an exhaustion $\mathfrak{g} = \cup_n \mathfrak{g}_n$, where $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ is a root injection of the form $sl(i) \subset s\ell(i+1)$, $so(i) \subset so(i+2)$, or $sp(2i) \subset sp(2i+2)$, for $\mathfrak{g}$ isomorphic respectively to $s\ell(\infty), so(\infty)$ or $sp(\infty)$. Then each $\mathfrak{g}_n$ is reductive in $\mathfrak{g}$ and $C_{\mathfrak{g}}(\mathfrak{g}_n) \simeq \mathfrak{g}$ for $\mathfrak{g} \simeq so(\infty), sp(\infty)$, and $C_{\mathfrak{g}}(\mathfrak{g}_n) \simeq g\ell(\infty)$ for $\mathfrak{g} = s\ell(\infty)$. Moreover, for a fixed n, the subalgebra $\mathfrak{g}_n \oplus C_{\mathfrak{g}}(\mathfrak{g}_n)$ has the property that its intersections with $\mathfrak{g}_{n'}$ for all $n' > n$ are symmetric subalgebras.

We fix next a reductive in $\mathfrak{g}_1$ subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}_1$, a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ and a $\mathfrak{t}_0$-compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$, and let $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{k}_0$. For instance, for $\mathfrak{g} \simeq s\ell(\infty)$, $\mathfrak{p}$ can be a maximal proper subalgebra of $\mathfrak{g}$, whose intersection with $\mathfrak{g}_n$ for $n > 1$ equals a maximal parabolic subalgebra of $\mathfrak{g}_n$ containing $C_{\mathfrak{g}_n}(\mathfrak{g}_1)$. Note that

$$\mathfrak{m}_0 \oplus C_{\mathfrak{g}}(\mathfrak{g}_1) \subset \mathfrak{k} \cap \mathfrak{m}. \tag{9}$$

Let $E = \cup_n E_n$, where, for n large enough, each E_n is a simple $\mathfrak{m}_n$-submodule of a tensor power $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ for fixed k, a, b, c (when $\mathfrak{g} \simeq so(\infty), sp(\infty)$, there is an isomorphism $V_n \simeq V_n^*$).

Proposition 5.1 *$M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type.*

Proof. According to (9), it suffices to show that $M(\mathfrak{p}, E)$ is an $\mathfrak{m}_0 \oplus C_{\mathfrak{g}}(\mathfrak{g}_1)$-module of finite type. The argument is very similar to that in the proof of Proposition 4.5. Consider $\mathrm{Gr} M(\mathfrak{p}, E) \simeq S^{\cdot}(\bar{\mathfrak{n}}) \otimes E$ and note that only finitely many $\mathfrak{t}_0$-weights occur in E, and that each $\mathfrak{t}_0$-weight of $S^{\cdot}(\bar{\mathfrak{n}})$ will occur only in finitely many symmetric powers of $\bar{\mathfrak{n}}$. Hence it suffices to show that each fixed tensor product $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $C_{\mathfrak{g}}(\mathfrak{g}_1)$-module of finite length. However, a direct verification based on the definition of $\mathfrak{g}_1$ shows that for each $n > 1$, $\bar{\mathfrak{n}} \cap \mathfrak{g}_n$ is a $C_{\mathfrak{g}}(\mathfrak{g}_1) \cap \mathfrak{g}_n$-submodule of a fixed tensor power $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$, where V_n is the natural representation of $C_{\mathfrak{g}}(\mathfrak{g}_1) \cap \mathfrak{g}_n$, and $a, b, c \in \mathbb{Z}_{>0}$. Hence, for each fixed t, $S^t(\bar{\mathfrak{n}} \cap \mathfrak{g}_n) \otimes E_n$ is a submodule of an analogous fixed tensor power, and by Proposition 2.1, $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $C_{\mathfrak{g}}(\mathfrak{g}_1)$-module of finite length. $\square$

In the remainder of this section we concentrate on the case $\mathfrak{k}_0 = \mathfrak{g}_1$, assuming that $\mathfrak{g}_1$ is non-abelian. In this case $\mathfrak{k}_n = (\mathfrak{g}_1 \oplus C_{\mathfrak{g}}(\mathfrak{g}_1)) \cap \mathfrak{g}_n$ is a symmetric subalgebra of $\mathfrak{g}_n$

for $n \geq 2$ and the existing literature on Harish-Chandra modules enables us to prove a stronger version of our main result under slightly different conditions on the compatible parabolic subalgebra $\mathfrak{p}$ and the $\mathfrak{p}$-module E. More precisely, let $\mathfrak{p}$ equal $\bigoplus_{\sigma\geq 0} \mathfrak{g}_h^\sigma$ for some real diagonal matrix $h \in \mathfrak{t}_0$, and $\mathfrak{m} := C_{\mathfrak{g}}(h)$. Then $\mathfrak{m}$ is the direct sum of a reductive in $\mathfrak{k}_0$ subalgebra $\mathfrak{m}'$ and an infinite dimensional subalgebra $\mathfrak{m}''$ isomorphic to $g\ell(\infty)$, $so(\infty)$ or $sp(\infty)$. Note that $\mathfrak{m}'' \supseteq C_{\mathfrak{g}}(\mathfrak{k}_0)$ and that $(\mathfrak{m}_n, \mathfrak{k}_n \cap \mathfrak{m}_n)$ is a symmetric pair for each n.

Theorem 5.2 *For $\mathfrak{g}$ and $\mathfrak{k}$ as above, let the $\mathfrak{p}$-module E satisfy the condition of Theorem 3.2(c). In addition, assume that, for some $N \in \mathbb{Z}_{\geq 0}$, E_N is a simple finite dimensional $\mathfrak{m}_N$-module such that $A(\mathfrak{p}_N, E_N)$ is a simple strict $(\mathfrak{g}_N, \mathfrak{k}_N)$-module with non-zero bottom layer. Let $v \in A(\mathfrak{p}, E)$ be a non-zero vector in the image of the bottom layer of $A(\mathfrak{p}_N, E_N)$ (the existence of v follows from Theorem 3.2(c)) and let X_v be a simple quotient of $U(\mathfrak{g})\cdot v$. Then*

(a) X_v is a strict $(\mathfrak{g}, \mathfrak{k})$-module;

(b) if, for all n, E_n has finite length as a $(\mathfrak{m}_n, \mathfrak{k}_n \cap \mathfrak{m}_n)$-module, $X_v = \cup_n (X_v)_n$ where each $(X_v)_n$ is a Harish-Chandra $(\mathfrak{g}_n, \mathfrak{k}_n)$-module.

Proof.

(a) Let $\pi : U(\mathfrak{g}) \cdot v \to X_v$ be the projection which defines X_v, and let $\kappa : A(\mathfrak{p}_N, E_N) \to A(\mathfrak{p}, E)$ be the functorially induced map of $(\mathfrak{g}_N, \mathfrak{k}_N)$-modules. By our assumptions, $(\pi \circ \kappa)(v) \neq 0$ and, as $A(\mathfrak{p}_N, E_N)$ is simple, $\pi \circ \kappa \neq 0$ is injective. It follows that $\mathfrak{g}_N[A(\mathfrak{p}_N, E_N] \supseteq \mathfrak{g}[X_v] \cap \mathfrak{g}_N$. Since $\mathfrak{g}_N[A(\mathfrak{p}_N, E_N)] = \mathfrak{k}_N$ and X_v is a $(\mathfrak{g}, \mathfrak{k})$-module we conclude that $\mathfrak{g}[X_v] \cap \mathfrak{g}_N = \mathfrak{k}_N$.

The inclusion $\mathfrak{g}[X_v] \supset \mathfrak{k}$ implies the following possibilities for $\mathfrak{g}[X_v]$. If $\mathfrak{g} = so(\infty), sp(\infty)$ $\mathfrak{g}[X_v]$ equals $\mathfrak{k}$ or $\mathfrak{g}$ as $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$, and if $\mathfrak{g} = s\ell(\infty)$ there are four possibilities for $\mathfrak{g}[X_v]$: $\mathfrak{g}$, the two opposite parabolic subalgebras $\mathfrak{q}^\pm$ containing $\mathfrak{k}$, and the subalgebra $\mathfrak{k}$. However, in all cases the only possibility compatible with the equality $\mathfrak{g}[X_v] \cap \mathfrak{g}_N = \mathfrak{k}_N$ is $\mathfrak{g}[X_v] = \mathfrak{k}$. This proves (a).

(b) Define $(X_v)_n$ as the image of the functorial map of $A(\mathfrak{p}_n, E_n)$ to X_v. We have $A(\mathfrak{p}_n, E_n) = R^s\Gamma_{\mathfrak{k}_0, \mathfrak{m}_0}(\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n)$, $\mathfrak{k}_n = \mathfrak{k}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$, and $\mathfrak{k}_n \cap \mathfrak{m}_n = \mathfrak{m}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$. The comparison principle yields an isomorphism of $(\mathfrak{g}_n, \mathfrak{k}_n)$-modules

$$A(\mathfrak{p}_n, E_n) \cong R^s\Gamma_{\mathfrak{k}_n, \mathfrak{k}_n \cap \mathfrak{m}_n}(\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n).$$

Since $(\mathfrak{m}_n, \mathfrak{k}_n \cap \mathfrak{m}_n)$ and $(\mathfrak{g}_n, \mathfrak{k}_n)$ are finite dimensional symmetric pairs, any $(\mathfrak{g}_n, \mathfrak{k}_n)$-module (respectively $(\mathfrak{m}_n, \mathfrak{k}_n \cap \mathfrak{m}_n)$)-module) of finite length is also of finite type, and hence is a Harish-Chandra module. Moreover, results in [KV, Ch.$\underline{\mathrm{V}}$] imply that if E_n has finite length, then $A(\mathfrak{p}_n, E_n)$ likewise has finite length. Hence $(X_v)_n$ itself has finite length, i.e. is a Harish-Chandra module. □

It is easy to construct $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-modules E which satisfy both the assumptions of Proposition 5.1 and Theorem 5.2. To satisfy the assumption of Theorem 5.2, we can take E to be the union $\cup_n E_n$ of finite dimensional simple $\mathfrak{m}_n$-modules under appropriate inclusions of $\mathfrak{m}_n$-modules $E_n \hookrightarrow E_{n+1}$. For a fixed N, we can take E_N (for instance $E_N = \mathbb{C}_{\lambda_{\mathfrak{p}_N}}$, see Theorem 6.1 below) so that $A(\mathfrak{p}_N, E_N)$ is simple with non-zero bottom layer. It is also clear that each E_n can be chosen to be a simple submodule of $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ for some fixed $a, b, c, k \in \mathbb{Z}_{\geq 0}$. Indeed, one can fix a, b, c, k so that the already chosen $\mathfrak{m}_N$-module E_N be a submodule of $T^k(V_N^a \oplus (V_N^*)^b \oplus \mathbb{C}^c)$ and then, for $n \geq N$, recursively choose E_n as a simple submodule of $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ for which there is an injection of $\mathfrak{m}_{n-1}$-modules $E_{n-1} \to E_n$. Such a module E_n clearly exists.

Corollary 5.3 *If $\mathfrak{g} = s\ell(\infty), so(\infty), sp(\infty)$ and $\mathfrak{k}_0 = \mathfrak{g}_1$ where $\mathfrak{g}_1$ is not abelian, then $\mathfrak{k} = \mathfrak{k}_0 \oplus C_{\mathfrak{g}}(\mathfrak{k}_0)$ is a primal subalgebra of $\mathfrak{g}$, and moreover there exists a simple strict $(\mathfrak{g}, \mathfrak{k})$-module X of finite type such that $X = \cup_n X_n$, where X_n are Harish-Chandra $(\mathfrak{g}_n, \mathfrak{k} \cap \mathfrak{g}_n)$-modules.*

6 Appendix: The Fernando-Kac subalgebra of a Vogan-Zuckerman module

Our aim in this appendix is to relate some of the basic literature on applications of cohomological induction with Section 5 of this paper. More precisely, we recall the definition of a class of Harish-Chandra modules known as the Vogan-Zuckerman modules, [VZ], and compute the Fernando-Kac subalgebra of a Vogan-Zuckerman module.

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra (over $\mathbb{C}$), $\mathfrak{k}$ be a symmetric subalgebra of maximal rank, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and let $\mathfrak{p}$ be a $\mathfrak{t}$-compatible parabolic subalgebra of $\mathfrak{g}$. Fix a Levi decomposition $\mathfrak{p} = \mathfrak{m} \supset\!\!\!\!+ \mathfrak{n}$ of $\mathfrak{p}$ with $\mathfrak{t} \subseteq \mathfrak{m}$, and also a $\mathfrak{t}$-compatible Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{p}$. Then $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ and $\mathfrak{b} \cap \mathfrak{m}$ is a Borel subalgebra of $\mathfrak{m}$. Relative to $\mathfrak{b}$, let $w_{\mathfrak{g}}$ be the longest element in the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$; relative to $\mathfrak{b} \cap \mathfrak{m}$ let $w_{\mathfrak{m}}$ be the longest element in the Weyl group of $\mathfrak{t}$ in $\mathfrak{m}$. Finally, let $\lambda_{\mathfrak{p}} := w_{\mathfrak{g}} \circ w_{\mathfrak{m}}^{-1}(\rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}}$. Note that $\lambda_{\mathfrak{p}|\mathfrak{t} \cap [\mathfrak{m},\mathfrak{m}]} = 0$, so that $\lambda_{\mathfrak{p}}$ defines a one-dimensional $\mathfrak{p}$-module $\mathbb{C}_{\lambda_{\mathfrak{p}}}$.

The induced $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{C}_{\lambda_{\mathfrak{p}}}$ and the $(\mathfrak{g}, \mathfrak{k})$-module $A_{\mathfrak{p}} := R^s\Gamma_{\mathfrak{k}, \mathfrak{k} \cap \mathfrak{m}}(\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{C}_{\lambda_{\mathfrak{p}}})$ have the same central character as the trivial $\mathfrak{g}$-module. (Here, as usual, $s = \frac{1}{2} \dim(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{m})$.) More generally, if $F := V_{\mathfrak{g}}(\tilde{\lambda})$ and $\tilde{\lambda} := w_{\mathfrak{g}} \circ w_{\mathfrak{m}}^{-1}(\lambda + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}}$, then the induced $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_{\mathfrak{m}}(\tilde{\lambda}))$ and the $(\mathfrak{g}, \mathfrak{k})$-module $A_{\mathfrak{p}}(F) := R^s\Gamma_{\mathfrak{k}, \mathfrak{k} \cap \mathfrak{m}}(\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_{\mathfrak{m}}(\tilde{\lambda}))$ have the same central character as F. We call $A_{\mathfrak{p}}(F)$ the *Vogan-Zuckerman module* attached to the pair $(\mathfrak{p}, F)$. (This definition can be extended to the case rank $\mathfrak{k} <$ rank $\mathfrak{g}$, but we do not consider this generalization here.)

Theorem 6.1

(a) The bottom layer of $A_{\mathfrak{p}}$ is simple, in particular non-zero.

(b) $A_{\mathfrak{p}}(F)$ is a simple $(\mathfrak{g}, \mathfrak{k})$-module, which is infinite dimensional if $\mathfrak{p}$ is proper in $\mathfrak{g}$.

Proof.

(a) By Lemma 3.1, the bottom layer of $A_{\mathfrak{p}}$ is isomorphic to $V_{\mathfrak{k}}(\lambda_{\mathfrak{p}}^{\vee})$. This implies that the bottom layer of $A_{\mathfrak{p}}$ is simple if non-zero. To ensure that it is indeed non-zero, we need to verify that $\lambda_{\mathfrak{p}}^{\vee}$ is dominant with respect to $\mathfrak{k}$. This follows from [VZ, Section 3], where it is established that $V_{\mathfrak{k}}(\lambda_{\mathfrak{p}}^{\vee})$ is a non-zero constituent of the $\mathfrak{k}$-module $\Lambda^{\cdot}(\mathfrak{k}^{\perp})$.

(b) For the simplicity of $A_{\mathfrak{p}}(F)$ see Theorem 8.2 on p. 550 in [KV]. When $\mathfrak{p}$ is proper, it is shown in [VZ, Section 2] that $A_{\mathfrak{p}}$ has a non-trivial $\mathfrak{k}$-submodule. Since $A_{\mathfrak{p}}$ has the central character of the trivial $\mathfrak{g}$-module, $\dim A_{\mathfrak{p}} = \infty$. By using the translation functor one shows that $A_{\mathfrak{p}}(F)$ is likewise infinite dimensional. $\square$

From now on we assume that $[\mathfrak{g}, \mathfrak{g}]$ is simple and that $\mathfrak{p}$ is proper in $\mathfrak{g}$. We want a formula for the Fernando-Kac subalgebra associated to $A_{\mathfrak{p}}(F)$. If $\mathfrak{k}$ is maximal in $\mathfrak{g}$, clearly $A_{\mathfrak{p}}(F)$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module under our assumptions. If $\mathfrak{k}$ is not maximal, then its orthogonal complement $\mathfrak{k}^{\perp} \subset \mathfrak{g}$ is reducible as a $\mathfrak{k}$-module: $\mathfrak{k}^{\perp} = \mathfrak{r} \oplus \bar{\mathfrak{r}}$, where $\mathfrak{r}$ and $\bar{\mathfrak{r}}$ are abelian subalgebras of $\mathfrak{g}$, and $\mathfrak{k} \niplus \mathfrak{r}$ and $\mathfrak{k} \niplus \bar{\mathfrak{r}}$ are parabolic subalgebras of $\mathfrak{g}$. Moreover, there are precisely four subalgebras of $\mathfrak{g}$ containing $\mathfrak{k}$: $\mathfrak{k}, \mathfrak{k} \niplus \mathfrak{r}, \mathfrak{k} \niplus \bar{\mathfrak{r}}, \mathfrak{g}$.

Theorem 6.2 *Assume $[\mathfrak{g}, \mathfrak{g}]$ is simple, $\mathfrak{k}$ is not maximal and $\mathfrak{p}$ is proper in $\mathfrak{g}$.*

(a) $\mathfrak{g}[A_{\mathfrak{p}}(F)] = \mathfrak{k} \niplus \mathfrak{r}$ *if* $\bar{\mathfrak{r}} \cap \mathfrak{n} = 0$.

(b) $\mathfrak{g}[A_{\mathfrak{p}}(F)] = \mathfrak{k} \niplus \bar{\mathfrak{r}}$ *if* $\mathfrak{r} \cap \mathfrak{n} = 0$.

(c) $\mathfrak{g}[A_{\mathfrak{p}}(F)] = \mathfrak{k}$ *if* $\mathfrak{r} \cap \mathfrak{n}$ *and* $\bar{\mathfrak{r}} \cap \mathfrak{n}$ *are both nonzero.*

The proof of Theorem 6.2 is based on a lemma relating $\mathfrak{g}[A_{\mathfrak{p}}]$ with $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\cdot\cdot}(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_{\mathfrak{p}})$, where $\Lambda^{\cdot\cdot}$ stands for bigraded exterior algebra. Set $a := \dim \bar{\mathfrak{r}} \cap \mathfrak{n}$ and $b := \dim \mathfrak{r} \cap \mathfrak{n}$. Then, according to the key Proposition 6.19 of [VZ], $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\cdot\cdot}(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_{\mathfrak{p}})$ is concentrated in bidegrees of the form $(a + j, b + j)$.

Lemma 6.3

(a) $\mathfrak{g}[A_{\mathfrak{p}}] = \mathfrak{k} \niplus \mathfrak{r} \Leftrightarrow a = 0$.

(b) $\mathfrak{g}[A_{\mathfrak{p}}] = \mathfrak{k} \niplus \bar{\mathfrak{r}} \Leftrightarrow b = 0$.

(c) $\mathfrak{g}[A_{\mathfrak{p}}] = \mathfrak{k} \Leftrightarrow a \neq 0$ *and* $b \neq 0$.

Proof of Lemma 6.3.

(a) $\mathfrak{g}[A_{\mathfrak{p}}] = \mathfrak{k} \niplus \mathfrak{r}$ if and only if there exists a simple finite dimensional $\mathfrak{k}$-module V such that $A_{\mathfrak{p}}$ is isomorphic to the unique irreducible quotient $L(\mathfrak{k} \niplus \mathfrak{r}, V)$ of $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k} \oplus \mathfrak{r}} V$. But the central character of $A_{\mathfrak{p}}$ is trivial and this constrains V to a finite set: V must be a $\mathfrak{k}$-type in $\Lambda^{\cdot}(\bar{\mathfrak{r}})$. Hence, $\mathfrak{g}[A_{\mathfrak{p}}] = \mathfrak{k} \niplus \mathfrak{r}$ implies $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\cdot}(\bar{\mathfrak{r}}), A_{\mathfrak{p}}) \neq 0$ which in turn implies $a = 0$.

Conversely, suppose $a = 0$. Let, for some simple finite dimensional $\mathfrak{k}$-module V, the V-isotypic subspace $A_{\mathfrak{p}}[V]$ of $A_{\mathfrak{p}}$ be in the bottom layer of $A_{\mathfrak{p}}$. Theorem 2.5 in [VZ] gives a necessary condition for a simple $\mathfrak{k}$-module V to occur in the restriction of $A_{\mathfrak{p}}$ to $\mathfrak{k}$. This condition implies that $\mathfrak{r} \cdot A_{\mathfrak{p}}[V] = 0$. Hence $A_{\mathfrak{p}} \cong L(\mathfrak{k} \niplus \mathfrak{r}, V)$.

(b) Repeat proof of (a) but substitute $\bar{\mathfrak{r}}$ for $\mathfrak{r}$.

(c) Follows from the combination of (a) and (b) and the statement above about $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\cdot\cdot}(\mathfrak{r} \oplus \bar{\mathfrak{r}}), A_{\mathfrak{p}})$. $\square$

Proof of Theorem 6.2 First we reduce to the case $F = \mathbb{C}$, $\lambda = 0$: for any F we have a pair of translation functors φ_λ and ψ_λ such that $A_{\mathfrak{p}}(F) \cong \varphi_\lambda(A_{\mathfrak{p}})$ and $A_{\mathfrak{p}} \cong \psi_\lambda(A_{\mathfrak{p}}(F))$ (see [KV, Ch.VII,Thm.7.237]). Since $\varphi_\lambda(A_{\mathfrak{p}})$ is a direct summand of $F \otimes A_{\mathfrak{p}}$, we have $\mathfrak{g}[A_{\mathfrak{p}}(F)] \supseteq \mathfrak{g}[A_{\mathfrak{p}}]$. Likewise, $\psi_\lambda(A_{\mathfrak{p}}(F))$ is a direct summand of $F^* \otimes A_{\mathfrak{p}}(F)$. Hence, $\mathfrak{g}[A_{\mathfrak{p}}] \supseteq \mathfrak{g}[A_{\mathfrak{p}}(F)]$. Thus, $\mathfrak{g}[A_{\mathfrak{p}}(F)] = \mathfrak{g}[A_{\mathfrak{p}}]$. $\square$

Example. Let $\mathfrak{g} = sl(n)$ with $n = p + q$, $p > 1$ and $q > 0$, and $\mathfrak{k} = s(gl(p) \oplus gl(q))$, the traceless matrices in the subalgebra $gl(p) \oplus gl(q)$ embedded in the standard fashion in $gl(n)$. We have $\mathfrak{k} = sl(p) \oplus gl(q)$, where $gl(q)$ is embedded as the centralizer of $sl(p)$ in $\mathfrak{g}$. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be the diagonal matrices; $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{g}$. Choose any real nonzero matrix $h \in \mathfrak{t} \cap sl(p)$ and let $\mathfrak{p}$ be the $\mathfrak{t}$-compatible parabolic subalgebra associated to $h \in \mathfrak{t}$. The subalgebra $\mathfrak{k}$ is not maximal and we have a triangular decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{k} \oplus \bar{\mathfrak{r}}$, where $\mathfrak{r}$ and $\bar{\mathfrak{r}}$ are nonzero simple $\mathfrak{k}$-submodules of $\mathfrak{g}$. Furthermore, since h has both positive and negative diagonal values, $\mathfrak{p} \cap \mathfrak{r} \neq 0$ and $\mathfrak{p} \cap \bar{\mathfrak{r}} \neq 0$. Therefore, for any simple finite dimensional $\mathfrak{g}$-module F, Theorem 6.2(c) implies that $A_{\mathfrak{p}}(F)$ is a strict simple $(\mathfrak{g}, \mathfrak{k})$-module.

References

[B] A. Baranov, Complex finitary simple Lie algebras, Arch. Math. **72**(1999), 101-106.

[BZh] A. Baranov, A. Zhilinski, Diagonal direct limits of simple Lie algebras, Comm. Algebra **27** (1998), 2749-2766.

[DP] I. Dimitrov, I. Penkov, Weight modules of direct limit Lie algebras, IMRN 1999, No. 5, 223-249.

[DMP] I. Dimitrov, O. Mathieu, I. Penkov, On the structure of weight modules, Trans. Amer. Math. Soc. **352** (2000), 2857-2869.

[EW] T. Enright, N. Wallach, Notes on homological algebra and representations of Lie algebras, Duke Math. J. **47** (1980), 1-15.

[F] S. L. Fernando, Lie algebra modules with finite-dimensional weight spaces, Trans. Amer. Math. Soc. **322** (1990), 2857-2869.

[Ha] A. Habib, Direct limits of Zuckerman derived functor modules, J. of Lie Theory **11** (2001), 339-353.

[HTW] R. Howe, E. Tan, J. Willenbring, Stable branching rules for classical symmetric pairs, Trans. Amer. Math. Soc. **357**(2005), 1601-1629.

[KV] A. Knapp, D. Vogan, Jr., Cohomological induction and unitary representations, Princeton University Press, Princeton, NJ, 1995.

[M] O. Mathieu, Classification of irreducible weight modules, Ann. Inst. Fourier (Grenoble) **50** (2000), 537-592.

[Na] L. Natarajan, Unitary highest weight modules of inductive limit Lie algebras and groups, J. of Algebra **167** (1994), 9-28.

[Ne] K.-H. Neeb, Holomorphic highest weight representations of infinite dimensional complex classical groups $SO_0(n, \infty)$ as $n \to \infty$, Funct. Anal. Appl. **20** (1988), 292-301.

[NO] Y. Neritin, G. I. Olshankii, Boundary values of holomorphic functions, singular unitary representations of $O(p, q)$ and their limits as $q \to \infty$, J. of Math. Sciences **87** (1997), 3982-4035.

[NP] K.-H. Neeb, I. Penkov, Cartan subalgebras of $g\ell(\infty)$, Canad. Math. Bull. **46** (2003), 597-616.

[NRW] L. Natarajan, E. Rodrigues-Carrington, J.A. Wolf, The Bott-Borel-Weil Theorem for direct limit Lie groups, Trans. Amer. Math. Soc. **353** (2001), 4583-4622.

[O1] G. I. Olshankii, Representations of infinite-dimensional classical groups: in Topics in Representation Theory, Advances in Sov. Math. **2**, Amer. Math. Soc., 1991.

[O2] G. I. Olshanskii, Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe: in Representations of Lie groups and related topics, Gordon and Breach, 1990, 269-464.

[PSZ] I. Penkov, V. Serganova, G. Zuckerman, On the existence of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type, Duke Math. J. **125** (2004), 329-349.

[PZ1] I. Penkov, G. Zuckerman, Generalized Harish-Chandra modules: a new direction of the structure theory of representations, Acta Applic. Math. **81**(2004), 311-326.

[PZ2] I. Penkov, G. Zuckerman, Generalized Harish-Chandra modules with generic minimal $\mathfrak{k}$-type, Asian J. Math. **8** (2004), 795-812.

[V] D. Vogan, Jr., Representations of real reductive Lie groups, Progress in Math. 15, Birkhauser, Boston, 1981.

[VZ] D. Vogan, Jr., G. Zuckerman, Unitary representations with non-zero cohomology, Compositio Math. **53** (1984), 51-90.

[W] J. A. Wolf, Principal series representations of direct limit groups, Compositio Math. **141** (2005), 1504-1530.

[WZ] J. Willenbring, G. Zuckerman, Small semisimple subalgebras of semisimple Lie algebras, in: Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory: In Honor of Roger E. Howe; Lecture Notes Series Volume 12, Institute for Mathematical Sciences, National University of Singapore, 2007.

Ivan Penkov
Jacobs University Bremen
Campus Ring 1
D-28759 Bremen, Germany
email: i.penkov@jacobs-university.de

Gregg Zuckerman
Department of Mathematics
Yale University
10 Hillhouse Avenue, P.O. Box 208283
New Haven, CT 06520-8283, USA
email: gregg@math.yale.edu