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The Gelfand–Kirillov conjecture and Gelfand–Tsetlin modules for finite W-algebras

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Abstract

We address two problems with the structure and representation theory of finite W-algebras associated with general linear Lie algebras. Finite W-algebras can be defined using either Kostant's Whittaker modules or a quantum Hamiltonian reduction. Our first main result is a proof of the Gelfand–Kirillov conjecture for the skew fields of fractions of finite W-algebras. The second main result is a parameterization of finite families of irreducible Gelfand–Tsetlin modules using Gelfand–Tsetlin subalgebra. As a corollary, we obtain a complete classification of generic irreducible Gelfand–Tsetlin modules for finite W-algebras.

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1. Introduction

The development of finite W-algebras can be traced to a paper by Kostant [26] that studies Whittaker modules and the subsequent generalization of this study by Lynch [28]. An alternative construction of W-algebras is also possible using a quantum Hamiltonian reduction based on

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the works of Feigin and Frenkel [13], Kac et al. [25], Kac and Wakimoto [24] and De Sole and Kac [36]. D'Andrea et al. [36] and Arakawa [3] have shown that the two definitions of finite W-algebras are equivalent.

Let $\mathfrak{g} = \mathfrak{gl}_m$ denote the general linear Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0; this formulation will be used throughout this paper. A finite W-algebra can be associated with a fixed nilpotent element $f \in \mathfrak{g}$ as follows. A $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ is called a *good grading* for f if $f \in \mathfrak{g}_2$ and the linear map

$$\operatorname{ad} f : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$$

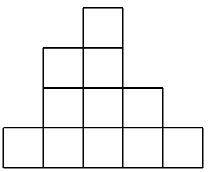
is injective for $j \leqslant -1$ and surjective for $j \geqslant -1$. A complete classification of good gradings for simple Lie algebras has been given by Elashvili and Kac [11]. A nondegenerate invariant symmetric bilinear form $(.,.)$ on $\mathfrak{g}$ induces a nondegenerate skew-symmetric form on $\mathfrak{g}_{-1}$ defined by $\langle x, y\rangle = ([x, y], f)$. Let $\mathcal{I} \subset \mathfrak{g}_{-1}$ be a maximal isotropic subspace, and set $\mathfrak{t} = \bigoplus_{j\leqslant -2} \mathfrak{g}_j \oplus \mathcal{I}$. Now let $\chi : U(\mathfrak{t}) \to \mathbb{C}$ be the one-dimensional representation such that $x \mapsto (x, f)$ for any $x \in \mathfrak{t}$. Let $I_\chi = \operatorname{Ker} \chi$ and $Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})I_\chi$. The corresponding finite W-algebra is defined by

$$W(\chi) = \operatorname{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

If the grading on $\mathfrak{g}$ is *even*, i.e., $\mathfrak{g}_j = 0$ for all odd j, then $W(\chi)$ is isomorphic to the subalgebra of $\mathfrak{t}$-twisted invariants in $U(\mathfrak{p})$ for the parabolic subalgebra $\mathfrak{p} = \bigoplus_{j\geqslant 0} \mathfrak{g}_j$. Note that according to Elashvili and Kac [11], it is sufficient to consider only even good gradings.

The growing interest in finite W-algebras theory is due, on the one hand, to their geometric realizations as quantizations of Slodowy slices (Premet [34] and Gan and Ginzburg [19]) and, on the other hand, to their close connections with Yangian theory, which was initially proposed by Ragoucy and Sorba [35], and was developed in full by Brundan and Kleshchev [7]. The latter of these pieces may well be regarded as a substantial step forward in understanding the structure of finite W-algebras associated with $\mathfrak{gl}_m$. These algebras are isomorphic to certain quotients of *shifted Yangians*, which makes possible their presentation in terms of generators and defining relations, thus opening the way for developing representation theory for finite W-algebras [8].

Following [11], consider a *pyramid* π, which is a unimodal sequence $(q_1, q_2, \dots, q_l)$ of positive integers with $q_1 \leqslant \cdots \leqslant q_k$ and $q_{k+1} \geqslant \cdots \geqslant q_l$ for some $0 \leqslant k \leqslant l$. Such a pyramid can be visualized as a diagram of bricks (unit squares) that consists of q_1 bricks stacked in the first (leftmost) column, q_2 bricks stacked in the second column, and so on. The pyramid π defines the tuple $(p_1, \dots, p_n)$ of the diagram's row lengths, where p_i is the number of bricks in the i-th row of the pyramid, so that $1 \leqslant p_1 \leqslant \cdots \leqslant p_n$. The figure illustrates the pyramid with columns $(1, 3, 4, 2, 1)$ and rows $(1, 2, 3, 5)$:



If the total number of bricks in the pyramid π is m, then the finite W-algebra $W(\pi)$ associated with $\mathfrak{gl}_m$ corresponds to the nilpotent matrix $f \in \mathfrak{gl}_m$ of Jordan type $(p_1, \dots, p_n)$; see Section 2 for the precise definition of and the relationship of $W(\pi)$ to the shifted Yangian. One of the surprising consequences of the results of [7] is that the isomorphism class of $W(\pi)$ depends only on the sequence of row lengths $(p_1, \dots, p_n)$ of π. Therefore, we may assume without restricting generality that the rows of π are left-justified.

The first problem we address in this paper is the *Gelfand–Kirillov conjecture* for the algebras $W(\pi)$. The original conjecture states that the universal enveloping algebra of an algebraic Lie algebra over an algebraically-closed field is "birationally" equivalent to some Weyl algebra over a purely transcendental extension of $\Bbbk$. That is, its skew field of fractions is a Weyl field. This conjecture was proven by Gelfand and Kirillov [20] for nilpotent Lie algebras and for $\mathfrak{gl}_m$ and $\mathfrak{sl}_m$; see also [21], where its weaker form was proven. For solvable Lie algebras, the proof for the conjecture was provided by Borho et al. [4], Joseph [23] and McConnell [29]. Some mixed cases have been considered by Nghiem [32], while Alev et al. [2] proved the conjecture for all Lie algebras of dimension at most eight. However, counterexamples to the conjecture exist for certain semi-direct products; see [1]. We refer the reader to Brown and Goodearl [6] and the references therein for generalizations of the Gelfand–Kirillov conjecture for quantized enveloping algebras.

For an associative algebra A, we denote its skew field of fractions as $D(A)$, if it exists. Let A_k be the k-th Weyl algebra over $\Bbbk$ and $D_k = D(A_k)$ its skew field of fractions. Let $\mathcal{F}$ be a pure transcendental extension of $\Bbbk$ of degree m, and let $A_k(\mathcal{F})$ be the k-th Weyl algebra over $\mathcal{F}$. Denote by $D_{k,m}$ the skew field of fractions of $A_k(\mathcal{F})$.

The Gelfand–Kirillov problem for $W(\pi)$: Does $D(W(\pi)) \simeq D_{k,m}$ for some k, m?

Our first main result is the positive solution of this problem.

Theorem I. *The Gelfand–Kirillov conjecture holds for $W(\pi)$:*

$$D\bigl(W(\pi)\bigr) \simeq D_{k,m},$$

where $k = \sum_{i=1}^{l} q_i(q_i - 1)/2$ and $m = q_1 + \cdots + q_l$.

Note that m is the number of bricks in the pyramid π, while k can be interpreted as the sum of all leg lengths of the bricks. Hence, k and m can be expressed in terms of the rows as $k = (n-1)p_1 + \cdots + p_{n-1}$ and $m = p_1 + \cdots + p_n$. For the case of a one-column pyramid $(1, \dots, 1)$ of height m, we reproduce the result presented in [20] for $\mathfrak{gl}_m$. One of the key points in the proof of Theorem I is a positive solution of the *noncommutative Noether problem* for the symmetric group S_k:

The noncommutative Noether problem for S_k: Does $D_k^{S_k} \simeq D_k$?

Here, S_k acts naturally on A_k and on D_k, through simultaneous permutations of variables and derivations.

The second problem that we address in this paper is the classification problem regarding irreducible Gelfand–Tsetlin modules (sometimes also called Harish-Chandra modules) for finite W-algebras with respect to the Gelfand–Tsetlin subalgebra. Given a pyramid π with the left-justified rows $(p_1, \dots, p_n)$, for each $k \in \{1, \dots, n\}$ we let π_k denote the pyramid $(p_1, \dots, p_k)$. We have the chain of natural subalgebras

$$W(\pi_1) \subset W(\pi_2) \subset \cdots \subset W(\pi_n) = W(\pi). \tag{1.1}$$

Denote by Γ the commutative subalgebra of $W(\pi)$ generated by the centers of the subalgebras $W(\pi_k)$ for $k = 1, \dots, n$. Note that the structure of the center of the algebra $W(\pi)$ is described in [8, Theorem 6.10]. Following the terminology of that paper, we call Γ the *Gelfand–Tsetlin subalgebra* of $W(\pi)$.

A finitely generated module M over $W(\pi)$ is called a *Gelfand–Tsetlin module* with respect to Γ if

$$M = \bigoplus_{\mathbf{m} \in \operatorname{Specm} \Gamma} M(\mathbf{m})$$

as a Γ-module, where

$$M(\mathbf{m}) = \{x \in M \mid \mathbf{m}^k x = 0 \text{ for some } k \geqslant 0\},$$

$\operatorname{Specm} \Gamma$ denotes the set of maximal ideals of Γ. In the case of the one-column pyramids π, this reduces to the definition of the Gelfand–Tsetlin modules for $\mathfrak{gl}_m$ [9]. Note also that the *admissible* $W(\pi)$-modules of [8] are Gelfand–Tsetlin modules.

An irreducible Gelfand–Tsetlin module M is said to be *extended* from $\mathbf{m} \in \operatorname{Specm} \Gamma$ if $M(\mathbf{m}) \neq 0$. The set of isomorphism classes of irreducible Gelfand–Tsetlin modules extended from $\mathbf{m}$ is called the *fiber* of $\mathbf{m} \in \operatorname{Specm} \Gamma$. Equivalently, this is the set of left maximal ideals of $W(\pi)$ containing $\mathbf{m}$. An important problem in the theory of Gelfand–Tsetlin modules is the determination of the cardinality of the fiber of an arbitrary $\mathbf{m}$. In the case in which fibers consist of single isomorphism classes, the corresponding irreducible Gelfand–Tsetlin modules are parameterized by the elements of $\operatorname{Specm} \Gamma$. This problem was solved for the particular cases of one-column pyramids [33] ($\mathfrak{gl}_n$ case) and two-row rectangular pyramids [17], which is restricted Yangian for $\mathfrak{gl}_2$. We extend these results to arbitrary finite W-algebras of type A. The technique used in this paper is quite different, as it is based on the properties of the *Galois orders* developed in [15] and [16]. Our second main result is the following theorem.

Theorem II. *The fiber of any* $\mathbf{m} \in \operatorname{Specm} \Gamma$ *in the category of Gelfand–Tsetlin modules over* $W(\pi)$ *is nonempty and finite.*

Clearly, the same irreducible Gelfand–Tsetlin module can be extended from the different maximal ideals of Γ; such ideals are called *equivalent*. Hence, Theorem II provides a parametrization of finite families of irreducible Gelfand–Tsetlin modules over $W(\pi)$ using the equivalence classes of characters of the Gelfand–Tsetlin subalgebra. Moreover, this provide a classification of the irreducible *generic* Gelfand–Tsetlin modules. To explain this theorem, one must recall that a nonempty set $X \subset \operatorname{Specm} \Gamma$ is called *massive* if X contains the intersection of countably many dense open subsets. If the field $\Bbbk$ is uncountable, then a massive set X is dense in $\operatorname{Specm} \Gamma$.

Theorem III. *There exists a massive subset* $\Omega \subset \operatorname{Specm} \Gamma$ *such that:*

(i) *For any* $\mathbf{m} \in \Omega$*, there exists a unique (up to isomorphism) irreducible module* $L_{\mathbf{m}}$ *over* $W(\pi)$ *in the fiber of* $\mathbf{m}$.
(ii) *For any* $\mathbf{m} \in \Omega$*, the extension category generated by* $L_{\mathbf{m}}$ *contains all indecomposable modules with support containing* $\mathbf{m}$. *This category is equivalent to the category of modules over the algebra of formal power series in* $np_1 + (n-1)p_2 + \cdots + p_n$ *variables.*

We also make the following conjecture about the size of fibers in general:

Conjecture 1. *Let $(p_1, \dots, p_n)$ be the rows of π. For any $\mathbf{m} \in \operatorname{Specm} \Gamma$, the fiber of $\mathbf{m}$ consists of at most $p_1!(p_1+p_2)!\dots(p_1+\dots+p_{n-1})!$ isomorphism classes of irreducible Gelfand–Tsetlin $W(\pi)$-modules. The same bound holds for the dimension of the subspace of $\mathbf{m}$-nilpotents $V(\mathbf{m})$ in any irreducible Gelfand–Tsetlin module V.*

It follows immediately from Theorem 5.3(iii) in [16] that this conjecture is a consequence of the following conjecture.

Conjecture 2. *$W(\pi)$ is free as a left (or right) module over the Gelfand–Tsetlin subalgebra.*

These conjectures are known to be true for the particular cases of one-column pyramids [33] and two-row rectangular pyramids [17]. We prove both conjectures for arbitrary two-row pyramids, i.e., for finite W-algebras associated with $\mathfrak{gl}_2$.

2. Shifted Yangians, finite W-algebras and their representations

As in [7], given a pyramid π with rows $p_1 \leqslant \dots \leqslant p_n$, let the corresponding *shifted Yangian* $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ be the associative algebra over $\Bbbk$ defined by generators

$$\begin{aligned} &d_i^{(r)}, \quad i = 1, \dots, n, \ r \geqslant 1, \\ &f_i^{(r)}, \quad i = 1, \dots, n-1, \ r \geqslant 1, \\ &e_i^{(r)}, \quad i = 1, \dots, n-1, \ r \geqslant p_{i+1} - p_i + 1, \end{aligned} \tag{2.1}$$

subject to the following relations:

$$\big[d_i^{(r)}, d_j^{(s)}\big] = 0,$$

$$\big[e_i^{(r)}, f_j^{(s)}\big] = -\delta_{ij} \sum_{t=0}^{r+s-1} d_i'^{(t)} d_{i+1}^{(r+s-t-1)},$$

$$\big[d_i^{(r)}, e_j^{(s)}\big] = (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-t-1)},$$

$$\big[d_i^{(r)}, f_j^{(s)}\big] = (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)} d_i^{(t)},$$

$$\big[e_i^{(r)}, e_i^{(s+1)}\big] - \big[e_i^{(r+1)}, e_i^{(s)}\big] = e_i^{(r)} e_i^{(s)} + e_i^{(s)} e_i^{(r)},$$

$$\big[f_i^{(r+1)}, f_i^{(s)}\big] - \big[f_i^{(r)}, f_i^{(s+1)}\big] = f_i^{(r)} f_i^{(s)} + f_i^{(s)} f_i^{(r)},$$

$$\big[e_i^{(r)}, e_{i+1}^{(s+1)}\big] - \big[e_i^{(r+1)}, e_{i+1}^{(s)}\big] = -e_i^{(r)} e_{i+1}^{(s)},$$

$$\big[f_i^{(r+1)}, f_{i+1}^{(s)}\big] - \big[f_i^{(r)}, f_{i+1}^{(s+1)}\big] = -f_{i+1}^{(s)} f_i^{(r)},$$

$$[e_i^{(r)}, e_j^{(s)}] = 0 \quad \text{if } |i-j| > 1,$$
$$[f_i^{(r)}, f_j^{(s)}] = 0 \quad \text{if } |i-j| > 1,$$
$$[e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]] = 0 \quad \text{if } |i-j| = 1,$$
$$[f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]] = 0 \quad \text{if } |i-j| = 1,$$

for all admissible i, j, r, s, t, where $d_i^{(0)} = 1$ and the elements $d_i'^{(r)}$ are obtained from the relations

$$\sum_{t=0}^{r} d_i^{(t)} d_i'^{(r-t)} = \delta_{r0}, \quad r = 0, 1, \dots.$$

Note that the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ depends only on the differences $p_{i+1} - p_i$, and our definition corresponds to the left-justified pyramid π, as compared to [7]. In the case of a rectangular pyramid π with $p_1 = \dots = p_n$, the algebra $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ is isomorphic to the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$; see [31] for a description of its structure and representations. Moreover, for an arbitrary pyramid π, the shifted Yangian $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ can be regarded as a natural subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$.

Given the main result presented in [7], the *finite W-algebra* $W(\pi)$, associated with $\mathfrak{gl}_m$ and the pyramid π, can be defined as the quotient of $\mathrm{Y}_\pi(\mathfrak{gl}_n)$ by the two-sided ideal generated by all elements $d_1^{(r)}$ with $r \geqslant p_1 + 1$. We refer the reader to [7,8] for a description of the structure of the algebra $W(\pi)$, which includes an analog of the Poincaré–Birkhoff–Witt theorem as well as a construction of algebraically independent generators of the center.

2.1. The Gelfand–Tsetlin basis for finite-dimensional representations

The explicit construction of a family of finite-dimensional irreducible representations of $W(\pi)$, as given in [18], will play an important role in the arguments presented in this paper. We reproduce some of the formulas here.

Denote a formal generating series in u^{-1} with coefficients in $W(\pi)$ by

$$d_i(u) = 1 + \sum_{r=1}^{\infty} d_i^{(r)} u^{-r}, \qquad f_i(u) = \sum_{r=1}^{\infty} f_i^{(r)} u^{-r},$$
$$e_i(u) = \sum_{r=p_{i+1}-p_i+1}^{\infty} e_i^{(r)} u^{-r}$$

and let

$$A_i(u) = u^{p_1}(u-1)^{p_2} \dots (u-i+1)^{p_i} a_i(u)$$

for $i = 1, \dots, n$ with $a_i(u) = d_1(u) d_2(u-1) \dots d_i(u-i+1)$, and

$$B_i(u) = u^{p_1}(u-1)^{p_2} \dots (u-i+2)^{p_{i-1}} (u-i+1)^{p_{i+1}} a_i(u) e_i(u-i+1),$$
$$C_i(u) = u^{p_1}(u-1)^{p_2} \dots (u-i+1)^{p_i} f_i(u-i+1) a_i(u)$$

for $i = 1, \dots, n-1$. According to [18], $A_i(u)$, $B_i(u)$, and $C_i(u)$ are polynomials in u, and their coefficients are generators of $W(\pi)$. Define the elements $a_r^{(k)}$ for $r = 1, \dots, n$ and $k = 1, \dots, p_1 + \cdots + p_r$ through the expansion

$$A_r(u) = u^{p_1+\cdots+p_r} + \sum_{k=1}^{p_1+\cdots+p_r} a_r^{(k)} u^{p_1+\cdots+p_r-k}.$$

Thus, the elements $a_r^{(k)}$ generate the Gelfand–Tsetlin subalgebra Γ of $W(\pi)$ as defined in the Introduction.

Recall the definitions and results presented in [8] regarding representations of $W(\pi)$. Fix an n-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ of monic polynomials in u, where $\lambda_i(u)$ has degree p_i. We let $L(\lambda(u))$ denote the irreducible highest weight representation of $W(\pi)$ with the highest weight $\lambda(u)$. Then $L(\lambda(u))$ is generated by a nonzero vector ξ (that is, the highest vector) such that

$$B_i(u)\xi = 0 \quad \text{for } i = 1, \dots, n-1, \quad \text{and}$$
$$u^{p_i} d_i(u)\xi = \lambda_i(u)\xi \quad \text{for } i = 1, \dots, n.$$

Let

$$\lambda_i(u) = \bigl(u + \lambda_i^{(1)}\bigr)\bigl(u + \lambda_i^{(2)}\bigr) \dots \bigl(u + \lambda_i^{(p_i)}\bigr), \quad i = 1, \dots, n.$$

We assume that the parameters $\lambda_i^{(k)}$ satisfy the conditions

$$\lambda_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+, \quad i = 1, \dots, n-1,$$

for any value $k \in \{1, \dots, p_i\}$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers. In this case, the representation $L(\lambda(u))$ of $W(\pi)$ is finite-dimensional. We only consider a certain family of representations of $W(\pi)$ by imposing the condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \quad \text{for all } i, j \text{ and all } k \neq m.$$

The *Gelfand–Tsetlin pattern* $\mu(u)$ associated with the highest weight $\lambda(u)$ is an array of rows $(\lambda_{r1}(u), \dots, \lambda_{rr}(u))$ of monic polynomials in u for $r = 1, \dots, n$, where

$$\lambda_{ri}(u) = \bigl(u + \lambda_{ri}^{(1)}\bigr) \dots \bigl(u + \lambda_{ri}^{(p_i)}\bigr), \quad 1 \leqslant i \leqslant r \leqslant n,$$

with $\lambda_{ni}^{(k)} = \lambda_i^{(k)}$, such that the top row coincides with $\lambda(u)$, and

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \quad \text{and} \quad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p_i$ and $1 \leqslant i \leqslant r \leqslant n-1$.

The following theorem was proven in [18]; it will play a key role in the arguments below, as it allows us to realize $W(\pi)$ as a Galois algebra; see Section 3.4. Let $l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1$.

Theorem 2.1. *The representation $L(\lambda(u))$ of the algebra $W(\pi)$ allows a basis $\{\xi_\mu\}$ parameterized by all patterns $\mu(u)$ associated with $\lambda(u)$ such that the action of the generators is given by the formulas*

$$A_r(u)\xi_\mu = \lambda_{r1}(u)\dots\lambda_{rr}(u-r+1)\xi_\mu, \tag{2.2}$$

for $r = 1, \dots, n$, and

$$\begin{aligned} B_r\bigl(-l_{ri}^{(k)}\bigr)\xi_\mu &= -\lambda_{r+1,1}\bigl(-l_{ri}^{(k)}\bigr)\dots\lambda_{r+1,r+1}\bigl(-l_{ri}^{(k)}-r\bigr)\xi_{\mu+\delta_{ri}^{(k)}}, \\ C_r\bigl(-l_{ri}^{(k)}\bigr)\xi_\mu &= \lambda_{r-1,1}\bigl(-l_{ri}^{(k)}\bigr)\dots\lambda_{r-1,r-1}\bigl(-l_{ri}^{(k)}-r+2\bigr)\xi_{\mu-\delta_{ri}^{(k)}}, \end{aligned} \tag{2.3}$$

for $r = 1, \dots, n-1$, where $\xi_{\mu\pm\delta_{ri}^{(k)}}$ corresponds to the pattern obtained from $\mu(u)$ by replacing $\lambda_{ri}^{(k)}$ by $\lambda_{ri}^{(k)} \pm 1$, while the vector ξ_μ is set at zero if $\mu(u)$ is not a pattern.

Note that the action of operators $B_r(u)$ and $C_r(u)$ for an arbitrary value of u can be calculated using the Lagrange interpolation formula.

3. The skew group structure of finite W-algebras

3.1. Skew group rings

Let R be a ring, $\mathcal{M}$ a subgroup of $\operatorname{Aut} R$, and $R * \mathcal{M}$ the corresponding skew group ring, i.e., the free left R-module with the basis $\mathcal{M}$ and the multiplication

$$(r_1m_1)\cdot(r_2m_2) = \bigl(r_1r_2^{m_1}\bigr)(m_1m_2), \quad m_1, m_2 \in \mathcal{M},\ r_1, r_2 \in R.$$

If $x \in R * \mathcal{M}$ and $m \in \mathcal{M}$, then denote by x_m the element of R such that $x = \sum_{m\in\mathcal{M}} x_m m$. Let

$$\operatorname{supp} x = \{m \in \mathcal{M} \mid x_m \neq 0\}.$$

If a finite group G acts by automorphisms on R and by conjugations on $\mathcal{M}$, then G acts on $R * \mathcal{M}$. Denote by $(R * \mathcal{M})^G$ the subring of invariants under this action. Then $x \in (R * \mathcal{M})^G$ if and only if $x_{m^g} = x_m^g$ for $m \in \mathcal{M}$, $g \in G$.

For $\varphi \in \operatorname{Aut} R$ set $H_\varphi = \{h \in G \mid \varphi^h = \varphi\}$ and for an H_φ-invariant $a \in R$ set

$$[a\varphi] := \sum_{g\in G/H_\varphi} a^g\varphi^g \in (R * \mathcal{M})^G, \tag{3.1}$$

where the sum is taken over the representatives of the cosets and does not depend on their choice.

3.2. Galois algebras

Let Γ be a commutative domain, K the field of fractions of Γ, $K \subset L$ a finite Galois extension, $G = \operatorname{Gal}(L/K)$ the corresponding Galois group, and $\mathcal{M} \subset \operatorname{Aut} L$ a subgroup. Assume that G belongs to the normalizer of $\mathcal{M}$ in $\operatorname{Aut} L$ and $\mathcal{M} \cap G = \{e\}$. Then G acts on the skew group algebra $L * \mathcal{M}$ by the authomorphisms: $(am)^g = a^g m^g$, $g \in G$, $a \in L$, $m \in \mathcal{M}$, where the action on $\mathcal{M}$ is through conjugation. Denote by $(L * \mathcal{M})^G$ the subalgebra of G-invariants in $L * \mathcal{M}$.

Definition 3.1. (See [15].) A subring $U \subset (L * \mathcal{M})^G$ finitely generated over Γ is called a *Galois ring over* Γ if $KU = UK = (L * \mathcal{M})^G$.

We always assume that both Γ and U are $\Bbbk$-algebras and that Γ is noetherian. In this case, we say that a Galois ring U over Γ is a *Galois algebra over* Γ.

Denote by $\bar{\Gamma}$ the integral closure of Γ in L. Let $S_* = \{S_1 \subset S_2 \subset \cdots \subset S_N \subset \cdots\}$ be an increasing chain of finite sets. Then the growth of S_* is defined as

$$\operatorname{growth}(S_*) = \varlimsup_{N \to \infty} \log_N |S_N|. \tag{3.2}$$

Let $\mathcal{M}_1 = \mathcal{O}_{\varphi_1} \cup \cdots \cup \mathcal{O}_{\varphi_n}$ be a set of generators of $\mathcal{M}$, where $\mathcal{O}_\varphi = \{\varphi^g \mid g \in G\}$. For $N \geqslant 1$, let $\mathcal{M}_N$ be the set of words $w \in \mathcal{M}$ such that $l(w) \leqslant N$, where l is the length of w, that is,

$$\mathcal{M}_{N+1} = \mathcal{M}_N \cup \left(\bigcup_{\varphi \in \mathcal{M}_1} \varphi \cdot \mathcal{M}_N \right). \tag{3.3}$$

Let $\mathcal{M}_* = \{\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_N \subset \cdots\}$. Then the growth of $\mathcal{M}$ is, by definition, $\operatorname{growth}(\mathcal{M}_*)$, which we denote as $\operatorname{growth}(\mathcal{M})$. For a ring R, we denote by $\operatorname{GKdim} R$ its Gelfand–Kirillov dimension.

Proposition 3.2. *(See [15, Theorem 6.1].) Let $U \subset L * \mathcal{M}$ be a Galois algebra over noetherian Γ, $\mathcal{M}$ a group of finite growth such that for every finite-dimensional $\Bbbk$-vector space $V \subset \bar{\Gamma}$, the set $\mathcal{M} \cdot V$ is contained in a finite-dimensional subspace of $\bar{\Gamma}$. Then*

$$\operatorname{GKdim} U \geqslant \operatorname{GKdim} \Gamma + \operatorname{growth}(\mathcal{M}). \tag{3.4}$$

3.3. PBW Galois algebras

Let U be an associative algebra over $\Bbbk$, endowed with an increasing, exhausting finite-dimensional filtration $\{U_i\}_{i \in \mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, such that $U_i U_j \subset U_{i+j}$ and $\operatorname{gr} U = \bigoplus_{i=0}^{\infty} U_i / U_{i-1}$ is the associated graded algebra. An algebra U is called a *PBW algebra* if $\operatorname{gr} U$ is a polynomial algebra. In particular, U is a noetherian affine $\Bbbk$-algebra. We have the following sufficient conditions for a PBW algebra to be a Galois algebra.

Theorem 3.3. *(See [15, Theorem 7.1].) Let U be a PBW algebra generated by the elements $u_1, \ldots, u_k$ over Γ, $\operatorname{gr} U$ a polynomial algebra in n variables, $\mathcal{M} \subset \operatorname{Aut} L$ a group, and $f : U \to (L * \mathcal{M})^G$ a homomorphism such that $\bigcup_i \operatorname{supp} f(u_i)$ generates $\mathcal{M}$. If*

$$\operatorname{GKdim} \Gamma + \operatorname{growth}(\mathcal{M}) = n,$$

then f is an embedding, and U is a Galois algebra over Γ.

3.4. Finite W-algebras as Galois algebras

Now consider the Gelfand–Tsetlin subalgebra Γ of the algebra $W(\pi)$, as defined in Section 2. Let Λ be the polynomial algebra with the variables x_{ri}^{k}, $1 \leqslant i \leqslant r \leqslant n$, $k = 1, \dots, p_i$. Consider the $\Bbbk$-homomorphism $\imath : \Gamma \to \Lambda$ defined by

$$\imath\big(a_r^{(k)}\big) = \sigma_{r,k}\big(x_{r1}^{1}, \dots, x_{r1}^{p_1}, \dots, x_{rr}^{1}, \dots, x_{rr}^{p_r}\big), \quad k = 1, \dots, p_1 + \dots + p_r, \tag{3.5}$$

where $\sigma_{r,j}$ is the j-th elementary symmetric polynomial in $p_1 + \dots + p_r$ variables. The map $\imath$ is injective according to the theory of symmetric polynomials, and we identify the elements of Γ with their images in Λ. Let $G = S_{p_1} \times S_{p_1+p_2} \times \dots \times S_{p_1+\dots+p_n}$. Then Γ consists of the invariants in Λ with respect to the natural action of G. Let $\mathcal{L} = \operatorname{Specm} \Lambda$. It can be identified with $\Bbbk^s$, where $s = np_1 + (n-1)p_2 + \dots + p_n$.

Let $\mathcal{M} \subseteq \mathcal{L}$, $\mathcal{M} \simeq \mathbb{Z}^{(n-1)p_1+\dots+p_{n-1}}$, be the free abelian group generated by $\delta_{ri}^{k} = (h_{pq}^{m}) \in \Bbbk^{(n-1)p_1+\dots+p_{n-1}}$ for $k = 1, \dots, p_i$, $1 \leqslant i \leqslant r \leqslant n-1$, where $h_{ri}^{k} = 1$ and all other entries are zero. Define an action $\mathcal{M}$ on $\mathcal{L}$ by the shifts $\delta_{ri}^{k}(\ell) := \ell + \delta_{ri}^{k}$ for $\ell = (\ell_{pq}^{m}) \in \mathcal{L}$, so that ℓ_{ri}^{k} is replaced with $\ell_{ri}^{k} + 1$, while all other coordinates remain unchanged. The group G acts on $\mathcal{L}$ by permutation and on $\mathcal{M}$ by conjugation.

Let K be the field of fractions of Γ, and let L be the field of fractions of Λ. Then $K \subset L$ is a finite Galois extension with the Galois group G, that is, $K = L^G$. Similarly to the above analysis, define the action of $\mathcal{M}$ on L, with the skew group algebra $L * \mathcal{M}$ and its invariant subalgebra $(L * \mathcal{M})^G$.

Recall the polynomials $A_i(u)$, $B_k(u)$, $C_k(u)$ in u, $i = 1, \dots, n$ and $k = 1, \dots, n-1$, with coefficients in $W(\pi)$, which were defined in Section 2.1. Consider the polynomials $\tilde{A}_i(u)$, $\tilde{B}_k(u)$, $\tilde{C}_k(u)$ in u, which are obtained by replacing the nonzero coefficients of the polynomials $A_i(u)$, $B_k(u)$, $C_k(u)$ by independent variables in such a way that the new polynomials have the same degrees as their respective counterparts and the polynomials $\tilde{A}_i(u)$ are monic. Introduce the free associative algebra T over $\Bbbk$ generated by the coefficients of these polynomials. Let $L[u] * \mathcal{M}$ be the skew group algebra over the ring of polynomials $L[u]$, and let e be the identity element of $\mathcal{M}$. Note that $A_i(u) \in L[u] * \mathcal{M}$, $i = 1, \dots, n$. Introduce an algebra homomorphism $\mathrm{t} : T \mapsto L[u] * \mathcal{M}$ using the formulas

$$\mathrm{t}\big(\widetilde{A}_j(u)\big) = A_j(u)e,$$

$$\mathrm{t}\big(\widetilde{B}_r(u)\big) = \sum_{(s,j)} X_{rsj}^{+}[u]\delta_{rj}^{s},$$

$$\mathrm{t}\big(\widetilde{C}_r(u)\big) = \sum_{(s,j)} X_{rsj}^{-}[u]\big(\delta_{rj}^{s}\big)^{-1},$$

where

$$X_{rsj}^{+}[u] = -\frac{\prod_{(k,i)\neq(s,j)}(u + x_{ri}^{k})}{\prod_{(k,i)\neq(s,j)}(x_{ri}^{k} - x_{rj}^{s})} \prod_{m,q}\big(x_{r+1,q}^{m} - x_{rj}^{s}\big),$$

$$X_{rsj}^{-}[u] = \frac{\prod_{(k,i)\neq(s,j)}(u + x_{ri}^{k})}{\prod_{(k,i)\neq(s,j)}(x_{ri}^{k} - x_{rj}^{s})} \prod_{m,q}\big(x_{r-1,q}^{m} - x_{rj}^{s}\big),$$

$j = 1, \ldots, r$ and $s = 1, \ldots, p_j$. The products (k, i) associated with the variables of the form x_{ri}^k run over the pairs with $i = 1, \ldots, r$ and $k = 1, \ldots, p_i$.

In the following lemma, we use notation (3.1).

Lemma 3.4. *We have*

$$\mathrm{t}\big(\widetilde{B}_r(u)\big) = \big[X_{r11}^{+}[u]\delta_{r1}^{1}\big], \qquad \mathrm{t}\big(\widetilde{C}_r(u)\big) = \big[X_{r11}^{-}[u]\big(\delta_{r1}^{1}\big)^{-1}\big].$$

In particular, t *defines a homomorphism from T to $(L * \mathcal{M})^G$.*

Proof. Note that $H_{\delta_{r1}^1} \subset G$ consists of permutations of G that fix 1, and that $X_{r11}^{\pm}$ are fixed points of $H_{\delta_{r1}^1}$. Then, for $g \in G$, such that $g(1) = p_1 + \cdots + p_{i-1} + k$, $0 < k \leqslant p_i$, the equality $(\delta_{r1}^1)^g = \delta_{ri}^k$ holds, and $(X_{r11}^{\pm})^g = X_{rki}^{\pm}$, which completes the proof. □

Denote by $\pi : T \to W(\pi)$ the projection defined by

$$\widetilde{A}_r(u) \mapsto A_r(u), \qquad \widetilde{B}_r(u) \mapsto B_r(u), \qquad \widetilde{C}_r(u) \mapsto C_r(u).$$

Lemma 3.5. *There exists a homomorphism of algebras $i : W(\pi) \to (L * \mathcal{M})^G$ such that the diagram*

$$\begin{tikzcd} T \arrow[rr, "\pi"] \arrow[dr, "\mathrm{t}"'] & & W(\pi) \arrow[dl, "i"] \\ & (L * \mathcal{M})^G & \end{tikzcd}$$

commutes.

Proof. Let V be a finite-dimensional $W(\pi)$-module with a basis $\{\xi_\mu\}$. It induces a module structure over T via the homomorphism π. Moreover, due to Theorem 2.1, V has a right module structure over $\mathrm{t}(T) \subset (L * \mathcal{M})^G$. If $z \in T$ and $\mathrm{t}(z) = \sum_{i=1}^{s}[a_i m_i]$, $m_i \in \mathcal{M}$, $a_i \in L$, then $\xi_\mu \cdot \mathrm{t}(z) = \sum_{i=1}^{s} a_i(\mu)\xi_{m_i+\mu}$, where $a_i(\mu)$ indicates the evaluation of the rational function $a_i \in L$ in μ. Suppose now that $z \in \operatorname{Ker}\pi$, and consider $\mathrm{t}(z)$. There exists a dense subset $\Omega(z)$ consisting of μs such that ξ_μ is a basis vector of some finite-dimensional $W(\pi)$-module V and $\xi_\mu \cdot \mathrm{t}(z)$ is defined. Moreover, for any $\mu \in \Omega(z)$, $\xi_\mu \cdot \mathrm{t}(z) = 0$, and hence, $a_i(\mu) = 0$ for all i. Since each a_i is a rational function on $\operatorname{Specm}\Lambda$, this implies that $a_i = 0$, and hence, $z \in \operatorname{Ker}\mathrm{t}$. Therefore, there exists a homomorphism $i : W(\pi) \to (L * \mathcal{M})^G$ such that the diagram commutes. □

Theorem 3.6. *$W(\pi)$ is a Galois algebra over Γ.*

Proof. First note that $W(\pi)$ is a PBW algebra, and $\dim_{\Bbbk} \mathcal{M} \cdot v < \infty$ for any $v \in \Lambda$. In addition,

$$\operatorname{GKdim} W(\pi) = (2n-1)p_1 + (2n-3)p_2 + \cdots + 3p_{n-1} + p_n = \operatorname{GKdim}\Gamma + \operatorname{growth}\mathcal{M}.$$

Since $\bigcup_r \operatorname{supp}\mathrm{t}(\widetilde{B}_r(u))$ and $\bigcup_r \operatorname{supp}\mathrm{t}(\widetilde{C}_r(u))$ contain all generators of group $\mathcal{M}$, all conditions of Theorem 3.3 are satisfied. Hence, we conclude that $i : W(\pi) \to (L * \mathcal{M})^G$ is an embedding, and $W(\pi)$ is a Galois algebra over Γ. □

Recall that a commutative subalgebra A of an associative algebra B is called a *Harish-Chandra subalgebra* if for any $b \in B$, the A-bimodule AbA if finitely generated both as a left and as a right A-module [10].

Corollary 3.7. *Γ is a Harish-Chandra subalgebra of $W(\pi)$.*

Proof. Since $\mathcal{M} \cdot \Lambda \subset \Lambda$ and $W(\pi)$ is a Galois algebra over Γ, the proof follows from [15, Proposition 5.2]. $\square$

Let $\imath : K \to L$ be a canonical embedding, $\phi \in \operatorname{Aut} L$, $\jmath = \phi\imath$. Consider a (K, L)-bimodule $\widetilde{V}_\phi = KvL$, where $av = v\phi(a)$ for all $a \in K$. Let V_ϕ be the set of $\mathrm{St}(\jmath)$-invariant elements of $\widetilde{V}_\phi$.

Corollary 3.8. *Let $S = \Gamma \setminus \{0\}$. Then*

(i) *S is an Ore set and*

$$W(\pi)\left[S^{-1}\right] \simeq (L * \mathcal{M})^G \simeq \left[S^{-1}\right]W(\pi).$$

(ii) *$K \otimes_\Gamma W(\pi) \otimes_\Gamma K \simeq (L * \mathcal{M})^G$ as K-bimodules.*
(iii) *$W(\pi)[S^{-1}] \simeq \bigoplus_{\phi \in \mathcal{M}/G} V_\phi$ as K-bimodules, where the sum is taken over representatives of the cosets.*

Proof. The proof follows from Theorem 3.6 and [15, Theorem 3.2(5)]. $\square$

4. The noncommutative Noether problem

If A is a noncommutative domain that satisfies the Ore conditions, then it allows a skew field of fractions, which we denote by $D(A)$.

The n-th Weyl algebra A_n is generated by x_i, ∂_i, $i = 1, \ldots, n$ subject to the relations

$$x_i x_j = x_j x_i,$$
$$\partial_i \partial_j = \partial_j \partial_i, \tag{4.1}$$
$$\partial_i x_j - x_j \partial_i = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{4.2}$$

This algebra is a simple noetherian domain with the skew field of fractions $D_n = D(A_n)$. The symmetric group S_n acts on A_n and hence on D_n via simultaneous permutations of x_i's and ∂_i's.

In this section, we prove the noncommutative Noether problem for S_n:

Theorem 4.1. $D_n^{S_n} \simeq D_n$.

4.1. Symmetric differential operators

If $P = \Bbbk[x_1, \ldots, x_n]$, then we identify the Weyl algebra A_n with the ring of differential operators $\mathcal{D}(P)$ on P by identifying x_i with the operator of multiplication on x_i and ∂_i with the operator of partial derivation by x_i, $i = 1, \dots, n$. If A is a localization of P, then $\mathcal{D}(A)$ is generated over A through $\partial_1, \dots, \partial_n$ with obvious relations.

It is well known that $A_n^{S_n}$ is not isomorphic to A_n, and hence, $\mathcal{D}(P)^{S_n}$ is not isomorphic to $\mathcal{D}(P^{S_n})$ if $n > 1$. For any $i = 1, \dots, n$ let σ_i denote the i-th symmetric polynomial in the variables $x_1, \dots, x_n$. Then $P^{S_n} = \Bbbk[\sigma_1, \dots, \sigma_n] \subset P$. Let $\delta = \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)$, and $\Delta = \delta^2 \in P^{S_n}$. Denote by P_Δ and by $P_\Delta^{S_n}$ the localizations of the corresponding algebras of the multiplicative set generated by Δ. The canonical embedding $i : P_\Delta^{S_n} \to P_\Delta$ induces a homomorphism of algebras

$$i_\Delta : \mathcal{D}(P_\Delta)^{S_n} \to \mathcal{D}\bigl(P_\Delta^{S_n}\bigr).$$

Let $\mathbb{A}^n$ be the n-dimensional affine space over $\Bbbk$. The algebra $\mathcal{D}(P_\Delta)$ is merely the ring of differential operators on $X = \operatorname{Specm} P_\Delta \subset \mathbb{A}^n$, which is open and S_n-invariant. The geometric quotient $X/S_n = \operatorname{Specm} \Bbbk[\sigma_1, \dots, \sigma_n]_\Delta$ is rational, and the projection $X \to X/S_n$ is etale. Since the action of S_n on X is free, i_Δ is an isomorphism.

Proposition 4.2. *The following isomorphisms hold:*

(i) *If A is an Ore domain, $S \subset A$ is an Ore subset, then $D(A_S) \simeq D(A)$.*
(ii) $\mathcal{D}(P_\Delta)^{S_n} \simeq (\mathcal{D}(P)^{S_n})_\Delta$.
(iii) $(P^{S_n})_\Delta \simeq (P_\Delta)^{S_n}$.
(iv) $\mathcal{D}(P_\Delta)^{S_n} \simeq \mathcal{D}(P^{S_n})_\Delta$.

Proof. The first statement is obvious. Note that $\mathcal{D}(P_S) \simeq \mathcal{D}(P)_S$ for a multiplicative set S; see [30, Theorem 15.1.25]. If $d \in \mathcal{D}(P_\Delta)^{S_n}$, then $d_1 = \Delta^k d \in \mathcal{D}(P)^{S_n}$ for some $k \geqslant 0$, implying (ii). The third statement is obvious, and (iv) follows from the previous statements. □

4.2. Proof of Theorem 4.1

$$\begin{aligned} D_n^{S_n} &\simeq D^{S_n}\bigl(\mathcal{D}(P)\bigr) \simeq D^{S_n}\bigl(\mathcal{D}(P)_\Delta\bigr) \simeq D^{S_n}\bigl(\mathcal{D}(P_\Delta)\bigr) \simeq D\bigl(\mathcal{D}(P_\Delta)^{S_n}\bigr) \simeq D\bigl(\mathcal{D}\bigl((P_\Delta)^{S_n}\bigr)\bigr) \\ &\simeq D\bigl(\mathcal{D}\bigl(\Bbbk[\sigma_1, \dots, \sigma_n]_\Delta\bigr)\bigr) \simeq D\bigl(\mathcal{D}\bigl(\Bbbk[\sigma_1, \dots, \sigma_n]\bigr)_\Delta\bigr) \simeq D\bigl(\mathcal{D}\bigl(\Bbbk[\sigma_1, \dots, \sigma_n]\bigr)\bigr) \simeq D_n. \end{aligned}$$

Hence $D_n^{S_n} \simeq D_n$.

5. Gelfand–Kirillov conjecture

Since $W(\pi)$ is a noetherian integral domain with a polynomial graded algebra, it satisfies the Ore conditions according to the Goldie theorem. Let $D_\pi(n) = D(W(\pi))$ be the skew field of fractions of $W(\pi)$. Recall that $W(\pi)$ is a Galois algebra over Γ: $W(\pi) \subset (L * \mathcal{M})^G$, where L is a field of rational functions in x_{ij}^k, $j = 1, \dots, i$, $k = 1, \dots, p_i$, $i = 1, \dots, n$. Then $D_\pi(n) \simeq D((L * \mathcal{M})^G)$ by Corollary 3.8(i). Moreover, we see below that $L * \mathcal{M}$ has a skew field of fractions, and thus, $D_\pi(n) \simeq D(L * \mathcal{M})^G$ [12, Theorem 1]. Since Γ is a Harish-Chandra subalgebra (Corollary 3.7), according to [15, Theorem 4.1], we find

Proposition 5.1. *The center $\mathcal{Z}$ of $D_\pi(n)$ is isomorphic to $K^{\mathcal{M}}$.*

Let Λ be the polynomial ring in variables x_{ij}^k, $j=1,\dots,i$, $k=1,\dots,p_j$, $i=1,\dots,n$. Denote by L_i (respectively Λ_i) the field of rational functions (respectively the polynomial ring) in x_{ij}^k with fixed i. Then

$$(\Lambda * \mathcal{M})^G \simeq \bigotimes_{i=1}^{n-1} \bigl(\Lambda_i * \mathbb{Z}^{p_1+\dots+p_i}\bigr)^{S_{p_1+\dots+p_i}} \otimes \Lambda_n^{S_{p_1+\dots+p_n}}.$$

Proposition 5.2. *For every $i=1,\dots,n$*

$$D\bigl(L_i * \mathbb{Z}^{p_1+\dots+p_i}\bigr) \simeq D\bigl(A_{p_1+\dots+p_i}(\Bbbk)\bigr).$$

Proof. Let $B_i = \Bbbk[t_1,\dots,t_i] * \mathbb{Z}^i$, where $\mathbb{Z}^i$ is generated by σ_k, $k=1,\dots,i$ and $\sigma_k(t_m) = t_m - \delta_{km}$. Then B_i is isomorphic to the localization $\mathcal{A}_i$ of the i-th Weyl algebra with respect to $x_1,\dots,x_i$. This isomorphism is given as follows:

$$x_k \mapsto \sigma_k, \qquad \partial_k \mapsto t_k\sigma_k^{-1}.$$

Hence, a subring $\Lambda_i * \mathbb{Z}^{p_1+\dots+p_i}$ of $L_i * \mathbb{Z}^{p_1+\dots+p_i}$ is isomorphic to a localization of $A_{p_1+\dots+p_i}(\Bbbk)$, which completes the proof. $\square$

Since $D(A_k)^{S_k} \simeq D(A_k^{S_k})$, we obtain the isomorphism

$$D\bigl((L*\mathcal{M})^G\bigr) = D\bigl((\Lambda*\mathcal{M})^G\bigr) \simeq \bigotimes_{i=1}^{n-1} D\bigl(\bigl(A_{p_1+\dots+p_i}(\Bbbk)\bigr)^{S_{p_1+\dots+p_i}} \otimes D(T_n)\bigr),$$

where $T_n = \Lambda_n^{S_{p_1+\dots+p_n}}$ is a polynomial ring isomorphic to Λ_n. Moreover, by applying Theorem 4.1, we obtain the isomorphism

$$D\bigl((L*\mathcal{M})^G\bigr) \simeq D\bigl(A_{(n-1)p_1+\dots+p_{n-1}}(\Bbbk) \otimes D(T_n)\bigr).$$

Since $D(T_n)$ is a pure transcendental extension of $\Bbbk$ of degree $p_1+\dots+p_n$ and since $D((L*\mathcal{M})^G) \simeq D(W(\pi))$, we have thus proved the Gelfand–Kirillov conjecture (Theorem I):

$$D\bigl(W(\pi)\bigr) \simeq D\bigl(A_{(n-1)p_1+\dots+p_{n-1}}\bigl(D(T_n)\bigr)\bigr) = D_{k,m},$$

$k=(n-1)p_1+\dots+p_{n-1}$, $m=p_1+\dots+p_n$.

Recall that the *Miura transform* [8] is an injective homomorphism

$$\tau : W(\pi) \to \bigotimes_{i=1}^{l} U(\mathfrak{gl}_{q_i}).$$

Observe that $D(\bigotimes_{i=1}^{l} U(\mathfrak{gl}_{q_i})) \simeq D_{k,m}$, since $k=\sum_{i=1}^{l} q_i(q_i-1)/2$ and $m=\sum_{i=1}^{l} q_i$. Hence, we have proved the following corollary.

Corollary 5.3. *The Miura transform extends to an isomorphism of the corresponding skew fields of fractions.*

6. Fibers of characters

6.1. Galois orders

Let $U \subset (L * \mathcal{M})^G$ be a Galois ring over an integral domain Γ.

Definition 6.1. (See [15].) A Galois ring U over Γ is called a *Galois order* if for any finite-dimensional right (or left) K-subspace $V \subset U[S^{-1}]$ (or $V \subset [S^{-1}]U$), $V \cap U$ is a finitely generated right (or left) Γ-module.

The concept of a Galois order over Γ is a natural noncommutative generalization of the classical notion of Γ-order in the skew group ring $(L * \mathcal{M})^G$. If Γ is a noetherian $\mathbb{k}$-algebra, then a Galois order over Γ is called an *integral Galois algebra*. Note that in particular, a Galois ring U over Γ is a Galois order if U is a projective right and left Γ-module.

The following criterion for Galois orders was established in [15, Corollary 5.6].

Proposition 6.2. *Let $U \subset L * \mathcal{M}$ be a Galois algebra over a noetherian normal $\mathbb{k}$-algebra Γ. Then the following statements are equivalent.*

(i) *U is an integral Galois algebra over Γ.*
(ii) *Γ is a Harish-Chandra subalgebra, and if for $u \in U$ there exists a nonzero $\gamma \in \Gamma$ such that $\gamma u \in \Gamma$ or $u\gamma \in \Gamma$, then $u \in \Gamma$.*

Suppose now that U is a PBW Galois algebra over Γ with the polynomial associated graded algebra $\operatorname{gr} U = A$. Then both U and A are endowed with degree function deg with obvious properties. For $u \in U$ denote $\bar{u} \in A$ as the corresponding homogeneous element. Also denote $\operatorname{gr} \Gamma$ the image of Γ in A. Then we have the following graded version of Proposition 6.2.

Lemma 6.3. *Let $U \subset L * \mathcal{M}$ be a PBW Galois algebra over a noetherian normal $\mathbb{k}$-algebra Γ with a polynomial graded algebra $\operatorname{gr} U$. Then the following statements are equivalent.*

(i) *U is an integral Galois algebra over Γ.*
(ii) *Γ is a Harish-Chandra subalgebra, and for $\gamma, \gamma' \in \Gamma \setminus \{0\}$, it follows from $\bar{\gamma}' = \bar{\gamma} a$, $a \in A$ that $a \in \operatorname{gr} \Gamma$.*

Proof. Suppose $\gamma' = \gamma u \neq 0$, $\gamma', \gamma \in \Gamma$, $u \in U \setminus \Gamma$, and $\deg \gamma'$ is the minimal possible. Then $\bar{\gamma}' = \bar{\gamma}\bar{u} \neq 0$ in A. By the assumption $\bar{u} = \bar{\gamma}''$ for some $\gamma'' \in \Gamma$, then either $\gamma'' = u$, or $\gamma_2 = \gamma u_1 \in \Gamma$, where $u_1 = u - \gamma''$, $\gamma_2 = \gamma' - \gamma\gamma''$. Since in the second case $\deg \gamma_2 < \deg \gamma_1$, this contradicts the minimality assumption. Therefore, $\gamma'' = u \in \Gamma$. The case $\gamma' = u\gamma \neq 0$ is proven analogously. Hence, the statement (6.2) of Proposition 6.2 holds, which implies the integrality of the Galois algebra U. $\square$

Representation theory of Galois algebras was developed in [16]. For $\mathbf{m} \in \operatorname{Specm} \Gamma$, denote by $F(\mathbf{m})$ the fiber of $\mathbf{m}$ consisting of isomorphism classes of irreducible Gelfand–Tsetlin U-modules M with respect to Γ and with $M(\mathbf{m}) \neq 0$.

Let E be the integral extension of Γ such that $\Gamma = E^G$, and assume that Γ is noetherian. Then the fibers of the surjective map $\varphi : \operatorname{Specm} E \to \operatorname{Specm} \Gamma$ are finite. Let $\mathbf{m} \in \operatorname{Specm} \Gamma$ and $l_{\mathbf{m}} \in \operatorname{Specm} E$ such that $\varphi(l_{\mathbf{m}}) = \mathbf{m}$. Let

$$\mathrm{St}_{\mathcal{M}}(\mathbf{m}) = \{x \in \mathcal{M} \mid x \cdot l_{\mathbf{m}} = l_{\mathbf{m}}\}.$$

Clearly the set $\mathrm{St}_{\mathcal{M}}(\mathbf{m})$ does not depend on the choice of $l_{\mathbf{m}}$ up to G-conjugations.

Theorem 6.4. *(See [16, Theorem A].) Let U be an integral Galois algebra over the noetherian Γ, $\mathbf{m} \in \operatorname{Specm} \Gamma$. If the set $\mathrm{St}_{\mathcal{M}}(\mathbf{m})$ is finite, then the fiber $F(\mathbf{m})$ is nontrivial and finite.*

6.2. Finite W-algebras as integral Galois algebras

In this section, we show that $W(\pi)$ is an integral Galois algebra over Γ.

Following [8, Section 2.2], for $1 \leqslant i \leqslant j \leqslant n$, define the higher root elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ of $W(\pi)$ inductively using the formulas $e_{i,i+1}^{(r)} = e_i^{(r)}$ for $r \geqslant p_{i+1} - p_i + 1$,

$$e_{ij}^{(r)} = \left[e_{i,j-1}^{(r-p_j+p_{j-1})}, e_{j-1}^{(p_j-p_{j-1}+1)}\right] \quad \text{for } r \geqslant p_j - p_i + 1,$$

and

$$f_{i+1,i}^{(r)} = f_i^{(r)}, \qquad f_{j,i}^{(r)} = \left[f_{j-1}^{(1)}, f_{j-1,i}^{(r)}\right] \quad \text{for } r \geqslant 1.$$

Furthermore, let

$$e_{ij}(u) = \sum_{r=p_j-p_i+1}^{\infty} e_{ij}^{(r)} u^{-r}, \qquad f_{ji}(u) = \sum_{r=1}^{\infty} f_{ji}^{(r)} u^{-r},$$

and define a power series

$$t_{ij}(u) = \sum_{r \geqslant 0} t_{ij}^{(r)} u^{-r} = \sum_{k=1}^{\min\{i,j\}} f_{ik}(u) d_k(u) e_{kj}(u)$$

for some elements $t_{ij}^{(r)} \in W(\pi)$. Due to [8, Lemma 3.6], an ascending filtration on $W(\pi)$ can be defined by letting $\deg t_{ij}^{(k)} = k$. Let $\overline{W}(\pi) = \operatorname{gr} W(\pi)$ denote the associated graded algebra, and let $\bar{t}_{ij}^{(r)}$ denote the image of $t_{ij}^{(r)}$ in the r-th component of $\operatorname{gr} W(\pi)$. Then $\overline{W}(\pi)$ is a polynomial algebra with the variables

$$\bar{t}_{ij}^{(r)} \quad \text{with } i \geqslant j,\ 1 \leqslant r \leqslant p_j \quad \text{and} \quad \bar{t}_{ij}^{(r)} \quad \text{with } i < j,\ p_j - p_i + 1 \leqslant r \leqslant p_j.$$

By [8, Theorem 3.5], the series

$$T_{ij}(u) = u^{p_j} t_{ij}(u), \quad 1 \leqslant i, j \leqslant n,$$

are polynomials in u. Consider the matrix $T(u) = (T_{ij}(u-j+1))_{i,j=1}^{n}$ and its *column determinant*

$$\operatorname{cdet} T(u) = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot T_{\sigma(1)1}(u)T_{\sigma(2)2}(u-1)\dots T_{\sigma(n)n}(u-n+1). \tag{6.1}$$

This is a polynomial in u, and the coefficients $d_s \in W(\pi)$ of the powers $u^{p_1+\dots+p_n-s}$, $s = 1, \dots, p_1 + \dots + p_n$ are algebraically independent generators of the center of $W(\pi)$ (see [5]).

For $F = \sum_i f_i u^i \in W(\pi)[u]$, let $\overline{F} = \sum_i \overline{f}_i u^i \in \overline{W}(\pi)[u]$. Also, we let $X_{ij}^k = \bar{t}_{ij}^{(k)}$ for $k \geqslant 1$, and $X_{ij}^0 = \delta_{ij}$. Let $X_{ij}(u) = \overline{T}_{ij}(u)$, and $X(u) = (X_{ij}(u))_{i,j=1}^{n}$. Since $\overline{T_{ij}(u-\lambda)} = X_{ij}(u)$ for any $\lambda \in \Bbbk$, one can easily check that $\operatorname{gr}\operatorname{cdet} T(u) = \det X(u)$.

Then

$$\bar{d}_s = \sum_{k_1+\dots+k_n=s} \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot X_{\sigma(1)1}^{k_1} \dots X_{\sigma(n)n}^{k_n} \tag{6.2}$$

is just the coefficient of $u^{p_1+\dots+p_n-s}$ in $\det X(u)$.

Fix r, $1 \leqslant r \leqslant n$, and let $X_r(u) = (X_{ij}(u))_{i,j=1}^{r}$. Then

$$d_{rs} = \sum_{k_1+\dots+k_r=s} \sum_{\sigma \in S_r} \operatorname{sgn}\sigma \cdot X_{\sigma(1)1}^{k_1} \dots X_{\sigma(r)r}^{k_r} \tag{6.3}$$

is the coefficient of $u^{p_1+\dots+p_r-s}$ in $\det X_r(u)$, and the elements

$$\{d_{rs},\ r = 1, \dots, n,\ s = 1, \dots, p_1 + \dots + p_r\}$$

are the generators of the algebra $\operatorname{gr}\Gamma$.

We use the idea of a weighted polynomial order on $\overline{W}(\pi)$ [22]. Let $S = \{X_{ij}^k \mid i, j = 1, \dots, n;\ k = 1, \dots, p_j\}$, $w : S \to \mathbb{N}$ be a function. Define the degree of each variable X_{ij}^k as $w(X_{ij}^k)$ and then define the degree of any monomial in these variables as the sum of the degrees of the variables occurring in the monomial. We will denote this degree associated with w by $\deg_w$. Note that this degree coincides with the usual polynomial degree if $w(X_{ij}^k) = 1$ for all i, j, k. Also, this coincides with the degree in $\overline{W}(\pi)$ if $w(X_{ij}^k) = k$ for all i, j. Fixing an order on S, we define a lexicographic order on the monomials. For the monomials m_1 and m_2, define $m_1 >_w m_2$, provided that $\deg_w(m_1) > \deg_w(m_2)$ or $\deg_w(m_1) = \deg_w(m_2)$ and $m_1 > m_2$ in the lexicographical order. This allows us to define the leading monomial of $f \in \overline{W}(\pi)$ with respect to w. Denote the leading monomial of f by $\operatorname{lm}(f)$. The coefficient of $\operatorname{lm}(f)$ in f is denoted by $\operatorname{lc}(m) = \operatorname{lc}(f)$. Note that the weighted polynomial order $\deg_w$ and the concepts of $\operatorname{lm}(f)$ and $\operatorname{lc}(f)$ naturally extend to $W(\pi)$ and Γ.

Lemma 6.5.

(1) *There exists a weight function w such that for any $r = 1, \dots, n$ and $s = 1, \dots, p_1 + \dots + p_r$ the leading monomial m_{rs} of d_{rs} contains a variable $X(r,s) = X_{ij}^k$ $(i, j, k = i(r,s), j(r,s), k(r,s))$, that does not enter in leading monomials $m_{r's'}$ for $(r',s') \neq (r,s)$. In addition, the variable $X(r,s)$ enters in m_{rs} in degree* 1.

(2) *For any* $\gamma \in \operatorname{gr}\Gamma$, *there exist* $f = \prod_{r,s} d_{rs}^{k_{r,s}}$ *and* $\lambda \in \Bbbk$ *such that* $\gamma >_w (\gamma - \lambda f)$.
(3) *If* $\gamma, \gamma_1 \in \operatorname{gr}\Gamma$ *and* $\operatorname{lm}(\gamma_1) | \operatorname{lm}(\gamma)$, *then there exists* $\gamma_2 \in \operatorname{gr}\Gamma$, *such that*

$$\operatorname{lm}(\gamma) = \operatorname{lm}(\gamma_1)\operatorname{lm}(\gamma_2).$$

Proof. Define a function v on S with values in $\mathbb{Z}$ satisfying the following conditions:

(i) $v(X_{i+1\,i}^{p_i}) = i + 1$, $i = 1, \dots, n-1$;
(ii) $v(X_{ij}^k) = -N$, where $N > 2n^2$, if $i < j$, $i, j = 1, \dots, n$;
(iii) $v(X_{ii}^k)$ are significantly smaller than those above, $v(X_{ii}^k) > v(X_{jj}^l)$ if $i > j$ or $i = j, k > l$;
(iv) For $i - j \geqslant 2$ or $j = i - 1$, $k < p_{i-1}$ the values $v(X_{ij}^k)$ are negative, and its absolute values are significantly larger than the absolute values of those above.

In particular, if a monomial m from (6.3) contains X_{ij}^k satisfying (iv), then $v(m) < v(m')$ for any m' that does not contain such a variable.

First, we construct a required monomial for the weight function v. Fix $r \in \{1, \dots, n\}$ and $s \in \{1, \dots p_1 + \cdots + p_r\}$.

If $s \leqslant p_r$, then let

$$y_{r,s} = X_{rr}^s.$$

Suppose $p_r < s \leqslant p_r + p_{r-1}$, and consider

$$y_{r,s} = X_{r,r-1}^{p_{r-1}} X_{r-1,r}^{s-p_{r-1}}.$$

Note that $s - p_{r-1} \leqslant p_r$ and $p_r - p_{r-1} < s - p_{r-1}$. Generalizing, suppose that

$$p_r + \cdots + p_{r-t+1} < s \leqslant p_r + \cdots + p_{r-t},$$

for some t, $2 \leqslant t \leqslant r - 1$ (note that such t is uniquely defined for a given r and s). In this case, let

$$y_{r,s} = X_{r,r-1}^{p_{r-1}} X_{r-1,r-2}^{p_{r-2}} \cdots X_{r-t+1,r-t}^{p_{r-t}} X_{r-t,r}^k,$$

where $k = s - (p_{r-1} + \cdots + p_{r-t})$. In particular $p_r - p_{r-t} < k \leqslant p_r$.

It is easy to see that the defined monomials $y_{r,s}$ belong to d_{rs}. Moreover, any other monomial in d_{rs} has a weight smaller than $y_{r,s}$. Indeed, the condition (iv) shows that if a leading monomial in d_{rs} contains X_{ij}^k, where $i > j$, then $i = j + 1$ and $k = p_j$. Hence $y_{r,s}$ is the leading monomial of d_{rs} if $s \leqslant p_r$. For the case $s > p_r$, the conditions (iii) and (iv) show that the leading monomial of d_{rs} contains only $X_{i+1\,i}^{p_i}$ and X_{ij}^b for $i < j$. According to condition (i), we have

$$v\big(X_{r,r-1}^{p_{r-1}}\big) > v\big(X_{r-1,r-2}^{p_{r-2}}\big) > \cdots > v\big(X_{21}^{p_1}\big),$$

and hence, $X_{i+1\,i}^{p_i}$ enters the leading monomial with the largest possible value of i. It is clear now that $y_{r,s}$ is the leading monomial of d_{rs}.

Now choose a sufficiently large integer $l > 0$ such that $v(x_{ij}^k) + kl \in \mathbb{N}$ for all possible i, j, k. We can define the required function $w : S \to \mathbb{N}$ by $w(x_{ij}^k) = v(x_{ij}^k) + kl$. Since d_{rs} are homogeneous, their leading monomials do not change after the shift of gradation. We conclude that with respect to the function w, the elements

$$\{y_{r,s} \mid r = 1, \dots, n;\ s = 1, \dots, p_1 + \dots + p_r\}$$

are the leading monomials of the generators of $\operatorname{gr} \Gamma \subset \overline{W}(\pi)$.

Note that $y_{r,s} \neq y_{r',s'}$ for different pairs r, s and r', s'. Given r and s let t be such that $0 \leqslant t \leqslant r-1$ and $p_r + \dots + p_{r-t+1} < s \leqslant p_r + \dots + p_{r-t}$. Set $X(r,s) = X^k_{r-t,r}$, $k = s - (p_{r-1} + \dots + p_{r-t})$ if $t > 0$, and $X(r,s) = X^s_{rr}$ if $t = 0$. Then $X(r,s)$ satisfies (1).

For any $\gamma \in \operatorname{gr} \Gamma$, the number of occurrences of d_{rs} in $\operatorname{lm}(\gamma)$ equals the number of occurrences of $X(r,s)$ in $\operatorname{lm}(\gamma)$. Denote this number by k_{rs}, and let $f = \prod_{r,s} d_{rs}^{k_{r,s}}$. Let $\lambda = \operatorname{lc}(\gamma)$. Then

$$\deg_w(\gamma) > \deg_w(\gamma - \lambda f),$$

implying (2) and thus (3). $\square$

Theorem 6.6. *Let the $\Gamma \subset W(\pi)$ be the Gelfand–Tsetlin subalgebra of $W(\pi)$. Then $W(\pi)$ is an integral Galois algebra over Γ.*

Proof. First, recall that Γ is a Harish-Chandra subalgebra. Assume that $\gamma a \in \operatorname{gr} \Gamma$ for some $\gamma \in \operatorname{gr} \Gamma$ and $a \in \overline{W}(\pi)$. Let w be the function constructed in Lemma 6.5. Then $\operatorname{lm}(\gamma a) = \operatorname{lm}(\gamma) \operatorname{lt}(a)$. Following Lemma 6.5(3), there exists $\gamma' \in \operatorname{gr} \Gamma$, such that $\operatorname{lm}(\gamma') = \operatorname{lt}(a)$. Consider $a' = a - \gamma'$. Then we have $\gamma a' \in \operatorname{gr} \Gamma$ and $\deg_w(a') < \deg_w(a)$. Applying induction in $\deg_w(a)$, we conclude that $a' \in \operatorname{gr} \Gamma$, and hence, $a \in \operatorname{gr} \Gamma$. Lemma 6.3 can then be applied. $\square$

Since $W(\pi)$ is an integral Galois algebra over Γ and Γ is noetherian, then $W(\pi) \cap K \subset L$ is an integral extension of Γ according to [15, Theorem 5.2]. Since $W(\pi)$ is a Galois algebra over Γ, then $K \cap W(\pi)$ is a maximal commutative $\Bbbk$-subalgebra in $W(\pi)$ according to [15, Theorem 4.1]. However, Γ is integrally closed in K. Hence, we find that

Corollary 6.7. *Γ is a maximal commutative subalgebra in $W(\pi)$.*

6.3. Proof of Theorem II

We are now in a position to prove our main result regarding Gelfand–Tsetlin modules, as explicated in the Introduction. Since the Gelfand–Tsetlin subalgebra is a polynomial ring, $W(\pi)$ is an integral Galois algebra by Theorem 6.6, and since for any $\mathbf{m} \in \operatorname{Specm} \Gamma$, the set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ is finite, Theorem II follows immediately from Theorem 6.4. Therefore, every character $\chi : \Gamma \to \Bbbk$ of the Gelfand–Tsetlin subalgebra defines an irreducible Gelfand–Tsetlin module, which is a quotient of $W(\pi)/W(\pi)\mathbf{m}$, $\mathbf{m} = \operatorname{Ker} \chi$. Of course, different characters can provide isomorphic irreducible modules. In such cases, we say that these characters are equivalent. Therefore, we obtain a classification of irreducible Gelfand–Tsetlin modules using the equivalence classes of characters of Γ up to a certain finiteness. This finiteness corresponds to the finite fibers of irreducible Gelfand–Tsetlin modules with a given character of Γ.

7. Category of Gelfand–Tsetlin modules

For a Γ-bimodule V, denote by ${}_{\mathbf{n}}\hat{V}_{\mathbf{m}}$ the I-adic completion of $\Gamma \otimes_{\Bbbk} \Gamma$-module V, where $I \subset \Gamma \otimes \Gamma$ is a maximal ideal $I = \mathbf{n} \otimes \Gamma + \Gamma \otimes \mathbf{m}$, that is,

$$ {}_{\mathbf{n}}\hat{V}_{\mathbf{m}} = \varinjlim_{n,m} {}_{\mathbf{n}^n} V_{\mathbf{m}^m}, $$

here ${}_{\mathbf{n}^n} V_{\mathbf{m}^m} = V/(\mathbf{n}^n V + V\mathbf{m}^m)$. Let $F(W(\pi))$ be the set of finitely generated Γ-subbimodules in $W(\pi)$.

Define a category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ with the set of objects $\operatorname{Ob}\mathcal{A} = \operatorname{Specm}\Gamma$ and with the space of morphisms $\mathcal{A}(\mathbf{m}, \mathbf{n})$ from $\mathbf{m}$ to $\mathbf{n}$, where

$$ \mathcal{A}(\mathbf{m}, \mathbf{n}) = \varinjlim_{V \in F(W(\pi))} {}_{\mathbf{n}}\hat{V}_{\mathbf{m}}. $$

Consider the completion $\Gamma_{\mathbf{m}} = \varprojlim_n \Gamma/\mathbf{m}^n$ of Γ by the ideal $\mathbf{m} \in \operatorname{Specm}\Gamma$. Then the space $\mathcal{A}(\mathbf{m}, \mathbf{n})$ has a natural structure of $(\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{m}})$-bimodule. The category $\mathcal{A}$ is naturally endowed with an inverse limit topology. Consider the category $\mathcal{A}$-mod_d of continuous functors $M : \mathcal{A} \to \Bbbk$-mod [10, Section 1.5], where $\Bbbk$-mod is endowed with a discrete topology.

Let $\mathbb{H}(W(\pi), \Gamma)$ denote the category of Gelfand–Tsetlin modules with respect to the Gelfand–Tsetlin subalgebra Γ for the finite W-algebra $W(\pi)$. Since Γ is a Harish-Chandra subalgebra by Corollary 3.7, then by [10, Theorem 17] (see also [16, Theorem 3.2]), the categories $\mathcal{A}$-mod_d and $\mathbb{H}(W(\pi), \Gamma)$ are equivalent.

A functor that determines this equivalence can be defined as follows. For $N \in \mathcal{A}$-mod_d, let

$$ \mathbb{F}(N) = \bigoplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} N(\mathbf{m}) $$

and for $x \in N(\mathbf{m})$, $a \in U$ set

$$ ax = \sum_{\mathbf{n} \in \operatorname{Specm}\Gamma} a_{\mathbf{n}} x, $$

where $a_{\mathbf{n}}$ is the image of a in $\mathcal{A}(\mathbf{m}, \mathbf{n})$. If $f : M \to N$ is a morphism in $\mathcal{A}$-mod_d, then set $\mathbb{F}(f) = \bigoplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} f(\mathbf{m})$. Hence, we obtain the functor

$$ \mathbb{F} : \mathcal{A}\text{-}\mathrm{mod}_d \to \mathbb{H}\big(W(\pi), \Gamma\big). $$

For $\mathbf{m} \in \operatorname{Specm}\Gamma$, denote by $\hat{\mathbf{m}}$ the completion of $\mathbf{m}$. Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \operatorname{Specm}\Gamma$, and let $\mathcal{A}_W = \mathcal{A}/I$.

Let $\mathbb{H}W(W(\pi), \Gamma)$ be the full subcategory of *weight* Gelfand–Tsetlin modules M such that $\mathbf{m}v = 0$ for any $v \in M(\mathbf{m})$. Clearly, the categories $\mathbb{H}W(W(\pi), \Gamma)$ and $\mathcal{A}_W$-mod are equivalent.

For a given $\mathbf{m} \in \operatorname{Specm}\Gamma$, denote by $\mathcal{A}_{\mathbf{m}}$ the indecomposable block of the category $\mathcal{A}$, which contains $\mathbf{m}$.

An embedding $\imath : \Gamma \to \Lambda$ induces the epimorphism

$$ \imath^* : \mathcal{L} \to \operatorname{Specm}\Gamma. $$

Denote by $\widetilde{\Omega} \subset \mathcal{L}$ the set of generic parameters $\mu = (\mu_{ij}^k,\ i = 1, \ldots, n;\ j = 1, \ldots, i;\ k = 1, \ldots, p)$ such that

$$\mu_{ij}^k - \mu_{i,s}^q \notin \mathbb{Z}, \qquad \mu_{r+1,j}^{(m)} - \mu_{ri}^{(k)} \notin \mathbb{Z}$$

for all r, i, j, m, k.

Theorem 7.1. *Let* $\mathbf{m} \in \operatorname{Specm} \Gamma$, $\mu \in (\imath^*)^{-1}(\mathbf{m})$. *Suppose* $\mu \in \widetilde{\Omega}$. *Then:*

(i) *All objects of* $\mathcal{A}_{\mathbf{m}}$ *are isomorphic, and for every* $\mathbf{n} \in \mathcal{A}_{\mathbf{m}}$,

$$\mathcal{A}(\mathbf{n}, \mathbf{n}) \simeq \hat{\Gamma}_{\mathbf{n}}.$$

(ii) *Let* $M_{\mathbf{m}} = \mathcal{A}_{\mathbf{m}} / \mathcal{A}_{\mathbf{m}} \hat{\mathbf{m}}$. *Then there is a canonical isomorphism*

$$\mathbb{F}(M_{\mathbf{m}}) \simeq W(\pi)/W(\pi)\mathbf{m}.$$

(iii) *The category* $\mathbb{H}(W(\pi), \Gamma, \mathbf{m})$, *which consists of modules that have support belonging to* $\mathcal{A}_{\mathbf{m}}$, *is equivalent to the extension category generated by module* $\mathbb{F}(M_{\mathbf{m}})$. *Moreover, this category is equivalent to the category* $\hat{\Gamma}_{\mathbf{m}}$-mod.

Proof. Since $\mathcal{M}$ acts freely on $\widetilde{\Omega}$ and $\mathcal{M} \cdot \mu \cap G \cdot \mu = \{\mu\}$, all statements follow from Theorem 6.6 and [16, Theorem 5.3, Theorem B]. □

Since for $\mathbf{m}$ as in Theorem 7.1, $\hat{\Gamma}_{\mathbf{m}}$ is isomorphic to the algebra of formal power series in $\operatorname{GKdim} \Gamma$ variables, we obtain Theorem III.

8. W-algebras associated with $\mathfrak{gl}_2$

In this section, we consider the case of W-algebras associated with $\mathfrak{gl}_2$: $W(\pi)$, where π has rows (p_1, p_2). We show that $W(\pi)$ is free over the Gelfand–Tsetlin subalgebra. A particular case $p_1 = p_2$ is considered in [17].

The shifted Yangian $W(\pi)$ is generated by $t_{11}^{(k)}$, $t_{21}^{(k)}$, $k = 1, \ldots, p_1$, $t_{22}^{(r)}$, $r = 1, \ldots, p_2$ and $t_{12}^{(m)}$, $m = p_2 - p_1 + 1, \ldots, p_2$.

We will denote by $\bar{t}_{11}^{(k)}$, $\bar{t}_{21}^{(k)}$, $\bar{t}_{22}^{(k)}$, $\bar{t}_{12}^{(k)}$ the images of the generators of $W(\pi)$ in the graded algebra $\overline{W}(\pi)$.

Let

$$T_{11}(u) = \sum_{i=0}^{p_1} t_{11}^{(i)} u^{p_1 - i}, \qquad T_{22}(u) = \sum_{i=0}^{p_2} t_{22}^{(i)} u^{p_2 - i},$$

$$T_{21}(u) = \sum_{i=1}^{p_1} t_{21}^{(i)} u^{p_1 - i}, \qquad T_{12}(u) = \sum_{i=1}^{p_1} t_{12}^{(s+i)} u^{p_1 - i}$$

and

$$D_1(u) = T_{11}(u), \qquad D_2(u) = T_{11}(u+1) T_{22}(u) - T_{21}(u+1) T_{12}(u).$$

The coefficients $d_1^{(1)},\dots,d_{p_1}^{(1)}$ of $D_1(u)$ and $d_1^{(2)},\dots,d_{p_1+p_2}^{(2)}$ of $D_2(u)$ are generators of the Gelfand–Tsetlin subalgebra Γ. Denote by $\bar{d}_i^{(j)}$ their images in the graded algebra.

Recall that a sequence $x_1,\dots,x_n$ of elements of some commutative ring R is called *regular* if for all $i=1,\dots,n$, the multiplication by x_i is injective on

$$R/\langle x_1,\dots,x_{i-1}\rangle R$$

and $R/\langle x_1,\dots,x_n\rangle R\neq 0$.

Since $W(\pi)$ is a *special filtered* algebra in the sense of [14], according to [14, Theorem 1.1], we only need to show that $\bar{d}_1^{(1)},\dots,\bar{d}_{p_1}^{(1)},\bar{d}_1^{(2)},\dots,\bar{d}_{p_1+p_2}^{(2)}$ is a regular sequence in $\overline{W}(\pi)$ to obtain the freeness of $W(\pi)$ as a Γ-module.

The following is standard.

Lemma 8.1. *A sequence of the form $x_1,\dots,x_r,y_1,\dots,y_t$, where $y_1,\dots,y_t$ are homogeneous elements of $A=\Bbbk[x_1,\dots,x_q]$, $q>r$, is regular in A if and only if the sequence $\tilde{y}_1,\dots,\tilde{y}_t$ is regular in $\Bbbk[x_{r+1},\dots,x_q]$, where $\tilde{y}_i(x_{r+1},\dots,x_q)=y_i(0,\dots,0,x_{r+1},\dots,x_q)$.*

Applying Lemma 8.1, we reduce this problem to the problem of regularity of the sequence of images $\bar{d}_1^{(2)},\dots,\bar{d}_{p_1+p_2}^{(2)}$ in $\overline{W}(\pi)/(D_1(u))$. Consider the first p_2 elements in this sequence. Then the image of $\bar{d}_i^{(2)}$ coincides with $\bar{t}_{22}^{(i)}$, $i=1,\dots,p_2$. Hence, applying again Lemma 8.1, we reduce the problem to the regularity of the sequence of images of $\bar{d}_{p_2+1}^{(2)},\dots,\bar{d}_{p_1+p_2}^{(2)}$ in $\overline{W}(\pi)/(T_{11}(u),T_{22}(u))$. We denote these elements as $z_{p_2+1},\dots,z_{p_1+p_2}$.

Consider the restricted Yangian $\mathrm{Y}_{p_2}(\mathfrak{gl}_2)$ of level p_2 (see [17]) generated by the coefficients of the polynomials

$$T'_{11}(u)=\sum_{i=0}^{p_2}t_{11}^{(i)}u^{p_2-i},\qquad T'_{22}(u)=\sum_{i=0}^{p_2}t_{22}^{(i)}u^{p_2-i},$$
$$T'_{21}(u)=\sum_{i=1}^{p_2}t_{21}^{(i)}u^{p_2-i},\qquad T'_{12}(u)=\sum_{i=1}^{p_2}t_{12}^{(i)}u^{p_2-i}.$$

Let $D(u)'=T'_{21}(u+1)T'_{12}(u)$. Let $y_1,\dots,y_{p_1+p_2}$ be the graded images of the coefficients of $D(u)'$ in $\mathrm{Y}_{p_2}(\mathfrak{gl}_2)/(T'_{11}(u),T'_{22}(u))$. The Yangian $\mathrm{Y}_{p_2}(\mathfrak{gl}_2)$ is free over its Gelfand–Tsetlin subalgebra generated by $T_{11}(u)'$ and $T'_{11}(u+1)T'_{22}(u)'-D(u)'$ according to [17, Theorem 3.4]. Since the sequence $y_1,\dots,y_{p_1+p_2}$ is obtained from a regular sequence in $\bar{\mathrm{Y}}_{p_2}(\mathfrak{gl}_2)$ by substituting zeros for some generators, then $y_1,\dots,y_{p_1+p_2}$ is regular according to Lemma 8.1. Hence, its subsequence $y_{p_2+1},\dots,y_{p_1+p_2}$ is also regular. Thus, the variety $V(y_{p_2+1},\dots,y_{p_1+p_2})\subset\Bbbk^{2p_2}$ is equidimensional. Now, project this variety on the subspace $\Bbbk^{2p_1}$ by substituting zeros for t_{12}^i, $i=1,\dots,s$ and t_{21}^i, $i=p_1+1,\dots,p_2$. The resulting variety is again equidimensional of pure dimension p_1. Moreover, this variety coincides with the variety $V(z_{p_2+1},\dots,z_{p_1+p_2})$, and therefore, the sequence $z_{p_2+1},\dots,z_{p_1+p_2}$ is regular.

Hence, we have proved the following.

Theorem 8.2. *$W(\pi)$ is free as a right (or left) module over the Gelfand–Tsetlin subalgebra.*

Consider the following analog of the *Kostant–Wallach map* [27]

$$KW : \operatorname{Specm} \overline{W}(\pi) \simeq \mathbb{k}^{3p_1+p_2} \to \operatorname{Specm} \bar{\Gamma} \simeq \mathbb{k}^{2p_1+p_2}.$$

In particular, we have shown that

Corollary 8.3. *The map KW is surjective, and the variety $KW^{-1}(0)$ is equidimensional of pure dimension p_1.*

We also obtain an estimate of the size of the fiber for any $\mathbf{m} \in \operatorname{Specm}$.

Theorem 8.4. *Let (p_1, p_2) be the rows of π. For any $\mathbf{m} \in \operatorname{Specm} \Gamma$, the fiber of $\mathbf{m}$ consists of at most $p_1!$ isomorphism classes of irreducible Gelfand–Tsetlin $W(\pi)$-modules. Moreover, the dimension of the subspace of $\mathbf{m}$-nilpotents in any such module is bounded by $p_1!$.*

Proof. Since $W(\pi)$ is free over Γ and Γ is a polynomial ring, then all conditions of [16, Theorem 5.3(iii)] are satisfied. Hence, the fiber of $\mathbf{m}$ consists of at most $p_1!(p_1 + p_2)!$ isomorphism classes of irreducible Gelfand–Tsetlin $W(\pi)$-modules. However, this bound can be improved by following [16, Corollary 6.1(2)]. Let $\bar{\Gamma}$ be the integral closure of Γ in L. If $\ell \in \operatorname{Specm} \bar{\Gamma}$ projects to $\mathbf{m} \in \operatorname{Specm} \Gamma$, then we write $\ell = \ell_{\mathbf{m}}$. Note that given $\mathbf{m} \in \operatorname{Specm} \Gamma$, the number of different $\ell_{\mathbf{m}}$ is finite. Moreover, for any $\mathbf{m} \in \operatorname{Specm} \Gamma$ and some fixed $\ell_{\mathbf{m}}$, there exists at most $p_1!$ elements $s \in \mathbb{Z}^{p_1}$ such that $\ell_{\mathbf{m}}$ and $\ell_{\mathbf{m}} + s$ differ through the action $G = S_{p_1} \times S_{p_1+p_2}$. This proves the statement regarding the bound for the fiber. The same number bounds the dimension of the subspace of $\mathbf{m}$-nilpotents according to [16, Corollary 6.1(1)]. $\square$

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