

CONSEQUENCES OF THE LAKSHMIBAI-SANDHYA THEOREM: THE UBIQUITY OF PERMUTATION PATTERNS IN SCHUBERT CALCULUS AND RELATED GEOMETRY

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ABSTRACT. In 1990, Lakshmibai and Sandhya published a characterization of singular Schubert varieties in flag manifolds using the notion of pattern avoidance. This was the first time pattern avoidance was used to characterize geometrical properties of Schubert varieties. Their results are very closely related to work of Haiman, Ryan and Wolper, but Lakshmibai-Sandhya were the first to use that language exactly. Pattern avoidance in permutations was used historically by Knuth, Pratt, Tarjan, and others in the 1960's and 1970's to characterize sorting algorithms in computer science. Lascoux and Schützenberger also used pattern avoidance to characterize vexillary permutations in the 1980's. Now, there are many geometrical properties of Schubert varieties that use pattern avoidance as a method for characterization including Gorenstein, factorial, local complete intersections, and properties of Kazhdan-Lusztig polynomials. These are what we call consequences of the Lakshmibai-Sandhya theorem. We survey the many beautiful results, generalizations, and remaining open problems in this area. We highlight the advantages of using pattern avoidance characterizations in terms of linear time algorithms and the ease of access to the literature via Tenner's Database of Permutation Pattern Avoidance. This survey is based on lectures by the second author at Osaka, Japan 2012 for the Summer School of the Mathematical Society of Japan based on the topic of Schubert calculus.

1. INTRODUCTION

Modern Schubert calculus is the study of effective methods to compute the expansion coefficients for the cup product of cohomology classes of Schubert varieties:

$$[X_u] \cdot [X_v] = \sum c_{u,v}^w [X_w].$$

These coefficients $c_{u,v}^w$ are called structure constants with respect to the Schubert classes $[X_w]$, and it is known that the structure constants are non-negative integers. In fact, each $c_{u,v}^w$ is the *intersection number* of three Schubert varieties X_u, X_v and X_{w_0w} ; they count the number of points of the intersection of those three varieties placed in generic positions. Observe that this is both a combinatorial and a geometrical statement.

For Schubert varieties in Grassmannians, we already have many tools for computing the structure constants for the cup product: *Littlewood-Richardson tableaux*, *Yamanouchi words*, *Knutson-Tao puzzles*, *Vakil's toric degenerations*. In general, we have not found analogs of all these beautiful tools for other types of Schubert varieties. We need to understand both the combinatorics and geometry of Schubert varieties in order to do Schubert calculus for all types of Schubert varieties.

In this article, we will focus on the combinatorics and geometry related to the tangent spaces of Schubert varieties and characterizations of smoothness and rational smoothness. The mathematical tools we will use also arise in Schubert calculus, but we will not make the connections explicit. For the record, the most explicit connection between characterizations of

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smoothness and Schubert calculus come from Kumar's criterion and the Kostant polynomials. See [8, 63, 93] for more details.

We begin with a review of Schubert varieties in flag manifolds. Then we will present the celebrated Lakshmibai-Sandhya Theorem characterizing smooth Schubert varieties using permutation pattern avoidance. We will give a total of 10 properties of Schubert varieties in flag manifolds that are completely characterized by pattern avoidance or a variation on that theme. We describe a method for extending permutation pattern avoidance to all Coxeter groups and discuss some geometrical properties characterized by Coxeter pattern avoidance more generally. We give pointers to some useful computational tools for studying Schubert geometry and beyond. Finally, we present many open problems in this area.

We want to highlight the fact that there are computational advantages of using permutation patterns to characterize interesting properties such as smoothness of Schubert varieties. Naively, avoiding a finite set of patterns of length at most k leads to a polynomial time algorithm of $O(n^k)$ by brute force testing of all k -subsets. As k and n get large, such algorithm is intractable. In fact, deciding if one permutation is contained in another is an NP-complete problem [18]. Remarkably, Guillemot and Marx [51] recently showed that for every permutation $v \in S_k$ there exists an algorithm to test if $w \in S_n$ contains v which runs in *linear time*, $O(n)$! This is a major improvement over brute force verification. It is often far from obvious that an $O(n)$ time algorithm exists for the geometric or algebraic properties characterized by pattern avoidance in this paper.

Another major advantage of permutation pattern characterizations is that they provide efficient fingerprints for theorems [15]. Tenner's Database of Permutation Pattern Avoidance (DPPA) provides a growing collection of known properties characterized by patterns with references to the literature [92]. This allows researchers to connect new theorems and conjectures with known results in a format free of language or notational differences.

2. PRELIMINARIES

2.1. The Flag Manifold.

Definition 2.1. A *complete flag* $F_\bullet = (F_1, \dots, F_n)$ in \mathbb{C}^n is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \leq i \leq n$. A flag F_\bullet is determined by an ordered basis $\langle f_1, f_2, \dots, f_n \rangle$ where $F_i = \text{span}\langle f_1, \dots, f_i \rangle$.

Let e_1, e_2, \dots, e_n be the standard basis for \mathbb{C}^n . The *base flag* is $E_\bullet = (E_1, E_2, \dots, E_n)$ where $E_i = \langle e_1, e_2, \dots, e_i \rangle$. Let F_\bullet be any flag given by the ordered basis $\langle f_1, f_2, \dots, f_n \rangle$. Writing each basis element f_i as a column vector in terms of the e_i 's, we obtain an $n \times n$ -non-singular matrix whose column vectors are the basis f_1, \dots, f_n . In this presentation, we can multiply the matrix by a non-zero scalar or we can add the i -th column to the j -th column where $i < j$ and it still represents the same flag. So, a flag can always be presented by a matrix in *canonical form*; the lowest non-zero entry of each column is 1, and the entries to its right are all zeros.

Example 2.2. The following two matrices represent the same flag $F_\bullet = \langle 2e_1 + e_2, 2e_1 + e_3, 7e_1 + e_4, e_1 \rangle$:

$$\begin{bmatrix} 6 & 4 & 9 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 7 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The right hand side is the canonical form.

It also follows that two non-singular matrices represent the same flag if and only if one is the other multiplied by an upper triangular matrix. That is, we have an identification $\mathcal{F}l_n(\mathbb{C}) = GL_n(\mathbb{C})/\tilde{B}$ where $\tilde{B} \subset GL_n(\mathbb{C})$ is the set of invertible upper triangular matrices. Similarly, we can rescale any invertible matrix by the inverse of its determinant and get another matrix representing the same flag. Hence, letting B be the set of upper triangular matrices in $SL_n(\mathbb{C})$, we see that

$$\mathcal{F}l_n(\mathbb{C}) = GL_n(\mathbb{C})/\tilde{B} = SL_n(\mathbb{C})/B.$$

2.2. Flags and Permutations. If a flag is written in canonical form, the leading 1's form a permutation matrix. This matrix is called *the position* of the flag F_\bullet with respect to the base flag E_\bullet , and is denoted by $\text{position}(E_\bullet, F_\bullet)$.

Example 2.3.

$$F_\bullet = \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4, \quad e_1 \rangle \approx \begin{bmatrix} 2 & 2 & 7 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$$

Note that there are many ways to represent a permutation; as a bijection from $[n] := \{1, 2, \dots, n\}$ to itself, matrix notation, two-line notation, one-line notation, rank table, diagram, string diagram, reduced word etc. Each of these representations is useful in some way or another for the study of Schubert varieties so we advise the reader to become comfortable with all of them simultaneously and choose the right one for the proof at hand. Note, we have not found much use for cycle notation for permutations in this context so we will not ever use that notation here.

To be precise, we use the following notation: for a permutation $w: [n] \rightarrow [n]$ in the symmetric group S_n , we denote by the same symbol $w = w_1 w_2 \dots w_n$ the permutation matrix which has 1's in the (w_j, j) -th entries for $1 \leq j \leq n$ and 0's elsewhere. Permutation multiplication is consistent with matrix multiplication using this notation. In particular, if t_{ij} is the transposition interchanging i and j , then the one-line notation for wt_{ij} agrees with w in all positions except i and j where the entries are switched. The permutation $t_{ij}w$ has the values i and j switched.

The *rank table* $rk(w)$ is obtained from the matrix w by setting

$$rk(w)[i, j] = \#\{h \in [j] : w(h) \in [i]\},$$

i.e. the rank of the submatrix of w with lower right corner $[i, j]$ and upper left corner $[1, 1]$.

A *string diagram* of a permutation for w is a braid with the strings proceeding from the initial ordering to the permuted order given by $w = w_1 w_2 \dots w_n$ in such a way that no three strings cross at any point. A wiring diagram is a string diagram with exactly one crossing on each row. A wiring diagram in which no two strings cross twice is said to be *reduced*. Starting at the top of a reduced wiring diagram, one can read off the index of the first string in each crossing to obtain a corresponding *reduced word*. All reduced words for w have the same length, denoted $\ell(w)$. Furthermore, the length of w is the number of *inversions* for w , $\ell(w) = \#\{w(i) > w(j) : i < j\}$.

The *diagram of a permutation* w is obtained from the matrix of w^{-1} by removing all cells in an $n \times n$ array which are weakly to the right or below a 1 in w^{-1} . The remaining cells form the diagram $D(w)$. The cells of $D(w)$ are in bijection with the inversions of w . One can recover w either from its diagram or its inversion set. It is unfortunate that the diagram is defined in terms of w^{-1} , but that is the most common convention in the literature [70].

Example 2.4.

$$\begin{array}{cccc}
 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} & = 2341 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \\
 \text{matrix} & & \text{two-line} & \text{one-line} \\
 \text{notation} & & \text{notation} & \text{notation} \\
 & & & \text{rank} \\
 & & & \text{table} \\
 * & \cdot & \cdot & \cdot \\
 * & \cdot & \cdot & \cdot \\
 * & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot \\
 & & 1234 & \\
 & & \text{string diagram} & = (1, 2, 3) \\
 & & 2341 & \\
 \text{diagram of a} & & \text{string diagram} & \text{reduced} \\
 \text{permutation} & & & \text{word}
 \end{array}$$

2.3. Schubert Cells and Schubert Varieties in $\mathcal{Fl}_n(\mathbb{C})$. For a permutation $w \in S_n$, the Schubert cell $C_w(E_\bullet) \subset \mathcal{Fl}_n(\mathbb{C})$ is the set of all flags F_\bullet with $\text{position}(E_\bullet, F_\bullet) = w$. Equivalently, we can write $C_w(E_\bullet)$ as

$$C_w(E_\bullet) = \{F_\bullet \in \mathcal{Fl}_n(\mathbb{C}) \mid \dim(E_i \cap F_j) = \text{rk}(w)[i, j] \text{ for all } 1 \leq i, j \leq n\}.$$

Note, the flag w_\bullet represented by the permutation matrix for w is in C_w by the rank conditions.

Example 2.5.

$$F_\bullet = \begin{bmatrix} 2 & 2 & 7 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} x & y & z & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$$

It is easy to observe the following properties for each permutation w .

- (i) The dimension of a Schubert cell is $\dim_{\mathbb{C}}(C_w) = \ell(w)$.
- (ii) The indeterminates for the canonical matrices in C_w all lie in the entries of the diagram $D(w^{-1})$.
- (iii) $C_w = \tilde{B} \cdot w_\bullet$ is a \tilde{B} -orbit using the left \tilde{B} action on flags given by multiplication of matrices. See Example 2.6.

Example 2.6. For arbitrary $b_{i,j}$'s with $b_{1,1}, b_{2,2}, b_{3,3}, b_{4,4}$ non-zero, we have

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & 0 & b_{4,4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} b_{1,2} & b_{1,3} & b_{1,4} & b_{1,1} \\ b_{2,2} & b_{2,3} & b_{2,4} & 0 \\ 0 & b_{3,3} & b_{3,4} & 0 \\ 0 & 0 & b_{4,4} & 0 \end{bmatrix} \in C_{2341}.$$

Definition 2.7. The Schubert variety $X_w(E_\bullet)$ of a permutation w is defined to be the closure of $C_w(E_\bullet)$ under the Zariski topology. As in the case for Schubert cells, $X_w(E_\bullet)$ can be written by the rank conditions:

$$X_w(E_\bullet) = \{F_\bullet \in \mathcal{Fl}_n \mid \dim(E_i \cap F_j) \geq \text{rk}(w)[i, j] \text{ for all } 1 \leq i, j \leq n\}.$$

Example 2.8.

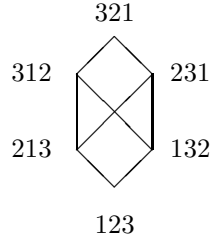
$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & * & * & \textcircled{1} \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\}}$$

2.4. Combinatorics and Geometry. Since Schubert cells are \tilde{B} -orbits, Schubert varieties are \tilde{B} -invariant by their definition. So each Schubert variety is equal to a disjoint union of Schubert cells

$$(1) \quad X_w = \bigcup_{v \leq w} C_v.$$

Thus, the containment relation on Schubert varieties $X_v \subset X_w$ defines a partial order on permutations $v \leq w$. This partial order has a nice description: for a permutation w and integers $1 \leq i < j \leq n$, we say $w < wt_{ij}$ if $w(i) < w(j)$. *Bruhat order* (discovered by Ehresmann 1934 [39], see also Chevalley 1958 [29]) is defined to be the transitive closure of this relation.

Example 2.9. The following is the Hasse diagram of the Bruhat order on permutations in S_3 .



The Hasse diagram of S_n is self dual, rank symmetric and rank unimodal.

Example 2.10. The Hasse diagram of S_4 is drawn in Figure 1.

One of the benefits of Bruhat order is a description of the *Poincaré polynomials* of Schubert varieties. More precisely, the Poincaré polynomial for $H^*(X_w)$ is given by

$$P_w(t) = \sum_{v \leq w} t^{2l(v)}.$$

Because only even exponents appear in the Poincaré polynomials above, we often abuse notation and define

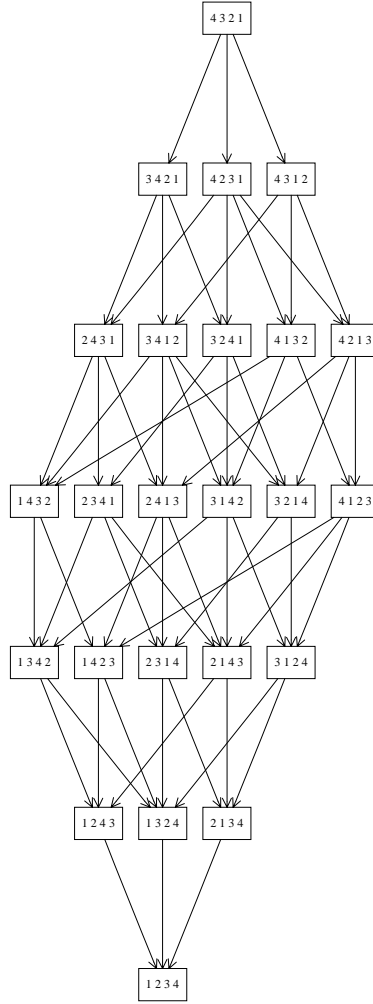
$$P_w(t) = \sum_{v \leq w} t^{l(v)}.$$

Example 2.11. For $w = 3412$, the following permutations are in the interval below 3412 in Bruhat order.

$$\begin{aligned}
 4 : & \quad (3412) \\
 3 : & \quad (3142)(3214)(1432)(2413) \\
 2 : & \quad (3124)(1342)(2143)(2314)(1423) \\
 1 : & \quad (2134)(1243)(1324) \\
 0 : & \quad (1234)
 \end{aligned}$$

So $P_{3412}(t) = 1 + 3t + 5t^2 + 4t^3 + t^4$. One can see that the Schubert variety X_{3412} is not smooth since its Poincaré polynomial is not symmetric (palindromic) which implies that Poincaré duality does not hold for $H^*(X_{3412})$.

There are several interesting things about Bruhat order. We will encounter some of them in the rest of the paper. We will focus on the relationship between singularities of Schubert varieties and pattern avoidance of permutations.

FIGURE 1. The Hasse diagram of S_4

We leave to the reader the following exercises.

- (1) The boundary of X_w has irreducible components given by the Schubert varieties X_v such that $v < w$ such that $\ell(v) = \ell(w) - 1$.
- (2) C_w is a dense open set in X_w .
- (3) X_w embeds into a product of projective spaces via Plücker coordinates. A matrix is mapped under this embedding to the list all of its lower left minors in a given order.
- (4) If $w_0 = [n, n-1, \dots, 1]$, then $GL_n/\tilde{B} = X_{w_0}$.
- (5) The point w_0 has an affine neighborhood C_{w_0} of dimension $\binom{n}{2}$ and a local coordinate system. A generic point g has an affine neighborhood $gw_0C_{w_0}$ in $\mathcal{F}l_n$.
- (6) GL_n acts transitively on the points in the flag manifold so it is a manifold and a projective variety.
- (7) The flag manifold is smooth (i.e. non-singular at every point).

3. SMOOTH SCHUBERT VARIETIES

Say we wish to determine which Schubert varieties are smooth and which are not. There are several combinatorial and geometrical observations which makes this determination easier to characterize than a typical variety.

First, an affine variety is smooth at a point if the dimension of its tangent space equals the dimension of the variety near that point. If the variety is given in terms of the vanishing of certain polynomials, then one can check the dimension of the tangent space by computing the rank of the Jacobian matrix for those polynomials evaluated at the point. The rank is smaller than expected if and only if all minors of a certain size vanish. Thus, the set of points where the variety is not smooth is itself a variety called the *singular locus*.

A priori, to determine if a variety is smooth at every point, one must check the dimension of the tangent space at every point. For Schubert varieties, we make an easy observation. A point $p \in C_v \subset X_w$ is singular in X_w if and only if every point in C_v is singular in X_w since the Schubert cell C_v is a \tilde{B} -orbit. Recalling that the singular locus of a variety is a closed set, the equality (1) implies that each Schubert variety X_w is smooth if and only if X_w is smooth at the identity matrix I . One can check the singularity at the identity by writing down the defining equations of X_w around an affine neighborhood of X_w around I (for example, $X_w \cap w_0 C_{w_0}$) and check the rank of the Jacobian matrix of the defining polynomials. However, there is another way which provides a more unified tool for the study of the singularity of Schubert varieties using Lie algebras.

3.1. Lie algebras and tangent spaces of Schubert varieties. Recall from Section 2.1 that the flag variety can be identified with the quotient of a semisimple algebraic group:

$$Fl_n = GL_n(\mathbb{C})/\tilde{B} = SL_n(\mathbb{C})/B$$

where \tilde{B} is the set of upper triangular matrices in $GL_n(\mathbb{C})$ and $B = \tilde{B} \cap SL_n(\mathbb{C})$. The tangent space of SL_n is isomorphic as a vector space to its Lie algebra, which is known to be the $n \times n$ trace zero matrices over \mathbb{C} . The Lie algebra of B is the subalgebra of upper triangular matrices with trace zero. Let $G = SL_n(\mathbb{C})$, $\mathfrak{g} = \text{Lie}(SL_n)$ and $\mathfrak{b} = \text{Lie}(B)$. Then the tangent space of G/B at the identity matrix is isomorphic to $\mathfrak{g}/\mathfrak{b}$. Denoting by $E_{i,j}$ the matrix with 1 in the (i,j) -entry and 0's elsewhere, we obtain a basis for $\mathfrak{g}/\mathfrak{b}$ by

$$\mathfrak{g}/\mathfrak{b} = \text{span}\{E_{j,i} : 1 \leq i < j \leq n\}.$$

Observe that there is a natural bijection between the basis elements $\{E_{j,i} : i < j\}$ and $R := \{t_{i,j} : i < j\}$ the set of reflections.

More generally, for any $v \in S_n$, the tangent space to G/B at v is given by

$$(2) \quad T_v(G/B) = v(\mathfrak{g}/\mathfrak{b})v^{-1} = \text{span}\{E_{v(j),v(i)} : i < j\}.$$

Why? Because, G/vBv^{-1} , is an isomorphic copy of the flag manifold G/B but with respect to the base flag v_\bullet . Here the flag $v_\bullet = vB$ is fixed by the left action of vBv^{-1} .

It is an easy exercise to check

$$\begin{aligned} v E_{ij} v^{-1} &= E_{v(i),v(j)}, \\ t_{v(i),v(j)} v &= v t_{ij} \end{aligned}$$

for any $1 \leq i < j \leq n$. The next theorem gives us an explicit description of a basis of the tangent space of each Schubert variety.

Theorem 3.1. (Lakshmibai-Seshadri [65]) *For $v \leq w \in S_n$, the tangent space of X_w at v is given by*

$$T_v(X_w) \cong \text{span}\{E_{v(j),v(i)} : i < j, \quad vt_{ij} \leq w\},$$

and hence we obtain

$$\dim T_v(X_w) = \#\{(i < j): vt_{ij} \leq w\}.$$

Proof. Recall from the definition of a Lie algebra that $E_{v(j),v(i)} \in T_v(X_w)$ is equivalent to $(I + \varepsilon E_{v(j),v(i)})v \in X_w$ for infinitesimal $\varepsilon > 0$ where we can assume $\varepsilon^2 = 0$. Think of $(I + \varepsilon E_{v(j),v(i)})$ as a matrix in G acting on the left of the flag v_\bullet by moving the flag a little bit in the direction of $E_{v(j),v(i)}$. In particular, $(I + \varepsilon E_{v(j),v(i)})v = v + \varepsilon E_{v(j),v(i)}v = v + \varepsilon E_{v(j),i} \in X_v$ if and only if $v(i) > v(j)$ which is equivalent to $vt_{ij} \leq v$. Since $v \leq w$ implies $T_v(X_v) \subset T_v(X_w)$ we see that $E_{v(j),v(i)}$ is in $T_v(X_w)$ whenever $v(i) > v(j)$.

On the other hand, if $v(i) < v(j)$ then $v + \varepsilon E_{v(j),i} \in C_{vt_{ij}}$ and so $E_{v(j),v(i)} \in T_v(X_w)$ if and only if $vt_{ij} \leq w$. Thus, in either case $E_{v(j),v(i)} \in T_v(X_w)$ if and only if $vt_{ij} \leq w$. Thus, $\dim T_v(X_w) \geq \#\{(i < j): vt_{ij} \leq w\}$.

To prove $\dim T_v(X_w) \leq \#\{(i < j): vt_{ij} \leq w\}$, assume there exists coefficients $a_{i,j}$ for $1 \leq i < j \leq n$ such that $v + \varepsilon \sum a_{i,j} E_{v(j),i} \in X_w$. Say $v + \varepsilon \sum a_{i,j} E_{v(j),i} \in C_{v'}$ for some $v' \leq w$. Since $\varepsilon \ll 1$, none of the minors in v which are nonzero will vanish in $v + \varepsilon \sum a_{i,j} E_{v(j),i}$, so the rank table for $v + \varepsilon \sum a_{i,j} E_{v(j),i}$ dominates the rank table for v in every position. Hence, $v \leq v' \leq w$. Thus, for each $a_{i,j} \neq 0$, we have $v + \varepsilon E_{v(j),i} \in X_w$ so $\sum a_{i,j} E_{v(j),v(i)}$ is in the span of the independent set of $E_{v(j),v(i)}$ already known to be in $T_v(X_w)$. \square

Corollary 3.2. X_w is smooth at $v \in S_n$ if and only if

$$\dim T_v(X_w) := \#\{(i < j): vt_{ij} \leq w\} = l(w)$$

or equivalently if and only if

$$\#\{(i < j): v < vt_{ij} \leq w\} = l(w) - l(v).$$

Example 3.3. Consider the case $n = 4$. The Schubert variety X_{4231} is not smooth at the point $v = 2143$. For all 6 transpositions, $vt_{ij} \leq w$, but $\ell(w) = 5$. Also, $6 = \#\{t_{ij} \leq 4231\} = \dim T_{id}(X_{4231}) > \ell(4231) = 5$. See Figure 2 to verify these statements. Similarly, one can check X_{3412} is not smooth at $v = 1324$ and is smooth at all $v' \leq w$ such that $v' \not\leq v$. It follows that

$$\text{Sing}(X_{4231}) = X_{2143}$$

$$\text{Sing}(X_{3412}) = X_{1324}.$$

Note that 3412 is the reverse of 2143 and 4231 is the reverse of 1324. All other Schubert varieties X_w for w in S_4 are smooth.

3.2. Bruhat graphs.

Definition 3.4. For a permutation w , the *Bruhat graph* for w is a graph whose vertex set is $\{v \in S_n: v \leq w\} = [id, w]$ and there is an edge between v and vt_{ij} if and only if both $v, vt_{ij} \leq w$.

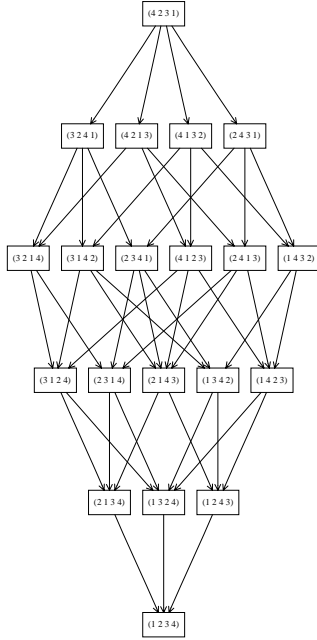
For example, the Bruhat graph of $w = 4321$ is drawn in Figure 3. Observe that the degree of v (i.e. the number of edges connected to v) in the Bruhat graph for w is $\dim T_v(X_w)$.

The Bruhat graph for w has a geometric interpretation: it is the *moment graph* of the Schubert variety X_w . Let $T \subset GL_n$ be the set of invertible diagonal matrices, then the permutation matrices in GL_n/\tilde{B} are exactly the T -fixed points.

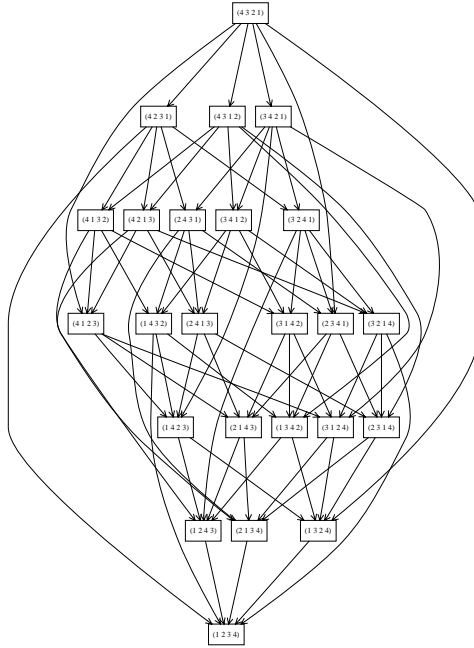
- (i) The permutations in $[id, w]$ are in bijection with the T -fixed points of X_w .
- (ii) If $v, vt_{ij} \leq w$, then the edge between v and vt_{ij} in the Bruhat graph for w , is realized as the corresponding curve passing through the flags corresponding to v and vt_{ij}

$$L_v = \{v + zE_{v(j),i}: z \in \mathbb{C}\} \cup \{vt_{ij}\} \approx \mathbb{P}^1.$$

This curve is T -invariant, and pointwise fixed by a torus $T' \subset T$ of codimension 1.

FIGURE 2. The interval $[id, 4231]$

Schubert varieties are examples of *GKM-spaces* studied by Goresky-Kottwitz-MacPherson [48] and others. It turns out that much of the T -equivariant topology or geometry of GKM spaces can be described in terms of their moment graph.

FIGURE 3. The Bruhat graph of $w = 4321$

3.3. Lakshmibai-Sandhya Theorem. There exists a simple criterion for characterizing smooth Schubert varieties using permutation pattern avoidance. Pattern avoidance first appeared in work by Knuth [62], Pratt [81] and Tarjan [90] related to computer sorting algorithms in the 1960's and 1970's. Today, many families of permutations are characterized by pattern avoidance or variations on that idea. We discuss one of the key results that brought this technique into the study of Schubert varieties.

Lakshmibai-Sandhya proved the following criterion for the singularity of Schubert varieties in 1990. See also the mutually independent work by Haiman (unpublished), Ryan [86], and Wolper [98].

Theorem 3.5. (*Lakshmibai-Sandhya [64]*) X_w is singular if and only if w has a subsequence with the same relative order as 3412 or 4231.

More generally, given any sequence of distinct real numbers $r_1 \dots r_m$ define $fl(r_1 \dots r_m)$ to be the permutation $v \in S_m$ such that $r_i < r_j$ if and only if $v_i < v_j$. Recall that a permutation is uniquely defined by its inversion set, so this condition uniquely defines v . The fl operator flattens the sequence. Then, a permutation $w = w_1 w_2 \dots w_n \in S_n$ contains a pattern $v = v_1 v_2 \dots v_m \in S_m$ for $m < n$ if there exists $i_1 < i_2 < \dots < i_m$ such that $fl(w_{i_1} w_{i_2} \dots w_{i_m}) = v$. Otherwise, w avoids v .

Example 3.6. The permutation $w = 625431$ contains the subsequence 6241 which flattens into a 4231-pattern. Hence, X_{625431} is singular. Also, $w = 612543$ avoids the patterns 4231 and 3412 which implies that X_{612543} is non-singular.

Let us sketch one approach to proving Theorem 3.5 by applying Theorem 3.1. Say w contains a 3412 or 4231 pattern in positions $i_1 < i_2 < i_3 < i_4$. Let v be the permutation obtained from w by rearranging the numbers $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ according to the pattern for the corresponding singular locus in S_4 . Specifically, if $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ is a 4231 then replace $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ by the 2143 pattern $w_{i_2} w_{i_4} w_{i_1} w_{i_3}$ in the same positions. If $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ is a 3412 then replace $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ by the 1324 pattern $w_{i_3} w_{i_1} w_{i_4} w_{i_2}$ in the same positions. For example, if $w = 625431$ and we use the 6241 instance of the pattern 4231, then $v = 215634$ which contains a 2143 pattern among the values 1, 2, 4, 6.

We claim that X_w is singular at the point v by construction. The proof proceeds by comparing $\ell(w) - \ell(v)$ with the number of t_{ij} such that $v < vt_{ij} \leq w$. For $i, j \in \{i_1, i_2, i_3, i_4\}$, we know there will be strictly more such transpositions than the length difference in these positions. A key lemma now states that if two permutations v and w agree in position i , then $v \leq w$ if and only if $fl(v_1 \dots \widehat{v_i} \dots v_n) \leq fl(w_1 \dots \widehat{w_i} \dots w_n)$ [6, Lemma 2.1]. This follows from looking at the rank tables of two permutations differing by a transposition. Next, note that vt_{ij} and w differ in at most 6 positions. Thus, by a computer verification on permutations of length 6 one can show that

$$\#\{t_{ij} : v < vt_{ij} \leq w\} > \ell(w) - \ell(v).$$

In the other direction, assume that w avoids the patterns 4231 and 3412. Lakshmibai and Sandhya show that avoiding these patterns is equivalent to an equidimensionality property of certain projections which implies smoothness.

Haiman's proof also contained the following enumerative formula as a corollary. Since his paper was never published, it wasn't until 2007 that this result had a proof in the literature due to Bousquet-Mélou and Butler.

Corollary 3.7. [19] *There is a closed form for the generating function for the sequence v_n counting the number of smooth Schubert varieties for $w \in S_n$:*

$$\begin{aligned} (3) \quad V(t) &= \frac{1 - 5t + 3t^2 + t^2 \sqrt{1 - 4t}}{1 - 6t + 8t^2 - 4t^3} \\ &= 1 + t + 2t^2 + 6t^3 + 22t^4 + 88t^5 + 366t^6 + 1552t^7 + 6652t^8 + O(t^9). \end{aligned}$$

Note that by the Lakshmibai-Sandhya theorem, testing for smoothness of Schubert varieties can be done naively in polynomial time, $O(n^4)$, based on the characterization of avoiding 3412 and 4231. As we pointed out in the introduction, the Guillemot-Marx [51] construction leads to a linear time algorithm in n for testing if a permutation in S_n contains either a 3412 or 4231 pattern.

Historically, there were some incremental results leading up to the linear time algorithm to detect pattern avoidance by Guillemot and Marx. These other algorithms might still have useful applications, so we mention a couple of them here. In [71], Madras and Liu study the 4231-avoiding permutations. They point out that using Knuth's original characterization of stack-sortable permutations in linear time, one can find a 4231 pattern in $O(n^2)$ time. In fact, Albert-Aldred-Atkinson-Holton show that every length 4 pattern can be detected in $O(n \log n)$ time [3].

4. 10 PATTERN AVOIDANCE PROPERTIES

In this section, we exhibit the ubiquity of pattern avoidance as a tool to characterize important properties in Schubert geometry and related areas. We give 10 distinct properties which are characterized by pattern avoidance. Each property will have a description in terms of avoiding certain patterns. Often these permutation families have other distinguishing features as well.

The first family of permutations defined by pattern avoidance is the 3412- and 4231-avoiding permutations appearing in the Lakshmibai-Sandhya Theorem. It is a family rich in structure. For the record, we state all the properties equivalently characterized by these two patterns. The history, citations, and some definitions follow the statement.

Pattern Avoidance Property 1. *The following are equivalent for $w \in S_n$.*

- (1) *The one-line notation for w avoids 3412 and 4231.*
- (2) *X_w is smooth.*
- (3) *$\ell(w) = \#\{t_{ij} \leq w\}$.*
- (4) *The Bruhat graph for w is regular and every vertex has degree $\ell(w)$.*
- (5) *The Poincaré polynomial for w , $P_w(t) = \sum_{v \leq w} t^{\ell(v)}$ is palindromic.*
- (6) *The Poincaré polynomial for w factors as*

$$P_w(t) = \prod_{i=1}^k (1 + t + t^2 + \cdots + t^{e_i})$$

for some positive integers $\{e_1, e_2, \dots, e_k\}$ such that $\ell(w) = \sum e_i$.

- (7) *The Poincaré polynomial $P_w(t)$ is equal to the generating function $R_w(t)$ for the number of regions r in the complement of the inversion hyperplane arrangement weighted by the distance of each region to the fundamental region. In symbols,*

$$R_w(t) = \sum_r t^{d(r)} = \sum_{v \leq w} t^{\ell(v)} = P_w(t).$$

Here, $d(r)$ is the number of hyperplanes crossed in a walk starting at the fundamental region and going to the region r .

- (8) *The inversion arrangement for w is free and the number of chambers of the arrangement is equal to the size of the Bruhat interval $[id, w]$.*
- (9) *The Kazhdan-Lusztig polynomial $P_{x,w}(t) = 1$ for all $x \leq w$.*
- (10) *The Kazhdan-Lusztig polynomial $P_{id,w}(t) = 1$.*

We have already discussed the equivalence of the first three items. Items (4), (5), and (10) are due to Carrell and Peterson [27]. Note, Carrell is the sole author on the paper cited, but he always acknowledges Peterson as a collaborator on this work so we give them both

credit. The term *palindromic* refers to the sequence of coefficients of the polynomial, so the coefficient of t^i equals the coefficient of $t^{\ell(w)-i}$ in a palindromic Poincaré polynomial.

Item (6) about factoring Poincaré polynomials is due to Gasharov [44]. This factorization implies that the geometry of smooth Schubert varieties has particularly nice structure in terms of iterated fiber bundles over Grassmannians [46, 84, 84, 86, 98].

Example 4.1. The permutation $w = 4321$ avoids the patterns 3412 and 4231. It has a palindromic Poincaré polynomial that also factors nicely,

$$\begin{aligned} P_{4321}(t) &= (1+t)(1+t+t^2)(1+t+t^2+t^3) \\ &= 1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6. \end{aligned}$$

Example 4.2. The permutation 3412 is one of the two cases in S_4 where the Poincaré polynomial does not have the nice factorization, nor the palindromic property. Here

$$P_{3412}(t) = 1 + 3t + 5t^2 + 4t^3 + t^4.$$

Item (7) about the inversion hyperplane arrangement is due to Oh-Postnikov-Yoo [78]. This arrangement is given by the collection of hyperplanes defined by $x_i - x_j = 0$ for all $i < j$ such that $w(i) > w(j)$. This generalizes the notion of the Coxeter arrangement of type A_{n-1} given by all the hyperplanes $x_i - x_j = 0$ for all $i < j$, so it is the inversion arrangement for w_0 . The Coxeter arrangement has $n!$ regions corresponding to all the permutations. In this case, the statistic $d(w)$ equals $\ell(w)$. Note no explicit bijective proof of Item (7) is known. The inversion arrangement comes up again in Property 5 below.

Item (8) is due to Slofstra [87]. Here a central hyperplane arrangement in a Euclidean space V is said to be *free* if the module of derivations of the complexified arrangement is free as a module over the polynomial ring $\mathbb{C}[V_{\mathbb{C}}]$. We refer the reader to this paper for more background. Note it also gives an algebraic interpretation for the generalized exponents e_1, e_2, \dots, e_k in terms of degrees of a homogeneous basis for the module of derivations.

Items (9) and (10) concern the Kazhdan-Lusztig polynomials [59]. These polynomials play an important role in the study of the singularities of Schubert varieties and in representation theory. We recall the definitions here, highlight some important developments, and refer the reader to the textbooks by Humphreys [54] and Björner-Brenti [17] for more details.

The *Hecke algebra* \mathcal{H} associated with S_n is an algebra over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ generated by $\{T_i : 1 \leq i \leq n-1\}$ with the relations

- (1) $(T_i)^2 = (q-1)T_i + q$,
- (2) $T_i T_j = T_j T_i$ if $|i-j| > 1$,
- (3) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for all $1 \leq i < n$.

This definition is patterned after the definition of the symmetric group S_n written in terms of its generating set of adjacent transpositions and their relations. In fact, if we take the specialization $q = 1$, then the resulting algebra is the group algebra of S_n . The relations (2) and (3) are called the *braid relations*. The braid relations imply that $T_w = T_{i_1} T_{i_2} \cdots T_{i_p}$ is well defined for any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_p}$. We will use the notation $T_{id} = 1 \in \mathcal{H}$ for the empty product of generators.

An easy observation is that $\{T_w : w \in S_n\}$ is a linear basis for \mathcal{H} over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. One can also observe that the T_w 's are invertible over $\mathbb{Z}[q, q^{-1}]$ which can be seen as follows. First check that $(T_i)^{-1} = q^{-1}T_i - (1 - q^{-1})$ by multiplying by T_i and using the stated relations. Then, we have $(T_w)^{-1} = (T_{i_p})^{-1} \cdots (T_{i_1})^{-1}$ for a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_p}$.

Next, let us review the *Kazhdan-Lusztig involution*. Consider the \mathbb{Z} -linear transformation $i : \mathcal{H} \rightarrow \mathcal{H}$ sending T_w to $(T_{w^{-1}})^{-1}$ and q to q^{-1} , respectively.

Theorem 4.3. (Kazhdan-Lusztig [59]) *There exists a unique basis $\{C'_w : w \in S_n\}$ for the Hecke algebra \mathcal{H} over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ such that*

- (i) $i(C'_w) = C'_w$.
- (ii) The change of basis matrix from $\{C'_w\}$ to $\{T_w\}$ is upper triangular when the elements of S_n are listed in a total order respecting Bruhat order, and the expansion coefficients $P_{x,w}(q)$ in

$$C'_w = q^{-\frac{1}{2}\ell(w)} \sum_{x \leq w} P_{x,w}(q) T_x$$

have the properties $P_{w,w} = 1$ and for all $x < w$, $P_{x,w}(q) \in \mathbb{Z}[q]$ with degree at most $\frac{\ell(w) - \ell(x) - 1}{2}$.

The basis $\{C'_w : w \in S_n\}$ is called the *Kazhdan-Lusztig basis* for \mathcal{H} , and $P_{x,w}(q)$ is the *Kazhdan-Lusztig polynomial* for $x, w \in S_n$. This theorem easily generalizes to all Coxeter groups for the reader familiar with that topic.

Example 4.4. We exhibit some computations with the Kazhdan-Lusztig basis indexed by permutations with the aid of Theorem 4.3. First, it is easy to see

$$C'_{s_i} = q^{-\frac{1}{2}}(1 + T_i) = q^{\frac{1}{2}}(1 + T_i^{-1}).$$

Then, for $i \neq j$, the computation

$$C'_{s_i} C'_{s_j} = q^{-1}(1 + T_i)(1 + T_j) = q^{-1}(1 + T_i + T_j + T_i T_j)$$

shows that $C'_{s_i s_j} = C'_{s_i} C'_{s_j}$ for $i \neq j$. Also, in the computation

$$\begin{aligned} C'_{s_1} C'_{s_2} C'_{s_1} &= q^{-\frac{3}{2}}(1 + T_1)(1 + T_2)(1 + T_1) \\ &= q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + T_1^2 + T_1 T_2 T_1) \\ &= q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + ((q-1)T_1 + q) + T_1 T_2 T_1), \end{aligned}$$

one notices that $qT_1 + q$ which comes from T_1^2 should not appear for $C'_{s_1 s_2 s_1}$ because the degree of the polynomial coefficient of T_1 and T_{id} are too large. We need a correction term. Since $C'_{s_i} = q^{-\frac{1}{2}}(1 + T_i)$ one can check that $C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1}$ by Theorem 4.3.

Example 4.5. If $i_1, \dots, i_k \in [n-1]$ are distinct, then one can check that

$$C'_{s_{i_1} \dots s_{i_k}} = C'_{s_{i_1}} \dots C'_{s_{i_k}}.$$

More generally, a permutation $w \in S_n$ is called *Deodhar* if $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \dots C'_{s_{i_p}}$ for some reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_p}$. We will return to the Deodhar permutations in Property 6.

Example 4.6. The Kazhdan-Lusztig polynomials $P_{id,w}$ for $w \in S_5$ are completely determined from the following table and the fact that $P_{id,w} = 1$ if and only if w is 3412 and 4231 avoiding.

w	$P_{id,w}$
(14523) (15342) (24513)	$q + 1$
(25341) (34125) (34152)	
(35124) (35142) (35241)	
(35412) (41523) (42315)	
(42351) (42513) (42531)	
(43512) (45132) (45213)	
(51342) (52314) (52413)	
(52431) (53142) (53241)	
(53421) (54231)	
(34512) (45123)	$2q + 1$
(45231) (53412)	
(52341)	$q^2 + 2q + 1$
(45312)	$q^2 + 1$

The reader might notice that all coefficients of Kazhdan-Lusztig polynomials shown so far are non-negative integers. In their 1979 paper, this property was stated as a conjecture for all Kazhdan-Lusztig polynomials. In 1980, Kazhdan and Lusztig proved their own conjecture using *intersection homology* as introduced by Goresky and MacPherson in 1974, see [49] as a good starting point for that theory.

Theorem 4.7. (*Kazhdan-Lusztig* [60]) *If W is a Weyl group or affine Weyl group then*

$$P_{x,w}(q) = \sum \dim \mathcal{IH}_x^i(X_w) q^i.$$

Corollary 4.8. *The coefficients of $P_{x,w}(q)$ are non-negative integers with constant term 1.*

The big news in Kazhdan-Lusztig theory is the recent proof that all Kazhdan-Lusztig polynomials for all Coxeter groups have non-negative integer coefficients. This proof is due to Elias and Williamson [40]. They give an algebraic structure (Soergel bimodules) which plays the same role as intersection homology of Schubert varieties in the original proof.

As stated in Property 1, Kazhdan-Lusztig polynomials can be used to determine smoothness of Schubert varieties (in type A). There are several other interesting properties of Kazhdan-Lusztig polynomials that have emerged since they were defined in 1979. We cover some of them here and recommend the Wikipedia page [97] for a very nice survey.

- (1) In 1981, Beilinson–Bernstein, and independently Brylinski–Kashiwara, proved another important conjecture due to Kazhdan and Lusztig. They showed that the multiplicities which appear when expressing the formal character of a Verma module in terms of the formal character for the corresponding simple highest weight module are determined by evaluating Kazhdan-Lusztig polynomials at $q = 1$ ([5, 25]).
- (2) The coefficients of Kazhdan-Lusztig polynomials are increasing as one goes down in Bruhat order, while keeping the second index fixed. Specifically, if $x \leq y \leq w$, then $\text{coef}_{q^k} P_{x,w}(q) \geq \text{coef}_{q^k} P_{y,w}(q)$. This monotonicity property was first published in 1988 by Ron Irving [56]. Irving’s proof is based on the socle filtration of a Verma module. In 2001, Braden and MacPherson gave a different proof using intersection homology [20, Cor. 3.7].
- (3) Every polynomial with constant term 1 and nonnegative integer coefficients is the Kazhdan-Lusztig polynomial of some pair of permutations. This is due to Patrick Polo, published in 1999 [80]. He gives an explicit construction of the pair of permutations for a given polynomial. This was a surprising result because from the small data that we can compute, say for $n \leq 8$, the polynomials seem quite special. They must get increasingly complex as n grows.
- (4) Let $\mu(x, w)$ be the coefficient of $q^{\frac{\ell(w) - \ell(x) - 1}{2}}$ in $P_{x,w}$. Note, $\mu(x, w)$ can be 0. For $x, w \in S_9$, $\mu(x, w) \in \{0, 1\}$. MacLarnen and Warrington found an example in S_{10} where $\mu(x, w) = 5$ [75]. Prior to their publication in 2003, this was referred to as the “0-1 Conjecture for Kazhdan-Lusztig polynomials.” This again demonstrates the increasing complexity as n grows. The reader might be wondering how anyone could have believed the 0-1 Conjecture after seeing Polo’s theorem in (3). However, Polo’s theorem does not contradict the 0-1 Conjecture because in his construction the length difference between w and x is large enough that the leading term in $P_{x,w}(q)$ is typically not the μ -coefficient.
- (5) There exists a formula for $P_{x,w}(q)$ which only depends on the abstract interval $[id, w]$ in Bruhat order. See the work of du Cloux (2003) [35], Brenti (2004) [21] and Brenti-Caselli-Marietti (2006) [22].

There are two interesting but difficult open problems in Kazhdan-Lusztig theory. There are many partial answers to these questions in the literature, but we don’t know of a complete source at this time. Perhaps there is a need for someone to start a wiki page.

Question 1. (Lusztig) Can one compute $P_{x,w}(q)$ using only the abstract poset given by the interval $[x, w]$ in Bruhat order? In other words, $P_{u,v}(q) = P_{x,y}$ whenever $[u, v]$ and $[x, y]$ are isomorphic as posets.

Question 2. Can one compute the coefficients of $P_{x,w}(q)$ by counting combinatorially defined objects?

The next pattern property connects the 3412 and 4231 patterns to the determination of the singular locus of a Schubert variety. Recall from Section 3, the singular locus of a Schubert variety X_w is a union of Schubert varieties X_v with $v < w$. Thus to determine the irreducible components of the singular locus, we just need to give the maximal permutations $v < w$ such that v determines a singular point in X_w .

Pattern Avoidance Property 2. (Billey-Warrington, Manivel, Kassel-Lascoux-Reutenauer, and Cortez [16, 72, 58, 32]) X_v is an irreducible component of the singular locus of X_w if and only if

$$v = w \cdot (1\text{-cycle permutation})$$

corresponding to a 4231 or 3412 or 45312 pattern from Figure 4 such that the shaded region contains no additional 1's except in the 45312 case where they must appear in the central region in decreasing order. Here \circ 's denote 1's in w , \bullet 's denote 1's in v .

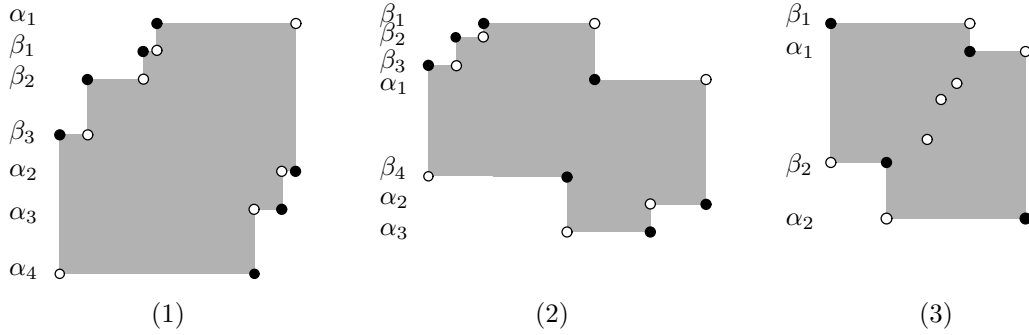


FIGURE 4. Patterns for the singular locus of X_w in the 4231, 3412, and 45312 cases respectively.

This result was found around 2000 by 7 authors in 4 papers, plus Gasharov proved on direction of the conjecture [45] around the same time. It must have been ripe for discovery. It refined and proved a conjecture due to Lakshmibai and Sandhya [64]. For the sake of history, we note that the authors of [16] were the first to report this result to Lakshmibai.

Corollary 4.9. *The codimension of the singular locus of a Schubert variety X_w is at least 3 for any $w \in S_n$.*

The corollary is in fact true for all simply laced types. However, it is not true in type B_n . The codimension of the singular locus of a Schubert variety in that case can be 2.

Inspired by the Lakshmibai-Sandhya Theorem and the construction of the singular locus of a Schubert variety in Property 2, Woo and Yong [102] defined the notion of interval pattern avoidance. Given permutations $u < v \in S_m$ and $x < y \in S_n$ for $m < n$, say $[u, v]$ *interval pattern embeds* into $[x, y]$ provided

- (1) There exist indices $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that $\text{fl}(x_{i_1}, \dots, x_{i_m}) = u$ and $\text{fl}(y_{i_1}, \dots, y_{i_m}) = v$ respectively.
- (2) The permutations x, y agree in all positions other than $1 \leq i_1 < i_2 < \dots < i_m \leq n$.
- (3) The Bruhat intervals $[u, v]$ and $[x, y]$ are isomorphic as posets.

In fact, if x, y agree everywhere outside of the indices $1 \leq i_1 < i_2 < \dots < i_m \leq n$ and $u = fl(x_{i_1}, \dots, x_{i_m})$, $v = fl(y_{i_1}, \dots, y_{i_m})$ then $[u, v]$ interval embeds in $[x, y]$ if and only if $\ell(v) - \ell(u) = \ell(y) - \ell(x)$ [102, Lemma 2.1]. Furthermore, for all $w \in S_n$ such that $x < w < y$, then w agrees everywhere with y outside of the sequence and $[fl(w_{i_1}, \dots, w_{i_m}), v]$ also interval embeds in $[x, y]$ [102, Lemma 2.4].

Observe that the condition from Figure 4 that the shaded region have no additional 1's in the permutation matrices implies that the length $\ell(w) - \ell(v)$ is equal to the corresponding length drop in each of the 4231, 3412 or 45312 cases. Thus, the maximal singular locus of a Schubert variety is determined by interval pattern conditions.

Another example of the power of interval pattern embeddings is the following result supporting Question 1. More examples will follow, but the reader is encouraged to see [102] for more details.

Theorem 4.10. [102, Cor. 6.3] *Suppose $[u, v] \subset S_m$ interval pattern embeds into $[x, y] \in S_n$, then the Kazhdan-Lusztig polynomials $P_{u,v}(q)$ and $P_{x,y}(q)$ are equal.*

Next, recall by a theorem due to Zariski that a variety X is smooth if and only if the local ring at every point is regular. A variety X is *factorial* at a point if the local ring at that point is a unique factorization domain. Note that a smooth variety is factorial at every point since any regular local ring is a unique factorization domain. The following property was conjectured by Woo-Yong and proved by Bousquet-Mélou and Butler in 2007.

Pattern Avoidance Property 3. (Bousquet-Mélou-Butler [19]) *Let $w \in S_n$, then the following are equivalent.*

- (1) *The Schubert variety X_w is factorial at every point.*
- (2) *The permutation w avoids 4231 and 3412 where 3412 means that the 4 and 1 must be adjacent in the one-line notation for w .*
- (3) *The permutation w avoids 4231, and for every $v < w$ differing in exactly 4 positions, the interval $[v, w]$ is not isomorphic to [3142, 3412]. Thus, one says w interval avoids [3142, 3412].*

Compare the generating function below with Corollary 3.7 which is the generating function for the number of smooth Schubert varieties in $\mathcal{F}l_n$.

Theorem 4.11. [19] *There is a closed form for the generating function for the sequence f_n counting the factorial Schubert varieties for $w \in S_n$:*

$$(4) \quad F(t) = \frac{(1-t)(1-4t-2t^2) - (1-5t)\sqrt{1-4t}}{2(1-5t+2t^2-t^3)} \\ = t + 2t^2 + 6t^3 + 22t^4 + 89t^5 + 379t^6 + 1661t^7 + 7405t^8 + \dots$$

Note, the term $\sqrt{1-4t}$ appears in both (3) and (4). This term is familiar in combinatorics because it also appears in the generating function for the Catalan numbers, $c_n = \frac{1}{n+1} \binom{2n}{n}$. In particular, as a power series $\sqrt{1-4t} = 1 + \sum_{n \geq 1} \frac{-2}{n} \binom{2n-2}{n-1} t^n$ by Newton's generalized binomial theorem. Thus, the generating function for the Catalan numbers is

$$\sum_{n \geq 1} c_n t^n = \frac{1 - \sqrt{1-4t}}{2t}.$$

There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance. Recall that a variety X is *Gorenstein* if it is Cohen-Macaulay and its canonical sheaf is a line bundle. Woo and Yong characterized the Gorenstein condition by using pattern avoidance.

Pattern Avoidance Property 4. (Woo-Yong [101]) *A Schubert variety X_w is Gorenstein if and only if the following two conditions are satisfied :*

- (i) w avoids 35142 and 42513 with Bruhat restrictions $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$, and
- (ii) for each descent d in w , the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

Later, Woo-Yong [102, Thm. 6.6] also gave a characterization of Gorenstein Schubert varieties in terms of an interval pattern avoidance using an infinite number of intervals.

We note that in the paper [101], the theorem states that w should avoid 31542 and 24153 which is twisted by w_0 from the permutations written above. The difference is that they are labeling Schubert varieties in such a way that the codimension of X_w is $\ell(w)$ which works better for computing products of Schubert classes.

The proof of this result due to Woo and Yong relates the Gorenstein property to Schubert classes for the flag manifold and Monk's formula. Since the topic of the conference in Osaka is "Schubert Calculus", we outline this proof to show the logical relationship. The steps are due to Woo and Yong unless otherwise mentioned.

Sketch of proof.

Step 1: Schubert varieties are all Cohen-Macaulay. (Ramanathan, 1985)

Step 2: Testing if X_w is Gorenstein reduces to a comparison using the Weil divisor class group and the Cartier class group for X_w . (Brion, Knutson, Kumar)

Step 3: The Weil divisor class group is generated by the set of all $[X_v] \in H^*(G/B)$ such that $v < w$ and $\ell(v) = \ell(w) - 1$. In this case we say w covers v in Bruhat order, denoted $v \triangleleft w$. If $v \triangleleft w$, then $w = vt_{ij}$ but $t_{i,j}$ does not need to be an adjacent transposition.

Step 4: The Cartier class group is generated by $[X_{w_0 s_i}][X_w]$ and

$$[X_{w_0 s_i}][X_w] = \sum [X_v]$$

summed over all $v = wt_{ab}$: $a \leq i < b$, $\ell(v) = \ell(w) - 1$ by Monk's formula.

Step 5: The Schubert variety X_w is Gorenstein if and only if there exists an integral solution $(\alpha_1, \dots, \alpha_{n-1})$ to

$$\sum_{i=1}^{n-1} \alpha_i [X_{w_0 s_i}][X_w] = \sum_{v \triangleleft w} [X_v].$$

For the details of the proof, see [101]. □

A Schubert variety $X_w(E_\bullet)$ is *defined by inclusions* if it can be described as the set of all flags F_\bullet where $F_i \subset E_j$ or $E_i \subset F_j$ for some collection of pairs i, j .

Pattern Avoidance Property 5. (Gasharov-Reiner [46]) *A Schubert variety X_w is defined by inclusions if and only if w avoids 4231, 35142, 42513, 351624.*

The four patterns appearing in this property have two other interesting and unexpected connections found using Tenner's Database of Permutation Pattern Avoidance.

Theorem 4.12. (Hultman-Linusson-Shareshian-Sjöstrand [53]) *The number of regions in the inversion arrangement for w is at most the number of elements below w in Bruhat order. The two quantities are equal if and only if w avoids 4231, 35142, 42513, 351624.*

Given a subset S of $[n] \times [n]$, let $\text{mat}_q(n, S, r)$ be the number of $n \times n$ matrices over \mathbb{F}_q with rank r , none of whose nonzero entries lie in S . For example, if $S = \emptyset$, then

$$\text{mat}_q(n, \emptyset, n) = q^{\binom{n}{2}}(q-1)^n \prod_{i=1}^n (1+q+\dots+q^{i-1}) = q^{\binom{n}{2}}(q-1)^n P_{w_0}(q)$$

where w_0 is the longest element of S_n and $P_{w_0}(q)$ is the Poincaré polynomial for X_{w_0} .

Theorem 4.13. (*Lewis-Morales [68]*) Fix a permutation w in S_n , and let $D(w)$ be its permutation diagram. We have that

$$\text{mat}_q(n, D(w), n)/(q-1)^n = q^{n(n-1)-\text{inv}(w)} P_{w_{w_0}}(q^{-1})$$

If and only if w avoids 1324, 24153, 31524, and 426153 (the reverses of the patterns in Property 5).

The theorem above was originally part of a more general conjecture by Klein-Lewis-Morales. We state the part that is still open.

Conjecture 4.1. (*Klein-Lewis-Morales [61, Conj. 5.1 and Conj 6.6]*) Using the notation above, $\text{mat}_q(n, D(w), n)/(q-1)^n$ is a polynomial function of q which is coefficient-wise less than or equal to $q^{\binom{n}{2}-\text{inv}(w)} P_w(q)$.

Recently, Albert and Brignall have shown that the enumeration of Schubert varieties defined by inclusions has a nice generating function and recurrence relation. Once again, it is interesting to compare this formula with (3) and (4).

Theorem 4.14. (*Albert-Brignall [4]*) Let $f(n)$ be the number of permutations in S_n which avoid 4231, 35142, 42513, and 351624. Then, we have the generating function

$$\sum f(n)t^n = \frac{1-3t-2t^2-(1-t-2t^2)\sqrt{1-4t}}{1-3t-(1-t+2t^2)\sqrt{1-4t}}.$$

Gasharov-Reiner give a nice description of the cohomology rings of Schubert varieties defined by inclusions. This result has been extended by Reiner-Woo-Yong in a beautiful way which relates to Fulton's essential set which is a subset of the diagram of a permutation. In order to describe it here, let us first recall Carrell's result on the cohomology of Schubert varieties.

Theorem 4.15. (*Carrell [26]*) $H^*(X_w) \approx H^*(G/B)/I_w$ where I_w is generated by all $[X_v]$ such that $v \not\leq w$.

A permutation x is called *Grassmannian* if x has at most 1 descent. Also, x is *bigrassmannian* if both x and x^{-1} are Grassmannian. We denote by $\text{Des}(x)$ the set of descents in x . In 1992, Akyildiz-Lascoux-Pragacz gave a description of the ideal I_w which was then further refined by Reiner-Woo-Yong.

Theorem 4.16. (*Akyildiz-Lascoux-Pragacz [2]*) I_w is generated by the set of all $[X_v]$ such that $v \not\leq w$ and v is Grassmannian.

Following [83], for a permutation w , let $E(w)$ be the set of permutations which are minimal elements in Bruhat order in the complement of the interval $[id, w]$. The set $E(w)$ is called the *essential set* of w . Clearly, this notion of essential set generalizes to all Coxeter groups.

Theorem 4.17. (*Lascoux-Schützenberger and Geck-Kim [47]*) The elements in $E(w)$ are bigrassmannian.

Theorem 4.18. (*Reiner-Woo-Yong [83]*) There exists a bijection between $E(w)$ and Fulton's essential set which is defined as the cells in the diagram of the permutation $D(w)$ which have no cell directly to their right or below.

Theorem 4.19. (*Reiner-Woo-Yong [83]*) I_w is generated by the set of all $[X_v]$ such that $v \not\leq w$, v is Grassmannian and there exists some bigrassmannian $x \in E(w)$ such that $x \leq v$ and $\text{Des}(x) = \text{Des}(v)$.

Reiner-Woo-Yong point out that this generating set for I_w is still not minimal in general. This leads to some interesting open questions.

Question 3. Find a minimal set of generators for I_w for all $w \in S_n$. (See [83]).

Question 4. What is the relationship between $E(w)$ and the defining equations for Schubert varieties in other types?

The next property relates the Bott-Samelson resolution for a singular Schubert variety and the Kazhdan-Lusztig basis elements to pattern avoidance. A resolution of a singular variety is called a *small resolution* if for every $r > 0$, the space of points of X where the fiber over the point in the resolution has dimension r is of codimension greater than $2r$. In words, the singular points where the resolution has to blow up the dimension a lot are rare in a small resolution. One reason that people care about small resolutions is that the intersection homology of a variety is just the homology of a small resolution of the variety.

Pattern Avoidance Property 6. (Deodhar [34], Billey-Warrington [12]) *The following are equivalent.*

- (1) $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}}$ for some (or any) reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_p}$.
- (2) The Bott-Samelson resolution of X_w is small.
- (3) $\sum_{v \leq w} t^{l(v)} P_{v,w}(t) = (1+t)^{l(w)}$.
- (4) For each $v \leq w$, the Kazhdan-Lusztig polynomial can be written as

$$P_{v,w}(t) = \sum_{\sigma \in E(v,w)} t^{\text{defect}(\sigma)}.$$

- (5) w is 321-hexagon avoiding, that is, w avoids

$$321, 56781234, 56718234, 46781235, 46718235.$$

The equivalence of the first four properties was given by Deodhar [34]. Showing these properties have a pattern avoidance characterization is due to Billey-Warrington [12]. Deodhar's theorem extends to all Weyl groups and in each case there is again a pattern avoidance characterization due to Billey-Jones [14].

We should explain Deodhar's terminology $\text{defect}(\sigma)$ and $E(v, w)$ because we believe that they might have important implications for answering Question 2. First, fix a reduced expression for w . This corresponds with a string diagram S for w . Think of each crossing in the string diagram as optional. Then $E(v, w)$ is the set of all string diagrams σ for v obtained from S by choosing some subset of the crossings. The *defect* of σ is the number of times two strings come together that have previously crossed an odd number of times in the string diagram, as one progresses vertically. Thus,

$$P_{v,w}(t) = \sum_{\sigma \in E(v,w)} t^{\text{defect}(\sigma)}$$

is precisely the sort of combinatorial formula for the Kazhdan-Lusztig polynomials we would like to have. Deodhar has shown that for every pair $v, w \in S_n$ there exists a set of string diagrams $E(v, w)$ for which the same formula holds. The only drawback is that in order to find $E(v, w)$ one must basically compute $P_{v,w}$ using another method first.

The next pattern property due to Tenner concerns a subset of the 321-hexagon avoiding permutations.

Pattern Avoidance Property 7. (Tenner [91]) *The principal order ideal below w in Bruhat order is isomorphic to a Boolean lattice if and only if w is 321 and 3412 avoiding. Equivalently, the Bott-Samelson resolution of X_w is isomorphic to X_w .*

Thus, a permutation is called *Boolean* if it is 321 and 3412 avoiding. These permutations give rise to a familiar enumerative sequence.

Theorem 4.20. (*Fan [42], West [96]*) *The number of Boolean permutations in S_n is the Fibonacci number F_{2n-1} , e.g. $F_1 = 1, F_3 = 2, F_5 = 5$.*

The next property relates Kazhdan-Lusztig polynomials to a filtration on permutations. It was conjectured by Billey-Braden [13] and proved by Woo [99].

Pattern Avoidance Property 8. (*Woo-Billey-Weed [99]*) *The Kazhdan-Lusztig polynomial $P_{id,w}(1) = 2$ if and only if w avoids 653421 , 632541 , 463152 , 526413 , 546213 , and 465132 and the singular locus of X_w has exactly 1 component.*

To define a filtration on permutations in a similar way, let's make the following definition.

Definition 4.21. Let $KL_m = \{w \in S_\infty \mid P_{id,w}(1) \leq m\}$.

For example, we know from Property 1 that KL_1 is the set of permutations avoiding 3412 and 4231 . Similarly Billey-Weed used Woo's theorem to show that KL_2 is characterized by the 66 permutation patterns of length ≤ 8 below. This result is in an appendix to [99].

(4 5 1 2 3) (3 4 5 1 2) (5 3 4 1 2) (5 2 3 4 1) (4 5 2 3 1)
 (3 5 1 6 2 4) (5 2 3 6 1 4) (5 2 6 3 1 4) (6 2 4 1 5 3) (5 2 4 6 1 3)
 (4 6 2 5 1 3) (5 2 6 4 1 3) (5 4 6 2 1 3) (3 6 1 4 5 2) (4 6 1 3 5 2)
 (3 6 4 1 5 2) (4 6 3 1 5 2) (5 3 6 1 4 2) (4 6 5 1 3 2) (4 2 6 3 5 1)
 (6 3 2 5 4 1) (6 3 5 2 4 1) (6 4 2 5 3 1) (6 5 3 4 2 1)
 (3 6 1 2 7 4 5) (6 2 3 1 7 4 5) (6 2 4 1 7 3 5) (3 4 1 6 7 2 5)
 (4 2 3 6 7 1 5) (4 2 6 3 7 1 5) (4 2 6 7 3 1 5) (3 7 1 2 5 6 4)
 (7 2 3 1 5 6 4) (3 7 1 5 2 6 4) (3 7 5 1 2 6 4) (7 5 2 3 1 6 4)
 (6 2 5 1 7 3 4) (7 2 6 1 4 5 3) (3 4 1 7 5 6 2) (3 5 1 7 4 6 2)
 (4 5 1 7 3 6 2) (4 2 3 7 5 6 1) (5 3 4 7 2 6 1) (4 2 7 5 6 3 1)
 (3 4 1 2 7 8 5 6) (4 2 3 1 7 8 5 6) (3 4 1 7 2 8 5 6)
 (4 2 3 7 1 8 5 6) (4 2 7 3 1 8 5 6) (3 5 1 2 7 8 4 6)
 (5 2 3 1 7 8 4 6) (5 2 4 1 7 8 3 6) (3 4 1 2 8 6 7 5)
 (4 2 3 1 8 6 7 5) (3 4 1 8 2 6 7 5) (4 2 3 8 1 6 7 5)
 (4 2 8 3 1 6 7 5) (3 4 1 8 6 2 7 5) (4 2 3 8 6 1 7 5)
 (4 2 8 6 3 1 7 5) (3 5 1 2 8 6 7 4) (5 2 3 1 8 6 7 4)
 (3 6 1 2 8 5 7 4) (6 2 3 1 8 5 7 4) (5 2 4 1 8 6 7 3)
 (6 2 5 1 8 4 7 3)

A local ring R is a *local complete intersection (LCI)* if it is the quotient of some regular local ring by an ideal generated by a regular sequence. A variety is *LCI* if every local ring is LCI.

Pattern Avoidance Property 9. (*Úlfarsson-Woo [95]*) *A Schubert variety X_w is LCI if and only if w avoids 53241 , 52341 , 52431 , 35142 , 42513 , and 426153 .*

Since regular local rings are LCI, smooth varieties are automatically LCI. Furthermore, LCI varieties are Gorenstein and hence Cohen-Macaulay. Thus, being LCI can be viewed as saying that the singularities are in some sense mild. Compare the above criterion with Property 1 (for smoothness) and Property 4 (for Gorenstein property).

A permutation is *vexillary* if it avoids 2143, introduced by Lascoux-Schützenberger in 1982 [67]. The word vexillary is related to flags, hence the choice. We say w is *covexillary* if w avoids 3412. There are so many interesting things to say related to vexillary and covexillary permutations so the tenth property has 3 parts.

Pattern Avoidance Property 10-1.

- (1) (Edelman-Greene [38]) *The number of reduced words for a vexillary permutation v is equal to the number of standard tableaux of shape determined by sorting the lengths of the rows of the diagram of v .*
- (2) (Edelman-Greene [38]) *The Stanley symmetric function F_v is a Schur function if and only if v is vexillary. Here*

$$F_v = \sum_{\mathbf{a}=a_1 a_2 \dots a_k \in R(v)} \sum_{i_1 \leq \dots \leq i_k \in C(\mathbf{a})} x_{i_1} x_{i_2} \dots x_{i_k}$$

where $R(v)$ are the reduced words for v and $C(\mathbf{a})$ are the weakly increasing sequences of positive integers such that $i_j < i_{j+1}$ if $a_j < a_{j+1}$.

- (3) (Tenner [91]) *The permutation v is vexillary if and only if for every permutation w containing v , there exists a reduced decomposition $\mathbf{a} \in R(w)$ containing a shift of some $\mathbf{b} \in R(v)$ as a factor.*

The next is a list of properties of vexillary permutations related to geometry of Schubert varieties.

Pattern Avoidance Property 10-2.

- (1) (Fulton [43]) *Recall, Fulton's essential set for w is the collection of cells in the diagram of w with no neighbor directly east or south. If w is vexillary, these cells lie on an increasing piecewise linear curve.*
- (2) (Lascoux [66]) *There exists a combinatorial approach to computing the Kazhdan-Lusztig polynomials $P_{v,w}$ when w is covexillary.*
- (3) (Li-Yong [69]) *There exists a combinatorial rule for computing multiplicities for X_w when w is covexillary.*

We say a permutation w is *k-vexillary* if its Stanley symmetric function F_w has at most k terms of Schur functions in its expansion. For example, $F_{2143} = s_{(2)} + s_{(1,1)}$, so 2143 is 2-vexillary.

Pattern Avoidance Property 10-3. (Billey-Pawlowski [9]) *The k-vexillary permutations are characterized by a finite set of patterns for all k .*

For example, if w is a permutation, then the following hold.

- (1) w is 2-vexillary if and only if w avoids 35 patterns in S_5, S_6, S_7, S_8 .
- (2) w is 3-vexillary if and only if w avoids 91 patterns in S_5, S_6, S_7, S_8 .

The list of 2-vexillary patterns is given as follows:

(3 2 1 5 4) (2 1 5 4 3) (2 1 4 3 6 5) (2 4 1 3 6 5) (3 1 4 2 6 5) (3 1 2 6 4 5) (2 1 4 6 3 5) (2 4 1 6 3 5)
 (2 3 1 5 6 4) (2 1 5 3 6 4) (3 1 5 2 6 4) (4 2 6 1 5 3) (5 2 7 1 4 3 6) (5 1 7 3 2 6 4) (4 2 6 5 1 7 3) (2
 5 4 7 1 6 3) (5 4 7 2 1 6 3) (5 2 7 6 1 4 3) (6 1 8 3 2 5 4 7) (2 6 4 8 1 5 3 7) (6 4 8 2 1 5 3 7) (2 6 5 8
 1 4 3 7) (6 5 8 2 1 4 3 7) (5 1 7 3 6 2 8 4) (5 1 7 6 3 2 8 4) (6 1 8 3 7 2 5 4) (6 1 8 7 3 2 5 4) (2 5 4 7
 6 1 8 3) (5 4 7 2 6 1 8 3) (5 4 7 6 2 1 8 3) (2 6 4 8 7 1 5 3) (6 4 8 7 2 1 5 3) (2 6 5 8 7 1 4 3) (6 5 8 2
 7 1 4 3) (6 5 8 7 2 1 4 3).

We have given 10+ properties of Schubert varieties which are amenable to pattern avoidance in their characterization. This is just the beginning of all the consequences for the Lakshmibai-Sandhya Theorem. In the next section, we will discuss how pattern avoidance extends to other Lie types and Coxeter groups.

There are two further directions/consequences concerning special families of varieties we should note. First is the $GL_p \times GL_q$ -orbits in the flag variety for GL_{p+q} with rationally smooth closure. These varieties are special cases of the symmetric varieties studied by Springer. McGovern has characterized which symmetric varieties in this case are rationally smooth by using patterns involving a multiset of numbers and $+$ and $-$ signs. See [73] for further details. Similar results are given in type C by McGovern and Trapa [74].

Second, pattern avoidance also comes up in the study of Peterson varieties. The Peterson variety for \mathbb{C}^n is the collection of complete flags F_\bullet such that $N \cdot F_i \subset F_{i+1}$ for all $1 \leq i < n$ where N is a fixed regular nilpotent matrix. Up to isomorphism, the variety is independent of the choice of N . Insko and Yong gave a combinatorial description of the singular locus of the Peterson variety which involves the patterns 123 and 2143 among other conditions [55].

5. PATTERN AVOIDANCE FOR COXETER GROUPS

In this section, we study pattern avoidance properties for Coxeter groups. First, we recall the definition of Coxeter groups and their basic properties. For details, see [17, 54].

5.1. A quick review on Coxeter groups. A *Coxeter graph* is a simple graph with vertices $\{1, 2, \dots, n\}$ and edges labeled by $\mathbb{Z}_{\geq 3} \cup \infty$. The Coxeter group associated to a Coxeter graph G is the group generated by $S = \{s_1, s_2, \dots, s_n\}$ with relations

- (1) $s_i^2 = 1$.
- (2) $s_i s_j = s_j s_i$ if i, j not adjacent in G .
- (3) $\underbrace{s_i s_j s_i \cdots}_{m(i,j) \text{ generators}} = \underbrace{s_j s_i s_j \cdots}_{m(i,j) \text{ generators}}$ if i, j connected by edge labeled $m(i, j) < \infty$.

Since a Coxeter group is completely determined by its Coxeter graph, we simply need to draw the graph to refer to the associated Coxeter group. Conventionally, we drop the label 3 from any edge in pictures for simplicity.

$$\bullet_1 \xrightarrow{4} \bullet_2 \xrightarrow{3} \bullet_3 \xrightarrow{3} \bullet_4 \approx \bullet_1 \xrightarrow{4} \bullet_2 \text{ --- } \bullet_3 \text{ --- } \bullet_4$$

Example 5.1. The following are examples of Coxeter groups.

- (1) Dihedral groups: Dih_{10} is $\bullet_1 \xrightarrow{5} \bullet_2$
- (2) Symmetric groups: S_5 is $\bullet_1 \text{ --- } \bullet_2 \text{ --- } \bullet_3 \text{ --- } \bullet_4$
- (3) Hyperoctahedral groups: B_4 is $\bullet_1 \xrightarrow{4} \bullet_2 \text{ --- } \bullet_3 \text{ --- } \bullet_4$
- (4) The exceptional Weyl groups: E_8 is
$$\begin{array}{cccccccc} \bullet_1 & \text{---} & \bullet_2 & \text{---} & \bullet_3 & \text{---} & \bullet_4 & \text{---} & \bullet_5 & \text{---} & \bullet_6 & \text{---} & \bullet_7 \\ & & & & | & & & & & & & & \\ & & & & \bullet_8 & & & & & & & & \end{array}$$

Curiously, the exceptional Weyl group E_8 appears in string theory and in chemistry related to the symmetry group of the C_{60} molecule and buckyballs [31].

Fix a Coxeter group W with Coxeter graph G . The set of *reflections* $R \subset W$ is the set of all conjugates of the generators,

$$R = \bigcup_{w \in W} w S w^{-1}.$$

A *reduced expression* of an element $w \in W$ is an expression $w = s_{i_1} \cdots s_{i_k}$ as a product of generators in which k is the minimum among such expressions. The *length* of $w \in W$ is the

length of a reduced expression for w , denoted $\ell(w)$ again. Bruhat order on the Coxeter group W is the transitive closure of the following relation

$$x \leq y \quad \text{if} \quad \ell(x) < \ell(y) \text{ and } xy^{-1} \in R.$$

It was observed by Chevalley that $x \leq y$ if and only if for any reduced expression $y = s_{i_1} s_{i_2} \dots s_{i_p}$ there exists a subexpression which is a reduced expression for x , in symbols $x = s_{i_1}^{\sigma_1} s_{i_2}^{\sigma_2} \dots s_{i_p}^{\sigma_p}$ for some *mask* $\sigma_1 \dots \sigma_p \in \{0, 1\}^p$ [30].

There are many expressions for any $w \in W$ as a product of generators, but it is a well known hard problem to tell when two expressions are equal in a group using only generators and relations. Luckily, there is an algorithm of finding a canonical representative for each element of W , called the *Mozes numbers game*. See Mozes 1990, Eriksson-Eriksson 1998, Björner-Brenti [17, 41, 77]. Let us briefly explain this game/algorithm here.

Replace each edge (i, j) of G by two opposing directed edges labeled $f_{ij} > 0$ (for the edge $i \rightarrow j$) and $f_{ji} > 0$ (for the edge $j \rightarrow i$) so that $f_{ij}f_{ji} = 4\cos^2\left(\frac{\pi}{m(i,j)}\right)$ or $f_{ij}f_{ji} = 4$ if $m(i, j) = \infty$. These labels are fixed in the game once chosen. The following is a useful choice since the labels are all integers.

$m(i, j)$	f_{ij}	f_{ji}
3	1	1
4	2	1
6	3	1
∞	4	1

Assume that we are given an element $w = s_{i_1} s_{i_2} \dots s_{i_p} \in W$. The canonical presentation of w is obtained as follows. We first assign value 1 to each vertex G . Next, *fire the vertex* s_{i_1} . Here, firing the vertex s_i is an operation done by adding to the value of each neighbor vertex j , the current value at the vertex i multiplied by f_{ij} , and then negating the sign of the value of the vertex i . We continue to fire the vertices $s_{i_2}, s_{i_3}, \dots, s_{i_p}$ consecutively. The resulting assignment of values for vertices of G , denoted by $G(w)$, provides a canonical presentation of the given w . In fact, this algorithm satisfies the following properties:

- (1) $G(w)$ only depends on the product $s_{i_1} s_{i_2} \dots s_{i_p}$ and not on the particular choice of expression.
- (2) The vertex i is negative in $G(w)$ if and only if $ws_i < w$.
- (3) The vertex i never has value 0.

Note, the map G is injective but not surjective on the set of all integer assignments to the nodes of the Coxeter graph.

Remark 5.2. For $I \subset S$, it is possible to modify the game to get representatives for W/W_I by starting with initial value 0 on vertices in I and 1's elsewhere. Then $ws_i = w$ if and only if the vertex i has value 0 in W/W_I . This is useful for Schubert geometry of Grassmannians and affine Grassmannians.

For a Coxeter group W , we can associate to it its *root system* $\Phi \subset V = \mathbb{R}^{|S|}$ where $\{\alpha_s : s \in S\}$ forms a basis of V [54, Section 5.4]. W acts linearly on V and Φ is W -invariant. We denote by Φ_+ and Φ_- the set of *positive roots* and the set of *negative roots*, respectively:

$$\Phi_+ = \{\alpha \in \Phi : \alpha = \sum c_s \alpha_s, \ c_s \geq 0, \forall s \in S\},$$

$$\Phi_- = \{\alpha \in \Phi : \alpha = \sum c_s \alpha_s, \ c_s \leq 0 \forall s \in S\}.$$

It follows that $\Phi = \Phi_+ \cup \Phi_-$ (disjoint union). There is a natural bijection between R and Φ_+ which we will denote by $r \rightarrow \alpha_r$. Then, for $r \in R, w \in W$, we have

$$wr > w \quad \text{if and only if} \quad w\alpha_r \in \Phi_+.$$

Example 5.3. Let e_1, \dots, e_n be the standard orthonormal basis of \mathbb{R}^n . Then the root system of the Weyl groups of classical types are determined by the following description of Φ_+ .

$$A_{n-1} : \quad \Phi_+ = \{e_i - e_j : 1 \leq i < j \leq n\}$$

$$B_n : \quad \Phi_+ = \{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\}$$

$$C_n : \quad \Phi_+ = \{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$$

$$D_n : \quad \Phi_+ = \{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i < j \leq n\}$$

The *inversion set* of $w \in W$ is defined to be $w\Phi_+ \cap \Phi_-$. In type A_{n-1} , these roots are in bijection with the inversion set of w defined originally. For a linear function $H : V \rightarrow \mathbb{R}$, we let

$$\Pi_H = \{\alpha \in \Phi : H(\alpha) > 0\}.$$

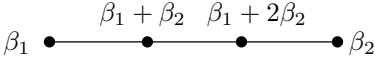
This is an intersection of the set of roots with a half space. We say H is *generic* if $H(\alpha) \neq 0$ for all $\alpha \in \Phi$.

Example 5.4. If $H_1 : V \rightarrow \mathbb{R}$ is defined by $H_1(\alpha_s) = 1$ for all $s \in S$, then $\Pi_{H_1} = \Phi_+$.

Definition 5.5. For each $w \in W$, set $H_w = H_1 \circ w^{-1}$. Then, we have $\Pi_{H_w} = w\Phi_+$.

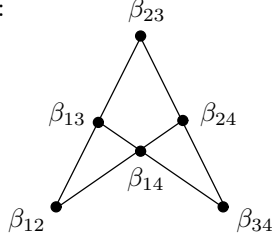
A key fact is that, if H is generic, then $\Pi_H = w\Phi_+$ for some unique $w \in W$. That is, every generic half space determines a unique $w \in W$ whose inversion set is exactly the negative roots in the given half space. Below are the positive roots for two types of Coxeter groups drawn projectively in 2 dimensions. We denote by $\beta_{ij} = e_i - e_j$ for A_3 .

$B_2 :$



β_1 and $\beta_1 + 2\beta_2$ are long roots.
 $\beta_1 + \beta_2$ and β_2 are short roots.

$A_3 = S_4 :$



$\beta_{12} = \beta_1, \beta_{23} = \beta_2, \beta_{34} = \beta_3,$
 $\beta_{13} = \beta_1 + \beta_2, \beta_{24} = \beta_2 + \beta_3,$
 $\beta_{14} = \beta_1 + \beta_2 + \beta_3.$

For example, if a given half space Π_H contains all the positive roots β_{ij} 's except for β_{23} and β_{24} , then $\Pi_H = \Pi_{H_w}$ for $w = 2431$.

5.2. Coxeter patterns. Each subset $I \subset S$ generates a subgroup W_I . A subgroup $W' \subset W$ which is conjugate to W_I for some I is called a *parabolic* subgroup. The W_I 's themselves are known as *standard* parabolic subgroups.

A parabolic subgroup $W' = xW_Ix^{-1}$ of W is again a Coxeter group, with simple reflections $S' = xIx^{-1}$ and reflections $R' = R \cap W'$. Note that $S' \not\subset S$ unless W' is standard.

We denote the length function and the Bruhat-Chevalley order for (W', S') by l' and \leq' , respectively. If $W' = W_I$ then

$$l' = l|_{W'} \text{ and } \leq' = \leq|_{W' \times W'},$$

but in general we only have $l'(w) \leq l(w)$ and $x \leq' y \implies x \leq y$. For instance, if $W' \subset S_4$ is generated by the reflections $r_{23} = 1324$ and $r_{14} = 4231$, then $r_{23} \leq r_{14}$ although they are not comparable for \leq' .

The following theorem/definition generalizes the flattening function for permutations to all Coxeter groups. The following theorem is closely related to a theorem due to Dyer on reflection subgroups [36, Thm. 1.4].

Theorem 5.6. [13] *Let $W' \subset W$ be a parabolic subgroup. There is a unique function $\text{fl}: W \rightarrow W'$, the pattern map for W' , satisfying the following two properties.*

- (a) *The map fl is W' -equivariant: $\text{fl}(wx) = w \text{fl}(x)$ for all $w \in W'$, $x \in W$.*
- (b) *If $\text{fl}(x) \leq' \text{fl}(wx)$ for some $w \in W'$, then $x \leq wx$.*

In particular, fl restricts to the identity map on W' .

If $W' = W_I$ is a standard parabolic, then (b) can be strengthened to “if and only if”. In this case the result is well-known.

To show uniqueness, note that (a) implies that fl is determined by the set $\text{fl}^{-1}(1)$, and (b) implies that $\text{fl}^{-1}(1) \cap W'x$ is the unique minimal element in $W'x$. Existence is more subtle; it is not immediately obvious that the function so defined satisfies (b). We give a construction of a function fl that satisfies (a) and (b).

Recall V is the real vector space spanned by the roots in the root system Φ associated to the Coxeter group W . If $U \subset V$ is a linear subspace, then we use the following notations:

$$\begin{aligned}\Phi^U &:= \Phi \cap U, \text{ a root subsystem of } \Phi, \\ W^U &\text{ is the group generated by reflections } r_\alpha \text{ for } \alpha \in \Phi^U, \\ R^U &:= R \cap W^U.\end{aligned}$$

One can show that W^U is a parabolic subgroup of W assuming W is finite, see [54, §1.12]. Note that not all subgroups of W generated by reflections are parabolic subgroups. For example, for B_3 , the group generated by reflections over the e_i 's is not parabolic.

By the uniqueness statement in Theorem 5.6, we can use the sets Π_H defined earlier to realize $\text{fl}: W \rightarrow W^U$. In fact, $\text{fl}(w)$ is the unique element $x \in W^U$ such that

$$\begin{aligned}w\Phi_+ \cap U &= \{\alpha \in U \cap \Phi: H_w(\alpha) > 0\} \\ &= \{\alpha \in \Phi^U: H'(\alpha) > 0\} \text{ where } H' = H_w|_U \\ &= x\Phi_+^U.\end{aligned}$$

This realization of the flattening map for Weyl groups was first given by Billey-Postnikov [10] even though it was published later than [13]. The delay is explained below.

Example 5.7. Let $U = \text{span}\langle \beta_{23}, \beta_{34} \rangle$. Then $\text{fl}_U(2431) = 243$. See the picture in the previous example.

5.3. Applications of Coxeter Patterns.

- G : a semisimple simply-connected complex Lie group,
- $B \subset G$: a Borel subgroup,
- $T \subset B$: a maximal torus,
- $W = N(T)/T$: the Weyl group (a finite Coxeter group),
- $\Phi \subset V$: the associated root system

where $N(T)$ is the normalizer of T in G . The finite Weyl groups (or root systems) that arise this way have been completely classified into types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. The *Bruhat decomposition* enables us to partition G using the Borel subgroup and the Weyl group:

$$G = \bigcup_{w \in W} BwB.$$

The quotient G/B is called the (*generalized*) *flag manifold*, and Schubert cells and Schubert varieties of G/B are

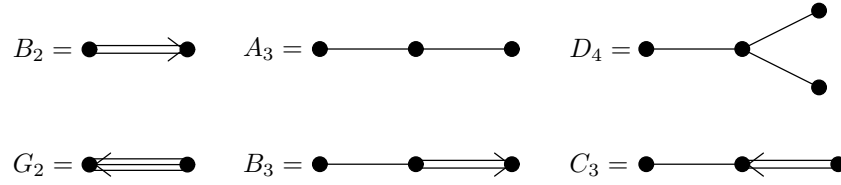
$$C_w = B \cdot w, \quad X_w = \overline{B \cdot w}$$

for each $w \in W$, respectively.

The next theorem characterizes all smooth Schubert varieties for any semisimple simply-connected complex Lie group G . To state the theorem, we need a few more definitions.

Definition 5.8. A Coxeter group W is *stellar* if its Coxeter graph has one central vertex and all other vertices are only adjacent to it.

The stellar Coxeter groups corresponding to the Weyl groups of types A, B, C, D, E, F and G (except for A_2) are drawn below where a double edge and a triple edges mean that the label of the corresponding edge is 4 and 6, respectively. Note, that the Weyl groups of types B_n and C_n are isomorphic, but the pattern map works slightly differently on each so we list their Dynkin diagram instead of their Coxeter graph.



Theorem 5.9. (Billey-Postnikov [10]) *A Schubert variety X_w is smooth if and only if for every stellar parabolic subgroup W^U , the Schubert variety X_v for $v = \text{fl}_U(w)$ is smooth in G^U/B^U .*

Here G^U is a semisimple Lie group with Weyl group W^U and B^U is one of its Borel subgroups. We remark that W^U might not be the same type as W . For example, for Weyl groups of type C and D there will appear W^U of type A . In fact, the type A singular patterns are most common. If the Coxeter graph of W has only edges labeled by 3's, we say W is *simply laced*. If W is simply laced, then all of its parabolic subgroups are also simply laced.

It turns out that there are very few patterns for which the corresponding Schubert varieties are singular among stellar reduced, irreducible Weyl groups; 2 patterns in A_3 , 1 pattern in B_2 , 6 patterns in B_3 and C_3 , 1 pattern in D_4 , 5 patterns in G_2 . Note that all Schubert varieties of type A_2 are smooth.

Example 5.10. In type B_n using the classical pattern avoidance on signed permutations, the smooth Schubert varieties are classified by avoiding the following 17 patterns

$$\begin{aligned}
 &(-2, -1), \\
 &(1, 2, -3), (1, -2, -3), (-1, 2, -3), (2, -1, -3), (-2, 1, -3), (3, -2, 1) \\
 &(2, -4, 3, 1), (-2, -4, 3, 1), (3, 4, 1, 2), (3, 4, -1, 2), (-3, 4, 1, 2) \\
 &(4, 1, 3, -2), (4, -1, 3, -2), (4, 2, 3, 1), (4, 2, 3, -1), (-4, 2, 3, 1).
 \end{aligned}$$

All length 4 patterns come from A_3 root subsystems.

Example 5.11. In type D_4 , there are 49 singular Schubert varieties, and the only element which does not comes from A_3 root subsystems is $w = s_2 \cdot s_1 s_3 s_4 \cdot s_2 = \bar{1}4\bar{3}2$. Thus, for all simply laced types, there are only 3 bad patterns to consider: 3412, 4231, and $\bar{1}4\bar{3}2$. It is instructive to look at the singular locus of the Schubert varieties for each of these 3 patterns:

$$\begin{aligned}
 \text{Sing } X_{s_2 s_1 s_3 s_2} &= X_{s_2} & (3412 \text{ case}), \\
 \text{Sing } X_{s_3 s_1 s_2 s_1 s_3} &= X_{s_1 s_3} & (4231 \text{ case}), \\
 \text{Sing } X_{s_2 s_1 s_3 s_4 s_2} &= X_{s_2} & (\bar{1}4\bar{3}2 \text{ case}).
 \end{aligned}$$

As we mentioned in Section 4, the definition of a Kazhdan-Lusztig polynomial $P_{v,w}(t)$ easily generalizes to all Coxeter groups. We use these polynomials to define the notion of a rationally smooth Schubert variety. This avoids the more general definition in terms of étale cohomology.

Definition 5.12. A point $v \in X_w$ is *rationally smooth* if and only if $P_{v,w}(t) = 1$. A Schubert variety X_w is *rationally smooth* if every point of X_w is rationally smooth.

The following theorem as stated is due to Carrell and Peterson. Related results also appear in Jantzen's book [57, Ch.5] in slightly different language.

Theorem 5.13. [27] *The following are equivalent.*

- (1) X_w is rationally smooth at v .
- (2) $P_{v,w}(t) = 1$
- (3) The Bruhat graph on $[v, w]$ is regular of degree $l(w) - l(v)$.

In the next theorem, the third condition is due to Carrell-Peterson [27]. The fourth condition combines work of Garsharov [44] in type A , [6] for types B and C , then it was conjectured to hold for all Weyl groups by McGovern and proved by Akyildiz-Carrell [1] for types D and E . It can be checked by computer for F_4 and G_2 can be done easily. The next condition is due to Oh-Yoo [79]. The last condition is due to Slofstra [87].

Theorem 5.14. *The following are equivalent for all Weyl groups.*

- (1) X_w is rationally smooth.
- (2) $P_{id,w}(t) = 1$
- (3) $P_w(t) = \sum_{v \leq w} t^{\ell(v)}$ is palindromic.
- (4) $P_w(t) = \prod (1 + t + t^2 + \cdots + t^{e_i})$
- (5) The Poincaré polynomial $P_w(t)$ is equal to the generating function $R_w(t)$ for the number of regions r in the complement of the inversion hyperplane arrangement for w weighted by the distance of each region to the fundamental region.
- (6) The inversion arrangement for w is free and the number of chambers of the arrangement is equal to the size of the Bruhat interval $[id, w]$.

For all finite Weyl groups, rational smoothness can be characterized by pattern avoidance.

Theorem 5.15. (Billey-Postnikov [10]) X_w is rationally smooth if and only for every stellar parabolic subgroup W^U , X_v for $v = \text{fl}_U(w)$ is rationally smooth in G^U/B^U .

Note that there are only 2 patterns in A_3 , 6 patterns of type B_3 and C_3 , 1 pattern of type D_4 which should be avoided by w in order for X_w to be rationally smooth. The Coxeter pattern map made a very large reduction in the number of patterns one needs to remember for both smoothness and rational smoothness.

Remark 5.16. Smoothness implies rational smoothness. In terms of the patterns characterization, the difference between smoothness and rational smoothness for all Weyl group types is just 6 additional patterns, 1 pattern in B_2 and 5 patterns of type G_2 .

Outline of proof of Theorems 5.9 and 5.15

- Step 1: For classical types B, C, D , use Lakshmibai's characterization of the tangent space basis to get the general smoothness results.
- Step 2: Use an analog of Gasharov's theorem to the factor of Poincaré polynomial for any signed permutation not containing a singular pattern to get the rational smoothness of B, C, D which extends to all finite types.

- Step 3: Use Kumar's criterion for (rational) smoothness in the nil-Hecke ring to test G_2 and F_4 by computer.
- Step 4: Run a massive parallel computation on the 696,729,600 elements $w \in E_8$.
 - If w has a pattern from type A or D , calculate the coefficient of t^1 and $t^{\ell(w)-1}$ and compare, if different, w is done. If not, calculate the coefficient of t^2 and $t^{\ell(w)-2}$, etc. Eventually one pair differed in every case.
 - If w avoids all patterns from type A or D , use analog of Gasharov's algorithm for factoring $P_w(t)$.

□

Note that smoothness of X_w automatically implies rational smoothness. Deodhar proved the following property for type A , and later Peterson proved that it also holds for type D and E (unpublished). See [28] for a proof. A proof for all finite Weyl group types except E_6, E_7, E_8 follows easily from Theorem 5.9 and Theorem 5.15. For E_6, E_7, E_8 , the Peterson theorem is used in the proof of these two theorems.

Theorem 5.17. (*Deodhar, Peterson, Carrel-Kuttler*) *For types A, D, E , a Schubert variety X_w is smooth if and only if it is rationally smooth.*

A new proof of Theorem 5.17 has recently been announced by Richmond and Slofstra [85]. In fact, they show that every rationally smooth Schubert variety in any finite Lie type is an iterated fibre bundle of Grassmannians. This generalizes the work in type A by Ryan [86], Wolper [98], and Gasharov-Reiner [46] mentioned in Pattern Property 1.

Note that smoothness and rational smoothness are not equivalent for affine type \tilde{A}_n by Mitchell [76] and Billey-Crites [7].

The definition of the Coxeter pattern map also has applications to the geometry of Schubert varieties for Weyl groups and affine Weyl groups. Once again, let $U \subset V$ be a linear subspace. We denote by $M(x, w; U)$ the set of maximal elements in $[id, w] \cap W^U x$ with respect to a new partial order \leq_x defined by

$$wx \leq_x w'x \quad \text{if} \quad \text{fl}(wx) \leq^U \text{fl}(w'x).$$

Theorem 5.18. (*Billey-Braden* [13]) *If $x, w \in W$, then*

$$P_{x,w}(1) \geq \sum_{y \in M(x,w;U)} P_{y,w}(1) P_{\text{fl}(x), \text{fl}(y)}^U(1).$$

Corollary 5.19. [13] *For all $x < w$, $P_{x,w}(1) \geq P_{\text{fl}(x), \text{fl}(w)}^U(1)$.*

Historically, Theorem 5.18 and Corollary 5.19 were the first application of simultaneous pattern embedding/flattening on two Coxeter group elements. For $u, v \in S_m$ and $x, y \in S_n$, if $[u, v]$ interval pattern embeds into $[x, y]$ using indices $1 \leq i_1 < \dots < i_m \leq n$ then one can construct a subspace U such that $u = \text{fl}_U(x), v = \text{fl}_U(y)$ by considering all the roots indexed by values in the set $\{x_{i_1}, \dots, x_{i_m}\}$. Thus, Corollary 5.19 implies one direction of the stronger type A result in Theorem 4.10. From this point of view, Theorem 5.18 and Corollary 5.19 were precursors to the notion of interval pattern avoidance introduced in [102].

Woo [100] extended the notion of interval pattern avoidance to other Weyl groups and proved that many of the nice properties in [102] continue to hold. In particular, the analog of Theorem 4.10 holds for all Weyl groups [100, Cor. 3.3]. Furthermore, Woo relates interval pattern embeddings with isomorphism of Richardson varieties which are intersections of two Schubert varieties with respect to two generic flags.

Theorem 5.20. [100, Thm. 3.1] *Let W', W be Weyl groups. Suppose there is some root subsystem embedding which embeds $[u, v] \subset W'$ in $[x, w] \subset W$. Then the Richardson varieties X_v^u and X_w^x are isomorphic.*

Corollary 5.19 also gives rise to filtrations on permutations.

Corollary 5.21. *For each m , $KL_m = \{w \in S_\infty \mid P_{id,w}(1) \leq m\}$ is closed under taking patterns.*

It is interesting to ask for a geometrical explanation for why (rational) smoothness of Schubert varieties can be characterized by Coxeter patterns. The following theorem proves one direction of this. The other direction is still open: namely, why are patterns from stellar Coxeter graphs enough.

Theorem 5.22. (Billey-Braden [13]) *If $X_{\mathfrak{H}(w)}^U$ is singular, then X_w is singular.*

Outline of proof.

- Realize G^U/B^U as the fixed points of a certain torus action.
- Use a theorem of Fogarty-Norman saying that for all smooth algebraic T -schemes X the fixed point scheme X^T is smooth.

□

Several other nice pattern avoidance properties in Coxeter groups are also known:

- (1) (Stembridge [89]) The fully commutative elements in types B and D are characterized with signed patterns.
- (2) (R.Green [50]) The fully commutative elements in the affine Weyl group of type A are exactly the 321-avoiding elements.
- (3) (Reading [82]) Coxeter-sortable elements are characterized and it is shown that they are equinumerous with clusters and with noncrossing partitions.
- (4) (Billey-Jones [14]) Deodhar elements for all Weyl groups are characterized.
- (5) (Billey-Crites [7]) The rationally smooth Schubert varieties in the affine type A flag manifold are characterized as 3412, 4231 avoiding plus one extra family of twisted spiral varieties. Crites also studied the enumeration of affine permutations indexing rationally smooth Schubert varieties in [33].
- (6) (Chen-Crites-Kuttler, manuscript) A Schubert variety X_w of affine type A is smooth if and only if $w \in \tilde{S}_n$ avoids 3412 and 4231. Furthermore, the tangent space to X_w at the identity can be described in terms of reflection over real and imaginary roots.
- (7) (Matthew Dyer, manuscript) Smooth and rationally smooth Schubert varieties can be detected using rank 2 subvarieties [37].
- (8) (Matthew Samuel, manuscript) Affine Schubert varieties for all types can be characterized by patterns using a new version of pattern avoidance for Coxeter groups based on reflection groups.

6. COMPUTER TOOLS FOR SCHUBERT GEOMETRY

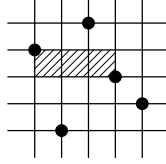
In the lecture series that gave rise to this article, we discussed some computer tools for the study of geometry of Schubert varieties and for more general topics in mathematics. The video file of the lecture devoted to the contents of this section is available at the following website.

<http://mathsoc.jp/en/videos/2012msj-si.html>

The main ideas presented are pertaining to computer proofs, the Online Encyclopedia of Integer Sequences, the Database of Permutation Pattern Avoidance and Sage. The demos in the lecture are best seen online so we will not include that discussion here.

We do want to highlight one of the Sage demos discussed, because it is related to some recent developments on marked mesh patterns which unify the descriptions of several pattern avoidance properties for permutations using the language of *marked mesh patterns*.

Definition 6.1. (Brändén and Claesson [24]) A *mesh pattern* is a permutation matrix with shaded regions between certain entries.



The dots represents 1's in the permutation matrix.

Definition 6.2. (Úlfarsson [94]) A *marked mesh pattern* is a mesh pattern with numbers in the shaded regions.

The next theorem states that we can also use marked mesh patterns for characterizing Schubert varieties. See [94] for details.

Theorem 6.3. (Úlfarsson [94]) *The smooth, Gorenstein, factorial, defined by inclusions, and 321-hexagon avoiding permutations can be described by marked mesh patterns.*

7. OPEN PROBLEMS

In addition to the open problems we have mentioned in the text, there are some more open problems concerning pattern avoidance properties. We hope that computer experiments will help the reader to study those problems.

Question 5. (Woo-Yong) Characterize the Gorenstein, LCI and factorial locus of X_w using patterns.

Question 6. (From Úlfarsson) Is there a nice generating function to count the number of Gorenstein/LCI permutations or Schubert varieties defined by inclusions, etc.

Question 7. Find a geometric explanation why a finite number of patterns suffice in all cases above.

Question 8. What nice properties does the inversion arrangement have for other pattern avoiding families?

Question 9. KL_m is closed under taking patterns by [13]. Can it always be described by a finite set of patterns? Conjectured to be yes by Billey-Weed-Woo.

Question 10. Conjecture (Woo): The Schubert varieties with multiplicity ≤ 2 can be characterized by pattern avoidance. Can this be extended to a pattern avoidance characterization of Schubert varieties with multiplicity $\leq k$? Note, Woo-Yong showed that $\text{mult}_x(X_y) = \text{mult}_u(X_v)$ if $[u, v]$ interval pattern embeds into $[x, y]$ [102, Cor. 6.15].

Question 11. What other filtrations on the set of all permutations can be characterized by (generalized) patterns?

Question 12. Describe the maximal singular locus of a Schubert variety for other semisimple Lie groups using Coxeter patterns.

Question 13. Give a pattern based algorithm to produce the factorial and/or Gorenstein locus of a Schubert variety in other types.

Question 14. Is there a nice generating function to count the number of smooth, factorial and/or Gorenstein permutations in other types?

Question 15. What is the right notion of patterns for GKM spaces?

Question 16. Say X_w is *combinatorially smooth* if $\ell(w) = \#\{t_{ij} : t_{ij} \leq w\}$. In S_n combinatorially smooth is equivalent to smooth by the Lakshmibai-Seshadri Theorem. However, for other Weyl groups this is a weaker notion than rational smoothness. Characterize the combinatorially smooth Schubert varieties by generalized pattern avoidance.

Question 17. Can Lakshmibai's characterization of the tangent space basis for B , C and D be translated into signed patterns or a signed variation on marked mesh patterns.

Question 18. What is the analog of marked mesh patterns for other types?

Question 19. What is the Möbius function for the poset of pattern containment on S_∞ ? See the excellent survey by Einar Steingrímsson [88, Sect. 5] for more details on this and other pattern related problems.

Question 20. Which of the many pattern avoidance related theorems on Schubert varieties have analogs for other interesting families of varieties such as the $GL_p \times GL_q$ -orbit closures of the flag manifold or the Peterson varieties as mentioned at the end of Section 4? See [23, 52, 55, 73, 74] for further details.

Question 21. More generally, what other types of theorems have canonical representations which might lead to more computer database tools? For example, how about hypergeometric series, integer sequences, patterns? See [11] for more discussion on this topic.

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