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Hopf actions on filtered regular algebras

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We study finite dimensional Hopf algebra actions on so-called filtered Artin–Schelter regular algebras of dimension n, particularly on those of dimension 2. The first Weyl algebra is an example of such an algebra with $n = 2$, for instance. Results on the Gorenstein condition and on the global dimension of the corresponding fixed subrings are also provided.

0. Introduction

The main motivation for this paper (as well as for [4,3]) is to classify all finite dimensional Hopf algebras which act on a given algebra R. By understanding the Hopf algebras H which act on R, we can further study other structures related to R, such as the fixed ring R^H and the smash product $R \# H$. The prototype of this problem is classical: the classification of finite subgroups G of $SL_2(\mathbb{C})$ (that act faithfully on the polynomial ring $\mathbb{C}[u, v]$) prompted the connection between the McKay quiver of G and the geometric features of the plane quotient singularity $\mathrm{Spec}(\mathbb{C}[u, v]^G)$. In our setting, the algebra R is allowed to be noncommutative and the Hopf algebras are allowed to be noncocommutative. More precisely, we study finite dimensional Hopf algebra actions on *filtered Artin–Schelter regular*

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algebras of dimension d. These are filtered algebras whose associated graded algebras are Artin–Schelter regular algebras of global dimension d. Our emphasis will be on the case of dimension 2.

Here, we assume that the base field k is algebraically closed of characteristic zero, unless otherwise stated. Examples of filtered Artin–Schelter regular algebras of dimension 2 include the first Weyl algebra $A_1(k) = k\langle u, v\rangle/(vu - uv - 1)$, quantum Weyl algebras $k\langle u, v\rangle/(vu - quv - 1)$ for some $q \in k^\times$, and other deformations of Artin–Schelter regular algebras of dimension 2.

The invariant theory of $A_1(k)$ by finite groups is already interesting. For example, the fixed subrings of $A_1(k)$ by finite groups actions are completely classified and studied by Alev, Hodges and Velez in [1]. Thus, it is natural to ask if there are any non-trivial finite dimensional Hopf algebra (H-)actions on the first Weyl algebra. By a "non-trivial" H-action, we mean that H is neither commutative nor cocommutative Hopf algebra (or neither a dual of a group algebra nor group algebra, respectively). We give a negative answer to this question in Theorem 0.1 below.

Recall that a left H-module M is called *inner-faithful* if $IM \neq 0$ for any nonzero Hopf ideal I of H. Let N be a right H-comodule with comodule structure map $\rho : N \to N \otimes H$. We say that this coaction is inner-faithful if $\rho(N) \nsubseteq N \otimes H'$ for any proper Hopf subalgebra $H' \subsetneq H$ [4, Definition 1.2]. We say that *a Hopf algebra H (co)acts on an algebra R* if R is a left H-(co)module algebra. Moreover, if the H-(co)module R is inner-faithful, then we say that H (co)acts on R *inner-faithfully*.

Theorem 0.1. *Let R be a non-PI filtered Artin–Schelter regular algebra of dimension 2 and let H be a finite dimensional Hopf algebra acting on R inner-faithfully. If the H-action preserves the filtration of R, then H is a group algebra.*

In particular, if R is the first Weyl algebra $A_1(k)$, then H is a group algebra.

Theorem 0.1 can be viewed as an extension of [4, Theorem 5.10] from the non-PI graded case to the non-PI filtered case. Most results in this work concern actions on non-PI AS regular algebras. In particular, combining Theorem 0.1 with [1, Proposition, p. 84], one classifies all finite dimensional Hopf algebras acting inner-faithfully on $A_1(k)$ with respect to the standard filtration (Corollary 5.7). Similarly, all finite dimensional Hopf algebras actions on the quantum Weyl algebras $k\langle u, v\rangle/(vu - quv - 1)$, for q not a root of unity, are classified (Corollary 5.8(a)). On the other hand, if R is PI filtered AS regular, then there are many interesting finite dimensional Hopf algebras (which are not group algebras) which act on R; see Examples 1.4 and 3.4 for instance.

Regarding the higher dimensional Weyl algebras, it is natural to ask the following question.

Question 0.2. Let $A_n(k)$ be the n-th Weyl algebra and let H be a finite dimensional Hopf algebra acting on $A_n(k)$ inner-faithfully. Is then H a group algebra?

If we assume that the H-action is filtration preserving, then the answer is yes if $n = 1$ (Theorem 0.1) or if H is pointed (Theorem 0.3).

Theorem 0.3. *Let H be a finite dimensional Hopf algebra acting on the n-th Weyl algebra $A_n(k)$ inner-faithfully which preserves the standard filtration of $A_n(k)$. Then H is semisimple. If, in addition, H is pointed, then H is a group algebra.*

In the setting of H-actions on graded algebras, we have the following result of Kirkman, Kuzmanovich and Zhang. Suppose that H is a semisimple finite dimensional Hopf algebra and R is an AS regular algebra. If H acts on R preserving the grading with trivial homological determinant, then the fixed subring R^H is AS Gorenstein [7, Theorem 0.1]. We can obtain a filtered analogue of the above by considering the induced H-action on $\operatorname{gr}_F R$.

Theorem 0.4. *Let R be a filtered AS regular algebra of dimension 2 and let H be a semisimple Hopf algebra acting on R. If the H-action is not graded and it preserves the filtration of R, then the fixed subring R^H is filtered AS Gorenstein.*

Note that if R is a graded H-module algebra, the fixed ring R^H can fail to be AS Gorenstein. This is well known if $R = k[u, v]$ by the existence of non-Gorenstein quotient singularities. See also [6, Proposition 6.5(2)] for a noncommutative example.

On the other hand, we also study the regularity of fixed rings when H is not necessarily semisimple. In the graded case, [3, Proposition 0.7] states that if R is graded AS regular of global dimension 2, and the H-action on R has trivial homological determinant, then R^H is <u>never</u> AS regular provided that $H \neq k$. Therefore, the following result is quite surprising.

Theorem 0.5. *Let R be a non-PI filtered AS regular algebra of dimension 2 and let H be a finite dimensional Hopf algebra acting inner-faithfully on R. If the H-action on R is not graded and preserves the filtration on R, then the fixed subring R^H has global dimension 1 or 2.*

This result is well known when R is the first Weyl algebra $A_1(k)$ as the corresponding fixed subrings are all hereditary [1]. On the other hand, it would be interesting to prove versions of Theorems 0.4 and 0.5 in the higher dimensional case.

Remark 0.6. Suppose that H is semisimple and $A_n(k) \# H$ is simple. Then by [8, Corollary 4.5.5], we have a Morita equivalence between $A_n(k)^H$ and $A_n(k) \# H$. As a consequence, $\operatorname{gldim} A_n(k)^H = \operatorname{gldim} A_n(k) \# H = \operatorname{gldim} A_n(k)$.

Moreover, observe that since R is non-PI in Theorem 0.5, there are no non-trivial Hopf actions on R by Theorem 0.1.

The paper is organized as follows. We define basic terminology and we also discuss certain properties of Hopf actions on filtered algebras in Section 1. In Section 2, we make several initial computations on the structure of a Hopf algebra H and a filtered AS regular algebra R of dimension 2, particularly when the H-(co)action on R is so-called *proper*. We provide preliminary results about the fixed subring R^H in Section 3, and we also prove Theorem 0.4 here. The proof of Theorem 0.3 is presented in Section 4. In Section 5, we prove Theorem 0.1 and we use it to classify Hopf actions on non-PI filtered AS regular algebras of dimension 2. Finally in Section 6, we prove Theorem 0.5 with use of Galois extensions.

1. Definitions

In this work, we study Hopf actions on filtered algebras whose associated graded algebras are Artin–Schelter (AS) regular algebras of global dimension 2. We refer to [4, Definition 1.1] for the definition of AS regular algebras in general, but in global dimension 2 (and generated in degree 1), we have that such algebras are isomorphic to either:

(i) $k_J[u, v] := k\langle u, v\rangle/(vu - uv - u^2)$ (*the Jordan plane*); or
(ii) $k_q[u, v] := k\langle u, v\rangle/(vu - quv)$ for some $q \in k^\times$ (*the skew polynomial ring*).

For filtered algebras, we have the following definition.

Definition 1.1. An algebra R is called *filtered AS regular of dimension d* (respectively, *filtered AS Gorenstein*) if the following conditions hold:

(a) R is generated by a finite dimensional subspace U with $1 \notin U$, and
(b) for $F_n = (k1 + U)^n$, the associated graded ring

$$\operatorname{gr}_F R := \bigoplus_{n \geqslant 0} F_n/F_{n-1}$$

is an AS regular algebra of global dimension d (respectively, is an AS Gorenstein algebra).

We define below *actions* of Hopf algebras on filtered regular algebras, and in particular, actions that are so-called *proper*.

Notation 1.2. We denote by R a filtered AS regular algebra of dimension d. If $d = 2$, then the generating vector space U has dimension 2 and we use $\{u, v\}$ for its basis. Unless otherwise stated, H and K are finite dimensional Hopf algebras. Here, the Hopf algebras have Hopf structure denoted by the standard notation $H = (H, m, \Delta, u, \epsilon, S)$. Moreover, for the following definition, we denote the left H-action on R by $\cdot : H \otimes R \to R$, and right K-coaction on R by $\rho : R \to R \otimes K$.

Since R is a filtered algebra, we require Hopf actions (or Hopf coactions) to preserve the given filtration of R. We give only the following definitions for Hopf actions, but similar definitions can be made for Hopf coactions with the obvious changes.

Definition 1.3. We say a Hopf algebra H *acts on a filtered algebra* R if

(a) R is a left H-module algebra, and
(b) $k1 + U$ is a left H-module.

We say the H-action on R is *proper* if, further,

(c) there is a choice of U as in Definition 1.1 such that U is a left H-module.

Let us provide an example of a non-proper Hopf coaction on a PI filtered AS regular algebra.

Example 1.4. Let R be the quantum Weyl algebra $k\langle u, v\rangle/(vu + uv - 1)$ (where the parameter q equals -1), and let K be Sweedler's non-semisimple 4-dimensional Hopf algebra $k\langle g, f\rangle/(fg + gf, g^2 - 1, f^2)$. Here, g is grouplike and f is $(1, g)$-primitive, to say,

$$\Delta(g) = g \otimes g \quad \text{and} \quad \Delta(f) = 1 \otimes f + f \otimes g.$$

Moreover, $\epsilon(g) = 1$, $\epsilon(f) = 0$, $S(g) = g$, and $S(f) = -fg$. Define a K-coaction on R by

$$\rho(u) = u \otimes g \quad \text{and} \quad \rho(v) = v \otimes g + 1 \otimes f.$$

Then, K coacts on R inner-faithfully. We show that the induced $K^* = H$-action is not proper, hence the K-coaction above is not proper.

Let e_1, e_g, e_f, e_{gf} denote the dual basis of H. Then $\gamma := e_1 - e_g$ and $\delta := e_f - e_{gf}$ generate H as an algebra. Moreover $\gamma \cdot 1 = 1$, $\gamma \cdot u = -u$, $\gamma \cdot v = -v$ and $\delta \cdot 1 = \delta \cdot u = 0$, $\delta \cdot v = 1$. By linear algebra, there is no 2-dimensional δ-invariant subspace of $F_1 = k1 \oplus kv \oplus ku$ not containing 1. So the H-action is not proper.

We require (co)actions that are 'faithful' in our setting. We refer to [4, Section 1] for a discussion of inner-faithful Hopf (co)actions, and we repeat some of these results here.

Lemma 1.5. *Let H be a Hopf algebra that acts on a filtered algebra R.*

(a) *If H is semisimple, then every H-action on R is proper.*
(b) [4, Lemma 1.3(c)] *The H-action on R is inner-faithful if and only if the H-module $k1 + U$ is inner-faithful.*
(c) *If the H-action is proper, the H-module $k1 + U$ is inner-faithful if and only if the H-module U is inner-faithful.*
(d) [4, Lemma 1.3(a)] *If H is finite dimensional with Hopf algebra dual H°, then the H-action is inner-faithful if and only if the H°-coaction is inner-faithful.*

Proof. (a) Since H is semisimple, $k1+U=k1\oplus U'$ which is a direct sum of left H-modules $k1$ and U' where U' is a finite dimensional generating subspace of R. Replacing U by U' gives the assertion.

(b, c, d) These are straightforward. □

We also work with Hopf actions that are *not graded* as in the following definitions.

Definition 1.6. Let R be a filtered AS regular of dimension d and let H act on R.

(a) We say that R *is not graded* if R is not isomorphic to $\mathrm{gr}_F R$ for any choice of U in Definition 1.1.
(b) We say that the H*-action is not graded*, if for any choice of U in Definition 1.3, R is not isomorphic to $\mathrm{gr}_F R$ as left H-module algebras.

Next, we recall the definition of the *homological determinant* of an H-action on a graded algebra A.

Definition 1.7. (See [7, Definitions 3.3 and 6.2].) Let A be a noetherian connected graded AS Gorenstein algebra and let H be a finite dimensional Hopf algebra acting on A that preserves the grading of A. Let $\mathfrak{e}$ denote the lowest degree nonzero homogeneous component of the d-th local cohomology $H^d_{\mathfrak{m}_A}(A)^*$, where $d=\mathrm{injdim}(A)<\infty$. Then there is an algebra homomorphism $\eta: H\to k$ such that

$$\mathfrak{e}\cdot h=\eta(h)\mathfrak{e}$$

for all $h\in H$.

(1) The composite map $\eta\circ S: H\to k$ is called the *homological determinant* of the H-action on A, denoted by $\mathrm{hdet}_H A$.
(2) We say that $\mathrm{hdet}_H A$ is *trivial* if $\mathrm{hdet}_H A=\epsilon$, where ϵ is the counit of H.

Dually, if a finite dimensional Hopf algebra K coacts on A from the right, then K coacts on $k\mathfrak{e}$ and

$$\rho(\mathfrak{e})=\mathfrak{e}\otimes \mathsf{D}^{-1}$$

for some grouplike element D in K.

(3) The *homological codeterminant* of the K-coaction on A is defined to be $\mathrm{hcodet}_K A=\mathsf{D}$.
(4) We say that hcodet_K is *trivial* if $\mathrm{hcodet}_K=1_K$.

2. Initial analysis

In this section, we compute the structure of the pair (H,R) for R a filtered AS regular algebra of dimension 2 and H a finite dimensional Hopf algebra so that H acts on R under various conditions. We do this particularly when the H-action on R is proper (Lemma 2.8, Remark 2.9, Corollary 2.10). Moreover, we end this section by showing that if the associated graded algebra of a certain filtered AS regular algebra R of dimension 2 is PI, then so is R (Lemma 2.11).

Notation 2.1. Let H be a Hopf algebra. Denote by $G:=G(H)$ the set of grouplike elements in H, and let kG be the corresponding group algebra, which is a Hopf subalgebra of H. For $g\in G$, denote by $\eta_g(h)$ the element $g^{-1}hg$ for any $h\in H$. For a polynomial $p(t)=\sum_{s=0}^n a_st^s$, denote by $(p\circ\eta_g)(h)$ the element $\sum_{s=0}^n a_s\eta_g^s(h)$. Let $\mathbb{U}_n$ be the set of primitive n-th roots of unity for $n\geqslant 2$, and put $\mathbb{U}:=\bigcup_{n\geqslant 2}\mathbb{U}_n$.

Consider the following preliminary results.

Lemma 2.2. *Let H be a finite dimensional Hopf algebra and $K := H^\circ$. If T is a 1-dimensional right K-comodule, then $T \cong kg$ for some grouplike element $g \in G(K)$.*

Proof. Take a nonzero basis element t of T. Now $\rho(t) = t \otimes g$, and by coassociativity,

$$t \otimes \Delta(g) = (1 \otimes \Delta) \circ \rho(t) = (\rho \otimes 1) \circ \rho(t) = t \otimes g \otimes g.$$

Hence, $\Delta(g) = g \otimes g$. $\square$

Lemma 2.3. *Let H be a finite dimensional Hopf algebra, and $g, h \in G$. Suppose that f is a $(1, g)$-primitive element not in kG. Then, the following statements hold.*

(a) *We have that $g \neq 1$ and there is no nonzero primitive element in H.*
(b) *If $\eta_h(f) - qf \in kG$ for some $q \in k$, then $q \in \mathbb{U} \cup \{1\}$.*
(c) *If $\eta_g(f) - qf \in kG$ for some $q \in k$, then $q \in \mathbb{U}$.*

Proof. (a) Suppose that $g = 1$. Then f is a primitive element and the Hopf subalgebra generated by f is infinite dimensional. This yields a contradiction. Therefore, $g \neq 1$.

(b) First, by induction we have that $\eta_h^s(f) - q^s f \in kG$ for every $s \geqslant 1$. Now h has finite order as H is finite dimensional. So $f - q^m f = (1 - q^m) f \in kG$ where m is the order of h. Since $f \notin kG$, we obtain that $q^m = 1$.

(c) By part (b), it suffices to show that $q \neq 1$. Suppose that $q = 1$ and that $\eta_g(f) - f \in kG$. Note that $\eta_g(f) - f$ is also $(1, g)$-primitive, so $\eta_g(f) - f = \alpha(1 - g)$ for some $\alpha \in k$. By induction, one sees that

$$\eta_g^i(f) = f + i\alpha(1 - g)$$

for all $i \geqslant 1$. Let m denote the order of g, then

$$f = \eta_g^m(f) = f + m\alpha(1 - g).$$

Since $g \neq 1$ by part (a) and $m > 0$, we have $\alpha = 0$. This implies $[g, f] = 0$, so the sub-Hopf algebra S generated by f, g is commutative. In particular, S is cosemisimple. Since $\operatorname{char}(k) = 0$, we have that S is also semisimple. Now S is generated by a grouplike element and a skew primitive element, so it is pointed. Therefore S is a group algebra which contradicts $f \notin kG$. Hence, $q \neq 1$. $\square$

We impose the following hypotheses for the next several results.

Hypothesis 2.4. *Let R be a filtered AS regular algebra of dimension 2, and let $F = \{F_n \mid n \geqslant 0\}$ be the filtration of R given in Definition 1.1. Let H be a finite dimensional Hopf algebra that acts on R inner-faithfully such that the H-action preserves the filtration of R. Moreover, we assume that the H-action on R is proper and not graded.*

Now, we prove several preliminary results that we use throughout this paper.

Lemma 2.5. *Let R and H be as in Hypothesis 2.4. Then the relation r of R is of the form*

$$vu - quv - \lambda u^2 + au + bv + c$$

where $\{u, v\}$ is a suitable basis of U and where $q \in k^\times$, $\lambda = 0$ or 1, and $a, b, c \in k$.

Proof. The relation of $\mathrm{gr}_F R$ is of the form $vu - quv - \lambda u^2$, so the assertion follows. $\square$

Lemma 2.6. *Let R and H be as in Hypothesis 2.4, and r be the relation of R. If U is a simple left H-module, then $a = b = 0$.*

Proof. We can view the relation r in Lemma 2.5 as an element of $U^{\otimes 2} \oplus U \oplus k \subset k\langle U\rangle$. Since kr is a 1-dimensional H-module, the image of kr under the projection map $\pi : U^{\otimes 2} \oplus U \oplus k \to U$ is an H-module of dimension at most 1. Since U is a simple 2-dimensional H-module, we have that $\pi(r) = 0$. Hence, $a = b = 0$. $\square$

If $(a, b) = (0, 0)$, then the following result discusses the structure of R and the homological determinant of the H-action on $\mathrm{gr}_F R$.

Lemma 2.7. *Let R and H be as in Hypothesis 2.4. Assume that $H \neq k$ and consider the relation r of R given by Lemma 2.5. If $a = b = 0$, then $c \neq 0$ and the homological determinant of the H-action on $\mathrm{gr}_F R$ is trivial.*

Proof. Since H-action is not graded, if $a = b = 0$, then $c \neq 0$. Since kr is a 1-dimensional H-module, for each $h \in H$, the equation

$$h \cdot \big(vu - quv - \lambda u^2 + c\big) = \phi(h)\big(vu - quv - \lambda u^2 + c\big)$$

defines an algebra map $\phi : H \to k$. Since $h \cdot c = \epsilon(h)c$, we see that $\phi = \epsilon$. Hence

$$h \cdot \big(vu - quv - \lambda u^2\big) = \epsilon(h)\big(vu - quv - \lambda u^2\big).$$

By [3, Theorem 2.1], the homological determinant of the H-action on $\mathrm{gr}_F R$ is now trivial as desired. $\square$

On the other hand, if $(a, b) \neq (0, 0)$, then we have the following result.

Lemma 2.8. *Let R and H be as in Hypothesis 2.4 with $H \neq k$. Consider the relation r of R as in Lemma 2.5. If $(a, b) \neq (0, 0)$ (for any choice of basis $\{u, v\}$), then we have the following statements.*

(a) *U is a direct sum of two 1-dimensional H-modules: $U \cong T_1 \oplus T_2$.*
(b) *Here, $T_1 \cong k$ and $T_2 \ncong k$, or $T_1 \ncong k$ and $T_2 \cong k$.*
(c) *$H \cong kC_m$, a cyclic group algebra for $m \geqslant 2$.*
(d) *The relation r of R is of the form $vu - uv - v$ up to a change of basis.*
(e) *The homological determinant of the H-action on the associated graded ring $\mathrm{gr}_F R$ is non-trivial. (Equivalently, if $\mathrm{hdet}_H \mathrm{gr}_F R = \epsilon$, then $(a, b) = (0, 0)$.)*

Proof. (a) We work with the coaction of $K := H^\circ$ instead of the action of H. Suppose to the contrary that U is indecomposable. We will show that in this case the K-coaction $\rho : R \to R \otimes K$ on R is graded, thus producing a contradiction.

Since $(a, b) \neq (0, 0)$, by Lemma 2.6, the K-comodule U is not simple. So there is a non-split exact sequence of K-comodules

$$0 \to T_1 \to U \to T_2 \to 0$$

where T_1 and T_2 are 1-dimensional. Choose a basis $\{u, v\}$ of U such that $u \in T_1$ with $v \in U \setminus T_1$, then

$$\begin{cases} \rho(u) = u \otimes g_1, \\ \rho(v) = v \otimes g_2 + u \otimes h. \end{cases} \tag{E2.8.1}$$

We claim that g_1, g_2 are grouplike and h is (g_1, g_2)-primitive. Applying the coassociativity equation $(\rho \otimes 1)\rho = (1 \otimes \Delta)\rho$ to (E2.8.1) gives

$$u \otimes \Delta(g_1) = u \otimes g_1 \otimes g_1,$$
$$v \otimes \Delta(g_2) + u \otimes \Delta(h) = v \otimes g_2 \otimes g_2 + u \otimes h \otimes g_2 + u \otimes g_1 \otimes h.$$

The claim then follows by comparing coefficients.

With respect to this basis $\{u, v\}$ of U, write the relation r of R as

$$r = a_{11}u^2 + a_{12}uv + a_{21}vu + a_{22}v^2 + au + bv + c,$$

for some scalars a_{ij}, $a, b, c \in k$. A simple calculation shows that

$$\begin{aligned}\rho(r) = {} & u^2 \otimes \left(a_{11}g_1^2 + a_{12}g_1h + a_{21}hg_1 + a_{22}h^2\right) + v^2 \otimes a_{22}g_2^2 \\ & + uv \otimes (a_{12}g_1g_2 + a_{22}hg_2) + vu \otimes (a_{21}g_2g_1 + a_{22}g_2h) \\ & + u \otimes (ag_1 + bh) + v \otimes bg_2 + 1 \otimes c.\end{aligned}$$

Since $\rho(r) = r \otimes g$ for some grouplike element g by Lemma 2.2, we have the following equations

$$a_{11}g = a_{11}g_1^2 + a_{12}g_1h + a_{21}hg_1 + a_{22}h^2, \tag{E2.8.2}$$
$$a_{12}g = a_{12}g_1g_2 + a_{22}hg_2, \tag{E2.8.3}$$
$$a_{21}g = a_{21}g_2g_1 + a_{22}g_2h, \tag{E2.8.4}$$
$$a_{22}g = a_{22}g_2^2, \tag{E2.8.5}$$
$$ag = ag_1 + bh, \tag{E2.8.6}$$
$$bg = bg_2, \tag{E2.8.7}$$
$$cg = c. \tag{E2.8.8}$$

If $b \neq 0$, then by (E2.8.6), we have $h = ab^{-1}(g - g_1) \in kG$, so $K \cong kG$, a contradiction. If $a_{22} \neq 0$, then by (E2.8.3) we have that $h = a_{22}^{-1}(a_{12}gg_2^{-1} - a_{12}g_1) \in kG$, which again is a contradiction. Hence we have $b = a_{22} = 0$. By hypothesis $a \neq 0$, so by (E2.8.6) we have $g = g_1$.

If $a_{12} = 0$ or $a_{21} = 0$, then $\mathrm{gr}_F R$ fails to be a domain. Therefore $a_{12}a_{21} \neq 0$, hence by (E2.8.3), we have that $g = g_1g_2$. Since $g = g_1$, we conclude $g_2 = 1$.

Now if $c \neq 0$, then by (E2.8.8) we have $g = 1$, so $g_1 = 1$. Thus h is a primitive element, which contradicts finite dimensionality of K (Lemma 2.3(a)). Hence $c = 0$.

Since $a_{21} \neq 0$, without loss of generality, we can take $a_{21} = 1$. Moreover, write $r = vu - quv - \lambda u^2 + au$ with $a_{11} = -\lambda$ and $a_{12} = -q$. Now (E2.8.2) yields $-\lambda g_1 = -\lambda g_1^2 - qg_1h + hg_1$, so

$$\eta_{g_1}(h) - qh = \lambda(g_1 - 1) \in kG.$$

Since h is $(g_1, 1)$-primitive, by Lemma 2.3(c), we have that $q \neq 1$ or $h \in kG$. In the latter case, $K \cong kG$, which yields a contradiction, so U is decomposable. On the other hand, if $q \neq 1$, let $v' = v + a(1-q)^{-1}$. Then $r = v'u - quv' - \lambda u^2$. Since $g_2 = 1$, we have that $\rho(v') = v' \otimes 1 + u \otimes h$. Thus, the K-coaction on R is graded when using the basis $\{u, v'\}$. This contradicts the hypothesis, so again U is decomposable.

(b, c) By part (a), we have an isomorphism $U \cong ku \oplus kv$ of K-comodules. Thus $\rho(u) = u \otimes g_1$, $\rho(v) = v \otimes g_2$, and by the argument after (E2.8.1), the elements g_1, g_2 are grouplike. Note that (E2.8.2)–(E2.8.8) hold and $h = 0$ in this case. Since $(a, b) \neq (0, 0)$ we can assume, by symmetry, that $a \neq 0$. Then, $g = g_1$ by (E2.8.6). Since $\operatorname{gr}_F R$ is a domain, either a_{11}, or a_{12}, or a_{21} is nonzero. So one of Eqs. (E2.8.2)–(E2.8.4) implies that $g_1 = 1$ or $g_2 = 1$. Since K-coaction on R is inner-faithful, we see that K is generated by a single grouplike element. So $K \simeq kC_m$ for some $m \geqslant 2$.

(d, e) Since a, b are not both zero, we may assume $b \neq 0$ by symmetry, which implies that $g = g_2$ by (E2.8.7). Recall that $h = 0$ in this case. By part (b), we have either $g_2 = 1$ or $g_1 = 1$. So we have two cases to consider:

Case 1: $g = g_2 = 1$ and $g_1 \neq 1$. By (E2.8.2)–(E2.8.8), we have that $a_{12} = a_{21} = a = 0$. So we have that $r = a_{11}u^2 + a_{22}v^2 + bv + c$. Replacing v with a change of variables, we can make $a = b = 0$. This contradicts the hypothesis.

Case 2: $g = g_2 \neq 1$ and $g_1 = 1$. By (E2.8.2)–(E2.8.8), we have $a_{11} = a_{22} = a = c = 0$. Up to scaling, $r = vu - quv + b'v$ for $b' \in k^\times$. If $q \neq 1$, we may replace u by $u' := u + b'(1-q)^{-1}$ so that the relation becomes $vu' - qu'v$. Since $g_1 = 1$, then K coacts on the new relation and whence the K-coaction on R is graded, yielding a contradiction. Therefore $q = 1$, and the assertion in (d) follows.

Since $g_1 = 1$, $\rho(r) = r \otimes g_2$. This means that $\operatorname{hcodet}_K \operatorname{gr}_F R = g_2 \neq 1$. Hence, part (e) follows. □

Remark 2.9. Suppose H acts on the filtered algebra $R = k\langle u, v\rangle/(r + s)$ where $r \in F_2$ and $s \in F_1$. Then by Lemmas 2.7 and 2.8(e), the induced H-action on $\operatorname{gr}_F R$ has trivial homological determinant if and only if $s \in k$. In this case, an H-action on $\operatorname{gr}_F R$ lifts uniquely to an H-action of R, so there is a bijective correspondence between H-actions on R and H-actions on $\operatorname{gr}_F R$ with trivial homological determinant.

Moreover, as a result of Lemma 2.8, we can classify the relations of R so that H acts on R under Hypothesis 2.4.

Corollary 2.10. *The relation r of R under Hypothesis 2.4 is in one of the following forms*:

$$vu - quv - 1, \qquad vu - uv - u^2 - 1, \quad \text{or} \quad vu - uv - v,$$

for $q \in k^\times$, up to a change of basis of U.

Proof. Apply Lemma 2.7 and Lemma 2.8(d, e). □

Now we turn our attention to PI algebras (algebras that satisfy a polynomial identity). The following result provides a sufficient condition for a filtered AS regular algebra of dimension 2 to be PI.

Lemma 2.11. *Let R be a filtered AS regular algebra of dimension 2 with relation $r = a_{11}u^2 + a_{12}uv + a_{21}vu + au + bv + c$ for $a_{ij}, a, b, c \in k$. Suppose that $a_{12} + a_{21} \neq 0$ and $\operatorname{gr}_F R$ is PI. Then R is PI. Equivalently, if R is not PI, then either $\operatorname{gr}_F R$ is not PI or $\operatorname{gr}_F R$ is commutative.*

Proof. Since $\operatorname{gr}_F R$ is a domain, we have $a_{12}a_{21} \neq 0$. So we may assume $a_{21} = 1$ and $a_{12} = -q \neq 0$. By the first hypothesis, we have that $q \neq 1$. By replacing v with $v - a_{11}(1-q)^{-1}u$ we can assume $a_{11} = 0$, so $r = vu - quv + au + bv + c$. Now by replacing u with $u - b(1-q)^{-1}$ and v with $v - a(1-q)^{-1}$, we can assume that $r = vu - quv + c$. Since $\operatorname{gr}_F R$ is PI, we see that q is an n-th root of unity for $n \geqslant 2$. Therefore u^n and v^n are central, so R is module finite over its center. Hence, R is PI.

Now if R is not PI, we just showed that $\operatorname{gr}_F R$ is not PI or $a_{12} + a_{21} = 0$, with $a_{12}, a_{21} \neq 0$. Thus if $\operatorname{gr}_F R$ is PI, it is isomorphic to $k\langle u, v\rangle/(r_\lambda)$ where $r_\lambda = vu - uv + \lambda u^2$ for some $\lambda \in k$. For all $\lambda \neq 0$, the algebra $k\langle u, v\rangle/(r_\lambda)$ is isomorphic to the Jordan plane, which is not PI. This shows that $\lambda = 0$, so $\operatorname{gr}_F R$ is commutative. □

3. Fixed subrings and the proof of Theorem 0.4

In this section, we provide several results about the fixed ring R^H corresponding to a finite dimensional Hopf algebra H acting on a filtered AS regular algebra R. In particular, we prove Theorem 0.4 and a weakened version of Theorem 0.5 (see Proposition 3.3 and the material after). We end the section by computing examples of fixed rings of Hopf actions on PI algebras (Example 3.4).

Given any algebra A that is not necessarily filtered AS regular, suppose that a Hopf algebra H acts on A. Then the *fixed subring* of the H-action is defined to be

$$A^H = \{a \in A \mid h \cdot a = \epsilon(h)a \text{ for all } h \in H\}.$$

Now let A be a filtered algebra with a nonnegative exhaustive filtration $\{F_nA\}_{n\geqslant 0}$. For any $x \in F_iA$, we use $\bar{x}$ (or $\bar{x}_i$) for the corresponding image of x in $A_i := F_iA/F_{i-1}A$. Suppose that H acts on A such that each F_nA is a left H-module. For each $h \in H$, and for each homogeneous element $\bar{x} \in A_i$ where $x \in F_iA$ is any lift of $\bar{x}$, define

$$h \cdot \bar{x} = \overline{(h \cdot x)}_i.$$

It is possible that $h \cdot x \in F_{i-1}A$, but we want to consider the image of $h \cdot x$ in A_i. It is easy to check that H acts on $\operatorname{gr}_F A$ so that $\operatorname{gr}_F A$ is a left H-module algebra. We record this without proof as part (a) of the following lemma.

Lemma 3.1. *Let A be a filtered algebra with filtration $\{F_nA\}_{n\geqslant 0}$. Suppose that H acts on A such that each F_nA is a left H-module. Then, the following statements hold.*

(a) *H acts on $\operatorname{gr}_F A$ naturally.*
(b) *A^H has an induced exhaustive filtration F' such that $\operatorname{gr}_{F'}(A^H)$ is a subalgebra of $(\operatorname{gr}_F A)^H$.*
(c) *If H is semisimple (whence finite dimensional), then $\operatorname{gr}_{F'}(A^H) = (\operatorname{gr}_F A)^H$.*

Proof. (b) Let $F'_n = A^H \cap F_nA$. Then $\{F'_n\}_{n\geqslant 0}$ is a nonnegative exhaustive filtration of the subalgebra A^H of A. Clearly, $\operatorname{gr}_{F'}(A^H)$ is a subalgebra of $\operatorname{gr}_F A$. For any nonzero homogeneous element $\bar{x} \in \operatorname{gr}_{F'}(A^H)$, we pick a lift $x \in A^H$. By definition,

$$h \cdot \bar{x} = (\overline{h \cdot x})_i = \left(\overline{\epsilon(h)x}\right)_i = \epsilon(h)\bar{x}.$$

Hence $\bar{x} \in (\operatorname{gr}_F A)^H$. The assertion follows.

(c) Consider the induced subfiltration $F'_n = A^H \cap F_nA$ of A^H. By part (b), it suffices to show that $\operatorname{gr}_{F'}(A^H) \supseteq (\operatorname{gr}_F A)^H$. Since H is semisimple, we may choose a left integral element $\int \in H$ with $\epsilon(\int) = 1$. Moreover, a left trace function $\operatorname{tr}: A \to A^H$, defined by $a \mapsto \int \cdot a$, is surjective as H is semisimple. Hence, $A^H = \int \cdot A$. For every nonzero homogeneous element $\bar{x} \in (\operatorname{gr}_F A)^H \subset \operatorname{gr}_F A$ of degree i,

$$\bar{x} = \int \cdot \bar{x} = \overline{\int \cdot x}.$$

This means that $x - \int \cdot x$ in $F_{i-1}A$. So we may replace x by $\int \cdot x$ and assume that $x \in A^H$. Therefore, $\bar{x}$ is in $\operatorname{gr}_{F'}(A^H)$. $\square$

Part (c) of the lemma above need not hold if H is not semisimple; we illustrate this in the following example. We also provide an example of an inner-faithful H-action on an algebra A so that the induced action on $\operatorname{gr}_F A$ is not inner-faithful.

Example 3.2. Here, we do not assume that k is of characteristic zero. Let H be Sweedler's 4-dimensional Hopf algebra $k\langle g, f\rangle/(fg+gf,\ g^2-1,\ f^2)$. (See Example 1.4 for the coalgebra structure and antipode of H.) Let $A=k[u]$ and define a left H-action on A by

$$f\cdot u=1 \quad \text{and} \quad g\cdot u=-u.$$

It is easy to check that A is a left H-module algebra. The following statements are also easy to check.

(a) The H-action on A is not proper.
(b) The H-action on A is inner-faithful.
(c) By induction, we have that

$$f\cdot u^n=\begin{cases}0, & n=\text{even},\\ u^{n-1}, & n=\text{odd}\end{cases}$$

for all n. As a consequence, $A^H=k[u^2]\neq k[u]=A$.
(d) Let F be the filtration defined by $F_n=(k+ku)^n$. Then $\operatorname{gr}_F A\cong k[u]$ with $\deg u=1$. Take $\bar{u}$ to be the image of $u\in F_1A$ in A_1. Hence, $f\cdot\bar{u}=(\overline{f\cdot u})_1=\bar{1}_1=0$. Likewise for g, we see that the H-action on $\operatorname{gr}_F A$ is determined by $f\cdot\bar{u}=0$ and $g\cdot\bar{u}=-\bar{u}$. As a consequence, the H-action on $\operatorname{gr}_F A$ is <u>not</u> inner-faithful.
(e) Let $F'=F\cap A^H$ be the induced filtration of A^H. If $\operatorname{char} k\neq 2$, then $(\operatorname{gr}_F A)^H=\operatorname{gr}_{F'} A^H$.
(f) If $\operatorname{char} k=2$, then the H-action on $\operatorname{gr}_F A$ is trivial. So $(\operatorname{gr}_F A)^H=\operatorname{gr}_F A$. As a consequence, $(\operatorname{gr}_F A)^H\ncong\operatorname{gr}_{F'} A^H$.

Next, we prove weakened versions of Theorems 0.4 and 0.5.

Proposition 3.3. *Let H be a semisimple Hopf algebra that acts on a filtered AS regular algebra R of dimension 2, inner-faithfully and preserving filtration, with the H-action not graded. Note that the H-action on R is proper by Lemma 1.5(a). If the homological determinant of the H-action of $\operatorname{gr}_F R$ is not trivial, then R^H has global dimension 2.*

Proof. Since the homological determinant of the H-action on $\operatorname{gr}_F R$ is not trivial, $(a,b)\neq(0,0)$ by Lemma 2.7. Now by Lemma 2.8(c, d), the relation of R is of the form $r=vu-uv-v$, the Hopf algebra is $H=kC_m=k\langle h\rangle$, and the action of H on R is given by

$$h\cdot u=u, \qquad h\cdot v=\xi v$$

for some primitive m-th root of unity ξ. It is easy to see that the fixed subring R^H is

$$R^H=k\bigl\langle u, v^m\bigr\rangle/\bigl(v^m u-(u+m)v^m\bigr).$$

So R^H has global dimension 2, since it is isomorphic to the Ore extension $k[u][v^m,\sigma]$ where $\sigma\in\operatorname{Aut}(k[u])$ is given by $\sigma(u)=u+m$. □

Now we are ready to prove Theorem 0.4.

Proof of Theorem 0.4. Since H is semisimple, the H-action on R is proper by Lemma 1.5(a). If the homological determinant of the H-action on $\operatorname{gr}_F R$ is not trivial, then the assertion follows from Proposition 3.3. If the homological determinant of the H-action on $\operatorname{gr}_F R$ is trivial, then by [7, Theorem 0.1], $(\operatorname{gr}_F R)^H$ is AS Gorenstein. By Lemma 3.1(c), $\operatorname{gr}_F(R^H)\cong(\operatorname{gr}_F R)^H$, which is AS Gorenstein, and by definition, R^H is filtered AS Gorenstein. □

Furthermore, we compute two examples of fixed subrings of Hopf actions on PI algebras.

Example 3.4. Let R be the algebra $k\langle u,v\rangle/(u^2+v^2-1)$. We will consider two different Hopf actions on R, and compute the corresponding fixed subrings R^H. Note that $\mathrm{gr}_F R$ is AS regular, and the following actions of H on R are filtered AS regular actions.

(1) Let H_8 be the (unique) 8-dimensional noncommutative and noncocommutative semisimple Hopf algebra. It is generated as an algebra by x, y, z and subject to the relations:

$$x^2=1, \qquad y^2=1, \qquad z^2=\frac{1}{2}(1+x+y-xy), \qquad xz=zy, \qquad yz=zx, \qquad xy=yx.$$

The coalgebra structure and antipode of H_8 is determined by

$$\Delta(x)=x\otimes x, \qquad \Delta(y)=y\otimes y, \qquad \Delta(z)=\frac{1}{2}(1\otimes 1+1\otimes x+y\otimes 1-y\otimes x)z\otimes z,$$
$$\epsilon(x)=1, \qquad \epsilon(y)=1, \qquad \epsilon(z)=1, \qquad S(x)=x, \qquad S(y)=y, \qquad S(z)=z.$$

Consider the H_8-action on R is given by

$$x\cdot u=-u, \qquad y\cdot u=u, \qquad z\cdot u=v,$$
$$x\cdot v=v, \qquad y\cdot v=-v, \qquad z\cdot v=u.$$

Denote

$$a=(uv)^2-(vu)^2, \qquad b=u^4-u^2+\frac{1}{4}, \qquad c=\left(u^2-\frac{1}{2}\right)\left((uv)^2+(vu)^2\right).$$

It is not hard to check that the fixed subring R^{H_8} is a commutative ring which is isomorphic to $k[a,b,c]/(c^2-b(a^2+4(b-\frac{1}{4})^2))$.

Note that R^{H_8} is Gorenstein by Theorem 0.4. Moreover, R^{H_8} has isolated singularities at $(a,b,c)=(0,\frac{1}{4},0)$ and $(\pm\sqrt{-1}/2,0,0)$. These are Kleinian singularities both of type A_1.

(2) Let $H=(kD_{2n})^\circ$ where D_{2n} is the dihedral group of order $2n$. Since the H-action is equivalent to the H°-coaction, we will consider the kD_{2n}-coaction on R.

By definition, $D_{2n}=\langle x,y \mid x^2=y^2=(xy)^n=1\rangle$. Define a comodule structure map $\rho: R\to R\otimes kD_{2n}$ by

$$\rho(u)=u\otimes x \quad \text{and} \quad \rho(v)=v\otimes y.$$

By a simple calculation, the fixed subring $R^H=R^{co(kD_{2n})}$ is a commutative ring isomorphic to $k[a,b,c]/(bc-a^n(1-a)^n)$ generated by $a=u^2$, $b=(uv)^n$ and $c=(vu)^n$. By Theorem 0.4, R^H is Gorenstein. If $n\geqslant 2$, then R^H has isolated singularities at $(a,b,c)=(0,0,0)$ and $(1,0,0)$. These are also Kleinian singularities both of type A_{n-1}.

4. The proof of Theorem 0.3

This section is dedicated to the proof of Theorem 0.3. First, we introduce some notation and we provide some preliminary results.

Let $F=\{F_nA\}_{n\geqslant 0}$ denote a filtration of A. The *Rees ring* of A with respect to F is defined to be

$$\mathrm{Rees}_F A=\bigoplus_{n\geqslant 0}(F_nA)t^n.$$

We begin by analyzing the Rees ring of the n-th Weyl algebra $A_n(k)$ with respect to the standard filtration. Here, $A_n(k)$ is generated by $u_1, \dots, u_n, v_1, \dots, v_n$ subject to relations

$$[u_i, u_j] = [v_i, v_j] = 0 \quad \text{and} \quad [v_i, u_j] = \delta_{ij}.$$

Moreover, we refer to [4, Definition 1.7] for the definition of the Calabi–Yau property in terms of Nakayama automorphisms.

Lemma 4.1. *Let $A_n(k)$ be the n-th Weyl algebra with the standard filtration F. Then the following statements regarding $B := \operatorname{Rees}_F A_n(k)$ hold.*

(a) *B is generated by $u_1, \dots, u_n, v_1, \dots, v_n, t$ subject to the relations:*

$$[u_i, u_j] = [v_i, v_j] = [u_i, t] = [v_i, t] = 0, \quad \textit{and} \quad [v_i, u_j] = \delta_{ij} t^2$$

for $1 \leqslant i, j \leqslant n$. We call $\{u_1, \dots, u_n, v_1, \dots, v_n, t\}$ the standard basis of B.

(b) *B is a Koszul AS regular algebra of global dimension $2n + 1$.*

(c) *B is Calabi–Yau.*

(d) *Let $\{u_1^*, \dots, u_n^*, v_1^*, \dots, v_n^*, t^*\}$ be the k-linear dual of the standard basis. If $\operatorname{char} k = 0$ (or if $\operatorname{char} k > n$), then in the Koszul dual $B^!$ of B, we have that $(t^*)^{2n+1} \neq 0$.*

(e) *If $f = t^* + \sum_{i=1}^n (a_i u_i^* + b_i v_i^*)$ is in $B^!$ for some scalars $a_i, b_i \in k$, then $f^{2n+1} = (t^*)^{2n+1}$.*

Proof. (a, b) These are well known.

(c, d) Note that the Koszul dual $B^!$ of B is generated by the k-linear dual of the standard basis, subject to the relations

$$u_i^* u_j^* + u_j^* u_i^* = 0, \qquad v_i^* v_j^* + v_j^* v_i^* = 0, \qquad u_i^* v_j^* + v_j^* u_i^* = 0, \qquad (u_i^*)^2 = 0, \qquad (v_i^*)^2 = 0,$$
$$u_i^* t^* + t^* u_i^* = 0, \qquad v_i^* t^* + t^* v_i^* = 0, \qquad (t^*)^2 + \sum_{i=1}^n v_i^* u_i^* = 0,$$

for all $1 \leqslant i, j \leqslant n$. Hence, $B^!$ is isomorphic to the exterior algebra

$$\Lambda(u_1^*, \dots, u_n^*, v_1^*, \dots, v_n^*, t^*)$$

as a graded vector space. In particular, $\mathfrak{e} := v_1^* u_1^* v_2^* u_2^* \cdots v_n^* u_n^* t^*$ is a nonzero element in the highest degree of $B^!$ (degree $= 2n + 1$). Using the relations $(u_i^*)^2 = (v_i^*)^2 = 0$ and $(t^*)^2 = -\sum_{i=1}^n v_i^* u_i^*$ for all i, we have that $(t^*)^{2n+1} = (-1)^n n! \mathfrak{e} \neq 0$. Hence, part (d) holds.

It is easy to check that $ab = ba$ if $a \in B^!$ has degree 1 and $b \in B^!$ has degree $2n$. Therefore, the Nakayama automorphism of $B^!$ is the identity; refer to [4, Section 3]. By [2, Theorem 6.3], the Nakayama automorphism of B is the identity. Now B is Calabi–Yau by definition.

(e) This follows by a direct computation. $\square$

The following lemma is clear, so we omit the proof.

Lemma 4.2. *Let H be a finite dimensional Hopf algebra. Let A be a filtered algebra so that H acts on A and each $F_n A$ is a left H-module. Then the following statements hold.*

(a) *There is an induced H-action on $\operatorname{Rees}_F A$ such that $\operatorname{Rees}_F A$ is a left H-module algebra with each homogeneous component of $\operatorname{Rees}_F A$ being a left H-module.*

(b) *The quotient map* $\mathrm{Rees}_F\, A \to \mathrm{Rees}_F\, A/(t-1) = A$ *is an* H*-module algebra homomorphism.*
(c) *The quotient map* $\mathrm{Rees}_F\, A \to \mathrm{Rees}_F\, A/(t) = \mathrm{gr}_F\, A$ *is an* H*-module algebra homomorphism.*
(d) $(\mathrm{Rees}_F\, A)^H = \mathrm{Rees}_{F'}\, A^H$ *where* F' *is the induced filtration on* A^H*.*

Since $B = \mathrm{Rees}_F\, A_n(k)$ is a left H-module algebra, we have that $K = H^\circ$ coacts on $B^!$ from the left [4, Remark 1.6(d)]. Here, the K-comodule structure map is denoted by $\rho : B^! \to K \otimes B^!$.

We now define an algebra $\tilde{B}^!$ that will aid in the study of the K-coaction on $B^!$. Let $\Lambda = \Lambda(z_1, \dots, z_{2n})$ denote the exterior algebra in $2n$ variables and define $\tilde{B}^! = \Lambda[y; \sigma]$, where $\sigma(z_i) = -z_i$ for all $i = 1, \dots, 2n$. By recalling the presentation of $B^!$ from the proof of Lemma 4.1(c), we see that there is a k-algebra isomorphism

$$B^! \cong \tilde{B}^! \Big/ \left(y^2 + \sum_{i=1}^{n} z_i z_{n+i} \right)$$

given by $t^* \mapsto y$, $v_i^* \mapsto z_i$ and $u_i^* \mapsto z_{n+i}$ for all $i = 1, \dots, n$. For convenience, we also use $z_1, \dots, z_{2n}$ and y as the corresponding generators for both $B^!$ and $\tilde{B}^!$.

Lemma 4.3. *Let* $\{c_i\}_{i=1}^{2n}$ *be a subset of* K *and let* $s := c_1 \otimes z_1 + \cdots + c_{2n} \otimes z_{2n}$*. For any* $r > 0$*, the following identity holds in* $K \otimes \tilde{B}^!$

$$(1 \otimes y + s)^{2r} = \sum_{j=0}^{r} \binom{r}{j} \sum_{\sigma \in S_{2j}} (-1)^{\mathrm{sgn}(\sigma)} c_{i_{\sigma(1)}} \cdots c_{i_{\sigma(2j)}} \otimes \sum_{i_1 < \cdots < i_{2j}} y^{2(r-j)} z_{i_1} \cdots z_{i_{2j}}.$$

Proof. Since $1 \otimes y$ and s skew-commute, we have $(1 \otimes y + s)^2 = 1 \otimes y^2 + s^2$. Moreover, $1 \otimes y^2$ and s^2 commute, so we can use the usual binomial theorem to get

$$(1 \otimes y + s)^{2r} = \sum_{j=0}^{r} \binom{r}{j} \left(1 \otimes y^{2(r-j)}\right) s^{2j}.$$

Since $s \in K \otimes \Lambda$ and Λ is the exterior algebra, the formula follows by expanding s^{2j} in the algebra $K \otimes \Lambda$. $\square$

Let $c \in K$, then left multiplication by c defines a k-vector space endomorphism $\mu_c : K \to K$. We define $\mathrm{tr}(c)$ to be the trace of μ_c as an element of $\mathrm{End}_k(K)$.

Lemma 4.4. *Let* $\rho : B^! \to K \otimes B^!$ *be a* K*-coaction of* $B^!$ *with* $\rho(y) = 1 \otimes y + \sum_{i=1}^{2n} c_i \otimes z_i$ *for* $\{c_i\}_{i=1}^{2n}$ *a subset of* K*, then*

$$\rho(y)^{2n+1} = (1 + p) \otimes y^{2n+1}$$

where $p \in K$ *with* $\mathrm{tr}(p) = 0$*.*

Proof. Working with algebras $\Lambda \subset B^!$, we have $y^2 = -\sum_{i=1}^{n} z_i z_{n+i} \in \Lambda_2$. By Lemma 4.1(d), $y^{2n} \neq 0$, so take it as a basis element for Λ_{2n}. For $1 \leqslant i_1 < i_2 < \cdots < i_{2j} \leqslant 2n$, we have that $y^{2(n-j)} z_{i_1} \cdots z_{i_{2j}} \in \Lambda_{2n} = k y^{2n}$. Write

$$y^{2(n-j)} z_{i_1} \cdots z_{i_{2j}} = \lambda_{i_1 \cdots i_{2j}} y^{2n}$$

for some $\lambda_{i_1\cdots i_{2j}} \in k$. So by the previous lemma, we have that

$$\rho(y)^{2n} = \sum_{j=0}^{n} \binom{n}{j} \sum_{i_1<\cdots<i_{2j}} \lambda_{i_1\cdots i_{2j}} \left(\sum_{\sigma\in S_{2j}} (-1)^{\sigma} c_{i_{\sigma(1)}} \cdots c_{i_{\sigma(2j)}} \right) \otimes y^{2n}.$$

For $j = 0$, the above expression is $1 \otimes y^{2n}$. If $j > 1$, then

$$\begin{aligned} \operatorname{tr}\left(\sum_{\sigma\in S_{2j}} (-1)^{\sigma} c_{i_{\sigma(1)}} \cdots c_{i_{\sigma(2j)}} \right) &= \operatorname{tr}\left(\sum_{\sigma\in S_{2j}} (-1)^{\sigma} c_{i_{\sigma(2j)}} c_{i_{\sigma(1)}} \cdots c_{i_{\sigma(2j-1)}} \right) \\ &= -\operatorname{tr}\left(\sum_{\sigma\in S_{2j}} (-1)^{\sigma} c_{i_{\sigma(1)}} \cdots c_{i_{\sigma(2j)}} \right). \end{aligned}$$

The above is zero since $\operatorname{char}(k) \neq 2$. This shows that $\rho(y)^{2n} = (1+p) \otimes y^{2n}$ where $\operatorname{tr}(p) = 0$. Finally, note that $y^{2n} z_i = z_i y^{2n} = 0$, so

$$\rho(y)^{2n+1} = (1 \otimes y)\rho(y)^{2n} = (1+p) \otimes y^{2n+1}.$$

This completes the proof. □

Now we are ready to prove Theorem 0.3.

Proof of Theorem 0.3. Let $B := \operatorname{Rees}_F A_n(k)$, and let $B^!$ denote the Koszul dual of B. Since B is Calabi–Yau (Lemma 4.1(c)), it suffices to show that the homological determinant of the left H-action on B is trivial [4, Theorem 0.6]. Equivalently, we show that the left $K = H^{\circ}$-coaction on $B^!$ has trivial homological codeterminant.

Note that since H acts on $A_n(k)$ preserving the filtration, H acts on B inner-faithfully. So K coacts on B inner-faithfully. Hence, K coacts on $B^!$ inner-faithfully [4, Proposition 2.5(c)]. Let ρ denote the K-coaction on $B^!$. Note that $T := kt$ is a trivial K-sub-comodule of $W = kt \oplus \bigoplus_{i=1}^{n}(ku_i \oplus kv_i)$ by Lemma 4.2(b). We have a K-comodule map $\pi : W^* \to T^*$ which sends u_i^* and v_i^* to zero. Since T^* is also a trivial K-comodule, that is $(1 \otimes \pi)\rho(t^*) = 1 \otimes \pi(t^*)$, we have that

$$\rho\big(t^*\big) = 1 \otimes t^* + \sum_{i=1}^{n} (a_i \otimes u_i + b_i \otimes v_i)$$

for some $a_i, b_i \in K$. By Lemma 4.3, we have

$$\rho\big(\big(t^*\big)^{2n+1}\big) = (1+p) \otimes \big(t^*\big)^{2n+1}$$

where $\operatorname{tr}(p) = 0$. By definition, the homological codeterminant D of the K-coaction on $B^!$ is $1+p$. Moreover, D is a grouplike element. Since K is finite dimensional, D also has finite order. Now $\operatorname{tr}(p) = 0$ implies that $\operatorname{tr}(\mathsf{D}) = \operatorname{tr}(1) = \dim K$. Since D has finite order, we have that $\mathsf{D} = 1$. Therefore the K-coaction on B has trivial homological codeterminant. Dually, the H-action on B has trivial homological determinant as desired. Thus, $H = K^{\circ}$ is semisimple.

Since $\operatorname{char} k = 0$, H is also cosemisimple, that is, H equals its coradical H_0. If H is pointed, then $H_0 = kG(H)$. Hence, H is a group algebra. □

5. The proof of Theorem 0.1 and consequences

We return to the study of Hopf algebra actions on filtered AS regular algebras R of dimension 2. In this section, we prove Theorem 0.1, and as a consequence, we classify the possible Hopf algebra actions when R is non-PI.

Note that if R is a non-PI filtered AS regular algebra of dimension 2, then it follows from Lemma 2.11 that the associated graded ring $\mathrm{gr}_F R$ is either non-PI or commutative. We provide preliminary results for these cases separately.

We have the following setup. Let K be a finite dimensional Hopf algebra coacting on a non-PI filtered AS regular algebra R. Let ρ denote the coaction and $R \simeq k\langle u, v\rangle/(r)$. Since ρ preserves the filtration, we can write

$$\rho(u) = u \otimes e_{11} + v \otimes e_{21} + 1 \otimes f_1, \tag{E5.0.1}$$

$$\rho(v) = u \otimes e_{12} + v \otimes e_{22} + 1 \otimes f_2 \tag{E5.0.2}$$

for some $e_{ij}, f_j \in K$, $i, j = 1, 2$. Using coassociativity of the coaction, we have

$$\Delta(e_{ij}) = \sum_{l=1}^{2} e_{il} \otimes e_{lj}, \tag{E5.0.3}$$

$$\Delta(f_1) = f_1 \otimes e_{11} + f_2 \otimes e_{21} + 1 \otimes f_1, \tag{E5.0.4}$$

$$\Delta(f_2) = f_1 \otimes e_{12} + f_2 \otimes e_{22} + 1 \otimes f_2, \tag{E5.0.5}$$

$$\epsilon(e_{ij}) = \delta_{ij} \quad \text{and} \quad \epsilon(f_i) = 0. \tag{E5.0.6}$$

5.1. $\mathrm{gr}_F R$ is non-PI

We need the following well-known lemma.

Lemma 5.1. *Suppose G is a finite group acting on $A = k_J[u, v]$ or $k_q[u, v]$ for $q \neq \pm 1$. Then G is abelian and the action is diagonal with respect to the basis $\{u, v\}$.*

Lemma 5.2. *Suppose that $\mathrm{gr}_F R$ is non-PI. Then H is a commutative group algebra.*

Proof. Let K' be the Hopf subalgebra of K generated by $\{e_{ij}\}_{i,j=1}^2$. Then by definition K' coacts on $\mathrm{gr}_F R$ inner-faithfully. Since the K'-coaction on $\mathrm{gr}_F R$ is inner-faithful, by [4, Theorem 5.10], we have that K' is the dual of a finite group algebra. By Lemma 5.1, the coaction ρ' is diagonal with respect to $\{u, v\}$ where $\mathrm{gr}_F R = k_q[u, v]$ for $k_J[u, v]$. Hence we can write

$$\rho(u) = u \otimes e_{11} + 1 \otimes f_1 \quad \text{and} \quad \rho(v) = v \otimes e_{22} + 1 \otimes f_2$$

where e_{ii} is grouplike (by Lemma 2.2) and f_i is $(1, e_{ii})$-primitive for $i = 1, 2$.

Suppose that $\mathrm{gr}_F R = k\langle u, v\rangle/(vu - quv)$. Then the relation r of R is of the form $r = vu - quv + au + bv + c$. Note that $\rho(r) = r \otimes g$ for some grouplike element g. So, we have that

$$\begin{aligned}\rho(r) = {} & vu \otimes e_{22}e_{11} - quv \otimes e_{11}e_{22} + u \otimes (f_2 e_{11} - q e_{11} f_2 + a e_{11}) \\ & + v \otimes (e_{22} f_1 - q f_1 e_{22} + b e_{22}) + 1 \otimes (f_2 f_1 - q f_1 f_2 + a f_1 + b f_2 + c).\end{aligned}$$

By comparing the coefficients of u, we have that $ag = f_2e_{11} - qe_{11}f_2 + ae_{11}$. In particular, $\eta_{e_{11}}(f_2) - qf_2 \in kG(K)$. Since $q \notin \mathbb{U} \cup \{1\}$, by Lemma 2.3(b), we have that $f_2 \in kG(K)$. Similarly $f_1 \in kG(K)$, hence K is a group algebra.

Finally, by comparing coefficients of vu and uv, we get $e_{22}e_{11} = g$ and $-qg = -qe_{11}e_{22}$, so $[e_{11}, e_{22}] = 0$. Since $f_i \in kG(K)$ is $(1, e_{ii})$-primitive we have $f_i \in k(1 - e_{ii})$ for $i = 1, 2$. Hence K is commutative and generated by grouplike elements e_{11} and e_{22}. Therefore, $H = K^\circ$ is a commutative group algebra.

Now suppose $\operatorname{gr}_F R = k\langle u, v\rangle/(vu - uv - u^2)$. Then $r = vu - uv - u^2 + au + bv + c$. We have

$$\begin{aligned}\rho(r) = {} & u^2 \otimes \left(-e_{11}^2\right) + vu \otimes e_{22}e_{11} - uv \otimes e_{11}e_{22} \\ & + u \otimes \left([f_2, e_{11}] - e_{11}f_1 - f_1e_{11} + ae_{11}\right) \\ & + v \otimes \left([e_{22}, f_1] + be_{22}\right) + 1 \otimes \left([f_2, f_1] - f_1^2 + af_1 + bf_2 + c\right).\end{aligned}$$

Again $\rho(r) = r \otimes g$ for some grouplike element g by Lemma 2.2. By comparing the coefficients of u^2 and vu, we have that $e_{11}^2 = g = e_{22}e_{11}$. Hence $e_{11} = e_{22}$. Comparing the coefficients of u and v gives

$$ag = [f_2, e_{11}] - e_{11}f_1 - f_1e_{11} + ae_{11}, \tag{E5.2.1}$$

$$bg = [e_{22}, f_1] + be_{22}. \tag{E5.2.2}$$

Rearranging Eq. (E5.2.2) gives $\eta_{e_{22}}(f_1) - f_1 \in kG(K)$, so by Lemma 2.3(c) we have that $f_1 \in kG(K)$. Now we can rearrang (E5.2.1) to give $\eta_{e_{11}}(f_2) - f_2 \in kG(K)$, so similarly $f_2 \in kG(K)$. Now f_i is $(1, e_{ii})$-primitive, so $f_i \in k(1 - e_{ii})$ for $i = 1, 2$. Since $e_{11} = e_{22}$, we see that K is generated by a single grouplike element, so $K = kC_m$. Consequently, H is a commutative group algebra. $\square$

5.2. $\operatorname{gr}_F R$ is commutative

Now we study the case where $\operatorname{gr}_F R \cong k[u, v]$.

Lemma 5.3. *If R is non-PI and $\operatorname{gr}_F R$ is isomorphic to $k[u, v]$, then R is isomorphic to either $A_1(k)$ or $k\langle u, v\rangle/(vu - uv - v)$.*

Proof. Since $\operatorname{gr}_F R \cong k[u, v]$, the relation r of R is of the form $r = vu - uv + au + bv + c$. If $a = b = 0$, then $R \cong A_1(k)$. If either a or b is nonzero, then by a change of variables, $R \cong k\langle u, v\rangle/(vu - uv - v)$ as desired. $\square$

Now suppose that H acts on $R = k\langle u, v\rangle/(vu - uv - v)$.

Lemma 5.4. *Retain the notation from the beginning of the section. Then $e_{11} = 1$, $e_{12} = 0$, and up to linear transformation $e_{21} = 0$.*

Proof. Consider the following computation:

$$\begin{aligned}\rho(r) &= \rho(vu - uv - v) \\ &= u^2 \otimes (e_{12}e_{11} - e_{11}e_{12}) + uv \otimes (e_{12}e_{21} - e_{11}e_{22}) \\ &\quad + vu \otimes (e_{22}e_{11} - e_{21}e_{12}) + v^2 \otimes (e_{22}e_{21} - e_{21}e_{22}) \\ &\quad + u \otimes (f_2e_{11} + e_{12}f_1 - e_{11}f_2 - f_1e_{12} - e_{12}) \\ &\quad + v \otimes (f_2e_{21} + e_{22}f_1 - e_{21}f_2 - f_1e_{22} - e_{22}) \\ &\quad + 1 \otimes (f_2f_1 - f_1f_2 - f_2).\end{aligned}$$

Since $\rho(r) = r \otimes g$ for some grouplike element g by Lemma 2.2, we have that

$$\begin{aligned} 0 &= e_{12}e_{11} - e_{11}e_{12}, & 0 &= f_2e_{11} + e_{12}f_1 - e_{11}f_2 - f_1e_{12} - e_{12}, \\ g &= e_{11}e_{22} - e_{12}e_{21}, & g &= -f_2e_{21} - e_{22}f_1 + e_{21}f_2 + f_1e_{22} + e_{22}, \\ g &= e_{22}e_{11} - e_{21}e_{12}, & 0 &= f_2f_1 - f_1f_2 - f_2, \\ 0 &= e_{22}e_{21} - e_{21}e_{22}. \end{aligned}$$

Here, the four equations in the left column are given as

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Now,

$$\begin{pmatrix} S(e_{11}) & S(e_{12}) \\ S(e_{21}) & S(e_{22}) \end{pmatrix} = \begin{pmatrix} e_{22}g^{-1} & -e_{12}g^{-1} \\ -e_{21}g^{-1} & e_{11}g^{-1} \end{pmatrix},$$

and by Eqs. (E5.0.3)–(E5.0.6) and the antipode axiom, we have that

$$\begin{aligned} S(f_1) &= -f_1e_{22}g^{-1} + f_2e_{21}g^{-1}, \\ S(f_2) &= f_1e_{12}g^{-1} - f_2e_{11}g^{-1}. \end{aligned}$$

Consider the seven relations above. By applying the antipode to the four equations in the left column above, and by using appropriate substitutions from the first two equations in the right column above, we obtain that

$$\begin{aligned} S^2(f_1) &= g(e_{11} + f_1 - 1)g^{-1}, \\ S^2(f_2) &= g(e_{12} + f_2)g^{-1}. \end{aligned}$$

Let η_g be the conjugation $a \mapsto g^{-1}ag$. Since $\eta_g \circ S^2(f_2) = e_{12} + f_2$ and $\eta_g \circ S^2(e_{12}) = e_{12}$, we have that $(\eta_g \circ S^2)^n(f_2) = f_2 + ne_{12}$. Since K is finite dimensional, both η_g and S^2 have finite order. Now there exists $m \geqslant 1$ such that $(\eta_g \circ S^2)^m = \mathrm{Id}_K$. Thus $me_{12} = 0$ and $e_{12} = 0$ as claimed.

On the other hand, we have that $(\eta_g \circ S^2)^n(f_1) = f_1 + n(e_{11} - 1)$. A similar argument shows that $e_{11} = 1$.

As a consequence, e_{22} is grouplike and e_{21} is $(e_{22}, 1)$-primitive. Since $e_{22}e_{21} = e_{21}e_{22}$, we have that $\eta_{e_{22}}(e_{21}) - e_{21} = 0$. Lemma 2.3(c) implies that $e_{22}^{-1}e_{21} \in kG$. Therefore, $e_{21} = c(1 - e_{22})$ for some $c \in k$. Replacing v by $v + cu$, we have $e_{21} = 0$. □

Lemma 5.5. *Let $R = k\langle u, v\rangle/(r)$ where $r = vu - uv - v$. Then H is a commutative group algebra.*

Proof. By Lemma 5.4, we may assume that $e_{12} = e_{21} = 0$ and $e_{11} = 1$. Since $e_{11} = 1$, f_1 is a primitive element, so $f_1 = 0$ (Lemma 2.3(a)). Therefore we have that

$$\rho(r) = vu \otimes e_{22} - uv \otimes e_{22} - v \otimes e_{22} - 1 \otimes f_2.$$

Since $\rho(r) = r \otimes g$ for some grouplike element g by Lemma 2.2, we have $g = e_{22}$ and $f_2 = 0$. Hence, K is generated by e_{22}, which is a commutative group algebra. □

5.3. Proof of Theorem 0.1

Here, we prove Theorem 0.1 and list an immediate consequence afterward.

Proof of Theorem 0.1. If R is non-PI, then $\mathrm{gr}_F R$ is non-PI or $\mathrm{gr}_F R \cong k[u, v]$. If $\mathrm{gr}_F R$ is non-PI, then the conclusion follows from Lemma 5.2. If $\mathrm{gr}_F R \cong k[u, v]$, then by Lemma 5.3, R is isomorphic to either $k\langle u, v\rangle/(vu - uv - v)$ or $A_1(k)$. In the first case, the conclusion follows from Lemma 5.5. We see that in each case (where $R \ncong A_1(k)$), K (and H) is a commutative group algebra. Finally assume that $R = A_1(k)$. By Theorem 0.3, H is semisimple. Therefore the H-action is proper (Lemma 1.5(a)). By Lemma 1.5(b, c), the induced H-action on $\mathrm{gr}_F R$ is inner-faithful. Since $\mathrm{gr}_F R \cong k[u, v]$, H is a group algebra by [4, Proposition 0.7]. $\square$

Corollary 5.6. *Let R be a filtered AS regular algebra of dimension 2. Suppose the H-action on R is inner-faithful and preserves the filtration of R. If the H-action on R is non-proper, then R is PI.*

5.4. Additional consequences of Theorem 0.1

In the rest of this section, we give more information about H-actions on filtered AS regular algebras R of dimension 2 which are non-PI. Note that Theorem 0.1 does not provide any information about which groups occur in the case where $R \cong A_1(k)$. Fortunately, this has been done in [1, Proposition on p. 84].

Corollary 5.7. *Let $k = \mathbb{C}$ and let H be a finite dimensional Hopf algebra acting on $A_1(\mathbb{C})$ inner-faithfully. Then $H = kG$ where G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, which is conjugate to one of the following special subgroups*:

(1) *a cyclic group of order n,*
(2) *a binary dihedral group of order $4n$,*
(3) *a binary tetrahedral group of order 24,*
(4) *a binary octahedral group of order 48, or*
(5) *a binary icosahedral group of order 120.*

The following result classifies the H-module algebra structures on filtered AS regular algebras of dimension 2 which are non-PI. The following is well known after we have shown that H is a group algebra.

Corollary 5.8. *Let R be a non-PI filtered AS regular algebra of dimension 2 and let H be a finite dimensional Hopf algebra acting on R inner-faithfully and preserving the filtration of R. Suppose that the H-action is not graded and that $R \ncong A_1(k)$. Then one of the following occurs.*

(a) *$R \cong k\langle u, v\rangle/(vu - quv - 1)$ for $q \in k^\times$ not a root of unity, and $H = kG$ where $G = C_m$ with a generator $\sigma \in \mathrm{Aut}(R)$ determined by*

$$\sigma(u) = \xi u, \qquad \sigma(v) = \xi^{-1} v$$

for some primitive m-th root of unity ξ.

(b) *$R \cong k\langle u, v\rangle/(vu - uv - v)$ and $H = kG$ where $G = C_m$. Up to a change of basis, a generator $\sigma \in \mathrm{Aut}(R)$ is determined by*

$$\sigma(u) = u, \qquad \sigma(v) = \xi v$$

for some primitive m-th root of unity ξ.

(c) $R \cong k\langle u, v\rangle/(vu - uv - u^2 - 1)$ *and* $H = kC_2$ *with a generator* $\sigma \in \operatorname{Aut}(R)$ *determined by*

$$\sigma(u) = -u, \qquad \sigma(v) = -v.$$

Proof. By Theorem 0.1, $H = kG$ for some finite group G. As a consequence, the H-action is proper by Lemma 1.5(a). By hypothesis, the H-action on R is not graded. By Corollary 2.10, $R \cong k\langle u, v\rangle/(r)$ where r has the form:

$$\begin{aligned} &\text{(a)} \quad r = vu - quv - 1, \\ &\text{(b)} \quad r = vu - uv - v, \quad \text{or} \\ &\text{(c)} \quad r = vu - uv - u^2 - 1. \end{aligned}$$

Case (a): Since R is not PI, q is either 1 or not a root of unity by Lemma 2.11. If $q = 1$, $R \cong A_1(k)$, which is excluded by hypothesis. Then, q is not a root of unity. It is easy to check that every filtered algebra automorphism σ of R is of the form

$$\sigma(u) = \xi u, \qquad \sigma(v) = \xi^{-1} v$$

for some primitive m-th root of unity ξ.

Case (b): See the proof of Lemma 5.5. Case (c): The assertion can be proved similarly and is omitted. $\square$

6. Galois extensions and the proof of Theorem 0.5

The goal of this section is to prove Theorem 0.5 via the use of Galois extensions.

Definition 6.1. (See [5, Definition 1.1].) Let K be a Hopf algebra and A be a right K-comodule algebra with structure $\rho : A \to A \otimes K$. Let $B = A^{co\,K}$. We say that $B \subset A$ is a (right) K-Galois extension if the map $\beta : A \otimes_B A \to A \otimes K$ given by

$$\beta(a \otimes b) = (a \otimes 1)\rho(b)$$

is surjective.

The following lemmas are well known. Lemma 6.4 is a consequence of Lemma 6.2. We use the convention that $K := H^\circ$, for H a finite dimensional Hopf algebra.

Lemma 6.2. *(See [5, Theorems 1.2 and 2.2].) Let H be a finite dimensional Hopf algebra and A a left H-module algebra. Then the following statements are equivalent.*

(a) *$A^H \subset A$ is right K-Galois.*
(b) *The map $A \# H \to \operatorname{End}(A_{A^H})$ is an algebra isomorphism and A is a finitely generated projective right A^H-module.*
(c) *A is a left $A \# H$-generator.*

Suppose $A^H \subset A$ is right K-Galois. Then the following statements are equivalent.

(d) *For any nonzero left integral element t, the corresponding trace function $\hat{t} : A \to A^H$ is surjective. (This holds if H is semisimple.)*

(e) *A is a generator for the category of right A^H-modules.*
(f) *A is a finitely generated projective left $A \# H$-module.*

In this case, A^H and $A \# H$ are Morita equivalent.

In the case that H is semisimple, if $A^H \subset A$ is K-Galois, then A^H is Morita equivalent to $A \# H$. Also, $\operatorname{gldim} A \# H = \operatorname{gldim} A$ since H is semisimple. We have the following remark.

Remark 6.3. When H is semisimple and when A^H is Morita equivalent to $A \# H$, we have that

$$\operatorname{gldim} A^H = \operatorname{gldim} A \# H = \operatorname{gldim} A.$$

On the other hand, if K is a group algebra kG, then A is a K-comodule algebra if and only if A is a G-graded algebra. Consider the following result.

Lemma 6.4. *(See [9].) Let $H = kG$ and A be a right H-comodule algebra. Let u be the identity of G. Then $A_u \subset A$ is kG-Galois if and only if A is strongly G-graded, if and only if $A_g A_{g^{-1}} = A_u$ for all $g \in G$.*

Now we break the proof of Theorem 0.5 into two cases: when $\operatorname{gr}_F R \cong k[u, v]$ and when $\operatorname{gr}_F R \ncong k[u, v]$. The first case is handled in the proposition below.

Proposition 6.5. *Assume Hypothesis 2.4 and suppose that $\operatorname{gr}_F R \cong k[u, v]$. Then, up to isomorphism, (H, R) occurs as one of the following.*

(a) *$H = kC_m$, a cyclic group algebra, and $R \cong k\langle u, v\rangle/(vu - uv - v)$. Moreover, $R^H \subset R$ is not K-Galois and the global dimension of R^H is 2.*
(b) *[1] $H = kG$ where G is a finite subgroup of $\mathrm{SL}_2(k)$ and $R = A_1(k) = k\langle u, v\rangle/(vu - uv - 1)$. Also, $R^H \subset R$ is K-Galois and R^H is simple of global dimension 1.*

Proof. By Corollary 2.10, the relation is of the form $r = vu - uv - 1$ or $r = vu - uv - v$.

(a) If $r = vu - uv - v$, then by Lemma 2.8(c) we have $H \cong kC_m$ and by Lemma 2.8(e), the homological determinant of the H-action on $\operatorname{gr}_F R$ is non-trivial. By Proposition 3.3, R^H has global dimension 2.

Let $\deg v = 1 \in C_m$ and $\deg u = 0$ and write $R = \bigoplus_{s=0}^{m-1} R_s$ as a C_m-graded algebra with respect to the degree defined as above. It is easy to check that $1 \notin R_1 R_{m-1}$. Thus R is not strongly C_m-graded, so $R^H = R_0 \subset R$ is not K-Galois by Lemma 6.4.

(b) If $r = vu - uv - 1$, then R is the Weyl algebra $A_1(k)$. By Theorem 0.1, H is a group algebra kG for a finite group G. By Lemma 2.7, the homological determinant of the H-action on $\operatorname{gr}_F R$ is trivial. This means that $G \subset \mathrm{SL}_2(k)$. Then classical results imply that $A_1(k)^G$ is simple and has global dimension 1 [1, p. 83]. Since $A_1(k)$ is simple and since G is finite and does not contain any non-trivial inner automorphisms, it is well known that $A_1(k) \# G \cong \operatorname{End}(A_1(k)_{A_1(k)^G})$. Moreover as Lemma 6.2(b) holds, we have that $A_1(k)^G \subset A_1(k)$ is $(kG)^\circ$-Galois by Lemma 6.2(a). $\square$

Proposition 6.6. *Assume Hypothesis 2.4 and suppose that $\operatorname{gr}_F R \ncong k[u, v]$ and that R is non-PI. Then, up to isomorphisms, (H, R) occurs as one of the following.*

(a) *$H = kC_2$ and $R \cong k\langle u, v\rangle/(vu - uv - u^2 - 1)$, and $R^H \subset R$ is K-Galois. The global dimension of R^H is 2.*
(b) *$H = kC_m$ and $R \cong k\langle u, v\rangle/(vu - quv - 1)$ where q is not a root of unity. Here, $R^H \subset R$ is K-Galois and the global dimension of R^H is 2.*

Proof. Similar to the beginning of the proof of the last proposition, we may assume that H is a group algebra kG and that the relation is of the form $r = vu - uv - u^2 - 1$ or $r = vu - quv - 1$ (Corollary 2.10).

(a) In this case $\mathrm{gr}_F R = k_J[u, v]$. The induced H-action on $\mathrm{gr}_F R$ is inner-faithful by Lemma 1.5(a, c). To avoid the trivial case, we assume that $H \neq k$. Now $H = kC_2 = k\langle\sigma\rangle$ by Corollary 5.8(c). Furthermore, the σ-action on U is given by $\sigma(u) = -u$ and $\sigma(v) = -v$.

Now $K \cong kC_2$ and write $R = R_{\sigma+} \oplus R_{\sigma-}$ where $R_{\sigma\pm} = \{f \in R \mid \sigma(f) \pm f = 0\}$. It is easy to see that $1 \in R^2_{\sigma-}$. Hence R is strongly C_2-graded. By Lemmas 6.2 and 6.4, $R^H \subset R$ is H°-Galois and R^H is Morita equivalent to $R \# H$. Therefore R^H has global dimension 2 by Remark 6.3.

(b) The remaining case is when $\mathrm{gr}_F R = k_q[u, v]$ where q is not a root of unity. Similarly, we may assume that H-action on $U = ku \oplus kv$ and hence on $\mathrm{gr}_F R$ is inner-faithful. Since the homological determinant of the H-action is trivial, we have that $r = vu - quv - 1$. By Corollary 5.8(a), $H = kC_m$ with a generator $\sigma \in C_m$ where $\sigma(u) = \xi u$, $\sigma(v) = \xi^{-1} v$ and ξ is a primitive m-th root of unity. The dual Hopf algebra $K(:= H^\circ)$ is also isomorphic to kC_m with a generator τ such that $\rho(u) = u \otimes \tau$ and $\rho(v) = v \otimes \tau^{-1}$. The assertion is verified by the following lemma. $\square$

Lemma 6.7. *Let $R = k\langle u, v\rangle/(vu - quv - 1)$ and let $C_m = \langle\tau\rangle$ coact on R by $\rho(u) = u \otimes \tau$ and $\rho(v) = v \otimes \tau^{-1}$. If the order of q is at least m, then $R^{co\, C_m} \subset R$ is a Galois extension and $R^{co\, C_m}$ has global dimension 2.*

Proof. Write $R = \bigoplus_{s=0}^{m-1} R_s$ be the C_m-graded decomposition where

$$R_s = \bigl\{f \in R \bigm| \rho(f) = f \otimes \tau^s\bigr\}.$$

In particular, $R_0 = R^{co\, C_m}$. An easy computation shows that R_0 is generated by $a := u^m$, $b := v^m$ and $c := uv$.

Note that R_1 is generated by u and v^{m-1}, and that R_{m-1} is generated by u^{m-1} and v. Thus $R_1 R_{m-1}$ contains elements

$$u^m,\ v^m,\ v^{m-1}u^{m-1}, \quad \text{and} \quad uv =: c.$$

Using the relation $vu = quv + 1$, we obtain that, for each $m > s \geqslant 1$, $v^s u^s = f_s(c)$ for some polynomial $f_s(t) \in k[t]$ of degree s. Moreover, $cu = u(qc + 1)$ implies that $c^n u = u(qc + 1)^n$, so

$$\begin{aligned} f_s(c) &= v^s u^s = v\bigl(v^{s-1}u^{s-1}\bigr)u \\ &= v f_{s-1}(c) u = vu f_{s-1}(qc + 1) \\ &= (qc + 1) f_{s-1}(qc + 1). \end{aligned}$$

By induction, we have that

$$f_s(c) = \prod_{i=1}^{s} \bigl(q^i c + [i]_q\bigr)$$

where $[i]_q = 1 + q + \cdots + q^{i-1}$. If the order of q is at least m, then $f_{m-1}(0) \in k^\times$. Recall that $c = uv \in R_1 R_{m-1}$. Since $f_{m-1}(0) = f_{m-1}(c) - c(g(c)) \in R_1 R_{m-1}$ for some $g(c) \in R_0$, we have that $1 \in R_1 R_{m-1}$ and $R_0 = R_1 R_{m-1}$.

For any $l \geqslant 1$, we have by induction that $R_l R_{m-l} \supseteq (R_1)^l (R_{m-1})^l = R_0$. This shows that R is strongly C_m-graded. By Lemmas 6.2 and 6.4, $R^{co\, C_m} \subset R$ is C_m-Galois and $R^{co\, C_m}$ is Morita equivalent to $R \# (kC_m)$. As a consequence, $R^{co\, C_m}$ has global dimension 2 by Remark 6.3. $\square$

We are now ready to prove Theorem 0.5.

Proof of Theorem 0.5. By Theorem 0.1, H is semisimple. Hence the H-action is proper (Lemma 1.5(a)). Therefore Hypothesis 2.4 holds. If $\mathrm{gr}_F R \cong k[u, v]$, the assertion follows from Proposition 6.5. If $\mathrm{gr}_F R \ncong k[u, v]$, then the result follows from Proposition 6.6. $\square$

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