Non-commutative Algebraic Geometry

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1991 *Mathematics Subject Classification.* 16-XX, 14-XX.

ABSTRACT. This is a work in progress and, as such, is full of shortcomings. I would be delighted to hear your suggestions, big and small, for improvements, corrections, et cetera. In particular, I'd like to know your favorite examples and exercises. The goal of the book is to make non-commutative algebraic geometry more easily accessible than it presently is. So you are the ideal critic. If you are frustrated by the present version of this book, let me know, and I will do what I can to improve it in light of your comments.

This is a big project, certainly bigger than I anticipated. Part of the problem is that the subject is moving rapidly. Among the topics which I must include are the work of Artin-Stafford on curves, Van den Bergh's blowing up, and the material needed to support that, Patrick's ruled surfaces,... I have deliberately avoided the derived category, but it seems that each passing month brings one more reason why I should use that language.

I hope to teach a graduate course on some of this material in 1998-99, and that will give me an opportunity to tie up a lot of the loose ends.

This version is dated September 1, 1997.

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CHAPTER 1

Solutions to non-commutative polynomial equations

This introductory chapter fixes some terminology and notation and discusses the fundamental problem of non-commutative algebra: classify the matrix solutions to a system of non-commutative polynomial equations. We show this is equivalent to classifying the finite dimensional modules over a ring determined by those equations. The effectiveness of the theory developed in the rest of the book may be judged in terms of its usefulness in attacking this fundamental problem.

The representation theories of groups and Lie algebras are historically important examples. In both cases the finite dimensional representation theory is equivalent to the theory of solutions to the system of non-commutative polynomial equations which arises when the group or Lie algebra is presented by generators and relations. The algebra associated to these equations is, in the first case, the group algebra, and in the second case, the universal enveloping algebra.

The reader should skip section 1. The free algebra and tensor algebra are introduced in Section 2. This allows us to define non-commutative polynomials as elements of the free algebra. It is also shown that the $d \times d$ solutions to a system of equations in a finite number of equations is an affine algebraic variety. The connection between solutions and modules appears in Section 4, and some issues concerning classification of modules are discussed. Section 5 briefly discusses finitely presented algebras.

The base field. In this book we work over a fixed commutative base ring k. Often we will require that k be a field, and we assume it is algebraically closed whenever that is convenient.

1. Rings, modules, and algebras

Although the reader is no doubt familiar with the basic ideas of ring theory and module theory, this section allows us to fix notation and terminology, and to provide some gentle reminders.

Rings. All rings are required to contain an identity element, denoted 1, which is required to be different from zero. Moreover, homomorphisms between rings will be required to send the identity to the identity. Thus, the map which sends an element α of a field k to the 2×2 matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is not a ring homomorphism. Although it respects both multiplication and addition, the image of the identity in k is not the identity in $M_2(k)$. Given a pair of rings $R \subset S$ having the same identity, we call R a subring of S, and call S an overring of R.

If R is a ring, $M_n(R)$ denotes the ring of $n \times n$ matrices with entries from R. We will denote by e_{ij} the matrix with 1 in the ij^{th} position and zeroes elsewhere. We call the e_{ij} matrix units.

If R is a ring its opposite ring is the ring, denoted R^{op}, which is R as an abelian group, but with new multiplication $x * y = yx$ where yx is the product in R. A ring homomorphism $f : R^{\text{op}} \to S$ is also a map $f : R \to S$ which satisfies $f(xy) = f(y)f(x)$; such a map is called a ring anti-homomorphism from R to S.

The center of a ring R is the set of all $z \in R$ such that $za = az$ for all $a \in R$. It is a subring having the same identity, and is usually denoted by $Z(R)$ (from the German 'zentrum').

Modules. A module over a ring R will also be called a representation of R. Most modules will be left modules.

We say that an R module M is generated by a subset S of M if every element of M can be written in the form $r_1m_1 + \dots r_nm_n$ for some $r_i \in R$ and some $m_i \in S$. We write $Rm_1 + \dots Rm_n$ for the submodule of M generated by $\{m_1, \dots, m_n\}$. If M is generated by a finite set we say M is finitely generated. If M is generated by a single element we say that M is cyclic. In this case, if $M = Rm$, then the map $\varphi : R \to M$ defined by $\varphi(m) = rm$ is surjective so $M \cong R/\ker\varphi$. Hence the cyclic R-modules are, up to isomorphism, the quotients of R by left ideals.

Let R be a ring and M a left R-module. Since M is an abelian group, we may consider the ring of all abelian group homomorphisms from M to itself, namely $\operatorname{Hom}_{\mathbb{Z}}(M, M)$. The R-module action is a map $R \times M \to M$. Define $\rho : R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$ by

$$\rho(r)(m) = r.m. \tag{1-1}$$

The module axioms imply that ρ is a ring homomorphism. Conversely, given a ring homomorphism $\rho : R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$, formula (1-1) endows M with an R-module structure. Thus a left R-module structure on M is the same thing as a ring homomorphism $R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$.

The preceding sentence is false for right R-modules! If M is a right R-module, the map $\rho : R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$ defined by $\rho(x)(m) = m.x$, for $m \in M$ and $x \in R$, is a ring *anti-homomorphism*; the point is that $\rho(xy) = \rho(y) \circ \rho(x)$. Thus, a right R-module structure on M is the same thing as a ring anti-homomorphism $R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$. Another point of view is that right (resp. left) R-modules are the same thing as left (resp. right) R^{op}-modules.

Homomorphisms. If M and N are R-modules, $\operatorname{Hom}_R(M, N)$ denotes the space of all R-module homomorphisms from M to N. It is an abelian group under $+$: $(\varphi+\psi)(m) := \varphi(m)+\psi(m)$. In case $M = N$, we write $\operatorname{End}_R M = \operatorname{Hom}_R(M, M)$ and call this the endomorphism ring of M. It is a ring with addition as above, and multiplication defined by composition of functions, namely $\varphi\psi = \varphi \circ \psi$. It has an identity, namely the identity map $\mathbb{1}_M : M \to M$.

Left multiplication by $x \in R$ is an endomorphism of R as a right R-module. This leads to an isomorphism $R \cong \operatorname{End}(R_R)$. Similarly there is an isomorphism $R^{\text{op}} \cong \operatorname{End}({}_RR)$.

Annihilators. For any module M, the annihilator of $m \in M$ is $\operatorname{Ann}(m) := \{x \in R \mid xm = 0\}$. It is clear that $\operatorname{Ann}(m)$ is the kernel of the R-module homomorphism $\varphi : R \to M$ defined by $\varphi(x) = xm$, so it is a left ideal of R and $Rm \cong R/\operatorname{Ann}(m)$.

The annihilator of an R-module M is the set $\operatorname{Ann}_R M = \{x \in R \mid xm = 0 \text{ for all } m \in M\}$. It is an easy exercise to see that $\operatorname{Ann}_R M = \ker\rho$ where $\rho : R \to \operatorname{Hom}_{\mathbb{Z}}(M, M)$ is the ring map corresponding to the module structure. In particular, $\operatorname{Ann} M$ is a two sided ideal of R. If I is a two-sided ideal contained in $\ker\rho$ then

ρ factors through the quotient R/I, thus giving M an R/I-module structure. In particular, M is an $R/\operatorname{Ann} M$-module.

Simple modules. A simple R-module, or an irreducible representation of R, is a module, M say, whose only submodules are 0 and M itself. Clearly, M is simple if and onlyif $Rm = M$ for all $0 \neq m \in M$. If M is simple, and $0 \leq n \in M$, then $M \cong R/\operatorname{Ann}(m)$, and $\operatorname{Ann}(m)$ is a maximal left ideal of R. Thus every simple R-module is of the form R/I for some maximal left ideal of R.

THEOREM 1.1 (Schur's Lemma). *If M is a simple R-module, then $\operatorname{End}_R M$ is a division ring.*

Algebras. If k is a commutative ring, and R is a ring, then a ring homomorphism $f : k \to R$ whose image is in $Z(R)$, is said to give R the structure of a k-algebra. We call f the structure map of the algebra. For example, every ring has the structure of $\mathbb{Z}$-algebra, the structure map being $f(n) = n.1$ for $n \in \mathbb{Z}$. Notice in particular, that f is not required to be injective.

We are mainly interested in algebras over a field, in which case the structure map $f : k \to R$ is injective. It is common practice to identify k with its image in R; from this point of view a k-algebra is a just a ring containing a central subring which is isomorphic to k. A ring may be an algebra over several fields. For example, $\mathbb{C}$ is an algebra over $\mathbb{Q}$, over $\mathbb{R}$, and over $\mathbb{C}$ itself.

The commutative polynomial ring $k[t_1, \dots, t_n]$, and the ring of $n \times n$ matrices $M_n(k)$ are k-algebras. The ring of real quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is an $\mathbb{R}$-algebra, but is not a $\mathbb{C}$-algebra, even though the subalgebra $\mathbb{R} \oplus \mathbb{R}i$ is isomorphic to $\mathbb{C}$; the point is that this subalgebra is not in the center of $\mathbb{H}$.

If R_1 and R_2 are k-algebras, with structure maps $f_i : k \to R_i$, then a k-algebra homomorphism $g : R_1 \to R_2$ is a ring homomorphism satisfying $gf_1 = f_2$. That is the composition $k \to R_1 \to R_2$ gives R_2 a k-algebra structure which coincides with that obtained from $f_2 : k \to R_2$. When we consider (the category of) k-algebras it is usual to be interested only in the k-algebra maps between them.

Let R be a k-algebra. Every quotient ring, R/I say, inherits a k-algebra structure and the natural map $R \to R/I$ is a k-algebra homomorphism. If $f : R \to S$ a ring homomorphism, then S does not necessarily inherit the structure of a k-algebra.

Every R-module (left or right) becomes a k-module, with k-action given by $c.m = f(c)m$ for $c \in k$ and $m \in M$, where $f : k \to R$ is the structure map. In particular, if R is a k-algebra then an R-module inherits the structure of a k-vector space. The R-module will be said to be finite dimensional if it is finite dimensional as a vector space over k.

If R is a k-algebra, then an R-module, M say, inherits a k-module structure; it is a k-vector space. Since k lies in the center of R, the action of $r \in R$ on M is k-linear. Therefore the map ρ defined by (1-1) actually has its image in $\operatorname{Hom}_k(M, M)$. Moreover, $\operatorname{Hom}_k(M, M)$ is a k-algebra, and ρ is a k-algebra homomorphism. Hence for a k-algebra R, specifying an R-module M is the same thing as specifying a k-algebra homomorphism $\rho : R \to \operatorname{Hom}_k(M, M)$.

If M is an R-module, then multiplication by $z \in Z(R)$ is an R-module homomorphism $\rho_z : M \to M$, $\rho_z(m) = zm$. This gives a ring homomorphism $\rho : Z(R) \to \operatorname{End}_R M$, the image of which is contained in the center of $\operatorname{End}_R M$. In particular, if A is a k-algebra, then the restriction of ρ to k is an injective map from k to the center of $\operatorname{End}_R M$. Thus $\operatorname{End}_R M$ is a k-algebra.

A k-algebra A is a finite dimensional k-algebra if $\dim_k A < \infty$. It is clear that a finite dimensional k-algebra is artinian. Its simple modules are finite dimensional so its artinian modules are precisely the finite dimensional ones.

If M and N are A-modules, every A-module map $\varphi : M \to N$ is k-linear, so there is an inclusion $\operatorname{Hom}_A(M,N) \subset \operatorname{Hom}_k(M,N)$. In particular, $\operatorname{End}_A M \subset \operatorname{End}_k M$. Hence if $\dim_k M < \infty$, then $\operatorname{End}_A M \subset M_n(k)$, and $\operatorname{End}_A M$ is a finite dimensional k-algebra.

Let A be a finite dimensional k-algebra, and M a simple A-module. Since $\dim_k M < \infty$, $\operatorname{End}_A M$ is a finite dimensional k-algebra. Hence the center of the division ring $\operatorname{End}_A M$ is a finite dimensional field extension of k. Thus, if k is algebraically closed, that field extension must be k itself. This proves that *if k is algebraically closed, and A is a finite dimensional k-algebra, then* $\operatorname{End}_A M \cong k$ *for every simple A-module M.*

Tensor Product of k-algebras. If R and S are k-algebras then $R \otimes_k S$ is made into a ring by defining

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = r_1 r_2 \otimes s_1 s_2,$$

and extending linearly in the obvious way. The identity element is $1 \otimes 1$. The map $\alpha \mapsto \alpha \otimes 1$ gives a homomorphism $k \to R \otimes_k S$, thus making $R \otimes_k S$ a k-algebra. Notice that $R \otimes k \cong R$ as k-algebras. If k is a field there are injective k-algebra homomorphisms $R \to R \otimes_k S$ and $S \to R \otimes_k S$ given by $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$. The copies of R and S inside $R \otimes S$ commute with one another, so $R \otimes S$ is commutative if both R and S are. The intersection of the copies of R and S is not $1 \otimes 1$, but $k = k \otimes k$, because $R \otimes 1 = R \otimes k$, and $1 \otimes S = k \otimes S$.

If M is a left R-module, and N is a left S-module, then $M \otimes_k N$ is an $R \otimes_k S$-module where the action is $(r \otimes s) \cdot (m \otimes n) = rm \otimes sn$.

Given a k-algebra T, subalgebras R and S are linearly disjoint if whenever $\{r_\lambda \mid \lambda \in \Lambda\}$ and $\{s_\mu \mid \mu \in \Gamma\}$ are linearly independent subsets of R and S, then $\{r_\lambda s_\mu \mid \lambda \in \Lambda, \mu \in \Gamma\}$ is linearly independent.

Lemma 1.2. *Let T be a k-algebra. Suppose that T contains subalgebras R and S such that $R \cap S = k$ and elements of R commute with elements of S. Then the map $\psi : R \otimes S \to T$ given by $\psi(r \otimes s) = rs$ is an algebra homomorphism. Furthermore ψ is an isomorphism onto its image if and only if R and S are linearly disjoint.*

Proposition 1.3. *$M_n(k) \otimes M_m(k) \cong M_{nm}(k)$.*

Proof. Let e_{ij} denote the usual matrix units. If $1 \le i,j \le n$ and $1 \le k,l \le m$ define $\epsilon_{(ik)(jl)} := e_{ij} \otimes e_{kl}$. Then the $\epsilon_{(ik)(jl)}$ are a k-basis for $M_n(k) \otimes M_m(k)$. Moreover, their multiplication is given by

$$\begin{aligned}\epsilon_{(ik)(jl)} \cdot \epsilon_{(pr)(qs)} &= (e_{ij} \otimes e_{kl}) \cdot (e_{pq} \otimes e_{rs}) \\ &= \delta_{jp}\delta_{lr} e_{iq} \otimes e_{ks} \\ &= \delta_{(jl),(pr)} \epsilon_{(ik)(qs)}\end{aligned}$$

which is precisely the multiplication table for $M_{nm}(k)$ with matrix units $\{\epsilon_{(ik)(jl)} \mid 1 \le i,j \le n \text{ and } 1 \le k,l \le m\}$. □

Example 1.4. Let R be a k-algebra. We write $M_n(R)$ for the ring of $n \times n$ matrices with entries in R. It can be characterized as being a free left R-module with basis $\{e_{ij} \mid 1 \le i,j \le n\}$ and multiplication given by

$$\left(\sum_{i,j} r_{ij}e_{ij}\right) \cdot \left(\sum_{p,q} s_{pq}e_{pq}\right) = \sum_{i,j,p,q} \delta_{jp}r_{ij}s_{pq}e_{iq}.$$

Then $R \otimes M_n(k) \cong M_n(R)$.

EXAMPLE 1.5. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be indeterminates over the field k. Then the tensor product of the polynomial rings $k[X_1, \dots, X_m] \otimes k[Y_1, \dots, Y_n]$ is isomorphic to the polynomial ring in $m+n$ indeterminates $k[t_1, \dots, t_{m+n}]$ via the map $X_i \otimes 1 \mapsto t_i$ and $1 \otimes Y_j \mapsto t_{m+j}$. Much more is true. If X and Y are affine varieties over the field k, then $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y)$. Thus the tensor product of algebras is the algebraic analogue of the cartesian product of varieties.

Extending the base. Let A be a k-algebra and K a commutative k-algebra. Then the natural map $K \to A \otimes_k K$ makes $A \otimes_k K$ a K-algebra. We say that $A \otimes_k K$ is obtained from A by extending the base or by extending scalars.

Bimodules. Let R and S be rings. An S-R-bimodule is an abelian group M which is simultaneously a left S-module and a right R-module such that the actions satisfy $s.(m.r) = (s.m).r$ for all $s \in S$, $r \in R$, $m \in M$. We write ${}_SM_R$ to indicate that M is an S-R-bimodule.

Let M be an R-S-bimodule. We make M into a left $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$-module, by defining $(r \otimes s) \cdot m = rms$. This *is* a left module structure because

$$[(r \otimes s) \cdot (r' \otimes s')] \cdot m = (rr' \otimes s's) \cdot m = rr'ms's = r(r'ms')s = (r \otimes s) \cdot [(r' \otimes s') \cdot m].$$

It is easy to show that R-S-bimodules are the same things as left $R \otimes_{\mathbb{Z}} S$-modules.

A ring R is an R-R bimodule through the multiplication in R, the condition in the definition being satisfied as a consequence of the associativity of multiplication in R. Every two sided ideal of R is an R-R-bimodule. These are the only sub-bimodules of ${}_RR_R$, so R is simple as a bimodule (i.e. has no sub-bimodules apart from itself and 0) if and only if it is a simple ring.

Let M be a right R-module, and set $E = \operatorname{End}_R M$. Make M a left E-module by defining $\vartheta \cdot m = \vartheta(m)$ for $\vartheta \in E$, and $m \in M$. Now M can be given the structure of an E-R-bimodule by defining $\vartheta \cdot m \cdot r = \vartheta(mr) = \vartheta(m)r$ for $m \in M$, $r \in R$ and $\vartheta \in E$. Another point of view on this is that the maps $\psi_x : M \to M$ defined by $\psi_x(m) = mx$ for $x \in R$ are E-module endomorphisms of M, so the map $\Psi : R \to \operatorname{End}_E M$ defined by $\Psi(x) = \psi_x$ is a ring homomorphism.

EXERCISES

1.1 Show that the transpose mapping, $a \mapsto a^{\mathsf{T}}$, is an isomorphism $M_n(R) \to M_n(R^{\mathrm{op}})^{\mathrm{op}}$.

1.2 Show that the ring $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{smallmatrix}\right)$ is right artinian and right noetherian but not left artinian or left noetherian.

1.3 Let k be a commutative ring, and suppose that R is a ring endowed with the structure of a k-module. Show that this structure makes R a k-algebra if and only if the multiplication map $R \times R \to R$ is a k-module homomorphism.

1.4 Give an example of a k-algebra R, with structure map $f : k \to R$, and a ring homomorphism $g : R \to S$ for which the composition $gf : k \to S$ does not give S a k-algebra structure.

1.5 Show that R-S-bimodule structures on an abelian group are the same things as pairs of maps $f : R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$, $g : S \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$ such that (i) f is a ring homomorphism, (ii) g is an anti-homomorphism and (iii) $f(R)$ commutes with $g(S)$.

1.6 Show that if M is an R-S-bimodule, then M may be made into a left $R \otimes S^{\mathrm{op}}$-module, and conversely.

1.7 Let A, B, C, D be k-algebras, M an A-B-bimodule and N a C-D-bimodule; in brief, ${}_AM_B$ and ${}_CN_D$. Show that $\mathrm{Hom}_k(M, N)$ is a right A-module, a left B-module, a left C-module and a right D-module. In other words $\mathrm{Hom}_k(M, N)$ is a $B \otimes_k C$-$A \otimes_k D$-bimodule or a left module over $A^{\mathrm{op}} \otimes B \otimes C \otimes D^{\mathrm{op}}$. Explicitly, the action is given by

$$(a \otimes b \otimes c \otimes d.f)(m) = cf(amb)d.$$

1.8 Consider the rings $\mathbb{C}$ and $\mathbb{H}$ (the quaternions) as $\mathbb{R}$-algebras (so that $\otimes$ denotes $\otimes_{\mathbb{R}}$ in the following exercises). Make $R = \mathbb{C} \otimes \mathbb{C}$ into a ring as above. Show that R is isomorphic to the semisimple ring $\mathbb{C} \oplus \mathbb{C}$. [*Hint:* find idempotents in R.]

1.9 Show that $\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$. [*Hint:* find matrix units.]

1.10 Show that $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$.

1.11 Let k^2 have ordered basis e_1, e_2, and let $k^2 \otimes k^2$ have ordered basis $v_1 = e_1 \otimes e_1$, $v_2 = e_1 \otimes e_2$, $v_3 = e_2 \otimes e_1$, $v_4 = e_2 \otimes e_2$. Consider the linear transformations h and f of k^2, which are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis e_1, e_2. What are the 4×4 matrices which represent the linear transformations

$$f \otimes 1,\ 1 \otimes f,\ f \otimes f,\ h \otimes f,\ f \otimes f,\ h \otimes h,\ h \otimes 1,\ 1 \otimes h$$

with respect to the ordered basis $v_1, \ldots, v_4$ of $k^2 \otimes k^2 \cong k^4$.

1.12 Let V be a finite dimensional vector space and let V^* be its dual. Give $V \otimes_k V$ a ring structure by defining

$$(u \otimes \alpha).(v \otimes \beta) := \alpha(v)u \otimes \beta,$$

whenever $u \otimes \alpha, v \otimes \beta \in V \otimes V^*$, and extend this to finite sums using distributivity. If $\{u_i\}$ is a basis for V, and α_i is its dual basis, show that the element $\sum u_i \otimes \alpha_i$ is the identity for this ring structure, and therefore is independent of the choice of basis. Make $V \otimes V^*$ a k-algebra through the map $k \to V \otimes V^*$, $\alpha \mapsto \alpha.1$. Show that the map $\Phi : V \otimes V^* \to \mathrm{End}_k(V)$ defined by $\Phi(\sum_i v_i \otimes f_i)(u) = \sum_i f_i(u)v_i$ is a k-algebra isomorphism.

1.13 If U and V are k-vector spaces show that there is a k-algebra isomorphism $\mathrm{End}_k(U) \otimes \mathrm{End}_k(V) \to \mathrm{End}_k(U \otimes V)$.

1.14 Let K/k be an extension field. Let $k[X_1, \ldots, X_n]$ be the commutative polynomial ring. Show that $K \otimes_k k[X_1, \ldots, X_n] \cong K[X_1, \ldots, X_n]$.

2. The free algebra and the tensor algebra

Definition 2.1. Given a set X, write X^* for the set of all finite sequences of elements from X, including the empty sequence which is denoted by 1. The elements of X^* are called words. Juxtaposition of words defines an associative multiplication on X^* making it a monoid with an identity element 1. Let $k\langle X\rangle$ denote the k-vector space with basis the elements of X^*. The multiplication in X^* is extended to a k-linear multiplication on $k\langle X\rangle$ which is associative and distributive. Thus $k\langle X\rangle$ becomes a ring.

The map $k \to k\langle X\rangle$ ending α to $\alpha.1$ is an injective ring homomorphism. We will identify k with its image in $k\langle X\rangle$, and write α for $\alpha.1$. Since 1 is an identity in X^* it is also an identity in $k\langle X\rangle$. Thus $k\langle X\rangle$ is a ring with identity, and is a k-algebra. We call $k\langle X\rangle$ the free k-algebra on X.

EXAMPLE 2.2. If $X = \emptyset$, then $X^* = \{1\}$, so $k\langle X\rangle = k.1$ and is isomorphic to the base ring k.

If $X = \{x\}$, then $X^* = \{1, x, xx, xxx, \dots\}$. It is usual to denote these elements by $\{1, x, x^2, x^3, \dots\}$. Thus $k\langle X\rangle$ is isomorphic to the polynomial ring in one variable.

Suppose that $X = \{x, y\}$. Then

$$X^* = \{1, x, y, xx, xy, yx, yy, xxx, xxy, xyx, xyy, yxx, yxy, yyx, yyy, \dots\}. \tag{2-1}$$

The multiplication in X^* is not commutative, whence $k\langle X\rangle$ is non-commutative. It is usual to write a word such as $xxxyyxyyy$ as $x^3y^2xy^3$. The elements of $k\langle X\rangle$ are finite k-linear combinations of words. For example, $1 + \alpha x^2 - \beta yxy \in k\langle X\rangle$ whenever $\alpha, \beta \in k$.

NOTATION . If $X = \{x_1, \dots, x_n\}$ it is usual to write $k\langle X\rangle$ as $k\langle x_1, \dots, x_n\rangle$, and to call it the free algebra on $x_1, \dots, x_n$. The elements of $k\langle x_1, \dots, x_n\rangle$ are called non-commutative polynomials in $x_1, \dots, x_n$.

PROPOSITION 2.3. *Let A be a k-algebra and $\psi : X \to A$ a map of sets. Then there is a unique k-algebra homomorphism $\Psi : k\langle X\rangle \to A$ such that $\Psi(x) = \psi(x)$ for all $x \in X$.*

PROOF. (Existence) Since X^* is a basis for $k\langle X\rangle$, there is a unique k-linear map $\Psi : k\langle X\rangle \to A$ defined by $\Psi(1) = 1$, and

$$\Psi(x_1 \dots x_n) := \psi(x_1) \dots \psi(x_n)$$

whenever $x_1, \dots, x_n \in X$. It follows from the definitions of $k\langle X\rangle$ and Ψ, that Ψ is a ring homomorphism. Since Ψ is k-linear and $\Psi(1) = 1$, it is also a k-algebra map.

(Uniqueness) For any k-algebra, two ring homomorphisms which agree on k and the generators agree everywhere. Hence Ψ is completely determined by the fact that $\Psi(x) = \psi(x)$ for all $x \in X$. $\square$

COROLLARY 2.4. *Every k-algebra is a quotient of a free k-algebra.*

PROOF. In Proposition 2.3, take $X = A$ and ψ to be the identity. $\square$

Definition 2.5. If $x_1, \dots, x_n \in X$, the degree of $w = x_1 \dots x_n \in X^*$ is defined to be $\deg w = n$. If $0 \neq f \in k\langle X\rangle$ the degree of f is

$$\deg f = \max\{\deg w \mid w \in X^* \text{ appears in } f \text{ with a non-zero coefficient}\}$$

and we define $\deg 0 = -\infty$.

This notion of degree has the usual properties, namely $\deg fg = \deg f + \deg g$ (cf. Exercise 1) and $\deg(f + g) \leq \max\{\deg f, \deg g\}$.

Definition 2.6. Fix an ordering on X. For words of a fixed degree, n say, define $x_1 \dots x_n < y_1 \dots y_n$ if either $x_1 < y_1$ or $x_1 \dots x_m = y_1 \dots y_m$ but $x_{m+1} < y_{m+1}$ for some m. This ordering is called the degree and dictionary ordering.

For example, if $X = \{a, b, c, d, \cdots\}$ with $a < b < c < d < \cdots$, then the words in increasing order are

$$1, a, b, c, \cdots, aa, ab, ac, \cdots, aaa, aab, aac, \cdots, aba, abb, abc, \cdots.$$

Thus, the words are ordered first by degree, then in alphabetical order. In particular, if $X = \{x, y\}$ with $x < y$ the elements of X^* are listed in (2-1) in increasing order.

Definition 2.7. Having fixed an ordering on X, and hence an ordering on X^* as in Definition 2.6 the leading term of $0 \neq f \in k\langle X\rangle$ is αw, where $\alpha \in k$ and $w \in X^*$, if all the words occurring in $f - \alpha w$ with non-zero coefficient are strictly less than w. This α is called the leading coefficient of f.

The tensor algebra. We now present a basis free appoach to the free algebra.

Definition 2.8. Let V be a free k-module. Write

$$V^{\otimes n} = \underbrace{V \otimes \dots \otimes V}_{n \text{ copies}}, \qquad \text{and} \qquad T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

with the convention that $V^{\otimes 0} = k$. Thus

$$T(V) = k \ \oplus \ V \ \oplus \ (V \otimes V) \ \oplus \dots .$$

Make $T(V)$ into a ring by defining the product of

- $\alpha \in V^{\otimes 0} = k$ and $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ to be
$$(\alpha).(v_1 \otimes \dots \otimes v_n) = (\alpha v_1) \otimes \dots \otimes v_n = (v_1 \otimes \dots \otimes v_n).(\alpha);$$
- $u_1 \otimes \dots \otimes u_m \in V^{\otimes m}$ and $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ to be
$$(u_1 \otimes \dots \otimes u_m).(v_1 \otimes \dots \otimes v_n) = u_1 \otimes \dots \otimes u_m \otimes v_1 \otimes \dots \otimes v_n.$$

Extending this product by linearity gives $T(V)$ the structure of a ring. It has an identity element, namely $1 \in k = V^{\otimes 0}$, and becomes a k-algebra.

We call $T(V)$ the tensor algebra on V.

PROPOSITION 2.9. *Let X be a set, and V a free k-module. Every set map $\psi : X \to V$ extends to a unique k-algebra homomorphism $\Psi : k\langle X\rangle \to T(V)$. The map Ψ is injective if and only if $\{\psi(x) \mid x \in X\}$ is linearly independent, and is surjective if and only if $\psi(X)$ spans V. In particular, Ψ is an isomorphism if and only if ψ maps X injectively to a basis for V.*

PROOF. The existence of Ψ is guaranteed by Theorem 4.1.

By definition $k\langle X\rangle$ has basis X^* and, if $x_1, \dots, x_n \in X$, then

$$\Psi(x_1 \dots x_n) = \psi(x_1) \dots \psi(x_n) = \psi(x_1) \otimes \dots \otimes \psi(x_n) \in V^{\otimes n}.$$

Hence $\psi(X)$ is linearly independent if and only if $\{\Psi(x_1 \dots x_n) \mid x_1, \dots, x_n \in X\}$ is linearly independent. The theorem follows from this in a straightforward way. $\square$

The action of $GL(V)$ on V extends to an action of $GL(V)$ as automorphisms of $T(V)$. In particular, if V has basis $x_1, \dots, x_n$ then the symmetric group action on $\{x_1, \dots, x_n\}$ extends to an action of S_n as automorphisms of $T(V)$.

The construction of $T(V)$ is functorial; this is a consequence of Proposition 2.9. That is, a k-linear map $f : U \to V$ extends to a k-algebra homomorphism. Hence the automorphism action of $GL(V)$ is a consequence of the functoriality of $T(-)$.

EXERCISES

2.1 Show that the free algebra $k\langle x_1, \dots, x_n\rangle$ is a domain.

2.2 In the free algebra $k\langle x, y\rangle$, show that the two-sided ideal (y) is not finitely generated as a left ideal.

2.3 Show that the free algebra $k\langle x, y\rangle$ is not noetherian by
(a) exhibiting an ascending chain of *left* ideals which does not terminate;
(b) exhibiting an ascending chain of *two-sided* ideals which does not terminate.

2.4 Show that the subalgebra of $k\langle x, y\rangle$ generated by $x, xy, xy^2, \dots, xy^n$ is isomorphic to the free algebra on $n+1$ variables.

2.5 Let R be a k-algebra which is a domain. Suppose that $0 \neq a, b \in R$ are such that $aR \cap bR = 0$. Show that $k[ab, b^2]$, the subalgebra generated by ab and b^2, is isomorphic to the free algebra on two variables.

2.6 Let $R = k\langle X\rangle/(\mathcal{F})$. Without loss of generality we may, and do, assume that the leading coefficient of each $f \in \mathcal{F}$ is 1. Write $\mathcal{F} = \{f_1, f_2, \dots\}$. For each i, let $w_i \in X^*$ be the leading term of f_i. Show that the image in R of

$$\{w \in X^* \mid w \text{ does not contain any } w_i \text{ as a subword}\}$$

spans R. [*Hint:* In R we have $f_i = 0$, which may be rewritten as $w_i = g_i$ with all the words occurring in g_i being $< w_i$.]

2.7 Fix $0 \neq \lambda \in k$ and consider $R = k\langle x, y\rangle/(yx - \lambda xy)$. Show that $\{x^i y^j \mid i, j \geq 0\}$ is a basis for R. [*Hint:* use the result in Exercise 6.]

3. Non-commutative polynomial equations

The seminal problem of non-commutative algebra is the following: given a system of non-commutative polynomial equations $f_1 = \ldots = f_r = 0$ in $k\langle x_1, \dots, x_n\rangle$ find all n-tuples of $d \times d$ matrices, with entries from k, which are solutions.

EXAMPLE 3.1. If $r = 0$ we have no equations, so all n-tuples are solutions.

If we are given the single equation $f = xy - yx \in k\langle x, y\rangle$ the solutions are all pairs of commuting matrices $x, y \in M_d(k)$.

Each equation, f say, involving unknowns $x_1, \dots, x_n$, is an element of the free algebra $k\langle x_1, \dots, x_n\rangle$. To evaluate the non-commutative polynomial f at matrices $a_1, \dots, a_n \in M_d(k)$, means to substitute, in the expression for f, the matrix a_i wherever x_i occurs. Doing this, one obtains an expression which is a sum of products of matrices and scalars (the coefficients in the original expression for f). This expression may be evaluated by using the addition and multiplication in $M_d(k)$. The process of replacing each x_i by a_i gives a map

$$\Psi : k\langle x_1, \dots, x_n\rangle \to M_d(k),$$

and $a_1, \dots, a_n$ provides a solution to the equation $f = 0$ if and only if $\Psi(f) = 0$. The map Ψ is a ring homomorphism: it is the unique k-algebra homomorphism for which $\Psi(x_i) = a_i$ (cf. Proposition 2.3). Given a system of equations, possibly infinite, and possibly involving infinitely many unknowns, we may do a similar thing.

Definition 3.2. Let $\mathcal{F}$ be a set of non-commutative polynomial equations in a set of unknowns X. Thus $\mathcal{F} \subset k\langle X\rangle$. Let $\psi : X \to M_d(k)$ be a map and let $\Psi : k\langle X\rangle \to M_d(k)$ be the k-algebra homomorphism extending ψ. We say that ψ is a solution to the system of equations $\mathcal{F}$, if $\Psi(f) = 0$ for all $f \in \mathcal{F}$.

Notice that the set of solutions to the system of equations $\mathcal{F} = 0$ is the same as the set of solutions to the system $\mathcal{G} = 0$ where $\mathcal{G}$ is the ideal generated by $\mathcal{F}$.

The action of $GL_d(k)$ on the space of solutions. If $(a_1, \dots, a_n) \in M_d(k)^n$ is a solution to a given system of equations so is $(ga_1g^{-1}, \dots, ga_ng^{-1})$ for every $g \in GL_d(k)$. Two solutions related in such a way are essentially the same, differing only by a choice of basis. This suggests the next definition.

Definition 3.3. Two solutions $\psi_i : X \to M_d(k)\,(i = 1, 2)$ to the system of equations $\mathcal{F} \subset k\langle X\rangle$ are equivalent if there exists $g \in GL_d(k)$ such that $\psi_2(x) = g\psi_1(x)g^{-1}$ for all $x \in X$.

Suppose that $X = \{x_1, \dots, x_n\}$ is finite. Then a $d \times d$ solution to $\mathcal{F} = 0$ is a point $(a_1, \dots, a_n)$ in the nd^2-dimensional vector space $M_d(k)^n$, consisting of ordered n-tuples of $d \times d$ matrices. Write S for the subset of $M_d(k)^n$ consisting of solutions. We will see shortly that S is an algebraic variety. As we just saw, there is an action of the group $GL_d(k)$ on S, given by

$$GL_d(k) \times S \to S, \qquad g.(a_1, \dots, a_n) := (ga_1g^{-1}, \dots, ga_ng^{-1}).$$

The orbits of $GL_d(k)$ on S are the equivalence classes of solutions and it is really these which we wish to classify.

The relation to commutative algebra. Our concern is solutions in $M_d(k)$ for *arbitrary* d, but the case $d = 1$ is classical; then one is seeking solutions in k, so all the solution matrices must commute with one another. Therefore the set of solutions in k to the original system, say $f_1 = \dots = f_r = 0$, is the same as the set of solutions to the system of *commutative* polynomial equations $g_1 = \dots = g_r = 0$ where g_j is obtained from f_j by assuming the x_i commute. To make this precise, suppose that each f_i is in $k\langle x_1, \dots, x_n\rangle$. Let $k[t_1, \dots, t_n]$ be the polynomial ring in n variables. Now define the map $\Psi : k\langle x_1, \dots, x_n\rangle \to k[t_1, \dots, t_n]$ by $\Psi(x_i) = t_i$, and define $g_j = \Psi(f_j)$. Then $(a_1, \dots, a_n) \in k^n$ is a solution to $f_1 = \dots = f_r = 0$ if and only if it is a solution to $g_1 = \dots = g_r = 0$.

The variety of solutions. Consider a (possibly infinite) set of equations $\mathcal{F} = 0$ in a *finite* set of unknowns $X = \{x_1, \dots, x_n\}$, and let $S \subset M_d(k)^n$ denote the set of $d \times d$ solutions. Write each x_i as a $d \times d$ matrix of unknown scalars, say $x_i = (y^{(i)}_{pq})$ with $1 \le p, q \le d$. Each equation $f \in \mathcal{F}$ can be expressed as d^2 equations, one for each matrix position, in terms of the nd^2 unknowns $y^{(i)}_{pq}$. As f runs over $\mathcal{F}$, one obtains a collection, $\mathcal{G}$ say, of equations in the $y^{(i)}_{pq}$'s, and solutions in k to the system $\mathcal{G} = 0$ are in bijection with solutions in $M_d(k)$ to the system $\mathcal{F} = 0$.

Since each $y^{(i)}_{pq} \in k$, we may view the equations in $\mathcal{G}$ as elements of the commutative polynomial ring

$$k[y^{(i)}_{pq} \mid 1 \le i \le n, 1 \le p, q \le d].$$

Let I be the ideal generated by $\mathcal{G}$. Thus solutions to $\mathcal{F} = 0$ are in bijection with the points of k^{nd^2} where I vanishes. That is, $S = \mathcal{V}(I)$, the affine algebraic variety which is the zero locus of I. Hilbert's Basis Theorem ensures that I is generated by a finite subset of $\mathcal{G}$, and hence the set of $d \times d$ solutions to $\mathcal{F} = 0$ is the same as the set of $d \times d$ solutions to some *finite* subset $\mathcal{F}_d \subset \mathcal{F}$ of equations.

The action of $GL_d(k)$ on S is an algebraic action of an algebraic group on an algebraic variety, so there is a rather well-developed theory to deal with questions concerning the orbits of $GL_d(k)$.

The fact that the set of solutions forms an affine algebraic variety is not used as the first step in finding the solutions by algebraic geometric means, but rather it allows discussion of some algebraic geometric features such as dimension, the number of irreducible components, whether the $GL_d(k)$-orbits are open or closed etc. The fact that it is a variety gives the set of solutions more structure.

EXERCISES

3.1 Let $k[t]$ be the polynomial ring. Let $P : k[t] \to k[t]$ be the k-linear map $P(f) = df/dt$. Let $Q : k[t] \to k[t]$ be the k-linear map $Q(f) = tf$, the product of f with t.
  (a) Show that $PQ - QP$ is the identity operator. Thus, although the equation $xy - yx = 1$ has no solutions in $M_n(k)$, for any n, when $\operatorname{char}(k) = 0$ there are solutions in the ring of linear operators on an *infinite* dimensional vector space; for example, P and Q.
  (b) Show that the only subspaces of $k[t]$ which are stable under the action of P and Q are 0 and $k[t]$.

3.2 Suppose that $\operatorname{char}(k) = p > 0$. Show that there exist $p \times p$ matrices such that $xy - yx = 1$. [*Hint:* the operators P and Q of the previous exercise still make sense when $\operatorname{char}(k) = p$, but $k[t]$ will now have proper subspaces which are stable under P and Q. Look for such subspaces, and consider the action of P and Q on the quotient, which will be finite dimensional.]

3.3 Suppose that $\operatorname{char}(k) \neq 2$. Show that $x = y = z = 0$ is the only solution to the system of equations

$$yz - zy = -y,\ zx - xz = x,\ xy - yx = z.$$

[*Hint:* rewrite the 3 equations as $zy = \dots, zx = \dots, yx = \dots$, and compute zyx in two different ways, namely as $(zy)x$ and $z(yx)$.] Notice that your proof shows that $x = y = z = 0$ is the only solution in *any ring which contains* $1/2$.

3.4 Consider the system of equations $he - eh = 2e, ef - fe = h, hf - fh = -2f$.
  (a) Show that the matrices in that example really are a solution to the system of equations.
  (b) That solution is presented with respect to a particular ordered basis for k^d, say $\{v_1, \dots, v_d\}$. Express the action of the ring $k\langle e, f, h\rangle$ on k^d by expressing each of $e.v_i$, $f.v_i$, $h.v_i$ as a linear combination of the v_j's.
  (c) Can you find another 1×1 solution? What about other 2×2 and 3×3 solutions?

3.5 Let $A = k[x, y]$ be the commutative polynomial ring and, for each $n \geq 0$, let A_n be the subspace with basis $\{x^i y^j \mid i+j = n\}$. The linear operators $e = x\partial/\partial y, f = y\partial/\partial x, h = x\partial/\partial x - y\partial/\partial y$ act on A in the usual way. Each A_n is an invariant subspace. Use the given basis for A_n to represent each of e, f, h as a matrix.

3.6 Find some solutions to the equation $xy + yx = 1$.

4. Modules and solutions

Consider a system of non-commutative polynomial equations $\mathcal{F}$ in a set of unknowns X; that is, $\mathcal{F} \subset k\langle X\rangle$. Associate to this the k-algebra

$$R = k\langle X\rangle/(\mathcal{F})$$

where $(\mathcal{F})$ denotes the two-sided ideal generated by the elements of $\mathcal{F}$. Let $\pi : k\langle X\rangle \to R$ denote the quotient map. We will show there is a bijection between solutions in $M_d(k)$ to $\mathcal{F} = 0$ and d-dimensional R-modules.

THEOREM 4.1. *There is a bijection between the set of all k-algebra homomorphisms $\rho : R \to M_d(k)$ and the set of solutions in $M_d(k)$ to $\mathcal{F} = 0$. The bijection is given by $\rho \leftrightarrow \rho \circ \pi|_X$.*

PROOF. If ρ is such an algebra map then $\Psi := \rho \circ \pi$ satisfies $\Psi(f) = 0$ for all $f \in \mathcal{F}$, simply because $\mathcal{F} \subset \ker \pi$. Hence $\Psi|_X$ gives a solution. Conversely, if $\psi : X \to M_d(k)$ is a solution, its extension to $k\langle X \rangle$, Ψ say, vanishes on $\mathcal{F}$. Therefore $\Psi = \rho \circ \pi$ for some ρ. □

We will often write x for the image in R of $x \in X$.

THEOREM 4.2. *Let $\mathcal{F} \subset k\langle X \rangle$ and define $R = k\langle X \rangle/(\mathcal{F})$. There is a bijection between solutions $\psi : X \to M_d(k)$ to the system of equations $\mathcal{F} = 0$, and d-dimensional left R-modules. The R-module corresponding to ψ is k^d with action given by $r.v = \rho(r)(v)$ for $r \in R$ and $v \in k^d$, where $\rho : R \to M_d(k)$ is the unique k-algebra homorphism such that $\rho|_X = \psi$.*

PROOF. Fix a basis for k^d, and use it to identify $M_d(k)$ with $\operatorname{Hom}_k(k^d, k^d)$.

Given a solution ψ, the algebra map $\rho : R \to M_d(k) = \operatorname{Hom}_k(k^d, k^d)$, associated to ψ as in Theorem 4.1 gives k^d a left R-module structure. Conversely, a left R-module structure on k^d is the same thing as a ring homomorphism $\rho : R \to \operatorname{Hom}_k(k^d, k^d)$. Therefore, if $\pi : k\langle X \rangle \to R$ is the quotient map, applying Theorem 4.1 to the composition $\rho \circ \pi : k\langle X \rangle \to M_d(k)$, yields a solution to $\mathcal{F} = 0$. □

Thus the problem of understanding the solutions to $\mathcal{F} = 0$ is equivalent to the problem of understanding the finite dimensional $k\langle X \rangle/(\mathcal{F})$-modules. The next result shows that equivalent solutions, in the sense of Definition 3.3, correspond to isomorphic modules, so the problem stated at the beginning of Section 3 may be rephrased as that of understanding finite dimensional $k\langle X \rangle/(\mathcal{F})$-modules up to isomorphism.

PROPOSITION 4.3. *Two solutions $\psi_i : X \to M_d(k)\,(i = 1, 2)$ to $\mathcal{F} = 0$ are equivalent if and only if the corresponding R-modules are isomorphic.*

PROOF. Let $\rho_i : R \to M_d(k)$ be the k-algebra maps corresponding to ψ_i, and let V_i be the corresponding R-modules.

The modules are isomorphic if and only if there is a k-linear isomorphism $\phi : V_1 \to V_2$ such that $\phi(x.v) = x.\phi(v)$ for all $x \in R$ and all $v \in V_1$. Since X generates R, it suffices to check this equality for all $x \in X$. Now $V_1 = V_2 = k^d$, so the linear isomorphism ϕ corresponds, after a choice of basis, to a $g \in GL_d(k)$ such that $\phi(v) = g.v$ for all $v \in k^d$. Thus

$$\begin{aligned}
\phi(x.v) = x.\phi(v) \quad &\Leftrightarrow \quad g\rho_1(x)(v) = \rho_2(x)g.v \text{ for all } v \in V_1 \text{ and all } x \in X,\\
&\Leftrightarrow \quad g\rho_1(x) = \rho_2(x)g \qquad \text{for all } x \in X,\\
&\Leftrightarrow \quad g\rho_1(x)g^{-1} = \rho_2(x) \quad \text{for all } x \in X,\\
&\Leftrightarrow \quad g\psi_1(x)g^{-1} = \psi_2(x) \quad \text{for all } x \in X.
\end{aligned}$$

□

It follows from Theorems 4.1 and 4.2, and the discussion at the end of the previous section, that the set of d-dimensional modules over a finitely generated k-algebra may be given the structure of an affine algebraic variety on which $GL_d(k)$ acts. Moreover, by Proposition 4.3 two modules are isomorphic if and only if they lie in the same orbit. That is, the orbit space classifies isomorphism classes of modules.

EXERCISES

4.1 Let k be an algebraically closed field. Fix $\lambda \in k\backslash\{0,1\}$ and consider the ring $R = k\langle x,y\rangle/(yx - \lambda xy)$ corresponding to the equation $yx = \lambda xy$. Recall that R has basis $\{x^i y^j \mid i,j \geq 0\}$ (Exercise 2.7).

(a) Let M be a left R-module. Show that $\{m \in M \mid xm = 0\}$ is a submodule.

(b) Show that, if λ is not a root of unity, every finite dimensional simple R-module is 1-dimensional. [*Hint:* consider determinants.]

(c) Assume that λ is not a root of unity, and find all 1-dimensional R-modules.

(d) Assume that λ is not a root of unity. Show that if M is a d-dimensional R-module then $(xy)^d M = 0$.

(e) Fix $\alpha \in k$, and define the R-module $M(\alpha)$ as follows: as a k-vector space $M = k[t]$ is the polynomial ring, and the R-module action on $f = f(t) \in M$ is defined by

$$x.f = \alpha t f \quad \text{and} \quad y.f = tf(\lambda t).$$

Show that the ideals (t^r) are R-submodules.

(f) Show that $M(\alpha) \cong R/R(x - \alpha y)$.

4.2 Continue the previous exercise, but assume that λ is a primitive n^{th} root of 1.

(a) Show that $M(\alpha)$ (see Exercise 1) is isomorphic to its submodule (t^n) via the R-module map defined by $\varphi(f) = t^n f$.

(b) Since $\varphi \in \operatorname{End}_R M(\alpha)$ so too is $1 - \beta\varphi \in \operatorname{End}_R M(\alpha)$ for all $\beta \in k\backslash\{0\}$. The image of this is a submodule, so

$$N(\alpha,\beta) := M(\alpha)/(1-\beta\varphi)(M(\alpha))$$

is an n-dimensional R-module. In particular, the actions of x and y on this yield solutions in $M_n(k)$ to $yx = \lambda xy$. What are the matrices for x and y with respect to the basis for $N(\alpha,\beta)$ given by the images of $1, t, \dots, t^{n-1}$?

(c) Show that $N(\alpha,\beta)$ is simple if and only if $\alpha \neq 0$.

4.3 Continue the previous exercise. Let $Z(R)$ denote the center of R.

(a) Show that the subalgebra $k[x^n, y^n]$ generated by x^n and y^n lies in $Z(R)$.

(b) Show that R is a free module over $k[x^n, y^n]$ of rank n^2.

(c) Show that $Z(R) = k[x^n, y^n]$.

(d) Show that if $\mathfrak{m}$ is a maximal ideal of $Z(R)$, then $\dim_k(R/\mathfrak{m}R) = n^2$.

4.4 Continue the previous exercise.

(a) If M is a finite dimensional simple R-module, show that $\operatorname{End}_R M = k$, and hence that M is annihilated by a maximal ideal of $Z(R)$.

(b) Show that every finite dimensional simple R-module is of dimension $\leq n$. [*Hint:* If M is a simple R-module of dimension d, then $R/AnnM \cong M_d(k)$ (see Appendix A,????).]

(c) What is the maximal ideal of the center which annihilates $N(\alpha,\beta)$?

(d) Show that the 1-dimensional modules together with the modules $N(\alpha,\beta)$ with $\alpha\beta \neq 0$ are all the finite dimensional simple R-modules.

4.5 Let $\psi_i : X \to M_{d_i}(k)$ $(i = 1,2)$ be two solutions to the system of equations $\mathcal{F} \subset k\langle X\rangle$. The **direct sum** of ψ_1 and ψ_2 is $\psi_1 \oplus \psi_2 : k\langle X\rangle \to M_{d_1+d_2}(k)$ defined by

$$(\psi_1 \oplus \psi_2)(x) = \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \psi_2(x) \end{pmatrix}$$

for all $x \in X$. Show that $\psi_1 \oplus \psi_2$ is also a solution to $\mathcal{F} = 0$, and that the corresponding $k\langle X\rangle/(\mathcal{F})$-module is the direct sum of the modules corresponding to ψ_1 and ψ_2.

4.6 Show that the set of solutions to the system of equations $\mathcal{F}$ is the same as the set of solutions to the system of equations in the ideal $(\mathcal{F})$.

5. Finitely presented algebras

A system of non-commutative polynomial equations consisting of only a finite number of equations involves only a finite number of unknowns, so the algebra associated to it will be finitely presented. Since finite systems of equations are the most common we tend to concentrate on finitely presented algebras.

Definition 5.1. A k-algebra, R say, is finitely generated as a k-algebra, if there exist a finite number of elements $a_1, \dots, a_n \in R$ such that the smallest subring of R containing k and $a_1, \dots, a_n$ is R itself.

Clearly, a k-algebra R is finitely generated if and only if it is isomorphic to a quotient of a free algebra on a finite set.

Definition 5.2. A k-algebra R is finitely presented if there exists a surjective k-algebra map $f : k\langle x_1, \dots, x_n\rangle \to R$ with the property that $\ker f$ is finitely generated as a two-sided ideal. The elements of $\ker f$ are called relations of R. An algebra, R say, which is defined in the form $k\langle x_1, \dots, x_n\rangle/(f_1, \dots, f_r)$ is said to be given by generators and relations; the f_i are called defining relations, and this is said to be a presentation of R.

There is no uniqueness to a presentation of an algebra. For example, the polynomial ring $k[x]$ may presented in this way, and also as $k\langle x, y\rangle/(y)$. The two-sided ideal (y) in $k\langle x, y\rangle$ is not finitely generated as a left ideal (Exercise 2), so it is important to note the requirement in the definition that the defining ideal be finitely generated as a *two-sided ideal.*

Generally speaking it is not known how to use the hypothesis of finite presentation effectively. More precisely, the structural consequences of having only a finite number of defining equations for an algebra are unknown. This is despite the fact that there exist several examples illustrating strange phenomena which can occur for infinitely presented algebras, but are not known not to occur for finitely presented algebras.

In general it is a difficult problem to construct a basis for an algebra given by generators and relations: for any particular algebra one tends to rely on a trick suited to that situation, but there are no methods which seem to be universally effective. However, see the section on the Diamond Lemma in Chapter ???

CHAPTER 2

Noetherian rings

In this chapter we develop the standard localization methods for noetherian rings, the key result being Goldie's Theorem, which states that a semiprime noetherian ring has a ring of fractions which is semisimple artinian.

1. Prime and semiprime ideals

Definition 1.1. An ideal I in a ring R is

- prime if every product of non-zero ideals in R/I is non-zero;
- semiprime if it is an intersection of prime ideals.

A ring is

- prime if 0 is a prime ideal.
- semiprime if 0 is a semiprime ideal.

PROPOSITION 1.2. *An ideal, I say, is semiprime if and only if, whenever J is an ideal such that $J^2 \subset I$, then $J \subset I$.*

PROPOSITION 1.3. [**167**, Proposition 3.9] *Let R be a right noetherian algebra, and $M \in \mathsf{mod}R$. If M is critical, then*

1. $\operatorname{Ann} M$ *is a prime ideal;*
2. *all non-zero submodules of M have the same annihilator.*

PROOF. Using the right noetherian hypothesis, let $\mathfrak{p}$ be maximal amongst the annihilators of non-zero submodules of M; say, $\mathfrak{p} = \operatorname{Ann}(N)$. If I and J are ideals such that $IJ \subset \mathfrak{p}$, then $IJN = 0$. If $JN = 0$ then $J \subset \mathfrak{p}$; if $JN \neq 0$ then $I \subset \mathfrak{p}$, by maximality of $\mathfrak{p}$. Thus $\mathfrak{p}$ is prime.

The map $\mathfrak{p} \times M/N \to \mathfrak{p}M$, defined by $(x, [m + N]) \mapsto xm$, is well-defined and bi-additive so there is a surjective map $\mathfrak{p} \otimes_A (M/N) \to \mathfrak{p}M$. Since A is right noetherian, $\mathfrak{p}$ is a quotient (as a right module) of a finite direct sum of shifts of A. Hence $\mathfrak{p} \otimes_A (M/N)$, and therefore $\mathfrak{p}M$, is a quotient of a finite direct sum of shifts of M/N. Hence, by Lemma 2.4,

$$\operatorname{GKdim}(\mathfrak{p}M) \leq \operatorname{GKdim}(M/N) < \operatorname{GKdim} M.$$

Therefore, since M is critical, $\mathfrak{p}M = 0$. Thus $\mathfrak{p} = \operatorname{Ann} M$, which proves (1).

Finally, if N is any non-zero submodule of M then $\mathfrak{p} \subset \operatorname{Ann}(N)$, but the argument above shows that $\operatorname{Ann}(N)$ can be no larger than $\mathfrak{p}$, whence (2). □

EXAMPLE 1.4. The noetherian hypothesis in Proposition 1.3 is essential. To see this, consider the algebra $A = k[x, y]$ with defining relations $y^2 = yx = 0$, which is right but not left noetherian (Exercise 8.3). Then A has basis $\{x^i, x^i y \mid i \geq 0\}$. It is easy to show that A/Ax is 1-critical and its left annihilator is zero. However, A is not a prime ring since $yAx = 0$.

LEMMA 1.5. *Let R be a semiprime, left noetherian ring and N a left (or right) ideal. If every element of N is nilpotent, then $N = 0$.*

Definition 1.6. An element x, in a ring R, is

- left regular if $ax \neq 0$ whenever $0 \neq a \in R$;
- right regular if $xa \neq 0$ whenever $0 \neq a \in R$;
- regular if it is both left and right regular.

Definition 1.7. Let x be an element of a ring R. The

- left annihilator of x is $\ell(x) := \{a \in R \mid ax = 0\}$;
- right annihilator of x is $r(x) := \{a \in R \mid xa = 0\}$.

The left annihilator of any subset of R is a left ideal. The left annihilator of a left ideal is a two-sided ideal.

EXERCISES

1.1 Show that a prime ring with no nilpotent elements is a domain.

2. Rings of fractions

Localization is an essential tool for non-commutative rings, but the general theory is more delicate than in the commutative case. The delicacy arises when one tries to rewrite a fraction ax^{-1} with the inverse on the left.

We begin with general considerations. It is always possible to invert elements, simply by adopting a universal construction but, as with other universal objects, it may be difficult to say much about the object obtained. However, in the noetherian case, which is the case of interest to us, there is a good and effective theory.

Definition 2.1. Let $\mathcal{S}$ be a subset of the ring R. A ring homomorphism $f : R \to R'$ is $\mathcal{S}$-inverting if $f(s)$ is a unit in R' for all $s \in \mathcal{S}$.

PROPOSITION 2.2. *Given a subset $\mathcal{S}$ of a ring R, there is a ring $R_{\mathcal{S}}$ and an $\mathcal{S}$-inverting ring homomorphism $\lambda : R \to R_{\mathcal{S}}$ which is universal in the following sense:*

- *if $f : R \to R'$ is $\mathcal{S}$-inverting, there is a unique $f' : R_{\mathcal{S}} \to R'$ such that $f = f' \circ \lambda$.*

PROOF. Let $\mathbb{Z}\langle s' \mid s \in \mathcal{S}\rangle$ denote the free $\mathbb{Z}$-algebra on the elements s' indexed by $\mathcal{S}$, and define

$$R_{\mathcal{S}} := \frac{R \coprod \mathbb{Z}\langle s' \mid s \in \mathcal{S}\rangle}{(ss' = s's = 1)} \tag{2-1}$$

to be the quotient of the free coproduct modulo the ideal generated by the elements $\{ss' - 1, s's - 1 \mid s \in \mathcal{S}\}$. The obvious ring homomorphism $\lambda : R \to R_{\mathcal{S}}$ is $\mathcal{S}$-inverting.

If $f : R \to R'$ is $\mathcal{S}$-inverting, we may define $f' : R_{\mathcal{S}} \to R'$ by $f'(\lambda(r)) := f(r)$ for $r \in R$, and $f'(s') = f(s)^{-1}$ for $s \in \mathcal{S}$; this map is defined on the coproduct, and vanishes on elements $ss' - 1$ and $s's - 1$, so gives a well-defined map on $R_{\mathcal{S}}$ satisfying $f' \circ \lambda = f$. $\square$

The ring $R_{\mathcal{S}}$ may be the zero ring (for example, when $0 \in \mathcal{S}$), and in general it might be difficult to probe the structure of $R_{\mathcal{S}}$. Example 2.9 illustrates the malicious behavior of $R_{\mathcal{S}}$ by exhibiting a domain R for which $R_{\mathcal{S}}$ is *not* a division ring when $\mathcal{S} = R \backslash \{0\}$.

A class of domains having a good localization theory are the Ore domains.

Definition 2.3. A subset $\mathcal{S} \subset R$ is a right denominator set if

- $\mathcal{S}$ is multiplicatively closed;
- $\mathcal{S}$ satisfies the right Ore condition: if $x \in R$ and $s \in \mathcal{S}$, then $sR \cap x\mathcal{S} \neq \emptyset$;
- if $x \in R$ and $s \in \mathcal{S}$ satisfy $sx = 0$, then $xt = 0$ for some $t \in \mathcal{S}$.

There is a similar definition with 'left' in place of 'right'.

The non-zero elements in the free algebra $R = k\langle x, y\rangle$ do not satisfy the right Ore condition because $xR \cap yR = 0$.

Definition 2.4. A right Ore domain is a domain whose non-zero elements form a right denominator set.

PROPOSITION 2.5. *If $\mathcal{S}$ is a right denominator set, then $R_{\mathcal{S}}$ may be constructed as follows:*

- *define an equivalence relation on $R \times \mathcal{S}$ by $(x, s) \sim (y, t)$ if there exist $u, v \in R$ such that $xu = yv$ and $su = tv$;*
- *there is a ring structure on the equivalence classes such that the natural map $R \to R \times \mathcal{S}/\sim$ sending r to $(r, 1)$ makes $R \times \mathcal{S}/\sim \cong R_{\mathcal{S}}$;*
- *$\lambda(s)^{-1} = (1, s)$ for $s \in \mathcal{S}$.*

Thus elements of $R_{\mathcal{S}}$ are of the form $\lambda(x)\lambda(s)^{-1}$ with $x \in R$ and $s \in \mathcal{S}$. Furthermore

$$\ker(\lambda) = \{x \in R \mid xs = 0 \text{ for some } s \in \mathcal{S}\}.$$

PROOF. $\square$

PROPOSITION 2.6. *A right noetherian domain R is a right Ore domain, and if $\mathcal{S} = R \backslash \{0\}$, then $R_{\mathcal{S}}$ is a division ring, and the map $R \to R_{\mathcal{S}}$ is injective.*

PROOF. If R is not a right Ore domain, then $xR \cap yR = 0$ for some non-zero x and y. Therefore

$$xR \oplus yR \supset xR \oplus yxR \oplus y^2R \supset xR \oplus yxR \oplus y^2xR \oplus y^3R \supset \cdots .$$

This gives an infinite ascending chain of ideals

$$xR \subset xR \oplus yxR \subset xR \oplus yxR \oplus y^2xR \subset \cdots$$

thus contradicting the noetherian hypothesis. Thus we conclude that $xR \cap yR \neq 0$, whence R is a right Ore domain.

It follows from Proposition 2.5 that $R \to R_{\mathcal{S}}$ is injective. Since non-zero elements of $R_{\mathcal{S}}$ are of the form xs^{-1}, they are units (the inverse is sx^{-1}). $\square$

The proof shows that a domain either contains an infinite direct sum of non-zero right ideals or is a right Ore domain. The next result gives a further dichotomy.

PROPOSITION 2.7. *A domain R with center k is either a right Ore domain or contains a free subalgebra $k\langle x, y\rangle$.*

PROOF. Suppose R is not a right Ore domain; say $xR \cap yR = 0$ with $0 \neq x, y \in R$. We will show that the k-subalgebra generated by x and y is free. If not, we may choose a relation of minimal degree in R, say

$$\alpha + xa + yb = 0,$$

with $\alpha \in k$ and at least one of $a, b \in k[x,y]$ non-zero. If $b \neq 0$, then $ybx = -x(\alpha + ax)$ is a non-zero element of $xR \cap yR$, contradicting the hypothesis. If it is a which is non-zero, there is a similar contradiction, so we conclude there can be no such relation, whence $k[x,y]$ is free. □

Definition 2.8. If $\mathcal{S}$ is a subset of R, we call $R_{\mathcal{S}}$ a right ring of fractions, or right localization, of R with respect to $\mathcal{S}$ if

- each element of $R_{\mathcal{S}}$ is of the form $\lambda(x)\lambda(s)^{-1}$ for some $x \in R$ and $s \in \mathcal{S}$;
- $\ker(\lambda) = \{x \in R \mid xs = 0 \text{ for some } s \in \mathcal{S}\}$.

There is a similar version on the left.

If $\mathcal{S}$ is the set of all regular elements in R, and $R_{\mathcal{S}}$ is a right ring of fractions, we call it *the* (right) ring of fractions of R and denote it by $\operatorname{Fract} R$.

It follows from the definition that the right ring of fractions of a domain is a division ring, provided it exists.

EXAMPLE 2.9. We will show that the free algebra $R = k\langle x, y\rangle$ does not have a ring of fractions by proving that $R_{\mathcal{S}}$ is not a division ring, where $\mathcal{S} = R\backslash\{0\}$. This will be done by exhibiting injective $\mathcal{S}$-inverting maps $f_n : R \to D_n$ into non-isomorphic division rings, such that $f_n(R)$ generates D_n as a division ring; it follows from this that the maps $f'_n : R_{\mathcal{S}} \to D_n$ satisfying $f_n = f'_n \circ \lambda$ are surjective and hence isomorphisms if $R_{\mathcal{S}}$ is a division ring.

Fix an integer $n > 0$, and let $\sigma : k[t] \to k[t]$ be the endomorphism defined by $\sigma(t) = t^n$. Let $k[t][x;\sigma]$ be the corresponding Ore extension, and observe that the right ideals generated by x and tx have zero intersection. Thus, by the proof of Proposition 2.7, the subalgebra $k[x, tx]$ is free; define $f_n : k[x,y] \to k[t][x;\sigma]$ by $f_n(x) = x$ and $f_n(y) = tx$. Now $k[t][x;\sigma]$ embeds in $k(t)[x;\sigma]$ which is a left Ore domain, being a principal left ideal domain (Proposition 16.7.3), so embeds in the division ring of $D_n := \operatorname{Fract} k(t)[x;\sigma]$. This gives the map $f_n : R \to D_n$. It is clear that D_n is generated as a division ring by x and tx.

$\boxed{\text{Paul}}$ why are the D_n non-isomorphic?

3. Goldie's theorems

A weak version of Goldie's Theorem says that a semiprime noetherian ring has a semisimple artinian ring of fractions. Here's a simple illustration in the commutative context.

EXAMPLE 3.1. Let $A = k[x,y]$ be the commutative ring with defining relation $xy = 0$. Then A is semiprime, the intersection of the prime ideals (x) and (y) being zero. We show directly that $\operatorname{Fract} A \cong k(x) \oplus k(y)$.

Let $k[X,Y]$ be the polynomial ring. We identify A with the subalgebra of $M_2(k[X,Y])$ generated by $x = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 0 & Y \end{smallmatrix}\right)$. Thus, A consists of those elements in $\left(\begin{smallmatrix} k[X] & 0 \\ 0 & k[Y] \end{smallmatrix}\right)$ having the same constant term in the 11- and 22-positions. In fact, A is of codimension 1 in this ring, a complement being $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The regular elements of A are those of rank 2, so we can view $\operatorname{Fract} A$ as a subalgebra of $M_2(k(X,Y))$. Now,

$$(x+y)^{-1} = \begin{pmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix}$$

so

$$\frac{x}{x+y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \frac{y}{x+y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\operatorname{Fract} A = \begin{pmatrix} k(X) & 0 \\ 0 & k(Y) \end{pmatrix} \cong k(X) \oplus k(Y).$$

Definition 3.2. A ring is left Goldie if it contains no infinite direct sum of non-zero left ideals, and satisfies the ascending chain condition on left annihilators.

Definition 3.3. A module is uniform if any pair of non-zero submodules has non-zero intersection. The uniform dimension of a module M is the largest integer n such that M contains a direct sum of n uniform submodules. If no such n exists, the uniform dimension of M is infinite.

A submodule $N \subset M$ is essential in M if $N \cap N' \neq 0$ for all non-zero submodules N'.

A noetherian module, M say, contains a uniform submodule: if N is a submodule of M, maximal subject to *not* being essential, and L a submodule of M such that $L \cap N = 0$, then L must be uniform; it follows that M contains an essential submodule which is a direct sum of uniform modules.

LEMMA 3.4. *A left Goldie ring has finite (left) uniform dimension.*

PROOF. □

PROPOSITION 3.5. *Let x be an element of a left noetherian (?? Goldie) ring, R say. There exists an integer n such that*

1. $\ell(x) \cap Rx^n = 0$;
2. $\ell(x) \oplus Rx^n$ *is an essential left ideal.*

Definition 3.6. A multiplicatively closed subset, $\mathcal{S}$ say, of a ring R

- satisfies the left Ore condition if, for all $s \in \mathcal{S}$ and all $a \in R$, $\mathcal{S}a \cap Rs \neq \emptyset$;
- is a left denominator set if it satisfies the left Ore condition and, whenever $s \in \mathcal{S}$ and $x \in R$ satisfy $xs = 0$, there exists $t \in \mathcal{S}$ such that $tx = 0$.

PROPOSITION 3.7. *The set of regular elements in a semi-prime left Goldie ring is a left denominator set.*

PROPOSITION 3.8. *A left ideal in a semi-prime left Goldie ring is essential if and only it contains a left regular element.*

PROPOSITION 3.9. *A left regular element in a semiprime left Goldie ring is also right regular.*

THEOREM 3.10. *[Goldie's Theorem] A ring is a left order in a semi-simple (resp., simple) artinian ring if and only if it is semi-prime (resp., prime) and left Goldie.*

PROPOSITION 3.11. *If R is a semi-prime left and right Goldie ring then the left ring of fractions of R is also a right ring of fractions.*

The semisimple artinian ring in Goldie's Theorem is a ring of fractions of R, and is the unique such up to isomorphism—we denote it by Fract R and call it the ring of fractions of R.

Definition 3.12. Let R be a semi-prime left Goldie ring, and $M \in \mathsf{Mod}R$.

- An element $m \in M$ is torsion if $cm = 0$ for some regular $c \in R$.
- The torsion submodule of M is

$$\tau M := \{m \in M \mid m \text{ is torsion}\};$$

  that τM *is* a submodule follows from Propositions 3.9 and 3.8, and the fact that the set of regular elements satisfies the left Ore condition.
- If $M = \tau M$, we call M a torsion module.

Warning: The word 'torsion' is used in Chapter 19 to mean something else; when we wish to distinguish the two notions we will refer to the present one as 'torsion in the classical sense'.

Since regular elements in R are units in $Q = \operatorname{Fract} R$, Q is torsion-free and Q/R is torsion.

PROPOSITION 3.13. *Let R be a semi-prime left Goldie ring. The torsion modules form a dense subcategory of* $\mathsf{Mod}R$.

PROOF. This follows from the fact that Fract R is flat as a right R-module, but here's another proof. First submodules and quotient modules of a torsion module are again torsion. Conversely, let $N \subset M$, and suppose that N and M/N are torsion. If $m \in M$, then $dm \in N$ for some regular element d and, since N is torsion, $cdm = 0$ for some regular c; but cd is regular, so m is torsion. Thus M is torsion, as required. □

LEMMA 3.14. *Let R be a semi-prime left Goldie ring.*

1. *If N is an essential submodule of $M \in \mathsf{mod}R$, then M/N is torsion.*
2. *An essential extension of a torsion module is torsion.*

PROOF. (1) Let $m \in M$; we must show that the left ideal $J := \{x \in R \mid xm \in N\}$ is essential. Let I be a left ideal of R. If $Im \neq 0$, then $Im \cap N \neq 0$, so $I \cap J \neq 0$. On the other hand, if $Im = 0$, then $I \subset J$. Thus J is essential.

(2) This follows from (1) and the density of the subcategory of torsion modules. □

Localisation of modules $Q \otimes_R M$.

There is an exact sequence $0 \to \tau M \to M \xrightarrow{f} Q \otimes_R M \to C \to 0$, with C torsion, and f essential. In particular, M is torsion if and only if $Q \otimes_R M = 0$.

$Q \otimes_R Q \cong Q$.

Q is flat as a left and right R-module.

Q is the injective envelope of R.

LEMMA 3.15. *If S is a left localization of R, then S is a flat right R-module, and $S \otimes_R S \cong S$ as an S-S-bimodule.*

LEMMA 3.16. *Let S be a left localization of the ring R. If $M \in \mathsf{mod}(S)$, then $M \cong S \otimes_R M'$ for some $M' \in \mathsf{mod}R$.*

PROOF. Let M' be a finitely generated R-submodule of M such that $M = SM'$. Then

$$S \otimes_R M \cong S \otimes_R (S \otimes_S M) \cong (S \otimes_R S) \otimes_S M \cong M.$$

Because S_R is flat, there is an exact sequence $0 \to S \otimes_R M' \to S \otimes_R M \to S \otimes_R (M/M') \to 0$. However, if $m \in M$, then $m = x^{-1}ym'$ for some $x \in S$, $y \in R$ and $m' \in M'$, whence $x \otimes \bar{m} = 0$ in $S \otimes (M/M')$; but x is a unit in S, so $1 \otimes \bar{m} = 0$; thus $S \otimes_R (M/M') = 0$. Therefore $S \otimes_R M' \cong S \otimes M \cong M$. □

4. Consequences of Goldie's Theorems

LEMMA 4.1. *Let R be prime noetherian, and let $Q = \operatorname{Fract} R$. Then*

1. *if L and N are non-zero left and right R-submodules of Q respectively, then $L \cap N \neq 0$;*
2. *a non-zero R-R-subbimodule of Q is essential as both a left a right submodule.*

PROOF. (1) Since R is essential as both a left, and as a right, submodule of Q, $L \cap R \neq 0$ and $N \cap R \neq 0$. Let $0 \neq x \in N \cap R$ and $0 \neq y \in L \cap R$. Since R is prime $xRy \neq 0$; but $xRy \subset L \cap N$.

(2) This follows at once from (1). □

EXERCISES

4.1 Let R be semiprime noetherian, and Q its ring of fractions. Show that Q is the injective envelope of R. Show that Q/R is injective if $\operatorname{gldim} R = 1$.

5. Reduced rank

Definition 5.1. Let R be a semi-prime left Goldie ring and set $Q = \operatorname{Fract} R$. The reduced rank of a left R-module M is

$$\rho(M) := \operatorname{length}(Q \otimes_R M),$$

the length of the induced module over the semi-simple artinian ring Q.

We extend the definition to an arbitary left noetherian ring R as follows. If $M \in \mathsf{Mod}R$ and N denotes the nilpotent radical of R, let $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a chain of submodules such that each M_i/M_{i+1} is annihilated by N. The reduced rank of M is

$$\rho(M) := \sum_{i=0}^{n-1} \rho_{R/N}\left(\frac{M_i}{M_{i+1}}\right).$$

This is well-defined by [**73**, Lemma 10.1].

REMARK 5.2. We do not require that M be finitely generated, so ρ may take the value ∞; notice that M need not be finitely generated in order that $\rho(M)$ be finite—for example, $\rho(Q) = n$ if $Q \cong M_n(D)$, where D is a division ring.

A function $f : \mathsf{Mod}R \to G$, taking values in an abelian group, is additive if, whenever $0 \to X \to Y \to Z \to 0$ is exact, $f(Y) = f(X) + f(Z)$. If f is additive and $0 \to X_0 \to \cdots \to X_n \to 0$ is exact, then $\sum_{i=0}^{n}(-1)^i f(X_i) = 0$. Since $\operatorname{length}(-)$ is an additive function and Q_R is flat, ρ is an additive function.

If M is torsion-free, then $\rho(M) = 1$ if and only if M is uniform: to prove this, use the fact that M is essential in $Q \otimes_R M$. It follows that $\rho(M)$ equals the uniform dimension of M.

A module M is torsion if and only if $Q \otimes_R M = 0$, whence M is torsion if and only if $\rho(M) = 0$. Therefore, by Lemma 3.14, $\rho(N) = \rho(M)$ whenever N is an essential submodule of M.

LEMMA 5.3. *Let R be a semi-prime left noetherian (?? Goldie) ring. If N is a submodule of a finitely generated left R-module M, then there exists a submodule L of M such that $L \cap N = 0$ and $\rho(L) + \rho(N) = \rho(M)$.*

PROOF. Choose L maximal such that $L \cap N = 0$. Then $L \oplus N$ is essential in M, so $\rho(M) = \rho(L \oplus N) = \rho(L) + \rho(N)$ as required. □

LEMMA 5.4. *Let R be a right noetherian ring and M a finitely generated right R-module.*

1. *If $0 \to L \to M \to N \to 0$ is exact, then $\rho(M) = \rho(L) + \rho(N)$.*
2. *$\rho(M) = 0$ if and only if for each $m \in M$ there exists $c \in \mathcal{C}(N)$ such that $mc = 0$.*
3. *A prime $\mathfrak{p} \in \operatorname{Spec} R$ is minimal if and only if $\rho(R/\mathfrak{p}) > 0$.*
4. *If M is torsion-free and $0 \neq N \subset M$, then $\rho(M/N) < \rho(M)$.*

PROOF. (1) This follows from the fact that $Q \otimes_R -$ is exact, and length is additive on short exact sequences of Q-modules.
(2)
(3)
(4) Since N is torsion-free, $\rho(N) \neq 0$. □

LEMMA 5.5. *Let M and N be finitely generated torsion-free modules over a prime, left Goldie ring R. If $\rho(N) = \rho(M)$, then M and N contain isomorphic copies of each other.*

PROOF. Write $r = \rho(M)$. Thus $Q \otimes_R M = S_1 \oplus \cdots \oplus S_r$ is a direct sum of exactly r simple Q-modules, and M is an R-submodule of this. Let $m_1, \dots, m_\ell$ be R-module generators of M and, for each i, write

$$m_i = s_{i1} + \cdots + s_{ir},$$

where $s_{ij} \in S_j$. Define $M_j = \sum_{i=1}^{\ell} R s_{ij}$ for $j = 1, \dots, r$. Then $M \subset M_1 \oplus \cdots \oplus M_r$, and each M_j is a finitely generated torsion-free R-module. Also, N contains a direct sum of r uniform torsion-free submodules, say $N_1 \oplus \cdots \oplus N_r$. To prove the result it suffices to show that, for each j, there is an injective map $M_j \to N_j$. Hence we may, and will, assume that M and N are both uniform (and torsion-free).

Now, observe that a torsion-free uniform module, L say, is isomorphic to an R-submodule of Q: L embeds in $Q \otimes_R L$ which is a simple Q-module so is isomorphic to a left ideal of Q. So, we may assume that M and N are uniform submodules of Q. The two-sided ideal $NR \cap R$ is non-zero because R is essential in Q, so contains a regular element $c = \sum_i x_i r_i$ say, where $x_i \in N$. If $\ell(x_i) \cap M \neq 0$ for each i then, since M is uniform, there exists $0 \neq a \in M$ such that $ax_i = 0$ for all i; but then $ac = 0$, which is absurd, so we conclude that $\ell(x) \cap M = 0$ for some $x \in N$.Hence the map $f : M \to N$ defined by $f(a) = ax$ is an injective R-module map. □

EXERCISES

5.1 Let N be an essential submodule of a torsion-free module M over a left Goldie ring R. If $f : M \to M$ is an R-module map such that $f|_N = \mathbb{1}_N$, show that $f = \mathbb{1}_M$.

6. Prime ideals in noetherian rings

The first point to be noted is that in contrast to the commutative case there may be very few primes. At the extreme one has simple noetherian rings such as the Weyl algebras. One might also have prime noetherian rings having only one non-zero prime, but such a ring need not resemble a 1-dimensional commutative local ring. For example, consider the ring $U(_2)/(\Omega)$. Here the only non-zero ideal $\mathfrak{m}$ is idempotent, $\mathfrak{m}^2 = \mathfrak{m}$. Thus height one primes need not behave well either.

$\boxed{\text{Paul}}$ Discuss localizing at primes, links, when height one primes are reflexive, invertible etc.

PROPOSITION 6.1 (Chamarie). *A reflexive prime ideal in a prime noetherian maximal order is classically localizable.*

PROOF. [**85**, pages 210-211] Let R be the ring, and $\mathfrak{p}$ the reflexive prime ideal. □

7. Reflexive modules

Information about reflexive modules over a commutative ring may be found in [**58**, Chapter 3]; the proofs of the first few results below follow their proofs.

NOTATION . If M is an R-module, we write

$$M^* := \operatorname{Hom}_R(M, R).$$

The rule $M \mapsto M^*$ is implemented by the functor $\operatorname{Hom}_R(-, R)$ from left to right R-modules. The analogous functor from right to left modules is also denoted by *.

There is a natural left R-module homomorphism

$$\Phi : M \to M^{**}$$

sending $m \in M$ to the map $\Phi_m : M^* \to R$ defined by $\Phi_m(f) = f(m)$ for $f \in M^*$. More formally, there is a natural transformation from the identity functor to the composition **. Replacing M by M^*, there is a similar natural right R-module map

$$\Psi : M^* \to M^{***}$$

sending $f \in M^*$ to Ψ_f, the map defined by $\Psi_f(\alpha) = \alpha(f)$ for $\alpha \in M^{**}$.

Definition 7.1. A module M is reflexive if the natural map $\Phi : M \to M^{**}$ is an isomorphism.

If F is a finitely generated free module, it is clear that F^* is free of the same rank as F, whence F is reflexive. It follows that a finitely generated projective module is reflexive.

Since the functor $M \mapsto M^*$ is contravariant left exact, if R is left noetherian, and M is finitely generated, then M^* is finitely generated; to see this, write M as a quotient of a finitely generated free module, F say, and observe that M^* is a submodule of F^*. Without the noetherian hypothesis it is possible for M to be finitely generated but M^* not finitely generated. given a surjective map $R^n \to M$, there is an injective map $M^* \to (R^n)^* \cong R^n$, but we need to assume that R is noetherian in order to conclude that M^* is finitely generated. $\boxed{\text{Paul}}$ example

Proposition 7.2. *Let M be a noetherian module over a noetherian ring R. Then the natural map $\Phi : M \to M^{**}$ is injective if and only if M is a first syzygy (equivalently, M is a submodule of a free module).*

Proof. ($\Rightarrow$) There is a surjective map $\theta : F \to M^*$ for some finitely generated free module F. Since θ^* is injective so is the map $\theta^* \circ \Phi : M \to F^*$, whence M is a first syzygy.

($\Leftarrow$) By hypothesis, there is an injective map $\varphi : M \to F$ with F finitely generated free. Since the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\Phi"'] & F \arrow[d] \\ M^{**} \arrow[r, "\varphi^{**}"'] & F^{**} \end{tikzcd}$$

commutes, and $F \to F^{**}$ is an isomorphism, it follows that Φ is injective. $\square$

Corollary 7.3. *If M is a noetherian module over a noetherian ring R, the natural map $M^* \to M^{***}$ is injective.*

Proof. There is a surjective map $F \to M$ with F finitely generated free, and hence an injective map $M^* \to F^*$, so M^* is a first syzygy, whence the result. $\square$

Exercise 1 shows that the map $M^* \to M^{***}$ need not be not surjective.

Proposition 7.4. *Let M be a noetherian module over a noetherian ring R. Then*

1. *if M is reflexive it is also a second syzygy;*
2. *if R is ??????, the converse is true.*

Proof. (1) Since M is finitely generated, so is M^*. Let $\theta : F \to M^*$ be a surjection with F a finite rank free module. By hypothesis, the natural map $\Phi : M \to M^{**}$ is an isomorphism. Since θ^* is injective, so is the composition $\varphi = \theta^* \circ \Phi : M \to F^*$. Let $C = \operatorname{coker} \varphi$, We will see in a moment that φ^* is surjective, from which it follows that there are exact sequences $0 \to M \xrightarrow{\varphi} F^* \to C \to 0$, and $0 \to C^* \to F^{**} \xrightarrow{\varphi^*} M^* \to 0$. The vertical maps in the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[d] & M^* \arrow[d, "\Phi^*"] \\ F^{**} \arrow[r, "\theta^{**}"'] & M^{***} \end{tikzcd}$$

are isomorphisms, so θ^{**} is surjective; therefore $\varphi^* = \Phi^* \circ \theta^{**}$ is surjective. The first two vertical maps in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & F^* \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M^{**} \arrow[r] & F^{***} \arrow[r] & C^{**} & \end{tikzcd}$$

are isomorphisms, so the map $C \to C^{**}$ is injective. Hence, by Proposition 7.2, C is a first syzygy, and therefore M is a second syzygy.

(2) Suppose that M is a second syzygy. Let $0 \to M \to P \xrightarrow{\varphi} Q$ be exact with P and Q finitely generated projectives. Consider the exact sequence $0 \to \varphi(P)^* \to P^* \xrightarrow{\varphi^*} M^*$; set $V := \varphi^*(P^*) \subset M^*$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & \varphi(P) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^* & \longrightarrow & P^{**} & \longrightarrow & \varphi(P)^{**} & &
\end{array}$$

Since $\varphi(P)$ is a submodule of Q it is a first syzygy, so the last vertical map is injective. The map $P \to P^{**}$ is an isomorphism so, by the Snake Lemma, the map $M \to V^*$ is an isomorphism. Hence by Corollary 7.3, the map $M \to M^{**}$ is injective.

???? surjective??? □

Exercise 1 shows that some hypothesis is necessary in order that a second syzygy be reflexive.

LEMMA 7.5. *Let R be a semi-prime left and right Goldie ring. Then $\rho(M) = \rho(M^*)$ for all finitely presented left (and right) R-modules M.*

PROOF. The argument for right modules is similar to that for left modules, so we assume that M is a finitely presented left R-module.

Let $Q = \operatorname{Fract} R$. If S is a simple left Q-module, then $\operatorname{Hom}_Q(S, Q)$ is a simple right Q-module. Hence

$$\begin{aligned}
\rho(M) &= \operatorname{length}(Q \otimes_R M) \\
&= \operatorname{length}\big(\operatorname{Hom}_Q(Q \otimes_R M, Q)\big) \\
&= \operatorname{length}\big(\operatorname{Hom}_R(M, R) \otimes_R Q\big) \\
&= \rho(M^*),
\end{aligned}$$

as required. □

PROPOSITION 7.6. *Let R be a semi-prime left and right Goldie ring and $M \in \mathsf{Mod} R$.*

1. *M is torsion if and only if $M^* = 0$.*
2. *M^* is torsion-free.*
3. *The map $M \to M/\tau M$ induces an isomorphism $(M/\tau M)^* \to M^*$.*
4. *There is an exact sequence $0 \to \tau M \to M \xrightarrow{\Phi} M^{**} \to C \to 0$, and C is torsion.*
5. *The natural map $M^* \to M^{***}$ is an isomorphism.*

PROOF. (1) ($\Rightarrow$) If $m \in M$, then $cm = 0$ for some regular c whence, for $f \in M^*$,

$$0 = f(cm) = c.f(m),$$

so $f(m) = 0$ since c is regular; thus $f = 0$.

(2) If $f \in M^*$ is torsion, then $f.c = 0$ for some regular $c \in R$. Thus, for $m \in M$, we have $0 = (f.c)(m) = f(m).c$, whence $f(m) = 0$ because c is regular. Thus $f = 0$.

(3) Since * is left exact, there is an exact sequence $0 \to (M/\tau M)^* \to M^* \to (\tau M)^*$, the last term of which is zero by (1).

(4) Let $K = \ker(\Phi : M \to M^{**})$. There is an exact sequence $0 \to (M/K)^* \to M^* \to K^*$, the final map of which sends $f \in M^*$ to its restriction $f|_K$. By definition of K, this restriction is zero, so the image of M^* in K^* is zero. Thus $(M/K)^* \cong M^*$, so $\rho(M/K) = \rho(M) = \rho(M/\tau M)$. Since $(\tau M)^* = 0$, the restriction of $f \in M^*$ to τM is zero, whence $\tau M \subset K$. Hence M/K is a quotient of $M/\tau M$, and $\rho(M/K) = \rho(M/\tau M)$. We conclude that $K = \tau M$. Thus we have an exact sequence

$$0 \to \tau M \to M \xrightarrow{\Phi} M^{**} \to \operatorname{coker}(\Phi) \to 0. \tag{7-1}$$

Since $\rho(M) = \rho(M^{**})$, and ρ is additive, $\rho(\operatorname{coker}\Phi) = 0$ as required.

(1) ($\Leftarrow$) If $M^* = 0$, it follows from the exact sequence in (4) that $\tau M = M$.

(5) By (4) applied to M^* in place of M, $\ker(\Psi : M^* \to M^{***})$ is the torsion-submodule of M^*, which is zero by (2). That is, Ψ is injective.

Since $*$ sends torsion modules to zero, it follows from (7-1) that there is an injective map

$$\Theta = \Phi^* : M^{***} \to M^*,$$

given by $\Theta(\lambda)(m) = \lambda(\Phi(m))$, for $\lambda \in M^{***}$ and $m \in M$; that is, $\Theta(\lambda) = \lambda \circ \Phi$.

Finally we show that $\Theta : M^{***} \to M^*$ and $\Psi : M^* \to M^{***}$ are mutual inverses. First, if $f \in M^*$, then $(\Theta \circ \Psi)(f) = \Psi(f) \circ \Phi = \Psi_f \circ \Phi$, so

$$(\Theta \circ \Psi)(f)(m)(\Psi_f \circ \Phi)(m) = \Psi_f(\Phi_m) = \Phi_m(f) = f(m),$$

whence $\Theta \circ \Psi = \mathbb{1}_{M^*}$. This also says that Θ is surjective; but we already showed Θ is injective, so it is an isomorphism with inverse Ψ. $\square$

Since M^{**} is torsion-free and an extension of $M/\tau M$, if $\alpha \in M^{**}$, there is a regular $c \in R$ such that $0 \neq c\alpha \in M/\tau M$. Therefore M^{**} is an *essential* extension of $M/\tau M$ (Exercise 1 shows this may be false without the semiprime hypothesis); it is by no means the largest essential extension of $M/\tau M$ such that the quotient is torsion—$Q \otimes_R M$ is.

If R is semi-prime left Goldie, and I is a left R-submodule of $Q = \operatorname{Fract} R$, then we define

$$I^+ := \{q \in Q \mid Iq \subset R\}.$$

Clearly I^+ is a right R-submodule of Q. Define $\Psi : I^+ \to I^*$ by $q \mapsto \Psi_q$, where $\Psi_q(a) = aq$ for $a \in I$ and $q \in I^+$. This is a right R-module homomorphism.

Now suppose that I contains a unit, c say, of Q. Then $cq \neq 0$ for all q, so Ψ is injective. On the other hand, if $f \in I^*$, define $q := c^{-1}f(c)$. Now let $a \in I$. There exist regular elements $u, v \in R$, and $a' \in R$ such that $c = uv^{-1}$ and $a = a'v^{-1}$; now, by the Ore condition, there exist $b \in R$ and a regular $d \in R$ such that $da' = bu$, whence $da = bc$. It follows that

$$d.f(a) = f(da) = f(bc) = b.f(c) = bcq = daq,$$

whence $f(a) = aq = \Psi_q(a)$. Hence Ψ is also surjective.

Thus, if I contains a unit, the map $\Psi : I^+ \to I^*$ is an isomorphism; in this situation it is common to identify I^* with I^+. It is tedious, but easy, to check that this identification is compatible with the functoriality of *.

EXERCISES

7.1 Let $k[x,y]$ be the polynomial ring, define $R = k[x,y]/(x,y)^2$ and let $M = R/(x,y)$. Show that
(a) $M^* \cong M \oplus M$;
(b) no two $M^{**\cdots**}$ are isomorphic;
(c) the image of $\Phi : M \to M^{**}$ is not an essential submodule of M^{**};
(d) there is an exact sequence $0 \to M \oplus M \to R \xrightarrow{x} R \to R/(x) \to 0$, and hence that $M \oplus M$ is a second syzygy which is not a reflexive module.

8. Maximal orders

A prime maximal order is the non-commutative analogue of an integrally closed domain (Proposition 8.4). However, maximal orders are generally much less well-behaved than integrally closed domains. For example, it is not known if a prime noetherian ring is contained in a maximal order equivalent to it. In contrast to the uniqueness of the integral closure of a commutative domain, there may be several maximal orders containing a given prime noetherian ring (Example 8.10. A prime noetherian ring of finite global dimension need not be a maximal order (Example 8.10).

Definition 8.1. A ring of fractions is a ring in which every regular element is a unit.

In particular, an artinian ring is a ring of fractions.

Definition 8.2. Let Q be a ring of fractions and R a subring. We say that R is

- a right order in Q if $Q = \{ac^{-1} \mid a, c \in R\}$;
- a left order in Q if $Q = \{c^{-1}a \mid a, c \in R\}$;
- an order in Q if it is both a left and a right order in Q;
- a maximal order in Q if it is an order in Q and, whenever S is a ring such that $R \subset S \subset Q$ and $aSb \subset R$ for some units $a, b \in Q$, then $S = R$.

Two left orders in Q, say R and S, are equivalent if there are units $a, b, c, d \in Q$ such that $aSb \subset R$ and $cRd \subset S$. (It is trivial to see this is an equivalence relation.)

We are interested in orders in simple artinian rings. By Theorem ???, such an order is a prime Goldie ring.

Let R be prime noetherian, and set $Q = \operatorname{Fract} R$. We simply say that R is a maximal order to mean it is a maximal order in Q. In that case, if $R \subset S \subset Q$ and $aSb \subset R$ for units $a, b \in Q$, then there exist regular elements $a', b' \in R$ such that $a'Sb' \subset R$: to see this, write $a = c^{-1}a'$ and $b = (b')^{-1}d$ for suitable regular elements $a', b', c, d \in R$. I believe it is an open question whether there exists a maximal order, S say, such that $R \subset S \subset Q$ and S_R is finitely generated!!

EXAMPLE 8.3. A simple noetherian ring, R say, is a maximal order: if $R \subset S \subset Q = \operatorname{Fract} R$ and $aSb \subset R$, then

$$R \supset RaSbR = (RaR)S(RbR) = RSR = S.$$

Prime maximal orders are the non-commutative analogues of integrally closed domains.

PROPOSITION 8.4. *A commutative noetherian domain is a maximal order if and only if it is integrally closed.*

Proof. Let R be the ring in question, and $\bar{R}$ its integral closure in $\operatorname{Fract} R$. Then $\bar{R}/R$ is a finitely generated torsion R-module, so $x\bar{R} \subset R$ for some $0 \neq x \in R$. Thus, if R is a maximal order, $\bar{R} = R$. On the other hand, if R is not a maximal order, there exists $S \supset\neq R$ and a unit a such that $aS \subset R$, so aS, and hence S, is a finitely generated R-module, whence R is not integrally closed. □

Lemma 8.5. *If R is a subring of an artinian ring Q, the following are equivalent:*

1. *R is a left order in Q;*
2. *R is a right order in Q;*
3. *R is an order in Q;*
4. *Q is both the right and the left ring of quotients of R.*

Definition 8.6. Suppose that R is either a right or left order in Q. Then

- a fractional left R-ideal is a submodule I of ${}_RQ$ such that $Ra \subset I$ and $Ib \subset R$ for some units $a, b \in Q$;
- a fractional right R-ideal is a submodule I of Q_R such that $aR \subset I$ and $bI \subset R$ for some units $a, b \in Q$;
- a fractional R-ideal is an R-R bimodule $I \subset Q$ which is both a fractional right and a fractional left R-ideal.

If I is a fractional left (or right) R-ideal of Q, the right order and left order of I are

$$\mathrm{O}_r(I) = \{q \in Q \mid Iq \subset I\} \qquad \text{and} \qquad \mathrm{O}_\ell(I) = \{q \in Q \mid qI \subset I\}$$

respectively. These are subrings of Q and, if I is a fractional ideal, they contain R.

An essential left ideal of a prime left noetherian ring R is a fractional left R-ideal of $\operatorname{Fract} R$, because it contains a regular element.

There is a natural ring anti-homomorphism $\mathrm{O}_r(I) \to \operatorname{Hom}_R({}_RI, {}_RI)$ sending q to the map ρ_q, right multiplication by q; that is, $\rho_q(x) = xq$. If I contains a right regular element it is injective.

Lemma 8.7. *If R is a left order in Q, and I is a fractional right or left R-ideal, then $\mathrm{O}_r(I)$ and $\mathrm{O}_\ell(I)$ are left orders in Q, which are equivalent to R.*

Proof. Suppose that I is a fractional left ideal, and that $a, b \in Q$ are units such that $Ra \subset I$ and $Ib \subset R$. Then $a \in I$, so

$$\mathrm{O}_\ell(I)ab \subset Ib \subset R \subset \mathrm{O}_\ell(I),$$

and

$$a\mathrm{O}_r(I)b \subset Ib \subset R, \qquad \text{and} \qquad bRa \subset \mathrm{O}_r(I).$$

It is clear that $\mathrm{O}_\ell(I)$ is an order; if $q \in Q$, then $b^{-1}qb = xy^{-1}$ for some $x, y \in R$, and there exist $x', y' \in \mathrm{O}_r(I)$ such that $x = b^{-1}x'a^{-1}$ and $y = b^{-1}y'a^{-1}$, whence $\mathrm{O}_r(I)$ is also an order in Q. The above equations also show that R is equivalent to both $\mathrm{O}_\ell(I)$ and $\mathrm{O}_r(I)$. A similar proof works for fractional right ideals. □

The significance of the rings $\mathrm{O}_r(I)$ and $\mathrm{O}_\ell(I)$ in the present context is explained by the next result.

Proposition 8.8. *If R is prime noetherian, the following are equivalent:*

1. *R is a maximal order;*
2. *for every non-zero ideal I in R, $\mathrm{O}_r(I) = \mathrm{O}_\ell(I) = R$.*

Proof. (1) $\Rightarrow$ (2) First, $R \subset \mathrm{O}_r(I)$. Since I is essential in R it contains a regular element, c say. By definition of $\mathrm{O}_r(I)$, we have $c\mathrm{O}_r(I) \subset I \subset R$, whence $R = \mathrm{O}_r(I)$. Similarly $R = \mathrm{O}_\ell(I)$.

(2) $\Rightarrow$ (1) Suppose that S is a subring of $\operatorname{Fract} R$ containing R, and $a, b \in R$ are regular elements such that $aSb \subset R$. Using hypothesis (2), we have

$$SbaR \subset \mathrm{O}_r(RaR) = R \qquad \text{and} \qquad RbaS \subset \mathrm{O}_\ell(RbR) = R,$$

whence $I = SbaR$ is a two-sided ideal of R such that $S \subset \mathrm{O}_\ell(I)$. Therefore $S = R$, and we conclude that R is a maximal order. $\square$

Proposition 8.9. *If R is a prime noetherian ring which is not a maximal order, then it contains a prime ideal, I say, such that either*

- *I is reflexive as a right R-module, and $\mathrm{O}_\ell(I) \supsetneq R$, or*
- *I is reflexive as a left R-module, and $\mathrm{O}_r(I) \supsetneq R$.*

Proof. By Proposition 8.8, there is a non-zero ideal I such that either $\mathrm{O}_\ell(I) \neq R$ or $\mathrm{O}_r(I) \neq R$; suppose the first possibility occurs. We will show that the first conclusion holds; a similar proof deals with the other possibility.

Choose a non-zero two sided ideal of R, I say, which is maximal subject to $\mathrm{O}_\ell(I) \neq R$. Write $S = \mathrm{O}_\ell(I)$, and define $J := \{x \in Q \mid Sx \subset R\}$. Clearly J is a right ideal of R. However, $SJ \subset J$ because $S.SJ = SJ \subset R$, so J is a two-sided ideal and $S \subset \mathrm{O}_\ell(J)$. Moreover, $I \subset J$ because $SI \subset I \subset R$, so $I = J$ by maximality. In other words $I = ({}_RS)^*$, so I is reflexive as a *right* R-module.

Now suppose that A and B are ideals of R strictly containing I, and $AB \subset I$. Then $(SA)B \subset I \subset B$, so $SA \subset \mathrm{O}_\ell(B)$. The maximality of I implies that $\mathrm{O}_\ell(B) = R$, whence $SA \subset R$. Thus $A \subset J = I$. Thus I is prime. $\square$

Example 8.10. The ring

$$A = \begin{pmatrix} k[x] & (x) \\ k[x] & k[x] \end{pmatrix}$$

is prime, noetherian, and $\operatorname{gldim} A = 1$. But A is not a maximal order. There are two distinct maximal orders containing it, namely $S = M_2(k[x])$ and

$$S' = \begin{pmatrix} k[x] & x^{-1}k[x]//(x) & k[x] \end{pmatrix}$$

Notice that $S \cong S'$ via an isomorphism which restricts to an automorphism of A; the automorphism is conjugation by $\begin{pmatrix} 0 & x//1 & 0 \end{pmatrix}$.

There simple A-modules are as follows. Since A is finite over its center, every simple module is killed by a maximal ideal of the center $k[x]$. If $0 \neq \lambda \in k$, then $A/(x-\lambda) \cong M_2(k)$; this gives a family of 2-dimensional simples parametrized by $k\backslash\{0\}$. Since $A/(x)$ is isomorphic to upper triangular matrices there are two 1-dimensional representations killed by x. One way to view this is to view the central embedding $k[x] \to A$ as giving a geometric map $\operatorname{Spec} A \to \operatorname{Spec} k[x]$, which is an isomorphism except at 0 where the fiber consists of two points.

9. Measures of size

It is important to have ways of measuring the size of a module; for example this permits induction arguments, and allows more delicate structural analysis.

Definition 9.1. A function $\delta : \mathsf{mod} R \to \mathbb{R}$ is called a dimension function if,

- for each $p \in \mathbb{R}$, $\mathcal{C}_p := \{M \mid \delta(M) \leq p\}$ is a dense subcategory of $\mathsf{mod}R$ or, equivalently,
- $\delta(M) = \max\{\delta(N), \delta(M/N)\}$ whenever N is a submodule of M.

Suppose that δ is a dimension function taking values in $\mathbb{N} \cup \{0\}$; hence there is a chain $\mathcal{C}_{-1} = \{0\} \subset \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots$ with abelian quotient categories $\mathcal{C}_p/\mathcal{C}_{p-1}$. If the objects in $\mathcal{C}_p/\mathcal{C}_{p-1}$ have finite length, we define the p-length or multiplicity $\ell_p(M)$ for $M \in \mathcal{C}_p$, to be its length in $\mathcal{C}_p/\mathcal{C}_{p-1}$. Notice that $\ell_p(-)$ is additive on short exact sequences in $\mathcal{C}_p$.

We have already met one example of this – over a semiprime Goldie ring R we may take $\mathcal{C}_0$ to be the torsion modules (these need not be of finite length!), and $\mathcal{C}_1 = \mathsf{mod}R$; the quotient category $\mathcal{C}_1/\mathcal{C}_0$ is equivalent to $\mathsf{mod}(Q)$, where $Q = \operatorname{Fract} R$, so objects in $\mathcal{C}_1/\mathcal{C}_0$ have finite length, and the multiplicity function in this case is the reduced rank $\rho(-)$.

An important example is Krull dimension.

Definition 9.2. Define full subcategories $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots$ of $\mathsf{mod}R$ as follows:

- $\mathcal{K}_0$ consists of all finite length objects in $\mathsf{mod}R$;
- once $\mathcal{K}_n$ has been defined, $\mathcal{K}_{n+1}$ consists of all objects in $\mathsf{mod}R$ which have finite length in the quotient category $\mathsf{mod}R/\mathcal{K}_n$.

The Krull dimension of $M \in \mathsf{mod}R$, denoted $\operatorname{Kdim}(M)$, is the smallest n such that $M \in \mathcal{K}_n$; if there is no such n we say that M has infinite Krull dimension.

LEMMA 9.3. 1. *Each $\mathcal{K}_n$ is a dense subcategory of* $\mathsf{mod}R$.
2. *If $0 \to L \to M \to N \to 0$ is an exact sequence in* $\mathsf{mod}R$*, then*

$$\operatorname{Kdim} M = \max\{\operatorname{Kdim} L, \operatorname{Kdim} N\}.$$

PROOF. In any abelian category the objects of finite length form a dense subcategory, and the quotient category is again abelian. So (1) is proved by an induction argument beginning with the fact that $\mathcal{K}_0$ is dense.

(2) If $L, N \in \mathcal{K}_n$, then $M \in \mathcal{K}_n$ too (because $\mathcal{K}_n$ is dense), whence $\operatorname{Kdim} M \leq \max\{\operatorname{Kdim} L, \operatorname{Kdim} N\}$. Conversely, if $M \in \mathcal{K}_n$, so are L and N, whence $\operatorname{Kdim} M \geq \max\{\operatorname{Kdim} L, \operatorname{Kdim} N\}$. □

It is convenient to define $\mathcal{K}_{-1}$ to be the subcategory of $\mathsf{mod}R$ consisting just of the zero object, and then observe that $\mathcal{K}_0$ can be defined inductively as the objects in $\mathsf{mod}R$ which have finite length in $\mathsf{mod}R/\mathcal{K}_{-1}$.

Now consider an arbitrary chain of dense subcategories $0 = \mathcal{C}_{-1} \subset \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots$ such that the objects in each $\mathcal{C}_p/\mathcal{C}_{p-1}$ have finite length. By induction, $\mathcal{C}_p \subset \mathcal{K}_p$, so Krull dimension is a sort of extremal dimension function, and we are particularly interested in other, more subtle, dimension functions. For example, GK-dimension is more subtle in that simple modules may have different GK-dimensions (see Stafford's Weyl algebra examples).

If $\mathcal{G}_i$ denotes the modules of GK-dimension $\leq i$, and $\mathcal{C}_p$ the modules of grade $\geq p$, then $\mathcal{C}_p = \mathcal{G}_{n-p}$ when R is Cohen-Macaulay of GK-dimension n.

LEMMA 9.4. *Let $0 = \mathcal{C}_{-\infty} \subset \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset$ $\mathsf{mod}R$ be dense subcatgeories. If B is a finitely generated R-R-bimodule and $I = \ell\operatorname{ann} B$, then $B \in \mathcal{C}_p$ if and only if $R/I \in \mathcal{C}_p$.*

Proof. By hypothesis B is a quotient of $(R/I)^n$ for some I, so by density, if $R/I \in \mathcal{C}_p$, so is B. Conversely, suppose that $B \in \mathcal{C}_p$. Write $B = b_1R + \cdots + b_nR$, and let $I_j = \ell\mathrm{ann}(b_j)$; thus $I = I_1 \cap \cdots \cap I_n$. Since R/I_j embeds in B, each $R/I_j \in \mathcal{C}_p$. Each of the successive quotients in the chain

$$R \supset I_1 + I_2 \supset I_1 \supset I_1 \cap I_2$$

belongs to $\mathcal{C}_p$, whence so does $R/I_1 \cap I_2$. It follows by induction that $R/I_1 \cap \cdots \cap I_n \in \mathcal{C}_p$ too, as required. $\square$

If the chain of $\mathcal{C}_p$'s in Lemma 9.4 is used to define a dimension function $\delta(M) := \min\{p \mid M \in \mathcal{C}_p\}$, the result may be rephrased as saying that $\delta(B) = \delta(R/I)$.

Warning. $\boxed{\text{Paul}}$ Kdim $M = 0$ is not the same as artinian since $k[x, x^{-1}]/k[x]$ is not of Kdim 0 as a $k[x]$-module. See Gabriel dimension.

CHAPTER 3

Presentations of algebras by generators and relations

In this chapter k denotes a field, and all algebras are connected k-algebras.

1. The free algebra

In this section we write $F = k\langle X\rangle$ for the free algebra over k on a set X of elements of positive degree. We give F the induced grading. Thus F is a connected algebra. In Chapter 1 we observed that F has a basis consisting of the words formed from the letters in X; we will refer to this as the standard basis for F. An immediate consequence of this basis is that F is a domain.

Proposition 1.1. *The free algebra on n generators is isomorphic to a subalgebra of the free algebra on two generators.*

Proof. Let $B = k\langle x, y\rangle$ be the free algebra on two generators, and let $A = k\langle x_1, \dots, x_n\rangle$ be the free algebra on n generators. Let $\mathcal{A}$ and $\mathcal{B}$ be the standard bases for A and B consisting of the words in the generators. Define $\varphi : A \to B$ by $\varphi(x_i) = xy^i$. Since φ sends $\mathcal{A}$ to $\mathcal{B}$ it suffices to show that φ is injective on $\mathcal{A}$. But this is clear, since we can recover each integer i_p from $w = \varphi(x_1^{i_1} \cdots x_n^{i_n}$ as the number of times $xy^p x$ occurs as a subword of w. □

It is useful to notice that a domain having two non-zero left ideals which intersect in zero must contain a free algebra (on ≥ 2 generators). To see this, let R be the domain in question, and suppose that $Rx \cap Ry = 0$ for some non-zero x and y. Let $\varphi : F = k\langle X, Y\rangle \to R$ be the algebra map defined by $\varphi(X) = x$ and $\varphi(Y) = y$. Choose $0 \neq \alpha + aX + bY \in \ker\varphi$ of least degree, with $\alpha \in k$. Write $\bar{a} = \varphi(a)$ and $\bar{b} = \varphi(b)$. Then

$$\alpha y + y\bar{b}y = -y\bar{a}x \in Rx \cap Ry = 0,$$

so $\bar{a} = 0$, and $\alpha + y\bar{b} = 0$, whence $\alpha = 0$ and $\beta = 0$. But a and b have smaller degree than $\alpha + aX + bY$; this is a contradiction, so we conclude that φ is injective.

Proposition 1.2. *The center of the free algebra on ≥ 2 generators is the base field.*

Proof. Suppose the free generators are $x_1, \dots, x_n$. It suffices to show that the only homogenous central elements are the scalars. Let $z = x_1a_1 + \cdots + x_na_n$ be a homogeneous central element of degree > 0. From the equality $zx_i = x_iz$. it follows that $a_j = 0$ for all $j \neq i$. By varying i we see that all a_j are zero. □

Notation . Each homogeneous element $f \in F$ of positive degree can be written uniquely as

$$f = \sum_{x \in X} f_x x.$$

LEMMA 1.3. *Suppose that a_i and b_i are non-zero homogeneous elements of F such that*

- $\deg a_i + \deg b_i$ *is the same for all i,*
- *the elements are labelled so that* $\deg a_1 \leq \cdots \leq \deg a_n$*, and*
- $a_1 b_1 + \cdots + a_n b_n = 0$.

Then there are homogeneous elements c_i which are either zero, or of degree $\deg a_n - \deg a_i$*, such that*

$$a_n = a_1 c_1 + \cdots + a_{n-1} c_{n-1}.$$

PROOF. If $b_n \in k$, the result is obvious. If not, then $(b_n)_x \neq 0$ for some $x \in X$. Now

$$0 = (a_1 b_1 + \cdots + a_n b_n)_x = a_1 (b_1)_x + \cdots + a_n (b_n)_x.$$

Now $\deg (b_n)_x < \deg b_n$, so the result follows by induction on the degree of b_n. $\square$

It is clear from the standard basis for the free algebra that the right (and the left) ideal generated by the generators of the free algebra is a free module with those generators as a basis. Indeed, every right (and left) ideal is a free module.

PROPOSITION 1.4. *Every left ideal in the free algebra is a free module.*

PROOF. Let I be a left ideal. Index the elements of I by some ordinal in such a way that $\deg b_i \leq b_j$ whenever $i \leq j$. Let $\bar{b}_i$ denote the leading homogeneous component of b_i, and let J be the left ideal generated by those $\bar{b}_j$ which are not in the left ideal generated by $\{\bar{b}_i \mid i < j\}$; it follows from Lemma 1.3 that such a $\bar{b}_j$ satisfies $F\bar{b}_j \cap (\sum_{i<j} F b_i) = 0$, and hence that J is free on the $\bar{b}_j$. But $J = \operatorname{gr} I$, so by ????, I is freely generated by the corresponding b_j. $\square$

PROPOSITION 1.5. [**74**] *Let I be a two-sided ideal in F, and let let $A = F/I$. If U and V are, respectively, a right and a left ideal containing I, then*

$$\operatorname{Tor}^A_{2n}(F/U, F/V) \cong \frac{I^n \cap U I^{n-1} V}{U I^n + I^n V}$$

and

$$\operatorname{Tor}^A_{2n+1}(F/U, F/V) \cong \frac{U I^n \cap I^n V}{U I^n V + I^{n+1}}.$$

PROOF. Consider the exact sequence $0 \to I \to F \to A \to 0$. Since UI^n is a free right F-module, the sequence

$$0 \to UI^n \otimes_F I \to UI^n \to UI^n \otimes_F A \to 0$$

is exact; the last term is a free right A-module, so UI^n/UI^{n+1} is a free right A-module. In particular, specializing U to I, I^n/I^{n+1} is a free right A-module.

For each $n \geq 0$, there are exact sequences

$$0 \to UI^n/I^{n+1} \to I^n/I^{n+1} \to I^n/UI^n \to 0$$

and

$$0 \to I^{n+1}/UI^{n+1} \to UI^n/UI^{n+1} \to UI^n/I^{n+1} \to 0$$

of right A-modules, in which the middle terms are free. Hence the syzygies of the right A-module F/U are

$$\Omega^{2n}(F/U) \cong I^n/UI^n \qquad \text{and} \qquad \Omega^{2n+1}(F/U) \cong UI^n/I^{n+1}.$$

We will compute the Tor-groups using the fact that $\mathrm{Tor}_s^A(M,N) \cong \mathrm{Tor}_1^A(\Omega^{s-1}M,N)$ whenever $s \geq 1$. For example, $\mathrm{Tor}_{2n}^A(F/U,F/V)$ is isomorphic to

$$\begin{aligned}
\mathrm{Tor}_1^A(UI^{n-1}/I^n,F/V) &\cong \ker\left(\frac{I^n}{UI^n} \otimes_A \frac{F}{V} \sharp \longrightarrow \frac{UI^{n-1}}{UI^n} \otimes_A \frac{F}{V}\right) \\
&\cong \ker\left(\frac{I^n}{UI^n + I^nV} \sharp \longrightarrow \frac{UI^{n-1}}{UI^n + UI^{n-1}V}\right) \\
&= \frac{I^n \cap UI^{n-1}V}{UI^n + I^nV}.
\end{aligned}$$

Similarly, $\mathrm{Tor}_{2n+1}^A(F/U,F/V)$ is isomorphic to

$$\begin{aligned}
\mathrm{Tor}_1^A(I^n/UI^n,F/V) &\cong \ker\left(\frac{UI^n}{I^{n+1}} \otimes_A \frac{F}{V} \sharp \longrightarrow \frac{I^n}{I^{n+1}} \otimes_A \frac{F}{V}\right) \\
&\cong \ker\left(\frac{UI^n}{UI^nV + I^{n+1}} \sharp \longrightarrow \frac{I^n}{I^nV}\right) \\
&= \frac{UI^n \cap I^nV}{UI^nV + I^{n+1}}.
\end{aligned}$$

This completes the proof. □

COROLLARY 1.6. *Let $\mathfrak{m}$ be the augmentation ideal in the free algebra F. If $I \neq F$ is a homogeneous ideal, and $A = F/I$, then*

$$\mathrm{Tor}_{2n}^A(k,k) \cong \frac{I^n \cap \mathfrak{m}I^{n-1}\mathfrak{m}}{\mathfrak{m}I^n + I^n\mathfrak{m}}$$

and

$$\mathrm{Tor}_{2n+1}^A(k,k) \cong \frac{\mathfrak{m}I^n \cap I^n\mathfrak{m}}{\mathfrak{m}I^n\mathfrak{m} + I^{n+1}}.$$

2. Bergman's Diamond Lemma

If an algebra is given in terms of generators and relations, it is usually difficult to say much about it. It may even be difficult to decide if the algebra is non-zero or not. For example, although it is reasonable to expect that the free algebra modulo a single relation of degree ≥ 1 is non-zero, this is only known in characteristic zero, and even then the proof is complicated [**113**], [**114**].

A first step towards understanding an algebra presented by generators and relations might be to obtain a basis for it. We describe an algorithm to do this, but please note in advance that the algorithm may not terminate in a finite time. The basic idea is apparent from the next example.

EXAMPLE 2.1. Let $A = k[x,y]$ with defining relation $yx^2 = x^2y$. Think of A as a quotient of the free algebra $F = k\langle x,y\rangle$. The goal is to find a subset of the standard basis for F which maps injectively to a basis for A. To this end, we may interpret the relation as saying that any word in x and y which has the subword yx^2, say $w = uyx^2v$ is not needed as a basis element for A since it is equal in A to ux^2yv. Hence A is be spanned by (the images of) those words which do not contain yx^2 as a subword. It is reasonable to guess that the words which do not contain yx^2 as a subword will give a basis for A. This is the case. These words are those which can be obtained from the standard basis for F by replacing a word of the form uyx^2v by the word ux^2yv, and repeating this process as often as possible. We call this process a 'replacement rule'.

Now suppose that $A = k[x, y]$ has two defining relations, $yx^2 = x^2y$ and $y^2x = xy^2$. Now there are two replacement rules, and either may be applied a word which contains y^2x^2 as a subword. This introduces a potential ambiguity, and some thought is required to see that the order of the replacement rules does not affect the final outcome.

Let $F = k\langle x_1, \dots, x_d\rangle$ be a free algebra, and let $\mathcal{B}$ be the standard basis consisting of the words in the x_i's. We may define a total ordering on $\mathcal{B}$, the degree and dictionary ordering, by declaring that $u < v$ if either

- $\deg u < \deg v$, or
- $\deg u = \deg v$, say $u = x_{i_1} \cdots x_{i_n}$ and $v = x_{j_1} \cdots x_{j_n}$, and for some q, we have $i_1 = j_1, \cdots, i_q = j_q$, but $i_{q+1} < j_{q+1}$.

Actually, the only two properties of this ordering which will be used in the following analysis are

- if $u < v$, and $a, b \in \mathcal{B}$, then $aub < avb$, and
- for each $v \in \mathcal{B}$, $\{u \in \mathcal{B} \mid u < v\}$ is finite.

Let $\pi : F \to A$ be a surjective homomorphism. Suppose that $\ker \pi$ is generated as an ideal by

$$\{r_\lambda \mid \lambda \in \Lambda\},$$

and, for each λ, write

$$r_\lambda = w_\lambda - s_\lambda,$$

where w_λ is the highest monomial (in the sense of $<$) occurring in r_λ.

Lemma 2.2. *The relations r_λ may be chosen so that if λ and μ are distinct elements of Λ, then w_μ is not a subword of w_λ.*

Proof. If w_μ is a subword of w_λ, we may write $w_\lambda = aw_\mu b$. Hence we have the following equality of ideals:

$$(r_\mu, r_\lambda) = (w_\mu - s_\mu, w_\lambda - s_\lambda) = (w_\mu - s_\mu, s_\lambda - as_\mu b).$$

Each monomial occurring in $s_\lambda - as_\mu b$ is $< w_\lambda$. If we replace r_λ by $s_\lambda - as_\mu b$, then we still have a set of generators for $\ker \pi$. We may repeat this process, and we will eventually stop because $(\mathcal{B}, <)$ has the descending chain property. When the process stops we will have obtained a set of defining relations satisfying the conditions in the lemma. $\square$

From now on, assume that the relations $R = \{r_\lambda \mid \lambda \in \Lambda\}$ are chosen as in the lemma. We associate linear maps $F \to F$ to these relations as follows: for each $\lambda \in \Lambda$, and each pair $(a, b) \in \mathcal{B} \times \mathcal{B}$, define

$$\rho^\lambda_{ab} : F \to F$$

to be the linear map which on $\mathcal{B}$ is given by

$$\rho^\lambda_{ab}(w) = \begin{cases} as_\lambda b & \text{if } w = aw_\lambda b. \\ w & \text{otherwise.} \end{cases}$$

We call ρ^λ_{ab} a simple reduction. A composition of simple reductions, arising from various λ and various (a, b), is called a reduction. An element $x \in F$ is irreducible (with respect to R) if $\rho(x) = x$ for all reductions. We write $\mathcal{I}rr$ for the set of irreducible elements. An element $x \in F$ is said to be reduction unique if there is a *unique* $y \in \mathcal{I}rr$ such that $\rho(x) = y$ for some reduction ρ; in this case we write $x_T = y$.

LEMMA 2.3. *Given the above situation,*

1. *$\mathcal{I}rr$ is a subspace of F with basis $\mathcal{B} \cap \mathcal{I}rr = \{w \in \mathcal{B} \mid$ no w_λ is a subword of $w\}$;*
2. *if $f \in F$, then $\rho(f) \in \mathcal{I}rr$ for some reduction ρ;*
3. *if ρ is a reduction, then $\rho(f) - f \in \ker \pi$ for all reductions ρ;*
4. *$F = \mathcal{I}rr + \ker \pi$;*
5. *if $f \in F$, and the reductions $\rho_1(f)$ and $\rho_2(f)$ are both irreducible, then $\rho_1(f) - \rho_2(f) \in \mathcal{I}rr \cap \ker \pi$;*
6. *the set of reduction unique elements is a subspace of F.*

PROOF. (1) Since reductions are linear maps, $\mathcal{I}rr$ is a subspace. If $f \in \mathcal{I}rr$, then $\rho(f) = f$ for all simple reductions ρ. Since a simple reduction changes at most one monomial occurring in f, $\rho(w) = w$ for all monomials w occurring in f. Hence each such w is irreducible, and f is in the span of $\mathcal{B} \cap \mathcal{I}rr$.

(2) A simple reduction changes at most one monomial occurring in f, and replaces it with a strictly smaller monomial. Since $(\mathcal{B}, <)$ has the descending chain condition, repeating this process will eventually stabilize, thus giving an irreducible element.

(3) If ρ is a simple reduction, then $\rho(w) - w \in \ker \pi$ for all $w \in \mathcal{B}$. Summing over the words appearring in f, gives $\rho(f) - f \in \ker \pi$. Suppose, inductively, that if ρ' is a composition of n simple reductions, then $\rho'(f) - f \in \ker \pi$ for all f. If ρ is a simple reduction, then

$$\rho\rho'(f) - f = \rho\rho'(f) - \rho'(f) + \rho'(f) - f,$$

so this is in $\ker \pi$.

(4) This follows at once from (2) and (3).

(5) We may write $\rho_1(f) - \rho_2(f) = \big(\rho_1(f) - f\big) - \big(\rho_2(f) - f\big)$, so the result follows from (1) and (3).

(6) A scalar multiple of a reduction unique element is reduction unique because reductions are linear maps. Now let x and x' be reduction unique. By (2), $\rho(x+x')$ is irreducible for some reduction ρ. Now $\rho(x)$ can be reduced to an irreducible element, but that element is also a reduction of x, so must equal x_T; say $\sigma\rho(x) = x_T$. Similarly, since x' is reduction unique, there is a reduction, τ say, such that $\tau\sigma\rho(x') = x'_T$. Notice that $\tau(x_T) = x_T$, because x_T is irreducible. Because $\rho(x+x')$ is irreducible,

$$\rho(x + x') = \tau\sigma\rho(x + x') = x_T + x'_T,$$

which proves the uniqueness of any irreducible element to which $x + x'$ can be reduced. In particular, $(x + x')_T = x_T + x'_T$. □

REMARK 2.4. The word 'diamond' refers to the idea of reduction uniqueness. If x is reduction unique, and $\rho_1(x)$ and $\rho_2(x)$ are reductions, then there exist reductions σ_1 and σ_2 such that $\sigma_1\rho_1(x) = \sigma_2\rho_2(x)$. We can draw a picture to illustrate this with the arrows going down to represent the idea that the reduction of an element is smaller with respect to $<$.

$$\begin{tikzcd} & \overset{a}{\bullet} \arrow[dl] \arrow[dr] & \\ b\ \bullet \arrow[dr] & & \bullet\ c \arrow[dl] \\ & \underset{d}{\bullet} & \end{tikzcd}$$

A word $abc \in \mathcal{B}$ is ambiguous if $ab = w_\lambda$ and $bc = w_\mu$ for some $\lambda, \mu \in \Lambda$. This ambiguity is resolvable if there are reductions ρ and ρ' such that $\rho(f_\lambda c) = \rho'(af_\mu)$. If all ambiguities are resolvable, we say that $\{r_\lambda \mid \lambda \in \Lambda\}$ is a complete set of relations, or that the relations satisfy the diamond condition.

Although the ambigous words are not the only ones that can be reduced in more than one way, they are the minimal such ones, and the next theorem says that we need only concern ourselves with those; this is good because if $\{r_\lambda \mid \lambda \in \Lambda\}$ is finite there are only finitely many ambiguities.

THEOREM 2.5 (The Diamond Lemma). [**31**] *The following are equivalent:*

1. $\{r_\lambda \mid \lambda \in \Lambda\}$ *is a complete set of reductions;*
2. *all elements of F are reduction unique;*
3. $F = \mathcal{I}rr \oplus \ker \pi$, *so the images of the irreducible words give a basis for A.*

PROOF. (1) $\Rightarrow$ (2) By Lemma 2.3(6), it suffices to prove that each $w \in \mathcal{B}$ is reduction unique. Suppose inductively that every word $< w$ is reduction unique. Let $\rho_1(w)$ and $\rho_2(w)$ be two distinct reductions. Each can be reduced to an irreducible element, and we must show that these two irreducible elements are the same. It suffices to do this when ρ_1 and ρ_2 are simple reductions. If $\rho_1 = \rho^\lambda_{**}$ and $\rho_2 = \rho^\mu_{**}$, then the proof depends on the relative positions of w_λ and w_μ in w.

Case 1. Suppose that $w = aw_\mu bw_\lambda c$ with $a, b, c \in \mathcal{B}$, so that $\rho_1(w) = \rho^\lambda_{bc}(w) = aw_\mu bs_\lambda c$ and $\rho_2(w) = \rho^\mu_{ab}(w) = as_\mu bw_\lambda c$. Since these elements are linear combinations of words $< w$, they are reduction unique by the induction hypothesis; both of them can be reduced to $as_\mu bs_\lambda c$, and hence to $(as_\mu bs_\lambda c)_T$. Thus $\rho_1(w)$ and $\rho_2(w)$ are reduction unique, and can be reduced to the same irreducible element, namely $(as_\mu bs_\lambda c)_T$.

Case 2. Suppose that $w = abcde$ with $a, b, c, d, e \in \mathcal{B}$, and $w_\mu = bc$ and $w_\lambda = cd$. Then $\rho_1(w) = \rho^\lambda_{be}(w) = abs_\lambda e$ and $\rho_2(w) = \rho(w)^\mu_{ad}(w) = as_\mu de$. As in case 1, these elements are reduction unique. By hypothesis (1), the ambiguity bcd is resolvable, so $s_\mu d$ and bs_λ can be reduced to a common element; hence so can $as_\mu de$ and $abs_\lambda e$; again, these elements are reduction unique, so we conclude that w is reduction unique.

(2) $\Rightarrow$ (3) By hypothesis (2), there is a well-defined linear map $\rho_T : F \to \mathcal{I}rr$ which sends an element to the *unique* irreducible element to which it can be reduced. If $r \in \ker \pi$, then r is a linear combination of elements of the form $a(w_\lambda - s_\lambda)b$ with $a, b \in \mathcal{B}$. Since ρ^λ_{ab} kills this element, so does ρ_T. Thus $\ker \pi \subset \ker \rho_T$. If $f \in \ker \pi \cap \mathcal{I}rr$, then $f = \rho_T(f) = 0$. Hence by Lemma 2.3(4), $F = \mathcal{I}rr \oplus \ker \pi$.

(3) $\Rightarrow$ (1) Fix $w \in \mathcal{B}$. If $\rho_1(w)$ and $\rho_2(w)$ are irreducible reductions of w, then their difference is in $\mathcal{I}rr \cap \ker \pi$ by Lemma 2.3(5) so, by hypothesis (3), this difference is zero. Hence w is reduction unique. $\square$

If there is an ambiguity, say w, which can be reduced to distinct irreducible elements, say $\rho_1(w)$ and $\rho_2(w)$, then $R = \{r_\lambda \mid \lambda \in \Lambda\}$ is not a complete set of relations. If we form $R' = R \cup \{r = \rho_1(w) - \rho_2(w)\}$, then $(R') = (R) = \ker\pi$, but w is a resolvable ambiguity with respect to R'. So replace R by R'. Unfortunately, doing this will often introduce new ambiguities, and in repeating the process there is no guarantee that one will end up with a complete set of relations.

EXAMPLE 2.6. Let $A = k[x, y, z]$ with defining relations

$$zy = -y - yz,\ zx = x - xz,\ yx = z - xy. \tag{2-1}$$

If we take the degree and dictionary ordering $x < y < z$, then (2-1) gives the relations in the form $w_\lambda = s_\lambda$. The only ambiguity is zyx, and if we repeatedly apply the simple reductions to this we get

$$(zy)x = -yx - yzx = -z + xy - y(x - xz) = -2z + 2xy + z^2 - xyz,$$

and

$$z(yx) = z^2 - zxy = z^2 - xy + xzy = z^2 - 2xy - xyz.$$

Hence, if $\operatorname{char} k \neq 2$, this ambiguity is not resolvable. Taking the difference of these two irreducible elements leads to a new relation $xy = \frac{1}{2}z$, which we may adjoin to the original three relations. Now the ambiguity zyx is resolvable. But doing this introduces new ambiguities, namely xyx, yxy, and zxy. For example, repeatedly applying simple reductions to zxy gives

$$(zx)y = xy - xzy = xy - x(-y - yz) = 2xy + xyz = z + \tfrac{1}{2}z^2,$$

and

$$z(xy) = \tfrac{1}{2}z^2.$$

Thus, we have a non-resolvable ambiguity, and we may adjoin the further relation $z = 0$. Now one sees from the relations (2-1), together with the relation $z = 0$, that A is defined by the three relations $x = 0$, $y = 0$, $z = 0$. Hence $A \cong k$.

3. Order ideals and monomial algebras

Let $\mathcal{B}$ be the free monoid with identity on the letters X; thus $\mathcal{B}$ is the standard basis for the free algebra $k\langle X\rangle$. The elements of X are called letters, and the elements of $\mathcal{B}$ are called words. We say that b is a subword of w, and write $b \subset w$, if there are words a and c such that $w = abc$. Thus every word has 1 as a subword. The length of a word is the number of letters it contains.

Definition 3.1. A subset $\mathcal{M}$ of $\mathcal{B}$ is called an order ideal if every subword of a word in $\mathcal{M}$ belongs to $\mathcal{M}$.

The next result says that every algebra has a basis consisting of words belonging to a suitable order ideal.

LEMMA 3.2. *Let $\pi : k\langle X\rangle \to A$ be a surjection of k-algebras. Then there is an order ideal which π sends injectively to a basis for A, namely*

$$\mathcal{M} := \{w \in \mathcal{B} \mid \pi(w) \text{ is not a linear combination of various } \pi(v) \text{ with } v < w\}.$$

PROOF. It is clear that π is injective on $\mathcal{M}$, and that $\pi(\mathcal{M})$ is a basis for A. To see that $\mathcal{M}$ is an order ideal, suppose that $w = aw'b \in \mathcal{M}$. If w' were not in $\mathcal{M}$, it would be a in the linear span of smaller words, whence so too would w, contradicting the fact that it is in $\mathcal{M}$. □

Definition 3.3. A subset $V \subset \mathcal{B}$ is an anti-chain no element of V is a subword of any other element of V. The set of obstructions for an order ideal $\mathcal{M}$ is

$$V_{\mathcal{M}} := \{v \notin \mathcal{M} \mid \text{every proper subword of } v \text{ belongs to } \mathcal{M}\}.$$

Notice that $V_{\mathcal{M}}$ is an anti-chain.

LEMMA 3.4. *There is a bijection between anti-chains and order ideals, implemented as follows.*

$$\begin{aligned} V \mapsto &\{w \in \mathcal{B} \mid \textit{no subword of } w \textit{ is in } V\} \\ \mathcal{M} \mapsto &V_{\mathcal{M}} \end{aligned}$$

PROOF. It is easy to see that these rules set up maps between order ideals and anti-chains. All we must do is show they are mutually inverse to one another.

First, if V is an anti-chain, and $\mathcal{M}$ the order ideal it determines, we must show that $V = V_{\mathcal{M}}$. If $v \in V$, then certainly $v \notin \mathcal{M}$; however, if u is a proper subword of v, then every subword of u is also a subword of V, so fails to be in V, whence $u \in \mathcal{M}$; thus $v \in V_{\mathcal{M}}$. Conversely, let $v \in V_{\mathcal{M}}$, and suppose, contrary to what we wish to show, that $v \notin V$. Since $v \notin \mathcal{M}$, some $u \in V$ is a subword of v. However, since v is an obstruction, such u belongs to $\mathcal{M}$; thus $u \in V \cap \mathcal{M}$; but this is absurd because $V \cap \mathcal{M} = \emptyset$ by the very definition of M. Therefore, we conclude that our original assumption was false, meaning that $v \in V$.

Let $\mathcal{M}$ be an order ideal. We must show that $\mathcal{M}$ equals the order ideal $\mathcal{N} := \{w \in \mathcal{B} \mid \text{no element of } V_{\mathcal{M}} \text{ is a subword of } w\}$. Let $w \in \mathcal{B}$, and let $v \in V_{\mathcal{M}}$. If $w \in \mathcal{M}$, and $v \subset w$, then every subword of v is a subword of w, so belongs to $\mathcal{M}$, contradicting the fact that v is an obstruction; hence we conclude that $w \in \mathcal{N}$. Conversely, suppose that $w \notin \mathcal{M}$, and let $w' \subset w$ be of minimal length such that $w' \notin \mathcal{M}$. Then w' is an obstruction, so $w \notin \mathcal{N}$. Thus $\mathcal{N} \subset \mathcal{M}$. □

Combining Lemmas 3.2 and 3.4, we see that given an algebra A, and a surjective map $\pi : k\langle X\rangle \to A$, there is an anti-chain V such that π sends

$$\{w \in \mathcal{B} \mid \text{no subword of } w \text{ belongs to } V\}$$

injectively to a basis for A. The obvious question is how does one determine V. Given a set of defining relations $\mathcal{R}$ for A, construct a complete set of relations $\mathcal{R}'$, using the method described in section 2 by repeatedly resolving ambiguities. The set of leading terms of the elements of $\mathcal{R}'$ is then an anti-chain, say V. The set of irreducible words is then a basis for A, by Theorem 2.5.

Definition 3.5. If S is a set of words in the letters X, the algebra $k\langle X\rangle/(S)$ is called a monomial algebra.

If S is a set of monomials and $s, t \in S$ with s a subword of t, then the ideal generated by S is the same as the ideal generated by $S\backslash\{t\}$; hence, the defining relations of $k\langle X\rangle/(S)$ may be taken to be an anti-chain. Hence, when discussing a monomial algebra, we may always assume that S is an anti-chain.

PROPOSITION 3.6. *Let S be a set of words, and $\pi : k\langle X\rangle \to k\langle X\rangle/(S) = A$ the natural map to the associated monomial algebra. Then $\pi(b) = 0$ if and only if b is a linear combination of words, each of which has a subword belonging to S. Thus, a basis of A is given by $\{\pi(w) \mid w \in \mathcal{M}\}$ where $\mathcal{M}$ is the order ideal $\{w \in \mathcal{B} \mid$ no subword of w is in $S\}$.*

PROOF. □

The elements of an order ideal $\mathcal{M}$ may be put in bijection with the paths in a directed graph as follows. Suppose that S is a set of words of degree $m+1$. The vertices of G are the words of degree m. There is an arrow $u \to v$ if $ux = yv$ for some $x, y \in X$. We then label that arrow by the word ux.

Definition 3.7. The incidence matrix of a directed graph has its rows and columns labeled by the vertices, and the (u,v)-entry is 1 if there is an arrow $u \to v$, and 0 otherwise. The Hilbert series of a finite directed graph is the formal power series

$$\sum_{n=0}^{\infty} p_n t^n,$$

where p_n is the number of paths of length n.

PROPOSITION 3.8. *Let M be the incidence matrix of a graph. Then the (u,v) entry in M^n is the number of paths of length n from u to v.*

PROOF. □

Hence the total number of paths of length n is the sum of all the entries in M^n.

PROPOSITION 3.9. *The Hilbert series of a finite directed graph is a rational function of the form $f(t)/g(t)$, where $f, g \in \mathbb{Z}[t]$.*

PROOF. □

EXAMPLE 3.10. Consider the monomial algebra $k[x,y]$ with defining relations $yx^2 = y^2x = 0$. Thus $S = \{yx^2, y^2x\}$, and the associated graph is

$$\begin{tikzcd} x^2 \arrow[r] & xy \arrow[r] \arrow[d, shift right] & y^2 \\ & yx \arrow[u, shift right] & \end{tikzcd}$$

$\boxed{\text{Paul}}$ Need loops at x^2 and y^2. The incidence matrix for this graph, with the ordering x^2, xy, yx, y^2 of the vertices is

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One computes that

$$M^{2i} = \begin{pmatrix} 1 & i & i & i^2 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M^{2i+1} = \begin{pmatrix} 1 & i+1 & i & i(i+1) \\ 0 & 0 & 1 & i+1 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the number of paths of length $2i$ is $(i+2)^2$ and the number of paths of length $2i+1$ is $(i+2)(i+3)$. Therefore

$$\dim A_n = \begin{cases} (i+1)^2 & \text{if } n = 2i, \\ (i+1)(i+2) & \text{if } n = 2i+1. \end{cases}$$

(These formulas are valid for all $n \geq 0$.)

4. Anick's resolution

The bar resolution (Chapter 7, Section 5) gives an explicit resolution of the trivial module over a connected algebra, but it is usually so large it is impossible to work with effectively. At the other extreme, the minimal resolution may be extremely difficult to compute. There is a resolution due to Anick which generally lies between these two extremes, and can be obtained directly from generators and relations.

Definition 4.1. Let V be an anti-chain in $\mathcal{B}$, and suppose that $1 \notin V$. Let n be a positive integer. An n-chain (with respect to V) is a word w which can be written in the form $a_1b_1a_2b_2 \cdots a_nb_n$ such that

- $a_1b_1 \in V$,
- $b_ia_ib_{i+1} \in V$ for $i = 1, \dots, n-1$,
- n is minimal with respect to the preceding conditions.

Briefly, w can be expressed as overlapping words, each of which is in V.

THEOREM 4.2. *Let $\pi : F \to A$ be a surjective graded algebra homomorphism. Let $\mathcal{M}$ be the order ideal giving a basis for A, let V be the set of obstructions for $\mathcal{M}$, and let $V^{(n)}$ be the set of n-chains on V. Then there is a free resolution of ${}_Ak$ of the form*

$$\cdots \to A \otimes kV^{(3)} \to\to A \otimes kV^{(2)} \to\to A \otimes kV \to\to A \otimes A_1 \to k \to 0, \tag{4-1}$$

PROOF. $\square$

The proof of Anick's theorem is not very explicit about the differential; indeed, what the result tells us is not the maps in the minimal resolution, but rather the individual terms. Moreover, as the proof makes clear, these terms depend not so much on the defining relations for the algebra, as on the high degree terms in the relations. This helps explain why the minimal resolutions of the trivial module over so many different connected algebras can have the same 'shape'.

EXAMPLE 4.3. We will construct Anick's resolution for the third generic Clifford algebra. It is generated by elements x_1, x_2, x_3, and the defining relations may be succinctly expressed by saying that each $x_ix_j + x_jx_i$ is central. There are nine such relations, but their linear span is only 8-dimensional. (We will find it convenient to write 33211 instead of $x_3^2x_2x_1^2$, since we are interested in the monoid generated by the x_i's.) The relations are

$$\begin{aligned}
332 &= 223,\\
331 &= 133,\\
322 &= 223,\\
321 &= 123 - 231 + 132,\\
312 &= 213 - 132 + 231,\\
311 &= 113,\\
221 &= 122,\\
211 &= 112.
\end{aligned}$$

The six 'short' relations express the centrality of the elements x_i^2; the three relations expressing the fact that $x_i x_j + x_j x_i$ commutes with x_h whenever $\{h, i, j\} = \{1, 2, 3\}$ are linearly dependent, and are spanned by the two 'long' relations.

To see that these do not form a complete set of relations, consider the two resolutions of the ambiguity 3321, namely

$$\begin{aligned}
3(321) &= 3(123 - 231 + 132) = (312)3 - 3231 + 3132\\
&= (213 - 132 + 231)3 - 3231 + 3132 = 2133 - 1323 + 2313 - 3231 + 3132
\end{aligned}$$

and

$$(332)1 = (233)1 = 2(331) = 2(133).$$

This gives an additional relation, namely

$$3231 = 2313 - 1323 + 3132.$$

The highest degree terms in the nine relations are

$$3231,\ 332,\ 331,\ 322,\ 321,\ 312,\ 311,\ 221,\ 211.$$

Let V denote the set of these; it is an anti-chain. These are also the 1-chains with respect to V. The 2-chains are

$$3322,\ 3321,\ 3312,\ 3311,\ 3221,\ 3211,\ 2211,$$

and

$$33231,\ 32312,\ 32311,\ 32221,\ 31221,\ 31211.$$

Notice that these are simply the ambiguities in the sense of section 2. The 3-chains are

$$332211,\ 331221,\ 331211,\ 322211,\ 312211,$$

and the only 4-chain is 3312211. Notice that 33221 is a 2-chain, but not a 3-chain, although it can be written by overlapping 332, 322, and 221; the problem is that if done in this way 332 overlaps 221.

CHAPTER 4

Filtered and Graded Rings

Most k-algebras can not be given a graded structure other than the stupid one whereby the whole algebra is given degree zero. But a k-algebra may always be made into a filtered algebra, to which is associated a graded algebra. A filtration on an algebra is an ascending chain of subspaces subject to some axioms, one of which is that their union is the whole algebra. A typical example occurs when the algebra is already graded, the filtration subspaces being $\{x \mid \deg x \leq n\}$; in general the filtration subspaces are analogous to these, and lead to a notion of degree having the property that $\deg xy \leq \deg x + \deg y$. There is quite a close relation between a filtered algebra and its associated graded algebra; for example, under a mild hypothesis, if the latter is noetherian, so is the former. The most important examples are the enveloping algebras of Lie algebras. These have a natural filtration, and the associated graded ring is the symmetric algebra on the Lie algebra.

Given a filtered ring, R say, there is an obvious notion of a filtered R-module, and one may associate to this a graded module over the associated graded ring of R. Study of the graded module sheds light on the original module. This method is very effective for modules over enveloping algebras of Lie algebras, because one may apply the techniques of commutative algebra and algebraic geometry to the associated graded ring and the associated graded modules.

Many of the results in this chapter can be found in [**127**] and [**126**], and we acknowledge our debt.

1. Filtered rings and modules

Definition 1.1. A filtration on an abelian group V is a sequence $FV = \{F_nV = V_n \mid n \in \mathbb{Z}\}$ of subgroups

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots$$

such that $V = \bigcup_{i\in\mathbb{Z}} V_i$. The degree of $v \in V$, with respect to the given filtration, is the smallest n such that $v \in V_n$; if there is no such n we set $\deg v = -\infty$. The filtration is

- bounded below if $V_n = 0$ for $n \ll 0$;
- separated if $\bigcap_i V_i = 0$.

A filtration which is bounded below is separated.

Definition 1.2. The degree of a homomorphism $f : U \to V$ of filtered abelian groups is the smallest integer d such that $f(U_i) \subset V_{i+d}$ for all $i \in \mathbb{Z}$. The homomorphisms of degree $\leq p$ form a subgroup $F_p \operatorname{Hom}_{\mathbb{Z}}(U, V) \subset \operatorname{Hom}_{\mathbb{Z}}(U, V)$, and their union is a filtered subgroup of $\operatorname{Hom}_{\mathbb{Z}}(U, V)$. The composition of homomorphisms of degrees $\leq p$ and $\leq q$ is of degree $\leq p + q$.

Definition 1.3. A filtered k-algebra is a k-algebra R endowed with a filtration $\cdots \subset R_{-1} \subset R_0 \subset R_1 \subset \cdots$ by k-submodules such that

$$R_i R_j \subset R_{i+j}$$

for all i and j, and $\deg 1 = 0$. A filtered R-module is a module M with a filtration M_i, $i \in \mathbb{Z}$, by k-submodules such that $R_i M_j \subset M_{i+j}$ for all i and j.

If R is a filtered ring, the category of filtered R-modules, which is denoted $\operatorname{Filt} R$, has objects the filtered R-modules, and morphisms the R-module homomorphisms $f : M \to N$ such that $f(F_n M) \subset F_n N$ for all $n \in \mathbb{Z}$; the set of all such morphisms is denoted $\operatorname{Hom}_{FR}(M, N)$.

The filtered k-algebras form a category in which the morphisms are the k-algebra homomorphisms which are morphisms of filtered abelian groups.

Definition 1.4. Let FM be a filtered R-module, and let L be an arbitrary submodule of M. The

- induced filtration on L is defined by setting $F_n L := L \cap F_n M$;
- the induced filtration on $N = M/L$ is defined by setting $F_n N := F_n M + L/L$.

The natural maps $L \to M$ and $M \to M/L$ are then morphisms in $\operatorname{Filt} R$.

Example 1.5. A graded k-algebra A can be made into a filtered k-algebra by defining $F_n A = \oplus_{i \le n} A_i$.

Example 1.6. Suppose that R is presented as a quotient of the free algebra, say $R = k\langle x_1, \cdots, x_n \rangle / J$. If the free algebra is given its standard grading and is made into a filtered algebra as in Example 1.5, then R with the induced filtration becomes a filtered k-algebra. The natural map $k\langle x_1, \cdots, x_n \rangle \to R$ is then a homomorphism of filtered algebras.

Example 1.7. Let R be a filtered k-algebra, and M an R-module. If V is a subspace of M which generates it as an R-module, then $F_n M := R_n V$ gives M the structure of a filtered R-module. Since there are many choices of V, the forgetful functor $\operatorname{Filt} R \to \mathsf{Mod} R$ is not faithful.

Lemma 1.8. *$\operatorname{Filt} R$ is a pre-additive category, but is not abelian. In particular, a bijective map in $\operatorname{Hom}_{FR}(M, N)$ need not be an isomorphism.*

Proof. Kernels and cokernels exist in $\operatorname{Filt} R$. If $f \in \operatorname{Hom}_{FR}(M, N)$ then $\{m \in M \mid f(m) = 0\}$ with the induced filtration is the kernel of f in $\operatorname{Filt} R$, and $N/\operatorname{Im} f$ with the filtration induced from N is the cokernel of f in $\operatorname{Filt} R$.

To see that $\operatorname{Filt} R$ is not abelian, let FM be a non-zero filtered R-module, and endow it with a new filtration as follows: to distinguish the two modules we will write $N = M$ with filtration defined by $F_i N = F_{i+1} M$. Then the identity map $M \to M$ is in $\operatorname{Hom}_{FR}(N, M)$ but not in $\operatorname{Hom}_{FR}(M, N)$, so as a map from N to M the identity has zero kernel and cokernel, but is not an isomorphism. □

The next lemma explains when a bijection in $\operatorname{Filt} R$ is an isomorphism.

Definition 1.9. A homomorphism $f \in \operatorname{Hom}_{FR}(M, N)$ is strict if $f(F_i M) = \operatorname{Im} f \cap F_i N$ for all i. A sequence $L \to M \to N$ is strict exact if it is exact and both maps are strict.

Lemma 1.10. *If $f \in \operatorname{Hom}_{FR}(M, N)$ is bijective, then f is an isomorphism in $\operatorname{Filt} R$ if and only if it is strict.*

Proof. (⇒) Because f and f^{-1} are filtered maps, we have

$$F_iM = f^{-1}f(F_iM) \subset f^{-1}(F_iN) \subset F_iM.$$

Thus $F_iM = f^{-1}(F_iN)$, whence $f(F_iM) = F_iN$, showing f is strict.

(⇐) Strictness says that $f(F_iM) = F_iN$, so f^{-1} is a filtered map. □

If N is a filtered R-module and $f : M \to N$ is an arbitrary homomorphism, then f is strict if M is filtered by defining $F_nM := f^{-1}(f(M) \cap F_nM)$.

Suppose that FM is given, and L is an arbitrary submodule. If L is given the induced filtration then the inclusion $L \to M$ is strict. If M/L is given the induced filtration, then the quotient $M \to M/L$ is strict. The zero map is strict, so we say that a sequence $0 \to L \to M \to N \to 0$ in $\operatorname{Filt} R$ is a strict short exact sequence if it is exact and all the maps are strict.

Remark 1.11. The filtrations discussed above have all been *ascending* filtrations, meaning that as the index increases, the subgroup gets bigger. One also encounters *descending* filtrations (for example, see the definition of a filtered complex in Chapter 8; or consider the chain of ideals $R \supset I \supset I^2 \supset \cdots$). Some notational chicanery allows us to treat a descending filtration as an ascending one: if V is an abelian group with a filtration $\cdots \supset V_{n+1} \supset V_n \supset V_{n-1} \supset \cdots$, we may define $F_nV := V_{-n}$, so that $\cdots \subset F_{n-1}V \subset F_nV \subset F_{n+1}V \subset \cdots$ is an ascending filtration.

EXERCISES

1.1 Show that a composition of strict maps need not be strict.

1.2 If k is a commutative ring and V a filtered k-module, show that $R = \bigcup_n F_n \operatorname{Hom}_k(V, V)$ is a filtered k-algebra, and that V is a filtered R-module.

2. Graded Rings and Modules

3. Associated graded rings and modules

In this section R denotes a filtered k-algebra.

Definition 3.1. If FV is a filtered abelian group, the associated graded abelian group is

$$\operatorname{gr} V := \bigoplus_{n\in\mathbb{Z}} F_nV/F_{n-1}V$$

with the obvious group structure. If $\deg x = i$, we write

$$\bar{x} = [x + F_{i-1}V] \in F_iV/F_{i-1}V.$$

The map $x \to \bar{x}$ is called the principal symbol map, and is *not* additive in general.

Notice that $\deg x = \deg \bar{x}$, where the right hand side is degree in the graded abelian group $\operatorname{gr} V$.

If $\bar{x} = \bar{y}$, then $\deg(x - y) < \deg x = \deg y$.

Definition 3.2. If R is a filtered ring, the associated graded ring is $\operatorname{gr} R$ with multiplication defined by

$$\bar{x}\bar{y} = [xy + R_{i+j-1}]$$

whenever $\deg x = i$ and $\deg y = j$. If FM is a filtered R-module, then the associated graded module $\operatorname{gr} M$ is made into a graded $\operatorname{gr} R$-module by defining

$$\bar{x}\bar{m} = [xm + F_{i+j-1}M]$$

whenever $\deg x = i$ and $\deg m = j$. If $f \in \operatorname{Hom}_{FR}(FM, FN)$, we define $\bar{f} : \operatorname{gr} M \to \operatorname{gr} N$ by

$$\bar{f}(\bar{m}) = [f(m) + F_{i-1}N]$$

whenever $\deg m = i$.

If we removed the requirement that $\deg 1 = 0$ in the definition of a filtered ring, then $\operatorname{gr} R$ would not have an identity.

PROPOSITION 3.3. *The rules $M \mapsto \operatorname{gr} M$ and $f \mapsto \bar{f}$ define a functor*

$$\operatorname{gr} : \operatorname{Filt} R \to \mathsf{GrMod}(\operatorname{gr} R).$$

EXAMPLE 3.4. If R is a graded ring which is filtered by $F_n R = R_{\le n}$, then there is an isomorphism of graded rings $\operatorname{gr} R \cong R$.

The next example shows that the functor gr is not faithful, and Example 3.6 shows it is not exact.

EXAMPLE 3.5. If $R = k[x, x^{-1}]$ with its standard grading is filtered by the subspaces $F_n R = \{f \mid \deg f \le n\}$, then there is only one way to make $M = R/(x+1)$ a filtered R-module; moreover, $\operatorname{gr} M = 0$. To see this, first notice that $F_n M = M$ for some n, whence $M = xM = xF_nM \subset F_{n+1}M$ and $M = x^{-1}M = x^{-1}F_nM \subset F_{n-1}M$, so $F_iM = M$ for all i; thus $\operatorname{gr} M = 0$. This example also shows that $\operatorname{gr} M$ may be bounded below even if FM is not; of course, if M is bounded below, then $\operatorname{gr} M$ is.

EXAMPLE 3.6. Let $V = ke_0 \oplus ke_1$, and filter V by $F_0V = ke_0$, and $F_1V = V$. Also let $U = V$, and filter it by $F_0U = U$. Then the identity map $f : U \to V$ is a morphism in $\operatorname{Filt} k$, but $\bar{f}(e_1) = 0$. Thus, although $0 \to U \to V \to 0$ is exact in $\operatorname{Filt} k$, $0 \to \operatorname{gr} U \to \operatorname{gr} V \to 0$ is not exact in $\mathsf{GrMod} k$.

Nevertheless, if M is a filtered R-module, with a submodule L and quotient $N = M/L$, if L and N are given the induced filtrations, then $0 \to \operatorname{gr} L \to \operatorname{gr} M \to \operatorname{gr} N \to 0$ is exact. This is a special case of Lemma 3.9(1).

Warnings: 1. If M is a filtered R-module, and $x \in R$ and $m \in M$, then

$$\bar{x}\bar{m} = \begin{cases} 0 & \text{if } \deg xm < \deg x + \deg m, \\ \overline{xm} & \text{if } \deg xm = \deg x + \deg m. \end{cases}$$

2. If $f \in \operatorname{Hom}_{FR}(M, N)$, then

$$\bar{f}(\bar{m}) = \begin{cases} 0 & \text{if } \deg f(m) < \deg m, \\ \overline{f(m)} & \text{if } \deg f(m) = \deg m. \end{cases}$$

LEMMA 3.7. *Suppose that $M \in \operatorname{Filt} R$ is bounded below. If $\{m_\alpha\} \subset M$ and $\{\bar{m}_\alpha\}$ generates $\operatorname{gr} M$ as a $\operatorname{gr} R$-module, then $\{m_\alpha\}$ generates M as an R-module.*

PROOF. Write $N = \sum Rm_\alpha$. We prove by induction on i that $F_iM \subset N$. Since M is finitely generated, $F_iM = 0$ for $i \ll 0$, so we may assume that $F_{i-1}M \subset N$. Let $m \in F_iM$. By hypothesis, $\bar{m} = \sum \bar{x}_\alpha \bar{m}_\alpha$ for some $x_\alpha \in R_{i(\alpha)}$, whence $m - \sum_\alpha x_\alpha m_\alpha \in F_{i-1}M$; therefore $m \in N$. □

The hypothesis that FM be bounded below is necessary in Lemma 3.7: in Example 3.5, $\operatorname{gr} R$ is generated by $\overline{x+1}$ as a $\operatorname{gr} R$-module, but $x+1$ does not generate R as an R-module. Also notice that although the complex $0 \to R(x+1) \to R \to 0$ in $\operatorname{Filt} R$ is not exact, the complex $0 \to \operatorname{gr}(R(x+1)) \to \operatorname{gr} R \to 0$ is exact. The next two results throw some light on the relation between complexes in $\operatorname{Filt} R$ and complexes in $\mathsf{GrMod}(\operatorname{gr} R)$.

LEMMA 3.8. *Let $f \in \operatorname{Hom}_{FR}(M, N)$. Then*

1. *if FM is separated and $\bar{f}$ is injective, then f is injective;*
2. *if FN is bounded below and $\bar{f}$ is surjective, then f is surjective.*

PROOF. (1) If $f(m) = 0$, then $\bar{f}(\bar{m}) = 0$ also, so $\bar{m} = 0$, whence $m = 0$ because FM is separated.

(2) Let $n \in N$. If $n = 0$ then $n \in \operatorname{Im} f$, so suppose that $n \neq 0$; hence $\bar{n} \neq 0$ because FN is separated. By hypothesis, $\bar{n} = \bar{f}(\bar{m})$ for some $m \in M$, whence $\bar{n} = \overline{f(m)}$; thus $\deg(n - f(m)) < \deg n$. Since FN is bounded below, $F_iN \subset \operatorname{Im} f$ for $i \ll 0$, so we may argue by induction on $\deg n$ and suppose that $n - f(m) \in \operatorname{Im} f$, whence $n \in \operatorname{Im} f$. □

In Lemma 3.8(2) it is not enough to assume that FN is separated. If we filter $R = k[x, x^{-1}]$ as in Example 3.5, and $f : R \to R$ is multiplication by $x+1$, then $\bar{f}$ is surjective but f is not.

LEMMA 3.9. *Let*

$$L \xrightarrow{f} M \xrightarrow{g} N \tag{3-1}$$

be a complex in $\operatorname{Filt} R$*, and let*

$$\operatorname{gr} L \xrightarrow{\bar{f}} \operatorname{gr} M \xrightarrow{\bar{g}} \operatorname{gr} N \tag{3-2}$$

be the associated graded complex in $\mathsf{GrMod}(\operatorname{gr} R)$*. Then*

1. *if (3-1) is strict exact, then (3-2) is exact;*
2. *if (3-2) is exact, then g is strict;*
3. *if (3-2) is exact and FM is bounded below, then f is strict;*
4. *if FM is bounded below, then (3-2) is exact if and only if (3-1) is strict exact;*
5. *if FM is bounded below, then a map $f : L \to M$ (respectively, $g : M \to N$) is surjective (respectively, injective) and strict if and only if $\bar{f}$ is surjective (respectively, injective).*

PROOF. (1) Let $m \in M$ be such that $\bar{g}(\bar{m}) = 0$. Suppose that $\deg m = i$. Then $g(m) \in F_{i-1}N$. But g is strict, so $g(m) = g(m')$ for some $m' \in F_{i-1}M$. Because (3-1) is exact, $m - m' = f(\ell)$ for some $\ell \in F_iL$; in this equation $\deg m' < \deg m = i$, so $\bar{m} = \overline{f(\ell)}$, whence $\bar{m} = \bar{f}(\bar{\ell})$; thus $\ker \bar{g} \subset \operatorname{Im} \bar{f}$.

(2) To show that g is strict, consider $n \in \operatorname{Im} g$ and suppose that $\deg n = i$. Choose $m \in M$ of minimal degree such that $g(m) = n$. If $\deg m > i = \deg g(m)$, then $\bar{g}(\bar{m}) = 0$; because (3-2) is exact $\bar{m} = \bar{f}(\bar{\ell}) = \overline{f(\ell)}$ for some $\ell \in L$; it follows that $\deg(m - f(\ell)) < \deg m$, and $g(m - f(\ell)) = g(m) = n$, which contradicts the choice of m. So we conclude that $\deg m \leq i$, as required.

(3) To show f is strict, consider $m \in \operatorname{Im} f$ and suppose that $\deg m = i$. Since FM is bounded below, if $i \ll 0$, $m = 0$, in which case $m \in f(F_iL)$. So we argue by ascending induction on i. Since $m \in \operatorname{Im} f$, $g(m) = 0$, which implies that

$\bar{g}(\bar{m}) = 0$. Since (3-2) is exact, there exists $\ell \in F_iL$ such that $\bar{m} = \bar{f}(\bar{\ell}) = \overline{f(\ell)}$; thus $m - f(\ell) \in \operatorname{Im} f \cap F_{i-1}M$, so by induction $m - f(\ell) \in f(F_{i-1}L)$; hence $m \in f(F_iL)$ as required.

(4) ($\Rightarrow$) By (2) and (3), both f and g are strict, so it remains to show that $\ker g \subset \operatorname{Im} f$; to this end, suppose that $\deg m = i$ and $g(m) = 0$. If $i \ll 0$, then $m = 0$ because FM is bounded below, so we argue by induction on i. Now $\bar{g}(\bar{m}) = 0$ so, as in the proof of (2) with $n = 0$, there is $\ell \in L$ such that $\deg(m - f(\ell)) < \deg m$, and $m - f(\ell) \in \ker g$ also. By the induction hypothesis, $m - f(\ell) \in \operatorname{Im} f$, whence $m \in \operatorname{Im} f$, as required. The reverse implication was proved in part (1).

(5) This is a special case of (4) because the zero map is strict. □

The condition that FM be bounded below in Lemma 3.9 is easily satisfied if R is bounded below: if $R = R_{\geq c}$ and M is generated by $\{m_\alpha\}$, then the filtration defined by $M_n := \sum_\alpha R_n m_\alpha$ satisfies $M = M_{\geq c}$.

Definition 3.10. A filtered R-module FP is filtered-free on $\{p_\alpha\}$ if

- $\{p_\alpha\}$ is a basis for P as an R-module, and
- there are integers d_α such that, for all n, $F_nP = \oplus_\alpha R_{n-d_\alpha} p_\alpha$.

Since $\deg 1 = 0$, $\deg p_\alpha = d_\alpha$.

LEMMA 3.11. *Let $M, P \in \operatorname{Filt} R$.*

1. *If FP is filtered-free on $\{p_\alpha\}$, then $\operatorname{gr} P$ is a free module on the basis $\{\bar{p}_\alpha\}$.*
2. *If FR and FP are bounded below and $\operatorname{gr} P$ is free on $\{\bar{p}_\alpha\}$, then P is filtered-free on $\{p_\alpha\}$.*
3. *If $M \in \operatorname{Filt} R$ is bounded below, then there is a strict surjection $f : FQ \to FM$ from some filtered-free module FQ; moreover, if FR is bounded below, then FQ can be chosen to be bounded below.*

PROOF. (1) Since $F_nP = \oplus R_{n-d_\alpha} p_\alpha$,

$$(\operatorname{gr} P)_n = F_nM/F_{n-1}M \cong \bigoplus_\alpha \left(\frac{R_{n-d_\alpha}}{R_{n-d_\alpha-1}} \right) \bar{p}_\alpha = \bigoplus_\alpha (\operatorname{gr} R)_{n-d_\alpha} \bar{p}_\alpha,$$

as required.

(2) Let $d_\alpha = \deg p_\alpha$. Let Q be a filtered-free R-module on basis $\{q_\alpha\}$, with filtration $F_nQ = \oplus R_{n-d_\alpha} q_\alpha$, and define $f : Q \to P$ by $f(q_\alpha) = p_\alpha$. Consider the complexes

$$0 \to Q \xrightarrow{f} P \to 0 \qquad \text{and} \qquad 0 \to \operatorname{gr} Q \xrightarrow{\bar{f}} \operatorname{gr} P \to 0.$$

By hypothesis $\bar{f}$ is an isomorphism, and both Q and P are bounded below; so by Lemma 3.9(5), f is strict and bijective, and hence an isomorphism by Lemma 1.10.

(3) Choose $\{m_\alpha\}$ such that $\{\bar{m}_\alpha\}$ generates $\operatorname{gr} M$, and let $d_\alpha = \deg m_\alpha$. Let Q be filtered-free on $\{q_\alpha\}$ with the filtration $F_nQ = \oplus_\alpha R_{n-d_\alpha} q_\alpha$. Thus $\deg q_\alpha = d_\alpha$. Let $f : Q \to M$ be the R-module map defined by $f(q_\alpha) = m_\alpha$. Clearly $\operatorname{gr} Q = \oplus R \bar{q}_\alpha$ is free, and

$$\bar{f}(\bar{q}_\alpha) = [f(q_\alpha) + F_{d_\alpha - 1}M] = \bar{m}_\alpha,$$

so $\bar{f}$ is surjective. Hence f is surjective and strict. Finally, since M is bounded below there is a lower bound on the d_α, whence Q is bounded below if R is. □

LEMMA 3.12. *Let $P, M \in \operatorname{Filt} R$. If FP is filtered-free, then the natural map $\operatorname{Hom}_{FR}(P, M) \to \operatorname{Hom}_{\operatorname{gr} R}(\operatorname{gr} P, \operatorname{gr} M)$ is surjective.*

Proof. Suppose that P is filtered-free on $\{p_\alpha\}$ and write $d_\alpha = \deg p_\alpha$. Let $g \in \operatorname{Hom}_{\operatorname{gr} R}(\operatorname{gr} P, \operatorname{gr} M)$ and choose $m_\alpha \in M$ such that $g(\bar{p}_\alpha) = \bar{m}_\alpha$ and $\deg m_\alpha = d_\alpha$. If $f \in \operatorname{Hom}_{FR}(P, M)$ is defined by $f(p_\alpha) = m_\alpha$, then $\bar{f}(\bar{p}_\alpha) = \bar{m}_\alpha$ because $\deg p_\alpha = \deg m_\alpha$, whence $\bar{f} = g$. $\square$

Proposition 3.13. *Suppose that R and $P \in \operatorname{Filt} R$ are bounded below. If $\operatorname{gr} P$ is projective in $\mathsf{GrMod}(\operatorname{gr} R)$, then P is projective in $\mathsf{Mod}R$.*

Proof. By Lemma 3.11, there is a filtered-free module Q which is bounded below and a strict surjection $\pi \in \operatorname{Hom}_{FR}(Q, P)$. Let $K = \ker \pi$, and let $i : K \to Q$ be the inclusion. Thus $0 \to K \to Q \to P \to 0$ is strict exact. Hence

$$0 \to \operatorname{gr} K \xrightarrow{\bar{i}} \operatorname{gr} Q \xrightarrow{\bar{\pi}} \operatorname{gr} P \to 0$$

is exact in $\mathsf{GrMod}(\operatorname{gr} R)$; it splits, so there is a map $g \in \operatorname{Hom}_{\operatorname{gr} R}(\operatorname{gr} Q, \operatorname{gr} K)$ such that $g \circ \bar{i} = \mathbb{1}$. By Lemma 3.12, $g = \bar{f}$ for some $f \in \operatorname{Hom}_{FR}(Q, K)$. Since $\bar{f} \circ \bar{i} = \overline{f \circ i}$ is an isomorphism, and K is bounded below because Q is, Lemma 3.9(5) shows that $f \circ i$ is bijective, hence an isomorphism by Lemma 1.10. Therefore the map $K \to Q$ splits, whence P is projective. $\square$

Lemma 3.14. *Let M be a filtered R-module, and let*

$$0 \to K \to P_n \to \cdots \to P_0 \to \operatorname{gr} M \to 0 \tag{3-3}$$

be an exact sequence in $\mathsf{GrMod}(\operatorname{gr} R)$ with each P_i a free graded module. Then there is an exact sequence

$$0 \to L \to Q_n \to \cdots \to Q_0 \to M \to 0 \tag{3-4}$$

in $\operatorname{Filt} R$ such that each Q_i is filtered-free, and the associated graded complex of (3-4) is (3-3).

Proof. Consider the short exact sequence $0 \to K_0 \to P_0 \to \operatorname{gr} M \to 0$ at the right hand end of (3-3). Let Q_0 be a filtered-free module with a basis in bijection with a homogeneous basis for P_0, and corresponding basis elements having the same degree. Then $\operatorname{gr} Q_0 \cong P_0$. By Lemma 3.12, there exists $f \in \operatorname{Hom}_{FR}(Q_0, M)$ such that $\bar{f}$ is the given map $P_0 \to \operatorname{gr} M$. Let $L_0 = \ker f$ and give it the filtration induced from Q_0.Then $0 \to L_0 \to Q_0 \to M \to 0$ is strict exact, whence $0 \to \operatorname{gr} L_0 \to \operatorname{gr} Q_0 \to \operatorname{gr} M \to 0$ is exact. Thus $\operatorname{gr} L_0 \cong K_0$. Now repeat the construction for $0 \to K_1 \to P_1 \to K_0 \to 0$, eventually getting an exact sequence of the form (3-4) which has the required properties. $\square$

Proposition 3.15. *Suppose that R is a filtered ring which is bounded below, and let $M \in \mathsf{Mod}R$. Then there is a filtration on M such that*

$$\operatorname{pdim}_R M \le \operatorname{pdim}_{\operatorname{gr} R} \operatorname{gr} M.$$

Proof. Choose any subset $V \subset M$ which generates M as an R-module and define $F_i M = R_i V$; then M is bounded below with respect to this filtration. Let $n = \operatorname{pdim}_{\operatorname{gr} R} \operatorname{gr} M$. If $n = \infty$ the result is true, so suppose that $n < \infty$. Then there is a projective resolution for $\operatorname{gr} M$ in $\mathsf{GrMod}(\operatorname{gr} R)$ of the form

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to \operatorname{gr} M \to 0 \tag{3-5}$$

with each P_i a free graded module. Moreover, each P_i may be taken to be bounded below since $\operatorname{gr} M$ and $\operatorname{gr} R$ are. Hence K is also bounded below. By Lemma 3.14, there is an exact sequence in $\operatorname{Filt} R$ of the form

$$0 \to L \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0, \tag{3-6}$$

such that each Q_i is filtered-free, and the associated graded complex of (3-6) is (3-5). By construction each Q_i is bounded below, hence so is L. But $\operatorname{gr} L \cong K$ so, by Proposition 3.13, L is a projective R-module. Hence $\operatorname{pdim}_R M \le n$. □

COROLLARY 3.16. *If R is a filtered ring which is bounded below, then*

$$\operatorname{gldim} R \le \text{gr-gldim}(\operatorname{gr} R).$$

This corollary fails without the hypothesis that R is bounded below: if $R = k[x, x^{-1}]$ with filtration $F_n R = \{f \in R \mid \deg f \le n\}$, then $\operatorname{gr} R \cong R$ and $\text{gr-gldim}(\operatorname{gr} R) = 0$, but $\operatorname{gldim} R = 1$.

THEOREM 3.17. *If A is $\mathbb{N}$-graded, then* $\operatorname{gldim} A = \text{gr-gldim} A$.

PROOF. As a filtered algebra with $F_n A = A_{\le n}$, A is bounded below, so the result follows from the inequality in the Corollary 3.16 together with the reverse inequality established in Proposition 7.7.5. □

EXERCISES

3.1 Let $R = k[x, y]$ with defining relation $xy = yx = 0$. Show that R is a filtered ring via

$$R_n = \begin{cases} kx^n + kx^{n-1} + \cdots + kx + k + ky^1 + ky^2 + \cdots & \text{if } n \ge 0, \\ ky + ky^2 + \cdots & \text{if } n < 0. \end{cases}$$

Show that
(a) $\operatorname{gr} R$ is bounded below, although R is not;
(b) $\operatorname{gr} R$ is a domain, although R is not;
(c) $P = xR$ with the induced filtration is not filtered-free on x, although FP is bounded below and $\operatorname{gr} P$ is a free $\operatorname{gr} R$-module on $\bar{x}$.

4. Transfer of properties from $\operatorname{gr} R$ to R

PROPOSITION 4.1. *Let $M \in \operatorname{Filt} R$. If $\operatorname{gr} M$ is a left noetherian $\operatorname{gr} R$-module, then M is a left noetherian R-module.*

PROOF. □

COROLLARY 4.2. *Suppose that a ring S is generated by a subring R and elements $x_1, \ldots, x_n$ which satisfy*
- *$R + x_i R = R + x_i R$ for all i, and*
- *$x_i x_j - x_j x_i \in R + \sum_{p=1}^n R x_p$ for all i and j.*

If R is left noetherian, so is S.

PROOF. Our proof follows that in [**117**, Section 1.6], where S is called an almost normalizing extension of R.

We make S a filtered ring by defining $F^0 S = R$, and $F^1 S = R + \sum_{p=1}^n R x_i$, and $F^m S = (F^1 S)^m$. We will show that
- $\operatorname{gr} S$ is generated by R and $\bar{x}_1, \ldots, \bar{x}_n$,
- in $\operatorname{gr} S$, $R\bar{x}_i = \bar{x}_i R$, and
- $\bar{x}_i \bar{x}_j = \bar{x}_j \bar{x}_i$,

then show this implies that $\operatorname{gr} S$ is left noetherian. The proof is then completed by invoking Proposition 4.1.

To see that the first of these claims holds, let $\bar{y} \in (\operatorname{gr} S)_m$, where $y \in F^m S$ has degree m. Then, up to elements of degree $< m$, y is a finite sum of terms of the form

$$s = r_1 x_{i_1} r_2 x_{i_2} \cdots r_m x_{i_m}.$$

But, either $\bar{s} = 0$ or

$$\bar{s} = \bar{r}_1 \bar{x}_{i_1} \bar{r}_2 \bar{x}_{i_2} \cdots \bar{r}_m \bar{x}_{i_m}$$

and this proves the first claim. The second and third claims are obvious.

There is a chain of subalgebras $R \subset R[\bar{x}_1] \subset R[\bar{x}_1][\bar{x}_2] \subset \cdots \subset \operatorname{gr} S$, so to prove that $\operatorname{gr} S$ is left noetherian it suffices to prove that if R is left noetherian, and $R\bar{x} = \bar{x}R$, then $R[\bar{x}]$ is left noetherian. But this is a consequence of Proposition 11.8.3, since $R[\bar{x}]/(\bar{x})$ is a quotient of R, and therefore left noetherian. □

A typical application of Corollary 4.2 occurs in Example 23.1.5.

Proposition 4.3. *If* $\operatorname{gr} R$ *is a domain, so is* R.

Proof. □

The converse is false: if $R = k[x]$ is made into a filtered ring by defining $R_i = k + kx + \cdots + kx^{2i}$, then $\bar{x}\bar{x} = 0$.

A module over a filtered ring may be given the structure of a filtered R-module in many ways, but some filtrations are better than others.

Definition 4.4. If M is a filtered R-module, the filtration on M is said to be good if $\operatorname{gr} M$ is a finitely generated $\operatorname{gr} R$-module.

Lemma 4.5. *Every* $M \in \mathsf{mod} R$ *has a good filtration.*

Proof. Let $\{m_\alpha\}$ be a finite set of generators for M, and define a filtration $F_i M := \sum_\alpha R_i m_\alpha$ making M a filtered R-module. If $\deg m = i$ and $m = \sum x_\alpha m_\alpha$ with $x_\alpha \in R_i$, then

$$\bar{m} = \sum \bar{x}_\alpha \bar{m}_\alpha.$$

so

????? □

Let $M, N \in \operatorname{Filt} R$. The degree of a homomorphism $f \in \operatorname{Hom}_R(M, N)$ is defined in Definition 1.2, and $F_p \operatorname{Hom}_R(M, N)$ denotes those maps of degree $\leq p$. We define

$$\underline{\operatorname{FHom}}_R(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Hom}_R(M, N).$$

It is easy to construct examples showing that this union may not be all of $\operatorname{Hom}_R(M, N)$. However, if M is finitely generated it is (cf. Proposition 11.2.2).

If R is left noetherian and M is finitely generated, then M has a resolution in $\operatorname{Filt} R$ in which the terms are all finitely generated free modules. It follows that $\operatorname{Ext}^i_R(M, N)$ is filtered by subspaces $F_p \operatorname{Ext}^i_R(M, N)$, and it is natural to expect that the associated graded object $\operatorname{gr} \operatorname{Ext}^i_R(M, N)$ is related to $\underline{\operatorname{Ext}}^i_{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N)$. This is the case, as we now proceed to show.

There is a natural map

$$\varphi : \operatorname{gr} \underline{\operatorname{FHom}}_R(M, N) \to \underline{\operatorname{Hom}}_{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N)$$

defined as follows: if $f \in \underline{\mathrm{FHom}}_R(M,N)$ with $\deg f = d$, and $\bar{f} = [f + F_{d-1}\operatorname{Hom}_R(M,N)]$ then

$$\varphi(\bar{f})(\bar{m}) := [f(m) + F_{d+i-1}N] \in (\operatorname{gr} N)_{d+i}$$

whenever $m \in M$ with $\deg m = i$.

LEMMA 4.6. *The map φ is injective, and is an isomorphism if M is filtered-free.*

PROOF. Let $0 \neq f \in \underline{\mathrm{FHom}}_R(M,N)$. By definition of $\deg f$, there exists $m \in M$ such that $\deg f(m) = \deg f + \deg m$. Thus $\varphi(\bar{f})(\bar{m}) \neq 0$, whence $\varphi(\bar{f}) \neq 0$. Thus φ is injective. An adaptation of the proof of Lemma 3.12 shows that φ is surjective when M is filtered free. □

THEOREM 4.7. *Let M and N be finitely generated modules over a filtered left noetherian ring R. Then there is a convergent spectral sequence*

$$E_1^{pq} = \underline{\mathrm{Ext}}^{p+q}_{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N)_p \Rightarrow \operatorname{Ext}^{p+q}_R(M,N).$$

PROOF. By Lemma 3.14, there is a resolution

$$\cdots \to Q_1 \to Q_0 \to M \to 0$$

by finitely generated filtered-free modules such that

$$\cdots \to \operatorname{gr} Q_1 \to \operatorname{gr} Q_0 \to \operatorname{gr} M \to 0$$

is a free resolution of $\operatorname{gr} M$ in $\mathsf{grmod}(\operatorname{gr} R)$. The individual terms in the complex

$$C^* : \qquad 0 \to \operatorname{Hom}_R(Q_0,N) \to \operatorname{Hom}_R(Q_1,N) \to \cdots$$

have an ascending filtration, and the differential is a map of filtered abelian groups. By the usual notational chicanery we may raise indices and replace them by their negatives to obtain descending filtrations, thus making C^* a filtered complex in the sense of Definition 8.2.1. The E_1-page of the spectral sequence associated to C^* has terms $E_1^{pq} = H^{p+q}(F_pC/F_{p-1}C)$. But

$$F_pC^n/F_{p-1}C^n = F_p\operatorname{Hom}_R(Q_n,N)/F_{p-1}\operatorname{Hom}_R(Q_n,N)$$

which is isomorphic to $\underline{\mathrm{Hom}}_{\operatorname{gr} R}(\operatorname{gr} Q_n, \operatorname{gr} N)_p$ by Lemma 4.6. Thus

$$E_1^{pq} \cong h^{p+q}(\underline{\mathrm{Hom}}_{\operatorname{gr} R}(\operatorname{gr} Q_\bullet, \operatorname{gr} N)_p \cong \underline{\mathrm{Ext}}^{p+q}_{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} N)_p,$$

as required. On the other hand,

$$E_\infty^{pq} = F_p h^{p+q}(C)/F_{p-1}h^{p+q}(C) = \operatorname{Ext}^{p+q}_R(M,N).$$

This completes the proof.

Convergence ???????????? □

COROLLARY 4.8. *Let R be a filtered ring, and $M \in \operatorname{Filt} R$. Then there is a natural filtration on $\operatorname{Ext}^i_R(M,R)$, and $\operatorname{gr}\operatorname{Ext}^i_R(M,R)$ is isomorphic to a subfactor of $\operatorname{Ext}^i_{\operatorname{gr} R}(\operatorname{gr} M, \operatorname{gr} R)$.*

5. The Rees ring construction

The first result in this section shows that an arbitrary k-algebra can be written as a quotient of an $\mathbb{N}$-graded k-algebra by a single central element. This is analogous to expressing an affine variety as a dense open subset of a projective variety.

PROPOSITION 5.1. *If R is a k-algebra, there exists a graded k-algebra $\tilde{R}$ and a central element $z \in \tilde{R}_1$ such that $\tilde{R}/(z-1) \cong R$.*

PROOF. Choose a subspace $V \subset R$ which generates R as a k-algebra. Then there is a surjective k-algebra map $\varphi : T(V) \to R$. Let $U = V \oplus kz$, where z is a new variable, and consider $T(V)$ as a subalgebra of $T(U)$. Give $T(U)$ its standard grading, namely $\deg U = 1$. If $f = \sum_i f_i \in T(V)$ with $\deg f_i = i$ and $\deg f = n$, define $\tilde{f} := \sum_i f_i z^{n-i}$. Thus $\tilde{f}$ is a homogeneous element of degree n in $T(U)$. Now define

$$\tilde{R} := T(U)/\tilde{I}$$

where I is the ideal generated by

$$\{\tilde{f} \mid f \in \ker \varphi\} \bigcup \{vz - zv \mid v \in V\}.$$

It is clear that $\tilde{R}$ has the required properties. □

There are many choices of the generating subspace V in the proof of Proposition 5.1, so there is no uniqueness to the ring $\tilde{R}$. To make the dependence of $\tilde{R}$ on the choice of V explicit, we may construct $\tilde{R}$ directly from the filtration $F_nR = k + V + \cdots + V^n$ as follows.

Definition 5.2. The Rees ring of a filtered ring R is the subring $\tilde{R}$ of the commuting Laurent extension $R[X, X^{-1}]$ defined by

$$\tilde{R} := \bigoplus_{n\in\mathbb{Z}} R_n X^n.$$

We view $\tilde{R}$ as a graded ring with $\deg R_nX^n = n$; that is R is concentrated in degree zero, and $\deg X = 1$. If M is a filtered R-module, we define the Rees module of M to be

$$\tilde{M} := \bigoplus_{n\in\mathbb{Z}} M_n X^n \subset M[X, X^{-1}].$$

This is a graded $\tilde{R}$ module in the obvious way, with $\deg M_nX^n = n$.

LEMMA 5.3. *Let $\tilde{R} \subset R[X, X^{-1}]$ be the Rees ring of a filtered ring R. Then X is a central regular element in $\tilde{R}_1$, and*

$$\tilde{R}/(X-1) \cong R, \qquad \textit{and} \qquad \tilde{R}/(X) \cong \operatorname{gr} R.$$

PROOF. Exercise. □

PROPOSITION 5.4. *Let R be a filtered ring, and let $\tilde{R}$ be the associated Rees ring. There is a commutative diagram of categories and functors*

$$\begin{array}{ccccc} \operatorname{Filt} R & \longrightarrow & \mathsf{GrMod}(\tilde{R}) & \longrightarrow & \mathsf{GrMod}(\operatorname{gr} R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Mod}R & \longrightarrow & \mathsf{Mod}(\tilde{R}) & \longrightarrow & \mathsf{Mod}(\operatorname{gr} R) \end{array} \tag{5-1}$$

where

- *the vertical maps are forgetful functors, and ....*
- *the horizontal maps consist of adjoint pairs of functors,*
- *the functor* $\operatorname{Filt} R \to \mathsf{GrMod}\tilde{R}$*, which is given by* $M \mapsto \tilde{M}$*, is an equivalence between* $\operatorname{Filt} R$ *and the full subcategory of* $\mathsf{GrMod}\tilde{R}$ *consisting of the* X*-torsionfree modules.*

Proof. Write $\mathcal{F}_X$ for the full subcategory of $\mathsf{GrMod}\tilde{R}$ consisting of the X-torsionfree modules. If $M \in \operatorname{Filt} R$, then $\tilde{M}$ is torsionfree because, if $m \in F_n M$ and $X.mX^n = 0$, then $mX^{n+1} \in F_{n+1}MX^{n+1} = 0$, which means that $m = 0$. Thus the functor $M \to \tilde{M}$ sends M to an X-torsionfree module. Now, suppose that $N \in \mathsf{GrMod}(\tilde{R})$ is X-torsionfree. Muliplication by X gives a directed system $\cdots \to N_n \to N_{n+1} \to \cdots$. Define $M = \lim_{\to} N_n$, and make it into a filtered R-module as follows: F_nM is the canonical image of N_n in M; if $r \in F_iR$ and $m \in F_jM$, let $m' \in N_j$ represent m, and set rm denote the image of $rX^i.m' \in N_{i+j}$ in $F_{i+j}M$. Since the X acts injectively on N, the natural maps $F_iM \to F_{i+1}M$ are injective.

The rest is straightforward. □

The localization of $\tilde{R}$ at $\{1, X, X^2, \cdots\}$, say $\tilde{R}_X$, is isomorphic to $R[X, X^{-1}]$; similarly, $\tilde{M}_X = \tilde{R}_X \otimes_{\tilde{R}} \tilde{M} \cong M[X, X^{-1}]$. The functors

$$\tilde{R}/(X) \otimes_{\tilde{R}} - : \mathcal{F}_X \to \mathsf{Mod}(\operatorname{gr} R),$$

and

$$\tilde{R}/(X-1) \otimes_{\tilde{R}} - : \mathsf{GrMod}(\tilde{R}) \to \operatorname{Filt} R,$$

are both exact.

$\boxed{\text{Paul}}$ Homogenizations of enveloping algebras

CHAPTER 5

Gelfand-Kirillov dimension and growth of algebras

In this chapter k is a field, and all rings are k-algebras.

1. Gelfand-Kirillov dimension

Definition 1.1. Let R be a finitely generated k-algebra. A frame for R is a subspace V, containing 1, which generates R as a k-algebra.

LEMMA 1.2. *Let R and S be k-algebras, and M an R-S-bimodule. If ${}_RM$ is finitely generated, then*

1. $\operatorname{GKdim} M_S = \operatorname{GKdim} S/\operatorname{Ann} M$*;*
2. $\operatorname{GKdim} M_S \leq \operatorname{GKdim} {}_RM$*;*
3. *if M_S is finitely generated, then* $\operatorname{GKdim} M_S = \operatorname{GKdim} {}_RM$.

PROOF. [**92**, Lemma 5.3] □

The following fundamental result of Bergman is proved in [**92**, Theorem ??].

THEOREM 1.3 (Bergman). *If R is a finitely generated k-algebra, then*

- $\operatorname{GKdim} R \in \{0,1\} \cup [2,\infty)$;
- *if* $\operatorname{GKdim} R = 1$*, then for every frame V, $f_V(n) \leq cn$ for some $c \in \mathbb{R}$.*

2. Gelfand-Kirillov dimension for graded algebras

Gelfand-Kirillov dimension, denoted $\operatorname{GKdim}(-)$, measures the 'size'— more accurately, the 'rate of growth'—of a locally finite graded vector space. It is defined in terms of the rate of growth of the function $n \mapsto \dim_k V_n$. We will see that $\operatorname{GKdim} V = 0$ if and only if $\dim_k V < \infty$, that the Gelfand-Kirillov dimension of the polynomial ring on n indeterminates, with its usual grading, is n, and that the free algebra on more than one variable has infinite Gelfand-Kirillov dimension. Typically, the graded algebras we consider are roughly the same size as polynomial rings—in particular, they have finite GK-dimension.

All graded vector spaces in this section will be assumed to be locally finite. The general considerations will be applied to finitely generated modules over a locally finite graded algebra.

Definition 2.1. Let V be a locally finite graded k-vector space. The Gelfand-Kirillov dimension, or GK-dimension, of V is

$$\operatorname{GKdim} V := \inf\{\rho \in \mathbb{R}^+ \mid \dim_k V_n \leq n^{\rho-1} \text{ for all } n \gg 0\}.$$

If this set is empty we say V has infinite GK-dimension, and write $\operatorname{GKdim} V = \infty$. We also write $d(V) = \operatorname{GKdim} V$.

Warning. This definition of GK-dimension differs from the usual one, although they coincide in good situations. For example, GK-dimension is not defined for all graded modules, just the locally finite ones.

Shifting does not alter GK-dimension: that is, $\operatorname{GKdim} V[r] = \operatorname{GKdim} V$ for all $r \in \mathbb{Z}$.

If V is a graded vector space we call $V_{\geq n}$ the tail of V; strictly speaking, we should call this *a* tail. It is an important consequence of the definition that $\operatorname{GKdim} V$ depends just on the tail of V.

LEMMA 2.2. *Let V be a locally finite graded k-vector space which is bounded below. Then*

1. $\operatorname{GKdim} V = 0$ *if and only if* $\dim_k V < \infty$;
2. *if V is not right bounded then* $\operatorname{GKdim} V \geq 1$;
3. $\operatorname{GKdim} V$ *takes values in* $\{0\} \cup [1, \infty) \cup \{\infty\}$;
4. *if $\dim V_n$ is bounded, but V is not right bounded, then* $\operatorname{GKdim} V = 1$.

PROOF. (1) If $d(V) = 0$ then, for all $0 < \rho < 1$, $\dim V_n < n^{\rho-1}$ for all $n \gg 0$. Since $n^{\rho-1} \to 0$ as $n \to \infty$, $\dim V_n \to 0$ too. Hence $\dim V_n = 0$ for $n \gg 0$, so V is finite dimensional. Conversely, if V is finite dimensional then, for any $\rho > 0$, $\dim V_n = 0 < n^{\rho-1}$ for all $n \gg 0$; thus $d(M) = 0$.

(2) For any $\rho < 1$, $n^{\rho-1} \to 0$ as $n \to \infty$ so, if V is not right bounded, $\dim V_n \geq n^{\rho-1}$ for infinitely many n; thus $\operatorname{GKdim} V \geq 1$.

(4) If $\dim V_n$ is bounded but V is not right bounded then, for any $\rho > 1$, $\dim V_n len^{\rho-1}$ for $n \gg 0$, so $\operatorname{GKdim} V \leq 1$; hence $\operatorname{GKdim} V = 1$. □

EXAMPLE 2.3. There exists a graded vector space V of GK-dimension 1 with $\dim V_n$ unbounded. It is a standard example in real analysis that

$$\lim_{n\to\infty} \frac{\log n}{n^{\epsilon}} = 0$$

for all $\epsilon > 0$. Hence, if $\dim V_n = [\log(n)]$ then, for all $\epsilon > 0$, $\dim V_n \leq n^{\epsilon}$ for all $n \gg 0$. Thus $\operatorname{GKdim} V = 1$.

LEMMA 2.4. *All vector spaces in this Lemma are assumed to be graded and locally finite.*

1. *If* $0 \to U \to V \to W \to 0$ *is an exact sequence in* $\mathsf{GrMod}(k)$, *then*
$$\operatorname{GKdim} V = \max\{\operatorname{GKdim} U, \operatorname{GKdim} W\}.$$
2. *If U is a finite direct sum of shifts of V then* $\operatorname{GKdim} U = \operatorname{GKdim} V$.
3. *If M is a finitely generated graded A-module then*
$$\operatorname{GKdim} M \leq \operatorname{GKdim} A.$$

PROOF. (1) (This does not hold for the usual definition of GK-dimension.) Since the sequence splits we may assume that $\operatorname{GKdim} U \leq \operatorname{GKdim} W = \alpha$, say. Let $\epsilon > 0$. Then, for $n \gg 0$,

$$\dim V_n = dim U_n + \dim W_n \leq 2n^{\alpha+\epsilon-1}.$$

Since $\lim_{n\to\infty} n^{\epsilon} = \infty$, $\dim V_n \leq n^{\alpha+2\epsilon-1}$ for $n \gg 0$. Since this is true for all $\epsilon > 0$, $\operatorname{GKdim} V \leq \alpha$. The reverse inequality holds because, for $n \gg 0$, $n^{\alpha-\epsilon-1} \leq \dim W_n \leq \dim V_n$.

(2) If (1) is interpreted as saying that

$$\operatorname{GKdim}(U \oplus W) = \max\{\operatorname{GKdim} U, \operatorname{GKdim} W\},$$

then this follows from repeated application of (1), taking note of the fact that GK-dimension is not changed by shifting the grading.

(3) Since M is finitely generated, it is a quotient of a finite direct sum of shifts of A. Now apply (2) and (1). □

By the Lemma, for a graded module M over a graded algebra A, one has

$$\operatorname{GKdim} M = \max\{\operatorname{GKdim} N, \operatorname{GKdim} M/N\}$$

for every graded submodule $N \subset M$.

LEMMA 2.5. *Let V be an infinite dimensional, locally finite, bounded below, graded vector space. If $H_V(t)$ has radius of convergence r, then*

1. $r \leq 1$, *and*
2. $r = 1$ *if and only if* $\operatorname{GKdim} V < \infty$.

PROOF. We may suppose that $V = V_{\geq 0}$. Write $H_V(t) = \sum_{n\geq 0} a_n t^n$. Since $\dim V = infty$, the series $\sum_{n\geq 0} a_n$ diverges, which proves (1).

Suppose that (2) is false. That is, suppose $r < 1$ and $\operatorname{GKdim} V < rho < \infty$. Then there exists $\epsilon > 0$ such that $\sum a_n(1-\epsilon)^n$ diverges. By hypothesis, there exists an integer N such that $a_n \leq n^\rho$ for all $n > N$. Therefore

$$\sum_{n\geq N} a_n(1-\epsilon)^n \leq \sum_{n\geq N} n^\rho(1-\epsilon)^n.$$

The ratio of the $(n+1)^{\text{th}}$ term to the n^{th} term in the second series is $(1+\frac{1}{n})^\rho(1-\epsilon)$; there exists $\delta < 1$ such that this ratio is $\leq \delta$ for sufficiently large n, whence the series converges (by d'Alemberts Test). Thus $\sum a_n(1-\epsilon)^n$ converges; this contradiction completes the proof. □

LEMMA 2.6. *Let V be a locally finite graded vector space which is bounded below and suppose that* $\operatorname{GKdim} V < \infty$. *If $H_V(t)$ is algebraic over the field $\mathbb{Q}(t)$ then, for some $f(t) \in \mathbb{Z}[t, t^{-1}]$,*

$$H_V(t) = f(t)\prod_{i=1}^{s}(1-t^{d_i})^{-1}.$$

PROOF. □

EXAMPLE 2.7. If $r > 1$ the free algebra $A = k\langle x_1, \dots, x_r\rangle$ (with its usual grading) has infinite GK-dimension because $\dim A_n = r^n$ is not bounded by n^ρ for any $\rho \in \mathbb{R}$.

Definition 2.8. If V is a graded vector space for which there exists a polynomial $f(n)$ such that $\dim V_n \leq f(n)$ whenever $n \gg 0$, we sya that V has polynomial growth. If, on the other hand there exists $\alpha > 1$ such that $\dim V_n \geq \alpha^n$ for $n \gg 0$, we say that V has exponential growth.

EXAMPLE 2.9. The algebra $A = k[x, y]$ with defining relation $y^2 = 0$ has exponential growth.

A basis for A_n is given by those words of length n in x and y which do not contain y^2 as a subword. This basis may be written as the disjoint union $\mathcal{B}_n \cup \mathcal{B}'_n$, where $\mathcal{B}_n$ are those basis words beginning with x and $\mathcal{B}'_n$ are those beginning with y. Thus, for $n \geq 1$,

$$\mathcal{B}_{n+1} = x(\mathcal{B}_n \cup \mathcal{B}'_n) \qquad \text{and} \qquad \mathcal{B}'_{n+1} = y\mathcal{B}_n = yx(\mathcal{B}_{n-1} \cup \mathcal{B}'_{n-1})$$

whence, by induction, $a_{n+1}) = a_n + a_{n-1}$ where $a_n = \dim A_n$. Hence $\dim A_n$ is given by the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$. In particular, $a_{n+2} \geq 2a_n$ so $a_{2n} \geq 2^{n-1}a_2$, which proves the claim that A has exponential growth.

Usually we restrict attention to graded algebras of polynomial growth; since we intend to develop a non-commutative version of projective algebraic geometry, and the commutative algebras arising there have finite GK-dimension, this is a reasonable restriction. The graded modules of interest will be finitely generated, so they too will have finite GK-dimension by Lemma 2.4.

If A is a graded quotient of a polynomial ring then $\operatorname{GKdim} A$ equals the dimension of the affine variety determined by A. Thus in some sense the GK-dimension of a non-commutative graded algebra is equal to the dimension of the (non-existent) affine variety corresponding to it!

We further refine the way in which we measure the size of a graded vector space: we want to compare the size of two graded vector spaces having the same GK-dimension: the idea is that if $\dim V_n = 2n^5$ whereas $\dim V'_n = 3n^5$ for large n, then V' grows faster that V.

Definition 2.10. Suppose that V is a locally finite graded vector space which is bounded below. If $H_V(t)$ has a pole at $t = 1$ of order d, we define the multiplicity of V to be

$$e(V) := (1 - t)^d H_V(t)\,|_{t=1}.$$

Shifting the grading does not change the multiplicity since $H_{V[-r]}(t) = t^r H_V(t)$. If $\operatorname{GKdim} V \geq 1$, then $e(V)$ depends only on the tail of V.

For a graded algebra A and $M \in \mathsf{GrMod}(A)$ it might be better to measure $e(M)$ relative to $e(A)$; perhaps $\epsilon(M) := e(M)/e(A)$ is a better invariant. Of course if we restrict attention to modules over a single graded algebra it makes little difference if we use $e(M)$ or $\epsilon(M)$, but if we wish to compare two different algebras it does. For example, let $A = k[T]$ with $\deg T = r > 0$, and let $M = A$; then $\epsilon(M) = 1$ but $e(M) = \frac{1}{r}$.

Multiplicity behaves well on short exact sequences.

LEMMA 2.11. *Suppose that $0 \to U \to V \to W \to 0$ is an exact sequence in $\mathsf{GrMod}(k)$ and that each of these vector spaces has GK-dimension $< \infty$.*

1. *If* $\operatorname{GKdim} U = \operatorname{GKdim} V = \operatorname{GKdim} W$ *then* $e(V) = e(U) + e(W)$.
2. *If* $\operatorname{GKdim} W < \operatorname{GKdim} V$ *then* $e(U) = e(V)$.
3. *If* $\operatorname{GKdim} U < \operatorname{GKdim} V$ *then* $e(W) = e(V)$.

PROOF. This is straightforward. $\square$

Definition 2.12. A locally finite graded A-module, M say, is d-critical if

- $\operatorname{GKdim} M = d$, and
- $\operatorname{GKdim}(M/N) < d$ for all non-zero graded submodules N.

By Lemma 2.4(1), if M is critical and $0 \neq N \subset M$, then $\operatorname{GKdim} N = \operatorname{GKdim} M$.

A noetherian module, M say, always has a critical quotient module: if N is maximal, then M/N is critical. More generally, if M has a quotient module of GK-dimension d, then it has a d-critical quotient: if N is maximal such that $\operatorname{GKdim}(M/N) = d$, then M/N is d-critical.

LEMMA 2.13. *If R is noetherian, and M a finitely generated bimodule, then* $\operatorname{GKdim}(A/\operatorname{Ann} M) = \operatorname{GKdim} M$.

PROOF. Let I denote the left annihilator of M. Since M is a finitely generated left R/I-module, $\operatorname{GKdim} M \leq \operatorname{GKdim}(R/I)$. On the other hand, if $M = m_1R + \cdots + m_nR$, and $I_j = \operatorname{Ann}(m_j)$, then $\operatorname{GKdim}(R/I_j) \leq \operatorname{GKdim} M$, and $I = \cap I_j$. In particular, $\operatorname{GKdim}(R/I) = \max\{\operatorname{GKdim}(R/I_j) \mid 1 \leq j \leq n\} \leq \operatorname{GKdim} M$. □

EXERCISES

2.1 PROPOSITION 2.14. *Let V be a graded vector space. Suppose there exist polynomials $f_1, f_2 \in \mathbb{R}[x]$ of degree $d-1$ such that*

$$0 \leq f_1(n) \leq \dim_k V_n \leq f_2(n)$$

for all $n \gg 0$. Then $\operatorname{GKdim} V = d$.

PROOF. For any $\epsilon > 0$ we have

$$n^{d-1-\epsilon} \leq f_1(n) \qquad \text{and} \qquad f_2(n) \leq n^{d-1+\epsilon}$$

for $n \gg 0$, as one sees on dividing by n^{d-1} and passing to the limit. Thus, for all $n \gg 0$, $n^{d-1-\epsilon} \leq \dim V_n \leq n^{d-1+\epsilon}$; the result now follows. □

2.2 Show that the Hilbert series of $k[x, y]$ with defining relation $yx = 0$ is the same as that of the polynomial ring on two indeterminates.

2.3 Show that a 1-critical module, which is bounded below, is noetherian. (Remember that a 1-critical module is, by definition, locally finite.)

2.4 $\boxed{\text{Paul}}$ If A is a locally finite, $\mathbb{N}$-graded k-algebra, and z a central homogeneous element of positive degree, is $\operatorname{GKdim}(A[z^{-1}]_0) = \operatorname{GKdim} A - 1$?

There are several open questions concerning GK-dimension. Suppose that A is a locally finite and finitely generated noetherian $\mathbb{N}$-graded k-algebra; then

- is $\operatorname{GKdim} A < \infty$?
- is $\operatorname{GKdim} A$ an integer?
- is $\operatorname{GKdim} M$ an integer for all $M \in \mathsf{grmod}(A)$?

No one has any idea how to tackle these questions. It is possible that ideas which are useful for attacking them would be related to ideas which could be used to tackle another famous open problem: if R is a finitely generated noetherian k-algebra, is R finitely presented?

Gelfand-Kirillov dimension may be defined more generally; if R is any finitely generated k-algebra and M a finitely generated R-module then one may define

$$\operatorname{GKdim} M = \overline{\lim}_{n\to\infty} \frac{\log(\dim V^n X)}{\log n}$$

where the supremum is taken over all possible finite dimensional subspaces V of R which generate it as a k-algebra and all possible finite dimensional subspaces X of M such that $RX = M$.

Some work is required to show that the definition does not depend on the choice of V or X.

$\boxed{\text{Paul}}$ what are the precise conditions for equality, and a simple example to show they are different in general? If z is a central, regular, homogeneous element is $\operatorname{GKdim}(A[z^{-1}]_0) = \operatorname{GKdim}(A) - 1$?

It is a famous Theorem of Gromov that, if G is a finitely generated group, then $\text{GKdim}(kG) < \infty$ if and only if G contains a nilpotent normal subgroup of finite index. If G is taken to be the fundamental group of a Riemannian manifold M then the existence of a nilpotent normal subgroup of finite index is related to the curvature of M; thus if one is given a manifold it is feasible to compute its fundamental group, the GK-dimension of the group algebra and deduce properties of possible Riemannian structures on the manifold.

There exist finitely generated groups G, having no nilpotent subgroup of finite index, for which kG is noetherian; such kG shows that one may have a finitely generated noetherian algebra of infinite GK-dimension. Thus in the question above the graded hypothesis is essential.

3. Graded algebras of geometric type

We introduce a class of graded algebras which has good properties. The first three conditions in the next definition are rather innocent, but the last one is not; it is an (important) open question whether it follows from the others.

Definition 3.1. A graded k-algebra A is geometric of dimension d if

- A is connected,
- $\text{GKdim}(A) = d$,
- A is left noetherian,

and there exist integers d_i such that, for every $M \in \mathsf{grmod}(A)$,

- there exists $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$H_M(t) = \frac{f(t)}{\prod_{i=1}^{s}(1 - t^{d_i})}.$$

$\boxed{\text{Paul}}$ Should I delete the connected hypothesis and replace the noetherian hypothesis by finitely generated.

This definition seems to include all the graded algebras to which one might wish to apply the theory being developed in this book. However, I do not know if this last axiom excludes some algebras which one might not want excluded; it certainly deserves further thought.

Proposition 3.2 shows there is a reasonably large supply of geometric algebras; in particular, quotients of polynomial rings are geometric. Proposition 3.3 describes some of the good properties enjoyed by geometric algebras. In particular, the GK-dimension of a finitely generated module is an integer, and such a module cannot have a proper descending chain of submodules with all the subfactors having the same GK-dimension as that of the original module—this last property is analogous to the artinian property (indeed, the modules having GK-dimension $< d$ form a dense subcategory of $\mathsf{grmod}(A)$ and the modules having GK-dimension d give objects of finite length in the quotient, the irreducible ones being those coming from d-critical modules).

Proposition 3.2. *If A is a graded quotient ring of a left noetherian, connected k-algebra of finite global dimension then A is geometric.*

Proof. Let B be a left noetherian, connected k-algebra of which A is a graded quotient. Then A satisfies the first and third axioms of Definition 3.1 because B does. Every finitely generated A-module is a finitely generated B-module, so has a Hilbert series of the prescribed form by Theorem 4.5. Thus the fourth axiom is satisfied. The second axiom is now a consequence of Theorem 4.5. □

In particular, every graded quotient of a commutative polynomial ring (with generators in arbitrary positive degree) is geometric.

It is not known whether a left noetherian, connected k-algebra is a quotient of another such algebra of finite global dimension. This seems to me to be an important question, particularly if it is the case.

PROPOSITION 3.3. *Let A be geometric of dimension d. Then*

1. $\operatorname{GKdim} M \in \mathbb{Z}$ *for all* $M \in \mathsf{grmod}(A)$*;*
2. *for each* r*,* $\min\{e(M) \mid \operatorname{GKdim} M = r\}$ *exists;*
3. *if* $M \in \mathsf{grmod}(A)$ *has GK-dimension* r *and* $0 = M^0 \subset M^1 \subset \ldots \subset M^n = M$ *is a chain of graded submodules then at most finitely many factors* M^{i+1}/M^i *have GK-dimension* r*;*

PROOF. (1) By viewing M as a module over a left noetherian, connected k-algebra mapping onto A, this is just a restatement of Theorem 4.5(2).

(2) A finitely generated graded A-module M of GK-dimension r has a Hilbert series of the form

$$H_M(t) = \frac{f_M(t)}{(1-t)^r g(t)}$$

where $f_M(t) \in \mathbb{Z}[t, t^{-1}]$ and $g(t)$ is independent of M and satisfies $g(1) \neq 0$. Thus $e(M)$ is is a positive integer multiple of $g(1)^{-1}$.

(3) By Lemma 2.11, the sum of $e(M^{i+1}/M^i)$, over those i for which the factor has GK-dimension r, equals $e(M)$. By (2), there exists $\epsilon > 0$ such that all these multiplicities are $\geq \epsilon$, so there are only finitely many terms in the sum. □

EXAMPLE 3.4. Let A be the commutative ring

$$k[x_1, x_2, x_4, x_8, \ldots]/(x_1^2, x_2^2, x_4^2, \ldots)$$

with grading defined by $\deg x_i = i$. Then $\dim A_n = 1$ for all $n \geq 0$; a basis element for A_n is obtained by expressing n in binary notation—for example, $A_{73} = kx_{64}x_8x_1$. Hence $H_A(t) = (1-t)^{-1}$. For each n define the ideal

$$J_n = (x_1, x_2, \ldots, x_{2^{n-1}});$$

thus we obtain an ascending chain of ideals

$$0 \subset J_1 \subset J_2 \subset \ldots \subset A. \tag{3-1}$$

As k-algebras, $A/J_n \cong A$ but this is not an isomorphism of graded algebras since

$$H_{A/J_n}(t) = (1 - t^{2^n})^{-1}.$$

It is easy to see that $J_n/J_{n-1} \cong A/J_n[-2^{n-1}]$. Hence each quotient J_n/J_{n-1} occuring in the chain (3-1) has GK-dimension 1, as is A itself. The reason the lemma does not apply here is that $e(A/J_n) = 2^{-n}$. (Question: is it possible to find a noetherian or finitely generated algebra with this sort of behavior?)

Notice also that A is isomorphic to each of its Veronese subalgebras $A^{(2^n)}$.

By Proposition 3.3, the situation in the previous example does not occur for geometric algebras (no fractals allowed!!).

EXERCISES

3.1 If A is of geometric type with $H_A(t) = (1-t)^{-n}$ show that any finitely generated module of GK-dimension 1 has a Hilbert series which is eventually constant, namely $\dim M_n = e(M)$ for $n \gg 0$.

$\boxed{\text{Paul}}$ If $\operatorname{GKdim} A = 1$, is A geometric? What if A is a finite module over its center? What if A satisfies a PI?

CHAPTER 6

Examples

Some of the most common, best-behaved, and best understood, non-commutative algebras are the iterated Öre extensions. We introduce these in Section 1. They are rather like polynomial rings: they have a basis of the form $x_1^{i_1} \dots x_n^{i_n}$, they are (usually) noetherian domains, and have good homological properties. They are built up from a chain of subalgebras, starting with the base field k and adjoining one generator at a time, subject to relations of a simple kind. This method of construction makes them amenable to induction arguments. Non-commutative algebras which are not iterated Öre extensions, or closely related to them, tend to be less tractable, but also tend have a richer structure—important examples are enveloping algebras of semisimple Lie algebras and the Sklyanin algebras (see Chapter ???).

1. Öre extensions

Öre extensions are among the simplest non-commutative analogues of polynomial rings.

Definition 1.1. Let R be an algebra over a commutative ring k. A k-linear derivation on R is a k-linear map $\delta : R \to R$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. We write $\operatorname{Der}_k R$ for the space of k-linear derivations on R.

If $\sigma : R \to R$ is a k-linear ring endomorphism, a σ-derivation on R is a k-linear map $\delta : R \to R$ such that $\delta(xy) = \delta(x)y + \sigma(x)\delta(y)$ for all $x, y \in R$. (Thus a derivation is simply a σ-derivation with $\sigma = \mathbb{1}_R$.)

We will say that (σ, δ) is a skew-derivation to mean that σ is an endomorphism of R and δ is a σ-derivation.

NOTATION . If $\sigma : R \to R$ is a ring endomorphism we will write x^σ for the image of $x \in R$ under σ.

The space of all k-linear automorphisms of a k-algebra R is denoted by $\operatorname{Aut}_k R$.

PROPOSITION 1.2. *Let $R = k[x_1, \dots, x_n]$ denote the commutative polynomial ring. The k-linear derivations on R are precisely the maps*

$$a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

where $a_1, \dots, a_n$ are arbitrary elements of R.

PROOF. That such a linear map is a derivation of R is simply Leibniz's rule for differentiation of a product. On the other hand, any derivation on any ring is determined by its action on a set of generators for the ring so, if δ is an arbitrary derivation and we define $a_i := \delta(x_i)$, then δ agrees on $x_1, \dots, x_n$ with the derivation $\sum a_i \partial/\partial x_i$, whence they are equal. $\square$

A σ-derivation on a k-algebra R is completely determined by its action on a set of algebra generators for R; in general (e.g., in contrast to what happens in Proposition 1.2), the action of a σ-derivation on a set of generators may not be arbitrarily specified—the relations amongst the generators must be respected.

PROPOSITION 1.3. *Let σ an endomorphism of a ring R, and let δ be a σ-derivation of R. Then there exists a ring S such that*

1. *R is a subring of S,*
2. *S is a free left R-module with a basis of the form $\{1, x, x^2, \dots\}$, and*
3. *if $a \in R$ then $xa = a^\sigma x + \delta(a)$.*

These properties determine S uniquely.

PROOF. Let V be a free left R-module with basis $\{e_0, e_1, \dots\}$. There is a natural action of R on V by left multiplication, and we will identify R with its image in $\operatorname{End}_{\mathbb{Z}} V$. Define $\theta \in \operatorname{End}_{\mathbb{Z}} V$ by

$$\theta(ye_i) = y^\sigma e_{i+1} + \delta(y)e_i$$

for all $y \in R$ and all $i \geq 0$. Let S be the subring of $\operatorname{End}_{\mathbb{Z}} V$ generated by R and θ. We now prove that S has the desired properties, with θ playing the role of x.

First, if $a \in R$ then $\theta a = a^\sigma \theta + \delta(a)$, as one sees from the calculation

$$(\theta a - a^\sigma \theta)(ye_i) = (ay)^\sigma e_{i+1} + \delta(ay)e_i - a^\sigma\big(y^\sigma e_{i+1} + \delta(y)e_i\big) = \delta(a)ye_i.$$

Second, we show that $S = R + R\theta + R\theta^2 + \cdots$. By its very definition, S contains the right hand side so, to prove equality, it suffices to show that the right hand side is a ring; i.e., it is closed under multiplication. By the previous paragraph, $\theta R \subset R\theta + R$, so induction gives $\theta^i R \subset R\theta^i + R\theta^{i-1} + \cdots + R$, from which the result follows.

Third, we show that $\{1, \theta, \theta^2, \dots\}$ is a basis for S as a left R-module. If $a_0, \dots, a_n \in R$ are such that $a_0 + a_1\theta + \cdots + a_n\theta^n = 0$, then

$$0 = (a_0 + a_1\theta + \cdots + a_n\theta^n)(e_0) = a_0e_0 + a_1e_1 + \cdots + a_ne_n,$$

whence $a_0 = a_1 = \cdots = a_n = 0$. □

Definition 1.4. The ring S constructed in Proposition 1.3 is denoted $R[x; \sigma, \delta]$ and is called the Öre extension of R with respect to σ and δ. If $\sigma = \mathbb{1}_R$ is the identity we just write $R[x; \delta]$ for simplicity, and if $\delta = 0$ we simply write $R[x; \sigma]$. If $\sigma = \mathbb{1}_R$ and $\delta = 0$, we write $R[x]$ for the Ore extension, and call it the polynomial ring over R—x is central in $R[x]$ and the elements may be viewed as polynomials in x with (possibly non-commuting) coefficients in R.

There is an isomorphism

$$R[x; \sigma, \delta] \cong R \coprod k[x]/I$$

where $R \coprod k[x]$ is the free coproduct, and I is the ideal generated by all elements $xr - r^\sigma x - \delta(r)$ as r ranges over R. If R is generated by $r_1, \dots, r_n$, then I is generated by the n elements $xr_i - r_i^\sigma x - \delta(r_i)$. We sometimes say that the relations for $R[x; \sigma, \delta]$ are those for R together with these n additional relations.

Definition 1.5. Let σ be an automorphism of a ring R. The skew Laurent polynomial ring, denoted $R[x, x^{-1}; \sigma]$, is defined to be the ring which is the free left R-module on $\{x^i \mid i \in \mathbb{Z}\}$ and the multiplication defined by

$$rx^i sx^j = rs^{\sigma^i} x^{i+j}$$

for all $r, s \in R$. This is simply the localization of the Öre extension $R[x;\sigma]$ at the powers of x.

PROPOSITION 1.6. *Let $S = R[x, x^{-1}; \sigma]$ be a skew Laurent extension of a domain. Suppose that $\sigma^n = 1$, and that σ^i is not an inner automorphism of R whenever $0 < i < n$. Then*

$$Z(S) = Z(R)^\sigma[x^n, x^{-n}].$$

PROOF. Since R^σ commutes with x, and x^n commutes with R, $Z(R)^\sigma[x^n, x^{-n}]$ is central. By Proposition 11.1.7, $Z(S)$ is graded. Suppose that $z = ax^i$ is central, with $a \in R$. For every $r \in R$, $rz = zr$, whence $ra = ar^{\sigma^i}$; therefore n divides i, which implies that x^i is central, and hence that a is central. But $ax = xa$ implies that $a = a^\sigma$, whence $a \in Z(R)^\sigma$. □

The Öre extension $S = R[x;\sigma,\delta]$ has an obvious universal property. If $\psi : R \to T$ is a ring homomorphism, and there exists $t \in T$ such that $t\psi(r) = \psi(\delta(r)) + \psi(r^\sigma)t$ for all $r \in R$, then there exists a unique $\rho : S \to T$ such that ψ is the composition $R \to S \xrightarrow{\rho} T$.

LEMMA 1.7. *Let T be a ring which is generated by a subring R and an element t. Suppose that Rt is a free left R-module, that $R \cap Rt = 0$, and that $tR \subset R + Rt$. Then T is a quotient of an Öre extension $R[x;\sigma,\delta]$.*

PROOF. Since Rt is a free left R-module there are well-defined maps $\sigma : R \to R$ and $\delta : R \to R$ defined by the requirement that

$$tr = \delta(r) + r^\sigma t$$

for all $r \in R$. If one now computes $t(ab) = (ta)b$, for $a, b \in R$, one sees that σ is an automorphism of R, and that δ is a σ-automorphism. □

Note that, if σ is an automorphism, then $\{1, x, x^2, \dots\}$ is also a basis for $R[x;\sigma,\delta]$ as a right R-module.

PROPOSITION 1.8. *Let $S = R[x;\sigma,\delta]$ be an Ore extension.*

1. *If R is a domain and σ is injective then S is a domain.*
2. *If σ is an automorphism, then S is left noetherian (respectively, right noetherian) if R is.*

PROOF. (1) If σ is injective then the coefficient of x^{n+m} in the product $(a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n)$ is $a_mb_n^{\sigma^m}$ which is non-zero if $a_m \neq 0$ and $b_n \neq 0$; thus S is a domain.

(2) We only prove that S is right noetherian if R is; the left noetherian case may be proved similarly, or it follows from the right noetherian result by using Exercise 1 below. We define the *degree* of an element $\sum_{i=0}^n a_ix^i \in S$ with $a_n \neq 0$ to be n.

Let J be a right ideal of S—we will show it is finitely generated. Define

$$I := \{a \in R \mid \text{there exists } f \in J \text{ such that } f = ax^n + (\text{lower degree terms})\}.$$

Since σ is an automorphism, I is a right ideal of R (to see this, consider fR in the definition of I). By hypothesis, we may write $I = a_1R + \cdots a_mR$. For each a_i, choose $f_i \in J$ such that

$$f_i = a_ix^{n_i} + (\text{lower degree terms}).$$

We may choose the f_i to be of the same degree, say d: just replace f_i by $f_i x^{d-n_i}$. Define

$$J' = f_1S + \cdots f_mS \subset J.$$

Let $f \in J$, say $f = ax^n$+(lower degree terms). Then $a \in I$, so we may write $a = a_1b_1 + \cdots + a_mb_m$ for some $b_i \in R$. If $n \geq d$ then

$$f - \sum_{i=1}^{m} f_i x^{n-d} (b_i)^{\sigma^{-n}}$$

is of degree $< n$. By repeating this, subtracting elements of J', we eventually find that we may write $f = g + h$ with $g \in J'$ and $\deg h < d$. Thus $h \in J \cap M$, where $M = R + Rx + \cdots + Rx^{d-1}$. We have shown that $J \subset J' + (J \cap M)$, whence $J = J' + (J \cap M)S$. Since σ is an automorphism, $M = R + xR + \cdots + x^{d-1}R$, so $J \cap M$ is a finitely generated right R-module. It follows that $J' + (J \cap M)S$ is a finitely generated right S-module, as required. □

Example 1.9. If σ is not an automorphism, then an Ore extension $R[x; \sigma, \delta]$ of a noetherian ring need not be noetherian. Let $R = k[t]$ be the polynomial ring, define σ by $t^\sigma = 0$, and let $S = R[x; \sigma]$; thus $xt = 0$.

Then S is neither left nor right noetherian: the partial sums of $St + Stx + Stx^2 + Stx^3 + \cdots$ give a strictly ascending chain of left ideals, and the partial sums of $xS + txS + t^2xS + \cdots$ give a strictly ascending chain of right ideals.

Example 1.10. If an Öre extension $R[x; \sigma, \delta]$ is a domain, then R is a domain. However, it is possible for $R[x; \sigma, \delta]$ to be a prime ring even though R is not. For example, let $R = k \oplus k$, and let $\sigma \in \operatorname{Aut} R$ switch the two simple modules. Then $R[x; \sigma]$ is prime, although R is not. To see that $R[x; \sigma]$ is prime, first we present it as $k[t, x]$ where $t^2 = 1$, and $xt = -tx$. We will show it is prime by invoking Goldie's theorem. First we embed $k[x, t]$ in $M_2(k[u])$ via

$$t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad x = \tfrac{1}{2} \begin{pmatrix} u+1 & u-1 \\ 1-u & -1-u \end{pmatrix}.$$

It is easy to check that this is an embedding and that $x^2 = \begin{pmatrix} u & 0//0 & u \end{pmatrix} = u$. Thus the ring of fractions of $k[x, t]$ contains $k(u)$; however, as is easily checked, $1, x, t, xt$ are linearly independent over $k(u)$, so $\operatorname{Fract} k[x, t] = M_2(k(u))$; it follows from Goldie's theorem 2.3.10 that $k[x, t]$ is prime.

A graded version of the previous result is given by the ring $k[x, y, z]$ with relations

$$zx - xz = zy - yz = xy + yx = y^2 - z^2 = 0.$$

This can be presented as an Öre extension $k[y, z][x; \sigma]$. It is the homogenized version of Example **??**.

Proposition 1.11. *If D is a division ring, then every left ideal of $D[x; \sigma, \delta]$ is principal, and every right ideal is principal if σ is an automorphism.*

Proof. Let I be a non-zero left ideal, and choose

$$0 \neq a = a_nx^n + a_{n-1}x^{n-1} + \cdots$$

of minimal degree in I. Suppose the result is false, and let $0 \neq b = b_mx^m + b_{m-1}x^{m-1} + \cdots$ be an element of I of smallest degree such that it is not in the left ideal generated by a. But

$$b_m\sigma^{m-n}(a_n^{-1})x^{m-n}.a_nx^n = b_mx^m,$$

so $b - b_m\sigma^{m-n}(a_n^{-1})x^{m-n}.a$ is an element of I of degree $< m$, whence is in the left ideal generated by a. Hence, so is b, contradicting its choice.

To get the result for right ideals, one may apply Exercise 1. □

EXAMPLE 1.12. The Öre extension $k[t][x;\sigma]$ with $t^\sigma = t^2$ contains a free subalgebra, namely $k[x,tx]$; this can be proved directly using the fact that $xS \cap txS = 0$. It is a special case of Proposition 2.2.7, which says that if S is a domain, and $0 \neq x, y \in S$, then $k[x,y]$ is a free algebra.

The Öre extension construction may be repeated.

Definition 1.13. An **iterated Öre extension**, of a ring R, is a ring of the form

$$R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \tag{1-1}$$

where, for each i, σ_{i+1} is an automorphism of $R[x_1;\sigma_1,\delta_1]\cdots[x_i;\sigma_i,\delta_i]$ and δ_{i+1} is a σ_{i+1}-derivation of this ring.

The only k-linear endomorphism of k is the identity, and the only k-linear derivation of k is the zero map so, in an iterated Öre extension of the form (1-1), the first step, namely $k[x_1;\sigma_1,\delta_1]$, is just the polynomial ring $k[x_1]$.

PROPOSITION 1.14. *Let $S = k[x_1][x_2;\sigma_2,\delta_2]\cdots[x_n;\sigma_n,\delta_n]$ be an iterated Öre extension of the field k. Suppose that each σ_i is a k-linear automorphism and that each δ_i is k-linear. Then S is a two-sided noetherian domain, having basis*

$$\{x_1^{i_1}\ldots x_n^{i_n} \mid i_1,\ldots,i_n \geq 0\}, \tag{1-2}$$

and $\binom{n}{2}$ defining relations

$$\{x_jx_i = x_i^{\sigma_j}x_j + \delta_j(x_i) \mid 1 \leq i < j \leq n\}.$$

PROOF. The fact that S is a noetherian domain, follows from Proposition 1.8 by induction. Since an Öre extension $R[x;\sigma,\delta]$ is a free left R-module with basis $\{x^i \mid i \geq 0\}$, an induction argument shows S has a basis of the prescribed form.

Suppose that R is generated as a k-algebra by $a_1,\ldots,a_m$. It follows from the universal property that a full set of defining relations for $R[x;\sigma,\delta]$ consists of the defining relations for R together with n additional relations, namely

$$xa_i = a_i^\sigma x + \delta(a_i).$$

Consequently S has $0 + 1 + \cdots + (n-1) = \binom{n}{2}$ relations, and these may be taken to be of the stated form. □

The basis for S looks like that for the commutative polynomial ring (which is a special case of an iterated Öre extension). It is common to call a ring having such a basis a **skew polynomial ring**, the adjective 'skew' suggesting the possible non-commutativity.

Two comments concerning skew polynomial rings are in order. First, a k-algebra having a basis of the form (1-2) need not be an iterated Öre extension. For example, if $\mathfrak{g}$ is a finite dimensional Lie algebra with basis $x_1,\ldots,x_n$ then its enveloping algebra $U(\mathfrak{g})$ has a basis as in (1-2), but amongst the simple Lie algebras over $\mathbb{C}$, only $U(\mathfrak{sl}(2,\mathbb{C}))$ is an iterated Öre extension. Second, an iterated

Öre extension may be presented in a such a way that it is not at all obvious that it *is* an iterated Öre extension; the next example illustrates this.

EXAMPLE 1.15. We consider some graded algebras which arise as Öre extensions of the polynomial ring $k[y, z]$.

Fix a non-zero $\lambda \in k$. Define $\sigma \in \operatorname{Aut} k[y, z]$ by $y^\sigma = \lambda y$ and $z^\sigma = z$. Any linear map $\delta : ky + kz \to ky + kz$ such that $\delta(z) = 0$ extends uniquely to a σ-derivation of $k[y, z]$ because

$$\delta(y)z + y^\sigma \delta(z) = \delta(z)y + z^\sigma \delta(y).$$

Hence we may define the Öre extension $A = k[y, z][x; \sigma, \delta]$. It has defining relations

$$yz = zy, \ xz = zx, \ xy = \lambda yx + \delta(y).$$

In particular, z is central. Now we add the additional constraint that $\delta(y) = z$. It follows from the third relation above that A is generated by just x and y, subject to the two relations which are obtained by substituting $xy - \lambda yx$ for z in the first two relations. Simplifying, these relations are

$$xy^2 - (\lambda + 1)yxy + \lambda y^2 x = x^2 y - (\lambda + 1)xyx + \lambda yx^2 = 0.$$

Hence, if x and y are given degree one (which forces $\deg z = 2$), A is defined by two homogeneous cubic relations.

Two special cases should be noticed. If $\lambda = 1$, then A is isomorphic to the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. If $\lambda = -1$, then A is isomorphic to the second generic Clifford algebra (which is defined and studied in section 5.

Iterated Öre extensions provide us with a source of non-commutative examples on which to test the theory being developed in this book. In particular, consider an iterated Öre extension $S = k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$ as in Proposition 1.14. In order for S to be a graded algebra, with $\deg x_i = 1$ for all i, each of the relations

$$x_j x_i = x_i^{\sigma_j} x_j + \delta_j(x_i) \qquad (1 \leq i < j \leq n)$$

must be homogenous. Since $\delta_j(x_i) \in k[x_1, \ldots, x_{j-1}]$ it cannot involve any x_j terms; thus S is graded if and only if, for all $1 \leq i < j \leq n$, $x_i^{\sigma_j}$ is homogeneous of degree one and $\delta_j(x_i)$ is homogeneous of degree two. This leaves an enormous number of possibilities, some of which will be considered later.

EXAMPLE 1.16. Let $\alpha_{ij} \in k$ be arbitrary scalars satisfying

$$\alpha_{ij} = \alpha_{ji}^{-1} \qquad \text{and} \qquad \alpha_{ii} = 1$$

for all i and j. Then the algebra $A = k[x_1, \ldots, x_n]$ with defining relations

$$x_j x_i = \alpha_{ij} x_i x_j,$$

for all i and j, is an iterated Öre extension (proof by induction on n—see Exercise 5). It is called a quasi-polynomial ring.

The following special case is important—in the theory of quantum groups, it plays the role of the natural homogenous space for quantum $\mathrm{GL}(n)$.

Definition 1.17. Fix $0 \neq q \in k$. The coordinate ring of quantum affine n-space is the algebra $A = k[x_1, \ldots, x_n]$, with defining relations

$$x_j x_i = q x_i x_j \qquad (1 \leq i < j \leq n).$$

EXERCISES

1.1 Show that $R[x;\sigma,\delta]^{\mathrm{op}} \cong R^{\mathrm{op}}[x,\sigma^{-1},-\delta\sigma^{-1}]$.

1.2 Let σ be an endomorphism of $k[x]$ and let δ be a σ-derivation. Show that $k[x][y;\sigma,\delta]$ is two-sided noetherian if and only if σ is an automorphism.

1.3 Show that an Öre extension $R[x;\sigma]$ is a domain if and only if R is a domain and σ is injective.

1.4 Let S be a ring which is generated by a subring R and an element x such that $R+Rx = xR+R$. Show that S is left noetherian if R is left noetherian.

1.5 Let R be the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[T])$ generated by the differentiation operator $x := d/dT$ and the operator y which is multiplication by T. Show that R is an iterated Öre extension, and hence determine its defining relations. This ring is known as the ring of differential operators on the affine line, and is often called the (first) Weyl algebra.

1.6 Show that the ring $R = k[x,y]$ with defining relation $yx - xy = x$ is an iterated Öre extension. This ring is isomorphic to the enveloping algebra of the 2-dimensional solvable Lie algebra. (Over an algebraically closed field, the enveloping algebra of every finite dimensional solvable Lie algebra is an iterated Öre extension—this allows the use of inductive arguments.)

1.7 Show that the ring $R = k[e,f,h]$ with defining relations

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f$$

is an iterated Öre extension. This is the enveloping algebra of the Lie algebra $\mathfrak{sl}(2,k)$.

1.8 Show that the algebra A in Example 1.4 is an iterated Öre extension of k.

1.9 Verify the details of Example 1.2.

1.10 Verify the remarks after Example 1.2 by showing that the ring of fractions of $k[x,y,z]$ is $M_2(k(u,v))$ where u and v are commuting indeterminates and

$$x = \tfrac{1}{2}\left(u+1 \quad v(u-1)//(1-u)v^{-1} \quad -u-1\right), \quad y = \left(0 \quad v^2//1 \quad 0\right), \quad z = \left(v \quad 0//0 \quad v\right).$$

2. More on Öre extensions

Definition 2.1. Let R be a k-algebra and $\sigma \in \mathrm{Aut}_k R$. For each $b \in R$, we define the inner σ-derivation $\mathrm{ad}_\sigma\, b$ by

$$(\mathrm{ad}_\sigma\, b)(r) := br - r^\sigma b$$

for all $r \in R$.

NOTATION . If u is a unit in the ring R, we write $\mathrm{Ad}\, u$ for the inner automorphism which is conjugation by u, namely

$$(\mathrm{Ad}\, u)(r) = uru^{-1}$$

for all $r \in R$.

LEMMA 2.2. *Let R be a k-algebra, $\sigma \in \mathrm{Aut}_k R$ and δ a σ-derivation of R.*

1. *If u is a unit in R, then $R[x;\sigma,\delta] \cong R[y;(\mathrm{Ad}\, u)\sigma,\delta]$.*
2. *If $b \in R$, then $R[x;\sigma,\delta] \cong R[y;\sigma,\delta + \mathrm{ad}_\sigma\, b]$.*

In particular,

- *if σ is an inner automorphism $R[x;\sigma,\delta] \cong R[y;\delta]$, and*
- *if δ is a σ-inner derivation $R[x;\sigma,\delta] \cong R[y;\sigma]$.*

PROOF. We begin by treating both cases together. Let $y = ux + b$. Then, for $r \in R$,

$$\begin{aligned} yr &= ur^\sigma x + u\delta(r) + br \\ &= (\operatorname{Ad} u)(r)(ux+b) + (u\delta + \operatorname{ad}_\sigma b)(r) - (\operatorname{Ad} u)(r^\sigma)b + r^\sigma b. \end{aligned}$$

It is clear that $R[x;\sigma,\delta] = R+Ry+Ry^2+\cdots$, that Ry is free, and that $yR \subset R+Ry$. Now apply Lemma 1.7 with $b=0$ to obtain (1), and with $u=1$ to obtain (2). □

Global Dimension An iterated Öre extension in which all the σ_i are automorphisms has finite global dimension, as can be seen by an induction argument using the next result.

THEOREM 2.3. *If $S = R[x;\sigma,\delta]$ is an Öre extension in which σ is an automorphism, then*

$$\operatorname{gldim} R \leq \operatorname{gldim} S \leq \operatorname{gldim} R + 1.$$

PROOF. [**117**, Section 5, Chapter 7]. □

Since our main concern is with graded algebras, this result is not quite in the form we need; we will restrict attention to graded algebras and prove a stronger result in Theorem 18.1.4 and Corollary 18.1.5): the global dimension of a graded iterated Öre extension, say

$$A = k[x_1][x_2;\sigma_2,\delta_2]\cdots[x_n;\sigma_n,\delta_n],$$

where each σ_i is an automorphism, and each x_i is of positive degree, is n. It is useful to see a special case which reinforces the idea that the coordinate rings of the quantum and Jordan planes are non-commutative analogues of the polynomial ring in two variables.

EXAMPLE 2.4. The coordinate rings of the quantum and Jordan planes are of global dimension 2. For the quantum plane $A = k[x,y]$ with $yx = qxy$, the trivial module ${}_Ak = A/Ax + Ay$ has minimal resolution

$$0 \to A[-2] \xrightarrow{(y,-qx)} A[-1]^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

Here we are writing elements of $A[-1]^2$ as row vectors, and the maps are to be viewed as right multiplication by the given matrices. To see this is exact, first note that it is a complex because $(y,-qx)\binom{x}{y} = 0$, then do a simple calculation to show that $\ker(\binom{x}{y})$ is cyclic, and finally observe that $\ker((y,-qx)) = 0$ because A is a domain.

A similar argument shows that, for the Jordan plane ($yx - xy = x^2$), the trivial module has minimal resolution

$$0 \to A[-2] \xrightarrow{(y-x,x)} A[-1]^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

EXERCISES

2.1 Show that $\mathrm{ad}_\sigma b$ really is a σ-derivation on R.

2.2 Let (σ, δ) be a skew-derivation on R. Show that

$$\delta(r^n) = \sum_{i=0}^{n-1} \sigma(a)^i \delta(a) \alpha^{n-i-1}$$

for all $r \in R$. [This and the next exercise are taken from [**69**].]

2.3 Let (σ, δ) be a skew-derivation on R. Let $\mathcal{S}$ be a right denominator set in R such that $\sigma(\mathcal{S}) \subset \mathcal{S}$. Show that there is a unique extension of (σ, δ) to the localization $R_{\mathcal{S}}$, and it satisfies

$$\sigma(rs^{-1}) = r^\sigma \sigma(s)^{-1} \qquad \text{and} \qquad \delta(rs^{-1}) = \delta(r)s^{-1} - \sigma(rs^{-1})\delta(s)s^{-1}$$

for all $r \in R$ and $s \in \mathcal{S}$.

2.4 Let δ be a derivation on a division algebra D over a field k, and let $S = D[x; \delta]$. Show that
(a) every left ideal of S is principal;
(b) S is a simple ring if δ is not inner;
(c) $S \cong D \otimes_k k[X]$ if δ is inner;
(d) if δ is inner, every two-sided ideal of S is generated by its intersection with the center of S, and hence that $\mathrm{Spec}(S)$ is homeomorphic to $\mathrm{Spec}(K[T])$, where K is the center of D.

3. Enveloping algebras of solvable Lie algebras

EXERCISES

3.1 Show that the enveloping algebra $U(\mathfrak{sl}(2)) = k[e, f, h]$, is a graded k-algebra with $\deg e = 1$, $\deg f = -1$ and $\deg h = 0$.

4. Lie superalgebras

Definition 4.1. A Lie superalgebra over a field k is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with a bilinear bracket

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the following conditions:

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for $i, j \in \mathbb{Z}_2$;
- $[x, y] = (-1)^{ij}[y, x]$ if $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$;
- if $x \in \mathfrak{g}_h$, $y \in \mathfrak{g}_i$ and $z \in \mathfrak{g}_j$, then

$$(-1)^{hj}[x, [y, z]] + (-1)^{hi}[y, [z, x]] + (-1)^{ij}[z, [x, y]] = 0. \tag{4-1}$$

We call (4-1) the Jacobi identity. The universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$ is the algebra

$$U(\mathfrak{g}) := T(\mathfrak{g})/I$$

where I is the ideal generated by the elements

$$\{xy - (-1)^{ij}yx - [x, y] \mid x \in \mathfrak{g}_i,\ y \in \mathfrak{g}_j\}.$$

LEMMA 4.2. *Let $\mathfrak{g}$ be a Lie superalgebra.*

1. *If $U(\mathfrak{g})$ is made into a filtered algebra by setting $U(\mathfrak{g})_0 = k$, and $U(\mathfrak{g})_1 = k \oplus \mathfrak{g}$, then $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g}_0) \otimes_k \Lambda(\mathfrak{g}_1)$.*
2. *If $\dim \mathfrak{g} < \infty$, then $U(\mathfrak{g})$ is right and left noetherian.*

3. *If* $x_1, \dots, x_n$ *is a basis for* $\mathfrak{g}_0$ *and* $y_1, \dots, y_m$ *is a basis for* $\mathfrak{g}_1$*, then a basis for* $U(\mathfrak{g})$ *is given by the elements*
$$x_1^{i_1} \cdots x_n^{i_n} y_{j_1} \cdots y_{j_p}$$
where $(i_1, \dots, i_n) \in \mathbb{N}^n$*, and* $1 \leq j_1 < \cdots < j_m \leq n$.

5. Weyl algebras

6. Rings of differential operators

7. Quantum groups

CHAPTER 7

Homological matters

1. Injective modules

Lemma 1.1. *Let $\mathcal{S}$ be a left denominator set of regular elements in a left noetherian ring R, and write $R_{\mathcal{S}} = \mathcal{S}^{-1}R$. If E is an injective R-module, then the natural map $E \to E_{\mathcal{S}}$ is surjective.*

Proof. If $x \in \mathcal{S}$, then $Rx \cong R$; hence, if $e \in E$, then $f(x) = e$ for some $f \in \operatorname{Hom}_R(Rx, E)$; but f extends to R since E is injective, whence $e \in xE$; thus $xE = E$.

Write $Q = R_{\mathcal{S}}$. There is an exact sequence $R \otimes_R E \to Q \otimes_R E \to (Q/R) \otimes_R E \to 0$, so it suffices to show that $Q/R \otimes_R E = 0$. If $q \in Q$, then $qx \in R$ for some $x \in \mathcal{S}$. If $e \in E$, then $e = xe'$ for some $e' \in E$, so $\bar{q} \otimes e = \bar{q} \otimes xe' = \bar{q}x \otimes e' = 0$. It follows that $Q/R \otimes_R E = 0$. □

Lemma 1.2. *Over a noetherian ring, the direct limit of a directed system of injective modules is injective.*

Proof. Let $E = \varinjlim E_\alpha$ with each E_α injective. We use Baer's criterion to show E is injective. By the noetherian hypothesis, every left ideal I of R is finitely presented, so $\operatorname{Hom}_R(I, E) \cong \varinjlim \operatorname{Hom}_R(I, E_\alpha)$ by Proposition A.8.9. Therefore any map $I \to E$ will factor as $I \to E_\beta \to E$ for some β. But E_β is injective so this factors as $I \to R \to E_\beta \to E$, as required. □

Lemma 1.3. *Let $x \in R$ be a regular normal element, and let E be an injective R-module. If $N = \{e \in E \mid xe = 0\}$, then E/N is an injective R-module.*

Proof. As in the proof of Lemma 1.1, $xE = E$. If φ denotes 'multiplication by x', then there is an exact sequence $0 \to N \to E \to E \to 0$ of abelian groups. If σ is the automorphism of R defined by $xr = r^\sigma x$ for $r \in R$, then $\varphi : E \to E^\sigma$ becomes an R-module homomorphism, whence $E/A \cong E^\sigma$. But $E^\sigma \cong \sigma^* E$, and σ^* is an automorphism of the category $\mathsf{Mod}(R)$, so E^σ is injective. □

Proposition 1.4. *Let $\mathcal{S}$ be a left denominator set of regular normal elements in a left noetherian ring R, and write $R_{\mathcal{S}} = \mathcal{S}^{-1}R$. If E is an injective R-module, then $E_{\mathcal{S}}$ is an injective $R_{\mathcal{S}}$-module.*

Proof. Write $N = \{e \in E \mid xe = 0 \text{ for some } x \in \mathcal{S}\}$, and $N_x = \{e \in E \mid xe = 0\}$. Tensoring the exact sequence $0 \to N \to E \to E/N \to 0$, and observing that each $x \in \mathcal{S}$ acts bijectively on E/N, it follows that $E_{\mathcal{S}} \cong E/N$.

If $x, y \in \mathcal{S}$, then $rx = zy$ for some $r \in R$ and $z \in \mathcal{S}$, whence $N_x \cup N_y \subset N_{zy}$. Hence the N_x form a directed system of submodules of E, whose direct limit is N. Therefore, by Proposition A.8.13, E/N is the direct limit of the directed system E/N_x. By Lemma 1.3, each E/N_x is an injective R-module, hence so is $E/N = E_{\mathcal{S}}$

because R is left noetherian. Therefore $E_{\mathcal{S}}$ is injective as an $R_{\mathcal{S}}$-module (Exercise 2). $\square$

LEMMA 1.5. *Let $\theta : R \to S$ be a ring homomorphism. If E is an injective R-module, then* $\operatorname{Hom}_R(S, E)$ *is an injective S-module.*

PROOF. If $M \in \mathsf{Mod}(S)$, then $\operatorname{Hom}_S(M, \operatorname{Hom}_R(S, E)) \cong \operatorname{Hom}_R(S \otimes_S M, E)$, so there is a natural equivalence of

$$\operatorname{Hom}_S(-, \operatorname{Hom}_R(S, E)) \cong \operatorname{Hom}_R(-, E)$$

of functors on $\mathsf{Mod}(S)$. By hypothesis, this functor is exact, thus proving the result. $\square$

LEMMA 1.6. *Let R and S be k-algebras. If E is an injective $R \otimes_k S$-module, then E is injective as an R-module.*

PROOF. Let $M \in \mathsf{Mod}(R)$. By the adjointness of Hom and $\otimes$, we have

$$\begin{aligned}\operatorname{Hom}_R(M, E) &\cong \operatorname{Hom}_R(M, \operatorname{Hom}_{R\otimes S}(R \otimes S, E)) \\ &\cong \operatorname{Hom}_{R\otimes S}((R \otimes S) \otimes_R M, E) \\ &\cong \operatorname{Hom}_{R\otimes S}(M \otimes_k S, E).\end{aligned}$$

Thus $\operatorname{Hom}_R(-, E)$ is naturally equivalent to $\operatorname{Hom}_{R\otimes S}(-, E) \circ (- \otimes_k S)$, so is an exact functor as claimed. $\square$

EXERCISES

1.1 [**76**, Chapter III, Exercise 3.8] Lemma 1.1 is false without the noetherian hypothesis. Let $A = k[x_0, x_1, \dots]$ be the commutative ring with relations $x_0^n x_n = 0$ for all $n \geq 1$. Let E be the injective envelope of A. Show that the map $E \to E_{x_0}$ is not surjective.

1.2 Let E be an indecomposable injective in an abelian category. Show that its endomorphism ring is local; i.e., it has a unique maximal two-sided ideal, and the quotient is a division ring.

2. Injective dimension

LEMMA 2.1. *If E is an injective right module over a left noetherian ring R, then there is a natural isomorphism of functors*

$$\operatorname{Tor}_n^R(E, -) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^n(-, R), E)$$

on finitely generated left R-modules.

PROOF. First we show that $E \otimes_R M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), E)$ for all $M \in \mathsf{mod}(R)$. There is a presentation $P \to Q \to M \to 0$ with P and Q finitely generated free. Hence there are exact sequences $0 \to \operatorname{Hom}_R(M, R) \to Q^\vee \to P^\vee$ and $\operatorname{Hom}_R(P^\vee, E) \to \operatorname{Hom}_R(Q^\vee, E) \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), E) \to 0$. However, the first part of this sequence coincides with $E \otimes_R P \to E \otimes_R Q$ since the result holds for finitely generated free modules. But the cokernel of this last map is $E \otimes_R M$, so the result follows.

For $n > 0$, let $M \in \mathsf{mod}(R)$, and let $P_\bullet \to M$ be a resolution by finitely generated projectives. Then

$$\operatorname{Tor}_n^R(E, M) = h^n\big(E \otimes_R P_\bullet\big) \cong h^n\big(\operatorname{Hom}_R(P_\bullet^\vee, E)\big) \cong \operatorname{Hom}_R(h^n(P_\bullet^\vee), E)$$

because $\operatorname{Hom}_R(-, E)$ is exact. The result follows from Exercise 4. $\square$

LEMMA 2.2. *If I is an ideal in a ring R, then* $\operatorname{Hom}_R(R/I, -) : \mathsf{Mod}(R) \to \mathsf{Mod}(R/I)$ *preserves essential monomorphisms and injectives.*

PROOF. By Lemma 1.5, $\operatorname{Hom}_R(R/I, E)$ is injective in $\mathsf{Mod}(R/I)$. To see that essential monomorphisms are preserved, just use the fact that $\operatorname{Hom}_R(R/I, N) \cong \{n \in N \mid In = 0\}$. □

LEMMA 2.3. [**22**, Theorem 2.2] *Let $x \in R$ be a normal, regular, non-unit acting faithfully on a module M. Let*

$$0 \to M \to E^0 \xrightarrow{d^0} E^1 \to \cdots$$

be a minimal injective resolution, and define $D = \operatorname{Im} d^0$. Then

$$0 \to \operatorname{Hom}_R(R/xR, D) \to \operatorname{Hom}_R(R/xR, E^1) \to \cdots \tag{2-1}$$

is a minimal injective resolution of M/xM in $\mathsf{Mod}(R/xR)$.

PROOF. $\boxed{\text{Paul}}$ This proof only works for x central, need some twisting for the normal case.

Let $N \in \mathsf{Mod}(R)$. First, since x is normal, xN and $\{n \in N \mid xn = 0\}$ are submodules of N. Second, if x acts faithfully on N, then $\operatorname{Ext}^1_R(R/xR, N) \cong N/xN$, as one sees by applying $\operatorname{Hom}_R(-, N)$ to $0 \to R \xrightarrow{x} R \to R/xR \to 0$.

Since multiplication by x is injective on M, it is also injective on E^0. The exact sequence $0 \to M \to E^0 \to D \to 0$ yields the exact sequence

$$\operatorname{Hom}_R(R/xR, E^0) \to \operatorname{Hom}_R(R/xR, D) \to \operatorname{Ext}^1_R(R/xR, M) \to 0.$$

The first term is zero since x acts injectively on E^0, whence $\operatorname{Hom}_R(R/xR, D) \cong M/xM$.

The complex (2-1) is obtained by applying $\operatorname{Hom}_R(R/xR, -)$ to the minimal injective resolution $0 \to D \to E^1 \to E^2 \to \cdots$, so the homology of (2-1) is $\operatorname{Ext}^*_R(R/xR, D)$. Since $\operatorname{pdim}(R/xR) = 1$, $\operatorname{Ext}^i_R(R/xR, D) = 0$ for $i \geq 2$. Since $\operatorname{Ext}^1_R(R/xR, E^0) = 0$, $xE^0 = E^0$, whence $D = xD$ also, so $\operatorname{Ext}^1_R(R/xR, D) = 0$. Hence (2-1) is exact. Therefore, by Lemma 2.2, (2-1) is a minimal injective resolution of M/xM in $\mathsf{Mod}(R/xR)$. □

LEMMA 2.4. *Let $N \in \mathsf{Mod}(R)$. Let $0 \to N \to E^0 \xrightarrow{d} E^1 \to \cdots$ be a minimal injective resolution and let Ω_i denote the i^{th}-cosyzygy of N. If $M \in \mathsf{Mod}(R)$, then*

1. $\operatorname{Ext}^{i+1}_R(M, N) \cong \operatorname{Ext}^1_R(M, \Omega_i)$ *for all $i \geq 0$;*
2. *if* $\operatorname{Hom}_R(M, E^{i-1}) = 0$, *then* $\operatorname{Ext}^i_R(M, N) \cong \operatorname{Hom}_R(M, \Omega_i)$.

PROOF. There are exact sequences $0 \to \Omega_i \to E^i \to \Omega_{i+1} \to 0$. The long exact sequence for $\operatorname{Ext}^*_R(M, -)$ therefore gives an exact sequence

$$0 \to \operatorname{Hom}_R(M, \Omega_i) \to \operatorname{Hom}_R(M, E^i) \to \operatorname{Hom}_R(M, \Omega_{i+1}) \to \operatorname{Ext}^1_R(M, \Omega_i) \to 0,$$

and isomorphisms $\operatorname{Ext}^j_R(M, \Omega_{i+1}) \cong \operatorname{Ext}^{j+1}_R(M, \Omega_i)$ for $j \geq 1$. The result follows. □

The next result says that the injective dimension of a module can be computed by looking only at finitely generated modules.

LEMMA 2.5. *If N is an R-module, then*

$$\operatorname{injdim} N = \sup\{n \mid \operatorname{Ext}^n_R(M, N) \neq 0,\ M \in \mathsf{mod}(R)\}.$$

Proof. By Lemma 2.4, $\operatorname{Ext}_R^{i+1}(-,N) \cong \operatorname{Ext}_R^1(-,\Omega_i)$, where Ω_i is a suitable coszyygy of N. So, $\operatorname{injdim} N \leq i$ if and only if Ω_i is injective; but, by Baer's Theorem, the injectivity of Ω_i can be checked on finitely generated modules. Hence the result. □

There is a graded version of the previous result. In particular, if A is graded, and $\operatorname{injdim} A = n < \infty$, then $\underline{\operatorname{Ext}}^n(M,A) \neq 0$ for some $M \in \mathsf{grmod}(A)$.

Lemma 2.6. *Let R be noetherian, and $0 \neq M \in \mathsf{mod}(R)$. If $\operatorname{pdim} M < \infty$, then*

$$\operatorname{pdim} M = \max\{n \mid \operatorname{Ext}_R^n(M,R) \neq 0\}.$$

Proof. By definition, $\operatorname{pdim} M$ is $t := \max\{i \mid \operatorname{Ext}_R^i(M,N) \neq 0 \text{ for some } N\}$. However, if $\operatorname{Ext}_R^t(M,N) \neq 0$, and F is a free module mapping onto N, then there is an exact sequence $\operatorname{Ext}_R^t(M,F) \to \operatorname{Ext}_R^t(M,N) \to 0$, so $\operatorname{Ext}_R^t(M,F) \neq 0$. Since M is finitely generated, it has a projective resolution by finitely generated modules; since $\operatorname{Hom}_R(P,-)$ commutes with arbitrary direct sums it follows that $\operatorname{Ext}_R^t(M,F)$ is a direct sum of copies of $\operatorname{Ext}_R^t(M,R)$, so this is non-zero as claimed. □

EXERCISES

2.1 There is a general principle at work in Lemma 1.5. If (F,G) is an adjoint pair of functors between abelian categories, with F exact, show that G preserves injectives. This applies to Lemma 1.5 with $F = \operatorname{Hom}_S(S,-)$ the restriction functor, and $G = \operatorname{Hom}_R(S,-)$. (The adjointness of these is Exercise A.5.4.)
(This idea may be used to prove the existence of injective envelopes: given a ring R, and $M \in \mathsf{Mod}R$, if E is the injective envelope of M as a $\mathbb{Z}$-module, then $\operatorname{Hom}_{\mathbb{Z}}(R,E)$ is an injective R-module containing M.)

2.2 Let $\mathcal{S}$ be a left denominator set in a ring R, and let E be a left $R_{\mathcal{S}}$-module. Show that E is injective as an R-module if and only if it is injective as an $R_{\mathcal{S}}$-module.

3. Rings of finite self-injective dimension

Definition 3.1. The injective dimension of R as a left R-module is called the self-injective dimension of R or, when there is no room for confusion, the injective dimension of R.

Definition 3.2. The flat dimension of a right R-module N, denoted $\operatorname{fdim} N$, is the least integer n such that $\operatorname{Tor}_{n+1}^R(N,-) = 0$.

Since N is flat if and only if $\operatorname{Tor}_1^R(N,R/I) = 0$ for all left ideals I of R ([**141**, Lemma 9.18]), flat dimension can be computed by examining only the finitely generated modules.

Lemma 3.3. *If N is a right R-module, then $\operatorname{fdim} N \leq n$ if and only if for all finitely generated ${}_RM$ one has $\operatorname{Tor}_R^{n+1}(N,M) = 0$.*

Proof. Tensor products, being right exact functors, commute with direct limits; therefore Tor commutes with direct limits. The result follows. □

Lemma 3.4. *If R is a left noetherian ring, then*

$$\operatorname{injdim}{}_RR = \sup\{\operatorname{fdim} E_R \mid E \textit{ is injective}\}.$$

Proof. Write $n = \operatorname{injdim}{}_RR$, and $p = \sup\{\operatorname{fdim} E_R \mid E \text{ is injective}\}$.

First we show that $n \le p$. By Lemma 2.5, $\operatorname{Ext}^n_R(M,R) \ne 0$ for some finitely generated module ${}_RM$. Let E be the injective envelope of the right R-module $\operatorname{Ext}^n_R(M,R)$. By Lemma 2.1, $\operatorname{Tor}^R_n(E,M) \cong \operatorname{Hom}_R(\operatorname{Ext}^n_R(M,R),E) \ne 0$, so $\operatorname{fdim} E \ge n$.

Now we show that $n \ge p$. Let E_R be injective. If $m > n$, then $\operatorname{Ext}^m_R(M,R) = 0$ for all ${}_RM$, whence, for finitely generated M, $\operatorname{Tor}^R_m(E,M) = 0$ by Lemma 2.1. Thus $\operatorname{fdim} E \le n$ by Lemma 3.3. □

Theorem 3.5. [**195**] *If the left and right injective dimensions of a noetherian ring are both finite, then they are equal.*

Proof. (The noetherian hypothesis is required so that Lemma 2.1 can be invoked.) Let R be the ring in question. Define

$$\alpha := \sup\{\operatorname{fdim}{}_RM \mid \operatorname{fdim} M < \infty\}$$
$$\beta := \sup\{\operatorname{fdim} N_R \mid \operatorname{fdim} N < \infty.$$

We will prove that $\alpha \le \operatorname{injdim}{}_RR \le \beta$. Interchanging the roles of right and left modules, this also gives $\beta \le \operatorname{injdim} R_R \le \alpha$. The only way for both these inequalities to hold is for α to equal β, from which it follows that $\alpha = \operatorname{injdim}{}_RR = \operatorname{injdim} R_R = \beta$.

Write $n = \operatorname{injdim}{}_RR$. It is obvious that $\sup\{\operatorname{fdim} E_R \mid E \text{ is injective}\} \le \beta$, so it follows from Lemma 3.4 that $n \le \beta$. It remains to show that $\alpha \le n$. Suppose that ${}_RM$ is a finitely generated module with $\operatorname{fdim} M = m$. Then $\operatorname{Tor}^R_m(N,M) \ne 0$ for some N_R. Let E be the injective envelope of N. There is an exact sequence

$$0 = \operatorname{Tor}^R_{m+1}(E/N,M) \to \operatorname{Tor}^R_m(N,M) \to \operatorname{Tor}^R_m(E,M),$$

so $\operatorname{Tor}^R_m(E,M) \ne 0$. Thus $\operatorname{Ext}^m_R(M,R) \ne 0$, by Lemma 2.1. Hence $m \le \operatorname{injdim}{}_RR$. It follows that $\alpha \le n$. □

Proposition 3.6. *If* $\operatorname{gldim} R = n < \infty$*, then* $\operatorname{injdim} R = n$.

Proof. By hypothesis, $\operatorname{Ext}^n_R(N,M) \ne 0$ for some $M, N \in \mathsf{Mod}(R)$. Let $0 \to K \to F \to M \to 0$ be exact with F free. Since $\operatorname{gldim} R < n+1$, the long exact sequence for $\operatorname{Ext}^*_R(N,-)$ gives an exact sequence

$$\cdots \to \operatorname{Ext}^n(N,K) \to \operatorname{Ext}^n(N,F) \to \operatorname{Ext}^n(N,M) \to 0.$$

Hence $\operatorname{Ext}^n_R(N,F) \ne 0$, whence $\operatorname{injdim} R \ge n$; but $\operatorname{Ext}^{n+1}_R(-,R) = 0$, so there is equality. □

The converse of Proposition 3.6 is false. For example, $k[x]/(x^2)$ is injective as a module over itself, but has infinite global dimension. Typically, the homogeneous coordinate ring of a projective variety has *infinite global* dimension (it does unless the variety is a linear subspace), but it can have *finite injective* dimension (it does when the variety is a complete intersection). For this, and other reasons, rings of finite self-injective dimension are important.

There are plenty of rings of finite injective dimension; for example, by Corollary 8.4.3, if $x \in R$ is a normal, regular non-unit, then $\operatorname{injdim} R/(x) \le \operatorname{injdim} R - 1$, so the coordinate ring of a complete intersection has finite injective dimension.

Corollary 3.7. *The quotient* $k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ *of a commutative polynomial ring by a regular sequence* $f_1, \dots, f_r$ *has injective dimension* $n - r$.

EXAMPLE 3.8. The commutative ring $R = k[x, y]/(x^2, xy)$ has infinite injective dimension. One way to prove this is to notice that R is not a Cohen-Macaulay ring (see Definition 15.5.2, and Example 15.6.5), and hence not Gorenstein, then invoke Proposition 15.6.8.

LEMMA 3.9. [**23**, Corollary 2.3] *If R is a commutative noetherian ring, then*

$$\begin{aligned}\operatorname{injdim}_R M &= \sup\{\operatorname{injdim}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{injdim}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max} R\}.\end{aligned}$$

PROPOSITION 3.10. *If R is noetherian and* $\operatorname{injdim} R = 0$, *then R is artinian.*

PROOF. □

EXERCISES

3.1 Verify the claims in Example 4.4.

3.2 Let R be a semiprime noetherian ring. Show that
1. $Q := \operatorname{Fract} R$ is the injective envelope of R, and
2. Q/R is an injective R-module if and only if $\operatorname{injdim} R = 1$.

3.3 Repeat Exercise 2 for a graded ring A, and Q its homogeneous ring of fractions.

4. The grade of a module

Definition 4.1. The grade of $0 \neq M \in \mathsf{Mod}(R)$ is

$$j(M) := \inf\{n \mid \operatorname{Ext}^n_R(M, R) \neq 0\},$$

or ∞ if no such n exists.

LEMMA 4.2. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules, and write $\ell = j(L)$, $m = j(M)$, and $n = j(N)$. Then*
1. $\ell \geq \min\{m, n - 1\}$;
2. $m \geq \min\{\ell, n\}$;
3. $n \geq \min\{\ell + 1, m\}$.

In particular, if $m < \ell$ then $n = \ell + 1$, if $m = \ell$ then $n \geq m$, and if $m > \ell$ then $n = m$.

PROOF. To show that $j(X) \geq \min\{p, q\}$, one must show that $\operatorname{Ext}^i_A(X, R) = 0$ if $i < p$ and $i < q$. The result follows from the long exact sequence for $\operatorname{Ext}^*_R(-, R)$. □

EXAMPLE 4.3. $\boxed{\text{Paul}}$ If $R =$, then $j(R/\mathfrak{m}) = \infty$.

EXAMPLE 4.4. The ring of upper triangular matrices illustrates some of the ways in which the homological behavior of a non-commutative ring can differ from that of a commutative ring. Let

$$R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

There are two simple left modules, namely $N = R/J$ and $M = R/I$, where

$$J = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{gldim} R = \operatorname{injdim} R = 1$. The modules M and N have different homological properties: first, $\operatorname{pdim} N = 0$ and $\operatorname{pdim} M = 1$; second, $\operatorname{injdim} N = 1$ and $\operatorname{injdim} M = 0$; third, $j(N) = 0$ and $j(M) = 1$. In chapter 9, we will show that over an Auslander-Gorenstein ring, $j(L) = \min\{j(K), j(L/K)\}$ whenever K is a submodule of L; however, this fails in general as one sees by considering the exact sequence $0 \to N \to J \to M \to 0$.

The injective envelope, E say, of R may be realized as $M_2(k)$, with R-module structure inherited from the inclusion $R \subset M_2(k)$. Moreover,

$$E(R) = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix} \cong E(N) \oplus E(N),$$

and E/R is annihilated from the left by I, so the minimal injective resolution of R is $0 \to R \to E \to M \to 0$.

The right and left grades of a bimodule may differ. Because I is a two sided ideal, M is a bimodule; we have $j({}_RM) = 1$ and $j(M_R) = 0$.

PROPOSITION 4.5. *Let R be noetherian. If $0 \neq M \in \mathsf{mod}(R)$, then*

1. *$j(M) \leq \operatorname{pdim} M$, and*
2. *$j(M) \leq \operatorname{injdim} R$.*

PROOF. (1) This is obvious when $\operatorname{pdim} M = \infty$, and when $\operatorname{pdim} M < \infty$ the result is given by Lemma 2.6.

(2) If $\operatorname{injdim} R = \infty$, the result holds. Suppose that $\operatorname{injdim} R < \infty$. Then there is at least one non-zero term on the E_2 page of the convergent spectral sequence (5-4); i.e., $\operatorname{Ext}^q_R(M, R) \neq 0$ for some q. □

COROLLARY 4.6. *If $\operatorname{injdim} R < \infty$, then $j(M) = \infty$ if and only if $M = 0$.*

Typically the inequality in Proposition 4.5(1) is strict: for example, a non-zero left ideal of a ring has grade zero, but its projective dimension need not be zero.

LEMMA 4.7. *If M is a submodule of the i^{th} cosyzygy of R, then $j(M) \leq i$.*

PROOF. By hypothesis, there is a non-split exact sequence $0 \to \Omega_{i-1} \to X \to M \to 0$, so $\operatorname{Ext}^1_R(M, \Omega_{i-1}) \neq 0$, whence the result follows from Lemma 2.4(1). □

5. The bar resolution

In this section A is a connected k-algebra.

Definition 5.1. The bar resolution of a module $M \in \mathsf{GrMod}(A)$ is the complex

$$B_*(M): \qquad \cdots \to A \otimes \mathfrak{m}^{\otimes n} \otimes M \to \cdots \to A \otimes \mathfrak{m} \otimes M \to A \otimes M \to M \to 0$$

where

- all tensor products are over k,
- $B_n(M) = A \otimes \mathfrak{m}^{\otimes n} \otimes M$,
- the augmentation $\varepsilon : A \otimes M \to M$ is $\varepsilon(x \otimes m) = xm$,
- the differential $d_n : B_{n+1} \to B_n$ is

$$d_n(x_0 \otimes \cdots \otimes \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}.$$

Each $B^n(M)$ is given the tensor product grading, and is a graded left A-module. The differential is an A-module map and preserves degree. We can write $B_*(M) = A \otimes T(\mathfrak{m}) \otimes M$; this has a bigrading, where the degree (n, m) component is

$$\begin{aligned} B_n(M)_m &= \sum_{p+q+r=m} A_p \otimes (\mathfrak{m}_q^{\otimes n}) \otimes M_r, \\ &= \sum_{p+q_1+\cdots+q_n+r=m} A_p \otimes (\mathfrak{m}_{q_1} \otimes \cdots \otimes \mathfrak{m}_{q_n}) \otimes M_r. \end{aligned}$$

It is common, and convenient, to save space by using the notation

$$x_0|x_1|\cdots|x_n = x_0 \otimes x_1 \otimes \cdots \otimes x_n.$$

I suppose this is the origin of the terminology 'bar' resolution.

Proposition 5.2. *The bar resolution is a projective resolution of M.*

Proof. First, $d^2 = 0$ because

$$\begin{aligned} d^2(x_0|\cdots|x_{n+1}) &= \sum_{i=0}^{n} (-1)^i d(x_0|\cdots|x_i x_{i+1}|\cdots|x_{n+1}) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i-2} (-1)^{i+j} x_0|\cdots|x_j x_{j+1}|\cdots|x_i x_{i+1}|\cdots|x_{n+1} \\ &\quad + \sum_{i=0}^{n} (-1)^{i+i-1} x_0|\cdots|x_{i-1}x_i x_{i+1}|\cdots|x_{n+1} \\ &\quad\quad + \sum_{i=0}^{n} (-1)^{i+i} x_0|\cdots|x_i x_{i+1} x_{i+2}|\cdots|x_{n+1} \\ &\quad\quad\quad + \sum_{i=0}^{n}\sum_{j=i+2}^{n} (-1)^{i+j-1} x_0|\cdots|x_i x_{i+1}|\cdots|x_j x_{j+1}|\cdots|x_{n+1}, \end{aligned}$$

which is zero since the first and fourth sums cancel, and the second and third sums cancel. This proves that the bar resolution is a complex. To see that the complex is exact, we exhibit a contracting homotopy. Let $s_0 : M \to A \otimes M$ be the map $s_0(m) = 1 \otimes m$. For $n \geq 1$, define $s_n : B_{n-1}(M) \to B_n(M)$ by

$$s_n(x_0|\cdots|x_{n-1}) = 1|(x_0 - \bar{x_0})|x_1|\cdots|x_{n-1},$$

where the map $x \mapsto \bar{x}$ is the augmentation map followed by the algebra structure map. Now

$$\begin{aligned} ds(x_0|\cdots|x_n) = (x_0 - \bar{x_0})|x_1|\cdots|x_n \quad &- \quad 1|(x_0 - \bar{x_0})x_1|x_2|\cdots|x_n \\ + \sum_{i=1}^{n-1} &(-1)^{i+1} 1|(x_0 - \bar{x_0})|x_1|\cdots|x_i x_{i+1}|\cdots|x_n \end{aligned}$$

and

$$\begin{aligned} sd(x_0|\cdots|x_n) &= \sum_{i=0}^{n-1} (-1)^i s(x_0|\cdots|x_i x_{i+1}|\cdots|x_n) \\ &= 1|x_0 x_1 - \overline{(x_0 x_1)}|x_2|\cdots|x_i x_{i+1}|\cdots|x_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i 1|(x_0 - \bar{x_0})|\cdots|x_i x_{i+1}|\cdots|x_n \end{aligned}$$

Thus $(ds+sd)(x_0|\cdots|x_n)$ equals

$$(x_0-\bar{x_0})|x_1|\cdots|x_n \quad - \quad 1|(x_0-\bar{x_0})x_1|x_2|\cdots|x_n \quad + \quad 1|(x_0x_1-x_0\bar{x}_1)|x_2|\cdots|x_n$$

But $x_0\bar{x}_1=0$ because $x_1 \in \mathfrak{m}$, so this sum equals $x_0|\cdots|x_n$. Thus $sd+ds$ is the identity map, as required. □

When $M=k$, the bar resolution is $B^*(k)=A\otimes T(\mathfrak{m})$. Thus $\mathrm{Tor}_*^A(k,k)$ is the homology of the complex $k\otimes_A B_*(k)\cong T(\mathfrak{m})$, and $\underline{\mathrm{Ext}}_A^*(k,k)$ is the homology of $\underline{\mathrm{Hom}}_A(B_*(k),k)\cong \underline{\mathrm{Hom}}_k(T(\mathfrak{m}),k)=T(\mathfrak{m})^*$. Recall that the tensor algebra on a bar space V has a bialgebra structure. Thus these complexes which compute $\mathrm{Tor}_*^A(k,k)$ and $\underline{\mathrm{Ext}}_A^*(k,k)$ may be given bialgebra structures. The differential interacts nicely.

6. The Yoneda Product

In this section, R is an arbitrary ring.

Definition 6.1. Let X, Y, and Z be left R-modules. The Yoneda product is, by definition, a map

$$\mathrm{Ext}_R^p(Y,Z)\otimes \mathrm{Ext}_R^q(X,Y)\to \mathrm{Ext}_R^{p+q}(X,Z). \tag{6-1}$$

It is defined as follows. Let $P_\bullet \to X$ and $Q_\bullet\to Y$ be projective resolutions. Suppose that $[\beta]\in \mathrm{Ext}_R^i(Y,Z)$ and $[\alpha]\in\mathrm{Ext}_R^j(X,Y)$ are represented by $\beta\in\mathrm{Hom}_R(Q_i,Z)$ and $\alpha\in\mathrm{Hom}_R(P_j,Y)$. There is a lift $\alpha_i : P_{i+j}\to Q_i$ of α, so $\beta\circ\alpha_i : P_{i+j}\to Z$ gives a class in $\mathrm{Ext}_R^{i+j}(X,Z)$. By definition $[\beta][\alpha]:=[\beta\circ\alpha_i]$. The following picture illustrates what is happening:

$$\begin{tikzcd}
P_{i+j} \arrow[r] \arrow[d, "\alpha_i"'] & \cdots \arrow[r] & P_j \arrow[r] \arrow[d, "\alpha_0"'] & \cdots \arrow[r] & X \arrow[r] & 0 \\
Q_i \arrow[r] \arrow[d, "\beta"'] & \cdots \arrow[r] & Q_0 \arrow[r] & Y \arrow[r] & 0 & \\
Z & & & & &
\end{tikzcd} \tag{6-2}$$

Lemma 6.2. *The Yoneda product is associative.*

Proof. □

Thus, $\mathrm{Ext}^*(Y,Y)$ becomes a graded algebra, with degree p-component $\mathrm{Ext}_R^p(Y,Y)$, and $\mathrm{Ext}_R^*(X,Y)$ becomes a graded left $\mathrm{Ext}_R^*(Y,Y)$-module with degree q component $\mathrm{Ext}_R^q(X,Y)$.

The foregoing applies to a graded algebra A. If $X,Y,Z\in\mathsf{GrMod}(A)$, then the Yoneda product respects the grading on each Ext-group; that is,

$$\mathrm{Ext}_A^p(Y,Z)_i\otimes\mathrm{Ext}_A^q(X,Y)_j\to\mathrm{Ext}_A^{p+q}(X,Z)_{i+j}. \tag{6-3}$$

Thus $\underline{\mathrm{Ext}}_A^*(Y,Y)$ becomes a bigraded algebra and $\underline{\mathrm{Ext}}_A^*(X,Y)$ becomes a bigraded left $\underline{\mathrm{Ext}}_A^*(Y,Y)$-module.

For an augmented k-algebra R, the Yoneda product on $\mathrm{Ext}_R^*(k,k)$ has a direct description using the syzygy functors. First, $\mathrm{Ext}_R^n(k,k)\cong\mathrm{Hom}_R(\Omega^n k,k)$ for each $n\geq 0$. Second, there is a natural map

$$\mathrm{Hom}_R(\Omega^i k,k)\times\mathrm{Hom}_R(\Omega^j k,k)\to\mathrm{Hom}_R(\Omega^{i+j}k,k),$$

namely $(\alpha, \beta) \mapsto \alpha \circ \Omega^i(\beta)$, which puts an algebra structure on

$$\bigoplus_{i \geq 0} \mathrm{Hom}_R(\Omega^i k, k) \cong \mathrm{Ext}_R^*(k, k).$$

It is easy to show that this agrees with the product defined earlier.

LEMMA 6.3. *If A is a locally finite, connected, graded k-algebra, then there is an isomorphism of graded algebras*

$$\underline{\mathrm{Ext}}_A^*(k_A, k_A) \cong \underline{\mathrm{Ext}}_A^*(k, k)^{\mathrm{op}}.$$

PROOF. (Zhang) Let ${}_A\mathcal{C}$ denote the full subcategory of $\mathsf{GrMod}(A)$ consisting of the locally finite modules M such that $M_n = 0$ for $n \ll 0$. Let $\mathcal{D}_A$ denote the full subcategory of $\mathsf{GrMod}(A^{\mathrm{op}})$ consisting of the locally finite modules M such that $M_n = 0$ for $n \gg 0$. The functor $\mathrm{Hom}_{\mathrm{Gr}}(-, k)$ is an equivalence of categories ${}_A\mathcal{C} \to \mathcal{D}_A^{\mathrm{op}}$, sending k to k_A.

The modules in the minimal projective resolution of k belong to ${}_A\mathcal{C}$, so ${}_A\mathcal{C}$ has enough projectives, and $\underline{\mathrm{Ext}}_{\mathcal{C}}^*(k, k) \cong \underline{\mathrm{Ext}}_A^*(k, k)$. Similarly, the modules in the minimal injective resolution of k_A belong to $\mathcal{D}_A$, so $\underline{\mathrm{Ext}}_{\mathcal{D}}^*(k_A, k_A) \cong \underline{\mathrm{Ext}}_A^*(k_A, k_A)$. The equivalence $\mathrm{Hom}_{\mathrm{Gr}}(-, k)$ ensures that $\underline{\mathrm{Ext}}_{\mathcal{C}}^*(k, k) \cong \underline{\mathrm{Ext}}_{\mathcal{D}^{\mathrm{op}}}^*(k_A, k_A)$; the result now follows from the fact that $\underline{\mathrm{Ext}}_{\mathcal{D}^{\mathrm{op}}}^*(k_A, k_A) \cong \underline{\mathrm{Ext}}_{\mathcal{D}}^*(k_A, k_A)^{\mathrm{op}}$. $\square$

7. Ext and Tor for $A \otimes B$ and $A \circ B$

CHAPTER 8

Spectral sequences

This chapter gathers together the spectral sequences which we will use later on. For the most part, these results can be found in several books, so we keep our treatment brief.

A spectral sequence is a complicated machine for doing homological calculations. The way in which spectral sequences arise from filtered complexes is discussed in section 1. Those which we need tend to arise from bicomplexes, so in section 3 we explain how a bicomplex gives a spectral sequence.

Among the simplest spectral sequences are those in section 4 which are related to a ring homomorphism $f : R \to S$. These can be found in Rotman's book [**141**].

In sections 5 and 6 are spectral sequences involving Tor and Ext groups of modules over a single ring. These play an important role in this book. In particular, the most important tool in the study of rings of finite injective dimension is the double-Ext spectral sequence introduced in section 5.

In section 8 we give graded versions of the spectral sequences appearing in the earlier sections.

Convention. In this chapter we deal with cochain complexes rather than chain complexes. Thus complexes are written with superscript indices, and the differential increases the index:

$$\cdots \to C^{n-1} \to C^n \to C^{n+1} \to \cdots . \tag{0-1}$$

The morphisms are labelled so that $d^n : C^n \to C^{n+1}$. There is no essential difference between cochain and chain complexes if we adopt the convention that chain complexes should be indexed by subscripts, and the differential lowers the index; thus, a chain complex

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots , \tag{0-2}$$

can be rewritten as a cochain complex of the form (0-1) if we define $C^n := C_{-n}$.

1. The basics

Definition 1.1. A spectral sequence is a sequence $(E_r, d_r)_{r\geq 0}$, each term of which is called a page, consisting of $\mathbb{Z}^2$-graded groups

$$E_r = \bigoplus_{p,q} E_r^{pq}$$

and differentials $d_r : E_r^{pq} \to E_r^{p+r\, q-r+1}$, such that E_{r+1} is the homology of E_r; that is

$$E_{r+1}^{pq} = \frac{\ker(E_r^{pq} \to E_r^{p+r\, q-r+1})}{\operatorname{Im}(E_r^{p-r\, q+r-1} \to E_r^{pq})}.$$

If each E_r^{pq} is a module over a ring R, and each d_r is an R-module map, we speak of a spectral sequence of R-modules.

We speak of first, second, third, or fourth quadrant spectral sequences if E_r^{pq} is zero whenever $(p,q) \in \mathbb{R}^2$ is outside the first, second, third, or fourth quadrant respectively.

One should keep in mind a picture of each page with the individual terms E_r^{pq} placed at the lattice points $(p,q) \in \mathbb{Z}^2$. We speak of the 'rows' and 'columns' of each page, indexed by p and q respectively. The differential on the E_0 page goes up the columns. The differential on the E_1 page goes to the right along rows. The differential on the E_2 page is a knights move two steps to the right and one step down.

Definition 1.2. The limit E_∞ of a spectral sequence is defined as follows. We define

$$B_r^{pq} := \operatorname{Im} d_r \cap E_r^{pq} \qquad \text{and} \qquad Z_r^{pq} := \ker d_r \cap E_r^{pq},$$

whence $E_{r+1}^{pq} = Z_r^{pq}/B_r^{pq}$. Each E_{r+1} is a subquotient of E_r and, with some careful identifications, we can consider the subspaces

$$B_1^{pq} \subset B_2^{pq} \subset \cdots \subset Z_2^{pq} \subset Z_1^{pq} \subset E_1^{pq}.$$

We define

$$Z_\infty^{pq} = \bigcap_r Z_r^{pq} \qquad \text{and} \qquad B_\infty^{pq} = \bigcup_r B_r^{pq}.$$

The limit term is $E_\infty^{pq} := Z_\infty^{pq}/B_\infty^{pq}$.

Definition 1.3. A bounded filtration on a graded module $H = \oplus H^n$ consists of, for each n,

- a chain of submodules $H^n \supset \cdots \supset F^{p-1}H^n \supset F^pH^n \supset F^{p+1}H^n \supset \cdots$ such that
- $F^sH^n = H^n$ for $s \ll 0$ and $F^tH^n = 0$ for $t \gg 0$.

Definition 1.4. A spectral sequence (E_r, d_r) converges to a graded module $H = \oplus H^n$, denoted

$$E_2^{pq} \Rightarrow_p H^n$$

if there is a bounded filtration $H = F^0H \supset F^1H \supset F^2H \supset \dots$ such that $E_\infty^{pq} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$ for all p,q.

The boundedness condition means that for each n, if $p \gg 0$, and $q = n-p$, then

$$E_\infty^{pq} = E_\infty^{p+1\,q-1} = \cdots = 0.$$

Understanding convergence is the key to using spectral sequences.

Proposition 1.5. *If $E_2^{pq} \Rightarrow_p H^n$ is a first quadrant spectral sequence, then $H^0 \cong E_2^{00}$, and there is a five term exact sequence*

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \to E_2^{20} \to H^2.$$

If $E_2^{pq} = 0$ for $q \neq 0,1$, then the five term exact sequence extends to a long exact sequence

$$\begin{aligned} 0 \to &E_2^{10} \to H^1 \to E_2^{01} \to \\ \to &E_2^{20} \to H^2 \to E_2^{11} \to \\ \to &E_2^{30} \to H^2 \to E_2^{21} \to \cdots \end{aligned}$$

PROOF. The slices of the filtration $H^0 = F^0H^0 \supset F^1H^0 \supset \cdots$ are

$$\begin{aligned} F^0H^0/F^1H^0 &\cong E_\infty^{00}, && \text{and} \\ F^pH^0/F^{p+1}H^0 &\cong E_\infty^{p,-p} = 0 && \text{if } p \geq 1. \end{aligned}$$

It follows at once that $H^0 \cong E_\infty^{00}$. However, the differential through E_2^{00} is $0 \to E_2^{00} \to 0$ so, taking homology, $E_3^{00} \cong E_2^{00}$. Repeating this reasoning, we see that $E_\infty^{00} \cong E_2^{00}$, whence $H^0 \cong E_2^{00}$.

The slices of the filtration on H^1 are

$$\begin{aligned} F^0H^1/F^1H^1 &\cong E_\infty^{01}, \\ F^1H^1/F^2H^1 &\cong E_\infty^{10}, \quad \text{and} \\ F^pH^1/F^{p+1}H^1 &\cong E_\infty^{p,1-p} = 0 \quad \text{if } p \geq 2. \end{aligned}$$

Hence there is a short exact sequence $0 \to E_\infty^{10} \to H^1 \to E_\infty^{01} \to 0$, which we can represent by the picture:

$$\begin{array}{l} \bullet \\ | \quad \}E_\infty^{01} \\ \bullet \\ | \quad \}E_\infty^{10} \\ \bullet \end{array}$$

The differential through E_2^{10} is $0 \to E_2^{10} \to 0$, so $E_2^{10} \cong E_3^{10} \cong \cdots \cong E_\infty^{10}$. Hence there is an exact sequnce $0 \to E_2^{10} \to H^1 \to E_\infty^{01} \to 0$. The differential through E_2^{01} is $0 \to E_2^{01} \to E_2^{20} \to 0$, so there is an exact sequence $0 \to E_3^{01} \to E_2^{01} \to E_2^{20}$. The differential on the E_3 page is $0 \to E_3^{01} \to 0$ so $E_3^{01} \cong E_\infty^{01}$. Hence there is an exact sequence $0 \to E_\infty^{01} \to E_2^{01} \to E_2^{20}$. Thus we get an exact sequence

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \to E_2^{20}.$$

The filtration on H^2 looks like

$$\begin{array}{l} \bullet \\ | \quad \}E_\infty^{02} \\ \bullet \\ | \quad \}E_\infty^{11} \\ \bullet \\ | \quad \}E_\infty^{20} \\ \bullet \end{array}$$

On the E_2 page the differential looks like $0 \to E_2^{01} \to E_2^{20} \to 0$, so there is an exact sequence $E_2^{01} \to E_2^{20} \to E_3^{20}$. But $E_3^{20} \cong E_\infty^{20}$ which embeds in H^2, so there is an exact sequence $E_2^{01} \to E_2^{20} \to H^2$. Combining this with the exact sequence at the end of the previous paragraph gives the five term exact sequence.

Now suppose that $E_2^{pq} = 0$ for $q \neq 0, 1$. Then for each p there is an exact sequence

$$0 \to E_\infty^{p1} \to E_2^{p1} \to E_2^{p+2\,0} \to E_\infty^{p+2\,0} \to 0.$$

The filtration on each H^n has only two non-zero slices which are given by the exact sequence

$$0 \to E_\infty^{n0} \to H^n \to E_\infty^{n-1\,1} \to 0.$$

Splicing together these sequences and the previous ones gives the long exact sequence. $\square$

EXAMPLE 1.6. The simplest type of convergent spectral sequence is one in which there is a single row. Let's say we have a first quadrant sequence $E_2^{pq} \Rightarrow_p H^n$, and the only non-zero row is the $q = 0$ one. The first thing to observe is that all pages are the same; in particular, $E_\infty \cong E_2$. Therefore $H^n \cong E_2^{0n}$ for all n.

2. The spectral sequence asociated to a filtered complex

The basic object giving rise to a spectral sequence is a complex on which there is a filtration compatible with the differential.

Definition 2.1. A filtered complex is a complex $(C^\bullet, d)$ such that each C^n is endowed with a filtration

$$\cdots \supset F^{p-1}C^n \supset F^pC^n \supset F^{p+1}C^n \supset \cdots$$

satisfying $d(F^pC^n) \subset F^pC^{n+1}$ for all p and n.

Given such a filtered complex, there is an induced filtration on the homology groups $H^n(C)$ because, for each p, the inclusion $F^pC \to C$ gives a homomorphism $H^n(F^pC) \to H^n(C)$; we define $F^pH^n(C)$ to be the image of this map. We introduce the notation

$$F^{pq}C := F^pC^{p+q}.$$

Hence there are inclusions $F^{p+1\,q-1}C \to F^{pq}C$, and we may define

$$E_0^{pq}C = F^{pq}C/F^{p+1\,q-1}C.$$

Forgetting the differential, a complex $C = \oplus C^n$ is a graded abelian group so, as in Definition 1.3, we may speak of a bounded filtration on the complex.

THEOREM 2.2. *A filtered complex (FC, d) determines a spectral sequence in which*

$$\begin{aligned}
E_0^{pq} &= F^pC^{p+q}/F^{p+1}C^{p+q} \\
E_1^{pq} &= H^{p+q}(F^pC/F^{p+1}C), \\
E_\infty^{pq} &= F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C),
\end{aligned}$$

If the filtration is bounded, then
1. *for each p, q, $E_\infty^{pq} = E_r^{pq}$ for $r \gg 0$, and*
2. *$E_2^{pq} \Rightarrow_p H^{p+q}(C)$.*

REMARK 2.3. So far we have only discussed *descending* filtrations, meaning that as the index increases, the subgroup gets smaller. One also encounters *ascending* filtrations (for example, see Chapter 4, and Theorem 4.4.7). Some notational chicanery, similar to that which allows chain complexes to be viewed as cochain complexes, allows us to treat an ascending filtration as a descending one: if V is an abelian group with a filtration $\cdots \subset V_{n-1} \subset V_n \subset V_{n+1} \subset \cdots$, we may define $F^nV := V_{-n}$, so that $\cdots \supset F^{n-1}V \supset F^nV \supset F^{n+1}V \supset \cdots$ is a descending filtration.

3. The spectral sequences associated to a bicomplex

Definition 3.1. A bicomplex of R-modules is a $\mathbb{Z}^2$-graded R-module

$$B := \bigoplus_{p,q,\in\mathbb{Z}} B^{pq}$$

together with differentials $d : B^{pq} \to B^{p+1\,q}$ and $\delta : B^{pq} \to B^{p\,q+1}$ satsifying $d^2 = \delta^2 = d\delta + \delta d = 0$. The associated total complex is $(B^\bullet, D)$ defined by

$$B^n := \bigoplus_{p+q=n} B^{pq}$$

and $D = d + \delta$.

There are two filtrations on $(B^\bullet, D)$ defined by

$$'F^pB^n = \bigoplus_{i+q=n, i\geq p} B^{iq}$$
$$''F^qB^n = \bigoplus_{p+j=n, j\geq q} B^{pj}$$

By Theorem 2.2, each of these filtrations gives a spectral seqences associated to the bicomplex B. If B is a first or third quadrant bicomplex, these filtrations are bounded so the spectral sequences converge. Now we describe the E_2 pages of these two spectral sequences.

4. Change of rings

Associated to a ring homomorphism $f : R \to S$ is an adjoint pair of functors (f^*, f_*) defined as follows:

$$f^* : \mathsf{Mod}(R) \to \mathsf{Mod}(S), \qquad f^*M = S \otimes_R M,$$
$$f_* : \mathsf{Mod}(S) \to \mathsf{Mod}(R), \qquad f_*N = \operatorname{Hom}_S(S, N)$$

where the R-module structure on f_*N is induced by the right action of R on S; thus, $f_*N = N$ as an abelian group and $r \in R$ acts like $f(r) \in S$. The associated adjoint isomorphism is

$$\operatorname{Hom}_S(S \otimes_R M, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(S, N)) \cong \operatorname{Hom}_R(M, N).$$

This leads to the first spectral sequence in the next theorem.

There is also a right adjoint to f_*, namely $\operatorname{Hom}_R(S, -) : \mathsf{Mod}(R) \to \mathsf{Mod}(S)$, and this too leads to a spectral sequence via the adjoint isomorphism

$$\operatorname{Hom}_S(N, \operatorname{Hom}_R(S, M)) \cong \operatorname{Hom}_R(\operatorname{Hom}_S(S, N), M)) \cong \operatorname{Hom}_R(N, M).$$

The associated spectral sequence is the second one in the next theorem.

The third spectral sequence in the next theorem arises from the isomorphism $L \otimes_S S \otimes_R M \cong L \otimes_R M$.

Theorem 4.1. *Let $f : R \to S$ be a ring homomorphism. Let $L \in \mathsf{Mod}(S^{\mathrm{op}})$, $N \in \mathsf{Mod}(S)$, $M \in \mathsf{Mod}(R)$; view N as an R-module in the obvious way. Then there are convergent spectral sequences*

$$\operatorname{Ext}^p_S(\operatorname{Tor}^R_q(S, M), N) \Rightarrow_p \operatorname{Ext}^n_R(M, N).$$
$$\operatorname{Ext}^p_S(N, \operatorname{Ext}^q_R(S, M)) \Rightarrow_p \operatorname{Ext}^n_R(N, M),$$
$$\operatorname{Tor}^S_p(L, \operatorname{Tor}^R_q(S, M)) \Rightarrow_p \operatorname{Tor}^R_n(L, M),$$

Proof. Let $P_\bullet \to M$ and $M \to I^\bullet$ be respectively a projective, and an injective, resolution of M as an R-module. Let $Q_\bullet \to N$ and $N \to E^\bullet$ be respectively a projective, and an injective resolution, of N as an S-module. Let $X_\bullet \to L$ be a projective resolution of L as an S-module.

For the first spectral sequence, consider the bicomplex

$$\begin{tikzcd}
& \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}_R(P_1, E^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_1, E^1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(P_0, E^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_0, E^1) \arrow[r] \arrow[u] & \cdots \\
& 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\mathrm{Hom}_R(P_p, -)$ is exact, all rows are exact except at the 0^{th} column; therefore, the homology along the rows yields a single non-zero column, namely

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \cdots ,$$

the homology of which is $\mathrm{Ext}^n_R(M, N)$. On the other hand, before taking homology up the columns of the bicomplex, rewrite the p^{th} column as

$$0 \to \mathrm{Hom}_S(S \otimes_R P_0, E^p) \to \mathrm{Hom}_S(S \otimes_R P_1, E^p) \to \cdots ;$$

since $\mathrm{Hom}_S(-, E^p)$ is exact, the homology of this at position pq is $\mathrm{Hom}_S(\mathrm{Tor}^R_q(S, M), E^p)$; now taking homology along the rows gives $\mathrm{Ext}^p_S(\mathrm{Tor}^R_q(S, M), N)$. The first spectral sequence follows.

For the second spectral sequence, consider the bicomplex

$$\begin{tikzcd}
& \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}_R(Q_0, I^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(Q_1, I^1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(Q_0, I^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(Q_1, I^0) \arrow[r] \arrow[u] & \cdots \\
& 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\mathrm{Hom}_R(-, I^q)$ is exact, all rows are exact except at the 0^{th} column; therefore, the homology along the rows yields a single non-zero column, namely

$$0 \to \mathrm{Hom}_R(N, I^0) \to \mathrm{Hom}_R(N, I^1) \to \cdots ,$$

the homology of which is $\mathrm{Ext}^n_R(N, M)$. On the other hand, before taking homology up the columns of the bicomplex, rewrite the p^{th} column as

$$0 \to \mathrm{Hom}_S(Q_p, \mathrm{Hom}_R(S, I^0)) \to \mathrm{Hom}_S(Q_p, \mathrm{Hom}_R(S, I^1)) \to \cdots ;$$

since $\mathrm{Hom}_S(Q_p, -)$ is exact, the homology of this at position pq is $\mathrm{Hom}_S(Q_p, \mathrm{Ext}^q_R(S, M))$; now taking homology along the rows gives $\mathrm{Ext}^p_S(N, \mathrm{Ext}^q_R(S, M), N)$, thus giving the second spectral sequence.

For the third spectral sequence, consider the bicomplex

$$\begin{tikzcd}
 & 0 & 0 & \\
\cdots \arrow[r] & X_1 \otimes_R P_0 \arrow[u] \arrow[r] & X_0 \otimes_R P_0 \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & X_1 \otimes_R P_1 \arrow[u] \arrow[r] & X_0 \otimes_R P_1 \arrow[u] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Since $- \otimes_R P_q$ is exact, all rows are exact except at the 0^{th} column; therefore, the homology along the rows yields a single non-zero column, namely

$$0 \to L \otimes_R P_0 \to L \otimes_R P_1 \to \cdots ,$$

the homology of which is $\operatorname{Tor}_n^R(N, M)$. On the other hand, before taking homology up the columns of the bicomplex, rewrite the p^{th} column as

$$0 \longleftarrow X_p \otimes_S (S \otimes_R P_0) \longleftarrow X_p \otimes_S (S \otimes_R P_1) \longleftarrow \cdots ;$$

since $X_p \otimes_S -$ is exact, the homology of this at position pq is $X_p \otimes \operatorname{Tor}_q^R(S, M))$; now taking homology along the rows gives $\operatorname{Tor}_p^S(L, \operatorname{Tor}_q^R(S, M))$, thus giving the third spectral sequence. □

If S is a projective R-module these give isomorphisms

$$\begin{aligned}
\operatorname{Ext}_S^p(S \otimes_R M, N) &\cong \operatorname{Ext}_R^n(M, N), \\
\operatorname{Ext}_S^p(N, \operatorname{Hom}_R(S, M)) &\cong \operatorname{Ext}_R^p(N, M), \\
\operatorname{Tor}_p^S(L, S \otimes_R M) &\cong \operatorname{Tor}_p^R(L, M).
\end{aligned}$$

Another important case occurs when $S = R/(x)$, and x is a normal, regular non-unit.

Proposition 4.2 (Rees' Lemma). *Let $x \in R$ be a normal regular non-unit acting faithfully on an R-module M. If N is an R-module annihilated by x, then*

$$\operatorname{Ext}_{R/(x)}^p(N, M/xM) \cong \operatorname{Ext}_R^{p+1}(N, M).$$

Proof. The projective resolution of $S = R/(x)$ as a left R-module is of the form $0 \to R \to R \to S \to 0$, where the first map is right multiplication by x. Therefore $\operatorname{Ext}_R^q(S, M) = 0$ if $q > 1$. Moreover, $\operatorname{Hom}_R(S, M) = 0$, and $\operatorname{Ext}_R^1(S, M) \cong M/xM$, so the second spectral sequence in Theorem 4.1 collapses to give the required isomorphism. □

Corollary 4.3. *Let $x \in R$ be a normal regular element. Then*

1. $\operatorname{gldim} R/(x)$ *is either infinite or* $\leq \operatorname{gldim} R - 1$*;*
2. $\operatorname{injdim} R/(x) \leq \operatorname{injdim} R - 1$.

PROOF. Write $S = R/(x)$. We can assume that x is not a unit because if it were, then $S = 0$, so there is nothing to do.

(1) Suppose that $\operatorname{gldim} S \neq \infty$. Let N be an S-module, and set $n = \operatorname{pdim}_S N$. Then $\operatorname{Ext}^n_S(N, S) \neq 0$ by Lemma 7.2.6, whence $\operatorname{Ext}^{n+1}_R(N, R) \neq 0$ by Rees' lemma. Thus $\operatorname{pdim}_R N \geq n + 1$, or $n \leq \operatorname{pdim}_R N - 1 \leq \operatorname{gldim} R - 1$. The result follows.

(2) If N is an S-module, then $\operatorname{Ext}^p_S(N, S) \cong \operatorname{Ext}^{p+1}_R(N, R)$, so the result follows. □

This result is best possible. For example, $\operatorname{gldim} k[x]/(x^2) = \infty$, and if R is the enveloping algebra of the 3-dimensional Heisenberg Lie algebra, then $\operatorname{gldim} R = 3$, but $\operatorname{gldim} R/(z - 1) = 1$ where z is a non-zero central element in the Lie algebra.

Corollaries: $\operatorname{gldim} R, R[x]$; and injdim

5. The double-Ext spectral sequence

An essential tool for the analysis of rings of finite self-injective dimension is the double-Ext spectral sequence; in its original form it is due to Grothendieck [**140**], with later developments due to Ischebeck [**83**] and Levasseur [**97**]. Grothendieck's spectral sequence arises from the bidualizing complex, whereas Ischebeck's arises from another bicomplex; that the two spectral sequences coincide under appropriate hypotheses is due to Levasseur.

THEOREM 5.1. *Let R and S be rings. Let M be a left R-module, and ${}_SN_R$ an S-R-bimodule. Suppose that M has a resolution by finitely generated projectives, and that* $\operatorname{injdim} N_R < \infty$. *Then there is a fourth quadrant convergent spectral sequence of left S-modules*

$$E_2^{pq} = \operatorname{Ext}^p_R(\operatorname{Ext}^{-q}_R(M, R), N) \Rightarrow \operatorname{Tor}^R_{-q-p}(N, M). \tag{5-1}$$

PROOF. Let $N \to E^\bullet$ be an injective resolution of N_R, and let $P_\bullet \to M$ be a projective resolution of ${}_RM$ with each P_j finitely generated. The Ischebeck complex is the fourth quadrant bicomplex $L^{pq} = E^p \otimes_R P_{-q}$, which looks like

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & E^0 \otimes P_0 \arrow[r] \arrow[u] & E^1 \otimes P_0 \arrow[r] \arrow[u] & E^2 \otimes P_0 \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & E^0 \otimes P_1 \arrow[r] \arrow[u] & E^1 \otimes P_1 \arrow[r] \arrow[u] & E^2 \otimes P_1 \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{5-2}$$

Since each P_j is finitely generated $E^p \otimes_R P_{-q} \cong \operatorname{Hom}_R(P^\vee_{-q}, E^p)$.

Consider the first spectral sequence associated to L. Taking homology down the columns, the rows of $H^{**}_{\mathrm{II}}(L)$ are

$$\cdots \to \operatorname{Tor}^R_q(E^p, M) \to \operatorname{Tor}^R_q(E^{p+1}, M) \to \cdots \tag{5-3}$$

However, by Lemma 7.2.1, $\operatorname{Tor}^R_q(E^p, M) \cong \operatorname{Hom}_R(\operatorname{Ext}^q_R(M, R), E^p)$, so (5-3) is obtained by applying the functor $\operatorname{Hom}_R(\operatorname{Ext}^q_R(M, R), -)$ to an injective resolution of N_R. Hence the homology of (5-3) is

$$'E_2^{pq} = \operatorname{Ext}_R^p(\operatorname{Ext}_R^{-q}(M,R),N).$$

The second spectral sequence collapses: since $-\otimes_R P_{-q}$ is exact, the rows L^{*q} are exact except in the left-most column L^{0*} where the homology is $N \otimes P_*$. Taking homology down the columns of $H_{\mathrm{I}}^{**}(L)$, we get

$$''E_2^{pq} = H^q(H_{\mathrm{I}}^{p*}(L)) = \begin{cases} \operatorname{Tor}_{-q}^R(N,M) & \text{if } p=0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

Therefore $\mathbb{H}^{p+q}(L) = \operatorname{Tor}_{-p-q}^R(N,M)$.

Combining the results of the last two paragraphs, we obtain a spectral sequence

$$\operatorname{Ext}_R^p(\operatorname{Ext}_R^q(M,R),N) \Rightarrow \operatorname{Tor}_{q-p}^R(N,M).$$

Since $\operatorname{injdim}(N_R) < \infty$, we may arrange for the injective resolution $N \to E^\bullet$ to be finite. It follows that the spectral sequence converges.

Finally, since an injective $S \otimes R^{\mathrm{op}}$-module is injective as an R^{op}-module (Lemma 1.6), we can arrange for $N \to E^\bullet$ to be an injective resolution of N as an S-R-bimodule. In that case the bicomplex (5-2) consists of left S-modules and S-module homomorphisms, so the spectral sequence lives in the category of left S-modules. □

Corollary 5.2. *Let R be a noetherian ring of finite injective dimension, and M a finitely generated R-module. Then there is a fourth quadrant convergent spectral sequence of left R-modules*

$$E_2^{pq} = \operatorname{Ext}_R^p(\operatorname{Ext}_R^{-q}(M,R),R) \Rightarrow \mathbb{H}^{p+q}(M) = \begin{cases} M & \text{if } p+q=0, \\ 0 & \text{if } p+q \neq 0. \end{cases} \tag{5-4}$$

We call (5-4) the double-Ext spectral sequence; its E_2 page looks like

$$\begin{matrix} E^{00}(M) & E^{10}(M) & E^{20}(M) & \cdots \\ E^{0,-1}(M) & E^{1,-1}(M) & E^{2,-1}(M) & \cdots \\ E^{0,-2}(M) & E^{1,-2}(M) & E^{2,-2}(M) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \tag{5-5}$$

where $E^{p,-q}(M)$ denotes $\operatorname{Ext}_R^p(\operatorname{Ext}_R^q(M,R),R)$. The differential d_2 has bidegree $(2,-1)$; i.e., it is a knight's move two steps to the right and one step down. Some authors prefer to display the E_2 page by reflecting (5-5) in the x-axis and using non-negative indices. Convergence of the spectral sequence means that there is a finite filtration $M = F^0M \supset F^1M \supset \cdots \supset F^{\mu+1}M = 0$ by R-submodules, such that $F^pM/F^{p+1}M \cong E_\infty^{p,-p}$ which is a subquotient of $\operatorname{Ext}_R^p(\operatorname{Ext}_R^p(M,R),R)$.

This filtration on M is called the canonical filtration.

6. The Tor-Ext spectral sequence

Theorem 6.1. *Let R be a left noetherian k-algebra, and S and T arbitrary k-algebra. Let ${}_RM_S$ and ${}_RN_T$ be bimodules, and assume that N is finitely generated as a left R-module. Then there is a convergent second quadrant spectral sequence*

$$\operatorname{Tor}_{-p}^R(\operatorname{Ext}_R^q(M,R),N) \Rightarrow \operatorname{Ext}_R^{q+p}(M,N), \tag{6-1}$$

of S-T-bimodules.

PROOF. Take a resolution $P_\bullet \to M$ by projective $R \otimes_k S^{\mathrm{op}}$-modules, and take a resolution $Q_\bullet \to N$ by finitely generated projective R-modules. Each P_j is a projective R-module, so we write $P_j^\vee = \mathrm{Hom}_R(P_j, R)$; although P_j need not be a finitely generated R-module, we may still conclude that $P_j^\vee$ is a flat R-module by Lemma 11.4.4(3).

Consider the second quadrant bicomplex

$$\begin{tikzcd}
 & \vdots & \vdots \\
 & {} & {} \\
\cdots \arrow[r] & P_1^\vee \otimes_R Q_1 \arrow[u] \arrow[r] & P_1^\vee \otimes_R Q_0 \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P_0^\vee \otimes_R Q_1 \arrow[u] \arrow[r] & P_0^\vee \otimes_R Q_0 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd} \tag{6-2}$$

The term in position pq is $L^{pq} = P_q^\vee \otimes_R Q_{-p}$. The homology along the $-i^{\text{th}}$ column is

$$h^q(P_\bullet^\vee \otimes_R Q_i) \cong h^q(P_\bullet^\vee) \otimes_R Q_i \cong \mathrm{Ext}_R^q(M, R) \otimes_R Q_i,$$

because $- \otimes_R Q_i$ is an exact functor. Now, taking homology along the q^{th} row gives

$$E_2^{pq} = h^p(\mathrm{Ext}_R^q(M, R) \otimes_R Q_\bullet) \cong \mathrm{Tor}_{-p}^R(\mathrm{Ext}_R^q(M, R), N).$$

On the other hand, the homology along the q^{th} row of the bicomplex is

$$h^p(P_q^\vee \otimes_R Q_\bullet) \cong P_q^\vee \otimes_R h^p(Q_\bullet) = \begin{cases} 0 & \text{if } p \neq 0, \\ P_q^\vee \otimes_R N & \text{if } p = 0, \end{cases}$$

where the first isomorphism arises because $P_q^\vee$ is flat. Now only the 0^{th} column is non-zero, and its homology is

$$h^q(P_\bullet^\vee \otimes_R N) \cong h^q(\mathrm{Hom}_R(P_\bullet, N)) \cong \mathrm{Ext}_R^q(M, N),$$

where the first isomorphism comes from Lemma 11.4.4(1). Hence we have a spectral sequence as claimed. □

7. Normal subalgebras

We follow [**43**, Chapter XVI, Section 6].

Definition 7.1. Let A be a connected k-algebra. If $a_1, \cdots, a_n \in A$ are homogeneous elements of positive degree such that the left ideal they generate coincides with the right ideal they generate, we call $B = k[a_1, \cdots, a_n]$ a normal subalgebra of A. We write $A//B$ for the quotient ring $A/(a_1, \ldots, a_n)$.

PROPOSITION 7.2. *Let B be a normal subalgebra of A, and write $C = A//B$. Suppose that A is free as a left B-module. Let $L \in \mathsf{GrMod}(C^{\mathrm{op}})$, $M \in \mathsf{GrMod}(C)$, $N \in \mathsf{GrMod}(A)$. Then there are spectral sequences*

$$\underline{\mathrm{Ext}}_C^p(M, \underline{\mathrm{Ext}}_B^q(k, N)) \Rightarrow_p \underline{\mathrm{Ext}}_A^n(M, N),$$

and

$$\mathrm{Tor}_p^C(\mathrm{Tor}_q^B(L, k), N) \Rightarrow_p \mathrm{Tor}_n^A(L, N),$$

where the right action of C on k is used to make $\underline{\mathrm{Ext}}^q_B(k,N)$ a left C-module, and $\mathrm{Tor}^B_q(L,k)$ a right C-module.

PROOF. □

EXAMPLE 7.3. If $x \in A$ is a normal regular element of positive degree, then $B = k[x]$ is a normal subalgebra of A, and $A//B = A/(x)$. Since x is regular, A is a torsion-free $k[x]$-module, and hence free since it is positively graded. Taking $N = A$, the spectral sequence gives an isomorphism

$$\underline{\mathrm{Ext}}^p_{A/(x)}(M, A/(x)) \cong \underline{\mathrm{Ext}}^{p+1}_A(M, A)$$

whenever $M \in \mathsf{GrMod}(A)$ is annihilated by x. This is a special case of Rees' Lemma (8.1).

EXAMPLE 7.4. Let B be connected, and $\sigma \in \mathrm{Aut}_k(B)$, and δ a homogeneous σ-derivation of B of non-negative degree, and form the Ore extension $A = B[y;\sigma,\delta]$. Then B is a normal subalgebra of A and $A//B \cong k[y]$. Moreover, A is a free B-module, so there is a spectral sequence

$$\underline{\mathrm{Ext}}^p_{k[y]}(M, \underline{\mathrm{Ext}}^q_B(k,N)) \Rightarrow_p \underline{\mathrm{Ext}}^n_A(M,N),$$

whenever $M \in \mathsf{GrMod}(A)$ is annihilated by the augmentation ideal of B. In particular, if $M = k$ and $N = A$, then

$$\underline{\mathrm{Ext}}^1_{k[y]}(k, \underline{\mathrm{Ext}}^q_B(k,A)) \cong \underline{\mathrm{Ext}}^{p+1}_A(k,A).$$

8. Graded versions of the spectral sequences

In this section all rings are graded.

PROPOSITION 8.1 (Rees' Lemma). *Let $x \in A_d$ be a normal regular non-unit acting faithfully on $M \in \mathsf{GrMod}(A)$. If $N \in \mathsf{GrMod}(A)$ is annihilated by x, then*

$$\underline{\mathrm{Ext}}^p_{A/(x)}(N, M/xM)[d] \cong \underline{\mathrm{Ext}}^{p+1}_A(N,M).$$

PROOF. The projective resolution of $B = A/(x)$ as a left A-module is of the form $0 \to A[-d] \to A \to B \to 0$, where the first map is right multiplication by x. Therefore $\underline{\mathrm{Hom}}_A(B,M) = 0$, $\underline{\mathrm{Ext}}^1_A(B,M) \cong M[d]/xM[d]$, and $\underline{\mathrm{Ext}}^q_A(B,M) = 0$ if $q > 1$. The second spectral sequence in Theorem 4.1 collapses to give the required isomorphism. □

It is also important to consider what happens when x does not act faithfully on M. This situation is considered in the chapter on Koszul algebras.

CHAPTER 9

Auslander-Gorenstein rings

This chapter concerns an important technical property which is a automatically satisfied by any commutative noetherian ring of finite injective dimension; however, for non-commutative rings this property does not follow from the ring having finite injective dimension.

Let R be a commutative domain which is a finitely generated k-algebra—thus, R is the coordinate ring of an irreducible affine variety, X say. Then R has finite global dimension if and only if X is non-singular. Unfortunately, for non-commutative algebras the requirement that a ring have finite global dimension is not a particularly strong hypothesis. The problem is that several technical consequences of this which hold in the commutative case, no longer hold in the non-commutative case—we will comment on specific examples of this phenomenon as they arise in the course of this chapter. Accordingly, if we seek a class of non-commutative algebras which behave in some respects like commutative algebras of finite global dimension, we need hypotheses which for commutative algebras are already consequences of finite global dimension.

1. Basic properties

Definition 1.1. Let R be a noetherian ring.

- A left R-module M satisfies the Auslander condition if for all submodules $N \subset \mathrm{Ext}^j_R(M,R)$, $\mathrm{Ext}^i_R(N,R) = 0$ whenever $i < j$.
- If every $M \in \mathsf{mod}(R)$ satisfies the Auslander condition we say that R satisfies the Auslander condition.
- If R satisfies the Auslander condition and $\mathrm{injdim}\, R < \infty$, we say that R is Auslander-Gorenstein.
- If R satisfies the Auslander condition and $\mathrm{gldim}\, R < \infty$, we say that R is Auslander-regular.

When we say a ring is Auslander-Gorenstein or Auslander-regular we tacitly imply that it is both left and right noetherian.

For the remainder of this section R denotes an Auslander-Gorenstein ring.

If R is noetherian and has finite injective dimension, and $M \in \mathsf{mod}(R)$ satisfies the Auslander condition, then the E_2-page of the double-Ext spectral sequence (Chapter 8, Section 5) for M looks like

$$\begin{array}{ccc} E^{00} & E^{10} & E^{20}\cdots \\ 0 & E^{1,-1} & E^{2,-1}\cdots \\ 0 & 0 & E^{2,-2}\cdots \\ \vdots & \vdots & \vdots \end{array} \tag{1-1}$$

where $E^{p,-q} = \mathrm{Ext}^p_R(\mathrm{Ext}^q_R(M,R),R)$.

THEOREM 1.2. *Let R be Auslander-Gorenstein with* $\operatorname{injdim} R = n$. *The canonical filtration*

$$M = F^0 M \supset F^1 M \supset \cdots \supset F^{\mu+1} M = 0, \tag{1-2}$$

on $M \in \mathsf{mod}(R)$ has the following properties:

1. *for each p there is an exact sequence*
$$0 \to F^p M / F^{p+1} M \to \operatorname{Ext}^p_R(\operatorname{Ext}^p_R(M,R),R) \to Q(p) \to 0; \tag{1-3}$$
2. *each $Q(p)$ has a filtration for which the 'composition factors' are R-module subquotients of $\oplus_{r\geq 2} \operatorname{Ext}^{p+r}_R(\operatorname{Ext}^{p+r-1}_R(M,R),R)$;*
3. *$j(F^p M) \geq p$;*
4. *$j(Q(p)) \geq p+2$;*
5. *if $m = j(M)$, then $M = F^0 M = \cdots = F^m M \neq F^{m+1} M$.*

PROOF. The canonical filtration (1-3) is determined by the double-Ext spectral sequence (5-4) in Corollary 8.5.2; that is $F^p M / F^{p+1} M \cong E^{p,-p}_\infty$.

(1) The subdiagonal zeroes in (1-1) imply that $E^{p,-p}_\infty \subset E^{p,-p}_2$, which gives the injection in (1). By definition $Q(p)$ is the cokernel of this injection.

(2) Now $E^{p,-p}_\infty \subset \cdots \subset E^{p,-p}_3 \subset E^{p,-p}_2$, so $Q(p)$ has a filtration with composition factors $E^{p,-p}_r / E^{p,-p}_{r+1}$ for $r \geq 2$. By definition, $E^{p,-p}_{r+1}$ is the kernel of the map $E^{p,-p}_r \to E^{p+r,-p-r+1}_r$, so $E^{p,-p}_r / E^{p,-p}_{r+1}$ is isomorphic to a submodule of $E^{p+r,-p-r+1}_r$, which is itself a subquotient of $E^{p+r,-p-r+1}_2$. The description of $Q(p)$ in (2) follows.

(3) By descending induction, beginning with $p = \mu + 1$, we can assume that $j(F^{p+1}M) \geq p+1$. The exact sequence $0 \to F^{p+1}M \to F^p M \to F^p M/F^{p+1}M \to 0$ and Lemma 7.4.2 imply $j(F^p M) \geq \min\{j(F^{p+1}M, j(F^p M/F^{p+1}M)\}$. However, by (1-3) and the Auslander condition, $j(F^p M/F^{p+1}M) \geq p$, so $j(F^p M) \geq p$.

(4) The Auslander condition implies that the composition factors of $Q(p)$ in (2) all have grade $\geq p+2$, so the result follows from Lemma 7.4.2(2).

(5) By (3), $j(F^{m+1}M) \geq m+1$, so $M \neq F^{m+1}M$. On the other hand, if $p < m$, then $\operatorname{Ext}^p_R(\operatorname{Ext}^p_R(M,R),R) = 0$, so $F^{p+1}M = F^p M$ by (1-3). □

The Auslander condition says that submodules of $\operatorname{Ext}^i_R(M,R)$ have grade $\geq i$; in particular, $j(\operatorname{Ext}^i(M,R)) \geq i$ whence $j(M^\vee) \geq j(M)$. In fact, the next result shows that this last inequality is an equality.

COROLLARY 1.3. *Let R be Auslander-Gorenstein. If $M \in \mathsf{mod}(R)$, then*

$$j(M^\vee) = j(M).$$

PROOF. Suppose the result is false, and write $m = j(M)$. The Auslander condition implies that $j(M^\vee) > m$, whence $\operatorname{Ext}^m_R(\operatorname{Ext}^m_R(M,R),R) = 0$. Also $\operatorname{Ext}^i_R(\operatorname{Ext}^i_R(M,R),R) = 0$ for $i < m$ so, by (1-3), $M = F^0 M = \cdots = F^{m+1}M$. Therefore $j(F^{m+1}M) = j(M) = m$, contradicting Theorem 1.2(3), so we conclude that the result is true. □

COROLLARY 1.4. *If R is Auslander-Gorenstein and $0 \to L \to M \to N \to 0$ is exact in $\mathsf{mod}(R)$, then*

$$j(M) = \min\{j(L), j(N)\}.$$

Proof. Write $\ell = j(L)$, $m = j(M)$, and $n = j(N)$. First, we will show that $\ell \geq m$. If not, then $\operatorname{Ext}^\ell_R(M,R) = 0$, so $L^\vee = \operatorname{Ext}^\ell_R(L,R)$ is a submodule of $\operatorname{Ext}^{\ell+1}(N,R)$, whence $j(L^\vee) \geq \ell+1$; this contradicts Corollary 1.3, so we conclude that $\ell \geq m$. But $n \geq \min\{\ell+1, m\}$ by Lemma 4.2, so $n \geq m$, whence $\min\{\ell, n\} \geq m$; the reverse inequality is established in Lemma 4.2, so the result follows. □

Proposition 1.5. *A noetherian ring R has the Auslander property if and only if R^{op} does.*

Proof. □

Proposition 1.6. *Let $0 \to R \to E^\bullet$ be a minimal injective resolution of R_R. Then R satisfies the Auslander condition if and only if $\operatorname{fdim} E^n \leq n$ as a right module for all $n \geq 0$.*

Proof. ($\Rightarrow$) We argue by induction on n. If the result is false for $n = 0$, then $\operatorname{Tor}^R_1(E^0, M) \neq 0$ for some $M \in \mathsf{mod}(R)$. By Lemma 7.2.1, $\operatorname{Hom}_R(\operatorname{Ext}^1_R(M,R), E^0) \neq 0$. Let $f : \operatorname{Ext}^1_R(M,R) \to E^0$ be a non-zero map. Since R is essential in E^0, there exists a cyclic submodule $N \subset \operatorname{Ext}^1_R(M,R)$ such that $0 \neq f(N) \subset R$. This contradicts the Auslander condition, so we conclude the result is true for $n = 0$.

Now we show that $\operatorname{fdim} E^n \leq n$. Suppose to the contrary that $M \in \mathsf{mod}(R)$ is such that $0 \neq \operatorname{Tor}^R_{n+1}(E^n, M) \cong \operatorname{Hom}_R(\operatorname{Ext}^{n+1}_R(M,R), E^n)$. Write Ω^{-n} for the co-syzygies of R; thus there are exact sequences $0 \to \Omega^{-n+1} \to E^{n-1} \to \Omega^{-n} \to 0$ for all n. Since Ω^{-n} is an essential submodule of E^n, there is a finitely generated submodule $N \subset \operatorname{Ext}^{n+1}_R(M,R)$ such that $\operatorname{Hom}_R(N, \Omega^{-n}) \neq 0$. By the induction hypothesis $0 = \operatorname{Tor}_{n+1}(E^{n-1}, M) \cong \operatorname{Hom}_R(\operatorname{Ext}^{n+1}_R(M,R), E^{n-1})$, so the injectivity of E^{n-1} implies that $\operatorname{Hom}_R(N, E^{n-1}) = 0$. Hence the last term in the exact sequence

$$0 \to \operatorname{Hom}(N, \Omega^{-n+1}) \to \operatorname{Hom}(N, E^{n-1}) \to \operatorname{Hom}(N, \Omega^{-n}) \to \operatorname{Ext}^1(N, \Omega^{-n+1}) \to 0$$

is non-zero, whence $\operatorname{Ext}^n_R(N,R) \neq 0$. This contradicts the Auslander condition, so we conclude that the result is true.

($\Leftarrow$) Fix $M \in \mathsf{mod}(R)$, integers $i < j$, and a submodule $N \subset \operatorname{Ext}^j_R(M,R)$. Since $\operatorname{fdim} E^i \leq i$, $\operatorname{Tor}^R_j(E^i, M) = 0$, whence $\operatorname{Hom}_R(\operatorname{Ext}^j(M,R), E^i) = 0$ by Lemma 2.1. Therefore $\operatorname{Hom}_R(N, E^i) = 0$ also, so the complex $\operatorname{Hom}_R(N, E^\bullet)$ has no homology in positions $0, \dots, j-1$; thus $\operatorname{Ext}^i_R(N,R) = 0$ for all $i < j$. □

Proposition 1.7. *If A is connected and Auslander-Gorenstein, then A is Gorenstein. Therefore, if $n = \operatorname{injdim} A$, then, for some $\ell \in \mathbb{Z}$,*

$$\underline{\operatorname{Ext}}^i_A(k,A) \cong \begin{cases} 0 & \text{if } i \neq n, \\ k[\ell] & \text{if } i = n. \end{cases}$$

Proof. $\boxed{\text{Paul}}$ Should be a simpler proof using Theorem 15.6.6.

The injective dimension of A is the same on both sides, so there exists $N \in \mathsf{grmod}(A^{\mathrm{op}})$ such that $\operatorname{Ext}^n_A(N,A) \neq 0$; since $\operatorname{Ext}^n_A(N,A) \in \mathsf{grmod}(A)$, k is a quotient of it, whence $j(k) \geq j(\operatorname{Ext}^n_A(N,A)) = n$. Thus $j(k) = n$. It follows that the only non-zero term on the E_2 page of the double-Ext spectral sequence for k is $k^{\vee\vee} = \operatorname{Ext}^n_A(\operatorname{Ext}^n_A(k,A),A)$, and this must be isomorphic to k.

Now consider the exact sequence $0 \to X \to k^\vee \to k_A \to 0$; each term in it has grade at least $j(k^\vee)$, so all have grade n. Hence there is an exact sequence $0 \to$

$\operatorname{Ext}^n(k_A, A) \to k^{\vee\vee} \to \operatorname{Ext}^n_A(X, A) \to 0$. But the middle term is isomorphic to k, and the first term is non-zero, so we conclude that $\operatorname{Ext}^n_A(k_A, A) \cong k$. Interchanging the roles of k and k_A gives the result. □

2. Purity

Definition 2.1. Let M be a non-zero left R-module. Then

- M is n-pure if $j(L) = n$ for all finitely generated submodules $0 \neq L \subset M$;
- $M^\vee := \operatorname{Ext}^{j(M)}_R(M, R)$ – this notation is compatible with the earlier notation $P^\vee = \operatorname{Hom}_R(P, R)$ for a projective module P.

We define $j(0) = \infty$ and $0^\vee = 0$.

Lemma 2.2. *A noetherian module of grade p has a p-pure quotient.*

Proof. Let M be the module in question. Choose $L \subset M$ maximal such that $j(M/L) = p$. Let X/L be a non-zero submodule of M/L, and consider the sequence $0 \to X/L \to M/L \to M/X \to 0$. By choice of L, $j(M/X) > p$. By parts (1) and (2) of Lemma 7.4.2, $j(X/L) = p$. Thus M/L is p-pure. □

However, a module of grade p need not have a p-pure submodule, as the next example shows.

Example 2.3. An essential extension of a finitely generated p-pure module need not be p-pure.

Let $R = U(\mathfrak{sl}_2)$ be the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. Then R is Auslander-regular and Cohen-Macaulay; in particular, $\operatorname{GKdim} M + j(M) = \operatorname{GKdim} R = 3$ for all $M \in \mathsf{mod}(R)$. There is a non-split exact sequence $0 \to L \to M \to N \to 0$ in which L and N are simple modules, $\operatorname{GKdim} L = 0$, and $\operatorname{GKdim} N = 1$; here M is the dual of a Verma module having a finite dimensional simple quotient. Thus M is an essential extension of L, and $3 = j(L) < j(M) = 2$. The point is that $j(L/M) \ngeq j(M) + 2$.

Lemma 2.4. *Let R be Auslander-Gorenstein and $M \in \mathsf{mod}(R)$. The following are equivalent*

1. *M is pure;*
2. $\operatorname{Ext}^p_R(\operatorname{Ext}^p_R(M, R), R) = 0$ *for all* $p \neq j(M)$*;*
3. $j(\operatorname{Ext}^p_R(M, R)) \geq p + 1$ *for all* $p \neq j(M)$.

Proof. (1) ⇒ (2) Write $E^{pp} = \operatorname{Ext}^p_R(\operatorname{Ext}^p_R(M, R), R)$. If $p < j(M)$, then certainly $E^{pp} = 0$. If $p > j(M)$, then $F^p M = 0$ by the purity hypothesis, so $j(E^{pp}) \geq p + 2$ by Theorem 1.2; if $E^{pp} \neq 0$, then $j(\operatorname{Ext}^p_R(M, R)) = p$, so by Corollary 1.3, $j(E^{pp}) = p$. This contradiction shows that $E^{pp} = 0$.

(2) ⇒ (3) By the Auslander condition, $j(\operatorname{Ext}^p_R(M, R)) \geq p$, so the result is immediate.

(3) ⇒ (1) If M is not pure there is an exact sequence $0 \to L \to M \to N \to 0$ with $j(L) = p > j(M)$. By hypothesis, $j(\operatorname{Ext}^p_R(M, R)) > p$ so the first term in the exact sequence

$$\operatorname{Ext}^p(M, R) \to \operatorname{Ext}^p(L, R) \xrightarrow{\varphi} \operatorname{Ext}^{p+1}_R(N, R) \to \cdots$$

is zero. Thus $\ker \varphi$ is a quotient of $\operatorname{Ext}^p_R(M, R)$, so has grade $> p$ also. Since $\operatorname{coker} \varphi$ embeds in $\operatorname{Ext}^{p+1}_R(N, R)$, it has grade $> p$ too. Applying Corollary 1.4 to the exact sequence $0 \to \ker \varphi \to L^\vee = \operatorname{Ext}^p_R(L, R) \to \operatorname{coker} \varphi \to 0$, we conclude that $j(L^\vee) > p$; but this violates Corollary 1.3, so we conclude that M is pure. □

Theorem 2.5. *If R is Auslander-Gorenstein, and $M \in \mathsf{mod}(R)$, then*

1. $M^\vee$ *is pure;*
2. *if* $\mathrm{Ext}^p(\mathrm{Ext}^p(M,R),R)$ *is non-zero it equals* $\mathrm{Ext}^p_R(M,R)^\vee$*;*
3. $\mathrm{Ext}^p(\mathrm{Ext}^p(M,R),R)$ *is pure;*
4. *the quotients $F^pM/F^{p+1}M$ in the canonical filtration are either zero or p-pure.*

Proof. (1) Let $p = j(M)$. By Lemma 2.4, we must show that $j(E^{ii}(M^\vee) = 0$ for $i \neq p$. Certainly this is true if $i < p$, so suppose that $i > p$. Since $E^{ii}(M^\vee) = E^i(E^{ip}M)$, it suffices to show that $j(E^{ip}M) > i$. Since it is an off-diagonal term, $E^{ip}_\infty M = 0$. Hence there is a finite filtration

$$E^{ip}_2 M \supset E^{ip}_3 M \supset \cdots,$$

with embeddings of the successive quotients

$$E^{ip}_r M/E^{ip}_{r+1}M \to E^{i+r,p+r-1}_r M$$

for $r \geq 2$. The right hand side of this has grade $\geq i+r \geq i+2$, hence so does the left hand side; this in turn gives $j(E^{ip}_2 M) \geq i+2$, by exactness of grade. But this is what we needed to prove.

(2) Since $\mathrm{Ext}^i_R(\mathrm{Ext}^p(M,R),R) = 0$ whenever $i < p$, if $\mathrm{Ext}^p_R(\mathrm{Ext}^p_R(M,R),R) \neq 0$, then $j(\mathrm{Ext}^p_R(M,R)) = p$, whence the result.

(3) This follows from (1) and (2).

(4) This is because $F^pM/F^{p+1}M$ is a submodule of $\mathrm{Ext}^p_R(\mathrm{Ext}^p_R(M,R),R)$. □

Corollary 2.6. *Cohen-Macaulay modules over an Auslander-Gorenstein ring are pure.*

Proof. Such a module M is isomorphic to $M^{\vee\vee}$. □

Proposition 2.7. *Let R be Auslander-Gorenstein. If $M \in \mathsf{mod}(R)$ is p-pure and $M \cong M^{\vee\vee}$, then $j(\mathrm{Ext}^q_R(M,R)) \geq q+2$ whenever $q \neq p$.*

Proof. There is nothing to prove if $q < p$. By Lemma 2.4, $j(E^qM) \geq q+1$ so, if the result fails $j(E^qM) = q+1$ for some $q > p$. In that case, $E^{q+1\,q}M \neq 0$ so $j(E^{q+1\,q}M) = q+1$; we will show this does not happen.

Since it is an off-diagonal term $E^{q+1\,q}_\infty M = 0$, so there is a finite filtration

$$E^{q+1\,q}_2 M \supset E^{q+1\,q}_3 M \cdots$$

with factors $E^{q+1\,q}_r M/E^{q+1\,q}_{r+1}M$ embedding in $E^{q+r+1\,q+r-1}_r M$. But this latter module has grade $\geq q+r+1 \geq q+3$, hence so do its submodules. It follows that $j(E^{q+1\,q}_2 M) \geq q+3$ also, whence the result. □

The next result gives an intrinsic description of the canonical filtration.

Corollary 2.8. *Let R be Auslander-Gorenstein. If $M \in \mathsf{mod}(R)$, then F^pM is the largest submodule of M having grade $\geq p$; indeed, it is the sum of all submodules having grade $\geq p$.*

Proof. Let L be a submodule of M with $j(L) \geq p$; we must show that $L \subset F^pM$. If not, then the largest i such that $L \subset F^iM$ is $< p$, whence $j(L) \geq i+1$. By choice of i, $L + F^{i+1}M/F^{i+1}M$ is a non-zero quotient of $L \oplus F^{i+1}M$, so

$$j(L + F^{i+1}M) \geq j(L \oplus F^{i+1}M) = \min\{j(L), j(F^{i+1}M)\} = i+1.$$

But this contradicts the fact that $L+F^{i+1}M/F^{i+1}M$ is a submodule of the i-pure module $F^iM/F^{i+1}M$. Hence $L \subset F^pM$, proving the result. $\square$

By Corollaries 1.4 and 2.8, $\psi(F^pM) \subset F^pN$ for any module homomorphism $\psi : M \to N$.

One might suspect that over an Auslander-Gorenstein ring R, if B is a finitely generated R-R-bimodule, and $M \in \mathsf{mod}(R)$, then $j(B \otimes_R M)$ is bounded below by $j(M)$ (a sort of weak-ideal invariance property); this might be suggested by the fact that if J is a two-sided ideal of R, then $R/J \otimes_R M \cong M/JM$, so $j(R/J \otimes_R M) \geq j(M)$. But this fails over the ring R of upper triangular matrices: if $M = R/I$ is the non-projective simple left R-module, and B is the nilpotent radical of R, then $B \otimes_R M \cong B/BI = B$, so $j(B \otimes_R M) = 0 < j(M)$.

$\boxed{\text{Paul}}$ If R is Auslander-Gorenstein, and $0 \leq n \leq \operatorname{injdim} R$, does there exist $M \in \mathsf{mod}(R)$ such tha $j(M) = n$? I think this is an open question even if $\dim_k R < \infty$.

In general a finitely generated R-R-bimodule can have different grades as a left and right module, but it is an open question whether this can happen for a bimodule over a prime connected Auslander-Gorenstein algebra–it would be great if it could not happen!

EXERCISES

2.1 None yet

3. The functor $M \mapsto M^{\vee\vee}$

Throughout this section R denotes an Auslander-Gorenstein ring.

Definition 3.1. Let R be Auslander-Gorenstein. For each integer $p \geq 0$ define the full subcategory

$$\mathcal{C}_p := \{M \in \mathsf{mod}(R) \mid j(M) \geq p\}.$$

of $\mathsf{mod}(R)$.

By Theorem 1.4, there is a chain of dense subcategories

$$\{0\} = \mathcal{C}_\infty \subset \mathcal{C}_n \subset \cdots \subset \mathcal{C}_1 \subset \mathcal{C}_0 = \mathsf{mod}(R)$$

with abelian quotient categories $\mathcal{D}_p := \mathcal{C}_p/\mathcal{C}_{p+1}$. We will show in Corollary 3.4 that objects in $\mathcal{C}_{p+1}/\mathcal{C}_p$ have finite length, and will subsequently use the dimension function associated to this chain.

Proposition 2.7.6(1) says that a finitely generated module M over a semiprime left noetherian ring is torsion if and only if $M^* = 0$; this may be restated as

$$\mathcal{C}_1 = \{M \mid M \text{ is torsion}\}; \tag{3-1}$$

thus the $\mathcal{C}_p$'s encourage a finer analysis of the structure of torsion modules.

Proposition 3.2. *Let R be Auslander-Gorenstein.*

1. *For each $p \geq 0$, the rule $M \mapsto M^\vee$ extends to a left exact contravariant functor $\mathcal{C}_p(R) \to \mathcal{C}_p(R^{\mathrm{op}})$.*
2. *There is a natural transformation from the identity functor on $\mathcal{C}_p$ to the functor $M \mapsto M^{\vee\vee}$.*
3. *The natural map $M \to M^{\vee\vee}$ given by (2) coincides with the map induced by the spectral sequence in Theorem 1.2(1).*

PROOF. (1) The Auslander condition ensures that the contravariant functor $G = \operatorname{Ext}_R^p(-, R)$ sends $\mathcal{C}_p(R)$ to $\mathcal{C}_p(R^{\mathrm{op}})$. If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathcal{C}_p(R)$, there is an exact sequence

$$\operatorname{Ext}_R^{p-1}(L, R) = 0 \to GN \to GM \to GL \to \operatorname{Ext}_R^{p+1}(N, R),$$

so the restriction of G to $\mathcal{C}_p(R)$ is left exact. Hence on $\mathcal{C}_p$, we define $M^\vee := \operatorname{Ext}_R^p(M, R)$.

(2) (3) Suppose that $j(M) = d$. Then $M = F^d M$, so Theorem 1.2(1) gives a map $M \to M^{\vee\vee}$; we will show that this yields the required natural transformation. Suppose then that $f : M \to N$ is a homomorphism between modules of grade $\geq d$. Then f induces a morphism $P_\bullet(M) \to P_\bullet(N)$ between the projective resolutions, and then induces a morphism between the double complexes $L^{**}(M) \to L^{**}(N)$ constructed in Theorem 5.1. By [**43**, page 332], there is an induced map

$$f_2^{pq} : {}'E_2^{pq}(M) = \operatorname{Ext}_R^p(\operatorname{Ext}_R^q(M, R), R) \longrightarrow {}'E_2^{pq}(N) = \operatorname{Ext}_R^p(\operatorname{Ext}_R^q(N, R), R)$$

and analogous induced maps on the second spectral sequence. But the second spectral sequence associated to L^{pq} collapses with ${}''E^{pq}(M) = \operatorname{Tor}_0(R, M) = M$, so the maps $F^p M \to F^p N$ are the restrictions of $f : M \to N$. Putting all this together, the result follows. $\square$

Warning. Earlier we defined $M^\vee := \operatorname{Ext}_R^{j(M)}(M, R)$, but when we wish to examine a fixed $\mathcal{C}_p$ we will use $M^\vee$ to denote $\operatorname{Ext}_R^p(M, R)$ *even if* $j(M) > p$. This is done so we have a functor defined on all of $\mathcal{C}_p$.

COROLLARY 3.3. *The functor* $^\vee$ *is a duality from* $\mathcal{D}_p \to \mathcal{D}_p(R^{\mathrm{op}})$, *and* $^\vee \circ {}^\vee$ *is naturally equivalent to the identity.*

PROOF. The functor is faithful because $M^\vee = 0$ in $\mathcal{D}_p$ if and only if $j(M^\vee) > p$; but $j(M^\vee) = j(M)$, so $M^\vee = 0$ if and only if $M = 0$. If $j(M) = p$, then the first and last terms of the exact sequence $0 \to F^{p+1}M \to M \to M^{\vee\vee} \to Q(p) \to 0$ belong to $\mathcal{C}_{p+1}$, so M is isomorphic to $M^{\vee\vee}$ in $\mathcal{C}_p/\mathcal{C}_{p+1}$. $\square$

COROLLARY 3.4. *Objects in* $\mathcal{C}_p/\mathcal{C}_{p+1}$ *have finite length.*

PROOF. Since R is noetherian, objects in $\mathcal{D}_p = \mathcal{C}_p/\mathcal{C}_{p+1}$ have the ascending chain condition. On the other hand, by the duality in Corollary 3.3, objects in $\mathcal{D}_p$ also have the descending chain property. $\square$

Definition 3.5. Let R be Auslander-Gorenstein, and let $M \in \mathcal{C}_p$. Then

- the p-length of M is $\varepsilon_p(M) :=$ the length of M in $\mathcal{C}_p/\mathcal{C}_{p+1}$;
- M is p-simple if $\varepsilon_p(M) = 1$;
- a p-simple series on M is a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each M_{i+1}/M_i is p-simple.

LEMMA 3.6. *Let* $M \in \mathcal{C}_p$.

1. *The length of a* p*-simple series for* M *is* $\varepsilon_p(M)$.
2. *If* $0 \to L \to M \to N \to 0$ *is exact, then* $\varepsilon_p(M) = \varepsilon_p(L) + \varepsilon_p(N)$.
3. *If* $j(M) = p$, *then* M *and* $M^{\vee\vee}$ *are isomorphic in* $\mathsf{mod}(R)/\mathcal{C}_{p+2}$.
4. $\varepsilon_p(M) = \varepsilon_p(M^\vee)$.

PROOF. (1) The image in $\mathcal{C}_p/\mathcal{C}_{p+1}$ of a p-simple series for a module is a composition series for the image of the module, so the result follows from standard arguments about composition series in an abelian category.

(2) The quotient functor $\mathsf{mod}(R) \to \mathsf{mod}(R)/\mathcal{C}_{p+1}$ is exact, so sends the exact sequence to an exact sequence of finite length objects in $\mathsf{mod}(R)/\mathcal{C}_{p+1}$; the result follows from the usual properties of length.

(3) This is proved in Corollary 3.3.

(4) Since $^\vee$ is a faithful exact functor on $\mathcal{D}_p$, $\varepsilon_p(M^\vee) \geq \varepsilon_p(M)$, whence

$$\varepsilon_p(M^{\vee\vee}) \geq \varepsilon_p(M^\vee) \geq \varepsilon_p(M).$$

But M and $M^{\vee\vee}$ are isomorphic in $\mathcal{D}_p$, so have the same p-length; hence these inequalities are equalities. □

The next result and its corollary were proved by J-E. Roos [**34**].

PROPOSITION 3.7. *If M is a finitely generated R-module, then*

$$\operatorname{Kdim} M + j(M) \leq \operatorname{injdim} R.$$

PROOF. Consider the sequence of dense subcategories

$$\{0\} = \mathcal{C}_\infty \subset \mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \cdots \subset \mathcal{C}_0 = \mathsf{mod}(R),$$

where $n = \operatorname{injdim} R$. Since objects in $\mathcal{C}_p/\mathcal{C}_{p+1}$ have finite length, a downwards induction argument on p shows that $\mathcal{C}_p \subset \mathcal{K}_{n-p}$ for all $0 \leq p \leq n$. In other words, if $j(M) \geq p$, then $\operatorname{Kdim} M \leq n - p$, whence the result. □

COROLLARY 3.8. *If R is Auslander-Gorenstein, then* $\operatorname{Kdim} R \leq \operatorname{injdim} R$.

The inequalities in Proposition 3.7 and Corollary 3.8 cannot be replaced by equalities: for example, the ring of 2×2 upper triangular matrices has Krull dimension zero, injective dimension 1, and has a simple module of grade 0. Even if R is a domain, the inequality may be strict, as one sees by considering $U(\mathfrak{sl}(2))/(\Omega)$. By considering upper triangular matrices, one sees that one cannot improve Proposition 3.7 by replacing $\operatorname{injdim} R$ by $\operatorname{Kdim} R$.

$\boxed{\text{Paul}}$ There are two possible improvements to Proposition 3.7. First, can one show that $\operatorname{GKdim} M + j(M) \leq \operatorname{GKdim} R$? Second, is $\operatorname{Kdim} M + \operatorname{pdim} M \leq \operatorname{injdim} R$ when $\operatorname{pdim} M < \infty$?

PROPOSITION 3.9 (Gabber's Lemma). *Let R be Auslander-Gorenstein. If E is an essential extension of a p-pure module $M \in \mathsf{mod}(R)$, then*

1. *there is a unique maximal member of the set*

$$\mathcal{S} := \{M' \in \mathsf{mod}(R) \mid M \subset M' \subset E \text{ and } j(M'/M) \geq p+2\};$$

2. *if E is an injective envelope of M, then the maximal member of $\mathcal{S}$ is isomorphic to $M^{\vee\vee} = \operatorname{Ext}^p_R(\operatorname{Ext}^p_R(M,R),R)$.*

PROOF. First, each $M' \in \mathcal{S}$ is p-pure: if $0 \neq N \subset M'$, then $N \cap M \neq 0$, so

$$j(M') \leq j(N) \leq j(N \cap M) = p,$$

and $j(M') = \min\{j(M), j(M'/M)\} = p$, whence $j(N) = p$.

If $M' \in \mathcal{S}$, let $f : M \to M'$ be the inclusion, and consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & M' \arrow[d] \\ M^{\vee\vee} \arrow[r, "f^{\vee\vee}"] & (M')^{\vee\vee} \end{tikzcd}$$

arising from Proposition 3.2(2). Since M and M' are pure, it follows from Theorem 1.2(1) that the vertical maps are essential embeddings. The functor $M \mapsto M^{\vee\vee}$ is left exact, so $f^{\vee\vee}$ is injective. Furthermore, since $j(\operatorname{coker} f) \geq p+2$, $f^{\vee\vee}$ is surjective, and hence an isomorphism.

To prove the result, it suffices to show that a chain $M \subset M_1 \subset M_2 \subset \cdots$ in $\mathcal{S}$ is eventually constant. By the previous paragraph, there is a commutative diagram

$$\begin{array}{ccccccc} M & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ M^{\vee\vee} & \longrightarrow & M_1^{\vee\vee} & \longrightarrow & M_2^{\vee\vee} & \longrightarrow & \cdots \end{array}$$

where the bottom row consists of isomorphisms. Hence the chain may be considered as a chain of submodules of $M^{\vee\vee}$, so stabilizes because $M^{\vee\vee}$ is noetherian. $\square$

COROLLARY 3.10. *Let R be Auslander-Gorenstein, and $M \in \mathsf{mod}(R)$. Then*
1. $M^{\vee\vee\vee} \cong M^{\vee\vee\vee\vee\vee\vee}$;
2. *if M is pure, then* $M^{\vee\vee} \cong M^{\vee\vee\vee\vee}$.

PROOF. (2) Suppose M is p-pure; then $M^{\vee\vee}$ is also p-pure. By Gabber's Lemma, $M \subset M^{\vee\vee} \subset M^{\vee\vee\vee\vee} \subset E(M)$, and $j(M^{\vee\vee\vee\vee}/M^{\vee\vee}) \geq p+2$. Also $j(M^{\vee\vee}/M) \geq p+2$, whence $j(M^{\vee\vee\vee\vee}/M) \geq p+2$ so, by the characterization of $M^{\vee\vee}$ in Gabber's Lemma, $M^{\vee\vee\vee\vee} \subset M^{\vee\vee}$ thus giving the required equality.

(1) Apply (2) to $M^{\vee}$, which is pure. $\square$

The necessity of the Auslander-Gorenstein hypothesis is apparent from Exercise 1 below and Exercise 2.7.1.

PROPOSITION 3.11. *Let R be Auslander-Gorenstein with* $\operatorname{injdim} R = n$. *If $M \in \mathsf{mod}R$ and $j(M) \geq n-2$, then $M^{\vee}$ and $M^{\vee\vee}$ are both Cohen-Macaulay.*

PROOF. The result for $M^{\vee\vee}$ is proved by applying the first result to the right module $M^{\vee}$; this can be done because $j(M^{\vee}) = j(M) \geq n-2$.

If $j(M) = n$, then $M^{\vee} = \operatorname{Ext}^n_R(M,R)$, whence $\operatorname{Ext}^i_R(M^{\vee},R)$ is zero for $i < n$ by the Auslander condition, and is zero for $i > n$ because $\operatorname{injdim} R = n$. Thus $M^{\vee}$ is Cohen-Macaulay.

Suppose that $j(M) = n-1$. Then $\operatorname{Ext}^i_R(M^{\vee},R) = \operatorname{Ext}^i_R(\operatorname{Ext}^{n-1}_R(M,R),R)$ is zero for $i > n$ because $\operatorname{injdim} R = n$, and is zero for $i < n-1$ by the Auslander condition, so it remains to prove it is zero for $i = n$. The double-Ext spectral sequence for M has only two potentially non-zero rows, namely

$$\begin{array}{cccc} E^{0,-n+1}(M) & E^{1,-n+1}(M) & \cdots & E^{n,-n+1}(M) \\ E^{0,-n}(M) & E^{1,-n}(M) & \cdots & E^{n,-n}(M) \end{array}$$

where $E^{p,-q}(M)$ denotes $\operatorname{Ext}^p_R(\operatorname{Ext}^q_R(M,R),R)$. Since the differential has bidegree $(2,-1)$, $E^{n,-n+1}$ survives to the E_∞ page, so is zero.

Suppose that $j(M) = n-2$. Then $\operatorname{Ext}^i_R(M^{\vee},R) = \operatorname{Ext}^i_R(\operatorname{Ext}^{n-2}_R(M,R),R)$ is zero for $i > n$ and for $i < n-2$, so we only need prove this is zero for $i = n-1, n$. The double-Ext spectral sequence for M has only three potentially non-zero rows

$$\begin{array}{cccc} E^{0,-n+2}(M) & E^{1,-n+2}(M) & \cdots & E^{n,-n+2}(M) \\ E^{0,-n+1}(M) & E^{1,-n+1}(M) & \cdots & E^{n,-n+1}(M) \\ E^{0,-n}(M) & E^{1,-n}(M) & \cdots & E^{n,-n}(M) \end{array}$$

Since the differential has bidegree $(2,-1)$ both $E^{n,-n+2}$ and $E^{n-1,-n+2}$ survive to the E_∞ page, so are zero. □

A Cohen-Macaulay module over an Auslander-Gorenstein ring is pure: if M is Cohen-Macaulay of grade p, then $M \cong F^pM/F^{p+1}M$ by Theorem 1.2, and this is pure by Theorem 2.5. The converse is not true: a pure module need not be Cohen-Macaulay— $\boxed{\text{Paul}}$ give a commutative connected graded example where

k itself is not Cohen-Macaulay—maybe $A = k[x^4, x^3y, xy^3, y^4]$.

THEOREM 3.12. [**98**] *If R is Auslander-regular and $K_0(R) \cong \mathbb{Z}$, then R is a domain.*

PROOF. If not, there is an exact sequence $0 \to I \to R \to Rx \to 0$ for some $0 \neq x \in R$, and a non-zero left ideal I. Since R/I embeds in R, $j(R/I) = 0$, whence $\varepsilon_0(R/I) \geq 1$. However, R/I has a finite resolution by finitely generated free modules, so $\varepsilon_0(R/I)$ is a non-zero integer multiple of $\varepsilon_0(R)$. This is impossible since $\varepsilon_0(R/I) = \varepsilon_0(R) - \varepsilon_0(Rx) < \varepsilon_0(R)$. □

The finite global dimension hypothesis is essential in Theorem 3.12. For example, $R = k[x]/(x^2)$ is Auslander-Gorenstein with $\operatorname{injdim} R = 0$, and $K_0(R) \cong \mathbb{Z}$, but is not a domain. The noetherian hypothesis is also essential: the ring $R = k\langle x, y\rangle/(xy)$ has global dimension 2, and $K_0(R) \cong \mathbb{Z}$, but is not a domain.

$\boxed{\text{Paul}}$ show it satisfies the Auslander condition.

PROPOSITION 3.13. *The minimal injective resolution of a semiprime Auslander-Gorenstein ring R begins $0 \to R \to E^0 \to E^1$ with each E^p being p-pure.*

PROOF. Recall that $E^0 = \operatorname{Fract} R$. If $M = Rq_1 + \cdots + Rq_n \subset E^0$, then there exists a regular element $c \in R$ such that each $q_ic \in R$. Hence the map $x \mapsto xc$ embeds M in R; that is, $j(M) = 0$, so E^0 is 0-pure.

By Lemma 2.3.14, an essential extension of a torsion R-module is torsion, so E^1, which is the injective envelope of $(\operatorname{Fract} R)/R$, is torsion. Hence, if M is a non-zero finitely generated submodule of E^1, $j(M) \geq 1$ by (3-1). Let $M' = M \cap (E^0/R)$, and write $M' = X/R$ for some finitely generated $X \subset E^0$. If $j(M) \geq 2$, then $j(M') \geq 2$ so, by Gabber's Lemma, $X \subset R^{\vee\vee} = R$, whence $M' = 0$, contradicting the fact that E^0/R is essential in E^1. Thus $j(M) \leq 1$, as required. □

LEMMA 3.14. *If $L \subset M$ and $L \cong M$, then $j(M) < j(M/L)$.*

PROOF. If $j(M) = j(M/L) = p$, then $\varepsilon_p(M) = \varepsilon_p(M/L) + \varepsilon_p(L)$ implies that $\varepsilon_p(M) > \varepsilon_p(L)$, contradicting $M \cong L$. □

Actually, the Lemma works equally well if L is isomorphic to the image of M under some auto-equivalence of $\mathsf{mod}(R)$; for example, if $L = xM$ where x is a normal regular element of R acting injectively on M.

EXERCISES

3.1 Let $R = k[x,y]/(x,y)^2$ (see Exercise 2.7.1). Show that
(a) the minimal injective resolution of k is of the form
$$0 \to k \to E \to E^2 \to E^4 \to \cdots$$
where $E = R^*$ is the injective envelope of k;
(b) there is an exact sequence $0 \to R \to E \oplus E \to k^3 \to 0$;
(c) $\operatorname{injdim} R = \infty$;
(d) R is not Auslander-Gorenstein.

3.2 $\boxed{\text{Paul}}$ What about progenerators in $\mathcal{C}_p/\mathcal{C}_{p+1}$?

3.3 Show that $M \in \mathcal{C}_p$ is p-simple if and only if for every $L \subset M$, either $j(L) > p$ or $j(M/L) > p$.

3.4 Show that $\varepsilon_p(M)$ is the largest n for which there is a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules with $j(M_i/M_{i-1}) = p$ for each $i \geq 1$.

3.5 Let M be a finitely generated module over an Auslander-Gorenstein ring R. If $\psi : M \to M$ is an injective R-module map, show that $j(M/\psi M) \geq j(M) + 1$.

4. Examples of Auslander-Gorenstein rings

For a commutative noetherian ring R of finite self-injective dimension, the Auslander condition automatically holds [**23**]. The next example shows this is no longer the case for non-commutative rings.

EXAMPLE 4.1. Let k be a field, fix $n > 0$ and let $V \cong k^n$. Define
$$R = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}.$$
It is easy to check that $\operatorname{gldim} A = 1$. We will show that R is Auslander-Gorenstein if and only if $n = 1$.

$\boxed{\text{DETAILS}}$

In part, the divergence from the commutative behavior in the previous example is due to the fact that R has simple modules of different (finite) projective dimension. This will be discussed in the section on 'smoothness'.

THEOREM 4.2. *A localization of an Auslander-Gorenstein ring is Auslander-Gorenstein.*

PROOF. Let S be a localization of an Auslander-Gorenstein ring R. First, S is noetherian because R is. Second, if $M \in \mathsf{mod}(S)$, then
$$\operatorname{Ext}^i_S(M,S) = \operatorname{Ext}^i_S(S \otimes_R M, S \otimes_R R) \cong \operatorname{Ext}^i_R(M,R) \otimes_R S,$$
whence $\operatorname{injdim} S \leq \operatorname{injdim} R < \infty$.

The key point is to show that S has the Auslander property. Let $M \in \mathsf{mod}(S)$ and N an S-submodule of $\operatorname{Ext}^j_S(M,S)$. For some $M' \in \mathsf{mod}(R)$, $M \cong S \otimes_R M'$, whence $\operatorname{Ext}^j_S(M,S) \cong \operatorname{Ext}^j_R(M',R) \otimes_R S$. It follows that there is an R-submodule $N' \subset \operatorname{Ext}^j_R(M',R)$ such that $N = \bar{N}S$ where $\bar{N}$ is the image of N' under the map $\operatorname{Ext}^j_R(M',R) \to \operatorname{Ext}^j_R(M',R) \otimes_R S$. Since $N \cong \bar{N} \otimes_R S$, we have $\operatorname{Ext}^i_S(N,S) \cong S \otimes_R \operatorname{Ext}^i_R(\bar{N},R)$, whence $j(N_S) \geq j(\bar{N}) \geq j(N') \geq j$ as required. □

THEOREM 4.3. *If R is an Auslander-Gorenstein ring with an increasing $\mathbb{N}$-filtration, then the Rees ring of R is also Auslander-Gorenstein.*

Proof. See Li-Huishi or Ekstrom. □

Theorem 4.4. *If R is Auslander-Gorenstein so is an Ore extension $R[x;\sigma,\delta]$.*

Proof. See Li-Huishi or Ekstrom. □

Theorem 4.5. *If R is a filtered k-algebra such that the associated graded ring $\operatorname{gr} R$ is Auslander-Gorenstein, then R is Auslander-Gorenstein.*

Proof. ??? □

Corollary 4.6. *The enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra is Auslander-Gorenstein.*

5. Low dimensional rings

Proposition 5.1. *Let A be a finite dimensional connected k-algebra. If* $\operatorname{injdim} A < \infty$*, then* $\operatorname{injdim} A = 0$.

Proof. Since $\dim_k A < \infty$, $\underline{\operatorname{Hom}}_A(k, A) \neq 0$, whence $j(k) = 0$.

Let $d = \operatorname{injdim} A$. Then $\underline{\operatorname{Ext}}^d_A(M, A) \neq 0$ for some cyclic graded A-module M. But k embeds in M, so $\underline{\operatorname{Ext}}^d_A(k, A) \neq 0$.

By induction on length, $j(M) = 0$ for all non-zero finite dimensional modules M; thus, $M^\vee = \underline{\operatorname{Hom}}_A(M, A)$. Given an exact sequence $0 \to L \to M \to N \to 0$, we get an exact sequence $0 \to N^\vee \to M^\vee \to L^\vee$; however, by Lemma 3.6, $\dim_k M = \dim_k M^\vee$, whence the map $M^\vee \to L^\vee$ must be surjective. Thus, $\underline{\operatorname{Hom}}_A(-, A)$ is exact, showing that A is an injective object in $\mathsf{GrMod}(A)$, and hence an injective A-module by [**97**, Lemma 3.3]. □

6. Cohen-Macaulay Auslander-Gorenstein rings

Suppose that R is Auslander-Gorenstein and Cohen-Macaulay. If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathsf{mod}(R)$, then $j(M) = \max\{j(L), j(N)\}$, so $\operatorname{GKdim} M = \min\{\operatorname{GKdim} L, \operatorname{GKdim} N\}$. Furthermore, $\operatorname{GKdim} R$ is an integer, so $\operatorname{GKdim} M$ is integer valued whenever $M \in \mathsf{mod}(R)$. Because a finitely generated R-R-bimodule has the same left and right GK-dimension, it has the same grade as a right and as a left module.

The subcategories $\mathcal{C}_p = \{M \in \mathsf{mod}(R) \mid j(M) \geq p\}$ are dense in $\mathsf{mod}(R)$, so there are quotient categories and exact functors $\mathsf{mod}(R) \to \mathsf{mod}(R)/\mathcal{C}_p$. If M and N are R-bimodules, finitely generated on each side, then the spectral sequence $\operatorname{Tor}^R_{-p}(\operatorname{Ext}^q_R(M, R), N) \Rightarrow \operatorname{Ext}^{q+p}_R(M, N)$ established in Theorem 8.6.1 is a spectral sequence of R-R-bimodules. Because the functors to the quotient categories are exact, this spectral sequence is sent to a spectral sequence in each $\mathsf{mod}(R)/\mathcal{C}_p$. We will use this idea several times in the proof of Theorem 6.2; what tends to happen is that for a particular p, there is a single row of the E_2-page in which terms of grade $< p$ occur, so the induced spectral sequence in $\mathsf{mod}(R)/\mathcal{C}_p$ has only a single non-zero row on the E_2-page; hence one obtains information about the limit terms in the original spectral sequence.

When viewing a bimodule M as a right module we will use a subscript R^{op} for emphasis; for example we will write $\operatorname{Ext}^*_{R^{\mathrm{op}}}(M, -)$.

LEMMA 6.1. *Let R be Auslander-regular and Cohen-Macaulay. If M is a finitely generated R-R-bimodule which is p-pure as a left module, then M is also p-pure as a right module; moreover,*

$$\operatorname{Ext}_R^p(\operatorname{Ext}_R^p({}_RM, {}_RR), R_R) \cong \operatorname{Ext}_R^p(\operatorname{Ext}_R^p(M_R, R_R), {}_RR)$$

as R-R-bimodules, via an isomorphism which restricts to the identity on the canonical copy of M inside each of these.

PROOF. To keep the notation simple, we will write

$$U = \operatorname{Ext}_R^p(\operatorname{Ext}_R^p(M_R, R_R), {}_RR) \qquad \text{and} \qquad V = \operatorname{Ext}_R^p(\operatorname{Ext}_R^p({}_RM, {}_RR), R_R).$$

Let $\operatorname{GKdim} R = n$ and $\operatorname{GKdim} M = d$.

First, we prove that M is p-pure as a *right* module. By Lemma 5.1.2, a finitely generated R-R-bimodule has the same right and left GK-dimension, and hence the same right and left grade by the Cohen-Macaulay hypothesis. Let $N \subset M$ be a nonzero right submodule. As a right module RN is a homomorphic image of a finite direct sum of copies of N, so $\operatorname{GKdim} RN = \operatorname{GKdim} N$. Thus $j(N) = j(RN) = j(M)$, where the last equality follows from the purity of M as a left module.

By Gabber's Lemma, V as a left module, is the largest finitely generated essential extension of ${}_RM$ satsifying $j(V/M) \geq p+2$. Because R is Cohen-Macaulay this can be rephrased in terms of GK-dimension. Now V also has a *right* module structure coming from the right action of R on M, but it is not clear that it is finitely generated as a right module. Consider V as the union, hence the direct limit, of a chain of finitely generated right submodules of V containing M; say

$$M = V_0 \subset V_1 \subset \cdots \subset \bigcup V_i = V.$$

Using GK-dimension, it follows that $j(V_i/M) \geq p+2$, so the inclusion $M \to V_i$ extends to an isomorphism $U \to V_i^{\vee\vee}$. This gives right module maps $V_i \to U$ which extend to $V \to U$ by the universal property of the direct limit.

$\square$

In particular, if I is a two-sided ideal in a Cohen-Macaulay Auslander-Gorenstein ring, then I is reflexive as a left module if and only if it is reflexive as a right module, and there is no ambiguity when we speak of I^{**}.

THEOREM 6.2. [**171**] *Let R be Auslander-regular and Cohen-Macaulay. If $K_0(R) \cong \mathbb{Z}$, then R is a maximal order in* $\operatorname{Fract}(R)$.

PROOF. By Theorem 9.3.12, R is a domain. Let $Q = \operatorname{Fract} R$. We will write ε for the function ε_1 measuring the 1-length of modules having grade ≥ 1 (see Definition 9.3.5).

Suppose the result is false. By Proposition 2.8.9, after replacing R by R^{op} if necessary, there is a prime ideal I, reflexive as a left R-module, such that $S := \{q \in Q \mid Iq \subset I\}$ properly contains R. Thinking of I as a left R-module, we have $S \cong \operatorname{Hom}_R({}_RI, I)$ as right R-modules (whence S_R is finitely generated); we will write $I^* = \operatorname{Hom}_R({}_RI, R)$.

<u>Step 1.</u> We show that R/I is 1-pure. Since R/I is torsion, $j(R/I) \neq 0$. Since S/R is a left R/I-module, if $j(R/I) \geq 2$, then $j(S/R) \geq 2$, whence $S = R$ by Proposition 9.3.9. The purity of R/I follows from the fact that it is a prime noetherian ring.

Step 2. We show that $\varepsilon(I/I^2) = \varepsilon(R/I)$. Since $I = I^{**}$, by Proposition 9.2.7, $j(\operatorname{Ext}^q_{R^{\mathrm{op}}}(I^*, R)) \geq 2$ for $q \neq 0$. Hence in $\mathsf{mod}(R)/\mathcal{C}_2$, there is at most one non-zero row on the E_2-page of the spectral sequence induced by

$$\operatorname{Tor}^R_{-p}(\operatorname{Ext}^q_{R^{\mathrm{op}}}(I^*, R), R/I)) \Rightarrow \operatorname{Ext}^{q-p}_R(I^*, R/I),$$

namely the $q = 0$ row. Hence, for $p < 0$, in $\mathsf{mod}(R)/\mathcal{C}_2$, $\operatorname{Tor}^R_{-p}(I, R/I) \cong \operatorname{Ext}^p_R(I^*, R) \cong 0$. Back in $\mathsf{mod}(R)$, this means that $j(\operatorname{Tor}^R_p(I, R/I)) \geq 2$ if $p \neq 0$; equivalently $\varepsilon(\operatorname{Tor}^R_p(I, R/I)) = 0$ when $p \neq 0$.

Let $P_\bullet \to I$ be a finite resolution of I as a right R-module by finitely generated free modules. Then

$$\begin{aligned}
\varepsilon(I/I^2) &= \varepsilon(\operatorname{Tor}^R_0(I, R/I)) \\
&= \sum_i (-1)^i \varepsilon(\operatorname{Tor}^R_i(I, R/I)) \\
&= \sum_i (-1)^i \varepsilon(P_i \otimes_R R/I) \\
&= \varepsilon(R/I) \sum_i (-1)^i \operatorname{rank}(P_i) \\
&= \varepsilon(R/I).
\end{aligned}$$

Step 3. We show that $j(S/SI + R) = 1$. Since R is a domain and I is non-zero, $\operatorname{Hom}_R(R/I, R) = 0$, whence $S/R \cong \operatorname{Ext}^1_R(R/I, R)$ as right R-modules; we are viewing I and R/I as left modules. By Lemma 9.2.4, $j(\operatorname{Ext}^q_R(R/I, R)) \geq 2$ whenever $q \neq 1$ because R/I is 1-pure. Therefore, in $\mathsf{mod}(R)/\mathcal{C}_2$, only the $q = 1$ row of the E_2 page of the spectral sequence $\operatorname{Tor}^R_{-p}(\operatorname{Ext}^q_R(R/I, R), R/I) \Rightarrow \operatorname{Ext}^{q-p}_R(R/I, R/I)$ is non-zero. Taking $p = 0$ and $q = 1$, this gives $S/R \otimes R/I \cong \operatorname{Ext}^1_R(R/I, R/I)$ in $\mathsf{mod}(R)/\mathcal{C}_2$. However, $\operatorname{Ext}^1_R(R/I, R/I) \cong \operatorname{Hom}_{R/I}(I/I^2, R/I)$; by Step 2, $j(I/I^2) = j(R/I)$, so I/I^2 is a torsion-free module over the prime noetherian ring R/I, whence $\operatorname{Hom}_{R/I}(I/I^2, R/I)$ is a torsion-free R/I-module, so has the same grade as R/I. It follows that $j(\operatorname{Ext}^1_R(R/I, R/I)) = 1$. But $S/R \otimes R/I \cong S/SI + R$, so $j(S/SI + R) = 1$.

Step 4. Now we show that $SI \subset R$. Since $I(SI + R) \subset I$, $(SI + R)^*$ contains I. If $I = (SI + R)^*$, then $S \subset I^* = (SI + R)^{**}$, so Gabber's Lemma gives $j(S/SI + R) \geq 2$; this contradicts Step 3, so we must have $I \neq (SI + R)^*$; thus $R/(SI + R)^*$ is a torsion R/I-module, so has grade $\geq j(R/I) + 1 = 2$; hence, by Gabber's Lemma, $R \subset (SI + R)^{***} = (SI + R)^*$, whence $SI \subset R$.

Step 5. We show that $S^{**} = S$. To do this we need to show that $IS^{**} \subset I$. Now

$$IS^{**}/I = IS^{**}/IS \cong I \otimes_R (S^{**}/S),$$

which is a homomorphic image, as a right module to a direct sum of copies of S^{**}/S. But $j(S^{**}/S) \geq 2$ by Gabber's Lemma. Hence applying Gabber's Lemma to the right module IS^{**}/I, we have $IS^{**} \subset I^{**} = I$, whence $S^{**} \subset S$. Thus $S = S^{**}$.

Step 6. We show that $\varepsilon(S/I) = \varepsilon(R/I)$. Because $S = S^{**}$, $j(\operatorname{Ext}^q_R(S^*, R)) \geq 2$ whenever $q \neq 0$, from which it follows that the E_2-page of the image in $\mathsf{mod}(R)/\mathcal{C}_2$ of the spectral sequence for $\operatorname{Tor}^R_{-p}(\operatorname{Ext}^q_R(S^*, R), R/I)$ has only one non-zero row, and from this we have $\operatorname{Tor}^R_{-p}(S, R/I) \cong \operatorname{Ext}^p_R(S^*, R/I)$ in $\mathsf{mod}(R)/\mathcal{C}_2$. In particular, back in $\mathsf{mod}(R)$, $j(\mathrm{Tor}_p^R(S, R/I)) \geq 2$ whenever $p \neq 0$. An argument like that in Step 2 will therefore show that $\varepsilon(S/I) = \varepsilon(R/I)$.

Step 7. Because $\varepsilon(S/I) = \varepsilon(R/I)$, it follows that $\varepsilon(S/R) = 0$. In other words, $j(S/R) \geq 2$ whence, by Gabber's Lemma, $S \subset R^{**} = R$, by Gabber's Lemma. This final contradiction completes the proof, showing that R is a maximal order. $\square$

LEMMA 6.3. *Suppose that R is Auslander-Gorenstein, Cohen-Macaulay, and* $\mathrm{GKdim}\, R = n$. *Then the n^{th} term in the minimal injective resolution of R is $\oplus E(S)$, the direct sum of the injective envelopes of the simple R-modules of GK-dimension zero.*

PROOF. Let E^n be the n^{th} term in the minimal injective resolution of R. If S is simple and $\mathrm{GKdim}\, S = 0$, then $j(S) = n$ by the Cohen-Macaulay property, whence $\mathrm{Ext}_R^n(S, A) \neq 0$; in particular, E^n contains a copy of S, and hence a copy of $E(S)$. Let $E' \subset E^n$ be the sum of all these copies of $E(S)$. We must show that $E' = E^n$. Let $0 \neq M \subset E^n$ be a submodule. Since $\mathrm{Ext}^n(M, R) \neq 0$, so is $L := \mathrm{Ext}_R^n(\mathrm{Ext}_R^n(M, R), R) \neq 0$. However, $j(N) \leq n$ for all N, so it follows from Theorem 1.2 that L is a submodule of M. But $j(L) = n$, so $\mathrm{GKdim}\, L = 0$, whence M contains a simple submodule of GK-dimension zero. Thus $M \cap E' \neq 0$, from which it follows that $E' = E^n$. $\square$

Warning. The term E^n in the lemma might be zero: for example, it is possible that $\mathrm{injdim}\, R < \mathrm{GKdim}\, R$; this happens when R is a Weyl algebra.

7. PI rings

Work of Stafford and Zhang on homological properties of PI rings of finite global dimension.

8. Graded Auslander-Gorenstein rings

CHAPTER 10

Rings satisfying a polynomial identity

Throughout the chapter we will write $X = \{x_1, x_2, \dots\}$.

1. Identities of rings and T-ideals

Definition 1.1. Let R be a k-algebra and $0 \neq f(x_1, \dots, x_n) \in k\langle X\rangle$. We say that f is a non-trivial identity of R and that R satisfies a polynomial identity if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in R$. If R has a non-trivial identity we call R a polynomial identity ring.

EXAMPLE 1.2. Every commutative ring is a polynomial identity ring, satisfying the identity $xy - yx = 0$.

EXAMPLE 1.3. The Amitsur-Levitzki Theorem (1.2.8) shows that $M_n(k)$ is a polynomial identity ring. More generally and subring of a matrix ring over a commutative ring is a polynomial identity ring (in effect, the proof of the Amitsur-Levitzki Theorem proves this stronger result).

Thus, in some sense rings which satisfy a polynomial identity are not too far from commutative rings. The results in this chapter provide more specific evidence of this fact.

Definition 1.4. An ideal J in $k\langle X\rangle$ is a T-ideal if $\varphi(J) \subset J$ for every k-algebra endomorphism φ of $k\langle X\rangle$.

PROPOSITION 1.5. 1. *The polynomial identities of a k-algebra form a T-ideal.*
2. *Every T-ideal is the full set of identities of some k-algebra.*

PROOF. (a) Let J be the ideal in $k\langle X\rangle$ of identities of R, and let $f(x_1, \dots, x_n) \in J$. If $\varphi : k\langle X\rangle \to k\langle X\rangle$ write $y_i = \varphi(x_i)$. If $a_1, \dots, a_n \in R$ write $b_i = y_i(a_1, \dots, a_n)$. Then

$$\varphi(f)(a_1, \dots, a_n) = f(y_1, \dots, y_n)(a_1, \dots, a_n) = f(b_1, \dots, b_n) = 0.$$

(b) If J is a T-ideal, set $R = k\langle X\rangle/J$. Then every element of J is an identity of R, and conversely if $f(x_1, \dots, x_n) \in k\langle X\rangle$ is an identity of R, then evaluation at of f at the images of x_i in R shows that $f \in J$. More precisely, if y_i is the image of x_i in R then $f(y_1, \dots, y_n) = 0$ implies that $f \in J$. $\square$

NOTATION . We write $\mathbb{T}(R)$ for the T-ideal of identities of the ring R.

Definition 1.6. We say that $f(x_1, \dots, x_n) \in k\langle x_1, \dots, x_n\rangle$ is multilinear if each x_i appears with degree ≤ 1 in each word which appears in f with non-zero coefficient.

Multilinear polynomials are useful because if $f(x_1, \dots, x_n)$ is multilinear, and vanishes whenever $x_1, \dots, x_n \in \{e-1, \dots, e_n\} \subset R$, then f vanishes whenever $x_1, \dots, x_n \in ke_1 + \dots ke_n$. Thus to check whether R satisfies a given multilinear polynomial one may work with a convenient basis of R.

PROPOSITION 1.7.

1. *Every T-ideal is graded. Oh, not true in char p e.g $R = \mathbb{F}_2$ satisfies $x^2 - x$!!!!!!*
2. *If a given T-ideal contains an element of degree r it contains a homogeneous multilinear element of degree r.*

PROOF. (b) By (a) the given T-ideal, J say, contains a homogeneous element of degree r, $f = f(x_1, \dots, x_n)$ say. Suppose that x_1 occurs in f with degree s, and that all other x_i occur in f with degree $\leq s$. If $s = 1$ we are done. Otherwise consider

$$g(x_1, \dots, x_n, x_{n+1}) =$$
$$f(x_1 + x_{n+1}, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n) - f(x_{n+1}, x_2, \dots, x_n).$$

Then x_1 and x_{n+1} both occur in g with degree $< s$, and all other x_i occur in g with degree $\leq s$. Proceeding inductively, adjoining new variables as necessary, one eventually obtains a polynomial in which all the x_i have degree ≤ 1, that is, a multilinear polynomial.

It remains to see that the resulting polynomial is homogeneous of degree r and belongs to J. It suffices to do this for the specific g above. Since $J = \mathbb{T}(R)$ for some R, f is an identity for R. It is clear that g is also an identity for R, whence $g \in J$. It is also clear that g is homogeneous. To see that g is non-zero,

takes a little care see e.g. Procesi □

COROLLARY 1.8. *A k-algebra satisfies a polynomial identity of degree r if and only if it satisfies a multilinear identity of degree r.*

PROPOSITION 1.9. *There are no identities of $M_d(k)$ of degree $< 2d$.*

PROOF. It is clear that if $M_d(k)$ satisfies a multilinear identity of degree $< 2d-1$ then it satisfies one of degree exactly $2d-1$, so let f be such an identity. It contains a term of the form $x_1 x_2 \dots x_{2n-1}$, perhaps after some relabelling.

The $2d-1$ matrix units

$$e_{11},\ e_{12},\ e_{22},\ e_{23}, \dots, e_{d-1\,d},\ e_{dd}$$

of $M_d(k)$ have the property that the only order they can be multiplied in to give a non-zero product is the order in which we have listed them.

???????? □

EXERCISES

1.1 Let k be a field of characteristic p. Show that T-ideal generated by $x^p - x$ is not a graded ideal.

1.2 Let k be a field of characteristic p. Show that T-ideal generated by x^p is not generated by multilinear polynomials.

2. Matrix identities

Consider the problem of finding, for a fixed d, all $f \in k\langle x_1, \dots, x_n\rangle$ for which every n-tuple of $d \times d$ matrices is a solution to the equation $f = 0$. Every solution in k to a system of equations $\mathcal{F} \subset k\langle x_1, \dots, x_n\rangle$ is also a solution to the system of equations $\{x_i x_j - x_j x_i = 0 \mid 1 \le i, j \le n\}$, so enlarging $\mathcal{F}$ by adjoining these extra equations does not change the set of solutions. Equivalently, the solutions to $\mathcal{F} = 0$ in k correspond to 1-dimensional modules over $k\langle x_1, \dots, x_n\rangle/(\mathcal{F}, x_i x_j - x_j x_i)$. But $k\langle x_1, \dots, x_n\rangle/(x_i x_j - x_j x_i) \cong k[t_1, \dots, t_n]$ so, if $\mathcal{G}$ is the image of $\mathcal{F}$ in this polynomial ring, the set of solutions in k to $\mathcal{F} = 0$ is the same as the set of solutions in k to the system of polynomial equations $\mathcal{G} = 0$.

Something similar may be done for $d \times d$ solutions, but now the quotient of $k\langle x_1, \dots, x_n\rangle$ is not the polynomial ring, but $k\langle x_1, \dots, x_n\rangle/I_d$ where I_d is the ideal of all identities of $d \times d$ matrices.

Definition 2.1. If $f \in k\langle x_1, \dots, x_n\rangle$ satisfies $f(a_1, \dots, a_n) = 0$ for all n-tuples of matrices $a_j \in M_d(k)$, then f is called an identity of $d \times d$ matrices. The identities in $k\langle x_1, \dots, x_n\rangle$ of $d \times d$ matrices form an ideal, which we denote by I_d.

PROPOSITION 2.2. *If I_d denotes the ideal of identities of $d \times d$ matrices, then $k\langle x_1, \dots, x_n\rangle$ and $k\langle x_1, \dots, x_n\rangle/I_d$ have exactly the same d-dimensional modules.*

PROOF. If V is a d-dimensional $k\langle x_1, \dots, x_n\rangle$-module, then the action of each x_i on V may be represented by a $d \times d$ matrix, say A_i. By the definition of I_d, $f(A_1, \dots, A_n) = 0$ for every $f \in I_d$, so f annihilates V. Hence V is a module over $k\langle x_1, \dots, x_n\rangle/I_d$. □

Proposition 1.9 showed that I_d has no elements of degree $< 2d$. However, I_2 contains an element of degree 5: $(xy - yx)^2 z - z(xy - yx)^2$ is an identity of 2×2 matrices (Exercise 2). Lemma 2.4 shows that I_d has an element of degree $d^2 + 1$. The main goal in this section is Theorem 2.8 which exhibits an element in I_d of degree $2d$.

Definition 2.3. Let S_n denote the symmetric group on n letters, and for each $\sigma \in S_n$ write $(-1)^\sigma$ for the sign of σ. The standard identity of degree n is

$$f_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma 1} x_{\sigma 2} \dots x_{\sigma n}.$$

The next result shows that f_{d^2+1} is an identity of $d \times d$ matrices.

LEMMA 2.4. *If R is a k-algebra of dimension $\le n-1$ and $a_1, \dots, a_n \in R$, then $f_n(a_1, \dots, a_n) = 0$.*

PROOF. It is clear that if two of the a_i are the same then $f_n(a_1, \dots, a_n) = 0$. But, if $e_1, \dots, e_m$ is a basis for R then $f_n(a_1, \dots, a_n)$ can be expanded as a linear combination of terms $f_n(e_{i_1}, \dots, e_{i_n})$. Since $m < n$, at least two of the e_{i_j}'s must coincide. □

The Amitsur-Levitzki Theorem, to be proved below, shows that $M_n(k)$ satisfies the standard identity f_{2n}. First we need some facts about symmetric functions and characteristic polynomials.

Symmetric functions. The obvious action of the symmetric group S_n on $\{x_1 \dots, x_n\}$ extends to an action of S_n as $\mathbb{Z}$-algebra automorphisms of the commutative polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$: if $\sigma \in S_n$ and $f \in \mathbb{Z}[x_1, \dots, x_n]$ we define

$$f^\sigma(x_1, \dots, x_n) = f(x_{\sigma_1}, \dots, x_{\sigma_n}).$$

For example, $(x_1^2 + x_2x_3)^{(123)} = x_2^2 + x_1x_3$.

Definition 2.5. A polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ is symmetric if $f^\sigma = f$ for all $\sigma \in S_n$. The elementary symmetric functions $e_1, \dots, e_n$ are defined by

$$e_r = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r},$$

and we adopt the convention that $e_0 = 1$.

For example

$$\begin{aligned} e_1 &= x_1 + \cdots + x_n, \\ e_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \\ e_n &= x_1 \cdots x_n. \end{aligned}$$

The elementary symmetric functions are also characterized by the property

$$\prod_{i=1}^n (T + x_i) = \sum_{r=0}^n e_r T^{n-r}.$$

Lemma 2.6 (Newton's Identities). *For each $r \ge 0$, define the symmetric function $s_r := \sum_{i=1}^n x_i^r$. Then*

(a) $s_r - s_{r-1}e_1 + s_{r-2}e_2 - \cdots + (-1)^r r e_r = 0$ *if $1 \le r < n$;*
(b) $s_r - s_{r-1}e_1 + s_{r-2}e_2 - \cdots + (-1)^n s_{r-n}e_n = 0$ *if $r \ge n$.*

Proof. Consider the functions:

$$E(T) = \prod_{i=1}^n (1 + x_iT) = \sum_{j=0}^n e_j T^j \quad \text{and} \quad S(T) = \sum_{j=0}^\infty s_{j+1}T^j.$$

Then

$$\begin{aligned} S(T) &= \sum_{i=1}^n \sum_{j=0}^\infty x_i^{j+1} T^j \\ &= \sum_{i=1}^n x_i(1 - x_iT)^{-1} \\ &= \sum_{i=1}^n -\tfrac{d}{dT} \log(1 - x_iT) \\ &= -\tfrac{d}{dT} \log E(-T) \\ &= E'(-T)/E(-T). \end{aligned}$$

It follows that $S(T)E(-T) = E'(-T)$. The coefficient of T^{r-1} on the left hand side is

$$s_r - s_{r-1}e_1 + s_{r-2}e_2 - \cdots + (-1)^{r-1}s_1e_{r-1},$$

and the coefficient of T^{r-1} on the right hand side is $(-1)^{r-1}re_r$, with the convention that $e_j = 0$ if $j > n$. Equating these gives the result. $\square$

LEMMA 2.7. *Let C be a commutative $\mathbb{Q}$-algebra and let $A \in M_n(C)$. The coefficients in the characteristic polynomial of A are polynomial functions, with rational coefficients and zero constant term, in the traces of A^r, $r = 1, \dots, n$. In particular, if $\mathrm{Tr}(A^i) = 0$ for all $1 \le i \le n$, then $A^n = 0$.*

PROOF. First we argue that it suffices to prove this for matrices with entries in an algebraically closed field of characteristic zero. This argument involves an important general principle, "Permanence of Identities" (see [Artin, Ch 12, §3] for discussion).

Let $R = \mathbb{Z}[t_{11}, \dots, t_{nn}]$ be the commutative polynomial ring on n^2 indeterminates and define $\varphi : R \to C$ by $\varphi(t_{ij}) = a_{ij}$, the ij^{th} entry of A, for all i and j. Then φ extends to a ring homomorphism

$$\varphi : M_n(R) \to M_n(C),$$

sending the matrix $T = (t_{ij})$ to A. Moreover, $\varphi(\mathrm{Tr}(T^r)) = \mathrm{Tr}(A^r)$, and by looking at the definition of the determinant as a sum of products of entries of a matrix, we see that φ sends the characteristic polynomial of T to that of A. That is, if

$$\det(xI - T) = x^n + d_1 x^{n-1} + \cdots + d_n$$

then

$$\det(xI - A) = x^n + \varphi(d_1)x^{n-1} + \cdots + \varphi(d_n).$$

Hence it suffices to prove that each d_j, $1 \le j \le n$, is a polynomial function in $\mathrm{Tr}(T), \dots, \mathrm{Tr}(T^n)$ with rational coefficients and zero constant term. The entries of T belong to the algebraic closure of the field of fractions of R, so it suffices to prove the result for matrices whose entries belong to that field.

Hence we suppose that $A \in M_n(k)$ where k is algebraically closed of characteristic zero. We may choose a basis for k^n such that

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Hence

$$\begin{aligned} \det(xI - A) &= \prod_{i=1}^{n} (x - \lambda_i) \\ &= x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n \end{aligned}$$

where $e_1, \dots, e_n$ are the elementary symmetric functions in $\lambda_1, \dots, \lambda_n$. By Lemma 2.6 the e_j's may be expressed as polynomials with rational coefficients in the expressions

$$s_r = \sum_{i=1}^{n} \lambda_i^r = \mathrm{Tr}(A^r).$$

Looking more closely at e_j as a function of $s_1, \dots, s_n$ it follows that these polynomials have zero constant term. Thus $\mathrm{Tr}(A^i) = 0$ for $i = 1, \dots, n$ implies that $\det(xI - A) = x^n$. By the Cayley-Hamilton Theorem A satisfies its characteristic polynomial, whence $A^n = 0$. $\square$

THEOREM 2.8 (Amitsur-Levitski Theorem). *For all n, the standard identity f_{2n} is an identity of $n \times n$ matrices.*

PROOF. Let $A_1, \dots, A_{2n} \in M_n(k)$. We must show that $f_{2n}(A_1, \dots, A_{2n}) = 0$. Suppose first that k contains $\mathbb{Q}$. Let $\Lambda = \Lambda(v_1, \dots, v_{2n})$ be the exterior algebra on $v_1, \dots, v_{2n}$. Define the following matrix in $M_n(\Lambda) \cong M_n(k) \otimes \Lambda$:

$$A = A_1 v_1 + \cdots + A_{2n} v_{2n}.$$

Then

$$A^m = \sum_{1 \le i_1 < \cdots < i_m \le 2n} f_m(A_{i_1}, \dots, A_{i_m}) v_{i_1} \cdots v_{i_m}.$$

In particular, when $m = 2n$, this sum has only one non-zero term, namely

$$A^{2n} = f_{2n}(A_1, \dots, A_{2n}) v_1 \cdots v_{2n}.$$

By Lemma 4.3, $v_1 \cdots v_{2n} \neq 0$, so it suffices to prove that $A^{2n} = 0$.

Now $A^{2n} = (A^2)^n$, and the entries in A^2 belong to the commutative subring $C = k[v_1 v_2, \dots, v_{2n-1} v_{2n}]$ of Λ. By Lemma 2.7 it suffices to show that $(A^2)^r$ has trace zero for all $r \ge 1$. But

$$\operatorname{Tr}(A^{2r}) = \sum_{1 \le i_1 < \cdots < i_{2r} \le 2n} \operatorname{Tr}(f_{2r}(A_{i_1}, \dots, A_{i_{2r}})) v_{i_1} \cdots v_{i_{2r}}$$

so it suffices to show that $f_{2r}(B_1, \dots, B_{2r})$ has trace zero for any $B_1, \dots, B_{2r} \in M_n(k)$. But this is obvious since the trace of $B_1 B_2 \cdots B_{2r}$ cancels with the trace of $B_2 \cdots B_{2r} B_1$. This proves the result when k contains $\mathbb{Q}$.

Hence the result holds for matrices having entries in the polynomial ring over $\mathbb{Z}$ in any number of variables, because the field of fractions of the polynomial ring contains $\mathbb{Q}$. Hence the result holds for matrices having entries in any commutative ring, because every commutative ring is a quotient of a polynomial ring over $\mathbb{Z}$. □

EXERCISES

2.1 Let e_r denote the r^{th} elementary symmetric polynomial in the variables $x_1, \dots, x_n$. For $1 \le r \le n$ write e_r as a polynomial function of the symmetric functions $s_j = \sum_{1 \le i \le n} x_i^j$.

2.2 Show that $(xy - yx)^2 z - z(xy - yx)^2 = 0$ for all $x, y, z \in M_2(k)$. [*Hint:* Don't calculate! Notice that $xy - yx$ has trace zero, and that the square of such a matrix is central.]

2.3 Find a matrix $A \in M_n(\mathbb{F}_q)$ such that $\operatorname{Tr}(A^i) = 0$ for all $i > 0$ but $A^j \neq 0$ for all $j > 0$. This shows the necessity of the hypothesis that C be a $\mathbb{Q}$-algebra in Lemma 2.7.

3. The ring of generic matrices

Definition 3.1. Let $C = k[y_{ij}^{(r)}]$ be the nd^2-dimensional commutative polynomial ring on the indeterminates $\{y(r)_{ij} \mid 1 \le r \le n, 1 \le i, j \le n\}$. The subring

$$C_{n,d} := k[Y_1, \dots, Y_n]$$

of $M_d(C)$, generated by the matrices

$$Y_r = (y_{ij}^{(r)})_{1 \le i,j \le d},$$

is called the ring of n generic $d \times d$ matrices. Each Y_r is called a generic matrix.

PROPOSITION 3.2. *If $a_1, \dots, a_n \in M_d(k)$ then there is a unique surjective k-algebra homomorphism $\Phi : C_{n,d} \to k[a_1, \dots, a_n]$ such that $\Phi(Y_i) = a_i$ for all i.*

PROOF. Any algebra homomorphism for which $\Phi(Y_i) = a_i$ is certainly unique and surjective, so it suffices to prove existence.

???????????

□

4. Structure of PI rings

THEOREM 4.1 (Posner's Theorem). *Let R be a prime PI ring. Then its center, Z say, is a domain and, if $K = \operatorname{Fract} Z$, then $\operatorname{Fract} R \cong R \otimes_Z K$. Moreover, $\operatorname{Fract} R$ is a central simple K-algebra.*

COROLLARY 4.2. *A ring is prime PI if and only if it is an order in a central simple algebra.*

PROPOSITION 4.3. *(MR 13.6.11) Let A be a prime PI ring with center Z. Then Z is noetherian if and only if A_Z is finitely generated and A is right (or left) noetherian.*

PROOF. □

Example. (MR 5.3.7) The center of a prime noetherian PI ring need not be noetherian.

5. Rings finite over their centers

The center of a ring is denoted by $Z(R)$. If R is a finitely generated $Z(R)$-module we say that R is finite over its center. As already remarked, a ring finite over its center satsifies a polynomial identity. The converse is false.

EXAMPLE 5.1. Let $\mathfrak{g}$ be the 3-dimensional Heisenberg Lie algebra over a field k of characteristic zero. Take a basis x, y, z with z central. The ring $R := U(\mathfrak{g})/(z^2)$ satisfies the identity $(x_1x_2 - x_2x_1)^2 = 0$. However, the center of R is $k[z]/(z^2)$, and R is clearly not a finite module over this.

In fact, R is not a finite module over any commutative subring (see [**142**, Exercise 1, §6.3]).

If R is finite over its center, then its structure is controlled quite tightly by its center. One may view R as a family of finite dimensional algebras parametrized by $\operatorname{Spec} Z(R)$, and the representation theory is constant over a non-empty open subset of $Z(R)$. This

THEOREM 5.2 (Artin-Tate Lemma). *Let R be a finitely generated k-algebra which is finite over its center. Then*

1. *$Z(R)$ is a finitely generated k-algebra, hence noetherian;*
2. *R is noetherian.*

PROOF. Write $Z = Z(R) = k[z_1, \dots, z_m]$ and $R = Za_1 + \dots + Za_n$. There is a finite set

$$\{\alpha_{pqr}, \beta_{st}\} \subset Z$$

such that

$$a_pa_q = \sum_{r=1}^{m} \alpha_{pqr}a_r \qquad \text{and} \qquad z_s = \sum_{t=1}^{n} \beta_{st}a_t.$$

Since $Z' := k[\alpha_{pqr}, \beta_{st}]$ is finitely generated and commutative it is noetherian. Since R is generated as a Z'-module by $a_1, \dots, a_n$, R is a noetherian Z'-module and hence a noetherian ring. Finally, since $Z' \subset Z \subset R$, Z is a finitely generated Z'-module, whence a finitely generated k-algebra, and thus a noetherian ring. □

THEOREM 5.3. *Let R be a finitely generated k-algebra which is finite over its center. If M is a simple R-module, then*

1. $\dim_k M < \infty$;
2. *M is annihilated by a maximal ideal of $Z(R)$;*
3. *if k is algebraically closed, then* $\operatorname{End}_R M = k$.

PROOF. Set $\bar{R} = R/\operatorname{Ann} M$; then $\bar{R}$ is also finite over its center, and is prime. The center of a prime ring is a field, so $Z(\bar{R}) = L$ is a field extension of k. Since $\dim_L(\bar{R}) < \infty$, $\bar{R}$ is artinian, and hence simple. Since L is a finitely generated k-algebra and a field, $\dim_k L < \infty$. The image of Z in $\bar{R}$ is contained in L so is a domain, and is finite dimensional over k so is a field. But the kernel of the map $Z \to \bar{R}$ is $\operatorname{Ann} M \cap Z$, so we conclude that this is a maximal ideal of Z, which proves (2). Since $\dim_L R < \infty$ and $[L : k] < \infty$, $\dim_k \bar{R} < \infty$. It follows that $\dim_k M < \infty$, and that $\operatorname{End}_R M$ is finite dimensional over k. But M is simple, so $\operatorname{End}_R M$ is a division algebra over k so, if k is algebraically closed it equals k. □

6. Representation theory of Clifford algebras

Let R be a finitely generated commutative algebra over an algebraically closed field k, and consider the Clifford algebra $C(M, q)$ of a free quadratic R-module (M, q). The classification of simple $C(M, q)$-modules is quite tractable, and the result is rather typical of the representation theory of a ring finite over its center.

Since R is in the center of $C(M, q)$, a simple $C(M, q)$-module is annihilated by some maximal ideal, $\mathfrak{m}$ say, of R. Hence, the simple module is a module over $R/\mathfrak{m} \otimes_R C(M, q)$; by Lemma 6.5.4, this tensor product is isomorphic to $C(M/\mathfrak{m}M, \bar{q})$, the Clifford algebra of a finite dimensional quadratic vector space over the field $R/\mathfrak{m}$. Hence, we must examine the simple modules over a finite dimensional Clifford algebra.

THEOREM 6.1. *Let (V, q) be a non-degenerate n-dimensional quadratic vector space. Let Q be the symmetric $n \times n$ matrix associated to (V, q), and define*

$$\delta := (-1)^{\frac{1}{2}n(n-1)} \det Q \in k^*.$$

Then

1. *if n is even, $C(V, q)$ is a central simple k-algebra;*
2. *if n is odd and $\delta \notin (k^*)^2$, then $C(V, q)$ is a central simple $k(\sqrt{\delta})$-algebra of dimension 2^{n-1};*
3. *if n is odd and $\delta \in (k^*)^2$, then $C(V, q)$ is the direct sum of two central simple k-algebras of dimension 2^{n-1}.*

$\boxed{\text{Paul}}$ Classify their f.diml repns via rank stratification. Follow Le Bruyn [**100**].

7. A sheaf of algebras

Let A be an $\mathbb{N}$-graded k-algebra which is finite over its center Z. Let S denote the scheme associated to the commutative graded algebra Z [**76**, Chapter II, Section 2]. Recall that the closed subsets of S are the sets

$$\mathcal{V}(I) = \{\text{graded prime ideals of } Z \text{ containing } I, \text{ but not containing } Z_{\geq 0}\},$$

where I is a graded ideal of Z. There is basis of open sets of the form

$$S_{(c)} := \text{the complement of } \mathcal{V}(cZ)$$

where $c \in Z$ is a homogenous element of positive degree. The sections of the structure sheaf $\mathcal{O}_S$ above $S_{(c)}$ are the elements of the ring $Z[c^{-1}]_0$.

We now define the sheaf $\mathcal{A}$ of $\mathcal{O}_S$-algebras by

$$\mathcal{A}(S_{(c)}) = A[c^{-1}]_0,$$

for each homogeneous $c \in Z$ of positive degree. The center $\mathcal{Z}$ of $\mathcal{A}$ is defined to be the subsheaf

$$\mathcal{Z}(S_{(c)}) = Z(A[c^{-1}]_0).$$

LEMMA 7.1. *Let A be a $\mathbb{Z}$-graded k-algebra. Suppose that $Z(A) = k[z_1, ..., z_m]$, with each z_i being a homogeneous non-zero-divisor, of degree n_i, say. Let $b :=$ $\gcd(n_1, ..., n_m)$. Suppose A is finite over $Z(A)$. Let $S = \operatorname{Proj}(Z(A))$ and let $\mathcal{A}$ be the sheaf of $\mathcal{O}_S$-algebras such that $\mathcal{A}(S_{(z_i)}) = A[z_i^{-1}]_0$ for each i. Let $\mathcal{Z}$ denote the center of $\mathcal{A}$. Then*

$$\mathbf{Spec}\,\mathcal{Z} \cong \operatorname{Proj}(Z(A^{(b)})).$$

PROOF. Fix an i and write $z = z_i$ and $n = n_i$. Since S is covered by the open affine sets $S_{(z_i)}$, we must show that

$$Z(A[z^{-1}]_0) = Z(A^{(b)})[z^{-1}]_0.$$

Since the degree of z^{-1} is divisible by b, we have $A[z^{-1}]_0 = A^{(b)}[z^{-1}]_0$. Hence, replacing A by $A^{(b)}$ and dividing all degrees by b, we can assume $b = 1$ and $A = A^{(b)}$. Similarly, $A[z^{-1}]_0 = A^{(n)}[z^{-1}]_0$, and $Z(A)[z^{-1}]_0 = (Z(A)^{(n)})[z^{-1}]_0$.

So we must show that, if $\gcd(n_j) = 1$, then $Z((A[z^{-1}])_0) = (Z(A[z^{-1}]))_0$. This follows from the fact that an element $y \in A[z^{-1}]$ which commutes with elements of degree 0 is in the center of that ring. To show that such a y commutes with an element x of degree d, say, it is enough to exhibit a central non-zero-divisor u of degree $-d$, for then y will commute with xu and hence with x. To construct u, write $-d = a_1 n_1 + ... + a_i n_i + ... + a_m n_m$, with $a_j \geq 0$ for $j \neq i$ and put $u = z_1^{a_1} \ldots z_i^{a_i} \ldots z_m^{a_m}$. □

THEOREM 7.2. *If A is an $\mathbb{N}$-graded algebra which is finite over its center, then the categories $\mathsf{Mod}(\mathcal{A})$ and $\operatorname{Proj}(A)$ are equivalent.*

PROOF. □

Definition 7.3. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras. A simple $\mathcal{A}$-module is a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules whose only submodules are zero and $\mathcal{F}$ itself.

PROPOSITION 7.4. *There is a bijection between the simple $\mathcal{A}$-modules and the points of $\operatorname{Proj}(A)$.*

PROOF. □

Proposition 7.5. *Let A be a graded k-algebra which is generated over A_0 by A_1. If z is a homogenous, regular, normalizing element of non-zero degree, then the categories $\mathsf{Mod}(A[z^{-1}]_0)$ and $\mathsf{GrMod}(A[z^{-1}]$ are equivalent.*

Proof. By Proposition 16.6.4, $A[z^{-1}]$ is strongly graded so this result is a restatement of Theorem 16.6.7. □

Let z be a homogenous, central, regular element of positive degree. The simple $\mathcal{A}$-modules having support outside $\mathcal{V}(z)$ are in bijection with the simple $A[z^{-1}]_0$-modules. But the categories $\mathsf{Mod}(A[z^{-1}]_0)$ and $\mathsf{GrMod}(A[z^{-1}])$ are equivalent, so such a simple $\mathcal{A}$-module corresponds to a 1-critical graded module over $A[z^{-1}]$; these modules in turn are obtained from 1-critical graded A-modules which have no z-torsion.

CHAPTER 11

Graded algebras and graded modules

In this chapter we work over an arbitrary base field k.

We will work with left A-modules—the same ideas apply to right modules.

In section 6 we show that the usual homological machinery involving Ext and Tor operates in $\mathsf{GrMod}(A)$. Sections 4 and 5 examine projectives and injectives in $\mathsf{GrMod}(A)$, showing amongst other things that there are enough of each. Sections 2 and 4 treat some matters which are not specific to graded rings. There are preliminaries in section 2 on injective dimension, and the grade of a module is defined and studied in section 4.

1. Graded algebras

Definition 1.1. A k-algebra A is graded if it is endowed with a k-vector space decomposition $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that

$$A_i A_j \subset A_{i+j}$$

for all i and j. If $A_i = 0$ for all $i < 0$, we say that A is $\mathbb{N}$-graded. If A is a graded k-algebra, a graded A-module, M say, is an A-module with a k-vector space decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$, such that

$$A_i M_j \subset M_{i+j}$$

for all i and j. Elements of A_i and elements of M_i are said to be homogeneous of degree i. We call M_i the degree i component of M. If $m = \sum_i m_i \in M$ with each $m_i \in M_i$, we call m_i the degree i component of m.

If $M = M_n$ we say that M is concentrated in degree n.

EXAMPLE 1.2. The field k will always be considered as a graded algebra concentrated in degree zero. With this grading, we call a graded k-module a graded vector space; thus a graded vector space is nothing more than a vector space decomposition, $V = \oplus_{n \in \mathbb{Z}} V_n$.

Let U and V be graded vector spaces. A linear map $f : U \to V$ has degree d if $f(U_i) \subset V_{i+d}$ for all i. The set of all degree d maps from U to V is denoted by $\mathrm{Hom}_k(U, V)_d$. We also introduce the notation

$$\underline{\mathrm{Hom}}_k(U, V) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_k(U, V)_d.$$

Thus, if V is concentrated in degree n, its dual V^* is concentrated in degree $-n$. It is possible for $\underline{\mathrm{Hom}}_k(U, V)$ to be a proper subspace of $\mathrm{Hom}_k(U, V)$ (see Exercise 3 below).

The tensor product of graded vector spaces is made into a graded vector space by defining

$$(U \otimes V)_n := \bigoplus_{i+j=n} U_i \otimes V_j.$$

We call this the tensor product grading.

Graded algebras and graded modules are, in the first instance, graded vector spaces. The two conditions in Definition 1.1 may be rephrased as saying that the multiplication maps $A \otimes_k A \to A$ and $A \otimes_k M \to M$ are degree zero maps, when the left hand sides are given the tensor product grading.

EXAMPLE 1.3. The commutative polynomial ring $A = k[t_1, \dots, t_n]$ on the indeterminates $t_1, \dots, t_n$ becomes a graded k-algebra by setting $A_0 = k$, $A_1 = kt_1 + \cdots + kt_n$ and extending this in the obvious way: since A_0 and A_1 generate A the *only* way to make A a graded algebra is by defining A_d to be the linear span of

$$\{t_1^{i_1} \dots t_n^{i_n} \mid i_1 + \dots + i_n = d\}.$$

In particular, the definition of degree in Definition 1.1 agrees with the usual one. However, one may also assign the degrees of the t_i arbitrarily, say $d_i = \deg t_i$, and then make A into a graded algebra by defining A_d to be the span of all $t_1^{i_1} \dots t_n^{i_n}$ such that $i_1 d_1 + \dots + i_n d_n = d$.

EXAMPLE 1.4. The free algebra $k\langle X\rangle$ may be made into a graded algebra by assigning the degrees of the elements of X arbitrarily, and extending this to elements of X^* by defining $\deg(x_1 \dots x_n) = \deg x_1 + \dots + \deg x_n$.

If V is a k-vector space, one may make $T(V)$ a graded algebra by defining $T(V)_n := V^{\otimes n}$. However, if V is a graded vector space to begin with, then it is sometimes more useful and natural to extend this grading to $T(V)$ as follows: if $v_1, \dots, v_n$ are homogeneous elements of V, then $\deg(v_1 \otimes \dots \otimes v_n) = \deg v_1 + \dots + \deg v_n$.

EXAMPLE 1.5. Let A be a graded algebra and $z \in A$ a homogeneous central regular element of degree s. We may form the localization $A[z^{-1}]$, the elements of which are of the form az^{-i}, where $a \in A$ and $i \in \mathbb{Z}$. If $a \in A_r$ we define $\deg az^{-i} := r - is$. This definition does not depend on how the element az^{-i} is represented, so there are well-defined k-vector spaces

$$A[z^{-1}]_n := \{x \in A[z^{-1}] \mid \deg x = n\}.$$

Now $A[z^{-1}] = \bigoplus_{n\in\mathbb{Z}} A[z^{-1}]_n$, and this gives $A[z^{-1}]$ the structure of a graded algebra. It is clear that

$$A[z^{-1}]_n = \sum_{i\in\mathbb{Z}} A_{n+is} z^{-i}.$$

More generally, if A is a graded algebra and $\mathcal{S} \subset A$ is a multiplicatively closed set of homogeneous elements satisfying the Ore condition (3.6), there is a unique graded algebra structure on the localization $A[\mathcal{S}^{-1}] = \{as^{-1} \mid a \in A,\ s \in \mathcal{S}\}$ which extends the grading on A.

EXAMPLE 1.6. The matrix algebra $M_n(k)$ may be made into a graded k-algebra by defining

$$\deg e_{ij} := i - j, \tag{1-1}$$

where $\{e_{ij} \mid 1 \le i, j \le n\}$ are the usual matrix units. It is trivial to check this makes $M_n(k)$ a graded algebra. We will explain why this is natural.

Let $V = kv_1 \oplus \cdots \oplus kv_n$ be the graded k-vector space with

$$\deg v_j = j,$$

for all j. Define $e_{ij} \in \mathrm{Hom}_k(V,V)$ by

$$e_{ij}(v_l) = \begin{cases} v_i & \text{if } j = l, \\ 0 & \text{if } j \neq l. \end{cases}$$

Then $e_{ij} \in \mathrm{Hom}_k(V,V)_{i-j}$, so this grading on $\mathrm{Hom}_k(V,V) \cong M_n(k)$ is that defined by (1-1).

A slightly different point of view obtains from consideration of the dual vector space

$$V^* = \mathrm{Hom}_k(V,k),$$

where k is given degree zero. Let $\varphi_1, \dots, \varphi_n$ be the basis for V^* dual to $v_1, \dots, v_n$. Then $\deg \varphi_j = -j$. Let $V \otimes_k V^*$ have the tensor product grading. The natural isomorphism

$$V \otimes_k V^* \to \mathrm{Hom}_k(V,V)$$

therefore induces a grading on $M_n(k)$, which coincides with that defined by (1-1) since e_{ij} is the image of $v_i \otimes \varphi_j$.

The grading on $M_n(k)$ is easy to visualize: the diagonal matrices have degree 0, elements in the first upper diagonal have degree -1, elements in the first lower diagonal have degree $+1$, elements in the second lower diagonal have degree $+2$, etc.

The non-degenerate pairing on $A = M_n(k)$, defined by $(\alpha, \beta) \mapsto \mathrm{Tr}(\alpha\beta)$, restricts to a non-degenerate pairing $A_m \times A_{-m} \to k$ for each m.

PROPOSITION 1.7. *The center of a graded k-algebra is a graded k-algebra.*

PROOF. Let z be a central element, and z_i its degree i component. If $a \in A_j$, then the degree $i+j$ component of $za - az$ is $z_i a - az_i$, and this must be zero. Hence z_i is central. Thus $Z(A) = \sum_i Z(A) \cap A_i$. □

Definition 1.8. The category of graded k-algebras has objects the graded k-algebras and morphisms the k-algebra homomorphisms $f : A \to B$ such that $f(A_n) \subset B_n$ for all n.

If A is a graded k-algebra, a graded left, right or two-sided ideal is just a left, right or two-sided ideal, I say, such that $I = \oplus_{n \in \mathbb{Z}} I \cap A_n$. The quotient by a two-sided graded ideal inherits the structure of a graded algebra. In particular, A itself is isomorphic as a graded algebra to a quotient of a free algebra, endowed with an appropriate grading, by a graded ideal.

NOTATION . Let A be an $\mathbb{N}$-graded k-algebra. We write $A_{\geq n} := \bigoplus_{i=n}^{\infty} A_i$ for each $n \geq 0$. This is a two-sided ideal of A.

Definition 1.9. If A is a graded k-algebra and r a positive integer, the r^{th} Veronese subalgebra of A is the graded algebra denoted by $A^{(r)}$ and having degree i component

$$A_i^{(r)} := A_{ri}.$$

If $r \neq 1$, the inclusion $A^{(r)} \to A$ is not a morphism in the category of graded algebras.

Definition 1.10. The Segre product of two graded k-algebras, A and B say, is the subalgebra of $A \otimes_k B$, denoted $A \circ B$, whose degree n component is the k-linear span of $\{a \otimes b \mid a \in A_n, b \in B_n\}$.

EXERCISES

1.1 Let $A = k[X, X^{-1}]$ be a graded k-algebra with $\deg X = 1$.
(a) Show that $\deg X^n = n$ for all $n \in \mathbb{Z}$.
(b) Show that the only graded ideals of A are 0 and A.

1.2 Let $A = k[x, y]$ be the commutative polynomial ring with its usual grading. Show that the Veronese subalgebra $A^{(2)}$ is isomorphic to $k[x_1, x_2, x_3]/(x_1x_3 - x_2^2)$. This corresponds to embedding $\mathbb{P}^1$ in $\mathbb{P}^2$ as a non-degenerate conic.

1.3 Let $A = k[x_1, x_2]$ and $B = k[y_1, y_2]$ be commutative polynomial rings with their usual gradings. Show that $A \odot B \cong k[t_0, t_1, t_2, t_3]/(t_0t_3 - t_1t_2)$. This corresponds to embedding $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ as a smooth quadric.

1.4 Let $A = k[u, v]$ with defining relations $vu^2 = u^2v$ and $v^2u = uv^2$. Show that $A^{(2)}$ is isomorphic to $k[w, x, y, z]/(xy - wz)$, the homogeneous coordinate ring of a smooth quadric in $\mathbb{P}^3$. (The ring A is the second generic Clifford algebra—see Example 6.5.11.)

1.5 Let $A = k[X, Y, Z]$ be the commutative polynomial ring with X and Y of degree 1, and Z of degree 2. Show that $A^{(2)}$ is isomorphic to $k[w, x, y, z]/(x^2 - wy)$, the homogeneous coordinate ring of a singular quadric in $\mathbb{P}^3$.

1.6 Let k be a field of characteristic zero and let $A = k[X_1, \dots, X_n]$ be the polynomial ring in n indeterminates. The ring of differential operators on $\mathbb{A}^n_k$ is the subalgebra of $\operatorname{End}_k(A)$ generated by $X_1, \dots, X_n$ acting by multiplication, and the partial derivatives $\partial/\partial X_1, \dots, \partial/\partial X_n$; that is
$$\mathcal{D}(\mathbb{A}^n) = k[X_1, \dots, X_n, \partial/\partial X_1, \dots, \partial/\partial X_n].$$
Show that $\mathcal{D}(\mathbb{A}^n)$ is a graded k-algebra with each X_i of degree one and each $\partial/\partial X_i$ of degree -1.

1.7 Show that the k-algebra $k[x, y]/(\alpha^2x^2 - y^2)$ is isomorphic to the k-algebra $k[x, y]/(xy + yx)$ for every $0 \neq \alpha \in k$.

2. The categories GrMod and grmod of graded modules

Definition 2.1. Let A be a graded k-algebra. If N and M are graded A-modules, we define the following homomorphism groups:

$$\operatorname{Hom}_A(N, M) := A\text{-module homomorphisms in } \mathsf{Mod}(A)$$

$$\operatorname{Hom}_A(N, M)_d := \{f \in \operatorname{Hom}_A(M, N) \mid f(N_i) \subset M_{i+d} \text{ for all } i\}$$

$$\underline{\operatorname{Hom}}_A(N, M) := \bigoplus_{d=-\infty}^{\infty} \operatorname{Hom}_A(N, M)_d$$

$$\operatorname{Hom}_{\mathrm{Gr}}(N, M) := \operatorname{Hom}_A(N, M)_0.$$

Define the categories
- $\mathsf{GrMod}(A)$, having objects the graded left A-modules and morphisms $\operatorname{Hom}_{\mathrm{Gr}}(N, M)$;
- $\mathsf{grmod}(A)$, the full subcategory of $\mathsf{GrMod}(A)$ consisting of the finitely generated graded A-modules.

Let $M \in \mathsf{GrMod}(A)$. A submodule N of M is a graded submodule if $N = \sum_{n\in\mathbb{Z}}(M_n \cap N)$. By defining $N_n := M_n \cap N$, such an N becomes a graded module.

Furthermore, the quotient module becomes a graded module by defining

$$(M/N)_n := M_n + N/N \cong M_n/N_n.$$

The kernel and image of a map $f \in \operatorname{Hom}_A(M,N)_d$ are graded submodules of M and N respectively.

If A is not left noetherian, then $\mathsf{grmod}(A)$ is not abelian: it is not closed under kernels. We will usually assume that A is left noetherian when considering $\mathsf{grmod}(A)$; in that case the category $\mathsf{GrMod}(A)$ is completely determined by $\mathsf{grmod}(A)$ (see [**131**, Section 5.8, Theorem 8.9] and [**65**, Chapter II, Section 4, Theorem 1]) so little is lost by restricting attention to finitely generated modules.

By forgetting the grading on a module we have a functor $\mathsf{GrMod}(A) \to \mathsf{Mod}(A)$ (which is generally not faithful).

PROPOSITION 2.2. *If $M \in \mathsf{grmod}(A)$ and $N \in \mathsf{GrMod}(A)$, then*

$$\underline{\operatorname{Hom}}_A(M,N) = \operatorname{Hom}_A(M,N).$$

PROOF. We will write y_i for the degree i component of $y \in N$. Let $f : M \to N$ be an arbitrary A-module map. For each $j \in \mathbb{Z}$, define a k-linear map $f_j : M \to N$ by $f_j(m) = f(m)_{i+j}$ for $m \in M_i$; thus $f_j(M_i) \subset N_{i+j}$ for all i. The map f_j is A-linear because, if $a \in A_p$ and $m \in M_q$, then

$$f_j(am) = f(am)_{j+p+q} = (a.f(m))_{j+p+q} = a.(f(m)_{j+q}) = a.f_j(m).$$

Hence $f_j \in \operatorname{Hom}_A(M,N)_j$.

Let $m_1, \dots, m_n$ be a homogeneous basis for M; choose $r, s \in \mathbb{N}$ such that each $m_i \in M_{-r} \oplus \cdots \oplus M_r$ and each $f(m_i)$ is contained in $N_{-s} \oplus \cdots \oplus N_s$. Then f and $f_{-r-s} + \cdots + f_{r+s}$ are A-module maps which coincide on each m_i, whence $f = f_{-r-s} + \cdots + f_{r+s}$ so we conclude that $f \in \underline{\operatorname{Hom}}_A(M,N)$. $\square$

The shift functor. A particularly important, albeit simple, operation on graded modules is the 'shift functor': it just changes the grading on a module. Since morphisms in $\mathsf{GrMod}(A)$ are required to be of degree zero, it is not surprising that a change in the grading might have serious consequences.

Definition 2.3. Let A be a graded algebra and M a graded A-module. If $d \in \mathbb{Z}$, we define $M[d]$ to be the graded A-module which is equal to M as an ungraded A-module, but has grading $M[d]_i = M_{d+i}$; thus $[d]$ just lowers the degree of an element by d. We call $M[d]$ a shift of M, and say that M is equal to $M[d]$ up to shifting. Graded modules M and N are shift equivalent if $M[d] \cong N$ for some $d \in \mathbb{Z}$.

The following facts are easy to establish:

- $M[d] \cong A[d] \otimes_A M$;
- $\underline{\operatorname{Hom}}_A(N,M)[d] \cong \underline{\operatorname{Hom}}_A(N,M[d]) \cong \underline{\operatorname{Hom}}_A(N[-d],M)$ as graded vector spaces, whence $\underline{\operatorname{Hom}}_A(N,M) \cong \oplus_{d\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{Gr}}(N,M)$;
- if $f : M \to N$ is a degree d homomorphism then there is an induced isomorphism $M/\ker f \cong \operatorname{Im} f[d]$.

It is convenient to explicitly define the covariant functors s and s^{-1} from $\mathsf{GrMod}(A)$ to itself:

$$sM = M[1] \qquad \text{and} \qquad s^{-1}M = M[-1].$$

It is clear that $s \circ s^{-1} = \mathbb{1}d = s^{-1} \circ s$, so each of these is an automorphism of $\mathsf{GrMod}(A)$. Of course, s is also an automorphism of $\mathsf{grmod}(A)$.

Definition 2.4. Let M be a graded k-vector space. Define $M_{\geq n} := \oplus_{i \geq n} M_i$. We say that

- M is left bounded or bounded below, if $M_n = 0$ for all $n \ll 0$;
- M is right bounded or bounded above, if $M_n = 0$ for all $n \gg 0$;
- the lower bound of M is the least n for which $M_n \neq 0$ (we allow the values $\pm\infty$);
- the upper bound of M is the greatest n for which $M_n \neq 0$ (we allow the values $\pm\infty$);
- M is bounded if its upper and lower bounds are both finite.

If $a, b \in \mathbb{Z}$, we denote by $[a, b]$ the collection of all graded k-modules M such that $M_{<a} = M_{>b} = 0$. We allow the notation $[a, \infty]$ and $[-\infty, b]$. We allow $a < b$, in which case the zero module is the only member of $[a, b]$.

A finitely generated module over an $\mathbb{N}$-graded algebra is bounded below.

It is straightforward to show that $[\ell, r]$ is a dense subcategory of $\mathsf{GrMod}(A)$. If M and N are graded k-vector spaces with $M \in [a, b]$ and $N \in [c, d]$, then

- $\underline{\mathrm{Hom}}_k(M, N) \in [c - b, d - a]$,
- $M[e] \in [a - e, b - e]$, and
- $M \otimes_k N \in [a + c, b + d]$.

Definition 2.5. A graded k-vector space $M_{\geq n} := \oplus_{i \geq n} M_i$ locally finite, if $\dim_k(M_i) < \infty$ for all i.

A finitely generated k-algebra is locally finite if all its generators are in positive degree. If A is locally finite, so is every finitely generated graded A-module. In the study of locally finite algebras, algebras which are not locally finite intervene; for example, the commutative polynomial ring $k[x, y]$ is locally finite, but its localization $k[x, y, y^{-1}]$ is not: its degree zero component is the polynomial ring $k[xy^{-1}]$.

Products and Coproducts. Suppose we have a family of modules $M^\alpha \in \mathsf{GrMod}(A)$. Their direct sum, or coproduct, in $\mathsf{Mod}(A)$ can be made into a graded module by defining $(\coprod M^\alpha)_i := \coprod M_i^\alpha$. The natural maps $M^\alpha \to \coprod M^\alpha$ are degree zero. Because $\coprod M^\alpha$ has the appropriate universal property in $\mathsf{Mod}(A)$, $\coprod M^\alpha$ is the coproduct of the M^α in $\mathsf{GrMod}(A)$. Hence there will be no confusion with refering to coproducts of graded modules. The forgetful functor $\mathsf{GrMod}(A) \to \mathsf{Mod}(A)$ commutes with taking coproducts.

However, the situation for products is more complicated. To see this, consider a family $V^i \in \mathsf{GrMod}(k)$, where $V^i \cong k$ is concentrated in degree i. The product $\prod V^i$ in $\mathsf{Mod}(k)$ cannot be graded in such a way that the projection maps $\prod V^i \to V^j$ are degree zero. The best we can do is to put a grading on the subspace of $\prod V^i$ consisting of those elements having only finitely many non-zero components, the degree i component being those elements whose components outside V^i are zero. This suggests the next definition.

Definition 2.6. The product of a collection $M^\alpha \in \mathsf{GrMod}(A)$ is the graded k-module whose degree j component is $\prod(M_j^\alpha)$. We will denote this by $\prod M^\alpha$.

Warning: When taking a product of graded modules we need to specify whether that product is taken in $\mathsf{Mod}(A)$ or in $\mathsf{GrMod}(A)$. As the example just showed, these are usually different. In other words, the forgetful functor does not commute with taking products.

The product we have just defined *is* a categorical product in $\mathsf{GrMod}(A)$. To see this, suppose that $M \in \mathsf{GrMod}(A)$, and that there are maps $q_\alpha \in \mathrm{Hom}_{\mathrm{Gr}}(M, M^\alpha)$. Then there is a degree zero lifting $q : M \to \prod M^\alpha$.

Similar considerations apply to limits and colimits. The colimit in $\mathsf{Mod}(A)$ of a diagram in $\mathsf{GrMod}(R)$ can be graded to give a colimit in $\mathsf{GrMod}(A)$; but the limit in $\mathsf{Mod}(A)$ of a diagram in $\mathsf{GrMod}(A)$ cannot usually be graded; however, there is a submodule which can be graded, and is a limit in $\mathsf{GrMod}(A)$. For example, the direct limit in $\mathsf{GrMod}(A)$ of a system M^α has degree i component $\varinjlim(M_i^\alpha)$, where this is the direct limit in $\mathsf{Mod}(A_0)$.

EXERCISES

2.1 Let $A = k[X, X^{-1}]$ with $\deg X = 1$.
 (a) Show that $A[n] \cong A$ for all n.
 (b) Show this is false if $\deg X = 2$.

2.2 Let A be a graded algebra and M a graded A-module.
 (a) Show that each A_i is an A_0-A_0-bimodule.
 (b) Show that each M_j is a left A_0-module.
 (c) Show that the multiplication map $A_i \otimes_k M_j \to M_{i+j}$ is a left A_0-module homomorphism.

2.3 Let $U = \oplus_{i \geq 0} ke_i$ with $\deg e_i = i$. Let $f : U \to k$ be the linear map defined by $f(e_i) = 1$ for all i. Show that $f \notin \underline{\mathrm{Hom}}_k(U, k)$.

2.4 Prove the properties of the shift functor stated after Definition 2.3.

2.5 Let $A = k[X, X^{-1}]$ be graded with $\deg X = 1$. Let M be a graded A-module.
 (a) Show that $M = A.M_0$.
 (b) Show that the multiplication map $A \otimes_k M_0 \to M$ is an isomorphism of A-modules.
 (c) Show that the category $\mathsf{GrMod}(A)$ is equivalent to $\mathcal{V}ec_k$, the category of k-vector spaces. [*Hint:* the functors are $M \mapsto M_0$ and $V \mapsto A \otimes_k V$.]
 (d) Show directly that every short exact sequence of graded A-modules splits, whence, in particular, A is injective in $\mathsf{GrMod}(A)$.

2.6 Let $A = k[x]$ be the polynomial ring with its usual grading, and let $M \cong A$.
 (a) Show that $M^* \cong k[x, x^{-1}]/xA$ as a right A-module.
 (b) Show that $M^* \cong \varinjlim(A/x^rA)[r-1]$ where the direct limit is taken with respect to the maps $\varphi_j^i : A/(x^i) \to A/(x^j)$ defined by $\varphi_j^i(a + (x^i)) = ax^{j-i} + (x^j)$ for $i \leq j$.

2.7 Let M be a graded A-module and suppose that M is finitely presented as an ungraded module; i.e., there is an exact sequence $A^p \to A^q \to M \to 0$ in $\mathsf{Mod}(A)$ for some integers p and q. Show that M is finitely presented in $\mathsf{GrMod}(A)$; i.e., show that there exists an exact sequence $P \to Q \to M \to 0$ in $\mathsf{GrMod}(A)$ with P and Q being finitely generated free graded A-modules. [*Hint:* use Schanuel's Lemma.]

2.8 Let A be a locally finite, $\mathbb{N}$-graded k-algebra. Show that $A_{\geq j}$ is a finitely generated left ideal for all $j > 0$ if a single one is. (Notice that any A which is generated by A_1 as a k-algebra has this property – indeed, any finitely generated algebra has this property – what about non-finitely generated algebras.)

2.9 Let A be an $\mathbb{N}$-graded k-algebra. Show that the irreducible objects in $\mathsf{GrMod}(A)$ (i.e., the modules which are simple as graded modules) are in bijection with the simple A_0-modules. Moreover, if $\dim_k A_0 < \infty$, then all finite length graded modules are finite dimensional.

3. Tensor products, graded bimodules, and bigraded bimodules

The simplest way to define a graded structure on a tensor product of graded modules is as follows. If M_A and ${}_AN$ are graded right and left A-modules respectively, the **tensor product grading** on $M \otimes_A N$ is defined by declaring that

$\deg m \otimes n = \deg m + \deg n$ for all homogeneous $m \in M$ and $n \in N$. This makes sense because $M \otimes_A N$ is the quotient of the $\mathbb{Z}$-graded k-module $M \otimes_k N$ by the submodule generated by the all elements $ma \otimes n - m \otimes an$, and each of these elements is a sum of homogeneous elements of the same form; thus $M \otimes_A N$ is a quotient of a graded k-module by a graded submodule.

More generally, M_A and ${}_AN$ may be endowed with bimodule structures; with this in mind we make the following definition.

Definition 3.1. Let A and B be graded k-algebras. A

- graded A-B-bimodule is a $\mathbb{Z}$-graded k-module M which is an A-B-bimodule and satisfies
$$A_p.M_i.B_q \subset M_{p+i+q}.$$
- bigraded A-B-bimodule is a $\mathbb{Z}^2$-graded k-module M which is an A-B-bimodule and satisfies
$$A_p.M_{(i,j)}.B_q \subset M_{(p+i,q+j)}.$$

Let M be a bigraded A-B-bimodule. Then
$$M_{(i,*)} := \sum_j M_{(i,j)}$$
is a graded B-module with degree j component $M_{(i,j)}$. There is a similar definition of $M_{(*,j)}$ which is a graded A-module. Typically a bigraded bimodule is not finitely generated on either side. For example, $A \otimes_k B$ is a bigraded A-B-bimodule in an obvious way, as are its two-sided ideals, and quotient rings.

LEMMA 3.2. *Let L be a graded A-B-bimodule, M a graded A-C-bimodule, and N a graded B-C bimodule. Then*

1. *$L \otimes_B N$ is a graded A-C-bimodule under the grading described prior to Definition 3.1;*
2. *$\underline{\mathrm{Hom}}_A(L, M)$ is a graded B-C-bimodule under the grading described in section 2.*

LEMMA 3.3. *Let L be a bigraded A-B-bimodule, M a bigraded A-C-bimodule, and N a bigraded B-C bimodule. Then*

1. *$L \otimes_B N$ is a bigraded A-C-bimodule via*
$$(L \otimes_B N)_{(p,q)} = L_{(p,*)} \otimes_B N_{(*,q)}.$$
2. *$\underline{\mathrm{Hom}}_A(L, M)$ is a bigraded B-C-bimodule via*
$$\underline{\mathrm{Hom}}_A(L, M)_{(p,q)} = \underline{\mathrm{Hom}}_A(L_{(*,p)}, M_{(*,q)}).$$

PROOF. □

The next result is the graded analogue of the adjointness of Hom and $\otimes$.

PROPOSITION 3.4. *If A and B are graded k-algebras and M is a graded A-B-bimodule, then*
$$\underline{\mathrm{Hom}}_A(M \otimes_B N, L) \cong \underline{\mathrm{Hom}}_B(N, \underline{\mathrm{Hom}}_A(M, L)); \tag{3-1}$$

PROOF. The isomorphism is implemented by the map Φ defined by $\Phi(f)(n)(m) = f(m \otimes n)$ for $n \in N$ and $m \in M$. The proof is like that in $\mathsf{Mod}(R)$, but a little extra care required to check surjectivity: if $\theta \in \underline{\mathrm{Hom}}_A(M, L)$, then $\theta = \sum \theta_i$ with θ_i of degree i, whence the map f defined by $f(m \otimes n) = \theta(n)(m)$ is a sum of homogeneous maps f_i corresponding to θ_i. □

If in addition N is a graded B-C-bimodule and L is a graded A-D-bimodule, then (3-1) is an isomorphism of graded C-D-bimodules.

Not only may we tensor two bigraded bimodules as in Lemma 3.3, but we may also tensor a bigraded bimodule with a graded module. If L is a bigraded A-B-bimodule, and N is a graded B-module, then $L \otimes_B N$ is a graded A-module with degree i component

$$(L \otimes_B N)_i = L_{(i,*)} \otimes_B N.$$

There is another graded k-module structure on $L \otimes_B N$; to see this, first forget the A-module structure on L and view it just as a B-module with degree n component $L_{(*,n)}$. Then $L \otimes_B N$ has the tensor product grading, the degree j-component being the image of $\sum_n L_{(*,n)} \otimes_k N_{j-n}$. Thus $L \otimes_B N$ has a graded left A-module structure, and a graded k-module structure, *but it is not a bigraded A-k-bimodule.* Nevertheless, we will denote the $\mathbb{Z}^2$-grading on it by $(L \otimes_B N)_{(i,j)}$, this being the image of $\sum_n L_{(i,n)} \otimes_k N_{j-n}$.

Next we consider the graded analogue of Watt's Theorem (A.12.1), which says that a right exact functor $\mathsf{Mod}(R) \to \mathsf{Mod}(S)$ is naturally equivalent to $L \otimes_R -$ for some S-R-bimodule L. If L is a bigraded A-B-bimodule, define $F_L : \mathsf{GrMod}(B) \to \mathsf{GrMod}(A)$ by

$$F_L(N) = (L \otimes_B N)_{(*,0)}$$

with the grading $F_L(N)_p = (L \otimes_B N)_{(p,0)}$.

THEOREM 3.5 (Zhang). *A right exact functor $\mathsf{GrMod}(B) \to \mathsf{GrMod}(A)$ is naturally equivalent to F_L for some bigraded A-B-bimodule L.*

PROOF. Let F be the right exact functor in question. Define

$$V := \bigoplus_{n=-\infty}^{\infty} B(n) \qquad \text{and} \qquad L := F(V).$$

Each $F(B(n))$ is a graded left A-module, and hence so is L. Define a $\mathbb{Z}^2$-grading on L by $L_{(p,q)} = F(B(q))_p$. If $b \in B_n$, define $\rho_b^i \in \mathrm{Hom}_{\mathrm{Gr}}(B(i), B(i+n))$ by $\rho_b^i(x) = xb$. Define $\rho_b := \oplus_i \rho_b^i \in \mathrm{Hom}_{\mathrm{Gr}}(V,V)$. Hence $F(\rho_b) : L \to L$ is a left A-module map sending $L_{(p,q)}$ to $L_{(p,q+n)}$. So, if $b \in B_n$ is made to act on L via $F(\rho_b)$, this gives an action of B on L making it a bigraded A-B-bimodule.

Let $N \in \mathsf{GrMod}(B)$. If $y \in N_{-q}$ define $\phi_y : B(q) \to N$ by $\phi_y(b) = by$; this is a degree zero B-module map, so there is a graded A-module map $F(\phi_y) : F(B(q)) \to F(N)$. We now define a natural transformation

$$\tau : F_L \to F$$

as follows: if $u \in F(B(q))_p$ and $y \in N_{-q}$, then $u \otimes y \in F_L(N)$, and we define

$$\tau_N(u \otimes y) := F(\phi_y)(u) \in F(N)_p;$$

thus $\tau_N : F_L(N) \to F(N)$ is a graded A-module map. If $N = B$, then

$$F_L(B)_p = (L \otimes_B B)_{(p,0)} = L_{(p,0)} = F(B(0))_p,$$

so $F_L(B) = F(B)$. Since F is right exact, taking a free presentation of a $N \in \mathsf{GrMod}(B)$, it follows that $F_L(N) = N$ also. The remaining details are as in Watt's Theorem. $\square$

EXAMPLE 3.6. Consider the identity functor on $\mathsf{GrMod}(A)$ in the context of the previous Theorem. The bigraded bimodule associated to it in the proof is $L = \oplus_n A[n]$. The action of $x \in A_i$ on an element $a \in L_{(p,q)} = A[q]_p$ is as follows: acting from the left, $x.a = xa \in A[q]_{p+i}$, whereas, acting form the right, $a.x = ax \in A[q+i]_p$.

EXAMPLE 3.7. A graded (not a bigraded) A-B-bimodule, X say, gives a right exact functor $\mathsf{GrMod}(B) \to \mathsf{GrMod}(A)$, namely $N \mapsto X \otimes_B N$. The bigraded bimodule L which is associated to this functor in the proof of Theorem 3.5 is $L = \oplus_n X[n]$, with $L_{(p,q)} = X[q]_p = X_{p+q}$. One can now explicitly compute

$$\begin{aligned}
F_L(N) &= (L \otimes_B N)_{(*,0)} \\
&= \sum_i (L \otimes_B N)_{(i,0)} \\
&= \sum_i \mathrm{Im}\Big(\sum_n L_{(i,n)} \otimes_k N_{-n}\Big) \\
&= \sum_i \mathrm{Im}\Big(\sum_n X[n]_i \otimes_k N_{-n}\Big) \\
&= \sum_i \mathrm{Im}\Big(\sum_n X_{n+i} \otimes_k N_{-n}\Big) \\
&= \sum_i \mathrm{Im}\Big(\sum_{r+s=i} X_r \otimes_k N_s\Big) \\
&= \sum_i (X \otimes_B N)_i
\end{aligned}$$

Hence $F_L(N) \cong X \otimes_B N$ as a left A-module.

The next result shows that those right exact functors $\mathsf{GrMod}(B) \to \mathsf{GrMod}(A)$ which are given by tensoring with a graded A-B bimodule, as in the previous example, may be characterized as the right exact functors which commute with the shift functors.

LEMMA 3.8. *Suppose that $F : \mathsf{GrMod}(B) \to \mathsf{GrMod}(A)$ is a right exact functor such that $F \circ [1] = [1] \circ F$. Then F is naturally equivalent to a functor of the form $X \otimes_B -$ where X is a graded A-B-bimodule.*

PROOF. Suppose that F is a functor such that $F \circ [1] = [1] \circ F$. Define $X = F(B)$. This is a graded left A-module. Let $b \in B_n$. Then right multiplication by b is a degree zero left B-module map $\rho : B \to B[n]$, so $F(\rho) : X = F(B) \to F(B[n]) = X[n]$. In particular, $F(\rho) : X_i \to X[n]_i = X_{i+n}$. So, X becomes a graded A-B-bimodule with b acting from the right as $F(\rho)$. It is now easy to show that F is equivalent to $X \otimes_B -$. □

4. Projectives in GrMod

We begin by showing that a module is projective in $\mathsf{GrMod}(A)$ if and only if it is projective in $\mathsf{Mod}(A)$; the analogous statement for injectives is false (Example 5.1). Consequently, there is a close relation between the homological properties of $M \in \mathsf{GrMod}(A)$ and the homological properties of M as an object of $\mathsf{Mod}(A)$.

NOTATION . If P is a projective left R-module, we write $P^\vee = \mathrm{Hom}_R(P, R)$. Thus $P^\vee$ is a projective right R-module, finitely generated if P is, in which case there is a natural equivalence of functors $\mathrm{Hom}_{R^{\mathrm{op}}}(P^\vee, -) \cong - \otimes_R P$.

Definition 4.1. A module $M \in \mathsf{GrMod}(A)$ is

- free if it has a basis consisting of homogeneous elements;
- projective if the functor $\mathrm{Hom}_{\mathrm{Gr}}(M, -)$ is exact.

The free A-modules are precisely the direct sums of the shifts $A[n]$, $n \in \mathbb{Z}$; remember that $A[n]$ is generated by a single element in degree $-n$. It is easy to see that a free module in $\mathsf{GrMod}(A)$ is projective, so $\mathsf{GrMod}(A)$ has enough projectives: every graded A-module is a quotient of a suitable direct sum of copies of the free module

$$P := \bigoplus_{n \in \mathbb{Z}} A[-n]. \tag{4-1}$$

More precisely, if M is generated by homogeneous elements $\{m_\lambda \mid \lambda \in \Lambda\}$, then M is a quotient of

$$\bigoplus_{\lambda \in \Lambda} A[-\deg m_\lambda].$$

In other words, if V is a graded vector subspace of M, which generates M as an A-module, and $A \otimes_k V$ is given the tensor product grading, then the multiplication map $A \otimes_k V \to M$ presents M as the image of a free graded A-module under a degree zero A-module map.

Since $\mathsf{GrMod}(A)$ has enough projectives, we can form projective resolutions in $\mathsf{GrMod}(A)$.

NOTATION . When we say 'let $P_\bullet$ be a projective resolution of M', we mean that there is a complex $\cdots \to P_1 \to P_0 \to 0$ belonging to the category under consideration, which is exact except at position zero where the homology group is M. If we need to specify the differential we will write $(P_\bullet, d_\bullet)$, where the indices are such that $d_n : P_{n+1} \to P_n$.

We emphasize that the morphisms in a projective resolution in $\mathsf{GrMod}(A)$ are degree zero maps; for example, the trivial module over $A = k[X]$ has a projective resolution $0 \to A[-1] \to A \to k \to 0$ in which the first map is multiplication by X.

Let $(P_\bullet, d_\bullet)$ be a complex of free *left* modules. We may choose bases for each P_n, view elements of P_n as *row* vectors with entries in A, and represent each d_n as *right* multiplication by a matrix with entries in A. If, instead, we have a complex of free *right modules*, the elements of each module may be viewed as *column* vectors, and the maps as *left* multiplication by matrices. For example, suppose $P_\bullet$ is a complex of free left modules with some fixed choice of bases; taking the dual bases for each $\underline{\mathrm{Hom}}_A(P_i, A)$, the matrices representing the differential in $\underline{\mathrm{Hom}}_A(P_\bullet, A)$ are the same as those representing the differential in $P_\bullet$ but now they are viewed as acting by left multiplication rather than by right multiplication.

PROPOSITION 4.2. *Let $P \in \mathsf{GrMod}(A)$. The following are equivalent:*

1. *P is projective in $\mathsf{GrMod}(A)$;*
2. *P is a direct summand of a free module in $\mathsf{GrMod}(A)$;*
3. *P is projective in $\mathsf{Mod}(A)$.*

Proof. Let $F \in \mathsf{GrMod}(A)$ be free, and let $\alpha \in \mathrm{Hom}_{\mathrm{Gr}}(F, P$ be surjective.

(1)⇒(2) By hypothesis the induced map $\alpha^* : \mathrm{Hom}_{\mathrm{Gr}}(P, F) \to \mathrm{Hom}_{\mathrm{Gr}}(P, P)$ is surjective, so there is $\varphi \in \mathrm{Hom}_{\mathrm{Gr}}(F, P)$ such that $\alpha\varphi = \mathbb{1}_P$. Thus φ is injective and $F = \ker \alpha \oplus \varphi(P)$.

(2)⇒(3) Simply forget the grading.

(3)⇒(2) By hypothesis there exists $\psi \in \mathrm{Hom}_A(P, F)$ such that $\alpha\psi = \mathbb{1}_P$; ψ may not be a graded homomorphism. Define $\varphi : P \to F$ as follows: for each $n \in \mathbb{Z}$, and each $p \in P_n$ define

$$\varphi(p) = \text{the degree } n \text{ component of } \psi(p).$$

Then $\varphi \in \mathrm{Hom}_A(P, F)$ and, for $p \in P_n$, we have

$$\begin{aligned} \alpha\varphi(p) &= \alpha(\text{degree } n \text{ component of } \psi(p)) \\ &= \text{degree } n \text{ component of } \alpha\psi(p) \\ &= \text{degree } n \text{ component of } p \\ &= p. \end{aligned}$$

Thus $\alpha\varphi = \mathbb{1}_P$.

(2)⇒(1) If the free module $F = P \oplus Q$ is a direct sum of graded submodules, then $\mathrm{Hom}_{\mathrm{Gr}}(F, -) = \mathrm{Hom}_{\mathrm{Gr}}(P, -) \oplus \mathrm{Hom}_{\mathrm{Gr}}(Q, -)$. The exactness of $\mathrm{Hom}_{\mathrm{Gr}}(P, -)$ follows from the exactness of $\mathrm{Hom}_{\mathrm{Gr}}(F, -)$. □

Definition 4.3. Let A be a graded k-algebra and $M \in \mathsf{GrMod}(A)$. Let $(P_\bullet, d_\bullet)$ be a projective resolution of M in $\mathsf{GrMod}(A)$. The j^{th} syzygy of M is the module

$$\Omega^j M := \operatorname{coker} d_j \cong \operatorname{Im}(d_{j-1} : P_j \to P_{j-1}).$$

Hence there are exact sequences $0 \to \Omega^{j+1} M \to P_j \to \Omega^j M \to 0$.

In general $\Omega^j M$ is only defined up to direct sums of projectives; however, by defining $\Omega^j M$ as an object of the stable category $\underline{\mathrm{GrMod}}(A)$, the quotient of $\mathsf{GrMod}(A)$ by the maps which factor through projectives, it becomes well-defined and one can even make Ω^j a functor.

Lemma 4.4. *Let $M, P \in \mathsf{GrMod}(A)$. Suppose that P is projective and write $P^\vee = \underline{\mathrm{Hom}}_A(P, A)$. Then*

1. *$P^\vee \otimes_A M \cong \underline{\mathrm{Hom}}_A(P, M)$ if either P is finitely generated or M is finitely presented;*
2. *$P^\vee$ is projective in $\mathsf{GrMod}(A^{\mathrm{op}})$ if P is finitely generated;*
3. *$P^\vee$ is flat in $\mathsf{GrMod}(A^{\mathrm{op}})$ if A is left noetherian.*

Proof. (1) If P is finitely generated this is standard (for example, see [**141**, Lemma 3.59]). On the other hand, if $X \to Y \to M \to 0$ is a presentation of M in $\mathsf{GrMod}(A)$ with X and Y finitely generated frees, then there is a commutative diagram

$$\begin{array}{ccccccc} P^\vee \otimes_A X & \longrightarrow & P^\vee \otimes_A Y & \longrightarrow & P^\vee \otimes_A M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \underline{\mathrm{Hom}}_A(P, X) & \longrightarrow & \underline{\mathrm{Hom}}_A(P, Y) & \longrightarrow & \underline{\mathrm{Hom}}_A(P, M) & \longrightarrow & 0 \end{array}$$

where the rows are exact because P is projective. Write $X = \coprod X_\alpha$ with each $X_\alpha \cong A[i_\alpha]$ for some i_α. Then

$$P^\vee \otimes_A X = P^\vee \otimes (\coprod_\alpha X_\alpha) \cong \coprod_\alpha (P^\vee \otimes X_\alpha) \cong \coprod_\alpha \underline{\mathrm{Hom}}(P, X_\alpha)$$

by definition of $P^\vee$. Since this coproduct is finite this is isomorphic to

$$\prod_\alpha \underline{\mathrm{Hom}}_A(P, X_\alpha) \cong \underline{\mathrm{Hom}}(P, \prod_\alpha X_\alpha) = \underline{\mathrm{Hom}}_A(P, X).$$

Hence the first two vertical maps in the commutative diagram are isomorphisms, so the last one is too.

(2) This is standard.

(3) Write P as a graded direct summand of a graded free module, F say. Then $P^\vee$ is a direct summand of $F^\vee$, so it suffices to prove that $F^\vee$ is flat. Write $F = \coprod_\alpha A[i_\alpha]$. Then

$$F^\vee = \underline{\mathrm{Hom}}_A(\coprod_\alpha A[i_\alpha], A) \cong \prod_\alpha \underline{\mathrm{Hom}}(A[i_\alpha], A) \cong \prod_\alpha A[-i_\alpha].$$

However, A is right noetherian, so this product of flat right modules is flat by [**2**, Theorem 19.20]. □

Projective generators. The module P, defined in (4-1), is a projective generator in $\mathsf{GrMod}(A)$. However, because P is not finitely generated the functor $\mathrm{Hom}_{\mathrm{Gr}}(P, -)$ does not commute with direct sums (see Exercise A.**??**.8); thus, Theorem A.11.17 may not be applied to conclude that $\mathsf{GrMod}(A)$ is equivalent to the category of right modules over $\mathrm{Hom}_{\mathrm{Gr}}(P, P)$. The problem is that P is not finitely generated—if A is $\mathbb{N}$-graded, *no* finitely generated module can be a generator: a finitely generated $P \in \mathsf{GrMod}(A)$ is generated by $P_{\le n}$ for some n, so $\mathrm{Hom}_{\mathrm{Gr}}(P, A[-n-1]) = 0$.

However, sometimes $\mathsf{GrMod}(A)$ does have a finitely generated projective generator. For example, if $A[r] \cong A$ for some $r > 0$, which happens if there is an element in A_r having a left inverse, then $P = A \oplus A[1] \oplus \cdots \oplus A[r-1]$ is a progenerator to which Theorem A.11.17 applies; that is, $\mathsf{GrMod}(A)$ is equivalent to $\mathsf{Mod}(R^{\mathrm{op}})$ where $R = \mathrm{Hom}_{\mathrm{Gr}}(P, P)$. View elements of P as row vectors on which $R \subset M_r(A)$ acts by right multiplication. Thus

$$R = \begin{pmatrix} A_0 & A_1 & \dots & A_{r-1} \\ A_{-1} & A_0 & \dots & A_{r-2} \\ \vdots & & & \vdots \\ A_{-r+1} & A_{-r+2} & \dots & A_0 \end{pmatrix}.$$

For example, if $A = k[X, X^{-1}]$ with $\deg X = r$, then $\mathsf{GrMod}(A) \cong \mathsf{Mod}(k \oplus \cdots \oplus k) = \mathsf{Mod}(k^r)$.

EXERCISES

4.1 In Proposition 4.2, one may not replace the word ‘projective’ by ‘free’. Let $A = k \oplus kx$ with $\deg x = 0$ and $x^2 = -1$. Let $P = ke_0 \oplus ke_1$ with $\deg e_i = i$, and make P into a graded A-module by $x.e_0 = e_0$ and $x.e_1 = -e_1$.
  (a) Show that P is a graded A-module, and is free in $\mathsf{Mod}(A)$.
  (b) Show that P is *not* free in $\mathsf{GrMod}(A)$.
  (c) Show that P *is* projective in $\mathsf{GrMod}(A)$ and exhibit it as a direct summand of a free graded module.

4.2 Let $(P_\bullet, d_\bullet) : \cdots \to P_n \to \cdots \to P_0 \to M \to 0$ be any complex in $\mathsf{GrMod}(A)$ with each P_i projective, and let $(\Lambda_\bullet, \partial_\bullet) : \cdots \to L_n \to \cdots \to L_0 \to N \to 0$ be any acyclic complex in $\mathsf{GrMod}(A)$. If $\varphi \in \operatorname{Hom}_{\mathrm{Gr}}(M, N)$ show that there exists a lifting of φ, say Φ, to a map of complexes $\Phi : P_\bullet \to L_\bullet$ which extends φ. If Ψ is another such lift show that there is a collection of maps $s_i : P_i \to L_{i+1}$ such that $\Phi_i - \Psi_i = \partial_{i+1} s_i + s_{i-1} d_i$ for all i.
The point is that all the maps Φ_i and s_i are required to be degree zero maps.

4.3 Let $\cdots P_{n+1} \to P_n \to P_{n-1} \to \cdots$ be a complex of projective left R-modules, and suppose that the homology groups $H_n(P_\bullet)$ are projective R-modules. Show that the homology groups of the complex $\cdots \to P_{n-1}^\vee \to P_n^\vee \to P_{n+1}^\vee \to \cdots$, obtained by applying $\operatorname{Hom}_R(-, R)$ to the original complex, are isomorphic to $H_n(P_\bullet)^\vee$.

4.4 If $C^\bullet$ is a complex of abelian groups and F is an exact functor (either covariant or contravariant), show that $F(H^i(C^\bullet)) \cong H^i(FC^\bullet)$ for all i.

5. Injectives in GrMod

Before we show that $\mathsf{GrMod}(A)$ has enough injectives notice that, in sharp contrast to projectives, injectives in $\mathsf{GrMod}(A)$ need not be injective in $\mathsf{Mod}(A)$.

EXAMPLE 5.1. Let $A = k[X, X^{-1}]$. Then $A \cong A[1]$, so A is a progenerator in $\mathsf{GrMod}(A)$. But $\operatorname{Hom}_{\mathrm{Gr}}(A, A) \cong A_0 = k$, so $\mathsf{GrMod}(A)$ is equivalent to $\mathsf{Mod}(k)$, whence every graded A-module is injective in $\mathsf{GrMod}(A)$. However, A is *not* injective in $\mathsf{Mod}(A)$.

Definition 5.2. A graded module M is an essential extension of a graded submodule N, if for each non-zero graded submodule L of M, $L \cap N \neq 0$; we also say that N is an essential submodule of M.

LEMMA 5.3. *A graded module is injective in* $\mathsf{GrMod}(A)$ *if and only if it has no proper essential extensions in* $\mathsf{GrMod}(A)$.

PROOF. Let E be a graded module which is injective in $\mathsf{GrMod}(A)$, and suppose M is an essential extension of E. Let $f : E \to M$ be the inclusion. Since $\operatorname{Hom}_{\mathrm{Gr}}(-, E)$ is right exact there is a degree zero map $g : M \to E$ such that $gf = \mathbb{1}_E$. If g were not injective, then $\ker g \cap E \neq 0$, whence the restriction of gf to E could not be the identity—this is absurd, so g is injective and hence an isomorphism.

Conversely, suppose E is a graded A-module having no proper essential extensions in $\mathsf{GrMod}(A)$. Let X be a graded submodule of a graded module Y, and let $\alpha : X \to E$. We must extend α to Y. Define

$$Z = Y \oplus E / \{(x, -\alpha(x)) \mid x \in X\},$$

and let $\beta : Y \to Z$ be the obvious map. It is clear that the natural map $E \to Z$ is injective, so we will consider E as a submodule of Z. By Zorn's Lemma, we may choose a graded submodule, W say, of Z, which is maximal subject to $E \cap W = 0$. It follows that Z/W is an essential extension of E so, by the hypothesis on E, $E \oplus W = Z$. Let $\gamma : Z \to E$ be the projection onto E with kernel W. If $x \in X$, then

$$(\gamma \circ \beta)(x) = \gamma(x, 0) = \gamma(0, \alpha(x)) = \alpha(x),$$

whence $\gamma \circ \beta : Y \to E$ is an extension of α, as required. $\square$

Definition 5.4. An injective envelope of $M \in \mathsf{GrMod}(A)$ is a graded module which is both injective in $\mathsf{GrMod}(A)$ and an essential extension of M in $\mathsf{GrMod}(A)$.

Theorem 5.5. *A graded A-module has an injective envelope in* $\mathsf{GrMod}(A)$.

Proof. Let $N \in \mathsf{GrMod}(A)$, and let I be an injective envelope of N in $\mathsf{Mod}(A)$. Let $\mathcal{S}$ denote the set of submodules of I which contain N, and may be endowed with a grading extending that on N. Define $M \le M'$ if $M \subset M'$ and the gradings are compatible. By Zorn's Lemma, $\mathcal{S}$ contains a maximal member, E say.

Suppose E is not injective in $\mathsf{GrMod}(A)$. By Lemma 5.3, there is a proper essential extension $E \subset E'$ in $\mathsf{GrMod}(A)$. Since I is injective there is an A-module map $f : E' \to I$ extending the inclusion $E \subset I$. We will show that f is injective.

Suppose f is not injective. Let $0 \neq x = x_i + \cdots + x_j \in \ker f$, where $x_i \in E'_i$ and $x_i \neq 0$. We will show by induction on $j - i$ that there is a homogeneous $a \in A$ such that $0 \neq ax \in E$. If $j - i = 0$, x is homogeneous, so the existence of a follows from the fact that E' is an essential extension of E as *graded* modules. Suppose $j-i > 0$. Choose a homogeneous $c \in A$ such that $0 \neq cx_i \in E$. If $c(x-x_i) = 0$, then $cx = cx_i$, and we set $a = c$. Otherwise, by the induction hypothesis on $j - i$, there exists a homogeneous $b \in A$ such that $0 \neq bc(x - x_i) \in E$, whence $0 \neq bcx \in E$; set $a = bc$. Now $f(ax) = 0$ which contradicts the fact that $f|_E$ is injective, so f is injective.

Since f is injective, $f(E')$ may be given the grading $f(E')_i = f(E'_i)$, whence $E \le f(E')$. This contradicts the maximality of E, so E must be injective. □

An injective envelope is unique up to isomorphism, so we will usually refer to *the* injective envelope.

Example 5.6. If $A = k[X]$, the injective envelope of A is $E = k[X, X^{-1}]$. Obviously A is an essential submodule of E, so we need only show that E is injective in $\mathsf{GrMod}(A)$. If N is a graded submodule of M and $f \in \mathrm{Hom}_{\mathrm{Gr}}(N, E)$, then f extends to M: for a homogeneous $m \in M$, define

$$g(m) := \begin{cases} 0 & \text{if } Am \cap N = 0, \\ X^{-i} f(X^i m) & \text{if } X^i m \in N. \end{cases}$$

Then g is a well-defined element of $\mathrm{Hom}_{\mathrm{Gr}}(M, E)$ whose restriction to N agrees with f. Hence E is injective.

Definition 5.7. An injective resolution of a module M is an exact sequence of the form $0 \to M \to E^0 \xrightarrow{d} E^1 \to \cdots$ in which each E^i is injective. The resolution is minimal if

- E^0 is an injective envelope of M, and
- for each $j \ge 1$, E^j is an injective envelope of dE^{j-1}.

The injective dimension of M, denoted $\operatorname{injdim} M$, is is the smallest integer n such that M has an injective resolution $0 \to M \to E^0 \to \cdots \to E^n \to 0$. If no such n exists, we say that M has infinite injective dimension.

The i^{th} cosyzygy of M is $\Omega_i M := \ker(E^i \to E^{i+1})$.

Every graded A-module has a minimal injective resolution in $\mathsf{GrMod}(A)$.

It is easy to show that

$$\operatorname{injdim} M = \sup\{j \mid \mathrm{Ext}^i_R(N, M) = 0 \text{ for all } i > j \text{ and all } N \in \mathsf{Mod}(R)\}.$$

The next result is a further illustration of the fact that properties of injectives in $\mathsf{GrMod}(A)$ are similar to properties of injectives in $\mathsf{Mod}(A)$.

THEOREM 5.8 (Baer's Criterion). *An A-module E is injective in $\mathsf{GrMod}(A)$ if and only if the restriction map $\mathrm{Hom}_{\mathrm{Gr}}(A, E) \to \mathrm{Hom}_{\mathrm{Gr}}(J, E)$ is surjective for all graded left ideals J.*

PROOF. The proof is similar to the ungraded case; for example, see [**141**, Theorem 3.20]. $\square$

EXERCISES

5.1 Show that the injective envelope of $M \in \mathsf{GrMod}(A)$ is unique up to isomorphism.

5.2 Show that the injective envelope of a locally finite module need not be locally finite. For example, find an essential extension of the polynomial ring $A = k[X, Y]$ which is not locally finite.

6. Ext and Tor in GrMod

Let $M \in \mathsf{GrMod}(A)$ and $N \in \mathsf{GrMod}(A^{\mathrm{op}})$. Then the functors $- \otimes_A M$ and $N \otimes_A -$ sending graded A-modules to graded vector spaces are right exact, and their left derived functors may be defined. Thus we obtain bifunctors $\mathrm{Tor}_n^A(-, -)$ having the usual properties. The Tor groups may be computed from these in the usual way by taking a flat resolution of either module.

Let $N \in \mathsf{GrMod}(A)$. Since $\mathsf{GrMod}(A)$ has enough injectives, we may define the functors $\mathrm{Ext}^i_{\mathrm{Gr}}(N, -)$ as the right derived functors of the left exact functor $\mathrm{Hom}_{\mathrm{Gr}}(N, -)$, and they may be computed by using an injective resolution in $\mathsf{GrMod}(A)$ of the argument. The general theory of derived functors gives a long exact sequence when $\mathrm{Ext}^\bullet_{\mathrm{Gr}}(N, -)$ is applied to a short exact sequence.

We will write

$$\mathrm{Ext}^q_{\mathrm{Gr}}(N, M), \quad \mathrm{Ext}^q_A(N, M), \quad \underline{\mathrm{Ext}}^q_A(N, M), \quad \mathrm{Ext}^q_A(N, M)_d$$

for the right derived functors of

$$\mathrm{Hom}_{\mathrm{Gr}}(N, M), \quad \mathrm{Hom}_A(N, M), \quad \underline{\mathrm{Hom}}_A(N, M), \quad \mathrm{Hom}_A(N, M)_d.$$

These Ext-groups have all the usual properties.

Keeping track of the grading on $\mathrm{Ext}^*_A(M, N)$ requires careful bookkeeping. Elements of $\mathrm{Ext}^1_{\mathrm{Gr}}(N[-n], M)$ give extensions

$$0 \to M \to X \to N[-n] \to 0$$

in $\mathsf{GrMod}(A)$, and there are equalities:

$$\mathrm{Ext}^j_{\mathrm{Gr}}(N[-n], M) = \mathrm{Ext}^j_A(N[-n], M)_0 = \mathrm{Ext}^j_A(N, M)[n]_0 = \mathrm{Ext}^j_A(N, M)_n$$

so, for example, an element in $\mathrm{Ext}^1_A(k, M)_n$ corresponds to an extension which differs from M only in degree n.

We may also compute $\mathrm{Ext}^\bullet_{\mathrm{Gr}}(N, M)$ by taking a graded projective resolution of N. By Proposition 4.2, a projective module in $\mathsf{GrMod}(A)$ is projective in $\mathsf{Mod}(A)$, so a projective resolution in $\mathsf{GrMod}(A)$ is also a projective resolution in $\mathsf{Mod}(A)$. Under reasonable hypotheses this implies that $\underline{\mathrm{Ext}}^\bullet_A(N, M)$ coincides with $\mathrm{Ext}^\bullet_A(N, M)$.

PROPOSITION 6.1. *Let A be graded, and $M, N \in \mathsf{GrMod}(A)$. If N has a resolution by finitely generated projective modules, then*

$$\underline{\mathrm{Ext}}^q_A(N, M) \cong \mathrm{Ext}^q_A(N, M)$$

for all $q \geq 0$.

PROOF. Let $P_\bullet \to N$ be a projective resolution with each P_i finitely generated. Then

$$\underline{\mathrm{Ext}}^q_A(N, M) = h^q(\underline{\mathrm{Hom}}_A(P_\bullet, M)) = h^q(\mathrm{Hom}_A(P_\bullet, M))$$

where the last equality is given by Proposition 11.2.2. But the last term computes $\mathrm{Ext}^q_A(N, M)$, proving the result. □

If A, B and D are graded k-algebras, and ${}_AN_B$ and ${}_AM_D$ are graded bimodules, then $\underline{\mathrm{Ext}}^i_A(N, M)$ obtains a graded B-D-bimodule structure as follows.

Let $0 \to M \to E^0 \xrightarrow{d} E^1 \to \cdots$ be an injective resolution of A in $\mathsf{GrMod}(A)$. For each i, $\underline{\mathrm{Hom}}_A(N, E^i)$ has a graded left B-module structure via $(b.f)(x) = f(xb)$ (see also Exercise 1.7). (Notice that $\mathrm{Hom}_{\mathrm{Gr}}(N, E^i)$ is *not* a B-module.) Moreover, the differential $d^* : \underline{\mathrm{Hom}}_A(N, E^i) \to \underline{\mathrm{Hom}}_A(N, E^{i+1})$ is a degree zero homomorphism of graded B-modules; i.e., $\underline{\mathrm{Hom}}_A(N, E^\bullet)$ is a complex of graded left B-modules, whence so is its homology. The D-module structure is obtained in a similar way by taking a projective resolution of N.

Moreover, if $0 \to N' \to N \to N'' \to 0$ is an exact sequence of A-B-bimodules, then the long exact sequence obtained by applying the functor $\underline{\mathrm{Ext}}^\bullet_A(-, M)$ is a sequence of graded B-D-bimodules.

The next result says, roughly speaking, that $\underline{\mathrm{Ext}}^i_A(N, M)$ inherits properties from M when N is finitely generated.

PROPOSITION 6.2. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and let $M \in \mathsf{GrMod}(A)$ and $N \in \mathsf{grmod}(A)$.*

1. *If M is left (respectively, right) bounded, so is $\underline{\mathrm{Ext}}^i_A(N, M)$.*
2. *If M is locally finite, so is $\underline{\mathrm{Ext}}^i_A(N, M)$.*
3. *If B is a right noetherian graded k-algebra, and M is a graded A-B-bimodule such that $M \in \mathsf{grmod}(B^{\mathrm{op}})$, then $\underline{\mathrm{Ext}}^i_A(N, M)$ is a finitely generated right B-module.*

PROOF. Let $P_\bullet \to N$ be a free resolution. By the noetherian hypothesis, we may assume that each P_i is finitely generated since N is, so is a finite direct sum of shifts of A. Since $\underline{\mathrm{Hom}}_A(A, M) \cong M$, each $\underline{\mathrm{Hom}}_A(P_i, M)$ is a finite direct sum of shifts of M. But $\underline{\mathrm{Ext}}^i_A(N, M)$ is a subquotient of $\underline{\mathrm{Hom}}_A(P_i, M)$, so has the stated properties because M does. □

7. Comparison of graded and ungraded homological properties.

A graded module can be considered as an object in either $\mathsf{Mod}(A)$ or in $\mathsf{GrMod}(A)$. Standard filtered/graded methods allow a comparison of homological properties in $\mathsf{Mod}(A)$ and $\mathsf{GrMod}(A)$.

Definition 7.1. The projective dimension of a module $M \in \mathsf{GrMod}(A)$ is the least length of a projective resolution of it in $\mathsf{GrMod}(A)$.

PROPOSITION 7.2. *If $M \in \mathsf{GrMod}(A)$, then its projective dimension as an object of $\mathsf{GrMod}(A)$ is the same as its projective dimension as an object of $\mathsf{Mod}(A)$.*

PROOF. Suppose the projective dimension of M in $\mathsf{Mod}(A)$ is n. If $Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0$ is an exact sequence in $\mathsf{GrMod}(A)$ with each Q_i projective, then the kernel of the leftmost map is projective in $\mathsf{Mod}(A)$, and is in $\mathsf{GrMod}(A)$, so is projective in $\mathsf{GrMod}(A)$. Hence the projective dimension of M in

$\mathsf{GrMod}(A)$ is at most n. The reverse inequality is clear, since a projective resolution of M in $\mathsf{GrMod}(A)$ is a projective resolution in $\mathsf{Mod}(A)$. $\square$

Definition 7.3. The **graded global dimension** of a graded ring A is

$$\text{gr-gldim}\, A := \sup\{\text{pdim}\, M \mid M \in \mathsf{GrMod}(A)\}.$$

The global dimension of a graded ring may differ from its graded global dimension. For example, $\text{gldim}\, k[x, x^{-1}] = 1$, but $\mathsf{GrMod}(k[x, x^{-1}])$ is equivalent to $\mathsf{Mod}(k)$, so $\text{gr-gldim}\, k[x, x^{-1}] = 0$.

LEMMA 7.4. *Let A be a graded k-algebra and give the polynomial extension $A[x]$ the tensor product grading with* $\deg x = 1$. *Let $f : A[x] \to A$ be the algebra map defined by $f(x) = 1$. Then the functor $f^* : \mathsf{GrMod}(A[x]) \to \mathsf{Mod}(A)$ defined by $A \otimes_{A[x]} -$ is exact.*

PROOF. The functor sends M to $M/(x-1)M$. Certainly f^* is right exact, so we only need to show that if $N \subset M$ are graded $A[x]$-modules, then $(x-1)N = N \cap (x-1)M$. Suppose that $(x-1)m \in N \cap (x-1)M$, and write $m = m_n + m'$, where $0 \neq m_n \in M_n$ and $m' \in M_{>n}$. Thus $(x-1)m = xm_n - m_n + xm' - m' \in N$; the degree n component in this expression is $-m_n$ so, as N is graded, $m_n \in N$; it follows that $(x-1)m' \in N$ also; by an induction argument on the number of homogeneous components in m, the case $m = 0$ being trivial, we may assume that $(x-1)m' \in (x-1)N$. Hence $(x-1)m \in (x-1)N$. The reverse inclusion is trivial, so we obtain the result. $\square$

The functor $f^* : \mathsf{Mod}(A[x]) \to \mathsf{Mod}(A)$ is not exact, so the previous result is rather surprising.

PROPOSITION 7.5. *If A is a graded ring, then*

$$\text{gldim}\, A - 1 \leq \text{gr-gldim}\, A \leq \text{gldim}\, A.$$

PROOF. Let x be an indeterminate, and consider the polynomial extension $A[x]$ and the map $f : A[x] \to A$ as in Lemma 7.4. Let $L \in \mathsf{Mod}(A)$. Define a grading on $M := L \otimes_k k[x]$ by concentrating L in degree zero, and taking $\deg x = 1$. Now make M a graded $\mathsf{GrMod}(A[x])$-module by defining $(a \otimes x^j).(\ell \otimes x^n) = a\ell \otimes x^{i+j+n}$ if $a \in A_i$. Then $L \cong f^*M$. If $P_\bullet \to M$ is a projective resolution of M in $\mathsf{GrMod}(A[x])$ then $f^*P_\bullet \to f^*M$ is a projective resolution of L in $\mathsf{Mod}(A)$, so $\text{pdim}_A L$ is no larger than the length of a projective resolution of M in $\mathsf{GrMod}(A[x])$. Thus $\text{gldim}\, A \leq \text{gr-gldim}\, A[x]$. But $\text{gr-gldim}\, A[x] = \text{gr-gldim}\, A + 1$.

The second inequality follows from the fact that the projective dimension of a module in $\mathsf{GrMod}(A)$ is the same as its projective dimension in $\mathsf{Mod}(R)$. $\square$

THEOREM 7.6. *If A is $\mathbb{N}$-graded, then* $\text{gldim}\, A = \text{gr-gldim}\, A$.

PROOF. The proof is postponed until after the standard filtered/graded methods have been developed in Chapter 4 (see Theorem 4.3.17). $\square$

THEOREM 7.7 (Auslander's Theorem). [**141**, Theorem 9.12] *If A is a graded ring, then*

$$\text{gr-gldim}\, A = \sup\{\text{pdim}(A/I) \mid I \textit{ is a graded left ideal}\}.$$

8. Noetherian properties

Most algebras of interest to us are noetherian. The next result shows that one need only verify that graded left ideals are finitely generated in order to prove that an $\mathbb{N}$-graded algebra is noetherian.

PROPOSITION 8.1. *A graded module which is bounded below is noetherian if and only if every graded submodule is finitely generated.*

PROOF. Let M be a graded A-module which is bounded below, and suppose that every graded submodule is finitely generated. Let N be an arbitrary submodule. Then N determines a graded submodule L as follows. For each $n \in \mathbb{Z}$, define

$$L_n := \{x_n \mid \text{there exists } x \in N \cap M_{\leq n} \text{ with degree } n \text{ component } x_n\}.$$

In other words, L_n consists of the leading terms of the degree n elements in N. Since N is a submodule, it follows that $L := \oplus L_n$ is a graded submodule.

The map $N \mapsto L$ from arbitrary submodules to graded submodules is *not* generally injective but, if N is a proper submodule of N', then L is a proper submodule of L'. To see this, let n be minimal such that there exists

$$y \in N' \cap M_{\leq n} \qquad \text{but} \qquad y \notin N \cap M_{\leq n}.$$

Thus $y_n \in L'$ but, if $y_n \in L$, there exists $x \in N$ with $x_n = y_n$. Therefore $y - x \in N' \cap M_{<n}$; by choice of n, it follows that $y - x \in N$, whence $y \in N$. This contradicts the choice of y, so we conclude that $L \neq L'$.

An ascending chain of submodules $N \subset N' \subset N'' \subset \dots$ gives rise to an ascending chain of graded submodules $L \subset L' \subset L'' \subset \dots$ which stabilizes, by hypothesis. Hence, by the previous paragraph, the chain $N \subset N' \subset \dots$ stabilizes, so M is noetherian.

The reverse implication is clear. □

There is no analogue of this result for artinian modules: for example, $A = k[x, x^{-1}]$ is artinian, indeed irreducible, in $\mathsf{GrMod}(A)$ but is not artinian in $\mathsf{Mod}(A)$.

Definition 8.2. An element z in a ring R is normalizing if $zR = Rz$.

PROPOSITION 8.3. *Let A be an $\mathbb{N}$-graded ring, $z \in A$ a homogeneous normalizing element of positive degree, and M a graded A-module which is bounded below. Then M is noetherian if and only if M/zM is.*

PROOF. First observe that although the map $\theta : M \to M$ defined by $\theta(m) = zm$ is not necessarily a left A-module homomorphism, $\ker \theta$ and $\operatorname{Im} \theta$ are both A-submodules of M. Furthermore, θ induces a bijection between submodules of $M/\ker \theta$ and submodules of M containing $\ker \theta$.

Suppose the result is false. Using Zorn's Lemma, we may choose a graded submodule N which is maximal with respect to the property that it is not finitely generated. Thus M/N is a noetherian A-module. Let $\theta : M/N \to M/N$ be 'left multiplication by z' and write $K = \ker \theta$. Define maps $\psi : K \to N/zN$ by $\psi(x + N) = zx + zN$, and $\chi : N/zN \to M/zM$ by $\chi(y + zN) = y + zM$. It is clear that ψ sends submodules to submodules, and that χ is an A-module map. The sequence

$$K \xrightarrow{\psi} N/zN \xrightarrow{\chi} M/zM$$

is exact. Since M/zM is noetherian, so is $\operatorname{Im}\chi$. Since M/N is a noetherian A-module, so are K and $K/\ker\psi$. Since the lattices of A-submodules of $K/\ker\psi$ and $\operatorname{Im}\psi$ are isomorphic via ψ, $\operatorname{Im}\psi$ is also noetherian. Therefore, since $\ker\chi = \operatorname{Im}\psi$ and $(N/zN)/\ker\chi \cong \operatorname{Im}\chi$, we conclude that N/zN is noetherian.

Thus N contains a finitely generated graded submodule, L say, such that $N = L + zN$. By induction, $N = L + z^n N$ for all $n \geq 0$. Since $\deg z \geq 1$ and N is bounded below it follows that $N_{n-1} = L_{n-1}$ for all $n \in \mathbb{Z}$. In other words, $N = L$ is finitely generated. Now apply Proposition 8.1. □

Corollary 8.4. *Let A be an $\mathbb{N}$-graded k-algebra and $z \in A$ a homogeneous normalizing element of positive degree. Then A is noetherian if and only if $A/(z)$ is.*

Example 8.5. Here's how Corollary 8.4 can be used to show that the algebra $A = k[x, y]$ with defining relation $yx = \lambda xy$, where $0 \neq \lambda \in k$, is left and right noetherian. Let $w \in A$ be a word in x and y. By repeatedly replacing yx by λxy, we may show that $yw \in Ay$. By repeatedly replacing xy by $\lambda^{-1}yx$, we may show that $wy \in yA$. Hence $yA = Ay$; that is, y is normalizing. Since $A/(y) \cong k[x]$, the polynomial ring in one indeterminate, Corollary 8.4 implies that A is right and left noetherian. If $\lambda = 0$, then A is neither right nor left noetherian (see Exercise 2 below).

EXERCISES

8.1 Let M be a graded module. Show that the following are equivalent:
  (a) every graded submodule of M is finitely generated;
  (b) every collection of graded submodules of M contains maximal members;
  (c) every ascending chain of graded submodules of M is eventually constant.

8.2 Let $A = k[x, y]$ with defining relation $yx = 0$. Show that A is neither left nor right noetherian by showing that
  (a) $Ax + Axy + Axy^2 + \cdots$ is not finitely generated, and
  (b) $yA + xyA + x^2yA + \cdots$ is not finitely generated.
  [*Hint:* it suffices to show that the chain of left ideals $Ax \subset Ax + Ayx \subset \ldots$ is strictly increasing, and similarly for (b).]

8.3 Let $A = k[x, y]$ with defining relations $yx = 0$ and $y^2 = 0$. Show that A is left noetherian but not right noetherian. [*Hint:* For the first part consider A as a left $k[x]$-module.]

8.4 Paul, is there a version of Prop 8.3 when A is $\mathbb{Z}$-graded and $\bigcap_{n\in\mathbb{Z}} A_n = 0$?

9. Hilbert series

The Hilbert series of a graded vector space is a bookkeeping device: it is a generating function for the dimensions of the homogenous components. Despite its simplicity it is a remarkably useful tool.

Definition 9.1. Let M be a locally finite graded k-vector space. The Hilbert series of M is the formal series

$$H_M(t) := \sum_{i=-\infty}^{\infty} \dim(M_i)t^i.$$

We will implicitly assume that graded vector spaces are locally finite whenever we refer to their Hilbert series.

Shifting degree affects the Hilbert series as follows: $H_{M[d]}(t) = t^{-d}H_M(t)$.

LEMMA 9.2. *All graded vector spaces in this lemma are assumed to be locally finite.*

1. *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence of graded vector spaces and degree 0 maps, then* $H_Y = H_X + H_Z$.
2. *Let* $\cdots \to X^{r-1} \to X^r \to X^{r+1} \to \cdots$ *be a complex of graded vector spaces, and write* Y_i *for the homology at the* i^{th} *position. Then* $\sum_i (-1)^i H_{X^i}(t) = \sum_i (-1)^i H_{Y^i}(t)$.
3. *If* X *and* Y *are graded vector spaces and* $X \otimes_k Y$ *is given the tensor product grading, then* $H_{X \otimes Y} = H_X . H_Y$.

PROOF. (1) For each integer n, there is an exact sequence of the degree n components of X, Y and Z, so $\dim Y_n = \dim X_n + \dim Z_n$. Multiplying this by t^n and summing over all n yields the result.

(2) This is clear.

(3) By definition $(X \otimes Y)_n = \sum_i X_i \otimes Y_{n-i}$. Since the sum on the right is direct, we have $\dim(X \otimes Y)_n = \sum_i \dim X_i \dim Y_{n-i}$. Multiplying this by t^n and summing over all n yields the result. □

The statement in (2) does not make sense unless the complex is assumed to have only a finite number of non-zero terms.

EXAMPLE 9.3. Let $A = k[X_1, \ldots, X_n]$ be the commutative polynomial ring on n indeterminates, with grading defined by $\deg X_i = 1$ for all i. Since

$$k[X_1, \ldots, X_n] \cong k[X_1] \otimes_k \ldots \otimes_k k[X_n]$$

is a tensor product of polynomial rings on a single indeterminate, Lemma 9.2 implies that $H_A(t)$ is the n^{th} power of $H_{k[X]}(t)$. Since $\dim(k[X]_i) = 1$ for all $i \geq 0$ we have

$$H_{k[X]}(t) = 1 + t + t^2 + \cdots = (1-t)^{-1}$$

whence

$$H_A(t) = (1-t)^{-n}.$$

The coefficients in this power series may be computed by differentiating the power series expansion of $(1-t)^{-1}$ a total of $n-1$ times, giving

$$\dim(k[X_1, \ldots, X_n]_r) = \binom{n+r-1}{n-1}.$$

See Exercise 1 below for an alternative derivation of these dimensions.

EXAMPLE 9.4. The Hilbert series of the free algebra on d letters of degree 1 is $(1-dt)^{-1}$ since $\dim_k (k^d)^{\otimes n} = d^n$. The Hilbert series of the exterior algebra on d letters of degree 1 is $(1+t)^d$ (see Chapter 1, Lemma 4.2). (See Exercise 2 for the Hilbert series of the free algebra and exterior algebra when the generators are not in degree one.)

EXERCISES

9.1 Let $A = k[X_1, \dots, X_n]$ be the polynomial ring with its usual grading. Use the following method to prove $\dim_k(A_r) = \binom{n+r-1}{n-1}$. Show that the words of length r, in the letters $X_1, \dots, X_n$, are in bijection with sequences of length $r + n - 1$ consisting of r zeroes and $n - 1$ stars $*$; such a sequence is completely determined by the position of the $*$s. Given such a sequence, the corresponding word is constructed as follows: read the sequence of from left to right; a 0 means write the same letter as last time, $*$ means stop writing the letter you just wrote, and proceed to the next letter; if the term in the sequence is 0 write X_1. For example, in $k[X_1, \dots, X_5]_6$, the sequence $00 * *0 * 000*$ corresponds to the word $X_1X_1X_3X_4X_4X_4 = X_1^2X_3X_4^3$, and the sequence $* * 0000 * *00$ corresponds to $X_3^4X_5^2$.

9.2 Let V be a locally finite graded k-vector space and write $f(t) = H_V(t)$. Show that $H_{T(V)}(t) = (1 - f(t))^{-1}$ and that $H_{\Lambda(V)}(t) = \prod_i (1 + d_i t)$ where $d_i = \dim(V_i)$.

9.3 What is the Hilbert series of $k[x, y]$ with defining relation $x^2 = 0$?

9.4 Let A be a finite dimensional graded algebra. Show that the Hilbert series of $A \otimes_k k[X, X^{-1}]$ is periodic, where $\deg X = d$.

9.5 Let $P \in \mathsf{grmod}(A)$ be free. Let $P^\vee = \underline{\mathrm{Hom}}_A(P, A)$. If $H_P(t) = f(t)H_A(t)$, show that $H_{P^\vee}(t) = f(t^{-1})H_A(t)$.

CHAPTER 12

Examples of graded algebras

Section 2 introduces the Veronese subalgebras of a graded algebra. These correspond to the Veronese embeddings (or d-uple embeddings) of a variety in various projective spaces. This notion reappears later in this book since the category $\mathsf{Tails}A$ is unchanged if A is replaced by one of its Veronese subalgebras (provided that A is generated in degree one).

Under reasonable conditions, a suitably high Veronese subalgebra is generated in degree 1 and has its defining relations in degree 2. This leads to the notion of a quadratic algebra, and such algebras are examined in section 3. Almost all the algebras in this Chapter are quadratic algebras; indeed, many important algebras are quadratic.

1. Graded Öre extensions

Proposition 1.1. *Let $S = R[x, x^{-1}; \sigma]$ be a skew Laurent extension of a domain. Suppose that $\sigma^n = 1$, and that σ^i is not an inner automorphism of R whenever $0 < i < n$. Then*

$$Z(S) = Z(R)^\sigma[x^n, x^{-n}].$$

Proof. Since R^σ commutes with x, and x^n commutes with R, $Z(R)^\sigma[x^n, x^{-n}]$ is central. By Proposition 11.1.7, $Z(S)$ is graded. Suppose that $z = ax^i$ is central, with $a \in R$. For every $r \in R$, $rz = zr$, whence $ra = ar^{\sigma^i}$; therefore n divides i, which implies that x^i is central, and hence that a is central. But $ax = xa$ implies that $a = a^\sigma$, whence $a \in Z(R)^\sigma$. □

Example 1.2. If an Öre extension $R[x; \sigma, \delta]$ is a domain, then R is a domain. However, it is possible for $R[x; \sigma, \delta]$ to be a prime ring even though R is not. For example, let $R = k \oplus k$, and let $\sigma \in \operatorname{Aut} R$ switch the two simple modules. Then $R[x; \sigma]$ is prime, although R is not. To see that $R[x; \sigma]$ is prime, first we present it as $k[t, x]$ where $t^2 = 1$, and $xt = -tx$. We will show it is prime by invoking Goldie's theorem. First we embed $k[x, t]$ in $M_2(k[u])$ via

$$t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad x = \tfrac{1}{2}\begin{pmatrix} u+1 & u-1 \\ 1-u & -1-u \end{pmatrix}.$$

It is easy to check that this is an embedding and that $x^2 = \big(u \quad 0//0 \quad u\big) = u$. Thus the ring of fractions of $k[x, t]$ contains $k(u)$; however, as is easily checked, $1, x, t, xt$ are linearly independent over $k(u)$, so $\operatorname{Fract} k[x, t] = M_2(k(u))$; it follows from Goldie's theorem 2.3.10 that $k[x, t]$ is prime.

A graded version of the previous result is given by the ring $k[x, y, z]$ with relations

$$zx - xz = zy - yz = xy + yx = y^2 - z^2 = 0.$$

This can be presented as an Öre extension $k[y,z][x;\sigma]$. It is the homogenized version of Example **??**.

EXAMPLE 1.3. We consider some graded algebras which arise as Öre extensions of the polynomial ring $k[y,z]$.

Fix a non-zero $\lambda \in k$. Define $\sigma \in \operatorname{Aut} k[y,z]$ by $y^\sigma = \lambda y$ and $z^\sigma = z$. Any linear map $\delta : ky + kz \to ky + kz$ such that $\delta(z) = 0$ extends uniquely to a σ-derivation of $k[y,z]$ because

$$\delta(y)z + y^\sigma \delta(z) = \delta(z)y + z^\sigma \delta(y).$$

Hence we may define the Öre extension $A = k[y,z][x;\sigma,\delta]$. It has defining relations

$$yz = zy, \; xz = zx, \; xy = \lambda yx + \delta(y).$$

In particular, z is central. Now we add the additional constraint that $\delta(y) = z$. It follows from the third relation above that A is generated by just x and y, subject to the two relations which are obtained by substituting $xy - \lambda yx$ for z in the first two relations. Simplifying, these relations are

$$xy^2 - (\lambda+1)yxy + \lambda y^2 x = x^2 y - (\lambda+1)xyx + \lambda yx^2 = 0.$$

Hence, if x and y are given degree one (which forces $\deg z = 2$), A is defined by two homogeneous cubic relations.

Two special cases should be noticed. If $\lambda = 1$, then A is isomorphic to the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. If $\lambda = -1$, then A is isomorphic to the second generic Clifford algebra (which is defined and studied in section 5.

Iterated Öre extensions provide us with a source of non-commutative examples on which to test the theory being developed in this book. In particular, consider an iterated Öre extension

$$S = k[x_1][x_2;\sigma_2,\delta_2]\cdots[x_n;\sigma_n,\delta_n]$$

as in Proposition 1.14. In order for S to be a graded algebra, with $\deg x_i = 1$ for all i, each of the relations

$$x_j x_i = x_i^{\sigma_j} x_j + \delta_j(x_i) \qquad (1 \le i < j \le n)$$

must be homogenous. Since $\delta_j(x_i) \in k[x_1,\dots,x_{j-1}]$ it cannot involve any x_j terms; thus S is graded if and only if, for all $1 \le i < j \le n$, $x_i^{\sigma_j}$ is homogeneous of degree one and $\delta_j(x_i)$ is homogeneous of degree two. This leaves an enormous number of possibilties, some of which will be considered later.

EXAMPLE 1.4. Let $\alpha_{ij} \in k$ be arbitrary scalars satisfying

$$\alpha_{ij} = \alpha_{ji}^{-1} \qquad \text{and} \qquad \alpha_{ii} = 1$$

for all i and j. Then the algebra $A = k[x_1,\dots,x_n]$ with defining relations

$$x_j x_i = \alpha_{ij} x_i x_j,$$

for all i and j, is an iterated Öre extension (proof by induction on n—see Exercise 5). It is called a quasi-polynomial ring.

The following special case is important—in the theory of quantum groups, it plays the role of the natural homogenous space for quantum $\mathrm{GL}(n)$.

Definition 1.5. Fix $0 \neq q \in k$. The coordinate ring of quantum affine n-space is the algebra $A = k[x_1, \dots, x_n]$, with defining relations

$$x_j x_i = q x_i x_j \qquad (1 \le i < j \le n).$$

EXAMPLE 1.6. The coordinate rings of the quantum and Jordan planes are of global dimension 2. For the quantum plane $A = k[x, y]$ with $yx = qxy$, the trivial module ${}_A k = A/Ax + Ay$ has minimal resolution

$$0 \to A[-2] \xrightarrow{(y,-qx)} A[-1]^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

Here we are writing elements of $A[-1]^2$ as row vectors, and the maps are to be viewed as right multiplication by the given matrices. To see this is exact, first note that it is a complex because $(y, -qx)\binom{x}{y} = 0$, then do a simple calculation to show that $\ker(\binom{x}{y})$ is cyclic, and finally observe that $\ker((y, -qx)) = 0$ because A is a domain.

A similar argument shows that, for the Jordan plane $(yx - xy = x^2)$, the trivial module has minimal resolution

$$0 \to A[-2] \xrightarrow{(y-x,x)} A[-1]^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

The next lemma lists the simplest non-commutative analogues of a polynomial ring—all except the last of them are iterated Öre extensions.

LEMMA 1.7. *Let k be an algebraically closed field, and let A be a connected k-algebra. Suppose that A is generated by two homogeneous elements of degrees a and b, with $a \le b$. If the generators satisfy a single relation of degree $a + b$, then there is a choice of generators $x \in A_a$ and $y \in A_b$ such that the relation is one of the following:*

- *$yx - qxy$ for some $0 \neq q \in k$;*
- *$yx - xy - x^{(a+b)/a}$;*
- *yx;*
- *$x^{(a+b)/a}$.*

The second and fourth cases can only occur if a divides b.

PROOF. First, suppose that $a = b$. Choose any generators $x \in A_a$ and $y \in A_b$. The relation may be written as

$$\alpha x^2 + \beta xy + \gamma yx + \delta y^2,$$

where $\alpha, \beta, \gamma, \delta \in k$. If $\lambda \in k$, then A is generated by $x - \lambda y$ and y so we may replace x by the $x - \lambda y$. Doing this, the relation becomes

$$\alpha x^2 + (\beta + \delta\lambda)xy + (\gamma + \delta\lambda)yx + (\alpha\lambda^2 + \beta\lambda + \gamma\lambda + \delta)y^2.$$

If $\alpha \neq 0$, then we may choose λ such that the relation becomes, after relabelling, $\alpha x^2 + \beta xy + \gamma yx$. If $\alpha = 0$, we may interchange x and y, once again getting a relation of the form $\alpha x^2 + \beta xy + \gamma yx$. Now, replacing y by $y - \mu x$, the relation becomes

$$(\alpha + (\beta + \gamma)\mu)x^2 + \beta xy + \gamma yx.$$

If $\beta + \gamma \neq 0$, then we may choose μ so the relation becomes $\beta xy + \gamma yx$. If $\beta + \gamma = 0$, then the relation is of the form $\alpha x^2 + \beta(xy - yx)$. Hence, if x and y are replaced by appropriate scalar multiples, the relation can be put in the form claimed.

Now suppose that $a < b$. Choosing $x \in A_a$ and $y \in A_b$, the relation is of the form
$$\alpha x^j + \beta xy + \gamma yx,$$
where $\alpha, \beta, \gamma \in k$, $j = (a+b)/a$, and $\alpha = 0$ if a does not divide b. Replacing y by $y - \lambda x^j$ the relation becomes
$$(\alpha + (\beta + \gamma)\mu)x^j + \beta xy + \gamma yx.$$
Now proceed as at the end of the previous paragraph. □

The algebras in the previous lemma having both generators in degree one are named as follows:

Relation	Name
$yx - qxy \quad (0 \neq q \in k)$	quantum plane
$xy - yx - y^2$	Jordan plane
yx	degenerate plane
x^2	Fibonnaci algebra

As noted after Definition 1.13, the first interesting iterated Öre extensions are the two step ones $k[x][y; \sigma, \delta]$. The quantum and Jordan planes are of particular importance.

Example 1.8 (The quantum plane). Fix $0 \neq q \in k$. The ring $A = k[x, y]$ with defining relation
$$yx = qxy$$
is called a coordinate ring of the quantum (affine) plane. It is an iterated Öre extension, namely
$$A = k[x][y; \sigma]$$
where $\sigma \in \mathrm{Aut}_k\, k[x]$ is defined by $x^\sigma = qx$. By Proposition 1.14, A is

- a two-sided noetherian ring,
- a domain, and
- $\{x^i y^j \mid i, j \geq 0\}$ is a basis for A.

We may make A a graded k-algebra with $\deg x = \deg y = 1$, whence $H_A(t) = (1-t)^{-2}$.

Example 1.9 (The Jordan plane). The ring $A = k[x, y]$ with defining relation
$$yx - xy = x^2$$
is called the coordinate ring of the Jordan plane. It is an iterated Öre extension, namely
$$A = k[x][y; \delta]$$
where $\delta \in \mathrm{Der}_k\, k[x]$ is defined by $\delta(x) = x^2$. By Proposition 1.14, A is

- a two-sided noetherian ring,
- a domain, and
- $\{x^i y^j \mid i, j \geq 0\}$ is a basis for A.

We may make A a graded k-algebra with $\deg x = \deg y = 1$, whence $H_A(t) = (1-t)^{-2}$.

NOTATION . Let t be an indeterminate and n an integer. We define

$$[n] := \frac{t^n - t^{-n}}{t - t^{-1}}$$
$$[n]! := [1][2] \dots [n]$$
$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[n-m]![m]!}.$$

These are all polynomials in $\mathbb{Z}[t]$. If we wish to indicate their evaluation at an element q in a ring R we will usually write $[n]_q$ or $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q$.

LEMMA 1.10. *Let q be a unit in a ring R. Then*

$$\begin{bmatrix} n \\ i \end{bmatrix}_q + q^{n+1} \begin{bmatrix} n \\ i-1 \end{bmatrix}_q = q^i \begin{bmatrix} n+1 \\ i \end{bmatrix}_q.$$

PROOF. This follows from the identity $q^i[j]_q + q^{-j}[i]_q = [i+j]_q$. □

LEMMA 1.11. *Let x and y be elements of a ring R such that $yx = q^2xy$ for some central unit q. Then*

$$(x+y)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{i(n-i)} x^i y^{n-i}.$$

PROOF. This is an easy consequence of the previous Lemma. □

LEMMA 1.12. *Let (σ, δ) be a skew-derivation on R such that $\delta\sigma = q^2\sigma\delta$ for some central unit $q \in R$.*

EXERCISES

1.1 Fix $0 \neq q \in k$ and $\alpha \in k$. Define $\sigma \in \operatorname{Aut}_k k[x]$ by $x^\sigma = qx$, and define the σ-derivation δ by $\delta(x) = \alpha x^2$. Let $A = k[x][y; \sigma, \delta]$. Show that
(a) if $q \neq 1$ then A is isomorphic to the coordinate ring of a quantum plane;
(b) if $q = 1$ and $\alpha \neq 0$, then A is isomorphic to the coordinate ring of the Jordan plane.

1.2 Let $A = k[x][y; \sigma, \delta]$ be an Öre extension of the polynomial ring $k[x]$. Suppose that A is graded with $\deg x = \deg y = 1$, and that A is two-sided noetherian. Show that A is isomorphic to the coordinate ring of either the Jordan plane or a quantum plane.

1.3 Let k be an algebraically closed field. Let $A = k[x, y]$ be a graded algebra with $\deg x = \deg y = 1$, and suppose that A has a single defining relation of degree 2. Show that up to a change of basis this relation is one of the following:

$$x^2, \quad xy, \quad yx - qxy, \quad yx - xy - x^2.$$

[*Hint:* if $V = kx + ky$ then the problem reduces to classifying the non-zero elements of $V \otimes V$ up to the action of $\mathrm{GL}(V) \times k^*$, where k^* acts by scalar multiplication.]

1.4 Show that the Hilbert series of the algebra $A = k[x, y]$ with defining relations $x^2y - 2xyx + yx^2 = y^2x - 2yxy + xy^2 = 0$, which appears as Example 1.3 above, is the same as that for the polynomial ring on 3 indeterminates, two of which are in degree 1, and the other in degree 2.

1.5 Show that the algebra A in Example 1.4 is an iterated Öre extension of k.

1.6 Verify the details of Example 1.2.

1.7 Verify the remarks after Example 1.2 by showing that the ring of fractions of $k[x, y, z]$ is $M_2(k(u, v))$ where u and v are commuting indeterminates and

$$x = \tfrac{1}{2}\begin{pmatrix} u+1 & v(u-1)//(1-u)v^{-1} & -u-1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & v^2//1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} v & 0//0 & v \end{pmatrix}.$$

2. Veronese subalgebras

Recall from Definition 11.1.9 that the d^{th} Veronese subalgebra of a graded algebra A is the algebra $A^{(d)}$, whose degree n-component is A_{nd}. The terminology comes from algebraic geometry.

Definition 2.1. For each $n \geq 0$ and each $d \geq 1$, the d^{th} Veronese embedding, or the d-uple embedding, is the morphism

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^N,$$

where $N = \binom{n+d}{d} - 1$, defined by

$$(x_0, \dots, x_n) \mapsto (x_0^d, \dots, x_n^d),$$

where the entries on the right run through all words of degree d in $x_0, \dots, x_n$. The restriction of ν_d to a closed subvariety $X \subset \mathbb{P}^n$ is called the d^{th} Veronese embedding, or d-uple embedding of X in $\mathbb{P}^N$—it is an isomorphism of varieties $X \to \nu_d(X)$.

If $k[x_0, \dots, x_n]$ is the homogeneous coordinate ring of $\mathbb{P}^n$, then the homogeneous coordinate ring of $\nu_d(\mathbb{P}^n)$ is $k[x_0^d, x_0^{d-1}x_1, \dots, x_n^d] = k[x_0, \dots, x_n]^{(d)}$, the d^{th} Veronese subalgebra.

The case $n = 1$, $d = 2$ embeds $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$. The image of the 2-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ is called the Veronese surface. Explicitly, it is the degree 4 surface consisting of the points $(x^2, xy, xz, y^2, yz, z^2)$.

From the point of view of projective algebraic geometry, there is no difference between A and $A^{(d)}$. We will see later that this simply means that the categories $\mathsf{Tails}A$ and $\mathsf{Tails}A^{(d)}$ are equivalent.

Proposition 2.2. [**121**, p. 282] *If A is finitely generated, commutative, and connected, then $A^{(d)}$ is generated in degree 1 for $d \gg 0$.*

Proof. Suppose that $x_1, \dots, x_r$ are homogeneous elements generating A, and set $d_i = \deg x_i$. Define $d' = d_1 \dots d_r$ and, for each i, write $d_i e_i = d'$. Define $d = rd'$. Now A_n is spanned by monomials

$$x_1^{f_1} \dots x_r^{f_r}$$

such that $n = \sum d_i f_i$. If $n \geq d$ then $f_i \geq e_i$ for some i, whence

$$x_1^{f_1} \dots x_r^{f_r} = (x_i^{e_i}) x_1^{f_1} \dots x_i^{f_i - e_i} \dots x_r^{f_r}.$$

Repeating this process, any monomial of degree $n \in d\mathbb{N}$ is a product of terms $x_i^{e_i}$, each of degree d', and a 'remainder' monomial of degree d. By grouping together the terms of degree d', we may write it as product of monomials of degree d; thus $A^{(d)}$ is generated by A_d. □

As the next example shows, Proposition 2.2 is false if the commutativity hypothesis is dropped—this is disappointing because algebras which are generated in degree 1 are technically easier to handle.

Example 2.3. [**173**, Corollary 3.2] There is a connected, noetherian k-algebra, none of whose Veronese subalgebras is generated in degree one.

Fix $0 \neq q \in k$ and suppose that q is not a root of unity. Let $B = k[x, y]$ with $\deg x = \deg y = 1$, and defining relation

$$xy - qyx = y^2. \tag{2-1}$$

This relation may be rewritten as $(x+\frac{1}{q-1}y)y = qy(x+\frac{1}{q-1}y)$, so B is the coordinate ring of a quantum plane, but for our present purposes (2-1) is preferable to the usual relation. Thus B is an iterated Öre extension, and a noetherian domain. It follows from the relations that B has basis $\{x^i y^j \mid i, j \geq 0\}$.

Let $A = k + xB$. We will establish the following properties of A:

- $\{1, x^{i+1}y^j \mid i, j \geq 0\}$ is a basis for A;
- $A = k[x, xy]$;
- B/A has basis $\{\bar{y}^{i+1} \mid i \geq 0\}$, and as a left A-module is isomorphic to a sum of shifts of the trivial module, namely
$$_A(B/A) \cong k[-1] \oplus k[-2] \oplus \cdots ;$$
- $B = A + yA$ (thus B_A is finitely generated, but $_AB$ is not);
- $A \cong k\langle u, v\rangle/(r_2, r_3)$, where $k\langle u, v\rangle$ is the free algebra on u and v, with degrees 1 and 2 respectively, and r_2, r_3 are the relations
$$r_n = u^n v - q^n v u^n - (n)_q v u^{n-2} v, \qquad (n = 2, 3), \tag{2-2}$$
of degrees 4 and 5 respectively, where $(n)_q = 1 + q + \cdots + q^{n-1}$;
- no Veronese subalgebra of A is generated in degree one;
- A is right and left noetherian.

To prove these facts, we first make note of some useful identities. For each integer $n \geq 1$, define
$$(n)_q = \sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1},$$
which is non-zero since q is not a root of unity. The following identities hold in B:
$$xy.xy^n - q^{-1}x.xy^{n+1} = -q^{-1}xy^{n+2} \qquad (n \geq 0) \tag{2-3}$$
$$xy^n - qyxy^{n-1} = y^{n+1} \qquad (n \geq 1) \tag{2-4}$$
$$y^n x = q^{-n}xy^n - q^{-n}(n)_q y^{n+1} \qquad (n \geq 1) \tag{2-5}$$
$$x^n y = q^n y x^n + (n)_q y x^{n-1} y \qquad (n \geq 1). \tag{2-6}$$
To obtain the first, multiply (2-1) on the left by x, and on the right by y^n; to obtain the second multiply (2-1) on the right by y^{n-1}; the third and fourth are proved by induction on n, making use of the identity $y^{-1}xy = qx + y$.

The basis for A follows at once from its definition. Certainly $x, xy \in A$; induction on n, using (2-3), shows that $xy^n \in k[x, xy]$ for all $n \geq 0$, whence $A = k[x, xy]$. Comparing bases for A and B yields the basis for B/A. Since $A_{\geq 1} = xB$, it is clear that $A_{\geq 1}.(B/A) = 0$, whence B/A is the indicated sum of trivial A-modules. It follows from (2-4) that $B = A + yA$.

The kernel of the surjective graded algebra map $k\langle u, v\rangle \to A$, defined by $u \mapsto x$ and $v \mapsto xy$, contains r_2 and r_3, as is seen by multiplying the identity (2-6) on the left by x, and setting $n = 2, 3$. Thus A is a quotient of $R := k\langle u, v\rangle/(r_2, r_3)$. The description of A via generators and relations will follow once we show that $\dim_k R_n \leq \dim_k A_n$ for all n. Use (2-2) to define $r_n \in k\langle u, v\rangle$ for all $n \geq 2$. A straightforward calculation gives
$$r_{n+2} = u^2 r_n + q^n r_2 u^n - (2)_q v r_n + (n)_q r_2 u^{n-2} v.$$

Thus the image in R of r_n is zero for all $n \geq 2$, whence, since $(n)_q \neq 0$, the image in R of $vu^{n-2}v$ belongs to the span of the images of uv^n and vu^n. Therefore, for $n \geq 1$, R_n is spanned by the n elements

$$\{u^n,\ u^i v u^j \mid i,j \geq 0,\ i+j = n-2\}. \tag{2-7}$$

Since $\dim_k A_n = \dim_k B_n - 1 = n$, for $n \geq 1$, $A \cong k\langle u,v\rangle/(r_2, r_3)$.

We can now prove that no Veronese subalgebra of A is generated in degree one. Write $A = k[u,v]$ with $u = x$ and $v = xy$. By the previous paragraph, the set in (2-7) is a basis for A_n. Hence $A_n.A_n$, which is that part of the degree 2 component of $A^{(n)}$ which is generated by the degree one component of $A^{(n)}$, is spanned by the elements

$$u^{2n},\ \ u^{n+i}vu^j,\ \ u^\ell v u^{m+n},\ \ u^i v u^j . u^\ell v u^m,$$

where $i,j,\ell,m \geq 0$ satisfy $i+j = \ell+m = n-2$. As noted above, since $r_{j+\ell+2} = 0$ in A, $vu^{j+\ell}v$ is a linear combination of $vu^{j+\ell+2}$ and $u^{j+\ell+2}v$, whence the last element in this list is already in the span of the others. Since $j \leq n-2$ and $\ell \leq n-2$, it follows that $u^{n-1}vu^{n-1} \notin A_n.A_n$. But $u^{n-1}vu^{n-1} \in A_2^{(n)}$, so we conclude that $A^{(n)}$ is not generated by $A_1^{(n)}$.

Finally, we show that A is right and left noetherian. Let J be a right ideal of A. There is an inclusion $JxB \subset J \subset JB$ of right A-modules; both JxB and JB are finitely generated right ideals of B, and hence finitely generated right A-modules since $B = A + yA$. Thus, to show that J is finitely generated, it suffices to show that JB/JxB is a noetherian right A-module. Since JB/JxB is a homomorphic image, as a B-module, of a finite direct sum of copies of B/xB, it suffices to show that B/xB, or even B/A, is a noetherian right A-module. Since B/A is graded and bounded below, by Proposition 11.8.1, we only need consider graded submodules, so it suffices to show that $\dim_k(B/bA + A) < \infty$ for a homogenous $b \in B\backslash A$. Since $B = A \oplus yk[y]$, it suffices to show that $\dim_k(B/y^nA + A) < \infty$. By (2-5), $y^nA + A$ contains $y^n x - q^{-n}xy^n = q^{-n}(n)_q y^{n+1} \neq 0$ so, by induction on n, $y^nA + A \supset k[y]y^n$. Thus $\dim_k(B/y^nA + A) = n$, which completes the proof that A is right noetherian.

The same sort of argument which showed that A is right noetherian, will show that $k + Bx$ is left noetherian. However, since B is a domain, there is a well-defined algebra isomorphism $\theta : A = k + xB \to k + Bx$ given by $\theta(xb) = bx$. Hence A is left noetherian.

$\boxed{\text{Paul}}$ Compute $\operatorname{gldim} A$.

The phenomena exhibited by this example are strictly *non-commutative*; if $S = k[X,Y]$ is the commutative polynomial ring, then $k + XS$ is *not* noetherian.

Definition 2.4. The d^{th} Veronese of a graded A-module M is the graded $A^{(d)}$-module $M^{(d)}$, defined by

$$M_i^{(d)} := M_{di}.$$

The rule $M \mapsto M^{(d)}$ is an exact covariant functor $\mathsf{GrMod}A \to \mathsf{GrMod}A^{(d)}$.

Proposition 2.5. *Let M be a graded left A-module. If M is a noetherian A-module, then $M^{(d)}$ is a noetherian $A^{(d)}$-module.*

Proof. If N is a submodule of $M^{(d)}$ then $N = AN \cap M^{(d)}$. Hence any proper ascending chain of submodules in $M^{(d)}$ would give a proper ascending chain of left ideals in M, by left multiplying by A. Since M contains no such chain, neither does $M^{(d)}$. □

COROLLARY 2.6. *If A is noetherian so is $A^{(d)}$, and A is a finitely generated left and right $A^{(d)}$-module.*

PROOF. Applying Proposition 2.5 to $M = A$, it follows that $A^{(d)}$ is right and left noetherian. If $N = A \oplus A[1] \oplus \cdots \oplus A[d-1]$, then $N^{(d)} \cong A$ as a left and as a right $A^{(d)}$-module, so applying Proposition 2.5 to N gives the rest of the result. □

It is possible for $A^{(d)}$ to be noetherian even if A is not: if $A = k[x,y]$ with defining relations $x^2 = xy = 0$, and $\deg x = 1$ and $\deg y = 2$, then A is not right noetherian, but $A^{(2)} = k[y]$ is noetherian.

EXAMPLE 2.7. The Veronese subalgebra $A^{(d)}$ of the coordinate ring of a quantum plane, $A = k[x,y]$ with $yx = qxy$, is of the form $k[u_0,\dots,u_d]$ $\quad (u_i = x^i y^{d-i})$, and there are obvious relations $u_j u_i = q^{d(i-j)} u_i u_j$, so $A^{(d)}$ is a quotient of the quasi-polynomial ring in Example 1.4.

There is another interesting aspect concerning the case $d = 2$. Recall that A is a non-commutative analogue of the coordinate ring of $\mathbb{P}^1$, so $A^{(2)}$ should be a non-commutative analogue of the Veronese embedding of $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$. If $R = k[u_0,u_1,u_2]$ has defining relations

$$u_1 u_0 = q^2 u_0 u_1,\ u_2 u_1 = q^2 u_1 u_2,\ u_2 u_0 = q^4 u_0 u_2,$$

then there is a surjective map $\varphi : R \to A^{(2)}$ defined by $\varphi(u_i) = x^i y^{2-i}$. Comparing Hilbert series, there exists $0 \neq \Omega \in R_2$ such that $\varphi(\Omega) = 0$, so $A^{(2)}$ is a quotient of $R/R\Omega$. Since R is a domain, $H_{R/R\Omega}(t) = (1-t)^{-3}(1-t^2)$. On the other hand, a simple computation shows that this is also the Hilbert series of $A^{(2)}$, whence $\ker\varphi = R\Omega$. A similar argument shows that $\ker\varphi = \Omega R$. Hence Ω is a normalizing element in R and $A^{(2)} \cong R/(\Omega)$.

We should view R as the homogenous coordinate ring of a non-commutative projective plane, and Ω as the defining equation of a conic contained in that plane.

From the point of view of *affine* algebraic geometry there is a great difference between A and $A^{(d)}$. For example, if A is a polynomial ring, then $\operatorname{Spec}(A)$ is smooth whereas $\operatorname{Spec}(A^{(d)})$ is singular if $d \geq 2$ (except in the 1-dimensional case).

The relationship between A and $A^{(d)}$ may also be expressed in terms of group actions. Let μ_d denote the group of d^{th} roots of unity in k, and assume there is a primitive d^{th} root of unity, ω say, whence $\mu_d \cong \mathbb{Z}/d\mathbb{Z}$. Let μ_d act on A_1 through scalar multiplication. This extends to an action of G on A as k-algebra automorphisms; that is, $g = \omega^i$ acts on $x \in A_j$ by $g.x = \omega^{ij}x$. The subalgebra of μ_d-invariants is simply $A^{(d)}$. It is well understood that there is a close relationship between A and its ring of invariants under a finite group; one should think of the analogue of covering spaces where G plays the role of the fundamental group.

EXERCISES

2.1 Fix d and n in $\mathbb{N}$, and let $k[x_0,\dots,x_n]$ denote the commutative polynomial ring on n indeterminates. Let A denote the commutative polynomial ring on the indeterminates X_I, where I runs through the set of all $(n+1)$-tuples $I = (i_0,\dots,i_n)$ such that $i_j \geq 0$ for all j and $i_0 + \ldots + i_n = d$. Show that the kernel of the map $\varphi : A \to k[x_0,\dots,x_n]^{(d)}$, defined by $\varphi(X_I) = x^I$, is generated by all $X_I X_J - X_K X_L$ where $I + J = K + L$.

2.2 Are the Veronese subalgebras of the coordinate ring of the Jordan plane quotients of iterated Öre extensions? If so, is there an analogue of Example 2.7 for the Jordan plane? Give an explicit description of the element Ω arising in Example 2.7.

2.3 Consider the ring R defined in Example 2.7. Determine the element Ω arising there. Show that Ω is not the only normalizing element in degree 2.

2.4 Find a full set of defining relations for the d-Veronese subalgebras of the coordinate ring of a quantum plane (cf. Example 2.7).

2.5 Let $A = k[x, y]$ be the coordinate ring of the Jordan plane, say $yx - xy = x^2$. Determine a set of defining relations for $A^{(2)}$. Is it a quotient of an iterated Öre extension of GK-dimension 3?

2.6 Let $B = k[X, Y]$ be the commutative polynomial ring and set $A = k + XB$. Show that A is not noetherian, and hence not a finitely generated k-algebra. (This should be contrasted with Example 2.3.)

2.7 This exercise concerns the algebra in Example 2.3.
(a) Show that $B/A \cong A/(y + qx)A$ as right A-modules.
(b) Use Theorem 14.4.2 to show that $\operatorname{gldim} A = \infty$.
(c) Show that if $q^n = 1$, then x^n and y^n are central in A.
(d) Show that A is not noetherian if q is a root of unity.

2.8 (Zhang) Show that the Hilbert series of $A^{(n)}$ is

$$\frac{1}{n}\sum_{i=1}^{n} H_A(\zeta^i t)$$

where ζ is a primitive n^{th} root of unity.

2.9 (Research Problem) Is a Veronese subalgebra of an iterated Öre extension a quotient of an iterated Öre extension—I guess it is, but the proof is probably rather tedious. I have in mind writing the original Öre extension as $R = k[x_1, \dots, x_n]$ with the variables adjoined in that order, and then adjoining the words of length d in lexicographic order to form $R^{(d)}$, and using Lemma 1.7.

3. Quadratic algebras

Definition 3.1. Let A be a connected graded k-algebra. Then A is a quadratic algebra if A is defined by homogeneous relations of degree 2; that is,

$$A \cong T(A_1)/(R)$$

where $R \subset A_1 \otimes A_1$. We will usually denote by R_A, the subspace of $A_1 \otimes A_1$ consisting of the defining relations.

Theorem 3.3 below says that, for any subvariety $X \subset \mathbb{P}^n$, for large r, the r^{th} Veronese $\nu_r(X)$ is an intersection of quadrics in $\mathbb{P}^N$.

LEMMA 3.2. *Let $X_1, \dots, X_n$ be vector spaces, and $Y_i \subset X_i$ subspaces. Then*

$$(X_1/Y_1) \otimes \cdots \otimes (X_n/Y_n) \cong X_1 \otimes \cdots \otimes X_n/Z,$$

where $Z := \sum_{j=1}^{n} X_1 \otimes \cdots \otimes X_{j-1} \otimes Y_j \otimes X_{j+1} \otimes \cdots \otimes X_n$.

PROOF. The kernel of the obvious map from $X_1 \otimes \cdots \otimes X_n$ to the left hand side is simply Z. □

THEOREM 3.3. [**18**] *Let $A = T(V)/I$ be a finitely presented graded algebra. If I is generated in degree $\leq d$, then*
1. *$A^{(r)}$ is a finitely presented algebra for all $r \geq 1$;*
2. *$A^{(r)}$ has defining relations of degree $\leq [2 + \frac{d-2}{r}]$;*
3. *if $r \geq d - 1$, then $A^{(r)}$ is a quadratic algebra.*

PROOF. (1) Write $B = A^{(r)}$, whence $B_n = A_{nr}$. Since A is generated by A_1, for all m and n, $A_{m+n} = A_m A_n$. Hence B is generated by B_1. We set $W = B_1$, and write $B = T(W)/J$.

(2) Fix $n \geq 1$. By definition, $W^{\otimes n} = (V^{\otimes r}/I_r)^{\otimes n}$ and $J_n = \ker(W^{\otimes n} \to A_{rn})$. By Lemma 3.2, $W^{\otimes n} = V^{\otimes rn}/Q_n$, where

$$Q_n = \sum_{j=1}^{n} V^{\otimes r(j-1)} \otimes I_r \otimes V^{\otimes r(n-j)}.$$

Since $I_{rn} = \ker(V^{\otimes rn} \to A_{rn})$, it follows that

$$\begin{aligned} J_n &= \ker(V^{\otimes rn}/Q_n \to A_{rn}) \\ &= Q_n + I_{rn}/Q_n. \end{aligned}$$

Clearly, $W \otimes J_n + J_n \otimes W \subset J_{n+1}$, and our goal is to prove there is equality if $n \geq [2 + (d-2)/r]$.

Observe that

$$V^{\otimes r} \otimes Q_n + Q_n \otimes V^{\otimes r} \ \subset \ Q_{n+1} \ \subset \ V^{\otimes r} \otimes I_{rn} + I_{rn} \otimes V^{\otimes r},$$

so

$$W \otimes J_n + J_n \otimes W = (V^{\otimes r} \otimes I_{rn} + I_{rn} \otimes V^{\otimes r})/Q_{n+1}.$$

Now suppose that $n \geq [2 + (d-2)/r]$. It follows that $nr \geq d + r - 1$, so

$$I_{r(n+1)} = \sum_{j=0}^{r(n+1)-d} V^{\otimes j} \otimes I_d \otimes V^{\otimes r(n+1)-d-j}.$$

If $j < r$, then $d + j \leq rn$, so $V^{\otimes j} \otimes I_d \subset I_{rn}$. If $j \geq r$, then

$$I_d \otimes V^{\otimes r(n+1)-d-j} \subset I_{rn}.$$

Therefore

$$I_{r(n+1)} \subset I_{rn} \otimes V^{\otimes r} + V^{\otimes r} \otimes I_{rn}$$

so

$$J_{n+1} = Q_{n+1} + I_{r(n+1)}/Q_{n+1} \ \subset \ W \otimes J_n + J_n \otimes W \ \subset \ J_{n+1},$$

and there is equality as required.

(3) Just apply (2) with $r \geq d - 1$, so $2 < 2 + (d-2)/r < 3$. □

Let A be a quadratic algebra on two generators with one relation, over an algebraically closed field k. By Lemma 1.7, $A = k[x, y]$ with defining relation one of the following:

$$x^2, \quad yx, \quad xy - yx - x^2, \quad yx - qxy \ (0 \neq q \in k).$$

The algebra defined by $x^2 = 0$ has exponential growth (Example 11.2.9), and the algebra defined by $yx = 0$ is neither left nor right noetherian (Exercise 11.8.2). The other algebras in this list are noetherian domains with the same Hilbert series as the polynomial ring on 2 variables. Thus, when we concentrate our attention on these algebras (i.e., the quantum and Jordan planes), we are really considering all quadratic algebras on two generators except those that have properties which are pathological when compared to the polynomial ring on two variables.

Example 3.4. [**18**, page 90] Without the finitely presented hypothesis, Theorem 3.3 is false. Let $A = k\langle x, y\rangle/I$ where I is the ideal generated by

$$\{xy, y^2, zx, zy, z^2\} \bigcup \{xy^i z \mid i \geq 0\}.$$

For $n \geq 1$, $\{x^n, x^{n-1}y, zx^{n-1}\}$ is a basis for A_n. It is easy to show that

$$A \cong A^{(d)}$$

for all $d \geq 1$.

The next result says, roughly speaking, that a generic quadratic algebra is finite dimensional.

Theorem 3.5. *Let V be a finite dimensional k-vector space of dimension ≥ 3 and fix $r \geq ?$. Then, for a dense open set of subspaces $R \in \mathbb{G}_r(V \otimes V)$, $\dim_k T(V)/(R) < \infty$.*

Proof. $\square$

Theorem 3.5 suggests that the algebras of interest to us, say infinite dimensional noetherian algebras of finite GK-dimension, arise from a restricted set of R's. If we fix a dimension, say r, for R, and view R as a point in the Grassmannian $\mathbb{G}_r(V \otimes V)$, then the requirement that A have these properties probably constrains R to lie on some proper subvariety of this. So in some sense R must be rather special. We would like to present an algebra in such a way that some of these special properties of R, whatever they might be, are more immediately apparent.

$\boxed{\text{Paul}}$ Insert examples related to R-matrices, as in Brown-Goodearl.

4. Exterior algebras

The exterior algebra is a fundamental object occurring in a wide variety of contexts. It is the cohomology ring of the torus $S^1 \times \ldots \times S^1$; it occurs in the minimal projective resolution of the trivial module over the commutative polynomial ring (the Koszul complex); it appears in the context of de Rham cohomology of a manifold. It is used in a natural way to embed the Grassmanian of d-dimensional subspaces of an n-dimensional vector space in an appropriate projective space.

For the moment we will use the it as a vehicle to illustrate a method for constructing a basis for an algebra defined by generators and relations. First we find a set which spans it, and then show this set is a basis by constructing a particular cyclic module for the algebra.

Definition 4.1. The exterior algebra over k, on $x_1, \ldots, x_n$ is defined to be

$$\Lambda(x_1, \ldots, x_n) = k\langle x_1, \ldots, x_n\rangle/J$$

where J is the ideal generated by

$$\{x_i x_j + x_j x_i; |; 1 \leq i < j \leq n\} \cup \{x_i^2 \mid 1 \leq i \leq n\}.$$

Lemma 4.2. *$\Lambda(x_1, \ldots, x_n)$ is spanned by the 2^n elements*

$$\mathcal{B} := \{x_{i_1} x_{i_2} \cdots x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\} \cup \{1\}.$$

PROOF. Let $w \in \Lambda$ be a non-zero word in the letters $x_1, \dots, x_n$. Using the relation $x_j x_i = -x_i x_j$ for $j > i$, we may rewrite w as $w = \alpha x_{i_1} \cdots x_{i_r}$ with $\alpha = \pm 1$ and $i_1 \leq \cdots \leq i_r$; for example, $x_4 x_2 x_1 = -x_2 x_4 x_1 = +x_2 x_1 x_4 = -x_1 x_2 x_4$). But $x_i^2 = 0$ for all i, so w cannot contain any x_i more than once. Therefore w belongs to $\mathcal{B}$. Since the set of non-zero words spans Λ, so does $\mathcal{B}$.

The ascending subsequences of $1, 2, \dots, n$ are given by deciding for each j whether to include it or exclude it from the sequence; there are 2^n such decisions, and therefore 2^n such sequences, including the empty sequence which corresponds to $1 \in \mathcal{B}$. Hence the cardinality of $\mathcal{B}$ is 2^n. $\square$

LEMMA 4.3. *$\mathcal{B}$ is a basis for $\Lambda(x_1, \dots, x_n)$.*

PROOF. Since $\mathcal{B}$ spans Λ, $\dim_k \Lambda \leq 2^n$, so it suffices to exhibit a cyclic Λ-module of dimension 2^n.

Fix a basis e_1, e_2 for k^2, and define linear operators f and h on k^2 by

$$fe_1 = e_2, \; fe_2 = 0, \quad \text{and} \quad he_1 = e_1, \; he_2 = -e_2.$$

With respect to this basis, $h = \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}$, and $f = \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$. Let

$$V = \underbrace{k^2 \otimes \cdots \otimes k^2}_{n \text{ copies}} \cong k^{2^n}.$$

Define linear operators $a_1, \dots, a_n$ on V by

$$a_j := \underbrace{h \otimes \cdots \otimes h}_{j-1} \otimes f \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j},$$

where 1 is the identity map. Since $hf + fh = f^2 = 0$, we have

$$a_i a_j + a_j a_i = a_i^2 = 0$$

for all i, j. Hence the map $x_j \to a_j$ makes V into a Λ-module of dimension 2^n. We now show that V is a cyclic module, generated by $e_1 \otimes \cdots \otimes e_n$.

A basis for V consists of elements which are tensor products of e_1 and e_2. Suppose that v is such a basis element with e_2's occurring in positions $i_1 < \cdots < i_r$, and e_1 occurring in all other positions. We claim that $v = \pm a_{i_1} \cdots a_{i_r}(e_1 \otimes \cdots \otimes e_1)$. To see this consider a basis element, u say, having an e_1 in the j^{th} position. It follows from the definition of a_j that $a_j \cdot u = \pm u'$ where u' differs from u only in the j^{th} position, where an e_2 now occurs. Now argue inductively to verify the claim. Thus V is a cyclic Λ-module generated by $e_1 \otimes \cdots \otimes e_1$. Since $\dim_k V = 2^n$, $\dim_k \Lambda \geq 2^n$, so the result follows from Lemma 4.2. $\square$

Before ending this section, we note that the elements $x_i x_j$ are central in $\Lambda(x_1, \dots, x_n)$.

PROPOSITION 4.4. *The exterior algebra on a finite dimensional vector space is a Frobenius algebra. In particular, it is self-injective.*

PROOF. $\square$

EXERCISES

4.1 Show that the kernel of the map

$$f : k\langle x_1, \dots, x_n\rangle \to k[t_1, \dots, t_n],$$

given by $f(x_i) = t_i$, is generated as a two-sided ideal by the elements $x_ix_j - x_jx_i$ for $i < j$.

4.2 Show that the rational function field $k(t)$ is not a finitely generated k-algebra for any field k.

4.3 Use Exercise 2.6 to show that the exterior algebra $\Lambda(x_1, \dots, x_n)$ is spanned by the set $\mathcal{B}$ given in Lemma 4.3.

5. Clifford algebras

Throughout this section all rings have characteristic *not 2.*

Definition 5.1. Let M be a module over a commutative ring R. An R-valued quadratic form on M is a map $q : M \to R$ such that

- $q(\lambda v) = \lambda^2 v$ for all $\lambda \in R$ and $v \in M$, and
- the function $(u, v) := \frac{1}{2}(q(u + v) - q(u) - q(v))$ is R-bilinear.

For brevity we call (M, q) a quadratic R-module.

The orthogonal sum of two quadratic R-modules (M_1, q_1) and (M_2, q_2), denoted $(M_1 \perp M_2, q_1 \perp q_2)$ is the direct sum $M_1 \oplus M_2$ endowed with the quadratic form $q(m_1, m_2) = q_1(m_1) + q_2(m_2)$.

There is a category having as objects the quadratic R-modules, and as morphisms the R-module maps $f : M_1 \to M_2$ such that $q_2 \circ f = q_1$.

Definition 5.2. Let R be a commutative ring and (M, q) a quadratic R-module. The Clifford algebra associated to (M, q) is

$$C(M, q) := T(M)/(x \otimes x - q(x) \mid x \in M),$$

where $T(M) = R \oplus M \oplus M \otimes_R M \oplus$ is the tensor algebra of M over R.

Notice that $C(M, q)$ is an R-algebra.

The next result shows that the Clifford algebra is the solution to a universal problem. To state the result we give a name to the natural map $j : M \to C(M, q)$.

THEOREM 5.3. *Let R be a commutative ring, and (M, q) a quadratic R-module. If S is an R-algebra, $\psi : M \to S$ is an R-module homomorphism such that $\psi(m)^2 = q(m).1_S$ for all $m \in M$, then there is a unique R-algebra map $\rho : C(M, q) \to S$ such that $\psi = \rho \circ j$.*

PROOF. □

LEMMA 5.4. *Let (M, q) be a quadratic R-module, and S a commutative R-algebra. Then $(S \otimes_R M, 1 \otimes q)$ is a quadratic S-module, and*

$$C(S \otimes_R M, 1 \otimes q) \cong S \otimes_R C(M, q).$$

PROOF. By the universal property, the obvious map $S \otimes_R M \to S \otimes_R C(M, q)$ extends to an algebra homomorphism $C(S \otimes_R M, 1 \otimes q) \to S \otimes_R C(M, q)$. On the other hand, the map $M \otimes S \otimes_R M$, $m \mapsto 1 \otimes m$, yields a map $M \to C(S \otimes_R M, 1 \otimes q)$ which, by the universal property, extends to an algebra homomorphism $C(M, q) \to C(S \otimes_R M, 1 \otimes q)$; since the image lies in an S-algebra, this extends to an algebra homomorphism $S \otimes_R C(M, q) \to C(S \otimes_R M, 1 \otimes q)$. It is easy to see that these homomorphisms are mutual inverses. □

LEMMA 5.5. *Let $(M,q) = (M_1,q_1) \perp (M_2,q_2)$ be the orthogonal sum of two quadratic R-modules. Then*

$$C(M,q) \cong C(M_1,q_1)\hat{\otimes}_R C(M_2,q_2).$$

PROOF. □

EXAMPLE 5.6. Suppose that $M = Re$ is free of rank 1, and suppose that M is endowed with the quadratic form $q(e) = a$. It follows easily from the definition that $C(M,q) \cong R[x]/(x^2 - a)$.

Now suppose that $M = Re_1 \oplus Re_2$ is the free R-module of rank 2 endowed with the quadratic form $q(a_1e_1 + a_2e_2) = r_1^2a_1 + r_2^2a_2$, and $(e_1,e_2) = 0$. By Lemma 5.5,

$$C(M,q) \cong R[x,y]/(x^2 - a_1, y^2 - a_2, xy + yx).$$

We are mostly interested in the case where M is a free R-module. Fix an R-module basis for M, say $e_1,\dots,e_n$, and define $V = ke_1 + \cdots + ke_n$. Define $\alpha_{ij} = (e_i,e_j)$, and $M = (\alpha_{ij})$; then M is an R-valued $n \times n$ symmetric matrix, and the form $(-,-)$ is given by

$$(u,v) = u^{\mathsf{T}}Mv$$

where $u = (u_1,\dots,u_n) = \sum u_ie_i$ and $v = (v_1,\dots,v_n) = \sum v_ie_i$.

Notice that $T(M) \cong R \otimes_k T(ke_1 + \cdots + ke_n)$.

LEMMA 5.7. *Let (M,q) be a quadratic R-module, and suppose that M is a free R-module with basis $\{e_1,\dots,e_n\}$. Then*

$$\{e_{i_1}\cdots e_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$$

is an R-module basis for $C(M,q)$.

PROOF. First, the result is obviously true if $\operatorname{rank} M = 1$. It follows from Lemma 5.5 that the result is true for an orthogonal sum of quadratic modules if it is tue for each module individually. Hence, by induction, the result is true if M has an orthogonal basis $e_1,\dots,e_n$. Therefore the result is true if R is field of characteristic not 2.

Now suppose that R is a domain whose characteristic is not 2. Let K be the quotient field of R. Thus the result is true for $(K \otimes_R M, 1 \otimes q)$. It therefore follows from Lemma 5.4 that the result is true for R.

Now let R be arbitrary, and let R' be a domain mapping onto R (for example, take R' to be a suitably large polynomial ring over $\mathbb{Z}$). Let (M',q') be a free quadratic R'-module such that $(M,q) \cong (R \otimes_{R'} M', 1 \otimes q')$; for example, let M' be free on $e_1,\dots,e_n$ and let each $q'(e_i)$ be a preimage of $q(e_i)$. The result is true for $C(M',q')$ by the previous paragraph. The result now follows fom Lemma 5.4. □

PROPOSITION 5.8. *Let (M,q) be a quadratic R-module, and suppose that $M \cong R \otimes_k V$. Then there is a filtration on $C(M,q)$ such that $\operatorname{gr} C \cong \Lambda(V) \otimes_k R$.*

PROOF. Define the filtration by writing $M = R \otimes_k V$, giving R degree zero, and V degree one. Fix a basis $e_1, e_2, \dots$ for V. The relations for $C(M,q)$ are of two types, namely $uv + vu = (u,v)$ for $u,v \in V$, and $rv = vr$ for $r \in R$ and $v \in V$. Passing to the associated graded algebra, relations of the second kind are unchanged, and those of the first kind are replaced by relations $uv + vu = 0$. Hence there is a homomorphism $\Lambda(V) \otimes R \to \operatorname{gr} C$. But both sides are free R-modules with bases of the form $\{e_{i_1}\cdots e_{i_p} \mid 1 \le i_1 < \cdots < i_p\}$ and the map sends the first basis to the second, so the result follows. □

Definition 5.9. If (M,q) is a free quadratic R-module of rank 2, we call the Clifford algebra $C(M,q)$ a generalized quaternion algebra.

The usual quaternion algebra over $\mathbb{R}$ is such an example: let $V = \mathbb{R}e \oplus \mathbb{R}f$ be the rank 2 quadratic module endowed with the bilinear form defined by $q(e) = q(f) = -1$, and $(e,f) = 0$.

The generic Clifford algebra. The coordinate ring of the variety of $n \times n$ matrices is the polynomial ring

$$k[z_{ij} \mid 1 \leq i,j \leq n]$$

where z_{ij} is the function giving the ij^{th} entry of the matrix. Let R be the coordinate ring of the subvariety of symmetric $n \times n$ matrices; thus, R is the quotient by the ideal generated by all $z_{ij} - z_{ji}$; we may think of R as the polynomial ring $k[z_{ij} \mid 1 \leq i \leq j \leq n]$ on $\frac{1}{2}n(n+1)$ indeterminates, with the convention that $z_{ij} = z_{ji}$ if $i > j$.

Let M be the free R-module with basis $X_1, \ldots, X_n$ and define the R-valued quadratic form q on M by

$$q\Big(\sum_i u_i X_i, \sum_j u_j X_j\Big) = \sum_{ij} z_{ij} u_i u_j.$$

We call $C(M,q)$ the n^{th} generic Clifford algebra over k. It is generic in the sense that if (M', q') is a quadratic module over a commutative k-algebra S, and $m_1, \ldots, m_n \in M'$, then there is a homomorphism $\varphi : C(M,q) \to C(M',q')$ sending R to S, and M to M' via $X_i \mapsto m_i$, the latter being a homomorphism of R-modules such that $q'(\varphi(v_1), \varphi(v_2)) = \varphi(q(v_1,v_2))$ for all $v_1, v_2 \in M$.

Notice that $C(M,q) = k[X_1, \ldots, X_n]$ and $z_{ij} = X_iX_j + X_jX_i$. If the X_i's are put in degree one, then the z_{ij}'s have degree 2. There are degree three relations $X_p z_{ij} = z_{ij} X_p$. By Lemma 5.7, the Hilbert series of $C(M,q)$ is

$$(1-t^2)^{-\frac{1}{2}n(n+1)}(1+t)^n$$

THEOREM 5.10. *The n^{th} generic Clifford algebra is an n-step iterated Ore extension of the polynomial ring on $\frac{1}{2}n(n-1)$ indeterminates.*

PROOF. [**100**, Theorem 3.1, page 36] Let $R = k[z_{ij} \mid 1 \leq i < j \leq n]$. For $d = 1, \ldots, n$, we define inductively the Ore extensions $R_d = R_{d-1}[y_d; \sigma_d, \delta_d]$, where

- $R_0 = R$,
- $\sigma_d \in \operatorname{Aut}(R_{d-1})$ is defined by $\sigma_d(y_i) = -y_i$, and the restriction of σ_d to R is the identity;
- δ_d is the σ_d-derivation defined by $\delta_d(y_i) = 2z_{id}$, and the restriction of δ_d to R is zero.

We need to check that σ_d *does* extend to an automorphism of R_{d-1}, and that δ_d *does* extend to a σ_d-derivation. To see this, first notice that $R_d = R_{d-1}[y_d]$ subject to the relations $y_dy_j + y_jy_d = 2z_{jd}$, and $ry_d = y_dr$ if $r \in R_0$. The only point that requires a calculation is

$$\delta_d(y_i)y_j + \sigma_d(y_i)\delta_d(y_j) + \delta_d(y_j)y_i + \sigma_d(y_j)\delta_d(y_i) = 2z_{id}y_j - 2y_iz_{jd} + 2z_{jd}y_i - 2y_jz_{id}$$

which is zero because the z's are central. If we declare that $\deg z_{ij} = 2$ and $\deg y_i = 1$, then the Hilbert series of R_n is

$$(1-t^2)^{-\frac{1}{2}n(n-1)}(1-t)^{-n}.$$

The elements $z_{ii} := y_i^2$ are central, and the ring $C = R[z_{11}, \dots, z_{nn}]$ is a polynomial ring. Let M be the free C-module with basis $y_1, \dots, y_n$ and quadratic form $q(y_i) = z_{ii}$. Then $C(M,q)$ is the generic Clifford algebra, and there is an obvious homomorphism $C(M,q) \to R_n = C[y_1, \dots, y_n]$ of graded rings. Since these two rings have the same Hilbert series this map is an isomorphism. $\square$

Example 5.11. The first Clifford algebra is isomorphic to $k[X]$, the polynomial ring in a single variable. The second generic Clifford algebra is more interesting. In this case $R = k[z_{12}]$, and the Clifford algebra is

$$C = k[z][y_1][y_2; \sigma_2, \delta_2],$$

where $z = z_{12}$ in the notation of the previous proof. The defining relations are $zy_i = y_i z$, and $y_1y_2 + y_2y_1 = 2z$.

But C is generated by the two degree one elements y_1 and y_2, subject to two cubic relations

$$y_1^2y_2 - y_2y_1^2 = y_2^2y_1 - y_1y_2^2 = 0;$$

these relations express the fact that $y_1y_2 + y_2y_1$ is central. The elements $z_{11} = y_1^2$ and $z_{22} = y_2^2$ are also central.

Let $S = k[X_1, X_2]$ be the two-dimensional polynomial ring, and let $M = Sm_1 \oplus Sm_2$ be a rank 2 free module with S-valued quadratic form $q(m_1) = a_1$ and $q(m_2) = a_2$. Then the Clifford algebra $C(M,q)$ is isomorphic to $k[y_1, y_2]/(z_{11} - b_1, z_{22} - b_2)$ where $b_i \in k[z_{11}, z_{22}]$ is the image of a_i under the isomorphism $k[X_1, X_2] \to k[z_{11}, z_{22}]$ sending X_i to z_{ii}.

We now consider situations where R is graded. In that case we call a quadratic module (M,q) a **graded quadratic module** if $\deg q(m) = 2 \deg m$ whenever m is a homogeneous element of M. In this case $T(M)$ has a graded structure coming from the grading on M, and each $m \otimes m - q(m)$ is homogeneous, so $C(M,q)$ is a graded algebra.

For example, if $V = ke_1 + \cdots + ke_n$ is a graded vector space with each e_i homogeneous, and $M = R \otimes_k V$ is given the tensor product grading, and $\deg(e_i, e_j) = \deg e_i + \deg e_j$, then $C(M,q)$ is graded, and $\{e_{i_1} \dots e_{1_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$ is a homogeneous basis for it over R. The simplest case occurs when V is concentrated in degree one, and each $(e_i, e_j) \in R_2$. In any case, the Hilbert series of $C(M,q)$ is $(1-t)^n H_R(t)$.

EXERCISES

5.1 Let C denote the n^{th} generic Clifford algebra. If $n = 2$, show that $C^{(2)}$ is commutative, and describe it in terms of generators and relations. If $n > 2$, show that $C^{(d)}$ is not commutative for any choice of d.

5.2 Let $C(M,q)$ be the n^{th} generic Clifford algebra, and let $X_1, \dots, X_n$ be a basis for M. Show that the automorphism group of $C(M,q)$ is isomorphic to $\mathrm{GL}(n)$ by showing that every automorphism of $kX_1 + \cdots + kX_n$ extends uniquely to an algebra automorphism.

$\boxed{\text{Paul}}$ What other algebras A generated by A_1 have the property that $\mathrm{Aut}(A) \cong \mathrm{GL}(A_1)$? Obvious examples are $T(A_1)$, $S(A_1)$, and $\Lambda(A_1)$. There are some obvious examples gotten by modding out $A_{\geq} n$ from these, but there may well be some more interesting examples.

6. Enveloping algebras of Lie algebras and Lie superalgebras

Some enveloping algebras can be graded.....

Consider the Clifford algebra associated to a quadratic R-module $(R \otimes_k V, q)$. Let $W \subset R$ be the subspace spanned by $q(V)$, and write $\mathfrak{g} = V \oplus W$. We make $\mathfrak{g}$ a Lie superalgebra by defining $\mathfrak{g}_0 = W$, $\mathfrak{g}_1 = V$, and defining the Lie bracket by $[W, V] = [W, W] = 0$, and $[u, v] = (u, v)$ if $u, v \in V$. The only condition that we need to check is the Jacobi identity (4-1); since $[w, -] = 0$ for all $w \in W$, if any one of x, y, z is in $\mathfrak{g}_0$, the identity holds; however, if all of x, y, z are in $\mathfrak{g}_1$, then the bracket of any two of them is in $\mathfrak{g}_0$, so each individual term in (4-1) is zero. Hence, by the universal property of the enveloping algebra, there is a k-algebra homomorphism $U(\mathfrak{g}) \to C(M, q)$.

EXAMPLE 6.1. Let V be a finite dimensional vector space, and let S^2V denote its second symmetric power. Define $q : V \to S^2V$ by $q(x) = x \otimes x$. Then $\mathfrak{g} = V \oplus S^2V$ is a Lie superalgebra in the obvious way. Its enveloping algebra is isomorphic to the Clifford algebra $C(M, q)$ where $R = S(S^2V)$, the symmetric algebra on S^2V, and $M = R \otimes_k V$ with the obvious extension of q.

For example, if $\dim V = 1$, then $U(\mathfrak{g}) \cong k[x]$, where $V = kx$ and $S^2V = kx^2$. If $\dim V = 2$, say $V = kx+ky$, then $S^2V = kx^2 \oplus k(xy+yx) \oplus ky^2$, and $U(\mathfrak{g}) = k[x, y]$ with defining relations saying that $x^2, xy + yx, y^2$ are central. Actually, these are all consequences of the relations saying that $xy + yx$ is central; thus there are two relations, namely $x(xy + yx) = (xy + yx)x$ and $y(xy + yx) = (xy + yx)y$; these simplify to the two relations

$$x^2y - yx^2 = xy^2 - y^2x = 0.$$

Comparing this with Example 5.11, we see this is the second generic Clifford algebra. More generally, if $\dim V = n$, this is the n^{th} generic Clifford algebra.

EXERCISES

6.1 Show that the enveloping algebra $U(\mathfrak{sl}(2)) = k[e, f, h]$, is a graded k-algebra with $\deg e = 1$, $\deg f = -1$ and $\deg h = 0$.

7. Twisting by a 2-cocycle

Definition 7.1. Let G be an abelian group, with group operation written multiplicatively. A G-graded k-algebra is a k-algebra A endowed with a k-vector space decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that

$$A_g A_h \subset A_{gh}$$

for $g, h \in G$.

Many of the ideas in Chapter 11 extend in a natural way to the category of G-graded k-algebras. For example, there is a category of G-graded A-modules et cetera.

EXAMPLE 7.2. The commutative polynomial ring $k[x_1, \dots, x_n]$ is $\mathbb{Z}^n$-graded. Let G be the free abelian group of rank n with basis $e_1, \dots, e_n$, and define $\deg x_j = e_j$. Each homogeneous component of A is of dimension 1 (or zero), and the only graded ideals are those generated by words in the x_i. In particular, amongst these, the only ones which are prime are those generated by a subset of $\{x_1, \dots, x_n\}$. Thus at first glance the category of G-graded modules does not appear very rich. On the other hand, the trivial module has lots of shifts, namely $k[g]$ for each $g \in \mathbb{Z}^n$.

It is useful to observe that the Koszul complex is a complex in the category of G-graded modules.

Definition 7.3. Let G be a group and M a left G-module. A 2-cocycle on G, taking values in M, is a map

$$c : G \times G \to M$$

such that, for all $x, y, z \in G$,

$$c(xy, z) - c(x, yz) + c(x, y) - x.c(y, z) = 0.$$

We say that c is normalized if $c(1, 1) = 0$. The space of 2-cocycles is denoted by $Z^2(G, M)$—it has an abelian group structure induced by that on M.

A 2-cocycle c is a coboundary if there exists a map $f : G \to M$ such that

$$c(x, y) = f(x) - f(xy) + x.f(y)$$

for all $x, y \in G$. The set of coboundaries is denoted by $B^2(G.M)$—they form a subgroup of $Z^2(G, M)$. The quotient is denoted by $H^2(G, M)$, and is called the second cohomology group of G with values in M. One has $H^2(G, M) \cong \operatorname{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial $\mathbb{Z}G$-module structure.

Two cocycles are cohomologous if they differ by a coboundary.

Every 2-cocycle is cohomologous to a normalized one.

Definition 7.4. Let G be an abelian group and A a G-graded k-algebra. If c is a k^*-valued normalized 2-cocycle, the twist of A by c is the G-graded algebra $A(c)$ which, as a graded vector space, equals A, and has multiplication

$$a * b := c(x, y)ab$$

for $a \in A_x$ and $b \in A_y$. The 2-cocycle condition ensures that the multiplication is associative (the normalized hypothesis ensures that $A(c)_1 \cong A_1$—remember, we are writing G multiplicatively!).

$\boxed{\text{Paul}}$ examples– quantum affine spaces and quantum Weyl algebras.

PROPOSITION 7.5. *The category of right G-graded A-modules is equivalent to the category of right G-graded $A(c)$-modules.*

PROOF. Let M be a G-graded A-module. Define $M(c)$ to be equal to M as a G-graded vector space and define an action of $A(c)$ on M by

$$m * b = c(x, y)mb,$$

for $m \in M(c)_x$ and $b \in A(c)_y$. □

PROPOSITION 7.6. *If c is a coboundary, then $A(c) \cong A$ as G-graded algebras. Conversely, if c is normalized and $A(c) \cong A$, then c is a coboundary. Hence there is a bijection between the isomorphism classes of 2-cocycle twists of A and elements of $H^2(G, k^*)$.*

PROOF. □

Our main examples below will be twists of polynomial rings endowed with a grading by a free abelian group. With that in mind, we compute $H^2(\mathbb{Z}^n, k^*)$.

PROPOSITION 7.7. *Let G be the free abelian group on generators $e_1, \dots, e_n$. Let $c \in Z^2(G, k^*)$ and define*

$$r_{ij}(c) = \frac{c(e_i, e_j)}{c(e_j, e_i)}.$$

Then the map $c \mapsto r(c)$ gives a bijection between $H^2(G, k^*)$ and the space of $n \times n$ matrices r satisfying $r_{ij} r_{ji} = 1$ and $r_{ii} = 1$.

PROOF. □

$\boxed{\text{Paul}}$
Give examples of twists of polynomial rings, and coordinate ring of $m \times n$ matrices with $\mathbb{Z}^m \times \mathbb{Z}^n$ grading.

CHAPTER 13

$\mathbb{N}$-graded algebras

In this chapter A is an $\mathbb{N}$-graded k-algebra, and k is a field.

In section 4 we introduce the technical conditions χ_i and χ_i°. These conditions are satsified by every commutative noetherian algebra, but not by every non-commutative noetherian algebra. However, as we will see in chapters 19 and 20 they are necessary if we are to have a non-commutative algebraic geometry which develops along the same lines as the commutative theory. In particular, see Serre's Finiteness Theorem for cohomology groups (20.1.4), and the results on ampleness such as Proposition 20.3.2. Large classes of non-commutative algebras do satisfy the condition χ: in particular, quotients of Gorenstein rings do.

1. Local finiteness

Over a noetherian $\mathbb{N}$-graded algebra whose degree zero component is finite dimensional, all finitely generated modules are locally finite.

LEMMA 1.1. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Then*

1. *A is generated by A_0 and a finite number of elements of positive degree;*
2. *if $M \in \mathsf{grmod}(A)$, then each M_n is a finitely generated A_0-module.*

PROOF. (1) Write $\mathfrak{m} = A_{\geq 1}$. Since A is noetherian, $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated module over $A/\mathfrak{m} = A_0$. Let V be a finitely generated left A_0-submodule of A such that $\mathfrak{m} = V + \mathfrak{m}^2$. By induction on n, we obtain $\mathfrak{m}^n = V^n + \mathfrak{m}^{n+1}$ for all $n \geq 1$. By induction on n, we obtain

$$\mathfrak{m} = V + V^2 + \ldots + V^n + \mathfrak{m}^{n+1}$$

for all $n \geq 1$. Hence $A_n \subset V + V^2 + \ldots + V^n$ for all $n \geq 1$. In other words, A is generated as a k-algebra by A_0 and the finite set of elements which generate V as a left A_0-module.

(2) Since A is left noetherian, each $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely generated left $A/\mathfrak{m}$-module. Hence each A_n is a finitely generated left A_0-module. Since a finitely generated A-module is a quotient of a finitely generated free A-module its homogenous components are finitely generated A_0-modules too. $\square$

Despite this result, for a noetherian, finitely generated, k-algebra it need not be true that A_iA_j equals A_{i+j} for all sufficiently large i and j (Example 6.2.3 has the property that $A_nA_n \neq A_{2n}$ for all $n \geq 1$).

2. The duality $M \mapsto M^*$

Definition 2.1. If A is a graded k-algebra and $M \in \mathsf{GrMod}(A)$, we define its dual to be the graded right A-module

$$M^* := \underline{\mathrm{Hom}}_k(M, k)$$

with A-action defined by $(f.x)(m) := f(xm)$, and grading defined by $(M^*)_{-n} = \operatorname{Hom}_k(M_n, k)$.

The rule $M \mapsto M^*$ is a contravariant functor

$$\underline{\operatorname{Hom}}_k(-,k) : \mathsf{GrMod}(A) \to \mathsf{GrMod}(A^{\mathrm{op}}).$$

It is obviously an exact functor. There is a functor in the reverse direction too, namely $N_A \mapsto \underline{\operatorname{Hom}}_k(N,k)$, which we also denote by $N \mapsto N^*$.

PROPOSITION 2.2. *Let A be a graded k-algebra, and $M \in \mathsf{GrMod}(A)$. Then*

1. *the natural map $M \to M^{**}$ is an isomorphism if and only if M is locally finite;*
2. *if P is projective in $\mathsf{GrMod}(A)$, then P^* is injective in $\mathsf{GrMod}(A^{\mathrm{op}})$;*
3. *if N is locally finite, then $\underline{\operatorname{Ext}}^i_A(M,N) \cong \underline{\operatorname{Ext}}^i_{A^{\mathrm{op}}}(N^*,M^*)$ for all $i \geq 0$.*
4. *if $N \in \mathsf{GrMod}(A^{\mathrm{op}})$, then $\underline{\operatorname{Ext}}^i_A(M,N^*) \cong \operatorname{Tor}^A_i(N,M)^*$ for all $i \geq 0$.*

PROOF. (1) Since $M^* = \oplus M_n^*$, the map $M \to M^{**}$ is an isomorphism if and only if for each n the map $M_n \to M_n^{**}$ is an isomorphism.

(2) Let P be a projective object in $\mathsf{GrMod}(A^{\mathrm{op}})$. Then

$$\underline{\operatorname{Hom}}_A(M,P^*) \cong \underline{\operatorname{Hom}}_A(M,\underline{\operatorname{Hom}}_k(P,k)) \cong \underline{\operatorname{Hom}}_k(P \otimes_A M, k),$$

so $\operatorname{Hom}_A(-,P^*)$ is the composition of the exact functors $P \otimes_A -$ and $\underline{\operatorname{Hom}}_k(-,k)$, so is exact too; hence the functor $\operatorname{Hom}_{\mathrm{Gr}}(-,P^*) = \underline{\operatorname{Hom}}_A(-,P^*)_0$ is exact, showing that P^* is injective in $\mathsf{GrMod}(A)$.

(3) Suppose that $i = 0$. The map $\Phi : \underline{\operatorname{Hom}}_A(M,N) \to \underline{\operatorname{Hom}}_{A^{\mathrm{op}}}(N^*,M^*)$ defined by $\Phi(f)(\alpha)(m) = \alpha(f(m))$ for $\alpha \in N^*$ and $m \in M$ is k-linear and injective; if N is locally finite, then Φ is also surjective.

For general i, suppose that $P_\bullet \to M$ is a projective resolution in $\mathsf{GrMod}(A)$. Then

$$\underline{\operatorname{Ext}}^i_A(M,N) = h^i(\underline{\operatorname{Hom}}_A(P_\bullet,N)) \cong h^i(\underline{\operatorname{Hom}}_{A^{\mathrm{op}}}(N^*,P_\bullet^*)), \tag{2-1}$$

where the last isomorphism is given by the case $i = 0$; however, the complex $M^* \to P_\bullet^*$ is an injective resolution of M^* in $\mathsf{GrMod}(A^{\mathrm{op}})$, so the final term in (2-1) equals $\underline{\operatorname{Ext}}^i_{A^{\mathrm{op}}}(N^*,M^*)$. Hence the result.

(4) When $i = 0$, we have

$$\underline{\operatorname{Hom}}_A(M,N^*) = \underline{\operatorname{Hom}}_A(M,\underline{\operatorname{Hom}}_k(N,k)) \cong \underline{\operatorname{Hom}}_k(N \otimes_A M, k) = (N \otimes_A M)^*.$$

In general, we take a projective resolution $P_\bullet \to N$ in $\mathsf{GrMod}(A^{\mathrm{op}})$; then $N^* \to P_\bullet^*$ is an injective resolution in $\mathsf{GrMod}(A)$, so

$$\underline{\operatorname{Ext}}^i_A(M,N^*) \cong h^i(\underline{\operatorname{Hom}}_A(M,P_\bullet^*)) \cong h^i((P_\bullet \otimes_A M)^*) = h^i(P_\bullet \otimes_A M)^*$$

which is isomorphic to $\operatorname{Tor}^A_i(N,M)^*$, as required □

Generally $M \mapsto M^*$ does not send finitely generated modules to finitely generated modules; for example, if $A = M = k[X]$, then $M^* \cong k[X,X^{-1}]/Xk[X]$. However, if M is locally finite M^* is too, so $M \mapsto M^*$ sets up a duality between the categories of left and right locally finite graded A-modules.

COROLLARY 2.3. *Let A be left noetherian and locally finite. If $M, N \in \mathsf{grmod}(A)$, then there is a convergent spectral sequence*

$$\underline{\operatorname{Ext}}^{-p}_A(M,\underline{\operatorname{Ext}}^q_A(N,A)^*) \Rightarrow \underline{\operatorname{Ext}}^{q+p}_A(N,M)^*. \tag{2-2}$$

PROOF. By Theorem 8.6.1, there is a convergent spectral sequence

$$E_2^{pq} = \operatorname{Tor}^A_{-p}(\underline{\operatorname{Ext}}^q_A(N,A),M) \Rightarrow \underline{\operatorname{Ext}}^{q+p}_A(N,M).$$

By Lemma 2.2(4), $(E_2^{pq})^* \cong \underline{\operatorname{Ext}}^{-p}_A(M,\underline{\operatorname{Ext}}^q_A(N,A)^*)$; the hypotheses ensure that E_2^{pq} is locally finite, we can apply * to obtain the result claimed. $\square$

COROLLARY 2.4. *Suppose that A is locally finite. If $M \in \mathsf{GrMod}(A)$, then $\underline{\operatorname{Hom}}_A(M,A^*) \cong M^*$ as graded right A-modules.*

PROOF. The isomorphism as graded k-vector spaces is a special case of part (3) of Proposition 2.2. The A-module structure on $\underline{\operatorname{Hom}}_A(M,A^*)$ is inherited from the right A-module action on A^*, namely $(f.x)(m) = f(m).x$.

There are several possible proofs. One could may show that the map $\Phi : \underline{\operatorname{Hom}}_A(M,A^*) \to M^*$ defined by $\Phi(f)(m) = f(m)(1)$ for $m \in M$ and $f : M \to A^*$ is an A-module isomorphism. Alternatively,

$$\underline{\operatorname{Hom}}_A(M,A^*) = \underline{\operatorname{Hom}}_A(M,\underline{\operatorname{Hom}}_k(A,k)) \cong \underline{\operatorname{Hom}}_k(A \otimes_A M,k) \cong M^*.$$

Or, $F : M \mapsto M^*$ is a left exact contravariant functor so, by Watt's Theorem, is isomorphic to $\underline{\operatorname{Hom}}_A(-,FA)$; then just check that the grading is respected. $\square$

Corollary 2.4 is a special case of the isomorphism $\underline{\operatorname{Hom}}_A(M,P^*) \cong (P \otimes_A M)^*$ arising from the isomorphisms

$$(P \otimes_A M)^* = \underline{\operatorname{Hom}}_k(P \otimes_A M,k) \cong \underline{\operatorname{Hom}}_A(M,\underline{\operatorname{Hom}}_k(P,k)) = \underline{\operatorname{Hom}}_A(M,P^*).$$

EXAMPLE 2.5. The graded dual of a polynomial ring has a nice description. Suppose $\operatorname{char} k = 0$. Let $A = k[x_1,\dots,x_n]$ be the polynomial ring with the standard grading. We will use the following standard multi-index notation: if $I = (i_1,\dots,i_n) \in \mathbb{Z}^n$, we write

$$\begin{aligned} |I| &:= i_1 + \cdots + i_n, \\ x^I &:= x_1^{i_1} \dots x_n^{i_n}, \\ I! &:= i_1! \dots i_n! \qquad \text{if each } i_j \geq 0. \end{aligned}$$

Let $\partial_i = \partial/\partial x_i$ denote the partial derivatives, and let $D = k[\partial_1,\dots,\partial_n]$ denote the subalgebra of $\operatorname{Hom}_k(A,A)$ they generate. Since $\partial_i(A_r) \subset A_{r-1}$, $\deg(\partial_i) = -1$ for all i. Thus D is a commutative, graded subalgebra of $\underline{\operatorname{Hom}}_k(A,A)$. We define a pairing

$$D \times A \to k$$

by $(\delta,a) \mapsto \delta(a)$ evaluated at $0 \in k^n$; that is $(\delta,a) \mapsto \varepsilon(\delta(a))$ where $\varepsilon : A \to k$ is the augmentation. For each $r \geq 0$, this map restricts to a non-degenerate pairing

$$D_{-r} \times A_r \to k,$$

with respect to which $\{x^I \mid |I| = r\}$ and $\{\partial^I/I! \mid |I| = r\}$ are dual bases. Hence, as graded vector spaces, $D \cong A^*$. This also shows that the ∂^I are linearly independent, and hence that D is a polynomial ring on the indeterminates $\partial_1,\dots,\partial_n$.

To see the A-module structure on A^* it is simpler to interchange the roles of A and D. That is, D is a polynomial ring, as graded vector spaces $D^* \cong A$, and the natural action of D on A, namely $\delta.a = \delta(a)$, satisfies

$$(\delta_1\delta_2,a) = (\delta_1,\delta_2.a),$$

so $A \cong D^*$ as graded D-modules (recall that the A-module structure on A^* is given by $\langle ab,f\rangle = \langle a,b.f\rangle$).

Although this is the most natural description of A^*, it is sometimes helpful to identify A^* with the polynomial ring $B := k[x_1^{-1}, \dots, x_n^{-1}]$, where the pairing $B \times A \to k$ is given by

$$(b, a) \mapsto \text{ the coefficient of 1 in } ba$$

when ba is written as a linear combination of the elements $\{x^I \mid I \in \mathbb{Z}^n\}$.

EXERCISES

2.1 Let M be a graded A-A-bimodule. Show that M^* has a bimodule structure defined by $(x.\alpha)(y) = \alpha(yx)$ and $(\alpha.x)(y) = \alpha(xy)$ for $\alpha \in M^*$ and $x, y \in A$.

2.2 Use Proposition 2.2 to show that $\operatorname{injdim} M^* \leq t$ if $\operatorname{pdim} M \leq t$. Give an example showing that the converse is false.

3. Torsion

Definition 3.1. Let M be a graded A-module. An element $m \in M$ is torsion if $A_{\geq n} m = 0$ for some n. If 0 is the only torsion element, M is said to be torsion-free. Thus $m \in M$ is torsion if and only if Am is right bounded.

LEMMA 3.2. *Let M be a graded A-module, and let $\{M_\alpha \mid \alpha \in I\}$ be the collection of all graded submodules of M such that M/M_α is torsion-free. Then $\bigcap_{\alpha \in I} M_\alpha$ is the smallest graded submodule of M such that the quotient is torsion-free.*

PROOF. It suffices to show that $M/\cap M_\alpha$ is torsion-free. If the image of $m \in M$ in $M/\cap M_\alpha$ is torsion, then so is its image in each M/M_α, whence m is contained in each M_α. □

Definition 3.3. The torsion submodule of $M \in \mathsf{GrMod}(A)$, denoted τM, is the smallest graded submodule such that $M/\tau M$ is torsion-free. If $M = \tau M$ we say that M is a torsion module.

Warning. The torsion submodule must contain all torsion elements, but may contain non-torsion elements if A is not noetherian (see Example 3.4). However, if A is noetherian, then the torsion submodule consists of precisely the torsion elements (3.8).

NOTATION . We denote by

- $\operatorname{Tors}(A)$ the full subcategory of $\mathsf{GrMod}(A)$ consisting of the torsion modules, and by
- $\operatorname{tors}(A)$ the full subcategory of $\mathsf{grmod}(A)$ consisting of the torsion modules.

We will show in Proposition 3.5 that $\operatorname{Tors}(A)$ and $\operatorname{tors}(A)$ are dense subcategories of $\mathsf{GrMod}(A)$ and $\mathsf{grmod}(A)$, respectively.

EXAMPLE 3.4. The torsion submodule may contain elements which are not torsion. Let $R = k[x_1, x_2, \dots]$ denote the commutative polynomial ring on a countable number of indeterminates, each of degree 1. Let J denote the graded ideal generated by the homogeneous elements $\{x_n^{n+1},\, x_i x_j \mid n \geq 1,\, 1 \leq i < j\}$. Define $A = R/J$ and give it the grading inherited from R.

Every element of $\mathfrak{m}$ is torsion: in particular, $\mathfrak{m}$ is generated as an ideal by $\{x_n \mid n \geq 1\}$ and x_n is annihilated by $A_{\geq n}$; since the generators of $\mathfrak{m}$ are torsion, so is every element of $\mathfrak{m}$. Thus τA contains $\mathfrak{m}$, so is either $\mathfrak{m}$ or A. But $A/\mathfrak{m}$ is not torsion-free, so we conclude that $\tau A = A$, so contains the element 1 which is not torsion.

Notice that $\tau M = 0$ if and only if M is torsion-free, whence $\tau(M/\tau M) = 0$.

A submodule of a torsion-free module is torsion-free.

Every element of a right bounded module is torsion, so a right bounded module is a torsion module. More generally, a direct limit of right bounded modules is torsion, but the converse is false as Example 3.4 shows.

A finite dimensional module is torsion and Proposition 3.8 shows that, if A is left noetherian and locally finite, the finitely generated torsion modules are precisely the finite dimensional modules.

If A is connected, the direct sum of all negative shifts of the trivial module, $k \oplus k[-1] \oplus k[-2] \oplus \dots$, is torsion, but neither right bounded nor finite dimensional.

The torsion submodule of A itself is a two-sided ideal: it is a left ideal by definition, and is a right ideal because a homomorphic image of a torsion module is torsion. If A is a prime ring and $\mathbb{N}$-graded, then $\tau A = 0$.

PROPOSITION 3.5. *$\mathrm{Tors}(A)$ is a dense subcategory of $\mathsf{GrMod}(A)$.*

PROOF. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{GrMod}(A)$. We must show that M is torsion if and only if both L and N are.

Suppose that M is torsion. Then every non-zero quotient of M contains a non-zero torsion element. Therefore, N also has this property, so is a torsion module. Now consider L. If $M = 0$, then L is certainly torsion, so suppose that $M \neq 0$. Suppose, by way of contradiction, that $\tau L \neq L$. Replacing M by $M/\tau L$ and L by $L/\tau L$, we may assume that $\tau L = 0$ and that $\tau M = M$. By Zorn's Lemma, we may choose a graded submodule $X \subset M$, maximal such that $X \cap L = 0$. Since L embeds in M/X, replacing M by M/X, we may assume that $\tau L = 0$, $\tau M = M$ and that L is essential in M. Now let $0 \neq m \in M$ be torsion. Since $Am \cap L \neq 0$ and since every element of Am is torsion, $\tau L \neq 0$. This contradiction shows that we must have had $L = \tau L$ to begin with. Thus, we have shown that if M is torsion, so is every quotient and submodule of M.

Conversely, suppose that both L and N are torsion. Consider the exact sequence

$$0 \longrightarrow \frac{L + \tau M}{\tau M} \longrightarrow \frac{M}{\tau M} \longrightarrow \frac{N}{\mathrm{Im}(\tau M)} \longrightarrow 0.$$

Since $M/\tau M$ is torsion-free, so is $(L+\tau M)/\tau M$. But this is isomorphic to $L/\tau M \cap L$ which is a quotient of the torsion module L, and therefore torsion by the first part of the proof. Hence $(L+\tau M)/\tau M = 0$, whence $L \subset \tau M$. Thus $M/\tau M$ is a quotient of $M/L \cong N$. But N is torsion so, by the first part of the proof, $M/\tau M$ is torsion. Since the only torsion module which is torsion-free is 0, $M = \tau M$, as required. $\square$

LEMMA 3.6. *If M is a graded A-module, then τM is the largest torsion module contained in M.*

PROOF. Certainly τM contains every torsion submodule of M because, if $X \subset M$ is torsion, then the image of X in $M/\tau M$ is also torsion, hence zero as $M/\tau M$ is torsion-free. It remains to show that τM is torsion. If the image of $m \in M$ in $M/\tau(\tau M)$ were torsion, then its image in $M/\tau M$ would also be torsion, whence $m \in \tau M$. Now the image of m in $\tau M/\tau(\tau M)$ would be torsion, and hence zero. Thus $\tau(\tau M) = \tau M$, as required. $\square$

The general theory of torsion is substantially easier in the two most important cases, namely when A is left noetherian, and when A is locally finite.

LEMMA 3.7. *Let A be a graded k-algebra and suppose that $M \in \mathsf{grmod}(A)$ is noetherian. Then M is torsion if and only if it is right bounded.*

PROOF. Suppose that M is torsion, and choose $N \subset M$ maximal subject to being right bounded. Then M/N cannot contain a non-zero right bounded submodule, so M/N cannot contain a non-zero torsion element. But M/N is torsion, so $M/N = 0$, whence M is right bounded. The converse is trivial. $\square$

PROPOSITION 3.8. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and $M \in \mathsf{GrMod}(A)$. Then*

1. *τM is the sum of the right bounded submodules of M;*
2. *$\tau M = \{m \in M \mid m \text{ is torsion}\}$;*
3. *if M is noetherian, τM is the largest right bounded submodule of M;*
4. *if M is noetherian, then M is torsion if and only if it is right bounded.*

If A is locally finite, rather than left noetherian, then (1), (3) and (4) remain true with 'right bounded' replaced by 'finite dimensional'.

PROOF. (1) Let T denote the sum of all right bounded submodules of M. Each element $m \in T$ belongs to a *finite* sum of right bounded modules, so Am is right bounded, whence m is torsion. Thus $T \subset \tau M$. To see that M/T is torsion-free, suppose that the image of m in M/T is torsion. Then $A_{\geq n}m \subset T$ for some n. But Am is noetherian, so $A_{\geq n}m$ is finitely generated, and therefore contained in a finite sum of right bounded modules. Thus $A_{\geq n}m$ is right bounded. It follows that Am is too, so $m \in T$. Thus $T = \tau M$.

Parts (2), (3) and (4) follow easily from (1).

When A is locally finite, every finitely generated A-module is locally finite and left bounded, so is finite dimensional if and only if it is right bounded. The results in the locally finite case now follow easily. $\square$

Example 3.4 shows that, over a locally finite $\mathbb{N}$-graded algebra, a finitely generated torsion module need not be right bounded, so we can not replace the noetherian hypothesis in parts (3) and (4) of Proposition 3.8 with the hypothesis that M is finitely generated.

If A is left noetherian, then $\tau M = \varinjlim \underline{\mathrm{Hom}}_A(A/\mathfrak{m}^i, M)$, so τ is a left exact functor.

PROPOSITION 3.9. *Taking torsion is a left exact functor*

$$\tau : \mathsf{GrMod}(A) \to \mathrm{Tors}(A).$$

PROOF. Since $\mathrm{Tors}(A)$ is dense in $\mathsf{GrMod}(A)$, if $f \in \mathrm{Hom}_{\mathrm{Gr}}(X, Y)$, then $f(\tau X) \subset \tau Y$. Therefore, if we define τf to be the restriction of f to τX, then τ becomes a functor $\mathsf{GrMod}(A) \to \mathrm{Tors}(A)$.

Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathsf{GrMod}(A)$. Then $0 \to \tau X \to \tau Y \to \tau Z$ is certainly a complex. It is exact at τX so it remains to prove exactness at τY. Now the kernel of the map $\tau Y \to \tau Z$ is $X \cap \tau Y$, so we must prove that $X \cap \tau Y = \tau X$.

The module

$$\frac{X}{X \cap \tau Y} \cong \frac{X + \tau Y}{\tau Y}$$

is torsion-free since it is a submodule of $Y/\tau Y$ which is torsion-free. Thus $\tau X \subset X \cap \tau Y$, by definition of τX. By Lemma 3.6, τY is a torsion module, hence so is $X \cap \tau Y$. Thus $X \cap \tau Y \subset \tau X$ as required. □

By Lemma A.13.13, the injective envelope of a torsion-free module is torsion-free.

PROPOSITION 3.10. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and let E be an injective envelope of M in* $\mathsf{GrMod}(A)$. *Then*

1. *the right bounds of E and M are equal;*
2. *E is torsion if and only if M is torsion;*
3. *E is torsion-free if and only if M is torsion-free.*

PROOF. (1) We only need to prove that $M_{\geq n} = 0$ implies $E_{\geq n} = 0$, so suppose that $M_{\geq n} = 0$ and $e \in E_{\geq n}$ is homogenous. As A is $\mathbb{N}$-graded, $Ae = (Ae)_{\geq n}$, so $Ae \cap M = (Ae)_{\geq n} \cap M = 0$, whence $e = 0$ because M is essential in E. Thus $E_{\geq n} = 0$.

(2) It suffices to show that M torsion implies E is torsion, so suppose M is torsion, and let $e \in E$. Since $Ae \cap M$ is noetherian and torsion, it is right bounded. Hence, for large n, $A_{\geq n}e \cap M = 0$, whence $A_{\geq n}e = 0$. Thus e is torsion, as required.

(3) This is true without any hypothesis on A; see Lemma A.13.13. □

PROPOSITION 3.11. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Then, every injective in* $\mathsf{GrMod}A$ *is a direct sum of a torsion injective and a torsion-free injective.*

PROOF. Let E be injective, and let I be an injective envelope of τE. By Proposition 3.10, I is torsion. Since E is injective, there is a factorization $\tau E \to I \xrightarrow{g} E$ of the inclusion $\tau E \to E$. Since τE is essential in I, g is injective; identify I with $g(I)$. Since I is injective, there exists a submodule Q of E such that $E = Q \oplus I$. But a direct summand of an injective module is injective, whence Q is injective. Finally $\tau Q \subset Q \cap \tau E = 0$, so Q is torsion-free. □

NOTATION . Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. If M is a graded A-module, write $E^\bullet M$ for a minimal resolution of M, and let $I^j M$ denote the torsion submodule of $E^j M$. Hence there is a short exact sequence of complexes $0 \to I^\bullet \to E^\bullet \to Q^\bullet \to 0$, where each $Q^j M$ is torsion-free.

PROPOSITION 3.12. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Let $M \in$* $\mathsf{GrMod}A$, *and suppose that T is a torsion module. Then*

$$\underline{\mathrm{Ext}}^i_A(T, M) = \mathrm{h}^i(\underline{\mathrm{Hom}}_A(T, I^\bullet M));$$

where $I^\bullet M$ is the torsion subcomplex of a minimal injective resolution of M.

PROOF. If $E^\bullet M$ is the minimal injective resolution of M, then

$$\underline{\mathrm{Ext}}^i_A(N, M) = \mathrm{h}^i(\underline{\mathrm{Hom}}_A(N, E^\bullet M)).$$

Applying the covariant left exact functor $F = \underline{\mathrm{Hom}}_A(T, -)$ to the exact sequence of complexes $0 \to I^\bullet M \to E^\bullet M \to \bar{Q}^\bullet M \to 0$ gives a long exact sequence in cohomology, namely

$$\cdots \to \mathrm{h}^{i-1}(F\bar{Q}^\bullet) \to \mathrm{h}^i(FI^\bullet) \to \mathrm{h}^i(FE^\bullet) \to h^i(F\bar{Q}^\bullet) \to \mathrm{h}^{i+1}(FI^\bullet) \to \cdots$$

But $\bar{Q}^\bullet$ is torsion-free, so $F\bar{Q}^\bullet = 0$. The result follows. □

LEMMA 3.13. *Let $f : A \to B$ be a homomorphism of $\mathbb{N}$-graded k-algebras such that B_A is finitely generated. If M is a torsion A-module, then $B \otimes_A M$ is a torsion B-module.*

PROOF. Write $B = \sum_{i=1}^{m} b_i A$, where b_i is homogeneous of degree d_i and set $d = \max\{d_i\}$. It follows that $B_{\geq n+d} \subset \sum b_i A_{\geq n}$.

Let X denote the torsion submodule of $B \otimes_A M$. Define the A-module map $g : M \to B \otimes_A M$ by $g(m) = 1 \otimes m$. Suppose that $m \in M$ gives a torsion element $\bar{m} \in M/g^{-1}X$. Thus, for some n, $g(A_{\geq n} m) \subset X$; in other words, $1 \otimes A_{\geq n} m \subset X$, which implies that $B_{\geq n+d}(1 \otimes m) \subset X$. But $B \otimes_A M/X$ is torsion-free, so $1 \otimes m \in X$, whence $\bar{m} = 0$. We have shown that $M/g^{-1}X$ is torsion-free, whence $g^{-1}X = M$ and $X = B \otimes_A M$, as required. □

LEMMA 3.14. *Let M be a finitely generated left module over a noetherian k-algebra R.*

1. *If N is a finite dimensional right R-module, then $\dim_k \operatorname{Tor}_i^R(N, M) < \infty$ for all $i \geq 0$.*
2. *If $\psi : N \to N'$ is a homomorphism of finitely generated right R-modules such that $\ker \psi$ and $\operatorname{coker} \psi$ are finite dimensional, then $\ker(\psi \otimes M)$ and $\operatorname{coker}(\psi \otimes M)$ are finite dimensional.*

PROOF. (1) Let $P_\bullet \to M$ be a free resolution of M by finitely generated free R-modules; since $\operatorname{Tor}^i_R(N, M)$ is a subquotient of the finite dimensional module $N \otimes_R F_i$, the result follows.

(2) Write $X = \ker \psi$ and $Y = \operatorname{coker} \psi$. Since $N \otimes M \to N' \otimes M \to Y \otimes M \to 0$ is exact, $\operatorname{coker}(\psi \otimes M) = \operatorname{coker}(\psi) \otimes M$, which is finite dimensional by (1). The short exact sequences $0 \to X \to N \to N/X \to 0$ and $0 \to N/X \to N' \to Y$ give exact sequences

$$X \otimes M \xrightarrow{\alpha} N \otimes M \to N/X \otimes M \to 0$$

and

$$\operatorname{Tor}_1^R(Y, M) \to N/X \otimes M \xrightarrow{\beta} N' \otimes M \to Y \otimes M \to 0.$$

By (1), $\dim(X \otimes M) < \infty$, so $\ker \alpha$ is finite dimensional. Similarly, by (1) applied to the second sequence, $\ker \beta$ is finite dimensional. It follows that $\ker(N \otimes M \to N' \otimes M)$ is finite dimensional, as required. □

EXERCISES

3.1 Let M be a graded A-module such that $\tau M \neq 0$. Show that the torsion elements form an essential submodule of τM. If A is $\mathbb{N}$-graded, show that $L := \{m \in M \mid A_{\geq 1} m = 0\}$ is an essential submodule of τM.

3.2 In the non-noetherian case, is an essential extension of a torsion module torsion?

4. The condition χ

We now encounter the technical conditions χ_i° and χ_i. They are unavoidable if one wants a theory for non-commutative algebras which resembles the commutative theory. These conditions are invisible in the commutative theory because every noetherian commutative algebra satisfies them. In contrast, there are rather nice non-commutative algebras which do not—see Lemma 5.2 and Example 5.3.

Nevertheless, Section 5 shows that large classes of non-commutative algebras do satisfy these conditions.

The condition χ reappears when we lay the foundations of non-commutative projective geometry in chapters 19 and 20; see for example Theorem 19.4.4, and Propositions 20.3.2 and 20.3.3. It is shown there that the non-commutative analogue of the ampleness of Serre's twisting sheaf $\mathcal{O}_X(1)$ is closely related to the condition χ_1°. Second, Theorem 20.4.2 shows that if A is left noetherian and ???, then $\mathsf{Tails}(A)$ is equivalent to $\mathsf{Tails}(B)$ for some B which satisfies χ_1°. Finally, if A is left noetherian, locally finite and $\mathbb{N}$-graded, then the conditions χ_i and χ_i° are equivalent (4.4) so, at a first reading one might assume that one is in this situation (which is, after all, the most natural one).

Lemma 4.1. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Let M be a graded A-module such that $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M) \in [\ell', r']$ for all $j \leq i$. If $T \in [\ell, r]$, then, for all $j \leq i$,*

$$\underline{\mathrm{Ext}}^j_A(T, M) \in [\ell' - r, r' - \ell].$$

Proof. The result is obviously true when $j = 0$, so suppose that $j \geq 1$. Suppose that T is concentrated in degree zero. There is an exact sequence $0 \to K \to F \to T \to 0$, where F is a direct sum of shifts of $A/A_{\geq 1}$. Since $\underline{\mathrm{Ext}}^j_A(F, M)$ is a direct product of copies of $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$, it is in $[\ell', r']$. By induction on j, the case $j = 1$ being trivial, we may assume that $\underline{\mathrm{Ext}}^{j-1}_A(K, M) \in [\ell', r']$. Therefore, it follows from the exact sequence

$$\cdots \to \underline{\mathrm{Ext}}^{j-1}_A(K, M) \to \underline{\mathrm{Ext}}^j_A(T, M) \to \underline{\mathrm{Ext}}^j_A(F, M) \to \cdots$$

that $\underline{\mathrm{Ext}}^j_A(T, M) \in [\ell', r']$ also.

If T is concentrated in degree d, then $\underline{\mathrm{Ext}}^j_A(T, M) \in [\ell' - d, r' - d]$ because

$$\underline{\mathrm{Ext}}^j_A(T, M) = \underline{\mathrm{Ext}}^j_A(T[d][-d], M) = \underline{\mathrm{Ext}}^j_A(T[d], M)[d].$$

Now suppose that $\ell < r$, and let $T \in [\ell, r]$. We will prove the result by induction on $r - \ell$, the case $r - \ell = 0$ having been dealt with above. There is a short exact sequence $0 \to T' \to T \to T'' \to 0$ with T' concentrated in degree ℓ and $T'' \in [\ell + 1, r]$. The induction hypothesis implies that $\underline{\mathrm{Ext}}^j_A(T'', M) \in [\ell' - r, r' - \ell - 1]$. By the previous paragraph, $\underline{\mathrm{Ext}}^j_A(T', M) \in [\ell' - \ell, r' - \ell]$, so consideration of the exact sequence

$$\cdots \to \underline{\mathrm{Ext}}^j_A(T'', M) \to \underline{\mathrm{Ext}}^j_A(T, M) \to \underline{\mathrm{Ext}}^j_A(T', M) \to \cdots$$

completes the proof. □

A special case of Lemma 4.1 shows that, if $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M) = 0$, then so is $\underline{\mathrm{Ext}}^j_A(T, M) = 0$ for all bounded T: just take $\ell' \gg r'$. Of course, this is easy to prove directly; the proof of the Lemma follows the same lines as that for this special case, with some additional careful bookkeeping.

Definition 4.2. Let A be an $\mathbb{N}$-graded k-algebra and $M \in \mathsf{GrMod}(A)$. We say that

- $\chi_i^\circ(M)$ holds if $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$ is right bounded for all $j = 0, 1, \ldots, i$;
- χ_i° holds for A, or A satisfies χ_i°, if $\chi_i^\circ(M)$ holds for all $M \in \mathsf{grmod}(A)$;
- χ° holds for A, or A satisfies χ°, if χ_i° holds for all i.

We say that

- $\chi_i(M)$ holds if, for each $d \in \mathbb{Z}$, $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)_{\geq d}$ is a finitely generated A-module for all $j = 0, 1, \dots, i$ and for all $n \gg 0$;
- χ_i holds for A, or A satisfies χ_i, if $\chi_i(M)$ holds for all $M \in \mathsf{grmod}(A)$;
- χ holds for A, or A satisfies χ, if χ_i holds for all i.

The conditions χ are about *right* bounds on Ext groups since if M is left bounded, then so is $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$ by Proposition 11.6.2(1).

If $\chi^\circ_i(M)$ holds, then Lemma 4.1 applies with $\ell' = -\infty$: in particular, if T is bounded, then $\underline{\mathrm{Ext}}^j_A(T, M)$ is right bounded for all $j \leq i$.

Since $\underline{\mathrm{Hom}}_A(A/A_{\geq n}, M)_{\geq d}$ identifies with the submodule of $(\tau M)_{\geq d}$ which is annihilated by $A_{\geq n}$, $\chi_0(M)$ holds for all noetherian $M \in \mathsf{GrMod}(A)$.

PROPOSITION 4.3. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra and let $M \in \mathsf{GrMod}(A)$. Then*

1. *$\chi^\circ_0(M)$ holds;*
2. *if $\chi^\circ_i(M)$ holds, and $N \in \mathsf{GrMod}(A)$ is left bounded, then, for all $j \leq i$, $\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ has a right bound which is independent of n.*
3. *if $M \in \mathsf{grmod}(A)$ satisfies $\chi_i(M)$, then $\chi^\circ_i(M)$ holds;*
4. *if M is locally finite and satisfies $\chi^\circ_i(M)$, then $\chi_i(M)$ holds.*

PROOF. (1) By Proposition 3.8, τM is right bounded, hence so is its submodule $\underline{\mathrm{Hom}}_A(A/A_{\geq 1}, M)$.

(2) Fix $d \in \mathbb{Z}$. Consider the exact sequence $0 \to T \to N/N_{\geq n+1} \to N/N_{\geq n} \to 0$, and the associated exact sequence

$$\underline{\mathrm{Ext}}^{j-1}_A(T, M) \to \underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M) \to \underline{\mathrm{Ext}}^j_A(N/N_{\geq n+1}, M) \to \underline{\mathrm{Ext}}^j_A(T, M).$$

Since $\chi^\circ_i(M)$ holds and $T \in [n, n]$, Lemma 4.1 implies that the first and last terms of this sequence are right bounded; moreover, their right bounds tend to $-\infty$ as $n \to \infty$. Hence, there exists r such that $\underline{\mathrm{Ext}}^{j-1}_A(T, M)_{\geq d} = \underline{\mathrm{Ext}}^j_A(T, M)_{\geq d} = 0$ for all $n \geq r$, which implies that $\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)_{\geq d} \to \underline{\mathrm{Ext}}^j_A(N/N_{\geq n+1}, M)_{\geq d}$ is an isomorphism.

Finally, since $N/N_{\geq n}$ is left bounded and $\chi^\circ_i(M)$ holds, Lemma 4.1 implies that $\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ is right bounded—the isomorphism in the previous paragraph ensures that there is a right bound independent of n.

(3) By (1), the result is trivially true when $i = 0$. So we suppose that $i > 0$, and proceed by induction. Since $\chi_i(M)$ holds, so does $\chi_{i-1}(M)$, whence $\chi^\circ_{i-1}(M)$ holds by the induction hypothesis.

Let $d \in \mathbb{Z}$, and choose n such that $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)_{\geq d}$ is finitely generated for all $j \leq i$. Consider the exact sequence

$$\underline{\mathrm{Ext}}^{j-1}_A(A_{\geq 1}/A_{\geq n}, M)_{\geq d} \to \underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)_{\geq d} \to \underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)_{\geq d}.$$

Since M is left bounded and $\chi^\circ_{i-1}(M)$ holds, (2) implies that the left hand term is bounded. Since the right hand term is finitely generated and torsion, it is right bounded. Hence the middle term is right bounded; that is, $\chi^\circ_i(M)$ holds.

(4) We must show that $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)_{\geq d}$ is finitely generated for $n \gg 0$. However, it is locally finite because M is (11.6.2), and right bounded by (2), but also left bounded. Hence it is finite dimensional, so certainly finitely generated. □

COROLLARY 4.4. *Let A be a left noetherian, locally finite, $\mathbb{N}$-graded k-algebra. Then χ_i holds for A if and only if χ°_i does.*

PROOF. Just combine parts (3) and (4) of Proposition 4.3. $\square$

COROLLARY 4.5. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Suppose that $M \in \mathsf{grmod}(A)$ satisfies $\chi_i^\circ(M)$, and that $N \in \mathsf{GrMod}(A)$ is left bounded. Fix $d \in \mathbb{Z}$. Then the natural maps*

$$\begin{aligned}\underline{\mathrm{Ext}}_A^j(N/N_{\geq n}, M)_{\geq d} &\to \underline{\mathrm{Ext}}_A^j(N/N_{\geq n+1}, M)_{\geq d} \quad \textit{and}\\ \underline{\mathrm{Ext}}_A^{j-1}(N_{\geq n}, M)_{\geq d} &\to \underline{\mathrm{Ext}}_A^{j-1}(N_{\geq n+1}, M)_{\geq d}\end{aligned}$$

are isomorphisms for $n \gg 0$.

PROOF. The first of these isomorphisms was established while proving part (2) of Proposition 4.3. The proof of the second is similar. The exact sequence $0 \to N_{\geq n+1} \to N_{\geq n} \to X \to 0$ gives an exact sequence

$$\underline{\mathrm{Ext}}_A^j(X, M) \to \underline{\mathrm{Ext}}_A^j(N_{\geq n}, M) \to \underline{\mathrm{Ext}}_A^j(N_{\geq n+1}, M) \to \underline{\mathrm{Ext}}_A^{j+1}(X, M).$$

Since X is concentrated in degree n, Lemma 4.1 shows there exists r such that, if $n \geq r$, the first and fourth terms in this sequence are zero in degree $\geq d$; the isomorphism follows. $\square$

COROLLARY 4.6. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. Suppose that $M \in \mathsf{grmod}(A)$ satisfies $\chi_i^\circ(M)$, and that $N \in \mathsf{GrMod}(A)$ is left bounded. If $j \leq i$, then $\varinjlim \underline{\mathrm{Ext}}_A^j(N/N_{\geq n}, M)$ is right bounded.*

PROOF. By the first isomorphism in Corollary 4.5, there exists r such that

$$\varinjlim \underline{\mathrm{Ext}}_A^j(N/N_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}_A^j(N/N_{\geq r}, M)_{\geq d}.$$

But this is right bounded by Proposition 4.3(2). $\square$

COROLLARY 4.7. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and let $M \in \mathsf{grmod}(A)$. Then $\chi_i^\circ(M)$ holds if and only if $\varinjlim \underline{\mathrm{Ext}}_A^j(A/A_{\geq n}, M)$ is right bounded for all $j \leq i$.*

PROOF. ($\Rightarrow$) This is a special case of Corollary 4.6.

($\Leftarrow$) The case $i = 0$ is trivial, since $\underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) \subset \tau M$, which is right bounded because M is noetherian. We now suppose that $i > 0$, and proceed by induction, assuming that $\chi_{i-1}^\circ(M)$ holds. To prove that $\chi_i^\circ(M)$ holds, it suffices to prove that $\underline{\mathrm{Ext}}_A^i(A/A_{\geq 1}, M)$ is right bounded. Consider the exact sequence

$$\underline{\mathrm{Ext}}_A^{i-1}(A_{\geq 1}/A_{\geq n}, M) \to \underline{\mathrm{Ext}}_A^i(A/A_{\geq 1}, M) \to \underline{\mathrm{Ext}}_A^i(A/A_{\geq n}, M). \tag{4-1}$$

By hypothesis, the direct limit of the right hand term is right bounded. By the induction hypothesis and Corollary 4.6, the direct limit of the left hand term is right bounded. Hence, so is the direct limit of the middle term; i.e., $\underline{\mathrm{Ext}}_A^i(A/A_{\geq 1}, M)$ is right bounded, as required. $\square$

5. Rings satisfying condition χ

Commutative noetherian algebras satisfy χ. A ring which is a finite module over a ring satisfying χ also satisfies χ. The condition χ behaves well with respect to taking Veronese subalgebras. Nevertheless, χ can fail for algebras which are, in some respects, reasonably well-behaved (see Proposition 2.12 and Example 5.3).

PROPOSITION 5.1. *A commutative, noetherian, $\mathbb{N}$-graded k-algebra satisfies condition χ.*

PROOF. Let M and N be left A-modules. Then $\mathrm{Ext}^i_A(N, M)$ has two A-module structures: a left A-module structure by viewing N as an A-A-bimodule, and using the right action of A on N, and a right A-module structure by viewing M as an A-A-bimodule, and using the right action of A on M. These two structures coincide.

Hence, if M is finitely generated, then, using the A-action on M, $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$ is finitely generated by Proposition 11.6.2(3). However, using the right action of A on $A/A_{\geq 1}$, it is annihilated by $A_{\geq 1}$, whence it is a finitely generated A_0-module, and therefore bounded. □

In Chapter 15, we will show that noetherian Cohen-Macaulay algebras satisfy condition χ, and that this class includes all graded iterated Ore extensions

$$A = k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n],$$

where each σ_i is an graded automorphism, and each x_i is of positive degree.

The next lemma gives a class of algebras which fail to satisfy χ_1, and an explicit example follows the lemma.

LEMMA 5.2. [**173**, Lemma 2.13] *Let $A \subset B$ be left noetherian, locally finite, $\mathbb{N}$-graded k-algebras. Suppose that $\dim_k(B/A) = \infty$, that B_A is finitely generated, and that ${}_A(B/A)$ is torsion. Then A does not satisfy χ_1.*

PROOF. Write $B = b_1A + \cdots + b_rA$. If $M \in \mathsf{GrMod}(B)$ and $m \in M$, then $Bm = b_1Am + \cdots + b_rAm$, so $\dim_k(Bm) < \infty$ if and only if $\dim_k(Am) < \infty$. Hence, by Proposition 3.8, the torsion submodule of M is the same whether M is considered as an A-module or as a B-module.

Write $\bar{A} = A/\tau A$ and $\bar{B} = B/\tau B$. By Proposition 3.9, $\tau A = A \cap \tau B$, whence there is an inclusion of graded algebras $\bar{A} \subset \bar{B}$. Since τA and τB are finite dimensional, $\dim_k(\bar{B}/\bar{A}) = \infty$. If $0 \neq b \in \bar{B}$, then $A_{\geq n}b \subset \bar{A}$ for some n, since B/A is torsion. However, $\bar{B}$ is torsion-free, so $A_{\geq n}b \neq 0$, whence $\bar{A}$ is an essential submodule of $\bar{B}$. Therefore, the inclusion of $\bar{A}$ in its injective envelope $E(\bar{A})$ extends to an inclusion $\bar{B} \subset E(\bar{A})$. Hence, by its definition, $\omega\pi A$ contains a copy of $\bar{B}$, and $\mathrm{coker}(A \to \omega\pi A)$ contains a copy of $\bar{B}/\bar{A}$. But $\bar{B}/\bar{A}$ is left bounded, locally finite, and infinite dimensional so is not right bounded. Therefore, $\mathrm{coker}(A \to \omega\pi A)$ is not right bounded, whence $\chi_1(A)$ fails to hold by Theorem 4.4. □

EXAMPLE 5.3. [**173**] There exists a noetherian, locally finite, $\mathbb{N}$-graded k-algebra A, and $M \in \mathsf{grmod}(A)$ such that $\chi_1(M)$ does not hold.

Let A be the algebra defined in Example 6.2.3; that is $A = k + xB$ where $B = k[x, y]$ with defining relation $yx - qxy = y^2$, and $0 \neq q \in k$ is not a root of unity. It was shown there that A and B are both two-sided noetherian, and that B_A is finitely generated. Moreover, $\dim_k(B/A) = \infty$, and B/A is torsion as a left A-module, since $A_{\geq 1}B = xB \subset A$. Hence, by Lemma 5.2, A does not satisfy χ_1.

By Corollary **??**, B satisfies χ. Thus, B is very well behaved—indeed, B is a twist of a polynomial ring, so $\mathsf{GrMod}(B)$ is equivalent to the category of graded modules over the commutative polynomial ring in two variables. Moreover, $\mathsf{Tails}(A) \cong \mathsf{Tails}(B)$ by Theorem 2.12, which makes the fact that A does not satisfy χ_1 rather striking.

Here is a direct proof that $\operatorname{Ext}^1_A(k, A) \cong B/A$; that is, $\chi_1(A)$ does not hold. From the exact sequence $0 \to xB \to A \to k \to 0$, it follows that $\operatorname{Ext}^1_A(k, A) \cong V/A$ where

$$V = \{q \in \operatorname{Fract}(A) \mid xBq \subset A\}.$$

It is clear that $B \subset V$, so it remains to prove the reverse inclusion. Let $q \in V$. Then $BxBq \subset B$. However, $BxB \supset Bx + xB = B_{\geq 2}$, whence $B_{\geq 2}q \subset B$. Thus, we obtain an exact sequence

$$0 \to B \to B + Bq \to B + Bq/B \to 0,$$

which is an extension of B by a torsion module. But B is Artin-Schelter regular, so $\operatorname{Ext}^1_B(k, B) = 0$, whence this sequence splits. The splitting gives a submodule of $Bq + B$ which is torsion, and hence zero since $\operatorname{Fract} A$ is a domain. Therefore, $B + Bq/B = 0$ and $q \in B$. Thus $V = B$, as claimed.

We now investigate condition χ in the presence of a homomorphism $f : A \to B$ between $\mathbb{N}$-graded algebras.

LEMMA 5.4. *Let $f : A \to B$ be a homomorphism of left noetherian, $\mathbb{N}$-graded k-algebras. Write I_n for the left ideal of B generated by $f(A_{\geq n})$.*

1. *If B_A is finitely generated, then (I_n) and $(B_{\geq n})$ are cofinal in the sense that, given n, then for all $r \gg 0$, $I_n \supset B_{\geq r}$ and $B_{\geq n} \supset I_r$.*
2. *If either*
   - *A is noetherian, or*
   - *$\ker f$ and $\operatorname{coker} f$ are bounded,*

   then $\operatorname{Tor}^A_q(B, A/A_{\geq n})$ is bounded for all q and n.

PROOF. (1) By hypothesis B_A is a quotient of a finite direct sum of shifts of A, so $B/I_n \cong B \otimes_A (A/A_{\geq n})$ is bounded. hence $B_{\geq r} \subset I_n$ for $r \gg 0$. Similarly, since $B/B_{\geq n}$ is bounded it is annihilated from the right by $A_{\geq r}$ for $r \gg 0$, whence $B_{\geq n} \supset I_r$ for $r \gg 0$.

(2) Suppose A is noetherian. As an A-module B_A has a free resolution, each term of which is a finite direct sum of shifts of A. Applying $- \otimes_A (A/A_{\geq n})$ to this resolution yields a complex in which each term is bounded. Hence the homology groups are bounded, as required.

Now suppose that $\ker f$ and $\operatorname{coker} f$ are bounded. Since A is left noetherian, taking a free resolution of $A/A_{\geq n}$ as a left module, and applying the argument in the previous paragraph, all the $\operatorname{Tor}^A_q(\ker f, A/A_{\geq n})$ and $\operatorname{Tor}^A_q(\operatorname{coker} f, A/A_{\geq n})$ are bounded. In addition, all $\operatorname{Tor}^A_q(A, A/A_{\geq n})$ are bounded. The result now follows by considering the long exact sequences for $\operatorname{Tor}^A_q(-, A/A_{\geq n})$ related to the exact sequence $0 \to \ker f \to A \to B \to \operatorname{coker} f \to 0$. $\square$

LEMMA 5.5. *Let $f : A \to B$ be a homomorphism of left noetherian, $\mathbb{N}$-graded k-algebras. Suppose that B_A is finitely generated. Write I_n for the left ideal of B generated by $f(A_{\geq n})$. If $M \in \mathsf{grmod}(B)$, then*

1. *$\chi^\circ_i(M)$ holds if and only if for each $j \leq i$, $\underline{\operatorname{Ext}}^j_B(B/I_n, M)$ is bounded for $n \gg 0$;*
2. *if $\chi^\circ_i(M)$ holds, then for each $j \leq i$, and each $d \in \mathbb{Z}$,*

   $$\underline{\operatorname{Ext}}^j_B(B/I_n, M)_{\geq d} \cong \underline{\operatorname{Ext}}^j_B(B/B_{\geq n}, M)_{\geq d}$$

   for $n \gg 0$.

PROOF. Since B_A is finitely generated, Lemma 5.4 applies.

(1) ($\Rightarrow$) Since B/I_n is bounded, the result is given by Lemma 4.1.

($\Leftarrow$) We proceed by induction on i, the case $i = 0$ being trivially true. Choose n large enough that $I_n \subset B_{\geq 1}$. The long exact sequence for Ext gives an exact sequence

$$\underline{\mathrm{Ext}}_B^{i-1}(B_{\geq 1}/I_n, M) \to \underline{\mathrm{Ext}}_B^i(B/B_{\geq 1}, M) \to \underline{\mathrm{Ext}}_B^i(B/I_n, M). \tag{5-1}$$

Since B_A is finitely generated, $B_{\geq 1}/I_n$ is bounded; hence, by the induction hypothesis and Lemma 4.1, the first term in (5-1) is bounded. By hypothesis, the last term in (5-1) is bounded too, hence so is the middle term; but that is precisely the condition that $\chi_i^\circ(M)$ holds.

(2) Fix n, and choose $r = r(n)$ such that $I_r \subset B_{\geq n}$. There is an exact sequence

$$\underline{\mathrm{Ext}}_B^{j-1}(B_{\geq n}/I_r, M) \to \underline{\mathrm{Ext}}_B^j(B/B_{\geq n}, M) \to \underline{\mathrm{Ext}}_B^j(B/I_r, M) \to \underline{\mathrm{Ext}}_B^j(B_{\geq n}/I_r, M).$$

By Lemma 4.1, the right bound of the first and last terms tend to $-\infty$ as $n \to \infty$. Hence, for a fixed $d \in \mathbb{Z}$, the middle map is bijective in degree $\geq d$ for $n \gg 0$. (Of course $r = r(n)$ will increase as n does.) By Corollary 4.5, $\underline{\mathrm{Ext}}_B^j(B/B_{\geq n}, M)_{\geq d}$ is independent of n for large n, so choosing n large enough,

$$\underline{\mathrm{Ext}}_B^j(B/I_{r(n)}, M) \cong \underline{\mathrm{Ext}}_B^j(B/B_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}_B^j(B/B_{r(n)}, M)_{\geq d}$$

as required. □

THEOREM 5.6. *Let $f : A \to B$ be a homomorphism of left noetherian, locally finite, $\mathbb{N}$-graded k-algebras.*

1. *If $\ker f$ and $\operatorname{coker} f$ are bounded, then A satisfies χ_i if and only if B does.*
2. *If ${}_AB$ and B_A are finitely generated and A satisfies χ_i, so does B.*

PROOF. Consider the change of rings spectral sequence

$$\underline{\mathrm{Ext}}_B^p(\mathrm{Tor}_q^A(B, A/A_{\geq n}), M) \Rightarrow \underline{\mathrm{Ext}}_A^n(N, M).$$

By Lemma 5.4, each $\mathrm{Tor}_q^A(B, A/A_{\geq n})$ is bounded.

??????? □

COROLLARY 5.7. *Let A be a noetherian, locally finite, $\mathbb{N}$-graded k-algebra. If A satisfies χ_i, so does every quotient A/I.*

COROLLARY 5.8. *Let A be a noetherian, locally finite, $\mathbb{N}$-graded k-algebra. Then A satisfies χ_i if and only if A/P does for all minimal primes P of A.*

THEOREM 5.9. *Let A be a left noetherian, graded k-algebra, and let $A^{(r)}$ be a Veronese subalgebra over which A is finitely generated both as a left, and as a right, module.*

1. *If $A^{(r)}$ is generated in degree one, then $A^{(r)}$ satisfies χ_i if and only if A does.*
2. *If A is noetherian, then $A^{(r)}$ satisfies χ if and only if A does.*

THEOREM 5.10. *Let A be a noetherian, $\mathbb{N}$-graded k-algebra. If z is a homogeneous normal element of positive degree, then A satisfies χ if and only if $A/(z)$ does.*

THEOREM 5.11. *Let A be a left noetherian, graded k-algebra, which satisfies a polynomial identity. The following are equivalent:*

1. *A is right noetherian;*

2. *A satisfies* χ_1*;*
3. *A satisfies* χ*.*

PROOF. □

EXERCISES

5.1 Prove the claim in Proposition 5.1 that, over a commutative ring R, the two R-module structures on $\mathrm{Ext}^i_R(N, M)$ are the same. [*Hint:* consider a projective resolution $P_\bullet \to N$ and an injective resolution $M \to E^\bullet$. Let $c \in \mathrm{Ext}^i_R(N, M)$, and suppose that c is represented both by $f : P_i \to M$ and $g : N \to E^i$. Show that there is a morphism of complexes $\varphi : P_\bullet \to E^\bullet$, sending P_j to E^{i-j-1}, which is f and g in the appropriate positions.]

CHAPTER 14

Connected algebras

In this chapter k denotes a fixed field.

Connected algebras are analogous to commutative local rings. First, there is an analogue of Nakayama's Lemma; second, projectives are free; third, homological properties are largely determined by the behavior of the trivial module.

1. Basic properties

Definition 1.1. An $\mathbb{N}$-graded k-algebra A is connected if $A_0 = k$. In this case the quotient map $\varepsilon : A \to k$ is called the augmentation and $\mathfrak{m} := \ker \varepsilon = A_{\geq 1}$ is called the augmentation ideal. We call $A/\mathfrak{m}$ the trivial A-module, and usually denote it by k (or ${}_Ak$, or k_A, if we need to specify on which side A acts).

For the rest of this section A denotes a connected k-algebra.

By Exercise 11.9, k is the unique irreducible object in $\mathsf{GrMod}(A)$.

By Lemma 13.1.1, a noetherian connected algebra is locally finite; we will show in Corollary 3.4 that such an algebra has a finite number of defining relations.

Lemma 1.2 (Nakayama's Lemma). *Let A be an $\mathbb{N}$-graded k-algebra, M a graded A-module which is bounded below, and let $\mathfrak{m} := A_{\geq 1}$. The following are equivalent:*

1. $M = 0$;
2. $\mathfrak{m}M = M$;
3. $A/\mathfrak{m} \otimes_A M = 0$.

Hence, $m_1, \dots, m_n$ generate M if their images generate $M/\mathfrak{m}M$.

Proof. The equivalence of (2) and (3) is clear since $A/\mathfrak{m} \otimes M \cong M/\mathfrak{m}M$. If $M \neq 0$ then, by hypothesis, there exists n such that $M = M_{\geq n}$ and $M_n \neq 0$. Therefore $\mathfrak{m}M \subset M_{\geq n+1} \neq M$, proving that (2) implies (1); the reverse implication is trivial. The final statement is proved by applying the implication (3) $\Rightarrow$ (1) to the module $M/Rm_1 + \cdots + Rm_n$. $\square$

Proposition 1.3. *If $M \in \mathsf{GrMod}(A)$ is bounded below, then M is free if and only if $\operatorname{Tor}_1^A(k, M) = 0$.*

Proof. Let V be a graded subspace of M such that $M = V \oplus \mathfrak{m}M$. Then $k \otimes_A (M/AV) = 0$ so, by Nakayama's Lemma, $M = AV$. If $A \otimes_k V$ is given the tensor product grading, there is a degree zero surjection $\psi : A \otimes_k V \to M$ defined by $\psi(x \otimes v) = xv$. Because $\operatorname{Tor}_1^A(k, M) = 0$, there is an exact sequence

$$0 \to k \otimes_A \ker \psi \to k \otimes_A A \otimes_k V \xrightarrow{1 \otimes \psi} k \otimes_A M \to 0$$

Identifying each of the last two terms in this sequence with V, $1 \otimes \psi$ becomes identified with the identity map. Hence $k \otimes_A \ker \psi = 0$. But $\ker \psi$ is bounded below since M is, so $\ker\psi = 0$ by Nakayama's lemma. Thus ψ is an isomorphism, showing that M is free. □

The hypothesis that M be bounded below is necessary: for example, $k[X, X^{-1}]$ is a flat $k[X]$-module which is not free.

Corollary 1.4. *A left bounded module is projective if and only if it is free.*

For connected algebras the injective envelope of the trivial module plays an important role. It is obtained by applying * to the surjection $A \to k$.

Proposition 1.5. *If A is connected, then the injective envelope of the trivial module is isomorphic to A^* with left A-module action $(x.f)(a) = f(ax)$.*

Proof. By Proposition 2.2, A^* is injective. It contains a copy of the trivial module, namely $k\varepsilon$ where ε is the augmentation. Moreover, A^* is an essential extension of $k\varepsilon$ because, if $0 \neq f \in A^*$ is of degree $-n$, then $f(x) \neq 0$ for some $x \in A_n$, so $0 \neq x.f \in A^*_0$. Hence $A^* \cong E(k)$ as claimed. □

Lemma 1.6. *Let A be a left noetherian, connected, k-algebra. Then every torsion injective A-module is a direct sum of shifts of A^*.*

Proof. Let I be a torsion injective in $\mathsf{GrMod}(A)$. If $0 \neq M \in \mathrm{Tors}(A)$, then $\underline{\mathrm{Hom}}_A(k, M) \neq 0$. We may consider $S = \underline{\mathrm{Hom}}_A(k, I)$ as a submodule of I; it is a (possibly infinite) direct sum of shifts of ${}_Ak$. If M is a non-zero submodule of I then, since M is torsion, $\mathrm{Hom}_A(k, M) \neq 0$, whence $M \cap S \neq 0$, so S is essential in I; thus $I = E(S)$. Since A is left noetherian, a direct sum of injective modules is injective [**2**, Proposition 18.13], whence $E(S)$ is a (possibly infinite) direct sum of shifts of $E({}_Ak) \cong A^*$. □

Lemma 1.7. *Let A be a connected, left noetherian algebra, and let $M \in \mathsf{GrMod}(A)$. Then*

$$\underline{\mathrm{Ext}}^j_A(k, M) \cong \underline{\mathrm{Hom}}_A(k, I^j M),$$

where $I^\bullet M$ is the torsion subcomplex of a minimal injective resolution of M.

Proof. Let $E^\bullet M$ be a minimal injective resolution of M. We noted in section 5 of chapter 11 that $\underline{\mathrm{Ext}}^j_A(k, M) = \underline{\mathrm{Hom}}_A(k, E^j M)$, so the result follows immediately from the decomposition of $E^j M$ as a direct sum of its torsion and torsion-free parts (Proposition 13.refprop.injs.decompose). □

Proposition 1.8. *Let A be a connected, left noetherian algebra, and let $M \in \mathsf{grmod}(A)$. Let $I^\bullet M$ denote the torsion subcomplex of a minimal injective resolution of M. Then $\chi_j(M)$ holds if and only if $I^j M$ is a finite direct sum of shifts of A^*.*

Proof. Since M is locally finite and left bounded, so is $\underline{\mathrm{Ext}}^j_A(k, M)$. Hence it is right bounded (equivalently, M satisfies χ_j) if and only if it is finite dimensional. By Lemma 1.7, $\underline{\mathrm{Ext}}^j_A(k, M) \cong \underline{\mathrm{Hom}}_A(k, I^j M)$. Since the injective envelope of k is A^*, the result follows. □

Definition 1.9. Let A be connected. The socle of $M \in \mathsf{GrMod}(A)$ is $\mathrm{soc}(M) := \{m \in M \mid \mathfrak{m}m = 0\}$. Equivalently, $\mathrm{soc}(M) = \underline{\mathrm{Hom}}_A(k, M)$.

The socle of A is a two sided ideal.

2. Minimal projective resolutions

In this section A denotes a connected k-algebra.

Definition 2.1. A complex $(P_\bullet, d_\bullet)$ is minimal if $\operatorname{Im} d_n \subset \mathfrak{m}P_n$ for all $n \geq 0$. A minimal projective resolution of a module $M \in \mathsf{GrMod}(A)$ is a projective resolution $P_\bullet \to M$ in which the complex $P_\bullet$ is minimal. (The minimal injective resolution is not minimal in this sense.)

Consider a complex of free modules. If each module is given a basis, the differential can be represented by matrices. Then the complex is minimal if and only if all the matrix entries belong to $\mathfrak{m}$.

If $(P_\bullet, d_\bullet)$ is a minimal complex, then the differential in the complexes $k \otimes_A P_\bullet$ and $\underline{\operatorname{Hom}}_A(P_\bullet, k)$ are zero. The next example shows that the differential in $\underline{\operatorname{Hom}}_A(k, P_\bullet)$ need not be zero (but see also the proof of Theorem 5.10(1)).

EXAMPLE 2.2. Let $A = k[x, y]$ with $yx = x^2 = 0$. By Exercise 11.3, A is right but not left noetherian. Its left socle is kx, but its right socle is zero. The complex $d : A \to A$ defined by $d(a) = ay$ is minimal. If $f : k \to A$ is a non-zero map into the left socle, then $\operatorname{Im}(df) = d(kx) = kxy \neq 0$. Hence, if $\underline{\operatorname{Hom}}_A(k, -)$ is applied to a minimal complex, the differential in the resulting complex need not be zero.

EXAMPLE 2.3. If $f : L \to M$ is minimal, then $f(L) \subset \mathfrak{m}M$ implies that the map $k \otimes_A L \to k \otimes_A M$ is zero. However, the map $\operatorname{Tor}_i^A(k, L) \to \operatorname{Tor}_i^A(k, M)$ may be non-zero for $i > 0$. This happens when $A = k[x]$, $L = A/(x)[-1]$, $M = A/(x^2)$, and $f : L \to M$ is the inclusion. Because $\operatorname{gldim} A = 1$, the map $\operatorname{Tor}_1^A(k, L) \to \operatorname{Tor}_1^A(k, M)$ is injective, but $\operatorname{Tor}_1^A(k, L) \neq 0$.

LEMMA 2.4. *Every left bounded module has a minimal projective resolution.*

PROOF. Let A be the algebra and M a left bounded graded A-module. Let V_0 be a graded subspace of M such that $M = V_0 \oplus \mathfrak{m}M$. Let $P_0 := A \otimes_k V_0$ have the tensor product grading and define $\epsilon : P_0 \to M$ by $\epsilon(x \otimes v) = xv$. Nakayama's Lemma shows that $\operatorname{coker} \epsilon = 0$ and, by definition of V_0, $\ker \epsilon \subset \mathfrak{m} \otimes V_0 = \mathfrak{m}P_0$. Let V_1 be a graded subspace of $\ker \epsilon$ such that $\ker \epsilon = V_1 \oplus \mathfrak{m} \ker \epsilon$, and define $d_0 : P_1 : A \otimes_k V_1 \to P_0$ by $d_0(x \otimes v) = xv$. Then $\ker d_0 \subset \mathfrak{m}P_1$, so we may continue in this way to obtain a minimal resolution of M. □

A minimal resolution is unique up to isomorphism of complexes (see Exercise 3), and any other resolution is a direct sum of the minimal resolution and a resolution of the zero module. Over a connected algebra the syzygy modules $\Omega^i M$ can be defined up to isomorphism by defining them in terms of a minimal resolution.

PROPOSITION 2.5. *Let $(P_\bullet, d_\bullet)$ be a minimal projective resolution of M. If $(Q_\bullet, \delta_\bullet) \to M$ is any complex in $\mathsf{GrMod}(A)$, there is a morphism of complexes of graded A-modules $\psi_\bullet : P_\bullet \to Q_\bullet$ lifting the identity on M. In particular, any two minimal resolutions of M are isomorphic as chain complexes.*

We will speak of *the* minimal resolution of a module.

PROPOSITION 2.6. *Let $A \otimes_k V_\bullet$ and $E^\bullet$ be minimal projective and injective resolutions respectively of $M \in \mathsf{GrMod}(A)$. Then, for all $j \geq 0$, there are isomorphisms of graded vector spaces*

$$V_j \cong \underline{\operatorname{Ext}}_A^j(M, k)^* \cong \operatorname{Tor}_j^A(k_A, M)$$

and

$$\underline{\mathrm{Ext}}^j_A(k,M) \cong \underline{\mathrm{Hom}}_A(k,E^j).$$

Proof. The first two isomorphisms are clear from the remark after Definition 2.1. If $m \in \mathrm{soc}(E^j)$, then Am is 1-dimensional, so must belong to the essential submodule dE^{j-1} of E^j. Therefore, the maps in the complex $\underline{\mathrm{Hom}}_A(k,E^\bullet)$ are all zero, so the final isomorphism follows. □

Theorem 2.7. *If* $M \in \mathsf{GrMod}(A)$ *is bounded below, then*

1. $\mathrm{pdim}\, M \le n$ *if and only if* $\mathrm{Tor}^A_{n+1}(k_A,M)=0$;
2. $\mathrm{pdim}\, k_A = \max\{n \mid \mathrm{Tor}^A_n(k_A,k) \neq 0\} = \mathrm{pdim}\, k$*;*
3. $\mathrm{pdim}\, k_A = \max\{n \mid \underline{\mathrm{Ext}}^n_A(k,k) \neq 0\} = \mathrm{pdim}\, k$*;*
4. $\mathrm{gldim}\, A = \mathrm{pdim}\, k$.

Proof. Let $A \otimes V_\bullet \to M$ be a minimal resolution.

(1) Since $\mathrm{Tor}_{n+1}(k_A,M) \cong V_{n+1}$, the result is clear.

(2) The second equality follows from (1), and the first follows from the second by replacing A by A^{op}.

(3) This follows from (2) and Proposition 2.6.

(4) It is clear that $\mathrm{gldim}\, A \ge \mathrm{pdim}\, k$. Conversely, Auslander's Theorem (7.7.7) gives $\mathrm{gldim}\, A = \sup\{\mathrm{pdim}(A/I) \mid I \text{ is a graded left ideal}\}$. But $\mathrm{pdim}(A/I) \le \mathrm{pdim}\, k$ by (1) and (2). □

Theorem 2.7(1) can be rephrased as

$$\mathrm{pdim}\, M = \max\{n \mid \mathrm{Tor}^A_n(k_A,M) \neq 0\} = \max\{n \mid \mathrm{Ext}^n_A(M,k) \neq 0\}.$$

If $\mathrm{gldim}\, A = \infty$, then $\mathrm{injdim}\, k = \infty$ because $\mathrm{Tor}^A_i(k_A,k) \cong \underline{\mathrm{Ext}}^i_A(k,k)^*$.

Lemma 2.8. *If A is a connected, noetherian k-algebra of finite global dimension, then* $\mathrm{injdim}\, M = \mathrm{gldim}\, A$ *for all* $0 \neq M \in \mathsf{grmod} A$.

Proof. By Theorem 2.7(3), if $\mathrm{gldim}\, A = n$, then $\mathrm{Ext}^n_A(k,k) \neq 0$. However, if $M \in \mathsf{grmod}(A)$, there is a surjection $M \to k$, and hence an exact sequence $\mathrm{Ext}^n_A(k,M) \to \mathrm{Ext}^n_A(k,k) \to 0$, so $\mathrm{injdim}\, M \ge n$. □

Lemma 2.9. *Let* $P,Q \in \mathsf{grmod}(A)$ *be free, and* $f \in \underline{\mathrm{Hom}}_A(P,Q)$.

1. *If $k \otimes_A f$ is injective, then f is a split monomorphism.*
2. *Suppose that* $\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(k,A) \neq 0$*. If f is injective, then it splits.*

Proof. (1) Consider the factorization $M \xrightarrow{\alpha} fM \xrightarrow{\beta}$. Since $k \otimes_A f$ is injective, $k \otimes_A \alpha$ is bijective, and $k \otimes_A \beta$ is injective. From the exat sequence $0 \to k \otimes_A \ker f \to k \otimes_A M \to k \otimes_A fM \to 0$, it follows that $k \otimes_A \ker f = 0$. By Nakayama's lemma, $\ker f = 0$. Hence we have an exact sequence $0 \to M \to N \to \mathrm{coker}\, f \to 0$. Because $k \otimes_A \beta$ is injective, the first tem in the exact sequence $0 \to \mathrm{Tor}^A_1(k, \mathrm{coker}\, \beta) \to k \otimes_A fM \to k \otimes_A N$ is zero. Hence $\mathrm{coker}\, \beta = \mathrm{coker}\, f$ is free, whence f splits.

(2) Let $i : k \to A$ be a non-zero map of *right* modules. Then there is a commutative diagram

$$\begin{tikzcd} k \otimes_A P \arrow[r, "i \otimes P"] \arrow[d, "k\otimes f"'] & A \otimes_A P \cong P \arrow[d, "f"] \\ k \otimes_A Q \arrow[r, "i \otimes Q"] & A \otimes_A Q \cong Q \end{tikzcd}$$

The horizontal maps are injective since $-\otimes P$ and $-\otimes Q$ are exact functors. By hypothesis, the right hand vertical map is injective, hence so is the left hand one, $k \otimes f$. Therefore f splits by (1). $\square$

Notice that the hypothesis in part (2) of Lemma 2.9 is on *right* modules. It is not enough to make the hypothesis on left modules: if $A = k[x, y]$ with relations $x^2 = yx = 0$, then there is a left module map $k \to A$, but the injective $f : A \to A$ given by $f(a) = ay$ does not split. However, if A is noetherian, then it is sufficient to assume that there is a *left* module map $k \to A$, because that then implies the existence of a *right* module map $k \to A$; if $\operatorname{Hom}_A(k, A) \neq 0$, then the left torsion submodule of A is non-zero, finitely generated, hence finite dimensional, and is a two-sided ideal, so is contained in the *right* torsion submodule. Thus, in the next result we are able to assume that $\operatorname{Hom}_A(k, A) \neq 0$, and still apply Lemma 2.9.

COROLLARY 2.10. *If A is noetherian and $\underline{\operatorname{Hom}}_A(k, A) \neq 0$, then every non-free module in $\mathsf{GrMod}(A)$ has infinite projective dimension.*

PROOF. It suffices to show that no finitely generated module has projective dimension one; but such a module is the cokernel of an injective map $f : P \to Q$ between free modules, so by Lemma 2.9 is free. $\square$

COROLLARY 2.11. *If A is finite dimensional and $A \neq k$, then* $\operatorname{gldim} A = \infty$.

PROOF. This follows from the previous corollary because such an A possesses a non-free module. $\square$

THEOREM 2.12. *Let A be finitely generated, commutative, and connected. Then* $\operatorname{gldim} A < \infty$ *if and only if A is a polynomial ring.*

PROOF. Suppose $\operatorname{gldim} A < \infty$, and let $x_1, \dots, x_n$ be a minimal set of homogeneous generators. Then the local ring $B = A_{\mathfrak{m}}$ also has finite global dimension, and $\mathfrak{m}/\mathfrak{m}^2 = kx_1 + \cdots + kx_n$, so $\operatorname{gr} B \cong k[x_1, \dots, x_n]$, the polynomial ring. Hence there are no relations between the x_i in B, and hence none in A (other than the commutativity relations). So A is a polynomial ring. $\square$

EXERCISES

2.1 Show that the minimal projective resolution of a module is unique up to isomorphism.

2.2 Let $P_\bullet$ be a minimal resolution of M. If r is the lower bound for M (i.e., $M_r \neq 0$ and $M_i = 0$ for all $i < r$), show that each P_n is bounded below, and its lower bound is $\geq r + n$.

2.3 Use Exercise 2 to show that if $P_\bullet \to M \to 0$ is a minimal projective resolution in $\mathsf{GrMod}(A)$ and $Q_\bullet \to M \to 0$ is an arbitrary projective resolution in $\mathsf{GrMod}(A)$, then $Q_\bullet = P_\bullet \oplus X_\bullet$ where $X_\bullet$ is an acyclic complex of projectives.

2.4 Show the necessity of the hypothesis that M be left bounded in Lemma 2.4 by showing that $M = k[X, X^{-1}]$ does not have a minimal resolution as a $k[X]$-module.

2.5 Suppose that $M, N \in \mathsf{GrMod}(A)$ with $M/\mathfrak{m}M \in [a, b]$ and $N \in [c, d]$. Show that $\underline{\operatorname{Hom}}_A(M, N) \in [c - b, a - d]$.

2.6 Suppose that $M, N \in \mathsf{GrMod}(A)$ with $M \in [a, b]$ and $N \in [c, d]$.
(a) If $P_\bullet \to N$ is a minimal resolution, show that $P_i \in [a + i, \infty]$.
(b) Show that $\underline{\operatorname{Ext}}^i_A(M, N) \in [-\infty, d - a - i]$.
(c) If $\underline{\operatorname{Ext}}^i_A(k, k) \in [e, f]$, and M and N are finite dimensional, show that $\underline{\operatorname{Ext}}^i_A(M, N) \in [e + c - b, f + d - a]$.

2.7 Suppose that $\operatorname{pdim} M = d$. Show that $\underline{\mathrm{Ext}}_A^d(M,N) \neq 0$ for all $0 \neq N \in \mathsf{grmod}(A)$.

3. Generators and relations

In this section A denotes a connected k-algebra.

Definition 3.1. Let A be an $\mathbb{N}$-graded k-algebra. A subset $\mathcal{G} \subset A$ of homogeneous elements is a generating set if it generates A as a k-algebra, and is a minimal generating set if no proper subset of it does.

If $\mathcal{G}$ generates A, and $A \cong k\langle\mathcal{G}\rangle/I$, then a subset $\mathcal{R} \subset I$ of homogeneous elements is minimal set of relations for A, with respect to $\mathcal{G}$, if $\mathcal{R}$ generates I as an ideal, but no proper subset of $\mathcal{R}$ does.

The next result says that minimal sets of generators and relations are, to some extent, unique; a minimal set of generators can be identified with a homogeneous basis for $\mathrm{Tor}_1^A(k,k)$, and a minimal set of relations (with respect to a minimal set of generators) can be identified with a homogeneous basis for $\mathrm{Tor}_2^A(k,k)$.

PROPOSITION 3.2. *Let $R = k\langle x_1,\dots,x_n\rangle$ be a connected free algebra, and let $f_1,\dots,f_r$ be homogeneous elements of R, each of degree ≥ 2. Define $A = R/(f_1,\dots,f_r)$, and let $\Psi : R \to A$ be the natural map. Write $f_i = \sum_{j=1}^n m_{ij}x_j$ and define the $r \times n$ matrix $L = (\Psi(m_{ij}))$ over A. If $\{f_1,\dots,f_r\}$ is a minimal set of relations for A, then the minimal resolution of k begins*

$$A^r \xrightarrow{L} A^n \xrightarrow{\underline{x}} A \to k \to 0, \tag{3-1}$$

where $\underline{x} = (x_1,\dots,x_n)^{\mathsf{T}}$.

PROOF. Since $L\underline{x} = 0$ in A, (3-1) is a complex. It is clearly exact at A. We now show it is exact at A^n. Write $I = Rx_1 + \dots + Rx_n$ and $J = (f_1,\dots,f_r)$. Suppose $(a_1,\dots,a_n) \in \ker \underline{x}$ and let $r_1,\dots,r_n \in R$ be preimages of the a_i's. Then $\sum_{i=1}^n r_ix_i \in J$. Since $R = k \oplus I$, for any $f \in R$, $(f) = Rf + (f)x_1 + \dots + (f)x_n$, whence $J = Rf_1 + \dots + Rf_r + Jx_1 + \dots + Jx_n$. Hence

$$\sum_{i=1}^n r_ix_i = \sum_{i=1}^r s_if_i + \sum_{i=1}^n t_ix_i = \sum_{i=1}^r s_i \sum_{j=1}^n m_{ij}x_j + \sum_{i=1}^n t_ix_i$$

with each $t_i \in J$. But $Rx_1 + \dots + Rx_n$ is a *direct* sum so, for each j,

$$r_j = \sum_{i=1}^r s_im_{ij} + t_j.$$

Applying Ψ to this equality gives $(a_1,\dots,a_n) = (b_1,\dots,b_r)L$ where $b_i = \Psi(s_i)$, as required.

To see that (3-1) is the start of a minimal resolution, we must show that the rows of L, say $\ell_1,\dots,\ell_r \in A^n$, are a basis for a complement to $\mathfrak{m}.\ker\underline{x}$ in $\ker\underline{x}$. We already know they span a complement by the proof of exactness. If they are not linearly independent, then there are scalars $\alpha_i \in k$, not all zero, such that $\sum_{i=1}^r \alpha_i\ell_i \in \mathfrak{m}\ell_1 + \dots + \mathfrak{m}\ell_r$. Lifting this back to R, and multiplying on the right by $\underline{x}$, we obtain

$$\sum_{i=1}^r \alpha_if_i \in R_{\geq 1}f_1 + \dots + R_{\geq 1}f_r + (f_1,\dots,f_r)R_{\geq 1}. \tag{3-2}$$

Suppose $\alpha_j \neq 0$, and let $d = \deg(f_j)$. Taking the degree d components of (3-2), it follows that f_j is in the ideal generated by the other f_i, contradicting the fact that $\{f_1, \dots, f_r\}$ is a minimal set of relations for A. $\square$

COROLLARY 3.3. *The number and the degrees of a minimal set of generators and a minimal set of relations for a connected algebra are uniquely determined by the algebra. In particular,* $\dim_k \operatorname{Tor}_1^A(k,k)_i$ *and* $\dim_k \operatorname{Tor}_2^A(k,k)_i$ *are the minimal number of generators and relations of degree* i.

PROOF. First, a minimal resolution of the trivial module is unique up to isomorphism and, second, by Proposition 3.2, this data may be read off from a minimal resolution. $\square$

COROLLARY 3.4. *A connected noetherian k-algebra is finitely presented.*

PROOF. The number of generators and relations required to define a connected algebra A are given by $\dim_k \operatorname{Tor}_1^A(k,k)$ and $\dim_k \operatorname{Tor}_2^A(k,k)$. Since A is noetherian, and k is finitely generated, these are finite dimensional (say, by Proposition 2.5). $\square$

4. Hilbert series and GK-dimension

In this section A denotes a connected k-algebra.

If A is left noetherian, then every $M \in \mathsf{grmod}(A)$ is locally finite by Lemma 11.1.1, so has a Hilbert series.

Definition 4.1. Let A be connected, and suppose $M \in \mathsf{grmod}(A)$ has a minimal projective resolution of the form

$$\cdots \to \bigoplus_{j=1}^{r_d} A[-\ell_{dj}] \to \cdots \to \bigoplus_{j=1}^{r_0} A[-\ell_{0j}] \to M \to 0. \tag{4-1}$$

The function

$$q_M(t) := \sum_{i=0}^{\infty} (-1)^i \sum_{j=0}^{r_i} t^{\ell_{ij}} \tag{4-2}$$

is called the characteristic polynomial of M.

THEOREM 4.2. *Let A be left noetherian, connected, and $M \in \mathsf{grmod}(A)$. Then*
1. $q_M(t) := H_M(t).H_A(t)^{-1}$;
2. *if* $\operatorname{pdim} M < \infty$, *then* $q_M(t) \in \mathbb{Z}[t, t^{-1}]$;
3. *if* $\operatorname{pdim} M < \infty$ *and N is locally finite, then*

$$\sum_i (-1)^i H_{\underline{\operatorname{Ext}}^i(M,N)}(t) = q_M(t^{-1}) H_N(t);$$

4. *If* $H_A(t)^{-1} \notin \mathbb{Z}[t]$ *then* $\operatorname{gldim} A = \infty$.

PROOF. (1) If $P_\bullet \to M$ is the minimal resolution, then $H_M(t) = \sum_{i=0}^{\infty} (-1)^i H_{P_i}(t)$, so the result follows immediately.

(2) Suppose $\operatorname{pdim} M = n$. Then only finitely many P_i are non-zero, and because M is finitely generated and A is noetherian, the numbers r_i in (4-2) are finite, so $q_M(t) \in \mathbb{Z}[t, t^{-1}]$.

(3) Applying $\underline{\mathrm{Hom}}_A(-,N)$ to the projective resolution (4-1) yields the complex

$$\cdots \leftarrow \bigoplus_{j=1}^{r_d} N[\ell_{dj}] \leftarrow \cdots \leftarrow \bigoplus_{j=1}^{r_1} N[\ell_{1j}] \leftarrow \bigoplus_{j=1}^{r_0} N[\ell_{0j}] \leftarrow 0, \tag{4-3}$$

the homology of which is $\underline{\mathrm{Ext}}^*_A(M,N)$. The result now follows from Lemma 11.9.2.

(4) This follows at from (3) because $H_A(t)^{-1} = q_k(t)$; however, also notice that no negative powers of t occur in $q_k(t)$ because k being concentrated in degree zero forces all ℓ_{ij} in (4-1) to be non-negative. □

If $N_i \neq 0$ for all $i \geq a$, then $N[\ell]_a \neq 0$ for all $\ell \geq 0$, so if $\operatorname{pdim} M = \infty$ then each term in (4-3) would have a non-zero component of degree a, and the product on the right hand side of Theorem 4.2(3) would not make sense.

EXAMPLE 4.3. If $A = k[x_1, \cdots, x_n]$ is the polynomial ring with its standard grading, and $0 \neq f \in A_d$ with $d \geq 2$, then $\operatorname{gldim} A/(f) = \infty$. There is an exact sequence $0 \to (f) \to A \to A/(f) \to 0$, and $f \in A_d$, so $fA \cong A[-d]$ whence $H_{fA}(t) = t^d H_A(t)$, and

$$H_{A/(f)}(t) = (1-t^d)H_A(t) = (1-t^d)(1-t)^{-n}.$$

Since $d \geq 2$, the inverse of this is not in $\mathbb{Z}[t]$, so Theorem 4.2 shows that $\operatorname{gldim} A/(f) = \infty$.

EXAMPLE 4.4. There is a connected, noetherian, domain which is not a quotient of any connected noetherian algebra of finite global dimension. Let B be the ring of Stafford and Zhang in Example 6.2.3. It has Hilbert series $(1-t+t^2)(1-t)^{-2}$. If B is a quotient of a connected noetherian algebra A of finite global dimension, then $H_B(t)$

THEOREM 4.5. *Let A be a left noetherian, connected and of finite global dimension. If $M \in \mathsf{grmod}(A)$, then*

1. *there exist $d_i \in \mathbb{N}$ and $f(t) \in \mathbb{Z}[t,t^{-1}]$ such that*

$$H_M(t) = f(t)\prod_{i=1}^{s}(1-t^{d_i})^{-1}$$

2. $\mathrm{GKdim}\, M$ *is the order of the pole of $H_M(t)$ at $t=1$.*

PROOF. (1) Taking $M = k$ in the formula $H_M(t) = H_A(t).q_M(t)$, we see that $H_A(t) \in \mathbb{Q}(t)$, whence $H_M(t) \in \mathbb{Q}(t)$ for all $M \in \mathsf{grmod}(A)$. By Lemma 2.6, $H_M(t)$ has the form claimed.

(2) By repeated application of Lemma 5.2.4(3), we may replace $H_M(t)$ by

$$H(t) := H_M(t)\prod_{i=1}^{s}(1+t+\cdots+t^{d_i-1});$$

note that $H_M(t)$ and $H(t)$ have poles of the same order, say d, at $t=1$. Now

$$H(t) = t^{-s}f(t)(1-t)^{-d},$$

where $f(t) \in \mathbb{Z}[t]$ satisfies $f(1) \neq 0$ and $s \geq 0$. Write $f(t) = \sum_{i=0}^{p} b_i t^i$ and $H(t) = \sum a_n t^n$. The coefficients of the power series

$$(1-t)^{-d} = \sum_{n=0}^{\infty}\binom{d+n-1}{d-1}t^n$$

are polynomials of degree $d-1$ in n. Since

$$a_n = b_0\binom{n+d-1}{d-1} + \cdots + b_p\binom{n+d-p-1}{d-1}$$

for $n \gg 0$, a_n is also a polynomial of degree $d-1$ for large n. Therefore, if $\epsilon > 0$ is given, then $a_n \le n^{d+\epsilon-1}$ for $n \gg 0$, and $a_n \ge n^{d-\epsilon-1}$ for infinitely many n. Thus $\operatorname{GKdim} M = d$, as required. □

PROPOSITION 4.6. *Let A be a finitely generated, connected, commutative k-algebra. Then* $\operatorname{GKdim} A$ *is an integer. Moreover,*

$$\operatorname{GKdim} A = \operatorname{Kdim} A = \operatorname{trdeg}_k A.$$

PROOF. Let $B = k[x_1, \dots, x_r]$ be the polynomial ring with $\deg(x_i) = d_i \ge 1$. Then

$$H_B(t) = \prod_{i=1}^{r}(1-t^{d_i})^{-1}.$$

This has a pole of order r at $t = 1$, so $\operatorname{GKdim} B = r$ by Theorem 4.5.

Now suppose A is a graded quotient of a commutative polynomial algebra having generators in arbitrary positive degree. By the graded version of Noether Normalization there exist algebraically independent, homogeneous elements $b_1, \dots, b_r \in A$ such that A is a finitely generated module over $k[b_1, \dots, b_r]$. By the previous paragraph $\operatorname{GKdim} k[b_1, \dots, b_r] = r$ whence $\operatorname{GKdim} A = r$ by Lemma 2.4.

Since $\operatorname{trdeg}_k(k[b_1, \dots, b_r]) = r$ and B is a finite module over this subring, $\operatorname{trdeg}_k(B) = r$ also. Hence the equalities. □

THEOREM 4.7. *Let A be a finitely generated, commutative, connected k-algebra which is generated in degree 1. Then* $\operatorname{gldim} A < \infty$ *if and only if A is a polynomial ring. (The hypothesis that the algebra be generated in degree one is unnecessary by Theorem 2.12.)*

PROOF. Suppose $\operatorname{gldim} A < \infty$. Let $x_1, \dots, x_n$ be a basis for A_1, let $R = k[X_1, \dots, X_n]$ be the polynomial ring on n variables with its usual grading, and write $A = R/J$ where J is a graded ideal. Since $\operatorname{gldim} R < \infty$, Theorem 4.2 applies to A viewed as an R-module; thus

$$H_A(t) = (1-t)^{-n}q(t)$$

for some $q(t) \in \mathbb{Z}[t, t^{-1}]$. Therefore, by Theorem 4.2(4), $H_A(t)^{-1} \in \mathbb{Z}[t, t^{-1}]$, whence

$$(1-t)^n = q(t)f(t)$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$. This implies that $q(t) = t^r(1-t)^s$ for some $r \in \mathbb{Z}$ and some $s \ge 0$, so

$$H_A(t) = t^r(1-t)^{s-n} = t^r(1+(n-s)t+\dots).$$

Since $\dim A_0 = 1$ and $\dim A_1 = n$ it follows that $r = s = 0$, whence $H_A(t) = (1-t)^{-n} = H_R(t)$. Hence the surjective map $R \to A$ is an isomorphism. □

We now introduce two ways of comparing the size of two modules of the same Gelfand-Kirillov dimension.

Definition 4.8. Let $M \in \mathsf{GrMod}(A)$, and suppose that $\operatorname{GKdim} M = d$.

- The multiplicity of M is
$$e(M) := \lim_{t\uparrow 1} (1-t)^d H_M(t);$$
- the d-length of M is the largest integer ℓ such that there is a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0$, with $\operatorname{GKdim}(M_i/M_{i+1}) = d$ for all i. We denote the d-length of M by $\ell_d(M)$.

The d-length is an additive function on exact sequences of modules of GK-dimension $\geq d$ (it is simply length in the quotient category of $\mathsf{GrMod}(A)$ modulo the modules of GK-dimension $< d$).

LEMMA 4.9. *Let $M \in \mathsf{grmod}(A)$.*

1. *If A is noetherian and* $\operatorname{gldim} A < \infty$*, then $e(M)$ is a positive integer multiple of $e(A)$.*
2. *The order of the zero at $t=1$ of $q_M(t)$ equals* $\operatorname{GKdim} A - \operatorname{GKdim} M$*.*
3. *If $0 \to L \to M \to N \to 0$ is exact and* $\operatorname{GKdim} L = \operatorname{GKdim} M = \operatorname{GKdim} N$*, then*
$$e(M) = e(L) + e(N).$$

PROOF. (1) Suppose $\operatorname{GKdim} M = d$ and $\operatorname{GKdim} A = n$. Then
$$\frac{e(M)}{e(A)} = \lim_{t\uparrow 1}(1-t)^{d-n} q_M(t) \tag{4-4}$$
is an integer because $q_M(t) \in \mathbb{Z}[t, t^{-1}]$. On the other hand, the limit of $H_M(t)$ as t approaches 1 from below is the dimension of M, which is positive, so $e(M)$ is positive. So is $e(A)$, whence $e(M)/e(A) > 0$.

(2) This follows at once from (4-4).

(3) This follows from the additivity of Hilbert series on exact sequences. □

LEMMA 4.10. *Let* $\operatorname{GKdim} A = n$*, and suppose that*
$$0 \to \oplus A[-i]^{b_i} \xrightarrow{f} \oplus A[-i]^{a_i} \to M \to 0$$
is a minimal projective resolution by finite rank free modules. Then

1. $\operatorname{GKdim} M \geq n-1$*;*
2. $\operatorname{GKdim} M = n-1$ *if and only if* $\sum a_i = \sum b_i$*;*
3. *if* $\operatorname{GKdim} M = n-1$*, then* $\sum b_i \leq \sum i(b_i - a_i) = e(M)/e(A)$*.*

PROOF. Since the map f is minimal, we have for each j,
$$f\left(\bigoplus_{i\leq j} A[-i]^{b_i}\right) \subset \bigoplus_{i<j} A[-i]^{a_i}.$$
If X is the quotient of these, then $H_X(t)/H_A(t) = \sum_{i<j} a_i t^i - \sum_{i\leq j} b_i t^i$. Letting t approach 1 from below, we see that $\sum_{i<j} a_i - \sum_{i\leq j} b_i \geq 0$, whence
$$b_j \leq \sum_{i<j}(a_i - b_i)$$

for all j. Choose $m \gg 0$ such that $a_i = b_i = 0$ for $i > m$. Then

$$\begin{aligned}\sum_j b_j = \sum_{j \le m} b_j &\le \sum_{j\le m}\sum_{i<j}(a_i - b_i)\\ &= \sum_{i<m}(m-i)(a_i-b_i)\\ &= m\Big(\sum a_i - \sum b_i\Big) \; + \; \sum i(b_i - a_i).\end{aligned}$$

By Lemma 4.9, $\operatorname{GKdim} A - \operatorname{GKdim} M$ is the order of the zero at $t = 1$ of $q_M(t) = \sum(a_i - b_i)t^i$. Thus $\operatorname{GKdim} M < n$ if and only if $\sum a_i = \sum b_i$.

Now suppose that $\operatorname{GKdim} M < n$. Then $\sum i(b_i - a_i) \ge \sum b_j > 0$. But

$$\sum i(b_i - a_i) = \lim_{t\uparrow 1} \frac{q_M(t)}{1-t},$$

whence $\operatorname{GKdim} M = n-1$, and $e(M)/e(A) = \sum i(b_i - a_i)$. □

PROPOSITION 4.11. *Let A be connected noetherian. If $\operatorname{GKdim} A = n$ and $\operatorname{gldim} A < \infty$, then*

1. *the n-length of A is one;*
2. *if N is the largest ideal of A such that $\operatorname{GKdim} N < n$, then A/N is a domain;*
3. $\operatorname{pdim} N \le \operatorname{gldim} A - 2$.

PROOF. (1) By the noetherian hypothesis there is some $M \in \mathsf{grmod}(A)$ such that $\ell_n(M) = 1$. But M has a finite projective resolution by modules which are finite direct sums of shifts of A, so the additivity of $\ell_n(-)$ forces $\ell_n(M)$ to be an integer multiple of $\ell_n(A)$, whence $\ell_n(A) = 1$.

(2) First notice that the sum of all left ideals of A of GK-dimension $< n$ is a two-sided ideal having GK-dimension $< n$. Now suppose that $0 \ne \bar{a} \in A/N$, and let $J = \{x \in A \mid x\bar{a} = 0\}$. Since $a \notin N$, $\operatorname{GKdim}(A\bar{a}) = n$, whence $\ell_n(A\bar{a}) \ge 1$. From the exact sequence $0 \to J \to A \to A\bar{a} \to 0$, it follows that $\ell_n(J) = 0$, whence $\operatorname{GKdim} J < n$. Thus $J \subset N$, which shows that $\bar{a}$ is a left regular element of A/N. Hence A/N is a domain.

(3) Let $I = \{x \in A \mid Nx = 0\}$. By Lemma 5.2.13, $\operatorname{GKdim}(A/I) = \operatorname{GKdim} N < n$, whence $\operatorname{GKdim} I = n$. Thus $I \not\subset N$. Choose $x \in I \backslash N$, and define $\rho : A \to A$ by $\rho(a) = ax$. Then $N \subset \ker\rho$; but A/N is a domain so $\ker\rho \subset N$. Hence there is an exact sequence $0 \to N \to A \xrightarrow{\rho} A \to A/Ax \to 0$, from which the result follows. □

EXERCISES

4.1 Let A be a left noetherian connected k-algebra. Show that the graded A-modules of GK-dimension $< d$, for a fixed d, form a dense subcategory of $\mathsf{grmod}(A)$. Hence a d-critical module gives an irreducible object in the quotient category.

4.2 Show that the only finite dimensional connected algebra of finite global dimension is k.

Suppose the connectedness hypothesis is weakened, and we assume that A is $\mathbb{N}$-graded and A_0 is a semisimple k-algebra. Set $\mathfrak{m} = A_{\ge 1}$. Then Definition 2.1 and Lemma 2.4 may be extended to cover this situation—it is still true that every left bounded module has a minimal resolution; the main point to be altered in the proof of Lemma 2.4 is to choose V_0 to be a graded A_0-submodule such that $M = V_0 \oplus \mathfrak{m}M$, and define $P_0 = A \otimes_{A_0} V_0$.

CHAPTER 15

Homological properties of connected algebras

In this chapter A denotes a connected k-algebra.

In section 1 we prove a non-commutative version of the Auslander-Buchsbaum formula. This is due to Jorgensen [**88**], where a more general version is proved for complexes in the derived category.

In section 5 we show that the condition χ can be expressed in terms of local cohomology, and we use this to show that left noetherian Cohen-Macaulay rings satisfy condition χ.

1. Depth

Definition 1.1. If $M \in \mathsf{grmod}(A)$, define

$$\operatorname{depth} M := \inf\{i \mid \underline{\operatorname{Ext}}^i_A(k, M) \neq 0\},$$

or ∞ if no such i exists.

Notice that a module is torsion-free if and only if its depth is ≥ 1. Thus depth allows a more subtle classification and investigation of modules.

A key point is to decide whether all non-zero finitely generated modules have finite depth. Unfortunately, this is not always so (example???), but it is for large classes of naturally occurring algebras. Indeed, depth can be finite for all non-zero finitely generated modules, even though the algebra itself has infinite global dimension; this allows depth to be used for inductive arguments. For the moment, we just note the following result.

Proposition 1.2. *If* $\operatorname{gldim} A < \infty$*, then* $\operatorname{depth} M \leq \operatorname{gldim} A$ *for all* $0 \neq M \in \mathsf{grmod}(A)$.

Proof. If $\operatorname{gldim} A = n$, then $\underline{\operatorname{Ext}}^n(k, k) \neq 0$ by Theorem 2.7 and there is a surjection $M \to k$ which yields a surjection $\underline{\operatorname{Ext}}^n_A(k, M) \to \underline{\operatorname{Ext}}^n_A(k, k)$. □

The next result, which will be referred to as the Auslander-Buchsbaum formula, is the analogue of the corresponding result for noetherian commutative local rings.

Theorem 1.3. *Suppose that A is left noetherian, and satisfies condition χ. Then* $\operatorname{depth} A < \infty$ *and, if* $M \in \mathsf{grmod}(A)$ *has finite projective dimension, then*

$$\operatorname{pdim} M + \operatorname{depth} M = \operatorname{depth} A.$$

Proof. We will use the convergent spectral sequence

$$\operatorname{Tor}^A_{-p}(\underline{\operatorname{Ext}}^q_A(k, A), M) \Rightarrow \underline{\operatorname{Ext}}^{p+q}_A(k, M).$$

Taking $M = k$, the right hand side is non-zero for $p + q = 0$, so some term on the left hand side is non-zero, whence $\operatorname{depth} A \neq \infty$. Write $t = \operatorname{pdim} M$ and

$n = \operatorname{depth} A$. Notice that $\operatorname{Tor}_t^A(\underline{\operatorname{Ext}}_A^n(k, A), M)$ survives to the E_∞ page, so is isomorphic to $\underline{\operatorname{Ext}}_A^{n-t}(k, M)$; to see that this is non-zero, first notice that $\underline{\operatorname{Ext}}_A^n(k, A)$ is finite dimensional because A satisfies χ, and is non-zero by definition of n; second, $\operatorname{Tor}_t^A(k, M) \neq 0$ by Proposition 14.2.6, so $\operatorname{Tor}_t^A(N, M) \neq 0$ for all finite dimensional N; in particular, $0 \neq \operatorname{Tor}_t^A(\underline{\operatorname{Ext}}_A^n(k, A), M) \cong \underline{\operatorname{Ext}}_A^{n-t}(k, M) \neq 0$. Therefore $\operatorname{depth} M \leq n - t = \operatorname{depth} A - \operatorname{pdim} M$. On the other hand, if $p + q < n - t$, then either $-p > t$ or $q < n$, so $\operatorname{Tor}_{-p}^A(\underline{\operatorname{Ext}}_A^q(k, A), M) = 0$, whence $\underline{\operatorname{Ext}}_A^{p+q}(k, M) = 0$. Therefore $\operatorname{depth} M \geq n - t = \operatorname{depth} A - \operatorname{pdim} M$. □

COROLLARY 1.4. *Suppose that A is left noetherian, and satisfies condition χ. If* $\operatorname{gldim} A < \infty$, *then* $\operatorname{depth} A = \operatorname{gldim} A$.

PROOF. Since $\operatorname{depth} k = 0$ and $\operatorname{pdim} k = \operatorname{gldim} A$, the result follows from the Auslander-Buchsbaum formula applied to the module k. □

LEMMA 1.5. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in* $\mathsf{GrMod}(A)$. *Write $\ell = \operatorname{depth} L$, $m = \operatorname{depth} M$, and $n = \operatorname{depth} N$. Then*

1. $\ell \geq \min\{m, n+1\}$;
2. $m \geq \min\{\ell, n\}$;
3. $n \geq \min\{\ell - 1, m\}$.

PROOF. To show that $\operatorname{depth} X \geq \min\{p, q\}$, we must show that $\underline{\operatorname{Ext}}_A^i(k, X)$ is zero if $i < p$ and $i < q$. The result follows from careful examination of the long exact sequence for $\underline{\operatorname{Ext}}_A^*(k, -)$. □

2. Depth for commutative local rings

In this section R denotes a commutative ring.

The notion of depth in the previous section is the obvious analogue of the commutative notion, which we briefly explore in this section. The main result is to show that the depth, as defined in the next definition, can be expressed in terms of regular sequences.

Definition 2.1. If M is a module over a commutative local ring $(R, \mathfrak{m}, k)$, the depth of M is

$$\operatorname{depth} M = \inf\{i \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

Definition 2.2. If M is an R-module, we call $x_1, \dots, x_n \in R$ an M-sequence if $M \neq x_1M + \cdots + x_nM$ and, for all i, x_{i+1} acts injectively on $M/x_1M + \cdots + x_iM$.

Definition 2.3. A prime ideal $\mathfrak{p}$ in R is associated to an R-module M if $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$. The set of associated primes of M is denoted $\operatorname{Ass}(M)$.

If $\mathfrak{p} = \operatorname{Ann}(m)$, then $R/\mathfrak{p} \cong Rm$, so $M_\mathfrak{p} \neq 0$, whence

$$\operatorname{Ass}(M) \subset \operatorname{Supp}(M).$$

In particular, $\operatorname{Ass}(M)$ is finite if R and M are noetherian.

PROPOSITION 2.4. *Let R be noetherian, I an ideal, and M an R-module. Then I contains an M-regular element if and only if* $\operatorname{Hom}_R(R/I, M) = 0$.

PROOF. ($\Rightarrow$) If $\varphi : R/I \to M$ is an R-module map, and $x \in I$, then $x.\varphi(1) = \varphi(x) = 0$, so $\varphi = 0$.

($\Leftarrow$) Suppose to the contrary that $0 \neq x \in I$ and $0 \neq m \in M$ satisfy $xm = 0$. Let $0 \neq m' \in M$ be such that $\mathrm{Ann}(m')$ is as large as possible subject to containing x; then $\mathfrak{p} := \mathrm{Ann}(m')$ is prime (see Exercise 1), hence in $\mathrm{Ass}(M)$. Thus

$$I \subset \bigcup \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M)\}.$$

But this is a finite union of primes, so I is contained in one of them, say $I \subset \mathfrak{p}$. Since $R/\mathfrak{p}$ embeds in M, there is a non-zero map $R/I \to M$. $\square$

The previous proof yields the following characterization of the associated primes.

COROLLARY 2.5. *Let R be noetherian, and M an R-module. Then*

$$\{x \in R \mid x \textit{ is not } M\textit{-regular}\} = \bigcup \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M)\}.$$

THEOREM 2.6 (Rees). *Let R be a commutative noetherian ring, $M \in \mathsf{mod}(R)$, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have length equal to*

$$\inf\{i \mid \mathrm{Ext}^i_R(R/I, M) \neq 0\}.$$

PROOF. Let $x_1, \ldots, x_n \in I$ be a maximal M-sequence. We argue by induction on n. If $n = 0$, then I has no M-regular element, so the result is given by Proposition 2.4. Suppose now that $n \geq 1$, and set $N := M/x_1M$; notice that $x_2, \ldots, x_n$ is a maximal N-sequence in I.

The exact sequence $0 \to M \xrightarrow{x_1} M \to N \to 0$ yields a long exact sequence for $\mathrm{Ext}^*_R(R/I, -)$, in which the maps $\mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M)$ are induced by multiplication by x_1; but $x_1 \in I$ which annihilates these Ext groups, so the long exact sequence breaks into short exact sequences

$$0 \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, N) \to \mathrm{Ext}^{i+1}_R(R/I, M) \to 0.$$

If $i \leq n-2$, the middle term of this is zero by the induction hypothesis, whence $\mathrm{Ext}^i_R(R/I, M) = 0$ for $i \leq n-1$. If $i = n-1$, the middle term is non-zero by induction, and we just saw that the first term is zero, so $\mathrm{Ext}^{i+1}_R(R/I, M) \neq 0$. $\square$

EXERCISES

2.1 If M is an R-module, and I is an ideal of R maximal subject to being the annihilator of some non-zero element of M, then I is a prime ideal. (This fact was used in Proposition 2.4.)

2.2 Show that the inequality $j(M) \leq \mathrm{Kdim}\, R - \mathrm{Kdim}\, M$ in Proposition **??** cannot be improved to show that $\mathrm{pdim}\, M$ is bounded above by $\mathrm{Kdim}\, R - \mathrm{Kdim}\, M$. (Hint: suppose M is the maximal ideal in a regular local ring.) The point is that the difference $\mathrm{pdim}\, M - j(M) \geq 0$ can be large.

2.3 Show that $k[x^3, x^2y, xy^2, y^3]$ is Cohen-Macaulay.

2.4 Show that $k[x^4, x^3y, x^2y^2, xy^3, y^4]$ is Cohen-Macaulay.

3. Local cohomology

In this section A is a connected k-algebra.

If $M \in \mathsf{GrMod}(A)$, then

$$\tau M \cong \varinjlim \underline{\mathrm{Hom}}_A(A/\mathfrak{m}^i, M).$$

The local cohomology functors are the right derived functors of the left exact functor $M \mapsto \tau M$ sending a module to its torsion submodule.

Definition 3.1. The n^{th} local cohomology group of $M \in \mathsf{GrMod}(A)$ is

$$H^n_{\mathfrak{m}}(M) := \varinjlim \underline{\mathrm{Ext}}^n_A(A/\mathfrak{m}^i, M).$$

The right action of A on $A/\mathfrak{m}^i$ makes each $\underline{\mathrm{Ext}}^n_A(A/\mathfrak{m}^i, M)$ a left A-module, so $\mathrm{H}^n_{\mathfrak{m}}(M)$ is a graded left A-module. Each $\underline{\mathrm{Ext}}^n_A(A/\mathfrak{m}^i, M)$ is artinian because it is annihilated by $\mathfrak{m}^i$, so $H^n_{\mathfrak{m}}(M)$ is artinian too. But $H^n_{\mathfrak{m}}(M)$ is rarely noetherian (see Lemma 5.6(3)).

If $0 \to L \to M \to N \to 0$ is exact, then there is a long exact sequence

$$0 \to H^0_{\mathfrak{m}}(L) \to H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(N) \to H^1_{\mathfrak{m}}(L) \to H^1_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(N) \to \cdots .$$

In other words, the functors $H^i_{\mathfrak{m}}(-)$ form a covariant δ-functor (see [**76**, Chapter III, Section 1].

If $M \to E^\bullet$ is an injective resolution, then $H^*_{\mathfrak{m}}(M) \cong h^n(\tau E^\bullet)$ because injectives in $\mathsf{GrMod}(A)$ are acyclic for local cohomology. More generally, one has the following result.

Proposition 3.2. *If $M \in \mathsf{GrMod}(A)$, then*

$$\lim_{\rightarrow} \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M) = \mathrm{h}^i(I^\bullet M),$$

where $I^\bullet M$ is the torsion subcomplex of the minimal injective resolution of M.

Proof. Since $I^\bullet M$ is torsion, $I^\bullet M = \lim_{\rightarrow} \underline{\mathrm{Hom}}(A/A_{\geq n}, I^\bullet(M))$; since $\lim_{\rightarrow}$ commutes with taking homology, the result follows. □

Proposition 3.3. *If $p > 0$, then $H^p_{\mathfrak{m}}(k) = 0$.*

Proof. An elementary computation shows that $\underline{\mathrm{Ext}}^1_A(A/\mathfrak{m}^i, k) \cong (\mathfrak{m}^i/\mathfrak{m}^{i+1})^*$, so $\mathrm{H}^1_{\mathfrak{m}}(k)$ is the direct limit of the directed system

$$\cdots \to (\mathfrak{m}^i/\mathfrak{m}^{i+1})^* \to (\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2})^* \to \cdots ,$$

with the maps being given by restriction. Hence each map is zero, so $H^1_{\mathfrak{m}}(k) = 0$.

For $p > 1$, let Ω^{p-1} be the $(p-1)^{\text{th}}$ cosyzygy of k. Becuase injectives are acyclic for local cohomology, $H^p_{\mathfrak{m}}(k) \cong H^1_{\mathfrak{m}}(\Omega^{p-1})$. But Ω^{p-1} is a direct limit of finite dimensional modules because the injective envelope of k is, so $H^1_{\mathfrak{m}}(\Omega^{p-1}) = 0$ by the previous paragraph. □

Lemma 3.4. *Let R be left noetherian, and $M \in \mathsf{mod}(R)$. Then*

$$\mathrm{Ext}^i_R(M, \varinjlim N_\alpha) \cong \varinjlim \mathrm{Ext}_R(M, N_\alpha),$$

whenever the direct limit is taken over a directed set.

Proof. Take a projective resolution $P_\bullet \to M$ in $\mathsf{mod}(R)$. Since each projective is finitely presented we may apply Proposition A.8.9, to obtain

$$\mathrm{Ext}^i_R(M, \varinjlim N_\alpha) = h^i(\mathrm{Hom}_R(P_\bullet, \varinjlim N_\alpha)) \cong h^i(\varinjlim \mathrm{Hom}_R(P_\bullet, N_\alpha)).$$

However, by Proposition A.8.13, taking direct limits over directed sets commutes with taking homology, so the last term is isomorphic to $\varinjlim h^i(\mathrm{Hom}(P_\bullet, N_\alpha)) \cong \varinjlim \mathrm{Ext}^i_R(M, N_\alpha)$, as required. □

Theorem 3.5. *Let A be left noetherian. If $M \in \mathsf{grmod}(A)$, then there is a convergent spectral sequence*

$$\underline{\mathrm{Ext}}^{-p}_A(M, H^q_{\mathfrak{m}}(A)^*) \Rightarrow H^{q+p}_{\mathfrak{m}}(M)^*. \tag{3-1}$$

Proof. By Corollary 13.2.3, there is a spectral sequence

$$\underline{\mathrm{Ext}}_A^{-p}(M, \underline{\mathrm{Ext}}_A^q(A/\mathfrak{m}^i, A)^*) \Rightarrow \underline{\mathrm{Ext}}_A^{q+p}(A/\mathfrak{m}^i, M)^*.$$

The direct limit over i of the right hand side is $H_{\mathfrak{m}}^{q+p}(M)$. Applying Lemma 3.4 to the direct limit of the terms on the left hand side, we may take the direct limit inside, thus obtaining

$$\underline{\mathrm{Ext}}_A^{-p}(M, \varinjlim \underline{\mathrm{Ext}}_A^q(A/\mathfrak{m}^i, A)^*) \Rightarrow \varinjlim \underline{\mathrm{Ext}}_A^{q+p}(A/\mathfrak{m}^i, M)^*,$$

from which the result follows. □

Proposition 3.6. *Let $f : A \to B$ be a homomorphism of noetherian connected algebras such that B_A is finitely generated. Then the local cohomology of a B-module is the same whether computed over A or B; that is, the two local cohomology groups are isomorphic as B-modules..*

Proof. Let $M \in \mathsf{GrMod}(B)$. Write $\mathfrak{m}$ and $\mathfrak{n}$ for the augmentation ideals of A and B respectively. Because B_A is finitely generated, $B/B\mathfrak{m} \cong B \otimes_A A/\mathfrak{m}$ is finite dimensional, whence $\mathfrak{n}^i \subset B\mathfrak{m}$ for some i. Therefore, if $m \in M$ is torsion with respect to A, it is torsion with respect to B; the converse is also true because $f(\mathfrak{m}) \subset \mathfrak{n}$. Hence the torsion submodule of M is the the same over both A and B. Thus $H_{\mathfrak{m}}^0(-) = H_{\mathfrak{n}}^0(-)$. Since $H_{\mathfrak{n}}^*(-)$ can be computed by taking an injective B-module resolution, it sufices to show that an injective $I \in \mathsf{GrMod}(B)$ is acyclic for $H_{\mathfrak{m}}^*(-)$; it is enough to prove this for indecomposable injectives. The change of rings spectral sequence

$$\underline{\mathrm{Ext}}_B^p(\mathrm{Tor}_q^A(B, A/\mathfrak{m}^i), I) \Rightarrow \underline{\mathrm{Ext}}_A^n(A/\mathfrak{m}^i, I)$$

collapses because I is injective, giving an isomorphism

$$\underline{\mathrm{Hom}}_B(\mathrm{Tor}_q^A(B, A/\mathfrak{m}^i), I) \cong \underline{\mathrm{Ext}}_A^q(A/\mathfrak{m}^i, I).$$

Since A is noetherian, B_A has a projective resolution by finitely generated A-modules, so $\mathrm{Tor}_q^A(B, A/\mathfrak{m}^i)$ is a finite dimensional B-module; therefore the only indecomposable injective to which it can map non-trivially is the injective envelope of k. Thus $\underline{\mathrm{Ext}}_A^q(A/\mathfrak{m}^i, I) = 0$ if I is not the injective envelope of k, whence $H_{\mathfrak{m}}^q(I) = 0$. On the other hand, if I is the injective envelope of k then I is a direct limit of finite-dimensional modules, so $H_{\mathfrak{m}}^n(I) = 0$ for $n > 0$ by Proposition 3.3. Therefore injective B-modules are acyclic for local cohomology over A. □

To see that the noetherian hypothesis is essential in Proposition 3.6, consider the free algebra: it has global dimension 1, so $H_{\mathfrak{m}}^2(-) = 0$.

4. Local dimension

The next result shows that depth may be defined in terms of local cohomology.

Lemma 4.1. $\operatorname{depth} M = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}$.

Proof. Write $d = \operatorname{depth} M$ and $e = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}$. Suppose that $e < d$. Then $\underline{\mathrm{Ext}}_A^e(k, M) = 0$, and by induction on j, $\underline{\mathrm{Ext}}_A^e(A/\mathfrak{m}^i, M) = 0$ for all $i \geq 1$; therefore $H_{\mathfrak{m}}^e(M) = 0$, contradicting the definition of e. Thus $e \geq d$.

Conversely, because $\underline{\mathrm{Ext}}_A^{d-1}(k, M) = 0$, $\underline{\mathrm{Ext}}_A^{d-1}(\mathfrak{m}^i/\mathfrak{m}^j, M) = 0$ whenever $j > i$. Hence the natural map $\underline{\mathrm{Ext}}_A^d(A/\mathfrak{m}^i, M) \to \underline{\mathrm{Ext}}_A^d(A/\mathfrak{m}^j, M)$ is injective; thus $H_{\mathfrak{m}}^d(M)$ is the union of the submodules $\underline{\mathrm{Ext}}_A^d(A/\mathfrak{m}^i, M)$; this is non-zero when $i = 0$, so $H_{\mathfrak{m}}^d(M) \neq 0$. Thus $e \leq d$. □

Definition 4.2. The local dimension of a non-zero module $M \in \mathsf{grmod}(A)$ is

$$\operatorname{ldim} M := \sup\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

LEMMA 4.3. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{GrMod}(A)$. Write $\ell = \operatorname{ldim} L$, $m = \operatorname{ldim} M$, and $n = \operatorname{ldim} N$. Then*

1. $\ell \leq \max\{m, n-1\}$;
2. $m \leq \max\{\ell, n\}$;
3. $n \leq \max\{\ell + 1, m\}$.

PROOF. To show that a module X has depth $\leq \max\{p, q\}$, we must show that $\mathrm{H}^i_{\mathfrak{m}}(X)$ is zero if $i > p$ and $i > q$. Hence the result follows from careful examination of the long exact sequence for local cohomology. □

PROPOSITION 4.4. *Let $x \in A$ be a homogeneous normal element of positive degree acting faithfully on $M \in \mathsf{GrMod}(A)$. Then*

1. $\operatorname{depth} M = \operatorname{depth}(M/xM) + 1$;
2. *if A is left noetherian, satisfies condition χ, and M is finitely generated, then* $\operatorname{ldim} M = \operatorname{ldim}(M/xM) + 1$.

PROOF. Let $d = \deg x$.

(1) By Rees' Lemma, $\underline{\mathrm{Ext}}^p_{A/(x)}(k, M/xM) \cong \underline{\mathrm{Ext}}^{p+1}_A(k, M)[-d]$, so the result follows.

(2) Left multiplication by x yields an exact sequence $0 \to M^\sigma[-d] \to M \to M/xM \to 0$. If $n = \operatorname{ldim} M$, then the long exact sequence in local cohomology looks like

$$\cdots \to H^{n-1}_{\mathfrak{m}}(M/xM) \to H^n_{\mathfrak{m}}(M^\sigma[-d]) \xrightarrow{f} H^n_{\mathfrak{m}}(M) \to H^n_{\mathfrak{m}}(M/xM) \to 0.$$

Since $H^n_{\mathfrak{m}}(M)$ and $H^n_{\mathfrak{m}}(M^\sigma)$ are isomorphic as graded k-modules, and are right bounded by Corollary 13.4.7, the map f is not injective (because $d > 0$). Hence $H^{n-1}_{\mathfrak{m}}(M/xM) \neq 0$. We must show that $H^n_{\mathfrak{m}}(M/xM)$ is zero. □

For a noetherian module $M \in \mathsf{grmod}(A)$ over a commutative noetherian ring A, the local dimension equals the Krull dimension. We can replace A by $A/\operatorname{Ann} M$, so assume that M is a faithful module. If $\operatorname{Kdim} A = n$, and $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ is a proper chain of primes, and $x_i \in \mathfrak{p}_i \backslash \mathfrak{p}_{i-1}$, then $x_1, \ldots, x_n$ is a regular sequence on A, whence $\operatorname{ldim} A \geq n$. ?????????

5. Cohen-Macaulay modules and algebras

Definition 5.1. A module $M \in \mathsf{GrMod}(A)$ is

- Cohen-Macaulay if $\operatorname{depth} M = \operatorname{ldim} M$;
- maximal Cohen-Macaulay if $\operatorname{depth} M = \operatorname{ldim} M = \operatorname{ldim} A$.

If B_A is a finitely generated module over a noetherian algebra A, then the depth and local dimension of a B-module do not depend on whether it is viewed as an A-module or as a B-module; these invariants are intrinsic to the module. In particular, a module is Cohen-Macaulay as a B-module if and only if it is Cohen-Macaulay as an A-module.

Since A is a bimodule, each $H^i_{\mathfrak{m}}(A)$ is an A-A-bimodule. If $\operatorname{ldim} A = a < \infty$, we call $H^a_{\mathfrak{m}}(A)^*$ the canonical A-module.

Definition 5.2. Let A be a left noetherian, connected algebra such that $\operatorname{ldim} A = a < \infty$. We say that A is Cohen-Macaulay if there is a graded A-A-bimodule ω_A such that, for all $i \geq 0$, there is a functorial isomorphism

$$\underline{\operatorname{Ext}}^i_A(M, \omega_A) \cong H^{a-i}_{\mathfrak{m}}(M)^* \tag{5-1}$$

of graded right A-modules for all $M \in \mathsf{GrMod}(A)$. The module ω_A is called a dualizing module for A. (Thus when we say a ring is Cohen-Macaulay, it is implicit that it is left noetherian.)

There is some potential ambiguity to this definition given Definition 5.1; the next result shows that there is no ambiguity.

PROPOSITION 5.3. *An algebra A is Cohen-Macaulay in the sense of Definition 5.2 if and only if A is Cohen-Macaulay as a left module over itself.*

PROOF. If A satisfies Definition 5.2, then, by putting $M = A$ in (5-1), we see that $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq a$. Hence ${}_AA$ is a Cohen-Macaulay module. Conversely, if ${}_AA$ is a Cohen-Macaulay module, then the spectral sequence $\underline{\operatorname{Ext}}^{-p}_A(M, H^q_{\mathfrak{m}}(A)^*) \Rightarrow H^{p+q}_{\mathfrak{m}}(M)^*$ collapses to give (5-1). □

PROPOSITION 5.4. *Let A be a left noetherian, and Cohen-Macaulay. If $\omega_A \in \mathsf{grmod}(A)$, then A satisfies condition χ.*

PROOF. By Corollary 13.4.7, $\chi_i(M)$ holds if and only if $H^j_{\mathfrak{m}}(M)$ is right bounded for all $j \leq i$. Since A is Cohen-Macaulay, this is the same as $\underline{\operatorname{Ext}}^j_A(M, \omega_A)$ being left bounded; by Proposition 11.6.2, this is left bounded because ω_A is. □

EXAMPLE 5.5. There is a noetherian A which has finite global dimension, but is not Cohen-Macaulay [**173**].

For any A with $\operatorname{ldim} A = a$, $H^a_{\mathfrak{m}}(-)^*$ is a contravariant left exact functor, so is representable by Watt's Theorem; in particular, it is represented by $H^a_{\mathfrak{m}}(A)^*$, so $\omega_A \cong H^a_{\mathfrak{m}}(A)^*$ if A is Cohen-Macaulay. Because the various $H^{a-i}_{\mathfrak{m}}(-)^*$ form a contravariant δ-functor, there are natural transformations

$$\underline{\operatorname{Ext}}^i_A(-, H^a_{\mathfrak{m}}(A)^*) \to H^{a-i}_{\mathfrak{m}}(-)^*.$$

The hypothesis that A is Cohen-Macaulay says that the these are natural equivalences.

LEMMA 5.6. *Suppose that A is Cohen-Macaulay and set $a = \operatorname{ldim} A$. Then*

1. $\operatorname{injdim} \omega_A = a$;
2. *if T is torsion, then, as left A-modules,* $\underline{\operatorname{Ext}}^i_A(T, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq a, \\ T^* & \text{if } i = a, \end{cases}$
3. *as left A-modules,* $H^p_{\mathfrak{m}}(\omega_A) \cong \begin{cases} 0 & \text{if } p \neq a, \\ A^* & \text{if } p = a; \end{cases}$
4. *ω_A is a maximal Cohen-Macaulay module;*
5. $\underline{\operatorname{Hom}}_A(\omega_A, \omega_A) \cong A$ *as right A-modules, where the homomorphisms are computed for ω_A as a left A-module.*

PROOF. (1) Since $0 \neq H^0_{\mathfrak{m}}(k) \cong \underline{\operatorname{Ext}}^a_A(k, \omega_A)$, $\operatorname{injdim} \omega_A \geq a$. Conversely, $\underline{\operatorname{Ext}}^i_A(M, \omega_A) = 0$ for $i > a$, so $\operatorname{injdim} \omega_A \leq a$.

(2) Let $0 \to \omega_A \to E^0 \to E^1 \to \cdots \to E^a \to 0$ be a minimal injective resolution. By Proposition 14.2.6, $\underline{\mathrm{Ext}}^i_A(k,\omega_A) \cong \underline{\mathrm{Hom}}_A(k,E^i)$ so, for $i \neq a$, we have $0 = \underline{\mathrm{Hom}}_A(k,E^i) = \underline{\mathrm{Hom}}_A(T,E^i)$. Thus $\underline{\mathrm{Ext}}^i_A(T,\omega_A) = 0$ if $i \neq a$.

On the other hand, $\underline{\mathrm{Hom}}_A(k,E^a) \cong \underline{\mathrm{Ext}}^a_A(k,\omega_A) \cong H^0_{\mathfrak{m}}(k) \cong k$, so E^a contains a copy of the injective envelope of k. Therefore $E^a \cong A^* \oplus Q$ and $\underline{\mathrm{Hom}}_A(k,Q) = 0$. Hence

$$\underline{\mathrm{Ext}}^a_A(T,\omega_A) \cong \underline{\mathrm{Hom}}_A(T,E^a) = \underline{\mathrm{Hom}}_A(T,A^*) \cong T^*,$$

where the final isomorphism is given by Corollary 13.2.4.

(3) By (2), $\underline{\mathrm{Ext}}^p_A(A/\mathfrak{m}^i,\omega_A) = 0$ if $p \neq a$, from which the first possibility follows. On the other hand, using (2) again,

$$H^n_{\mathfrak{m}}(\omega_A) = \varinjlim \underline{\mathrm{Ext}}^n_A(A/\mathfrak{m}^i,\omega_A) \cong \varinjlim (A/\mathfrak{m}^i)^* \cong A^*,$$

from which the other possibility follows.

(4) This follows at once from (3).

(5) By (3), $\mathrm{H}^a_{\mathfrak{m}}(\omega_A)^* \cong A$, so $\underline{\mathrm{Hom}}_A(\omega_A,\omega_A) \cong A$ by the definition of the dualizing module (5-1). □

Proposition 5.7. *If A is Cohen-Macaulay, then the last term in the minimal injective resolution of ω_A is isomorphic to A^*.*

Proof. As in the proof of Proposition 5.6, $E^n \cong A^* \oplus Q$. If $Q \neq 0$, let M be a non-zero finitely generated graded submodule. Then

$$0 \neq \underline{\mathrm{Ext}}^1_A(M,\Omega_{n-1}) \cong \underline{\mathrm{Ext}}^n_A(M,\omega_A) \cong H^0_{\mathfrak{m}}(M)^*,$$

so $\tau M \neq 0$. Thus $\underline{\mathrm{Hom}}_A(k,Q) \neq 0$; but this forces $H^0_{\mathfrak{m}}(k) \neq k$. Hence $Q = 0$. □

Theorem 5.8. *Suppose that ω_A is both a left and a right dualizing module for A. If $M \in \mathsf{grmod}(A)$ has a projective resolution by finitely generated A-modules, then there is a fourth quadrant convergent spectral sequence of left A-modules*

$$E_2^{pq} = \underline{\mathrm{Ext}}^p_A(\underline{\mathrm{Ext}}^{-q}_A(M,\omega_A),\omega_A) \Rightarrow \begin{cases} M & \text{if } p+q=0, \\ 0 & \text{if } p+q \neq 0. \end{cases} \tag{5-2}$$

Proof. The proof is like that of Theorem 8.5.1. One forms the bicomplex $\underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_A(P_\bullet,\omega),E^\bullet)$, where $\omega \to E^\bullet$ is an injective resolution of ω_A as a right A-module, and $P_\bullet \to M$ is a projective resolution of M with each P_j finitely generated. As in Theorem 8.5.1, taking homology up columns followed by homology along the rows, gives

$$E_2^{pq} = \underline{\mathrm{Ext}}^p_A(\underline{\mathrm{Ext}}^{-q}_A(M,\omega),\omega).$$

On the other hand, if we write $P_{-q} = A \otimes_k V_{-q}$, and take homology along the $-q^{\text{th}}$ row, we get

$$\begin{aligned} h^p(\underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_A(P_q,\omega_A),E^\bullet)) &\cong h^p(\underline{\mathrm{Hom}}_A(V^*_{-q} \otimes_k \omega_A, E^\bullet)) \\ &\cong h^p(\underline{\mathrm{Hom}}_A(\omega_A,E^\bullet)) \otimes_k V^*_{-q}) \\ &\cong \underline{\mathrm{Ext}}^p_A(\omega_A,\omega_A) \otimes_k V^*_{-q}. \end{aligned}$$

But ω_A is a Cohen-Macaulay module on the right, so the only one of these homology groups which is non-zero is $\underline{\mathrm{Ext}}^0_A(\omega_A,\omega_A) \cong A$. Thus

$$h^p(\underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_A(P_q,\omega_A),E^\bullet)) \cong \begin{cases} 0 & \text{if } p \neq 0 \\ A \otimes_k V^*_{-q} \end{cases}$$

Now take homology of this column, to get $h^q(A \otimes_k V_\bullet) = M$ for $q = 0$, and zero otherwise. Hence the second spectral sequence collapses. $\square$

If A is Cohen-Macaulay, we define

$$j'(M) := \inf\{i \mid \underline{\mathrm{Ext}}^i_A(M, \omega_A) \neq 0\}$$

for $0 \neq M \in \mathsf{GrMod}(A)$.

Corollary 5.9. *If A is Cohen-Macaulay, left noetherian, and $0 \neq M \in \mathsf{grmod}(A)$, then*

1. $\operatorname{depth} M \leq \operatorname{ldim} M \leq \operatorname{ldim} A$*;*
2. $j'(M) + \operatorname{ldim} M = \operatorname{ldim} A$*;*
3. *if* $\operatorname{pdim} M < \infty$*, then M is Cohen-Macaulay if and only if* $j'(M) = \operatorname{pdim} M$.

Proof. (1) and (2). The existence of the spectral sequence (5-2) shows that $\underline{\mathrm{Ext}}^i_A(M, \omega_A) \neq 0$ for some i. Therefore (2) is a tautology because $\underline{\mathrm{Ext}}^i_A(M, \omega_A) \cong H^{a-i}_{\mathfrak{m}}(M)^*$. Now (1) follows from (2).

(3) If M is Cohen-Macaulay, then $j'(M) + \operatorname{depth} M = \operatorname{depth} A$ and, by the Auslander-Buchsbaum formula, $\operatorname{depth} A = \operatorname{pdim} M + \operatorname{depth} M$, so $j'(M) = \operatorname{pdim} M$. Conversely, if $j'(M) = \operatorname{pdim} M$, then only a single $\underline{\mathrm{Ext}}^i_A(M, \omega)$ is non-zero, so M is Cohen-Macaulay. $\square$

Proposition 5.10. *Suppose A is Cohen-Macaulay on both sides, and let $a = \operatorname{ldim} A$. Let $0 \neq M \in \mathsf{grmod}(A)$.*

1. $\operatorname{ldim} M = 0$ *if and only if* $\dim_k M < \infty$.
2. $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M/\tau M)$ *for all* $i > 0$.

Proof. (1) ($\Leftarrow$) This is Lemma 5.6(2).

($\Rightarrow$) Since $\operatorname{depth} M = \operatorname{ldim} M$, there is only one non-zero row on the E_2 page of the spectral sequence (5-2), the $q = a$ one. It survives to the E_∞ page, so can have only one non-zero entry, namely $\underline{\mathrm{Ext}}^a_A(\underline{\mathrm{Ext}}^a_A(M, \omega_A), \omega_A) \cong M$. Hence $H^i_{\mathfrak{m}}(M) = 0$ for $i > 0$. Now consider the exact sequence $0 \to \tau M \to M \to \bar{M} \to 0$, and the associated long exact sequence in local cohomology. Since $H^i_{\mathfrak{m}}(\tau M) = 0$ for $i > 0$, it follows that $H^i_{\mathfrak{m}}(\bar{M}) = 0$ for $i > 0$. But $H^0_{\mathfrak{m}}(\bar{M}) \cong \tau\bar{M} = 0$, so $H^i_{\mathfrak{m}}(\bar{M}) = 0$ for all i. By Corollary 5.9, $\bar{M} = 0$, whence $M = \tau M$; this is finite dimensional because M is finitely generated.

(2) This follows from the long exact sequence for local cohomology since $H^i_{\mathfrak{m}}(\tau M) = 0$ for $i > 0$. $\square$

I don't know if depth and ldim are finite on non-zero modules over a general A. This is a crucial question.

Proposition 5.11. *Let $f : A \to B$ be a homomorphism of noetherian connected algebras such that B_A is finitely generated.*

1. *B is a Cohen-Macaulay algebra if and only if it is Cohen-Macaulay as an A-module.*
2. *If A is Cohen-Macaulay, and $0 \neq M \in \mathsf{GrMod}(B)$, then*
$$\operatorname{depth} M \leq \operatorname{ldim} M \leq \operatorname{ldim} B.$$
3. *If A and B are Cohen-Macaulay, then $\omega_B \cong \underline{\mathrm{Ext}}^t_A(B, \omega_A)$, where $t = \operatorname{ldim} A - \operatorname{ldim} B = \operatorname{depth} A - \operatorname{depth} B$.*

PROOF. (1) This is because the local cohomology of a B-module is independent of whether it is computed over A or B. (2) The first inequality follows from Lemma 4.1 and the definition of local dimension. Viewing M and B as A-modules, we have

$$j'(M) + \operatorname{ldim} M = \operatorname{ldim} A = j'(B) + \operatorname{ldim} B.$$

If $q < j(B)$, then $\underline{\mathrm{Ext}}^q_A(B, \omega_A) = 0$ so, by the change of rings spectral sequence $\underline{\mathrm{Ext}}^p_B(M, \underline{\mathrm{Ext}}^q_A(B, \omega_A)) \Rightarrow \underline{\mathrm{Ext}}^n_A(M, \omega_A)$, $\underline{\mathrm{Ext}}^q_A(M, \omega_A) = 0$ also. Thus $j'(M) \geq j'(B)$. The result follows.

(3) Write $a = \operatorname{ldim} A$, $b = \operatorname{ldim} B$, and $\mathfrak{m}$ and $\mathfrak{n}$ for the augmentation ideals of A and B respectively. Since the local cohomology of a B-module is the same when computed over either A and B, we have

$$\omega_B \cong H^b_{\mathfrak{n}}(B)^* \cong H^{a-t}_{\mathfrak{m}}(B)^* \cong \underline{\mathrm{Ext}}^t_A(B, \omega_A),$$

as required. □

EXAMPLE 5.12. The commutative algebra $B = k[w, x, y, z]/(w, x) \cap (y, z)$ is noetherian, reduced, and of Krull dimension 2, but is not Cohen-Macaulay. Now $x + z$ is a regular element of B, and $B/(x + z) \cong k[w, x, y]/(wx, wy, xy, x^2)$. Since x is a torsion element in this quotient, $\operatorname{depth} B/(x + z) = 0$, whence $\operatorname{depth} B = 1$.

$\boxed{\text{Paul}}$ Suppose that A has a finitely generated bimodule ω_A of finite injective dimension such that

$$\underline{\mathrm{Ext}}^i_A(k, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq \operatorname{ldim} A, \\ k & \text{if } i = \operatorname{ldim} A. \end{cases}$$

Does this imply A is CM?

If the local ring at a point on a variety is Cohen-Macaulay, it is appropriate to think of the singularity as being rather mild. Many important singularities are of this kind. One can continue the idea, and use local cohomology as a measure of how bad a singularity is. For example, an important class of rings which are not Cohen-Macaulay are the Buchsbaum rings. A commutative noetherian local ring $(R, \mathfrak{m}, k)$ is Buchsbaum if the canonical maps $\mathrm{Ext}^i_R(k, R) \to H^i_{\mathfrak{m}}(R)$ are surjective for all $i < \dim R$ [**177**, Corollary 2.16, page 87]. There is an analogue for connected algebras. An important class of Buchsbaum rings are those of the form $\bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$, where $\mathcal{L}$ is a very ample line bundle on an abelian variety [**177**, Theorem 2.5, page 235]; these rings are not usually Cohen-Macaulay.

6. Gorenstein algebras

Definition 6.1. A Cohen-Macaulay algebra A is Gorenstein if, as a left module, $\omega_A \cong A[-\ell]$ for some $\ell \in \mathbb{Z}$.

PROPOSITION 6.2. *Let A be Gorenstein of injective dimension a.*

1. $\operatorname{injdim} A = \operatorname{ldim} A$.
2. *If* $0 \neq M \in \mathsf{GrMod}(A)$, *then* $j(M) + \operatorname{ldim} M = \operatorname{ldim} A$.
3. *If* $M \in \mathsf{GrMod}(A)$, *then* $\underline{\mathrm{Ext}}^a_A(M, A) \cong (\tau M)^*[\ell]$.
4. *As left A-modules,* $H^i_{\mathfrak{m}}(A[-\ell]) \cong \begin{cases} 0 & \text{if } i \neq a, \\ A^* & \text{if } i = a. \end{cases}$

PROOF. Parts (1) and (4) are given in Lemma 5.6. Part (2) follows from Proposition 5.9 because the function $j'(-)$ defined there is simply the grade. Part (3) is given by $\underline{\mathrm{Ext}}^a_A(M, A[-\ell] \cong H^0_{\mathfrak{m}}(M) = \tau M$. $\square$

The following result is often taken as the defining property of Gorenstein rings (sometimes called Artin-Schelter Gorenstein rings).

PROPOSITION 6.3. *A connected algebra A is* Gorenstein *if and only if* injdim A *is finite, say n, and there is an integer ℓ such that*

$$\underline{\mathrm{Ext}}^i_A(k, A[-\ell]) \cong \begin{cases} 0 & \text{if } i \neq n, \\ k_A & \text{if } i = n. \end{cases}$$

PROOF. ($\Rightarrow$) By Lemma 5.6, $\operatorname{injdim} A < \infty$. The other part is a special case of Proposition 5.6(3).

($\Leftarrow$) The same argument as in Lemma 5.6 shows that

$$H^i_{\mathfrak{m}}(A[-\ell]) \cong \begin{cases} 0 & \text{if } i \neq n, \\ A^* & \text{if } i = n. \end{cases}$$

Hence the spectral sequence $\underline{\mathrm{Ext}}^{-p}_A(M, H^n_{\mathfrak{m}}(A)^*) \Rightarrow H^{n+p}_{\mathfrak{m}}(M)^*$ collapses giving $\underline{\mathrm{Ext}}^p_A(M, H^n_{\mathfrak{m}}(A)^*) \cong H^{n-p}_{\mathfrak{m}}(M)^*$. Therefore the dualizing module is $H^n_{\mathfrak{m}}(A)^* \cong A[-\ell]$, as required. $\square$

EXAMPLE 6.4. Let $A = k[x]$ be the polynomial ring on a single indeterminate fo degree d. By Rees' Lemma, $\underline{\mathrm{Ext}}^1_A(k, A) \cong \underline{\mathrm{Hom}}_k(k, k)[d]$, so A is Gorenstein, with $\omega_A \cong A[-d]$. It is straightforward to show that if A and B are Gorenstein rings, then so is $A \otimes_k B$, and its dualizing module is isomorphic to $\omega_A \otimes_k \omega_B$. It follows by induction that the polynomial ring $k[x_1, \dots, x_n]$ is Gorenstein, and its dualizing module is isomorphic to $A[-\sum \deg x_i]$.

EXAMPLE 6.5. The commutative ring $A = k[x, y]/(x^2, xy, yx)$ is not Cohen-Macaulay. To see this first observe that $\tau A = kx \neq 0$, so $\operatorname{depth} A = 0$. If A is viewed as a quotient of the polynomial ring $R = k[X, Y]$, then it has a projective resolution

$$0 \to R \xrightarrow{(y,-x)} R^2 \xrightarrow{\begin{pmatrix} x^2 \\ xy \end{pmatrix}} R \to A \to 0,$$

and it follows easily that $\underline{\mathrm{Ext}}^1_R(A, R) \cong R/xR \neq 0$, whence $H^1_{\mathfrak{m}}(A) \neq 0$.

Our use of the word 'Gorenstein' deviates slightly from the standard usage in commutative algebra; a commutative noetherian ring is said to be Gorenstein if it has finite injective dimension. By Proposition 5.4, a left noetherian Gorenstein ring satisfies condition χ. Since a commutative noetherian local ring satisfies the condition χ, the next result shows that our usage only deviates a little from the standard one.

THEOREM 6.6. *Suppose that A is noetherian of finite injective dimension. The following conditions are equivalent:*

1. *A satisfies condition χ on both sides;*
2. *$\underline{\mathrm{Ext}}^i_A(k, A)$ and $\underline{\mathrm{Ext}}^i_A(k_A, A)$ are finite dimensional for all i;*
3. *A is Gorenstein on both sides.*

PROOF. (1) $\Rightarrow$ (2) This is part of the definition of condition χ.

(2) $\Rightarrow$ (3) Let $d = \operatorname{depth} A$, and define $c = \sup\{i \mid \underline{\operatorname{Ext}}^i_A(k_A, A) \neq 0\}$. It follows from the double-Ext spectral sequence

$$E_2^{pq} = \underline{\operatorname{Ext}}^p_A(\underline{\operatorname{Ext}}^{-q}_A(k, A), A) \Rightarrow \begin{cases} 0 & \text{if } p+q \neq 0 \\ k & \text{if } p+q = 0 \end{cases}$$

that some $\underline{\operatorname{Ext}}^i_A(k, A) \neq 0$, so d is finite. Since $\operatorname{injdim} A < \infty$, so is c finite. We also have $0 \neq E_2^{c,-d} = E_\infty^{c,-d}$, so $c = d$. The same argument applies with the roles of k and k_A reversed, so we conclude that $\underline{\operatorname{Ext}}^i_A(k, A)$ and $\underline{\operatorname{Ext}}^i_A(k_A, A)$ are only non-zero for a single value of i, namely $d = \operatorname{depth} A$. Moreover, the spectral sequence shows that $\underline{\operatorname{Ext}}^d_A(k, A) \cong k_A[\ell]$ for some $\ell \in \mathbb{Z}$. The same argument as in Lemma 5.6 shows that

$$H^i_{\mathfrak{m}}(A[-\ell]) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A^* & \text{if } i = d. \end{cases}$$

Thus $d = \operatorname{ldim} A$, so the spectral sequence $\underline{\operatorname{Ext}}^{-p}_A(M, H^q_{\mathfrak{m}}(A)^*) \Rightarrow H^{p+q}_{\mathfrak{m}}(M)^*$ collapses to give

$$\underline{\operatorname{Ext}}^p_A(M, H^d_{\mathfrak{m}}(A)^*) \cong H^{d-p}_{\mathfrak{m}}(M)^*.$$

Therefore A is Cohen-Macaulay with dualizing module $H^d_{\mathfrak{m}}(A)^* \cong A[\ell]$, and hence Gorenstein.

(3) $\Rightarrow$ (1) This is a consequence of Proposition 5.4. □

PROPOSITION 6.7. *A finitely generated module over a Gorenstein ring has finite injective dimension if and only if it has finite projective dimension.*

PROOF. (Zhang) Let A be the ring, and M the module. If $\operatorname{pdim} M < \infty$, then M has a finite free resolution; since the terms in the resolution have finite injective dimension, so does M. Conversely, if $\operatorname{injdim} M < \infty$, then there is a convergent spectral sequence

$$\underline{\operatorname{Ext}}^p_A(\underline{\operatorname{Ext}}^{-q}_A(k, A), M) \Rightarrow \operatorname{Tor}^A_{-p-q}(k, M).$$

Since $\operatorname{injdim} A < \infty$, $E_2^{pq} = 0$ for $q \ll 0$. Hence $\operatorname{Tor}_n(k, M) = 0$ for $n \gg 0$, whence $\operatorname{pdim} M < \infty$. □

PROPOSITION 6.8. *A commutative, noetherian ring is Gorenstein if and only if it has finite injective dimension.*

7. Regular algebras

Definition 7.1. A connected algebra is regular if it is Gorenstein (hence left noetherian) and has finite global dimension.

If A is regular of global dimension n, it follows from Corollary 13.2.3 that

$$\underline{\operatorname{Ext}}^p_A(M, A/\mathfrak{m}^i[\ell]) \cong \underline{\operatorname{Ext}}^{n-p}_A(A/\mathfrak{m}^i, M)^*.$$

The next result gives an elementary proof of this, avoiding spectral sequences.

PROPOSITION 7.2. *Suppose that A is regular of global dimension n. If $M \in \mathsf{grmod}(A)$, then*

$$\underline{\operatorname{Ext}}^p_A(k[\ell], M) \cong \underline{\operatorname{Ext}}^{n-p}_A(M, k)^*.$$

Proof. Let $0 \to P_n \to \cdots \to P_0 \to k$ be the minimal resolution of k. Then $P_{n-\bullet}^\vee \to \underline{\mathrm{Ext}}_A^n(k, A)$ is a projective resolution of $k_A[\ell]$; thus $(P_{n-\bullet}^\vee)^* \to (k_A[\ell])^* \cong k[-\ell]$ is an injective resolution. Therefore

$$\underline{\mathrm{Ext}}^p(k, M) = h^p(\underline{\mathrm{Hom}}_A(P_\bullet, M)) \cong h^p(P_\bullet^\vee \otimes_A M) \cong h^p(\underline{\mathrm{Hom}}_A(M, (P_\bullet^\vee)^*)^*),$$

where the last isomorphism comes from Proposition 13.2.2 because each $P_j^\vee \otimes_A M$ is locally finite. Finally, we have

$$h^p(\underline{\mathrm{Hom}}_A(M, (P_\bullet^\vee)^*))^* \cong \underline{\mathrm{Ext}}_A^{n-p}(M, k[-\ell])^* \cong \underline{\mathrm{Ext}}_A^{n-p}(M, k)^*[\ell]$$

from which the result follows. $\square$

Example 7.3. The only regular algebra of global dimension zero is k; indeed k is the only connected algebra of global dimension zero. If A is regular of global dimension one, it is a polynomial ring in one variable: the minimal resolution of k looks like $0 \to P \to A \to k \to 0$, and by Proposition 7.2, $P \cong A[-\ell]$, so A has one generator (in degree ℓ), and no relations, whence $A \cong k[X]$.

Theorem 7.4. (J. Zhang) *Let A be regular of global dimension n. If $H_A(t) = (1-t)^{-n}$, then A is Koszul.*

Proof. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a minimal projective resolution of k. Write $P_i = A \otimes_k V_i$ and $P_i^\vee = \underline{\mathrm{Hom}}_A(P_i, A) \cong V_i^* \otimes A$. Then

$$H_A(t)^{-1} = \sum_{i=0}^{n} (-1)^i H_{V_i}(t). \tag{7-1}$$

Let a_i (resp. b_i) denote the least (resp. largest) degree of a component of V_i. Since the resolution is minimal $0 = b_0 = a_0 < a_1 < \cdots < a_n$. In particular, $a_n \geq n$.

Since A is Gorenstein, $0 \to P_0^\vee \to \cdots \to P_n^\vee \to k_A[\ell] \to 0$ is a projective resolution of $k_A[\ell]$ for some integer ℓ. The minimality of $P_\bullet$ ensures that $P_\bullet^\vee$ is also minimal, so $P_n^\vee \cong A[\ell]$, whence $\ell = a_n = b_n \geq n$, and $-b_n < \cdots < -b_1 < -b_0 = 0$.

By hypothesis, the highest degree term occuring in $H_A(t)^{-1}$ is t^n. Since $P_n \cong A[-\ell]$ contributes a term $(-1)^n t^\ell$ to the sum (4-1), which cannot be cancelled out by any other terms in the sum because $\ell = b_n > b_i$ for all $i \neq n$, it follows that $\ell \leq n$. Therefore $\ell = n$, and $a_i = i = b_i$ for all i. In other words, $P_\bullet$ is a linear resolution of k. $\square$

The next example shows that the hypothesis on the Hilbert series in Theorem 4.2 is necessary.

Example 7.5. [**165**, Example 6.5] There is a regular quadratic algebra of global dimension 5 which is not Koszul.

8. Cohen-Macaulay modules

Proposition 8.1. *Suppose that A is noetherian, and that ω_A is a bidualizing module. Then $M \mapsto M^\vee$ is a duality setting up a bijection between left and right Cohen-Macaulay modules of a fixed depth. Moreover, for a Cohen-Macaulay module $M \cong M^{\vee\vee}$.*

Proof. If M is Cohen-Macaulay, then the spectral sequence (5-2) has a single non-zero row, and each term on that row survives to the E_∞ page. But on the E_∞ page only the diagonal terms can be non-zero, so there is only one non-zero term on the E_2 page, and it must be isomorphic to M. Thus $M \cong M^{\vee\vee}$, and $M^\vee$ is Cohen-Macaulay. □

LEMMA 8.2. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in which* $\operatorname{depth} M = \operatorname{depth} N + 1$. *Then*

1. *if* M *and* N *are Cohen-Macaulay, so is* L*;*
2. *if* L *and* M *are Cohen-Macaulay, so is* N*.*

PROOF. Look at the long exact sequence for local cohomology. □

THEOREM 8.3. *Let* A *be a finitely generated, commutative, connected* k*-algebra, and* $M \in \mathsf{grmod}(A)$*. The following are equivalent:*

1. M *is a Cohen-Macaulay module;*
2. *if* $\operatorname{Kdim} M = n$*, then* M *is a finitely generated free module over a polynomial subalgebra* $k[x_1, \dots, x_n]$ *of* $A/\operatorname{Ann} M$*, generated by homogeneous elements* $x_1, \dots, x_n$*;*
3. *if* $x_1, \dots, x_n$ *are homogeneous elements generating a polynomial subalgebra* $k[x_1, \dots, x_n]$ *of* $A/\operatorname{Ann} M$ *over which* M *is finitely generated, then* M *is a free* $k[x_1, \dots, x_n]$*-module.*

PROOF. □

EXERCISES

8.1 Missing

9. Problems and questions

The following questions are motivated by a wish to prove non-commutative versions of the results for a noetherian commutative local ring $(R, \mathfrak{m}, k)$.

Definition 9.1. Let A be left noetherian and $M \in \mathsf{grmod}(A)$. Then

- the Bass numbers of M are $\mu_i(M) := \dim_k \underline{\operatorname{Ext}}^i_A(k, M)$;
- the Betti numbers of M are $\beta_i(M) := \dim_k \operatorname{Tor}^A_i(k, M)$;
- the type of M is $\operatorname{type}(M) := \mu_d(M)$, where $d = \operatorname{depth} M$.

The $\beta_i(M)$ are finite for finitely generated modules over a noetherian ring because $\beta_i(M)$ is the rank of the i^{th} term in the minimal projective resolution of M. The $\mu_i(M)$ may not be finite – this is related to $\chi_i(M)$. It is obvious that $\operatorname{pdim} M = \sup\{i \mid \beta_i(M) \neq 0\}$; is $\operatorname{injdim} M = \sup\{i \mid \mu_i(M) \neq 0\}$? This would follow from (*) below.

Questions

- (*) Jorgensen shows $\operatorname{depth} N + \sup\{i \mid \underline{\operatorname{Ext}}^i_A(N, M) \neq 0\} = \operatorname{injdim} M$. Prove it without derived cats.
- If A satisfies χ, is A a finite module over a Gorenstein algebra?
- Proposition 7.2 shows that $\mu_p(M) = \beta_{n-p}(M)$; is there an analogue for CM algebras.
- Is local dimension a dimension function ? Relate it to the generalized Auslander condition.
- If A is left CM, is it right CM?
- Is ω_A also a dualizing module on the right?
- Is $H^a_{\mathfrak{m}}(A) \cong H^a_{\mathfrak{m}^{\mathrm{op}}}(A)$ as bimodules?

- Is $\sum_i (-1)^i \beta_i(M) = \sum_i (-1)^i \mu_i(H^i_{\mathfrak{m}}(M)^*)$? See Roberts.
- If $\operatorname{gldim} A < \infty$, and A is left noetherian, and A satisfies χ, then A is Gorenstein; can we weaken the gldim hypothesis and conclude CM?
- Analogue of Stanley's numerical result.
- Is A CM if and only if $j'(M) + \operatorname{ldim} M = \operatorname{ldim} A$ for all $M \neq 0$?
- Use the dualizing spectral sequence to get a functional equation for a Cohen-Macaulay module M.
- If A is prime CM, is ω_A faithful? Use that the natural map $A \to \operatorname{Hom}_A(\omega_A, \omega_A$ is an isomorphism.
- Is $\operatorname{ldim} M = 0$ if and only if $\dim_k M < \infty$?
- When is ldim the same on both sides of a bimodule?
- Show that $H^d_{\mathfrak{m}}(M)$ is not noetherian if $d = \operatorname{ldim} M > 0$. It would be not noetherian if it were not left bounded.
- Is $H^i_{\mathfrak{m}}(M)$ locally finite?
- Show $\operatorname{ldim} H^i_{\mathfrak{m}}(M) \leq i$ for all $0 \leq i < \operatorname{ldim} M$ (omit $i = \operatorname{ldim} M$ since then $H^i_{\mathfrak{m}}(M)$ is not f.gend). Relate to the Auslander condition.
- Why does comm ring satisfy the Auslander condition?

$\boxed{\text{Paul}}$ Let $\mathcal{C}$ be the modules of $\operatorname{ldim} \leq d$. If $L, N \in \mathcal{C}$, so is M. Converse ???? Auslander-condition. Want to show $\operatorname{ldim} H^i_{\mathfrak{m}}(M)^* \leq a - i$ where $a = \operatorname{ldim} A$.

CHAPTER 16

Graded ring theory

This chapter concerns basic ring theoretic issues in the setting of graded algebras. For example, sections 3 and 4 consider the analogue of the Morita theorems in answer to the question 'what is the relation between two algebras having equivalent categories of graded modules?' The structure implicit in the grading adds an extra level of complication. Before tackling this question, at the beginning of section 1 we explain how to recover A from the triple $(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [1])$; the rest of that section considers this proceedure in the abstract giving a construction which associates to such a triple $(\mathcal{C}, \mathcal{O}, s)$ a graded algebra. After these preparations the question of the relation between algebras having equivalent module categories is tackled. As the Morita theorems suggest, projective bimodules play a key role, but there is a new ingredient in the graded case, namely the the notion of a twisting system: a twist of a graded algebra A is a new algebra A' which is equal to A as a graded vector space, but is endowed with a new multiplication. The algebras A and A' have equivalent categories of graded modules. The twisting proceedure is studied in section 4. An automorphism of an algebra determines a twisting system, and these special twisting systems are studied in section 5.

Section 6 examines strongly graded algebras; it is proved that if A is strongly graded, then $\mathsf{GrMod}A$ is equivalent to $\mathsf{Mod}A_0$.

In section 7 we consider the structure of a graded algebra A which is an artinian object in $\mathsf{GrMod}A$. In particular, a graded analogue of Schur's lemma shows that $\underline{\mathrm{End}}_A M$ is a graded division ring (meaning that every non-zero homogeneous element is a unit) when M is an irreducible object of $\mathsf{GrMod}A$. The structure of graded division rings is described, and we also prove a graded analogue of the Artin-Wedderburn theorem describing the structure of a graded artinian ring having no proper graded two-sided ideals.

1. The graded algebras $B(\mathcal{C}, \mathcal{O}, s)$

A ring R can be recovered from the pair $(\mathsf{Mod}R^{\mathrm{op}}, R_R)$ as the endomorphism ring of the distinguished object R_R. In trying to recover a graded algebra from its category of graded modules we need more data than just a category and a distinguished object: the most we can recover from the pair $(\mathsf{GrMod}A^{\mathrm{op}}, A_A)$ is the ring $\mathrm{Hom}_{\mathrm{Gr}}(A, A) \cong A_0$. But we *can* recover A from the categorical data $(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [1])$ because

$$\bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{Gr}}(A, A[n]) = \mathrm{Hom}_A(A_A, A_A) \cong A.$$

The ring structure on this direct sum is defined as follows: the product of $f \in \mathrm{Hom}_{\mathrm{Gr}}(A, A[m])$ and $g \in \mathrm{Hom}_{\mathrm{Gr}}(A, A[n])$ is

$$f.g := f[n] \circ g \in \mathrm{Hom}_{\mathrm{Gr}}(A, A[n+m]), \tag{1-1}$$

where $f[n]$ is simply f considered as a map $A[n] \to A[n+m]$.

More generally, given a triple $(\mathcal{C}, \mathcal{O}, s)$ consisting of an abelian category $\mathcal{C}$, a distinguished object $\mathcal{O}$, and a functor $s : \mathcal{C} \to \mathcal{C}$, we will describe how to construct a graded algebra $B(\mathcal{C}, \mathcal{O}, s)$.

Definition 1.1. We define ATrip, the category of algebraic triples, the objects of which are triples $(\mathcal{C}, \mathcal{O}, s)$ consisting of

- a k-linear abelian category $\mathcal{C}$,
- a fixed object $\mathcal{O}$, and
- an automorphism $s : \mathcal{C} \to \mathcal{C}$, which we call a shift functor. (We will often write $\mathcal{F}(n) = s^n\mathcal{F}$.)

A morphism in ATrip is a triple

$$(F, \theta, \mu) : (\mathcal{C}_1, \mathcal{O}_1, s_1) \to (\mathcal{C}_2, \mathcal{O}_2, s_2)$$

consisting of

- a k-linear functor $F : \mathcal{C}_1 \to \mathcal{C}_2$,
- an isomorphism $\theta : F\mathcal{O}_1 \to \mathcal{O}_2$, and
- a natural transformation of functors $\mu : F \circ s_1 \to s_2 \circ F$.

Construction of the algebra $B(\mathcal{C}, \mathcal{O}, s)$**.** Define

$$B(\mathcal{C}, \mathcal{O}, s)_n := \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{O})$$

and

$$B(\mathcal{C}, \mathcal{O}, s) := \bigoplus_{n\in\mathbb{Z}} B(\mathcal{C}, \mathcal{O}, s)_n.$$

We will write B for brevity. The k-linearity of $\mathcal{C}$ and s ensures that B is a graded k-vector space. The product of $a \in B_m$ and $b \in B_n$ is defined to be

$$ab := s^n(a) \circ b; \tag{1-2}$$

that is, the product is the composition

$$\mathcal{O} \xrightarrow{b} s^n\mathcal{O} \xrightarrow{s^n(a)} s^{n+m}\mathcal{O}.$$

This product is associative since s is a functor, and is k-bilinear since s is k-linear. The identity map on $\mathcal{O}$ becomes the identity element of B. Thus B is a $\mathbb{Z}$-graded k-algebra with identity.

Proposition 1.2. *The rule $(\mathcal{C}, \mathcal{O}, s) \mapsto B(\mathcal{C}, \mathcal{O}, s)$ is a covariant functor from the category* ATrip *to the category of graded k-algebras.*

Proof. Suppose that $(F, \theta, \mu) : (\mathcal{C}, \mathcal{O}, s) \to (\mathcal{C}', \mathcal{O}', t)$ is a morphism of triples. We define a homomorphism of graded k-algebras

$$\Phi : B(\mathcal{C}, \mathcal{O}, s) \to B(\mathcal{C}', \mathcal{O}', t)$$

as follows: if $a \in B(\mathcal{C}, \mathcal{O}, s)_\ell = \mathrm{Hom}(\mathcal{O}, s^\ell\mathcal{O})$, then

$$\Phi(a) := t^\ell(\theta) \circ \mu^\ell_{\mathcal{O}} \circ F(a) \circ \theta^{-1}$$

is the composition

$$\mathcal{O}' \xrightarrow{\theta^{-1}} F\mathcal{O} \xrightarrow{F(a)} Fs^\ell(\mathcal{O}) \xrightarrow{\mu^\ell_{\mathcal{O}}} t^\ell F(\mathcal{O}) \xrightarrow{t^\ell(\theta)} t^\ell\mathcal{O}',$$

where μ^ℓ is the natural transformation $F\circ s^\ell \to t^\ell \circ F$ defined by

$$\mu^n_X := t^{n-1}(\mu_X)\circ t^{n-2}(\mu_{sX})\circ\cdots \quad \cdots\circ t(\mu_{s^{n-2}X})\circ \mu_{s^{n-1}X}.$$

We will use the fact that

$$\mu^{i+j}_X = t^i(\mu^j_X)\circ \mu^i_{s^jX}$$

for all i and j and all $X\in\mathcal{C}$.

We now show that Φ respects the products in the two rings. Let a and b be homogeneous elements of degrees ℓ and m in $B(\mathcal{C},\mathcal{O},s)$. Then

$$\begin{aligned}
\Phi(a)\Phi(b) &= t^m\big(t^\ell(\theta)\circ\mu^\ell_{\mathcal{O}}\circ F(a)\circ\theta^{-1}\big)\circ\big(t^m(\theta)\circ\mu^m_{\mathcal{O}}\circ F(b)\circ\theta^{-1}\big)\\
&= t^{m+\ell}(\theta)\circ t^m(\mu^\ell_{\mathcal{O}})\circ t^mF(a)\circ\mu^m_{\mathcal{O}}\circ F(b)\circ\theta^{-1}\\
&= t^{m+\ell}(\theta)\circ t^m(\mu^\ell_{\mathcal{O}})\circ \mu^m_{s^\ell\mathcal{O}}\circ Fs^m(a)\circ F(b)\circ\theta^{-1}\\
&= t^{m+\ell}(\theta)\circ t^m(\mu^\ell_{\mathcal{O}})\circ \mu^m_{s^\ell\mathcal{O}}\circ F(a.b)\circ\theta^{-1}\\
&= t^{m+\ell}(\theta)\circ \mu^{m+\ell}_{\mathcal{O}}\circ F(a.b)\circ\theta^{-1}\\
&= \Phi(a.b).
\end{aligned}$$

Thus Φ is a ring homomorphism. $\square$

Construction of the module $\Gamma\mathcal{F}$. We now associate to each object in $\mathcal{C}$ a graded right $B(\mathcal{C},\mathcal{O},s)$-module. If $\mathcal{F}$ is an object of $\mathcal{C}$, define

$$(\Gamma\mathcal{F})_n := \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{F})$$

and

$$\Gamma\mathcal{F} := \bigoplus_{n\in\mathbb{Z}}(\Gamma\mathcal{F})_n.$$

This is a graded k-vector space. The action of $b\in B_n$ on $g\in(\Gamma\mathcal{F})_m$ is defined by

$$gb := s^n(g)\circ b; \tag{1-3}$$

this is the composition

$$\mathcal{O}\xrightarrow{b} s^n\mathcal{O}\xrightarrow{s^n(g)} s^{n+m}\mathcal{F}.$$

It is an easy exercise to check that this makes $\Gamma\mathcal{F}$ a graded right B-module.

We make Γ into a covariant functor from $\mathcal{C}$ to graded right B-modules by defining the action of Γ on a morphism $f:\mathcal{F}\to\mathcal{G}$ in $\mathcal{C}$. The functor s^n yields a morphism $s^nf: s^n\mathcal{F}\to s^n\mathcal{G}$ and hence a k-linear map

$$(\Gamma f)_n : \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{F})\to \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{G});$$

explicitly, if $\alpha\in(\Gamma\mathcal{F})_n$, then

$$(\Gamma f)_n(\alpha) = (s^nf)\circ\alpha. \tag{1-4}$$

Summing over all n, we obtain a map of graded vector spaces

$$\Gamma f:\Gamma\mathcal{F}\to\Gamma\mathcal{G}.$$

Notice that $\deg(\Gamma f)=0$.

For example, on the triple $(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [1])$, $\Gamma M\cong M$.

PROPOSITION 1.3. *Γ is a covariant functor $\mathcal{C}\to\mathsf{GrMod}B^{\mathrm{op}}$.*

PROOF. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathcal{C}$. We must show that Γf is a B-module map. Let $b \in B_n$ and $g \in (\Gamma\mathcal{F})_m$. Then

$$\begin{aligned}(\Gamma f)(g.b) &= (\Gamma f)_{n+m}(s^n(g) \circ b) \\ &= (s^{n+m} f) \circ (s^n g) \circ b \\ &= s^n(s^m(f) \circ g) \circ b \\ &= ((s^m f) \circ g).b \\ &= \big((\Gamma f)_m(g)\big).b \\ &= \big((\Gamma f)(g)\big).b\end{aligned}$$

as required.

We now show that $(\Gamma g) \circ (\Gamma f) = \Gamma(g \circ f)$ whenever $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$. If $\alpha \in (\Gamma\mathcal{H})_n = \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{H})$, then

$$(\Gamma g \circ \Gamma f)(\alpha) = (\Gamma g)(s^n f \circ \alpha) = s^n g \circ s^n f \circ \alpha = s^n(g \circ f) \circ \alpha = (\Gamma(g \circ f))(\alpha),$$

as required. $\square$

$\boxed{\text{Paul}}$ When is Γ an equivalence of categories? (cf. [**12**])

LEMMA 1.4. *Write $B = B(\mathcal{C}, \mathcal{O}, s)$.*

1. $\Gamma(s^r\mathcal{O}) \cong B[r]$;
2. *If $x \in B_r$, then $\Gamma(s^{-r}x) : B[-r] \to B$ is left multiplication by x.*

PROOF. (1) This is obvious since $(\Gamma(s^r\mathcal{O}))_n = \operatorname{Hom}(\mathcal{O}, s^{n+r}\mathcal{O}) = B_{n+r}$. Of course one should also check that the right action of B is as expected.

(2) By definition, $B_r = \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^r\mathcal{O})$, so $s^{-r}x \in \operatorname{Hom}_{\mathcal{C}}(s^{-r}\mathcal{O}, \mathcal{O})$. Hence, by (1), $\Gamma(s^{-r}x)$ is a right B-module map from $B[-r]$ to B; but such a map is left multiplication by the image of $1 \in B[-r]_r$. However,

$$\Gamma(s^{-r}x)(1) = (\Gamma(s^{-r}x))_r(1) = (s^{r-r}x) \circ 1\!\!1_{\mathcal{O}} = x,$$

where the first equality comes from the fact that $\deg(1) = r$, the second comes from (1-4) and the third from (1-2). $\square$

In general, neither $B(\mathcal{C}, \mathcal{O}, s)$ nor the modules $\Gamma\mathcal{F}$ is left bounded. Set $B_+ = B(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ and define the subfunctor Γ_+ of $\Gamma : \mathcal{C} \to \mathsf{GrMod}B^{\mathrm{op}}$ by the rule

$$\Gamma_+\mathcal{F} = \bigoplus_{n \geq 0} \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, s^n\mathcal{F})$$

and

$$(\Gamma_+ f)(\alpha) = \bigoplus_{n \geq 0} (s^n f) \circ \alpha.$$

Warning: Although $\Gamma\mathcal{O}(r) \cong B[r]$, the modules $\Gamma_+\mathcal{O}(r)$ and $B_+[r]$ are not usually isomorphic—the situation is as follows:

$$\begin{aligned}\Gamma_+\mathcal{O}(r) &= (\Gamma\mathcal{O})_{\geq r}[r], \quad \text{and} \\ \Gamma_+\mathcal{O}[r] &= (\Gamma\mathcal{O})_{\geq 0}[r],\end{aligned}$$

so there are injections

$$\begin{aligned}\Gamma_+\mathcal{O}(r) \to \Gamma_+\mathcal{O}[r] &\qquad \text{if } r \geq 0, \text{ and} \\ \Gamma_+\mathcal{O}[r] \to \Gamma_+\mathcal{O}(r) &\qquad \text{if } r \leq 0;\end{aligned}$$

in both cases the cokernel is bounded, so $\pi\Gamma_+\mathcal{O}(r) \cong \pi\Gamma_+\mathcal{O}[r]$ in $\mathsf{Tails}(B_+^{\mathrm{op}})$.

EXERCISES

1.1 Fill in the details required to show that $B(\mathcal{C}, \mathcal{O}, s)$ is a graded k-algebra and that $\Gamma\mathcal{F}$ is a graded right B-module.

1.2 Show there is an isomorphism of graded B-modules $\Gamma\mathcal{F}(1) \cong (\Gamma\mathcal{F})[1]$ for all $\mathcal{F}$ in $\mathcal{C}$.

1.3 Show that $(\Gamma\mathcal{F})_{\geq r}[r] \cong \Gamma_+\mathcal{F}(r)$.

1.4 Let $(\mathcal{C}, \mathcal{O}, \sigma)$ be an algebraic triple, and let $F : \mathcal{C} \to \mathsf{Mod}\mathbb{Z}$ be a functor. Define a functor $\underline{F} : \mathcal{C} \to \mathsf{GrMod}\mathbb{Z}$ by

$$(\underline{F}M)_i := \begin{cases} (F \circ s^i)M & \text{if } F \text{ is covariant,} \\ (F \circ s^{-i})M & \text{if } F \text{ is contravariant.} \end{cases}$$

Show this notation is consistent with the earlier notation $\underline{\mathrm{Hom}}_A$ by showing
(a) if $F = \mathrm{Hom}_{\mathrm{Gr}}(N, -)$, then $\underline{F} = \underline{\mathrm{Hom}}_A(N, -)$, and
(b) if $G = \mathrm{Hom}_{\mathrm{Gr}}(-, M)$, then $\underline{G} = \underline{\mathrm{Hom}}_A(-, M)$.
Thus $\Gamma = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{O}, -)$.

2. Examples of the B-construction

In this section A is a graded ring.

Example 2.1. Prior to Definition 1.1 we saw that

$$B(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [1]) \cong A.$$

More generally,

$$B(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [n]) \cong A^{(n)}.$$

Given A, we can obtain various matrix algebras over A through the B-construction.

Definition 2.2. Let $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$. We define the graded matrix algebra $M_n(A)(\underline{d})$ through its homogeneous components

$$M_n(A)(\underline{d})_j = \begin{pmatrix} A_j & A_{j+d_2-d_1} & \cdots & A_{j+d_n-d_1} \\ A_{j+d_1-d_2} & A_j & \cdots & A_{j+d_n-d_2} \\ \vdots & & & \vdots \\ A_{j+d_1-d_n} & A_{j+d_2-d_n} & \cdots & A_j \end{pmatrix}.$$

Notice that $\deg(e_{ij}) = d_i - d_j$.

Lemma 2.3. *Let $M \in \mathsf{grmod}A^{\mathrm{op}}$, and let $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$. If $P := M[d_1] \oplus \cdots \oplus M[d_n]$, then*

$$B(\mathsf{GrMod}A^{\mathrm{op}}, P, [1]) \cong \underline{\mathrm{End}}_A P \cong M_n(E)(\underline{d})$$

as graded k-algebras, where $E = \underline{\mathrm{End}}_A M$.

Proof. Write $P := M[d_1] \oplus \cdots \oplus M[d_n]$. Since P is finitely generated $\underline{\mathrm{End}}_A M = \mathrm{End}_A M \cong M_n(E)$. We only need to check that the grading on $M_n(\underline{d})$ is that induced by the grading on P. But $P_i = M_{i+d_1} \oplus \cdots \oplus M_{i+d_n}$, so left multiplication by $M_n(E)(\underline{d})_j$ sends elements of P_i to elements of P_{i+j}; hence the grading is the correct one. □

EXAMPLE 2.4. Let $A = k[x, x^{-1}]$ with $\deg(x) = 3$ and let $\underline{d} = (2, 7)$. Then $Q = M_2(k[x, x^{-1}])(\underline{d})$ has the following homogeneous components:

$$Q_{3n} = \begin{pmatrix} kx^n & 0 \\ 0 & kx^n \end{pmatrix}, \quad Q_{3n+1} = \begin{pmatrix} 0 & kx^{n+2} \\ 0 & 0 \end{pmatrix}, \quad Q_{3n+2} = \begin{pmatrix} 0 & 0 \\ kx^{n-1} & 0 \end{pmatrix}.$$

Also notice that $Q = J \oplus J'$, where $J = \left(\begin{smallmatrix} * & 0 \\ * & 0 \end{smallmatrix}\right)$ and $J' = \left(\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right)$. Both J and J' have periodic Hilbert series which look like $\dots, 1, 0, 1, 1, 0, 1, 1, 0, 1, \dots$; more precisely, $H_J(t) = (1 + t^2)(1 - t^3)^{-1}$ and $H_{J'}(t) = (1 + t)(1 - t^3)^{-1}$. The Hilbert series of Q itself looks like $\dots, 2, 1, 1, 2, 1, 1, 2, 1, 1, \dots$.

EXAMPLE 2.5. Skew Laurent extensions can be obtained through the B-functor. If σ is an automorphism of the ring R and $s = \sigma^*$, then

$$B(\mathsf{Mod}R^{\mathrm{op}}, R, s) \cong R[x, x^{-1}; \sigma].$$

.

EXAMPLE 2.6. We want to describe $B(\mathcal{C}, \mathcal{O}, s)$ in the following situation. Let $\mathcal{C} = \mathsf{Mod}R^{\mathrm{op}}$, where

$$R = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

is the direct sum of two copies of k. Let $L_1 = (0\ k)$ and $L_2 = (k\ 0)$ be the two simple right R-modules on which R acts by right multiplication. A typical R-module may be written $V = n_1 L_1 \oplus n_2 L_2$, meaning the direct sum of n_1 copies of L_1 and n_2 copies of L_2. Define $s : \mathcal{C} \to \mathcal{C}$ by

$$s(n_1 L_1 \oplus n_2 L_2) = n_2 L_1 \oplus n_1 L_2,$$

and $\mathcal{O} := L_1 \oplus 2L_2$.

Define $\mathcal{O}^* = \mathrm{Hom}_R(\mathcal{O}, R)$. Since $\mathcal{O}$ is a progenerator, the functor $- \otimes_R \mathcal{O}^*$ sets up an equivalence $(\mathsf{Mod}R^{\mathrm{op}}, \mathcal{O}) \xrightarrow{\sim} (\mathsf{Mod}T^{\mathrm{op}}, T)$, where $T = \mathcal{O} \otimes_R \mathcal{O}^* \cong \mathrm{Hom}_R(\mathcal{O}, \mathcal{O})$. Now, transferring s via this equivalence, we have

$$B(\mathcal{C}, \mathcal{O}, s) \cong B(\mathsf{Mod}T^{\mathrm{op}}, T_T, s)$$

which is a special case of Example 6.14. If we write

$$T = \begin{pmatrix} k & 0 & 0 \\ 0 & k & k \\ 0 & k & k \end{pmatrix}$$

then the bimodule representing s is

$$X = \begin{pmatrix} 0 & k & k \\ k & 0 & 0 \\ k & 0 & 0 \end{pmatrix}.$$

From this it follows that B can be described as the subalgebra of $M_3(k) \otimes_k k[t]$,

$$B = \begin{pmatrix} E & U & U \\ U & E & E \\ U & E & E \end{pmatrix}$$

where $E = k[t^2]$ and $U = tk[t^2]$. The grading on $M_3(k) \otimes k[t]$ is the tensor product grading with $\deg(t) = 1$ and $\deg(M_3(k)) = 0$.

EXAMPLE 2.7. If $\mathcal{L}$ is a line bundle on a scheme Y and $s = \mathcal{L} \otimes_{\mathcal{O}_Y} -$, then

$$B(\mathsf{Coh}(\mathcal{O}_Y), \mathcal{O}_Y, s) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(Y, \mathcal{L}^{\otimes n}).$$

The functoriality of B has the following interpretation. If $f : X \to Y$ is a morphism of schemes there is a morphism of triples

$$(f^*, \mathbb{1}, \mathbb{1}) : (\mathsf{Coh}(\mathcal{O}_Y), \mathcal{O}_Y, s) \to (\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, s')$$

where $s' = f^*\mathcal{L} \otimes_{\mathcal{O}_X} -$ and hence, by Proposition 1.2, a homomorphism of the corresponding graded algebras. On the degree n component this is the natural map $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \to \mathrm{H}^0(X, f^*\mathcal{L}^{\otimes n})$.

A Generalization

The construction of $B(\mathcal{C}, \mathcal{O}, s)$ makes sense even if the functor $s : \mathcal{C} \to \mathcal{C}$ is not an auto-equivalence.

EXAMPLE 2.8. Consider the triple $(\mathsf{Mod}k, k, s)$ where $s = V \otimes_k -$ is tensoring with a fixed vector space. Then $B(\mathsf{Mod}k, k, s)_{\geq 0} = T(V)$ the tensor algebra on V.

EXAMPLE 2.9. Consider the co-syzygy functor $s = \Omega_{-1}$ on $\mathsf{Mod}R$. We have $\mathrm{Ext}^n_R(X, Y) \cong \mathrm{Hom}_R(X, s^nY)$, and thus $B(\mathsf{Mod}R, X, s) \cong \mathrm{Ext}^*_R(X, X)$ with its Yoneda product, and the Γ functor makes each $\mathrm{Ext}^*_R(X, Y)$ a right module over $\mathrm{Ext}^*_R(X, X)$.

3. Equivalences of module categories GrMod A

Eventually, when we define and study maps between non-commutative schemes, the issue of isomorphism arises, and in part this reduces to understanding when two graded rings can have equivalent module categories. The graded analogue of Watt's Theorem (11.3.5) says that any right exact functor $\mathsf{GrMod}A \to \mathsf{GrMod}B$ naturally equivalent to $M \otimes_A -$ for a suitable bigraded bimodule ${}_AM_B$. We now consider the case where M implements an equivalence of categories.

Definition 3.1. A bimodule P over a ring R is invertible if there exists a bimodule Q with the property that $P \otimes_R Q \cong Q \otimes_R P \cong R$, the isomorphisms being as R-R-bimodules.

If P is an R-R-bimodule, then the functor $P \otimes_R - : \mathsf{Mod}R \to \mathsf{Mod}R$ is an equivalence if and only if P is invertible.

EXAMPLE 3.2. Let $\sigma, \tau \in \mathrm{Aut}_k(R)$ be algebra automorphisms. Define the R-R-bimodule ${}^\sigma R^\tau$ to be R as a k-vector space, with R-R-action defined by

$$a.x.b = a^\sigma x b^\tau,$$

for $a, b \in R$, $x \in {}^\sigma R^\tau$, where the right hand side is the usual product in R. There is a bimodule isomorphism

$$\theta : {}^\sigma R^\tau \otimes_R {}^\mu R^\nu \cong {}^\lambda R^\rho, \tag{3-1}$$

where $\lambda, \rho \in \mathrm{Aut}_k(R)$ are any automorphisms which satisfy $\lambda^{-1}\rho = \sigma^{-1}\tau\mu^{-1}\nu$, and θ is explicitly given by

$$\theta(x \otimes y) = x^\alpha y^\beta$$

where $\alpha := \lambda\sigma^{-1}$ and $\beta := \rho\nu^{-1}$. The isomorphism (3-1) shows that the map $\mathrm{Aut}_k(R) \to \mathrm{Pic}(R)$, defined by

$$\sigma \mapsto {}^1R^\sigma,$$

is a group homomorphism with kernel the subgroup of inner automorphisms of R, the inner automorphisms being those given by conjugation by units in R. There are also isomorphisms

$$^{\sigma}R^{\tau} \cong {}^{1}R^{\sigma^{-1}\tau}, \qquad {}^{\sigma}R^{1} \cong {}^{1}R^{\sigma^{-1}}, \qquad {}^{\sigma}R^{\tau} \cong {}^{\nu\sigma}R^{\nu\tau}.$$

The functor

$$\Phi_\sigma = {}^{\sigma}R^{1} \otimes_R -$$

is an auto-equivalence of $\mathsf{Mod}R$ with inverse $\Phi_{\sigma^{-1}}$. (A simple interpretation of Φ_σ is given in Exercise 6.)

If $\sigma : R \to R$ is an algebra automorphism, then the functors σ^* and σ_* are certainly exact, so which bimodules do they correspond to? It is easy to see that σ_* is naturally equivalent to ${}^{\sigma}R^{1} \otimes_R -$, and σ^* is naturally equivalent to ${}^{1}R^{\sigma} \otimes_R -$, and

Definition 3.3. An invertible bimodule over a graded algebra A is a graded A-A-bimodule, X say, such that, there is another graded A-A-bimodule, Y say, with the property that $Y \otimes_A X \cong X \otimes_A Y \cong A$ as graded A-A-bimodules.

LEMMA 3.4. *Let A be connected, and M a graded A-A-bimodule such that $M \cong A[\ell]$ as both a left and as a right module. Then $M \cong {}^{1}A^{\sigma}[\ell]$ for some $\sigma \in \operatorname{Aut}_k A$.*

PROOF. Replacing M by $M[-\ell]$, we may assume that $\ell = 0$. Choose $0 \neq e \in M_0$. By hypothesis $M = Ae = eA$. Define $\sigma : A \to A$ by $\sigma(x)e = ex$. It is easy to check that σ is a graded algebra automorphism. Now define $\varphi : {}^{1}A^{\sigma} \to M$ by $\varphi(a) = ae$. This is a degree zero k-linear bijection. It is obviously a left module homomorphism; also, if $x \in {}^{1}A^{\sigma}$, then $\varphi(x.a) = \varphi(xa^{\sigma}) = xa^{\sigma}e = xea = \varphi(x)a$, so φ is a right module homomorphism. □

4. Twisting systems

We will show that two graded algebras having equivalent categories of graded modules are twists of each other (Theorem 4.7), and conversely.

Definition 4.1. A twisting system on a graded k-algebra A is a collection $\theta = (\theta_n)_{n\in\mathbb{Z}}$ of graded vector space isomorphisms $\theta_n : A \to A$ such that, for all $\ell, m, n \in \mathbb{Z}$, and all $a_\ell \in A_\ell$ and $a_m \in A_m$,

$$\theta_n(\theta_m(a_\ell)a_m)) = \theta_{m+n}(a_\ell)\theta_n(a_m). \tag{4-1}$$

We define the twist of A by θ to be the graded algebra ${}^{\theta}A$, which is A as a graded vector space, and multiplication $*$ defined by

$$a_\ell * a_m = \theta_m(a_\ell)a_m,$$

for $a_\ell \in A_\ell$ and $a_m \in A_m$. Condition (4-1) ensures that $*$ is associative.

A twisting system is a kind of generalized automorphism. In fact, if $\sigma \in \operatorname{Aut}_k A$ is a graded algebra automorphism, then (σ^n) is a twisting system. Twists of this special type are discussed in Section 5.

The next result says that twisting may be interpreted as an equivalence relation on the class of graded algebra structures on a given graded vector space.

THEOREM 4.2. *Write $A \sim B$ if B is a twist of A with respect to a twisting system. Then $\sim$ is an equivalence relation.*

PROOF. Before beginning the proof, observe that the associativity of the product in ${}^{\theta}A$ is equivalent to the relation (4-1).

The relation is reflexive: take $\theta = (\theta_n)$ with $\theta_n = \mathbb{1}_A$ for all n.

The relation is symmetric. If $B = {}^{\theta}A$, define $\psi = (\psi_n)$ with $\psi_n = \theta_n^{-1}$ for all n. To see that ψ is a twisting system on B it suffices, by the first paragraph of the proof, to show that the product defined by

$$(b_m, b_n) \mapsto b_m * \psi_m(b_n),$$

for $b_m \in B_m$ and $b_n \in B_n$, is associative. By definition of ψ and $*$, this equals $b_m b_n$, the product of b_m and b_n in A itself. Hence it is associative.

The relation is transitive. Let $B = {}^{\theta}A$ and $C = {}^{\psi}B$. Define $\varphi = (\varphi_n)$ by $\varphi_n = \theta_n\psi_n$ for all n. The product on ${}^{\varphi}A$ is defined by

$$(a_m, a_n) \mapsto a_m\varphi_m(a_n),$$

for $a_m \in A_m$ and $a_n \in A_n$, whereas the product on C is defined by

$$(b_m, b_n) \mapsto b_m * \psi_m(b_n) = b_m\theta_m\psi_m(b_n) = b_m\varphi_m(b_n),$$

for $b_m \in B_m$ and $b_n \in B_n$. Thus $C = {}^{\varphi}A$. Since the product in C is associative φ is a twisting system on A. $\square$

LEMMA 4.3. *A graded algebra B is isomorphic to a twist of A if and only if there are graded vector space isomorphisms $\phi_n : A \to B$ such that*

$$\phi_n(ab) = \phi_{m+n}(a)\phi_n(b) \tag{4-2}$$

for all $\ell, m, n \in \mathbb{Z}$ and all $a \in A_\ell$, $b \in A_m$.

PROOF. Let $\theta = (\theta_n)$ be a twisting system on A and $f : B \to {}^{\theta}A$ a graded algebra isomorphism. Define $\phi_n = \theta_n \circ f$. Then $\phi_n(ab)$ equals

$$\theta_n(f(a) * f(b)) = \theta_n(\theta_m(f(a))f(b)) = \theta_{m+n}(f(a))\theta_n(f(b)) = \phi_{m+n}(a)\phi_n(b).$$

Conversely, given maps ϕ_n satisfying (4-2), we will show that the maps $\theta_n := \phi_n \circ \phi_0^{-1}$ define a twisting system, and that $\phi_0 : B \to {}^{\theta}A$ is an algebra isomorphism. Let $a \in A_\ell$ and $b \in A_m$, and write $y = \phi_{m+n}(a)$ and $z = \phi_n(b)$. Applying ϕ_n^{-1} to both sides of (4-2), we have

$$\phi_n^{-1}(yz) = \phi_{m+n}^{-1}(y)\phi_n^{-1}(z) \tag{4-3}$$

for all $y \in A_\ell$ and $z \in A_m$. Now

$$\begin{aligned}
\theta_n(\theta_m(y)z) &= \phi_n(\phi_m^{-1}(\theta_m(y))\phi_0^{-1}(z)) && \text{by (4-3) with } n = 0,\\
&= \phi_n(\phi_0^{-1}(y)\phi_0^{-1}(z))\\
&= \phi_{m+n}(\phi_0^{-1}(y))\phi_n(\phi_0^{-1}(z)) && \text{by (4-2)}\\
&= \theta_{m+n}(y)\theta_n(z),
\end{aligned}$$

so (θ_n) is a twisting system. Finally, if $*$ denotes the product in ${}^{\theta}A$, and $a \in A_\ell$ and $b \in A_m$, then

$$\phi_0(ab) = \phi_m(a)\phi_0(b) = \theta_m(\phi_0(a))\phi_0(b) = \phi_0(a) * \phi_0(b)$$

so ϕ_0 is an algebra homomorphism, and hence isomorphism. $\square$

PROPOSITION 4.4. *Let $(\mathcal{C}, \mathcal{O}, s)$ and $(\mathcal{C}', \mathcal{O}', t)$ be triples and suppose that $F : \mathcal{C} \to \mathcal{C}'$ is a fully faithful functor such that $F(s^n\mathcal{O}) \cong t^n\mathcal{O}'$ for all n. Then $B(\mathcal{C}', \mathcal{O}', t)$ is isomorphic to a twist of $B(\mathcal{C}, \mathcal{O}, s)$.*

Proof. We will exhibit isomorphisms $\phi_n : B \to B'$ satisfying (4-2). To this end, fix isomorphisms $\psi_n : F(s^n\mathcal{O}) \to t^n\mathcal{O}'$, let $a \in B_\ell = \operatorname{Hom}(\mathcal{O}, s^\ell\mathcal{O})$, and consider the diagram

$$\begin{array}{ccc} F(s^n\mathcal{O}) & \xrightarrow{F(s^n a)} & F(s^{\ell+n}\mathcal{O}) \\ {\scriptstyle\psi_n}\big\downarrow & & \big\downarrow{\scriptstyle\psi_{\ell+n}} \\ t^n\mathcal{O}' & & t^{\ell+n}\mathcal{O}' \end{array}$$

We define

$$\phi_n(a) := t^{-n}\big(\psi_{\ell+n} \circ F(s^n a) \circ \psi_n^{-1}\big) \in B'_\ell = \operatorname{Hom}(\mathcal{O}', t^\ell\mathcal{O}')$$

For $a \in B_\ell$ and $b \in B_m$, we have

$$\begin{aligned} \phi_n(ab) &= t^{-n}\big(\psi_{\ell+m+n} \circ F(s^n(ab)) \circ \psi_n^{-1}\big) \\ &= t^{-n}\big(\psi_{\ell+m+n} \circ F(s^n(s^m(a) \circ b)) \circ \psi_n^{-1}\big) \\ &= t^{-n}\big(\psi_{\ell+m+n} \circ Fs^{n+m}(a) \circ Fs^n(b) \circ \psi_n^{-1}\big) \\ &= t^{-n}\big(\psi_{\ell+m+n} \circ Fs^{n+m}(a) \circ \psi_{m+n}^{-1} \circ \psi_{m+n} \circ Fs^n(b) \circ \psi_n^{-1}\big) \\ &= t^{-n}\big(\psi_{\ell+m+n} \circ Fs^{n+m}(a) \circ \psi_{m+n}^{-1}\big) \circ t^{-n}\big(\psi_{m+n} \circ Fs^n(b) \circ \psi_n^{-1}\big) \\ &= t^m(\phi_{m+n}(a)) \circ \phi_n(b) \\ &= \phi_{m+n}(a)\phi_n(b), \end{aligned}$$

so the result follows from Lemma 4.3. □

There is a version of twisting on the other side. To distinguish the two notions we call the first type a left twist and the new type a right twist.

Definition 4.5. A right twisting system on a graded k-algebra A is a collection $\tau = (\tau_n)_{n\in\mathbb{Z}}$ of graded vector space isomorphisms $\tau_n : A \to A$ such that, for all $\ell, m, n \in \mathbb{Z}$, and all $a \in A_\ell$ and $b \in A_m$,

$$\tau_n(a\tau_\ell(b)) = \tau_n(a)\tau_{\ell+n}(b). \tag{4-4}$$

We define the right twist of A by τ to be the graded algebra A^τ, which is A as a graded vector space, and multiplication $\odot$ defined by

$$a \odot b = a\tau_\ell(b)$$

for $a \in A_\ell$ and $b \in A_m$. Condition (4-4) ensures that $\odot$ is associative.

The next lemma shows that algebras obtained by right twists are the same as those obtained by left twists, so there are versions of the earlier results for both right and left twisting systems. The reason we introduce both twists is that formulae involving right (resp. left) modules are simpler if we work with right (resp. left) twists.

Lemma 4.6. *Let $\tau = (\tau_n)$ be a right twisting system for A, and define*

$$\nu_n(y) := \tau_{-m-n}\tau_{-m}^{-1}(y)$$

whenever $y \in A_m$. Then $\nu = (\nu_n)$ is a left twisting system and ${}^\tau A \cong A^\nu$.

PROOF. Let $a \in A_\ell$ and $b \in A_m$. Then

$$\begin{aligned}\nu_n(\nu_m(a)b) &= \tau_{-\ell-m-n}\tau^{-1}_{-\ell-m}\big(\tau_{-m-\ell}(\tau^{-1}_{-\ell}(a))b\big)\\ &= \tau_{-\ell-m-n}\tau^{-1}_{-\ell}(a)\tau_{-\ell-m-n}(\tau^{-1}_{-\ell-m}(b)\\ &= \nu_{m+n}(a)\nu_n(b),\end{aligned}$$

so ν is a twisting system.

Write $*$ for the multiplication in A^ν and $\odot$ for the multipication in ${}^\tau A$. Define $f : {}^\tau A \to A^\nu$ by $f(a) := \tau_{-\ell}(a)$ for $a \in A_\ell$. Then

$$\begin{aligned}f(a) * f(b) &= \nu_m(f(a))f(b)\\ &= \tau_{-\ell-m}(a)\tau_{-m}(b)\\ &= \tau_{-\ell-m}(a\tau_\ell(b)) \qquad \text{by (4-4)}\\ &= f(a\tau_\ell(b))\\ &= f(a \odot b),\end{aligned}$$

so f is an algebra isomorphism. □

THEOREM 4.7. *Let A and B be connected, graded k-algebras with $A_1 \neq 0$. Then $\mathsf{GrMod}A^{\mathrm{op}}$ is equivalent to $\mathsf{GrMod}B^{\mathrm{op}}$ if and only if B is isomorphic to a twist of A.*

PROOF. ($\Leftarrow$) Without loss of generality we may suppose that B is a twist of A, say $B = A^\tau$. For each graded right A-module N, define N^τ to be N as a graded vector space, and define a left action of B on N^τ by

$$n \odot x = n\tau_\ell(x)$$

for $n \in N_\ell$ and $x \in B_i$. This makes N^τ into a graded right B-module. Define a covariant functor $\Theta : \mathsf{GrMod}A^{\mathrm{op}} \to \mathsf{GrMod}B^{\mathrm{op}}$ by

$$\Theta(N) = N^\tau \qquad \text{and} \qquad \Theta(f) = f,$$

for $f \in \mathrm{Hom}_{\mathrm{Gr}}(N, N')$. It is easy to see that f is a homomorphism of B-modules, so Θ really is a functor between these categories.

By Theorem 4.2, $A \cong B^\sigma$ for the twisting system $\sigma = (\sigma_n)$ defined by $\sigma_n = \tau_n^{-1}$, so there is a similar functor $\Psi : \mathsf{GrMod}B^{\mathrm{op}} \to \mathsf{GrMod}A^{\mathrm{op}}$. It is an easy matter to check that Θ and Ψ are inverse to one another, thus giving the desired equivalence of categories.

($\Rightarrow$) Let $F : \mathsf{GrMod}A^{\mathrm{op}} \to \mathsf{GrMod}B^{\mathrm{op}}$ be the equivalence. Since A is connected the indecomposable projectives in $\mathsf{GrMod}A$ are the shifts $A(n)$. Hence there is a bijection on $\mathbb{Z}$, $n \leftrightarrow n'$, such that $F(A[n]) \cong B[n']$ for all n. Since shifting by a fixed degree is an automorphism of GrMod, we can replace F by $[-0'] \circ F$, and assume that $FA \cong B$. This gives a new bijection which we still denote by $n \leftrightarrow n'$. Since $\mathrm{Hom}_{\mathrm{Gr}}(A[n], A[n+1]) \cong A_1 \neq 0$, so is $\mathrm{Hom}_{\mathrm{Gr}}(B[n'], B[(n+1)']) \neq 0$, whence $n' \leq (n+1)'$. Therefore, since $0' = 0$, the bijectivity implies that $n' = n$ for all n. Thus $F(A[n]) \cong B[n]$ for all n. The conclusion now follows from Proposition 4.4 because $B(\mathsf{GrMod}A^{\mathrm{op}}, A_A, [1]) \cong A$. □

The equivalence between $\mathsf{GrMod}A$ and $\mathsf{GrMod}({}^\theta A)$ matches up A with ${}^\theta A$, so the pairs $(\mathsf{GrMod}A, A)$ and $(\mathsf{GrMod}({}^\theta A), {}^\theta A)$ are equivalent in an obvious sense. Thus, A can not be recaptured simply from knowledge of the category $\mathsf{GrMod}A$—one needs the extra information implicit in the shift functor.

$\boxed{\text{MORE}}$

The following properties are twisting invariants: domain, noetherian, semiprime when the ring is noetherian, the number of defining relations and their degrees (use Section 1 of Chapter 7 ??), Homological properties. For a polynomial ring, all twisting systems are twists by an autom....

EXAMPLE 4.8. The property of being a prime ring is *not* a twisting invariant. Let $A = k\langle x,y\rangle/(xy,yx)$ and define $\sigma \in \operatorname{Aut}_k A$ by $x^\sigma = y$ and $y^\sigma = x$. Then $A^\sigma = k\langle x,y\rangle/(x^2,y^2)$. Although A^σ is prime, A is not.

EXERCISES

4.1 Let $\sigma \in \operatorname{Aut}_k A$. Show that, for all d,

$$(A^\sigma)^{(d)} \cong (A^{(d)})^{\sigma^d}.$$

5. Twisting an algebra by an automorphism

If $\theta \in \operatorname{Aut}_k A$, then (θ^n) is a right and a left twisting system. Thus the products in $(A^\theta, \odot)$ and $({}^\theta A, *)$ are given by

$$x \odot y := xy^{\theta^\ell} \qquad \text{and} \qquad x * y = x^{\theta^m} y$$

respectively, for $x \in A_\ell$ and $y \in A_m$. The isomorphism in Lemma 4.6 gives

$${}^\theta A \cong A^{\theta^{-1}}.$$

Proposition 5.10 shows that every twist of the commutative polynomial ring is isomorphic to a twist of this special type.

LEMMA 5.1. *If $\sigma \in \operatorname{Aut}_k A$ is a graded algebra automorphism, then*

$$B(\mathsf{GrMod} A^{\mathrm{op}}, A_A, [1] \circ \sigma^*) \cong A^\sigma,$$

the twist of A with respect to the automorphism σ.

PROOF. We give two proofs, each of which is illuminating.

First proof. Write $R = {}^\sigma A$. We will show that there is an isomorphism of triples

$$(F, \mathbb{1}, \mu) : (\mathsf{GrMod} A^{\mathrm{op}}, A_A, [1] \circ \sigma^*) \to (\mathsf{GrMod} R^{\mathrm{op}}, R_R, [1]),$$

so the result will follow from the functoriality of B, since $B(\mathsf{GrMod} R^{\mathrm{op}}, R_R, [1]) \cong R$. Let $F : \mathsf{GrMod} A^{\mathrm{op}} \to \mathsf{GrMod} R$ be the functor defined by $FM = M$, as graded vector spaces, endowed with the action $m * x = mx^{\sigma^j}$ for $x \in R = {}^\sigma A$, and $m \in (FM)_j = M_j$. We need only show that there is a natural equivalence $\mu : F \circ [1] \circ \sigma^* \to [1] \circ F$. Now $\sigma^* M = M$ as a graded vector space, but is endowed with a new A-action, $m.a = ma^\sigma$. Hence, if $m \in (F \circ [1] \circ \sigma^* M)_j = M_{j+1}$, and $x \in R$, then

$$m * x = m.x^{\sigma^j} = mx^{\sigma^{j+1}}.$$

On the other hand, if $m \in ([1] \circ FM)_j = M_{j+1}$, then $m * x = mx^{\sigma^{j+1}}$. Hence the identity map $M \to M$ induces an equivalence of functors, as required.

Second proof. Define $\Phi : B \to A^\sigma$ by $\Phi(b) = b(1)$; this makes sense since if $b \in B_\ell = \mathrm{Hom}_{\mathrm{Gr}}(A, s^\ell A)$, then $b(1) \in (s^\ell A)_0 = A_\ell = (A^\sigma)_\ell$. Now let $b \in B_\ell$ and $c \in B_m$. Then

$$\begin{aligned}
\Phi(b.c) &= \Phi(s^m(b) \circ c) \\
&= s^m(b)(c(1)) && \text{where } c(1) \in s^m A \\
&= s^m(b)(1 * c(1)^{\sigma^{-m}}) && \text{since } s^m(b) \text{ is a right module map } s^m A \to s^{\ell+m} A \\
&= s^m(b)(1) * c(1)^{\sigma^{-m}} && \text{using the right action of } c(1) \text{ on } s^{\ell+m} A \\
&= b(1) * c(1)^{\sigma^{-m}} \\
&= b(1)c(1)^{\sigma^\ell}.
\end{aligned}$$

But $b(1)c(1)^{\sigma^\ell}$ is the product of $b(1)$ and $c(1)$ in A^σ. Hence the result. $\square$

The twisting construction generalizes the Öre extension construction.

LEMMA 5.2. *Let A be a graded k-algebra, and $\sigma \in \mathrm{Aut}_k A$ a graded algebra autmorphism. Extend σ to the polynomial extension $A[t] = A \otimes_k k[t]$ by*

$$\sigma(\sum_j a_j t^j) = \sum_j a_j^\sigma t^j.$$

If $A[t]$ is given the tensor product grading, then

$$A[t]^\sigma \cong A[t; \sigma]$$

as graded algebras.

PROOF. The map $\Phi : A[t]^\theta \to A[x; \sigma]$, defined by $\Phi(\sum_j a_j t^j) = \sum_j a_j x^j$, is an algebra isomorphism. $\square$

Lemma 5.2 implies that the coordinate rings of the quantum and Jordan planes can be realized as twists of the commutative polynomial ring on two indeterminates. The next example shows this in an explicit way.

EXAMPLE 5.3. Let $A = k[x, y]$ be the commutative polynomial ring, fix $0 \neq q \in k$ and let $\theta : A \to A$ be the algebra automorphism defined by

$$x^\theta = x \qquad \text{and} \qquad y^\theta = qy.$$

It follows that $x * y = qxy$ and $y * x = yx$, so

$$y * x = q^{-1} x * y. \tag{5-1}$$

The k-algebra generated by x and y, and having defining relation (5-1), is a coordinate ring of a quantum affine plane and has Hilbert series the same as the polynomial ring on 2 indeterminates; therefore $(A^\theta, *) \cong k\langle x, y\rangle/(xy - qyx)$.

On the other hand, if $\varphi : A \to A$ is defined by

$$x^\varphi = x \qquad \text{and} \qquad y^\varphi = y - x,$$

then $y * x = yx$ and $x * y = xy - x^2$, so $y * x - x * y = x^2$. A Hilbert series argument shows that $(A^\varphi, *) \cong k\langle x, y\rangle/(yx - xy - x^2)$, which is the coordinate ring of the Jordan plane.

EXAMPLE 5.4. Although the coordinate rings of the quantum and Jordan planes are twists of the polynomial ring by suitable automorphisms, they are not twists of one another with respect to any automorphism. Thus, twisting by an automorphism is not an equivalence relation.

To see this, suppose that $A = k[x, y]$ with $yx = qxy$ $(q \neq 1)$, and let $\theta \in \operatorname{Aut}_k A$. Since an automorphism sends a normal element to another normal element, θ sends $\{x, y\}$ to $\{\alpha x, \beta y\}$ for some $\alpha, \beta \in k$; moreover, since θ preserves the relation, $x^\theta \in kx$ and $y^\theta \in ky$. Hence, in A^θ, $x * y$ and $y * x$ are scalar multiples of one another; thus A^θ is another quantum plane, *not* a Jordan plane. However, since the coordinate rings of the quantum and Jordan planes are both twists of a polynomial ring, the equivalence in Theorem 4.2 shows they are twists of one another with respect to a twisting system.

If k is algebraically closed, then the coordinate rings of the quantum planes and Jordan planes are the only twists, up to isomorphism, of the polynomial ring $k[x, y]$ with its usual grading. To see this, let $\theta \in \mathrm{GL}(A_1)$ and put θ in Jordan normal form: then θ is a scalar multiple of either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. By the next example, the scalar multiple does not affect the isomorphism class of the twist, and the previous example shows that these two matrices give the coordinate rings of the quantum and Jordan planes respectively.

EXAMPLE 5.5. Let A be a graded k-algebra and let $0 \neq \lambda \in k$. Define the automorphism $\psi = \psi_\lambda$ by $\psi(a) = \lambda^n a$, for $a \in A_n$. Then the map $\Phi : A \to (A^\psi, *)$ defined by

$$\Phi(a) = \lambda^{\frac{1}{2}n(n-1)} a \quad \text{if } a \in A_n,$$

is an algebra isomorphism. Thus twisting by ψ_λ does not change the isomorphism class of the algebra.

It is important to observe that the functor $\Theta : \mathsf{GrMod} A \to \mathsf{GrMod} A^\theta$ giving the equivalence of categories in Theorem 4.7 does not generally commute with the shift functors in the two categories; the next example illustrates this.

EXAMPLE 5.6. We continue Example 5.3, with $A = k[x, y]$ the polynomial ring and ${}^\theta A$ the quantum plane with relation $y \odot x = qx \odot y$. Let $M = A/A(x - \lambda y)$. We will show that ${}^\theta(M[-1])$ is not isomorphic to $({}^\theta M)[-1]$; i.e., if s_1 and s_2 denote the shift functors in $\mathsf{GrMod} A$ and $\mathsf{GrMod}({}^\theta A)$ then $\Theta s_1 \neq s_2 \Theta$ (where Θ is as in the proof of Theorem 4.7).

Both ${}^\theta(M[-1])$ and $({}^\theta M)[-1]$ are cyclic, since M is, and generated in degree 1, since M is generated in degree 0 (i.e., $M = AM_0$). Therefore, any ${}^\theta A$-module isomorphism between them would send a generator of one to a generator of the other; in particular, the degree one elements in one module would have the same annihilator as the degree one elements in the other module. We will show this does not happen.

If e_j denotes the image of y^j in M, then

$$M = ke_0 \oplus ke_1 \oplus \cdots$$

where $\deg(e_j) = j$, and the action of A is given by

$$xe_j = \lambda e_{j+1} \qquad \text{and} \qquad ye_j = e_{j+1}.$$

Since ${}^{\theta}(M[-1])$ is generated by e_0 which is of degree 1, the action of $x, y \in {}^{\theta}A$ is given by

$$x \odot e_0 = x^{\theta} e_0 = \lambda e_1 \qquad \text{and}$$
$$y \odot e_0 = y^{\theta} e_0 = q e_1.$$

Hence the generator of ${}^{\theta}(M[-1])$ is annihilated by $qx - \lambda y$.

Now $({}^{\theta}M)[-1]$ is also generated by e_0 which is of degree 1, *but the action of $x, y \in {}^{\theta}A$ is computed before shifting the degree—that is, with e_0 in degree zero.* Thus

$$x \odot e_0 = x e_0 = \lambda e_1 \qquad \text{and}$$
$$y \odot e_0 = y e_0 = e_1.$$

Hence the generator of $({}^{\theta}M)[-1]$ is annihilated by $x - \lambda y$. Thus, if $\lambda \neq 0$, then ${}^{\theta}(M[-1]) \ncong ({}^{\theta}M)[-1]$, as claimed.

The next result describes a useful trick to turn a normalizing regular element into a central element (but note that elements which are central in the original algebra may no longer be central in the twisted algebra).

LEMMA 5.7. *Let A be a graded k-algebra and suppose that $y \in A_1$ is a left and right regular, normalizing element. Then there is a twist of A, A^{θ} say, such that y is central in $(A^{\theta}, *)$.*

PROOF. Define $\theta : A \to A$ as follows: for each $a \in A$ let a^{θ} be the unique element such that $ay = ya^{\theta}$. Since y is regular and normalizing, θ is a well-defined k-linear algebra automorphism. Moreover, $y^{\theta} = y$. Hence, if $a \in A_n$ then $a * y = ay^{\theta^n} = ay = ya^{\theta} = y * a$, as required. □

If $A = T(V)/I$ and $\sigma \in \mathrm{GL}(V)$ induces an automorphism of A, then $A^{\sigma} \cong T(V)/J$ where $J_{n+1} = (\sigma^n \otimes \cdots \otimes \sigma \otimes 1)(I_{n+1})$.

PROPOSITION 5.8. *Let V be a finite dimensional vector space over an algebraically closed field k, and let $\sigma \in \mathrm{GL}(V)$. Then*

1. *$S(V)^{\sigma}$ is an iterated Öre extension of k, and*
2. *if $x_1, \ldots, x_n$ is a basis for V, $S(V)^{\sigma}$ has defining relations*

$$x_j^{\sigma} \otimes x_i - x_i^{\sigma} \otimes x_j.$$

PROOF. If $x, y \in V$ then $x * y = xy^{\sigma}$ so, in $S(V)^{\sigma}$, we have the relation

$$x^{\sigma} * y = y^{\sigma} * x. \tag{5-2}$$

Since k is algebraically closed, V has a basis $x_1, \ldots, x_n$ such that, for all j,

$$x_j^{\sigma} \in kx_1 + \cdots + kx_j.$$

Hence, we may write $x_j^{\sigma} = \lambda_j x_j + w_j$ with $0 \neq \lambda_j \in k$ and $w_j \in kx_1 + \cdots + kx_{j-1}$. It follows from (5-2) that

$$x_j * x_i = \lambda_j^{-1}(\lambda_i x_i + w_i) * x_j - \lambda_j^{-1} w_j * x_i \tag{5-3}$$

for all i and j.

We now prove the result by induction on n. It is true for $n = 1$, so suppose it is true for $n-1$. Hence the subalgebra $R = k[x_1, \ldots, x_{n-1}]$ of $S(V)^{\sigma}$, which is of the form $S(U)^{\sigma}$ for a codimension one subspace $U \subset V$, is an iterated Öre extension. Moreover, $S(V)^{\sigma} = R[x_n]$. By (5-3), $x_n R \subset Rx_n + R$. Since $S(V)$ is a domain, so is $S(V)^\sigma$, whence Rx_n is a free left R-module. Hence, by Lemma 6.1.7, $S(V)^\sigma$ is a quotient of an Öre extension of R. But, $H_R(t) = (1-t)^{n-1}$, so the Öre extension of R has Hilbert series $(1-t)^n$; since this is also the Hilbert series of $S(V)^\sigma$ we conclude that $S(V)^\sigma$ is an Öre extension of R. The proof is now complete. □

EXAMPLE 5.9. The simplest case of Proposition 5.8 is for a diagonalizable $\sigma \in \mathrm{GL}(V)$. Choose a basis $x_1, \dots, x_n$ of eigenvectors for σ, and suppose that $x_i^\sigma = \lambda_i x_i$. Then $S(V)^\sigma = k[x_1, \dots, x_n]$ with defining relations

$$x_j x_i = \lambda_j^{-1} \lambda_i x_i x_j$$

for $1 \le i, j \le n$. These are the only quasi-polynomial rings (see Example 1.4) which are twists of a polynomial ring (Exercise 4).

PROPOSITION 5.10. *Let $A = k[x_1, \dots, x_n]$ be the polynomial ring with its standard grading,* $\deg(x_i) = 1$. *If (θ) is a twisting system on A, then*

$$A^{(\theta)} \cong A^\sigma,$$

where $\sigma \in \mathrm{Aut}_k A$ is defined by $\sigma(x) = \theta_1(x)$ for $x \in A_1$. Then

PROOF. Proposition 5.8 showed that A^σ is generated by A_1 with defining relations $x^\sigma * y = y^\sigma * x$ for $x, y \in A_1$. If the product in $A^{(\theta)}$ is denoted by $\bullet$, then $A^{(\theta)}$ is generated by A_1 and for $x, y \in A_1$ we have

$$\theta_1(x) \bullet y = \theta_1(x)\theta_1(y) = \theta_1(y)\theta_1(x) = \theta_1(y) \bullet x.$$

But $\theta_1 = \sigma$ on A_1, so there is a surjective algebra homomorphism $A^\sigma \to A^{(\theta)}$; Hilbert series considerations force this map to be injective too, and hence an isomorphism as claimed. □

LEMMA 5.11. *Let A be a graded k-algebra and $\sigma \in \mathrm{Aut}_k A$. Then*

1. *there is an isomorphism*

$$\varphi : {}^{\theta^{-1}}A \xrightarrow{\sim} A^\theta,$$

   defined by $\varphi(a) = \theta^n(a)$ for $a \in A_n$;
2. *there is an isomorphism*

$$\left(A^\sigma\right)^{\mathrm{op}} \cong \left(A^{\mathrm{op}}\right)^{\sigma^{-1}}.$$

PROOF. (1) This is straightforward.
(2) From the definition of the right and left twists it follows that

$$A^\sigma \cong \left({}^\sigma\left(A^{\mathrm{op}}\right)\right)^{\mathrm{op}}$$

so the result follows from (1). □

EXERCISES

5.1 Fill in the details in Example 5.5 to show that the map Φ is an algebra isomorphism.

5.2 Let $A = k[x, y]$ be the commutative polynomial ring with $\deg(x) = 1$ and $\deg(y) = r > 0$. Describe the algebras which arise as twists of A.

5.3 Prove the remark prior to Proposition 5.8.

5.4 Consider a quasi-polynomial ring $A = k[x_1, \dots, x_n]$ with defining relations $x_j x_i = \alpha_{ij} x_i x_j$ where $\alpha_{ij}\alpha_{ji} = 1$ and $\alpha_{ii} = 1$. Show that A is a twist of a polynomial ring if and only if $\operatorname{rank}(\log \alpha_{ij}) = 2$.

5.5 Show that every left (respectively, right) ideal of A remains a left (respectively, right) ideal in the twist ${}^{\theta}A$ (respectively A^{θ}).

5.6 Let θ be a graded algebra automorphism of A.
(a) Show that θ is an algebra automorphism of $(A^{\theta}, *)$.
(b) Show that the twist of A^{θ} with respect to θ^{-1} is isomorphic to A with its original multiplication.
Thus, twisting by an automorphism yields a relation which is reflexive, but not transitive (see Example 5.4).

6. Strongly graded algebras

In this section A is a graded algebra over a field k.

Definition 6.1. A graded algebra A is strongly graded if $A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{Z}$.

EXAMPLE 6.2. Let A be a strongly graded algebra and $\sigma \in \operatorname{Aut}_k A$ a graded k-algebra automorphism. Then the skew Laurent extension $A[x, x^{-1}; \sigma]$, with $\deg(x) = 1$, is strongly graded. By induction, an iterated skew Laurent extension $k[x_1, x_1^{-1}][x_2, x_2^{-1}, \sigma_2] \dots [x_n, x_n^{-1}; \sigma_n]$ which is graded with $\deg(x_i) = 1$ for all i, is strongly graded.

LEMMA 6.3. *A is strongly graded if and only if $1 \in A_{-i}A_i$ for all $i \in \mathbb{Z}$.*

PROOF. If A is strongly graded, then $A_0 = A_{-i}A_i$ certainly contains 1. Conversely,
$$A_{i+j} = 1.A_{i+j} \subset A_i A_{-i} A_{i+j} \subset A_i A_j \subset A_{i+j},$$
so equality holds. $\square$

Strongly graded algebras arise naturally as localizations of graded algebras generated in degree 1.

PROPOSITION 6.4. *Suppose that A is generated over A_0 by A_1. If z is a homogenous, regular, normalizing element of non-zero degree, then $A[z^{-1}]$ is strongly graded.*

PROOF. Given $j \in \mathbb{Z}$ choose $m, n \in \mathbb{Z}$ such that $md - j > 0$ and $j - nd > 0$. Then $R_j \supset A_{j-nd} z^n \neq 0$ and $R_{-j} \supset A_{md-j} z^{-m} \neq 0$ so
$$R_{-j} R_j \supset A_{md-j} A_{j-nd} z^{n-m} = A_{(m-n)d} z^{n-m},$$
where we used the fact that A is generated in degree 1. But $z^{m-n} \in A_{(m-n)d}$, so $1 \in R_{-j}R_j$. Hence by Lemma 6.3, R is strongly graded. $\square$

LEMMA 6.5. *If A is strongly graded, then each A_j is a finitely generated projective A_0-module (on both the right and the left).*

Proof. Fix j. Since $A_{-j}A_j = A_0$, we may write $1 = \sum x_i y_i$ with each $x_i \in A_{-j}$ and $y_i \in A_j$. Set $L := \sum A_0 y_i$ and define the A_0-module maps $\theta_i : L \to A_0$ by $\theta_i(a) = ax_i$. For each $a \in L$, $\sum \theta_i(a) y_i = a$, so the Dual Basis Lemma implies that L is a projective left A_0-module. It is also finitely generated. However, since $1 = \sum x_i y_i$, $A_0 = A_{-j}L$, whence $A_j = A_j A_0 = A_j A_{-j} L = A_0 L = L$, so A_j is a finitely generated, projective, left A_0-module. The right module result is analogous. □

Lemma 6.6. *If A is strongly graded, and $M \in \mathsf{GrMod}A$, then $M = AM_i$ for all i.*

Proof. Notice that $M_{i+j} = A_0 M_{i+j} = A_j A_{-j} M_{i+j} \subset A_j M_i$. □

Theorem 6.7. *If A is strongly graded, the categories $\mathsf{GrMod}A$ and $\mathsf{Mod}A_0$ are equivalent via the functors $M \mapsto M_0$ and $N \mapsto A \otimes_{A_0} N$.*

Proof. Denote these functors by $F(M) = M_0$ for $M \in \mathsf{GrMod}A$ and $G(N) = A \otimes_{A_0} N$ for $N \in \mathsf{Mod}A$; of course GN is given the tensor product grading with N concentrated in degree zero. It is clear that $FG(N) \cong N$, so FG is naturally equivalent to the identity functor on $\mathsf{Mod}A$.

Let $t_M : GF(M) = A \otimes_{A_0} M_0 \to M$ be the map $t_M(a \otimes m) = am$. By Lemma 6.6, t_M is surjective. On the other hand, $\ker t_M$ is generated by its degree zero component by Lemma 6.6, but this is clearly zero, so t_M is injective. Thus t_M gives a natural equivalence $t : GF \xrightarrow{\sim} \mathbb{1}d$, thus completing the proof. □

Corollary 6.8. *If M is a graded module over a strongly graded algebra A, then $M[i] \cong A \otimes_{A_0} M_i$.*

Proof. The degree zero component each of these modules is isomorphic to M_i so, by Theorem 6.7, the modules are isomorphic. □

Corollary 6.9. *If A is strongly graded and there is a unit $u \in A_1$, then $A \cong A_0[x, x^{-1}; \sigma]$ as graded algebras, where $\sigma \in \operatorname{Aut}_k A_0$ is defined by $a^\sigma = u^{-1}au$.*

Proof. If we set $\deg(x) = 1$, then the map $\varphi : A_0[x, x^{-1}; \sigma] \to A$ defined by $\varphi(ax^i) = au^i$ for $a \in A_0$, is a homomorphism of graded algebras. It is surjective because $A_0 \supset A_i u^{-i}$, which implies that $A_i = A_0 u^i$ for all i. It is clear that $(\ker \varphi)_0 = 0$, whence $\ker \varphi = 0$ because $A_0[x, x^{-1}; \sigma]$ is strongly graded. □

Proposition 6.10. *If A is strongly graded, then the multiplication map $A_i \otimes_{A_0} A_j \to A_{i+j}$ is an isomorphism of A_0-A_0-bimodules for all $i, j \in \mathbb{Z}$.*

Proof. We establish the isomorphism as left A_0-modules, the right module version being similar. By the equivalence of categories, it suffices to show that the induced map $A \otimes_{A_0} (A_i \otimes_{A_0} A_j) \to A \otimes_{A_0} A_{i+j}$ is an isomorphism in $\mathsf{GrMod}A$. But this is immediate since, by Corollary 6.8, the left side is isomorphic to $A[i][j]$ and the right side is isomorphic to $A[i+j]$. □

Proposition 6.11. *If A is strongly graded the map $n \mapsto A_n$ defines a group homomorphism $\mathbb{Z} \to \operatorname{Pic} A_0$.*

Proof. By Lemma 6.5, each A_i is finitely generated. By Proposition 6.10, each A_i is invertible, and the map from $\mathbb{Z}$ is a group homomorphism. □

EXAMPLE 6.12. Let R be a k-algebra and σ an automorphism of R. The skew Laurent extension $A = R[x, x^{-1}; \sigma]$ is a strongly graded algebra if we set $\deg(x) = 1$ and $\deg(R) = 0$. The invertible R-R bimodule $A_n = Rx^n$ is isomorphic to ${}^1R^{\sigma^n}$. This example illustrates the next result.

THEOREM 6.13. *For any k-algebra R, there is a bijection between* $\mathrm{Pic}(R)$ *and isomorphism classes of strongly graded k-algebras A such that $A_0 \cong R$.*

PROOF. Fix an invertible R-bimodule, X say. We now construct a strongly graded k-algebra, A say, such that $A_0 = R$ and $A_1 \cong X$ as an R-R-bimodule. Define $X^\vee := \mathrm{Hom}_R(X_R, R_R)$, and

$$\begin{aligned} A_n &:= X^{\otimes n} \quad \text{for } n > 0, \\ A_0 &:= R, \\ A_{-n} &:= (X^\vee)^{\otimes n} \quad \text{for } n > 0. \end{aligned}$$

Care is required to define an associative multiplication on A. Let $f : X^\vee \otimes_R X \to R$ be the natural R-R-bimodule isomorphism, namely $f(y \otimes x) = y(x)$; by [**24**, Chapter II, Section 3] there is an R-R-bimodule isomorphism $g : X \otimes_R X^\vee \to R$ such that the following two diagrams commute:

$$\begin{array}{ccc} X \otimes X^\vee \otimes X & \xrightarrow{1 \otimes f} & X \otimes R \\ {\scriptstyle g\otimes 1}\big\downarrow & & \big\downarrow \\ R \otimes X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X^\vee \otimes X \otimes X^\vee & \xrightarrow{1 \otimes g} & X^\vee \otimes R \\ {\scriptstyle f\otimes 1}\big\downarrow & & \big\downarrow \\ R \otimes X^\vee & \longrightarrow & X^\vee \end{array}$$

It is convenient to denote the maps f and g simply by juxtaposition; thus the commutativity of the diagrams is equivalent to the associativity identities

$$x_1(yx_2) = (x_1y)x_2 \qquad \text{and} \qquad y_1(xy_2) = (y_1x)y_2,$$

where the x's are in X and the y's are in $X^\vee$. One may now define the product in A in an obvious way—we avoid writing it down since that would make it appear more complicated than it really is. It is clear that A is strongly graded. $\square$

EXAMPLE 6.14. Let s be an arbitrary auto-equivalence of $\mathsf{Mod}R^{\mathrm{op}}$, and let X be the invertible bimodule, unique up to isomorphism, such that $s \cong - \otimes_R X$. Then $B(\mathsf{Mod}R^{\mathrm{op}}, R, s)$ is isomorphic to the strongly graded algebra

$$\cdots \oplus (X^\vee)^{\otimes 2} \oplus X^\vee \oplus R \oplus X \oplus X^{\otimes 2} \oplus \cdots$$

defined in Theorem 6.13.

What are invertible bimodules over $R = M_n(k)$? Probably ${}_1R_\sigma$

EXERCISES

6.1 Let X, Y and Z be R-R-bimodules, and suppose there are bimodule isomorphisms $X \otimes_R Y \cong Z \otimes_R X \cong R$.
  (a) Show that $Y \cong Z \cong X^\vee := \mathrm{Hom}_R(X_R, R_R)$ as R-R-bimodules.
  (b) Show that the natural map $X^\vee \otimes_R X \to R$, namely $y \otimes x \mapsto y(x)$, is an R-R-bimodule isomorphism.

6.2 Let X be an invertible R-R-bimodule. Show that X is
  (a) a finitely generated, projective R-module on both sides;
  (b) a generator in the category of left (and right) R-modules.

6.3 Show that there is a bijection between elements of $\mathrm{Pic}(R)$ and auto-equivalences of $\mathsf{Mod}R$ up to natural equivalence.

6.4 Check that the map θ in (3-1) really is an isomorphism, as claimed. One must first check that θ really is well-defined.

6.5 Show that there is a left R-module isomorphism ${}^{\sigma}R^{\tau} \to R$ defined by $x \mapsto x^{\sigma^{-1}}$, and a right R-module isomorphism ${}^{s}R^{\tau} \to R$ defined by $x \mapsto x^{\tau^{-1}}$. Specializing to the case $\sigma = \tau$, there is an isomorphism $\theta : {}^{\sigma}R^{s} \to R$ of R-R-bimodules defined by $\theta(x) = x^{\sigma^{-1}}$.

6.6 If $M \in \mathsf{Mod}R$, define ${}^{\sigma}M \in \mathsf{Mod}R$ to be ${}^{\sigma}M = M$ as a graded vector space, and R-action

$$a.m = a^{\sigma}m$$

for $m \in {}^{\sigma}M$, where the action on the right is that on M. Show that $\Phi_{\sigma}M \cong {}^{\sigma}M$. [*Hint:* Since the identity map ${}^{\sigma}R^{1} \to R$ is a right R-module isomorphism, we may make the identification $\Phi_{\sigma}M = {}^{\sigma}R^{1} \otimes M \equiv M$ via $1 \otimes m \equiv m$; thus, the left R-module action on ${}^{\sigma}R^{1} \otimes M$ transfers to a new action of R on M.]

6.7 If $f : R \to S$ is a ring homomorphism, write $f^* : \mathsf{Mod}R \to \mathsf{Mod}S$ and $f_* : \mathsf{Mod}S \to \mathsf{Mod}R$ for the induction and restriction functors. Let $\sigma \in \mathrm{Aut}(R)$. Show that
(a) $\sigma^* \cong (\sigma^{-1})_*$;
(b) $\sigma^{-1*}M \cong {}^{\sigma}M$.

6.8 Check that the kernel of the map $\mathrm{Aut}_k(R) \to \mathrm{Pic}(R)$ is as claimed.

7. Graded artinian rings

The first result describes the graded analogues of division algebras; as might be expected such algebras arise when one inverts the homogeneous elements of a graded noetherian domain (Theorem 8.9.3).

Definition 7.1. A graded division algebra is a graded algebra in which every non-zero homogenous element is a unit.

A graded division algebra is not necessarily a division algebra: $k[x, x^{-1}]$ is such an example. By the next result, the only way a division algebra can be graded is if it is concentrated in degree zero.

PROPOSITION 7.2. *Let D be a graded k-algebra. The following are equivalent:*
1. *D is a graded division algebra;*
2. *D has no proper graded left ideals;*
3. *D has no proper graded right ideals;*
4. *D_0 is a division algebra, and either $D = D_0$ or $D \cong D_0[x, x^{-1}; \sigma]$, the twisted Laurent extension with $\sigma \in \mathrm{Aut}(D_0)$ and x homogeneous of positive degree.*

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) It is clear that (1) implies (2) and (3). Suppose that (2) is true. Let u be a non-zero homogeneous element of D. Since $Du = D$, there exists a homogeneous v such that $uv = 1$. The left annihilator of $vu - 1$ is a graded left ideal containing u, so equals D by hypothesis (2), which shows that $vu = 1$, so u is a unit. Thus (2) implies (1). The case (3) implies (1) is analogous.

(1) $\Rightarrow$ (4) Certainly D_0 is a division algebra, so suppose that $D \neq D_0$. Let $0 \neq u \in D$ be homogeneous of minimal positive degree, n say. Since u is a unit, there is a well-defined automorphism $a \mapsto u^{-1}au$ of D_0, which we call σ. Define

$$\varphi : D_0[x, x^{-1}; \sigma] \to D$$

by $\varphi(ax^i) = au^i$ for $a \in D_0$. This is clearly a graded algebra homomorphism if we set $\deg(x) = n$.

Since u is a unit, $D_{in} = D_0u^i$ for all $i \in \mathbb{Z}$. On the other hand if $j \notin n\mathbb{Z}$, there exists i such that $0 < j - in < n$, whence $D_ju^{-i} = 0$ by the minimality of $\deg(u)$; thus $D_j = 0$. We have shown that $D = \oplus_{i\in\mathbb{Z}} D_0u^i$, from which it follows that φ is an isomorphism.

(4) $\Rightarrow$ (1) This is obvious. □

The next result describes in more detail the structure of graded division algebras from the ungraded point of view.

PROPOSITION 7.3. *Let D be a division algebra over k, let $\sigma \in \operatorname{Aut}_k D$ and define $S = D[x, x^{-1}; \sigma]$. Then*

1. *every left and right ideal of S is principal;*
2. *S is a simple ring if and only if no power of σ is an inner automorphism;*
3. *if $\sigma^n = 1$, and σ^i is not inner for any $0 < i < n$, then $Z(S) = Z(D)^\sigma[x^n, x^{-n}]$.*

PROOF. (1) This is proved in Proposition 6.1.11.

(2) First suppose that σ^n is inner; suppose that $u \in D$ is such that $a^{\sigma^n} = u^{-1}au$ for all $a \in D$. Notice that this forces $u^{\sigma^n} = u$. Since $x^nax^{-n} = a^{\sigma^n}$ also, it follows that $w := u^{-1}x^n$ commutes with all of D. If we define $x^\sigma = x$, then $w^\sigma, \dots, w^{\sigma^{n-1}}$ also commute with D. It follows that

$$z := (1 - w)(1 - w^\sigma) \dots (1 - w^{\sigma^{n-1}})$$

also commutes with D; but z also commutes with x, so is central in S. It is clear that $z \neq 0$ and is not a unit since $1 - w$ is not. Thus z generates a proper two-sided ideal of S.

Now suppose that no power of σ is inner. Let I be a non-zero two sided ideal of S. Choose $f = 1 + a_1x + \dots + a_nx^n \in I$ with each $a_j \in D$, $a_n \neq 0$ and n minimal subject to these conditions. We will show that $n = 1$, whence $I = S$. First, I contains

$$(xfx^{-1} - f)x^{-1} = (1 + a_1^\sigma x + \cdots a_n^\sigma x^n - f)x^{-1}.$$

This must be zero by the minimality of n, so we conclude that $a_j = a_j^\sigma$ for all j. If $b \in D$, then I also contains

$$bf - fb = (ba_1 - a_1b^\sigma)x + \dots + (ba_n - a_nb^{\sigma^n}).$$

By the minimality of n, $ba_j = a_jb^{\sigma^j}$ for all j and all $b \in D$. If $a_j \neq 0$, then $a_j^{-1}ba_j = b^{\sigma^j}$, so σ^j is inner. Therefore all $a_j = 0$, whence $f = 1$, as required.

(3) This is a special case of Proposition **??**.1.1. □

Graded division algebras typically arise as follows.

LEMMA 7.4 (Schur's Lemma). *If M is an irreducible object in $\mathsf{GrMod}A$, then $\underline{\operatorname{End}}_A M$ is a graded division algebra.*

PROOF. If $0 \neq \varphi \in (\underline{\operatorname{End}}_A M)_n$, then $\ker\varphi$ and $\operatorname{Im}\varphi$ are graded submodules of M, so equal 0 and M respectively, since M is irreducible. Thus φ is an A-module isomorphism of degree n, so $\varphi^{-1} \in (\underline{\operatorname{End}}_A M)_{-n}$, as required. □

The building blocks for semisimple artinian rings are matrices over division rings. The analogous building blocks in the graded theory are matrices over graded division algebras. The possible gradings on the matrix ring add an extra level of complication.

PROPOSITION 7.5. *Let D be a graded division algebra, and let $\underline{d} = (d_1, \dots, d_n)$ be an n-tuple of integers. Then $M_n(D)(\underline{d})$ has no proper two-sided graded ideals and is artinian in the category of graded modules over itself. Conversely, a graded algebra having these properties is isomorphic to some such $M_n(D)(\underline{d})$.*

The next result is the graded analogue of the Artin-Wedderburn Theorem.

THEOREM 7.6. *Suppose that the graded algebra A is a direct sum of graded right ideals, each of which is irreducible in $\mathsf{GrMod}A$. If A has no proper two-sided graded ideals, then $A \cong M_n(D)(\underline{d})$ for some graded division algebra D.*

PROOF. Write $A = J_1 \oplus \cdots \oplus J_n$, where each J_i is a graded right ideal which is irreducible in $\mathsf{GrMod}A$. The J_i must all be isomorphic to shifts of one another for otherwise a sum of isotypic components yields a proper two-sided graded ideal. Thus we fix a particular M and integers d_i such that $J_i \cong M[d_i]$ for each i. Write $D = \underline{\operatorname{End}}_A M$; by Lemma 7.4, D is a graded division algebra. Left multiplication by A on itself induces an isomorphism of graded algebras,

$$A \to \underline{\operatorname{End}}\, A_A = \underline{\operatorname{End}}_A(M[d_1] \oplus \cdots \oplus M[d_n]) \cong M_n(D)(\underline{d}),$$

where $\underline{d} = (d_1, \dots, d_n)$. $\square$

$\boxed{\text{Paul}}$ Too many hypotheses?? Just want analogues of simple artinian, then deduce from that the direct sum condition! Also want a semisimple artinian version.

8. Öre extensions of artinian rings

We now consider some special cases of Öre extensions which commonly arise. In particular, suppose that R is semisimple, hence a direct sum of matrix rings over division algebras, say

$$R = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r).$$

Any automorphism of R restricts to an automorphism of the center of R. The center is a direct sum of fields $Z(R) = K_1 \oplus \cdots \oplus K_r$. There are two extreme cases which we consider. The first, in which R is a single division algebra, is covered by Proposition 7.3. The second is when R is a direct sum of r isomorphic matrix algebras and σ acts as a cyclic permutation.

See DeConcini-Procesi paper for second case.

9. Localization of graded algebras

Recall Goldie's Theorem that a noetherian semi-prime ring has a semisimple artinian ring of fractions. We require a version of this for graded rings in which only homogenous elements are inverted. Although the graded ring of fractions is not usually artinian, it is artinian as an object in the category of graded modules over itself.

If R is a graded ring and $\mathfrak{p}$ a graded ideal, then $\mathfrak{p}$ is prime if, whenever I and J are graded ideals of R such that $IJ \subset \mathfrak{p}$, then either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

Definition 9.1. If A is a graded ring, we define $\operatorname{Fract}_{\mathrm{Gr}} A$, the graded ring of fractions of A, to be the subring of $\operatorname{Fract} A$ generated by A and the inverses of the regular homogeneous elements of A.

The following example shows that the graded analogue of Goldie's Theorem is not completely obvious.

EXAMPLE 9.2. Suppose that $A = k[x, y]$ has defining relations $xy = yx = 0$, and $\deg x = 1$ and $\deg y = -1$. Then A is semiprime, and noetherian. The homogeneous elements are the scalar multiples of $\{x^i, y^i \mid i \geq 0\}$, so the homogeneous regular elements in A are already units, whence $A = \mathrm{Fract}_{\mathrm{Gr}} A$. Thus $\mathrm{Fract}_{\mathrm{Gr}} A$ is not graded artinian.

If the grading is changed so that $\deg x = \deg y = 1$, then there are homogenous regular elements of non-zero degree, and $\mathrm{Fract}_{\mathrm{Gr}} A \cong k[X, X^{-1}] \oplus k[Y, Y^{-1}]$ by Example ???. One difference between these two cases is that the graded ideal (x, y), which is essential, contains no homogenous regular elements in the first case, but does in the second.

THEOREM 9.3. *Let A be a graded k-algebra which is semiprime and noetherian. Suppose that one of the following holds:*

- *A is $\mathbb{N}$-graded and has a homogeneous regular element of positive degree;*
- *A is $\mathbb{N}$-graded and the minimal prime ideals do not contain $A_{\geq 1}$;*
- *A has a homogenous, regular, central element of positive degree.*

Then the graded ring of fractions $Q = \mathrm{Fract}_{\mathrm{Gr}} A$ exists, and

- *Q is a semisimple artinian object in $\mathsf{GrMod}(Q)$;*
- *if A is prime, then $Q \cong M_n(D)(\underline{d})$ for some graded division algebra D and some $\underline{d} = (d_1, \dots, d_n)$, so has no proper two-sided graded ideals;*
- *if A is a domain, then $Q \cong D[z, z^{-1}; \sigma]$ is a skew-Laurent extension of a division algebra.*

PROOF. □

PROPOSITION 9.4. *Let A be an $\mathbb{N}$-graded domain of finite GK-dimension. Then* $\mathrm{Fract}_{\mathrm{Gr}} A \cong D[z, z^{-1}; \sigma]$ *is a skew-Laurent extension of a division algebra.*

PROOF. Let $\mathcal{S}$ denote the non-zero homogeneous elements in A. Since $\mathrm{GKdim}\, A < \infty$, A does not contain a free subalgebra, so $Ax \cap Ac \neq 0$ for all non-zero elements $x, c \in A$. It is easy to see that if x and c are homogeneous, then $\mathcal{S}x \cap Ac \neq 0$, whence $\mathcal{S}$ is an Ore set. Thus $\mathrm{Fract}\, A = A_{\mathcal{S}}$ exists and, as in Theorem 9.3, is isomorphic to $D[x, x^{-1}; \sigma]$ for some division algebra D. □

CHAPTER 17

Koszul Algebras

There are several ways to approach the material in this chapter, depending on how one wihes to define a Koszul algebra. First a Koszul algebra will be a connected algebra; it may be convincingly argued that this is unnecessarily restricted, but it will suffice for us. Beyond that, one must decide whether to define a Koszul algebra as a quadratic algebra subject to various conditions, or to give a definition which will later be used to prove that a Koszul algebra is necessarily quadratic. There are other considerations too, and in Theorem ??? we gather several equivalent defining characteristics of a Koszul algebra.

1. The Koszul complex

In this section A is an $\mathbb{N}$-graded k-algebra.

Definition 1.1. A graded k-algebra is quadratic if $A = T(V)/(R)$ where V is a graded k-vector space concentrated in degree one, and (R) is the ideal generated by a subspace $R \subset V \otimes V$.

Let A be a quadratic algebra. We identify V with A_1. For each $n \geq 0$ define left A-module maps

$$d_n : A \otimes V^{\otimes n} \to A \otimes V^{\otimes n-1}$$

by

$$a \otimes v_1 \otimes \cdots \otimes v_n \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_n.$$

The diagram $\cdots \to A \otimes V^{\otimes n} \to \cdots \to A \otimes V \to A \to 0$ obtained is a complex if and only if the product uv is zero for all $u, v \in V = A_1$; thus it is rarely a complex. However, if, for each $n \geq 2$, we replace $A \otimes V^{\otimes n}$ by the submodule $A \otimes R \otimes V^{\otimes n-2}$, the resulting diagram is a complex (simply because the image of R in A is zero). The Koszul complex is a subcomplex of this.

Definition 1.2. Let $A = T(V)/(R)$ be a quadratic algebra. The Koszul complex $K_\bullet(A)$ has terms

$$K_n(A) := A \otimes \left(\bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right)$$

and differential d_n defined above.

Thus, the Koszul complex ends

$$\cdots \to A \otimes (R \otimes V \cap V \otimes R) \to A \otimes R \to A \otimes V \to A \to 0.$$

EXAMPLE 1.3. Suppose that $A = k[x, y, z]$ is the commutative polynomial ring on three variables. Then A is a quadratic algebra, and R is the space of skew symmetric tensors in $V \otimes V$ having basis

$$xy - yx,\ yz - zy,\ zx - xz$$

(it is convenient to omit the $\otimes$ sign, but at the cost of vigilance as to whether the product xy should be interpreted in $T(V)$ or in A). A simple computation shows that $R \otimes V \cap V \otimes R$ contains the element

$$x(yz - zy) + y(zx - xz) + z(xy - yx) = (yz - zy)x + (zx - xz)y + (xy - yx)z.$$

It is not too difficult to show that this element spans $R \otimes V \cap V \otimes R$; notice that the symmetric group S_3 acts on $V^{\otimes 3}$ by permuting the 3 terms, and that the element above is the image of xyz under the action of

$$1 - (23) + (123) - (12) + (132) - (13) \in kS_3.$$

It is somewhat tedious to show that

$$R \otimes V \otimes V \cap V \otimes R \otimes V \cap V \otimes V \otimes R = 0,$$

and hence $K_n(A) = 0$ for $n \geq 4$. Taking x, y, z as a basis for V, and bases for R and $R \otimes V \cap V \otimes R$ as indicated above, the Koszul complex takes the form

$$0 \to A[-3] \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A[-2]^3 \xrightarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}} A[-1]^3 \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \to 0.$$

This complex is exact, except at A where the homology is A modulo the image of $A \otimes V$, namely $A/AV = k$, the trivial A module. Thus the Koszul complex is a projective resolution of the trivial module. The reader is encouraged to check that the Koszul complex is also a projective resolution of the trivial module for the polynomial rings in one and two variables also.

Definition 1.4. If the Koszul complex is exact except at $K_0(A) = A$, we call A a Koszul algebra.

The next result is clear.

Lemma 1.5. *The kernel of the restriction of d_n to*

$$A_1 \otimes \left(\bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \subset K_n(A)$$

is $\bigcap_{i+j=n-1} V^{\otimes i} \otimes R \otimes V^{\otimes j}$.

The lemma says that A is a Koszul complex provided that the only elements in the kernel of each d_n are the 'obvious' ones.

If V is a graded vector space then the tensor algebra $T(V)$ inherits a bigrading, namely

$$T(V)_{(p,q)} = \text{the degree } q \text{ component of } V^{\otimes p}.$$

For example, if A is a graded algebra, then $\underline{\mathrm{Ext}}^*_A(k,k)$ is a bigraded algebra, and there is a natural map of bigraded algebras

$$T(\underline{\mathrm{Ext}}^1_A(k,k)) \to \underline{\mathrm{Ext}}^*_A(k,k).$$

Definition 1.6. The dual of the quadratic algebra $T(V)/(R)$ is $A^! := T(V^*)/(R^\perp)$, where

$$R^\perp = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

We identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) = \alpha(u)\beta(v)$ for $\alpha, \beta \in V^*$ and $u, v \in V$ (not all authors adopt the same convention).

If V is concentrated in degree > 0, then V^*, and hence $A^!$, is concentrated in degree < 0. All the results for $\mathbb{N}$-graded algebras and connected algebras apply to A if we change the grading. We will *not* do that though.

It is an easy exercise to see that

$$(A^!)_{-n} \;=\; V^{*\otimes n}/\left(\bigcap_{i+j=n-2} V^{\otimes i}\otimes R\otimes V^{\otimes j}\right)^{\perp} \;\cong\; \left(\bigcap_{i+j=n-2} V^{\otimes i}\otimes R\otimes V^{\otimes j}\right)^{*}.$$

The dual of the surjection $(V^*)^{\otimes n} \equiv (V^{\otimes n})^* \to A^!_n$ is an inclusion $(A^!_n)^* \to V^{\otimes n}$, the image of which is

$$\bigcap_{i=0}^{n-2} V^{\otimes i}\otimes R\otimes V^{\otimes n-i-2}.$$

REMARK 1.7. If in the definition of $A^!$ we identify $(V\otimes V)^*$ with $V^*\otimes V^*$ by defining $(\alpha\otimes\beta)(u\otimes v)=\alpha(v)\beta(u)$, then $\underline{\mathrm{Ext}}^*_A(k,k)$ is isomorphic to $(A^!)^{\mathrm{op}}$.

Definition 1.8. Let A be a quadratic algebra. The Koszul complex is

$$K(A) := A\otimes_k (A^!)^*,$$

where the n^{th} term is $K_n(A) = A\otimes(A^!_n)^*$, which is naturally a subspace of $A\otimes V^{\otimes n}$, and the differential

$$d_n : K_n(A) \to K_{n-1}(A)$$

is the restriction of the map $A\otimes V^{\otimes n}\to A\otimes V^{\otimes n-1}$ defined by

$$a\otimes v_1\otimes\cdots\otimes v_n \mapsto av_1\otimes v_2\otimes\cdots\otimes v_n.$$

We view each K_n as a graded free left A-module with the tensor product grading, where $\deg(A^!_n)^* = n$.

Since $K_0(A) = A\otimes k$, there is an augmentation map $\varepsilon : K_0(A)\to k$.

LEMMA 1.9. *The Koszul complex really is a complex of left A-modules.*

PROOF. It is clear that each $K_n(A)$ is a free left A-module, and that d is a module homomorphism of degree zero. We must show that $d^2 = 0$. Since $(A^!_n)* \subset R\otimes V^{\otimes n-2}$, and the image of R in A is zero, this is clear. □

REMARK 1.10. If $\{x_\lambda\}$ is a basis for A_1 and $\{\xi_\lambda\}$ its dual basis in A_1^*, then the element $e := \sum_\lambda x_\lambda\otimes\xi_\lambda$ is independent of the choice of basis. Since $K(A) = A\otimes(A^!)^*$ is a right $A\otimes A^!$-module, e acts on it from the right, and as such is a left A-module homomorphism. It is easy to show that this action of e is the same as the action of the differential ∂.

PROPOSITION 1.11. [**111**] *Let A be a quadratic k-algebra. Then the subalgebra of $\underline{\mathrm{Ext}}^*_A(k,k)$ generated by $\underline{\mathrm{Ext}}^1_A(k,k)$ equals $\bigoplus_{p\geq 0}\underline{\mathrm{Ext}}^p_A(k,k)_{-p}$, and is isomorphic to $A^!$.*

THEOREM 1.12. *Let A be a quadratic algebra. The following are equivalent:*

1. *A is Koszul;*
2. *A^{op} is Koszul;*
3. *$A^!$ is Koszul;*
4. *$\underline{\mathrm{Ext}}^*_A(k,k)\cong(A^!)$ as graded k-algebras;*
5. *the augmented Koszul complex is a minimal resolution of k;*

6. $\underline{\mathrm{Ext}}^p_A(k,k)_j = 0$ *if* $p+j \neq 0$.

Further, if A is Koszul, then

$$H_A(t)H_{A^!}(-t) = 1. \tag{1-1}$$

PROOF. (1) $\Leftrightarrow$ (2) By considering $\mathrm{Tor}^A_*(k_A, k)$, it follows from Lemma **??** that k_A has a linear resolution if and only if k does.

(1) $\Leftrightarrow$ (3) Certainly A has a linear resolution, so this follows from Theorem 5.3(4) below (the proof of which is independent of the fact that $A^!$ is Koszul) since, in the notation there, $A^\dagger = {}_{A^!}k$.

(1) $\Leftrightarrow$ (6) This is the equivalence of statements (1) and (4) in Lemma **??**.

(4) $\Leftrightarrow$ (5) The hypothesis ensures that k has a linear minimal resolution. Since there is a morphism from that resolution to the Koszul complex, comparing homology of the two complexes, it follows that that morphism is an isomorphism.

(4) $\Leftrightarrow$ (6) This is Proposition 1.11.

(5) $\Rightarrow$ (6) If the Koszul complex is exact, then it is a linear resolution of k.

Equation (1-1) is a restatement of the final remark in Lemma **??**. $\square$

THEOREM 1.13. [**161**] *Let A be a Koszul algebra, and suppose that $z \in A_2$ is a normal regular element. Then*

1. $B := A/(z)$ *is Koszul, and*
2. $A^! \cong B^!/(\omega)$ *where $\omega \in B^!_2$ is regular, and central if z is central.*

PROOF. Consider the following situation: A is a connected algebra, $x \in A_d$ is a normal regular non-unit, $B = A/(x)$, $M \in \mathsf{GrMod}(A)$, and $N \in \mathsf{GrMod}(A)$ is annihilated by x. In other words, we have the hypotheses in Rees' Lemma, except that we do not assume that x acts faithfully on M. In this case the spectral sequence

$$\underline{\mathrm{Ext}}^p_B(N, \underline{\mathrm{Ext}}^q_A(B,M)) \Rightarrow \underline{\mathrm{Ext}}^n_A(N,M)$$

no longer degenerates. Nevertheless, there are only non-zero rows on the E_2 page are the $q=0$ and $q=1$ ones. We have

$$\underline{\mathrm{Ext}}^q_A(B,M) \cong \begin{cases} \{m \in M \mid xm = 0\} & \text{if } q=0, \\ (M/xM)[d] & \text{if } q=1. \end{cases}$$

Write $L = \{m \in M \mid xm=0\}$, and $\bar{M} = M/xM$. The non-zero rows on the E_2 page are

$$\begin{array}{lllll} q=1 \text{ row} & \underline{\mathrm{Ext}}^0_B(N,\bar{M})[d] & \underline{\mathrm{Ext}}^1_B(N,\bar{M})[d] & \underline{\mathrm{Ext}}^2_B(N,\bar{M})[d] & \underline{\mathrm{Ext}}^3_B(N,\bar{M})[d] \\ q=0 \text{ row} & \underline{\mathrm{Ext}}^0_B(N,L) & \underline{\mathrm{Ext}}^1_B(N,L) & \underline{\mathrm{Ext}}^2_B(N,L) & \underline{\mathrm{Ext}}^3_B(N,L) \end{array}$$

By the discussion in Example 8.**??** there is a long exact sequence of the form

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \to E_2^{20} \to H^2 \to E_2^{11} \to E_2^{30} \to H^3 \to \cdots$$

where $H^i = \underline{\mathrm{Ext}}^i_A(N,M)$.

Take $M = N = k$ in the analysis above. Then the non-zero part of the E_2 page looks like

$$\begin{array}{lllll} q=1 \text{ row} & \underline{\mathrm{Ext}}^0_B(k,k)[2] & \underline{\mathrm{Ext}}^1_B(k,k)[2] & \underline{\mathrm{Ext}}^2_B(k,k)[2] & \underline{\mathrm{Ext}}^3_B(k,k)[2] \\ q=0 \text{ row} & \underline{\mathrm{Ext}}^0_B(k,k) & \underline{\mathrm{Ext}}^1_B(k,k) & \underline{\mathrm{Ext}}^2_B(k,k) & \underline{\mathrm{Ext}}^3_B(k,k) \end{array}$$

Hence there is a long exact sequence

$$\begin{aligned} 0 \to \underline{\mathrm{Ext}}^1_B(k,k) \to \underline{\mathrm{Ext}}^1_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^0_B(k,k)[2] \to \underline{\mathrm{Ext}}^2_B(k,k) \to \underline{\mathrm{Ext}}^2_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^1_B(k,k)[2] \to \underline{\mathrm{Ext}}^3_B(k,k) \to \underline{\mathrm{Ext}}^3_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^2_B(k,k)[2] \to \underline{\mathrm{Ext}}^4_B(k,k) \to \cdots\cdots \end{aligned}$$

Since A is Koszul, $\underline{\mathrm{Ext}}^i_A(k,k)$ is concentrated in degree $-i$. Furthermore, $\underline{\mathrm{Hom}}_B(k,k)$ is concentrated in degree zero, $\underline{\mathrm{Ext}}^1_B(k,k)$ is concentrated in degree -1, and $\underline{\mathrm{Ext}}^2_B(k,k)$ is concentrated in degree -2. An easy induction argument using the long exact sequence shows that $\underline{\mathrm{Ext}}^i_B(k,k)$ is concentrated in degree $-i$, so B is Koszul. □

PROPOSITION 1.14. *Let A be a quadratic algebra. If $x \in A_1$ is a normal, regular element, and $A/(x)$ is Koszul, then A is Koszul.*

PROOF. The analysis is similar to that in the previous proposition, but with the long exact sequence

$$\begin{aligned} 0 \to \underline{\mathrm{Ext}}^1_B(k,k) \to \underline{\mathrm{Ext}}^1_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^0_B(k,k)[1] \to \underline{\mathrm{Ext}}^2_B(k,k) \to \underline{\mathrm{Ext}}^2_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^1_B(k,k)[1] \to \underline{\mathrm{Ext}}^3_B(k,k) \to \underline{\mathrm{Ext}}^3_A(k,k) \to \\ \to \underline{\mathrm{Ext}}^2_B(k,k)[1] \to \underline{\mathrm{Ext}}^4_B(k,k) \to \underline{\mathrm{Ext}}^4_A(k,k) \to \\ \cdots \end{aligned}$$

□

THEOREM 1.15. [**106**] *Let A be a Koszul algebra, and suppose that $z \in A_1$ is a normal regular element such that $B := A/(z)$ is Koszul. Let R and R' denote the quadratic relations for A and B respectively. Then A is Koszul if and only if the natural map $(A_1 \otimes R) \cap (R \otimes A_1) \to (B_1 \otimes R') \cap (R' \otimes B_1)$ is surjective.*

2. Koszul criteria

Definition 2.1. A lattice $(\mathcal{L}, +, \cap)$ is distributive if

$$(X+Y) \cap Z = X \cap Y + X \cap Z$$

for all $X, Y, Z \in \mathcal{L}$.

The basic example of a distributive lattice is $(\mathcal{S}, \cup, \cap)$, where $\mathcal{S}$ is all subsets of a fixed set S, and $\cup$ and $\cap$ denote union and intersection.

LEMMA 2.2. *Let $X_1, \dots, X_n$ be subspaces of a fixed vector space W. The following are equivalent:*

1. *the lattice generated by $X_1, \dots, X_n$ through sums and intersections is distributive;*
2. *there is a direct sum decomposition $W = \oplus_{\lambda \in \Lambda} W_\lambda$ such that each X_i is the sum of some of the W_λ.*

PROOF. (1) ⇒ (2)

(2) ⇒ (1) Let $\mathcal{S}$ denote the set of all subsets of Λ. Let $\mathcal{L}'$ be the lattice generated by the W_λ. For each $s \in \mathcal{S}$ define

$$W_s := \bigoplus_{\lambda \in s} W_\lambda.$$

It is easy to see that $W_s \cap W_t = W_{s \cap t}$ and $W_s + W_t = W_{s \cup t}$. Hence there is an isomorphism between the lattices $\mathcal{S}$ and $\mathcal{L}'$. But $\mathcal{S}$ is distributive, so $\mathcal{L}'$ is too. Since $\mathcal{L}$ is a sub-lattice of $\mathcal{L}'$ it is distributive too. □

LEMMA 2.3. *Let $X_1, \dots, X_n$ be subspaces of a vector space W. Suppose that for each i the collection of subspaces $X_1, \dots, \hat{X}_i, \dots, X_n$ obtained by deleting one of the X_i's is distributive. Then the following are equivalent:*

1. *the complex*
$$W \to \bigoplus_s W/X_s \to \bigoplus_{s<t} W/(X_s + X_t) \to \cdots \to W/\sum_j X_j \to 0$$
is exact except at the leftmost term;
2. *the complex*
$$0 \to \bigcap_s X_s \to \cdots \to \bigoplus_{s<t} X_s \cap X_t \to \bigoplus_s X_s \to W$$
is exact except at the rightmost term;
3. *the lattice generated by $X_1, \dots, X_n$ is distributive.*

PROOF. □

THEOREM 2.4. *A quadratic algebra $T(V)/(R)$ is Koszul if and only if, for each $n \geq 2$, the lattice generated by the subspaces*

$$V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2} \ \subset \ V^{\otimes n}$$

is distributive.

PROOF. □

COROLLARY 2.5. *Let V be a vector space concentrated in degree 1, with a fixed basis $\{x_1, x_2, \dots\}$. If R is a subspace of $V \otimes V$ spanned by any collection of words $x_i x_j$, then $T(V)/(R)$ is Koszul.*

PROOF. Fix $n \geq 2$. There is a direct sum decomposition of $V^{\otimes n}$ into 1-dimensional subspaces spanned by the individual words of degree n in the letters $\{x_1, x_2, \dots\}$. It is clear that if $0 \leq i \leq n-2$, then $V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2}$ is the sum of certain of these 1-dimensional spaces. Hence by Lemma 2.2, and Theorem 2.4, $T(V)/(R)$ is Koszul. □

We now want to prove that deformations of Koszul algebras are Koszul [**132**]. The key result is the next lemma, due to Kempf [**91**].

LEMMA 2.6. *Let R be a discrete valuation ring, with residue field k and quotient field K. Let $B = R \oplus B_1 \oplus \cdots$ be a finitely generated graded R-algebra. Then $B \otimes_R k$ is Koszul if and only if $B \otimes_R K$ is Koszul.*

PROOF. □

THEOREM 2.7. [**3**, Theorem 1.4] *Let A be a quadratic algebra. If all ambiguities are resolvable, then A is Koszul.*

Proof. Since the ambiguities are all resolvable, all the obstructions have degree two. Hence every n-chain has degree $n+1$, so Anick's resolution (3.4.2) is a linear resolution. $\square$

Proof. Write $A = T(V)/(R)$ with $R \subset V \otimes V$, and choose a basis for V so that R has a basis $\{r_\lambda = w_\lambda - s_\lambda\}$, where w_λ is the highest word occurring in r_λ.

Let R be the discrete valuation ring $k[t]_{(t)}$, and let $K = k(t)$. Let $B = R \oplus B_1 \oplus \cdots$ be the graded R-algebra defined by relations $w_\lambda - ts_\lambda$. Then $B \otimes_R k$ is defined by monomial relations, so is Koszul by Corollary 2.5. Hence $B \otimes_R K$ is Koszul.

Now replace R by the ring $R' = k[t]_{(t-1)}$, and form $B' = R' \oplus B'_1 \oplus \cdots$ with defining relations $w_\lambda - ts_\lambda$. Then $B' \otimes_{R'} K \cong B \otimes_R K$, so this is Koszul, whence $B' \otimes_{R'} k$ is also Koszul by Lemma 2.6. But this last ring is isomorphic to A. $\boxed{\text{Paul}}$ Explain this – use the hypothesis of resolvable ambiguities to show A and B' have the same basis, hence the same Hilbert series etc etc. $\square$

Proposition 2.8. *Let A be a Koszul algebra of finite global dimension. Then A is Gorenstein if and only if $A^!$ is Frobenius.*

Proof. Since $\operatorname{gldim}(A) < \infty$, $A^!$ is finite dimensional. Let n denote the global dimension of A.

The groups $\underline{\operatorname{Ext}}^i_A(k, A)$ are the homology groups of the complex obtained by applying $\underline{\operatorname{Hom}}_A(-, A)$ to the Koszul complex for A. That is, they are the homology groups of the complex

$$0 \to A \xrightarrow{d} A^!_1 \otimes A \xrightarrow{d} \cdots \xrightarrow{d} A^!_n \otimes A \to 0 \tag{2-1}$$

of right A-modules, where the differential d is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Therefore A is Gorenstein if and only if (2-1) is exact except at the final position where its homology is $k_A[n]$. But A^{op} is Koszul, so k_A has a minimal resolution given by the Koszul complex

$$0 \to (A^!_n)^* \otimes A \xrightarrow{\delta} \cdots \xrightarrow{\delta} (A^!_1)^* \otimes A \xrightarrow{\delta} A \to k_A \to 0 \tag{2-2}$$

where δ is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Thus A is Gorenstein if and only if (2-1) and (2-2) are isomorphic as complexes of right A-modules.

Thus A is Gorenstein if and only if there is an isomorphism $\Phi : A^! \to (A^!)^*[-n]$ of graded vector spaces such that $\Phi \otimes \mathbb{1}_A$ is an isomorphism of complexes; that is, such that $\delta \circ (\Phi \otimes \mathbb{1}) = (\Phi \otimes \mathbb{1}) \circ d$. Given the above descriptions of d and δ, if $\alpha \otimes a \in A^!_i \otimes A$, then

$$\big(\delta \circ (\Phi \otimes \mathbb{1})\big)(\alpha \otimes a) = \sum_\lambda \xi_\lambda \Phi(\alpha) \otimes x_\lambda a$$

and

$$\big((\Phi \otimes \mathbb{1}) \otimes d\big)(\alpha \otimes a) = \sum_\lambda \Phi(\xi_\lambda \alpha) \otimes x_\lambda a.$$

Hence A is Gorenstein if and only if there exists an isomorphism $\Phi : A^! \to (A^!)^*[-n]$ of graded vector spaces such that $\xi_\lambda \Phi(\alpha) = \Phi(\xi_\lambda \alpha)$ for all $\alpha \in A^!$ and all λ. But this is precisely the requirement that Φ be a left $A^!$-module isomorphism so, by Lemma **??**, the existence of such a Φ is equivalent to the condition that $A^!$ is Frobenius. $\square$

Proposition 2.9. *A twist of a Koszul algebra is Koszul.*

PROOF. The categories of graded modules over an algebra and its twist are equivalent. But being Koszul can be phrased in terms of the category. (Alternatively, if A is replaced by its twist, then $\underline{\mathrm{Ext}}^*_A(k,k)$ is replaced by an appropriate twist, and being generated by $\underline{\mathrm{Ext}}^1_A(k,k)$ is unchanged by twisting.) $\square$

3. Examples

As already remarked, the commutative polynomial ring with its standard grading is a Koszul algebra. Its dual is the exterior algebra. The free algebra is a Koszul algebra too. Next we show that the connected algebras on two relations with a single quadratic relation are all Koszul. Recall that over an algebraically closed field, there are four such algebras, namely $k[x,y]$ with defining relation one of the following:

$$x^2, \quad yx, \quad yx - qxy, \quad xy - yx - y^2$$

where $0 \neq q \in k$.

The Jordan and quantum planes are twists of the commutative polynomial ring, so are Koszul algebras.

EXAMPLE 3.1. Let $A = k\langle x,y\rangle/(yx)$. The elements x^iy^j are a basis for A. The minimal resolution of k is

$$0 \to A[-2] \xrightarrow{(y,0)} A[-1]^2 \xrightarrow{\begin{pmatrix} x//y \end{pmatrix}} A \to k \to 0.$$

It follows that A is Koszul.

EXAMPLE 3.2. Let $A = k\langle x,y\rangle/(y^2)$. To compute the Hilbert series of this, let $\mathcal{A}_i$ be the basis for A_i consisting of monomials; thus we can write $\mathcal{A}_i = \mathcal{B}_i \cup \mathcal{C}_i$ as a disjoint union, where $\mathcal{B}_i$ is the words of length i ending in x, and $\mathcal{C}_i$ is the words of length i ending in y. Thus $\mathcal{B}_{i+1} = \mathcal{A}_i x$ and $\mathcal{C}_{i+1} = \mathcal{B}_i y$; hence if $a_i = \sharp\mathcal{A}_i$, $b_i = \sharp\mathcal{B}_i$, and $c_i = \sharp\mathcal{C}_i$, then $b_{i+1} = a_i$ and $c_{i+1} = b_i = a_{i-1}$, whence $a_{i+1} = a_i + a_{i-1}$; since $a_0 = 1$ and $a_1 = 2$, we see that $a_0, a_1, a_2, \dots$ is the Fibonacci sequence $1,2,3,5,\dots$.

The minimal resolution of k as a left A-module looks like

$$\cdots \xrightarrow{y} A \xrightarrow{y} A \xrightarrow{y} \cdots \xrightarrow{y} A \xrightarrow{(0,y)} A^2 \xrightarrow{\binom{x}{y}} A \to k \to 0.$$

It is easy to check this is exact using the monomial basis for A described above. Thus k has a linear resolution, showing that A is Koszul.

$\boxed{\text{Paul}}$ The Koszul dual of the previous example is $B = k[x,y]/(x^2,xy,yx)$. This is the standard example of a commutative ring having infinite injective dimension. What are the groups $\underline{\mathrm{Ext}}^i_B(k,B)$? What does the minimal injective resolution of B look like? Recall $\underline{\mathrm{Ext}}^i_B(k,B)$ counts the number of copies of B^* in the i^{th} term of the minimal injective resolution of B.

The minimal projective resolution of ${}_Bk$ is of the form

$$\cdots \to B[-4]^8 \to B[-3]^5 \to B[-2]^3 \to B[-1]^2 \to B \to k \to 0.$$

EXAMPLE 3.3. The Koszul complex for the 3-dimensional polynomial ring, say $A = k[x,y,z]$, is

$$0 \to A \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ y & -x & 0 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \to k \to 0.$$

The elements of A^3 are viewed as row vectors and the maps are given by right multiplication by the indicated matrices. (Although right and left A-modules are the same thing since A is commutative, we are really treating the terms in the complex as *left* modules.)

To compute $\mathrm{Ext}^*_A(k, A)$, apply the functor $\mathrm{Hom}_A(-, A)$ and take homology. Applying this functor to A^m, we have $\mathrm{Hom}_A(A^m, A) \cong A^m$, but we should now think of this as a *right* A-module whose elements are column vectors, and maps are given by left multiplication by matrices. Thus we obtain the complex

$$0 \longleftarrow A \xleftarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A^3 \xleftarrow{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ y & -x & 0 \end{pmatrix}} A^3 \xleftarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \longleftarrow 0.$$

where the maps are given by left multiplication by the given matrices. However, this is simply the deleted Koszul complex which is the projective resolution for k_A, the trivial module viewed as a right module.

4. Linear resolutions

Definition 4.1. A module $M \in \mathsf{GrMod}(A)$ has a linear resolution if it has a projective resolution of the form

$$\cdots \to P_r \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{4-1}$$

such that P_r is generated in degree r for all r.

LEMMA 4.2. *A module $M \in \mathsf{GrMod}(A)$ has a linear resolution if and only if any of the following equivalent conditions holds:*

1. *its minimal projective resolution is linear;*
2. $\mathrm{Tor}^A_p(k, M)$ *is concentrated in degree p for all p;*
3. $\mathrm{Ext}^p_A(M, k)$ *is concentrated in degree $-p$ for all p.*

PROOF. (1) This is clear since any resolution of M contains the minimal resolution as a direct summand.

(2), (3) These both follow from the fact that the p^{th} term of the minimal resolution is isomorphic to $A \otimes_k \mathrm{Tor}^A_p(k, M)$ and to $A \otimes_k \underline{\mathrm{Hom}}_k(\underline{\mathrm{Ext}}^p_A(M, k), k)$. $\square$

Definition 4.3. An connected k-algebra A is Koszul if k has a linear resolution.

Reformulating this interms of the grading on $\mathrm{Tor}^A_*(k, k)$, it follows that A is Koszul if and only if A^{op} is.

PROPOSITION 4.4. *A Koszul algebra is a quadratic algebra.*

PROOF. The hypothesis may be rephrased as saying that $\underline{\mathrm{Ext}}^p_A(k, k[n]) = 0$ unless $n = p$. $\square$

5. A duality for modules having a linear resolution

We will show that, for a Koszul algebra A, the functor $\underline{\mathrm{Ext}}^*_A(-, k)$ is a duality between $\mathrm{Lin}(A)$ and $\mathrm{Lin}(A^!)$.

NOTATION . Let R be an augmented k-algebra. We denote the functor $\mathrm{Ext}^*_R(-, k)$: $\mathsf{Mod}(R) \to \mathsf{Mod}(\mathrm{Ext}^*_R(k, k))$ by

$$M \mapsto M^\dagger := \mathrm{Ext}^*_R(M, k).$$

For example, $R^\dagger \cong k$ and $k^\dagger \cong \mathrm{Ext}^*_R(k,k)$. For connected graded algebras, $M \mapsto M^\dagger$ sends graded modules to graded modules.

LEMMA 5.1. *Let A be a connected graded k-algebra. Suppose that L, M and N are graded A-modules having linear resolutions, and that $0 \to L[-1] \to M \to N \to 0$ is an exact sequence. Then there is an exact sequence of left $\underline{\mathrm{Ext}}^*_A(k,k)$-modules*

$$0 \to L^\dagger[-1] \to N^\dagger \to M^\dagger \to 0.$$

PROOF. Applying $\underline{\mathrm{Hom}}_A(-,k)$ to the initial exact sequence gives an exact sequence

$$\begin{aligned}\cdots \to \underline{\mathrm{Ext}}^{p-1}_A(M,k)_{-p} \to \underline{\mathrm{Ext}}^{p-1}_A(L[-1],k)_{-p} \to \underline{\mathrm{Ext}}^{p}_A(N,k)_{-p} \to \\ \to \underline{\mathrm{Ext}}^{p}_A(M,k)_{-p} \to \underline{\mathrm{Ext}}^{p}_A(L[-1],k)_{-p} \to \cdots .\end{aligned} \tag{5-1}$$

However, since M and L have linear resolutions, the first and last terms in (5-1) are zero, whence we have an exact sequence

$$0 \to L^\dagger_{-p-1} \to N^\dagger_{-p} \to M^\dagger_{-p} \to 0.$$

Summing over all p gives the result. □

For quadratic algebras the left action of $A^!$ on $M^\dagger$ has a simple description in terms of the differential on the minimal resolution of M.

PROPOSITION 5.2. *Let A be a quadratic algebra and $(A \otimes V_\bullet, d)$ a minimal resolution of $M \in \mathsf{GrMod}(A)$. There is a commutative diagram*

$$\begin{array}{ccc} A^!_1 \otimes V^*_i & \xrightarrow{d^*} & V^*_{i+1} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Ext}}^1_A(k,k) \otimes \underline{\mathrm{Ext}}^i_A(M,k) & \xrightarrow{\rho} & \underline{\mathrm{Ext}}^{i+1}_A(M,k) \end{array} \tag{5-2}$$

where ρ denotes the Yoneda product and the vertical maps are induced by the isomorphism $V^*_i \to \underline{\mathrm{Ext}}^i_A(M,k)$, $\alpha \mapsto [\varepsilon \otimes \alpha]$.

PROOF. Let $\xi \in A^!_1$ and $\alpha \in V^*_i$. The image of $\xi \otimes \alpha$ going in the counterclockwise direction is $\rho(\xi \otimes [\alpha])$, namely $[(\varepsilon \otimes \xi) \circ \alpha_1]$ where α_1 is the lift of α in the following commutative diagram:

$$\begin{array}{ccccc} V_{i+1} & \xrightarrow{d} & A_1 \otimes V_i & & \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_0 = 1 \otimes \alpha} & & \\ A_0 \otimes A_1 & \xrightarrow[\partial]{} & A_1 \otimes A_0 & \xrightarrow[\varepsilon \otimes 1]{} & k \\ {\scriptstyle \varepsilon \otimes \xi}\downarrow & & & & \\ k & & & & \end{array}$$

(We have ommitted those parts of the resolutions which are irrelevant to the present proof.) Since $\partial(1 \otimes a) = a \otimes 1$ for $1 \otimes a \in A \otimes A_1$, $\varepsilon \otimes \xi = (\xi \otimes \varepsilon) \circ \partial$. Therefore

$$\begin{aligned}
\rho(\xi \otimes [\alpha]) &= [(\varepsilon \otimes \xi) \circ \alpha_1] \\
&= [(\xi \otimes \varepsilon) \circ \partial \circ \alpha_1] \\
&= [(\xi \otimes \varepsilon) \circ (1 \otimes \alpha) \circ d] \\
&= [(\xi \otimes \alpha) \circ d] \\
&= [d^*(\xi \otimes \alpha)].
\end{aligned}$$

However, $[d^*(\xi \otimes \alpha)]$ is the image of $\xi \otimes \alpha$ under the clockwise composition of maps, thus proving the commutativity of the diagram. □

THEOREM 5.3. *Let A be a Koszul algebra, and let $M \in \mathrm{Lin}(A)$.*

1. *The minimal projective resolution of M is $A \otimes_k (M^\dagger_\bullet)^*$ with differential given by right multiplication by e (see Remark 1.10);*
2. *$H_M(t) = H_A(t) H_{M^\dagger}(-t)$;*
3. *If $A \otimes D \to A \otimes C \to M$ is the start of a minimal resolution of M, then the minimal resolution of $M^\dagger$ begins $A^! \otimes_k D^\perp \to A^! \otimes_k C^* \to M^\dagger$;*
4. *$M^\dagger$ has a linear resolution as a left $A^!$-module, and $M^{\dagger\dagger} \cong M$.*

PROOF. (1) If L is a left $A^!$-module with structure map $\rho : A^! \otimes L \to L$, it is a triviality that $\rho(\xi \otimes \ell) = \sum_\lambda x_\lambda(\xi)\xi_\lambda.\ell = e.(\xi \otimes \ell)$. Hence the map ρ in (5-2) is left multiplication by e, and the dual of (5-2) is the commutative diagram

$$\begin{tikzcd}
V_{i+1} \arrow[r, "d"] \arrow[d] & A_1 \otimes V_i \arrow[d] \\
(M^\dagger_{i+1})^* \arrow[r, "e"] & A_1 \otimes (M^\dagger_i)^*
\end{tikzcd} \tag{5-3}$$

where the bottom map denotes *right* multiplication by e, and the vertical maps are isomorphisms. Hence the vertical maps induce an isomorphism of complexes of graded A-modules

$$\begin{tikzcd}
\cdots \arrow[r] & A \otimes V_{i+1} \arrow[r, "d"] \arrow[d] & A \otimes V_i \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A \otimes (M^\dagger_{i+1})^* \arrow[r, "e"] & A \otimes (M^\dagger_i)^* \arrow[r] & \cdots
\end{tikzcd} \tag{5-4}$$

Therefore $A \otimes (M^\dagger_\bullet)^*$, with differential e, is a minimal resolution of M, as claimed.

(2) This is a trivial consequence of (1).

(3) From the minimal resolution of M, one sees that $M_0 = C$ and that there is an exact sequence $0 \to D \to A_1 \otimes C \to M_1 \to 0$. Hence $M_0^* = C^*$ and $M_1^* \cong D^\perp \subset (A_1 \otimes C)^* \cong A_1^! \otimes C^*$. By (1) and (3), the minimal resolution of $M^\dagger$ begins $A^! \otimes M_1^* \to A^! \otimes M_0^* \to M^\dagger \to 0$, whence the result.

(4) We will show that the minimal resolution of $M^\dagger$ is given by the complex $(A^! \otimes M^*_\bullet, \partial)$, where ∂ is right multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$; that is, we will prove the exactness of

$$\cdots A^! \otimes (M_2)^* \to A^! \otimes (M_1)^* \to A^! \otimes (M_0)^* \to M^\dagger \to 0.$$

By part (3), the right hand end of this is the beginning of a minimal resolution of $M^\dagger$.

Define the bicomplex $(C^{\bullet\bullet}, d', d'')$, where

$$C^{pq} = A^!_p \otimes A^* \otimes M^\dagger_q,$$

and the differentials are

$$d' : C^{pq} \to C_{p+1\,q} = \text{right multiplication by } \sum(\xi_\lambda \otimes x_\lambda \otimes 1)$$

and

$$d'' : C^{pq} \to C^{p\,q+1} = \text{left multiplication by } \sum(1 \otimes x_\lambda \otimes \xi_\lambda).$$

We consider the two associated spectral sequences.

We begin with the first filtration. The p^{th} column is $A^!_p \otimes (A \otimes (M^\dagger_\bullet)^*)^*$, where $A \otimes (M^\dagger_\bullet)^*$ is the projective resolution of M in part (1), so the columns are exact except in the 0^{th} row. Therefore

$$'E_2^{pq} = H^p_I H^q_{II}(C^{\bullet\bullet}) = \begin{cases} H^p(A_\bullet \otimes M^*, \partial) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0 \end{cases}$$

and $H^n(\operatorname{Tot}(C^{\bullet\bullet})) \cong H^n(A^!_\bullet \otimes M^*, \partial)$. With respect to the second filtration,

$$''E_2^{pq} = H^q_{II}(H^p(A^!_\bullet \otimes A^*) \otimes M^\dagger_\bullet) = \begin{cases} H^q({}_Ak \otimes M^\dagger_\bullet) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

because $A^!_\bullet \otimes A^*$ with differential right multiplication by $\sum \xi_\lambda \otimes x_\lambda$ is the graded k-dual of the Koszul complex (which is a projective resolution of k_A). The differential on ${}_Ak \otimes M^\dagger_\bullet$ induced by d'' is left multiplication by $\sum x_\lambda \otimes \xi_\lambda$; this is zero since ${}_Ak$ is killed by x_λ. Therefore $H^q({}_Ak \otimes M^\dagger_\bullet) = M^\dagger_q$.

Comparing the two spectral sequences $H^n(A^!_\bullet \otimes M^*) = H^n(\operatorname{Tot}(C^{\bullet\bullet})) = M^\dagger_n$, which is what we needed to prove. □

COROLLARY 5.4. *If A is a Koszul algebra, then $\underline{\operatorname{Ext}}^*_A(-,k) : \operatorname{Lin}(A) \to \operatorname{Lin}(A^!)$ is a duality.*

6. Koszul duality

The functor $\underline{\operatorname{Ext}}^*_A(-,k)$ is closely related to the functor in [**26**] which is used to establish an equivalence between certain derived categories over a Koszul algebra and its dual. The results in [**26**] are not used in the rest of this paper; this section is intended to place the methods we use in a broader context.

Warning: Our definition of $A^!$ in (1.1) is *not* the same as that given in [**26**, page 5]. Their $A^!$ is the *opposite* of ours. In what follows we will use $A^!$ the same way as we have been, so when we quote results from [**26**] we will need to replace their use of $A^!$ by $(A^!)^{\text{op}}$. (See Remark 1.7.)

Definition 6.1. Let B be an $\mathbb{N}$-graded k-algebra. If (M, ∂) is a complex of graded B-modules we will write $M = \oplus M^i_j$ where i denotes the position in the complex, and j denotes the degree for the B-module action; thus $\partial : M^i_j \to M^{i+1}_j$.

We define the following categories:

- $C(B)$ = the homotopy category of complexes of graded B-modules, morphisms being homotopy classes of maps of complexes;
- $C^+(B)$ = the full subcategory of $C(B)$ consisting of complexes $M^\bullet_\bullet$ such that $M^i_j = 0$ if $i \gg 0$ or $i + j \ll 0$;

- $C^-(B)$ = the full subcategory of $C(B)$ consisting of complexes $M^\bullet_\bullet$ such that $M^i_j = 0$ if $i \ll 0$ or $i + j \gg 0$;
- $D^+(B)$ and $D^-(B)$ are the quotient categories of $C^+(B)$ and $C^-(B)$ obtained by localizing at the quasi-isomorphisms (they are triangulated categories).

Theorem 6.2. [**26**] *If A is a Koszul algebra, then there is an equivalence of categories $D^+(A) \to D^-((A^!)^{\mathrm{op}})$.*

The equivalence of categories is induced by the functor by $F : C^+(A) \to C^-((A^!)^{\mathrm{op}})$ defined as follows: if $(M, \partial) \in C^+(A)$, then

$$FM = ((A^!)^{\mathrm{op}} \otimes M, d)$$

where

$$d(a \otimes m) = a \otimes \partial m + (-1)^{i+j} \sum_\lambda \xi_\lambda a \otimes x_\lambda m, \tag{6-1}$$

whenever $a \otimes m \in (A^!)^{\mathrm{op}} \otimes M^i_{i+j}$, and FM is viewed as a complex of left $(A^!)^{\mathrm{op}}$-modules, which in position p is

$$(FM)^p = \bigoplus_{i+j=p} A^! \otimes M^i_j. \tag{6-2}$$

(In (6-1) the product $\xi_\lambda a$ is computed in $A^!$; in [**26**], it appears as $a\xi_\lambda$, using the product in $(A^!)^{\mathrm{op}}$.)

There is a full embedding of $\mathsf{GrMod}(A)$ in $D^+(A)$ sending a module M to the complex which is M in position zero, and zero elsewhere. When F is applied to a single module M, we obtain the complex

$$\cdots \to A^! \otimes M_p \to A^! \otimes M_{p+1} \to \cdots, \tag{6-3}$$

with differential being left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Sometimes this is quasi-isomorphic to a single $A^!$-module.

Proposition 6.3. *Let A be a Koszul algebra of finite global dimension. Furthermore, suppose that A is Gorenstein with $\underline{\mathrm{Ext}}^n_A(k, A) \cong k[n]$. If $M \in \mathsf{GrMod}(A)$, then FM is*

1. *quasi-isomorphic to the complex, with zero differential, which in the p^{th} position is $\oplus \mathrm{Ext}^j_A(k, M)_{p-j}$;*
2. *quasi-isomorphic to a single $A^!$-module if and only if $M[r] \in \mathrm{Lin}(A)$ for some r.*

Proof. Consider $\underline{\mathrm{Ext}}^*_A(k, M)$. This is computed by taking homology after applying $\underline{\mathrm{Hom}}_A(-, M)$ to $A \otimes (A^!_\bullet)^*$; that is, it is the homology of the complex

$$0 \to M \to A^!_1 \otimes M \to A^!_2 \otimes M \to \cdots, \tag{6-4}$$

with differential left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Since $\deg(A^!_j) = -j$, it follows that $\underline{\mathrm{Ext}}^p(k, M)_q$ is a subquotient of $A^!_p \otimes M_{p+q}$. We can rearrange the complex (6-4) so it is precisely the complex (6-3). Hence, taking homology of $(FM)^\bullet$, which is the same thing as homology of (6-3), we obtain $\oplus_j \underline{\mathrm{Ext}}^j_A(k, M)_{p-j}$ in position p.

Therefore, FM is quasi-isomorphic to a single $A^!$-module if and only if, for some p,

$$\underline{\mathrm{Ext}}_A^*(k, M[p]) = \bigoplus_j \underline{\mathrm{Ext}}_A^j(k, M[p])_{-j}.$$

By Corollary **??**, this is equivalent to the requirement that $M[p]$ have a linear resolution. $\square$

CHAPTER 18

Artin-Schelter regular algebras

The class of Artin-Schelter regular algebras was first identified in a joint paper of Artin and Schelter in 1987 [**6**]. Subsequent papers by Artin, Tate and van den Bergh ([**9**] and [**10**]) showed that this was a rich class of rings which could be reasonably considered good non-commutative generalizations of polynomial rings.

The 3-dimensional Artin-Schelter regular algebras are the smallest ones which exhibit interesting new features; they give rise to non-commutative analogues of the projective plane which will be investigated in detail in chapter 24.

1. Basic results

We will show that commutative polynomial algebras are Artin-Schelter regular, and that Artin-Schelter regular algebras have many of its good properties.

Definition 1.1. A locally finite, connected, graded, k-algebra, A say, is Artin-Schelter regular of dimension n if

1. the minimal resolution of k is of the form $0 \to P_n \to \cdots \to P_0 \to k \to 0$, with each P_i finitely generated, and
2. $\mathrm{GKdim}(A) < \infty$, and
3. A is Gorenstein.

Clearly such an A has global dimension n.

By Theorem 14.2.12, the only commutative Artin-Schelter regular algebras are the polynomial rings.

Proposition 1.2. *If A is Artin-Schelter regular so is A^{op}.*

Proof. Let $0 \to P_n \to \cdots \to P_1 \to P_0 \to k \to 0$ be a minimal resolution. By Proposition 14.2.6, $P_i \cong A \otimes_A \mathrm{Tor}_i^A(k,k)$, so condition (1) is equivalent to saying that the $\mathrm{Tor}_i(k,k)$ are finite dimensional and zero for $i > n$. By interchanging the roles of k and k_A, the truth of this for A implies its truth for A^{op}. So (1) holds for A^{op}. Obviously (2) holds for A^{op}.

The Gorenstein hypothesis implies that if $\underline{\mathrm{Hom}}_A(-,A)[-\ell]$ is applied to the minimal resolution of k, then one obtains a minimal resolution of k_A of the form

$$0 \leftarrow k_A \leftarrow P_n^\vee[-\ell] \leftarrow \cdots \leftarrow P_0^\vee[-\ell] \leftarrow 0.$$

To compute $\underline{\mathrm{Ext}}^*_A(k_A,A)$ apply $\underline{\mathrm{Hom}}_A(-,A)$ to this complex; because each P_i is finitely generated, $P_i^{\vee\vee} \cong P_i$, so one recovers a shift of the minimal resolution of k, from which it follows that A^{op} is Gorenstein. Thus (3) holds for A^{op}. $\square$

The Hilbert series of an Artin-Schelter regular algebra satisfies a functional equation.

Proposition 1.3. *If A is Artin-Schelter regular and $\underline{\mathrm{Ext}}^n_A(k,A) \cong k[\ell]$, then*

1. $H_A(t) = (-1)^n t^{-\ell} H_A(t^{-1})$;
2. *for any* $M \in \mathsf{grmod}(A)$, $\sum(-1)^i \, H_{\underline{\mathrm{Ext}}^i_A(M,A)}(t) = (-1)^n t^{-\ell} H_M(t^{-1})$;
3. *the poles of* $H_A(t)$ *are roots of unity;*
4. *the product of the poles of* $H_A(t)$ *is* $(-1)^{n+\ell}$.

PROOF. Notice that (1) is obtained from (2) with $M = A$. By Theorem 14.4.2,

$$\sum_i (-1)^i H_{\underline{\mathrm{Ext}}^i_A(M,A)}(t) = q_M(t^{-1}) H_A(t). \tag{1-1}$$

Substituting $M = k$, and using the fact that $q_k(t) = H_A(t)^{-1}$, we obtain

$$(-1)^n t^{-\ell} = H_A(t^{-1})^{-1} \, H_A(t),$$

which proves (1). Substituting (1) into the right hand side of (1-1) gives (2).

Properties (3) and (4) may be translated into statements about the zeroes of $q_k(t) = H_A(t)^{-1}$; this is a polynomial by Theorem 14.4.2. The leading term of $q_k(t)$ is $(-1)^n t^\ell$, and its lowest term is 1 because the minimal resolution of k looks like

$$0 \to A[-\ell] \to \cdots \to \bigoplus_{j=1}^{r_1} A[-\ell_{1j}] \to A \to k \to 0. \tag{1-2}$$

Hence the zeroes of $q_k(t)$ are algebraic integers. Because $\mathrm{GKdim}\, A < \infty$, all zeroes of $q_k(t)$ have absolute value ≤ 1 (11.2.5). But the functional equation implies that z is a zero if and only if z^{-1} is, so these zeroes must lie *on* the unit circle. However, the roots of unity are the only algebraic integers all of whose conjugates lie on the unit circle. This proves (3), and (4) is clear because $q_k(t) = (-1)^n t^\ell + \cdots + 1$. $\square$

$\boxed{\text{Paul}}$

Say something about GKdim being an integer. Is multiplicity well behaved? Yes

THEOREM 1.4. *Let R be a left noetherian, n-dimensional Artin-Schelter regular algebra. Let $S = R[x; \sigma, \delta]$ be a graded Öre extension with $\sigma \in \mathrm{Aut}(R)$ a graded algebra automorphism, and* $\deg(x) > 0$. *Then S is Artin-Schelter regular of dimension $n + 1$.*

PROOF. Suppose that $\deg(x) = d > 0$. Then $S_n = R_n \oplus R_{n-d}x \oplus \cdots$, so $\mathrm{GKdim}(S) = \mathrm{GKdim}(R) + 1$.

Let $I = R_{\ge 1}$. Since σ is an automorphism $SI = IS$ is a two-sided ideal, and $S/IS = k[X]$, a polynomial ring. Thus, as a left S-module, $k[X] \cong S \otimes_R k$, and as a right S-module, $k[X] \cong k \otimes_R S$.

Let $P_\bullet \to k$ be a minimal resolution of ${}_Rk$ as a left R-module. Since S is free, and hence flat, as a right R-module, $S \otimes_R P_\bullet \to k[X]$ is a free resolution of $k[X]$ as a left S-module. Thus $\mathrm{pdim}_S \, k[X] \le n$ as a left S-module, and similarly as a right S-module. The exact sequence

$$0 \to k[X] \xrightarrow{\rho} k[X] \to k \to 0 \tag{1-3}$$

of left S-modules, where ρ is multiplication by X, will play a key role in the rest of the proof.

If N is a right S-module, then

$$\mathrm{Tor}^S_\bullet(N, k[X]) \cong \mathrm{Tor}^R_\bullet(N, k), \tag{1-4}$$

since both are computed as the homology of $N \otimes_S S \otimes_R P_\bullet \cong N \otimes_R P_\bullet$. Similarly, if M is a left S-module, then

$$\operatorname{Ext}^\bullet_S(k[X], M) \cong \operatorname{Ext}^\bullet_R(k, M), \tag{1-5}$$

since both are the homology of $\operatorname{Hom}_S(S \otimes_R P_\bullet, M) \cong \operatorname{Hom}_R(P_\bullet, M)$.

Applying the functor $\operatorname{Tor}^S_\bullet(k, -)$ to (1-3), gives the exact sequence

$$\cdots \to \operatorname{Tor}^S_{j+1}(k, k) \to \operatorname{Tor}^S_j(k, k[X]) \to \operatorname{Tor}^S_j(k, k[X]) \to \operatorname{Tor}^S_j(k, k) \to \cdots .$$

By (1-4), $\operatorname{Tor}^S_j(k, k[X]) \cong \operatorname{Tor}^R_j(k, k)$. But $\operatorname{gldim}(R) = n$, so $\operatorname{Tor}^R_j(k, k) = 0$ for $j \geq n + 1$, whence $\operatorname{Tor}^S_j(k, k) = 0$ if $j \geq n + 1$. Hence, by Theorem 11.2.7, $\operatorname{gldim}(S) \leq n + 1$ (equality will emerge later in the proof).

It remains to show that S satisfies the Gorenstein condition. Applying the functor $\operatorname{Ext}^\bullet_S(-, S)$ to (1-3) gives the exact sequence

$$\cdots \to \operatorname{Ext}^j_S(k, S) \to \operatorname{Ext}^j_S(k[X], S) \to \operatorname{Ext}^j_S(k[X], S) \to \operatorname{Ext}^{j+1}_S(k, S) \to \cdots .$$

Since R is left noetherian, each P_i is finitely generated, so $\operatorname{Ext}^j_R(k, -)$ commutes with direct sums. Therefore, by (1-5),

$$\operatorname{Ext}^j_S(k[X], S) \cong \operatorname{Ext}^j_R(k, S) \cong \operatorname{Ext}^j_R(k, \oplus_{i \geq 0} Rx^i) = \oplus \operatorname{Ext}^j_R(k, R)x^i.$$

The Gorenstein property for R, together with the previous exact sequence, implies that $\operatorname{Ext}^j_S(k, S) = 0$ if $0 \leq j \leq n - 1$. It remains to determine the first and last terms in the exact sequence

$$0 \to \operatorname{Ext}^n_S(k, S) \to \operatorname{Ext}^n_S(k[X], S) \xrightarrow{\alpha} \operatorname{Ext}^n_S(k[X], S) \to \operatorname{Ext}^{n+1}_S(k, S) \to 0, \tag{1-6}$$

where α is induced by the map ρ in (1-3). We wissh to show that $\ker(\alpha) = 0$ and that $\operatorname{coker}(\alpha) = k$.

To compute α, first lift ρ to a morphism of S-module complexes

$$\varphi : S \otimes_R P_\bullet \to S \otimes_R P_\bullet,$$

which we may assume is homogeneous of degree d, since ρ is. Then α is induced by $\varphi^\vee : \operatorname{Hom}_S(S \otimes_R P_n, S) \to \operatorname{Hom}_S(S \otimes_R P_n, S)$. Since R is Artin-Schelter regular, P_n is free of rank 1, so we make the identifications $P_n = R$ and $S \otimes_R P_n = S$. To take account of the grading, suppose that $\underline{\operatorname{Ext}}^n_R(k, R) = k[\ell]$, whence $P_n = R[-\ell]$. Thus $\operatorname{Hom}_S(S \otimes_R P_n, S) = S[\ell]$, and $\varphi^\vee$ is right multiplication by an element of S_d, say $a_d + a_0 x$, where $a_i \in R_i$. Taking account of the graded structures, (1-5) gives

$$\begin{aligned} \underline{\operatorname{Ext}}^n_S(k[X], S) &\cong \underline{\operatorname{Ext}}^n_R(k, S) \\ &\cong \underline{\operatorname{Ext}}^n_R(k, R \oplus R[d] \oplus \cdots) \\ &\cong \underline{\operatorname{Ext}}^n_R(k, R) \oplus \underline{\operatorname{Ext}}^n_R(k, R)[d] \oplus \cdots \\ &\cong k[\ell] \oplus k[\ell + d] \oplus \cdots . \end{aligned} \tag{1-7}$$

In order to transfer α to the right hand side of (1-7), recall that these isomorphisms are induced by the isomorphisms

$$\begin{aligned} \underline{\operatorname{Hom}}_S(S \otimes_R P_n, S) &\cong \underline{\operatorname{Hom}}_R(P_n, S) \\ &\cong R[\ell] \oplus Rx[\ell] \oplus \cdots \\ &\cong R[\ell] \oplus R[\ell + d] \oplus \cdots . \end{aligned}$$

Since (1-3) is really $0 \to k[X][-d] \to k[X] \to k \to 0$, α takes values in $\underline{\mathrm{Ext}}^n_S(k[X], S)[d]$. The effect of right multiplication by $a_d + a_0 x$ on the right hand side of (1-7), shows that α transfers to the map

$$k[\ell] \oplus k[\ell + d] \oplus \cdots \to k[\ell + d] \oplus k[\ell + 2d] \oplus \cdots$$

defined by

$$(\gamma_0, \gamma_1, \dots) \mapsto (0, \gamma_0 a_0, \gamma_1 a_0, \dots).$$

To show that $\ker(\alpha) = 0$ and $\operatorname{coker}(\alpha) = k$, we must prove that $a_0 \neq 0$.

Apply the functor $\mathrm{Tor}^S_\bullet(k[X], -)$ to (1-3) to obtain

$$0 = \mathrm{Tor}^S_{n+1}(k[X], k) \to \mathrm{Tor}^S_n(k[X], k[X]) \xrightarrow{\beta} \mathrm{Tor}^S_n(k[X], k[X]) \to \mathrm{Tor}^S_n(k[X], k),$$

where the first term is zero because $\operatorname{pdim}_S k[X] \leq n$, and β is induced by ρ. Since $\mathrm{Tor}^S_n(k[X], k[X]) \cong \mathrm{Tor}^R_n(k[X], k) \neq 0$, $\beta \neq 0$. Now, β is induced by φ, namely by the map

$$S/IS \otimes_S (S \otimes_R P_n \xrightarrow{\varphi} S \otimes_R P_n).$$

Thus β is the map $k[X] \to k[X]$ given by right multiplication by $a_d + a_0 x$; however, a_d annihilates $k[X] = S/IS$, so the map is right multiplication by $a_0 X$. Since $\beta \neq 0$, we conclude that $a_0 \neq 0$. □

Corollary 1.5. *A graded iterated Öre extension, say*

$$A = k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n],$$

where $\deg(x_i) > 0$ *for all* i*, and each* σ_i *is a graded algebra automorphism, is Artin-Schelter regular of dimension* n*, and*

$$\underline{\mathrm{Ext}}^n_A(k, A) = k[\deg(x_1) + \cdots + \deg(x_n)].$$

Proof. This follows by induction; the only point not immediately clear is the degree, but the proof of Theorem 1.4 showed, in the notation there, that $\underline{\mathrm{Ext}}^{n+1}_S(k, S) \cong \underline{\mathrm{Ext}}^n_R(k, R)[\deg(x)]$. □

The requirement that the σ_i be automorphisms is necessary, as the next example shows.

Example 1.6. Here is an iterated Öre extension which is not Artin-Schelter regular. Let $A = k[x, y]$ with defining relation $yx = 0$, and $\deg(x) = \deg(y) = 1$; this is the Öre extension in Example 6.1.9. The sequence of left A-modules

$$0 \to A \xrightarrow{\begin{pmatrix} y & 0 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A \to k \to 0, \tag{1-8}$$

where the maps are right multiplication by the given matrices, is a minimal resolution of the trivial module. The groups $\mathrm{Ext}^\bullet_A(k, A)$ are the homology groups of the complex

$$0 \leftarrow A \xleftarrow{\begin{pmatrix} y & 0 \end{pmatrix}} A^2 \xleftarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A \leftarrow 0,$$

of *right* A-modules, with maps being *left* multiplication by the given matrices. It follows at once that $\mathrm{Ext}^2_A(k, A) \cong A/yA$, so A is not Gorenstein.

For completeness, we also compute

$$\operatorname{Ext}^1_A(k,A) = \frac{\left\{\begin{pmatrix} a \\ b\end{pmatrix} \in A^2 \;\middle\|\; \begin{pmatrix} y & 0\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = 0\right\}}{\left\{\begin{pmatrix} x \\ y\end{pmatrix} a \;\middle\|\; a \in A\right\}}$$
$$= \begin{pmatrix} xA \\ A\end{pmatrix} \Big/ \left\{\begin{pmatrix} xa \\ ya\end{pmatrix} \;\middle\|\; a \in A\right\}.$$

The identity $\left(\begin{smallmatrix} xa \\ b\end{smallmatrix}\right) = \left(\begin{smallmatrix} xa \\ ya\end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ b-ya\end{smallmatrix}\right)$ leads to the decomposition

$$\begin{pmatrix} xA \\ A\end{pmatrix} = \left\{\begin{pmatrix} xa \\ ya\end{pmatrix} \;\middle\|\; a \in A\right\} \bigoplus \begin{pmatrix} 0 \\ A\end{pmatrix},$$

so $\operatorname{Ext}^1_A(k,A) \cong A$.

EXERCISES

1.1 Compute the groups $\operatorname{Ext}^i_A(k,k)$, where k is the trivial module over each of the following rings:
(a) $A = k[x,y]$ with $yx = qxy \quad (0 \neq q \in k)$;
(b) $A = k[x,y]$ with $yx - xy = x^2$;
(c) $A = k[x,y]$ with $yx = 0$;
(d) $A = k[x,y]$ with $x^2 = 0$.

1.2 Show that a twist of an Artin-Schelter regular algebra is Artin-Schelter regular.

2. Artin-Schelter regular algebras of dimension ≤ 2

It is easy to find the Artin-Schelter regular algebras of low dimension. First, if $\operatorname{gldim}(A) = 0$, then the minimal resolution of the trivial module must be $0 \to A \to k \to 0$, whence $A = k$. The final term in the resolution of the trivial module is $A[-\ell]$ so, if $\operatorname{gldim}(A) = 1$, the minimal free resolution of the trivial module is $0 \to A[-\ell] \to A \to k \to 0$; therefore A is generated by a single element of degree ℓ and there are no relations—that is, $A = k[X]$ with $\deg(X) = \ell$. The 2-dimensional case is slightly less trivial.

PROPOSITION 2.1. *Let A be an Artin-Schelter regular algebra over an algebraically closed field k. If $\operatorname{gldim}(A) = 2$, then $A = k[x,y]$ with $\deg(x) = a$, $\deg(y) = b$ and a single defining relation, which is either of the form*

- *$yx - qxy$ for some $0 \neq q \in k$, or*
- *$yx - xy - x^{(a+b)/a}$.*

The second possibility can only arise if a divides b.

PROOF. The minimal resolution of the trivial module is of the form

$$0 \to A[-\ell] \to \oplus_{i=1}^n A[-d_i] \to A \to k \to 0.$$

In particular, A has n generators and one relation. The Hilbert series of A is therefore

$$H_A(t) = p_A(t)^{-1} = \left(1 - \sum_{i=1}^n t^{d_i} + t^\ell\right)^{-1}.$$

Since GKdim $A < \infty$, $p_A(t)$ cannot have a real zero > 1. But $p_A(1) = 2 - n$ and $p_A(t) \to \infty$ as $t \to \infty$, whence $n = 2$. Thus A has two generators. It also has a single relation, so is one of the algebras listed in Lemma 6.1.7.

Among those algebras, $k[x,y]$ with relation x^j, for some $j > 1$, has infinite GK-dimension, and $k[x,y]$ with relation $yx = 0$ does not satisfy the Gorenstein condition. This leaves the two cases mentioned. Both these are iterated Öre extensions—the first is the quantum plane, and the second is $k[x][y;\delta]$ with $\delta(x) = x^{(a+b)/a}$. These are Artin-Schelter regular by Theorem 1.4. $\square$

3. Artin-Schelter regular algebras of dimension 3

The 3-dimensional Artin-Schelter regular algebras are the first really interesting ones—half of them are treated in Chapter 24 under the section on quantum $\mathbb{P}^2$'s—in particular, they need not be iterated Öre extensions. We will postpone proving that the 3-dimensional Artin-Schelter regular algebras are noetherian until Chapter ???, since the methods in the proof of that result fit better there; the proof of that result is independent of the material in this chapter, so we will use the noetherian result here; this is essential in order to prove that the 3-dimensional Artin-Schelter algebras have most of the good properties of the commutative polynomial ring: in particular, they are domains, Auslander-Gorenstein, and Cohen-Macaulay.

First, there is the following dichotomy.

Theorem 3.1. *Let A be Artin-Schelter regular of dimension 3. If A is generated by A_1 as a k-algebra, then either*

- *A has three generators and three quadratic relations,*
- *$H_A(t) = (1-t)^{-3}$, and*
- *the minimal resolution of ${}_Ak$ is*

$$0 \to A[-3] \to A[-2]^3 \to A[-1]^3 \to A \to k \to 0,$$

or

- *A has two generators and two cubic relations,*
- *$H_A(t) = (1-t)^{-2}(1-t^2)^{-1}$, and*
- *the minimal resolution of ${}_Ak$ is*

$$0 \to A[-4] \to A[-3]^2 \to A[-1]^2 \to A \to k \to 0.$$

Proof. Take a minimal resolution of the trivial module ${}_Ak$, say

$$0 \longrightarrow A^t \longrightarrow A^s \longrightarrow A^r \longrightarrow A \longrightarrow k \longrightarrow 0. \tag{3-1}$$

By Proposition 3.2, $r = \dim A_1$ and s is the minimal number of defining relations. By the Gorenstein hypothesis, applying $\operatorname{Hom}_A(-, A)$ to (3-1) gives a minimal resolution of k_A, namely

$$0 \longleftarrow k_A \longleftarrow A^t \longleftarrow A^s \longleftarrow A^r \longleftarrow A \longleftarrow 0. \tag{3-2}$$

Hence $t = 1$. By Proposition 3.2, $s = \dim A_1$ and r is the minimal number of defining relations. Hence $r = s$. We may rewrite (3-1) as

$$0 \longrightarrow A \xrightarrow{\underline{y}^{\mathsf{T}}} A^r \xrightarrow{L} A^r \xrightarrow{\underline{x}} A \longrightarrow k \longrightarrow 0 \tag{3-3}$$

where $\underline{x} = (x_1, \dots, x_r)^{\mathsf{T}}$, $\underline{y}^{\mathsf{T}} = (y_1, \dots, y_r)$, $L = (m_{ij})$ is some $r \times r$ matrix, and $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_r\}$ are bases for A_1. The defining relations for A are the entries in the matrix $L\underline{x}$, and also the entries in $\underline{y}^{\mathsf{T}}L$; write $f_i := \sum_{j=1}^r m_{ij}x_j$

and $g_j := \sum_{i=1}^r y_i m_{ij}$. Since (3-1) begins $A[-1]^r \to A \to k \to 0$, the Gorenstein hypothesis implies that the minimal resolution of ${}_Ak$ is

$$0 \to A[-d-1] \xrightarrow{\underline{y}^\mathsf{T}} A[-d]^r \xrightarrow{L} A[-1]^r \xrightarrow{\underline{x}} A \to k \to 0, \tag{3-4}$$

where $d = \deg(f_i) = \deg(g_j)$ for all i and j.

Write $H_A(t) = \sum_n a_n t^n$ for the Hilbert series of A. Taking the degree n component of (3-4), we obtain

$$a_{n-d-1} - ra_{n-d} + ra_{n-1} - a_n + \delta_{n0} = 0. \tag{3-5}$$

Multiplying this by t^n and summing over all n, $H_A(t) = p(t)^{-1}$ where

$$p(t) = t^{d+1} - rt^d + rt - 1.$$

Now $p(0) = -1$, $p(1) = 0$ and $p'(1) = 2 - (r-1)(d-1)$. If $(r-1)(d-1) > 2$, then $p'(1) < 0$, whence $p(t)$ has a zero in the interval $(0,1)$; therefore the radius of convergence of $H_A(t)$ is < 1, which implies that $\operatorname{GKdim}(A) = \infty$ by Lemma 11.2.5. This possibility is excluded by hypothesis, so $(r-1)(d-1) \leq 2$. Thus $r \leq 3$ and $d \leq 3$.

We cannot have $r = 1$, else A is a quotient of the polynomial ring in one variable (and hence of the form $k[X]/(X^d)$, so not of global dimension 3). Also $d \neq 1$, else (3-4) is not a minimal resolution. Hence $r \geq 2$ and $d \geq 2$. However, if $r = d = 2$, then a simple recursion using (3-5) yields $a_5 = -1$, which is absurd. Hence either $(r,d) = (3,2)$ or $(r,d) = (2,3)$ which completes the proof. □

The polynomial ring on 3 indeterminates with its usual grading is an example of the first kind, and the algebra $A = k[x,y]$ with relations $x^2y + \lambda xyx + yx^2 = y^2x + \lambda yxy + xy^2 = 0$ (6.1.3) is an example of the second kind; notice that this algebra is the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. The Hilbert series for the second class of examples is that of the polynomial ring with two generators in degree one and one generator in degree two—if we had not insisted that the algebra be generated in degree one, the polynomial ring with this grading would have fallen into the second class of Artin-Schelter regular algebras of dimension three (perhaps this is a reason for dropping the requirement that the algebra be generated in degree one).

More can be said about the minimal resolutions in Theorem 3.1, and this is taken up in Chapter 24 (see especially Proposition 24.).

To say more about the 3-dimensional regular algebras we first need to prove they are noetherian. The ideas used to prove this are more appropriately developed at a later stage (see Chapters ???). However,ome of the consequences of the noetherian property use methods which fit well in the present chapter. So we will delay the proof of the noetherian property, but will use that result now.

Theorem 3.2. *A 3-dimensional Artin-Schelter regular algebra is noetherian.*

Proof. This is Theorem ??? in Chapter ???. □

Theorem 3.3. *A 3-dimensional Artin-Schelter regular algebra is a domain.*

Proof. Let A be the ring in question, and let N be the largest ideal in A of GK-dimension ≤ 2. By Proposition 14.4.11, A/N is a domain, so it suffices to show that $N = 0$. Suppose that $N \neq 0$. By (14.4.11) again, $\operatorname{pdim} N \leq 1$ so $\operatorname{GKdim} N = 2$ by Lemma 14.4.10. Thus $H_N(t)$ has a pole of order 2 at $t = 1$.

Write $N^\vee = \underline{\mathrm{Ext}}^1_A(N, A)$, and $N^* = \underline{\mathrm{Hom}}_A(N, A)$. By Proposition 1.3,

$$H_{N^*}(t) - H_{N^\vee}(t) = (-1)^3 t^{-\ell} H_N(t^{-1}). \tag{3-6}$$

By the proof of Proposition 14.4.11, N is the second syzygy of some A/Ax, so $N^\vee = \underline{\mathrm{Ext}}^1_A(N, A) \cong \underline{\mathrm{Ext}}^3_A(A/Ax, A)$, which is finite dimensional because A is Gorenstein (14.**??**). In particular, $H_{N^\vee}(t)$ has no pole at $t = 1$. By (3-6) $H_{N^*}(t)$ has a pole of order 2 at $t = 1$, so

$$\lim_{t\uparrow 1}(1-t)^2 H_{N^*}(t) = -\lim_{t\uparrow 1}(1-t)^2 H_N(t^{-1});$$

that is, $e(N^*) = -e(N)$. But this is absurd since $e(N)/e(A)$ is a positive integer, so we conclude that $N = 0$, as required. $\square$

We now exploit the double-Ext spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_A(\mathrm{Ext}^{-q}_A(M, A), A) \Rightarrow \mathbb{H}^{p+q}(M) = \begin{cases} M & \text{if } p+q = 0, \\ 0 & \text{if } p+q \neq 0. \end{cases} \tag{3-7}$$

It is notationally convenient to label the spectral sequence

$$E_2^{pq} = \underline{\mathrm{Ext}}^p_A(\underline{\mathrm{Ext}}^q_A(M, A), A).$$

We will tend to drop the subscript from E_2^{pq} when it is clear we are discussing the terms on the E_2 page.

Lemma 3.4. *If R is prime noetherian and $N \in \mathsf{mod}(R)$, the following are equivalent:*

1. $\mathrm{GKdim}\, N = \mathrm{GKdim}\, R$*;*
2. $j(N) = 0$*;*
3. $Q \otimes_R N \neq 0$*, where* $Q = \mathrm{Fract}\, R$.

In particular, $\mathrm{Fract}\, R$ *is* 0*-pure.*

Proof. Conditions (2) and (3) are equivalent over a semiprime noetherian ring by Proposition 2.7.6.

(1) $\Rightarrow$ (3) There is some $m \in N$ such that $\mathrm{GKdim}(Rm) = \mathrm{GKdim}\, R$, so $\mathrm{Ann}(m)$ cannot contain a regular element. Thus $Q \otimes_R N \neq 0$.

(2) $\Rightarrow$ (1) All uniform left ideals of R are sub-isomorphic (2.5.5), so have the same GK-dimension, which must be that of R. So if $j(N) = 0$, then $\mathrm{GKdim}\, N = \mathrm{GKdim}\, R$. $\square$

Theorem 3.5. *Let A be a 3-dimensional Artin-Schelter regular algebra and let $M \in \mathsf{grmod}(A)$. Then the E_2-table of the double-Ext spectral sequence for M looks like*

$$\begin{matrix} E^{00} & E^{10} & 0 & 0 \\ 0 & E^{11} & E^{21} & E^{31} \\ 0 & 0 & E^{22} & E^{32} \\ 0 & 0 & 0 & E^{33} \end{matrix}$$

Proof. Since $\mathrm{gldim}\, A = 3$, the double-Ext spectral sequence is zero outside the region depicted. Therefore the E^{20} and E^{30} terms survive to the E_∞-page. But any non-zero terms on the E_∞ page must lie on the diagonal, so $E^{20} = E^{30} = 0$. By Corollary 14.**??**, $\underline{\mathrm{Ext}}^3_A(M, A)$ is finite dimensional so, by Lemma 14.**??**, $E^{p3} = 0$ for $p < 3$. This explains the zeroes in the top and bottom rows.

Since A is a noetherian domain it has a division ring of fractions, Q say, which is flat as both a left and as a right A-module. Now $\operatorname{gldim} Q = 0$ so for $i > 0$,

$$0 = \operatorname{Ext}^i_Q(Q \otimes_A M, Q \otimes_A A) \cong \underline{\operatorname{Ext}}^i_A(M, A) \otimes_A Q.$$

Applying Proposition 2.7.6(1) to $N = \underline{\operatorname{Ext}}^i_A(M, A)$, we obtain $E^{01} = E^{02} = E^{03} = 0$, giving the zeroes in the left-most column.

It remains to show that $E^{12} = 0$. Set $L = \underline{\operatorname{Ext}}^2_A(M, A)$; if $L = 0$ there is nothing to do, so suppose that $L \neq 0$. Since $\ker(E^{12} \to E^{33})$ survives to the E_∞ page, it must be zero. By Corollary 14.**??**, E^{33} is finite dimensional, so $\dim_k E^{12} < \infty$; that is, $\dim_k \underline{\operatorname{Ext}}^1_A(L, A) < \infty$.

Now consider the spectral sequence for L. If $\underline{\operatorname{Ext}}^1_A(L, A) = 0$, we are finished, so suppose otherwise. Since $L \otimes_A Q \cong \underline{\operatorname{Ext}}^2_Q(Q \otimes_A M, Q \otimes_A A) = 0$, $\operatorname{Hom}_A(L, A) = 0$, so the top two rows of E_2-page of the spectral sequence for L look like

$$\begin{matrix} 0 & 0 & 0 & 0 \\ E^{01} & E^{11} & E^{21} & E^{31} \end{matrix}$$

Therefore E^{31} survives to the E_∞ page, so must be zero. However, $\underline{\operatorname{Ext}}^1_A(L, A)$ is non-zero and finite dimensional, so $\underline{\operatorname{Ext}}^3_A(\underline{\operatorname{Ext}}^1_A(L, A), A) \neq 0$ by Lemma 14.**??**. This is a contradiction, so we conclude that $\underline{\operatorname{Ext}}^1_A(L, A) = 0$, as required. □

Next we prove that A is Cohen-Macaulay. That is, we prove that $j(M) + \operatorname{GKdim} M = 3$ for all $0 \neq M \in \mathsf{grmod}(A)$. The next result shows this equality holds if $\operatorname{GKdim} M = 3$, and Lemma 14.**??** establishes it if $\operatorname{GKdim} M = 0$, so it remains for us to examine the modules of GK-dimension 1 and 2.

Lemma 3.6. *Let A be a 2-dimensional Artin-Schelter regular algebra, and suppose that $M \in \mathsf{grmod}(A)$. If $j(M) = 1$, then $M^\vee$ is Cohen-Macaulay,* $\operatorname{pdim} M^\vee = 1$, *and* $\operatorname{GKdim} M^\vee = 2$.

Proof. Since $j(M) = 1$, the E_2-page of the spectral sequence for M looks like

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & E^{11} & E^{21} & E^{31} \\ 0 & 0 & E^{22} & E^{32} \\ 0 & 0 & 0 & E^{33} \end{matrix}$$

Thus E^{21} and E^{31} survive to the E_∞-page, so they are zero. Since $M^\vee = \underline{\operatorname{Ext}}^1_A(M, A)$ is non-zero, some $\underline{\operatorname{Ext}}^1_A(M^\vee, A)$ is non-zero, so we conclude that $E^{11} \neq 0$. Thus $j(M^\vee) = 1$, and $M^\vee$ is Cohen-Macaulay. Therefore $\operatorname{pdim} M^\vee = 1$ too. Since $j(M^\vee) > 0$, we conclude that $\operatorname{GKdim} M^\vee \leq 2$. However, by Lemma 14.4.10, $\operatorname{GKdim} M^\vee \geq 2$, so there is equality. □

Theorem 3.7. *If A is 3-dimensional Artin-Schelter regular then A is Cohen-Macaulay; that is, if $0 \neq M \in \mathsf{grmod}(A)$, then*

$$j(M) + \operatorname{GKdim} M = 3.$$

Proof. The condition $\operatorname{GKdim} M = 0$ is equivalent to the condition $\dim_k M < \infty$ so, by Corollary 14.**??**, $\operatorname{GKdim} M = 0$ if and only if $j(M) = 3$.

By Lemma 3.4, $\operatorname{GKdim} M = 3$ if and only if $j(M) = 0$.

Now suppose that $\operatorname{GKdim} M = 1$. By Lemma **??**, $j(M) \geq 1$. The top row of the double-Ext spectral sequence for M consists of zeroes, so $E^{21} = E^{31} = 0$.

Hence the E_2-page looks like

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & E^{11} & 0 & 0 \\ 0 & 0 & E^{22} & E^{32} \\ 0 & 0 & 0 & E^{33} \end{array}$$

Hence the canonical filtration is $M = F^0M = F^1M \supset F^2M \supset \cdots$; in particular, there is a surjective map

$$M \to F^1M/F^2M = E_\infty^{11} = \ker(E^{11} \to E^{32}).$$

In other words, there is an exact sequence $M \to E^{11} \to E^{32}$. Since $\dim_k(E^{32}) < \infty$, it follows that $\mathrm{GKdim}(E^{11}) \le 1$. If $E^{11} \neq 0$, then $j(\underline{\mathrm{Ext}}_A^1(M,A)) = 1$ so, by Lemma 3.6, $\mathrm{GKdim}\, \underline{\mathrm{Ext}}_A^1(M,A)^\vee = 2$; that is, $\mathrm{GKdim}(E^{11}) = 2$, which contradicts the foregoing. So we must have $E^{11} = 0$. But then the second row of the E_2-page is zero, so $\underline{\mathrm{Ext}}_A^1(M,A) = 0$, and $j(M) \ge 2$. However, $\jmath(M) \neq 3$ because $\dim_k M = \infty$, so $j(M) = 2$.

Now suppose that $\mathrm{GKdim}\, M = 2$. By what we have already said, $j(M)$ is either 1 or 2. Suppose that $j(M) = 2$; we seek a contradiction. Because $j(\tau M) = 3$, it follows that $j(M/\tau M) = 2$; of course $\mathrm{GKdim}(M/\tau M) = 2$ also, so we may replace M by $M/\tau M$. Doing this, we have $\tau M = 0$, whence $\underline{\mathrm{Ext}}_A^3(M,A) = 0$. By Proposition 1.3,

$$H_{M^\vee}(t) = -t^{-3}H_M(t^{-1}).$$

Since $\mathrm{GKdim}\, M = 2$, each side of this equation has a pole of order 2 at $t = 1$, so

$$\lim_{t\uparrow 1}(1-t)^2 H_{M^\vee}(t) = -\lim_{t\uparrow 1}(1-t)^2 H_M(t^{-1});$$

that is, $e(M^\vee) = -e(M)$. But this is absurd since $e(M)/e(A)$ is a positive integer, so we conclude that $j(M) \neq 2$, as required. □

Corollary 3.8. *A 3-dimensional Artin-Schelter regular algebra is Auslander-Gorenstein.*

Proof. Let A be the ring in question, let $0 \neq M \in \mathsf{grmod}(A)$, and let N be a non-zero graded A-submodule of $\underline{\mathrm{Ext}}_A^i(M,A)$. By Theorem 3.5 the E_2 page is upper triangular, so $j(\underline{\mathrm{Ext}}_A^i(M,A)) \ge i$, whence $\mathrm{GKdim}(\underline{\mathrm{Ext}}_A^i(M,A)) \le 3-i$ because A is Cohen-Macaulay. Hence $\mathrm{GKdim}\, N \le 3-i$ and, by the Cohen-Macaulay property again, $j(N) \ge i$, which proves that A has the Auslander property. □

4. Quantum polynomial rings

The algebras in this section are non-commutative analogues of polynomial rings. They will be used later to define quantum projective spaces, the non-commutative analogues of $\mathbb{P}^n$.

Definition 4.1. A connected k-algebra A is an n-dimensional quantum polynomial ring if

- it is Gorenstein;
- it is right and left noetherian;
- it is generated by A_1 as a k-algebra;
- $\mathrm{gldim}\, A = n$;
- $H_A(t) = (1-t)^{-n}$.

Notice that a quantum polynomial ring is Artin-Schelter regular.

A natural modification of this definition would be to weaken the requirement that A have the same Hilbert series as the polynomial ring with its standard grading, and to allow the Hilbert series to agree with that of a polynomial ring having generators of different degrees; one could still require that A be generated by A_1 or, if one wishes drop that condition. Notice that the 3-dimensional Artin-Schelter regular algebras on two generators have the same Hilbert series as the polynomial ring generated by two elements of degree one, and one of degree two.

When $n = 3$, the noetherian condition follows from the others—in Chapter 24 we will prove that a 3-dimensional Artin-Schelter regular algebra is noetherian.

We conjecture that the noetherian property follows from the other conditions. We also conjecture that a quantum polynomial ring is Auslander-Gorenstein and Cohen-Macaulay; if this conjecture is false it would probably be appropriate to change the definition to include these properties—this would ensure, amongst other things, that quantum polynomial rings are domains.

The next result shows that quantum polynomial rings are Koszul.

THEOREM 4.2. (J. Zhang) *Let A be a graded k-algebra. Suppose that A is noetherian, Gorenstein,* $\operatorname{gldim}(A) = n$, *and $H_A(t) = (1-t)^{-n}$. Then A is Koszul.*

PROOF. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a minimal projective resolution of k. Write $P_i = A \otimes_k V_i$ and $P_i^\vee = \underline{\operatorname{Hom}}_A(P_i, A) \cong V_i^* \otimes A$. Then

$$H_A(t)^{-1} = \sum_{i=0}^{n} (-1)^i H_{V_i}(t). \tag{4-1}$$

Let a_i (resp. b_i) denote the least (resp. largest) degree of a component of V_i. Since the resolution is minimal $0 = b_0 = a_0 < a_1 < \cdots < a_n$. In particular, $a_n \geq n$.

Since A is Gorenstein, $0 \to P_0^\vee \to \cdots \to P_n^\vee \to k_A[\ell] \to 0$ is a projective resolution of $k_A[\ell]$ for some integer ℓ. The minimality of $P_\bullet$ ensures that $P_\bullet^\vee$ is also minimal, so $P_n^\vee \cong A[\ell]$, whence $\ell = a_n = b_n \geq n$, and $-b_n < \cdots < -b_1 < -b_0 = 0$.

By hypothesis, the highest degree term occuring in $H_A(t)^{-1}$ is t^n. Since $P_n \cong A[-\ell]$ contributes a term $(-1)^n t^\ell$ to the sum (4-1), which cannot be cancelled out by any other terms in the sum because $\ell = b_n > b_i$ for all $i \neq n$, it follows that $\ell \leq n$. Therefore $\ell = n$, and $a_i = i = b_i$ for all i. In other words, $P_\bullet$ is a linear resolution of k. □

CHAPTER 19

The category Tails

The main object of study in algebraic geometry is the category $\mathsf{Coh}X$ of coherent sheaves of modules over the structure sheaf of a scheme X. The main object of study in *non-commutative* algebraic geometry is the category $\text{tails}\, A$ associated to a left noetherian, locally finite, $\mathbb{N}$-graded algebra A. This is justified by Serre's Theorem (20.4.4): if A is the homogeneous coordinate ring of a closed subscheme $X \subset \mathbb{P}^n$, then the categories $\text{tails}\, A$ and $\mathsf{Coh}X$ are equivalent.

The category $\text{tails}(A)$, and the larger category $\mathsf{Tails}A$, are defined as the quotient categories of $\mathsf{grmod}(A)$ and $\mathsf{GrMod}(A)$, modulo the dense subcategory of *torsion* modules. Torsion modules were defined and studied in chapter 13. If A is left noetherian, locally finite, and $\mathbb{N}$-graded, then the torsion modules in $\mathsf{grmod}(A)$ are the finite dimensional modules. This is the main case of interest to us, and is substantially simpler than the general case. For example, two noetherian modules, say M and M', give isomorphic objects in $\text{tails}(A)$ if and only if their tails are isomorphic; i.e., if and only if $M_{\geq n} \cong M'_{\geq n}$ for some n.

The basic properties of the category $\mathsf{Tails}A$ are treated in section 1. A key point is that morphisms in $\mathsf{Tails}A$ are easier to handle when one has reasonable finiteness conditions on A (for example, if A is noetherian and/or locally finite). If $f : A \to B$ is a homomorphism of graded algebras, restriction yields a functor $f_* : \mathsf{Tails}B \to \mathsf{Tails}A$ and, under suitable hypotheses, induction yields a functor $f^* : \mathsf{Tails}A \to \mathsf{Tails}B$. Section 2 discusses some of the functorial behavior of $\mathsf{Tails}(-)$. A key point is that $\mathsf{Tails}A$ does not determine A—$\mathsf{Tails}A$ is unchanged if A is replaced by either a Veronese subalgebra $A^{(r)}$, or a twist of itself, or by a subalgebra which differs from A by a finite dimensional piece. Section 3 discusses injectives in $\mathsf{Tails}A$: first, there are enough, which allows computation of Ext groups via injective resolutions. The Ext groups in $\mathsf{Tails}A$ are used in section 1 of chapter 20 to define cohomology groups for non-commutative schemes (generalizing the usual sheaf cohomology groups).

A key point in studying $\mathsf{Tails}A$ is that the quotient functor $\pi : \mathsf{GrMod}(A) \to \mathsf{Tails}A$ has a right adjoint $\omega : \mathsf{Tails}A \to \mathsf{GrMod}(A)$, and this satisfies $\pi \circ \omega \cong \mathbb{1}_{\mathsf{Tails}}$. To use ω as an effective computational tool, Theorem 4.1 shows that, if A is left noetherian and $\mathbb{N}$-graded, then $\omega\pi M \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, M)$, and the natural transformation $\mathbb{1}_{\mathsf{GrMod}} \to \omega \circ \pi$ yields an exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0.$$

Direct limits of such Ext-groups are central to the study of $\mathsf{Tails}A$, and we have already treated discussed these in section 4 of chapter 13 where the technical conditions χ_i and χ_i° were introduced. For example, if M is a left noetherian module over a locally finite, left noetherian, $\mathbb{N}$-graded algebra A, then $\chi_1(M)$ holds if and only if $\operatorname{coker}(M \to \omega\pi M)$ is right bounded (equivalently, if and only if $(\omega\pi M)_{\geq d}$) is finitely generated for all $d \in \mathbb{Z}$).

1. Tails

Definition 1.1. Let A be a graded k-algebra. Define the quotient categories

- $\mathsf{Tails}A := \mathsf{GrMod}(A)/\operatorname{Tors}(A)$, and
- $\operatorname{tails}(A) := \mathsf{grmod}(A)/\operatorname{tors}(A)$.

NOTATION . We will write $\pi : \mathsf{GrMod}(A) \to \mathsf{Tails}(A)$ and $\pi : \mathsf{grmod}(A) \to \operatorname{tails}(A)$ for the natural functors. Thus, if M is a graded A-module, we write πM when we consider it as an object of $\mathsf{Tails}(A)$. We will use script letters (such as $\mathcal{M}$) to denote objects in $\mathsf{Tails}(A)$; this reinforces the idea that objects in $\mathsf{Tails}(A)$ are like sheaves of modules (the aptness of this analogy will be discussed later). In particular, we will write

$$\mathcal{A} = \pi A$$

for the image of A in $\mathsf{Tails}(A)$.

The subcategory $\operatorname{Tors}(A)$ is stable under the shift functor because M is torsion if and only if $M[1]$ is. Hence there is an induced automorphism on $\mathsf{Tails}(A)$, which we still denote by $[1]$, and call the shift functor; there is no ambiguity in writing $\pi M[1]$.

The morphisms in $\mathsf{Tails}(A)$, and in its full subcategory $\operatorname{tails}(A)$, will be denoted by

$$\operatorname{Hom}_{\mathsf{Tails}}(-,-).$$

We also make use of the shift functor to define, for $\mathcal{F}, \mathcal{G} \in \mathsf{Tails}(A)$,

$$\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G}) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}[n]).$$

In this way $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})$ is a graded k-vector space with $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})_n := \operatorname{Hom}(\mathcal{F}, \mathcal{G}[n])$.

Definition 1.2. We say that $M \in \mathsf{GrMod}(A)$ is finitely generated up to torsion if $\pi M \in \operatorname{tails}(A)$—that is, if $\pi M \cong \pi N$ for some $N \in \mathsf{grmod}(A)$.

The terminology 'Tails' is explained by Theorem 1.6: a submodule $M_{\geq n}$ of M is called a tail of M, and Theorem 1.6 says that, if M and N are noetherian, then $\pi M \cong \pi N$ if and only if M and N have isomorphic tails. Ultimately, the category $\mathsf{Tails}(A)$ is more important to us than $\mathsf{GrMod}(A)$, so it is not just a case of the tail wagging the dog—the tail *is* the dog!

EXAMPLE 1.3. If A is strongly graded, then $A.A_{\geq n} = A$, so $\operatorname{Tors}(A)$ consists only of the zero module, whence $\mathsf{Tails}(A) \cong \mathsf{GrMod}(A) \cong \mathsf{Mod}(A_0)$.

We now recall some of the basic properties of quotient categories which are proved in section 13 of Appendix A.

Since $\mathsf{GrMod}(A)$ is a k-linear abelian category, so are $\mathsf{Tails}(A)$ and $\operatorname{tails}(A)$. By Theorem A.13.6, π is an exact functor. The objects in the quotient category are the same as those in the original category—they are of the form πM—but there are more morphisms in the quotient category. In particular, if $f : N \to M$ is a degree 0 homomorphism of graded A-modules, such that $\ker(f)$ and $\operatorname{coker}(f)$ are finite dimensional, then πf is an isomorphism in $\mathsf{Tails}(A)$. Thus, up to isomorphism, every object in $\mathsf{Tails}(A)$ is of the form πN for some torsion-free $N \in \mathsf{GrMod}(A)$.

By definition of the quotient category, if $N, M \in \mathsf{GrMod}(A)$, then

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N', M/M')$$

where the direct limit is taken over the directed set, I say, consisting of all pairs (N', M') of submodules of N and M respectively, for which N/N' and M' are torsion modules. The quasi-ordering on I is defined by

$$(N', M') \leq (N'', M'') \qquad \text{if } N'' \subset N' \text{ and } M' \subset M''.$$

Since I is directed, every morphism $\pi N \to \pi M$ is of the form πf for some $f \in \operatorname{Hom}_{\mathrm{Gr}}(N', M/M')$ some $(N', M') \in I$; that is, every morphism in $\mathsf{Tails}(A)$ is the image, in the appropriate direct limit, of a morphism in $\mathsf{GrMod}(A)$.

The next several results show that under reasonable hypotheses this description of the morphisms in $\mathsf{Tails}(A)$ may be simplified.

Proposition 1.4. *Let N and M be graded A-modules. Then*

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N', M/\tau M)$$

where the direct limit is taken over the submodules $N' \subset N$ such that N/N' is torsion.

Proof. By Lemma 3.6, the set $(N', \tau M)$ is cofinal in the index set I defined above, so the result follows from Proposition A.8.6. $\square$

Proposition 1.5. *Let A be an $\mathbb{N}$-graded, k-algebra. Suppose that $M \in \mathsf{GrMod}(A)$ and that N is a noetherian module.*

1. *If A is left noetherian or locally finite,*

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) \;=\; \varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M/\tau M).$$

2. *If A is left noetherian,*

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi M, \pi N) = \varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M).$$

Proof. (1) If N/N' is torsion, then N/N' is right bounded by Proposition 3.8, so $N_{\geq n} \subset N'$ for $n \gg 0$. Hence $\{(N_{\geq n}, \tau M) \mid n \in \mathbb{Z}\}$ is cofinal in the index set I defined above, so the equality follows from Proposition A.8.6.

(2) First we prove this when M is finitely generated. By Proposition A.8.13, the direct limit over n of the exact sequences

$$0 \to \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \tau M) \to \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M) \to \\ \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M/\tau M) \to \operatorname{Ext}^1_{\mathrm{Gr}}(N_{\geq n}, \tau M)$$

remains exact. Since M is noetherian, τM is right bounded by Proposition 3.8, so $\operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \tau M) = 0$ for $n \gg 0$. Since a minimal free resolution of $N_{\geq n}$ is zero in degree $< n$, it also follows that $\operatorname{Ext}^1_{\mathrm{Gr}}(N_{\geq n}, \tau M) = 0$ for $n \gg 0$. Therefore, the direct limits of the first and last terms are zero, so the direct limits of the middle two terms are isomorphic. Hence the result follows from (1).

Now suppose that M is arbitrary, and write $M = \varinjlim M_\alpha$ as a direct limit of finitely generated graded modules. The index set is directed. By Proposition 3.9, $\tau M_\alpha = M_\alpha \cap \tau M$, whence $\varinjlim(\tau M_\alpha) = \tau M$. Therefore, taking the direct limit of the exact sequences $0 \to \tau M_\alpha \to M_\alpha \to M_\alpha/\tau M_\alpha \to 0$, Proposition A.8.13

implies that $\varinjlim(M_\alpha/\tau M_\alpha) = M/\tau M$. Since $N_{\geq n}$ is finitely generated, the functor $\operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, -)$ commutes with direct limits. Thus

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) &= \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M/\tau M) \\
&= \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \varinjlim_\alpha (M_\alpha/\tau M_\alpha)) \\
&= \varinjlim_n \varinjlim_\alpha \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M_\alpha/\tau M_\alpha) \\
&= \varinjlim_\alpha \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M_\alpha/\tau M_\alpha) \\
&= \varinjlim_\alpha \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M_\alpha) \\
&= \varinjlim_n \varinjlim_\alpha \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M_\alpha) \\
&= \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \varinjlim_\alpha M_\alpha) \\
&= \varinjlim_n \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, M)
\end{aligned}$$

as required. □

Theorem 1.6. *Let A be $\mathbb{N}$-graded. If $M, N \in \mathsf{grmod}(A)$ are noetherian, then $\pi N \cong \pi M$ if and only if $N_{\geq n} \cong M_{\geq n}$ for some n.*

Proof. ($\Leftarrow$) If $f : N_{\geq n} \xrightarrow{\sim} M_{\geq n}$ is an isomorphism in $\mathsf{GrMod}(A)$, then πf is an isomorphism in $\mathsf{Tails}(A)$. But $N/N_{\geq n}$ and $M/M_{\geq n}$ are torsion, so $\pi N \cong \pi(N_{\geq n}) \cong \pi(M_{\geq n}) \cong \pi M$.

($\Rightarrow$) Suppose that $\pi N \cong \pi M$. This isomorphism is of the form πf for a suitable $f : N' \to M/M'$ with N/N' and M' torsion. By Proposition A.13.5, $\ker(f)$ and $\operatorname{coker}(f)$ are torsion; but they are also noetherian since M and N are, so they are right bounded. Hence f restricts to an isomorphism $N'_{\geq n} \to (M/M')_{\geq n}$ if $n \gg 0$. But N/N' and M' are torsion and noetherian, so right bounded, whence $N_{\geq n} = N'_{\geq n}$ and $M_{\geq n} = (M/M')_{\geq n}$ for $n \gg 0$. Thus, for large n, f induces an isomorphism $N_{\geq n} \to M_{\geq n}$, as required. □

The noetherian hypothesis on M is essential to the previous result: if $A = k$ and $M = \oplus_{n \leq 0} k[n]$, then $\pi M \cong 0$, but $M_{\geq n} \ncong 0$ for any n.

Corollary 1.7. *Let A be a locally finite, $\mathbb{N}$-graded k-algebra, and N and M noetherian, graded, left A-modules. Suppose that $\pi N \cong \pi M$. Then* $\operatorname{GKdim}(N) = \operatorname{GKdim}(M)$ *and, if $\pi N \neq 0$, then $e(N) = e(M)$.*

Proof. There exists n such that $N_{\geq n} \cong M_{\geq n}$ so, as GK-dimension just depends on the behavior of a module in large degree, $\operatorname{GKdim}(N) = \operatorname{GKdim}(M)$. If N is not finite dimensional, then $e(N) = e(N_{\geq n})$ so $e(N) = e(M)$. □

Our study of $\mathsf{Tails}(A)$ is simplified by the fact that π has a right adjoint, which we denote by ω—we just take note of the existence of ω for now, and leave a more detailed examination of it until section 4 (but see also Proposition 3.1).

Theorem 1.8. *The functor $\pi : \mathsf{GrMod}(A) \to \mathsf{Tails}(A)$ has a right adjoint,*

$$\omega : \mathsf{Tails}(A) \to \mathsf{GrMod}(A).$$

PROOF. This is a consequence of Theorem A.13.14: $\mathsf{GrMod}(A)$ has enough injectives (Theorem 11.5.5) and every graded module has a maximal torsion submodule (Lemma 3.6). $\square$

We will make frequent use of the adjoint isomorphism

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \mathcal{F}) \cong \operatorname{Hom}_{\mathrm{Gr}}(N, \omega\mathcal{F}). \tag{1-1}$$

Definition 1.9. We call $\omega\pi M$ the saturation of M, and say that M is saturated if the natural map $M \to \omega\pi M$ is an isomorphism.

EXERCISES

1.1 Suppose that A is a finite dimensional graded k-algebra. Show that every object in $\mathsf{Tails}(A)$ is isomorphic to zero. Find weaker conditions on A for which the conclusion still holds.

1.2 Let $A = k[X]$ with $\deg(X) = 1$. Show that $\mathrm{tails}(A)$ contains a unique irreducible object and that every object is isomorphic to a finite direct sum of copies of this irreducible object.

1.3 Let $A = k[X]$ with $\deg(X) = r > 0$. Show that $\mathrm{tails}(A)$ contains r irreducible objects up to isomorphism, that all these are shifts of a single one, and that every object is isomorphic to a finite direct sum of copies of these irreducible objects.

1.4 Let $A = k[x, y]/(f)$ where $k[x, y]$ is the commutative polynomial ring with its usual grading, and f is a homogeneous polynomial of degree $n \geq 1$ which is a product of n distinct linear terms. Show that $\mathsf{Tails}(A)$ has n non-isomorphic irreducible objects, and that every object is isomorphic to a direct sum of various irreducible objects. Are any of these irreducible objects shifts of other ones?

1.5 Let $A = k[x, y]/(y^2)$ be commutative with $\deg(x) = \deg(y) = 1$. Show that there is a unique irreducible object in $\mathsf{Tails}(A)$ and show that there exists an object of length two in $\mathsf{Tails}(A)$ which is not a direct sum of copies of this irreducible object.

1.6 If $\mathcal{F}$ is a non-zero irreducible object in $\mathsf{Tails}(A)$ show that $\omega\mathcal{F}$ is critical.

2. Functorial properties of Tails

In this section we compare $\mathsf{Tails}(-)$ for related algebras. First we show that Tails is unchanged by twisting and taking Veronese subalgebras. Second, we examine the relation between $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$ when there is a graded algebra homomorphism $f : A \to B$. For example, as one might expect, if the kernel and cokernel of f are finite dimensional (i.e., A and B differ by a finite dimensional piece), the categories $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$ are equivalent—see Corollary 2.13.

PROPOSITION 2.1. *Let θ be a twisting system on the graded algebra A. Then the categories $\mathsf{Tails}(A)$ and $\mathsf{Tails}({}^{\theta}A)$ are equivalent. Furthermore, if A is left noetherian, $\mathrm{tails}(A)$ and $\mathrm{tails}({}^{\theta}A)$ are equivalent.*

PROOF. This is an immediate consequence of Theorem 16.4.7. The functors $\Theta : \mathsf{GrMod}(A) \to \mathsf{GrMod}({}^{\theta}A)$ and $\Psi : \mathsf{GrMod}({}^{\theta}A) \to \mathsf{GrMod}(A)$ in the proof of that Theorem, which give equivalences of categories, send torsion modules to torsion modules, so induce functors between $\mathsf{Tails}(A)$ and $\mathsf{Tails}({}^{\theta}A)$, which are also equivalences. Since ${}^{\theta}A$ is noetherian if A is, and Θ and Ψ send finitely generated modules to finitely generated modules, the equivalence between $\mathrm{tails}(A)$ and $\mathrm{tails}({}^{\theta}A)$ follows. $\square$

Under reasonable hypotheses, taking a Veronese subalgebra does not change the category Tails—this is the analogue of the fact that the d-uple embedding is indeed an embedding. Notice that, if $d \neq 1$, the inclusion $A^{(d)} \to A$ is *not* a morphism of algebras in the graded category, since it sends elements of degree i to elements of degree di.

PROPOSITION 2.2. *Let A be a graded k-algebra. Then the rule $M \mapsto M^{(d)}$ induces a functor $\mathsf{Tails}(A) \to \mathsf{Tails}(A^{(d)})$. If A is left noetherian, $\mathbb{N}$-graded, and generated in degree 1, then*

1. $\mathsf{Tails}(A)$ *is equivalent to* $\mathsf{Tails}(A^{(d)})$*, and*
2. $\mathrm{tails}(A)$ *is equivalent to* $\mathrm{tails}(A^{(d)})$*.*

PROOF. The exact covariant functor $f_* : \mathsf{GrMod}(A) \to \mathsf{GrMod}(A^{(d)})$, defined by $f_*M = M^{(d)}$, sends $\mathrm{Tors}(A)$ to $\mathrm{Tors}(A^{(d)})$, so induces a well-defined functor $\mathsf{Tails}(A) \to \mathsf{Tails}(A^{(d)})$. We call the induced functor f_* also.

(1) Define $f^* : \mathsf{GrMod}(A^{(d)}) \to \mathsf{GrMod}(A)$ by

$$f^*N = A \otimes_{A^{(d)}} N,$$

with the grading defined by $\deg(A_j \otimes N_i) = j + di$. To show that f^* induces a functor at the level of Tails, it suffices to show that, if N is torsion, then so is f^*N. By Proposition 13.3.8, an A-module is torsion if and only if it is a sum of finite dimensional submodules. Hence, as f^* is right exact, it suffices to show that, if $\dim_k(N) < \infty$, then $\dim_k(f^*N) < \infty$. The hypotheses ensure that A is a finitely generated right $A^{(d)}$-module, so $A \otimes_{A^{(d)}} N$ is a quotient of $X \otimes_{A^{(d)}} N$ for some finitely generated free right $A^{(d)}$-module X. It follows at once that f^*N is finite dimensional if N is, and hence f^* induces a functor $\mathsf{Tails}(A^{(d)}) \to \mathsf{Tails}(A)$, which we denote by f^* also.

If N is a graded $A^{(d)}$-module, then $f_*f^*(N) = N$ so f_*f^* is naturally equivalent to $\mathbb{1}_{\mathsf{Tails}(A^{(d)})}$.

We will prove that f^*f_* is naturally equivalent to $\mathbb{1}_{\mathsf{Tails}(A)}$ by showing that the multiplication map $\varphi : f^*f_*(M) = A \otimes_{A^{(d)}} M^{(d)} \to M$ induces an isomorphism in $\mathsf{Tails}(A)$. First, φ is a degree zero map, so it suffices to show that $\ker\varphi$ and $\operatorname{coker}\varphi$ are torsion. The cokernel is $M/A.M^{(d)}$. If $m \in M_j$, then, for some i, $A_i m \subset M^{(d)}$, whence $A.A_i m \subset A.M^{(d)}$. But A is generated in degree 1, so $A.A_i = A_{\geq i}$; therefore the image of m in $M/A.M^{(d)}$ is torsion.

Before considering $\ker\varphi$, notice that A decomposes as a direct sum of $A^{(d)}$-bimodules, namely

$$A = \bigoplus_{j=0}^{d-1} A^{(d)+j},$$

where $A^{(d)+j} = \sum_{i\in\mathbb{Z}} A_{j+di}$. Since A is generated in degree 1, $A^{(d)+j} = A_j A^{(d)}$.

Let $x = \sum a_i \otimes m_i$ be a homogeneous element of $\ker\varphi$. Without loss of generality, we may assume each a_i and each m_i is homogeneous. Since each m_i has degree a multiple of d, there exists j, $0 \leq j \leq d-1$, such that $a_i \in A^{(d)+j}$ for all i. Hence, if $b \in A_{d-j}$, then

$$bx = \sum ba_i \otimes m_i = \sum 1 \otimes ba_i m_i = 0.$$

Thus $A_{d-j}x = 0$; but $A_{\geq d-j} = A.A_{d-j}$ since A is generated by A_1, so $A_{\geq d-j}x = 0$. Thus $\ker\varphi$ is torsion, as required.

(2) By Proposition 6.2.5, $M^{(d)}$ is finitely generated whenever M is. Therefore f_* restricts to an equivalence between $\text{tails}(A)$ and $\text{tails}(A^{(d)})$. $\square$

The next example shows the necessity of the hypothesis in Proposition 2.2 that A is generated in degree one.

EXAMPLE 2.3. If $A = k[x,y]$ is the polynomial ring with $\deg(x) = 1$ and $\deg(y) = 2$, the functor $f^* : \mathsf{GrMod}(A) \to \mathsf{GrMod}(A^{(2)})$ given by $f^*M = M^{(2)}$ does not induce an equivalence $\mathsf{Tails}(A) \to \mathsf{Tails}(A^{(2)})$. For brevity, write $R = A^{(2)}$; thus $R = k[x^2, y]$ is also a polynomial ring. Consider the inclusion $g : xA[1] \to A[1]$; the induced map in $\mathsf{Tails}(A)$ is *not* an isomorphism since A/xA is not a torsion module. However, since $A = R \oplus xR$, $f^*(xA[1]) = xR[1] = f^*(A[1])$, whence $f^*(g)$ is an isomorphism in $\mathsf{Tails}(R)$.

It is worthwhile to observe that the projective schemes associated to the algebras A and $A^{(2)}$ are both isomorphic to $\mathbb{P}^1$. (This is *not* at variance with Serre's Theorem 20.4.4 since Serre's Theorem concerns algebras generated in degree one.)

Let $f : A \to B$ be a homomorphism of graded k-algebras. There is the usual pair of adjoint functors

$$f^* : \mathsf{GrMod}(A) \to \mathsf{GrMod}(B) \tag{2-1}$$

defined by

$$f^*M = B \otimes_A M,$$

where B is viewed as a right A-module via f and $B \otimes_A M$ is given the tensor product grading, and

$$f_* : \mathsf{GrMod}(B) \to \mathsf{GrMod}(A) \tag{2-2}$$

is defined by

$$f_*N = N,$$

viewed as an A-module via f. We have the adjoint isomorphism

$$\mathrm{Hom}_{\mathrm{Gr}}(f^*M, N) \cong \mathrm{Hom}_{\mathrm{Gr}}(M, f_*N)$$

because $f_*N \cong \mathrm{Hom}_B(B, N)$, where the left A-module action is that induced from the right A-action on B (cf. Exercise 1.1.7).

To decide whether f^* and f_* induce functors between $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$ we must examine their behavior on torsion modules.

PROPOSITION 2.4. *Let $f : A \to B$ be a homomorphism of graded k-algebras. Then*

1. *f induces an exact covariant functor $f_* : \mathsf{Tails}(B) \to \mathsf{Tails}(A)$;*
2. *if ${}_AB$ is finitely generated up to torsion, then $f_* : \text{tails}(B) \to \text{tails}(A)$.*

PROOF. (1) If N is a torsion B-module, then f_*N is a torsion A-module, since $f(A_{\geq i}) \subset B_{\geq i}$ for all i. Thus the composition

$$\mathsf{GrMod}(B) \xrightarrow{f_*} \mathsf{GrMod}(A) \to \mathsf{Tails}(A)$$

is an exact covariant functor which sends each object in $\mathrm{Tors}(B)$ to zero. Hence by Theorem A.13.7, there is a unique functor $\mathsf{Tails}(B) \to \mathsf{Tails}(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{GrMod}(B) & \xrightarrow{f_*} & \mathsf{GrMod}(A) \\ \big\downarrow & & \big\downarrow \\ \mathsf{Tails}(B) & \longrightarrow & \mathsf{Tails}(A). \end{array}$$

We write f_* for the functor $\mathsf{Tails}(B) \to \mathsf{Tails}(A)$; it is exact by Theorem A.13.7.

(2) If ${}_AB$ is finitely generated up to torsion, so is every finitely generated B-module. Hence f_* sends $\text{tails}(B)$ to $\text{tails}(A)$. □

It is easy to check that if $f : A \to B$ and $g : B \to C$ are graded algebra homomorphisms, then $(gf)_* = g_* \circ f_*$, whence $A \mapsto \mathsf{Tails}(A)$ is functorial—see section 1 of chapter 20 for a fuller discussion about morphisms in the category to which $\mathsf{Tails}(A)$ belongs.

PROPOSITION 2.5. *Let $u \in A$ be a homogeneous, regular, normalizing element of degree d, and let $f : A \to A[u^{-1}]$ be the inclusion. Then the induced functor $f_* : \text{tails}(A[u^{-1}]) \to \text{tails}(A)$ is fully faithful.*

$\boxed{\text{DETAILS}}$

PROOF. □

PROPOSITION 2.6. *Let I be a graded ideal of A.*

1. *The natural map $f : A \to A/I$ induces fully faithful embeddings $f_* : \mathsf{Tails}(A/I) \to \mathsf{Tails}(A)$. and $f_* : \text{tails}(A/I) \to \text{tails}(A)$.*
2. *If I is torsion as a left A-module, then f_* is an equivalence of categories.*

PROOF. A graded A/I-module is torsion as an A/I-module if and only if it is torsion as an A-module, so f_* is faithful (Proposition A.11.12). It is obvious that f_* is full.

Now suppose that I is torsion. To show that f_* is an equivalence of categories it suffices, by Theorem A.5.12, to show that every object of $\mathsf{Tails}(A)$ is isomorphic to $f_*\mathcal{N}$ for some $\mathcal{N} \in \mathsf{Tails}(A/I)$. If M is a graded A-module, then IM is torsion since I is (to see this, write IM as a quotient of $I \otimes_A F$ for some free-module F). Therefore, $\pi M \cong \pi(M/IM)$, whence $\pi M \cong f_*\mathcal{N}$ where $\mathcal{N} = \pi(M/IM) \in \mathsf{Tails}(A/I)$. □

We will usually view $\mathsf{Tails}(A/I)$ as a full subcategory of $\mathsf{Tails}(A)$.

In general, we cannot expect f^* to induce a functor from $\mathsf{Tails}(A)$ to $\mathsf{Tails}(B)$. The problem is that if M is a torsion A-module, then $f^*M = B \otimes_A M$ need not be a torsion B-module, so f^* does not map $\text{Tors}(A)$ to $\text{Tors}(B)$; equivalently, the composition $\mathsf{GrMod}(A) \to \mathsf{GrMod}(B) \to \mathsf{Tails}(B)$ need not send torsion A-modules to zero. A simple example illustrates this.

EXAMPLE 2.7. Let $f : A \to B$ be the inclusion of $A = k[x_0, x_1]$ in the polynomial ring $B = k[x_0, x_1, x_2]$. If $M = A/(x_0, x_1)$, then $f^*M = B/(x_0, x_1)$ is not torsion.

The geometric interpretation of this is illuminating. Although f induces a morphism of affine varieties $\mathbb{A}^3 \to \mathbb{A}^2$, namely the projection $(\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0, \alpha_1)$, this linear map does not send lines to lines: it collapses the line $\mathcal{V}(x_0, x_1)$ to the single point $(0, 0)$. Hence f does not induce a morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$: the first two coordinates of $(0,0,1) \in \mathbb{P}^2$ do not yield a point in $\mathbb{P}^1$. However, if we remove $(0,0,1)$ from $\mathbb{P}^2$, there is a well-defined morphism on its complement. This is 'the projection with center $(0,0,1)$'— see [**147**, page 40] for details.

The fact that f^* does not give a functor $\mathsf{Tails}(A) \to \mathsf{Tails}(B)$ should be interpreted as the algebraic analogue of the geometric fact that f does not induce a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. In terms of f_*, the problem is that f_* sends $B/(x_0, x_1)$ to the zero object in $\mathsf{Tails}(A)$.

In the previous example, B is not a finitely generated A-module.

PROPOSITION 2.8. *Let $f : A \to B$ be a homomorphism of graded k-algebras. If either*

- *B_A is finitely generated, or*
- *$\operatorname{coker} f$ is right bounded,*

then there are right exact covariant functors

$$f^* : \mathsf{Tails}(A) \to \mathsf{Tails}(B)$$

and

$$f^* : \mathrm{tails}(A) \to \mathrm{tails}(B).$$

PROOF. It suffices to show that, if $M \in \mathrm{Tors}(A)$, then $B \otimes_A M \in \mathrm{Tors}(B)$, because then the functor $f^* : \mathsf{GrMod}(A) \to \mathsf{GrMod}(B)$ will induce a functor $\mathsf{Tails}(A) \to \mathsf{Tails}(B)$; once f^* exists for the category Tails, it automatically restricts to tails, since $M \in \mathsf{grmod}(A)$ implies $B \otimes_A M \in \mathsf{grmod}(B)$.

When B_A is finitely generated, Lemma 13.3.13 provides the result.

Now suppose that $\operatorname{coker} f$ is right bounded. Let $L \in \mathsf{GrMod}(B)$. Since $f(A_{\geq n}) = B_{\geq n}$ for $n \gg 0$, an element of L is torsion with respect to A if and only if it is torsion with respect to B. Thus L is torsion as a B-module if and only if it is torsion as an A-module.

The first and last terms in the exact sequence

$$M = A \otimes_A M \xrightarrow{f \otimes \mathbb{1}_M} B \otimes_A M \to B/f(A) \otimes_A M \to 0$$

is in $\mathrm{Tors}(A)$, whence so is the middle term, since $\mathrm{Tors}(A)$ is dense. Hence, by the previous paragraph, $B \otimes_A M$ is torsion as a B-module, as required. $\square$

The hypothesis in Proposition 2.8 that $\operatorname{coker}(f)$ be right bounded cannot be weakened to the requirement that it simply be a torsion A-module. To see this, let f be the inclusion of the subalgebra $A = k + xB$ in the polynomial ring $B = k[x, y]$. Since B/A is annihilated by $A_{\geq 1}$, $\operatorname{coker}(f)$ is torsion, but $B \otimes_A (A/A_{\geq 1}) \cong B/xB$, so f^* does not send torsion modules to torsion modules; that is, f^* does not induce a functor from $\mathsf{Tails}(A)$ to $\mathsf{Tails}(B)$.

The adjointness of f^* and f_* ensures that there are natural transformations

$$f^* f_* \to \mathbb{1}_{\mathsf{GrMod}(B)} \qquad \text{and} \qquad \mathbb{1}_{\mathsf{GrMod}(A)} \to f_* f^*$$

induced by the maps

$$B \otimes_A N \to N \qquad \text{and} \qquad M \to B \otimes_A M$$

defined by

$$b \otimes n \mapsto n \qquad \text{and} \qquad m \mapsto 1 \otimes m$$

where $M \in \mathsf{GrMod}(A)$ and $N \in \mathsf{GrMod}(B)$. The next result and its Corollary show that, under suitable hypotheses, these natural transformations induce natural equivalences at the level of $\mathsf{Tails}(-)$.

Proposition 2.9. [**12**, Proposition 2.5] *Let $f : A \to B$ be homomorphism of graded k-algebras such that* $\ker f$ *is torsion and* $\operatorname{coker} f$ *is right bounded. Then* $\mathsf{Tails}(A) \cong \mathsf{Tails}(B)$ *and* $\operatorname{tails}(A) \cong \operatorname{tails}(B)$.

Proof. By Proposition 2.6, we can replace A by $A/\ker f$. By Propositions 2.4 and 2.8, f_* and f^* induce functors between $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$. We will show these are equivalences; under the hypotheses of the Proposition, f^* and f_* restrict to the subcategories tails, so will then induce equivalences between $\operatorname{tails}(A)$ and $\operatorname{tails}(B)$ also.

Let M be a left A-module. The first and last terms in the exact sequence

$$\operatorname{Tor}_1^A(B/A, M) \to A \otimes_A M \xrightarrow{f\otimes 1\!\!1_M} B \otimes_A M \to B/A \otimes_A M \to 0$$

are in $\operatorname{Tors}(A)$, so the map $M = A \otimes_A M \to B \otimes_A M$ induces an isomorphism in $\mathsf{Tails}(A)$. Hence the natural transformation $1\!\!1_{\mathsf{Tails}(A)} \to f_* f^*$ is a natural equivalence.

To show that the natural transformation $f^* f_* \to 1\!\!1_{\mathsf{Tails}(B)}$ is a natural equivalence it suffices to show that $\pi(B \otimes_A N) \to \pi N$ is an isomorphism for all $N \in \mathsf{GrMod}(B)$. Since $\operatorname{coker}(B \otimes_A N \to N) = 0$, it suffices to show that the kernel is torsion. In the commutative diagram

$$\begin{array}{ccc} A \otimes_A N & \xrightarrow{f\otimes 1\!\!1_N} & B \otimes_A N \\ \downarrow & & \downarrow \\ N & \xrightarrow{1\!\!1_N} & N \end{array}$$

the horizontal and left hand maps give isomorphisms in $\mathsf{Tails}(A)$, hence so does the right hand map. Thus $\ker(B \otimes_A N \to N)$ is torsion as an A-module, whence torsion as a B-module by the proof of Proposition 2.8. This completes the proof of the equivalence between $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$. $\square$

As in the remark after Proposition 13.3.8, the hypothesis in Proposition 2.9 that $\operatorname{coker}(f)$ be right bounded cannot be replaced by the assumption that it merely be torsion as an A-module.

Corollary 2.10. *Let A be a graded k-algebra. The inclusion $A_{\geq 0} \subset A$ induces equivalences of categories* $\mathsf{Tails}(A_{\geq 0}) \cong \mathsf{Tails}(A)$ *and* $\operatorname{tails}(A_{\geq 0}) \cong \operatorname{tails}(A)$.

Example 2.11. Let $A = k[X]$, the polynomial ring in one variable, with $\deg(X) = 1$. We will show that $\mathsf{Tails}(A)$ is equivalent to $\mathcal{V}ec(k)$, the category of vector spaces over k. By Corollary 2.10, if $B = k[X, X^{-1}]$, then $\mathsf{Tails}(A) \cong \mathsf{Tails}(B)$. However, $\operatorname{Tors}(B)$ consists only of the zero module, since X is a unit in B. Thus $\mathsf{Tails}(B) \cong \mathsf{GrMod}(B)$. Moreover, B is strongly graded so, by Theorem 16.6.7, $\mathsf{GrMod}(B) \cong \mathsf{Mod}(B_0) = \mathsf{Mod}(k)$, which proves the claim.

Proposition 2.12. [**173**, Proposition 2.7] *Let $A \subset B$ be left noetherian, $\mathbb{N}$-graded k-algebras such that B/A is a finitely generated right A-module and a torsion left A-module. Then* $\operatorname{tails}(A) \cong \operatorname{tails}(B)$.

Proof. Let $f : A \to B$ be the inclusion. By Proposition 2.9, f_* and f^* produce an equivalence between $\mathsf{Tails}(A)$ and $\mathsf{Tails}(B)$.

Write $B = \sum_{i=1}^m b_i A$. Since A is left noetherian, Proposition 13.3.8 ensures that every element of B/A is torsion, so there exists n such that $A_{\geq n} b_i \subset A$ for all i; in fact, $A_{\geq n} b_i \subset A \cap B_{\geq n} = A_{\geq n}$. Therefore

$$A_{\geq n} B \;=\; \sum_{i=1}^{m} A_{\geq n} b_i A \;\subset\; A_{\geq n},$$

whence $A_{\geq n}$ is an A-B-bimodule.

It is clear that f^* sends finitely generated modules to finitely generated modules. On the other hand, if $N \in \mathsf{grmod}(B)$, then N need not be a finitely generated A-module. However, if $N = Bn_1 + \cdots + Bn_s$ and $N' = An_1 + \cdots + An_s$, then $A_{\geq n}$ annihilates N/N'. Thus $\pi N \cong \pi N' \in \operatorname{tails}(A)$. Hence f_* sends $\operatorname{tails}(B)$ to $\operatorname{tails}(A)$, so f^* and f_* give an equivalence between $\operatorname{tails}(A)$ and $\operatorname{tails}(B)$. □

Corollary 2.13. *Let $f : A \to B$ be a homomorphism of graded k-algebras. If B_A is finitely generated and* $\ker(f)$ *and* $\operatorname{coker}(f)$ *are torsion as left A-modules, then* $\mathsf{Tails}(A)$ *and* $\mathsf{Tails}(B)$ *are equivalent.*

Proof. This follows from Propositions 2.6 and 2.12. □

We end this section with an example showing that if $f : A \to B$ is a graded algebra homomorphism, then it is a delicate question whether or not the functors f^* and f_* on GrMod induce functors between $\operatorname{tails}(A)$ and $\operatorname{tails}(B)$.

Example 2.14. Let A and B be the algebras defined in Example 6.2.3; that is $A = k + xB$ where $B = k[x, y]$ with defining relation $yx - qxy = y^2$, and $0 \neq q \in k$ is not a root of unity. We also define the polynomial extensions $A[z]$ and $B[z]$. Consider the inclusions

$$f : A \to B \qquad \text{and} \qquad g : A[z] \to B[z].$$

The following properties were established in 6.2.3:

- A, B, $A[z]$ and $B[z]$ are all two-sided noetherian;
- B_A is finitely generated;
- ${}_AB$ is not finitely generated, but is finitely generated up to torsion;
- $B[z]_{A[z]}$ is finitely generated;
- ${}_{A[z]}B[z]$ is not finitely generated, nor is it finitely generated up to torsion.

The results in this section have the following consequences:

- $f^* : \operatorname{tails}(A) \to \operatorname{tails}(B)$ and $f_* : \operatorname{tails}(B) \to \operatorname{tails}(A)$ exist, by (2.8) and (2.4);
- for *right* modules $f^* : \operatorname{tails}(A^{\mathrm{op}}) \to \operatorname{tails}(B^{\mathrm{op}})$ does *not* exist because although A/xA is torsion, $(A/xA) \otimes_A B \cong B/xB$ is not torsion, but $f_* : \operatorname{tails}(B^{\mathrm{op}}) \to \operatorname{tails}(A^{\mathrm{op}})$ does exist because B_A is finitely generated;
- $g^* : \operatorname{tails}(A[z]) \to \operatorname{tails}(B[z])$ exists because $B[z]_{A[z]}$ is finitely generated (2.8), but $g_* : \operatorname{tails}(B[z]) \to \operatorname{tails}(A[z])$ does not exist because ${}_{A[z]}B[z]$ is not finitely generated up to torsion.

3. Ext groups in Tails

The development of homological machinery in the category $\mathsf{Tails}(A)$ requires an understanding of injectives—they are related in a simple way to the injectives in $\mathsf{GrMod}(A)$. First, recall that $\mathsf{GrMod}(A)$ has enough injectives (Theorem 11.11.14).

PROPOSITION 3.1. *Let A be a graded k-algebra.*

1. $\mathsf{Tails}(A)$ *has enough injectives.*
2. *If $\mathcal{Q} \in \mathsf{Tails}(A)$ is injective, then $\omega\mathcal{Q}$ is injective and torsion-free.*
3. *If $Q \in \mathsf{GrMod}(A)$ is injective and torsion-free, then πQ is injective, and $Q \cong \omega\pi Q$.*

PROOF. This is Theorem A.13.16; the isomophism $Q \cong \omega\pi Q$ is not stated explicitly there but follows from the definition of ω. $\square$

By Proposition 13.3.11, graded injective modules over a left noetherian $\mathbb{N}$-graded algebra decompose as the direct sum of their torsion submodule, and a torsion-free module.

Since $\mathsf{Tails}(A)$ has enough injectives, we may define the right derived functors of $\operatorname{Hom}_{\mathsf{Tails}}(\mathcal{F}, -)$ as follows.

Definition 3.2. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Tails}(A)$, and let $\mathcal{G} \to \mathcal{E}^\bullet$ be an injective resolution. Define

$$\operatorname{Ext}^i_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}) := \mathrm{h}^i(\operatorname{Hom}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{E}^\bullet)),$$

$$\underline{\operatorname{Hom}}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}[n]),$$
$$\underline{\operatorname{Ext}}^i_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Ext}^i_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}[n]).$$

We will usually omit the subscript $_{\mathsf{Tails}}$ when it is clear from the context what category we are in.

The next result expresses the Ext groups in $\mathsf{Tails}(A)$ in terms of the Ext groups in $\mathsf{GrMod}(A)$. For example, the adjoint isomorphism gives

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \operatorname{Hom}_{\mathrm{Gr}}(N, \omega\pi M).$$

By Proposition 13.3.11, if $M \to E^\bullet M$ is a minimal injective resolution, then there is an exact sequence of complexes $0 \to I^\bullet M \to E^\bullet M \to Q^\bullet M \to 0$ where $I^i M$ is the torsion submodule of $E^i M$, which is again injective, and $Q^i M$ is the torsion-free complement.

LEMMA 3.3. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, $M \in \mathsf{GrMod}(A)$ and $N \in \mathsf{grmod}(A)$. If $\mathcal{N} = \pi N$ and $\mathcal{M} = \pi M$, then*

1. $\operatorname{Ext}^i(\mathcal{N}, \mathcal{M}) = \mathrm{h}^i(\operatorname{Hom}_{\mathrm{Gr}}(N, Q^\bullet M))$*, where $Q^\bullet M$ is the torsion-free part of the minimal injective resolution of M;*
2. $\underline{\operatorname{Ext}}^i(\mathcal{N}, \mathcal{M}) \cong \varinjlim \underline{\operatorname{Ext}}^i_A(N_{\geq n}, M)$ *for $i \geq 0$.*

PROOF. (1) Adopt the notation prior to the statement of the result. By Proposition 3.1(3), each $\mathcal{Q}^j := \pi Q^j$ is injective in $\mathsf{Tails}(A)$ and $\omega\mathcal{Q}^i \cong Q^i$. Since π is an exact functor, $\mathcal{M} \to \mathcal{Q}^\bullet$ is an injective resolution in $\mathsf{Tails}(A)$. By the adjoint isomorphism, we have

$$\operatorname{Hom}_{\mathrm{Gr}}(N, Q^\bullet) = \operatorname{Hom}_{\mathrm{Gr}}(N, \omega\mathcal{Q}^\bullet) \cong \operatorname{Hom}_{\mathsf{Tails}}(\mathcal{N}, \mathcal{Q}^\bullet),$$

so the result follows.

(2) In the general case we have

$$\begin{aligned}
\varinjlim \underline{\mathrm{Ext}}^i_A(N_{\geq n}, M) &= \varinjlim \mathrm{h}^i(\underline{\mathrm{Hom}}_A(N_{\geq n}, Q^\bullet \oplus I^\bullet)) \\
&= \mathrm{h}^i(\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, Q^\bullet \oplus I^\bullet)) \\
&= \mathrm{h}^i(\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, Q^\bullet)) \bigoplus \mathrm{h}^i(\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, I^\bullet)) \\
&= \mathrm{h}^i(\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, Q^\bullet)) \quad \text{by the } i=0 \text{ case} \\
&= \mathrm{h}^i(\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{Q}^\bullet)) \quad \text{by Proposition 1.5} \\
&= \underline{\mathrm{Ext}}^i(\mathcal{N}, \mathcal{M}),
\end{aligned}$$

as required. □

The technical condition χ_i, introduced in chapter 13, allows us to obtain much stronger information than that in Lemma 3.3. In particular, we have the following result.

Proposition 3.4. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra satisfying χ_i. Let $M, N \in \mathsf{grmod}(A)$, and write $\mathcal{N} = \pi N$ and $\mathcal{M} = \pi M$. If $j \leq i-1$, then*

1. *for all d, $\underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})_{\geq d} \cong \underline{\mathrm{Ext}}^j_A(N_{\geq r}, M)_{\geq d}$ for $r \gg 0$;*
2. *the kernel and cokernel of the natural map $\underline{\mathrm{Ext}}^j_A(N, M) \to \underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})$ are right bounded;*
3. *if A is locally finite, then $\underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})$ is locally finite.*

Proof. (1) For $r \gg 0$, there are isomorphisms

$$\underline{\mathrm{Ext}}(\mathcal{N}, \mathcal{M})_{\geq d} \cong \varinjlim \underline{\mathrm{Ext}}^j_A(N_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}^j_A(N_{\geq r}, M)_{\geq d},$$

the first given by Lemma 3.3, and the second by Corollary 4.5.

(2) It suffices to show that the kernel and cokernel of

$$\underline{\mathrm{Ext}}^j_A(N, M)_{\geq d} \to \underline{\mathrm{Ext}}^j(\mathcal{N}, \mathcal{M})_{\geq d}$$

are right bounded for some d. By (1), we may replace the right hand side by $\underline{\mathrm{Ext}}^j_A(N_{\geq r}, M)_{\geq d}$ for a suitably large r. Thus the map in question is the middle map in the exact sequence

$$\begin{aligned}
\underline{\mathrm{Ext}}^j_A(N/N_{\geq r}, M)_{\geq d} \to \underline{\mathrm{Ext}}^j_A(N, M)_{\geq d} &\to \\
&\underline{\mathrm{Ext}}^j_A(N_{\geq r}, M)_{\geq d} \to \underline{\mathrm{Ext}}^{j+1}_A(N/N_{\geq r}, M)_{\geq d}.
\end{aligned}$$

Since $\chi_i(M)$ holds, and $N/N_{\geq r}$ is bounded, the first and last terms are right bounded by Lemma 13.4.1. Hence the kernel and cokernel of the middle map are right bounded.

(3) Since $M \in \mathsf{grmod}(A)$, it is locally finite if A is, whence $\underline{\mathrm{Ext}}^j_A(N_{\geq r}, M)$ is locally finite by Lemma 11.6.2(2). Hence the result follows from (1). □

4. The right adjoint of π

In this section, A is always a left noetherian, $\mathbb{N}$-graded k-algebra.

We saw in section 1 that π has a right adjoint ω. Among the results established in section 13 of Appendix A are the following:

- $\omega\pi M$ is isomorphic to the largest graded submodule, H say, of the injective envelope of $\bar{M} = M/\tau M$ such that $\bar{M} \subset H$ and $H/\bar{M}$ is torsion;
- for each $\mathcal{F} \in \mathsf{Tails}(A)$, $\pi\omega\mathcal{F} \cong \mathcal{F}$;
- for each $\mathcal{F} \in \mathsf{Tails}(A)$, $\omega\mathcal{F}$ is torsion-free.

The first result in this section gives a more useful description of $\omega\pi M$, which has the advantage of allowing an explicit description of $\operatorname{coker}(M \to \omega\pi M)$. The rest of this section is devoted to the question of when this cokernel is right bounded.

THEOREM 4.1. *Suppose that A is a left noetherian, $\mathbb{N}$-graded k-algebra. If $M \in \mathsf{GrMod}(A)$, then*

1.
$$\omega\pi M \cong \varinjlim \underline{\operatorname{Hom}}_A(A_{\geq n}, M);$$

2. *there is an exact sequence*
$$0 \to \tau M \to M \xrightarrow{f} \omega\pi M \to \varinjlim \underline{\operatorname{Ext}}^1_A(A/A_{\geq n}, M) \to 0. \tag{4-1}$$

PROOF. (1) The result is given by the following equalities:

$$\begin{aligned}
\omega\pi M &= \operatorname{Hom}_A(A, \omega\pi M) \\
&= \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{Gr}}(A, \omega\pi M[j]) \quad \text{since } {}_AA \text{ is finitely generated} \\
&= \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{Tails}}(\pi A, \pi M[j]) \quad \text{by adjointness} \\
&= \bigoplus_{j\in\mathbb{Z}} \varinjlim \operatorname{Hom}_{\mathrm{Gr}}(A_{\geq n}, M[j]) \quad \text{by Proposition (1.5)} \\
&= \varinjlim \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{Gr}}(A_{\geq n}, M[j]) \\
&= \varinjlim \underline{\operatorname{Hom}}_A(A_{\geq n}, M).
\end{aligned}$$

(2) For each n, there is an exact sequence $0 \to A_{\geq n} \to A \to A/A_{\geq n} \to 0$. Applying the functor $\underline{\operatorname{Ext}}^\bullet_A(-, M)$ gives an exact sequence

$$0 \to \underline{\operatorname{Hom}}_A(A/A_{\geq n}, M) \to M \to \underline{\operatorname{Hom}}_A(A_{\geq n}, M) \to \underline{\operatorname{Ext}}^1_A(A/A_{\geq n}, M) \to 0$$

of left A-modules. Since A is left noetherian, every element of τM is torsion (13.3.8), so is in the image of the map $\underline{\operatorname{Hom}}_A(A/A_{\geq n}, M) \to M$ for some n. Hence the direct limit of these exact sequences is (4-1), which is exact because $\varinjlim$ is an exact functor when the index set is directed. $\square$

EXAMPLE 4.2. If $A = k[X]$, then $E(A) \cong k[X, X^{-1}]$ (Example 11.5.6). It is clear that E/A is torsion, so $\omega\pi A = k[X, X^{-1}]$. Notice that $\operatorname{coker}(A \to \omega\pi A)$ is right bounded, but not left bounded.

In contrast, if A is the polynomial ring in two or more variables, $\omega\pi A \cong A$; this is a special case of the next result.

COROLLARY 4.3. *If A is an Artin-Schelter regular k-algebra of global dimension ≥ 2, then $\omega\pi A = A$.*

PROOF. First, $\underline{\mathrm{Hom}}_A(k, A) = 0$, so $\tau A = 0$. Second, $\underline{\mathrm{Ext}}^1_A(k, A) = 0$, from which it follows that $\underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, A) = 0$ for all $n > 0$. Thus the first and last terms in the exact sequence (4-1) are zero. The middle arrow is therefore an isomorphism, as required. □

The left A-module structure on the final term in the exact sequence (4-1) arises because $A/A_{\geq n}$ is an A-A-bimodule; in particular, the right structure induces a left $A/A_{\geq n}$-module structure on $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)$, so it is torsion as a left A-module.

It is important to notice that ω is defined on $\mathsf{Tails}(A)$, *not* on the subcategory tails(A). As Example 4.2 shows, the restriction of ω to tails(A) will not generally take values in $\mathsf{grmod}(A)$. The point is that neither the injective envelope of M, nor $\omega\pi M$, need be finitely generated even if M is. The most we can ask for is that if M is finitely generated, then so is $(\omega\pi M)_{\geq d}$ for some d. Unfortunately this is not true in full generality (see Example 5.3).

The next theorem shows how the technical condition χ_1° (Definition 13.4.2) is related to the good behavior of the functor $\omega\pi$.

THEOREM 4.4. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and let $M \in$ $\mathsf{grmod}(A)$. Then*

1. $\chi_1^\circ(M)$ *holds if and only if* $\operatorname{coker}(M \to \omega\pi M)$ *is right bounded;*
2. $\chi_1(M)$ *holds if and only if* $(\omega\pi M)_{\geq d}$ *is finitely generated for all* $d \in \mathbb{Z}$.

If A is also locally finite, these four conditions are equivalent.

PROOF. We will use the exact sequence (4-1) to identify $\operatorname{coker}(M \to \omega\pi M)$ with $\varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)$.

(1) ($\Leftarrow$) By Proposition 13.4.3(2), $\underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)$ has a right bound which is independent of n, so the direct limit of these is right bounded.

($\Rightarrow$) This is part of Corollary 13.4.7.

(2) ($\Rightarrow$) Let $f : M \to \omega\pi M$ be the natural map. By Proposition 13.4.3(3), $\chi_1^\circ(M)$ holds so, by Corollary 13.4.5, $\operatorname{coker}(f)_{\geq d} \cong \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)_{\geq d}$ for $n \gg 0$. But $\chi_1(M)$ holds, so this latter module is finitely generated, and hence noetherian. Hence the middle term in $M_{\geq d} \to (\omega\pi M)_{\geq d} \to \operatorname{coker}(f)_{\geq d}$ is also noetherian, as claimed.

($\Leftarrow$) Since $(\omega\pi M)_{\geq d}$ is finitely generated, so is its image $\varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)$. But this is a torsion module, so it is right bounded. Taking the direct limit of the sequences (13.4-1) above, it follows that $\chi_1^\circ(M)$ holds. Hence, by Corollary 13.4.5, $\varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)_{\geq d}$ for $n \gg 0$. Therefore, this is finitely generated, as claimed. □

I like to think of a module M as a mutilated version of $\omega\pi M$, and the functor $M \mapsto \omega\pi M$ as being like plastic surgery which grafts back onto M those pieces of $\omega\pi M$ which have been chopped off. If $M_{\geq n} = (\omega\pi M)_{\geq n}$, this mutilation occurs only at the top of M, and the tail of M remains intact.

A special case concerns A itself. We will see that, if A is $\mathbb{N}$-graded and satisfies some other reasonable hypotheses, then $B := (\omega\pi A)_{\geq 0}$ has a graded algebra structure, B satisfies χ_1, and A is of finite codimension in B; this ensures that $\mathsf{Tails}(A) \cong \mathsf{Tails}(B)$. Thus one should think of B as an improved version of A and, to the extent that the category $\mathsf{Tails}(A)$ is concerned, we can replace A by the *better* algebra B.

EXERCISES

4.1 Show that $\underline{\mathrm{Ext}}^1_A(N, \omega\pi M) = 0$ for all torsion modules N.

5. What is non-commutative projective geometry?

The informal discussion in this section is an attempt to provide some context for the material in the this and subsequent chapters.

We begin by reviewing some features of geometry and explain how these are translated into the non-commutative setting. First, the objects of geometry. Let A be the commutative polynomial ring on $n+1$ indeterminates, viewed as the homogeneous coordinate ring of $\mathbb{P}^n$. The closed subschemes of $\mathbb{P}^n$ correspond to ideals in A, or, equivalently, to cyclic A-modules. There are other important geometric objects such as vector bundles on subvarieties of $\mathbb{P}^n$, and these too correspond to A-modules. Thus modules play the role of the geometric objects. Second, geometry concerns incidence relations. Incidence relations correspond to homomorphisms between modules: if $X \subset Y$ are closed subvarieties of $\mathbb{P}^n$, then the homogeneous coordinate ring of Y maps onto that of X, the surjective map being the algebraic counterpart of the geometric inclusion. Thus one is led to consider the category of A-modules and A-module homomorphisms. However, our concern is projective geometry, so the modules should be graded. Thus, the category $\mathsf{GrMod}(A)$ is a first approximation to the geometry. There is a further important detail: if I is an ideal of finite codimension in A, then $\mathcal{V}(I)$ is empty, so finite dimensional modules correspond to the empty set. This provides the motivation for passing to the quotient category $\mathsf{Tails}(A)$. Thus, geometric objects are the same thing as objects in $\mathsf{Tails}(A)$ and incidence relations are morphisms in $\mathsf{Tails}(A)$. The following result is a formal justification of these heuristics.

THEOREM 5.1 (Serre's Theorem). [**144**] *Let A be the quotient of the commutative polynomial ring $k[x_0, \dots, x_n]$, generated in degree one, by a graded ideal I, let X be the subscheme of $\mathbb{P}^n$ defined by the ideal, and $\mathcal{O}_X$ denote the sheaf of regular functions on X. Then the category $\mathsf{Coh}(X)$ of sheaves of coherent $\mathcal{O}_X$-modules is equivalent to the category of finitely generated graded A-modules modulo the full subcategory of finite dimensional modules.*

If A is any graded k-algebra, not necessarily commutative, we may still define the categories $\mathsf{Tails}(A)$ and tails(A). We are led to the following definition. Given a left noetherian, connected, graded k-algebra A, the projective scheme of A is the pair $(\mathsf{Tails}(A), \pi A)$. It is also natural to consider the triple $(\mathsf{Tails}(A), \pi A, [1])$, which we call a polarized scheme.

We will now describe the equivalence of categories in Serre's Theorem. First there are functors

$$\begin{aligned} \tilde{} &: \mathsf{GrMod}A \to \mathsf{QCoh}X \\ \Gamma_* &: \mathsf{QCoh}X \to \mathsf{GrMod}A, \end{aligned}$$

which are analogous to $\pi : \mathsf{GrMod}(A) \to \mathsf{Tails}(A)$ and its right adjoint ω. The equivalence in Serre's Theorem is implemented by $\pi \circ \Gamma_* : \mathsf{QCoh}X \to \mathsf{Tails}(A)$.

$$\begin{tikzcd}
\mathsf{GrMod}(A) \arrow[r, shift left, "\pi"] \arrow[d, shift left, "\sim"] & \mathsf{Tails}(A) \arrow[l, shift left, "\omega"] \\
\mathsf{QCoh}(X) \arrow[u, shift left, "\Gamma_*"] &
\end{tikzcd}$$

Now, we define the functors Γ_* and $\sim$ when A is a domain; that is, when X is an irreducible Zariski closed subset of $\mathbb{P}^n$. First, X is covered by the open affine sets

$$X_f := \{x \in X \mid f(x) \neq 0\},$$

where $f \in A$ ranges over the homogenous elements of positive degree. A sheaf on X can be described by describing it on each X_f, together with the glueing isomorphisms on the intersections X_{fg}. (The glueing data is obvious in what follows, so we supress it.) For example, the structure sheaf, $\mathcal{O}_X$ is defined by

$$\mathcal{O}(X_f) = A[f^{-1}]_0,$$

the degree zero component of the localization, which consists of the functions in $k(X)$, the field of rational functions on X, having no poles on X_f. Recall that $k(X)$ is the subfield of $\mathrm{Fract}(A)$ consisting of elements F/G for which $F, G \in A$ are homogeneous of the same degree.

If $M \in \mathsf{GrMod}(A)$, then

$$\widetilde{M}(X_f) = (A[f^{-1}] \otimes_A M)_0 = M[f^{-1}]_0.$$

Clearly, $\widetilde{M}(X_f)$ is a module over $\mathcal{O}(X_f)$, so $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. A graded A-module homomorphism $M \to N$ induces homomorphisms $M[f^{-1}] \to N[f^{-1}]$ which restrict to the degree zero components, thus yielding an $\mathcal{O}_X$-module map $\widetilde{M} \to \widetilde{N}$. Therefore the rule $M \mapsto \widetilde{M}$ is a covariant functor from graded A-modules to quasi-coherent sheaves of $\mathcal{O}_X$-modules.

Before defining Γ_* we need to introduce the sheaf $\mathcal{O}_X(1)$; the notation for it is a little misleading since its depends on A, not just on X as an abstract scheme. For each $n \in \mathbb{Z}$, define

$$\mathcal{O}(n) := \widetilde{A[n]},$$

and for each $\mathcal{O}_X$-module $\mathcal{F}$ define

$$\mathcal{F}(n) := \mathcal{O}(n) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Thus $\mathcal{O}(n)(X_f) = A[f^{-1}]_n$, and $\widetilde{M}(n) = \widetilde{M[n]} = M[f^{-1}]_n$. We call $\mathcal{O}(1)$ Serre's twisting sheaf. Notice that $\mathcal{O}(0) = \mathcal{O}$, and that $\mathcal{O}(m) \otimes_{\mathcal{O}} \mathcal{O}(n) \cong \mathcal{O}(m+n)$. If $\mathcal{F} \in \mathsf{QCoh}X$, define

$$\Gamma_* \mathcal{F} := \bigoplus_{n=-\infty}^{\infty} \mathrm{H}^0(X, \mathcal{F}(n)),$$

with grading $(\Gamma_*\mathcal{F})_n := \mathrm{H}^0(X, \mathcal{F}(n))$; if $a \in A_p$ and $m \in \mathrm{H}^0(X, \mathcal{F}(q))$, then $a.m$ is defined as follows: since a is a global section of $\mathcal{O}(p)$, $a \otimes m$ is a global section of $\mathcal{O}(p) \otimes \mathcal{F}(q) = \mathcal{F}(p+q)$, so

$$a.m := a \otimes m \in \mathrm{H}^0(\mathcal{F}(p+q)).$$

This makes $\Gamma_*\mathcal{F}$ a graded A-module. If $f : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{O}_X$-module homomorphism then there is an induced map $\Gamma_* f : \Gamma_*\mathcal{M} \to \Gamma_*\mathcal{N}$, so Γ_* is a functor from quasi-coherent $\mathcal{O}_X$-modules to graded A-modules.

If M is a finite dimensional graded A-module then, for all homogeneous $f \in A$ of positive degree, $f^r M = 0$ for $r \gg 0$, so $M[f^{-1}] = 0$, whence $\widetilde{M} = 0$. Hence $\widetilde{\ }$ factors through $\mathsf{Tails}(A)$. We will show in the next chapter that the factorization map $\mathsf{Tails}(A) \to \mathsf{QCoh}X$ is an equivalence of categories, thus proving Serre's Theorem.

One of the main tools of algebraic geometry, namely Čech cohomology, can be formulated in the non-commutative setting. In [**144**], Serre defines the Čech cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ with respect to an open covering of X. He proves these are the right derived functors of the global section functor $\mathrm{H}^0(X, -)$, and establishes their basic properties:

1. $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{Ext}^i_{\mathsf{QCoh}X}(\mathcal{O}_X, \mathcal{F})$;
2. $\dim_k \mathrm{H}^i(X, \mathcal{F}) < \infty$;
3. $\mathrm{H}^i(X, \mathcal{F}(n)) = 0$ for all $i > 0$ and all $n \gg 0$;
4. $\mathcal{O}_X(1)$ is ample;
5. $A_n = \mathrm{H}^0(X, \mathcal{O}(n))$ for $n \gg 0$.

If A is not commutative, we will use (1) as the *definition* of the cohomology groups: if $\mathcal{F} \in \mathrm{tails}(A)$, and $\mathcal{A} = \pi A$, then we define $\mathrm{H}^i(\mathcal{F}) := \mathrm{Ext}^i_{\mathsf{Tails}}(\mathcal{A}, \mathcal{F})$. Of course, since $\mathsf{Tails}(A)$ has enough injectives we make sense of this by taking homology after applying $\mathrm{Hom}_{\mathsf{Tails}}(\mathcal{A}, -)$ to an injective resolution of $\mathcal{F}$. We will prove that (2)-(5) hold, when A satisfies the property χ. An immediate consequence of defining $\mathrm{H}^i(-)$ by (1) is that, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathsf{Tails}(A)$, then there is a long exact sequence

$$0 \to \mathrm{H}^0(\mathcal{F}') \to \cdots \quad \cdots \to \mathrm{H}^i(\mathcal{F}') \to \mathrm{H}^i(\mathcal{F}) \to \mathrm{H}^i(\mathcal{F}'') \to \mathrm{H}^{i+1}(\mathcal{F}') \to \cdots .$$

Serre's proof that the Čech cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ have the properties (1)-(5) above begins with an algebraic interpretation of these groups. If $\mathcal{F} = \widetilde{M}$, then $\mathrm{H}^i(X, \mathcal{F}) \cong \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq j}, M)$ [**144**, Corollaire 2, page 263]; we prove exactly this result in Proposition 20.1.3. In any case, this explains why the properties of direct limits of such Ext groups are examined in chapter 19.

One other point should be mentioned. Although there is a natural ring structure on $A' := \Gamma_*\mathcal{O}_X$ making it a graded k-algebra, A' may be strictly larger than A (see for example [**76**, Chapter II, Ex. 5.14] and also ??? below).

The adjective 'quantum' is commonly used to denote a non-commutative analogue of a commutative object; we followed this convention when speaking of the quantum affine plane, and quantum affine n-space. This is a convenient heuristic, which we will continue to use. If A is a noetherian, quadratic k-algebra, having Hilbert series $(1-t)^{-n}$ we will call $(\mathsf{Tails}(A), \pi A, [1])$ a quantum projective space. Thus, if A is the coordinate ring of a quantum plane or a Jordan plane we will refer to this pair as a quantum projective line; iterated Ore extensions provide a large class of quantum $\mathbb{P}^n$'s. It is not known whether every non-commutative scheme is (isomorphic to) a closed subscheme of some quantum $\mathbb{P}^n$.

One goal of non-commutative projective geometry is to characterize, describe and classify the quantum projective spaces. To date, this has only been carried out for $\mathbb{P}^1$ and $\mathbb{P}^2$, and only under some further (natural) hypotheses on A. Although we know several examples of quantum $\mathbb{P}^n$s for $n > 2$, we are far from having any idea of their typical features. A second, and related, goal is to understand the intrinsic geometry of quantum $\mathbb{P}^n$'s. This involves understanding the subschemes of quantum $\mathbb{P}^n$'s. Several interesting examples have been studied in some detail, but it is probably fair to say that we have not yet sailed far from the commutative shores. A third goal is to understand the low dimensional non-commutative varieties. The dimension of the projective scheme of A is defined as $\mathrm{GKdim}(A) - 1$. Non-commutative curves are defined as those projective schemes arising from the A's of GK-dimension 2; Artin and Stafford are presently studying these. It appears that a complete classification is possible and that non-commutative curves correspond, roughly speaking, to orders over a (commutative) projective curve. Non-commutative surfaces ($\mathrm{GKdim}(A) = 3$) therefore seem likely to provide the first examples of non-commutative schemes which are far from being commutative. As mentioned before, the quantum $\mathbb{P}^2$'s have been classified and at least one family of these is far from the commutative theory, namely the projective schemes of the 3-dimensional Sklyanin algebras; we will call the associated projective scheme a Sklyanin $\mathbb{P}^2$.

What is a point of a non-commutative scheme? The answer is not clear yet. This is already a delicate question in the context of *commutative* schemes—the reader is referred to [**121**, pages 155-163] and to [**57**, pages 119-126] for a discussion of this issue, and how it was resolved by Grothendieck. We will simplify matters by assuming that A is defined over an algebraically closed field k. By a k-valued point we mean the projective scheme of $k[T]$, the polynomial ring in the indeterminate T, graded with $\deg(T) = 1$; thus we adopt the usual notion of point, and accordingly we will write $\mathrm{Spec}(k) = (\mathsf{Tails}(k[T]), \pi k[T])$. The delicacy really arises when we ask what we mean by a point of the scheme $X = (\mathsf{Tails}(A), \pi A,)$—the answer should be some kind of morphism $\mathrm{Spec}(k) \to X$.

Unfortunately, it is not yet clear what is the appropriate definition of a morphism in the category of non-commutative schemes. Similarly, it is not clear what the appropriate definition of a subscheme is, although the projective scheme of a quotient of A should, of course, be a closed subscheme of the projective scheme of A. The commutative theory provides some guidance.

A morphism of schemes is a pair $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consisting of a continuous map $f : X \to Y$ and a homomorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y; that is, for each open set $V \subset Y$, there is a ring homomorphism $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$, subject to some obvious compatibility conditions. Now, $(f, f^\sharp)$ induces a pair of functors f_*, the direct image functor, sending $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules, and f^*, the inverse image functor, sending $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules. Moreover, these are adjoints:

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Thus, in non-commutative algebraic geometry a morphism should probably be defined as a functor $\mathsf{Tails}(A) \to \mathsf{Tails}(B)$ which has a left adjoint, and is subject to some further conditions. Also recall [**76**, Lemma 2.10]: if $f : X \to Y$ is the inclusion of a closed subscheme of Y, or more generally an affine morphism of noetherian schemes, and $\mathcal{F}$ is a sheaf on X, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ where $f_*\mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside X.

Theorem 23.1.4 shows that, if X is a projective variety over an algebraically closed field, then the points of X are in bijection with the irreducible objects in $\mathsf{Tails}(A)$, where A is a homogeneous coordinate ring of X. Thus it would be reasonable to define the points of $\mathsf{Tails}(A)$ to be simply the irreducible objects in this category. If $\mathcal{F}$ is such an irreducible object, there is certainly a functor $\varphi : \mathsf{Tails}(k[T]) \to \mathsf{Tails}(A)$ sending $\pi k[T]$ to $\mathcal{F}$. However, it is by no means clear that this has a left adjoint.

By Exercise 1.2, $\mathsf{Tails}(k[T])$ has a unique irreducible object, and every object is a direct sum of copies of that irreducible object.

The comments above on behaviour of cohomology under the direct image functor suggest that a point of $\mathsf{Tails}(A)$ have no cohomology in dimension ≥ 1. This is because a point, being an irreducible object should correspond to a morphism $f_* : \mathsf{Tails}(k[T]) \to \mathsf{Tails}(A)$ and cohomology over $\mathsf{Tails}(k[T])$ vanishes in dimension ≥ 1.

EXERCISES

5.1 Let A be a graded quotient algebra of the commutative polynomial ring $k[x_0, \dots, x_n]$ with its usual grading. Let M be a finitely generated graded A-module.
- (a) If $x \in A_1$ is a regular element, show that $M[x^{-1}]_0$ is a finitely generated $A[x^{-1}]_0$-module.
- (b) Suppose now that A is the homogeneous coordinate ring of an irreducible projective variety $X \subset \mathbb{P}^n$. Show that $\widetilde{M}$ is a coherent $\mathcal{O}_X$-module.

CHAPTER 20

Non-commutative projective schemes

NOTATION . For a graded k-algebra A, we will write $\mathcal{A}$ for the distinguished object πA in $\mathsf{Tails}A$. We emphasize that $\mathcal{A}$ is *an object in* $\mathsf{Tails}A$, not a ring of any sort.

Let X be a scheme, $\mathcal{O}_X$ its structure sheaf, $\mathsf{QCoh}X$ the category of quasi-coherent $\mathcal{O}_X$-modules, and $\mathsf{Coh}X$ the category of coherent $\mathcal{O}_X$-modules. The main object of study in algebraic geometry is the pair $(\mathsf{QCoh}X, \mathcal{O}_X)$; the ring structure on $\mathcal{O}_X$ is irrelevant since it is the objects in the category $\mathsf{QCoh}X$ which are of interest, but we do require the distinguished object $\mathcal{O}_X$, so we can, for example, define the global sections of an $\mathcal{O}_X$-module $\mathcal{F}$ as $\mathrm{Hom}(\mathcal{O}_X, \mathcal{F})$, and its cohomology groups via the derived functors of $\mathrm{Hom}(\mathcal{O}_X, -)$. Guided by this, we define the projective scheme of a graded k-algebra A to be the pair $\mathrm{Proj}\, A := (\mathsf{Tails}A, \mathcal{A})$; we also define $\mathrm{proj}\, A := (\mathrm{tails}\, A, \mathcal{A})$, which is analogous to $(\mathsf{Coh}X, \mathcal{O}_X)$. If A is a finitely generated commutative graded k-algebra, generated in degree one, and X the scheme it determines, Serre's Theorem says that $(\mathsf{Coh}X, \mathcal{O}_X)$ and $(\mathrm{tails}\, A, \mathcal{A})$ are isomorphic. This result allows us to define the subject matter of algebraic geometry in terms which continue to make sense in the non-commutative setting. That is, we do not need a 'space' to do algebraic geometry, either commutative or non-commutative.

In section 1, the definition of the projective scheme of a graded algebra is followed by the definition of a map between such schemes, and several examples. We also treat *polarized* schemes—the polarized scheme of a graded algebra A is the triple $(\mathsf{Tails}A, \mathcal{A}, [1])$. The shift functor $[1]$ is an auto-equivalence of $\mathsf{Tails}A$; the results in (2.10) and (2.11) on Veronese subalgebras and twists suggest that it is natural, even essential, to consider other auto-equivalences on $\mathsf{Tails}A$. Section 3 begins with the definition of an ample auto-equivalence, which generalizes the usual notion of ampleness—if $\mathcal{L}$ is an ample $\mathcal{O}_X$-module, then the auto-equivalence $\mathcal{L} \otimes_{\mathcal{O}_X} -$ of $\mathsf{Coh}X$ is ample. For a commutative graded algebra, $[1]$ is always ample, but this is not so for a non-commutative algebra, and the ampleness of $[1]$ is related to the condition χ introduced in chapter 13.

1. Schemes

Definition 1.1. Let k be a commutative ring. A k-scheme is a pair $X = (\mathcal{C}, \mathcal{O})$, consisting of a small k-linear abelian category $\mathcal{C}$ and a distinguished non-zero object $\mathcal{O}$ in $\mathcal{C}$. The objects in $\mathcal{C}$ are called sheaves on X, and $\mathcal{O}$ is called the structure sheaf. We say that X is noetherian if every object in $\mathcal{C}$ is noetherian. The underlying category and object will often be denoted $\mathcal{C}_X$ and $\mathcal{O}_X$.

If X is a scheme in the usual sense of algebraic geometry [**76**, Chapter II, Section 2] we will call X a classical scheme. A classical scheme over a commutative ring k is a scheme in the sense of Definition 1.1, if by X we mean either of the pairs $(\mathsf{QCoh}X, \mathcal{O}_X)$ or $(\mathsf{Coh}X, \mathcal{O}_X)$.

Definition 1.2. For a graded k-algebra A,

- the general projective scheme of A is the pair $\operatorname{Proj} A := (\mathsf{Tails}A, \mathcal{A})$;
- the noetherian projective scheme of A is the pair $\operatorname{proj} A := (\operatorname{tails} A, \mathcal{A})$.

Definition 1.3. The affine schemes associated to a k-algebra R are $\operatorname{Spec} R := (\mathsf{Mod}R, {}_RR)$, and $\operatorname{spec} R := (\mathsf{mod}R, {}_RR)$.

PROPOSITION 1.4. *Let $X = (\mathcal{C}, \mathcal{O})$ be a k-scheme. Then there is a unique k-scheme $\bar{X}$, and a full, exact, embedding of categories $\mathcal{C} \to \mathcal{C}_{\bar{X}}$ such that*

1. *every sheaf on $\bar{X}$ is a direct limit of sheaves on X,*
2. *$\bar{X}$ has direct limits,*
3. *injective envelopes exist in $\bar{X}$,*
4. *if X is noetherian, then $\bar{X}$ is locally noetherian; i.e., all its objects are direct limits of noetherian objects, and the sheaves on X are precisely the sheaves on $\bar{X}$ which are noetherian;*
5. *if $Y = (\mathcal{D}, \mathcal{O}_Y)$ is another k-scheme, and $F : \mathcal{C} \to \mathcal{D}$ is a k-linear functor, then there is a unique functor $\bar{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ extending F.*

PROOF. Let $\overline{\mathcal{C}}$ denote the category of all contravariant left exact functors from $\mathcal{C}$ to $\mathsf{Mod}k$, and let $j_* : \mathcal{C} \to \overline{\mathcal{C}}$ be the Yoneda embedding $\mathcal{F} \mapsto \operatorname{Hom}_{\mathcal{C}}(-, \mathcal{F})$. □

A given projective scheme may arise from different commutative algebras. This leads to the notion of a polarization of a scheme; that is, the given scheme together with a particular embedding of it in some ambient projective space. As will become apparent, this idea is intimately related to the shift functor [1] arising from a graded algebra.

Definition 1.5. A polarized scheme is a triple $(\mathcal{C}, \mathcal{O}, s)$ consisting of

- a k-scheme $(\mathcal{C}, \mathcal{O})$, and
- an auto-equivalence $s : \mathcal{C} \to \mathcal{C}$, which we call a shift functor.

We will write $\mathcal{F}(n)$ to denote $s^n\mathcal{F}$ whenever this causes no confusion.

For us, the main examples satisfying the conditions in Definition 1.5 are the triples $(\mathsf{Tails}A, \mathcal{A}, s)$, $(\mathsf{QCoh}X, \mathcal{O}_X, s)$, and $(\mathsf{Mod}R, {}_RR, s)$, where s is an arbitrary shift functor. If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then $s = \mathcal{L} \otimes_{\mathcal{O}_X} -$ is a shift functor with inverse $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} -$, where $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$.

Definition 1.6. If A is a graded k-algebra, we will call the triples

$$(\mathsf{Tails}A, \mathcal{A}, [1]) \qquad \text{and} \qquad (\operatorname{tails} A, \mathcal{A}, [1])$$

the canonical polarizations of $\operatorname{Proj} A$ and $\operatorname{proj} A$, and refer to them as polarized schemes.

Definition 1.7. The dimension of a scheme X is defined as the Krull dimension of the category $\mathcal{C}_X$.

2. Maps between schemes

Definition 2.1. A morphism $f : X \to Y$ between k-schemes is a triple $(f^*, f_*, f^\sharp)$ consisting of

- k-linear functors $f^* : \mathcal{C}_Y \to \mathcal{C}_X$ and $f_* : \mathcal{C}_X \to \mathcal{C}_Y$, such that

- f^* is left adjoint to f_*, and
- an isomorphism $f^\sharp : f^*\mathcal{O}_Y \to \mathcal{O}_X$ in $\mathcal{C}_X$.

We call f^* the inverse image functor and f_* the direct image functor of f. If f^* is an equivalence of categories we say the schemes are isomorphic.

The morphisms from X to Y are themselves the objects in a category; the arrows $f \to g$ in this category are the natural transformations

$$\tau : f^* \Rightarrow g^*$$

for which the composition $f^*\mathcal{O}_Y \to g^*\mathcal{O}_Y \to \mathcal{O}_X$ equals $f^\sharp$. The natural equivalence class of a morphism is called a map from X to Y.

EXAMPLE 2.2. Let R and S be k-algebras and $f : R \to S$ a k-algebra homomorphism. Then there is an induced map $f : \operatorname{Spec} S \to \operatorname{Spec} R$. The functors in question are induction $f^* : \mathsf{Mod}R \to \mathsf{Mod}S$, $f^*M = S \otimes_R M$, and restriction f_*. These are an adjoint pair, and the multiplication map $S \otimes_R R \to S$ gives an isomorphism $f^*R \cong S$.

A particular case of this arises from the structure map $k \to R$, giving a map $i : \operatorname{Spec} k \to \operatorname{Spec} R$. Generally R is not a finite dimensional vector space, so there is not a map $\operatorname{spec} k \to \operatorname{spec} R$, even if R is noetherian.

EXAMPLE 2.3. Let $X = (\mathcal{C}, \mathcal{O})$ be a k-scheme. Suppose that $\mathcal{C}$ contains all small coproducts of $\mathcal{O}$. Then the structure map

$$i : X \to \operatorname{Spec}(k)$$

is defined as follows. First the direct image functor is

$$i_* = \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, -).$$

By Theorem A.7.11, i_* commutes with all small limits, so by Theorem A.9.4, it has a left adjoint, and we define i^* to be a left adjoint (recall that an adjoint is determined only up to natural equivalence). The map $k \to \operatorname{Hom}_X(\mathcal{O}, \mathcal{O})$ sending 1 to the identity map is an element of $\operatorname{Hom}_k(k, i_*\mathcal{O}) \cong \operatorname{Hom}_X(i^*k, \mathcal{O})$, so corresponds to a map $i^\sharp : i^*k \to \mathcal{O}$.

EXAMPLE 2.4. Do the points of $X = (\mathsf{Tails}A, \mathcal{A})$ correspond to maps $\operatorname{Spec}(k) \to X$? Let $\mathcal{F}$ be a point of X. Define $f^*\mathcal{G} = \operatorname{Hom}_X(\mathcal{G}, \mathcal{F})^*$. This is a right exact functor. Define $f^\sharp : f^*\mathcal{O}_X = \operatorname{Hom}(\mathcal{O}_X, \mathcal{F})^* \to k$ by ?????

On the other hand, suppose that $f : \operatorname{Spec}(k) \to X$ is a map. Is f_*k an irreducible object in X?

EXAMPLE 2.5. The requirement that $f^\sharp$ be an isomorphism in Definition 2.1 is forced on us by the following consideration. A morphism of schemes is a pair

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y),$$

where $f : X \to Y$ is a continuous map, and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of rings on Y. This data induces the inverse image functor $f^* : \mathsf{QCoh}Y \to \mathsf{QCoh}X$, and its right adjoint, the direct image functor $f_* : \mathsf{QCoh}X \to \mathsf{QCoh}Y$. Moreover, by the adjointness property $f^\sharp$ corresponds to an isomorphism $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ of $\mathcal{O}_Y$-modules. Thus we obtain a map $(\mathsf{QCoh}Y, \mathcal{O}_Y) \to (\mathsf{QCoh}X, \mathcal{O}_X)$ in the sense of Definition 2.1. Notice that the rule $X \mapsto (\mathsf{QCoh}X, \mathcal{O}_X)$ is *contravariant.*

EXAMPLE 2.6. Our definition allows too many maps. Let X be a projective scheme over the field k, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $H^0(\mathcal{F}) = k$. Define $f : \operatorname{spec}(k) \to X$ by $f_*(V) = \mathcal{F} \otimes_k V$, $f^*(\mathcal{G}) = \operatorname{Hom}_X(\mathcal{G}, \mathcal{F})^*$ the k-dual, and $f^\sharp : H^0(\mathcal{O}_X) \to k$ is any fixed isomorphism. It is easy to check this is a map, the adjointness coming from the isomorphisms

$$\begin{aligned}\operatorname{Hom}_k(f^*\mathcal{G}, V) &\cong \operatorname{Hom}_k(f^*\mathcal{G}, k) \otimes V \cong \operatorname{Hom}_X(\mathcal{G}, \mathcal{F}) \otimes_k V \\ &\cong \operatorname{Hom}_X(\mathcal{G}, \mathcal{F} \otimes_k V) \cong \operatorname{Hom}_X(\mathcal{G}, f_*V).\end{aligned}$$

However, there is no morphism of schemes $g : \operatorname{spec}(k) \to X$ realizing f unless $\mathcal{F} \cong \mathcal{O}_X/\mathcal{I}_p$ for some k-rational point p, because the inverse image $g^*\mathcal{F}$ must be isomorphic to k. For example, if $X = \mathbb{P}^n$ and p is a point, then the map of topological spaces $g : \operatorname{spec}(k) \to X$ with image p gives $g^*(\mathcal{O}_X/\mathcal{I}_p) \cong k(p)$, the function field of k.

One additional restriction which should perhaps be placed on a map is the requirement that $H^q(f_*\mathcal{M}) \cong H^q(\mathcal{M})$.

Definition 2.7. A map of polarized schemes is a pair

$$(f, \mu) : (\mathcal{C}_1, \mathcal{O}_1, s_1) \to (\mathcal{C}_2, \mathcal{O}_2, s_2)$$

consisting of

- a map of k-schemes $f : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$, and
- a natural equivalence of functors $\mu : f^* \circ s_2 \to s_1 \circ f^*$.

This should be compared with the notion of a morphism of algebraic triples in Definition 16.1.1.

If $f : X \to Y$ is a morphism of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, then $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, and there is a morphism

$$(f, \mathbb{1}) : (\mathsf{QCoh}Y, \mathcal{O}_Y, \mathcal{L}) \to (\mathsf{QCoh}X, \mathcal{O}_X, f^*\mathcal{L}),$$

where we have denoted the shift functors by the line bundles which determine them.

The results in chapter 19 on the functorial behavior of $\mathsf{Tails}(-)$ translate into results on maps between schemes.

PROPOSITION 2.8. *Let $f : A \to B$ be a homomorphism of graded k-algebras such that either*

- *B is a finitely generated right A-module, or*
- $\operatorname{coker} f$ *is right bounded.*

Then there are induced maps

$$\operatorname{Proj} B \to \operatorname{Proj} A \qquad \textit{and} \qquad (\operatorname{Proj} B, [1]) \to (\operatorname{Proj} A, [1]),$$

which are isomorphisms if $\ker f$ *is torsion and* $\operatorname{coker} f$ *is right bounded.*

PROOF. By Proposition 19.2.4, f induces a functor $f_* : \mathsf{Tails}B \to \mathsf{Tails}A$; by Proposition 19.2.8, there is also an induced functor $f^* : \mathsf{Tails}A \to \mathsf{Tails}B$. Moreover, $f^*\mathcal{A} \cong \mathcal{B}$, so we obtain a morphism as claimed; moreover, if $\ker f$ is torsion and $\operatorname{coker} f$ is right bounded, then f^* is an equivalence (19.2.9), so the schemes are isomorphic.

The result for polarized schems follows because the tensor product grading on $f^*M = B \otimes_A M$ is such that $f^* \circ [1] = [1] \circ f^*$. □

Corollary 2.9. *If A is a graded k-algebra then*

$$(\operatorname{Proj} A, [1]) \cong (\operatorname{Proj} A_{\geq 0}, [1]).$$

Proposition 2.10. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra which is generated in degree one, and let $A^{(d)}$ be a Veronese subalgebra. Then there is an isomorphism of polarized schemes*

$$(\operatorname{Proj} A, [d]) \cong (\operatorname{Proj}(A^{(d)}), [1]).$$

Proof. By Proposition 19.2.2, the functor $f_* : \mathsf{GrMod}A \to \mathsf{GrMod}(A^{(d)})$, defined by $f_*M = M^{(d)} = \oplus_{i\in\mathbb{Z}}M_{id}$ and $(f_*M)_i = M_{id}$, induces an equivalence of categories

$$f_* : \mathsf{Tails}A \to \mathsf{Tails}(A^{(d)}).$$

Since $f_*A = A^{(d)}$, there is an isomorphism as claimed. It is easy to see that $(f^* \circ [d])M_i = M_{(i+1)d} = M^{(d)}[1]_i$ which implies that $f^* \circ [d] = [1] \circ f^*$. $\square$

Another illuminating example concerns the twist of an algebra, as defined in chapter 16; before discussing it, we consider invertible bimodules (see section 6 in chapter 16) over a graded algebra and the shift functors which they give rise to.

If $\sigma, \tau \in \operatorname{Aut}_k A$ are graded algebra automorphisms, then the bimodules ${}^\sigma A^\tau$ defined in Example 16.3.2 are graded A-A bimodules. recall that ${}^\sigma A^\tau$ is A as a graded k-vector space, but with A-A-action defined by $a.x.b = a^\sigma x b^\tau$, for $a, b \in A$, $x \in {}^\sigma A^\tau$, where the right hand side is the usual product in A. There are bimodule isomorphisms

$$ {}^\sigma A^\tau \otimes_A {}^\mu A^\nu \cong {}^\lambda A^\rho. \tag{2-1}$$

The functor

$$\Phi_\sigma = {}^\sigma A^1 \otimes_A -$$

is an auto-equivalence of $\mathsf{GrMod}A$ with inverse $\Phi_{\sigma^{-1}}$ and, since $\operatorname{Tors} A$ is stable under Φ_σ,

$$\Phi_\sigma : \mathsf{Tails}A \to \mathsf{Tails}A$$

is an auto-equivalence. These auto-equivalences arise in response to the questions at the end of the next paragraph.

If ${}^\sigma A$ denotes the twisted algebra of A with respect to σ, as defined in section 4 of chapter 16, there is an equivalence of categories

$$\Theta : \mathsf{Tails}A \to \mathsf{Tails}({}^\sigma A)$$

sending the object A to ${}^\sigma A$. Example 16.5.6 showed that, in general, $\Theta \circ [1]$ is not equivalent to $[1] \circ \Theta$. However, there is an isomorphism of polarized schemes.

Proposition 2.11. *There is an isomorphism of polarized schemes*

$$(\Theta, \mu) : (\operatorname{Proj} A, \Phi_\sigma \circ [1]) \to (\operatorname{Proj}({}^\sigma A), [1]).$$

Proof. Write $s = \Phi_\sigma \circ [1]$. The only point to check is the existence of a natural equivalence between $\Theta \circ s$ and $[1] \circ \Theta$. If $M \in \mathsf{GrMod}A$, then

$$(\Theta \circ s)M = (\Theta \circ \Phi_\sigma \circ [1])M = {}^\sigma({}^\sigma A^1 \otimes_A M[1]), \quad \text{and}$$
$$([1] \circ \Theta)M = ({}^\sigma M)[1].$$

Define $\nu : ([1] \circ \Theta)M \to (\Theta \circ s)M$ as follows: for $m \in ({}^\sigma M)[1]_j = {}^\sigma M_{j+1} = M_{j+1}$, define

$$\nu(m) = 1 \otimes m.$$

This is a degree zero map since $1 \otimes m$ is a degree j element of ${}^{\sigma}({}^{\sigma}A^1 \otimes M[1])$ whenever $m \in M_{j+1}$. Moreover, ν is a ${}^{\sigma}A$-module map, since, if $a \in {}^{\sigma}A$, then

$$a \odot m = a^{\sigma^{j+1}}$$

and

$$a \odot (1 \otimes m) = a^{\sigma^j}(1 \otimes m) = (a^{\sigma^j}.1) \otimes m = a^{\sigma^{j+1}} \otimes m = 1 \otimes a^{\sigma^{j+1}} m.$$

Hence $t_M := \nu^{-1}$ gives the required natural equivalence (the diligent reader may check that the appropriate diagram commutes). □

A simple but striking consequence of the fact that $\operatorname{proj} A \cong \operatorname{proj}(A^\sigma)$ is that one cannot recognize whether or not A is prime simply from the pair $(\operatorname{tails} A, \mathcal{A})$. By Example 16.4.8, there is a twist of $A = k[x,y]/(xy, yx)$ is the homogeneous coordinate ring of two reduced points, namely $A^\sigma \cong k[x,y]/(x^2, y^2)$, which *is* prime. On the other hand, being a domain *is* a twisting invariant by ????, so $\boxed{\text{Paul}}$ can we define a good notion of irreducible.

By Watt's Theorem (A.12.1), a right exact functor $\mathsf{Mod}R \to \mathsf{Mod}R$ is naturally equivalent to a functor of the form $X \otimes_R -$ where X is an R-R-bimodule, which is flat as a right R-module. Moreover, if $X \otimes_R -$ is an equivalence, there is another bimodule Y such that $X \otimes_R Y \cong Y \otimes_R X \cong R$. Thus one should think of a shift functor as playing a role analogous to an invertible bimodule. Indeed, we have the following result.

PROPOSITION 2.12. *Let R be a k-algebra, and s an arbitrary auto-equivalence of $\mathsf{Mod}R$. Then there exists an $\mathbb{N}$-graded k-algebra A such that*

$$(\operatorname{Spec} R, s) \cong (\operatorname{Proj} A, [1]).$$

PROOF. If A is a strongly graded k-algebra, then there is an equivalence of categories $f^* : \mathsf{GrMod}A \to \mathsf{Mod}(A_0)$ (Theorem 16.6.7), defined by $f^*M = M_0$; in particular, $f^*A = A_0$. Moreover, if A is strongly graded, the only torsion module is the zero module, so $\mathsf{Tails}A$ is equivalent to $\mathsf{GrMod}A$. Hence there is an equivalence $f^* : \mathsf{Tails}A \to \mathsf{Mod}(A_0)$ such that $f^*\mathcal{A} = A_0$.

Let X be an invertible R-R-bimodule such that the functors $X \otimes_R -$ and s are naturally equivalent. In section 6 of chapter 16 we describe how to construct a strongly graded k-algebra associated to the pair (R, X); this algebra, A say, satisfies $A_0 = R$ and $A_1 \cong X$ as an R-R-bimodule. Hence, if $M \in \mathsf{GrMod}A$, then $f^*(M[1]) = (A[1] \otimes_A M)_0 = X \otimes_R M_0 \cong (s \circ f^*)(M)$. That is, $s \circ f^* \cong f^* \circ [1]$. Hence we have an isomorphism

$$(f, \mu) : (\mathsf{Tails}A, \mathcal{A}, [1]) \to (\mathsf{Mod}R, R, s). \tag{2-2}$$

Finally, $\mathsf{Tails}A \cong \mathsf{Tails}(A_{\geq 0})$ by Corollary 19.2.10, so we may relace A by $A_{\geq 0}$ in the isomorphism (2-2) to obtain an $\mathbb{N}$-graded algebra, as required. □

To be precise, the algebra A in Proposition 2.12 is $R \oplus X \oplus X^{\otimes 2} \oplus \cdots$ with the obvious multiplication. It is easy to see that, if s is the identity functor, then $A \cong R[x]$; more generally, if s is induced by $\sigma \in \operatorname{Aut}_k R$, that is, $s \cong {}^1R^\sigma \otimes_R -$, then A is isomorphic to the Ore extension $R[x; \sigma]$.

EXERCISES

2.1 Check that $\operatorname{Tors} A$ is stable under the functors Φ_σ.

2.2 Show the embedding of polynomial rings $f : k[x_0, x_1] \to k[x_0, x_1, x_2]$ does not induce a functor f^* at the level of Tails, nor is there an induced morphism $\mathbb{P}^2 \to \mathbb{P}^1$.

3. Ample shift functors

If $\mathcal{L}$ is a line bundle on a scheme X, then $\mathcal{L} \otimes_{\mathcal{O}_X} -$ is a shift functor. Amongst the line bundles, the ample ones are of particular importance. This section extends the notion of ampleness to a general shift functor, and it is shown that, just as Serre's twisting sheaf $\mathcal{O}_X(1)$ is an ample line bundle, the shift functor [1], on the non-commutative scheme $(\operatorname{tails} A, \mathcal{A})$, is ample *provided A satisfies the condition χ_1 introduced in Definition 13.4.2.*

Definition 3.1. Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple as in Definition 2.7. For each object $\mathcal{F}$ in $\mathcal{C}$, we define

- $\mathrm{H}^0(\mathcal{F}) := \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F})$, and
- $\mathcal{F}(n) := s^n(\mathcal{F})$ for $n \in \mathbb{Z}$.

We say that s is ample, if the following two conditions hold:

1. for each object $\mathcal{F}$, there is a set of positive integers $n_1, \dots, n_p$ and an epimorphism
$$\bigoplus_{i=1}^{p} \mathcal{O}(-n_i) \to \mathcal{F};$$
2. for each epimorphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$, the induced map $\mathrm{H}^0(\mathcal{F}(n)) \to \mathrm{H}^0(\mathcal{G}(n))$ is surjective for all $n \gg 0$.

Condition (1) in Definition 3.1 says that $\mathcal{O}$ is something like a generator in $\mathcal{C}$ (cf. Definition A.11.15 and Proposition A.11.16), and that $\mathcal{F}$ satisfies a certain finiteness condition (we will only discuss ampleness in $\operatorname{tails} A$, not in $\mathsf{Tails}A$). Condition (2) says that $(\mathcal{O}, s)$ behaves somewhat like a projective object.

It is important that the shifts in condition (1) be *negative* shifts. If A is $\mathbb{N}$-graded, then [1] is *not* ample for $(\mathsf{grmod}A, A, [1])$ because $A[1]$, being generated in degree -1, cannot be a homomorphic image of a sum of *negative* shifts of A.

For a classical scheme X, and $\mathcal{F} \in \mathsf{QCoh}X$, we have $\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$, the global sections of $\mathcal{F}$, so the notation $\mathrm{H}^0(-)$ in Definition 3.1 is consistent with the usual notation in algebraic geometry, namely $\mathrm{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Now consider $(\operatorname{tails} A, \mathcal{A}, [1])$, a module $M \in \mathsf{grmod}A$, and write $\mathcal{F} = \pi M$. Using the adjoint isomorphism, we have
$$\mathrm{H}^0(\mathcal{F}) = \operatorname{Hom}_{\text{tails}}(\pi A, \pi M) \cong \operatorname{Hom}_{\mathrm{Gr}}(A, \omega\pi M) = (\omega\pi M)_0.$$
Thus $\mathrm{H}^0(\mathcal{F}[n]) = (\omega\pi M)_n$. Therefore, condition (2) in Definition 3.1 has the following interpretation: a morphism $f : \mathcal{F} \to \mathcal{G} = \pi N$ induces morphisms $f[n] : \mathcal{F}[n] \to \mathcal{G}[n]$ which correspond, via the adjoint isomorphism, to maps $(\omega\pi M)_n \to (\omega\pi N)_n$ and condition (2) says that, if f is an epimorphism, then these maps are surjective for $n \gg 0$.

Proposition 3.2. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra satisfying χ_1°. Then [1] is an ample shift functor on* $\operatorname{proj} A$.

PROOF. We must prove that [1] is ample in (tails $A, \mathcal{A}$). Let $\mathcal{F} = \pi M$ be an object of tails A, where $M \in \mathsf{grmod} A$. Since A is $\mathbb{N}$-graded, $M_{\geq 1}$ is an A-module and $M/M_{\geq 1}$ is torsion, so $\mathcal{F} \cong \pi(M_{\geq 1})$. Since A is noetherian, $M_{\geq 1}$ is finitely generated, so is a quotient of a free module $\oplus_{i=1}^{p} A[-n_i]$ for some positive integers n_i. Thus $\mathcal{F} \cong \pi M$ is a quotient of $\oplus_i \mathcal{A}(-n_i)$. This proves condition (1) in Definition 3.1.

Now let $f : \mathcal{F} \to \mathcal{G}$ be an epimorphism in tails A. Write $\mathcal{F} = \pi M$ and $\mathcal{G} = \pi N$ with $M, N \in \mathsf{grmod} A$. By the discussion above, on the meaning of the ampleness condition, we must show that the induced map $\omega\pi M \to \omega\pi N$ is surjective in large degree. Since χ_1° holds, Theorem 19.4.4 shows that $(\omega\pi M)_{\geq n} = M_{\geq n}$ and $(\omega\pi N)_{\geq n} = N_{\geq n}$ for $n \gg 0$, so we need only show that f induces a surjective map $M_{\geq n} \to N_{\geq n}$ for $n \gg 0$.

By Proposition 19.1.5, f is of the form $\pi\varphi$ for some $\varphi \in \operatorname{Hom}_{\mathrm{Gr}}(M', N)$ for some submodule M' of M such that M/M' is torsion; in particular, $M_{\geq n} = M'_{\geq n}$ for $n \gg 0$. Since f is an epimorphism, $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ are both torsion so, for large n, $\varphi : M'_{\geq n} = M_{\geq n} \to N_{\geq n}$ is surjective, which completes the proof. $\square$

The next result shows that Proposition 3.2 requires some hypothesis like χ_1.

PROPOSITION 3.3. *Let A be a left noetherian, connected graded k-algebra which does not satisfy χ_1 (for example, the algebra in Example 19.5.3). Let $A[z]$ be the polynomial extension. Then [1] is not an ample shift functor on* $\operatorname{proj}(A[z])$.

PROOF. The idea behind the proof is simple: if $\varphi : F \to M$ is a surjective map between modules such that $\chi_i^\circ(F)$ holds but $\chi_1^\circ(M)$ does not, then $\pi\varphi : \pi F \to \pi M$ will be an epimorphism for which condition (2) in Definition 3.1 fails.

To avoid any possible confusion, we will only use the notation ω and π for $\mathsf{GrMod}(A[z])$ and $\mathsf{Tails}(A[z])$. Since $A \cong A[z]/(z)$ an A-module is an $A[z]$-module in a canonical way.

Since A does not satisfy χ_1, which is equivalent to the condition χ_1° in this situation (19.4.4), Theorem 19.4.4 implies there exists $M \in \mathsf{grmod} A$ such that $\bar{M} = M/\tau M$ has an essential extension, H say, such that $H/\bar{M}$ is torsion but not right bounded (we have used the definition of the adjoint functor $\mathsf{Tails} A \to \mathsf{GrMod} A$). But M and H are $A[z]$-modules, and the torsion submodule of M is independent of whether M is considered as an A-module or as an $A[z]$-module, so H is an essential extension of $\bar{M}$ in $\mathsf{GrMod}(A[z])$ and $H/\bar{M} \in \operatorname{Tors}(A[z])$. Therefore $\operatorname{coker}(M \to \omega\pi M)$ is not right bounded.

There exists a free $A[z]$-module of finite rank, F say, and a surjective map $\varphi : F \to M$. Write $\mathcal{F} = \pi F$, $\mathcal{G} = \pi M$, and $f = \pi\varphi$. Then $f : \mathcal{F} \to \mathcal{G}$ is an epimorphism in $\mathsf{Tails}(A[z])$, since π is exact. By [**141**, Theorem 11.68], $\underline{\operatorname{Ext}}^1_{A[z]}(k, A[z]) \cong \underline{\operatorname{Hom}}_A(k, A)$. This is contained in τA, so is right bounded. Hence $\chi_1^\circ(A[z])$ holds, so $(\omega\mathcal{F})_{\geq n} = (\omega\pi F)_{\geq n} = F_{\geq n}$ for $n \gg 0$. Hence, for $n \gg 0$, the image of the map

$$f[n] : H^0(\mathcal{F}[n]) = (\omega\pi F)_n \to H^0(\mathcal{G}[n]) = (\omega\pi M)_n$$

is contained in the image of $F_{\geq n}$ which belongs to $M_{\geq n} \neq (\omega\pi M)_{\geq n}$. Thus condition (2) of Definition 3.1 is not satisfied. $\square$

One further comment on this result is appropriate. Consider the algebra A in Example 19.5.3, and the algebra B containing A. Although B/A is a torsion A-module, $B[z]/A[z]$ is *not* a torsion $A[z]$-module; thus, although $(\omega_A\pi_A A)_{\geq 0} = B$,

$(\omega\pi A[z]) \neq B[z]$. This also has the following interpretation. Let $f : A \to B$ and $g : A[z] \to B[z]$ be the inclusions. Although $f_* : \operatorname{Proj} B \to \operatorname{Proj} A$ is an isomorphism, $g_* : \operatorname{Proj}(B[z]) \to \operatorname{Proj}(A[z])$ is *not* an isomorphism. I do not know whether there is an ample shift functor on $\operatorname{proj}(A[z])$.

EXAMPLE 3.4. Let $A = k[X]$. The exact sequence $0 \to A[1] \to A \to k \to 0$ in $\mathsf{GrMod}A$ shows that $\pi A \cong \pi A[1]$ in $\mathsf{Tails}A$. Hence, if $M \in \mathsf{GrMod}A$, applying $[1]$ to a free resolution of M shows that $\pi M \cong \pi M[1]$. Hence, on $\mathsf{Tails}A$, $[1]$ is the identity functor. In fact, $\mathsf{Tails}A$ is equivalent to $\mathcal{V}ec(k)$.

Let X be a noetherian scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and $s = \mathcal{L} \otimes_{\mathcal{O}_X} -$ the associated shift on $\mathsf{Coh}X$. Recall that $\mathcal{L}$ is ample if, for every $\mathcal{F} \in \mathsf{Coh}X$, $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ is generated by its global sections for $n \gg 0$; that is, $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ is a quotient of the free $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_k \mathrm{H}^0(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F})$, which is the same as saying that the natural map $\mathcal{O}_X \otimes_k \mathrm{H}^0(X, \mathcal{L}^{\otimes n} \otimes \mathcal{F}) \to \mathcal{L}^{\otimes n} \otimes \mathcal{F}$ is an epimorphism in $\mathsf{Coh}X$. It is not difficult to show that $\mathcal{L}$ is ample if and only if s is ample in the sense of Definition 3.1.

$\boxed{\text{DETAILS}}$

4. Open and closed subschemes

Should we define a closed subscheme through $f^*f_* \sim \mathbb{1}$, and $f_*(\mathcal{C})$ being a full subcat closed under kernels and cokernels? What about open subschemes? Is this the place to treat the case of an open affine determined by a normal regular element?

A map $f : (\mathsf{C}', \mathcal{O}') \to (\mathsf{C}, \mathcal{O})$ is a closed embedding if $f(\mathsf{C}')$ is a full subcategory of C closed under kernels and cokernels.

PROPOSITION 4.1. *Let u be a homogeneous normal regular element of positive degree in an $\mathbb{N}$-graded algebra A. If A is generated in degree one, then there is a map*

$$f : \operatorname{Spec} A[u^{-1}]_0 \to \operatorname{Proj} A.$$

PROOF. Set $R = A[u^{-1}]_0$. By Proposition 16.6.4, $A[u^{-1}]$ is strongly graded, so there is an equivalence of categories $\mathsf{GrMod}A[u^{-1}] \cong \mathsf{Mod}R$. On the other hand there is the exact localization functor $\mathsf{GrMod}A \to \mathsf{GrMod}A[u^{-1}]$. Composing these gives an exact functor $f^* : \mathsf{GrMod}A \to \mathsf{Mod}R$. It is a localization functor so has a right adjoint f_*. Moreover, f^* sends a module M to $M[u^{-1}]_0$, so $f^*A \cong R$. Since a torsion A-module is locally finite, each element of it is killed by a power of u, whence f^* kills torsion modules. Thus f^* descends to $\mathsf{Tails}A \to \mathsf{Mod}R$. As such f^* still has a right adjoint which we continue to call f_*. Finally, as $f^*\mathcal{A} \cong R$, we obtain a map as claimed. □

5. The Picard group

Definition 5.1. Let $X = (\mathcal{C}, \mathcal{O})$ be a k-scheme. The automorphism group of X, denoted $\operatorname{Aut}_k X$ or $\operatorname{Aut}_k(\mathcal{C}, \mathcal{O})$ is the group of natural equivalence classes of k-linear auto-equivalences $s : \mathcal{C} \to \mathcal{C}$ such that $s\mathcal{O} \cong \mathcal{O}$. There is a natural map $\operatorname{Aut}_k X \to \operatorname{Aut}_k \mathcal{C}$ to the group of natural equivalence classes of k-linear auto-equivalences of $\mathcal{C}$. The Picard group of X, denoted $\operatorname{Pic} X$ or $\operatorname{Pic}(\mathcal{C}, \mathcal{O})$, is the cokernel of this map. Hence there is an exact sequence

$$1 \to \operatorname{Aut} X \to \operatorname{Aut} \mathcal{C} \to \operatorname{Pic} X \to 1.$$

For example, if σ is an automorphism of a classical scheme X, then σ^* is an auto-equivalence of $\mathsf{Coh}X$ such that $\sigma^*\mathcal{O}_X \cong \mathcal{O}_X$, so $\sigma^* \in \mathrm{Aut}_k X$ according to the definition above. On the other hand, if $\mathcal{L}$ is a non-trivial invertible $\mathcal{O}_X$-module then $s = \mathcal{L} \otimes -$ is an auto-equivalence of $\mathsf{Coh}X$ which does not preserve $\mathcal{O}_X$, so s gives a non-trivial element of $\mathrm{Pic}\, X$ according to our definition above.

It is an important problem to determine $\mathrm{Aut}\, X$ and $\mathrm{Pic}\, X$ for non-commutative schemes, and to determine the ample elements of $\mathrm{Pic}\, X$.

6. The Segre product

Given classical schemes X and Y over a base scheme S, they have a product, and there are projection maps $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$. We want to construct the non-commutative analogues of these maps; that is we want an adjoint pair of functors pr_i^* and pr_{i*}.

For simplicity, we will restrict attention to projective k-schemes $X \subset \mathbb{P}^n_k$ and $Y \subset \mathbb{P}^m_k$; the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ is such that the homogeneous coordinate ring of $X \times Y$ is the Segre product of the homogeneous coordinate rings of X and Y.

Definition 6.1. Let U and V be graded vector spaces over a field k. The Segre product of U and V is the vector space

$$U \circ V := \bigoplus_{n \in \mathbb{Z}} U_n \otimes V_n \;\subset\; U \otimes_k V,$$

with grading defined by $(U \circ V)_n = U_n \otimes V_n$. The Segre product of graded k-algebras, A and B say, is $A \circ B$ with the algebra structure inherited from that on $A \otimes B$.

If $M \in \mathsf{GrMod}A$ and $N \in \mathsf{GrMod}B$, then $M \circ N$ is a graded $A \circ B$-module under the action $a \otimes b.(m \otimes n) = am \otimes bn$. This provides a functor

$$\mathsf{GrMod}A \times \mathsf{GrMod}B \to \mathsf{GrMod}A \circ B.$$

The next result gives the analogue of $\mathrm{pr}_1^* : \mathsf{QCoh}X \to \mathsf{QCoh}X \times_S Y$.

LEMMA 6.2. *Let A and B be graded k-algebras. The rule $M \mapsto M \circ B$ defines an exact functor*

$$\mathrm{pr}_1^* : \mathsf{Tails}A \to \mathsf{Tails}A \circ B,$$

and yields a map $\mathrm{Proj}\, A \circ B \to \mathrm{Proj}\, A$.

PROOF. The rule $M \mapsto M \circ B$ is an exact functor $\mathsf{GrMod}A \to \mathsf{GrMod}A \circ B$. We need to show that it sends torsion modules to torsion modules.

$\boxed{\text{DETAILS}}$

$\square$

The algebraic analogue of pr_{1*} is not immediately obvious—the image of the algebra homomorphism $A \to A \otimes B$, $a \mapsto a \otimes 1$, is not contained in $A \circ B$, so it is not clear that there *is* a functor $\mathsf{Tails}A \circ B \to \mathsf{Tails}A$. To describe the algebraic analogue of pr_{1*} we must first digress.

There are three useful gradings on the tensor product of two graded vector spaces, U and V say. First, there is the usual tensor product grading defined by

$$(U \otimes V)_n := \bigoplus_{i+j=n} U_i \otimes V_j. \tag{6-1}$$

Second, there is a $\mathbb{Z}^2$-grading defined by

$$(U \otimes V)_{(i,j)} := U_i \otimes V_j; \tag{6-2}$$

i.e., the homogeneous components are 'located' at the lattice points of $\mathbb{Z}^2 \subset \mathbb{R}^2$, with $U \circ V$ lying on the main diagonal—we call $U \circ V$ the diagonal of $U \otimes_k V$. Third, there is a grading defined by

$$(U \otimes V)_i = \bigoplus_{j \in \mathbb{Z}} U_{i+j} \otimes V_j, \tag{6-3}$$

the homogenous components of which are the lines parallel to the main diagonal; $U \circ V$ is the degree zero component of $U \otimes V$ under this grading. We refer to this as the diagonal grading on $U \otimes V$.

In particular, a tensor product of graded algebras, say $A \otimes_k B$, may be endowed with any one of these gradings and, if $M \in \mathsf{GrMod}A$ and $N \in \mathsf{GrMod}B$, $M \otimes N$ endowed with the corresponding grading becomes a graded $A \otimes B$-module.

If $A \otimes B$ is given the $\mathbb{Z}^2$-grading, we will write $\mathrm{BiGrMod}\, A \otimes B$ for the corresponding category of graded $A \otimes B$-modules: thus the objects are the $A \otimes B$-modules L endowed with a $\mathbb{Z}^2$-grading

$$L = \bigoplus_{(i,j) \in \mathbb{Z}^2} L_{(i,j)}$$

such that

$$(A_i \otimes B_j).L_{(r,s)} \subset L_{(i+r,j+s)}.$$

Read BiGrMod as 'bigraded modules'.

NOTATION . Write

$$\begin{aligned} A_+ &:= \bigoplus_{i \geq 1} A_i, \\ B_+ &:= \bigoplus_{i \geq 1} B_i, \\ T &:= A \otimes_k B, \\ T_{\geq(p,q)} &:= \bigoplus_{i \geq p, j \geq q} A_i \otimes B_j, \\ T_{++} &:= T_{\geq(1,1)} = A_+ \otimes B_+. \end{aligned}$$

If A and B are generated in degree one, then $(T_{++})^n = T_{\geq(n,n)}$.

We may define a dense subcategory of $\mathrm{BiGrMod}\, A \otimes B$ of torsion modules as follows. If $L \in \mathrm{BiGrMod}\, A \otimes B$, then $m \in L$ is torsion if it is annihilated by a power of the ideal T_{++}. If 0 is the only torsion element, we say L is torsion-free. Proceeding in the obvious way, we define torsion modules, and write $\mathrm{Tors}(T)$ for the full subcategory of such modules; it is a dense subcategory, so we may define the quotient category.

LEMMA 6.3. *Let $S = A \circ B$ be the Segre product of $\mathbb{N}$-graded k-algebras A and B, which are generated in degree one. Consider $T = A \otimes_k B$ with its $\mathbb{Z}^2$-grading, and with the $\mathbb{Z}$-grading $T_n = \sum_{i \in \mathbb{Z}} A_{i+n} \otimes B_i$. Let $L \in \mathrm{BiGrMod}(T)$. Then*

1. *$T_{\geq(n,n)} = T.S_{\geq n} = S_{\geq n}.T$ for all $n \geq 1$;*
2. *$T_n T_{-n} = T_{-n} T_n = S_{\geq n}$ for all $n \geq 1$;*

3. *if* $L_n := \sum_{i\in\mathbb{Z}} L_{i+n}$, *then* L *is a* $\mathbb{Z}$*-graded* T*-module;*
4. *if* $L_0 = 0$, *then each* L_n *is a torsion* S*-module, namely* $S_{\geq |n|}.L_n = 0$*;*
5. *if* $L_0 = 0$, *then* $T_{\geq (m,m)}L = 0$ *for all* $m \gg 0$*; that is,* $L \in \mathrm{Tors}(T)$.

Proof. Notice that $T_0 = S$, whence each T_n is an S-module. Since A and B are generated in degree one, so is S, whence $(S_+)^n = S_{\geq n}$ for $n \geq 1$.

(1) Trivial.

(2) The cases T_nT_{-n} and $T_{-n}T_n$ are similar, so we only treat the first. By definition,

$$T_nT_{-n} = \sum_{i,j} A_{i+n}A_{j-n} \otimes B_iB_j. \tag{6-4}$$

When $i = 0$ and $j \geq n$ then, since A and B are generated in degree one, the right hand side of (6-4) equals $A_j \otimes B_j = S_j$, whence the result.

(3) This is trivial.

(4) This follows from the computation

$$S_{\geq |n|}.L_n = T_nT_{-n}L_n \subset T_n.L_0 = 0.$$

(5) Given a finite set of generators for L, there exists a single m such that each generator is annihilated by $S_{\geq m}$, and hence by $TS_{\geq m} = S_{\geq m}T$. Thus L is annihilated by $S_{\geq m}$, and hence by $(T_{++})^m$. □

In the situation of Lemma 6.3, each T_0/T_nT_{-n} is a torsion module over the graded algebra $S = T_0$; it is reasonable to think of T as being 'almost strongly graded'. With this in mind, the next Theorem is analogous to Theorem 16.6.7.

Theorem 6.4. [**185**] *Let* A *and* B *be* $\mathbb{N}$*-graded* k*-algebras generated in degree one. Then there is an equivalence of categories*

$$\mathrm{BiGrMod}(A \otimes B)/\,\mathrm{Tors}(A \otimes B) \cong \mathsf{Tails}(A \circ B).$$

Proof. We will write $T = A \otimes_k B$. The subcategory $\mathrm{Tors}(T)$ is as described prior to Lemma 6.3. The diagonal grading on T is given by $T_n = \sum_i A_{i+n} \otimes B_i$, and $M \in \mathrm{BiGrMod}(T)$ may also be considered as a $\mathbb{Z}$-graded T-module via $M_n := \sum_{i\in\mathbb{Z}} M_{i+n,i}$. Notice that $S = T_0$, and that each M_n is an S-module.

Define functors

$$F : \mathrm{BiGrMod}(A \otimes B) \to \mathsf{GrMod}(A \circ B)$$

and

$$G : \mathsf{GrMod}(A \circ B) \to \mathrm{BiGrMod}(A \otimes B),$$

by

$$F(M) = M_0 = \sum_{i\in\mathbb{Z}} M_{ii} \qquad \text{and} \qquad G(N) = T \otimes_S N.$$

The grading on FM is specified by $(FM)_i = M_{ii}$. The grading on GN is defined by the rule that $(GN)_{ij}$ is the sum over $m \in \mathbb{Z}$ of the images of $T_{i-m,j-m} \otimes_k N_m$.

The diagonal grading on GN is therefore

$$\begin{aligned}(GN)_p &= \sum_{i\in\mathbb{Z}}(GN)_{i+p,i} \\ &= \sum_{i\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} T_{i+p-m,i-m}\otimes N_m \\ &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} T_{j+p,j}\otimes N_m \\ &= T_p\otimes_S N.\end{aligned}$$

It follows that $F(GN) = (GN)_0 = N$.

Now we check that F and G send torsion modules to torsion modules. If M is a torsion bigraded T-module, then given $m \in M$, $T_{\geq(n,n)}m = 0$ for $n \gg 0$. Since $S_{\geq n} \subset T_{\geq(n,n)}$, $S_{\geq n}m = 0$ too, so m is a torsion element of M viewed as an S-module. Thus $FM \in \operatorname{Tors}(S)$. Conversely, suppose that $N \in \operatorname{Tors}(S)$. Then $T_{\geq(n,n)}.GN = T.S_{\geq n}\otimes_S N = T\otimes_S S_{\geq n}N$, whence $GN \in \operatorname{Tors}(T)$. Thus F an G are well-defined functors between $\mathsf{Tails}(T)$ and $\mathsf{Tails}(S)$. Since $F(GN) = N$, $F\circ G \cong \mathbb{1}d_{\mathsf{Tails}(A\circ B)}$.

Now define $\phi : GF(M) = T\otimes_S M_0 \to M$ by $\phi(t\otimes m) = tm$. This is a homomorphism of bigraded T-modules, of bi-degree $(0,0)$. Its kernel and cokernel are bigraded T-modules, and may be given the diagonal gradings. The restriction of ϕ to the degree zero components for the diagonal grading is the natural isomorphism $(GFM)_0 = T_0\otimes_S M_0 \to M_0$, whence the degree zero components of $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are zero. Hence, by Lemma 6.3, $\ker(\phi)$ and $\operatorname{coker}(\phi)$ both belong to $\operatorname{Tors}(T)$, whence ϕ is an isomorphism in $\mathsf{Tails}(T)$, which proves that $G\circ F$ is naturally equivalent to the identity functor on the quotient category. □

We may now describe the analogue of $\mathrm{pr}_{1*} : \mathsf{QCoh}X\times_S Y \to \mathsf{QCoh}X$. It is the composition

$$\operatorname{Proj}(A\circ B) \xrightarrow{\sim} \operatorname{BiGrMod}(A\otimes B)/\operatorname{Tors}(A\otimes B) \longrightarrow \operatorname{Proj} A,$$

where the isomorphism is that in Theorem 6.4, and the second functor is induced from the functor

$$\operatorname{BiGrMod}(A\otimes B) \to \mathsf{GrMod}A$$

defined by $M \mapsto M$, but with grading $M_n = \sum_i M_{i+n,i}$. To show that this gives a map, we need to show that pr_1^* is a left adjoint to this functor.

???????????????

$\boxed{\text{Paul}}$ Does this lead to a product in the category of non-commutative schemes?

cf. $\operatorname{Hom}_S(Z,X)\times\operatorname{Hom}_S(Z,Y) = \operatorname{Hom}_S(Z,X\times_S Y)$ for S-schemes X,Y,Z. Take a naive definition of the category of non-commutative projective schemes over k: the objects are all $\operatorname{Proj} A = (\mathsf{Tails}A,\mathcal{A})$ arising from graded k-algebras A (maybe one wants a restriction on A, like finite GK-dimension etc.?), and the morphisms are the maps. The embedding of the category of commutative schemes in this category is contravariant, so we should seek a co-product in this category as the analogue to the product $X\times Y$ of schemes. So, do the maps $\operatorname{Proj} A \to \operatorname{Proj}(A\circ B)$ and $\operatorname{Proj} B \to \operatorname{Proj}(A\circ B)$ satisfy the appropriate universal property? That is, if $\operatorname{Proj} A \to \operatorname{Proj} R$

and $\operatorname{Proj} B \to \operatorname{Proj} R$ are maps, is there a map $\operatorname{Proj}(A \circ B) \to \operatorname{Proj} R$ making the appropriate diagrams commute? Probably not— to get a product in the category of non-commutative schemes, we probably need to use the free coproduct rather than the tensor product in the construction of $A \circ B$.

– –

$\boxed{\text{Paul}}$

If $\mathcal{F} \in \mathsf{Mod}(\mathcal{O}_X)$ and $\mathcal{G} \in \mathsf{Mod}(\mathcal{O}_Y)$, then we define the $\mathcal{O}_{X\times Y}$-module

$$\mathcal{F} \boxtimes \mathcal{G} := \mathrm{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^* \mathcal{G}.$$

If A and B are homogeneous coordinate rings of X and Y respectively, and $M \in \mathsf{GrMod} A$ and $N \in \mathsf{GrMod} B$ correspond to $\mathcal{F}$ and $\mathcal{G}$, then $M \circ N$ is the $A \circ B$-module corresponding to $\mathcal{F} \boxtimes \mathcal{G}$. We have

$$\mathrm{H}^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^0(Y, \mathcal{G}).$$

Is it easy to prove this algebraically via the $\omega\pi$ functor?

If A and B are integrally closed commutative domains, is $A \circ B$ integrally closed?

If A and B are prime noetherian maximal orders, is $A \circ B$ too ?

What is known about properties of $A \otimes_k B$ inherited from A and B, say for finitely generated k-algebras of finite GK-dim? (perhaps with k algebraically closed to avoid separability issues)? Not much, I think!

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We also note that $(A \circ B)[1] \cong A[1] \circ B[1]$ as graded $A \circ B$-modules.

EXERCISES

6.1 Let σ and τ be automorphisms of graded algebras A and B. Show there is an induced automorphism $\sigma \circ \tau$ of $A \circ B$, and that

$$A^\sigma \circ B^\tau \cong (A \circ B)^{\sigma \circ \tau}.$$

7. Comments and Questions

Almost all the definitions in this chapter are tentative. We have not yet examined sufficiently many deep examples to know what the appropriate definitions should be.

In the definition of a scheme the condition that $\mathcal{O} \neq 0$ is imposed so that $\mathcal{O}$ is not ridiculously small—after all, it is supposed to be like the structure sheaf. However, we should probably impose a stronger condition on $\mathcal{O}$ which makes it more the right size so that there is something like an ample shift functor; maybe that given any $\mathcal{F} \neq 0$, there is an auto-equivalence s of $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(s\mathcal{O}, \mathcal{F}) \neq 0$.

CHAPTER 21

Cohomology groups for projective quasi-schemes

Let A be a graded k-algebra, and $X = \operatorname{Proj} A$ the associated projective quasi-scheme. Section 1 defines cohomology groups of X-modules $\mathcal{F}$, namely

$$\mathrm{H}^q(X, \mathcal{F}) := \mathrm{Ext}^q_{\mathsf{Tails}}(\mathcal{A}, \mathcal{F}).$$

Serre proved that these coincide with the usual Čech cohomology groups in the classical case. A version of Serre's Finiteness Theorem holds in the non-commutative setting (the condition χ is relevant here too).

In section 4 we prove a generalization of Serre's Theorem that $\mathsf{Coh}X$ is equivalent to tails A. More precisely, the goal of that section is to understand the functor $\Gamma_* : \mathsf{Coh}X \to \mathsf{GrMod}A$ and its non-commutative generalization.

Section 3 is mainly preparation for section 4.

Almost all this chapter comes directly from Artin and Zhang's paper [**12**].

1. Cohomology

Since $\mathsf{Tails}A$ has enough injectives, we may define the right derived functors of the left exact functor $\mathrm{Hom}_{\mathsf{Tails}}(\mathcal{A}, -)$ by taking an injective resolution of the argument. Doing so gives rise to cohomology groups.

Definition 1.1. Let X be a quasi-scheme with structure sheaf $\mathcal{O}_X$, and let $\mathcal{F} \in \mathsf{Mod}X$. For each $i \geq 0$ we define the cohomology group

$$\mathrm{H}^i(X, \mathcal{F}) := \mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{F}).$$

Given also a polarization s on X, we define the cohomology module

$$\underline{\mathrm{H}}^i(X, \mathcal{F}) := \underline{\mathrm{Ext}}^i_{\mathcal{C}}(\mathcal{O}, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathcal{C}}(\mathcal{O}, \mathcal{F}(n)),$$

where $\mathcal{F}(n) = s^n \mathcal{F}$. It has a natural grading, namely

$$\underline{\mathrm{H}}^i(X, \mathcal{F})_n = \mathrm{H}^i(X, \mathcal{F}(n)) = \mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{F}(n)).$$

The cohomology group $\mathrm{H}^i(X, \mathcal{F})$ does not depend on the shift functor, but the cohomology module $\underline{\mathrm{H}}^i(X, \mathcal{F})$ does.

If we begin with a graded algebra A, we will just write $H^i(\pi M)$ for $H^i(\operatorname{Proj} A, \pi M)$, and $\underline{\mathrm{H}}^i(\pi M)$ for the cohomology module over the polarized scheme $(\mathsf{Tails}A, \mathcal{A}, [1])$.

PROPOSITION 1.2. *Let $X = (\mathcal{C}, \mathcal{O})$ be a scheme. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathcal{C}$, then there is a long exact cohomology sequence*

$$0 \to \mathrm{H}^0(\mathcal{F}') \to \mathrm{H}^0(\mathcal{F}) \to \mathrm{H}^0(\mathcal{F}'') \to \mathrm{H}^1(\mathcal{F}') \to \cdots.$$

PROOF. The usual proof works, viz., take injective resolutions of each term and use the Snake Lemma. $\square$

PROPOSITION 1.3. *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra, and let $M \in$ $\mathsf{GrMod}A$. Then*

$$\underline{\mathrm{H}}^0(\pi M) = \omega\pi M; \tag{1-1}$$

$$\underline{\mathrm{H}}^i(\pi M) = \lim_{\rightarrow} \underline{\mathrm{Ext}}^i_A(A_{\geq n}, M); \tag{1-2}$$

$$\underline{\mathrm{H}}^i(\pi M) = \lim_{\rightarrow} \underline{\mathrm{Ext}}^{i+1}_A(A/A_{\geq n}, M) \quad \text{if } i \geq 1; \tag{1-3}$$

$$\underline{\mathrm{H}}^i(\pi M) = h^{i+1}(I^\bullet M) \qquad\qquad \text{if } i \geq 1. \tag{1-4}$$

If A is connected, and $i \geq 1$, then $\underline{\mathrm{H}}^i(\pi M)$ is isomorphic to the local cohomology module $H^{i+1}_{\mathfrak{m}}(M)$.

PROOF. The first equality follows from the definition of $\underline{\mathrm{H}}^0(-)$, and the adjoint property of π and ω: if we write $\mathcal{F} = \pi M$, then

$$\underline{\mathrm{H}}^0(\mathcal{F}) = \oplus_n \mathrm{Hom}_{\mathsf{Tails}}(\mathcal{A}, \mathcal{F}[n]) = \oplus_n \mathrm{Hom}_{\mathrm{Gr}}(A, \omega\mathcal{F}[n]) = \underline{\mathrm{Hom}}_A(A, \omega\mathcal{F}),$$

which equals $\mathrm{Hom}_A(A, \omega\mathcal{F}) = \omega\mathcal{F}$, since A is a finitely generated A-module. The second equality follows from Lemma 19.3.3(2), namely

$$\underline{\mathrm{H}}^i(\pi M) = \underline{\mathrm{Ext}}^i(\pi A, \pi M) = \varinjlim \underline{\mathrm{Ext}}^i_A(A_{\geq n}, M).$$

The third equality follows from the long exact sequence for Ext, which gives $\underline{\mathrm{Ext}}^i_A(A_{\geq n}, M) \cong$ $\underline{\mathrm{Ext}}^{i+1}(A/A_{\geq n}, M)$ whenever $i \geq 1$. The fourth equality follows from the previous one and Proposition 19.3.12. The final statement for connected algebras is an immediate consequence of (1-3). $\square$

It follows from (1-2) that $\underline{\mathrm{H}}^i(\mathcal{F})$ has a natural left A-module structure, and from (1-4) that $\underline{\mathrm{H}}^i(\mathcal{F})$ is a torsion module if $i \geq 1$.

THEOREM 1.4 (Serre's Finiteness Theorem). *Let A be a left noetherian, $\mathbb{N}$-graded k-algebra. If A satisfies χ then, for all $\mathcal{F} \in \mathrm{tails}\, A$,*

1. *$\mathrm{H}^i(\mathcal{F})$ is a finitely generated A_0-module, for all i;*
2. *if $i \geq 1$, then $\mathrm{H}^i(\mathcal{F}[n]) = 0$ for $n \gg 0$.*

If (1) and (2) hold for all $\mathcal{F} \in \mathrm{tails}\, A$, and A satisfies χ_1, then A satisfies χ.

PROOF. Write $\mathcal{F} = \pi M$ with $M \in \mathsf{grmod}A$. By Proposition 19.4.3(3), if $\chi_i(M)$ holds, so does $\chi_i^\circ(M)$.

(1) By Proposition 1.3, $\mathrm{H}^0(\mathcal{F}) = (\omega\pi M)_0$. Because $\chi_1(M)$ holds, $(\omega\pi M)_{\geq 0}$ is a finitely generated A-module by Theorem 19.4.4, whence $(\omega\pi M)_0$ is a finitely generated A_0-module (11.1.1).

Now suppose that $i > 0$. Let $d \in \mathbb{Z}$. Since $\chi^\circ_{i+1}(M)$ holds, Corollary 19.4.5 gives, for $r \gg 0$,

$$\underline{\mathrm{Ext}}^{i+1}_A(A/A_{\geq r}, M)_{\geq d} \cong \varinjlim \underline{\mathrm{Ext}}^{i+1}_A(A/A_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{H}}^i(\mathcal{F})_{\geq d},$$

Since $\chi_{i+1}(M)$ holds, the left hand term is a finitely generated A-module for large r; it is also torsion. Therefore $\underline{\mathrm{H}}^i(\mathcal{F})_{\geq d}$ is finitely generated and right bounded. Both (1) and (2) now follow because d is arbitrary.

Now suppose that (1) and (2) hold for all $\mathcal{F} \in \mathrm{tails}\, A$, and that χ_1 holds. We will show that $\chi_i(M)$ holds for all $M \in \mathsf{grmod}A$.

First, we will prove, by induction on i, that $\chi^\circ_i(M)$ holds. By Proposition 19.4.3, $\chi^\circ_0(M)$ holds and, since we are assuming that $\chi_1(M)$ holds, $\chi^\circ_1(M)$ holds too. Now suppose that $i \geq 2$, and that $\chi^\circ_{i-1}(M)$ holds. Condition (2) ensures the right boundedness of $\underline{\mathrm{H}}^{i-1}(\mathcal{F}) = \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M)$; but this, together with $\chi^\circ_{i-1}(M)$ holding, implies that $\chi^\circ_i(M)$ holds (19.4.7).

Thus $\chi^\circ_i(M)$ holds for all i. We will now show that χ_i holds for all i. Of course, $\chi_0(M)$ holds because A is noetherian, and $\chi_1(M)$ holds by hypothesis, so we may assume that $i \geq 2$. Let $d \in \mathbb{Z}$. Corollary 19.4.5 ensures that, for $r \gg 0$,

$$\underline{\mathrm{Ext}}^i_A(A/A_{\geq r}, M)_{\geq d} \cong \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{H}}^{i-1}(\pi M)_{\geq d},$$

and condition (1) ensures that each homogenous component of this is a finitely generated A_0-module. It is obvious that $\underline{\mathrm{Ext}}^i_A(A/A_{\geq r}, M)$ is left bounded, and by Proposition 19.4.3(2) it is right bounded. But a bounded A-module, each component of which is a finitely generated A_0-module, is a finitely generated A-module. Thus, for $r \gg 0$, $\underline{\mathrm{Ext}}^i_A(A/A_{\geq r}, M)_{\geq d}$ is finitely generated; that is, $\chi_i(M)$ holds. □

In algebraic geometry, the first example for which one explicitly computes cohomology is projective space—that is, one computes $\mathrm{H}^i(\mathbb{P}^n, \mathcal{O}(d))$. For example, see [**76**, Chapter III, Theorem 5.1], where this is done using Čech cohomology. We now carry out this computation using algebraic methods. Actually, we will treat a more general case, that of a regular algebra, which includes the polynomial ring as a special case. The result shows that regular algebras are a reasonable non-commutative generalization of polynomial rings.

EXAMPLE 1.5. We compute the cohomology groups of $\mathcal{A}[m]$ when A is a regular algebra of global dimension n. By Proposition 1.3,

$$\underline{\mathrm{H}}^i(\mathcal{A}) \cong \begin{cases} \varinjlim \underline{\mathrm{Ext}}^i_A(A_{\geq n}, A) & \text{if } i \geq 0, \\ H^{i+1}_{\mathfrak{m}}(A) & \text{if } i \geq 1. \end{cases}$$

First we deal with the low dimensional cases. The only regular algebras of global dimension 0 and 1 are k and $k[X]$ by Example 15.7.3. The only object in $\mathsf{Tails}(k)$ is 0, so all cohomology groups are zero. If $A = k[X]$, then $\underline{\mathrm{H}}^0(\mathcal{A}) = \omega\pi A = k[X, X^{-1}]$ by Example 19.4.2, and the higher cohomology groups are all zero.

Now suppose that $n \geq 2$.

By Proposition 1.3, $\underline{\mathrm{H}}^0(\mathcal{A}) = \omega\pi A$, which may be computed from the exact sequence

$$0 \to \tau A \to A \to \omega\pi A \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, A) = H^1_{\mathfrak{m}}(A) \to 0. \tag{1-5}$$

First, since A is regular and hence Gorenstein, of dimension ≥ 2, $\underline{\mathrm{Ext}}^1_A(k, A) = \underline{\mathrm{Hom}}_A(k, A) = 0$ by Proposition 6.3. So the first and last terms of (1-5) are zero. Thus $\omega\pi A = A$, and $\underline{\mathrm{H}}^0(\mathcal{A}) = A$.

If $i \geq 1$, then by Proposition 1.3, and Proposition 15.6.2,

$$\underline{\mathrm{H}}^i(\mathcal{A}) \cong H^{i+1}_{\mathfrak{m}}(A) \cong \begin{cases} 0 & \text{if } i \neq n-1, \\ A^*[\ell] & \text{if } i = n-1. \end{cases}$$

Thus, as graded vector spaces,

$$\underline{\mathrm{H}}^i(\mathcal{A}) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } 0 < i < n-1, \\ A^*[\ell] & \text{if } i = n-1, \\ 0 & \text{if } i > n-1. \end{cases}$$

This data may also be written as

$$\underline{\mathrm{H}}^i(\mathcal{A})_m = \mathrm{H}^i(\mathcal{A}[m]) \cong \begin{cases} A_m & \text{if } i=0, \\ 0 & \text{if } 0<i<n-1, \\ (A_{-\ell-m})^* & \text{if } i=n-1, \\ 0 & \text{if } i>n-1. \end{cases}$$

In particular, if $A = k[x_1, \dots, x_n]$ is a graded iterated Ore extension, where each σ_i is an automorphism, and $\deg x_i = 1$ for all i, then $\ell = n$, so the non-zero cohomology groups are $\mathrm{H}^0(\mathcal{A}[m]) = A_m$ and $\mathrm{H}^{n-1}(\mathcal{A}[-n-m]) = A_m^*$, and there is a natural non-degenerate pairing

$$\mathrm{H}^0(\mathcal{A}[m]) \times \mathrm{H}^{n-1}(\mathcal{A}[-n-m]) \to \mathrm{H}^{n-1}(\mathcal{A}[-n]) = k.$$

Thus, the cohomological properties of $\mathcal{A}[m]$ are just like those of the line bundles $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$.

Definition 1.6. The cohomological dimension of a scheme $X = (\mathcal{C}, \mathcal{O})$ is

$$\operatorname{cd} X := \max\{i \mid \mathrm{H}^i(X, \mathcal{F}) \neq 0 \text{ for some } \mathcal{F} \in \mathcal{C}\}.$$

The cohomological dimension of a non-commutative scheme is like the dimension of a commutative scheme—for commutative schemes it *is* the dimension. It is not known whether the projective scheme of a noetherian algebra has finite cohomological dimension; this is an important problem.

PROPOSITION 1.7. *Let A be a graded k-algebra.*

1. $\operatorname{cd}(\operatorname{proj} A) \leq \operatorname{gldim} A - 1$.
2. *If* $\operatorname{cd}(\operatorname{proj} A)$ *is finite, then*

$$\operatorname{cd}(\operatorname{proj} A) = \max\{i \mid \underline{\mathrm{H}}^i(\mathcal{A}) \neq 0\} \leq \operatorname{injdim} A - 1.$$

PROOF. (1) This follows from (1-2).

(2) Let $d = \operatorname{cd}(\operatorname{proj} A)$, and suppose that $\mathcal{F} \in \operatorname{tails} A$ with $\underline{\mathrm{H}}^d(\mathcal{F}) \neq 0$. There is a short exact sequence $0 \to \mathcal{G} \to \oplus_{i=1}^p \mathcal{A}[n_i] \to \mathcal{F} \to 0$. The long exact sequence in cohomology gives

$$\cdots \to \oplus_{i=1}^p \underline{\mathrm{H}}^d(\mathcal{A})[n_i] \to \underline{\mathrm{H}}^d(\mathcal{F}) \to \underline{\mathrm{H}}^{d+1}(\mathcal{G}) = 0,$$

whence $\underline{\mathrm{H}}^d(\mathcal{A}) \neq 0$. This proves the equality in (2).

One the other hand, if $p \geq 1$ then $\underline{\mathrm{H}}^p(\mathcal{A}) = \varinjlim \underline{\mathrm{Ext}}_A^{p+1}(A/A_{\geq n}, A)$, which equals zero if $p \geq \operatorname{injdim} A$. $\square$

$\boxed{\text{Paul}}$ What happens in (2) if $\operatorname{injdim} A = 0$? Is $H^0(-) = 0$? i.e. $\operatorname{cd} = -1$.

PROPOSITION 1.8. *Let $f : A \to B$ be a homomorphism of noetherian algebras such that ${}_AB$ and B_A are finitely generated. Suppose that B satisfies χ°. If $\mathcal{M} \in \operatorname{tails} B$, then*

$$\mathrm{H}^i(\mathcal{M}) = \mathrm{H}^i(f_*\mathcal{M}).$$

In particular, $\operatorname{cd}(\operatorname{proj} B) \leq \operatorname{cd}(\operatorname{proj} A)$.

$\boxed{\text{Paul}}$ Is there a version of Proposition 1.8 when $f : X \to Y$ is a finite morphism of schemes?

EXERCISES

1.1 Let A be an Artin-Schelter regular algebra of global dimension n. Let $M \in \mathsf{grmod}A$ and write $\mathcal{F} = \pi M$. Show that, for $0 \le i \le n-2$, $\underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{A}) = \underline{\mathrm{Ext}}^i_A(M, A)$. [*Hint:* Use the Gorenstein property to show that all the maps in the direct system used to compute $\underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{A})$ are isomorphisms.]

2. $\mathsf{Mod}\mathbb{P}^1$

Throughout this section k is an algebraically closed field.

In this section we describe the structure of $\mathbb{P}^1$-modules by using the presentation

$$\mathsf{Mod}\mathbb{P}^1 = \mathsf{GrMod}k[x, y]/\operatorname{Tors}$$

This approach is different from the method in most algebraic geometry texts which uses the open affine cover of $\mathbb{P}^1$ by two copies of $\mathbb{A}^1$. That approach utilizes the fiber product presentation

$$\begin{tikzcd} \mathsf{Mod}\mathbb{P}^1 \arrow[r] \arrow[d] & \mathsf{Mod}k[t] \arrow[d] \\ \mathsf{Mod}k[t^{-1}] \arrow[r] & \mathsf{Mod}k[t, t^{-1}]. \end{tikzcd}$$

Each approach has its advantages. Our approach also works for $\mathsf{GrMod}_{\mathbb{Z}^2} k[x, y]/\operatorname{Fdim}$ where $k[x, y]$ is given the $\mathbb{Z}^2$-grading defined by $\deg x = (1, 0)$ and $\deg y = (0, 1)$.

Since $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, there is a quotient functor $\mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}\mathbb{A}^1$ killing a single simle module, so one expects a close relationship between $\mathsf{Mod}\mathbb{P}^1$ and $\mathsf{Mod}k[t]$. A finitely generated $k[t]$-module splits as a direct sum of a torsion and a torsion-free module; the torsion module is a direct sum of various $k[t]/(t-\lambda)^n$ and the torsion-free part is a direct sum of copies of $k[t]$, so we expect something similar for $\mathbb{P}^1$-modules. We will show that a noetherian $\mathbb{P}^1$-module is a direct sum of a torsion part, which is a direct sum of various $\mathcal{O}/\mathcal{I}_p^n$s, and a torsion-free part, which is a direct sum of various line bundles $\mathcal{O}(n)$. The fact that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles has been reproved many times by Grothendieck, Birkhoff, Hilbert, Dedekind and Weber, ... (see [**130**, page 44]. The proof of Dedekind and Weber is phrased in terms of commuting pairs of matrices, and is intimately related to the connection with finite dimensional algebras discussed in 25.6.

Set $A = k[x, y]$ with its standard $\mathbb{Z}$-grading. Since $\operatorname{Kdim} A = 2$, and Tors is the full subcategory of modules of Krull dimension zero, $\operatorname{Kdim} \mathsf{Mod}\mathbb{P}^1 = 1$. It is convenient to call a $\mathbb{P}^1$-module **torsion** if it is a direct limit of finite length modules, and **torsion-free** if it has no such submodules except zero.

We denote the image of A in $\mathsf{Mod}\mathbb{P}^1$ by $\mathcal{O}$. If $0 \neq a \in A_1$ is a linear form vanishing at $p \in \mathbb{P}^1$, we also write p for the image of A/Aa in $\mathsf{Mod}\mathbb{P}^1$, and $\mathcal{I}_p$ for the image of Aa. Thus we have a short exact sequence

$$0 \to \mathcal{I}_p \to \mathcal{O} \to p \to 0.$$

Since $\mathcal{I}_p$ is the image of $Aa \cong A(-1)$, $\mathcal{I}_p \cong \mathcal{O}(-1)$.

LEMMA 2.1. 1. *The simple $\mathbb{P}^1$-modules are $\{p = \mathcal{O}/\mathcal{I}_p \mid p \in \mathbb{P}^1\}$.*
2. *Shifting the simple modules does not change them; that is $p(n) = p$ for all $n \in \mathbb{Z}$.*
3. *If $p, q \in \mathbb{P}^1$, then* $\operatorname{Ext}_{\mathbb{P}^1}(p, q) \cong \begin{cases} k & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$.

4. $H^r(\mathbb{P}^1, p) \cong \begin{cases} k & \text{if } r = 0, \\ 0 & \text{if } r = 1. \end{cases}$.
5. $\operatorname{Ext}_{\mathbb{P}^1}(\mathcal{O}(n), p) = 0$ *for all* $n \in \mathbb{Z}$ *and all* $p \in \mathbb{P}^1$.

PROOF. (2) If a is a linear form, then $A/Aa \cong k[t]$, the polynomial ring in one variable, so multiplication by t yields an exact sequence $0 \to A/Aa(-1) \to A/Aa \to k \to 0$, whence $p \cong p(-1)$.

(1) Since every proper quotient of $k[t] \cong A/Aa$ is finite dimensional, p is a simple $\mathbb{P}^1$-module. Now let πM be a simple $\mathbb{P}^1$-module. Without loss of generality, we may assume that M has no non-zero finite dimensional submodule. Certainly M has a cyclic submodule $A/Af(n)$ for some f; since M is simple, $f \neq 0$. Writing f as a product of linear terms, we see there is an injection $A/Aa(m) \to M$ for some $0 \neq a \in A_1$. Thus $p(m)$ embeds in πM, whence $\pi M \cong p(m) \cong p$ for some $p \in \mathbb{P}^1$.

(3) Write $p = \pi M$ and $q = \pi N$, where $M = A/Aa$ and $N = A/Ab$ for some $0 \neq a, b \in A_1$. By **??**,

$$\operatorname{Ext}^1_{\mathbb{P}^1}(p, q) \cong \varinjlim \operatorname{Ext}^1_A(M_{\geq n}, N) \cong \varinjlim \operatorname{Ext}^1_A(M(-n), N).$$

A simple computation shows that $\operatorname{Ext}^1_A(M(-n), N) \cong (N/aN)_{n+1}$ which is zero for $n \geq 1$ if $p \neq q$ and is k if $p = q$.

(4) The exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to p \to 0$ gives an exact sequence in cohomology

$$\begin{aligned} 0 \to & H^0(\mathbb{P}^1, \mathcal{O}(-1)) \to H^0(\mathbb{P}^1, \mathcal{O}) \to H^0(\mathbb{P}^1, p) \\ \to & H^1(\mathbb{P}^1, \mathcal{O}(-1)) \to H^1(\mathbb{P}^1, \mathcal{O}) \to H^1(\mathbb{P}^1, p) \to 0. \end{aligned}$$

The result follows at once from the computation of the cohomology groups in Example 1.5.

(5) This is an immediate consequence of (2) and (4). □

If $p \neq q$, there is a non-split extension $0 \to A/Aab \to A/Aa \oplus A/Ab \to k \to 0$ in $\mathsf{GrMod}A$.

THEOREM 2.2. 1. *Every torsion-free noetherian module is a direct sum of various* $\mathcal{O}(n)$*s.*
2. *Every noetherian* $\mathbb{P}^1$*-module is a direct sum of a torsion and a torsion-free module.*
3. *Every torsion noetherian module is a direct sum of various* $\mathcal{O}/\mathcal{I}_p^n$.

PROOF. (1) Let $K = \operatorname{Fract} A$. Let $\mathcal{M}$ be a torsion-free noetherian module. Then $\mathcal{M} = \pi M$ for some noetherian A-module M which we may assume has no non-zero submodule of Krull dimension ≤ 1. Therefore the natural map $M \to K \otimes_A M$ is injective. Thus M embeds in a free A-module, so $\mathcal{M}$ embeds in a direct sum of various $\mathcal{O}(n)$s. Thus we have an exact sequence $0 \to \mathcal{M} \to \oplus\mathcal{O}(n) \to \mathcal{N} \to 0$. Taking cohomology gives a long exact sequence

$$0 \to \omega\mathcal{M} \xrightarrow{\theta} \omega(\oplus\mathcal{O}(n)) \cong \oplus A(n) \to \omega\mathcal{N} \to \underline{\mathrm{H}}^1(\mathbb{P}^1, \mathcal{M}) \to \cdots.$$

Let $G = \operatorname{coker}\theta$. Since $\omega\mathcal{N}$ has no finite dimensional submodules except zero, $\operatorname{pdim} G \leq 1$ by Proposition **??**. Hence $\operatorname{pdim}\omega\mathcal{M} = 0$; but projectives are free, so ωM is isomorphic to a direct sum of shifts of A; since $M = \pi\omega\mathcal{M}$, we obtain (1).

(2) This follows from (1) and Lemma 2.1.

(3) By Lemma 2.1.3, a finite length $\mathbb{P}^1$ module, say M, splits as a direct sum of various M_p's where M_p is a module all of whose composition factors are isomorphic to p. If $\pi(A/Aa) \cong p$, then $\pi(A/Aa^n) \cong \mathcal{O}/\mathcal{I}_p^n$ is a length n module all of whose slices are isomorphic to p. To prove the result it suffices to show this is indecomposable, and for this it suffices to show that $\omega(\mathcal{O}/\mathcal{I}_p^n)$ is indecomposable (cf. Exercise A.6). This follows from the fact that this module appears inside the injective envelope of A/Aa which is indecomposable. ($\boxed{\text{Paul}}$ Explain this better!) □

The splitting result in part (1) of Theorem 2.2 does not mean that every short exact sequence of torsion-free $\mathbb{P}^1$-modules splits. Indeed, applying π to the minimal resolution

$$0 \to A(-2) \to A(-1) \oplus A(-1) \to A \to k \to 0$$

gives a non-split sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0$$

corresponding to the fact that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-2)) = k$.

3. Homogeneous coordinate rings of projective schemes

We wish to extend the notion of a homogeneous coordinate ring of a classical scheme to the schemes defined in Chapter 20.

How can we recover A from $(\mathsf{Tails}A, \mathcal{A}, [1])$? In general, this is not possible; the best we can do is recover $\omega\pi A$ with a graded algebra structure. This is done by using the B functor described in section 1 of chapter 16. The question is like asking how to recover a homogeneous coordinate ring for a scheme from the scheme itself.

Example 3.1. If $\mathcal{L}$ is a line bundle on a scheme Y and $s = \mathcal{L} \otimes_{\mathcal{O}_Y} -$, then

$$B(\mathsf{Coh}Y, \mathcal{O}_Y, s) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(Y, \mathcal{L}^{\otimes n}).$$

The functoriality of B has the following interpretation. If $f : X \to Y$ is a morphism of schemes there is a morphism of triples

$$(f^*, \mathbb{1}, \mathbb{1}) : (\mathsf{Coh}Y, \mathcal{O}_Y, s) \to (\mathsf{Coh}X, \mathcal{O}_X, s')$$

where $s' = f^*\mathcal{L} \otimes_{\mathcal{O}_X} -$ and hence, by Proposition 16.1.2, a homomorphism of the corresponding graded algebras. On the degree n component this is the natural map $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \to \mathrm{H}^0(X, f^*\mathcal{L}^{\otimes n})$.

Example 3.2. By definition,

$$B(\mathsf{Tails}A^{\mathrm{op}}, \mathcal{A}, [1]) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathsf{Tails}}(\mathcal{A}, \mathcal{A}[n]) \cong \omega\pi A$$

where the last isomorphism is given by ???. Thus $\omega\pi A$ has a natural graded algebra structure, and the functoriality of B applied to the morphism of triples

$$(\pi, \mathbb{1}, \mathbb{1}) : (\mathsf{GrMod}A^{\mathrm{op}}, A, [1]) \to (\mathsf{Tails}A^{\mathrm{op}}, \mathcal{A}, [1])$$

shows that the natural map $A \to \omega\pi A$ is a graded algebra homomorphism.

We will call $\omega\pi A$ the torsion closure of A.

EXAMPLE 3.3. Let A be a finitely generated, $\mathbb{N}$-graded, commutative k-algebra, and suppose that A is a domain. Write

$$A' = B(\operatorname{tails} A, \mathcal{A}, [1])_{\geq 0}.$$

The canonical map $f : A \to A' \cong (\omega\pi A)_{\geq 0}$ is injective, because $\ker f = \tau A = 0$ since A is a domain. Moreover, since A is commutative, it satisfies χ and also χ°. Thus A' is a finitely generated A-module and the quotient module A'/A is bounded. It follows that $A'_n = A_n$ for $n \gg 0$, which implies that $A \subset A' \subset \operatorname{Fract} A$. Finally since, A' is a finitely generated A-module, A' is contained in the integral closure of A. In particular, if A is integrally closed, then $A' = A$. This should be compared with [**76**, Exercise 5.14, Chapter II].

The simplest explicit example arises with $A = k[t^2, t^3] \subset A' = k[t]$. In this case A' *is* the integral closure of A, but this is not usually true: the integral closure of $A = k[t^2, t^3, s]$ is the polynomial ring $k[t, s]$, but this is not A' since $k[t, s]/A$ is not bounded.

EXAMPLE 3.4. Consider the triple $(\mathsf{Tails}A^{\mathrm{op}}, \mathcal{A}, [1])$, and an object $\mathcal{F} = \pi M$, where $M \in \mathsf{GrMod}A$. Then

$$\Gamma M = \bigoplus \operatorname{Hom}_{\mathsf{Tails}}(\mathcal{A}, \mathcal{M}[n]) \cong \omega\pi M.$$

Thus $\omega\pi M$ becomes a graded right $\omega\pi A$-module. The graded algebra homomorphism $A \to \omega\pi A$ then induces an action of A on $\omega\pi M$; it is easily checked that this is the natural action of A.

EXERCISES

3.1 Let A and B be graded k-algebras, and write A' and B' for their torsion closures. Let $f : A \to B$ be a graded algebra homomorphism such that either B_A is finitely generated or $\operatorname{coker} f$ is right bounded. Thus there is an induced map $f^* : (\operatorname{Proj} A, [1]) \to (\operatorname{Proj} B, [1])$. By functoriality of $B(-)$, there is an induced algebra map $f : A' \to B'$. Show that this extends f. In particular, $\sigma \in \operatorname{Aut}_k A$ extends to an automorphism of $\omega\pi A$.
Some hypotheses are neccessary for f to extend to the torsion closures (take the polynomial rings $A = k[x]$ and $B = k[x, y]$).

$\boxed{\text{Paul}}$
Is $\omega\pi A$ its own torsion closure?
What is the relation between $\omega\pi(A \circ B)$ and $(\omega\pi A) \circ (\omega\pi B)$?
Notice that $\omega\pi(A \otimes_k B) \ncong (\omega\pi A) \otimes_k (\omega\pi B)$; take $A = B = k[x]$.

4. The non-commutative version of Serre's theorem

In this section all schemes are defined over a fixed commutative base ring k.

The main result in this section is due to M. Artin and J. Zhang. There are two interpretations of it. First, it is an abstract characterization of the triples $(\mathcal{C}, \mathcal{O}, s)$ which can arise as the projective scheme of a noetherian graded k-algebra satisfying condition χ_1. Second, it shows to what extent a noetherian algebra A can be recaptured from the triple $(\operatorname{tails} A^{\mathrm{op}}, \mathcal{A}, [1])$.

LEMMA 4.1. *Let $X = (\mathcal{C}, \mathcal{O}, s)$ be a polarized k-scheme. Given an object $\mathcal{F}$ and an integer r, there exist integers $n_1, \dots, n_p$, all greater than r, and an epimorphism*

$$\bigoplus_{i=1}^{p} \mathcal{O}(-n_i) \to \mathcal{F}.$$

PROOF. Applying the definition of ampleness to $\mathcal{F}(r)$, there exist integers $\ell_i > 0$ and an epimorphism $\oplus_{i=1}^{p}\mathcal{O}(-\ell_i) \to \mathcal{F}(r)$. Now apply the shift functor s^r to this and define $n_i = \ell_i + r$. $\square$

THEOREM 4.2. *Let $X = (\mathcal{C}, \mathcal{O}, s)$ be a polarized k-scheme and define $A = B(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$. Suppose that*

1. *s is ample,*
2. *$\mathcal{O}$ is a noetherian object in $\mathcal{C}$,*
3. *$A_0 = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O})$ is right noetherian, and*
4. *$\mathrm{H}^0(X, \mathcal{F})$ is a finitely generated right A_0-module for all $\mathcal{F}$ in $\mathcal{C}$.*

Then A is a right noetherian, $\mathbb{N}$-graded k-algebra satisfying χ_1, and

$$(\mathcal{C}, \mathcal{O}, s) \cong (\mathrm{tails}\, A^{\mathrm{op}}, \mathcal{A}, [1]). \tag{4-1}$$

PROOF. We begin with some preliminary considerations. Write $B = B(\mathcal{C}, \mathcal{O}, s)$, and recall that there is a commutative diagram

$$\begin{tikzcd} \mathsf{GrMod} B^{\mathrm{op}} \arrow[r] \arrow[d, "\pi"'] & \mathsf{GrMod} A^{\mathrm{op}} \arrow[d, "\pi"] \\ \mathsf{Tails} B^{\mathrm{op}} \arrow[r, "\sim"] & \mathsf{Tails} A^{\mathrm{op}} \end{tikzcd}$$

with the bottom arrow an isomorphism (19.2.10). As in the previous section, we write $\Gamma : \mathcal{C} \to \mathsf{GrMod} A^{\mathrm{op}}$ for the functor

$$\Gamma = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^n -);$$

strictly speaking Γ takes values in $\mathsf{GrMod} B^{\mathrm{op}}$. The isomorphism in (4-1) will be implemented by the functor

$$\pi \circ \Gamma : \mathcal{C} \longrightarrow \mathsf{Tails} A^{\mathrm{op}}.$$

We will also consider the subfunctor

$$\Gamma_+ = \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^n -)$$

of Γ. Since A is $\mathbb{N}$-graded, $\Gamma_+\mathcal{F}$ is an A-submodule of $\Gamma\mathcal{F}$. The inclusion $\Gamma_+\mathcal{F} \subset \Gamma\mathcal{F}$ amounts to a natural transformation $\Gamma_+ \to \Gamma$ which induces a natural equivalence $\pi\Gamma_+ \to \pi\Gamma$ because $\Gamma\mathcal{F}/\Gamma_+\mathcal{F}$ is torsion. Hence we may, and will, use either $\pi\Gamma$ or $\pi\Gamma_+$ in the proof. In particular, we will show that $\Gamma_+\mathcal{F}$ is a finitely generated A-module and hence that $\pi\Gamma$ takes values in tails A^{op}.

The proof is broken into nine steps.

Step 1. $\pi\Gamma$ is an exact functor.

Proof. First, $\pi\Gamma$ is left exact because s is exact, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, -)$ is left exact, and π is exact. To see that $\pi\Gamma$ is right exact, let $f : \mathcal{F} \to \mathcal{G}$ be an epimorphism in $\mathcal{C}$. Since s is ample, $\Gamma f : \Gamma\mathcal{F} \to \Gamma\mathcal{G}$ is surjective in high degree, whence $\pi\Gamma f : \pi\Gamma\mathcal{F} \to \pi\Gamma\mathcal{G}$ is an epimorphism in $\mathsf{Tails} A^{\mathrm{op}}$.

Step 2. If $\mathcal{G}$ is a subobject of $\mathcal{F}$, then $\Gamma\mathcal{F}/\Gamma\mathcal{G}$ is torsion-free.

Proof. Let $x \in (\Gamma\mathcal{F})_r$, and suppose that $x.A_{\geq n} \subset \Gamma\mathcal{G}$. By definition, $x : \mathcal{O} \to \mathcal{F}(r)$; we will show that $\mathrm{Im}(x) \subset \mathcal{G}(r)$, whence $x \in \Gamma\mathcal{G}$, thus proving the result.

Let $u : \oplus_{i=1}^{p} \mathcal{O}(-n_i) \to \mathcal{O}$ be an epimorphism, with each $n_i \geq n$. Then u is a sum of morphisms $u_i : \mathcal{O}(-n_i) \to \mathcal{O}$; thus $s^{n_i} u_i \in A_{\geq n}$, so $x.(s^{n_i} u_i) \in \Gamma\mathcal{G}$. But $x.(s^{n_i} u_i)$ is the composition

$$\mathcal{O} \xrightarrow{s^{n_i} u_i} \mathcal{O}(n_i) \xrightarrow{s^{n_i} x} \mathcal{F}(r + n_i),$$

so $\operatorname{Im}(s^{n_i}(x \circ u_i)) \subset \mathcal{G}(r + n_i)$. Hence $\operatorname{Im}(x \circ u) = \oplus_{i=1}^{p} \operatorname{Im}(x \circ u_i) \subset \mathcal{G}(r)$. Since u is an epimorphism, $\operatorname{Im}(x) \subset \mathcal{G}(r)$.

Step 3. $\pi\Gamma$ is faithful.

Proof. Let $\mathcal{F}$ be a non-zero object of $\mathcal{C}$. Ampleness ensures that, for some $n > 0$, there is a non-zero morphism $\mathcal{O}(-n) \to \mathcal{F}$ and hence a non-zero morphism $\mathcal{O} \to \mathcal{F}(n)$. Hence $\Gamma\mathcal{F} \neq 0$, so by Step 2, with $\mathcal{G} = 0$, $\Gamma\mathcal{F}$ is not torsion. Thus $\pi\Gamma\mathcal{F} \neq 0$; the faithfulness now follows from Proposition 11.12 in Appendix A.

Step 4. For all d, $(\Gamma\mathcal{F})_{\geq d}$ is a finitely generated A-module. In particular, $\pi\Gamma_+ : \mathcal{C} \to \mathsf{tails}\, A^{\mathrm{op}}$.

Proof. Since $(\Gamma\mathcal{F})_{\geq d} = \Gamma_+\mathcal{F}(d)$ as ungraded A-modules, it suffices to show that $\Gamma_+\mathcal{F}$ is finitely generated. We do this first for $\mathcal{F} = \mathcal{O}(-r)$ with $r > 0$. As noticed in the previous section (1), there is an inclusion of graded A-modules

$$A[-r] = \Gamma_+\mathcal{O}[-r] \subset \Gamma_+\mathcal{O}(-r), \tag{4-2}$$

with the quotient being bounded. By hypothesis (3), this quotient is a finitely generated A_0-module, whence $\Gamma_+\mathcal{O}(-r)$ is a finitely generated A-module.

Now let $\mathcal{F}$ be arbitrary. Since s is ample, there is an epimorphism $f : \mathcal{P} \to \mathcal{F}$ from a finite sum of negative shifts of $\mathcal{O}$. By ampleness, $\operatorname{coker}(\Gamma f)$ is right bounded, whence $\operatorname{coker}(\Gamma_+ f)$ is bounded, and therefore a finitely generated A-module because each component is a finitely generated A_0-module. By the previous paragraph $\Gamma_+\mathcal{P}$ is a finitely generated A-module, hence so is $\Gamma_+\mathcal{F}$.

Step 5. A is right noetherian.

Proof. It suffices to show that each graded right ideal J of A is finitely generated. If X is a finite homogeneous subset of J, define

$$\mathcal{P}_X := \bigoplus_{x \in X} \mathcal{O}(-\deg x)$$

and

$$f_X := \bigoplus_{x \in X} s^{-\deg x} x \ : \ \mathcal{P}_X \to \mathcal{O}.$$

Since $\mathcal{O}$ is noetherian we may choose a finite set X such that $\operatorname{Im}(f_X)$ is maximal amongst the images over all possible X. We fix such an X, and write $f = f_X$ and $\mathcal{P} = \mathcal{P}_X$. Define

$$P := \Gamma_+\mathcal{P} \qquad \text{and} \qquad P' := \bigoplus_{x \in X} A[-\deg x].$$

By (4-2), $P' \subset P$ and P/P' is a finitely generated A_0-module.

By Lemma 1.4, $\Gamma_+(s^{-\deg x} x)$ is left multiplication by x, so $(\Gamma_+ f)(P')$ is the right ideal of A generated by X. By maximality of $\operatorname{Im}(f)$, $J \subset (\Gamma f)(P)$. Therefore

$$(\Gamma_+ f)(P') \subset J \subset (\Gamma_+ f)(P).$$

Since P/P' is a finite A_0-module, the quotient of the two outer terms is a finite A_0-module, hence so is $J/(\Gamma_+ f)(P')$ because A_0 is right noetherian. But P' is a finitely generated A-module, whence J is too.

Step 6. If $M \in \mathsf{grmod} A^{\mathrm{op}}$, there exists some $\mathcal{F}$ in $\mathcal{C}$ such that $\pi\Gamma_+\mathcal{F} \cong \pi M$.

Proof. Since $\pi M \cong \pi M_{\geq 1}$ we can, and will, replace M by $M_{\geq 1}$. Since A is right noetherian, there is an exact sequence

$$P \xrightarrow{f} Q \to M \to 0$$

with P and Q finite direct sums of *negative* shifts of A. Let $\mathcal{P}$ and $\mathcal{Q}$ be the corresponding sums of negative shifts of $\mathcal{O}$. A module map $A[i] \to A[j]$ is left multiplication by an element of $A_{j-i} = \mathrm{Hom}(\mathcal{O}, s^{j-i}\mathcal{O}) \cong \mathrm{Hom}(s^i\mathcal{O}, s^j\mathcal{O})$, so corresponds to a map $\mathcal{O}(i) \to \mathcal{O}(j)$. Hence there is a well-defined map $\varphi : \mathcal{P} \to \mathcal{Q}$ corresponding to f. Let $\mathcal{M} = \operatorname{coker}\varphi$, and apply $\pi\Gamma_+$ to the exact sequence

$$\mathcal{P} \xrightarrow{\varphi} \mathcal{Q} \to \mathcal{M} \to 0;$$

since $\pi\Gamma_+\mathcal{O}(-n) \cong A[-n]$ by (4-2), this gives the exact sequence

$$\pi P \xrightarrow{\pi f} \pi Q \to \pi\Gamma_+\mathcal{M} \to 0,$$

so $\pi\Gamma_+\mathcal{M} \cong \pi M$.

Step 7. $\Gamma\mathcal{M} \cong \omega\pi\Gamma\mathcal{M}$ for all $\mathcal{M}$ in $\mathcal{C}$.

Proof. Let $M = \Gamma\mathcal{M}$. By Step 2, with $\mathcal{G} = 0$, M is torsion-free. Hence, to prove that $M \cong \omega\pi M$, it suffices to show that if M' is an essential extension of M, then M'/M is torsion-free.

Let M' be an essential extension of M. Then M' is also torsion-free by Proposition 19.3.10. Suppose that $x \in M'_r$ and that $xA_{\geq n} \subset M$.

By Lemma 4.1, there is an exact sequence

$$\mathcal{P} = \bigoplus \mathcal{O}(-m_{ij}) \xrightarrow{f} \mathcal{Q} = \bigoplus \mathcal{O}(-n_i) \xrightarrow{g} \mathcal{O}(-r) \to 0.$$

with each $n_i \geq n+r$. Write $g = (g_i)$ and $f = (f_{ij})$ where $g_i : \mathcal{O}(-n_i) \to \mathcal{O}(-r)$ and $f_{ij} : \mathcal{O}(-m_{ji}) \to \mathcal{O}(-n_i)$. Thus $\sum_i g_i f_{ij} = 0$ for all j. Thus $s^{n_i}g_i \in A_{n_i - r} \subset A_{\geq n}$, so $x.s^{n_i}g_i \in M_{n_i}$. We can assume, by increasing the number of terms in $\mathcal{Q}$ if necessary, that $\{s^{n_i}g_i\}$ generates $A_{\geq d}$ for some large d. Define $\Phi := \oplus_i \phi_i$, where

$$\phi_i := s^{-n_i}(x.s^{n_i}g_i) : \mathcal{O}(-n_i) \to \mathcal{M}.$$

Thus $\phi \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{Q}, \mathcal{M})$ and $\phi \circ f = 0$. By the left exactness of $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{M})$ applied to the exact sequence above, there exists $\psi : \mathcal{O}(-r) \to \mathcal{M}$ such that $\phi = \psi \circ g$; i.e., $\psi \circ g_i = \phi_i$. If $y = s^r\psi \in M_r$, then

$$y.s^{n_i}g_i = s^{n_i - r}y \circ s^{n_i}g_i = s^{n_i}(\psi \circ g_i) = x.s^{n_i}g_i,$$

so $(y - x).s^{n_i}g_i = 0$. Therefore $(y - x)A_{\geq d} = 0$; since M' is torsion-free $y = x$. Thus $x \in M$, showing that $M' = M$ as required.

Step 8. $\pi \circ \Gamma_+ : \mathcal{C} \to \operatorname{tails} A^{\mathrm{op}}$ is an equivalence, and induces an isomorphism $(\mathcal{C}, \mathcal{O}, s) \cong (\operatorname{tails} A^{\mathrm{op}}, \mathcal{A}, [1])$.

Proof. Once we have shown that $\pi\Gamma_+$ is an equivalence, the rest is clear, since $\pi\Gamma_+\mathcal{O} = \pi A = \mathcal{A}$, and $\pi\Gamma_+\mathcal{F}(1) \cong \pi\Gamma\mathcal{F}(1) \cong (\pi\Gamma\mathcal{F})[1]$, so $\pi\Gamma_+ \circ s \cong [1] \circ \pi\Gamma_+$.

Since Step 6 showed that every object in $\operatorname{tails} A^{\mathrm{op}}$ is isomorphic to an object in the image of $\pi\Gamma_+$, by Theorem A.5.12, it remains to show that $\pi\Gamma_+$ is fully faithful; that is, we must show that, if $\mathcal{M}$ and $\mathcal{N}$ are objects of $\mathcal{C}$, then the functorial map

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M}) \to \mathrm{Hom}_{\mathsf{Tails}}(\pi\Gamma\mathcal{N}, \pi\Gamma\mathcal{M})$$

is an isomorphism. By the ampleness hypothesis, there are finite direct sums of shifts of $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{Q}$ say, and an exact sequence $\mathcal{P} \to \mathcal{Q} \to \mathcal{N} \to 0$. Using the exactness of $\pi\Gamma$ and the left exactness of the Hom functor, the rows of the following commutative diagram are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(\mathcal{N},\mathcal{M}) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(\mathcal{Q},\mathcal{M}) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(\mathcal{P},\mathcal{M}) \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & \operatorname{Hom}(\pi\Gamma\mathcal{N},\pi\Gamma\mathcal{M}) & \longrightarrow & \operatorname{Hom}(\pi\Gamma\mathcal{Q},\pi\Gamma\mathcal{M}) & \longrightarrow & \operatorname{Hom}(\pi\Gamma\mathcal{P},\pi\Gamma\mathcal{M}).
\end{array}$$

So it suffices to show that the two rightmost vertical maps are isomorphisms. This reduces to proving the result for $\mathcal{N} = \mathcal{O}$.

Write $M = \Gamma\mathcal{M}$, and notice that $\omega\pi M \cong M$ by Step 7. By definition $\operatorname{Hom}_{\mathcal{C}}(\mathcal{O},\mathcal{M}) = M_0$, and this equals $\operatorname{Hom}_{\mathrm{Gr}}(A, M) = \operatorname{Hom}_{\mathrm{Gr}}(A, \omega\pi M)$. Using the adjoint isomorphism, this is isomorphic to $\operatorname{Hom}_{\mathsf{Tails}}(\pi A, \pi M)$, whence

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}(\mathcal{O},\mathcal{M}) &\cong \operatorname{Hom}_{\mathsf{Tails}}(\pi A, \pi M) \\
&= \operatorname{Hom}_{\mathsf{Tails}}(\pi\Gamma_+\mathcal{O}, \pi\Gamma\mathcal{M}) \\
&= \operatorname{Hom}_{\mathsf{Tails}}(\pi\Gamma\mathcal{O}, \pi\Gamma\mathcal{M}),
\end{aligned}$$

as required. Thus $\pi\Gamma_+$ is an equivalence.

Step 9. A satisfies χ_1.

Proof. By Theorem 19.4.4, it suffices to show that $(\omega\pi M)_{\geq d}$ is a finitely generated A-module for all $d \in \mathbb{Z}$, and all $M \in \mathsf{grmod} A^{\mathrm{op}}$. By Step 6, $\pi M \cong \pi\Gamma_+\mathcal{F} \cong \pi\Gamma\mathcal{F}$ for some $\mathcal{F}$ in $\mathcal{C}$, so $\omega\pi M \cong \omega\pi\Gamma\mathcal{F} \cong \Gamma\mathcal{F}$ by Step 7. Step 4 showed that $(\Gamma\mathcal{F})_{\geq d}$ is finitely generated, so the result holds.

The proof of the Theorem is now complete. □

THEOREM 4.3. *Suppose that A is a right noetherian, $\mathbb{N}$-graded k-algebra satisfying χ_1. Then the triple* $(\operatorname{tails} A^{\mathrm{op}}, \mathcal{A}, [1])$ *satisfies the hypotheses of Theorem 4.2, and there is a natural graded algebra homomorphism $f : A \to B := B(\operatorname{tails} A^{\mathrm{op}}, \mathcal{A}, [1])$ such that*

1. $B \cong \omega\pi A$ *as a right A-module,*
2. $\ker f = \tau A$ *and* $\operatorname{coker} f$ *are right bounded,*
3. f *induces an isomorphism* $(\operatorname{proj} B_{\geq 0}, [1]) \cong (\operatorname{proj} A, [1])$.

PROOF. First the canonical map

$$A_n = \operatorname{Hom}_{\mathrm{Gr}}(A, A[n]) \xrightarrow{\pi} \operatorname{Hom}_{\mathsf{Tails}}(\mathcal{A}, \mathcal{A}[n]) = B_n$$

induces the algebra homomorphism $f : A \to B$. As noted in (1-1), $B \cong \omega\pi A$. Hence $\ker f = \tau A$, and this is right bounded because A is right noetherian. Also, A satisfies χ_1° by Proposition 19.4.3, so $\operatorname{coker} f$ is right bounded by Theorem 19.4.4.

It remains to prove that the hypotheses of Theorem 4.2 are satisfied. By Proposition 3.2, [1] is ample. Since A is right noetherian, $\mathcal{A}$ is a noetherian object of $\operatorname{tails} A^{\mathrm{op}}$; thus condition (1) in (4.2) holds.

Since A is right noetherian, so is its quotient ring A_0. If $\mathcal{F} \in \operatorname{tails} A^{\mathrm{op}}$, then $\mathcal{F} \cong \pi M$ for some $M \in \mathsf{grmod} A^{\mathrm{op}}$, whence $H^0(X, \mathcal{F}) = (\omega\pi M)_0$; but $(\omega\pi M)_{\geq 0}$ is a finitely generated A-module, so its degree zero component is a finitely generated A_0-module. That is, $H^0(X, \mathcal{F})$ is a finitely generated A_0-module. In particular, $B_0 = \mathrm{H}^0(X, \mathcal{A})$ is a finitely generated A_0-module, hence a right noetherian ring. This proves that condition (2) holds; condition (3) follows because $H^0(X, \mathcal{F})$ being a finitely generated A_0-module certainly implies it is a finitely generated B_0-module. Thus all the hypotheses of Theorem 4.2 hold.

The isomorphism of polarized schemes is now a consequence of Theorem 4.2. That isomorphism is induced by the functor $\pi_B\Gamma$. It is not difficult to see that f_* is an adjoint to this. $\square$

If A is a prime, noetherian, locally finite, $\mathbb{N}$-graded k-algebra, then $\tau A = 0$, so $A \subset A' := (\omega\pi A)_{\geq 0}$. If A also satisfies χ_1, then A'/A is finite dimensional, so $A'_n = A_n$ for $n \gg 0$, whence $A \subset A' \subset \operatorname{Fract} A$. hence if A is a maximal order, then $A = A'$.

THEOREM 4.4 (Serre's Theorem [**144**]). *Let $A = k[x_0, \dots, x_n]/I$ be a graded quotient of the commutative polynomial ring, and let $X \subset \mathbb{P}^n$ be the subscheme defined by I. There is an equivalence of categories*

$$(\mathsf{Coh} X, \mathcal{O}_X, (1)) \cong (\operatorname{tails} A, \mathcal{A}, [1]),$$

where (1) *denotes the shift functor* $\mathcal{F} \mapsto \mathcal{F}(1) = \mathcal{O}(1) \otimes_{\mathcal{O}_X} \mathcal{F}$.

PROOF. It is probably best to prove this directly using Serre's approach, but one can also use Theorem 4.2. The latter approach requires one to first show that $(\mathsf{Coh} X, \mathcal{O}_X, (1))$ satisfies the hypotheses of Theorem 4.2, whence one obtains from that result

$$(\mathsf{Coh} X, \mathcal{O}_X, (1)) \cong (\operatorname{tails} A', \mathcal{A}', [1])$$

where $A' = B(\mathsf{Coh} X, \mathcal{O}_X, (1))_{\geq 0}$. Next one shows that there is a map $f : A \to A'$ with torsion kernel and cokernel, so

$$(\operatorname{tails} A, \mathcal{A}, [1]) \cong (\operatorname{tails} A', \mathcal{A}', [1]). \tag{4-3}$$

We'll start by proving the second fact when I is a prime ideal for simplicity. Recall that

$$A' = \bigoplus_{d \geq 0} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}(d)) = \bigoplus_{d \geq 0} \mathrm{H}^0(X, \mathcal{O}(d)).$$

Now, $\mathcal{O}(d) = \widetilde{A[d]}$, so on the open set $U_i = \{p \in X \mid x_i(p) \neq 0\}$, $\mathcal{O}(d)(U_i) = A[x_i^{-1}]_d$. We will identify $\mathrm{H}^0(X, \mathcal{O}(d))$ with $\bigcap_{i=0}^n A[x_i^{-1}]_d$ of $\operatorname{Fract} A$. Thus $A \subset A' = \bigcap_{i=0}^n A[x_i^{-1}] \subset \operatorname{Fract} A$. Let $f \in A'_d$. Choose m such that $x_i^m f \in A$ for all i. Since $(x_0^m, \dots, x_n^m)$ is of finite codimension in A, for large r, $A_{\geq r} f \subset A$; in fact, since $\deg f = d \geq 0$, $A_{\geq r} f \subset A_{\geq r}$ so it follows that $A_{\geq r} f^j \subset A$ for all $j \geq 0$. Hence, $A_{\geq r}.A[f] \subset A$; if $y \in A_{\geq r}$, then $A[f] \subset Ay^{-1}$. In particular, $A[f]$ is a finitely generated A-module, so belongs to the integral closure $\bar{A}$ of A in $\operatorname{Fract} A$. Since $\bar{A}$ is a finitely generated A-module, so is A'. Hence A'/A is a finitely generated A-module. It is also torsion, so is finite dimensional. Thus the inclusion $A \to A'$ has finite dimensional kernel and cokernel, so induces the isomorphism (4-3). $\square$

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$\boxed{\text{Paul}}$

Suppose $(f, \theta, \mu) : (\mathcal{C}, \mathcal{O}, s) \to (\mathcal{C}', \mathcal{O}', s')$ gives a map. Functoriality yields a graded algebra homomorphism $g : B \to B'$. Suppose that Theorem 4.2 applies to both $\mathcal{C}$ and $\mathcal{C}'$. If B' is a finitely generated B-module on both sides, then we have an adjoint pair $g^* : \operatorname{tails} B \cong \mathcal{C} \to \operatorname{tails} B' \cong \mathcal{C}'$ and $g_* : \operatorname{tails} B' \cong \mathcal{C}' \to \operatorname{tails} B \cong \mathcal{C}$. Is g^* naturally equivalent to f??? If so, then f has a right adjoint; this would show that in some nice situations all maps according to Definition 2.1 are induced by algebra homomorphisms, and have right adjoints.

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5. Serre duality

We begin with Serre duality in the commutative setting. Let k be a field.

THEOREM 5.1 (Serre). *Let X be an n-dimensional projective scheme over k.*

1. *There is a coherent $\mathcal{O}_X$-module ω_X^0 and a k-linear map $t : H^n(X, \omega_X^0) \to k$ such that for all $\mathcal{F} \in \mathsf{Coh}X$ the natural pairing*
$$\operatorname{Hom}(\mathcal{F}, \omega_X^0) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^0)$$
followed by t gives an isomorphism
$$\theta^0 : \operatorname{Hom}(\mathcal{F}, \omega_X^0) \xrightarrow{\sim} H^n(X, \mathcal{F})^*.$$
2. *For all $i \geq 0$, and all $\mathcal{F} \in \mathsf{Coh}X$, there is a natural functorial map*
$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X^0) \to H^{n-i}(X, \mathcal{F})^*.$$
3. *The following conditions are equivalent:*
   (a) *X is Cohen-Macaulay and equi-dimensional;*
   (b) *for each locally free $\mathcal{F} \in \mathsf{Coh}X$, and for all $i < n$, $H^i(X, \mathcal{F}(-q)) = 0$ for all $q \gg 0$;*
   (c) *all θ^i are isomorphisms.*

REMARK 5.2. The sheaf ω_X^0 is called the dualizing sheaf for X, and it is unique up to isomorphism: if ω is another dualizing sheaf then (1) says that $\operatorname{Hom}(-, \omega_X^0) \cong H^n(X, -)^* \cong \operatorname{Hom}(-, \omega)$ so, by Yoneda's Lemma, $\omega_X^0 \cong \omega$. The map t is called the trace morphism. The pair $\omega_X^0, t)$ is unique up to isomorphism.

When $X = \mathbb{P}^0 \cong \operatorname{Spec}(k)$ is a single reduced point, the duality in (1) reduces to the duality for finite dimensional vector spaces.

The proof of Serre duality in Hartshorne's book [**76**, Chapter III, Section 7] first establishes (1) for $\mathbb{P}_k^N$, then uses an embedding $X \subset \mathbb{P}^N$ to establish (1) for X. The proof of (2) follows from (1) by general functorial principles. For the non-commutative version of Serre duality the strategy is to prove (1) directly for $\operatorname{proj} A$, then (2) follows in the same way. Conditions (b) and (c) are equivalent in the non-commutative setting, and these may be used to define a 'non-commutative equi-dimensional Cohen-Macaulay scheme'. Thus the non-commutative version of Serre duality we prove below encompasses all of the commutative version with the exception of condition (a).

To have a non-commutative version of Serre duality we certainly need that $H^i(X, -) = 0$ for $i \gg 0$; however, as discussed in section 1, it is not known for a general noetherian A whether $\operatorname{proj} A$ has this property; for that reason, we require a hypothesis that $\operatorname{proj} A$ have finite cohomological dimension.

THEOREM 5.3 (Zhang). *Let A be an $\mathbb{N}$-graded left noetherian, locally finite k-algebra satisfying χ, and suppose that* $\operatorname{cd}(\operatorname{proj} A) = n < \infty$. *Suppose that $\underline{\mathrm{H}}^n(\mathcal{A})^* \in \mathsf{grmod}A$, and set $\omega^0 := \pi\, \underline{\mathrm{H}}^n(\mathcal{A})^*$. Then*

1. *for each i there is a natural transformation $\eta_i : \operatorname{Ext}^i(-, \omega^0) \to H^{n-i}(-)^*$, and η_0 is a natural isomorphism;*
2. *η_i is an isomorphism for all i if and only if $\underline{\mathrm{H}}^{n-i}(\mathcal{A})$ is finite dimensional whenever $1 \leq i < n$, and $\underline{\mathrm{H}}^0(\mathcal{A})$ is left bounded.*

Definition 5.4. Let A be a graded noetherian k-algebra. We write $\mathbb{D}(A)$ for the derived category of $\mathsf{GrMod}A$, and $\mathbb{D}^b_f(A)$ for the derived category of bounded complexes with finitely generated homology groups.

We write $A^e = A \otimes_k A^{\mathrm{op}}$. A complex $R^\bullet \in \mathbb{D}^b(A^e)$ is $\mathsf{dualizing}$ if the functors $\mathbb{R}\operatorname{Hom}_{\operatorname{Gr} A}(-, R^\bullet)$ and $\mathbb{R}\operatorname{Hom}_{\operatorname{Gr} A^{\mathrm{op}}}(-, R^\bullet)$ induce dualities between $\mathbb{D}^b_f(A)$ and $\mathbb{D}^b_f(A^{\mathrm{op}})$.

CHAPTER 22

Twisted homogeneous coordinate rings

In this chapter we fix a field k, and all schemes will be k-schemes.

If $\mathcal{L}$ is an ample line bundle on a scheme X, we call

$$\bigoplus_{d=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}^{\otimes d})$$

a homogeneous coordinate ring of X. The graded algebras $B(X, \sigma, \mathcal{L})$ we construct and study in this chapter, depending on a scheme X, an automorphism σ, and a line bundle $\mathcal{L}$, are non-commutative generalizations of these; indeed, $B(X, \mathbb{1}, \mathcal{L}) = \oplus_{d=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}^{\otimes d})$. If the data $(X, \sigma, \mathcal{L})$ satisfies the conditions in Definition 1.6 $B(X, \sigma, \mathcal{L})$ is called a twisted homogeneous coordinate ring of X. The relation between $B(X, \mathbb{1}, \mathcal{L})$ and $B(X, \sigma, \mathcal{L})$ is analogous to the relation between A and A^σ, the twist defined in chapter 16; indeed, sometimes $B(X, \sigma, \mathcal{L}) \cong B(X, \mathbb{1}, \mathcal{L})^\sigma$.

Section 1 begins with a definition of the algebras $B(X, \sigma, \mathcal{L})$ and their basic properties. We may associate to $(X, \sigma, \mathcal{L})$ a triple $(\mathcal{C}, \mathcal{O}, s)$ in the category GTrip, and it is shown that $B(X, \sigma, \mathcal{L}) \cong B(\mathcal{C}, \mathcal{O}, s)$. This provides one route for understanding $B(X, \sigma, \mathcal{L})$ via Theorem 20.4.2, at least when s is ample; the ampleness of s is equivalent to the condition that $\mathcal{L}$ be σ-ample (Definition 1.6).

Some elementary examples in section 2 show how the $B(X, \sigma, \mathcal{L})$ are related to the twisting construction in chapter 16, and to Ore extensions.

Section 3 describes a proceedure which associates to a connected algebra a sequence of schemes which, in some situations, is sufficient to characterize the algebra. From that data one may construct a graded algebra B and a map from the original algebra to B; sometimes the algebra B is of the form $B(X, \sigma, \mathcal{L})$, so the construction can be viewed as a generalization of the twisted homogeneous coordinate rings.

Section 4 examines the structure of $B(X, \sigma, \mathcal{L})$ in more detail, particularly when σ is of finite order. In that case, B is a finite module over its center.

Section 5 shows that, under suitable hypotheses on $(X, \sigma, \mathcal{L})$, the category of graded $B(X, \sigma, \mathcal{L})$-modules is equivalent to $\mathsf{Mod}(\mathcal{O}_X)$; compare this with the fact that $\mathsf{GrMod}(A)$ and $\mathsf{GrMod}(A^\sigma)$ are equivalent. This is a very powerful and useful result. It eliminates much of the mystery surrounding the non-commutative algebra $B(X, \sigma, \mathcal{L})$, and allows one to bring to bear the machinery of algebraic geometry.

Section 6 examines the structure of $B(X, \sigma, \mathcal{L})$ when X is an elliptic curve.

1. Definition and basic properties

NOTATION . If X is a k-scheme and $\sigma \in \operatorname{Aut}_k X$, we write p^σ for the image of a point $p \in X$ under σ. We write $\mathcal{K}$ for the sheaf of total quotient rings of $\mathcal{O}_X$; that is, $\mathcal{K}$ is the sheaf of rings associated to the presheaf $U \mapsto \operatorname{Fract} \mathcal{O}_X(U)$

where U ranges over all open affine subsets of X. If $f \in \mathrm{H}^0(X, \mathcal{K})$, we write f^σ for the corresponding element of $\mathrm{H}^0(X, \mathcal{K}^\sigma)$. Thus, if X is an integral k-scheme, then $f \mapsto f^\sigma$ is an automorphism of the field $K = \mathrm{H}^0(X, \mathcal{K})$ of rational functions of X, and for closed points $p \in X$, we hav we have $f^\sigma(p) = f(p^\sigma)$.

Definition 1.1. The category of geometric triples, denoted GTrip, has as its objects triples $(X, \sigma, \mathcal{L})$ consisting of a k-scheme X, a k-automorphism σ of X and an invertible $\mathcal{O}_X$-module $\mathcal{L}$. A morphism in the category is a pair

$$(f, u) : (X, \sigma, \mathcal{L}) \to (X', \sigma', \mathcal{L}')$$

consisting of a morphism of k-schemes $f : X \to X'$ such that $\sigma' f = f\sigma$, and an $\mathcal{O}_X$-module homomorphism $u : f^*\mathcal{L}' \to \mathcal{L}$. If $(g, v) : (X', \sigma', \mathcal{L}') \to (X'', \sigma'', \mathcal{L}'')$ is another morphism, then $(g, v) \circ (f, u) = (gf, u \circ f^*v)$.

Definition 1.2. Let $(X, \sigma, \mathcal{L})$ be a geometric triple. We make the following definitions:

$$\mathcal{L}_n := \begin{cases} \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}} & \text{if } n > 0, \\ \mathcal{O}_X & \text{if } n = 0, \\ \mathcal{L}^{-\sigma^n} \otimes \ldots \otimes \mathcal{L}^{-\sigma^{-2}} \otimes \mathcal{L}^{-\sigma^{-1}} & \text{if } n < 0. \end{cases}$$

$$B_n := \mathrm{H}^0(X, \mathcal{L}_n),$$

$$B(X, \sigma, \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} B_n.$$

Since $\mathcal{L}_m \otimes \mathcal{L}_n^{\sigma^m} = \mathcal{L}_{m+n}$, there is a map

$$\mathrm{H}^0(X, \mathcal{L}_m) \otimes \mathrm{H}^0(X, \mathcal{L}_n) \to \mathrm{H}^0(X, \mathcal{L}_{m+n})$$

defined by $u \otimes v \mapsto u \otimes v^{\sigma^m}$, which yields a map $B_m \times B_n \to B_{m+n}$ which is easily seen to be associative. Thus $B(X, \sigma, \mathcal{L})$ is an $\mathbb{N}$-graded k-algebra.

In particular

$$B(X, \mathbb{1}, \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{L}^{\otimes n}).$$

LEMMA 1.3. *The rule $(X, \sigma, \mathcal{L}) \mapsto B(X, \sigma, \mathcal{L})$ is a contravariant functor from GTrip to graded k-algebras. If $(f, u) : (X, \sigma, \mathcal{L}) \to (X', \sigma', \mathcal{L}')$ is a morphism, the induced algebra homomorphism $\varphi : B(X', \sigma', \mathcal{L}') \to B(X, \sigma, \mathcal{L})$ is defined by*

$$\varphi(b) = (u \otimes u^\sigma \otimes \ldots \otimes u^{\sigma^{n-1}})(f^*(b))$$

for $b \in B(X', \sigma', \mathcal{L}')_n$ if $n > 0$.

PROOF. Straightforward. □

Taking Veronese subalgebras behaves well under the B functor, as does taking the opposite algebra.

LEMMA 1.4. *If $(X, \sigma, \mathcal{L})$ is a geometric triple, then*
1. *$B(X, \sigma, \mathcal{L})^{(r)} \cong B(X, \sigma^r, \mathcal{L}_r)$ for all $r \geq 1$;*
2. *$B(X, \sigma, \mathcal{L})^{\mathrm{op}} \cong B(X, \sigma^{-1}, \mathcal{L})$.*

PROOF. □

The category of geometric triples is closely related to the category of algebraic triples in Definition 16.1.1.

PROPOSITION 1.5. *There is a contravariant functor*

$$\mathsf{GTrip} \to \mathsf{ATrip},$$

which on objects is given by the rule

$$(X, \sigma, \mathcal{L}) \mapsto (\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*),$$

where $(1) = \mathcal{L} \otimes_{\mathcal{O}_X} -$. *Moreover,*

$$B(X, \sigma, \mathcal{L}) \cong B(\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*).$$

PROOF. We must define the functor on a morphism

$$(f, u) : (X, \sigma, \mathcal{L}) \to (Y, \tau, \mathcal{L}').$$

As usual, f^* and f_* denote the adjoint pair 'inverse image' and 'direct image'. The morphism of schemes $f : X \to Y$ is really a pair $(f, f^\sharp)$ with $f : X \to Y$ a continuous map, and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ a homomorphism of sheaves of rings on Y. Under the adjoint isomorphism $f^\sharp$ corresponds to a homomorphism of $\mathcal{O}_X$-modules $\theta : f^*\mathcal{O}_Y \to \mathcal{O}_X$; actually, θ is an isomorphism. Since $\tau f = f\sigma$, we have $f^*\tau^* = \sigma^* f^*$. We now define a natural transformation

$$\mu : f^* \circ (1_Y) \circ \tau^* \to (1_X) \circ \sigma^* \circ f^*$$

as follows. If $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_Y)$, then $\mu_\mathcal{F}$ is the composition of maps

$$\begin{aligned}(f^* \circ (1_Y) \circ \tau^*)(\mathcal{F}) &\cong f^*\mathcal{L}' \otimes_{\mathcal{O}_X} (f^* \circ \tau^*)\mathcal{F} \\ &\cong f^*\mathcal{L}' \otimes_{\mathcal{O}_X} (\sigma^* \circ f^*)\mathcal{F} \\ &\xrightarrow{u \otimes \mathbb{1}} \mathcal{L} \otimes_{\mathcal{O}_X} (\sigma^* \circ f^*)\mathcal{F} = ((1_X) \circ \sigma^* \circ f^*)(\mathcal{F}).\end{aligned}$$

We now define the value of the functor on the morphism (f, u) to be

$$(f^*, \theta, \mu) : (\mathsf{Coh}(\mathcal{O}_Y), \mathcal{O}_Y, (1_Y) \circ \tau^*) \to (\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1_X) \circ \sigma^*)$$

It is easy, but tedious, to check this is a functor, as claimed.

We have

$$B(X, \sigma, \mathcal{L})_n = \mathrm{H}^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}})$$

and

$$\begin{aligned}B(\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*)_n &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, ((1) \circ \sigma^*)^n \mathcal{O}_X) \\ &\cong \mathrm{H}^0(X, ((1) \circ \sigma^*)^n \mathcal{O}_X).\end{aligned}$$

Write $s = (1) \circ \sigma^*$. It is easy to see that

$$s^n \mathcal{O}_X \cong \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \otimes \mathcal{O}_X^{\sigma^n}.$$

Hence there is a natural map

$$B(X, \sigma, \mathcal{L})_n \to B(\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*)_n$$

sending b to $b \otimes 1$. To see this is an algebra homomorphism, we must show that if $a \in B_m$ and $b \in B_n$, then $a \otimes b^{\sigma^m} \otimes 1 = s^n(a \otimes 1).(b \otimes 1)$. OOPS!! $\boxed{\text{DETAILS}}$

□

Definition 1.6. If $\sigma \in \mathrm{Aut}_k X$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is σ-ample if, for every $\mathcal{F} \in \mathsf{Coh}(\mathcal{O}_X)$, and each $q > 0$, $\mathrm{H}^q(X, \mathcal{L}_n \otimes \mathcal{F}) = 0$ for $n \gg 0$.

We call $B(X, \sigma, \mathcal{L})$ a twisted homogeneous coordinate ring of X if $\mathcal{L}$ is σ-ample.

When $\sigma = \mathbb{1}_X$, this definition of σ-ample coincides with the usual definition of ample.

EXAMPLE 1.7. A σ-ample sheaf need not be ample. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{L} = \mathcal{O}(1,0) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1)$.

THEOREM 1.8. [**11**] *Let $(X, \sigma, \mathcal{L})$ be a geometric triple, and write $B = B(X, \sigma, \mathcal{L})$. If $\mathcal{L}$ is σ-ample, then*

1. *B is right and left noetherian;*
2. *B satisfies χ_1;*
3. *B is a finitely generated algebra;*
4. *there is an equivalence of categories* $\mathrm{tails}(B^{\mathrm{op}}) \cong \mathsf{Coh}(\mathcal{O}_X)$.

PROOF. By Proposition 1.5, we may interpret $B(X, \sigma, \mathcal{L})$ as $B(\mathcal{C}, \mathcal{O}, s)$ where

$$(\mathcal{C}, \mathcal{O}, s) := (\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*).$$

The key point is to verify that $\mathcal{L}$ being σ-ample implies that the shift functor for $(\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*)$ is ample. Then we may apply Theorem 20.4.2 to obtain the isomorphism

$$(\mathrm{tails}(B^{\mathrm{op}}_{\geq 0}), \mathcal{B}, [1]) \cong (\mathsf{Coh}(\mathcal{O}_X), \mathcal{O}_X, (1) \circ \sigma^*).$$

In particular, this gives the equivalence of categories in (4). The other conclusions of Theorem 20.4.2 show that B is right noetherian, and satisfies the condition χ_1 on the right.

$\boxed{\text{MORE}}$

□

2. Elementary examples

The following notation will be used in this section.

NOTATION . Fix a finite dimensional k-vector space V and set $\mathbb{P} = \mathbb{P}(V^*)$. All automorphisms of $\mathbb{P}$ as a k-scheme are linear [**76**, Chapter II, Example 7.1.1]; i.e., they are induced by elements of $\mathrm{GL}(V)$. We will think of $\mathrm{GL}(V)$ acting on V and $S(V)$ from the right, and on V^* from the left. If $\sigma \in \mathrm{GL}(V)$, $p \in V^*$, and $f \in S(V)$, we will write $p^\sigma = \sigma(p)$ and $f^\sigma(p) = f(p^\sigma)$.

PROPOSITION 2.1. *If $\sigma \in \mathrm{GL}(V)$, then*

$$B(\mathbb{P}, \sigma, \mathcal{O}(1)) \cong S(V)^\sigma.$$

PROOF. Let $*$ denote the multiplication in $S(V)^\sigma$. By definition, if $x, y \in V$, $x * y = xy^\sigma$, so $x^\sigma * y - y^\sigma * x = 0$. By ???, as x and y vary over V, these are a full set of defining relations for $S(V)^\sigma$, so

$$S(V)^\sigma = T(V)/(\{x^\sigma \otimes y - y^\sigma \otimes x \mid x, y \in V\}).$$

Since $V = \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = B_1$, there is an algebra map $T(V) \to B(\mathbb{P}, \sigma, \mathcal{O}(1))$. The map $V \otimes V \to B_2$ is, by definition,

$$\begin{gathered}\mathrm{H}^0(\mathcal{O}(1)) \otimes_k \mathrm{H}^0(\mathcal{O}(1)) \to \mathrm{H}^0(\mathcal{O}(1)) \otimes_k \mathrm{H}^0(\mathcal{O}(1)^\sigma) \to \mathrm{H}^0(\mathcal{O}(1) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}(1)^\sigma) \\ x \otimes y \mapsto x \otimes y^\sigma,\end{gathered}$$

so $\{x^\sigma \otimes y - y^\sigma \otimes x \mid x, y \in V\}$ is contained in $\ker(V \otimes V \to B_2)$, whence the map from $T(V)$ factors

$$S(V)^\sigma \xrightarrow{\varphi} B(\mathbb{P}, \sigma, \mathcal{O}(1)). \tag{2-1}$$

Notice that $S(V) \cong B(\mathbb{P}^n, \mathbb{1}, \mathcal{O}(1))$. Now, observe that the right hand side of (2-1) is generated in degree one because $\mathcal{O}(1)^\sigma \cong \mathcal{O}(1)$, and because the map

$$\mathrm{H}^0(\mathbb{P}, \mathcal{O}(i)) \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(j)) \to \mathrm{H}^0(\mathbb{P}, \mathcal{O}(i+j))$$

is surjective for all $i, j \geq 0$. Hence φ is surjective. However, the two sides of (2-1) have the same Hilbert series, so φ is an isomorphism. □

There is a more general version of Proposition 2.1.

PROPOSITION 2.2. *Let $i : X \to \mathbb{P} = \mathbb{P}(V^*)$ be the inclusion of a closed subscheme of $\mathbb{P}$, let I be the ideal of $S(V)$ vanishing on X, and let $A = S(V)/I$ denote the homogeneous coordinate ring of X. Let $\sigma \in \mathrm{Aut}_k(\mathbb{P})$, and suppose that σ restricts to an automorphism of X. Then*

1. *σ induces an automorphism of A;*
2. *I is an ideal of $S(V)^\sigma$;*
3. *$A^\sigma \cong S(V)^\sigma/I$;*
4. *there is a commutative diagram of graded algebra homomorphisms*

$$\begin{array}{ccc} S(V)^\sigma & \xrightarrow{\ \varphi\ } & B(\mathbb{P}, \sigma, \mathcal{O}_{\mathbb{P}}(1)) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ A^\sigma & \xrightarrow{\ \psi\ } & B(X, \sigma, i^*\mathcal{O}_{\mathbb{P}}(1)). \end{array} \tag{2-2}$$

PROOF. (1) Since X is σ-stable, $\sigma(I) \subset I$, so σ induces an automorphism of $A = S(V)/I$.

(2) Let $f \in I_n$ and let $x \in S(V)_n$. Then, in $S(V)^\sigma$, we have $f * x = f x^{\sigma^n} \in I$ and $x * f = x f^\sigma$ which is also in I because I is σ-stable.

(3) The algebra homomorphism $S(V) \to A$ may be considered as a map $S(V)^\sigma \to A^\sigma$; as such, it is a graded algebra homomorphism. By its very definition, I is the kernel of this map so the result follows. (This also gives an alternative proof of (2)).

(4) Write $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}}(1)$. The map φ is the isomorphism in Proposition 2.1. The map β is induced by the morphism $(i, \mathbb{1}) : (X, \sigma, \mathcal{L}) \to (\mathbb{P}^n, \sigma, \mathcal{O}_{\mathbb{P}^n}(1))$ of triples. On the degree n component, the composition $\beta \circ \varphi$, is

$$\mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to \mathrm{H}^0(X, i^*\mathcal{O}_{\mathbb{P}}(n)) \cong \mathrm{H}^0(X, \mathcal{L}_n),$$

so has kernel I_n. Hence $\beta \circ \varphi$ factors through $S(V)^\sigma/I = A^\sigma$, as required. □

The map ψ in (2-2) is injective. It is surjective if $B(X, \sigma, \mathcal{L})$ is generated in degree 1 and the natural map $\mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \to \mathrm{H}^0(X, \mathcal{L})$ is surjective.

PROPOSITION 2.3. *Let A be the homogeneous coordinate ring of a closed irreducible subvariety $X \subset \mathbb{P}$. Let $K = k(X)$ be the field of rational functions on X. Suppose that $\sigma \in \mathrm{GL}(V)$ satisfies $\sigma(X) = X$. Fix $v \in V$ which does not vanish identically on X, and define $U := \{\frac{x}{v} \mid x \in V\} \subset K$. Define the graded subalgebra*

$$k[Ut] := k + Ut + UU^\sigma t^2 + UU^\sigma U^{\sigma^2} t^3 + \cdots$$

of the Ore extension $K[t;\sigma]$, *with the grading determined by* $\deg(t) = 1$ *and* $\deg(K) = 0$. *Then, as graded* k*-algebras,*

$$A^\sigma \cong k[Ut].$$

PROOF. Define $\varphi : V \to Ut$ by $\varphi(x) = \frac{x}{v}t$. A full set of defining relations for $S(V)^\sigma$ is given by $\{x^\sigma y - y^\sigma x \mid x, y \in V\}$ and, in $K[t;\sigma]$, we have

$$\left(\frac{x^\sigma}{v}t\right)\left(\frac{y}{v}t\right) - \left(\frac{y^\sigma}{v}t\right)\left(\frac{x}{v}t\right) = \left(\frac{x^\sigma}{v}\right)\left(\frac{y^\sigma}{v^\sigma}\right)t^2 - \left(\frac{y^\sigma}{v}\right)\left(\frac{x^\sigma}{v^\sigma}\right)t^2 = 0,$$

so φ extends to a graded algebra homomorphism

$$\varphi : S(V)^\sigma \to k[Ut].$$

Let $f \in S(V)^\sigma_n$, and write

$$f \ = \underbrace{\sum \alpha_J x_{j_1} * \cdots * x_{j_n}}_{\text{in } S(V)^\sigma} \ = \underbrace{\sum \alpha_J x_{j_1} x_{j_2}^\sigma \cdots x_{j_n}^{\sigma^{n-1}}}_{\text{in } S(V)}.$$

Then

$$\varphi(f) \ = \ \sum \alpha_J \left(\frac{x_{j_1}}{v}t\right) \cdots \left(\frac{x_{j_n}}{v}t\right) \ = \ \sum \alpha_J \left(\frac{x_{j_1} x_{j_2}^\sigma \cdots x_{j_n}^{\sigma^{n-1}}}{v v^\sigma \cdots v^{\sigma^{n-1}}}\right) t^n.$$

Thus $\varphi(f) = 0$ if and only if the denominator

$$\sum \alpha_J x_{j_1} x_{j_2}^\sigma \cdots x_{j_n}^{\sigma^{n-1}}$$

vanishes on X. But this denominator is f itself, so $\ker(\varphi) = I$, the ideal generated by the homogeneous polynomials vanishing on X. Hence $k[Ut] \cong S(V)^\sigma/I \cong A^\sigma$, as required. □

EXAMPLE 2.4. Consider $B(\eta, \sigma, \mathcal{L})$, where $\eta = \operatorname{Spec}(K)$ is the scheme of a field extension K of k, σ is a k-linear automorphism of K, and $\mathcal{L}$ is the unique invertible $\mathcal{O}_\eta$-module, namely K itself (cf. Example B.8.5). Since $\operatorname{Spec}(K)$ is affine, we may work with global sections in place of $\mathcal{L}$. We will write $L = \mathrm{H}^0(\eta, \mathcal{L})$; of course, as a K-module, $L \cong K$.

As in Example B.8.5, there is an isomorphism $\sigma^* L \to L$ given by $x \mapsto x^\sigma$, so $\mathcal{L}_n \cong K$, and $B_n \cong K$ for all n. Multiplication $B_1 \otimes B_1 \to B_2$ is given by

$$\begin{gathered} L \otimes L \to L \otimes L^\sigma \to K \\ x \otimes y \mapsto x \otimes y^\sigma \mapsto x y^\sigma. \end{gathered}$$

It follows that there is an isomorphism $\Phi : B(\eta, \sigma, \mathcal{L}) \to K[t;\sigma]$ given by $\Phi(x) = xt^n$ for $x \in B_n$.

Under quite mild restrictions a natural localization of $B(X, \sigma, \mathcal{L})$ is an Ore extension of the function field $K = k(X)$. Let X be an irreducible and reduced projective k-scheme, let $\xi = \operatorname{Spec}(K)$ denote the generic point of X, let σ denote the restriction of σ to ξ, and let $\mathcal{L}_\xi$ denote the inverse image of $\mathcal{L}$ on ξ. The obvious morphism of triples $(\xi, \sigma, \mathcal{L}_\xi) \to (X, \sigma, \mathcal{L})$ induces an injective algebra homomorphism $B(X, \sigma, \mathcal{L}) \to B(\xi, \sigma, \mathcal{L}_\xi)$, and we obtain the following result.

PROPOSITION 2.5. *Let* $(X, \sigma, \mathcal{L})$ *be a geometric triple and suppose that* X *is an irreducible and reduced projective* k*-scheme. There is an isomorphism* $B(\xi, \sigma, \mathcal{L}_\xi) \cong K[t, t^{-1};\sigma]$, *and if* $\mathcal{L}$ *is* σ*-ample, then*

1. *$B(\xi,\sigma,\mathcal{L}_\xi)$ is a localization of $B(X,\sigma,\mathcal{L})$;*
2. *if σ is of finite order, $B(\xi,\sigma,\mathcal{L}_\xi)$ is a central localization of $B(X,\sigma,\mathcal{L})$.*

PROOF. Since ξ is an integral scheme, $B(\xi,\sigma,\mathcal{L}_\xi)$ is a domain. In particular, the multiplication map in $B(\xi,\sigma,\mathcal{L}_\xi)$ gives each $B(\xi,\sigma,\mathcal{L}_\xi)_i$ the structure of a 1-dimensional vector space over $B(\xi,\sigma,\mathcal{L}_\xi)_0 = K$. If $0 \neq x \in B(\xi,\sigma,\mathcal{L}_\xi)_1$ then $B(\xi,\sigma,\mathcal{L}_\xi)_i = Kx^i$. Hence there is an algebra isomorphism $\varphi_x : B(\xi,\sigma,\mathcal{L}_\xi) \to K[t,t^{-1};\sigma]$ given by $\varphi_x(fx^i) = ft^i$ for all $i \in \mathbb{Z}$ and $f \in K$.

(1) By [**11**, Proposition 3.2(iii)] we may choose an n such that $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$ are very ample. Thus B_n and B_{n+1} are both non-zero, so Fract$B(X,\sigma,\mathcal{L})$ contains a non-zero element of degree 1. Let $0 \neq u \in \mathrm{H}^0(X,\mathcal{L}_n)$ and set $D = (u)$. Then $B(X,\sigma,\mathcal{L})[u^{-1}]$ contains $H^0(X,\mathcal{O}_X(D))$ and hence Fract$B(X,\sigma,\mathcal{L})$ contains the subfield of K generated by $H^0(X,\mathcal{O}_X(D))$. But this subfield is K since $\mathcal{O}_X(D)$ is very ample. It follows that $B(\xi,\sigma,\mathcal{L}_\xi)$ is the localisation of $B(X,\sigma,\mathcal{L})$ at the non-zero homogeneous elements.

(2) If σ is of finite order, n say, then $K[t,t^{-1};\sigma]$ has center $K^{\langle\sigma\rangle}[t^n,t^{-n}]$, so $B(\xi,\sigma,\mathcal{L}_\xi)$ is finite over its center. Thus $B(X,\sigma,\mathcal{L})$ is a prime ring satisfying a polynomial identity, whence by Posner's Theorem (10.4.1), Fract$B(X,\sigma,\mathcal{L})$ is a central localization of $B(X,\sigma,\mathcal{L})$. Thus, if $r \in B(\xi,\sigma,\mathcal{L}_\xi)$ is homogeneous, then $rz \in B(X,\sigma,\mathcal{L})$ for some non-zero central element $z \in B(X,\sigma,\mathcal{L})$. But each homogeneous component of z is also central, so we may assume that z is homogeneous. Thus $r \in B(X,\sigma,\mathcal{L})[z^{-1}] \subset B(\xi,\sigma,\mathcal{L}_\xi)$, so $B(\xi,\sigma,\mathcal{L}_\xi)$ is a central localization as claimed. □

COROLLARY 2.6. *If X is an integral scheme, then $B(X,\sigma,\mathcal{L})$ is a domain.*

The next result generalizes Proposition 2.1, and is a precursor of the results in the next section. Later on it will be used in the following way. Suppose that $A = T(V)/I$ is a quadratic algebra, and that I_2 vanishes on the graph of an automorphism, σ say, of some closed subscheme $X \subset \mathbb{P}$. Then, Lemma 2.7 says there is a graded algebra map $A \to B(X,\sigma,\mathcal{L})$, where $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to X. The point of this is that algebro-geometric methods yield information about $B(X,\sigma,\mathcal{L})$, which then sheds light on A. For example, this method is crucial to understanding the quantum $\mathbb{P}^2$'s in chapter 24.

LEMMA 2.7. *Let $i : X \to \mathbb{P}$ be a closed immersion, $\sigma \in \mathrm{Aut}_k X$, and let $\Gamma \subset \mathbb{P}\times\mathbb{P}$ be the graph of σ. Then there is a graded algebra homomorphism*

$$\psi : T(V)/(R) \to B(X,\sigma,i^*\mathcal{O}_{\mathbb{P}}(1)),$$

where $R = \{f \in V\otimes V \mid f|_\Gamma = 0\}$.

PROOF. Write $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}}(1)$ and $B = B(X,\sigma,\mathcal{L})$. The map $V = \mathrm{H}^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(1)) \to \mathrm{H}^0(X,\mathcal{L})$ induces an algebra homomorphism $T(V) \to B$; we must show that $R \subset \ker(V\otimes V \to B_2)$. The map $V\otimes V \to B_2$ is the composition

$$\mathrm{H}^0(X,\mathcal{L})\otimes\mathrm{H}^0(X,\mathcal{L}) \to \mathrm{H}^0(X,\mathcal{L})\otimes\mathrm{H}^0(X,\mathcal{L}^\sigma) \to \mathrm{H}^0(X,\mathcal{L}\otimes\mathcal{L}^\sigma).$$

The image in B_2 of $f = \sum_{i,j}\alpha_{ij}x_i\otimes x_j \in V\otimes V$ is $\sum_{i,j}\alpha_{ij}x_ix_j^\sigma$; the value at $p \in X$ of this section of $\mathcal{L}\otimes\mathcal{L}^\sigma$ is

$$\sum_{i,j}\alpha_{ij}x_i(p)x_j^\sigma(p) = \sum_{i,j}\alpha_{ij}x_i(p)x_j(p^\sigma) = f(p,p^\sigma).$$

But, if $f \in R$ then $f(p,p^\sigma) = 0$ by definition of R, so f is in the kernel. □

EXERCISES

2.1 Show that the diagram in Proposition 2.2 is really part of the following larger commutative diagram

$$\begin{array}{ccccc} S(V)^\sigma & \xrightarrow{\varphi} & B(\mathbb{P}, \mathbb{1}, \mathcal{O}_{\mathbb{P}}(1))^\sigma & \longrightarrow & B(\mathbb{P}, \sigma, \mathcal{O}_{\mathbb{P}}(1)) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ A^\sigma & \xrightarrow{\psi} & B(X, \mathbb{1}, i^*\mathcal{O}_{\mathbb{P}}(1))^\sigma & \longrightarrow & B(X, \sigma, i^*\mathcal{O}_{\mathbb{P}}(1)). \end{array} \tag{2-3}$$

2.2 Although it is already implicit in Proposition 2.3, prove that up to isomorphism of graded algebras, $k[Ut]$ does not depend on the choice of v. [*Hint:* if v is replaced by v' and the resulting subalgebra of $K[t;\sigma]$ is denoted by $k[U't]$, set $f = \frac{v}{v'}$, and define $\psi : k[Ut] \to k[U't]$ by $\psi(b_n) = ff^\sigma \dots f^{\sigma^{n-1}} b_n$.]

3. Geometric data associated to an algebra

Fix a finite dimensional k-vector space V, and write $\mathbb{P} = \mathbb{P}(V^*)$.

In this section we associate to a connected graded algebra, say $A = T(V)/I$, a sequence of subschemes $\Gamma_n \subset \mathbb{P}^{\times n}$, and from this data we construct a graded algebra B for which there is a natural map $A \to B$. The data Γ_n is susceptible to algebro-geometric analysis which yields information about B, which in turn gives information about A.

It is often uninformative to describe an algebra by explicit generators and relations. For example, it may be difficult to study an algebraic variety simply in terms of its defining equations; it may even be impractical or unnecessary to determine the defining equations. It is also difficult in general to determine whether two algebras given by generators and relations are isomorphic, or whether an algebra given by a particular set of generators and relations is an iterated Ore extension. It may even be difficult to decide if an algebra given by generators and relations is non-zero (Exercise 1.3.3). This suggests that one should seek other ways of defining an algebra, or for methods which associate more tractable data (possibly geometric) to generators and relations—this is one of the issues motivating this section.

Definition 3.1. Let $A = T(V)/I$ be a graded k-algebra. The n^{th} homogeneous component I_n, of I, is a subspace of $V^{\otimes n}$, so its elements are linear maps $(V^*)^{\otimes n} \to k$ or, equivalently, n-multilinear maps

$$V^* \times \cdots \times V^* \to k.$$

We define an inverse system of schemes $\Gamma = (\Gamma_n)_{n \geq 1}$ by

$$\Gamma_n := \mathcal{V}(I_n) \ \subset \ \mathbb{P} \times \cdots \times \mathbb{P} = \mathbb{P}^{\times n},$$

and morphisms $\pi_n^m : \Gamma_m \to \Gamma_n$ for $m \geq n$ which are the restrictions of the projections $\mathbb{P}^{\times m} \to \mathbb{P}^{\times n}$ onto the first n copies. (The next Lemma shows that this is an inverse system.)

LEMMA 3.2. *Let $A = T(V)/I$ be a connected, graded algebra. Then*

$$\Gamma_{n+1} \subset (\mathbb{P} \times \Gamma_n) \cap (\Gamma_n \times \mathbb{P}),$$

with equality if $I_{n+1} = V \otimes I_n + I_n \otimes V$.

PROOF. It is clear that $\mathcal{V}(V \otimes I_n) = \mathbb{P} \times \Gamma_n$ and that $\mathcal{V}(I_n \otimes V) = \Gamma_n \times \mathbb{P}$. Since $V \otimes I_n + I_n \otimes V \subset I_{n+1}$ the result follows. $\square$

EXAMPLE 3.3. We compute the spaces Γ_n for the tensor algebra, symmetric algebra, and exterior algebra on a vector space V. We will denote the algebra in question by $T(V)/I$ and write $\mathbb{P} = \mathbb{P}(V^*)$. In all cases $I_1 = 0$, so $\Gamma_1 = \mathbb{P}$.

For the tensor algebra $T(V)$, $I_n = 0$ for all n, so $\Gamma_n = \mathbb{P}^{\times n}$.

For the symmetric algebra, $I_2 = \{x \otimes y - y \otimes x \mid x, y \in V\}$ so Γ_2 contains the diagonal $\{(p,p) \mid p \in \mathbb{P}\}$. Conversely, if $f(p,q) = 0$ for all $f \in I_2$ then, choosing x such that $x(p) \neq 0$, we obtain $y(q) = \alpha y(p)$ for all $y \in V$, where $\alpha = x(q)x(p)^{-1}$; hence $q = p$. Thus $\Gamma_2 = \{(p,p) \mid p \in \mathbb{P}\}$. By Lemma 3.2, $\Gamma_{n+1} \subset (\mathbb{P} \times \Gamma_n) \cap (\Gamma_n \times \mathbb{P})$ so, by induction on n, $\Gamma_n \subset \{(p, \dots, p) \mid p \in \mathbb{P}\}$. On the other hand, $I_n = \sum_{i=0}^{n-2} V^{\otimes i} \otimes I_2 \otimes V^{\otimes n-i-2}$, so I_n vanishes on all $(p, \dots, p)$. Thus $\Gamma_n = \{(p, \dots, p) \mid p \in \mathbb{P}\}$ for all $n \geq 1$.

For the exterior algebra, $\Lambda(V) = T(V)/I$, $I_2 = \{x \otimes y + y \otimes x \mid x, y \in V\}$. If $f(p,q) = 0$ for all $f \in I_2$ then, choosing x and y such that $x(p) \neq 0$ and $y(p) = 0$, we obtain $y(q) = 0$ also; this holds for all y vanishing at p, so $q = p$, whence $2x(p)y(p) = 0$ for all $x, y \in V$. Hence, if $\operatorname{char}(k) \neq 2$, $\Gamma_2 = \emptyset$ and by Lemma 3.2, $\Gamma_n = \emptyset$ for all $n \geq 2$.

The computation of Γ_n from Γ_2 for the tensor, symmetric, and exterior algebras is a special case of the following general principle.

LEMMA 3.4. *Let $A = T(V)/I$ be a quadratic algebra, and suppose that $\Gamma_2 = \mathcal{V}(I_2)$ is the graph of an automorphism σ of some subscheme $X \subset \mathbb{P}(V^*)$. Then, for all $n \geq 2$,*

$$\Gamma_n = \bigcap_{i=0}^{n-2} \mathbb{P}^{\times i} \times \Gamma \times \mathbb{P}^{n-i-2},$$

the scheme-theoretic intersection. In particular, if X is a variety, then

$$\Gamma_n = \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in X\}.$$

PROOF. Since $I_n = \sum_{i=0}^{n-2} V^{\otimes i} \otimes I_2 \otimes V^{\otimes n-i-2}$, it is clear that I_n vanishes on the given points. On the other hand, an induction argument, using Lemma 3.2, shows that Γ_n must belong to this set. Hence there is equality, as claimed. $\square$

The next example is a more interesting application of Lemma 3.4.

EXAMPLE 3.5. Let $S(V)^\sigma$ be the twist of the polynomial ring with respect to $\sigma \in \mathrm{GL}(V)$. We will show that $\Gamma_n = \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in \mathbb{P}\}$.

By Proposition 16.5.8, $S(V)^\sigma \cong T(V)/I$, where I is generated $\{x^\sigma \otimes y - y^\sigma \otimes x \mid x, y \in V\}$; in particular I is generated by I_2. Write $I_2 = (\sigma \otimes 1)(R)$ where R is the linear span of $\{x \otimes y - y \otimes x \mid x, y \in V\}$. Thus $T(V)/(R) = S(V)$. We will show that Γ_2 equals the shifted diagonal

$$\Delta_\sigma := \{(p, p^\sigma) \mid p \in \mathbb{P}\} \ \subset \ \mathbb{P} \times \mathbb{P}.$$

The computation in Example 3.3 for the symmetric algebra established this when $\sigma = 1$. If $f \in V \otimes V$, then

$$\begin{aligned} f(\Delta_\sigma) = 0 \ &\Leftrightarrow \ (1 \otimes \sigma)(f)(\Delta_1) = 0 \\ &\Leftrightarrow \ (1 \otimes \sigma)(f) \in R \\ &\Leftrightarrow \ f \in I_2, \end{aligned}$$

where the last equivalence is because $R = (\sigma \otimes \sigma)(R)$. Hence $\Gamma_2 = \Delta_\sigma$ as claimed, and by Lemma 3.4, the result follows.

The geometric data $\{\Gamma_n \mid n \geq 1\}$ does not determine the algebra from which it comes: for example, the exterior algebra $\Lambda(V)$ and $T(V)/(V \otimes V)$ determine the same sequence of Γ_n's. However, we may associate to the Γ_n's a graded k-algebra as in the next definition.

Before defining this algebra, we realize $V^{\otimes n}$ as the global sections of a line bundle on $\mathbb{P}^{\times n}$. Let $\mathrm{pr}_i : \mathbb{P}^{\times n} \to \mathbb{P}$ be the projection onto the i^{th} component, and define

$$\begin{aligned} \mathcal{O}(1, \dots, 1) &:= \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \mathrm{pr}_n^* \mathcal{O}_{\mathbb{P}}(1) \\ &\cong \mathcal{O}_{\mathbb{P}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}}(1). \end{aligned}$$

Thus $V^{\otimes n} = \mathrm{H}^0(\mathbb{P}^{\times n}, \mathcal{O}(1, \dots, 1))$.

Definition 3.6. Let $\Gamma = (\Gamma_n)_{n \geq 1}$ be a sequence of subschemes $\Gamma_n \subset \mathbb{P}^{\times n}$ such that, for all $n \geq 1$,

$$\Gamma_{n+1} \subset (\mathbb{P} \times \Gamma_n) \cap (\Gamma_n \times \mathbb{P})$$

scheme-theoretically. Let $j_n : \Gamma_n \to \mathbb{P}^{\times n}$ and $i_{mn} : \Gamma_{m+n} \to \Gamma_m \times \Gamma_n$ be the inclusions. Define

$$\begin{aligned} \mathcal{B}_n &:= j_n^* \mathcal{O}(1, \dots, 1) \quad \text{for } n \geq 0, \\ B_n &:= \mathrm{H}^0(\Gamma_n, \mathcal{B}_n), \\ B(\Gamma) &:= \bigoplus_{n=0}^{\infty} B_n. \end{aligned}$$

If the Γ_n's are clear from the context, we just write B for $B(\Gamma)$. We give B a graded algebra structure: the multiplication map $B_m \times B_n \to B_{m+n}$ is the composition

$$\mathrm{H}^0(\Gamma_m, \mathcal{B}_m) \times \mathrm{H}^0(\Gamma_n, \mathcal{B}_n) \xrightarrow{\sim} \mathrm{H}^0(\Gamma_m \times \Gamma_n, \mathcal{B}_m \boxtimes \mathcal{B}_n) \to \mathrm{H}^0(\Gamma_{m+n}, \mathcal{B}_{m+n}),$$

where the second map is induced by the $\mathcal{O}_{\Gamma_{m+n}}$-module map $i_{mn}^*(\mathcal{B}_m \boxtimes \mathcal{B}_n) \to \mathcal{B}_{m+n}$ as in (8-3) of Appendix B. The product on B is associative because inverse image is functorial.

EXAMPLE 3.7. If each $\Gamma_n = \mathbb{P}^{\times n}$, then $\mathcal{B}_n = \mathcal{O}(1, \dots, 1)$ so $B_n = V^{\otimes n}$ and $B = T(V)$.

If each $\Gamma_n = \{(p, \dots, p) \mid p \in \mathbb{P}\}$, then $B = S(V)$.

If $\sigma \in \mathrm{Aut}_k \mathbb{P}$, and $\Gamma_n = \{(p, p^{\sigma}, \dots, p^{\sigma^{n-1}}) \mid p \in \mathbb{P}\}$, then $B = S(V)^{\sigma}$.

PROPOSITION 3.8. *Let $\Gamma = (\Gamma_n)_{n \geq 1}$ be the sequence of subschemes of $\mathbb{P}^{\times n}$ determined by an algebra $A = T(V)/I$. Then there is a graded algebra homomorphism $A \to B(\Gamma)$.*

PROOF. The natural map $V \to \mathrm{H}^0(\Gamma_1, j_1^* \mathcal{O}_{\mathbb{P}}(1)) = B_1$ induces an algebra homomorphism $T(V) \to B$; in degree n this is $V^{\otimes n} \to \mathrm{H}^0(\Gamma_n, j_n^* \mathcal{O}(1, \dots, 1))$ (given by (8-3) in Appendix B). The image of I_n under this map is zero because I_n vanishes on Γ_n, so the map $T(V) \to B$ factors through A. $\square$

Initially, it may be difficult to get a good feel for B, but there is a first approximation to B which can be defined more directly, namely the algebra $T(V)/J$ where $J = \oplus J_n$ and $J_n := \{f \in V^{\otimes n} \mid f(\Gamma_n) = 0\}$. Notice that J is an ideal: since $\Gamma_{n+1} \subset (\Gamma_n \times \mathbb{P}) \cap (\mathbb{P} \times \Gamma_n)$, if $f \in J_n$ and $a \in V$ then $f \otimes a$ and $a \otimes f$ both vanish on Γ_{n+1} so belong to J_{n+1}. By definition of Γ_n, $I_n \subset J_n$ so there is a surjective map $A = T(V)/I \to T(V)/J$. If B is generated in degree one and the map $V \to B_1$

is surjective, then $B = T(V)/J$. In fact, for all the examples in this section, with the exception of the exterior algebra and the next example, $A \cong T(V)/J$.

EXAMPLE 3.9. Consider $A = k[x, y, z]$ with defining relations

$$zy = \alpha yz, \quad zx = \beta xz, \quad yx = \gamma xy$$

where $\alpha, \beta, \gamma \in k$ satisfy $\alpha\beta^{-1}\gamma \neq 1$. (If $\alpha\beta^{-1}\gamma = 1$ then A is a twist of the polynomial ring in 3 variables by Exercise 16.4.4.) Write $V = kx + ky + kz$ and $A = T(V)/I$.

Let $X = \mathcal{V}(xyz)$ be the 3 'coordinate axes' in $\mathbb{P}^2$ and define $\sigma : X \to X$ by

$$\begin{aligned}(0, y, z) &\mapsto (0, \alpha y, z),\\ (x, 0, z) &\mapsto (\beta x, 0, z),\\ (x, y, 0) &\mapsto (\gamma x, y, 0).\end{aligned}$$

(This automorphism of X does not extend to a linear automorphism of $\mathbb{P}^2$ because $\alpha\beta^{-1}\gamma \neq 1$.) Write Γ for the graph of this automorphism of X; that is,

$$\Gamma := \{(p, p^\sigma) \mid p \in X\}.$$

We will show that $\Gamma_2 = \Gamma$, that $I_2 = J_2$, but $I_3 \neq J_3$.

It is clear that I_2 vanishes on Γ, so $\Gamma \subset \Gamma_2$. Conversely, suppose that $(p, q) \in \Gamma_2$. Then

$$z(p)y(q).\beta x(p)z(q).y(p)x(q) = \alpha y(p)z(q).z(p)x(q).\gamma x(p)y(q) \tag{3-1}$$

so, since $\alpha\gamma \neq \beta$, either p or q must lie on $\mathcal{V}(xyz)$. Suppose, for argument's sake, that $x(p) = 0$. Then

$$z(p)x(q) = y(p)x(q) = 0$$

whence $x(q) = 0$. Now, from the equality $z(p)y(q) = \alpha y(p)z(q)$, it follows that $q = p^\sigma$. The other cases arising from (3-1) are similar. Therefore $\Gamma_2 = \Gamma$.

Next, we show that $J_2 \subset I_2$. Let $f \in J_2$ and assume that f is written as a linear combination of tensors of the form $x \otimes x$, $x \otimes y$, ... et cetera. Since Γ_2 contains $((1, 0, 0), (1, 0, 0))$, it follows that $x \otimes x$ does not occur in f. Similarly, neither $y \otimes y$ nor $z \otimes z$ occurs. Now write $f = x \otimes a + b \otimes x + \lambda y \otimes z - \nu z \otimes y$. Evaluating f at $((0, 1, 1), (0, \alpha, 1))$, it follows that $\lambda y \otimes z - \nu z \otimes y$ is a multiple of $z \otimes y - \alpha y \otimes z$ and hence in I_2. A similar analysis applied to $x \otimes a + b \otimes x$ finally yields $f \in I_2$.

By Lemma 3.4, $\Gamma_3 = \{(p, p^\sigma, p^{\sigma^2}) \mid p \in X\}$, so $x \otimes y \otimes z$ vanishes on Γ_3, and hence belongs to J_3. However, xyz is a non-zero element of A (because A is an iterated Ore extension with basis $x^i y^j z^k$).

$\boxed{\text{Paul}}$ If $X \subset \mathbb{P}$ is a closed subscheme, $\sigma \in \operatorname{Aut}_k X$, and $\Gamma_n := \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in X\}$, then $B(\Gamma) \cong B(X, \sigma, \mathcal{L})$, where $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to X.

$\boxed{\text{Paul}}$ does $B = T(V)/J$ if the latter satisfies χ_1? do they differ by a finite piece, does B satisfy χ_1?. Oh, assume that $\mathcal{B}_n$ is an ample sequence.

EXERCISES

3.1 For each $r \geq 1$, determine the Γ_n for the commutative ring $k\langle x, y\rangle/(xy - yx, x^r)$.

3.2 Show that the Γ_n associated to the homogeneous coordinate ring of $X \subset \mathbb{P}(V^*)$ are $\{(p, \dots, p) \mid p \in X\}$ for large n (but not necesarily for small n).

3.3 Let $X \subset \mathbb{P} = \mathbb{P}(V^*)$ be a subvariety such that its homogeneous coordinate ring, A say, is a quadratic algebra. Suppose that $\sigma \in \mathrm{Aut}_k(\mathbb{P})$ restricts to an automorphism of X, and let A^σ be the twist of A; A^σ is a quadratic algebra. Show that
(a) the schemes associated to A^σ are $\Gamma_n = \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in X\}$, and
(b) the ideal J determined by the Γ_n is the ideal of relations for A^σ.

3.4 (Research Problem) Let A be an iterated Ore extension which is also a quadratic algebra. Let $R \subset A_1 \otimes A_1$ be its space of defining relations. Does $\mathcal{V}(R)$ have a nice description? How is it obtained in terms of the automorphisms and derivations? Is $R = \mathcal{R}(\mathcal{V}(R))$? If not, how can one single out the subspace of $\mathcal{R}(\mathcal{V}(R))$ which is R?

3.5 Does B satisfy χ_1 if $\mathcal{B}_n$ is an ample sequence? What would ample mean in this context?

4. The structure of $B(X, \sigma, \mathcal{L})$

PROPOSITION 4.1. [**10**, pages 374-375] *Suppose that X is projective, reduced and irreducible, and that σ^s fixes the class of $\mathcal{L}$ in $\mathrm{Pic}(X)$. Let $u : \mathcal{L}^{\sigma^s} \xrightarrow{\sim} \mathcal{L}$ be an isomorphism, and define the isomorphism $v : \mathcal{L}_s^\sigma \xrightarrow{\sim} \mathcal{L}_s$ by*

$$v(x_1 \otimes \ldots \otimes x_s) = u(x_s) \otimes x_1 \otimes \ldots \otimes x_{s-1}.$$

By functoriality $(\sigma^s, u) \in \mathrm{Aut}(X, \sigma, \mathcal{L})$, and $(\sigma, v) \in \mathrm{Aut}(X, \sigma^s, \mathcal{L}_s)$ induce algebra automorphisms

$$\rho \in \mathrm{Aut}\, B(X, \sigma, \mathcal{L}) \quad \textit{and} \quad \tau \in \mathrm{Aut}\, B(X, \sigma^s, \mathcal{L}_s).$$

Let $x, y \in B_i$, $z \in B_j$ and $w \in B_{ms}$ where $i + j = ms$. Then
1. *$x^{\rho^m} zy = y^{\rho^m} zx$;*
2. *$\tau^i(zy) = y^{\rho^m} z$;*
3. *if $B(X, \sigma^s, \mathcal{L}_s)$ is identified with $B(X, \sigma, \mathcal{L})^{(s)}$, then the restriction of ρ to $B(X, \sigma, \mathcal{L})^{(s)}$ equals τ^s;*
4. *$w^{\tau^i} y = y^{\rho^m} w$.*

PROOF. Both (σ^s, u) and (σ, v) extend to automorphisms of the triples $(\xi, \sigma, \mathcal{L}_\xi)$ and $(\xi, \sigma^s, (\mathcal{L}_s)_\xi)$, so ρ and τ extend to automorphisms of $B(\xi, \sigma, \mathcal{L}_\xi)$ and $B(\xi, \sigma, \mathcal{L}_\xi)^{(s)}$ respectively. Therefore it suffices to prove the Proposition for $B(\xi, \sigma, \mathcal{L}_\xi)$ which has the advantage of being generated over its degree 0 component by the elements of degree 1 and -1. We will only consider the cases $i, j \geq 0$.

(1) The case $i = m = 1$ is true because

$$x^\rho zy = (u \otimes 1 \otimes 1)(x^{\sigma^s} \otimes z^\sigma \otimes y^{\sigma^s}) = (u \otimes 1 \otimes 1)(y^{\sigma^s} \otimes z^\sigma \otimes x^{\sigma^s}) = y^\rho zx.$$

For $i = 1$ and a general m we proceed by induction. Suppose that $z = z_1 z_2 z_3$ with $z_1 \in B_{s-1}$, $z_2 \in B_1$, $z_3 \in B_{(m-1)s-1}$. Then

$$x^{\rho^m} zy = x^{\rho^m} z_1 z_2 z_3 y = z_2^\rho z_1 x^{\rho^{m-1}} z_3 y = z_2^\rho z_1 y^{\rho^{m-1}} z_3 x = y^{\rho^m} z_1 z_2 z_3 x = y^{\rho^m} zx.$$

We now prove the general case by induction on i. Suppose $x = x_1 x_2$ and $y = y_1 y_2$ with $x_1, y_1 \in B_{i-1}$ and $x_2, y_2 \in B_1$. Then

$$x^{\rho^m} zy = (x_1 x_2)^{\rho^m} zy_1 y_2 = x_1^{\rho^m} y_2^{\rho^m} zy_1 x_2 = y_1^{\rho^m} y_2^{\rho^m} zx_1 x_2 = y^{\rho^m} zx.$$

(2) The result is true for $i=1$ and $m=1$ because, if $z_1,\dots,z_s \in B_1$ then

$$\begin{aligned}\tau(z_1\dots z_s) &= v((z_1\dots z_s)^\sigma)\\ &= v(z_1^\sigma \otimes z_2^{\sigma^2}\otimes\dots\otimes z_s^{\sigma^s})\\ &= (u\otimes 1\otimes 1\otimes\dots\otimes 1)(z_s^{\sigma^s}\otimes z_1^\sigma\otimes\dots\otimes z_{s-1}^{\sigma^{s-1}})\\ &= u(z_s^{\sigma^s})\otimes z_1^\sigma\otimes\dots\otimes z_{s-1}^{\sigma^{s-1}}\\ &= z_s^\rho z_1\dots z_{s-1}.\end{aligned}$$

We now prove the case $i=1$ and general m by induction. Suppose that $z = z_1\dots z_{ms-1}$ with each $z_l \in B_1$. Then

$$\begin{aligned}\tau(zy) &= \tau(z_1\dots z_s)\tau(z_{s+1}\dots z_{ms-1}y)\\ &= z_s^\rho z_1\dots z_{s-1}y^{\rho^{m-1}}z_{s+1}\dots z_{ms-1}\\ &= y^{\rho^m}z_1\dots z_{ms-1}\\ &= y^\rho z.\end{aligned}$$

For a general i and general m we proceed by induction on i. Suppose that $y=y_1y_2$ with $y_1 \in B-1$ and $y_2 \in B_{i-1}$. Then

$$\tau^i(zy) = \tau\big(\tau^{i-1}(zy_1y_2)\big) = \tau(y_2^{\rho^m}zy_1) = y_1^{\rho^m}y_2^{\rho^m}z = y^{\rho^m}z.$$

(3) This follows by iterating s times the case $i=m=1$ of (b).

(4) Suppose that $w=w_1w_2$ with $w_1 \in B_{ms-i}$ and $w_2 \in B_i$. Then, using (a) and (b),

$$w^{\tau^i}y = \tau^i(w_1w_2)y = w_2^{\rho^m}w_1y = y^{\rho^m}w_1w_2$$

as required. $\square$

COROLLARY 4.2. *Suppose that X is a reduced and irreducible projective k-scheme, σ is of finite order and $\mathcal{L}$ is σ-ample. Then $B(X,\sigma,\mathcal{L})$ is finite over its center.*

PROOF. By the arguments in [**11**] $B=B(X,\sigma,\mathcal{L})$ is a finite $B^{(m)}$-module for $m \gg 0$.

$\boxed{\text{DETAILS}}$

Hence we can choose n such that $\sigma^n=1$ and B is a finite $B^{(n)}$-module. Taking $s=n$ and $u=\mathrm{Id}$ in Proposition 4.1 gives $\rho=1$ and $\tau\in\operatorname{Aut}B^{(n)}$ satisfies $\tau^n=1$. By Proposition **??**(4), the τ-invariants in $B^{(n)}$ are central in B. But $B^{(n)} \cong B(X,1,\mathcal{L}_n)$ is commutative, so is a finite module over its subring of τ-invariants. Hence the result. $\square$

If we are in the situation considered in Proposition 4.1 then, by (4.1(1)) $x^\rho y = y^\rho x$ for $x,y\in B(X,\sigma,\mathcal{L})_s$. Hence the subalgebra of $B(X,\sigma,\mathcal{L})^{(s)}$ generated by the elements of degree s is a quotient of a twist of a polynomial ring.

Define $\mathbb{P}' = \mathbb{P}(\mathrm{H}^0(E/G',\mathcal{L}')^*) \cong \mathbb{P}^{d-1}$ and let $j: E/G' \to \mathbb{P}'$ be the inclusion. Then $\mathcal{L}' = j^*\mathcal{O}_{\mathbb{P}'}(1)$ and we can make the identification

$$\mathrm{H}^0(\mathbb{P}',\mathcal{O}(1)) = \mathrm{H}^0(E/G',\mathcal{L}') = B(E/G',\sigma^s,\mathcal{L}')_s^{G'}.$$

The resulting action of ρ on $\mathrm{H}^0(\mathbb{P}', \mathcal{O}(1))$ induces an automorphism μ of $\mathbb{P}'$ which extends the automorphism σ^s on E'. Let $w : j^*\mathcal{O}(1) \to \mathcal{L}'$ be the natural isomorphism. Then $(j, w) : (E/G', \sigma^s, \mathcal{L}') \to (\mathbb{P}', \mu, \mathcal{O}(1))$ is a morphism of triples, so determines an algebra homomorphism

$$\alpha : B(\mathbb{P}', \mu, \mathcal{O}(1)) \to B(E/G', \sigma^s, \mathcal{L}').$$

Proposition 4.3. *Identify $B(E/G', \sigma^s, \mathcal{L}')$ with $B(E, \sigma, \mathcal{L})^{(s)G'}$ as in (2.6). Let $\beta : k[u_1, \ldots, u_d] \to B(E/G', \sigma^s, \mathcal{L}')$ be the restriction of the surjection $A \to B$. There is a unique surjective algebra homomorphism*

$$\gamma : B(\mathbb{P}', \mu, \mathcal{O}(1)) \to k[u_1, \ldots, u_d]$$

such that $\alpha = \beta \circ \gamma$.

Proof. Suppose that the good basis $x_1, \ldots, x_d$ for $B(E, \sigma, \mathcal{L})_1$ satisfies $x_i^\rho = \eta_i^{-1} x_i$. By (3.7b) u_i is $(\lambda_{\eta_i} \circ \rho)$-normalizing, and therefore $(\lambda_{\eta_i} \circ \rho)$-invariant. It follows that $u_i u_j = (\eta_i \eta_j^{-1})^s u_j u_i$, or equivalently $u_i^\rho u_j = u_j^\rho u_i$. Therefore $u^\rho v = v^\rho u$ for all $u, v \in k[u_1, \ldots, u_d]_s$.

There exists $d : \mathcal{O}(1)^\mu \xrightarrow{\sim} \mathcal{O}(1)$ such that the automorphism of $B(\mathbb{P}', \mu, \mathcal{O}(1))$ induced by (d, μ) agrees in degree 1 with the action of ρ on $\mathrm{H}^0(\mathbb{P}', \mathcal{O}(1))$. Hence ρ extends to an automorphism of $B(\mathbb{P}', \mu, \mathcal{O}(1))$.

The maps α and β commute with the action of ρ and are isomorphisms in degree 1. Since $B(\mathbb{P}', \mu, \mathcal{O}(1))$ is generated in degree 1 and has defining relations $x^\rho y = y^\rho x$ for x, y of degree 1, it follows that the map $\beta^{-1}\alpha$ in degree 1 extends to an algebra homomorphism γ. $\square$

Proposition 4.4. *Consider a triple $(\mathbb{P}^n, \sigma, \mathcal{O}(1))$. Let $w : \mathcal{O}(1)^\sigma \xrightarrow{\sim} \mathcal{O}(1)$ be an isomorphism, and let $\theta \in AutB(\mathbb{P}^n, 1, \mathcal{O}(1))$ be determined by $(\sigma, w) \in Aut(\mathbb{P}^n, 1, \mathcal{O}(1))$. Then the identity map $B(\mathbb{P}^n, 1, \mathcal{O}(1))_1 \to B(\mathbb{P}^n, \sigma, \mathcal{O}(1))_1$ extends to an algebra isomorphism*

$$\varphi : B(\mathbb{P}^n, 1, \mathcal{O}(1))^\theta \xrightarrow{\sim} B(\mathbb{P}^n, \sigma, \mathcal{O}(1)).$$

If $\sigma^m = 1$ then w may be chosen such that $\theta^m = 1$.

Proof. Since $B(\mathbb{P}^n, 1, \mathcal{O}(1))$ is a polynomial ring, its twist $B(\mathbb{P}^n, 1, \mathcal{O}(1))^\theta$ is generated in degree 1 and its ideal of relations is generated by the relations of degree 2 (chapter 16 section ????). If $x, y \in B(\mathbb{P}^n, 1, \mathcal{O}(1))_1^\theta$ then $x^\theta * y = x^\theta y^\theta = y^\theta x^\theta = y^\theta * x$. Letting x and y run through a basis for $B(\mathbb{P}^n, 1, \mathcal{O}(1))$ we obtain $\binom{n+1}{2}$ linearly independent relations of the form $x^\theta \otimes y - y^\theta \otimes x$. Hence these are defining relations for $B(\mathbb{P}^n, 1, \mathcal{O}(1))^\theta$.

Notice that $(\sigma, w) \in \mathrm{Aut}(\mathbb{P}^n, \sigma, \mathcal{O}(1))$ also, and as such it determines $\rho \in \mathrm{Aut}B(\mathbb{P}^n, \sigma, \mathcal{O}(1))$. Furthermore, ρ and θ agree on $B(\mathbb{P}^n, 1, \mathcal{O}(1))_1^\theta = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) = B(\mathbb{P}^n, \sigma, \mathcal{O}(1))_1$ since $\theta(x) = w(x^\sigma) = \rho(x)$ for $x \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$. By (4.1), if $x, y \in B(\mathbb{P}^n, \sigma, \mathcal{O}(1))_1$ then $x^\rho y = y^\rho x$ so the identity map does indeed extend to an algebra homomorphism φ. Since $B(\mathbb{P}^n, \sigma, \mathcal{O}(1))$ is generated in degree 1, φ is surjective, and since the two rings have the same Hilbert series, φ is an isomorphism.

As in the discussion after (4.1) we may replace any particular w by a suitable scalar multiple such that θ and σ have the same order. $\square$

5. Sheaves of bimodule algebras

Definition 5.1. A sequence $\mathcal{B}_n$, $n \geq 0$, of $\mathcal{O}_X$-bimodules, is ample if, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$

- $\mathcal{B}_n \otimes \mathcal{F}$ is generated by its global sections, and
- for every $q > 0$, $\mathrm{H}^q(X, \mathcal{B}_n \otimes \mathcal{F}) = 0$

for $n \gg 0$.

Warning. We may define a sheaf $\mathcal{B} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ where $\mathcal{L}_n$ is defined as in Definition (1.2). Thus $B(X, \sigma, \mathcal{L}) = \mathrm{H}^0(X, \mathcal{B})$. However, $\mathcal{B}$ is *not* a sheaf of rings in general; if U is an open set in X, then $\mathcal{B}(U)$ is not usually a ring. The product is defined *globally*, and the best one can do locally is observe that there is a natural map $\mathcal{B}_i(U) \times \mathcal{B}_j(\sigma^i U) \to \mathcal{B}_{i+j}(U)$, which is compatible with the product on B.

THEOREM 5.2. *If $\mathcal{L}_n$ is σ-ample, then* $\mathrm{tails}(\mathcal{B}) \cong \mathrm{tails}(B)$.

$\boxed{\text{Paul}}$ Is $\mathcal{B}$ some kind of twist of $\mathcal{A} = \oplus\mathcal{L}^{\otimes n}$?

6. The case of elliptic curves

In preparation for the study of Sklyanin algebras, we now consider $B(E, \sigma, \mathcal{L})$ when E is an elliptic curve.

As remarked before Lemma 2.7, the study of an algebra A may lead to knowledge about a graded algebra homomorphism $\psi : A \to B(X, \sigma, \mathcal{L})$. We need to know the kernel and cokernel of ψ and something about the structure of B in order to obtain information about A. Relevant information about B includes its Hilbert series, and the degrees of its generators and relations. The last of these problems reduces, in large part, to understanding the kernel and cokernel of the multiplication maps $B_m \otimes B_n \to B_{m+n}$. Hence we make the following definition.

Definition 6.1. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Define $R(\mathcal{F}, \mathcal{G})$ and $C(\mathcal{F}, \mathcal{G})$ to be the kernel and cokernel of the canonical map $\mu(\mathcal{F}, \mathcal{G})$ in the exact sequence

$$0 \to R(\mathcal{F}, \mathcal{G}) \to \mathrm{H}^0(\mathcal{F}) \otimes \mathrm{H}^0(\mathcal{G}) \xrightarrow{\mu(\mathcal{F},\mathcal{G})} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{G}) \to C(\mathcal{F}, \mathcal{G}) \to 0.$$

LEMMA 6.2. *Let X be a k-scheme, and let $\mathcal{F}$ and $\mathcal{G}$ be locally free coherent $\mathcal{O}_X$-modules. Let $\mathcal{F}'$ be the kernel of the natural map $\mathcal{O}_X \otimes_k \mathrm{H}^0(X, \mathcal{F}) \to \mathcal{F}$. Suppose that $\mathcal{F}$ is generated by its global sections and that $\mathrm{H}^1(X, \mathcal{G}) = 0$. Then*

1. $R(\mathcal{F}, \mathcal{G}) \cong \mathrm{H}^0(X, \mathcal{F}' \otimes \mathcal{G})$ *and* $C(\mathcal{F}, \mathcal{G}) \cong \mathrm{H}^1(X, \mathcal{F}' \otimes \mathcal{G})$*;*
2. $\mathcal{F}'$ *is locally free and* $\mathrm{rank}(\mathcal{F}') = \mathrm{h}^0(\mathcal{F}) - \mathrm{rank}\,\mathcal{F}$*;*
3. $\det(\mathcal{F}') = (\det \mathcal{F})^*$*;*
4. *if $\mathcal{F}$ is invertible*

PROOF. (1) Since $\mathcal{G}$ is locally free, tensoring the exact sequence

$$0 \to \mathcal{F}' \to \mathrm{H}^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F} \to 0 \tag{6-1}$$

with $\mathcal{G}$ yields an exact sequence; taking cohomology gives a long exact sequence

$$0 \to \mathrm{H}^0(\mathcal{F}' \otimes \mathcal{G}) \to \mathrm{H}^0(\mathcal{F}) \otimes \mathrm{H}^0(\mathcal{G}) \xrightarrow{\mu} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{G}) \to \mathrm{H}^1(\mathcal{F}' \otimes \mathcal{G}) \to \mathrm{H}^0(\mathcal{F}) \otimes \mathrm{H}^1(\mathcal{G}).$$

By hypothesis, the last term in the long exact sequence is zero. The result follows immediately.

(2) This is obvious from the short exact sequence (6-1).

(3) This is the global version of the following fact. If $0 \to U \to V \to W \to 0$ is a short exact exact sequence of vector spaces, with $\dim U = \ell$, $\dim V = m$, and $\dim W = n$, then the product in the exterior algebra $\Lambda(V)$ gives a non-degenerate pairing $\Lambda^\ell U \times \Lambda^n W \to \Lambda^m V \cong k$, so $\Lambda^\ell U \cong (\Lambda^n W)^*$ in a natural way. $\square$

If $H^0(X, \mathcal{O}_X) = k$, then the cohomology sequence obtained from (6-1) gives $\mathrm{H}^0(\mathcal{F}') = 0$. Moreover, if X is a curve, the long exact sequence in the proof of the lemma continues $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{G}) \to 0$, so $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{G}) = 0$ (still under the hypothesis that $\mathcal{F}$ is generated by its global sections and $H^1(X, \mathcal{G}) = 0$.

Some of the information we need about $R(\mathcal{F}, \mathcal{G})$ and $C(\mathcal{F}, \mathcal{G})$ is given by the following results of Mumford.

THEOREM 6.3. [**123**, Theorem 2, page 41] *Let X be a variety, and suppose that $\mathcal{L}$ is a base-point free ample invertible $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module such that*

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{-i}) = 0$$

for $i \geq 1$. Then

1. $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^j) = 0$ *if $i \geq 1$ and $i + j \geq 0$;*
2. $C(\mathcal{F} \otimes \mathcal{L}^i, \mathcal{L}) = 0$ *if $i \geq 0$.*

THEOREM 6.4. [**123**, Theorem 4, page 47] *Let X be a projective variety, and let $\mathcal{N}$ and $\mathcal{L}$ be base-point free ample invertible $\mathcal{O}_X$-modules. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, and suppose that*

$$\mathrm{H}^{i+j-1}(X, \mathcal{M} \otimes \mathcal{N}^{-i} \otimes \mathcal{L}^{-j}) = 0$$

if $i, j \geq 1$. Then the natural map

$$R(\mathcal{M}, \mathcal{N}) \otimes \mathrm{H}^0(\mathcal{L}) \to R(\mathcal{M} \otimes \mathcal{L}, \mathcal{N})$$

is surjective.

THEOREM 6.5. [**123**, Theorem 6, page 53] *Let X be a smooth projective curve of genus g. If $\mathcal{L}$ and $\mathcal{M}$ are invertible $\mathcal{O}_X$-modules such that*

$$\deg(\mathcal{L}) \geq 2g + 1 \qquad \text{and} \qquad \deg(\mathcal{M}) \geq 2g,$$

then $C(\mathcal{L}, \mathcal{M}) = 0$.

THEOREM 6.6. [**123**, Theorem 8, page 58] *Let X be a smooth projective curve of genus g. If $\mathcal{L}, \mathcal{M}$ and $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules such that*

$$\deg(\mathcal{M}) \geq 3g + 1 \qquad \text{and} \qquad \deg(\mathcal{L}), \deg(\mathcal{N}) \geq 2g + 2,$$

then the natural map

$$\mathrm{H}^0(\mathcal{L}) \otimes R(\mathcal{M}, \mathcal{N}) \to R(\mathcal{L} \otimes \mathcal{M}, \mathcal{N})$$

is surjective.

Next we apply these results to the twisted coordinate rings of an elliptic curve. Before doing so, observe that if σ is an automorphism of E, the group it generates, say G, acts on the abelian group $\mathrm{Pic}(E)$, so $\mathrm{Pic}(E)$ becomes a module over the group algebra $\mathbb{Z}G$; for example $(2\sigma - 1).\mathcal{L} = \mathcal{L}^\sigma \otimes \mathcal{L}^\sigma \otimes \mathcal{L}^*$.

THEOREM 6.7. *Let E be an elliptic curve, $\sigma \in \mathrm{Aut}(E)$, $\mathcal{L}$ a line bundle of degree $r \geq 3$, and write $B = B(E, \sigma, \mathcal{L})$. Then*

1. *B is a connected graded k-algebra generated in degree one;*
2. $\dim_k(B_n) = rn$ *for* $n \geq 1$*;*
3. *if* $r > 3$*, then B is defined by* $r^2 - 2r$ *quadratic relations;*
4. *if* $r = 3$*, and* $(\sigma - 1)^2\mathcal{L} \cong \mathcal{O}_E$ *(which happens when σ is a translation), then B is defined by three quadratic relations and a single cubic relation;*
5. *if* $r = 3$*, and* $(\sigma - 1)^2\mathcal{L} \ncong \mathcal{O}_E$*, then B is defined by three quadratic relations.*

PROOF. We will use the notation $\mathcal{F}'$ in Lemma 6.2.

(1) In the terminology of Definition 6.1, B is generated in degree one if and only if $C(\mathcal{L}_n, \mathcal{L}^{\sigma^n}) = 0$ for all $n \geq 1$. However, by Theorem 6.5, this holds because $\deg(\mathcal{L}) \geq 3$.

(2) This follows from Riemann-Roch (B.12.2), since $\deg(\mathcal{L}_n) = rn$.

(3) Express B as a quotient of the free algebra $T = T(B_1)$ by the graded ideal J. We must show that $J_{m+1} = T_1 J_m + J_m T_1$ for $m \geq 2$. By Theorem 6.6, there is a surjective map

$$\mathrm{H}^0(\mathcal{L}) \otimes R(\mathcal{L}^{\sigma}_{m-1}, \mathcal{L}^{\sigma^m}) \to R(\mathcal{L} \otimes \mathcal{L}^{\sigma}_{m-1}, \mathcal{L}^{\sigma^m})$$

since $\deg(\mathcal{L}) \geq 4$. Given the definition of R, this simply expresses the surjectivity of the multiplication map

$$B_1 \otimes \ker(B_{m-1} \otimes B_1 \to B_m) \longrightarrow \ker(B_m \otimes B_1 \to B_{m+1}). \tag{6-2}$$

However,

$$\begin{aligned}
\ker(B_s \otimes B_1 \to B_{s+1}) &= \ker\left(\frac{T_s}{J_s} \otimes T_1 \to \frac{T_{s+1}}{J_{s+1}}\right) \\
&= \ker\left(\frac{T_{s+1}}{J_s T_1} \to \frac{T_{s+1}}{J_{s+1}}\right) \\
&= \frac{J_{s+1}}{J_s T_1}.
\end{aligned}$$

Hence (6-2) says that the multiplication map

$$T_1 \otimes \frac{J_m}{J_{m-1}T_1} = \frac{T_1 J_m}{T_1 J_{m-1} T_1} \longrightarrow \frac{J_{m+1}}{J_m T_1}$$

is surjective. That is, $J_{m+1} = T_1 J_m + J_m T_1$ as required.

(4) By (1) and (2), the multiplication map $B_1 \otimes B_1 \to B_2$ is a surjective map from a 9-dimensional to a 6-dimensional space, so B has 3 defining relations in degree 2. So, it remains to show that

$$\dim_k\left(\frac{J_{n+2}}{T_1 J_{n+1} + J_2 T_n}\right) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases}$$

which will prove that J is generated by J_2 and J_3, and that there is a single cubic relation.

Write $\mathcal{M} = \mathcal{L}^{\sigma}$ and $\mathcal{N} = \mathcal{L}_n^{\sigma^2}$. We consider the following commutative diagram, in which the rows are exact, and the description of the kernels follows from the definition of $R(\mathcal{L}, \mathcal{M})$ and Lemma 6.2:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0(\mathcal{L}' \otimes \mathcal{M}) \otimes \mathrm{H}^0(\mathcal{N}) & \longrightarrow & B_1 \otimes B_1 \otimes B_n & \longrightarrow & B_2 \otimes B_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(\mathcal{L}' \otimes \mathcal{M} \otimes \mathcal{N}) & \longrightarrow & B_1 \otimes B_{n+1} & \longrightarrow & B_{n+2} & \longrightarrow & 0.
\end{array}$$

Using the fact that $B_s = T_s/J_s$, this diagram may also be written as

$$\begin{tikzcd}
0 \arrow[r] & J_2 \otimes \frac{T_n}{J_n} \arrow[r] \arrow[d] & T_1 \otimes T_1 \otimes \frac{T_n}{J_n} \arrow[r] \arrow[d] & \frac{T_2}{J_2} \otimes \frac{T_n}{J_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{J_{n+2}}{T_1 J_{n+1}} \arrow[r] & T_1 \otimes \frac{T_{n+1}}{J_{n+1}} \arrow[r] & \frac{T_{n+2}}{J_{n+2}} \arrow[r] & 0.
\end{tikzcd}$$

Comparing the cokernels of the leftmost columns of these two diagrams, it follows that

$$\frac{J_{n+2}}{T_1 J_{n+1} + J_2 T_n} = \mathrm{H}^1((\mathcal{L}' \otimes \mathcal{M})' \otimes \mathcal{N}),$$

where the right hand side is the cokernel of the first vertical map by Lemma 6.2. For brevity, write $\mathcal{F} = (\mathcal{L}' \otimes \mathcal{M})' \otimes \mathcal{N}$. We must show that $\mathrm{h}^1(\mathcal{F})$ is 1 when $n = 1$, and 0 when $n \geq 2$. The second of these is obvious since $\deg(\mathcal{F}) = 3(n-1)$.

To handle the case $n = 1$, we must identify $(\mathcal{L}' \otimes \mathcal{M})'$. First, $\operatorname{rank} \mathcal{L}' = 2$, so $\operatorname{rank}(\mathcal{L}' \otimes \mathcal{M}) = 2$ also. Second, $\mathrm{H}^0(\mathcal{L}' \otimes \mathcal{M}) = R(\mathcal{L}, \mathcal{M}) \cong J_2$ is 3-dimensional, so $\operatorname{rank}(\mathcal{L}' \otimes \mathcal{M})' = 1$. Thus

$$(\mathcal{L}' \otimes \mathcal{M})' \cong \det(\mathcal{L}' \otimes \mathcal{M})' \cong \left(\det(\mathcal{L}' \otimes \mathcal{M})\right)^*$$

by Lemma 6.2. Since $\operatorname{rank} \mathcal{L}' = 2$, $\det(\mathcal{L}' \otimes \mathcal{M}) = \det(\mathcal{L}') \otimes \mathcal{M} \otimes \mathcal{M}$. Combining this with the observation in Lemma 6.2 that $\det \mathcal{L}' \cong (\det \mathcal{L})^*$, we obtain

$$(\mathcal{L}' \otimes \mathcal{M})' \cong \mathcal{L} \otimes \mathcal{M}^* \otimes \mathcal{M}^*.$$

Therefore, when $n = 1$,

$$\mathcal{F} \cong \mathcal{L} \otimes (\mathcal{L}^\sigma \otimes \mathcal{L}^\sigma)^* \otimes \mathcal{L}^{\sigma^2} = (\sigma - 1)^2 \mathcal{L}.$$

Hence, if $(\sigma - 1)^2 \mathcal{L} \cong \mathcal{O}_E$, then $\mathrm{h}^1(\mathcal{F}) = \mathrm{h}^1(\mathcal{O}_E) = 1$; the Theorem of the Square ensures that this happens if σ is a translation. This proves (3). On the other hand, if $\mathcal{F} \not\cong \mathcal{O}_E$ then $\mathrm{h}^1(\mathcal{F}) = 0$ because $\deg(\mathcal{F}) = 0$, which proves (4). $\square$

EXERCISES

6.1 Let E be an elliptic curve, $\sigma \in \operatorname{Aut}_k(E)$, and $\mathcal{L}$ an invertible $\mathcal{O}_E$-module. Suppose that $\deg(\mathcal{L}) = 2$. Show that
(a) if $\mathcal{L} \not\cong \mathcal{L}^\sigma$, then $B(E, \sigma, \mathcal{L})$ is generated in degree one.
(b) if $\mathcal{L} \cong \mathcal{L}^\sigma$, then $B(E, \sigma, \mathcal{L})$ is not generated in degree one.
In both cases find the degrees of a the defining relations, and in the second case find the degrees of the generators.

6.2 What about curves of higher genus, even plane curves where $\mathcal{L} = i^* \mathcal{O}_{\mathbb{P}^2}(1)$?

6.3 Consider the proof of Theorem 6.7 for $\deg(\mathcal{L}) = 3$. Use Theorem 6.4 to show that $J_{m+1} = T_1 J_m + J_m T_1$ if $m \geq 4$. However, show that Theorem 6.4 does not imply this for $m = 3$ [*Hint:* $(\sigma^3 - \sigma^2 - \sigma + 1).\mathcal{L} \cong \mathcal{O}_E$ when σ is a translation–why?]. In particular, Theorem 6.4 is not best possible.

7. Additional Remarks

For each $r \geq 0$, there is map $\Gamma_{n+r} \to \Gamma_n$ defined by

$$(p_{n+r-1}, \dots, p_0) \mapsto (p_{n-1}, \dots, p_0). \tag{7-1}$$

These maps make Γ_n an inverse system (see Appendix A, section 7); we do not pursue this here, but it is important in the context of pint modules (see chapter 23).

$\boxed{\text{Paul}}$ If $\theta \in \mathrm{Aut}_k(X)$ is such that $\theta\sigma = \sigma\theta$ and $\mathcal{L}^\theta \cong \mathcal{L}$, then θ induces an automorphism of $B(X, \sigma, \mathcal{L})$; is s $B(X, \sigma, \mathcal{L})^\theta \cong B(X, \sigma\theta, \mathcal{L})$?

The algebra $T(V)/J$ isn't readily susceptible to algebro-geometric methods, so we replace it by a slightly different and more tractable algebra (which sometimes equals $T(V)/J$). This should be compared with the fact that the homogeneous coordinate ring is not well-behaved from a functorial standpoint, and it is better to work with the algebras $\bigoplus \mathrm{H}^0(X, \mathcal{L}^{\otimes d})$.

CHAPTER 23

Points and point modules

In $\text{Proj}(A)$, as in any abelian category, the irreducible objects play a key role. Amongst all irreducible objects we single out those which are of the form πM for some graded A-module M of GK-dimension 1—these are called the *points* of $\text{Proj}(A)$ (see Definition 1.2). Often the points are all the irreducible objects in $\text{Proj}(A)$. This chapter studies the basic properties of the points in $\text{Proj}(A)$.

The most accessible points in $\text{Proj}(A)$ are those which come from the point modules for A; these are defined and discussed in Section 2.

We will see that for several classes of algebras the points in $\text{Proj}(A)$ are parametrized by a projective variety and that, in effect, the shift functor [1] acts as an automorphism of that variety. This geometric data is an important invariant of A—it is susceptible to the methods of algebraic geometry, and it often yields important information about A.

The relation between points and finite dimensional simple modules is a further reason for their importance; this is discussed in Section 4. The result there is that the finite dimensional simples are obtained in a natural way from those points in $\text{Proj}(A)$ which have a finite orbit under the shift functor [1].

It should be emphasized that in general there may be relatively few point modules (probably none), and although the task of finding these may be tractable, it is usually far more difficult to find and/or classify those points in $\text{Proj}(A)$ which do not arise from point modules. There are no general methods, but some important examples will be discussed in subsequent chapters.

1. Points in Proj(A)

A non-zero object in an abelian category is irreducible if its only subobjects are isomorphic to itself and 0.

Lemma 1.1. *If M is 1-critical then πM an irreducible object in* $\text{Proj}(A)$.

Proof. Any morphism to πM is of the form πf for some $f : N' \to M/M'$ where N/N' and M' are torsion. Since M is 1-critical, $\text{coker}\, f$ is torsion, whence πf is either zero or surjective. Thus πM is irreducible. □

Definition 1.2. Let A be a graded k-algebra. An object $\mathcal{F} \in \text{Proj}(A)$ is called a point of $\text{Proj}(A)$ if $\mathcal{F}$ is irreducible, and $\mathcal{F} \cong \pi M$ for some (locally finite) 1-critical $M \in \mathsf{GrMod}(A)$. Two isomorphic points are identified; thus a point is an isomorphism class in $\text{Proj}(A)$.

Lemma 1.3. *Suppose that A is locally finite and $\mathbb{N}$-graded, and let M be a graded left A-module.*

1. *If M is 1-critical, then M is torsion-free.*

2. *Given a point* $\mathcal{F} \in \operatorname{Proj}(A)$ *there exists a 1-critical, torsion-free, cyclic A-module M such that* $\mathcal{F} \cong \pi M$.
3. *If M is noetherian and πM is a point, then* $\operatorname{GKdim} M = 1$.
4. *If $\mathcal{F}$ and $\mathcal{G}$ are points then* $\operatorname{Hom}_{\operatorname{Proj}}(\mathcal{F}, \mathcal{G})$ *is either a division algebra or zero, depending on whether* $\mathcal{F} \cong \mathcal{G}$ *or not.*

PROOF. (1) If $0 \neq m \in M$ is torsion, then Am is finite dimensional because A is locally finite. Thus $\operatorname{GKdim} Am = 0$, so $\operatorname{GKdim} M/Am = 1$ by Lemma 11.2.4. Since M is critical it follows that $m = 0$.

(2) If $\mathcal{F}$ is a point then, by definition, $\mathcal{F} \cong \pi N$ for some 1-critical $N \in \mathsf{GrMod}(A)$. Let $0 \neq m \in N$ be homogeneous. Then $\operatorname{GKdim} N/Am = 0$ so N/Am is finite dimensional and hence torsion. Thus $\pi N \cong \pi(Am)$. Since N is 1-critical so is Am, whence Am is torsion-free. Hence the result holds with $M = Am$.

(3) If πM is a point, then $\pi M \cong \pi N$ for some 1-critical, cyclic N. Since N is bounded below it is noetherian. Hence $\operatorname{GKdim} M = 1$ by Corollary 1.7.

(4) This is Schur's Lemma (for irreducible objects in an abelian category). □

The noetherian hypothesis is essential in part (3) of the Lemma because a torsion module can have arbitrarily large GK-dimension.

THEOREM 1.4. *Assume that k is algebraically closed, and let A denote the homogeneous coordinate ring of a projective algebraic variety $X \subset \mathbb{P}^n_k$. Then there is a bijection*

$$\{\textit{points in } X\} \leftrightarrow \{\textit{points in } \operatorname{Proj}(A)\}$$

given by

$$p \leftrightarrow M(p) := A/I(p)$$

where $I(p)$ is the ideal generated by the homogeneous $f \in A$ such that $f(p) = 0$.

PROOF. Choose homogeneous coordinate functions $X_0, \dots, X_n$ on $\mathbb{P}^n$. Let $p = (\alpha_0, \dots, \alpha_n) \in X$. Then $I(p)$ is generated by $\{\alpha_i X_j - \alpha_j X_i \mid 0 \leq i, j \leq n\}$. We may assume that $\alpha_0 = 1$, whence X_0 generates $A/I(p)$. Thus

$$A/I(p) \cong k[T],$$

the polynomial ring in one variable (with $\deg T = 1$, this is an isomorphism of graded algebras). Since every proper quotient ring of $k[T]$ is finite dimensional, $M(p)$ is 1-critical. Thus $\pi M(p)$ is a point.

Before proving that every point in $\operatorname{Proj}(A)$ is of this form we make a further observation. Write $M(p) = ke_0 + ke_1 + \cdots$ where $\deg e_i = i$. Under the isomorphism $M(p) \cong k[T]$, $Ae_1 \cong (T)$, whence $Ae_1 \cong M(p)[-1]$. However, $\pi M(p) \cong \pi(Ae_1)$ so $\pi M(p) \cong \pi M(p)[-1]$. Hence $\pi M(p) \cong \pi M(p)[r]$ for all $r \in \mathbb{Z}$.

Now let πM be a point of $\operatorname{Proj}(A)$. We may assume that M is finitely generated and 1-critical, hence torsion-free. Choose a graded ideal which is maximal amongst those which annihilate some non-zero homogeneous element of M; say $I = \operatorname{Ann}(m)$ with $0 \neq m \in M_r$. It follows that I is a prime ideal: if not, there exist homogeneous elements x and y such that $xym = 0$ but neither xm nor ym is zero—hence $\operatorname{Ann}(ym)$ is strictly larger that I, contradicting the choice of I.

Since M is torsion-free $I \neq \mathfrak{m}$, the augmentation ideal of A. Hence $\mathcal{V}(I) \subset X$ is non-empty. If $p \in \mathcal{V}(I)$ then $I \subset I(p)$, whence $M(p)$ is a quotient of $A/I \cong Am[r]$. But $\pi(Am) \cong \pi M$ since M is 1-critical, so $\pi M(p)[r]$ is isomorphic to a quotient of πM; but πM is irreducible so $\pi M \cong \pi M(p)[r] \cong \pi M(p)$ as required. □

This is reassuring: our new definition of a point agrees with the traditional definition in the case of a projective variety.

EXAMPLE 1.5. Points may not be the only irreducible objects in $\operatorname{Proj}(A)$—here is a 2-critical module M for which πM is irreducible.

Let k be a field of characteristic zero, and let $\mathcal{D}(k^2) = k[x, y, \partial_x, \partial_y]$ be the ring of differential operators on k^2. This may be graded by declaring

$$\deg x = \deg y = 1, \qquad \deg \partial_x = \deg \partial_y = -1.$$

If the polynomial ring $k[x, y]$ is given its usual grading then, under the natural action, $k[x, y]$ is a graded $\mathcal{D}(k^2)$-module. The subring $A = k[x, y, y^2\partial_x, x^2\partial_y]$. is connected, generated in degree 1, has GK-dimension 4, and $M = k[x, y]$ is a graded A-module. We will show that πM is irreducible in $\operatorname{Proj}(A)$ by proving that every non-zero graded A-submodule of M is of finite codimension. Let $N \subset M$ be a graded submodule. Then N is an ideal of $k[x, y]$ since $k[x, y] \subset A$. Let $0 \neq f \in N$. Applying $y^2\partial_x$ and $x^2\partial_y$ repeatedly to f eventually gives $x^n, y^m \in N$ for some n, m. But an ideal containing x^n and y^m is of finite codimension, so πM is irreducible in $\operatorname{Proj}(A)$.

To show that A is noetherian, observe that $A = R[\delta_1, \dots, \delta_5]$, where $R = k[x, y]$, and

$$\delta_1 = x^2\partial_y,\ \delta_2 = x^4\partial_x,\ \delta_3 = y^4\partial_y,\ \delta_4 = xy(x\partial_x - y\partial_y),\ \delta_5 = y^2\partial_x,$$

and these satisfy

- $R + \delta_i R = R + R\delta_i$ for all i, and
- $\delta_i\delta_j - \delta_j\delta_i \in R + \sum_{n=1}^5 R\delta_n$ for all i and j.

The δ_i satisfy the first of these properties because they are derivations, and the other property is verified by the following computations

$$\begin{aligned}
&[\delta_1, \delta_2] = -2x^3\delta_1,\ [\delta_1, d_3] = 4y^3\delta_1,\ [\delta_1, \delta_4] = 4xy\delta_1 - \delta_2, && [\delta_1, \delta_5] = 2\delta_4,\\
&\qquad\qquad [\delta_2, \delta_3] = 0, \qquad [\delta_2, \delta_4] = -x^2y^2\delta_1 - 2xy\delta_2, && [\delta_2, \delta_5] = -4x^3\delta_5,\\
&\qquad\qquad\qquad\qquad [\delta_3, \delta_4] = x^2y^2\delta_5 + 2xy\delta_3, && [\delta_3, \delta_5] = 2y^3\delta_5,\\
&&& [\delta_4, \delta_5] = \delta_3 - 4xy\delta_5.
\end{aligned}$$

Therefore the noetherian property for A follows from Corollary 4.4.2.

$\boxed{\text{Paul}}$

Zhang pointed out a nice example of the above. Let $B(E, \sigma, \mathcal{L})$ be a twisted homogeneous coordinate ring of an elliptic curve with σ of infinite order and $\deg \mathcal{L} = 3$. View B as a left $A \otimes_k A^{\mathrm{op}}$-module where A is the 3-dimensional Sklyanin algebra mapping onto B. By [ATV], a non-zero two-sided ideal of B is of finite codimension, so πB is irreducible in $\operatorname{Proj}(A \otimes A^{\mathrm{op}})$. Moreover, $A \otimes A^{\mathrm{op}}$ has all the usual good homological properties.

PROPOSITION 1.6. *Let A be a graded k-algebra. The natural map* $\operatorname{Proj}(A) \to \operatorname{Proj}(A^{(r)})$ *sends points to points.*

PROOF. Let $\mathcal{F} = \pi M$ be a point in $\operatorname{Proj}(A)$, with M 1-critical. If $0 \neq m \in M^{(r)}$ then $M^{(r)}/A^{(r)}m$ embeds in M/Am, since $AN \cap M^{(r)} = N$ for all submodules N of $M^{(r)}$. Since M is 1-critical, $\dim M/Am < \infty$, so $M^{(r)}$ is also 1-critical; that is, $\pi(M^{(r)}$ is a point, as required. $\square$

LEMMA 1.7. *Let I be a graded ideal of A, $f : A \to A/I$ be the natural map, and $f_* : \operatorname{proj}(A/I) \to \operatorname{proj}(A)$ the corresponding functor. If $\mathcal{F} \in \operatorname{tails}(A/I)$, then $\mathcal{F}$ is a point for A/I if and only if $f_*\mathcal{F}$ is a point for A. In particular, f_* sends points to points.*

PROOF. ($\Rightarrow$) There is a 1-critical A/I-module M such that $\mathcal{F} = \pi_{A/I}M$. But M is also a 1-critical A-module, so $\pi_A M = f_*\mathcal{F}$ is a point for A.

($\Leftarrow$) Since f_* is fully faithful and exact, if $\mathcal{F}$ is not a point, neither is $f_*\mathcal{F}$. $\square$

$\boxed{\text{Paul}}$ More generally, if $f : A \to B$, does f_* send points to points? First it is possible for f_* to send a point to zero if B_A is not finitely generated—if $A = k[X, Y] \subset B = k[X, Y, Z]$ and $M = B/BX + BY$ then M is torsion as an A-module, so $f_*(\pi M) = 0$. If $_AB$ is finitely generated up to torsion, then $f_* : \operatorname{tails}(B) \to \operatorname{tails}(A)$. Under reasonable hypotheses f_* will be faithful (e.g. if $_AB$ and B_A are f.gend, or if $_AB$ is f.gend and A is left noeth).

The action of [1] on points.

If πM is a point in $\operatorname{Proj}(A)$, so is $\pi M[1]$—thus $[1]$ is an automorphism of the points of $\operatorname{Proj}(A)$.

PROPOSITION 1.8. *If A is commutative and generated in degree 1, then* $[1]$ *acts as the identity on points of* $\operatorname{Proj}(A)$.

PROOF. Let M be 1-critical. There exists $x \in A_1$ such that $xM \neq 0$. Hence by Proposition 11.1.3, $xm \neq 0$ for all $0 \neq m \in M$. Therefore the degree zero map $f : M \to M[1]$ defined by $f(m) = xm$ is an injective A-module map (since x is central in A!). Since $M[1]$ is 1-critical, $\operatorname{coker} f$ has GK-dimension zero, so is finite dimensional and hence torsion. Thus πf is an isomorphism; i.e., $\pi M \cong \pi M[1]$. $\square$

If A is not commutative, then the action of $[1]$ on points is typically not the identity (see Section 2), and it provides a useful invariant for A.

LEMMA 1.9. *Suppose that A is of geometric type. If $M \in \mathsf{grmod}(A)$ and* $\operatorname{GKdim} M = 1$*, then πM is of finite length.*

PROOF. By Proposition 11.3.3(3), there exists an integer N such that, for any sequence $0 = M^0 \subset M^1 \subset \ldots \subset M^n = M$ of graded submodules, the number of subfactors M^{i+1}/M^i having GK-dimension 1 is at most N. Hence we may choose a chain with the maximum possible number of such subfactors; this hypothesis implies that if $\operatorname{GKdim}(M^{i+1}/M^i) = 1$ then M^{i+1}/M^i is 1-critical. Passing to $\operatorname{Proj}(A)$, this gives a composition series for πM, as required. $\square$

Definition 1.10. If A is of geometric type and $M \in \mathsf{grmod}(A)$ has GK-dimension 1, the $\mathsf{support}$ of πM, denoted $\operatorname{Supp}(\pi M)$, is the set of composition factors of πM. Thus $\operatorname{Supp}(\pi M)$ is a finite set of points in $\operatorname{Proj}(A)$. We may, if we wish, count these points with multiplicity (i.e., count the number of times a given point occurs in the composition series). We sometimes abuse this notation and terminology by writing $\operatorname{Supp}(M)$ and calling it the support of M.

PROPOSITION 1.11. *Let A be a left noetherian, graded k-algebra of finite global dimension, and suppose that $H_A(t) = (1-t)^{-n}$. If M is a finitely generated A-module such that* $\operatorname{GKdim} M = 1$ *then, for large r,* $\dim M_r = e(M)$.

Proof. By Theorem 11.4.2, $H_M(t) = q(t)(1-t)^{-n}$ for some $q(t) \in \mathbb{Z}[t,t^{-1}]$ so, since $H_M(t)$ has a pole of order one at $t=1$, there exists $f(t) \in \mathbb{Z}[t,t^{-1}]$ such that $H_M(t) = f(t)(1-t)^{-1}$. Multiplying this out, one sees that $\dim M_r = f(1) = e(M)$ for $r \gg 0$. □

As far as I know it is an open question whether the global dimension of an algebra A, which satisfies the hypotheses of the Proposition, *must equal* n. This would be a wonderful result if it is true.

Example 1.12. Here is an example of a 1-critical module having a periodic Hilbert series. Let

$$A = \begin{pmatrix} k[T^2] & Tk[T^2] & T^2k[T^2] \\ Tk[T^2] & k[T^2] & Tk[T^2] \\ T^2k[T^2] & Tk[T^2] & k[T^2] \end{pmatrix},$$

with its grading defined by viewing A as a subalgebra of $M_3(k) \otimes_k k[T]$, endowed with the tensor product grading with $M_3(k)$ concentrated in degree zero and $\deg T = 1$. Notice that A is right and left noetherian, and $\operatorname{GKdim} A = 1$, since it is a finitely generated module over its central subring $k[T^2]$, and is prime because $\operatorname{Fract}(A) \cong M_3(k(T^2))$.

Let $N = k[T]$ be the 3×1 column vectors considered as a left A-module in the obvious way. This contains a direct sum of submodules

$$\begin{pmatrix} k[T] \\ k[T] \\ k[T] \end{pmatrix} \supset \begin{pmatrix} k[T^2] \\ Tk[T^2] \\ T^2k[T^2] \end{pmatrix} \oplus \begin{pmatrix} Tk[T^2] \\ k[T^2] \\ Tk[T^2] \end{pmatrix} = A.\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus A.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The submodules $C := A.(1,0,0)^{\mathsf{T}}$ and $D := A.(0,1,0)^{\mathsf{T}}$ are 1-critical, and their Hilbert series are

$$H_C(t) = 1,1,2,1,2,1,2,\ldots \qquad \text{and}$$
$$H_D(t) = 1,2,1,2,1,2,1,\ldots .$$

Since the Hilbert series of C and D are not eventually the same, the points πC and πD are distinct. However, one has

$$\pi C \cong \pi C[2] \cong \pi D[1].$$

To obtain an example over a connected algebra, replace A by its subring in which the diagonal entries of the matrix are required to have the same constant term.

EXERCISES

1.1 Suppose that X, Y and Z are graded A-modules whose images in $\operatorname{Proj}(A)$ are of finite length. Show that, if there is an exact sequence $0 \to \pi X \to \pi Y \to \pi Z \to 0$ in $\operatorname{Proj}(A)$, then $\operatorname{Supp}(Y) = \operatorname{Supp}(X) \cup \operatorname{Supp}(Z)$.

$\boxed{\text{Paul}}$ Give an example with $\mathcal{F}$ and $\mathcal{G}$ distinct points in $\operatorname{Proj}(A)$ and $\operatorname{Ext}_{\operatorname{Proj}}(\mathcal{F},\mathcal{G}) \neq 0$.

2. Point modules

In this section A will denote a graded k-algebra, which is generated by A_1 as a k-algebra; in particular, A is connected.

Point modules are an accessible class of graded modules which give irreducible objects in $\mathrm{Proj}(A)$. If A is commutative and k is algebraically closed, the point modules give all the irreducible objects in $\mathrm{Proj}(A)$ and they are in bijection with the points in the projective variety determined by A. For a rather large class of non-commutative graded algebras (including most of the algebras arising from quantum groups) it is not too difficult to determine all the point modules. Later on we will discuss some of these algebras in detail.

Definition 2.1. Let A be a graded k-algebra which is generated in degree 1. A point module is a graded A-module M such that

- M is cyclic, and
- $H_M(t) = (1-t)^{-1}$.

If M is a point module, then $A.M_n = M_{\geq n}$. It follows that a point module is noetherian (some hypothesis on the generators of A is necessary for this result, since Example 11.3.4 gives a non-noetherian ring which is a point module when viewed as a module over itself).

Let A be a graded algebra generated in degree 1. If M is a point module, then so is $M_{\geq 1}[1]$. Since $\pi M_{\geq 1}[1] \cong \pi M[1]$ the action of $[1]$ on $\mathrm{Proj}(A)$ restricts to give an action of $[1]$ on $\{\pi M \mid M \text{ is a point module}\}$.

LEMMA 2.2. *Let A be a graded k-algebra, generated in degree 1. If M is a point module then*

1. *M is 1-critical, and*
2. *πM is a point in* $\mathrm{Proj}(A)$.

PROOF. (1) If N is a non-zero submodule then N contains some M_i, so contains M_j for all $j \geq i$ (since A is generated in degree 1) whence the quotient is finite dimensional.

(2) This is just Lemma 1.1. □

The terminology is motivated by the classical case of points on projective varieties, which is discussed next—also recall Theorem 1.4 which proved that, if A is the homogeneous coordinate ring of a projective variety X over an algebraically closed field, then there is a bijection between the points of $\mathrm{Proj}(A)$ and the points of X, the point $p \in X$ corresponding to the point $\pi M(p) \in \mathrm{Proj}(A)$).

EXAMPLE 2.3. (We do not assume that k is algebraically closed.) Let A be the homogeneous coordinate ring of a projective variety $X \subset \mathbb{P}^n$. Then there is a bijection

$$\{\text{points of } X\} \leftrightarrow \{\text{isomorphism classes of point modules}\}.$$

Each point $p \in X$ determines a point module $M(p) = A/I(p)$ where $I(p)$ is the ideal generated by the homogeneous functions vanishing at p. On the other hand, if M is a point module then it is of the form A/I where $I = \mathrm{Ann}(M_0)$. The only graded k-algebra which is generated in degree 1 and has Hilbert series $(1-t)^{-1}$ is the polynomial ring $k[T]$ with $\deg T = 1$, whence $A/I \cong k[T]$. Thus $I = I(p)$ for some point $p \in X$.

If we assume in addition that k is algebraically closed then, by Theorem 1.4, we also have a bijection

$$\{\text{points in } \operatorname{Proj}(A)\} \leftrightarrow \{\text{isomorphism classes of point modules}\}.$$

If k is *not* algebraically closed this need not be true: for example, if $A = \mathbb{R}[x, y]$ is the polynomial ring, then $M = A/(x^2 + y^2)$ gives an irreducible object πM in $\operatorname{Proj}(A)$ (since M is a domain, and thus a 1-critical module), but $H_M(t) = (1 + t)(1 - t)^{-1}$ so M is not a point module.

Example 2.4. If A is a commutative algebra which is *not* generated in degree one, there may be irreducible objects in $\operatorname{Proj}(A)$ which do not arise from point modules. Let $A = k[x, y]$ be the polynomial ring over an algebraically closed field, with the grading defined by

$$\deg x = 1 \quad \text{and} \quad \deg y = 2.$$

The point modules for A are the modules $N_\alpha := A/(\alpha x^2 - y)$, where $\alpha \in k$ is arbitrary. The module $N_\infty = A/(x)$ is also 1-critical and πN_∞ is not isomorphic to any πN_α. It is not difficult to see that the set of irreducible objects in $\operatorname{Proj}(A)$ is $\{\pi N_\infty\} \cup \{\pi N_\alpha \mid \alpha \in k\}$.

Example 2.3 shows that the point modules over the commutative polynomial ring in two variables, with its usual grading, are parametrized by $\mathbb{P}^1$. The coordinate rings $k[x, y]$ of the quantum ($yx = qxy$) and Jordan ($yx - xy = x^2$) planes also have their point modules parametrized by $\mathbb{P}^1$—this is a special case of the next result.

Proposition 2.5. *Let σ be an automorphism of the homgeneous coordinate rings, A say, of a projective variety $X \subset \mathbb{P}^n$. Then*

1. *the point modules for ${}^\sigma A$ are in bijection with the points of X via*
$$p \leftrightarrow M(p) = {}^\sigma A/{}^\sigma Ax_1 + \cdots + {}^\sigma Ax_n$$
where $x_1, \ldots, x_n \in A_1$ satisfy $\mathcal{V}(x_1, \ldots, x_n) = \{p\}$.
2. *$M(p)_{\geq 1} \cong M(p^\sigma)[-1]$.*

Moreover, if k is algebraically closed then the $\pi M(p)$ are all the points in $\operatorname{Proj}({}^\sigma A)$.

Proof. The equivalence of categories $\mathsf{GrMod}(A) \to \mathsf{GrMod}({}^\sigma A)$ (Theorem 6.4.7) establishes a bijection between the point modules over A and ${}^\sigma A$: M is a point module if and only if ${}^\sigma M$ is.

For each $p \in X$ define

$$M(p) := {}^\sigma(A/I(p)),$$

where $I(p)$ is generated by the homogeneous polynomials vanishing at p. Write $M(p) = ke_0 + ke_1 + \cdots$, where $\deg e_j = j$. If $x \in {}^\sigma A_1$, then

$$\begin{aligned} x \odot e_0 &= xe_0 = x(p)e_1, && \text{and} \\ x \odot e_1 &= x^\sigma e_1 = x^\sigma(p)e_2 = x(p^\sigma)e_2. \end{aligned}$$

It follows from the first of these calculations that the kernel of the surjection ${}^\sigma A \to M(p)$ contains $I(p)$, and hence is equal to that by comparison of Hilbert series. But $I(p)$ is generated as a left ideal of both ${}^\sigma A$ and A by the same elements, from which (1) follows. The second of the above calculations gives (2). The truth of the final statement follows from the fact that the equivalence of the categories GrMod induces an equivalence of categories $\operatorname{Proj}(A) \to \operatorname{Proj}({}^\sigma A)$. □

In the context of Example 2.3, if the projective space in question is $\mathbb{P}(V)$, then A is a quotient of the polynomial ring generated in degree 1 by V^*; thus A_1 is a quotient of V^*, and V contains a copy of A_1^*. In particular, $X \subset \mathbb{P}(A_1^*)$; that is, the point modules for A are naturally in bijection with the points of a projective subvariety of $\mathbb{P}(A_1^*)$.

Even if A is not commutative, a point module still determines a point of the projective space $\mathbb{P}(A_1^*)$; in fact, it determines a sequence of points.

Definition 2.6. Let M be a point module over an algebra A generated by A_1. The sequence of points $p_0, p_1, \dots$ in $\mathbb{P}(A_1^*)$, defined by

$$p_j := \mathcal{V}(\{x \in A_1 \mid xM_j = 0\}),$$

is called the point module sequence determined by M.

The definition makes sense: since $M = A.M_0$ and A is generated by A_1, $M_{j+1} = A_1 M_j$ for all $j \geq 0$, whence the kernel of the map $A_1 \to \operatorname{Hom}_k(M_j, M_{j+1})$ is of codimension 1 in A_1.

By Proposition 2.5(2), the point module sequence associated to $M(p)$, over the twisted homogeneous coordinate ring ${}^{\sigma}A$, is $p, p^{\sigma}, p^{\sigma^2}, \dots$.

PROPOSITION 2.7. *Let A be a graded k-algebra, generated in degree 1. Then*

1. *two point modules are isomorphic if and only if they determine the same point module sequence;*
2. *if the point module M determines the sequence $p_0, p_1, \dots$, then the point module $M_{\geq 1}[1]$ determines the sequence $p_1, p_2, \dots$;*
3. *if M and N are point modules then $\pi M \cong \pi N$ if and only if their associated point module sequences are eventually the same;*
4. *if $A = T(A_1)$ is the free algebra, every sequence in $\mathbb{P}(A_1^*)$ is a point module sequence;*
5. *if A is commutative, every point module sequence is constant.*

PROOF. (1) Exercise 2.

(2) This is a straightforward consequence of the definition of a point module sequence.

(3) If M and N are point modules then, since they are noetherian, $\pi M \cong \pi N$ if and only if $M_{\geq n} \cong N_{\geq n}$ for some n, by Theorem 1.6; but this is equivalent to the two point module sequences eventually being the same.

(4) Let $\lambda_0, \lambda_1, \dots$ be a sequence of non-zero elements in A_1^* and define an action of $x \in A_1$ on $M := ke_0 \oplus ke_1 \oplus \cdots$ by

$$x.e_j = \lambda_j(x) e_{j+1}. \tag{2-1}$$

By the universal property of $T(A_1)$, the map $A_1 \to \operatorname{End}_k(M)$ defined by (2-1) extends to an algebra map $T(A_1) \to \operatorname{End}_k(M)$, thus making M a left $T(A_1)$-module. Indeed, M becomes a graded module with Hilbert series $(1-t)^{-1}$. Since each λ_j is non-zero, M is cyclic and hence a point module.

(5) By Example 2.3, if M is a point module then $M \cong A/I$ for some ideal I for which $A/I \cong k[T]$, the commutative polynomial ring. Thus $\operatorname{Ann}(M_n) = I$ for all N, whence $p_n = \mathcal{V}(I)$ for all n. $\square$

If M is a point module whose point module sequence is $p_0, p_1, p_2, \dots$ we will sometimes say, imprecisely, that there is a basis $e_j \in M_j$ such that, for all $x \in A_1$,

$$x.e_j = x(p_j) e_{j+1}.$$

The next example shows that point module sequences need not be determined by their initial terms (in contrast to what happens for the quantum and Jordan planes); it also shows that non-isomorphic point modules need not give distinct points of $\mathrm{Proj}(A)$— there are infinitely many point modules up to isomorphism, but $\mathrm{Proj}(A)$ has only two distinct points.

EXAMPLE 2.8. Let $A = k[x, y]$ with defining relation $yx = 0$. Recall that A is not noetherian; in particular, $xA = Ax + Axy + Axy^2 + \cdots$ is not a finitely generated left ideal. We also note that xA and Ay are two-sided ideals.

We will show that the point module sequences for A are

- the constant sequence $(0,1)$, $(0,1)$, $\ldots$ corresponding to A/xA, and
- the sequences
$$\underbrace{(0,1),\ (0,1), \ldots, (0,1)}_{n \text{ times}},\ p,\ (1,0,), (1,0), \ldots,$$
where $(0,1) \neq p \in \mathbb{P}^1$ is arbitrary, corresponding to the point module
$$A/Ax + Axy + \cdots + Axy^{n-1} + Aay^n,$$
where $0 \neq a \in A_1$ satisfies $\mathcal{V}(a) = p$ and $n \geq 0$.

It follows from this that the point modules for A give only two distinct points in $\mathrm{Proj}(A)$, namely $\pi(A/Ay)$ and $\pi(A/xA)$. We will also show that these are the only points in $\mathrm{Proj}(A)$.

To check these claims, first note that A/xA is isomorphic, as a ring, to a polynomial ring in 1 variable, so is a point module. Second, if $a \in A_1 \backslash kx$ then ay^n is left regular so, since $Ax = kx + kx^2 + \cdots$, the Hilbert series of the left ideal $J := Ax + Axy + \cdots + Axy^{n-1} + Aay^n$ is
$$(t + t^2 + \cdots + t^n)(1-t)^{-1} + t^{n+1}(1-t)^{-2} = t(1-t)^{-2};$$
thus A/J is a point module.

We now show these are all the point modules. Let $M = ke_0 + ke_1 + \cdots$ be a point module. If $xy^{n-1}e_0 = 0$ for all $n \geq 1$, then $M = A/Ax$. On the other hand, suppose that
$$xe_0 = xye_0 = \cdots = xy^{n-1}e_0 = 0 \qquad \text{but} \qquad xy^n e_0 \neq 0.$$
Therefore, there exists $a \in A_1 \backslash kx$ such that $ay^n e_0 = 0$. Thus M is a quotient of $A/Ax + Axy + \cdots + Axy^{n-1} + Aay^n$; but we just showed this was a point module, so there is equality.

Finally, we show that $\mathrm{Proj}(A)$ has only two points, namely $\pi(A/xA)$ and $\pi(A/Ay)$. The key point is that $A/(x)$ and $A/(y)$ are both polynomial rings in 1 variable so Proj of each contains a single point. Let M be a 1-critical A-module. Then $yxM = 0$. If $xM = 0$ then M is an $A/(x)$-module, hence $\pi M \cong \pi(A/(x))$. If $xM \neq 0$ then, since $Ax \subset xA$, xM is a non-zero A-module, so $\pi M \cong \pi(xM)$; but xM is an $A/(y)$-module, so $\pi(xM) \cong \pi(A/(y))$.

We now consider how to determine the point module sequences. No general methods are available, but Proposition 2.10 below, which is our immediate goal, is useful in a variety of situations.

With this in mind, let A be a graded k-algebra which is generated by A_1, and write $A = T(A_1)/I$. Define the schemes
$$\Gamma_n := \mathcal{V}(I_n) \subset \mathbb{P}(A_1^*)^{\times n},$$

as in Section 3 of Chapter 22.

PROPOSITION 2.9. *Let A be a graded k-algebra, generated in degree 1. Then $p_0, p_1, \ldots \in \mathbb{P}(A_1^*)$ is a point module sequence if and only if, for all $i \geq 0$, and all $n \geq 1$,*

$$(p_{i+n-1}, \ldots, p_i) \in \mathcal{V}(I_n).$$

PROOF. Let $\lambda_0, \lambda_1, \ldots \in A_1^*$ be representatives of the points $p_0, p_1, \ldots$. Let $M = \sum_{i=0}^{\infty} ke_i$ be the point module for $T(A_1)$ associated to this sequence by defining $x.e_i = \lambda_i(x)e_i$ for $x \in A_1$. Then $p_0, p_1, \ldots$ is a point module sequence for A if and only if M is an A-module; that is, if and only if $IM = 0$ or, more explicitly, $I_n M_i = 0$ for all $i \geq 0$ and all $n \geq 1$.

Let $f = \sum \alpha_{j_1 \cdots j_n} x_{j_1} \otimes \cdots \otimes x_{j_n} \in I_n$. Then

$$\begin{aligned} f.e_i &= \sum \alpha_{j_1 \cdots j_n} \lambda_{i+n-1}(x_{j_1}) \cdots \lambda_i(x_{j_n}) e_{i+n} \\ &= \sum \alpha_{j_1 \cdots j_n} (x_{j_1} \cdots \otimes \cdots x_{j_n})(\lambda_{i+n-1}, \ldots, \lambda_i) e_{i+n}. \end{aligned}$$

Thus, $f.e_i = 0$ if and only if

$$\sum \alpha_{j_1 \cdots j_n} (x_{j_1} \otimes \cdots \otimes x_{j_n})(p_{i+n-1}, \ldots, p_i) = 0.$$

Hence $I_n.M_i = 0$ if and only if $f(p_{i+n-1}, \ldots, p_i) = 0$ for all $f \in I_n$. $\square$

We observed in Lemma 22.3.2 that

$$\Gamma_{d+1} \subset (\mathbb{P}(A_1^*) \times \Gamma_d) \cap (\Gamma_d \times \mathbb{P}(A_1^*)),$$

with equality if $I_{d+1} = A_1 \otimes I_d + I_d \otimes A_1$. Moreover the Γ_n form an inverse system of k-schemes: if $r \geq 0$, there is morphism $\Gamma_{n+r} \to \Gamma_n$ defined by

$$(p_{n+r-1}, \ldots, p_0) \mapsto (p_{n-1}, \ldots, p_0). \tag{2-2}$$

We define

$$\Gamma := \varprojlim \Gamma_n.$$

The closed points of Γ are in bijection with the set of point module sequences, whence the point modules are parametrized by this set, which is an inverse limit of projective schemes.

Inverse limits are decidedly easier to handle when the morphisms defining the inverse system are eventually isomorphisms—the next Proposition examines one such case: roughly speaking it examines circumstances in which the point module sequences are determined by their initial terms. This should be contrasted with Example 2.8.

PROPOSITION 2.10. *Let $A = T(A_1)/I$ be a graded algebra. Suppose that, for some d, I is generated in degree $\leq d+1$, and that the projection $\pi : \Gamma_{d+1} \to \Gamma_d$ defined by*

$$\pi(p_d, \ldots, p_0) = (p_{d-1}, \ldots, p_0)$$

is injective. Write $E := \pi(\Gamma_{d+1})$ and define $\sigma : E \to \Gamma_d$ by

$$\sigma(p_{d-1}, \ldots, p_0) = (p_d, \ldots, p_1)$$

where $(p_d, \ldots, p_0) = \pi^{-1}(p_{d-1}, \ldots, p_0)$.

If $\sigma(E) \subset E$, then

1. *for all $n \geq d+1$, the map $\Gamma_n \to E$ defined by*
$$(p_{n-1}, \dots, p_0) \mapsto (p_{d-1}, \dots, p_0)$$
is bijective, and
2. *there is a bijection $\Gamma \to E$ defined by*
$$p_0, p_1, \dots \mapsto (p_{d-1}, \dots, p_0);$$
3. *this bijection has the property that $(p_{i+d-1}, \dots, p_i) = \sigma^i(p_{d-1}, \dots, p_0)$ for all $i \geq 0$.*

Proof. First, note that σ is well-defined because π is injective.

(1) We prove injectivity by induction on n. The case $n = d+1$ is true by hypothesis. We suppose the result is true for n and prove it for $n+1$. Let $(p_n, \dots, p_0), (q_n, \dots, q_0) \in \Gamma_{n+1}$ and suppose that $(p_{d-1}, \dots, p_0) = (q_{d-1}, \dots, q_0)$. Now $(p_{n-1}, \dots, p_0), (q_{n-1}, \dots, q_0) \in \Gamma_n$ so, by the induction hypothesis, $(p_{n-1}, \dots, p_0) = (q_{n-1}, \dots, q_0)$. In particular, $(p_d, \dots .p_1) = (q_d, \dots, q_1)$ since $n-1 \geq d$. But these are elements of E since $(p_n, \dots, p_1)$ and $(q_n, \dots, q_1)$ belong to Γ_n. Therefore, by the induction hypothesis, $(p_n, \dots, p_1) = (q_n, \dots, q_1)$. This proves the injectivity.

Before proving the surjectivity, we make an observation. For brevity, write $\mathbb{P} = \mathbb{P}(A_1^*)$. First, since $\sigma(E) \subset E$, $\Gamma_{d+1} = (\mathbb{P} \times E) \cap (E \times \mathbb{P})$. Now, since I is generated in degree $\leq d+1$, it follows from Lemma 3.2 applied inductively, that, for $n \geq d+1$,
$$\Gamma_n = (\mathbb{P}^{\times(n-d)} \times E) \cap (\mathbb{P}^{\times(n-d-1)} \times E \times \mathbb{P}) \cap \dots$$
$$\dots \cap (\mathbb{P} \times E \times \mathbb{P}^{\times(n-d-1)}) \cap (E \times \mathbb{P}^{\times(n-d)}). \quad (2\text{-}3)$$

We now prove that $\Gamma_n \to E$ is surjective if $n \geq d+1$. Let $p = (p_{d-1}, \dots, p_0) \in E$. Define $p_d, \dots, p_{n-1}$ inductively by
$$(p_{i+d-1}, \dots, p_i) = \sigma^i(p)$$
for $0 \leq i \leq n-d$. To see that this is a good definition we must check that the last $d-1$ terms in $\sigma^{i+1}(p)$ agree with the first $d-1$ terms of $\sigma^i(p)$. However, $\sigma^i(p) \in E$ so, by definition of σ, and its application to $\sigma^i(p)$, this is true. Thus $(p_{n-1}, \dots, p_0)$ belongs to the right hand side of (2-3), and hence to Γ_n. This proves the surjectivity, and hence the bijectivity.

(2) and (3). Since the maps $\Gamma_{n+1} \to E$ are bijective for $n \geq d+1$, so are the maps $\Gamma_{n+1} \to \Gamma_n$. Hence the inverse limit of this inverse system is in bijection with E, and has the form described. $\square$

The next section treats the simplest case of Proposition 2.10, namely the case $d = 1$; that is, I is generated by I_2.

Example 2.11. We illustrate Proposition 2.10 by computing the point module sequences for the algebra $A = k[x, y]$ with defining relations
$$x^2y - \lambda xyx + yx^2 = 0,$$
$$y^2x - \lambda yxy + xy^2 = 0$$
where $0 \neq \lambda \in k$. We adopt the notation of Proposition 2.10. First, Γ_3 consists of the points $((x_2, y_2), (x_1, y_1), (x_0, y_0)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ satisfying the equations
$$x_2x_1y_0 - \lambda x_2y_1x_0 + y_2x_1x_0 = 0,$$
$$y_2y_1x_0 - \lambda y_2x_1y_0 + x_2y_1y_0 = 0.$$

That is,

$$\begin{pmatrix} x_2 & y_2 \end{pmatrix} \begin{pmatrix} x_1y_0 - \lambda y_1x_0 & y_1y_0 \\ x_1x_0 & -lx_1y_0 + y_1x_0 \end{pmatrix} = 0. \tag{2-4}$$

Thus $((x_1, y_1), (x_0, y_0))$ is in $E := \pi(\Gamma_3)$ if and only if the determinant of this matrix vanishes; that is, if and only if

$$x_1^2y_0^2 - \lambda x_0x_1y_0y_1 + x_0^2y_1^2 = 0. \tag{2-5}$$

Choosing μ such that $\lambda = \mu + \mu^{-1}$, and factorizing (2-5), we see that $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the union of the two lines

$$\ell_+ = \mathcal{V}(x_1y_0 - \mu x_0y_1) \qquad \text{and} \qquad \ell_- = \mathcal{V}(x_1y_0 - \mu^{-1}x_0y_1),$$

except when $\lambda = 2$, in which case E is a double line. If we define the automorphism τ of $\mathbb{P}^1$ by

$$(x, y)^\tau = (\mu x, y),$$

then

$$E = \{(p^\tau, p) \mid p \in \mathbb{P}^1\} \cup \{(p^{\tau^{-1}}, p) \mid p \in \mathbb{P}^1\}.$$

It is easy to see that the matrix (2-4) is non-zero for points of E, so given a point in E there is a unique solution to (2-4), namely

$$(x_2, y_2) = \begin{cases} (x_0, y_0)^{\tau^2} & \text{if } (x_0, y_0) \in \ell_+, \\ (x_0, y_0)^{\tau^{-2}} & \text{if } (x_0, y_0) \in \ell_-. \end{cases}$$

Thus

$$\Gamma_3 = \{(p^{\tau^2}, p^\tau, p) \mid p \in \mathbb{P}^1\} \cup \{(p^{\tau^{-2}}, p^{\tau^{-1}}, p) \mid p \in \mathbb{P}^1\}.$$

It is clear that all the hypotheses of Proposition 2.10 are satisfied.

EXERCISES

2.1 Check that in Example 2.4 we did in fact find all the irreducible objects in $\mathrm{Proj}(A)$. Notice that the points in $\mathrm{Proj}(A)$ are parametrized by $\mathbb{P}^1$.

2.2 Give a detailed proof of part (1) of Proposition 2.7.

2.3 Suppose that the hypotheses of Proposition 2.10 hold. Let L be a left ideal of A such that A/L is a point module. Show that L is generated, as a left ideal, in degree $\leq d$. [*Hint:* consider the truncated point modules of length $d+1$ which can arise as quotients of $A/A.L_{\leq d}$—also see the proof of Theorem 3.1 below.]

2.4 Let A be a graded k-algebra, generated in degree 1. If M is a point module for A, show that $M^{(r)}$ is a point module for the Veronese subalgebra $A^{(r)}$.

2.5 Show that there is a bijection

$$\{\text{constant point module sequences}\} \leftrightarrow \{\text{2-sided ideals } I \text{ such that } A/I \cong k[T]\}$$

with the correspondence being given by

$$p, p, \ldots \leftrightarrow \mathrm{Ann}(M),$$

where M is the point module corresponding to the given sequence.

3. Examples

In this section the point modules are determined for several examples, all of which are quadratic algebras.

NOTATION . (The following notation will be kept throughout this section.) Fix a finite dimensional vector space V, write $\mathbb{P} = \mathbb{P}(V^*)$, and write $\mathrm{pr}_i : \mathbb{P}\times\mathbb{P} \to \mathbb{P}$ $(i = 1, 2)$ for the projections onto the first and second components. Let $A = T(V)/I$ be a quadratic algebra, with I generated by I_2, and let $\Gamma_2 \subset \mathbb{P} \times \mathbb{P}$ be the subscheme cut out by I_2.

THEOREM 3.1. *Suppose that* $\mathrm{pr}_1(\Gamma) = \mathrm{pr}_2(\Gamma)$, *write* $E = \mathrm{pr}_i(\Gamma)$, *and suppose that both projections* $\mathrm{pr}_i : \Gamma \to E$ *are injective. Then*

1. *there is an automorphism* $\sigma : E \to E$, *such that* Γ *is the graph of* σ, *and*
2. *there are bijections*

$$E \leftrightarrow \{\textit{point module sequences}\} \leftrightarrow \{\textit{isomorphism classes of point modules}\}$$

$$p \leftrightarrow (p, p^\sigma, p^{\sigma^2}, \dots) \leftrightarrow M(p)$$

where $M(p) = A/Ax_1 + \cdots + Ax_n$ *where* $x_i \in V$ *satisfy* $\mathcal{V}(x_1, \dots , x_n) = \{p\}$.

PROOF. Most of this is just a restatement of Proposition 2.10 with $d = 1$: since A is quadratic, I is generated in degree $\leq d + 1$; since $\Gamma = \Gamma_{d+1}$, the E notation is consistent with that in (2.10), and $\mathrm{pr}_2 = \pi$ so the injectivity of $\pi : \Gamma_{d+1} \to \Gamma_1 = \mathbb{P}$ is ensured; the map $\sigma : E \to E$, which may be defined as $\mathrm{pr}_1 \circ \mathrm{pr}_2^{-1}$, is bijective since each pr_i is. Thus Γ consists of the points (p^σ, p) with $p \in E$.

The bijection between E and the point module sequences, and the explicit description of this bijection, is part (2) of Proposition 2.10. The bijection with the isomorphism classes of point modules is Proposition 2.7.

Now we explicitly describe $M(p)$, the point module corresponding to the sequence $p, p^\sigma, p^{\sigma^2}, \dots$. Let $x_1, \dots , x_n \in A_1$ be such that $\mathcal{V}(x_1, \dots , x_n) = \{p\}$. Since $M(p) = A.M(p)_0$, and $x_i.M(p)_0 = 0$ for all i, $M(p)$ is a quotient of $N := A/Ax_1 + \ldots + Ax_n$. Suppose to the contrary that the (surjective) map $N \to M(p)$ is not injective. Then, after shifting, N has a cyclic subquotient, L say, with $\dim L_0 = \dim L_1 = 1$ and $\dim L_2 = 2$. Choose elements e_0, e_1, e_{21}, e_{22} such that $L_0 = ke_0, L_1 = ke_1, L_2 = ke_{21} + ke_{22}$. Then there are points $p, q_1, q_2 \in A_1^*$ such that $x.e_0 = x(p)e_1$ and $x.e_1 = x(q_1)e_{21} + x(q_2)e_{22}$ for all $x \in A_1$. Since L is cyclic, q_1 and q_2 are linearly independent. Since L/ke_{22} and L/ke_{21} are truncated point modules of length 3 the points (p, q_1) and (p, q_2) are both in Γ. But $q_1 \neq q_2$, which contradicts the fact that pr_1 is injective. □

EXAMPLE 3.2. Let $A = S(V)$ be the commutative polynomial ring with its usual grading. Then A is a quadratic algebra, and by Example 22.3.3, $\Gamma_2 = \{(p, p) \mid p \in \mathbb{P}\}$. Hence Theorem 3.1 applies and confirms what we already know, namely that the point modules are parametrized by the points of $\mathbb{P}$.

The next example also confirms what we already know, but we carry it out explicitly for practice.

EXAMPLE 3.3. Let $A = k[x, y]$ with $yx = qxy$ where $0 \neq q \in k$ and $\deg x = \deg y = 1$. Then A is a quadratic algebra with $I_2 = k(y\otimes x - qx\otimes y)$. Now consider $\mathcal{V}(I_2) \subset \mathbb{P}^1 \times \mathbb{P}^1$ where $\mathbb{P}^1 = \mathbb{P}(A_1^*)$.

Define $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ by $(\alpha, \beta)^\sigma = (q\alpha, \beta)$; this is the automorphism induced by the action of $\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on k^2. The graph of this automorphism is

$$\Delta_\sigma := \{((\alpha, \beta), (q\alpha, \beta)) \mid (\alpha, \beta) \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Since $y \otimes x - qx \otimes y$ vanishes on Δ_σ, $\Delta_\sigma \subset \mathcal{V}(I_2)$. Conversely, if $(p_1, p_0) \in \mathcal{V}(I_2)$, with $p_i = (\alpha_i, \beta_i)$, then $\beta_1\alpha_0 = q\alpha_1\beta_0$; that is, $(p_1, p_0) \in \Delta_\sigma$. Thus

$$\mathcal{V}(I_2) = \Delta_\sigma.$$

In particular, Theorem 3.1 applies and gives a description of the point modules—compare this with the description of the point modules obtained for this algebra in Example 7.1.

Now we make some remarks concerning the hypotheses in Theorem 3.1.

Let $A = T(V)/(I_2)$ be a quadratic algebra, and fix bases $x_1, \dots, x_n$ for V and $f_1, \dots, f_r$ for I_2. For brevity, we introduce the notation

$$\underline{x} = (x_1, \dots, x_n)^\mathsf{T} \qquad \text{and} \qquad \underline{f} = (f_1, \dots, f_r)^\mathsf{T}$$

where $(\ )^\mathsf{T}$ denotes the transpose. Then there is an $r \times n$ matrix $L \in M_{r,n}(V)$, having entries in V, and an $n \times r$ matrix $N \in M_{n,r}(V)$ having entries in V, such that

$$\underline{f} = L\underline{x} \tag{3-1}$$

and

$$\underline{f}^\mathsf{T} = \underline{x}^\mathsf{T} N. \tag{3-2}$$

These are equations involving matrices over the free algebra $T(V)$.

It is not necessarily true that N equals L^T—the familiar equality $(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}$ depends on the fact that the entries in A commute with those in B. One can already see this for the commutative polynomial ring $k[x, y, z]$; equation (3-1) is

$$\begin{pmatrix} xy - yx \\ yz - zy \\ zx - xz \end{pmatrix} = \begin{pmatrix} -y & x & 0 \\ 0 & -z & y \\ z & 0 & -x \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and equation (3-2) is

$$\begin{pmatrix} xy - yx & yz - zy & zx - xz \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} y & 0 & -z \\ -x & z & 0 \\ 0 & -y & x \end{pmatrix}.$$

Lemma 3.4. *With the notation above,*

$$\operatorname{pr}_1(\Gamma) = \{p \in \mathbb{P} \mid \operatorname{rank} L(p) < n\},$$
$$\operatorname{pr}_2(\Gamma) = \{p \in \mathbb{P} \mid \operatorname{rank} N(p) < n\}.$$

In particular, pr_1 *is injective if and only if* $\operatorname{rank} L(p) = n - 1$ *for all* $p \in \operatorname{pr}_1(\Gamma)$*, and* pr_2 *is injective if and only if* $\operatorname{rank} N(p) = n - 1$ *for all* $p \in \operatorname{pr}_2(\Gamma)$*.*

Proof. Let $p \in \mathbb{P}$. Then $p \in \operatorname{pr}_1(\Gamma)$ if and only if there exists $q \in \mathbb{P}$ such that $(p, q) \in \Gamma$; that is, if and only if $L(p)\underline{x}(q) = 0$. Hence $p \in \operatorname{pr}_1(\Gamma)$ if and only if the matrix equation $L(p)\underline{x} = 0$ has a non-trivial solution in k^n. The description of $\operatorname{pr}_1(\Gamma)$ follows. The $\operatorname{pr}_2(\Gamma)$ case is similar.

Second, pr_1 fails to be injective if and only if, for some $p \in \operatorname{pr}_1(\Gamma)$ there exist two distinct points $q, q' \in \mathbb{P}$ such that $L(p)\underline{x}(q) = L(p)\underline{x}(q') = 0$; that is, if and only if $L(p)$, viewed as a linear map $k^n \to k^r$, has a kernel of dimension ≥ 2. The rank condition for injectivity follows. $\square$

In general it is not easy to check whether the hypotheses of Theorem 3.1 and/or Lemma 3.4 are satisfied—this is especially true if the algebra is given by generators and relations. However, there are some low dimensional examples, algebras on three generators having three quadratic relations, which can be treated by elementary means. We now look at some of these examples; one salient feature is that the point modules are parametrized by a curve of degree 3 in $\mathbb{P}^2$.

EXAMPLE 3.5. Fix $0 \neq q, \alpha \in k$ and suppose that $\alpha q^2 \neq 1$. Let $A = k[x, y, z]$ have defining relations

$$yx = qxy, \quad zx = q^{-1}xz, \quad zy = \alpha yz + x^2.$$

We will show that the point modules for A are parametrized by a cubic curve $E = C \cup \ell$, in $\mathbb{P}^2 = \mathbb{P}(A_1^*)$, consisting of a non-degenerate conic C and a line ℓ which meets C at two distinct points. The geometric configuration $C \cup \ell$ is related to the algebraic properties of A in several ways.

First, there are normal elements in A corresponding to ℓ and C, namely $x \in A_1$ and $\Omega \in A_2$ which is defined by (3-3) below. We have $\ell = \mathcal{V}(x)$ and, if $p \in E$, then $p \in \ell$ if and only if $x.M(p) = 0$. Similarly, if $p \in E$, then $p \in C$ if and only if $\Omega.M(p) = 0$. Keeping in mind our philosophy, we should interpret the fact that x and Ω annihilate the point modules corresponding to the points of ℓ and C as meaning that x and Ω 'are the defining equations of this line and conic in the quantum $\mathbb{P}^2$ determined by A'.

Second, the quotient rings $A/(x)$ and $A/(\Omega)$ are, in some sense, 'the homogeneous coordinate rings of the non-commutative line and conic in this quantum $\mathbb{P}^2$'. In fact, $A/(x) = k[y, z]$ is isomorphic to a coordinate ring of a quantum $\mathbb{P}^1$, and $A/(\Omega)$ is isomorphic to a 2-Veronese subalgebra of a homogenous coordinate ring of a quantum $\mathbb{P}^1$; this gives a Veronese embedding of a quantum $\mathbb{P}^1$ in a quantum $\mathbb{P}^2$.

Adopting the notation set up prior to Lemma 3.4, we have

$$L = \begin{pmatrix} y & -qx & 0 \\ z & 0 & -q^{-1}x \\ -x & z & -\alpha y \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -qy & -q^{-1}z & -x \\ x & 0 & -\alpha z \\ 0 & x & y \end{pmatrix}.$$

By Lemma 3.4, $\mathrm{pr}_1(\Gamma) = \mathcal{V}(\det L)$, and a calculation gives

$$\begin{aligned} \mathrm{pr}_1(\Gamma) &= \mathcal{V}(x) \cup \mathcal{V}(x^2 + \beta yz) \\ &= \ell \cup C, \end{aligned}$$

where $\beta = \alpha q - q^{-1}$. A similar calculation shows that this is also the defining equation of $\mathrm{pr}_2(\Gamma)$, so we set $E = \mathrm{pr}_1(\Gamma) = \mathrm{pr}_2(\Gamma)$. It is routine to check that $\operatorname{rank} L(p) = \operatorname{rank} N(p) = 2$ for all $p \in E$, whence all the hypotheses of Theorem 3.1 are satisfied. Hence E parametrizes the point modules and there is an automorphism σ of E such that Γ is the graph of σ^{-1}. For each $p \in E$ we will write $M(p) = ke_0 + ke_1 + \cdots$ for the corresponding point module.

To compute σ explicitly we must consider each component of E separately. Recall that $p^{\sigma^{-1}}$ is, by definition, the unique zero in $\mathbb{P}^2$ of $L(p)$. Solving the equation $L(p)\underline{x} = 0$, we obtain

$$\sigma^{-1}|_\ell : (0, y, z) \mapsto (0, \alpha y, z)$$

and

$$\sigma^{-1}|_C : (x,y,z) \mapsto (qx, y, q^2 z).$$

Both $\sigma^{-1}|_\ell$ and $\sigma^{-1}|_C$ extend to automorphisms of the ambient $\mathbb{P}^2$, and have the following representatives in $\mathrm{GL}(A_1^*)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}.$$

The element Ω referred to above may be computed in a simple way from the defining equation of C; that is, from the fact that

$$(x^2 + \beta yz)(p) = 0$$

for all $p \in C$. Let $p \in C$; using the explicit description of σ above, and the fact that $Ae_1 \cong M(p^\sigma)[-1]$, we have

$$x^2.e_0 = x(p^\sigma)x(p)e_2 = (q^{-1}x^2)(p)e_2$$

and

$$yz.e_0 = y(p^\sigma)z(p)e_2 = (yz)(p)e_2.$$

Hence the element

$$\Omega := qx^2 + \beta yz \tag{3-3}$$

annihilates $M(p)$ for all $p \in C$. It is possible to show directly, using the defining equations for A, that Ω is a normalizing element. However, a less error-prone, and more illuminating, method is to show that $A/A\Omega$, $A/\Omega A$ and $A/(\Omega)$ all have the same Hilbert series. To see this, first note that, since A is a domain, $A/A\Omega$ and $A/\Omega A$ both have Hilbert series $(1-t)^{-3}(1-t^2)$. Second, note that $A/(\Omega)$ has defining relations

$$yx = qxy, \qquad zx = q^{-1}xz, \qquad zy = q^{-2}yz,$$
$$qx^2 + \beta yz = 0.$$

The first three of these relations define an iterated Ore extension (cf. Example 6.1.4): let $R = k[u_0, u_1, u_2]$ with defining relations

$$u_1u_0 = qu_0u_1, \quad u_2u_1 = q^{-1}u_1u_2, \quad u_2u_1 = q^{-2}u_1u_2.$$

Since $\lambda_0 u_0^2 + \lambda_2 u_1 u_2$ is a normal element in R for all $(\lambda_0, \lambda_1) \in \mathbb{P}^1$, the quotient $R/(\lambda_0 u_0^2 + \lambda_1 u_1 u_2)$ has Hilbert series $(1-t)^{-3}(1-t^2)$. But this quotient of R, with $(\lambda_0, \lambda_1) = (q, \beta)$, is isomorphic to $A/(\Omega)$; hence the three quotients of A have the same Hilbert series, whence Ω is normalizing in A.

Now we show that $A/(\Omega)$ is isomorphic to the 2-Veronese subalgebra of the homogeneous coordinate ring of a quantum $\mathbb{P}^1$. Let $p \in k$ be such that $p^2 = q$, and define $B := k[u,v]$ with defining relation $vu = puv$. Define $\psi : A \to B$ by

$$\psi(x) = uv, \quad \psi(y) = -(p\beta)^{-1}v^2, \quad \psi(z) = u^2.$$

It is straightforward to check that this is a well-defined algebra homomorphism, and that $\psi(\Omega) = 0$. Furthermore, $\operatorname{Im}\psi = B^{(2)}$, the 2-Veronese subalgebra of B. Since the Hilbert series of $B^{(2)}$ is the same as that of $A/(\Omega)$, we conclude that $A/(\Omega) \cong B^{(2)}$ (cf. Example 6.2.7).

If N is a point module for B then $N^{(2)}$ is a point module for $B^{(2)}$ and hence a point module for A which is killed by Ω. Thus the functor $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$, implemented by the composition $A \to A/(\Omega) \to B^{(2)}$ and the functor $\mathrm{Proj}(B) \to$

$\operatorname{Proj}(B^{(2)})$, is a non-commutative analogue of the 2-uple embedding of $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$.

The next example computes the point modules for a family of algebras of GK-dimension $n+1$; the case $n=0$ is the polynomial ring in 1 variable, and the case $n=1$ is the coordinate ring of the quantum affine plane. It would be impractical to determine the point modules by the method of Lemma 3.4 since the matrices in question would be too large for hand computations.

EXAMPLE 3.6. Fix $0 \neq q \in k$ and assume that $q \neq \pm 1$. Let $A = k[x_0, \dots, x_n]$ be the coordinate ring of quantum affine $(n+1)$-space (Definition 6.1.5). It has defining relations

$$x_j x_i = q x_i x_j \qquad (0 \le i < j \le n),$$

Let is an iterated Ore extension of k, and therefore a noetherian domain with basis $\{x_0^{i_0} \cdots x_n^{i_n} \mid i_0, \dots, i_n \ge 0\}$.

As usual we write $\mathbb{P} = \mathbb{P}(A_1^*) \cong \mathbb{P}^n$.

First, note that A has many quotients which are homogeneous coordinate rings of quantum planes. For each pair (i,j) with $0 \le i, j \le n$, define

$$I(i,j) := Ax_0 + \cdots + \widehat{Ax_i} + \cdots + \widehat{Ax_j} + \cdots + Ax_n,$$

where the hat $\hat{}$ means 'omit this term'. Each $I(i,j)$ is a two-sided ideal and $A/I(i,j)$ is isomorphic to the homogeneous coordinate ring of a quantum $\mathbb{P}^1$, say $k[x,y]$, with defining relation $yx = qxy$. The point modules for $A/I(i,j)$ are naturally parametrized by the points of the line

$$\ell_{ij} := \mathcal{V}(x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n).$$

We will now show that $\Gamma = \mathcal{V}(R) \subset \mathbb{P} \times \mathbb{P}$ is the graph of an automorphism, σ^{-1}, of the variety

$$E := \bigcup_{0 \le i < j \le n} \ell_{ij}.$$

To this end, let $p_1 = (\alpha_0, \dots, \alpha_n)$ and $p_0 = (\beta_0, \dots, \beta_n)$ be points of $\mathbb{P}$, and suppose that $(p_1, p_0) \in \Gamma$. Then

$$\alpha_j \beta_i = q \alpha_i \beta_j$$

for all $0 \le i < j \le n$. Fix j such that $\alpha_j \neq 0$ and define $\mu = \beta_j \alpha_j^{-1}$. Then

$$\beta_i = \begin{cases} q\mu\alpha_i & \text{if } i < j, \\ \mu\alpha_i & \text{if } i = j, \\ q^{-1}\mu\alpha_i & \text{if } i > j, \end{cases} \tag{3-4}$$

so

$$p_0 = (q^2\alpha_0, \dots, q^2\alpha_{j-1}, q\alpha_j, \alpha_{j+1}, \dots, \alpha_n).$$

But this holds for all j such that $\alpha_j \neq 0$, so at most two α_j are non-zero. Thus

$$\begin{aligned} p_1 &= (0, \dots, 0, \alpha_i, 0, \dots, 0, \alpha_j, 0, \dots, 0) \qquad \text{and} \\ p_0 &= (0, \dots, 0, q\alpha_i, 0, \dots, 0, \alpha_j, 0, \dots, 0) \end{aligned}$$

for some $i < j$. We now use this to define $\sigma : \ell_{ij} \to \ell_{ij}$ by $p_0^\sigma = p_1$. Hence $\Gamma \subset \{(p^\sigma, p) \mid p \in E\}$. Conversely, if $p \in \ell_{ij}$, then the defining relations of A obviously vanish on (p^σ, p), whence $\Gamma = \{(p^\sigma, p) \mid p \in E\}$.

When $n = 2$ the point module variety is the triangle $\mathcal{V}(x_0x_1x_2)$, when $n = 3$ it is a tetrahedron, etc.

EXERCISES

3.1 Let α, β, γ be non-zero elements of k. Classify the point modules over the algebra $A = k[x, y, z]$ with defining relations

$$zy = \alpha yz, \quad zx = \beta xz, \quad yx = \gamma xy$$

and grading determined by $\deg x = \deg y = \deg z = 1$.

3.2 (Research Problem) Let $(\alpha_{ij}) \in M_{n+1}(k)$ with $\alpha_{ii} = 1$ and $\alpha_{ij} = \alpha_{ji}^{-1}$, and define $A = k[x_0, \dots, x_n]$ with defining relations

$$x_j x_i = \alpha_{ij} x_i x_j \qquad (0 \le i, j \le n).$$

Show that $\Gamma = \{(p^\sigma, p) \mid p \in E\}$ where $E \subset \mathbb{P}$ is a union of linear subspaces of $\mathbb{P}$, and σ is an automorphism of E. Determine E.

I suspect a key issue is the rank of the matrix $B := (\log(\alpha_{ij}))$; the larger the rank, the fewer point modules. At one extreme is the case $A = S(V)^\sigma$, a twist of a polynomial ring with respect to a diagonal automorphism σ, in which case the rank is 2 and $E = \mathbb{P}$. Another extreme is the case of the quantum affine space (Example 3.6), where the rank is $n+1$ and E is a union of lines. In both these cases $\dim E + \operatorname{rank} B = n+2$.

I'm not interested so much in the actual answer, but rather the *form* of the answer. What is an elegant way to describe E in terms of the initial data (α_{ij})?

4. Finite dimensional simple modules and points

Let A be an $\mathbb{N}$-graded k-algebra. This section examines the close relation between points in $\operatorname{Proj}(A)$ and finite dimensional simple A-modules (such simple modules are usually *not* graded). The relation is good enough that it is feasible, and sometimes sensible, to find the finite dimensional simple A-modules by first finding the points in $\operatorname{Proj}(A)$.

Definition 4.1. Let A be an $\mathbb{N}$-graded k-algebra. A module $M \in \mathsf{GrMod}(A)$ is trivial if $A_{\ge 1} M = 0$.

Suppose that A is $\mathbb{N}$-graded. Then the subcategory of $\mathsf{GrMod}(A)$ consisting of the trivial modules is equivalent to the category of A_0-modules; in particular, if A is connected, then the trivial modules are just the direct sums of shifts of the trivial module. All the composition factors of $A/A_{\ge n}$ are trivial. A finite dimensional graded module is annihilated by $A_{\ge n}$ for some n, so all its composition factors are trivial.

Let A be a locally finite, $\mathbb{N}$-graded k-algebra, and M a non-trivial finite dimensional simple A-module. We will prove that there is a bijection between equivalence classes (Definition 4.11) of non-trivial, finite dimensional, simple A-modules and finite [1]-orbits of points in $\operatorname{Proj}(A)$; the orbit corresponding to M will be denoted by O_M. The bijection is defined/established by proving that if N is a finitely generated, 1-critical graded A-module, then M is a quotient of N if and only if πN belongs to O_M.

This bijection is a natural extension of the commutative result. Suppose that k is algebraically closed, let A be the homogeneous coordinate ring of a projective variety $X \subset \mathbb{P}^n$, and let $Y \subset \mathbb{A}^{n+1}$ be the affine cone over X. The non-trivial finite dimensional simple A-modules correspond to the points of $Y \backslash \{0\}$, and two such simples are equivalent if and only if the corresponding points of Y span the same line; i.e., if and only if they determine the same point of X. But the points of X are in bijection with the points of $\operatorname{Proj}(A)$, each of which is fixed by the shift [1]. Thus the equivalence class of simple modules corresponding to a point $p \in \operatorname{Proj}(A)$

consists of the simple A-modules corresponding to the points of the line in Y above p.

PROPOSITION 4.2. *Let A be a graded k-algebra. There is a faithful, exact functor*

$$\tilde{\ } : \{\textit{finite dimensional } A\textit{-modules}\} \to \mathsf{GrMod}(A)$$

defined by

- $M \mapsto \tilde{M} := M \otimes_k k[T, T^{-1}]$ *where T is an indeterminate, and*
    - *the grading is defined by* $\tilde{M}_n = M \otimes_k kT^n$,
    - *if $a \in A_i$, then* $a.(m \otimes T^j) = (a.m) \otimes T^{i+j}$;
- *if $f : M \to M'$, then $\tilde{f} : \tilde{M} \to \tilde{M}'$ is* $\tilde{f} := f \otimes 1\!\!1_{k[T,T^{-1}]}$.

If A is $\mathbb{N}$-graded and M a finite dimensional A-module, then

1. *$\tilde{M}_{\geq 0}$ is a finitely generated A-module if all the composition factors of M are non-trivial;*
2. *if M is a non-trivial simple module, then $\tilde{M}$ is torsion-free;*
3. *$\pi\tilde{M} = 0$ if and only if all composition factors of M are trivial.*

PROOF. The existence, exactness, and faithfulness of the functor is routine.

(1) Since M is not trivial $A_1M + A_2M + \cdots$ is a non-zero submodule of M, so equals M. Thus $A_1M + \cdots + A_nM = M$ for some n. Hence, for any $r \geq 0$,

$$A_rM = A_r(A_1M + \cdots + A_nM) \subset A_{r+1}M + \cdots + A_{r+n}M.$$

By induction, it follows that $A_{r+1}M + \cdots + A_{r+n}M = M$ for all $r \geq 0$. Now let $X = \tilde{M}_0 + \cdots + \tilde{M}_{n-1}$. The degree $n + r$ component of AX is

$$\begin{aligned} A_{n+r}\tilde{M}_0 + \cdots + A_{r+1}\tilde{M}_{n-1} &= (A_{n+r}M + \cdots + A_{r+1}M) \otimes T^{n+r} \\ &= M \otimes T^{n+r}. \end{aligned}$$

Thus $\tilde{M}$ is generated by the finite dimensional vector space X, as required.

(2) If $\tilde{M}$ is not torsion-free, say $A_{\geq n}(m \otimes T^j) = 0$, then $A_{\geq n}m = 0$, so M is a quotient of $A/A_{\geq n}$, whence M is trivial. The result follows.

(3) If M is a trivial module, then $\tilde{M} = \cdots \oplus M[-1] \oplus M \oplus M[1] \oplus \cdots$ is a torsion module, so $\pi\tilde{M} = 0$. Conversely, if N is a non-trivial composition factor of M, then $\pi\tilde{N} \neq 0$ by (2), whence $\pi\tilde{M} \neq 0$ because $\pi \circ (\tilde{\ })$ is an exact functor. □

If A is $\mathbb{N}$-graded and M is a non-trivial 1-dimensional A-module, then $\tilde{M}_{\geq 0}$ is a point module.

Definition 4.3. Let M be a finite dimensional A-module. Define

$$\varepsilon : \tilde{M} \to M$$

by $\varepsilon(m \otimes T^j) = m$. Then ε is surjective and its kernel is $(T - 1)\tilde{M}$.

If A is $\mathbb{N}$-graded, then M is also a quotient of $\tilde{M}_{\geq 0}$.

The next result says that $\tilde{M}$ must play a key role in any relation between a finite dimensional module and graded modules.

LEMMA 4.4. *Let M be a finite dimensional A-module and N a graded A-module. If $f : N \to M$ is an A-module homomorphism, then there is a unique graded module homomorphism $\hat{f} : N \to \tilde{M}$ such that $f = \varepsilon \circ \hat{f}$.*

Proof. Define $\hat{f}(y) = f(y) \otimes T^j$ for $y \in N_j$. Then $\hat{f} \in \mathrm{Hom}_{\mathrm{Gr}}(N, \tilde{M})$ and $f = \varepsilon\hat{f}$. If $g \in \mathrm{Hom}_{\mathrm{Gr}}(N, \tilde{M})$ satisfies $f = \varepsilon g$, then $\mathrm{Im}(\hat{f} - g) \subset \ker \varepsilon$; however, $(T-1)\tilde{M}$ does not contain a non-zero homogeneous element, so $\mathrm{Im}(\hat{f} - g) = 0$, whence $\hat{f} = g$, thus proving the uniqueness. □

The grading on $\tilde{M}$ is the tensor product grading, where M is concentrated in degree 0, and $k[T, T^{-1}]$ is graded with $\deg T = 1$. Furthermore, $\tilde{M}$ is an A-$k[T, T^{-1}]$-bimodule, so the action of A on $\tilde{M}$ is equivalent to an algebra homomorphism $A \to \mathrm{End}_{k[T,T^{-1}]}(\tilde{M})$. In fact, this is a homomorphism of graded algebras, where this endomorphism ring is given the natural grading. There is also an isomorphism of graded algebras

$$\mathrm{End}_{k[T,T^{-1}]}(\tilde{M}) \cong \mathrm{End}_k(M) \otimes_k k[T, T^{-1}],$$

where $\mathrm{End}_k(M)$ is concentrated in degree zero, so we obtain a graded algebra homomorphism

$$A \to \mathrm{End}_k(M) \otimes_k k[T, T^{-1}].$$

If A is $\mathbb{N}$-graded, the image of this map lies in $\mathrm{End}_k(M) \otimes k[T]$.

Lemma 4.5. *Let A be a graded k-algebra, and M a non-trivial finite dimensional simple A-module. Let P be the largest graded ideal in A such that $P.M = 0$. Then*

1. *P is a prime ideal;*
2. *$P = \mathrm{Ann}\,\tilde{M}$;*
3. *$\mathrm{GKdim}(A/P) = 1$;*
4. *$\dim_k(A/P)_n \leq (\dim_k M)^2$ for all n.*

Proof. First, there *is* a largest such graded ideal, namely the sum of all the graded ideals which annihilate M.

(1) If P is not prime, then there exist strictly larger graded ideals, I and J say, such that $IJ \subset P$; that is, $IJM = 0$. But $JM \neq 0$, so $JM = M$, whence $IM = 0$, so $I \subset P$. This is a contradiction.

(2) If $x \in P_n$ and $m \in M$, then $x.(m \otimes T^i) = xm \otimes T^{i+n} = 0$, whence $x\tilde{M} = 0$. Conversely, $\mathrm{Ann}\,\tilde{M}$ is a graded ideal and, since M is a quotient of $\tilde{M}$, it annihilates M, so is contained in P.

(3) (4) Certainly $\dim_k(A/P) = \infty$, else all its composition factors, one of which is M, would be trivial. By the remarks prior to the Proposition, A/P is isomorphic to a graded subalgebra of

$$\mathrm{End}_k(M) \otimes_k k[T, T^{-1}] \cong M_d(k[T, T^{-1}]),$$

where $d = \dim_k M$, so $\dim_k(A/P)_n \leq d^2$ and $\mathrm{GKdim}\, A/P \leq 1$. □

By Lemma 4.5, every non-trivial, finite dimensional, simple A-module is a quotient of a graded, prime quotient ring A/P of GK-dimension 1. Consequently, our study of finite dimensional simple modules entails a preliminary study of graded algebras of GK-dimension 1. This has already been carried out in Chapter 2; the main result which we require is Theorem 2.2.5, which states that a finitely generated k-algebra, R say, which is semi-prime and of GK-dimension 1, satisfies a polynomial identity, is noetherian and is a finite module over its center (which is also noetherian).

LEMMA 4.6. *Suppose that A is a left noetherian, prime, locally finite, $\mathbb{N}$-graded k-algebra. Let M and N be finitely generated, critical, graded A-modules such that*

$$\operatorname{GKdim} M = \operatorname{GKdim} N = \operatorname{GKdim} A > 0.$$

Then, given $r_0 \in \mathbb{Z}$, there exists $r \geq r_0$ such that $M[-r]$ embeds in N.

PROOF. Since $\operatorname{GKdim} M = \operatorname{GKdim} A$, M is torsion-free (in the classical sense). Since M is critical it is uniform. Hence, by 2.5.5, M and N contain isomorphic copies of one another. However, these embeddings of M and N in one another may not respect the gradings. Nevertheless, $\operatorname{Hom}_A(M, N)$ is the sum of its homogeneous components (11.2.2), so there exists a non-zero graded map $f : M \to N$. Since M and N are critical of the same GK-dimension, f is injective. If $\deg f = r$ then there is a degree zero injective map $M[-r] \to N$.

To see that r may be chosen arbitrarily large, fix n such that $M_n \neq 0$, and let $\ell \in \mathbb{Z}$ be arbitrary. The hypotheses on N ensure that $\dim(N/N_{\geq \ell}) < \infty$, so $N_{\geq \ell}$ satisfies the same hypotheses as N. By the first part of the proof, there exists r such that $M[-r]$ embeds in $N_{\geq \ell}$. Since $M[-r]_{n+r} \neq 0$, it follows that $(N_{\geq \ell})_{n+r} \neq 0$. Thus $n + r \geq \ell$; since n was fixed, and ℓ was arbitrary, this shows that we may choose r arbitrarily large. □

Definition 4.7. Graded modules, M and N say, are shift-equivalent if, for some $n \in \mathbb{Z}$,

$$\pi M[n] \cong \pi N;$$

that is, if πM and πN generate the same [1]-orbit in $\operatorname{Proj}(A)$.

PROPOSITION 4.8. *Suppose that A is a finitely generated, locally finite, $\mathbb{N}$-graded k-algebra. Let M and N be finitely generated, critical A-modules, and suppose that*

$$\operatorname{GKdim} A/\operatorname{Ann} M = \operatorname{GKdim} A/\operatorname{Ann} N = 1.$$

Then $\operatorname{Ann} M = \operatorname{Ann} N$ if and only if M and N are shift-equivalent.

PROOF. ($\Rightarrow$) The common annihilator, say P, is prime since M is critical (11.1.3). By Theorem 2.2.5, A/P is noetherian. Without loss of generality, we may assume that $P = 0$. Thus A is prime, noetherian, M and N are noetherian modules, and $\operatorname{GKdim} M = \operatorname{GKdim} N = \operatorname{GKdim} A = 1$. By Lemma 4.6, $M[n]$ embeds in N for some $n \in \mathbb{Z}$. Since N is 1-critical, the cokernel of this embedding is finite dimensional, so $\pi M[n] \cong N$.

($\Leftarrow$) The hypotheses ensure that $A/\operatorname{Ann} M$ is noetherian, whence M is a noetherian module; so is N. If $\pi M[n] \cong \pi N$, then by Theorem 19.1.6, $M[n]$ and N have a common non-zero submodule. Since they are critical, their annihilators are the same as the annihilator of that submodule (Theorem 11.1.3). □

THEOREM 4.9. *Suppose that A is a finitely generated, prime, locally finite $\mathbb{N}$-graded k-algebra. If $\operatorname{GKdim} A = 1$, then there exists a finitely generated, 1-critical, graded A-module C such that*

1. *C is generated in degree zero;*
2. *there exists $p > 0$ such that the points in $\operatorname{Proj}(A)$ are*

$$\{\pi C[n] \mid 0 \leq n < p\};$$

3. *the Hilbert series of C is eventually periodic.*

Proof. By Theorem 2.2.5, A is noetherian. First A possesses a 1-critical, cyclic, graded module which is generated in degree zero: if L is maximal subject to being a graded left ideal for which $\dim A/L = \infty$, then A/L satisfies these conditions.

If D is any 1-critical A-module, then $\mathrm{Ann}(D) = 0$, since the noetherian hypothesis ensures that any proper quotient ring of A (for example, $A/\operatorname{Ann} D$) is finite dimensional. Now let C be a finitely generated, 1-critical, graded module, which is generated in degree 0. By Proposition 4.8, $\pi D \cong C[r]$ for some r, so A has a unique 1-critical module up to shift-equivalence.

By Lemma 4.6, there is an injective degree 0 map $\varphi : C \to C_{\geq 1}[p]$ for some $p \geq 1$, whence $\pi C \cong \pi C[p]$. It follows that $\pi C[i] \cong \pi C[i+p]$ for all $i \in \mathbb{Z}$. Let $p \in \mathbb{N}$ be minimal such that $\pi C \cong \pi C[p]$. By Theorem 19.1.6, $\dim C_n = \dim C_{n+p}$ for all $n \gg 0$, so the Hilbert series of C is eventually periodic. $\square$

Before the next proof, we note that if A is as in Theorem 4.9, and M is a finitely generated, 1-critical, graded A-module then $\pi M \cong \pi C[n]$ for some n, so M and $C[n]$ have isomorphic tails (Theorem 19.1.6), whence C and M have the same multiplicity (Chapter 11, Section 2), namely $e(M) = e(C)$.

Corollary 4.10. *Let A be a finitely generated, $\mathbb{N}$-graded k-algebra. Let M be a non-trivial, finite dimensional, simple A-module. Then M is a quotient of a finitely generated, 1-critical, graded A-module.*

Proof. Without loss of generality, we may, and do, replace A by $A/\operatorname{Ann}(\tilde{M})$. Thus A is prime, noetherian, locally finite, and $\operatorname{GKdim} A = 1$.

We construct a chain of graded left ideals $A = I_0 \supset I_1 \supset \cdots$ inductively by choosing I_{j+1} to be maximal in I_j such that $\dim_k(I_j/I_{j+1}) = \infty$. Thus each I_j/I_{j+1} is 1-critical. If C is chosen as in Theorem 4.9, then $e(C) = e(I_j/I_{j+1})$ for all j. But $e(A)$ is finite, and $e(-)$ is additive on short exact sequences (Lemma 11.2.11(1)), so this chain must be finite. Since A is prime, $\tau A = 0$, so the chain must terminate at zero.

Since M is a quotient of A, it must be a quotient of one of the I_j/I_{j+1}. $\square$

Consider the integer p occurring in Theorem 4.9(2). The points of $\mathrm{Proj}(A)$ are in bijection with the elements of $\mathbb{Z}/\mathbb{Z}p$ and form a single [1]-orbit. If $0 \neq z \in A_d$ is a central element of positive degree, then multiplication by z is an injective map $C \to C[d]$; it follows that $\pi C \cong \pi C[d]$, whence $d \in \mathbb{Z}p$.

Definition 4.11. Let $0 \neq \lambda \in k$. Define $\varphi_\lambda \in \mathrm{Aut}(A)$ by $\varphi_\lambda(x) = \lambda^n x$ for all $x \in A_n$. If M is an A-module, define M^λ the twist of M by λ by

- $M^\lambda = M$ as a k-vector space, and
- $x * m = \varphi_\lambda(x)m$ for $x \in A$ and $m \in M^\lambda$.

Two A-modules, say M and M', are equivalent if $M' \cong M^\lambda$ for some $\lambda \in k$.

This twisting operation induces an automorphism of the category $\mathsf{Mod}(A)$.
If M is a graded module, then $M \cong M^\lambda$.

Suppose that k is algebraically closed. Let A be the homogeneous coordinate ring of a projective variety $X \subset \mathbb{P}^n$, and let $Y \subset \mathbb{A}^{n+1}$ be the cone over X. As remarked in the opening paragraphs of this section, if $p \in Y\backslash\{0\}$, then the simple module corresponding to p is equivalent to each of the simple modules corresponding to the other points on the line kp. Moreover, the point module corresponding to this line (i.e., the point in X), namely $A/I(p)$ where $I(p)$ is the ideal generated by the homogeneous polynomials vanishing at p, has a 1-parameter family of (non-trivial) simple quotients, namely the simple modules corresponding to the points on the line kp. The following result is an appropriate non-commutative analogue of this.

PROPOSITION 4.12. [**?**] *Suppose that k is algebraically closed. Let A be a finitely generated, locally finite, $\mathbb{N}$-graded k-algebra. Let M_1 and M_2 be non-trivial, finite dimensional, simple A-modules. Let X be a graded A-module such that πX is a point in* $\operatorname{Proj}(A)$. *If M_1 and M_2 are quotients of X, they are equivalent.*

PROOF. Let $P = \operatorname{Ann}(\tilde{M}_1)$. By Theorem 2.2.5, $B := A/P$ is a finite module over its center. Hence there exists a central element $0 \neq z \in B_d$ of positive degree such that $\operatorname{Fract}_{\mathrm{Gr}}(B) = B[z^{-1}]$. Hence by Theorem 2.2.6, $B[z^{-1}] \cong M_n(k[x, x^{-1}])(\underline{d})$, for some $\underline{d} = (d_1, \dots, d_n)$.

Write $Q = M_n(k[x, x^{-1}])(\underline{d})$. If V is a simple Q-module, then $\operatorname{End}_Q(V) \cong k$, since k is algebraically closed, whence a central element of Q must act on V as multiplication by some scalar. Thus, a non-trivial simple Q-module is annihilated by $x - \nu$, for some $0 \neq \nu \in k$, so is a module over $Q/(x - \nu) \cong M_n(k)$. The left Q-modules $Q/(x - \nu)$, $\nu \neq 0$, are all equivalent to one another as Q-modules, and hence as B-modules. It is easy to see that this ring has a single equivalence class of simple modules. It is also a relatively simple matter to see that this implies that B itself has only one equivalence class of simple modules. □

Definition 4.13. Let A be a finitely generated, $\mathbb{N}$-graded k-algebra and M a non-trivial, finite dimensional, simple A-module. Define

$$\mathrm{O}_M := \{\text{points in } \operatorname{Proj}(A) \text{ annihilated by } \operatorname{Ann} \tilde{M}\}.$$

By Theorem 4.9, O_M is a single [1]-orbit of points in $\operatorname{Proj}(A)$. Recall that $\operatorname{Proj}(A/\operatorname{Ann} \tilde{M})$ may be identified with a full subcategory of $\operatorname{Proj}(A)$. Thus O_M is, roughly speaking, that subcategory.

LEMMA 4.14. *Let A be a finitely generated, locally finite, $\mathbb{N}$-graded algebra over an algebraically closed field k. Let M be a non-trivial, finite dimensional, simple A-module and let X be a finitely generated, graded A-module such that πX is a point. Then M is a quotient of X if and only if $\pi X \in \mathrm{O}_M$.*

PROOF. ($\Rightarrow$) If M is a quotient of X, then, by Lemma 4.4, X embeds in $\tilde{M}$, whence X is a module over $A/\operatorname{Ann} \tilde{M}$. Thus $\pi X \in \mathrm{O}_M$, by definition.

($\Leftarrow$) Let C be a finitely generated, 1-critical, graded module mapping onto M. Then $\pi C \in \mathrm{O}_M$ also, so X and C are shift-equivalent. Thus, by Proposition 4.8, $X[-r]$ embeds in C, for some $r \in \mathbb{Z}$. Since $C/X[-r]$ is a torsion module, it follows that the restriction to $X[-r]$ of the surjective map $C \to M$ remains surjective. □

THEOREM 4.15. *Let A be a finitely generated, locally finite, $\mathbb{N}$-graded algebra over an algebraically closed field k. Then the correspondence*

$$M \leftrightarrow \mathrm{O}_M$$

is a bijection between equivalence classes of non-trivial, finite dimensional, simple A-modules and finite [1]*-orbits of points in* $\operatorname{Proj}(A)$.

PROOF. We must show that every finite [1]-orbit occurs as some O_M, and that $\mathrm{O}_M = \mathrm{O}_{M'}$ if and only if M and M' are equivalent.

Suppose that $\pi X[n] \cong \pi X$; we will show that X has a finite dimensional simple quotient, thus proving that every finite [1]-orbit occurs as some O_M. By Theorem 19.1.6, $X_{\geq n+s} \cong X_{\geq s}$ for some s. Let $\varphi : X_{\geq s} \to X_{\geq n+s}$ be a (degree n) isomorphism, and consider $1 - \varphi \in \operatorname{End}_A(X_{\geq s})$.

There are no non-zero homogeneous elements in $\operatorname{Im}(1-\varphi)$ because, if $0 \neq x \in X_{\geq s}$ has highest and lowest degree components x_h and x_l, then the highest and lowest degree components of $(1-\varphi)(x)$ are $-\varphi(x_h)$ and x_l.

If $x \in X_{\geq n+s}$, there exists $y \in X_{\geq s}$ such that $\deg y = \deg x - n$ and $x = y - (1-\varphi)(y)$; hence, by induction on degree,

$$X_{\geq s} = X_s + X_{s+1} + \cdots + X_{s+n-1} + \operatorname{Im}(1-\varphi).$$

Thus $X_{\geq s}/\operatorname{Im}(1-\varphi)$ is finite dimensional. (We have in effect used a standard filtered and associated graded argument.)

It follows from the last two paragraphs that at least one of the composition factors of $X_{\geq s}/\operatorname{Im}(1-\varphi)$ is non-trivial: if all were trivial then $\operatorname{Im}(1-\varphi)$ would contain the homogeneous elements in $A_{\geq m}X_{\geq s}$ for some m. It follows that there is a graded submodule, X' say, of X such that X' has a finite dimensional, non-trivial, simple quotient, say M. Since $\pi X \cong \pi X'$, $X[-r]$ embeds in X' for some r, whence M is a quotient of $X[-r]$, since all the composition factors of $X'/X[-r]$ are trivial. Thus M is a quotient of X, as required.

Next, we show that equivalent simple modules give the same [1]-orbit. Let M be a non-trivial finite dimensional simple, and suppose that M is a quotient of a finitely generated, 1-critical, graded module, say X. The existence of a surjection $X \to M$ implies the existence of a surjection $X^\lambda \to M^\lambda$ for all $0 \neq \lambda \in k$; but $X^\lambda \cong X$, since X is graded, so the modules equivalent to M are also quotients of X. Thus O_M depends only on the equivalence class of M.

Finally, suppose that $\mathrm{O}_M = \mathrm{O}_{M'}$. Let X be a finitely generated, 1-critical, graded modules mapping onto M. By Lemma 4.14, $\pi X \in \mathrm{O}_M$ whence, by the same Lemma, M' is also a quotient of X. Hence by Proposition 4.12, M and M' are equivalent. □

Corollary 4.16. *Let A be a finitely generated, locally finite, $\mathbb{N}$-graded algebra over an algebraically closed field k. Let X be a finitely generated, 1-critical A-module. The following are equivalent:*

1. *X has a non-trivial, finite dimensional, simple quotient;*
2. $\operatorname{GKdim} A/\operatorname{Ann} X = 1$*;*
3. *the* [1]*-orbit of πX is finite.*

Proof. (1) ⇒ (2) If M is a finite dimensional, non-trivial, simple quotient of X, then X embeds in $\tilde{M}$, so $\operatorname{Ann}(X) \supset \operatorname{Ann}(\tilde{M})$. Thus $\operatorname{GKdim} A/\operatorname{Ann} X \leq 1$; but $\operatorname{GKdim} A/\operatorname{Ann} X \neq 0$ since $\dim_k X = \infty$.

(2) ⇒ (3) This is part of Theorem 4.9.

(3) ⇒ (1) This is part of Lemma 4.14. □

Example 4.17. Let $A = k[x, y]$ with $\deg x = \deg y = 1$ and defining relations $x^2 = y^2 = 0$. It is easy to see that A has a basis

$$\{1, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \dots\}$$

whence $\operatorname{GKdim} A = 1$. There is another description of A as follows. Consider the commutative polynomial ring $k[t]$ with $\deg t = 2$. Put a grading on the ring $M_2(k)$ of 2×2 matrices by declaring that the matrix unit e_{ij} have degree $i - j$.

Put the tensor product grading on $M_2(k) \otimes k[t]$. Then the k-subalgebra generated by $x := e_{21}$ and $y := te_{12}$ is isomorphic to A. It is easy to see that A is prime and noetherian with center generated by $t = xy + yx$. The only point modules are $C := A/Ax$ and $D := A/Ay$. It is easy to see that $\mathrm{Hom}_A(C, C) \cong k[\theta]$ with $\deg \theta = 2$, and $\mathrm{Hom}_A(C, D) = \psi.k[\theta]$ with $\deg \psi = 1$. It follows that $C[1] \sim D$ and the points in $\mathrm{Proj}(A)$ are C and $C[1]$; notice that $C[2] \cong C$.

It is easy to see that

$$A = \begin{pmatrix} k[t] & \langle t \rangle \\ k[t] & k[t] \end{pmatrix} \subset M_2(k[t]).$$

Example 4.18. Let $\theta \in \mathrm{GL}(V)$ and let $A = S(V)^\theta$, the twist of the symmetric algebra with respect to θ. We will determine the finite dimensional simple A-modules. First the categories $\mathsf{GrMod}(A)$ and $\mathsf{GrMod}(S(V))$ are equivalent, so the points in $\mathrm{Proj}(A)$ are given by the point modules $M(p)$, $p \in \mathbb{P}(V^*)$. Now $\pi M(p)[1] \cong M(p^\theta)$, so the finite [1]-orbits of points in $\mathrm{Proj}(A)$ are given by the finite θ-orbits in $\mathrm{Proj}(A)$. Hence there is a bijection between

$$\{\text{equivalence classes of finite dimensional simple } A\text{-modules}\}$$

and

$$\{\text{finite } \theta\text{-orbits in } \mathbb{P}(V^*)\}.$$

Example 4.19. Reconsider the algebra in Example 1.12, namely

$$A = \begin{pmatrix} k[T^2] & Tk[T^2] & T^2k[T^2] \\ Tk[T^2] & k[T^2] & Tk[T^2] \\ T^2k[T^2] & Tk[T^2] & k[T^2] \end{pmatrix}.$$

A finite dimensional simple A-module is annihilated by a maximal ideal of the center of A, namely the diagonal copy of $k[T^2]$. If $\lambda \neq 0$, then $A/(T^2 - \lambda) \cong M_3(k)$, which gives a single equivalence class of 3-dimensional simple modules. On the other hand, $A/(T^2) \cong k \times k \times k$, which gives rise to three 1-dimensional simple modules, the trivial simple modules.

We finish this Section by using the ideas we have just developed to clarify a small point which arose earlier. Example 2.8 shows that non-isomorphic point modules may give the same point in $\mathrm{Proj}(A)$. However, this phenomenon does not occur over a noetherian PI ring. The following more general result holds.

Proposition 4.20. *Let M be a point module over a noetherian k-algebra A, and suppose that* $\mathrm{GKdim}\, A/\mathrm{Ann}\, M = 1$. *If N is a point module such that $\pi M \cong \pi N$, then $M \cong N$.*

Proof. By Theorem 19.1.6, $M_{\geq n} \cong N_{\geq n}$ for some sufficiently large n. Since N and M are critical, and A is 2-sided noetherian, $\mathrm{Ann}\, N = \mathrm{Ann}(N_{\geq n}) = \mathrm{Ann}(M_{\geq n}) = \mathrm{Ann}(M)$. By Lemma 4.8, suitable shifts of N and $M[1]$ embed in M via degree zero maps. Thus for some $i, j > 0$ both $N[-i]$ and $M[-j]$ embed in M. Therefore, $M[-ij]$ and $N[-ij]$ embed in M. But the image of both these embeddings is $M_{\geq ij}$ whence $M[-ij] \cong N[-ij]$, as required. $\square$

We summarize the point of the results in this Section. The problem of finding all finite dimensional simple A-modules reduces to the following steps:

- find the finite [1]-orbits of points in $\mathrm{Proj}(A)$;
- determine the dimension of the simple quotients.

EXERCISES

4.1 Prove that if A is the homogeneous coordinate ring of a projective variety $X \subset \mathbb{P}^n$, then two non-trivial A-modules are equivalent if and only if the points in $\mathbb{A}^{n+1}$ to which they correspond have the same image in X.

4.2 Let M_1 and M_2 be non-trivial, finite dimensional, simple A-modules. Show that M_1 and M_2 are equivalent if and only if, for some $0 \neq \lambda \in k$, there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_\lambda"] \arrow[d, "\psi_1"'] & A \arrow[d, "\psi_2"] \\ M_{d_1}(k) \arrow[r, "\sim"] & M_{d_2}(k) \end{tikzcd},$$

where $\psi_i : A \to \operatorname{End}_k(M_i) \cong M_{d_i}(k)$ is the map induced by the action of A on M_i.

4.3 If M is a graded A-module, show that $M \cong M^\lambda$ for all $0 \neq \lambda \in k$.

4.4 Grade $A = k[T]$ by setting $\deg T = 1$. Show, for every A-module M, that $M^\lambda \cong M^{-\lambda}$.

4.5 Show that the inclusion $A/\operatorname{Ann}\tilde{M} \to \operatorname{End}_k(M)\otimes k[T, T^{-1}]$ need not induce an isomorphism between the rings of fractions [*Hint:* suppose that A is concentrated in even degree. What if A is non-zero in every degree?]

4.6 Show there is a bijection between the following three sets:

$$\{\text{point modules } M \text{ such that } \pi M \cong \pi M[1]\}$$

and

$$\{\text{graded ideals } J \text{ such that } A/J \cong k[T]\}$$

and

$$\{\text{equivalence classes of non-trivial 1-dimensional modules}\}$$

4.7 Give an example of a non-trivial, finite dimensional, simple module M for which $\tilde{M}$ is not critical.

4.8 Give an example of a point module having no non-trivial, finite dimensional, simple quotient.

4.9 Show that there is a bijection between the set of graded prime ideals P such that $\operatorname{GKdim} A/P = 1$ and the set of finite [1]-orbits of points in $\operatorname{Proj}(A)$.

5. The point module functor

Definition 5.1. Let A be an $\mathbb{N}$-graded k-algebra and R a commutative k-algebra. Let R be concentrated in degree zero, and give $R\otimes_k A$ the tensor product grading. A point module with values in R or an R-valued point module is a graded $R\otimes A$-module, M say, such that

- M is generated in degree zero;
- $M_0 \cong R$ as an R-module;
- each M_i is a locally free R-module of rank 1.

Notice that a k-valued point module is a point module in the sense of Definition 2.1. If M is an R-valued point module then, for each maximal ideal $\mathfrak{m}$ of R such that $R/\mathfrak{m} \cong k$, we obtain a point module $R/\mathfrak{m} \otimes_R M$. Thus an R-valued point module is a family os point modules parametrized by $\operatorname{Spec}(R)$.

PROPOSITION 5.2. *Each $\mathbb{N}$-graded k-algebra A gives a covariant functor*

$$\{\textit{commutative } k\textit{-algebras}\} \to \mathsf{Set}$$

which sends R to the set of R-valued point modules. If $A = T(V)/I$, this is a representable functor, represented by the pro-scheme $\Gamma = \operatorname{proj\,lim} \mathcal{V}(I_n)$.

PROOF. It is clear that this is a functor since, if $f : R \to S$ is a k-algebra map and M is an R-valued point module, then $S \otimes_R M$ is an S-valued point module.

$\boxed{\text{MORE}}$

□

Definition 5.3. Let A be a graded k-algebra, generated in degree 1. A *truncated point module of length* $n+1$ is a graded A-module M such that

- M is cyclic, and
- $H_M(t) = 1 + t + t^2 + \cdots + t^n$.

PROPOSITION 5.4. *Let A be a graded k-algebra, generated in degree 1. There is a bijection*

$$\{\textit{truncated point modules of length } n+1\} \leftrightarrow \Gamma_n.$$

PROOF. □

We should really define Γ_d to be the scheme $\mathcal{V}(I_d)$.

Definition 5.5. If the hypotheses of Proposition 2.10 hold, we call E the point module variety for A.

Recall that a point module over D is a graded left D-module $M_0 \oplus M_1 \oplus \cdots$ generated in degree zero such that $\dim M_i = 1$ for all $i \geq 0$. This definition may, in an obvious way, be extended to families so as to define a functor from Sch$/k$ to Sets (see [**9**]). Our definition of $\mathcal{P}_D$ may then be justified by the following specialization of a result in [**9**].

THEOREM 5.6. *Suppose that $\mathcal{P}_D = \mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$ and that Γ_D is the graph of an automorphism $\sigma_D : \mathcal{P}_D \to \mathcal{P}_D$. Then*

1. *$\mathcal{P}_D$ represents the functor of point modules, and the truncation functor $M \mapsto M_{\geq 1}(1)$ is represented by σ^{-1}.*
2. *Every point module of D is of the form $D/Dy_1 + \cdots + Dy_n$ where $(y_i)_i \in D_1$. The corresponding point in $\mathcal{P}_D$ is given by the common zero of $(y_i)_i$.*

PROOF. 1. The first statement is given in [**9**, Corollary 3.13] and the remarks thereafter.

□

If the hypotheses of Theorem 5.6 apply (and they always will in the examples we consider in this paper) then we will call the pair $(\mathcal{P}_D, \sigma_D)$ the *point variety* of D. If $p \in \mathcal{P}_D$ then the corresponding point module will be denoted $M(p)$. Thus (5.6.1) says that $M(p)_{\geq 1}(1) \cong M(p^{\sigma^{-1}})$.

LEMMA 5.7. *Suppose that $(\Gamma_D)_{\mathrm{red}}$ defines an isomorphism between $\mathrm{pr}_1(\Gamma_D)_{\mathrm{red}}$ and $\mathrm{pr}_2(\Gamma_D)_{\mathrm{red}}$. Then Γ_D defines an isomorphism between $\mathrm{pr}_1(\Gamma_D)$ and $\mathrm{pr}_2(\Gamma_D)$.*

PROOF. This is a consequence of [**9**, Proposition 3.6]. If $p \in \mathrm{pr}_1(\Gamma_D)$ then the preimage of p is (scheme-theoretically) a linear space, and by hypothesis is 0-dimensional. Hence the preimage consists of a single reduced point. Hence by the argument in [**9**] pr_1 is an isomorphism in a neighbourhood of p. The same argument applies to pr_2, whence both projections are isomorphisms from Γ_D. □

6. Normal elements and open affine subsets

If A is a homogeneous coordinate ring of an irreducible projective variety X, and $f \in A$ is a regular element of positive degree, then X is the disjoint union of $\mathcal{V}(f)$, the zero locus of f, and the open affine set $X_f = \{p \in X \mid f(p) \neq 0\}$, and to some extent the study of X can be split into these two separate pieces. The rings $A/(f)$ and $A[f^{-1}]_0$ are respectively the homogeneous coordinate ring of $\mathcal{V}(f)$ and the coordinate ring of X_f. This section examines a non-commutative analogue of this proceedure. Suppose that A is a graded k-algebra and that u is a homogenous, regular, normalizing element of positive degree—to what extent is $\operatorname{Proj}(A)$ composed of $\operatorname{Proj}(A/(u))$ and $\mathsf{Mod}(A[u^{-1}]_0)$? Under mild hypotheses we will show that the set of points for A is the union of those for $A/(u)$ and $A[u^{-1}]$, and that the set of points for $A[u^{-1}]$ is naturally in bijection with the set of finite dimensional simple modules over $A[u^{-1}]_0$.

Notation . The following notation applies throughout this section:

- A is a graded k-algebra, and
- $u \in A$ is a homogeneous, regular, normalizing element of degree $d > 0$.

The following is a standard result on quotient categories when π has a right adjoint.

Lemma 6.1. *Let $M \in \mathsf{GrMod}(A)$ and suppose that M is torsion-free. Then $M = \omega\pi M$ if and only if $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, M) = 0$.*

Proof. ($\Rightarrow$) We will show that $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, \omega\pi N) = 0$ for all $N \in \mathsf{GrMod}(A)$. Suppose that $0 \to \omega\pi N \to L \to A/A_{\geq 1}[n] \to 0$ is non-split. There is a submodule $L' \subset L$ which is a proper essential extension of $\omega\pi N$, and hence of $N/\tau N$. Moreover, it is an extension by a torsion module because there is an exact sequence $0 \to (\omega\pi N)/N \to L'/N \to L'/\omega\pi N \to 0$ with the first and last terms torsion. But $\omega\pi N$ is the largest essential extension of $N/\tau N$ by a torsion module, so $L' \subset \omega\pi N$. This is a contradiction, so we conclude that $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, \omega\pi N) = 0$.

($\Leftarrow$) If $\omega\pi M \neq M$, then M has a proper essential extension by a torsion module, so $\underline{\operatorname{Ext}}^1_A(T, M) \neq 0$ for some quotient T of $A/A_{\geq 1}$. Since M is torsion-free this implies that $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, M) \neq 0$ also, a contradiction. Thus $M = \omega\pi M$. □

Lemma 6.2. *If $M \in \mathsf{GrMod}(A[u^{-1}])$, then $\omega_A\pi_A M \cong M$.*

Proof. Since $u \in A_{\geq 1}$ acts injectively on M, M is a torsion-free A-module, so it suffices to show that $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, M) = 0$.

Suppose that $0 \to M \to L \to A/A_{\geq 1}[n] \to 0$ is exact in $\mathsf{GrMod}(A)$. Write $L = M + A_0 m'$ with $\deg m' = -n$. Since $u \in A_{\geq 1}$, $um' \in M = uM$, whence $um' = um$ for some $m \in M_{-n}$. Therefore $uA(m-m') = 0$ because u is normalizing. Since M is u-torsion-free, $A(m-m') \cap M = 0$, whence $L = A(m-m') \oplus M$. Thus the extension splits, so the Ext group is zero, as required. □

Lemma 6.3. *Suppose that for every $M \in \mathsf{grmod}(A)$ having Gelfand-Kirillov dimension 1, $\dim_k(M_n)$ is bounded. If such an M is u-torsion-free, then $\omega\pi M \cong M[u^{-1}]$.*

Proof. Since $\dim_k(M_n)$ is bounded, and left multiplication $u^i : M \to M$ is injective, $u^i M_n = M_{n+id}$ for $i \gg 0$. Thus $M/u^i M$ is a torsion module, and hence so is $M[u^{-1}]/M$. Thus $\omega\pi M \cong \omega\pi M[u^{-1}]$, which is isomorphic to $M[u^{-1}]$ by Lemma 6.2. □

LEMMA 6.4. *Let $u \in A$ be a homogeneous, regular, normalizing element of degree d. Define $\sigma : A \to A$ by $u\sigma(a) = au$. Then*

1. *σ is an automorphism and σ_* is an auto-equivalence of $\mathsf{GrMod}(A)$;*
2. *if $M \in \mathsf{GrMod}(A)$ is u-torsion-free, then $\sigma_* M \cong uM[d]$.*

PROOF. (1) Already done ???? where

(2) By definition, $\sigma_* M = M$ as a graded vector space, and the A-action is given by $a.m = \sigma(a)m$. It is easy to see that $\Phi : \sigma_* M \to uM$, defined by $\Phi(m) = um$ is an A-module map, and is bijective because M has no u-torsion. Thus Φ is the required isomorphism. $\square$

LEMMA 6.5. *Let $M \in \mathsf{GrMod}(A)$ be critical. Let u be a homogeneous, regular, normalizing element. Then either $uM = 0$ or $\operatorname{Ann}_M(u) = 0$.*

PROOF. Define $\sigma : A \to A$ by $u\sigma(a) = au$. Since M is critical, so is $\omega_* M$, and it has the same GK-dimension. Consider the exact sequence

$$0 \to \operatorname{Ann}_M(u) \to \sigma_* M \xrightarrow{\Phi} uM \to 0$$

where $\Phi(m) = um$. If $\operatorname{Ann}_M(u) \neq 0$, then $\operatorname{GKdim} uM < \operatorname{GKdim} \sigma_* M = \operatorname{GKdim} M$, whence $uM = 0$. $\square$

Now we consider the homomorphisms $f : A \to A/(u)$ and $g : A \to A[u^{-1}]$, and the induced functors $f_* : \operatorname{proj}(A/(u)) \to \operatorname{proj}(A)$ and $g_* : \operatorname{proj}(A[u^{-1}]) \to \operatorname{proj}(A)$, both of which are fully faithful.

PROPOSITION 6.6. *Suppose that for every $M \in \mathsf{grmod}(A)$ having Gelfand-Kirillov dimension 1, $\dim_k(M_n)$ is bounded. Let $\mathcal{F} \in \operatorname{proj}(A)$ be a point for A. Then*

1. *$\mathcal{F}$ is in the image of either f_* or g_*;*
2. *if $\mathcal{G} \in \operatorname{proj}(A[u^{-1}])$, then $\mathcal{G}$ is a point for $A[u^{-1}]$ if and only if $g_* \mathcal{G}$ is a point for A;*
3. *the set of points in $\operatorname{proj}(A)$ is the disjoint union of the set of points in $\operatorname{proj}(A/(u))$ and $\operatorname{proj}(A[u^{-1}])$.*

PROOF. (1) We may write $\mathcal{F} = \pi M$ with $M \in \mathsf{grmod}(A)$ and $\operatorname{GKdim} M = 1$. By Lemma 6.5, either $uM = 0$ or M is u-torsion-free. If $uM = 0$, then $\mathcal{F} \in \operatorname{Im} f_*$. If M is u-torsion-free, then $\omega\pi M \cong M[u^{-1}]$ by Lemma 6.3, whence $\mathcal{F} = \pi M \cong \pi(M[u^{-1}]) \in \operatorname{Im} g_*$.

(2) ($\Rightarrow$) Write $\mathcal{G} = \pi_{A[u^{-1}]} M$ with $M \in \mathsf{GrMod}(A[u^{-1}])$; we can assume that $\operatorname{GKdim} M = 1$. Suppose that $g_* \mathcal{G}$ is not a point for A. Then there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{GrMod}(A)$ with $\pi L \neq 0$ and $\pi N \neq 0$. Hence there is an exact sequence $0 \to \omega\pi L \to \omega\pi M \to \omega\pi N \to 0$. Since M is an $A[u^{-1}]$-module it is u-torsion-free, hence so is L. Hence this sequence is of the form $0 \to L[u^{-1}] \to M \to \omega\pi N \to 0$. Since the first two modules are $A[u^{-1}]$-modules, so is $\omega\pi N$. Therefore, applying $\pi_{A[u^{-1}]}$ to this sequence, it follows that $\mathcal{G}$ is not irreducible, and hence not a point.

($\Leftarrow$) Suppose that $\mathcal{G}$ is not a point for $A[u^{-1}]$. Write $\mathcal{G} = \pi_{A[u^{-1}]} M$; then $g_* \mathcal{G} = \pi M$, so we can assume that $\operatorname{GKdim} M = 1$. Since $\mathcal{G}$ is not a point, there is an exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$. Since f_* is faithful, $f_* \mathcal{G}$ cannot be a point either.

(3) This follows from (1) and (2) together with Lemma 1.7 which says that f_* is compatible with the notion of a point. $\square$

Lemma 6.7. *Suppose that $M \in \mathsf{grmod}(A)$ is u-torsionfree and 1-critical. Then*
1. *$M[u^{-1}]$ is an irreducible object in $\mathsf{grmod}(A[u^{-1}])$;*
2. *$M[u^{-1}]_0$ is a simple module over $A[u^{-1}]_0$;*
3. *$\dim_k(M_j)$ is bounded;*
4. *for $i \gg 0$, $\dim_k(M[u^{-1}]_0) = \dim_k(M_{id})$.*

Proof. (1) Let L be a proper quotient of $M[u^{-1}]$. Since M is an essential A-submodule of $M[u^{-1}]$, the image of M in L is a proper quotient of M, hence finite dimensional, and thus u-torsion. But u acts bijectively on L, so the image of M in L must be zero. Since M generates $M[u^{-1}]$, L must be zero.

(2) If $0 \neq x \in M[u^{-1}]_0$, then $A[u^{-1}]x = M[u^{-1}]$ by (1), so $A[u^{-1}]_0 x = M[u^{-1}]_0$ as required.

(3) The module M/uM is finite dimensional, so $M_{j+d} = M_j$ for $j \gg 0$.

(4) By (3), there is $e \in \mathbb{N}$ such that $\dim M_{id} = e$ for all $i \gg 0$; we will show that $\dim_k(M[u^{-1}]_0 = e$. Since every element of $M[u^{-1}]/M$ is u-torsion, if V is a finite dimensional subspace of $M[u^{-1}]_0$, $u^i V \subset M_{id}$ for some large i. Thus $\dim V \leq e$, whence $\dim M[u^{-1}]_0 \leq e$. Since $u^{-i}M_{id} \subset M[u^{-1}]_0$, the reverse equality also holds. $\square$

Since $A[u^{-1}]$ contains a unit of positive degree, its only torsion module is zero. Hence $\mathsf{Tails}(A[u^{-1}])$ is equivalent to $\mathsf{GrMod}(A[u^{-1}])$. If A is generated by A_0 and A_1, then $A[u^{-1}]$ is strongly graded, so $\mathsf{GrMod}(A[u^{-1}]) \cong \mathsf{Mod}(A[u^{-1}]_0)$ by Proposition 16.6.4 and Theorem 16.6.7 The functor implementing the equivalence is $M \mapsto M_0$. Under this equivalence irreducible objects in $\mathsf{Tails}(A[u^{-1}])$ correspond to simple modules over $A[u^{-1}]_0$.

Hence, if $M \in \mathsf{grmod}(A)$ is u-torsionfree and $\mathcal{F} = \pi M$ is a point for A, then $\mathcal{F}$ corresponds to the simple $A[u^{-1}]_0$-module

$$M[u^{-1}]_0 = \cdots + uM_{-d} + M_0 + u^{-1}M_d + \cdots .$$

Since $\dim_k(M_n)$ is bounded, $M[u^{-1}]_0 = u^{-n}M_{nd}$ for $n \gg 0$. Thus $M[u^{-1}]_0$ is finite dimensional. Conversely, if S is a finite dimensional simple $A[u^{-1}]_0$-module, then $A[u^{-1}] \otimes_{A[u^{-1}]_0} S$ is a critical $A[u^{-1}]$-module and has GK-dimension 1 since its Hilbert series is periodic. In summary, points of $\operatorname{proj}(A)$ which do not lie in the closed subscheme $\operatorname{proj}(A/(u))$ are in bijection with finite dimensional simple modules over $A[u^{-1}]_0$.

Proposition 6.8. *Let A be an $\mathbb{N}$-graded algebra and $u \in A_d$ a normalizing regular element. Write $B = A/(u)$ and $R = A[u^{-1}]_0$. Let φ be the automorphism of A defined by $u\varphi(a) = au$ for $a \in A$, and let $\sigma \in \operatorname{Aut}(B)$ be induced by φ. Filter R by the ascending sequence of subspaces $F_iR = A_{id}u^{-i}$, and let $\operatorname{gr}(R)$ denote the associated graded algebra. Then there is an isomorphism*

$$\operatorname{gr}(R) \cong \left(B^{(d)}\right)^{\sigma},$$

where the right hand side is the twist of the d^{th} Veronese subalgebra of B.

Proof. Let $\pi : A \to B$ be the natural map. Thus $\sigma(\pi(a)) = \pi(\varphi(a))$. Define $\Psi : \operatorname{gr}(R) \to B$ by

$$\Psi(\operatorname{gr}(x)) = \pi(a),$$

if $x = au^{-i} \in F_iR \backslash F_{i-1}R$ and $a \in A_{id}$. The map Ψ is well-defined: if $\operatorname{gr}(x) = \operatorname{gr}(x')$ with $x' = a'u^{-i}$, then $(a - a')u^{-i} \in F_{i-1}R$, so $a - a' \in (u)$ whence $\pi(a) = \pi(a')$.

We now show that

$$\Psi(\mathrm{gr}(x)\,\mathrm{gr}(y)) = \Psi(\mathrm{gr}(x)) * \Psi(\mathrm{gr}(y)) \tag{6-1}$$

where $*$ is the product in $\left(B^{(d)}\right)^\sigma$; that is, $u * v = u\,\sigma^i(v)$ if $u \in B_i^{(d)} = B_{id}$. Let $x = au^{-i} \in F_iR\backslash F_{i-1}R$ where $a \in A_{id}$, and $y = bu^{-j} \in F_jR\backslash F_{j-1}R$ where $b \in A_{jd}$. A calculation gives

$$\Psi(\mathrm{gr}(x)) * \Psi(\mathrm{gr}(y)) = \pi(a) * \pi(b) = \pi(a)\sigma^i(\pi(b)) = \pi(a\varphi^i(b)). \tag{6-2}$$

Either $\mathrm{gr}(x)\,\mathrm{gr}(y) = 0$ or $\mathrm{gr}(x)\,\mathrm{gr}(y) = \mathrm{gr}(xy)$, the first possibility occurring if and only if $xy \in F_{i+j-1}R$. Since $xy = au^{-i}bu^{-j} = a\varphi^i(b)u^{-(i+j)}$, $\mathrm{gr}(x)\,\mathrm{gr}(y) = 0$ if and only if $a\varphi^i(b) \in uA_{(i+j-1)d}$. Thus, if $\mathrm{gr}(x)\,\mathrm{gr}(y) = 0$, then (6-1) follows from (6-2). On the other hand, if $\mathrm{gr}(x)\,\mathrm{gr}(y) \neq 0$, then

$$\Psi(\mathrm{gr}(x)\,\mathrm{gr}(y)) = \Psi(\mathrm{gr}(xy)) = \pi(a\varphi^i(b)) = \Psi(\mathrm{gr}(x)) * \Psi(\mathrm{gr}(y)),$$

as required. □

If M is a line module for A which is u-torsionfree, and $\deg u = 1$, then I think the associated graded module of $M[u^{-1}]_0$ will be a point module for $\mathrm{gr}(R)$. To see this, just look at $M_0 \subset u^{-1}M_1 \subset u^{-2}M_2 \cdots$ in which dimensions increase by 1. So one should be able to get some info about linear modules for A by a study of $\mathrm{gr}(R)$. $\boxed{\text{Paul}}$ is the point module for $\mathrm{gr}(R)$ which is associated to this line module related to M/uM and/or the point where the line meets $\mathcal{V}(u)$? Yes, I think so, and it is probably related to the general theory of Rees rings.

7. Additional Remarks

$\boxed{\text{Paul}}$ if $\mathrm{Proj}(A) \cong \mathrm{Proj}(A')$ are the points the same ??? i.e., is $\mathrm{GKdim}\,\pi M = 1$ meaningful?

Gelfand-Kirillov dimension is not defined on $\mathrm{Proj}(A)$. If $\mathcal{F} \in \mathrm{Proj}(A)$ we can't define the GK-dimension of $\mathcal{F}$ until we choose a graded module M such that $\pi M \cong \mathcal{F}$; but choosing an M requires remembering that $\mathrm{Proj}(A)$ comes from a particular algebra A—but A is not part of the structure of $\mathrm{Proj}(A)$. This suggests that sometimes it is necessary to consider the pair $(\mathrm{Proj}(A), A)$.

There is an example due to Artin and van den Bergh [**11**] of graded algebras A and B such that $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$ but $\mathrm{GKdim}\,A = 5$ and $\mathrm{GKdim}\,B = 3$. These algebras are not pathological: B is the homogeneous coordinate ring of a projective surface S, and A is a twisted homogenous coordinate ring of S. Thus $\mathrm{Proj}(A)$ is equivalent to $\mathsf{Coh}(S)$.

The definition of a point module (2.1 should be modified if A is not generated in degree 1. This does not seem to be a serious problem though—all we really want is some class of A-modules which give representatives of the points in $\mathrm{Proj}\,A$ having smallest multiplicity, so we might just take the modules $(\omega\mathcal{F})_{\geq 0}$ where $\mathcal{F} \in \mathrm{Proj}\,A$ is such a point.

Example 7.1. The point modules over the coordinate rings of the quantum plane and the Jordan plane are parametrized by $\mathbb{P}^1$. Let $A = k[x, y]$ be either

- the coordinate ring of a quantum plane, with defining relation $yx = qxy$ where $0 \neq q \in k$, or
- the coordinate ring of the Jordan plane, with defining relation $yx - xy = x^2$.

We make A a graded algebra with $\deg x = \deg y = 1$. Since A is an iterated Ore extension of k, it is a noetherian domain, having Hilbert series $(1-t)^{-2}$, and basis $\{x^i y^j \mid i, j \geq 0\}$. If $0 \neq a \in A_1$, then $Aa \cong A[-1]$, whence

$$H_{A/Aa}(t) = H_A(t) - H_{Aa}(t) = (1-t)^{-2}(1-t) = (1-t)^{-1}.$$

Therefore A/Aa is a point module. Moreover, as we will show next, these are all the point modules. If M is a point module, there is a surjective map $f : A \to M$ and, since $\dim A_1 = 2$ whereas $\dim M_1 = 1$, $\ker f$ contains some $0 \neq a \in A_1$. Hence M is a quotient of A/Aa; but M and A/Aa have the same Hilbert series, so $M \cong A/Aa$.

Finally, observe that $A/Aa \cong A/Ab$ if and only if $ka = kb$, so the isomorphism classes of point modules are in bijection with the 1-dimensional subspaces of A_1; that is, with the points of $\mathbb{P}^1$.

The next two examples determine all the points in $\operatorname{Proj} A$ for the quantum and Jordan planes. (We are being slightly dishonest here by presenting an ad hoc proof rather than one based on general principles: we will show later that the various categories $\operatorname{Proj} A$ for the quantum and Jordan planes are all equivalent to one another; this is because all such A are twists of a single ring, namely the commutative polynomial ring $k[X_1, X_2]$, and twisting an algebra by an automorphism does not change the category $\mathsf{GrMod} A$.)

Example 7.2. We determine the point module sequences for $A = k[x, y]$, the coordinate ring of either a quantum plane or the Jordan plane. This is a continuation of Example 7.1. Using the results there, we may label the point modules for A as follows: for each point $p = (\alpha, \beta) \in \mathbb{P}(A_1^*)$, define

$$M(p) = A/A(\beta x - \alpha y).$$

The first point in the sequence associated to $M(p)$ is

$$p_0 = \mathcal{V}(\{a \in A_1 \mid a.M(p)_0 = 0\}) = p.$$

The second point in the sequence is $\mathcal{V}(a)$ where $0 \neq a \in A_1$ satisfies $a.M(p)_1 = 0$. We treat the two cases separately.

- For the quantum plane, $yx = qxy$, we have
$$\begin{aligned} (q\beta x - \alpha y)x &= qx(\beta x - \alpha y) \quad \text{and} \\ (q\beta x - \alpha y)y &= y(\beta x - \alpha y), \end{aligned}$$
so $p_1 = (\alpha, q\beta)$.
- For the Jordan plane, $yx = xy + x^2$, we have
$$\begin{aligned} ((\alpha+\beta)x - \alpha y)x &= x(\beta x - \alpha y) \qquad \text{and} \\ ((\alpha+\beta)x - \alpha y)y &= (y - x)(\beta x - \alpha y), \end{aligned}$$
so $p_1 = (\alpha, \alpha + \beta)$.

In both cases, there is an automorphism, σ say, of $\mathbb{P}(A_1^*)$ such that $p_1 = p_0^\sigma$. Since $M_{\geq n} \cong M(p_n)[-n]$, p_{n+1} is determined by p_n in the same way that p_1 is determined by p_0, whence $p_{n+1} = p_n^\sigma$. Thus

$$p_n = p^{\sigma^n}$$

for all $n \geq 0$—equivalently, $M(p)_{\geq n} \cong M(p^{\sigma^n})[-n]$ and, in $\operatorname{Proj} A$, $\pi M(p)[n] \cong \pi M(p^{\sigma^n})$. For the quantum plane, σ is the automorphism determined by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ acting on A_1^* (with respect to the ordered basis x, y), and for the Jordan plane σ is the automorphism determined by $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ acting on A_1^* (with respect to the ordered basis x, y).

EXAMPLE 7.3. Suppose that k is algebraically closed and let A be the coordinate ring of either a quantum or Jordan plane. We use the results in the previous example to show that for the quantum and Jordan planes every point in $\operatorname{Proj} A$ is of the form πM for some point module M. Hence the points in $\operatorname{Proj} A$ are parametrized by $\mathbb{P}^1$.

First observe that two point module sequences $(p, p^{\sigma}, p^{\sigma^2}, \dots)$ are eventually the same if and only if they are the same. Hence non-isomorphic point modules give distinct points in $\operatorname{Proj}(A)$.

Now let πM be a point in $\operatorname{Proj}(A)$. By Lemma 1.3, we may assume that M is 1-critical and finitely generated. Hence by Proposition 1.11, $\dim M_r$ is constant for $r \gg 0$. If $\dim M_r = \dim M_{r+1} = e$ then, choosing suitable isomorphisms, we may view the map $A_1 \to \operatorname{Hom}_k(M_r, M_{r+1})$ as a map $A_1 \to M_e(k)$. Composing this map with the determinant, $\det : M_e(k) \to k$, gives a homogeneous polynomial function $A_1 \to k$ of degree $e \geq 1$. Since k is algebraically closed this polynomial has a non-trivial zero, say $0 \neq a \in A_1$. Thus, there exists $0 \neq m \in M_r$ such that $a.m = 0$, whence there is a non-zero map $A/Aa \to M[r]$. Since A/Aa and M are 1-critical (whence M is torsion-free), this map is injective, so $\pi(A/Aa) \cong \pi M[r]$. That is, $\pi M \cong \pi(A/Aa)[-r]$. In the notation of the previous example, we have $A/Aa \cong M(p)$, whence $\pi(A/Aa)[r] \cong \pi M(p)[r] \cong \pi M(p^{\tau^r})$. Thus πM, and hence every point in $\operatorname{Proj}(A)$ is of the form πN for some point module N. By the previous paragraph, the points in $\operatorname{Proj}(A)$ are parametrized by $\mathbb{P}^1$.

CHAPTER 24

Quantum projective spaces

This chapter concerns the non-commutative analogues of the projective spaces $\mathbb{P}^n$. If A is an $(n+1)$-dimensional quantum polynomial ring (Definition 18.4.1), we call $X = (\operatorname{tails} A, \mathcal{A}, [1])$ a quantum $\mathbb{P}^n$.

Section 1 deals with generalities. The first rich examples are the quantum $\mathbb{P}^2$'s (we call them quantum planes for short), and we study them in Sections 3,

Quantum planes, or rather there homogeneous coordinate rings, first appeared in a paper of Artin and Schelter [**6**] in 1987, and they gave a classification of the generic ones. A complete classification of their homogeneous coordinate rings was given by Artin, Tate and van den Bergh [**9**] in 1989 and subsequently they examined the geometry of the quantum planes in detail [**10**]. These two papers introduced a wealth of new ideas which were seminal for the development of non-commutative algebraic geometry.

1. Quantum $\mathbb{P}^n$'s

Definition 1.1. Let A be an $(n+1)$-dimensional quantum polynomial ring (Definition 18.4.1). We call $X = (\operatorname{tails} A, \mathcal{A}, [1])$ a quantum $\mathbb{P}^n$, and call A the homogeneous coordinate ring of X.

For $n \geq 1$, one has $B(\operatorname{tails} A, \mathcal{A}, [1]) \cong A$ by ???, so calling A the homogeneous coordinate ring of X is compatible with the use of this terminology in Chapter ???, Section ???

Quantum $\mathbb{P}^n$'s include the usual (commutative) projective spaces, because a commutative polynomial ring generated in degree 1 is a quantum polynomial ring.

The only 1-dimensional quantum polynomial ring is $k[x]$, so $\mathbb{P}^0$ is the only quantum $\mathbb{P}^0$.

A 2-dimensional quantum polynomial rings is of the form $A = k[x, y]$ with a single relation which is either of the form $yx - qxy$ with $0 \neq q \in k^*$, or of the form $xy - yx - y^2$. We saw in ??? that $\operatorname{proj}(A) \cong \mathbb{P}^1$, so the quantum $\mathbb{P}^1$'s are the commutative $\mathbb{P}^1$'s *but with a different polarization* coming from tHe shift functor on A.

Ore extensions provide a ready source of quantum polynomial rings (Corollary 7.1.5), and consequently a source of quantum $\mathbb{P}^n$'s.

PROPOSITION 1.2. *Let $A = k[x_0][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be an iterated Ore extension which is a graded algebra with $\deg(x_i) = 1$ for each i. If each σ_i is an automorphism, then A is the coordinate ring of a quantum $\mathbb{P}^n$.*

PROOF. It is clear that A has the correct Hilbert series and is generated by A_1. By Corollary 6.1.5, A is Artin-Schelter regular of global dimension $n+1$. By ??? A is noetherian. □

This result applies to the polynomial ring on $n+1$ variables with its standard grading, and its twists (either because they are iterated Ore extensions by Proposition 16.5.8, or because the conditions in Definition 18.4.1 are conditions on the category GrMod which is unchanged by twisting). These examples are not too interesting since they all give rise to the same scheme $\text{Proj}(A)$, which is isomorphic to the classical $\mathbb{P}^n$. Example 23.3.5 (to which Proposition 1.2 applies) is an example of a non-classical quantum $\mathbb{P}^2$.

2. Linear modules

Definition 2.1. A graded A-module M is a *d-linear module* if

- M is cyclic, and
- $H_M(t) = (1-t)^{-d}$.

The terminology is is explained by the next result.

PROPOSITION 2.2. *Let $A = S(A_1)$ be the homogeneous coordinate ring of $\mathbb{P}(A_1^*)$, and let M be a graded A-module. The following are equivalent:*

1. *M is a shifted $(d+1)$-linear module;*
2. *$M \cong A/\langle y_1, \ldots, y_{n-d}\rangle$ for some linearly independent $y_1, \ldots, y_{n-d} \in A_1$;*
3. *M is a Cohen-Macaulay module with $d(M) = d+1$ and $e(M) = 1$.*

Thus there is a bijection

$$\{d\text{-dimensional linear subspaces of } \mathbb{P}(A_1^*)\ \} \leftrightarrow \{(d+1)\text{-linear modules up to shifting}\}$$

The only 0-linear module is the trivial module A/A_+. The 1-linear modules are precisely the point modules. A 2-linear module is called a *line module* and a 3-linear module is called a *plane module*. Notice that the Hilbert series of a $(d+1)$-linear module is the same as that of the polynomial ring in $d+1$ indeterminates (which is the homogeneous coordinate ring of $\mathbb{P}^d$).

Even in the non-commutative case, each $(d+1)$-linear module determines a d-dimensional linear subspace of $\mathbb{P}(A_1^*)$, namely $\mathcal{V}(\text{Ann}_{A_1}(M_0) = \big(\text{Ann}_{A_1}(M_0)\big)^{\perp}$.

3. Homogeneous coordinate rings of quantum $\mathbb{P}^2$s.

The main result in this section is that the homogenous coordinate ring of a quantum plane, A say, determines, and is determined by, a pair (E,σ) consisting of a subscheme E of $\mathbb{P}^2$ and an automorphism σ of E. In particular, there is a surjective map $A \to B(E,\sigma,\mathcal{L})$ where $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ to E, and A and A is defined by the quadratic relations in $B(E,\sigma,\mathcal{L})$. Moreover, E is either all of $\mathbb{P}^2$ or a cubic divisor. In the first case, the quantum $\mathbb{P}^2$ is classical, meaning that its homogeneous coordinate ring is a twist of a polynomial ring. The second case is more interesting.

It is already implicit in Theorem 7.3.1 that a quantum plane is defined by three quadratic relations, but the next result adds a little precision.

PROPOSITION 3.1. *Let A be the homogeneous coordinate ring of a quantum $\mathbb{P}^2$. Then*

1. *A is a quadratic algebra with 3 defining relations;*

2. *the minimal resolution of the trivial module ${}_Ak$ is of the form*

$$0 \to A[-3] \xrightarrow{\underline{x}^\mathsf{T}} A[-2]^3 \xrightarrow{L} A[-1]^3 \xrightarrow{\underline{x}} A \to k \to 0, \tag{3-1}$$

where $\underline{x} = (x_1, x_2, x_3)^\mathsf{T}$ with $\{x_1, x_2, x_3\}$ a basis for A_1, and L is a matrix of linear forms;

3. *there is a choice of basis $\{f_1, f_2, f_3\}$ for the defining relations, such that*

$$L\underline{x} = \underline{f} \qquad \text{and} \qquad \underline{x}^\mathsf{T} L = (Q\underline{f})^\mathsf{T},$$

for some $Q \in \mathrm{GL}(3, k)$, where $\underline{f} = (f_1,\ f_2,\ f_3)^\mathsf{T}$.

Proof. The homogeneous coordinate ring of a quantum plane is an Artin-Schelter regular algebra of global dimension 3 so, by Theorem 7.3.1, the minimal resolution of the trivial module looks like

$$0 \to A \xrightarrow{\underline{y}^\mathsf{T}} A^3 \xrightarrow{L} A^3 \xrightarrow{\underline{x}} A \to k \to 0. \tag{3-2}$$

where $\underline{x} = (x_1, x_2, x_3)^\mathsf{T}$, $\underline{y} = (y_1, y_2, y_3)^\mathsf{T}$, $L = (m_{ij})$ is a matrix of linear forms, and $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are bases for A_1. Changing the basis in the first copy of A^3 in the resolution, we may assume that $\underline{y} = \underline{x}$. Moreover, the defining relations for A are the entries in the matrix $\underline{f} := L\underline{x}$, and also the entries in $\underline{g}^\mathsf{T} := \underline{x}^\mathsf{T} L$, namely $f_i := \sum_j m_{ij} x_j$ and $g_j := \sum_i x_i m_{ij}$. Since $\{f_1, f_2, f_3\}$ and $\{g_1, g_2, g_3\}$ span the same 3-dimensional vector space, $\underline{x}^\mathsf{T} L = \underline{g} = (Q\underline{f})^\mathsf{T}$ for some $Q \in \mathrm{GL}(3, k)$. $\square$

We now describe the subscheme $\Gamma_2 \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ cut out by the relations defining A. This result can also be interpreted as a computation of the point modules for A, but that point of view is postponed until Section 6.

Theorem 3.2. *Let A be a quantum $\mathbb{P}^2$. Write $A = T(V)/(R)$, and $\mathbb{P} = \mathbb{P}(A_1^*)$. Let $\Gamma_2 = \mathcal{V}(R) \subset \mathbb{P} \times \mathbb{P}$. Then Γ_2 is the graph of an automorphism, σ say, of*

$$E := \mathcal{V}(\det L), \tag{3-3}$$

where L is the 3×3 matrix of linear forms defined in Proposition 3.1, and E is either equal to $\mathbb{P}$, or is a cubic divisor in $\mathbb{P}$.

Proof. We retain the notation used in Proposition 3.1; by that result, the defining relations for A are the three entries in the matrix

$$\underline{f} = L\underline{x} = \underline{x}^\mathsf{T} N,$$

where L and N are appropriate 3×3 matrices of linear forms. Let pr_1 and pr_2 be the projections of $\mathbb{P} \times \mathbb{P}$ onto its first and second components. By the results in Chapter 23, Section 3,

$$\mathrm{pr}_1(\Gamma) = \mathcal{V}(\det L) \qquad \text{and} \qquad \mathrm{pr}_2(\Gamma) = \mathcal{V}(\det N).$$

By Proposition 3(3), there is a matrix $Q \in \mathrm{GL}(3, k)$ such that $NQ^\mathsf{T} = L$, so $\mathrm{pr}_1(\Gamma) = \mathrm{pr}_2(\Gamma)$. We will denote this common image by E. Since the entries in L are linear forms, E is either a cubic curve or all of $\mathbb{P}^2$.

The rest of the proof is devoted to showing that Γ is the graph of an automorphism of E. Suppose not. Then there exists a point $p \in E$ for which $\operatorname{rank} L(p) = \operatorname{rank} N(p) < 2$, so Γ contains $p \times \ell$, for some line ℓ in $\mathbb{P}$.

Before proceeding, we write $\{x, y, z\}$ for a basis for A_1, and write

$$L = \begin{pmatrix} a_1x + a_2y + a_3z & a_4x + a_5y + a_6z & a_7x + a_8y + a_9z \\ b_1x + b_2y + b_3z & b_4x + b_5y + b_6z & b_7x + b_8y + b_9z \\ c_1x + c_2y + c_3z & c_4x + c_5y + c_6z & c_7x + c_8y + c_9z \end{pmatrix}.$$

Thus the defining relations for A are

$$\begin{aligned} f_1 &= a_1x^2 + a_2yx + a_3zx + a_4xy + a_5y^2 + a_6zy + a_7xz + a_8yz + a_9z^2, \\ f_2 &= b_1x^2 + b_2yx + b_3zx + b_4xy + b_5y^2 + b_6zy + b_7xz + b_8yz + b_9z^2, \\ f_3 &= c_1x^2 + c_2yx + c_3zx + c_4xy + c_5y^2 + c_6zy + c_7xz + c_8yz + c_9z^2. \end{aligned}$$

By the Gorenstein hypothesis, applying $\underline{\mathrm{Hom}}_A(-, A)$ to (3-1) yields a minimal resolution of k_A, so R is also spanned by the elements g_1, g_2, g_3 defined by $(g_1,\ g_2,\ g_3) = (x,\ y,\ z)L$. Explicitly,

$$\begin{aligned} g_1 &= a_1x^2 + b_1yx + c_1zx + a_2xy + b_2y^2 + c_2zy + a_3xz + b_3yz + c_3z^2, \\ g_2 &= a_4x^2 + b_4yx + c_4zx + a_5xy + b_5y^2 + c_5zy + a_6xz + b_6yz + c_6z^2, \\ g_3 &= a_7x^2 + b_7yx + c_7zx + a_8xy + b_8y^2 + c_8zy + a_9xz + b_9yz + c_9z^2, \end{aligned}$$

There are two cases depending on whether or not p lies on ℓ.

Case 1. Suppose $p \notin \ell$. We may choose the basis x, y, z for A_1 such that $x(p) = y(p) = 0$ and $z(\ell) = 0$. The elements of R vanish on $p \times \ell$ so, evaluating these at $((0, 0, 1), (1, 0, 0))$ and $((0, 0, 1), (0, 1, 0))$, the coefficients of zx and zy in the f_i's and g_i's are zero. That is

$$a_3 = b_3 = c_3 = a_6 = b_6 = c_6 = c_1 = c_4 = c_7 = c_2 = c_5 = c_8 = 0.$$

Therefore $f_3 = c_9z^2$. But an algebra having the relation $c_9z^2 = 0$ cannot be a quantum $\mathbb{P}^2$ because then $\ker(L)$ contains $(0,\ 0,\ z)$, which contradicts the fact that $\ker(L) = A(x,\ y,\ z)$. Hence this case cannot occur.

Case 2. Suppose that $p \in \ell$. We may assume that $x(p) = y(p) = 0$ and $x(\ell) = 0$. By evaluating the relations at the points $((0, 0, 1), (0, 0, 1))$ and $((0, 0, 1), (0, 1, 0))$, we see that the coefficients of z^2 and zy in the f_i's and g_i's are zero. Thus

$$a_9 = b_9 = c_9 = a_6 = b_6 = c_6 = c_3 = c_2 = c_5 = c_8 = 0.$$

Hence the relations look like

$$\begin{aligned} f_1 &= a_1x^2 + a_2yx + a_3zx + a_4xy + a_5y^2 + a_7xz + a_8yz \\ f_2 &= b_1x^2 + b_2yx + b_3zx + b_4xy + b_5y^2 + b_7xz + b_8yz \\ f_3 &= c_1x^2 + c_4xy + c_7xz \\ g_1 &= a_1x^2 + b_1yx + c_1zx + a_2xy + b_2y^2 + a_3xz + b_3yz \\ g_2 &= a_4x^2 + b_4yx + c_4zx + a_5xy + b_5y^2 \\ g_3 &= a_7x^2 + b_7yx + c_7zx + a_8xy + b_8y^2 \end{aligned}.$$

Now we suppose that $c_7 \neq 0$, and deduce a contradiction. Then $a_3 \neq 0$, and if f_3 is expressed as a linear combination of the g_i's, the coefficient of g_1 is non-zero. Therefore $b_3 = 0$, which implies that $b_8 = a_8 = 0$. We may still perform a change of coordinates which preserves the conditions $x(p) = y(p) = 0$ and $x(\ell) = 0$; in particular we may substitute $d_1x + d_2y + z$ for z. We may do this in such a way that the coefficients c_1 and c_4, in f_3, become zero. Hence the relations are

$$\begin{array}{rcl}
f_1 & = & a_1x^2 + a_2yx + a_3zx + a_4xy + a_5y^2 + a_7xz \\
f_2 & = & b_1x^2 + b_2yx + b_4xy + b_5y^2 + b_7xz \\
f_3 & = & c_7xz \\
g_1 & = & a_1x^2 + b_1yx + a_2xy + b_2y^2 + a_3xz \\
g_2 & = & a_4x^2 + b_4yx + a_5xy + b_5y^2 \\
g_3 & = & a_7x^2 + b_7yx + c_7zx \quad .
\end{array}$$

If $b_5 = 0$ also, then we conclude in turn that

$$b_2 = a_5 = a_2 = b_1 = b_4 = b_7 = 0,$$

which implies that $f_2 = 0$, contradicting the fact that $\dim(R) = 3$. Thus $b_5 \neq 0$. Now, making a suitable substitution $d_3x + y$ for y, we may eliminate the coefficient b_2 in f_2. Since $c_7 \neq 0$, f_3 is a linear combination of g_1 and g_2, so the vectors (a_1, b_1, a_2, b_2) and (a_4, b_4, a_5, b_5) are proportional. But $b_5 \neq 0$ and $b_2 = 0$, so $a_1 = b_1 = a_2 = 0$. Hence the coefficients of x^2 and yx are zero in all f_i's, and hence in all g_i's. Therefore $a_4 = b_4 = a_7 = b_7 = 0$. Thus $f_2 = b_5y^2$, and as in Case 1, A cannot be a quantum $\mathbb{P}^2$. This is the contradiction we sought, so we conclude that $c_7 = 0$.

We must have $c_4 \neq 0$, else $f_3 = c_1x^2$, which leads to the same contradiction as before. By a suitable substitution for y of the form $d_3x + y$, we may eliminate the coefficient c_1 in f_3, so the relations are of the form

$$\begin{array}{rcl}
f_1 & = & a_1x^2 + a_2yx + a_3zx + a_4xy + a_5y^2 + a_7xz + a_8yz \\
f_2 & = & b_1x^2 + b_2yx + b_3zx + b_4xy + b_5y^2 + b_7xz + b_8yz \\
f_3 & = & c_4xy \\
g_1 & = & a_1x^2 + b_1yx + a_2xy + b_2y^2 + a_3xz + b_3yz \\
g_2 & = & a_4x^2 + b_4yx + c_4zx + a_5xy + b_5y^2 \\
g_3 & = & a_7x^2 + b_7yx + a_8xy + b_8y^2 \quad .
\end{array}$$

Since the coefficient of zx in g_2 is non-zero, if b_3 were zero, it would follow that $a_3 \neq 0$ (and, looking at the coefficients of yz, that $a_8 = b_8 = 0$), whence f_3 would be a linear combination of g_2 and g_3, hence a multiple of g_3 alone—but the coefficient of xy in g_3 is zero, so this is impossible. Thus $b_3 \neq 0$, whence f_3 is a linear combination of g_2 and g_3—in fact, it must be a multiple of g_3 since $c_4 \neq 0$. Thus $a_7 = b_7 = b_8 = 0$. By considering the coefficients of xz, $a_3 = 0$ also. By making a suitable substitution $d_1x + d_2y + z$ for z, we may eliminate b_1 and b_2 in f_2. Hence the relations look like

$$\begin{array}{rcl}
f_1 & = & a_1x^2 + a_2yx + a_4xy + a_5y^2 + a_8yz \\
f_2 & = & b_3zx + b_4xy + b_5y^2 \\
f_3 & = & c_4xy \\
g_1 & = & a_1x^2 + a_2xy + b_3yz \\
g_2 & = & a_4x^2 + b_4yx + c_4zx + a_5xy + b_5y^2 \\
g_3 & = & a_8xy \quad .
\end{array}$$

Since $b_3 \neq 0$, $a_8 \neq 0$ also. Since g_1 has no zx term, it is a linear combination of f_1 and f_3. Since g_2 has no yz term, it is a linear combination of f_2 and f_3. Thus, comparing f_1 and g_1, $a_2 = a_5 = a_4 = 0$. Comparing f_2 and g_2, $b_4 = 0$. Hence we obtain

$$L = \begin{pmatrix} a_1x & 0 & a_8y \\ b_3z & b_5y & 0 \\ 0 & c_4x & 0 \end{pmatrix}$$

and the relations are

$$f_1 = a_1x^2 + a_8yz$$
$$f_2 = b_3zx + b_5y^2$$
$$f_3 = c_4xy.$$

Since zx is a multiple of y^2 and $xy = 0$ in A, $y(zx) = (zx)y = 0$. Since yz is a multiple of x^2 and $xy = 0$, $yzy = 0$. Therefore $(yz, 0, 0)L = 0$. But $\ker L = A(x, y, z)$, so we must have $(yz, 0, 0) = -a_1a_8^{-1}x(x, y, z)$. Thus $a_1xz = 0$, which forces $a_1 = 0$. Therefore $(x, 0, 0)L = 0$; but this is impossible since $(x, 0, 0) \notin A(x, y, z)$. This contradiction shows that Case 2 cannot occur.

Hence our supposition that Γ_2 is not a graph of an automorphism of E is false. □

We now show that a quantum plane, A say, is determined by the geometric data (E, σ) associated to it: the main point here, and in all subsequent investigations, is that $B(E, \sigma, \mathcal{L})$ is a quotient of A.

Corollary 3.3. *Let A be a quantum plane, (E, σ) the associated geometric data, and Γ the graph of σ. Let $\mathcal{L}$ denote the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to E. Then*

1. *there is a surjective graded algebra homomorphism $A \to B(E, \sigma, \mathcal{L})$, which is an isomorphism in degree one;*
2. *$A \cong T(A_1)/(R)$ where $R = \{f \in A_1 \otimes A_1 \mid f|_\Gamma = 0\}$;*
3. *the defining relations for A are the quadratic relations for $B(E, \sigma, \mathcal{L})$.*

Proof. The existence of a map $A \to B(E, \sigma, \mathcal{L})$ is proved in Lemma 22.2.7. ????????? □

Corollary 3.4. *Let A be a quantum plane, and (E, σ) the associated geometric data. If $E = \mathbb{P}(A_1^*)$ then*

$$A \cong S(A_1)^\theta,$$

the twist of a 3-dimensional polynomial ring by an automorphism $\theta \in GL(A_1)$ which induces σ on $\mathbb{P}(A_1^*)$.

Proof. By Corollary 3.3, there is a map $A \to B(\mathbb{P}^2, \sigma, \mathcal{O}(1))$ which is surjective in degree one. However, by Proposition 22.2.1, $B(\mathbb{P}^2, \sigma, \mathcal{O}(1)) \cong S(A_1)^\theta$. This algebra is generated in degree 1, so the map $A \to S(A_1)^\theta$ is surjective. But these algebras have the same Hilbert series, so they are isomorphic. □

Although θ is only determined by σ up to a scalar multiple, Corollary 3.4 is unambiguous because, by Example 6.5.5, the isomorphism class of the twisted algebra $S(V)^\theta$ is not changed when θ is replaced by a scalar multiple of itself.

Corollary 3.4 shows that the quantum planes giving rise to (E, σ) with $E = \mathbb{P}(A_1^*)$ are rather dull. The more interesting case is when E is a cubic divisor, particularly when E is a smooth cubic—any cubic may occur as E, with two exceptions, namely the nodal and cuspidal cubics. Example 3.5 in Chapter 23 is a quantum $\mathbb{P}^2$ with E being the union of a conic and line.

EXERCISES

3.1 Let $A = k[x, y, z]$ with relations $zx = xz$, $zy = yz$ and $xy - yx = z^2$. Show that A is a quantum $\mathbb{P}^2$ with $E = \mathcal{V}(z^3)$. Notice that $A/(z-1) \cong A[z^{-1}]_0$ is a Weyl algebra. This is the most degenerate quantum $\mathbb{P}^2$ which is not a twisted polynomial ring.

4. The noetherian property

5. The 3-dimensional Sklyanin algebras

Definition 5.1. Let $E \subset \mathbb{P}^2 = \mathbb{P}(V^*)$ be a smooth cubic curve. Fix $\tau \in E$ and define

$$\Gamma := \{(p, p+\tau) \mid p \in E\}.$$

Let $R = \{f \in V \otimes V \mid f|_\Gamma = 0\}$. The 3-dimensional Sklyanin algebra associated to the pair (E, τ) is the algebra

$$A(E, \tau) := T(V)/(R).$$

It is useful to define the space of relations R in more geometric terms. To do this, first observe that the inclusion $E \subset \mathbb{P}(V^*)$ determines an invertible $\mathcal{O}_E$-module, say $\mathcal{L}$, the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to E. Thus $V = \mathrm{H}^0(E, \mathcal{L})$, and $V \otimes V$ may be identified either with $\mathrm{H}^0(E, \mathcal{L}) \otimes \mathrm{H}^0(E, \mathcal{L})$, or with $\mathrm{H}^0(E \times E, \mathcal{L} \boxtimes \mathcal{L})$. With this last interpretation, R is the kernel of the restriction map

$$\mathrm{H}^0(E \times E, \mathcal{L} \boxtimes \mathcal{L}) \to \mathrm{H}^0(\mathcal{G}, j^*(\mathcal{L} \boxtimes \mathcal{L})),$$

where $j : \mathcal{G} \to E \times E$ is the inclusion. Thus, $R = \mathrm{H}^0(E \times E, \mathcal{O}_{E \times E}(-\Gamma))$.

LEMMA 5.2. *Let E be an elliptic curve over k, $\sigma \in \mathrm{Aut}_k(E)$ a translation, and $\mathcal{L}$ and $\mathcal{L}'$ line bundles on E of the same degree. Then $B(E, \sigma, \mathcal{L}) \cong B(E, \sigma, \mathcal{L}')$.*

PROOF. Because the B-construction is functorial, it suffices to find an isomorphism of triples $(f, u) : (E, \sigma, \mathcal{L}) \to (E', \sigma, \mathcal{L})$. Hence we need a morphism $f : E \to E$ such that $f \circ \sigma = \sigma \circ f$ and $f^*\mathcal{L}' \cong \mathcal{L}$.

Let $\sum_{i=1}^r m_i(p_i)$ and $\sum_{j=1}^s n_j(q_j)$ be the divisors associated to $\mathcal{L}$ and $\mathcal{L}'$. Since E is a divisible group, we may choose $\tau \in E$ such that $\sum_{i=1}^r m_i p_i + (\sum_{i=1}^r m_i)\tau = \sum_{j=1}^s n_j q_j$. Now define $f : E \to E$ by $f(p) = p+\tau$. If $D \in \mathrm{Div}(E)$, then $f^*\mathcal{O}_E(D) = \mathcal{O}_E(f^{-1}D)$, so $f^*\mathcal{L}' \cong \mathcal{L}$. Finally f and σ commute since they are translations. □

$A(E, -\tau) \cong A(E, \tau)^{\mathrm{op}}$

$\boxed{\text{Paul}}$ What if σ is not a translation?

The next result is *not* saying that a 3-dimensional Sklyanin algebra is a quantum plane; that is the substance of Theorem 5.4 which follows the Proposition.

PROPOSITION 5.3. *Let A be a quantum plane, and (E, σ) the associated geometric data. If E is an elliptic curve, and σ is translation by τ, then*

$$A \cong A(E, \tau).$$

PROOF. By Corollary 3.3, there is a graded algebra homomorphism $A \to B(E, \sigma, \mathcal{L})$ where $\deg(\mathcal{L}) = 3$. By Theorem 22.6.7, B is generated in degree one, so this map is surjective. Moreover, the defining relations for A are the quadratic relations for B. But these are the elements of the kernel of the map

$$\mathrm{H}^0(\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{L}) \to \mathrm{H}^0(\mathcal{L} \otimes \mathcal{L}^\sigma). \tag{5-1}$$

The results now follow by the comments after Definition 5.1. □

THEOREM 5.4. *Let $A = A(E,\tau)$ be a 3-dimensional Sklyanin algebra. Then*

1. *A is a quantum $\mathbb{P}^2$;*
2. *there is a regular normalizing element $g \in A_3$ such that*

$$A/(g) \cong B(E,\sigma,\mathcal{L}),$$

where σ is translation by τ, and $\mathcal{L}$ is a degree 3 line bundle on E;

3. *A is a domain.*

PROOF. (2) By definition of $A(E,\tau)$ there is a map $A(E,\tau) \to B(E,\sigma,\mathcal{L})$, and the defining relations for A are the quadratic relations for B. This map is surjective in degree one. By Theorem 7.6.7, $B(E,\sigma,\mathcal{L})$ is generated in degree one and is defined by its quadratic relations and a single cubic relation, so $B \cong A/(g)$ for some $g \in A_3$.

Write $B = T(B_1)/J$. In proving Theorem 22.6.7, we showed that $J_4 = T_1J_3 + J_2T_2$; a similar argument may be used to show that $J_4 = J_3T_1 + T_2J_2$ also. The equality $T_1J_3 + J_2T_2 = J_3T_1 + T_2J_2$ implies that g is a normalizing element of A. The other part of (2), that g is regular, will be established in the course of proving (1), which we do next.

(1) The minimal projective resolution of ${}_Ak$ begins

$$A[-2]^3 \xrightarrow{L} A[-1]^3 \xrightarrow{\underline{x}} A \to k \to 0,$$

where $L \in M_3(A_1)$ and $\underline{x} = (x_1,x_2,x_3)^{\mathsf{T}}$ for some x_i which span A_1. By Theorem 22.6.7, $\dim(B_3) = 9$, so $\dim(A_3) = 10$. Hence the degree 3 part of the above exact sequence is $(k^3)^3 \to (k^6)^3 \to k^{10} \to 0$. Therefore the first of these maps is not injective; that is, there exists $0 \neq \underline{y}^{\mathsf{T}} = (y_1,y_2,y_3) \in A_1^3$ such that $\underline{y}L = 0$. Hence there is a complex

$$0 \to A[-3] \xrightarrow{\underline{y}^{\mathsf{T}}} A[-2]^3 \xrightarrow{L} A[-1]^3 \xrightarrow{\underline{x}} A \to k \to 0, \tag{5-2}$$

which we refer to as a 'potential resolution' of ${}_Ak$. In fact, we will show that (5-2) *is* exact, from which it follows that $\operatorname{gldim}(A) = 3$, and $H_A(t) = (1-t)^{-3}$. Let P and Q denote the homology groups at positions $A[-3]$ and $A[-2]^3$ in (5-2).

Write $(1-t)^{-3} = \sum_{n=0}^{\infty} c_nt^n$. Theorem 22.6.7 showed that the Hilbert series of B is the same as that of a quotient of the polynomial ring in three variables by an ideal generated by a non-zero cubic form. Hence $H_B(t) = (1-t^3)(1-t)^{-3}$; that is,

$$\dim_k(B_n) = c_n - c_{n-3}. \tag{5-3}$$

Define $Z = \{a \in A \mid ga = 0\}$ and $Z' = \{a \in A \mid ag = 0\}$. Left multiplication by g gives an exact sequence

$$0 \to Z_{n-3} \to A_{n-3} \to A_n \to B_n \to 0; \tag{5-4}$$

there is a similar exact sequence involving Z', so Z and Z' have the same Hilbert series.

Define

$$\begin{aligned} a_n &= \dim_k(A_n), \\ b_n &= \dim_k(B_n), \\ p_n &= \dim_k(P_n), \\ q_n &= \dim_k(Q_n), \\ z_n &= \dim_k(Z_n), \\ \delta_n &= a_n - c_n. \end{aligned}$$

Our goal is to prove that $p_n = q_n = z_n = \delta_n = 0$ for all $n \geq 0$. This will show that (5-2) is exact, that $H_A(t) = (1-t)^{-3}$, and that g is right regular, and hence left regular.

From (5-2), we obtain

$$a_{n-3} - 3a_{n-2} + 3a_{n-1} - a_n + \delta_{n0} = p_{n-3} - q_{n-2}. \tag{5-5}$$

Either by direct calculation, or by appealing to the exactness of the Koszul complex for the polynomial ring, we have

$$c_{n-3} - 3c_{n-2} + 3c_{n-1} - c_n + \delta_{n0} = 0. \tag{5-6}$$

Subtracting (5-6) from (5-5) gives

$$\delta_{n-3} - 3\delta_{n-2} + 3\delta_{n-1} - \delta_n = p_{n-3} - q_{n-2}. \tag{5-7}$$

It follows from (5-4) and (5-3) that $0 = z_{n-3} - a_{n-3} + a_n - (c_n - c_{n-3})$, whence

$$z_{n-3} = \delta_{n-3} - \delta_n. \tag{5-8}$$

Combining (5-7) and (5-8) gives

$$z_{n-3} - 3\delta_{n-2} + 3\delta_{n-1} = p_{n-3} - q_{n-2}. \tag{5-9}$$

We will prove by induction on n that the statement

$$S_n \quad : \quad \begin{cases} p_i = 0 & \text{if } i \leq n-3, \\ q_i = 0 & \text{if } i \leq n-2, \\ z_i = 0 & \text{if } i \leq n-3, \\ \delta_i = 0 & \text{if } i \leq n-1 \end{cases}$$

holds for all $n \in \mathbb{Z}$. For $n \leq 0$ it holds trivially.

The key step is to observe that, if $z_{j-2} = p_{j-3} = 0$, then $p_j = 0$. To see this, let $u \in P_j$. Then $uy_1 = uy_2 = uy_3 = 0$ in A. But B is a domain, so $u \in (g) = gA$; say $u = gv$, $v \in A_{j-3}$. Thus $gvy_1 = gvy_2 = gvy_3 = 0$, with each $vy_i \in A_{j-2}$. But $Z_{j-2} = 0$, so each $vy_i = 0$; that is, $v \in P_{j-3}$. But $P_{j-3} = 0$, so $v = 0$, whence $u = 0$.

Now assume that S_n holds. From (5-8), $\delta_n = 0$. From (5-9), with $n+1$ in place of n, $z_{n-2} = p_{n-2} - q_{n-1}$. But $p_{n-2} = 0$ by the previous paragraph, so $z_{n-2} = -q_{n-1}$. However, these numbers are vector space dimensions, so non-negative, whence $z_{n-2} = q_{n-1} = 0$. Therefore S_{n+1} holds.

We have now shown that g is regular, that $H_A(t) = (1-t)^{-3}$, and that (5-2) is exact, whence $\operatorname{gldim}(A) = 3$. To complete the proof that A is a quantum plane, it remains to verify the Gorenstein condition.

$\boxed{\text{DETAILS}}$

(3) Since g is regular, normalizing of positive degree, and $A/(g)$ is a domain, it follows that A is a domain. □

Relations for $A(E, \tau)$

One may choose generators for such that $A(E, \tau) = k[x, y, z]$ with the following 3 defining relations:

$$\begin{aligned} ax^2 + byz + czy &= 0 \\ ay^2 + bzx + cxz &= 0 \\ az^2 + bxy + cyx &= 0, \end{aligned}$$

where $(a, b, c) \in \mathbb{P}^2$ depends on the pair (E, τ).

6. Points and lines in a quantum plane

The reader may have already noticed that the point modules for a quantum plane have already been classified, at least implicitly, in Section 3.

PROPOSITION 6.1. *Let A be a quantum plane. Then there exists a subscheme $E \subset \mathbb{P}(A_1^*)$ and an automorphism σ of E such that the point module sequences for A are*

$$p, p^\sigma, p^{\sigma^2}, \dots$$

for $p \in E$. Furthermore, E is either a cubic divisor, or equal to $\mathbb{P}(A_1^*)$.

PROOF. Theorem 23.3.1 classifes the point modules: they correspond to point module sequences $p, p^\sigma, p^{\sigma^2}, \dots$ where $p \in E$. □

Definition 6.2. A line module over a graded k-algebra is a cyclic, graded module M such that $H_M(t) = (1-t)^{-2}$. That is, M has the same Hilbert series as the commutative polynomial ring on 2 indeterminates (which is the homogeneous coordinate ring of the projective line, thus accounting for the terminology).

PROPOSITION 6.3. *Let A be the homogeneous coordinate ring of a closed subvariety $X \subset \mathbb{P}^n$. There is a bijection between*

$$\{\textit{line modules for } A\} \leftrightarrow \{\textit{lines in } \mathbb{P}^n \textit{ which lie on } X\}.$$

PROOF. First consider $\mathbb{P}^n$ itself and its homogeneous coordinate ring $R = k[x_0, \dots, x_n]$. If M is a line module for R then, since $\dim(R_1) = n+1$ and $\dim(M_1) = 2$, there is a surjection

$$R/(y_1, \dots, y_{n-1}) \to M,$$

for some linearly independent $y_1, \dots, y_{n-1} \in R_1$. However, this quotient ring of R is a polynomial ring on 2 variables, so has the same Hilbert series as M, so this map is an isomorphism. We associate to M the line $\mathcal{V}(y_1, \dots, y_{n-1})$. The bijection is now clear for this case.

Now suppose that $A = R/I$. Then the line modules for A are precisely those line modules for R which are annihilated by I. Since I is the ideal of X, the line module corresponding to a line ℓ is an A-module if and only if $\ell \subset X$. This gives the bijection for A. □

$\boxed{\text{Paul}}$ In the next Lemma, we use the fact that A is a domain, even though we have not proved it yet; to prove it we must first show that A has good homological properties. It is consequence of the Auslander-Gorenstein property - we need to know A is noetherian to apply this result.

LEMMA 6.4. *If A is a quantum $\mathbb{P}^2$ there is a bijection between the isomorphism classes of line modules for A and the lines in $\mathbb{P}(A_1^*)$.*

PROOF. Since A is a domain, if $0 \neq a \in A_1$, A/Aa is a line module. Conversely, if M is a line module then, since $\dim(A_1) = 3$ and $\dim(M_1) = 2$, there is a surjective map $A/Aa \to M$ for some $0 \neq a \in A_1$. Since these modules have the same Hilbert series it follows that the line modules for A are precisely the modules A/Aa for $0 \neq a \in A_1$. By associating to this line module the line $\mathcal{V}(a)$ in $\mathbb{P}(A_1^*)$ we establish the bijection. $\square$

NOTATION . We will write $M(p)$ for the point module corresponding to $p \in E$. That is, $M(p)$ has the point module sequence $p, p^\sigma, \dots$. If $\ell = \mathcal{V}(a)$ is a line in $\mathbb{P}(A_1^*)$, we will write $M(\ell)$ for the corresponding line module.

Definition 6.5. Let $\mathcal{F} \in \operatorname{Proj}(A)$ be a point, and ℓ a line in $\mathbb{P}(A_1^*)$. We say that $\mathcal{F}$ lies on the line ℓ if there is a non-zero degree zero A-module homomorphism $\varphi : M(\ell) \to M$, for some 1-critical $M \in \mathsf{grmod}(A)$ such that $\pi M \cong \mathcal{F}$.

PROPOSITION 6.6. *Let $p \in E$ and ℓ a line in $\mathbb{P}(A_1^*)$. Then $\pi M(p)$ lies on ℓ if and only if $p \in \ell$.*

PROOF. Let the line module in question be $M(\ell) = A/Aa$. There is a non-zero map $M(\ell) \to M(p)$ if and only if $a(p) = 0$; i.e., if and only if $p \in \ell$. $\square$

PROPOSITION 6.7. *Let ℓ be a line in $\mathbb{P}(A_1^*)$ and suppose that $p \in \ell \cap E$. Then there is a short exact sequence*

$$0 \to M(\ell')[-1] \to M(\ell) \to M(p)$$

for some line ℓ'.

PROOF. By Chapter 23, Theorem 3.1, $M(p) \cong A/Ax + Ay$ where $x, y \in A_1$ satisfy $\mathcal{V}(x, y) = p$. Without loss of generality, $\ell = \mathcal{V}(x)$. Hence there is a short exact sequence $0 \to A/J[-1] \to M(\ell) \to M(p) \to 0$, where $J = \{a \in A \mid ay \in Ax\}$ and $H_{A/J}(t) = (1-t)^{-2}$. Therefore, to prove the result it suffices to show that A/J is a quotient of a line module. i.e., that J contains a non-zero element of degree 1. But $\dim(A/Ax + Ay)_2 \geq 1$, and A is a domain, so $A_1 x \cap A_1 y \neq 0$, whence $J \cap A_1 \neq 0$, as required. $\square$

PROPOSITION 6.8. *Every point in $\operatorname{Proj}(A)$ lies on a line.*

It would not be unreasonable to think of the other class of algebras in Theorem 7.3.1. as quantum versions of quadric surfaces in $\mathbb{P}^3$.

7. Quantum $\mathbb{P}^3$s

CHAPTER 25

Non-commutative Curves

Throughout this chapter k is an algebraically closed field.

A basic goal of non-commutative algebraic geometry is to classify and describe the structure of curves defined over k. We begin with a disclaimer. It is not yet clear what the boundary of the definition of a curve should be. Nevertheless, we know many examples which should fall within the scope of any reasonable definition. For example, any finitely presented k-algebra of GK-dimension one will be the coordinate ring of an affine curve. If A is a graded noetherian domain of GK-dimension 2, then $\operatorname{Proj} A$ should be a curve.

1. Basic facts and definitions

We must exclude the first Weyl algebra – this is a surface.

2. Rings of GK-dimension 1

All rings in this section are algebras over a field k.

LEMMA 2.1. *Let V be a finite dimensional frame for a k-algebra R. Let W_1 and W_2 be frames for a a finitely generated R-module M, and define $g_i(n) = \dim_k(V^nW_i)$. Then*

1. *there is an integer r such that $g_1(n-r) \le g_2(n) \le g_1(n+r)$ for all n;*
2. *if $\dim_k M = \infty$, then $g_i(n)$ is strictly increasing;*
3. *if $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathsf{mod}R$, and g_L, g_M, g_N are the growth functions associated to any subframes, then there exists $r \in \mathbb{Z}$ such that $g_M(n) \ge g_L(n-r) + g_N(n-r)$ for all n.*

PROOF. (1) Since $W_2 \subset V^rW_1$ for some r, one has $V^nW_2 \subset V^{n+r}W_1$ for all n, which implies $g_2(n) \le g_1(n+r)$. Now switch the roles of W_1 and W_2.

(2) This is obvious since $V^nW = V^{n+1}W$ implies that $M = V^nW$, and hence $\dim_k M < \infty$.

(3) By (1) the statement to be proved does not depend on the choice of frames. Fix a frame W in M, and let $\bar{W}$ be its image in N. Let g_M and g_N be the growth functions for these two frames. If we define $L_n = L \cap V^nW$, then $0 \to L_n \to V^nW \to V^n\bar{W} \to 0$ is exact, so $g_M(n) = g_N(n) + \dim_k L_n$. Therefore, it suffices to show that $\dim_k L_n \ge g_L(n-r) = \dim_k(V^{n-r}U)$ for some frame $U \subset L$. Since $U \subset M$, $U \subset V^rW$ for some r, whence $V^nU \subset V^{n+r}W \cap L$; thus $g_L(n) \le \dim_k L_{n+r}$, as required. $\square$

COROLLARY 2.2. *Let R be a finitely generated k-algebra and suppose that the growth function of $M \in \mathsf{mod}R$ is bounded by a linear function. If $0 = M_0 \subset M_1 \subset \cdots$ is a chain of submodules of M, then $\dim_k(M_{i+1}/M_i) < \infty$ for large i.*

Proof. Suppose the result is false. Then there is a chain of submodules with each $N_i = M_i/M_{i-1}$ infinite dimensional. Thus $g_{N_i}(n) \geq n$, so $g_{M_i}(n) \geq g_{M_{i-1}}(n-r) + (n-r)$ for some r. Thus, by induction $g_{M_i}(n)$ is not bounded above by in. In particular, there is no c such that $g_M(n) \leq cn$ for all n, contradicting the hypothesis. □

Lemma 2.3. *Let $I \subset J$ be left ideals in a prime k-algebra R. Suppose that I is a left annihilator and $\dim_k(J/I) < \infty$. Then either $I = J$ or $\dim_k R < \infty$.*

Proof. Let $I = \ell\mathrm{ann}(X)$. If $I \neq J$, then there is $x \in X$ such that $Jx \neq 0$. But Jx is a quotient of J/I, so is finite dimensional. Let $A = \ell\mathrm{ann}(Jx)$ and $B = \text{r-ann}\, A$. Then $\dim_k(R/A) < \infty$, and $AB = 0$. But $Jx \subset B$, so $B \neq 0$, whence $A = 0$. Thus R embeds in $\mathrm{End}_k(Jx)$, so is finite dimensional. □

Proposition 2.4. *A finitely generated prime k-algebra of GK-dimension 1 is a Goldie ring.*

Proof. □

Theorem 2.5. [**154**] *Let R be a finitely generated k-algebra of GK-dimension 1. Then R satisfies a polynomial identity. If R is also semiprime, then it is noetherian and a finite module over its center (which is also noetherian).*

Theorem 2.6. *Let A be a finitely generated, $\mathbb{N}$-graded algebra over an algebraically closed field k. If A is prime of GK-dimension 1, then*

$$\mathrm{Fract}_{\mathrm{Gr}}\, A \cong M_n(k[x, x^{-1}])(\underline{d}).$$

Proof. By Theorem 2.5, the center of A contains an element of positive degree, which is necessarily regular since A is prime. Hence, by Theorem 9.3, $Q := \mathrm{Fract}_{\mathrm{Gr}}\, A$ exists, and $Q \cong M_n(D)(\underline{d})$ for some graded division algebra D. Since $A_n \neq 0$ for infinitely many n, D is not concentrated in degree zero. Hence, by Proposition 16.7.2, $D \cong D_0[x, x^{-1}; \sigma]$ is a skew Laurent extension of some division algebra D. However, since $\mathrm{GKdim}\, A = 1$, $\dim_k D_0 < \infty$, which forces $D_0 = k$ since k is algebraically closed. Hence $\sigma = \mathbb{1}$ also, which completes the proof. □

In the situation which is relevant to us, the classical notion of torsion coincides with the notion introduced in Chapter 19.

Lemma 2.7. *Let A be a prime, noetherian, locally finite, $\mathbb{N}$-graded k-algebra of GK-dimension 1. Then*

1. *If $x \in A$ is homogeneous of positive degree, then x is regular if and only if $\dim_k A/Ax < \infty$;*
2. *if M is a graded A-module, then M is torsion-free in the classical sense if and only if it is torsion-free in the sense of Chapter 19.*

Proof. (1) ($\Rightarrow$)

□

EXERCISES

2.1 Let k be an algebraically closed field, and let A be a graded k-algebra with $A_0 = k$. Suppose that A is a commutative domain of GK-dimension 1. Show that $A = k[x^{i_1}, \dots, x^{i_n}] \subset k[x, x^{-1}]$ for some integers $i_1, \dots, i_n$. Furthermore, if A has a unit of non-zero degree, then $A \cong k[x, x^{-1}]$. [*Hint:* Consider a subalgebra $k[x, y]$ of A generated by homogeneous elements x and y of non-zero degree.]

3. Graded noetherian domains of GK-dimension 2

4. Graded noetherian prime rings of GK-dimension 2

5. The curves of Smith and Zhang

6. Curves arising from finite dimensional algebras

We begin with a provocative example.

EXAMPLE 6.1. There is a left exact functor

$$\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1), -) : \mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}^{\mathrm{op}}.$$

Since $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, and $\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(-1), \mathcal{O}) \cong k^2$, it is clear that

$$\operatorname{End}_{\mathbb{P}^1} \mathcal{O} \oplus \mathcal{O}(1) \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix},$$

so there is a functor as claimed. Denote the functor by F. We write $F\mathcal{M}$ as a row vector

$$F\mathcal{M} = (\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{M}), (\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(-1), \mathcal{M}))$$

and view $\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$ as acting by right multiplication. The k^2 should be viewed as $kx + ky$ where x and y are homogeneous coordinate functions on $\mathbb{P}^1$, and multiplication by them gives a pair of maps

$$\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(-1), \mathcal{M}) \to \operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{M}).$$

Thus F sends $\mathsf{Mod}\mathbb{P}^1$ to representations of the quiver

$$\bullet \;\; \substack{\to \\ \to} \;\; \bullet.$$

One computes that

$$F\mathcal{O}(n) \cong \begin{pmatrix} k^n \\ k^{n+1} \end{pmatrix}$$

The functor F commutes with products because it is a Hom-functor, so has a left adjoint. This adjoint sends modules over the finite dimensional algebra to $\mathbb{P}^1$-modules. In this section we give a result which associates to each finite dimensional algebra R (satisfying some hypotheses) a curve X, and a right exact functor $\mathsf{Mod}R \to \mathsf{Mod}X$.

$\boxed{\text{Paul}}$ 1. Compare example with description in [**17**] of indecomposable modules over $\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$ Use results in section 2 of chapter 21 which classify the irreducible $\mathbb{P}^1$-modules.

2. Consider the example. Set $\mathcal{L} = \mathcal{O} \oplus \mathcal{O}(-1)$, and let $s = (1)$. What is $B(\mathbb{P}^1, \mathcal{L}, s)$? WE have

$$B_n = \begin{pmatrix} k^{n+1} & k^{n+2} \\ k^n & k^{n+1} \end{pmatrix}$$

so maybe $B \cong \begin{pmatrix} k[x,y] & k[x,y]//(x) & k[x,y] \end{pmatrix}$. Work it out!

The Auslander-Reiten translation is the functor

$$\tau = D \circ \mathrm{Tr}$$

where

$$D = \mathrm{Hom}_k(-, k) : \mathsf{mod}\Lambda^{\mathrm{op}} \to \mathsf{mod}\Lambda,$$

and

$$\mathrm{Tr}\, M : \mathsf{mod}\Lambda \to \mathsf{mod}L^{\mathrm{op}}$$

We follow [**20**]. Let Λ be a finite dimensional hereditary k-algebra. We will assume that Λ has infinite representation type. Let $P_1, \dots, p_n$ be the indecomposable projective Λ-modules. The post-projective Λ-modules are those which are direct sums of the modules $\tau^{-n} P_i$ where $n \geq 0$; thus an indecomposable module M is post-projective if and only if $\tau^n M$ is a non-zero projective for some $n > 0$; thus an arbitrary module M is post-projective if $\tau^n M = 0$ for some $n \geq 0$. A more intrinsic description is that a module P is post-projective if $\mathrm{Hom}_\Lambda(M, P) \neq 0$ for only finitely many indecomposable M. A module M is pre-injective if $\tau^{-n} M = 0$ for some $n \geq 0$.

The functor $D = \mathrm{Hom}_k(-, k)$ is a duality from the full subcategory of $\mathsf{mod}\Lambda$ consisting of the post-projectives to the full subcategory of $\mathsf{mod}\Lambda^{\mathrm{op}}$ consisting of the pre-injectives.

Warning. What we call post-projective used to be called pre-projective! The reason for the change is the following. In drawing the Auslander-Reiten quiver one draws the arrows going from left to right, corresponding to the almost split sequences $0 \to \tau M \to E \to M \to 0$; thus non-zero maps of Λ-modules can only go from a module to something appearing to the right of it in the AR-quiver; now the projectives must, of necessity, appear at the left-hand edge of the AR-quiver, and the injectives at the right-hand edge; the AR-quiver is composed of three basic pieces, those in the left hand piece appear to the right of the projectives, and are called post-projectives, those on right appear to the left of the injectives, and are called pre-injectives, while those in the middle are called regular.

There is a distinguished group homomorphism

$$\mathrm{rank} : K_0(\Lambda) \to \mathbb{Z}$$

called the rank. An indecomposable module M is post-projective, regular, or pre-injective according as $\mathrm{rank}\, M$ is > 0, $= 0$, or < 0. Sometimes the closure under direct sums of these indecomposables are denoted by $\mathsf{mod}^+\Lambda$, $\mathsf{mod}^0\Lambda$, and $\mathsf{mod}^-\Lambda$. The regular part, $\mathsf{mod}^0\Lambda$ decomposes into an infinite direct sum of uniserial categories $R_x(\Lambda)$; each $R_x(\Lambda)$ has only finitely many simple objects called the simple regular modules.

We follow [**17**, Chapter VIII]. If Λ has infinite representation type there are indecomposables which are neither post-projective or pre-injective. These are called regular modules; more generally is all the indecomposable summands of M are regular, we call M regular. Write $\mathsf{reg}\Lambda$ for the full subcategory of $\mathsf{mod}\Lambda$ consisting of the regular modules.

PROPOSITION 6.2. [**17**, page 280] *Let Λ be a hereditary artin algebra of infinite representation type. Then τ and τ^{-1} are mutually inverse auto-equivalences of $\mathsf{reg}\Lambda$. They preserve exact sequences, irreducible morphisms, and almost split sequences.*

Because Λ is hereditary, $\tau^{-1}M \cong \operatorname{Ext}^1_\Lambda(DM,\Lambda)$. Define the post-projective algebra of Λ to be

$$B = \oplus_{n\geq 0} \operatorname{Hom}_\Lambda(\Lambda, \tau^{-n}\Lambda);$$

this is a $\mathbb{Z}$-graded algebra in the usual way. Thus $B_0 \cong \Lambda$. In [**20**, Proposition 3.1] it is shown that $B(\Lambda)$ is isomorphic to the tensor algebra of the Λ-Λ-bimodule $B_1 \cong \operatorname{Ext}^1_\Lambda(D\Lambda,\Lambda)$.

THEOREM 6.3. [**20**, Theorem 6.5] *The post-projective algebra $B(\Lambda)$ is a finitely generated, prime, noetherian ring of global dimension, and Krull dimension two, and satisfies a polynomial identity.*

There is a bijection between the height one homogeneous prime ideals in $B(\Lambda)$ and the regular components $R_x(\Lambda)$.

Assume that Λ is tame, hereditary, connected. Let F denote the category of finitely presented covariant functors $\mathsf{mod}^+\Lambda \to \mathsf{Ab}$. Let F_0 denote the full subcategory consisting of the finite length objects. Denote by $(M,-]$ and $\operatorname{Ext}^1(M,-]$ the restrictions of the functors $\operatorname{Hom}_\Lambda(M,-)$ and $\operatorname{Ext}^1_\Lambda(M,-)$ to $\mathsf{mod}^+\Lambda$. The category of finite length objects in F/F_0 is equivalent to $\mathsf{mod}^0\Lambda$ via the functor $R \mapsto \operatorname{Ext}^1(DR,-]$.

CHAPTER 26

Commutative surfaces

A central theme in the development of many branches of mathematics is classification of small, or low-dimensional objects. In algebraic geometry the classification of curves, surfaces, 3-folds, et cetera, has played just such a role. The analogous problem for non-commutative algebraic geometry is also proving fruitful. Before we turn to non-commutative surfaces in the next chapter, we recall the basic facts concerning commutative surfaces.

Let k be an algebraically closed field of characteristic zero. If K is a finitely generated extension field of k, Hironaka's Theorem tells us that there is a smooth projective variety X such that $K \cong k(X)$. (This is not known if $\operatorname{char} k > 0$, but for $\operatorname{trdeg} K \leq 2$ it remains true.) The variety X is called a smooth model for K. If $\operatorname{trdeg} K = 1$, K has a *unique* smooth model. If $\operatorname{trdeg} K \geq 2$, this is no longer true. For example, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ have the same function field, but are not isomorphic. Varieties X and Y are birationally equivalent, birational for short, if $k(X) \cong k(Y)$. It is elementary to show that X and Y are birational if and only if they have isomorphic dense open subvarieties; a morphism between dense open subvarieties is often denoted by a dotted arrow $X - \to Y$.

1. Intersection theory for surfaces

A basic problem is to understand how curves on a surface intersect; for example, what minimal data is needed to compute the number of points of intersection of two curves? For example, Bézout's theorem says that two curves C and D on $\mathbb{P}^2$ meet at $\deg C \cdot \deg D$ points provided the curves have no common component, and intersections are counted with appropriate multiplicity; thus, one only needs to know the degree. Bézout's theorem can be expressed in terms of a function

$$\operatorname{Div} \mathbb{P}^2 \times \operatorname{Div} \mathbb{P}^2 \to \mathbb{Z},$$

the intersection pairing; to be precise, there is a symmetric bilinear map $(C, D) \mapsto C \cdot D := \deg C \cdot \deg D$ which gives the number of intersection points when C and D have no common component. More generally, we have the following result.

Theorem 1.1. *Let X be a smooth projective surface over an algebraically closed field k. There is a unique map*

$$\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}, \quad (C, D) \mapsto C \cdot D,$$

such that

1. *if C and D are smooth curves meeting transversally, then $C \cdot D$ is the number of points in $C \cap D$;*
2. *$C \cdot D = D \cdot C$;*
3. *$(C + C') \cdot D = C \cdot D + C' \cdot D$;*
4. *if C and C' are linearly equivalent, then $C \cdot D = C' \cdot D$.*

The only condition needing further explanation is the last one. Think of the surface X embedded in some projective space, and suppose that C is cut out by a hyperplane H. As H moves to a new hyperplane H', it meets X at a new curve C'; however, $C \cap D = H \cap D$ and $C' \cap D = H' \cap D$, so the number of intersection points is simply the degree of D in the ambient projective space; thus the cardinality of $C \cap D$ should stay constant as C moves within a fixed linear equivalence class.

The key step in proving the theorem is the following.

LEMMA 1.2. *Let X be a smooth projective surface over an algebraically closed field k. Let C be a smooth irreducible curve on X, and let D be any curve meeting C transversally. Then*

$$|C \cap D| = \deg \mathcal{O}_X(D)|_C = \deg \mathcal{O}_C \otimes \mathcal{O}_X(D).$$

This result shows how to define the intersection pairing in such a way that it extends to $\operatorname{Div} X$. Thus, for any curve C, and any divisor D, we define

$$C \cdot D := \deg \mathcal{O}_C \otimes \mathcal{O}_X(D).$$

Having defined the intersection pairing, we can define the self-intersection of a curve, namely $c^2 = C \cdot C$. It is possible for a curve (i.e., an effective divisor) to have negative self-intersection. For example, each of the 27 lines on a cubic surface satisfies $C^2 = -1$; such a curve is sometimes called a -1-curve or an exceptional curve. Such curves are central to blowing up and down.

Let C be a curve on X with defining ideal $\mathcal{I} \cong \mathcal{O}(-C)$. Write $i : C \to X$ for the inclusion. Thus there is a short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$. (To be precise we should write $i_*\mathcal{O}_C$ not $\mathcal{O}_C$.) Tensoring this with $\mathcal{O}(-C)$, we see that $\mathcal{O}(-C) \otimes \mathcal{O}_C \cong \mathcal{I}/\mathcal{I}^2$, which is the conormal bundle of the embedding $C \to X$. The normal bundle is, by definition, the dual of the conormal bundle; that is,

$$\begin{aligned}
\mathcal{N}_{C/X} =&\mathcal{H}om_C(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \\
\cong&\mathcal{H}om_X(\mathcal{O}(-C) \otimes \mathcal{O}_C, \mathcal{O}_C) \\
\cong&\mathcal{H}om_X(\mathcal{O}_C, \mathcal{O}(C) \otimes \mathcal{O}_C) \\
\cong&\mathcal{O}(C) \otimes \mathcal{O}_C.
\end{aligned}$$

Thus $C^2 = \deg \mathcal{N}_{C/X} = -\deg \mathcal{I}/\mathcal{I}^2$.

2. Blowing up and blowing down

THEOREM 2.1. *Let X be a smooth surface and $p \in X$. Then there is a surface $\tilde{X}$ and a morphism $\pi : \tilde{X} \to X$ with the following properties:*

1. *$\tilde{X}$ is smooth;*
2. *$E = \pi^{-1}(p) \cong \mathbb{P}^1$;*
3. *the restriction of π gives an isomorphism $\tilde{X} \backslash E \to X \backslash \{p\}$;*
4. *$E^2 = -1$;*
5. *$K_{\tilde{X}} = \pi^* K_X + E$.*

Properties (1), (2), and (3) determine $\tilde{X}$ up to isomorphism.

The surface $\tilde{X}$ is called the blowup of X at p, and E is called the exceptional curve.

THEOREM 2.2 (Castelnuovo). *Let X be a smooth projective surface over an algebraically closed field k. If $C \cong \mathbb{P}^1$ is a -1-curve on X, then there is a smooth* *surface X and a morphism $f : X \to X'$ which sends C to a single point, and is an isomorphism outside C.*

Theorem 2.3. [**76**, Chapter V, Theorem 3.2] *Let p be a point on a smooth surface X, and let $\pi : \tilde{X} \to X$ be the blowup at p. Then*

1. $\operatorname{Pic}\tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}.E$*;*
2. $\pi^*C.\pi^*D = C.D$ *if* $C, D \in \operatorname{Pic} X$*;*
3. $\pi^*C.E = 0$ *if* $C \in \operatorname{Pic} X$*;*
4. *if* $\pi_* : \operatorname{Pic}\tilde{X} \to \operatorname{Pic} X$ *denotes projection to the first factor, then* $\pi^*C.D = C.\pi_*D$ *for* $C \in \operatorname{Pic} X$ *and* $D \in \operatorname{Pic}\tilde{X}$*.*

Let $\pi : \tilde{X} \to X$ be the blowup at p. If C is an irreducible curve on X, then $\pi^{-1}(C)$ is the union of E (provided $p \in C$) and one other irreducible component which we denote by $\tilde{C}$ and call the strict transform of C. Let C and D be curves on X passing through the point p; in $\operatorname{Pic}\tilde{X}$, we have $\pi^*C = \tilde{C} + E$, so

$$\tilde{C}.\tilde{D} = (\pi^*C - E).(\pi^*D - E) = C.D - 1.$$

In particular, $\tilde{X}$ may have other (-1)-curves than E, namely those $\tilde{C}$ where $C^2 = 0$. For example, if L_1 and L_2 are the two lines on $X = \mathbb{P}^1 \times \mathbb{P}^1$ passing through p, then $\tilde{L}_i{}^2 = -1$, so there are three exceptional curves on $\tilde{X}$, namely $\tilde{L}_1$, $\tilde{L}_2$, and E. Notice that each $\tilde{L}_i$ meets E, but $\tilde{L}_1 \cap \tilde{L}_2 = \emptyset$. Blowing down $\tilde{L}_1$ and $\tilde{L}_2$ successively gives $\mathbb{P}^2$.

Theorem 2.4. *If X and Y are birationally equivalent smooth projective surfaces, then there are sequences of blow ups*

$$X \leftarrow X_1 \leftarrow ... \leftarrow X_n$$

and

$$Y \leftarrow Y_1 \leftarrow ... \leftarrow Y_m$$

such that $X_n \cong Y_m$.

3. Classification

There are several possibilities as to what a classification of surfaces might mean. For example, one might begin by classifying smooth projective surfaces, ignoring the problem of describing how the singular surfaces are gotten from the smooth ones. A fundamental result in the classification of curves is the following.

Definition 3.1. We say that X and Y are birationally equivalent, or simply birational, if $k(X) \cong k(Y)$. We say that X is rational if X is birational to a projective space, i.e., $k(X) \cong k(\mathbb{P}^n) = k(\mathbb{A}^n) = \operatorname{Fract} k[T_1, ..., T_n]$ for some n.

Theorem 3.2. *Let K be a finitely generated field over k. If $\operatorname{trdeg}_k K = 1$, there is a unique smooth projective curve X such that $K \cong k(X)$.*

This can be rephrased as follows.

Corollary 3.3. *(a) There is a bijection between transcendence degree 1 field extensions of k and isomorphism classes of smooth projective curves over k.*

(b) Two smooth projective curves X and Y are birationally equivalent if and only if they are isomorphic.

There is no analogous result for surfaces as the next example shows.

LEMMA 3.4. *$\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent but not isomorphic.*

PROOF. Note that $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q := \mathcal{V}(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$. Define a rational map $\varphi : Q - \to \mathbb{P}^2$ by $\varphi(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2)$; φ is defined at all points of Q except $p = (0, 0, 0, 1)$. Define $\psi : \mathbb{P}^2 - \to Q$ by $\psi(x_0, x_1, x_2) = (x_0^2, x_0x_1, x_0x_2, x_1x_2)$; ψ is defined at all points of $\mathbb{P}^2$ except $q_1 = (0, 0, 1)$ and $q_2 = (0, 1, 0)$. Since φ and ψ are mutually inverse, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally equivalent.

Any two curves on $\mathbb{P}^2$ intersect, but two distinct lines $p_1 \times \mathbb{P}^1$ and $p_2 \times \mathbb{P}^1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ do not, so $\mathbb{P}^2 \ncong \mathbb{P}^1 \times \mathbb{P}^1$. □

This suggests that one might classify surfaces in two steps: first, up to birational equivalence, then within each birational equivalence class. Theorem 2.4 explains the relationship between two surfaces in the same birational equivalence class.

For example, the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a single point is isomorphic to the $\mathbb{P}^2$ at two distinct points. The reverse process to blowing up is called blowing down, so the theorem could be restated as saying that one can pass from X to Y by a sequence of blowing ups and blowing downs.

Definition 3.5. On each birational equivalence class of smooth projective surfaces, define a partial order by $X_1 \geq X_2$ if there is a surjective morphism $X_1 \to X_2$. We call X minimal if it is minimal with respect to this partial order in its birational equivalence class.

For example, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are minimal surfaces. By the previous theorem, a surface is minimal if and only if it can't be blown down. Now the problem becomes to classify all minimal smooth surfaces, because every smooth surface is gotten by blowing up a minimal surface.

THEOREM 3.6. *Each birational equivalent class of smooth surfaces has a unique minimal model unless the class is that of $C \times \mathbb{P}^1$ for some curve C.*

Definition 3.7. Let C be a smooth curve. A ruled surface over C is a smooth surface S together with a morphism $\pi : S \to C$ such that $\pi^{-1}(p) \cong \mathbb{P}^1$ for all $p \in C$, i.e., S is a $\mathbb{P}^1$-bundle over C.

For example, $\mathbb{P}^1 \times \mathbb{P}^1$ is a ruled surface.

THEOREM 3.8.
1. *Every ruled surface is minimal (with one exception).*
2. *Ruled surfaces are all the minimal surfaces birationally equivalent to $C \times \mathbb{P}^1$ for some curve C (with one exception).*

If we blow up $\mathbb{P}^2$ at a single point, we get a ruled surface which is not minimal, which is the exception for (1). On the other hand, $\mathbb{P}^2$ is not a ruled surface but is birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$, so $\mathbb{P}^2$ is the exception for (2).

4. Ruled Surfaces

Ruled surfaces play a central role in the classification of surfaces. There is a partial ordering on each birational equivalence class: we write $S_1 \geq S_2$ and say S_1 dominates S_2 if there is a morphism $S_1 \to S_2$ inducing a birational isomorphism. Each birational class has minimal members, surfaces which do not dominate any other one; such a surface is called a minimal model. Thus every smooth surface is gotten by blowing up a minimal model some finite number of times. Now apart from the birational class of $k(C \times \mathbb{P}^1)$, every birational class has a unique minimal model! In the birational class of $C \times \mathbb{P}^1$, the minimal models are precisely the ruled surfaces, with the sole exception of the ruled surface $\mathbb{F}_1$ in the birational class of $\mathbb{P}^2$—it should be replaced by $\mathbb{P}^2$ which is gotten from $\mathbb{F}_1$ by blowing down a single -1-curve.

Definition 4.1. Let C be a smooth curve. A ruled surface over C is a surface S and a smooth morphism $\pi : S \to C$ such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$ for sufficiently small open sets $U \subset C$.

n the previous definition it is not enough to simply require all fibers $\pi^{-1}(p)$ to be isomorphic to $\mathbb{P}^1$, because this would allow the quadric cone in $\mathbb{P}^3$, namely $\mathcal{V}(x_0^2 - x_2x_3)$, to be included. If we insisted first that S be smooth, then it would be possible to simply require that the fibers be isomorphic to $\mathbb{P}^1$ (see [**25**, Definition III.3] and [**25**, Theorem III.4]). Hartshorne's definition [**76**, Chapter V] includes the requirement that π have a section, $\sigma : C \to S$ such that $\pi\sigma = \mathrm{id}_C$, but this is a consequence of the other hypotheses.

If S is a ruled surface over C, then $k(S) \cong k(C \times \mathbb{P}^1)$, so all ruled surfaces over C belong to a single birational equivalence class. Moreover, if $C \ncong \mathbb{P}^1$, then the ruled surfaces over C are precisely the minimal models of $C \times \mathbb{P}^1$ [**25**, Theorem III.10].

THEOREM 4.2. [**25**, Theorem III.7], [**76**, Proposition V.2.2] *Every ruled surface over C is isomorphic to $\mathbb{P}_C(\mathcal{E})$ for some rank 2 vector bundle $\mathcal{E}$ on C. Conversely, if $\mathcal{E}$ is such a bundle then $\mathbb{P}_C(\mathcal{E})$ is a ruled surface over C, and $\mathbb{P}_C(\mathcal{E}) \cong \mathbb{P}_C(\mathcal{E}')$ if andl only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on C.*

5. Rational surface scrolls

This section concerns the ruled surfaces over $C = \mathbb{P}^1$. The simplest one is $\mathbb{P}^1 \times \mathbb{P}^1$; it is a ruled surface in two distinct ways.

Before constructing our rational ruled surfaces as scrolls we put ourselves in the right frame of mind by recalling two constructions. First, $\mathbb{P}^n$ is the quotient $(\mathbb{A}^{n+1}\backslash\{0\})/k^\times$ for the action of the multiplicative group $k^\times$ acting by

$$\lambda.(x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n).$$

Second,, $\mathbb{P}^n \times \mathbb{P}^m$ is the quotient $(\mathbb{A}^{n+1}\backslash\{0\}) \times (\mathbb{A}^{m+1}\backslash\{0\})/(k^\times \times k^\times)$ where $(\lambda, \mu) \in k^\times \times k^\times$ acts via

$$(\lambda, \mu).(x_0, \dots, x_n; y_0, \dots, y^m) = (\lambda x_0, \dots, \lambda x_n; \mu y_0, \dots, \mu y_m).$$

Subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$ are defined by the vanishing of bihomogeneous polynomials; that is, by polynomials which are homogeneous in both the x's and y's. Rational functions on $\mathbb{P}^n \times \mathbb{P}^m$ are ratios of bihomogeneous polynomials which have the same bidegree.

Definition 5.1. We define the quotient variety

$$\mathbb{F}(a, b) \ := \ (\mathbb{A}^2\backslash\{0\}) \times (\mathbb{A}^2\backslash\{0\})/(k^\times \times k^\times),$$

where $(\lambda, \mu) \in k^\times \times k^\times$ acts via

$$(\lambda, \mu).(x_0, x_1; y_0, y_1) = (\lambda x_0, \lambda x_1; \mu\lambda^{-a} y_0, \mu\lambda^{-b} y_1). \tag{5-1}$$

We will denote points of $\mathbb{F}(a, b)$ by $(x_0, x_1; y_0, y_1)$ understanding that this represents the same point as $(\lambda x_0, \lambda x_1; \mu\lambda^{-a} y_0, \mu\lambda^{-b} y_1)$.

Since the ratio of the x coordinates is unchanged by the action of (λ,μ), there is a well defined map

$$\pi:\mathbb{F}(a,b)\to\mathbb{P}^1,\qquad (x_0,x_1;y_0,y_1)\mapsto(x_0,x_1).$$

The fibers are isomorphic to $\mathbb{P}^1$. Indeed, $\mathbb{F}(a,b)$ is covered by open pieces of the form $U\times\mathbb{P}^1=\pi^{-1}(U)$, so is a ruled surface over $\mathbb{P}^1$. In particular, it is smooth and rational.

It follows from (5-1) that $\mathbb{F}(a,b)$ depends only on the difference $a-b$ because replacing (a,b) by $(a+c,b+c)$ leaves the action of (λ,μ) on (x_0,x_1) unchanged, and changes the action on (y_0,y_1) by a factor λ^{-c} on both y_0 and y_1, and this factor can be absorbed into μ. Thus $\mathbb{F}(a,b)\cong\mathbb{F}(a+c,b+c)$. We therefore define

$$\mathbb{F}_n=\mathbb{F}(0,n).$$

We also retain the $\mathbb{F}(a,b)$ notation. The fibers of π do not depend on the choice of (a,b), so there is a well-defined map $\pi:\mathbb{F}_n\to\mathbb{P}^1$.

Put a $\mathbb{Z}^2$-grading on $A=k[x_0,x_1;y_0;y_1]$ by declaring

$$\deg x_0=\deg x_1=(1,0),\ \deg y_0=(-a,1),\ \deg y_1=(-b,1).$$

Write $S=k[x_0,x_1]$, and let S_j be the usual degree j-component. Thus $S_j=S_{(j,0)}$ in the $\mathbb{Z}^2$-grading, and $A_{(0,1)}=S_ay_0+S_by_1$. A bihomogeneous polynomial viewed as a function on $\mathbb{A}^2\times\mathbb{A}^2$ is constant on the orbits of $k^\times\times k^\times$, so its zero locus in $\mathbb{F}(a,b)$ is well-defined. We can think of A as a bihomogeneous coordinate ring of $\mathbb{F}(a,b)$.

Since $\mathbb{F}(a,b)$ depends only on $a-b$, the following results gives embeddings $\mathbb{F}_n\to\mathbb{P}^N$.

Lemma 5.2. *If a and b are positive, there is an embedding $\varphi:\mathbb{F}(a,b)\to\mathbb{P}^{a+b+1}$ defined by*

$$(x_0,x_1;y_0,y_1)\mapsto(x_0^ay_0,x_0^{a-1}x_1y_0,\dots,x_1^ay_0,x_0^by_1,x_0^{b-1}x_1y_1,\dots,x_1^by_1),$$

which sends each fiber $\pi^{-1}(p)$ to a line in $\mathbb{P}^{a+b+1}$. The image is the variety cut out by the equations

$$\operatorname{rank}\begin{pmatrix}u_0 & u_1 & \dots & u_{a-1} & u_{a+1} & \dots & u_{a+b}\\ u_1 & u_2 & \dots & u_a & u_{a+2} & \dots & u_{a+b+1}\end{pmatrix}\le 1.$$

Proof. The fiber $\pi^{-1}(\alpha,\beta)$ is sent to the line cut out by the $a+b$ linear forms

$$\beta u_i-\alpha u_{i+1},\qquad i=0,\dots,a-1,a+1,\dots,a+b.$$

$\square$

It is sometimes easier to think of φ as the following map

$$(1,x;y_0,y_1)\mapsto\begin{pmatrix}y_0, & xy_0 & \dots & x^{a-1}y_0 & y_1 & xy_1 & \dots & x^{b-1}y_1\\ xy_0 & x^2y_0 & \dots & x^ay_0 & xy_1 & x^2y_1 & \dots & x^by_1\end{pmatrix}$$

For example, φ embeds $\mathbb{F}(1,1)\cong\mathbb{P}^1\times\mathbb{P}^1$ as the quadric surface $u_0u_3-u_1u_2=0$ in $\mathbb{P}^3$.

Example 5.3 (The cubic scroll in $\mathbb{P}^4$). Lemma 5.2 embeds $\mathbb{F}(1,2)$ in $\mathbb{P}^4$ as the matrices

$$\operatorname{rank}\begin{pmatrix}u_0 & u_1 & u_3\\ u_1 & u_2 & u_4\end{pmatrix}\le 1.$$

To see that this is a cubic surface observe that the intersection of it with the plane spanned by

$$\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\end{pmatrix},$$

consists of those matrices

$$\begin{pmatrix}\alpha&\beta&\gamma\\\beta&\gamma&\alpha\end{pmatrix},$$

such that $\alpha^2-\beta\gamma=\beta^2-\alpha\gamma=\gamma^2-\alpha\beta=0$; the scheme theoretic locus cut out by these equations consists of three points.

There is a projection from the Veronese surface $\mathbb{P}^2\cong\mathbb{V}\subset\mathbb{P}^5$ to $\mathbb{F}(1,2)$. If $\mathbb{V}$ is realized as the 3×3 symmetric matrices of rank ≤ 1, the projection is

$$\begin{pmatrix}u_0&u_1&u_3\\u_1&u_2&u_4\\u_3&u_4&u_5\end{pmatrix}\mapsto\begin{pmatrix}u_0&u_1&u_3\\u_1&u_2&u_4\end{pmatrix}.$$

Alternatively one can see the map as

$$\begin{pmatrix}x^2&xy&xz\\xy&y^2&yz\\xz&yz&z^2\end{pmatrix}\mapsto\begin{pmatrix}x^2&xy&xz\\xy&y^2&yz\end{pmatrix}.$$

We now want to realize $\mathbb{F}(1,2)$ as $\mathbb{P}^2$ blown up at a single point. Define

$$\psi:\mathbb{F}(1,2)\to\mathbb{P}^2,\qquad\begin{pmatrix}u_0&u_1&u_3\\u_1&u_2&u_4\end{pmatrix}\mapsto(u_0,u_1,u_3)\equiv(u_1,u_2,u_4);$$

since the two rows of the matrix are linearly dependent this is a well-defined morphism. Notice that

$$\psi^{-1}(0,0,1)=\begin{pmatrix}0&0&*\\0&0&*\end{pmatrix}=\varphi(*,*;1,0),$$

where φ is the embedding defined in Lemma 5.2. The fiber over (x_0,x_1) of the structure map $\pi:\mathbb{F}(1,2)\to\mathbb{P}^1$ is the line $(x_0,x_1;*,*)$; these lines define the ruling on $\mathbb{F}(1,2)$. The map $\psi:\mathbb{F}(1,2)\to\mathbb{P}^2$ sends these lines to the lines in $\mathbb{P}^2$ passing through $(0,0,1)$.

Although the lemma as stated does not apply when $a=0$, there is still a morphism $\varphi:\mathbb{F}(0,b)\to\mathbb{P}^{b+1}$, namely

$$(x_0,x_1;y_0,y_1)\mapsto(y_0,x_0^by_1,\dots,x_1^by_1).$$

When $b=1$, we recover the map $\mathbb{F}(0,1)\to\mathbb{P}^2$ discussed in the previous example. When $b=2$, we get a map $\mathbb{F}(0,2)\to\mathbb{P}^3$, the image of which is the quadric cone. This quadric cone may be realized as $\mathbb{P}^2/\mathbb{Z}_2$, the quotient variety for the cyclic group action. More generally, consider $\mathbb{P}^2/\mathbb{Z}_n$, realized as $\operatorname{Proj}k[x,y,z^n]$ where $k[x,y,z]$ is the polynomial ring with its standard grading. Taking the n^{th} Veronese, we get an embedding $\mathbb{P}^2/\mathbb{Z}^n\to\mathbb{P}^{n+1}$ realizing $\mathbb{P}^2/\mathbb{Z}^n$ as a cone over a degree n rational normal curve. The map $\mathbb{F}(0,n)\to\mathbb{P}^{n+1}$ is a resolution of this cone which contracts the $(-n)$-curve to the vertex.

We now match up these surfaces with the ruled surfaces defined as $\mathbb{P}_C(\mathcal{E})$. Recall that every vector bundle on $\mathbb{P}^1$ splits as a

direct sum of line bundles, so taking into account the isomorphisms in Theorem 4.2, the follwing result is not a surprise.

THEOREM 5.4. *The ruled surfaces over* $\mathbb{P}^1$ *are precisely*

$$\mathbb{F}_n \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)),$$

for $n \geq 0$.

PROPOSITION 5.5. [**25**, Proposition III.18] *Let* $\pi : S \to C$ *be a ruled surface with* $S \cong \mathbb{P}_C(\mathcal{E})$. *Then* $\operatorname{Pic} S \cong \pi^* \operatorname{Pic} C \oplus \mathbb{Z}h$ *where* h *is the class of* $\mathcal{O}_S(1)$, *and* $h^2 = \deg \Lambda^2 \mathcal{E}$.

LEMMA 5.6. *A fiber* F *of a surjective morphism* $S \to C$ *from a smooth surface to a smooth curve satisfies* $F^2 = 0$.

PROPOSITION 5.7. *Consider the rational ruled surfaces* $\mathbb{F}_n$.

1. $\operatorname{Pic} \mathbb{F}_n = \mathbb{Z}f \oplus \mathbb{Z}h$ *where* f *is the class of a fiber,* h *is the class of a section, and* $f^2 = 0$, $f.h = 1$, *and* $h^2 = n$.
2. *If* $n > 0$, *there is a unique irreducible curve on* $\mathbb{F}_n$ *with negative self-intersection. Its class in* $\operatorname{Pic} \mathbb{F}_n$ *is* $h - nf$, *and* $b^2 = -n$.
3. *Every* $\mathbb{F}_n$ *is minimal except* $\mathbb{F}_1$ *which is isomorphic to* $\mathbb{P}^2$ *with a single point blown up.*

We can also write $\operatorname{Pic} \mathbb{F}_n = \mathbb{Z}C_n \oplus \mathbb{Z}F$, where C_n is the unique irreducible $(-n)$-curve on $\mathbb{F}_n$, $C^2 = -n$, and $C.F = 1$. Then the canonical divisor on $\mathbb{F}_n$ is given by $K \sim -2C - (n+2)F$. Thus $K^2 = -4n + 4(n+2) + (n+2)^2 = 8 + (n+2)^2$.

PROPOSITION 5.8. *Let* $n \geq 1$. *There is a surjective morphism* $\mathbb{F}_n \to \mathbb{P}^2/\mu_n$ *where* μ_n *is the cyclic group of order* n *acting on* $\mathbb{P}^2$ *by*

$$(\alpha, \beta, \gamma) \mapsto (\alpha, \beta, \zeta\gamma)$$

where ζ *is a primitive* n^{th} *root of 1. The map is an isomorphism outside* $(0, 0, 1)$, *the fiber over that being the unique irreducible curve on* $\mathbb{F}_n$ *having negative self-intersection.*

There is an explicit description of the $\mathbb{F}_n$'s given in Harris's book [**?**].

As remarked above all ruled surfaces over C belong to a single birational equivalence class, so Theorem **??** suggests the problem of understanding how to get from one such surface to another by blowing up and down (see [**25**, Exercises III.24]).

6. Rational Surfaces

THEOREM 6.1 (Castelnuovo). *A smooth surface* X *is rational if and only if* $p_a = P_2 = 0$, *where* $p_a = \dim H^2(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$ *is the arithmetic genus, and* $P_2 = \dim H^0(X, \mathcal{O}(2K))$ *is the second plurigenus.*

CHAPTER 27

Non-commutative surfaces

APPENDIX A

Categories

We assume the reader is comfortable using category theory as a language, or framework, in which to express basic results about algebraic objects such as groups, rings, modules, et cetera. At that basic level the definitions of category theory take center stage, the theorems playing a small supporting role. Certainly that is the role category theory plays in much non-commutative ring theory, with the notable exception of D-modules and finite dimensional algebras. In commutative algebra and algebraic geometry the deeper aspects of category theory come into play, and this also happens in non-commutative algebraic geometry. With that in mind, this appendix briefly introduces the categorical prerequisites for the rest of the book.

Our treatment is terse. Since category theory provides a framework for a variety of subjects it is necessarily rather technical; we will provide some focus by illustrating the ideas with examples relevant to non-commutative algebraic geometry. For a more comprehensive account the reader may consult [**112**].

We begin with the definition of a category and several examples related to rings and modules. Functors are introduced in the same spirit. Most categories in this book are abelian, and we introduce the main ideas for this: kernels, cokernels, exact functors, Mitchell's Embedding Theorem et cetera. Our treatment of localization and quotient categories in Section 13 follows Gabriel's exposition [**65**].

1. Definitions and examples

Definition 1.1. A category C consists of the following data:

- a set $\mathrm{Ob}(\mathsf{C})$ whose members are called the objects of C;
- for every pair of objects X and Y, a set $\mathrm{Hom}_{\mathsf{C}}(X, Y)$, whose elements are called morphisms from X to Y;
- for every triple of objects X, Y, Z, a map
$$\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Z) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$$
called the composition law.

In addition these data are required to satisfy the following conditions:

- the composition law is associative;
- for every object X, there exists an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ h = h$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and all $h \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$, and all objects Y. The element id_X is called the identity.

Definition 1.2. A subcategory of C is a category D such that $\mathrm{Ob}(\mathsf{D}) \subset \mathrm{Ob}(\mathsf{C})$ and for every pair $X, Y \in \mathrm{Ob}(\mathsf{D})$, $\mathrm{Hom}_{\mathsf{D}}(X, Y) \subset \mathrm{Hom}_{\mathsf{C}}(X, Y)$. It is also required that the composition law in D agree with that in C, and that the identity morphisms be the same. If $\mathrm{Hom}_{\mathsf{D}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathsf{D})$, we call D a full subcategory of C.

Notation . It is usual to denote a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ by an arrow: either $f : X \to Y$ or $X \xrightarrow{f} Y$. It is permissible to drop the subscript from $\mathrm{Hom}_{\mathsf{C}}$ if the category in question is clear.

For most categories of interest to us it is a routine matter to check that the axioms are satisfied.

Example 1.3. The category Set has as its objects all sets, and as its morphisms all maps between sets. The identity id_X is the map such that $\mathrm{id}_X(x) = x$ for all $x \in X$.

Foundations. In order to avoid paradoxes involving 'the set of all sets which are not members of themselves' (see Example 3.5), we fix some universe, that is, some suitably large set of sets, in which to work—see [**112**, Chapter I, Section 6] for the definition of a universe. One axiom is that if a set belongs to the universe, then so does its power set. Sets belonging to our fixed universe, are called small sets. With this in mind, we modify the above definition of the category Set: its objects are all *small* sets. It is crucial to realize that Set is *not* itself a small set. Categories of algebraic objects will be subcategories of Set; in other words, algebraic objects such as modules will be small sets with the appropriate additional structure.

A category is small if its class of objects and its set of morphisms are both small sets. Most categories we encounter will be small with the notable exception of the category consisting of all small categories (Definition 4.7). We will also make the requirement that all index sets are small.

Many important categories are subcategories of Set; that is, the objects are small sets, the morphisms are maps between these sets, and the identity maps id_X are identity maps on the underlying sets; the morphisms in Examples 1.14 and 1.15 are not of this form.

Example 1.4. The category of abelian groups, denoted Ab, has as its objects all abelian groups, and as its morphisms all group homomorphisms.

In this example the Hom-sets have an additive structure making them abelian groups, and the composition of morphisms $(f, g) \mapsto f \circ g$ is bilinear. This will be the case in almost all the examples we encounter. A category with such a structure is called pre-additive.

Example 1.5. If R is a ring, then we may form a category with one object, say $*$, and morphisms $\mathrm{Hom}(*, *) = R$ with composition being the product in R. The additive structure on R makes this a pre-additive category. Indeed, a pre-additive category with a single object is of this form. Thus a pre-additive category can be seen as a generalization of a ring.

Example 1.6. If R is a graded ring, we may form a category with objects the set of integers, and morphisms $\mathrm{Hom}(m, n) = R_{m-n}$ with composition the product in R. This is a pre-additive category. One can picture the category as being laid out in the plane, with the homogeneous components of R ditributed over the lattice points; each component R_i appears infinitely often.

Example 1.7. The category of vector spaces over a field k, denoted $\mathcal{V}ec_k$, has as objects all k-vector spaces, and as morphisms all k-linear maps.

Example 1.8. The category of graded vector spaces over a field k, denoted GrVec_k, has as objects all k-vector spaces, V say, which are endowed with a decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$ as a direct sum of distinguished subspaces. (A single vector space endowed with two different decompositions gives *two distinct objects* in GrVec_k.) The elements of $\mathrm{Hom}(U, V)$ are the k-linear maps $f : U \to V$ such that $f(U_n) \subset V_n$ for all $n \in \mathbb{Z}$.

Example 1.9. The category of rings, denoted Ring, has as objects all rings, and as morphisms all ring homomorphisms. In keeping with the convention adopted in Appendix A, the objects of Ring are required to have an identity which differs from zero, and morphisms are required to send 1 to 1.

Example 1.10. If k is a commutative rings the category Alg_k has as objects k-algebras, and as morphisms the k-algebra ring homomorphisms. In keeping with the convention adopted in Appendix A, we do not allow the zero map to be a morphism in this category, whence Alg_k does *not* contain the zero ring.

Example 1.11. Let R be a ring. The set of all left R-modules, together with all R-module homomorphisms forms a category, denoted $\mathsf{Mod}R$. When R is left noetherian the full subcategory $\mathsf{mod}R$ consisting of the finitely generated modules plays an important role. The category of right R-modules will be denoted $\mathsf{Mod}R^{\mathrm{op}}$; that is, it will be realized as the category of left modules over the opposite ring.

Example 1.12. Fix a group G. A G-set is a set X endowed with an action of G such that $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in X$, and such that $1.x = x$ for all $x \in X$. The G-sets form a category Set_G with morphisms the maps $f : X \to Y$ satisfying $f(g.x) = g.f(x)$ for all $g \in G$ and all $x \in X$. The morphisms are called G-equivariant maps.

Example 1.13. Let X be a fixed topological space. The objects of the category Open_X are the open subsets of X, including the empty set and X itself. Let U and V be open subsets of X. If $U \not\subset V$ then $\mathrm{Hom}(U, V)$ is empty. If $U \subset V$ then $\mathrm{Hom}(U, V)$ consists of a single morphism, namely the inclusion map $i_U^V : U \to V$.

Example 1.14. Here is a category having just one object, and morphisms which are not defined as set maps: let G be a group, denote the single object by $*$, and define $\mathrm{Hom}(*, *) = G$, with composition of morphisms being the product in G. The morphisms are *not* set maps from $*$ to $*$. Groups are just categories with a single object in which every morphism is an isomorphism. More generally, a category with a single object is the same thing as a monoid.

Example 1.15. Here is another example where the morphisms are not set maps. A correspondence from a set X to a set Y is a subset $C \subset X \times Y$ such that $\mathrm{pr}_1(C) = X$ where $\mathrm{pr}_1 : X \times Y \to X$ is the projection map. We say that $x \in X$ corresponds to those $y \in Y$ such that $(x, y) \in C$. The category Corres of correspondences has the same objects as Set, and the morphisms from X to Y, denoted $\mathsf{Corres}(X, Y)$, are the correspondences from X to Y. The composition of correspondences $X \xrightarrow{C} Y \xrightarrow{D} Z$ is

$$D \circ C := \{(x, z) \mid \text{there exists } y \in Y \text{ such that } (x, y) \in C \text{ and } (y, z) \in D\}.$$

The identity correspondence on a set X is the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$.

EXAMPLE 1.16. A category $\mathcal{P}$ in which all Hom-sets have at most one element is called a preorder. We may define a binary relation $\leq$ on the objects of $\mathcal{P}$ by saying $p \leq q$ if $\mathrm{Hom}_{\mathcal{P}}(p,q) \neq \emptyset$; i.e., if there is a morphism $p \to q$. Among the preorders are the partial orders, namely those preorders in which $p \leq q$ and $q \leq p$ implies $p = q$.

Definition 1.17. Let C be a category. The dual category, denoted C^{op}, is defined by $\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathsf{C})$ and

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathsf{C}}(Y,X).$$

The composition

$$\cdot : \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(Y,Z) \times \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X,Y) \to \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X,Z)$$

is defined by $f \cdot g := g \circ f$, where $g \circ f$ is the composition in C. The identity morphisms remain the same.

EXAMPLE 1.18. The category of affine schemes $\mathcal{S}ch$ may be defined as the dual of the category of commutative rings. However, the objects of $\mathcal{S}ch$ are usually represented as pairs $(\mathrm{Spec}\, R, \mathcal{O})$, where $\mathrm{Spec}\, R = \{\text{prime ideals of } R\}$ is endowed with the Zariski topology, and $\mathcal{O}$ is the sheaf of rings on $\mathrm{Spec}\, R$ whose stalks are the localizations $R_{\mathfrak{p}}$.

EXERCISES

1.1 Show that the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X,X)$ is unique.

1.2 Show that the category Open_X of open subsets of the topological space X, defined in Example 1.13, satisfies the category axioms.

1.3 Show that the rule which assigns to a set X the free k-algebra $k\langle X\rangle$ may be made into a functor from the category of sets to the category of k-algebras.

1.4 Show that the rule which assigns to a k-vector space V the tensor algebra $T(V)$, may be made into a functor from the category $\mathcal{V}ec_k$ to the category of k-algebras. Do the same with the symmetric algebra $S(V)$ in place of $T(V)$

1.5 Can the rule which assigns to a vector space V the projective space $\mathbb{P}(V)$ be made into a functor from $\mathcal{V}ec_k$ to $\mathcal{V}ar_k$?

1.6 Check that the composition of correspondences is associative, and hence that Corres is a category.

2. Special types of morphisms and objects

Definition 2.1. A morphism $f : X \to Y$ in C is

- a monomorphism if, whenever $g_1, g_2 : W \to X$ are morphisms in C such that $fg_1 = fg_2$, then $g_1 = g_2$;
- an epimorphism if, whenever $g_1, g_2 : Y \to Z$ are morphisms in C such that $g_1 f = g_2 f$, then $g_1 = g_2$;
- an isomorphism if there exists $g \in \mathrm{Hom}_{\mathsf{C}}(Y,X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If such a g exists it is unique, and is denoted by f^{-1}; we call it the inverse of f. Objects X and Y are isomorphic in C if there exists and isomorphism $f : X \to Y$ in C.

EXAMPLE 2.2. A map $f : X \to Y$ in Set is a monomorphism (respectively, an epimorphism) if and only if it is injective (respectively, surjective).

Example 2.3. In general, an epimorphism need not be surjective. In the category of topological Hausdorff spaces the inclusion $f : \mathbb{Q} \to \mathbb{R}$ of the rationals in the reals, both being given their usual topology, is an epimorphism. To see this, suppose that $g_1, g_2 : \mathbb{R} \to Z$ are continuous maps such that $g_1 f = g_2 f$.

We put the product topology on products of spaces. The Hausdorff hypothesis ensures that the diagonal

$$\Delta := \{(z, z) \mid z \in Z\} \subset Z \times Z$$

is closed. The map $g := (g_1, g_2) : \mathbb{R} \times \mathbb{R} \to Z \times Z$ is continuous, so $g^{-1}(\Delta)$ is closed. By hypothesis $g^{-1}(\Delta)$ contains $\Delta_{\mathbb{Q}} := \{(q, q) \mid q \in \mathbb{Q}\}$. Hence g^{-1} contains the closure of $\Delta_{\mathbb{Q}}$ which is $\Delta_{\mathbb{R}} := \{(r, r) \mid r \in \mathbb{R}\}$. Thus $g_1(r) = g_2(r)$ for all $r \in \mathbb{R}$, so $g_1 = g_2$.

The general principle illustrated by this example is that the inclusion of a dense subspace is an epimorphism.

Example 2.4. Here is an algebraic example showing that an epimorphism need not be surjective. The inclusion $f : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism. If $g_1, g_2 : \mathbb{Q} \to R$ and $g_1(n) = g_2(n)$ for all $n \in \mathbb{Z}$ then, for $n \neq 0$ we have

$$1 = g_1(n.\frac{1}{n}) = g_1(n)g_1(\frac{1}{n}),$$

from which it follows that $g_1(\frac{1}{n}) = g_2(\frac{1}{n})$ for all $n \neq 0$. From this, it follows that $g_1(m/n) = g_2(m/n)$ for all $m \in \mathbb{Z}$. That is, $g_1 = g_2$. (Exercise 4 generalizes this example.)

In Section 11, we will see that, in an abelian category, the terms 'monomorphism' and 'epimorphism' are equivalent to the terms 'injective' and 'surjective'.

The notion of isomorphism of objects depends on the category in question: two objects in a subcategory $\mathsf{D} \subset \mathsf{C}$ may be isomorphic as objects of C but non-isomorphic as objects of D. This happens when $\mathrm{Hom}_{\mathsf{D}}(X, Y)$ is a proper subset of $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ which does not contain the morphism implementing the isomorphism.

An isomorphism is both a monomorphism and an epimorphism (Exercise 6) but, as Examples 2.3 and 2.4 show, a morphism which is both a monomorphism and an epimorphism need not be an isomorphism.

Definition 2.5. Let X be an object in a category C. A

- subobject of X is a pair (A, α) consisting of an object A and a monomorphism $\alpha : A \to X$;
- quotient object of X is a pair (B, β) consisting of an object B and an epimorphism $\beta : X \to B$.

Definition 2.6. An object Z in a category C is

- an initial object if $\mathrm{Hom}_{\mathsf{C}}(Z, X)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a terminal object if $\mathrm{Hom}_{\mathsf{C}}(X, Z)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a zero object if it is both an initial and a terminal object.

A zero object is denoted by 0 and, for every pair of objects X and Y, the composition of morphisms $X \to 0 \to Y$ is called the zero morphism and is denoted by 0, or 0_{XY} if necessary.

$\boxed{\text{Paul}}$ Define initial and terminal objects as equivalence classes!

EXAMPLE 2.7. Let k be a commutative ring. In the category Alg_k of k-algebras, k itself is an initial object, but there is no terminal object.

EXAMPLE 2.8. The empty set is an initial object in Set, with $\mathrm{Hom}(\emptyset, X)$ consisting of the 'empty function', and any singleton set is a terminal object in Set.

It is easy to check that initial, terminal and zero objects are all unique up to unique isomorphism.

Definition 2.9. An object X in a category C is

- noetherian if any increasing sequence of subobjects $X_1 \subset X_2 \subset \dots$ of X is eventually stationary;
- artinian if any decreasing sequence of subobjects $X_1 \supset X_2 \supset \dots$ of X is eventually stationary;
- of finite length if it is both artinian and noetherian;
- simple or irreducible if it is non-zero and its only subobjects are 0 and X.

If X has finite length a composition series for X is a finite sequence of subobjects $0 = X_0 \subset X_1 \subset \dots \subset X_n = X$ such that each X_i/X_{i-1} is simple. The quotients X_i/X_{i-1} are called the composition factors of X; they are determined up to isomorphism by X.

EXERCISES

2.1 Show that the composition of two monomorphisms (respectively, epimorphisms) is a monomorphism (respectively, an epimorphism).

2.2 If $g : W \to X$ and $f : X \to Y$ are morphisms show that
(a) f is an epimorphism if fg is;
(b) g is a monomorphism if fg is.

2.3 Let $f : X \to Y$ be a morphism.
(a) Show that f is a monomorphism if and only if the induced map $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ is injective for all W.
(b) Show that f is an epimorphism if and only if the induced map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective for all Z.

2.4 Let R be a commutative ring and $\mathcal{S}$ a multiplicatively closed subset consisting of regular elements. Show that the inclusion $f : R \to R_{\mathcal{S}}$ of R in the localization is an epimorphism in the category of rings (cf. Example 2.4).

2.5 Let $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$. Show that f is a monomorphism if and only if f, viewed as a morphism $f^{\mathrm{op}} : Y \to X$ in the dual category, is an epimorphism.

2.6 Show that an isomorphism is both a monomorphism and an epimorphism. Show the converse is false. [*Hint:* why are the maps f in Examples 2.3 and 2.4 not isomorphisms?]

2.7 In the category Open_X of open subsets of a topological space X, show that X is a terminal object and the empty set an initial object.

2.8 Suppose that C has a zero object, let W, X, Y and Z be objects of C, and let $f : W \to X$ and $g : Y \to Z$ be morphisms. Show that $0_{XY} \circ f = 0_{WY}$ and $g \circ 0_{XY} = 0_{XZ}$.

2.9 Show that an object is noetherian if and only if it is artinian as an object in the dual category.

2.10 Show that an object X is noetherian if and only if any set of subobjects of X has a maximal member with respect to the ordering by inclusion.

2.11 Show that a functor need not send irreducible objects to irreducible objects. [*Hint:* consider the linearization functor $L : \mathsf{Set}_G \to \mathsf{Mod}kG$ in Example 5.5.] See also Exercise **??**.1.

2.12 Show that the irreducible objects in the category of affine schemes are the spectra of fields $(\operatorname{Spec} k, k)$ (i.e., the reduced points).

3. Products and coproducts

Definition 3.1. Let $\{X_\alpha \mid \alpha \in I\}$ be an indexed set of objects in a category C. A product of the X_α is an object $\prod X_\alpha$ together with distinguished morphisms $p_\alpha : \prod X_\alpha \to X_\alpha$ called projections such that, if $Y \in \mathrm{Ob}(\mathsf{C})$ and $q_\alpha : Y \to X_\alpha$ are any morphisms, then there is a unique morphism $f : Y \to \prod X_\alpha$ such that the following diagram commutes for all $\alpha \in I$:

$$\begin{tikzcd} Y \arrow[r, "f"] & \prod X_\alpha \arrow[d, "p_\alpha"] \\ & X_\alpha \end{tikzcd}$$

If the index β is such that $\mathrm{Hom}_{\mathsf{C}}(X_\beta, X_\alpha) \neq \emptyset$ for each α, then the projection p_β must be an epimorphism. For example, in Set, each p_β is surjective.

EXAMPLE 3.2. Suppose that $X \prod X$ exists. By its universal property there is a unique map $\Delta : X \to X \prod X$, called the diagonal, such that its composition with each projection $X \prod X \to X$ is the identity id_X.

Definition 3.3. Let $\{X_\alpha \mid \alpha \in I\}$ be an indexed set of objects in a category C. A coproduct (or direct sum) of the X_α is an object $\coprod X_\alpha$ together with distinguished morphisms $i_\alpha : X_\alpha \to \coprod X_\alpha$ called injections such that, if $Y \in \mathrm{Ob}(\mathsf{C})$ and $j_\alpha : X_\alpha \to Y$ are any morphisms, then there is a unique morphism $g : \coprod X_\alpha \to Y$ such that the following diagram commutes for all $\alpha \in I$:

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] & \coprod X_\alpha \arrow[d, "g"] \\ & Y \end{tikzcd} \tag{3-1}$$

REMARK 3.4. The definitions are *not* asserting that products and coproducts exist. However, if a product and/or a coproduct exists it is unique up to isomorphism, so we shall speak of *the* product and *the* coproduct. The uniqueness up to isomorphism may be proved directly, or as a consequence of Yoneda's Lemma (Example 6.6), or as a consequence of the fact that a product is a special case of a limit, and hence a terminal object in an appropriate category (Section 7).

Warning: It is now that foundational issues arise. Recall that we are working in a fixed universe, and that the objects of Set are those sets belonging to the universe.

EXAMPLE 3.5. Set does not contain the product of all sets in Set. Suppose to the contrary that $P \in \mathrm{Ob}(\mathsf{Set})$ is a product of all sets. Then the power set, Q say, of P also belongs to Set, so there is a projection $\pi : P \to Q$, which is surjective. The subset $X = \{x \in P \mid x \notin \pi(x)\}$ of P is a member of Q, so there is some $y \in P$ such that $\pi(y) = X$. If $y \in X$, then by definition of X, $y \notin \pi(y) = X$, a contradiction. However, if $y \notin X$, then by definition of X, $y \in \pi(y) = X$, a contradiction. Thus we conclude that P cannot belong to Set.

The problem lies in the fact that the index set for this product, which is Set itself, is too large: it is not a small set.

If every indexed collection of objects in C has a product (respectively, coproduct) we say that products (resp., coproducts) exist in C. If products exist for all finite (or small) index sets, we say that finite (or small) products exist. We use similar terminology for coproducts.

EXAMPLE 3.6. Small products and arbitrary coproducts exist in Set. The product of a family of sets X_α is the cartesian product and the maps $p_\alpha : \prod X_\alpha \to X_\alpha$ are the obvious projections. The coproduct is the disjoint union and the maps $i_\alpha : X_\alpha \to \coprod X_\alpha$ are the obvious inclusions.

EXAMPLE 3.7. Products and coproducts exist in the category $\mathsf{Mod}R$. The product is the cartesian product made into an R-module by $r.(x_\alpha) = (rx_\alpha)$, and the p_α are the projections. The coproduct is the submodule of $\prod X_\alpha$ consisting of those elements (x_α) for which x_α is non-zero for only finitely many α. That is, the coproduct is the direct sum of the X_α, denoted $\bigoplus X_\alpha$. It is important to note that if the index set is finite, then $\prod X_\alpha \cong \coprod X_\alpha$.

In the full subcategory of finitely generated R-modules products and coproducts do not always exist: for example, the product of infinitely many non-zero vector spaces is not finite dimensional.

EXAMPLE 3.8. Products and finite coproducts exist in both the category Alg_k of k-algebras and in the category of *commutative* k-algebras. In both cases the product is the cartesian product with component-wise addition and multiplication, and the k-algebra structure arising from the diagonal embedding of k in the product. In the category of commutative k-algebras the coproduct is the tensor product over k, whereas in Alg_k the coproduct is the free coproduct, defined as follows. If $A = k\langle X\rangle/I$ and $B = k\langle Y\rangle/J$ are written as quotients of free algebras, then

$$A \coprod_k B := k\langle X \textstyle\coprod Y\rangle/(I, J).$$

(This is independent of the presentation of A and B.)

EXERCISES

3.1 Prove that $(X \prod Y) \prod Z \cong X \prod (Y \prod Z)$ whenever these products exist.

3.2 If T is a terminal object in C, prove that $X \prod T \cong T \prod X \cong X$ for all $X \in \mathrm{Ob}(\mathsf{C})$.

3.3 Show that the product of a collection $\{X_\alpha \mid \alpha \in I\}$ in C is the coproduct of the X_α in C^{op}.

3.4 Suppose we are in an abelian category. Consider a product $X = \prod_I X_\alpha$ and the projections $p_\alpha : X \to X_\alpha$. Show that if Y is a non-zero subobject of X, then the restriction of some p_α to Y is non-zero. Hence show that if every X_α is simple, every simple subobject of X is isomorphic to some X_α.

3.5 Suppose that $J \subset I$ are index sets. If the products exist, show there are morphisms $\theta : \prod_I X_\alpha \to \prod_J X_\alpha$ and $\varphi : \prod_J X_\alpha \to \prod_I X_\alpha$ such that $\theta\varphi = \mathrm{id}$. Hence show that if I is the disjoint union of J and K, then $\prod_I X_\alpha \cong (\prod_J X_\alpha) \prod (\prod_K X_\alpha)$.

3.6 Show that any finite length subobject of $\prod_I X_\alpha$ naturally embeds in $\prod_J X_\alpha$ for some finite subset $J \subset I$.

3.7 Verify the claims in Example 3.8 regarding the existence and description of the product and coproduct in the categories of commutative k-algebras and all k-algebras.

3.8 What is wrong with the following argument? Let C denote the category of commutative k-algebras, and let $A_1, \dots, A_n$ be in C. For each $\alpha = 1, \dots, n$ define $f_\alpha : A_\alpha \to \prod_\alpha A_\alpha$ by $f_\alpha(x) = (0, \dots, 0, x, 0, \dots, 0)$, where the x is in the A_α-position. By the universal property of the coproduct, there is a morphism

$$g : \coprod A_\alpha = \bigotimes_a A_\alpha \to \prod A_\alpha$$

such that $f_\alpha = g i_\alpha$, where $i_\alpha : A_\alpha \to \coprod A_\alpha$ is the map $i_\alpha(x) = 1 \otimes \dots 1 \otimes x \otimes 1 \dots \otimes 1$. Therefore $g(1 \otimes \dots \otimes x \otimes \dots \otimes 1) = (0, \dots, x, \dots, 0)$ for $x \in A_\alpha$. But this is ambiguous if $x = 1$.

3.9 One must be careful when taking productsto pay attention to the category in which the product is taken. Let Tors denote the full subcategory of Ab consisting of the torsion groups. Let X_α be a collection of torsion groups. Show that the product of the X_α in Tors is *not the same* as their product in Ab. In particular, show that their product in Tors is the torsion subgroup of their product in Ab.

3.10 Show that $\operatorname{Hom}(Y, \prod X_\alpha) \cong \prod \operatorname{Hom}(Y, X_\alpha)$ and $\operatorname{Hom}(\coprod X_\alpha, Y) \cong \prod \operatorname{Hom}(X_\alpha, Y)$ whenever these products and coproducts exist.

4. Functors

A recurrent theme in mathematics is to assign to the objects being investigated objects in another category, the assigned object being in some sense an invariant of the original one. The classical example is the fundamental group of a topological space. It is even better if, in addition, one assigns to morphisms in the original category morphisms in the other category. This idea is formalized by the notion of a functor. Moreover, functors are the appropriate morphisms in the category whose objects are themselves categories.

Definition 4.1. A covariant functor $F : \mathsf{C} \to \mathsf{C}'$ between two categories consists of the following data:

- a map $F : \operatorname{Ob}(\mathsf{C}) \to \operatorname{Ob}(\mathsf{C}')$;
- for all $X, Y \in \operatorname{Ob}(\mathsf{C})$ a map $F_{XY} : \operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{C}'}(FX, FY)$, the image of $f \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$ being denoted by $F(f)$.

This data is subject to the conditions:

- if f and g are morphisms in C, then $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined;
- $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ for all $X \in \operatorname{Ob}(\mathsf{C})$.

A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ from the dual category. That is, if $f : X \to Y$ then $F(f) : FY \to FX$, and the obvious analogues of the conditions for a covariant functor are satisfied.

If C and D are pre-additive categories, and $F(f + g) = F(f) + F(g)$ for all morphisms f and g in C, we call F and additive functor.

Example 4.2. Spec is a contravariant functor from the category of commutative rings to the category of topological spaces.

Example 4.3. Let X be a scheme over $\mathbb{Z}$. For each commutative ring R, write

$$X(R) = \operatorname{Mor}(\operatorname{Spec} R, X)$$

for the set of morphisms of schemes $\operatorname{Spec} R \to X$. This gives a covariant functor from commutative rings to sets. We call $X(R)$ the R-valued points of X. By the Yoneda Lemma (see section 6), this functor completely determines X as a $\mathbb{Z}$-scheme.

Hence we can think of schemes as certain types of functors from the category of commutative rings to Set. It is often an important problem to recognize whether a given functor is of the form $X(-)$, and if so to describe X as completely as possible.

Definition 4.4. A covariant functor $F : \mathsf{C} \to \mathsf{D}$ is

- faithful if all F_{XY} are injective;
- full if all F_{XY} are surjective;
- fully faithful if it is both full and faithful.

The same teminology is used for contravariant functors.

If C is a subcategory of D, the inclusion $\mathsf{C} \to \mathsf{D}$ is faithful.

EXAMPLE 4.5 (Hom functors). Fix an object X in a category C. Define the covariant functor $F_X : \mathsf{C} \to \mathsf{Set}$ by $F_X(Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and for $f \in \mathrm{Hom}_{\mathsf{C}}(Y_1, Y_2)$ define $F_X(f) : \mathrm{Hom}_{\mathsf{C}}(X, Y_1) \to \mathrm{Hom}_{\mathsf{C}}(X, Y_2)$ by $F_X(f)(g) = f \circ g$. We also use the notation $\mathrm{Hom}_{\mathsf{C}}(X, -)$ for F_X. There is a similar contravariant functor $\mathrm{Hom}_{\mathsf{C}}(-, X)$. Notice that $\mathrm{Hom}_{\mathsf{C}}(X, -)$ preserves terminal objects, and $\mathrm{Hom}_{\mathsf{C}}(-, X)$ sends initial objects to terminal objects.

EXAMPLE 4.6. Associated to a ring homomorphism $f : R \to S$ are two functors, extension and restriction, denoted by

$$f^* : \mathsf{Mod}R \to \mathsf{Mod}S,$$

and

$$f_* : \mathsf{Mod}S \to \mathsf{Mod}R.$$

Define $f^*M := S \otimes_R M$ and $f^*(g) := 1 \otimes g$ whenever g is an R-module map. Define $f_*(N) := N$ with the R-action defined as follows: if $r \in R$ and $n \in N$ then $r.n = f(r)n$. An S-module homomorphism is automatically an R-module homomorphism, so we define f_* to send an S-module map to the same map viewed as an R-module map.

More generally, if ${}_SB_R$ is a bimodule over the rings R and S, then there are functors $B \otimes_R - : \mathsf{Mod}R \to \mathsf{Mod}S$ and $\mathrm{Hom}_S(B, -) : \mathsf{Mod}S \to \mathsf{Mod}R$. When $f : R \to S$, we may take $B = S$, thus obtaining f^* and f_*.

Definition 4.7. The category Cat has as its objects the set of all small categories, and as morphisms the functors between them.

EXAMPLE 4.8. There is a functor $\mathsf{Ring} \to \mathsf{Cat}$ sending a ring R to $\mathsf{Mod}R$, the category of left R-modules, and a ring homomorphism $f : R \to S$ to the functor f^*, defined in Example 4.6. There is also a contravariant functor $\mathsf{Ring} \to \mathsf{Cat}$ which sends R to $\mathsf{Mod}R$, and sends a homomorphism f to f_*.

EXAMPLE 4.9. Let R be a ring, and view it as a category, still denoted R, with one object $*$, and $\mathrm{Hom}(*, *) = R$. Let $F : R \to \mathsf{Ab}$ be a covariant functor. Then $F(*)$ is an abelian group; let's call it M. If $r \in R$, then $F(r) : M \to M$ is an abelian group homomorphism; if we write rm for $F(r)(m)$, then $r(sm) = (rs)m$. If we also assume that F is an additive functor, then $F(r + s) = F(r) + F(s)$, so $(r+s)m = rm+sm$, whence M becomes a left R-module. Conversely, if we are given a left R-module M, then it determines an additive functor $R \to \mathsf{Ab}$ in an obvious way. Thus left R-modules are the same things as covariant functors $R \to \mathsf{Ab}$; *right* R-modules correspond in a similar way to *contravariant* functors $R \to \mathsf{Ab}$. It is an easy exercise to see that from this point of view R-module homomorphisms are the same things as natural transformations between the functors involved.

The previous example, together with our experience that modules are useful, suggests that for any additive category C it might be useful to consider the additive functors $\mathsf{C} \to \mathsf{Ab}$. Amongst other things, this point of view is that taken in the representation theory of quivers, where the category has objects the nodes and morphisms the arrows.

EXAMPLE 4.10. Let Open_X be the category of open subsets of X as in Example 1.13, and let $F : \mathsf{Open}_X \to \mathsf{Ab}$ be a contravariant functor For each open $U \subset X$ define $\mathcal{F}(U) := F(U)$, and write $\rho_U^V = F(i_U^V) : \mathcal{F}(V) \to \mathcal{F}(U)$ whenever $U \subset V$. Then $\mathcal{F}$ together with the maps ρ_U^V gives $\mathcal{F}$ the structure of a presheaf of abelian groups on X. Conversely a presheaf of abelian groups on X gives a contravariant functor $F : \mathsf{Open}_X \to \mathsf{Ab}$. Therefore such contravariant functors are the same things as presheaves of abelian groups on X.

Definition 4.11. Let $F, G : \mathcal{A} \to \mathcal{B}$ be covariant functors. A natural transformation $t : F \to G$ is a class of morphisms $t_A : FA \to GA$, one for each object $A \in \mathcal{A}$, such that, for each $f \in \operatorname{Hom}_{\mathcal{A}}(A, A')$ the diagram

$$\begin{array}{ccc} FA & \xrightarrow{F(f)} & FA' \\ \Big\downarrow{\scriptstyle t_A} & & \Big\downarrow{\scriptstyle t_{A'}} \\ GA & \xrightarrow{G(f)} & GA' \end{array}$$

commutes. If each t_A is an isomorphism, t is said to be a natural equivalence, F and G are said to be naturally equivalent, and we write $F \simeq G$. We write $\operatorname{Nat}(F, G)$ for the set of natural transformations from F to G.

Categories C and D are equivalent if there are covariant functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $F \circ G \simeq \operatorname{Id}_{\mathsf{D}}$, and $G \circ F \simeq \operatorname{Id}_{\mathsf{C}}$.

There are similar definitions for contravariant functors, except that if $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ are contravariant functors such that $FG \simeq \mathrm{id}$ and $GF \simeq \mathrm{id}$, we say that F is a duality.

EXAMPLE 4.12. The standard example of a natural equivalence is duality of finite dimensional vector spaces. Let $\mathsf{mod}k$ be the category of finite dimensional vector spaces over the field k, and define $* : \mathsf{mod}k \to \mathsf{mod}k$ to be the contravariant functor sending a vector space to its dual, and a linear map $\varphi : V \to W$ to its transpose $\varphi^* : W^* \to V^*$ defined by $\varphi^*(f)(u) = f(\varphi(u))$. Composing this functor with itself yields a covariant functor $F : \mathsf{mod}k \to \mathsf{mod}k$ sending V to V^{**}, and sending φ to φ^{**} which is given by $\varphi^{**}(\alpha)(f) = \alpha(\varphi^*(f))$ where $u \in V$, $f \in W^*$ and $\alpha \in V^{**}$. It is an easy exercise to show that the rule $t_V : V \to V^{**}$, defined by $t_V(u)(f) = f(u)$ for $u \in V$ and $f \in V^*$, yields a natural equivalence $t : F \to \mathrm{id}_{\mathsf{mod}k}$.

The functor $V \mapsto V^*$ is a duality from $\mathsf{mod}k$ to itself.

Natural equivalence allows one to recognize two categories of different sizes as being essentially the same. Before elaborating on this, we first define a skeleton of a category C to be a full subcategory D such that each object in C is isomorphic to a unique object in D. For example, the full subcategory of $\mathsf{mod}k$ consisting of all k^n is a skeleton. If D is a skeleton for C, then the inclusion $F : \mathsf{D} \to \mathsf{C}$ is an equivalence of categories. Since we do not usually want to distinguish between isomorphic objects in a category, equivalence is a more useful notion than isomorphism of categories: categories C and D are isomorphic if there exist functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $F \circ G = \mathrm{Id}$ and $G \circ F = \mathrm{Id}$.

EXAMPLE 4.13. Paul: Morita equiv repres functor (the proj implementing equiv)

Morita equivalence... skeleton of Cat...lack of faithfulness of $R \mapsto \mathsf{Mod}R$

EXERCISES

4.1 (Compare with Exercise 1.1.7.)
Let A, B, C, D be k-algebras. Show that $\mathrm{Hom}_k(-,-)$ gives a covariant functor

$$\mathrm{Hom}_k(-,-) : \mathsf{mod}(A \otimes B^{\mathrm{op}})^{\mathrm{op}} \times \mathsf{mod}(C \otimes D^{\mathrm{op}}) \to \mathsf{mod}(A^{\mathrm{op}} \otimes B \otimes C \otimes D^{\mathrm{op}}).$$

4.2 Show that a category is equivalent to any one of its skeletons, an equivalence being induced by the inclusion.

4.3 A group is the same thing as a category with a single object in which all morphisms are isomorphisms (see Example 1.14). Show that a functor from such a category to the category of vector spaces is the same thing as a representation of the group.
More generally a representation of a category is a functor to the category of vector spaces.

4.4 Let $\mathrm{id} = \mathrm{id}_{\mathsf{Mod}R}$ denote the identity functor on the category of left R-modules. Show that $\mathrm{Nat}(\mathrm{id}, \mathrm{id})$ is a ring isomorphic to the center of R.

4.5 Fill in the details in Example 4.10.

4.6 Fill in the details in Example 4.12 to show that the double dual functor is naturally equivalent to the identity functor on the category of finite dimensional vector spaces.

4.7 Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose that A_α are objects in A. Show there are natural morphisms $F(\prod A_\alpha) \to \prod F(A_\alpha)$, and $\coprod F(A_\alpha) \to F(\coprod A_\alpha)$. Give examples to show these morphisms need not be isomorphisms even when F is fully faithful.

5. Adjoint pairs of functors

Definition 5.1. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be covariant functors. We say that F is a left adjoint of G and that G is a right adjoint of F if, for each pair of objects X in C and Y in D, there is a bijection

$$\tau_{XY} : \mathrm{Hom}_{\mathsf{C}}(X, GY) \to \mathrm{Hom}_{\mathsf{D}}(FX, Y)$$

which yields a morphism of bifunctors $\tau : \mathrm{Hom}_{\mathsf{C}}(X, GY) \to \mathrm{Hom}_{\mathsf{D}}(FX, Y)$ which is a natural transformation in each variable. For brevity we call (F, G) an adjoint pair; we will use the convention that the left adjoint is written first.

The naturality condition on τ can be expressed as follows. If $f \in \mathrm{Hom}_{\mathsf{C}}(X, X')$ and $g \in \mathrm{Hom}_{\mathsf{D}}(Y, Y')$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X', GY) \arrow[r, "\tau_{X'Y}"] \arrow[d, "(-)\circ f"'] & \mathrm{Hom}_{\mathsf{D}}(FX', Y) \arrow[d, "(-)\circ (Ff)"] \\
\mathrm{Hom}_{\mathsf{C}}(X, GY) \arrow[r, "\tau_{XY}"'] \arrow[d, "(Gg)\circ(-)"'] & \mathrm{Hom}_{\mathsf{D}}(FX, Y) \arrow[d, "g\circ(-)"] \\
\mathrm{Hom}_{\mathsf{C}}(X, GY') \arrow[r, "\tau_{XY'}"'] & \mathrm{Hom}_{\mathsf{D}}(FX, Y')
\end{tikzcd}$$

commutes. More succinctly, if $f : X \to X'$, $g : Y \to Y'$, and $h : X' \to GY$, then

$$\tau(h \circ f) = \tau(h) \circ Ff \qquad \text{and} \qquad \tau(Gg \circ h) = g \circ \tau(h). \tag{5-1}$$

EXAMPLE 5.2. If ${}_SB_R$ is an S-R-bimodule, then $B \otimes_R -$ is a left adjoint to $\mathrm{Hom}_S(B,-)$. In particular,

$$\mathrm{Hom}_R(X, \mathrm{Hom}_S(B,Y)) \cong \mathrm{Hom}_S(B \otimes_R X, Y)$$

for all left R-modules X and all left S-modules Y; the isomorphism sends f to the map $b \otimes x \mapsto (f(x))(b)$, and from this explicit formula one may check that the diagrams in Definition 5.1 commute. The extension and restriction functors (f^*, f_*) associated to a ring homomorphism $f : R \to S$ (see Example 4.6) are a special case of this, the bimodule in question being ${}_SS_R$.

A forgetful functor is one which simply forgets some of the structure on the objects in a category. The next few examples illustrate the adage that left adjoints to forgetful functors solve universal problems.

EXAMPLE 5.3. The left adjoint to the forgetful functor $\mathsf{Mod}R \to \mathsf{Set}$ sending a module to its underlying set of elements is the functor F sending a set to the free R-module with that set as basis. That is, for an R-module M and set X, $\mathrm{Hom}_{\mathsf{Set}}(X,M) \cong \mathrm{Hom}_R(FX, M)$.

EXAMPLE 5.4. If R is a k-algebra, we may send R to the k-Lie algebra which is R itself as a k-vector space endowed with the Lie bracket $[a,b] = ab - ba$. The left adjoint to this functor sends a Lie algebra to its universal enveloping algebra.

EXAMPLE 5.5. Let G be a group, kG its group algebra, and Set_G the category of G-sets (Example 1.12). The forgetful functor $F : \mathsf{Mod}kG \to \mathsf{Set}_G$ which forgets the linear structure has a left adjoint, namely the linearization functor L sending a G-set X to the vector space with basis the elements of X endowed with the kG-action linearly extending the G-action. That is $\mathrm{Hom}_G(X, FV) \cong \mathrm{Hom}_{kG}(LX, V)$.

EXAMPLE 5.6. The forgetful functor sending an abelian group to its underlying semigroup has a left adjoint. When applied to the semigroup of finitely generated projective modules over a ring R (with direct sum as the operation), this gives the Grothendieck group $K_0(R)$. The adjoint functor can either be constructed explicitly, or obtained as a consequence of the Adjoint Functor Theorem.

EXAMPLE 5.7. Let $\mathcal{S}h(X)$ and $\mathsf{Pre}\mathcal{S}h(X)$ be the categories of sheaves and presheaves of abelian groups on a topological space X. The left adjoint of the forgetful functor $\mathcal{S}h(X) \to \mathsf{Pre}\mathcal{S}h(X)$ assigns to a presheaf its sheafification.

EXAMPLE 5.8. Let D denote the category of commutative domains with morphisms being the *injective* ring homomorphisms. There is a (forgetful) fully faithful embedding $F : \mathcal{F} \to \mathsf{D}$ of the category of fields into D. This has a left adjoint $Q : \mathsf{D} \to \mathcal{F}$ which sends a domain D to its field of fractions $Q(D)$.

$\boxed{\text{Paul}}$ – what happens to this example if we take non-commutative domains — why is there is no left adjoint???

PROPOSITION 5.9. *If (F,G) is an adjoint pair, then there are natural transformations $\varepsilon : FG \to \mathrm{id}_{\mathsf{D}}$ and $\eta : \mathrm{id}_{\mathsf{C}} \to GF$.*

PROOF. For each $Y \in \mathrm{Ob}(\mathsf{D})$ the isomorphism $\tau_{GY,Y} : \mathrm{Hom}_{\mathsf{C}}(GY, GY) \to \mathrm{Hom}_{\mathsf{D}}(FGY, Y)$ allows us to define $\varepsilon_Y = \tau_{GY,Y}(\mathrm{id}_{GY}) : FGY \to Y$. In a similar way, if $X \in \mathrm{Ob}(\mathsf{C})$, one uses $\tau_{X,FX}$ to define η. The details are left to the reader. $\square$

PROPOSITION 5.10. *Let (F, G) be an adjoint pair, with $F : \mathsf{C} \to \mathsf{D}$. Then*

1. *F is faithful if and only if $\tau^{-1}(\mathrm{id}_{FX}) : X \to GF(X)$ is a monomorphism for all X;*
2. *G is faithful if and only if $\tau(\mathrm{id}_{GY}) : FG(Y) \to Y$ is an epimorphism for all Y.*

PROOF. We use the fact that the adjunction bijection τ yields a natural transformation of bifunctors. Setting $h = \tau^{-1}(\mathrm{id}_{FX})$ in the first equality in (5-1) gives

$$\tau^{-1}(\mathrm{id}_{FX}) \circ f_1 = \tau^{-1}(\mathrm{id}_{FX}) \circ f_2 \Longleftrightarrow Ff_1 = Ff_2,$$

and (1) follows from this. Similarly, setting $h = \mathrm{id}_{GY}$ in the second equality in (5-1) gives

$$Gg_1 = Gg_2 \Longleftrightarrow g_1 \circ \tau(\mathrm{id}_{GY}) = g_2 \circ \tau(\mathrm{id}_{GY}),$$

and (2) follows from this. □

COROLLARY 5.11. *Let $f_* : \mathsf{C} \to \mathsf{D}$ be a functor having a left adjoint f^*, and a right adjoint $f^!$. The following are equivalent:*

1. *f_* is faithful;*
2. *$X \to f^! f_* X$ is monic for all $X \in \operatorname{Ob} \mathsf{C}$;*
3. *$f^* f_* X \to X$ is epic for all $X \in \operatorname{Ob} \mathsf{C}$.*

THEOREM 5.12. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. The following are equivalent:*

1. *F is an equivalence of categories;*
2. *F is fully faithful and every object of D is isomorphic to an object of the form FX;*
3. *F is fully faithful, and has a fully faithful left adjoint;*
4. *F is fully faithful, and has a fully faithful right adjoint.*

PROOF. [**112**, Theorem 1, page 99] or [**131**, Chapter 1, Theorem 5.3]. □

EXERCISES

5.1 Let $F_1 : \mathsf{C} \to \mathsf{D}$ and $F_2 : \mathsf{D} \to \mathsf{E}$ be functors. Suppose that (F_1, G_1) and (F_2, G_2) are adjopint pairs. Show that (F_2F_1, G_1g_2) is an adjoint pair.

5.2 If (F, G) is an adjoint pair, show that F preserves initial objects, and G preserves terminal objects.

5.3 Let $f : A \to B$ be a map of sets. Let $\mathcal{A}$ and $\mathcal{B}$ be the categories of all subsets of A and B, and morphisms the inclusions. Show that we have an adjoint pair (f_*, f^*), where $f_* : \mathcal{A} \to \mathcal{B}$ is defined by $f_*(X) = \{f(x) \mid x \in X\}$, and $f^* : \mathcal{B} \to \mathcal{A}$ is defined by $f^*(Y) = \{x \in A \mid f(x) \in Y\}$. We call f^* the inverse image functor, and f_* the direct image functor.
Compare this with the adjoint pair (f^*, f_*) associated to a ring homomorphism $f : R \to S$. In particular, if R and S are commutative rings, then there is a natural map $\operatorname{Spec} S \to \operatorname{Spec} R$. This is all related to the fact that if (F, G) is an adjoint pair, then so is $(G^{\mathrm{op}}, F^{\mathrm{op}})$.

5.4 Let $f : R \to S$ be a ring homomorphism, and (f^*, f_*) the associated adjoint pair (Example 4.6). Show that f_*, which is $\operatorname{Hom}_S(S, -)$ has a *right* adjoint, namely $f^! = \operatorname{Hom}_R(S, -)$.

5.5 Let (F, G) be an adjoint pair, and η and ε the associated natural transformations as in Proposition 5.9. Show that the maps $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$ and $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$ are both the identity map.

5.6 Show that a covariant functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence if and only if it is fully faithful and has a fully faithful right adjoint.

5.7 Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be covariant functors. Show that G is a right adjoint to F if and only if the covariant functors $\mathrm{Hom}_{\mathsf{C}}(-, G-)$ and $\mathrm{Hom}_{\mathsf{D}}(F-, -)$, taking $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$, are naturally equivalent.

5.8 Show that the right adjoint of a functor is only determined up to natural equivalence.

5.9 Complete the proof of Proposition 5.9.

5.10 Prove Lemma **??**.

5.11 Let Tors denote the full subcategory of Ab consisting of torsion abelian groups. Let $i_* :$ Tors $\to \mathsf{Ab}$ be the inclusion functor, and $i^!$ the functor sending a group to its torsion subgroup. Show that $(i_*, i^!)$ is an adjoint pair. Show that i_* does not have a left adjoint [Hint: Exercise 2.9.]

6. Representable functors and Yoneda's lemma

Definition 6.1. A covariant (resp. contravariant) functor $F : \mathsf{C} \to \mathsf{Set}$ is representable if there is an object X in C such that F is naturally equivalent to $\mathrm{Hom}_{\mathsf{C}}(X, -)$ (resp. F is naturally equivalent to $\mathrm{Hom}_{\mathsf{C}}(-, X)$. We say that F is represented by X, or that X is a representing object.

EXAMPLE 6.2. If $(X, \mathcal{O}_X)$ is a scheme, the global section functor $\Gamma(X, -)$, defined on the category of quasi-coherent $\mathcal{O}_X$-modules is representable; it is naturally equivalent to $\mathrm{Hom}(\mathcal{O}_X, -)$.

EXAMPLE 6.3. The duality $\mathsf{mod}k \to \mathsf{mod}k$ which sends a finite dimensional vector space to its dual and a morphism to its transpose is representable: a representing object is the 1-dimensional vector space k. For this reason k is called a dualizing object for $\mathsf{mod}k$.

The functor category. Let C and D be categories. We define

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}),$$

the category of covariant functors from C to D to have objects the covariant functors $\mathsf{C} \to \mathsf{D}$, and morphisms the natural transformations $\mathrm{Nat}(F, G)$.

In order for $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ to be a category the Hom-sets $\mathrm{Nat}(F, G)$ are required to be small; since $\mathrm{Nat}(F, G)$ naturally embeds in $\prod_{C \in \mathrm{Ob}\,\mathsf{C}} \mathrm{Hom}_{\mathsf{C}}(FC, GC)$, the smallness of C ensures that $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is a category.

Two functors F and G are isomorphic in $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ if and only if they are naturally equivalent.

The contravariant functor $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$. We extend the rule $X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, -)$ to a functor. If $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ the natural transformation

$$\hat{f} : \mathrm{Hom}_{\mathsf{C}}(Y, -) \to \mathrm{Hom}_{\mathsf{C}}(X, -)$$

is defined as follows: for each $A \in \mathrm{Ob}(\mathsf{C})$ let

$$\hat{f}_A : \mathrm{Hom}_{\mathsf{C}}(Y, A) \to \mathrm{Hom}_{\mathsf{C}}(X, A)$$

be the map $\hat{f}_A(h) := hf$. It is easy to check that is a natural transformation, and hence that the rule

$$X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, -) \qquad \text{and} \qquad f \mapsto \hat{f} \tag{6-1}$$

defines a contravariant functor $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$.

THEOREM 6.4 (Yoneda's Lemma). *The functor* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$ *defined by* $X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, -)$ *is fully faithful.*

Proof. Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor, and let $X \in \mathrm{Ob}(\mathsf{C})$. For each $\xi \in FX$ and each $A \in \mathrm{Ob}(\mathsf{C})$, define $\tilde{\xi}_A : \mathrm{Hom}_{\mathsf{C}}(X, A) \to FA$ by $\tilde{\xi}_A(g) := (Fg)(\xi)$. It suffices to prove

- $\tilde{\xi} : \mathrm{Hom}_{\mathsf{C}}(X, -) \to F$ is a natural transformation, and
- the rule $\xi \mapsto \tilde{\xi}$ is a bijection between FX and $\mathrm{Nat}(\mathrm{Hom}_{\mathsf{C}}(X, -), F)$;
- in particular, for each $Y \in \mathrm{Ob}(\mathsf{C})$, the rule $\xi \mapsto \tilde{\xi}$ is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(Y, X) \to \mathrm{Nat}(\mathrm{Hom}(X, -), \mathrm{Hom}(Y, -)).$$

The details can be found in several books. □

Corollary 6.5. *The functors* $\mathrm{Hom}_{\mathsf{C}}(X, -)$ *and* $\mathrm{Hom}_{\mathsf{C}}(Y, -)$ *are naturally equivalent if and only if* $X \cong Y$. *In particular, an object representing a functor is unique up to isomorphism.*

Example 6.6. Products and coproducts can be defined by using Yoneda's Lemma and the fact that products exist in Set. To see this, let X_α be an indexed set of objects in a category C. If the contravariant functor $\mathsf{C} \to \mathsf{Set}$ defined by

$$Y \mapsto \prod \mathrm{Hom}_{\mathsf{C}}(Y, X_\alpha)$$

is representable, we define $\prod X_\alpha$ to be a representing object; that is,

$$\mathrm{Hom}(Y, \prod X_\alpha) \cong \prod \mathrm{Hom}(Y, X_\alpha).$$

Similarly, if the covariant functor $Y \mapsto \prod \mathrm{Hom}_{\mathsf{C}}(X_\alpha, Y)$ is representable we define $\coprod X_\alpha$ to be a representing object; that is,

$$\mathrm{Hom}(\coprod X_\alpha, Y) \cong \prod \mathrm{Hom}(X_\alpha, Y).$$

Hence the uniqueness up to isomorphism of a product or coproduct follows from the uniqueness up to isomorphism of a representing object (Corollary 6.5).

Proposition 6.7. *A functor* $F : \mathsf{C} \to \mathsf{D}$ *has a right adjoint if and only if, for each* Y *in* D, *the functor* $X \mapsto \mathrm{Hom}_{\mathsf{D}}(FX, Y)$ *is representable.*

Proof. If G is a right adjoint of F then the functor is represented by GY.

Conversely, suppose that the functor is representable. For each Y in D let GY be a representing object (GY is only determined up to isomorphism, so we just make some choice) and let $\varphi_Y : \mathrm{Hom}_{\mathsf{C}}(-, GY) \to \mathrm{Hom}_{\mathsf{D}}(F-, Y)$ be a natural equivalence. If $f : Y \to Y'$ is a morphism in D, we define $Gf : GY \to GY'$ to be the unique morphism making the following diagram commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{C}}(-, GY) & \xrightarrow{\varphi_Y} & \mathrm{Hom}_{\mathsf{D}}(F-, Y) \\
{\scriptstyle \mathrm{Hom}_{\mathsf{C}}(-,Gf)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{D}}(F-,f)} \\
\mathrm{Hom}_{\mathsf{C}}(-, GY') & \xrightarrow{\varphi_{Y'}} & \mathrm{Hom}_{\mathsf{D}}(F-, Y').
\end{array}$$

It is now straightforward to check that G is a right adjoint to F. □

The following is a nice application of the Yoneda embedding.

Lemma 6.8. *Let* $i_* : \mathsf{B} \to \mathsf{A}$ *be the inclusion of a full subcategory.*

1. *If* i_* *has a left adjoint* i^*, *then the adjunction* $i^* i_* \to \mathrm{id}_{\mathsf{B}}$ *is a natural equivalence.*

2. *If i_* has a right adjoint $i^!$, then the adjunction* $\mathrm{id}_{\mathsf{B}} \to i^! i_*$ *is a natural equivalence.*

PROOF. (1) Let $B, B' \in \operatorname{Ob} \mathsf{B}$. By adjointness, $\operatorname{Hom}_{\mathsf{B}}(i^* i_* B, B') \cong \operatorname{Hom}_{\mathsf{A}}(i_* B, i_* B')$. Because B is full, this is naturally isomorphic to $\operatorname{Hom}_{\mathsf{B}}(B, B')$ (via i_*). Hence $\operatorname{Hom}_{\mathsf{B}}(i^* i_* B, B') \cong \operatorname{Hom}_{\mathsf{B}}(B, B')$ for all B', so Yoneda gives $i^* i_* B \cong B$.

(2) The proof is similar, starting from $\operatorname{Hom}_{\mathsf{B}}(B, i^! i_* B')$. $\square$

EXERCISES

6.1 Fill in the details to prove Yoneda's Lemma. Show that $\hat{f}$ really is a natural transformation, then show that $\widehat{f_1 f_2} = \hat{f}_2 \hat{f}_1$, to prove that (6-1) does define a contravariant functor.

6.2 Let F be the functor from commutative k-algebras to groups such that $F(R) = SL_n(R)$, the group of $n \times n$ matrices, with entries in R, which have determinant 1. Show that F is a representable functor, represented by the k-algebra

$$A = k[x_{11}, \dots, x_{nn}]/(\det -1)$$

where det denotes the determinant of the $n \times n$ generic matrix $X = (x_{ij})$.

6.3 Let $\mathcal{F} \subset k\langle X \rangle$ be a set of non-commutative polynomials, and let F be the functor from the category of k-algebras to the category of sets given by

$$F(S) = \{\text{solutions in } S \text{ to the system of equations } \mathcal{F} = 0\}.$$

Show that F is represented by $k\langle X \rangle / (\mathcal{F})$.

7. Limits and colimits

The notions of limit and colimit subsume some familiar ideas. For example, products, kernels, pullbacks, and inverse limits are special types of limits, and coproducts, cokernels, pushouts, and direct limits are special types of colimits.

We begin this section by formalizing the notion of a diagram in a category—a diagram will be a certain sort of functor, but it is really just a fancy way of saying that we have a collection of objects and morphisms between them satisfying certain commutativity rules. The formal definition of a diagram is similar to the functorial definition of a representation of a quiver.

Definition 7.1. A directed graph consists of

- a set of vertices, and
- for each ordered pair of vertices (α, β) a set E_β^α of edges from α to β.

We denote the graph by $G = (V, E)$ where V is the set of vertices, and E the set of all edges. All our graphs will be small, meaning that both V and E are small sets.

If e is an edge from α to β, we call α the start of e and β the end of e, and indicate this by writing $\alpha \xrightarrow{e} \beta$.

A directed path from a vertex α to a vertex β is a finite sequence of edges $e_1, \dots, e_n$ such that the start of e_1 is α, the end of e_n is β and, for each $i = 1, \dots, n-1$, the end of e_i is the start of e_{i+1}; we write $e_n \cdots e_2 e_1$ for the path. For each vertex α, we define the empty path e_α which begins and ends at α; we declare that $e e_\alpha = e$ for any edge e starting at α, and $e_\alpha e' = e'$ for any edge e' ending at α.

REMARK 7.2. A directed graph can be viewed as a category: the objects are the vertices, the morphisms from α to β are the directed paths from α to β, and the identity id_α is the empty path e_α. Composition of morphisms is concatenation of paths. There is a category Graph, in which the objects are graphs and the morphisms are the functors from one to the other. There is a forgetful functor $F : \mathsf{Cat} \to \mathsf{Graph}$.

Definition 7.3. Let C be a category. Let G be a graph with associated category $\mathcal{G}$. A diagram in C of shape G is a functor $D : \mathcal{G} \to \mathsf{C}$.

Let D be a diagram in C. A tuple (Z, ψ_α) consisting of an object Z in C, and morphisms $\psi_\alpha : Z \to D(\alpha)$, one for each vertex α, is called a cone over D, or a cone from Z to D, if, for all edges $\alpha \xrightarrow{e} \beta$, $D(e) \circ \psi_\alpha = \psi_\beta$. A morphism between two cones, say $\theta : (Z', \psi'_\alpha) \to (Z, \psi_\alpha)$, is a morphism $\theta \in \mathrm{Hom}_{\mathsf{C}}(Z', Z)$ such that $\psi_\alpha \circ \theta = \psi'_\alpha$ for all vertices α. The collection of all cones over D is a category $\mathsf{Cone}(D)$.

A limit of the diagram D is a terminal object in $\mathsf{Cone}(D)$; we denote it by $\lim D$, or $\lim_\alpha D(\alpha)$, if it exists; being a terminal object, it is unique up to isomorphism.

A small diagram is one arising from a small graph. A small limit is one associated to a small diagram.

Although a limit consists of an object together with morphisms, we will often refer to the object itself as the limit.

The terminology 'cone over D' should evoke a picture of the diagram lying in a horizontal plane, with Z sitting above the plane, and maps ψ_α down to each vertex of the diagram in such a way that each triangle having Z as a vertex commutes.

EXAMPLE 7.4 (Products). If $G = (V, \emptyset)$ is a graph with no edges, then a diagram, D say, in C of shape G is just a collection of objects D_α indexed by V. The limit of this diagram is therefore the product of the objects,

$$\lim D \cong \prod_\alpha D_\alpha.$$

Now let D be an arbitrary diagram in C; since there are morphisms $\lim D \to D_\alpha$, the universal property of the product implies there is a morphism

$$\lim D \to \prod_\alpha D_\alpha.$$

EXAMPLE 7.5 (Equalizers and kernels). A diagram of shape

$$\begin{tikzcd} \bullet \arrow[d, shift left] \arrow[d, shift right] \\ \bullet \end{tikzcd}$$

consists of a pair of objects X and Y, and two morphisms $f_1, f_2 : X \to Y$. The limit of this is a triple (L, g, h) consisting of an object L and two morphisms $g : L \to X$ and $h : L \to Y$ such that $f_1 \circ g = f_2 \circ g = h$. Notice that the morphism h carries no extra information: we could say that the limit is a pair (L, g) where $g : L \to X$ satisfies $f_1 \circ g = f_2 \circ g$ and the appropriate universal property holds. We call L the equalizer of f_1 and f_2. In Set the equalizer exists and is the subset of X where f_1 and f_2 agree. In $\mathsf{Mod}R$ the equalizer exists and is the kernel of $f_1 - f_2$; in particular, the kernel of a module homomorphism f is a special type of limit, namely the equalizer of the pair $(f, 0)$. That is, if $f : X \to Y$, then $\ker f$ is the pair (L, g) where L is an R-module, and $g : L \to X$ satisfies $fg = 0$ and, if $g' : L' \to X$ satisfies $fg' = 0$, then there is a unique $h : L' \to L$ such that $g' = gh$.

EXAMPLE 7.6 (Pullbacks). A pullback is a limit over a diagram of shape

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The pullback or fiber product of a diagram

$$\begin{tikzcd} & X \arrow[d, "f_1"] \\ Y \arrow[r, "f_2"'] & S \end{tikzcd}$$

in C is a triple $(X \times_S Y, g_1, g_2)$ consisting of an object $X \times_S Y$ in C, and morphisms $g_1 : X \times_S Y \to X$ and $g_2 : X \times_S Y \to Y$ such that the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "g_1"] \arrow[d, "g_2"'] & X \arrow[d, "f_1"] \\ Y \arrow[r, "f_2"'] & S \end{tikzcd}$$

commutes and, whenever (Z, h_1, h_2) is another triple in C with $h_1 : Z \to X$, $h_2 : Z \to Y$ and $f_1 \circ h_1 = f_2 \circ h_2$, there is a unique map $\theta : Z \to X \times_S Y$ such that $h_1 = g_1 \circ \theta$ and $h_2 = g_2 \circ \theta$. The requirement that $\lim D$ be a terminal object in $\mathsf{Cone}(D)$ coincides with the universal property the pullback is required to possess.

THEOREM 7.7. Set *has all small limits.*

PROOF. Let $\mathcal{G}$ be the category associated to a small graph G, and let $D : \mathcal{G} \to \mathsf{Set}$ be a diagram. Fix a singleton set $*$, and let L be the set of all cones over D of the form $(*, \varphi_\alpha)$. Note that L is small since G is, so L is an object in Set. For each vertex α, define $\psi_\alpha : L \to D(\alpha)$ by $\psi_\alpha((*, \varphi_\alpha)) = \varphi_\alpha(*)$. Thus L is a cone over D. We will show it is the limit of D.

Suppose that (Z, ρ_α) is a cone over D. For each $z \in Z$, define $\iota_z : * \to Z$ by $\iota_z(*) = z$. It is easy to see that $(*, \rho_\alpha \circ \iota_z)$ is a cone over D, and hence an element of L, so we may define $\theta : Z \to L$ by $\theta(z) = (*, \rho_\alpha \circ \iota_z)$. Therefore

$$(\psi_\alpha \circ \theta)(z) = \psi_\alpha((*, \rho_\alpha \circ \iota_z)) = \rho_\alpha(i_z(*)) = \rho_\alpha(z),$$

whence $\rho_\alpha = \psi_\alpha \circ \theta$, thus showing that (L, ψ_α) is a terminal object in $\mathsf{Cone}(D)$, and hence a limit of D. $\square$

Colimits. Colimits are like limits, except that they are initial objects defined in terms of cones having vertex below a horizontal plane containing a diagram and morphisms going down from the diagram to the vertex of the cone.

Definition 7.8. Let D be a diagram in C. A tuple (Z, φ_α) consisting of an object Z in C, and morphisms $\varphi_\alpha : D(\alpha) \to Z$, one for each vertex α, is called a cone under D, or a cone from D to Z, if for all edges $\alpha \xrightarrow{e} \beta$, $\varphi_\alpha = \varphi_\beta \circ D(e)$. A morphism between two cones, say $\theta : (Z', \varphi'_\alpha) \to (Z, \varphi_\alpha)$, is a morphism $\theta \in \mathrm{Hom}_{\mathsf{C}}(Z', Z)$

such that $\varphi_\alpha = \theta \circ \varphi'_\alpha$ for all vertices α. The collection of all cones under D is a category $\mathsf{Cone}(D)$.

A colimit of the diagram D is an initial object in $\mathsf{Cone}(D)$, and we denote it by $\operatorname{colim} D$ if it exists.

REMARK 7.9. The terminology 'cone under D' should evoke a picture of the diagram lying in a horizontal plane, with Z sitting below the plane, and maps φ_α down to Z from each vertex of the diagram in such a way that each triangle with Z as one vertex commutes.

The arguments showing that products, kernels, and pullbacks are special types of limits have analogues showing that coproducts, cokernels, and pushouts are special types of colimits. For example, if $f : X \to Y$, $\operatorname{coker} f$ is the colimit over the diagram

$$\begin{tikzcd} X \arrow[d, shift right, "f"'] \arrow[d, shift left, "0"] \\ Y \end{tikzcd}$$

and a pushout is a colimit over a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

A coproduct is a colimit over a diagram with no edges; for any diagram D, there is a morphism $\coprod_\alpha D(\alpha) \to \operatorname{colim} D$.

Definition 7.10. A covariant functor F

- preserves, or commutes with, limits if $F(\lim X_\alpha) \cong \lim FX_\alpha$ whenever $\lim X_\alpha$ exists;
- preserves, or commutes with colimits if $F(\operatorname{colim} X_\alpha) = \operatorname{colim} FX_\alpha$ whenever $\operatorname{colim} X_\alpha$ exists;

In section 11 we will show that left (respectively, right) exact functors between abelian categories preserve finite limits (respectively, colimits). Furthermore, left (respectively, right) exact contravariant functors change finite colimits (respectively, limits) to limits (respectively, colimits).

THEOREM 7.11. *The functor* $\operatorname{Hom}_{\mathsf{C}}(X, -)$ *preserves limits, and* $\operatorname{Hom}_{\mathsf{C}}(-, X)$ *sends colimits to limits.*

PROOF. Let $D : \mathcal{G} \to \mathsf{C}$ be a diagram with limit (L, ψ_α). If (Z, φ_α) is a cone in Set over the diagram $\operatorname{Hom}(X, -) \circ D$, then each φ_α is a set map $Z \to \operatorname{Hom}_{\mathsf{C}}(X, D(\alpha))$. Thus, for each $z \in Z$, the maps $\varphi_\alpha(z) : X \to D(\alpha)$ make $(X, \varphi_\alpha(z))$ a cone over D (if $\alpha \xrightarrow{e} \beta$, then $\varphi_\beta(z) = D(e) \circ \varphi_\alpha(z)$ because $\varphi_\beta = D(e) \circ \varphi_\alpha$). Hence there is a unique morphism $\theta_z : X \to L$ in C such that $\varphi_\alpha(z) = \psi_\alpha \circ \theta_z$ for all vertices α. Now, defining $\theta : Z \to \operatorname{Hom}_{\mathsf{C}}(X, L)$ by $\theta(z) = \theta_z$, we obtain $\varphi_\alpha = \psi_\alpha \circ \theta$ for all α. The uniqueness of θ follows from the uniqueness of θ_z. Hence $(\operatorname{Hom}_{\mathsf{C}}(X, L), \operatorname{Hom}(X, \psi_\alpha))$ is a limit of $\operatorname{Hom}(X, -) \circ D$.

If we write $F = \operatorname{Hom}_{\mathsf{C}}(-, X)$, then $\bar{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is the covariant functor $\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, -)$, which commutes with limits. But the limit of a diagram $D : \mathcal{G} \to \mathsf{C}^{\mathrm{op}}$ is the colimit of the 'same' diagram $D : \mathcal{G} \to \mathsf{C}$. Hence the result. $\square$

The isomorphisms in Example 6.6 are special cases of the next result.

COROLLARY 7.12. *If X_α is a family of objects in* C, *and* $X \in \mathrm{Ob}(\mathsf{C})$, *then*

$$\mathrm{Hom}_{\mathsf{C}}(X, \prod X_\alpha) \cong \prod \mathrm{Hom}_{\mathsf{C}}(X, X_\alpha)$$

and

$$\mathrm{Hom}_{\mathsf{C}}(\coprod X_\alpha, X) \cong \prod \mathrm{Hom}_{\mathsf{C}}(X_\alpha, X),$$

where $\mathrm{Hom}_{\mathsf{C}}(-,-)$ *is viewed as an object in* Set.

Using Yoneda's Lemma, the existence of limits in Set (Theorem 7.7), Theorem 7.11, and its Corollary we can define limits and colimits as the objects representing suitable functors.

COROLLARY 7.13. *If X is an object in* C, *and* $D : \mathcal{G} \to \mathsf{C}$ *is a diagram, then*

1. $\lim D$ *represents the functor* $X \mapsto \lim(\mathrm{Hom}(X,-) \circ D)$, *and*
2. $\operatorname{colim} D$ *represents the functor* $Y \mapsto \lim(\mathrm{Hom}(-,Y) \circ D)$.

COROLLARY 7.14. *If (F, G) is an adjoint pair of functors, then F preserves colimits, and G preserves limits.*

PROOF. Let $F : \mathsf{C} \to \mathsf{D}$, and let $D : \mathcal{G} \to \mathsf{C}$ be a diagram such that $\operatorname{colim} D$ exists. We must show that $F(\operatorname{colim} D) \cong \operatorname{colim}(F \circ D)$. For an arbitrary Y in D, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}}(F(\operatorname{colim} D), Y) &\cong \mathrm{Hom}_{\mathsf{C}}(\operatorname{colim} D, GY) && \text{by adjointness,}\\
&\cong \lim \mathrm{Hom}_{\mathsf{C}}(-, GY) \circ D\\
&= \lim \mathrm{Hom}_{\mathsf{C}}(D(\alpha), GY)\\
&\cong \lim \mathrm{Hom}_{\mathsf{D}}(F(D(\alpha)), Y) && \text{by adjointness,}\\
&\cong \mathrm{Hom}_{\mathsf{D}}(\operatorname{colim}(F \circ D), Y) && \text{by Corollary 7.13.}
\end{aligned}$$

Therefore, by Yoneda's Lemma $\operatorname{colim}(F \circ D) \cong F(\operatorname{colim} D)$.

The proof that G commutes with limits is similar (also see Exercise 7). □

Thus, if (F, G) is an adjoint pair of functors between abelian categories, F is right exact, and G is left exact. For example, $M \otimes_R -$ is right exact.

A morphism $f : X \to Y$ of schemes is affine if for every open affine subset of Y, $f^{-1}(V)$ is affine (see [**?**, Exercise II.5.17]). If X is noetherian, then f is affine if and only if f_* is exact [**?**, Exercise III.8.2] (f_* is always left exact because it is a right adjoint). Thus, f_* cannot have a right adjoint unless f is affine. If f is affine then f_* does have a right adjoint $f^!$. In the special case where X is a scheme over a field k, and $Y = \operatorname{Spec} k$, and f is the structure map then $f_* = H^0(X, -)$.

Definition 7.15. A category C is

- complete if it has all limits;
- cocomplete if it has all colimits.

(Remembering our convention that all index sets be small, it might be better to say "small-complete" rather than "complete".)

Definition 7.16. A category C is filtering if

- for any two objects $i, j \in \mathsf{C}$, there exists an object $k \in \mathsf{C}$ and morphisms $i \to k$ and $j \to k$, and

- given any two morphisms $\alpha_1, \alpha_2 : i \to j$, there exists an object k and a morphism $\beta : j \to k$ such that $\beta \circ \alpha_1 = \beta \circ \alpha_2$.

In particular, a directed category if filtering, but not conversely, since there may be more than one morphism between two objects.

EXERCISES

7.1 Consider the empty graph having one vertex and no edges. Show that a limit over this graph is a terminal object, and a colimit is an initial object.

Pullbacks and pushouts in $\mathsf{Mod}R$. Let A, B, C be R-modules.

(a) Show that the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "g_2"'] & B \\ C & \end{tikzcd}$$

is the cokernel of the map $g_1 \coprod g_2 : A \to B \coprod C$, together with the obvious maps from B and C to it.

(b) Show that the pullback of the diagram

$$\begin{tikzcd} . & C \arrow[d, "g_2"] \\ A \arrow[r, "g_1"] & B \end{tikzcd}$$

is the kernel of the map $g_1 \prod g_2 : A \prod C \to B$, together with the obvious maps to A and C from it.

8. Direct and inverse limits

The terms 'direct limit' and 'inverse limit' are becoming obsolete, replaced by the words 'colimit' and 'limit' with which they were once synonomous. There is still a tendency amongst algebraists to retain the words 'direct limit' and 'inverse limit' for colimits and limits which are taken over directed sets. We will adopt this convention; thus, in this book direct and inverse limits are *always taken over a directed set*. As such they are special kinds of colimits and limits.

Definition 8.1. A set I with a reflexive and transitive binary relation $\leq$ is quasi-ordered. (It is possible for $\alpha \leq \beta$ and $\beta \leq \alpha$ with $\alpha \neq \beta$, so I is not necessarily partially ordered.) If, in addition, for each pair $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$, we say that I is directed. A subset $J \subset I$ is cofinal in I if, for each $\alpha \in I$, there exists $\gamma \in J$ with $\alpha \leq \gamma$.

A quasi-ordered set I can be viewed as a category: the objects are the elements of I, and the morphisms are

$$\mathrm{Hom}(\alpha, \beta) = \begin{cases} \text{singleton} & \text{if } \alpha \leq \beta, \\ \emptyset & \text{otherwise;} \end{cases}$$

if $\alpha \leq \beta$ write ι^α_β for the unique morphism $\alpha \to \beta$, and define the composition $\iota^\beta_\gamma \circ \iota^\alpha_\beta = \iota^\alpha_\gamma$ whenever $\alpha \leq \beta \leq \gamma$. Conversely, if $\mathcal{I}$ is a category such that for every pair of objects α and β there is at most one element in $\mathrm{Hom}_{\mathcal{I}}(\alpha, \beta) \cup \mathrm{Hom}_{\mathcal{I}}(\beta, \alpha)$, then $\mathcal{I}$ is equivalent to a category of the form in the previous paragraph; we will say that $\mathcal{I}$ is a quasi-ordered category.

Definition 8.2. Let C be a category, and I a directed set viewed as a category. A directed system in C, with index set I, is a covariant functor $I \to \mathsf{C}$.

Thus a directed system consists of objects X_α, indexed by $\alpha \in I$, and morphisms $\varphi^\alpha_\beta : X_\alpha \to X_\beta$ whenever $\alpha \leq \beta$, which satisfy:

- $\varphi^\alpha_\alpha = \mathrm{id}_{X_\alpha}$, and
- $\varphi^\beta_\gamma \circ \varphi^\alpha_\beta = \varphi^\alpha_\gamma$ whenever $\alpha \leq \beta \leq \gamma$.

Definition 8.3. Let $(X_\alpha, \varphi^\alpha_\beta)$ be a directed system in C, indexed by I. A direct limit of this system, is an object $\varinjlim X_\alpha$ in C and a set of morphisms

$$\varphi_\alpha : X_\alpha \to \lim_{\to} X_\alpha$$

such that

- $\varphi_\alpha = \varphi_\beta \varphi^\alpha_\beta$ whenever $\alpha \leq \beta$, and
- if Y is any object, and $\psi_\alpha : X_\alpha \to Y$ are morphisms such that $\psi_\alpha = \psi_\beta \psi^\alpha_\beta$ whenever $\alpha \leq \beta$, then there exists a unique morphism $\rho : \varinjlim X_\alpha \to Y$ such that $\psi_\alpha = \rho \circ \varphi_\alpha$ for all α.

If every directed system in C has a direct limit in C we say that C has direct limits.

EXAMPLE 8.4. Let R be a commutative domain, and $\mathfrak{p}$ a prime ideal. Then the localization $R_\mathfrak{p}$ can be realized as the direct limit of the localizations $R[u^{-1}]$, where the limit is taken over all $u \notin \mathfrak{p}$, and $u \leq v$ if and only if $v \in uR$.

PROPOSITION 8.5. *Every directed system in* $\mathsf{Mod}R$ *has a direct limit. In particular, retaining the notation in Definition 8.3,*

$$\lim_{\to} X_\alpha = \coprod X_\alpha / Y$$

where Y is the submodule generated by all elements $\varphi^\alpha_\beta(x_\alpha) - x_\alpha$ whenever $\alpha \leq \beta$ (we are identifying X_α with its image in $\coprod X_\alpha$).

PROPOSITION 8.6. *Let $(X_\alpha, \varphi^\alpha_\beta)$ be a direct system in C, indexed by I. If J is cofinal in I, then*

$$\lim_{\substack{\to \\ I}} X_\alpha \cong \lim_{\substack{\to \\ J}} X_\alpha.$$

In particular, if there is a unique element $\omega \in I$ such that $\alpha \leq \omega$ for all $\alpha \in I$, then $\lim_{\to} X_\alpha \cong X_\omega$.

Elements in $\lim_{\to} X_\alpha$ are images of elements in the X_α's, and it is easy to recognize when the image of such an element is zero in the direct limit.

PROPOSITION 8.7. *Let $(X_\alpha, \varphi^\alpha_\beta)$ be a directed system in* $\mathsf{Mod}R$*. If we identify* $\lim_{\to} X_\alpha$ *with* $\coprod X_\alpha / Y$*, as in Proposition 8.5, then*

1. *every element in* $\lim_{\to} X_\alpha$ *is an image of an element in some X_α, that is, of the form $x_\alpha + Y$ for some $x_\alpha \in X_\alpha$;*
2. *$x_\alpha + Y = 0$ if and only if $\varphi^\alpha_\beta x_\alpha = 0$ for some $\beta \geq \alpha$.*

COROLLARY 8.8. *Let $(X_\alpha, \varphi^\alpha_\beta)$ be a direct system, indexed by an ordered set I. If there exists $\gamma \in I$ such that φ^γ_δ is an isomorphism for all $\gamma \leq \delta$, then*

$$\varinjlim_{\alpha} X_\alpha \cong X_\gamma.$$

PROPOSITION 8.9. *If M is a finitely presented R-module, then* $\mathrm{Hom}_R(M, -)$ *commutes with direct limits.*

Proof. Let X_α be a directed system in $\mathsf{Mod}R$. The maps $X_\alpha \to \varinjlim X_\alpha$ yield maps $\mathrm{Hom}_R(M, X_\alpha) \to \mathrm{Hom}_R(M, \varinjlim X_\alpha)$, so the universal property yields a map $\Phi : \varinjlim \mathrm{Hom}_R(M, X_\alpha) \to \mathrm{Hom}_R(M, \varinjlim X_\alpha)$: an element of the left hand side is the image of some $f : M \to X_\beta$, and $\Phi(f)$ is the composition $M \to X_\beta \to \varinjlim X_\alpha$. We must prove that Φ is an isomorphism. Injectivity is easy to prove, so we only prove surjectivity. Let $f : M \to \varinjlim X_\alpha$.

By hypothesis, there is an exact sequence $0 \to K \to F \to M \to 0$ with F finitely generated free, and K finitely generated. Since F is finitely generated free, and the system is directed we can factor the map $F \to M \to \varinjlim X_\alpha$ through $F \to X_\beta$ for some β. Because the composition $K \to F \to X_\beta \to \varinjlim X_\alpha$ is zero, and K is finitely generated, there is some $\gamma \geq \alpha$ such that $K \to F \to X_\beta \to X_\gamma$ is zero. This yields a factorization of the original map $M \to X_\gamma \to \varinjlim X_\alpha$. If $f' : M \to X_\gamma$ is this map, then $\Phi(f') = f$. $\square$

Definition 8.10. An object $X \in \mathsf{C}$ is of finite type if $\mathrm{Hom}_{\mathsf{C}}(X, -)$ preserves coproducts.

It follows from Proposition 8.9 that a finitely presented module is a small object of $\mathsf{Mod}R$; this is because a coproduct $\coprod_I X_\alpha$ is the direct limit of the directed system $\{X_\alpha \mid \alpha \in I\} \cup \{X_\omega\}$ where $X_\omega = \coprod_I X_\alpha$, and the only relations are $\alpha \leq \omega$ for $\alpha \in I$, and the map $X_\alpha \to X_\omega$ is the natural one.

Definition 8.11. Let I be a directed set viewed as a category, and let C be a category. An inverse system in C, with index set I, is a contravariant functor $I \to \mathsf{C}$; that is, it consists of objects X_α, indexed by the elements of I, and morphisms $\psi_\alpha^\beta : X_\beta \to X_\alpha$ whenever $\alpha \leq \beta$ satisfying:

- $\psi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$, and
- $\psi_\alpha^\beta \circ \psi_\beta^\gamma = \psi_\alpha^\gamma$ whenever $\alpha \leq \beta \leq \gamma$.

An inverse limit of this inverse system, is an object $\lim_{\leftarrow} X_\alpha$ in C and a set of morphisms

$$\psi_\alpha : \varprojlim X_\alpha \to X_\alpha$$

such that

- $\psi_\alpha = \psi_\alpha^\beta \psi_\beta$ whenever $\alpha \leq \beta$, and
- if Y is any object, and $\psi_\alpha : Y \to X_\alpha$ are morphisms such that $\psi_\alpha = \psi_\alpha^\beta \psi_\beta$ whenever $\alpha \leq \beta$, then there exists a unique morphism $\rho : Y \to \lim_{\leftarrow} X_\alpha$ such that $\psi_\alpha = \psi_\alpha \circ \rho$ for all α.

If every inverse system has an inverse limit we say that inverse limits exist in C.

There is always a morphism $\lim_{\leftarrow} X_\alpha \to \prod_I X_\alpha$, and this is an isomorphism if the index set has the trivial ordering; in particular, a product is a special kind of inverse limit.

Example 8.12. Since small limits exist in Set, so do small inverse limits. Explicitly, if $(X_\alpha, \psi_\alpha^\beta)$ is an inverse system then

$$\varprojlim X_\alpha = \{(x_\alpha) \in \prod X_\alpha \mid \psi_\alpha^\beta(x_\beta) = x_\alpha \text{ whenever } \alpha \leq \beta\}.$$

Direct limits need not exist in Set (see Exercise 9). However if $(X_\alpha, \varphi_\beta^\alpha)$ is a direct system indexed by a *directed* set I then $\lim_{\rightarrow} X_\alpha$ exists. Define an equivalence relation $\sim$ on the disjoint union $\coprod X_\alpha$ by $x_\alpha \sim x_\beta$ ($x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$) if

$\varphi_\gamma^\alpha(x_\alpha) = \varphi_\gamma^\beta(x_\beta)$ for some γ such that $\alpha \le \gamma$ and $\beta \le \gamma$. Notice that the hypothesis that I is directed is necessary to prove the transitivity of $\sim$. Now define $\lim_\rightarrow X_\alpha = \coprod X_\alpha / \sim$ and define $\varphi_\alpha : X_\alpha \to \lim_\rightarrow X_\alpha$ to be the obvious map.

PROPOSITION 8.13. *Suppose we have morphisms of direct systems*

$$(X_\alpha, \varphi_\beta^\alpha) \to (Y_\alpha, \psi_\beta^\alpha) \to (Z_\alpha, \phi_\beta^\alpha)$$

in $\mathsf{Mod}R$. *Suppose further that for each* α *the sequence* $0 \to X_\alpha \to Y_\alpha \to Z_\alpha \to 0$ *is exact.*

1. *There is an exact sequence*

$$\lim_{\rightarrow} X_\alpha \to \lim_{\rightarrow} Y_\alpha \to \lim_{\rightarrow} Z_\alpha \to 0.$$

2. *If* I *is directed, then there is an exact sequence*

$$0 \to \lim_{\rightarrow} X_\alpha \to \lim_{\rightarrow} Y_\alpha \to \lim_{\rightarrow} Z_\alpha \to 0.$$

PROOF. [**141**, Theorem 2.18] □

The situation for inverse systems is more delicate.

EXAMPLE 8.14. The analogue of Proposition 8.13 for inverse systems fails. If R is a discrete valuation ring with maximal ideal $\mathfrak{m}$ there are inverse systems (with obvious maps)

$$\cdots \to \mathfrak{m}^n \to \cdots\cdots \to \mathfrak{m}^2 \to \mathfrak{m},$$
$$\cdots \to R \to \cdots\cdots \to R \to R,$$
$$\cdots \to R/\mathfrak{m}^n \to \cdots\cdots \to R/\mathfrak{m}^2 \to R/\mathfrak{m}.$$

For each $n \ge 1$ there is an exact sequence

$$0 \to \mathfrak{m}^n \to R \to R/\mathfrak{m}^n \to 0. \tag{8-1}$$

Now $\lim_\leftarrow \mathfrak{m}^n = 0$ since $\bigcap \mathfrak{m}^n = 0$, and $\lim_\leftarrow R/\mathfrak{m}^n = \hat{R}$, the $\mathfrak{m}$-adic completion of R. Hence taking direct limits of the individual terms in the short exact sequences (8-1) gives a complex $0 \to 0 \to R \to \hat{R} \to 0$; this is not exact unless R is complete.

Nevertheless, there is an analogue of Proposition 8.13 for *some* inverse systems.

Definition 8.15. An inverse system of abelian groups $(X_\alpha, \varphi_\alpha^\beta, \alpha \in \mathbb{N})$, satisfies the Mittag-Leffler condition if, for each n, there exists $n_0 \ge n$ such that the image of $M_i \to M_n$ equals the image of $M_j \to M_n$ whenever $i, j \ge n_0$.

PROPOSITION 8.16. *Suppose that there are morphisms of inverse systems of abelian groups*

$$(X_\alpha, \varphi_\alpha^\beta) \to (Y_\alpha, \psi_\alpha^\beta) \to (Z_\alpha, \phi_\alpha^\beta),$$

where each system is indexed by $\alpha \in \mathbb{N}$. *Further, suppose that, for each* α, *the sequence* $0 \to X_\alpha \to Y_\alpha \to Z_\alpha \to 0$ *is exact.*

1. *If* (Y_α) *satisfies the Mittag-Leffler condition, so does* (Z_α).
2. *If* (X_α) *and* (Z_α) *both satisfy the Mittag-Leffler condition, so does* (Y_α).
3. *If* (X_α) *satisfies the Mittag-Leffler condition, then the sequence*

$$0 \to \lim_{\leftarrow} X_\alpha \to \lim_{\leftarrow} Y_\alpha \to \lim_{\leftarrow} Z_\alpha \to 0$$

is exact.

PROOF. □

EXERCISES

8.1 Show that there is a category Graph whose objects are the directed graphs and a morphism $f : (V_1, E_1) \to (V_2, E_2)$ is a set map sending vertices to vertices and paths to paths in such a way that $f(E^{\alpha}_{\beta}) \subset E^{f(\alpha)}_{f(\beta)}$.

8.2 Let X be a topological space and let Open_X be the category of open subsets of X, as described in Example 1.13.
(a) Show that Open_X forms a directed set if we define $V \le U$ whenever $U \subset V$. [*Hint:* if U and V are open so is $U \cap V$.]
(b) If $x \in X$ show that the subcategory consisting of those open sets which contain x is a directed system.

8.3 Let p be a prime number. For $m \le n$ define $\varphi^n_m : \mathbb{Z}/(p^n) \to \mathbb{Z}/(p^m)$ to be the natural projection. Show this is an inverse system and that its inverse limit exists. This inverse limit is defined to be the ring of p-adic integers.

8.4 Do Exercise 2 in [**84**, Volume II, Section 2.6]. This gives a simple example where a direct limit does not exist.

8.5 Inverse limits in Ring.

8.6 Show that an R-module is the direct limit of its finitely generated submodules, and that this gives a directed system.

8.7 Show that the dual of a quasi-ordered category is quasi-ordered. Deduce that direct systems in a category C are identical to inverse systems in the dual category C^{op}.

8.8 As in Example 7.4, if X_α is a direct system, there is a morphism $g : \coprod X_\alpha \to \lim_{\to} X_\alpha$ such that the diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{i_\alpha} & \coprod X_\alpha \\ & & \downarrow g \\ & & \varinjlim X_\alpha \end{array}$$

commutes. Now suppose that the quasi-ordering on I satisfies $\alpha \le \beta$ if and only if $\alpha = \beta$; this is called the trivial quasi-ordering. Thus the only maps in the direct system are $\varphi^\alpha_\alpha = \mathrm{id}_{X_\alpha}$. Show that the universal properties defining the direct limit and the coproduct of the X_α coincide so, provided they exist, the map g gives an isomorphism

$$\coprod X_\alpha \cong \lim_{\to} X_\alpha.$$

Thus coproducts are a special case of direct limits. In particular, direct limits do not always exist in the category of finitely generated modules over a ring.

8.9

9. Existence of adjoints

We showed in section 7 that if (F, G) is an adjoint pair then F preserves colimits and G preserves limits. Hence, if a functor F is to have a right (respectively, left) adjoint it is necessary that F preserve colimits (respectively, limits). Often this condition is also sufficient.

THEOREM 9.1 (Freyd's Adjoint Functor Theorem). *Let D be a complete category. A functor $G : \mathsf{D} \to \mathsf{C}$ has a left adjoint if and only if*

1. *it preserves small limits, and*
2. *it satisfies the solution set condition: for each $C \in \mathrm{Ob}(\mathsf{C})$ there is a small set of pairs $(D_\alpha, f_\alpha : C \to GD_\alpha)$ such that every morphism $g : C \to GX$*

may be factorized as $C \xrightarrow{f_\alpha} GD_\alpha \to GX$ *for some* α *and some morphism* $GD_\alpha \to GX$.

PROOF. [**112**, Theorem 2, page 117]. □

REMARK 9.2. If G has a left adjoint, F say, then $(FC, \eta_C : C \to GFC)$ is a solution set because the adjunction isomorphism $\mathrm{Hom}_{\mathsf{D}}(FC, X) \cong \mathrm{Hom}_{\mathsf{C}}(C, GX)$ sends a morphism $h : FC \to X$ to $(Gh) \circ \eta_C : C \to GFC \to GX$.

The solution set condition cannot in general be satisfied by taking the set of all pairs $(D, f : C \to D)$ because this set is not usually small (cf. Example 3.5).

Definition 9.3. A category C is well-powered if the subobjects of each object in C can be indexed by a small set.

Set is well-powered because one of the axioms of a universe is that the power set of a small set is a small set.

THEOREM 9.4 (Special Adjoint Functor Theorem). *Let* D *be a small-complete well-powered category with small Hom-sets and a small cogenerating set. Let* C *be a category with small Hom-sets. A functor* $G : \mathsf{D} \to \mathsf{C}$ *has a left adjoint if and only if it preserves small limits.*

PROOF. [**112**, Corollary, page 120]. □

COROLLARY 9.5. *Let* C *be a co-well-powered category having small Hom-sets and small colimits. If* C *has a generator, then a contravariant functor* $F : \mathsf{C} \to \mathsf{D}$ *has a right adjoint if an only if it preserves small colimits.*

PROOF. Consider F as a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$. □

Special cases of the adjoint functor theorems applied to module categories yield Watt's theorems.

THEOREM 9.6. *If* $G : \mathsf{Mod}R \to \mathsf{Ab}$ *is a contravariant left exact functor, then* $G \cong \mathrm{Hom}_R(-, M)$ *where* $M \cong GR$.

THEOREM 9.7. *If* $G : \mathsf{Mod}R \to \mathsf{Ab}$ *is a covariant functor preserving inverse limits, then* $G \cong \mathrm{Hom}_R(M, -)$ *for some* M.

Definition 9.8. A dualizing object in a category C is an object E with the property that $\mathrm{Hom}_{\mathsf{C}}(-, E) : \mathsf{C} \to \mathsf{D}$ is a duality for a suitable category D.

Matlis duality: $(R, \mathfrak{m}, k)$ complete Noetherian local, E the injective envelope of k. Then $\mathrm{Hom}_R(-, E)$ is a duality between the artinian R-modules and the noetherian R-modules.

Freyd [**62**, page 84] refers to the next result as 'electrifying'!

PROPOSITION 9.9. *An abelian category with a generator is well-powered.*

PROOF. If P is a generator, then there is a bijection between subobjects of X and subsets of $\mathrm{Hom}(P, X)$: the subobject $X' \to X$ corresponds to the subset $\mathrm{Hom}(P, X') \subset \mathrm{Hom}(P, X)$. Since Set is well-powered, the result follows. □

10. Additive categories

Definition 10.1. A category C is pre-additive if all Hom sets are abelian groups and composition is bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ between pre-additive categories is additive if each $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is a group homomorphism.

EXAMPLE 10.2. The categories Set, Group and Ring are not pre-additive. Although there is no obvious abelian group structure on the Hom spaces, a little thought is required to see that none can be imposed in such a way as to make composition of morphisms bilinear.

By definition a group is non-empty, but $\mathrm{Hom}_{\mathsf{Ring}}(R, S)$ may be empty, so Ring is not pre-additive.

Suppose to the contrary that Set is pre-additive. In Set, let X be any set, and $Y = \{s, t\}$ a set with two distinct elements. Since $\mathrm{Hom}_{\mathsf{Set}}(X, \{s\})$ has one element it is the trivial group so, since composition is bilinear $X \to \{s\} \to Y$ must be the identity element in the group $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$. But the same argument applies to $X \to \{t\} \to Y$, whence $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$ has two distinct identity elements! This is absurd.

There is an obvious generalization of pre-additive categories.

Definition 10.3. Let k be a commutative ring. A category C is k-linear if each $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ has a k-module structure, and the composition $(f, g) \to f \circ g$ is k-bilinear. A functor F between k-linear categories is k-linear if all the maps F_{XY} are k-linear.

NOTATION . The group operation in the Hom-sets of a k-linear category will be written additively. Thus the identity elements will be denoted by 0; there is some potential for confusion with many different zeroes so we will write 0_{XY} for the zero element in $\mathrm{Hom}(X, Y)$ when appropriate.

Definition 10.4. A category C is additive if

- it is $\mathbb{Z}$-linear (i.e., pre-additive),
- it has a zero object, and
- it has all finite products and coproducts.

REMARK 10.5. In Set the empty set $\emptyset$ is an initial object, and any singleton is a terminal object. Therefore a product in Set indexed by the empty set is a singleton, and the coproduct in Set indexed by the empty set is the empty set. Hence in an additive category a coproduct over the empty family is an initial object and a product over the empty family is a terminal object. Therefore the axiom that an additive category has a zero object already follows from the other axioms.

In an additive category the bilinearity of the composition

$$\mathrm{Hom}(0, Y) \times \mathrm{Hom}(X, 0) \to \mathrm{Hom}(X, Y)$$

implies that 0_{XY} is the composition $X \to 0 \to Y$.

EXAMPLE 10.6. The module category $\mathsf{Mod}R$ is additive: the abelian group structure on $\mathrm{Hom}_R(M, N)$ is that induced from the abelian structure on N, and the existence of products and coproducts was explained in Example 3.7. The full subcategory $\mathsf{mod}R$ of finitely generated modules is also additive since only *finite* products and coproducts are required to exist.

PROPOSITION 10.7. *In an additive category the product and coproduct of any finite set of objects are isomorphic.*

PROOF. Let X_α be a finite set of objects in an additive category. Let

$$p_\alpha : \prod X_\alpha \to X_\alpha \qquad \text{and} \qquad i_\alpha : X_a \to \coprod X_\alpha$$

be the morphisms occuring in the definitions of the product and coproduct. For each pair (α, β) define $\delta^\alpha_\beta : X_\alpha \to X_\beta$ by

$$\delta^\alpha_\beta = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \mathrm{id}_{X_\alpha} & \text{if } \alpha = \beta. \end{cases}$$

Fixing α and letting β vary, by the universal property of the product, there exist morphisms $f_\alpha : X_\alpha \to \prod X_\alpha$ such that $\delta^\alpha_\beta = p_\beta f_\alpha$. Fixing β and letting α vary, by the universal property of the coproduct, there exist morphisms $g_\beta : \coprod X_\alpha \to X_\beta$ such that $\delta^\alpha_\beta = g_\beta i_\alpha$.

For each α we have

$$\left(\sum_\beta i_\beta g_\beta\right) \circ i_\alpha = \mathrm{id}_{\coprod X_\alpha} \circ i_\alpha$$

so, by the uniqueness of the map g in (3-1),

$$\sum_\beta i_\beta g_\beta = \mathrm{id}_{\coprod X_\alpha}.$$

Now we will show that $(\coprod X_\alpha, g_\alpha)$ has the required universal property to be a product. Let $q_\alpha : Y \to X_\alpha$ be morphisms. Define

$$\rho = \sum_\alpha i_\alpha q_\alpha : Y \to \coprod X_\alpha.$$

Then

$$g_\beta \circ \rho = \sum_\alpha g_\beta i_\alpha q_\alpha = \sum_\alpha \delta^\alpha_\beta q_\alpha = q_\beta.$$

If $\rho' : Y \to \coprod X_\alpha$ also satisfies $g_\beta \circ \rho' = q_\beta$ for all β, then

$$\rho' = \mathrm{id}_{\coprod X_\alpha} \circ \rho' = \left(\sum_\alpha i_\alpha g_\alpha\right) \circ \rho' = \sum_\alpha i_\alpha q_\alpha = \rho.$$

This completes the proof. □

COROLLARY 10.8. *Let f be a morphism in an additive category. Then f is a monomorphism (respectively, an epimorphism) if and only if $fg = 0$ implies $g = 0$ (respectively, $gf = 0$ implies $g = 0$).*

PROOF. Exercise 3. □

If X is an object in an additive category C, then $\mathrm{Hom}_{\mathsf{C}}(X, X)$ is a ring with identity, and the associativity and trilinearity of the composition maps

$$\mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{C}}(W, X) \to \mathrm{Hom}_{\mathsf{C}}(W, Z)$$

give each $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ the structure of a $\mathrm{Hom}_{\mathsf{C}}(Y, Y)$-$\mathrm{Hom}_{\mathsf{C}}(X, X)$-bimodule.

Definition 10.9. Let $f : X \to Y$ be a morphism in an additive category $\mathcal{A}$.

A kernel of f is a pair (A, α), consisting of an object A and a morphism $\alpha : A \to X$ such that $f\alpha = 0$ and, if $\alpha' : A' \to X$ is a morphism for which $f\alpha' = 0$, then there is a unique morphism $\rho : A' \to A$ such that $\alpha' = \alpha\rho$.

A cokernel of f is a pair (B, β), consisting of an object B and a morphism $\beta : Y \to B$ such that $\beta f = 0$ and, if $\beta' : Y \to B'$ is a morphism for which $\beta' f = 0$, then there is a unique morphism $\rho : B \to B'$ such that $\beta' = \rho\beta$.

The uniqueness of kernels and cokernels follows from the uniqueness of limits and colimits: if $f : X \to Y$, then $\ker f$ is the inverse limit of the inverse system $(X \xrightarrow{f} Y)$, and $\operatorname{coker} f$ is the direct limit of the same system.

LEMMA 10.10. *Let $f : X \to Y$ be a morphism in an additive category. Then*

1. *f is a monomorphism if and only if* $\ker f = (0 \to X)$*;*
2. *f is an epimorphism if and only if* $\operatorname{coker} f = (Y \to 0)$.

PROOF. (1) ($\Rightarrow$) Let $\alpha' : A' \to X$ be a morphism such that $f\alpha' = 0$. Since f is a monomorphism, $\alpha' = 0$. Hence there is a unique morphism $\rho : A' \to 0$ such that $\alpha' = \rho \circ 0$, namely $\rho = 0$. Thus the pair $(0, 0 : 0 \to X)$ satisfies the required universal property to be a kernel.

($\Leftarrow$) If $g_1, g_2 : W \to X$ are morphisms such that $fg_1 = fg_2$ then $f(g_1 - g_2) = 0$ so, by the universal property of a kernel, there is a (unique) morphism $\rho : W \to 0$ such that $g_1 - g_2 = 0 \circ \rho$. That is, $g_1 - g_2 = 0$, showing that f is a monomorphism.

(2) Exercise. □

PROPOSITION 10.11. *Let $f : X \to Y$ be a morphism in an additive category $\mathcal{A}$. Then*

1. *if* $\ker f$ *exists, it is a subobject of X;*
2. *if* $\operatorname{coker} f$ *exists, it is a quotient object of Y;*
3. *there is a natural map* $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$*, if these objects exist.*

PROOF. (1) Let $(A, \alpha) = \ker f$ and suppose there exist morphisms $\rho_1, \rho_2 : W \to A$ such that $\alpha\rho_1 = \alpha\rho_2$. Since $f\alpha\rho_1 = 0$, the uniqueness of ρ in the definition of the kernel implies that $\rho_1 = \rho_2$.

(2) Let $(B, \beta) = \operatorname{coker} f$ and suppose there exist morphisms $\rho_1, \rho_2 : B \to Z$ such that $\rho_1\beta = \rho_2\beta$. Since $\rho_1\beta f = 0$, the uniqueness of ρ in the definition of the cokernel implies that $\rho_1 = \rho_2$.

(3) Retain the earlier notation. Provided all the required objects exist, there is a diagram:

$$\begin{array}{ccccccc} A & \sharp\xrightarrow{\alpha} & X & \sharp\xrightarrow{f} & Y & \sharp\xrightarrow{\beta} & B. \\ & & \gamma \downarrow & & \uparrow \delta & & \\ & & \operatorname{coker}\alpha & & \ker\beta & & \end{array}$$

We will construct a morphism $\mu : \operatorname{coker}\alpha \to \ker\beta$ making the rectangle commute. Since $f\alpha = 0$, the defining property of $\operatorname{coker}\alpha$ guarantees the existence of a morphism $\rho : \operatorname{coker}\alpha \to Y$ such that $f = \rho\gamma$. Thus $\beta\rho\gamma = \beta f = 0$. But γ is an epimorphism, so $\beta\rho = 0$. The defining property of $\ker\beta$ guarantees the existence of a morphism $\mu : \operatorname{coker}\alpha \to \ker\beta$ such that $\rho = \delta\mu$. Therefore the morphism μ makes the diagram commute. □

EXERCISES

10.1 Show that in an additive category $\mathrm{Hom}_{\mathcal{A}}(X, X)$ is a ring.

10.2 Let R be a ring with identity. Define the category $\mathcal{R}$ to have a single object R and morphisms the elements of R, with composition being the multiplication in R. The addition in R makes $\mathcal{R}$ a $\mathbb{Z}$-linear category. Show that $\mathcal{A}dd(\mathcal{R}, \mathsf{Ab})$, the category of additive functors from $\mathcal{R}$ to Ab, is equivalent to $\mathsf{Mod}R$.

10.3 Prove Corollary 10.8.

10.4 Prove Lemma 10.10(2).

10.5 Show that the kernel and cokernel of a morphism $f : X \to Y$ are, respectively, a subobject of X, and a quotient of Y.

10.6 In an additive category show that
(a) $\ker(X \xrightarrow{0} Y) = (X, \mathrm{id}_X)$;
(b) $\mathrm{coker}(X \xrightarrow{0} Y) = (Y, \mathrm{id}_Y)$;
(c) $f = 0$ if and only if $\mathrm{Im}\, f = 0$.

10.7 Show that, in an abelian category, every morphism $f : X \to Y$ may be factored as $f = \beta \circ \alpha$ with α an epimorphism and β a monomorphism. [*Hint:* use the proof of Proposition 10.11 to show there are morphisms $X \to \mathrm{coker}\ker f \xrightarrow{\sim} \ker\mathrm{coker} f \to Y$.]

10.8 Clarify the remark after Definition 10.9: for a fixed object X, define a notion of equivalence of pairs (A, α) consisting of an object A and a morphism $\alpha : A \to X$, and show that any two kernels of $f : X \to Y$ are equivalent. Do the same for cokernels.

10.9 Use the proof of Proposition 10.7 to show that in an additive category the maps $p_\alpha : \prod X_\alpha \to X_\alpha$ and $i_\alpha : X_\alpha \to \coprod X_\alpha$ arising in the definition of the product and coproduct are, respectively, an epimorphism and a monomorphism.

11. Abelian categories

Definition 11.1. An additive category is abelian if every morphism $f : X \to Y$ has a kernel and cokernel, and the natural map $\mathrm{coker}\ker f \to \ker\mathrm{coker} f$ is an isomorphism.

EXAMPLE 11.2. Kernels and cokernels exist in $\mathsf{Mod}R$: if $f : X \to Y$ then $\ker f = \{x \in X \mid f(x) = 0\}$ (together with its natural inclusion in X) and $\mathrm{coker} f = Y/\{f(x) \mid x \in X\}$ (together with the natural surjection from Y). However, if R is not left noetherian then the full subcategory of finitely generated modules is *not* abelian since it does not have kernels: if J is a left ideal which is not finitely generated then the kernel of $R \to R/J$ does not exist in this subcategory.

THEOREM 11.3. *The category* $\mathsf{Mod}R$ *of left modules over a ring* R *is abelian. If* R *is left noetherian, then the full subcategory* $\mathsf{mod}R$ *consisting of the finitely generated modules is also abelian.*

EXAMPLE 11.4. The full subcategory of $\mathsf{Mod}R$ consisting of the projective modules is not usually abelian since the cokernel of a map between projectives need not be projective.

EXAMPLE 11.5. The category of sheaves of abelian groups on a topological space is abelian.

EXAMPLE 11.6. The category $\mathsf{Filt}R$ of filtered left modules over a filtered ring R is not usually abelian, although it does have kernels and cokernels. Let $f : M \to$

N be a morphism of filtered modules. The filtration on the cokernel of f is given by $F_i(\text{coker } f) = F_iN + f(M)/f(M)$, and the filtration on $X = \ker \text{coker } f$ is given by $F_iX = F_iN \cap f(M)$. The filtration on $Y = \text{coker} \ker f$ is given by $F_iY = f(F_iM)$, but the natural isomorphism $X \cong Y$ of unfiltered modules does not respect the filtrations on X and Y, so $\ker \text{coker } f \ncong \text{coker} \ker f$ in general.

Definition 11.7. In an abelian category the image of a morphism f is defined to be

$$\operatorname{im}(f) := \ker \text{coker } f.$$

LEMMA 11.8. *Let $f : X \to Y$ be a morphism in an abelian category $\mathcal{A}$. If f is a monomorphism and an epimorphism then f is an isomorphism.*

PROOF. There is an isomorphism $\text{coker} \ker f \xrightarrow{\sim} \ker \text{coker } f$ since $\mathcal{A}$ is abelian. By Lemma 10.10, $\ker f = (0 \to X)$ and $\text{coker } f = (Y \to 0)$. But $\text{coker}(0 \to X) = X$ and $\ker(Y \to 0) = Y$ by Exercise 6 below, whence the result. □

Definition 11.9. A sequence $\ldots \to X_{i-1} \to X_i \to X_{i+1} \to \ldots$ in an abelian category is

- exact if $\operatorname{im}(X_{i-1} \to X_i) = \ker(X_i \to X_{i+1})$ for all i;
- a complex if, for each i, the composition $X_{i-1} \to X_i \to X_{i+1}$ is zero.

PROPOSITION 11.10. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then F is*

1. left exact *if and only if, for every exact sequence $0 \to X \to Y \to Z$, the sequence $0 \to FX \to FY \to FZ$ is exact;*
2. right exact *if, for every exact sequence $X \to Y \to Z \to 0$, the sequence $FX \to FY \to FZ \to 0$ is exact;*
3. exact *if and only if, for every exact sequence $X \to Y \to Z$, the sequence $FX \to FY \to FZ$ is exact.*

PROPOSITION 11.11. *Let A be an object in an abelian category $\mathcal{A}$ and let R denote the ring $\operatorname{Hom}_{\mathcal{A}}(A, A)$. Then $\operatorname{Hom}_{\mathcal{A}}(A, -)$ and $\operatorname{Hom}_{\mathcal{A}}(-, A)$ are left exact functors taking values in $\mathsf{Mod}R^{\text{op}}$ and $\mathsf{Mod}R$ respectively.*

PROOF. We will just treat the covariant functor $F = \operatorname{Hom}_{\mathcal{A}}(A, -)$; by considering the contravariant functor $G = \operatorname{Hom}_{\mathcal{A}}(-, A)$ as a covariant functor on the dual category $\mathcal{A}^{\text{op}}$ the result for F implies the result for G.

It is easy to check that, for an object X, the map

$$\operatorname{Hom}_{\mathcal{A}}(A, X) \times \operatorname{Hom}_{\mathcal{A}}(A, A) \to \operatorname{Hom}_{\mathcal{A}}(A, X),$$

given by composition of morphisms in $\mathcal{A}$, endows FX with the structure of a right R-module. Moreover, if $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ then $Ff : FX \to FY$, which is given by $Ff(\alpha) = f\alpha$, is a right R-module map because composition of morphisms is associative (and $\mathcal{A}$ is a $\mathbb{Z}$-linear category).

It remains to show that F is left exact. Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be exact in $\mathcal{A}$. Then $0 \to FX \xrightarrow{\alpha^*} FY \xrightarrow{\beta^*} FZ$ is a complex of right R-modules. Since $\alpha^*(f) = \alpha f$, the fact that α is a monomorphism implies that α^* is injective; hence this complex is exact at FX. Now suppose that $\beta^*(g) = 0$; that is, $\beta g = 0$. Since (X, α) is the kernel of β, there exists a unique $\rho : A \to X$ such that $g = \alpha\rho$; that is, $g \in \operatorname{Im} \alpha^*$, which proves exactness at FY. □

PROPOSITION 11.12. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a exact functor between abelian categories. Then F is faithful if and only if, for every non-zero object A in $\mathcal{A}$, $FA \ncong 0$.*

PROOF. ($\Rightarrow$) Suppose F is faithful and $A \neq 0$. Then the identity map id_A is non-zero, so $F(\mathrm{id}_A) \neq 0$. But this equals id_{FA}, so $FA \neq 0$.

($\Leftarrow$) Let $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Since F is exact, the sequence

$$0 \to F(\ker f) \to FX \xrightarrow{Ff} FY \to F(\operatorname{coker} f) \to 0$$

is exact in $\mathcal{A}'$. It follows that $F(\ker f) = \ker(Ff)$ and that $F(\operatorname{coker} f) = \operatorname{coker}(Ff)$. If we now suppose that $f \neq 0$ then $\operatorname{Im} f = \ker \operatorname{coker} f \neq 0$. But we have just shown that F commutes with taking kernels and cokernels, so $\operatorname{Im}(Ff) = F(\operatorname{Im} f)$. Hence, by the hypothesis on F, $\operatorname{Im}(Ff) \neq 0$, as required. □

Definition 11.13. We say that an object X in an abelian category $\mathcal{A}$ is

- injective if $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact;
- projective if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is exact.

We say that $\mathcal{A}$

- has enough injectives if every object in $\mathcal{A}$ is isomorphic to a subobject of some injective object;
- has enough projectives if every object in $\mathcal{A}$ is isomorphic to a quotient object of some projective object.

THEOREM 11.14. *If R is a ring, the category* $\mathsf{Mod}R$ *has enough injectives.*

PROOF. See [**2**, Theorem 18.10] or [**141**, pages 69-71]. □

Let X be a projective variety of dimension ≥ 1 and let $\mathcal{O}_X$ be the sheaf of regular functions on X. Then $\mathcal{O}_X$ is not a projective object in the category of coherent $\mathcal{O}_X$-modules. The functor $\mathrm{Hom}(\mathcal{O}_X, -)$ coincides with the global sections functor $\Gamma(X, -)$. This is not right exact: its derived functors are the sheaf cohomology groups $H^q(X, -)$.

Definition 11.15. An object X in a category C is a generator if, for every pair of distinct morphisms $f_1, f_2 : Y \to Z$ in C, there exists $g : X \to Y$ such that $f_1 g \neq f_2 g$. In other words, X is a generator if the functor $\mathrm{Hom}_{\mathsf{C}}(X, -)$ is faithful. An object Y in C is a cogenerator if the functor $\mathrm{Hom}_{\mathsf{C}}(-, Y)$ is faithful.

PROPOSITION 11.16. *A projective object P in an abelian category $\mathcal{A}$ is a generator if and only if $\mathrm{Hom}_{\mathcal{A}}(P, X) \neq 0$ for every $X \neq 0$ in $\mathcal{A}$.*

PROOF. Since $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor, the result follows from Proposition 11.12. □

Suppose that X_α are objects in an abelian category $\mathcal{A}$, that their coproduct $\coprod_\alpha X_\alpha$ exists, and that F is a covariant functor on $\mathcal{A}$. By definition of the coproduct there are morphisms $X_\alpha \to \coprod X_\alpha$ and hence morphisms $FX_\alpha \to F(\coprod X_\alpha)$. Hence, if $\coprod FX_\alpha$ exists, there is a morphism $\coprod FX_\alpha \to F(\coprod X_\alpha)$. In general, this morphism need not be an isomorphism (Exercise 8); this explains the necessity of one of the hypotheses in the next result.

THEOREM 11.17. *Let $\mathcal{A}$ be an abelian category having coproducts. Let P be a projective generator in $\mathcal{A}$, and define the ring $R := \mathrm{Hom}_{\mathcal{A}}(P, P)$. Suppose that,* *for every collection of objects X_α in $\mathcal{A}$, the natural map $\coprod_\alpha \operatorname{Hom}_{\mathcal{A}}(P, X_\alpha) \to \operatorname{Hom}_{\mathcal{A}}(P, \coprod X_\alpha)$ is an isomorphism. Then*

$$\operatorname{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathsf{Mod} R^{\mathrm{op}}$$

is an equivalence of categories.

PROOF. Write $F = \operatorname{Hom}_{\mathcal{A}}(P, -)$. Since P is projective F is exact and, since P is a generator, F is faithful. By Theorem 5.12, it remains to show that every right R-module is isomorphic to a module of the form FX for some X in $\mathcal{A}$, and that F is full, i.e., that the map

$$F : \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_R(FX, FY) \tag{11-1}$$

is surjective for every X and Y in $\mathcal{A}$. If $\alpha \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ then $F(\alpha)$ is the right R-module map $F(\alpha)(f) = \alpha \circ f$ for $f \in FX = \operatorname{Hom}_{\mathcal{A}}(P, X)$.

Fix an object Y in $\mathcal{A}$. We consider the two sides of (11-1) as contravariant functors

$$G := \operatorname{Hom}_{\mathcal{A}}(-, Y) \ : X \mapsto \operatorname{Hom}_{\mathcal{A}}(X, Y)$$

and

$$H := \operatorname{Hom}_R(-, FY) \circ F \ : X \mapsto \operatorname{Hom}_R(FX, FY),$$

whence F may be viewed as a natural transformation $t : G \to H$; thus, we must show that $t_X : GX \to HX$ is an isomorphism for all X (i.e., that t is a natural equivalence).

First t_P is an isomorphism since, for each $\alpha \in \operatorname{Hom}_{\mathcal{A}}(P, P)$, $t_P(\alpha)$ is left multiplication by α, and the map $R \to \operatorname{Hom}_R(R_R, R_R)$ sending $\alpha \in R$ to left multiplication by α is an isomorphism.

Now let X be arbitrary. Since P is a generator there is an exact sequence $Q_1 \to Q_0 \to X \to 0$ with Q_1 and Q_0 being direct sums of copies of P. Applying G and H yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & GX \arrow[r] \arrow[d, "t_X"'] & GQ_0 \arrow[r] \arrow[d, "t_{Q_0}"'] & GQ_1 \arrow[d, "t_{Q_1}"'] \\
0 \arrow[r] & HX \arrow[r] & HQ_0 \arrow[r] & HQ_1
\end{tikzcd}$$

of abelian groups with exact rows. By the previous paragraph, and the fact that F commutes with direct sums, the second and third vertical maps are isomorphisms. We must show that the first vertical map is an isomorphism; we already know it is injective since F is faithful, so it remains to show it is surjective. This results from standard diagram chasing. Thus t_X is an isomorphism.

Finally, if M is an R-module there is an exact sequence

$$M_1 \xrightarrow{\psi} M_0 \xrightarrow{\varphi} M \to 0 \tag{11-2}$$

with M_1 and M_0 being free R-modules. There exist objects Q_1 and Q_0 in $\mathcal{A}$, both of which are direct sums of copies of P, such that $FQ_1 = M_1$ and $FQ_0 = M_0$. Since the map in (11-1) is surjective, $\psi = F\alpha$ for some $\alpha \in \operatorname{Hom}_{\mathcal{A}}(Q_1, Q_0)$. Applying F to the exact sequence $Q_1 \to Q_0 \to X \to 0$, where $X = \operatorname{coker} \alpha$, yields an exact sequence $FQ_1 \xrightarrow{F\alpha} FQ_0 \to FX \to 0$. Comparing this with (11-1), it follows that $M \cong FX$. $\square$

EXAMPLE 11.18. An abelian category having no progenerator... $\mathsf{Coh}\mathbb{P}^1$, but is there a simpler example.

THEOREM 11.19 (Mitchell's Theorem). *Let $\mathcal{A}$ be a small abelian category (i.e., the objects of $\mathcal{A}$ form a set). Then there exists a ring R and a fully faithful exact covariant functor $\mathcal{A} \to \mathsf{Mod}R$.*

PROOF. See [**131**, Theorem 11.6, Chapter 4] for the details. The basic idea is to embed $\mathcal{A}$ in a larger abelian category which has a projective generator and to apply the last result to the larger category. □

Mitchell's Theorem allows us to think of the objects in a small abelian category as modules over some ring and the morphisms are homomorphisms between the modules. In particular, objects may be thought of as having elements, and diagrams are therefore susceptible to 'diagram chasing' arguments. We will also abuse terminology by writing $X \subset Y$ for a subobject X of an object Y. All the categories which concern us are abelian, so we will make frequent implicit use of Mitchell's Theorem.

If we identify an abelian category $\mathcal{A}$ with a suitable subcategory of modules over some ring, then the notions of monomorphism, epimorphism, kernel and cokernel become the familiar ones. That is, a morphism $f : X \to Y$ is a monomorphism if and only if it is injective, and an epimorphism if and only if it is surjective, has kernel $\{x \in X \mid f(x) = 0\}$ and cokernel $Y/f(X)$.

EXERCISES

11.1 Show that the dual of an abelian category is abelian.

11.2 Let $\mathcal{A}$ be an abelian category, fix an object X in C, and define $R := \operatorname{Hom}_{\mathsf{C}}(X, X)$. Show that $F := \operatorname{Hom}_{\mathsf{C}}(X, -)$ is a covariant left exact functor from C to the category of right R-modules.

11.3 Let $\mathcal{A}$ be an abelian category having coproducts. Show that $X \in \operatorname{Ob}(\mathcal{A})$ is a generator if and only if, for each $Y \in \operatorname{Ob}(\mathcal{A})$, there exists an epimorphism $\coprod_I X \to Y$ defined on some coproduct of copies of X.

11.4 Let E be an injective object in an abelian category $\mathcal{A}$.
(a) Show that E is a cogenerator if and only if $\operatorname{Hom}_{\mathcal{A}}(X, E) \neq 0$ for each $X \neq 0$ in $\mathcal{A}$.
(b) Suppose that $\mathcal{A} = \mathsf{Mod}R$. Show that E is a cogenerator if and only if it contains a copy of every simple R-module.

11.5 Let R be an artinian ring, and let E be the direct sum of the injective envelopes of the simple left R-modules. Let D denote the functor $\operatorname{Hom}_R(-, E) : \mathsf{mod}R \to \mathsf{mod}R^{op}$. Show that D is a duality.
This generalizes the duality $V \mapsto V^*$ for finite dimensional vector spaces. We call E a **dualizing module**.

11.6 Let $\mathcal{A}$ be an abelian category having products. Show that $Y \in \operatorname{Ob}(\mathcal{A})$ is a cogenerator if and only if, for each $X \in \operatorname{Ob}(\mathcal{A})$, there exists a monomorphism $X \to \prod_I Y$ to some product of copies of Y.

11.7 Let $\mathcal{A}$ be an abelian category. Show that an object X is projective if and only if it is injective when considered as an object in the dual category $\mathcal{A}^{\mathrm{op}}$.

11.8 If P is a projective object in an abelian category $\mathcal{A}$ then $\operatorname{Hom}_{\mathcal{A}}(P, -)$ need not commute with arbitrary direct sums. In particular, if $\mathcal{A}$ is the category of k-vector spaces, and V is an infinite dimensional vector space with basis $\{e_\lambda\}$, show that $\mathrm{id}_V \notin \oplus \operatorname{Hom}_{\mathcal{A}}(V, ke_\lambda)$.

11.9 If M is a finitely generated R-module, show that $\operatorname{Hom}_R(M, -)$ commutes with direct sums.

11.10 If $\{X_\alpha \mid \alpha \in I\}$ is a collection of projective (respectively, injective) objects in $\mathcal{A}$, show that $\coprod X_\alpha$ is projective (respectively, $\prod X_\alpha$ is injective).

11.11 An R-module D is divisible if given $d \in D$ and $0 \neq r \in R$, there exists $c \in D$ such that $rc = d$. Let R be a commutative principal ideal domain. Show that an R-module is injective if and only if it is divisible.

11.12 Let (F, G) be an adjoint pair of functors between abelian categories. Show that
(a) F preserves projectives if G is exact, and
(b) G preserves injectives if F is exact.

12. Morita equivalence

THEOREM 12.1 (Watt's Theorem). *Let R and S be rings, and $F : \mathsf{Mod}R^{\text{op}} \to \mathsf{Mod}S^{\text{op}}$ a right exact additive functor commuting with direct sums. Then FR has the structure of an R-S-bimodule and F is naturally equivalent to $- \otimes_R FR$.*

PROOF. Write B for the right S-module FR. The map

$$x \mapsto \lambda_x = \text{left multiplication by } x$$

is a ring isomorphism $R \to \operatorname{Hom}_R(R_R, R_R)$. Since F is additive the map $F : \operatorname{Hom}_R(R_R, R_R) \to \operatorname{Hom}_S(B, B)$ is a ring homomorphism, thus making B an R-S-bimodule with the action of $x \in R$ defined by $x.b = (F\lambda_x)(b)$ for $b \in B$.

Now fix a right R-module M. For each $m \in M$ define $\varphi_m \in \operatorname{Hom}_R(R, M)$ by $\varphi_m(x) = mx$; thus $F\varphi_m \in \operatorname{Hom}_S(B, FM)$. If $b \in B$, $m \in M$ and $x \in R$ then

$$(F\varphi_{mx})(b) = F(\varphi_m \circ \lambda_x)(b) = (F\varphi_m \circ F\lambda_x)(b) = (F\varphi_m)(xb).$$

Hence the rule $t_M(m \otimes b) := (F\varphi_m)(b)$ gives a well-defined map

$$t_M : M \otimes_R B \to FM;$$

it is a right S-module map since $F\varphi_m$ is. It is routine to show that $t : - \otimes_R B \to F$ is a natural transformation, so it remains to show that t_M is an isomorphism for all M.

Since F commutes with direct sums, if $Q = \oplus_I R$ then $t_Q : Q \otimes_R B \xrightarrow{\sim} FQ$. Now, for an arbitary M, there is an exact sequence $Q_1 \to Q_0 \to M \to 0$ with each Q_i a free R-module. In the commutative diagram

$$\begin{array}{ccccccc}
Q_1 \otimes_R B & \longrightarrow & Q \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\
{\scriptstyle t_{Q_1}}\big\downarrow & & {\scriptstyle t_{Q_0}}\big\downarrow & & {\scriptstyle t_M}\big\downarrow & & \\
FQ_1 & \longrightarrow & FQ_0 & \longrightarrow & FM & \longrightarrow & 0
\end{array}$$

the first two vertical maps are isomorphisms, and the rows are exact, so a diagram chase shows that t_M is an isomorphism. $\square$

13. Quotient categories

This section follows closely the exposition in Gabriel's paper [**65**], and the reader is referred there for proofs of (13.5), (13.6) and (13.7).

Definition 13.1. A non-empty full subcategory C of an abelian category $\mathcal{A}$ is dense if, for all short exact sequences $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, both X' and X'' belong to C if and only if X does. We will call objects of C torsion objects; if the only subobject of Y belonging to C is 0, we say that Y is torsion-free. Thus 0 is the only object which is both torsion and torsion-free.

For the rest of this section $\mathcal{A}$ will denote an abelian category and C a dense subcategory.

For the applications we have in mind $\mathcal{A}$ will be a category of modules.

If $\mathcal{A}$ is a category of modules, and X_1, X_2 are submodules of $X \in \mathrm{Ob}(\mathcal{A})$, both of which are torsion, then so is their sum $X_1 + X_2$ since it is a quotient of $X_1 \oplus X_2$, which itself occurs in an exact sequence $0 \to X_1 \to X_1 \oplus X_2 \to X_2 \to 0$. However, the sum of *all* torsion submodules of X may not be torsion. For example, this happens if $\mathcal{A}$ is the category of k-vector spaces and C is the full subcategory of finite dimensional vector spaces.

EXAMPLE 13.2. If R is a commutative ring, and $\mathcal{S}$ a multiplicatively closed subset, then the $\mathcal{S}$-torsion modules form a dense subcategory of $\mathsf{Mod}R$. More generally, suppose that R is a ring having ring of fractions $\mathrm{Fract}\, R$ and S is an intermediate ring, $R \subset S \subset \mathrm{Fract}\, R$. If S_R is flat, then $\{M \in \mathsf{Mod}R \mid S \otimes_R M = 0\}$ is a dense subcategory of $\mathsf{Mod}R$.

Definition 13.3. Let $\mathcal{A}$ be an abelian category and C a dense subcategory. The quotient category $\mathcal{A}/\mathsf{C}$ is defined as follows:

- its objects are the objects of $\mathcal{A}$;
- if X and Y are objects of $\mathcal{A}$ then
$$\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, Y) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y'),$$
where the direct limit is taken over all subobjects X' of X and all subobjects Y' of Y with the property that X/X' and Y' are torsion;
- the composition of morphisms in $\mathcal{A}/\mathsf{C}$ is induced by that in $\mathcal{A}$ (see below).

There are several points to be checked in this definition.

First, the direct limit makes sense. For a fixed pair of objects, X and Y, let I denote the set of all pairs (X', Y') of subobjects $X' \subset X$ and $Y' \subset Y$, such that X/X' and Y' are torsion. We define
$$(X', Y') \le (X'', Y'')$$
if $X'' \subset X'$ and $Y' \subset Y''$. Thus I is a quasi-ordered set. If $(X', Y') \le (X'', Y'')$, the natural morphisms $X'' \to X'$ and $Y/Y' \to Y/Y''$ induce maps
$$\mathrm{Hom}_{\mathcal{A}}(X', Y/Y') \to \mathrm{Hom}_{\mathcal{A}}(X'', Y/Y') \to \mathrm{Hom}_{\mathcal{A}}(X'', Y/Y'').$$
Thus $\mathrm{Hom}(X', Y/Y')$ is a direct system indexed by I. Since direct limits exist in the category of abelian groups (8.5), the definition of $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}$ makes sense. Observe that $(X, 0) \in I$, whence $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ appears in the direct system. It is useful to observe that I is a directed set: given $(X_i', Y_i) \in I$, $i = 1, 2$, then
$$(X_i', Y_i') \le (X_1' \cap X_2', Y_1' + Y_2') \in I.$$
Therefore, by Proposition 8.7, every morphism in $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, Y)$ is the image of a morphism in $\mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$ for some $(X', Y') \in I$.

Second, there is a well-defined composition of morphisms in $\mathcal{A}/\mathsf{C}$: we will state the main steps required to verify this, leaving the details to the reader. For $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$, the composition
$$\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, Z)$$
is defined as follows. Let
$$\bar{f} \in \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(Y, Z) \qquad \text{and} \qquad \bar{g} \in \mathrm{Hom}_{\mathcal{A},\mathsf{C}}(X, Y).$$

Since I is directed, $\bar{f}$ and $\bar{g}$ are images of morphisms $g : X' \to Y/Y'$ and $f : Y'' \to Z/Z'$ in $\mathcal{A}$ where

$$X/X',\ Y',\ Y/Y'',\ Z' \in \mathrm{Ob}(\mathsf{C}).$$

Define $X'' := g^{-1}(Y' + Y''/Y')$, check that X/X'' is torsion, and define

$$g' : X'' \to Y' + Y''/Y'$$

to be the restriction of g to X''. Both $f(Y' \cap Y'')$ and $Z'' := Z' + f(Y' \cap Y'')$ are torsion. Now define

$$f' : Y''/Y' \cap Y'' \to Z/Z''$$

to be the map induced by f. Define h to be the composition

$$X'' \xrightarrow{g'} (Y' + Y''/Y') \xrightarrow{\sim} (Y''/Y' \cap Y'') \xrightarrow{f'} Z/Z'',$$

where the middle map is the natural isomorphism. Finally, one checks that $\bar{h}$, the image of h in $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, Z)$, depends only on $\bar{f}$ and $\bar{g}$ and not on a choice of representatives f and g.

Third, $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(X, X)$ contains an identity morphism, namely the image of id_X in the direct limit.

Notation . We define the functor $\pi : \mathcal{A} \to \mathcal{A}/\mathsf{C}$ by

- $\pi X = X$ on objects, and
- $\pi f =$ the image of f in the direct limit, on morphisms.

It follows immediately from the definitions that $\mathcal{A}/\mathsf{C}$ is an additive category and that $\pi : \mathcal{A} \to \mathcal{A}/\mathsf{C}$ is an additive functor.

Lemma 13.4. *If $X \in \mathrm{Ob}(\mathcal{A})$, then $\pi X \cong 0$ if and only if X is torsion.*

Proof. Write I for the directed set used in defining $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi X)$.

($\Rightarrow$) Since $\pi X \cong 0$, $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi X) = 0$. In particular, the image in the direct limit of the identity morphism id_X is zero. Since I is directed, this says that there exists $(X', Y') \in I$ such that the map $X' \to X/Y'$ induced by id_X is the zero map. But the image of this map is $X' + Y'/Y'$, so $X' \subset Y'$. Thus, since C is dense, X' is torsion whence X is torsion.

($\Leftarrow$) It suffices to show that $\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi X) = 0$. Every morphism in this set is of the form πf for some $f : X' \to X/Y'$ and some $(X', Y') \in I$. But $(X', Y') \leq (0, Y') \in I$ since X is torsion, and the image of f in $\mathrm{Hom}_{\mathcal{A}}(0, Y')$ is zero, so $\pi f = 0$. $\square$

Proposition 13.5. *Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then*

1. *πf has a kernel and cokernel in $\mathcal{A}$, namely*
   $\ker(\pi f) = \pi(\ker f)$ *and* $\mathrm{coker}(\pi f) = \pi(\mathrm{coker}\, f)$;
2. *πf is zero if and only if $\mathrm{Im}\, f$ is torsion;*
3. *πf is a monomorphism if and only if $\ker f$ is torsion;*
4. *πf is an epimorphism if and only if $\mathrm{coker}\, f$ is torsion;*
5. *πf is an isomorphism if and only if both $\ker f$ and $\mathrm{coker}\, f$ are torsion.*

Theorem 13.6. *The category $\mathcal{A}/\mathsf{C}$ is abelian and π is exact.*

Theorem 13.7. *Let $\mathcal{A}$ be an abelian category, C a dense subcategory and D another abelian category.*

1. *If $F : \mathcal{A} \to \mathsf{D}$ is a covariant exact functor such that $FX = 0$ for all torsion objects X, then there is a unique functor $G : \mathcal{A}/\mathsf{C} \to \mathsf{D}$ such that $F = G\pi$; that is, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} & & \\ {\scriptstyle\pi}\Big\downarrow & & F \\ \mathcal{A}/\mathsf{C} & \xrightarrow[G]{\hspace{2em}} & \mathsf{D} \end{array}$$

2. *If $G : \mathcal{A}/\mathsf{C} \to \mathsf{D}$ is a functor, then G is exact if and only if $G\pi$ is exact.*

There are analogous results for contravariant functors.

Proposition 13.8. [**?**, Corollaire 1, page 368] *If $0 \to L' \to M' \to N' \to 0$ is an exact sequence in A/C, then there is an exact sequence $0 \to L \to M \to N \to 0$ in A, and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi L \arrow[r] & \pi M \arrow[r] & \pi N \arrow[r] & 0 \end{tikzcd}$$

such that α, β, γ are isomorphisms in A/C.

Let $\varphi \in \operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y)$. As remarked earlier, φ is the image of a map $f \in \operatorname{Hom}_{\mathcal{A}}(X', Y/Y')$ for some $(X', Y') \in I$. Write $s_1 : X' \to X$ for the inclusion, and $s_2 : Y \to Y/Y'$ for the surjection. Both πs_1 and πs_2 are isomorphisms, so we may write

$$\varphi = (\pi s_2)^{-1} \circ (\pi f) \circ (\pi s_1)^{-1}.$$

This point of view may be used as the starting point for the definition of a quotient category. That is, rather than starting with a class of objects, the dense subcategory, one begins with a class of morphisms which are to be inverted. This latter point of view is more general, and leads to the notion of a category of fractions (see [**1**] for details). This point of view is required to develop the notion of universal localization (see [**47**, Chapter 7] and [**143**, Chapter 4]).

In general, it is difficult to compute the morphisms in the quotient category. However, $\operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y)$ is more tractable if, amongst the subobjects of Y belonging to C, there is a unique maximal one.

Definition 13.9. If an object Y in $\mathcal{A}$ has a largest subobject which belongs to C, that subobject is denoted by τY and is called the torsion subobject of Y. We will often indicate the existence of a largest torsion subobject by saying 'suppose that τY exists'.

Let X and Y be objects in $\mathcal{A}$ and suppose that τY exists. Then

$$J := \{(X', \tau Y) \mid X' \subset X,\ X/X' \in \operatorname{Ob}(\mathsf{C})\}$$

is cofinal in I, so

$$\operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(X, Y) = \varinjlim \operatorname{Hom}_{\mathcal{A}}(X', Y/\tau Y). \tag{13-1}$$

A right adjoint to π

The next two results concern the consequences of the existence of a right adjoint to π. Proposition 13.11 shows that, if π has a right adjoint, then τY exists for all

Y in $\mathcal{A}$. In most situations of interest to us π has a right adjoint, which we denote by ω. The existence of ω greatly simplifies the analysis of the relation between $\mathcal{A}$ and $\mathcal{A}/\mathsf{C}$.

PROPOSITION 13.10. *Suppose that π has a right adjoint $\omega : \mathcal{A}/\mathsf{C} \to \mathcal{A}$. Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{A}/\mathsf{C})$. Then*

1. *$\omega\mathcal{F}$ is torsion-free;*
2. *if $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and πf is an isomorphism, then the map*
$$\mathrm{Hom}_{\mathcal{A}}(Y, \omega\mathcal{F}) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}),$$
defined by $f^*(g) = g \circ f$, is an isomorphism;
3. *for each $X \in \mathrm{Ob}(\mathcal{A})$, the map $f \mapsto \pi f$ is an isomorphism*
$$\mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi\omega\mathcal{F});$$
4. *every exact sequence in $\mathcal{A}$ of the form $0 \to \omega\mathcal{F} \xrightarrow{f} Y \to Z \to 0$, with $Z \in \mathrm{Ob}(\mathsf{C})$, splits;*
5. *$\pi\omega \cong \mathrm{id}_{\mathcal{A}/\mathsf{C}}$.*

PROOF. Let $Y \in \mathrm{Ob}(\mathcal{A})$ be such that $\mathcal{F} = \pi Y$.

(1) By the adjoint property $\mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y)$ for every $X \in \mathrm{Ob}(\mathcal{A})$. In particular, if X is torsion then $\mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) = 0$, since $\pi X \cong 0$. Thus $\omega\mathcal{F}$ is torsion-free.

(2) By the adjoint property there is a commutative diagram as follows, with the horizontal maps being isomorphisms:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(Y, \omega\mathcal{F}) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi Y, \mathcal{F}) \\
{\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle (\pi f)^*} \\
\mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \mathcal{F}).
\end{array}$$

By Proposition 13.5(4), πf is an isomorphism, so the right hand vertical map is an isomorphism, whence the left one is too.

(3) Since $\omega\mathcal{F}$ is torsion-free, $\tau(\omega\mathcal{F})$ exists—it is zero. Thus, by (13-1), The map $f \mapsto \pi f$ is the natural map

$$\mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) \to \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', \omega\mathcal{F}) \tag{13-2}$$

where the direct limit is taken over the $X' \subset X$ such that X/X' is torsion. By (2), all the maps $\mathrm{Hom}_{\mathcal{A}}(X', \omega\mathcal{F}) \to \mathrm{Hom}_{\mathcal{A}}(X'', \omega\mathcal{F})$ in the direct system are isomorphisms, whence so is (13-2).

(4) By (2) the map $\mathrm{Hom}_{\mathcal{A}}(Y, \omega\mathcal{F}) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(\omega\mathcal{F}, \omega\mathcal{F})$ is an isomorphism, so there exists $g : Y \to \omega\mathcal{F}$ such that $f \circ g = \mathrm{id}_Y$.

(5) Let $t : \pi\omega \to \mathrm{id}_{\mathcal{A}/\mathsf{C}}$ be the natural transformation induced by the adjointness. We must show that, for each $\mathcal{F} \in \mathrm{Ob}(\mathcal{A}/\mathsf{C})$, the map $t_{\mathcal{F}} : \pi\omega\mathcal{F} \to \mathcal{F}$ is an isomorphism. By Yoneda's Lemma, it suffices to prove that

$$(t_{\mathcal{F}})^* : \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\mathcal{G}, \pi\omega\mathcal{F}) \to \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\mathcal{G}, \mathcal{F})$$

is an isomorphism for all $\mathcal{G} \in \mathrm{Ob}(\mathcal{A}/\mathsf{C})$. Any such $\mathcal{G}$ is of the form πX, for some $X \in \mathrm{Ob}(\mathcal{A})$, so we must show show that the bottom map in the following diagram is an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(X, \omega\mathcal{F}) & & \\ \pi\Big\downarrow & & \\ \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X \pi\omega\mathcal{F}) & \xrightarrow[(t_{\mathcal{F}})^*]{} & \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \mathcal{F}) \end{array}$$

The vertical map is $f \mapsto \pi f$, the diagonal map is the isomorphism coming from adjointness. By (3) the vertical map is an isomorphism, so the commutativity of the diagram implies that the bottom map is an isomorphism too. $\square$

PROPOSITION 13.11. *Suppose that* π *has a right adjoint* ω, *let* $X \in \mathrm{Ob}(\mathcal{A})$,*and let* $f : X \to \omega\pi X$ *be the morphism induced by the natural transformation* $\mathrm{id}_{\mathcal{A}} \to \omega\pi$. *Then*

1. $\ker f$ *is the largest subobject of* X *which is torsion;*
2. $\mathrm{coker}\, f$ *is torsion;*
3. $\omega\pi X$ *is an essential extension of* $f(X)$.

PROOF. (1) and (2). Writing $W = \ker f$ and $Z = \mathrm{coker}\, f$ we have an exact sequence

$$0 \to W \to X \xrightarrow{f} \omega\pi X \to Z$$

in $\mathcal{A}$ and, since π is exact, an exact sequence

$$0 \to \pi W \to \pi X \xrightarrow{\pi f} \pi\omega\pi X \to \pi Z \to 0$$

in $\mathcal{A}/\mathsf{C}$. In the commutative diagram (13) in the previous proof, take $\mathcal{F} = \pi X$; then we have the following correspondences between maps

$$\begin{array}{ccc} (X \to \omega\pi X) & & \\ \pi\Big\downarrow & & \\ (\pi X \to \pi\omega\pi X) & \xrightarrow[(t_{\mathcal{F}})^*]{} & (\mathrm{id} : \pi X \to \pi X) \end{array}$$

Therefore πf is an isomorphism, whence $\pi W \cong 0$ and $\pi Z \cong 0$. Thus, by Proposition 13.5, W and Z are torsion. This proves (2).

By Proposition 13.10(1), $\omega\pi X$ is torsion-free, so W contains every torsion subobject of X. Thus W is the largest torsion subobject of X. This proves (1).

(3) If T is a subobject of $\omega\pi X$ such that $T \cap f(X) = 0$, then T embeds in Z, so is torsion. But $\omega\pi X$ is torsion-free, so $T = 0$. Thus $f(X)$ is essential in $\omega\pi X$. $\square$

Theorem 13.14 gives conditions under which π has a right adjoint, but first we need the following lemmas.

LEMMA 13.12. *IF* τY *exists, then* $\mathrm{Hom}_{\mathcal{A}}(X, Y/\tau Y) = 0$ *for all torsion objects* X. *In particular,* $Y/\tau Y$ *is torsion-free.*

PROOF. Suppose that X is in C and that $f : X \to Y/\tau Y$. Write Y' for the kernel of the composition $Y \to Y/\tau Y \to \mathrm{coker}\, f$. OThen there is an exact sequence $0 \to \tau Y \to Y' \to Y'/\tau Y \cong \mathrm{Im}\, f \to 0$. Since X is torsion so is $\mathrm{Im}\, f$, and hence so is Y' as C is dense. Since τY is the largest torsion subobject of Y, $Y' \subset \tau Y$. Therefore $\mathrm{Im}\, f = 0$, whence $f = 0$ as required. $\square$

LEMMA 13.13. *An essential extension of a torsion-free object is torsion-free.*

Proof. Let Q be an essential extension of a torsion-free object Y. If $X \subset Q$ is a torsion object, so is $X \cap Y$. Therefore $X \cap Y = 0$, whence $X = 0$. $\square$

Theorem 13.14. *Suppose that $\mathcal{A}$ has enough injectives and that τX exists for all X in $\mathcal{A}$. Then*

1. *$\pi : \mathcal{A} \to \mathcal{A}/\mathsf{C}$ has a right adjoint, ω say;*
2. *for each X in $\mathcal{A}$, $\omega\pi X$ is isomorphic to the largest subobject of the injective envelope of $X/\tau X$ which extends $X/\tau X$ by a torsion object;*
3. *for each $X \in \mathrm{Ob}(\mathcal{A})$, there is an exact sequence*
$$0 \to \tau X \to X \xrightarrow{f} \omega\pi X \to \operatorname{coker} f \to 0$$
with $\operatorname{coker} f$ *a torsion object.*

Proof. Fix Y in $\mathcal{A}$, write $\bar{Y} = Y/\tau Y$, and let $\alpha : \bar{Y} \to E$ denote the inclusion of Y in an injective envelope. Let H denote the kernel of the morphism
$$E \to \operatorname{coker} \alpha \to \operatorname{coker} \alpha/\tau(\operatorname{coker} \alpha).$$
This gives rise to an exact sequence
$$0 \to \tau Y \to Y \xrightarrow{f} H \to \operatorname{coker} f \to 0$$
in which $\ker f$ and $\operatorname{coker} f$ are both torsion. In particular, $\pi f : \pi Y \xrightarrow{\sim} \pi H$ is an isomorphism in $\mathcal{A}/\mathsf{C}$.

By Lemma 13.12, $\bar{Y}$ is torsion-free, hence so is H by Lemma 13.13. Moreover, $E/H \cong \operatorname{coker} \alpha/\tau(\operatorname{coker} \alpha)$ is also torsion-free by Lemma 13.12.

To show that π has a right adjoint it suffices, by Proposition 6.7, to show that the functor
$$X \mapsto \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y)$$
is representable for each Y in $\mathcal{A}$. Consider the diagram
$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{A}}(X, H) \arrow[d, "\pi"] \\ \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi H) \end{tikzcd}$$
where the horizontal map is induced by composing with the isomorphism πf. Since H is torsion-free,
$$\mathrm{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi H) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', H)$$
where the direct limit is taken over those $X' \subset X$ for which X/X' is torsion.

Suppose that $X' \subset X$ and X/X' is torsion; we will show that the natural map $\mathrm{Hom}_{\mathcal{A}}(X, H) \to \mathrm{Hom}_{\mathcal{A}}(X', H)$ is an isomorphism. Since $\mathrm{Hom}_{\mathcal{A}}(-, H)$ is left exact and X/X' is torsion whereas H is torsion-free, it follows from Lemma 13.12 that this map is injective, so it remains to prove it is surjective. To see this, let $f' \in \mathrm{Hom}_{\mathcal{A}}(X', H)$ and consider the diagram
$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[r] & X/X' \arrow[r] & 0 \\ 0 \arrow[r] & H \arrow[r] & E \arrow[r] & E/H \arrow[r] & 0. \end{tikzcd}$$
Since E is injective there is a morphism $f : X \to E$ extending the composition $X' \to H \to E$. It follows that there exists a morphism $g : X/X' \to E/H$ making

the diagram commute. But E/H is torsion-free and X/X' is torsion, so $g = 0$ by Lemma 13.12. Therefore $\operatorname{Im} f \subset H$ and f' is the image of f.

Therefore the maps in diagram 13 are isomorphisms, and they induce an isomorphism $\operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y) \to \operatorname{Hom}_{\mathcal{A}}(X, H)$ showing that H represents the functor $X \mapsto \operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Y)$. Thus π has a right adjoint, ω say, which on objects is defined by $\omega\pi Y = H$. □

Definition 13.15. If $\pi : \mathcal{A} \to \mathcal{A}/\mathsf{C}$ has a right adjoint, ω say, then ω is called the section functor and C is called a localizing subcategory of $\mathcal{A}$.

A comparison of homological issues in $\mathcal{A}$ and $\mathcal{A}/\mathsf{C}$ requires an understanding of the relation between injectives in $\mathcal{A}$ and $\mathcal{A}/\mathsf{C}$.

THEOREM 13.16. *Suppose that $\mathcal{A}$ has enough injectives, and that π has a right adjoint.*

1. *If $\mathcal{Q}$ is an injective in $\mathcal{A}/\mathsf{C}$, then $\omega\mathcal{Q}$ is injective in $\mathcal{A}$.*
2. *The injectives in $\mathcal{A}/\mathsf{C}$ are $\{\pi Q \mid Q$ is a torsion-free injective in $\mathcal{A}\}$.*
3. *$\mathcal{A}/\mathsf{C}$ has enough injectives.*

PROOF. (1) Let X be an object in $\mathcal{A}$. By the adjoint property,

$$\operatorname{Hom}_{\mathcal{A}}(X, \omega\mathcal{Q}) \cong \operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \mathcal{Q});$$

In other words,

$$\operatorname{Hom}_{\mathcal{A}}(-, \omega\mathcal{Q}) = \operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(-, \mathcal{Q}) \circ \pi.$$

Both functors on the right are exact, hence so is their composition; thus $\omega\mathcal{Q}$ is injective.

(2) Let Q be a torsion-free injective in $\mathcal{A}$; we will show that πQ is injective. By Lemma 13.12, the exact functor $\operatorname{Hom}_{\mathcal{A}}(-, Q)$ from $\mathcal{A}$ to Ab vanishes on C; hence by Theorem 13.7, the rule

$$\pi X \mapsto \operatorname{Hom}_{\mathcal{A}}(X, Q) \tag{13-3}$$

defines an exact functor on $\mathcal{A}/\mathsf{C}$. By Theorem 13.14(2), $Q \cong \omega\pi Q$ so, by Proposition 13.10(3),

$$\operatorname{Hom}_{\mathcal{A}}(X, Q) \cong \operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(\pi X, \pi Q).$$

Therefore the functor defined by (13-3) is $\operatorname{Hom}_{\mathcal{A}/\mathsf{C}}(-, \pi Q)$; since this is exact, πQ is injective.

Now we show that every injective in $\mathcal{A}/\mathsf{C}$ is πQ for some torsion-free injective Q. Let $\mathcal{Q}$ be an injective in $\mathcal{A}/\mathsf{C}$. Then $\omega\mathcal{Q}$ is injective by (1), and is torsion-free by Proposition 13.10(1). Moreover, $\pi\omega\mathcal{Q} \cong \mathcal{Q}$ by Proposition 13.10(5), thus completing the proof.

(3) Let $\mathcal{F}$ be an object in $\mathcal{A}/\mathsf{C}$. Let $f : \omega\mathcal{F} \to Q$ be the inclusion of $\omega\mathcal{F}$ in its injective envelope. Since $\omega\mathcal{F}$ is torsion-free, so is Q by Lemma 13.13. By Proposition 13.5(3), πf is injective, so πQ is an injective containing $\pi\omega\mathcal{F} \cong \mathcal{F}$. □

EXERCISES

13.1 Let $f : R \to S$ be a homomorphism of rings. Show that the subcategory, $\mathsf{C} \subset \mathsf{Mod}R$, consisting of those M for which $S \otimes_R M = 0$, is dense if and only if S_R is flat.

13.2 Fill in the details required to show that the composition of morphisms in $\mathcal{A}/\mathsf{C}$ is well-defined.

13.3 If P is an object in an abelian category $\mathcal{A}$, define the full subcategory $P^\perp := \{M \mid \operatorname{Hom}(M, P) = 0\}$. Show that $P^\perp$ need not be dense, and find conditions on P which ensure that $P^\perp$ is dense.

13.4 Let $\mathcal{A}$ denote the category of k-vector spaces, and let C be the full subcategory consisting of the finite dimensional vector spaces. Show that C is dense, but the functor $\pi : \mathcal{A} \to \mathcal{A}/\mathsf{C}$ does not have a right adjoint.

13.5 Show that the artinian (respectively, noetherian) objects in an abelian category form a dense subcategory.

13.6 Suppose that π has a right adjoint ω. Show that
(a) M indecomposable does not imply πM indecomposable (Hint: consider $M = k[x, y]/(xy)$ and it image in $\mathsf{Mod}\mathbb{P}^1$);
(b) $\omega\pi M$ indecomposable implies πM indecomposable;
(c) M indecomposable does not imply $\omega\pi M$ indecomposable.

14. Functor categories

Let C be a pre-additive category. We write $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$ for the category of additive covariant functors $\mathsf{C} \to \mathsf{Ab}$; the morphisms are the natural transformations.

Let C be a pre-additive category. A contravariant functor $\mathsf{C} \to \mathsf{Ab}$ is called a representation of C, and we denote by $\mathsf{Mod}\mathsf{C}$ the category of such representations; the morphisms are the natural transformations.

Lemma 14.1. $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$ *is an abelian category.*

Proof. The kernel of a natural transformation $\tau : F \to G$ is the functor $K : \mathsf{C} \to \mathsf{Ab}$ defined by $KX = \ker(\tau_X : FX \to GX)$, and if $f : X \to Y$ is a morphism in C, then $Kf : KY \to KX$ is the restriction to KY of the morphism $Ff : FY \to FX$; more precisely, the composition $KY \to FY \to FX \to GX$ is the same as the composition $KY \to FY \to GY \to GX$, so is zero, whence, by the universal property of the monomorphism $KX \to FX$, there is a morphism $KY \to KX$, and this we define to be Kf. The cokernel, say L, of $F \to G$ is defined in a similar way; on objects LX is the cokernel of $FX \to GX$. $\square$

Hence we have the following result.

Proposition 14.2. *A sequence $F \to G \to H$ in* $\mathsf{Mod}\mathsf{C}$ *is exact at G if and only if the sequence $FX \to GX \to HX$ is exact in* Ab *for all objects X in* C.

It follows from the Yoneda Lemma that the representable functors $(-, X) = \operatorname{Hom}_{\mathsf{C}}(-, X)$ are projective objects in $\mathsf{Mod}\mathsf{C}$. To see this, first notice that if F is a representation, then there is an isomorphism $\mathsf{Mod}((-, X), F) \to FX$ defined by $\tau \mapsto \tau_X(\mathrm{id}_X)$; the inverse sends $p \in FX$ to the natural transformation $\tau : (-, X) \to F$ which is, for each object Y in C, the map $\tau_Y : (Y, X) \to FY$ sending $f : Y \to X$ to $(Ff)(p) \in FY$. Hence, if we apply the functor $\mathsf{Mod}((-, X), -)$ to a sequence $F \to G \to H$ in $\mathsf{Mod}\mathsf{C}$, we obtain a sequence in Ab which is isomorphic to $FX \to GX \to HX$; however, if $F \to G \to H$ is exact at G, so is $FX \to GX \to HX$, whence $\mathsf{Mod}((-, X), -)$ is an exact functor. Thus $(-, X)$ is projective.

Let C be an abelian category. The Yoneda map $C \mapsto \mathrm{Hom}_{\mathsf{C}}(-, C)$ is a fully faithful covariant functor $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$, realizing C as the full subcategory of representable functors.

We denote by $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ the full subcategory of $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$ consisting of the left exact functors $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$.

It is not immediately clear that $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ is abelian. Let F and G be left exact functors and $\tau : F \to G$ a natural transformation. Although $\ker \tau$ is left exact, $\operatorname{coker} \tau$ need not be.

To show that $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ is abelian we proceed as follows. Suppose that C has enough injectives, and let I be the full subcategory of C consisting of the injectives. Let $F : \mathsf{I} \to \mathsf{Ab}$ be any functor. We extend F to C as follows. Let C be an object in C and take an exact sequence $0 \to C \to I_0 \to I_1$ with I_0 and I_1 injective; define $F'C := \ker(FI_0 \to FI_1)$. It is proved in [**?**, Chapitre I, Section 9] that F' is well-defined, and left exact. In fact, there is an equivalence of categories between $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ and $\mathsf{Fun}(\mathsf{I}, \mathsf{Ab})$ from which it follows that $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ is abelian. One should note that the composition $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab}) \to \mathsf{Fun}(\mathsf{I}, \mathsf{Ab}) \to \mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ sends a functor F to its 0^{th} right derived functor R^0F.

Thus $R^0(\operatorname{coker} \tau)$ is the cokernel of τ in $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$.

Proposition 14.3. $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ *is an abelian category. The kernel in* $\mathsf{Lex}(\mathsf{C}, \mathsf{Ab})$ *of* $\tau : F \to G$ *coincides with its kernel in* $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$. *However, the* $\operatorname{coker} \tau$ *is not usually the same as its cokernel in* $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$.

EXERCISES

14.1 Let R be a ring, M and M' non-isomorphic simple left R-modules, and suppose that M is not projective. For each left R-module N define

$$FN := \{f \in \mathrm{Hom}_R(N, M) \mid f \text{ is not a split epimorphism}\}.$$

Show that F extends to a functor $\mathsf{Mod}R \to \mathsf{Ab}$, and hence that F is a proper subfunctor of $\mathrm{Hom}_R(-, M)$. [*Hint:* check that $F(M' \oplus M) = 0$ but $\mathrm{Hom}_R(M' \oplus M, M) \neq 0$, and that $FX \neq 0$ for some X.]

This exercise shows that the functor $\mathsf{Mod}R \to \mathsf{Fun}(\mathsf{Mod}R, \mathsf{Ab})$ does not in general send irreducible objects to irreducible objects.

APPENDIX B

Algebraic geometry

This appendix provides some of the algebraic geometric background which is needed for the main text.

1. Projective varieties

Any three points on $\mathbb{P}^1$ can be sent to any other three points by an automorphism. There is a single invariant of four points, the cross-ration; a set of four distinct points can be sent to another set of four points by an automorphism if and only if the cross-ratios are the same.

Let f be an irreducible form of degree d. The degree of a hypersurface $\mathcal{V}(f)$ in $\mathbb{P}^n$ is defined to be d. If L is a line in $\mathbb{P}^n$, the restriction of f to $\mathbb{P}^1$ gives a form on L of degree d; this form will have d zeroes on L, counted with multiplicity. For a general L these zeroes will be distinct, so we see that if X is a hypersurface, then $\deg X$ is the number of points in $L \cap X$ for a general line L. More generally, if X is an irreducible subvariety of $\mathbb{P}^n$ we define $\deg X$ to be the number of points in $X \cap L$ where L is a linear subspace of $\mathbb{P}^n$ of dimension $n - \dim X$.

Homogeneous Coordinate Rings Let X be a closed subscheme of $\mathbb{P}(V^*)$. Let $S(V)$ denote the symmetric algebra on V. If I is the ideal of $S(V)$ generated by the homogeneous polynomials which vanish on X, we call $S(V)/I$ a homogeneous coordinate ring of X. A homogenous coordinate ring of a projective scheme X depends on an embedding of X in some ambient $\mathbb{P}^n$.

$\boxed{\text{Paul}}$ regular embeddings

2. Plane curves

There are several good books on plane curves. For example, Fulton's book on algebraic curves [**64**] is a particularly good introduction for someone having a background in commutative algebra. Reid's book [**134**] is a lively, easy read which keeps the geometry in the foreground.

In this section we work over a fixed algebraically closed field k. We will take coordinate functions x, y, z, or x_0, x_1, x_2, on the projective plane $\mathbb{P}^2$, and usually will write x and y for the coordinate functions on the affine plane $\mathbb{A}^2$.

We must distinguish between the smooth and singular points on a curve. A finer measure of singularity is given by the notion of multiplicity. Let p be a point on an affine curve $C \subset \mathbb{A}^2$ defined by $f \in k[x, y]$. If $p = (\alpha, \beta)$, the Taylor expansion of f at p is a polynomial in $x - \alpha$ and $y - \beta$, say

$$f = f_1 + f_2 + \cdots + f_n,$$

where f_i is homogeneous of degree i in $x - \alpha$ and $y - \beta$. The multiplicity of C at p is defined to be the smallest integer i such that $f_i \neq 0$; we denote it by $m_p(C)$

or $m_p(f)$. Thus $m_p(C) \geq m$ if and only if the first $m-1$ partial derivatives of f vanish at p; thus p is a smooth point of C if and only if $m_p(C) = 1$. We may, if we want, take this as a definition of smoothness.

Suppose that p has multiplicity m on C. Because k is algebraically closed f_m factors as a product of m, not necessarily distinct, degree 1 polynomials, each of which vanishes at p. The tangent lines to C at p are the lines cut out by the linear factors of f_m.

The fundamental result regarding intersection of curves in $\mathbb{P}^2$ is Bézout's theorem. It is the prototype for the intersection theory of curves on surfaces which is discussed in section **??**. The heart of Bézout's theorem is that points of intersection must be counted with multiplicity.

Suppose that C and D are curves in $\mathbb{A}^2$, cut out by the equations f and g; suppose further that f and g have no common factor (equivalently, C and D have no common component). Then $k[x,y]/(f,g)$ is finite dimensional, so splits as a direct sum of local rings, say

$$k[x,y]/(f,g) = R_1 \oplus \cdots \oplus R_n. \tag{2-1}$$

The distinct maximal ideals in this ring, equivalently the summands in this decomposition, correspond to the distinct points in $\mathcal{V}(f,g) = C \cap D$. If $p \in C \cap D$, we define the intersection multiplicity of C and D at p to be

$$I(C,D,p) := \dim_k R,$$

where R is the local summand of $k[x,y]/(f,g)$ corresponding to p in the decomposition (2-1). This definition is transferred to curves in $\mathbb{P}^2$ by taking an open affine neighbourhood of $p \in C \cap D$.

One has $I(C,D,p) \geq m_p(C)m_p(D)$, with equality if and only if C and D meet transversally at p; that is, if and only if the tangent lines to C at p are distinct from the tangent lines to D at p.

Theorem 2.1 (Bézout's Theorem). *Let C and D be plane curves having no common component. Then C meets D at $\deg C \cdot \deg D$ points counted with multiplicity. That is,*

$$\sum_{p \in C \cap D} I(C,D,p) = \deg C \cdot \deg D.$$

Let C be a plane curve defined by the form $f \in k[x,y,z]$. We write $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, and $f_z = \partial f/\partial z$. The tangent line at a smooth point $p \in C$ is given by the equation

$$f_x(p)x + f_y(p)y + f_z(p)z = 0.$$

We say $p \in C$ is an inflection point if it is a smooth point, and the tangent line to C at p meets C at p with multiplicity ≥ 3. It is easy to show that the inflection points of $C = \mathcal{V}(f)$ are the points of C lying on the curve cut out by the determinant of the Hessian matrix, namely

$$\det(\partial^2 f/\partial x_i \partial x_j) = 0.$$

If $\deg C = d$, this determinant has degree $3(d-2)$, so C will have $3d(d-2)$ inflection points. Thus a cubic has 9 inflection points.

3. Plane cubics

A good reference for the material in this section is the book of Silverman and Tate [**164**]. See also [**150**]. We continue to work over a fixed algebraically closed field k.

By Bézout's theorem, two irreducible cubics meet at 9 points. Since the cubics in $k[x, y, z]$ form a 10-dimensional vector space, the space of plane cubics forms a $\mathbb{P}^9$. If $p \in \mathbb{P}^2$, the cubics passing through p form a hyperplane in that $\mathbb{P}^9$ (the coefficients of a basis for $k[x, y, z]_3$ give homogeneous coordinates on this $\mathbb{P}^9$, and the condition that a cubic pass through p translates to a linear condition on these coefficients). Thus, if we take 9 points in general position in $\mathbb{P}^2$, there will be a unique cubic passing through them (given by the intersection of nine general hyperplanes in $\mathbb{P}^9$).

PROPOSITION 3.1. *Let C_1 and C_2 be plane cubics with $C_1 \cap C_2 = \{p_1, \dots, p_9\}$ (these points need not be distinct, but are counted with multiplicity). If D is a cubic passing through $\{p_1, \dots, p_8\}$, then $p_9 \in D$ also.*

The group law on a smooth cubic. Let E be a smooth cubic. We will say that three, not necessarily distinct, points of E are collinear if they are the points of $L \cap E$ for some line $L \subset \mathbb{P}^2$, counted with multiplicity. Fix a point on E and label it 0. If $p \in E$ there is a unique line which meets E at p and 0; if $p = 0$ this is the tangent line at p; by Bézout's theorem this line meets E at a third point which we label $\tilde{p}$. Thus $0, p, \tilde{p}$ are collinear; it follows that $(\tilde{p})\tilde{} = p$, and, if 0 is an inflection point $(\tilde{0})\tilde{} = 0$. If $a, b \in E$, we define

$$a + b = \tilde{c} \tag{3-1}$$

where c is the third point where the line through a and b meets E. Thus $a, b, \widetilde{a+b}$ are collinear. Notice that $p + \tilde{p} = \tilde{0}$. This construction makes E an abelian group. The proof of associativity is intricate, but the rest is straightforward. For example, the abelian property is immediate from (3-1); furthermore, since $0, p, \tilde{p}$ are collinear, $0 + p = (\tilde{p})\tilde{} = p$, so 0 is the identity element; if $p', p, 0$ are collinear, then $p + p' = 0$, so inverses exist (and are unique).

The group law has a particularly simple geometric description if 0 is an inflection point. In that case $\tilde{0} = 0$, so $\tilde{p} = -p$, whence $p + q + r = 0$ if and only if p, q, r are collinear. In particular, $3p = 0$ if and only if p is an inflection point, so the 9 inflection points make up the 3-torsion subgroup of E. A nice consequence of this fact is that the line through any two inflection points meets the curve at a third inflection point. Thus the 9 inflection points lie on 12 lines, and each of those lines passes through 3 of the inflection points–this is a classical configuration. It is easy to show that the group structure on E does not depend on the choice of 0; the groups determined by two different choices of 0 are isomorphic.

It is not immediately apparent that the addition law $E \times E \to E$ is a morphism of algebraic varieties. One may choose an explicit equation for E, take two points $p, q \in E$ and compute to see that the coordinates of $p + q$ are given by ratios of polynomials in the coordinates of p and q [**150**, Chapter III].

Equations. Suppose that $\operatorname{char} k \neq 2, 3$. We may choose coordinates so that $(0, 1, 0)$ is an inflection point, and the tangent line at this point is $z = 0$. By making some further judicious choices one may ensure that the equation defining E can be put in the form $y^2 z = g(x, z)$, where g is a cubic of the form $x^3 + \cdots$; thus the affine equation for E can be put in Weierstrauss normal form

$$y^2 = x^3 + \alpha x + \beta \qquad \text{or} \qquad y^2 = x(x-1)(x-\lambda); \tag{3-2}$$

the smoothness of E ensures that $4\alpha^3 - 27\beta^2 \neq 0$, and that $\lambda \neq 0, 1$; if these conditions are satisfied, then the (projectivization of) equation (3-2) cuts out a smooth cubic. It is easy to show that as λ varies there are no isomorphisms between the E's that arise.

Suppose that E is given by an equation as in (3-2). Take $(0, 1, 0)$ to be the identity $0 \in E$; this is the only point on E lying on the line at infinity. If $p = (\alpha, \beta, 1) \in E$, then the line through p and 0 meets E again at the point $(\alpha, -\beta, 1)$, so this is $-p$. Thus the involution $p \mapsto -p$ on E is simply reflection in the x-axis.

Consider the configuration of inflection points, and the lines through them. Such a configuration is given by the following points:

$$\begin{matrix} (0,-1,1) & (0,\omega,1) & (0,\omega^2,1) \\ (1,0,-1) & (1,0,\omega) & (1,0,\omega^2) \\ (-1,1,0) & (\omega,1,0) & (\omega^2,1,0) \end{matrix}$$

where ω is a primitive cube root of -1. These nine points are the intersection of the two cubics $xyz = 0$ and (the Fermat curve) $x^3 + y^3 + z^3$ so they lie on all the cubics of the form

$$(x^3 + y^3 + z^3) - 3\lambda xyz. \tag{3-3}$$

The curve defined by (3-3) is singular if and only if $\lambda \in \{\infty, 1, \omega, \omega^2\}$. Every non-singular cubic arises in this way, and this equation is called the Hesse normal form.

Suppose that E is given by the equation (3-3), and take $0 = (0, -1, 1)$ to be the identity. The fact that the group law $E \times E \to E$ is a morphism is equivalent to the fact that for each $q \in E$ the map 'translation by q', $p \mapsto p + q$, is an automorphism of E as an algebraic variety. Suppose that $q = (a, b, c)$, and write σ for translation by q. Thus $\sigma(0, -1, 1) = (a, b, c)$. Since $q \in E$, we have $3\lambda = (a^3 + b^3 + c^3)/abc$; equivalently the equation of E is

$$abc(x^3 + y^3 + z^3) - (a^3 + b^3 + c^3)xyz. \tag{3-4}$$

EXERCISES

3.1 In this and subsequent exercises E denotes a smooth cubic curve. Show that the 2-torsion subgroup of E is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3.2 Use the previous exercise to show that given $q \in C$ there are 4 distinct lines through q which meet E tangentially.

3.3 Fix a line L and intersect the 4 tangent lines arising in the previous exercise with L. Show that λ appearing in the Weierstrauss normal form for E can be recovered as "the" cross-ratio of these points.

3.4 Show that the equation of the cubic curve in (3-4) may be written in the form

$$\det \begin{pmatrix} ax & cz & by \\ bz & ay & cx \\ cy & bx & az \end{pmatrix} = 0.$$

3.5 Let σ denote translation by $q = (a, b, c)$ as in the discussion above. Show that

$$\begin{aligned} \sigma(x,y,z) &= (a^2yz - bcx^2, c^2xy - abz^2, b^2xz - acy^2) \\ &= (acz^2 - b^2xy, bcy^2 - a^2xz, abx^2 - c^2yz) \\ &= (aby^2 - c^2xz, acx^2 - b^2yz, bcz^2 - a^2xy). \end{aligned}$$

[Hint: use the minors of the matrix in the previous exercise.]

3.6 If E is cut out by an equation in Hesse normal form, show that the involution $p \mapsto -p$ is given by $(x, y, z) \mapsto (x, z, y)$.

4. Grassmannians

Throughout this section k is a field.

Definition 4.1. The Grassmanian $G(d, n)$ is the set of d-dimensional subspaces of k^n. We write $G(d, V)$ for the d-dimensional subspaces of a vector space V.

Projective spaces are Grassmanians because $\mathbb{P}^n$ is $G(1, n+1)$. Moreover, every Grassmanian can be made into a projective variety in a natural way.

The Plücker embedding. Let V be an n-dimensional vector space, and consider the projective space of lines in the d^{th} exterior power of V, namely $\mathbb{P}(\Lambda^d V) \cong \mathbb{P}^{\binom{n}{d}-1}$. Define

$$\Psi : G(d, n) \to \mathbb{P}(\Lambda^d V)$$

by $W = kw_1 \oplus \cdots \oplus kw_d \mapsto w_1 \wedge \cdots \wedge w_d$; changing the basis for W by some $g \in GL(d)$ changes $\Psi(W)$ by a constant multiple $\det g$, so $\Psi(W)$ is a well-defined element of $\mathbb{P}(\Lambda^d V)$. This map is injective because it has a left inverse sending $\Psi(W)$ to $\{v \in V \mid v \wedge \Psi(W) = 0\} = W$.

The image of Ψ is a subvariety of $\mathbb{P}(\Lambda^d V)$. To see this, notice that $\omega \in \mathbb{P}(\Lambda^d V)$ is in the image of Ψ if and only if $\{v \in V \mid v \wedge \omega = 0\}$ is d-dimensional, so $\omega \in \operatorname{Im} \Psi$ if and only if the map $\varphi_\omega : V \to \Lambda^{d+1} V$ given by $v \mapsto v \wedge \omega$ has rank $n - d$; but the rank of φ_ω can't be less than $n - d$, so

$$\operatorname{Im} \Psi = \{\omega \mid \operatorname{rank} \varphi_\omega \leq n - d\}.$$

But the map $\varphi : \Lambda^d V \to \operatorname{Hom}_k(V, \Lambda^{d+1} V)$ is linear, so the result follows. Explicitly, $\operatorname{Im} \Psi$ is given by the vanishing of the $(n-d+1) \times (n-d+1)$ minors of the matrix representing φ_ω.

From now on we will think of the Grassmanian as a projective variety. The d-dimensional subspaces of k^n correspond to the $d-1$-dimensional linear subspaces of $\mathbb{P}^{n-1}$; when we think of $G(d, n)$ in this way we will deote it by $\mathbb{G}(d-1, n-1)$.

The cellular decomposition. A Grassmanian may be written as a disjoint union of copies of affine spaces $\mathbb{A}^i$ in a way which generalizes the decomposition

$$\mathbb{P}^n = \mathbb{A}^0 \cup \mathbb{A}^1 \cup \cdots \cup \mathbb{A}^n.$$

To do this, first fix a full flag

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} = k^{n+1}$$

with $\dim V_i = i$. For each d-tuple $\underline{a} = (a_1, \ldots, a_d)$ we define

$$X_{\underline{a}} = \{W \in G(d, n+1) \mid \dim W \cap V_{n-d+i-a_i} = i \text{ for all } i\}.$$

It is clear that $G(d, n+1)$ is the disjoint union of the various $X_{\underline{a}}$.

Example 4.2. The first Grassmanian which is not a projective space is $G(2, 4)$. The Plucker embedding embeds it in $\mathbb{P}^5$. There are 6 cells in the cellular decomposition labelled by $\underline{a} = 00$, 10, 11, 20, 21, 22. Thinking of $G(2, 4)$ as $\mathbb{G}(1, 3)$, the flag consists of a point, a line, and a plane in $\mathbb{P}^3$, say $p \in \ell \subset h$. The following picture describes the inclusion relations amongst the cells. The dimension of each cell is written down to the right

$$\begin{tikzcd}[column sep=small]
 & \{\text{lines not in } h\} \arrow[d] & & 4 \\
 & \{\text{lines meeting } \ell\} \arrow[dl] \arrow[dr] & & 3 \\
\{\text{lines through } p\} \arrow[dr] & & \{\text{lines on } h\} \arrow[dl] & 2 \\
 & \{\text{lines on } h \text{ through } p\} \arrow[d] & & 1 \\
 & \{\ell\} & & 0
\end{tikzcd}$$

The Grassmanian as a homogeneous space. Let V be an n-dimensional vector space over k. The action of $GL(V)$ on V sends d-dimensional subspaces to d-dimensional subspaces, thus giving an action of $GL(V)$ on $G(d,n)$. Since any two bases of V are conjugate under the action of $GL(V)$, $GL(V)$ acts transitively on $G(d,n)$. It turns out that this is an algebraic action (one can see this by writing the action explicitly making use of the Plucker embedding). Since the action is transitive, $G(d,n) \cong \mathrm{GL}(V)/P$, where P is the stabilizer of a particular d-dimensional subspace. In fact, since $GL(V)$ is an algebraic group, P is an algebraic subgroup, and $GL(V)/P$ inherits the structure of an algebraic variety and $G(d,n)$ and $GL(V)/P$ are isomorphic as algebraic varieties.

To identify P, fix a basis $\{e_1,\ldots,e_n\}$ for V, and let $W = ke_1+\cdots+ke_d$. Then $g \in GL(V)$ sends W to itself if and only if the entries in the lower left $(n-d)\times d$ corner of g are all zero. Thus

$$P = \begin{pmatrix} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & & & & & \vdots \\ * & * & * & \cdots & * & * \\ 0 & 0 & 0 & \cdots & * & * \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & * & * \end{pmatrix}$$

Since $\dim GL(V) - \dim P = d(n-d)$, $\dim G(d,n) = d(n-d)$.

The cohomology of the Grassmanian. If we view $G(d,n)$ as the homogeneous subspace $GL(n)/P$, then we may compute its cohomology ring. Indeed, for any semisimple algebraic group G over $\mathbb{C}$, and any parabolic subgroup P, $H^*(G/P,\mathbb{Z})$ has been computed [**32**]. Then

$$H^*(G/P,\mathbb{Z}) \cong S(\mathfrak{h}^*)^{W_P}/I$$

where $\mathfrak{h}^*$ is the dual of the Cartan subalgebra of the Lie algebra of G, W denotes the Weyl group, W_P its subgroup generated by the simple reflections determined by the roots in P, $S(\mathfrak{h}^*)^{W_P}$ the invariants, and I the ideal generated by the the W-invariants of positive degree in $S(\mathfrak{h}^*)$.

For example, consider $G(2,4) \cong GL(4)/P$ where P is the subgroup

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

The Weyl group naturally identifies with the symmetric group S_4, and W_P is the subgroup generated by (12) and (34). We may write $S(\mathfrak{h}^*) = k[t_1, t_2, t_3, t_4]$, and W acts in the obvious way by permuting the subscripts. It is clear that

$$S(\mathfrak{h}^*)^{W_P} = k[t_1 + t_2, t_3 + t_4, t_1t_2, t_3t_4]$$

and I is generated by the symmetric polynomials. If we write

$$w = t_1 + t_2,\ x = t_3 + t_4,\ y = t_1t_2,\ z = t_3t_4,$$

then $H^*(G(2,4), \mathbb{Z})$ is generated by w, x, y, z subject to the relations

$$\begin{aligned} w + x &= t_1 + t_2 + t_3 + t_4 = 0, \\ wx + y + z &= \sum t_i t_j = 0, \\ wz + xy &= \sum t_h t_i t_j = 0, \\ yz &= t_1t_2t_3t_4. \end{aligned}$$

A basis for the cohomology ring is therefore $1, w, y, w^2, wy, w^3, w^2y, y^2, w^4$, and the Hilbert series is

$$1 + t + 2t^2 + 2t^3 + 3t^4.$$

5. Quadrics

6. Sheaves of abelian groups

Definition 6.1. Let $f : X \to Y$ be a continuous map of topological spaces. We define functors $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ and $f^{-1} : \mathcal{S}h(Y) \to \mathcal{S}h(X)$ as follows. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y respectively. The

- direct image sheaf $f_*\mathcal{F}$ on Y is defined by
$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V) \tag{6-1}$$
for each open set $V \subset Y$.
- inverse image sheaf $f^{-1}\mathcal{G}$ on X is the sheaf associated to the presheaf
$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$
for each open set $U \subset X$, where the direct limit is taken over all open sets $V \subset Y$ containing $f(U)$; note that $f(U)$ need not be open, so $\mathcal{G}(f(U))$ does not make sense.

If f is the inclusion of a closed subspace, we call $f^{-1}\mathcal{G}$ the restriction of $\mathcal{G}$ to X, and denote it by $\mathcal{G}|_X$; the stalks of $\mathcal{G}$ and $f^{-1}\mathcal{G}$ coincide at points of X.

Proposition 6.2. *The functors f_* and f^{-1} are adjoint to one another:*

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}). \tag{6-2}$$

7. Ringed spaces

Definition 7.1. A ringed space is a pair $(X, \mathcal{R})$ consisting of a topological space X and a sheaf of rings $\mathcal{R}$ on X. Thus, for each open subset U of X, we have a ring $\mathcal{R}(U)$, and if $U \subset V$ are open subsets, there is a ring homomorphism $\rho_U^V : \mathcal{R}(V) \to \mathcal{R}(U)$ such that $\rho_U^V \circ \rho_V^W = \rho_U^W$ whenever $U \subset V \subset W$. A sheaf of $\mathcal{R}$-modules is a sheaf of abelian groups $\mathcal{M}$ on X such that each $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module, and each restriction map $\mathcal{M}(V) \to \mathcal{M}(U)$ is an $\mathcal{R}(V)$-module homomorphism, where $\mathcal{M}(U)$ is viewed as a $\mathcal{R}(V)$-module via the map $\mathcal{R}(V) \to \mathcal{R}(U)$.

There is no requirement in the definition that the rings $\mathcal{R}(U)$ be commutative. If $(X, \mathcal{R})$ is a ringed space, then the $\mathcal{R}$-modules form an abelian category.

The construction can be generalized considerably. The topological space X can be replaced by a category $\mathcal{X}$, the open sets being replaced by the objects in $\mathcal{X}$, and the inclusions between the open sets being replaced by the morphisms in $\mathcal{X}$. To each object in $\mathcal{X}$ we associate a ring, and to each morphism in $\mathcal{X}$ we associate a ring homomorphism. This is done in a functorial way; that is, we take a contravariant functor $\mathcal{R} : \mathcal{X} \to \mathsf{Ring}$. (Paul, what replaces the sheaf axiom?) We can then define an $\mathcal{R}$-module as a contravariant functor $\mathcal{M} : \mathcal{X} \to \mathsf{Ab}$ such that each $\mathcal{M}(U)$ is endowed with an $\mathcal{R}(U)$-module structure, and the restriction maps $\mathcal{M}(V) \to \mathcal{M}(U)$ are $\mathcal{R}(V)$-module homomorphisms in the obvious way.

Even this can be generalized. We replace the category of rings with a new category, a sort of souped up version of the category of rings. The objects in the new category are pairs $(\mathsf{Mod}R, R)$, and a morphism $f : (\mathsf{Mod}R, R) \to (\mathsf{Mod}S, S)$ is a functor $f_* : \mathsf{Mod}S \to \mathsf{Mod}R$. Thus, the objects are affine schemes.

Example 7.2. Consider a pair of rings R and S together with an R-S-bimodule B. Form the upper triangular matrix ring

$$T = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}.$$

A left T-module is a triple (M, N, φ) consisting of a left R-module M, a left S-module N, and a left R-module homomorphism $\varphi : B \otimes_S N \to M$; this triple should be viewed as a column

$$\begin{pmatrix} M \\ N \end{pmatrix}$$

on which T acts by left multiplication in the obvious way. Let $\mathcal{X}$ be the category with objects $0, 1$ and a single morphism $0 \to 1$. Let $\mathcal{R}(1) = (\mathsf{Mod}R, R)$ and $\mathcal{R}(0) = (\mathsf{Mod}S, S)$. The bimodule B yields a functor $f_* = B \otimes_S - : \mathsf{Mod}S \to \mathsf{Mod}R$.

8. Schemes

All schemes will be k-schemes; that is, schemes over a fixed base field k. If X and Y are k-schemes we write $\mathrm{Mor}_k(X, Y)$ for the morphisms $f : X \to Y$ in the category of k-schemes.

Definition 8.1. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a space X and a sheaf $\mathcal{O}_X$ of rings on X. A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is a pair $(f, f^\sharp)$ consisting of a continuous map $f : X \to Y$ of topological spaces and a homomorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ becomes an $f_*\mathcal{O}_X$-module—this follows at once from the definition of f_* (6-1). Using the homomorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, $f_*\mathcal{F}$ becomes a $\mathcal{O}_Y$-module. Hence we have the direct image functor

$$f_* : \mathsf{Mod}\mathcal{O}_X \to \mathsf{Mod}\mathcal{O}_Y.$$

The definition of the inverse image functor on $\mathcal{O}_Y$-modules is less straightforward. First, a special case of the adjoint isomorphism is $\mathrm{Hom}_Y(\mathcal{O}_Y, f_*\mathcal{O}_X) \cong \mathrm{Hom}_X(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$, so $f^\sharp$ corresponds to a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, which is a homomorphism of sheaves of rings on X. Now, if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we define

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X. \tag{8-1}$$

In particular, $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module, so we have the inverse image functor

$$f^* : \mathsf{Mod}\mathcal{O}_Y \to \mathsf{Mod}\mathcal{O}_X.$$

PROPOSITION 8.2. *The functors f^* and f_* are an adjoint pair:*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \tag{8-2}$$

General principles show that f^* is right exact and f_* left exact. We also note that f^* commutes with tensor product. If f is a map between affine schemes then f_* also has a right adjoint (Exercise A.4), so is exact. But on a projective scheme f_* is not right exact, and its right derived functors R^if_* play an important role. For example, if $f : X \to \operatorname{Spec} k$ is the structure morphism, then $R^if_* = H^i(X, -)$ are the usual cohomology functors.

It follows immediately from (8-1) that $f^*\mathcal{O}_Y \cong \mathcal{O}_X$, so (8-2) gives an isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, f_*\mathcal{F})$; that is,

$$\mathrm{H}^0(X, \mathcal{F}) \cong \mathrm{H}^0(Y, f_*\mathcal{F}).$$

We also take note of the morphism $\mathcal{G} \to f_*f^*\mathcal{G}$ arising from (8-2) with $\mathcal{F} = f^*\mathcal{G}$; this gives a map

$$\mathrm{H}^0(Y, \mathcal{G}) \to \mathrm{H}^0(Y, f_*f^*\mathcal{G}) \cong \mathrm{H}^0(X, f^*\mathcal{G}). \tag{8-3}$$

LEMMA 8.3 (Projection Formula). *Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F} \in \mathsf{Mod}\mathcal{O}_X$ and $\mathcal{G} \in \mathsf{Mod}\mathcal{O}_Y$. If $\mathcal{G}$ is locally free, then*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

PROOF. $\square$

Definition 8.4. A closed subscheme, X say, of a scheme Y is a morphism $i : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $i(X)$ is a closed subset of Y and $i^\sharp : \mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective. We say i is a closed immersion if i is an isomorphism of ringed spaces from $(X, \mathcal{O}_X)$ to a closed subscheme of Y.

If $i : X \to Y$ is the inclusion of a closed subscheme, we define the sheaf of ideals

$$\mathcal{I}_X = \ker(i^\sharp : \mathcal{O}_Y \to i_*\mathcal{O}_X).$$

Hence there is an exact sequence $0 \to \mathcal{I}_X \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0$; but $\mathcal{O}_X \cong i^*\mathcal{O}_Y$, so

$$i_*i^*\mathcal{O}_Y \cong \mathcal{O}_Y/\mathcal{I}_X.$$

Inverse images under automorphisms play an important role in Chapter 22 on twisted homogeneous coordinate rings. An automorphism of a scheme X is a pair $(\sigma, \sigma^\sharp)$ where $\sigma : X \to X$ and $\sigma^\sharp : \mathcal{O}_X \to \sigma_*\mathcal{O}_X$; thus, for each open $U \subset X$ we have a map $\sigma^\sharp_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma^{-1}U)$, which we will denote by $f \mapsto f^\sigma$. We will think of σ acting on X from the left and on $\mathcal{O}_X$ from the right: if $f \in \mathcal{O}(U)$, then $f^\sigma := f \circ \sigma \in \mathcal{O}_X(\sigma^{-1}U)$.

Let $\mathcal{G} \in \mathsf{Mod}\mathcal{O}_X$. We will write $\mathcal{G}^\sigma$ for $\sigma^*\mathcal{G}$. It is useful to observe that $\mathcal{G}^\sigma \cong \sigma_*^{-1}\mathcal{G}$: if $\mathcal{F} \in \mathsf{Mod}\mathcal{O}_X$, then

$$\begin{aligned}\operatorname{Hom}(\sigma^*\mathcal{G}, \mathcal{F}) &\cong \operatorname{Hom}(\mathcal{G}, \sigma_*\mathcal{F})\\ &\cong \operatorname{Hom}((\sigma^{-1})_*\mathcal{G}, (\sigma^{-1})_*\sigma_*\mathcal{F})\\ &\cong \operatorname{Hom}((\sigma^{-1})_*\mathcal{G}, \mathcal{F}),\end{aligned}$$

so $\sigma^*\mathcal{G} \cong (\sigma^{-1})_*\mathcal{G}$ by Yoneda's Lemma (Corollary A.6.5). We will identify $\mathcal{G}^\sigma$ and $\sigma_*^{-1}\mathcal{G}$. Hence, on an open set U,

$$\mathcal{G}^\sigma(U) \cong \mathcal{G}(\sigma U).$$

To keep the notation (somewhat) clear, if $s \in \mathcal{G}(\sigma U)$, we will write s^σ for s considered as an element of $\mathcal{G}^\sigma(U)$. We also think of this as an isomorphism

$$\begin{aligned}\mathcal{G}(\sigma U) &\to \mathcal{G}^\sigma(U)\\ s &\mapsto s^\sigma.\end{aligned} \tag{8-4}$$

On global sections, the map $s \mapsto s^\sigma$ is an isomorphism $\mathrm{H}^0(X, \mathcal{G}) \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{G}^\sigma)$.

Let $f \in \mathcal{O}_X(\sigma U)$ and $s \in \mathcal{G}(\sigma U)$. Consider $f^\sigma.s^\sigma$, which is an element of $\mathcal{G}^\sigma(U)$: the cO_X-module structure of $\mathcal{G}^\sigma = \sigma_*^{-1}\mathcal{G}$ is obtained via the map $(\sigma^{-1})^\sharp : \mathcal{O}_X \to \sigma_*^{-1}\mathcal{O}_X$, $f \mapsto f^{\sigma^{-1}}$, so

$$f^\sigma.s^\sigma = [((\sigma^{-1})^\sharp f^\sigma).s]^\sigma = (fs)^\sharp.$$

EXAMPLE 8.5. Let K be an extension field of k and let σ be a k-linear automorphism of K. We consider $\eta = \operatorname{Spec} K$ as a k-scheme, and σ as an automorphism of this scheme; we write $\mathcal{O}_\eta$ for the structure sheaf of $\operatorname{Spec} K$. Since this is an affine scheme we will denote sheaves of $\mathcal{O}_\eta$ modules by their global sections; thus $\mathcal{O}_\eta$ is denoted by K, and $\mathcal{O}_\eta$-modules are simply K-vector spaces. We will write L for K considered as a module over itself (and reserve the letter K for K considered as a ring). Thus $L = K$ with the action $x.a = xa$ for $x \in K$ and $a \in L$.

As a ringed space this scheme is $(\eta, \mathcal{O}_\eta) = (\eta, K)$. The automorphism of this scheme induced by σ is denoted by f; thus, $f = (\mathbb{1}, \sigma) : (\eta, K) \to (\eta, K)$, where $f^\sharp = \sigma : K \to f_*K = K$.

We want to consider the functor f^* acting on L; everything we are about to do is trivial because f^*L must again be 1-dimensional whence $f^*L \cong L$. But, we want to be careful about what this isomorphism is; this care is necessary in Chapter 22. First, notice that $f^{-1}K = K$ and that $f^{-1}L = L$. The tricky point in the definition of $f^*L := K \otimes_{f^{-1}K} f^{-1}L$ is the structure of K as a $f^{-1}K$-module—this is obtained from the morphism $f^{-1}K \to K$ which corresponds to $f^\sharp$

under the adjoint isomorphism $\mathrm{Hom}_\eta(f^{-1}K, K) \cong \mathrm{Hom}_\eta(K, f_*K)$; that is simply $\sigma : f^{-1}K = K \to K$, so the module action is

$$x.y = xy^\sigma \qquad \text{for } x \in K \text{ and } y \in f^{-1}K = K.$$

Therefore, f^*L has elements $1 \otimes a$ for $a \in L$ and the action of K is given by

$$x.(1 \otimes a) = x \otimes a = 1.x^{\sigma^{-1}} \otimes a = 1 \otimes x^{\sigma^{-1}} a.$$

Hence, the K-module isomorphism $\psi : f^*L \to L$ which sends $1 \otimes 1$ to 1 must also satisfy

$$\psi(x.(1 \otimes 1)) = x.\psi(1 \otimes 1) = x.1;$$

that is,

$$\psi(1 \otimes x^{\sigma^{-1}}) = x,$$

or

$$\psi(1 \otimes x) = x^\sigma$$

Thus if σ^*L is identified with K, the isomorphism $\sigma^*L \to L$ is given by $x \mapsto x^\sigma$.

9. Divisors

In this section X denotes a smooth irreducible variety.

Definition 9.1. A prime divisor on X is a closed irreducible subvariety Z, of codimension one. We write $\mathrm{Div}(X)$ for the free abelian group with basis the prime divisors; its elements are called (Weil) divisors. A divisor is written as $D = n_1Z_1 + \cdots + n_rZ_r$, where the $n_i \in \mathbb{Z}$ and each Z_i is a prime divisor. If all $n_i = 0$, we write $D = 0$. If all $n_i \geq 0$, we write $D \geq 0$ and say that D is effective.

If X is an affine variety, the prime divisors are in bijection with the height one primes. If X is affine and its coordinate ring R is a UFD, the height one primes are principal, so the prime divisors are in bijection with the irreducible elements of R; this explains the terminology.

On a curve the prime divisors are simply the points, so a divisor is a formal $\mathbb{Z}$-linear combination of points. On $\mathbb{P}^2$ an effective divisor is simply the zero locus of some homogenous polynomial of positive degree.

Definition 9.2. Let X be an irreducible projective variety over a field k, and let $k(X)$ be its field of rational functions. We say that X is regular in codimension one if the local rings $\mathcal{O}_{X,Z}$ are regular, and hence discrete valuation rings, for all irreducible codimension 1 subvarieties $Z \subset X$.

Assume X is regular in codimension one. Write ν_Z for the valuation on $k(X)$ associated to a prime divisor Z on X. If $0 \neq f \in k(X)$, we say that f has a pole along Z if $\nu_Z(f) < 0$, and a zero along Z if $\nu_Z(f) > 0$. The principal divisor associated to f is

$$\mathrm{div}(f) = (f) := \sum \nu_Z(f)[Z];$$

this is a *finite* sum. The map $f \mapsto (f)$ is a homomorphism $k(X)^* \to \mathrm{Div}(X)$. The cokernel, namely

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/\{\text{principal divisors}\}$$

is called the divisor class group. Thus we have an exact sequence

$$1 \to k^* \to k(X)^* \to \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0.$$

EXAMPLE 9.3. $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.

EXAMPLE 9.4. Let Q be a smooth quadric in $\mathbb{P}^3$; thus $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathrm{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let X be an irreducible variety all of whose local rings are UFDs. Let Z be a prime divisor on X. Then, each $x \in X$ has an open affine neighbourhood U_x such that $Z \cap U_x$ is the zero locus of a single $f_x \in \mathcal{O}(U_x)$. Since X is quasi-compact, there is a finite open affine cover U_i for X, and elements $f_i \in \mathcal{O}(U_i)$ such that $Z \cap U_i = \mathcal{V}(f_i)$ for all i. On $U_{ij} := U_i \cap U_j$, f_i and f_j have the same zero locus, so f_i/f_j is a unit in $\mathcal{O}(U_{ij})$. These give rise to an element of $\mathrm{H}^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of units in $\mathcal{O}_X$.

Conversely, given a finite open affine cover U_i and elements $f_i \in \mathcal{O}(U_i)$ such that $f_i f_j^{-1} \in \mathcal{O}(U_{ij})^*$, there is an associated divisor $\sum_Z n_Z[Z]$ where $n_Z = \nu_Z(f_i)$ if $U_i \cap Z \neq \emptyset$ and $n_Z = 0$ if $U_i \cap Z = \emptyset$.

10. Invertible $\mathcal{O}_X$-modules

Over a commutative ring R, a module M is invertible if there exists another R-module N such that $M \otimes_R N \cong R$. One may show that this is equivalent to the requirement that $M_\mathfrak{p} \cong R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Also recall that M is projective if and only if $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 10.1. An $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if X can be covered by open sets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module for each U. The rank of $\mathcal{F}$ on such a U is the rank of the free module $\mathcal{F}|_U$. A locally free $\mathcal{O}_X$-module of rank 1 (everywhere) is called an invertible $\mathcal{O}_X$-module.

PROPOSITION 10.2. *Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules. Then*
1. *$\mathcal{L} \otimes \mathcal{M}$ is invertible;*
2. *$\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is invertible and $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.*

Definition 10.3. The Picard group of a variety X, $\mathrm{Pic}(X)$, has elements the isomorphism classes of invertible $\mathcal{O}_X$-modules and multiplication the tensor product.

Definition 10.4. Let D be a divisor on an irreducible projective variety X. Define the sheaf $\mathcal{O}(D)$ by

$$\mathcal{O}(D)(U) := \{f \in k(X) \mid f = 0 \text{ or } (f) + D \geq 0 \text{ on } U\}.$$

EXAMPLE 10.5. $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, isom $r \mapsto \mathcal{O}(r)$.

Let $f : X \to Y$ be a morphism. Since $f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} \cong f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})$ and $f^*\mathcal{O}_Y \cong \mathcal{O}_X$, it follows that f^* sends invertible $\mathcal{O}_Y$-modules to invertible $\mathcal{O}_X$-modules. In terms of divisors, $f^*\mathcal{O}_Y(D) \cong \mathcal{O}_X(f^{-1}D)$.

Definition 10.6. Let X be a k-scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is generated by its global sections if there is a set of global sections $\{s_i\} \in \mathrm{H}^0(X, \mathcal{F})$ such that for each open set $U \subset X$, $\mathcal{F}(U)$ is generated as an $\mathcal{O}_X(U)$-module by the images of the s_i (equivalently, if each stalk $\mathcal{F}_x$ is generated by the images of the s_i).

For any $\mathcal{O}_X$-module $\mathcal{F}$ there is a natural $\mathcal{O}_X$-module homomorphism

$$\mathrm{H}^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F} \tag{10-1}$$

sending $s \otimes 1$ to s. This map is surjective if and only if $\mathcal{F}$ is generated by global sections. Thus, if $\mathcal{F}$ is generated by its global sections, it is a quotient of a free $\mathcal{O}_X$-module. Conversely, if $\mathcal{F}$ is a quotient of a free $\mathcal{O}_X$-module, it is generated by its global sections (because an epimorphism $\mathcal{G} \to \mathcal{F}$ gives surjective maps $\mathcal{G}_x \to \mathcal{F}_x$ for all $x \in X$). It follows from this last characterization that, if $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of rank n. We define

- the determinant line bundle
$$\det(\mathcal{F}) := \Lambda^n \mathcal{F};$$
- the dual $\mathcal{L}^* := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$;
- if X is a curve, $\deg(\mathcal{F}) := \deg(\det \mathcal{F})$.

If X is a smooth curve, the Riemann-Roch Theorem for locally free $\mathcal{O}_X$-modules states that
$$\mathrm{h}^0(\mathcal{F}) - \mathrm{h}^1(\mathcal{F}) = \deg(\mathcal{F}) + (1-g)\operatorname{rank}\mathcal{F},$$
where $g := \mathrm{h}^1(\mathcal{O}_X)$ is the arithmetic genus of X.

$\boxed{\text{Paul}}$ There is no apparent non-commutative analog of the isomorphism $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$. The problem is that $X = \operatorname{Proj} A$ in the non-commutative case has no 'open subsets' so we can't define a sheaf of abelian groups on X; all we have is the category $\mathsf{Tails}(A)$.

11. Ample line bundles and immersions $X \to \mathbb{P}^n$

This section discusses the relation between invertible $\mathcal{O}_X$-modules and morphisms $X \to \mathbb{P}^n$, where X is a fixed k-scheme.

The following result motivates the material in this section.

Proposition 11.1. [**57**, Corollary III-18, page 97] *If R is a commutative k-algebra, then there is a bijection between $\operatorname{Mor}_k(\operatorname{Spec} R, \mathbb{P}^n_k)$ and*
$$\frac{\text{invertible } R\text{-modules } M \text{ together with a surjection } R^{n+1} \to M}{\text{units of } R \text{ acting as automorphisms of } M}.$$

Theorem 11.2. [**57**, Theorem III-13] *If X is a k-scheme, then there is a bijection between $\operatorname{Mor}_k(X, \mathbb{P}^n_k)$ and*
$$\frac{\text{invertible } \mathcal{O}_X\text{-modules } \mathcal{L} \text{ together with a surjection } \mathcal{O}_X^{n+1} \to \mathcal{L}}{\text{units of } \mathcal{O}_X(X) \text{ acting as automorphisms of } \mathcal{L}}.$$

Theorem 11.3. [**76**, Chapter II, Theorem 7.1] *Let X be a scheme over the base field k.*

1. *If $f : X \to \mathbb{P}^n$ is a morphism, then $f^*\mathcal{O}_{\mathbb{P}}(1)$ is an invertible $\mathcal{O}_X$-module which is generated by its global sections $\{s_i = f^*(x_i) \mid 0 \le i \le n\}$.*
2. *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, and $s_0, \dots, s_n \in \mathrm{H}^0(X, \mathcal{L})$ generate $\mathcal{L}$, then there is a unique morphism $f : X \to \mathbb{P}^n$ such that $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}}(1)$ (and $s_i = f^*(x_i)$ under this isomorphism).*

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $s \in \mathrm{H}^0(X, \mathcal{L})$ and $x \in X$, we can evaluate s at x via an isomorphism $\mathcal{L}|_U \cong \mathcal{O}_X|_U$ in some open neighbourhood U of x; the conditions $s(x) = 0$ and $s(x) \neq 0$ are independent of the choice of local isomorphism, so the next definition makes sense.

Definition 11.4. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A point $x \in X$ is a base point of $\mathcal{L}$, or of $\mathrm{H}^0(X, \mathcal{L})$, if $s(x) = 0$ for all $s \in \mathrm{H}^0(X, \mathcal{L})$. If $\mathcal{L}$ has no base points we say that $\mathcal{L}$ is base point free.

If $\mathcal{L}$ is base point free, we define the morphism

$$\phi_{\mathcal{L}} : X \to \mathbb{P} = \mathbb{P}(H^0(X, \mathcal{L})^*)$$

by

$$\phi_{\mathcal{L}}(x) = \{s \in \mathrm{H}^0(X, \mathcal{L}) \mid s(x) = 0\}.$$

We are identifying points of $\mathbb{P}$ with hyperplanes in $H^0(X, \mathcal{L})$. If $s_0, \ldots, s_n$ is a basis for $\mathrm{H}^0(X, \mathcal{L})$, then $\phi_{\mathcal{L}}(x)$ is the point with homogeneous coordinates $(s_0(x), \ldots, s_n(x))$.

Definition 11.5. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is very ample if

- $\mathcal{L}$ is base point free, and
- $\phi_{\mathcal{L}}$ is an immersion.

We say $\mathcal{L}$ is ample if $\mathcal{L}^{\otimes d}$ is very ample for some d.

There are several important characterizations of ampleness.

THEOREM 11.6. *Let $\mathcal{L}$ be a line bundle on a projective scheme X over k. The following are equivalent:*

1. *$\mathcal{L}$ is ample;*
2. *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$,*
$$\mathrm{H}^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$
for all $q > 0$ and all $n \gg 0$;
3. *for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections for $n \gg 0$.*

If $\mathcal{L}$ is very ample, the immersion $\phi_{\mathcal{L}} : X \to \mathbb{P}^n$ is such that $\phi_{\mathcal{L}}^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}$; more generally, $\phi_{\mathcal{L}}^* \mathcal{O}_{\mathbb{P}^n}(d) \cong \mathcal{L}^{\otimes d}$.

Section rings. We may associate to a closed subscheme $X \subset \mathbb{P}(V^*)$ its homogeneous coordinate ring $S(V)/I$ where I is generated by the functions defining X. Homogeneous coordinate rings are rather badly behaved from a functorial point of view, so it is better to work with the section ring

$$B(X, \mathcal{L}) := \bigoplus_{d=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}^{\otimes d}),$$

where $\mathcal{L} = i^* \mathcal{O}_{\mathbb{P}}(1)$ is the ample line bundle determining the inclusion $i : X \to \mathbb{P} = \mathbb{P}(V^*)$.

If $\mathcal{L}$ is any $\mathcal{O}_X$-module, we may endow $B(X, \mathcal{L})$ with the structure of a graded k-algebra; in fact, $\oplus_{d \geq 0} \mathcal{L}^{\otimes d}$ is a sheaf of $\mathcal{O}_X$-algebras with an obvious multiplication coming from the isomorphism $\mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes j} \to \mathcal{L}^{\otimes i+j}$. The simplest case is given by $\mathbb{P}^n$ with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$. In that case, if $V = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then

$$\bigoplus_{d=0}^{\infty} \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong S(V),$$

the homogeneous coordinate ring of $\mathbb{P}^n$.

We are particularly interested in the case when $\mathcal{L}$ is very ample. Let $f : X \to \mathbb{P}^n$ be a morphism, set $V = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n}(1)$. It follows from (8-3) that there is a map of graded vector spaces

$$\varphi : S(V) \to \bigoplus_{d=0}^{\infty} H^0(X, \mathcal{L}^{\otimes d}), \tag{11-1}$$

which is easily verified to be a graded algebra homomorphism. Its kernel is the ideal of functions vanishing on $f(X)$. If f is the inclusion of a closed subscheme $X \subset \mathbb{P}^n$, then $S(V)/\ker(\varphi)$ is, by definition, the homogeneous coordinate ring of X.

By [**76**, Chapter III, Exercise 5.5], the map in (11-1) is surjective if X is a complete intersection.

LEMMA 11.7. *If $i : X \to \mathbb{P}^n$ is the inclusion of a closed subscheme, then the map φ in (11-1) embeds the homogeneous coordinate ring of X in $\oplus \mathrm{H}^0(X, \mathcal{L}^{\otimes d})$. Moreover, these rings are equal in high degree.*

PROOF. The only point to be verified is that φ is surjective in high degree. First, by the projection formula (8.3) with $\mathcal{F} = \mathcal{O}_X = i^*\mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^n}(d)$,

$$i_*i^*\mathcal{O}_{\mathbb{P}^n} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d) \cong i_*i^*\mathcal{O}_{\mathbb{P}^n}(d).$$

Tensoring the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X = i_*i^*\mathcal{O}_{\mathbb{P}^n} \to 0$$

with the locally free sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ yields the exact sequence

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^n}(d) \to i_*i^*\mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

Since $\mathcal{O}_{\mathbb{P}^n}(d)$ is ample, $\mathrm{H}^1(\mathbb{P}, \mathcal{I}_X(d)) = 0$ for large d, whence the long exact sequence in cohomology gives a surjective map

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to \mathrm{H}^0(\mathbb{P}^n, i_*i^*\mathcal{O}_{\mathbb{P}^n}(d)).$$

But this last term is isomorphic, by adjointness, to $\mathrm{H}^0(X, i^*\mathcal{O}_{\mathbb{P}^n}(d)) = \mathrm{H}^0(X, \mathcal{L}^{\otimes d})$. Thus, we obtain the required surjection for $d \gg 0$. $\square$

EXAMPLE 11.8. Let $X = \mathbb{P}^1$, and $\mathcal{L} = \mathcal{O}(n)$ with $n \geq 1$. The embedding determined by $\mathcal{L}$ is the n-uple embedding $\mathbb{P}^1 \to \mathbb{P}^n$ defined by

$$(x, y) \mapsto (x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n).$$

The homogeneous coordinate ring is isomorphic to the n^{th} Veronese subalgebra $k[x, y]^{(n+1)}$ of the polynomial ring. The dimension of the degree d component is $nd + 1$. On the other hand $B(\mathbb{P}^1, \mathcal{L})_d = H^0(\mathbb{P}^1, \mathcal{L}^{\otimes d}) = H^0(\mathbb{P}^1, \mathcal{O}(nd))$ is also of dimension $nd + 1$ so, by Lemma 11.7, $B(X, \mathcal{L})$ coincides with the homgeneous coordinate ring.

EXERCISES

11.1 Let $i : X \to \mathbb{P}(V^*)$ be the inclusion of a closed subvariety and define $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}}(1)$. In general, the natural map $\varphi : V \to \mathrm{H}^0(X, \mathcal{L})$ is neither injective nor surjective.

(a) If X does not span $\mathbb{P}$ show that φ is not injective.

(b) Let $X \subset \mathbb{P}^3$ be the image of the morphism $f : \mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$f(x, y) = (x^4, x^3y, xy^3, y^4).$$

The homogeneous coordinate ring of X is $A = k[x^4, x^3y, xy^3, y^4]$, the (codimension 1) subalgebra of the Veronese $k[x, y]^{(4)}$ of the polynomial ring. Show directly, by computing the local sections on the 4 open affines $X_i \neq 0$ covering $\mathbb{P}^3$, that $\mathrm{H}^0(X, \mathcal{L})$ has basis $x^4, x^3y, x^2y^2, xy^3, y^4$.

12. Line bundles on curves

In this section X is a fixed smooth irreducible curve over an algebraically closed field k.

For a divisor D on X, we define

$$\mathrm{h}^i(D) = \dim_k \mathrm{H}^i(X, \mathcal{O}_X(D)).$$

There is a distinguished divisor, the canonical divisor on X, which we denote by K. The genus of X is by definition,

$$g(X) := \mathrm{h}^0(K).$$

There is a natural isomorphism $\mathrm{H}^0(X, \mathcal{O}(D)) \cong \mathrm{H}^1(X, \mathcal{O}(K - D))$.

Lemma 12.1. *If* $\deg(D) < 0$*, then* $\mathrm{h}^0(D) = 0$.

Theorem 12.2 (Riemann-Roch). *Let D be a divisor on a smooth irreducible curve X over an algebraically closed field k. Then*

$$\mathrm{h}^0(D) - \mathrm{h}^1(D) = 1 + \deg(D) - g(X).$$

$\boxed{\text{Paul}}$ Is there a non-commutative version of Riemann-Roch? Maybe one can get a proof along the following lines. First, we define genus as $\mathrm{h}^1(\mathcal{O}_X)$. Second, the function $\chi(\mathcal{F}) := \mathrm{h}^0(\mathcal{F}) - \mathrm{h}^1(\mathcal{F})$ is additive on short exact sequences, so can be considered as a function $\chi : K_0(X) \to \mathbb{Z}$. Third, for any divisor D and any point p, there is a short exact sequence $0 \to \mathcal{O}(D) \to \mathcal{O}(D+p) \to \mathcal{F}_p \to 0$, and $\mathrm{h}^0(\mathcal{F}_p) = 1$ and $\mathrm{h}^1(\mathcal{F}_p) = 0$, so $\chi(D+p) = \chi(D)+1$. Similarly, if $p \in \operatorname{Supp} D$, then $\chi(D-p) = \chi(D) - 1$. Hence if $D = \sum n_j p_j$, then $\chi(D) = \chi(0) + \sum n_j = 1 - g + \deg(D)$. Fourth, Serre duality, $h^1(D) = h^0(K - D)$, so

$$h^0(D) - h^0(K - D) = \chi(D) = 1 - g + \deg(D).$$

Corollary 12.3. *Let D be a divisor on a smooth, irreducible curve of genus g. If* $\deg(D) \geq 2g - 1$*, then* $\mathrm{h}^0(D) = \deg(D) + 1 - g$.

13. Elliptic curves

A smooth, irreducible curve of genus 1 is called an elliptic curve.

Proposition 13.1. *Let D be a divisor on an elliptic curve. Then*

$$\mathrm{h}^0(D) = \begin{cases} 0 & \text{if } \deg(D) \leq 0 \text{ and } D \neq 0, \\ 1 & \text{if } D = 0, \\ \deg(D) & \text{if } \deg(D) \geq 0 \text{ and } D \neq 0 \end{cases} \tag{13-1}$$

and

$$\mathrm{h}^1(D) = \begin{cases} -\deg(D) & \text{if } \deg(D) \leq 0 \text{ and } D \neq 0, \\ 1 & \text{if } D = 0, \\ 0 & \text{if } \deg(D) \geq 0 \text{ and } D \neq 0. \end{cases} \tag{13-2}$$

Proof. First, Riemann-Roch gives $\mathrm{h}^0(D) - \mathrm{h}^1(D) = \deg(D)$. For $D = 0$, the $\mathrm{h}^1(D)$ case is simply the definition of genus. □

THEOREM 13.2 (Theorem of the square). *Let $\mathcal{L}$ be a line bundle on an elliptic curve E, and let $x, y \in E$. Let σ_x denote the translation automorphism $p \mapsto p + x$. Then*

$$\sigma_{x+y}^* \mathcal{L} \otimes \mathcal{L} \cong \sigma_x^* \mathcal{L} \otimes \sigma_y^* \mathcal{L}.$$

An important special case (relevant to the Sklyanin algebra) occurs when $x = y$: if $\sigma : E \to E$ is a translation, then

$$\mathcal{L}^{\sigma^2} \otimes \mathcal{L} \cong \mathcal{L}^{\sigma} \otimes \mathcal{L}^{\sigma}.$$

One can re-phrase this as follows: if G denotes the group generated by σ, then the abelian group $\mathrm{Pic}(E)$ is a $\mathbb{Z}G$-module which is annihilated by $(\sigma - 1)^2$.

The group law on an elliptic curve

Automorphisms of elliptic curves

If $\tau \in E$, then the map $p \to p+\tau$ is an automorphism of E called the translation by τ.

The map $p \mapsto -p$ is an automorphism of E called the reflection.

Complex multiplications.

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