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Limits of Gaudin Algebras, Quantization of Bending Flows, Jucys–Murphy Elements and Gelfand–Tsetlin Bases

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Abstract. Gaudin algebras form a family of maximal commutative subalgebras in the tensor product of n copies of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$. This family is parameterized by collections of pairwise distinct complex numbers $z_1, \ldots, z_n$. We obtain some new commutative subalgebras in $U(\mathfrak{g})^{\otimes n}$ as limit cases of Gaudin subalgebras. These commutative subalgebras turn to be related to the Hamiltonians of bending flows and to the Gelfand–Tsetlin bases. We use this to prove the simplicity of spectrum in the Gaudin model for some new cases.

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1. Introduction

The Gaudin model was introduced in [26] as a spin model related to the Lie algebra sl_2, and generalized to the case of arbitrary semisimple Lie algebras in [27, 13.2.2]. The generalized Gaudin model has the following algebraic interpretation. Let V_λ be an irreducible representation of a semisimple (reductive) Lie algebra $\mathfrak{g}$ with the highest weight λ. For any collection of integral dominant weights $(\lambda) = \lambda_1, \ldots, \lambda_n$, let $V_{(\lambda)} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. For any $x \in \mathfrak{g}$, consider the operator $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ (x stands at the ith place), acting on the space $V_{(\lambda)}$. Let $\{x_a\}$, $a = 1, \ldots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to Killing form, and let $z_1, \ldots, z_n$ be pairwise distinct complex numbers. The Hamiltonians of the

Gaudin model are the following commuting operators acting in the space $V_{(\lambda)}$:

$$H_i = \sum_{k\neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{1}$$

We can treat the H_i as elements of the universal enveloping algebra $U(\mathfrak{g})^{\otimes n}$. In [22], the existence of a large commutative subalgebra $\mathcal{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ containing H_i was proved. For $\mathfrak{g} = sl_2$, the algebra $\mathcal{A}(z_1, \dots, z_n)$ is generated by H_i and the central elements of $U(\mathfrak{g})^{\otimes n}$. In other cases, the algebra $\mathcal{A}(z_1, \dots, z_n)$ has also some new generators known as higher Gaudin Hamiltonians. Their explicit construction for $\mathfrak{g} = gl_r$ was obtained in [46], see also [10,12] (for $\mathfrak{g} = gl_3$ see [13]). The construction of $\mathcal{A}(z_1, \dots, z_n)$ uses the quite nontrivial fact [21] that the completed universal enveloping algebra of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level has a large center $Z(\hat{\mathfrak{g}})$. There is a natural homomorphism from the center $Z(\hat{\mathfrak{g}})$ to the enveloping algebra $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$. To any collection $z_1, \dots, z_n$ of pairwise distinct nonzero complex numbers, one can naturally assign the evaluation homomorphism $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(\mathfrak{g})^{\otimes n}$. The image of the center under the composition of the above homomorphisms is $\mathcal{A}(z_1, \dots, z_n)$. It turns out that the subalgebra $\mathcal{A}(z_1, \dots, z_n)$ does not change under simultaneous affine transformations of the parameters $z_i \mapsto az_i + b$ and hence we can assume that $z_1, \dots, z_n$ is an arbitrary collection of pairwise distinct complex numbers (not necessarily nonzero). We will call $\mathcal{A}(z_1, \dots, z_n)$ the Gaudin algebra.

The main problem in Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin Hamiltonians. The bibliography on this problem is enormous (cf. [4,22–25,33,35,38]). It follows from the [22] construction that all elements of $\mathcal{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ are invariant with respect to the diagonal action of $\mathfrak{g}$, and therefore it is sufficient to diagonalize the algebra $\mathcal{A}(z_1, \dots, z_n)$ in the subspace $V_{(\lambda)}^{\text{sing}} \subset V_{(\lambda)}$ of singular vectors with respect to $\text{diag}_n(\mathfrak{g})$ (i.e., with respect to the diagonal action of $\mathfrak{g}$). The standard conjecture says that, for generic z_i, the algebra $\mathcal{A}(z_1, \dots, z_n)$ has simple spectrum in $V_{(\lambda)}^{\text{sing}}$. This conjecture is proved in [38] for $\mathfrak{g} = sl_r$ and λ_i equal to ω_1 or ω_{r-1} (i.e., for the case when every V_{λ_i} is the standard representation of sl_r or its dual) and in [45] for $\mathfrak{g} = sl_2$ and arbitrary λ_i.

In the present paper, we consider some limits of the Gaudin algebras when some of the points $z_1, \dots, z_n$ glue together. We obtain some new commutative subalgebras this way. We consider, in particular, the "most degenerate" subalgebra of this type. In the case of $\mathfrak{g} = gl_N$, this subalgebra gives a quantization of the (higher) "bending flows Hamiltonians", introduced in [16]. Original bending flows were introduced in [28], and are related with $SU(2)$. Subsequent developments [2,15,17–19] lead in particular to the construction of a set of (classical) integrals of motion, not equivalent to the "standard" set associated with the ring of spectral invariants of the classical Lax matrix $L = \sum_{i=1}^n \frac{L^{(i)}}{z - z_i}$. One of the results of this paper is that this alternative set of Hamiltonians can be quantized. It should also be noticed that, more recently, applications of systems related with "partial glueings" of the poles of the Gaudin Lax matrix are being considered in the literature (see, e.g., [30–32]).

Further, we establish a connection of this subalgebra to the Gelfand–Tsetlin bases via the results of Mukhin, Tarasov and Varchenko on (gl_N, gl_M) duality (cf. [33,35]). This result was obtained first in [15] by different methods. We use this to prove the simple spectrum conjecture for $\mathfrak{g}=gl_N$ and $\lambda_i=m_i\omega_1,\ m_i\in\mathbb{Z}_+$.

The paper is organized as follows. In Section 2 we collect some well-known facts on Gaudin algebras. In Section 3, we describe some limits of Gaudin algebras. In Section 4, we obtain quantum Hamiltonians of bending flows as limits of higher Gaudin Hamiltonians. In Sections 5 and 6, we establish a connection between quantum bending flows Hamiltonians and Gelfand–Tsetlin theory. Finally, in Section 7, we apply our results and prove the simple spectrum conjecture.

2. Preliminaries

2.1. CONSTRUCTION OF GAUDIN SUBALGEBRAS

Let $\mathfrak{g}$ be a semisimple (reductive) Lie algebra. We fix an invariant scalar product on $\mathfrak{g}$ and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via this scalar product. Consider the infinite-dimensional pro-nilpotent Lie algebra $\mathfrak{g}_- := \mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$ – it is a "half" of the corresponding affine Kac–Moody algebra $\hat{\mathfrak{g}}$. The universal enveloping algebra $U(\mathfrak{g}_-)$ bears a natural filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(\mathfrak{g}_-)$ by the Poincaré–Birkhoff–Witt theorem. The commutator operation on $U(\mathfrak{g}_-)$ defines the Poisson–Lie bracket $\{\cdot,\cdot\}$ on $S(\mathfrak{g}_-)$: for the generators $x, y\in\mathfrak{g}_-$ we have $\{x,y\}=[x,y]$. For any $g\in\mathfrak{g}$, we denote the element $g\otimes t^m\in\mathfrak{g}_-$ by $g[m]$.

The Poisson algebra $S(\mathfrak{g}_-)$ contains a large Poisson-commutative subalgebra $A\subset S(\mathfrak{g}_-)$. This subalgebra can be constructed as follows.

Consider the following derivations of the Lie algebra $\mathfrak{g}_-$:

$$\partial_t(g[m])=mg\otimes t^{m-1}\quad \forall g\in\mathfrak{g},\ \ m=-1,-2,\dots \tag{2}$$
$$t\partial_t(g[m])=mg\otimes t^{m}\quad \forall g\in\mathfrak{g},\ \ m=-1,-2,\dots \tag{3}$$

The derivations (2), (3) extend to the derivations of the associative algebras $S(\mathfrak{g}_-)$ and $U(\mathfrak{g}_-)$. The derivation (3) induce a grading of these algebras.

Let $i_{-1}: S(\mathfrak{g})\hookrightarrow S(\mathfrak{g}_-)$ be the embedding, which maps $g\in\mathfrak{g}$ to $g[-1]$. The algebra of invariants, $S(\mathfrak{g})^{\mathfrak{g}}$, is known to be a free commutative algebra with $\operatorname{rk}\mathfrak{g}$ generators. Let $\Phi_l,\ l=1,\dots,\operatorname{rk}\mathfrak{g}$ be some set of free generators of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$.

Fact 1 (1) [3,22,25] *The subalgebra $A\subset S(\mathfrak{g}_-)$ generated by the elements $\partial_t^n\overline{S_l}$, $l=1,\dots,\operatorname{rk}\mathfrak{g}$, $n=0,1,2,\dots$, where $\overline{S_l}=i_{-1}(\Phi_l)$, is Poisson-commutative.*

(2) *There exist commuting elements $S_l\in U(\mathfrak{g}_-)$, homogeneous with respect to $t\partial_t$, such that $\operatorname{gr} S_l=\overline{S_l}$.*

(3) *The subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ generated by $\partial_t^n S_l$, $k=1,\dots,\operatorname{rk}\mathfrak{g}$, $n=0,1,2,\dots$, is a free commutative algebra (i.e., all the $\partial_t^n S_l$ are algebraically independent and pairwise commute).*

Remark. The generators of the subalgebra $A \subset S(\mathfrak{g}_-)$ can be described in the following equivalent way. Let $i(z): S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$ be the embedding depending on the formal parameter z, which maps $g \in \mathfrak{g}$ to $\sum_{k=1}^{\infty} z^{k-1} g[-k]$. Then the coefficients of the power series $\overline{S_l}(z) = i(z)(\Phi_l)$ in z freely generate the subalgebra $A \subset S(\mathfrak{g}_-)$.

Remark. The subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ comes from the center of $U(\hat{\mathfrak{g}})$ at the critical level by the AKS-scheme (see [12,14,22] for details).

The Gaudin subalgebra $\mathcal{A}(z_1,\dots,z_n) \subset U(\mathfrak{g})^{\otimes n}$ is the image of the subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ under the homomorphism $U(\mathfrak{g}_-) \to U(\mathfrak{g})^{\otimes n}$ of specialization at the points $z_1,\dots,z_n$ (see [14,22]). Namely, let $U(\mathfrak{g})^{\otimes n}$ be the tensor product of n copies of $U(\mathfrak{g})$. We denote the subspace $1 \otimes \cdots \otimes 1 \otimes \mathfrak{g} \otimes 1 \otimes \cdots \otimes 1 \subset U(\mathfrak{g})^{\otimes n}$, where $\mathfrak{g}$ stands at the ith place, by $\mathfrak{g}^{(i)}$. Respectively, for any $f \in U(\mathfrak{g})$ we set

$$f^{(i)} = 1 \otimes \cdots \otimes 1 \otimes f \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes n}. \tag{4}$$

Let $\operatorname{diag}_n : U(\mathfrak{g}_-) \hookrightarrow U(\mathfrak{g}_-)^{\otimes n}$ be the diagonal embedding [i.e., for $f \in \mathfrak{g}_-$, we have $\operatorname{diag}_n(f) = \sum_{i=1}^n f^{(i)}$]. To any nonzero $z \in \mathbb{C}$, we assign the homomorphism $\varphi_z : U(\mathfrak{g}_-) \to U(\mathfrak{g})$ of evaluation at the point z [i.e., for $g \in \mathfrak{g}$, we have $\varphi_z(g \otimes t^m) = z^m g$]. For any collection of pairwise distinct nonzero complex numbers z_i, $i = 1,\dots,n$, we have the following homomorphism:

$$\varphi_{z_1,\dots,z_n} = (\varphi_{z_1} \otimes \cdots \otimes \varphi_{z_n}) \circ \operatorname{diag}_n : U(\mathfrak{g}_-) \to U(\mathfrak{g})^{\otimes n}. \tag{5}$$

More explicitly, we have

$$\varphi_{z_1,\dots,z_n}(g \otimes t^m) = \sum_{i=1}^n z_i^m g^{(i)}.$$

Set

$$\mathcal{A}(z_1,\dots,z_n) = \varphi_{z_1,\dots,z_n}(\mathcal{A}) \subset U(\mathfrak{g})^{\otimes n}$$

Furthermore, there is a homomorphism

$$\varphi_\infty : U(\hat{\mathfrak{g}}_-) \to S(\mathfrak{g}), \quad g \otimes t^{-1} \mapsto g, \;\; g \otimes t^m \mapsto 0, \;\; m = -2,-3,\dots \tag{6}$$

We set

$$\mathcal{A}(z_1,\dots,z_n,\infty) = (\varphi_{z_1,\dots,z_n} \otimes \varphi_\infty) \circ \operatorname{diag}_{n+1}(\mathcal{A}) \subset U(\mathfrak{g})^{\otimes n} \otimes S(\mathfrak{g})$$

2.2. QUANTUM MISHCHENKO–FOMENKO "SHIFT OF ARGUMENT" SUBALGEBRAS

Consider the subalgebra $\mathcal{A}(z_1, z_2) \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The associated graded algebra of $\mathcal{A}(z_1, z_2)$ with respect to the second tensor factor is a commutative subalgebra $\overline{\mathcal{A}(z_1, z_2)} \subset U(\mathfrak{g}) \otimes S(\mathfrak{g})$. Any element $\mu \in \mathfrak{g}^* = \operatorname{Spec} S(\mathfrak{g})$ gives the evaluation homomorphism $\mathrm{id} \otimes \mu : U(\mathfrak{g}) \otimes S(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes(n-1)}$. The image of $\overline{\mathcal{A}(z_1, z_2)}$ under this homomorphism is a commutative subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$, which does not depend on z_1, z_2 [in particular, this subalgebra can be obtained as $(\mathrm{id} \otimes \mu)(\mathcal{A}(z_1, \infty))$]. This subalgebra is a quantum version of the Mishchenko–Fomenko "shift of argument" subalgebra $A_\mu \subset S(\mathfrak{g})$ (see [12,23,36,41]). The latter is generated by the derivatives (of any order) along μ of the generators of the Poisson center $S(\mathfrak{g})^{\mathfrak{g}}$, (or, equivalently, generated by central elements of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ shifted by $t\mu$ for all $t \in \mathbb{C}$) [29]. In [23,41] it is shown, that $\operatorname{gr} \mathcal{A}_\mu = A_\mu$ for μ regular. The algebra A_μ is a free commutative algebra with $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ generators, and hence has the maximal possible transcendence degree (see [29] for the case of regular semisimple μ, and [7,40] for the general case).

2.3. THE GENERATORS OF $\mathcal{A}(z_1, \dots, z_n)$

For our purposes, we need some specific set of generators of $\mathcal{A}(z_1, \dots, z_n)$. Let us describe it.

Consider the following $U(\mathfrak{g})^{\otimes n}$-valued functions in the variable w

$$S_l(w; z_1, \dots, z_n) := \varphi_{w-z_1, \dots, w-z_n}(S_l).$$

Let $S_l^{i,m}(z_1, \dots, z_n)$ be the coefficients of the principal part of the Laurent series of $S_l(w; z_1, \dots, z_n)$ at the point z_i, i.e.,

$$S_l(w; z_1, \dots, z_n) = \sum_{m=1}^{m=\deg \Phi_l} S_l^{i,m}(z_1, \dots, z_n)(w - z_i)^{-m} + O(1) \quad \text{as } w \to z_i.$$

The following assertion is standard.

PROPOSITION 1. (1) *The elements $S_l^{i,m}(z_1, \dots, z_n) \in U(\mathfrak{g})^{\otimes n}$ are homogeneous under simultaneous affine transformations of the parameters $z_i \mapsto a z_i + b$ (i.e., $S_l^{i,m}(a z_1 + b, \dots, a z_n + b)$ is proportional to $S_l^{i,m}(z_1, \dots, z_n)$).*

(2) *The subalgebra $\mathcal{A}(z_1, \dots, z_n)$ is a free commutative algebra generated by the elements $S_l^{i,m}(z_1, \dots, z_n) \in U(\mathfrak{g})^{\otimes n}$, where $i = 1, \dots, n-1$, $l = 1, \dots, \operatorname{rk} \mathfrak{g}$, $m = 1, \dots, \deg \Phi_l$, and $S_l^{n, \deg \Phi_l}(z_1, \dots, z_n) = \Phi_l^{(n)} \in U(\mathfrak{g})^{\otimes n}$, where $l = 1, \dots, \operatorname{rk} \mathfrak{g}$.*

(3) *All the elements of $\mathcal{A}(z_1, \dots, z_n)$ are invariant with respect to the diagonal action of $\mathfrak{g}$.*

(4) *The center of the diagonal $\operatorname{diag}_n(U(\mathfrak{g})) \subset U(\mathfrak{g})^{\otimes n}$ is contained in $\mathcal{A}(z_1, \dots, z_n)$.*

Remark. Note that the assertions (1–2) provide a way to define $\mathcal{A}(z_1, \ldots, z_n)$ for any collection of pairwise distinct complex numbers $z_1, \ldots, z_n$, not necessarily non-zero.

Proof. (1) The Laurent coefficients of the functions $S_l(w; z_1+b, \ldots, z_n+b)$ at z_i+b are equal to those of $S_l(w-b; z_1, \ldots, z_n)$ at z_i, hence, our generators are stable under simultaneous transformations of the parameters $z_i \mapsto z_i + b$.

Now consider simultaneous transformations $z_i \mapsto az_i$. The Laurent coefficients of the functions $S_l(w; az_1, \ldots, az_n)$ at az_i are proportional to those of $S_l(aw; az_1, \ldots, az_n)$ at z_i. Since the elements $S_l \in U(\mathfrak{g}_-)$ are homogeneous with respect to $t\partial_t$, we see that $\exp((\log a)t\partial_t)S_l$ is proportional to S_l. Note that $S_l(aw; az_1, \ldots, az_n) = \varphi_{w-z_1,\ldots,w-z_n}(\exp((\log a)t\partial_t)S_l)$. This means that the Laurent coefficients of the functions $S_l(aw; az_1, \ldots, az_n)$ at z_i are proportional to those of $S_l(w; z_1, \ldots, z_n)$ at z_i.

(2) The algebra $\mathcal{A}(z_1, \ldots, z_n)$ is generated by the elements $\varphi_{z_1,\ldots,z_n}(\partial_t^m S_l)$. The latter are (up to proportionality) Taylor coefficients of $S_l(w; z_1, \ldots, z_n) = \varphi_{w-z_1,\ldots,w-z_n}(S_l)$ about $w=0$. This follows from the identity $\varphi_{w-z_1,\ldots,w-z_n}(\partial_t S) = \partial_w(\varphi_{w-z_1,\ldots,w-z_n}(S))$ for any $S \in U(\mathfrak{g}_-)$: indeed, the mth Taylor coefficient of $S_l(w; z_1, \ldots, z_n)$ is $\frac{1}{m!}\partial_w^m(\varphi_{w-z_1,\ldots,w-z_n}(S))|_{w=0} = \frac{1}{m!}\varphi_{-z_1,\ldots,-z_n}(\partial_t^m S_l) = \frac{1}{m!}(-1)^{\deg S_l + m}$ $\varphi_{z_1,\ldots,z_n}(\partial_t^m S_l)$ since S_l is homogeneous with respect to $t\partial_t$. The function $S_l(w; z_1, \ldots, z_n)$ is meromorphic in w having a zero of order $\deg \Phi_l$ at ∞. Since the poles of $S_l(w)$ are exactly $z_1, \ldots, z_n$, the Taylor coefficients of $S_l(w)$ about $w=0$ are linear expressions in the coefficients of the principal part of the Laurent series for the same function about $z_1, \ldots, z_n$. Since the function $S_l(w)$ has zero of order $\deg \Phi_l$ at ∞, for each $m = 1, \ldots, \deg \Phi_l - 1$ the Laurent coefficient $S_l^{n,m}(z_1, \ldots, z_n) = \mathrm{Res}_{w=z_n}(w-z_n)^{m-1} S_l(w; z_1, \ldots, z_n)$ is a linear combination of the Laurent coefficients at $z_1, \ldots, z_{n-1}$.

Now, it remains to check that the generators $S_l^{i,m}$ are algebraically independent. Equivalently, we need to prove that the transcendence degree of $\mathcal{A}(z_1, \ldots, z_n)$ is $\frac{n-1}{2}(\dim \mathfrak{g} + \mathrm{rk}\,\mathfrak{g}) + \mathrm{rk}\,\mathfrak{g}$. We shall deduce this from the maximality of quantum Mishchenko–Fomenko subalgebras in a sequence of steps as follows.

Let $U(\mathfrak{g})_s$ be the algebra having $\mathfrak{g}$ as the space of generators, with defining relations $xy - yx = s[x, y]$. For any $s \neq 0$, the map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto s^{-1}x$ induces the associative algebra isomorphism

$$\psi_s : U(\mathfrak{g}) \tilde{\to} U(\mathfrak{g})_s. \tag{7}$$

We also fix an isomorphism of vector spaces $S(\mathfrak{g}) \to U(\mathfrak{g})_s$ via the symmetrization map, which sends $x^n \in S(\mathfrak{g})$ to $x^n \in U(\mathfrak{g})_s$ for all $x \in \mathfrak{g}$, $n \in \mathbb{Z}_{\geq 0}$.

For $s = 0$, we have $U(\mathfrak{g})_0 = S(\mathfrak{g})$. Let us denote, for $s \neq 0$ by $\mathcal{A}_s(z_1, \ldots, z_{n-1}, z_n)$ the algebra

$$(\mathrm{id}^{\otimes(n-1)} \otimes \psi_s)(\mathcal{A}(s^{-1}z_1, \ldots, s^{-1}z_{n-1}, s^{-1}z_n)).$$

By item (2) of this Proposition and by the definition of ψ_s, we have

$$\mathcal{A}_s(z_1,\dots,z_n)=(\mathrm{id}^{\otimes(n-1)}\otimes\psi_s)(\mathcal{A}(z_1,\dots,z_{n-1},z_n))\simeq\mathcal{A}(z_1,\dots,z_n). \tag{8}$$

Let us further denote by $\mathcal{A}(z,\infty)^{(i)}\subset U(\mathfrak{g})^{\otimes(n-1)}\otimes S(\mathfrak{g})$ the image of the subalgebra $\mathcal{A}(z,\infty)\subset U(\mathfrak{g})\otimes S(\mathfrak{g})$ under the homomorphism sending $a\otimes b\in U(\mathfrak{g})\otimes S(\mathfrak{g})$ to $a^{(i)}\otimes b\in U(\mathfrak{g})^{\otimes(n-1)}\otimes S(\mathfrak{g})$. Let us show (essentially reproducing the proof of Theorem 2 of [41]) that the limit subalgebra (this limit is well-defined due to the vector space isomorphism $S(\mathfrak{g})\to U(\mathfrak{g})_s$)

$$\mathcal{A}_0(z_1,\dots,z_n)\equiv\lim_{s\to 0}(\mathrm{id}^{\otimes(n-1)}\otimes\psi_s)(\mathcal{A}(s^{-1}z_1,\dots,s^{-1}z_{n-1},s^{-1}z_n))$$

contains the subalgebra $\mathcal{A}(0,\infty)^{(1)}\cdots\mathcal{A}(0,\infty)^{(n-1)}\subset U(\mathfrak{g})^{\otimes(n-1)}\otimes S(\mathfrak{g})$.

The coefficients of the Laurent expansion of $S_k(w;s^{-1}z_1,\dots,s^{-1}z_n)$ at any point $s^{-1}z_i$ are equal to the Laurent coefficients of $S_k(w+s^{-1}z_i;s^{-1}z_1,\dots,s^{-1}z_n)$ at the point 0. By Lemmas 2 and 3[1] of [41], for $z\neq 0$ we have $\lim_{s\to 0}\varphi_{s^{-1}z}=\varepsilon$, where ε: $U(\hat{\mathfrak{g}}_-)\to\mathbb{C}\cdot 1\subset U(\mathfrak{g})$ is the co-unit, and $\lim_{s\to 0}\psi_s\circ\varphi_{s^{-1}z}=\varphi_\infty$. Due to the assertion (1) of this Proposition, we can assume that all z_i are nonzero. Hence, we have for $i=1,\dots,n-1$:

$$\begin{aligned}&\lim_{s\to 0}S_k(w+s^{-1}z_i;s^{-1}z_1,\dots,s^{-1}z_n)=\\ &\quad=\lim_{s\to 0}(\mathrm{id}^{\otimes(n-1)}\otimes\psi_s)\circ\varphi_{w-s^{-1}(z_1-z_i),\dots,w,\dots,w-s^{-1}(z_n-z_i)}(S_k)=\\ &\quad=(\varepsilon\otimes\cdots\otimes\varepsilon\otimes\varphi_w\otimes\varepsilon\otimes\cdots\otimes\varepsilon\otimes\varphi_\infty)\circ\mathrm{diag}_n(S_k)=S_k(w;0,\infty)^{(i)}.\end{aligned}$$

This means that the generators of $(\mathrm{id}^{\otimes(n-1)}\otimes\psi_s)(\mathcal{A}(s^{-1}z_1,\dots,s^{-1}z_{n-1},s^{-1}z_n))$ give the generators of $\mathcal{A}(0,\infty)^{(1)}\cdots\mathcal{A}(0,\infty)^{(n-1)}$ as the limit. Hence, we conclude

$$\lim_{s\to 0}(\mathrm{id}^{\otimes(n-1)}\otimes\psi_s)(\mathcal{A}(s^{-1}z_1,\dots,s^{-1}z_{n-1},s^{-1}z_n))\supset\mathcal{A}(0,\infty)^{(1)}\cdots\mathcal{A}(0,\infty)^{(n-1)}.$$

Any element $\mu\in\mathfrak{g}^*=\mathrm{Spec}\,S(\mathfrak{g})$ gives the evaluation homomorphism $\mathrm{id}^{\otimes(n-1)}\otimes\mu$: $\underline{U(\mathfrak{g})^{\otimes(n-1)}\otimes S(\mathfrak{g})}$. This homomorphism sends the central generators $\overline{S_l^{n,\deg\Phi_l}(z_1,\dots,z_n)}=\Phi_l^{(n)}\in\overline{\mathcal{A}(z_1,\dots,z_n)}\subset U(\mathfrak{g})^{\otimes(n-1)}\otimes S(\mathfrak{g})$ to constants, where $\overline{\mathcal{A}(z_1,\dots,z_n)}$ is the associated graded quotient of $\mathcal{A}(z_1,\dots,z_n)$ with respect to the PBW filtration on the nth tensor factor. The algebra $(\mathrm{id}^{\otimes(n-1)}\otimes\mu)(\mathcal{A}_0(z_1,\dots,z_n))$ contains the tensor product of quantum Mishchenko–Fomenko subalgebras, $\mathcal{A}_\mu^{(1)}\otimes\cdots\otimes\mathcal{A}_\mu^{(n-1)}$, which is known to be of the transcendence degree $\frac{n-1}{2}(\dim\mathfrak{g}+\mathrm{rk}\,\mathfrak{g})$ (see [29]) for μ regular). Thus, the limit algebra $\mathcal{A}_0(z_1,\dots,z_n)$ has the transcendence degree $\frac{n-1}{2}(\dim\mathfrak{g}+\mathrm{rk}\,\mathfrak{g})$ over $\mathbb{C}[\Phi_1^{(n)},\dots,\Phi_{\mathrm{rk}\,\mathfrak{g}}^{(n)}]=S(\mathfrak{g})^{\mathfrak{g}}$. The same is true for

[1]The proof of these two Lemmas can be directly achieved verifying their validity on the generators. For instance, for Lemma 2, one has

$$\lim_{z\to\infty}\varphi_z(g[m])=\lim_{z\to\infty}z^m g=0\quad\forall g\in\mathfrak{g},\ \ m=-1,-2,\dots$$

which implies the statement. Lemma 3 of [41] can be proven similarly.

generic values of s, that is for $\mathcal{A}_s(z_1, \dots, z_{n-1}, z_n)$ and hence due to Equation (8) the assertion is proven.

(3) The elements $S_l \in U(\mathfrak{g}_-)$ are $\mathfrak{g}$-invariant, and the homomorphisms $\varphi_{w-z_1,\dots,w-z_n}$ are $\mathfrak{g}$-equivariant for any w, hence, the images of S_l under the homomorphisms $\varphi_{w-z_1,\dots,w-z_n}$ are $\mathfrak{g}$-invariant as well.

(4) The $\deg \Phi_l$th Taylor coefficient of $S_l(w; z_1, \dots, z_n)$ at ∞ is the lth generator of the center of $\mathrm{diag}_n(U(\mathfrak{g})) \subset U(\mathfrak{g})^{\otimes n}$. $\square$

2.4. TALALAEV'S FORMULA

In [46] D. Talalaev constructed explicitly some elements of $U(\mathfrak{g})^{\otimes n}$ commuting with the quadratic Gaudin Hamiltonians for the case $\mathfrak{g} = gl_N$. Actually, the formulas of [46] are universal, i.e., they describe a commutative subalgebra of $U(\mathfrak{g}_-)$ which gives a commutative subalgebra of $U(\mathfrak{g})^{\otimes n}$ as the image of the specialization homomorphism at the points $z_1, \dots, z_n$ (see [12]). However, here, we will restrict ourselves to the case $\mathfrak{g} = gl_N$, since it is enough for our purposes (and, namely, for the explicit constructions to be done in Section 4).

Set

$$L(z) = \sum_{1 \le i,j \le N} \sum_{n=1}^{\infty} z^{n-1} e_{ij}[-n] \otimes e_{ji} \in U(\mathfrak{g}_-) \otimes \mathrm{End}\, \mathbb{C}^N, \tag{9}$$

where z is a formal parameter, and consider the following differential operator in z with the coefficients from $U(\mathfrak{g}_-)$:

$$D = (1 \otimes \mathrm{Tr}) A_N \prod_{i=1}^{N} (L(z)^{(i)} - \partial_z) = (-1)^N \partial_z^N + \sum_{k=1}^{N} (-1)^{N-k} \sum_{n=1}^{\infty} QH_{n,k} z^{n-1} \partial_z^{N-k}. \tag{10}$$

Here, we denote by $L(z)^{(i)} \in U(\mathfrak{g}_-) \otimes (\mathrm{End}\, \mathbb{C}^N)^{\otimes N}$ the element obtained by putting $L(z)$ in the ith tensor factor, and A_N denotes the projector onto $U(\mathfrak{g}_-) \otimes \mathrm{End}(\Lambda^N \mathbb{C}^N) \subset U(\mathfrak{g}_-) \otimes (\mathrm{End}\, \mathbb{C}^N)^{\otimes N}$. The operators $QH_{n,k}$ defined by the LHS of Equation (10) are "good" explicit quantum Hamiltonians for the gl_N Gaudin systems, in view of the following

PROPOSITION 2. *The elements $QH_{n,k} \in U(\mathfrak{g}_-)$ commute, and generate the commutative subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ defined in Section 2, Fact 1.*

Proof. In [46] a distinguished commutative subalgebra in $U(gl_N)^{\otimes K}$ was constructed. (We use the formulation from [10], formula 4, p. 3.) To prove our first assertion we show, following ideas contained in [43], why the same construction works for the case $U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}])$. We shall proceed in a series of steps.

- We consider the evaluation homomorphisms $ev_{z_m} : U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(gl_N)$ defined on generators as follows: $e_{ij}[-n] \to e_{ij}(z_m^{-n})$, where z_m is an arbitrary nonzero complex number. Using the comultiplication $\Delta^K: U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}])^{\otimes K}$ followed by the tensor product $ev_{z_1} \otimes \cdots \otimes ev_{z_K}$ one obtains the homomorphisms: $ev_{z_1,\dots,z_k}: U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(gl_N)^{\otimes K}$.

- ("Universality"). The Lax matrix $L(z)$ is mapped by $ev_{z_1,\dots,z_K}$ to $-(L_{\text{Gaudin}}(z))^t$, where $L_{\text{Gaudin}}(z)$ is precisely the Lax matrix used in [10,46]. It is clear since it is just a summation of geometric series: $\sum_{n=1,\dots,\infty} z^{n-1}/z_m^n = (1/z_m)(1-z/z_m)^{-1} = -1/(z-z_m)$.
  The minus sign and the transposition here are essential: $L(z) \mapsto -L_{\text{Gaudin}}(z)^t$. Also note that $\Psi^{\otimes K}(L_{\text{Gaudin}}(z)) = -L_{\text{Gaudin}}(z)^t$, where $\Psi: gl_N \to gl_N$ is an automorphism of gl_N: $e_{ij} \mapsto -e_{ji}$. Since Ψ is an automorphism one can use in Talalaev's formula $L_{\text{Gaudin}}(z)$ or $-L_{\text{Gaudin}}(z)^t$ on equal footing.
- ("Faithfulness"). These homomorphisms are asymptotically faithful, i.e., $\forall g \neq 0 \in U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}])$, $\exists K, z_1, \dots, z_K: ev_{z_1,\dots,z_K}(g) \neq 0$.
  To ascertain this, we can make use of the following arguments. For $K \in \mathbb{Z}$ consider the homomorphisms $\text{cut}_K: U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}]) \to U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}]/t^{-K})$; they are clearly asymptotically faithful. On the other hand the homomorphisms $ev_{z_1,\dots,z_K}$ and cut_K are factorizations of $U(gl_N \otimes t^{-1}\mathbb{C}[t^{-1}])$ by the ideals generated by $g \otimes (\prod_i (t^{-1} - (z_i)^{-1}))$ and $g \otimes t^{-K}$, $g \in gl_N$, respectively. So one obtains the second one by limit $z_i \to \infty$ of the first ones. Indeed, if we assume that $\forall z_i: ev_{z_1,\dots,z_K}(Q) = 0$, taking the limit $z_i \to \infty$ one obtains $\text{cut}_K(Q) = 0$. This contradicts the faithfulness of cut_K.
- Now we are ready to finish the proof.
  Assume that $[QH_{m,n}, QH_{a,b}] = Q \neq 0$; then there should exist $z_1, \dots, z_K$ such that $ev_{z_1,\dots,z_K}(Q) \neq 0$. The map $ev_{z_1,\dots,z_K}$ sends $\operatorname{Tr} A_r \prod_{i=1}^r (L(z)^{(i)} - \partial_z)$ to $\operatorname{Tr} A_r \prod_{i=1}^r ((-L_{\text{Gaudin}}(z)^t)^{(i)} - \partial_z)$ due to item 3 and homomorphism property. So we conclude that $[QH_i(z), QH_j(w)] \neq 0$, since its coefficient at $z^{m-1}\partial_z^{r-n} w^{a-1}\partial_w^{r-b}$ equals to $ev_{z_1,\dots,z_K}(Q) \neq 0$. This contradicts Talalaev's theorem, so the commutativity of the $QH_{i,j}$ is proved.
  To show the second statement, we rely on [43], where it is shown that there is a unique commutative subalgebra in $U(\mathfrak{g}_-)$ whose associated graded is $\operatorname{gr}\mathcal{A} \subset S(\mathfrak{g}_-)$, and hence, the elements $QH_{n,k} \in U(\mathfrak{g}_-)$ generate the same commutative subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$. The proof is achieved. □

Remarks. It was first observed in [34], that one can use simply the column determinant to define D, $D = \det^{\text{col}}(L(z) - \partial_z)$, where $\det^{\text{col}} M = \sum_\sigma (-1)^{|\sigma|} \prod M_{\sigma(i),i}$, and hence the quantum Hamiltonians $QH_{n,k}$. We explicitly remark that in these summands, the order in which the matrix elements appear in the product is

$$\prod M_{\sigma(i),i} = M_{\sigma(1),1} \cdot M_{\sigma(2),2} \cdot M_{\sigma(3),3} \cdots M_{\sigma(r),r}.$$

This picture becomes natural due to an observation of [8], that $L(z) - \partial_z$ is a "Manin matrix", (that is elements in the same column commute, and the elements of any 2×2 submatrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfy $[A,D]=[C,B]$) so the use of column determinant is dictated by Manin's general theory.

In complete analogy one can define a commutative subalgebra in $U(gl_N[t])$.

3. Limits of Gaudin Algebras

The algebra $U(\mathfrak{g})^{\otimes n}$ has an increasing filtration by finite-dimensional spaces, $U(\mathfrak{g})^{\otimes n}=\bigcup_{k=0}^{\infty}(U(\mathfrak{g})^{\otimes n})_{(k)}$ (by degree with respect to the generators). For any one-parameter family of subalgebras $B(s)\subset U(\mathfrak{g})^{\otimes n}$ depending on the parameter algebraically, we define the limit $\lim_{s\to 0}B(s)$ as $\bigcup_{k=0}^{\infty}\lim_{s\to 0}B(s)\cap(U(\mathfrak{g})^{\otimes n})_{(k)}$. This limit is well-defined since the Grassmannian of $(U(\mathfrak{g})^{\otimes n})_{(k)}$ is compact and $B(s)$ depends algebraically on s. It is clear that the limit of a family of *commutative* subalgebras is a commutative subalgebra. It is also clear that passage to the limit commutes with homomorphisms of filtered algebras (in particular, with the projection onto any factor and with finite-dimensional representations).

We shall consider the limits of Gaudin subalgebras when some of the points $z_1,\dots,z_n$ glue together. More precisely, let $z_1,\dots,z_k$ be independent on s, and $z_{k+i}=z+su_i$, $i=1,\dots,n-k$, where $z_1,\dots,z_k,z\in\mathbb{C}$ are pairwise distinct and $u_1,\dots,u_{n-k}\in\mathbb{C}$ are pairwise distinct. Let us describe the limit subalgebra $\lim_{s\to 0}\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})\subset U(\mathfrak{g})^{\otimes n}$.

Consider the following homomorphisms

$$D_{k,n}:=\mathrm{id}^{\otimes k}\otimes\mathrm{diag}_{n-k}:U(\mathfrak{g})^{\otimes(k+1)}\hookrightarrow U(\mathfrak{g})^{\otimes n},$$

and

$$I_{k,n}:=1^{\otimes k}\otimes\mathrm{id}^{\otimes(n-k)}:U(\mathfrak{g})^{\otimes(n-k)}\hookrightarrow U(\mathfrak{g})^{\otimes n},$$

where $\mathrm{id}:U(\mathfrak{g})\to U(\mathfrak{g})$ is the identity map, $\mathrm{diag}_{n-k}:U(\mathfrak{g})\hookrightarrow U(\mathfrak{g})^{\otimes(n-k)}$ is the diagonal embedding. Clearly, the image of $[U(\mathfrak{g})^{\otimes(n-k)}]^{\mathfrak{g}}$ under the homomorphism $I_{k,n}$ commutes with the image of the homomorphism $D_{k,n}$.

Let $z_1,\dots,z_k,z$ and $u_1,\dots,u_{n-k}$ be two collections of pairwise distinct complex numbers. We assign to these data a commutative subalgebra

$$\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}:=D_{k,n}(\mathcal{A}(z_1,\dots,z_k,z))\cdot I_{k,n}(\mathcal{A}(u_1,\dots,u_{n-k}))\subset U(\mathfrak{g})^{\otimes n}.$$

PROPOSITION 3. *The subalgebra $\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}$ is a free commutative algebra generated by the elements $D_{k,n}(S_l^{i,m}(z_1,\dots,z_k,z))$, with $i=1,\dots,k$, $l=1,\dots,$ $\mathrm{rk}\,\mathfrak{g}$, $m=1,\dots,\deg\Phi_l$, $I_{k,n}(S_l^{i,m}(u_1,\dots,u_{n-k}))$, with $i=1,\dots,n-k-1$, $l=1,\dots,\mathrm{rk}\,\mathfrak{g}$, $m=1,\dots,\deg\Phi_l$ and $I_{k,n}(S_l^{n-k,\deg\Phi_l}(u_1,\dots,u_{n-k}))$, where $l=1,\dots,\mathrm{rk}\,\mathfrak{g}$.*

Proof. Note that by the assertion (4) of Proposition 1 the center of $\mathrm{id}^{\otimes k}\otimes\mathrm{diag}_{n-k}(U(\mathfrak{g}))$ is contained in $I_{k,n}(\mathcal{A}(u_1,\dots,u_{n-k}))$. Hence, by (1) of Proposition 1, the elements defined above generate the algebra $\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}$. Thus, it remains to show that these elements are algebraically independent. But this is so due to the same argument as (1) of Proposition 1. □

THEOREM 1. $\lim_{s\to 0}\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})=\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}\subset U(\mathfrak{g})^{\otimes n}$.

Proof. Let us choose the generators of $\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})$ as in Proposition 1. The coefficients of the Laurent expansion of $S_l(w;z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})$ at $z+su_j$ are proportional to the Laurent coefficients of $S_l(w;\frac{z_1-z}{s},\dots,\frac{z_k-z}{s},u_1,\dots,u_{n-k})$ at the point u_j. On the other hand, still by Lemma 2 of [41] which states that $\lim_{z\to\infty}\varphi_z$ yields the counit $\varepsilon: U(\mathfrak{g}_-)\to\mathbb{C}\cdot 1\subset U(\mathfrak{g})$, we have

$$\begin{aligned}&\lim_{s\to 0} S_l(w;\frac{z_1-z}{s},\dots,\frac{z_k-z}{s},u_1,\dots,u_{n-k})=\\&\quad=\lim_{s\to 0}\varphi_{w-\frac{z_1-z}{s},\dots,w-\frac{z_k-z}{s},w-u_1\dots,w-u_{n-k}}(S_l)=\\&\quad=(\varepsilon\otimes\dots\otimes\varepsilon\otimes\varphi_{w-u_1}\otimes\dots\otimes\varphi_{w-u_{n-k}})\circ\mathrm{diag}_n(S_l)=I_{k,n}S_l(w;u_1,\dots,u_{n-k}).\end{aligned}$$

Therefore, $\lim_{s\to 0}S_l^{i+k,m}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})=I_{k,n}(S_l^{i,m}(u_1,\dots,u_{n-k}))$ for $i=1,\dots,n-k$.

Now let us compute the limits of the coefficients of the Laurent expansion of $S_l(w;z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})$ at z_i.

LEMMA 1. $\lim_{s\to 0}\varphi_{z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k}}=D_{k,n}\circ\varphi_{z_1,\dots,z_k,z}$.

Proof. It is sufficient to check this on the generators. We have

$$\begin{aligned}&\lim_{s\to 0}\varphi_{z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k}}(g[m])=\\&\quad=\lim_{s\to 0}(\sum_{i=1}^{k}z_i^m g^{(i)}+\sum_{i=k+1}^{n}(z+su_{i-k})^m g^{(i)})=\\&\quad=D_{k,n}\circ\varphi_{z_1,\dots,z_k,z}(g[m]).\end{aligned}$$

□

By Lemma 1, we have $\lim_{s\to 0}S_l^{i,m}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})=D_{k,n}(S_l^{i,m}(z_1,\dots,z_k,z))$ for $i=1,\dots,k$.

Thus, we have $\lim_{s\to 0}\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})\supset\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}$. On the other hand, from Propositions 1 and 3, it follows that the algebras $\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})$ and $\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}$ have the same Poincaré series. Hence, $\lim_{s\to 0}\mathcal{A}(z_1,\dots,z_k,z+su_1,\dots,z+su_{n-k})=\mathcal{A}_{(z_1,\dots,z_k,z),(u_1,\dots,u_{n-k})}$

4. Quantum Hamiltonians of Bending Flows

In this section, we shall consider the algebra $\mathfrak{g} = gl_N$, identified with its dual gl_N^* by means of the trace form. In [16][2], a system of Poisson-commuting elements in the Poisson algebra $S(gl_N)^{\otimes n}$ (that is, in the phase space of gl_N–valued classical Gaudin system with n sites) was constructed. Namely, the following functions on $gl_N \oplus \cdots \oplus gl_N$:

$$\overline{H}_{l,k}^{(\alpha)}(X_1, \ldots, X_n) := \mathrm{Res}_{z=0} \frac{1}{z^{\alpha+1}} \mathrm{Tr}\left(X_k + z\left(\sum_{i=k+1}^{n} X_i\right)\right)^l, \quad X_i \in gl_N, \qquad (11)$$

commute with respect to the standard (product) Lie-Poisson bracket on $gl_N \oplus \cdots \oplus gl_N$. The range of the integers α, l, k in (11) is, respectively,

$$k = 1, \ldots, n-1, \quad l = 1, \ldots, N, \quad \alpha = 0, \ldots, l.$$

Together with suitable invariants associated with the diagonal gl_N action, the "Hamiltonians" $\overline{H}_{l,k}^{(\alpha)}$ form a complete set of mutually Poisson-commuting elements. Note that $\overline{H}_{l,k}^{(\alpha)}(X_1, \ldots, X_n)$ with l, k fixed form a basis of the Poisson algebra generated by the elements $\mathrm{Tr}(w_1 X_k + w_2(\sum_{i=k+1}^n X_i))^l$ with any $w_1, w_2 \in \mathbb{C}$.

To start making a connection with the results of the previous section, we notice that one can iterate the limiting procedure described in the previous section to obtain some new commutative subalgebras in $U(\mathfrak{g})^{\otimes n}$. In particular, we can obtain the following subalgebra $\mathcal{A}_{(z_{1,1}, z_{1,2}), \ldots, (z_{n-1,1}, z_{n-1,2})} \subset U(\mathfrak{g})^{\otimes n}$, which is generated by

$$D_{1,n}(\mathcal{A}(z_{1,1}, z_{1,2})), 1 \otimes D_{1,n-1}(\mathcal{A}(z_{2,1}, z_{2,2})), \ldots, 1^{\otimes(n-2)} \otimes \mathcal{A}(z_{n-1,1}, z_{n-2,2}).$$

By Proposition 1 assertion (2), this subalgebra is independent on $z_{i,j}$. We denote it by $\mathcal{A}_{\lim}$.

PROPOSITION 4. *The associated graded* $\mathrm{gr}\,\mathcal{A}_{\lim}$ *is generated by the Hamiltonians of bending flows* $\overline{H}_{l,k}^{(\alpha)}$ *with* $k = 1, \ldots, n$, $l = 1, \ldots, N$, $\alpha = 0, \ldots, l$.

Proof. The associated graded subalgebra $\mathrm{gr}\,\mathcal{A}(z_1, z_2) \subset S(\mathfrak{g}) \otimes S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]$ is generated by the elements $\mathrm{gr}\, S_l(w; z_1, z_2)$, $l = 1, \ldots, \mathrm{rk}\,\mathfrak{g}$ for all possible $w \in \mathbb{C}$. Note that

$$\mathrm{gr}\, S_l(w; z_1, z_2)(X_1, X_2) = \Phi_l\left(\frac{1}{w - z_1} X_1 + \frac{1}{w - z_2} X_2\right), \qquad (12)$$

where, as before, Φ_l is a complete set of free generators of $S(\mathfrak{g})^{\mathfrak{g}}$. Hence, the subalgebra $\mathrm{gr}\,\mathcal{A}(z_1, z_2) \subset S(\mathfrak{g}) \otimes S(\mathfrak{g})$ is generated by the elements $\Phi_l(w_1 X_1 + w_2 X_2)$ for all possible $w_1, w_2 \in \mathbb{C}$. Taking $\mathrm{Tr}(X^l)$ as $\Phi_l(X)$, we see that the subalgebra

[2] Technically, the quoted paper treated the case $\mathfrak{g} = sl_N$, and a different ordering was chosen for the X_i's, but this is inessential.

$\mathrm{gr}\,\mathcal{A}(z_1,z_2)\subset S(\mathfrak{g})\otimes S(\mathfrak{g})$ is generated by the elements $\mathrm{Tr}(X_1+zX_2)^l$ for all possible $z\in\mathbb{C}$ and $l=1,\ldots,N$. Hence, the algebra $\mathrm{gr}\,D_{1,n-k+1}(\mathcal{A}(z_1,z_2))$ is generated by $\overline{H}_{l,k}^{(\alpha)}$, with $l=1,\ldots,N$, $\alpha=0,\ldots,l$ and k fixed. This means that the algebra $\mathcal{A}_{\lim}$ contains the elements $H_{l,k}^{(\alpha)}\in U(gl_N)^{\otimes n}$ such that $\mathrm{gr}\,H_{l,k}^{(\alpha)}=\overline{H}_{l,k}^{(\alpha)}$. The proposition is thus proved. □

The meaning of this proposition is that, for $\mathfrak{g}=gl_N$, the algebra $\mathcal{A}_{\lim}$ gives a quantization of the classical gl_N–"bending flows" system discussed in [16] as a generalization of the bending flows on the moduli space of polygons in $\mathbb{R}^3$ introduced in [28].

We next want to provide explicit formulas for the quantum integrals of bending flows. To this end, we think it is useful to rephrase (part of) the content of Proposition 4 as follows. The Hamiltonians $\overline{H}_{l,k}^{(\alpha)}(X_1,\ldots,X_n)$ of Equation (11) are the so-called spectral invariants of the Lax matrices

$$\widetilde{L}_k(z)=X_k+z\sum_{i=k+1}^{n}X_i, \tag{13}$$

that were introduced in [16] to provide a Lax representation for the (classical) generalized Bending flows. The ring of spectral invariants of matrices (13) coincides with the ring of the spectral invariants of matrices

$$L_k(w)=\frac{X_k}{w-z_1}+\frac{\sum_{i=k+1}^{n}X_i}{w-z_2}, \tag{14}$$

for any choice of (nonzero) different complex numbers z_1,z_2 [see, in particular, Equation (12)]. This holds thanks to the fact that we can find a rational transformation $z=z(w)$ that sends $\widetilde{L}_k(z)$ to $f(w)L_k(w)$ for some rational coefficient $f(w)$. Thus, we are allowed to replace, as Hamiltonians of the (classical) gl_N-bending flows, the quantities $H_{l,k}^{(\alpha)}(X_1,\ldots,X_n)$ of Equation (11) by the quantities

$$\begin{aligned}&\overline{P}_{l,k}^{(\alpha)}(X_1,\ldots,X_n)=\mathrm{Res}_{w=z_1}(z-z_1)^{\alpha-1}\,\mathrm{Tr}\,(L_k(w))^l\,,\\ &\quad k=1,\ldots,n-1,\quad l=1,\ldots,N,\quad \alpha=0,\ldots,l,\end{aligned} \tag{15}$$

and consider the problem of explicit "quantization" of these Hamiltonians.

That is, we aim at constructing explicit (mutually commuting) elements $P_{l,k}^{(\alpha)}(X_1,\ldots,X_n)$ in $U(gl_N)^{\otimes n}$ such that

$$\mathrm{gr}(P_{l,k}^{(\alpha)}(X_1,\ldots,X_n))=\overline{P}_{l,k}^{(\alpha)}(X_1,\ldots,X_n)\in S(gl_N)^{\otimes n}. \tag{16}$$

To this end, let us – as it is customary in the theory of quantum spin systems – introduce the operators $\widehat{X}_k$ as

$$\widehat{X}_k=\sum_{i,j=1}^{N}e_{ij}^{(k)}\otimes e_{ji}\in U(\mathfrak{g})^{\otimes n}\otimes\mathrm{End}(\mathbb{C}^N), \tag{17}$$

where $e_{ij}^{(k)} = \mathbf{1} \otimes \cdots \underbrace{e_{ij}}_{k\text{th place}} \cdots \otimes \mathbf{1}$. Let us also consider "quantum" Lax matrices

$$\widehat{L}_k(w) = \frac{\widehat{X}_k}{w - z_1} + \frac{\sum_{i=k+1}^n \widehat{X}_i}{w - z_2} \in U(gl_N)^{\otimes n} \otimes \mathrm{End}(\mathbb{C}^N)((w)), \tag{18}$$

and define their "quantum powers" $\widehat{L}_k^{[i]}(w)$ by the iterative relation

$$\widehat{L}_k^{[0]}(w) = Id, \qquad \widehat{L}_k^{[i]}(w) = \widehat{L}_k^{[i-1]}(w)\widehat{L}_k(w) - \frac{\partial}{\partial w}(\widehat{L}_k^{[i-1]}(w)). \tag{19}$$

PROPOSITION 5. *The quantum Hamiltonians defined as*

$$P_{l,k}^{(\alpha)} = \mathrm{Res}_{w=z_1} (w - z_1)^{\alpha-1} (1 \otimes \mathrm{Tr}) \widehat{L}_k(w)^{[l]}, \tag{20}$$

provide a commutative family of elements in $U(gl_N)^{\otimes n}$ that satisfy the relation (16).

Proof. We start with considering the operators $QH_{l,k}^{(\alpha)}$ defined as

$$\sum_{l=0,\ldots,N} QH_{l,k}^{(\alpha)} (\partial_w)^l := \mathrm{Res}_{w=z_1} (w - z_1)^{\alpha-1} \mathrm{det}^{\mathrm{col}} (\widehat{L}_k(w) - \partial_w). \tag{21}$$

The operators $QH_{l,k}^{(\alpha)}$ generate the algebra $\mathcal{A}_{\mathrm{lim}}$, since $\widehat{L}_k(w) = D_{1,n-k+1}(\varphi_{z_1,z_2}(\widehat{L}(w)))$, where $\widehat{L}_k(w)$ is given by (9).

Due to Talalaev's results (see Section 2.4), the quantum Hamiltonians (21) commute. The quantum Hamiltonians (20) generate the same commutative subalgebra generated by the Hamiltonians (21). As we briefly discuss below, this follows from Theorem 5 of [8] (see [9] for more details about its proof, and [11] for the general case $U_-(gl_N)$), about the Newton identities for Manin matrices. The definition of Manin matrix was briefly reminded at the end of Section 2.

To see why our assertion holds, we have to recall that, among other important properties, Manin matrices satisfy Newton identities between the coefficients of their characteristic polynomial $\mathrm{det}^{\mathrm{col}}(M - t)$ and traces of their kth powers. These identities can be, in analogy with the classical commutative case, collected in the equality

$$t\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{det}^{\mathrm{col}}(M - t)) = \mathrm{det}^{\mathrm{col}}(M - t) \sum_{k=0}^{\infty} \frac{1}{t^k} \mathrm{Tr}(M^k). \tag{22}$$

We can apply this formula to the Manin matrix $M = \widehat{L}_k(w) - \partial_w$ whose entries are in the ring $U(gl_N)^{\otimes n}((w)) \otimes \mathrm{Diff}_w$, where with Diff_w we denote the ring of differential operators in the formal variable w. Keeping the same notation as in

Formula (10), we get from (22) the equality

$$t\frac{\mathrm{d}}{\mathrm{d}t}(1\otimes \det{}^{\mathrm{col}})(L(w)-(\partial_w+t)) = (1\otimes \det{}^{\mathrm{col}})(L(w)-(\partial_w+t))\times \\ \times\sum_{k=0}^{\infty}\frac{1}{t^k}(1\otimes \mathrm{Tr})\left((L(w)-\partial_w)^k\right). \tag{23}$$

We notice that we can compactly write Talalaev's formula (10) for the matrix $L_k(w)-(\partial_w+t)$ as

$$(1\otimes \det{}^{\mathrm{col}})(L_k(w)-(\partial_w+t)) = \sum_{j=0}^{N}(-1)^{N-j}QH_{j,k}(w)(\partial_w+t)^{N-j}, \\ \text{with } QH_0=1. \tag{24}$$

On the other hand, the quantum powers $\widehat{L}_k^{[i]}(w)$, $i=0,1,\dots$ satisfy the relations

$$(\widehat{L}_k(w)-\partial_w)^m = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}\widehat{L}_k^{[j]}(w)\partial_w^{m-j}, \quad m=0,1,\dots. \tag{25}$$

Thus, the Newton identities (23) can be translated into relations between the generating functions of Talalev's quantum Hamiltonians $QH_{k,i}(w)$ and the new generating functions

$$QT_{m,k}(w) := (1\otimes Tr)\widehat{L}_k^{[m]}(w). \tag{26}$$

These relations are easily seen to express $QT_{m,k}$ as

$$QT_{m,k} = m(-1)^{m+1}QH_{m,k}+\Delta_{m,k}, \quad m\le N,$$

where $\Delta_{m,k}$ is a differential polynomial in the $QH_{k,p}(w)$, with $p<m$. For instance, in the case $N=3$, we have the following explicit equalities ($'=\frac{\mathrm{d}}{\mathrm{d}w}$):

$$\begin{aligned} QT_{1,k}(w) &= QH_{1,k}(w), \quad QT_{2,k}(w) = -2\,QH_{2,k}(w)+\left(QH_{1,k}(w)\right)^2-3\,QH'_{1,k}(w) \\ QT_{3,k}(w) &= 3\,QH_{3,k}(w)+\left(QH_{1,k}(w)\right)^3-3\,QH_{2,k}(w)QH_{1,k}(w)- \\ &\quad -7\,QH_{1,k}(w)QH'_{1,k}(w)+6\left(QH''_{1,k}(w)+QH'_{2,k}(w)\right). \end{aligned} \tag{27}$$

Since the quantum Hamiltonians $QH_{m,k}(w)$ – together with their w-derivatives – commute, the same is true for the quantum Hamiltonians $QT_{m,k}(w)$ as well as also for their residues at the singular point $w=z_1$ [see Equation (20)],

$$P_{l,k}^{(\alpha)} = \mathrm{Res}_{w=z_1}(w-z_1)^{\alpha-1}(1\otimes \mathrm{Tr})\widehat{L}_k(w)^{[l]}.$$

Finally, $\mathrm{gr}\,P_{l,k}^{(\alpha)} = \overline{P}_{l,k}^{(\alpha)}$ since $\mathrm{gr}\,\widehat{L}_k(w)^{[l]} = (L_k(w))^l$. □

Remark. One could also prove the commutativity of the above quantum Hamiltonians directly, without reference to the limiting procedure of the Gaudin algebras. Let us consider, e.g., the family (21). The commutativity of the $QH^{\alpha}_{l,k}$ with $QH^{\alpha'}_{l',k'}$ for $k>k'$ basically follows from global gl_n invariance of H_k, which clearly holds (see, also [19]). For $k=k'$ commutativity follows from Talalaev's theorem. The same arguments hold for the Hamiltonians (20).

We would like to point out that our construction (for the case $\mathfrak{g}=gl_N$) can be shown to work, *mutatis mutandis*, for any of the limit algebras $\mathcal{A}_{(z_1,\ldots,z_k,z),(u_1,\ldots,u_{n-k})}$ defined in Section 3.

5. Schur–Weyl Duality and Jucys–Murphy Elements

Let N be a natural number, and consider $\mathfrak{g}=gl_N$, and $V_{(\lambda)}=\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N$. By Schur–Weyl duality, the centralizer of the diagonal $\mathfrak{g}$ in $\mathrm{End}(V_{(\lambda)})$ is the image of the group algebra $\mathbb{C}[S_n]$. Equivalently, the space $V^{\mathrm{sing}}_{(\lambda)}=[\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N]^{\mathrm{sing}}$ decomposes into the sum of irreducible S_n-modules with multiplicities 0 or 1. Since the elements of $\mathcal{A}(z_1,\ldots,z_n)$ commute with the diagonal action of $\mathfrak{g}$, we can treat them as commuting elements of the image of $\mathbb{C}[S_n]$ in $\mathrm{End}(\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N)$. In particular, one can rewrite quadratic Gaudin Hamiltonians (1) as follows:

$$H_i=\sum_{j\neq i}\frac{(i,j)}{z_i-z_j}.$$

For large N, the homomorphism $\mathbb{C}[S_n]\to\mathrm{End}(\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N)$ is an embedding, hence, H_i are commuting elements of $\mathbb{C}[S_n]$.

The quadratic elements of $\mathcal{A}_{\mathrm{lim}}$ can be rewritten as follows

$$H_i=\sum_{j<i}(i,j).$$

The latter are known as Jucys–Murphy elements. By [39], these elements generate the Gelfand–Tsetlin subalgebra in $\mathbb{C}[S_n]$ (in other words, this subalgebra is generated by the centers of the group subalgebra $\mathbb{C}[S_{n-1}]\subset\mathbb{C}[S_n]$, $\mathbb{C}[S_{n-2}]\subset\mathbb{C}[S_{n-1}]\subset\mathbb{C}[S_n]$ and so on). This algebra has a simple spectrum in any irreducible representation of S_n. We can obtain from this the following result of Mukhin and Varchenko.

PROPOSITION 6. [38] *Suppose* $\mathfrak{g}=gl_N$ *and consider* $V_{(\lambda)}=\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N$. *The Gaudin algebra* $\mathcal{A}(z_1,\ldots,z_n)$ (*and, moreover, its quadratic part*) *has simple joint spectrum in* $V^{\mathrm{sing}}_{(\lambda)}$ *for generic values of the parameters* $z_1,\ldots,z_n$.

Proof. The Gelfand–Tsetlin subalgebra in $\mathbb{C}[S_n]$ has a simple spectrum in any irreducible representation of S_n. This means that the algebra $\mathcal{A}_{\mathrm{lim}}$ has a simple spectrum in $V_{(\lambda)}^{\mathrm{sing}} = [\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N]^{\mathrm{sing}}$ (since the latter is multiplicity-free as an S_n-module). Note that the condition that the Gaudin algebra $\mathcal{A}(z_1, \ldots, z_n)$ has simple spectrum on the given finite-dimensional representation is an open condition on $z_1, \ldots, z_n$. Since the subalgebra $\mathcal{A}_{\lim}$ belongs to the closure of the family of Gaudin subalgebras $\mathcal{A}(z_1, \ldots, z_n)$, for generic values of z_i the algebra $\mathcal{A}(z_1, \ldots, z_n)$ has simple spectrum in $V_{(\lambda)}^{\mathrm{sing}}$ as well. □

6. (gl_N, gl_M) Duality and Gelfand–Tsetlin Algebra

Consider the space $W := \mathbb{C}^N \otimes \mathbb{C}^M$ with the natural action of the Lie algebra $gl_N \oplus gl_M$. The universal enveloping algebra $U(gl_N \oplus gl_M) = U(gl_N) \otimes U(gl_M)$ acts on the symmetric algebra $S(W) = \mathbb{C}[W^*]$ by differential operators. Let $D(W)$ be the algebra of differential operators on W^*. We shall use the following classical result.

Fact 2 *The image of $U(gl_M)$ is the centralizer of the image of $U(gl_N)$ in $D(W)$. In particular, the space $S(W)^{\mathrm{sing}}$ of singular vectors with respect to the diagonal gl_N-action is multiplicity-free as gl_M-module.*

We can treat the space W as the direct sum of M copies of $\mathbb{C}^N$, and hence, we have the action of $U(gl_N \oplus \cdots \oplus gl_N) = U(gl_N)^{\otimes M}$ on $S(W)$. The elements of the Gaudin subalgebra $\mathcal{A}(z_1, \ldots, z_M) \subset U(gl_N)^{\otimes M}$ commute with the diagonal gl_N, and hence, they can be rewritten as the elements of the image of $U(gl_M)$. For large N, the homomorphism $U(gl_M) \to D(W)$ is an embedding, hence, we obtain a commutative subalgebra in $U(gl_M)$. Let us describe this subalgebra explicitly. Consider the diagonal $M \times M$-matrix Z with the diagonal entries equal to $z_1, \ldots, z_M$. To any diagonal matrix Z one can naturally assign a quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_Z \subset U(gl_M)$. For generic Z, the subalgebra $\mathcal{A}_Z \subset U(gl_M)$ is the centralizer of the following family of commuting quadratic elements [42]:

$$Q_Z := \left\{ \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, h \rangle}{\langle \alpha, Z \rangle} e_\alpha e_{-\alpha} | h \in \mathfrak{h} \right\}, \tag{28}$$

where $\mathfrak{h} \subset gl_M$ is the subalgebra of diagonal matrices, Δ is the root system of gl_M, Δ_+ is the set of positive roots, and e_α are certain nonzero elements of the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$. The following assertion is due to Mukhin, Tarasov and Varchenko.

Fact 3 [35] *The image of the Gaudin subalgebra $\mathcal{A}(z_1, \ldots, z_M) \subset U(gl_N)^{\otimes M}$ in* $\mathrm{End}\, S(W)$ *coincides with the image of $\mathcal{A}_Z \subset U(gl_M)$. The space of quadratic Gaudin Hamiltonians coincides with the image of $Q_Z \subset U(gl_M)$.*

Remark. The quasi-classical version of this fact goes back to [1] (see also [20] Section 5.4 and especially formula 5.27, p. 23). But the relation with the (gl_N, gl_M) duality was not understood and the full picture was not developed. (See also [5,6] for some related considerations.) In our opinion, these facts might possibly be traced back to the fact, well-known in the theory of integrable systems, that the n-site Toda chain has two Lax representations: one is by 2×2 matrices another by $n\times n$ matrices. Working out this idea is outside the size of this paper.

Remark. The (quantum) $(gl_N,\ gl_M)$ duality for quadratic Hamiltonians was discovered by Toledano Laredo in [48]. Fact 3 can also be deduced from the duality for quadratic Hamiltonians and the centralizer result for shift of argument subalgebras [42].

In [44] Shuvalov described the closure of the family of subalgebras $A_Z\subset S(\mathfrak{g})$ under the condition $Z\in\mathfrak{h}^{reg}$ (i.e., for regular Z in the fixed Cartan subalgebra).

Fact 4 *Suppose that* $Z(t)=Z_0+tZ_1+t^2Z_2+\cdots\in\mathfrak{h}^{reg}$ *for generic* t. *Set* $\mathfrak{z}_k=\bigcap_{i=0}^k\mathfrak{z}_\mathfrak{g}(Z_i)$ [*where* $\mathfrak{z}_\mathfrak{g}(Z_i)$ *is the centralizer of* Z_i *in* $\mathfrak{g}$], $\mathfrak{z}_{-1}=\mathfrak{g}$. *Then, we have*

(1) *the subalgebra* $\lim_{t\to 0}A_{Z(t)}\subset S(\mathfrak{g})$ *is generated by all elements of* $S(\mathfrak{z}_k)^{\mathfrak{z}_k}$ *and their derivatives* (*of any order*) *along* Z_{k+1} *for all* k.
(2) $\lim_{t\to 0}A_{Z(t)}$ *is a free commutative algebra.*
(3) *for the case* $\mathfrak{g}=gl_M$, $\lim_{t\to 0}\mathcal{A}_{Z(t)}$ *is the Gelfand–Tsetlin commutative subalgebra* $\mathcal{A}_{G-Ts}\subset U(gl_M)$ [*i.e., it is generated by the center of* $U(gl_{M-1})\subset U(gl_M)$, *the center of* $U(gl_{M-2})\subset U(gl_{M-1})\subset U(gl_M)$, *etc.*]

Items 1 and 2 are due to [44]. Item 3 on the level of Poisson algebras was discovered by E. Vinberg in [49], 6.1–6.4. On the other hand in [47], it is proved that there is a unique lifting of such limit subalgebra to $U(gl_M)$. Thus, the limit of $\mathcal{A}_Z\subset U(gl_M)$ is Gelfand–Tsetlin commutative subalgebra $\mathcal{A}_{G-Ts}\subset U(gl_M)$.

Now we ready to prove the main result of this section, which can be informally formulated that: quantum bending flows integrable system is $(gl_N,\ gl_M)$ dual to quantum Gelfand–Tsetlin one. It has been first discovered in [15].

THEOREM 2. *The image of the quantum bending flows commutative subalgebra* $\mathcal{A}_{\lim}\subset U(gl_N)^{\otimes M}$ *in* $\mathrm{End}\,S(W)$ *coincides with the image of the quantum Gelfand–Tsetlin commutative subalgebra* $\mathcal{A}_{G-Ts}\subset U(gl_M)$.

Proof. Indeed, by Fact 3 the image of the quantum Gaudin subalgebra $\mathcal{A}(z_1,\ldots,z_M)$ coincides with the image of the quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_Z\subset U(gl_M)$. By Fact 3, we know that:

$$\lim_{z_1\to z_M}\left(\cdots\left(\lim_{z_{M-2}\to z_M}\left(\lim_{z_{M-1}\to z_M}\mathcal{A}_Z\right)\right)\cdots\right)$$

is the Gelfand–Tsetlin subalgebra. On the other hand by proposition 4 we know that: the same limit of the quantum Gaudin subalgebra $\mathcal{A}(z_1,\ldots,z_M)$ is the quantum bending flows commutative subalgebra $\mathcal{A}_{\lim}$. □

7. Simplicity of the Joint Spectrum of the Gaudin Algebras in the gl_N Case

Let V be the tautological representation of $\mathfrak{g} = gl_N$ (of the highest weight ω_1) and $V_{(\lambda)} = \bigotimes_{i=1}^{n} S^{m_i} V$ (or, equivalently, $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = m_i \omega_1$, $m_i \in \mathbb{Z}_+$). We shall show[3] in this section, that the joint spectrum of the Gaudin algebra $\mathcal{A}(z_1, \ldots, z_n)$ in the space $V_{(\lambda)}^{\text{sing}}$ is simple for generic values of the parameters $z_1, \ldots, z_n$.

THEOREM 3. *Consider $V_{(\lambda)} = \bigotimes_{i=1}^{n} S^{m_i} V$. The Gaudin algebra $\mathcal{A}(z_1, \ldots, z_n)$ has simple spectrum in $V_{(\lambda)}^{\text{sing}}$ for generic values of the parameters $z_1, \ldots, z_n$.*

Proof. The Gelfand–Tsetlin subalgebra in $U(gl_M)$ has simple spectrum in any irreducible representation of gl_M. Since the image of $\mathcal{A}_{G-Ts} \subset U(gl_M)$ in $D(W)$ coincides with the image of $\mathcal{A}_{\text{lim}}$ in $D(W)$, and $S(W)^{\text{sing}}$ is multiplicity-free as gl_M-module, the algebra $\mathcal{A}_{\text{lim}}$ has a simple spectrum in $S(W)^{\text{sing}}$.

Note that the representation $V_{(\lambda)}$ occurs in $S(W)$. Thus, the algebra $\mathcal{A}_{\text{lim}}$ has simple spectrum in $V_{(\lambda)}^{\text{sing}}$.

Since the subalgebra $\mathcal{A}_{\text{lim}}$ belongs to the closure of the family of Gaudin subalgebras $\mathcal{A}(z_1, \ldots, z_n)$, (repeating the deformation argument of Proposition 6) there is a Zariski open subset $\Omega \subset \mathbb{C}^n$ such that for each $(z_1, \ldots, z_n) \in \Omega$ the algebra $\mathcal{A}(z_1, \ldots, z_n)$ has simple spectrum on $V_{(\lambda)}^{\text{sing}}$. Hence the assertion. □

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[3]After the first version of the present paper was submitted, the paper [37], dealing with the simplicity of the spectrum of $\mathcal{A}(z_1, \ldots, z_n)$ appeared in the Math-archive.

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