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TORIC DEGENERATIONS OF SCHUBERT VARIETIES

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Abstract. Let G be a simply connected semisimple complex algebraic group. We prove that every Schubert variety of G has a flat degeneration into a toric variety. This provides a generalization of results of [9], [7], [6]. Our basic tool is Lusztig's canonical basis and the string parametrization of this basis.

0. Introduction

0.1. Let G be a simply k connected semisimple complex algebraic group. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. Let W the Weyl group of G relative to T. For any w in W, let $X_w = \overline{BwB/B}$ denote the Schubert variety corresponding to w. This article is concerned with the following problem.

Degeneration Problem. *Is there a flat family over* $\operatorname{Spec}\mathbb{C}[t]$, *such that the general fiber is* X_w *and the special fiber is a toric variety?*

The existence of such a degeneration was obtained by N. Gonciulea and V. Lakshmibai for the flag variety G/B when $G = \mathrm{SL}_n$; see [9]. Their proof is based on the theory of standard monomials. The corner stone of their proof is the following: fundamental weights are minuscule weights, hence, a basis of every fundamental representation is endowed with a structure of distributive lattice.

A complete study of the degeneration problem was made in the case when G has rank two [6]. Note also that the A_n case was studied by R. Dehy and R. Yu for a class of elements w in the Weyl group [7], [8]. The proofs rely on the theory of standard monomials as well. A natural question would be: is there a (flat) toric degeneration of the flag variety G/B which restricts to a toric degeneration of the Schubert varieties X_w for any w in the Weyl group? In [5], R. Chirivi gives a degeneration of the flag variety which restricts into semitoric degenerations of the Schubert varieties, i.e., finite unions of irreducible toric varieties. An explanation of this fact was given to us by O. Mathieu: intersections of irreducible toric varieties are irreducible toric varieties, but intersection of Schubert varieties can be a union of several Schubert varieties. Hence, the answer to the previous question is negative. In [5], the degeneration problem is solved with toric replaced by semitoric.

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0.2. Our approach of the problem is based on the canonical/global base of Lusztig/Kashiwara and the so-called string parametrization of this base studied by A. Berenstein and A. Zelevinsky [1], [2] and P. Littelmann [12].

Fix w in W. Let P^+ be the semigroup of dominant weights. For all λ in P^+, let $\mathcal{L}_\lambda$ be the line bundle on G/B corresponding to λ. Then, the direct sum of global sections $R_w := \bigoplus_{\lambda \in P^+} H^0(X_w, \mathcal{L}_\lambda)$ carries a natural structure of P^+-graded $\mathbb{C}$-algebra. Moreover, there exists a natural action of T on R_w. Our principal result can be stated as follows:

Theorem. *Fix w in W. There exists a filtration $(\mathcal{F}_m^w)_{m \in \mathbb{N}}$ of R_w such that* (i) *for all m in $\mathbb{N}$, $\mathcal{F}_m^w$ is compatible with the P^+-grading of R_w;* (ii) *for all m in $\mathbb{N}$, $\mathcal{F}_m^w$ is compatible with the action of T;* (iii) *the associated graded algebra is the $\mathbb{C}$-algebra of the semigroup of integral points in a rational convex polyhedral cone.*

This cone (and the filtration) depends on the choice of a reduced decomposition $\tilde{w}_0$ of the longest element w_0 of the Weyl group. Explicit equations for the faces of this cone can be obtained from [12] for so-called nice decompositions $\tilde{w}_0$. More generally, those equations were obtained in [2] from $\tilde{w}_0$-trails in fundamental Weyl modules of the Langlands dual of G.

This theorem gives a positive answer to the degeneration problem. Indeed, let λ be a regular dominant weight. Then the line bundle $\mathcal{L}_\lambda$ is ample and X_w is the projective spectrum of $\bigoplus_{m \in \mathbb{N}} H^0(X_w, \mathcal{L}_{m\lambda})$. Moreover, the spectrum of a noetherian graded algebra associated to a filtration of a noetherian algebra R is a flat degeneration of Spec(R). This is proved as follows by a standard argument: let t be an indeterminate and consider the filtration $(R_n)_{n \in \mathbb{N}}$ of R. Then, the $\mathbb{C}[t]$-algebra $R^t = \oplus_n R_n t^n$ is flat over $\mathbb{C}[t]$ and it verifies $R^t/(t - t_0)R^t \simeq R$ for $t_0 \neq 0$ and $R^t/tR^t \simeq \mathrm{Gr}R$.

Let U be the maximal unipotent subgroup of B. Hence, R_{w_0} is the algebra $\mathbb{C}[G/U]$ of regular functions on G/U. For any w, the algebras R_w are quotients of this algebra. Let I_w be the ideal such that $R_w = R_{w_0}/I_w$. If $\tilde{w}_0$ is adapted to w in the sense of Definition 2.4, then the filtration $(\mathcal{F}_m^w)_{m \in \mathbb{N}}$ is the quotient filtration of $(\mathcal{F}_m^{w_0})_{m \in \mathbb{N}}$ defined by $\mathcal{F}_m^w := \overline{\mathcal{F}}_m^{w_0} = \mathcal{F}_m^{w_0} + I_w$. For a general reduced decomposition $\tilde{w}_0$, the filtration $(\overline{\mathcal{F}}_m^{w_0})_{m \in \mathbb{N}}$ of R_w provides a graded associated algebra whose spectrum is a semitoric variety.

The proof of the theorem is based on two facts.

Let U^- be the maximal unipotent subgroup of G which is opposite to U. Set $B^- = TU^-$. Then the algebra $\mathbb{C}[G/U]$ embeds in $\mathbb{C}[B^-]$. Moreover, we can embed the (specialized) dual of the canonical base in the algebra $\mathbb{C}[U^-]$. We prove that this dual has good multiplicative properties inherited from the quantum case; see Theorem 2.3. This part of the article is inspired by [15]. But here, we do not use the positivity arguments or the elaborate Hall algebra model, which is only true for the simply laced case.

In a second step we show how to restrict from flag variety to Schubert varieties: this part relies on the compatibility of the canonical base with the Demazure modules [11], [12].

1. Notation and recollection on the dual canonical basis

1.1. Denote by G a semisimple simply connected complex Lie group. Fix a torus T of G and let B be a Borel subgroup such that $T \subset B \subset G$. Denote by U the unipotent radical of B. Let B^- be the opposite Borel subgroup and let U^- be its unipotent radical. Let $\mathfrak{g}$, resp. $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{b}$, $\mathfrak{n}^-$, $\mathfrak{b}^-$, be the Lie $\mathbb{C}$-algebra of G, resp. T, U, B, U^-, B^-. Let n be the rank of $\mathfrak{g}$. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let $\{\alpha_i\}_i$ be a basis of the root system Δ corresponding to this decomposition. Let P be the weight lattice generated by the fundamental weights ϖ_i, $1 \leq i \leq n$, and let $P^+ := \sum_i \mathbb{N}\varpi_i$ be the semigroup of integral dominant weights. Let W be the Weyl group, generated by the reflections s_{α_i} corresponding to the simple roots α_i, and let w_0 be the longest element of W. We denote by $(\, , \,)$ the W-invariant form on P.

1.2. Let d be an integer such that $(P,P) \subset (2/d)\mathbb{Z}$. Let q be an indeterminate and set $\mathbb{K} = \mathbb{C}(q^{1/d})$. Let $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra on $\mathbb{K}$, as defined in [4], i.e., it is the standard quantized enveloping algebra on $\mathbb{K}$ whose Cartan subalgebra is isomorphic to the group algebra $\mathbb{K}[P]$. Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$ for all i. Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the subalgebra generated by the canonical generators E_i, resp. F_i, of weight, α_i, resp. $-\alpha_i$, and the quantum Serre relations. For all λ in P, let K_λ be the element corresponding to λ in the algebra $U_q^0 = \mathbb{K}[P]$ of the Cartan subalgebra of $U_q(\mathfrak{g})$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. We set

$$U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \otimes U_q^0, \qquad U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U_q^0.$$

We endow $U_q(\mathfrak{g})$ with the structure of a Hopf algebra and the comultiplication Δ; the antipode S and the augmentation ε are given by

$$\Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,\ \Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i,\ \Delta K_\lambda = K_\lambda \otimes K_\lambda,$$
$$S(E_i) = -K_{\alpha_i}^{-2} E_i,\ S(F_i) = -F_i K_{\alpha_i}^2, S(K_\lambda) = K_{-\lambda}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \varepsilon(K_\lambda) = 1.$$

If M is a U_q^0-module and $\gamma \in P$, we set $M_\gamma := \{m \in M,\ K_\lambda.m = q^{(\lambda,\gamma)}m\}$.

For n a nonnegative integer, we set: $[n]_i = \frac{1-q_i^n}{1-q_i}$, $[n]_i! = [n]_i[n-1]_i \ldots [1]_i$.

1.3. The dual $U_q(\mathfrak{g})^*$ is endowed with a structure of left, resp. right, $U_q(\mathfrak{g})$-module by $u.c(a) = c(au)$, resp. $c.u(a) = c(ua)$, u, $a \in U_q(\mathfrak{g})$, $c \in U_q(\mathfrak{g})^*$. If M is a finite dimensional left $U_q(\mathfrak{g})$-module, we endow the dual M^* with a structure of left $U_q(\mathfrak{g})$-module by $u\xi(v) = \xi(S(u)v)$, $u \in U_q(\mathfrak{g})$, $\xi \in M^*$, $v \in M$.

For all λ in P^+, let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight λ. We can embed $V_q(\lambda)^* \otimes V_q(\lambda)$ in $U_q(\mathfrak{g})^*$ by setting $\xi \otimes v(u) = \xi(u.v)$, $u \in U_q(\mathfrak{g})$, $\xi \in V_q(\lambda)^*$, $v \in V_q(\lambda)$. Let v_λ be a highest weight vector of $V_q(\lambda)$. For all integral dominant weights λ, let $C(\lambda)$, resp. $C^+(\lambda)$, be the subspace of $U_q(\mathfrak{g})^*$ generated by the $\xi \otimes v$, resp. $\xi \otimes v_\lambda$, $\xi \in V_q(\lambda)^*$, $v \in V_q(\lambda)$. We set $\mathbb{C}_q[G] = \bigoplus_{\lambda \in P^+} C(\lambda)$, $\mathbb{C}_q[G/U] = \bigoplus_{\lambda \in P^+} C^+(\lambda)$. Then, $\mathbb{C}_q[G]$ and $\mathbb{C}_q[G/U]$ are subalgebras of the Hopf dual of $U_q(\mathfrak{g})$. $\mathbb{C}_q[G]$, resp. $\mathbb{C}_q[G/U]$, is the algebra of quantum regular functions on G, resp. on the quotient G/U.

1.4. There exists a unique bilinear form $(\, , \,)$ on $U_q(\mathfrak{b}) \times U_q(\mathfrak{b}^-)$; see [16], [17], [4], such that:

$$(u^+, u_1^- u_2^-) = (\Delta(u^+), u_1^- \otimes u_2^-), \qquad u^+ \in U_q(\mathfrak{b})\,;\, u_1^-, u_2^- \in U_q(\mathfrak{b}^-), \tag{1.4.1}$$
$$(u_1^+ u_2^+, u^-) = (u_2^+ \otimes u_1^+, \Delta(u^-)), \qquad u^- \in U_q(\mathfrak{b}^-)\,;\, u_1^+, u_2^+ \in U_q(\mathfrak{b}), \tag{1.4.2}$$
$$(K_\lambda, K_\mu) = q^{-(\lambda,\mu)}, \qquad \lambda, \mu \in P, \tag{1.4.3}$$
$$(K_\lambda, F_i) = 0, \qquad \lambda \in P, 1 \leq i \leq n, \tag{1.4.4}$$

$$(E_i,\, K_\lambda) = 0\,, \qquad \lambda \in P\,, 1 \le i \le n, \tag{1.4.5}$$

$$(E_i,\, F_i) = \delta_{ij}(1-q_i^2)^{-1}\,, \qquad 1 \le i,j \le n. \tag{1.4.6}$$

where the bilinear form on $(U_q(\mathfrak{b})\otimes U_q(\mathfrak{b}))\times(U_q(\mathfrak{b}^-)\otimes U_q(\mathfrak{b}^-))$ is defined by $(u\otimes v, u'\otimes v') := (u,u')(v,v')$, $u,v \in U_q(\mathfrak{b})$, $u',v' \in U_q(\mathfrak{b}^-)$. For all β in $Q^+ := \sum_i \mathbb{N}\alpha_i$, let $U_q(\mathfrak{n})_\beta$, resp. $U_q(\mathfrak{n}^-)_{-\beta}$, be the subspace of $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, with weight β, resp. $-\beta$. The form $(\,,\,)$ is nondegenerate on $U_q(\mathfrak{n})_\beta \times U_q(\mathfrak{n}^-)_{-\beta}$, $\beta \in Q^+$. We have, by (1.4.1–1.4.5):

$$(XK_\lambda,\, YK_\mu) = q^{-(\lambda,\mu)}(X,\, Y)\,, \qquad X \in U_q(\mathfrak{n}),\, Y \in U_q(\mathfrak{n}^-). \tag{1.4.7}$$

We can define a bilinear form $\langle\,,\,\rangle$ on $U_q(\mathfrak{g}) \times U_q(\mathfrak{g})$ by:

$$\langle X_1K_\lambda S(Y_1),\, Y_2K_\mu S(X_2)\rangle = (X_1,\, Y_2)(X_2,\, Y_1)q^{-(\lambda,\mu)/2}, \tag{1.4.8}$$

where $X_1, X_2 \in U_q(\mathfrak{n})$, $Y_1, Y_2 \in U_q(\mathfrak{n}^-)$, $\lambda, \mu \in P$. This form is nondegenerate.

There exists an algebra isomorphism from $U_q(\mathfrak{n})$ to $U_q(\mathfrak{n}^-)$ which maps E_i on F_i for all i. Via this isomorphism, the restriction of the form $(\,,\,)$ on $U_q(\mathfrak{n}) \times U_q(\mathfrak{n}^-)$ coincides with the one defined by Lusztig in [13, par 1].

1.5. Define the maps:

$$\beta : U_q(\mathfrak{b}) \to U_q(\mathfrak{b}^-)^*, \quad \beta(u)(v) = (u,\, v), \quad \zeta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^*, \quad \zeta(u)(v) = \langle u,\, v\rangle.$$

We have the following lemma from 1.4.

Lemma. *With the previous notations, we have:* (i) *β, ζ are injective,* (ii) *β is an antihomomorphism of algebras.*

Denote by ρ the restriction homomorphism from $U_q(\mathfrak{g})^*$ onto $U_q(\mathfrak{b}^-)^*$. We know, see [3, Proposition 3.4]:

Proposition. *The restriction of ρ to $\mathbb{C}_q[G/U]$ is injective. Moreover, for all λ in P^+, we have:* (i) *for all e in $U_q(\mathfrak{n})$, $\rho(\zeta(eK_{-2\lambda})) = \beta(eK_{-\lambda})$,* (ii) *there exists a (unique) subspace E_λ of $U_q(\mathfrak{n})$ such that $\zeta(E_\lambda K_{-2\lambda}) = C^+(\lambda)$.* □

1.6. Let $u \mapsto \overline{u}$ be the $\mathbb{K}$-antihomomorphism of $U_q(\mathfrak{g})$ such that $\overline{E_i} = E_i$, $\overline{K_\lambda} = K_{-\lambda}$, $\overline{F_i} = F_i$. It is easily seen that $\langle u,v\rangle = (u,v) = (\overline{u},\overline{v}) = \langle\overline{u},\overline{v}\rangle$, $u \in U_q(\mathfrak{n})$, $v \in U_q(\mathfrak{n}^-)$.

Let $\mathcal{B}$ be Lusztig's canonical basis of $U_q(\mathfrak{n}^-)$; see [13], which coincides with Kashiwara's global basis [11]. Let $\mathcal{B}^* \subset U_q(\mathfrak{n})$ be the dual basis in $U_q(\mathfrak{n})$, i.e., $(b^*, b') = \delta_{b,b'}$. Note that what is usually called "dual canonical basis" lies in the dual space $U_q(\mathfrak{n}^-)^*$. To be more precise, the common dual canonical basis is the image of $\beta(\mathcal{B}^*)$ by the restriction morphism $U_q(\mathfrak{b}^-)^* \to U_q(\mathfrak{n}^-)^*$. Let $\tilde{E}_i, \tilde{F}_i : U_q(\mathfrak{n}^-) \to U_q(\mathfrak{n}^-)$ be the Kashiwara operators [11]. For $b \in \mathcal{B}$, $\tilde{E}_i(b)$, resp. $\tilde{F}_i(b)$, equals some b' in $\mathcal{B} \cup \{0\}$, modulo $q^{-1}\mathbb{Z}[q^{-1}]\mathcal{B}$. The assignment $b \mapsto b'$ defines maps $\tilde{e}_i$ and $\tilde{f}_i$ from $\mathcal{B}$ to $\mathcal{B} \cup \{0\}$. For $b \in \mathcal{B}$, $1 \le i \le n$, set $\varepsilon_i(b) = \mathrm{Max}\{r,\, \tilde{e}_i^r(b) \neq 0\}$.

Let L_i, $1 \le i \le n$, be the adjoint of the left multiplication operator $u \mapsto F_iu$ for the form $(\,,\,)$ on $U_q(\mathfrak{n}) \times U_q(\mathfrak{n}^-)$. Then L_i is a quantum derivation of $U_q(\mathfrak{n})$; see [13, par 1]: $L_i(e_\alpha u) = L_i(e_\alpha)u + q^{(\alpha,\alpha_i)}e_\alpha L_i(u)$, $u \in U_q(\mathfrak{n})$, $e_\alpha \in U_q(\mathfrak{n})_\alpha, \alpha \in Q$. Set $L_i^{(r)} = \frac{1}{[r]_i!}L_i^r$.

The following is a recollection of results about the canonical basis and its dual. Assertions (i) and (ii) can be read in [13, 14.4.13,14.4.14]. Assertion (iii) is a standard consequence of [13, 14.3.2 (c)] by dualization.

Theorem 1. *For b, b' in $\mathcal{B}$, we have* (i) $bb' \in \mathbb{Z}[q,q^{-1}]\mathcal{B}$, (ii) $b^*b'^* \in \mathbb{Z}[q,q^{-1}]\mathcal{B}^*$, (iii) $L_i^{(\varepsilon_i(b))}(b^*) = (\tilde{e}_i^{\varepsilon_i(b)}(b))^*$, (iv) $\overline{b} \in \mathcal{B}$. □

The following theorem states precisely the compatibility of the canonical basis with the finite dimensional highest weight modules. It can be found in [11, Proposition 8.2].

Theorem 2. *Fix λ in P^+, $\lambda = \sum_i \lambda_i \varpi_i$. Set*

$$\mathcal{B}_\lambda := \{b \in \mathcal{B},\ \varepsilon_i(\overline{b}) \leq \lambda_i, 1 \leq i \leq n\}.$$

Then $\mathcal{B}_\lambda = \{b \in \mathcal{B},\ b.v_\lambda \neq 0\}$ and $\mathcal{B}_\lambda.v_\lambda$ is a basis of $V_q(\lambda)$. Moreover, the correspondence $\pi_\lambda : b \mapsto bv_\lambda$ is injective on $\mathcal{B}_\lambda$. □

When b belongs to $\mathcal{B}_\lambda$, and if no confusion occurs, we shall use the same symbol for b and $b.v_\lambda$, i.e., we set $b = \pi_\lambda(b) = b.v_\lambda$.

1.7. Let $A = \mathbb{C}[q,q^{-1}]$. Let $U_A(\mathfrak{n}^-)$ be the A-submodule of $U_q(\mathfrak{n}^-)$ generated by $\mathcal{B}$. Then, $U_A(\mathfrak{n}^-)$ is a free A-space and an A-algebra. Indeed, $U_A(\mathfrak{n}^-)$ is the A-algebra generated by the $\frac{1}{[m]_i!}F_i^m$, see [10, Theorem 11.10 (b)]. Let U_A be the sub-A-algebra of $U_q(\mathfrak{g})$ generated by the $\frac{1}{[m]_i!}F_i^m$, and the $\frac{1}{[m]_i!}E_i^m$. Set $V_A(\lambda) = U_A(\mathfrak{n}^-).v_\lambda \subset V_q(\lambda)$. Then $V_A(\lambda)$ is the (free) A-module generated by $\mathcal{B}_\lambda$. By [10, Theorem 11.19], we know that $\mathcal{B}$ is compatible with specialization $\mathbb{C} \otimes_A U_A(\mathfrak{n}^-) \simeq U(\mathfrak{n}^-)$, $\mathbb{C} \otimes_A V_A(\lambda) \simeq V(\lambda)$, $\lambda \in P^+$, where $\mathbb{C} = A/(q-1)A$ as an A-module, $U(\mathfrak{n}^-)$ is the (classical) enveloping $\mathbb{C}$-algebra of $\mathfrak{n}^-$, and $V(\lambda)$ is the classical Weyl module with highest weight λ.

Let $V_A(\lambda)^*$ be the A-dual of $V_A(\lambda)$. Then, it has a natural U_A-module structure by $u.v^*(m) = v^*(S(u).m)$, $u \in U_A$, $v \in V_A(\lambda)^*$, $m \in V_A(\lambda)$. The module $V_A(\lambda)^*$ specializes at $q = 1$ onto the dual $\mathfrak{g}$-module $V(\lambda)^*$.

1.8. Fix λ in P^+ and w in W. We know that $V_q(\lambda)$ verifies the Weyl character formula; we denote by $v_{w\lambda}$ an extremal vector of weight $w\lambda$. Then the $U_q(\mathfrak{b})$-module $V_{q,w}(\lambda) := U_q(\mathfrak{n})v_{w\lambda}$ verifies the Demazure character formula. We know [11, Theorem 12.4], [12, 5.3–5.4], the following.

Theorem. *There exists a subset $\mathcal{B}_w$ of $\mathcal{B}$ such that $V_{q,w}(\lambda)$ is spanned by $\mathcal{B}_w.v_\lambda$. Moreover, if b is in $\mathcal{B}_w$, then $\Delta(b) \in \langle\mathcal{B}_w\rangle \otimes \langle K_\mu.\mathcal{B}_w, \mu \in P\rangle$, where $\langle E\rangle$ denotes the subspace generated by the part E of $U_q(\mathfrak{g})$.* □

In particular, the orthogonal $V_{q,w}(\lambda)^\perp$ in $V_q(\lambda)^*$ of the Demazure module $V_{q,w}(\lambda)$ is generated as a space by $(\mathcal{B}_\lambda \cap \mathcal{B}\backslash\mathcal{B}_w)^*$ and the dual $V_{q,w}(\lambda)^*$ is generated by the image of $(\mathcal{B}_\lambda \cap \mathcal{B}_w)^*$ by the quotient morphism.

As in 1.7, this allows us to define A-forms for Demazure modules. We denote by $V_{A,w}(\lambda)$ the A-module generated by $\mathcal{B}_\lambda \cap \mathcal{B}_w$. It specializes for $q = 1$ to the classical Demazure module $V_w(\lambda)$.

2. A multiplicative property and string parametrizations of the dual canonical basis

2.1. For λ in P^+ and b in $\mathcal{B}_\lambda$, let $\pi_\lambda(b)^*$ be the element of $V_q(\lambda)^*$ such that $\pi_\lambda(b)^*(b'.v_\lambda) = \delta_{b,b'}$, $b' \in \mathcal{B}_\lambda$, where δ means the Kronecker symbol.

Lemma. *For all λ in P^+ and b in $\mathcal{B}_\lambda$, whe have $\zeta(b^*K_{-2\lambda}) = \pi_\lambda(b)^* \otimes v_\lambda$, $\beta(b^*K_{-\lambda}) = \rho(\pi_\lambda(b)^* \otimes v_\lambda)$, where ζ, β, and ρ are defined as in* 1.5.

Proof. By Lemma 1.5, we only need to prove that $\langle b^*K_{-2\lambda}, u\rangle = \pi_\lambda(b)^*(u.v_\lambda)$, $u \in U_q(\mathfrak{g})$. As $U_q(\mathfrak{g}) = U_q(\mathfrak{b}^-)\oplus(U_q(\mathfrak{b}^-)\otimes U_q(\mathfrak{n})\cap\mathrm{Ker}(\varepsilon))$, we only need to prove this formula for $u \in U_q(\mathfrak{b}^-)$. Remark that a basis of $U_q(\mathfrak{b}^-)$ can be given as $(b'K_\mu,\ b' \in \mathcal{B}, \mu \in P)$. By (1.4.7) and (1.4.8), $\langle b^*K_{-2\lambda}, b'K_\mu\rangle = (b^*, b')q^{(\lambda,\mu)} = \delta_{b,b'}q^{(\lambda,\mu)} = \pi_\lambda(b)^*(b'K_\mu.v_\lambda)$. This implies the lemma. □

Remark. By Proposition 1.5, the lemma implies that E_λ is spanned by a part of $\mathcal{B}^*$, namely $\mathcal{B}^*_\lambda$.

Let $A_q[G/U]$ be the sub-A-module of $\mathbb{C}_q[G]$ generated by the $\pi_\lambda(b)^* \otimes v_\lambda$, $\lambda \in P^+$ and $b \in \mathcal{B}_\lambda$. Let $d^{b''}_{b,b'}$ be the coefficient of b''^* in the product $b^*b'^*$.

Proposition. *We have*

(i) $\mathbb{K} \otimes_A A_q[G/U] = \mathbb{C}_q[G/U]$.
(ii) $A_q[G/U]$ *is an A-algebra.*
(iii) *If $b \in \mathcal{B}_\lambda$, $b' \in \mathcal{B}_{\lambda'}$ and $d^{b''}_{b',b} \neq 0$, then $b'' \in \mathcal{B}_{\lambda+\lambda'}$.*
(iv) $A_q[G/U]/(q-1)A_q[G/U] \simeq \mathbb{C}[G/U]$, *where $\mathbb{C}[G/U]$ is the $\mathbb{C}$-algebra of regular functions on G/U.*

Proof. The A-basis $\pi_\lambda(b)^*$ of $V_A(\lambda)^*$ is a $\mathbb{K}$-basis of $V_q(\lambda)^*$. Hence (i) is clear.

From Lemma 1.5, Proposition 1.5, and the previous lemma, we have the following equalities $\beta^{-1}[(\pi_\lambda(b)^*\otimes v_\lambda).(\pi_{\lambda'}(b')^*\otimes v_{\lambda'})] = \beta^{-1}(\pi_{\lambda'}(b')^*\otimes v_{\lambda'}).\beta^{-1}(\pi_\lambda(b)^*\otimes v_\lambda) = (b'^*K_{-\lambda'}).(b^*K_{-\lambda}) = q^{-(\lambda',\omega(b^*))}b'^*b^*K_{-\lambda-\lambda'} = q^{-(\lambda',\omega(b^*))}(\sum d^{b''}_{b',b}b''^*)K_{-\lambda-\lambda'}$. We know that $(\pi_\lambda(b)^* \otimes v_\lambda).(\pi_{\lambda'}(b')^* \otimes v_{\lambda'})$ belongs to $C^+(\lambda+\lambda')$. Hence the previous formula and Remark 2.1 imply (iii). Moreover, applying β, we obtain

$$(\pi_\lambda(b)^* \otimes v_\lambda).(\pi_{\lambda'}(b')^* \otimes v_{\lambda'}) = q^{-(\lambda',\omega(b^*))}\sum d^{b''}_{b',b}(\pi_{\lambda+\lambda'}(b'')^* \otimes v_{\lambda+\lambda'}). \tag{2.1.1}$$

This gives (ii). Now, (iv) is clear by 1.7 and the fact that specialization commutes with tensor product. □

Corollary. *Fix b, b', b'' in $\mathcal{B}$ with $d^{b''}_{b,b'}$ nonzero. Then, for all i, $1 \leq i \leq n$, we have $\varepsilon_i(b'') \leq \varepsilon_i(b) + \varepsilon_i(b')$.*

Proof. By applying the antiautomorphism $^-$, see 1.6, we obtain that $d^{\overline{b}''}_{\overline{b}',\overline{b}} = d^{b''}_{b,b'}$. Set $\lambda = \sum_i \varepsilon_i(b)\varpi_i$ and $\lambda' = \sum_i \varepsilon_i(b')\varpi_i$. As $^-$ is an involution, we deduce from [1.6, Theorem 2] that $\overline{b} \in \mathcal{B}_\lambda$ and $\overline{b}' \in \mathcal{B}_{\lambda'}$. This gives $\overline{b}'' \in \mathcal{B}_{\lambda+\lambda'}$ by (iii). Hence we obtain the corollary from [1.6, Theorem 2]. □

Remark. In the simply laced case, this corollary is easily obtained by the positivity property of the dual canonical basis, i.e., $d^{b''}_{b,b'} \in \mathbb{N}[q,q^{-1}]$ by [13]. For general $\mathfrak{g}$, we can conclude by a standard argument, given by E. Vasserot. It is based on the realization of $d^{b''}_{b,b'}$ in terms of traces of an automorphism of a diagram on spaces arising from perverse sheaves; see [13, 14.4.14].

2.2. We introduce the string parametrization of the (dual) canonical basis. Fix a reduced decomposition of the longest element of the Weyl group w_0: $\tilde{w}_0 = s_{i_1}\dots s_{i_N}$, where $N = \dim\mathfrak{n}$. For all u in $U_q(\mathfrak{n})$ and $1 \leq i \leq n$, set

$$a_i(u) = \mathrm{Max}\{r,\ L_i^r(u) \neq 0\},\quad \Lambda_i(u) = L_i^{(a_i(u))}(u).$$

For all b in $\mathcal{B}$, set

$$A_{\tilde{w}_0}(b) = (a_{i_N}(b^*), a_{i_{N-1}}(\Lambda_{i_N}(b^*)), \dots, a_{i_1}(\Lambda_{i_2} \cdots \Lambda_{i_N}(b^*))) \in \mathbb{N}^N.$$

This parametrization can be found in [12, par 1]. It coincides with the parametrization in [2, 3.2] by [1.6, Theorem 1 (iii)]. We now present a theorem due to Littelmann [12, par 1]; see also [2, 3.10].

Theorem. *The map $A_{\tilde{w}_0}$ embeds $\mathcal{B}$ into $\mathbb{N}^N$. Let $\mathcal{C}$ be its image. Then $\mathcal{C}$ is the set of integral points of a rational convex polyhedral cone of $\mathbb{R}^N$. Moreover, set $\Gamma_{\tilde{w}_0} := \{(\lambda, \psi) \in P^+ \times \mathcal{C},\ \psi \in A_{\tilde{w}_0}(\mathcal{B}_\lambda)\}$. Then, $\Gamma_{\tilde{w}_0}$ is the set of integral points of a rational convex polyhedral cone of $\mathbb{R}^{n+N}$.* $\square$

Note that equations of this cone are given in [2, 3.10]. For all ψ in $\mathcal{C}$, set $b^\psi = b^\psi_{\tilde{w}_0} = A^{-1}_{\tilde{w}_0}(\psi)$. We also set $d^{\psi''}_{\psi,\psi'} := d^{b^{\psi''}}_{b^\psi, b^{\psi'}}$.

2.3. Let $\prec$ be the lexicographical ordering of $\mathbb{N}^N$. We have the following.

Theorem. *Fix a reduced decomposition $\tilde{w}_0$ of the longest element of the Weyl group. Let b, b', b'' be in $\mathcal{B}$, $A_{\tilde{w}_0}(b) = \psi$, $A_{\tilde{w}_0}(b') = \psi'$ Then, $d^{b''}_{b,b'}$ nonzero implies $A_{\tilde{w}_0}(b'') \prec \psi + \psi'$. Moreover, $d^{\psi+\psi'}_{\psi,\psi'}$ is a power of q.*

Proof. First remark that $a_i(uv) = a_i(u) + a_i(v)$, u, v in $U_q(\mathfrak{n})$, by the quantized Leibniz rule. Write

$$b^* b'^* = \sum d^{b''}_{b,b'} b''^*. \tag{$*$}$$

Let $\psi'' = A_{\tilde{w}_0}(b'')$, for b'' in the sum. Then, $\psi''_1 \leq \psi_1 + \psi'_1$ by Corollary 2.1 and [1.6, Theorem 1 (iii)]. This gives the first step of the induction. Now, applying $L_{i_N}^{\psi_1+\psi'_1}$ and the Leibniz rule gives $\xi_1 \Lambda_{i_N}(b^*)\Lambda_{i_N}(b'^*) = \sum d^{b''}_{b,b'} \Lambda_{i_N}(b''^*)$, where the sum is taken over the elements b'' such that $\psi''_1 = \psi_1 + \psi'_1$, and where ξ_1 is a power of q. Now remark that, by [1.6, Theorem 1 (iii)], $\Lambda_{i_N}(b^*)$, $\Lambda_{i_N}(b'^*)$, and the $\Lambda_{i_N}(b''^*)$ all belong to the dual of the canonical basis. So, we can proceed as the first step by induction. We then have the first assertion of the theorem. For conclusion, note that we obtain $\xi_N \Lambda_{i_1} \cdots \Lambda_{i_N}(b^*)\Lambda_{i_1} \cdots \Lambda_{i_N}(b'^*) = \sum d^{b''}_{b,b'} \Lambda_{i_1} \cdots \Lambda_{i_N}(b''^*)$, where the sum is taken over the elements b'' such that $\psi''_k = \psi_k + \psi'_k$, for all k and where ξ_N is a power of q. Hence by Theorem 2.2 there is at most one element in the sum and for this element we have $A_{\tilde{w}_0}(b'') = \psi + \psi'$. By [12, par 1], we have:

Lemma. *For all elements B of the canonical basis, one has $\Lambda_{i_1} \cdots \Lambda_{i_N}(B^*) = 1$.* $\square$

Hence the coefficient $d^{\psi+\psi'}_{\psi,\psi'}$ is a power of q. This finishes the proof of the theorem.

Remark. The theorem and the lemma appeared for type A in [1].

2.4. By 1.8, the results of 2.1, 2.2 can be easily generalized to quotients of $\mathbb{C}_q[G/U]$ which correspond to Demazure modules. Indeed, fix w in W and let $\mathcal{B}_{\overline{w}}$ be the complement of $\mathcal{B}_w$ in $\mathcal{B}$. Set

$$I_{A,w} := \bigoplus\nolimits_{\lambda \in P^+,\, b \in \mathcal{B}_{\overline{w}} \cap \mathcal{B}(\lambda)} Ab^* \otimes v_\lambda = \bigoplus\nolimits_\lambda V_{A,w}(\lambda)^\perp \otimes v_\lambda.$$

Then, $I_{A,w}$ is the orthogonal $\langle \mathcal{B}_w \rangle^\perp$ of $\langle \mathcal{B}_w \rangle$ in $A_q[G/U]$. By Theorem 1.8, $I_{A,w}$ is an ideal of $A_q[G/U]$. We have the following decomposition for the quotient algebra:

$$A_q[G/U]/I_{A,w} = \bigoplus\nolimits_\lambda V_{A,w}(\lambda)^* \otimes v_\lambda.$$

Definition 1. Fix an element w in the Weyl group. Let $\tilde{w} = s_{i_1} \dots s_{i_p}$ be a reduced decomposition of w. Then there exists a reduced decomposition of w_0 such that $\tilde{w}_0 = s_{i_N} \dots s_{i_p} \dots s_{i_1}$. Such a reduced decomposition of $\tilde{w}_0$ will be regarded as adapted for w.

From [12, par 1], we have:

Theorem. *Fix w in W and let $\tilde{w}_0$ be a reduced decomposition of w_0 which is adapted to w. Then $A_{\tilde{w}_0}(B_w) = \mathcal{C} \cap (\mathbb{N}^p \times \{0\}^{N-p})$.* □

3. Specialization

3.1. Fix a reduced decomposition $\tilde{w}_0$ of w_0. At this stage we can construct a $\mathbb{N}^N$-filtration of the algebra $\mathbb{C}[G/U]$ such that the associated graded algebra is the algebra of the semigroup $\Gamma_{\tilde{w}_0}$. To be more precise, let $b^*_{\lambda,\psi}$ in $\mathbb{C}[G/U]$ be the image of the element $(b^{\psi}_{\lambda})^* \otimes v_\lambda$ by the morphism of specialization at $q = 1$; see Proposition 2.1 (iv). We have by Theorem 2.3 and (2.1.1):

Proposition. *The spaces $\mathcal{F}_\psi := \langle b^*_{\lambda,\phi},\ (\lambda,\phi) \in \Gamma_{\tilde{w}_0},\ \phi \prec \psi \rangle$, $\psi \in C$ define a filtration of the algebra $\mathbb{C}[G/U]$. The graded associated algebra is naturally isomorphic to the $\mathbb{C}$-algebra of the semigroup $\Gamma_{\tilde{w}_0}$.* □

3.2. What results from Proposition 3.1 is that there exists a finite sequence of degenerations of the flag variety which ends in a toric variety but what we want is a "degeneration in one step". Hence we need a linear form $\mathbb{N}^N \to \mathbb{N}$, which satisfies some strict inequalities, and which transforms the $\mathbb{N}^N$-filtration of $\mathbb{C}[G/U]$ into an $\mathbb{N}$-filtration. This is made possible because the cone $\Gamma_{\tilde{w}_0}$ is convex polyhedral and hence has a finite presentation. We start by a lemma.

Lemma. *Let S be a finite set of points in $\mathbb{N}^N$. Then there exists a linear form $e : \mathbb{N}^N \to \mathbb{N}$ such that $e(\phi) < e(\psi)$ whenever $\phi \prec \psi$ in S.*

Proof. Let $a_1, \dots, a_N$ be the coordinate linear forms on $\mathbb{N}^N$. Let C be an upper bound for all coordinates of all points in S. Choose positive integer $\varepsilon_1, \dots, \varepsilon_N$ such that $\varepsilon_k > C(\varepsilon_{k+1} + \dots + \varepsilon_N)$ for $1 \le k < N$. Then the linear form $e = \sum_k \varepsilon_k a_k$ obviously has the desired property. □

As a consequence of [2], see also [12], $\Gamma_{\tilde{w}_0}$ is the set of integral points of a polyhedral convex cone. Hence the semigroup $\Gamma_{\tilde{w}_0}$ has a (unique) minimal set of generators and a finite presentation. Let (λ_i, ψ_i), $1 \le i \le K$, be the minimal set of generators of $\Gamma_{\tilde{w}_0}$ and $(\sum_i n_i^k(\lambda_i,\psi_i) - \sum_i m_i^k(\lambda_i,\psi_i))$, n_i^k, $m_i^k \in \mathbb{N}$, be the (finite) minimal set of generators of the relations. Then, from Proposition 3.1, we obtain:

Proposition. *The (commutative) algebra $\mathbb{C}[G/U]$ is defined by generators $b^*_{\lambda_i,\psi_i}$, $1 \le i \le K$, and relations*

$$\prod b^{*n_i^k}_{\lambda_i,\psi_i} = \prod b^{*m_i^k}_{\lambda_i,\psi_i} + \sum c^k_{\mu,\phi} b^*_{\mu,\phi}, \quad \text{with } c^k_{\mu,\phi} \in \mathbb{C} \tag{3.2.1}$$

and $c^k_{\mu,\phi} \neq 0 \Rightarrow \phi \prec \sum_i n_i^k(\lambda_i,\psi_i) = \sum_i m_i^k(\lambda_i,\psi_i)$.

Proof. The algebra $\mathbb{C}[G/U]$ has a natural P^+-grading defined by $\mathbb{C}[G/U]_\lambda = V(\lambda)^* \otimes v_\lambda$. By construction, $\mathcal{F}_\psi = \oplus_\lambda \mathcal{F}_{\psi,\lambda}$, where $\mathcal{F}_{\psi,\lambda} = \mathcal{F}_\psi \cap \mathbb{C}[G/U]_\lambda$. Clearly, $\mathcal{F}_{\psi,\lambda}$ is finite dimensional. Hence by induction, we obtain that generators of the $\mathcal{C}$-graded algebra of Proposition 3.1 lift to generators of the algebra $\mathbb{C}[G/U]$. Hence we have the first part of the proposition.

Now Proposition 3.1 implies the relations (3.2.1). Let $\mathbb{C}[X_i, 1 \le i \le K]$ be the polynomial algebra with K indeterminates. There exists a surjective morphism φ : $\mathbb{C}[X_i, 1 \le i \le K]/J \to \mathbb{C}[G/U]$, where J is the ideal generated by the relations resulting from (3.2.1). This morphism maps X_i to $b^*_{\lambda_i,\psi_i}$ for all i. Endow $\mathbb{C}[X_i, 1 \le i \le K]/J$ with the quotient filtration $\langle \prod X_i^{m_i} + J,\ \sum m_i\psi_i \prec \psi \rangle_{\psi \in \mathcal{C}}$. Then, the associated graded algebra is defined by generators $\mathrm{Gr}X_i$, $1 \le i \le K$, and relations $\prod \mathrm{Gr}X_i^{n_i^k} = \prod \mathrm{Gr}X_i^{m_i^k}$. Now, endow $\mathbb{C}[G/U]$ with its $\mathcal{C}$-filtration, see 3.1. By Proposition 3.1, $\mathrm{Gr}\,\varphi$ is an isomorphism from $\mathrm{Gr}\,\mathbb{C}[X_i, 1 \le i \le K]/J$ onto $\mathrm{Gr}\,\mathbb{C}[G/U]$. This implies that φ is an isomorphism. This finishes the proof of the proposition. □

Let $\psi^k := \sum_i n_i^k \psi_i$, and for each k, let the $\phi^{k,l}$ be the elements ϕ occurring in (3.2.1). Fix an $\mathbb{N}$-form e as in Lemma 3.2 associated to the finite set the ψ^k and $\phi^{k,l}$.

Corollary. *The spaces $\mathcal{F}_m(\mathbb{C}[G/U]) = \langle \prod b_{\lambda_i,\psi_i}^{s_i^k},\ e(\sum_i s_i^k \psi_i)' \le m \rangle$, $m \in \mathbb{N}$, define a filtration of $\mathbb{C}[G/U]$. The graded associated algebra is $\mathbb{C}[\Gamma_{\tilde{w}_0}]$.* □

3.3. We now give some analogous results for Demazure modules. Fix w in W with length p. Set $\Gamma^w_{\tilde{w}_0} := A_{\tilde{w}_0}(\mathcal{B}_w)$.

Lemma. *For all reduced decompositions $\tilde{w}_0$ of w_0, the set $\Gamma^w_{\tilde{w}_0}$ is a finite union of rational convex polyhedral cones. Moreover, if $\tilde{w}_0$ is adapted to w, see Definition 2.4, then $\Gamma^w_{\tilde{w}_0}$ is a p-dimensional face of $\Gamma_{\tilde{w}_0}$. In particular, it is a rational convex polyhedral cone.*

Proof. Fix w in W and fix a reduced decomposition $\tilde{w}_0$ adapted to w. By Theorem 2.4, $\Gamma^w_{\tilde{w}_0}$ is a p-dimensional face of $\Gamma_{\tilde{w}_0}$ and so it is a convex polyhedral cone. Let $\tilde{w}_0'$ be another reduced decomposition of w_0 which is not assumed to be adapted to w. Then, by [2, 3.3], $\Gamma^w_{\tilde{w}_0'}$ is the image of $\Gamma^w_{\tilde{w}_0}$ by a continuous piecewise linear map l. To be more precise, there exists a finite set of (convex) cones $\mathcal{C}_i$ in $\mathbb{R}^{n+N}$ such that $\bigcup \mathcal{C}_i = \mathbb{R}^{n+N}_{\ge 0}$ and such that l is linear on $\Gamma^w_{\tilde{w}_0} \cap \mathcal{C}_i$. Hence we have the lemma. □

We still denote by v_λ a highest weight vector of the classical Weyl module $V(\lambda)$. The algebra $A_q[G/U]$ specializes for $q = 1$ onto the algebra $\mathbb{C}[G/U] = \bigoplus_\lambda V(\lambda)^* \otimes v_\lambda$. The algebra $A_q[G/U]/I_{A,w}$ specializes for $q = 1$ onto the quotient algebra $\mathbb{C}[G/U]/I_w = \bigoplus_\lambda V_w(\lambda)^* \otimes v_\lambda$, where $I_w := \bigoplus_{\lambda \in P^+} V_w(\lambda)^\perp \otimes v_\lambda$.

Let $\tilde{w}_0$ be adapted to w. By the previous lemma, $\Gamma^w_{\tilde{w}_0}$ is generated as a semigroup by a part of the minimal set of generators (λ_i, ψ_i) of $\Gamma_{\tilde{w}_0}$ and with the corresponding relations. This implies the following

Theorem. *Choose a reduced decomposition $\tilde{w}_0$ adapted to w. Then, the graded associated algebra of the quotient filtration of $(\mathcal{F}_m)_{m \in \mathbb{N}}$ on $\mathbb{C}[G/U]/I_w$ is the $\mathbb{C}$-algebra $\mathbb{C}[\Gamma^w_{\tilde{w}_0}]$ of the semigroup $\Gamma^w_{\tilde{w}_0}$.* □

Remark. Note that this filtration (and hence the corresponding degeneration) depends on the choice of the $\mathbb{N}$-form e of Lemma 3.2, and of the reduced decomposition $\tilde{w}_0$ adapted to w. For a general reduced decomposition $\tilde{w}_0'$ of w_0, Lemma 3.3 implies that the spectrum of the associated graded algebra is a union of irreducible components which are toric varieties.

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