Math. Z. 207, 341–358 (1991)

Mathematische Zeitschrift

Graded modules over G-sets II

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Received June 10, 1989; in final form May 15, 1990

Introduction

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring and A a left G-set. In the paper [8] the category of A-graded R-modules. (G, A, R)-gr was introduced and studied. An object of this category is a left R-module M together with a direct sum decomposition $M = \bigoplus_{x \in A} M_x$, where $\{M_x \mid x \in A\}$ is a family of subgroups of the additive group of M such that $R_\sigma M_x \subseteq M_{\sigma x}$ for all $\sigma \in G$ and $x \in A$. When A is the G-set G (G acts on itself by left translation) then (G, A, R)-gr is exactly the category R-gr of all graded R-modules (extensively studied in [6] and other works). An important example of G-set is $A = G/H$, where $H < G$ is a subgroup and G/H denotes the cosets of H in G (G acts on G/H by left translation). In this case, (G, A, R)-gr is denoted by $(G/H, R)$-gr. This category was studied for the first time by E. Dade in [2].

In the case of the category R-gr, a powerful tool for the study of this category is the ring denoted by $\tilde{R} \# G^*$, called the smash product of R by G. This ring was introduced for the case where G is a finite group by M. Cohen and S. Montgomery in [1], and extended for the general case by D. Quinn in [9]. The utility of this ring has been emphasized, in [5] and [7] and other recent publications.

In the paper [8] the smash product $R \# A$ is defined in case R is a G-graded ring and A is a finite G-set. In the first section of the present paper, we introduce the ring $R \# A$ in case A is an arbitrary G.set and R is a G-graded ring. When A is the G-set G, one obtains the smash product $\tilde{R} \# G^*$.

If the ring $\tilde{R} \# G^*$ proved to be a powerful tool for studying the category R-gr, in the case of the category (G, A, R)-gr, unfortunately, the smash product $R \# A$ is not a satisfactory tool. This happens mainly because in this generality one does not obtain a duality theorem of the Cohen-Montgomery type.

The main purpose of this paper is to show that a good tool for studying the category $(G/H, R)$-gr is provided by the ring $R\{H\} = (\tilde{R} \# G^*)\bar{H}$ which is obtained by adjoining to the smash product $\tilde{R} \# G^*$ a group of invertible elements of the matrix ring $M_G(R)$ isomorphic to the subgroup H of G. The idea for using this ring was provided by the constructions performed in [5], as well as by the Corollary 2.17 of [8], which stated that if G is a finite group, then the rings $(R\# G/H)$ and $R\{H\}$ are Morita equivalent.

The paper contains four sections. In the first section we define the smash product $R\# A$ for an arbitrary G-set. The construction follows the one given by D. Quinn for the smash product $\tilde{R}\# G^*$ in [9]. The section ends with an example showing that in this case the Cohen-Montgomery Duality Theorem does not function in general.

In Sect. 2 we introduce the ring $R\{H\}$ (which was introduced in fact by D. Quinn in [9]) and we define the functor $(-)^{\#,H}$: $(G/H, R)$-gr $\to R\{H\}-$mod. A series of properties of this functor is given.

In Sect. 3 we construct a right adjoint for the functor $(-)^{\#,H}$ and we show that the category $(G/H, R)$-gr is equivalent with a certain localizing subcategory of $R\{H\}$-mod. The main result of this section is Theorem 3.1. The constructions in Sects. 2 and 3 are inspired from the paper [5].

Section 4 contains a series of applications of Theorem 3.1. The main results of this section are Theorems 4.1 and 4.2.

1 Preliminaries and remarks on the smash product

All rings considered in this paper will be unitary. If R is a ring, by an R-module we will mean a left R-module, and we will denote the category of R-modules by R-mod. Let G be a multiplicative group with identity "1". A G-graded ring is a ring with identity 1, together with a direct sum decomposition (as additive subgroups) $R=\bigoplus_{\sigma\in G} R_\sigma$ such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau\in G$. If A is a left G-set by an A-graded module we will mean an R-module M together with a direct sum decomposition $M=\bigoplus_{x\in A} M_x$ such that $R_\sigma M_x \subseteq M_{\sigma x}$ for all $\sigma\in G$ and $x\in A$.

If $M=\bigoplus_{x\in A} M_x$ and $N=\bigoplus_{x\in A} N_x$ are A-graded R-modules, then a morphism $f\colon M\to N$ is an R-linear map such that $f(M_x)\subseteq N_x$ for all $x\in A$. The category (G, A, R)-gr consists of left A-graded R-modules with the morphisms defined as above. It is known [8] that (G, A, R)-gr is a Grothendieck catagory.

If $A=G$ with the natural left action of G on itself, then the category (G, A, R)-gr is exactly the category R-gr of all left G-graded R-modules. If H is a subgroup of G and $A=G/H$ is the set of all left H-cosets in G with the usual G-action on it (left translation), then we write $(G/H, R)$-gr for the category $(G, G/H, R)$-gr.

If A is a singleton with G acting trivially on it, then the category (G, A, R)-gr is exactly the category R-mod.

We note that if A is a left G-set, then the category (G, A, R)-gr is a direct product of categories of the form $(G/H, R)$-gr, where H is some subgroup of G, [8].

Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring an A a left G-set. Following the construction given by D. Quinn in [9] for the case where $A=G$, we define now the smash product of the ring R by the G-set A (the construction in the case where A is finite was done in [8]).

We denote by $M_A(R)$ the set of row and column finite matrices over R, with rows and columns indexed by the elements of A.

$M_A^*(R)$ is the ideal of $M_A(R)$ consisting of the matrices with only finitely many non-zero entries. Clearly, if A is finite, then $M_A^*(R)=M_A(R)$. If $\alpha\in M_A(R)$, then we write $\alpha(x, y)$ for the entry in the (x, y) position of α. For α, $\beta\in M_A(R)$, the matrix product is given by

$$(\alpha\beta)(x, y)=\sum_{z\in A}\alpha(x, z)\,\beta(z, y).$$

If x, $y\in A$, then we let $e_{x,y}$ denote the matrix with 1 in the (x, y)-position and zero elswehere. Let $p_x=e_{x,x}$. Define $\eta: R\to M_A(R)$ ba $\eta(r)=\tilde{r}$, where $\tilde{r}=\sum_{\sigma\in G}\sum_{x\in A} r_\sigma e_{\sigma x, x}$, $r=\sum_{\sigma\in G} r_\sigma$, $r_\sigma\in R_\sigma$ for any $\sigma\in G$.

We show that η is a ring homomorphism. Indeed, if $r_\sigma\in R_\sigma$, $r_\tau\in R_\tau$, then we have $\eta(r_\sigma r_\tau)=r_\sigma r_\tau\sum_{x\in A} e_{\sigma\tau x, x}$. But $\eta(r_\sigma)\,\eta(r_\tau)=r_\sigma r_\tau\sum_{x\in A} e_{\sigma x, x}\sum_{y\in A} e_{\tau y, y}$ $=r_\sigma r_\tau\sum_{x\in A}\sum_{y\in A} e_{\sigma x, x}e_{\tau y, y}=r_\sigma r_\tau\sum_{x\in A} e_{\sigma x, \tau^{-1}x}=r_\sigma r_\tau\sum_{y\in A} e_{\sigma\tau y, y}=\eta(r_\sigma r_\tau)$. We prove now that η is a monomorphism. Indeed, let r, $s\in R$, $r=\sum_{\sigma\in G} r_\sigma s=\sum_{\sigma\in G} s_\sigma$, r_σ, $s_\sigma\in R_\sigma$, such that $\eta(r)=\eta(s)$. Fix $y\in A$. Then for any $x\in A$ we denote by

$$G_{y,x}=\{\sigma\in G\,|\,\sigma x=y\}.$$

We have for $x\neq x'$ that $G_{y,x}\cap G_{y,x'}=\phi$ and $G=\bigcup_{x\in A} G_{y,x}$. Since $\eta(r)=\eta(s)$, then for any $x\in A$ we obtain that $\sum_{\sigma\in G_{y,x}} r_\sigma=\sum_{\sigma\in G_{y,x}} s_\sigma$ (the equality holds even if $G_{x,y}=\phi$ if we agree that a sum indexed by an empty family is zero). Since $r=\sum_{x\in A}\sum_{\sigma\in G_{y,x}} r_\sigma$, then $r=s$.

We denote by $\tilde{R}=\eta(R)$ and by $R\# A$ the subring of $M_A(R)$ generated by $\tilde{R}$ and the set $\{p_x\,|\,x\in A\}$.

If, r, $s\in R$, we have the equality:

$$(\tilde{r}p_x)(\tilde{s}p)=\begin{cases}(\sum\limits_{\substack{\sigma\in G\\ \sigma y=x}} rs_\sigma)\,p_y=r(\sum\limits_{\substack{\sigma\in G\\ \sigma y=x}} s_\sigma)\,p_y\\ 0\quad\text{if there is no }\sigma\text{ such that }\sigma y=x\end{cases}.$$

Indeed, it is sufficient to prove that

$$p_x(\tilde{s}p_y)=(\sum_{\substack{\sigma\in G\\ \sigma y=x}} s_\sigma)\,p_y.$$

Suppose there exists $\sigma \in G$ such that $\sigma y = x$. Then

$$\begin{aligned} p_x(\tilde{s} p_y) &= p_x\big(\sum_{\sigma\in G}\sum_{x\in A} s_\sigma e_{\sigma x' x}\big) p_y \\ &= e_{x,x}\big(\sum_{\sigma\in G}\sum_{z\in A} s_\sigma e_{\sigma z, z}\big) p_y \\ &= \big(\sum_{\sigma\in G} s_\sigma \sum_{z\in A} e_{x,x} e_{\sigma z, z}\big) p_y \\ &= \big(\sum_{\sigma\in G} s_\sigma e_{x,\sigma^{-1}x}\big) e_{y,y} \\ &= \big(\sum_{\substack{\sigma\in G\\ \sigma y = x}} s_\sigma\big) p_y. \end{aligned}$$

In particular, if $r_\sigma \in R_\sigma$, $x \in A$, we have

$$p_x \tilde{r}_\sigma = \tilde{r}_\sigma p_{\sigma^{-1}x}.$$

If $r \in R$ such that $\tilde{r} p_x = 0$ or $p_x \tilde{r} = 0$, then $r = 0$. Indeed, if $\tilde{r} p_x = 0$, then $(\sum_{\sigma\in G}\sum_{y\in A} r_\sigma e_{\sigma y, y}) e_{x,x} = 0$, and therefore $\sum_{\sigma\in G} r_\sigma e_{\sigma x, x} = 0$.

Thus $\sum_{\sigma\in G_{y,x}} r_\sigma = 0$ for all $y \in A$, and hence $r = \sum_{y\in A}\sum_{\sigma\in G_{y,x}} r_\sigma = 0$. We have that $G_{x,y} \cap G_{y',x} = 0$ for $y \neq y'$, and $G = \bigcup_{y\in A} G_{y,x}$.

Now if $p_x \tilde{r} = 0$, then we obtain $0 = p_x \sum_{\sigma\in G}\sum_{z\in A} r_\sigma e_{\sigma z, z} = \sum_{\sigma\in G} r_\sigma e_{x,\sigma^{-1}x}$, and hence $\sum_{\sigma\in G_{y,\sigma^{-1}x}} r_\sigma = 0$, so $r = 0$.

Thus the set of orthogonal idempotents $\{p_x \mid x \in A\}$ is a free set on the left and on the right over the ring $\tilde{R}$. Hence, if A is an infinite set, then

$$R \# A = \tilde{R} \oplus \big(\bigoplus_{x\in A} \tilde{R} p_x\big).$$

We denote by $\mathrm{Aut}_G(A)$ the group of G-automorphisms of the G-set A. If $\varphi \in \mathrm{Aut}_G(A)$ we define $\bar{\varphi} \in M_A(R)$ by $\bar{\varphi} = \sum_{x\in A} e_{x,\varphi(x)}$. If $\varphi \in \mathrm{Aut}_G(A)$ and $\alpha \in M_A(R)$, then for any $x, y \in A$ we have

$$(\bar{\varphi}^{-1}\alpha\bar{\varphi})(x, y) = \alpha(\bar{\varphi}^{-1}(x), \bar{\varphi}^{-1}(y)).$$

Indeed, if $\beta = \alpha\bar{\varphi}$, then

$$\beta = \sum_{v\in A} \alpha e_{v,\varphi(v)},$$

and therefore

$$\beta(x, y) = \sum_{v\in A}\sum_{z\in A} \alpha(x,) e_{v,\varphi(v)}(z, y) = \alpha(x, z) \quad \text{with } y = \varphi(z),$$

and hence $\beta(x, y) = \alpha(x, \bar{\varphi}^{-1}(y))$.

Now

$$\begin{aligned}(\bar{\varphi}^{-1}\alpha\bar{\varphi})(x,y)&=\sum_{u\in A}\sum_{z\in A}e_{u,\varphi^{-1}(u)}(x,z)\,\beta(z,y)\\&=\sum_{u\in A}\sum_{z\in A}e_{u,\varphi^{-1}(u)}(x,z)\,\alpha(z,\bar{\varphi}^{-1}(y))\\&=\sum_{z\in A}\sum_{u\in A}e_{u,\varphi^{-1}(u)}(x,z)\,\alpha(z,\bar{\varphi}^{-1}(y))\\&=\alpha(\varphi^{-1}(x),\varphi^{-1}(y)).\end{aligned}$$

If we denote by $R\{\mathrm{Aut}_G(\mathrm{A})\}$ the subring of $M_A(R)$ generated by $R\# A$ and the set $\{\bar{\varphi}\mid\varphi\in\mathrm{Aut}_G(A)\}$, then $R\{\mathrm{Aut}_G(\mathrm{A})\}=\sum_{\varphi\in\mathrm{Aut}_G(A)}(R\# A)\,\bar{\varphi}$, and the set $\{\bar{\varphi}\mid\varphi\in\mathrm{Aut}_G(A)\}$ is a set of normalizing elements for the ring $R\# A$.

Unfortunately, we do not have a Duality Theorem in the sense of Cohen-Montgomery over the ring $R\# A$, i.e. in general

$$R\{\mathrm{Aut}_G(A)\}\neq M_A(R).$$

For example, if $H<G$ is a subgroup of G such that $N(H)=H(N(H)$ is the normalizer of H in G) then by [8] we have $\mathrm{Aut}_G(G/H)\simeq N(H)/H=\{e\}$. Thus $R\{\mathrm{Aut}_G(\mathrm{A})\}=R\# A$, but $R\# A\neq M_A(R)$.

Using the same arguments as in [5] it may be shown that if A is an arbitrary G-set, then (G, A, R)-gr is isomorphic to some localizing subcategory of $R\# A$-mod. If A is finite, then (G, A, R)-gr is isomorphic to $R\# A$-mod (this result was obtained using different methods in [8]).

2 The functor $(.)^{\#,H}$

Let $R=\sum_{\sigma\in G}R_\sigma$ be a G-graded ring and H is a subgroup of G. Let $\tilde{R}\# G^*$ denote the smash product associated to the graded ring R, i.e. the smash product when the G-set A is the group G with the natural left action on itself.

$\tilde{R}\# G^*$ is a subring of $M_G(R)$. For any $g\in G$ we consider the element $\bar{g}\in M_G(R)$, $\bar{g}=\sum_{x\in G}e_{x,xg}$. Then $\bar{g}$ is a unit in $M_G(R)$, and $\bar{G}=\{\bar{g}\mid g\in G\}$ is a group isomorphic to G.

We denote by $R\{H\}$ the subring of $M_G(R)$ defined by the equality

$$R\{H\}=(\tilde{R}\# G^*)\,\bar{H}=\sum_{h\in H}(\tilde{R}\# G^*)\,\bar{h}.$$

It is well known from [9] that

$$R\{H\}=\sum_{h\in H}\tilde{R}\bar{h}\oplus R^*\{H\}$$

where $R^*\{H\}=\{\alpha\in M_G^*(R) \mid \alpha(x,y)\in R\langle xHy^{-1}\rangle\}$ (if X is a subset of G, then $R\langle X\rangle$ denotes the set $R\langle X\rangle=\bigoplus_{x\in X} R_x$).

We have the inclusions

$$\tilde{R}\# G^*\subset R\{H\}\subset R\{G\}\subset M_G(R).$$

When G is a finite group, we have $R\{G\}=M_G(R)$, which is exactly the Duality Theorem of Cohen and Montgomery.

We have the functor

$$\mathrm{Col}_G(-)\colon R\text{-mod}\to R\{G\}\text{-mod}$$

which is defined as follows: if $M\in R$-mod, then $\mathrm{Col}_G(M)$ is the set of column matrices over M with elements indexed by G and with finitely many non-zero entries. In fact, $\mathrm{Col}_G(M)$ is a left $M_G(R)$-module, and by restriction of scalars it is a left $R\{G\}$-module. This functor is exact. By [5] we have the functor

$$(-)^{\#}\colon R\text{-gr}\to\tilde{R}\# G^*-\text{mod}$$

which is defined as follows: if $M\in R$-gr, then M has a natural structure of on $\tilde{R}\# G^*$-module (see [5]) if for $m\in M$, $m=\sum_{x\in G} m_x (m_x\in M_x$ are the homogeneous components of m) and $\tilde{r}\in\tilde{R}$ we put $\tilde{r}m=rm$ and $p_x m=m_x$. We denote the module M considered with this structure by $M^{\#}$. The correspondence $M\to M^{\#}$ defines an exact functor $(-)^{\#}\colon R\text{-gr}\to\tilde{R}\# G^*$-mod.

If we denote by

$$(R\text{-gr})^{\#}=\{M\in\tilde{R}\# G^* \mid M=\bigoplus_{x\in G} p_x M\}$$

then by [5], $(R\text{-gr})^{\#}$ is a localizing subcategory of $\tilde{R}\# G^*$-mod (i.e. it is closed under subobjects, quotient objects, extensions and arbitrary direct sums). Since $(R\text{-gr})^{\#}$ is a localizing subcategory, then for any $\tilde{R}\# G^*$-module N there exists a largest $\tilde{R}\# G^*$-submodule $t_{\#}N$ of N such that $t_{\#}N\in(R\text{-gr})^{\#}$. In fact $t_{\#}N=\sum_{x\in G} p_x N$. Now if $N\in(R\text{-gr})^{\#}$, then N has a natural structure of a graded R-module if we consider N as an R-module via the morphism

$$\eta\colon R\to\tilde{R}\# G^*$$

and with the grading $N_x=p_x N$.

Thus we obtain a new functor

$$(-)_{\mathrm{gr}}\colon \tilde{R}\# G^*-\text{mod}\to R\text{-gr}$$

which sends $N\in\tilde{R}\# G^*$-mod to $t_{\#}N$ considered as a graded R-module.

It is showed in [5] that the functor $(-)_{\mathrm{gr}}$ is a right adjoint of the functor $(-)^{\#}$. Moreover, the functors $(-)^{\#}$ and $(-)_{\mathrm{gr}}$ define an equivalence (in fact an isomorphism) between the categories R-gr and $(R\text{-gr})^{\#}$.

We define now the functor

$$(-)^{\#,H}: (G/H, R)\text{-gr} \to R\{H\}\text{-mod}$$

as follows: if $M = \bigoplus_{C \in G/H} M_C \in (G/H, R)$-gr, then $M^{\#,H}$ is the subset of $\mathrm{Col}_G(M)$ such that the entries in the x-position belong to M_C if $x \in C$.

Lemma 2.1 *With the above notation, $M^{\#,H}$ is an $R\{H\}$-submodule of* $\mathrm{Col}_G(M)$.

Proof. Let $v \in M^{\#,H}$ and $\alpha \in R\{H\}$. We must prove that $\alpha v \in M^{\#,H}$. We can assume that $v_x \in M_C$, where $x \in C$ (v_x in the entry in the x-position of v), and $v_y = 0$ for $y \neq x$. Now if $\alpha = p_y$, we have

$$p_y v = \begin{cases} 0 & \text{if } y \neq x \\ v & \text{if } y = x \end{cases}$$

and therefore $p_y v \in M^{\#,H}$.

If $\alpha = \tilde{r}_\sigma$, $r_\sigma \in R_\sigma$, then we have

$$(\tilde{r}_\sigma v_t) = \sum_{y \in G} \tilde{r}_\sigma(t, y)\, v_y = \tilde{r}_\sigma(t, x)\, v_x.$$

But $\tilde{r}_\sigma(t, x) = r_\sigma$ if $\sigma = t x^{-1}$ and $\tilde{r}_\sigma(t, x) = 0$ if $\sigma \neq t x^{-1}$. Hence $(\tilde{r}_\sigma v)_t = r_\sigma v_x$ if $t = \sigma x$ and $(\tilde{r}_\sigma v)_t = 0$ if $t \neq \sigma x$. But $r_\sigma v_x \in M_{\sigma C}$, $\sigma x \in \sigma C$ and $\sigma C \in G/H$, therefore $\tilde{r}_\sigma v \in M^{\#,H}$.

Let now $\alpha = \bar{h}$, $h \in H$. Then we have $(\bar{h} v)_t = \sum_{y \in G} \bar{h}(t, y)\, v_y = \bar{h}(t, x)\, v_x$. Since $\bar{h} = \sum_{z \in G} e_{z, zh}$, we have $\bar{h}(t, x) = 1$ if $t = x h^{-1}$ and $h(t, x) = 0$ of $t \neq x h^{-1}$.

$$\text{Hence } (\bar{h} v)_t = \begin{cases} v_x & \text{if } t = x h^{-1} \\ 0 & \text{if } t \neq x h^{-1} \end{cases}$$

and thus $(\bar{h} v)_{x h^{-1}} = v_x$ and $(\bar{h} v)_t = 0$ if $t \neq x h^{-1}$, since $x \in C$, then $C = xH$, and we note that $x h^{-1} H = xH = C$, i.e. $x h^{-1} \in C$. Hence $\bar{h} v \in M^{\#,H}$, and therefore $M^{\#,H}$ is an $R\{H\}$-submodule of $\mathrm{Col}_G(M)$.

Remark 2.1 If $H = \{1\}$, then $M^{\#,H}$ is exactly the $\tilde{R} \# G^*$-module $M^\#$. If $H = G$, then $M^{\#,G}$ is exactly the $R\{G\}$-module $\mathrm{Col}_G(M)$.

Lemma 2.2 *The correspondence $M \to M^{\#,H}$ defines an exact functor* $(-)^{\#,H}: (G/H, R)\text{-gr} \to R\{H\}\text{-mod}$.

Proof. If $M, N \in (G/H, R)$-gr and $f \in \mathrm{Hom}_{(G/H, R)\text{-gr}}(M, N)$, then it is obvious that $\mathrm{Col}_G(f)(M^{\#,H}) \subseteq N^{\#,H}$, and hence the correspondence $M \to M^{\#,H}$ defines a covariant functor. It is obvious that this functor is exact.

Let now $\varphi: G \to G/H$ be the canonical map $\varphi(g) = gH$ for $g \in G$, which is a morphism of G-sets. By [8] we can associate to the map the canonical functor

$$(-)_\varphi: R\text{-gr} \to (G/H, R)\text{-gr}$$

defined as follows: if $M\in R$-gr, $M=\bigoplus_{x\in G} M_x$, then $M_\varphi=M$ as R-modules, and M_φ has the G/H-grading $M=\bigoplus_{C\in G/H} M_C$ where $M_C=\bigoplus_{x\in C} M_x$.

By Theorem 3.1 of [8] we have that $(-)_\varphi$ has a right adjoint. This right adjoint may be constructed using the functor $(-)^{\#,H}$ as follows:

Proposition 2.1 *The functor* $(-)_{\mathrm{gr}}\circ i_*\circ(-)^{\#,H}\colon (G/H, R)\text{-gr}\to R\text{-gr}$ *is a right adjoint of the functor* $(-)_\varphi$. *(Here* i_* *denotes the restriction of scalars functor asociated to the inclusion morphism* $i\colon \tilde{R}\# G^*\hookrightarrow R\{H\}$*).*

Proof. By the construction of the right adjoint of the functor $(-)_\varphi$ given in Theorem 3.1 of [8] it is easy to see that this is exactly the functor $(-)_{\mathrm{gr}}\circ i_*\circ(-)^{\#,H}$.
Assume now that H is a finite subgroup of G. If $M=\bigoplus_{C\in G/H} M_C$ is an object of $(G/H, R)$-gr, then these exists a canonical map

$$\alpha_M\colon M\to M^{\#,H}$$

defined as follows: if $m\in M_C$, then $\alpha_M(m)$ is the column matrix such that for any $x\in C$ the entry in the x-position is m and the other entries are zero. Since C is a finite set then α_M is well defined.

Lemma 2.3 *Assume that H is a finite subgroup of G. Then $\alpha_M\colon M\to M^{\#,H}$ is an R-monomorphism (here $M^{\#,H}$ is considered as an R-module via the morphisms of rings $R\xrightarrow{\eta}\tilde{R}\# G^*\xrightarrow{i} R\{H\}$).*

Proof. Let $m\in M_C$ and $a_\sigma\in R_\sigma$. If $C=xH$, then $\sigma C=\sigma xH$ and thus if we denote by $v=\alpha_M(a_\sigma m)$, then for any $y\in\sigma C$ we have $v_y=a_\sigma m$ and if $y\notin\sigma C$, then $v_y=0$.

On the other hand, $a_\sigma\alpha_M(m)=\tilde{a}_\sigma(\ldots, m, \ldots)^t\to x(x\in C)$. But $\tilde{a}_\sigma(\ldots, m, \ldots)^t-x=(\ldots, a_\sigma m, \ldots)^t-\sigma x$, and therefore $\alpha_M(a_\sigma m)=a_\sigma\alpha_M(m)$. It is obvious that α_M is injective.

Remark 2.2 In fact α_M is morphism in the category $(G/H, R)$-gr, of we consider $M^{\#,H}$ as an object of $(G/H, R)$-gr via the functors $R\{H\}\text{-mod}\xrightarrow{i_*}\tilde{R}\# G^*-\text{mod}\xrightarrow{(-)_{\mathrm{gr}}} R\text{-gr}\xrightarrow{(-)_\varphi}(G/H, R)\text{-gr}$ where $\varphi\colon G\to G/H$ is the canonical map of G-sets.

Lemma 2.4 *Let $K<H<G$ be subgroups, $\phi\colon G/K\to G/H$ the canonical morphism of G-sets ($\phi(\sigma K)=\sigma H$, $\sigma\in G$). If $M\in(G/K, R)$-gr, then M may be viewed as on object from $(G/H, R)$-gr via the canonical functor $(-)_\phi\colon (G/K, R)\text{-gr}\to(G/H, R)\text{-gr}$. We have that*

$$M^{\#,H}\simeq R\{H\}\bigotimes_{R\{K\}} M^{\#,K}\quad\text{as } R\{H\}\text{-modules.}$$

Proof. We define the map

$$\gamma\colon R\{H\}\bigotimes_{R\{K\}} M^{\#,K}\to \mathrm{Col}_G(M)$$

by the equality $\gamma(\alpha\otimes v)=\alpha v$, $\alpha\in R\{H\}$, $v\in M^{\#,K}$. First we mut prove that $\gamma(\alpha\otimes v)\in M^{\#,H}$. It is obvious that we may assume that the entry of v in the x-position is $v_x\in M_{xK}$ and all the other entries of v are zero. If $\alpha=p_y$, then

$$\alpha v=\begin{cases} v & \text{if } y=x \\ 0 & \text{if } y\neq x \end{cases}$$

and $v\in M^{\#,H}$, since $M_{xK}\subseteq M_{xH}$ by the definition of the functor $(-)_\phi$. If $\alpha=\tilde{a}_\sigma$, then the entry in the t-position of $\tilde{a}_\sigma v$ is

$$(\tilde{a}_\sigma v)_t=\begin{cases} a_\sigma v_x & \text{if } t=\sigma x \\ 0 & \text{if } t\neq\sigma x. \end{cases}$$

But $a_\sigma v_x\in M_{\sigma xK}\subseteq M_{\sigma xH}$, hence $\tilde{a}_\sigma v\in M^{\#,H}$. Now if $\alpha=\bar{h}$, $h\in H$, then the entry in the t-position of $\bar{h}v$ is

$$(\bar{h}v)_t=\begin{cases} v_x & \text{if } t=xh^{-1} \\ 0 & \text{if } t\neq xh^{-1}. \end{cases}$$

But $v_x\in M_{xK}\subseteq M_{xH}=M_{xh^{-1}H}$ because $h\in H$, so $\bar{h}v\in M^{\#,H}$. In conclusion, $\gamma(\alpha\otimes v)\in M^{\#,H}$, and so γ may be considered as a map from $R\{H\}\bigotimes_{R\{K\}}M^{\#,K}$ to $M^{\#,H}$, which is clearly a morphism of $R\{H\}$-modules.

We prove now that γ is bijective. In order to show that γ is injective, let us remark first that if $\{\sigma_i\}_{i\in I}$ is a set of representatives for the left cosets of K in H, then clearly we have that $R\{H\}=\sum_{i\in I}\bar{\sigma}_i R\{K\}$. Now if $\sum_{i\in I}\bar{\sigma}_i a_i=0$, where $a_i\in R\{K\}$, then we can write $a_i=\sum_{j\in I_i}\bar{k}_{ij}a_{ij}$, where $k_{ij}\in K$, $k_{ij}\neq k_{ij}$, for $j\neq j'$, and $a_{ij}\in\tilde{R}\#G^*$. But $0=\sum i\in I\,\bar{\sigma}_i a_i=\sum_{i\in I}\sum_{j\in I_i}\bar{\sigma}_i\bar{k}_{ij}a_{ij}$. For $i\neq i'$ we have that $\sigma_i k\neq\sigma_i k'$ for any $k,k'\in K$, and for $j\neq j'$ we have that $\sigma_i k_{ij'}$. Since the set $\{\bar{g}\,|\,g\in G\}$ is free over $\tilde{R}\#G^*$, then $a_{ij}=0$ for any $i\in I$, $j\in I_i$. Hence $a_i=0$ for any $i\in I$. So $R\{H\}=\bigoplus_{i\in I}\bar{\sigma}_i R\{K\}$. Therefore every element from $R\{H\}\bigotimes_{R\{K\}}M^{\#,H}$ has the form $u=\sum_{i\in I}\bar{\sigma}_i\otimes v_i$, where $v_i\in M^{\#,K}$.

Assume that $\gamma(u)=0$. Then $\sum_{i\in I}\bar{\sigma}_i v_i=0$. If $x\in G$, then the entry of $\sum_{i\in I}\bar{\sigma}_i v_i$ in the x-position is the sum (indexed $by\, i\in I$) of the entries in the x-position of $\sigma_i v_i$. The entry in the x-position of $\bar{\sigma}_i v_i$ is equal to the entry in the $x\sigma_i$-position of v_i. Since for $i\neq j$ we have $x\sigma_i\neq x\sigma_j$, we obtain that the entry of v_i in the $x\sigma_i$-position is zero for any $i\in I$. It follows that if we fix $i\in I$, then the entry of v_i in the $x\sigma_i$-position is zero for all $x\in G$. If $y\in G$, then there exists $x\in G$ such that $y=x\sigma_i$, so the entry of v_i in the y-position is zero. It follows that $v_i=0$, and so $u=0$, i.e. γ is injective. We prove now that γ is surjective. Let $v\in M^{\#,H}$. We can assume that the entry of v in the x-position ($x\in G$) is $m\in M_{xH}$ and all the other entries of v are zero. Now we have that $m\in M_{xH}=\bigoplus_{i\in I}M_{x\sigma_i K}$, so $m=\sum_{i\in I}m_i$, where $m_i\in M_{x\sigma_i K}$. We put $v_i\in M^{\#,K}$ as follows: the entry of v_i

in the $x\sigma_i$ position is m_i, and all the other entries are zero. Then $\gamma(u)=\sum_{i\in I}\bar{\sigma}_i v_i$. We let $u=\sum_{i\in I}\bar{\sigma}_i\otimes v_i$. If we denote for each $w\in M^{\#,H}$ and $t\in G$ by w_t the entry of W in the t-position, then $\gamma(u)_x=(\sum_{i\in I}\bar{\sigma}_i v_i)_x=\sum_{i\in I}(\bar{\sigma}_i v_i)_x=\sum_{i\in I}(v_i)_{x\sigma_i}=\sum_{i\in I}m_i=m$, and $\gamma(u)_y=\sum_{i\in I}(v_i)_{y\sigma_i}=0$ for $y\neq x$. So $\gamma(u)=v$ and hence γ is surjective too.

Remarks 2.3 a) By Lemma 2.4, if $K<H<G$ are subgroups, ϕ: $G/K\to G/H$ is the canonical map of G-sets, $(-)_\phi$: $(G/K, R)$-gr $\to(G/H, R)$-gr is the associated functor, then we have the following commutative diagram of functors:

$$\begin{array}{ccc}
(G/K, R)\text{-gr} & \xrightarrow{(-)_\phi} & (G/H, R)\text{-gr} \\
{\scriptstyle (-)^{\#,K}}\downarrow & & \downarrow{\scriptstyle (-)^{\#,H}} \\
R\{K\}\text{-mod} & \xrightarrow{R\{H\}\otimes_{R\{K\}}-} & R\{H\}\text{-mod}
\end{array}$$

b) Taking $H=G$ we obtain that for any $M\in(G/K, R)$-gr-mod we have $R\{G\}\bigotimes_{R\{K\}}M^{\#,K}\simeq\mathrm{Col}_G(M)$ as $R\{G\}$-modules, so the following diagram of functors is commutative

$$\begin{array}{ccc}
(G/K, R)\text{-gr} & \xrightarrow{U} & R\text{-mod} \\
{\scriptstyle (-)^{\#,K}}\downarrow & & \downarrow{\scriptstyle \mathrm{Col}_G(-)} \\
R\{K\}\text{-mod} & \xrightarrow{R\{K\}\otimes_{R\{K\}}-} & R\{G\}\text{-mod}
\end{array}$$

where U is the forgetful functor.

c) Taking $K=\{1\}$ we obtain that for any $M\in R$-gr we have $R\{H\}\bigotimes_{\tilde{R}\#G^*}M^\#\simeq M^{\#,H}$ as $R\{H\}$-modules, so the following diagram of functors is commutative:

$$\begin{array}{ccc}
R\text{-gr} & \xrightarrow{(-)_\phi} & (G/H, R)\text{-gr} \\
{\scriptstyle (-)^{\#}}\downarrow & & \downarrow{\scriptstyle (-)^{\#,H}} \\
\tilde{R}\#G^*\text{-mod} & \xrightarrow{R\{H\}\otimes_{\tilde{R}\#G^*}-} & R\{H\}\text{-mod}
\end{array}$$

where φ: $G\to G/H$ denotes the canonical map.

d) If $\{\sigma_i\}_{i\in I}$ denotes as in the proof of Lemma 2.4 a set of representatives for the left cosets of K in H, then from the proof of the Lemma 2.4 we get that $R\{H\}\bigotimes_{R\{K\}}M^{\#,K}\simeq\bigoplus_{i\in I}\bar{\sigma}_i\otimes M^{\#,K}$ as $R\{K\}$-modules whenever K is a normal subgroup of H. Indeed, if $K\lhd H$, then $R\{K\}\bar{\sigma}_i=\bar{\sigma}_i R\{K\}$, and so $\bar{\sigma}_i\otimes M^{\#,K}$ is an $R\{K\}$-module for each $i\in I$.

In particular, if $K=\{1\}$, then we obtain that $R\{H\}\bigotimes_{\tilde{R}\#G^*}M^*\simeq\bigoplus_{h\in H}\bar{h}\otimes M^\#$ as $\tilde{R}\#G^*$-modules.

We recall now that if $M=\bigoplus_{x\in G} M_x$ is an object of the category R-gr and $\sigma\in G$, then $M(\sigma)$ denotes the graded module obtained from M putting $M(\sigma)_\lambda = M_{\lambda\sigma}$. The graded module $M(\sigma)$ is called the σ-suspension of M.

Lemma 2.5 *Assume that $M\in R$-gr and $H<G$. Then*

$$M^{\#,H} \simeq \bigoplus_{h\in H} M(h)^{\#} \quad \textit{as } \tilde{R}\#G^*\text{-modules}$$

Proof. By Lemma 2.4 and Remarks 2.3, c), d) we have that

$$M^{\#,H} \simeq R\{H\} \bigotimes_{\tilde{R}\#G^*} M^{\#} \simeq \bigoplus_{h\in H} \bar{h}\otimes M^{\#}, \quad \text{as} \quad \tilde{R}\#G^*\text{-modules}$$

(each $\bar{h}\otimes M^{\#}$ is an $\tilde{R}\#G^*$-module). We show that for any $h\in H$ we have that $\bar{h}\otimes M^{\#} \simeq M(h)^{\#}$.

Indeed, we define the map $\delta\colon \bar{h}\otimes M^{\#} \to \mathrm{Col}_G(M)$ by

$$\delta(\bar{h}\otimes v) = \bar{h}v \qquad \text{for any } v\in M^{\#}.$$

It is easy to see that $\delta(\bar{h}\otimes M^{\#}) \subseteq M(h)^{\#}$ and that δ is bijective. We prove that δ is an $\tilde{R}\#G^*$-morphism. Indeed, we have $\delta(p_x(\bar{h}\otimes v)) = \delta(p_x\bar{h}\otimes v) = \delta(\bar{h}p_{xh}\otimes v) = \delta(\bar{h}\otimes p_{xh}v) = \bar{h}p_{xh}v = p_x\bar{h}v = p_x\delta(\bar{h}\otimes v)$, and $\delta(\tilde{a}_\sigma(\bar{h}\otimes v)) = \delta(\tilde{a}_\sigma\bar{h}\otimes v) = \delta(\bar{h}\tilde{a}_\sigma\otimes v) = \delta(\bar{h}\otimes\tilde{a}_\sigma v) = \bar{h}(\tilde{a}_\sigma v) = \tilde{a}_\sigma\bar{h}v = \tilde{a}_\sigma\delta(\bar{h}\otimes v)$.

3 The construction of a right adjoint functor of the functor $(-)^{\#,H}$

In this section we show that the functor $(-)^{\#,H}$ has a right adjoint and that the category $(G/H, R)$-gr is equivalent, via the functor $(-)^{\#,H}$ with some subcategory of $R\{H\}$-mod.

We denote by

$$\mathscr{C}^{\#,H} = \{M\in R\{H\}\text{-mod} \mid M = \sum_{x\in G} p_x M\}.$$

It is easy to see that $\mathscr{C}^{\#,H}$ is a localizing subcategory of $R\{H\}$-mod. In particular, if $H=\{1\}$, then we obtain the localizing subcategory $\mathscr{C}^{\#}$ of $\tilde{R}\#G^*$-mod which is studied in [5]. In fact $\mathscr{C}^{\#}$ is exactly $(R\text{-gr})^{\#}$ defined in § 2. Since $\tilde{R}\#G^* \subseteq R\{H\}$ we have the canonical functors between the categories $\mathscr{C}^{\#}$ and $\mathscr{C}^{\#,H}$

$$\mathscr{C}^{\#} \underset{i_*}{\overset{R\{H\}\bigotimes_{\tilde{R}\#G^*} -}{\rightleftarrows}} \mathscr{C}^{\#,H}$$

where i_* is the functor "restriction of scalars".

Indeed if $M\in\mathscr{C}^{\#,H}$, then it is obvious that $i_*(M)\in\mathscr{C}^{\#}$. Now if $N\in\mathscr{C}^{\#}$, we denote $M=R\{H\}\bigotimes_{\tilde{R}\#G^*} N$. If $m\in M$, then $m=\sum_{i\in I}\bar{h}_i\otimes n_i$, where h_i, where $h_i\in H$, $n_i\in N$.

Thus $p_x m = \sum_{i\in I} p_x \bar{h}_i \otimes n_i = \sum_{i\in I} \bar{h}_i p_{xh_i} \otimes n_i = \sum_{i\in I} \bar{h}_i \otimes p_{xh_i} n_i$, and therefore $\sum_{x\in G} p_x m = \sum_{i\in I} \bar{h}_i \otimes \sum_{x\in G} p_{xh_i} n_i$.

Since $N \in \mathscr{C}^{\#}$, then $n_i = \sum_{x\in G} p_{xh_i} n_i$ and therefore $\sum_{x\in G} p_x m = \sum_{i\in I} \bar{h}_i \otimes n_i = m$. Hence $M \in \mathscr{C}^{\#,H}$. The following result will be very useful in the sequel:

Lemma 3.1 *Let $M \in \mathscr{C}^{\#,H}$, $x \in G$ and $h \in H$. Then there exists a canonical isomorphism*

$$\varphi_{x,h}: p_x M \to p_{xh} M.$$

Proof. If $m \in M$ we define $\varphi_{x,h}(p_x m) = p_{xh} \bar{h}^{-1} m$. We check that $\varphi_{x,h}$ is well defined. Indeed, if $p_x m = p_x m'$, then $\bar{h}^{-1} p_x m = \bar{h}^{-1} p_x m'$. Since $\bar{h}^{-1} p_x = p_{xh} \bar{h}^{-1}$, then we obtain that $p_{xh} \bar{h}^{-1} m = p_{xh} \bar{h}^{-1} m'$.

Now it is easy to see that $\varphi_{x,h}$ is bijective. We have that $\varphi_{x,h}^{-1}: p_{xh} M \to p_x M$ is the map given by $\varphi_{x,h}^{-1}(p_{xh} m) = p_x \bar{h} m$. Let now $\{\sigma_i\}_{i\in I}$ be a left transversal for H in G. If $M \in \mathscr{C}^{\#,H}$. We define the abelian group $M_0 = \bigoplus_{i\in I} p_{\sigma i} M$. We introduce now an R-module structure on M_0. Let $a_\lambda \in R_\lambda$, $m \in M$. Then we have $\tilde{a}_\lambda p_{\sigma_i} m = p_{\lambda\sigma_i} \tilde{a}_\lambda m$. Since $\lambda\sigma_i = \sigma_j h$ where $h \in H$ and σ_j is uniquely determined by σ_i and λ, we have that $\tilde{a}_\lambda p_{\sigma_i} m = p_{\sigma_j} h \tilde{a}_\lambda m$. By Lemma 3.1 we have the isomorphism $\varphi_{\sigma_j,h}^{-1}: p_{\sigma_j h} M \to p_{\sigma_j} M$. Hence we can define the product

$$a_\lambda * p_{\sigma_i} m = \varphi_{\sigma_j,h}^{-1}(p_{\sigma_j h} \tilde{a}_\lambda m) = p_{\sigma_j} \bar{h} \tilde{a}_\lambda m.$$

Now if $a_\mu \in R_\mu$ and $a_\lambda \in R_\lambda$, then $a_\mu * (a_\lambda * p_{\sigma_i} m) = a_\mu * (p_{\sigma_i} \bar{h} \tilde{a}_\lambda m)$ if $\lambda\sigma_i = \sigma_i h$. Hence $a_\mu * (p_{\sigma_j} \bar{h} \tilde{a}_\lambda m) = p_{\sigma_k} \bar{h}' \tilde{a}_\mu \bar{h} \tilde{a}_\lambda m$ if $\mu\sigma_j = \sigma_k h'$, so $a_\mu * (a_\lambda * p_{\sigma_i} m) = p_{\sigma_k} \tilde{a}\mu \tilde{a}\lambda \bar{h}' \bar{h} m = p_{\sigma_k} \overline{h' h} \tilde{a}_\mu \tilde{a}_\lambda m$. Since $\lambda\sigma_i = \sigma_j h$, then $\mu\lambda\sigma_i = \mu\sigma_j h = \sigma_k h' h$, and so we have $(a_\mu a_\lambda) * (p_{\sigma_i} m) = p_{\sigma_k} \overline{h' h} \tilde{a}_\mu \tilde{a}_\lambda m$.

Hence $a_\mu * (a_\lambda * p_{\sigma_i} m) = (a_\mu a_\lambda) * p_{\sigma_i} m$.

Thus M_0 has a canonical structure of a left R-module. Moreover, M_0 is an object of the category $(G/H, R)$-gr. Indeed, if $C \in G/H$, then there exists a unique σ_i such that $C = \sigma_i H$. We put $(M_0)_c = p_{\sigma_i} M$. Hence $M_0 = \bigoplus_{C\in G/H} (M_0)_C$. If $a_\lambda \in R_\lambda$ we have $a_\lambda * (M_0)_C = a_\lambda * p_{\sigma_i} M = p_{\lambda\sigma_i} \tilde{a}_\lambda M = p_{\sigma_i} \bar{h} \tilde{a}_\lambda M$. Since $\lambda C = \lambda\sigma_i H = \sigma_j h H = \sigma_j H$, we have that $a_\lambda * (M_0)_C \subseteq (M_0)_{\lambda C}$, and therefore M is an object of $(G/H, R)$-gr.

Since $\mathscr{C}^{\#,H}$ is a localizing subcategory, if $M \in R\{H\}$-mod, then there exists the largest $R\{H\}$-submodule of M, $t_{\#,H}(M)$, such that $t_{\#,H}(M) \in \mathscr{C}^{\#,H}$. It is obvious that $t_{\#,H}(M) = \sum_{x\in G} p_x M = \bigoplus_{x\in G} p_x M$.

Now if $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is an exact sequence in $R\{H\}$-mod, then we have exact sequence

$$0 \to t_{\#,H}(M') \xrightarrow{u} t_{\#,H}(M) \xrightarrow{v} t_{\#,H}(M'') \to 0.$$

Indeed, let $m''\in t_{\#,H}(M'')$. Then $m''=\sum_{i=1}^{n} p_{x_i}m_i''$. If $m_i\in M$ such that $v(m_i)=m_i''$ (v is surjective), then we put $m=\sum_{i=1}^{n} p_{x_i}m_i$. We have clearly that $m\in t_{\#,H}(M)$ and $v(m)=m''$.

Now we can define the functor

$$(-)_{G/H}\colon R\{H\}\text{-mod}\to(G/H,\,R)\text{-gr}$$

by $(M)_{G/H}=M_0$ defined as above. It is easy to see that the functor $(-)_{G/H}$ is exact. We have the main result of this section:

Theorem 3.1 *With the above notation, $(-)_{G/H}$ is a right adjoint of the functor $(-)^{\#,H}$. Moreover, the functors $(-)^{\#,H}$ and $(-)_{G/H}$ define an equivalence between the categories* (G/H, R)-gr *and $\mathscr{C}^{\#,H}$.*

Proof. We define the functorial morphisms

$$\operatorname*{Hom}_{R\{H\}\text{-mod}}((-)^{\#,\mathrm{H}},-)\underset{\beta}{\overset{\alpha}{\rightleftarrows}}\operatorname*{Hom}_{(G/H,\,R)\text{-gr}}(-,(-)_{\mathrm{G/H}})$$

as follows: if $M\in(G/H,R)$-gr and $N\in R\{H\}$-mod, then

$$\alpha(M,N)\colon \operatorname*{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)\to\operatorname*{Hom}_{(G/H,\,R)\text{-gr}}(M,N_{G/H})$$

is defined in the following way: if $u\in\operatorname*{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)$, then $u(M^{\#,H})\subseteq t_{\#,H}(N)$. If $\{\sigma_i\}_{i\in I}$ is a left transversal of H in G, then $M=\bigoplus_{i\in I}M_i$, where $M_i=M_{\sigma_i H}$. If $m\in M$, then $m=\sum_{i\in I}m_i$, $m_i\in M_i$. We put $\alpha(M,N)(u)(m)=\sum_{i\in I}u(\tilde m_i)$, where $\tilde m_i=(0,\ldots,0,m_i,\ldots,0)^t-\sigma_i$.

We have that $\alpha(M,N)(u)(m)\in N_{G/H}$, since $u(\tilde m_i)=u(p_{\sigma_i}\tilde m_i)=p_{\sigma_i}u(\tilde m_i)\in p_{\sigma_i}N$.

If $a_\lambda\in R_\lambda$, then $a_\lambda m_i\in M_{\lambda\sigma_i H}=M_{\sigma_j hH}=M_{\sigma_j H}$ if $\lambda\sigma_i=\sigma_j h$.

We have that $\tilde a_\lambda\tilde m_i=(0,\ldots,0,a_\lambda m_i,0,\ldots)^t-\lambda\sigma_i$ and do $\bar h\tilde a_\lambda\tilde m_i=(0,\ldots,0,a_\lambda m_i,\ldots,0)^t-\sigma_j=\widetilde{a_\lambda m_i}$. Now $\alpha(M,N)(u)(a_\lambda m)=\sum_{i\in I}u(\widetilde{a_\lambda m_i})=\sum_{i\in I}u(\bar h\tilde a_\lambda\tilde m_i)=\sum_{i\in I}\bar h\tilde a_\lambda u(\tilde m_i)=\sum_{i\in I}p_{\sigma_j}\bar h\tilde a_\lambda u(\tilde m_i)=\sum_{i\in I}a_\lambda*\alpha(M,N)(u)(m_i)=a_\lambda*\alpha(M,N)(u)(m)$. Hence $\alpha(M,N)(u)\in\operatorname*{Hom}_{(G/H,\,R)\text{-gr}}(M,N_{G/H})$. We define now

$$\beta(M,N)\colon \operatorname*{Hom}_{(G/H,\,R)\text{-gr}}(M,N_{G/H})\to\operatorname*{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)$$

as follows:

if $v\in\operatorname*{Hom}_{(G/H,\,R)\text{-gr}}(M,N_{G/H})$, then it will be sufficient to define $\beta(M,N)(v)$ on elements of $M^{\#,H}$ of the form $\tilde m=(0,\ldots,0,m_x,\ldots,0)^t-x$ where $m_x\in M_C$, $C=xH$. There exists $i\in I$ such that $C=\sigma_i H$, i.e. $x=\sigma_i h$ for some $h\in H$. We have that $v(m_x)\in(N_{G/H})_C=p_{\sigma_i}N$. By Lemma 3.1 we have the canonical isomorphism $\varphi_{\sigma_i,h}\colon p_{\sigma_i}N\to p_xN$, so we put $\beta(M,N)(v)(\tilde m)=\varphi_{\sigma_i,h}(v(m_x))=p_x\bar h^{-1}v(m_x)$.

We must prove that $\beta(M,N)(v)\in \operatorname{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)$. If $\alpha\in R\{H\}$ and $\alpha=p_y$ then

$$p_y\tilde{m}=\begin{cases}\tilde{m} & \text{if } y=x\\ 0 & \text{if } y\neq x.\end{cases}$$

So for $y=x$, $\beta(M,N)(v)(p_y\tilde{m})=\beta(M,N)(v)(\tilde{m})=\varphi_{\sigma_i,h}(v(m_x))$.
But $p_y\beta(M,N)(v)(\tilde{m})=p_y\varphi_{\sigma_i,h}(v(m_x))=\varphi_{\sigma_i,h}(v(m_x))$.

If $\alpha=\tilde{a}_\lambda$ then $\tilde{n}=\tilde{a}_\lambda\tilde{m}=(0,\ldots,0,a_\lambda m_x,\ldots,0)^t-\lambda x$. Since $\lambda x=\lambda\sigma_i h=\sigma_j h'h$, where $\lambda\sigma_i=\sigma_j h'$, then $\beta(M,N)(v)(\tilde{a}_\lambda\tilde{m})=\varphi_{\sigma_{j'}h'h}(v(a_\lambda m_x))=\varphi_{\sigma_{j'}h'h}(a_\lambda * v(m_x))$ $=\varphi_{\sigma_{j'}h'h}(p_{\sigma_j}\bar{h}'\tilde{a}_\lambda v(m))=p_{\lambda x}\bar{h}^{-1}\bar{h}'^{-1}p_{\sigma_j}h'$. $\tilde{a}_\lambda v(m_x)=p_{\lambda x}\bar{h}^{-1}p_{\sigma_j h'}\tilde{a}_\lambda v(m_x)$ $=p_{\lambda x}\bar{h}^{-1}p_{\lambda\sigma_i}\tilde{a}_\lambda v(m_x)=p_{\lambda x}\bar{h}^{-1}\tilde{a}_\lambda$. $p_{\sigma_i}v(m_x)=p_{\lambda x}\tilde{a}_\lambda\bar{h}^{-1}p_{\sigma_i}v(m_x)=\tilde{a}_\lambda p_x\bar{h}^{-1}p_{\sigma_i}v(m_x)$. Since $v(m_x)\in(N_{G/H})_C=p_{\sigma_i}N$, then $p_{\sigma_i}v(m_x)=v(m_x)$, and thus $\beta(M,N)(v)(\tilde{a}_\lambda\tilde{m})$ $=\tilde{a}_\lambda p_x\bar{h}^{-1}v(m_x)=\tilde{a}_\lambda\beta(M,N)(v)(\tilde{m})$.

Now if $\alpha=\bar{k}$, where $k\in H$, then $\bar{k}\tilde{m}=(0,\ldots,0,m_x,\ldots,0,\ldots)^t-xk^{-1}$. Since $x=\sigma_i h$, then $xk^{-1}=\sigma_i hk^{-1}$, and so $\beta(M,N)(v)(\bar{k}\tilde{m})=p_{xk^{-1}}(\overline{hk^{-1}})^{-1}v(m_x)$ $=p_{xk}\bar{k}\bar{h}^{-1}v(m_x)=\bar{k}p_x\bar{h}^{-1}v(m_x)=\bar{k}\beta(M,N)(v)(\tilde{m})$. Therefore $\beta(M,N)(v)$ $\in\operatorname{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)$. We prove now that $\beta(M,N)0\alpha(M,N)=1$. Indeed, if $u\in\operatorname{Hom}_{R\{H\}\text{-mod}}(M^{\#,H},N)$ and $\tilde{m}=(0,\ldots,0,m_x,0)^t-x$ where $m_x\in M_{\sigma_i H}$, and $h\in H$ is such that $x=\sigma_{ih}$, then
$\beta(M,N)(\alpha(M,N)(u))(\tilde{m})=p_x\bar{h}^{-1}\alpha(M,N)(u)(m_x)=p_x\bar{h}^{-1}u(0,\ldots,0,m_x,0)^t-\sigma_i$
$=p_x\bar{h}^{-1}u(\ldots,0,m_x,0,\ldots)^t-xh^{-1}=p_x u(\bar{h}^{-1}(\ldots,0,m_x,0,\ldots)^t-xh^{-1})$
$=p_x u(\ldots,0,m_x,0,\ldots)^t-x=u(p_x(\ldots,0,m_x,0,\ldots)^t-x)=u(\tilde{m})$.

We prove now that $\alpha(M,N)\circ\beta(M,N)=1$. Let $v\in\operatorname{Hom}_{(G/H,R)\text{-gr}}(M,N_{G/H})$, and $m\in M_{\sigma_i H}$. Then $\alpha(M,N)(\beta(M,N)(v))(m)=\beta(M,N)(v)(\ldots,0,m,0,\ldots)^t-\sigma_i$ $=p_{\sigma_i}\bar{l}^{-1}v(m)=p_{\sigma_i}v(m)=v(m)$, since $v(m)\in(N_{G/H})_{\sigma_i H}=p_{\sigma_i}N$.

We prove now the second part of the statement. The functorial morphisms α and β define the canonical functorial morphisms γ and δ as follows: for $M\in(G/H,R)$-gr, $\gamma(M)\colon M\to(M^{\#,H})_{G/H}$. If $m\in M_{\sigma H}$, then $\gamma(M)(m)=(\ldots,0,m,0,\ldots)^t$ $-\sigma_i=p_{\sigma_i}(0,\ldots,0,m,0,\ldots)^t-\sigma_i\in[(M^{\cdot H})_{G/H}]_{\sigma_i H}$.

For $N\in R\{H\}$-mod $\delta(N)\colon(N_{G/H})^{\#,H}\to N$. Let $\tilde{n}=(\ldots,0,n,0,\ldots)^t-x$ $\in(N_{G/H})^{\#,H}$, so $n\in(N_{G/H})_{\sigma_i H}=\rho_{\sigma_i}N$, $x=\sigma_i h$. Then $\delta(N)(\tilde{n})=p_x\bar{h}^{-1}n$.

It is easy to see that γ is a functorial isomorphism, and the if $N\in\mathscr{C}^{\#,H}$, then $\delta(N)$ is an isomorphism. Therefore $(G/H,R)$-gr is equivalent with the category $\mathscr{C}^{\#,H}$.

Corollary 3.1 [8] *Assume that G is a finite group. Then the category* $(G/H,R)$-gr *is equivalent with the category* $R\{H\}$-mod. *In particular, it follows that the rings* $\bar{R}\# G/H$ and $R\{H\}$ *are Morita equivalent.*

Proof. Since G is finite, then the set $\{p_x\mid x\in G\}$ is finite and therefore $\mathscr{C}^{\#,H}$ $=R\{H\}$-mod. Now our assertion follows from Theorem 3.1.

Corollary 3.2 *Assume that G is an infinite group. Then the quotient category* $R\{H\}$-mod$/\mathscr{C}^{\#,H}$ *is equivalent with the category* $R[H]$-mod, *where* $R[H]$ *is the ordinary group ring associated to the ring R and to the group H.*

Proof. The same as the Corollary 1.2 (assertion 2) of [5].

4 Applications

Using Theorem 3.1 we obtain some applications to the study of the objects of the category $(G/H, R)$-gr. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring and $H < G$ a subgroup. In the paper [8] there is an example (Example 2.3) which shows that if $M \hookrightarrow N$ is an essential extension in the category $(G/H, R)$-gr, then it does not follow that $M \hookrightarrow N$ remains essential in R-mod. Now, using Theorem 3.1 we can give a sufficient condition for this result to hold:

Theorem 4.1 *Let H be a finite subgroup of G with $n = |H|$ and $M \hookrightarrow N$ be an essential extension in the category $(G/H, R)$-gr. If N is n-torsion free, then $M \hookrightarrow N$ is an essential extension in R-mod.*

Proof. We have the commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\alpha_M"'] & N \arrow[d, "\alpha_N"] \\ M^{\#,H} \arrow[r] & N^{\#,H} \end{tikzcd}$$

By Theorem 3.1 we have that $M^{\#,H} \hookrightarrow N^{\#,H}$ is an essential extension in the category $R\{H\}$-mod. Since the ring $R\{H\}$ is a skew group ring $(\tilde{R} \# G^*)\tilde{H}$, then by Maschke's Theorem (see [4]) it follows that $M^{\#,H} \hookrightarrow N^{\#,H}$ is an essential extension in $\tilde{R} \# G^*$-mod. Since the functor $(-)$-gr:$(R\text{-gr})^\# \to R$-gr is an equivalence of categories, then $M^{\#,H} \hookrightarrow N^{\#,H}$ is an essential extension in the category R-gr. Let $n \in N$, $n \neq 0$. Since $\alpha_N(n) \in N^{\#,H}$ and $\alpha_N(n) \neq 0$, then by Lemma 3.3.18 [6] there exists $\lambda_0 \in R_\sigma$ such that $\lambda_\sigma \alpha_N(n) \in M^{\#,H}$ and $\lambda_\sigma \alpha_N(n) \neq 0$.

If $n = \sum_{C \in G/H} n_C$, where $n_C \in N_C$, then we have $\lambda_\sigma \alpha_N(n) = \sum_{C \in G/H} \alpha_N(\lambda_\sigma n_C)$. For $C, C' \in G/H$, $C \neq C'$, we have $\sigma C \neq \sigma C'$ and since $\lambda_\sigma \alpha_N(n) \in M^{\#,H}$, then for any $y \in \sigma C$ there exists $m_y \in M_{\sigma C}$ such that $\lambda_\sigma n_C = m_y$. If $y' \in \sigma C$ is another element, we obtain the element $m_{y'} \in M_{\sigma C}$ such that $m_{y'} = \lambda_\sigma n_C$. Hence $m_y = m_{y'}$ for any $y, y' \in \sigma C$. We denote by $m_{\sigma C}$ this common element m_y. Thus $m_{\sigma C} \in M_{\sigma C}$. Since the map $C \mapsto \sigma C$ is bijective then we can define the element $m = \sum_{C \in G/H} m_{\sigma C}$. Thus $m \in M$ and $\alpha_M(m) = \alpha_N(\lambda_\sigma n)$. Since $\alpha_M(m) = \alpha_N(m)$ and α_N is injective, then $m = \lambda_\sigma n$. Hence $\lambda_\sigma n \in M$ and $\lambda_\sigma n \neq 0$. Therefore $M \hookrightarrow N$ is an essential extension in R-mod.

Theorem 4.2 *Let H be a finite subgroup of G and $M \in (G/H, R)$-gr. If M is a Noetherian (resp. Artinian, has Krull dimension, has Gabriel dimension, simple) object, then there exists an object $N \in R$-gr with the same property such that M is isomorphic with a submodule of N.*

Proof. Since M is a Noetherian (resp. Artinian, etc.) object of the category $(G/H, R)$-gr then $M^{\#,H}$ has the same property in $R\{H\}$-mod. Since $R\{H\}$ is a finite normalizing extension of the ring $\tilde{R} \# G^*$, then $M^{\#,H}$ is Noetherian (resp. Artinian, etc.) in $\tilde{R} \# G^*$-mod. Now since the functor $(-)$-gr: $(R\text{-gr})^\# \to R$-gr is an equivalence, then $M^{\#,H}$ is gr-Noetherian (resp. Artinian, etc.). Since H is a finite subgroup, there exists a monomorphism $\alpha_M : M \to M^{\#,H}$. Thus we can put $N = M^{\#,H}$.

Recall that a group G is said to be polycyclic-by-finite if G has a subnormal series $\{1\}=G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$, where $G_{i/G_{i-1}}$ is either finite a cyclic for all $i=1, \ldots, n$. The number of infinite cyclic factors which occur in this series is called the Hirsch number of G and is written $h(G)$. Since any two series have isomorphic refinements, $h(G)$ is a well-defined non-negative integer invariant of G.

Corollary 4.1 *Assume that G is polycyclic-by-finite group and H is a finite subgroup of G. If M is Noetherian in the category $(G/H, R)$-gr, then M is Noetherian in R-mod.*

Moreover, $\underset{(G/H,R)\text{-gr}}{K.\dim}(M) \leqq \underset{R}{K.\dim}(M) \leqq \underset{(G/H,R)\text{-gr}}{K.\dim}(M)+h(G)$ *(where* $\underset{(G/H,R)\text{-gr}}{K.\dim}(M)$ *(resp.* $\underset{R}{K.\dim}(M)$*) denotes the Krull dimension of M in the category $(G/H, R)$-gr (resp. R-mod)).*

Proof. With the notation of Theorem 4.2 we have that N is Noetherian in R-gr and $\underset{(G/H,R)\text{-gr}}{K.\dim}(M) = \underset{R\text{-gr}}{K.\dim}(N)$. By [9] it follows that N is Noetherian in R-mod and $\underset{R}{K.\dim}(N) \leqq \underset{R\text{-gr}}{K.\dim}(N)+h(G)$.

Therefore M is Noetherian and $\underset{R}{K.\dim}(M) \leqq \underset{(G/H,R)\text{-gr}}{K.\dim}(M)+h(G)$. The inequality $\underset{(G/H,R)\text{-gr}}{K.\dim}(M) \leqq \underset{R}{K.\dim}(M)$ is obvious.

Let $H<G$ be a subgroup. We say that G is polycyclic-by-finite with respect to H if there exists a subnormal series of the form

$$H=G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

such that G_i/G_{i-1} is finite cyclic for $i=1, \ldots, n$. We remark that for $H=\{1\}$ we obtain the notion of a polycyclic-by-finite group. We denote by $h_{G/H}$ the number of infinite cyclic factors in the series. The following result is a consequence of Theorem 3.1:

Corollary 4.2 *Let $K<H<G$ be subgroups of G such that H is polycyclic-by-finite with respect to K. If M is a Noetherian object of $(G/K, R)$-gr, then M is Noetherian in $(G/H, R)$-gr. Moreover, we have* $\underset{(G/K,R)\text{-gr}}{K.\dim}(M) \leqq \underset{(G/H,R)\text{-gr}}{K.\dim}(M) \leqq \underset{(G/K,R)\text{-gr}}{K.\dim}(M)+h_{H/K}$.

Proof. By Lemma 2.4 we have the following commutative diagram.

$$\begin{tikzcd}
(G/K, R)\text{-gr} \arrow[r, "(-)_\phi"] \arrow[d, "(-)^{*,K}"'] & (G/H, R)\text{-gr} \arrow[d, "(-)^{*,H}"] \\
R\{K\}\text{-mod} \arrow[r, "R\{H\} \otimes_{R\{K\}} -"] & R\{H\}\text{-mod}
\end{tikzcd}$$

Moreover, we remark that since $(-)_{G/H^\circ}(\ )^{*,H}$ is the identity functor of $(G/H, R)$-gr, then $M \simeq (R\{H\} \underset{R\{K\}}{\otimes} M^{*,H})_{G/H}$ in $(G/H, R)$-gr.

Now $M^{\#,H}$ is Noetherian in $R\{K\}$-mod by Theorem 3.1, and $K.\dim_{(G/K,R)\text{-gr}}(M) = K.\dim_{R\{K\}}(M^{\#,H})$.

We now note that if $H_1 \lhd H_2 < G$, then $R\{H_2\}$ is a strongly graded ring of type H_2/H_1. Indeed, if $C \in H_2/H_1$, i.e. $C = \sigma H_1 = H_1 \sigma$ for some $\sigma \in H_2$, then we define $R\{H_2\}_C = \sum_{\mathscr{C} \in C} R\{H_1\}\mathscr{C} = R\{H_1\}\bar{\sigma}$. Thus $R\{H_2\} = \bigoplus_{c \in G/H} R\{H_2\}_c$, and $R\{H_2\}_c R\{H_2\}_{c'} = R\{H_2\}_{cc'}$ for any $c, c' \in H_2/H_1$.

We proceed now by induction on the length of the series $K = H_0 \lhd H_1 \lhd \ldots \lhd H_n = H$, and using Theorem II.3.8 of [6] the result follows.

Corollary 4.3 *If G is polycyclic-by-finite with respect to the subgroup H, and $M \in (G/H, R)$-gr is a Noetherian object, then M is Noetherian in R-mod. Moreover,* $K.\dim_{(G/H,R)\text{-gr}}(M) \leqq K.\dim_R(M) \leqq K.\dim_{(G/H,R)\text{-gr}}(M) + h_{G/H}$.

Proof. Take $K = H$ and $H = G$ in Corollary 4.2.

Corollary 4.4 *If H is a polycyclic-by-finite subgroup of the group G, and $M \in R$-gr is gr-Noetherian, then M is Neotherian in $(G/H, R)$-gr. Moreover,* gr-$K.\dim(M) \leqq K.\dim_{(G/H,R)\text{-gr}}(M) \leqq$ gr-$K.\dim_R(M) + h(H)$ *(here* gr-$K.\dim_R(M) = K.\dim_{R\text{-gr}}(M)$. ^{R}

Proof. Take $K = \{1\}$ in Corollary 4.2.

Corollary 4.5 *Let $M \in R$-gr be a graded module such that for each R-submodule $N \leqq M$ there exists a polycyclic-by-finite subgroup H of G such that N is an object of $(G/H, R)$-gr. Then if M is gr-Noetherian it follows that M is Noetherian in R-mod.*

Proof. If M is not Noetherian, let $N \leqq M$ be an R-submodule of M which is not finitely generated, and let H be a polycyclic-by-finite subgroup of G such that $N \in (G/H, R)$-gr. It follows that N is not finitely generated as an object of $(G/H, R)$-gr. But M is a Noetherian object of $(G/H, R)$-gr by Corollary 4.4, and this is a contradiction.

Corollary 4.6 *Let $K \lhd H < G$ be subgroups of G such that $n = |H/K| < \infty$. Suppose that $M \in (G/K, R)$-gr is a semisimple object which has no n-torsion. Then M is a semisimple object in the category $(G/H, R)$-gr.*

Proof. By Lemma 2.4 we have the commutative diagram.

$$\begin{array}{ccc} (G/K, R)\text{-gr} & \xrightarrow{(-)_\Phi} & (G/H, R)\text{-gr} \\ {\scriptstyle (-)^{\#,K}}\downarrow & & \downarrow{\scriptstyle (-)^{\#,H}} \\ R\{K\}\text{-mod} & \xrightarrow[R\{K\}]{R\{H\}\otimes -} & R\{H\}\text{-mod} \end{array}$$

It is clear that we may assume that M is a simple object of $(G/K, R)$-gr. By Theorem 3.1 $M^{\#,H}$ is simple in $R\{K\}$-mod and it is clear that n is invertible on $M^{\#,H}$ (i.e. $M^{\#,H} = nM^{\#,H}$). By Remark 2.3.d) $M^{\#,H}$ is semisimple over $R\{K\}$ and we can apply Proposition 2.1 of [4].

Corollary 4.7 *Let $H<G$ be a finite subgroup, $n=|H|$. Suppose that $M\in R$*-gr *is a semisimple object and that it has no n-torsion. Then M is semisimple in* $(G/H, R$-gr.

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