ANNALS OF MATHEMATICS
Vol. 47, No. 3, July, 1946

ON THE COHOMOLOGY THEORY FOR ASSOCIATIVE ALGEBRAS

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(Received October 9, 1945)

Introduction.

The cohomology theory for associative algebras may be regarded as a superstructure built upon the classical representation theory. Let us consider an algebra $\mathfrak{A}$ over a field Φ. Suppose that we are given a finite dimensional linear space $\mathfrak{P}$ over Φ which is simultaneously a representation space and an antirepresentation space for $\mathfrak{A}$. Then, to each element a of $\mathfrak{A}$, there correspond two linear transformations $u \to u \cdot a$ and $u \to a \cdot u$ of $\mathfrak{P}$. If, in addition to the usual requirements, these transformations commute with each other, i.e. if $a_1 \cdot (u \cdot a_2) = (a_1 \cdot u) \cdot a_2$, for all $a_1, a_2 \in \mathfrak{A}$ and all $u \in \mathfrak{P}$, we say that $\mathfrak{P}$ is an $\mathfrak{A}$-module. The notions of the cohomology theory are based on this set-up, as follows:

The set of all m-linear mappings of $\mathfrak{A}$ into $\mathfrak{P}$, when regarded as an additive group in the usual way, is called the group of the m-dimensional $\mathfrak{P}$-cochains (or $(m, \mathfrak{P})$-cochains) of $\mathfrak{A}$. We shall denote it by $C^m(\mathfrak{A}, \mathfrak{p})$. The group $C^0(\mathfrak{A}, \mathfrak{p})$ is identified with the additive group of $\mathfrak{p}$. The coboundary operator, δ, as defined by Eilenberg and MacLane, maps each $C^m(\mathfrak{A}, \mathfrak{p})$ linearly into $C^{m+1}(\mathfrak{A}, \mathfrak{p})$, as follows: Let $f \in C^m(\mathfrak{A}, \mathfrak{p})$, $a_1, \cdots, a_m \in \mathfrak{A}$; then $(\delta f)\{a_1, \cdots, a_{m+1}\} = a_1 \cdot f\{a_2, \cdots, a_{m+1}\} + \sum_{i=1}^{m} (-1)^i f\{a_1, \cdots, a_i a_{i+1}, \cdots, a_{m+1}\} + (-1)^{m+1} f\{a_1, \cdots, a_m\} \cdot a_{m+1}$. For any cochain f, we have $\delta(\delta f) = 0$. Hence the cohomology groups may be defined by the well known procedure of combinatorial topology. An element $f \in C^m(\mathfrak{A}, \mathfrak{p})$ is called an m-dimensional cocycle if $\delta f = 0$. An element of the form δg, where $g \in C^{m-1}(\mathfrak{A}, \mathfrak{p})$ is called an m-dimensional coboundary. These form a subgroup $B^m(\mathfrak{A}, \mathfrak{p})$ of the group $Z^m(\mathfrak{A}, \mathfrak{p})$ of the m-dimensional cocycles. The m-dimensional cohomology group $H^m(\mathfrak{A}, \mathfrak{p})$ is defined as the quotient group $Z^m(\mathfrak{A}, \mathfrak{p})/B^m(\mathfrak{A}, \mathfrak{p})$.

For $m = 1$ and $m = 2$, these groups have been interpreted with reference to classical notions of structure in a previous paper.[1] Although no such interpretation has as yet been found for the higher dimensional groups it is possible to make a rough forecast of the rôle which the cohomology theory will play in the study of the structure of algebras. There is good evidence that it will provide appropriate methods for investigating a variety of structural characteristics which seem to lie outside the domain of the classical representation theory.

The trend which takes its logical departure in Wedderburn's structure theorems has been to intensity and refine the structural analysis only at the level of maximal "regularity."[2] Thus, the radical of an algebra is split off by Wedder-

[1] G. HOCHSCHILD, "*On the Cohomology Groups of an Associative Algebra*," Annals of Mathematics, Vol. 46 (1945), No. 1, pp. 58–67. This paper will henceforth be referred to as "CG."

[2] With the notable exception of the study of Frobenusian and related algebras by Nakayama and Nesbitt.

burn's decomposition. Then the semi-simple part is decomposed into its simple components. When these are represented as full matrix algebras over skew fields the structural analysis is reduced to the study of skew fields to which the most powerful available methods, such as those of group theory, Galois theory, and arithmetic, are applied. This approach largely ignores the structural ties between the radical and the semi-simple part, and contributes nothing to the study of nilpotent algebras.

In a sense, the cohomology theory is applied in the opposite direction. Already at the level of semi-simplicity (or separability, if the groundfield is of characteristic p), the cohomology groups are trivial. More precisely, for dimension $m \geqq 1$, all the cohomology groups of a separable algebra (i.e. an algebra which is semi-simple over every extension field of the groundfield) are zero. A significant feature of the theory is that it leads to a classification of algebras which differentiates between algebras with a non-zero radical.

Denote by $K_m(\Phi)$, $m = 1, 2, \cdots$, the class of all algebras over the field Φ whose m-dimensional cohomology groups are all zero.[3] Then, referring to results obtained in CG, we have $K_m(\Phi) \subseteqq K_{m+1}(\Phi)$, for all m. $K_1(\Phi)$ is the class of all separable algebras over Φ. In order to characterize the class $K_2(\Phi)$ we recall the definition of an extension of an algebra: If $\mathfrak{B}$ is an algebra over Φ, and if ω is a homomorphism of $\mathfrak{B}$ onto another algebra $\mathfrak{A}$ over Φ, we say that the pair $(\mathfrak{B}, \omega)$ is an extension of $\mathfrak{A}$. If $\mathfrak{B}$ contains a subalgebra $\mathfrak{A}^*$ such that ω maps $\mathfrak{A}^*$ isomorphically onto $\mathfrak{A}$, we say that $\mathfrak{A}$ is segregated in $(\mathfrak{B}, \omega)$. We call an algebra absolutely segregated if it is segregated in every extension. In these terms, $K_2(\Phi)$ may be characterized as the class of all absolutely segregated algebras over Φ. It will be shown by an example that $K_2(\Phi)$ is strictly larger than $K_1(\Phi)$, and it is noteworthy that the algebras of $K_2(\Phi)$ which do not belong to $K_1(\Phi)$ are neither semi-simple nor nilpotent. It seems very probable that $K_m(\Phi)$ is a proper subclass of $K_{m+1}(\Phi)$ for every m, although this question is unsettled for $m > 1$.

The first part of this paper is devoted to a further development of the general theory, with a view to eliminating the influence of inessential structural characteristics on the cohomology groups. Thus, it will be shown that the adjunction of an identity element to an algebra does not change its cohomology groups. The cohomology groups of the direct sum of two algebras with identity elements are shown to be the direct sums of the cohomology groups of the components. We show also that the cohomology groups of an algebra with an identity determine the cohomology groups of its Kronecker product with any separable algebra.

In the second part we apply the cohomology theory to the study of the extensions of an algebra. The fundamental correspondence between these and the two dimensional cohomology groups was dealt with in CG and constitutes our point of departure here. Our results in this connection will be outlined in section 6.

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[3] $K_0(\Phi)$ is the null class.

1. Preliminaries.

Let $\mathfrak{A}$ be an algebra over the field Φ, and $\mathfrak{P}$ an $\mathfrak{A}$-module. If we introduce the m-fold Kronecker products $\mathfrak{A}^{(m)}$ of the linear space underlying $\mathfrak{A}$ we may define an element of $C^m(\mathfrak{A}, \mathfrak{P})$ as a linear mapping of $\mathfrak{A}^{(m)}$ into $\mathfrak{P}$.

From an algebra $\mathfrak{A}$ and an $\mathfrak{A}$-module $\mathfrak{P}$, we can construct a new $\mathfrak{A}$-module $(\mathfrak{A}:\mathfrak{P})$, as follows: As a linear space $(\mathfrak{A}:\mathfrak{P})$ is identical with the set of the one dimensional $\mathfrak{P}$-cochains of $\mathfrak{A}$, i.e. with $C^1(\mathfrak{A}, \mathfrak{P})$. If $f \in (\mathfrak{A}:\mathfrak{P})$, and $a, a' \in \mathfrak{A}$, we define the transforms $a*f$ and $f*a$ by setting

$$(a*f)\{a'\} = a\cdot f\{a'\};\ (f*a)\{a'\} = f\{aa'\} - f\{a\}\cdot a'.$$

These transformations make $(\mathfrak{A}:\mathfrak{P})$ an $\mathfrak{A}$-module. Corresponding to an element $f \in C^m(\mathfrak{A}, \mathfrak{P})$, $m \geqq 1$, we have an element $\bar{f} \in C^{m-1}(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$ which is defined by the relation $(\bar{f}\{a_1, \cdots, a_{m-1}\})\{a_m\} = f\{a_1, \cdots, a_m\}$. As was shown in CG, the mapping $f \rightarrow \bar{f}$ is such that $\overline{\delta f} = \delta\bar{f}$, and induces an isomorphism of $H^{m+1}(\mathfrak{A}, \mathfrak{P})$ onto $H^m(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$, for $m \geqq 1$. This result enables us to use complete induction in many of our proofs.

If an algebra $\mathfrak{A}$ has an identity element, 1, we may confine our attention to those modules in which 1 operates as the identity operator. Precisely, let $\mathfrak{P}$ be an arbitrary $\mathfrak{A}$-module. Denote the identity operator on $\mathfrak{P}$ by I. We can decompose $\mathfrak{P}$ into the direct sum of the two $\mathfrak{A}$-invariant subspaces $\mathfrak{P}\cdot 1$ and $\mathfrak{P}\cdot(I - 1)$. If $f \in C^m(\mathfrak{A}, \mathfrak{P})$, with $m \geqq 1$, define an element $g \in C^{m-1}(\mathfrak{A}, \mathfrak{P})$ by setting

$$g\{a_1, \cdots, a_{m-1}\} = f\{a_1, \cdots, a_{m-1}, 1\}\cdot(I - 1).$$

Then

$$(\delta g)\{a_1, \cdots, a_m\} = (\delta f)\{a_1, \cdots, a_m, 1\}\cdot(I-1) + (-1)^{m+1}f\{a_1, \cdots, a_m\}\cdot(I - 1).$$

In particular, if f is a cocycle, we have $(\delta g)\{a_1, \cdots, a_m\} = (-1)^{m+1}f\{a_1, \cdots, a_m\}\cdot(I - 1)$. For any $f \in C^n(\mathfrak{A}, \mathfrak{P})$, define $f' \in C^n(\mathfrak{A}, \mathfrak{P}\cdot 1)$ by setting

$$f'\{a_1, \cdots, a_n\} = f(a_1, \cdots, a_n\}\cdot 1.$$

Then $\delta f' = (\delta f)'$, for all $n \geqq 0$. Hence the mapping $f \rightarrow f'$ induces a homomorphism of $H^m(\mathfrak{A}, \mathfrak{P})$ onto $H^m(\mathfrak{A}, \mathfrak{P}\cdot 1)$. The relation just proved for the cocycles shows that this is actually an isomorphism when $m \geqq 1$, for then we have $f = f' + (-1)^{m+1}\delta g$. By symmetry, we obtain the following result:

THEOREM 1. *If $m \geqq 1$, and if $\mathfrak{A}$ has an identity, 1, then $H^m(\mathfrak{A}, \mathfrak{P}) \cong H^m(\mathfrak{A}, \mathfrak{P}\cdot 1) \cong H^m(\mathfrak{A}, 1\cdot\mathfrak{P}) \cong H^m(\mathfrak{A}, 1\cdot\mathfrak{P}\cdot 1)$.*

2. Adjunction of an Identity Element.

Let $\mathfrak{A}$ be an algebra, $\mathfrak{P}$ an $\mathfrak{A}$-module. If $\mathfrak{A}^*$ is the algebra obtained from $\mathfrak{A}$ by adjoining an identity element 1, we can make $\mathfrak{P}$ an $\mathfrak{A}^*$-module simply by making the element 1 of $\mathfrak{A}^*$ act as the identity operator on $\mathfrak{P}$. We denote this $\mathfrak{A}^*$-module by $\mathfrak{P}^*$. Now we shall prove the following result:

THEOREM 2. *$H^m(\mathfrak{A}^*, \mathfrak{P}^*) \cong H^m(\mathfrak{A}, \mathfrak{P})$, for all m.*

For $m = 0$, $H^m(\mathfrak{A}, \mathfrak{P})$ is simply the additive group consisting of all $u \in \mathfrak{P}$ such that $a \cdot u = u \cdot a$, for every $a \in \mathfrak{A}$. This is clearly identical with $H^0(\mathfrak{A}^*, \mathfrak{P}^*)$.

In general, it is clear that the restriction of the $(n, \mathfrak{P}^*)$ cocycles of $\mathfrak{A}^*$ to $\mathfrak{A}$ gives a natural homomorphism of $H^m(\mathfrak{A}^*, \mathfrak{P}^*)$ into $H^m(\mathfrak{A}, \mathfrak{P})$. We shall show first that this homomorphism actually maps $H^m(\mathfrak{A}^*, \mathfrak{P}^*)$ onto $H^m(\mathfrak{A}, \mathfrak{P})$.

Indeed, if $f \in C^m(\mathfrak{A}, \mathfrak{P})$ we may define an element $f^* \in C^m(\mathfrak{A}^*, \mathfrak{P}^*)$ by setting $f^*\{a_1^*, \cdots, a_m^*\} = f\{a_1, \cdots, a_m\}$, where the a_i are the components in $\mathfrak{A}$ of the elements $a_i^* \in \mathfrak{A}^*$. It can be verified directly that $\delta f^* = (\delta f)^*$, and it follows that the mapping $f \to f^*$ maps cocycles into cocycles. Since the restriction of f^* to $\mathfrak{A}$ coincides with f our assertion is proved.

In order to complete the proof of our theorem we require the following lemma:

LEMMA 1. *Let $\mathfrak{A}$ be an algebra with an identity element,* 1. *Let $\mathfrak{P}$ be an $\mathfrak{A}$-module. Then, given any $f \in Z^m(\mathfrak{A}, \mathfrak{P})$, $m \geqq 1$, there exists a $g \in C^{m-1}(\mathfrak{A}, \mathfrak{P})$ such that $(f - \delta g)\{a_1, \cdots, a_m\} = 0$, whenever one of the a_i is equal to* 1.

We prove this by induction on m: If $m = 1$, set $g = 1 \cdot f\{1\} - f\{1\} \cdot 1$. Then

$$(f - \delta g)\{a\} = f\{a\} - a \cdot f\{1\} + a \cdot f\{1\} \cdot 1 + 1 \cdot f\{1\} \cdot a - f\{1\} \cdot a.$$

Since $f\{1\} = 1 \cdot f\{1\} + f\{1\} \cdot 1$, we have $1 \cdot f\{1\} \cdot 1 = 0$, and $(f - \delta g)\{1\} = 0$.

Now suppose that the lemma has been proved for all $m \leqq n - 1$, and consider an $f \in Z^n(\mathfrak{A}, \mathfrak{P})$. If $\bar{f}$ is the corresponding element of $Z^{n-1}(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$ there is, by our inductive hypothesis, an element $\bar{g} \in C^{n-2}(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$ such that $(\bar{f} - \delta\bar{g})\{a_1, \cdots, a_{n-1}\} = 0$, whenever one of the a_i is equal to 1. Define $g \in C^{n-1}(\mathfrak{A}, \mathfrak{P})$ by setting $g\{a_1, \cdots, a_{n-1}\} = (\bar{g}\{a_1, \cdots, a_{n-2}\})\{a_{n-1}\}$. Then $(f - \delta g)\{a_1, \cdots, a_n\} = 0$, whenever one element of the set $a_1, \cdots, a_{n-1}$ is equal to 1. Now write $f' = f - \delta g$, and set $g'\{a_1, \cdots, a_{n-1}\} = f'\{a_1, \cdots, a_{n-1}, 1\} - 2f'\{a_1, \cdots, a_{n-1}, 1\} \cdot 1$. Using the fact that f' is a cocycle, we see that

$$(\delta g')\{a_1, \cdots, a_n\} = (-1)^{n+1} f'\{a_1, \cdots, a_n\} + (-1)^n [f'\{a_1, \cdots, a_n\} \cdot 1 - f'\{a_1, \cdots, a_{n-1}, 1\} \cdot a_n].$$

It follows that the element $g + (-1)^{n+1} g'$ satisfies the requirement of the lemma for $m = n$.

The proof of our theorem can now be completed quite easily: Let $F \in Z^m(\mathfrak{A}^*, \mathfrak{P}^*)$ be such that the restriction f of F to $\mathfrak{A}$ is a coboundary. Then there exists an element $g \in C^{m-1}(\mathfrak{A}, \mathfrak{P})$ such that $f = \delta g$. By the lemma, there exists an element $H \in C^{m-1}(\mathfrak{A}^*, \mathfrak{P}^*)$ such that $(F - \delta H)\{a_1, \cdots, a_m\} = 0$, whenever one of the a_i is equal to 1. Let h be the restriction of H to $\mathfrak{A}$. Then $(f - \delta h)^* = F - \delta H$, i.e. $(\delta g - \delta h)^* = F - \delta H$, whence $F = \delta(H - h^* + g^*)$. Hence the mapping $F \to f$ induces an isomorphism of $H^m(\mathfrak{A}^*, \mathfrak{P}^*)$ onto $H^m(\mathfrak{A}, \mathfrak{P})$.

Given an $\mathfrak{A}^*$ module $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}$ the $\mathfrak{A}$ module defined by $\mathfrak{P}$. By combining Theorem 2 with Theorem 1 we obtain the following result:

THEOREM 3. *If $m \geqq 1$,*[4] $H(\mathfrak{A}^*, \mathfrak{P}) \cong H(\mathfrak{A}, \overline{1 \cdot \mathfrak{P} \cdot 1})$.

[4] From now on, the 0-dimensional groups will no longer be considered. They are rather trivial and their inclusion would merely complicate our statements. If a theorem is stated for dimension m it is to be understood that that $m \geqq 1$. Theorem 3 does not hold for $m = 0$.

Note that it is not generally true that $H^m(\mathfrak{A}^*, \mathfrak{P})$ is isomorphic with $H^m(\mathfrak{A}, \bar{\mathfrak{P}})$. For example, take for $\mathfrak{P}$ any linear space over Φ, and define $a^* \cdot u = u \cdot a^* = 0$, for all $a^* \in \mathfrak{A}^*$ and $u \in \mathfrak{P}$. Take $\mathfrak{A}$ such that $\mathfrak{A}^2 \neq \mathfrak{A}$. Select any linear mapping f of $\mathfrak{A}/\mathfrak{A}^2$ into $\mathfrak{P}$ which is not identically zero. If we denote the coset modulo $\mathfrak{A}^2$ of an element $a \in \mathfrak{A}$ by $\bar{a}$, and set $F\{a\} = f\{\bar{a}\}$, then F is a non-cobounding $(1, \bar{\mathfrak{P}})$-cocycle of $\mathfrak{A}$. On the other hand, Theorem 3 shows that $H^1(\mathfrak{A}^*, \mathfrak{P}) = \{0\}$.

3. Direct Sums.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras over Φ, and assume that $\mathfrak{B}$ has an identity element, say 1_β. We wish to express the cohomology groups of the direct sum $\mathfrak{A} + \mathfrak{B}$ in terms of the cohomology groups of $\mathfrak{A}$ and $\mathfrak{B}$. Our basic tool will be the following generalization of Lemma 1:

LEMMA 2. *Let $\mathfrak{P}$ be an $(\mathfrak{A} + \mathfrak{B})$-module, $f \in Z^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$. Then there exists $g \in C^{m-1}(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$ such that $(f - \delta g)\{c_1, \cdots, c_m\} = 0$, whenever one of the $c_i(\in \mathfrak{A} + \mathfrak{B})$ is equal to 1_β.*

In order to prove this we merely have to make the same constructions as in the proof of Lemma 1, replacing "1" by "1_β".

We shall denote the set of all $u \in \mathfrak{P}$ for which $\mathfrak{B} \cdot u = \{0\} = u \cdot \mathfrak{B}$ by $\mathfrak{P}^\beta$. The purpose of this section is to prove the following theorem:

THEOREM 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras, $\mathfrak{P}$ an $(\mathfrak{A} + \mathfrak{B})$-module. If $\mathfrak{B}$ has an identity then*

$$H^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P}) \cong H^m(\mathfrak{A}, \mathfrak{P}^\beta) + H^m(\mathfrak{B}, \mathfrak{P}).$$

Denote by I the identity operator on $\mathfrak{P}$. If f is any element of $C^n(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$ we define, for $a_1, \cdots, a_n \in \mathfrak{A}$, and $b_1, \cdots, b_n \in \mathfrak{B}$,

$$f_\alpha\{a_1, \cdots, a_n\} = (I - 1_\beta) \cdot f\{a_1, \cdots, a_n\} \cdot (I - 1_\beta),$$

and

$$f_\beta(b_1, \cdots, b_n\} = 1_\beta \cdot f\{b_1, \cdots, b_n\} \cdot 1_\beta .$$

It is easily verified that these mappings are such that $(\delta f)_\alpha = \delta f_\alpha$, and $(\delta f)_\beta = \delta f_\beta$. It follows that the mapping $f \to f_\alpha + f_\beta$ induces a homomorphism of $H^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$ into

$$H^m(\mathfrak{A}, \mathfrak{P}^\beta) + H^m(\mathfrak{B}, 1_\beta \cdot \mathfrak{P} \cdot 1_\beta).$$

Furthermore, if $f_1 \in Z^m(\mathfrak{A}, \mathfrak{P}^\beta)$ and $f_2 \in C^m(\mathfrak{B}, 1_\beta \cdot \mathfrak{P} \cdot 1_\beta)$, and we define $f\{a_1 + b_1, \cdots, a_m + b_m\} = f_1\{a_1, \cdots, a_m\} + f_2\{b_1, \cdots, b_m\}$, then $f \in Z^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$, $f_\alpha = f_1$, and $f_\beta = f_2$. Hence our mapping is a homomorphism onto.

Now consider an $f \in Z^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$ such that f_α and f_β are coboundaries. Then there exists $g_1 \in C^{m-1}(\mathfrak{A}, \mathfrak{P}^\beta)$ and $g_2 \in C^{m-1}(\mathfrak{B}, 1_\beta \cdot \mathfrak{B} \cdot 1_\beta)$ such that $f_\alpha = \delta g_1$ and $f_\beta = \delta g_2$. On the other hand, applying Lemma 2 to f, we obtain an $h \in C^{m-1} (\mathfrak{A} + \mathfrak{B}, \mathfrak{P})$ such that $(f - \delta h)\{c_1, \cdots, c_m\} = 0$, whenever one of the elements c_i is equal to 1_β. It is easily seen that this implies that

$$(f - \delta h)\{a_1 + b_1, \cdots, a_m + b_m\} = (f - \delta h)_\alpha\{a_1, \cdots, a_m\} + (f - \delta h)_\beta\{b_1, \cdots, b_m\},$$

i.e.

$$(f - \delta h)\{a_1 + b_1, \cdots, a_m + b_m\} = (\delta g_1 - \delta h_\alpha)\{a_1, \cdots, a_m\} + (\delta g_2 - \delta h_\beta)\{b_1, \cdots, b_m\}.$$

Now define

$$g^*\{a_1 + b_1, \cdots, a_{m-1} + b_{m-1}\} = g_1\{a_1, \cdots, a_{m-1}\} + g_2\{b_1, \cdots, b_{m-1}\}$$

and

$$h^*\{a_1 + b_1, \cdots, a_{m-1} + b_{m-1}\} = h_\alpha\{a_1, \cdots, a_{m-1}\} + h_\beta\{b_1, \cdots, b_{m-1}\}.$$

Then $f - \delta h = (\delta(g^* - h^*))_\alpha + (\delta(g^* - h^*))_\beta = \delta(g^* - h^*)$, or $f = \delta(h - h^* + g^*)$. Thus, our homomorphism is an isomorphism. Finally, by Theorem 1, $H^m(\mathfrak{B}, 1_\beta \cdot \mathfrak{P} \cdot 1_\beta) \cong H^m(\mathfrak{B}, \mathfrak{P})$.

If $\mathfrak{A}$ has also an identity element, 1_α, we have $1_\alpha \cdot \mathfrak{P}^\beta \cdot 1_\alpha = 1_\alpha \cdot \mathfrak{P} \cdot 1_\alpha$, whence $H^m(\mathfrak{A}, \mathfrak{P}^\beta) \cong H^m(\mathfrak{A}, \mathfrak{P})$, so that

$$H^m(\mathfrak{A} + \mathfrak{B}, \mathfrak{P}) \cong H^m(\mathfrak{A}, \mathfrak{P}) + H^m(\mathfrak{B}, \mathfrak{P}).$$

4. Generic Cocycles.

Given any algebra $\mathfrak{A}$, we shall construct certain $\mathfrak{A}$-modules $\mathfrak{P}_n$ $(n = 1, 2, \cdots)$ and, for each n, a generic $(n, \mathfrak{P}_n)$-cocycle F_n such that every $(n, \mathfrak{P})$-cocycle of $\mathfrak{A}$ is the image of F_n by an operator homomorphism of $\mathfrak{P}_n$ into $\mathfrak{P}$. Then $\mathfrak{A} \in K_n(\Phi)$ if and only if F_n is a coboundary.

First we construct $\mathfrak{A}^* = (\mathfrak{A}, 1)$ and take the underlying linear space of $\mathfrak{P}_n$ to be the Kronecker product $\mathfrak{A}^{(n)} \times \mathfrak{A}^*$. We define the operations on $\mathfrak{P}_n$ by setting

$$(a_1 \times \cdots \times a_n \times \overset{*}{a}_{n+1}) \cdot a = a_1 \times \cdots \times a_n \times \overset{*}{a}_{n+1} a,$$

and

$$a \cdot (a_1 \times \cdots \times a_n \times \overset{*}{a}_{n+1}) = aa_1 \times a_2 \times \cdots \times a_n \times \overset{*}{a}_{n+1} + \sum_{i=1}^{n-1} (-1)^i a \times a_1 \times \cdots \times a_i a_{i+1} \times \cdots \times a_n \times \overset{*}{a}_{n+1} + (-1)^n a \times a_1 \times \cdots \times a_n \overset{*}{a}_{n+1}.$$

This makes $\mathfrak{P}_n$ an $\mathfrak{A}$-module.

Let $F_n\{a_1, \cdots, a_n\} = a_1 \times \cdots \times a_n \times 1$. Then it is easily verified that $F_n \in Z^n(\mathfrak{A}, \mathfrak{P}_n)$.

Now suppose that $\mathfrak{P}$ is any $\mathfrak{A}$-module, and let $f \in Z^n(\mathfrak{A}, \mathfrak{P})$. Define $u \cdot 1 = u$, for all $u \in \mathfrak{P}$, and $\mu_f\{a_1 \times \cdots \times a_n \times \overset{*}{a}_{n+1}\} = f\{a_1, \cdots, a_n\} \cdot \overset{*}{a}_{n+1}$. If μ_f is extended by linearity to the whole of $\mathfrak{P}_n$ it gives an operator-homomorphism of $\mathfrak{P}_n$ into $\mathfrak{P}$. Also $\mu_f F_n = f$, and our assertions follow directly from these facts.

By considering $\mathfrak{P}_1$ we obtain the following criterion for separability:

Theorem 5. *A necessary and sufficient condition for an algebra $\mathfrak{A}$ to be separable is that it have an identity 1_α, and that, for any basis $(a_1, \cdots, a_n)$ of $\mathfrak{A}$ over Φ, we can find dual elements $\bar{a}_1, \cdots, \bar{a}_n$ in $\mathfrak{A}$ such that*

(i) $\sum_{i=1}^{n} a_i \bar{a}_i = 1_\alpha$, and

(ii) $\sum_{i=1}^{n} aa_i \times \bar{a}_i = \sum_{i=1}^{n} a_i \times \bar{a}_i a$, for every $a \in \mathfrak{A}$.

The existence of an identity element is well known to be a necessary condition. Furthermore, by Theorem 4.1 of CG, a necessary and sufficient condition is that every one dimensional cocycle of $\mathfrak{A}$ be a coboundary. By the properties of F_1 we have, therefore, as a necessary and sufficient condition for separability, that F_1 be a coboundary. Making use of Theorem 1, we see that we may confine ourselves to considering the module $\mathfrak{P}_1 \cdot 1_\alpha$ and the cocycle F_1', where $F_1'\{a\} = F_1\{a\} \cdot 1_\alpha$. Hence, a necessary and sufficient condition is that there exist an element $u \in \mathfrak{P}_1 \cdot 1_\alpha$ such that $F_1' = \delta u$. If such an element exists we may write it in the form $u = -\sum_{i=1}^{n} a_i \times \bar{a}_i$, where the $\bar{a}_i \in \mathfrak{A}$. The condition now becomes

$$a \times 1_\alpha = -\sum_{i=1}^{n} aa_i \times \bar{a}_i + \sum_{i=1}^{n} a \times a_i \bar{a}_i + \sum_{i=1}^{n} a_i \times \bar{a}_i a.$$

For $a = 1_\alpha$ we obtain $1_\alpha \times 1_\alpha = 1_\alpha \times \sum_{i=1}^{n} a_i \bar{a}_i$, i.e. (i) above, and the identity reduces to (ii). Conversely, if the $\bar{a}_i$ exists, and we set $u = -\sum_{i=1}^{n} a_i \times \bar{a}_i$, we have $F_1' = \delta u$. This completes the proof of Theorem 5.

As an example, consider the elements $g_1, \cdots, g_n$ of a finite group. They constitute a linear basis for the group ring over the field Φ. If the characteristic of Φ does not divide n, the elements $(1/n)g_1^{-1}, \cdots, (1/n)g_n^{-1}$ are duals in the sense of Theorem 5, which shows that such a group ring is separable.

5. Kronecker Products.

Let $\mathfrak{A}$ be an algebra with an identity element 1_α, and let $\mathfrak{B}$ be a separable algebra. We consider the Kronecker product $\mathfrak{A} \times \mathfrak{B}$. Our analysis of the cohomology groups of $\mathfrak{A} \times \mathfrak{B}$ is based on the following lemma:

LEMMA 3. *If $f \in C^m(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$, where $\mathfrak{P}$ is an $\mathfrak{A} \times \mathfrak{B}$-module, and if (δf) $\{a^1 \times b^1, \cdots, a^{m+1} \times b^{m+1}\} = 0$, whenever one of the a^i is equal to 1_α, there exists an element g of $C^{m-1}(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$ such that $(f - \delta g)\{a^1 \times b^1, \cdots, a^m \times b^m\} = 0$, whenever one of the a^i is equal to 1_α.*[5]

In order to prove this, select a basis $(b_1, \cdots, b_N)$ of $\mathfrak{B}$ over Φ, and dual elements $\bar{b}_1, \cdots, \bar{b}_N$, in accordance with Theorem 5. If $f \in C^1(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$ define

$$g = f\{1_\alpha \times 1_\beta\} - f\{1_\alpha \times 1_\beta\} \cdot (1_\alpha \times 1_\beta) - \sum_{i=1}^{N} f\{1_\alpha \times b_i\} \cdot (1_\alpha \times \bar{b}_i).$$

Using the property (i) in Theorem 5 and the property of δf, we obtain

$(a \times b) \cdot g - g \cdot (a \times b) = f\{a \times b\}$

[5] The superscripts are not exponents.

$$+ \sum_{i=1}^{N} [f\{1_\alpha \times b_i\} \cdot (a \times \bar{b}_i b) - f\{a \times bb_i\} \cdot (1_\alpha \times \bar{b}_i)].$$

It follows immediately from the property (ii) in Theorem 5 that g satisfies the requirement of our lemma.

Now suppose that the lemma has been proved for $m \leqq n - 1$, and consider an $f \in C^n(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$. If we apply the inductive hypothesis to the corresponding element $\bar{f}$ of $C^{n-1}(\mathfrak{A} \times \mathfrak{B}, (\mathfrak{A} \times \mathfrak{B}:\mathfrak{P}))$ we conclude that there exists $g \in C^{n-1}(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$ such that $(f - \delta g)\{a^1 \times b^1, \cdots, a^n \times b^n\} = 0$, whenever one of the elements $a^1, \cdots, a^{n-1}$ is equal to 1_α. Write $f' = f - \delta g$ and set

$$\begin{aligned} g'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}\} &= f'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}, 1_\alpha \times 1_\beta\} \cdot (1_\alpha \times 1_\beta) \\ &\quad - f'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}, 1_\alpha \times 1_\beta\} \\ &\quad + \sum_{i=1}^{N} f'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}, 1_\alpha \times b_i\} \cdot (1_\alpha \times \bar{b}_i). \end{aligned}$$

Using the property (i) in Theorem 5 and the property of δf, we can verify by a straightforward computation that

$$\begin{aligned} (\delta g')\{a^1 \times b^1, \cdots, a^n \times b^n\} &= (-1)^n f'\{a^1 \times b^1, \cdots, a^n \times b^n\} \\ &\quad + (-1)^n \sum_{i=1}^{N} [f'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}, 1_\alpha \times b_i\} \cdot (a^n \times \bar{b}_i b^n) \\ &\quad - f'\{a^1 \times b^1, \cdots, a^{n-1} \times b^{n-1}, a^n \times b^n b_i\} \cdot (1_\alpha \times \bar{b}_i)]. \end{aligned}$$

Taking account of the property (ii) in Theorem 5 and the properties of f' we see that the cochain $g'' = g + (-1)^n g'$ satisfies the requirement of the lemma for $m = n$.

Now we are in a position to prove the following result:

THEOREM 6. *Let $\mathfrak{A}$ be an algebra with an identity element, $\mathfrak{B}$ a separable algebra. Then, if $\mathfrak{P}$ is any $\mathfrak{A} \times \mathfrak{B}$-module, we have*

$$H^m(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P})) \cong H^m(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P}),$$

where $Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P})$ is regarded as an $\mathfrak{A}$-module in the natural way.

If f is any element of $C^n(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$ we may define an element $f^* \in C^n(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$ by setting

$$f^*(a^1 \times b^1, \cdots, a^n \times b^n\} = (1_\alpha \times b^1 b^2 \cdots b^n) \cdot f\{a^1, \cdots, a^n\}.$$

It is easily verified that $\delta f^* = (\delta f)^*$. Hence the mapping $f \to f^*$ induces a homomorphism of $H^m(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$ into $H^m(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$. Now, if $f \in Z^m(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$ is such that $f\{a^1 \times b^1, \cdots, a^m \times b^m\} = 0$, whenever one of the a^i is equal to 1_α, it is easy to verify that $f\{a^1 \times 1_\beta, \cdots, a^m \times 1_\beta\} \in Z^\circ(1_\alpha \times \mathfrak{B}, \mathfrak{P})$, and that $f\{a^1 \times b^1, \cdots, a^m \times b^m\} = (1_\alpha \times b^1 \cdots b^m) \cdot f\{a^1 \times 1_\beta, \cdots, a^m \times 1_\beta\}$. Therefore, if we define an element g of $Z^m(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$ by setting $g\{a^1, \cdots, a^m\} = f\{a^1 \times 1_\beta, \cdots, a^m \times 1_\beta\}$ we have $f = g^*$. This, in conjunction with Lemma 3, show that our homomorphism maps $H^m(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$ onto $H^m(\mathfrak{A} \times \mathfrak{B}, \mathfrak{P})$.

Now consider an element $f \in Z^m(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$ such that f^* is a coboundary, say $f^* = \delta G$. By Lemma 1, we may suppose that $f\{a^1, \cdots, a^m\} = 0$, whenever one of the a_i is equal to 1_α. Then we have $f^*\{a^1 \times b^1, \cdots, a^m \times b^m\} = 0$, whenever one of the a^i is equal to 1_α, and $f^*\{a^1 \times 1_\beta, \cdots, a^m \times 1_\beta\} = f\{a^1, \cdots, a^m\}$. When $m = 1$, and there is an element $G \in \mathfrak{P}$ such that $f^*\{a \times b\} = (a \times b)\cdot G - G\cdot(a \times b)$, then $(1_\alpha \times b)\cdot G - G\cdot(1_\alpha \times b) = f^*\{1_\alpha \times b\} = 0$, i.e. $G \in Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P})$, and we conclude that our homomorphism is actually an isomorphism.

In general (for $m \geqq 2$), we see from Lemma 3 that G may be chosen such that $G\{a^1 \times b^1, \cdots, a^{m-1} \times b^{m-1}\} = 0$, whenever one of the a^i is equal to 1_α. It follows then that

$$G\{a^1 \times b^1, \cdots, a^{m-1} \times b^{m-1}\} = (1_\alpha \times b^1 \cdots b^{m-1})\cdot G\{a^1 \times 1_\beta, \cdots, a^{m-1} \times 1_\beta\},$$

and that, if we define $g\{a^1, \cdots, a^{m-1}\} = G\{a^1 \times 1_\beta, \cdots, a^{m-1} \times 1_\beta\}$, we have $g \in C^{m-1}(\mathfrak{A}, Z^0(1_\alpha \times \mathfrak{B}, \mathfrak{P}))$. Clearly, $f = \delta g$, and we conclude that our homomorphism is an isomorphism.

The general problem of determining the cohomology groups of the Kronecker product of two arbitrary algebras appears to be rather difficult. We shall prove one relevant result which is related to Theorem 6.

Let $\mathfrak{B}$ be an algebra of N by N matrices with elements in an arbitrary algebra $\mathfrak{A}$. If $\mathfrak{P}$ is any $\mathfrak{A}$-module define $\tilde{\mathfrak{P}}$ as the direct sum of N^2 linear spaces $\mathfrak{P}_{ij}$, each isomorphic with $\mathfrak{P}$, so that every $\tilde{u} \in \tilde{\mathfrak{P}}$ can be written as an N by N matrix (u_{ij}), with $u_{ij} \in \mathfrak{P}$. If $b = (a_{ij}\{b\})$ is any matrix in $\mathfrak{B}$, we define

$$b\cdot(u_{ij}) = \left(\sum_{k=1}^{N} a_{ik}\{b\}\cdot u_{kj}\right) \quad \text{and} \quad (u_{ij})\cdot b = \left(\sum_{k=1}^{N} u_{ik}\cdot a_{kj}\{b\}\right).$$

Then $\tilde{\mathfrak{P}}$ is a $\mathfrak{B}$-module. In this notation, we can state our result as follows:

THEOREM 7. *If there exists an index q such that, for each $a \in \mathfrak{A}$, $\mathfrak{B}$ contains a diagonal matrix $(a_{ij}) = D\{a\}$, with $a_{qq} = a$, then there is an isomorphism of $H^m(\mathfrak{A}, \mathfrak{P})$ into $H^m(\mathfrak{B}, \tilde{\mathfrak{P}})$.*

Given any $f \in C^n(\mathfrak{A}, \mathfrak{P})$, we define an element $\tilde{f} \in C^n(\mathfrak{B}, \mathfrak{P})$ by setting $\tilde{f}\{b_1, \cdots, b_n\} = (\tilde{f}_{ij}\{b_1, \cdots, b_n\})$, where $\tilde{f}_{ij}\{b_1, \cdots, b_n\} = \sum_{\{k\}=1}^{N} f\{a_{ik_1}\{b_1\}, a_{k_1k_2}\{b_2\}, \cdots, a_{k_{n-1}j}\{b_n\}\}$. Then $\delta\tilde{f} = \tilde{\delta f}$, whence we see that the mapping $f \to \tilde{f}$ induces a homomorphism of $H^m(\mathfrak{A}, \mathfrak{P})$ into $H^m(\mathfrak{B}, \tilde{\mathfrak{P}})$. Now, for the diagonal matrices $D\{a_1\}, \cdots, D\{a_n\}$, it is clear that $\tilde{f}\{D\{a_1\}, \cdots, D\{a_n\}\}$ is a diagonal matrix, with $\tilde{f}_{qq}\{D\{a_1\}, \cdots, D\{a_n\}\} = f\{a_1, \cdots, a_n\}$. If $\tilde{f} = \delta\tilde{g}$, it is easily verified that $f = \delta g$, where $g\{a_1, \cdots, a_n\} = \tilde{g}_{qq}\{D\{a_1\}, \cdots, D\{a_n\}\}$, which proves Theorem 7.

This result implies that the class (in the sense of the introduction) of a Kronecker product of two algebras with identity elements is at least as high as that of either factor.

6. Extensions of Algebras.

It is trivial that every algebra possess extensions in which it is segregated. Our problem is to investigate under what conditions an algebra has extensions in which it is not segregated. It is known that a separable algebra is segregated in every extension. In fact, as was shown in CG, this is an easy consequence of the generalized form of Wedderburn's structure theorem which states that if an algebra $\mathfrak{A}$, with radical $\mathfrak{R}$, is such that $\mathfrak{A}/\mathfrak{R}$ is separable, then $\mathfrak{A}/\mathfrak{R}$ is segregated in $(\mathfrak{A}, \omega)$, where ω is the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}$. One of our results will show that the condition that $\mathfrak{A}/\mathfrak{R}$ be separable cannot be relaxed.[6] More precisely, if $\mathfrak{S}$ is any inseparable semi simple algebra, there exists an algebra $\mathfrak{A}$, with radical $\mathfrak{R}$, such that $\mathfrak{A}/\mathfrak{R}$ is isomorphic with $\mathfrak{S}$, but is not segregated in $(\mathfrak{A}, \omega)$.

At the other extreme, we show that every nilpotent algebra possesses (nilpotent) extensions in which it is not segregated. These results might suggest that the property of being absolutely segregated characterizes the separable algebras. However, it will be shown by an example that this is not the case. The property in question distinguishes a certain subclass of the class of algebras with a non-zero radical which might deserve some special attention.

A necessary and sufficient condition for an algebra to be absolutely segregated is that every two dimensional cocycle be a coboundary (CG, section 6). From an $\mathfrak{A}$-module $\mathfrak{P}$, and an $f \in Z^2(\mathfrak{A}, \mathfrak{P})$ we can construct an extension $(\mathfrak{B}, \omega)$ of $\mathfrak{A}$ in which $\mathfrak{A}$ is segregated if and only if f is a coboundary. The underlying linear space of $\mathfrak{B}$ is the direct sum $(\mathfrak{A}, \mathfrak{P})$ of $\mathfrak{A}$ and $\mathfrak{P}$, the elements of which we shall denote by (a, u), with $a \in \mathfrak{A}$ and $u \in \mathfrak{P}$. We define multiplication by the formula $(a_1, u_1)(a_2, u_2) = (a_1a_2, a_1 \cdot u_2 + u_1 \cdot a_2 + f\{a_1, a_2\})$. The associativity condition for this multiplication is exactly the condition that f be a cocycle. If we define $\omega\{(a, u)\} = a$, the pair $(\mathfrak{B}, \omega)$ is the desired extension of $\mathfrak{A}$.

7. Inseparable Semi Simple Algebras.

In this section, we wish to prove the following theorem:

THEOREM 8. *Let $\mathfrak{A}$ be an inseparable semi-simple algebra. Then $\mathfrak{A}$ is not absolutely segregated, or equivalently, there exists an $\mathfrak{A}$-module $\mathfrak{P}$ such that $H^2(\mathfrak{A}, \mathfrak{P}) \neq \{0\}$.*

Since $\mathfrak{A}$ is semi-simple it is the direct sum of simple two sided ideals. Since $\mathfrak{A}$ is inseparable the groundfield, Φ, must be of characteristic $p \neq 0$, and the center of at least one of the simple components of $\mathfrak{A}$, say $\mathfrak{A}_1$, must be an inseparable extension field, Ψ, of Φ. It is clearly sufficient to construct a non-cobounding two dimensional cocycle of $\mathfrak{A}_1$.

Let Σ be the largest field between Φ and Ψ which is separable over Φ. Then there exists an element $x \in \Psi$ which is of degree p over Σ and such that $x^p \in \Sigma$. We have $\Phi \subsetneqq \Sigma \subset \Sigma(x) \subsetneqq \Psi$. We define a differentiation D of $\Sigma(x)$ over Σ, and hence also over $\Phi \subsetneqq \Sigma$, by setting $D\{x^q\} = qx^{q-1}$, for $1 \leqq q \leqq p - 1$, and $D\{\sigma\} = 0$, for every $\sigma \in \Sigma$. Now let $f\{z_1, z_2\} = \sum_{r=1}^{p-1} (p-1)!/r!(p-r)! \cdot D^r\{z_1\}D^{p-r}\{z_2\}$, for $z_1, z_2 \in \Sigma(x)$. (Note that $(p-1)!/r!(p-r)!$ is an integer, since p is a prime). If we regard $\Sigma(x)$ as a $\Sigma(x)$-module for $\Sigma(x)$ over Φ, the

[6] The fact that this condition is not altogether superfluous is shown by an example in "*Algebren,*" by MAX DEURING, Ergebnisse der Mathematik, Vol. 4, Ch. 2, para. 11.

transformations being simply the multiplications in $\Sigma(x)$, it is easily verified that $f \in Z^2(\Sigma(x), \Sigma(x))$. We shall show, furthermore, that f is not a coboundary:

Suppose that this is false. Then there exists a Φ-linear mapping g of $\Sigma(x)$ into itself, such that

$$f\{z_1, z_2\} = z_1 g\{z_2\} - g\{z_1 z_2\} + g\{z_1\} z_2 .$$

Since $f\{\sigma_1, \sigma_2\} = 0$, for all $\sigma_1, \sigma_2 \in \Sigma$, we see that the restriction of g to Σ belongs to $Z^1(\Sigma, \Sigma(x))$. Since Σ is separable over Φ, this implies that $g\{\sigma\} = 0$, for all $\sigma \in \Sigma$. Since $D^s\{x^u\} = 0$, when $s > u$, we have $f\{x^u, x^v\} = 0$, whenever $u + v < p$. Hence $g\{x^{u+v}\} = x^u g\{x^v\} + g\{x^u\}x^v$, whenever $u + v < p$. It follows that $g\{x^s\} = sx^{s-1}g\{x\}$, for $1 \leqq s \leqq p - 1$. It is now evident that g must be a differentiation in $\Sigma(x)$, which implies that f is identically zero. On the other hand, we have $f\{x, x^{p-1}\} = (p - 1)! \neq 0$. This contradiction shows that f cannot be a coboundary.

Now the regular representation of $\mathfrak{A}_1$, regarded as an algebra over $\Sigma(x)$, provides us with an isomorphic mapping of the elements of $\mathfrak{A}_1$ into square matrices with elements in $\Sigma(x)$. By this mapping, the elements of $\Sigma(x)$ are mapped into the $\Sigma(x)$-multiples of the identity matrix. Therefore, we may apply Theorem 7 in order to complete the proof of our theorem.

8. Nilpotent Algebras.

If $\mathfrak{A}$ is any algebra we can construct an $\mathfrak{A}$-module whose underlying linear space is the Kronecker product $\mathfrak{A} \times \mathfrak{A}$ by defining, for $a, a_1, a_2 \in \mathfrak{A}$, $a \cdot (a_1 \times a_2) = aa_1 \times a_2$, and $(a_1 \times a_2) \cdot a = a_1 \times a_2 a$. Set $f\{a_1, a_2\} = a_1 \times a_2$. Then it is easily seen that f is a two dimensional $\mathfrak{A} \times \mathfrak{A}$-cocycle of $\mathfrak{A}$.

Now suppose that $\mathfrak{A}$ has an absolute divisor of zero, i.e. a non-zero element a_0 such that $a_0\mathfrak{A} = \{0\} = \mathfrak{A}a_0$. Then both the left and the right transformation of $\mathfrak{A} \times \mathfrak{A}$ by a_0 are zero. It follows that f cannot be a coboundary since this would imply that $f\{a_0, a_0\} = 0$. Therefore, we have the following theorem:

THEOREM 9. *Every algebra with an absolute divisor of zero has an extension in which it is not segregated.*

This applies, in particular, to nilpotent algebras, for, if $\mathfrak{A}^{m+1} = \{0\}$, and $\mathfrak{A}^m \neq \{0\}$, we may take for a_0 any non-zero element of $\mathfrak{A}^m$.[7]

9. Absolutely Segregated Algebras.

Every separable algebra is absolutely segregated. Theorem 8 implies that any other absolutely segregated algebra must have a non-zero radical. By Theorem 9, such an algebra cannot be nilpotent. The following is the simplest example of an algebra which is absolutely segregated, but not separable.

Let Φ be an arbitrary field, $\mathfrak{A}$ the two dimensional algebra with basis (a, r) over Φ, where $aa = a$, $rr = 0$, $ar = r$, $ra = 0$. Then, if f is any two dimensional

[7] Actually, a similar construction gives a stronger result for nilpotent algebras: If also $\mathfrak{A}^2 \neq \mathfrak{A}$ there is an $\mathfrak{A}$-module $\mathfrak{P}$ such that $H^3(\mathfrak{A}, \mathfrak{P}) \neq \{0\}$.

cocycle of $\mathfrak{A}$, we may define a linear mapping g of $\mathfrak{A}$ into the relevant $\mathfrak{A}$-module by setting

$$g\{a\} = 2f\{a, a\}\cdot a - f\{a, a\}, \text{ and}$$

$$g\{r\} = 2f\{r, a\}\cdot a - f\{r, a\} + f\{a, r\}\cdot a - f\{a, r\},$$

and it can be verified directly that $f = \delta g$. Thus, $\mathfrak{A}$ is absolutely segregated.

It follows immediately from our general results that the class of absolutely segregated algebras has certain closure properties:

THEOREM 10. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be absolutely segregated algebras, $\mathfrak{S}$ a separable algebra over Φ. Suppose that $\mathfrak{A}$ has an identity element, and denote by $\mathfrak{B}^*$ the algebra obtained from $\mathfrak{B}$ by adjoining an identity element. Then $\mathfrak{B}^*$, $\mathfrak{A} \times \mathfrak{S}$, and $\mathfrak{A} + \mathfrak{B}$ are absolutely segregated.*

If $\mathfrak{A}$ is absolutely segregated and $\mathfrak{T}$ is a two sided ideal in $\mathfrak{A}$, under what conditions is $\mathfrak{A}/\mathfrak{T}$ absolutely segregated? The answer is given in the following theorem:

THEOREM 11. *If $\mathfrak{A}$ is absolutely segregated then $\mathfrak{A}/\mathfrak{T}$ is absolutely segregated if and only if $H^2(\mathfrak{A}/\mathfrak{T}, \mathfrak{T}/\mathfrak{T}^2) = \{0\}$.*

The condition is obviously necessary. On the other hand, if $\mathfrak{P}$ is any $\mathfrak{A}/\mathfrak{T}$-module, and if we denote the coset modulo $\mathfrak{T}$ of an element $a \in \mathfrak{A}$ by $\bar{a}$, we may define $a*u = \bar{a}\cdot u$, and $u*a = u\cdot\bar{a}$, thus making $\mathfrak{P}$ an $\mathfrak{A}$-module. If $f \in Z^2(\mathfrak{A}/\mathfrak{T}, \mathfrak{P})$ we define an element $\check{f}$ of $Z^2(\mathfrak{A}, \mathfrak{P})$ by setting $\check{f}\{a_1, a_2\} = f\{\bar{a}_1, \bar{a}_2\}$. Since $\mathfrak{A}$ is absolutely segregated there exists $G \in C^1(\mathfrak{A}, \mathfrak{P})$ such that $f\{\bar{a}_1, \bar{a}_2\} = \bar{a}_1 \cdot G\{a_2\} - G\{a_1a_2\} + G\{a_1\}\cdot\bar{a}_2$. Let us select a linear mapping ρ of $\mathfrak{A}/\mathfrak{T}$ into $\mathfrak{A}$ such that $\overline{\rho\{\bar{a}\}} = \bar{a}$, for all $\bar{a} \in \mathfrak{A}/\mathfrak{T}$. Define $g\{a\} = G\{\rho\{\bar{a}\}\}$. Then $(\check{f} - \delta g)\{a_1, a_2\} = G\{-\rho\{\bar{a}_1\}\rho\{\bar{a}_2\} + \rho\{\bar{a}_1\bar{a}_2\}\}$. Since the expression $\bar{a}_1\cdot G\{a_2\} - G\{a_1a_2\} + G\{a_1\}\cdot\bar{a}_2$ depends only on $\bar{a}_1$ and $\bar{a}_2$ we must have $G\{\mathfrak{T}^2\} = \{0\}$. If $\varphi\{\bar{a}_1, \bar{a}_2\}$ denotes the coset modulo $\mathfrak{T}^2$ of the expression $\rho\{\bar{a}_1\}\rho\{\bar{a}_2\} - \rho\{\bar{a}_1\bar{a}_2\}$ it is easily seen that $\varphi \in Z^2(\mathfrak{A}/\mathfrak{T}, \mathfrak{T}/\mathfrak{T}^2)$.

If the condition of our theorem is satisfied there exists a linear mapping ψ of $\mathfrak{A}/\mathfrak{T}$ into $\mathfrak{T}/\mathfrak{T}^2$ such that $\varphi\{\bar{a}_1, \bar{a}_2\} = \bar{a}_1\cdot\psi\{\bar{a}_2\} - \psi\{\bar{a}_1\bar{a}_2\} + \psi\{\bar{a}_1\}\cdot\bar{a}_2$. Now select a linear mapping σ of $\mathfrak{A}/\mathfrak{T}$ into $\mathfrak{T}$ such that the coset modulo $\mathfrak{T}^2$ of $\sigma(\bar{a})$ is $\psi(\bar{a})$. Put $\rho^*\{\bar{a}\} = \rho\{\bar{a}\} - \sigma\{\bar{a}\}$. Since the cosets modulo $\mathfrak{T}^2$ of $\rho\{\bar{a}\}i$ and $i\rho\{\bar{a}\}$ are the transforms by $\bar{a}$ of the coset of i ($\in \mathfrak{T}$), the expression $\rho^*\{\bar{a}_1\}\rho^*\{\bar{a}_2\} - \rho^*\{\bar{a}_1\bar{a}_2\}$ is easily seen to be an element of T^2. It follows that if we define $g^*\{\bar{a}\} = G\{\rho^*\{\bar{a}\}\}$ we have $f = \delta g^*$. This completes the proof.

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