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On some commutative subalgebras of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$

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Abstract. For the Lie algebra $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{C})$ it is proved that the maximal commutative subalgebras of the Poisson algebra $P(\mathfrak{g})$ obtained by the method of shifting the invariants can be lifted to the enveloping algebra. Moreover, this lifting can be carried out by means of the symmetrization map.

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Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. The algebra $U(\mathfrak{g})$ has an ascending filtration $U(\mathfrak{g})=\bigcup_{k=0}^{\infty}U^k(\mathfrak{g})$ in which $U^k(\mathfrak{g})$ is the subspace of all elements representable as polynomials of degree $\leqslant k$ in elements of $\mathfrak{g}$. We denote by $P(\mathfrak{g})$ the associated graded algebra (the Poisson algebra), that is,

$$P(\mathfrak{g})=\operatorname{gr}U(\mathfrak{g})=\bigoplus_{k=0}^{\infty}P_k(\mathfrak{g}),\qquad P_k(\mathfrak{g})=U^k(\mathfrak{g})/U^{k-1}(\mathfrak{g}).$$

By the Poincaré–Birkhoff–Witt theorem $P(\mathfrak{g})$ is canonically isomorphic to the symmetric algebra of the space $\mathfrak{g}$. The algebra $P(\mathfrak{g})$ can be naturally endowed with a Lie operation (Poisson bracket) $\{\,\cdot\,,\cdot\,\}$ such that $\{P_k(\mathfrak{g}),P_l(\mathfrak{g})\}\subset P_{k+l-1}(\mathfrak{g})$. Namely, for

$$u\in U^k(\mathfrak{g}),\qquad v\in U^l(\mathfrak{g})$$

we set

$$\{u+U^{k-1}(\mathfrak{g}),v+U^{l-1}(\mathfrak{g})\}=[u,v]+U^{k+l-2}(\mathfrak{g}).$$

The *Leibniz identity* connects the Poisson bracket with multiplication:

$$\{x,yz\}=\{x,y\}z+y\{x,z\}.$$

The algebras $U(\mathfrak{g})$ and $P(\mathfrak{g})$ can be regarded as Lie algebras with respect to the commutation operation and Poisson bracket, respectively. The algebra $\mathfrak{g}$ can be canonically embedded in these Lie algebras. For $x\in\mathfrak{g}$ we identify the image of x in $P(\mathfrak{g})$ with x itself (and denote it by the same letter), and we denote the

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image of x in $U(\mathfrak{g})$ by $\widehat{x}$. We also extend the embedding $\mathfrak{g} \to U(\mathfrak{g})$ to a $\mathfrak{g}$-module isomorphism $P(\mathfrak{g}) \to U(\mathfrak{g})$ such that

$$x_1 \cdots x_k \mapsto \widehat{x_1 \cdots x_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \widehat{x}_{\sigma(1)} \cdots \widehat{x}_{\sigma(n)} \qquad (x_1, \dots, x_k \in \mathfrak{g}).$$

Let $\mathfrak{g} = \mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. We denote by e_{ij} $(1 \leqslant i, j \leqslant n)$ the matrix units forming a basis of the Lie algebra $\mathfrak{g}$, and define the 'principal minors of order p' as the elements

$$M_{i_1 \dots i_p} = \sum_{\sigma \in S_p} (\operatorname{sgn} \sigma) e_{i_1 i_{\sigma(1)}} \cdots e_{i_p i_{\sigma(p)}} \qquad (i_1 < \dots < i_p),$$

where multiplication is understood in the sense of an operation in the Poisson algebra $P(\mathfrak{g})$. The sums

$$F_p = \sum_{i_1 < \dots < i_p} M_{i_1 \dots i_p} \qquad (p = 1, \dots, n)$$

are algebraically independent and generate the centre Z of the Lie algebra $P(\mathfrak{g})$, see [1]. (As is known, Z consists of the invariants of the natural Lie algebra action of $\mathfrak{g}$ on $P(\mathfrak{g})$.) We note that the elements $\widehat{F}_r$ $(r = 1, \dots, n)$ generate the centre of $U(\mathfrak{g})$.

We write $N_{i_1 \dots i_k} := M_{1 \dots \hat{\imath}_1 \dots \hat{\imath}_2 \dots \hat{\imath}_k \dots n}$ $(i_1 < \dots < i_k)$, where the symbol $\hat{\jmath}$ indicates that the index j is missing. We have $N_{i_{\sigma(1)} \dots i_{\sigma(k)}} := N_{i_1 \dots i_k}$ for each $\sigma \in S_k$.

Let $\partial_{ij} \in \operatorname{Der} P(\mathfrak{g})$ be differentiation with respect to e_{ij}. For a matrix $a = (a_{ij})$ we set

$$\partial_a u = \sum_{i,j} a_{ij} \partial_{ij} u \qquad (u \in P(\mathfrak{g})).$$

Let $h = \operatorname{diag}(h_1, \dots, h_n)$, where $h_1, \dots, h_n$ are all distinct. In this case, by a general result of Fomenko and Mishchenko [1] the elements

$$F_{p,k}(h_1, \dots, h_n) = \partial_h^k F_p \qquad (p = 1, \dots, n; \quad k = 0, \dots, p-1)$$

of the Poisson algebra $P(\mathfrak{g})$ are algebraically independent and commute with respect to the Poisson bracket.

Let $L(h)$ be the commutative subalgebra of the Lie algebra $P(\mathfrak{g})$ generated by the elements $F_{p,k}(h_1, \dots, h_n)$. As is known [2], the transcendence degree of $L(h)$ is the largest among all commutative subalgebras of the Lie algebra $P(\mathfrak{g})$. It is also known [2] that if it is possible to lift the generators of $L(h)$ to commuting elements of $U(\mathfrak{g})$, then the subalgebra generated by these elements has the largest transcendence degree among the commutative subalgebras of the algebra $U(\mathfrak{g})$. The existence of such a lifting is proved in [3], but its explicit form is unknown to the authors. The aim of the present paper is to show that such a lifting can be realized by the map '$\widehat{\ }$'. Namely, we prove the following result.

Theorem. *For arbitrary* $h_1, \dots, h_n \in \mathbb{C}$ *the elements* $\widehat{F}_{p,k} \equiv \widehat{F}_{p,k}(h_1, \dots, h_n)$ $(p = 1, \dots, n;\ k = 0, 1, \dots, p-1)$ *commute.*

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Proof of the theorem

1. We shall find a necessary and sufficient condition for the elements $\widehat{F}_{p,k}$ ($p = 1, \dots, n$; $k = 0, \dots, p-1$) to commute for all $h_1, \dots, h_n \in \mathbb{C}$.

For each subset $I \subset \{1, \dots, n\}$ we denote the cardinality of this set by $|I|$. The notation $I \perp U$ means that $I \cap U = \varnothing$, and we write IU in place of $I \cup U$ if $I \perp U$. For $I = \{i_1, \dots, i_k\}$ we set $h_I := h_{i_1} \cdots h_{i_k}$.

It is clear that

$$\partial_h^k F_n = \sum_I h_I N_I,$$

and therefore

$$\widehat{\partial_h^k F_n} = \sum_I h_I \widehat{N}_I,$$

where the sum is taken over all k-tuples $I \subset \{1, \dots, n\}$.

In a similar way,

$$\widehat{\partial_h^k F_p} = \sum_I h_I \sum_{U \perp I} \widehat{N}_{IU},$$

where $|I| = k$ and $|U| = n - p$.

Hence

$$\begin{aligned} [\widehat{\partial_h^k F_p}, \widehat{\partial_h^l F_q}] &= \Bigl[\sum_I h_I \sum_{U \perp I} \widehat{N}_{IU}, \sum_J h_J \sum_{V \perp J} \widehat{N}_{JV}\Bigr] \\ &= \sum_I \sum_J h_I h_J \Bigl[\sum_{U \perp I} \widehat{N}_{IU}, \sum_{V \perp J} \widehat{N}_{JV}\Bigr], \end{aligned}$$

where $|I| = k$, $|U| = n - p$, $|J| = l$, and $|V| = n - q$. Collecting similar terms we see that the elements $\widehat{F}_{p,k}(h_1, \dots, h_n)$ commute for all $h_1, \dots, h_n$ if and only if

$$Y_{S,T}(k, l, p, q) := \sum_{\substack{I \cup J = S \\ I \cap J = T}} \Bigl[\sum_{U \perp I} \widehat{N}_{IU}, \sum_{V \perp J} \widehat{N}_{JV}\Bigr] = 0 \tag{1}$$

for all $S, T \subset \{1, \dots, n\}$ with $T \subset S$.

2. We shall prove the relations $Y_{S,T}(k, l, p, q) = 0$ by induction on n.

These relations obviously hold for $n = 1$. We assume that they hold for matrices of order $< n$ and prove that these relations hold also for matrices of order n.

Lemma 0. *If $T \neq \varnothing$, then $Y_{S,T}(k, l, p, q) = 0$.*

Proof. We note that for each matrix unit $\widehat{e}_{\gamma\delta}$ entering expression (1) the indices γ and δ do not belong to T. We denote by $\mathfrak{g}'$ the subalgebra of $\mathfrak{g}$ generated by such matrix units. Then $Y_{S,T}(k, l, p, q) = Y'_{S \setminus T, \varnothing}(k - t, l - t, p, q)$, where $t = |T|$ and the prime means that the corresponding object relates to the Lie algebra $\mathfrak{g}'$. Hence, if $T \neq \varnothing$, then $Y_{S,T}(k, l, p, q) = 0$ by the inductive assumption.

It remains to treat the case when $T = \varnothing$. We set

$$Y_S := Y_S(k, l, p, q) := Y_{S,\varnothing}(k, l, p, q) = \sum_{\substack{I \cup J = S \\ I \perp J}} \Bigl[\sum_{U \perp I} \widehat{N}_{IU}, \sum_{V \perp J} \widehat{N}_{JV}\Bigr], \tag{2}$$

where $|I| = k$, $|U| = n - p$, $|J| = l$, and $|V| = n - q$ as above. The indices in S are said to be *principal*, and the other indices are said to be *simple*.

3. We note that

$$Y_S = \sum_{\substack{I \cup J = S \\ I \perp J}} \sum_{U \perp S} \sum_{V \perp S} [\widehat{N}_{IU}, \widehat{N}_{JV}] + \sum_{\substack{I \cup J = S \\ I \perp J}} \sum_{\substack{U \perp I,\, V \perp J \\ (I \cap V) \cup (J \cap U) \neq \varnothing}} [\widehat{N}_{IU}, \widehat{N}_{JV}]. \qquad (3)$$

We denote by Z_S the first term and by X_S the second. Along with the relation $Y_S = 0$ we shall prove by induction the equality $Z_S = 0$ (the basis of the induction is obvious). We claim that X_S=0 if the inductive assumption holds.

We decompose X_S into several subsums by collecting together the terms with the same intersections $I_0 = I \cap V$ and $J_0 = J \cap U$. Each of these subsums is equal to $Z'_{S \setminus (I_0 \cup J_0)}(k - k_0, l - l_0, p - l_0, q - k_0)$, where $k_0 = |I_0|$, $l_0 = |J_0|$, and the prime means that our notation relates to the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ generated by the matrix units $\widehat{e}_{\gamma\delta}$ with indices γ, δ not belonging to $I_0 \cup J_0$. By the inductive assumption all these subsums are equal to zero. Hence $X_S = 0$, and therefore $Y_S = Z_S$.

4. Let M be an arbitrary set of indices, $M \subset \{1, \dots, n\}$. We denote by $\mathfrak{gl}(M)$ the subalgebra generated by the elements e_{ij}, $i, j \in M$, and by $\mathrm{GL}(M)$ the corresponding subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Lemma 1. *For arbitrary simple indices γ and δ the equality $[\widehat{e}_{\gamma\delta}, Y_S]$=0 holds.*

Proof. This is obvious: each sum in the commutators in (2) commutes with $\widehat{e}_{\gamma\delta}$ since each sum of minors before the lift is an invariant of the action of the group GL_{n-k}, where $k = |U|$ for the left-hand sum and $k = |V|$ for the right-hand one.

Lemma 2. *For arbitrary principal indices γ and δ, $[\widehat{e}_{\gamma\delta}, Z_S] = 0$ (and therefore $[\widehat{e}_{\gamma\delta}, Y_S] = 0$).*

Proof. We consider an auxiliary element

$$Z_{S'} = Z_{S'}(k-1, l-1, p-1, q-1) = \sum_{\substack{I' \cup J' = S' \\ I' \perp J'}} \Bigg[\sum_{U' \perp S'} \widehat{N}_{I'U'}, \sum_{V' \perp S'} \widehat{N}_{J'V'} \Bigg],$$

where $S' = S \setminus \{\gamma, \delta\}$. Since γ and δ are simple indices for $Z_{S'}$, it follows by Lemma 1 that $[\widehat{e}_{\gamma\delta}, Z_{S'}] = 0$.

For an arbitrary commutator of the form $[\widehat{N}_{I'U'}, \widehat{N}_{J'V'}]$ in $Z_{S'}$ we examine whether the indices γ and δ belong to the sets U' and V'. There are 16 cases of the possible arrangement (each of the two indices can either belong or not belong to each of the two sets). For each arrangement we denote by $Z(*, *)$ the sum of all the corresponding commutators replacing the first (the second) asterisk by the index γ or δ that belongs to U' (to V', respectively). We set $H(*, *) = [\widehat{e}_{\gamma\delta}, Z(*, *)]$.

It is clear that

$$Z_S = Z_1 + Z(\gamma, \delta) + Z(\delta, \gamma),$$

where Z_1 is the sum of commutators in Z_S such that either $\gamma, \delta \in I$ or $\gamma, \delta \in J$ (see (3)).

It is also clear that $e_{\gamma\delta}$ commutes with each unlifted minor that either contains both indices γ and δ or none. Consequently, $\widehat{e}_{\gamma\delta}$ commutes with Z_1; moreover, $H(\varnothing,\varnothing)=H(\varnothing,\gamma\delta)=H(\gamma\delta,\varnothing)=H(\gamma\delta,\gamma\delta)=0$.

We claim that $H(\gamma,\varnothing)+H(\delta,\varnothing)=0$. We note first that

$$Z(\gamma,\varnothing)+Z(\delta,\varnothing)=\sum_{\substack{I'\cup J'=S'\\ I'\perp J'}}\ \sum_{\substack{U'\perp S';\, U''=(U'\cup\delta)\setminus\gamma\\ \gamma\in U';\,\delta\notin U'}}\ \sum_{\substack{V'\perp S'\\ \gamma,\delta\notin V'}}[\widehat{N}_{I'U'}+\widehat{N}_{I'U''},\widehat{N}_{J'V'}]. \quad (4)$$

It is clear that $[\widehat{e}_{\gamma\delta},\widehat{N}_{I'U'}+\widehat{N}_{I'U''}]=0$ because the last sum of minors was invariant before lifting under the action of the subgroup $\mathrm{GL}(\{\gamma,\delta\})$.

We can prove in a similar way that

$$H(\varnothing,\gamma)+H(\varnothing,\delta)=H(\gamma,\gamma\delta)+H(\delta,\gamma\delta)=H(\gamma\delta,\gamma)+H(\gamma\delta,\delta)=0.$$

By the inductive assumption $H(\gamma,\gamma)+H(\gamma\delta,\gamma)+H(\gamma,\gamma\delta)+H(\gamma\delta,\gamma\delta)=H(\delta,\delta)+H(\gamma\delta,\delta)+H(\delta,\gamma\delta)+H(\gamma\delta,\gamma\delta)=0$.

Hence

$$\begin{aligned}0&=[H(\gamma,\gamma)+H(\gamma\delta,\gamma)+H(\gamma,\gamma\delta)+H(\gamma\delta,\gamma\delta)]\\&\qquad+[H(\delta,\delta)+H(\gamma\delta,\delta)+H(\delta,\gamma\delta)+H(\gamma\delta,\gamma\delta)]\\&=[H(\gamma,\gamma)+H(\delta,\delta)]+[H(\gamma,\gamma\delta)+H(\delta,\gamma\delta)]+[H(\gamma\delta,\gamma)+H(\gamma\delta,\delta)]+2H(\gamma\delta,\gamma\delta)\\&=H(\gamma,\gamma)+H(\delta,\delta).\end{aligned}$$

Finally,

$$\begin{aligned}[\widehat{e}_{\gamma\delta},Z_S]&=[\widehat{e}_{\gamma\delta},Z(\gamma,\delta)+Z(\delta,\gamma)]\\&=[\widehat{e}_{\gamma\delta},Z_{S'}]-H(\varnothing,\varnothing)-H(\varnothing,\gamma\delta)-H(\gamma\delta,\varnothing)-H(\gamma\delta,\gamma\delta)\\&\quad-[H(\gamma,\gamma)+H(\delta,\delta)]-[H(\gamma,\gamma\delta)+H(\delta,\gamma\delta)]-[H(\gamma\delta,\gamma)+H(\gamma\delta,\delta)]\\&\quad-[H(\varnothing,\gamma)+H(\varnothing,\delta)]-[H(\gamma,\varnothing)+H(\delta,\varnothing)]=0.\end{aligned}$$

5. Removing the parentheses in (2) we obtain a linear combination of products of the form $\widehat{e}_{p_1q_1}\cdots\widehat{e}_{p_sq_s}$ in which the number of occurrences of an arbitrary index as the first and as the second subscript is the same, and it is $\leqslant 1$ for principal and $\leqslant 2$ for simple indices. A possible reduction of the degree by virtue of relations of the form

$$\widehat{e}_{pq}\widehat{e}_{qr}-\widehat{e}_{qr}\widehat{e}_{pq}=\widehat{e}_{pr}-\delta_{pr}\widehat{e}_{qq}$$

preserves this property.

Let m be the least integer such that $Z_S\in U^m(\mathfrak{g})$, and let W_S be the projection of Z_S onto $P_m(\mathfrak{g})$. It is clear that $[e_{\gamma\delta},W_S]=0$ for arbitrary simple (principal) indices γ and δ.

Let $U(\mathfrak{g})^{\mathfrak{h}}$ be the subspace of $U(\mathfrak{g})$ formed by the elements commuting with $\mathfrak{h}$ (the algebra of diagonal matrices). Let us introduce the subspace $U'(\mathfrak{g})\subset U(\mathfrak{g})^{\mathfrak{h}}$ of elements representable as linear combinations of products of the form $\widehat{e}_{p_1q_1}\cdots\widehat{e}_{p_sq_s}$ in which each principal index occurs at most once as the first (or, accordingly, as the second) subscript and define in a similar way the subspace $P'(\mathfrak{g}) \subset P(\mathfrak{g})^{\mathfrak{h}}$. Then $\operatorname{gr} U'(\mathfrak{g}) = P'(\mathfrak{g})$. It is clear that $W_S \in P'_m(\mathfrak{g}) = P'(\mathfrak{g}) \cap P_m(\mathfrak{g})$.

6. Let τ be an anti-automorphism of the Lie algebra $U(\mathfrak{g})$ sending $\widehat{e}_{ij}$ to $\widehat{e}_{ji}$. It is clear that τ preserves the lifted minors, that is, $\tau(\widehat{N}_I) = \widehat{N}_I$. Hence τ multiplies the commutators of the minors $[\widehat{N}_{IU}, \widehat{N}_{JV}]$ by -1, therefore the same holds for the elements Y_S: $\tau(Y_S) = -Y_S$. We note also that τ preserves the filtration and therefore generates an automorphism of the Lie algebra $P(\mathfrak{g})$ such that $\tau(W_S) = -W_S$.

In the next subsections we prove that $\tau(W_S) = W_S$. This formula yields the relation $Y_S = 0$.

7. Let S be the set of principal indices and T the set of simple indices.

Main lemma. *Let W be an element of the space $P'_m(\mathfrak{g})$ that is invariant under the actions of the groups* $\mathrm{GL}(S)$ *and* $\mathrm{GL}(T)$. *Then $\tau(W) = W$.*

8. Definition. We denote by

$$P^R_m(\mathfrak{g}) \subset P'_m(\mathfrak{g})$$

the space of homogeneous elements with fixed set $R \subset S$ of principal indices (the same one for the first and second subscripts). A linear transformation π on $P^R_m(\mathfrak{g})$ is defined as the averaging over the permutations of equal parity of the first and second principal subscripts.

We note that one can define the transformation π on the entire space $P'_m(\mathfrak{g})$ by representing this space as a direct sum of subspaces of the form $P^R_m(\mathfrak{g})$.

Lemma 3. *The equality $\pi W = W$ holds.*

Lemma 3 is an obvious consequence of the next result.

Lemma 4. *If an element W contains a term of the form $ce_{i\gamma}e_{j\delta}f$ (where i and j are distinct principal indices, γ and δ are some indices, and f is the product of the other factors), then W contains also the term $-ce_{j\gamma}e_{i\delta}f$. A similar result holds for principal indices occurring as second subscripts.*

Proof. By the assumption of the main lemma $[\mathfrak{gl}(S), W] = 0$. In particular, $[e_{ij}, W] = 0$. However, $[e_{ij}, W] = [e_{ij}, ce_{i\gamma}e_{j\delta}f + \cdots] = ce_{i\gamma}e_{i\delta}f + \cdots$. Finally, the term $ce_{i\gamma}e_{i\delta}f$ must cancel out with some other terms after commutation, and this is possible only if the term $-ce_{j\gamma}e_{i\delta}f$ is present.

Obviously, $\pi\tau = \tau\pi$.

9. Lemma 5. Each element $x \in P(\mathfrak{g})$ invariant with respect to $\mathfrak{gl}(M)$, where $M \subset \{1, \ldots, n\}$, can be expressed as a polynomial in the variables

(1) $A_{ij} = e_{ij}$,
(2) $B_{ij}(l) = \sum_{\gamma_1,\ldots,\gamma_l} e_{i\gamma_1}e_{\gamma_2\gamma_3}\cdots e_{\gamma_l j}$, $l \geqslant 1$,
(3) $C(l) = \sum_{\gamma_1,\ldots,\gamma_l} e_{\gamma_1\gamma_2}e_{\gamma_2\gamma_3}\cdots e_{\gamma_l\gamma_1}$, $l \geqslant 1$,

where $i, j \in \{1, \ldots, n\} \setminus M$ and $\gamma_1, \ldots, \gamma_l \in M$.

Proof. The action of $\mathrm{GL}(M)$ on $\mathfrak{g}$ can be regarded as an action on a set consisting of a linear operator, $n-m$ linear forms, $n-m$ vectors, and $(n-m)^2$ scalars, where $m = |M|$. The lemma follows from the description of the invariants of the system of tensors of the group $\mathrm{GL}_m(\mathbb{C})$ (see [4]).

Definition. By a *decomposable invariant* we mean a product of elements of the form A_{ij}, $B_{ij}(l)$, and $C(l)$, where $i,j \in S$ and $\gamma_1, \gamma_2, \ldots \in T$.

By Lemma 5 an element W can be represented as a linear combination of decomposable invariants in which the collection of first principal subscripts and the collection of second principal subscripts are the same ($M = T$).

10. Proof of the main lemma. For a fixed term u in the expression for W we consider a decomposable invariant containing u. Assume that this decomposable invariant has the following form:

$$A_{i_1 j_1} \cdots A_{i_s j_s} B_{p_1 q_1}(l_1) \cdots B_{p_t q_t}(l_t) C(m_1) \cdots C(m_z).$$

Then

$$\begin{aligned} u = (e_{i_1 j_1} \cdots e_{i_s j_s})(e_{p_1 \gamma_1^1} e_{\gamma_1^1 \gamma_2^1} \cdots e_{\gamma_{l_1}^1 q_1} \cdots e_{p_t \gamma_1^t} e_{\gamma_1^t \gamma_2^t} \cdots e_{\gamma_{l_t}^1 q_t}) \\ \times (e_{\delta_1^1 \delta_2^1} \cdots e_{\delta_{m_1}^1 \delta_1^1} \cdots e_{\delta_1^z \delta_2^z} \cdots e_{\delta_{m_z}^z \delta_1^z}), \end{aligned}$$

where i_λ, j_λ, p_λ, and q_λ are principal indices and γ_λ^μ and δ_λ^μ are simple indices.

We can associate with the element u a unique term u' in the same decomposable invariant that looks as follows:

$$\begin{aligned} u' = (e_{i_1 j_1} \cdots e_{i_s j_s})(e_{p_1 \gamma_{l_1}^1} e_{\gamma_{l_1}^1 \gamma_{l_1-1}^1} \cdots e_{\gamma_1^1 q_1} \cdots e_{p_t \gamma_{l_t}^t} e_{\gamma_{l_t}^t \gamma_{l_t-1}^t} \cdots e_{\gamma_1^t q_t}) \\ \times (e_{\delta_1^1 \delta_{m_1}^1} \cdots e_{\delta_2^1 \delta_1^1} \cdots e_{\delta_1^z \delta_{m_z}^z} \cdots e_{\delta_2^z \delta_1^z}). \end{aligned}$$

We claim that $\pi u' = \pi\tau u$. For the proof we shall produce a permutation of the first and the second principal subscripts transforming u' into τu.

We write down the sequences of first and second principal subscripts for u' corresponding to the above order of factors in the factorization:

$\sigma_1 = (i_1 i_2 \ldots i_s p_1 \ldots p_t)$ for the first subscripts,

$\sigma_2 = (j_1 j_2 \ldots j_s q_1 \ldots q_t)$ for the second subscripts.

Consider now the permutation $\sigma := \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$. To obtain τu from u' we must apply σ to σ_1 and σ^{-1} to σ_2. Obviously, $\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma^{-1}) = 1$. Thus, the parity is as required and the relation $\pi u' = \pi\tau u$ follows from Lemma 3.

Considering in the same way other terms of our decomposable invariant we obtain the equality $\pi u' = \pi\tau u$.

Applying this relation to all decomposable invariants participating in the decomposition of W we see that $\pi W = \pi\tau W$. However, $\pi W = W$ and $\pi\tau = \tau\pi$. Thus, $W = \pi W = \pi\tau W = \tau\pi W = \tau W$.

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