Harish-Chandra Subalgebras and Gelfand-Zetlin Modules

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ABSTRACT. A new framework for the study of some modules over algebras is elaborated and applied to a new class of representations of Lie algebra $\mathcal{GL}(n)$.

1. Abstract Harish-Chandra situation

1.1. Cofinite spectrum of an algebra.

Through the whole chapter we fix a field K. All considered algebras and categories will be K-algebras and K-linear categories. Respectively, all homomorphisms and functors will be K-linear. We shall write Hom, $\otimes$, dim etc. instead of Hom_K, $\otimes_K$, $\dim_K$ etc. For any algebra or category A denote A^o the opposite algebra or category.

Denote $\mathrm{cfs}(\Gamma)$ the *cofinite spectrum* of an algebra Γ, i.e. the set of maximal ideals of finite codimension in Γ. If $\mathbf{m} \in \mathrm{cfs}(\Gamma)$, then $\Gamma/\mathbf{m} \simeq M_{\nu(\mathbf{m})}(K(\mathbf{m}))$ where $K(\mathbf{m})$ is a finite dimensional division algebra over K. In particular, if K is algebraically closed, then $K(\mathbf{m}) = K$. Let $S_{\mathbf{m}}$ be the only simple left $\Gamma/\mathbf{m}$-module and $DS_{\mathbf{m}} = \mathrm{Hom}(S_{\mathbf{m}}, K)$ the only simple right $\Gamma/\mathbf{m}$-module. Then $\mathbf{m} \mapsto S_{\mathbf{m}}$ (or $DS_{\mathbf{m}}$) is a 1-1 correspondence between $\mathrm{cfs}(\Gamma)$ and the set of isomorphism classes of simple left (or, resp., right) finite-dimensional Γ-modules.

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Put $\Gamma_{\mathbf{m}} = \varprojlim_n \Gamma/\mathbf{m}^n$, the $\mathbf{m}$-adique completion of Γ, and $J_{\mathbf{m}} = \varprojlim_n \mathbf{m}/\mathbf{m}^n$ (this is an ideal in $\Gamma_{\mathbf{m}}$).

Proposition 1.

1. $J_{\mathbf{m}} = \mathrm{Rad}\Gamma_{\mathbf{m}}$ *(the Jacobson radical).*
2. $\Gamma_{\mathbf{m}} \simeq M_{\nu(\mathbf{m})}(\Delta_{\mathbf{m}})$ *where* $\Delta_{\mathbf{m}}$ *is a local ring.*
3. $\Delta_{\mathbf{m}}/R_{\mathbf{m}} \simeq K(\mathbf{m})$ *where* $R_{\mathbf{m}} = \mathrm{Rad}\Delta_{\mathbf{m}}$.

The proof is evident.

Sometimes the following simple observation is useful.

Proposition 2. *If K is algebraically closed, then*

$$\mathrm{cfs}(\Gamma) \times \mathrm{cfs}(\Lambda) \simeq \mathrm{cfs}(\Gamma \otimes \Lambda)$$

Namely, this bijection is given by:

$$(\mathbf{m}, \mathbf{n}) \longmapsto \mathbf{m} \otimes \Lambda + \Gamma \otimes \mathbf{n}$$

Moreover, the corresponding simple left (right) $\Gamma \otimes \Lambda$-module is $S_{\mathbf{m}} \otimes S_{\mathbf{n}}$ (resp., $DS_{\mathbf{m}} \otimes DS_{\mathbf{n}}$).

The proof is immediately reduced to the finite-dimensional case, where it is quite evident.

1.2. Quasi-commutative algebras.

Call an algebra Γ *quasi-commutative* provided $\mathrm{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$ for all $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma), \mathbf{m} \neq \mathbf{n}$.

Example 3.

1. Of course, any commutative algebra as well as any semi-simple [1] one is quasi-commutative.
2. Let $\Gamma = U(\mathcal{G})$ be the universal envelopping algebra of a finite-dimensional Lie algebra $\mathcal{G}$. If $\mathrm{char}K = 0$ and $\mathcal{G}$ is either reductive or nilpotent, then Γ is quasi-commutative [1].

Proposition 4. *Let $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma), \mathbf{m} \neq \mathbf{n}$, and suppose that $\mathbf{m}$ is finitely generated as left ideal. Then the following conditions are equivalent:*

1. $\mathrm{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$

[1] "semi-simple" will always mean "semi-simple artinian".

2. $\mathbf{n} \cap \mathbf{m} = \mathbf{nm}$

3. $\mathbf{mn} \subseteq \mathbf{nm}$

Proof. Remark that $\mathbf{n} + \mathbf{m} = \Gamma$, whence $\mathbf{n} \cap \mathbf{m} = \mathbf{nm} + \mathbf{mn}$. Thus *2.* $\Longleftrightarrow$ *3.*

1. $\Longrightarrow$ *2.* Consider the exact sequence

$$0 \longrightarrow \mathbf{m}/\mathbf{nm} \longrightarrow \Gamma/\mathbf{nm} \longrightarrow \Gamma/\mathbf{m} \longrightarrow 0 \tag{1}$$

Here $\Gamma/\mathbf{m} \simeq \nu(\mathbf{m})S_{\mathbf{m}}$ and $\mathbf{m}/\mathbf{nm} \simeq kS_{\mathbf{n}}$ for some integer k. Hence, (1) splits and there are left ideals M, N in Γ such that:

$$M + N = \Gamma; \;\; M \cap N = \mathbf{nm}; \;\; \Gamma/M \simeq \nu(\mathbf{m})S_{\mathbf{m}}; \;\; \Gamma/N \simeq kS_{\mathbf{n}}$$

Therefore, $\mathbf{m} \subseteq M, \mathbf{n} \subseteq N$ and

$$\mathbf{n} \cap \mathbf{m} \subseteq N \cap M = \mathbf{nm} \subseteq \mathbf{n} \cap \mathbf{m}$$

i.e. $\mathbf{n} \cap \mathbf{m} = \mathbf{nm}$.

2. $\Longrightarrow$ *1.* Consider any exact sequence of the form:

$$0 \longrightarrow S_{\mathbf{n}} \longrightarrow M \longrightarrow S_{\mathbf{m}} \longrightarrow 0 \tag{2}$$

Evidently, $\mathbf{nm}M = 0$, i.e. M is a module over the algebra

$$\Gamma/\mathbf{nm} = \Gamma/\mathbf{n} \cap \mathbf{m} \simeq \Gamma/\mathbf{n} \times \Gamma/\mathbf{m}$$

which is semi-simple. Hence, (2) splits and $\mathrm{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$ Q.E.D.

Proposition 5. *If Γ is a finitely generated algebra and I is a left ideal of finite codimension in Γ, then I is finitely generated as left ideal.*

Proof. Let G be a generating set of Γ and B be a basis of Γ/I. For each $b \in B$ fix its representative $\overline{b} \in \Gamma$ and for any $x = \sum_i \lambda_i b_i$ with $\lambda_i \in K, b_i \in B$, put $\overline{x} = \sum_i \lambda_i \overline{b}_i$. Then it is easy to check that the set

$$\{ \; g\overline{b} - \overline{gb} \mid g \in G, b \in B \; \}$$

generates I as left ideal Q.E.D.

Corollary 6. *If Γ is a finitely generated algebra, then the following conditions are equivalent:*

1. *Γ is quasi-commutative.*

2. *If $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$ and $\mathbf{m} \neq \mathbf{n}$, then $\mathbf{m} \cap \mathbf{n} = \mathbf{nm}$.*

3. *If $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$, then $\mathbf{mn} = \mathbf{nm}$.*

Corollary 7. *If Γ is quasi-commutative, then so is Γ^o.*

1.3. Harish-Chandra subalgebras.

Let Γ be a subalgebra of an algebra A. Call Γ *quasi-central* (in A) if for any element $a \in A$ the bimodule $\Gamma a \Gamma$ is finitely generated both as left and as right Γ-module.

Proposition 8. *Suppose that Γ is noetherian and G is a set of generators of the algebra A. Then Γ is quasi-central in A if and only if $\Gamma g \Gamma$ is finitely generated both as left and as right Γ-module for each $g \in G$.*

The proof is evident as $\Gamma(ab)\Gamma \subseteq (\Gamma a \Gamma)(\Gamma b \Gamma)$ and $\Gamma(a+b)\Gamma \subseteq \Gamma a \Gamma + \Gamma b \Gamma$.

Example 9.

1. Of course, if Γ is central (i.e. contained in the centre of A), it is also quasi-central.
2. Let $A = U(\mathcal{G})$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{G}$ is a finite-dimensional Lie algebra and $\mathcal{H}$ its Lie subalgebra. Then one can easily check that $\Gamma\mathcal{G} = \mathcal{G}\Gamma$. By Proposition 8, Γ is quasi-central in A.

Now, call the subalgebra $\Gamma \subseteq A$ a *Harish-Chandra subalgebra* provided it is both quasi-central and quasi-commutative.

Example 10.

1. Any central subalgebra is a Harish-Chandra one.
2. Suppose that char$K = 0$. If $A = U(\mathcal{G})$ for a finite-dimensinal Lie algebra $\mathcal{G}$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{H}$ is either reductive or nilpotent Lie subalgebra of $\mathcal{G}$, then Γ is a Harish-Chandra subalgebra of A.
3. One more example - the *Gelfand-Zetlin subalgebra* - will be considered below.

From now on, let Γ be a Harish-Chandra subalgebra of A. Put $\Gamma^e = \Gamma \otimes \Gamma^o$. For any $a \in A$ consider the Γ-bimodule epimorphism $\phi_a : \Gamma^e \longrightarrow \Gamma a \Gamma$ mapping $\beta \otimes \gamma^o$ to $\beta a \gamma$. Let $I_a = \mathrm{Ker}\phi_a$ (it is a left ideal in Γ^e). Define the subset $X_a \subseteq \mathrm{cfs}(\Gamma)^2$ by the rule:

$$X_a = \{\ (\mathbf{m}, \mathbf{n}) \mid S_{\mathbf{n}} \text{ is a composition factor of } \Gamma a \Gamma / \Gamma a \mathbf{m} \text{ as of left } \Gamma\text{-module }\}$$

Proposition 11. *The following conditions are equivalent:*

1. $(\mathbf{m}, \mathbf{n}) \in X_a$.
2. *$DS_{\mathbf{m}}$ is a composition factor of $\Gamma a \Gamma / \mathbf{n} a \Gamma$ as of right Γ-module.*

3. $\mathbf{n}a\Gamma + \Gamma a\mathbf{m} \neq \Gamma a\Gamma$.

4. $\mathbf{n} \otimes \Gamma^o + \Gamma \otimes \mathbf{m}^o + I_a \neq \Gamma^e$.

Proof. Put $M = \Gamma a\Gamma/\Gamma a\mathbf{m}$. As $\Gamma a\Gamma$ is finitely generated right Γ-module and $\Gamma/\mathbf{m}$ is finite-dimensional, M is also finite-dimensional. Hence M considered as left Γ-module has a composition series with factors isomorphic to $S_{\mathbf{l}}$ for some ideals $\mathbf{l} \in \mathrm{cfs}(\Gamma)$. But as Γ is quasi-commutative, $S_{\mathbf{n}}$ is a composition factor of M if and only if it is isomorphic to a factor-module of M which means, of course, that $\mathbf{n}M \neq M$. Therefore, *1.* $\Longleftrightarrow$ *3.*

Quite analogousely, *2.* $\Longleftrightarrow$ *3.* At last, *3.* $\Longleftrightarrow$ *4.* is evident, Q.E.D.

Corollary 12. *For any* $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ *and* $a \in A$ *the set*

$$X_a(\mathbf{m}) = \{\ \mathbf{n} \in \mathrm{cfs}(\Gamma) \mid (\mathbf{m}, \mathbf{n}) \in X_a\ \}$$

is finite.

Denote $\prec$ the least preorder relation on $\mathrm{cfs}(\Gamma)$ containing all X_a (i.e. such that $(\mathbf{m}, \mathbf{n}) \in X_a$ implies $\mathbf{m} \prec \mathbf{n}$) and Δ the least equivalence relation containing all X_a. Put also $\nabla = \prec \cap \prec^{-1}$ (the equivalence relation associated with the preorder $\prec$). Let $\Delta\mathrm{m}$ (resp., $\nabla\mathrm{m}$) denotes the equivalence class of Δ (resp., ∇) containing $\mathbf{m}$ and $\Delta(A,\Gamma)$ (resp., $\nabla(A,\Gamma)$) denotes the set of all equivalence classes of Δ (resp., ∇).

1.4. Harish-Chandra modules.

Remind that we consider a fixed Harish-Chandra subalgebra $\Gamma \subseteq A$. For an A-module M and an ideal $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ put

$$M(\mathbf{m}) = \{\ x \in M \mid \exists k(\mathbf{m}^k x = 0)\ \}$$

Call M a *Harish-Chandra module (with respect to* Γ*)* if $M = \coprod_{\mathbf{m} \in \mathrm{cfs}(\Gamma)} M(\mathbf{m})$. Of course, as Γ is quasi-commutative, M is a Harish-Chandra module if and only if it is a sum of finite-dimensional Γ-submodules. Remark that any submodule or factor-module of a Harish-Chandra module is also a Harish-Chandra module.

Example 13. Let $\mathrm{char} K = 0$, $A = U(\mathcal{G})$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{G}$ is a finite-dimensional Lie algebra and $\mathcal{H}$ its semi-simple Lie subalgebra. Then the notion of Harish-Chandra modules coincides with the usual definition of Harish-Chandra $\mathcal{G}$-modules with respect to $\mathcal{H}$ (cf. [1]).

Denote $\mathbf{H}(A,\Gamma)$ the category of all Harish-Chandra A-modules with respect to Γ and $\mathrm{Irr}(A,\Gamma)$ the set of isomorphism classes of simple modules from$\mathbf{H}(A,\Gamma)$.

Proposition 14. *For any* $a \in A$ *and* $\mathbf{m} \in \mathrm{cfs}(\Gamma)$

$$aM(\mathbf{m}) \subseteq \coprod_{\mathbf{n} \in X_a(\mathbf{m})} M(\mathbf{n})$$

Proof. If $x \in M(\mathbf{m})$, then Γx has a composition series with all factors isomorphic to $S_{\mathbf{m}}$. Of course, $ax \in \Gamma a \Gamma x$. The last module is an epimorphic image of $\Gamma a \Gamma \otimes_\Gamma \Gamma x$. But $\Gamma a \Gamma \otimes_\Gamma S_{\mathbf{m}}$ has a composition series with the factors isomorphic to $S_{\mathbf{n}}$ for $\mathbf{n} \in X_a(\mathbf{m})$. Hence, the same is true for $\Gamma a \Gamma \otimes_\Gamma \Gamma x$ and for $\Gamma a \Gamma x$. As Γ is quasi-commutative, we obtain that

$$\Gamma a \Gamma x \subseteq \coprod_{\mathbf{n} \in X_a(\mathbf{m})} M(\mathbf{n})$$

Q.E.D.

For any $D \subseteq \mathrm{cfs}(\Gamma)$, put $M(D) = \coprod_{\mathbf{m} \in D} M(\mathbf{m})$. If $R \subseteq \mathrm{cfs}(\Gamma)^2$ is a relation on $\mathrm{cfs}(\Gamma)$, call D $R-$*closed* provided $\mathbf{m} \in D$ and $(\mathbf{m}, \mathbf{n}) \in R$ implies $\mathbf{n} \in D$. Call the *support* of M the set

$$\mathrm{Supp} M = \{ \mathbf{m} \in \mathrm{cfs}(\Gamma) \mid M(\mathrm{m}) \neq 0 \}$$

Corollary 15. *Let* $M \in \mathbf{H}(A, \Gamma)$.

1. *If* $D \subseteq \mathrm{cfs}(\Gamma)$ *is* $\prec$*-closed, then* $M(D)$ *is a submodule of* M.
2. $M = \coprod_{D \in \Delta(A,\Gamma)} M(D)$ *as* A*-module.*
3. *If* M *is indecomposable and* $M(\mathbf{m}) \neq 0$, *then* $\mathrm{Supp} M \subseteq \Delta \mathbf{m}$.
4. *If* M *is irreducible and* $M(\mathbf{m}) \neq 0$, *then* $\mathrm{Supp} M \subseteq \nabla \mathbf{m}$.

Denote $\mathbf{H}(A, \Gamma, D)$ the full subcategory of $\mathbf{H}(A, \Gamma)$ consisting of all modules M with $\mathrm{Supp}(M) \subseteq D$ and $\mathrm{Irr}(A, \Gamma, D)$ the set of isomorphism classes of simple modules from $\mathbf{H}(A, \Gamma, D)$.

Corollary 16.

1. $\mathbf{H}(A, \Gamma) = \coprod_{D \in \Delta(A,\Gamma)} \mathbf{H}(A, \Gamma, D)$ *(the direct sum of categories).*
2. $\mathrm{Irr}(A, \Gamma) = \bigsqcup_{D \in \nabla(A,\Gamma)} \mathrm{Irr}(A, \Gamma, D)$ *(the disjoint union of sets).*

1.5. Category $\mathcal{A}$.

Define a new category $\mathcal{A} = \mathcal{A}_{A,\Gamma}$ in the following way. The set of objects $\mathrm{Ob}\mathcal{A} = = \mathrm{cfs}(\Gamma)$. The set of morphisms from $\mathbf{m}$ to $\mathbf{n}$ is

$$\mathcal{A}(\mathbf{m},\mathbf{n}) = \varprojlim_{n,m} A/(\mathbf{n}^n A + A\mathbf{m}^m)$$

To define the multiplication $\mathcal{A}(\mathbf{n},\mathbf{l}) \times \mathcal{A}(\mathbf{m},\mathbf{n}) \longrightarrow \mathcal{A}(\mathbf{m},\mathbf{l})$, take any two elements $a, b \in A$ and consider the left Γ-module $M = \Gamma a\Gamma/\Gamma a\mathbf{m}^m$ and the right Γ-module $L = \Gamma b\Gamma/\mathbf{l}^\ell b\Gamma$. Both of them are finite-dimensional as Γ is quasi-central. Moreover, as Γ is quasi-commutative,

$$M = M_0 \oplus M_1 \text{ where } \mathbf{n}^n M_0 = 0 \text{ and } \mathbf{n}M_1 = M_1$$

and

$$L = L_0 \oplus L_1 \text{ where } L_0\mathbf{n}^n = 0 \text{ and } L_1\mathbf{n} = L_1$$

for some natural n.

Therefore, $a = a_0 + a_1$, $b = b_0 + b_1$ where:

$$\mathbf{n}^n a_0 \in A\mathbf{m}^m; \; a_1 \in \mathbf{n}^n A + A\mathbf{m}^m; \; b_0\mathbf{n}^n \in \mathbf{l}^\ell A; \; b_1 \in \mathbf{l}^\ell A + A\mathbf{n}^n$$

Now it is obvious that the class of $b_0 a_0$ in $A/(\mathbf{l}^\ell A + A\mathbf{m}^m)$ depends only on the classes of a and b in $A/(\mathbf{n}^n A + A\mathbf{m}^m)$ and in $A/(\mathbf{l}^\ell A + A\mathbf{n}^n)$ respectively. Of course, it makes possible to define the needed multiplication.

Suppose that M is a Harish-Chandra module. If $x \in M(\mathbf{m})$, then $\mathbf{m}^m x = 0$ for some m. For an element $a \in A$ and an ideal $\mathbf{n} \in \mathrm{cfs}(\Gamma)$ choose n as above. Then the projection of ax onto $M(\mathbf{n})$ again depends only on the class of a in $A/(\mathbf{n}^n A + A\mathbf{m}^m)$. Therefore, for any element $\alpha \in \mathcal{A}(\mathbf{m},\mathbf{n})$ we are able to define the product $\alpha x \in M(\mathbf{n})$. In other words, the correspondence $\mathbf{m} \longmapsto M(\mathbf{m})$ becomes a functor from the category $\mathcal{A}$ to the category **Vect** of vector spaces over K. Moreover, this functor is continuous if we consider the discrete topology on vector spaces and the natural topology of the inverse limite on the sets $\mathcal{A}(\mathbf{m},\mathbf{n})$. Call such functors *discrete $\mathcal{A}$-modules* or simply *$\mathcal{A}$-modules.*

If N is any $\mathcal{A}$-module, then we can construct the corresponding Harish-Chandra module as $\coprod_{\mathbf{m}} N(\mathbf{m})$. To define the product ax for $a \in A$, $x \in N(\mathbf{m})$, put $ax = \sum_{\mathbf{n}} a_{\mathbf{n}} x$ where $a_{\mathbf{n}}$ denotes the image of a in $\mathcal{A}(\mathbf{m},\mathbf{n})$. This sum is finite due to Corollary 12.

Hence, we obtain the following result.

Theorem 17. *The category $\mathbf{H}(A,\Gamma)$ of Harish-Chandra modules is equivalent to the category $\mathcal{A} - \mathbf{mod}$ of discrete $\mathcal{A}$-modules.*

Of course, the image of a in $\mathcal{A}(\mathbf{m},\mathbf{n})$ is non-zero if and only if $(\mathbf{m},\mathbf{n}) \in X_a$. Therefore,

$$\mathcal{A} = \coprod_{D\in\Delta(A,\Gamma)} \mathcal{A}(D)$$

where $\mathcal{A}(D)$ is the full subcategory of $\mathcal{A}$ consisting of all objects $\mathbf{m} \in D$.

The following result from *general nonsence* seems to be rather known though we have never seen it published.

Theorem 18. *For any object $\mathbf{m} \in \mathrm{Ob}\mathcal{A}$ let $\mathrm{Irr}(\mathbf{m})$ denotes the set of isomorphism classes of simple $\mathcal{A}$-modules M such that $M(\mathbf{m}) \neq 0$. Then there is a 1-1 correspondence between $\mathrm{Irr}(\mathbf{m})$ and the set $\mathrm{Irr}\mathcal{A}(\mathbf{m},\mathbf{m})$ of isomorphism classes of simple (discrete)* [2] *$\mathcal{A}(\mathbf{m},\mathbf{m})$-modules.*

Proof. Let M be an $\mathcal{A}$-module and let $U(\mathbf{m})$ be a non-trivial $\mathcal{A}(\mathbf{m},\mathbf{m})$-submodule of $M(\mathbf{m})$. Put $U(\mathbf{n}) = \mathcal{A}(\mathbf{m},\mathbf{n})U(\mathbf{m})$ for any object $\mathbf{n}$. Then we obtain a non-trivial submodule U of M. Hence, if M is simple and $M(\mathbf{m}) \neq 0$, then $M(\mathbf{m})$ is a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module.

On the other hand, let $N(\mathbf{m})$ be a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module. Put

$$N(\mathbf{n}) = \mathcal{A}(\mathbf{m},\mathbf{n}) \otimes_{\mathcal{A}(\mathbf{m},\mathbf{m})} N(\mathbf{m})$$

Then the set $\{N(\mathbf{n})\}$ can be evidently viewed as an $\mathcal{A}$-module N. We claim that N contains the only maximal submodule N' and $N'(\mathbf{m}) = 0$. Really, if $L \subseteq N$ is a submodule and $L(\mathbf{m}) \neq 0$, then $L(\mathbf{m}) = N(\mathbf{m})$ as the last one is a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module. But $N(\mathbf{m})$ generates N, hence, $L = N$. Therefore, if we denote N' the sum of all proper submodules $L \subset N$, then $N'(\mathbf{n}) = 0$ and N' is the only maximal submodule of N. Thus $M = N/N'$ is a simple $\mathcal{A}$-module with $M(\mathbf{m}) = N(\mathbf{m})$.

Moreover, if M' is any $\mathcal{A}$-module and $\phi : N(\mathbf{m}) \longrightarrow M'(\mathbf{m})$ is a homomorphism of $\mathcal{A}(\mathbf{m},\mathbf{m})$-modules, then it prolongs uniquely to a homomorphism of $\mathcal{A}$-modules $N \to M'$. In particular, if M' is simple with $M'(\mathbf{m}) \simeq M(\mathbf{m})$, then we obtain an epimorphism $\pi : N \to M'$. The kernel of π is a maximal submodule of N, hence it coincides with N' and $M' \simeq N/N' \simeq M$ Q.E.D.

Call the subalgebra Γ *big at the point* $\mathbf{m}$ provided $\mathcal{A}(\mathbf{m},\mathbf{m})$ is finitely generated as $\Gamma(\mathbf{m})$-module (left or right or as bimodule which is equivalent as Γ is quasi-central).

Corollary 19. *Suppose that Γ is big at the point $\mathbf{m}$. Then:*

1. *The set $\mathrm{Irr}(\mathbf{m})$ is finite.*

[2] in the same sence as above

2. *For any simple Harish-Chandra module M the vector space $M(\mathbf{m})$ is finite-dimensional.*

Proof. Put $B = \mathcal{A}(\mathbf{m}, \mathbf{m})$, $J = J_{\mathbf{m}}$ (cf. section 1.1). Then B/BJ is finite-dimensional, hence $J^n B \subseteq BJ$ for some n. If I is a maximal right ideal in B, then $I \supseteq J^n B$ (otherwise $I + BJ \supseteq I + J^n B = B$, whence $I = B$ by Nakayama's lemma). Therefore, $\mathrm{Rad} B \supseteq J^n B$ and $B/\mathrm{Rad} B$ is finite-dimensional, which implies both *1.* and *2.* Q.E.D.

2. Gelfand-Zetlin modules

2.1. Gelfand-Zetlin subalgebra.

In this section we suppose that K is algebraically closed of characteristic 0 and denote $\mathcal{G}_m = \mathcal{GL}(m, K)$, $U_m = U(\mathcal{G}_m)$ and Z_m the centre of U_m. Put $\mathcal{G} = \mathcal{G}_n$, $U = U_n$ and identify $\mathcal{G}_m$ for $m \leq n$ with the Lie subalgebra of $\mathcal{G}$ generated by the matrix units $\{ e_{ij} \mid i, j = 1..m \}$. Then we obtain the inclusions: $\mathcal{G}_1 \subset \mathcal{G}_2 \subset ... \subset \mathcal{G}_n = \mathcal{G}$ and $U_1 \subset U_2 \subset ... \subset U_n = U$. Let Γ be the subalgebra of U generated by $\{ Z_m \mid m = 1..n \}$. Call Γ the *Gelfand-Zetlin subalgebra* of U or *GZ-subalgebra.* In this case the Harish-Chandra U-modules with respect to Γ are called the *Gelfand-Zetlin modules* (or *GZ-modules*) [2]. Respecitvely, we shall denote $\mathbf{GZ}$ and $\mathbf{GZ}(D)$ the categorie of GZ-modules and that of GZ-modules with the support in D (where $D \subseteq \mathrm{cfs}(\Gamma)$). We shall also write in this case $\mathcal{U}$ for the category $\mathcal{A}_{U,\Gamma}$ (cf. section 1.5).

Proposition 20. *Z_m is the polynomial algebra in m variables $\{c_{km} \mid k = 1..m \}$ where*

$$c_{km} = \sum_{i_1, i_2, ..., i_k = 1..m} e_{i_1 i_2} e_{i_2 i_3} ... e_{i_k i_1}$$

(cf.[3]).

Put $\mathcal{L} = K^{n(n+1)/2}$. The elements of $\mathcal{L}$ will be called "tableaux" and considered as double indexed families:

$$\ell = (\ell_{im} \mid m = 1..n;\ i = 1..m)$$

Denote $\mathcal{L}^+$ the subset of $\mathcal{L}$ consisting of all tableaux ℓ such that $\ell_{im} \in \mathbf{Z}$ and $\ell_{im} \geq \ell_{i,m-1} > \ell_{i+1,m}$ for all possible values of i, m. For any vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in K^n$ let

$$\mathcal{L}_\alpha = \{ \ell \in \mathcal{L} \mid \ell_{in} = \alpha_i \text{ for } i = 1..n \}$$

and $\mathcal{L}_\alpha^+ = \mathcal{L}_\alpha \cap \mathcal{L}^+$. Clearly, $\mathcal{L}_\alpha^+ \neq \emptyset$ if and only if $\alpha_i \in \mathbf{Z}$ and $\alpha_i > \alpha_{i+1}$ for all possible i.

It is well-known that all finite-dimensional U-modules are Gelfand-Zetlin ones. Namely, the following statement holds (cf. [3]).

Proposition 21. *Let M be a finite-dimensional simple U-module. Then M possesses a base $\{ [\ell] \mid \ell \in \mathcal{L}_\alpha^+ \}$ for some $\alpha \in K^n$ such that:*

$$c_{km}[\ell] = c_{km}(\ell)[\ell],$$

$$E_m^{\pm}[\ell] = \sum_{i=1}^{m} a_{im}^{\pm}(\ell)[\ell \pm \delta^{im}]$$

where $E_m^+ = e_{m,m+1}$; $E_m^- = e_{m+1,m}$ $(m = 1..n-1)$:

$$c_{km}(\ell) = \sum_{i=1}^{m}(\ell_{im} + m)^k \prod_{j \neq i}\left(1 - \frac{1}{\ell_{im} - \ell_{jm}}\right)$$

$$a_{im}^{\pm}(\ell) = \mp \frac{\prod_j(\ell_{j,m\pm 1} - \ell_{im})}{\prod_{j\neq i}(\ell_{jm} - \ell_{im})}$$

(here $\delta^{im} \in \mathcal{L}$ is the Kronecker symbol: $\delta_{jk}^{im} = 1$ if $i = j, m = k$ and 0 otherwise).

This base is called the *Gelfand-Zetlin base* of M. To precise α, we shall denote $M = M^\alpha$. Remark that the dominant weight of M^α is $(\alpha_1 + 1, \alpha_2 + 2, ..., \alpha_n + n)$.

We shall also widely use the following *Harish-Chandra Theorem* (cf. [1]).

Proposition 22. *Let $u \in U$ is such that $uM = 0$ for any finite-dimensional simple U-module M. Then $u = 0$.*

Consider the polynomial algebra Λ in $n(n+1)/2$ variables λ_{im} where $m = 1..n$; $i = 1..m$. Identify Λ with the algebra of polynomial functions on $\mathcal{L}$ putting $\lambda_{im}(\ell) = \ell_{im}$. Then $\mathcal{L}$ is identified with $\mathrm{cfs}(\Lambda)$. PROPOSITION 21 allows to define the homomorphism $\iota : \Gamma \to \Lambda$ which maps

$$c_{km} \longmapsto \sum_{i=1}^{m}(\lambda_{im} + m)^k \prod_{j\neq i}\left(1 - \frac{1}{\lambda_{im} - \lambda_{jm}}\right)$$

It is not difficult to check that it is really a polynomial of degree k in λ_{im} of the form $\sum_i \lambda_{im}^k + h$ with $\deg h < k$.

The symmetric group S_m acts on Λ permuting λ_{im} ($i = 1..m$; m fixed). Thus the direct product $S = \prod_{m=1}^n S_m$ acts on Λ. As the power sums are algebraically independent and generate the algebra of the symmetric polynomials, we obtain the following

Corollary 23. *ι is an inclusion and its image coincides with the algebra of invariants Λ^S. In particular, Γ is the polynomial algebra in c_{km} ($m = 1..n$; $k = 1..m$).*

From now on identify Γ with its image in Λ. This inclusion induces the surjection $\pi : \mathcal{L} \to \mathrm{cfs}(\Gamma)$ which identifies $\mathrm{cfs}(\Gamma)$ with the orbit set $\mathcal{L}/S$. If M is a GZ-module, write $M(\ell)$ instead $M(\pi(\ell))$ for $\ell \in \mathcal{L}$, $\mathbf{GZ}(D)$ instead of $\mathbf{GZ}(\pi(D))$ for $D \subseteq \mathcal{L}$ etc.

Let $\mathcal{L}_0$ be the subgroup of $\mathcal{L}$ generated by all δ^{im} ($i = 1..m$; $m = 1..n-1$). For two elements $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$ put $\mathbf{m} \equiv \mathbf{n}$ provided there exist $\ell, \ell' \in \mathcal{L}$ such that $\mathbf{m} = \pi(\ell)$, $\mathbf{n} = \pi(\ell')$ and $\ell - \ell' \in \mathcal{L}_0$. Of course, it is an equivalence relation on $\mathrm{cfs}(\Gamma)$. Denote Ω the set of equivalence classes of $\equiv$. Define also two subsets, $\mathcal{L}_1$ and $\mathcal{L}_2$, in $\mathcal{L}$:

$$\mathcal{L}_1 = \{\ \ell \mid \ell_{im} - \ell_{jm} \notin \mathbf{Z} \text{ for all } i \neq j \text{ and } m = 2..n-1\ \}$$

$$\mathcal{L}_2 = \mathcal{L}_1 \cap \{\ \ell \mid \ell_{im} - \ell_{j,m+1} \notin \mathbf{Z} \text{ for all } i, j \text{ and } m = 1..n-1\ \}$$

Evidently, $\mathcal{L}_1$ and $\mathcal{L}_2$ are stable under the congruence modulo $\mathcal{L}_0$ and under the action of the group S. So their images in Ω are well-defined. Denote them Ω_1 and Ω_2 respectively. Remark that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are dense in Zarisky topology on $\mathcal{L}$. Moreover, if $K = \mathbf{C}$, they are dense in usual (euclidean) topology as well.

The main theorem of this chapter is the following one.

Theorem 24.

1. *The Gelfand-Zetlin subalgebra is a Harish-Chandra subalgebra of U.*
2. *$\mathcal{U} = \coprod_{D \in \Omega} \mathcal{U}(D)$*
3. *If $\ell \in \mathcal{L}_1$, then there exists the unique simple GZ-module M with $M(\ell) \neq 0$. Moreover, in this module $\dim(M(\ell)) = 1$.*
4. *If $D \in \Omega_2$, then there exists the unique simple GZ-module M in $\mathbf{GZ}(D)$. Moreover, $\mathrm{Supp}(M) = D$.*

2.2. Some identities in U.

For any element $x \in M^\alpha$ and any tableaux $\ell \in \mathcal{L}_\alpha^+$ let x_ℓ be its $[\ell]$-coefficient with respect to GZ-basis, i.e.

$$x = \sum_{\ell \in \mathcal{L}_\alpha^+} x_\ell [\ell]$$

(cf. PROPOSITION 21). For $u \in U$ denote $\mathcal{L}_u$ the set of all such tableaux $\delta \in \mathcal{L}$ that there exist $\ell \in \mathcal{L}^+$ and $\sigma \in S$ with $(u[\ell])_{\ell + \sigma(\delta)} \neq 0$. As U is generated by the elements $E_m^{\pm}$ $(m = 1..n-1)$, it follows from PROPOSITION 21 that $\mathcal{L}_u$ is finite and $\mathcal{L}_u \subseteq \mathcal{L}_0$. Say that u *relates* ℓ *with* ℓ' provided $\ell' = \sigma(\ell+\delta)$ for some $\sigma \in S$ and $\delta \in \mathcal{L}_u$. Denote $u(\ell)$ the set of all $\ell' \in \mathcal{L}$ such that u relates ℓ with ℓ'. Thus, for any $\ell \in \mathcal{L}^+$ we have:

$$u[\ell] = \sum_{\delta \in \mathcal{L}_u} \theta(u,\ell,\delta)[\ell+\delta]$$

for certain coefficients $\theta(u,\ell,\delta) \in K$ (some of them may be 0).

Any $\delta \in \mathcal{L}$ defines an automorphism $\lambda \mapsto \lambda^\delta$ of Λ where $\lambda_{im}^\delta = \lambda_{im} + \delta_{im}$.

For any $z \in \Gamma$ and $u \in U$ form the polynomial

$$F_{u,z}(T,\lambda) = \prod_{\delta \in \mathcal{L}_u} (T - z^\delta)$$

Clearly, $F_{u,z} \in \Gamma[T]$, as $\mathcal{L}_u$ is, by definition, stable under the action of S.

Lemma 25. *Let* $z \in \Gamma$ *and* $F_{u,z} = \sum_i T^i \phi_i$ *where* $\phi_i \in \Gamma$ *and* i *runs through all possible multy-indeces. Then* $\sum_i z^i u \phi_i = 0$.

Proof. By PROPOSITIONS 21 and 22, we need only to prove that $\sum_i z^i u\phi_i[\ell] = = 0$ for any $\ell \in \mathcal{L}^+$. But

$$\sum_i z^i u\phi_i[\ell] = \sum_i z^i u\phi_i(\ell)[\ell] = \sum_i z^i \phi_i(\ell) \sum_{\delta \in \mathcal{L}^u} \theta(u,\ell,\delta)[\ell+\delta] =$$

$$= \sum_{\delta \in \mathcal{L}^u} \theta(u,\ell,\delta) \sum_i z(\ell+\delta)^i \phi_i(\ell)[\ell+\delta] = \sum_{\delta \in \mathcal{L}^u} \theta(u,\ell,\delta) F_{u,z}(z^\delta(\ell),\ell)[\ell+\delta] = 0$$

Q.E.D.

Remark. The same result remains valid for $z \in Z_m$ if we replace $F_{u,z}$ by

$$F_{u,z,m}(T,\lambda_m) = \prod_{\delta \in \mathcal{L}_{u,m}} (T - z^\delta)$$

where $\lambda_m = (\lambda_{1m}, \lambda_{2m}, ..., \lambda_{mm})$ and $\mathcal{L}_{u,m}$ denotes the set of the m-th rows $(\delta_{1m}, \delta_{2m}, ..., \delta_{mm})$ of all elements $\delta \in \mathcal{L}_u$.

Corollary 26. Γ *is a Harish-Chandra subalgebra in* U.

Proof. Evidently, $F_{u,z} = T^k + \sum_{i<k} T^i \phi_i$ for $k = \mathrm{card}(\mathcal{L}_u)$. So, by LEMMA 25, $z^k u \in \sum_{i=1}^{k-1} z^i u\Gamma$. As Γ is a finitely generated algebra, it follows that $\Gamma u \Gamma$ is a finitely generated Γ-module. But the standard involution of U (mapping $g \in \mathcal{G}$ to $-g$, cf. [1]) maps Γ to Γ. So $\Gamma u \Gamma$ is also finitely generated as left Γ-module Q.E.D.

Corollary 27. *Suppose that* $(\mathbf{m}, \mathbf{n}) \in X_u$ *where* $\mathbf{m} = \pi(\ell)$, $\mathbf{n} = \pi(\ell')$. *Then* $\ell' \in u(\ell)$.

Proof. Let $\ell' \notin u(\ell)$, i.e. $\mathbf{n} \neq \pi(\ell + \delta)$ for all $\delta \in \mathcal{L}_u$. Then there exists $z \in \Gamma$ lying in all $\pi(\ell + \delta)$ but not in $\mathbf{n}$. As $(\mathbf{m}, \mathbf{n}) \in X_u$, there exists $v \in \Gamma u \Gamma / \Gamma u \mathbf{m}$ such that $v \neq 0$ and $\mathbf{n}v = 0$. But we have:

$$0 = \sum_i z^i u \phi_i = \sum_i z^i u(\phi_i - \phi_i(\ell)) + \sum_i z^i u \phi_i(\ell) =$$

$$= u_0 + \sum_i z^i \phi_i(\ell) u = u_0 + F_{u,z}(z, \ell) u$$

where $u_0 \in \Gamma u \mathbf{m}$, whence

$$0 = F_{u,z}(z, \ell) v = F_{u,z}(z(\ell'), \ell) v = \prod_{\delta \in \mathcal{L}_u} (z(\ell') - z(\ell + \delta)) v = z(\ell')^k v$$

This is a contradiction as $v \neq 0$, $z(\ell') \neq 0$ Q.E.D.

Corollary 28. $\Delta \subseteq \equiv$, *i.e.* $(\mathbf{m}, \mathbf{n}) \in \Delta$ *implies* $\mathbf{m} \equiv \mathbf{n}$. [3]

COROLLARY 26 coincides with *p.1.* of THEOREM 24 and COROLLARY 28 evidently implies *p.2.* of it. To prove the rest of the theorem, we need the following observations.

PROPOSITION 21 implies that the coefficients $\theta(u, \ell, \delta)$ are rational functions in ℓ_{im}. So they can be considered as elements of the field of fractions Q of Λ which we denote $\theta(u, \lambda, \delta)$. Moreover, the denominator of $\theta(u, \lambda, \delta)$ is a product of some of $\lambda_{im} + \lambda_{jm} - k$ $(i \neq j)$, where k is some integer. Thus $\theta(u, \ell, \delta)$ is defined for any $\ell \in \mathcal{L}_1$. Remark that $\theta(u, \ell, 0)$ is obviousely S-invariant. Hence, it lies in Q^S which is the field of fractions of $\Lambda^S = \Gamma$.

Lemma 29. *Let again* $z \in \Gamma$. *Put* $\theta_u = \theta(u, \lambda, 0) = \beta_u / \gamma_u$ *where* $\beta_u, \gamma_u \in \Gamma$ *and*

$$F^0_{u,z}(T) = \prod_{\delta \in \mathcal{L}_u \setminus 0} (T - z^\delta) = \sum_i T^i \psi_i$$

Then

$$\gamma_u \sum_i z^i u \psi_i = \beta_u \sum_i z^i \psi_i$$

The proof is quite the same as that of LEMMA 25, so we omit it.

[3] cf. section 1.4 for the definition of the relation Δ.

2.3. MODULES $\mathcal{M}(L)$.

Take a coset $L \in \mathcal{L}/\mathcal{L}_0$ and suppose that $L \subset \mathcal{L}_1$. Consider the vector space $\mathcal{M}(L)$ with the basis $\{ [\ell] \mid \ell \in L \}$ and put, for every $u \in U$:

$$u[\ell] = \sum_{\delta \in \mathcal{L}_u} \theta(u, \ell, \delta)[\ell + \delta]$$

PROPOSITIONS 21 and 22 evidently imply then that $\mathcal{M}(L)$ becomes a GZ-module over U with $\mathrm{Supp}\mathcal{M}(L) = L$ and $\dim\mathcal{M}(L)(\ell) = 1$ for all $\ell \in L$.

For any $\ell \in L$ denote $\mathcal{M}_\ell$ the submodule of $\mathcal{M}(L)$ generated by $[\ell]$.

Theorem 30. *There exists the unique maximal submodule $\mathcal{M}'_\ell \subset \mathcal{M}_\ell$ and the factor-module $V_\ell = \mathcal{M}_\ell/\mathcal{M}'_\ell$ is the unique simple GZ-module with $V_\ell(\ell) \neq 0$.*

Proof. As $\dim\mathcal{M}_\ell(\ell) = 1$, $N(\ell) = 0$ for any proper submodule $N \subset \mathcal{M}_\ell$ which implies the existence and uniqueness of $\mathcal{M}'_\ell$. Hence, V_ℓ is really a well-defined simple GZ-module with $\dim V_\ell(\ell) = 1$. Its uniqueness follows from THEOREM 18 and the next fact.

Proposition 31. *If $\mathbf{m} = \pi(\ell)$ and $\ell \in \mathcal{L}_1$, then $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is isomorphic to a factor-algebra of $\Gamma(\mathbf{m})$.* [4]

Proof. Take any $u \in U$. If $0 \notin \mathcal{L}_u$, then the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is zero by COROLLARY 27. If $0 \in \mathcal{L}_u$, find $z \in \Gamma$ such that $z^\delta \in \mathbf{m}^m$ for all $\delta \in \mathcal{L}_u \setminus 0$ and $z - 1 \in \mathbf{m}^n$. Use LEMMA 29. Here all $\psi_i \in \mathbf{m}^m$ except $\psi_k = 1$ for $k = \mathrm{card}(\mathcal{L}_u \setminus 0)$. So we have $\gamma_u z^k u = \beta_u z^k + u_0$ where $u_0 \in U\mathbf{m}^m$, whence in $U/(\mathbf{m}^n U + U\mathbf{m}^m)$ the images of $\gamma_u u$ and β_u coinside. But as $\ell \in \mathcal{L}_1$, the image of γ_u in $\Gamma(\mathbf{m})$ is invertible. Hence the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ coinsides with that of β_u/γ_u Q.E.D.

THEOREM 30 implies *p.3.* of THEOREM 24. At last, *p.4.* of it is now a consequence of the following theorem.

Theorem 32. *If $D \in \Omega_2$, then all objects in $\mathcal{U}(D)$ are isomorphic.*

Proof. Let $\eta \in \Gamma$ and $u = E^-_m \eta E^+_m$. Denote also $\theta = \theta(u, \lambda, 0) = \beta/\gamma$ with $\beta, \gamma \in \Gamma$. PROPOSITION 21 implies that

$$\theta = \sum_{i=1}^{m} a^-_{im}(\lambda + \delta^{im}) a^+_{im}(\lambda) \eta(\lambda + \delta^{im})$$

[4] Probably, in the case $\mathcal{U}(\mathbf{m}, \mathbf{m}) \simeq \Gamma(\mathbf{m})$ but we have no proof of it. At least, $\mathcal{U}(\mathbf{m}, \mathbf{m}) \neq 0$ as there exist GZ-modules M with $M(\ell) \neq 0$.

(cf. ibid. for notations). Suppose that $\ell \in \mathcal{L}_2$ and put $\mathbf{m} = \pi(\ell)$, $\mathbf{m}_i = \pi(\ell + \delta^{im})$. As $\ell \in \mathcal{L}_1$, we have $\gamma(\ell) \neq 0$. Moreover, the elments ℓ, $\ell + \delta^{im}$, $\ell + \delta^{jm}$, $\ell + \delta^{im} - \delta^{jm}$ $(j \neq i)$ lie in different S-orbits. Hence, they have different images under π and we are able to choose η and $z \in \Gamma$ such that:

$$\eta(\ell + \delta^{im}) = 1, \ \eta(\ell + \delta^{jm}) = 0 \text{ for } j \neq i$$

$$z(\ell) = 1, \ z(\ell + \delta^{im} - \delta^{jm}) = 0 \text{ for } j \neq i$$

Now use LEMMA 29. Remark that in our case $\mathcal{L}_u \setminus 0 = \{ \delta^{im} - \delta^{jm} \mid j \neq i \}$. Therefore, we obtain that $\gamma(\mathbf{m}) \neq 0$ and all $\psi_s(\mathbf{m}) = 0$ except $\psi_k(\mathbf{m}) = 1$ for $k = \text{card}(\mathcal{L}_u \setminus 0)$. Hence, the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is invertible. Denote e_i^+ the image of E_{im}^+ in $\mathcal{U}(\mathbf{m}, \mathbf{m}_i)$ and e_i^- the image of $E_{im}^- \eta$ in $\mathcal{U}(\mathbf{m}_i, \mathbf{m})$. It follows then (just as in the proof of PROPOSITION 31) that $e_i^- e_i^+$ is invertible and e_i^+ is left invertible. Quite analogouse calculation shows that e_i^+ is right invertible. Thus e_i^+ is invertible and $\mathbf{m} \simeq \mathbf{m}_i$ in $\mathcal{U}$. As i, m were arbitrary and $\mathcal{L}_0$ is generated by δ^{im} , it implies the statement Q.E.D.

Corollary 33. *If $L \subset \mathcal{L}_2$, the module $\mathcal{M}(L)$ is the unique simple GZ-module in $\mathbf{GZ}(L)$.*

Now THEOREM 24 is completely proved.

Conjecture. *For any $D \in \Omega$ the set* $\text{Irr}(U, \Gamma, D)$ *is finite.*

Really, this conjecture would follow from the following two:

1. For any $\mathbf{m} \in \text{cfs}(\Gamma)$ the subalgebra Γ is big at the point $\mathbf{m}$, hence the set $\text{Irr}(\mathbf{m})$ *is finite* (cf. COROLLARY 19). *2. For any $D \in \Omega$ there are only finitely many non-isomorphic objects in $\mathcal{U}(D)$.*

References

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[3] Zhelobenko D.P., *Compact Lie groups and their representations*, Nauka, Moscow, 1974.