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Prime ideals in the quantum grassmannian

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Abstract. We consider quantum Schubert cells in the quantum grassmannian and give a cell decomposition of the prime spectrum via the Schubert cells. As a consequence, we show that all primes are completely prime in the generic case where the deformation parameter q is not a root of unity. There is a natural torus action of $\mathcal{H} = (\mathbb{k}^*)^n$ on the quantum grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ and the cell decomposition of the set of $\mathcal{H}$-primes leads to a parameterisation of the $\mathcal{H}$-spectrum via certain diagrams on partitions associated to the Schubert cells. Interestingly, the same parameterisation occurs for the nonnegative cells in recent studies concerning the totally nonnegative grassmannian. Finally, we use the cell decomposition to establish that the quantum grassmannian satisfies normal separation and catenarity.

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Introduction

Let $m \leq n$ be positive integers and let $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ denote the quantum deformation of the affine coordinate ring on $m \times n$ matrices, with nonzero deformation parameter q in the base field. The quantum deformation of the homogeneous coordinate ring of the grassmannian, denoted $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$, is defined as the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ generated by the maximal quantum minors of the generic matrix of $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$. To simplify, these algebras will be referred to as the algebra of quantum matrices and the quantum grassmannian, respectively.

The main goal of this work is the study of the prime spectrum of the quantum grassmannian. As this algebra is naturally endowed with the action of a torus $\mathcal{H}$, it is natural to expect that the *stratification theory* as developed by Goodearl and Letzter (see [1]) will apply to this algebra. Recall that if A is an algebra and $\mathcal{H}$ a torus which acts on A by algebra automorphisms then the stratification theory predicts a partition of the prime spectrum of A into strata, each stratum being indexed by an $\mathcal{H}$-prime ideal (equivalently, a prime ideal invariant under the action of $\mathcal{H}$ in the cases that we will consider). When there are infinitely many $\mathcal{H}$-prime ideals, this partition turns out to be of limited use. However, when there are only finitely many $\mathcal{H}$-prime ideals, then the geometric nature of the prime spectrum of A is fully understood; there are only finitely many strata and each stratum is homeomorphic to the scheme of irreducible subvarieties of a torus. Moreover, still in the case where there are only finitely many $\mathcal{H}$-prime ideals, the primitive ideals of A turn out to be those prime ideals that are maximal within their $\mathcal{H}$-strata, and the Dixmier–Mœglin equivalence is known to be satisfied (under some mild extra assumptions). Thus the finiteness of the set of $\mathcal{H}$-prime ideals is a crucial result in the study of the whole prime spectrum of an algebra supporting a torus action.

For many algebras arising from the theory of quantum groups, general results have been proved about the finiteness of this set. For example, when such an algebra is a certain kind of iterated skew polynomial extension, general results show that it has only finitely many $\mathcal{H}$-primes. However, the algebra which interests us here is far from being such an extension and none of the existing general results can be applied to prove that $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ has only finitely many $\mathcal{H}$-primes. For this reason, we are led to use a very different approach which has a geometric flavour. Recall that a classical approach to the study of the grassmannian variety $G_{m,n}(\mathbb{k})$ is to use its partition into Schubert cells and their closures which are the so-called Schubert subvarieties of the grassmannian. Notice that, in this decomposition, Schubert cells are indexed by Young diagrams fitting in a rectangular Young diagram of size $m \times (n-m)$. Our method is inspired by this classical geometric setting. Indeed, we exhibit a partition of the prime spectrum of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ where each component plays the rôle the Schubert cells play in the classical geometric context. Further, the $\mathcal{H}$-stratification turns out to be a refinement of this partition. In this sense, our approach might be taken to be reminiscent of the approach discussed in [5] for the classical case. However, the results we obtain are more closely related to the non-negative geometry investigated in [17] (see our comment in Section 6).

Quantum analogues of Schubert varieties (or rather of their coordinate rings) were studied in [15] in order to show that the quantum grassmannian has a certain combinatorial structure, namely the stucture of a *quantum graded algebra with a straightening law*. Subsequently, some of their properties have been established in [16]. In this paper, we define quantum Schubert cells as noncommutative dehomogenisations of quantum Schubert varieties. Using the structure of a quantum graded algebra with a straightening law enjoyed by the quantum grassmannian, we are then in a position to define a partition of its $\mathcal{H}$-prime spectrum. This partition is called a *cell decomposition* since it turns out that the set of $\mathcal{H}$-primes of a given component is in natural one-to-one correspondence with the set of $\mathcal{H}$-primes of an associated quantum Schubert cell. Hence, the description of the $\mathcal{H}$-primes of the quantum grassmannian reduces to that of the $\mathcal{H}$-primes of each of its associated quantum Schubert cells. (Here, the actions of $\mathcal{H}$ on the quantum Schubert varieties and cells are naturally induced by its action on the quantum grassmannian.)

On the other hand, we can show that a quantum Schubert cell can be identified as a subalgebra of a quantum matrix algebra, with the variables that are included sitting naturally in the Young diagram associated to that cell; we call these subalgebras *partition subalgebras*. As a consequence, we can establish properties for quantum Schubert cells akin to known properties of quantum matrix algebras. For example, we are able to parameterise the $\mathcal{H}$-prime ideals of a quantum Schubert cell by *Cauchon diagrams* on the corresponding Young diagram, in the same way that Cauchon was able to parameterise the $\mathcal{H}$-prime ideals in quantum matrices (see [3]). This is achieved by using the theory of *deleting derivations* as developed by Cauchon in [2]. Recently, Cauchon diagrams on Young diagrams have appeared elsewhere in the literature under the name of Le-diagrams (see, for example, [17] and [19]).

By using this approach, we are able to show that there are only finitely many $\mathcal{H}$-prime ideals in $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$. More precisely, we show that such $\mathcal{H}$-primes are in natural one-to-one correspondence with Cauchon diagrams defined on Young diagrams fitting into a rectangular $m \times (n-m)$ Young diagram. Following on from this description, we are able to calculate the number of $\mathcal{H}$-prime ideals in the quantum grassmannian.

In addition, we are able to show that prime ideals in the quantum grassmannian are completely prime, and that this algebra satisfies normal separation and, hence, is catenary. Again, the method is to establish these properties for each quantum Schubert cell and then transfer them to the quantum grassmannian.

The paper is organised as follows. In Section 1 basic definitions and properties of the objects that interest us in this paper are introduced. These objects include the quantum grassmannian, quantum Schubert varieties and the notion of a quantum graded algebra with a straightening law. In Section 2 an explicit description of the set of $\mathcal{H}$-prime ideals is worked out in $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$, the first nontrivial case of a quantum grassmannian. This calculation, which illustrates the strategy underlying our approach to the general problem, can be done by hand; and so one is able to see the outline strategy without getting distracted by the technicalities of the general case. As mentioned above, quantum Schubert cells are crucial to our approach and turn out to be isomorphic to certain partition subalgebras of the algebra of quantum matrices. Section 3 is devoted to the study of these partition subalgebras. Notably, their spectrum is investigated and in particular their $\mathcal{H}$-prime ideals. Hence, to a large extent, this section is of preparatory nature. In Section 4 quantum Schubert cells are introduced. Their basic properties are studied and they are shown to be isomorphic to partition subalgebras. Section 5 is at the heart of the present work. Here, it is shown that the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ can be partitioned into cells in such a way that the set of $\mathcal{H}$-primes in a given cell is in bijection with the set of $\mathcal{H}$-primes in an associated Schubert cell. Hence, by using the results of the two preceding sections, we can establish properties of the spectrum of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ and, in particular, we can study its $\mathcal{H}$-prime spectrum. Section 6 concludes the paper by stressing a very interesting connection between our results in the present paper and recent results in the theory of total positivity.

1. Basic definitions

Throughout the paper, $\mathbb{k}$ is a field and q is a nonzero element of $\mathbb{k}$ that is not a root of unity. Occasionally, we will remind the reader of this restriction in the statement of results.

In this section, we collect some basic definitions and properties about the objects we intend to study. Most proofs will be omitted since these results already appear in [12, 15, 16]. Appropriate references will be given in the text.

To start with, recall the following basic definitions. Let R be a ring. A proper ideal P of R is said to be *prime* whenever, for all ideals I, J of R, if $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. In addition, an ideal P of R is said to be *completely prime* if R/P is an integral domain. As is well known, a completely prime ideal is prime but, in general, the converse does not hold. One simplifying factor in the context of this paper is that prime ideals are completely prime in the algebras that we discuss. An element a of R is said to be *normal* if the right ideal and the left ideal it generates in R coincide; that is, if $aR = Ra$.

Let m, n be positive integers. The quantisation of the coordinate ring of the affine variety $M_{m,n}(\mathbb{k})$ of $m \times n$ matrices with entries in $\mathbb{k}$ is denoted $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$. It is the $\mathbb{k}$-algebra generated by mn indeterminates x_{ij}, with $1 \le i \le m$ and $1 \le j \le n$, subject to the relations:

$$\begin{array}{ll} x_{ij}x_{il} = qx_{il}x_{ij} & \text{for } 1 \le i \le m, \text{ and } 1 \le j < l \le n; \\ x_{ij}x_{kj} = qx_{kj}x_{ij} & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j \le n; \\ x_{ij}x_{kl} = x_{kl}x_{ij} & \text{for } 1 \le k < i \le m, \text{ and } 1 \le j < l \le n; \\ x_{ij}x_{kl} - x_{kl}x_{ij} = (q - q^{-1})x_{il}x_{kj} & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j < l \le n. \end{array}$$

To simplify, we write $M_n(\mathbb{k})$ for $M_{n,n}(\mathbb{k})$ and $\mathcal{O}_q(M_n(\mathbb{k}))$ for $\mathcal{O}_q(M_{n,n}(\mathbb{k}))$. The $m \times n$ matrix $\mathbf{X} = (x_{ij})$ is called the *generic matrix* associated with $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$.

As is well known, there exists a $\mathbb{k}$-algebra *transpose isomorphism* between $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ and $\mathcal{O}_q(M_{n,m}(\mathbb{k}))$ (see [15, Remark 3.1.3]). Hence, from now on, we assume that $m \le n$, without loss of generality.

An *index pair* is a pair (I, J) such that $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ are subsets with the same cardinality. Hence, an index pair is given by an integer t such that $1 \le t \le m$ and ordered sets $I = \{i_1 < \cdots < i_t\} \subseteq \{1, \dots, m\}$ and $J = \{j_1 < \cdots < j_t\} \subseteq \{1, \dots, n\}$. To any such index pair we associate the *quantum minor*

$$[I|J] = \sum_{\sigma \in \mathfrak{S}_t} (-q)^{\ell(\sigma)} x_{i_{\sigma(1)}j_1} \cdots x_{i_{\sigma(t)}j_t}.$$

Definition 1.1. The *quantisation of the coordinate ring of the grassmannian of m-dimensional subspaces of* $\mathbb{k}^n$, denoted by $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ and informally referred to as the $(m \times n)$ *quantum grassmannian*, is the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ generated by the $m \times m$ quantum minors.

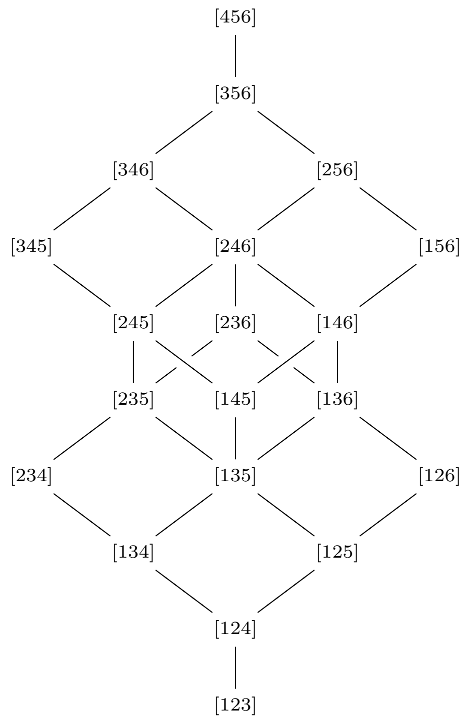


Fig. 1. The partial ordering $\leq_{\text{st}}$ on $\mathcal{O}_q(G_{3,6}(\Bbbk))$.

A maximal quantum minor corresponds to an index pair $(\{1,\dots,m\},J)$ with $J=\{j_1<\cdots<j_m\}\subseteq\{1,\dots,n\}$. We call such J *index sets* and denote the corresponding minor by $[J]$ in what follows. Thus, such a $[J]$ is an element of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. The set of all index sets is denoted by $\Pi_{m,n}$, or sometimes Π when m and n are understood. Since $\Pi_{m,n}$ is in one-to-one correspondence with the set of all maximal quantum minors of $\mathcal{O}_q(M_{m,n}(\Bbbk))$, we will often identify these two sets. We equip $\Pi_{m,n}$ with a partial order $\leq_{\text{st}}$ defined in the following way. Let $I=\{i_1<\cdots<i_m\}$ and $J=\{j_1<\cdots<j_m\}$ be two index sets. Then

$$I \leq_{\text{st}} J \;\Leftrightarrow\; i_s \leq j_s \text{ for } 1 \leq s \leq m.$$

For example, Figure 1 shows the partial ordering on generators of $\mathcal{O}_q(G_{3,6}(\Bbbk))$.

Let A be a noetherian $\Bbbk$-algebra, and assume that the torus $\mathcal{H} := (\Bbbk^*)^r$ acts rationally on A by $\Bbbk$-algebra automorphisms. (For details concerning rational actions of tori, see [1, Chapter II.2]. In particular, note that the action of $\mathcal{H}$ is semisimple and all $\mathcal{H}$-eigenvalues are rational.) A two-sided ideal I of A is called *$\mathcal{H}$-invariant* if $h\cdot I = I$ for all $h\in\mathcal{H}$. An *$\mathcal{H}$-prime ideal* of A is a proper $\mathcal{H}$-invariant ideal J of A such that whenever J contains the product of two $\mathcal{H}$-invariant ideals of A then J contains at least one of them. We denote by $\mathcal{H}$-Spec(A) the *$\mathcal{H}$-spectrum* of A, that is, the set of all $\mathcal{H}$-prime ideals of A. It follows from [1, Proposition

II.2.9] that every $\mathcal{H}$-prime ideal is prime when q is not a root of unity, so that in this case $\mathcal{H}$-Spec(A) coincides with the set of all $\mathcal{H}$-invariant prime ideals of A.

There are natural torus actions on the classes of algebras that we study here, including quantum matrices, partition subalgebras of quantum matrices and quantum grassmannians. These actions are rational, and so the remarks above apply.

First, there is an action of a torus $\mathcal{H} := (\mathbb{k}^*)^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ given by

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \circ x_{ij} := \alpha_i \beta_j x_{ij}.$$

In other words, one is able to multiply through rows and columns by nonzero scalars.

Next, there is an action of the torus $\mathcal{H} := (\mathbb{k}^*)^n$ on $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ which comes from the column action on quantum matrices. Thus, $(\alpha_1, \dots, \alpha_n) \circ [i_1, \dots, i_m] := \alpha_{i_1} \cdots \alpha_{i_m} [i_1, \dots, i_m]$. We shall be interested in prime ideals left invariant under the action of this torus. The set of such prime ideals is the $\mathcal{H}$-spectrum of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$.

We recall the definition of quantum Schubert varieties given in [16].

Definition 1.2. Let γ be an element of $\Pi_{m,n}$ and put $\Pi_{m,n}^{\gamma} = \{\alpha \in \Pi_{m,n} \mid \alpha \not\geq_{\mathrm{st}} \gamma\}$. The *quantum Schubert variety* $S(\gamma)$ associated to γ is

$$S(\gamma) := \mathcal{O}_q(G_{m,n}(\mathbb{k}))/\langle \Pi_{m,n}^{\gamma} \rangle.$$

(Note that $S(\gamma)$ was denoted by $\mathcal{O}_q(G_{m,n}(\mathbb{k}))_\gamma$ in [16].)

This definition is inspired by the classical description of the coordinate rings of Schubert varieties in the grassmannian. For more details about this matter, see [8, Section 6.3.4].

Note that each of the maximal quantum minors that generate $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is an $\mathcal{H}$-eigenvector. Thus, the $\mathcal{H}$-action on $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ transfers to the quantum Schubert varieties $S(\gamma)$.

In order to study properties of the quantum grassmannian, the notion of a quantum graded algebra with a straightening law (on a partially ordered set Π) was introduced in [15]. We now recall the definition of these algebras and mention various properties that we will use later.

Let A be an algebra and Π a finite subset of elements of A with a partial order $<_{\mathrm{st}}$. A *standard monomial* on Π is an element of A which is either 1 or of the form $\alpha_1 \cdots \alpha_s$, for some $s \geq 1$, where $\alpha_1, \dots, \alpha_s \in \Pi$ and $\alpha_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} \alpha_s$.

Definition 1.3. Let A be an $\mathbb{N}$-graded $\mathbb{k}$-algebra and Π a finite subset of A equipped with a partial order $<_{\mathrm{st}}$. We say that A is a *quantum graded algebra with a straightening law* (*quantum graded A.S.L.* for short) on the poset $(\Pi, <_{\mathrm{st}})$ if the following conditions are satisfied.

(1) The elements of Π are homogeneous with positive degree.
(2) The elements of Π generate A as a $\mathbb{k}$-algebra.
(3) The set of standard monomials on Π is a linearly independent set.
(4) If $\alpha, \beta \in \Pi$ are not comparable for $<_{\mathrm{st}}$, then $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\mathrm{st}} \mu$ and $\lambda <_{\mathrm{st}} \alpha, \beta$.

(5) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \mathbb{k}^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\rm st} \mu$ and $\lambda <_{\rm st} \alpha, \beta$.

By [15, Proposition 1.1.4], if A is a quantum graded A.S.L. on the partially ordered set $(\Pi, <_{\rm st})$, then the set of standard monomials on Π forms a $\mathbb{k}$-basis of A. Hence, in the presence of a standard monomial basis, the structure of a quantum graded A.S.L. may be seen as providing more detailed information on the way standard monomials multiply and commute.

Example. It is shown in [15, Theorem 3.4.4] that $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is a quantum graded algebra with a straightening law on $(\Pi_{m,n}, \leq_{\rm st})$.

From our point of view, one important feature of quantum graded A.S.L. is the following. Let A be a $\mathbb{k}$-algebra which is a quantum graded A.S.L. on the set $(\Pi, \leq_{\rm st})$. A subset Ω of Π will be called a Π-*ideal* if it is an ideal of the partially ordered set $(\Pi, \leq_{\rm st})$ in the sense of lattice theory; that is, if it satisfies the following property: if $\alpha \in \Omega$ and if $\beta \in \Pi$, with $\beta \leq_{\rm st} \alpha$, then $\beta \in \Omega$. We can consider the quotient $A/\langle\Omega\rangle$ of A by the ideal generated by Ω. Clearly, it is still a graded algebra and it is generated by the images in $A/\langle\Omega\rangle$ of the elements of $\Pi \setminus \Omega$. The important point here is that $A/\langle\Omega\rangle$ inherits from A a natural quantum graded A.S.L. structure on $\Pi \setminus \Omega$ (or, more precisely, on the canonical image of $\Pi \setminus \Omega$ in $A/\langle\Omega\rangle$). In particular, the set of homomorphic images in $A/\langle\Omega\rangle$ of the standard monomials of A which either equal 1 or are of the form $\alpha_1 \cdots \alpha_t$ ($t \in \mathbb{N}^*$) and $\alpha_1 \notin \Omega$ form a $\mathbb{k}$-basis for $A/\langle\Omega\rangle$. The reader will find all the necessary details in §1.2 of [15].

Example. Let γ be an element of $\Pi_{m,n}$. It is clear that the set $\Pi_{m,n}^{\gamma}$ introduced in Definition 1.2 is a $\Pi_{m,n}$-ideal. Hence, the discussion above shows that the quantum Schubert variety $S(\gamma)$ is a quantum graded A.S.L. on the canonical image in $S(\gamma)$ of $\Pi_{m,n} \setminus \Pi_{m,n}^{\gamma}$. In particular, the canonical images in $S(\gamma)$ of the standard monomials of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ which either equal to 1 or are of the form $[I_1] \cdots [I_t]$, for some $t \geq 1$ and with $\gamma \leq_{\rm st} [I_1]$, form a $\mathbb{k}$-basis of $S(\gamma)$.

Remark 1.4. Let γ be an element of $\Pi_{m,n}$. As mentioned above, the quantum Schubert variety $S(\gamma)$ is a quantum graded A.S.L. on the canonical image in $S(\gamma)$ of $\Pi_{m,n} \setminus \Pi_{m,n}^{\gamma}$. At this point, it is worth noting that the set $\Pi_{m,n} \setminus \Pi_{m,n}^{\gamma}$ has a single minimal element, namely γ, and that the image of γ is a normal non-zerodivisor in $S(\gamma)$, by [15, Lemma 1.2.1].

2. An example: the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$

In this section, we illustrate the stategy for the paper by computing the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$. Full details are not given, since the justification for most of our claims follows from the results proved in the rest of the paper. The computation of the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ was first obtained by Ewan Russell, and full details will appear in his PhD thesis [18].

The quantum grassmannian $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ is the $\mathbb{k}$-subalgebra generated by the 2×2 quantum minors of the 2×4 generic matrix of $\mathcal{O}_q(M_{2,4}(\mathbb{k}))$, that is, the subalgebra generated by the quantum minors $[12], [13], [14], [23], [24]$ and $[34]$. The following commutation relations can be checked by using the defining relations of $\mathcal{O}_q(M_{2,4}(\mathbb{k}))$:

$$[ij][ik] = q[ik][ij], \quad [ik][jk] = q[jk][ik] \quad \text{for } i < j < k,$$

and

$$[14][23] = [23][14], \quad [12][34] = q^2[34][12], \quad [13][24] = [24][13] + (q - q^{-1})[14][23].$$

There is also a quantum Plücker relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

The quantum Plücker relation may be rewritten as

$$[34][12] - q^{-1}[24][13] + q^{-2}[23][14] = 0$$

and one can also check that

$$[13][24] = q^2[24][13] + (q^{-1} - q)[12][34].$$

One can check directly from these relations that $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ is a quantum graded algebra with a straightening law on the poset $\Pi_{2,4}$ in Figure 2 (see Definition 1.3).

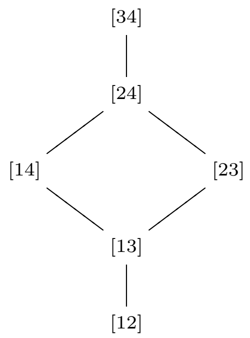
There is a natural way to associate a partition to any quantum minor and thus a Young diagram (see Theorem 4.8 for the general case). In the case of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ this association is illustrated in Figure 2.

Before starting, let us recall briefly the work of Cauchon on the $\mathcal{H}$-prime spectrum of quantum matrices in the generic case where q is not a root of unity. In [3], Cauchon proves that the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ are in bijection with the so-called Cauchon diagrams on the Young diagram with m rows and n columns. A *Cauchon diagram* on a Young diagram is a colouring of each square either black or white in such a way that if a square is coloured black then either each square to the left in the same row is black or each square above in the same column is black. It is easy to see that only two of the 16 possible black-white colourings of the Young diagram  fail this test and so there are 14 Cauchon diagrams on . Hence, there are 14 $\mathcal{H}$-primes in $\mathcal{O}_q(M_2(\mathbb{k}))$ in the generic case where q is not a root of unity.

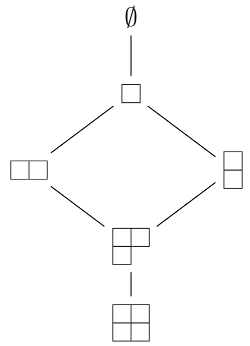
The major goal of this paper is to show that Cauchon's work can be extended to describe the $\mathcal{H}$-prime spectrum of the quantum grassmannian.

In Section 5, we will prove that each $\mathcal{H}$-prime ideal P of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is naturally associated to a (unique) quantum minor γ such that $\gamma \notin P$, but $\alpha \in P$ for any quantum minor $\alpha \not\geq_{\mathrm{st}} \gamma$. We use this to partition the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$.

We start by considering $\mathcal{H}$-prime ideals P of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ that do not contain the quantum minor $[12]$. Note that $[12]$ is a normal element of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ and



The partially ordered set $\Pi_{2,4}$



The partial ordering on corresponding partitions

FIG. 2. $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$

so we can consider the localisation $\mathcal{O}_q(G_{2,4}(\mathbb{k}))[[12]^{-1}]$ obtained by inverting [12]. The dehomogenisation homomorphism introduced in [12, Corollary 4.1] shows that

$$\mathcal{O}_q(G_{2,4}(\mathbb{k}))[[12]^{-1}] \cong \mathcal{O}_q(M_2(\mathbb{k}))[y, y^{-1}; \sigma],$$

where σ is an automorphism of $\mathcal{O}_q(M_2(\mathbb{k}))$ that arises from the commutation rules for the quantum minor [12] in $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$. There are $\mathcal{H}$-actions on all the algebras involved and this isomorphism is $\mathcal{H}$-equivariant. Thus, the $\mathcal{H}$-prime ideals P of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ that do not contain the quantum minor [12] are in natural bijection with the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_2(\mathbb{k}))[y, y^{-1}; \sigma]$ and these, in turn, are in natural bijection with the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_2(\mathbb{k}))$, by [14, Theorem 2.3]. As indicated above, there are 14 $\mathcal{H}$-prime ideals in $\mathcal{O}_q(M_2(\mathbb{k}))$, and this produces 14 $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ that do not contain [12]. Explicit generators of these primes can be calculated by using the known generators of the $\mathcal{H}$-prime ideals in $\mathcal{O}_q(M_2(\mathbb{k}))$ and the isomorphism above. Note that we can consider the generators of $\mathcal{O}_q(M_2(\mathbb{k}))$ as sitting naturally in the partition that is associated with the quantum minor [12].

Next, we consider $\mathcal{H}$-prime ideals P of $\mathcal{O}_q(G_{2,4}(\Bbbk))$ that contain the quantum minor [12] but do not contain [13]. Note that the relations above for $\mathcal{O}_q(G_{2,4}(\Bbbk))$ show that [13] is normal modulo the ideal generated by [12]. We start by factoring out this ideal and then proceed as above. Thus, we obtain a dehomogenisation isomorphism

$$\overline{\mathcal{O}_q(G_{2,4}(\Bbbk))}\,[\,\overline{[13]}^{-1}] \cong T[y, y^{-1}; \sigma],$$

where $\overline{\mathcal{O}_q(G_{2,4}(\Bbbk))}$ is the factor ring $\mathcal{O}_q(G_{2,4}(\Bbbk))/\langle[12]\rangle$, the automorphism σ arises from the commutation rules for $\overline{[13]}$ in $\overline{\mathcal{O}_q(G_{2,4}(\Bbbk))}$ and T is the dehomogenisation of $\overline{\mathcal{O}_q(G_{2,4}(\Bbbk))}$ at $\overline{[13]}$. Once again, there is a natural bijection between the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{2,4}(\Bbbk))$ that contain the quantum minor [12] but do not contain [13] and the $\mathcal{H}$-prime ideals of T. It only remains to identify T. The dehomogenisation map shows that T is generated by the following elements:

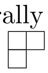
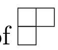
$$\overline{[14]}\,\overline{[13]}^{-1}, \quad \overline{[23]}\,\overline{[13]}^{-1}, \quad \overline{[24]}\,\overline{[13]}^{-1}, \quad \overline{[34]}\,\overline{[13]}^{-1}.$$

The second version of the quantum Plücker relation shows that

$$\overline{[24]}\,\overline{[13]} = q^{-1}\overline{[23]}\,\overline{[14]},$$

and from this one can see that T is in fact generated by

$$\widetilde{[14]} := \overline{[14]}\,\overline{[13]}^{-1}, \quad \widetilde{[23]} := \overline{[23]}\,\overline{[13]}^{-1}, \quad \widetilde{[34]} := \overline{[34]}\,\overline{[13]}^{-1}.$$

By examining the commutation relations between these three generators, we see that T can be considered to be a subalgebra of $\mathcal{O}_q(M_2(\Bbbk))$ with the generators sitting naturally in the positions determined by the partition corresponding to [13], that is, . Such an algebra is a quantum affine 3-space, and so has eight $\mathcal{H}$-prime ideals each of these being generated by a subset of $\{\widetilde{[14]}, \widetilde{[23]}, \widetilde{[34]}\}$. Each of these eight $\mathcal{H}$-prime ideals in T will produce a $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G_{2,4}(\Bbbk))$ that contains [12] but does not contain [13]. Explicit generators can be found by using the dehomogenisation isomorphism: as an example, it is instructive to consider the one coming from the subset $\{\widetilde{[14]}, \widetilde{[23]}\}$. Let P be the corresponding $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G_{2,4}(\Bbbk))$. Then $\overline{P}\,\overline{[13]}^{-1}$ contains $\overline{[14]}$ and $\overline{[23]}$, and so P contains [14] and [23]. Since P also contains [12] and does not contain [13], by definition, the quantum Plücker relation forces $[24] \in P$. It is then easy to check that P is the ideal generated by $[12], [14], [23], [24]$. Similar calculations produce all of the eight $\mathcal{H}$-primes explicitly. Note again that each black-white colouring of  is a Cauchon diagram, and there is a natural bijection between these Cauchon diagrams and the eight $\mathcal{H}$-primes being discussed.

At the next stage, we consider $\mathcal{H}$-prime ideals P of $\mathcal{O}_q(G_{2,4}(\Bbbk))$ that contain the quantum minors [12] and [13] but do not contain [14]. However, note that such a prime ideal is then forced by the quantum Plücker relation to contain [23]. This can be better stated by saying that P contains each quantum minor that is *not greater than or equal to* [14] (this is what happens in general, as we have stated above). We follow the procedure as above and produce four more $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{2,4}(\Bbbk))$.

In the same way, [23] produces four $\mathcal{H}$-primes, [24] produces two $\mathcal{H}$-primes, [34] produces one $\mathcal{H}$-prime and finally there is the *irrelevant ideal* generated by all of the quantum minors. Thus, we obtain $14 + 8 + 4 + 4 + 2 + 1 + 1 = 34$ $\mathcal{H}$-prime ideals in $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$.

Figure 3 shows a diagram of the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$ and the corresponding poset of Cauchon diagrams. In each case the $\mathcal{H}$-prime ideal is gener-

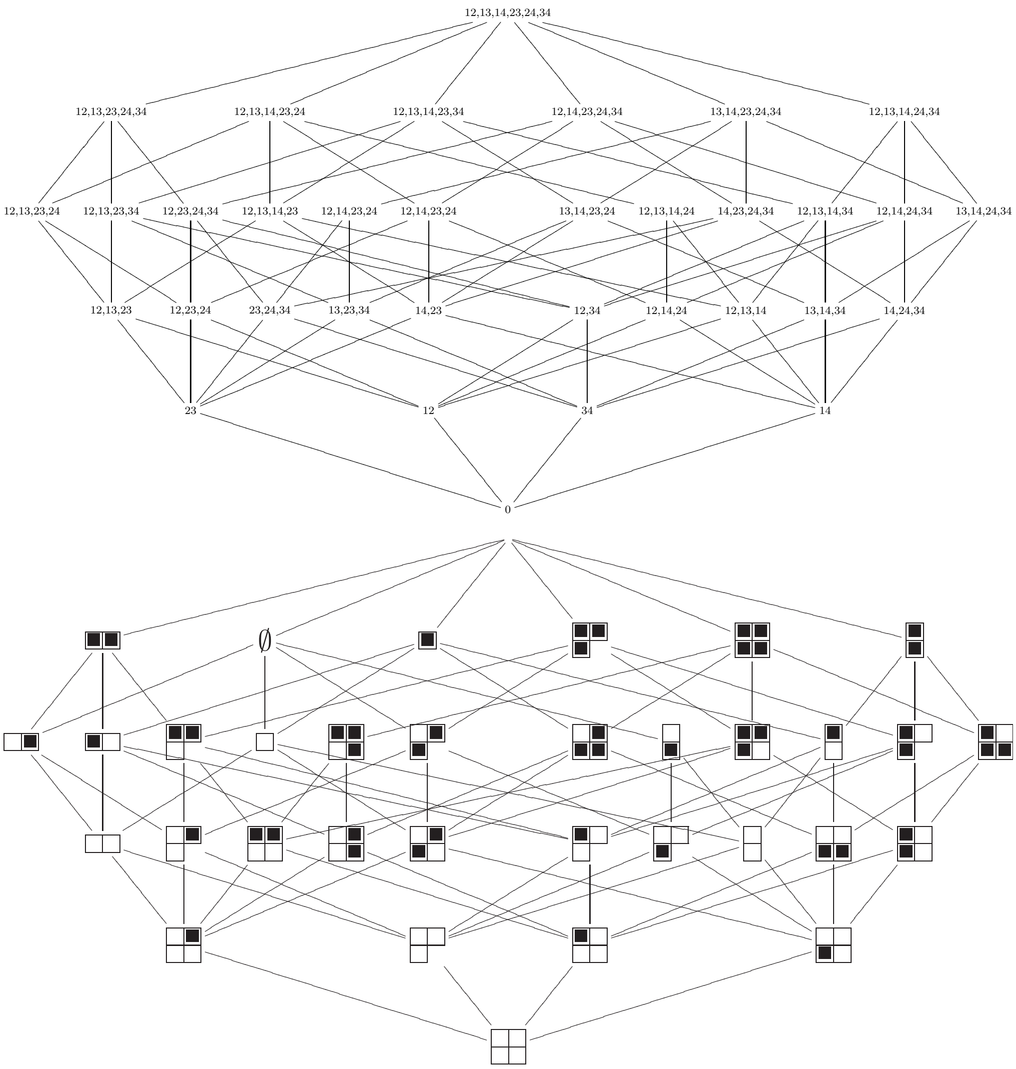


Fig. 3. The complete $\mathcal{H}$-prime spectrum of the quantum grassmannian $\mathcal{O}_q(G_{2,4}(\mathbb{k}))$.

ated by the quantum minors indicated. We conjecture that each $\mathcal{H}$-prime ideal in $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is generated by the $m \times m$ quantum minors that it contains.

We have considered the example of $\mathcal{O}_q(G_{2,4}(\Bbbk))$ in some detail because it illustrates most of the ideas that we develop in the rest of the paper. The outline strategy is as follows. Each $\mathcal{H}$-prime ideal P of $\mathcal{H}$-Spec$(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is associated with a (unique) quantum minor (or index set) γ such that $\gamma \notin P$, but $\alpha \in P$ for any quantum minor $\alpha \not\geq_{\mathrm{st}} \gamma$ (see Theorem 5.1). This gives a partition of $\mathcal{H}$-Spec$(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. In order to study the $\mathcal{H}$-prime ideals P that are associated with γ we pass first to the quantum Schubert variety $S(\gamma)$. This is the factor algebra of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ by the ideal generated by those minors $\alpha \not\geq_{\mathrm{st}} \gamma$. Quantum Schubert varieties inherit from $\mathcal{O}_q(G_{m,n}(\Bbbk))$ the structure of a quantum graded algebra with a straightening law. Next, we consider the noncommutative dehomogenisation of the Schubert variety at the image of the minor γ: this is the quantum Schubert cell $S^o(\gamma)$. Each of the $\mathcal{H}$-primes associated with γ "survives" in this quantum Schubert cell. Thus, the problem is to describe $\mathcal{H}$-Spec$(S^o(\gamma))$. In order to do this, we show that a natural set of generators for $S^o(\gamma)$ can be placed in the boxes of the Young diagram associated with γ in such a way that the commutation relations of these generators are the same of those of a corresponding subalgebra of quantum matrices. We call such subalgebras *partition subalgebras* of quantum matrices. It turns out that Cauchon's methods for quantum matrices work for partition subalgebras, and this enables us to describe $\mathcal{H}$-Spec$(S^o(\gamma))$.

In the following sections of the paper, we reverse the order in the strategy that we have just outlined. We start by studying properties of partition subalgebras (and hence quantum Schubert cells), then use these results to study quantum Schubert varieties, and finally deduce the results we are aiming for in the quantum grassmannian.

3. Partition subalgebras of quantum matrices

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. The *partition subalgebra* $\mathcal{A}_\lambda$ of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ is defined to be the subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the variables x_{ij} with $j \leq \lambda_i$. By looking at the defining relations for quantum matrices, it is easy to see that $\mathcal{A}_\lambda$ can be presented as an iterated Ore extension with the variables x_{ij} added in lexicographic order. As a consequence, partition subalgebras are noetherian domains. Recall that there is an action of a torus $\mathcal{H} := (\Bbbk^*)^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\Bbbk))$ given by

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \circ x_{ij} := \alpha_i \beta_j x_{ij}.$$

This action induces an action on $\mathcal{A}_\lambda$, by restriction. Our main aim in this section is to observe that the Goodearl–Letzter stratification theory and the Cauchon theory of deleting derivations apply to partition subalgebras of quantum matrices. As a consequence, we can then exploit these theories to obtain information about the prime and $\mathcal{H}$-prime spectra of partition subalgebras.

The conditions needed to use the theories have been brought together in the notion of a (torsion-free) CGL extension introduced in [14, Definition 3.1]; the definition is given below, for convenience.

Definition 3.1. An iterated skew polynomial extension

$$A = \mathbb{k}[x_1][x_2;\sigma_2,\delta_2]\dots[x_n;\sigma_n,\delta_n]$$

is said to be a *CGL extension* (after Cauchon, Goodearl and Letzter) provided that the following list of conditions is satisfied:

- With $A_j := \mathbb{k}[x_1][x_2;\sigma_2,\delta_2]\dots[x_j;\sigma_j,\delta_j]$ for each $1 \le j \le n$, each σ_j is a $\mathbb{k}$-algebra automorphism of A_{j-1}, each δ_j is a locally nilpotent $\mathbb{k}$-linear σ_j-derivation of A_{j-1}, and there exist non-roots of unity $q_j \in \mathbb{k}^*$ such that $\sigma_j\delta_j = q_j\delta_j\sigma_j$;
- For each $i < j$ there exists a $\lambda_{ji} \in \mathbb{k}^*$ such that $\sigma_j(x_i) = \lambda_{ji}x_i$;
- There is a torus $\mathcal{H} = (\mathbb{k}^*)^r$ acting rationally on A by $\mathbb{k}$-algebra automorphisms;
- The x_i for $1 \le i \le n$ are $\mathcal{H}$-eigenvectors;
- There exist elements $h_1,\dots,h_n \in \mathcal{H}$ such that $h_j(x_i) = \sigma_j(x_i)$ for $j > i$ and such that the h_j-eigenvalue of x_j is not a root of unity.

If, in addition, the subgroup of $\mathbb{k}^*$ generated by the λ_{ji} is torsion-free then we will say that A is a *torsion-free CGL extension.*

For a discussion of rational actions of tori, see [1, Chapter II.2].

It is easy to check that all of these conditions are satisfied for partition subalgebras (for exactly the same reasons that quantum matrices are CGL extensions).

Proposition 3.2. *Partition subalgebras of quantum matrix algebras are torsion-free CGL extensions when the parameter q is not a root of unity.*

Proof. It is only necessary to show that we can introduce the variables x_{ij} that define the partition subalgebra in such a way that the resulting iterated skew polynomial extension satisfies the list of conditions above. Lexicographic ordering is suitable. □

Corollary 3.3. *Let $\mathcal{A}_\lambda$ be a partition subalgebra of quantum matrices and suppose that $\mathcal{A}_\lambda$ is equipped with the induced action of $\mathcal{H}$. Suppose that the parameter q is not a root of unity. Then $\mathcal{A}_\lambda$ has only finitely many $\mathcal{H}$-prime ideals and all prime ideals of $\mathcal{A}_\lambda$ are completely prime.*

Proof. This follows immediately from the previous result and [1, Theorems II.5.12 and II.6.9]. □

In fact, we can be much more precise about the number of $\mathcal{H}$-primes. We will prove below that there exists a natural bijection between the $\mathcal{H}$-prime spectrum of $\mathcal{A}_\lambda$ and Cauchon diagrams defined on the Young diagram corresponding to the partition λ.

Suppose that Y_λ is the Young diagram corresponding to the partition λ. Then a *Cauchon diagram* on Y_λ is an assignment of a colour, either white or black, to each square of the diagram Y_λ in such a way that if a square is coloured black then either each square above is coloured black, or each square to the left is coloured black. These diagrams were first introduced by Cauchon [3] in his study of the $\mathcal{H}$-prime spectrum of quantum matrices. Recently, they have occurred with the name Le-diagrams in work of Postnikov [17] and Williams [19].

Lemma 3.4. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m > 0$. The number of $\mathcal{H}$-prime ideals in $\mathcal{A}_\lambda$ is equal to the number of Cauchon diagrams defined on the Young diagram corresponding to the partition λ.*

Proof. Let n_λ denote the number of $\mathcal{H}$-prime ideals in $\mathcal{A}_\lambda$. First, we obtain a recurrence relation for n_λ.

The $\mathcal{H}$-prime spectrum of $\mathcal{A}_\lambda$ can be written as a disjoint union

$$\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) = \{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \in J\} \sqcup \{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \notin J\}.$$

It follows from the complete primeness of every $\mathcal{H}$-prime ideal of $\mathcal{A}_\lambda$ that an $\mathcal{H}$-prime ideal J of $\mathcal{A}_\lambda$ that contains x_{m,λ_m} must also contain either x_{i,λ_m} for each $i \in \{1, \dots, m\}$ or $x_{m,\alpha}$ for each $\alpha \in \{1, \dots, \lambda_m\}$. Let I_1 be the ideal generated by x_{i,λ_m} for $i \in \{1, \dots, m\}$, and let I_2 be the ideal generated by $x_{m,\alpha}$ for $\alpha \in \{1, \dots, \lambda_m\}$. Set $I_3 := I_1 + I_2$. As

$$\frac{\mathcal{A}_\lambda}{I_1} \simeq \mathcal{A}_{(\lambda_1 - 1, \dots, \lambda_m - 1)}, \qquad \frac{\mathcal{A}_\lambda}{I_2} \simeq \mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1})}, \qquad \frac{\mathcal{A}_\lambda}{I_3} \simeq \mathcal{A}_{(\lambda_1 - 1, \dots, \lambda_{m-1} - 1)},$$

we obtain

$$\begin{aligned} n_\lambda = {} & n_{(\lambda_1 - 1, \dots, \lambda_m - 1)} + n_{(\lambda_1, \dots, \lambda_{m-1})} - n_{(\lambda_1 - 1, \dots, \lambda_{m-1} - 1)} \\ & + |\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \notin J\}|. \end{aligned}$$

(Even though the above isomorphisms are not always $\mathcal{H}$-equivariant, they preserve the property of being an $\mathcal{H}$-prime.)

As $\mathcal{A}_\lambda$ is a CGL extension, one can apply the theory of deleting derivations to this algebra. In particular, it follows from [2, Théorème 3.2.1] that the multiplicative system of $\mathcal{A}_\lambda$ generated by x_{m,λ_m} is an Ore set in $\mathcal{A}_\lambda$, and

$$\mathcal{A}_\lambda[x_{m,\lambda_m}^{-1}] \simeq \mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1)}[y^{\pm 1}; \sigma],$$

where σ is the automorphism of $\mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1)}$ defined by $\sigma(x_{i\alpha}) = q^{-1} x_{i\alpha}$ if $i = m$ or $\alpha = \lambda_m$, and $\sigma(x_{i\alpha}) = x_{i\alpha}$ otherwise. Denote this isomorphism by ψ, and note that $\psi(x_{m,\lambda_m}) = y$. As x_{m,λ_m} is an $\mathcal{H}$-eigenvector, the action of $\mathcal{H}$ on $\mathcal{A}_\lambda$ extends to an action of $\mathcal{H}$ on $\mathcal{A}_\lambda[x_{m,\lambda_m}^{-1}]$, and so on $\mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1)}[y^{\pm 1}; \sigma]$. It is easy to show that this action restricts to an action on $\mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1)}$ which coincides with the "natural" action of $\mathcal{H}$ on this algebra. Hence the isomorphism ψ induces a bijection from $\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \notin J\}$ to the set $\mathcal{H}\text{-Spec}(\mathcal{A}_{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m - 1)}[y^{\pm 1}; \sigma])$; and so it follows from [14, Theorem 2.3] that there exists a bijection between the sets $\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \notin J\}$ and $\mathcal{H}\text{-Spec}(\mathcal{A}_{(\lambda_1,\dots,\lambda_{m-1},\lambda_m-1)})$. Hence

$$|\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) \mid x_{m,\lambda_m} \notin J\}| = n_{(\lambda_1,\dots,\lambda_{m-1},\lambda_m-1)};$$

so that

$$\begin{aligned} n_\lambda = n_{(\lambda_1-1,\dots,\lambda_m-1)} &+ n_{(\lambda_1,\dots,\lambda_{m-1})} - n_{(\lambda_1-1,\dots,\lambda_{m-1}-1)} \\ &+ n_{(\lambda_1,\dots,\lambda_{m-1},\lambda_m-1)}. \end{aligned}$$

On the other hand, it follows from [19, Remark 4.2] that the number of Cauchon diagrams (equivalently, Le-diagrams) defined on the Young diagram corresponding to the partition λ satisfies the same recurrence. As the number of $\mathcal{H}$-prime ideals in $\mathcal{A}_{(1)}$ is equal to 2 which is also the number of Cauchon diagrams defined on the Young diagram corresponding to the partition $\lambda = (1)$, the proof is complete. □

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m > 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Let $\mathcal{C}_\lambda$ denote the set of Cauchon diagrams on the Young diagram Y_λ corresponding to the partition λ. We have just seen that the sets $\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)$ and $\mathcal{C}_\lambda$ have the same cardinality. In fact, there is a natural bijection between these two sets which carries over important algebraic and geometric information. This natural bijection arises by using Cauchon's theory of deleting derivations developed in [2] and [3].

As $\mathcal{A}_\lambda$ is a CGL extension, the theory of deleting derivations can be applied to the iterated Ore extension $\mathcal{A}_\lambda = \Bbbk[x_{1,1}] \dots [x_{m,\lambda_m}; \sigma_{m,\lambda_m}, \delta_{m,\lambda_m}]$ (where the indices are increasing for the lexicographic order). Before describing the deleting derivations algorithm, we introduce some notation. Denote by $\leq_{\text{lex}}$ the lexicographic ordering on $\mathbb{N}^2$ and set

$$E := \Big(\bigsqcup_{i=1}^{m} \{i\} \times \{1, \dots, \lambda_i\} \cup \{(m, \lambda_m + 1)\} \Big) \setminus \{(1,1)\}.$$

If $(j,\beta) \in E$ with $(j,\beta) \neq (m, \lambda_m + 1)$, then $(j,\beta)^+$ denotes the least element (relative to $\leq_{\text{lex}}$) of the set $\{(i,\alpha) \in E \mid (j,\beta) < (i,\alpha)\}$.

The deleting derivations algorithm constructs, for each $r \in E$, a family of elements $x_{i,\alpha}^{(r)}$ for $\alpha \leq \lambda_i$ of $F := \text{Frac}(\mathcal{A}_\lambda)$, defined as follows:

1. If $r = (m, \lambda_m + 1)$, then $x_{i,\alpha}^{(m,\lambda_m+1)} = x_{i,\alpha}$ for all (i,α) with $\alpha \leq \lambda_i$.
2. Assume that $r = (j,\beta) < (m, \lambda_m + 1)$, and that the $x_{i,\alpha}^{(r^+)}$ are already constructed. Then it follows from [2, Théorème 3.2.1] that $x_{j,\beta}^{(r^+)} \neq 0$ and, for all (i,α), we have

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r^+)} - x_{i,\beta}^{(r^+)} (x_{j,\beta}^{(r^+)})^{-1} x_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta, \\ x_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

It is interesting to notice that the above changes of variables, on which the theory of deleting derivations is based, can be reinterpreted using quasi-determinants as defined in [6] and [7]. Although we do not use this reinterpretation in the present paper, in the following example we illustrate it for the cases of $\mathcal{O}_q(M_2(\Bbbk))$ and $\mathcal{O}_q(M_3(\Bbbk))$.

Example. In the case where $\lambda = (n, \dots, n)$ (m times), one can express the elements $x_{i,\alpha}^{(1,2)}$ obtained at the end of this algorithm as quasi-determinants of the $m \times n$ matrix $\mathbf{X} = (x_{i,\alpha})$ and its submatrices. The reader is referred to [7] for the definition and the notation relative to quasi-determinants. In particular, when $\lambda = (2, 2)$, we are dealing with $\mathcal{O}_q(M_2(\Bbbk))$. In this case, there is only one nontrivial step in the deleting derivations process, and this involves replacing the entry x_{11} by $x_{11} - x_{12}x_{22}^{-1}x_{21}$. Note that $x_{11} - x_{12}x_{22}^{-1}x_{21} = (x_{11}x_{22} - qx_{12}x_{21})x_{22}^{-1}$, and that the term in brackets is the 2×2 quantum determinant. Hence, it is easy to check that

$$\begin{pmatrix} x_{11}^{(1,2)} & x_{12}^{(1,2)} \\ x_{21}^{(1,2)} & x_{22}^{(1,2)} \end{pmatrix} = \begin{pmatrix} |\mathbf{X}|_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

in the quasi-determinant notation.

As a further example when $\lambda = (3, 3, 3)$, it can be shown that

$$\begin{pmatrix} x_{11}^{(1,2)} & x_{12}^{(1,2)} & x_{13}^{(1,2)} \\ x_{21}^{(1,2)} & x_{22}^{(1,2)} & x_{23}^{(1,2)} \\ x_{31}^{(1,2)} & x_{32}^{(1,2)} & x_{33}^{(1,2)} \end{pmatrix} = \begin{pmatrix} |\mathbf{X}|_{11} & |\mathbf{X}^{31}|_{12} & x_{13} \\ |\mathbf{X}^{13}|_{21} & |\mathbf{X}^{11}|_{22} & x_{32} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

As in [2], we denote by $\overline{\mathcal{A}}_\lambda$ the subalgebra of $\mathrm{Frac}(\mathcal{A}_\lambda)$ generated by the indeterminates obtained at the end of this algorithm; that is, $\overline{\mathcal{A}}_\lambda$ is the subalgebra of $\mathrm{Frac}(\mathcal{A}_\lambda)$ generated by the $t_{ij} := x_{ij}^{(1,2)}$ for each (i, j) such that $j \leq \lambda_i$. Cauchon has shown that $\overline{\mathcal{A}}_\lambda$ can be viewed as the quantum affine space generated by indeterminates t_{ij} for $j \leq \lambda_i$ with relations $t_{ij}t_{il} = qt_{il}t_{ij}$ for $j < l$, while $t_{ij}t_{kj} = qt_{kj}t_{ij}$ for $i < k$, and all other pairs commute. Observe that the torus $\mathcal{H}$ still acts by automorphisms on $\overline{\mathcal{A}}_\lambda$ via $(a_1, \dots, a_m, b_1, \dots, b_n).t_{ij} = a_ib_jt_{ij}$. The theory of deleting derivations allows the explicit (but technical) construction of an embedding φ, called the *canonical embedding*, from $\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)$ into the $\mathcal{H}$-prime spectrum of $\overline{\mathcal{A}}_\lambda$. The $\mathcal{H}$-prime ideals of $\overline{\mathcal{A}}_\lambda$ are well-known: they are generated by the subsets of $\{t_{ij}\}$. If C is a Cauchon diagram defined on the Young diagram corresponding to λ, then we denote by K_C the (completely) prime ideal of $\overline{\mathcal{A}}_\lambda$ generated by the subset of indeterminates t_{ij} such that the square in position (i, j) is a black square of C.

Theorem 3.5. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m > 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Let $\mathcal{C}_\lambda$ denote the set of Cauchon diagrams defined on the Young diagram corresponding to λ. For every Cauchon diagram $C \in \mathcal{C}_\lambda$, there exists a unique $\mathcal{H}$-invariant* *(completely) prime ideal J_C of $\mathcal{A}_\lambda$ such that $\varphi(J_C) = K_C$. Moreover, there are no other $\mathcal{H}$-prime ideals in $\mathcal{A}_\lambda$, so that*

$$\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) = \{J_C \mid C \in \mathcal{C}_\lambda\}.$$

Proof. As the sets $\mathcal{H}$-Spec$(\mathcal{A}_\lambda)$ and $\mathcal{C}_\lambda$ have the same cardinality by the previous lemma, it is sufficient to show that $\varphi(\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)) \subseteq \{K_C \mid C \in \mathcal{C}_\lambda\}$. This inclusion can be obtained by following the arguments of [3, Lemmes 3.1.6 and 3.1.7]. The details are left to the interested reader. □

Remark 3.6. Theorem 3.5 provides more than just an explicit bijection between the $\mathcal{H}$-spectrum of $\mathcal{A}_\lambda$ and $\mathcal{C}_\lambda$. This natural bijection carries algebraic and geometric data. For example, it can be shown that the height of J_C is given by the number of black boxes of the Cauchon diagram C. Also, the dimension of the $\mathcal{H}$-stratum (in the sense of [1, Definition II.2.1]) associated to J_C can be read off from the Cauchon diagram C.

An algebra A is said to be *catenary* if for each pair of prime ideals $Q \subseteq P$ of A all saturated chains of prime ideals between Q and P have the same length. Our next aim is to show that partition subalgebras of quantum matrix algebras are catenary. The key property that we need to establish in order to prove catenarity is the property of normal separation. Two prime ideals $Q \subsetneq P$ are said to be *normally separated* if there is an element $c \in P \setminus Q$ such that c is normal modulo Q. The algebra is *normally separated* if each such pair of prime ideals is normally separated. In our case, a result of Goodearl (see [9, Section 5]) shows that it is enough to concentrate on the $\mathcal{H}$-prime ideals. Suppose that A is a $\Bbbk$-algebra with a torus $\mathcal{H}$ acting rationally. If Q is any $\mathcal{H}$-invariant ideal of A then an element c is said to be *$\mathcal{H}$-normal modulo Q* provided that there exists $h \in \mathcal{H}$ such that $ca - h(a)c \in Q$ for all $a \in A$. Goodearl observes that in this case one may choose the element c to be an $\mathcal{H}$-eigenvector. The algebra A has *$\mathcal{H}$-normal separation* provided that for each pair of $\mathcal{H}$-prime ideals $Q \subsetneq P$ there exists an element $c \in P \setminus Q$ such that c is $\mathcal{H}$-normal modulo Q.

A slightly weaker notion, also introduced by Goodearl, is that of normal $\mathcal{H}$-separation. The algebra A has *normal $\mathcal{H}$-separation* provided that for each pair of $\mathcal{H}$-primes $Q \subsetneq P$ there is an $\mathcal{H}$-eigenvector $c \in P \setminus Q$ which is normal modulo Q. Goodearl shows that in the situation that we are considering, normal $\mathcal{H}$-separation implies normal separation (see [9, Theorem 5.3]).

Notice that, as explained in paragraph 5.1 of [9], the action of $\mathcal{H}$ induces a grading on A by a suitable free abelian group. Using this grading, it is easy to see that A has normal $\mathcal{H}$-separation if and only if for each pair of $\mathcal{H}$-primes $Q \subsetneq P$ there is an element $c \in P \setminus Q$ whose image in A/Q is normal and an $\mathcal{H}$-eigenvector. This fact will be freely used in the following.

Recall, from [14, Definition 2.5], the definition of a Cauchon extension. Let A be a domain that is a noetherian $\Bbbk$-algebra and let $R = A[X; \sigma, \delta]$ be a skew polynomial extension of A. We say that $R = A[X; \sigma, \delta]$ is a *Cauchon extension* provided that:

- σ is a $\Bbbk$-algebra automorphism of A and δ is a $\Bbbk$-linear locally nilpotent σ-derivation of A. Moreover, we assume that there exists $q \in \Bbbk^*$ which is not a root of unity such that $\sigma \circ \delta = q\delta \circ \sigma$.
- There exists an abelian group $\mathcal{H}$ which acts on R by $\Bbbk$-algebra automorphisms such that X is an $\mathcal{H}$-eigenvector and A is $\mathcal{H}$-stable.
- σ coincides with the action on A of an element $h_0 \in \mathcal{H}$.
- Since X is an $\mathcal{H}$-eigenvector and since $h_0 \in \mathcal{H}$, there exists $\lambda_0 \in \Bbbk^*$ such that $h_0.X = \lambda_0 X$. We assume that λ_0 is not a root of unity.
- Every $\mathcal{H}$-prime ideal of A is completely prime.

Our next aim is to show that if a Cauchon extension satisfies $\mathcal{H}$-normal separation then so does its base ring. This will have important consequences later on.

Suppose that $R = A[X;\sigma,\delta]$ is a Cauchon extension. Moreover, assume that the group $\mathcal{H}$ is a torus and that the action of $\mathcal{H}$ on R is rational. First, note that $S = \{X^j \mid j \in \mathbb{N}\}$ is an Ore set in R, by [2, Lemme 2.1], and so we can form the Ore localization $\widehat{R} := RS^{-1} = S^{-1}R$. As X is an $\mathcal{H}$-eigenvector, the rational action of $\mathcal{H}$ on R extends to a rational action on $\widehat{R}$.

For each $a \in A$, set

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{-n} \in \widehat{R}.$$

(Note that $\theta(a)$ is a well-defined element of $\widehat{R}$, since δ is locally nilpotent, q is not a root of unity, and $0 \neq 1-q \in \Bbbk$.)

The following facts are established in [2, Section 2]. The map $\theta : A \to \widehat{R}$ is a $\Bbbk$-algebra monomorphism. Let $A[Y;\sigma]$ be a skew polynomial extension. Then θ extends to a monomorphism $\theta : A[Y;\sigma] \to \widehat{R}$ with $\theta(Y) = X$. Set $B = \theta(A)$ (and note that $B \cong A$) and set $T = \theta(A[Y;\sigma]) \subseteq \widehat{R}$. Then $T = B[X;\alpha]$, where α is the automorphism of B defined by $\alpha(\theta(a)) = \theta(\sigma(a))$. The element X is a normal element in T, and so the set S defined above is also an Ore set in T. Cauchon shows that $TS^{-1} = S^{-1}T = \widehat{R}$. Thus, $\widehat{R} = B[X^{\pm 1};\alpha]$. Also, the $\mathcal{H}$-action transfers to B via θ, by [14, Lemma 2.6]. Note, in particular, that α coincides with the action of an element of $\mathcal{H}$ on B.

Hence we have the following picture:

$$\begin{tikzcd}
 & \widehat{R} = B[X^{\pm 1};\alpha] & A[Y^{\pm 1};\sigma] \arrow[l, "\theta"', "\cong"] \\
A[X;\sigma,\delta] = R \arrow[ur] & T = B[X;\alpha] \arrow[u, "\subseteq"'] & A[Y;\sigma] \arrow[l, "\theta"', "\cong"] \arrow[u] \\
A \arrow[u] & \theta(A) = B \arrow[u, "\subseteq"'] & A \arrow[l, "\theta"', "\cong"] \arrow[u]
\end{tikzcd}$$

where all the maps are $\mathcal{H}$-equivariant.

Lemma 3.7. *Suppose that $R = A[X;\sigma,\delta]$ is a Cauchon extension. Moreover, assume that $\mathcal{H}$ is a torus and that the action of $\mathcal{H}$ on R is rational. If R has $\mathcal{H}$-normal separation then A has $\mathcal{H}$-normal separation.*

Proof. In order to prove this lemma, we will proceed in two steps.

First, we show that $\widehat{R}$ has $\mathcal{H}$-normal separation. Suppose that $Q \subsetneq P$ are $\mathcal{H}$-prime ideals of $\widehat{R}$. Then $Q \cap R \subsetneq P \cap R$ are distinct $\mathcal{H}$-prime ideals of R. Thus, there exist $c \in (P \cap R) \setminus (Q \cap R)$ and $h \in \mathcal{H}$ such that $cr - h(r)c \in Q \cap R$ for all $r \in R$. In particular, $cX - \lambda Xc = cX - h(X)c \in Q \cap R$ for some $\lambda \in \Bbbk^*$, as X is an $\mathcal{H}$-eigenvector. From this it is easy to calculate that $(\lambda X)^{-k}c - cX^{-k} \in Q$. Now, let $y = rX^{-k}$ be an element of $\widehat{R}$. Then, working modulo Q, we calculate

$$cy = crX^{-k} = h(r)(\lambda X)^{-k}c = h(r)h(X^{-k})c = h(rX^{-k})c = h(y)c,$$

so that $\widehat{R}$ has $\mathcal{H}$-normal separation, as claimed.

Secondly, we show that B has $\mathcal{H}$-normal separation. We already know that $\widehat{R} = B[X^{\pm 1};\alpha]$ has $\mathcal{H}$-normal separation by the first part of the proof. Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals of B. Set $\widehat{Q} = \bigoplus_{i\in\mathbb{Z}} QX^i$ and $\widehat{P} = \bigoplus_{i\in\mathbb{Z}} PX^i$. Then $\widehat{Q} \cap B = Q$ and $\widehat{P} \cap B = P$, and it follows that $\widehat{Q} \subsetneq \widehat{P}$ are $\mathcal{H}$-prime ideals in $B[X^{\pm 1};\alpha]$ (see [14, Theorem 2.3]). As $B[X^{\pm 1};\alpha]$ has $\mathcal{H}$-normal separation, there are $c \in \widehat{P} \setminus \widehat{Q}$ and $h \in \mathcal{H}$ such that $cs - h(s)c \in \widehat{Q}$ for each $s \in B[X^{\pm 1};\alpha]$. Now, write $c = \sum_{i\in\mathbb{Z}} c_iX^i$. Note that each c_i is in P and at least one c_i is not in Q, say $c_{i_0} \notin Q$. Let $b \in B$. Then $cb - h(b)c \in \widehat{Q}$. Therefore, $\sum_i c_iX^ib - h(b)c_iX^i \in \widehat{Q}$, and so

$$\sum_i (c_i\alpha^i(b) - h(b)c_i)X^i \in \widehat{Q}.$$

As $\widehat{Q} = \bigoplus_{i\in\mathbb{Z}} QX^i$, this forces $c_i\alpha^i(b) - h(b)c_i \in Q$ for each i, and, in particular, $c_{i_0}\alpha^{i_0}(b) - h(b)c_{i_0} \in Q$. As b was an arbitrary element of B, we may replace b by $\alpha^{-i_0}(b)$ to obtain

$$c_{i_0}b - h\alpha^{-i_0}(b)c_{i_0} \in Q.$$

As α coincides with the action of an element of $\mathcal{H}$ on B, this produces an element $h_{i_0} \in \mathcal{H}$ such that

$$c_{i_0}b - h_{i_0}(b)c_{i_0} \in Q,$$

as required to show that B has $\mathcal{H}$-normal separation.

As $A \cong B$ via an $\mathcal{H}$-equivariant isomorphism, this shows that A has $\mathcal{H}$-normal separation. □

In the next theorem, we use this result to show that partition subalgebras have $\mathcal{H}$-normal separation. The starting point is the result due to Cauchon that generic quantum matrices have $\mathcal{H}$-normal separation.

Theorem 3.8. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ has $\mathcal{H}$-normal separation.*

Proof. Let $\mu = (n, \dots, n)$ (m times), so that Y_μ is an $m \times n$ rectangle. Then $\mathcal{A}_\mu = \mathcal{O}_q(M_{m,n}(\Bbbk))$, and so $\mathcal{A}_\mu$ has $\mathcal{H}$-normal separation, by [3, Théorème 6.3.1]. We can construct $\mathcal{A}_\mu$ from $\mathcal{A}_\lambda$ by adding the missing variables x_{ij} in lexicographic order. At each stage, the extension is a Cauchon extension. Thus, $\mathcal{A}_\lambda$ has $\mathcal{H}$-normal separation, by repeated application of the previous lemma. $\square$

Corollary 3.9. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ has normal $\mathcal{H}$-separation and normal separation.*

Proof. We have already seen earlier that $\mathcal{H}$-normal separation implies normal $\mathcal{H}$-separation. Normal separation now follows from [9, Theorem 5.3]. $\square$

Corollary 3.10. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ is catenary.*

Proof. This follows from the previous results and [20, Theorem 0.1] which states that if A is a normally separated filtered $\Bbbk$-algebra such that $\mathrm{gr}(A)$ is a noetherian connected graded $\Bbbk$-algebra with enough normal elements then $\mathrm{Spec}(A)$ is catenary. (For the notion of an algebra with enough normal elements see [21].) $\square$

Note that it is also possible to deduce this result from [10, Theorem 1.6].

4. Quantum Schubert cells

Quantum Schubert cells in the quantum grassmannian are obtained from quantum Schubert varieties via the process of noncommutative dehomogenisation introduced in [12]. Recall that if $R = \bigoplus R_i$ is an $\mathbb{N}$-graded $\Bbbk$-algebra and x is a regular homogeneous normal element of R of degree one, then the *dehomogenisation* of R at x, written $\mathrm{Dhom}(R, x)$, is defined to be the zero degree subalgebra S_0 of the $\mathbb{Z}$-graded algebra $S := R[x^{-1}]$. If R is generated as a $\Bbbk$-algebra by $a_1, \dots, a_s$ and each a_i has degree one, then it is easy to check that $\mathrm{Dhom}(R, x)$ is the $\Bbbk$-subalgebra of S generated by $a_1x^{-1}, \dots, a_sx^{-1}$.

If σ denotes the automorphism of S given by $\sigma(s) = xsx^{-1}$ for $s \in S$ then σ induces an automorphism of S_0, also denoted by σ, and there is an isomorphism

$$\theta : \mathrm{Dhom}(R, x)[y, y^{-1}; \sigma] \to R[x^{-1}]$$

which is the identity on $\mathrm{Dhom}(R, x)$ and sends y to x.

Let γ be an element of $\Pi_{m,n}$. Recall from Remark 1.4 that

$$S(\gamma) = \mathcal{O}_q(G_{m,n}(\Bbbk))/\langle \Pi^\gamma_{m,n} \rangle$$

and that $\overline{\gamma}$ is a homogeneous regular normal element of degree one in $S(\gamma)$. It follows that we can form the localisation $S(\gamma)[\overline{\gamma}^{-1}]$ and that $S(\gamma) \subseteq S(\gamma)[\overline{\gamma}^{-1}]$.

Definition 4.1. The *quantum Schubert cell* associated to the quantum minor γ is denoted by $S^o(\gamma)$ and is defined to be $\mathrm{Dhom}(S(\gamma), \overline{\gamma})$.

Remark 4.2. In the classical case when $q = 1$, it can be seen that this definition coincides with the usual definition of Schubert cells, as discussed, for example, in [4, Section 9.4].

It follows from the definition that $S^o(\gamma)$ is generated by the elements $\overline{x}\,\overline{\gamma}^{-1}$ for $x \in \Pi_{m,n} \setminus (\Pi^{\gamma}_{m,n} \cup \{\gamma\})$. However, these elements are not independent, so we will pick out a better generating set for the quantum Schubert cell.

This is achieved by using the quantum ladder matrix algebras introduced in [16, Section 3.1]. (The "ladder" terminology is adapted from the classical case, as introduced, for example, in [8, Section 12.3].) Let us recall the definition. To each $\gamma = (\gamma_1, \dots, \gamma_m) \in \Pi_{m,n}$ with $1 \le \gamma_1 < \cdots < \gamma_m \le n$, we associate the subset $\mathcal{L}_\gamma$ of $\{1, \dots, m\} \times \{1, \dots, n\}$ defined by

$$\mathcal{L}_\gamma = \{(i,j) \in \{1, \dots, m\} \times \{1, \dots, n\} \mid j > \gamma_{m+1-i} \text{ and } j \ne \gamma_l \text{ for } 1 \le l \le m\},$$

which we call the *ladder* associated with γ.

Definition 4.3. Let $\gamma = (\gamma_1, \dots, \gamma_m) \in \Pi_{m,n}$ with $1 \le \gamma_1 < \cdots < \gamma_m \le n$. The *quantum ladder matrix algebra* associated with γ, denoted $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$, is the $\Bbbk$-subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the elements $x_{ij} \in \mathcal{O}_q(M_{m,n}(\Bbbk))$ such that $(i,j) \in \mathcal{L}_\gamma$.

The following example, taken from [16], will help clarify this definition.

Example. We put $(m,n) = (3,7)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3) = (1,3,6) \in \Pi_{3,7}$. In the 3×7 generic matrix $\mathbf{X} = (x_{ij})$ associated with $\mathcal{O}_q(M_{3,7}(\Bbbk))$, put a bullet on each row as follows: on the first row, the bullet is in column 6 because γ_3 is 6, on the second row, the bullet is in column 3 because γ_2 is 3, and on the third row, the bullet is in column 1 because $\gamma_1 = 1$. Now, in each position which is to the left of a bullet, or which is below a bullet, put a star. To finish, place x_{ij} in any position (i,j) that has not yet been filled. We obtain

$$\begin{pmatrix} * & * & * & * & * & \bullet & x_{17} \\ * & * & \bullet & x_{24} & x_{25} & * & x_{27} \\ \bullet & x_{32} & * & x_{34} & x_{35} & * & x_{37} \end{pmatrix}.$$

By definition, the quantum ladder matrix algebra associated with $\gamma = (1,3,6)$ is the subalgebra of $\mathcal{O}_q(M_{3,7}(\Bbbk))$ generated by the elements x_{17}, x_{24}, x_{25}, x_{27}, x_{32}, x_{34}, x_{35}, x_{37}.

Notice that if we rotate the matrix above through 180° then the x_{ij} involved in the definition of $\mathcal{O}_q(M_{3,7,\gamma}(\Bbbk))$ sit naturally in the Young diagram of the partition $\lambda = (4,3,1)$. We will return to this point later.

Consider the quantum minors m_{ij} defined by

$$m_{ij} := [\{\gamma_1, \dots, \gamma_m\} \setminus \{\gamma_{m+1-i}\} \cup \{j\}]$$

for each $(i,j) \in \mathcal{L}_\gamma$. These are the quantum minors that are above γ in the standard order and differ from γ in exactly one position (before rearranging the entries in ascending order). Denote the set of these quantum minors by $\mathcal{M}_\gamma$.

Proposition 4.4. *The quantum Schubert cell $S^o(\gamma)$ is the $\Bbbk$-subalgebra of $S(\gamma)$ generated by $\{\overline{m}_{ij}\overline{\gamma}^{-1} \mid m_{ij} \in \mathcal{M}_\gamma\}$.*

Proof. In the proof of [16, Theorem 3.1.6] it is shown that $S(\gamma)[\overline{\gamma}^{-1}]$ is generated by the elements $\overline{\gamma}, \overline{\gamma}^{-1}$ and the $\overline{m}_{ij}$. The Schubert cell $S^o(\gamma)$ is the degree zero part of this algebra. As $\overline{\gamma}$ and $\overline{m}_{ij}$ commute up to scalars (see [16, Lemma 3.1.4(v)]), it is easy to check that $S^o(\gamma)$ is generated by $\overline{m}_{ij}\overline{\gamma}^{-1}$, as required. $\square$

Set $\widetilde{m}_{ij} := \overline{m}_{ij}\overline{\gamma}^{-1}$.

Lemma 4.5. *There is an induced action of $\mathcal{H} = (\Bbbk^*)^n$ on $S^o(\gamma)$ given by*

$$(\alpha_1, \ldots, \alpha_n) \circ \widetilde{m}_{ij} := \alpha^{-1}_{\gamma_{m+1-i}} \alpha_j \widetilde{m}_{ij}.$$

Proof. This follows immediately from the fact that

$$\widetilde{m}_{ij} = \overline{[\{\gamma_1, \ldots, \gamma_m\} \setminus \{\gamma_{m+1-i}\} \cup \{j\}]}\;\overline{[\gamma_1, \ldots, \gamma_m]}^{-1}. \qquad \square$$

We now need to establish the commutation relations between the $\widetilde{m}_{ij}$.

Lemma 4.6. *The quantum Schubert cell $S^o(\gamma)$ is isomorphic to the quantum ladder matrix algebra $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$.*

Proof. Lemma 3.1.4 of [16] shows that the commutation relations for the m_{ij} are the same as the commutation relations for corresponding variables x_{ij} in the quantum ladder matrix algebra $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$. As $\gamma m_{ij} = q m_{ij}\gamma$ for each i, j, by [16, Lemma 3.1.4(v)], it follows that the $\widetilde{m}_{ij}$ satisfy the same relations. Thus there is an epimorphism from $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$ onto $S^o(\gamma)$. If this epimorphism were not also a monomorphism then we would obtain $\mathrm{GKdim}(S^o(\gamma)) < \mathrm{GKdim}(\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$, by [13, Proposition 3.15], and then a comparison of Gelfand–Kirillov dimensions similar to that used in [16, Theorem 3.1.6] would produce a contradiction. Thus, there is an isomorphism from $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$ to $S^o(\gamma)$, as required. $\square$

Theorem 4.7. *The quantum Schubert cell $S^o(\gamma)$ is (isomorphic to) a partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\Bbbk))$.*

Proof. For any n, there is an isomorphism $\delta : \mathcal{O}_q(M_n(\Bbbk)) \to \mathcal{O}_{q^{-1}}(M_n(\Bbbk))$ defined by $\delta(x_{ij}) = x_{n+1-i,n+1-j}$ (see the proof of [11, Corollary 5.9]). The isomorphism δ can be used to convert quantum ladder matrix algebras into partition subalgebras. As Schubert cells are isomorphic to quantum ladder matrix algebras, the result follows. $\square$

The previous theorem should be considered as a noncommutative analogue of the well-known result in the classical case that each Schubert cell is isomorphic to an affine space of the appropriate dimension (see, for example, the discussion in [4, Section 9.4]).

The isomorphism described in the previous result carries over the $\mathcal{H}$-action on $S^o(\gamma)$ to the partition subalgebra, and this induced action acts via row and column multiplications. After suitable renumbering of the summands of $\mathcal{H}$, this action coincides with the action discussed at the beginning of Section 3. As a consequence of Theorem 4.7, the results obtained in Section 3 apply to quantum Schubert cells. In particular, the following results hold.

Theorem 4.8. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the partition with $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ defined by $\lambda_i + \gamma_i = n - m + i$ and let Y_λ be the corresponding Young diagram. Then the $\mathcal{H}$-prime spectrum of $S^o(\gamma)$ is in bijection with the set of Cauchon diagrams on the Young diagram, Y_λ, as described in Theorem* 3.5.

Theorem 4.9. *The quantum Schubert cell $S^o(\gamma)$ has $\mathcal{H}$-normal separation, normal $\mathcal{H}$-separation and normal separation.*

Corollary 4.10. *The quantum Schubert cell $S^o(\gamma)$ is catenary.*

5. The prime spectrum of the quantum grassmannian

In this section, we use the quantum Schubert cells to obtain information concerning the prime spectrum of the quantum grassmannian. We show that, in the generic case, where q is not a root of unity, all primes are completely prime, and there are only finitely many primes that are invariant under the natural torus action on the quantum grassmannian. By using a result of Lauren Williams, we are able to count the number of $\mathcal{H}$-primes. Also, we are able to show that the quantum grassmannian is catenary.

Note that the following result is valid for any $q \neq 0$.

Theorem 5.1. *Let P be a prime ideal of $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ with $P \neq \langle \Pi \rangle$, that is, P is not the irrelevant ideal. Then there is a unique γ in Π with the property that $\gamma \notin P$ but $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$.*

Proof. If $\Pi \subseteq P$ then P is the irrelevant ideal. Otherwise, there exists $\gamma \in \Pi$ with $\gamma \notin P$. Choose such a γ that is minimal in Π with this property. Then $\lambda \in P$ for all $\lambda <_{\mathrm{st}} \gamma$.

Note that $\langle \{\lambda \mid \lambda <_{\mathrm{st}} \gamma\} \rangle \subseteq P$ and γ is normal modulo $\langle \{\lambda \mid \lambda <_{\mathrm{st}} \gamma\} \rangle$, by [15, Lemma 1.2.1], so that γ is normal modulo P.

Suppose that $\pi \not\geq_{\mathrm{st}} \gamma$. If $\pi <_{\mathrm{st}} \gamma$ then $\pi \in P$ by the choice of γ. If not, then π and γ are not comparable. Thus, we can write

$$\pi\gamma = \sum k_{\lambda\mu} \lambda\mu$$

with $k_{\lambda\mu} \in \mathbb{k}$ while $\lambda, \mu \in \Pi$ with $\lambda <_{\mathrm{st}} \gamma$, by [15, Theorem 3.3.8].

It follows that $\pi\gamma \in P$. Thus, $\pi \in P$, since $\gamma \notin P$ and γ is normal modulo P.

This shows that there is a γ with the required properties. It is easy to check that there can only be one such γ. $\square$

This theorem enables us to give a decomposition of the prime spectrum, $\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. Set $\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ to be the set of prime ideals P such that $\gamma \notin P$ while $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$. The previous result shows that

$$\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk))) = \bigsqcup_{\gamma\in\Pi} \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk))) \sqcup \langle\Pi\rangle.$$

We now reinstate our convention that q is not a root of unity.

Theorem 5.2. *Let q be a non-root of unity. Then all prime ideals of the quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are completely prime.*

Proof. Let P be a prime ideal of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. If $P = \langle\Pi\rangle$ then $\mathcal{O}_q(G_{m,n}(\Bbbk))/P \cong \Bbbk$; so P is completely prime.

Otherwise, suppose that $P \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. In this case, $\overline{P} = P/\langle\Pi^\gamma_{m,n}\rangle$ is a prime ideal in $S(\gamma) = \mathcal{O}_q(G_{m,n}(\Bbbk))/\langle\Pi^\gamma_{m,n}\rangle$ and it is enough to show that $\overline{P}$ is completely prime. Set $T := S(\gamma)[\overline{\gamma}^{-1}]$. Then $\overline{P}T$ is a prime ideal of T and $\overline{P}T \cap S(\gamma) = \overline{P}$. Thus $S(\gamma)/\overline{P} \subseteq T/\overline{P}T$ and so it is enough to show that $\overline{P}T$ is completely prime.

Now, the dehomogenisation isomorphism shows that $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$, where σ is the automorphism determined by the conjugation action of $\overline{\gamma}$ (see the beginning of Section 4).

We know that $S^o(\gamma)$ is a torsion-free CGL extension by Proposition 3.2 and Theorem 4.7. It is then easy to check that $S^o(\gamma)[y; \sigma]$ is also a torsion-free CGL extension. Thus, all prime ideals of $S^o(\gamma)[y; \sigma]$ are completely prime, by [1, Theorem II.6.9], and it follows that all prime ideals of $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ are completely prime, as required. □

Of course, the decomposition of $\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ above induces a similar decomposition of $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$:

$$\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk))) = \bigsqcup_{\gamma\in\Pi} \mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk))) \sqcup \langle\Pi\rangle,$$

where $\mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is the set of $\mathcal{H}$-prime ideals P such that $\gamma \notin P$ while $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$.

Our next task is to show that $\mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is in natural bijection with $\mathcal{H}\text{-}\mathrm{Spec}(S^o(\gamma))$ and hence in bijection with Cauchon diagrams on the associated Young diagram Y_λ. As a consequence, we are able to calculate the size of $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$.

Remark 5.3. Recall from the beginning of Section 4 that, for any $\gamma \in \Pi_{m,n}$, there is the dehomogenisation isomorphism

$$\theta : S^o(\gamma)[y, y^{-1}; \sigma] \to S(\gamma)[\overline{\gamma}^{-1}],$$

where σ is conjugation by $\overline{\gamma}$. Hence, the action of $\mathcal{H}$ on $S(\gamma)[\overline{\gamma}^{-1}]$ transfers, via θ, to an action on $S^o(\gamma)[y, y^{-1}; \sigma]$. By Lemma 4.5, $S^o(\gamma)$ is stable under this action and it is clear that y is an $\mathcal{H}$-eigenvector. Further, let $h_0 = (\alpha_1, \ldots, \alpha_n) \in \mathcal{H}$

be such that $\alpha_i = q^2$ if $i \notin \{\gamma_1, \dots, \gamma_m\}$ and $\alpha_i = q$ otherwise. Then, by using [16, Lemma 3.1.4(v)] and Lemma 4.5, it is easily verified that the action of h_0 on $S^o(\gamma)$ coincides with σ. In addition, $h_0(y) = q^m y$, since $h_0(\overline{\gamma}) = q^m \overline{\gamma}$. It follows that $S^o(\gamma)[y, y^{-1}; \sigma]$ satisfies Hypothesis 2.1 in [14].

Theorem 5.4. *Let $P \in \mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$, so that P is an $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ such that $\gamma \notin P$, while $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$. Set $T = S(\gamma)[\overline{\gamma}^{-1}] \cong S^o(\gamma)[y, y^{-1}; \sigma]$. Then the assignment $P \mapsto \overline{P}T \cap S^o(\gamma)$ defines an inclusion-preserving bijection from $\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ to $\mathcal{H}\text{-Spec}(S^o(\gamma))$, with inverse obtained by sending Q to the inverse image in $\mathcal{O}_q(G_{m,n}(\Bbbk))$ of $QT \cap S(\gamma)$. (Note we are treating the isomorphism above as an identification in these assignments.)*

Proof. This follows from the conjunction of two bijections. First, standard localisation theory shows that $\overline{P} = \overline{P}T \cap S(\gamma)$, and this gives a bijection between $\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ and $\mathcal{H}\text{-Spec}(T)$. For the second bijection, note that $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ and the automorphism σ is given by the action of an element of $\mathcal{H}$ (see Remark 5.3). Thus, it follows from [14, Theorem 2.3] that there is a bijection between $\mathcal{H}\text{-Spec}(T)$ and $\mathcal{H}\text{-Spec}(S^o(\gamma))$ given by intersecting an $\mathcal{H}$-prime of T with $S^o(\gamma)$. The composition of these two bijections produces the required bijection. □

Corollary 5.5. *$\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is in bijection with the Cauchon diagrams on Y_λ, where λ is the partition associated with γ.*

Proof. This follows from the previous theorem and Theorem 4.8. □

It follows from this corollary and the partition of the $\mathcal{H}$-spectrum of the quantum grassmannian that the $\mathcal{H}$-spectrum of the quantum grassmannian is finite. This finiteness is a crucial condition needed to investigate normal separation, Dixmier–Mœglin equivalence, etc. in the quantum case because of the stratification theory (see, for example, [9, Theorem 5.3], [1, Theorem II.8.4]). However, in this situation, we can say much more: we can say exactly how many $\mathcal{H}$-primes there are in the quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$. This is one more (the irrelevant ideal $\langle \Pi \rangle$) than the total number of Cauchon diagrams on the Young diagrams Y_λ corresponding to the partitions λ that fit into the partition $(n-m)^m$. This combinatorial problem has been solved by Lauren Williams in [19]. The following result is obtained by setting $q = 1$ in the formula for $A_{k,n}(q)$ in [19, Theorem 4.1].

Theorem 5.6.

$$|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))| \\ = 1 + \sum_{i=0}^{m-1} \binom{n}{i} \big((i-m)^i (m-i+1)^{n-i} - (i-m+1)^i (m-i)^{n-i}\big).$$

Proof. By using the results above, we see that, except for the irrelevant ideal, each $\mathcal{H}$-prime corresponds to a unique Cauchon diagram drawn on the Young diagram

Y_λ that corresponds to the partition λ associated to the quantum minor γ which determines the cell that P is in.

In [19, Theorem 4.1], Lauren Williams has counted the number of Cauchon diagrams on the Young diagrams Y_λ that fit into the partition $(n-m)^m$; and this count, plus one, gives the number of $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. $\square$

For example, $|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{2,4}(\Bbbk)))| = 34$ and $|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{3,6}(\Bbbk)))| = 884$. (These numbers can be seen from the table in [17, Figure 23.1].)

We now turn to the questions of normal separation and catenarity. In order to establish these properties for the quantum grassmannian, we need to use the dehomogenisation isomorphism. Recall that the methods of [14] are available because of Remark 5.3.

Lemma 5.7. *Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals in $S(\gamma)$ that do not contain $\overline{\gamma}$. Then there is an $\mathcal{H}$-eigenvector in $P \setminus Q$ that is normal modulo Q.*

Proof. Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals in $S(\gamma)$ that do not contain $\overline{\gamma}$. Set $T := S(\gamma)[\overline{\gamma}^{-1}]$ and observe that there is an induced action of the torus $\mathcal{H}$ on T, because γ is an $\mathcal{H}$-eigenvector. Note that $P = PT \cap S(\gamma)$ and $Q = QT \cap S(\gamma)$; so $QT \subsetneq PT$ are $\mathcal{H}$-prime ideals in T. Now, set $P_0 := PT \cap S^o(\gamma)$ and $Q_0 := QT \cap S^o(\gamma)$ (here, we are treating the isomorphism $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ as an identification) and note that $PT = \bigoplus_{n\in\mathbb{Z}} P_0\, y^n$ and $QT = \bigoplus_{n\in\mathbb{Z}} Q_0\, y^n$; so $Q_0 \subsetneq P_0$ are $\mathcal{H}$-prime ideals of $S^o(\gamma)$ (see Remark 5.3 and [14, Theorem 2.3]). These observations make it clear that

$$\frac{S^o(\gamma)}{Q_0}[y, y^{-1}; \sigma] \cong \frac{T}{QT} \cong \frac{S(\gamma)}{Q}[\overline{\gamma}^{-1}].$$

As usual, $\overline{S^o(\gamma)}$ will denote $S^o(\gamma)/Q_0$, etc.

The quantum Schubert cell $S^o(\gamma)$ has $\mathcal{H}$-normal separation, by Theorem 4.9. Thus, there exists an $\mathcal{H}$-eigenvector $c \in P_0 \setminus Q_0$ and an element $h \in \mathcal{H}$ such that $ca - h(a)c \in Q_0$ for all $a \in S^o(\gamma)$. Recall that the action of σ coincides with the action of an element h_y of $\mathcal{H}$, so that $yc = h_y(c)y = \lambda cy$ for some $\lambda \in \Bbbk^*$. It follows that $\overline{c}$ is normal in T/QT. Define $\sigma_c : T/QT \to T/QT$ by $\overline{c}\overline{t} = \sigma_c(\overline{t})\overline{c}$ for all $t \in T$. Note that $\sigma_c|_{\overline{S^o(\gamma)}} = h|_{\overline{S^o(\gamma)}}$ and $\sigma_c(y) = \lambda^{-1}y$.

We claim that $\sigma_c(S(\gamma)/Q) = S(\gamma)/Q$, so that σ_c induces an isomorphism on this algebra. In order to see this, note that $S(\gamma)/Q$ is generated as an algebra by the images of the quantum minors $[\alpha_1, \dots, \alpha_m]$ for $[\alpha_1, \dots, \alpha_m] \geq \gamma$. Now, $\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1} \in \overline{S^o(\gamma)}$, because $[\alpha_1, \dots, \alpha_m]\gamma^{-1}$ has degree zero in T so that $[\alpha_1, \dots, \alpha_m]\gamma^{-1} \in S^o(\gamma)$. Thus, recalling that $\overline{\gamma}$ is identified with y under the isomorphisms above,

$$\begin{aligned}
\sigma_c(\overline{[\alpha_1, \dots, \alpha_m]}) &= \sigma_c(\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1})\sigma_c(\overline{\gamma}) = h(\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1})(\lambda^{-1}y) \\
&= \mu\,\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1}(\lambda^{-1}y) = (\mu\lambda^{-1})\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1}y \\
&= (\mu\lambda^{-1})\overline{[\alpha_1, \dots, \alpha_m]},
\end{aligned}$$

where the existence of $\mu \in \Bbbk^*$ is guaranteed because h is acting as a scalar on the element $\overline{[\alpha_1, \dots, \alpha_m]}\,\overline{\gamma}^{-1} \in S^o(\gamma)/Q_0$. The claim follows.

There exists $d \geq 0$ such that $\overline{c}\,\overline{\gamma}^d \in S(\gamma)/Q$. It is obvious that $c\overline{\gamma}^d$ is an $\mathcal{H}$-eigenvector, because each of c and γ is an $\mathcal{H}$-eigenvector. Also, $c\overline{\gamma}^d \in P \setminus Q$. Finally, $\overline{c}\,\overline{\gamma}^d$ is normal in $S(\gamma)/Q$, because $S(\gamma)/Q$ is invariant under conjugation by each of $\overline{c}$ and $\overline{\gamma}$. □

Theorem 5.8. *The quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ has normal $\mathcal{H}$-separation and hence normal separation.*

Proof. Suppose that $Q \subsetneq P$ are $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. Suppose that $Q \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. If $\gamma \in P$, then P contains the $\mathcal{H}$-eigenvector γ.

Otherwise, $\gamma \notin P$ and $P \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. In this case, it is enough to show that there is an $\mathcal{H}$-eigenvector in $\overline{P} \setminus \overline{Q}$ which is normal modulo $\overline{Q}$, where $\overline{P} = P/\langle \Pi^\gamma_{m,n} \rangle$ and $\overline{Q} = Q/\langle \Pi^\gamma_{m,n} \rangle$ are $\mathcal{H}$-prime ideals in $S(\gamma)$. However, this has been done in the previous lemma. □

Theorem 5.9. *The quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is catenary.*

Proof. As in Corollary 3.10, this follows from the previous results and [20, Theorem 0.1]. □

Remark 5.10. It is obvious from the style of proof of the preceding results that there is now a good strategy for producing results concerning the quantum grassmannian: first, establish the corresponding results for partition subalgebras of quantum matrices, and then use the theory of quantum Schubert cells and noncommutative dehomogenisation to obtain the result in the quantum grassmannian. We leave any further developments for interested readers.

6. Concluding remark

We end this work by stressing some important connections between the results established in Section 5 above and recent work of Postnikov in total positivity (see [17]).

Let $M^+_{m,n}(\mathbb{R})$ denote the space of $m \times n$ real matrices of rank m and whose $m \times m$ minors are nonnegative. The group $GL^+_m(\mathbb{R})$ of $m \times m$ real matrices of positive determinant acts naturally on $M^+_{m,n}(\mathbb{R})$ by left multiplication. The corresponding quotient space $G^+_{m,n}(\mathbb{R}) = M^+_{m,n}(\mathbb{R})/GL^+_m(\mathbb{R})$ is the *totally nonnegative* grassmannian of m-dimensional subspaces in $\mathbb{R}^n$. One can define a cellular decomposition of $G^+_{m,n}(\mathbb{R})$ by specifying, for each element of $G^+_{m,n}(\mathbb{R})$, which $m \times m$ minors are zero and which are strictly positive. The corresponding cells are called the *totally nonnegative cells* of $G^+_{m,n}(\mathbb{R})$. In [17], Postnikov shows that totally nonnegative cells in $G^+_{m,n}(\mathbb{R})$ are in bijection with the Cauchon diagrams on partitions fitting into the partition $(n-m)^m$. For further details, see Sections 3 and 6 in [17].

Hence, by the results in Section 5 above, the set of totally nonnegative cells of $G^+_{m,n}(\mathbb{R})$ is in one-to-one correspondence with the set of $\mathcal{H}$-prime ideals of

$\mathcal{O}_q(G_{m,n}(\Bbbk))$ distinct from the augmentation ideal. We believe it would be interesting to understand this coincidence and we intend to pursue this theme in a subsequent paper.

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