Finite Dimensional Algebras and Related Topics

Edited by

V. Dlab and L. L. Scott

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PREFACE

These Proceedings report on a number of lecture series delivered at the 1992 Annual Canadian Mathematical Seminar / NATO Advanced Research Workshop held at Carleton University, Ottawa, Canada during August 10-18, 1992.

The Workshop was dedicated to the interaction of finite dimensional algebras with other areas of Mathematics, especially Lie theory, but including also algebraic geometry, automorphic forms, finite group representation theory and mathematical physics. The Scientific Organizing Committee consisted of M. Auslander, R. Bautista, C.W. Curtis, V. Dlab (Chairman), I.M. Gelfand, D. Handelman, G.O. Michler, L.L. Scott and H. Tachikawa. The intellectual predecessor of this conference was perhaps the Ottawa - Moosonee Conference organized by one of the Editors in August 1987. That conference included similar topics on a smaller scale, and some of the activity at this Workshop can be directly traced back to it. Nevertheless, while the serious interaction of finite dimensional algebras with many of these other areas was somewhat novel at the Ottawa-Moosonee Conference, it was clear at the present meeting that it has become an established feature of the mathematical landscape, both within finite dimensional algebras and without. This volume stands as a partial testimony to this new and welcome development.

The program of the Seminar/Workshop consisted of 27 lecture series, all listed separately in this volume. The Editors invited a number of the lecturers to submit articles for these Proceedings, with a preference to those lectures illustrating the interaction described above. It was expected that all submissions would be refereed as appropriate; in fact, most referees reported with enthusiasm. We would like to express our gratitude to the authors who submitted contributions, and to all referees for their assistance.

We wish to express our thanks for financial support to both the Natural Sciences and Engineering Research Council of Canada and the Scientific Division of the North Atlantic Treaty Organization. We also wish to extend our gratitude to Drs. István Ágoston and Erzsébet Lukács for their much appreciated secretarial support and efficiency; without their generous assistance the Workshop could not have run so smoothly.

Ottawa and Charlottesville, August 1993

Vlastimil Dlab and Leonard Scott

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Equivalences of Blocks of Group Algebras

MICHEL BROUÉ

Ecole Normale Supérieure, Paris

Three lectures given at the International Conference on Representations of Algebras
Ottawa, August 1992

Let $\mathcal{O}$ be a complete local noetherian ring, whose field of fractions has characteristic zero and residue field has non-zero characteristic. A block algebra over $\mathcal{O}$ is an indecomposable summand of the algebra of a finite group over $\mathcal{O}$.

We introduce briefly (and justify through examples) several types of equivalences. Three levels of equivalences between block algebras seem to be relevant: Morita equivalence, Rickard (derived) equivalence, stable equivalence of Morita type.

We give a classification of various classical "invariants" of block algebras (such as numerical defect, decomposition matrices, defect of irreducible characters, etc.) depending on the type of equivalence we consider between block algebras.

After recalling why, when switching from the algebra point of view to the group point of view, the source algebra is a suitable replacement for the basic algebra, we try to give suitable "group theoretic" refinements of the previous equivalences.

This is an introductory survey : almost no proof is given, the comments are brief and the applications short. We emphasize the "algebra-theoretic approach", which should be viewed as a first approximation to the methods used in block theory, as we try to explain in the last paragraph. In order to simplify the exposition, we restrict ourselves, most of the time, to the case of principal blocks.

1. BASIC CONTEXT AND NOTATION

Let A be a left and right noetherian ring.

We denote by ${}_A\mathbf{mod}$ the abelian category of finitely generated left A-modules, and by ${}_A\mathbf{proj}$ the category of finitely generated projective left A-modules. We denote by $\mathcal{R}(A)$ the Grothendieck group of ${}_A\mathbf{mod}$ and by $\mathcal{R}^{\mathrm{pr}}(A)$ the Grothendieck group of ${}_A\mathbf{proj}$. If X is an object of ${}_A\mathbf{mod}$ (resp. of ${}_A\mathbf{proj}$), we denote by $[X]$ its representative in $\mathcal{R}(A)$ (resp. in $\mathcal{R}^{\mathrm{pr}}(A)$).

We denote by $\mathbf{mod}_A$ the abelian category of finitely generated right A-modules, and by $\mathbf{proj}_A$ the category of finitely generated projective right A-modules. For B another ring, we denote by ${}_A\mathbf{mod}_B$ the category of finitely generated (A, B)-bimodules.

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Let G be a finite group, and let p be a prime number. Let K be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ which contains the $|G|$-th roots of unity. Thus the group algebra KG is a split semi-simple K-algebra. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_p$. We denote by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, and we set $k := \mathcal{O}/\mathfrak{p}$. If JkG denotes the Jacobson radical of the group algebra kG, the algebra kG/JkG is a split semi-simple k-algebra.

By extension of scalars we get two functors

$$_{\mathcal{O}G}\mathbf{mod} \to {}_{KG}\mathbf{mod} \quad \text{and} \quad {}_{\mathcal{O}G}\mathbf{mod} \to {}_{kG}\mathbf{mod}\,.$$

$$\begin{tikzcd}
K & & \\
& \mathcal{O} \arrow[ul, hook] \arrow[r, two heads] & k = \mathcal{O}/\mathfrak{p} \\
\mathbb{Q}_p \arrow[uu, hook] & & \\
& \mathbb{Z}_p \arrow[ul, hook] \arrow[uu, hook] \arrow[r, two heads] & \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p \arrow[uu, hook]
\end{tikzcd}$$

The Cartan–Decomposition triangle.

For the following classical facts, we refer the reader to [Se], part III.

The set $\mathrm{Irr}(KG)$ of representatives in $\mathcal{R}(KG)$ of the irreducible KG-modules is an orthonormal basis of $\mathcal{R}(KG)$ for the scalar product defined by

$$\langle [X], [X'] \rangle := \dim \mathrm{Hom}_{KG}(X, X')\,.$$

The set $\mathrm{Irr}(kG)$ of representatives in $\mathcal{R}(kG)$ of the irreducible kG-modules is a $\mathbb{Z}$-basis of $\mathcal{R}(kG)$, while the set $\mathrm{Pim}(kG)$ of representatives in $\mathcal{R}^{\mathrm{pr}}(kG)$ of the indecomposable projective kG-modules is a $\mathbb{Z}$-basis of $\mathcal{R}^{\mathrm{pr}}(kG)$. The pairing

$$\mathcal{R}^{\mathrm{pr}}(kG) \times \mathcal{R}(kG) \to \mathbb{Z}$$

defined by

$$\langle [P], [X] \rangle := \dim \mathrm{Hom}_{kG}(P, X)$$

(P an object of ${}_{kG}\mathbf{proj}$, X an object of ${}_{kG}\mathbf{mod}$) defines a duality between $\mathcal{R}^{\mathrm{pr}}(kG)$ and $\mathcal{R}(kG)$.

Let X be a finitely generated KG-module. Let X_0 be a finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module such that $X = K \otimes_{\mathcal{O}} X_0$. Then the corresponding element $[k \otimes X_0]$ in $\mathcal{R}(kG)$ depends only on X, and this defines the decomposition map

$$\mathrm{dec}^G \colon \mathcal{R}(KG) \to \mathcal{R}(kG)\,.$$

The reduction modulo $\mathfrak{p}$ defines an isomorphism $\mathcal{R}^{\mathrm{pr}}(\mathcal{O}G) \xrightarrow{\sim} \mathcal{R}^{\mathrm{pr}}(kG)$. Identifying $\mathcal{R}^{\mathrm{pr}}(\mathcal{O}G)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$ through this isomorphism, the adjoint of the decomposition map is the linear map ${}^t\mathrm{dec}^G \colon \mathcal{R}^{\mathrm{pr}}(\mathcal{O}G) \to \mathcal{R}(KG)$ which sends the representative of a projective $\mathcal{O}G$–module X onto the representative of the KG–module $K \otimes_{\mathcal{O}} X$.

Finally, the Cartan map $\mathrm{Car}^G \colon \mathcal{R}^{\mathrm{pr}}(kG) \to \mathcal{R}(kG)$ is the linear map which sends the representative in $\mathcal{R}^{\mathrm{pr}}(kG)$ of a projective kG–module X onto its representative in $\mathcal{R}(kG)$.

$$(\mathcal{T}(G)) \qquad \begin{tikzcd} \mathcal{R}(KG) \arrow[rr, "\mathrm{dec}^G"] & & \mathcal{R}(kG) \\ & \mathcal{R}^{\mathrm{pr}}(kG) \arrow[ul, "{}^t\mathrm{dec}^G"] \arrow[ur, "\mathrm{Car}^G"'] & \end{tikzcd}$$

1.1. Theorem.

(1) *The cokernel of* Car^G *is a finite p-group, whose exponent is the order of a Sylow p-subgroup of G.*
(2) *The map* dec^G *is onto, and the image of* ${}^t\mathrm{dec}^G$ *is a pure submodule of* $\mathcal{R}(KG)$.
(3) $\mathrm{Car}^G = \mathrm{dec}^G \cdot {}^t\mathrm{dec}^G$.

2. Blocks

The decomposition of the unity element of $\mathcal{O}G$ into a sum of orthogonal primitive central idempotents $1 = \sum e$ corresponds to the decomposition of the algebra $\mathcal{O}G$ into a direct sum of indecomposable two–sided ideals $\mathcal{O}G = \bigoplus A$ $(A = \mathcal{O}Ge)$, called the *blocks* of $\mathcal{O}G$. For A a block of $\mathcal{O}$, we set $KA := K \otimes_{\mathcal{O}} A$ and $kA := k \otimes_{\mathcal{O}} A$.

By reduction modulo $\mathfrak{p}$, a primitive central idempotent remains primitive central, and consequently $kG = \bigoplus kA$ is still a decomposition into a direct sum of indecomposable two–sided ideals, called the blocks of kG.

$$\begin{array}{ccccc} \mathcal{O}G & = & \bigoplus & A \\ \downarrow & & & \downarrow \\ kG & = & \bigoplus & kA \end{array}$$

The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by$A_0(\mathcal{O}G)$.

2.A. The invariants of a block.

Let e be a central idempotent of $\mathcal{O}G$ and let $A := \mathcal{O}Ge$ be the corresponding algebra (note that we are not assuming e necessarily primitive, so what follows applies to direct sums of blocks). The idempotent e is the unity element of the algebra A, and A is a symmetric $\mathcal{O}$–algebra for the form

$$t \colon A \to \mathcal{O} \ , \ \sum_{g \in G} a(g)g \mapsto a(1) \ .$$

Center and projective center. View A as an (A, A)–bimodule. The ring $\mathrm{End}_A(A)_A$ of its endomorphisms is the center $Z(A)$ of A. The set of projective endomorphisms (endomorphisms which factorize through a projective (A, A)–bimodule) is an ideal of $Z(A)$ which is denoted by $Z^{\mathrm{pr}}(A)$ and called *the projective center* of A.

2.1. Proposition. *We have* $Z^{\mathrm{pr}}(A) = \{\sum_{g\in G} gag^{-1} \mid (a \in A)\}$.

The set of projective endomorphisms of the (kA, kA)–bimodule kA is denoted by $Z^{\mathrm{pr}}(kA)$. It is equal to the image of $Z^{\mathrm{pr}}(A)$ through the reduction modulo $\mathfrak{p}$ $Z(A) \to Z(kA)$.

c-d-triangle and associated invariants. The Grothendieck groups $\mathcal{R}(KA)$, $\mathcal{R}(kA)$ and $\mathcal{R}^{\mathrm{pr}}(kA)$ are summands of the Grothendieck groups $\mathcal{R}(KG)$, $\mathcal{R}(kG)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$, and the maps dec^G, ${}^t\mathrm{dec}^G$, Car^G restrict to maps which define the "c-d-triangle" of the block A,

$$(\mathcal{T}(A))\qquad \begin{tikzcd} \mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] & & \mathcal{R}(kA) \\ & \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] & \end{tikzcd}$$

which we view as endowed with its "metric structure" given by the dualities

$$\mathcal{R}(KA) \times \mathcal{R}(KA) \to \mathbb{Z} \quad \text{and} \quad \mathcal{R}^{\mathrm{pr}}(kA) \times \mathcal{R}(kA) \to \mathbb{Z}\,.$$

We denote by $\mathrm{Irr}(KA)$ (resp. $\mathrm{Irr}(kA)$, $\mathrm{Pim}(kA)$) the set of representatives in the corresponding Grothendieck group of the irreducible KA–modules (resp. of the irreducible kA–modules, of the projective indecomposable kA–modules), called the *canonical basis* of the corresponding $\mathbb{Z}$–modules.

The matrix of Car^A on the canonical basis is called the Cartan matrix of A and denoted by C^A, while the matrix of dec^A on the canonical basis is called the decomposition matrix of A and denoted by D^A.

It is traditional to set

$$\mathrm{k}(A) := |\mathrm{Irr}(KA)| \quad \text{and} \quad \mathrm{l}(A) := |\mathrm{Irr}(kA)| = |\mathrm{Pim}(kA)|\,.$$

The $\mathcal{O}$–rank of $Z(A)$ equals $\mathrm{k}(A)$, while the rank of $Z^{\mathrm{pr}}(kA)$ equals the number of trivial invariant factors of the map Car^A. We set

$$\mathrm{l}^{\mathrm{pr}}(A) := \dim Z^{\mathrm{pr}}(kA)\,.$$

The exponent of the cokernel of Car^A divides the order of a Sylow p–subgroup of G and so has the shape $p^{d(A)}$. The integer $d(A)$ is called the *defect* of the block A.

Defects of irreducible KA–modules.

If X is an irreducible KA–module, we set $p^{d(X)} := \left(\dfrac{|G|}{\dim X}\right)_p$, and we call the integer $d(X)$ the *defect* of X.

2.2. Proposition.

(1) *We have* $d(A) = \sup\{d(X) \mid (X \in \mathrm{Irr}(KA))\}$.
(2) *Let* X_0 *be an* $\mathcal{O}$*-free* A*-module such that* $X = K \otimes_{\mathcal{O}} X_0$. *Let* $\mathrm{End}_A^{\mathrm{pr}}(X_0)$ *be the ideal of* $\mathrm{End}_A(X_0)$ *consisting of the projective endomorphisms of* X_0. *Then* $\mathrm{End}_A(X_0)/\mathrm{End}_A^{\mathrm{pr}}(X_0) = p^{d(A)-d(X)}\mathcal{O}$.

Remark.

Let P be a p–group. Then we have $J\mathcal{O}P = \mathfrak{p}\mathcal{O}P + \mathcal{A}\mathcal{O}P$ where $J\mathcal{O}P$ denotes the Jacobson radical of $\mathcal{O}P$ and $\mathcal{A}\mathcal{O}P$ denotes the augmentation ideal of $\mathcal{O}P$. So $\mathcal{O}P$ is itself a block and we have $\mathrm{l}(\mathcal{O}G) = 1$. The c-d-triangle is trivial :

$$\begin{tikzcd} \mathcal{R}(KP) \arrow[rr, "\dim"] & & \mathcal{R}(kP) \\ & \mathcal{R}^{\mathrm{pr}}(kP) \arrow[ul, "\mathrm{reg}"] \arrow[ur, "|P|"'] & \end{tikzcd}$$

where the map "reg" maps the generator $|P|$ of $\mathcal{R}^{\mathrm{pr}}(kP)$ onto the representative of the regular representation of KP. Notice that, on the other hand, the category ${}_{\mathcal{O}P}\mathbf{mod}$ is far from being trivial. If P is neither cyclic nor (for $p = 2$) dihedral, semidihedral or generalized quaternion, then the algebra $\mathcal{O}P$ is wild.

2.B. Problems of block theory.

Block theory, as introduced and developed by Richard Brauer, originated mainly in the problem of the classification of finite simple groups. As a first approximation, we may say that the main problem of block theory is *to compare the category* ${}_{\mathcal{O}G}\mathbf{mod}$ *to the "local" categories* ${}_{\mathcal{O}N_G(P)}\mathbf{mod}$*, where P runs over the set of non–trivial p–subgroups of G*, and $N_G(P)$ denotes the normalizer of P in G.

Remark.

• Let A_0 be the principal block of $\mathcal{O}G$. The structure of ${}_{A_0}\mathbf{mod}$ is closely related to the structure of the group G itself — more precisely, to the structure of $G/O_{p'}(G)$, where $O_{p'}(G)$ denotes the largest normal subgroup of G whose order is relatively prime to p.

For example, let P be a Sylow p–subgroup of G. The following assertions are equivalent:

(i) $\mathrm{l}(A_0) = 1$,
(ii) ${}_{A_0}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}P}\mathbf{mod}$,
(iii) G is p–nilpotent, *i.e.*, isomorphic to the semi–direct product $O_{p'}(G) \rtimes P$.

• The situation is more complicated for non–principal blocks. Indeed, there are blocks A of non–abelian simple groups G which satisfy one of the following equivalent properties ("defect zero") :

(i) There is $X \in \mathrm{Irr}(KA)$ such that $(\dim X)_p$ equals the order of a Sylow p–subgroup of G,
(ii) ${}_{A}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}}\mathbf{mod}$.

For example, the block defined by the Steinberg character of $\mathrm{GL}_n(p^m)$ has the above properties.

More generally, the "nilpotent blocks" A (see [BrPu], [Pu]) are such that ${}_{A}\mathbf{mod} \simeq {}_{\mathcal{O}P}\mathbf{mod}$ for a certain p–group P. We give here an example of such a block in $\mathrm{GL}_n(\ell^m)$ for $\ell \neq p$ (see [Br1] for more details). We view $\mathrm{GL}_n(\ell^m)$ as the group of fixed points of the algebraic group $\mathbf{G} := \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ under the action of the usual Frobenius endomorphism F. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ and $\theta \colon \mathbf{T}^F \to K^\times$ a character of $\mathbf{T}^F$ such that :

- the order of θ is prime to p,

- θ is in general position in $\mathrm{GL}_n(\ell^m)$ (*i.e.*, an element which normalizes $\mathbf{T}$ and fixes θ must centralize $\mathbf{T}$).

We denote by $R_{\mathbf{T}}^{\mathbf{G}}\colon \mathcal{R}(K\mathbf{T}^F) \to \mathcal{R}(K\mathbf{G}^F)$ the linear map defined by Deligne and Lusztig (see [DeLu]). There is a block $A(\mathbf{T},\theta)$ of $\mathcal{O}\mathrm{GL}_n(\ell^m)$ such that

$$\mathrm{Irr}(KA(\mathbf{T},\theta)) = \{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(\theta\eta)\}_\eta$$

where η runs over the set of characters of $\mathbf{T}^F$ whose order is a power of p. The category ${}_{A(\mathbf{T},\theta)}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}\mathbf{T}_p^F}\mathbf{mod}$ where $\mathbf{T}_p^F$ denotes the Sylow p-subgroup of $\mathbf{T}^F$.

3. Morita equivalences between blocks

From now on, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$. We set $A := \mathcal{O}Ge$ and $B := \mathcal{O}Hf$.

3.A. Preliminaries : bimodules and adjunctions.

We first recall in this context well known properties of functors induced by bimodules. Let M be an (A,B)–bimodule. Let X (resp. Y) be an A–module (resp. a B–module).

1. We have $\mathrm{Hom}_A(M \underset{B}{\otimes} Y, X) \simeq \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M,X))$ through the maps

$$\left(\alpha\colon M \underset{B}{\otimes} Y \to X\right) \mapsto (\hat{\alpha}\colon Y \to \mathrm{Hom}_A(M,X)\,,\, y \mapsto (m \mapsto \alpha(m\otimes y)))$$
$$(\beta\colon Y \to \mathrm{Hom}_A(M,X)) \mapsto \left(\hat{\beta}\colon m\otimes y \mapsto \beta(y)(m)\right)$$

2. Let us set $M^\vee := \mathrm{Hom}_A(M,A)$ viewed as an object of ${}_B\mathbf{mod}_A$. We denote by $<,>\colon M \times M^\vee \to A$ the natural A–pairing between M and $M^\vee$. Suppose that $M \in {}_A\mathbf{mod}_B \cap {}_A\mathbf{proj}$. Then the map

$$M^\vee \underset{A}{\otimes} X \to \mathrm{Hom}_A(M,X)$$
$$(m^\vee \underset{A}{\otimes} x) \mapsto (m \mapsto <m,m^\vee>x)$$

is an isomorphism in ${}_B\mathbf{mod}$.

3. Let us set $M^* := \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$, viewed as an object of ${}_B\mathbf{mod}_A$. Since the linear form $t_A\colon A \to \mathcal{O}\,,\, \sum_{g\in G} a(g)g \mapsto a(1)$ is a symmetrizing form for A, the maps

$$m^\vee \mapsto t_A \cdot m^\vee \quad \text{and} \quad m^* \mapsto \left(m \mapsto \sum_{g\in G} m^*(g^{-1}m)g\right)$$

are inverse isomorphisms (in ${}_B\mathbf{mod}_A$) between $M^\vee$ and M^*.

Suppose given a (B,A)–bimodule N and a duality $M \times N \to \mathcal{O}$ which is (A,B)–compatible, *i.e.*, $<m, bna> = <amb, n>$ for $a \in A$, $b \in B$, $m \in M$, $n \in N$. From what preceds, we deduce two explicit isomorphisms (respectively in ${}_A\mathbf{mod}_A$ and in ${}_B\mathbf{mod}_B$) :

$$\mathrm{Hom}_B(N,N) \xrightarrow{\sim} \mathrm{Hom}_A(M \underset{B}{\otimes} N, A)$$
$$\mathrm{Hom}_A(M,M) \xrightarrow{\sim} \mathrm{Hom}_B(B, N \underset{A}{\otimes} M)\,.$$

We denote by $\varepsilon_{M,N}$ the image of Id_M through the first isomorphism, and by $\eta_{M,N}$ the image of Id_N through the second isomorphism. The maps $\varepsilon_{M,N}$ and $\eta_{M,N}$ are called the *adjunctions* associated with the pair (M,N).

3.1. Proposition. *With the previous hypothesis and notation, the maps $\varepsilon_{M,N}$ and $\eta_{M,N}$ are computed as follows :*

$$\varepsilon_{M,N} \colon M \underset{B}{\otimes} N \to A\,,\; m \underset{B}{\otimes} n \mapsto \sum_{g \in G} <g^{-1}m, n>g$$

$$\eta_{M,N} \colon B \to N \underset{A}{\otimes} M\,,\; b \mapsto \sum_{i \in I} n_i \underset{A}{\otimes} m_i$$

where $\sum_{i\in I} n_i \underset{A}{\otimes} m_i$ is the element of $N \underset{A}{\otimes} M$ such that, for all $m \in M$,

$$\sum_{g\in G}\sum_{i\in I} <n_i g^{-1}, m>gn_i = bm\,.$$

3.B. Morita theorem and block invariants.

The following statement is a variation on Morita's theorem, applied in the particular case of symmetric algebras.

3.2. Theorem. *The following assertions are equivalent :*

(i) *The categories ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$ are equivalent.*

(ii) *There exist*

- *an (A,B)-bimodule M which is projective both as an A-module and as a module-B,*
- *a (B,A)-bimodule N which is projective both as a B-module and as a module-A,*
- *an (A,B)-compatible $\mathcal{O}$-duality between M and N*

such that $M \underset{B}{\otimes} N \simeq A$ in ${}_A\mathbf{mod}_A$ and $N \underset{A}{\otimes} M \simeq B$ in ${}_B\mathbf{mod}_B$.

Moreover, if the preceding statements are satisfied, then all of the adjunctions $\varepsilon_{M,N}$, $\eta_{M,N}$, $\varepsilon_{N,M}$, $\eta_{N,M}$ are isomorphisms.

Morita equivalence and triangle invariants. If (M,N), as above, defines an equivalence between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$, the pairs $(K\otimes_{\mathcal{O}} M, K\otimes_{\mathcal{O}} N)$ and $(k\otimes_{\mathcal{O}} M, k\otimes_{\mathcal{O}} N)$ define equivalences respectively between ${}_{KA}\mathbf{mod}$ and ${}_{KB}\mathbf{mod}$ and between ${}_{kA}\mathbf{mod}$ and ${}_{kB}\mathbf{mod}$. So the Morita equivalence defined by (M,N) induces bijections

$$\mathrm{Irr}(KA) \simeq \mathrm{Irr}(KB)\,,\; \mathrm{Irr}(kA) \simeq \mathrm{Irr}(kB)\,,\; \mathrm{Pim}(kA) \simeq \mathrm{Pim}(kB)\,.$$

Moreover, by construction of the c-d-triangles (see §1 above), it is clear that the induced isomorphisms $\mathcal{R}(KA) \simeq \mathcal{R}(KB)$, $\mathcal{R}(kA) \simeq \mathcal{R}(kB)$, $\mathcal{R}^{\mathrm{pr}}(kA) \simeq \mathcal{R}^{\mathrm{pr}}(kB)$, commute with the Cartan and the decomposition maps. To summarize :

3.3. Proposition. *A Morita equivalence between A and B induces an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$, which preserves the canonical basis.*

$$\begin{tikzcd}
& & & \mathcal{R}(KB) \arrow[rr, "\mathrm{dec}^B"] & & \mathcal{R}(kB) \\
\mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] \arrow[urrr, leftrightarrow] & & \mathcal{R}(kA) \arrow[urrr, leftrightarrow] & & \mathcal{R}^{\mathrm{pr}}(kB) \arrow[ul] \arrow[ur, "\mathrm{Car}^B"'] & \\
& \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] \arrow[urrr, leftrightarrow] & & & &
\end{tikzcd}$$

As a consequence, a Morita equivalence between A and B preserves all the invariants determined by the c-d-triangles and their canonical basis :

$$\mathrm{k}(A) = \mathrm{k}(B)\,,\, \mathrm{l}(A) = \mathrm{l}(B)\,,\, C^A = C^B\,,\, D^A = D^B\,, d(A) = d(B)\,.$$

Morita equivalence and centers. As it is well known, a Morita equivalence between A and B induces an algebra isomorphism between the centers $Z(A)$ and $Z(B)$, since $Z(A)$ may be viewed as the center of the category ${}_A\mathbf{mod}$. It also results from what follows (see [Ri2]), which also proves the preservation of projective centers.

We denote by A^{op} the opposite algebra of A, and $A^{\mathrm{en}} := A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ the "enveloping algebra". Assume that (M, N) induces a Morita equivalence between A and B. Then $M \otimes_{\mathcal{O}} N$ is endowed with a natural structure of $(A^{\mathrm{en}}, B^{\mathrm{en}})$–bimodule defined by

$$(a \otimes a')(m \otimes n)(b \otimes b') := amb \otimes b'na'$$

and similarly $N \otimes_{\mathcal{O}} M$ is endowed with a natural structure of $(B^{\mathrm{en}}, A^{\mathrm{en}})$–bimodule. The map

$$M \otimes_{\mathcal{O}} N \times N \otimes_{\mathcal{O}} M \to \mathcal{O} \quad , \quad (m \otimes n, n' \otimes m') \mapsto <m,n'><m',n>$$

is an $(A^{\mathrm{en}}, B^{\mathrm{en}})$–compatible duality.

3.4. Proposition. *With the previous notation, $(M \otimes_{\mathcal{O}} N, N \otimes_{\mathcal{O}} M)$ defines a Morita equivalence between A^{en} and B^{en} which exchanges A and B.*

3.5. Corollary. *A Morita equivalence between A and B induces an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.*

Morita equivalence and defects of irreducible KA-modules. By 3.3, a Morita equivalence induces a bijection $I\colon \mathrm{Irr}(KB) \xrightarrow{\sim} \mathrm{Irr}(KA)$, and it results from 2.2, (2), that I preserves the defects : for all $Y \in \mathrm{Irr}(KB)$, $d(I(Y)) = d(Y)$. This is also a consequence of what follows, which will be generalized later on to the case of a Rickard equivalence.

For $X \in \mathrm{Irr}(KA)$ (resp. $Y \in \mathrm{Irr}(KB)$) we denote by e_X (resp. f_Y) the corresponding primitive idempotent of ZKA (resp. of ZKB). We denote by e (resp.

f) the unity element of A (resp. B), so $A = \mathcal{O}Ge$ and $e = \sum_{X \in \mathrm{Irr}(KA)} e_X$ (resp. $B = \mathcal{O}Hf$ and $f = \sum_{Y \in \mathrm{Irr}(KB)} f_Y$). The set $\{e_X\}_{X \in \mathrm{Irr}(KA)}$ is a K-basis of ZKA, and if $\sum_X \lambda_X e_X \in Z(A)$, then $\lambda_X \in \mathcal{O}$ for all $X \in \mathrm{Irr}(KA)$.

Let (M, N) as in 3.2 which induces an equivalence between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$. It induces a bijection $\mathrm{Irr}(KA) \xrightarrow{\sim} \mathrm{Irr}(KB)$ which we denote by $X \mapsto Y_X$. The adjunctions $\eta_{N,M}$ and $\varepsilon_{M,N}$ are isomorphisms in ${}_A\mathbf{mod}_A$, hence $\eta_{N,M} \cdot \varepsilon_{M,N}$ is an automorphism of A in ${}_A\mathbf{mod}_A$ and it restricts to an automorphism of $Z(A)$ viewed as an $\mathcal{O}$-module.

3.6. Proposition. *With the previous notation,*

$$\eta_{N,M} \cdot \varepsilon_{M,N} \colon e \mapsto \sum_{X \in \mathrm{Irr}(KA)} \frac{|G|/\dim X}{|H|/\dim Y_X} e_X \,.$$

In particular, $\dfrac{|G|/\dim X}{|H|/\dim Y_X}$ *is invertible modulo p and the defects are preserved :* $d(X) = d(Y_X)$.

3.7. Corollary. *If e is primitive (i.e., if A is a block), then* $\dfrac{|G|/\dim X}{|H|/\dim Y_X}$ *is constant modulo p for $X \in \mathrm{Irr}(KA)$.*

3.C. Examples of Morita equivalences.

Clifford Theory. "Clifford theory" is the name of a set of theorems relating representations of a group G with representations of a normal subgroup N of G. It can be viewed as a series of Morita equivalences. We present here the first (and easy) part of Clifford theory : the "reduction to the inertial group".

Let N be a normal subgroup of G. Let f be a central primitive idempotent of $\mathcal{O}N$. We denote by H the stabilizer of f in G (which acts by conjugation on the set of central idempotents of $\mathcal{O}N$). Then f is a central idempotent of $\mathcal{O}H$. We set $B := \mathcal{O}Hf$.

We set $e := \sum_{g \in [G/H]} gfg^{-1}$, where $[G/H]$ denotes a set of representatives of the cosets of G modulo H. Then e is a central idempotent of $\mathcal{O}G$. We set $A := \mathcal{O}Ge$.

Let $M := e\mathcal{O}Gf = \mathcal{O}Gf$, endowed with left multiplication by A and right multiplication by B. Let $N := f\mathcal{O}Ge = f\mathcal{O}G$, viewed similarly as a (B, A)-bimodule. Since $M \underset{B}{\otimes} N \simeq A$ in ${}_A\mathbf{mod}_A$ and $N \underset{A}{\otimes} M \simeq B$ in ${}_B\mathbf{mod}_B$, and since the functors $M \underset{B}{\otimes} \bullet$ and $N \underset{A}{\otimes} \bullet$ are respectively $f \cdot \mathrm{Res}_H^G$ and Ind_H^G, we get :

3.8. Proposition. *The functors $f \cdot \mathrm{Res}_H^G$ and Ind_H^G are inverse equivalences between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$.*

p-nilpotent groups. Let G be a p-nilpotent group, *i.e.*, $G \simeq S \rtimes P$ where P is a p-group and S is a normal p'-subgroup (group with order prime to p) of G.

For every irreducible KS-module X, we denote by f_X the corresponding central primitive idempotent of KS (since S is a p'-group, $f_X \in Z\mathcal{O}S$) and by P_X the stabilizer of f_X in P.

3.9. Proposition. *We have*

$$ {}_{\mathcal{O}G}\mathbf{mod} \simeq \bigoplus_{X\in \mathrm{Irr}(KS) \bmod P} {}_{\mathcal{O}P_X}\mathbf{mod}\,. $$

Sketch of proof. Set $e_X := \sum_{[P/P_X]} gf_Xg^{-1}$, and $G_X := S \rtimes P_X$. By 3.8 above, we see that ${}_{\mathcal{O}Ge_X}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}G_Xf_X}\mathbf{mod}$. Since $\mathcal{O}G = \bigoplus_{X\in\mathrm{Irr}(KS) \bmod P} \mathcal{O}Ge_X$, it suffices to prove that ${}_{\mathcal{O}G_Xf_X}\mathbf{mod} \simeq {}_{\mathcal{O}P_X}\mathbf{mod}$.

We may assume $P_X = P$. Let X_0 be an $\mathcal{O}$–free $\mathcal{O}S$–module such that $X = K\otimes_{\mathcal{O}} X_0$ (note that X_0 is unique up to isomorphism). Because $\dim X$ and $|P|$ are relatively prime, there is an action of $S \rtimes P$ on X_0 which extends the action of S. This allows us to define on $M := \mathcal{O}G \underset{\mathcal{O}S}{\otimes} X_0$ a structure of $(\mathcal{O}Ge_X, \mathcal{O}P)$–bimodule as follows :

$$ g.(g_1 \underset{\mathcal{O}S}{\otimes} x).\pi := gg_1 \underset{\mathcal{O}S}{\otimes} \pi^{-1}(x) \quad \text{for } g, g_1 \in G\,,\ x\in X_0\,,\ \pi \in P\,. $$

Similarly, the module $N := X_0^* \underset{\mathcal{O}S}{\otimes} \mathcal{O}G$ is endowed with a structure of $(\mathcal{O}P, \mathcal{O}Ge_X)$–bimodule, and it is not difficult to check that (M,N) induces a Morita equivalence between ${}_{\mathcal{O}Ge_X}\mathbf{mod}$ and ${}_{\mathcal{O}P}\mathbf{mod}$. $\square$

Remark. There are lots of Morita equivalences in the theory of blocks of p–solvable groups, analogous (although sometimes far deeper) to the ones just described for the case of a p–nilpotent group. The classical "Fong reduction theorem" (see [Fo]) may be viewed as the description of a Morita equivalence between two blocks of two p–solvable groups (see for example [Pul]). The description of blocks of groups of p–length one relies on some highly non trivial Morita equivalences (see [Da]).

On the other hand, Morita equivalences between blocks seem far less frequent for non abelian simple groups. In this case, the equivalence must be weakened to what we call a Rickard equivalence.

4. Rickard equivalences between blocks

4.A. Complexes : Notation and conventions.

As in the previous section, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$.

Homomorphisms and tensor product of two complexes. The definitions we use here for the differentials of the homomorphisms and the tensor product of two complexes are slightly different from the usual ones (although they provide complexes isomorphic to the usual ones).

1. Let $X := \left(\cdots \to X^m \overset{d_X^m}{\to} X^{m+1} \to \cdots\right)$ and $Y := \left(\cdots \to Y^m \overset{d_Y^m}{\to} Y^{m+1} \to \cdots\right)$ be complexes in mod A. We set

$$ \mathrm{Hom}_A^{\cdot}(X,Y) := \left(\cdots \to \mathrm{Hom}_A^m(X,Y) \overset{d^m}{\to} \mathrm{Hom}_A^{m+1}(X,Y) \to \cdots\right) $$

where

$$\mathrm{Hom}^m_A(X,Y) := \prod_{i,j,j-i=m} \mathrm{Hom}_A(X^i,Y^j) \quad \text{and}$$

$$\begin{aligned} d^m \colon \mathrm{Hom}_A(X^i,Y^j) &\to \mathrm{Hom}_A(X^i,Y^{j+1}) \times \mathrm{Hom}_A(X^{i-1},Y^j) \\ \alpha &\mapsto \Big((-1)^m d^j_Y \cdot \alpha, -\alpha \cdot d^{i-1}_X\Big) \end{aligned}$$

We set $X^* := \mathrm{Hom}^{\cdot}_{\mathcal{O}}(X,\mathcal{O})$, viewed as a complex in $\mathbf{mod}_A$, and $d^m_{X^*} = -{}^t d_X^{-(m+1)}$.

2. Assume now that Y is a complex in $\mathbf{mod}_A$. We set

$$Y \underset{A}{\otimes} X := \Big(\cdots \to (Y \underset{A}{\otimes} X)^m \xrightarrow{d^m} (Y \underset{A}{\otimes} X)^{m+1} \to \cdots\Big)$$

where

$$(Y \underset{A}{\otimes} X)^m := \bigoplus_{i,j,i+j=m} (Y^j \underset{A}{\otimes} X^i) \quad \text{and}$$

$$\begin{aligned} d^m \colon (Y^j \underset{A}{\otimes} X^i) &\to (Y^{j+1} \underset{A}{\otimes} X^i) \oplus (Y^j \underset{A}{\otimes} X^{i+1}) \\ y \otimes x &\mapsto \Big((-1)^m d^j_Y(y) \otimes x\Big) \oplus \big(y \otimes d^i_X(x)\big) \end{aligned}$$

Some classical maps. We denote now by $M := \Big(\cdots \to M^i \xrightarrow{d^i_M} M^{i+1} \to \cdots\Big)$ a *bounded complex of (A,B)–bimodules.*

Let $N := \Big(\cdots \to N^j \xrightarrow{d^j_N} N^{j+1} \to \cdots\Big)$ be a complex in ${}_B\mathbf{mod}_A$. An (A,B)–compatible $\mathcal{O}$–duality between M and N is the following datum :

(1) for all i, an (A,B)–compatible duality $<,>\colon N^i \times M^{-i} \to \mathcal{O}$,

(2) such that the maps d^i_N and $-d_M^{-(i+1)}$ are adjoint for this duality.

- We denote by $\varepsilon_{M^i,N^{-i}} \colon M^i \underset{B}{\otimes} N^{-i} \to A$ the map defined by

$$\varepsilon_{M^i,N^{-i}}(m \otimes n) := \sum_{g\in G} <n, g^{-1}m> g\,,$$

and we denote by $\varepsilon_{M,N} \colon M \underset{B}{\otimes} N \to A$ the chain map defined by

$$\varepsilon_{M,N} := \sum_i (-1)^i \varepsilon_{M^i,N^{-i}}\,.$$

- We denote by $\tau_{N^{-j},M^i} \colon N^{-j} \underset{A}{\otimes} M^i \to \mathrm{Hom}_A(M^j,M^i)$ the map defined by

$$\tau_{N^{-j},M^i}(n \otimes m) \colon m' \mapsto \varepsilon_{M^j,N^{-j}}(m' \otimes n)m\,,$$

and we denote by $\tau_{N,M} : N \underset{A}{\otimes} M \to \mathrm{Hom}^{\bullet}_A(M, M)$ the chain map defined by the family $(\tau_{N^{-j},M^i})_{i,j}$.

• Finally, we denote by $\sigma^B_{M^i} : B \to \mathrm{Hom}_A(M^i, M^i)$ the morphism which defines the structure of module–B on M, and we denote by $\sigma^B_M : B \to \mathrm{Hom}^{\bullet}_A(M, M)$ the chain map defined by

$$b \mapsto \prod_i \sigma^B_{M^i}(b) .$$

The adjunctions. Let ${}_A\mathbf{com}$ be the category of complexes in ${}_A\mathbf{mod}$ with chain maps as morphisms, and let ${}_A\mathbf{com}_B$ be the category of complexes in ${}_A\mathbf{mod}_B$ with chain maps as morphisms.

Assume that, as above, M is a bounded complex of (A, B)–bimodules, N is a bounded complex of (B, A)–bimodules with a given (A, B)–compatible duality with M. Assume moreover that each component M^i of M is projective both as an A–module and as a module–B. Then the functors

$$M \underset{B}{\otimes} \bullet : {}_B\mathbf{com} \to {}_A\mathbf{com} \quad \text{and} \quad N \underset{A}{\otimes} \bullet : {}_A\mathbf{com} \to {}_B\mathbf{com}$$

are adjoint one to the the other on both sides.

The chain map $\tau_{N,M} : N \underset{A}{\otimes} M \to \mathrm{Hom}^{\bullet}_A(M, M)$ is an isomorphism in ${}_B\mathbf{com}_B$. We denote by $\eta_{M,N} : B \to N \underset{A}{\otimes} M$ the chain map defined by $\eta_{M,N} := \tau^{-1}_{N,M} \cdot \sigma^B_M$.

Definition. *We call adjunctions the two pairs of chain maps of complexes of bimodules*

$$\begin{aligned} \varepsilon_{M,N} : M \underset{B}{\otimes} N \to A \quad &\textit{and} \quad \eta_{M,N} : B \to N \underset{A}{\otimes} M \\ \varepsilon_{N,M} : N \underset{A}{\otimes} M \to B \quad &\textit{and} \quad \eta_{N,M} : A \to M \underset{B}{\otimes} N \end{aligned}$$

which define respectively adjunctions for the adjoint pairs $(M \underset{B}{\otimes} \bullet, N \underset{A}{\otimes} \bullet)$ *and* $(N \underset{A}{\otimes} \bullet, M \underset{B}{\otimes} \bullet)$.

4.B. Rickard equivalences and block invariants.

Definition. *We say that A and B are Rickard equivalent if there exists*

- *a bounded complex M in ${}_A\mathbf{com}_B$, each component of which is both projective as an A-module and as a module-B,*
- *a bounded complex N in ${}_B\mathbf{com}_A$, each component of which is both projective as an B-module and as a module-A,*
- *an (A, B)-compatible $\mathcal{O}$-duality between M and N,*

$$\textit{such that} \begin{cases} M \underset{B}{\otimes} N \textit{ is homotopy equivalent to } A \textit{ in } {}_A\mathbf{com}_A \\ N \underset{A}{\otimes} M \textit{ is homotopy equivalent to } B \textit{ in } {}_B\mathbf{com}_B \, . \end{cases}$$

In this case, the complexes M and N are called "Rickard tilting complexes" for A and B.

We denote by $\mathcal{D}^b(A)$ the derived bounded category of ${}_A\mathbf{mod}$.

The following theorem is due to J. Rickard ([Ri1], [Ri3]). Note that it may be viewed as a generalization of Morita theorem 3.2.

4.1. Theorem. *The following assertions are equivalent :*

(i) *The derived bounded categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*

(ii) *The algebras A and B are Rickard equivalent.*

Moreover, in this case, the adjunctions $\varepsilon_{M,N}$, $\eta_{M,N}$, $\varepsilon_{N,M}$, $\eta_{N,M}$ are all homotopy equivalences between the corresponding complexes of bimodules.

Like in the case of a Morita equivalence (see §3 above), any pair of complexes (M, N) which induces a Rickard equivalence between A and B defines pairs $(K \otimes_{\mathcal{O}} M, K \otimes_{\mathcal{O}} N)$ and $(k \otimes_{\mathcal{O}} M, k \otimes_{\mathcal{O}} N)$ which induce Rickard equivalences between respectively KA and KB, kA and kB.

An object of $\mathcal{D}^b(A)$ is called *perfect* (see [Gro]) if it is isomorphic to a bounded complex of projective A–modules. Let $\mathcal{D}^b_{\mathrm{perf}}(A)$ be the full subcategory of $\mathcal{D}^b(A)$ consisting of perfect complexes. A Rickard equivalence between A and B induces an equivalence of categories between $\mathcal{D}^b_{\mathrm{perf}}(A)$ and $\mathcal{D}^b_{\mathrm{perf}}(B)$.

Rickard equivalences and triangle invariants. The Grothendieck groups of the triangulated categories (see [Gro], and also [Ha]) $\mathcal{D}^b(KA)$, $\mathcal{D}^b(kA)$, $\mathcal{D}^b_{\mathrm{perf}}(kA)$ are respectively the groups $\mathcal{R}(KA)$, $\mathcal{R}(kA)$, $\mathcal{R}^{\mathrm{pr}}(kA)$. Hence a Rickard equivalence induces isomorphisms between these groups. By construction, these isomorphisms commute with the maps of the c-d-triangle (see §1).

The metric structure on the c-d-triangle may be defined in terms of the derived category. For example, the duality between $\mathcal{R}(kA)$ and $\mathcal{R}^{\mathrm{pr}}(kA)$ is defined as follows. For $P := (\cdots \to P^i \to P^{i+1} \to \cdots)$ a bounded complex of projective A-modules, object of $\mathcal{D}^b_{\mathrm{perf}}(kA)$, and $X := (\cdots \to X^i \to X^{i+1} \to \cdots)$ an object of $\mathcal{D}^b(kA)$, we have

$$<[P],[X]> := \sum_i (-1)^i \dim \mathrm{Hom}_{\mathcal{D}^b(kA)}(P[i], X)\,.$$

Hence the isomorphisms between triangles defined by a Rickard equivalence preserve the natural $\mathbb{Z}$–dualities.

Since $\mathrm{Irr}(KA)$ is an orthonormal basis of the $\mathbb{Z}$–module $\mathcal{R}(KA)$, it follows that the isomorphism $\mathcal{R}(KA) \xrightarrow{\sim} \mathcal{R}(KB)$ sends an element of $\mathrm{Irr}(KA)$ onto an element of $\{\pm Y \mid (Y \in \mathrm{Irr}(KB))\}$ (while a Morita equivalence induces a bijection between $\mathrm{Irr}(KA)$ and $\mathrm{Irr}(KB)$). There is no analogous property for $\mathrm{Irr}(kA)$ or $\mathrm{Pim}(kA)$.

4.2. Proposition. *A Rickard equivalence between A and B induces an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$ (viewed as endowed with their natural "metric").*

$$\begin{tikzcd}
 & & & \mathcal{R}(KB) \arrow[rr, "\mathrm{dec}^B"] & & \mathcal{R}(kB) \\
\mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] \arrow[urrr, leftrightarrow] & & \mathcal{R}(kA) \arrow[urrr, leftrightarrow] & & \mathcal{R}^{\mathrm{pr}}(kB) \arrow[ul] \arrow[ur, "\mathrm{Car}^B"'] & \\
 & \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] \arrow[urrr, leftrightarrow] & & & &
\end{tikzcd}$$

As a consequence, a Rickard equivalence between A and B preserves all the invariants determined by the c-d-triangles and their metrics :

- $\mathrm{k}(A) = \mathrm{k}(B)$, $\mathrm{l}(A) = \mathrm{l}(B)$, $d(A) = d(B)$,
- the Cartan matrices C^A and C^B are equivalent as matrices of quadratic forms over $\mathbb{Z}$ (in particular they have the same invariant factors, and $\mathrm{l}^{\mathrm{pr}}(A) = \mathrm{l}^{\mathrm{pr}}(B)$),
- the decomposition matrices are equivalent as follows : there exists an orthonormal matrix U in $\mathrm{Mat}_{\mathrm{k}(A)}(\mathbb{Z})$ and an invertible matrix V in $\mathrm{Mat}_{\mathrm{l}(A)}(\mathbb{Z})$ such that $D^B = UD^AV$ (and in particular D^A and D^B have the same invariant factors).

Rickard equivalence and centers. A Rickard equivalence between A and B induces an algebra isomorphism between the centers $Z(A)$ and $Z(B)$, since $Z(A)$ may be viewed as the center of the category $\mathcal{D}^b(A)$.

Like in the case of Morita equivalences (see §3), it also results from the following proposition.

4.3. Proposition. *If (M, N) induces a Rickard equivalence between A and B, then $(M \otimes_{\mathcal{O}} N, N \otimes_{\mathcal{O}} M)$ defines a Rickard equivalence between A^{en} and B^{en} which exchanges A and B.*

Since a morphism between two modules factorizes through a projective module if and only if it factorizes through a perfect complex, we get as a consequence :

4.4. Corollary. *A Rickard equivalence between A and B induces an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.*

Rickard equivalence and defects of irreducible KA-modules. Like in §3, for $X \in \mathrm{Irr}(KA)$, we denote by e_X the corresponding primitive idempotent of ZKA. We denote by e (resp. f) the unity element of A (resp. B).

Let (M, N) be a pair of complexes which induces a Rickard equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$. It sends an element $X \in \mathrm{Irr}(KA)$ onto $\varepsilon_X Y_X$ where $\varepsilon_X = \pm 1$ and $Y_X \in \mathrm{Irr}(KB)$. The adjunctions $\eta_{N,M}$ and $\varepsilon_{M,N}$ are homotopy equivalences between objects of ${}_A\mathbf{com}_A$, hence $\eta_{N,M} \cdot \varepsilon_{M,N}$ is an automorphism of A in ${}_A\mathbf{mod}_A$ and it restricts to an automorphism of $Z(A)$ viewed as an $\mathcal{O}$-module.

4.5. Proposition. *With the previous notation,*

$$\eta_{N,M} \cdot \varepsilon_{M,N} : e \mapsto \sum_{X \in \mathrm{Irr}(KA)} \frac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X} e_X .$$

In particular, $\dfrac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}$ is invertible modulo p and the defects are preserved : $d(X) = d(Y_X)$.

4.6. Corollary. *If e is primitive in $Z(A)$ (i.e., if A is a block), $\dfrac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}$ is constant modulo p for $X \in \mathrm{Irr}(KA)$.*

4.C. Examples of Rickard equivalences. Although derived equivalences are conjecturally very frequent in block theory, only a very small number of them is actually proved.

Groups with cyclic Sylow p–subgroups. The following result is a particular case of results proved by Rickard and Linckelmann (see [Ri2] and [Li]) as a consequence of the structure theorem of blocks with cyclic defect groups.

4.7. Theorem. *Assume that G has a cyclic Sylow p–subgroup P. Let us denote by A_0 and B_0 respectively the principal blocks of G and of the normalizer of P in G. Then A_0 and B_0 are Rickard equivalent.*

Remark. At this date (december 1992), no explicit construction of a Rickard tilting complex is known.

The principal 2–block of $\mathfrak{A}_5$ and $\mathfrak{A}_4$. Let us denote by G the alternating group $\mathfrak{A}_5$ on 5 letters, and by H the normalizer of a Sylow 2-subgroup of G, isomorphic to the alternating group $\mathfrak{A}_4$. Let $p = 2$. Let A_0 (resp. B_0) be the principal block of G (resp. H).

It is easy to check that A_0 and B_0 are not Morita equivalent (since, for example, they have different decomposition matrices). Nevertheless, the functors Ind_H^G and Res_H^G induce a stable equivalence between A_0 and B_0. The following unpublished result of Rickard shows that one can "twist" this stable equivalence to get a Rickard equivalence.

Let $\mathcal{A}\mathcal{O}G$ be the augmentation ideal of $\mathcal{O}G$ and let $P(\mathcal{A}\mathcal{O}G) \xrightarrow{d} \mathcal{A}\mathcal{O}G$ be the projective cover of $\mathcal{A}\mathcal{O}G$ viewed as a $(\mathcal{O}G, \mathcal{O}H)$-bimodule. We set

$$M := \left(\cdots \to 0 \to P(\mathcal{A}\mathcal{O}G) \xrightarrow{d} \mathcal{O}G \to 0 \to \cdots\right)$$

and $N := M^*$.

4.8. Theorem. *The pair (M, N) defines a Rickard equivalence between A_0 and B_0.*

A conjecture. The preceding two examples are particular cases of a conjectural general result (see [Br1]).

4.9. Conjecture. *Let G be a finite group whose Sylow p–subgroups are abelian. Let H be the normalizer of one of the Sylow p–subgroups of G. Then the principal blocks of G and H are Rickard equivalent.*

4.D. Perfect isometries.

The preceding conjecture seems hard to prove (or even to check on examples) at the moment. Nevertheless, one of its non–trivial consequences, which should be viewed as the "shadow", at the level of characters, of a Rickard equivalence, has already been checked on a long series of cases.

For the definitions and properties stated in this paragraph, see [Br1].

As in §3, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$. From now on, we identify $\mathcal{R}(KG)$ with the group of virtual characters of KG–representations, and $\mathrm{Irr}(KG)$ with the set of irreducible characters.

Let μ be a virtual character of $G\times H$, element of $\mathcal{R}(K[G\times H])$. Then μ corresponds to a linear map $I_\mu\colon \mathcal{R}(KH)\to\mathcal{R}(KG)$ as follows : for $\zeta\in\mathrm{Irr}(KH)$, the function $I_\mu(\zeta)$ is defined by $I_\mu(\zeta)(g):=\dfrac{1}{|H|}\sum_{h\in H}\mu(g,h^{-1})\zeta(h)$.

Definition. *We say that a virtual character μ of $G\times H$ is perfect if :*

(pe.1) *for all $g\in G$ and $h\in H$, $|C_G(g)|_p$ and $|C_H(h)|_p$ both divide $\mu(g,h)$,*

(pe.2) *if $\mu(g,h)\neq 0$, then either g and h are both p-regular, or g and h are both p-singular.*

If moreover the map I_μ defined by μ induces an isometric bijection from $\mathcal{R}(KB)$ to $\mathcal{R}(KA)$, we say that I_μ is a perfect isometry between B and A, and that A and B are perfectly isometric.

The connection with Rickard equivalences is made by the following statement.

4.10. Proposition. *Assume that M is a Rickard tilting complex for A and B. Let μ_M be the virtual character of $G\times H$ defined by*

$$\mu_M(g,h):=\sum_i(-1)^i\mathrm{tr}((g,h^{-1});M^i)\,.$$

Then μ_M defines a perfect isometry between B and A.

The point is that, if A and B are perfectly isometric, their invariants behave "as if" they were Rickard equivalent (see [Brl]) — compare with assertions 4.2, 4.4, 4.5 above.

4.11. Theorem. *Suppose that A and B are perfectly isometric.*

(1) There is an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$ (viewed as endowed with their natural "metric"). In particular,

- $\mathrm{k}(A)=\mathrm{k}(B)$, $\mathrm{l}(A)=\mathrm{l}(B)$, $d(A)=d(B)$,
- *the Cartan matrices C^A and C^B are equivalent as matrices of quadratic forms over $\mathbb{Z}$,*
- *the decomposition matrices are equivalent as follows : there exists an orthonormal matrix U in $\mathrm{Mat}_{\mathrm{k}(A)}(\mathbb{Z})$ and an invertible matrix V in $\mathrm{Mat}_{\mathrm{l}(A)}(\mathbb{Z})$ such that $D^B=UD^AV$.*

(2) There is an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.

(3) There is an automorphism of (A,A)-bimodules of $Z(A)$ such that, if $I_{\mu_M}^{-1}(X)=\varepsilon_XY_X$, then

$$e\mapsto\sum_{X\in\mathrm{Irr}(KA)}\frac{|G|/\dim X}{|H|/\varepsilon_X\dim Y_X}\,e_X\,.$$

In particular, $\dfrac{|G|/\dim X}{|H|/\varepsilon_X\dim Y_X}$ is invertible modulo p and the defects are preserved : $d(X)=d(Y_X)$.

The following conjecture is a weaker form of 4.9.

4.12. Conjecture. *Let G be a finite group whose Sylow p–subgroups are abelian. Let H be the normalizer of one of the Sylow p–subgroups of G. Then the principal blocks of G and H are perfectly isometric.*

The preceding conjecture is known to be true in the following cases :

- for all p, if G is p–solvable ;
- for $p = 2$, in all cases ([FoHa]) ;
- for all p, if G is a sporadic simple group ([Rou]) ;
- for all p, if G is a symmetric group ([Rou]) or an alternating group (Fong, *private communication*) ;
- if G is the group of rational points of a connected reductive algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$ and p is a prime number which does not divide q and which is good for $\mathbf{G}$ ([BMM], [BrMi]).

4.E. The case of the finite reductive groups.

In the case where G is a "finite reductive group", the conjecture 4.9 can be made more precise and closely linked with the underlying algebraic geometry (for more details, see [BrMa]).

Notation. In this paragraph, we temporarily change our notation to fit with the usual notation of finite reductive groups : our prime p (the characteristic of our field $k := \mathcal{O}/\mathfrak{p}$) is now denoted by ℓ, and q denotes a power of another prime $p \neq \ell$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, endowed with a Frobenius endomorphism F which defines a rational structure on $\mathbb{F}_q$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, with unipotent radical $\mathbf{U}$, and with F–stable Levi subgroup $\mathbf{L}$. We denote by $\mathrm{Y}(\mathbf{U})$ the associated Deligne–Lusztig variety defined (*cf.* for example [Lu]) by

$$\mathrm{Y}(\mathbf{U}) := \{g(\mathbf{U} \cap F(\mathbf{U})) \in \mathbf{G}/\mathbf{U} \cap F(\mathbf{U})\,;\, g^{-1}F(g) \in F(\mathbf{U})\}\,,$$

and we recall that $\mathbf{G}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by left multiplication while $\mathbf{L}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by right multiplication. It is known (*cf.* [Lu]) that $\mathrm{Y}(\mathbf{U})$ is an $\mathbf{L}^F$–torsor on a variety $\mathrm{X}(\mathbf{U})$, which is smooth of pure dimension equal to $\dim(\mathbf{U}/\mathbf{U} \cap F(\mathbf{U}))$, and which is affine (at least if q is large enough). In particular $\mathrm{X}(\mathbf{U})$ is endowed with a left action of $\mathbf{G}^F$. If $\mathcal{O}$ is a commutative ring, the image of the constant sheaf $\mathcal{O}$ on $\mathrm{Y}(\mathbf{U})$ through the finite morphism $\pi\colon \mathrm{Y}(\mathbf{U}) \to \mathrm{X}(\mathbf{U})$ is a locally constant constant sheaf $\pi_*(\mathcal{O})$ on $\mathrm{X}(\mathbf{U})$. We denote this sheaf by $\mathcal{F}_{\mathcal{O}\mathbf{L}^F}$.

Let ℓ be a prime number which does not divide q and let $\mathcal{O}$ be the ring of integers of a finite extension of the field $\mathbb{Q}_\ell$ of ℓ–adic numbers. For any $\mathbf{G}^F$–equivariant torsion free $\mathcal{O}$–sheaf $\mathcal{F}$ on $\mathrm{X}(\mathbf{U})$, we denote by $\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F})$ the algebra of endomorphisms of the "ℓ–adic cohomology" complex $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}), \mathcal{F})$ viewed as an element of the derived bounded category $\mathcal{D}^b(\mathcal{O}\mathbf{G}^F)$ of the category of finitely generated $\mathcal{O}\mathbf{G}^F$–modules.

We set $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U})) := \mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})$ and $\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U})) := \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})$.
Note that the algebra $\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))$ contains the group algebra $\mathcal{O}\mathbf{L}^F$ as a subalgebra.
For K an extension of $\mathcal{O}$, we set $\mathcal{H}_K(\mathrm{X}(\mathbf{U}), \mathcal{F}) := K \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F})$.

The data.

- Let ℓ be a prime number, $\ell \neq p$, which does not divide $|Z(\mathbf{G})/Z^o(\mathbf{G})|$ nor $|Z(\mathbf{G}^*)/Z^o(\mathbf{G}^*)|$, and which is good for $\mathbf{G}$. We assume that the Sylow ℓ–subgroups of $\mathbf{G}^F$ are *abelian*.

- Let $\mathcal{O}$ be the ring of integers of a finite unramified extension k of the field of ℓ–adic numbers $\mathbb{Q}_\ell$, with residue field k, such that the finite group algebra $k\mathbf{G}^F$ is split.
- Let $A = \mathcal{O}\mathbf{G}^F e$ be the principal block of $\mathcal{O}\mathbf{G}^F$. Let S be a Sylow ℓ–subgroup of $\mathbf{G}^F$, let $\mathbf{L} := C_{\mathbf{G}}(S)$, and let f be the principal block idempotent of $\mathcal{O}\mathbf{L}^F$.

The group $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$. We have $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{L})$, and we set $W_{\mathbf{G}^F}(\mathbf{L}) := N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$. The group S is a Sylow ℓ–subgroup of $Z(\mathbf{L})^F$, and ℓ does not divide $|W_{\mathbf{G}^F}(\mathbf{L})|$.

Conjectures. There exist
- a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$,
- a finite complex $\Upsilon = (\cdots \to \Upsilon^{n-1} \to \Upsilon^n \to \Upsilon^{n+1} \to \cdots)$ of $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodules, which are finitely generated projective as $\mathcal{O}\mathbf{G}^F$–modules as well as $\mathcal{O}\mathbf{L}^F$–modules,

with the following properties.

(C1) Viewed as an object of the category $\mathcal{D}^b({}_{\mathcal{O}\mathbf{G}^F}\mathbf{mod}_{\mathcal{O}\mathbf{L}^F})$, the complex Υ is isomorphic to $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U}))$. In particular, for each n, the n-th homology group of Υ is isomorphic, as an $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodule, to $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}), \mathbb{Z}_\ell)$.

(C2) The idempotent e acts as the identity on the complex $\Upsilon.f$,

(C3) • the structure of complex of $(A, \mathcal{O}\mathbf{L}^F f)$–bimodules of $\Upsilon.f$ extends to a structure of complex of $(A, f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f)$–bimodules, all of which are projective as right $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$–modules,
• the complexes $(\Upsilon.f \otimes_{f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f} f.\Upsilon^*)$ and A are homotopy equivalent as complexes of (A, A)–bimodules,
• the complexes $(f.\Upsilon^* \otimes_{\mathcal{O}\mathbf{G}^F e} \Upsilon.f)$ and $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$ are homotopy equivalent as complexes of $(f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f, f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f)$–bimodules.

(C4) The algebra $f\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))f$ is isomorphic to the principal block $\mathcal{O}N_{\mathbf{G}^F}(S)f$.

5. Stable equivalences of Morita type

5.A. Definition and first remarks.

An example. Some blocks may look very similar without being Morita equivalent nor even Rickard equivalent. This is often the case in the following situation :

(p–t.i.) We assume that the Sylow p–subgroups of G are t.i., *i.e.,* for S a Sylow subgroup, for all $g \in G$, one has $S \cap gSg^{-1} = \{1\}$ or S. We set $H := N_G(S)$ and we denote respectively by A and B the principal blocks of G and H.

It is easy to see that the functors Ind_H^G and Res_H^G induce inverse stable equivalences between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$. Such a stable equivalence has certain properties we shall formalize below : it is a "stable equivalence of Morita type".

The known examples show that, under hypothesis (p–t.i.), if S is non abelian and G is a non abelian simple group, the algebras A and B are not necessarily Morita nor Rickard equivalent, although they have the same numbers of irreducible characters : $\mathrm{k}(A) = \mathrm{k}(B)$ and $\mathrm{l}(A) = \mathrm{l}(B)$, according to Alperin's conjecture ([Al]).

For example, if $G = \mathrm{Sz}(8)$ and $p = 2$, the algebras A and B have non isomorphic centers (G. Cliff, *private communication*) and their Cartan matrices are not quadratically equivalent ([Br]).

Stable equivalences, Morita and Rickard equivalences. As before, we denote by G and H two finite groups, by e and f two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$. We denote by ${}_A\mathbf{stab}$ and ${}_B\mathbf{stab}$ the stable categories of A and B respectively. The stable categories have a natural structure of triangulated categories (see for example [Ha]).

Definition. *We say that A and B are "stably equivalent à la Morita" (or that there is a stable equivalence of Morita type between A and B) if there exist*

- *an (A, B)-bimodule M which is projective as an A-module and as a module-B,*
- *a (B, A)-bimodule N which is projective as a B-module and as a module-A,*
- *an (A, B)-compatible $\mathcal{O}$-duality between M and N,*

$$\textit{such that} \begin{cases} M \underset{B}{\otimes} N \textit{ is stably equivalent to } A \textit{ in } {}_A\mathbf{mod}_A \\ N \underset{A}{\otimes} M \textit{ is stably equivalent to } B \textit{ in } {}_B\mathbf{mod}_B\,. \end{cases}$$

The following statement is trivial.

5.1. Proposition. *A stable equivalence of Morita type between A and B induces an equivalence of triangulated categories between the stable categories ${}_A\mathbf{stab}$ and ${}_B\mathbf{stab}$*

It is obvious that if A and B are Morita equivalent, then they are stably equivalent à la Morita. Since the stable category ${}_A\mathbf{stab}$ is equivalent to the quotient category $\mathcal{D}^b(A)/\mathcal{D}^b_{\mathrm{perf}}(A)$ (see [Ri2]), we see that if A and B are Rickard equivalent, then they are stably equivalent. In fact, Rickard proved a more precise result ([Ri3], 5.5)[1]:

5.2. Proposition. *Assume that A and B are Rickard equivalent. Then A and B are stably equivalent à la Morita.*

5.B. Stable equivalences of Morita type and block invariants.

Stable triangle invariants. Let us set

$$\mathcal{R}^{\mathrm{st}}(KA) := \mathcal{R}(KA)/\mathrm{im}({}^t\mathrm{dec}^A) = \text{“}\mathcal{R}(KA)/\mathcal{R}^{\mathrm{pr}}(kA)\text{”}$$
$$\mathcal{R}^{\mathrm{st}}(kA) := \mathcal{R}(kA)/\mathrm{im}(\mathrm{Car}^A) = \text{“}\mathcal{R}(kA)/\mathcal{R}^{\mathrm{pr}}(kA)\text{”}\,.$$

The triangle $\mathcal{T}(A)$ defines by quotient a morphism of abelian groups :

$$(\mathcal{T}^{\mathrm{st}}(A)) \qquad \mathcal{R}^{\mathrm{st}}(KA) \xrightarrow{\mathrm{dec}^A_{\mathrm{st}}} \mathcal{R}^{\mathrm{st}}(kA)$$

Since $\mathcal{R}^{\mathrm{st}}(kA)$ is the Grothendieck group of the stable category of kA (viewed as triangulated category), a stable equivalence of Morita type induces an isomorphism between $\mathcal{R}^{\mathrm{st}}(kA)$ and $\mathcal{R}^{\mathrm{st}}(kB)$. More precisely :

5.3. Proposition. *A stable equivalence of Morita type between A and B induces*

(1) *an isomorphism between $\mathcal{T}^{\mathrm{st}}(A)$ and $\mathcal{T}^{\mathrm{st}}(B)$,*
(2) *an isometry between $\ker \mathrm{dec}^A$ and $\ker \mathrm{dec}^B$.*

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[1] Strictly speaking, this result concerns group algebras over a *field* ; but it can easily be extended to our context.

$$\begin{tikzcd}
& \mathcal{R}^{\mathrm{st}}(KB) \arrow[rr, "\mathrm{dec}_{\mathrm{st}}^{B}"] & & \mathcal{R}^{\mathrm{st}}(kB) \\
\mathcal{R}^{\mathrm{st}}(KA) \arrow[ur, leftrightarrow] \arrow[rr, "\mathrm{dec}_{\mathrm{st}}^{A}"] & & \mathcal{R}^{\mathrm{st}}(kA) \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

As a consequence, a stable equivalence of Morita type preserves the numerical invariants attached to the "stable triangles" $\mathcal{T}^{\mathrm{st}}$. For example :

$$\mathrm{k}(A) - \mathrm{l}(A) = \mathrm{k}(B) - \mathrm{l}(B)\,,\ \mathrm{l}(A) - \mathrm{l}^{\mathrm{pr}}(A) = \mathrm{l}(B) - \mathrm{l}^{\mathrm{pr}}(B)\,,\ d(A) = d(B)\,,$$

and the Cartan matrices C^A and C^B have the same non trivial invariant factors.

Stable equivalences of Morita type and centers. We set $Z^{\mathrm{st}}(A) := Z(A)/Z^{\mathrm{pr}}(A)$ and we call this algebra the *stable center* of A.

It is not true in general that the stable center of an algebra is the center of its stable category. Nevertheless :

5.4. Proposition. *A stable equivalence of Morita type between A and B induces an algebra isomorphism between $Z^{\mathrm{st}}(A)$ and $Z^{\mathrm{st}}(B)$.*

Proof. Let ${}_A\mathbf{stab}_A$ denote the stable category of A^{en}, and let ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ denote the full subcategory of ${}_A\mathbf{stab}_A$ whose objects are the (A,A)–bimodules which are projective as A–module and as module-A. Assume that (M,N) induces a stable equivalence of Morita type between A and B. Then the pair $(M \underset{\mathcal{O}}{\otimes} N\,,\, N \underset{\mathcal{O}}{\otimes} M)$ (where $M \underset{\mathcal{O}}{\otimes} N$ is viewed as an $(A^{\mathrm{en}}, B^{\mathrm{en}})$–bimodule and $N \underset{\mathcal{O}}{\otimes} M$ is viewed as a $(B^{\mathrm{en}}, A^{\mathrm{en}})$–bimodule, as in §3 above) induce inverse equivalences between ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ and ${}_B\mathbf{stab}_B^{\mathrm{pr}}$ which exchange A and B. The assertion follows from the fact that $Z^{\mathrm{st}}(A)$ is the algebra of endomorphisms of A in ${}_A\mathbf{stab}_A^{\mathrm{pr}}$. $\square$

Example. Let ΩA denote the kernel of the multiplication map

$$A \otimes_{\mathcal{O}} A \to A\ ,\ a \otimes a' \mapsto aa'\,.$$

Then the pair of (A,A)–bimodules $(\Omega A, (\Omega A)^*)$ induces a self stable equivalence of Morita type of A. Let X be an A–module and let $\pi : P \to X$ be a surjective morphism, where P is a projective A–module. Then there is a unique isomorphism $\ker \pi \xrightarrow{\sim} \Omega A \underset{A}{\otimes} X$ in ${}_A\mathbf{stab}$.

Remark. It is not known at the moment whether the existence of a stable equivalence of Morita type between A and B implies that $\mathrm{k}(A) = \mathrm{k}(B)$, and, if so, if there is a bijection between $\mathrm{Irr}(KA)$ and $\mathrm{Irr}(KB)$ which preserves the defects.

6. Inputting the group action

In all what has been stated so far, A and B might as well have been symmetric algebras over $\mathcal{O}$ — the groups themselves did not play an essential role. In what follows, we give brief indications on the actual methods of *group* representation theory.

6.A. Defect groups and source algebras.

Let $\Delta\colon G \to G\times G$ be the diagonal morphism. As $G\times G$–module, $\mathcal{O}G$ is isomorphic to $\mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{O}$.

6.1. Theorem–Definition. *(*[Gre], [Al2], [Pu2]*) Let A be a block of $\mathcal{O}G$.*

(1) The vertices of the $\mathcal{O}[G\times G]$–module A are the $G\times G$–conjugates of ΔD, where D is a p–subgroup of G. The G–conjugates of D are called the defect groups of A.

(2) Let S be an indecomposable summand of $\mathrm{Res}_{G\times D}^{G\times G} A$ with vertex ΔD. Such an S, viewed as an A–module, is unique up to isomorphism, and is a progenerator of ${}_A\mathbf{mod}$. We call source algebra of A the algebra $\mathrm{Sce}(A) := \mathrm{End}_{\mathcal{O}G}(S)$, viewed as endowed with the natural morphism $D \to \mathrm{Sce}(A)^\times$.

Thus in particular a source algebra of A is Morita equivalent to A. But the source algebra contains much more information than the Morita type of A. One can prove[2] that it contains all the "local information" of the block A, such as the category of subpairs ([AlBr]), the vertices and sources of indecomposable A–modules, and the generalized c-d-triangles (see below). The source algebra may be seen as the "*group* representation version" of the basic algebra.

6.B. Generalized c-d-triangles.

Definition. For x an element of finite order of a group, we let $\zeta_x \in \overline{\mathbb{Q}}$ be a root of unity of the same order as x. If A is any ring, we set $\mathcal{R}_x(A) := \mathbb{Z}[\zeta_x] \otimes_{\mathbb{Z}} \mathcal{R}(A)$.

Let G be a finite group. As in [Se], chap. 18, we identify now $\mathcal{R}(KG)$, $\mathcal{R}(kG)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$ with various subgroups of the group of $\mathcal{O}$–valued class functions on G.

Let x be a p–element of G. We denote by $C_G(x)$ its centralizer in G. The generalized decomposition map $\mathrm{dec}^{G,x}\colon \mathcal{R}_x(KG) \to \mathcal{R}_x(kC_G(x))$ is defined as follows :

For $\chi \in \mathcal{R}_x(KG)$, $\mathrm{dec}^{G,x}(\chi)$ is the class function on $C_G(x)$ defined by

$$\mathrm{dec}^{G,x}(\chi)(y) := \begin{cases} \chi(xy) & \text{if } y \text{ is a } p'\text{–element}\,, \\ 0 & \text{if not}\,. \end{cases}$$

The generalized c-d-triangle associated with x is

$$(\mathcal{T}(G,x)) \qquad \begin{tikzcd} \mathcal{R}_x(KG) \arrow[rr, "\mathrm{dec}^{G,x}"] & & \mathcal{R}_x(kC_G(x)) \\ & \mathcal{R}_x^{\mathrm{pr}}(kC_G(x)) \arrow[ul, "{}^t\mathrm{dec}^{G,x}"] \arrow[ur, "\mathrm{Car}^{C_G(x)}"'] & \end{tikzcd}$$

Notice that $\mathcal{T}(G,1) = \mathcal{T}(G)$.

The generalized decomposition matrix is the matrix of the map $\mathrm{dec}^{G,x}$ on the natural basis $\mathrm{Irr}(KG)$ and $\mathrm{Irr}(kC_G(x))$.

The triangle of a block. To simplify the exposition, we assume from now on that *A is the principal block of $\mathcal{O}G$*[3]. For x a p–element of G, we denote by A_x the

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[2] see for example [Br4] for a brief account and some bibliographical references of Puig's work along these lines.

[3] otherwise we would have to introduce the subpairs and the Brauer elements as in [AlBr].

principal block of $\mathcal{O}C_G(x)$. Then the combination of Brauer's Second and Third Main Theorems (see for example [Fe]) implies that the image of $\mathcal{R}_x(KA)$ through $\mathrm{dec}^{G,x}$ is contained in $\mathcal{R}_x(kA_x)$, from which one defines the corresponding triangle :

$$(\mathcal{T}(A,x)) \qquad \begin{tikzcd} \mathcal{R}_x(KA) \arrow[rr, "\mathrm{dec}^{A,x}"] & & \mathcal{R}_x(kA_x) \\ & \mathcal{R}_x^{\mathrm{pr}}(kA_x) \arrow[ul, "{}^t\mathrm{dec}^{A,x}"] \arrow[ur, "\mathrm{Car}^{A_x}"'] & \end{tikzcd}$$

Of course one has $\mathcal{T}(A,1) = \mathcal{T}(A)$.

6.C. Equivalences "with groups".

Puig equivalences.

Definition. *We say that two blocks A and B of two finite groups G and H are Puig equivalent*[4] *if, denoting by D (resp. E) a defect group of A (resp. B), there exist a group isomorphism $D \xrightarrow{\sim} E$ and an algebra isomorphism* $\mathrm{Sce}(A) \xrightarrow{\sim} \mathrm{Sce}(B)$ *such that the following diagram is commmutative*

$$\begin{tikzcd} \mathcal{O}D \arrow[r, "\sim"] \arrow[d] & \mathcal{O}E \arrow[d] \\ \mathrm{Sce}(A) \arrow[r, "\sim"] & \mathrm{Sce}(B) \end{tikzcd}$$

Example. Let us use again the notation introduced in §4.E above : our prime p (the characteristic of our field $k := \mathcal{O}/\mathfrak{p}$) is now denoted by ℓ, and q denotes a power of another prime $p \neq \ell$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, endowed with a Frobenius endomorphism F which defines a rational structure on $\mathbb{F}_q$. We assume for simplicity that $(\mathbf{G}, F)$ is split. Let W be the Weyl group of $\mathbf{G}$.

Assume that ℓ does not divide $|W|$ and divides $(q-1)$. Then the Sylow ℓ–subgroups of $\mathbf{G}^F$ are abelian, and the centralizer in $\mathbf{G}$ of a Sylow ℓ–subgroup is a Levi subgroup of an F–stable parabolic subgroup of $\mathbf{G}$. Let $\mathbf{H}$ be the normalizer in $\mathbf{G}$ of a Sylow ℓ–subgroup of $\mathbf{G}^F$.

Then ([Pu4]) the principal ℓ–blocks of $\mathbf{G}^F$ and of $\mathbf{H}^F$ are Puig equivalent.

6.2. Puig Conjecture. [5] *Given a finite p–group D, there exists only a finite number of interior D–algebras over $\mathcal{O}$ which are the source algebras of some block of some finite group.*

The validity of this conjecture would imply in particular that there is only a finite number of Morita types for blocks with a given defect group.

[4] Puig says "isomorphic".

[5] Stated in the conference on representation of finite group, Oberwolfach, 1982.

Puig equivalences as "equivalences with groups".

Let us first recall the definition of the "Brauer functor" (see [Br3]). For V an $\mathcal{O}G$–module and P a p–subgroup of G, we set

$$\mathrm{Br}_P(V) := V^P/(\sum_{Q<P} \mathrm{Tr}_Q^P(V^Q) + \mathfrak{p}V^P),$$

where V^P denotes the set of fixed points of V under P, and where $\mathrm{Tr}_Q^P(v) := \sum_{x\in[P/Q]} x(v)$ for Q a subgroup of P and $v \in V^Q$. It defines a functor

$$\mathrm{Br}_P \colon {}_{\mathcal{O}G}\mathbf{mod} \to {}_{k[N_G(P)/P]}\mathbf{mod}.$$

In particular, if V is a permutation P–module, $\mathrm{Br}_P(V)$ is a permutation $(N_G(P)/P)$–module.

From now on, to simplify the exposition, the following hypothesis will be in force :

(A1) G and H are two finite groups with a common Sylow p–subgroup D, and D is abelian,

(A2) $N_G(D)/C_G(D) \simeq N_H(D)/C_H(D)$ — note that this implies that the Frobenius categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$ (see for example [Br4]) are equivalent.

We denote by A and B the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$. Whenever P is a subgroup of D, we denote by A_P and B_P the principal blocks of $\mathcal{O}C_G(P)$ and $\mathcal{O}C_H(P)$.

Assume that A and B are Puig equivalent. Then there exists a family (M_P, N_P) (P runs over the set of subgroups of D) where, for each P, M_P is an (A_P, B_P)–bimodule and N_P is a (B_P, A_P)–bimodule such that :

(pu1) (M_P, N_P) induces a Morita equivalence between A_P and B_P.

(pu2) As an $\mathcal{O}[C_G(P) \times C_H(P)]$–module, M_P is a summand of $\mathrm{Ind}_{\Delta(D)}^{C_G(P)\times C_H(P)}\mathcal{O}$, where $\mathcal{O}$ is the trivial D–module.

(pu3) $k \otimes M_P \simeq \mathrm{Res}_{C_G(P)\times C_H(P)}^{N_{G\times H}(\Delta(P))} \mathrm{Br}_{\Delta(P)}(M_{\{1\}})$.

Such a family (M_P, N_P) induces in particular an isomorphism between all generalized "local" c-d-triangles

$$\mathcal{T}(A_P, x) \overset{\sim}{\longleftrightarrow} \mathcal{T}(B_P, x) \quad (\text{for all } x \in D)$$

which preserves the canonical basis.

Rickard equivalences with groups. As just seen, a Puig equivalence may be seen as a "Morita equivalence with groups" The preceding formulation of a Puig equivalence allows us to define (still under the hypothesis (A1) and (A2)) what is a "Rickard equivalence with groups".

We still denote by A and B the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ and, for P a subgroup of D, by A_P and B_P the principal blocks of $\mathcal{O}C_G(P)$ and $\mathcal{O}C_H(P)$.

We say that A and B are "Rickard equivalent with groups" if there exists a family (M_P, N_P) (P runs over the set of subgroups of D) where, for each P, M_P is a bounded complex of (A_P, B_P)–bimodules and N_P is a bounded complex of (B_P, A_P)–bimodules such that :

(ri1) (M_P, N_P) induces a Rickard equivalence between A_P and B_P.

(ri2) As $\mathcal{O}[C_G(P) \times C_H(P)]$–modules, M_P^n is a summand of $\operatorname{Ind}_{\Delta(D)}^{C_G(P)\times C_H(P)} X_P^n$, where X_P^n is a permutation D–module.

(ri3) $k \otimes M_P \simeq \operatorname{Res}_{C_G(P)\times C_H(P)}^{N_{G\times H}(\Delta(P))} \operatorname{Br}_{\Delta(P)}(M_{\{1\}})$.

Such a family (M_P, N_P) induces in particular an isometry between all generalized "local" c-d-triangles

$$\mathcal{T}(A_P, x) \overset{\sim}{\longleftrightarrow} \mathcal{T}(B_P, x) \quad (\text{for } x \in D)$$

corresponding to what is called an "isotypie" in [Br1].

Some unpublished work of J. Rickard shows the relevance of the preceding definition. In particular, complexes with properties (ri2) and (ri3) above occur naturally for finite reductive groups in the context of étale cohomology (see [Ri5]).

On stable equivalences. Let us end with a result which has been often used in applications to structure of finite groups. Consider a slighly more general situation than (A1) and (A2). Now G and H are two finite groups with a common Sylow p–subgroup D. The group D is not necessarily abelian, but we still assume that G and H have "the same fusion" on p–subgroups, *i.e.,* the embedding of D in both G and H defines an equivalence between the Frobenius categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$.

Let e and f be central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. We set $A := \mathcal{O}Ge$ and $B := \mathcal{O}Hf$. For P a subgroup of D, we set $\overline{e}_P := \operatorname{Br}_P(e)$, $\overline{f}_P := \operatorname{Br}_P(f)$, and $\overline{A}_P := kC_G(P)\overline{e}_P$, $\overline{B}_P := kC_H(P)\overline{f}_P$.

Let M be an (A, B)–bimodule and N be a (B, A)–bimodule. For each subgroup P of D, we set $\overline{M}_P := \operatorname{Br}_{\Delta(P)}(M)$ and $\overline{N}_P := \operatorname{Br}_{\Delta(P)}(N)$.

6.3. Theorem. *Assume that*

(st1) *M is a summand of $\operatorname{Ind}_{\Delta(D)}^{G\times H} X$, where X is a permutation D–module.*

(st2) *For each non trivial subgroup P of D, $(\overline{M}_P, \overline{N}_P)$ induces a Morita equivalence between $\overline{A}_P$ and $\overline{B}_P$.*

Then (M, N) induces a stable equivalence of Morita type between A and B.

Example. The following situation is a direct generalization of the (p–t.i.)–case mentioned in §5.

6.4. *Assume that H is a subgroup of G with index prime to p, and with the following property :*
(p–s.c.) whenever P is a p-subgroup of H, we have $N_G(P) = N_H(P)O_{p'}C_G(P)$.
Let A and B be the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively, with unity elements e and f. Then the functors $e.\operatorname{Ind}_H^G$ and $f.\operatorname{Res}_H^G$ induce inverse stable equivalences of Morita type between A and B.

The preceding statement has several applications to some "non–simplicity criteria" for finite groups. In this spirit, an important open question is to find a direct and "representation theoretic" proof to the Z_p^*–theorem for p odd, which would provide a significant simplification in the classification of finite simple groups.

6.5. Theorem. *Let H be a subgroup of G which controls the fusion of p–subgroups in G (i.e., the inclusion of H in G induces an equivalence between the Frobenius* *categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$). Assume that H is the centralizer in G of a p-subgroup of G. Then $G = HO_{p'}(G)$.*

For $p = 2$, the preceding theorem is due to Glauberman ([Gl]). For p odd, it is a consequence of the classification of finite simple groups. An important work of G. Robinson ([Ro1], [Ro2]) makes plausible to find a direct proof using representation theory.

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On the Endomorphism Algebras of Gelfand-Graev Representations

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ABSTRACT

The Gelfand-Graev representations of a finite reductive group G^F are multiplicity free representations, each of which contains a large share of the irreducible representations of G^F. The G^F-endomorphism algebra of a Gelfand-Graev representation γ is a commutative algebra, whose irreducible representations correspond to the irreducible components of γ. This note contains announcements of some results on the connection between the irreducible representations of these endomorphism algebras and the virtual representations $R_{T,\theta}$ of Deligne and Lusztig.

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1 Introduction

Let G be a connected, reductive algebraic group, defined over a finite field F_q, with Frobenius map $F : G \to G$, so that the group of fixed points under F, $G^F = \{g \in G : F(g) = g\}$, is finite. Let $\{B_0, T_0\}$ be a pair consisting of an F-stable Borel subgroup B_0, and an F-stable maximal torus T_0 contained in B_0. Then $B_0 = U_0T_0$ (semidirect product), with U_0 the unipotent radical of B_0. Let $N = N_G(T_0)$; then $W = N/T_0$ is the Weyl group of G. The root system of G with respect to T_0 is denoted by Φ, with Φ^+ the set of positive roots associated with B_0, and Π the set of simple roots in Φ^+. For the purposes of this note, it is assumed that G is of split type, so that F acts trivially on W and on the set of roots Φ. This means that $U_0 = \Pi_{\alpha\in\Phi^+} U_\alpha$, and the root subgroups $\{U_\alpha\}$ are F-stable.

Let K denote an algebraically closed field of characteristic zero. The Gelfand-Graev representations of G^F are K-representations defined as follows. Let $U_0^* = \Pi_{\beta\in\Phi^+\setminus\Pi} U_\beta$; then U_0^{*F} is the derived subgroup of U_0, and $U_0^F/U_0^{*F} \cong \Pi_{\alpha\in\Pi} U_\alpha^F$ is an abelian group. A one dimensional representation $\psi : U_0^F \to K$ is called nondegenerate if its restriction $\psi \mid U_\beta^F$ to a positive root subgroup U_β^F is nontrivial if and only if $\beta \in \Pi$. The nonsimple root subgroups U_β^F are automatically contained in the kernel of ψ if U_0^{*F} is the derived group of U_0^F, which is generally the case with a few exceptions.

Definition. A *Gelfand-Graev representation* γ of G^F is an induced representation $\gamma = \psi^{G^F}$ from a nondegenerate one dimensional representation ψ of U_0^F.

In case the center $Z(G)$ is connected, there is only one equivalence class of Gelfand-Graev representations γ, and the irreducible components of its character Γ were constructed by Deligne and Lusztig [4] as linear combinations of the virtual characters $\{R_{T,\theta}\}$. In general, the equivalence classes of Gelfand-Graev representations are parametrized by the elements of $H^1(F, Z(G))$, where $H^1(F, Z(G))$ denotes the set of F-conjugacy classes in the center $Z(G)$ (Digne, Lehrer and Michel [6]).

Each Gelfand-Graev representation γ contains, in a sense that will become clear later, a large share of the irreducible representations of G^F. The problem to be discussed in this note is the decomposition of a Gelfand-Graev representation $\gamma = \psi^{G^F}$ in terms of the representations of its endomorphism algebra, or Hecke algebra, $H = eKG^Fe$, where $e = |U_0^F|^{-1}\sum_{u\in U_0^F}\psi(u^{-1})\,u$ is a primitive idempotent in KU_0^F affording the nondegenerate one dimensional representation ψ.

Gelfand and Graev introduced the representation γ in two paper published in 1962 ([7], [8]). One of their main results was the assertion that the Gelfand-Graev representations γ are multiplicity free, which is equivalent to the statement that the Hecke algebra $H = eKG^Fe$ is commutative. They proved this result for the case $G = SL_n$. The commutativity of H was proved for adjoint Chevalley groups by Yokonuma [11], and in general by Steinberg [10].

The theory of Hecke algebras ([3] §11D) asserts that there is a bijection from the set of irreducible characters $\{\zeta\}$ of the group algebra KG^F which occur with positive multiplicity in a Gelfand-Graev character Γ to the set of irreducible characters of the Hecke algebra $H = eKG^Fe$. The irreducible character φ of H corresponding to an irreducible character ζ of KG^F occuring in Γ is the restriction $\varphi = \zeta \mid H$, where H is identified with a subalgebra of the group algebra KG^F.

The irreducible representations of the commutative Hecke algebras H of Gelfand-Graev representations were constructed previously in the following cases: for $G = SL_2$, by Gelfand and Graev [7]; for $G = GL_2$, by Helversen-Pasotto [9]; and for $G = GL_3$, by Chang [1].

In §§ 2, 3, we shall state some general results on the irreducible representations of the algebras H. A fuller discussion, with proofs of the theorems, is published elsewhere (Curtis [2]).

2 The Irreducible Representations of the Hecke Algebras of Gelfand-Graev Representations.

Let $\gamma = \psi^{G^F}$ be a fixed Gelfand-Graev representation, with character Γ, and let $H = eKG^F e$ denote its Hecke algebra. The key to the representation theory of H is the connection with the virtual characters $R_{T,\theta}$ of Deligne-Lusztig [4]. These are parametrized by pairs (T, θ), with T a maximal F-stable torus in G, and θ an irreducible character of the finite group T^F. This is not the place to recall their definition or properties in detail, but a few facts should be mentioned. The virtual characters $R_{T,\theta}$, viewed as class functions on G^F, take their values in the field $K = \overline{Q}_l$, the algebraic closure of the l-adic numbers, for some prime l different from the characteristic of F_q. They are orthogonal, in the sense that the inner products $(R_{T,\theta}, R_{T',\theta'}) = 0$, unless the pairs (T, θ) and (T', θ') are G^F-conjugate. Virtual characters may be orthogonal and still have irreducible characters in common. The disjointness theorem of Deligne and Lusztig states that $R_{T,\theta}$, and $R_{T',\theta'}$ contain no common irreducible characters with non-zero multiplicity, unless (T, θ) and (T', θ') are geometrically conjugate. This means, roughly speaking, that (T, θ) and (T', θ') are conjugate by the action of G, suitably interpreted. The first result is as follows. We shall use the parametrization of the Gelfand-Graev representations by the elements $z \in H^1(F, Z(G))$.

(2.1) *Theorem.* Let Γ_z be the character of the Gelfand-Graev representation γ_z, for an element $z \in H^1(F, Z(G))$, and let (T, θ) be a pair as above. Then there exists a unique irreducible character $\chi_{T,\theta,z}$ of G^F such that

$$(\chi_{T,\theta,z}, \Gamma_z) \neq 0 \text{ and } (\chi_{T,\theta,z}, R_{T,\theta}) \neq 0.$$

Every irreducible character occurring in Γ_z with positive multiplicity coincides with one of the characters $\chi_{T,\theta,z}$.

This follows at once from results in [4] in case $Z(G)$ is connected. The idea of the proof, in this case, is that by [4].

$$\Gamma_z = \sum \chi_{[T,\theta]},$$

with irreducible characters $\chi_{[T,\theta]}$ parametrized by the geometric conjugacy classes $[T,\theta]$. Each character $\chi_{[T,\theta]}$ is a linear combination of virtual characters $R_{T',\theta'}$, with (T',θ') in the geometric conjugacy class $[T,\theta]$. This implies that $(\chi_{[T,\theta]}, R_{T,\theta}) \neq 0$. The uniqueness of $\chi_{[T,\theta]}$ as a common constituent of Γ_z and $R_{T,\theta}$ follows from the disjointness theorem. The proof in the general case follows from a (somewhat less explicit) decomposition of Γ_z obtained in [6] (or [5]).

From the theory of Hecke algebras, it follows that the irreducible representations of the Hecke algebras $H_z = e_z KG^F e_z$ of γ_z correspond to the irreducible characters $\chi_{T,\theta,z}$ in Theorem 2.1. In order to describe them and the corresponding idempotents, it is useful to introduce the notation $a = \sum_{g \in G^F} \alpha(g^{-1})g$ for an element of the group algebra KG^F corresponding to a function α: $G^F \to K = \overline{Q}_l$.

(2.2) *Theorem.* Let H_z be the Hecke algebra associated with Γ_z for $z \in H^1(F, Z(G))$, and let (T,θ) be a pair as in Theorem 2.1. Let $r_{T,\theta} \in KG$ be the element corresponding to $R_{T,\theta}$, and e_z the idempotent in KU_0^F associated with a nondegenerate one dimensional representation ψ_z defining γ_z. Then:

(i) $Kr_{T,\theta}e_z$ is a Wedderburn component of the commutative algebra H_z;

(ii) the irreducible representation $f_{T,\theta,z} : H_z \to K$ corresponding to the character $\chi_{T,\theta,z}$ in Theorem 2.1 is given by

$$c \cdot r_{T,\theta}e_z = f_{T,\theta,z}(c) r_{T,\theta} \cdot e_z,\ c \in H_z;$$

and

(iii) every irreducible representation of the commutative algebra H_z coincides with one of the representations $f_{T,\theta,z}$.

Here is the idea of the proof. The element $r_{T,\theta}$ corresponding to the class function $R_{T,\theta}$ belongs to the center of KG^F, and is a linear combination of central primitive idempotents corresponding to irreducible characters of G^F. By Theorem 2.1, the primitive central idempotent $\epsilon_{T,\theta,z}$ corresponding to $\chi_{T,\theta,z}$ occurs in $r_{T,\theta}$ with a nonzero coefficient, while the other primitive central idempotents ϵ_ζ occuring in $r_{T,\theta}$ with nonzero coefficients, if any, correspond to characters ζ such that $(\zeta, \Gamma_z) = 0$. It follows that $r_{T,\theta}e_z$ is a nonzero multiple of $\epsilon_{T,\theta,z}e_z$, which is a primitive central idempotent in the Hecke algebra H_z associated with the irreducible character $\chi_{T,\theta,z}$. As H_z is commutative, $r_{T,\theta}e_z$ affords the irreducible representation $f_{T,\theta,z} : H_z \to K$ as in (ii).

(2.3) *Corollary.* The primitive idempotents in the Hecke algebra H_z are the elements

$$f_{T,\theta,z}(r_{T,\theta}e_z)^{-1}r_{T,\theta}e_z,$$

corresponding to the irreducible representations $f_{T,\theta,z}$ of H_z.

3 A Factorization of the Representations of H_z.

The elements of N^F are a set of representatives of the double $U_0^F \backslash G^F / U_0^F$. Each element $n \in N^F$ such that $e_z n e_z \neq 0$ defines a standard basis element $c_{n,z}$ of H_z, by [3, §11D], for $z \in H^1(F, Z(G))$.

(3.1) *Theorem.* Keep the notation of Theorem 2.2. Let T be an F-stable maximal torus of G. There exists a homomorphism of commutative algebras $f_{T,z} : H_z \to KT^F$, with the property that for each irreducible representation $\theta : T^F \to K$, the representation $f_{T,\theta,z}$ of H_z, defined in Theorem 2.2, has a factorization

$$f_{T,\theta,z} = \widetilde{\theta} \circ f_{T,z},$$

where $\widetilde{\theta}$ is the extension of θ to an irreducible representation of the group algebra KT^F. The coefficient at $t \in T^F$ of $f_{T,z}(c_{n,z}) \in KT^F$, for a standard basis element

$c_{n,z}$ of H_z, is given by the formula

$$f_{T,z}(c_{n,z})(t) = k \sum_{g\in G^F, u\in U_0^F} \psi_z\left(u^{-1}\right) Q_T^{C_G(t)^0}\left(\left(gung^{-1}\right)_{uni}\right),$$

where $(gung^{-1})_{ss} = t$ and

$$k = \operatorname{ind} n\left(Q_T^G, \Gamma_z\right)^{-1}\left|U_0^F\right|^{-1}\left|C_G(t)^{0F}\right|^{-1}$$

and Q_T^G is the Green function associated with T, as in [4].

The formula for $f_{T,z}(c_{n,z})$ is proved using Theorem 2.2 and the character formula for $R_{T,\theta}$ in [4].

3.2 *Example.* Let $G = SL_2$, with the usual BN-pair and Frobenius map. In this case there are two Gelfand-Graev characters if the characteristic of F_q is odd, and one if it is even. We have

$$U_0^F = \left\{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in F_q\right\} \cong F_{q,+}$$

Let ψ be a fixed nontrivial character of U_0^F; then

$$\psi\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \tau(\alpha),$$

for a nontrivial additive character τ of F_q. Then the induced representation $\gamma = \psi^{G^F}$ is a Gelfand-Graev representation of G^F, and we let H denote its Hecke algebra.

These are two G^F-conjugacy classes of maximal tori in G: T_0 and a twisted torus T_s corresponding to the nontrivial element $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of the Weyl group. We shall describe the homomorphism f_{T_s} from H to the group algebra KT_s, using Theorem 3.1. There is an isomorphism

$$T_s^F \cong T_0^{sF} \cong C$$

where

$$C = \left\{\xi \in F_{q^2} : \xi^{q+1} = 1\right\}.$$

The standard basis elements $\{c_n\}$ of H arise from the elements $\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, and $\begin{pmatrix}0 & \lambda\\ -\lambda^{-1} & 0\end{pmatrix}$ of N^F, with $\lambda \neq 0$ in F_q. Let c_λ denote the standard basis element of H corresponding to $\begin{pmatrix}0 & \lambda\\ -\lambda^{-1} & 0\end{pmatrix}$.

According to the formula in Theorem 3.1, the coefficient of $f_{T_s}(c_\lambda)$ at $\xi \in C \cong T_s^F$ is given by

$$f_{T_s}(c_\lambda)(\xi) = -\tau\left(\lambda\left(\xi+\xi^{-1}\right)\right).$$

The corresponding representation $f_{T_s,\theta}$ of H is given by

$$f_{T_s,\theta}(c_\lambda) = \widetilde{\theta} \circ f_{T_s}(c_\lambda) = -\sum_{\xi \in C} \tau\left(\lambda\left(\xi+\xi^{-1}\right)\right)\theta(\xi),$$

for each irreducible character θ of $C \cong T_s^F$. These representations of H correspond to cuspidal characters of G^F if $\theta \neq 1$. These formulas, and similar ones for the representations $f_{T_{0,\theta}}$, were obtained by Gelfand-Graev in [7], and they called the resulting functions on F_q Bessel functions over finite fields. For more details, we refer the reader to [2].

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Harish-Chandra Vertices, Green Correspondence in Hecke Algebras, and Steinbergs Tensor Product Theorem in Nondescribing Characteristic *

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Abstract

This is a survey on some recent results in representation theory of finite groups of Lie type which were derived in joint work with J.Du. We begin with two generalizations of Green's theory of vertices and sources: First by allowing as vertex of an indecomposable representation of a finite group G not only subgroups of G but also subfactors. In addition we derive a relative version of the notion of a vertex (and a source) by considering socalled Mackey systems of subfactors. As a consequence one gets for example the Harish-Chandra theory for irreducible ordinary representations of finite groups of Lie type as special case. Secondly we extend the classical theory of vertices and sources to Hecke algebras associated with Coxeter groups. We apply this to the representation theory of finite general linear groups G in nondescribing characteristic. In particular if we take as Mackey system the set of Levi subgroups of G, (considered as factor groups of the corresponding parabolic subgroups), we explain how vertices with respect to this Mackey system correspond to vertices in the associated Hecke algebra of type A. Using a version of Steinberg's tensor product theorem in the nondescribing characteristic case the vertices of the irreducible representation with respect to the Mackey system of Levi subgroups (Harish-Chandra vertices) are described.

0 Introduction

This is the third paper in a series of survey articles on the p-modular representation theory of finite reductive groups G defined over some finite field $GF(q)$ for primes p not dividing q, and its connections with Hecke algebras, and, in the case of general linear groups, with q-Schur algebras and quantum groups.

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In the first survey [8] the main theme was the connection between representations of finite general linear groups on the one side, and q-Schur algebras, hence quantum GL_n on the other side, given by certain functors. Moreover a notion of an analogous result to Steinberg's tensor product theorem for q-Schur algebras and general linear groups in non describing characteristic was formulated. The second article [9] gave an outline of a generalization of Green's vertex theory to Hecke algebras, and on its translation into Brauer homomorphisms of q-Schur algebras in the case of Hecke algebras of type A. It was then shown, that the tensor product theorem for q-Schur algebras is intimately related to the theory of vertices and sources in Hecke algebras.

In this third article we shall outline how the theory of vertices and sources for Hecke algebras translates into a new theory of vertices and sources for the corresponding finite reductive groups. In fact this can be generalized to arbitrary finite groups, leading to the notion of Harish-Chandra vertices and sources. The main difference to the classical theory is, that we allow as possible vertices not only subgroups, but subfactors of G with some the additional hypothesis. As a consequence we get a common generalization of Green's vertex theory and the Harish-Chandra theory for finite groups of Lie type. We shall relate our results to representations of these groups. In particular in the last section we shall apply the theory to general linear groups and show, how tensor product theorem can be used to compute the Harish-Chandra series of irreducible p-modular representations of finite general linear groups in nondescribing characteristic p

This is a survey, thus the emphasis is more on the big picture than technical details. It is based on a series of two lectures, which I gave during the CMS Annual Seminar on representations of algebras and related topics at Carleton University, Ottawa, and one lecture which I did not give at ICRA IV, and it reports on joint work with J. Du. I would like to thank CMS, but in particular the organizers of the meeting for the opportunity to take part in this very interesting conference.

1 Harish-Chandra vertices and sources

Our main reference for this section is [13]. Let G denote an arbitrary finite group. R is a commutative ring with identity. For simplicity we assume that R is a field or a complete discrete valuation ring. All RG-modules are free as R-modules of finite rank, that is they are RG-lattices, and G operates from the right, if not stated otherwise.

Green's theory of vertices and sources, which was developed by Green in the sixties and seventies, deals with the idea that R-representations of G may 'come' from subgroups of G. More precisely an RG-module M is said to be *H-projective* or *relative projective to H* for a subgroup H of G, if it is a direct summand of some to G induced RH-module. This idea leads to two invariants of indecomposable representations of G, namely the vertices and sources. A *vertex* of an indecomposable RG-module M is a minimal member among all subgroups H of G such that M is H-projective, and a *source* N of M (with respect to H) is an indecomposable RH-module such that M is a direct summand of the induced module $\mathrm{Ind}_H^G(N) = N \otimes_{RH} RG$. Both are unique up to conjugacy. Thus the class of indecomposable RG-modules is partitioned into subclasses parametrized by vertices and sources. It turns out that for an indecomposable RG-module the set of vertices is precisely a conjugacy class of p-subgroups of G for a prime p, if R itself or in case of a valuation ring R the residue field of R is a field of characteristic p.

The Mackey decomposition theorems are of central importance for the construction of these invariants of group representations. Here are the main ingredients which make Mackey decomposition and vertex theory work:

Ingredients 1.1 (The functors) *For a subgroup H of G we have the induction functor Ind_H^G taking representations of H to representations of G. This functor has an adjoint on both sides, the restriction functor Res_H^G. Moreover these functors are transitive on subgroups, that is if $K \leq H \leq G$ then $\mathrm{Ind}_K^G = Ind_H^G \circ \mathrm{Ind}_K^H$ and the corresponding statement holds for restriction.*

We remark that restriction is right adjoint to induction by general reasons. The left adjointness however is special for group algebras of finite groups, (see for example [2, 10.21] or [27, p.82]).

Ingredients 1.2 *A) The collection of subgroups of G is partially ordered by inclusion.*

B) Two subgroups can be intersected to produce a third subgroup which is less or equal each of the original subgroups.

C) The group G acts by conjugation on the set of subgroups.

D) We can combine conjugation and inclusion of subgroups to a new relation $\subseteq_G$ which means "conjugate to a subgroup". This is a preorder relation.

E) The equivalence relation on subgroups of G defined by the preorder in D), that is $H \quad K$ for $H, K \leq G$ if and only if $H \subseteq_G K$ and $K \subseteq_G H$, is obviously conjugation by elements of G.

Here is Mackey's decomposition theorem (see for example [27, p.90]).

Theorem 1.3 (Mackey's decomposition theorem) *Let H, K be subgroups of G and let N be an RK-module. Then*

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G(N) = \bigoplus_{g \in K\backslash G/H} \mathrm{Ind}_{K^g \cap H}^H(\mathrm{Res}_{K^g \cap H}^{K^g}(N^g)).$$

Here N^g denotes the conjugate module, and $K\backslash G/H$ a cross section of K-H-double coset representatives in G.

From the Mackey decomposition theorem one sees immediately, that vertices of indecomposable representations have to be conjugate: If M is an indecomposable RG-module which is relative projective to the subgroups H and K then it is $K^g \cap H$-projective for some $g \in G$, so D) and E) of our ingredients 1.2 show the claim.

Indeed the vertices of M consist precisely of a conjugacy class of p-subgroups, where p is the characteristic of R respectively of the residue field of R. To see that they are actually p-subgroups one needs one more ingredient:

Ingredients 1.4 (The norm map) *Let M be an RG-module and $H \leq G$. By $\mathrm{Fix}_H(N)$ we denote the R-subspace of H-fixpoints of M. The* **trace** *or* **norm** *map $\mathcal{N}_H^G : \mathrm{Fix}_H(M) \to \mathrm{Fix}_G(M)$ is given by*

$$\mathcal{N}_H^G(m) = \sum_{g \in H\backslash G} mg.$$

Here $H\backslash G$ denotes an arbitrary right transversal of H in G.

One applies this in particular to the space of R-linear mappings between RG-modules, on which G acts by conjugation. The H-fixpoints are precisely the RH-linear maps, and so $\mathcal{N}_H^G$ maps the H-homomorphisms into G-homomorphisms, and homorphisms in the image of this norm map are usually called H-projective or projective relative H. Higman's lemma (see for example [27, Cor. 2.4]) gives the connection between relative projective modules and homomorphisms. More precisely one shows that an RG-module M is H-projective if and only if the identity map on M is H-projective. If the index of H in G is not divisible by p one sees that the norm map is invertible. As a consequence M is relative projective to the Sylow p-subgroups of G and consequently vertices are always p-subgroups.

Finally in order to see that vertices consist of precisely all elements of a single G-conjugacy class of subgroups, one has only to observe the following trivial fact:

Ingredients 1.5 *If H and K are conjugate subgroups of G then the functors* Ind_H^G *and* Ind_K^G *are naturally equivalent. The same holds for the restriction functor.*

We want to introduce a generalization of this theory: First we want more groups from which we may induce modules. More precisely we want to replace the collection of subgroups by a suitable collection of all subfactors of G containing the subgroups. Then we want to restrict ourselves to subsets of this collection satisfying certain axioms, socalled Mackey systems. This second step goes back to the thesis of Grabmeier ([22]), where he used the Mackey system of Young subgroups of symmetric groups to investigate representations of symmetric groups and Schur algebras. We shall come back to this in the second section.

Obviously for our generalization we want to use ingredients as above. We cannot get all of them, but most, however in adjusted form. We note that a subfactor of G is uniquely determined by a pair (P, U) of subgroups of G, where U is normal in P. The subfactor is then given as the factor group $L = P/U$. So we let Θ be the collection of such pairs:

$$\Theta = \{(P, U) \mid P, U \leq G, U \trianglelefteq P\}.$$

Moreover we set

$$\tilde{\Theta} = \{L = P/U \mid (P, U) \in \Theta\}.$$

A word of warning is in order here: $\tilde{\Theta}$ is really a different notation for the set Θ, or with other words the notion of a subfactor of G is a group together with the way it is built into G. Consequently two subfactors are considered to be equal if and only if the associated pairs are equal. Later on we shall for example deal with Levi subgroups of finite groups of Lie type. In this case we may consider such a group L as a subfactor by choosing a parabolic subgroup P with unipotent radical U such that $P/U = L$. Now P is not unique and we could even choose the pair $(L, (1)$, since L is a subgroup of G as well. As subfactors, that is as elements of $\tilde{\Theta}$, we have to distuingish all these copies of L.

First we adjust induction and restriction 1.1:

Ingredients 1.6 (The functors) *For a subfactor L of G which is given by the pair (P, U) in Θ we define* **Harish-Chandra induction** *(HC-induction for short)* R_H^G *to be the functor which takes an L-module, lifts it first to a P-module with trivial U-action (that is applies the inflation functor* Inf_L^P*) and then induces the result to G. Thus*

$$\mathrm{R}_L^G = \mathrm{Ind}_P^G \circ \mathrm{Inf}_L^P.$$

The **Harish-Chandra restriction** *(or* **truncation**, *HC-restriction for short)* T_L^G *is defined to be the composite functor which restricts a G-module to the subgroup P and then takes U-fixpoints. The result is obviously an L-module.So*

$$\mathrm{T}_L^G = \mathrm{Fix}_U \circ \mathrm{Res}_P^G$$

Both HC-induction and truncation are transitive on subfactors.

Sometimes we want to indicate the pair $(P, U) \in \Theta$ with $P/U = L$ rather than L itself in our notation. We write then $\mathrm{R}_{(P,U)}^G$, $\mathrm{T}_{(P,U)}^G$ etc.

Note that if $L = P/U, (P, U) \in \Theta$ and if K is a subfactor of L given by the pair (A, E) with $E \trianglelefteq A \leq L$, then K is also a subfactor of G corresponding to the pair (AU, EU) in Θ. Thus transitivity of HC-induction written in terms of pairs is the following formula:

$$\mathrm{R}_{(AU,EU)}^G = \mathrm{R}_{(P,U)}^G \circ \mathrm{R}_{(A,E)}^L.$$

On the level of subfactors it simply states

$$\mathrm{R}_K^G = \mathrm{R}_L^G \circ \mathrm{R}_K^L.$$

For truncation we have analogous formulas.

We remark that the truncation functor T_L^G is right adjoint of HC-induction R_L^G. On the other hand HC-induction has a left adjoint as well. That is restriction followed by taking U-coinvariants (that is the maximal trivial factor module) instead of fixpoints (invariants). Obviously if the order $|U|$ of U is invertible in R, that is if the characteristic of R respectively of the residue field of R does not divide $|U|$, then U-invariants and U-coinvariants coincide naturally, and thus $\mathrm{T}_{(P,U)}^G$ is adjoint of $\mathrm{R}_{(P,U)}^G$ on both sides.

We have now to adjust the ingredients 1.2. In several places we state that two groups are canonical isomorphic. By this we mean that there exists an isomorphism which is composed of canonical isomorphisms built according to the second and third standard isomorphism theorems for finite groups.

Ingredients 1.7 *Let $(P, U), (Q, V) \in \Theta$ and let $L = P/U, K = Q/V$.*

A) We define a partial order $\sqsubseteq$ on $\tilde{\Theta}$ setting $L \sqsubseteq K$, if $V \leq U \leq P \leq Q$. We note that L is canonically isomorphic to a subfactor of K, if $L \sqsubseteq K$.

B) The **shift intersection** *on $\tilde{\Theta}$ is given by $L \cap K = (P \cap Q)V/(U \cap Q)V$. Note that $\cap$ is not a symmetric relation. However $L \cap K \sqsubseteq K$. Moreover $L \cap K$ and $K \cap L$ are canonically isomorphic.*

C) The conjugation operation of G on $\tilde{\Theta}$ is clear: We have $L^g = K$ if and only if $P^g = Q$ and $U^g = V$.

D) We define a preorder $\preceq_G$ on $\tilde{\Theta}$ setting $L \preceq_G K$ if there is an $g \in G$ such that $L^g \cap K = K$.

E) We define an equivalence relation $\ _G$ on $\tilde{\Theta}$ setting

$$L \quad {}_G K \quad \text{if} \quad K \preceq_G L \preceq_G K.$$

We say then that L and K are **associated** *and call the equivalence classes* **associativity** *classes. We note that conjugate subfactors are associated, but there are associated subfactors which are not conjugate. So associativity classes are unions of conjugacy classes of subfactors (or corresponding pairs).*

It is easy to formulate all the ingredients above directly for the corresponding pairs, that is for Θ.

It is clear how we proceed from here: For an indecomposable RG-module M and $L \in \tilde{\Theta}$ we say that M is *L-projective* or *relative projective* to L, if M is a direct summand of $\mathrm{R}_L^G \mathrm{T}_L^G(M)$, denoted by

$$M \mid \mathrm{R}_L^G \mathrm{T}_L^G(M).$$

We say that L is a *Harish-Chandra vertex* (HC-vertex for short) of M, if L is minimal with the property that M is L-projective, where minimality is taken with respect to the partial order $\sqsubseteq$ on $\tilde{\Theta}$. We want to apply a Mackey decomposition theorem and ingredients 1.7 to show that HC-vertices have to be associated.

Theorem 1.8 (Mackey's decomposition theorem) *Let $(P,U),(Q,V) \in \Theta$, and let L and K be the corresponding subfactors of G. Suppose that $|U|$ and $|V|$ are both invertible in R. Fix a cross section $\mathcal{D}_{L,K}$ of P-Q-double coset representatives in G. Then we have an isomorphism of functors:*

$$\mathrm{T}_K^G \circ \mathrm{R}_L^G = \bigoplus_{x \in \mathcal{D}_{L,K}} \mathrm{R}^K_{L^x \cap K} \circ \mathrm{T}^{L^x}_{K \cap L^x} \circ C^x,$$

where C^x denotes the conjugation functor by $x \in G$.

Obviously we made use of the canonical isomorphism between $L^x \cap K$ and $K \cap L^x$.

We remark in passing that there is for our setting similarly a generalization of the Mackey formula, which yields a decomposition of Hom-sets of induced modules.

Now Grabmeier's idea [22] comes in, providing a further generalization: We observe that in order to apply Mackey decomposition in the classical case we need only subgroups occuring in the Mackey decomposition, hence subgroups which come up by combining ingredients B) and C) of 1.2. Consequently we may replace the collection of all subgroups by a collection of subgroups which is closed under conjugation and intersection. Grabmeier called such a collection Mackey system including a further hypothesis, namely that the collection is closed under taking normalizers. This additional condition is needed to formulate and prove the full Green correspondence. This part cannot be generalized directly to our setting, mainly since there is no sensible definition of the normalizer of a subfactor which itself is a subfactor. Obviously the generalized Mackey decomposition theorem can be applied, even if we restrict to systems of subfactors closed under the ingredients B) and C) of 1.7.

Thus we define a *Mackey system* to be a subset of $\tilde{\Theta}$ (or Θ) which is closed under shift intersection $\cap$ and conjugation. To ensure that every indecomposable G-module has actually a vertex in $\mathcal{M}$ we require in addition that $\mathcal{M}$ contains the pair $(G,(1))$. We remark that for a prime p the system

$$\Theta_p = \{(P,U) \in \Theta \mid (p,|U|) = 1\}$$

is a Mackey system. The corresponding system of subfactors is denoted by $\tilde{\Theta}_p$. A Mackey system $\mathcal{M}$ is called *p-regular* if $\mathcal{M} \subseteq \tilde{\Theta}_p$.

For a classical Mackey system it is clear that intersection with a fixed subgroup H of G produces a Mackey system for H. The correct generalization is obvious: We replace intersection by shift intersection $\Cap$. Thus for a subfactor H of G and a Mackey system $\mathcal{M}$ of G we define the *inherited* Mackey system $\mathcal{M}_H$ to be

$$\mathcal{M}_H = \{L \Cap H \mid L \in \mathcal{M}\}.$$

Clearly $\mathcal{M}_H$ is a Mackey system for H. Moreover, if H itself is a member of $\mathcal{M}$, then $\mathcal{M}_H \subseteq \mathcal{M}$.Thus we may consider subfactors of H belonging to $\mathcal{M}_H$ in the natural way as subfactors of G belonging to $\mathcal{M}$, and therefore, if A is an H-module, $\mathrm{R}_H^G(A)$ is L-projective, if A is L-projective for $L \in \mathcal{M}_H \subseteq \mathcal{M}$. Note that $\mathcal{M}_H$ is p-regular, if $\mathcal{M}$ is, provided $H \in \mathcal{M}$.

Here are further examples of Mackey systems:

Example 1.9 *1. Let the characteristic of R respectively of the residue field of R be p. Define*

$$\Theta^{(1)} = \{(P,(1)) \mid P \leq G\}.$$

Then $\Theta^{(1)} \subseteq \Theta$ and one gets the classical theory back. The same holds for

$$\Theta^{(1)}(p) = \{(P,(1)) \mid P \text{ is a p-subgroup of } G\}.$$

2.) Let G be a finite group of Lie type. We fix a Borel subgroup B, and we let W be the corresponding Weyl group with associated system Δ of simple roots. For $I \subseteq \Delta$ we denote the corresponding parabolic subgroup of G by P_I with unipotent radical U_I and Levi kernel L_I. Set

$$\mathcal{L} = \mathcal{L}_G = \{(P_I, U_I)^g \mid g \in G\},$$

This is the **Mackey system of all parabolic subgroups** *of G.*

It is now obvious how the notion of HC-vertices and -sources has to be modified to $\mathcal{M}$-vertices and -sources, by considering minimal members of $\mathcal{M}$ such that a given indecomposable G-module is relative projective to it. From now an we fix a Mackey system $\mathcal{M}$ and we assume always, that $\mathcal{M}$ is p-regular, if R respectively the residue field of R has characteristic p.

The following theorem is immediate from the Mackey decomposition 1.8:

Theorem 1.10 *Let M be an indecomposable RG-module. Then $\mathcal{M}$-vertices of M are associated.*

Thus $\mathcal{M}$-vertex class of an indecomposable module is always contained in a single associativity class. We would like to have equality, as in the clasical case. But this is not true in general, one needs additional properties of $\mathcal{M}$, for example the straightforward generalization of ingredient 1.5: One chooses $\mathcal{M}$ such that Harish-Chandra inductions from associated subgroups are isomorphic functors. By adjointness this implies of course that the same holds for Harish-Chandra restriction. In important special cases these functors are indeed isomorphic, for example for $\Theta^{(1)}$ and $\Theta^{(1)}(p)$, yielding the classical theory, and for the Mackey system $\mathcal{L}$ in 1.9 by theorem [13, 5.2]. The latter was proved independently by Howlett and Lehrer in [25].

Here are some more examples: Taking $\mathcal{M} = \tilde{\Theta}$ the assumption above is not satisfied, e.g. if R is a field of charactristic zero. In this case one sees easily that if an RG-module M

is (P, U)- projective, $(P, U) \in \Theta$, then it is (U, U)-projective as well. Thus the Θ-vertices of M are of the form (H, H) for subgroups H of G. Now M is (H, H)-projective if and only if M is a constituent of the permutation representation on the cosets of H in G. In particular M is (G, G)-projective, if and only if it is the trivial G-module (being indecomposable, hence irreducible). On the other hand, one checks easily, that all pairs (H, H) with $H \leq G$ are associated, and we see, that for the Mackey system Θ in characteristic zero, vertex classes are in general proper subclasses of associativity classes.

If R is a field of characteristic p and M is an indecomposable finite dimensional RG-module, then the $\tilde{\Theta}_p$-vertices of M are p-groups. To show this, one needs a generalization of ingredient 1.4 and a statement analogous to Higman's lemma. Such a result holds ([13, 2.2]), although it does not generalize the classical theorem [27, Cor. 2.4] completely, only parts of it. We do not need this furthermore, therefore we refer to [13] for details.

Another example concerns finite groups of Lie type. Choosing $\mathcal{M} = \mathcal{L}$ and again R a field of characteristic zero, one gets the Harish-Chandra series (compare e.g. [3, §70], [1, 9.2], and [30, chapt.IV]). Its generalization to the p-modular case for a prime p different from the characteristic of the finite field of definition of the group - (*the nondescribing characteristic case*) - is our main application and we shall come back to it later.

We come to the notion of $\mathcal{M}$-sources. Classically, if M is an indecomposable RG-module with vertex D, then there exists an indecomposable RD-module N, uniquely determined up to conjugation in the normalizer $N_G(D)$, such that M is direct summand of the induced module $\mathrm{Ind}_D^G(N)$. We call N the *source* or *D-source* of M.

We first define the normalizer of subfactors: So let $(P, U) \in \Theta$ and let $L = P/U$ the corresponding element of $\tilde{\Theta}$. Then the *normalizer* $N_G(L) = N_G((P, U))$ is the subset of all elements x of G such that $L^x \cap L = L$. We note that $N_G(L)$ is not a subgroup or a subfactor of G but a union of P-P-double cosets in G. If M is an indecomposable RG-module with $\mathcal{M}$-vertex L, then there is an indecomposable RL-module N such that M is a direct summand of $\mathrm{R}_L^G(N)$. If N' is another such RL-module, then one concludes from the Mackey decomposition theorem 1.8 that there is an $x \in N_G(L)$ such that $N = N'^x$. Here we observe, that we may consider N'^x as L-module, since $L = L^x \cap L = L \cap L^x = L^x$ canonically. The latter equality holds, since $L \cap L^x$ is a subfactor of L^x, of order $|L| = |L^x|$. So we have the following theorem:

Theorem 1.11 *Let M be an indecomposable RG-module with $\mathcal{M}$-vertex L. Then the L-vertices of M are conjugate in $N_G(L)$.*

The generalized Harish-Chandra theory introduced above can be used, to subdivide the indecomposable RG-modules into series parametrized by $\mathcal{M}$-vertex and $\mathcal{M}$-sources. This works in particular well, when $\mathcal{M}$ has the nice property, that HC-induction from associated subfactors in $\mathcal{M}$ produces isomorphic functors. As in classical Green theory we have then the following procedure to determine the indecomposable RG-modules:

Procedure 1.12 *From each associativity class of subfactors in $\mathcal{M}$ choose a representative L. To be conjugate under $N_G(L)$ is an equivalence relation on the set of indecomposable RL-modules with $\mathcal{M}_L$-vertex L. From each equivalence class choose a representative N. Decompose $\mathrm{R}_L^G(N)$ into indecomposable direct summands. Not all of them have $\mathcal{M}$-vertex L, some may have smaller vertices. Take from each isomorphism class of indecomposable* *direct summands of* $\mathrm{R}_L^G(N)$ *with* $\mathcal{M}$*-vertex* L *a representative. We get so the indecomposable* RG*-modules partitioned into subsets parametrized by* $\mathcal{M}$*-vertices and -sources.*

Our goal so far was to subdivide the class of indecomposable RG-modules into sublists according to their $\mathcal{M}$-vertices and -sources. In order to get the irreducible RG-modules partitioned into series, we define now semisimple $\mathcal{M}$-vertices and -sources. The idea is to develop a HC-vertex and -source theory for the Grothendieck group of RG. Of course irreducible modules are in particular indecomposable, and have therefore $\mathcal{M}$-vertices and -sources. We remark that those are different from the semisimple $\mathcal{M}$-vertices, which we shall define below.

R is now a field of characteristic $p \geq 0$, and $\mathcal{M}$ is a p-modular Mackey system, (where the condition *p-modular* is empty for $p = 0$). In the Grothendieck group $\mathcal{G}(RG)$ of RG - (that is the free abelian group on RG-modules modulo short exact sequences) - the indecomposable objects are precisely the classes of the irreducible RG-modules, and indecomposable direct summands are given by the composition factors of an RG-module. For simplicity we denote the class of the RG-module M in $\mathcal{G}(RG)$ again by M. So the following statements will have always an interpretation in both, the module category and the Grothendieck group of RG. Let M be an irreducible RG-module, $L \in \mathcal{M}$. We say that M is *weakly* L*-projective*, if M is a composition factor of $\mathrm{R}_L^G\mathrm{T}_L^G(M)$. This is obviously the analog for $\mathcal{G}(RG)$ to the notion of L-projective modules. There are many equivalent formulations for weak relative projectivity ([13, thm. 4.3], [23, 4.1]). We mention some of them:

Theorem 1.13 *The following statements are equivalent:*

i) M *is weakly* L*-projective.*

ii) $\mathrm{T}_L^G(M) \neq (0)$.

iii) $M\hat{U} \neq (0)$, *where* $L = P/U$, $(P, U) \in \mathcal{M}$.

iv) $\mathrm{Hom}_{RG}(\mathrm{Ind}_U^G(R_U), M) \neq (0)$.

v) M *is a composition factor of the head of* $\mathrm{R}_L^G(\mathcal{P})$ *for some projective* RL*-module* $\mathcal{P}$.

Here $\hat{S}$ denotes for an arbitrary subset S of G with $|S|$ invertible in R the element $|S|^{-1}\sum_{s\in S} s$ of RG, and R_U is the trivial U-module. The *radical* $J(M)$ is the intersection of all maximal submodules of M, *the head* $\mathrm{hd}(M)$ is $M/J(M)$. We remark that by general assumption $\mathrm{Ind}_U^G(R_U)$ is a projective RG-module. Thus the implication *iv)* $\Rightarrow$ *v)* is obvious. This explains how the head of modules comes in here.

Let M be an irreducible RG-module. A *semisimple vertex (or semisimple* $\mathcal{M}$*-vertex)* $vx_{ss}(M)$ of M is a subfactor L in $\mathcal{M}$ which is minimal with the property, that M is weakly L-projective. A module with semisimple vertex G is called *cuspidal.* Thus M is cuspidal if and only if $M\hat{U} = (0)$ for all pairs (P, U) in $\mathcal{M}$ with $(1) \neq U$. Again Makey's decomposition theorem 1.8 implies that semisimple vertices of M are associated. We observe that an $\mathcal{M}$-vertex of M always contains a semisimple $\mathcal{M}$-vertex as subfactor, which then is contained in $\mathcal{M}_L$.

Let M as above. Obviously one can choose the projective RL-module $\mathcal{P}$ in 1.13 *v)* to be indecomposable. If L is a semisimple vertex and $\mathcal{P}$ is indecomposable, we call $\mathcal{P}$ a *projective*

L-source of M. If L is a semisimple vertex of M then there exists always an irreducible RL-module N such that M is composition factor of the head of $\mathrm{R}_L^G(N)$. We call such an N *semisimple L-source* of M. It can be shown that for such an N the projective cover $\mathcal{P}_N$ of N is a projective source of M. But the inverse is true only under additional hypotheses. For example it suffices again to assume, that HC-induction from associate subfactors yields isomorphic functors. Now 1.8 implies again, that projective and semisimple sources for a fixed semisimple vertex L of M are equivalent under the action of $N_G(L)$.

Here is the procedure, how one can calculate the Harish-Chandra series for a given group G and a p-regular Mackey system $\mathcal{M}$:

Procedure 1.14 *Assume that $\mathcal{M}$ has the property that HC-induction from associated subfactors in $\mathcal{M}$ yields isomorphic functors for RG. Choose from each associativity class in $\mathcal{M}$ a representative L. Being conjugate under $N_G(L)$ is an equivalence relation on the set of irreducible cuspidal (with respect to $\mathcal{M}_L$) RL-modules. Choose from each equivalence class a representative N. Then decompose the head* $\mathrm{hd}(\mathrm{R}_L^G(N))$ *of* $\mathrm{R}_L^G(N)$ *and determine its non-isomorphic irreducible constituents. Those are the irreducible RG-modules with semisimple vertex L and semisimple L-source N.*

Thus semisimple vertices and semisimple respectively projective sources provide a tool to subdivide the (finite) list of irreducible RG-modules into disjoint sublists, parametrized by the semisimple vertex and semisimple (projective) source. We call those sublists *Harish-Chandra series* or *$\mathcal{M}$-series.*

We observe, that for finite groups of Lie type with $\mathcal{M} = \mathcal{L}$ as in 1.9, we get the classical Harish-Chandra series of irreducible complex characters ([1], [3], [30]), if R is the field of complex numbers, and the Harish-Chandra series of irreducible Brauer characters ([23]) for fields R of positive non-describing characteristic.

Here are some more details: So let G be a finite group of Lie type and choose B, W and Δ as in 1.9. Let P be a parabolic subgroup with Levi decomposition $P = LU$. That means L is a Levi complement, U the unipotent radical of P. Here we have to use our setting with care: P is a semidirect product, and thus U has a complement. This as well as the factor group P/U are often denoted by L and identified. We have to keep them for the moment apart. The same Levi subgroup can be a Levi complement of non conjugate parabolic subgroups, and as such have to be distinguished as subfactors of G. Since the unipotent radical of a parabolic subgroup is unique, every pair in $\mathcal{M}$ is already uniquely determined by specifying the first component. Thus we can identify $\mathcal{L}$ with the set of all parabolic subgroups. The corresponding subfactors of G are now isomorphic (but not equal) to Levi subgroups of G. Fortunately one can show that associativity classes of parabolic subgroups are precisely given by the G-conjugacy classes of Levi subgroups. The conjugacy classes of parabolic subgroups are parametrized by the subsets of Δ. Let I, J be such subsets. Then the corresponding Levi subgroups L_I and L_J are conjugate in G if and only if there is a $w \in W$ such that $wI = J$. Thus associativity classes of $\mathcal{L}$ are labelled by equivalence classes of subsets of Δ, the equivalence relation on Δ given by $I \quad J$ if $wI = J$ for a $w \in W$.

As mentioned above we have now the following theorem, ([13], [25]):

Theorem 1.15 *Suppose that R has characteristic $p \geq 0$ different from the describing characteristic of G. Let (P, U), (Q, V) be associated pairs in $\mathcal{L}$. Then the functors* $\mathrm{R}_{(P,U)}^G$ *and* $\mathrm{R}_{(Q,V)}^G$ *are naturally isomorphic.*

By adjointness the same holds of course for HC-restriction. We remark that theorem 1.15 now allows us to relax our setting and really use Levi subgroups instead of Levi subfactors. In particular we may write R_L^G etc. for Levi subgroups L of G, but we have to distinguish this functor from ordinary induction Ind_L^G. From each P-P-double coset we may choose a representative (which is a preimage of a Weyl group element in G) such that the set of these representatives maps onto the normalizer of L in the Weyl group W. Thus $N_G(P,U)$-orbits on L-modules coincide with the orbit of the normalizer of the subgroups L of G on L-modules. Note that theorem 1.15 says that the hypothesis for the procedure 1.14 is satisfied, and therefore we may apply 1.14. One sees now easily, that we get indeed the classical Harish-Chandra theory and its modular extension.

Thus our general setting appears as common generalization of Green's theory of vertices and sources and Harish-Chandra theory.

For the describing characteristic p of G the unipotent radical of a parabolic subgroup is a p-group, and we cannot apply our theory. It should be noted that in this case 1.15 does not hold.

2 Hecke algebras

We come now to our second generalization of Green's vertex theory, namely the formulation of such a theory for Hecke algebras associated with Weyl groups. We shall use the Mackey system of parabolic subalgberas.

Our main reference for this section is [12]. However [9] contains a detailed survey on Green theory for Hecke algebras and its connections with Frobenius homomorphisms of quantum groups. Thus we restrict ourselves to a brief summary.

R is a commutative ring, preferably with some nice properties, e.g. it is a complete valuation ring, or a field. Let W be a Weyl group, $S = \{s_1, \ldots, s_k\}$ be a set of simple reflections generating W. For $w \in W$ the *length* $l(w)$ is the minimal number k such that w can be written as product of k simple reflections. Such a representation of w is then called *reduced expression* for w.

We fix an invertible element q of R. The *Hecke algebra* $\mathcal{H} = \mathcal{H}_{R,q}(W)$ is free as R-module with basis $\{T_w \mid w \in W\}$. The multiplication is given by the following relations: For $w \in W$ and $s \in S$ define:

$$T_sT_w = \begin{cases} T_{sw} & , \ if\ l(sw) = l(w) + 1 \\ qT_{vw} + (q-1)T_w, & if\ l(sw) = l(w) - 1 \end{cases}$$

$\mathcal{H}_{R,q}(W)$ may be considered as specialization $x \mapsto q$ of the generic Hecke algebra $\mathcal{H}_{\mathbb{Z}[x,x^{-1}],x}(W)$ over the ring of Laurant polynomials with integral coefficients.

In the case of group algebras one uses subalgebras, which come from subgroups, for inducing and restricting representations. Those have several advantages, for example is the group algebra free as module over subgroup algebras, and induction and restriction with respect to subgroup algebras are adjoint on both sides. The system of parabolic subalgebras of $\mathcal{H}$ has similar nice properties. The existence of distinguished coset representatives guarantee those. Here are some details:

For a subset I of S the *parabolic subgroup* of W generated by I is denoted by W_I. The *parabolic subalgebra* $\mathcal{H}_I$ is the algebra generated as an R-module by $\{T_w \mid w \in W_I\}$

or alternatively as an algebra by $\{T_s \mid s \in S\}$. Note that we insist that W_I is generated by subsets of the generator set S. Usually every subgroup conjugate to some W_I is called parabolic, but the $T_w's$ coming from those subgroups do not span a subalgebra in general. Observe that $\mathcal{H}_I = \mathcal{H}_{R,q}(W_I)$. Every right W_I-coset in W contains a unique element of minimal length. These special coset representatives are called *distinguished* ([1, 2.7]) and denoted by $\mathcal{D}_I$. Similarly the set $\mathcal{D}_I^{-1}$ is a set of distinguished left, and the set $\mathcal{D}_{IJ} = \mathcal{D}_I \cap \mathcal{D}_J^{-1}$ a set of distinguished W_I-W_J-double coset representatives, where J is another subset of S. We have $l(wd) = l(w) + l(d)$ for $w \in W_I$ and $d \in \mathcal{D}_I$. As a consequence $\mathcal{H}$ is free as left $\mathcal{H}_I$-module with basis $\{T_d \mid d \in \mathcal{D}_I\}$, [17, 2.4]. In particular induction $\mathrm{R}_{\mathcal{H}_I}^{\mathcal{H}}$ and restriction $\mathrm{T}_{\mathcal{H}_I}^{\mathcal{H}}$ are exact functors. Moreover the Nakayama relations hold, [17, 2.5 and 2.6], and thus we have again our ingredient 1.1: Induction and restriction are adjoint on both sides, the transitivity properties are obvious.

Let $I, J \subseteq S$, and $d \in \mathcal{D}_{I,J}$. Then $d^{-1}W_I d \cap W_J = W_{dI \cap J}$, and we have as a consequence ([17]):

$$T_{d^{-1}}\mathcal{H}_I T_d \cap \mathcal{H}_J = T_d^{-1}\mathcal{H}_I T_d \cap \mathcal{H}_J = \mathcal{H}_{dI \cap J}.$$

As said above we cannot use arbitrary conjugation, but only conjugation by distinguished double coset representatives. The formula above tells us, that we get a system of subgroups which is closed under conjugating by these special elements of W and intersecting the subgroups. But that is precisely what we need to apply Mackey's decomposition theorem 1.3. It is clear now how the ingredients 1.2 and the definition of a Mackey system have to be modified, to make things work here. We have then the following theorem:

Theorem 2.1 (Mackey's decomposition theorem) *Let $I, J \subseteq S$. Then there is a natural isomorphism of functors:*

$$\mathrm{T}_{\mathcal{H}_J}^{\mathcal{H}} \circ \mathrm{R}_{\mathcal{H}_I}^{\mathcal{H}} = \bigoplus_{d \in \mathcal{D}_{I,J}} \mathrm{R}_{\mathcal{H}_{dI \cap J}}^{\mathcal{H}_J} \circ \mathrm{C}^d,$$

where C^d denotes the conjugation functor by T_d.

Note that on a given $\mathcal{H}_I$-module one can define a conjugate operation turning it into an $\mathcal{H}_{dI \cap J}$-module.

An indecomposable $\mathcal{H}$-module M is *$\mathcal{H}_I$-projective* or *projective relative $\mathcal{H}_I$* if it is a direct summand of some induced module $\mathrm{R}_{\mathcal{H}_I}^{\mathcal{H}}(N)$ for some $\mathcal{H}_I$-module N. This can be choosen to be indecomposable. If $\mathcal{H}_I$ is minimal with this property, we call it *vertex* and the indecomposable N *source* of M. Again vertices are unique up to conjugation by elements of the form T_d where d is a distinguished W_I-W_J-double coset representative with $dI = J$. And similarly an $\mathcal{H}_J$-source of an $\mathcal{H}$-module is unique up to conjugation by elements T_d with $d \in \mathcal{D}_{JJ}$ satisfying $dJ = J$. (Here we need of course our general assumption from the beginning of this section that R is a field or a discrete complete valuation ring to ensure the Krull Schmidt theorem).

Next we want to generalize ingredient 1.4 and Higman's criterion:

Ingredients 2.2 (The norm map) *Let U and V be $\mathcal{H}$-modules, and let $J \subseteq S$. The* **trace** *or* **norm** *map $\mathcal{N}_J^S : \mathrm{Hom}_{\mathcal{H}_J}(U, V) \to \mathrm{Hom}_{\mathcal{H}}(U, V)$ is given by*

$$\mathcal{N}_J^S(f) = \sum_{d \in \mathcal{D}_J} q^{-l(d)} T_{d^{-1}} f T_d,$$

for $f \in \mathrm{Hom}_{\mathcal{H}_J}(U,V)$.

This norm map is a special case of the norm map, which Hoefsmith and Scott introduced in the late seventies in an unpublished manuscript. Homomorphisms in the image of $\mathcal{N}_J^S$ are called *induced.* We have now Higman's criterion which states that an indecomposable $\mathcal{H}$-module M is $\mathcal{H}_J$-projective precisely if the identity map on M is induced from an $\mathcal{H}_J$-linear endomorphism of M.

In [12] this was applied to Hecke algebras of type A, that is to Hecke algebras associated with symmetric groups $\mathfrak{S}_n$. Here we were in particular interested in socalled Young modules. Those correspond in the classical case to trivial source modules. They are precisely the indecomposable direct summands of the q-tensor space ([20]) on which $\mathcal{H}$ acts. The centralizer algebra of this action is the q-Schur algebra ([19]) and one can use Higman's lemma to translate the vertex and source theory for trivial source modules into results on q-Schur algebras. This was carried out in [12]. This reformulation in q-Schur algebras yields a certain algebra homomorphism between q-Schur algebras, called the Brauer homomorphism. This generalizes ideas of Scott [29], and the Brauer homomorphism turns out to be the dual to a Frobenius endomorphism of the associated quantized coordinate ring of general linear groups ([11], [28]). As a consequence, one gets a quantized version of Steinberg's tensor product theorem, [28], [12]. We shall come back to this later. We don't want to get into to much detail of the connection between q-Schur algebras and quantum groups here. In [11] it was shown that quantum GL_n (or rather its dual) is built from q-Schur algebras. A survey on that subject and its connections with representations of finite general linear groups in non-describing characteristic was given in [8].

We come back to the general case of an arbitrary Hecke algebra. Ingredient 1.5 carries over immediately again, with the understanding that we use for conjugation only distinguished double coset representatives.

We want to interrelate the results of section one with our results here. So let G be a finite group of Lie type and R of characteristic $p \geq 0$ and different from the describing characteristic of G. Again as Mackey system we choose the system $\mathcal{L}$ of Levi subfactors of G as defined in 1.9. We fix a Borel subgroup B with maximal split torus T, take $N = N_G(T)$ and $W = N/T$, the Weyl group of G. Our data determine a basis of the associated root system, and thus the set S of generators of W, consisting of basic reflections. For simplicity we restrict ourselves to non twisted groups G. Let the finite field $F = GF(q)$ be the field of definition of G, and set $\mathcal{H} = \mathcal{H}_{R,q}(W)$. The trivial B-module (over R) is denoted by R_B:

The key fact for the connection between representations of G and those of $\mathcal{H}$ is the following result of Iwahori [26] (and its extension due to Howlett and Lehrer [24]):

Theorem 2.3 (Iwahori 1964) *Keep the notation introduced above. Assume that* $R = \mathbb{C}$ *is the complex field. Then the endomorphism ring of the permutation representation* $\mathrm{Ind}_B^G(\mathbb{C}_B)$ *of* G *on the cosets of* B *is the Hecke algebra* $\mathcal{H} = \mathcal{H}_{\mathbb{C},q}(W)$ *associated with the Weyl group* W *of* G.

Obviously, since we are dealing with a permutation representation here, the restriction of R to the complex numbers $\mathbb{C}$ is not really necessary. Result 2.3 is true for every commutative ring R. However regarding the extension of 2.3 in [25] we have a different story. Howlett and Lehrer showed there, that the endomorphism algebra of an $\mathbb{C}G$-module M is a Hecke algebra associated with a certain (almost) reflection group, provided M has the form $M = R_L^G(N)$, where L is a Levi subgroup of G (considered as subfactor by choosing a parabolic subgroup P of G, of which L is the Levi complement), and N is a cuspidal irreducible $\mathbb{C}\mathbb{L}$-module. This result carries over to sufficiently nice commutative rings R under certain additional hypotheses (see [5], [6] for the case of finite general linear groups, and [15], [16] in general). To keep things simple we shall stick here to the special case of the theorem 2.3. In the next section however we shall give some more details in the general situation for the type A case.

Thus we take now $M = \mathrm{Ind}_B^G(R_B) = \mathrm{R}_T^G(R_T)$. We assume that R is one of the rings in a split p-modular system $(k, \mathcal{O}, K)$ for G. So $\mathcal{O}$ is a discrete complete valuation ring with quotient field K of characteristic zero, and residue field k of characteristic p. We assume that p is not the describing characteristic. Viewing M as left $\mathcal{H}$-module we have a functor

$$\check{\mathrm{H}} = \check{\mathrm{H}}_{R,q} : \mathrm{mod}\,\mathcal{H} \to \mathrm{mod}\,RG : V \mapsto V \otimes_{\mathcal{H}} M.$$

In [8, 2.9] it was shown that there is a left inverse functor

$$\mathrm{H} = \mathrm{H}_{R,q} : \mathrm{mod}\,RG \to \mathrm{mod}\,\mathcal{H}.$$

Thus

$$\mathrm{H} \circ \check{\mathrm{H}} = id_{\mathrm{mod}\,\mathcal{H}}.$$

A general theory of such a situation is investigated in [6], its applications to finite reductive groups and especially general linear groups will appear in [10]. In the survey article [8] the reader may find more details and further applications. Obviously these functors allow us to consider the module category $\mathrm{mod}\,\mathcal{H}$ of the Hecke algebra as part (direct summand) of the module category $\mathrm{mod}\,RG$. Thus it is natural to ask, how modules in this part of $\mathrm{mod}\,RG$ and their origin in $\mathrm{mod}\,\mathcal{H}$ relate concerning algebraic invariants, as for example vertices and sources. Here is a result which relates HC-induction and HC-restriction in G with induction and restriction in $\mathcal{H}$, where $\check{\mathrm{H}}_J : \mathrm{mod}\,\mathcal{H}_J \to \mathrm{mod}\,RL_J$ is the functor above for the Levi subgroup L_J, with $J \subseteq S$, viewing L_J as finite group of Lie type in its own right.

Theorem 2.4 *Keeping our notation we have two commutative diagrams:*

$$\begin{tikzcd} \mathrm{mod}\,\mathcal{H} \arrow[r, "\check{\mathrm{H}}"] & \mathrm{mod}\,RG \\ \mathrm{mod}\,\mathcal{H}_J \arrow[u, "\mathrm{R}_{\mathcal{H}_J}^{\mathcal{H}}"] \arrow[r, "\check{\mathrm{H}}_J"'] & \mathrm{mod}\,RL_J \arrow[u, "\mathrm{R}_{L_J}^G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{mod}\,\mathcal{H} \arrow[r, "\check{\mathrm{H}}"] \arrow[d, "\mathrm{T}_{\mathcal{H}_J}^{\mathcal{H}}"'] & \mathrm{mod}\,RG \arrow[d, "\mathrm{T}_{L_J}^G"] \\ \mathrm{mod}\,\mathcal{H}_J \arrow[r, "\check{\mathrm{H}}_J"'] & \mathrm{mod}\,RL_J \end{tikzcd}$$

Thus

$$\check{\mathrm{H}} \circ \mathrm{R}_{\mathcal{H}_J}^{\mathcal{H}} = \mathrm{R}_{L_J}^G \circ \check{\mathrm{H}}_J, \textit{ and } \check{\mathrm{H}}_J \circ \mathrm{T}_{\mathcal{H}_J}^{\mathcal{H}} = \mathrm{T}_{L_J}^G \circ \check{\mathrm{H}}.$$

A proof of theorem 2.4 as well as of the following consequence will appear in [14]:

Theorem 2.5 *Assume that R is of characteristic $p \geq 0$ not dividing q, where $GF(q)$ is the field of definition of G. Let N be an $\mathcal{H}$-module and $M = \check{\mathrm{H}}(N) \in \mathrm{mod}\,(RG)$. Then M is L_J-projective (with respect to $\mathcal{L}$) if and only if N is $\mathcal{H}_J$-projective, for $J \subseteq S$. So in particular if M is indecomposable, then N is indecomposable too, and L_J is HC-vertex of M precisely, if $\mathcal{H}_J$ is vertex of N.*

Thus determining vertices for $\mathcal{H}$-modules yields HC-vertices for the corresponding RG-modules under the functor $\check{\mathrm{H}}$. The obvious question is, if the image under the functor $\check{\mathrm{H}}$ describes a substantial portion of module category of RG. The answer is no, but fortunately there are many such functors. We do not want to go into details here, but give only some general indication how those functors arise.

For fields R of characteristic zero, e.g. $R = \mathbb{C}$, the indecomposable RG-modules are precisely the irreducible ones, and $\mathcal{L}$-vertices and semisimple $\mathcal{L}$-vertices coincide. Thus procedures 1.12 and 1.14 yield the same partition of the ordinary irreducible representations (or characters) of G into Harish-Chandra series.

We consider G now in the framework of algebraic groups, namely as set of fixed points under some Frobenius endomorphism $\mathcal{F}$ of an algebraic reductive group $\tilde{G}$ over the algebraic closure of $GF(q)$. For simplicity we assume that $\tilde{G}$ has connected center.

There is another partition of the irreducible complex characters of G into series, socalled geometric conjugacy classes. Those are labelled by semisimple conjugacy classes of the dual group G^*. Given a representative $s \in G^*$ of such a class, the corresponding geometric conjugacy class $\mathcal{E}(G,(s))$ consists of the constituents of the generalized characters $\mathrm{R}^G_{\tilde{T}^*}(s)$. Here $(s) = s^{G^*}$ is the conjugacy class of s, $\tilde{T}^*$ is an $\mathcal{F}$-stable maximal torus of $\tilde{G}^*$ containing s, and $\mathrm{R}^G_{\tilde{T}^*}$ is the Deligne-Lusztig operator [4]. We remark that Harish-Chandra induction with respect to $\mathcal{L}$ is indeed a special case of a functorial version of (generalized) Deligne-Lusztig operators, justifying the notation. For the definition of the dual group G^* and further details of the ordinary representation theory of finite groups of Lie type we refer to [1].

We take now a Levi subgroup L of G, and $s \in L^*$ semisimple. Let $\tilde{T}^*$ be an $\mathcal{F}$-stable maximal torus of $\tilde{L}^*$ which is not contained in any proper parabolic subgroup of $\tilde{G}$. Moreover we assume that the centralizer $C_{\tilde{G}^*}(s)$ of s in $\tilde{G}$ is $\tilde{T}^*$. Then $\mathrm{R}^L_{\tilde{T}^*}(s)$ is up to the sign the character of an irreducible cuspidal RL-module N, [15, 4.4]. But $\mathrm{R}^G_{\tilde{T}^*}(s) = \mathrm{R}^G_L(\mathrm{R}^L_{\tilde{T}^*}(s))$, hence the characters of the constituents of $\mathrm{R}^G_L(N)$ belong to the geometric conjugacy class $\mathcal{E}(G,(s))$. Here we have a functor $\check{\mathrm{H}} = \check{\mathrm{H}}_{\mathbb{C},\shortparallel,\sim} : \mathrm{mod}\,\mathcal{H}_s \to \mathrm{mod}\,\mathbb{C}\mathbb{G}$, coming as the original functor $\check{\mathrm{H}}$ defined above as tensoring with a suitable bimodule, where $\mathcal{H}_s = \mathcal{H}_{\mathbb{C},\shortparallel,\sim}$ is a Hecke algebra associated with a certain reflection group. For $s = 1$ we have $\check{\mathrm{H}} = \check{\mathrm{H}}_1$.

In the p-modular case we need additional hypotheses. Thus let $(k, \mathcal{O}, K)$ be a split p-modular system for G, and let R one of these rings. The prime p is asumed not to divide q as usual. We split the element s in its p-regular part s' and its p-singular part y. Thus s' has order not divivible by p, and y is a p-element. Moreover $s = s'y = ys'$. We may assume that $L = L_J$ for an $J \subseteq S$. The torus $\tilde{T}^*$ is a maximal split torus in the algebraic group $\tilde{G}^*$ hence the Weyl group W of $\tilde{G}^*$ (which is the same as the Weyl group of $\tilde{G}$) acts on it. We define

$$W_s = \{w \in W \mid wJ = J, ws =_L s\},$$

where the symbol '$=_L$' means 'conjugate in L'. It was shown in [16] that one has a functor

$\check{\mathrm{H}}_s = \check{\mathrm{H}}_{R,q,s}$ from mod $\mathcal{H}_{R,q,s}$ to mod RG, where $\mathcal{H}_{R,q,s}$ is a Hecke algebra associated with W_s defined above, which is indeed always a reflection group. This Hecke algebra might however involve varying powers of q. The functors defined above a compatible with 'reduction modulo p'. We remark that the additional hypothesis is trivially satisfied if s is a p-regular element. This is important since by a theorem of Geck and Hiss ([21]) the constituents of reductions mod p of the irreducible $\mathbb{C}G$-modules affording characters in $\mathcal{E}(G,(s))$ are precisely the irreducible kG-modules in a collection of p-blocks of G. We see that the images of all these functors of the form $\check{\mathrm{H}}_{R,q,s}$ contain a substantial part of the category of RG-modules. Theorems 2.4 and 2.5 carry over and provide a tool to calculate Harish Chandra vertices for modules in one of those images. [16] contains an extension of these results to groups, whose center is not necessarily connected.

For the special case of general linear groups G, geometric conjugacy classes and Harish-Chandra series coincide. Every irreducible RG-module, ($R = k$ or K), comes as the image of an irreducible Hecke algebra module under one of the functors above. However there are other important classes of indecomposable RG-modules, whose HC-vertex can be calculated using 2.5. For example there are the trival HC-source modules, that are the indecomposable direct summands of $\mathrm{R}_{L_J}^G(R_{L_J})$ for some $J \subseteq S$ or equivalently the indecomposable direct summands of the permutation representation of $G = GL_n(q)$ on the cosets of a Borel subgroup B of G (e.g. the subgroup of all invertible upper triangular matrices). The corresponding indecomposable $\mathcal{H}$-modules are precisely the indecomposable direct summands of q-tensor space viewed as $\mathcal{H}$-module. Their vertices were determined in [12] and they are given by l-p-adic decomposition of weights. For more details we refer to [9].

So far we dealt with HC-vertices and sources. In the last section we shall determine the semisimple vertices and sources of the irreducible kG-modules for finite general linear groups G.

3 General linear groups

Throughout this section G is a finite general linear group $GL_n(q)$. We take a split p-modular system $(k, \mathcal{O}, K)$, where p is a prime not dividing q, and if not stated otherwise, R is always one of those rings. We remark that $G^* = G$, and that for $R = K$ the Harish-Chandra series coincide with the geometric conjugacy classes. These are parametrized by conjugacy classes of semisimple elements of G. We take from each class a representative s. In the notation introduced in the last section, we have associated with s a Levi subgroup L_s of G, and an irreducible cuspidal KL_s-module C_s. The geometric conjugacy class $\mathcal{E}(G,(s))$ consists precisely of the characters afforded by the constituents of $M_s = \mathrm{R}_{L_s}^G(C_s)$. Moreover if $s \in G$ is a semisimple p-regular element then

$$\mathcal{E}_0(s) = \bigcup_y \mathcal{E}(G,(sy)),$$

where y runs through a set of G-conjugacy classes of p-elements of the centralizer of s in G, is a complete set of irreducible non-isomorphic characters in a collection of blocks.

To each geometric conjugacy class $\mathcal{E}(G,(s))$ we have three functors $\check{\mathrm{H}}_{R,s}$, corresponding to the three possibilities $R = k, \mathcal{O}, K$. One can identify the Weyl group W of G with the set $\mathfrak{S}_n$ of permutation matrices, and L can be choosen to be of the form

$$L = GL_{n_1}(q) \times GL_{n_2}(q) \times \ldots,$$

naturally embedded into G, (so $n_1 + n_2 + \ldots = n$). We take $\tilde{W}_L$ to be the stabilizer of L in W. Let s' be the p-regular part of s, as in the last section. We assume for simplicity that $W_s = W_{s'}$, where W_s consists of all elements of $\tilde{W}_L$ which centralize s. In this case the three functors are compatible, that is $R \otimes_{\mathcal{O}} \check{\mathrm{H}}_{\mathcal{O},s} = \check{\mathrm{H}}_{R,s}$ for $R = k$ or K, and they take $\mathcal{H}_{R,s}$-modules to RG-modules (in $\mathcal{E}_0(s')$). Here $\mathcal{H}_{R,s}$ is the Hecke algebra associated with the Weyl group of $C_G(s)$. Observing that

$$C_G(s) = GL_{k_1}(q^{d_1}) \times GL_{k_2}(q^{d_2}) \times \ldots,$$

we get

$$\mathcal{H}_{R,s} = \mathcal{H}_{R,q^{d_1}}(\mathfrak{S}_{k_1}) \times \mathcal{H}_{R,q^{d_2}}(\mathfrak{S}_{k_2}) \times \ldots.$$

If s does not satisfy the condition above, the three functors are not compatible and one has to take special precautions. Thus we shall assume this condition to hold for the semisimple elements which we are dealing with.

It was shown in [19] that associated with $\mathcal{E}(G,(s))$ is a q-Schur algebra (coming from $\mathcal{H}_{R,s}$ of course) and a functor similar to our $\check{\mathrm{H}}_{R,s}$, taking q-Schur algebra modules to RG-modules in the collection $\mathcal{E}_0(s')$ of p-blocks of G. For details we refer [8]. We collect the facts which we shall need: These functors take irreducible q-Schur algebra modules into irreducible RG-modules for $R = k$ or K. In this way one gets all irreducible KG-modules sorted out in Harish-Chandra series. One also gets all irreducible kG-modules, but with duplications: It turns out that already those coming from semisimple p-regular classes of G suffice. Note that for semisimple p-regular elements s of G the condition above is trivially satisfied. Restricting to these indeed produces a complete list of non-isomorphic irreducible kG-modules.

But the q-Schur algebras contain much more information. For example the decomposition matrices of these describe the decomposition numbers of the irreducible KG-modules in the corresponding geometric conjugacy classes $\mathcal{E}(G,(s))$, however with the 'wrong' labelling, if s is not p-regular. Thus the question arises, how the irreducible kG-modules in $\mathcal{E}(G,(s))$ can be identified with those in $\mathcal{E}(G,(s'))$, where again s' is the p-regular part of s. Here we say the irreducible kG-module S *belongs* to $\mathcal{E}(G,(s))$ if it is in the image of the functor associated with the corresponding q-Schur algebra. This identification was carried out the first time in [18] before the q-Schur algebras were discovered. In fact the identification is a version of Steinberg's tensor product theorem for the non-describing characteristic case, coming from a tensor product theorem for q-Schur algebras, or really for quantum GL_n. This in turn is closely related to the theory of vertices and sources for the associated Hecke algebras. This connection was shown in [12] and a survey of it may be found in [9].

The irreducible kG-modules in $\mathcal{E}(G,(s))$ are labelled by pairs (s,λ), where the *weights* λ parametrize the irreducible modules of the corresponding q-Schur algebra. Thus the irreducible kG-module $D(s,\lambda)$ associated with the pair (s,λ) is the image of the irreducible q-Schur algebra module of weight λ under the corresponding functor.

We shall describe now the tensor product theorem for the special case of the unipotent blocks of G, that is for $\mathcal{E}_0(1)$. Notationally it is much simpler than the general case. On the other hand it contains all information needed to guess the general result.

The geometric conjugacy class $\mathcal{E}(G,(1))$ consists of the unipotent characters, that is of the components of the permutation character on the cosets of a Borel subgroup B of G. The associated Hecke algebra is the Iwahori algebra

$$\mathcal{H}_{R,q}(\mathfrak{S}_n) = \mathrm{End}_{RG}(\mathrm{Ind}_B^G(R_B))$$

(compare 2.3). For $R = K$ the q-Schur algebra and Hecke algebra coincide (in general) and therefore the irreducible q-Schur algebra modules are parametrized by partitions λ of n, denoted by $\lambda \vdash n$. In general the decomposition matrices of q-Schur algebras are square matrices. In particular rows and columns can be labelled by the same combinatorial objects (vectors of partitions). Thus the irreducible kG-modules in $\mathcal{E}(G,(1))$ are of the form $D(1,\lambda)$, where the weight λ is a partition of n.

Let $D(\lambda) = D(1,\lambda)$, $\lambda \vdash n$. We denote the multiplicative order of q modulo p by d, thus q is an dth root of unity in k. We can decompose λ as follows:

$$\lambda = \lambda_{-1} + d\lambda_0 + dp\lambda_1 + \ldots + dp^m l_m \tag{1}$$

such that λ_{-1} is a d- and λ_i is p-restricted partition of some number n_{-1} respectively n_i for $0 \leq i \leq m$. Here a partition is e-restricted for some natural number e if the dual partition is e-regular, that is is of the form $(1^{r_1}, 2^{r_2}, \ldots)$ with $r_i \lneq e$ for all i. Moreover

$$n = n_{-1} + \sum_{i=0}^{m} dp^i n_i. \tag{2}$$

We call such a decompositon of λ *d-p-adic* and remark that it is uniquely determined.

We construct now a p-element y of G as follows: First we set $y_{-1} = 1$ and choose y_i to be a p-element of maximal order in $GL_{dp^i}(q)$. Then we define $y \in G$ as matrix direct sum:

$$y = \prod_{i=-1}^{m} y^{\times n_i}. \tag{3}$$

We note that equation (2) implies that y is indeed an element of G. One version of Steinberg's tensor product theorem simply states:

Theorem 3.1 *Keeping the notation introduced above there is an isomorphism between the irreducible kG-modules $D(1,\lambda)$ and $D(y,\bar{\lambda})$, where $\bar{\lambda}$ is the vector $(\lambda_{-1}, \lambda_0, \ldots, \lambda_m)$ of partitions given by the d-p-adic decomposition (1) of λ.*

This version of the tensor product theorem is different from the ones given in [8] and [9]. A proof of all versions will appear in [14]. We remark that a theorem similar to 3.1 holds for irreducible kG-modules belonging to geometric conjugacy classes $\mathcal{E}(G,(\tilde{y}))$ for other p-elements $\tilde{y}$ of G. This can be used to derive the 'identification' of irreducible kG-modules in the various geometric conjugacy classes with the same semisimple part.

We want to determine the semisimple vertex and source if the irreducible kG-module $D(1,\lambda)$. We note that y is actually element of the Levi subgroup L_1 of G defined as

$$L_1 = GL_1(q)^{\times n_{-1}} \times GL_d(q)^{\times n_0} \times \ldots \times GL_{dp^m}(q)^{\times n_m}. \tag{4}$$

The kL_1-module N_λ is irreducible and cuspidal:

$$N_\lambda = k_{GL_1(q)}^{\otimes n_{-1}} \otimes D(y_1,(1))^{\otimes n_0} \otimes \ldots \otimes D(y_m,(1))^{\otimes n_m}. \tag{5}$$

We define the Levi subgroup L_2 by:

$$L_2 = GL_{n_{-1}}(q) \times GL_{dn_0}(q) \times \ldots \times GL_{dp^m n_m}(q). \tag{6}$$

Then L_1 is a Levi subgroup of L_2 and we may consider the kL_2-module $M = \mathrm{R}_{L_1}^{L_2}(N_\lambda)$. Then it turns out that the following result holds:

Lemma 3.2 *The irreducible kL_2-module*

$$M_\lambda = D(1,\lambda_{-1}) \otimes D(y_0,\lambda_0) \otimes \ldots \otimes D(y_m,\lambda_m)$$

is a composition factor of the head of $M = \mathrm{R}_{L_1}^{L_2}(N_\lambda)$.

From procedure 1.14 and transitivity of Harish-Chandra induction we conclude immediately:

Lemma 3.3 *L_1 is the semisimple vertex and N_λ the semisimple source of M_λ.*

Here is another version of the tensor product theorem (compare [8] and [9]):

Theorem 3.4 (Steinberg's tensor product theorem) *Keeping our notation we have:*

$$D(\lambda) = D(1,\lambda) = \mathrm{R}_{L_2}^{G}(M_\lambda) = \mathrm{R}_{L_2}^{G}(D(1,\lambda_{-1}) \otimes \ldots \otimes D(y_m,\lambda_m)).$$

We have here a real tensor product theorem: We decompose the irreducible module according to the d-p-adic decomposition of its weight. The Frobenius twist turns into the choice of the p-element y.

Theorem 3.4 has some immediate consequences:

Corollary 3.5 *The semisimple vertex of M_λ hence of $D(\lambda)$ is L_1 and its semisimple L_1-source is N_λ.*

Similarly one determines the semisimple vertices and sources of arbitrary irreducible kG-modules.

Corollary 3.6 *The $\mathcal{L}$-vertex of $D(\lambda)$ is a a Levi subgroup of L_2 and contains L_1.*

We can now apply the first version 3.1 of the tensor product theorem to express the modules M_λ and N_λ again as irreducible modules in $\mathcal{E}(L_2,1)$ respectively in $\mathcal{E}(L_1,1)$

Theorem 3.7 *Let $\lambda \vdash n$. Then*

$$N_\lambda = k_{GL_1(q)}^{\otimes n_{-1}} \otimes D(1,(1^d))^{\otimes n_0} \otimes \ldots \otimes D(1,(1^{dp^m}))^{\otimes n_m},$$

and

$$M_\lambda = D(1,\lambda_{-1}) \otimes D(1,d\lambda_0) \otimes \ldots \otimes D(1,dp^m\lambda_m).$$

We have seen that the tensor product theorem 3.4 is a tool to determine the semisimple vertex of an irreducible kG-module. However from the discussion of the tensor product theorem in [9] it is not suprising that this comes up in Harish-Chandra vertex theory. After all it was shown there, that 3.4 (and all the other versions of 3.4) are based on the vertex theory in Hecke algebras, and this can be translated into tensor product theorems for q-Schur algebras. Finally we have seen in 2.4 that vertices for Hecke algebras correspond to Harish-Chandra vertices in G. Thus it is not so suprising that the tensor product theorems play a central role for the determination of the Harish-Chandra series of irreducible kG-modules.

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ON TILTING MODULES AND INVARIANTS FOR ALGEBRAIC GROUPS

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1. Polynomial invariants of matrices and of representations of quivers

The theme of this article is the calculation and description of generators of polynomial invariants for groups actions by means of trace functions. In this section we give our main applications of this point of view. In the second section we discuss the general framework and also the beautiful theory of conjugacy classes in algebraic groups due to Steinberg, where trace functions play an important role. This serves as as model for our approach in similar situations. In the third section we describe various results on tilting modules and saturated subgroups of algebraic groups. In Section 4 we explain how part of Steinberg's set-up may be adapted to the action by conjugation of a closed subgroup H of G, in the framework of "group pairs". The main result here is that the class functions relative to H are trace functions determined by tilting modules. We conclude with some appendices. In the main body of the text we have omitted all but the simplest proofs. Instead references to the literature are given for proofs where appropriate and to the Appendices (which are of a somewhat technical nature) where no reference is available.

Characteristic 0 readers are warned that we always have in mind a field of characteristic $p > 0$ in what follows (except where this is explicitly excluded) : though most of the results described here make perfect sense in characteristic zero many of them in this case are either well known or trivial (or both).

We begin with our applications. Let k be an algebraically closed field of arbitrary characteristic. If θ is a square matrix or an endomorphism of a finite dimensional vector space and $s \geq 1$ we denote by $\chi_s(\theta)$ the trace of $\bigwedge^s \theta$. (Thus, up to sign, the $\chi_s(\theta)$'s are coefficients of the characteristic polynomial of θ.) Let $M(n)$ be the set of $n \times n$ matrices over k viewed as affine n^2-space $\mathbb{A}^{n^2}$. Let $X = M(n)^m = M(n) \times M(n) \times \cdots \times M(n)$. Thus $k[M(n)^m]$ is the free polynomial algebra $k[c_{rij} : 1 \leq r \leq m, 1 \leq i,j \leq n]$, where c_{rij} is the function which takes $x = (x_1, x_2, \ldots, x_m)$ to the (i,j)-entry of the matrix x_r. The general linear group $G = \mathrm{GL}(n)$ over k acts on X by simultaneous conjugation : $g \cdot (x_1, x_2, \ldots, x_m) = (gx_1g^{-1}, gx_2g^{-1}, \ldots, gx_mg^{-1})$.

Theorem 1.1 *The algebra of invariants $k[M(n)^m]^{\mathrm{GL}(n)}$ is generated by the functions $(x_1, x_2, \ldots, x_m) \mapsto \chi_s(x_{i_1}x_{i_2}\ldots x_{i_r})$, for $r, s \geq 1$ and $1 \leq i_1, i_2, \ldots, i_r \leq m$.*

The theorem asserts that the algebra of invariants is generated by the coefficients of the characteristic polynomials of monomials in $x_1, x_2, \ldots, x_m$. (Though the result is quite satisfying the alternative would have been very exciting. The orbits of $\mathrm{GL}(n)$ on $M(n)^m$ correspond to the isomorphism classes of representations of of degree n of the free non-commutative algebra $k\langle X_1, X_2, \ldots, X_m\rangle$. So if the theorem were not true there would have to exist some "universal" polynomial invariant

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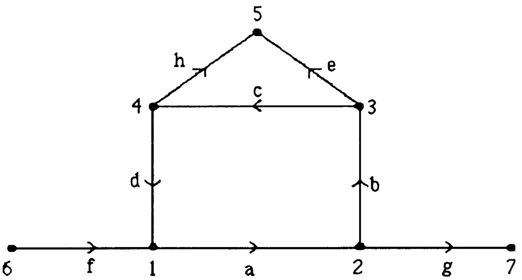
not derived from the trace function which would separate isomorphism classes of modules. Perhaps, therefore, the theorem may be regarded as an explanation of the ubiquitous role of the trace function in representation theory.) In characteristic 0 the result was obtained independently by Sibirski, [37] and Procesi, [33]. For arbitrary characteristic see [18].

We now describe a generalization of Theorem 1.1. By a quiver we mean a quadruple $Q = (V, A, h, t)$, consisting of the vertex set $V = \{1, 2, \ldots, n\}$, a finite set A of arrows and maps $h : A \to V$, $t : A \to V$ which assign to an arrow $a \in A$ its head, $h(a)$, and tail, $t(a)$.

Let $E_1, E_2, \ldots, E_n$ be finite dimensional k-vector spaces. Let $\alpha_i = \dim_k E_i$, $1 \le i \le n$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We write $\mathrm{GL}(\alpha)$ for $\mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_n)$. Then $R(Q,\alpha) = \prod_{a\in A} \mathrm{Hom}_k(E_{t(a)}, E_{h(a)})$ is the space of all k-representations of Q on the spaces $E_1, E_2, \ldots, E_n$. Now $\mathrm{GL}(\alpha)$ acts on $R(Q,\alpha)$ by $g \cdot (y_a)_{a\in A} = (g_{h(a)} y_a g_{t(a)}^{-1})_{a\in A}$, for $g = (g_1, g_2, \ldots, g_n) \in \mathrm{GL}(\alpha)$ and $(y_a)_{a\in A} \in R(Q,\alpha)$. Let $\theta_i \in \mathrm{End}_k(E_i)$ and let $\mathrm{GL}(E_i)_{\theta_i}$ be the centralizer of θ_i in $\mathrm{GL}(E_i)$, $1 \le i \le n$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathrm{End}_k(E_1) \oplus \mathrm{End}_k(E_2) \oplus \cdots \oplus \mathrm{End}_k(E_n)$ and let $\mathrm{GL}(\alpha)_\theta = \mathrm{GL}(E_1)_{\theta_1} \times \mathrm{GL}(E_2)_{\theta_2} \times \cdots \times \mathrm{GL}(E_n)_{\theta_n}$.

Theorem 1.2 *The algebra of invariants $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)_\theta}$ is generated by the functions $(y_a)_{a\in A} \mapsto \chi_s(\theta_{h(a_r)}^{q_r} y_{a_r} \cdots \theta_{h(a_2)}^{q_2} y_{a_2} \theta_{h(a_1)}^{q_1} y_{a_1})$, for $(a_1, a_2, \ldots, a_r)$ an oriented cycle, $q_1, q_2, \ldots, q_r \ge 0$ and $s \ge 0$.*

In particular, the $\mathrm{GL}(\alpha)$-invariants are generated by the coefficients of the characteristic polynomials of products over cycles, a result of Le Bruyn and Procesi in characteristic 0, [29]. Theorem 1.2 is deduced from a version of Theorem 1.1 in [21]. (We recover Theorem 1.1 by taking Q to be a quiver having only one vertex (with several loops) and $\mathrm{GL}(\alpha)_\theta = \mathrm{GL}(\alpha)$, i.e. taking each θ_i to be multiplication by a scalar.) For example, if Q is the quiver



if $\dim E_1 = 13$ and $\dim E_i \ge 13$ for $2 \le i \le 4$ then $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)} = k[t_1, t_2, \ldots, t_{13}]$, the free polynomial algebra on $t_1, t_2, \ldots, t_{13}$, where $t_i(y) = \chi_i(y_d y_c y_b y_a)$, for $1 \le i \le 13$.

Question. For which quivers Q and dimension vectors α is $k[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ a free polynomial algebra?

2. Invariants and class functions

Let k be an algebraically closed field of arbitrary characteristic. Suppose that we have an algebraic group G over k acting on an affine variety X over k. Then G acts as algebra automorphisms of the coordinate algebra $k[X]$. A basic problem in invariant theory is to describe the structure of the algebra of invariants $k[X]^G$, i.e. the subalgebra of $k[X]$ consisting of all functions which are constant on G-orbits. In general we know, thanks to Haboush's proof of the Mumford Conjecture, [25], that if G is reductive then $k[X]^G$ is finitely generated — in other words Hilbert's 14th problem has a positive solution in this case. A reasonable starting point for our problem would therefore be to find explicit generators for the algebra of invariants $k[X]^G$. The inclusion map $k[X]^G \to k[X]$

corresponds to a morphism of affine varieties $\pi : X \to Z$, where $Z = \mathrm{Spec}(k[X]^G)$ is the (categorical) quotient. The map π is constant on G-orbits — the fibres of $\pi : X \to Z$ are the equivalence classes for the relation $x \sim y$ if $f(x) = f(y)$ for every $f \in k[X]^G$ — and a first step in describing the orbit structure of X might be a study of the fibres of π.

We start with a brief review of certain aspects of Steinberg's theory of conjugacy classes in algebraic groups. In this case we have $X = G$ with G acting by conjugation. The theory is of central importance in the study of reductive algebraic groups, but there are two special reasons for discussing it here. In the first place it is, for the author, the paradigm of the group action, with enough detail to illustrate many of the features that crop up in general, but enough structure to make one feel that the situation is under control. The second reason is that the description of invariants as trace functions is our model for the description of invariants by "shifted trace functions" in the slightly more general set-up described in Section 4.

An element s of G is called semisimple if $R(s)$ is diagonalizable for some (and hence every) faithful representation $R : G \to \mathrm{GL}(V)$ of G. An element u of G is called unipotent if $R(u) - R(1)$ is nilpotent for some (and hence every) faithful representation R of G. An arbitrary element x of G may be written uniquely $x = x_s x_u$ as a product of commuting semisimple and unipotent elements x_s and x_u (the Jordan decomposition.) We denote by $R_u(G)$ the unipotent radical of G, i.e. the largest normal, connected subgroup of G which consists of unipotent elements. We now suppose that G is reductive, i.e. G is connected and $R_u(G) = \{1\}$.

We now briefly describe the standard framework of rational representation theory of G. For further details see, for example, [26]. If V is a finite dimensional kH-module, where H is an algebraic group over k, we call V *rational* if the corresponding representation $R : H \to \mathrm{GL}(V)$ is a morphism of algebraic groups. A kH-module of arbitrary dimension is rational if it is the union of its finite dimensional rational submodules. Let $\mathrm{mod}\, H$ denote the category of finite dimensional rational H-modules.

Let T be a maximal torus of G, that is T is a closed connected subgroup consisting of semisimple elements and maximal with respect to these conditions. Let r be the dimension of T as an algebraic variety. Then T is isomorphic to a direct product of r copies of $k^\times$, the multiplicative group of k, and any two maximal tori are conjugate in G. The rank of G is, by definition, r. The character group $X(T)$ of T is the abelian group of all algebraic group homomorphisms from T to $k^\times$. Thus $X(T)$ is free abelian of rank r. A rational T-module decomposes $V = \oplus_{\lambda \in X(T)} V^\lambda$ as a direct sum of weight spaces, where $V^\lambda = \{v \in V : tv = \lambda(t)v \text{ for all } t \in T\}$. We say that $\lambda \in X(T)$ is a weight of V if $V^\lambda \neq 0$. The non-zero weights of the Lie algebra $\mathrm{Lie}(G)$, under the adjoint action, are called roots; we denote by Φ the set of roots of G with respect to T. Let $W = N_G(T)/T$ be the Weyl group of G, with respect to T, and choose a positive-definite, symmetric, W-invariant, bilinear form $(\ ,\ )$ on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$. Then $(\mathbb{R} \otimes_{\mathbb{Z}} X(T), (\ ,\ ), \Phi)$ is a root system. Let B be a Borel subgroup (i.e. a maximal connected solvable subgroup) containing the maximal torus T. Then $B = T.U$, the semidirect product of T and $U = R_u(B)$. Let Φ^- be the set of non-zero weights of $\mathrm{Lie}(B)$ and let $\Phi^+ = \{-\alpha : \alpha \in \Phi^-\}$ — the system of positive roots in Φ for which B is the negative Borel subgroup. We let w_o be the longest element of the Weyl group W and, for $\lambda \in X^+(T)$ write λ^* for $-w_0(\lambda)$. There is a natural partial ordering on the character group : we write $\lambda \leq \mu$, for $\lambda, \mu \in X(T)$ if the difference $\mu - \lambda$ is a sum of positive roots.

A parabolic subgroup of G is a closed subgroup containing a Borel subgroup. A parabolic subgroup is conjugate to a unique parabolic subgroup containing B and the parabolic subgroups containing B are in $1-1$ correspondence with the subsets of the set of simple roots. A parabolic subgroup P is the semidirect subgroup of its unipotent radical $R_u(P)$ and a closed subgroup G_P. Such a subgroup G_P is known as a Levi subgroup of G and any two such subgroups of P are conjugate in P. In the case $G = \mathrm{GL}(n)$ any Levi subgroup has the form $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \cdots \times \mathrm{GL}(n_l)$ (for positive integers $n_1, n_2, \ldots, n_l$ with $n = n_1 + n_2 + \cdots + n_l$) realized as the group of block matrices of shape $n_1, n_2, \ldots, n_l$. Furthermore any Levi subgroup in $\mathrm{GL}(n)$ is conjugate to one of this form.

Let $X^+(T) = \{\lambda \in X(T) : (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Phi\}$. For each $\lambda \in X(T)$ we have a one dimensional

T-module k_λ and we inflate this to a B-module, also denoted k_λ (on which U acts trivially). For each $\lambda \in X(T)$ we have the induced module $\mathrm{Ind}_B^G k_\lambda$. This is finite dimensional and can be 0. For $\lambda \in X^+(T)$ we set $\nabla(\lambda) = \mathrm{Ind}_B^G k_\lambda$. Then $\nabla(\lambda)$ has a simple socle $L(\lambda)$, say, for $\lambda \in X^+(T)$ and $\{L(\lambda) : \lambda \in X^+(T)\}$ is a complete set of inequivalent irreducible rational G-modules. The dual module $L(\lambda)^*$, for $\lambda \in X^+(T)$, is isomorphic to $L(\lambda^*)$.

In the case $G = \mathrm{GL}(n)$ we make the following choices. For T we take the subgroup consisting of the diagonal matrices. The abelian group $X(T)$ is free abelian on $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where $\epsilon_i : T \to k^\times$ takes a diagonal matrix to its (i,i)-entry, $1 \le i \le n$. We identify an n-tuple of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with the element $\sum_i \lambda_i \epsilon_i$. The normalizer $N_G(T)$ is the semidirect product of T with the group of monomial matrices. We identify a permutation of $\{1, 2, \ldots, n\}$ with the corresponding monomial matrix and in this way identify the symmetric group S_n of degree n with the Weyl group $W = N_G(T)/T$. The action of W on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ is given by $\sigma\epsilon_i = \epsilon_{\sigma(i)}$, for a permutation σ and $1 \le i \le n$. We take the bilinear form on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ such that $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ is an orthonormal basis. For B we take the subgroup consisting of the lower triangular matrices. Thus U is the group of lower unitriangular matrices. The root system Φ is $\{\epsilon_i - \epsilon_j : 1 \le i, j \le n, i \ne j\}$ and $\Phi^+ = \{\epsilon_i - \epsilon_j : 1 \le i < j \le n\}$. The natural partial order on $X(T)$ (sometimes known, somewhat confusingly, as the dominance partial order) is given by $\lambda = (\lambda_1, \ldots, \lambda_n) \le \mu = (\mu_1, \ldots, \mu_n)$ if $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$ and $\sum_{i=1}^j \lambda_i \le \sum_{i=1}^j \mu_i$ for all $1 \le j \le n$. Let E be the natural module, of column vectors of length n. Some of the induced modules are known already from other contexts : for example, for $r \ge 0$ the induced module $\nabla(r\epsilon_1)$ is isomorphic to $S^r(E)$, the rth symmetric power and, for $1 \le r \le n$ the induced module $\nabla(\epsilon_1 + \cdots + \epsilon_r)$ is isomorphic to $\bigwedge^r(E)$, the r-th exterior power.

We now take G to be semi-simple and simply connected, for example G could be the special linear group $\mathrm{SL}(n)$ or symplectic group $\mathrm{Sp}(2n)$. An element $\varpi \in X^+(T)$ is called a fundamental dominant weight if ϖ cannot be written as the sum of two non-zero dominant weights. The character group $X(T)$ is free abelian on the fundamental dominant weight $\varpi_1, \varpi_2, \ldots, \varpi_r$ and $X^+(T)$ is the set of integer linear combinations of fundamental dominant weights, with non-negative coefficients.

Let $C(G)$ be the algebra of regular class functions, i.e. the k-subalgebra of the coordinate algebra $k[G]$ consisting of functions $f : G \to k$ such that $f(xyx^{-1}) = f(y)$ for all $x, y \in G$. Thus $C(G)$ is the algebra of invariants for the action of G on G by conjugation. Let $V \in \mathrm{mod}\, G$ afford the representation $R : G \to \mathrm{GL}(V)$. As is well known, the trace (or character) of a representation is a class function, so we get $\chi_V \in C(G)$ defined by $\chi_V(g) = \mathrm{trace}(R(g))$, $g \in G$. Let $C_0(G)$ be the k-span of all trace functions χ_V, as V varies over all rational G-modules. For modules U, V the character $\chi_{U \otimes V}$ is the product $\chi_U \cdot \chi_V$ so that $C_0(G)$ is a subalgebra of $C(G)$. Further, characters are additive on short exact sequences so that $C_0(G)$ is spanned by the characters of the irreducible rational G-modules. Let χ_i be the trace function (or character) defined by $L(\varpi_i)$, $1 \le i \le n$. In the case $G = \mathrm{SL}(n)$ we may take $L(\varpi_i) = \bigwedge^i(E)$, $1 \le i < n$, so that the fundamental characters are, up to sign, coefficients of the characteristic polynomial. The starting point of Steinberg's theory is the following.

Theorem 2.1 *The algebra of class functions $C(G)$ is the free polynomial algebra on $\chi_1, \chi_2, \ldots, \chi_r$, in particular $C(G) = C_0(G)$.*

Thus the variety $\mathrm{Spec}(C(G))$ defined by $C(G)$ is affine r-space $\mathbb{A}^r$ and the canonical map (the Steinberg map) $\pi : G \to \mathrm{Spec}(C(G)) = \mathbb{A}^r$ is $\pi(g) = (\chi_1(g), \chi_2(g), \ldots, \chi_r(g))$, $g \in G$. Elements g and h of G lie in the same fibre of π if and only if g and h have the same trace at every representation of G. The fibre containing 1 is the variety of all unipotent elements of G. (This is the most difficult fibre and many problems concerning general fibres may be reduced to this case.) We now summarize some properties of the Steinberg map. For a G-variety X and $n \ge 0$ the set $X(n)$ of elements $x \in X$ such that the orbit $G \cdot x$ has dimension $\le n$ is closed in X. An element $x \in X$ is regular if the dimension of the orbit $G \cdot x$ is maximal among orbits of elements of X. Thus the set of regular elements is open in X. In the present case $X = G$, the regular elements are the group elements $x \in G$ such that the centralizer $Z_G(x)$ has the smallest possible dimension (which is the rank, r).

Theorem 2.2 *Let F be a fibre of the natural map $G \to \mathrm{Spec}(C(G))$.*
(i) F contains a unique closed orbit and this is exactly the set of semisimple elements in F.
(ii) If x is an element of F then F consists of the elements $y \in G$ such that y_s is conjugate to x_s.
(iii) F is irreducible and has codimension r in G.
(iv) The simple (i.e. non-singular) points of F are exactly the regular elements and these form a single orbit (i.e. conjugacy class). F is regular in codimension 1 (i.e the singular locus, consisting of all non-simple points, has codimension ≥ 2).
(v) F is a normal variety.
(vi) F consists of finitely many conjugacy classes.

There is a very closely related work of Kostant, [28]. In Kostant's paper G is a complex Lie group acting on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ via the adjoint representation and $\mathfrak{g}$ is analyzed, as a G-variety, as in the present context, via the natural map $\mathfrak{g} \to \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$. It is a result of Chevalley that $\mathbb{C}[\mathfrak{g}]^G$ is a polynomial algebra. All of the properties listed above hold in this case too. Moreover, over $\mathbb{C}$, one can pass between the group and Lie algebra via differentiation. (In this case Kostant proved that there are only finitely many nilpotent orbits by showing that each nilpotent element lies in a "principal three dimensional Lie algebra" (isomorphic to $\mathrm{sl}(2,\mathbb{C})$). Later, Richardson,[34], found a method for establishing finiteness on very general grounds. Richardson's method also established finiteness in the modular case for fields of "good" characteristic.)

Suppose, for the moment, that k is the algebraic closure of the finite field $\mathbb{F}_p$ of cardinality p. Then $g \in G$ is unipotent if and only if g is a p-element, i.e. an element whose order is a power of the prime p. An element $g \in G$ is semisimple if and only if g is a p'-element, i.e. the order of g is prime to p. The Jordan decomposition is then the product (well known in finite group theory and called the p-decomposition) $g = su$ of commuting elements s and u with s a p'-element and u a p-element. The elements s and u are called the p'-part and p-part of g. Thus the first assertion above becomes the following.

(1) *Elements of G belong to the same fibre of the natural (Steinberg) map $G \to \mathrm{Spec}(C(G))$ if and only their p'-parts are conjugate.*

All but the last part of Theorem 2.2 were demonstrated in Steinberg's celebrated paper "Regular elements in semisimple groups", [40], published in 1965. (An exposition is also given in [41].) The final assertion, however, had to wait for Lusztig's proof in 1976,[30]. Here the result was obtained by using the newly developed Deligne-Lusztig theory to bound the number of conjugacy classes of p-elements in finite groups of Lie type by complex character theory.

Example. It is interesting to compare Steinberg's theory with Brauer theory of finite groups. So now we shall take G to be a finite group and k a field (algebraically closed) of characteristic p. We can again consider the space $C(G)$ of all k-valued class functions on G. Thus the k dimension of $C(G)$ is the number of conjugacy classes of G and in this case the fibres of the natural map $G \to \mathrm{Spec}(C(G))$ are just the conjugacy classes of G.

We can also consider the subalgebra $C_0(G)$ spanned by the trace functions (i.e. the natural characters) of the finite dimensional kG-modules. Note that $C_0(G)$ has a basis consisting of the natural characters of a complete set of inequivalent irreducible kG-modules. (These span $C_0(G)$ since the character is additive on short exact sequences and characters of non-isomorphic irreducibles are independent, as is shown by a result of Frobenius-Schur, see [9],(27.13).)

As above, inclusion $C_0(G) \to k[G]$ (where $k[G]$ is the algebra of all k-valued functions on G) corresponds to a surjective morphism $G \to \mathrm{Spec}(C_0(G))$ and $x, y \in G$ belong to the same "Brauer fibre" if and only if x and y have the same trace on all finite dimensional kG-modules.

If $R : G \to \mathrm{GL}(V)$ is a representation of G and $g \in G$ has p-decomposition $g = su$ we get

$$\chi_V(g) = \mathrm{trace}(R(su)) = \mathrm{trace}(R(s)) + \mathrm{trace}(R(s)(R(u) - R(1)).$$

However, $u^q = 1$ where q is some power of p. Thus $(R(u) - R(1))^q = 0$ so that $(R(u) - R(1))$ is nilpotent. Since $R(s)$ commutes with $R(u)$ we get that $R(s)(R(u) - R(1))$ is nilpotent and therefore has trace zero. Thus $\chi_V(g) = \chi_V(s)$. Therefore $f(g) = f(s)$ for all $f \in C_0(G)$ and we have:

(2) *Restriction* : $C_0(G) \to C(G_0)$ *is injective.*

Here G_0 denotes the set of p'-elements of G and $C(G_0)$ denotes the algebra of functions on G_0 which are constant on conjugacy classes. The dimension of $C(G_0)$ is the number of p'-conjugacy classes. However, the restriction map is also surjective. A sketch proof of this is as follows. First note that this is so if G is a p-elementary group, i.e. a direct product of a p-group and a cyclic p'-group. For any subgroup H of G we have the induction functor Ind_H^G from H-modules to G-modules and this gives rise to a linear map $\mathrm{ind}_H^G : C_0(H) \to C_0(G)$ with the property that $\mathrm{ind}_H^G(\chi_V) = \chi_{\mathrm{Ind}_H^G V}$ for an H-module V. Now let s be a p'-element and take $H = \langle s \rangle \times Q$, where Q is a Sylow p-subgroup of the centralizer $Z_G(s)$ of s. Let $f \in C_0(H)$ be the function which is 1 on the coset sQ and 0 off it. It is easy to check that $\mathrm{ind}_H^G(f)$ is non-zero on the class of s and 0 on all p'-elements which are not conjugate to s. We get :

(3) *Restriction* : $C_0(G) \to C(G_0)$ *is an isomorphism and hence the number of inequivalent irreducible kG-modules is equal to the number of conjugacy classes of p'-elements.*

Thus we have arrived at a proof à la Steinberg of the famous theorem of Brauer. (This is of course similar to one of the standard proofs, more usually executed using Brauer characters, see e.g. [38], n^o 18.2.)

Note that, from our point of view of trace functions in invariant theory, this is a badly behaved example. Since $C_0(G) \neq C(G)$ for G finite of order divisible by p we cannot compute the class functions (i.e. the invariants for the conjugation action) representation theoretically by finding trace functions defined by kG-modules.

However, one can see from this example that several aspects of reductive groups have parallels in finite groups. (It follows from (3), by the way, that elements x, y in a finite group G belong to the same "Brauer fibre" if and only $R(x)$ and $R(y)$ belong to the same Steinberg fibre in $GL(V)$ for every representation $R : G \to \mathrm{GL}(V)$.) One has, by Theorem 2.2, (i) and (iv), a bijection between regular classes and semisimple classes in reductive groups. Unfortunately, there seems to be no parallel of this in finite groups and, indeed, no analogue in a finite group of the notion of a regular element in a reductive group.

3. Tilting modules and saturated subgroups

We now describe the theory of tilting modules and saturated subgroups, which will play a crucial role in the invariant theory for certain group pairs. We continue to denote by G a reductive group and use the notation developed above. The following elegant result of Cline, Parshall, Scott and van der Kallen, on the vanishing of certain Hochschild cohomology groups, is of great importance and practical value — it is a key ingredient in several of the results in this section and is perhaps the origin of the recent development of the theory of quasi-hereditary algebras.

Theorem 3.1 *For $\lambda, \mu \in X^+(T)$ we have*

$$H^i(G, \nabla(\lambda) \otimes \nabla(\mu)) = \begin{cases} k, & \text{if } \lambda = \mu^* \text{ and } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

For a proof see [8],(3.3) (or [11],(2.1.4) and [10], Proposition 2.2).

A filtration $0 = V_0 \leq V_1 \leq \cdots$ of a rational G-module is declared to be *good* if, for every $i \geq 1$, the section V_i/V_{i-1} is either 0 or isomorphic to $\nabla(\lambda_i)$ for some $\lambda_i \in X^+(T)$. A simple consequence of Theorem 3.1 is the following (see [10], Proposition 2.2 or [11],(12.1.1)).

Theorem 3.2 *Let V be a rational G-module which admits a good filtration. Then, for $\lambda \in X^+(T)$, the multiplicity $(V : \nabla(\lambda))$ of $\nabla(\lambda)$ as a section in a good filtration of V is independent of the choice of good filtration, and in fact equal to $\dim(V \otimes \nabla(\lambda^*))^G$.*

Part (i) of the following was proved by Wang,[42], when the characteristic p of k is large compared with the root system. The theorem was proved, for most p, by the author, [11] is now known in complete generality thanks to Mathieu,[32].

Theorem 3.3 *(i) For all $\lambda, \mu \in X^+(T)$ the G-module $\nabla(\lambda) \otimes \nabla(\mu)$ has a good filtration, for $\lambda, \mu \in X^+(T)$.*
(ii) The restriction $\nabla(\lambda)|_L$ has a good filtration, for any $\lambda \in X^+(T)$ and any Levi subgroup L of G.

We call $V \in \operatorname{mod} G$ a *tilting module* if both V and its dual V^* admit good filtrations. We have the following basic result of Ringel,[36].

Theorem 3.4 *(i) For each $\lambda \in X^+(T)$ there is an indecomposable $M(\lambda) \in X^+(T)$ which has unique highest weight λ and such that λ occurs with multiplicity one.*
(ii) An arbitrary tilting module in $\operatorname{mod} G$ is a direct sum of $M(\lambda)$'s ($\lambda \in X^+(T)$).

A couple of remarks are in order at this point. In the first place, in [36], Ringel is working within the context of finite dimensional algebras which are quasi-hereditary in the sense of Cline, Parshall and Scott and we are working with rational modules. The connection between these contexts may be described as follows. Let π be a finite subset of $X^+(T)$ which is saturated, i.e. π has the property that whenever $\lambda \in \pi$ and $\mu \in X^+(T)$ satisfies $\mu \leq \lambda$ then $\mu \in \pi$. We shall say that $V \in \operatorname{mod} G$ *belongs to* π if the composition factors of V come from the set $\{L(\lambda) : \lambda \in \pi\}$. In [14] and [15] we study a finite dimensional algebra $S(\pi)$ whose module category is the category of rational G-modules belonging to π. The category of rational G-modules belonging to π is a highest weight category in the sense of [7], Definition 3.1 (this is the content of [10] §2 Remark (2)) or equivalently $S(\pi)$ is a quasi-hereditary algebra.

The construction of the algebra $S(\pi)$ is as follows. If V is a rational G-module we define $O_\pi(V)$ to be the largest submodule of V belonging to π. Thus we have a left exact functor O_π from rational modules to rational modules belonging to π. We have on $k[G]$ a natural left $G \times G$-module structure : for $(x, y) \in G \times G$ and $f \in k[G]$, $(x, y) \cdot f$ is the function taking $z \in G$ to $f(x^{-1}zy)$. We obtain the structure of a left G-module by restricting the action to the first factor. Let $A(\pi)$ be $O_\pi(k[G])$ for the left G-action. For each $g \in G$ we have an operator $R_g : k[G] \to k[G]$ given by the action of $(1, g)$. This operator commutes with the left G-action so that R_g is a module homomorphism. By functoriality, we have $R_g(O_\pi(k[G]) \subseteq O_\pi(k[G])$. Thus $A(\pi)$ is a $G \times G$-submodule of $k[G]$ and hence a subcoalgebra. The coalgebra $A(\pi)$ is finite dimensional and the modules category of the dual algebra $S(\pi)$ is naturally equivalent to the category of rational G-modules belonging to π.

For further explanation of this connection see [20],§1 (see especially §1,(1) where we essentially run through Ringel's argument for the existence of the $M(\lambda)$'s in the context of group schemes). We already considered modules with the defining properties of tilting modules in [13], where they were used to produce finite resolutions of certain rational modules. But it must be said that we only had the existence of the $M(\lambda)$'s under some unpleasant characteristic/root system restrictions and we had no uniqueness assertion.

Another point that should be made is that we are using the term "tilting module" for what would be called a partial tilting module in the corresponding finite dimensional algebra situation.

It is natural to ask if there is some interpretation of the multiplicities $(M(\lambda) : \nabla(\mu))$ in terms of other numbers associated with G. In the case $G = \mathrm{GL}(n)$ we do have such an interpretation. Let $\mathcal{P}$

be the set of partitions, i.e. sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_i = 0$ for $i \gg 0$. We denote the transpose of a partition λ by λ'. For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$ we set $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$. For $r \geq 0$ we set $\Lambda^+(n,r) = \{\lambda \in \mathcal{P} : |\lambda| = r \text{ and } \lambda_{n+1} = 0\}$ and identify $\Lambda^+(n,r)$ with the set of sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length n with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = r$. For $\lambda \in \Lambda^+(n,r)$ we have, as in §2, the induced module $\nabla(\lambda)$ and the irreducible module $L(\lambda)$ and the tilting module $M(\lambda)$. These modules are polynomial of degree r, in the sense of [24]. For $\omega = (1, 1, \ldots, 1) \in \Lambda^+(n,n)$ we have $\nabla(\omega) \cong L(\omega) \cong D$, the module affording the determinant representation. Furthermore, for an arbitrary dominant weight $\lambda \in X^+(T)$ and an integer a we have $\nabla(\lambda + a\omega) \cong \nabla(\lambda) \otimes D^{\otimes a}$, $L(\lambda + a\omega) \cong L(\lambda) \otimes D^{\otimes a}$ and $M(\lambda + a\omega) \cong M(\lambda) \otimes D^{\otimes a}$. For $\lambda \in X^+(T)$ and $a \geq 0$, $\lambda + a\omega$ is a partition so that all $\nabla(\lambda)$'s, $L(\lambda)$'s and $M(\lambda)$'s can be understood in terms of polynomial ones. For $\lambda, \mu \in \mathcal{P}$ with $|\lambda| = |\mu| = r$, say, and $n >> 0$, the composition multiplicity $[\nabla(\lambda) : L(\mu)]$ is independent of n, by [24], (6.6e) Theorem, and we denote this number by $[\lambda : \mu]$.

Theorem 3.5 *For $\lambda, \mu \in \Lambda^+(n,r)$ we have $(M(\lambda) : \nabla(\mu)) = [\mu' : \lambda']$.*

This is proved in [20],(3.10) Proposition. The key point in the proof is to regard $M(\lambda)$ as a module for the Schur algebra $S(n,r)$ (as in [24]) and show that, if $r \leq n$ then $S(n,r)$ is its own "Ringel dual", i.e. Morita equivalent to $\mathrm{End}_{S(n,r)}(\bigoplus_{\lambda \in \Lambda^+(n,r)} M(\lambda))$. Notice that in the Theorem the partitions λ' and μ' may have more than n parts so that to interpret the filtration multiplicities of the tilting modules for GL(n) it will be necessary to pass to larger general linear groups. (This makes it difficult to guess what the generalization of Theorem 3.5 to arbitrary reductive groups might be, for example starting with a group of type F_4 one cannot pass to a larger algebraic group of the same type.)

For more on tilting modules for general linear groups and Schur algebras see the article by Erdmann, [22], in these Proceedings.

An easy consequence of Theorem 3.3 is the following result (see [20]).

Theorem 3.6 *(i) If M and M' are tilting modules for G then $M \otimes M'$ is a tilting module.*
(ii) If M is a tilting module for G then the restriction $M|_L$ is a tilting module for L, for any Levi subgroup L of G.

We wish to discuss cases in which these tilting modules can be brought to bear in the calculation of invariants as in Section 2. For this we need to introduce the idea of a saturated subgroup of G. Let π be a saturated subset of $X^+(T)$. The following is [12],§1,(16) (if π is replaced by $X^+(T)$ this is also a result of Koppinen, [27]).

Theorem 3.7 *As a $G \times G$-module, $A(\pi)$ has a filtration with sections $\nabla(\lambda) \otimes \nabla(\lambda^*)$ with $\lambda \in \pi$, each occurring exactly once.*

Thus, for a closed subgroup H, the space of invariants of $A(\pi)$ for the left H action has dimension $\dim A(\pi)^H \leq \sum_{\lambda \in \pi} \dim \nabla(\lambda)^H . \dim \nabla(\lambda^*)$. We say that H is a *saturated* subgroup of G if this is equality for every finite saturated subset π of $X^+(T)$. The next result was proved in [16],§1.4 Proposition.

Theorem 3.8 *For a closed subgroup H of G the following are equivalent:*
(i) H is saturated;
(ii) the H fixed point functor is exact on short exact sequences of G-modules which admit good filtrations;
(iii) the induced module $\mathrm{Ind}_H^G k$ admits a good filtration.

In characteristic zero, by Weyl's Complete Reducibility Theorem, every exact sequence of rational G-modules is split and therefore, by (ii), every closed subgroup of G is saturated.

We have been interested in saturated subgroups for some time and so far have made the following collection.

Theorem 3.9 *(i) Any Levi subgroup is a saturated subgroup of G.*
(ii) The diagonal subgroup $\{(x, x, \dots, x) : x \in G\}$ is a saturated subgroup of $G \times G \times \cdots \times G$.
(iii) The unipotent radical of any parabolic subgroup is a saturated subgroup of G.
(iv) If G is semisimple, simply connected and p is a good prime then the centralizer of a regular element a is a saturated subgroup of G and the orbit map $G \to F$ (taking $g \in G$ to gag^{-1}) induces an isomorphism $k[F] \to \mathrm{Ind}_{Z_G(a)}^G k$, where F is the Steinberg fibre containing a.
(v) If G is a product of general linear groups and a is any element of G then $Z_G(a)$ is saturated.
(vi) The symplectic group $\mathrm{Sp}(2n)$ is saturated in $\mathrm{SL}(2n)$.

The first two assertions follow from Theorems 3.1 and 3.3 (using the criterion Theorem 3.8 (ii) for saturation). For (iii) see [16] (where the definition of saturated subgroup was introduced) and for (v) see [17]. We sketch proofs of (iv) and (vi) in Appendix A. For (vi) one proves that the restriction to $\mathrm{Sp}(2n)$ of any $\mathrm{GL}(2n)$-module with a good filtration has a good filtration. An explicit filtration for the restriction of each $\nabla(\lambda)$ is given in a preprint of Maliakas, [31]. The existence of such a filtration is also claimed by Andersen and Jantzen, in [3],pp. 508–509 but the argument given there is not correct. Our argument is a refinement of theirs.

Assume that G is as in (iv) and a is any regular element. We have the following curious consequence. For $\lambda \in X^+(T)$ we have $\nabla(\lambda)^{Z_G(a)} \cong (\mathrm{Ind}_{Z_G(a)}^G \nabla(\lambda))^G \cong (\mathrm{Ind}_{Z_G(a)}^G k \otimes \nabla(\lambda))^G$ by Frobenius reciprocity and the tensor identity. Thus, by (iv), we have $\nabla(\lambda)^{Z_G(a)} \cong (k[F] \otimes \nabla(\lambda))^G$. Now $k[F]$ has a good filtration so that $\dim(k[F] \otimes \nabla(\lambda))^G$ is the multiplicity of $\nabla(\lambda^*)$ in a good filtration of $k[F]$, by Theorem 3.2, and this is $\dim \nabla(\lambda^*)^T = \dim \nabla(\lambda)^T$, by [17], Proposition 1.3f. Thus $\dim \nabla(\lambda)^{Z_G(a)} = \dim \nabla(\lambda)^T$ and, by (iv), for any rational G-module M with a good filtration we have $\dim M^{Z_G(a)} = \dim M^T$ (for any regular element a).

4. Variations on a theme of Steinberg

We now consider a slight generalization of Steinberg's set-up. But we should say straight away that the generalization is only of the initial step in the theory, describing invariants as trace functions; so far we have no results on the deeper aspects of Steinberg's theory in this more general setting (but see the concluding remarks of this section).

By a *group pair* we mean a pair (G, H) where G is an algebraic group over k and H is a closed subgroup. We are interested in computing the algebra $C(G, H)$ of regular functions on G which are class functions relative to H : thus $C(G, H)$ consists of those elements f of $k[G]$ such that $f(hgh^{-1}) = f(g)$ for all $g \in G$ and $h \in H$. If we take $G = \mathrm{GL}(n) \times \mathrm{GL}(n) \times \cdots \times \mathrm{GL}(n)$ and take $H = \{(x, x, \dots, x) : x \in \mathrm{GL}(n)\}$, the diagonal embedding of $\mathrm{GL}(n)$, then $C(G, H)$ is the localization at the determinant of the polynomial functions on m-tuples of matrices which are invariants for the action of $\mathrm{GL}(n)$ by simultaneous conjugation. Thus, up to localization, we recover the matrix invariants problem of Theorem 1.1 as a special case.

As in Steinberg's case $G = H$, we can produce certain elements of $C(G, H)$ representations theoretically. Given $V \in \mathrm{mod}\, G$, affording the representation $R : G \to \mathrm{GL}(V)$, and an H-endomorphism θ we define $\chi_\theta : G \to k$ by the formula

$$\chi_\theta(g) = \mathrm{trace}(R(g) \circ \theta).$$

Notice that χ_θ is a class function relative to H. For $g \in G$ and $h \in H$ we have

$$\begin{aligned} \mathrm{trace}(R(hg) \circ \theta) &= \mathrm{trace}(\theta \circ R(hg)) = \mathrm{trace}(\theta \circ R(h) \circ R(g)) \\ &= \mathrm{trace}(R(h) \circ \theta \circ R(g)) = \mathrm{trace}(R(g) \circ R(h) \circ \theta) \\ &= \mathrm{trace}(R(gh) \circ \theta) \end{aligned}$$

since θ commutes with the action of H, so that $\chi_\theta(hg) = \chi_\theta(gh)$. Thus we get $\chi_\theta(hgh^{-1}) = \chi_\theta(g)$, for $g \in G$, $h \in H$ and so $\chi_\theta \in C(G,H)$. We let $C_0(G,H)$ denote the set of all functions of the form χ_θ, with $\theta \in \mathrm{End}_H(V)$ for some rational G-module V. Note that, for U, V rational G-modules, $\alpha \in \mathrm{End}_H(U)$, $\beta \in \mathrm{End}_H(V)$ and scalars $a, b \in k$, we have $a\alpha \oplus b\beta \in \mathrm{End}_H(U \oplus V)$ and $a\chi_\alpha + b\chi_\beta = \chi_{a\alpha\oplus b\beta}$. Also $\chi_\alpha \cdot \chi_\beta = \chi_{\alpha\otimes\beta}$ and $\alpha \otimes \beta \in \mathrm{End}_H(U \otimes V)$ and clearly $1 \in C_0(G,H)$ so that $C_0(G,H)$ is a k-subalgebra of $C(G,H)$. We shall call an element of $C_0(G,H)$ an *H-endofunction*, call $C_0(G,H)$ the algebra of *H-endofunctions* and call the fibres of the natural map $\pi : G \to \mathrm{Spec}(C_0(G,H))$ the *H-endoclasses* of G. I am grateful to Jon Alperin for suggesting this terminology, as an alternative to our earlier description of elements of $C_0(G,H)$ as "shifted trace functions".

We are most interested in cases in which $C(G,H) = C_0(G,H)$, so that invariants may be computed representation theoretically. From this standpoint the case G finite is badly behaved — a group theoretical description of the endoclasses in finite groups is given in Appendix B. Returning to the case in which G is reductive we have the following general result, which may be viewed as an extension of Theorem 2.1.

Theorem 4.1 *If H is a saturated subgroup of G then $C_0(G,H) = C(G,H)$ and in fact $C(G,H)$ is exactly the algebra of endofunctions defined by tilting modules.*

This is proved in [18]. A sketch is as follows. Take $f \in C(G,H)$. Then $f \in A(\pi) = O_\pi(k[G])$ for some finite saturated set π so that it suffices to show that $A(\pi)^H$ is contained in the algebra of all endofunctions defined by tilting modules. Let $\lambda \in \pi$ be maximal and let $\sigma = \pi\backslash\{\lambda\}$. Using Theorem 3.7 one can show that $A(\pi)/A(\sigma)$ is isomorphic to $\nabla(\lambda) \otimes \nabla(\lambda^*)$ and we can assume inductively that $A(\sigma)^H$ is contained in the algebra of all endofunctions defined by tilting modules. Now we have a map $F : \mathrm{End}_k(M(\lambda)) \to k[G]$, taking $\theta \in \mathrm{End}_k(M(\lambda))$ to χ_θ in $k[G]$, where $\chi_\theta(g) = \mathrm{trace}(R(g) \circ \theta)$ for $g \in G$ and $R : G \to \mathrm{GL}(M(\lambda))$ is the representation afforded by $M(\lambda)$. One shows that F induces an epimorhism $\mathrm{End}_k(M(\lambda)) \to A(\pi)/A(\sigma) \cong \nabla(\lambda) \otimes \nabla(\lambda^*)$ and the kernel of the induced map has a good filtration. But H is saturated so, by Theorem 3.3 (ii), we get an epimorphism $\mathrm{End}_H(M(\lambda)) \to (A(\pi)/A(\sigma))^H$. It follows that $A(\pi)^H = A(\sigma)^H + F(\mathrm{End}_H(M(\lambda))$, i.e. $A(\pi)^H$ is spanned by $A(\sigma)^H$ together with the endofunctions defined by $M(\lambda)$. This, together with the inductive hypothesis, gives the result.

Thus one can calculate $C(G,H)$ in two steps : first find the (partial) tilting modules and then find all endofunctions determined by the tilting modules. Clearly, one does not need a completely explicit description of all tilting modules. One needs, for each $\lambda \in X^+(T)$, a G-module $C(\lambda)$, say, in which $M(\lambda)$ occurs as a direct summand. Then every function χ_θ, with $\theta \in \mathrm{End}_H(M(\lambda))$ is equal to χ_α for some $\alpha \in \mathrm{End}_H(C(\lambda))$.

As an extension of Steinberg's theory, it is very natural to consider the group pair $(G \times G \times \cdots \times G, D(G))$, where G is reductive and $D(G)$ is the group G diagonally embedded in the product of m copies of G. Thus the case $G = \mathrm{GL}(n)$ is, more or less, the situation of Theorem 1.1. There is so far no concrete description of the generators of the relative class functions in the general case so one may ask whether the following analogue of Theorem 1.1 is true.

Question. *Is $C(G \times G \times \cdots \times G, D(G))$ spanned by the functions $(x_1, x_2, \ldots, x_m) \mapsto \mathrm{trace}(x_{i_1} x_{i_2} \ldots x_{i_r}, V)$, for $1 \le i_1, i_2, \ldots, i_r \le m$ and V a rational G-module?*

As well as the case $G = \mathrm{GL}(n)$ covered by Theorem 1.1 and the case $m = 1$ covered by Steinberg's theory (see Theorem 2.2) there is some extra evidence in characteristic 0. For G is a symplectic or orthogonal group (over a field of characteristic 0) it follows from results of Procesi,[33], that the answer is yes.

Fortunately the other parts of Steinberg's theory are in much better shape for this case thanks to the work of Richardson (see [35]) who managed to obtain analogues of many of the deeper aspects described in Section 2.

In applying Theorem 4.1 in proving Theorem 1.1 one first needs a collection of modules such that each tilting modules occurs as a component of one of these. It is quite easy to describe such a collection for $\mathrm{GL}(n)$. For a partition $\alpha = (\alpha_1, \alpha_2, \ldots)$ we set $\bigwedge^{\alpha}(E) = \bigwedge^{\alpha_1}(E) \otimes \bigwedge^{\alpha_2}(E) \otimes \cdots$ (where E is the natural $\mathrm{GL}(n)$-module). Then $\bigwedge^{\alpha}(E)$ is a (partial) tilting module and, for $\lambda \in \Lambda^+(n,r)$, the indecomposable tilting module $M(\lambda)$ occurs exactly once as a summand of $\bigwedge^{\lambda'}(E)$ and if X is any other summand then X is isomorphic to $M(\mu)$, for some $\mu \in \Lambda^+(n,r)$ with $\mu < \lambda$ (see [20],§3.4 Lemma (ii)). If $\underline{\alpha} = (\alpha(1), \alpha(2), \ldots, \alpha(m))$ is a sequence of partitions then $\bigwedge^{\underline{\alpha}}(E) = \bigwedge^{\alpha(1)}(E) \otimes \cdots \otimes \bigwedge^{\alpha(m)}(E)$ is a tilting modules for $\mathrm{GL}(n)^m = \mathrm{GL}(n) \times \cdots \times \mathrm{GL}(n)$ (for the component-wise action) and every indecomposable tilting module is a component of such a module tensored with tensor powers (positive or negative) of determinant modules. Thus we need to calculate $\mathrm{End}_{\mathrm{GL}(n)}(\bigwedge^{\alpha}(E))$, for $\alpha = (\alpha_1, \alpha_2, \ldots)$ a partition of r, say, and find the associated endofunctions on $\mathrm{GL}(n)^m$. For $r \leq n$ we have that $\mathrm{End}_{\mathrm{GL}(n)}(\bigwedge^{\alpha}(E))$ is naturally isomorphic to $\mathrm{End}_{S_r}(\mathrm{Ind}_{S_\alpha}^{S_r} k)$, by [20],§3, where S_r is the symmetric group of degree $r = |\alpha|$ and $S_\alpha \cong S_{\alpha_1} \times S_{\alpha_2} \times \cdots$ is the Young subgroup associated to α. One therefore gets a basis of $\mathrm{End}_{\mathrm{GL}(n)}(\bigwedge^{\alpha}(E))$ labelled by double cosets $D \in S_\alpha \backslash S_r / S_\alpha$.

It would be pleasant if one could remark at this point that each associated endofunction χ_D is obviously a polynomial in the trace functions described in Theorem 1.1. Unfortunately, in characteristic $p > 0$, this is far from clear, and established by means of a "reduction mod p" argument. This involves a careful study of the $\mathbb{Z}$-subalgebra $J(n)$ of $\mathbb{C}[M(n)^m]$ generated by the trace functions of Theorem 1.1. We consider the graded inverse limit J of the $J(n)$'s (m is fixed throughout). This is analyzed in the spirit of symmetric function theory, in particular we show that J is a free polynomial ring, [19],§3,(11).

In order to deduce Theorem 1.2 from Theorem 1.1, one proceeds as follows. For G any reductive group we mean by a *good pair* (X, A) an affine variety X on which G acts together with a G-stable closed set A such that both the coordinate algebra $k[X]$ and the defining ideal $I_A = \{f \in k[X] : f|_A = 0\}$ of $k[X]$ admit good filtrations. It follows from Theorem 3.1 that $H^i(G, I_A) = 0$ for all $i > 0$ (if (X, A) is a good pair). In particular $H^1(G, I_A) = 0$ so that the short exact sequence $0 \to I_A \to k[X] \to k[A] \to 0$ gives rise to a surjection $k[X]^G \to k[A]^G$. One now takes $G = \mathrm{GL}(\alpha)$ as in Theorem 1.2 and finds invariants for $k[M(n)^m]^G$ by a simple refinement of Theorem 1.1, where $n = \alpha_1 + \alpha_2 + \cdots$. Now embed $R(Q, \alpha)$ in $M(n)^m$, for suitable m, such that $(M(n)^m, R(Q, \alpha))$ is a good pair. One now gets the invariants for $k[R(Q, \alpha)]$ described in Theorem 1.2 by restriction $k[M(n)^m]^G \to k[R(Q, \alpha)]^G$.

We end with a subversive remark : It is not necessary to use the theory of tilting modules and endofunctions to obtain generators for the algebra of matrix invariants (as in Theorem 1.1). The coordinate algebra $k[M(n)^m]$ is a direct sum of tensor products of symmetric powers $S^{a_1}(E \otimes E^*) \otimes S^{a_2}(E \otimes E^*) \otimes \cdots \otimes S^{a_m}(E \otimes E^*)$. One can work out the $\mathrm{GL}(n)$-invariants by means of the modules $\bigwedge^{\alpha}(E)$ using the argument of Theorem 4.1, but without ever mentioning that $\bigwedge^{\alpha}(E)$ is a partial tilting module, or that the diagonal embedding of $\mathrm{GL}(n)$ in $\mathrm{GL}(n)^m$ is saturated. (Even so such a proof would rely heavily on properties of good filtrations and one would still be left with the task of showing that the invariants obtained in this way lie in the algebra of trace functions.) The set-up as described here is more appealing to the author since it seems quite natural and general and should have other application (e.g. to symplectic invariants).

Appendix A : Some saturated subgroups

Here we prove Theorem 3.9 (iv) and (vi). We start with (iv). Recall that for an indecomposable root system Φ the prime p is called bad if one of the following holds: Φ has type B, C or D and $p = 2$; Φ has type E_6, E_7, F_4 or G_2 and $p = 2$ or 3; Φ has type E_8 and $p = 2, 3$ or 5. Recall also that p is called good if it is not bad and very good if p is good and if Φ has type A_r then p does not divide $r+1$. Recall that p is good (resp. very good) for an arbitrary root system Φ if it is good (resp. very good) for each indecomposable component. A proof of (iv) runs as follows. We may assume that the root system of G is connected. Let a be a regular element of G. Assume that p is very good. Then, by a result of Slodowy, see [39], p.38, the Lie algebra $\mathrm{Lie}(Z_G(a))$, of the centralizer of a, is equal to the centralizer $Z_{\mathfrak{g}}(a)$ of a in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. By [4], p.128, the orbit map $G \to F$ (taking $g \in G$ to gag^{-1}) is separable and induces an isomorphism $G/Z_G(a) \to C(a)$, where $C(a)$ is the conjugacy class of a. The orbit map $G \to F$ induces an isomorphism $k[F] \to \mathrm{Ind}_{Z_G(a)}^G k$ as in [17], Corollary 2.1e (i) (i.e. by [23],§5, Lemma 1) and $k[F]$ has a good filtration by [17],Proposition 1.3f. Now let $G = \mathrm{GL}(n)$, $G_0 = \mathrm{SL}(n)$ and let $a \in G_0$. Then $k[F] \to \mathrm{Ind}_{Z_G(a)}^G k$ is an isomorphism by [17], Theorem 2.2a. We leave it to the reader to check that restriction of functions $\mathrm{Ind}_{Z_G(a)}^G k \to \mathrm{Ind}_{Z_{G_0}(a)}^{G_0} k$ is an isomorphism, giving the required result for $\mathrm{SL}(n)$.

We now move on to (vi). Let $G = \mathrm{GL}(n)$, with $n = 2m$ even, and make our standard choices for T, B etc. We put a skew linear form (,) on E, $(e_p, e_q) = \epsilon(p,q)$, $(1 \le p, q \le n)$ where $\epsilon(p,q)$ is $+1$ if $p+q = n+1$ with $p < q$, is -1 if $p+q = n+1$ with $p > q$, and is 0 otherwise. Let $G_0 = \mathrm{Sp}(n) = \{g \in G : (gu, gv) = (u,v) \text{ for all } u, v \in E\}$. Then $T_0 = T \cap G_0$ is a maximal torus and $B_0 = B \cap G_0$ is a Borel subgroup of G_0. We let $\nu_i = \epsilon_i|_{T_0}$ for $1 \le i \le n$. Then $X(T_0)$ is free abelian on $\nu_1, \nu_2, \ldots, \nu_m$ and $\Phi_0 = \{\nu_i - \nu_j : 1 \le i, j \le n, i \ne j\}$ is the root system of G_0 with respect to T_0. Further B_0 is the negative Borel for the choice $\Phi_0^+ = \{\nu_i - \nu_j : 1 \le i < j \le n\}$ of positive roots. The set of fundamental weights is $\{\nu_1 + \cdots + \nu_r : 1 \le r \le m\}$. We write $x_i = e_{2i-1}$, $y_i = e_{2i}$, $1 \le i \le m$. For $\lambda \in X^+(T_0)$, let $\nabla_0(\lambda) = \mathrm{Ind}_{B_0}^{G_0} k_\lambda$. Let $\lambda_r = \nu_1 + \cdots + \nu_r$, $1 \le r \le m$. By [3], penultimate paragraph of §4.9, we have the following.

(1) *There exists a G_0-module surjection $\bigwedge^r(E) \to \nabla_0(\lambda_r)$, for $1 \le r \le m$.*

Let $r \ge 0$. We define $z_r \in \bigwedge^{2r}(E)$ to be 1 if $r = 0$ and $\sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} y_{i_1} x_{i_2} y_{i_2} \ldots x_{i_r} y_{i_r}$ if $r > 0$, where products are taken in the exterior algebra $\bigwedge^*(E)$. Then $z_r \in \bigwedge^{2r}(E)^{G_0}$. For $r = 1$ this is the symplectic form defining G_0 (i.e. $G_0 = \{g \in \mathrm{GL}(E) : gz_1 = z_1\}$). In characteristic zero we get $z_r \in \bigwedge^{2r}(E)^{G_0}$ by raising z_1 to the rth power in $\bigwedge^*(E)$ and in the modular case by reduction modulo p. Let $r \ge h \ge 0$. We define $\bigwedge^{2r}(E)_h \le \bigwedge^{2r}(E)$ to be kz_r if $h = 0$ and $\bigwedge^{2r}(E)_h = \bigwedge^{2r}(E)_{h-1} + z_{r-h}\bigwedge^{2h}(E)$ if $h > 0$. Note that $z_i z_j$ is a multiple of z_{i+j}, for $i, j \ge 0$. Thus we get, for $r \ge g \ge 0, s \ge h \ge 0$:

(2) $\bigwedge^{2r}(E)_g \bigwedge^{2s}(E)_h \le \bigwedge^{2(r+s)}(E)_{g+h}$.

Notice also that:

(3) *all weights of $\bigwedge^{2r}(E)_h$ are $\le \lambda_{2h}$*

and

(4) *if $0 < h < r \le m/2$ then $z_{r-h} x_1 x_2 \ldots x_{2h}$ is a non-zero vector in $\bigwedge^{2r}(E)_h$ of weight λ_{2h} so that $\bigwedge^{2r}(E)_h / \bigwedge^{2r}(E)_{h-1}$ has a non-zero λ_{2h} weight space.*

We now prove:

(5) *for $m/2 \ge r \ge h > 0$ we have $\bigwedge^{2r}(E)_h / \bigwedge^{2r}(E)_{h-1} \cong \nabla_0(\lambda_{2h})$.*

We argue by induction on r. Suppose first that $h < r$. Let $\theta : \bigwedge^{2h}(E) \to \bigwedge^{2r}(E)_h$ be multiplication by z_{r-h}. By (2), $\theta(\bigwedge^{2h}(E)_{h-1}) \le \bigwedge^{2r}(E)_{h-1}$ so that θ induces $\bar{\theta} : \bigwedge^{2h}(E)/\bigwedge^{2h}(E)_{h-1} \to$

$\bigwedge^{2r}(E)_h/\bigwedge^{2r}(E)_{h-1}$ and, by the definition of $\bigwedge^{2r}(E)_h$, $\bar{\theta}$ is surjective. We assume, inductively, that $\bigwedge^{2h}(E)/\bigwedge^{2r}(E)_{h-1} \cong \nabla_0(\lambda_{2h})$, which has a simple socle of highest weight λ_{2h}. By (4), $\bar{\theta}$ is not zero on the socle. Hence $\bar{\theta}$ is injective and so is an isomorphism. Thus we have

$$\bigwedge^{2r}(E)_h/\bigwedge^{2r}(E)_{h-1} \cong \nabla_0(\lambda_{2h}) \tag{$*$}$$

for $h < r$. Let $\pi : \bigwedge^{2r}(E) \to \nabla_0(\lambda_{2r})$ be the surjection of (1). Now $\nabla_0(\lambda_{2r})$ has a simple socle of highest weight λ_{2r} and, by (3), λ_{2r} is not a weight of $\bigwedge^{2r}(E)_{r-1}$. Thus

$$\pi(\bigwedge^{2r}(E)_{r-1}) \cap \mathrm{soc}_G(\nabla_0(\lambda_{2r})) = 0$$

and so $\pi(\bigwedge^{2r}(E)_{r-1}) = 0$. Thus π induces a surjection $\bar{\pi} : \bigwedge^{2r}(E)/\bigwedge^{2r}(E)_{r-1} \to \nabla_0(\lambda_{2r})$. But (e.g. by [11],(4.1.1))

$$\dim \bigwedge^{2r}(E) = \dim \nabla_0(\lambda_{2r}) + \dim \nabla_0(\lambda_{2(r-1)}) + \cdots + \dim \nabla_0(0).$$

By (*) this is

$$\dim \nabla_0(\lambda_{2r}) + \dim \bigwedge^{2r}(E)_{r-1}/\bigwedge^{2r}(E)_{r-2} + \dim \bigwedge^{2r}(E)_{r-2}/\bigwedge^{2r}(E)_{r-3} + \cdots$$

so that $\dim \bigwedge^{2r}(E) = \dim \nabla_0(\lambda_{2r}) + \dim \bigwedge^{2r}(E)_{r-1}$. Hence $\bar{\pi}$ is an isomorphism and $\bigwedge^{2r}(E)/\bigwedge^{2r}(E)_{r-1} \cong \nabla_0(\lambda_{2r})$. This completes the proof of (5).

Therefore $\bigwedge^{2r}(E)$ has a good filtration as an $\mathrm{Sp}(n)$-module. For odd degree put $\bigwedge^{2r+1}(E)_h = E.\bigwedge^{2r}(E)_h$ and argue as above. We thus get that $\bigwedge^{r}(E)$ has a good filtration as an $\mathrm{Sp}(n)$-module for $r \le m$. For $m < r \le n = 2m$ the dual module $\bigwedge^{r}(E)^*$ is isomorphic to $\bigwedge^{n-r}(E^*)$ which is isomorphic to $\bigwedge^{n-r}(E)$, as an $\mathrm{Sp}(n)$-module (since E is self dual). Thus we have:

(6) *for $r \ge 0$ the G_0-module $\bigwedge^{r}(E)$ has a good filtration.*

But the fundamental modules for $SL(n)$ are exterior powers of E so we get the following, by the argument of [11],Proposition 3.5.4 (ii).

(7) *If V is any rational $SL(n)$-module with a good filtration then $V|_{\mathrm{Sp}(n)}$ has a good filtration.*

It follows that $\mathrm{Sp}(n)$ is saturated in $\mathrm{SL}(n)$ by Theorems 3.1 and 3.3 (and the criterion for saturation Theorem 3.8 (ii)).

Appendix B : Endoclasses in finite group pairs

We now give the determination of endoclasses in finite groups, as promised in Section 4. Let (G, H) be a group pair with G (and H) finite. Let $E_H(x, G)$ denote the H-endoclass in G of an element x of G. We write simply $E_H(x)$ for $E_H(x, G)$ if confusion is unlikely. We write $V \in \mathrm{mod}\, G$ to indicate that V is a finite dimensional left kG-module. For $V \in \mathrm{mod}\, G$ affording representation $R : G \to \mathrm{GL}(V)$ and $g \in G$ we shall often write simply g for the endomorphism $R(g)$ of V. In particular we write $\chi_\theta(g) = \mathrm{trace}(g\theta)$, for $\theta \in \mathrm{End}_H(V)$. For $V \in \mathrm{mod}\, G$ we set $C_0(G, H, V) = \{\chi_\theta : \theta \in \mathrm{End}_H(V)\}$. Thus $C_0(G, H) = \sum_{V \in \mathrm{mod}\, G} C_0(G, H, V)$ and

(1) $C_0(G, H, V).C_0(G, H, W) \le C_0(G, H, V \otimes W)$ *for* $V, W \in \mathrm{mod}\, G$.

We view $\mathrm{Ind}_H^G(k)$ as the permutation module on the left cosets of H. For a union of H double cosets D we have $\theta \in \mathrm{End}_H(\mathrm{Ind}_H^G(k))$ defined by

$$\theta(xH) = \begin{cases} H, & \text{if } x \in D; \\ 0, & \text{if } x \notin D. \end{cases}$$

Let $g, x \in G$. If $g \cdot \theta(xH) \neq 0$ then $xH \subseteq D$ and $g \cdot \theta(xH) = gH$. If $\mathrm{trace}(g \cdot \theta) \neq 0$ we therefore get $xH = gH$ for some $xH \subseteq D$. For $g \in D$ we have $\mathrm{trace}(g \cdot \theta) = \#\{xH \in G/H : xH = gH\} = 1$. Hence χ_θ is the characteristic function on D and so, from (1), we get

(2) *$C_0(G, H, V) \leq C_0(G, H, V \otimes \mathrm{Ind}_H^G(k))$ for any $V \in \mathrm{mod}\, G$.*

Now let K be a subgroup of G containing H and suppose $x \in K$. Then $E_H(x, G) \subseteq K$ by (2). Let $y \in E_H(x, K)$. For $\theta \in \mathrm{End}_H(V)$, $V \in \mathrm{mod}\, G$ the restriction $\chi_\theta|_K$ belongs to $C_0(K, H)$ so that $\chi_\theta(y) = \chi_\theta(x)$. Hence $y \in E_H(x, G)$ and so we have $E_H(x, K) \subseteq E_H(x, G)$.

Suppose, conversely, that $y \in E_H(x, G)$ and $\chi_\theta \in C_0(K, H)$ is the endofunction defined by $\theta \in \mathrm{End}_H(V)$. Now V is a component of $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(V))$, where $\mathrm{Res} : \mathrm{mod}\, G \to \mathrm{mod}\, H$ is the restriction functor, by Mackey's Theorem, [1], Chapter III, Lemma 7. Thus we have $C_0(K, H, V) \subseteq C_0(K, H, W)$, where $W = \mathrm{Res}_K^G(\mathrm{Ind}_K^G(V))$. Hence $\chi_\theta = \chi_\phi$ for some $\phi \in \mathrm{End}_H(W)$. Now χ_ϕ is the restriction to K of the endofunction f, say, defined by ϕ. Since $y \in E_H(x, G)$ we have $f(y) = f(x)$, i.e. $\chi_\theta(y) = \chi_\theta(x)$. Hence $y \in E_H(x, K)$ and we have shown:

(3) *If $H \leq K \leq G$ and $x \in K$ then $E_H(x, K) = E_H(x, G)$.*

Recall that for a subgroup $Q \leq H$ and $V \in \mathrm{mod}\, H$ we have the transfer map $\mathrm{Tr}_Q^H : V^Q \to V^H$ on fixed points defined by $\mathrm{Tr}_Q^H(v) = \sum_{i=1}^n t_i v$ where $H = \cup_{i=1}^n t_i Q$, a disjoint union of cosets. In particular we have $\mathrm{Tr}_Q^H : \mathrm{End}_Q(V) \to \mathrm{End}_H(V)$, for $V \in \mathrm{mod}\, H$ which satisfies

$$\mathrm{Tr}_Q^H(\theta)(v) = \sum_i t_i \theta(t_i^{-1} v)$$

for $\theta \in \mathrm{End}_Q(V)$, $v \in V$. Recall that the transfer map is onto if the index $|H : Q|$ is coprime to p. For $\theta \in \mathrm{End}_Q(V)$ we have

$$\begin{aligned}\chi_{\mathrm{Tr}_Q^H(\theta)}(g) &= \sum_{i=1}^n \mathrm{trace}(t_i \theta t_i^{-1} g) = \sum_{i=1}^n \mathrm{trace}(\theta t_i^{-1} g t_i) \\ &= \sum_{i=1}^n \chi_\theta(t_i^{-1} g t_i) \in C_0(G, H).\end{aligned}$$

Thus we get the following.

(4) *We have a k-linear map $\mathrm{tr}_Q^G : C_0(G, Q) \to C_0(G, H)$ defined by $\mathrm{tr}_Q^H(f)(g) = \sum_{i=1}^n f(t_i^{-1} g t_i)$. Moreover, for $V \in \mathrm{mod}\, G$, $\theta \in \mathrm{End}_Q(V)$, we have $\mathrm{tr}_Q^H(\chi_\theta) = \chi_{\mathrm{Tr}_Q^H(\theta)}$. If $|H : Q|$ is coprime to p then $\mathrm{tr}_Q^H : C_0(G, Q) \to C_0(G, H)$ is surjective.*

We can now reduce to the case in which H is a p-group.

(5) *If $Q \leq H$, $(|H : Q|, p) = 1$ and E is an H-endoclass of G then there exists an element $x \in E$ such that $E = \cup_{h \in H} h E_Q(x) h^{-1}$.*

Proof. An H-endofunction is constant on H-conjugacy classes of G. Also $C_0(G, H) \leq C_0(G, Q)$ so that $E_Q(y) \subseteq E_H(y)$ for $y \in G$ so we get $\cup_{h \in H} h E_Q(x) h^{-1} \subseteq E$, for any $x \in E$.

We write E as a disjoint union of Q-endoclasses $E = E_Q(x_1) \cup E_Q(x_2) \cup \cdots \cup E_Q(x_r)$. Let $f_i : G \to k$ be the characteristic function on $E_Q(x_i)$, $1 \leq i \leq r$. Then $\mathrm{tr}_Q^H(f_i)$ vanishes outside $\cup_{h \in H} h E_Q(x_i) h^{-1} \subseteq E$ so that $\mathrm{tr}_Q^H(f_i) = \lambda_i f$, for some $\lambda_i \in k$, where f is the characteristic function on E. Also, if g is the characteristic function on $E_Q(y) \not\subseteq E$ then $\mathrm{tr}_Q^H(g)$ vanishes on E. Since $\mathrm{tr}_Q^H : C_0(G, Q) \to C_0(G, H)$ is surjective, f is in the image of tr_Q^H and hence some $\lambda_i \neq 0$. Thus, for $y \in E$, $\mathrm{tr}_Q^H(f_i)(y) \neq 0$ and so some $h^{-1} y h \in E_Q(x_i)$. Thus $E = \cup_{h \in H} h E_Q(x_i) h^{-1}$, as required.

(6) *If H is a p-group and x normalizes H then $E_H(x) = Hx$.*

Proof. We have $E_H(x) \subseteq Hx$ by (2). On the other hand, for $\theta \in \mathrm{End}_H(V)$, $V \in \mathrm{mod}\, G$ and $u \in H$ we have

$$\begin{aligned}\chi_\theta(ux) &= \mathrm{trace}(ux\theta) = \mathrm{trace}(x(x^{-1}ux - 1)\theta) + \mathrm{trace}(x\theta) \\ &= \mathrm{trace}(x\theta) = \chi_\theta(x)\end{aligned}$$

as $x(x^{-1}ux - 1)\theta$ is a nilpotent endomorphism of V. (For x, y in the normalizer of H, for $\theta, \phi \in \mathrm{End}_H(V)$ and $a \in \omega^r$, $b \in \omega^s$, where ω is the augmentation ideal of kH, we have

$$\begin{aligned}xa\theta yb\phi &= xy(y^{-1}ay)(y^{-1}\theta y)b\phi \\ &= xy(y^{-1}ay)b(y^{-1}\theta y)\phi.\end{aligned}$$

We may write this expression $zc\psi$, where $z = xy$, $c = (y^{-1}ay)b$ and $\psi = (y^{-1}\theta y)\phi$. Moreover $c \in \omega^{r+s}$. Thus, for $m \geq 1$ we get $(xa\theta)^m = td\eta$ for some t in the normalizer of H, $d \in \omega^m$ and $\eta \in \mathrm{End}_H(V)$. Since the augmentation ideal over k of a finite p-group is nilpotent we obtain that $xa\theta$ is nilpotent. Thus the endomorphism $x(x^{-1}ux - 1)\theta$ of V is nilpotent, as required.)

We shall say that $x, y \in G$ are (H,p)-adjacent if either x and y are conjugate by an element of H or $y = ux$ for some $u \in Q$, where Q is a p-subgroup of H normalized by x. Let (H,p)-equivalence be the equivalence relation generated by adjacency.

(7) *The H-endoclasses are exactly the (H,p)-equivalence classes.*

Proof. By (6), each H-endoclass is a union of (H,p)-equivalence classes. It remains to show that if $x, y \in G$ are in the same H-endoclass E, say, then they are (H,p)-equivalent. We argue by induction on $|H|$. If $|H| = 1$ then the result is true by (2). Now suppose $|H| > 1$ and the result holds for all group pairs (G', H') with $|H'| < |H|$.

Let Q be a Sylow subgroup of H. Then $E = \cup_{h \in H} hE_Q(z)h^{-1}$ for some $z \in E$, by (5). Thus x and y are H-conjugate to elements x_1 and y_1, say, of $E_Q(z)$. If $Q \neq H$ we get, by the inductive hypothesis, that x_1 and y_1 are (Q,p)-equivalent and so (H,p)-equivalent and therefore x and y are (H,p)-equivalent. We may therefore assume that H is a p-subgroup of G. If x normalizes H we are done by (6) so we assume $x \notin N_G(H)$. Thus $|H \cap H^x| < |H|$. We know by (2) that $y \in HxH$.

Let $V \in \mathrm{mod}\, G$ and $\alpha \in \mathrm{End}_{H \cap H^x}(V)$. We define $\theta \in \mathrm{End}_k(V \otimes \mathrm{Ind}_H^G k)$ by

$$\theta(v \otimes zH) = \begin{cases} h_1\alpha(h_1^{-1}v) \otimes H, & \text{if } z = h_1xh_2 \text{ for some } h_1, h_2 \in H \\ 0, & \text{otherwise.}\end{cases}$$

We leave it as an exercise to show that θ is a well defined H-module map. Thus we have $\chi_\theta(x) = \chi_\theta(y)$. Now $y \cdot \theta(v \otimes gH)$ is zero unless $gH = hxH$ for some $h \in H$, and $y \cdot \theta(v \otimes hxH) = yh\alpha(h^{-1}v) \otimes yH$ which contributes 0 to the trace unless $hxH = yH$ and we get

$$\begin{aligned}\chi_\theta(y) &= \mathrm{trace}(yh\alpha h^{-1}) = \mathrm{trace}(h^{-1}yh\alpha) \\ &= \chi_\alpha(h^{-1}yh)\end{aligned}$$

where $hxH = yH$. Thus, for $y = h_1xh_2 \in HxH$, we get $\chi_\theta(y) = \chi_\alpha(xh_2h_1^{-1})$. Let $y' = h_1^{-1}yh_1 = xh$, where $h = h_2h_1$. Then (as $\chi_\theta \in C_0(G,H)$) we get

$$\begin{aligned}\chi_\theta(y) &= \chi_\theta(y') = \chi_\alpha(xh) \\ &= \chi_\alpha(y').\end{aligned}$$

In particular $\chi_\theta(x) = \chi_\alpha(x)$ so that $\chi_\alpha(y') = \chi_\alpha(x)$ for all $\alpha \in \mathrm{End}_{H \cap H^x}(V)$. Thus $y' \in E_{H \cap H^x}(x)$. By the inductive hypothesis y' and x are $(H \cap H^x, p)$-equivalent and therefore (H,p)-equivalent.

Exercise. Prove directly that elements x and y of G are (G,p)-equivalent if and only if x and y have conjugate p'-parts (cf §2,(1)).

(8)*Proposition* *Let E be an H-endoclass and let $\sigma \in E$. Then there is a p-subgroup Q, say, of H normalized by σ such that $E = \cup_{x\in H} xQ\sigma x^{-1}$.*

Proof. First suppose that H is a p-subgroup. Let Q be the maximal subgroup of H normalized by σ. Let $B = \cup_{x\in H} xQ\sigma x^{-1}$. Then B is contained in the (H,p)-equivalence class of σ so it suffices to show that every element (H,p)-equivalent to an element of B already belongs to B. Thus it suffices to show that if $b \in B$ and b normalizes a p-subgroup R of H then $Rb \subseteq B$. Now $b = xq\sigma x^{-1}$ for some $x \in H$, $q \in Q$. We have $R^{xq\sigma x^{-1}} = R$ so that $R^{x(q\sigma)} = R^x$. We claim:

(*) If $q\sigma$ normalizes a subgroup U of H then $U \leq Q$.

Proof of (*). We may assume that U is maximal with respect to being normalized by $q\sigma$. Now $Q^{q\sigma} = Q$ so we get $Q \leq U$. Hence $U^q = U$ and so $U^{q\sigma} = U^\sigma = U$ and $U \leq Q$ so $U = Q$.

Thus we get $R^x \leq Q$ so that $R \leq Q^{x^{-1}}$ and $Rb \leq xQx^{-1}.xq\sigma x^{-1} = xQq\sigma x^{-1}$, which is contained in B as required.

We now turn to the general case. Let P be a Sylow subgroup of H. By (5) we have $E = \cup_{x\in H} xE_P(\tau)x^{-1}$, for some $\tau \in E$. Hence $E = \cup_{x\in H} xQ\tau x^{-1}$, where Q is a subgroup of P normalized by τ, by the case already considered. Thus $\sigma = hq\tau h^{-1}$, for some $h \in H$, $q \in Q$, which normalizes $R = hQh^{-1}$ and $E = \cup_{x\in H} xR\sigma x^{-1}$.

If H is a p-subgroup we can say a bit more.

(9)*Proposition.* *(i) If H is a p-subgroup, $\sigma \in G$ and $Q \leq H$ is normalized by σ and is maximal with this property then $E_H(\sigma) = \dot\cup_{t\in T} tQ\sigma t^{-1}$, (a disjoint union) where T is a system of left coset representatives for Q in G.*
(ii) If H is a p-group then every H-endoclass has size $|H|$.

Proof. (i) Let $x, y \in H$ and suppose $xQ\sigma x^{-1} \cap yQ\sigma y^{-1} \neq \emptyset$. Let $a = x^{-1}y$, then $aQ\sigma a^{-1} \cap Q\sigma \neq \emptyset$, $aQ \cap Q\sigma a\sigma^{-1} \neq \emptyset$ so we have some equation

$$aq_1 = q_2\sigma a\sigma^{-1} = (q_2\sigma a\sigma^{-1}q_2^{-1})q_2$$

and $aq = \tau a\tau^{-1}$, where $q = q_1q_2^{-1}$, $\tau = q_2\sigma$. Thus $aQ = \tau a\tau^{-1}Q$ and so τ normalizes $\langle a, Q\rangle$. By (*) (in the proof of (8)) we have $\langle a, Q\rangle = Q$ and so $a \in Q$. Hence $xQ = yQ$. Thus we get $E_H(\sigma) = \dot\cup_{t\in T} tQ\sigma t^{-1}$, giving (i).
(ii) $|E_H(\sigma)| = |H/Q|.|Q| = |H|$.

Remarks. 1) $C_0(G,H)$ is exactly the algebra of functions such that $f(x) = f(y)$ whenever $x, y \in G$ belong to the same endoclass so the above gives an explicit description of such functions in terms of the group theory.

2) The property (9)(ii) is reminiscent of the fact that a "Young superclass" for a Young subgroup S_α of a symmetric group S_r has size $|S_\alpha|$, [19], §1,(3).

3) The only modules which we used in the determination of the endoclasses are permutation modules. It follows that the fibres of the natural map $G \to \mathrm{Spec}(\sum_{K\leq G} C_0(G,H,\mathrm{Ind}_K^G(k))$ are the H-endoclasses. Therefore we have $C_0(G,H) = \sum_{K\leq G} C_0(G,H,\mathrm{Ind}_K^G(k))$ and indeed, by (2) and Mackey decomposition we have $C_0(G,H) = \sum_{K\leq H} C_0(G,H,\mathrm{Ind}_K^G(k))$. In this respect the permutation modules on cosets of subgroups of H are like the tilting modules for G reductive (cf Theorem 4.1).

Appendix C : Divisibility of the Steinberg module

In [20], (2.9) Remark we gave an application of the theory of tilting modules to the question of virtual divisibility of the Steinberg module on restriction to a Levi subgroup. This however, is not a genuine application since a stronger result, namely actual divisibility, may be proved by more traditional means. Here we make amends by giving a tilting-free proof. Actual divisibility was first established by Zongzhu Lin (private communication) by applying work of Cline, Parshall and Scott, [6], Theorem 4.1. The proof given here (more or less dual to Lin's) uses instead the restricted enveloping algebra. The question arose in connection with work of Alperin and Mason,[2].

We assume that k has characteristic $p > 0$. For an algebraic group H over k let $u(\mathfrak{h})$ denote the restricted enveloping algebra of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$. Thus $\dim u(\mathfrak{h}) = p^{\dim H}$ and if $\mathfrak{h} = \mathfrak{a}\oplus\mathfrak{b}$ is the sum of restricted subalgebras then (by the restricted version of the Poincaré-Birkhoff-Witt Theorem) $u(\mathfrak{h}) = u(\mathfrak{a})\otimes u(\mathfrak{b})$. Now H acts on $u(\mathfrak{h})$ as algebra automorphisms via the adjoint action, $\mathrm{Ad} : H \longrightarrow \mathrm{GL}(u(\mathfrak{h}))$. Any rational H-module M is naturally a $u(\mathfrak{h})$-module with action satisfying $x(\gamma m) = \mathrm{Ad}(x)(\gamma)xm$, for $x \in G$, $\gamma \in u(\mathfrak{h})$ and $m \in M$, i.e. the action $u(\mathfrak{h}) \otimes M \longrightarrow M$ is a G-module map. If N is a closed normal subgroup of H and $\mathfrak{n} = \mathrm{Lie}(N)$ then $u(\mathfrak{n})$ is a subalgebra and H-submodule of $u(\mathfrak{h})$.

We now take G to be a simply connected, semisimple algebraic group and adopt our usual notational conventions except that we write N for the unipotent radical $R_u(B)$. Let $M = \mathrm{St}(G)$, the simple module of highest weight $(p-1)\rho$, where ρ is half the sum of the positive roots. Then $\dim \mathrm{St}(G) = p^{|\Phi^+|}$, see [26],II,3.18,(6). Let $0 \neq m^+$ be a highest weight vector. Let B^+ be the Borel subgroup opposite to B (i.e. the positive Borel subgroup). Then $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{b}^+$ and we have

$$M = u(\mathfrak{g})m^+ = u(\mathfrak{n})u(\mathfrak{b}^+)m^+ = u(\mathfrak{n})m^+ \qquad (*)$$

(since m^+ is a highest weight vector) and, by dimensions, M is freely generated by m^+ as a $u(\mathfrak{n})$-module. Let P be a parabolic subgroup containing B. Then $P = V.H$, for $V = R_u(P)$ and H a Levi subgroup, and $N = V.N_H$, where $N_H = N \cap H$. Thus $\mathfrak{n} = \mathfrak{v}\oplus\mathfrak{n}_H, u(\mathfrak{n}) = u(\mathfrak{v})\otimes u(\mathfrak{n}_H)$ and, by (*), multiplication $\phi : u(\mathfrak{v})\otimes S \longrightarrow M$ is a linear isomorphism, where $S = u(\mathfrak{n}_H)m^+$. Now $B_H^+ = B^+\cap H$ is a Borel subgroup of H, $\mathfrak{h} = \mathrm{Lie}(H) = \mathfrak{n}_H\oplus\mathfrak{b}_H^+$ and m^+ is a maximal vector so $S = u(\mathfrak{n}_H)m^+ = u(\mathfrak{n}_H).u(\mathfrak{b}_H^+)m^+ = u(\mathfrak{h})m^+$.

It follows that S is an H-submodule of M, by e.g. the argument of [4],6.2 Lemma (i) (used in Curtis's Theorem on the restriction of irreducible modules to the restricted enveloping algebra). Moreover, ϕ is the restriction of the action $u(\mathfrak{g})\otimes M \longrightarrow M$ and hence an H-module map. Thus ϕ is an isomorphism of H-modules. But S has highest weight $(p-1)\rho$ and $\dim S = \dim u(\mathfrak{n}_H) = \dim \mathrm{St}(H)$ so that $S \cong \mathrm{St}(H)$ and we have an H-module isomorphism

$$u(\mathfrak{n}_H)\otimes \mathrm{St}(H) \longrightarrow \mathrm{St}(G) \qquad (\dagger)$$

as required.

In particular, the restriction of $\mathrm{St}(G)$ to H is divisible by the Steinberg module for H. For $r > 1$ we have the higher Steinberg module $\mathrm{St}_r(G)$, i.e. the irreducible G-module of highest weight $(p^r-1)\rho$. By Steinberg's Tensor Product Theorem we have $\mathrm{St}_r(G) \cong \mathrm{St}(G)\otimes\mathrm{St}(G)^F\otimes\cdots\otimes\mathrm{St}(G)^{F^{r-1}}$, where $F : G \longrightarrow G$ is the Frobenius morphism. By tensoring together Frobenius twists of (†) we get that $\mathrm{St}_r(G)$ is divisible by $\mathrm{St}_r(H)$, for $r > 1$. One could also prove this by replacing $u(\mathfrak{g})$ by the rth hyperalgebra $u_r(\mathfrak{g})$ in the argument above. Thus we have the following.

Proposition. *For $r \geq 1$, the restriction of $\mathrm{St}_r(G)$ to H has the form $V_1\otimes V_2$ where V_1 is the simple H-module of highest weight $(p^r-1)\rho$ and V_2 is a rational H-module.*

Note that a precise description of such a module V_2 has been given in the course of the proof. We call a dominant weight λ restricted if we have $2(\lambda,\alpha)/(\alpha,\alpha) < p$ for every simple root α. Note also that the argument gives, for any restricted weight λ, an H-module epimorphism $u(\mathfrak{n}_H)\otimes L_H(\lambda) \longrightarrow L(\lambda)$, where $L_H(\lambda)$ is the simple H-module of highest weight λ.

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Harish-Chandra Subalgebras and Gelfand-Zetlin Modules

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Abstract. A new framework for the study of some modules over algebras is elaborated and applied to a new class of representations of Lie algebra $\mathcal{GL}(n)$.

1. Abstract Harish-Chandra situation

1.1. Cofinite spectrum of an algebra.

Through the whole chapter we fix a field K. All considered algebras and categories will be K-algebras and K-linear categories. Respectively, all homomorphisms and functors will be K-linear. We shall write Hom, $\otimes$, dim etc. instead of Hom_K, $\otimes_K$, $\dim_K$ etc. For any algebra or category A denote A^o the opposite algebra or category.

Denote $\mathrm{cfs}(\Gamma)$ the *cofinite spectrum* of an algebra Γ, i.e. the set of maximal ideals of finite codimension in Γ. If $\mathbf{m} \in \mathrm{cfs}(\Gamma)$, then $\Gamma/\mathbf{m} \simeq M_{\nu(\mathbf{m})}(K(\mathbf{m}))$ where $K(\mathbf{m})$ is a finite dimensional division algebra over K. In particular, if K is algebraically closed, then $K(\mathbf{m}) = K$. Let $S_\mathbf{m}$ be the only simple left $\Gamma/\mathbf{m}$-module and $DS_\mathbf{m} = \mathrm{Hom}(S_\mathbf{m}, K)$ the only simple right $\Gamma/\mathbf{m}$-module. Then $\mathbf{m} \mapsto S_\mathbf{m}$ (or $DS_\mathbf{m}$) is a 1-1 correspondence between $\mathrm{cfs}(\Gamma)$ and the set of isomorphism classes of simple left (or, resp., right) finite-dimensional Γ-modules.

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Put $\Gamma_{\mathbf{m}} = \varprojlim_n \Gamma/\mathbf{m}^n$, the $\mathbf{m}$-adique completion of Γ, and $J_{\mathbf{m}} = \varprojlim_n \mathbf{m}/\mathbf{m}^n$ (this is an ideal in $\Gamma_{\mathbf{m}}$).

Proposition 1.

1. $J_{\mathbf{m}} = \operatorname{Rad}\Gamma_{\mathbf{m}}$ *(the Jacobson radical).*
2. $\Gamma_{\mathbf{m}} \simeq M_{\nu(\mathbf{m})}(\Delta_{\mathbf{m}})$ *where* $\Delta_{\mathbf{m}}$ *is a local ring.*
3. $\Delta_{\mathbf{m}}/R_{\mathbf{m}} \simeq K(\mathbf{m})$ *where* $R_{\mathbf{m}} = \operatorname{Rad}\Delta_{\mathbf{m}}$.

The proof is evident.

Sometimes the following simple observation is useful.

Proposition 2. *If K is algebraically closed, then*

$$\operatorname{cfs}(\Gamma) \times \operatorname{cfs}(\Lambda) \simeq \operatorname{cfs}(\Gamma \otimes \Lambda)$$

Namely, this bijection is given by:

$$(\mathbf{m}, \mathbf{n}) \longmapsto \mathbf{m} \otimes \Lambda + \Gamma \otimes \mathbf{n}$$

Moreover, the corresponding simple left (right) $\Gamma \otimes \Lambda$-module is $S_{\mathbf{m}} \otimes S_{\mathbf{n}}$ (resp., $DS_{\mathbf{m}} \otimes DS_{\mathbf{n}}$).

The proof is immediately reduced to the finite-dimensional case, where it is quite evident.

1.2. Quasi-commutative algebras.

Call an algebra Γ *quasi-commutative* provided $\operatorname{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$ for all $\mathbf{m}, \mathbf{n} \in \operatorname{cfs}(\Gamma), \mathbf{m} \neq \mathbf{n}$.

Example 3.

1. Of course, any commutative algebra as well as any semi-simple [1] one is quasi-commutative.
2. Let $\Gamma = U(\mathcal{G})$ be the universal envelopping algebra of a finite-dimensional Lie algebra $\mathcal{G}$. If $\operatorname{char} K = 0$ and $\mathcal{G}$ is either reductive or nilpotent, then Γ is quasi-commutative [1].

Proposition 4. *Let $\mathbf{m}, \mathbf{n} \in \operatorname{cfs}(\Gamma), \mathbf{m} \neq \mathbf{n}$, and suppose that $\mathbf{m}$ is finitely generated as left ideal. Then the following conditions are equivalent:*

1. $\operatorname{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$

[1] "semi-simple" will always mean "semi-simple artinian".

2. $\mathbf{n} \cap \mathbf{m} = \mathbf{nm}$

3. $\mathbf{mn} \subseteq \mathbf{nm}$

Proof. Remark that $\mathbf{n} + \mathbf{m} = \Gamma$, whence $\mathbf{n} \cap \mathbf{m} = \mathbf{nm} + \mathbf{mn}$. Thus *2.* $\Longleftrightarrow$ *3.*

1. $\Longrightarrow$ *2.* Consider the exact sequence

$$0 \longrightarrow \mathbf{m}/\mathbf{nm} \longrightarrow \Gamma/\mathbf{nm} \longrightarrow \Gamma/\mathbf{m} \longrightarrow 0 \tag{1}$$

Here $\Gamma/\mathbf{m} \simeq \nu(\mathbf{m})S_{\mathbf{m}}$ and $\mathbf{m}/\mathbf{nm} \simeq kS_{\mathbf{n}}$ for some integer k. Hence, (1) splits and there are left ideals M, N in Γ such that:

$$M + N = \Gamma;\ \ M \cap N = \mathbf{nm};\ \ \Gamma/M \simeq \nu(\mathbf{m})S_{\mathbf{m}};\ \ \Gamma/N \simeq kS_{\mathbf{n}}$$

Therefore, $\mathbf{m} \subseteq M, \mathbf{n} \subseteq N$ and

$$\mathbf{n} \cap \mathbf{m} \subseteq N \cap M = \mathbf{nm} \subseteq \mathbf{n} \cap \mathbf{m}$$

i.e. $\mathbf{n} \cap \mathbf{m} = \mathbf{nm}$.

2. $\Longrightarrow$ *1.* Consider any exact sequence of the form:

$$0 \longrightarrow S_{\mathbf{n}} \longrightarrow M \longrightarrow S_{\mathbf{m}} \longrightarrow 0 \tag{2}$$

Evidently, $\mathbf{nm}M = 0$, i.e. M is a module over the algebra

$$\Gamma/\mathbf{nm} = \Gamma/\mathbf{n} \cap \mathbf{m} \simeq\ \Gamma/\mathbf{n} \times \Gamma/\mathbf{m}$$

which is semi-simple. Hence, (2) splits and $\mathrm{Ext}^1_\Gamma(S_{\mathbf{m}}, S_{\mathbf{n}}) = 0$ Q.E.D.

Proposition 5. *If Γ is a finitely generated algebra and I is a left ideal of finite codimension in Γ, then I is finitely generated as left ideal.*

Proof. Let G be a generating set of Γ and B be a basis of Γ/I. For each $b \in B$ fix its representative $\bar{b} \in \Gamma$ and for any $x = \sum_i \lambda_i b_i$ with $\lambda_i \in K, b_i \in B$, put $\bar{x} = \sum_i \lambda_i \bar{b}_i$. Then it is easy to check that the set

$$\{\ g\bar{b} - \overline{gb} \mid g \in G, b \in B\ \}$$

generates I as left ideal Q.E.D.

Corollary 6. *If Γ is a finitely generated algebra, then the following conditions are equivalent:*

1. *Γ is quasi-commutative.*

2. *If $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$ and $\mathbf{m} \neq \mathbf{n}$, then $\mathbf{m} \cap \mathbf{n} = \mathbf{nm}$.*

3. *If $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$, then $\mathbf{mn} = \mathbf{nm}$.*

Corollary 7. *If Γ is quasi-commutative, then so is Γ^o.*

1.3. Harish-Chandra subalgebras.

Let Γ be a subalgebra of an algebra A. Call Γ *quasi-central* (in A) if for any element $a \in A$ the bimodule $\Gamma a \Gamma$ is finitely generated both as left and as right Γ-module.

Proposition 8. *Suppose that Γ is noetherian and G is a set of generators of the algebra A. Then Γ is quasi-central in A if and only if $\Gamma g \Gamma$ is finitely generated both as left and as right Γ-module for each $g \in G$.*

The proof is evident as $\Gamma(ab)\Gamma \subseteq (\Gamma a \Gamma)(\Gamma b \Gamma)$ and $\Gamma(a+b)\Gamma \subseteq \Gamma a \Gamma + \Gamma b \Gamma$.

Example 9.

1. Of course, if Γ is central (i.e. contained in the centre of A), it is also quasi-central.
2. Let $A = U(\mathcal{G})$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{G}$ is a finite-dimensional Lie algebra and $\mathcal{H}$ its Lie subalgebra. Then one can easily check that $\Gamma\mathcal{G} = \mathcal{G}\Gamma$. By PROPOSITION 8, Γ is quasi-central in A.

Now, call the subalgebra $\Gamma \subseteq A$ a *Harish-Chandra subalgebra* provided it is both quasi-central and quasi-commutative.

Example 10.

1. Any central subalgebra is a Harish-Chandra one.
2. Suppose that char$K = 0$. If $A = U(\mathcal{G})$ for a finite-dimensinal Lie algebra $\mathcal{G}$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{H}$ is either reductive or nilpotent Lie subalgebra of $\mathcal{G}$, then Γ is a Harish-Chandra subalgebra of A.
3. One more example - the *Gelfand-Zetlin subalgebra* - will be considered below.

From now on, let Γ be a Harish-Chandra subalgebra of A. Put $\Gamma^e = \Gamma \otimes \Gamma^o$. For any $a \in A$ consider the Γ-bimodule epimorphism $\phi_a : \Gamma^e \longrightarrow \Gamma a \Gamma$ mapping $\beta \otimes \gamma^o$ to $\beta a \gamma$. Let $I_a = \mathrm{Ker}\phi_a$ (it is a left ideal in Γ^e). Define the subset $X_a \subseteq \mathrm{cfs}(\Gamma)^2$ by the rule:

$$X_a = \{\ (\mathbf{m}, \mathbf{n}) \mid S_{\mathbf{n}} \text{ is a composition factor of } \Gamma a \Gamma / \Gamma a \mathbf{m} \text{ as of left } \Gamma\text{-module}\ \}$$

Proposition 11. *The following conditions are equivalent:*

1. $(\mathbf{m}, \mathbf{n}) \in X_a$.
2. *$DS_{\mathbf{m}}$ is a composition factor of $\Gamma a \Gamma / \mathbf{n} a \Gamma$ as of right Γ-module.*

3. $\mathbf{n}a\Gamma + \Gamma a\mathbf{m} \neq \Gamma a\Gamma$.

4. $\mathbf{n} \otimes \Gamma^o + \Gamma \otimes \mathbf{m}^o + I_a \neq \Gamma^e$.

Proof. Put $M = \Gamma a\Gamma/\Gamma a\mathbf{m}$. As $\Gamma a\Gamma$ is finitely generated right Γ-module and $\Gamma/\mathbf{m}$ is finite-dimensional, M is also finite-dimensional. Hence M considered as left Γ-module has a composition series with factors isomorphic to $S_\mathbf{l}$ for some ideals $\mathbf{l} \in \mathrm{cfs}(\Gamma)$. But as Γ is quasi-commutative, $S_\mathbf{n}$ is a composition factor of M if and only if it is isomorphic to a factor-module of M which means, of course, that $\mathbf{n}M \neq M$. Therefore, *1.* $\iff$ *3.*

Quite analogousely, *2.* $\iff$ *3.* At last, *3.* $\iff$ *4.* is evident, Q.E.D.

Corollary 12. *For any* $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ *and* $a \in A$ *the set*

$$X_a(\mathbf{m}) = \{ \ \mathbf{n} \in \mathrm{cfs}(\Gamma) \mid (\mathbf{m}, \mathbf{n}) \in X_a \ \}$$

is finite.

Denote $\prec$ the least preorder relation on $\mathrm{cfs}(\Gamma)$ containing all X_a (i.e. such that $(\mathbf{m}, \mathbf{n}) \in X_a$ implies $\mathbf{m} \prec \mathbf{n}$) and Δ the least equivalence relation containing all X_a. Put also $\nabla = \prec \cap \prec^{-1}$ (the equivalence relation associated with the preorder $\prec$). Let $\Delta \mathrm{m}$ (resp., $\nabla \mathrm{m}$) denotes the equivalence class of Δ (resp., ∇) containing $\mathbf{m}$ and $\Delta(A, \Gamma)$ (resp., $\nabla(A, \Gamma)$) denotes the set of all equivalence classes of Δ (resp., ∇).

1.4. HARISH-CHANDRA MODULES.

Remind that we consider a fixed Harish-Chandra subalgebra $\Gamma \subseteq A$. For an A-module M and an ideal $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ put

$$M(\mathbf{m}) = \{ \ x \in M \mid \exists k(\mathbf{m}^k x = 0) \ \}$$

Call M a *Harish-Chandra module (with respect to* Γ*)* if $M = \coprod_{\mathbf{m} \in \mathrm{cfs}(\Gamma)} M(\mathbf{m})$. Of course, as Γ is quasi-commutative, M is a Harish-Chandra module if and only if it is a sum of finite-dimensional Γ-submodules. Remark that any submodule or factor-module of a Harish-Chandra module is also a Harish-Chandra module.

Example 13. Let $\mathrm{char} K = 0$, $A = U(\mathcal{G})$ and $\Gamma = U(\mathcal{H})$ where $\mathcal{G}$ is a finite-dimensional Lie algebra and $\mathcal{H}$ its semi-simple Lie subalgebra. Then the notion of Harish-Chandra modules coincides with the usual definition of Harish-Chandra $\mathcal{G}$-modules with respect to $\mathcal{H}$ (cf. [1]).

Denote $\mathbf{H}(A, \Gamma)$ the category of all Harish-Chandra A-modules with respect to Γ and $\mathrm{Irr}(A, \Gamma)$ the set of isomorphism classes of simple modules from$\mathbf{H}(A, \Gamma)$.

Proposition 14. *For any $a \in A$ and $\mathbf{m} \in \mathrm{cfs}(\Gamma)$*

$$aM(\mathbf{m}) \subseteq \coprod_{\mathbf{n}\in X_a(\mathbf{m})} M(\mathbf{n})$$

Proof. If $x \in M(\mathbf{m})$, then Γx has a composition series with all factors isomorphic to $S_{\mathbf{m}}$. Of course, $ax \in \Gamma a \Gamma x$. The last module is an epimorphic image of $\Gamma a \Gamma \otimes_\Gamma \Gamma x$. But $\Gamma a \Gamma \otimes_\Gamma S_{\mathbf{m}}$ has a composition series with the factors isomorphic to $S_{\mathbf{n}}$ for $\mathbf{n} \in X_a(\mathbf{m})$. Hence, the same is true for $\Gamma a \Gamma \otimes_\Gamma \Gamma x$ and for $\Gamma a \Gamma x$. As Γ is quasi-commutative, we obtain that

$$\Gamma a \Gamma x \subseteq \coprod_{\mathbf{n}\in X_a(\mathbf{m})} M(\mathbf{n})$$

Q.E.D.

For any $D \subseteq \mathrm{cfs}(\Gamma)$, put $M(D) = \coprod_{\mathbf{m}\in D} M(\mathbf{m})$. If $R \subseteq \mathrm{cfs}(\Gamma)^2$ is a relation on $\mathrm{cfs}(\Gamma)$, call D $R-$*closed* provided $\mathbf{m} \in D$ and $(\mathbf{m},\mathbf{n}) \in R$ implies $\mathbf{n} \in D$. Call the *support* of M the set

$$\mathrm{Supp}M = \{ \mathbf{m} \in \mathrm{cfs}(\Gamma) \mid M(\mathrm{m}) \neq 0 \}$$

Corollary 15. *Let $M \in \mathbf{H}(A,\Gamma)$.*

1. *If $D \subseteq \mathrm{cfs}(\Gamma)$ is $\prec$-closed, then $M(D)$ is a submodule of M.*
2. *$M = \coprod_{D\in\Delta(A,\Gamma)} M(D)$ as A-module.*
3. *If M is indecomposable and $M(\mathbf{m}) \neq 0$, then $\mathrm{Supp}M \subseteq \Delta\mathbf{m}$.*
4. *If M is irreducible and $M(\mathbf{m}) \neq 0$, then $\mathrm{Supp}M \subseteq \nabla\mathbf{m}$.*

Denote $\mathbf{H}(A,\Gamma,D)$ the full subcategory of $\mathbf{H}(A,\Gamma)$ consisting of all modules M with $\mathrm{Supp}(M) \subseteq D$ and $\mathrm{Irr}(A,\Gamma,D)$ the set of isomorphism classes of simple modules from $\mathbf{H}(A,\Gamma,D)$.

Corollary 16.

1. $\mathbf{H}(A,\Gamma) = \coprod_{D\in\Delta(A,\Gamma)} \mathbf{H}(A,\Gamma,D)$ *(the direct sum of categories).*
2. $\mathrm{Irr}(A,\Gamma) = \bigsqcup_{D\in\nabla(A,\Gamma)} \mathrm{Irr}(A,\Gamma,D)$ *(the disjoint union of sets).*

1.5. Category $\mathcal{A}$.

Define a new category $\mathcal{A} = \mathcal{A}_{A,\Gamma}$ in the following way. The set of objects $\mathrm{Ob}\mathcal{A} = = \mathrm{cfs}(\Gamma)$. The set of morphisms from $\mathbf{m}$ to $\mathbf{n}$ is

$$\mathcal{A}(\mathbf{m},\mathbf{n}) = \varprojlim_{n,m} A/(\mathbf{n}^n A + A\mathbf{m}^m)$$

To define the multiplication $\mathcal{A}(\mathbf{n},\mathbf{l}) \times \mathcal{A}(\mathbf{m},\mathbf{n}) \longrightarrow \mathcal{A}(\mathbf{m},\mathbf{l})$, take any two elements $a, b \in A$ and consider the left Γ-module $M = \Gamma a\Gamma/\Gamma a\mathbf{m}^m$ and the right Γ-module $L = \Gamma b\Gamma/\mathbf{l}^\ell b\Gamma$. Both of them are finite-dimensional as Γ is quasi-central. Moreover, as Γ is quasi-commutative,

$$M = M_0 \oplus M_1 \text{ where } \mathbf{n}^n M_0 = 0 \text{ and } \mathbf{n}M_1 = M_1$$

and

$$L = L_0 \oplus L_1 \text{ where } L_0\mathbf{n}^n = 0 \text{ and } L_1\mathbf{n} = L_1$$

for some natural n.

Therefore, $a = a_0 + a_1$, $b = b_0 + b_1$ where:

$$\mathbf{n}^n a_0 \in A\mathbf{m}^m;\ a_1 \in \mathbf{n}^n A + A\mathbf{m}^m;\ b_0\mathbf{n}^n \in \mathbf{l}^\ell A;\ b_1 \in \mathbf{l}^\ell A + A\mathbf{n}^n$$

Now it is obvious that the class of b_0a_0 in $A/(\mathbf{l}^\ell A + A\mathbf{m}^m)$ depends only on the classes of a and b in $A/(\mathbf{n}^n A + A\mathbf{m}^m)$ and in $A/(\mathbf{l}^\ell A + A\mathbf{n}^n)$ respectively. Of course, it makes possible to define the needed multiplication.

Suppose that M is a Harish-Chandra module. If $x \in M(\mathbf{m})$, then $\mathbf{m}^m x = 0$ for some m. For an element $a \in A$ and an ideal $\mathbf{n} \in \mathrm{cfs}(\Gamma)$ choose n as above. Then the projection of ax onto $M(\mathbf{n})$ again depends only on the class of a in $A/(\mathbf{n}^n A + A\mathbf{m}^m)$. Therefore, for any element $\alpha \in \mathcal{A}(\mathbf{m},\mathbf{n})$ we are able to define the product $\alpha x \in M(\mathbf{n})$. In other words, the correspondence $\mathbf{m} \longmapsto M(\mathbf{m})$ becomes a functor from the category $\mathcal{A}$ to the category **Vect** of vector spaces over K. Moreover, this functor is continuous if we consider the discrete topology on vector spaces and the natural topology of the inverse limite on the sets $\mathcal{A}(\mathbf{m},\mathbf{n})$. Call such functors *discrete $\mathcal{A}$-modules* or simply *$\mathcal{A}$-modules*.

If N is any $\mathcal{A}$-module, then we can construct the corresponding Harish-Chandra module as $\coprod_{\mathbf{m}} N(\mathbf{m})$. To define the product ax for $a \in A$, $x \in N(\mathbf{m})$, put $ax = \sum_{\mathbf{n}} a_{\mathbf{n}}x$ where $a_{\mathbf{n}}$ denotes the image of a in $\mathcal{A}(\mathbf{m},\mathbf{n})$. This sum is finite due to Corollary 12.

Hence, we obtain the following result.

Theorem 17. *The category $\mathbf{H}(A,\Gamma)$ of Harish-Chandra modules is equivalent to the category $\mathcal{A} - \mathbf{mod}$ of discrete $\mathcal{A}$-modules.*

Of course, the image of a in $\mathcal{A}(\mathbf{m},\mathbf{n})$ is non-zero if and only if $(\mathbf{m},\mathbf{n}) \in X_a$. Therefore,

$$\mathcal{A} = \coprod_{D\in\Delta(A,\Gamma)} \mathcal{A}(D)$$

where $\mathcal{A}(D)$ is the full subcategory of $\mathcal{A}$ consisting of all objects $\mathbf{m} \in D$.

The following result from *general nonsence* seems to be rather known though we have never seen it published.

Theorem 18. *For any object* $\mathbf{m} \in \mathrm{Ob}\mathcal{A}$ *let* $\mathrm{Irr}(\mathbf{m})$ *denotes the set of isomorphism classes of simple* $\mathcal{A}$*-modules* M *such that* $M(\mathbf{m}) \neq 0$. *Then there is a 1-1 correspondence between* $\mathrm{Irr}(\mathbf{m})$ *and the set* $\mathrm{Irr}\mathcal{A}(\mathbf{m},\mathbf{m})$ *of isomorphism classes of simple (discrete)* [2] $\mathcal{A}(\mathbf{m},\mathbf{m})$*-modules.*

Proof. Let M be an $\mathcal{A}$-module and let $U(\mathbf{m})$ be a non-trivial $\mathcal{A}(\mathbf{m},\mathbf{m})$-submodule of $M(\mathbf{m})$. Put $U(\mathbf{n}) = \mathcal{A}(\mathbf{m},\mathbf{n})U(\mathbf{m})$ for any object $\mathbf{n}$. Then we obtain a non-trivial submodule U of M. Hence, if M is simple and $M(\mathbf{m}) \neq 0$, then $M(\mathbf{m})$ is a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module.

On the other hand, let $N(\mathbf{m})$ be a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module. Put

$$N(\mathbf{n}) = \mathcal{A}(\mathbf{m},\mathbf{n}) \otimes_{\mathcal{A}(\mathbf{m},\mathbf{m})} N(\mathbf{m})$$

Then the set $\{N(\mathbf{n})\}$ can be evidently viewed as an $\mathcal{A}$-module N. We claim that N contains the only maximal submodule N' and $N'(\mathbf{m}) = 0$. Really, if $L \subseteq N$ is a submodule and $L(\mathbf{m}) \neq 0$, then $L(\mathbf{m}) = N(\mathbf{m})$ as the last one is a simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-module. But $N(\mathbf{m})$ generates N, hence, $L = N$. Therefore, if we denote N' the sum of all proper submodules $L \subset N$, then $N'(\mathbf{n}) = 0$ and N' is the only maximal submodule of N. Thus $M = N/N'$ is a simple $\mathcal{A}$-module with $M(\mathbf{m}) = N(\mathbf{m})$.

Moreover, if M' is any $\mathcal{A}$-module and $\phi : N(\mathbf{m}) \longrightarrow M'(\mathbf{m})$ is a homomorphism of $\mathcal{A}(\mathbf{m},\mathbf{m})$-modules, then it prolongs uniquely to a homomorphism of $\mathcal{A}$-modules $N \to M'$. In particular, if M' is simple with $M'(\mathbf{m}) \simeq M(\mathbf{m})$, then we obtain an epimorphism $\pi : N \to M'$. The kernel of π is a maximal submodule of N, hence it coincides with N' and $M' \simeq N/N' \simeq M$ Q.E.D.

Call the subalgebra Γ *big at the point* $\mathbf{m}$ provided $\mathcal{A}(\mathbf{m},\mathbf{m})$ is finitely generated as $\Gamma(\mathbf{m})$-module (left or right or as bimodule which is equivalent as Γ is quasi-central).

Corollary 19. *Suppose that* Γ *is big at the point* $\mathbf{m}$. *Then:*

1. *The set* $\mathrm{Irr}(\mathbf{m})$ *is finite.*

---

[2] in the same sence as above

2. *For any simple Harish-Chandra module M the vector space $M(\mathbf{m})$ is finite-dimensional.*

Proof. Put $B = \mathcal{A}(\mathbf{m},\mathbf{m})$, $J = J_{\mathbf{m}}$ (cf. section 1.1). Then B/BJ is finite-dimensional, hence $J^nB \subseteq BJ$ for some n. If I is a maximal right ideal in B, then $I \supseteq J^nB$ (otherwise $I + BJ \supseteq I + J^nB = B$, whence $I = B$ by Nakayama's lemma). Therefore, $\mathrm{Rad}B \supseteq J^nB$ and $B/\mathrm{Rad}B$ is finite-dimensional, which implies both *1.* and *2.* Q.E.D.

2. Gelfand-Zetlin modules

2.1. Gelfand-Zetlin subalgebra.

In this section we suppose that K is algebraically closed of characteristic 0 and denote $\mathcal{G}_m = \mathcal{GL}(m,K)$, $U_m = U(\mathcal{G}_m)$ and Z_m the centre of U_m. Put $\mathcal{G} = \mathcal{G}_n$, $U = U_n$ and identify $\mathcal{G}_m$ for $m \leq n$ with the Lie subalgebra of $\mathcal{G}$ generated by the matrix units $\{ e_{ij} \mid i,j = 1..m \}$. Then we obtain the inclusions: $\mathcal{G}_1 \subset \mathcal{G}_2 \subset ... \subset \mathcal{G}_n = \mathcal{G}$ and $U_1 \subset U_2 \subset ... \subset U_n = U$. Let Γ be the subalgebra of U generated by $\{ Z_m \mid m = 1..n \}$. Call Γ the *Gelfand-Zetlin subalgebra* of U or *GZ-subalgebra.* In this case the Harish-Chandra U-modules with respect to Γ are called the *Gelfand-Zetlin modules* (or *GZ-modules*) [2]. Respecitvely, we shall denote **GZ** and **GZ**(D) the categorie of GZ-modules and that of GZ-modules with the support in D (where $D \subseteq \mathrm{cfs}(\Gamma)$). We shall also write in this case $\mathcal{U}$ for the category $\mathcal{A}_{U,\Gamma}$ (cf. section 1.5).

Proposition 20. *Z_m is the polynomial algebra in m variables $\{c_{km} \mid k = 1..m \}$ where*

$$c_{km} = \sum_{i_1,i_2,...,i_k=1..m} e_{i_1i_2}e_{i_2i_3}...e_{i_ki_1}$$

(cf.[3]).

Put $\mathcal{L} = K^{n(n+1)/2}$. The elements of $\mathcal{L}$ will be called "tableaux" and considered as double indexed families:

$$\ell = (\ \ell_{im} \mid m = 1..n;\ i = 1..m\ )$$

Denote $\mathcal{L}^+$ the subset of $\mathcal{L}$ consisting of all tableaux ℓ such that $\ell_{im} \in \mathbf{Z}$ and $\ell_{im} \geq \ell_{i,m-1} > \ell_{i+1,m}$ for all possible values of i,m. For any vector $\alpha = (\alpha_1,\alpha_2,...,\alpha_n) \in K^n$ let

$$\mathcal{L}_\alpha = \{\ \ell \in \mathcal{L} \mid \ell_{in} = \alpha_i \text{ for } i = 1..n\ \}$$

and $\mathcal{L}_\alpha^+ = \mathcal{L}_\alpha \cap \mathcal{L}^+$. Clearly, $\mathcal{L}_\alpha^+ \neq \emptyset$ if and only if $\alpha_i \in \mathbf{Z}$ and $\alpha_i > \alpha_{i+1}$ for all possible i.

It is well-known that all finite-dimensional U-modules are Gelfand-Zetlin ones. Namely, the following statement holds (cf. [3]).

Proposition 21. *Let M be a finite-dimensional simple U-module. Then M possesses a base $\{ [\ell] \mid \ell \in \mathcal{L}_\alpha^+ \}$ for some $\alpha \in K^n$ such that:*

$$c_{km}[\ell] = c_{km}(\ell)[\ell],$$

$$E_m^{\pm}[\ell] = \sum_{i=1}^{m} a_{im}^{\pm}(\ell)[\ell \pm \delta^{im}]$$

where $E_m^+ = e_{m,m+1}$; $E_m^- = e_{m+1,m}$ $(m = 1..n-1)$;

$$c_{km}(\ell) = \sum_{i=1}^{m} (\ell_{im} + m)^k \prod_{j \neq i} \left(1 - \frac{1}{\ell_{im} - \ell_{jm}}\right)$$

$$a_{im}^{\pm}(\ell) = \mp \frac{\prod_j (\ell_{j,m\pm 1} - \ell_{im})}{\prod_{j \neq i} (\ell_{jm} - \ell_{im})}$$

(here $\delta^{im} \in \mathcal{L}$ is the Kronecker symbol: $\delta_{jk}^{im} = 1$ if $i = j, m = k$ and 0 otherwise).

This base is called the *Gelfand-Zetlin base* of M. To precise α, we shall denote $M = M^\alpha$. Remark that the dominant weight of M^α is $(\alpha_1 + 1, \alpha_2 + 2, \ldots, \alpha_n + n)$.

We shall also widely use the following *Harish-Chandra Theorem* (cf. [1]).

Proposition 22. *Let $u \in U$ is such that $uM = 0$ for any finite-dimensional simple U-module M. Then $u = 0$.*

Consider the polynomial algebra Λ in $n(n+1)/2$ variables λ_{im} where $m = 1..n$; $i = 1..m$. Identify Λ with the algebra of polynomial functions on $\mathcal{L}$ putting $\lambda_{im}(\ell) = \ell_{im}$. Then $\mathcal{L}$ is identified with cfs(Λ). PROPOSITION 21 allows to define the homomorphism $\iota : \Gamma \to \Lambda$ which maps

$$c_{km} \longmapsto \sum_{i=1}^{m} (\lambda_{im} + m)^k \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{im} - \lambda_{jm}}\right)$$

It is not difficult to check that it is really a polynomial of degree k in λ_{im} of the form $\sum_i \lambda_{im}^k + h$ with $\deg h < k$.

The symmetric group S_m acts on Λ permuting λ_{im} ($i = 1..m$; m fixed). Thus the direct product $S = \prod_{m=1}^{n} S_m$ acts on Λ. As the power sums are algebraically independent and generate the algebra of the symmetric polynomials, we obtain the following

Corollary 23. ι *is an inclusion and its image coincides with the algebra of invariants* Λ^S*. In particular,* Γ *is the polynomial algebra in* c_{km} $(m = 1..n;\ k = 1..m)$.

From now on identify Γ with its image in Λ. This inclusion induces the surjection $\pi : \mathcal{L} \to \mathrm{cfs}(\Gamma)$ which identifies $\mathrm{cfs}(\Gamma)$ with the orbit set $\mathcal{L}/S$. If M is a GZ-module, write $M(\ell)$ instead $M(\pi(\ell))$ for $\ell \in \mathcal{L}$, $\mathbf{GZ}(D)$ instead of $\mathbf{GZ}(\pi(D))$ for $D \subseteq \mathcal{L}$ etc.

Let $\mathcal{L}_0$ be the subgroup of $\mathcal{L}$ generated by all δ^{im} $(i = 1..m;\ m = 1..n-1)$. For two elements $\mathbf{m}, \mathbf{n} \in \mathrm{cfs}(\Gamma)$ put $\mathbf{m} \equiv \mathbf{n}$ provided there exist $\ell, \ell' \in \mathcal{L}$ such that $\mathbf{m} = \pi(\ell)$, $\mathbf{n} = \pi(\ell')$ and $\ell - \ell' \in \mathcal{L}_0$. Of course, it is an equivalence relation on $\mathrm{cfs}(\Gamma)$. Denote Ω the set of equivalence classes of $\equiv$. Define also two subsets, $\mathcal{L}_1$ and $\mathcal{L}_2$, in $\mathcal{L}$:

$$\mathcal{L}_1 = \{\ \ell \mid \ell_{im} - \ell_{jm} \notin \mathbf{Z} \text{ for all } i \neq j \text{ and } m = 2..n-1\ \}$$

$$\mathcal{L}_2 = \mathcal{L}_1 \cap \{\ \ell \mid \ell_{im} - \ell_{j,m+1} \notin \mathbf{Z} \text{ for all } i, j \text{ and } m = 1..n-1\ \}$$

Evidently, $\mathcal{L}_1$ and $\mathcal{L}_2$ are stable under the congruence modulo $\mathcal{L}_0$ and under the action of the group S. So their images in Ω are well-defined. Denote them Ω_1 and Ω_2 respectively. Remark that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are dense in Zarisky topology on $\mathcal{L}$. Moreover, if $K = \mathbf{C}$, they are dense in usual (euclidean) topology as well.

The main theorem of this chapter is the following one.

Theorem 24.

1. *The Gelfand-Zetlin subalgebra is a Harish-Chandra subalgebra of* U.
2. $\mathcal{U} = \coprod_{D \in \Omega} \mathcal{U}(D)$
3. *If* $\ell \in \mathcal{L}_1$*, then there exists the unique simple GZ-module* M *with* $M(\ell) \neq 0$*. Moreover, in this module* $\dim(M(\ell)) = 1$.
4. *If* $D \in \Omega_2$*, then there exists the unique simple GZ-module* M *in* $\mathbf{GZ}(D)$*. Moreover,* $\mathrm{Supp}(M) = D$.

2.2. Some identities in U.

For any element $x \in M^\alpha$ and any tableaux $\ell \in \mathcal{L}^+_\alpha$ let x_ℓ be its $[\ell]$-coefficient with respect to GZ-basis, i.e.

$$x = \sum_{\ell \in \mathcal{L}^+_\alpha} x_\ell[\ell]$$

(cf. Proposition 21). For $u \in U$ denote $\mathcal{L}_u$ the set of all such tableaux $\delta \in \mathcal{L}$ that there exist $\ell \in \mathcal{L}^+$ and $\sigma \in S$ with $(u[\ell])_{\ell+\sigma(\delta)} \neq 0$. As U is generated by the elements $E_m^{\pm}$ $(m = 1..n-1)$, it follows from PROPOSITION 21 that $\mathcal{L}_u$ is finite and $\mathcal{L}_u \subseteq \mathcal{L}_0$. Say that u *relates* ℓ *with* ℓ' provided $\ell' = \sigma(\ell + \delta)$ for some $\sigma \in S$ and $\delta \in \mathcal{L}_u$. Denote $u(\ell)$ the set of all $\ell' \in \mathcal{L}$ such that u relates ℓ with ℓ'. Thus, for any $\ell \in \mathcal{L}^+$ we have:

$$u[\ell] = \sum_{\delta \in \mathcal{L}_u} \theta(u, \ell, \delta)[\ell + \delta]$$

for certain coefficients $\theta(u, \ell, \delta) \in K$ (some of them may be 0).

Any $\delta \in \mathcal{L}$ defines an automorphism $\lambda \mapsto \lambda^\delta$ of Λ where $\lambda_{im}^\delta = \lambda_{im} + \delta_{im}$.

For any $z \in \Gamma$ and $u \in U$ form the polynomial

$$F_{u,z}(T, \lambda) = \prod_{\delta \in \mathcal{L}_u} (T - z^\delta)$$

Clearly, $F_{u,z} \in \Gamma[T]$, as $\mathcal{L}_u$ is, by definition, stable under the action of S.

Lemma 25. *Let* $z \in \Gamma$ *and* $F_{u,z} = \sum_i T^i \phi_i$ *where* $\phi_i \in \Gamma$ *and* i *runs through all possible multy-indeces. Then* $\sum_i z^i u \phi_i = 0$.

Proof. By PROPOSITIONS 21 and 22, we need only to prove that $\sum_i z^i u \phi_i[\ell] = 0$ for any $\ell \in \mathcal{L}^+$. But

$$\sum_i z^i u \phi_i[\ell] = \sum_i z^i u \phi_i(\ell)[\ell] = \sum_i z^i \phi_i(\ell) \sum_{\delta \in \mathcal{L}^u} \theta(u, \ell, \delta)[\ell + \delta] =$$

$$= \sum_{\delta \in \mathcal{L}^u} \theta(u, \ell, \delta) \sum_i z(\ell + \delta)^i \phi_i(\ell)[\ell + \delta] = \sum_{\delta \in \mathcal{L}^u} \theta(u, \ell, \delta) F_{u,z}(z^\delta(\ell), \ell)[\ell + \delta] = 0$$

Q.E.D.

Remark. The same result remains valid for $z \in Z_m$ if we replace $F_{u,z}$ by

$$F_{u,z,m}(T, \lambda_m) = \prod_{\delta \in \mathcal{L}_{u,m}} (T - z^\delta)$$

where $\lambda_m = (\lambda_{1m}, \lambda_{2m}, ..., \lambda_{mm})$ and $\mathcal{L}_{u,m}$ denotes the set of the m-th rows $(\delta_{1m}, \delta_{2m}, ..., \delta_{mm})$ of all elements $\delta \in \mathcal{L}_u$.

Corollary 26. Γ *is a Harish-Chandra subalgebra in* U.

Proof. Evidently, $F_{u,z} = T^k + \sum_{i<k} T^i \phi_i$ for $k = \mathrm{card}(\mathcal{L}_u)$. So, by LEMMA 25, $z^k u \in \sum_{i=1}^{k-1} z^i u \Gamma$. As Γ is a finitely generated algebra, it follows that $\Gamma u \Gamma$ is a finitely generated Γ-module. But the standard involution of U (mapping $g \in \mathcal{G}$ to $-g$, cf. [1]) maps Γ to Γ. So $\Gamma u \Gamma$ is also finitely generated as left Γ-module Q.E.D.

Corollary 27. *Suppose that* $(\mathbf{m},\mathbf{n}) \in X_u$ *where* $\mathbf{m} = \pi(\ell)$, $\mathbf{n} = \pi(\ell')$. *Then* $\ell' \in u(\ell)$.

Proof. Let $\ell' \notin u(\ell)$, i.e. $\mathbf{n} \neq \pi(\ell+\delta)$ for all $\delta \in \mathcal{L}_u$. Then there exists $z \in \Gamma$ lying in all $\pi(\ell+\delta)$ but not in $\mathbf{n}$. As $(\mathbf{m},\mathbf{n}) \in X_u$, there exists $v \in \Gamma u \Gamma / \Gamma u \mathbf{m}$ such that $v \neq 0$ and $\mathbf{n}v = 0$. But we have:

$$0 = \sum_i z^i u \phi_i = \sum_i z^i u(\phi_i - \phi_i(\ell)) + \sum_i z^i u \phi_i(\ell) =$$

$$= u_0 + \sum_i z^i \phi_i(\ell) u = u_0 + F_{u,z}(z,\ell)u$$

where $u_0 \in \Gamma u \mathbf{m}$, whence

$$0 = F_{u,z}(z,\ell)v = F_{u,z}(z(\ell'),\ell)v = \prod_{\delta \in \mathcal{L}_u} (z(\ell') - z(\ell+\delta))v = z(\ell')^k v$$

This is a contradiction as $v \neq 0$, $z(\ell') \neq 0$ Q.E.D.

Corollary 28. $\Delta \subseteq \equiv$, *i.e.* $(\mathbf{m},\mathbf{n}) \in \Delta$ *implies* $\mathbf{m} \equiv \mathbf{n}$. [3]

COROLLARY 26 coincides with *p.1.* of THEOREM 24 and COROLLARY 28 evidently implies *p.2.* of it. To prove the rest of the theorem, we need the following observations.

PROPOSITION 21 implies that the coefficients $\theta(u,\ell,\delta)$ are rational functions in ℓ_{im}. So they can be considered as elements of the field of fractions Q of Λ which we denote $\theta(u,\lambda,\delta)$. Moreover, the denominator of $\theta(u,\lambda,\delta)$ is a product of some of $\lambda_{im} + \lambda_{jm} - k$ $(i \neq j)$, where k is some integer. Thus $\theta(u,\ell,\delta)$ is defined for any $\ell \in \mathcal{L}_1$. Remark that $\theta(u,\ell,0)$ is obviousely S-invariant. Hence, it lies in Q^S which is the field of fractions of $\Lambda^S = \Gamma$.

Lemma 29. *Let again* $z \in \Gamma$. *Put* $\theta_u = \theta(u,\lambda,0) = \beta_u/\gamma_u$ *where* $\beta_u, \gamma_u \in \Gamma$ *and*

$$F^0_{u,z}(T) = \prod_{\delta \in \mathcal{L}_u \setminus 0} (T - z^\delta) = \sum_i T^i \psi_i$$

Then

$$\gamma_u \sum_i z^i u \psi_i = \beta_u \sum_i z^i \psi_i$$

The proof is quite the same as that of LEMMA 25, so we omit it.

[3] cf. section 1.4 for the definition of the relation Δ.

2.3. Modules $\mathcal{M}(L)$.

Take a coset $L \in \mathcal{L}/\mathcal{L}_0$ and suppose that $L \subset \mathcal{L}_1$. Consider the vector space $\mathcal{M}(L)$ with the basis $\{ [\ell] \mid \ell \in L \}$ and put, for every $u \in U$:

$$u[\ell] = \sum_{\delta \in \mathcal{L}_u} \theta(u, \ell, \delta)[\ell + \delta]$$

Propositions 21 and 22 evidently imply then that $\mathcal{M}(L)$ becomes a GZ-module over U with $\mathrm{Supp}\mathcal{M}(L) = L$ and $\dim\mathcal{M}(L)(\ell) = 1$ for all $\ell \in L$.

For any $\ell \in L$ denote $\mathcal{M}_\ell$ the submodule of $\mathcal{M}(L)$ generated by $[\ell]$.

Theorem 30. *There exists the unique maximal submodule $\mathcal{M}'_\ell \subset \mathcal{M}_\ell$ and the factor-module $V_\ell = \mathcal{M}_\ell / \mathcal{M}'_\ell$ is the unique simple GZ-module with $V_\ell(\ell) \neq 0$.*

Proof. As $\dim\mathcal{M}_\ell(\ell) = 1$, $N(\ell) = 0$ for any proper submodule $N \subset \mathcal{M}_\ell$ which implies the existence and uniqueness of $\mathcal{M}'_\ell$. Hence, V_ℓ is really a well-defined simple GZ-module with $\dim V_\ell(\ell) = 1$. Its uniqueness follows from THEOREM 18 and the next fact.

Proposition 31. *If $\mathbf{m} = \pi(\ell)$ and $\ell \in \mathcal{L}_1$, then $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is isomorphic to a factor-algebra of $\Gamma(\mathbf{m})$.* [4]

Proof. Take any $u \in U$. If $0 \notin \mathcal{L}_u$, then the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is zero by COROLLARY 27. If $0 \in \mathcal{L}_u$, find $z \in \Gamma$ such that $z^\delta \in \mathbf{m}^m$ for all $\delta \in \mathcal{L}_u \setminus 0$ and $z - 1 \in \mathbf{m}^n$. Use LEMMA 29. Here all $\psi_i \in \mathbf{m}^m$ except $\psi_k = 1$ for $k = \mathrm{card}(\mathcal{L}_u \setminus 0)$. So we have $\gamma_u z^k u = \beta_u z^k + u_0$ where $u_0 \in U\mathbf{m}^m$, whence in $U/(\mathbf{m}^n U + U\mathbf{m}^m)$ the images of $\gamma_u u$ and β_u coinside. But as $\ell \in \mathcal{L}_1$, the image of γ_u in $\Gamma(\mathbf{m})$ is invertible. Hence the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ coinsides with that of β_u / γ_u Q.E.D.

Theorem 30 implies *p.3.* of THEOREM 24. At last, *p.4.* of it is now a consequence of the following theorem.

Theorem 32. *If $D \in \Omega_2$, then all objects in $\mathcal{U}(D)$ are isomorphic.*

Proof. Let $\eta \in \Gamma$ and $u = E^-_m \eta E^+_m$. Denote also $\theta = \theta(u, \lambda, 0) = \beta/\gamma$ with $\beta, \gamma \in \Gamma$. PROPOSITION 21 implies that

$$\theta = \sum_{i=1}^{m} a^-_{im}(\lambda + \delta^{im}) a^+_{im}(\lambda) \eta(\lambda + \delta^{im})$$

[4] Probably, in the case $\mathcal{U}(\mathbf{m}, \mathbf{m}) \simeq \Gamma(\mathbf{m})$ but we have no proof of it. At least, $\mathcal{U}(\mathbf{m}, \mathbf{m}) \neq 0$ as there exist GZ-modules M with $M(\ell) \neq 0$.

(cf. ibid. for notations). Suppose that $\ell \in \mathcal{L}_2$ and put $\mathbf{m} = \pi(\ell)$, $\mathbf{m}_i = \pi(\ell + \delta^{im})$. As $\ell \in \mathcal{L}_1$, we have $\gamma(\ell) \neq 0$. Moreover, the elments ℓ, $\ell + \delta^{im}$, $\ell + \delta^{jm}$, $\ell + \delta^{im} - \delta^{jm}$ $(j \neq i)$ lie in different S-orbits. Hence, they have different images under π and we are able to choose η and $z \in \Gamma$ such that:

$$\eta(\ell + \delta^{im}) = 1,\ \eta(\ell + \delta^{jm}) = 0 \text{ for } j \neq i$$

$$z(\ell) = 1,\ z(\ell + \delta^{im} - \delta^{jm}) = 0 \text{ for } j \neq i$$

Now use LEMMA 29. Remark that in our case $\mathcal{L}_u \setminus 0 = \{\ \delta^{im} - \delta^{jm} \mid j \neq i\ \}$. Therefore, we obtain that $\gamma(\mathbf{m}) \neq 0$ and all $\psi_s(\mathbf{m}) = 0$ except $\psi_k(\mathbf{m}) = 1$ for $k = \operatorname{card}(\mathcal{L}_u \setminus 0)$. Hence, the image of u in $\mathcal{U}(\mathbf{m}, \mathbf{m})$ is invertible. Denote e_i^+ the image of E_{im}^+ in $\mathcal{U}(\mathbf{m}, \mathbf{m}_i)$ and e_i^- the image of $E_{im}^- \eta$ in $\mathcal{U}(\mathbf{m}_i, \mathbf{m})$. It follows then (just as in the proof of PROPOSITION 31) that $e_i^- e_i^+$ is invertible and e_i^+ is left invertible. Quite analogouse calculation shows that e_i^+ is right invertible. Thus e_i^+ is invertible and $\mathbf{m} \simeq \mathbf{m}_i$ in $\mathcal{U}$. As i, m were arbitrary and $\mathcal{L}_0$ is generated by δ^{im} , it implies the statement Q.E.D.

Corollary 33. *If $L \subset \mathcal{L}_2$, the module $\mathcal{M}(L)$ is the unique simple GZ-module in* $\mathbf{GZ}(L)$.

Now THEOREM 24 is completely proved.

Conjecture. *For any $D \in \Omega$ the set* $\operatorname{Irr}(U, \Gamma, D)$ *is finite.*

Really, this conjecture would follow from the following two:

1. For any $\mathbf{m} \in \operatorname{cfs}(\Gamma)$ *the subalgebra* Γ *is big at the point* $\mathbf{m}$, *hence the set* $\operatorname{Irr}(\mathbf{m})$ *is finite* (cf. COROLLARY 19). *2. For any $D \in \Omega$ there are only finitely many non-isomorphic objects in $\mathcal{U}(D)$.*

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ALGEBRAS ASSOCIATED TO BRUHAT INTERVALS AND POLYHEDRAL CONES

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ABSTRACT. For any Chevalley (Bruhat) interval or face lattice of a polyhedral cone such that the Kazhdan-Lusztig polynomial or g-vector associated to each subinterval equals one, we construct an integral quasi-hereditary algebra C over the symmetric algebra of a naturally associated real vector space U. Under extension of scalars to the reals, C becomes isomorphic to a quasi-hereditary Koszul algebra with quasi-hereditary quadratic dual which has been previously defined by the author. We study a class of Koszul algebras such that both the algebra and its quadratic dual have properties similar to those of integral graded quasi-hereditary algebras, and show that for general reasons, C is a specialization under $t \mapsto 1$ of a $S(U')$-algebra of this type, where $U' = U \oplus \mathbb{R}t$.

Introduction

Kazhdan and Lusztig [14,15] have associated to an arbitrary Coxeter group W a family of polynomials, now called Kazhdan-Lusztig polynomials, which (for crystallographic Coxeter groups) have many remarkable known or conjectural applications within representaion theory (e.g. of algebraic and Lie groups, semisimple complex and Kac-Moody Lie algebras) and geometry (Schubert varieties). These polynomials may be regarded as associated to closed intervals in Chevalley (Bruhat) order on W, and the construction of the polynomials may be extended to twisted versions of these orders defined in [7]. In another context, a similar family of polynomials has been associated to face lattices of polyhedral cones by Stanley [19].

It seems likely that many (known or conjectured) interesting common properties of these families of polynomials in general are combinatorial shadows of corresponding properties of a rich variety of previously unknown algebraic and geometric objects. Apart from its own great intrinsic interest, the problem of finding and studying these objects is important because of probable applications to contexts in which Kazhdan-Lusztig and Stanley polynomials already play a role.

The Kazhdan-Lusztig conjectures (now proved by Beilinson-Bernstein and Brylinski-Kashiwara) and work of Beilinson-Ginsburg-Soergel [1] imply that the Kazhdan-Lusztig polynomials for a finite Weyl group have a natural interpretation in the representation theory of a finite-dimensional graded algebra A. The ungraded version of A may be constructed as the endomorphism algebra of a minimal projective generator of a regular integral block of category $\mathcal{O}$ for the semisimple complex Lie algebra associated to the Weyl group. Let X

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denote the Weyl group in Chevalley order, and $K = \mathbb{C}$. The algebra A has the following properties (see [1])

(a) $A = \oplus_{n \in \mathbb{N}} A_n$ is a positively graded K algebra with A_0 isomorphic to a product of copies of K
(b) A has a graded algebra anti-involution fixing A_0 elementwise
(c) A is Koszul
(d) A is quasi-hereditary with weight poset X and its quadratic dual algebra is quasi-hereditary with weight poset X^{op}.

In [9], we called any algebra (over any field K) with these properties (for any finite poset X) a $\mathcal{O}$ algebra with weight poset X.

Let Γ denote either a suitable poset in one of the twisted orders on a Coxeter group or the face lattice of a polyhedral cone, and let $q_{x,y}$ for $x, y \in \Gamma$ denote the (suitably renormalized) Kazhdan-Lusztig-Stanley polynomial (see 1.2–1.3 for details).

The following conjecture was made in [9], and established in the "multiplicity one" situation in which each non-zero $q_{x,y}$ for $x, y \in \Gamma$ is a power of v.

Conjecture. *Let Γ be as in 1.2 or 1.3. Then there exist two (in general non-isomorphic) $\mathcal{O}$ algebras A, B over $\mathbb{R}$, each with weight poset Γ, which both have the property that for any $x, y \in \Gamma$,*

$$q_{x,y} = \sum_{n \in \mathbb{N}} [M(y) : L(x)\langle n \rangle] v^n$$

where $[M : L]$ denotes the multiplicity of L as a composition factor of M, $L(x)\langle n \rangle$ is the simple module corresponding to $x \in \Gamma$, graded so as to appear in degree n and $M(y)$ denotes a graded Verma (Weyl) module with $L(y)$ as quotient.

The most basic difficulty with this conjecture has been finding a usable construction of A and B in general. It appears very likely that A and B should both be obtained by extension of scalars from a single graded algebra C over $S(U) \otimes_{\mathbb{R}} S(U)^{\mathrm{op}}$, where $S(U)$ is the symmetric algebra of the vector space U of 1.2 or 1.3, graded so U appears in degree 2. In fact, this is true over $\mathbb{C}$ by Soergel's work [18] if Γ is a finite Weyl group in Chevalley order, taking A, B both to be the algebras of blocks of $\mathcal{O}$ for the corresponding complex semisimple Lie algebra and C to be the endomorphism ring of an explicitly defined $(S(U), S(U))$-bimodule. While this paper was in preparation, I have found a natural candidate for C in general. Regarded as $S(U)$-algebra with either the left or right $S(U)$-module action, C should be a graded quasi-hereditary $S(U)$ algebra of split type (in the sense of [4, 3.2, 4.3]) and the graded $\mathbb{R}$-algebras and $A := \mathbb{R} \otimes_{S(U)} C$, $B := C \otimes_{S(U)} \mathbb{R}$ should satisfy Conjecture 1 (see also 3.17).

In Section 1 of this paper, we construct such a algebras C for multiplicity one intervals. These algebras $C = C_X$ will be described both explicitly by generators and relations, and as subalgebras of the endomorphism ring of a suitable $(S(U), S(U))$-bimodule (C is actually the full endomorphism ring, but we don't show this here).

The rings C constructed here for multiplicity one intervals are not Koszul rings except in trivial situations. However, let $U' = U \oplus \mathbb{R}t$ be the direct sum of U with a one-dimensional vector space with basis element t. Grade $S(U')$ so $K = S(U')_0$ and $U' = S(U')_1$. There is a unique graded K-algebra homomorphism $S(U) \to S(U')$ such that $\chi \in U = S(U)_2$ maps to $\chi t \in S(U')_2$, and we regard $S(U')$ as a graded $S(U)$-algebra by means of this homomorphism. The graded rings $A' = S(U') \otimes_{S(U)} C$ and $B' = C \otimes_{S(U)} S(U')$ obtained by extension of the left and right $S(U)$-algebra structure on C are Koszul. They are (trivially) quasi-hereditary $S(U')$ algebras. For a proper understanding of A', B', one must also describe their quadratic dual algebras; though the duals are not integral quasi-hereditary (see 3.18), they have many analogous prperties.

Now the algebra C has a family $M(x)$ of "big" Verma modules (such that as for finite-dimensional graded quasi-hereditary algebras, the projective indecomposable C modules have filtrations with degree shifts of various $M(x)$ as successive subquotients). A main observation (3.15) of this paper is that the properties of C in the preceeding paragraph, as well as the facts that $A := \mathbb{R} \otimes_{S(U)} C$ and $B := C \otimes_{S(U)} \mathbb{R}$ are $\mathcal{O}$ algebras, all follow for general reasons from a slight refinement of the statement that C is quasi-hereditary, a homological purity condition on the modules $M(x)$ and the Koszulity of $S(U')$. The proof of this is given in Sections 2 and 3. In Section 2, we indicate the adjustments to the definition of integral quasi-hereditary algebra necessary to describe the quadratic duals of A', B'. In Section 3, we study the class of $\tilde{\mathcal{O}}$ algebras, obtained by performing the corresponding adjustment to (d) in the definition of $\mathcal{O}$ algebra.

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1. Some quasi-hereditary algebras over symmetric algebras

We begin by recalling one definition of (renormalized versions of) the polynomials defined by Kazhdan-Lusztig and Stanley. Then we discuss some natural labellings of the associated posets, and use them to construct a $(S(U), S(U))$-bimodule associated to the poset. Finally, in this section we show that in the case of a multiplicity one poset (i.e. all non-zero renormalized Kazhdan-Lusztig-Stanley polynomials are powers of the indeterminate) a certain subalgebra (which may be alternatively described by generators and relations) of the endomorphism algebra of this bimodule is quasi-hereditary and by extension of scalars, gives algebras A, B satisfying the conjecture in the introduction.

1.1. Let $\mathcal{R} = \mathbb{Z}[v, v^{-1}]$ denote the ring of integral Laurent polynomials in an indeterminate v. There is a ring involution of $\mathcal{R}$ determined by $\overline{v} = v^{-1}$. Now fix a finite poset Γ. The ring involution of $\mathcal{R}$ extends to a ring involution, also to be denoted $r \mapsto \overline{r}$, of the ring $\mathrm{Mat}_\Gamma(\mathcal{R})$ of $\Gamma \times \Gamma$-matrices over $\mathcal{R}$, defining the bar of a matrix entrywise. One has the following result, due to Gabber [16];

Proposition. *Suppose that $r \in \mathrm{Mat}_\Gamma(\mathcal{R})$ is upper unitriangular (i.e. $r_{x,x} = 1$ for all $x \in \Gamma$ and $r_{x,y} = 0$ unless $x \leq y$) and that $r^{-1} = \overline{r}$. Then there is a unique upper unitriangular $q \in \mathrm{Mat}_\Gamma(\mathcal{R})$ such that $\overline{q} = qr$ and $q_{x,y} \in v\mathbb{Z}[v]$ for $x < y$ in Γ.*

We now recall how the above construction applies in two situations of particular interest to obtain (up to some renormalization) the important families of polynomials defined by Kazhdan-Lusztig and by Stanley. It has been conjectured that in these cases, the polynomials $q_{x,y}$ have non-negative coefficients.

1.2. Suppose that Γ is the face lattice of a polyhedral cone C in a real Euclidean space U i.e Γ consists of the facets of C, ordered by inclusion. For a facet x, let $l(x)$ denote the dimension of the subspace spanned by x. We define

$$r_{x,y} = \begin{cases} (v^{-1} - v)^{l(y)-l(x)} & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Now its well-known that the simplicial complex of any non-empty open interval in Γ is a combinatorial sphere (for short, we call a poset with this property spherical). This implies that for $x \leq y$ in Γ, the Möbius function $\mu(x,y) = (-1)^{l(y)-l(x)}$, and this is easily seen to be equivalent to the condition $r^{-1} = \overline{r}$. The polynomials $q_{x,y}$ so obtained are, up to some renormalization, g-vectors as defined by Stanley [19].

1.3. Consider a Coxeter system (W,S) (with S finite), which we assume without loss of generality is realized in one of the standard ways (see e.g. [7]) as a group of linear transformations associated to a suitable (reduced, possibly infinite) root system in a finite-dimensional real vector space U. Let $\leq$ denote Chevalley (Bruhat order) or its reverse. There is by [14,15] a well-defined family of elements $r_{x,y}$ for $x \leq y$ in W, with $r_{x,y} = 0$ if $x \not\leq y$, satisfying the initial condition $r_{x,x} = 1$ for all $x \in W$ and recurrence formulae

$$r_{x,y} = r_{sx,sy} + (v^{-1} - v) r_{x,sy}, \qquad r_{sx,y} = r_{x,sy}$$

for $x, y \in W$ and $s \in S$ with $x \leq y$ and $x \leq sx$, $sy \leq y$. Then for any interval Γ in W, the matrix $\{r_{x,y}\}_{x,y \in \Gamma}$ is known to satisfy $r^{-1} = \overline{r}$. If $\leq$ is Chevalley order (resp., reverse Chevalley order), then, again up to different normalization, the $q_{x,y}$ are the inverse Kazhdan-Lusztig polynomials $Q_{x,y}$ (resp., Kazhdan-Lusztig polynomials $P_{y,x}$) constructed in [14,15] from the Iwahori-Hecke algebra of W.

More generally, take $\Gamma \neq \emptyset$ to be a finite closed spherical subset of W in one of the twisted orders on W defined in [7] (we define a closed subset of a poset to be one which is an intersection of an ideal and a coideal). We assume that Γ satisfies the orientability condition [9, (3.15)(b)] (which is automatically true if Γ is contained in some finite subinterval of W). For $x, y \in \Gamma$, define $r_{x,y}$ to be the element $R_{\preceq}$ of [7,3.1] with $q^{\frac{1}{2}}$ replaced by v. Then $r^{-1} = \overline{r}$ by [loc cit, 3.6(3)] and one can again define q.

The algebra which we associate in this section to (multiplicity-free) Γ as above will be defined from a natural labelling of the poset. We first describe the relevant properties of this labelling in general and then indicate how the labelling is defined for the posets Γ of 1.2 and 1.3. The discussion improves on earlier versions in [8,9] in which linear transformations were missing as an element of the labelling in the polyhedral cone case.

1.4. Let U denote a finite-dimensional vector space over a field K, and let $\langle , \rangle : U \times U^* \to K$ be the natural pairing with its dual. For $\alpha \in U$, $\alpha^\vee \in U^*$ with $\langle \alpha, \alpha^\vee \rangle \neq 1$, we define $s_{\alpha,\alpha^\vee} \in \mathrm{GL}(U)$ by $u \mapsto u - \langle u, \alpha^\vee \rangle \alpha$ and similarly define $s_{\alpha^\vee,\alpha} \in \mathrm{GL}(U^*)$. For $g \in \mathrm{GL}(U)$, let $g^t \in \mathrm{GL}(U^*)$ be the adjoint i.e. $\langle gu, \phi \rangle = \langle u, g^t \phi \rangle$ for all $u \in U$, $\phi \in U^*$.

Using the equations $x s_{\gamma,\gamma^\vee} x^{-1} = s_{x\gamma,(x^{-1})^t\gamma^\vee}$ for $x \in \mathrm{GL}(U)$ and $s^{-1}_{\gamma,\gamma^\vee} = s_{s^{-1}_{\gamma,\gamma^\vee}\gamma,-\gamma^\vee}$, one readily checks that for any $x, y \in \mathrm{GL}(U)$, $\gamma \in U$ and $\gamma^\vee \in U^*$ with $\langle \gamma, \gamma^\vee \rangle \neq 1$, the following four conditions are equivalent:

(a) $y = s_{\gamma,\gamma^\vee} x$
(b) $(x^{-1})^t = s_{\gamma^\vee,\gamma^\vee} (y^{-1})^t$
(c) $y^{-1} = s_{y^{-1}\gamma,-x^t\gamma^\vee} x^{-1}$
(d) $x^t = s_{x^t\gamma^\vee,-y^{-1}\gamma} y^t$.

Now let X be a finite partially ordered set. Let $E = \{(x,y) \in X \times X \mid x \lessdot y\}$ denote the edge set of the Hasse diagram of X (we always write $x \lessdot y$ to indicate $x \leq y$ and no $z \in X$ satisfies $x < z < y$).

Definition. We define a labelling L of X on U to be a triple $L = (\iota_L, \ell_L, \ell_L^\vee) = (\iota, \ell, \ell^\vee)$ where $\iota: X \to \mathrm{GL}(U)$ is an injective function, and $\ell: E \to U \setminus 0$ and $\ell^\vee: E \to U^* \setminus 0$ are functions such that for any $x \lessdot y \in X$, one has $y = s_{\alpha,\alpha^\vee} x$ where $\alpha = \ell(x,y)$ and $\alpha^\vee = \ell^\vee(x,y)$.

From (a)–(d) above, we then have a number of other labellings, for which we introduce some notation. First, $-L$ denotes the labelling $(\iota, -\ell, -\ell^\vee)$ of X on U where $(-\ell)(x,y) = -\ell(x,y)$ etc. Secondly, we have a labelling L^* of X^{op} on U^* with $\iota_{L^*}(x) = (\iota(x)^{\mathrm{t}})^{-1}$, $\ell_{L^*}(x,y) = \ell^\vee(y,x)$ and $\ell^\vee_{L^*}(x,y) = \ell(y,x)$. Also, we have a labelling L' of X on U with $\iota_{L'}(x) = \iota(x)^{-1}$, $\ell_{L'}(x,y) = y^{-1}\ell(x,y)$ and $\ell^\vee_{L'}(x,y) = -x^{\mathrm{t}}\ell(x,y)$. Note that $L'' = L$, $L^{**} = L$ and $L'^* = -(L^{*\prime})$.

1.5. We describe the two situations of interest. Take $K = \mathbb{R}$.

(a) Suppose that $X = \Gamma$ as in 1.2 is the face lattice of a polyhedral cone C in a real Euclidean space U. We identify U with U^* by means of the inner product on U. For a facet A of U, let $\iota(A) \in \mathrm{GL}(U)$ be the linear transformation which restricts to $-\mathrm{Id}$ on the subspace spanned by A and Id on the orthogonal complement of this subspace. The map $\iota: \Gamma \to \mathrm{GL}(U)$ so defined is injective since the facet A is the intersection of C with the (-1)-eigenspace of $\iota(A)$. For two facets $A \lessdot B$ of C, there is a unique unit vector α in the subspace spanned by B such that α is orthogonal to the linear span of A and has non-negative inner product with every element of B. Setting $\ell(A,B) = \alpha \in U$ and $\ell^\vee(A,B) = 2\alpha \in U^*$, one easily sees one has a labelling $(\iota, \ell, \ell^\vee)$.

(b) Now suppose instead that $X = \Gamma$ is as in 1.3. Take $\iota: \Gamma \to \mathrm{GL}(U)$ to be the inclusion. Let $\Phi \subseteq U$ and $\Phi^\vee \subseteq U^*$ be dual root systems of W on U, with the standard W-equivariant bijection $\alpha \mapsto \alpha^\vee: \Phi \to \Phi^\vee$. For any pair of elements x, y of X such that $x \lessdot y$, yx^{-1} is a reflection, necessarily of the form $s_{\gamma,\gamma^\vee}$ for a unique positive root $\gamma \in \Phi^+$, and we define $\ell(x,y) = \gamma$, $\ell^\vee(x,y) = \gamma^\vee$. This again gives a labelling.

1.6. We now fix a labelling $L = (\iota, \ell, \ell^\vee)$ of a finite poset X on U. We assume from now on that X is spherical and we are given a fixed "length function" $l: X \to \mathbb{Z}$ such that if $x \leq y$, the maximal length of any chain $x = x_0 < \ldots < x_n = y$ in the closed interval $[x,y]$ is $l(y) - l(x)$. In particular, any closed length 2 subinterval $[x,w]$ of X has exactly two atoms. We wish to impose some "non-degeneracy conditions" (a)–(b), (a′)–(b′) on L. First, we assume

- (a) Let I be a closed length 3 subinterval of X. If the subspace of U spanned by the labels $\ell(x,y)$ with $x, y \in I$ and $x \lessdot y$ is two-dimensional, then I has exactly two atoms (equivalently, exactly two coatoms).
- (a′) Same as (a) with ℓ replaced by $\ell^\vee$ and U by U^*.

Now consider a length two subinterval $[x,w]$ of X with atoms y, z. Write $\ell(x,y) = \gamma$, $\ell(x,z) = \delta$, $\ell(y,w) = \alpha$, $\ell(z,w) = \beta$ and similarly $\ell^\vee(x,y) = \gamma^\vee$ etc. We assume

- (b) Any two consecutive vectors in the list $\gamma, \alpha, \beta, \delta, \gamma$ are linearly independent over K.
- (b′) Same as (b) for $\gamma^\vee, \alpha^\vee, \beta^\vee, \delta^\vee, \gamma^\vee$.

(There are some dependencies amongst the various parts of these conditions.) If L satisfies all these conditions, (a)–(b), (a′)–(b′) we will say that L is non-degenerate.

1.7. Consider a spherical poset X with a non-degenerate labelling $(\iota, \ell, \ell^\vee)$. It will be notationally convenient to regard X as a subset of $\mathrm{GL}(U)$ by identifying $x \in X$ with $\iota(x) \in \mathrm{GL}(U)$, and to identify $g \in \mathrm{GL}(U)$ with its contragredient $(g^{\mathrm{t}})^{-1} \in \mathrm{GL}(U^*)$, so we may also regard $X \subseteq \mathrm{GL}(U^*)$.

Fix a length two subinterval $[x, w] = \{x, y, z, w\}$ of X and define $\alpha, \ldots, \delta^\vee$ as at the end of 1.6. Then $wx^{-1} = s_{\alpha,\alpha^\vee} s_{\gamma,\gamma^\vee} = s_{\beta,\beta^\vee} s_{\delta,\delta^\vee} \in \mathrm{GL}(U)$, so by the non-degeneracy conditions 1.6(b)–(b′), $(\mathrm{Id} - wx^{-1})\colon U \to U$ has as image the subspace $K\gamma + K\alpha = K\delta + K\beta$. So there are uniquely detrmined $a, b, c, d \in K$ with $ad - bc \neq 0$ and $bc \neq 0$ such that

(a) $$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Similarly, there exist $e, f, g, h \in K$ with $eh - fg \neq 0$ and $fg \neq 0$ and

(b) $$\begin{pmatrix} \alpha^\vee \\ \beta^\vee \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \gamma^\vee \\ \delta^\vee \end{pmatrix}.$$

We regard the above equation (a) as associated to the labelling L and (b) as the corresponding equation associated to L^*. Then (as we check below) the corresponding equations associated to L' and $L^{*\prime}$ are

(c) $$\begin{pmatrix} y^{-1}\gamma \\ z^{-1}\delta \end{pmatrix} = \begin{pmatrix} -e & g \\ f & -h \end{pmatrix} \begin{pmatrix} w^{-1}\alpha \\ w^{-1}\beta \end{pmatrix}$$

(d) $$\begin{pmatrix} y^{-1}\alpha^\vee \\ z^{-1}\beta^\vee \end{pmatrix} = \begin{pmatrix} -a & c \\ b & -d \end{pmatrix} \begin{pmatrix} x^{-1}\gamma^\vee \\ x^{-1}\delta^\vee \end{pmatrix}$$

and moreover, one has

(e) $$b = g, \quad c = f, \quad a + e = \langle \gamma, \alpha^\vee \rangle, \quad d + h = \langle \delta, \beta^\vee \rangle.$$

These imply easily that L' is also non-degenerate, as also are L^* and $L^{*\prime}$.

To check (d), for instance, note first that for any $\chi \in U$, one has $\chi - s_{\alpha,\alpha^\vee} s_{\gamma,\gamma^\vee} \chi = \chi - s_{\beta,\beta^\vee} s_{\delta,\delta^\vee} \chi$ i.e.

$$\langle \chi, \gamma^\vee \rangle \gamma + \langle \chi, s_{\gamma^\vee,\gamma} \alpha^\vee \rangle \alpha = \langle \chi, \delta^\vee \rangle \delta + \langle \chi, s_{\delta^\vee,\delta} \beta^\vee \rangle \beta.$$

Using

$$\begin{pmatrix} \delta \\ \beta \end{pmatrix} = \frac{1}{b} \begin{pmatrix} d & cb - ad \\ 1 & -a \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \end{pmatrix}$$

in this and linear independence of α, γ, one has since χ is arbitrary that

$$\begin{pmatrix} \gamma^\vee \\ s_{\gamma^\vee,\gamma} \alpha^\vee \end{pmatrix} = \frac{1}{b} \begin{pmatrix} d & 1 \\ cb - ad & -a \end{pmatrix} \begin{pmatrix} \delta^\vee \\ s_{\delta^\vee,\delta} \beta^\vee \end{pmatrix}$$

Now this gives

$$\begin{pmatrix} s_{\gamma^\vee,\gamma} \alpha^\vee \\ s_{\delta^\vee,\delta} \beta^\vee \end{pmatrix} = \begin{pmatrix} -a & c \\ b & -d \end{pmatrix} \begin{pmatrix} \gamma^\vee \\ \delta^\vee \end{pmatrix}.$$

Applying $x^{-1} \in \mathrm{GL}(U^*)$ to this gives (d), while writing $s_{\gamma^\vee,\gamma} \alpha^\vee = \alpha^\vee - \langle \gamma, \alpha^\vee \rangle \gamma^\vee$ in this and using (b) gives half of (e). The other half of (e) and (c) hold by symmetry.

Definition. We will say that the non-degenerate labelling L is symmetric if for each length two subinterval $[x,w]$ as above, one has $b=c$. It follows that all labellings L^*, L' and L'^* are also symmetric.

For example (see [9]) the labellings in 1.5(a)–(b) are symmetric and non-degenerate. These examples have positivity and convexity properties which play a role elsewhere but are not needed in this paper.

1.8. Henceforward, we fix a symmetric, non-degenerate labelling $L=(\iota,\ell,\ell^\vee)$ of a spherical poset X (with length function l) on a K-vector space U. As before, we regard $X\subseteq \mathrm{GL}(U)$. We write $x\xrightarrow{\gamma} y$ to indicate that $x<y$ in X and $\ell(x,y)\coloneqq\gamma$. We then write $\gamma^\vee$ for $\ell^\vee(x,y)$.

In considering bimodules for algebras over our fixed field K, we always assume that the left and right K-actions coincide. We now define a $(S(U),S(U))$-bimodule associated to the labelling L. In the case of a Coxeter group in Chevalley order, the following result follows from [13, (4.3)(b)]. The modules then arose more generally for twisted orders on Coxeter groups as a step in the construction of some modules for the dual nil Hecke ring.

Proposition. *There exists a $(S(U),S(U))$-bimodule N_X with the following properties:*

(a) *as right $S(U)$-module, N_X is free with basis $\{n_x\}_{x\in X}$*
(b) *for $\chi\in U$, one has*

$$\chi n_x = n_x(x^{-1}\chi)-\sum_{x\xrightarrow{\gamma}y}\langle\chi,\gamma^\vee\rangle n_y.$$

Proof. Let N_X be a right $S(U)$-module which is free with basis $\{n_x\}_{x\in X}$. Then N_X has a unique structure of $(T(U),S(U))$-bimodule, where $T(U)$ is the tensor algebra, such that (b) holds. One then just has to check that

(c) $$\chi_1\chi_2 n_x=\chi_2\chi_1 n_x$$

for any $\chi_i\in U$. By definition,

$$\begin{aligned}\chi_1\chi_2 n_x =&\chi_1\Big(n_x x^{-1}(\chi_2)-\sum_{x\xrightarrow{\gamma}y}\langle\chi_2,\gamma^\vee\rangle n_y\Big)\\ =&\Big(n_x x^{-1}(\chi_1)-\sum_{x\xrightarrow{\gamma}y}\langle\chi_1,\gamma^\vee\rangle n_y\Big)x^{-1}(\chi_2)\\ &-\sum_{x\xrightarrow{\gamma}y}\langle\chi_2,\gamma^\vee\rangle\Big(n_y y^{-1}(\chi_1)-\sum_{y\xrightarrow{\beta}z}\langle\chi_1,\beta^\vee\rangle n_z\Big)\\ =&n_x x^{-1}(\chi_1\chi_2)-\sum_{x\xrightarrow{\gamma}y}\Big(\langle\chi_2,\gamma^\vee\rangle n_y y^{-1}(\chi_1)+\langle\chi_1,\gamma^\vee\rangle n_y x^{-1}(\chi_2)\Big)\\ &+\sum_{x\xrightarrow{\gamma}y}\sum_{y\xrightarrow{\beta}z}\langle\chi_1,\beta^\vee\rangle\langle\chi_2,\gamma^\vee\rangle n_z.\end{aligned}$$

Note that $x^{-1}(\chi_2) = y^{-1}s_{\gamma,\gamma^\vee}(\chi_2) = y^{-1}(\chi_2) - \langle \chi_2, \gamma^\vee \rangle y^{-1}(\gamma)$. So to check (c), it is enough to show that for any $z \in X$ with $z > x$ and $l(z) = l(x) + 2$, one has that

$$A_z(\chi_1, \chi_2) := \sum_y \langle \chi_1, \beta^\vee \rangle \langle \chi_2, \gamma^\vee \rangle,$$

(sum over y with $x \xrightarrow{\gamma} y$ and $y \xrightarrow{\beta} z$) is equal to $A_z(\chi_2, \chi_1)$. Expressing the two labels $\ell(x, y)$ with $x < y < z$ as linear combinations of the two labels $\ell(y, z)$ with $x < y < z$ as in 1.7, one sees immediately that this follows from the symmetry condition on L.

1.9 Remarks. (a) The n_x are also a basis in the left $S(U)$-module structure.

(b) One easily checks that the $(S(U), S(U))$-bimodule obtained by swapping the left and right $S(U)$-actions on N_X is isomorphic to that obtained from X with labelling L'.

(c) A similar construction can be given under conditions weaker than the symmetry condition (with a slightly more complicated formula in place of 1.8(b)).

1.10. We recall that Boolean intervals are the posets isomorphic to the inclusion-ordered powerset of a finite set; we define dihedral intervals to be the posets isomorphic to a closed subinterval of $\mathbb{Z}$ in the non-standard partial order $\preceq$ with $x \prec y$ iff $| x | < | y |$.

Assumption. *We now assume for the remainder of this section that every closed length 3 subinterval of X is isomorphic either to a Boolean interval or a dihedral interval.*

We will call a poset satisfying the assumption above "multiplicity one." This is because for $X = \Gamma$ as in 1.2 or 1.3, it is equivalent to the "multiplicity one" requirement that all non-zero (renormalized) Kazhdan-Lusztig-Stanley polynomials for the interval be powers of v (see [9, 3.24, 3.25(a)]).

1.11. Before proceeding any further, lets set down some notation concerning graded K-algebras. For a $\mathbb{Z}$-graded K-algebra $A = \oplus_{n \in \mathbb{Z}}$ let gr A denote the category of $\mathbb{Z}$-graded A-modules $M = \oplus_{n \in \mathbb{Z}} M_n$, with $\mathrm{Hom}_{\mathrm{gr}\ A}(M, N)$ the degree zero homomorphisms. For $p \in \mathbb{Z}$, let $M\langle p \rangle$ denote the module M with grading shifted up by p i.e. $(M\langle p \rangle)_n = M_{n-p}$. For M, N in gr A, let $\mathrm{hom}_A(M, N)$ denote the graded K-module with $\mathrm{hom}_A(M, N)_p = \mathrm{Hom}_{\mathrm{gr}\ A}(M, N\langle -p \rangle)$. Similarly, one has a $\mathbb{Z}$-graded K-algebra $\mathrm{end}_A(M)$. Given a graded K-algebra k and a graded (k, k)-bimodule V (as always with left and right K-actions coinciding), we may form the tensor algebra

$$T_k(V) = \oplus_{n \in \mathbb{N}} V^{\otimes n} = k \oplus V \oplus (V \otimes_k V) \oplus \dots$$

which is again a graded K-algebra in a natural way.

1.12. Regard $S(U)$ as a graded K-algebra so $K = S(U)_0$, $S(U)_2 = U$. For each $x \in X$, let $N_{\geq x}$ be the $(S(U), S(U))$-bimodule associated to the obvious labelling induced by L on the coideal of X generated by x (we'll also use similar abbreviated notations $< x$ etc for subsets of X elsewhere). There is a unique graded $(S(U), S(U))$-bimodule structure on $N_{\geq x}$ which induces the ungraded one just defined and is such that for any $y \geq x$, $n_y \in N_{\geq x}$ has degree $-l(x) + 2l(y)$. Note that $N_{\geq y}\langle l(y) - l(x) \rangle$ is a graded $(S(U), S(U))$-subbimodule of $N_{\geq x}$ for $x \leq y$.

Definition. We form the direct sum $N := \oplus_{x\in X} N_{\geq x}$ of copies of the $N_{\geq x}$ and consider the graded algebra $\tilde{C}_X := \mathrm{end}_{S(U)\otimes_K S(U)^{\mathrm{op}}}(N)$. We regard $\tilde{C}_X$ as a graded $S(U)$-algebra via the left $S(U)$-module structure on N i.e. under the action $(\chi f)(n) = \chi f(n)$ for $\chi \in S(U)$, $f \in \tilde{C}_X$ and $n \in N$ (we call this the left $S(U)$-algebra structure). Similarly, one has a right graded $S(U)$-algebra structure and a graded $S(U)\otimes S(U)^{\mathrm{op}}$-algebra structure on $\tilde{C}_X$. When not otherwise specified, the $S(U)$-algebra structure on $\tilde{C}_X$ will be the left one.

1.13. We now define (by generators and relations) a graded $S(U)$-algebra C_X and give a graded $S(U)$-algebra monomorphism $\theta: C_X \to \tilde{C}_X$. Let k denote the graded (product) $S(U)$-algebra $k = S(U)^{\sharp(X)}$, and denote the primitive idempotents of $k_0 \cong K^{\sharp(X)}$ by $\{b_x\}_{x\in X}$. We consider graded (k,k)-bimodules V with left and right $S(U)$-actions coinciding i.e. graded $(k \otimes_{S(U)} k^{\mathrm{op}})$-modules. One then has $V = \oplus_{x,y\in X} b_x V b_y$ where $b_x V b_y$ is a graded $S(U)$-module. Conversely, given graded $S(U)$-modules $V_{x,y}$, there is a unique up to isomorphism graded $(k \otimes_{S(U)} k^{\mathrm{op}})$-module with $b_x V b_y = V_{x,y}$.

Now let V denote a graded (k,k)-bimodule such that the $S(U)$-module $b_x V b_y$ is free of rank one, generated by an element $b_{x,y}$ of degree one, if $x < y$ or $y < x$, and with $b_x V b_y = 0$ for other x, y in X. We usually abbreviate $b_{x,y}$ as b_{xy}.

Definition. The (positively) graded $S(U)$-algebra C_X is defined as the quotient algebra $C_X = T_k(V)/\langle W\rangle$ where $T_k(V)$ is the tensor algebra of V over k and W is the (k,k)-subbimodule of $k_2 + (V\otimes_k V)_2 = T_k(V)_2$ defined in the section below.

1.14. Suppose that $x,y,z,w \in X$ satisfy $x < y < w$, $x < z < w$ (we allow $y = z$). Let z' denote the atom of $[x,w]$ unequal to z and use 1.4(b) to write

$$\ell(x,y) = c_{zxy;w}\,\ell(z,w) + d_{zxy;w}\,\ell(z',w)$$

where $c_{zxy;w}$, $d_{zxy;w} \in K$. Note that by the symmetry condition,

(a) $$c_{zxy} = c_{xyz}.$$

For $x,y,z \in X$ satisfying $x < y$ and $x < z$, define $r_{zxy} \in T_k(V)_2$ by

(∗) $$r_{zxy} = \begin{cases} b_{zx}\otimes b_{xy} - \sum c_{zxy;w}\, b_{zw}\otimes b_{wy} & \text{if } y \neq z \\ b_{zx}\otimes b_{xy} + \sum c_{zxy;w}\, b_{zw}\otimes b_{wy} + \ell(x,y) b_y & \text{if } y = z \end{cases}$$

where the sum is over $w \in X$ satisfying $y < w$ and $z < w$.

Finally, let W denote the (k,k)-subbimodule of $T_k(V)_2$ spanned as $S(U)$-module by elements of the following types (b), (c):

(b) $$r_{zxy} \text{ for } x,y,z \in X \text{ with } x < y \text{ and } x < z.$$

(c) $$b_{xy_1}\otimes b_{y_1z} - b_{xy_2}\otimes b_{y_2z},\ b_{zy_1}\otimes b_{y_1x} - b_{zy_2}\otimes b_{y_2x},$$
$$\text{for } y_1 \neq y_2, x, z \in X \text{ with } x < y_i < z \ (i = 1,2).$$

1.15 Remark. By 1.7, the element r_{zxy} defined as in [9, 3.7(c)] from the labelling function $(x,y) \mapsto x^{-1}\ell^{\vee}(x,y)$ (for $x < y \in X$) may be obtained formally from the r_{zxy} defined in

1.14 above (from the labelling L of X on U) by omitting any terms $\ell(x,y)b_y$ and replacing each $b_{pq}\otimes b_{qr}$ by pqr to conform to the notation in [9].

1.16. We make some observations about $C = C_X$; all follow easily from the explicit presentation. Let $\pi = \pi_X : T_k(V) \to C_X$ be the natural surjection. We denote the images in C_X under π of the elements b_x, b_{xy} of $T_k(V)$ by b_x, b_{xy}. Note by 1.14(a) that C_X has a graded $S(U)$-algebra anti-involution ω fixing C_0 elementwise and interchanging b_{yx} and b_{xy}. For any $x \le y \ge z$ in X, there is a well defined element b_{xyz} of $C_{2l(y)-l(x)-l(z)}$ such that for any maximal chains $x = x_0 < \ldots < x_n = y$, $y = x_n > \ldots > x_{n+m} = y$, one has $b_{xyz} = b_{x_0x_1}b_{x_1x_2}\ldots b_{x_{n+m-1}x_{n+m}}$. Moreover, these elements b_{xyz} span C as $S(U)$-module. Recall $\tilde{C}_X = \mathrm{end}_{S(V)\otimes S(V)}(\oplus_{x\in X} N_{\geq x})$.

1.17 Theorem. *There is a unique graded $S(U)$-algebra homomorphism $\theta: C_X \to \tilde{C}_X$ with the following properties (a)–(c):*

(a) *for $x \in X$, $\theta(b_x)$ is the obvious (projection) map $N \to N_{\geq x} \to N$*
(b) *for $y < x$ in X, $\theta(b_{yx})$ is the map $N \to N_{\geq x} \to N_{\geq y} \to N$ in which $N_{\geq x} \to N_{\geq y}$ is the monomorphism $n_z \in N_{\geq x} \mapsto n_z \in N_{\geq y}$*
(c) *for $y < x$ in Y, $\theta(b_{xy})$ maps $n_y \in N_{\geq y}$ to $n_x \in N_{\geq x}$.*

Proof. For uniqueness, one has only to show that the conditions uniquely determine the bimodule homomorphisms $\theta(b_{yx})$ for $x < y$. Assume inductively this is so for all $x' < y'$ with $x' > x$. Let $q \geq x$. Then $\theta(b_{yx})(n_q) = n_y$ by (c) if $q = x$. If $q > x$, choose $x' > x$ with $q \geq x'$. By (b), one has that

$$\theta(b_{yx})(n_q) = \theta(b_{yx})(\theta(b_{xx'})(n_q)) = \Big((\theta\circ\pi)(b_{yx}\otimes b_{xx'} - r_{yxx'})\Big)(n_q)$$

is expressible as a linear combination of (known) elements $(\theta(b_{yw}b_{wx'}))(n_q)$ with $w > x'$, $w > y$ and (if $x' = y$) $\theta(\chi b_y)(n_y)$ with χ in U.

To show existence, we use induction on $\sharp(X)$. Suppose that x is a minimal element of X and that the proposition holds for $Y = X\setminus\{x\}$, so we have a homomorphism $\theta': C_Y \to \tilde{C}_Y$, a projection $\pi': T_{k'}(V') \to C_Y$ etc, where we set $e_Y = \oplus_{y\in Y} b_y$, an idempotent in k and identify k' with $e_Y k e_Y$, V' with $e_Y V e_Y \subseteq V$ and $T_{k'}(V')$ with the obvious subalgebra (with different identity) of $T_k(V)$. Let $y > x$. We define a left $S(U)$-module homomorphism $\theta(b_{yx}): N_{\geq x} \to N_{\geq y}$ as follows, by defining $\theta(b_{yx})(n_q)$ for each $q \geq x$ and extending by left $S(U)$-linearity. Set $\theta(b_{yx})(n_q) = n_y$ if $q = x$. If $q > x$, choose $x' > x$ with $q \geq x'$ and set $\theta(b_{yx})(n_q) = \big((\theta'\circ\pi')p_{yxx'}\big)(n_q)$ where we regard $p_{yxx'} := b_{yx}\otimes b_{xx'} - r_{yxx'} \in T_k(V)$ as an element of $T_{k'}(V')$. To see that the definition of $\theta(b_{yx})$ doesn't depend on the choice of the x', its enough (since the order complex of the open interval (x,q) is a combinatorial sphere) to check that $\big((\theta'\circ\pi')p_{yxx'}\big)(n_q) = \big((\theta'\circ\pi')p_{yxx''}\big)(n_q)$ in the case that x' and x'' are the two atoms of some interval $[x,w]$ with $x < w \leq q$ and $l(w) = l(x)+2$. For this, it would suffice to check that $\pi'(p_{yxx'}\otimes b_{x'w}) = \pi'(p_{yxx''}\otimes b_{x''w})$ in C_Y. Now by use of the relations of type 1.14(b) for C_Y, either side can be shown to be equal to an element of C_Y of the form $\sum_u c_u b_{yuw} + \chi_w b_{yw}$ where $c_w \in K$, $\chi_w \in U$, the second term is omitted unless $w > y$, and the sum is over $u \in Y$ with $u > w$ and $u > y$ (cf. [9, 3.11(f)–(g)]). From this, one sees that to check $c_u = 0$, one can assume without loss of generality that X actually is an interval $[x,u]$ with $l(u) = l(x)+3$. Then $c_u = 0$ by the remark 1.15 and the first sentence after (g) in [9, 3.11] (or by a direct calculation like the one below).

Similarly, to verify $\chi_w = 0$, one may assume $y < w$ and that $X = [x, w]$. One atom of $[x, w]$ is y and we adopt the notation of 1.6, denoting the other atom by z, $\ell(x, y) = \gamma$ etc. Then in C_Y, one has $\pi'(p_{yxy}b_{yw}) = (-\gamma b_y - ab_{ywy})b_{yw} = -\gamma b_{yw} + a\alpha b_{yw} = -b\beta b_{yw}$ and $\pi'(p_{yxz}b_{zw}) = cb_{ywz}b_{z,w} = -b\beta b_{yw}$ since $b = c$ by the symmetry condition on L.

We now have a well-defined left $S(U)$-linear map $b'_{yx} = \theta(b_{yx}): N_{\geq x} \to N_{\geq y}$, clearly homogeneous of degree one, for $y > x$. It is clear from the construction that for any $q \geq x$, $b_{yx}n_q$ is expressible as a left $S(U)$-linear combination of elements n_r with $r \geq q$ and $r \geq y$. Now we want to show that $b'_{yx} \in C_X$ i.e. for any $\chi \in U$, $q \geq x$ one has

$$\text{(d)} \qquad b'_{yx}(n_q\chi) = b'_{yx}(n_q)\chi.$$

This is immediate from the definition of b'_{yx} unless $q = x$, noting $n_q\chi$ is expressible as a left $S(U)$-linear combination of elements n_r with $r \geq q$. But if $q = x$, both sides of (d) are expressible as (left) $S(U)$-linear combinations of elements m_p with $p \geq y$, $l(p) \leq l(y) + 1$ and one sees that one may without loss of generality take X to be an interval of length at most 2. First consider the length one case. If $X = [x, y]$, $\ell(x, y) = \gamma$, then by definition b'_{yx} maps $n_x \mapsto n_y$, $n_y \mapsto -\gamma n_y$. Hence it maps $n_x\chi = x(\chi)n_x + \langle x(\chi), \gamma^\vee\rangle n_y$ to

$$x(\chi)n_y - \langle x(\chi), \gamma^\vee\rangle\gamma n_y = s_{\gamma,\gamma^\vee}x(\chi)n_y = y(\chi)n_y = n_y\chi = b'_{yx}(n_x)\chi$$

as required. For the length two case, we again take $X = [x, z]$ with notation as in 1.6. The definitions give that $b'_{y,x}$ maps $n_x \mapsto n_y$, $n_y \mapsto -an_w - \gamma n_y$, $n_z \mapsto bn_w$ and $n_w \mapsto -b\beta n_w$. Hence it maps $n_x\chi = x(\chi)n_x + \langle x(\chi), \gamma^\vee\rangle n_y + \langle x(\chi), \delta^\vee\rangle n_z$ to

$$\begin{aligned} & x(\chi)n_y + \langle x(\chi), \gamma^\vee\rangle(-an_w - \gamma n_y) + \langle x(\chi), \delta^\vee\rangle bn_w \\ = & s_{\gamma,\gamma^\vee}x(\chi)n_y + \langle y(\chi), \alpha^\vee\rangle n_w \\ = & n_y\chi = b'_{yx}(n_x)\chi \end{aligned}$$

as required, using 1.7(d) and $b = c$. Hence $b'_{yx} \in \tilde{C}_X$.

Now we have to check that the map θ defined on the generators b_u, $b_{u,v}$ of C_X as $S(U)$-algebra preserve the relations 1.14(b)–(c). This is clear by induction and the definitions of the $\theta(b_{yx})$ except for the relations of the form $b_{zy_1}b_{y_1x} = b_{zy_2}b_{y_2x}$ with $x \lessdot y_i \lessdot z$. These relations can be shown to be preserved by an easy argument using the anti-involution ω of C_X exactly as in [9, 3.11]. This completes the proof.

1.18. Corollary. *The elements b_{xyz} of C_X with $x \leq y \geq z$ form a $S(U)$-module basis of C_X, and $\theta: C_X \to \tilde{C}_X$ is a monomorphism.*

Proof. The proof is the same as that of [9, 3.13]. One just needs to show that the various $\theta(b_{xyz})$ are (left) $S(U)$-linearly independent. This follows at once since $\theta(b_{xyz})$ maps $n_z \in N_{\geq z}$ to $n_y \in N_{\geq x}$. Note that this argument also shows that the $\theta(b_{xyz})$ are right $S(U)$-linearly independent.

1.19. We'll now check that $\theta(C_X)$ is a $(S(U) \otimes_k S(U)^{\mathrm{op}})$ subalgebra of $\tilde{C}_X$ and hence C_X has a natural $(S(U) \otimes_k S(U)^{\mathrm{op}})$-algebra strucure. In fact, this is immediate from the following lemma.

1.20 Lemma. *For $\chi \in U$, $x \in X$ define $\chi_x = x(\chi)b_x + \sum_{x \xrightarrow{\gamma} y} \langle x(\chi), \gamma^\vee \rangle b_{xyx}$. Then $\theta(\chi_x)n = n\chi$ for all $n \in N_{\geq x}$.*

Proof. Its sufficient to check that $\theta(\chi_x)(n_z) = n_z\chi$ for all $n_z \in N_{\geq x}$. For $z = x$, its immediate from the definitions. For $z > x$, choose $y > x$ with $y \leq z$ and note its enough by induction to show that $\chi_x b_{xy} = b_{xy}\chi_y$ in C_X. One sees at once its sufficient to do this computation for an interval of length at most two, and we leave this to the reader.

1.21. From the lemma, there is a $(S(U) \otimes_k S(U)^{\mathrm{op}})$-algebra strucure on C_X in which the right action by $\chi \in U$ is just multiplication by the (central) element $\sum_{x \in X} \chi_x$. Writing the left and right actions on the appropriate side, one has (cf 1.8(b))

(a)
$$\chi b_x = b_x(x^{-1}\chi) - \sum_{x \xrightarrow{\gamma} y} \langle \chi, \gamma^\vee \rangle b_{xyx}.$$

From the proof of 1.18, the elements b_{xyz} there are a basis of C_X also in the right $S(U)$-algebra structure. One sees easily from this and 1.7 that if one defines a new $(S(U) \otimes_k S(U)^{\mathrm{op}})$-algebra strucure on C_X by interchanging the old (left and right) $S(U)$-actions, one obtains the $(S(U) \otimes_k S(U)^{\mathrm{op}})$-algebra defined by generators and relations in 1.13–1.14 from the labelling L' of X on U (and the correspondence matches the basis element b_{xyz} of one with b_{xyz} of the other).

1.22 Proposition. *Consider the graded $S(U) \otimes_K S(U)^{\mathrm{op}}$-algebra $C = C_X$ defined above where X is a multiplicity free spherical poset with a non-degenerate symmetric labelling on a K-vector space U. Then C (as left or right $S(U)$-algebra) is a graded quasi-hereditary $S(U)$-algebra (of split type).*

Proof. Its enough to show this for the left $S(U)$-structure. This is easy from 1.18. Let x be any maximal element of X. As right C-module, the ideal $J = Cb_xC$ is a direct sum $J = \oplus_{y \leq x} b_{yx}C$ and for $y \leq x$, the right C module $b_{yx}C$ is isomorphic to $b_xC\langle l(x) - l(y)\rangle$. Note also $b_xCb_x = S(U)b_x$ is isomorphic to $S(U)$. Hence $J^2 = J$, J is projective as right C-module and the ring of all right C-module endomorphisms of the ungraded C-module underlying J is a full matrix ring over $S(U)$. According to the definition [4, (3.1)], J is a heredity ideal (of split type) of C. Moreover, one sees at once that if $X \neq \{x\}$, then $C/J \cong C_{X \setminus \{x\}}$ which may be assumed quasi-hereditary (of split type) by induction. The result is then immediate from the definition [4, (3.2)].

1.23 Remark. Assume that X satisfies the orientability condition [9, (3.15)(b)]. One sees at once from 1.15, 1.18 and [9, 3.10(a)] that $K \otimes_{S(U)} C_X$ is isomorphic as K-algebra with the $\mathcal{O}$ algebra associated in [9] to the edge labelling $(x, y) \to x^{-1}\ell^\vee(x, y)$ (for $x < y \in X$). and similarly from 1.21 that $C_X \otimes_{S(U)} K$ is isomorphic to the $\mathcal{O}$ algebra associated in [9] to the edge-labelling $(x, y) \to \ell^\vee(x, y)$ for $x < y$ in X

Orientability was used in [9] in showing that the above algebras were $\mathcal{O}$ algebras and satisfied "Beilinson-Ginsburg duality." However, one can show by direct calculation that the above K-algebras satisfy 3.15(c) of this paper and hence are $\mathcal{O}$ algebras even without the orientability assumption.

2. A variation on quasi-hereditary rings

Our purpose in this section is to indicate the simple changes that can be made to standard definitions and results on quasi-hereditary k-algebras to accomodate rings similar to the quadratic duals of the algebras A', B' constructed in the introduction. In place of k, we will have a family of algebras $A(x)$ indexed by the weight poset. Another feature of the development here is there are two natural analogues $M(x)$, $\overline{M}(x)$ of the Verma (Weyl) modules for quasi-hereditary rings. All the arguments we need have been used (e.g. in [2–6, 10–11]) in the study of quasi-hereditary rings and algebras, so for proofs of results we will often just indicate a reference to a published proof that applies with trivial modifications to the situation here. We give the arguments only in the generality needed for our intended applications, and except in 2.10, we deal only with the graded case.

2.1. We fix a field K and consider a graded K-algebra $A = \oplus_{n\in \mathbb{N}} A_n$ such that $A_0 \cong K^m$ for some m and A_n is finite-dimensional for $n \geq 0$. Set $A_{>0} = \oplus_{n>0} A_n$ and let $\{e_x\}_{x\in X}$ denote the set of primitive idempotents of A_0. For $Y \subseteq X$, we denote the idempotent $\sum_{y\in Y} e_y \in A_0$ by e_Y. For each $x \in X$, there is a (unique up to isomorphism, one-dimensional) simple module $L(x)$ with $e_x L(x) \neq 0$ and $L(x) = L(x)_0$. Any simple graded A-module is isomorphic to $L(x)\langle n\rangle$ for a unique integer n and $x \in X$. We call a simple subquotient module of M a composition factor of M; the "multiplicity" of $L(y)\langle n\rangle$ as a composition factor of M is $\dim e_y M_n$. We also define $P(x) := Ae_x$, the projective cover of $L(x)$ in gr A. For any M in gr A, the cap (largest semisimple homomorphic image) of M is $M/A_{>0}M$; for instance, the cap of $P(x)$ is $L(x)$. If M is any module with simple cap $L(x)$ and $M_n = 0$ for $n << 0$, then $\dim M_0 = 1$ and $M = AM_0$; hence M is a quotient of $P(x)$ and $\mathrm{End}_{\mathrm{gr}\ A}(M) \cong K$. We write $\mathrm{Ext}_{\mathrm{gr}\ A}$ for the right derived functors of $\mathrm{Hom}_{\mathrm{gr}\ A}$.

2.2. Suppose that X is endowed with a partial order $\leq$. Following [10] with some modifications, we consider a family of modules (to be called big Verma modules) $\{M(x)\}_{x\in X}$ in gr A with the following properties (a)–(d):

(a) $M(x)$ has simple cap $L(x)$
(b) $M(x)_n = 0$ for $n << 0$
(c) $[M(w) : L(y)\langle k\rangle] = 0$ unless $y \leq w$
(d) for $y \in X$, $P(y)$ has a filtration $P(y) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^n = 0$ by graded A-modules F^i such that for each $i \geq 0$, one has $F^i/F^{i+1} \cong M(x_i)\langle n_i\rangle$ for some $n_i \in \mathbb{Z}$ and $x_i \geq y$ in Y.

2.3. Essentially the same arguments as in [10, 2.2] prove the following.

Proposition. *For a family $\{M(x)\}_{x\in X}$ as above, the following conditions are equivalent:*

(a) *in any filtration as in 2.2(d) above, one has $x_0 = y$ and $x_i > y$ for $i > 0$.*
(a′) *in some filtration as in 2.2(d) above, one has $x_0 = y$ and $x_i > y$ for $i > 0$.*
(b) *if M in* gr *A has $M_n = 0$ for $n << 0$, simple cap $L(y)$ and all composition factors of the form $L(z)\langle n\rangle$ with $z \not> y$, then M is a homomorphic image of $M(y)$*
(c) *any extension $0 \to M \to N \to M(y) \to 0$ in which every composition factor $L(z)\langle n\rangle$ of M has $z \not> y$ is split*
(d) $\mathrm{Ext}^1_{\mathrm{gr}\ A}(M(y), M(w)\langle n\rangle) = 0$ *for all $n \in \mathbb{Z}$ unless $y < w$.*

2.4. We assume here that the equivalent conditions of 2.3 hold. It is clear that

(a) $$\mathrm{Hom}_{\mathrm{gr}\ A}(M(y), M) = 0, \quad \text{if } yM_0 = 0$$

(b) if y is a maximal element of X, then $P(y) \cong M(y)$.

Consider M in gr A. If $y \in X$ and M has no composition factor $L(t)\langle p\rangle$ with $t > p$, then

(c) $$\mathrm{Ext}^i_{\mathrm{gr}\ A}(M(y), M) = 0 \quad \text{for } i > 0.$$

This follows by an inductive argument like [10, 4.4(ii)], starting with 2.3(c). In particular,

(d) $\mathrm{Ext}^i_{\mathrm{gr}\ A}(M(y), M(w)\langle p\rangle) = \mathrm{Ext}^i_{\mathrm{gr}\ A}(M(y), L(w)\langle p\rangle) = 0 \quad \text{for } i > 0,\ p \in \mathbb{Z},\ \ y \not< w.$

For $y \leq w$ in X, let $l(y,w)$ be the maximal length n of a chain $y = y_0 < y_1 < \ldots y_n = w$ in X. Let M be in gr A and d denote the maximum of $l(y,t)$ over $t \geq y$ in X such that M has a composition factor $L(t)\langle p\rangle$ for some p. An induction on d starting from (c) above shows

(e) $$\mathrm{Ext}^i_{\mathrm{gr}\ A}(M(y), M) = 0 \quad \text{for } i > d.$$

2.5. We wish to give another description of the algebras considered above, closely related to a standard characterization of quasi-hereditary algebras involving heredity ideals. First, a few simple observations useful for the proof.

Lemma. *Consider a graded K-algebra A, a left A-module M and an idempotent $e \in A_0$. Then the graded left A-module AeM is projective iff eM is a projective eAe-module and the multiplication map $Ae \otimes_{eAe} eM \to AeM$ is bijective.*

Proof. If AeM is projective, the multiplication map is bijective by [6, Lemma 2]. Also, there is a surjection $\oplus_{i\in I} Ae\langle n_i\rangle \to AeM$ which splits, so AeM is an A-module direct summand of $\oplus_i Ae\langle n_i\rangle$ and $eM = eAeM$ is an eAe-module summand of $\oplus eAe\langle n_i\rangle$. The other direction is trivial.

2.6. As a trivial consequence of the lemma, it follows that for a graded ring A as in 2.1 and idempotent $e \in A_0$, the following are equivalent:

(a) The left A-module AeA is isomorphic to a finite direct sum $\oplus_{i=1}^m Ae\langle n_i\rangle$ for some $n_i \in \mathbb{Z}$.

(b) The multiplication map $Ae \otimes_{eAe} eA \to AeA$ is bijective and the left eAe-module eA is isomorphic to $\oplus_{i=1}^m eAe\langle n_i\rangle$.

If in the above e is primitive, then for any idempotent $f \in A_0$ one has by the Krull-Schmidt theorem that $AeAf \cong \oplus_{j=1}^p Ae\langle d_j\rangle$ for some $d_j \in \mathbb{Z}$. Now one has

2.7 Proposition. *The following two conditions on a graded ring A as in 2.1 are equivalent:*

(a) *there is a total ordering $e_1 > \ldots > e_n$ of the set of primitive idempotents of A_0 such that for each i, the idempotent $e_i + J_{i-1} \in A/J_{i-1}$ (where J_{i-1} denotes the two-sided ideal $J_{i-1} := A(e_1 + \ldots + e_{i-1})A$ of A) satisfies the equivalent conditions 2.6(a)–(b)*

(b) *there is a partial order on the set X indexing the primitive idempotents of A_0 and a family of modules $\{M(x)\}_{x\in X}$ in gr A satisfying the conditions of 2.2 and the equivalent conditions of 2.3.*

Proof. The statement is similar to [2, 3.6] and the proof is essentially the same as there. It will show that the order on the idempotents in (a) can be taken to be any refinement to a total order of the order induced on them by a partial order of the indexing set X as in (b).

Suppose that (a) holds, and define a (total) order on X by $x \leq y$ iff $e_x \leq e_y$, for $x, y \in X$. Set $M(e_i) = (A/J_{i-1})e_i$ regarded as graded A-module. Then the conditions of 2.2(a)–(c), and also 2.3(b) hold. Now consider the indecomposable projective module $P(e_i) = Ae_i$. This has a filtration

$$0 = J_0 e_i \subseteq J_1 e_i \subseteq \ldots \subseteq J_n e_i = P(e_i)$$

and we claim that $J_j e_i / J_{j-1} e_i$ is isomorphic to a finite direct sum of modules $M(e_j)\langle p\rangle$ for various p. In fact, its enough to show this for $j = 1$, when its immediate from 2.6 above. Hence the filtration above can be refined to one as required in 2.2(d).

Conversely, suppose that (b) holds, and choose a maximal element w of X. We first observe that the algebra $A' = A/Ae_wA$ also satisfies the conditions 2.2 and 2.3 with respect to the induced partial order on $Y := X \setminus \{w\}$. For take $M(x)$ in gr A' to be the module $M(x)$ in gr A, but regarded as an A'-module. Then 2.2(a)–(c) hold. Let $J(x) = Ae_wAe_x$ in gr A. Then $P(x)/J(x)$ is the largest quotient in gr A of $P(x)$ with all composition factors of the form $L(y)\langle n\rangle$ for various $y \in Y$ and $n \in \mathbb{Z}$. A standard argument using 2.3(d) shows that $P(x)$ has a filtration as in 2.2(d) in which there exists some j with $x_i = w$ for $i \geq j$ and $x_i \neq w$ for $i < j$. It follows that $J(x) = F_j$, and identifying $P(x)/J(x)$ with $A'x$, one sees easily that 2.2(d) and 2.3(a) hold for A'.

Hence to prove that (a) holds, it will suffice (by induction on $\sharp(X)$) to show that the idempotent e_w satisfies the conditions of 2.6(b). Let $P = \oplus_{x \in X} P(x)$, and set $B = \mathrm{end}_A(P) \cong A^{\mathrm{op}}$. Under this isomorphism, the idempotent $e_x \in A^{\mathrm{op}}$, $x \in X$ corresponds to the obvious (projection) map $p_x \in B: P \to P(x) \to P$; set $p = p_w$. Let $J = \oplus_{x \in X} J(x) = Ae_wA \subseteq P$. From above, one may choose a left A-module decomposition $J = \oplus_{i \in 1}^{p} P(w)\langle d_i\rangle$. The inclusions in this direct sum induce maps $\lambda_i: P(w) \to J\langle -d_i\rangle \to P\langle -d_i\rangle$. Now the inclusion $J \to P$ induces an isomorphism $\mathrm{hom}_A(P(w), J) \to \mathrm{hom}_A(P(w), P)$ by 2.4(a). Hence the λ_i form a graded right $\mathrm{end}_A(P(w)) \cong pBp$-basis of $\mathrm{hom}_A(P(w), P) \cong Bp$. Also, it is clear that any $f \in BpB$ (i.e. which factors through $P(w)$) can be written $\sum_i \lambda_i \circ f_i$ for uniquely determined $f_i \in \mathrm{hom}_A(P, P(w))$. This implies that the multiplication map $Bp \otimes_{pBp} pB \to BpB$ is injective, hence bijective. Recalling $B \cong A^{\mathrm{op}}$, we've now checked 2.6(b) as required.

2.8 Proposition. *Suppose that the conditions 2.7(b) hold. If Y is an ideal of X, then $A/Ae_{X \setminus Y}A$ satisfies the conditions of 2.7(b) with respect to the induced order on Y. If Y is instead a coideal, the same is true of the centralizer ring $e_Y A e_Y$.*

Proof. If Y is an ideal, this is clear from the preceeding proof. For coideals Y, the argument in [2, 3.5] adapts easily to this situation (cf. also [5, Statement 10]).

2.9. Contrary to the situation for quasi-hereditary algebras, its not true that if A satisfies 2.7(b), then so does A^{op}. We wish to consider rings so both A, A^{op} satisfy 2.7(b); since in all intended applications there will be an anti-involution of the ring, we assume this from the start.

Definition. Let X be a finite poset. A graded K-algebra $A = \oplus_{n \in \mathbb{N}} A_n$ with $A_0 \cong K^m$ and all A_n finite-dimensional will be called a VQH ring with weight poset X if there is a (bijective) parametrization $\{e_x\}_{x \in X}$ of the primitive idempotents of A_0 such that

2.7(b) holds, and if there is also given a graded ring anti-involution ω of A which fixes A_0 elementwise.

2.10. We wish to record some additional facts about Ext groups of modules for VQH algebras A. The following result is a variant of [5, Statement 5], [4, (3.6)(a)] for which I could find no reference. It might as well be given for an arbitrary ring A. Consider ungraded right A-modules, and let Hom_A, Ext_A denote homomorphisms of ungraded right A-modules, and the corresponding derived functors. Let r.gl.dim. A denote the usual (right) global dimension of A.

Proposition. *Let e be an idempotent in a ring A and set $J = AeA$. Assume that J is projective as both left and right A-module, and let B denote the quotient ring $B = A/J$. Then*

$$\text{r.gl.dim. } A \leq \max(\text{r.gl.dim. } B + 2,\ \text{r.gl.dim. } eAe).$$

Proof. The proof partly follows that of [5, Statement 5]. We regard right B-modules as right A-modules annihilated by J, and let proj. dim. M_R denote the projective dimension of a right module M for a ring R. First, one has since J_A is projective (see e.g. [5, Statement 1]) that

(a) $$\text{proj. dim. } X_A \leq \text{proj. dim. } X_B + 1$$

for any right B-module X. Set $D = \text{r.gl.dim. } B$, $E = \text{r.gl.dim. } eAe$, $F = \max(D+2, E)$. Let M be a right A-module. We must show proj. dim. $M \leq F$. Assume without loss of generality that F is finite. Consider the exact sequence

$$0 \to MJ_A \to M_A \to M/MJ_A \to 0.$$

Since M/MJ may be regarded as a right B-module, one has proj. dim. $M/MJ_A \leq D+1 < F$ by (a), so by a standard argument involving the long exact Ext sequence, it will be sufficient to show proj. dim. $MJ_A \leq F$. Now $MJ = MeA$, and we consider the exact sequence

(b) $$0 \to N \to Me \otimes_{eAe} eA \to MeA \to 0$$

of right A-modules. Since r.gl.dim. $eAe \leq E$ and Me can be regarded as a right eAe-module, there is a resolution

$$0 \to P_E \to \ldots \to P_0 \to Me \to 0$$

with the P_i projective right eAe-modules. Now ${}_{eAe}eA$ is projective, by the ungraded version of 2.5, so an application of the exact functor $-\otimes_{eAe}eA$ to this resolution yields a resolution of length E of $Me\otimes_{eAe}eA_A$ by projective right A-modules, showing that proj. dim. $Me\otimes_{eAe} eA_A \leq E$. Now in (b), the bijectivity of the canonical map $Me \otimes_{eAe} eAe \to MeAe = Me$ implies that $N \subseteq MeA(1-e)$. Hence $NJ = NeA = 0$ and N may be regarded as a right B-module, so by (a) again, proj. dim. $N_A \leq D+1$. The proof is completed by a standard argument involving the long exact Ext sequences for (b) which shows that

$$\text{proj. dim. } MeA \leq \max(\text{proj. dim. } N_A + 1,\ \text{proj. dim. } Me \otimes_{eAe} eA_A) \leq F.$$

2.11. For the remainder of this section, let $A = \oplus_{n\in\mathbb{N}} A_n$ be a VQH ring with weight poset X. We now introduce graded rings $A(x)$ and small Verma modules $\overline{M}(x)$ for $x \in X$.

Fix $x \in X$ and let $B = A/Ae_{\not\leq x}A$. Define the (positively) graded ring $A(x) = e_x B e_x$; this has a graded K-algebra ant-involution induced by ω. Note that $M(x) = Be_x$ by 2.2(c), so $A(x) \cong \mathrm{End}_B(M(x))^{\mathrm{op}} \cong (\mathrm{End}_A(M(x))^{\mathrm{op}}$. Its easily checked that if all $A(x) \cong K$, then A is a finite-dimensional graded quasi-hereditary algebra over K.

In general, by 2.7 and 2.6, one has that

(a) $M(x)$ is graded free of finite rank as a right $A(x)$-module.

Regarding K as a (simple) $A(x)$-module in degree 0, we define $\overline{M}(x) = M(x) \otimes_{A(x)} K$ in gr A. Note that

(b) $\overline{M}(x)$ is finite-dimensional as K-space, has $L(x)$ as cap and has all other composition factors of the form $L(y)\langle p\rangle$ for $y < x$, $p \in \mathbb{Z}$.

2.12. For M in gr A, we define the graded right A-module M^t which is equal to M as graded vector space, and with $ma := \omega(a)m$ for $a \in A$, $m \in M$ where ω is the given anti-involution on A. One also has a contravariant exact functor $\delta\colon \mathrm{gr}\ A \to \mathrm{gr}\ A$ (continuous dual) defined on objects as follows. For any M in gr A, define δM in gr A by $(\delta M)_n = (M_{-n})^*$ with the left A-module action $(af)(m) = f(\omega(a)m)$ for $a \in A$, $m \in M$ and $f \in \delta M \subseteq M^*$. Let gr $'A$ denote the full subcategory consisting of those M in gr A with M_n finite-dimensional for all n and zero for $n << 0$. If M is in gr $'A$, then $\delta\delta M \cong M$. Moreover, M has a resolution by projectives of gr A which are in gr $'A$, and aplying δ gives a resolution of δM in gr A, by injectives I of gr A with δI in gr $'A$. If also N is in gr A, then $\mathrm{Hom}_{\mathrm{gr}\ K}(M, \delta N)$ is finite dimensional. One sees easily from these remarks that for M, N in gr A',

(a) $$\mathrm{Ext}^n_{\mathrm{gr}\ A}(M, \delta N) = \mathrm{Ext}^n_{\mathrm{gr}\ A}(N, \delta M)$$

and this is a finite-dimensional K vector space.

2.13. Recall (see e.g. [17]) that for any positively graded ring, the left (or right) projective dimension of A coincides with the corresponding graded projective dimension. If each ring $A(x)$ has finite left and right global dimension, then 2.10 implies that A has finite (left and right) global dimension and the same is true of each quotient ring and centralizer ring in 2.8. Under appropriate hypotheses, one then has a graded recollement result analogous to [2, 3.9], cf. also [4, 4.5]. For our purposes here, the following general fact will suffice.

2.14 Lemma. *Suppose that A is a VQH ring. Let $B = A/Ae_YA$ where Y is a coideal of X and regard graded B-modules as graded A-modules annihilated by e_Y. Then for $n \in \mathbb{N}$ and any two graded B-modules M, N, one has*

$$\mathrm{Ext}^n_{\mathrm{gr}\ B}(M, N) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A}(M, N).$$

Proof. If $Y = \{x\}$, a maximal element of X, the result is immediate from the graded version of [5, Statement 3], and 2.7, 2.6. In general, it then follows by induction on $\sharp(Y)$.

2.15. In case all $A(x) \cong K$, all parts (a)–(d) of the following lemma reduce to standard properties of quasi-hereditary algebras (and (a)–(c) all reduce to the same result [11, 6.4], [3, 2.2]).

Lemma. *Let A be a VQH ring. Then for any $x, y \in X$ and $p \in \mathbb{Z}$, one has*

(a) $$\mathrm{Ext}^n_{\mathrm{gr}\ A}(M(x), (\delta M(y))\langle p\rangle) = \begin{cases} A(x)_p & \text{if } n = 0,\ x = y \\ 0 & \text{otherwise.} \end{cases}$$

(b) $$\mathrm{Ext}^n_{\mathrm{gr}\ A}(M(x), (\delta \overline{M}(y))\langle p\rangle) = \begin{cases} K & \text{if } n = p = 0,\ x = y \\ 0 & \text{otherwise.} \end{cases}$$

(c) $$\mathrm{Ext}^n_{\mathrm{gr}\ A}(\overline{M}(x), (\delta \overline{M}(y))\langle p\rangle) = \begin{cases} \mathrm{Ext}^n_{\mathrm{gr}\ A(x)}(K, K\langle p\rangle) & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

(d) $\mathrm{Ext}^n_{\mathrm{gr}\ A}(\overline{M}(x), L(y)\langle p\rangle)$ *is zero unless $y \geq x$. Moreover, for $y = x$, it is isomorphic to* $\mathrm{Ext}^n_{\mathrm{gr}\ A(x)}(K, K\langle p\rangle)$.

Proof. Let $B = A/Ae_{>x}A$; we will make extensive use of 2.14 and 2.12(a). First, consider (a)–(c) in the case $n = 0$. It is then clear that the Hom spaces on the left vanish unless $x \leq y$, and by 2.12(a) for instance, also unless $y \leq x$. Hence, we may assume $x = y$. Then by 2.14, one may assume that x is maximal in X, and then the assertions are easily checked noting that $M(x) = Ae_x$ for (a)-(b), and using 2.2(b) for (c).

Henceforward, we assume that $n > 0$. To prove (a), we may assume by 2.11(b) that $y \not> x$. Then

$$\mathrm{Ext}^n_{\mathrm{gr}\ A}(M(x), \delta M(y)) \cong \mathrm{Ext}^n_{\mathrm{gr}\ B}(M(x), \delta M(y)) = 0$$

since $M(x)$ is projective in gr B (cf also 2.4(c)).

For (b) and (c), we will use the following fact: if M is a graded $(B, A(x))$-bimodule, projective as an object of both gr B and gr $A(x)^{\mathrm{op}}$, then for any N in gr B and P in gr $A(x)$, one has

$$\mathrm{Ext}^n_{\mathrm{gr}\ B}(M \otimes_{A(x)} P, N) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A(x)}(P, \mathrm{hom}_B(M, N)).$$

In fact, for $n = 0$ this is the standard adjoint isomorphism. For general n, take a projective resolution of P in gr $A(x)$. Tensoring with M gives a projective resolution of $M \otimes_{A(x)} P$ in gr B. If one uses these projective resolutions to compute the Ext groups, the resulting complexes are isomorphic by the adjoint isomorphism.

Now for (c). By 2.11(b), we may assume without loss of generality that $y \not> x$. Then

$$\mathrm{Ext}^n_{\mathrm{gr}\ A}(\overline{M}(x), (\delta \overline{M}(y))\langle p\rangle) \cong \mathrm{Ext}^n_{\mathrm{gr}\ B}(M(x) \otimes_{A(x)} K, (\delta \overline{M}(y))\langle p\rangle) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A(x)}(K, M')$$

where $M' = \mathrm{hom}_B(M(x), (\delta \overline{M}(y))\langle p\rangle)$ in gr $A(x)$ is zero unless $x = y$, in which case $M' = K\langle p\rangle$.

Finally we prove (b). If $y \not> x$, then $\mathrm{Ext}^n_{\mathrm{gr}\ A}(M(x), (\delta \overline{M}(y))\langle p\rangle) = 0$ as in the proof of (a). Also, for $y \not\geq x$, one has as in the proof of (c) that

$$\mathrm{Ext}^n_{\mathrm{gr}\ A}(M(y), (\delta \overline{M}(x))\langle p\rangle) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A}(\overline{M}(x), (\delta M(y))\langle p\rangle) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A(x)}(K, M')$$

where $M' = \mathrm{hom}_B(M(x), (\delta M(y))\langle p\rangle) = 0$. The similar proof of (d) is left to the reader.

2.16. We now have the following analogue of Brauer-Humphreys (BGG) reciprocity [11, 3.5], [2, 3.11].

Proposition. *Let A be a VQH algebra over the field K. Fix $x, y \in X$ and $n \in \mathbb{N}$. Then in any filtration $P(y) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^n = 0$ in* gr A *with successive subquotients $F^i/F^{i+1} \cong M(x_i)\langle n_i \rangle$ for $n_i \in \mathbb{Z}$ and $x_i \geq y$, the number of subquotients isomorphic to $M(x)\langle n \rangle$ is equal to the multiplicity $[\overline{M}(x): L(y)\langle n \rangle]$.*

Proof. Note that by 2.15(b), the number of occurences of $M(x)\langle n \rangle$ in the filtration is

$$\dim \operatorname{Hom}_{\mathrm{gr}\ A}(P(y), (\delta \overline{M}(x))\langle n \rangle) = \dim e_y(\delta \overline{M}(x))\langle n \rangle)_0 = \dim e_y \overline{M}(x)_n.$$

2.17. The above result can be interpreted in terms of matrix Poincaré seies. Define $X \times X$-matrix Poincaré series $p(A, v)$, $m(A, v)$, $\overline{m}(A, v)$ and $a(A, v)$ with entries in the formal power series ring $\mathbb{Z}[[v]]$ as follows: for $x, y \in X$ set

$$p_{x,y}(A, v) = \sum_{i \in \mathbb{N}} \dim(e_x P(y)_i) v^i$$

$$m_{x,y}(A, v) = \sum_{i \in \mathbb{N}} \dim(e_x M(y)_i) v^i$$

$$\overline{m}_{x,y}(A, v) = \sum_{i \in \mathbb{N}} \dim(e_x \overline{M}(y)_i) v^i$$

and let $a(A, V)$ be the diagonal matrix with (x, x)-entry $\sum_{i \in \mathbb{Z}} \dim A(x)_i v^i$. Note that with respect to a total ordering of X refining the given partial order, $m(A, v)$ is upper triangular (i.e. $m_{x,y}(A, v) = 0$ unless $x \leq y$) and $\overline{m}(A, v)$ is upper unitriangular. By 2.13 and 2.16 one has matrix equations (where T denotes matrix transpose)

(a) $$m(A, v) = \overline{m}(A, v) a(A, v), \qquad p(A, v) = m(A, v) \overline{m}(A, v)^T.$$

3. $\tilde{\mathcal{O}}$ algebras

In this section, we study a class of algebras which are Koszul and such that the algebra and its quadratic dual algebra are VQH algebras with opposite weight posets.

3.1. We first record one characterization of the class of Koszul rings. For equivalent definitions and many additional properties of Koszul rings one should consult [1].

Definition. A graded ring $A = \oplus_{n \in \mathbb{N}} A_n$ is Koszul if for any graded A-modules M, N with $M = M_p$ and $N = N_q$ for some $p, q \in \mathbb{Z}$, one has $\operatorname{Ext}^n_{\mathrm{gr}\ A}(M, N)$ is zero unless $q - p = n$.

In this paper, we will always require also of a Koszul ring A that it be a graded algebra over the field K, with $A_0 \cong K^n$ (product of copies of K) for some positive integer n and with A_1 a finite-dimensional K-space.

3.2. We define a notion of quadratic K-algebra. Consider a K-algebra $k \cong K^n$ for some positive integer n and let U be a finitely-generated $k \otimes_K k^{\mathrm{op}}$-module, regarded as (k, k)-bimodule. For any (k, k) subbimodule W of $U \otimes_k U$, define the algebra $A(k, U, W) =$

$T_k(U)/\langle W\rangle$. We define a graded K algebra to be quadratic if it is isomorphic to some $A(k,U,W)$ as above.

Now for a quadratic algebra $A \cong A(k,U,W)$, we can define the quadratic dual algebra $A^! \cong A(k,U^*,W^\perp)$ as follows. Give U^* the bimodule structure determined by $afb(u) = f(bua)$ for $a,b \in k$, $u \in U$ and $f \in U^*$. We identify $(U^{\otimes n})^*$ with $(U^*)^{\otimes n}$ by means of the non-degenerate pairing

$$\langle -,-\rangle : (U^*)^{\otimes n} \times U^{\otimes n} \to K$$

determined by $\langle f_n \otimes \ldots \otimes f_1, v_1 \otimes \ldots \otimes v_n\rangle = f_n(v_n)\ldots f_1(v_1)$. We define

$$W^\perp := \{ f \in (U^* \otimes_k U^*) \mid \langle f, w\rangle = 0 \text{ for all } w \in W \},$$

and then one has the quadratic dual $A^! := A(k,U^*,W^\perp) = T_k(U^*)/\langle W^\perp\rangle$.

From [1], a Koszul algebra A is quadratic i.e. it is isomorphic as graded algebra to some $A(k,V,W)$ as defined above. It is known that any Koszul algebra as in 3.1 is quadratic in the above sense, and that A^{op} and $A^!$ are both Koszul if A is Koszul. We may now make the following definition.

3.3 Definition. Let X be a finite poset. A $\tilde{\mathcal{O}}$ algebra (over a field K) with weight poset X is a graded K-algebra $A = \oplus_{n\in\mathbb{N}} A_n$ such that

(a) $A_0 \cong K^{\sharp(X)}$
(b) A is a Koszul algebra
(c) A is VQH with weight poset X and the quadratic dual algebra $A^!$ is VQH with weight poset X^{op}.

In (c) of the definition (as also in the definition of $\mathcal{O}$ algebra in the introduction) it is to be understood that the same parametrization $\{e_x\}_{x\in X}$ of the primitive idempotents of $A_0 = A^!_0$ is used for both A and $A^!$ in the definition of a VQH algebra. Also, we assume the anti-involution on $A^!$ is the one induced by the anti-involution on A. Of course, if A is a $\tilde{\mathcal{O}}$ algebra with weight poset X, then $A^!$ is a $\tilde{\mathcal{O}}$ algebra with weight poset X^{op}.

3.4. The $\mathcal{O}$ algebras defined in the introduction are precisely the $\tilde{\mathcal{O}}$ algebras A with all $A(x) \cong K$ (that $A^!(x) \cong K$ follows from the proposition below).

3.5. As a first indication of relations between the objects $A(x)$, $L(x)$, $\overline{M}(x)$, $M(x)$ and $P(x)$ for a $\tilde{\mathcal{O}}$ algebra A and the corresponding objects $A^!(x)$, $L_!(x)$, $\overline{M}_!(x)$, $M_!(x)$ and $P_!(x)$ for $A^!$, we have the following simple numerical fact.

Proposition. *One has matrix equations*

$$\overline{m}(A,v)\overline{m}(A^!,-v)^T = \mathrm{Id}, \qquad a(A,v)a(A^!,-v)^T = \mathrm{Id}$$

relating the Poincaré series defined in 2.17 for A and $A^!$.

Proof. Koszulity implies by [1, Lemma 7] that one has $p(A,v)p(A^!,-v)^T = \mathrm{Id}$. Now $p(A,v) = \overline{m}(A,v)a(A,v)\overline{m}(A,v)^T$ from 2.17(a), and there is a similar equation for $p(A^!,v)$. The result follows on noting as in 2.17 that with respect to a suitable total order on X, $\overline{m}(A,v)$ (resp., $\overline{m}(A^!,v)$) is upper (resp., lower) unitriangular.

3.6. The matrix equations above (and many others obtained by combining with 2.17 for A and $A^!$) have more conceptual explanations. For instance, the matrix equation for a in 3.5 suggests again by [1, Lemma 7] that $A(x)$ is Koszul with quadratic dual $A(x)^! \cong A^!(x)$. To establish this and other properties of $\tilde{\mathcal{O}}$ algebras, we now extend to $\tilde{\mathcal{O}}$ algebras some arguments from [9, Section 2]. First, we need to recall some notation used there.

Let A be a quadratic algebra as in 3.2. For a graded left $A^!$-module M and graded right A-module N, consider the K-vector space $\mathcal{K}_A(N,M) = \mathcal{K}(N,M) := N \otimes_{A_0} M$ endowed with the map $\partial: \mathcal{K}(N,M) \to \mathcal{K}(N,M)$ defined by

$$\partial(n \otimes m) = \sum_{\nu} ne_{\nu} \otimes e_{\nu}^{*}m$$

for $n \in N$ and $m \in M$, where $\{e_\nu\}$ is a basis of $A_1 = V$ and $\{e_\nu^*\}$ is the dual basis of $A_1^! = V^*$. It is easily checked that ∂ is independent of the choice of basis and is a differential (i.e. $\partial^2 = 0$; cf [1, 2.7]). Note that $\mathcal{K}$ may be regarded a functor of M and N, covariant and exact in each variable.

One may regard $\mathcal{K}(N,M)$ as a complex of graded vector spaces

$$\ldots \mathcal{K}(N,M)^p \to \mathcal{K}(N,M)^{p+1} \to \ldots$$

where for $p,q \in \mathbb{Z}$, $\mathcal{K}(N,M)_q^p = N_{p+q} \otimes M_p$. If N is a graded (A,A)-bimodule, $\mathcal{K}(N,M)$ may be regarded as a complex of graded left A-modules, and it has a decomposition $\mathcal{K}(N,M) = \oplus_{y\in Y} \mathcal{K}(N,Me_y)$ as a direct sum of complexes of graded vector spaces. If in addition M is a graded $(A^!,A^!)$-bimodule, one has $\mathcal{K}(N,M) = \oplus_{x,y\in Y} \mathcal{K}(e_xN,Me_y)$ where $\mathrm{H}^i(\mathcal{K}(e_xN,Me_y) \cong e_x\mathrm{H}^i(N,M)e_y$.

3.7. The Koszul complex of a quadratic algebra A is defined to be $\mathcal{K}(A,\delta(A^!))$ (see [1, Section 2]; we are assuming here A has a graded algebra anti-involution fixing A_0 elementwise). The Koszul complex may be regarded as a complex of graded projective left A-modules, with a natural augmentation $\mathcal{K}(A,\delta(A^!))^0 \cong A \to A_0$. It is known that a quadratic algebra A (with anti-involution) as in 3.2 is Koszul iff the augmented Koszul complex is acyclic.

3.8 Proposition. *Suppose A is a $\tilde{\mathcal{O}}$ algebra with weight poset X. If Y is a (non-empty, proper) coideal of X, then the centralizer ring e_YAe_Y is a $\tilde{\mathcal{O}}$ algebra with weight poset Y. Moreover, the quotient algebra A/Ae_YA is a $\tilde{\mathcal{O}}$ algebra with weight poset $Z = X \setminus Y$ and the quadratic dual of this quotient is isomorphic to the centralizer ring $e_ZA^!e_Z$.*

Proof. Set $e = e_Y$, $f = e_{X\setminus Y}$, $B := eAe$ and put $C := A^!/A^!fA^!$. We may assume without loss of generality that $f = e_x$, where x is a minimal element of X. The proof is essentially identical to that of the corresponding fact [9, 2.13] for $\mathcal{O}$ algebras, using facts from Section 2 in place of standard facts on quasi-hereditary rings used in [loc cit]. In outline, it proceeds as follows. From 2.8 one knows that B and C are VQH with weight posets Y and Y^{op} respectively. One then verifies that $\mathcal{K}(eA,\delta\, C)$ gives a (projective) resolution of B_0 in gr B, using the exact sequence of complexes

$$0 \to \mathcal{K}(eA,\delta\, C) \to \mathcal{K}(eA,\delta\, A^!) \to \mathcal{K}(eA,\delta\, A^!e_xA^!) \to 0$$

and the facts that $\mathrm{H}^i(\mathcal{K}(A,\delta\, P_!(x))$ is zero for $i \neq 0$ (by Koszulity of A) and that $A^!e_xA^!$ is isomorphic to a direct sum of degree shifts of $P_!(x)$ as left $A^!$-module by 2.7. It follows from the definition that B is Koszul, and one readily checks that C identifies with $B^!$.

3.9. From this proposition, one obtains for any non-empty closed subset Y of X a canonical (up to isomorphism) $\tilde{\mathcal{O}}$ algebra A_Y with weight poset Y. One has $A_Y \cong e_Y A e_Y$ if Y is a coideal of X, $A_Y \cong A/Ae_{X\setminus Y}A$ if Y is an ideal of X, $(A_Y)^! \cong (A^!)_Y$ and $(A_Y)_Z \cong A_Z$ for non-empty closed $Z \subseteq Y$.

3.10. The next few results concern homological "purity" properties of the standard graded modules for $\tilde{\mathcal{O}}$ algebras.

Lemma. *Let A be a $\tilde{\mathcal{O}}$ algebra with weight poset X. Then for any $x \neq y$ in X, the following complexes of graded vector spaces are acyclic:*

(a) $\mathcal{K}(M(x)^t, \delta M_!(y))$
(b) $\mathcal{K}(\overline{M}(x)^t, \delta M_!(y))$
(c) $\mathcal{K}(M(x)^t, \delta \overline{M}_!(y))$
(d) $\mathcal{K}(\overline{M}(x)^t, \delta \overline{M}_!(y))$.

Moreover, for $x = y$, (a), (c) and (d) are acyclic except in degree zero.

Proof. Consider $x \neq y$ in X. The proof of (a) is exactly the same as that of one part of [9, 2.15]. Let Y be any closed subset of X, and set $B = A_Y$. Note that for any $x, y \in X$, one may identify $\mathcal{K}_A(M_A(x)^t, \delta M_{A^!}(y))$ with $\mathcal{K}_B(M_B(x)^t, \delta M_{B^!}(y))$. We use this observation to prove that (a) holds by induction on $\sharp(Y)$. Fix distinct x, y in Y. Then $\mathcal{K}(M(x)^t, \delta M_!(y))$ is acyclic by induction and the above remark, unless perhaps x is a maximal element of X and y is a minimal element of X. But in this case,

$$\mathrm{H}^i(\mathcal{K}(M(x)^t, \delta M_!(y))) = \mathrm{H}^i(\mathcal{K}(P(x)^t, \delta P_!(y))) \cong e_x \mathrm{H}^i(\mathcal{K}(A, \delta A^!)) e_y$$

is zero if $i \neq 0$ and isomorphic to $e_x A_0 e_y = 0$ if $i = 0$.

Now the assertion (c) (for $A^!$ instead of A) is equivalent by duality to the assertion (b) for A. Hence it will suffice to show that acyclicity of (a) (resp., (c)) implies acyclicity of (b) (resp., (d)). Assume the complex in (a) is acyclic. Since $M(x)^t$ is a free left $A(x)$-module, the complex $\mathcal{K}(M(x)^t, \delta M_!(y))$ may be regarded as a complex of free (in particular, flat) left $A(x)$-modules. Since this complex is bounded above, it follows that the complex $K \otimes_{A(x)} \mathcal{K}(M(x)^t, \delta M_!(y)) \cong \mathcal{K}(\overline{M}(x)^t, \delta M_!(y))$ is acyclic. Hence acyclicity of (a) implies that of (b). Similarly, acyclicity of (c) implies that of (d). Now consider $x = y$. Then the complexes (c) and (d) are concentrated in degree zero, while the complex in (a) identifies naturally with the Koszul complex of the Koszul ring $A(x)$ (the Koszulity follows from 3.9 with $Y = \{x\}$).

3.11 Corollary. *Let A be a $\tilde{\mathcal{O}}$ algebra with weight poset X, and fix $x \in Y$. Then*

(a) *the complex $\mathcal{K}_1(x) := \mathcal{K}(A, \delta P_!(x))$ with augmentation $\mathcal{K}_1(x)^0 = P(x) \to L(x)$ is a projective resolution of $L(x)$ in* gr A
(b) *the complex $\mathcal{K}_2(x) := \mathcal{K}(A, \delta M_!(x))$ with augmentation $\mathcal{K}_2(x)^0 = P(x) \to \overline{M}(x)$ is a projective resolution of $\overline{M}(x)$ in* gr A
(c) *the complex $\mathcal{K}_3(x) := \mathcal{K}(A, \delta \overline{M}_!(x))$ with augmentation $\mathcal{K}_3(x)^0 = P(x) \to M(x)$ is a projective resolution of $M(x)$ in* gr A.

Proof. Assertion (a) follows from 3.7. We now give the proof of (b) following an argument in the proof of [9, 2.15] again. Choose a filtration $A = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^n = 0$ of A

by right A-modules with subquotients $F^{i-1}/F^i \cong \sigma^{k_i} M(x_i)^t$. Then one has a filtration $\mathcal{K}_2(x) = G^0 \supseteq G^1 \supseteq \ldots \supseteq G^n = 0$ of $\mathcal{K}_2(x)$ by subcomplexes $G^i := \mathcal{K}(F^i, \delta M_!(x))$ (of graded vector spaces), and the subquotient complexes $G^{i-1}/G^i \cong \mathcal{K}(\sigma^{k_i} M(x_i)^t, \delta M_!(x))$ are acyclic by 3.10 except perhaps in degree 0. Hence $\mathrm{H}^j(\mathcal{K}_2(x)) = 0$ for $j \neq 0$. One checks easily that $\mathrm{H}^0(\mathcal{K}_2(x)) \cong \overline{M}(x)$, proving (b). The proof of (c) is entirely similar.

3.12. Let $x, y \in X$ and consider any one of the following four pairs of modules $C(x)$ in gr A, $D(x)$ in gr $A^!$.

(a) $C(x) = P(x)$, $D(x) = L_!(x)$
(b) $C(x) = M(x)$, $D(x) = \overline{M}_!(x)$
(c) $C(x) = \overline{M}(x)$, $D(x) = M_!(x)$
(d) $C(x) = L(x)$, $D(x) = P_!(x)$.

Corollary. *For any of the four pairs of modules above, one has*

$$\mathrm{Ext}^n_{\mathrm{gr}\, A}(C(x), L(y)\langle p\rangle) \cong \begin{cases} (e_y D(x))_p & \text{if } n = p \\ 0 & \text{if } n \neq p. \end{cases}$$

Proof. The assertion (a) is trivial, while (b)–(d) follow immediately using the projective resolution of $C(x)$ given by the appropriate part of 3.11 to compute the Ext-groups.

In each case, the dimension of the space on the right may be computed from the "multiplicity matrices" $\overline{m}(A, v)$ and $a(A, v)$ using the formulae 3.5, 2.17. For $\mathcal{O}$ algebras, this reduces essentially to [9, 2.17], [11, 4.4 and 4.6] and [3, 3.7 and 3, Appendix]. As with $\mathcal{O}$ algebras, the above results have a natural interpretation in terms of "Koszul duality."

3.13. According to [1, Theorem 15], for a Koszul algebra A, the "Koszul duality functor" κ gives an equivalence

$$\kappa \colon D^{\downarrow}(A) \to D^{\uparrow}(A^!)$$

between certain (non-standard in general) derived categories of graded modules for A, $A^!$ defined there. By definition of κ, one has $\kappa(X[n]) \cong (\kappa X)[n]$, $\kappa(X\langle n\rangle) \cong (\kappa X)[-n]\langle -n\rangle$ for X in $D^{\downarrow}(A)$, where for a complex $X = (X^i, \partial)$, $X[n]$ is the usual shifted complex $(X[n])^i = X^{n+i}$ with differentials $(-1)^n \partial$ and $\langle n\rangle$ is induced by the grading shift on modules i.e. $(X\langle n\rangle)^i = (X^i)\langle n\rangle$ with differentials $\partial\langle n\rangle$.

If A is a $\tilde{\mathcal{O}}$ algebra, then Koszul duality κ maps $\delta P(x)$, $\delta M(x)$, $\delta\overline{M}(x)$, $\delta L(x)$ to $L_!(x)$, $\overline{M}_!(x)$, $M_!(x)$, $P_!(x)$ respectively. Indeed, for $P(x)$ and $L(x)$, this assertion is contained in [1, Theorem 15] and for $M(x)$, $\overline{M}(x)$ it follows immediately from 3.11 and the definition of κ in [loc cit, 2.11].

3.14. We now consider $\tilde{\mathcal{O}}$ algebras which are algebras over a commutative ring k.

Fix a graded commutative K-algebra k ($ab = ba$ for $a, b \in k$). Suppose A is a VQH algebra A (over K) with weight poset X, and that

(a) the K-algebra structure on A is induced by a k-algebra structure on A
(b) the anti-involution ω of A is k-linear
(c) for each $x \in X$, the map $k \to A(x)$ given by $a \mapsto ae_x \in e_x B e_x = A(x)$ (where $B = A/Ae_{\not\leq x}A$) is a graded K-algebra isomorphism.

It follows that $M(x)$ is graded free as left k-module, and the same is therefore true of the $P(y)$, which have filtrations with degree shifts of the $M(x)$ as successive subquotients. (This implies that if k is Noetherian, then A is a quasi-hereditary k-algebra, of split type, in the sense of [4, 3.2 and 4.3]). In general, $\overline{M}(x) \cong K \otimes_k M(x)$ for all $x \in X$.

Lemma. *Define the K-algebra $\overline{A} := K \otimes_k A \cong A/k_{>0}A$, and regard the graded A-modules $\overline{M}(x)$ and $L(x)$ also as graded $\overline{A}$-modules. Then under the above conditions (a)–(c), $\overline{A}$ is VQH with weight poset X, simple modules $L(x)$ and (big and small) Verma modules $\overline{M}(x)$. Since all $\overline{A}(x) = K$, $\overline{A}$ is even a finite-dimensional quasi-hereditary K-algebra.*

Proof. A filtration $P(y) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^n = 0$ of $P(y)$ by graded A-modules F^i with successive subquotients $F^i/F^{i+1} \cong M(x_i)\langle n_i\rangle$ for some $n_i \in \mathbb{Z}$ and $x_i \in Y$ induces a filtration $\overline{P}(y) := K \otimes_k P(y)K = K \otimes_k F^0 \supseteq \ldots \supseteq K \otimes_k F^n = 0$ of the projective cover $\overline{P}(y)$ of $L(y)$ in gr $\overline{A}$ with successive subquotients $\overline{M}(x_i)\langle n_i\rangle$ (since the $M(x)$ are free k-modules). Note also that $\overline{A}$ inherits a graded K-algebra anti-involution $\omega \otimes \mathrm{Id}_K$ fixing $\overline{A}_0$ elementwise from the given anti-involution ω of A. The other facts to be checked have already been observed in 2.11(b).

3.15. Let us call a graded module M for a positively graded graded ring R homologically pure if there is a resolution

$$0 \to P^n \to \ldots \to P^0 \to M \to 0$$

in gr R with P^i a finitely generated graded projective R-module generated by P^i_i. Note that if A is a $\tilde{\mathcal{O}}$ algebra with weight poset X, then for $x \in X$, $L(x)$, $\overline{M}(x)$ and $M(x)$ are homologically pure by 3.11, and $A(x)$ is Koszul by 3.9.

Theorem. *Let A, k and $\overline{A}$ be as in 3.14 above, and assume in addition that k is Koszul. Then the following conditions (a)–(d) are equivalent:*

(a) *A is a $\tilde{\mathcal{O}}$ algebra*
(b) *for $x \in X$, the A-module $M(x)$ is homologically pure*
(c) *for $x \in X$, the $\overline{A}$-module $\overline{M}(x)$ is homologically pure*
(d) *$\overline{A}$ is a $\mathcal{O}$ algebra.*

Proof. The remark preceeding the statement of the theorem gives that (a) implies (b). To show (b) implies (c), consider a resolution

$$0 \to P^n \to \ldots \to P^0 \to M(x) \to 0$$

showing $M(x)$ is homologically pure. If one applies $K \otimes_k -$, the resulting complex is still acyclic since $M(x)$ and all the P^i are k-free, and it shows $\overline{M}(x)$ is a homologically pure $\overline{A}$-module. By 3.14 above and a main result [9, 2.15, 2.12(b)], one has that (c) is equivalent to (d). Finally, we will show that (d) implies (a). Suppose that $\overline{A}$ is a $\mathcal{O}$ algebra. The proof that A is a $\tilde{\mathcal{O}}$ algebra will involve arguments very similar to those in the proof of [9, 2.16], in the derived categories of the categories of graded $\overline{A}$ and A-modules; we denote these by $\overline{D}$ and D respectively. Each is equipped both with the usual translation functors $X \mapsto X[n]$ and the functors $X \to X\langle n\rangle$ induced by the grading shift functors $\langle n\rangle$ (see 3.13). Moreover, the natural inclusion gr $\overline{A} \to$ gr A (recall $\overline{A} \cong A/k_{>0}A$) induces an (exact) functor $\iota\colon \overline{D} \to D$. Regard gr $\overline{A}$, gr A as full subcategories of $\overline{D}$, D respectively in the usual way (complexes with cohomology concentrated in degree 0). It will be convenient to denote the module $\delta\overline{M}(x)$ in gr A (or gr $\overline{A}$) by $\overline{N}(x)$, and the injective hull of $L(x)$ in gr A by $I(x) := \delta P(x)$.

Define E_L to be the smallest full additive subcategory of D which contains all objects $\overline{M}(x)[p]\langle p\rangle$ for all $x \in Y$ and $p \in \mathbb{Z}$, and is closed under extension (i.e if X_1, X_3 are objects of E_L and there is a distinguished triangle $X_1 \to X_2 \to X_3 \to$ in D, then X_2 is an object of E_L). Similarly, define E_R to be the full additive subcategory of D generated in the same way by the objects $\overline{N}(x)[p]\langle p\rangle$ with $x \in Y$ and $p \in \mathbb{Z}$. Now one has by 2.15(c) that

$$\mathrm{Hom}_D(\overline{M}(x), \overline{N}(y)[n]\langle p\rangle) \cong \mathrm{Ext}^n_{\mathrm{gr}\ A}(\overline{M}(x), \overline{N}(y)\langle p\rangle)$$

is zero unless $x = y$ in X, in which case it is isomorphic to $\mathrm{Ext}^n_{\mathrm{gr}\ k}(K, K\langle p\rangle)$, and this last is zero unless $n = p$ since k is Koszul. It follows by induction (since $\mathrm{Hom}_D(-,-)$ is cohomological in each variable) that for objects Y of E_L and Z of E_R, one has

(a) $$\mathrm{Hom}_D(Y, Z[n]\langle p\rangle) = 0 \text{ unless } n = p.$$

In turn, this implies that for any distinguished triangle $W \to V \to U \to$ with U, V, W in E_R, and any Z in E_L, the sequence

(b) $$0 \to \mathrm{Hom}_D(Z, W[n]\langle p\rangle) \to \mathrm{Hom}_D(Z, V[n]\langle p\rangle) \to \mathrm{Hom}_D(Z, U[n]\langle p\rangle) \to 0$$

is exact for any integers n, p.

We now use the following fact, which was the basis of the proof of [9, 2.16] (cf. also [3, 4.3]). For any $x \in X$, there are objects $Y_0 = L(x), Y_1, \dots, Y_n = 0$ (resp., $Z_0 = L(x), \dots, Z_m = 0$)in $\overline{D}$ and distinguished triangles

(c) $$Y_{n+1} \to Y_n \to \overline{N}(y_i)[p_i]\langle p_i\rangle \to \quad (\text{resp., } \overline{M}(z_i)[q_i]\langle q_i\rangle \to Z_n \to Z_{n+1} \to)$$

for some integers p_i with $p_0 = 0$ (resp., q_i with $q_0 = 0$) and some $y_i \in X$ with $y_0 = x$ and $y_i < x$ for $i > 0$ (resp., some $z_i \in X$ with $z_0 = x$ and $z_i < x$ for $i > 0$). Now apply $\iota\colon \overline{D} \to D$ to the objects Y_i, Z_i regarding them henceforward as objects of D. The triangles in (c) remain exact in D and show that all Y_i are in E_R and all Z_i are in E_L. In particular, $L(x)$ is in $E_L \cap E_R$ for any $x \in X$. Applying (a) with $Y = L(x)$, $Z = L(y)$ gives that $\mathrm{Ext}^n_{\mathrm{gr}\ A}(L(x), L(y)\langle p\rangle) = 0$ for $n \neq p \in \mathbb{Z}$, so A is Koszul by definition.

Finally, we give the proof that $A^!$ is VQH with weight poset X^{op}; it is essentially identical to the corresponding part of the proof of [9, 2.16]. First, let E denote the Yoneda Ext-algebra $E := \mathrm{Ext}^\bullet_A(A_0, A_0)$ (here, Ext_A denotes ordinary Ext of ungraded A-modules). Now E may be described in terms of Hom_D as follows. For M, N in gr A, define the graded K-vector space $e(M,N)_n := \oplus_{p\in\mathbb{Z}} \mathrm{Hom}_D(M, N[n]\langle p\rangle)$ for $n \in \mathbb{Z}$. Composition

$$\mathrm{Hom}_D(N[n]\langle p\rangle, P[n+m]\langle p+q\rangle) \times \mathrm{Hom}_D(M, N[n]\langle p\rangle) \to \mathrm{Hom}_D(M, P[n+m]\langle p+q\rangle)$$

induces bilinear maps $e(N,P)_m \times e(M,N)_n \to e(M,P)_{n+m}$ which are associative for triple products. In case M, N are in gr A, $\mathrm{Ext}^n_{\mathrm{gr}\ A}(M,N) \cong \mathrm{Hom}_D(M, N[n])$; moreover, if M has a resolution by finitely generated graded projective modules (e.g $M = A_0$) one has $\mathrm{Ext}^n_A(M,N) = \oplus_p \mathrm{Ext}^n_{\mathrm{gr}\,A}(M,N\langle p\rangle)$ and so one can identify $e(M,N)_n$ with $E = \mathrm{Ext}^n_A(M,N)$. In particular, this makes it clear that the (graded) Yoneda Ext-algebra E of A_0 may be identified with $e(A_0,A_0)$ (with multiplication induced by the above bilinear maps). We may define a covariant additive functor $\theta\colon D \to \mathrm{gr}\ E^{\mathrm{op}}$ by $\theta(N) := e(A_0,N)$ for N in D . As a consequence of (b) (since $A_0 \in E_L$), one has that for a distinguished triangle $N' \to N \to N'' \to$ in D with N', N and N'' all in E_R, the sequence

(d) $$0 \to \theta(N') \to \theta(N) \to \theta(N'') \to 0$$

in gr E^{op} is exact. We have $\theta(N)_n \cong \mathrm{Ext}^n_A(A_0,N)$ if N is in gr A.

Now by a general property of Koszul algebras [1, Theorem 2], there is a natural identification of $A^!$ with E^{op}, under which the idempotent $e_x \in (A_0^!)^{\mathrm{op}}$, for any $x \in X$, identifies with the projection $A_0 \to Ke_x \to A_0$. Henceforward, we identify $A^! = E^{\mathrm{op}}$ and regard graded right E-modules as graded left $A^!$-modules. It follows immediately that the simple left $A^!$-module not annihilated by the idempotent e_x is $L_!(x) := \theta(I(x))$ and that its projective cover is $P_!(x) = A^! e_x \cong \theta(L(x))$. We claim that the modules $M_!(x) := \theta(\overline{N}(x))$ satisfy the conditions 2.2(a)–(d) and 2.3(a′) for $A^!$ with respect to the weight poset X^{op}, which will complete the proof of the theorem. Observe first that the short exact sequences $0 \to \theta(Y_{n+1}) \to \theta(Y_n) \to \theta(\overline{N}(y_i))\langle -p_i\rangle \to 0$ (from (c)–(d)) in gr $A^!$ imply that $P_!(x)$ has a filtration as required in 2.2(d) and 2.3(a′); this also shows $M_!(x)$ is a (non-zero) quotient of $P_!(x)$, so 2.2(a) holds. Obviously $M_!(x)_n = 0$ for $n < 0$, so we have 2.2(b). Finally, one has $e_y M_!(x) \cong e(L(y),\overline{N}(x)) = 0$ unless $y \geq x$ by 2.15(d), so 2.2(c) holds, completing the proof. One easily sees as well that $\theta(\delta M(x)) \cong \overline{M}_!(x)$.

3.16. Suppose that U is a finite-dimensional K-vector space and C is a VQH algebra with weight poset X which is also a $k := S(U)$ algebra satisfying 3.14(a)–(c) (where k has the grading $k_0 = K$, $k_2 = U$). Assume moreover that the big Verma modules $M(x)$ of C are homologically pure.

Let $U' = U \oplus \mathbb{R}t$. Grade the symmetric algebra $k' = S(U')$ so $k'_0 = K$, $k'_1 = U'$. Arguments like that in the proof of 3.14 and that of (b) implies (c) in 3.15 show that $C' = k' \otimes_k C$ is a VQH algebra with weight poset X, satisfying 3.14(a)–(c) with k replaced by k', and that the big Verma modules $k' \otimes_k M(x)$ of C' are homologically pure. Since k' is Koszul, the theorem implies that C' is a $\tilde{\mathcal{O}}$ algebra with weight poset X and $K \otimes_k C \cong K \otimes_{k'} C'$ is a $\mathcal{O}$ algebra with weight poset X.

3.17. In a subsequent paper, for any $X = \Gamma$ as in 1.2 or 1.3, we will construct a naturally associated $S(U) \otimes S(U)^{\mathrm{op}}$ algebra C. It is expected that as left or right $S(U)$-algebra, C should have the properties in 3.16.

3.18. Finally, to show the reason for introducing the VQH algebras considered in Section 2, we return to the algebras constructed in Section 1 to give some explicit examples of $\tilde{\mathcal{O}}$ algebras.

Consider the graded $S(U)$-algebra $C = C_X$ defined in 1.19 where X is a multiplicity free spherical poset (satisfying the orientability condition [9, (3.15)(b)]) with a non-degenerate symmetric labelling on a K-vector space U. One sees easily from the proof of 1.22 that C is VQH, and (as either left or right k-algebra, where $k = S(U)$) satisfies the coditions 3.7(a)–(c). Using 1.23 and 3.15, the algebras $A' = S(U') \otimes_{S(U)} C$ and $B' = C \otimes_{S(U)} S(U')$ are $\tilde{\mathcal{O}}$ algebras with weight poset X. Let's consider A' (the same statements hold for B').

One has $A'(x) \cong S(U')$ for all $x \in X$. Also, $\overline{m}_{x,y}(A',v)$ is $v^{l(y)-l(x)}$ if $x \leq y$ and 0 otherwise, and $a(A',v) = (1-v)^{-n-1}\mathrm{Id}$ (where $n = \dim U$) is the scalar matrix with diagonal entries equal to the Poincaré (Hilbert) series $(1-v)^{-n-1} \in \mathbb{Z}[[v]]$ of $S(U')$. All results of this section and Section 2 apply to A' and its quadratic dual. The dual $(A')^!$ is a finite-dimensional K-algebra which is in general not quasi-hereditary as K-algebra; for X a point, $(A')^!$ is the exterior algebra $\Lambda((U')^*)$. In general, one has $(A')^!(x) \cong \Lambda((U')^*)$ for all $x \in X$, but $(A')^!$ fails to be an "algebra" over the exterior algebra, as the reader can check by examining the case of a length one interval.

For completeness, we check that C has the properties in 3.16. Consider the projective resolutions $\mathcal{K}_3(x) \to M(x)$ in gr A' in 3.11, for $x \in X$. Regard $S(U)$ as (ungraded) $S(U')$-algebra via $t \mapsto 1$. Since the above resolutions are $S(U')$-free, one obtains ungraded prjective resolutions $S(U) \otimes_{S(U')} \mathcal{K}_3(x) \to S(U) \otimes_{S(U')} M(x)$ of the big Verma modules for C. Its easy to see that the modules in these resolutions can be graded to give resolutions in gr C showing the big Verma modules for C are homologically pure.

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SYMMETRIC GROUPS AND QUASI–HEREDITARY ALGEBRAS

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Quasi–hereditary algebras were introduced by L. Scott [S] in order to study highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups, and important results were proved by Cline, Parshall and Scott (see [$CPS_{1,2}$]). These algebras can be defined entirely in ring–theoretic terms; and they were studied from this point of view by Dlab and Ringel (see [$DR_{1,2}$], [$R_{1,2}$]) and by others. In particular it turns out that quasi–hereditary algebras are quite common.

Suppose A is a quasi–hereditary algebra, then there is a partial order $(\Lambda, \leq)$ on the set of simple A–modules; and one studies the standard modules $\{\Delta(\lambda): \lambda \in \Lambda\}$ and the co–standard modules $\{\nabla(\lambda): \lambda \in \Lambda\}$ of A, see 1.2. Of main interest are the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of A–modules which have a Δ–filtration (respectively ∇–filtration). It was proved in [R_1] that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains only finitely many indecomposable modules, called canonical modules, which may be labelled as $T(\lambda)$, $\lambda \in \Lambda$, see 1.3. Moreover, let $T := \oplus\, T(\lambda)$, then T is a generalized tilting – and cotilting module, and $A' :=$ End $({}_A T)$ is again quasi–hereditary, where the partial order is the reverse order of $(\Lambda, \leq)$. The functor $\mathrm{Hom}_A(T, -)$ induces an equivalence $\mathcal{F}_A(\nabla) \to \mathcal{F}_{A'}(\Delta)$, taking $\Delta_A(\lambda)$ to $\nabla_{A'}(\lambda)$. The algebra $(A')'$ obtained by iterating this procedure be identified with A, as a quasi–hereditary algebra. We call the

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algebra A' the Ringel dual of A.

We consider representations of the symmetric group Σ_r over some field K of characteristic p. Amongst the $K\Sigma_r$–modules studied most often are the Specht modules and the Young modules, see 2.3, 2.4; and their structure is very similar to that of standard modules and canonical modules of some quasi–hereditary algebra. The aim of this paper is to show that these properties can in fact be understood in terms of quasi–hereditary algebras which arise naturally.

The representation theory of $K\Sigma_r$ is linked to the theory of polynomial representations of GL_n via the Schur algebras S(n, r), for each n. These algebras are quasi–hereditary, by [D_1, P, G_3]. Hence for each n, there is a quasi–hereditary algebra S(n,r)', the Ringel dual of S(n,r). Our result is that for each n, a certain quotient of $K\Sigma_r$ is isomorphic to algebra of the form eS'e where S' is Morita equivalent to S(n,r)' and e is a certain idempotent of S', see 4.3. Moreover Specht modules S^λ and Young modules Y^λ for partitions λ with at most n parts are identified with modules $e\Delta(\lambda)$ and $eT(\lambda)$.

The main results are 4.4 and 4.6. Namely, in good cases (if n is small relative to p) one can take e = 1, and then the quotient of $K\Sigma_r$ is quasi–hereditary, Morita–equivalent to the Ringel–dual S(n,r)'. In particular it follows that the category of $K\Sigma_r$–modules which have Specht filtrations by S^λ with partitions λ with $\leq$ n parts is equivalent to the category $\mathcal{F}(\Delta)$ of modules with Weyl filtration of S(n,r). In particular, this category has relative Auslander–Reiten sequences. The results have a block version (Theorem 4.6).

In general, the connection between $K\Sigma_r$ and eS'e gives an easy way to relate decomposition numbers to filtration multiplicities of canonical modules for the Schur algebra, as shown in 4.5. We include a short proof, due to S. Donkin, of a Theorem by G.D. James on the decomposition numbers of 2–part partitions of symmetric groups (see Chapter 6).

For $n \geq r$ there is also the well–known Schur functor which relates the quasi–hereditary algebra $S(n,r)$ to the group algebra $K\Sigma_r$. Namely there is an idempotent ξ of $S(n,r)$ such that $\xi S(n,r)\xi$ is isomorphic to $K\Sigma_r$, and the Schur functor $S(n,r)$–mod $\to$ $K\Sigma_r$–mod is defined by $M \to \xi M$. However this is something different, although we use the Schur functor as a tool.

In Chapter 1 we discuss quasi–hereditary algebras. We include some results which were proved for algebraic groups or Schur algebras and which generalize to arbitrary quasi–hereditary algebras. In Chapters 2 and 3 we summarise results on $K\Sigma_r$–modules and $S(n,r)$–modules respectively. In Chapters 4 and 5 we prove the main results, and the last chapter contains the result on filtration multiplicities and decomposition numbers of 2–part partitions.

This work depends very much on theorems obtained by S. Donkin , J. A. Green and also by G.D. James and others. I am especially grateful to S. Donkin, not only for proving many of the crucial results but also for valuable discussions, and for allowing me to include the result on decomposition numbers. I am also grateful to the referee for suggesting improvements to 4.3.

1. Quasi–hereditary algebras

1.1 Suppose A is a finite–dimensional K–algebra where K is a field which we assume to be algebraically closed. We denote by A–mod the category of all finite–dimensional left A–modules. If θ is a class of A–modules closed under isomorphisms then $\mathcal{F}(\theta)$ is defined to be the class of all A–modules which have a θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \dots \subset M_1 \subset M_0 = M$$

such that all factors M_{i-1}/M_i, $1 \leq i \leq t$, belong to θ.

1.2 Let $L(\lambda)$, $\lambda \in \Lambda$, be the simple A–modules, one from each isomorphism class. We assume that there is a fixed partial order $(\Lambda, \leq)$. Let also $P(\lambda)$ be the projective cover and $Q(\lambda)$ be the injective hull of $L(\lambda)$, for $\lambda \in \Lambda$.

We define the *standard module* $\Delta(\lambda)$ to be the largest factor module of $P(\lambda)$ with all composition factors of the form $L(\mu)$ for $\mu \leq \lambda$. Dually, the *co–standard module* $\nabla(\lambda)$ is defined to be the largest submodule of $Q(\lambda)$ with composition factors of the form $L(\mu)$ where $\mu \leq \lambda$.

The algebra A is said to be *quasi–hereditary* (with respect to $(\Lambda, \leq)$) if for all $\lambda \in \Lambda$ we have

(i) $\mathrm{End}_A(\Delta(\lambda)) \cong K$;

(ii) $P(\lambda)$ belongs to $\mathcal{F}(\Delta)$; and moreover $P(\lambda)$ has a Δ–filtration with quotients $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

1.2.1 Suppose A is quasi–hereditary, then we have the following:

(1) It is also true that $\mathrm{End}_A(\nabla(\lambda)) \cong K$; moreover the modules $Q(\lambda)$ belong to $\mathcal{F}(\nabla)$ and have a ∇–filtration with quotients $\nabla(\mu)$ for $\mu \geq \lambda$ in which $\nabla(\lambda)$ occurs

only once.

(2) $\Delta(\lambda)$ has a simple top isomorphic to $L(\lambda)$,and all other composition factors are of the form $L(\mu)$ for $\mu < \lambda$. A dual property holds for $\nabla(\lambda)$.

(3) Suppose $\{e_\lambda: \lambda \in \Lambda\}$ is a set of orthogonal idempotents such that $Ae_\lambda \cong P(\lambda)$. Let $U(\lambda) := \sum_{\mu > \lambda} Ae_\mu Ae_\lambda$, then we have that $Ae_\lambda / U(\lambda)$ is isomorphic to $\Delta(\lambda)$.

<u>1.3</u> Assume that A is quasi–hereditary. We will summarise results which are important in this context, proofs may be found in [CPS] and $[R_1]$.

(1) The intersection $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains exactly t indecomposable modules where $t = |\Lambda|$. They may be parametrized as $T(\lambda)$, $\lambda \in \Lambda$, such that the following holds:

There are exact sequences

(a) $0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0$ and $X(\lambda)$ is filtered by $\Delta(\mu)$'s for certain $\mu < \lambda$.

(b) $0 \to Z(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0$ and $Z(\lambda)$ is filtered by $\nabla(\mu)$'s for certain $\mu < \lambda$.

In particular $T(\lambda)$ has a unique composition factor isomorphic to $L(\lambda)$ and all other composition factors are of the form $L(\mu)$ for $\mu < \lambda$. We call the modules $T(\lambda)$ "canonical modules".

(2) Let $T := \bigoplus_{\Lambda} T(\lambda)$ which is denoted as the *characteristic module* in $[R_1]$, $[DR_2]$. This is a generalized tilting– and cotilting module in the sense of [Mi]. Let $A' := \mathrm{End}_A T$, then A' is again quasi–hereditary, with standard modules $\Delta'(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$ and where the partial order is reversed. The functor $\mathrm{Hom}_A(T, -)$ induces an equivalence of $\mathcal{F}_A(\nabla)$ and $\mathcal{F}_{A'}(\Delta)$, and moreover

$\mathrm{Hom}_A(T,\ Q(\lambda)) \cong T_{A'}(\lambda)$. The algebra A'' obtained by iterating this procedure is Morita equivalent to A, as a quasi–hereditary algebra with the same order, such that the standard– and costandard modules are identified. We call the algebra A' the "Ringel dual" of A.

(3) The category $\mathcal{F}(\Delta)$ is closed under direct sums, extensions and kernels of epimorphisms. Moreover, the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite in A–mod (see [AR], [R]). In particular $\mathcal{F}(\nabla)$ and $\mathcal{F}(\nabla)$ have relative Auslander–Reiten sequences, see [R_1] and [AS].

(4) As an important homological property we have that for all $n \geq 0$,

$$\mathrm{Ext}^n_A(\ \Delta(\lambda), \nabla(\mu)) \cong \begin{cases} K & \text{if } n = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

This implies that for M in $\mathcal{F}(\Delta)$ (or $\mathcal{F}(\nabla)$) the filtration multiplicity of $\Delta(\mu)$ (or $\nabla(\mu)$) depends only on M and not on the choice of the filtration. (For M in $\mathcal{F}(\Delta)$, the multiplicity of $\Delta(\mu)$ is equal to dim $\mathrm{Hom}_A(M,\ \nabla(\mu))$.) This multiplicity is denoted by $[M : \Delta(\mu)]$ (or $[M : \nabla(\mu)]$). (We also write $[M{:}L(\mu)]$ for the multiplicity of $L(\mu)$ as a composition factor of M.)

1.4 We will call more generally a module T a characteristic module if it is of the form $\bigoplus_{\Lambda} n_\lambda T(\lambda)$ with $n_\lambda \geq 1$, or equivalently if $\mathrm{add}(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. If this is the case then the endomorphism ring $\mathrm{End}_A(T)$ is Morita equivalent to the Ringel dual A'.

The main tools for the study of quasi–hereditary algebras in our context will be functors of the form $\mathrm{Hom}_A(Ae, -)$. Some of the important results for Schur algebras can be formulated for arbitrary quasi–hereditary algebras, which we will do since it seems to be of interest.

1.5 Let A be an arbitrary finite–dimensional algebra. Suppose e is an idempotent of A, this gives an exact functor $\mathrm{Hom}_A(Ae, -)$: A–mod $\to$ eAe–mod which takes V to eV. If L $\in$ A–mod is a simple module then eL is either zero or a simple module, and all simple eAe–modules are of this form. Suppose eL is non–zero, then eP(L) is the projective cover of eL and eQ(L) is its injective hull, as an eAe–module.(See for example [A, G_1].)

1.6 Assume that A is quasi–hereditary with partially ordered set $(\Lambda, \leq)$. Take a set of orthogonal primitive idempotents $\{ e_\lambda : \lambda \in \Lambda \}$ such that $Ae_\lambda \cong P(\lambda)$. Suppose Γ is a subset of Λ; we denote by e_Γ the idempotent $e_\Gamma = \sum_{\lambda \in \Gamma} e_\lambda$, and we set $A_\Gamma := e_\Gamma A e_\Gamma$. We are interested in subsets Γ such that A_Γ is again quasi–hereditary.

We say that a subset Γ of Λ is *saturated* if $\mu \in \Gamma$ and $\lambda \leq \mu$ implies $\lambda \in \Gamma$. This notion is taken from the theory of representations of algebraic groups. We have the following result (see also [G_1, 6.5]).

LEMMA *Assume A is quasi–hereditary with respect to* $(\Lambda, \leq)$, *and let* $\Gamma \subset \Lambda$ *be a subset such that* $\Lambda - \Gamma$ *is saturated. Then*

(i) A_Γ *is quasi–hereditary with standard modules* $\{e\Delta(\mu): \mu \in \Gamma\}$*; with partial order* $(\Gamma, \leq)$.

(ii) The costandard modules of A_Γ *are the modules* $e\nabla(\mu)$ *for* $\mu \in \Gamma$.

(iii) If $\lambda \notin \Gamma$ *then* $e\Delta(\lambda) = 0$ *and* $e\nabla(\lambda) = 0$.

Proof: Write $e = e_\Gamma$. By 1.5 the simple A_Γ–modules are given by $\{eL(\lambda): \lambda \in \Gamma\}$, and if $\lambda \in \Gamma$ then $eP(\lambda)$ is the projective cover of $eL(\lambda)$. For $\lambda \in \Gamma$, we write λ^* if the module parametrized is an A_Γ–module, so that $L(\lambda^*) = eL(\lambda)$

and $P(\lambda^*) = eP(\lambda)$.

(iii) Suppose $\lambda \notin \Gamma$, then for all $\mu \leq \lambda$ we have $\mu \notin \Gamma$ since $\Lambda - \Gamma$ is saturated. It follows now that in this case $e\Delta(\lambda) = 0$ and $e\nabla(\lambda) = 0$ (see 1.2).

(i) Let $\lambda \in \Gamma$. Then all composition factors of $e\Delta(\lambda)$ are isomorphic to $L(\mu^*)$ for certain $\mu \in \Gamma$ with $\mu \leq \lambda$, and hence $e\Delta(\lambda)$ is a quotient of $\Delta(\lambda^*)$. Since A is quasi–hereditary, there is an exact sequence

$0 \to U(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$ of A–modules, where $U(\lambda)$ is filtered by $\Delta(\mu)$ with $\mu > \lambda$. as in 1.2. This gives an exact sequence of A_Γ–modules $0 \to eU(\lambda) \to P(\lambda^*) \to e\Delta(\lambda) \to 0$, and it follows from (iii) that $eU(\lambda)$ is filtered by modules of the form $e\Delta(\mu)$ for $\mu > \lambda$ and μ in Γ. This implies now that $e\Delta(\lambda) \cong \Delta(\lambda^*)$ and moreover that $P(\lambda^*)$ satisfies 1.2(ii). It is also clear that 1.2(i) holds since $L(\lambda^*)$ is multiplicity–free as a composition factor of $\Delta(\lambda^*)$.

(ii) Dually one proves that $e\nabla(\lambda) \cong \nabla(\lambda^*)$ for $\lambda \in \Gamma$.

<u>1.7</u> The following result, for the case of algebraic groups, was proved in [D_3, 1.5].

PROPOSITION *Assume A is quasi–hereditary with respect to* $(\Lambda, \leq)$, *and assume* $\Gamma \subset \Lambda$ *such that* $\Lambda - \Gamma$ *is saturated.*

(i) Suppose $M \in \mathcal{F}(\Delta)$ *and* $N \in \mathcal{F}(\nabla)$, *then the map*

$Hom_A(M, N) \to Hom_{A_\Gamma}(eM, eN)$ *is surjective.*

(ii) Let $\lambda \in \Gamma$, *then* $eT(\lambda) \cong T(\lambda^*)$, *and for* $\lambda \notin \Gamma$ *we have* $eT(\lambda) = 0$.

Proof:(i) It follows from 1.6 that eM belongs to $\mathcal{F}_\Gamma(\Delta)$ and that eN lies in $\mathcal{F}_\Gamma(\nabla)$. Suppose $0 \to M_1 \to M \to M/M_1 \to 0$ is exact with M/M_1 and M_1 in

$\mathcal{F}(\Delta)$. Recall that $\mathrm{Ext}^1_A(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$, see 1.3(4); hence we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_A(M_1, N) & \to & \mathrm{Hom}_A(M, N) & \to & \mathrm{Hom}_A(M/M_1, N) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}_{A_\Gamma}(eM_1, eN) & \to & \mathrm{Hom}_{A_\Gamma}(eM, eN) & \to & \mathrm{Hom}_{A_\Gamma}(e(M/M_1),N) & \to & 0. \end{array}$$

If the outer vertical maps are surjective then so is the middle vertical map. This reduces to the case $M = \Delta(\mu)$; and similarly it suffices to consider $N = \Delta(\tau)$.

Suppose $\mathrm{Hom}_{A_\Gamma}(e\Delta(\mu), e\nabla(\tau))$ is non–zero, then $e\Delta(\mu) \cong \Delta(\mu^*)$ and $e\nabla(\tau) \cong \nabla(\tau^*)$. It follows from 1.3(4) that $\mu = \tau$, and that the map is an isomorphism.

(ii) Suppose $\lambda \in \Gamma$. Since $T(\lambda)$ is indecomposable, we have that $\mathrm{End}_A(T(\lambda))$ is a local algebra. By (i), $\mathrm{End}_{A_\Gamma}(eT(\lambda))$ is also local and therefore $eT(\lambda)$ is indecomposable if $eT(\lambda)$ is non–zero. Moreover, the module belongs to $\mathcal{F}_\Gamma(\Delta) \cap \mathcal{F}_\Gamma(\nabla)$ and hence is isomorphic to $T(\mu^*)$ for some $\mu \in \Gamma$. By 1.3(1) it is clear that $\lambda = \mu$. The last part follows from 1.3(1) and 1.6(iii).

1.7.1 Assume that $N = Q(\lambda)$ for $\lambda \in \Gamma$ and M lies in $\mathcal{F}(\Delta)$, then the map in 1.7(i) is an isomorphism.

To prove this, it suffices by 1.7 to show that both spaces have the same dimension. But $\dim \mathrm{Hom}_A(M, Q(\lambda)) = [M{:}L(\lambda)] = [eM : eL(\lambda)] = \dim \mathrm{Hom}_{A_\Gamma}(eM, Q(\lambda^*))$.

1.8 The following is a general version of $[D_1,\mathrm{II}(2.3)]$

PROPOSITION *Assume A is quasi–hereditary and e is an idempotent of A such that Ae is a faithful A–module. Assume also that* $\nabla(\mu)$ *is a homomorphic* *image of $D(A_A)$, for each μ in Λ. Suppose V, W belong to $\mathcal{F}(\nabla)$, then the map $Hom_A(V, W) \to Hom_{eAe}(eV, eW)$ is injective.*

Proof: (1) We may assume $W = \nabla(\lambda)$ and that $V = \nabla(\mu)$:
Suppose there is an exact sequence $0 \to W_1 \to W \to W/W_1 \to 0$ with W_1 and W/W_1 in $\mathcal{F}(\nabla)$. This gives a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_A(V, W_1) & \to & \mathrm{Hom}_A(V, W) & \to & \mathrm{Hom}_A(V, W/W_1) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Hom}_{A_0}(eV, eW_1) & \to & \mathrm{Hom}_{A_0}(eV, eW) & \to & \mathrm{Hom}_{A_0}(eV, e(W/W_1)) \end{array}$$

Suppose we know that the outer vertical maps are one–to–one; then it follows that also the middle vertical map is one–to–one. Similarly one reduces to the case when $V = \nabla(\mu)$.

(2) We may assume that $W = D(A_A)$: The module D(A) is an injective cogenerator, so there is a monomorphism $0 \to \nabla(\lambda) \to D(A)$; and this induces an inclusion of $\mathrm{Hom}_A(V, \nabla(\lambda))$ into $\mathrm{Hom}_A(V, D(A))$.

(4) We may assume that $V = D(A)$: By the hypothesis there is an epimorphism $D(A) \to \nabla(\mu)$ and this induces an inclusion $\mathrm{Hom}_A(\nabla(\mu), D(A)) \to \mathrm{Hom}_A(D(A), D(A))$.

Now, $\mathrm{Hom}_A(DA, DA)$ is isomorphic to ${}_AA$ by identifying $a \in A$ with the map φ_a, $a \in A$, where $(f)\varphi_a = fa$, $f \in DA$. We consider now the map $\mathrm{Hom}_A(DA, DA) \to \mathrm{Hom}_{eAe}(eDA, eDA)$. Suppose the functor takes φ_a to zero; that is, φ_a is zero on eD(A). For all f in D(A), we have $(ef)\varphi_a = (ef)a = 0$. That is, for all $x \in A$ we have $((ef)a)(x) = f(axe) = 0$, which implies $aAe = 0$. We assume that Ae is faithful, therefore $a = 0$.

2.On the representation theory of $K\Sigma_r$

2.1 Let Σ_r be the symmetric group on $\underline{r} = \{1, 2, ..., r\}$ and let K be a field. The modules we study are related to permutation modules.

(a) Take a finite–dimensional vector space E, then the r–fold tensor product $E^{\otimes r}$ is a permutation module for Σ_r if one lets Σ_r act on the right by place permutations. To be precise, suppose dim E = n, and fix a basis $\{\underline{e}_1, ..., \underline{e}_n\}$ for E. Then $E^{\otimes r}$ has a basis indexed by the set

$I(n, r) := \{ i = (i_1, i_2, ..., i_r): i_\nu \in \underline{n} \}$ where for $\underline{i} \in I(n, r)$ we write $\underline{e}_i = \underline{e}_{i_1} \otimes \underline{e}_{i_2} \otimes ... \otimes \underline{e}_{i_r}$.

The group Σ_r acts on I(n,r) on the right by the rule $i\pi = (i_{(1)\pi}, i_{(2)\pi}, ..., i_{(r)\pi})$ and $E^{\otimes r}$ is the corresponding permutation module.

(b) The module $E^{\otimes r}$ is a direct sum of transitive permutation modules, one for each Σ_r–orbit on I(n, r). Let $\Lambda(n, r)$ be the set of unordered partitions of r with at most n parts, so that an element $\lambda \in \Lambda(n, r)$ is of the form $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ with $\lambda_i \geq 0$ and $\Sigma\, \lambda_i = r$. This set $\Lambda(n, r)$ parametrizes the Σ_r–orbits on I(n, r). Namely λ labels the orbit consisting of all $\underline{i}$ in I(n, r) where $\underline{i}$ has λ_ν entries equal to ν, for $1 \leq \nu \leq n$. In this case we say that $\underline{i} \in$ I(n,r) belongs to λ.

Now let M^λ be the span of the set $\{\underline{e}_i$: i belongs to λ }. Then M^λ is the transitive permutation module corresponding to λ, and moreover $E^{\otimes r} = \oplus M^\lambda$ where the sum is taken over $\Lambda(n, r)$. It is clear that $M^\lambda \cong M^\mu$ if λ and μ give rise to the same ordered partition of r. Let $\Lambda^+(n, r)$ be the set of ordered partitions of r with at most n parts.

2.2 Suppose λ, μ are ordered partitions. The dominance order of partitions is the partial order on $\Lambda(n,r)$ is given as follows:

$$\lambda \leq \mu \text{ if and only if } \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \text{ for each } k \leq n.$$

Recall also that λ is p–regular if λ does not have p parts of the same (non–zero) length. We write $\Lambda^+(n,r)_p$ for the set of p–regular partitions with at most n parts, and $\lambda \vdash r$ means that λ is a proper partition of r. Moreover, λ' is partition conjugate to λ.

2.3 For each $\lambda \vdash r$, there is an explicitly defined submodule S^λ of M^λ, the *Specht module*, which has the following properties (see [JK Ch. 7] or [Ja_1]):

(1) The module is defined characteristic–free. If char K = 0 (or does not divide the order of Σ_r) then S^λ is simple, and $\{S^\lambda : \lambda \vdash r\}$ is a full set of pairwise non–isomorphic simple modules.

(2) Suppose λ is p–regular where p = char K. Then S^λ has a unique simple quotient D^λ and all other composition factors of S^λ are isomorphic to D^μ with $\mu \gneqq \lambda$. The modules D^λ for λ p–regular form a full set of pairwise non–isomorphic simple $K\Sigma_r$–modules.

The dual $(S^\lambda)^*$ for λ p–regular has a simple socle, isomorphic to D^λ, and all other composition factors are isomorphic to D^μ for some $\mu > \lambda$. (If G is a finite group and M a KG–right module then M^* is viewed as a right module via $(fg)(m) = f(g^{-1}m)$, for $f \in M^*$, $g \in G$ and $m \in M$).

2.3.1 (1) We will define the Specht module S^λ explicitly, using notation as in [G_1]. Fix a bijective λ–tableau T^λ and let $C(\lambda)$ be the column stabilizer of T^λ. Denote by $\{C(\lambda)\}$ the alternating sum $\{C(\lambda)\} = \sum_{C(\lambda)} s(\sigma)\sigma$, an element of $K\Sigma_r$. Define also $\ell \in I(n,r)$ to be the element $\ell = (i_1, i_2, ..., i_r)$ where $i_\nu = m$ if ν lies on the m–th row of T. Then ℓ belongs to λ. For example, if $\lambda = (3,2)$ and $T^\lambda = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{pmatrix}$ then we have that $C(\lambda) =$ $\mathrm{Sym}\,\{1, 4\} \times \mathrm{Sym}\,\{2, 5\}$. Moreover $\ell = (1, 1, 1, 2, 2)$.

Set now $f_\ell := e_\ell\{C(\lambda)\}$, this is an element on M^λ. The Specht module S^λ is now, by definition, $S^\lambda = f_\ell K\Sigma_r$.

(2) The module S^λ is isomorphic to a right ideal of $K\Sigma_r$. Namely let $R(\lambda)$ be the row stabilizer of T^λ and $[R(\lambda)] = \sum_{R(\lambda)} g$. Then S^λ is isomorphic to $[R(\lambda)]\{C(\lambda)\}K\Sigma_r$, see [$Ja_1$, G_1].

(3) If $\lambda = (r)$ then $S^\lambda \cong K$, and if $\lambda = (1^r)$ then $S^\lambda = K_a$, the alternating representation.

<u>2.3.2</u> Let $< , >$ be the bilinear form on M^λ such that the standard basis is an orthonormal basis. Then $< , >$ is symmetric and non–singular on M^λ and also Σ_r–invariant, that is $< v\pi, w\pi > = < v, w >$. It was proved by James (see [JK 7.1.7]) that for for any $K\Sigma_r$–submodule X of M^λ, either $S^\lambda \subset X$ or $X \subset (S^\lambda)^\perp$. It follows that if $S^\lambda \cap (S^\lambda)^\perp \underset{\neq}{\subset} S^\lambda$ then this is the unique maximal submodule of S^λ. This is the case if and only if λ is p–regular; if so then D^λ is by definition the simple quotient (see 2.3(2)). The modules M^λ and D^λ are self–dual.

<u>2.4</u> The *Young modules* may be defined as follows. Write M^λ as a direct sum of indecomposable $K\Sigma_r$–modules, say $M^\lambda = \bigoplus_{j=1}^{t} Y_j$. Then there is a unique j (1

$\leq j \leq t$) such that $S^\lambda \subset Y_j$: To see this, consider the bilinear form $< , >$ as in 2.3.2. Since $< , >$ is non–singular, there must be some j such that Y_j is not contained in $(S^\lambda)^\perp$. Now it follows from James' Theorem that $S^\lambda \subset Y_j$. Clearly Y_j is unique since the sum is direct. Define now $Y^\lambda :=$ the summand Y_j of M^λ that contains S^λ.

It is true that $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$. Also, every indecomposable direct summand of M^λ is isomorphic to Y^μ for some $\mu \geq \lambda$. Moreover the modules Y^λ are self–dual.

2.4.1 It has been proved in $[D_1II]$ that Young modules Y^λ have Specht filtrations. By construction we have $S^\lambda \subset Y^\lambda$ and moreover the quotient is filtered with S^μ for $\mu > \lambda$. We may write this in terms of an exact sequence $0 \to S^\lambda \to Y^\lambda \to X^\lambda \to 0$. Taking duals we obtain an exact sequence $0 \to Z^\lambda \to Y^\lambda \to (S^\lambda)^* \to 0$ where Z^λ is filtered with $(S^\mu)^*$ for $\mu > \lambda$. These are analogue to the characterization of the indecomposable modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ given in 1.3.

2.5 The properties of S^λ, Y^λ described in 2.3(2), 2.4.1 suggest that there may exist quotient algebras $A = K\Sigma_r/ I$ which are quasi–hereditary such that the modules $\Delta_A(\lambda)$ and $T_A(\lambda)$ are identified with Specht modules S^λ and Young modules Y^λ. Moreover, the partial order for any such A should be the reverse dominance order. Any such algebra A must necessarily be a proper quotient of $K\Sigma_r$, for example since it should have finite global dimension.

2.6(1) We started with $K\Sigma_r$–right modules since $E^{\otimes r}$ comes naturally with a right action. The modules occuring later will be $K\Sigma_r$–left modules; and given a

$K\Sigma_r$–right module M we must view it as a left module by setting

$$\pi m := m\pi^{-1} \ (\pi \in \Sigma_r,\ m \in M).$$

(2) We need the following properties of Young modules which can be proved by representation theory of finite groups. Namely, the module Y^λ is projective if and only if λ' is p–regular (see [Gb, Ch. 7]). Suppose λ is p–regular, then $Y^{\lambda'} \otimes K_a \cong P(D^\lambda)$.

(3) The blocks of $K\Sigma_r$ are parametrized by p–cores of r. By the "Nakayama conjecture" we have that S^λ and S^μ belong to the same block B of $K\Sigma_r$ if and only if λ and μ have the same p–cores. For details, see [JK, 6.1.21]. We write $\Lambda^+(n,r) \cap B$ for the set of partitions $\lambda \in \Lambda^+(n,r)$ where S^λ is a B–module.

3. Schur algebras

3.1 The Schur algebra $S(n, r)$ (or $S_K(n, r)$) is a finite–dimensional algebra, originally defined via GL_n, and the module category $S(n, r)$–mod is equivalent to the category of polynomial representations of $GL_n(K)$ which are homogeneous of degree r, see 3.2.2.

3.2 Most important here is the relationship to mod $K\Sigma_r$. Let E be the fixed n–dimensional vector space and $E^{\otimes r}$ the permutation module of Σ_r as described in Chapter 1. Then by a Theorem of Schur, the algebra $S_K(n, r)$ is isomorphic to $\mathrm{End}_{K\Sigma_r}(\ E^{\otimes r}\ )$, see $[G_1]$. We take this here as an identification. From now we fix the field K and integers n and r and we set $S = S_K(n, r)$.

3.2.1 (1) Since S is the endomorphism ring of a permutation module, it has a basis parametrized by the Σ_r–orbits on $I \times I$ where $I = I(n, r)$, where the action is given by $(i, j)\pi = (i\pi, j\pi)$. Namely, if θ is such an orbit and (i, j) lies in θ then we define an element $\xi_{i,j}$ of S by setting

(*) $\xi_{i,j}(\underline{e}_k) = \Sigma\, \underline{e}_h$ where the sum is taken over all $h \in I$ such that $(h, k) \in \theta$,

($= 0$ if k and j are not in the same orbit of Σ_r).

Then $\xi_{i,j} = \xi_{k,l}$ for (k, h) and (i, j) in the same orbit on $I \times I$. The algebra S has as K–basis the set of all distinct $\xi_{i,j}$.

(2) These maps are related to the direct sum decomposition $E^{\otimes r} = \oplus\ M^\lambda$ of 2.1. Suppose $i \in \mu$ and $j \in \lambda$. Then we have that $\xi_{i,j}$ maps M^ρ to zero if $\rho \neq \lambda$ and maps M^λ into M^μ. Moreover $\xi_{i,i}$ is the identity on M^λ. Hence $\xi_{i,i}$ is the projection with image M^λ and kernel $\bigoplus_{\mu\neq\lambda} M^\mu$. This shows that

(a) $\xi_{i,j}\xi_{k,l} = 0$ unless j and k are in the same Σ_r–orbit, and

(b) $\xi_{i,i}\xi_{i,j} = \xi_{i,j} = \xi_{i,j}\xi_{j,j}$.

In particular $\xi_{i,i}$ is an idempotent. It depends only on the orbit λ of i, so we write $\xi_{i,i} = \xi_\lambda$. Then

(c) $\xi_\lambda\xi_\rho = 0$ for $\lambda \neq \rho$, and $\sum_\lambda \xi_\lambda = 1_S$.

Note however that ξ_λ is usually not primitive, the module M^λ is very rarely indecomposable.

(3) Explicit formulae for the products $\xi_{i,j}\xi_{p,q}$ in general in terms of the permutation action of Σ_r may be found in [G_1 (2.3)] or [G_2].

<u>3.2.2</u> We will now describe how S–modules may be related to polynomial representations of $G = GL_n(K)$. Proofs may be found in [G_1].

(1) Let $c_{\mu\nu} : G \to K$ be the coordinate functions, and let A(n) be the algebra of polynomial functions $G \to K$, generated by the $c_{\mu\nu}$. This is the algebra of all polynomials in n^2 variables, for K infinite. Then A(n,r) is the subspace of A(n) consisting of the elements expressible as polynomials which are homogeneous of degree r in the $c_{\mu\nu}$. It is spanned by monomials which one writes as

$$c_{i,j} = c_{i_1j_1}c_{i_2j_2}\cdots c_{i_rj_r}$$

where $i = (i_1, i_2, \dots, i_r)$ and $j = (j_1, \dots, j_r)$ belong to I(n,r). Moreover $c_{i,j} = c_{k,l}$ if and only if (i,j) and (k,l) are in the same Σ_r–orbit. The distinct $c_{i,j}$ form a basis of A(n,r). It is true that A(n,r) has a coalgebra structure, and this gives rise to an algebra structure on the dual $A(n,r)^*$. This algebra is the same as S(n,r) if one identifies $\xi_{i,j}$ with the element $(c_{i,j})^*$ of the dual basis corresponding to the $\{c_{i,j}: i, j \in I(n,r)\}$.

(2) By definition, $M(n,r)$ ($= M_K(n,r)$) is the category of finite–dimensional G–modules whose coordinate functions belong to $A(n,r)$ ($= A_K(n,r)$). Let $g \in G$ and denote by e_g: $A(n,r) \to K$ the evaluation map, $e_g(c) = c(g)$. Then e_g belongs to $A(n,r)^* \cong S$, and the map e: $KG \to S$ is a surjective algebra homomorphism. This gives an equivalence of the categories $M(n,r)$ and S–mod; if V is a module in one of these categories then it can be viewed as an object in the other via the formula $gv = e_g v$, $g \in G$ and $v \in V$ (see $[G_1(2.4)]$.

(3) Given V in $M(n,r)$, then the action of S in terms of the $\xi_{i,j}$ may be described as follows: If $\{v_b\}$ is a basis of V and $gv_b = \Sigma\, r_{a,b}(g)v_a$ with $r_{a,b}$ in $A(n,r)$ then

$$(**)\quad \xi v_b = \Sigma\ \xi(r_{a,b})v_a.$$

Consider the example where $V = E^{\otimes r}$ and the basis is $\{\underline{e}_i: i \in I(n,r)\}$. Then the coordinate functions of G on V are the $c_{i,j}$ and for $\xi = \xi_{i,j}$ the formula (**) becomes the same as (*) in 3.2.1.

(4) Suppose V is an S–module, then $V = \underset{\alpha}{\oplus}\ \xi_\alpha V$, by 3.2.1(c). It is true that $\xi_\alpha V$ is equal to the α–weight space V^α in the usual sence of V as a GL_n–module, see $[G_1(3.2)]$, or [Sch, pp 6,7].

<u>3.3.1</u> (1) The simple S–modules are indexed as $\{L(\lambda): \lambda \in \Lambda^+(n, r)\ \}$. Here $L(\lambda)$ is characterized by the properties

(i) The λ–weight space $L(\lambda)^\lambda$ is 1–dimensional.

(ii) If $L(\lambda)^\mu$ is non–zero then $\mu \le \lambda$ (see $[G_1]$)

(2) If one views $L(\lambda)$ as a $GL_n(K)$–module then it is uniquely determined by its restriction to $SL_n(K)$ if r is fixed. Hence one may translate the labelling

into the notation in terms of fundamental dominant weights. For example if n = 2 and $\alpha = (\alpha_1, \alpha_2)$ lies in $\Lambda^+(2,r)$ then as an $SL_2(K)$–module, $L(\alpha)$ is the module usually denoted by $L(m)$ where $m = \alpha_1 - \alpha_2$.

In general, taking a simple SL_n–module $L(\mu)$, it belongs to $S(n,r)$–mod for a unique integer $r \geq 0$; and then $L(\mu) \otimes \mathrm{Det}^a$ belongs to $S(n, r+an)$–mod for any $a \geq 0$. Here Det is the determinant representation of $GL_n(K)$.

3.3.2 (1) If $a \geq 1$ is an integer we denote by $S^a(E)$ the a–th symmetric power of E. More generally, let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition of r then we define $S^\lambda(E)$ to be the tensor product of symmetric powers, namely $S^\lambda(E) = S^\lambda 1(E) \otimes ... \otimes S^\lambda n(E)$. Then $S^\lambda(E)$ is a $GL_n(K)$–module and hence a module for $S(n,r)$. Moreover, the canonical epimorphism $\psi: E^{\otimes r} \rightarrow S^\lambda(E)$ is an S–homomorphism.

(2) The module $D_{\lambda,K}$ in $[G_1(4.4)]$ may be defined as a submodule of $S^\lambda(E)$. Let T^λ be the bijective λ–tableau, as in 2.3.1, and let $C(\lambda)$ be the column stabilizer of T^λ and $\{C(\lambda)\} \in K\Sigma_r$ be the alternating sum. This operates on the right of $S^\lambda(E)$, and we define

$$D_{\lambda,K} := (S^\lambda(E))\{C(\lambda)\}.$$

This is clearly an S–submodule. Moreover, we have an exact sequence of S–modules

$$0 \rightarrow N \rightarrow E^{\otimes r} \xrightarrow{\varphi} D_{\lambda,K} \rightarrow 0$$

where φ takes $\underline{e}_i$ to $\psi(\underline{e}_i)\{C(\lambda)\}$ where $\psi(\underline{e}_i)$ is the image of $\underline{e}_i$ in $S^\lambda(E)$, and N = ker φ. This is the same as $[G_1(5.1a)]$.

(3) Now one defines $V_{\lambda,K}$ to be the orthogonal complement to N, relative to the form $<\ ,\ >$ on $E^{\otimes r}$. Then $V_{\lambda,K}$ is isomorphic to the contravariant dual $(D_{\lambda,K})^0$, see $[G_1]$.

<u>3.4</u> THEOREM (Donkin, Parshall, Green) *The algebra S is quasi–hereditary, with respect to the dominance order, and* $\Delta(\lambda) \cong V_{\lambda,K}$ *and* $\nabla(\lambda) \cong D_{\lambda,K}$.

This was already proved in [D_1II(2.2)]; for Parshall's proof, see [P]. More recently there is new proof by J. A. Green [G_3].

The canonical modules have also been determined, in [D_3].
Let $\alpha = (\alpha_1,\ldots, \alpha_t)$ be a partition of r. Define the module $\Lambda^{\alpha}(E)$ to be the tensor product of exterior powers, $\Lambda^{\alpha}(E) := \Lambda^{\alpha_1}(E) \otimes \ldots \otimes \Lambda^{\alpha_t}(E)$.
For example if $\alpha = (1, 1, \ldots, 1)$ then $\Lambda^{\alpha}(E) = E^{\otimes r}$.
<u>3.5</u> THEOREM (S. Donkin) *The canonical modules of S(n,r) are precisely the indecomposable summands of* $\Lambda^{\lambda'}(E)$ *for* $\lambda \in \Lambda^{+}(n, r)$. *Moreover,* $T(\lambda)$ *occurs in* $\Lambda^{\lambda'}(E)$ *with multiplicity one, and all other summands of* $\Lambda^{\lambda'}(E)$ *are isomorphic to* $T(\mu)$ *with* $\mu < \lambda$.

First one observes that $\Lambda^{a}(E) \cong \Delta(1^{a}) \cong \nabla(1^{a})$ for $1 \leq a \leq n$. Hence $\Lambda^{b}(E)$ belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ for all $b \geq 1$. Now one uses the Theorem that $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta)$ are closed under tensor products (see [D_2], [M]). By analysing weight spaces one sees that $\Lambda^{(\lambda')}(E)$ has a unique composition factor $\cong L(\lambda)$ and all other composition factors are of the form $L(\mu)$ for $\mu < \lambda$. Hence $T(\lambda)$ is a summand, with multiplicity one.

<u>3.6</u> The Ringel dual of S is Morita equivalent to a generalized Schur algebra, in the sense of [D_1I]. It is proved in [D_3] that for $n \geq r$, the Ringel dual S(n, r)' is Morita equivalent to S(n, r). For $n < r$, the algebras are different in general.

3.7 The modules $S^{\lambda}(E)$ are injective S–modules for $\lambda \in \Lambda^{+}(n,r)$. To see this, one observes that $S^{\lambda}(E)$ is isomorphic to a summand of $A(n,r)$ as an S–module. Now, S is isomorphic to $A(n,r)^{*}$ and so $D(S) = \mathrm{Hom}_K(S, K)$ is isomorphic to $A(n,r)$, clearly as a vector space, but also as an S–module (see $[G_1(4.4)]$), and $D(S)$ is injective. For example we have that $S^{(1^r)}(E) \cong E^{\otimes r}$ and hence $E^{\otimes r}$ is injective as an S–module if $n \geq r$.

3.8 One possible way to relate S–modules and $K\Sigma_r$–modules is via the Schur functor, see $[G_1, \S 6]$.

(1) Assume that $n \geq r$ and let $\omega \in \Lambda^{+}(n,r)$ be the weight $\omega = (1,1,\ldots,1, 0, ..0)$. Let $e := \xi_{\omega}$, then e is an idempotent of S; and the Schur functor, usually denoted by f, is defined to be the functor

$$f = \mathrm{Hom}_S(Se, -) : S\text{–mod} \to eSe\text{–mod}.$$

The idempotent e is chosen in this way because eSe is isomorphic to the group algebra $K\Sigma_r$. Let $u = (1, 2,\ldots, r) \in I(n,r)$, then u belongs to ω. The algebra eSe has a K–basis $\{\xi_{u\pi,\, u}: \pi \in \Sigma_r\}$, and the linear map with $\xi_{u\pi,\, u} \to \pi$ is an algebra isomorphism.

(2) We have that $f(E^{\otimes r}) \cong K\Sigma_r$ as a bimodule and hence if ϵ is a central idempotent of $K\Sigma_r$ then $f(E^{\otimes r}\epsilon) \cong K\Sigma_r\epsilon$: First note that $f(E^{\otimes r}) = eE^{\otimes r} = M^{\omega}$ (see 3.2.1(2)). This has K–basis $\{\underline{e}_{u\sigma}: \sigma \in \Sigma_r\}$.(here $\underline{e}_u = \underline{e}_1 \otimes \underline{e}_2 \otimes .. \otimes \underline{e}_r$). By 3.2.1(*) we obtain that $\xi_{u\pi,u}\underline{e}_{u\sigma} = e_{u\pi\sigma}$ $(\pi, \sigma \in \Sigma_r)$ and this is equal to $(\underline{e}_{u\pi})\sigma$.

(3) The simple eSe–modules are given by $\{fL(\lambda): \lambda \in \Lambda^{+}(n,r)$ and λ' is p–regular $\}$; and if λ' is not p–regular then $eL(\lambda) = 0$. This is related to the labelling of the simple $K\Sigma_r$–modules as in 2.3 by $fL(\lambda) \cong D^{\lambda'} \otimes K_a$.

(4) We have $f(\nabla(\lambda)) \cong S^{\lambda}$, the Specht module and $f(\Delta(\lambda)) \cong (S^{\lambda})^{*}$, see $[G_1(6.3b)]$. Moreover $f(T(\lambda)) \cong Y^{\lambda'} \otimes K_{a}$, by $[D_3]$.

(5) We want to show that the result 1.8 may be applied. We have that Se is isomorphic to $E^{\otimes r}$: Since $eE^{\otimes r} = M^{\omega}$ we have (by (2)) that

$$Se \cong \mathrm{Hom}_{K\Sigma_r}(eE^{\otimes r}{}_{K\Sigma_r}, E^{\otimes r}{}_{K\Sigma_r}) \cong \mathrm{Hom}_{K\Sigma_r}(K\Sigma_r, E^{\otimes r}) \cong E^{\otimes r}$$

(see also $[G_1(6.4f)]$). This is clearly a faithful S–module. Moreover $\nabla(\mu)$ is a homomorphic image of $D(S_S)$ since $\nabla(\mu)$ is a homomorphic image of $E^{\otimes r}$ (by 3.3); and $E^{\otimes r}$ is injective (see 3.7).

<u>3.9</u> The Schur functor is not defined for small n. This makes it necessary to relate Schur algebras S(N, r) and S(n, r) for $N \geq n$, and this can be done as follows (see $[G_1(6.5)]$).

(1) Take $E \subset F$ where F is a vector space with basis $\{\underline{e}_1, \ldots, \underline{e}_n, \underline{e}_{n+1}, \ldots, \underline{e}_N\}$. Let $e \in \overline{S} := \mathrm{Hom}_{\Sigma_r}(F^{\otimes r}, F^{\otimes r}) = S(N,r)$ be the projection with $eF^{\otimes r} = E^{\otimes r}$ and $e = 0$ on the space spanned by the $\underline{e}_i$ with $i \notin I(n,r)$. Then $e\overline{S}e \cong \mathrm{Hom}_{K\Sigma_r}(E^{\otimes r}, E^{\otimes r}) = S(n,r)$ and we have the functor

d: $\mathrm{Hom}_{\overline{S}}(\overline{S}e, -) : \overline{S}\text{–mod} \to S(n, r)\text{–mod}$ as described in $[G_1]$. We want to show that this satisfies the hypothesis of 1.6 and 1.7.

(2) Explicitly, we take for e the element $e = \Sigma\, \xi_{\beta}$, the sum taken over all β in Γ where

$$\Gamma := \{ \beta \in \Lambda^{+}(N,r) : \beta_{n+1} = \ldots \beta_N = 0\}.$$

We identify this with $\Lambda^{+}(n, r)$; and the simple $e\overline{S}e$ –modules are precisely the $\{eL(\lambda) : \lambda \in \Gamma \}$ and this is the same labelling as for S(n, r).

It is easy to see that $\Lambda^{+}(N,r) - \Gamma$ is a saturated subset of $\Lambda^{+}(N, r)$.

(3) For $N \geq n \geq r$ this functor d induces an equivalence of categories.

4. From $S(n, r)$ to $K\Sigma_r$

We have introduced the module $E^{\otimes r}$ as an S–$K\Sigma_r$–bimodule. Let $R = \mathrm{End}_S(E^{\otimes r})$, then $E^{\otimes r}$ is a balanced (S, R)–bimodule, and there is a canonical map ρ: $K\Sigma_r \to R$.

4.1 PROPOSITON *The canonical map $\rho : K\Sigma_r \to R$ is surjective.*

Proof: This is a special case of [CP. 4.1] or of [D_4, §2 Corollary].

In order to use this, we will now study the structure of the S–module $E^{\otimes r}$. As we noted above, we have that $E^{\otimes r}$ is isomorphic to $\Lambda^{(1^r)}(E)$, hence by 3.5 the indecomposable summands of $E^{\otimes r}$ are certain canonical modules $T(\mu)$. Actually, one can give a precise description; which is implicitly contained in [D_3].

4.2 PROPOSITION *The indecomposable summands of ${}_SE^{\otimes r}$ are precisely the $T(\lambda)$ where λ is p–regular, $\lambda \in \Lambda^+(n,r)$. Moreover, for such λ, the multiplicity of $T(\lambda)$ as a direct summand of $E^{\otimes r}$ is equal to the dimension of the simple $K\Sigma_r$–module D^λ.*

This generalizes the characteristic 0 situation: If char $K = 0$ or large then $T(\lambda)$ is isomorphic to the simple module $L(\lambda)$ and $D^\lambda \cong S^\lambda$, since all λ are p–regular. It is well–known that in this case $E^{\otimes r}$ is isomorphic to $\oplus\, n_\lambda L(\lambda)$ and $n_\lambda = \dim S^\lambda$; see for example [W, Th 4.4D]. The decomposition of $E^{\otimes r}$ for n = 2 and char K = p was already obtained in [K].

Proof of 4.2: By 3.5 we have that $E^{\otimes r} \cong \oplus n_\lambda T(\lambda)$ where the sum is taken over $\Lambda^+(n,r)$, where n_λ are multiplicities.

(1) Assume first that $n \geq r$, and let f: S–mod $\to$ $K\Sigma_r$–mod be the Schur functor, as in 3.8. This takes $T(\lambda)$ to the left $K\Sigma_r$–module $(Y^{\lambda'}) \otimes K_a$ (see 3.8, [D_3]). Hence $f(E^{\otimes r}) \cong \oplus n_\lambda f(T(\lambda)) \cong \oplus n_\lambda(Y^{\lambda'} \otimes K_a)$. On the other hand, by 3.8 we have that $f(E^{\otimes r}) \cong K\Sigma_r$, as a $K\Sigma_r$–left module, hence the modules $(Y^{\lambda'} \otimes K_a)$ and then also $Y^{\lambda'}$are projective. Conversely, if $Y^{\lambda'}$ is projective then so is $Y^{\lambda'} \otimes K_a$ and this is therefore isomorphic to a summand of $K\Sigma_r$. By 2.6(2) this is true if and only if λ is p–regular. Moreover, n_λ is the multiplicity of $Y^{\lambda'} \otimes K_a$ as a direct summand of $K\Sigma_r$. We have seen in 2.7 that for λ p–regular we have $Y^{\lambda'} \otimes K_a \cong P(D^\lambda)$ and hence $n_\lambda = \dim D^\lambda$.

(2) Now let n be arbitrary, and choose an integer $N \geq n$, r. Let $e \in S(N,r)$ be such that $eS(N,r)e = S(n,r)$, as in 3.9. With F and E as in 3.9 we have that $eF^{\otimes r} = E^{\otimes r}$. By (1) we know that $F^{\otimes r} \cong \oplus n_\lambda T(\lambda)$ where the sum is taken over all p–regular partitions of $\Lambda^+(N,r)$; and moreover by 3.9 (and 1.7) we know that $eT(\lambda)$ is zero for λ with more than n parts, and is the canonical module $T(\lambda)$ of $S(n,r)$ otherwise. This gives $E^{\otimes r} \cong eF^{\otimes r} \cong \oplus\ n_\lambda eT(\lambda) \cong \oplus n_\lambda T(\lambda)$, where the sum is taken over all p–regular partitions in $\Lambda^+(n,r)$.

<u>4.2.1</u> For any $\mu \in \Lambda^+(n, r)$ there is an epimorphism $E^{\otimes r} \to T(\mu)$, take the composition of $E^{\otimes r} \to \Lambda^{(\mu')}(E) \to T(\mu)$ (see 3.5).

<u>4.3</u> Let now $C := \oplus\ T(\mu)$ where the sum is taken over $\mu \in \Lambda^+(n, r)$ which are not p–regular, and set $T := E^{\otimes r} \oplus C$. Then T is a characteristic module, by 4.2

and 4.1. Now let S' := $\mathrm{End}_S(T)$, then S' is Morita equivalent to the Ringel dual S(n, r)'; in particular it is quasi–hereditary. Let F be the functor F = $\mathrm{Hom}_S(T, -)$; this takes $Q(\lambda)$ to $T_{S'}(\lambda)$ and $\nabla(\lambda)$ to $\Delta_{S'}(\lambda)$, see 1.3(2). Now take e to be the projection with kernel C and image $E^{\otimes r}$. Then $\mathrm{Hom}_S(E^{\otimes r}, -)$ = $\mathrm{Hom}_S(Te, -) \cong e\mathrm{Hom}_S(T, -)$, and this takes $Q(\lambda)$ to $eT_{S'}(\lambda)$ and takes also $\nabla(\lambda)$ to $e\Delta_{S'}(\lambda)$. We will now relate these to $K\Sigma_r$– modules.

PROPOSITION *Let ρ: $K\Sigma_r \to End_S(E^{\otimes r})$ be the canonical homomorphism. Then*

(a) The algebra $K\Sigma_r/\ker \rho$ is isomorphic to eS'e.

(b) The Specht module S^λ is identified with $e\Delta_{S'}(\lambda)$, and the Young module Y^λ with $eT_{S'}(\lambda)$. Moreover $(S^\lambda)^* \cong e\nabla_{S'}(\lambda)$.

(c) The simple module D^λ for $\lambda \in \Lambda^+(n,r)_p$ is identified with $eL_{S'}(\lambda)$.

(d) Assume that all ρ with $\rho \geq \lambda$ in the same block as λ are p–regular. Then

$$e\Delta_{S'}(\lambda) \cong \Delta_{S'}(\lambda) \text{ and } eT_{S'}(\lambda) \cong T_{S'}(\lambda).$$

Proof: Part (a) holds by 4.1 and 4.2.

(c) If $\lambda \in \Lambda^+(n,r)$ is p–regular then D^λ is a composition factor of $E^{\otimes r}$, as a $K\Sigma_r$–module, see 2.3(2) and 2.4. Hence D^λ is identified with some eS'e–module and then $D^\lambda \cong eL_{S'}(\lambda)$, by (b).

(d) Suppose $\mathrm{Hom}_S(T(\mu), \nabla(\lambda)) \neq 0$. Then $L(\lambda)$ is a composition factor of $T(\mu)$, and by 1.3 we have that $\lambda \leq \mu$. Now the hypotheses implies that $T(\mu)$ is a summand of Te. Hence $\mathrm{Hom}_S(C, \nabla(\lambda)) = 0$ and $\mathrm{Hom}_S(Te, \nabla(\lambda)) = \mathrm{Hom}_S(T, \nabla(\lambda)) \cong \Delta_{S'}(\lambda)$. Now suppose that $\mathrm{Hom}_S(T(\mu), Q(\lambda)) \neq 0$, then $L(\lambda)$ is a composition factor of $T(\mu)$ and it follows from 1.3 that then $\lambda \leq \mu$. Again, by the hypothesis $T(\mu)$ is a summand of Te and $\mathrm{Hom}_S(C, T(\lambda)) = 0$ and $\mathrm{Hom}_S(Te, Q(\lambda)) \cong T_{S'}(\lambda)$.

(b) We show now that the last statement follows from the first part of (b) and part (c). The characteristic module $T = E^{\otimes r} \oplus C$ is self–dual, hence admits a non–degenerate contravariant form (,). This form induces an anti–automorphism τ: S' $\to$ S' given by $(x\tau(s), y) = (x, ys)$ for $s \in S'$ and x, $y \in T$, which in turn gives rise in the usual manner to an anti–equivalence $V \to V^o$ on S'–mod.

Claim: We can choose this form so that the following conditions are satisfied:

(i) $L^o \cong L$ for each simple S'–module L.

(ii) The modules $E^{\otimes r}$ and C are orthogonal with respect to (,).

(iii) The restriction of τ to eS'e is the map induced by inversion in Σ_r.

Assuming this for the moment we can prove the result. It follows easily from (i) that $\Delta_{S'}(\lambda)^o \cong \nabla_{S'}(\lambda)$ for all λ. If $V \in$ S'–mod, it follows from (ii) and (iii), by considering a non–degenerate "τ–contravariant" form $V \times V^o \to K$, that $e(V^o) \cong (eV)^*$. The conclusion $e\nabla_{S'}(\lambda) \cong (S^\lambda)^*$ now follows from the first part of 4.3(b).

To prove the claim, take a non–degenerate contravariant form on each $T(\mu)$ for non p–regular μ, and on $E^{\otimes r}$ take the canonical form of [G_1; 2.7]. Putting these together gives a form on T satisfying (ii). This satisfies (iii) by the definition of the canonical form. Take a simple module $L = L_{S'}(\lambda)$. Condition (i) holds for non p–regular λ, since then the idempotent e_λ projecting onto $T(\lambda)$ with kernel $E^{\otimes r} \oplus (\bigoplus_{\mu \neq \lambda} T(\mu))$ is easily seen to be τ–fixed. On the other hand if λ is p–regular, (ii), (iii) and 4.3(c) together imply that $e(L^o) \cong (D^\lambda)^* \cong D^\lambda \cong eL$ and hence $L^o \cong L$.

The proof of the first two statements in (b) will take most of the next chapter.

Before going into this we will describe some consequences.

4.4 THEOREM *Suppose that all partitions in $\Lambda^+(n,r)$ are p–regular. Then $K\Sigma_r/\ker\rho$ is quasi–hereditary and Morita equivalent to S'. Moreover,*

$$\Delta_{S'}(\lambda) \cong S^\lambda \text{ and } T_{S'}(\lambda) \cong Y^\lambda.$$

Under this hypothesis we take $C = 0$ and $e = 1$ in 4.3, and the statement follows from 4.3. This implies by 1.3:

THEOREM *Suppose all $\lambda \in \Lambda^+(n,r)$ are p–regular. Then the category of $K\Sigma_r$–modules which have filtrations by $\{S^\lambda : \lambda \in \Lambda^+(n,r)\}$ is functorially finite in $K\Sigma_r$–mod. In particular it has relative Auslander–Reiten sequences.*

4.5 The decomposition numbers for symmetric groups are related to filtration multiplicities for Schur algebras. Namely, we have:

LEMMA *Assume $\lambda, \mu \in \Lambda^+(n,r)$ and λ is p–regular. Then $d_{\mu\lambda} := [S^\mu : D^\lambda] = [T(\lambda) : \Delta(\mu)]$.*

Proof: Let S' be the Ringel dual of S, recall that there is an equivalence $\mathcal{F}_{S'}(\nabla) \to \mathcal{F}_S(\Delta)$ if S is identified with S'' (see 1.3). This gives that $[Q_{S'}(\lambda) : \nabla_{S'}(\mu)] = [T(\lambda) : \Delta(\mu)]_S$. By the "BGG–reciprocity" proved in [CPS], the first number is equal to $[\Delta_{S'}(\mu) : L_{S'}(\lambda)]$ which is equal to $[e\Delta_{S'}(\mu) : eL_{S'}(\lambda)]$ where e is the idempotent as in 4.3. Now $L_{S'}(\lambda) = eL_{S'}(\lambda) \cong D^\lambda$ and $e\Delta_{S'}(\mu) \cong S^\mu$, by 4.3, hence the statement follows.

4.5.1 The result in 4.5 is essentially the same as [D_3; 3.10] where Donkin proves $[T(\lambda) : \nabla(\mu)] = [\nabla(\mu') : L(\lambda')]$. Namely, by duality we have

$[T(\lambda) : \Delta(\mu)] = [T(\lambda) : \nabla(\mu)]$. Moreover, we claim that $d_{\mu\lambda} = [\nabla(\mu') : L(\lambda')]$: By $[G_1$, (6.6e)], the multiplicities $[\nabla(\mu') : L(\lambda')]$ are independent of n, so one may assume $n \geq r$ and apply the Schur functor to get

$$[\nabla(\mu') : L(\lambda')] = [S^{\mu'} : D^{\lambda} \otimes K_a]$$

by 3.8(3) and (4). This proves the claim, since $(S^{\mu'})^* \cong S^{\mu} \otimes K_a$ by $[Ja_1$; (8.15)].

4.6 This theorem has a block version. Suppose B is a block of $K\Sigma_r$, so that $B = (K\Sigma_r)\epsilon$ for some central primitive idempotent of $K\Sigma_r$. Then we set $S_B := \mathrm{End}_{K\Sigma_r}(E^{\otimes r}\epsilon)$. This is a direct sum of blocks of S, in particular it is quasi–hereditary. We have the following results; for the proofs see Chapter 5.

(1) The canonical map ρ: $B \to \mathrm{End}_{S_B}(E^{\otimes r}\epsilon)$ is surjective.

(2) As an S– module $E^{\otimes r}\epsilon \cong \oplus\, n_\lambda T(\lambda)$ where the sum is taken over all λ in $\Lambda^+(n,r) \cap B$ which are p–regular.

(3) The simple modules of $S(n,r)_B$ are precisely the S–modules $\{L(\lambda)$: $\lambda \in \Lambda^+(n,r) \cap B\}$.

THEOREM *Let B be a block of $K\Sigma_r$. Suppose that all λ in $\Lambda^+(n, r) \cap B$ are p–regular. Then $B/\ker\rho \cap B$ is quasi–hereditary and Morita equivalent to the Ringel dual of $S(n,r)_B$. Moreover $\Delta'(\lambda) \cong S^\lambda$ and $T'(\lambda) \cong Y^\lambda$.*

5. On the functor $\mathrm{Hom}_S(E^{\otimes r}, -)$

In this section we study the left Σ_r–modules $\mathrm{Hom}_S(E^{\otimes r}, W)$ for $W = \nabla(\lambda)$ and $Q(\lambda)$. The aim to show that they are isomorphic to the Specht module S^λ and the Young module Y^λ respectively. Moreover we give the proof of 4.6.

Put $\widetilde{F} := \mathrm{Hom}_S(E^{\otimes r}, -)$. This functor preserves exactness of sequences in $\mathcal{F}(\nabla)$ and agrees with the Schur functor when $n \geq r$.

5.1 PROPOSITION *Let $S^\lambda(E)$ be the symmetric power of type λ where $\lambda \in \Lambda^+(n,r)$. Then we have isomorphisms of $K\Sigma_r$–modules*

(a) $\mathrm{Hom}_S(E^{\otimes r}, S^\lambda(E)) \cong M^\lambda$, and

(b) $\mathrm{Hom}_S(E^{\otimes r}, Q(\lambda)) \cong Y^\lambda$.

Proof: It is clear that $\mathrm{Hom}_S(-, S^\lambda(E))$ has the effect of taking λ–weight spaces, so $\widetilde{F}(S^\lambda(E)) \cong (E^{\otimes r})^\lambda = M^\lambda$, which is 5.1(a).

(b) By 3.7 we have $S^\lambda(E) \cong \oplus\, m_\mu Q(\mu)$, where the sum is taken over $\Lambda^+(n,r)$ and $0 \leq m_\mu$. We claim that m_μ is equal to the multiplicity of Y^μ as a direct summand of M^λ. Assume first that $n \geq r$. By [G_1; (6.3b)] we have $f(S^\lambda E) \cong M^\lambda$; and also $fQ(\mu) \cong Y^\mu$ (see 3.8 or [D_3]). Hence $M^\lambda \cong fS^\lambda(E) \cong \oplus\, m_\mu fQ(\mu)$, as required. In general choose $N \geq n$, r and use the functor in 3.9. The statement follows from 1.6(iii).

We have now $\oplus\, m_\mu Y^\mu \cong M^\lambda \cong \mathrm{Hom}_S(E^{\otimes r}, S^\lambda(E)) \cong \oplus\, m_\mu \mathrm{Hom}_S(E^{\otimes r}, Q(\mu))$. By 2.4 we know that $m_\lambda = 1$ and $m_\mu \neq 0$ only for $\mu \geq \lambda$. Hence one obtains the statement in (b) by induction on $\leq$, starting with λ maximal.

5.2 PROPOSITION *Let S^λ be the Specht module for the partition $\lambda \in \Lambda^+(n,r)$. We have $\mathrm{Hom}_S(E^{\otimes r}, \nabla(\lambda)) \cong S^\lambda$, as $K\Sigma_r$–modules.*

Proof: If $n \geq r$, $\widetilde{F}(\nabla(\lambda)) \cong f(\nabla(\lambda)) \cong S^\lambda$ by [G_1;6.3]. If $N \geq n$ and $\lambda \in \Lambda^+(n, r)$, we have by 3.9 and 1.7 a surjection

$$(*) \qquad \tilde{F}_N(\nabla(\lambda)) \to \tilde{F}_n(\nabla(\lambda)),$$

since if e is the idempotent in 3.9, $e\nabla_N(\lambda) \cong \nabla_n(\lambda)$ by [G_1;6.5]. (Here the subscripts N and n are used to discriminate between different Schur algebras.) We have

$$\dim M^\lambda = \sum_\mu [S^\lambda(E) : \nabla(\mu)] \dim \tilde{F}(\nabla(\mu)),$$

and $[S^\lambda(E) : \nabla(\mu)] = \dim \mathrm{Hom}_S(\Delta(\mu), S^\lambda(E)) = \dim \Delta(\mu)^\lambda$, which is independent of n. Thus (*) is an isomorphism, and $\tilde{F}(\nabla(\lambda)) \cong S^\lambda$, without restriction on n.

<u>5.3</u> If one takes a bijective λ–tableau T^λ such that one reads down the columns and from left to right then φ factors through the natural transformation $(-)^{\otimes r} \to \Lambda^{\lambda'}(-)$, and one has a commutative diagram

$$\begin{tikzcd} E^{\otimes r} \arrow[rr] \arrow[dr] & & \nabla(\lambda) \\ & \Lambda^{\lambda'}(E) \arrow[ur, dashed] & \end{tikzcd}$$

By 3.5, the canonical module $T(\lambda)$ is a summand of $\Lambda^{\lambda'}(E)$; and using 1.3 (and 3.5) one sees that $\pi_{|T(\lambda)}$ must be onto. There is also a general argument why φ should factor through $T(\lambda)$; namely the epimorphism $T(\lambda) \to \nabla(\lambda)$ is the right $\mathcal{F}(\Delta)$–approximation of $\nabla(\lambda)$, in the sense of [AR], see [R_1].

<u>5.4</u> We will now prove the block version 4.6. Since $\mathrm{Hom}_{K\Sigma_r}(E^{\otimes r}\epsilon, E^{\otimes r}(1-\epsilon)) = 0$ we have that S_B is a direct sum of connected components of the algebra S, hence it is quasi–hereditary. Moreover $E^{\otimes r}\epsilon$ viewed as an S–module is the same as viewed as an S_B–module.

(1) Let $\underline{x} : E^{\otimes r}\epsilon \rightarrow E^{\otimes r}\epsilon$ be an S_B–homomorphism, then it is an S–homomorphism and $\underline{x} = \rho(x)$ for some $x \in K\Sigma_r$, by 4.1. We have $x = \epsilon x + (1-\epsilon)x$ and it follows that $\rho(x) = \rho(\epsilon x)$.

This shows that $E^{\otimes r}\epsilon$ is a balanced bimodule with S_B acting on the left and $B/B \cap \ker \rho$ acting on the right.

(2) By 4.2 we have that $E^{\otimes r}\epsilon = \oplus\, n_\lambda T(\lambda)$ where the sum is taken over some set $\underline{\Lambda}$ of p–regular partitions in $\Lambda^+(n,r)$; and we want to show that $\underline{\Lambda} = \Lambda^+(n,r) \cap B$. Assume first that $n \geq r$, and let f be the Schur functor. We know that $fT(\mu) \cong P(D^\mu)$ (see 3.8(3) and 2.6(2)). On the other hand, by 3.8 we know that $f(E^{\otimes r}\epsilon) \cong K\Sigma_r\epsilon = B$ and this is the direct sum of $P(D^\lambda)$ for λ p–regular in $\Lambda^+(n,r) \cap B$, and the statement follows. The general case is done as in the proof of 4.2.

(3) Since the module $E^{\otimes r}\epsilon$ is balanced, the number of simple S_B–modules is the same as the number of summands of $E^{\otimes r}\epsilon$ as an Σ_r–module. This number is equal to $|\Lambda^+(n,r) \cap B|$, because $E^{\otimes r}\epsilon \cong \oplus\, m_\mu Y^\mu$ where the sum is over $\mu \in \Lambda^+(n,r) \cap B$ and $m_\mu \geq 1$, see 2.4 and 2.1(b). Hence it suffices to show that for $\lambda \in \Lambda^+(n,r) \cap B$, the simple module $L(\lambda)$ is an S_B–module. This is true if λ is p–regular, since in this case $T(\lambda)$ is an S_B–module by (2), and $L(\lambda)$ is a composition factor of $T(\lambda)$ (see 1.3).

In general, there is some $\mu \in \Lambda^+(n,r)_p \cap B$ such that $[S^\lambda:D^\mu] \neq 0$ (take a composition factor of S^λ). We deduce from 4.5 that $[T(\mu):\Delta(\lambda)] \neq 0$, and hence $L(\lambda)$ is a composition factor of $T(\mu)$ and is an S_B–module by (2). The Theorem 4.6 follows now from 4.3 and 4.6(1) to (3).

<u>5.5</u> (1) For $n \leq r$, the functors $\mathrm{Hom}_S(E^{\otimes r}, -)$ and the Schur functor are the same, by 3.8(5).

(2) Let $n = 2$ and $r = p^2$, and consider the principal block B of $K\Sigma_r$. By using [E] one can determine the Ringel dual of $S(2,r)_B$ explicitly by quiver and relations. One finds that it is of finite representation type and not Morita equivalent to a component of any Schur algebra.

For $p > 2$, all partitions in $\Lambda^+(2,r)$ are p–regular and hence the quasi–hereditary algebra $S(2,r)_B'$ is Morita equivalent to a quotient of the principal block of $K\Sigma_r$. If $p = 2$ then the partition $(2,2)$ is not 2–regular, and the quotient of $K\Sigma_4$ which is Morita equivalent to $eS(2,4)'e$ can be determined easily. It has infinite global dimension.

6. Decomposition numbers

6.1 In [D_3] a tensor product theorem for the canonical modules $T(\lambda)$ was proved, and this gives inductively information about the filtration multiplicities $[T(\lambda) : \Delta(\mu)]$, and these are equal to the decomposition numbers $d_{\mu\lambda}$ for $K\Sigma_r$, by 4.5. In the case when $n = 2$ one can determine these multiplicities completely and this gives a short proof of a Theorem by James on the decomposition numbers of partitions with at most two parts [$Ja_{2,3}$]. This proof is due to S. Donkin.

In order to study Δ–filtrations of canonical modules for $GL_2(K)$, it suffices to consider the restrictions to $SL_2(K) =:\overline{G}$, by [D_2, 3.2.7]. If $\lambda = (\lambda_1, \lambda_2) \in \Lambda^+(2,r)$ then λ corresponds to the weight $m = \lambda_1 - \lambda_2$ for $SL_2(K)$. We write $T(m) = T(\lambda)_{|\overline{G}}$ (and $\Delta(s) = \Delta(\mu)_{|\overline{G}}$ if $\mu = (\mu_1, \mu_2) \in \Lambda^+(2,r)$ with $s = \mu_1 - \mu_2$). We study the multiplicities $[T(m) : \Delta(s)]$, for all $m, s \geq 0$ with $m \equiv s$ (mod 2). For such m and s, the order $<$ translates into the natural order. Clearly, for $m < s$ we have that the multiplicity is zero, by 1.3.

6.2 We consider now the modules $T(m)$.

(1) If $0 \leq m \leq p-1$ then $\nabla(m) \cong \Delta(m) \cong L(m)$; this is well–known. Hence $T(m) \cong \Delta(m)$.

(2) If $m \geq p$ then write $m = kp + i$ where $0 \leq i \leq p-1$. The result in [D_3 (2.1) and Example 2] is that

$$T(m) \cong \begin{cases} T(p+i) \otimes T(k-1)^F & \text{if } i < p-1 \\ T(p-1) \otimes T(k)^F & \text{if } i = p-1 \end{cases}.$$

Here F is the Frobenius twist.

(3) This leaves us to consider the modules $T(p+i)$ with $0 \leq i \leq p-2$. We claim that there is an exact sequence

$$0 \to \Delta(p+i) \to T(p+i) \to \Delta(j) \to 0 \quad \text{where } i + j = p-2.$$

To prove this one shows that $\mathrm{Ext}^1_{GL_2}(\Delta(j), \Delta(p+i)) \cong K$, so there is a unique non–split exact sequence $0 \to \Delta(p+i) \to U \to \Delta(j) \to 0$. Moreover, it is well–known that $\Delta(p+i)$ has length two, with socle $L(j)$ and hence U is indecomposable. Then $U/L(j)$ must be isomorphic to $\nabla(p+i)$ since extensions between simple GL_2–modules are $\leq$ 1–dimensional; and also $L(j) \cong \nabla(j)$; hence there is also an exact sequence $0 \to \nabla(j) \to U \to \nabla(p+i) \to 0$. This shows that U belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and then necessarily $U \cong T(p+i)$, by 1.3(1).

(4) Let $0 \leq i, j$ such that $i+j = p-2$. Then there is an exact sequence $0 \to \Delta((s+1)p + i) \to \Delta(s)^F \otimes T(p+i) \to \Delta(sp + j) \to 0$, for $s \geq 0$. This was proved in [X, 7.1.2].

<u>6.3</u> We obtain now formulae for the filtration multiplicities, namely

(1) $[T(k) : \Delta(s)] = [T(kp + p-1) : \Delta(sp + p-1)]$,

(2) $[T(k-1) : \Delta(s)] = [T(kp+i) : \Delta(sp+j)] = [T(kp+i) : \Delta((s+1)p+i)]$ where $0 \leq i \leq p-2$, and $i + j = p-2$. .

Proof: (1) Let $m = kp + (p-1)$ and suppose that

(*) $0 \subset M_1 \subset \ldots \subset M_t \subset \ldots \subset T(k)$ is a Δ–filtration of $T(k)$.

Then by 6.2(2) the module $T(m)$ has a filtration

(**) $0 \subset M_1{}^F\otimes \Delta(p-1) \subset \ldots \subset M_t{}^F\otimes \Delta(p-1) \subset \ldots \subset T(m)$.

For each $\Delta(s)$ occuring as a quotient in (*), there is a quotient $\Delta(s)^F \otimes \Delta(p-1)$ of $T(m)$. This module is isomorphic to $\Delta(sp +(p-1))$, see [J, 3.8 Bem.

2]. This shows that (**) is a Δ–filtration for T(m), and that (1) holds.

(2) Let m = kp+i. Similarly as in (1), we obtain a filtration of T(m) with quotients $\Delta(s)^F \otimes T(p+i)$ where the $\Delta(s)$ are quotients of a Δ–filtration for T(k–1). The statement follows now by considering the exact sequence in 6.2(4).

6.4 We are left to give a general formula for the filtration multiplicities, or equivalently the decomposition matrix for the 2–part partitions. This is done by James as follows (see [Ja_3].

(1) Let $a \in \mathbb{N}$, and let $\lambda(a)$ be the length of a when written to base p. We say that a contains b to base p if $\lambda(b) < \lambda(a)$ and for each non–zero digit in the expansion of b written to base p, there is the same digit in the same place in the expansion of a written to base p.

(2) Suppose, inductively, that an x–1 by x–1 *type I* matrix has already been defined. Turn this into an x by x–1 matrix by adding 0 along the top. Then add 0's and 1's down a new first column according to Rule 1: There is a 1 in the bth place if and only if 2x contains b–1 to base p. This defines an x by x type I matrix.

Define an x by x *type II* matrix as above, replacing rule I by Rule 2. There is a 1 in the b–th place if and only if 2x–1 contains b–1 to base p.

THEOREM (G.D. James) *The decomposition matrix $[d_{\mu\lambda}]$ for λ, μ with at most two parts is*

(i) x by x type I if $r = 2x-1$.

(ii) $x+1$ by $x+1$ type II if $r = 2x$,

except that for $p = 2$ and $r = 2x$ the first column on the right must be deleted.

Proof: Consider the λ–th column of $[d_{\mu\lambda}]$. Write $\lambda = m$ in SL_2–notation; then the column is given by the numbers $d_{\mu\lambda} = [T(m) : \Delta(s)]$. For $m < s$ it is zero (see 1.3). The number $d_{\mu\lambda}$ in position b (counted as in (2) above) corresponds to s such that $m - s = 2(b-1)$.

Write m+1 and b–1 in p–adic expansion. We must show that $[T(m):\Delta(s)] = 1$ if m+1 contains b–1 and is zero otherwise.

We proceed by induction on m, and for small values of m this is true by 6.2. Now let $m \geq p$, and write $m = kp + i$ where $0 \leq i \leq p-1$.

Case 1: $i < p-1$. By 6.3(2), the filtration multiplicities of T(m) are determined by those of T(k–1). Suppose k has p–adic expansion $k = (a_0, ..., a_\nu)$ is the p–adic expansion. Then the p–adic expansion of $m + 1 = kp+i+1$ is $(i+1, a_0, a_1, ..., a_\nu)$. Hence b–1 is contained in k if and only if both p(b–1) and p(b–1)+i+1 are contained in m+1. The position p(b–1) gives the entry $[T(m) : \Delta((s+1)p+i)]$, and position p(b–1)+i+1 gives $[T(m) : \Delta(sp+j)]$. By the induction hypothesis and 6.3(2) the statement is true for m.

Case 2: $i = p-1$. By 6.3, the filtration multiplicities of T(m) are determined by those of T(k). Let $k+1 = (a_0, ..., a_\nu)$ be the p–adic expansion. Then the p–adic expansion of $m+1 = (k+1)p$ which is $(0, a_0, ..., a_\nu)$. We see now that b–1 is contained in k+1 if and only if p(b–1) is contained in m+1, and this gives by induction and 6.3(1) the statement.

If $p = 2$ and r is even then the first column must be deleted since $\lambda = (x, x)$ is not 2–regular.

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QUASITILTED ALGEBRAS

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Introduction

For classes of algebras defined in nonhomological terms it is often of interest to look for a homological description. For example the Auslander algebras, that is the algebras which are obtained as endomorphism rings of a direct sum of representatives from each isomorphism class of the indecomposable modules over a non semisimple algebra of finite representation type, are characterized by having global dimension and dominant dimension equal to 2. Another important class of artin algebras is the class of tilted algebras, and it is well known that they have global dimension at most 2. In addition, they have no indecomposable module having both projective and injective dimension equal to 2. However these properties do not characterize tilted algebras. But it turns out that they characterize a natural generalization of tilted algebras obtained by introducing the notion of tilting objects in hereditary abelian categories more general than module categories over hereditary algebras. The endomorphism algebras of these tilting objects are called quasitilted algebras. And this is precisely the class which is characterized by the homological properties above. Besides the tilted algebras, also the canonical algebras [Ri] belong to this class.

These endomorphism algebras came up as a special case of a more general tilting theory which we developed in abelian categories, where we tilt with respect to special torsion classes. A full account of this theory will be given in [HRS]. We point out that there exist different generalizations of tilting theory, see for example [H], [M] and [Ric].

The aim of this paper is to introduce and investigate the class of quasitilted algebras. In section 1 we give the basic definitions and explain how the ordinary tilting theory fits into the more general setting. In section 2 we state a characterization of when one-point extensions of hereditary algebras are quasitilted, and introduce and investigate the concepts of a module governing and dominating other modules. In section 3 we use some results from section 2 to characterize when the two-by-two lower triangular matrix ring over a hereditary algebra is quasitilted. For proofs not given we refer to [HRS].

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1. Tilting Objects in Abelian Categories

In this section we define the notion of a tilting torsion class and a tilting object in an abelian category. We introduce the class of quasitilted algebras and show the connection

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to tilted algebras. Moreover we deduce certain properties of abelian categories with tilting objects.

Let R be a commutative artin ring and $\mathcal{A}$ an abelian R-category. This is by definition an abelian category such that $\mathrm{Hom}(X,Y)$ is an R-module for all $X,Y \in \mathcal{A}$ and the composition of morphisms in $\mathcal{A}$ is R-bilinear. We assume throughout that for all $X,Y \in \mathcal{A}$ the R-module $\mathrm{Hom}(X,Y)$ is of finite length and that idempotents split in $\mathcal{A}$. This ensures that $\mathcal{A}$ is a Krull-Schmidt category.

Typical examples of abelian R-categories we are interested in are given by the category $\mathrm{mod}\,\Lambda$ of finitely generated left Λ-modules over an artin R-algebra Λ or by $\mathrm{coh}\,\mathbb{X}$, the category of coherent sheaves on a projective k-variety $\mathbb{X}$ where k is some algebraically closed field. The latter category can be thought of as follows. Let S be the $\mathbb{Z}$-graded coordinate ring of $\mathbb{X}$. Then $\mathrm{coh}\,\mathbb{X}$ is equivalent to the quotient category of finitely generated $\mathbb{Z}$-graded S-modules by the category of $\mathbb{Z}$-graded S-modules of finite length.

Recall that a pair $(\mathcal{T},\mathcal{F})$ of full subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathcal{A}$ is called a **torsion pair** if the following conditions are satisfied.

(i) $\mathrm{Hom}(X,Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.

(ii) For all $X \in \mathcal{A}$ there exists a short exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

Given a torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ the objects in $\mathcal{T}$ are called **torsion objects** while those in $\mathcal{F}$ are called **torsion free objects**. $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions. $\mathcal{T}$ is closed under factor objects and $\mathcal{F}$ is closed under subobjects. Let us give two examples of torsion pairs.

In this paper we say as in [HR] that a module T over an artin algebra Λ is a **tilting module** if the projective dimension $\mathrm{pd}_\Lambda T \leq 1$, $\mathrm{Ext}^1_\Lambda(T,T) = 0$ and there exists a short exact sequence $0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to 0$ where T^0, T^1 are in $\mathrm{add}\,T$. Here we denote by $\mathrm{add}_\Lambda T$ the full subcategory of $\mathrm{mod}\,\Lambda$ whose objects are direct sums of direct summands of ${}_\Lambda T$. Let $\mathcal{T} = \mathrm{Fac}\,T$ be the full subcategory of $\mathrm{mod}\,\Lambda$ containing those modules which are epimorphic images of modules in $\mathrm{add}\,T$. It can be shown [HR] that if T is a tilting module then $\mathcal{T} = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(T,X) = 0\}$, so $\mathcal{T}$ contains all indecomposable injective Λ-modules. Let $\mathcal{F} = \{Y \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}_\Lambda(X,Y) = 0 \text{ for all } X \in \mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is a torsion pair associated with the tilting module ${}_\Lambda T$. For a further discussion we refer to the original account in [HR].

For the second example let $\mathbb{X} = \mathbb{P}^1(k)$ be the projective line over an algebraically closed field k. Let $S = k[X,Y]$ be the polynomial ring in two variables over k endowed with the usual grading $\deg X = \deg Y = 1$. Consider the module $T = S[0] \amalg S[1]$. Observe that using the equivalence to coherent sheaves mentioned above T is identified with $\mathcal{O} \amalg \mathcal{O}(1)$, where $\mathcal{O}$ denotes the structure sheaf on $\mathbb{X}$ and $\mathcal{O}(1)$ is the twisted structure sheaf. So T is the standard example of a socalled tilting sheaf. For a more detailed account we refer to [Be], [GL]. As above we now define a torsion pair $(\mathcal{T},\mathcal{F})$ on $\mathrm{coh}\,\mathbb{X}$ by $\mathcal{T} = \mathrm{Fac}\,T$ and $\mathcal{F} = \{X \in \mathrm{coh}\,\mathbb{X} \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } X \in \mathcal{T}\}$.

Motivated by the first example we call the torsion class $\mathcal{T}$ of a torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ a **tilting torsion class** if $\mathcal{T}$ is a cogenerator for $\mathcal{A}$. Thus for each $X \in \mathcal{A}$ there is some

$T_X \in \mathcal{T}$ and a monomorphism $X \xrightarrow{\mu_X} T_X$.

If $\mathcal{A} = \operatorname{mod}\Lambda$ for an artin algebra Λ, then $\mathcal{T}$ is a tilting torsion class if and only if $\mathcal{T}$ contains the indecomposable injective Λ-modules.

Now we come to the main definition in this section. Let $T \in \mathcal{A}$. Then T is called a **tilting object** if there exists a torsion pair $(\mathcal{T}, \mathcal{F})$ satisfying the following properties:

(i) $\mathcal{T}$ is a tilting torsion class.

(ii) $\mathcal{T} = \operatorname{Fac} T$.

(iii) $\operatorname{Ext}^1(T, X) = 0$ for $X \in \mathcal{T}$, so T is Ext-projective in $\mathcal{T}$.

(iv) If $Z \in \mathcal{T}$ satisfies $\operatorname{Ext}^1(Z, X) = 0$ for all $X \in \mathcal{T}$, then $Z \in \operatorname{add} T$ where $\operatorname{add} T$ is the full subcateory of $\mathcal{A}$ whose objects are (finite) direct sums of direct summands of T and $\operatorname{Fac} T$ is the full subcategory of $\mathcal{A}$ containing those objects which are epimorphic images of objects in $\operatorname{add} T$.

(v) If $X \in \mathcal{A}$ with $\operatorname{Ext}^i(T, X) = 0$ for all $i \geq 0$, then X is 0.

This definition was motivated by tilting modules over an artin algebra Λ and we show below that these two concepts coincide in case $\mathcal{A} = \operatorname{mod}\Lambda$. But before doing that we have to recall some useful notions (see [AS1], [AR]). Let $X, Y \in \operatorname{mod}\Lambda$ and let $f : X \to Y$ be a map. Then f is said to be **right minimal** if the restriction of f to any non-zero direct summand of X is non-zero. Let $\mathcal{C} \subset \operatorname{mod}\Lambda$ be a full subcategory which we assume to be closed under direct sums, direct summands and isomorphisms. Let $X \in \mathcal{C}$ and $Y \in \operatorname{mod}\Lambda$ and let $f : X \to Y$ be a map. Then f is called a **minimal right $\mathcal{C}$-approximation** if f is right minimal and for all $Z \in \mathcal{C}$ the induced map $\operatorname{Hom}_\Lambda(Z, f)$ is surjective. The notions left minimal and minimal left $\mathcal{C}$-approximation are defined dually. Finally let us recall that a full subcategory $\mathcal{C} \subset \operatorname{mod}\Lambda$ is called **coresolving** if $\mathcal{C}$ is closed under extensions, contains the indecomposable injective Λ-modules and is closed under cokernels of monomorphisms in $\mathcal{C}$.

Lemma 1.1. *Let Λ be an artin algebra. The tilting objects in* $\operatorname{mod}\Lambda$ *are precisely the tilting modules.*

Proof. The proof is well known, see for example [HR] and [AS2]. For the convenience of the reader we sketch the argument.

If ${}_\Lambda T$ is a tilting module then we have defined above a torsion pair $(\mathcal{T}, \mathcal{F})$. So T satisfies properties (i) and (ii) in the definition of a tilting object. Since $\operatorname{pd}_\Lambda T \leq 1$ we have that ${}_\Lambda T$ is Ext-projective in $\mathcal{T} = \operatorname{Fac} T$. Finally let $Z \in \mathcal{T}$ be Ext-projective in $\mathcal{T}$. Let $f : T' \to Z$ be the minimal right $\operatorname{add} T$-approximation of Z. Since $Z \in \operatorname{Fac} T$ we have that f is surjective. Now $\operatorname{Ext}^1_\Lambda(T, T) = 0$ implies that $\operatorname{Ext}^1_\Lambda(T, \ker f) = 0$, hence $\ker f \in \mathcal{T}$ as observed above. So f is a split epi, thus $Z \in \operatorname{add} T$. This shows that T is a tilting object in $\operatorname{mod}\Lambda$.

Conversely assume that T is a tilting object in $\operatorname{mod}\Lambda$. By property (i) and the remark above $\mathcal{T} = \operatorname{Fac} T$ contains the indecomposable injective Λ-modules. Thus $\mathcal{T}$ is a coresolving category showing that $\operatorname{Ext}^i_\Lambda(T, X) = 0$ for all $X \in \mathcal{T}$ and $i \geq 1$ by using (iii). Now let $X \in \operatorname{mod}\Lambda$ and consider an exact sequence $0 \to X \to I \to I/X \to 0$ with I injective. So I and I/X are in $\mathcal{T}$. Thus $\operatorname{Ext}^2_\Lambda(T, X) = 0$, hence $\operatorname{pd}_\Lambda T \leq 1$. Using (iii) again this shows that T is a partial tilting module. Next we consider the minimal left $\operatorname{add} T$-approximation $\phi : \Lambda \to T'$ of ${}_\Lambda\Lambda$. It is easy to see that ϕ is actually a minimal left $\mathcal{T}$-approximation of ${}_\Lambda\Lambda$. Since $\mathcal{T}$ contains the injective Λ-modules, it follows that ϕ is injective. By construction coker ϕ belongs to $\mathcal{T}$ and is Ext-projective, so coker $\phi \in \operatorname{add} T$ by (iv), hence T is a tilting module. $\square$

In general it is not clear which abelian categories admit a tilting object. We give certain restrictions. It has turned out that by studying tilting modules over artin algebras the use of derived categories provided some new insight. This is true for tilting objects in abelian categories as well. So let $\mathcal{D}^b(\mathcal{A})$ be the bounded derived category of $\mathcal{A}$ (see [V]). If $\mathcal{A} = \operatorname{mod} \Lambda$ for an artin algebra Λ we denote $\mathcal{D}^b(\operatorname{mod} \Lambda)$ by $\mathcal{D}^b(\Lambda)$.

Theorem 1.2. *Let T be a tilting object in $\mathcal{A}$. Then $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(\operatorname{End}(T)^{op})$ are equivalent. In particular the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is free abelian of finite rank.*

This has the following immediate consequence. Let $\mathbb{X}$ be a smooth projective curve over an algebraically closed field k. If coh $\mathbb{X}$ admits a tilting object then $\mathbb{X}$ is a rational curve. This follows from Theorem 1.2 and the computation of $K_0(\mathbb{X})$, see for example [Har].

Let us come back to the second example of a torsion pair $(\mathcal{T}, \mathcal{F})$ in coh $\mathbb{P}^1(k)$ as given above. It is easy to see that T is a tilting object and $\operatorname{End}(T)^{op}$ is the path algebra $k(\cdot \leftleftarrows \cdot)$ of the Kronecker quiver.

We now define the class of quasitilted algebras. But first recall that an artin algebra Λ is called a **tilted algebra** if there exists a hereditary artin algebra H and a tilting module ${}_HT$ such that $\Lambda \simeq \operatorname{End}_H(T)^{op}$. Lemma 1.1 shows that the following is a direct generalization. Let $\mathcal{H}$ be a hereditary abelian R-category (i.e. $\operatorname{Ext}^2(X,Y) = 0$ for all $X, Y \in \mathcal{H}$). Let T be a tilting object in $\mathcal{H}$. Then $\Lambda = \operatorname{End}(T)^{op}$ is called a **quasitilted algebra**.

Let us stress that it can be shown that a quasitilted algebra of finite representation type is necessarily a tilted algebra. In other words new interesting examples of quasitilted algebras will be of infinite representation type. It seems an interesting problem to describe the quasitilted algebras of tame representation type which are not tilted.

It is not clear which hereditary abelian categories admit a tilting object. Let us point out some additional restrictions. It can be shown that an indecomposable hereditary abelian category $\mathcal{H}$ with tilting object either has enough projective objects or does not have any non-zero projective object. To formulate the next restriction let us recall the notion of an almost split sequence in an abelian category $\mathcal{A}$. A short exact sequence $0 \to X \xrightarrow{\mu} E \xrightarrow{\pi} Y \to 0$ is said to be **almost split** if it is non-split, X and Y are indecomposable and for $f \in \operatorname{Hom}(W,Y)$ not split epi there is some $g \in \operatorname{Hom}(W,E)$ such that $f = \pi g$. We say that $\mathcal{A}$ **has almost split sequences** if for all indecomposable non-projective objects Y there is an exact sequence $0 \to X \to E \to Y \to 0$ satisfying the conditions above.

Proposition 1.3. *Let $\mathcal{H}$ be a hereditary abelian R-category with tilting object T. Then $\mathcal{H}$ has almost split sequences.*

We sketch the proof. Let $\Lambda = \operatorname{End}_{\mathcal{H}}(T)^{op}$. It follows easily from Theorem 1.2 that gl. dim $\Lambda < \infty$. Thus $\mathcal{D}^b(\Lambda)$ has Auslander-Reiten triangles (see [H]). But then $\mathcal{D}^b(\mathcal{H})$ has Auslander-Reiten triangles by Theorem 1.2. It is well known and easy to see that an indecomposable object $X \in \mathcal{D}^b(\mathcal{H})$ is isomorphic to $H[i]$ for some $i \in \mathbb{Z}$ and some indecomposable object $H \in \mathcal{H}$, where $H[i]$ denotes the stalk complex concentrated in degree $-i$ with stalk H. Let $H \in \mathcal{H}$ be indecomposable and let $H'^{\bullet} \to E^{\bullet} \to H[0] \to H'^{\bullet}[1]$ be the Auslander–Reiten triangle. We show that if H is not projective then $H'^{\bullet} \simeq H'[0]$ for some $H' \in \mathcal{H}$. Otherwise, since $\mathcal{H}$ is hereditary, $H'^{\bullet} \simeq H'[-1]$ for some $H' \in \mathcal{H}$. Let $Z \in \mathcal{H}$. Using 3.8 in [H] and the derived equivalence between $\mathcal{D}^b(\mathcal{H})$ and $\mathcal{D}^b(\Lambda)$ it follows that $\mathrm{Ext}^1_{\mathcal{H}}(H,Z) \simeq \mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(Z, H'[-1]) = 0$, thus H is projective. So if H is not projective, then also $E^\bullet \simeq E[0]$ for some $E \in \mathcal{H}$ and the triangle induces an almost split sequence ending at H.

Let Λ be a quasitilted algebra, say $\Lambda \simeq \mathrm{End}(T)^{op}$ for a tilting object T over a hereditary abelian category $\mathcal{H}$. As shown above the derived equivalence $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(\Lambda)$ is quite useful. It is also the essential tool in deriving the homological properties in the theorem below.

Theorem 1.4. *Let Λ be an artin algebra. Then the following are equivalent.*
(i) Λ is quasitilted.
(ii) $\mathrm{gl.dim}\,\Lambda \leq 2$ *and an indecomposable Λ-module X satisfies* $\mathrm{pd}_\Lambda X \leq 1$ *or* $\mathrm{id}_\Lambda X \leq 1$.

We observe that the canonical algebras are quasitilted algebras. See [Ri] for a definition of the canonical algebra. It follows from the description of $\mathrm{mod}\, C$ in [Ri] that C satisfies the homological properties in the theorem above. Using the results in [GL] one may also verify the definition directly. In fact Geigle and Lenzing consider hereditary abelian categories associated with weighted projective curves and construct tilting objects similar to the one given in the second example above, whose endomorphism algebras are just the canonical algebras.

2. One-point extensions of hereditary algebras.

In this section we investigate when the one-point extension of a hereditary algebra H by an H-module M is quasitilted. The answer is especially nice when H is tame. In order to formulate our results we introduce the notion of a module M dominating a module or a class of modules. We investigate the concept of dominating more closely, including the relationship to elementary modules over wild hereditary algebras.

We start out by recalling some of the main features of the representation theory of hereditary algebras H (see [Ri]). We here assume that H is basic and indecomposable.

An indecomposable H-module X is called **preprojective** if there is some $n \geq 0$ such that $\mathrm{DTr}^n X$ is projective, and it is called **preinjective** if there is some $n \geq 0$ such that $\mathrm{TrD}^n X$ is injective. If X is neither preprojective nor preinjective it is called **regular**. An H-module Z is called preprojective (respectively preinjective, regular) if all the indecomposable direct summands of Z are preprojective (respectively preinjective, regular). H is of finite representation type if and only if there are no indecomposable regular modules, that is all indecomposable H-modules are preprojective and preinjective.

If H is not of finite representation type, then either all indecomposable regular H-modules are DTr-periodic or no indecomposable regular H-module is DTr-periodic. In the first case H is said to be tame, and in the second case wild. In case we assume that H is a k-algebra where k is an algebraically closed field, then H is isomorphic to the path algebra $k\vec{\Delta}$ of a finite connected quiver $\vec{\Delta}$ without oriented cycles. Then $k\vec{\Delta}$ is of finite representation type if and only if the underlying graph Δ of $\vec{\Delta}$ is a Dynkin diagram, and $k\vec{\Delta}$ is tame if and only if Δ is a Euclidean diagram.

If H is tame, the full subcategory $\mathcal{R} \subseteq \mathrm{mod}\, H$ formed by regular modules is an abelian category. The category $\mathcal{R}$ is a product of infinitely many serial categories $\mathcal{R}_i$, $i \in I$. With the exception of at most three, all $\mathcal{R}_i$ contain a unique simple object. The number of simple objects in the exeptional cases can be read off from the graph Δ. The Auslander-Reiten quiver of $\mathcal{R}_i$ is a tube $\mathcal{T}_i$ of the form $\mathbb{Z}A_\infty / < \tau^{r_i} >$ where r_i is the number of simple objects in $\mathcal{R}_i$ and is referred to as the rank of $\mathcal{T}_i$, and τ is the translation on the Auslander-Reiten quiver. The simple objects in $\mathcal{R}$ are called simple regular modules.

In order to study one-point extensions of hereditary algebras it is convenient to introduce some terminology. For a module M denote for $n \geq 1$ by M^n the direct sum of n copies of M. We say that an indecomposable module M **dominates** a module T if for any right minimal map $f : M^n \to T$ with $n \geq 1$ we have that $\ker f$ is projective. And we say that M dominates a class of modules $\mathcal{T}$ if M dominates T for each T in $\mathcal{T}$. We shall see later that it is not sufficient to consider the case $n = 1$. We have however the following characterization of this concept.

Proposition 2.1. *Let M be an indecomposable module over an artin algebra Λ with* $\operatorname{pd} M \leq 1$ *and* $\operatorname{End}(M)$ *a division ring. Then M dominates a module T if and only if in the minimal right* $\operatorname{add} M$*-approximation $h : M^n \to T$ we have that* $\ker h$ *is projective.*

We have borrowed the expression that M dominates T from [Ri], where it is used in a different but related way. In order to see the relationship we briefly explain the definition in [Ri]. Let W be a sincere indecomposable directing module over an artin algebra Λ satisfying an extra condition (of being a wing module in the sense of [Ri]). Since then Λ has finite global dimension, we have an associated Coxeter transformation Φ. Then a module M is said to dominate W if we have that $\underline{\dim} M = (1 - \Phi^{-1})\underline{\dim} W$ and for any non-zero map $f : M \to W$ we have $\operatorname{pd}_\Gamma(k, W, f) \leq 1$. Here $\underline{\dim} M$ and $\underline{\dim} W$ denote the associated dimension vectors, and the triple (k, W, f) is viewed as a module over the one-point extension algebra $\Gamma = \Lambda[M] = \left(\begin{smallmatrix} k & 0 \\ M & \Lambda \end{smallmatrix}\right)$. It is not hard to see that $\operatorname{pd}(k, W, f) \leq 1$ implies that $\ker f$ is projective, giving the connection with our definition.

We also recall that a module M over an artin algebra Λ is the **middle of a short chain** if there is an indecomposable Λ-module X with $\operatorname{Hom}_\Lambda(X, M) \neq 0$ and $\operatorname{Hom}_\Lambda(M, \operatorname{DTr} X) \neq 0$ (see [AR], [RSS]). If M is an indecomposable module over a hereditary algebra H, we denote by $\mathcal{L}_M$ the class of H-modules X having a nonzero regular summand and such that there is an indecomposable map $f : M^n \to X$ in $\operatorname{mod} H$ for some $n \geq 0$. We say that M **governs** $\mathcal{L}_M$ if $\ker f$ is projective for each indecomposable map $f : M^n \to X$ with X in $\mathcal{L}_M$. We then have the following result.

Theorem 2.2. *Let M be a module over a hereditary algebra H. Then $\Gamma = H[M]$ is quasitilted if and only if M is not the middle of a short chain or M is indecomposable regular and governs $\mathcal{L}_M$.*

If H is hereditary and M is a H-module which is not the middle of a short chain, then the quasitilted algebra $H[M]$ will always be tilted. For this one uses that a module which is not the middle of a short chain must be preprojective or preinjective. Hence in order to construct examples of quasitilted algebras which are not tilted using Theorem 2.2 it is necessary to start with an indecomposable regular module M.

Note that it follows directly from the definition that $\mathcal{L}_M$ contains all indecomposable regular H-modules. Hence if M is indecomposable regular and $H[M]$ is quasitilted, then M must dominate the indecomposable regular H-modules. However, that M is indecomposable regular and dominates the indecomposable regular modules is not sufficient for $H[M]$ to be quasitilted. We are grateful to Otto Kerner for the following example of H and M given in [K], where we can show that M dominates the regular modules but $H[M]$ is not quasitilted.

Let H be the path algebra of the quiver $\overset{1}{\bullet} \leftleftarrows \overset{2}{\bullet} \leftarrow \overset{3}{\bullet}$ over a field k and let M be the unique indecomposable H-module with dimension vector $(1,2,0)$. Then one can show that M dominates the regular modules. Further, letting the last coordinate correspond to the vertex added by the extension, one can show that there is an indecomposable $H[M]$-module with dimension vector $(2,4,2,2)$ having both projective and injective dimension equal to two. This shows that $H[M]$ is not quasitilted.

When M is an indecomposable regular module over a tame hereditary algebra H, it is easy to see that M is simple regular if and only if M dominates the indecomposable regular H-modules, so that $H[M]$ being quasitilted implies that M is simple regular. Since it can also be shown that $\mathcal{L}_M$ is contained in the regular H-modules when M is simple regular, we get the following.

Corollary 2.3. *Let H be a tame hereditary algebra and M a regular H-module. Then $H[M]$ is quasitilted if and only if M is a simple regular module.*

In particular it follows that if the hereditary algebra H is tame, there is always some regular module M such that $H[M]$ is quasitilted.

For tame hereditary algebras over an algebraically closed field it is known exactly which one-point extensions $H[M]$ are tilted [Ri]. Hence we can in this case describe exactly when a simple regular module M gives rise to a quasitilted algebra $H[M]$ which is not tilted as follows.

Let M be a simple regular module over a tame hereditary k-algebra $H = k\vec{\Delta}$ with k algebraically closed. Then $H[M]$ is quasitilted and not tilted exactly in the following cases.

(i) If Δ is of type $\tilde{E}_8$, then M is arbitrary.

(ii) If Δ is of type $\tilde{E}_7$, then M is in a tube of rank at most 3.

(iii) If Δ is of type $\tilde{E}_6$, then M is in a tube of rank at most 2.

(iv) If Δ is of type $\tilde{D}_n$, then M is in a tube of rank at most 2 if $n \geq 8$, and in a tube of rank 1 if $n \leq 7$.

(v) If Δ is of type $\tilde{A}_n$, with tubes of rank p, q and 1, with $p \geq q$, then M is in a tube of rank 1 if $q \geq 3$ or if $p + q \geq 9$.

We give a generalization of Corollary 2.3 to tame concealed algebras, which we shall need in the next section. Recall that a k-algebra Λ is tame concealed if $\Lambda \simeq \operatorname{End}_H(X)^{op}$ where X is a preprojective tilting module over a tame hereditary algebra H. For Λ there is also a subcategory of socalled regular modules, equivalent to the subcategory of regular H-modules. Using this close connection with hereditary algebras it is possible to prove the following.

Proposition 2.4. *Let M be a regular module over a tame concealed k-algebra Λ. Then the one-point extension $\Lambda[M]$ is quasitilted if and only if M is simple regular.*

Motivated by the connection with quasitilted algebras we investigate when a module dominates other modules, in particular when it dominates the regular modules. We recall that a module M is said to be a cogenerator of a module N if N can be embedded in a direct sum of copies of M. A module M is said to be a cogenerator of a class $\mathcal{X}$ of modules if M is a cogenerator for each module in $\mathcal{X}$. We have the following criterion.

Proposition 2.5. *Let H be a hereditary k-algebra and M an indecomposable regular H-module with $\operatorname{End}_H(M)$ a division ring. If X is a cogenerator for a class $\mathcal{C}$ of H-modules and M dominates X, then M dominates $\mathcal{C}$.*

The following example shows that the assumption that $\operatorname{End}_H(M)$ is a division ring can not be dropped in Proposition 2.5.

Let $H = k(\overset{1}{\cdot} \leftleftarrows \overset{2}{\cdot})$ and let M be an indecomposable regular module of regular length two. Then $\operatorname{End}_H(M) \simeq k[X]/(X^2)$. Denote by $I(j)$ the injective envelope of the simple module associated with the vertex j, and by $P(j)$ the projective cover. Now $I(1)$ is a cogenerator for the regular H-modules and there exists a surjective map $f \in \operatorname{Hom}(M, I(1))$ which is the minimal add M-approximation of $I(1)$. Clearly we have $P(1) = \ker f$, but M does not dominate the regular modules.

We illustrate Proposition 2.5 with the following example.

Let k be a field and $\vec{\Delta}$ be the quiver $0\cdot\begin{array}{l}\swarrow\cdot 1\\ \swarrow\cdot 2\\ \leftarrow\cdot 3\\ \nwarrow\cdot 4\\ \nwarrow\cdot 5\end{array}$ and M a representation of the form $k^2\begin{array}{l}\overset{f_1}{\swarrow} k\\ \overset{f_2}{\swarrow} k\\ \overset{f_3}{\leftarrow} k\\ \overset{f_4}{\nwarrow} k\\ \overset{f_5}{\nwarrow} k\end{array}$

where the $\operatorname{Im} f_i$ are distinct subspaces of k^2 for $1 \le i \le 5$. Then M is indecomposable and one proves that $\operatorname{End}_{k\vec{\Delta}}(M) \simeq k$. Let I denote the injective envelope of the simple projective $k\vec{\Delta}$-module S. Then I has dimension vector $1\begin{array}{l}\swarrow 1\\ \swarrow 1\\ \leftarrow 1\\ \nwarrow 1\\ \nwarrow 1\end{array}$ and $\mathrm{DTr}\, I$ has dimension vector $4\begin{array}{l}\swarrow 3\\ \swarrow 3\\ \leftarrow 3\\ \nwarrow 3\\ \nwarrow 3\end{array}$. $\mathrm{DTr}\, I$ is clearly a preinjective cogenerator for the regular modules. One can show that there is an exact sequence of the form $0 \to S^2 \to M^3 \xrightarrow{h} \mathrm{DTr}\, I \to 0$, where $h : M^3 \to \mathrm{DTr}\, I$ is a minimal right add M-approximation. Here we use that $\dim_k \operatorname{Hom}(M, \mathrm{DTr}\, I) = \dim_k \operatorname{Hom}(\mathrm{TrD}\, M, I) = 3$. Since S^2 is projective, M dominates $\mathrm{DTr}\, I$, and hence M dominates the regular $k\vec{\Delta}$-modules.

Note that M does not dominate the injective module I, even though I is a preinjective cogenerator for the regular modules. For one can show that there is an exact sequence $0 \to K \to M^2 \xrightarrow{g} I \to 0$ where $g : M^2 \to I$ is a minimal right add M-approximation and K is not projective. It is also easy to see that for any non-zero map $h : M \to I$ we have that $\ker h$ is projective, so that it is not enough to check whether M dominates a module I by only considering the non-zero maps $h : M \to I$.

The one-point extension $k\vec{\Delta}[M]$ is one of the canonical algebras. Our methods can be used to prove that all canonical algebras are qusitilted.

The property that an indecomposable regular H-module dominates the regular modules is closely related to elementary modules. Recall that an indecomposable regular module M over a hereditary algebra is **elementary** if there is no exact sequence $0 \to L \to M \to N \to 0$ where L and N are non-zero regular [KL].

Assume now that the hereditary algebra H is wild. It is proved in [KL] that there is only a finite number of Coxeter orbits of dimension vectors containing elementary modules, but there may be infinitely many nonisomorphic modules having the dimension vector of an elementary module. If an indecomposable regular H-module M governs $\mathcal{L}_M$ then

M dominates the indecomposable regular modules, which again clearly implies that M is elementary. Hence we get the following consequence of Theorem 2.2, generalizing a result in [KL] from tilted to quasitilted algebras.

Proposition 2.6. *Let H be a hereditary algebra and M an indecomposable regular H-module such that $H[M]$ is quasitilted. Then M is elementary.*

The DTr-orbit of an indecomposable module M consists of all modules of the form $\mathrm{DTr}^i\, M$ for an integer i. If M is elementary, then clearly all modules in the DTr-orbit of M are elementary. For the property of dominating the regular modules we have the following.

Proposition 2.7. *Let M be an indecomposable regular module over a wild hereditary algebra H.*

(a) If M dominates the regular modules, then $\mathrm{DTr}^i\, M$ *dominates the regular modules for all $i \geq 0$.*

(b) There is some N in the DTr*-orbit of M which does not dominate the regular modules.*

Proof. (a) Assume that M dominates the regular modules. If $\mathrm{DTr}\, M$ did not dominate the regular modules, we would have an exact sequence $0 \to K \to (\mathrm{DTr}\, M)^n \xrightarrow{h} X$ with K not projective and X regular and where h can be assumed to be surjective. Then $0 \to \mathrm{TrD}\, K \to M^n \to \mathrm{TrD}\, X \to 0$ is also exact since X is regular, and $\mathrm{TrD}\, K$ is not projective, giving a contradiction.

(b) Assume without loss of generality that M is of minimal length in its DTr-orbit. Then for some $i > 0$ there is a non-zero map $f : \mathrm{TrD}^i\, M \to M$ (see [Ba]). Then $\ker f$ must be non-zero. If $\ker f$ is not projective, the exact sequence $0 \to \ker f \to \mathrm{TrD}^i\, M \to \mathrm{Im}\, f \to 0$ shows that $\mathrm{TrD}^i\, M$ does not dominate the regular modules since $\mathrm{Im}\, f$ is clearly regular. If $\ker f$ is projective, then the exact sequence $0 \to \mathrm{TrD}(\ker f) \to \mathrm{TrD}^{i+1}\, M \to \mathrm{TrD}\,\mathrm{Im}\, f \to 0$ shows that $\mathrm{TrD}^{i+1}\, M$ does not dominate the regular modules. □

As a consequence of this result we see that M can be elementary without dominating the regular modules. The next example will even show that the DTr-orbit of an elementary module does not necessarily contain a module which dominates the regular modules.

Consider the path algebra Γ of the quiver $\vec{\Delta} = {}_0\!\left\langle\begin{smallmatrix}\cdot 1\\ \cdot 2\\ \cdot 3\\ \cdot 4\\ \cdot 5\\ \cdot 6\end{smallmatrix}\right.$ and let M be an indecomposable regular module with $\underline{\dim} M = {}_2\!\left\langle\begin{smallmatrix}0\\ 0\\ 1\\ 1\\ 1\\ 1\end{smallmatrix}\right.$ such that the subspaces $M(3)$, $M(4)$, $M(5)$ and $M(6)$ are pairwise different. Then M is elementary since any nonprojective submodule is maximal with a simple injective quotient module.

We next want to show that we have an exact sequence $0 \to K \xrightarrow{\mu} M^2 \xrightarrow{f} Y \to 0$ where f is a minimal right add M-approximation and K is indecomposable regular. Consider the indecomposable module ${}_\Gamma Y$ with $\underline{\dim} Y = {}_1\!\left\langle\begin{smallmatrix}0\\ 0\\ 1\\ 1\\ 1\\ 1\end{smallmatrix}\right.$, so ${}_\Lambda Y = {}_\Lambda I(0)$ where Λ is the support algebra of M. Then ${}_\Gamma Y$ is regular and $\dim_k \mathrm{Hom}_\Lambda(M, Y) = 2$. We consider the minimal add M-approximation f of Y. Clearly f is surjective, so we obtain the following short exact sequence

$$(*)\ 0 \to K \xrightarrow{\mu} M^2 \xrightarrow{f} Y \to 0.$$

We want to show that K is indecomposable regular. Let Z be the indecomposable Γ-module with $\underline{\dim} Z = 3 \!<\! \begin{smallmatrix}0\\0\\1\\1\\1\\1\end{smallmatrix}$. Note that $\underline{\dim} Z = \underline{\dim} K$ and that ${}_\Lambda Z = {}_\Lambda\ \mathrm{TrD}_\Lambda\, P(0)$ is preprojective as Λ-module. Applying $\mathrm{Hom}_\Gamma(\ ,Z)$ to $(*)$ yields the following long exact sequence

$$0 \to \mathrm{Hom}_\Gamma(K,Z) \to \mathrm{Ext}^1_\Gamma(Y,Z) \to \mathrm{Ext}^1_\Gamma(M^2,Z) \to \mathrm{Ext}^1_\Gamma(K,Z) \to 0.$$

Now $\dim_k \mathrm{Ext}^1_\Gamma(M^2,Z) = 4$ and $\dim_k \mathrm{Ext}^1_\Gamma(Y,Z) = 5$ shows that $\mathrm{Hom}_\Gamma(K,Z) \neq 0$. If $K \not\simeq Z$, then K has a non-zero projective indecomposable summand P. Also note that ${}_\Lambda K$ has no non-zero Λ-regular indecomposable summand since ${}_\Lambda M$ is simple regular. Applying $\mathrm{Hom}_\Lambda(\ ,M)$ to $(*)$ shows that

$$0 \to \mathrm{Hom}_\Lambda(M^2,M) \to \mathrm{Hom}_\Lambda(K,M) \to \mathrm{Ext}^1_\Lambda(Y,M) \to \mathrm{Ext}^1_\Lambda(M^2,M) \to 0$$

is exact. Since $\dim_k \mathrm{Ext}^1_\Lambda(Y,M) = \dim_k \mathrm{Ext}^1_\Lambda(M^2,M)$ we conclude that μ is the minimal left add M-approximation of K. This yields the following commutative diagram with exact rows, where π is a projection of K to P and we observe that for any projective Λ-module Q with $\mathrm{Hom}(Q,K) \neq 0$ we have a monomorphism $\alpha : Q \to M$.

$$\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{\mu} & M^2 & \xrightarrow{f} & Y & \to & 0 \\
 & & \downarrow \pi & & \downarrow \beta & & \downarrow & & \\
0 & \to & P & \longrightarrow & M & \longrightarrow & M/P & \to & 0
\end{array}$$

Now if $P = P(j)$ for $3 \le j \le 6$ we have that $\mathrm{Hom}_\Lambda(Y, M/P) = 0$, since ${}_\Lambda M/P = \mathrm{DTr}_\Lambda\, I(j)$ as can easily be verified. But then $\mathrm{Hom}_\Lambda(M^2, P) \neq 0$ which is a contradiction. so we get $P = P(0)$ and $Y \simeq M/P$. But then $\ker \pi \simeq \ker \beta \simeq M$ gives the desired contradiction. Hence we get that $K \simeq Z$. An easy calculation shows that Z is not preprojective, hence Z, and consequently K, is regular.

Since for any integer n we have that

$$0 \to \mathrm{DTr}^n_\Gamma K \to \mathrm{DTr}^n_\Gamma M^2 \to \mathrm{DTr}^n_\Gamma Y \to 0$$

is an exact sequence, we see that $\mathrm{DTr}^n_\Gamma M$ does not dominate the regular modules for all integers n.

Kerner has shown that if M is an elementary module which is a **stone**, that is $\mathrm{Ext}^1_\Lambda(M,M) = 0$, then any exact sequence $0 \to X \to M^n \to Y \to 0$ with X and Y regular splits [K]. The exact sequence $0 \to K \to M^2 \to Y \to 0$ above shows that this conclusion is not valid for any elementary module M, thus answering a question of Kerner [K].

3. Quasitilted triangular algebras

In this section we describe the quivers $\vec{\Delta}$ such that $\Gamma_{\vec{\Delta}} = \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$ is quasitilted when Λ is the path algebra of $\vec{\Delta}$ over an algebraically closed field k, including proofs. The motivation for this investigation is to illustrate techniques for proving that algebras are quasitilted and that algebras are not quasitilted. In particular we show how to apply results from section 2, and at the same time we provide more examples of quasitilted algebras. The main result is the following.

Theorem 3.1. *Let $\Gamma_{\vec{\Delta}}$ be as above. Then $\Gamma_{\vec{\Delta}}$ is quasitilted if and only if $\vec{\Delta}$ is in the following list of Dynkin quivers*
$\{A_1, A_2, A_3, A_4, D_4 \mid$ *any orientation* $\}$
$\{A_5 \mid$ *without linear orientation* $\}$.

The proof of this result is divided into several steps. Recall that if Λ is any artin algebra, then a module over the triangular matrix algebra $\Gamma = \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$ is a triple (M, N, f) with M and N Λ-modules and $f : M \to N$ a Λ-homomorphism. The projective Γ-modules are given as those triples (M, N, f) where M and N are projective Λ-modules and f is a split monomorphism and the injective Γ-modules are given dually as those triples (M, N, f) where M and N are injective and f is a split epimorphism. Using this description we have the following description of the Γ-modules of projective and injective dimension less than or equal to one respectively when $\Gamma = \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$ and Λ is hereditary.

Lemma 3.2. *Let Λ be a hereditary artin algebra and Γ the triangular matrix ring $\left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$. Then the following hold for a Γ-module (M, N, f).*
(a) $\operatorname{pd}_\Gamma(M, N, f) \leq 1$ if and only if the kernel of f is projective.
(b) $\operatorname{id}_\Gamma(M, N, f) \leq 1$ if and only if the cokernel of f is injective.

Proof. We shall only need the if part of these statements and since (b) is the dual of (a) we only give a proof of the needed implication in (a).

Let (M, N, f) be a Γ-module and assume that $\operatorname{pd}_\Gamma(M, N, f) \leq 1$. Let K be the kernel of f and C the cokernel of f with $i : K \to M$ and $p : N \to C$ the natural morphisms. Since Λ is hereditary we have projective Λ-resolutions $0 \to Q_1 \xrightarrow{\beta} Q_0 \xrightarrow{\alpha} M \to 0$ and $0 \to P_1 \xrightarrow{\delta} P_0 \xrightarrow{\gamma} C \to 0$. Since p is an an epimorphism there is some $\varepsilon : P_0 \to N$ such that $p\varepsilon = \gamma$. Hence we obtain the following exact commuting diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & Q_1 & \xrightarrow{\beta} & Q_0 & \xrightarrow{\alpha} & M & \to & 0 \\
 & & \downarrow g & & \downarrow \binom{1}{0} & & \downarrow f & & \\
0 & \to & A & \to & Q_0 \amalg P_0 & \xrightarrow{(f\alpha,\varepsilon)} & N & \to & 0 \\
 & & \downarrow & & \downarrow (0,1) & & \downarrow p & & \\
 & & B & \xrightarrow{\nu} & P_0 & \xrightarrow{\gamma} & C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with A the kernel of $(f\alpha, \varepsilon)$ and B the cokernel of g. Since $\operatorname{pd}_\Gamma(M, N, f) \leq 1$ we have that (Q_1, A, g) is a projective Γ-module and then Q_1 and A are projective Λ-modules with g a split monomorphism. Hence B is a projective Λ-module. We now have by the snake lemma a long exact sequence

$$0 \to K \to B \xrightarrow{\nu} P_0 \xrightarrow{\gamma} C \to 0$$

Since Λ is hereditary, K is projective. $\square$

The next result when applied to path algebras gives a severe limitation on the quivers needed to be considered.

Proposition 3.3. *Let Λ be a hereditary artin algebra and Γ the triangular matrix algebra over Λ. If there exists a Λ-module A with $\mathrm{DTr}_\Lambda^4(A) \neq 0$, then Γ is not quasitilted.*

Proof. Assume there exists a Λ-module A with $\mathrm{DTr}_\Lambda^4 A \neq 0$. Without loss of generality we may assume that A is indecomposable. We then have an almost split sequence

$$0 \to \mathrm{DTr}_\Lambda^3 A \to M \xrightarrow{f} \mathrm{DTr}_\Lambda^2 A \to 0$$

with $\mathrm{DTr}_\Lambda^3 A$ not projective and an almost split sequence

$$0 \to \mathrm{DTr}_\Lambda^2 A \xrightarrow{g} N \to \mathrm{DTr}_\Lambda A \to 0$$

with $\mathrm{DTr}_\Lambda A$ not injective. Consider now the Γ-module (M, N, gf). Clearly the kernel of gf is $\mathrm{DTr}_\Lambda^3 A$ which is not projective and the cokernel of gf is $\mathrm{DTr}\, A$ which is not injective. Hence both the projective and injective dimension of (M, N, gf) is greater than or equal to two by Lemma 3.4.

It remains to prove that (M, N, gf) is indecomposable. Decompose (M, N, gf) as $(M', N', h') \amalg (M'', N'', h'')$, i.e. $M = M' \amalg M''$, $N = N' \amalg N''$ and $gf = \begin{pmatrix} h' & 0 \\ 0 & h'' \end{pmatrix}$. But the image of gf is $\mathrm{DTr}^2(A)$ which is indecomposable. Hence h' or h'' is zero. Without loss of generality we may assume h' is zero. Hence M' is contained in the kernel of gf, and is therefore a summand of the kernel of gf. But as already observed the kernel of gf is $\mathrm{DTr}_\Lambda^3 A$, which is indecomposable and not a summand of M. It follows that M' is zero. By the dual argument N' is a summand of the cokernel of gf and again this implies that N' is zero. Hence (M, N, gf) is indecomposable, completing the proof of the proposition. $\square$

Using Proposition 3.3 it is clear that Γ is not quasitilted if Λ is of infinite type. One can also use Proposition 3.3 to prove by inspection that $\Gamma_{\vec{\Delta}}$ is not quasitilted for any path algebra $k\vec{\Delta}$ if $\vec{\Delta}$ contains D_6, E_6, E_7, E_8 or A_5 with linear orientation; A_6 with $\mathrm{rad}^3 kA_6 \neq 0$ or A_7 with $\mathrm{rad}^2 kA_7 \neq 0$.

We are left with considering the following diagrams.

(a) D_5 with any orientation,

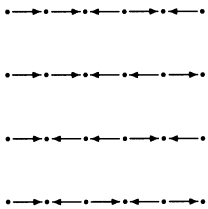
(b) A_7 with the two orientations which are dual of each other,

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

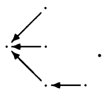
$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

(c) A_6 with the following five orientations up to duality.

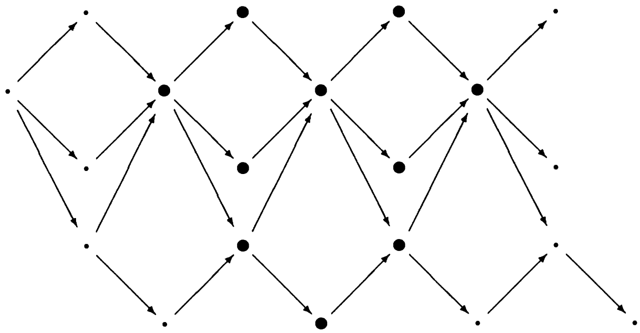
$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$



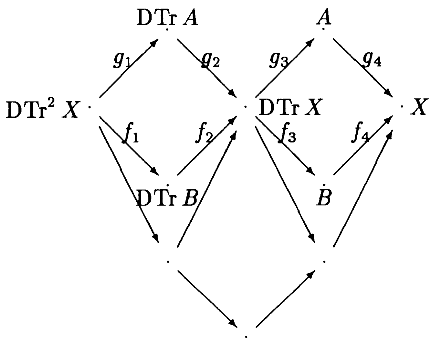
We first treat the case $\vec{\Delta} = D_5$ with the orientation



We obtain that the AR-quiver of $k\vec{\Delta}$ is as follows.



Considering the part of the AR-quiver marked with big dots, we obtain the following diagram.



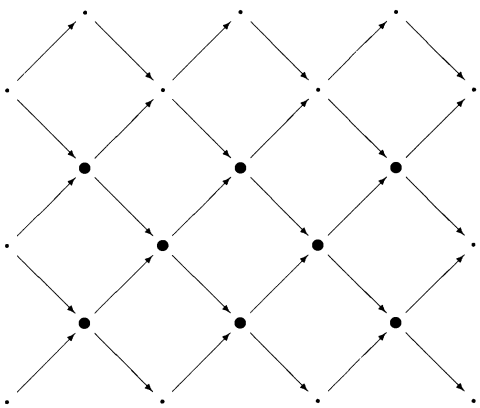
It is now possible to choose maps f_i and g_i where $i = 1, 2, 3, 4$ as indicated on the diagram such that f_i and g_i are irreducible morphisms with $f_2 f_1$ and $g_2 g_1$ linearly independent, $f_4 f_3$ and $g_4 g_3$ linearly independent, $f_3 f_2 = 0 = g_3 g_2$ and such that $g_4 g_3 f_2 f_1 \neq 0$. Now let $M = \mathrm{DTr}^2 X \amalg \mathrm{DTr}\, X$ and $N = \mathrm{DTr}\, X \amalg X$ and let $\psi : M \to N$ be given by the matrix $\psi = \begin{pmatrix} f_2f_1 & 0 \\ g_4g_3f_2f_1 & g_4g_3 \end{pmatrix}$. Any endomorphism of M may be represented by a matrix $\phi = \begin{pmatrix} a & 0 \\ bf_2f_1+cg_2g_1 & d \end{pmatrix}$ with a, b, c, $d \in k$ and any endomorphism of N may be represented by a matrix $\phi' = \begin{pmatrix} a' & 0 \\ b'f_4f_3+c'g_4g_3 & d' \end{pmatrix}$ with a', b', c', $d' \in k$. Now a Γ-map $F : (M, N, \psi) \to (M, N, \psi)$ is a pair (ϕ, ϕ') where $\phi : M \to M$ and $\phi' : N \to N$ are Λ-maps such that $\psi\phi = \phi'\psi$. A direct calculation shows that if the pair (ϕ, ϕ') given above is a Γ-map then $a = d = a' = d'$, and therefore $\mathrm{End}_\Gamma(M, N, \psi)$ is a local ring. Hence (M, N, ψ) is an indecomposable Γ-module.

Now $\binom{0}{g_2} : DTrA \to \mathrm{DTr}^2 X \amalg \mathrm{DTr}\, X$ has the property that $\phi\binom{0}{g_2} = 0$. Thus there is a non-zero morphism from $\mathrm{DTr}\, A$ to the kernel of ψ and therefore the kernel of ψ is not projective. It follows by Lemma 3.4 that (M, N, ψ) has projective dimension two. In the same way $(f_3, 0) : \mathrm{DTr}\, X \amalg X \to B$ has the property that $(f_3, 0)\psi = 0$. Since B is not injective, the cokernel of ψ is not injective and therefore (M, N, ψ) has also injective dimension two by Lemma 3.4. It follows that $\Gamma_{\vec{\Delta}}$ is not quasitilted when $\vec{\Delta}$ is D_5 with this orientation.

It turns out that for all other orientations of D_5 one obtains the same diagram inside the AR-quiver. Hence $\Gamma_{\vec{\Delta}}$ is not quasitilted for all orientations on D_5.

We now consider the cases A_6 and A_7. Consider the AR-quiver for the case

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$



Taking out the part marked with big dots we get the following diagram.

$$(*)\qquad \begin{tikzcd}
\mathrm{DTr}^2 X \arrow[dr, "f_1"] & & \mathrm{DTr}\, X \arrow[dr, "f_3"] & & X \\
& \mathrm{DTr}\, Z \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & & Z \arrow[ur, "f_4"] \arrow[dr, "g_4"'] & \\
\mathrm{DTr}^2 Y \arrow[ur, "g_1"'] & & \mathrm{DTr}\, Y \arrow[ur, "g_3"'] & & Y
\end{tikzcd}$$

The maps may now be chosen to be irreducible and such that $f_3f_2 = g_3g_2$, $g_2g_1 \neq 0 \neq f_2f_1$, $f_4f_3 \neq 0 \neq g_4g_3$ and $g_4f_3f_2f_1 \neq 0$. Now let $M = \mathrm{DTr}^2 X \amalg \mathrm{DTr}^2 Y \amalg \mathrm{DTr}\, X \amalg \mathrm{DTr}\, Y$, $N = \mathrm{DTr}\, X \amalg \mathrm{DTr}\, Y \amalg X \amalg Y$ and $\phi : M \to N$ be given by the matrix

$$\begin{pmatrix} f_2f_1 & f_2g_1 & 0 & 0 \\ g_2f_1 & g_2g_1 & 0 & 0 \\ 0 & 0 & f_4f_3 & f_4g_3 \\ g_4f_3f_2f_1 & 0 & g_4f_3 & g_4g_3 \end{pmatrix}$$

Letting $\beta : \mathrm{DTr}\, Z \to \mathrm{DTr}^2 X \amalg \mathrm{DTr}^2 Y \amalg \mathrm{DTr}\, X \amalg \mathrm{DTr}\, Y$ be given by $\beta = (0, 0, -f_2, g_2)^{tr}$ we get that $\phi\beta = 0$. Hence the kernel of ϕ is not projective. In the same way using $\alpha : \mathrm{DTr}\, X \amalg \mathrm{DTr}\, Y \amalg X \amalg Y \to Z$ given by $\alpha = (-f_3, g_3, 0, 0)$ we obtain that the cokernel of ϕ is not injective. Hence the module (M, N, ϕ) has both projective and injective dimension equal to two. When the endomorphisms of M and N are given as matrices one solves a system of equations which shows that $\mathrm{End}_\Gamma(M, N, \phi)$ is local. This shows that A_6 with the above orientation is not quasitilted.

By drawing the AR-quiver for the other orientations of A_6 as well as for the orientations of A_7 one can always produce a configuration as the one shown in $(*)$ with the same properties of the irreducible maps and with Z and $\mathrm{DTr}\, Z$ not injective and not projective respectively. The above analysis then gives that also in these cases Γ is not quasitilted.

In order to complete the proof of the theorem we have to show that in the remaining cases $\Gamma_{\vec{\Delta}}$ is quasitilted.

First observe that if $\Lambda = k\vec{\Delta}$ with $\vec{\Delta}$ a quiver, there is the following description of $\Gamma_{\vec{\Delta}} = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$. Form the quiver $\underline{\vec{\vec{\Delta}}}$ consisting of vertices $(\vec{\Delta}_0 \times 0) \cup (\vec{\Delta}_0 \times 1)$ and arrows $(\vec{\Delta}_1 \times 0) \cup (\vec{\Delta}_1 \times 1) \cup \vec{\Delta}_0$ where (α, i) is an arrow from (x, i) to (y, i) for each arrow α from x to y in $\vec{\Delta}$, $i = 0, 1$ and α_x is an arrow from $(x, 1)$ to $(x, 0)$ for each vertex x in $\vec{\Delta}_0$.

For each arrow α from x to y in $\vec{\Delta}$ we get the square

$$\begin{array}{ccc} (x,1) & \xrightarrow{(\alpha,1)} & (y,1) \\ \downarrow \alpha_x & & \downarrow \alpha_y \\ (x,0) & \xrightarrow{(\alpha,0)} & (y,0) \end{array}$$

Now Γ is the path algebra of $\underline{\vec{\vec{\Delta}}}$ modulo the ideal given by the commutativity relations obtained from the squares described above. We indicate this on the examples by inserting a dotted diagonal in the quivers as follows.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

When it is clear from the picture that we have commutativity relations we call the corresponding algebra just the algebra of the quiver for the rest of this section.

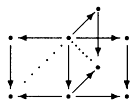
We use our criterion from section 2 except in one case to prove that if $\vec{\Delta}$ is A_1, A_2, A_3, A_4, D_4 or A_5 without linear orientation, then $\Gamma_{\vec{\Delta}}$ is quasitilted.

Case A_i, $i \leq 4$: It will follow that $\Gamma_{\vec{\Delta}}$ is quasitilted when $\vec{\Delta}$ is either A_1, A_2, A_3 or A_4 from the result for A_5 since Γ_{A_i} with $i \leq 4$ will then be obtained from some Γ_{A_5} for an A_5 without linear orientation by dividing out an ideal which preserves the property of being quasitilted.

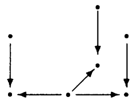
The cases A_1, A_2, A_3 and A_4 may also be done by using that Γ in these situations is of finite representation type [IPTZ] with a sincere indecomposable module and satisfies the separation condition of Bautista-Larrion [BL]. Hence Γ is by [Ri] a tilted algebra in these cases.

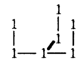
Case D_4: Up to duality we have the following two possibilities of orientations of D_4.

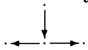
For the first case we get that Γ is the quiver algebra of the quiver

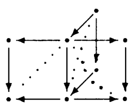


Thus Γ is then the one-point extension of the path algebra of the quiver

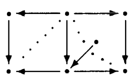


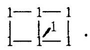
by the unique indecomposable module M given by the dimension vector  . Now an easy calculation shows that M is a simple regular module with $\mathrm{DTr}^2\, M \simeq M$. Hence Γ is a one-point extension of a tame hereditary algebra by a simple regular module and therefore Γ is quasitilted by Corollary 2.3.

Next consider the quiver  . Then $\Gamma_{\tilde{\Delta}}$ is given as the path algebra of the quiver



Then $\Gamma_{\tilde{\Delta}}$ is a one-point extension of the algebra given by the quiver



by the unique indecomposable module M with dimension vector  . This last algebra is a tame concealed algebra from the Bongartz-Happel-Vossieck list of such algebras (see [Ri]). Further the module M is simple regular with $\mathrm{DTr}^2\, M \simeq M$. Therefore Proposition 2.4 applies to show that $\Gamma_{\tilde{\Delta}}$ is quasitilted.

Case A_5: We are left with A_5, with nonlinear orientations. We have to consider the folowing cases. The other cases follow by duality.

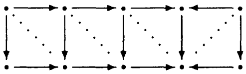
(i) $\bullet \to \bullet \to \bullet \to \bullet \leftarrow \bullet$

(ii) $\bullet \to \bullet \to \bullet \leftarrow \bullet \to \bullet$

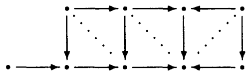
(iii) $\bullet \leftarrow \bullet \leftarrow \bullet \to \bullet \to \bullet$

(*iv*) $\bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet$

For the orientation (i), $\Gamma_{\tilde{\Delta}}$ is the algebra of the quiver

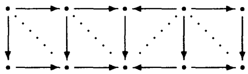


This is a one-point extension of the tame concealed algebra given by the quiver

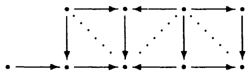


by the unique indecomposable module M with dimension vector $\begin{smallmatrix} & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$. M is a simple regular module with $\mathrm{DTr}^5 M \simeq M$. Proposition 2.4 gives that $\Gamma_{\tilde{\Delta}}$ is quasitilted.

The orientation (ii) gives that $\Gamma_{\tilde{\Delta}}$ is the algebra of the quiver

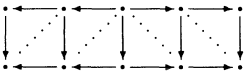


which is a one-point extension of the tame concealed algebra given by the quiver



by the unique indecomposable module M with dimension vector $\begin{smallmatrix} & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$. M is here also simple regular with $\mathrm{DTr}^5\, M \simeq M$. Again Proposition 2.4 implies that $\Gamma_{\tilde{\Delta}}$ is quasitilted.

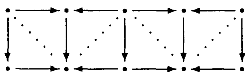
The orientation (iii) gives that $\Gamma_{\tilde{\Delta}}$ is the algebra of the quiver



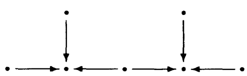
This is a one-point extension of the tame concealed algebra given by the quiver

by the unique indecomposable module M with dimension vector $\begin{smallmatrix} 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$. M is then simple regular with $\mathrm{DTr}^2\, M \simeq M$. Hence $\Gamma_{\tilde{\Delta}}$ is quasitilted also in this case.

For the last orientation we need a more elaborate argument. Here $\Gamma_{\tilde{\Delta}}$ corresponds to the quiver algebra



which is not a one-point extension of a tame concealed algebra. But it is a tubular extension of the hereditary algebra given by the quiver



by the three simple regular pairwise orthogonal modules M_1, M_2 and M_3 where M_1 has dimension vector $\begin{smallmatrix} & 1 & & 0 & \\ 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$, M_2 has dimension vector $\begin{smallmatrix} & 0 & & 1 & \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix}$ and M_3 has dimension vector $\begin{smallmatrix} & 1 & & 1 & \\ & | & & | & \\ 0 - & 1 - & 1 - & 1 - & 0 \end{smallmatrix}$. M_1 and M_2 belong to the rank 4 tube, and M_3 belongs to a rank 2 tube. So Γ is a tubular algebra of tubular type $(6,3,2)$ (see [Ri,4.7]). Theorem 5.2 (4) in [Ri] then implies that $\Gamma_{\hat{\Delta}}$ satisfies the homological characterization of a quasitilted algebra. This completes the proof of the theorem.

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TILTING THEORY AND DIFFERENTIAL GRADED ALGEBRAS

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ABSTRACT. We introduce the formalism of derived categories of differential graded algebras and demonstrate its usefulness for tilting theory in the sense of 'Morita theory for derived categories' (4).

1. The derived category of a differential graded algebra

1.1 DG algebras. Let k be a commutative ring and A a *differential graded k-algebra* (=DG algebra), i.e. a $\mathbf{Z}$-graded associative k-algebra

$$A = \coprod_{p \in \mathbf{Z}} A^p$$

endowed with a k-linear differential $d : A \to A$ which is homogeneous of degree 1 (i.e. $dA^p \subset A^{p+1}$ for each p) and satisfies the graded Leibniz rule

$$d(ab) = (da)\, b + (-1)^p a\, db\, ,\ \forall\, a \in A^p\, ,\ \forall\, b \in A.$$

It turns out to be convenient *not* to impose any a priori finiteness conditions on A.

Examples. a) If B is an 'ordinary' k-algebra, it gives rise to a DG algebra A defined by

$$A^p = \begin{cases} B & p = 0 \\ 0 & p \neq 0. \end{cases}$$

Conversely, any DG algebra A which is *concentrated in degree* 0 (i.e. A^p=0 for all $p \neq 0$) is obtained in this way from an 'ordinary' algebra.

V. Dlab and L. L. Scott (eds.), Finite Dimensional Algebras and Related Topics, 183–190.

b) If B is a k-algebra and

$$M = (\ldots \to M^i \xrightarrow{d} M^{i+1} \to \ldots)\,,\ i \in \mathbf{Z}\,,\ dd = 0$$

a complex of right B-modules, we consider the DG algebra $A = \mathcal{H}om_B\,(M, M)$ with the components

$$A^p = \prod_{-i+j=p} \operatorname{Hom}_B(M^i, M^j)$$

and the differential defined by

$$d(f^i) = (d \circ f^i - (-1)^p f^{i+1} \circ d)\,,\ (f^i) \in A^p.$$

Note that even if $M^i = 0$ for all $i \gg 0$, there will in general be non-vanishing components of A in arbitrarily small *and* arbitrarily large degrees.

1.2 DG modules. A *differential graded module over* A (=DG A-module) is a Z-graded right A-module

$$M = \coprod_{p \in Z} M^p$$

endowed with a k-linear differential $d : M \to M$ which is homogeneous of degree 1 and satisfies the graded Leibniz rule

$$d(ma) = (dm)\,a + (-1)^p m\,da\,,\ \forall\, m \in M^p\,,\ \forall\, a \in A.$$

Differential graded left A-modules are defined similarly. The Leibniz rule then reads

$$d(am) = (da)\,m + (-1)^p a\,(dm)\,,\ \forall\, a \in A^p\,,\ \forall\, m \in M.$$

A *morphism* of DG A-modules $f : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree 0 and commutes with the differential.

Examples. a) In the situation of example 1.1 a), the category of DG A-modules identifies with the category of differential complexes of right B-modules.

b) In the situation of example 1.1 b), each complex N of right B-modules gives rise to a DG A-module $\mathcal{H}om_B\,(M, N)$ endowed with the A-action $(g^j)(f^i) = (g^{i+p} \circ f^i)$, where $(g^j) \in \operatorname{Hom}_B(M, N)^q$ and $(f^i) \in A^p$. On the other hand, M becomes itself a DG left A-module for the action $(f^i)(m^j) = (f^i(m^i))$.

1.3 The homotopy category. Let $f : M \to N$ be a morphism of DG A-modules. We say that f *is null-homotopic* if we have $f = dr + rd$, where

$r : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree -1.

The *homotopy category* $\mathcal{H}A$ has the DG A-modules as *objects*. Its *morphisms* are classes $\overline{f}$ of morphisms f of DG A-modules modulo null-homotopic morphisms.

Define the *suspension functor* $S : \mathcal{H}A \to \mathcal{H}A$ by

$$(SM)^p = M^{p+1}\ ,\ d_{SM} = -d_M\ ,\ \mu_{SM}(m,a) = \mu_M(m,a)\,,$$

for $m \in M$ and $a \in A$, where μ_M and μ_{SM} are the multiplication maps of the respective modules. Define a *standard triangle of* $\mathcal{H}A$ to be a sequence

$$L \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} Cf \xrightarrow{\overline{h}} SL\,,$$

where $f : L \to M$ is a morphism of DG modules, $Cf = M \oplus SL$ as a graded k-module,

$$d_{Cf} = \begin{bmatrix} d_M & f \\ 0 & d_{SL} \end{bmatrix}\ ,\ \mu_{Cf}(\begin{bmatrix} m \\ l \end{bmatrix}, a) = \begin{bmatrix} ma - (-1)^p l(da) \\ la \end{bmatrix}\,,$$

for $m \in M$, $l \in L^p$, the morphism g is the canonical injection $M \to Cf$, and $-h$ (note the sign) is the canonical projection. As usual, Cf is called the *mapping cone* over f.

Lemma. *Endowed with the suspension functor S and the triangles isomorphic to standard triangles, the category $\mathcal{H}A$ becomes a triangulated category in the sense of Verdier* (7).

In the situation of example 1.1 a), the category $\mathcal{H}A$ identifies with the homotopy category of complexes of right B-modules. To prove the lemma, one may proceed as in (2). Alternatively (3), one can make the category of DG modules into a Frobenius category whose associated stable category identifies with $\mathcal{H}A$, which therefore automatically carries a triangulated structure (1).

1.4 The derived category. A morphism of DG A-modules $s : M \to M'$ is a *quasi-isomorphism* if the induced morphism in homology $\mathrm{H}^*s : \mathrm{H}^*M \to \mathrm{H}^*M'$ is invertible. Here, as always, H^*M denotes the Z-graded k-module with components

$$H^pM = \mathrm{Ker}\,(d : A^p \to A^{p+1})/dA^{p-1}.$$

By definition, the *derived category of A* is the localization (cf. (7))

$$\mathcal{D}A := (\mathcal{H}A)[\Sigma^{-1}]\,,$$

where Σ denotes the class of all (homotopy classes of) quasi-isomorphisms. In the situation of example 1.1 a), the category $\mathcal{D}A$ identifies with the (unbounded) derived category of the category of right B-modules.

It is not hard to check that $\mathcal{D}A$ *has infinite direct sums* and that these are given by the ordinary sums of DG A-modules.

Let A_A denote the free DG A-module on one generator. Let M be any A-module. Then it is easy to check that the map

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, N) \to \mathrm{H}^0 M \ , \ \overline{f} \mapsto \overline{f(1)}$$

is bijective. In particular, each quasi-isomorphism $s : M \to M'$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, M) \xrightarrow{\overline{s}_*} \mathrm{Hom}_{\mathcal{H}A}(A_A, M'). \tag{1}$$

As an immediate consequence, we have a bijection

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, M) \xrightarrow{\sim} \varinjlim \mathrm{Hom}_{\mathcal{H}A}(A_A, M') = \mathrm{Hom}_{\mathcal{D}A}(A_A, M). \tag{2}$$

Here $\varinjlim$ is taken over the filtering category of quasi-isomorphims $\overline{s} : M \to M'$ with domain M. We note in passing that this implies

$$\mathrm{Hom}_{\mathcal{D}A}(A_A, M) \xrightarrow{\sim} \mathrm{H}^0 M \tag{3}$$

A DG A-module sharing the two equivalent properties (1) and (2) with A_A is called *closed* ('having property (P)' in the terminology of (3)). We denote by $\mathcal{H}_p A$ the full subcategory of $\mathcal{H}A$ formed by the closed objects. Property (2) combined with the 5-lemma shows that the mapping cone over a morphism of closed objects is still closed. So $\mathcal{H}_p A$ is a triangulated subcategory of $\mathcal{H}A$.

Proposition.

a) *For each $M \in \mathcal{H}A$, there is a quasi-isomorphism*
$$\boldsymbol{p}M \to M$$
where $\boldsymbol{p}M$ is closed.

b) *The assignment $M \mapsto \boldsymbol{p}M$ may be completed to a triangle functor which commutes with infinite sums and induces a triangle equivalence $\mathcal{D}A \xrightarrow{\sim} \mathcal{H}_p A$.*

c) *$\mathcal{H}_p A$ is the smallest full triangulated subcategory of $\mathcal{H}A$ containing A_A and closed under infinite direct sums*

In the situation of example 1.1 a), if M is concentrated in degree 0, then $\boldsymbol{p}M$ may be chosen as a projective resolution of M^0. If M is a right bounded complex, $\boldsymbol{p}M$ is a 'projective resolution of the complex M' (cf. (2)). For arbitrary M over an 'ordinary' k-algebra, $\boldsymbol{p}M$ is a K-projective resolution in the sense of (6). The proof for an arbitrary DG algebra may be found in (3).

It follows from b) and c) that $\mathcal{D}A$ coincides with its smallest full triangulated subcategory containing A_A and closed under infinite sums. This gives rise to an 'induction principle' as illustrated by the following fact: If $\mathcal{T}$ is a triangulated category admitting infinite sums and $F_1, F_2 : \mathcal{D}A \to \mathcal{T}$ are two triangle functors commuting with infinite sums, then *a morphism* $\mu : F_1 \to F_2$ *of triangle functors is invertible if (and only if)* $\mu A_A : F_1 A_A \to F_2 A_A$ *is invertible.* Indeed, the full subcategory of $\mathcal{D}A$ formed by the DG modules U with invertible μU is a triangulated subcategory by the 5-lemma, contains A_A by assumption, and is closed under infinite sums since F_1 and F_2 commute with infinite sums.

1.5 Left derived tensor functors. Let A and B be DG algebras, and ${}_BX_A$ a DG *B-A-bimodule*, i.e.

$$X = \coprod_{p\in\mathbb{Z}} X^p$$

is a graded left B-module and a graded right A-module, the two actions commute and coincide on k, and X is endowed with a homogeneous k-linear differential d of degree 1 satisfying

$$d(bxa) = (db)\,xa + (-1)^p b\,(dx)\,a + (-1)^{p+q} bx\,(da)$$

for all $b \in B^p$, $x \in X^q$, $a \in A$. We define the DG *algebra* $B^{\mathrm{op}} \otimes A$ by

$$(B^{\mathrm{op}} \otimes A)^n = \coprod_{p+q=n} B^p \otimes A^q\,,$$
$$d(b \otimes a) = (db) \otimes a + (-1)^p b \otimes (da)\,,$$
$$(b \otimes a)(b' \otimes a') = (-1)^{qp'} bb' \otimes aa'\,,$$

for all $b \in B^p$, $a \in A^q$, $b' \in B^{p'}$, $a' \in A$. We may then view X as a (right) DG $B^{\mathrm{op}} \otimes A$-module via

$$x.(b \otimes a) = (-1)^{rp} bxa\,,\ \forall\, x \in X^r\,,\ \forall\, b \in B^p\,,\ \forall\, a \in A.$$

Let N be a DG B-module. We define $N \otimes_k X$ to be the DG A-module with the action of A on X and with the DG structure

$$(N \otimes_k X)^m = \coprod_{p+q=m} N^p \otimes_k X^q\,,\ d(n \otimes x) = (dn) \otimes x + (-1)^p n \otimes (dx)\,,$$

for all $n \in N^p$, $x \in X$. The k-submodule generated by all differences $nb \otimes x - n \otimes bx$ is stable under d and under multiplication by elements of A. So $N \otimes_B X$, the quotient modulo this submodule, is a well defined DG A-module, which is moreover functorial in N and X. The functor $? \otimes_B X$ yields a triangle functor $\mathcal{H}B \to \mathcal{H}A$ which will be denoted by the same symbol. We define the left derived functor

$$? \otimes_B^{\mathbf{L}} X : \mathcal{D}B \to \mathcal{D}A$$

by $N \otimes_B^{\mathbf{L}} X := (\boldsymbol{p}N) \otimes_B X$. Note that $? \otimes_B^{\mathbf{L}} X$ commutes with infinite sums since $\boldsymbol{p}$ and $? \otimes_B X$ do.

Lemma. *The functor $F = ? \otimes_B^{\mathbf{L}} X$ is an equivalence if and only if the following conditions hold*

a) *F induces bijections*
$$\mathrm{Hom}_{\mathcal{D}B}(B, S^n B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(X_A, S^n X_A)\,,\ \forall\, n \in \mathbf{Z}\,,$$
b) *the functor* $\mathrm{Hom}_{\mathcal{D}A}(X_A, ?)$ *commutes with infinite sums,*

c) *the smallest full triangulated subcategory of $\mathcal{D}A$ containing X_A and closed under infinite sums coincides with $\mathcal{D}A$.*

Proof. The conditions are necessary by section 1.4. We prove that they suffice. Consider the full subcategory $\mathcal{V}$ of objects V of $\mathcal{D}B$ for which the maps

$$\mathrm{Hom}_{\mathcal{D}B}(S^n B, V) \to \mathrm{Hom}_{\mathcal{D}A}(S^n X_A, FV)$$

are bijective for all $n \in \mathbf{Z}$. It is clearly closed under S and S^{-1}. By the 5-lemma, it is a triangulated subcategory. By b) it is closed under infinite sums and by a) it contains B_B. Thus we have $\mathcal{V} = \mathcal{D}B$. So the full subcategory $\mathcal{U}$ of objects U such that

$$\mathrm{Hom}_{\mathcal{D}B}(U, V) \to \mathrm{Hom}_{\mathcal{D}A}(FU, FV)$$

is bijective for all $V \in \mathcal{D}B$ contains B_B. It is clearly closed under S and S^{-1}, and under infinite sums. By the 5-lemma, it is a triangulated subcategory. Thus $\mathcal{V} = \mathcal{D}B$, and F is fully faithful. It follows then from c) that F is essentially surjective.

Example. Suppose that $\varphi : A \to B$ is a quasi-isomorphism, i.e. a morphism of DG algebras inducing an isomorphism $\mathrm{H}^*A \xrightarrow{\sim} \mathrm{H}^*B$. We claim that

$$? \otimes_B^{\mathbf{L}} B_A : \mathcal{D}B \to \mathcal{D}A$$

is an equivalence. Indeed B_A is isomorphic to A_A in $\mathcal{D}A$ so that conditions b) and c) of the lemma are clear. For a) we look at the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}B}(B, S^n B) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}A}(B_A, S^n B_A) \\
\sim\uparrow & & \uparrow\sim \\
\mathrm{H}^n B & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{H}A}(A, S^n B_A).
\end{array}$$

Similarly, we claim that

$$? \otimes_A^{\mathbf{L}} B_B : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence. Again conditions b) and c) are clear. For a) we look at the diagram

$$\begin{array}{ccc} \mathrm{H}^n A & \xrightarrow{\sim} & \mathrm{H}^n B \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Hom}_{\mathcal{D}A}(A, S^n A) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}B}(B, S^n B). \end{array}$$

1.6 Compositions. Keep the hypotheses of section 1.5 and assume in addition that C is another DG algebra and ${}_C Y_B$ a DG C-B-bimodule.

Lemma. *If A is flat as a k-module, we have*

$$(? \otimes^{\mathbf{L}}_C Y) \otimes^{\mathbf{L}}_B X \xrightarrow{\sim} ? \otimes^{\mathbf{L}}_C Z$$

for some DG C-A-bimodule Z.

Proof. Let $P = \boldsymbol{p}({}_B X_A)$, where $\boldsymbol{p}$ is the 'resolution functor' associated with the DG algebra $B^{\mathrm{op}} \otimes A$. We claim that the canonical morphism

$$? \otimes^{\mathbf{L}}_B P \to ? \otimes^{\mathbf{L}}_B X$$

is invertible. Indeed, this follows from the 'induction principle' of section 1.4, since the composition

$$P \xrightarrow{\sim} B \otimes_B P \to B \otimes_B X \xrightarrow{\sim} X$$

is clearly an isomorphism. We claim that the morphism

$$N \otimes^{\mathbf{L}}_B P = (\boldsymbol{p}N) \otimes_B P \to N \otimes_B P$$

is invertible in $\mathcal{D}A$ for each $N \in \mathcal{D}B$. This also follows from the 'induction principle' since P is closed over $B^{\mathrm{op}} \otimes_k A$ and the functor $? \otimes_B (B \otimes_k A) \xrightarrow{\sim} ? \otimes_k A$ preserves quasi-isomorphisms by the k-flatness of A. Thus we have

$$(L \otimes^{\mathbf{L}}_B Y) \otimes^{\mathbf{L}}_B X \xrightarrow{\sim} (L \otimes^{\mathbf{L}}_C Y) \otimes_B P \xrightarrow{\sim} (\boldsymbol{p}L) \otimes_C Y \otimes_B P = L \otimes^{\mathbf{L}}_C (Y \otimes_B P).$$

So the assertion holds with $Z = Y \otimes_B P$.

2. Application to tilting theory

Let k be a commutative ring, B a k-algebra and A a flat k-algebra. As an application of the techniques developped in section 1 we give a new proof of J. Rickard's

Theorem. (5) *There is a triangle equivalence $F : \mathcal{D}A \to \mathcal{D}B$ if and only if there is a derived equivalence, i.e. a triangle equivalence of the form $? \otimes^{\mathbf{L}}_B X : \mathcal{D}B \to \mathcal{D}A$ for some complex ${}_B X_A$ of B-A-bimodules.*

Proof. Let $F : \mathcal{D}B \to \mathcal{D}A$ be the given triangle equivalence. Put $T = \boldsymbol{p}(FB_B)$ (cf. proposition 1.4) and $\widetilde{B} = \mathcal{H}om_A(T,T)$ (cf. example 1.1 b). We have canonical isomorphisms

$$\mathrm{H}^n\widetilde{B} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}A}(T, S^nT)\,,\ n \in \mathbf{Z}.$$

Since T is closed in $\mathcal{H}A$, we also have

$$\mathrm{Hom}_{\mathcal{H}A}(T, S^nT) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(T, S^nT) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(B, S^nB).$$

Thus $\mathrm{H}^n\widetilde{B} = 0$ for $n \neq 0$ and $\mathrm{H}^0\widetilde{B}$ identifies with B. If we view A as a DG algebra concentrated in degree 0, we may view T as a DG $\widetilde{B}$-A-bimodule (cf. example 1.2 b). We claim that

$$?\otimes^{\mathbf{L}}_{\widetilde{B}} T : \mathcal{D}\widetilde{B} \to \mathcal{D}A$$

is an equivalence. Since F is an equivalence, conditions b) and c) of lemma 1.5 are clear. For a) we use the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}\widetilde{B}}(\widetilde{B}, S^n\widetilde{B}) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}A}(T, S^nT) \\
\sim\uparrow & & \uparrow\sim \\
\mathrm{H}^nB & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{H}A}(T, S^nT).
\end{array}$$

To establish a connexion between B and $\widetilde{B}$, we introduce the DG subalgebra $C \subset \widetilde{B}$ with $C^n = \widetilde{B}^n$ for $n < 0$, $C = \mathrm{Z}^0\,\widetilde{B}$ and $C^n = 0$ for $n > 0$. Note that we have canonical morphisms

$$B \leftarrow C \to \widetilde{B}$$

which are both quasi-isomorphisms (we view B as a DG algebra concentrated in degree 0). So by example 1.5, we have a chain of equivalences

$$\mathcal{D}B \xrightarrow{\otimes^{\mathbf{L}}_B B} \mathcal{D}C \xrightarrow{\otimes^{\mathbf{L}}_C \widetilde{B}} \mathcal{D}\widetilde{B} \xrightarrow{\otimes^{\mathbf{L}}_{\widetilde{B}} T} \mathcal{D}A.$$

Applying lemma 1.6 twice we get the required complex X of B-A-bimodules.

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Wild Canonical Algebras and Rings of Automorphic Forms

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ABSTRACT: This paper aims to show that the modern representation theory of finite dimensional algebras, usually judged to be of fairly recent origin, has strong roots within classical mathematics. We do justify this claim by exploring the links between

- tame hereditary algebras Σ and simple surface singularities (corresponding to the invariant theory of binary polyhedral groups),
- wild canonical algebras Λ and automorphic forms, respectively.

1 Introduction

While the link between tame hereditary algebras and simple surface singularities is known for some time, and summarized here mainly for analogy, motivation and setting of the proper framework, the objective of the present paper is to show that — for the base field of complex numbers — the representation theory of wild canonical algebras [24], [19] is directly linked to the classical topic of automorphic forms as developed by F. Klein, H. Poincaré and R. Fricke [16], [17] at the turn of the century. Automorphic forms continue to be a topic attracting interest in more recent years as shown by work of J. Milnor [22], W. Neumann [23], I. Dolgacev [9], P. Wagreich [28], [27] and others.

These links are established through an analysis of the behaviour of the Auslander-Reiten translations τ_Σ^- resp. τ_Λ at a suitable indecomposable projective module P over a tame hereditary algebra Σ (resp. a wild canonical algebra Λ). Technically speaking this means to derive from the Auslander-Reiten orbits $\mathcal{O}_- = \left\{\tau_\Sigma^{-n}(P) \mid n \geq 0\right\}$ (resp. $\mathcal{O}_+ = \{\tau_\Lambda^n(P) \mid n \geq 0\}$) a positively $\mathbb{Z}$-graded algebra R, which allows to reconstruct $\mathcal{O}_-$ (resp. $\mathcal{O}_+$) as a category together with the action $\tau_\Sigma^- : \mathcal{O}_- \to \mathcal{O}_-$ (resp. $\tau_\Lambda : \mathcal{O}_+ \to \mathcal{O}_+$). Depending on the case under review, this algebra R turns out to be isomorphic — as a $\mathbb{Z}$-graded algebra — either to the algebra of invariants of the action of a binary polyhedral group on the polynomial algebra $\mathbb{C}[X, Y]$ or else to the algebra of entire automorphic forms with regard to an action of a Fuchsian group of the first kind on the upper complex half plane.

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The notion, we are using here of the *algebra* $\mathbb{A}(F;X)$ *of an endofunctor* $F : \mathcal{A} \to \mathcal{A}$ *at an object* X *of* $\mathcal{A}$ (see section 2) can be thought of as generalizing the notion of the tensor algebra of a bimodule and thus seems to be of independent interest. This device has some tradition in algebraic geometry; in representation theory it was used in [3] to establish the general ring-theoretic properties of the *preprojective algebra*(s) (see [12], [8], [5]) attached to a tame hereditary algebra.

We will approach the topic from two antipodal points of view: Given a *weight sequence* $\mathbf{p} = (p_1, \ldots, p_t)$ and a *parameter sequence* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct elements from the projective line $\mathbf{P}_1(k)$, Section 4 deals with the representations of a canonical algebra $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ over a base field of arbitrary characteristic. Section 6 deals with the *marked Riemann surfaces* arising as quotients $\mathbf{X} = \mathbb{H}_+/G$ of an action of a *Fuchsian group* G *of the first kind* on the upper complex half plane $\mathbb{H}_+$. Such a quotient can be thought of as being a Riemann sphere $\mathbf{P}_1(\mathbb{C})$ with a finite set of "cusps" of orders $p_1, \ldots, p_t$ located in a set of (pairwise distinct) points $\lambda_1, \ldots, \lambda_t$. The link between both views is established through the notion of a weighted projective line [10] attached to the data $(\mathbf{p}, \underline{\lambda})$.

Our results underline the importance of the *canonical algebras*, a class of algebras introduced by C. M. Ringel [24], and further studied through the sheaf theory on the *attached weighted projective line* in [10] and [19]. The $\mathbb{Z}$-graded algebras $\mathbb{A}(\tau_\Sigma; P)$ corresponding to the wild canonical algebras can be formed over an algebraically closed base field of arbitrary characteristic and therefore might be viewed as proper analogues for *algebras of automorphic forms in characteristic* $p > 0$.

Throughout, D refers to formation of the k-dual $\mathrm{Hom}(-, k)$. For a finite dimensional algebra Λ we denote by $\mathrm{mod}(\Lambda)$ ($\underline{\mathrm{mod}}(\Lambda)$, $\overline{\mathrm{mod}}(\Lambda)$) the category of finite dimensional Λ-modules and the corresponding stable categories modulo projectives (resp. injectives); accordingly we denote the morphism spaces in the two stable categories by $\underline{\mathrm{Hom}}_\Lambda(M, N)$ and $\overline{\mathrm{Hom}}_\Lambda(M, N)$. $\mathrm{Tr} : \underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda)$ denotes the transpose.

For an abelian group H a H-graded k-algebra R is equipped with a decomposition $R = \bigoplus_{h \in H} R_h$ of finite dimensional k-spaces R_h such that $R_h R_l \subseteq R_{h+l}$ holds for all $h, l \in H$. A graded R-module M carries a decomposition $M = \bigoplus_{h \in H} M_h$ such that the M_h are finite dimensional k-spaces and $R_h M_l \subseteq M_{h+l}$ holds. H acts on graded modules through shift $(h, M) \mapsto M(h)$, where $M(h)_l = M_{h+l}$. The *companion category* $\mathcal{C} = [H; R]$ of R is the category whose objects are the elements from H, morphisms are given by the rule $\mathcal{C}(m, n) = R_{n-m}$, composition in $\mathcal{C}$ is given by the multiplication from R. For a subset $U \subseteq H$ we denote $[U; R]$, also referred to as companion category, the full subcategory of $[H; R]$ formed by all objects $u \in U$.

A two-sided graded notherian algebra R (a finitely generated graded R-module M) will be called *Cohen-Macaulay* (CM for short) if the condition $\mathrm{Ext}^i_R(S, R) = 0$ (resp. $\mathrm{Ext}^i_R(S, M) = 0$) is satisfied for $i = 0, 1$ and all graded simple R-modules S, where the Ext's are to be taken in the category of graded R-modules. $\mathrm{CM}^H(R)$ denotes the category of all H-graded Cohen-Macaulay modules over R. If additionally the minimal injective resolution of R in the category of graded R-modules is finite then R is called *graded Gorenstein*.

2 The local analysis of an endofunctor

Let k be a field. By a k-category $\mathcal{A}$ we mean an additive category whose Hom-sets carry the structure of (finite dimensional) k-vectorspaces and composition is k-bilinear. For the applications we have in mind, $\mathcal{A}$ could be primarily thought of as a full subcategory of either the category $\mathrm{mod}(\Lambda)$ of finite dimensional representations of a finite dimensional k-algebra Λ or else the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves over a projective — possibly weighted — projective variety.

For a k-linear endofunctor $F : \mathcal{A} \longrightarrow \mathcal{A}$ (k-linear means that the mappings $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$, $u \mapsto F(u)$, induced by F are k-linear) we fix an object A of $\mathcal{A}$ and study the local properties of F at A by forming the *F-orbit of A*, i.e. the full subcategory $\mathcal{O}_+$ of $\mathrm{mod}(\Lambda)$ formed by the sequence

$$A, FA, F^2A, \ldots, F^nA, \ldots.$$

A convenient tool for such an analysis is the *algebra of F at A*, which is defined to be the positively $\mathbb{Z}$-graded algebra given by

$$\mathbb{A}(F; A) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathcal{A}}(A, F^nA),$$

where the product of homogeneous elements $u : A \to F^nA$ of degree n and $v : A \to F^mA$ of degree m is given by the composition

$$v\,u = [A \xrightarrow{u} F^nA \xrightarrow{F^nv} F^{n+m}A].$$

Forming the *companion category* $[\mathbb{N}; R]$ of $R = \mathbb{A}(F; A)$ often leads back to the F-orbit $\mathcal{O}_+ = \{F^nA \mid n \geq 0\}$ of A. Assume that $\mathrm{Hom}(F^nA, F^mA) = 0$ for all $n > m$ and further F induces isomorphisms $\mathrm{Hom}(F^nA, F^mA) \to \mathrm{Hom}(F^{n+1}A, F^{m+1}A)$ for all $n \leq m$. In this case the correspondence $n \mapsto F^n(A)$ extends to an equivalence from $[\mathbb{N}; R]$ to $\mathcal{O}_+$; moreover under this equivalence the obvious action on $[\mathbb{N}; R]$, given on objects by the correspondence $n \mapsto n + 1$, corresponds to the action of F on $\mathcal{O}_+$.

As a result the graded algebra $\mathbb{A}(F; A)$ reflects the *local properties of F at A* sensibly; however, the actual calculation is usually difficult. In case $\Lambda \in \mathcal{A} \subseteq \mathrm{mod}(\Lambda)$, formation of $\mathbb{A}(F; \Lambda)$ moreover shows that $\bigoplus_{n=0}^{\infty} F^n(\Lambda)$ carries a natural structure of a $\mathbb{Z}$-graded algebra, an observation we are going to use repeatedly.

Example 2.1 $\mathcal{A} = \mathrm{mod}(k)$, the category of finite dimensional k-vectorspaces, F the identity functor. Then

$$\mathbb{A}(F; k) = k[X]$$

is the polynomial algebra in one indeterminate over k.

This is a particular case of a general result showing besides that in general $\mathbb{A}(F; X)$ is far from being commutative.

Example 2.2 $\mathcal{A} = \text{mod}(\Lambda)$ the category of finite dimensional modules over a finite dimensional k-algebra Λ, $F : \text{mod}(\Lambda) \longrightarrow \text{mod}(\Lambda)$ any *right exact functor*. Taking $M = F(\Lambda)$ we obtain a (Λ, Λ)-bimodule M such that F — up to isomorphism of functors — is the functor $F = - \otimes_R M$ given by tensoring.

The algebra of F at Λ is the *tensor algebra* of M

$$\mathsf{A}(F; \Lambda) = \bigoplus_{i=0}^{\infty} T^n M, \quad \text{where } T^n M = \overbrace{M \otimes M \otimes \cdots \otimes M}^{n \text{ times}}.$$

It is thus appropriate to view the algebras $\mathsf{A}(F; X)$ as a generalization of the tensor algebras of bimodules.

We next turn to a local analysis of the *Auslander-Reiten translates* $\tau_\Lambda = \text{DTr}$ and $\tau_\Lambda^- = \text{TrD}$ relating the end terms of almost-split sequences in $\text{mod}(\Lambda)$. Unfortunately, $\tau_\Lambda : \underline{\text{mod}}(\Lambda) \to \overline{\text{mod}}(\Lambda)$ and $\tau_\Lambda^- : \overline{\text{mod}}(\Lambda) \to \underline{\text{mod}}(\Lambda)$ do not have an interpretation as endofunctors $\text{mod}(\Lambda) \to \text{mod}(\Lambda)$ unless we deal with a *hereditary* algebra Σ. In this case $\tau_\Sigma = \text{D}\,\text{Ext}^1_\Sigma(-, \Sigma)$ and $\tau_\Sigma^- = \text{Ext}^1_\Sigma(\text{D}\,-, \Sigma)$. Recall that $M \in \text{mod}(\Sigma)$ is called *preprojective* if it is a direct sum of modules of the form $\tau_\Sigma^{-n} P$, P indecomposable projective. In case Σ is tame hereditary there is a linear form $\text{K}_0(\Lambda) \to \mathbb{Z}$, called *defect* on the Grothendieck group of Σ uniquely defined by the properties to be surjective and to be constant < 0 on each τ_Σ^--orbit of an indecomposable projective module [6].

Formation of $\mathsf{A}(\tau_\Sigma^-; \Sigma)$ (resp. $\mathsf{A}(\tau_\Sigma^-; P)$ if Σ is additionally tame and P indecomposable projective of defect -1) leads to the *preprojective algebras* $\Pi(\Sigma)$ (resp. $\Pi'(\Sigma)$) introduced by I. M. Gelfand and V. A. Ponomarev [12] and further studied in [8], [3], [11]. O. Kerner has recently obtained interesting results for the algebras $\mathsf{A}(\tau_\Sigma; M)$ for Σ wild hereditary and M an indecomposable regular Σ-module[14].

Remark 2.3 Assume Λ has finite global dimension. According to D. Happel [13] the derived category $\text{D}^b(\Lambda)$ of bounded complexes of $\text{mod}(\Lambda)$ then has Auslander-Reiten triangles, moreover the Auslander-Reiten translation τ, relating the end-terms of Auslander-Reiten triangles in $\text{D}^b(\Lambda)$, is an automorphism $\tau : \text{D}^b(\Lambda) \to \text{D}^b(\Lambda)$. Thus passing from $\text{mod}(\Lambda)$ to $\text{D}^b(\Lambda)$ seems to be the natural device to circumvent the difficulties viewing Auslander-Reiten translation as an endofunctor of $\text{mod}(\Lambda)$. Assuming Λ to be of finite global dimension thus allows to form algebras $\mathsf{A}(\tau; X)$ for any $X \in \text{D}^b(\Lambda)$, in particular for any $X \in \text{mod}(\Lambda)$. Actually the examples discussed in this and the next two sections would keep their full meaning passing to this new setting.

As a possible alternative we point to work of E. Green (for algebras given by quivers and relations, unpublished) and Auslander-Reiten (for arbitrary finite dimensional algebras) showing that it is possible — however not canonically — to lift Auslander-Reiten translation $\text{DTr} : \underline{\text{mod}}(\Lambda) \to \overline{\text{mod}}(\Lambda)$ to an endofunctor $\text{mod}(\Lambda) \to \text{mod}(\Lambda)$, which is moreover left-exact.

The theorem below summarises the main known features of $\Pi(\Sigma)$ for Σ finite dimensional hereditary.

Theorem 2.4 (**[3]**) *Let Σ be a hereditary (basic, connected) algebra over an arbitrary base field k and $\Pi = \Pi(\Sigma)$ be the algebra $\mathsf{A}(\tau_\Sigma^-; \Sigma)$ of the (inverse) Auslander-Reiten translation $\tau_\Sigma^- : \mathrm{mod}(\Sigma) \to \mathrm{mod}(\Sigma)$ at Σ. Then the following assertions hold:*

(*i*) *The algebra Π, viewed as a module over $\Pi_0 = \Sigma$, is the direct sum of a representative system of indecomposable preprojective Σ-modules*[1].

(*ii*) Σ *is* representation-finite *if and only if* $\Pi(\Sigma)$ *is finite-dimensional over* k.

(*iii*) Σ *is* tame *if and only if* Π *is two-sided noetherian. In this case* Π *and* $\Pi' = \Pi'(\Sigma)$ *have the following additional properties*

(*a*) Π *(resp. Π') is a finitely generated k-algebra satisfying a polynomial identity.*

(*c*) Π *(resp. Π') is graded Cohen-Macaulay of Krull dimension two, moreover the graded height one prime ideals of Π (resp. Π') are in natural one-to-one correspondence with the set of regular components of Σ.*

(*d*) Π *(resp. Π') satisfies the Ore condition for regular homogeneous elements. The corresponding ring of homogeneous fractions of degree zero is a full matrix algebra over a division ring D, finite dimensional over its center K, further K/k is an algebraic function field in one variable of genus zero.*

(*iv*) *The following assertions are equivalent:*

(*a*) Σ *is* wild*;*

(*b*) Π *does not satisfy a polynomial identity;*

(*c*) Π *is not noetherian.* □

Notice further that — assuming Σ tame hereditary — there is a major difference in the homological properties of Π and Π' because Π has global dimension two, whereas the global dimension of Π' is usually infinite. However, both Π and Π' always do have graded selfinjective dimension two, so are graded *Gorenstein*.

3 Simple singularities, viewed algebraically

For simplicity we assume that the base field k is algebraically closed; no assumption on the characteristic is made, unless explicitly stated otherwise. Note that $\Pi(\Sigma)$ is always generated by elements of degree one, an assertion not carrying over to $\Pi'(\Sigma)$ as is shown by example 3.1 below. As indicated by theorem 2.4, the study of $\Pi(\Sigma)$, accordingly of $\Pi'(\Sigma)$, seems to be especially promising in case Σ is tame hereditary.

Example 3.1 Let Σ be the *path algebra of the extended Dynkin quiver* $\tilde{\mathsf{E}}_8$ with orientation given by

$$\begin{tikzcd} & & \circ \arrow[d] & & & & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

[1]For this reason $\Pi(\Sigma)$ was termed by Dlab and Ringel [8] the *preprojective algebra of* Σ.

Let P be the simple projective right Σ-module attached to the marked vertex. Then surprisingly the graded algebra $\mathbb{A}(\tau_\Sigma^-; P)$ is commutative, indeed isomorphic to

$$k[x,y,z] = k[X,Y,Z]/(X^2+Y^3+Z^5), \quad \deg(x,y,z) = (15,10,6).$$

Hence $\mathbb{A}(\tau_\Sigma^-; P)$ is isomorphic to the algebra of invariants $\mathbb{C}[X,Y]^G$ of the natural action of the binary icosahedral group $G \subseteq \mathbb{SL}(2,\mathbb{C})$ on the binary polynomial algebra over $\mathbb{C}$. Correspondingly, the completion of $\mathbb{A}(\tau_\Sigma^-; P)$ is isomorphic to the *simple surface singularity* $k[[X,Y,Z]]/(X^2+Y^3+Z^5)$ of type $\mathbb{E}_8$.

This is a particular case of a theorem, dealing with all the Dynkin cases of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$.

Theorem 3.2 ([11]) *Let Δ be a Dynkin diagram and $\Sigma = k[\Gamma]$ be the path algebra of an oriented quiver Γ of extended Dynkin type $\tilde{\Delta}$. No assumption is made on the base field k. Let P be an (indecomposable) projective Σ module of defect -1, then the $\mathbb{Z}$-graded algebra $\mathbb{A}(\tau_\Sigma^-; P)$ of $\tau_\Sigma^- = \mathrm{TrD}$ at P has the form $k[x,y,z] = k[X,Y,Z]/(F)$, where the relation F and the degree triple of the generators x,y,y are shown in the following list:*

Dynkin type Δ	relation F	deg (x,y,z)	
$\mathbb{A}_{pq} \quad = (p,q)$	$X^{p+q} - Y\,Z$	$(1,p,q)$	
$\mathbb{D}_{2l+2} = (2,2,2l)$	$Z^2 + X(Y^2 + Y\,X^l)$	$(2,2l,2l+1)$	•
$\mathbb{D}_{2l+3} = (2,2,2l+1)$	$Z^2 + X(Y^2 + Z\,X^l)$	$(2,2l+1,2l+2)$	•
$\mathbb{E}_6 \quad = (2,3,3)$	$Z^2 + Y^3 + X^2\,Z$	$(3,4,6)$	•
$\mathbb{E}_7 \quad = (2,3,4)$	$Z^2 + Y^3 + X^3\,Y$	$(4,6,9)$	
$\mathbb{E}_8 \quad = (2,3,5)$	$Z^2 + Y^3 + X^5$	$(6,10,15)$	

Sketch of Proof. One first proves that Σ can be realized as a tilting bundle on a weighted projective line $\mathbb{X}$ [10] of weight type $(p_1,p_2,p_3) = \Delta$, whose sheaf theory is derived from the algebra $S = k[x_1,x_2,x_3] = k[X_1,X_2,X_3]/(X_1^{p_1}+X_2^{p_2}+X_3^{p_3})$ thought to be graded by the rank one abelian group $\mathbf{L}$ on generators $\vec{x}_1, \vec{x}_2, \vec{x}_3$ with relations $p_1\vec{x}_1 = p_2\vec{x}_2 = p_3\vec{x}_3 (= \vec{c})$. This $\mathbf{L}$-grading is achieved by setting $\deg(x_i) = \vec{x}_i$. From these facts it is straightforward to deduce that the category $\mathcal{P}'(\Sigma)$ consisting of a representative system of preprojective Σ-modules of defect -1 is — up to removal of finitely many objects — equivalent to the companion category $[\mathbf{L}_+, S]$ of S, where $\mathbf{L}_+ = \sum_{i=1}^3 \mathbb{N}\vec{x}_i$. Exploiting Serre duality on coh($\mathbb{X}$) it follows that (inverse) Auslander-Reiten translation $\tau_\Sigma^- : \mathcal{P}'(\Sigma) \to \mathcal{P}'(\Sigma)$ corresponds to the shift $\vec{x} \mapsto \vec{x} - \vec{\omega}$ in the companion category where $\vec{\omega} = \vec{c} - (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)$.

This proves that $\Pi'(\Sigma)$ is isomorphic to the subalgebra R of S obtained by restricting the grading to the subgroup generated by $(-\vec{\omega})$, i.e. $R = \bigoplus_{n=0}^{\infty} S_{-n\vec{\omega}}$. From the explicit structure of S it is then straightforward to derive in each case a triple of generators x,y,z for R

Dynkin type	generators (x,y,z)
(p,q)	$(x_0\,x_1, x_1^{p+q}, x_0^{p+q})$
$(2,2,2l)$	$(x_2^2, x_0^2, x_0\,x_1\,x_2)$
$(2,2,2l+1)$	$(x_2^2, x_0\,x_1, x_0^2\,x_2)$
$(2,3,3)$	$(x_0, x_1\,x_2, x_1^3)$
$(2,3,4)$	$(x_1, x_2^2, x_0\,x_2)$
$(2,3,5)$	(x_2, x_1, x_0)

having all the properties listed in the theorem. □

Recall that a *binary polyhedral group* G is a finite subgroup of $\mathbb{SL}(2,\mathbb{C})$, and — through the notion of the attached *McKay graph* [25] — the action $G \times \mathbb{C}^2 \to \mathbb{C}^2$ is determined — up to isomorphism — by one of the Dynkin types $\mathbb{A}_{pq}$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. The corresponding action of G on the symmetric algebra $\mathbb{C}[X,Y]$ of $\mathbb{C}^2$ preserves the grading thus leads to a $\mathbb{Z}$-graded algebra $\mathbb{C}[X,Y]^G$ of binary G-invariants, first determined by F. Klein [15]. We refer to the surveys [25] and [4] for further information on the link to singularity theory.

Corollary 3.3 *Assume $k = \mathbb{C}$. For a Dynkin diagram Δ let Σ be a tame hereditary algebra of extended Dynkin type $\tilde{\Delta}$. Then the algebra $\mathbb{A}(\tau_\Sigma^-; P)$, P projective over Σ of defect -1, as a positively $\mathbb{Z}$-graded algebra is isomorphic to the algebra of invariants $\mathbb{C}[X,Y]^G$, where $G \subseteq \mathbb{SL}(2,\mathbb{C})$ is a binary polyhedral group of Dynkin type Δ.*

Accordingly the completion of the graded algebra $\mathbb{A}(\tau_\Sigma^-; P)$ is isomorphic to the surface singularity of type Δ.

Proof. Let G^* denote the character group of G, let $\chi \in G^*$ and define $T_{(n,\chi)}$ to consist of all homogeneous polynomials $f \in \mathbb{C}[X,Y]$ of degree n satisfying $\sigma.f = \chi(\sigma)f$ for each $\sigma \in G$. The arising *algebra* $T = \mathbb{C}[X,Y]^{G,rel} = \bigoplus_{(n,\chi)} T_{(n,\chi)}$ *of relative G-invariants* thus has a natural $\mathbb{Z} \times G^*$-grading. Because G has Dynkin type $\Delta = (p_1, p_2, p_3)$, i.e. $1/p_1 + 1/p_2 + 1/p_3 > 1$, we can present G as the group on generators $\sigma_1, \sigma_2, \sigma_3$ with relations $\sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = \sigma_1\sigma_2\sigma_3$. Switching to additive notation shows that G^* is the (finite) abelian group on generators χ_1, χ_2, χ_3 with relations $p_1\chi_1 = p_2\chi_2 = p_3\chi_3 = \chi_1 + \chi_2 + \chi_3$. Accordingly the group $\mathbf{L}$ defined above can be viewed as a subgroup of $\mathbb{Z} \times G^*$ by mapping $\vec{x}_i$ to $(p/p_i, \chi_i)$, $p = \text{l.c.m.}(p_1, p_2, p_3)$, with $\mathbb{Z}\vec{\omega}$ corresponding to $\mathbb{Z} \times 0$ under this embedding. (Due to additive notation 0 means the trivial character on G.)

Setting $S = \mathbb{C}[x_1, x_2, x_3] = \mathbb{C}[X_1, X_2, X_3]/(X_1^{p_1} + X_2^{p_2} + X_3^{p_3})$ Klein [15] (see also [26]) has established an isomorphism $S \to T$, actually compatible with the gradings of S and T. Hence the algebra R obtained from S by restricting the grading to the subgroup $\mathbb{Z}\vec{\omega}$ of $\mathbf{L}$ is isomorphic to the algebra obtained from T by restricting the grading to the subgroup $\mathbb{Z} \times 0$, i.e. to the algebra $\mathbb{C}[X,Y]^G$ of binary G-invariants. □

The • in the table from theorem 3.2 marks those places where the form of the relations usually given for the singularities F ($\mathbb{E}_6$: $Z^2 + Y^3 + X^4$, $\mathbb{D}_n$: $Z^2 + X(Y^2 - X^n)$, see [15], [1] or [25]) deviates from that given here; in characteristic $\neq 2$ an easy parameter change shows that both forms are actually equivalent.

The relationship between Σ, Π and Π' is quite close as shown by the next proposition. Assume Σ to be tame hereditary, basic and connected. Let $\mathcal{P}(\Sigma)$ resp. $\mathcal{I}(\Sigma)$ denote the categories of indecomposable preprojective (resp. preinjective) Σ-modules. We form the full subcategory $\mathcal{B} = \mathcal{I}(\Sigma)[-1] \cup \mathcal{P}(\Sigma)$ of the derived category $\mathrm{D}^b(\Sigma)$ of bounded complexes. In elementary terms an object of $\mathcal{B}$ is either an indecomposable preprojective or an indecomposable preinjective module, morphisms are given by the rule

$$\mathrm{Hom}_{\mathcal{B}}(M,N) = \begin{cases} \mathrm{Hom}_\Sigma(M,N) & \text{if } M,\ N \text{ are both in } \mathcal{P}(\Sigma) \text{ resp. } \mathcal{I}(\Sigma), \\ 0 & \text{if } M \in \mathcal{P}(\Sigma) \text{ and } N \in \mathcal{I}(\Sigma), \\ \mathrm{Ext}^1_\Sigma(M,N) & \text{if } M \in \mathcal{I}(\Sigma) \text{ and } N \in \mathcal{P}(\Sigma), \end{cases}$$

and composition in $\mathcal{B}$ is given by Yoneda-composition. The Auslander-Reiten translation τ of $\mathrm{D}^b(\Sigma)$ induces an equivalence $\tau : \mathcal{B} \to \mathcal{B}$, that agrees on all objects from $\mathcal{B}$ with $\tau_\Sigma = \mathrm{DTr}$, except on indecomposable projective modules P, where τ acts as the Nakayama functor sending the projective hull of a simple Σ-module S to the injective hull of S.

Theorem 3.4 *Let Σ be the path algebra of an extended Dynkin quiver $\widetilde{\Delta}$, where $\Delta = (p_1, p_2, p_3)$ is a Dynkin diagram. Then the following five categories are equivalent:*

1. $\mathcal{B}(\Sigma) = \mathcal{I}(\Sigma)[-1] \cup \mathcal{P}(\Sigma)$ *thought of as a full subcategory of $\mathcal{D}^b\Sigma$,*
2. $\operatorname{ind}\mathrm{CM}^{\mathbf{L}}(S)$, *the category of indecomposable $\mathbf{L}$-graded CM-modules over S,*
3. $\operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(\Pi)$, *the category of indecomposable $\mathbb{Z}$-graded CM-modules over Π,*
4. $\operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(\Pi')$, *the category of indecomposable $\mathbb{Z}$-graded CM-modules over Π',*
5. $\operatorname{ind}\mathrm{vect}(\mathbb{X})$, *the category of indecomposable vector bundles on a weighted projective line $\mathbb{X}$ of weight type (p_1, p_2, p_3).*

Proof. By theorem 3.2 Π' arises by restriction of the $\mathbf{L}$-grading from S to the subgroup $\mathbb{Z}\vec{\omega}$; the corresponding restriction $\mathrm{CM}^{\mathbf{L}}(S) \to \mathrm{CM}^{\mathbb{Z}}(\Pi')$, $\bigoplus_{\vec{x}\in\mathbf{L}} M_{\vec{x}} \mapsto \bigoplus_{n\in\mathbb{Z}} M_{n(-\vec{\omega})}$, for CM-modules induces an equivalence [11]. By a combination of [3] and [11] the restriction $\mathrm{CM}^{\mathbb{Z}}(\Pi) \to \mathrm{CM}^{\mathbb{Z}}(\Pi')$ corresponding to inclusion $\Pi' \subseteq \Pi$ is an equivalence as well which proves that the categories labelled 2, 3, 4 are naturally equivalent.

Assume $M \in \mathcal{B}$; interpreting elements $r \in \Pi_n$ as morphisms $r : \tau^n\Sigma \to \Sigma$ in $\mathcal{B}$, we obtain on $\bar{M} = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(\tau^n\Sigma, M)$ the structure of a graded Π-module. This process induces an equivalence between $\mathcal{B}$ and the category $\operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(\Pi)$.

Finally sheafification $\mathrm{mod}^{\mathbf{L}}(S) \to \mathrm{coh}(\mathbb{X})$, $M \mapsto \widetilde{M}$, induces an equivalence $\mathrm{CM}^{\mathbf{L}}(S) \to \mathrm{vect}(\mathbb{X})$ [11]. □

The setting — set-up sofar — allows to extend the relationship between tame hereditary algebras Σ and attached surface singularities $\widehat{R}$, $R = \mathsf{A}(\tau_\Sigma^-; \Sigma)$ to a correspondence relating modules over Σ with Cohen Macaulay modules over the singularity $\widehat{R}$, viewing in this way the category $\mathcal{I}(\Sigma)[-1] \cup \mathcal{P}(\Sigma)$ of representations as a covering of the category $\mathrm{CM}(\widehat{R})$ of Cohen Macaulay modules with covering group $\mathbb{Z}$. This follows from the properties of the completion functor established by Auslander and Reiten [2].

Corollary 3.5 *Completion $\mathrm{CM}^{\mathbb{Z}}(\Pi') \to \mathrm{CM}(\widehat{\Pi'})$ leads to a functor $\pi : \mathcal{I}(\Sigma)[-1] \cup \mathcal{P}(\Sigma) \to \mathrm{CM}(\widehat{\Pi'})$ preserving indecomposability and Auslander-Reiten sequences. Moreover, π is dense i.e. surjective on isomorphism classes of objects, and $\pi(M) \cong \pi(N)$ holds if and only if $M \cong \tau^n N$ for some $n \in \mathbb{Z}$.* □

For an algebraically closed base field theorem 3.2 gives the complete structure of the preprojective algebras $\Pi'(\Sigma)$, Σ tame hereditary. The knowledge is much less complete if the base field is not algebraically closed. From the few cases known, we just mention the following two showing also that — in general — $\Pi'(\Sigma)$ needs not be commutative. The completions $\widehat{\Pi}$ belong to the class of two-dimensional tame orders studied in [5].

Example 3.6 Let $\Sigma = \begin{bmatrix} D & 0 \\ k & D \end{bmatrix}$ for a finite dimensional division algebra D over k.

(*i*) Let $k = \mathbb{R}$ and $D = \mathbb{H}$ be the algebra of real quaternions. Then Σ is tame hereditary, and [7]

$$\Pi'(\Sigma) = \mathbb{R}[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2), \quad \text{with } \deg(X_1, X_2, X_3) = (1, 1, 1).$$

(*ii*) Let $k = \mathbb{Q}$ and $D = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then Σ is tame hereditary and

$$\Pi'(\Sigma) = \mathbb{Q} < X, Y, Z > /(XY - YX, YZ - ZY, XZ + ZX, Z^2 + 3X^2 - 2Y^2)$$

where $\mathbb{Q} < X, Y, Z >$ denotes the free algebra in three indeterminates, and $\deg(X, Y, Z) = (1, 1, 1)$.

4 Modules of rank one over a canonical algebra

For most of this section k is an algebraically closed field of arbitrary characteristic.

To a *weight sequence* $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers and a *parameter sequence* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct elements of the projective line $\mathbf{P}_1(k) = k \cup \{\infty\}$ we associate a finite dimensional algebra $\Lambda = \Lambda(p, \underline{\lambda})$ as follows. Without loss of generality we assume that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$.

We introduce the rank one abelian group $\mathbf{L}(\mathbf{p})$ on generators $\vec{x}_1, \ldots, \vec{x}_t$ with relations $(c :=) p_1\vec{x}_1 = \cdots = p_t\vec{x}_t$ and consider right modules over the canonical algebra $\Lambda = \Lambda(p, \underline{\lambda})$, which is given by the quiver

$$\begin{tikzcd}
 & \vec{x}_1 \arrow[r, "x_1"] & \vec{x}_1 \arrow[r, "x_1"] & \cdots \arrow[r, "x_1"] & (p_1\text{-}1)\vec{x}_1 \arrow[dr, "x_1"] & \\
\vec{0} \arrow[ur, "x_1"] \arrow[r, "x_2"] \arrow[ddr, "x_t"'] & 1\vec{x}_2 \arrow[r, "x_2"] & 2\vec{x}_2 \arrow[r, "x_2"] & \cdots \arrow[r, "x_2"] & (p_2\text{-}1)\vec{x}_2 \arrow[r, "x_2"] & \vec{c} \\
 & \vdots & \vdots & & \vdots & \\
 & \vec{x}_1 \arrow[r, "x_t"] & 2\vec{x}_t \arrow[r, "x_t"] & \cdots \arrow[r, "x_t"] & (p_t\text{-}1)\vec{x}_t \arrow[uur, "x_t"'] &
\end{tikzcd}$$

subject to the $(t-2)$ relations $(*)$

$$\begin{array}{ccc} x_3^{p_3} & = & x_2^{p_2} - \lambda_3 x_1^{p_1} \\ \vdots & & \vdots \\ x_t^{p_t} & = & x_2^{p_2} - \lambda_t x_1^{p_1} \end{array}$$

Thus a right Λ-module may be viewed as a system of finite dimensional k-spaces and k-linear maps

$$\begin{tikzcd}
 & M_{\vec{x}_1} \arrow[dl, "x_1"'] & M_{2\vec{x}_1} \arrow[l, "x_1"'] & \cdots \arrow[l, "x_1"'] & M_{(p_1\text{-}1)\vec{x}_1} \arrow[l, "x_1"'] & \\
M_{\vec{0}} & M_{\vec{x}_2} \arrow[l, "x_2"'] & M_{2\vec{x}_2} \arrow[l, "x_2"'] & \cdots \arrow[l, "x_2"'] & M_{(p_2\text{-}1)\vec{x}_2} \arrow[l, "x_2"'] & M_{\vec{c}} \arrow[l, "x_2"'] \arrow[ul, "x_1"'] \arrow[ddl, "x_t"] \\
 & \vdots & \vdots & & \vdots & \\
 & M_{t\vec{x}_1} \arrow[uul, "x_t"] & M_{2\vec{x}_t} \arrow[l, "x_t"'] & \cdots \arrow[l, "x_t"'] & M_{(p_t\text{-}1)\vec{x}_t} \arrow[l, "x_t"'] &
\end{tikzcd}$$

subject to same system (*) of relations.

Definition 4.1 *The integer* $\dim_k M_{\vec{0}} - \dim_k M_{\vec{c}}$ *is called the* rank of M. *By* $\mathrm{mod}_+(\Lambda)$ *(*$\mathrm{mod}_0(\Lambda)$, $\mathrm{mod}_-(\Lambda)$*) we denote the full subcategory of* $\mathrm{mod}(\Lambda)$ *consisting of finite direct sums of indecomposable modules of rank* > 0 *(rank* $= 0$*, resp. rank* < 0*).*

Thus each indecomposable Λ-module is either in $\mathrm{mod}_+(\Lambda)$, $\mathrm{mod}_0(\Lambda)$ or $\mathrm{mod}_-(\Lambda)$. All the projective (resp. injective) modules ly in $\mathrm{mod}_+(\Lambda)$ (resp. $\mathrm{mod}_-(\Lambda)$). Clearly formation of the k-dual induces a duality $D : \mathrm{mod}_+(\Lambda) \to \mathrm{mod}_-(\Lambda)$. If $\mathbb{X}$ is the weighted projective line attached to the data $(\mathbf{p}, \underline{\lambda})$ — which may be thought to be the projective line $\mathbf{P}_1(k)$ with the points $\lambda_1, \ldots, \lambda_t$ marked by the weights $p_1, \ldots, p_t$ — the category $\mathrm{mod}_0(\Lambda)$ which is an exact (abelian) subcategory of $\mathrm{mod}(\Lambda)$ decomposes into an $\mathbb{X}$-parametrized family of uniserial categories $\mathcal{U}_\lambda$ (tubes), $\lambda \in \mathbb{X}$, where the number of simples objects in $\mathcal{U}_\lambda$ (=rank of the corresponding tube) equals the weight of λ which is 1 if $\lambda \notin \{\lambda_1, \ldots, \lambda_t\}$ and p_i for $\lambda = \lambda_i$. Moreover, there are no non-zero homomorphisms from $\mathrm{mod}_-(\Lambda)$ to $\mathrm{mod}_0(\Lambda)$ and $\mathrm{mod}_+(\Lambda)$, respectively from $\mathrm{mod}_0(\Lambda)$ to $\mathrm{mod}_+(\Lambda)$.

The three subcategories are closed under extensions and almost-split sequences, hence under the Auslander-Reiten translations τ_Λ and τ_Λ^-. The combinatorial structure of the Auslander-Reiten components for Λ is completely known [19]. By contrast, only little information is available on the properties of the Auslander-Reiten components as categories, except the growth behaviour [19].

Of particular importance are the Λ-modules of rank one. We view the algebra

$$S = k[x_1, \ldots, x_t] = k[X_1, \ldots, X_t]/(\rho_3, \ldots, \rho_t), \quad \rho_i = x_i^{p_i} - (x_2^{p_2} - \lambda_i x_1^{p_1})$$

as a $\mathbf{L}(\mathbf{p})$-graded algebra, with the grading specified by

$$\deg(x_i) = \vec{x}_i \quad \text{for } i = 1, \ldots, t.$$

Thus

$$S = \bigoplus_{\vec{x} \in \mathbf{L}(\mathbf{p})} S_{\vec{x}}, \quad S_{\vec{x}} = x_1^{\ell_1} \cdots x_t^{\ell_t} H_\ell$$

where in the formula above we assume that the element

$$\vec{x} = \sum_{i=1}^{t} \vec{x}_i + \ell\vec{c}, \quad 0 \le \ell_i < p_i, \ \ell \in \mathbb{Z},$$

is represented in socalled *normal form*, and where H_ℓ denotes the $\ell + 1$-dimensional k-space of homogeneous polynomials of degree ℓ in the indeterminates $T_1 = x_1^{p_1}$, $T_2 = x_2^{p_2}$. From this description of S it is straightforward to derive two important properties. Note that an element $\pi \in S_{\vec{x}}$ is called *homogeneous prime* if $S/(\pi)$ is a graded integral domain.

Proposition 4.2 ([10]) *(i)* S *is* $\mathbf{L}(\mathbf{p})$*-graded factorial, thus* S *is a* graded integral domain, *(meaning that the product of two non-zero homogeneous elements from* S *is non-zero), and* *moreover each homogeneous non-zero element from S is a product of homogeneous prime elements.*

(*ii*) *The relations $\rho_3, \ldots, \rho_t$ form a homgogeneous regular sequence, accordingly S is a* graded complete intersection *algebra of Krull dimension two. In particular, S is graded Cohen-Macaulay and graded Gorenstein.* □

For later reference we note that the minimal graded-injective resolution of S has the form

$$0 \to S \to K \to Q^1 \to \mathbf{E}(k)(\vec{\omega}) \to 0$$

where K is the graded quotient field of S, and $\mathbf{E}(k)$ is the injective envelope of $k = S/(x_1, \ldots, x_t)$ in the category of graded S-modules and $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$ is the *dualizing element* of $\mathbf{L}(\mathbf{p})$. This shows that S is graded Gorenstein with *dualizing module* $S(\vec{\omega})$.

For each $\vec{y} \in \mathbf{L}^+(\mathbf{p}) = \sum_{i=1}^{t} \mathbb{N}\vec{x}_i$ we introduce a rank one representation $\mathcal{O}(\vec{y})$ of Λ through the diagram:

$$\begin{tikzcd}
 & S_{\vec{y}-\vec{x}_1} \arrow[dl, "x_1"'] & S_{\vec{y}-2\vec{x}_1} \arrow[l, "x_1"'] & \cdots \arrow[l, "x_1"'] & S_{\vec{y}-(p_1-1)\vec{x}_1} \arrow[l, "x_1"'] & \\
S_{\vec{y}} & S_{\vec{y}-\vec{x}_2} \arrow[l, "x_2"'] & S_{\vec{y}-2\vec{x}_2} \arrow[l, "x_2"'] & \cdots \arrow[l, "x_2"'] & S_{\vec{y}-(p_2-1)\vec{x}_2} \arrow[l, "x_2"'] & S_{\vec{y}-\vec{c}} \arrow[l, "x_2"'] \arrow[ul, "x_1"'] \arrow[ddl, "x_t"] \\
 & \vdots & \vdots & & \vdots & \\
 & S_{\vec{y}-\vec{x}_t} \arrow[uul, "x_t"] & S_{\vec{y}-2\vec{x}_t} \arrow[l, "x_t"'] & \cdots \arrow[l, "x_t"'] & S_{\vec{y}-(p_t-1)\vec{x}_t} \arrow[l, "x_t"'] &
\end{tikzcd}$$

where an arrow with label x_i means the injective mapping "multiplication with x_i". If $\ell = \dim_k S_{\vec{y}}$ and $0 \le \ell_i < p_i$ marks the position in the i-th arm where the *dimension jump* occurs, then

$$\vec{y} = \sum_{i=0}^{t} \ell_i \vec{x}_i + \ell\vec{c}$$

is the normal form representation of $\vec{y}$.

Proposition 4.3 (*i*) *Each rank one Λ-module has the form $\mathcal{O}(\vec{y})$ for a uniquely determined $\vec{y} \in \mathbf{L}^+(\mathbf{p})$.*

(*ii*) *Multiplication from S induces* natural isomorphisms

$$S_{\vec{y}-\vec{x}} \longrightarrow \mathrm{Hom}_\Lambda(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})), \quad s \mapsto \bar{s}, \quad \text{for all } \vec{x}, \vec{y} \in \mathbf{L}^+(\mathbf{p}).$$

$$(iii)\ \mathrm{D}\,\mathrm{Ext}^1_\Lambda(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = \begin{cases} \mathrm{Hom}_\Lambda(\mathcal{O}(\vec{y}), \mathcal{O}(\vec{x}+\vec{\omega})) & \text{if } \vec{x}, \vec{x}+\vec{\omega} \in \mathbf{L}^+(\mathbf{p}), \\ 0 & \text{else.} \end{cases}$$

(*iv*) *Each $\mathcal{O}(\vec{y})$ is an* exceptional module, *i.e. has* trivial endomorphism ring k *and* no self-extensions, *i.e.* $\mathrm{Ext}^i_\Lambda(\mathcal{O}(\vec{y}), \mathcal{O}(\vec{y})) = 0$ *for all $i \ge 1$.*

(*v*) *Each $\mathcal{O}(\vec{x})$, $\vec{x} \in \mathbf{L}^+(\mathbf{p})$ has projective dimension ≤ 1.*

Proof. (i) Assume M is an indecomposable rank one module over Λ. All the k-linear mappings constituting M need to be injective [10],[24]. For each $i = 1, \dots, t$ let $0 \le \lambda_i < p_i$ denote the position in the i-th arm where the dimension jump occurs. Let $\ell = \dim_k M_{\vec{c}}$, and put $\vec{y} = \sum_{i=1}^{t} \lambda_i \vec{x}_i + \ell \vec{c}$. Keeping track of the relations (*) it is easily checked that $M \cong \mathcal{O}(\vec{y})$.

(iii) Since S is graded integral the mapping $s \mapsto \bar{s}$ is clearly injective, so assume a morphism $\varphi : \mathcal{O}(\vec{x}) \to \mathcal{O}(\vec{y})$ is given. From the commutativity of the diagrams

$$\begin{array}{ccc} S_{\vec{x}-j\vec{x}_i} & \xrightarrow{x_i^j} & S_{\vec{x}} \\ \downarrow \varphi & & \downarrow \varphi \\ S_{\vec{y}-j\vec{x}_i} & \xrightarrow{x_i^j} & S_{\vec{y}} \end{array} \qquad (1 \le i \le t, 1 \le j \le p_i)$$

we conclude that the quotients $\varphi(s)/s$ do not depend on the choice of $0 \ne s \in S_{\vec{x}}$. Correspondingly φ is given by the multiplication with a homogeneous element q from the graded quotient field K of S. By construction $qS_{\vec{x}} \subseteq S_{\vec{y}}$ holds, graded factoriality of S then implies $q \in S$.

(iii) is comparatively less obvious. The proof exploits that the relations $\rho_3, \dots, \rho_t$ for S form a regular homogeneous sequence in $k[X_1, \dots, X_t]$, and affords an analysis of the attached Koszul complex [10]. Alternatively the claim can be deduced from the fact that $S(\vec{\omega})$ serves as the dualizing module for S.

(iv) follows from (ii) and (iii). For (v) we refer to [10]. □

Remark 4.4 The following assertions can easily be derived from [10] or [19].

(i) From the $\mathbf{L}(\mathbf{p})$-action on $\mathrm{coh}(\mathbb{X})$, $\mathbb{X}$ the weighted projective line corresponding to Λ, the category $\mathrm{mod}_+(\Lambda)$, viewed as a full subcategory of $\mathrm{coh}(\mathbb{X})$, inherits an action of the semigroup $\mathbf{L}^+(\mathbf{p})$, called $\mathbf{L}^+(\mathbf{p})$-*shift*,

$$(\vec{x}, M) \mapsto M(\vec{x})$$

such that $(\mathcal{O}(\vec{y}))(\vec{x}) = \mathcal{O}(\vec{x} + \vec{y})$ holds for all $\vec{x}, \vec{y} \in \mathbf{L}^+(\mathbf{p})$. In particular the rank one modules are the $\mathbf{L}^+(\mathbf{p})$-shifts of the simple projective Λ-module $\mathcal{O} = \mathcal{O}(\vec{0})$. Further, the shift by $\vec{x} \in \mathbf{L}^+(\mathbf{p})$ is a full embedding $\mathrm{mod}_+(\Lambda) \hookrightarrow \mathrm{mod}_+(\Lambda)$ which *preserves the rank* of Λ-modules.

(ii) The rank one modules $\mathcal{O}(\vec{x})$ serve as *exceptional* building blocks for $\mathrm{mod}_+(\Lambda)$, since up to shift each $M \in \mathrm{mod}_+(\Lambda)$ has a finite filtration by modules of rank one. Since $\mathrm{mod}_0(\Lambda)$, and due to proposition 4.3 the morphisms and extensions between rank one modules are perfectly known, this provides a favourable setting — not so frequently encountered in representation theory — to analyse the properties of arbitrary modules over the wild algebra Λ.

5 Local study of the Auslander-Reiten translation

In this section we deal with modules over a canonical algebra $\Lambda = \Lambda(p, \underline{\lambda})$ of wild representation type [19], i.e. we assume $(t-2) - \sum_{i=1}^{t} 1/p_i > 0$.

Lemma 5.1 *The* Auslander-Reiten translation *induces a functor*

$$\tau_\Lambda : \mathrm{mod}_+(\Lambda) \longrightarrow \mathrm{mod}_+(\Lambda)$$

uniquely determined by adjointness

$$\mathrm{D}\,\mathrm{Ext}^1_\Lambda(M,N) = \mathrm{Hom}_\Lambda(N,\tau_\Lambda M), \quad \textit{for all } M,N \in \mathrm{mod}_+(\Lambda).$$

Proof. The category $\mathrm{mod}_+(\Lambda)$ is closed against the formation of Auslander Reiten sequences in $\mathrm{mod}(\Lambda)$ [19]. Since each injective indecomposable Λ-module has rank < 0, there are no non-zero homomorphisms from a positive rank module into an injective module. The claim now follows from the Auslander-Reiten formula $\mathrm{D}\,\mathrm{Ext}^1_\Lambda(M,N) = \overline{\mathrm{Hom}}_\Lambda(N,\tau_\Lambda M)$. □

Lemma 5.2 *On rank one modules the Auslander-Reiten translation is given by*

$$\tau_\Lambda \mathcal{O}(\vec{x}) = \begin{cases} 0 & \textit{if } \mathcal{O}(\vec{x}) \textit{ is projective,} \\ \mathcal{O}(\vec{x}+\vec{\omega}) & \textit{otherwise.} \end{cases}$$

Proof. The general theory [19] tells us that $\tau_\Lambda \mathcal{O}(\vec{x})$ is a module in $\mathrm{mod}_+(\Lambda)$ which is indecomposable or zero, and further

$$\mathrm{rk}\,\tau_\Lambda \mathcal{O}(\vec{x}) \leq \mathrm{rk}\,\mathcal{O}(\vec{x}) = 1$$

with equality if and only if $\tau_\Lambda \mathcal{O}(\vec{x}) = \mathcal{O}(\vec{x}+\vec{\omega})$. □

Thus if $\mathcal{O}(\vec{x})$ is *not preprojective* its AR-translates are given by the sequence

$$\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x}+\vec{\omega}), \mathcal{O}(\vec{x}+2\vec{\omega}), \ldots$$

and by a finite truncation of this sequence otherwise.

The next result is from joint work with T. Hübner:

Theorem 5.3 *Assume that $M = \mathcal{O}(\vec{x})$ is not preprojective. Exactly for the* minimal wild canonical algebras $(2,3,7)$, $(2,4,5)$, $(3,3,4)$ *and their close "neighbours" listed in the table below, the algebra $R = \mathsf{A}(\tau_\Lambda; M)$ can be generated by three homogeneous elements. In this case R has the form*

$$R = k[x,y,z] = k[X,Y,Z]/(F),$$

where the relation F, the degree-triple $\deg(x,y,z)$, *and* $\deg F$ *are displayed in the table below:*

Λ	$\deg(x,y,z)$	*relation F*	$\deg F$	
(2,3,7)	$(6,14,21)$	$Z^2+Y^3+X^7$	42	
$(2,3,8)$	$(6,8,15)$	$Z^2+X^5+XY^3$	30	
$(2,3,9)$	$(6,8,9)$	$Y^3+XZ^2+X^4$	36	
(2,4,5)	$(4,10,15)$	$Z^2+Y^3+X^5Y$	30	
$(2,4,6)$	$(4,6,11)$	$Z^2+X^4Y+XY^3$	22	
$(2,4,7)$	$(4,6,7)$	$Y^3+X^3Y+XZ^2$	18	
$(2,5,5)$	$(4,5,10)$	$Z^2+Y^2Z+X^5$	20	•
$(2,5,6)$	$(4,5,6)$	$XZ^2+Y^2Z+X^4$	16	
(3,3,4)	$(3,8,12)$	$Z^2+Y^3+X^4Z$	24	•
$(3,3,5)$	$(3,5,9)$	$Z^2+XY^3+X^3Z$	18	•
$(3,3,6)$	$(3,5,6)$	$Y^3+X^3Z+XZ^2$	15	•
$(3,4,4)$	$(3,4,8)$	$Z^2-Y^2Z+X^4Y$	16	•
$(3,4,5)$	$(3,4,5)$	$X^3Y+XZ^2+Y^2Z$	13	
$(4,4,4)$	$(3,4,4)$	$X^4-YZ^2+Y^2Z$	12	•

These 14 equations are equivalent to *Arnold's 14 exceptional unimodal singularities* in the theory of singularities of differentiable maps [1]. In the theory of *automorphic forms* the 14 equations do occur as the relations of exactly those rings of entire automorphic forms having *three generators* [27]. In the rows marked by • the two lists mentioned quote an expression for the "singularity" F, which is different from the expression given in the table, but — for $k = \mathbb{C}$ — easily seen to be equivalent.

Corollary 5.4 *Assume Λ is one of the canonical algebras displayed in the above table with weight type (p_1, p_2, p_3), degree triple (a, b, c), degree of the relation d, then the Hilbert-Poincaré series expressing the growth of the sequence $\tau_\Lambda^n M$ with M of rank one not preprojective, is given by*

$$\sum_{n=0}^{\infty} \dim_k \mathrm{Hom}_\Lambda(M, \tau_\Lambda^n M)\, T^n = \frac{T^d - 1}{(T^a-1)(T^b-1)(T^c-1)}.$$

□

For an arbitrary system of wights $\mathbf{p}$ the Hilbert-Poincaré series of Λ is given as (see [19], [27])

$$T + \frac{1}{1-T} + (t-2)\frac{T}{(1-T)^2} - \sum_{i=1}^{t} \frac{T}{(1-T)(1-T^{p_i})}.$$

The rest of the paper will deal with the following objectives:

- To show how to compute these algebras R and to establish their properties (*characteristic arbitrary*).
- To establish the link to automorphic forms (*characteristic zero*).

The next theorem makes no restriction on the the weight type $\mathbf{p} = (p_1, \ldots, p_t)$ of the wild canonical algebra Λ. Recall that a positively $\mathbb{Z}$-graded commutative k-algebra R is called *quasihomogeneous* if R is finitely k-generated and $R_0 = k$.

Theorem 5.5 *Let Λ be a wild canonical algebra. Then there exists a* quasihomogeneous *Cohen-Macaulay algebra R of Krull dimension two such that for each rank one Λ-module M we have*

$$\mathsf{A}(\tau_\Lambda; M) \cong \begin{cases} R & \textit{for } M \textit{ not preprojective,} \\ R/\bigoplus_{n \geq n_0} R_n & \textit{for the first } n_0 \textit{ with } \tau_\Lambda^{n_0} M = 0 \textit{ otherwise.} \end{cases}$$

Proof. We assume that $M = \mathcal{O}(\vec{x})$ is not preprojective and put $R = \mathsf{A}(\tau_\Lambda; \mathcal{O}(\vec{x}))$. In view of Lemma 5.2 we obtain

$$R_n = \mathrm{Hom}_\Lambda(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x} + n\vec{\omega})) = S_{n\vec{\omega}}.$$

Thus R is the subalgebra of S obtained by restriction to the subgroup $\mathbb{Z}\vec{\omega}$ of $\mathbf{L}(\mathbf{p})$:

$$R = S|_{\mathbb{Z}\vec{\omega}}.$$

Further

$$H := S|_{\mathbb{Z}\vec{c}} = k[T_1, T_2] \quad \text{with } T_1 = x_1^{p_1},\ T_2 = x_2^{p_2}.$$

Notice that $U = \mathbb{Z}\vec{\omega} \cap \mathbb{Z}\vec{c}$ has finite index in $\mathbf{L}(\mathbf{p})$, hence has finite index $d := [\mathbb{Z}\vec{c} : U]$ in $\mathbb{Z}\vec{c}$. We thus obtain the following set of subalgebras of S:

$$\begin{tikzcd}[arrows=dash]
 & S \arrow[dl] \arrow[dr, "\text{f.g.free}"] & \\
R = S|_{\mathbb{Z}\vec{\omega}} \arrow[dr] & & H := S|_{\mathbb{Z}\vec{c}} \arrow[dl, "\text{f.g.free}"] \\
 & S|_U \arrow[d, "\text{f.g.free}"] & \\
 & k[T_1^d, T_2^d] &
\end{tikzcd}$$

In the diagram we have noted where the larger algebra is easily seen to be a finitely generated free graded module over the smaller one. Denoting by $\{0\} \cup T$ a transversal of U in $\mathbf{L}(\mathbf{p})$, we obtain a direct decomposition

$$S = R \oplus \bigoplus_{t \in T} S|_{t+U}$$

of graded $S|_U$-modules, hence of graded $k[T_1^d, T_2^d]$-modules. It follows from the above that R viewed as a $\mathbb{Z}$-graded $k[T_1^{p_1}, T_2^{p_2}]$-module is finitely generated free, and therefore (graded) Cohen-Macaulay of Krull dimension two. $\square$

Corollary 5.6 *$R = R(\mathbf{p}, \underline{\lambda})$ is always graded* Gorenstein *with $R(1)$ serving as the dualizing module.*

Proof. As a graded complete intersection the $\mathbf{L}(\mathbf{p})$-graded algebra $S = S(\mathbf{p}, \underline{\lambda})$ is graded Gorenstein with $S(\vec{\omega})$ serving as the dualizing module. Restriction of the grading to the subgroup $\mathbb{Z}\vec{\omega}$ proves the claim. $\square$

Unlike the situation dealt with in theorem 5.3 — R is in general not a complete intersection.

The algebra $R(\mathbf{p}, \underline{\lambda})$ carries sufficient information to reconstruct the $(\mathbf{p}, \underline{\lambda})$, hence the canonical algebra $\Lambda(p, \underline{\lambda})$:

Theorem 5.7 *Let $R = R(\mathbf{p}, \underline{\lambda})$. Then the affine spectrum $\mathbf{Y} = \mathbf{Spec}(R)$ is a surface equiped with an action of the multiplicative group scheme $\mathbb{G}_m = \mathbf{Spec}(k[T, T^{-1}])$. Further the quotient*

$$\mathbf{Proj}^{\mathbb{Z}}(R) = (\mathbf{Y} \setminus \{\mathfrak{m}\})/\mathbb{G}_m$$

endowed with the $\mathbb{Z}$-graded sheaf theory induced by this action is isomorphic to the weighted projective line $\mathbb{X}(\mathbf{p}, \underline{\lambda})$ attached to the canonical algebra $\Lambda = \Lambda(p, \underline{\lambda})$.

If $k = \mathbb{C}$ this may be expressed in more concrete terms: We write $R = k[T_1, \ldots, T_h]/J$, where $\deg x_i = q_i$ and consider the surface in $\mathbb{C}^h$ given by

$$\left\{x \in \mathbb{C}^h \mid f(x) = 0 \quad \text{for every } f \text{ in } J\right\}.$$

$\mathbf{Y}$ is equiped with a $\mathbb{C}^*$-action given by

$$(\alpha, x_1, \ldots, x_t) \mapsto (\alpha^{q_1} x_1, \ldots, \alpha^{q_h} x_h).$$

Exactly for t $\mathbb{C}^*$-orbits $[y_1], \ldots, [y_t]$ from $\mathbf{Y}^* = \mathbf{Y} \setminus \{0\}$ this action has a non-trivial stabilizer group (which then is cyclic of order $p_1, \ldots, p_t$, resp.). Moreover there is a natural bijection

$$\mathbf{Y}^*/\mathbb{C}^* = \mathbf{P}_1(\mathbb{C})$$

sending the $[y_i]$ to the λ_i occurring in the relations $(*)$. Here, we keep to the convention $\lambda_1 = \infty$, $\lambda_2 = 0$.

Proof. We have to show that the category coh($\mathbf{Y}$) of $\mathbb{Z}$-graded coherent sheaves on $\mathbf{Y} = \mathbf{P}_{\mathbb{Z}}(R)$ is equivalent to the category of $\mathbf{L}(\mathbf{p})$-graded coherent sheaves on the $\mathbf{L}(\mathbf{p})$-graded prime spectrum $\mathbf{X} = \mathbf{Proj}^{\mathbf{L}(\mathbf{p})}(S)$ of S. This is done by *restriction from* $\mathbf{L}(\mathbf{p})$ *to* $\mathbb{Z}\vec{\omega}$:

$$\begin{array}{ccc} \operatorname{mod}^{\mathbf{L}(\mathbf{p})}(S) & \xrightarrow{\text{res}} & \operatorname{mod}^{\mathbb{Z}}(R) \\ \downarrow & & \downarrow \\ \operatorname{coh}(\mathbf{X}) & \xrightarrow{\cong} & \operatorname{coh}(\mathbf{Y}) \end{array}$$

The vertical maps denote the sheafification for graded modules. One has to show that a finitely generated $\mathbf{L}(\mathbf{p})$-graded S-module M has finite (graded) length if and only if its restriction to the subgroup $\mathbb{Z}\vec{\omega}$ has finite length as a $\mathbb{Z}$-graded R-module.

By noetherianness of R this can be reduced to the case that (up to $\mathbf{L}(\mathbf{p})$-shift) M has the form $S/\mathfrak{p}$, where $\mathfrak{p}$ is a homogeneous prime ideal in S. Due to the graded factoriality the homogeneous prime ideals of S are explicitly known. From this information it is not hard to verify the claim (see [11] for details). $\square$

Corollary 5.8 ([11]) *Sheafification induces equivalences*

$$\mathrm{CM}^{\mathbf{L}(\mathbf{p})}(S) \to \operatorname{vect}(\mathbf{X}), \quad M \mapsto \widetilde{M},$$

$$\mathrm{CM}^{\mathbb{Z}}(R) \to \operatorname{vect}(\mathbf{X}), \quad M \mapsto \widetilde{M}.$$

$\square$

As is further shown in [11] the knowledge of vect($\mathbf{X}$), hence of coh($\mathbf{X}$), allows to recover the data $(\mathbf{p}, \underline{\lambda})$ and therefore the algebra $\Lambda = \Lambda(p, \underline{\lambda})$ from $R = R(\mathbf{p}, \underline{\lambda})$.

As in section 3 we may now invoke completion

$$\mathrm{CM}^{\mathbb{Z}}(R) \to \mathrm{CM}(\widehat{R}), \quad M \mapsto \widehat{M}$$

to establish the relationship between Λ-modules and CM-modules over the isolated singularity $\widehat{R}$:

Corollary 5.9 *The categories* vect($\mathbf{X}$), $\mathrm{CM}^{\mathbf{L}(\mathbf{p})}(S)$, $\mathrm{CM}^{\mathbb{Z}}(R)$ *are (naturally) equivalent. Accordingly, completion* $\mathrm{CM}^{\mathbb{Z}}(R) \to \mathrm{CM}(\widehat{R})$ *leads to a functor*

$$\pi : \operatorname{vect}(\mathbf{X}) \to \mathrm{CM}(\widehat{R})$$

with the following properties

(*i*) *π preserves indecomposability and Auslander-Reiten sequences,*

(*ii*) *For indecomposable vector bundles X and Y we have $\pi(X) \cong \pi(Y)$ if and only if $Y \cong \tau_{\mathbf{X}}^{n} X$ for some $n \in \mathbb{Z}$, where $\tau_{\mathbf{X}}$ denotes the Auslander-Reiten translation for* coh($\mathbf{X}$).

Moreover, if τ is the Auslander-Reiten translation for $\mathrm{D}^b(\Lambda)$ *then* vect($\mathbf{X}$) *can be viewed as the full subcategory of* $\mathrm{D}^b(\Lambda)$ *consisting of all $\tau^n X$ with $X \in \operatorname{mod}_+(\Lambda)$.* $\square$

So as in section 3 we obtain that $\mathrm{mod}_-(\Lambda)[-1] \cup \mathrm{mod}_+(\Lambda) \subseteq \mathrm{vect}(\mathbb{X})$; in the wild situation — we are considering here — this inclusion is however alsways strict, requesting the more complicated argument given to construct $\mathrm{vect}(\mathbb{X})$ from module data. In more concrete terms the relationship between Λ-modules and $\widehat{R}$-modules can be described through introduction of the category $\mathrm{mod}'_+(\Lambda)$ consisting of all finite direct sums of indecomposable modules of rank > 0 which are not preprojective. Then restriction of π to $\mathrm{mod}'_+(\Lambda)$ leads to:

Corollary 5.10 *There is a functor $\pi' : \mathrm{mod}_+(\Lambda) \to \mathrm{CM}(\widehat{R})$ preserving indecomposability and Auslander-Reiten sequences such that $\pi'(M) \cong \pi'(N)$ if and only if $\tau_\Lambda^n M \cong \tau_\Lambda^m N$ for some $m, n \in \mathbb{N}$.* □

Unlike in section 3 the functor π (hence also the functor π') is not dense.

6 Algebras of automorphic forms

In case of the base field $\mathbb{C}$ we view a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \underline{\lambda})$ as the Riemann sphere with t marked points $\lambda_1, \ldots, \lambda_t$, furtheron called exceptional, given order $p_1, \ldots, p_t$. This motivates the introduction of the *fundamental group* and the *universal covering* of $\mathbb{X}$ to be given below. We refer the reader to [21], [22] and [20] for further information on the topological and the group theoretic setting. We recall from [10] that the complexity of the sheaf theory on $\mathbb{X}$ is largely determined by the (virtual) *genus* of $\mathbb{X}$ given by

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} p/p_i\right); \quad p = \mathrm{l.c.m.}(p_1, \ldots, p_t).$$

Let $\mathbf{Y}$ be a Riemann surface and let G be a group of automorphisms of $\mathbf{Y}$ onto itself. We say that the action of G is *discontinuous* (resp. *freely discontinuous*) at a point $y \in \mathbf{Y}$, if there is a neighborhood U of y, so that $\sigma(U) \cap U \neq \emptyset$ only for finitely many $\sigma \in G$ (resp. only for $\sigma = 1$).

To define the *fundamental group* $\pi_1(\mathbb{X})$ of the marked Riemann surface $\mathbb{X} = \mathbb{X}(\mathbf{p}, \underline{\lambda})$ we need a *modified notion of homotopy* taking care of the marking. Let E denote the set $\{\lambda_1, \ldots, \lambda_t\}$ of exceptional points of $\mathbb{X}$. We consider the monoid S of homotopy classes of paths in $\mathbb{X} \setminus \{\lambda_1, \ldots, \lambda_t\}$ and take the smallest congruence relation on S that additionally makes each of the following paths σ trivial: For $i = 1, \ldots, t$ take any neighborhood V of λ_i not containing any of the remaining exceptional points and take any closed path σ in $V \setminus \{\lambda_i\}$, whose winding index with respect to λ_i is a multiple of p_i. As the fundamental group of $\mathbf{P}_1(\mathbb{C}) \setminus E$ is the group on generators $\sigma_1, \ldots, \sigma_t$ with relation $\sigma_1\sigma_2\cdots\sigma_t = 1$, we obtain:

Proposition 6.1 *The fundamental group $\pi_1(\mathbb{X})$ is the group on generators $\overline{\sigma}_1, \ldots, \overline{\sigma}_t$ subject to the relations*

$$\overline{\sigma}_1^{p_1} = \overline{\sigma}_2^{p_2} = \cdots = \overline{\sigma}_t^{p_t} = 1 = \overline{\sigma}_1\overline{\sigma}_2\cdots\overline{\sigma}_t.$$

□

As any marked Riemann surface, $\mathbf{X}$ has a simply connected *branched universal covering* $\pi : \widetilde{\mathbf{X}} \to \mathbf{X}$ where p_i is the branch number at λ_i. Accordingly, $\overline{G} = \pi_1(\mathbf{X})$ acts discontinuously on $\widetilde{\mathbf{X}}$. This action commutes with the projection π inducing a bijection $\widetilde{\mathbf{X}}/\overline{G} = \mathbf{X}$, and is further freely discontinuous in all points not in $\pi^{-1}(E)$, whereas for $\pi(y) = \lambda_i$ the stabilizer group is cyclic of order p_i, referred to as the *ramification order* of λ_i.

Proposition 6.2 *Let* $\mathbf{X} = \mathbf{X}(\mathbf{p}, \underline{\lambda})$ *accordingly* $\Lambda = \Lambda(p, \underline{\lambda})$. *The branched universal cover* $\widetilde{\mathbf{X}}$ *is*

1. *the Riemann sphere* $\mathbf{P}_1(\mathbb{C})$ *if* $g_{\mathbf{X}} < 1$, *accordingly* $\mathbf{X}$ *is elliptic and* Λ *is tame domestic,*

2. *the omplex plane* $\mathbb{C}$ *if* $g_{\mathbf{X}} = 1$, *accordingly* $\mathbf{X}$ *is parabolic and* Λ *is tubular,*

3. *the upper complex half plane* $\mathbb{H}_+$ *if* $g_{\mathbf{X}} > 1$, *accordingly* $\mathbf{X}$ *is hyperbolic and* Λ *is wild.*

□

We notice that the elliptic case has been dealt with in section 3. We thus restrict the the further discussion to the case $g_{\mathbf{X}} > 1$.

Corollary 6.3 $\overline{G} = \pi_1(\mathbf{X})$ *is a discrete subgroup of* $\mathrm{Aut}(\mathbb{H}_+) = \mathbb{SL}(2, \mathbb{R})$, *acting discontinuously on* $\mathbb{H}_+$.

Topologically, the orbit space $\mathbb{H}_+/\overline{G}$ *is the Riemann sphere* $\mathbf{P}_1(\mathbb{C})$; *moreover the action of* $\overline{G}$ *on* $\mathbb{H}_+$ *has exactly t orbits*

$$\lambda_1 = [y_1], \ldots, \lambda_t = [y_t]$$

with non-trivial stabilizer group which are cyclic of order

$$p_1, \ldots, p_t, \quad \text{respectively.}$$

In particular, $\overline{G}$ *is a* Fuchsian group of the first kind *of signature* $(0; p_1, \ldots, p_t)$. □

For the later discussion we further need [22], [23] that the lift G of $\overline{G}$ in the universal covering group $\widetilde{\mathbb{SL}(2, R)}$ is a central extension of $\overline{G}$ on generators

$$\sigma_1, \ldots, \sigma_t$$

with relations

$$\sigma_1^{p_1} = \sigma_2^{p_2} = \cdots = \sigma_t^{p_t} := \sigma_0, \quad \sigma_0^{t-2} = \sigma_1\sigma_2\cdots\sigma_t.$$

The center $C = < \sigma_0 >$ of G is infinite cyclic, and $G/ < \sigma_0 > = \overline{G}$. Notice that G acts on $\mathbb{H}_+$ in an obvious way. The degree homomorphism $\delta : \mathbf{L}(\mathbf{p}) \to \mathbb{Z}$ is given on generators by $\delta(\vec{x}_i) = p/p_i$. As before $\vec{\omega}$ is the dualising element of $\mathbf{L}(\mathbf{p})$.

Lemma 6.4 (*i*) *The character group* G^* *of* G *is the* finite abelian group *on generators*

$$\chi_1, \ldots, \chi_t$$

with relations

$$p_1\chi_1 = \cdots = p_t\chi_t := \chi_0, \quad (t-2)\chi_0 = \sum_{i=1}^{t} \chi_i.$$

Accordingly there there is a surjection

$$\nu : \mathbf{L}(\mathbf{p}) \longrightarrow G^*, \quad \vec{x}_i \mapsto \chi_i$$

with kernel $\mathbb{Z}\vec{\omega}$.

(*ii*) *The homomorphism*

$$\varphi : \mathbf{L}(\mathbf{p}) \longrightarrow \mathbb{Q} \times G^*, \quad \vec{x} \mapsto (\frac{\delta(\vec{x})}{\delta(\vec{\omega})}, \nu(\vec{x}))$$

allows to view $\mathbf{L}(\mathbf{p})$ *as a subgroup of* $\mathbb{Q} \times G^*$.

(*iii*) *By means of this identification the subgroup* $\mathbb{Z}\vec{\omega}$ *of* $\mathbf{L}(\mathbf{p})$ *corresponds to the subgroup* $\mathbb{Z} \times \{0\}$ *of* $\mathbb{Z} \times G^*$. □

Let χ be a character of G and a be an integer. A χ-*automorphic form* of degree a with respect to G is a holomorphic function f on $\mathbb{H}_+$ such that

$$f(\sigma.z) = \left(\frac{d\sigma}{dz}\right)^a f(z) \quad \text{for all } \sigma \in G$$

A modified — but more technical — definition due to Milnor can also be given for a *rational exponent* a, see [22].

Theorem 6.5 (**[22], [23]**) *For a rational number* a *and a character* χ *of* G *the* $\mathbb{C}$*-space* $A_{a,\chi}$ *of* χ*-automorphic forms of degree* a *is finite dimensional. With respect to multiplication of automorphic forms*

$$A = \bigoplus_{(a,\chi)\in\mathbb{Q}\times G^*} A_{a,\chi}$$

is a $\mathbb{Q} \times G^*$*-graded algebra generated by* t *homogeneous elements*

$$x_1, \ldots, x_t, \quad \textit{with} \quad \deg(x_i) = (\frac{p}{\delta(\vec{\omega})p_i}, \chi_i)$$

subject to the $t-2$ *relations*

$$\begin{aligned} x_3^{p_3} &= x_2^{p_2} - \lambda_3 x_1^{p_1} \\ \vdots &= \quad \vdots \\ x_3^{p_t} &= x_2^{p_2} - \lambda_t x_1^{p_1}. \end{aligned}$$

□

The next assertions dealing with the link between canonical algebras and automorphic forms are now direct consequences of the results from section 5.

Corollary 6.6 *Identifying* $\mathbf{L}(\mathbf{p})$ *— as above — with a subgroup of* $\mathbb{Q} \times G^*$, *A can be viewed as an* $\mathbf{L}(\mathbf{p})$*-graded algebra. Moreover*

(i) $S = S(\mathbf{p}, \underline{\lambda})$ *and A are isomorphic as* $\mathbf{L}(\mathbf{p})$*-graded algebras.*

(ii) The algebra $R(\mathbf{p}, \underline{\lambda}) = S|_{\mathbb{Z}\vec{\omega}}$ *and the algebra of entire automorphic forms* $A_{*,0} = \bigoplus_{n \geq 0} A_{n,0}$ *are isomorphic as* $\mathbb{Z}$*-graded algebras.*

(iii) If $\Lambda = \Lambda(p, \underline{\lambda})$ *is the canonical algebra attached to the data* $(\mathbf{p}, \underline{\lambda})$ *and M is a rank-one* Λ*-module, which is not preprojective, then the algebra* $\mathbb{A}(\tau_\Lambda; M)$ *of the Auslander-Reiten translation* τ_Λ *at M is isomorphic to the algebra of entire automorphic forms* $A_{*,0}$.

Corollary 6.7 *Let* $\Lambda = \Lambda(p, \underline{\lambda})$. *Completion of* $\mathbb{Z}$*-graded* $A_{*,0}$*-modules leads to a functor*

$$\pi' : \mathrm{mod}'_+(\Lambda) \to \mathrm{CM}(\widehat{A_{*,0}})$$

preserving indecomposability and Auslander-Reiten sequences, morover $\pi'(M) \cong \pi'(N)$ *holds if and only if* $\tau_\Lambda^n(M) \cong \tau_\Lambda^m(N)$ *holds for some* $n, m \in \mathbb{N}$. □

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THE EXT ALGEBRA OF A HIGHEST WEIGHT CATEGORY

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ABSTRACT. In this paper, we survey some recent results of Cline, Parshall, Scott concerning the homological dual of a highest weight category. The main result gives conditions under which the homological dual is also a highest weight category. We also discuss the corresponding theory for graded highest weight categories.

Let $\mathcal{C}$ be a highest weight category over an algebraically closed field k. Assume that every object in $\mathcal{C}$ has finite length and that there are only finitely many simple objects $L(\lambda)$, indexed by a finite poset Λ. Thus, $\mathcal{C}$ is equivalent to the category of finite dimensional, right modules for a quasi-hereditary algebra S. Write $L = \oplus_{\lambda\in\Lambda} L(\lambda)$, and let $S^!$ denote the Yoneda Ext-algebra $\mathrm{Ext}^\bullet_{\mathcal{C}}(L, L)$. Similarly, $\mathcal{C}^!$ denotes the category of finite dimensional, right $S^!$-modules. In general, this *homological dual* $\mathcal{C}^!$ is *not* a highest weight category. However, in the most important circumstances arising from Lie theory, namely, highest weight categories associated to complex semisimple Lie algebras (the BGG categories $\mathcal{O}_{\mathrm{triv}}$), semisimple algebraic groups (and their various infinitesimal versions) in positive characteristic p, and quantum groups at a root of unity, $\mathcal{C}^!$ is a highest weight category, *under an important qualifying assumption.* That assumption, namely, that $\mathcal{C}$ has a *Kazhdan-Lusztig theory*, is either known to be true (as in the category $\mathcal{O}_{\mathrm{triv}}$ case) or is expected to hold. In any event, these examples suggest an investigation of the homological duals of highest weight categories which possess a Kazhdan-Lusztig theory. This program has been begun in earnest in [8], and this paper largely summarizes some of the results obtained there. (Further general results can be found in [16].) We hope the reader will be inspired to go to the references to learn more about this area.

In section 1, we discuss the notion of a Kazhdan-Lusztig theory as originally formulated in [5,7]. We sketch, in section 2, the argument that, if $\mathcal{C}$ has a Kazhdan-Lusztig theory, then $\mathcal{C}^!$ is a highest weight category. A few consequences of this

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theorem are also presented there. In section 3, we indicate some generalizations of the theory to include *graded* highest weight categories.

§1. Kazhdan-Lusztig theories

Let $\mathcal{C}$ be a highest weight category over an algebraically closed field k. For some of this section, it is *not* necessary to assume that the weight poset Λ of $\mathcal{C}$ is finite. However, we do assume that every object in $\mathcal{C}$ has finite length and that the opposite category $\mathcal{C}^{\mathrm{op}}$ is a highest weight category (with the same poset Λ). Thus, for $\lambda \in \Lambda$, both the "induced" object $A(\lambda)$ and the "Weyl" object $V(\lambda)$ are defined. Recall that if $I(\lambda)$ (resp., $P(\lambda)$) denotes the injective hull (resp., projective cover) of the simple object $L(\lambda)$, then $A(\lambda)$ (resp., $V(\lambda)$) is the unique maximal subobject (resp., quotient object) of $I(\lambda)$ (resp., $P(\lambda)$ have all composition factors $L(\nu)$ satisfying $\nu \leq \lambda$.

The injective object $I(\lambda)$ has a increasing filtration

$$0 = F_0(\lambda) \subset F_1(\lambda) \subset \cdots \subset F_{t(\lambda)}(\lambda)$$

in which

$$F_i(\lambda)/F_{i-1}(\lambda) \cong \begin{cases} A(\lambda), & i = 1 \\ A(\nu), \ \nu > \lambda, & i > 1. \end{cases}$$

Dually, the projective cover $P(\lambda)$ has a decreasing filtration

$$P(\lambda) = P^0(\lambda) \supset P^1(\lambda) \supset \cdots \supset P^{s(\lambda)}(\lambda) = 0$$

in which

$$P^i(\lambda)/P^{i+1}(\lambda) \cong \begin{cases} V(\lambda) & i = 0 \\ V(\nu), \ \nu > \lambda, & i > 1. \end{cases}$$

Now fix a function $l : \Lambda \to \mathbb{Z}$, usually called a *length function*. The concept of a Kazhdan-Lusztig theory can be described in several equivalent ways.

(1.1) Kazhdan-Lusztig theory via derived categories. In the bounded derived category $D^b(\mathcal{C})$ we introduce two full subcategories $\mathcal{E}^L$ and $\mathcal{E}^R$. Let $\mathcal{E}_0^L$ be the full additive subcategory consisting of objects which are finite direct sums objects of the form

$$V(\lambda)[s], \quad \lambda \in \Lambda, \quad s \equiv l(\lambda) \bmod 2.$$

Having defined $\mathcal{E}_i^L$ $(i = 0, 1, \cdots)$, let $\mathcal{E}_{i+1}^L$ be the full subcategory of $D^b(\mathcal{C})$ consisting of objects X fitting into a distinguished triangle

$$E_i \to X \to E_i' \to, \quad E_i, E_i' \in \mathrm{Ob}(\mathcal{E}_i^L).$$

Finally, let

$$\mathcal{E}^L = \bigcup_{i=0}^{\infty} \mathcal{E}_i^L. \tag{1.1.1}$$

The full subcategory $\mathcal{E}^R$ is defined by a dual construction, working with the $A(\lambda)$'s instead of the $V(\lambda)$'s.

We say that $\mathcal{C}$ has a Kazhdan-Lusztig theory relative to l provided

$$L(\lambda)[l(\lambda)] \in \mathrm{Ob}(\mathcal{E}^L) \cap \mathrm{Ob}(\mathcal{E}^R) \quad \forall \lambda \in \Lambda. \tag{1.1.2}$$

(1.2) Homological characterization of a Kazhdan-Lusztig theory. For $\lambda, \mu \in \Lambda$, we consider the statements

$$\mathrm{Ext}^n_{\mathcal{C}}(V(\lambda), L(\nu)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\nu) \mod 2. \tag{1.2.1}$$

$$\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), A(\nu)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\nu) \mod 2. \tag{1.2.2}$$

Then:

(1.2.3) Theorem. (*Recognition theorem* [5, Thm. (2.4)]) *Assume that $\mathcal{C}$ has a finite weight poset Λ. Then $\mathcal{C}$ has a Kazhdan-Lusztig theory relative to a length function l iff conditions (1.2.1), (1.2.2) hold for all $\lambda, \nu \in \Lambda$.*

(1.3) Consequences of a Kazhdan-Lusztig theory. We mention several striking implications of imposing a Kazhdan-Lusztig theory on a highest weight category.

(1.3.1) Theorem. ([5, Thm. (4.3)]) *Assume that $\mathcal{C}$ has a finite weight poset and has a Kazhdan-Lusztig theory relative to l. For $\lambda, \nu \in \Lambda$, the natural map*

$$Ext^\bullet_{\mathcal{C}}(L(\lambda), L(\nu)) \to Ext^\bullet_{\mathcal{C}}(L(\lambda), A(\nu)) \tag{1.3.1.1}$$

induced by the inclusion $L(\nu) \hookrightarrow A(\nu)$ is surjective. Dually, the natural map

$$Ext^\bullet_{\mathcal{C}}(L(\nu), L(\lambda)) \to Ext^\bullet_{\mathcal{C}}(V(\nu), L(\lambda))$$

induced by the surjection $V(\nu) \to L(\nu)$ is surjective for all λ, ν.

For more details, the reader is referred to [5] where a more general result is actually established.

(1.3.2) Theorem. ([5, Thm. (4.2)]) *Let $\mathcal{C}$ have a Kazhdan-Lusztig theory relative to a length function l. Suppose that $X \in Ob(\mathcal{C})$ satisfies $X[s] \in Ob(\mathcal{E}^L)$ and $X[t] \in Ob(\mathcal{E}^R)$ for integers s, t. Then X is completely reducible in $\mathcal{C}$.*

An important consequence of a Kazhdan-Lusztig theory is that it often permits the explicit calculation of Ext-groups between simple objects. In fact, we have the following result.

(1.3.3) Theorem. *([5, Thm. (3.5)]) Assume that $\mathcal{C}$ has a Kazhdan-Lusztig theory relative to l. For $\lambda, \nu \in \Lambda$,*
(1.3.3.1)
$$\dim \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), L(\nu)) = \sum_{\tau\in\Lambda}\sum_{i+j=n} \dim \mathrm{Ext}^i_{\mathcal{C}}(L(\lambda), A(\tau)) \cdot \dim \mathrm{Ext}^j_{\mathcal{C}}(V(\tau), L(\nu)).$$

Define *Poincaré polynomials* $p^L_{\tau,\lambda}$, $p^R_{\tau,\lambda} \in \mathbb{Z}[t]$ by

$$p^L_{\tau,\lambda} = \sum_n \dim \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), A(\tau))t^n \tag{1.3.4}$$

$$p^R_{\tau,\lambda} = \sum_n \dim \mathrm{Ext}^n_{\mathcal{C}}(V(\tau), L(\lambda)t^n. \tag{1.3.5}$$

Thus, the formula (1.3.3.1) can be expressed as

$$\sum \dim \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), L(\nu))t^n = \sum_\tau p^L_{\tau,\nu} \cdot p^R_{\tau,\nu}. \tag{1.3.6}$$

The *Kazhdan-Lusztig polynomials* $P_{\tau,\lambda} \in \mathbb{Z}[t, t^{-1}]$ are defined by:

$$P_{\tau,\lambda} \equiv P^L_{\tau,\lambda} = t^{l(\lambda)-l(\tau)}\bar{p}^L_{\tau,\lambda}. \tag{1.3.7}$$

(For $f \in \mathbb{Z}[t, t^{-1}]$, $\bar{f}$ is defined by the rule $\bar{f}(t) = f(t^{-1})$.) As discussed in [7], one expects these polynomials to be calculated recursively, so that the above theorem provides a recursive calculation of the Ext-groups between simples.

(1.4) Examples. We wish to illustrate what the expected situation is in certain concrete cases arising in Lie theory.

(1.4.1) The category $\mathcal{O}_\lambda$. Let $\mathcal{O}$ be the BGG category associated to a complex semisimple Lie algebra $\mathfrak{g}$. Fix a weight λ on the Cartan subalgebra such that $\lambda + \rho$ is anti-dominant (ρ = Weyl weight). Consider the full subcategory $\mathcal{O}_\lambda$ consisting of objects having composition factors with highest weight linked to λ. This is a highest weight category with weight poset $W^\lambda \cdot \lambda$, where W^λ is the set of distinguished left coset representatives for the "dot" stabilizer W_λ of λ in the Weyl group W of $\mathfrak{g}$. The Bruhat-Chevalley partial order on W^λ defines the poset structure on $\Lambda = W^\lambda \cdot \lambda$. Using [18], it is proved in [5, Thm. (3.8)] that $\mathcal{O}_\lambda$ has a Kazhdan-Lusztig theory relative to the length function $l : \Lambda \to \mathbb{Z}$, $w \cdot \lambda \to l(w)$, the usual length in the Weyl group W. As a corollary, it is established that for $y, w \in W^\lambda$,

$$\sum_n \dim \mathrm{Ext}^n_{\mathcal{O}}(L(y\cdot\lambda), L(w\cdot\lambda))t^n = \sum_{z\in W^\lambda}\sum_{a,b\in W_\lambda} t^{l(y)+l(w)-2l(z)}(-1)^{l(a)+l(b)}\bar{P}_{za,y}\bar{P}_{zb,w}$$

where $P_{c,d} \in \mathbb{Z}[t, t^{-1}]$ denotes the classical Kazhdan-Lusztig polynomial [11, Ch. 7]. Thus, we obtain, in this case, a complete calculation of Ext-groups between any two simple objects in the category $\mathcal{O}$.

(1.4.2) Algebraic groups in positive characteristic. Let G be a (almost) simple, simply connected algebraic group over an algebraically closed field k of positive characteristic p. We use the $\uparrow$ partial ordering on the set X_+ of dominant weights [12]. Assume that $p \geq h$, the Coxeter number of G; thus, the weight 0 lies in the bottom p-alcove for the root system of G. Let Γ denote the subposet of X_+ consisting of dominant weights $\mu \in W_p \cdot 0$ in the orbit of λ under the "dot" action of the affine Weyl group W_p which satisfy the (Janzten) condition $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+2)$, where ρ is the Weyl weight and α_0 is the maximal short root in the root system Φ of G. Let $\mathcal{C}$ denote the category of finite dimensional rational G-modules which have composition factors $L(\tau)$ having highest weight $\tau \in \Gamma$.

In the following result, $P_{u,v}, u, v \in W_p$, are the classical Kazhdan-Lusztig polynomials for the affine Coxeter group W_p, and $w_0 \in W$ is the long word. If $w \in W_p$, let $l(w)$ denote the usual length of w as an element in the Coxeter group W_p (relative to the standard generators).

(1.4.2.1) Theorem. ([7]) *The following statements are equivalent:*

(a) (Lusztig conjecture) For $w \cdot 0 \in \Gamma$

$$ch\, L(w \cdot \lambda) = \sum_{yw_0 \leq ww_0, y \cdot \lambda \in X(T)^+} (-1)^{l(y)-l(w)} P_{yw_0, ww_0}(-1) ch\, V(y \cdot \lambda).$$

(b) $\mathcal{C}$ *has a Kazhdan-Lusztig theory relative to* $l : \Gamma \to \mathbb{Z}$, $w \cdot \lambda \mapsto l(w)$.

(c) For $\tau, \sigma \in \Gamma$ *in adjacent* p*-alcoves,* $\mathrm{Ext}^1_G(L(\tau), L(\sigma)) \neq 0$.

(d) For $\tau, \nu \in \Gamma$ *satisfying* $l(\tau) \equiv l(\nu)$, *mod 2, we have* $\mathrm{Ext}^1_{\mathcal{C}}(V(\tau), L(\nu)) = 0$.

If these conditions hold, the polynomials (1.3.7) identify with the classical Kazhdan-Lusztig polynomials: $P_{y \cdot \lambda, w \cdot \lambda} = P_{w_0 y, w_0 w}$ $(w \cdot \lambda, y \cdot \lambda \in \Gamma(\lambda))$.

There is an interesting connection between the above theory and the infinitesimal representation theory of G. Suppose that G is defined an split over the prime field $\mathbb{F}_p$. Let G_1T be the pull-back of a fixed maximal split torus T through the Frobenius morphism $F : G \to G$. As is well-known, the irreducible G_1T-modules are indexed by the full character group X on T. Let $\mathcal{C}_0^1$ denote the category of finite dimensional rational G_1T-modules having composition factors $L(\nu)$, where $\nu \in W_p \cdot 0 \in X$. As discussed in [6], $\mathcal{C}_0^1$ is a highest weight category with naturally defined Weyl objects $V_1(\lambda)$ and induced objects $A_1(\lambda)$. The $\uparrow$ partial ordering extends to $W_p \cdot 0$, and there is a natural function $l_0^1 : W_p \cdot 0 \to \mathbb{Z}$ given by setting $l(w \cdot 0)$ equal to the number of hyperplanes separating the bottom p-alcove C from $w \cdot C$.

(1.4.2.3) Theorem. ([6]) *For the conditions below, we have (b)* $\Leftrightarrow$ *(c)* $\Rightarrow$ *(a). If* $p \geq 2h - 3$, *the statements are all equivalent.*

(a) $\mathcal{C}$ *in (1.4.2.2) has a Kazhdan-Lusztig theory;*
(b) $\mathcal{C}_0^1$ *has a Kazhdan-Lusztig theory relative to the function* l_0^1.
(c) *For any two weights* $\tau, \nu \in W \cdot 0$ *which lie in adjacent p-alcoves, we have*

$$Ext^1_{\mathcal{C}_0^1}(L_1(\tau), L_1(\nu)) \neq 0.$$

Furthermore, in [6, Thm. (5.8)], the Poincaré polynomials (1.3.4), (1.3.5) for $\mathcal{C}_0^1$ are determined in terms of the *generic* Kazhdan-Lusztig polynomials for the affine Weyl group W_p [14]. Thus, (1.3.3) gives a complete calculation of the groups $\mathrm{Ext}^\bullet_{\mathcal{C}_0^1}(L_1(\tau), L_1(\nu))$ for p-regular weights τ, ν.

(1.4.3) Quantum enveloping algebras at a root of unity.

Consider the standard arithmetic quantization $\mathfrak{U}_q = \mathfrak{U}_q(\mathfrak{g})$ of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ with parameter q. Assume that q is a primitive ℓth root of unity, where $\ell > 1$ is an odd integer (subject to certain restrictions as described in [17], for example). The category $\mathcal{C}_q$ of integral modules of type 1 forms a highest weight category with weight poset X_+. Then it is not difficult to extend the methods of [5] to obtain a quantum group analogue of (1.4.2.1), taking $\Gamma = X_+$ and replacing W_p by the affine Weyl group W_l. As to the validity of quantum Lusztig conjecture, see [13].

The quantum enveloping algebra $\mathfrak{U}_q$ has generators $E_i, E_i^{(l)}, F_i, F_i^{(l)}$ and $K_i^{\pm 1}$ for $i = 1, 2, \cdots, \mathrm{rank}\,\Phi$. Then $\hat{\mathbf{u}}_q$ denotes the subalgebra of $\mathfrak{U}_q$ generated by $\mathfrak{U}^0, E_i$ and F_i for $i = 1, 2, \cdots, \mathrm{rank}\,\Phi$. Here $\mathfrak{U}_q^0$ is a commutative and cocommutative sub-Hopf algebra of $\mathfrak{U}_q$ corresponding to the Cartan subalgebra of $\mathfrak{g}$. The category $\mathcal{C}_q^1$ of integral modules of type 1 $\hat{\mathbf{u}}_q$ also forms a highest weight category. Using [11], it is possible to extend the results of (1.4.2.3) to $\mathcal{C}_q^1$.

Again, these results give a complete calculation of Ext groups between simple objects in $\mathcal{C}_1$ or $\mathcal{C}_q^1$ having ℓ-regular high weights. (Actually, it is not difficult to obtain Ext-group calculations for $\hat{\mathbf{u}}_q$ directly from those for $\mathfrak{U}_q$, see [17].)

§2. The category $\mathcal{C}^!$

In this section, $\mathcal{C}$ is a highest weight category with a *finite* weight poset λ. Assume that the objects in $\mathcal{C}$ have finite length. Let $L = \oplus_{\lambda\in\Lambda} L(\lambda)$ denote the direct sum of the simple objects in $\mathcal{C}$. Put $S^! = \mathrm{Ext}^\bullet_{\mathcal{C}}(L, L)$. Yoneda multiplication endows the vector space $S^!$ with the structure of an algebra over k. Furthermore, $S^!$ is finite dimensional, since $\mathcal{C}$ has finite global dimension [3]. Let $\mathcal{C}^!$ denote the category of finite dimensional, right $S^!$-modules.

(2.1) Theorem. *([8, Thm. (3.1)]) Assume that* $\mathcal{C}$ *has a Kazhdan-Lusztig theory relative to a length function* $l : \Lambda \to \mathbb{Z}$. *Then* $\mathcal{C}^!$ *is a highest weight category with weight poset* Λ^{op}, *the opposite poset of* Λ.

Proof. We give a sketch of the argument, showing how the ideas of the previous section are applied.

Choose a listing $\lambda_1, \cdots, \lambda_t$ of Λ compatible with the partial ordering, i. e., $\lambda_i < \lambda_j \Rightarrow i < j$. Each $\lambda \in \Lambda$ defines an idempotent $r_\lambda = \mathrm{id}_{L(\lambda)} \in \mathrm{Hom}_{\mathcal{C}}(L(\lambda), L(\lambda)) \subset S^!$. Clearly, $\sum r_\lambda = 1$. Write $r_i = r_{\lambda_i}$.

For $1 \leq j \leq t$, let $f_j = r_1 + \cdots + r_j$, put $J^!_j = S^! f_j S^!$, giving the following chain of idempotent ideals:

$$0 = J^!_0 \subset J^!_1 \subset \cdots \subset J^!_t = S^!. \tag{2.1.1}$$

Also, put $S^!_j = S^!/J^!_{j-1}$. Clearly, for each j, we have surjective homomorphisms

$$S^!_j r_j \otimes_k r_j S^!_j \to S^!_j r_j \otimes_{r_j S^!_j r_j} r_j S^!_j \to S^!_j r_j S^!_j. \tag{2.1.2}$$

Also, it is clear that $S^!_j r_j$ is the image of a surjective homomorphism

$$\mathrm{Ext}^\bullet_{\mathcal{C}}(L(\lambda_j), L) \to S^!_j r_j. \tag{2.1.3}$$

Since $\mathcal{C}$ has a Kazhdan-Lusztig theory, (1.3.1) guarantees that there is a natural surjective homomorphism

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(L(\lambda_j), L) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(V(\lambda_j), L). \tag{2.1.4}$$

The key part of the argument amounts to showing that, again using the Kazhdan-Lusztig hypothesis, the homomorphism (2.1.3) factors through the homomorphism (2.1.4). (See [8] for exact details.) Thus,

$$\dim S^!_j r_j \leq \dim \mathrm{Ext}^\bullet_{\mathcal{C}}(V(\lambda_j), L), \ j = 1, \cdots, t. \tag{2.1.5}$$

Dually, we have inequalities

$$\dim r_j S^!_j \leq \dim \mathrm{Ext}^\bullet_{\mathcal{C}}(L, A(\lambda_j)), \ j = 1, \cdots, t. \tag{2.1.6}$$

Observe that

$$\sum_j (\dim S^!_j r_j) \cdot (\dim r_j S^!_j) \geq \sum_j \dim S^!_j r_j S^!_j = \sum_j \dim J^!_j / J^!_{j-1} = \dim S^!. \tag{2.1.7}$$

But

$$\sum_\nu (\dim \mathrm{Ext}^\bullet_{\mathcal{C}}(V(\nu), L)) \cdot (\dim \mathrm{Ext}^\bullet_{\mathcal{C}}(L, A(\nu))) = \dim S^!$$

by (1.3.3.1). It follows that the surjective homomorphisms in (2.1.2) are, in fact, isomorphisms. In particular, each $r_j S^!_j r_j \cong k$. We have therefore established that the sequence (2.1.1) of idempotent ideals is a defining sequence for $S^!$. Thus, $S^!$ is a quasi-hereditary algebra. $\square$

An important feature of a highest weight category $\mathcal{C}$ with a (finite) weight poset Λ concerns the fact that any ideal $\Gamma \subset \Lambda$ leads to a decomposition (or "recollement") of the associated derived category $D^b(\mathcal{C})$, in a way entirely analogous to the situation arising in perverse sheaf theory relative to a closed subspace; see [1]. More precisely, we have a recollement diagram

$$(2.2) \qquad \begin{tikzcd} D^b(\mathcal{C}[\Gamma]) \arrow[r, "i_*"] & D^b(\mathcal{C}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b(\mathcal{C}(\Omega)) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the properties [1, (1.4.3.1)–(1.4.3.5)]. Here Ω is the coideal $\Lambda\backslash\Gamma$, i_* is induced by the natural inclusion $\mathcal{C}[\Gamma] \to \mathcal{C}$, while j^* is induced by the quotient functor $\mathcal{C} \to \mathcal{C}(\Omega)$.

In the context of (2.1), we have the following result, which slightly strengthens [8, Th. (2.3)]. We assume the hypothesis and notation of (2.1). Let Γ be a non-empty ideal in Λ, put $\Omega = \Lambda\backslash\Gamma$, let $r_\Gamma = \sum_{\gamma\in\Gamma} r_\gamma$, and let $L_\Gamma = \oplus_{\gamma\in\Gamma} L(\gamma)$.

(2.3) Theorem. *With the above notation and assumptions, we have*

$$(2.3.1) \qquad Ext^\bullet_{\mathcal{C}[\Gamma]}(L_\Gamma, L_\Gamma) \cong r_\Gamma S^! r_\Gamma.$$

Also,

$$(2.3.2) \qquad S^!/S^! r_\Gamma S^! \cong Ext^\bullet_{\mathcal{C}(\Omega)}(j^*L, j^*L).$$

These induces natural equivalences

$$(2.3.3) \qquad \mathcal{C}[\Gamma]^! \cong \mathcal{C}^!(\Gamma^{\mathrm{op}}),\ \mathcal{C}(\Omega)^! \cong \mathcal{C}^![\Omega^{\mathrm{op}}]$$

of highest weight categories.

Proof. Statement (2.3.3) is established in [8, Thm. (2.3)]. Also, it is proved there that (2.3.1) holds. (This is essentially obvious from the definitions.) This also proves the first equivalence in (2.3.3), while the second equivalence is shown in [8] to be defined by the inflation functor Ψ^* corresponding to the natural algebra homomorphism $\Psi : S^!/S^! r_\Gamma S^! \to \mathrm{Ext}^\bullet_{\mathcal{C}(\Omega)}(j^*L, j^*L)$. Since Ψ^* is an equivalence, it is clear that Ψ is injective. It follows easily that Ψ is an isomorphism. (Observe that both sides of (2.3.2) have the same dimension, arguing as in the proof of (2.1).) $\square$

Thus, (2.3) shows how the recollement diagram (2.2) for $\mathcal{C}$ "dualizes" to give the corresponding recomment diagram for $\mathcal{C}^!$. Finally, we mention the following corollary of our development. It is closely related to the fact that if the quasi-hereditary algebra S is described by means of the recursive construction [15, §4], then $S^!$ is described by means of the recursive construction [9]. Details will appear elsewhere.

(2.4) Corollary. *Continue to assume the hypotheses of (2.3). Let $\lambda \in \Lambda$ be a maximal element, and set $L' = \bigoplus_{\gamma\neq\lambda} L(\gamma)$. Yoneda multiplication induces an isomorphism*

$$Ext^\bullet_S(L', L(\lambda)) \otimes_{\mathrm{Ext}^\bullet_S(L',L')} Ext^\bullet_S(L(\lambda), L') \xrightarrow{\sim} rad\ Ext^\bullet_S(L(\lambda), L(\lambda)).$$

Here rad denotes the radical of the indicated Ext algebra.

§3. Graded highest weight categories

Suppose that $S = \oplus_{n\in\mathbb{Z}^+} S_n$ is a positively graded, finite dimensional k-algebra. Let $\mathcal{C}_{\rm gr}$ denote the category of finite dimensional, graded, right S-modules. We regard the simple S-modules L as graded modules concentrated in degree 0. There is a shift operation on $\mathcal{C}_{\rm gr}$: if $M = \oplus_{n\in\mathbb{Z}} M_n \in \mathrm{Ob}(\mathcal{C}_{\rm gr})$ and $i \in \mathbb{Z}$, let $M(i)$ be the graded S-module obtained by shifting the grading i places to the right, i e., $M(i)_n = M_{n-i}$. Clearly, this shift also defines a operation $X \mapsto X(i)$ for objects X in $D^b(\mathcal{C}_{\rm gr})$, obtained by applying (i) to each terms of the complex X. If $X \in D^b(\mathcal{C}_{\rm gr})$, we write $X\{i\} \stackrel{\rm def}{=} X(i)[i]$.

The theory of graded quasi-hereditary algebras has been developed in [4]. In particular, if $0 = J_0 \subset J_1 \subset \cdots \subset J_t = S$ is a defining sequence of idempotent ideals in a graded quasi-hereditary algebra S, each J_i is generated by an idempotent $e_i \in S_0$ [4, (5.4)]. It follows that the induced, Weyl, etc. objects $A(\lambda), V(\lambda)$, etc. all carry a natural graded structure, and the axioms of a highest weight category (having finite weight poset) extend naturally to include this graded structure; see [8, (1.2)] for further details. The notion of a graded Kazhdan-Lusztig theory can also be introduced [5, 8]. The homological characterization goes as follows. Fix a length function $l : \Lambda \to \mathbb{Z}$. Then $\mathcal{C}_{\rm gr}$ has a graded Kazhdan-Lusztig theory iff for all $\lambda, \nu \in \Lambda$, we have

$$\mathrm{Ext}^n_{\mathcal{C}_{\rm gr}}(V(\lambda)(m), L(\nu)\{l(\nu\}) \neq 0 \Rightarrow n = -m \equiv l(\lambda), \bmod 2$$

$$\mathrm{Ext}^n_{\mathcal{C}}(L(\nu)\{l(\nu)\}, A(\lambda)(m)) \neq 0 \Rightarrow m = n \equiv l(\lambda) \bmod 2.$$

This is a very strong condition, as suggested by the theorem below. Observe that the graded structure of an algebra S defines a natural *bigrading* on the spaces $\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N)$, for $M, N \in \mathrm{Ob}(\mathcal{C}_{\rm gr})$, since

$$\mathrm{Ext}^\bullet_{\mathcal{C}}(M,N) \cong \bigoplus_{m,n\in\mathbb{Z}} \mathrm{Ext}^m_{\mathcal{C}_{\rm gr}}(M, N(n)).$$

Recall [18] that S is called a *Koszul algebra* provided, for every pair of simple S-modules L, L' (viewed as graded modules concentrated in degree 0) the bigrading on $\mathrm{Ext}^\bullet_{\mathcal{C}}(L, L')$ is concentrated along the diagonal $\{(m, m)\}$.

We can now state the following important result.

(3.1) Theorem. ([8, Thm. (3.9)] *Let S be a graded quasi-hereditary algebra, and let $l : \Lambda \to \mathbb{Z}$ be a length function on the weight poset Λ of $\mathcal{C} =$ mod-S. Then the graded highest weight category $\mathcal{C}_{\text{gr}}$ has a graded Kazhdan-Lusztig theory relative to l if and only if S is a Koszul algebra and $\mathcal{C}$ has a Kazhdan-Lusztig theory relative to l.*

The paper [8] contains a number of further results along these lines. Also, in [16], a criterion in terms of automorphisms is established which guarantees that $\mathcal{C}_{\text{gr}}$ has a graded Kazhdan-Lusztig theory. Making use of the theory of perverse sheaves, this is then applied to the category $\mathcal{O}_{\text{triv}}$. (See also [2].)

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Coxeter transformations and the representation theory of algebras

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Abstract. This work presents a survey (with some proofs) of the recent developments in the study of Coxeter transformations and their applications to representation theory of associative algebras. Let Q be a quiver and $C = C(Q)$ the associated Coxeter matrix. We consider results on the eigenvalues of C: multiplicities, distribution, bounds, relation with the symmetries of Q... We consider applications to the study of the indecomposable modules over the hereditary algebra $k[Q]$ (k a field): position of modules, growth of the Auslander-Reiten translates... Other applications concern growth of components of the representation-quiver of finite dimensional algebras, towers of algebras...

Let Δ be a finite connected graph without loops. Let $\Delta_0 = \{1, \ldots, n\}$ be the set of vertices of Δ. The *Cartan matrix* A of Δ is the $n \times n$-matrix with integer coefficients A_{ij} satisfying:

$$A_{ii} = 2;\ A_{ij} = -a_{ij}, \quad \text{where } a_{ij} \text{ is the number of edges between the vertices } i \text{ and } j \text{ in } \Delta.$$

We consider the real vector space $V = \mathbf{R}^n$ with canonical basis $\{e_1, \ldots, e_n\}$. For each $i \in \Delta_0$, there is a *reflection* R_i on V defined by

$$e_j R_i = e_j - A_{ji} e_i$$

for all $j \in \Delta_0$. Let $\pi \colon \Delta_0 \to \Delta_0$ be a permutation, then the product

$$C = C(\Delta, \pi) = R_{\pi(1)} \ldots R_{\pi(n)},$$

is called a *Coxeter transformation* for Δ.

Coxeter transformations and their properties play an important role in several areas of mathematics: Lie theory [8, 10]; Kac-Moody algebras [25]; the study of singularities [1]; towers of algebras [19]; representation theory of associative algebras and others.

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The purpose of this work is to survey the main properties of Coxeter transformations and their applications to the study of the representations of associative algebras. In the last years there has been several interesting developments: results concerning the study of the eigenvalues of the Coxeter transformation; the growth behaviour of the Auslander-Reiten translation for hereditary and canonical wild algebras...

We will not try to be exhaustive. In general, we will include the sketch or the main ideas of the proofs of the stated results. Some few results appear here for the first time. The setting of the problem is not the most general possible; namely, we do not consider generalized Cartan matrices (valued graphs and species instead of algebras) to avoid some technicalities.

This work is an extended version (of part) of the lectures we presented in the Canadian Mathematical Society Annual Seminar 1992 held in Ottawa in August of that year. We thank V. Dlab and coworkers for the organization and hospitality during the Seminar and ICRA VI.

1. The Coxeter elements of a Weyl group.

1.1. Let Δ be a finite graph as in the introduction and $A = (A_{ij})$ the associated $n \times n$-Cartan matrix. The matrix A satisfies one and only one of the following three properties (see for example [25]):

(Finite) $\det A \neq 0$; there exists $u > 0$ such that $uA > 0$

(Affine) $\det A = 0$; there exists $u > 0$ such that $uA = 0$

(Indefinite) there exists $u > 0$ such that $uA < 0$.

It turns out that A is of finite type if and only if Δ is a *Dynkin diagram*; A is of affine type if and only if Δ is an *affine* (also called *Euclidean*) *diagram.*

1.2. Let $V = \mathbf{R}^n$ and $\{e_1, \dots, e_n\}$ be the canonical basis. For each $i \in \Delta_0$, let R_i be a reflection as defined above. The *Weyl group* $W(\Delta)$ associated to Δ is the subgroup of $GL(V)$ generated by the reflectons $R_1, \dots, R_n$. For each pair $i, j \in \Delta_0$, the order m_{ij} of R_iR_j in $W(\Delta)$ is given in the following way:

$$m_{ii} = 1;\ m_{ij} = 3 \text{ (resp. } \infty) \text{ if } a_{ij} = 1 \text{ (resp. } a_{ij} \geq 2);\ m_{ij} = 2, \text{ otherwise.}$$

The group $W(\Delta)$ is a Coxeter group with presentation

$$\langle s_1, \dots, s_n | (s_is_j)^{m_{ij}} = 1 \rangle.$$

The Weyl group appears for the first time in 1935 in Weyl's approach to the study of semisimple Lie algebras. Although finite dimensional Lie algebras were described by Killing and Cartan at the end of last century, many concepts were developed by Weyl later. His influential 1935–lectures were given at Princeton and the notes were written by Brauer and Coxeter.

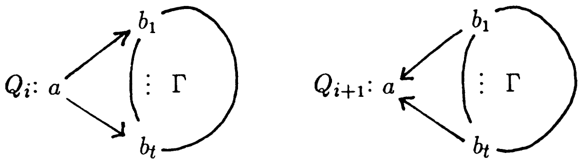
If the Cartan matrix A is of finite type, then the Weyl group $W(\Delta)$ is finite. If A is of affine type, then $W(\Delta)$ is a semidirect product $W' \ltimes T$, where

$W' = W(\Delta')$ is a finite Weyl group associated to a full subgraph Δ' of Δ and T is an abelian group. See [8, Chapter VI] and [25, 6.5].

1.3. Given a permutation $\pi\colon \Delta_0 \to \Delta_0$, the product $C = C(\Delta, \pi) = R_{\pi(1)} \cdots R_{\pi(n)} \in W(\Delta)$ is called a *Coxeter element* of $W(\Delta)$.

In general, the conjugation class of a Coxeter element depends on the permutation π. Observe that π also defines a quiver (= oriented graph) $Q = \Delta(\pi)$ whose underlying graph is Δ: set $i \to j$ for each edge between i and j provided $\pi^{-1}(i) < \pi^{-1}(j)$ and $A_{ij} \neq 0$. In fact, the Coxeter element $C(\Delta, \pi)$ depends only on Q, hence we also write $C(Q) = C(\Delta, \pi)$.

Given two quivers Q, Q' with the same underlying graph Δ, we say that Q and Q' have *equivalent orientations* if there is a sequence $Q = Q_1, Q_2, \ldots, Q_s = Q'$ of quivers with underlying graph Δ such that Q_{i+1} is obtained from Q_i by transforming a source into a sink as in the figure:



The following facts are easy to check:

a) If Q and Q' have equivalent orientations, then the Coxeter matrices $C(Q)$ and $C(Q')$ are conjugate.

b) If Δ is a tree (i.e. a graph without cycles) and Q, Q' are two quivers with underlying graph Δ, then Q and Q' have equivalent orientations.

1.4. Proposition. *Let Δ be a graph, $\pi\colon \Delta_0 \to \Delta_0$ a permutation and $Q = \Delta(\pi)$ the corresponding quiver.*
a) *If Δ is of Dynkin type, then all Coxeter elements in $W(\Delta)$ are conjugate and have the same finite order h (= Coxeter number of Δ).*
b) *The Coxeter element $C = C(Q)$ has an invariant vector $u \geq 0$ (that is $0 \neq u \geq 0$ and $uC = u$) if and only if Δ is of Euclidean type.*

The first statement of (a) follows directly from (1.3.b).

In (1.2) we mention that $W(\Delta)$ is a finite group in case Δ is of Dynkin type. In fact, there is a direct argument showing that $C = C(Q)$ has finite order: since the quadratic form defined by the Cartan matrix A is positive, it only admits finitely many *roots* (i.e. vectors v with $vAv^t = 1$). Since C preserves roots of A (which form a generating set of $\mathbf{R}^n$), then $C^h = 1$ for some $h \geq 1$.

Consider the quiver $Q = \Delta(1)$ with underlying graph Δ. We define the $n \times n$-matrix $M = (M_{ij})$ given by

$$M_{ii} = 1; \ M_{ij} = -a_{ij} \text{ if } i \to j \text{ in } Q; \quad M_{ij} = 0, \text{ otherwise.}$$

Observe that $A = M + M^t$. Moreover, $C = -MM^{-t}$.

For the proof of (b), let $0 \neq u$ be a non-negative vector, then $uC = u$ if and only if $uA = 0$ which happens exactly when Δ is of Euclidean type.

1.5. The characteristic polynomial $\chi_C(T) = \det(TI - C)$ is called the *Coxeter polynomial* of $C = C(Q)$.

In the finite and affine situations, the Coxeter polynomial is well known. If Δ is of Dynkin type and $Q = \Delta(\pi)$, then

$$\chi_C(T) = \prod_{j=1}^{n}(T - \exp(2i\pi m_j/h))$$

where $m_1, \ldots, m_n$ are integers such that $1 = m_1 < m_2 \leq \cdots \leq m_{n-1} < m_n = h-1$, and h is the Coxeter number of Δ. The order of $W(\Delta)$ is $(m_1+1)(m_2+1)\cdots(m_n+1)$. See [8, Chap. V].

If Δ is of Euclidean type and an extension of the Dynkyn diagram Δ' with $n-1$ vertices, then

$$\chi_C(T) = (T-1)^2 \prod_{j=1}^{n-2}(T - \exp(2i\pi m_j/h'))$$

where $1 \leq m_1 \leq \cdots \leq m_{n-2} \leq h' - 1$ are integers and h' is the Coxeter number of Δ'.

The following result is due to Howlett [23]; previous versions in the case Δ is a tree were shown in [1] and [41].

Theorem [23]. *Let $\Delta, A, Q = \Delta(\pi)$, $C = C(Q) \in W(\Delta)$ be as above. The following are equivalent:*
a) *$W(\Delta)$ is an infinite group.*
b) *C has infinite order.*
c) *C has a real eigenvalue greater or equal to 1.*
d) *The quadratic form associated to A is not positive.*

1.6. To indicate the main steps for the proof of Theorem (1.5), we need some concepts and lemmata.

Let $C = C(Q)$ be a Coxeter element in $W(\Delta)$. By renumbering the vertices of Δ, we may assume that $Q = \Delta(1)$. Define $M = (M_{ij})$ a $n \times n$-matrix as in (1.4). Then $A = M + M^t$ and $C = -MM^{-t}$. Observe that the matrix M has non-positive off-diagonal entries and positive principal minors (in fact, equal to 1). A matrix satisfying these conditions is called a *M-matrix.*

Lemma. i) *Let M be a real $n \times n$-matrix such that $M + M^t$ defines a positive quadratic form, then M is non-singular and $-MM^{-t}$ is diagonalizable over $\mathbf{C}$, all*

its eigenvalues having modulus 1.
ii) *Let M be a M-matrix such that $M + M^t$ defines a non-positive quadratic form. Then $-MM^{-t}$ has a real eigenvalue $\lambda \geq 1$. If $M + M^t$ does not define a non-negative form, then $\lambda > 1$. If $M + M^t$ defines a non-negative form, then all eigenvalues have modulus one, 1 is a repeated eigenvalue and $-MM^{-t}$ is not diagonalizable.* □

Sketch of the proof of (1.5): (a) $\Rightarrow$ (d): If A defines a positive quadratic form, then Δ is of Dynkin type and $W(\Delta)$ is finite.

(d) $\Rightarrow$ (c): Part (ii) of the Lemma.

(c) $\Rightarrow$ (d): If A defines a positive form, then 1 should be an eigenvalue of $C = -MM^{-t}$. By (1.4), Δ is of Euclidean type and A is singular, a contradiction.

(d) $\Rightarrow$ (b): By the lemma, either C has an eigenvalue $\lambda > 1$ or C is not diagonalizable. In both cases the order of C is infinite.

(b) $\Rightarrow$ (a): is obvious. □

1.7. The following interesting result is due to J. Tits:

Theorem [43]. *Let Δ be a graph not of Dynkin or Euclidean type and $n \geq 3$. Then the Weyl group $W(\Delta)$ contains a non-abelian free subgroup.*

The original result of Tits covers a more general situation. A shorter and more direct proof was given by P. de la Harpe [20]. We still simplified that proof in [30]. The proof relies on a careful study of the behaviour of the Coxeter elements of $W(\Delta)$.

The *spectral radius* $\rho(L)$ of a linear transformation L of $\mathbf{R}^n$ is the maximum of the absolute values of the eigenvalues of L. If L is non-singular, we say that L is *sharp* if it has an eigenvalue λ such that $\|\mu\| < \|\lambda\| (= \rho(L))$ for any other eigenvalue μ of L. We say that L is *very sharp* if both L and L^{-1} are sharp.

Let L and N be two very sharp linear transformations of $\mathbf{R}^n$. Let y^+ (resp. y^-, z^+, z^-) be an eigenvector of L (resp. L^{-1}, N, N^{-1}) with eigenvalue λ of norm $\|\lambda\| = \rho(L)$ (resp. $\rho(L^{-1}), \rho(N), \rho(N^{-1})$). Then L and N are said to be *in general position* if y^+ and y^- (resp. z^+ and z^-) are not eigenvectors of N (resp. of L).

Let Δ be a graph not of Dynkin or Euclidean type. By Ringel's recent result (2.1), any Coxeter element $C \in W(\Delta)$ is very sharp. It is not hard to see [30], there exists some vertex $j \in \Delta_0$ such that the transformations C and R_jCR_j are two very sharp elements in general position. Then the Theorem follows from the following result.

Proposition [20]. *Let L and N be two very sharp transformations in general position. Then there exists a number m_0 such that for any $m \geq m_0$, the subgroup* *of $GL(\mathbf{R}^n)$ generated by $\{L^m, N^m\}$ is free.*

Sketch of proof: Let y^+, y^-, z^+, z^- be eigenvectors as in the definition, we may assume that all four vectors have euclidean norm 1. Consider the images $\bar{L}, \bar{N}$ of L and N in $PGL(\mathbf{R}^n) = GL(\mathbf{R}^n)/\mathbf{R}^*$. Hence $y^+, y^- \in \mathbf{S}^n$ are fixed points of $\bar{L}$, moreover y^+ (resp. y^-) is an attractive (resp. repulsing) point of $\bar{L}$. Therefore, we may find disjoint compact sets K and K' with $y^+, y^- \in K$, $z^+, z^- \in K'$ and such that $\bar{L}^m(K') \subset K$, $\bar{N}^m(K) \subset K'$ for $|m| \geq m_0$.

Let $m \in \mathbf{Z}$ be such that $|m| \geq m_0$ and let $L_0 = L^m$ and $N_0 = N^m$. We shall prove that the group generated by $\{L_0, N_0\}$ is free. This is an application of an argument of Klein which de la Harpe names "ping-pong criterium". Indeed, assume that

$$L_0^{a_1} N_0^{b_1} L_0^{a_2} \cdots N_0^{b_{s-1}} L_0^{a_s} = 1.$$

If $a_1 \neq 0 \neq a_s$, then

$$K' = \bar{L}_0^{a_1} \bar{N}_0^{b_1} \cdots \bar{N}_0^{b_{s-1}} \bar{L}_0^{a_s}(K') \subset \bar{L}_0^{a_1} \bar{N}_0^{b_1} \cdots \bar{N}_0^{b_{s-1}}(K) \subset \cdots \subset \bar{L}_0^{a_1}(K') \subset K,$$

a contradiction. If $a_1 = 0 < a_s$, then $L_0^{-1} N_0^{b_1} L_0^{a_2} \cdots N_0^{b_{s-1}} L_0^{a_s+1} = 1$ and we are in the situation of the first case. The remaining cases are similar. □

2. Eigenvalues of a Coxeter transformation.

2.1. Let L be a linear transformations of $\mathbf{R}^n$. The *spectrum* Spec(L) of L is the set of all eigenvalues of L, that is, the roots of the characteristic polynomial $\chi_L(T)$; the *multiplicity* $m(\lambda)$ of an eigenvalue λ of L is its multiplicity as a root of $\chi_L(T)$. The *spectral radius* $\rho(L)$ is the positive real number

$$\rho(L) = \max\{\|\lambda\| : \lambda \in \operatorname{Spec}(L)\}.$$

The degree $d(\lambda)$ of an eigenvalue $\lambda \in \operatorname{Spec}(L)$ is the maximal size of the Jordan λ-blocks of L.

Let Δ be a graph and $C = C(Q) \in W(\Delta)$ be a Coxeter transformation. If Δ is of Dynkin type, we know that $\rho(C) = 1 \notin \operatorname{Spec}(C)$; if Δ is of Euclidean type, then $\rho(C) = 1 \in \operatorname{Spec}(C)$ and $m(\rho(C)) = 2$. The main result in this context was completed recently by Ringel.

Theorem [1, 38]. *Assume that Δ is neither of Dynkin nor of Euclidean type. Let $C = C(Q) \in W(\Delta)$ be a Coxeter transformation. Then*
i) *$\rho(C) > 1$ and $\rho(C)$ is an eigenvalue of C with multiplicity one.*
ii) *For any other $\mu \in \operatorname{Spec}(C)$, $\|\mu\| < \rho(C)$.*
iii) *There exists a vector y^+ with all its coordinates positive (we write $y^+ >> 0$) such that $y^+C = \rho(C)y^+$.*

Partial results were known before: in [1] and [41] a proof was given in case the quiver $Q = \Delta(\pi)$ is *bipartite* (i.e. any vertex of Q is either a source or a sink); in [13], it was shown that $\rho(C) \in \operatorname{Spec}(C)$ and for any other $\mu \in \operatorname{Spec}(C)$, with $\|\mu\| = \rho(C)$, $d(\mu) \leq d(\rho(C))$; finally, [42] showed that the eigenspace for the eigenvalue $\rho(C)$ is one-dimensional.

In order to sketch the proof of the Theorem, we first consider the bipartite case following [1].

2.2. Let Δ be a graph and $Q = \Delta(\pi)$ be a *bipartite quiver*. Assume that the vertices $1, \dots, m$ are the sources of Q and $m+1, \dots, n$ the sinks.

The *adjacency matrix* $B = B(\Delta)$ is the $n \times n$-matrix whose (i,j)-entry B_{ij} is the number of edges between i and j. Then B takes the form $B = N + N^t$, where

$$N = \begin{array}{cc} \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} & \begin{array}{l} \}m \\ \end{array} \\ \begin{array}{l} \underbrace{}_{m} \end{array} & \end{array}$$

Lemma [1]. *Let $C = C(Q)$ be a Coxeter transformation for Δ. Then*

$$\chi_C(T^2) = T^n \chi_B(T + T^{-1})$$

Proof: Let M be the M-matrix defined in (1.4) such that

$$A = M + M^t \qquad \text{and} \quad C = -MM^{-t}.$$

Hence $M = I - N$. Since $N^2 = 0$, then $M^{-1} = I + N$. Therefore

$$\begin{aligned} \det(T^2 I - C) &= \det(T^2 I + (I - N)(I + N)^t) \det(I - N^t) \\ &= \det(T^2 I - T^2 N^t + (I - N)) \\ &= T^n \det((T + T^{-1})I - TN^t - T^{-1}N) = T^n \det((T + T^{-1})I - B). \end{aligned}$$

The last equality follows from

$$(T + T^{-1})I - B = \begin{bmatrix} I & 0 \\ 0 & T^{-1}I \end{bmatrix} \begin{bmatrix} (T + T^{-1})I & -T^{-1}D \\ -TD^t & (T + T^{-1})I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & TI \end{bmatrix}. \qquad \square$$

2.3. Proposition [1]. *Let the notation be as in (2.2). Then*
i) *Given $0 \neq \lambda \in \mathbf{C}$, then $\lambda^2 \in \operatorname{Spec}(C)$ if and only if $\lambda + \lambda^{-1} \in \operatorname{Spec}(B)$*
ii) $\operatorname{Spec}(C) \subset \mathbf{S}^1 \cup \mathbf{R}^+$, *where* $\mathbf{S}^1 = \{\lambda \in \mathbf{C} \colon \|\lambda\| = 1\}$

iii) *If Δ is not of Dynkin type, there exists a number $\lambda \geq 1$ such that $\rho(B) = \lambda + \lambda^{-1}$ and $\rho(C) = \lambda^2$. If Δ is neither of Euclidean type, then $\lambda > 1$.*

Proof: i) follows directly from (2.2). For (ii), observe that $\mathrm{Spec}(B) \subset \mathbf{R}$, since B is symmetric. Assume $\mu = \lambda + \lambda^{-1} \in \mathbf{R}$ with $\lambda = a + ib \in \mathbf{C}$, then either $b = 0$ or $\|\lambda\| = 1$.

(iii): Since B is a positive real matrix, then Perron-Frobenius Theorem implies that $\rho(B) \in \mathrm{Spec}(B)$ (see (2.4) for more etails). Hence there is some $\mu \in \mathbf{C}$ with $\mu + \mu^{-1} = \rho(B)$ and $\mu^2 \in \mathrm{Spec}(C)$. Since Δ is not of Dynkin type, then $\rho(B) \geq 2$ and we may choose $\mu \geq 1$. Then it is easy to check that $\mu^2 = \rho(C)$. If Δ is neither of Euclidean type, then by (1.4) and the Perron-Frobenius Theorem, $\rho(C) > 1$ and hence $\mu > 1$. □

Proof of (2.1) *in the bipartite case:* (i): Let $\lambda > 1$ with $\rho(B) = \lambda + \lambda^{-1}$ and $\rho(C) = \lambda^2$. Since Δ is connected, the adjacency matrix is irreducible, then the Perron-Frobenius Theorem asserts that $\rho(B)$ has multiplicity one as eigenvalue of B. Using that $\chi_C(T^2) = T^n \chi_B(T + T^{-1})$, we get

$$
\begin{aligned}
0 &\neq \frac{1}{2} T^{n-1}(1 - T^{-2}) \frac{d}{d(T + T^{-1})}(\det((T + T^{-1})I - B))|_{T=\lambda} \\
&= \frac{d}{dT^2}(\det(T^2 I - C))|_{T=\lambda}
\end{aligned}
$$

(ii): Follows from $\rho(C) > 1$ and $\mathrm{Spec}(C) \subset \mathbf{S}^1 \cup \mathbf{R}^+$.

In (2.6) we will explicity construct all the eigenvectors of C, in particular showing (iii). □

2.4. Let $V = \mathbf{R}^n$. A closed subset K of V is called a *cone* if it satisfies: $K + K \subset K$; $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap (-K) = 0$. The interior of a cone K is denoted by K^0. The *cone* K is called *solid* if $K^0 \neq \emptyset$, or equivalently, if K contains a basis of V.

The *Perron-Frobenius Theorem* (see [18, 40]) in the way it was generalized by Birkhoff and Vandergraft [6, 44] states the following:

Let L be a linear transformation of V leaving a solid cone K invariant (that is, $KL \subset K$), then
i) *$\rho(L)$ is an eigenvalue of L.*
ii) *The degree $d(\rho(L)) \geq d(\lambda)$ for any $\lambda \in \mathrm{Spec}(L)$ with $\|\lambda\| = \rho(L)$.*
iii) *K contains an eigenvector corresponding to $\rho(L)$.*
iv) *If $(K \setminus \{0\})L^s \subset K^0$ for some $s \geq 1$, then the multiplicity of $\rho(L)$ is one as eigenvalue of L. Moreover, if $\rho(L) \neq \lambda \in \mathrm{Spec}(L)$, then $\|\lambda\| < \rho(L)$.*

The proof of (2.1) is an application of Perron-Frobenius Theorem's: given the Coxeter transformation $C = C(Q) \in W(\Delta)$ for a graph Δ not of Dynkin or Euclidean type, a basis $b_1, \dots, b_n$ of $\mathbf{R}^n$ is constructed such that the solid cone K generated by this basis satisfies $(K \setminus \{0\})C^s \subset K^0$, for some $s \geq 1$.

In view of (2.3), we may assume that Δ is not a tree. Let $Q = \Delta(\pi)$ be a quiver with underlying graph Δ. A *grip* for Q is a path $i_1 \to i_2 \to \cdots \to i_t$ in Q satisfying the following conditions:

(1) i_1 is the only source of Q and $\sum_{j=1}^{n} A_{ji_1} \leq -1$.

(2) There exists a path $i_1 = x_1 \to x_2 \to \cdots \to x_{s-1} \to x_s = i_t$ with $x_2 \neq i_2$ and $x_{s-1} \neq i_{t-1}$.

(3) For $1 < r < t$, there is only one path starting at i_1 and ending at i_r and only one path starting at i_r and ending at i_t.

If Δ is not a tree and not of (Euclidean) type $\tilde{\mathbf{A}}_n$ then there is a quiver $Q' = \Delta(\pi')$ having a grip and such that Q and Q' have equivalent orientations (see [38]).

For the *proof of* (2.1) we may assume that $Q = \Delta(\pi)$ has a grip $i_1 \to i_2 \to \cdots \to i_t$. We may assume that $i_r = r$ for $1 \leq r \leq t$.

The basis we want to consider is the following: for $t+1 \leq i \leq n$, let $b_i = e_i$, whereas $b_i = \sum_{j=i}^{t} e_j$ for $1 \leq i \leq t$. Let K be the set of all linear combinations $\sum_{i=1}^{n} \lambda_i b_i$ with $\lambda_i \geq 0$. Then K is a solid cone with interior K^0 formed by the linear combinations $\sum_{i=1}^{n} \lambda_i b_i$ with $\lambda_i > 0$. For this cone, Ringel shows that $KC \subset K$ and $(K \setminus \{0\})C^{h+t+2} \subset K^0$, where h is the length of the longest path in Q. We omit the details. This would show (i) and (ii) of (2.1).

Let $y^+ \in K$ be an eigenvector of C with eigenvalue $\rho(C)$. Therefore $y^+ \in K^0$ and the coordinates y_i^+ satisfy that $y_i^+ > 0$ for all i and $y_1^+ < y_2^+ < \cdots < y_t^+$. In particular, $y^+ >> 0$.

2.5. In the last paragraphs of the section we return to consider in more detail the *bipartite case*.

Let Δ be a graph and $Q = \Delta(\pi)$ be a bipartite quiver. As in (2.2) we assume that the vertices $1, \dots, m$ are sources of Q and $m+1, \dots, n$ are sinks. We define the adjacency matrix $B = N + N^t$ as before.

Let $C = C(Q)$. By (2.3), we know that $\operatorname{Spec}(C) \subset \mathbf{S}^1 \cup \mathbf{R}^+$. Following [41], we may get an explicit expression for the eigenvalues and eigenvectors of C.

Let $\nu_1, \dots, \nu_m$ be the eigenvalues of the symmetric positive matrix $E = DD^t$. Let $x_1, \dots, x_m$ be an orthonormal basis of eigenvectors for E, with ν_i the eigenvalue of x_i. Assume that $\nu_i \neq 0, 4$, for $1 \leq i \leq p$; $\nu_i = 4$ for $p+1 \leq i \leq q$ and $\nu_i = 0$ for $q+1 \leq i \leq m$. We obtain eigenvalues λ_{ij} and eigenvectors y_{ij} of C as follows:

$$\lambda_{i1} = \frac{1}{2}\nu_i - 1 + \frac{1}{2}\sqrt{\nu_i(\nu_i - 4)}, \qquad y_{i1} = (x_i, \frac{1}{\lambda_{i1}+1}x_iD)$$

$$\lambda_{i2} = \frac{1}{2}\nu_i - 1 - \frac{1}{2}\sqrt{\nu_i(\nu_i - 4)}, \qquad y_{i2} = (x_i, \frac{1}{\lambda_{i2}+1}x_iD)$$

with $y_{ij}C = \lambda_{ij}y_{ij}$, for $j = 1, 2$ and $1 \leq i \leq p$;

$$\lambda_{i1} = 1,\ y_{i1} = (x_i, \frac{1}{2}x_iD); \quad \lambda_{i2} = 1,\ y_{i2} = \frac{1}{4}(x_i, -\frac{1}{2}x_iD)$$

with $y_{i1}C = y_{i1}$ and $y_{i2}C = y_{i2} + y_{i1}$, for $p + 1 \leq i \leq q$;

$$\lambda_{i1} = -1,\ y_{i1} = (x_i, 0) \text{ with } x_iD = 0 \text{ for } q + 1 \leq i \leq m;$$

$$\lambda_{i2} = -1,\ y_{i2} = (0, x_i') \text{ with } 0 \neq x_i',\ x_i'D = 0 \text{ for } q + 1 \leq i \leq n - m$$

with $y_{ij}C = \lambda_{ij}y_{ij}$.

Hence for $1 \neq \lambda \in \operatorname{Spec}(C)$, the multiplicity of λ is the dimension of the λ-eigenspace of C. The multiplicity of 1 as eigenvalue of C is twice the dimension of the 1-eigenspace of C.

Problem 1: Is it true that for any graph Δ and a Coxeter transformation $C \in W(\Delta)$, we have $\operatorname{Spec}(C) \subset \mathbf{S}^1 \cup \mathbf{R}^+$?

It is known that the answer is negative for graphs with valuation [4].

2.6. Let Δ be a graph. For any vertex $i \in \Delta_0$, the *degree* $d(i)$ is the number of vertices $j \in \Delta_0$ which are joined with i by an edge. The vertex $i \in \Delta_0$ is a *ramification point* if $d(i) \geq 3$.

The following result was obtained in discussions with H. Lenzing:

Proposition. *Let Δ be a tree graph with s ramification points. Let $Q = \Delta(\pi)$ be a bipartite quiver and $C = C(Q)$ be a Coxeter transformation. Let $\lambda_1, \ldots, \lambda_t$ be the real positive eigenvalues of C and $m(\lambda_i)$ be the multiplicity of λ_i.*
Then $\sum_{i=1}^{t} m(\lambda_i) \leq 2s$.

Proof: For $s = 0$, the graph Δ is of (Dynkin) type $\mathbf{A}_n$. Hence by (1.5), C has no real positive eigenvalues. So we assume that $s \geq 1$.

Let B be the adjacency matrix of Δ (2.2). Then $\mu^2 \in \operatorname{Spec}(C)$ if and only if $\mu + \mu^{-1} \in \operatorname{Spec}(B)$. Hence for $\lambda = \mu^2 \in \operatorname{Spec}(C) \cap \mathbf{R}^+$, $2 \leq \mu_\lambda := \mu + \mu^{-1} \in \operatorname{Spec}(B)$. Since $\mu_\lambda = \mu_{\lambda^{-1}}$, then

$$|\operatorname{Spec}(C) \cap (\mathbf{R}^+ \setminus \{1\})| = 2|\operatorname{Spec}(B) \cap (2, \infty)|\,.$$

Moreover the multiplicity $m_C(\lambda)$ of λ as eigenvalue of C is the same that the multiplicity $m_B(\mu_\lambda)$ of μ_λ as eigenvalue of B, if $\lambda \neq 1$.

Indeed, let N be the matrix

$$N = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$$

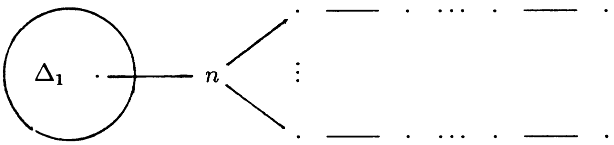
as in (2.2) such that $B = N + N^t$. For each eigenvalue $\nu \neq 0$ of the matrix $E = DD^t$ with eigenvector x, we may define eigenvectors and eigenvalues of B as follows:

$$\beta_1 = \sqrt{\nu},\ \beta_2 = -\sqrt{\nu},\ z_i = (x, \beta_i^{-1} x D) \text{ with } z_i B = \beta_i z_i,\ i = 1, 2.$$

By the description given in (2.5), we get $m_C(\lambda) = m_B(\mu_\lambda)$. This argument also shows that for $\lambda = 1$ (i.e. $\mu_\lambda = 2$), we have $m_C(1) = 2m_B(2)$. [A direct proof of this statements could be obtained using Lemma (2.2)].

Summarizing, we have $\sum\limits_{i=1}^{t} m_C(\lambda_i) = 2 \sum\limits_{\mu \in \mathrm{Spec}\,(B) \cap [2,\infty)} m_B(\mu)$. Hence, it remains to show that $\sum\limits_{\mu \in \mathrm{Spec}\,(B) \cap [2,\infty)} m_B(\mu) \leq s$.

We may consider the tree Δ in the following form



where $\Delta' = \Delta \setminus \{n\} = \Delta_1 \sqcup \Delta_2 \sqcup \ldots \sqcup \Delta_\ell$ with $\Delta_2, \ldots, \Delta_\ell$ linear graphs. Let B' be the adjacency matrix of Δ', then

$$B = \begin{bmatrix} B' & y^t \\ y & 0 \end{bmatrix}$$

Let $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n-1}$ be the eigenvalues of B' (considered with multiplicities). Since Δ' has only $s-1$ ramification points, then by induction hypothesis

$$\nu_k \geq 2 > \nu_{k+1} \text{ with } k \leq s - 1.$$

Let $\hat{\nu}_1 \geq \hat{\nu}_2 \geq \cdots \geq \hat{\nu}_n$ be the eigenvalues of B. Then by the *interlacing eigenvalues Theorem* (see [22, 4.3]), we have

$$\hat{\nu}_1 \geq \nu_1 \geq \hat{\nu}_2 \geq \nu_2 \geq \cdots \geq \nu_k \geq \hat{\nu}_{k+1} \geq \nu_{k+1} \geq \cdots \geq \nu_{n-1} \geq \hat{\nu}_n.$$

Therefore there are at most $(k+1) \leq s$ eigenvalues of B which are greater or equal than 2. $\square$

2.7. Let $C = C(Q) \in W(\Delta)$ be a Coxeter transformation and $\chi_C(T)$ the Coxeter polynomial of C.

We recall that a non-zero polynomial $f(T) = \sum_{i=0}^{n} a_i T^i \in \mathbf{C}[T]$ of degree n is said to be *reciprocal* if it satisfies one of the following equivalent conditions:

i) $f(T) = T^n f(T^{-1})$

ii) $a_i = a_{n-i}$ for all $i \in \mathbf{Z}$.

Lemma. *The Coxeter polynomial $\chi_C(T)$ is reciprocal. In particular, if $\alpha \in \mathbf{C}$ is a root of $\chi_C(T)$, then α^{-1} is also a root with the same multiplicity.*

Proof: Since $TI - C = -TC(T^{-1}I - C^{-1})$, we get that

$$\chi_C(T) = (-T)^n \det C \cdot \det(T^{-1}I - C^{-1})$$

On the other hand $C^t = M^{-1}C^{-1}M$ where M is defined in (1.4), implies that $\det(T^{-1}I - C^{-1}) = \chi_C(T^{-1})$. Since $\det C = (-1)^n$ we get the result. □

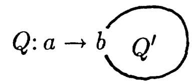
Consider the factorization of $\chi_C(T)$ in $\mathbf{Z}[T]$ into irreducible polynomials, $\chi_C(T) = p_1(T) \cdots p_r(T)$.

Problem 2: Are the irreducible factors $p_i(T)$ of $\chi_C(T)$ reciprocal polynomials?

In Section 4 we will consider some relations between the irreducible factorization of $\chi_C(T)$ and the symmetries of the quiver Q such that $C = C(Q)$.

2.8. For the explicit calculation of the Coxeter polynomial $\chi_C(T)$ of a Coxeter transformation $C = C(Q)$ the following *reduction formulas* are useful (see [7] for more results in this direction).

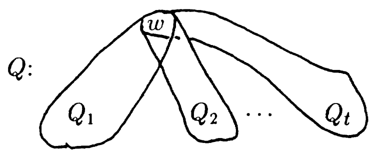
Assume Q is a one-point extension of Q' as follows



Let $C = C(Q)$, $C' = C(Q')$ and $C'' = C(Q'')$ where $Q'' = Q' \setminus \{b\}$, then

$$\chi_C(T) = (1+T)\chi_{C'}(T) - T\chi_{C''}(T).$$

Assume $Q_1, \ldots, Q_t$ are full subquivers of Q as in the picture

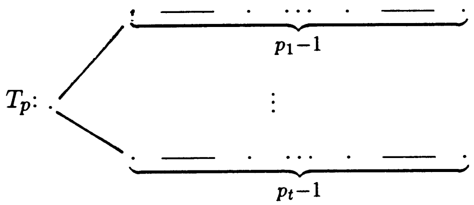


that is, every vertex of Q lies in some Q_i and the intersection of Q_i and Q_j, for $i \neq j$, is just the vertex w. Let $Q'_i = Q_i \setminus \{w\}$ and $C_i = C(Q_i)$, $C'_i = C(Q'_i)$, then we have

$$\chi_C(T) = \left(\prod_{i=1}^{t} \chi_{C'_i}(T)\right)\left(\sum_{j=1}^{t} \frac{\chi_{C_j}(T)}{\chi_{C'_j}(T)} - (t-1)(T+1)\right).$$

As a simple application of these formulas we show some features of the Coxeter polynomial of a star graph.

Let $p = (p_1, \ldots, p_t)$, $t \geq 3$, be a sequence of positive integers $p_1 \leq p_2 \leq \cdots \leq p_t$, and let $n = \sum_{i=1}^{t} p_i - t$. The *star graph* T_p is the graph



Lemma. *Let Q be any quiver whose underlying graph is T_p and let $C = C(Q)$ be a Coxeter transformation. Then the Coxeter polynomial $\chi_C(T)$ has the form*

$$\chi_C(T) = 1 + T - a_2T^2 - a_3T^3 - \cdots - a_{n-2}T^{n-2} + T^{n-1} + T^n$$

for some integers $a_i \geq 0$, $i = 2, \ldots, n-2$.

Proof: We consider succesively the following cases:

(a) $p = (2, 2, \ldots, 2)$; (b) $n \leq 7$; (c) the general case.

(a): Let $p_1 = \cdots = p_t = 2$ and $t \geq 3$. Let $\Delta = T_p$ and $C \in W(\Delta)$ be a Coxeter element. Then

$$\begin{aligned}
\chi_C(T) &= [1 + (2-t)T + T^2](1+T)^{t-1} \\
&= [1 - a_1T + T^2](1 + c_1T + \cdots + c_{t-2}T^{t-2} + T^{t-1}) \\
&= 1 + (c_1 - a_1)T + (1 + c_2 - a_1c_1)T^2 \\
&\quad + \cdots + (c_{i-2} + c_i - a_1c_{i-1})T^i + \cdots + (c_{t-2} - a_1)T^t + T^{t+1}
\end{aligned}$$

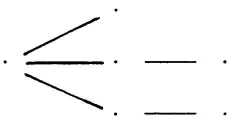
where $a_1 = t - 2$ and $c_i = \binom{t-1}{i}$.

Clearly, $c_1 - a_1 = 1 = c_{t-2} - a_1$. Moreover, for $2 \leq i \leq (t-1) - i$,

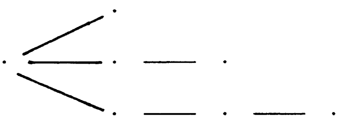
$$c_{i-2} + c_i - a_1c_{i-1} \leq \binom{t-1}{i-1}\left[\frac{t-1}{i} - (t-2)\right] \leq 0.$$

(b) We have only few cases to check:

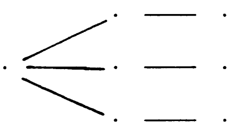
$$\chi_C(T) = 1 + T + T^{n-1} + T^n$$



$$\chi_C(T) = 1 + T - T^3 + T^5 + T^6$$



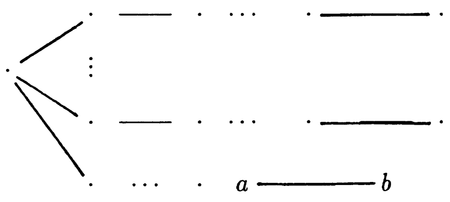
$$\chi_C(T) = 1 + T - T^3 - T^4 + T^6 + T^7$$



$$\chi_C(T) = 1 + T - 2T^3 - 2T^4 + T^6 + T^7.$$

(c) For the general case we may assume that $p_t > 2$ and $n > 7$. Let $\Delta = T_p$ be as in the picture

Δ:



Let $\Delta' = \Delta \setminus \{b\}$ and $\Delta'' = \Delta \setminus \{a, b\}$. Consider $C \in W(\Delta)$, $C' \in W(\Delta')$ and $C'' \in W(\Delta'')$ be Coxeter elements. By induction hypothesis $\chi_{C'}(T)$ and $\chi_{C''}(T)$ have the desired form:

$$\chi_{C'}(T) = 1 + T - a_2T^2 - \cdots - a_{n-2}T^{n-2} + T^{n-1} + T^n$$

$$\chi_{C''}(T) = 1 + T - b_2T^2 - \cdots - b_{n-3}T^{n-3} + T^{n-2} + T^{n-1}$$

with a_i, b_i non-negative integers. Moreover, we assume that $b_i \leq a_i$, for $i = 2, \ldots, n-3$, (a condition that can be easily checked in cases (a) and (b)).

Therefore $\chi_C(T) = (1+T)\chi_{C'}(T) - T\chi_{C''}(T) = 1 + T - a_2T^2 - (a_3 + a_2 - b_2)T^3 - \cdots - (a_i + a_{i-1} - b_{i-1})T^i - \cdots - a_{n-2}T^{n-1} + T^n + T^{n+1}$, which has the desired form and $a_i + (a_{i-1} - b_{i-1}) \geq a_i$. $\square$

Remark: In view of (2.6), a Coxeter transformation $C \in W(\Delta)$ associated to a star quiver Δ which is not of Dynkin nor of Euclidean type has exactly two positive real eigenvalues, namely $\rho(C)$ and $\rho(C)^{-1}$. This also follows from the lemma above using Descartes change of signs rule and the fact that $\chi_C(T)$ has only 2 changes of sign.

In connection with Problem 2, we observe that the Coxeter polynomial $\chi_C(T)$ factorizes as $\chi_C(T) = p(T)q_1(T)\dots q_s(T)$, where $p(T)$ is a reciprocal irreducible polynomial which has $\rho(C)$ and $\rho(C)^{-1}$ as roots and each $q_i(T)$ is a cyclotomic polynomial.

2.9. As we have seen (and will become clearer in Section 3) the spectral radius $\rho(C)$ of a Coxeter transformation $C \in W(\Delta)$ is an important invariant of C. Knowing bounds for $\rho(C)$ is useful.

Proposition. *Let Δ be a graph, $Q = \Delta(\pi)$ be a quiver and $C = C(Q) \in W(\Delta)$ be a Coxeter transformation. Let M_Δ be the maximal of the degrees of vertices of Δ. Then*
a) *If Δ is a tree, then $\rho(C) \leq M_\Delta^2 - 2$*
b) *If Δ is not a tree and it is not of Euclidean type $\tilde{\mathbf{A}}_n$, then $\nu_0 \leq \rho(C)$; where ν_0 is the largest root of the polynomial $x^3 - x - 1$.*
c) *If Δ is neither of Dynkin nor of Euclidean type, then $\mu_0 \leq \rho(C)$; where μ_0 is the largest root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.*
d) *If Δ does not have any vertex of degree one, then $M_\Delta - 1 \leq \rho(C)$.*

Bound (a) was observed in [34]. Bounds (b) and (c) were given in [45]; the number $\nu_0 = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{23}{108}}} \approx 1.3271$ first appeared in [46]; the number μ_0 is approximately 1.1762. Bound (d) was given in [36].

For the *proof* of (a), observe that $M_\Delta(1,1,\dots,1) \geq (1,1,\dots,1)B$, where B is the adjacency matrix of Δ. By the Perron-Frobenius theory, $M_\Delta \geq \rho(B)$. Clearly, we may assume that Δ is not of Dynkin type and find a number $\lambda \geq 1$ such that

$$\rho(B) = \lambda + \lambda^{-1} \text{ and } \rho(C) = \lambda^2.$$

Therefore, $\rho(C) \leq \lambda^2 + \lambda^{-2} = \rho(B)^2 - 2 \leq M_\Delta^2 - 2$.

For the proof of the other claims we may use a technique introduced in [36].

The *universal cover* $\pi\colon \tilde{\Delta} \to \Delta$ of a graph Δ is an onto morphism π of graphs satisfying:

(i) There is a group G of automorphisms of $\tilde{\Delta}$ acting freely on the vertices of $\tilde{\Delta}$, such that $\pi g = \pi$, for every $g \in G$.

(ii) For any vertex $x \in \tilde{\Delta}_0$, $\pi^{-1}(\pi(x)) = Gx$.

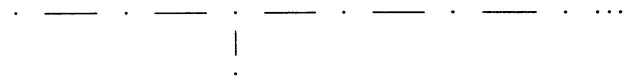
(iii) The (possibly infinite) graph $\tilde{\Delta}$ is a tree.

Let $(\Delta^i)_i$ be a sequence of finite connected induced subgraphs of $\tilde{\Delta}$, such that Δ^i is contained in Δ^{i+1}, for $i \in \mathbf{N}$ and $\tilde{\Delta}_0 = \bigcup \Delta_0^i$. Let $C_i \in W(\Delta^i)$ be a Coxeter transformations, then

$$\rho(C_i) \leq \rho(C_{i+1}) \text{ and the limit } \lim \rho(C_i) =: \rho(\tilde{\Delta}) \text{ exists.}$$

In [36] is shown that for any Coxeter transformation $C \in W(\Delta)$, we have $\rho(\tilde{\Delta}) \leq \rho(C)$.

For (b), observe that for any graph Δ which is not a tree and not of type $\tilde{\mathbf{A}}_n$, the universal covering $\tilde{\Delta}$ has a subgraph of type $\mathbf{E}_\infty$, where

$\mathbf{E}_\infty$: 

There is an ascending sequence (Δ^i) of subgraphs of $\mathbf{E}_\infty$ such that $\lim \rho(C_i) = \nu_0$, where $C_i \in W(\Delta^i)$ is a Coxeter transformation, $i \in \mathbf{N}$.

For (c), it is enough to consider the case where Δ is a tree. Since Δ contains an induced subgraph of type $\tilde{\mathbf{D}}_m$ $(m \leq 8)$, $\tilde{\tilde{\mathbf{E}}}_6$, $\tilde{\tilde{\mathbf{E}}}_7$ or $\tilde{\tilde{\mathbf{E}}}_8$, the problem is reduced to those cases. The minimal value of spectral radii of Coxeter transformations is reached for $\rho(C) = \mu_0$ when $C \in W(\tilde{\tilde{\mathbf{E}}}_8)$.

Claim (d) is shown in a similar way to (b). $\square$

Remark: Let Q' be a full induced subquiver of the bipartite quiver Q. Let C' (resp. C) be the Coxeter transformation of Q' (resp. Q). Then $\rho(C') \leq \rho(C)$, (a fact used in the above proof). Indeed let B' (resp. B) be the adjacency matrix of the underlying graph of Q' (resp. Q). By the Perron-Frobenius theory, $\rho(B') \leq \rho(B)$. Then (2.3) implies the claim.

3. Hereditary algebras.

3.1. Let Δ be a connected graph as above. Let Q be a quiver with underlying graph Δ.

Let k be an algebraically closed field. Then the *path algebra* $A = k[Q]$ is a finite dimensional, basic k-algebra (for definitions and basic properties, see [17]). Moreover, the algebra A is hereditary, that is all left ideals are projective A-modules.

By $\operatorname{mod} A$ we denote the category of finite dimensional left A-modules. We will identify these modules with the finite-dimensional representations of the quiver Q. For each vertex $j \in \Delta_0 = \{1, \dots, n\}$, we have several important associated A-modules: the simple module S_j; the projective cover P_j and the injective envelope I_j of S_j.

The isomorphism-classes of the indecomposable A-modules form the *representation-quiver* Γ_A which carries a translation τ (the *Auslander-Reiten translation*) defined for every non-projective vertex. The indecomposable A-modules are classified as preprojective, regular or preinjective according to their position on Γ_A.

3.2. For each A-module X, the dimension vector $\mathbf{dim}\,X \in \mathbf{Z}^n$ has at the i-th coordinate the number of times that the simple module S_i appears as composition factor of X.

Let $C = C(Q)$ be the Coxeter transformation associated to Q. The matrix C is characterized by the property $(\mathbf{dim}\,P_j)C = -\mathbf{dim}\,I_j \quad (j = 1, \ldots, n)$. The importance of C for the representation-theory of the algebra A, comes from the fact that

$$\mathbf{dim}\,\tau X = (\mathbf{dim}\,X)C$$

for every indecomposable non-projective A-module X.

Let M be the M-matrix defined in (1.6) such that $C = -MM^{-t}$. The *Euler-Poincaré characteristic* of A is the bilinear form $\langle x, y\rangle = xMy^t$. For any two A-modules X, Y we have

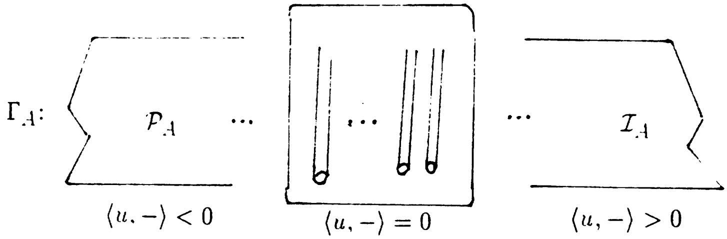
$$\langle \mathbf{dim}\,X, \mathbf{dim}\,Y\rangle = \dim_k \mathrm{Hom}_A(X,Y) - \dim_k \mathrm{Ext}^1_A(X,Y).$$

3.3. If Δ is of Dynkin type, then the algebra A is representation-finite, that is, there are only finitely many indecomposable A-modules up to isomorphy, [16]. In this case the quiver Γ_A is formed by a unique finite component, which is preprojective (that is, for every $X \in \Gamma_A$, there is some $m \in \mathbf{N}$ such that $\tau^m X$ is projective).

If Δ is of Euclidean type, the module category $\mathrm{mod}\,A$ may also be completely described. The representation-quiver Γ_A has three kind of components: a preprojective component $\mathcal{P}_A$ containing all the indecomposable projective modules; a preinjective component $\mathcal{I}_A$ containing all the indecomposable injective modules; and a set of *regular* components $\mathcal{R}_A = (T_\lambda)_{\lambda \in \mathbf{P}_1(k)}$, where for each λ, T_λ is a tubular component of rank n_λ (that is, for each $X \in T_\lambda$, $\tau^{n_\lambda} X = X, \tau X, \ldots, \tau^{n_\lambda - 1} X$ are pairwise non isomorphic modules). Moreover there are at most three $n_\lambda \neq 1$; assume $n_{\lambda_1}, \ldots, n_{\lambda_r}$ are those $n_\lambda \neq 1$, the star $T_{(n_{\lambda_1}, \ldots, n_{\lambda_r})}$ is a Dynkin diagram such that Δ is an extension of it.

Proposition [12]. *Let Δ be of Euclidean type. Let $0 << u \in \mathbf{Z}^n$ be such that $\langle u, u\rangle = 0$ (1.4). Consider the linear map $\partial = \langle u, -\rangle \colon \mathbf{Z}^n \to \mathbf{Z}$. Let X be an indecomposable Λ-module. Then X is preprojective (resp. regular, preinjective) if and only if $\partial(\mathbf{dim}\,X) < 0$ (resp. $= 0$, > 0).*

Graphically the representation-quiver Γ_A has the following shape

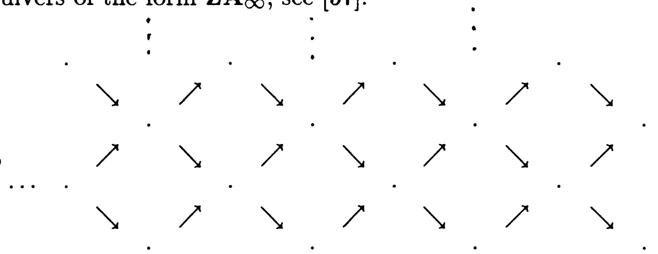


An algebra A as above is of *tame type,* since the indecomposable modules of every fixed dimension are classified by means of a one-parameter family.

3.4. Let now Δ be neither of Dynkin nor of Euclidean type. Then the algebra $A = k[Q]$ is of *wild type.* Indeed, there is a fully-faithful embedding $F\colon \operatorname{mod} W \to \operatorname{mod} A$, where W is the path algebra of the quiver with one vertex and two loops, see [9].

The representation-quiver has now the following structure: a preprojective componente $\mathcal{P}_A$, a preinjective component $\mathcal{I}_A$ and a set of regular components which are quivers of the form $\mathbf{ZA}_\infty$, see [37].

$\mathbf{ZA}_\infty$



Let $C = C(Q)$ be the Coxeter transformation. As we have seen in Section 2, the spectral radius $\rho(C) > 1$ and there exists a vector $y^+ >> 0$ such that $y^+C = \rho(C)y^+$. Similarly, there exists $y^- >> 0$ such that $y^-C = \rho(C)^{-1}y^-$. The *position* of the indecomposable modules on Γ_Λ can be given using the linear functionals $\langle -, y^+\rangle$ and $\langle y^-, -\rangle$ as follows:

Theorem [34]. *Let $A = k[Q]$ be a wild algebra. Keep the notation as above. Let X be an indecomposable A-module. Then*
a) *X is preprojective if and only $\langle y^-, \mathbf{dim} X\rangle < 0$. Moreover, if X is not preprojective, then $\langle y^-, \mathbf{dim} X\rangle > 0$.*
b) *X is preinjective if and only if $\langle \mathbf{dim} X, y^+\rangle < 0$. Moreover, if X is not preinjective, then $\langle \mathbf{dim} X, y^+\rangle > 0$.*
c) *X is regular if and only if $\langle y^-, \mathbf{dim} X\rangle > 0$ and $\langle \mathbf{dim} X, y^+\rangle > 0$.*

Proof: By duality, it is enough to show (a). First, we observe that the vector y^-

belongs to the *preprojective cone* $K_{\mathcal{P}}$ defined as the closure in $\mathbf{R}^n$ of the convex hull of the vectors $\mathbf{dim}\,X$ with X preprojective. Indeed, the solid cone $K_{\mathcal{P}}$ is invariant under C^{-1}; therefore the Perron-Frobenius Theorem (2.4) yields a vector $0 \neq z \in K_{\mathcal{P}}$ such that $zC^{-1} = \rho(C)z$. By Ringel's Theorem (2.1), the vectors y^- and z are linearly dependent. Hence $y^- \in K_{\mathcal{P}}$.

Hence there is a sequence $(u_m)_m$ of vectors such that $u_m = \sum_{i=1}^{\ell_m} \mu_i^{(m)} \mathbf{dim}\,V_i^{(m)}$, for some preprojective modules $V_i^{(m)}$, $\mu_i^{(m)} > 0$, and $y^- = \lim_{m\to\infty} u_m$.

Let X be an indecomposable preprojective module. Then there exists a number $m \geq 0$ and $s \in \Delta_0$ such that $(\mathbf{dim}\,X)C^m = \mathbf{dim}\,P_s$. Then

$$\langle y^-, \mathbf{dim}\,X\rangle = \langle y^- C^{-m}, \mathbf{dim}\,P_s\rangle = -\rho(C)^{-m-1} y^-(s) < 0.$$

Conversely, assume that $\langle y^-, \mathbf{dim}\,X\rangle < 0$. By continuity, there is some m such that $\langle u_m, \mathbf{dim}\,X\rangle < 0$. Therefore, there is some $1 \leq j \leq \ell_m$ with

$$\dim_k \mathrm{Hom}_A(V_j^{(m)}, X) - \dim_k \mathrm{Ext}_A^1(V_j^{(m)}, X) = \langle \mathbf{dim}\,V_j^{(m)}, \mathbf{dim}\,X\rangle < 0.$$

Thus, $\mathrm{Ext}_A^1(V_j^{(m)}, X) \neq 0$ and X is preprojective.

For X an indecomposable preinjective module with $(\mathbf{dim}\,X)C^{-m} = \mathbf{dim}\,I_s$, we have $\langle y^-, \mathbf{dim}\,X\rangle = \rho^m y^-(s) > 0$.

Let X be a regular, module and assume that $\langle y^-, \mathbf{dim}\,X\rangle = 0$. We may choose X to have the minimal possible dimension. Then X has no proper regular submodule. Indeed, consider

$$0 \to Y \to X \to C \to 0$$

an exact sequence with $0 \neq Y$ regular and $0 \neq C$ having all direct summands either regular or preinjective. Then

$$0 \leq \langle y^-, \mathbf{dim}\,Y\rangle = -\langle y^-, \mathbf{dim}\,C\rangle \leq 0. \text{ Hence } X = Y.$$

By an argument of Baer [2], there is a number $m \geq 0$ and an exact sequence

$$0 \to K \to \tau^{-m}X \to X \to 0$$

such that K has some preprojective direct summands.

[The module $T = \bigoplus_{i=0}^{n} \tau^{-2i}X$ is not tilting –too many non-isomorphic summands!–. Hence $0 \neq \mathrm{Ext}_A^1(T,T) \xrightarrow{\sim} D\mathrm{Hom}_A(T, \tau T)$. Then there is some $m \neq 0$ with $0 \neq \nu: \tau^{-m}S \to S$. Therefore ν is surjective. Since the vector $(\mathbf{dim}\ker\nu)C^m = \mathbf{dim}\,S - \mathbf{dim}\,\tau^m S$ has not all its coordinates non-negative, then $\ker\nu$ has a preprojective direct summand and $m > 0$].

Decompose $K = K_p \oplus K_r$, where $K_p \in \operatorname{add}(\mathcal{P}_A)$ and $K_r \in \operatorname{add}(\mathcal{R}_A)$. Since τ preserves monomorphisms $K_r = 0$. We get

$$0 > \langle y^-, \mathbf{dim}\, K_p \rangle = \langle y^-, \mathbf{dim}\, \tau^{-m} X \rangle - \langle y^-, \mathbf{dim}\, X \rangle = 0,$$

a contradiction. The proof is complete. □

3.5. For $A = k[Q]$ wild, we may describe the growth behaviour of the sequence of vectors $(\mathbf{dim}\, \tau^m X)_m$, for X indecomposable not preprojective.

Theorem [34, 46]. *Keep the notation as in (3.4). Let X be an indecomposable module. Then*
a) *If X is preprojective or regular, then* $\lim_{m\to\infty} \frac{1}{\rho(C)^m} \mathbf{dim}\, \tau^{-m} X = \lambda_X^- y^-$ *for some* $\lambda_X^- > 0$.
b) *If X is regular or preinjective, then* $\lim_{m\to\infty} \frac{1}{\rho(C)^m} \mathbf{dim}\, \tau^m X = \lambda_X^- y^+$ *for some* $\lambda_X^+ > 0$.

Proof: Let $\rho = \rho(C)$ and $y^+ = (y_1, \ldots, y_n)$. Consider the matrix

$$P = \frac{1}{\rho} DCD^{-1},$$

where $D = \operatorname{diag}(y_1, \ldots, y_n)$. By (2.1), 1 is a simple root of $\chi_P(T)$ and for every $\lambda \in \operatorname{Spec}(P)$, $\|\lambda\| < 1$. In this case the limit

$$P^\infty = \lim_{m\to\infty} P^m$$

exists and each row of P^∞ is a multiple of the vector $(1, 1, \ldots, 1)$.

Let X be a regular or preinjective module. Let $v = \mathbf{dim} X$. Therefore

$$\lim_{m\to\infty} \frac{vC^m}{\rho^m} = \lim_{m\to\infty} vD^{-1} P^m D = vD^{-1} P^\infty D = \lambda_X^+ y^+,$$

where $\lambda_X^+ = \sum_{i=1}^{n} v(i) \alpha_i y_i^{-1}$ with α_i the constant value of the i-th row of P^∞.

Clearly $\lambda_X^+ \geq 0$. Moreover,

$$\lambda_X^+ \langle y^-, y^+ \rangle = \lim_{m\to\infty} \frac{1}{\rho^m} \langle y^-, vC^m \rangle = \langle y^-, v \rangle > 0.$$

□

In particular, for an indecomposable not preprojective module X,

$$\lim_{m\to\infty} \sqrt[m]{\dim_k \tau^m X} = \rho(C), \qquad [13].$$

Due to these results, the number $\rho(C)$ is also called the *growth number* of A.

3.6. Let $A = k[Q]$ be wild. In the above paragraphs we have consider some properties of the representation-quiver Γ_A. Disgracefully, morphisms between modules are not well described by Γ_A. Indeed, we have the following important result.

Theorem. *Let X, Y be two indecomposable regular modules. Then there is a number $N \in \mathbf{N}$ such that for every $m \geq N$, we have:*
a) [2]: $\mathrm{Hom}_A(X, \tau^m Y) \neq 0$.
b) [26]: $\mathrm{Hom}_A(X, \tau^{-m} Y) = 0$.

In particular, given any two regular components C_1, C_2 we have $\mathrm{Hom}_A(C_1, C_2) \neq 0$. Inside a regular component most of the morphisms go in the direction opposite to the arrows.

We show (a) as an application of (3.4, 3.5):

For some $\lambda_Y^+ > 0$, we have

$$\lim_{m\to\infty} \frac{\mathbf{dim}\,\tau^m Y}{\rho(C)^m} = \lambda_Y^+ y^+.$$

Hence $0 < \lambda_Y^+ \langle \mathbf{dim}\,X, y^+ \rangle = \lim_{m\to\infty} \frac{1}{\rho(C)^m} \langle \mathbf{dim}\,X, \mathbf{dim}\,\tau^m Y \rangle$. □

Corollary. *Let X, Y be indecomposable regular modules. Then*

$$\lim_{m\to\infty} \frac{1}{\rho(C)^m} \dim_k \mathrm{Hom}_A(X, \tau^m Y) > 0. \qquad \square$$

3.7. Some properties studied for hereditary algebras can be used in more general situations. We consider here the problem of the *growth of components*.

Let Λ be a finite dimensional k-algebra. Let X be an indecomposable Λ-module. The *left growth number* of X is

$$\rho_\ell(X) = \overline{\lim}\, \sqrt[m]{\dim_k \tau^m X}.$$

Similarly, the right growth number is $\rho_r(X) = \overline{\lim}\, \sqrt[m]{\dim_k \tau^{-m} X}$.

If $\mathcal{C}$ is a component of the representation-quiver Γ_Λ, we define the *left growth number* of $\mathcal{C}$ as

$$\rho_\ell(\mathcal{C}) = \sup\{\rho_\ell(X) : X \in \mathcal{C}\}.$$

Similarly, we define the right growth number $\rho_r(\mathcal{C})$ of $\mathcal{C}$.

In some cases, the number $\rho_\ell(X)$ is independent of the chosen module X in a component $\mathcal{C}$. For example, if $\mathcal{C}$ is *left stable* (that is, $\tau^m X$ is well defined for every $X \in \mathcal{C}$ and $m \geq 0$), then $\rho_\ell(X) = \rho_\ell(Y)$ for any $X, Y \in \mathcal{C}$.

For some kind of components $\mathcal{C}$ of Γ_Λ, the value of $\rho_\ell(\mathcal{C})$ may be computed. For example, if $\mathcal{C}$ is a tubular component then $\rho_\ell \mathcal{C} = 1 = \rho_r(\mathcal{C})$. In general, the value $\rho_\ell(\mathcal{C})$ depends on the algebra Λ and not only on the shape of $\mathcal{C}$, see (3.8).

Proposition [30]. *Let $\Lambda = k[Q]/I$ be a k-algebra such that Q has no oriented cycles. Let $\mathcal{P}$ be a preprojective component of Γ_Λ. The following are equivalent:*
a) *The algebra $\Lambda_{\mathcal{P}} = \mathrm{End}_\Lambda(P)$ is tame, where $P = \bigoplus_{P_x \in \mathcal{P}} P_x$.*
b) *There is a constant $c > 0$ such that $\dim_k \tau^{-s} P_x \leq cs$, for every $P_x \in \mathcal{P}$ and $s \in \mathbf{N}$.*
c) *The right growth number $\rho_r(\mathcal{P}) \leq 1$.*

Sketch of proof: Since $\mathcal{P}$ is a component of $\Lambda_{\mathcal{P}}$, we may assume that $\Lambda = \Lambda_{\mathcal{P}}$.

a) $\Rightarrow$ b): Assume that Λ is tame. Then the *Tits form* q_Λ of Λ is weakly non-negative (see [30]). Then following [21] we may show that for every $X \in \mathcal{P}$ and $i \in Q_0$ we have

$$|\dim_k \tau^{-1} X(i) - \dim_k X(i)| \leq 2.$$

Let $m = \max\{\dim_k P_x : x \in Q_0\}$. Then $\dim_k \tau^{-s} P_x \leq 2ns + m$ for every $x \in Q_0$, $s \in \mathbf{N}$ and $n =$ number of vertices of Q.

b) $\Rightarrow$ c): This is clear.

c) $\Rightarrow$ a): Assume that Λ is not tame (therefore wild [11, 15]). We do not loose generality assuming that $\mathcal{P}$ does not contain injective modules (otherwise we pass to a convenient quotient of Λ). Then Λ is tilted algebra (see [38]). Let T be a tilting module associated with a *slice* in $\mathcal{P}$ and consider the wild hereditary algebra $A = \mathrm{End}_\Lambda(T)$.

Let $\sigma_T : K_0(\Lambda) \to K_0(A)$ be the isomorphism of the Grothendieck groups induced by T. If C_A is the Coxeter matrix associated to A, then $C_\Lambda = \sigma_T C_A \sigma_T^{-1}$ is called the *Coxeter matrix* of Λ and satisfies that $\mathbf{dim}\, \tau_\Lambda^{-1} X = (\mathbf{dim}\, X) C_\Lambda^{-1}$, for any X in $\mathcal{P}$ which receives paths from any indecomposable direct summand of T.

Choose $X \in \mathcal{P}$ as above. Thus $\mathrm{Ext}^1_\Lambda(T, X) = 0$. Then the A-module $Y = \mathrm{Hom}_\Lambda(T, X)$ is preprojective. A simple calculation shows that

$$\rho_r(\mathcal{P}) \geq \lim_{m \to \infty} \sqrt[m]{\dim_k \tau_\Lambda^{-m} X} = \lim_{m \to \infty} \sqrt[m]{\dim_k \tau_A^{-m} Y} = \rho(C_A) > 1. \qquad \square$$

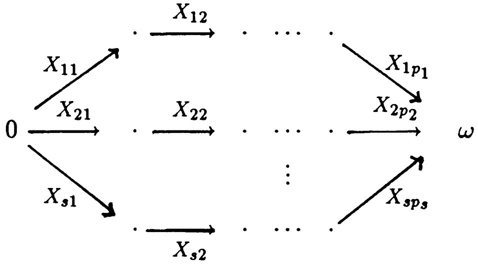
Problem 3: Let Λ be a tame algebra and let $\mathcal{C}$ be a component of the representation-quiver Γ_Λ. Then $\rho_r(\mathcal{C}) \leq 1$ and $\rho_\ell(\mathcal{C}) \leq 1$?

3.8. We give some examples of components and their growth numbers in the wild case.

a) Let $\Lambda = k[Q]$ be a wild hereditary algebra. Let C be the Coxeter matrix of Λ. Then

1) Let $\mathcal{P}$ be the preprojective component of Γ_Λ, then $\rho_r(\mathcal{P}) = \rho(C)$.
2) Let $\mathcal{C}$ be a regular component of Γ_Λ, then $\rho_r(\mathcal{C}) = \rho(C) = \rho_\ell(\mathcal{C})$.
3) $\rho(C) \geq \mu_0 \approx 1.1762$.

b) Let $p = (p_1, \ldots, p_s)$ be a sequence of natural numbers and $\lambda = (\lambda_3, \ldots, \lambda_s)$ be a sequence of pairwise different elements of k. Following [38], the *canonical algebra* $\Lambda = \Lambda(p, \lambda)$ is defined as the algebra with quiver

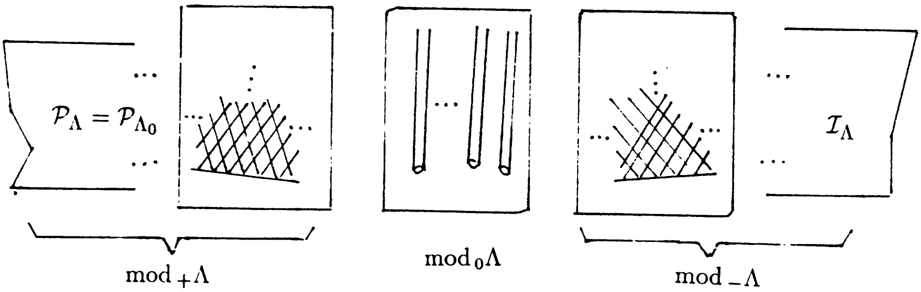


and satisfying the $s-2$ equations:

$$X_{ip_i} \cdots X_{i2} X_{i1} = X_{2p_2} \cdots X_{22} X_{21} - \lambda_i X_{1p_1} \cdots X_{12} X_{11}, \quad i = 3, \ldots, s.$$

The full subalgebra Λ_0 of Λ obtained by omiting the vertex 0 is hereditary and Λ is wild if and only if Λ_0 is wild.

Let Λ be of wild type. In [28] the following is shown. The structure of Γ_Λ is as follows,



where $\mathcal{P}_\Lambda$ is a preprojective component with $\rho_r(\mathcal{P}) = \rho(C_0) > 1$, where C_0 is the Coxeter matrix associated to Λ_0. The modules in $\operatorname{mod}_+\Lambda$ are those X with rank satisfying

$$0 < rk_\Lambda(X) = \dim_k X(\omega) - \dim_k X(0).$$

The components $\mathcal{C}$ of Γ_Λ contained in $\operatorname{mod}_+\Lambda$ are of the form $\mathbf{Z}\mathbf{A}_\infty$ (with the exception of one containing P_0). Any module $X \in \mathcal{C}$ satisfies

$$\lim_{m\to\infty} \frac{\mathbf{dim}\,\tau^{-m}X}{\rho(C_0)^m} = \lambda_X^- y_0^- \text{ and } \lim_{m\to\infty} \frac{\mathbf{dim}\,\tau^m X}{m} = \mu_X^+ w$$

where $0 \leq y_0^-$ is an eigenvector of C_0^{-1} with eigenvalue $\rho(C_0)$ and w is the vector with 1 in every coordinate; moreover, $\lambda_X^- > 0$ and $\mu_X^+ > 0$.

In particular,

$$\rho_r(\mathcal{C}) = \rho(C_0) \text{ and } \rho_\ell(\mathcal{C}) = 1.$$

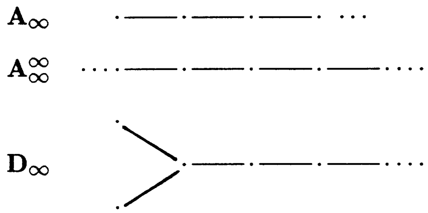
The modules X in Γ_Λ with $rk_\Lambda X = 0$ form a separating family of tubes. For a component $\mathcal{C}$ of Γ_Λ contained in mod $_-\Lambda$ (whose modules X have $rk_\Lambda(X) < 0$), we have

$$\rho_r(\mathcal{C}) = 1 \text{ and } \rho_\ell(\mathcal{C}) = \rho(C_0)$$

3.9. We recall that the number $\nu_0 \approx 1.3271$ is the maximal root of the polynomial $x^3 - x - 1$.

Theorem [46]. *Let $\mathcal{C}$ be a non-periodic regular component of the representation-quiver Γ_Λ of the algebra Λ. Assume that the growth numbers $\rho_\ell(\mathcal{C})$ and $\rho_r(\mathcal{C})$ are both smaller than ν_0, then $\mathcal{C}$ is of the form $\mathbf{Z}\mathbf{A}_\infty$, $\mathbf{Z}\mathbf{A}_\infty^\infty$ or $\mathbf{Z}\mathbf{D}_\infty$.*

We recall the shape of the graphs,



Sketch of the proof: By [47] (see also [29]), $\mathcal{C}$ is of the form $\mathbf{Z}\Delta$ where Δ is a quiver without oriented cycles. We identify $\mathcal{C} = \mathbf{Z}\Delta$. We will only treat the case that Δ is not a tree.

Consider the *additive function* $\ell: \mathcal{C} \to \mathbf{Z}$, $X \mapsto \dim_k X$ (that is, $\ell(X) + \ell(\tau X) = \sum_{i=1}^{s} \ell(Y_i)$, for $0 \to \tau X \to \bigoplus_{i=1}^{s} Y_i \to X \to 0$ an Auslander-Reiten sequence). Consider the universal covering $\pi: \tilde{\Delta} \to \Delta$ and a lifting $\tilde{\ell}: \mathbf{Z}\tilde{\Delta} \to \mathbf{Z}$, $(i, x) \mapsto \ell(i, \pi(x))$. Thus $\tilde{\ell}$ is again an additive function.

Let $(F_s)_s$ be a sequence of finite induced subquivers of $\tilde{\Delta}$ such that F_s is contained in F_{s+1} and $\lim F_s = \tilde{\Delta}$. Let ρ_s be the spectral radius of $C_s = C(F_s)$ a Coxeter matrix. Then

$$\nu_0 \leq \sup\{\rho_s : s \in \mathbf{N}\} \leq \rho := \rho(C(\Delta)), \qquad (2.9).$$

Consider the restriction $\ell^s: \mathbf{Z}F_s \to \mathbf{Z}$ of $\tilde{\ell}$. By [3], we get

$$\ell_i^s := \ell^s(i, -) = \ell_0^s C_s^{-i} + p_{is}$$

where p_{is} is a positive vector (corresponding to a preprojective $k[F_s]$-module).

As in (3.5), we have

$$\lim_{m\to\infty} \frac{\ell_0^s C_s^m}{\rho_s^m} = \lambda_s^+ y_s^+ \text{ and } \lim_{m\to\infty} \frac{\ell_0^s C_s^{-m}}{\rho_s^m} = \lambda_s^- y_s^-$$

for some vectores $y_s^-, y_s^+ \geq 0$ and numbers $\lambda_s^-, \lambda_s^+ \geq 0$ with at least one of them not zero. Assume that for an infinite set S of indices, we have $\lambda_s^- > 0$, $s \in S$. Then for each $s \in S$ and x a vertex in F_s, we get

$$\lim_{m\to\infty} \sqrt[m]{(\ell_0^s C_s^{-m})(x)} = \rho_s$$

Let X be a module in $\mathcal{C} = \mathbf{Z}\Delta$ in the position $(0, x_0)$ with $x_0 \in \Delta_0$. For any $\tilde{x}_0 \in \tilde{\Delta}_0$ with $\pi(\tilde{x}_0) = x_0$, if $\tilde{x}_0$ belongs to F_s with $s \in S$ we have

$$\rho_r(\mathcal{C}) = \lim_{m\to\infty} \sqrt[m]{\dim_k \tau^{-m} X} = \lim_{m\to\infty} \sqrt[m]{\ell_m^s(\tilde{x}_0)} \geq \lim_{m\to\infty} \sqrt[m]{(\ell_0 C_s^{-m})(\tilde{x}_0)} = \rho_s.$$

Hence $\rho_r(\mathcal{C}) \geq \nu_0$ contradicting the hypothesis. □

Problem 4: Is it enough in the last theorem to assume that $\rho_r(\mathcal{C}) \leq 1$ or $\rho_\ell(\mathcal{C}) \leq 1$?

4. Symmetries of quivers.

4.1. Let Q be a connected quiver without oriented cycles. Let $A = k[Q]$ be the corresponding hereditary algebra. A *symmetry* g of Q is a permutation of the set of vertices Q_0 which induces an automorphism of Q. By $\mathrm{Aut}(Q)$ we denote the group of all symmetries of Q.

Each symmetry g gives rise to a matrix $g \in GL(n)(= GL_{\mathbf{Q}}(K_0(A)))$, sending the simple module S_i to $S_{g(i)}$. This representation $\gamma: \mathrm{Aut}(Q) \to GL(n)$ is called the *canonical representation*.

Let $C = C(Q)$ be the Coxeter transformation associated to Q. Since clearly, $gC = Cg$ for every $g \in \mathrm{Aut}(Q)$, then the Coxeter matrix C is an automorphism of the canonical representation.

Let G be any subgroup of $\mathrm{Aut}(Q)$. The set of G-invariant vectors

$$\mathrm{Inv}_G(Q) = \{v \in K_0(A) \otimes_{\mathbf{Z}} \mathbf{Q} : vg = v \text{ for } g \in G\}$$

is a $\mathbf{Q}$-vector space of dimension $t_0(G)$, the number of orbits of Q under the action of G.

Proposition [32]. *Assume that $A = k[Q]$ is a wild algebra. Let $\rho = \rho(C)$ be the spectral radius of the Coxeter matrix $C = C(Q)$. Let $m_\rho(T) \in \mathbf{Z}[T]$ be the minimal* *polynomial of ρ (that is $m_\rho(T)$ is a monic irreducible polynomial with $m_\rho(\rho) = 0$, hence $m_\rho(T)$ is a factor of $\chi_C(T)$). Then*

$$\textit{degree } m_\rho(T) \leq t_0(\mathrm{Aut}\,(Q)).$$

In particular, $\chi_C(T)$ is not irreducible if $\mathrm{Aut}\,(Q)$ *is not trivial.*

Proof: Let j be a source in Q. The simple module S_j is injective and the vector

$$v = \sum_{g \in \mathrm{Aut}\,(Q)} \mathbf{dim}\, S_{g(j)},$$

belongs to the space $V = \mathrm{Inv}_{\,\mathrm{Aut}\,(Q)}(Q)$. Since V is invariant under the action of C, the matrix C takes the form

$$\begin{bmatrix} C_1 & * \\ 0 & C_2 \end{bmatrix}$$

where C_1 is the restriction of C to V. We get a factorization

$$\chi_C(T) = \chi_{C_1}(T)\chi_{C_2}(T),$$

where $\chi_{C_i}(T) \in \mathbf{Z}[T]$ is the characteristic polynomial of C_i, $i = 1, 2$.

By (3.5), the limit

$$\lim_{m \to \infty} \frac{vC^m}{\rho^m} = \lambda y^+$$

exists, with $0 << y^+$ and $0 < \lambda$. Therefore $y^+ \in V \otimes_{\mathbf{Q}} \mathbf{R}$ and ρ is a root of $\chi_{C_1}(T)$. Thus $m_\rho(T)$ is a factor of $\chi_{C_1}(T)$ and degree $m_\rho(T) \leq \dim_{\mathbf{Q}} V = t_0(\mathrm{Aut}\,(Q))$ as desired. □

Examples: a) For the star $\Delta = T_{(2,3,7)}$, the Coxeter polynomial $\chi_C(T) = 1 + T - T^3 - T^4 - T^5 - T^6 - T^7 + T^9 + T^{10}$ is irreducible.

b) For the star $\Delta = T_{(2,6,6)}$, the Coxeter polynomial is $\chi_C(T) = 1 + T - T^3 - 2T^4 - 3T^5 - 4T^6 - 3T^7 - 2T^8 - T^9 + T^{11} + T^{12} = (1+T)^2(1+T+T^2)(1 - T + T^2)(1 - T - T^3 - T^5 + T^6)$.

The degree $m_\rho(T) = 6 < 7 = t_0(\mathrm{Aut}\,(Q))$, defining Q as the quiver with underlying graph Δ and the ramification vertex as a unique source.

4.2. On the lines of (4.1) there are other results concerning the relations of the Coxeter polynomial $\chi_C(T)$ and the behaviour of indecomposable A-modules.

Let X be an indecomposable $A(= k[Q])$-module. Let $\mathcal{O}(X)$ be the subspace of $K_0(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ generated by $\{\mathbf{dim}\, \tau^m X : m \in \mathbf{Z}\}$.

Theorem [33]. *Let $A = k[Q]$ be a wild algebra. Then*
a) *For any indecomposable regular or preinjective module, we have*

$$\dim_{\mathbf{Q}} \mathcal{O}(X) \geq \textit{degree } m_\rho(T),$$

where $m_\rho(T)$ is the minimal polynomial of $\rho = \rho(C(Q))$ in $\mathbf{Z}[T]$.
b) *There exists an indecomposable regular A-module X such that*

$$t_0(\mathrm{Aut}\,(Q)) \geq \dim_{\mathbf{Q}} \mathcal{O}(X). \qquad \square$$

As an example we consider the quiver Q with symmetries g_i, $i = 2, \dots, 5$

$$Q\colon \begin{tikzcd}[row sep=tiny] & 1 \\ & 2 \\ 0 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & 3 \\ & 4 \\ & 5 \end{tikzcd} \qquad g_i\colon Q_0 \to Q_0, \quad j \mapsto \begin{cases} i & \text{if } j = 1 \\ 1 & \text{if } j = i \\ j & \text{else.} \end{cases}$$

For the injective module I_0 we have $\mathcal{O}(I_0) \subset \mathrm{Inv}_{\mathrm{Aut}\,(Q)}(Q)$ and hence $\dim_{\mathbf{Q}} \mathcal{O}(I_0) \leq 2$. Since on the other hand

$$\chi_C(T) = (1 - 3T + T^2)(1 + T)^4 \text{ and } m_\rho(T) = 1 - 3T + T^2,$$

then $\dim_{\mathbf{Q}} \mathcal{O}(I_0) = 2$.

4.3. Let G be a subgroup of $\mathrm{Aut}\,(Q)$. The irreducible decompositions of the restriction $\gamma\colon G \to GL(n)$ of the canonical representation (as $\mathbf{Q}$ or $\mathbf{C}$ representation) have interesting properties related with the structure of $\mathrm{Spec}\,(C)$, for $C = C(Q)$.

Let $R_1, \dots, R_\ell$ be a set of representatives of the *irreducible* $\mathbf{Q}$-*representations* of G. Let R_1 be the trivial representation. By Maschke Theorem, there exists an invertible rational matrix L such that the conjugate $\gamma^L (:= L\gamma L^{-1})$ has a decomposition

$$\gamma^L = \bigoplus_{\alpha=1}^{\ell} R_\alpha^{r(\alpha)}.$$

Since $C = C(Q)$ is an automorphism of γ, by Schur's Lemma, the conjugate C^L takes the block diagonal form

$$C^L = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_\ell \end{bmatrix}$$

where $C_\alpha\colon R_\alpha^{r(\alpha)} \to R_\alpha^{r(\alpha)}$ is an automorphism. Therefore we get a factorization

$$\chi_C(T) = \chi_{C_1}(T) \cdots \chi_{C_\ell}(T)$$

where $\chi_{C_i}(T) \in \mathbf{Z}[T]$ is the characteristic polynomial of C_i. In particular $\chi_C(T)$ has at least $|\{\alpha : r(\alpha) > 0\}|$ factors.

Let $S_1, \ldots, S_m$ be a set of representatives of the *irreducible* $\mathbf{C}$*-representations* of G. Let S_1 be the trivial representation. Then m is the number of conjugacy classes of G.

There is a conjugate γ^M of γ with a decomposition

$$\gamma^M = \bigoplus_{\beta=1}^{m} S_\beta^{n(\beta)}.$$

Theorem [32]. *Let $C = C(Q)$ be a Coxeter transformation. Then*
a) *The cardinality of* $\operatorname{Spec}(C)$ *is at most* $\sum\limits_{\beta=1}^{m} n(\beta)$.
b) *Let $\lambda_1, \ldots, \lambda_t$ be the set of eigenvalues of C with geometric multiplicity one and let $d(\lambda_i)$ be the size of the corresponding Jordan block. Then*

$$\sum_{i=1}^{t} \alpha(\lambda_i) \leq \sum_{\beta \in \mathcal{U}_0} n(\beta),$$

where $\mathcal{U}_0 = \{\beta : \dim S_\beta = 1\}$.
c) *Let $\lambda_1, \ldots, \lambda_s$ be those real eigenvalues of C with geometric multiplicity one. Then*

$$\sum_{i=1}^{m} d(\lambda_i) \leq \sum_{\beta \in \mathcal{U}_1} n(\beta)$$

where $\mathcal{U}_1 = \{\beta : \dim S_\beta = 1$ and $\chi_\beta^2 = 1\}$ and χ_β is the character *corresponding to S_β (that is, $\chi_\beta : \operatorname{Aut}(Q) \to \mathbf{C}^*$, $g \mapsto t_r S_\beta(g)$).* □

As an *example* we consider again the quiver Q in (4.2) and consider the alternating group A_5 as subgroup of $\operatorname{Aut}(Q)$.

Let $\gamma : A_5 \to GL(6)$ be the canonical representation with irreducible $\mathbf{C}$-decomposition $\gamma^M = \bigoplus\limits_{\beta=1}^{m} S_\beta^{n(\beta)}$.

In this case $m = 5$ and the *character table* of A_5 is

	x_1	x_2	x_3	x_4	x_5	: representative of a conjugacy class
	1	2	3	5	5	: order of the representative
$1 = \chi_1$	1	1	1	1	1	
χ_2	4	0	1	1	1	
χ_3	5	1	1	0	0	
χ_4	3	1	0	α_1	α_2	
χ_5	3	1	0	α_2	α_1	

with $\alpha_1 = (1+\sqrt{5})/2$ and $\alpha_2 = (1-\sqrt{5})/2$.
We calculate,

$$n(1) = (\chi_\gamma, \chi_1) = \frac{1}{60}\sum_{g\in G}\chi_\gamma(g) = 2(= t_0(A_5))$$

$$n(2) = (\chi_\gamma, \chi_2) = \frac{1}{60}\sum_{g\in G}\chi_\gamma(g)\overline{\chi_2(g)} = 1 \text{ and } \dim S_2 = 4.$$

Thus $n(3) = n(4) = n(5) = 0$ and $\gamma^T = S_1 \oplus S_1 \oplus S_2$ is also a **Q**-decomposition.

The Coxeter polynomial $\chi_C(T)$ has roots $\rho = (3+\sqrt{5})/2$, $\rho^{-1} = (3-\sqrt{5})/2$ and -1 with geometric multiplicity 4.

In this case the bounds of the Theorem are optimal.

4.4. In (4.3) we have gotten restrictions on Spec(C) knowing the group Aut(Q) and some representations. The knowledge of Spec(C) also gives interesting restrictions on Aut(Q).

Proposition [32]. *Let* Spec$(C) = \{\lambda_1, \ldots, \lambda_t\}$ *and assume that the eigenspace of* λ_i *has dimension* m_i. *Then* Aut(Q) *is a subgroup of the product*

$$\mathcal{U}(m_1) \times \cdots \times \mathcal{U}(m_t)$$

where $\mathcal{U}(m_i)$ *denotes the group of unitary* $m_i \times m_i$*-complex matrices.* □

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Translation Functors and Equivalences of Derived Categories for Blocks of Algebraic Groups

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ABSTRACT.
We prove the existence of many self-equivalences of the derived categories of blocks of reductive groups in prime characteristic that are not induced by self-equivalences of the module categories of the blocks. We conjecture that our examples are just the simplest special case of a much more general phenomenon. This is analogous to conjectures about derived categories of blocks of symmetric groups.

1. Introduction

In (3), Happel took the machinery of derived categories from algebraic geometry and introduced it into the representation theory of finite-dimensional algebras, showing that the tilting theory of algebras has an extremely elegant interpretation in terms of equivalences of derived categories of module categories. Since then, more general equivalences of derived categories have been studied in some detail. Perhaps the most intriguing situation where such equivalences have been discovered is in the modular representation theory of finite groups. Broué, in (1), made some deep conjectures about conditions that would ensure that two blocks have equivalent derived categories. If proved, these conjectures would have important consequences for the study of block invariants, such as the number of characters in a block, since most such invariants are actually invariants of the derived category.

One consequence of an equivalence of derived categories for blocks is that it induces a 'perfect isometry' between the groups of characters of the two blocks. One important class of examples of perfect isometries was found by Enguehard (2); he proved that any two blocks of (possibly different) symmetric groups in characteristic p are 'perfectly isometric' if they have the same defect group. This gives a strong indication that such blocks should have equivalent derived categories. At about the same time, Scopes (6) proved that many blocks of symmetric groups are actually Morita equivalent, and that in fact there are only finitely many Morita equivalence

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classes for a given defect group. The pairs of blocks for which Enguehard's perfect isometry can be taken to be a bijection between irreducible characters ***without signs*** are precisely the pairs of blocks which Scopes proves to be Morita equivalent.

Several people seem to have independently noticed an analogy between Scopes' equivalences and Jantzen's translation functors, which give many equivalences of categories between blocks of rational representations of reductive groups in positive characteristic. During an only partially successful attempt to generalize the ideas of Scopes to find equivalences of derived categories for symmetric groups, I realized that the analogy seemed to extend to these derived equivalences. In neither the case of symmetric groups nor the case of reductive groups have I so far been able to prove all that I would like, but in this paper I will describe one of the simplest cases for reductive groups.

Let us now recall some basic facts about blocks of reductive groups. Jantzen's book (4) is a convenient reference for all the facts we shall use. First we shall fix some notation.

- Let k be an algebraically closed field of prime characteristic p.
- Let G be a connected reductive group defined over k.
- Let T be a maximal torus of G.
- Let B be a Borel subgroup of G containing T.
- Let $X(T)$ be the group of characters of T.
- Let $X(T)_+$ be the set of dominant weights in $X(T)$.
- Let mod-G be the category of finite-dimensional rational G-modules.

1. The simple rational G-modules are parametrized by the set $X(T)_+$ of dominant weights: for each $\lambda \in X(T)_+$ there is a unique simple module $L(\lambda)$ with highest weight λ (4, II.2).
2. Each $\lambda \in X(T)$ determines a line bundle on G/B and the cohomology groups $H^i(\lambda)$ of these line bundles are rational G-modules (4, I.5). By Kempf's Vanishing Theorem (4, II.4), $H^i(\lambda) = 0$ for $i > 0$ if λ is dominant.
3. Each M in mod-G has a formal character $\mathrm{ch}(M)$ in the group ring $\mathbf{Z}[X(T)]$; the function taking M to $\mathrm{ch}(M)$ is additive on short exact sequences. The set of formal characters $\mathrm{ch}(L(\lambda))$ of the simple modules form a basis for the group generated by formal characters of all modules (i.e., $\mathrm{ch}(M)$ determines the composition factors of M). The formal characters $\mathrm{ch}(H^0(\lambda))$ for $\lambda \in X(T)_+$ form another basis.
4. For any $\lambda \in X(T)$, not necessarily dominant, we define an element $\chi(\lambda)$ of $\mathbf{Z}[X(T)]$ by
$$\chi(\lambda) = \sum_{i \geq 0} (-1)^i \mathrm{ch}(H^i(\lambda)),$$
so if λ is dominant, then $\chi(\lambda) = \mathrm{ch}(H^0(\lambda))$.
5. The Weyl group W of G acts as a reflection group on $X(T)$, and this action extends to an action of the affine Weyl group W_p which is generated by W

together with all translations by $p\beta$ for roots β. By shifting the origin of this action, we get a new action, the dot action, of W_p on $X(T)$, given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

where ρ is half the sum of the positive roots (4, II.6.1).

6. If $w \in W$, then there is a very simple formula (4, II.5.9) for the effect of w on the characters $\chi(\lambda)$:
$$\chi(w \cdot \lambda) = \det(w)\chi(\lambda).$$
7. Weyl's Character Formula (4, II.5.10) implies that $\chi(\lambda)$ is independent of the characteristic of k, and so (4, II.5.5) applied in characteristic zero implies that $\chi(\lambda) = 0$ unless λ is conjugate to a dominant weight under the dot action of W on $X(T)$. Hence $\chi(\lambda)$ is either zero or is $\pm\mathrm{ch}(H^0(\mu))$ for some $\mu \in X(T)_+$.
8. If we define
$$\overline{C}_{\mathbf{Z}} = \{\lambda \in X(T) : 0 \leq \langle \lambda + \rho, \beta^\vee \rangle \leq p \quad \text{for all positive roots } \beta\},$$
then $\overline{C}_{\mathbf{Z}}$ is a fundamental domain for the action of W_p on $X(T)$.
9. For each $\lambda \in X(T)$, we can define the full subcategory $\mathcal{M}_\lambda$ of mod-G to consist of those modules whose simple composition factors all have highest weights in $W_p \cdot \lambda$. The Linkage Principle (4, II.6.17) implies that every indecomposable rational G-module lies in $\mathcal{M}_\lambda$ for some λ. In other words, each $\mathcal{M}_\lambda$ is a sum of blocks (usually a single block) of G.

2. The Main Theorem

Let us first recall the definition of Jantzen's translation functors (4, III.7).

Let λ and μ be two weights in $\overline{C}_{\mathbf{Z}}$, and let pr_λ be the functor from mod-G to $\mathcal{M}_\lambda$ that takes a module M to the largest submodule of M that lies in $\mathcal{M}_\lambda$ (i.e., pr_λ is 'projection onto the block M_λ'), and let pr_μ be the similarly defined functor from mod-G to $\mathcal{M}_\mu$.

Since $X(T)_+$ is a fundamental domain for the action of W on $X(T)$, there is a unique dominant weight ν_1 in $W(\mu - \lambda)$. We now define the translation functor

$$T_\lambda^\mu : \mathcal{M}_\lambda \longrightarrow \mathcal{M}_\mu$$

by

$$T_\lambda^\mu(V) = pr_\mu(L(\nu_1) \otimes V).$$

This definition gives an exact functor, and one of the basic properties of these functors is that T_λ^μ is both left and right adjoint to T_μ^λ.

If λ and μ are in $\overline{C}_{\mathbf{Z}}$, then the effect of the translation functor T_λ^μ on formal characters is easy to describe (4, II.7.8). If V is in $\mathcal{M}_\lambda$, then

$$\mathrm{ch}(V) = \sum_{w \in W_p} a_w \chi(w \cdot \lambda)$$

for some coefficients $a_w \in \mathbf{Z}$ with almost all a_w zero. Then

$$\operatorname{ch}(T_\lambda^\mu(V)) = \sum_{w \in W_p} a_w \sum_{w_1} \chi(ww_1 \cdot \mu),$$

where w_1 runs through a set of coset representatives for

$$\mathrm{Stab}_{W_p}(\lambda)/(\mathrm{Stab}_{W_p}(\lambda) \cap \mathrm{Stab}_{W_p}(\mu)).$$

In particular, if λ and μ have the same stabilizer in W_p, then

$$\operatorname{ch}(T_\lambda^\mu(V)) = \sum_{w \in W_p} a_w \chi(w \cdot \mu),$$

from which the 'translation principle', saying that in this case T_λ^μ is an equivalence of categories from $\mathcal{M}_\lambda$ to $\mathcal{M}_\mu$, follows quite quickly (4, II.7.9).

More generally, if λ and μ are arbitrary elements of $X(T)$ with the same stabilizer in W_p, then

$$\theta : \sum_{w \in W_p} a_w \chi(w \cdot \lambda) \longmapsto \sum_{w \in W_p} a_w \chi(w \cdot \mu)$$

is still a well-defined isomorphism between the groups of characters of M_λ and of M_μ. However, when $\lambda, \mu \notin \overline{C}_{\mathbf{Z}}$ this map sends the character of a module to a virtual character that need not be the character of a module, so in general the map cannot be realized by an equivalence of module categories.

For those familiar with Broué's 'perfect isometries' (1) for characters of finite groups, however, the map θ has some suggestive properties. The characters $\chi(w \cdot \lambda)$ for dominant $w \cdot \lambda$ are the analogue for reductive groups of the ordinary irreducible characters for blocks of finite groups, so θ takes an 'ordinary irreducible' character to another 'ordinary irreducible' character multiplied by a sign ± 1 — just like a perfect isometry.

The philosophy of perfect isometries is that they frequently occur for blocks of finite groups because they are the character-theoretic 'shadow' of a structural phenomenon: an equivalence of triangulated categories between the derived categories of the blocks. So it is natural to ask whether the map θ comes from an equivalence between the derived categories of $\mathcal{M}_\lambda$ and M_μ. We shall show that this is so in at least some cases.

THEOREM 2.1. *Let G be a connected reductive group defined over k. Suppose that G has a simply-connected derived subgroup and that the root system of G has a Coxeter number $h \leq p$. Let T be a maximal torus of G, and let $\lambda, \mu \in X(T)$ be weights with trivial stabilizer in W_p. There is an equivalence of triangulated categories*

$$D^b(\mathcal{M}_\lambda) \longrightarrow D^b(\mathcal{M}_\mu)$$

that induces the map

$$\chi(w \cdot \lambda) \longmapsto \chi(w \cdot \mu)$$

on characters.

This theorem will follow easily from the following.

PROPOSITION 2.2. *Let G be a connected reductive group defined over k whose derived subgroup is simply-connected and whose root system has Coxeter number $h \leq p$. Let $\lambda \in \overline{C}_{\mathbf{Z}}$ be a weight with trivial stabilizer in W_p, let $s \in W_p$ be reflection in a wall of $\overline{C}_{\mathbf{Z}}$, and let $\mu = s \cdot \lambda$. Then there is an equivalence of triangulated categories*

$$D^b(\mathcal{M}_\lambda) \longrightarrow D^b(\mathcal{M}_\mu)$$

that induces the map

$$\chi(w \cdot \lambda) \longmapsto \chi(w \cdot \mu)$$

on characters.

The next section will be devoted to the proof of this proposition. Let us first show how it may be used to deduce Theorem 2.1.

Proof of Theorem 2.1: Since $\overline{C}_{\mathbf{Z}}$ is a fundamental domain for the action of W_p on $X(T)$, $\mu \in W_p \cdot \lambda'$ for some $\lambda' \in \overline{C}_{\mathbf{Z}}$. By the translation principle (4, II.7.9) there is an equivalence of categories

$$\mathcal{M}_\lambda \longrightarrow \mathcal{M}_{\lambda'}$$

that induces the map

$$\chi(w \cdot \lambda) \longmapsto \chi(w \cdot \lambda')$$

on characters, so without loss of generality we may and shall assume that $\lambda = \lambda'$.

The set of reflections in walls of $\overline{C}_{\mathbf{Z}}$ generates W_p as a group (4, II.6.3), so

$$\mu = s_1 s_2 \ldots s_n \cdot \lambda$$

for some sequence $s_1, \ldots s_n$ of reflections in walls of $\overline{C}_{\mathbf{Z}}$. By Proposition 2.2, for each reflection s_i there is an equivalence of triangulated categories

$$D^b(\mathcal{M}_\lambda) \longrightarrow D^b(\mathcal{M}_\lambda)$$

inducing the map

$$\chi(w \cdot \lambda) \longmapsto \chi(ws_i \cdot \lambda)$$

on characters. The theorem follows by taking a composition of such equivalences. □

We should remark that with λ and μ as in Theorem 2.1, both of these weights are conjugate under the dot action of W_p to weights in $\overline{C}_{\mathbf{Z}}$ which also have trivial stabilizer. Applying the translation principle for these weights, we immediately deduce that $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ are equivalent categories, and are both equivalent to the principal block $\mathcal{M}_0$ of G. So the fact that the corresponding derived categories are equivalent is not in itself surprising. The content of Theorem 2.1 is rather that there are many non-trivial self-equivalences of these derived categories that realize various natural automorphisms of the group of characters. If, as we suggested before we stated the main theorem, the obvious generalization of the theorem to weights with non-trivial stabilizer is also true, then this would provide equivalences of derived categories for pairs of blocks for which the translation principle does not give an equivalence of module categories.

3. Proof of Proposition 2.2

Since the derived subgroup of G is simply-connected and $h \leq p$, it follows from (4, II.6.3) that there is some $\nu \in X(T)$ on the wall in which s reflects.

Consider the bounded complex of functors

$$\ldots \longrightarrow 0 \longrightarrow T_\nu^\lambda T_\lambda^\nu \longrightarrow \mathrm{id} \longrightarrow 0 \longrightarrow \ldots$$

from $\mathcal{M}_\lambda$ to itself, where $T_\nu^\lambda T_\lambda^\nu$ is in degree zero and the map from $T_\nu^\lambda T_\lambda^\nu$ to the identity functor is the counit of the adjunction. Since the functors involved are all exact, this complex defines an exact functor

$$F : D^b(\mathcal{M}_\lambda) \longrightarrow D^b(\mathcal{M}_\lambda)$$

by applying the complex of functors to a complex of modules and then taking the total complex of the resulting double complex. We shall show that F is an equivalence of categories.

There is another bounded complex

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{id} \longrightarrow T_\nu^\lambda T_\lambda^\nu \longrightarrow 0 \longrightarrow \ldots$$

of functors, giving a functor $F^\vee$ from $D^b(\mathcal{M}_\lambda)$ to itself, where again $T_\nu^\lambda T_\lambda^\nu$ is in degree zero, and this time the map from the identity functor to $T_\nu^\lambda T_\lambda^\nu$ is the unit of the adjunction. The first thing we want to show is that $F^\vee$ is both left and right adjoint to F.

Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories and let

$$L, L' : \mathcal{C} \longrightarrow \mathcal{D} \quad \text{and} \quad R, R' : \mathcal{D} \longrightarrow \mathcal{C}$$

be functors and suppose we are given adjunctions making R right adjoint to L and R' right adjoint to L'. Then by (5, IV.7, Theorem 2), for every map of functors $L \longrightarrow L'$ there is a unique map of functors $R' \longrightarrow R$ (the *conjugate* map) such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(L'-,-) & \cong & \mathrm{Hom}_{\mathcal{C}}(-,R'-) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(L-,-) & \cong & \mathrm{Hom}_{\mathcal{C}}(-,R-) \end{array}$$

commutes.

It follows easily from the definition of conjugate maps that the conjugate of a composition of maps of functors is the composition of the conjugates of the individual maps, and that if $\mathcal{C}$ and $\mathcal{D}$ are additive categories then the conjugate of the zero map between additive functors is also the zero map. Hence, if

$$L^* = \ldots \longrightarrow L^n \longrightarrow L^{n+1} \longrightarrow \ldots$$

is a bounded complex of functors, where each L_n has a right adjoint R^n, then we get a bounded complex of functors

$$R^* = \ldots \longrightarrow R^{n+1} \longrightarrow R^n \longrightarrow \ldots$$

with degree i term R^{-i} by taking the conjugates of all the differentials of L^*. If X^* and Y^* are bounded complexes over $\mathcal{C}$ and $\mathcal{D}$ respectively, then the definition of conjugacy immediately gives a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X^*, R^*Y^*) \cong \mathrm{Hom}_{\mathcal{D}}(L^*X^*, Y^*)$$

of triple complexes. Since the degree zero homology groups of the total complexes of these triple complexes are the groups of chain homotopy classes of maps from X^* to the total complex of R^*Y^* and from the total complex of L^*X^* to Y^* respectively, we deduce that the functor given by L^* from the homotopy category $K^b(\mathcal{C})$ of complexes over $\mathcal{C}$ to $K^b(\mathcal{D})$ is left adjoint to the functor given by R^* from $K^b(\mathcal{D})$ to $K^b(\mathcal{C})$.

If, furthermore, $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and all the functors L^i and R^i are exact, then this adjunction of functors between homotopy categories provides an adjunction of functors between the derived categories $D^b(\mathcal{C})$ and $D^b(\mathcal{D})$. Returning now to our functors F and $F^\vee$, we see that to prove that $F^\vee$ is right adjoint to F, we just need to show that the unit

$$\mathrm{id} \longrightarrow T_\nu^\lambda T_\lambda^\nu$$

is conjugate to the counit

$$T_\nu^\lambda T_\lambda^\nu \longrightarrow \mathrm{id}.$$

This is true because if we regard $T_\nu^\lambda T_\lambda^\nu$ as a functor from mod-G to mod-G, vanishing on modules with no summand in M_λ and sending every module to an object of M_λ, then the counit map can be expressed as the composition of three maps: the inclusion of the direct summand

$$T_\nu^\lambda T_\lambda^\nu \longrightarrow - \otimes L(\alpha_1) \otimes L(\alpha_1)^*,$$

where α_1 is the unique dominant weight in $W(\nu - \lambda)$, the map

$$- \otimes L(\alpha_1) \otimes L(\alpha_1)^* \longrightarrow \mathrm{id}_G \cong - \otimes k$$

induced by the natural map $L(\alpha_1) \otimes L(\alpha_1)^* \longrightarrow k$, and the projection

$$\mathrm{id}_G \longrightarrow \mathrm{pr}_\lambda$$

onto a direct summand. But it is clear that the conjugate of the inclusion of a direct summand of a functor is just projection onto the corresponding direct summand of the adjoint, and that the conjugate of the natural map

$$- \otimes L(\alpha_1) \otimes L(\alpha_1)^* \longrightarrow - \otimes k$$

is just the natural map

$$- \otimes k \longrightarrow - \otimes L(\alpha_1) \otimes L(\alpha_1)^* .$$

The claim about the conjugate of the counit map now follows by composing the three conjugates. Hence $F^\vee$ is right adjoint to F, and similarly $F^\vee$ is also left adjoint to F.

Let us now consider the composite $F^\vee F$. This is induced by the complex of functors we get by taking the total complex of the double complex whose only non-zero terms are

$$\begin{array}{ccc} T_\nu^\lambda T_\lambda^\nu & \longrightarrow & \mathrm{id} \\ \big\downarrow & & \big\downarrow \\ T_\nu^\lambda T_\lambda^\nu T_\nu^\lambda T_\lambda^\nu & \longrightarrow & T_\nu^\lambda T_\lambda^\nu , \end{array}$$

where the left hand vertical map is the map

$$\epsilon \circ T_\nu^\lambda T_\lambda^\nu : \mathrm{id} \circ T_\nu^\lambda T_\lambda^\nu \longrightarrow T_\nu^\lambda T_\lambda^\nu T_\nu^\lambda T_\lambda^\nu$$

induced by the unit

$$\epsilon : \mathrm{id} \longrightarrow T_\nu^\lambda T_\lambda^\nu$$

of the adjunction, and the bottom horizontal map is the map

$$T_\lambda^\nu T_\nu^\lambda \circ \eta$$

induced by the counit η. But for any functor L with a right adjoint R, the composition

$$L \xrightarrow{L\epsilon} LRL \xrightarrow{\eta L} L$$

is the identity (5, IV.1, Theorem 1), where ϵ and η are respectively the unit and counit of the adjunction. Hence the left hand vertical map is a split monomorphism, and similarly the bottom horizontal map is a split epimorphism, so the total complex is split, and so is chain homotopy equivalent to a complex of functors concentrated in degree zero. Thus the functor $F^\vee F$ is isomorphic to a functor induced by an exact functor from the abelian category $\mathcal{M}_\lambda$ to itself. In particular, if we regard $\mathcal{M}_\lambda$ as a full subcategory of its derived category $D^b(\mathcal{M}_\lambda)$ in the usual way then $F^\vee F(V)$ is in $\mathcal{M}_\lambda$ for every V in $\mathcal{M}_\lambda$.

Let us next consider the maps

$$K_0(\mathcal{M}_\lambda) \longrightarrow K_0(\mathcal{M}_\lambda)$$

induced by F and $F^\vee$. Let us call these maps ϕ and $\phi^\vee$ respectively. These maps are easy to evaluate on characters of the form $\chi(w \cdot \lambda)$, since we know the effect of translation functors on characters of this form. So

$$\begin{aligned} \phi(\chi(w \cdot \lambda)) &= [\chi(w \cdot \lambda) + \chi(ws \cdot \lambda)] - \chi(w \cdot \lambda) \\ &= \chi(ws \cdot \lambda) \\ &= \chi(w \cdot \mu) \end{aligned}$$

and

$$\begin{aligned}\phi^\vee(\chi(w\cdot\lambda)) &= [\chi(w\cdot\lambda)+\chi(ws\cdot\lambda)] - \chi(w\cdot\lambda)\\ &= \chi(ws\cdot\lambda)\\ &= \chi(w\cdot\mu),\end{aligned}$$

since T_λ^ν maps $\chi(w\cdot\lambda)$ to $\chi(w\cdot\nu)$ and T_ν^λ maps $\chi(w\cdot\nu)$ to $\chi(w\cdot\lambda)+\chi(ws\cdot\lambda)$.

Therefore, since s^2 is the identity,

$$\phi^\vee\phi = \mathrm{id} = \phi\phi^\vee.$$

Since we already know that $F^\vee F$ maps modules to modules, and we now know that it preserves characters, it must send a simple module $L(w\cdot\lambda)$ to an isomorphic simple module. Since the unit $id \longrightarrow F^\vee F$ of the adjunction evaluates as a non-zero map on every object of $D^b(\mathcal{M}_\lambda)$, it must therefore evaluate as an isomorphism on all simple modules. Since the simple modules generate $D^b(\mathcal{M}_\lambda)$ as a triangulated category, the unit of the adjunction is an isomorphism of functors, so $F^\vee F \cong \mathrm{id}$. Similarly $FF^\vee \cong \mathrm{id}$, and so F and $F^\vee$ are equivalences of categories, completing the proof of the Proposition. □

4. Concluding Remarks

Our main theorem has variants that apply in slightly different situations. Indeed, an almost identical proof gives the obvious generalization of Theorem 2.1 to representations of blocks of G_rT or G_rB, the infinitesimal thickenings of a maximal torus or Borel subgroup of a reductive group G, since all the properties that we use for translation functors for G also hold for G_rT and G_rB (4, II.9).

As we indicated in Section 2, character calculations suggest that Theorem 2.1 generalizes to more general blocks. We make the following conjecture.

CONJECTURE 4.1. *Let G be a reductive group over k that has a simply-connected derived group, and whose Coxeter number is less than p. Let λ and μ be any two weights in $X(T)$ with the same stabilizer under the dot action of W_p. Then there is an equivalence of triangulated categories*

$$D^b(\mathcal{M}_\lambda) \longrightarrow D^b(\mathcal{M}_\mu)$$

that induces the map

$$\chi(w\cdot\lambda) \longmapsto \chi(w\cdot\mu)$$

on characters.

It is not clear to us whether or not the conditions on the derived subgroup and the Coxeter number are strictly necessary.

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Blocks with cyclic defect (Green orders) *

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1 Introduction

One of the goals of integral representation theory is to describe the integral group ring **ZG** or $\hat{\mathbb{Z}}_p G$ [1] for a finite group G, in terms of the following data: We have the inclusions

$$\mathbb{Z}G \subset \mathbb{Q}G = \sum_{1 \leq i \leq n} \mathbb{Q}G \cdot e_i \text{ with } \mathbb{Q}G \cdot e_i = (D_i)_{n_i}, \tag{1}$$

where the sum is taken over a complete set of central primitive idempotents and D_i are finite dimensional skewfields over $\mathbb{Q}$.

1. Describe $\Lambda_i := \mathbb{Z}G \cdot e_i \subset (D_i)_{n_i}$.
2. Describe Λ as a subdirect sum in $\prod_{1 \leq i \leq n} \Lambda_i$. This is usually done as iterated pullbacks with finite common quotients.

Let us point out, that the natural basis $\{g \in G\}$ does in general not reflect the arithmetical properties of $\mathbb{Z}G$, as is apparent for example in the description of blocks with defect p (cf. [6]).

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[1] Here $\hat{\mathbb{Z}}_p$ stands for the p-adic integers.

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We shall give here a description of an integral p-adic block B with cyclic defect. The referee has pointed out that Markus Linckelmann gives in [3] a description of these blocks by generators and relations. However, the present method of determining B seems to be easier and it is likely that this approach could simplify most of the arguments in [3].

At this point, I wish to express my thanks to the referee for his many illuminating comments; they have lead to a number of improvements in the presentation of this paper [2].

Let me start with an **example**.

With every group G we have associated the augmentation sequence

$$0 \longrightarrow I(G) \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{2}$$

Example 1 : In general we put $\mathbb{C}_{p^n} :=< c_{p^n} \,|\, c_{p^n}^{p^n} = 1 >$ and for $m < n$ we sometimes identify C_{p^m} with a subgroup of C_{p^n}, in which case $c_{p^m} = c_{p^n}^{p^{n-m}}$ and sometimes with a quotient, in which case c_{p^m} is the image of c_{p^n}. We consider $G := C_{p^2} =< c_{p^2} \,|\, c_{p^2}^{p^2} = 1 >$ and denote be $C_p =< c_p >$ the unique subgroup of order p. Then we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_p)\cdot \mathbb{Z}G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z}C_p & \longrightarrow & 0 \\ & & \| & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & (\zeta_{p^2}-1)\cdot R_{p^2} & \longrightarrow & R_{p^2} & \longrightarrow & \mathbb{F}_pC_1 & \longrightarrow & 0\,, \end{array} \tag{3}$$

where α is reduction modulo $\sum_{i=1}^{p} c_p^i$ and β is reduction modulo p, ζ_{p^i} is a primitive p^ith root of unity and $R_{p^i} = \mathbb{Z}[\zeta_{p^i}]$ and $\mathbb{F}_p$ is the field with p elements. Noting that $\mathbb{F}_p \simeq R_p/(\zeta_p)\cdot R_p$ we can write $\mathbb{Z}C_p$ as a pullback

$$\begin{array}{ccc} \mathbb{Z}C_p & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ R_p & \longrightarrow & \mathbb{F}_p\,. \end{array} \tag{4}$$

[2] Editors remark: The author chooses occasionally to quote from the referee's report. The editors wish to emphasize that the responsibility for the accuracy in all results and definitions remains with the author and not the referee, in this and other papers in this volume.

With these data we have described $\mathbb{Z}C_p^2$. Inductively this construction can be continued to give an explicit description of the group rings of cyclic groups in general and cyclic p-groups in particular.

In general though, very little is known for p-adic blocks. The aim of this note is to describe p-blocks with cyclic defect of the group ring RG, where R is a finite - not necessarily unramified - extension of $\mathbb{Z}$. The details can be found in [7]. Parts of the present results can be obtained from Plesken's Habilitationsschrift [5].

At the 4-th international conference on representations of Algebras in Ottawa (1979) independently H. Jacobinski and the author reported on the structure of blocks with defect p in SG, where S is an unramified extension of $\hat{\mathbb{Z}}_p$. The results here will generalize these structure theorems to the case of cyclic defect.

2 Blocks with normal cyclic defect and isotypic orders

Green orders generalize p-adic blocks with cyclic defect and still allow "Green's walk around the Brauer tree" [2][3]. These orders are determined more or less by a finite tree - though they can also be defined for graphs, even graphs with a valuation - and generalize the Bäckström orders [6]. The vertices correspond to orders and the edges to pullbacks and projective lattices respectively.

In order to make the situation more easily understood I shall discuss here in detail the situation of a principal block with normal cyclic defect group. This case is easily reduced to

$$G = C_{p^n} \rtimes C_q \text{ where } C_q =< c_q \,|\, c_q^q = 1 > \text{ and } q|(p-1). \qquad (5)$$

c_q acts as Galois automorphism on c_{p^n}. For $i < n$ the group $C_{p^i} =< c_{p^i} >$ is the normal subgroup of order p^i in C_{p^n}. Moreover we assume here that $R = \hat{\mathbb{Z}}_p$. As a typical example I think of $p = 7, n = 2$ and

[3]This farreaching result of Green is based on the work of Dade [1] and Thompson [8].

$q = 3$ with the action ${}^{c_3}c_{p^2} = c_{p^2}^{30}$. The group ring RC_{p^n} is then in a natural way an RG-module, where C_{p^n} acts by left multiplication and C_q acts by conjugation; it is even projective indecomposable since its restriction to C_{p^n} is RP. Its ring of endomorphisms is given by

$$\mathcal{E}_{p^n} := End_{RG}(RC_{p^n}) = H^0(C_q, RC_{p^n}) =: (RC_{p^n})^{C_q}\,, \tag{6}$$

the ring of fixed points, which can be computed inductively as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_p)\cdot RC_{p^n} & \longrightarrow & RC_{p^n} & \longrightarrow & RC_{p^{n-1}} & \longrightarrow & 0 \\ & & \| & & \downarrow\alpha & & \downarrow\beta & & \\ 0 & \longrightarrow & (\zeta_{p^n}^{p^{n-1}}-1)\cdot R_{p^n} & \longrightarrow & R_{p^n} & \longrightarrow & \mathbb{F}_pC_{p^{n-1}} & \longrightarrow & 0\,. \end{array} \tag{7}$$

Since $q\cdot R = R$, taking fixed points is an exact functor, and so we get the pullback diagram

$$\begin{array}{ccc} \mathcal{E}_{p^n} & \longrightarrow & \mathcal{E}_{p^{n-1}} \\ \downarrow & & \downarrow \\ R_{p^n}^{C_q} & \longrightarrow & (\mathbb{F}_pC_{p^{n-1}})^{C_q}\,. \end{array} \tag{8}$$

We note, that $R_{p^n}^{C_q}$ is totally ramified of degree $(p^{n-1}\cdot(p-1))/q$ and $(\mathbb{F}_qC_{p^{n-1}})^{C_q} \simeq \mathbb{F}_p[X]/X^{p^{n-2}\cdot(p-1)/q}$.

This inductively gives a description of $End_{RG}(RC_{p^n})$.

We now have described one projective module, and we shall now turn to the group ring RG, which we again describe as pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_{p^n})\cdot \mathbb{Z}G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z}C_q & \longrightarrow & 0 \\ & & \| & & \downarrow\alpha & & \downarrow\beta & & \\ 0 & \longrightarrow & (c_{p^n}-1)\cdot RG & \longrightarrow & \Gamma & \longrightarrow & (\mathbb{Z}/p^n\cdot\mathbb{Z})C_q & \longrightarrow & 0\,. \end{array} \tag{9}$$

Here α is reduction modulo the trace of C_{p^n} and β is reduction modulo p^n. Since $\hat{\mathbb{Z}}_p$ contains a primitive q^{th} root of unity - $q|(p-1)$ - we conclude

$$RC_q \simeq \oplus_q R \text{ and } \mathbb{Z}/p^n\cdot\mathbb{Z}C_q \simeq \oplus_q \mathbb{Z}/p^n\cdot\mathbb{Z}\,. \tag{10}$$

In order to describe Γ we have to recall from [7]:

Definition 1 : *Let D be a Dedekind domain with field of fractions K. A D-order Γ in a separable K-algebra A is called isotypic if there exists a two-sided Γ-ideal J such that*

1. $K \otimes_R J = A$,
2. *J is a projective left Γ-module,*
3. *Γ/J is a product of local R-algebras,*
4. *J is nilpotent modulo the Higman-ideal.*

Note 1 : Λ is an isotypic R-order if and only if this holds locally. Since the global order is the intersection of its localization, it suffices to describe the isotypic orders in the local situation. So we assume that D is a complete Dedekind domain. Then the Condition 4. is equivalent to $J \subset rad(\Gamma)$. The following is a description of the isotypic orders, which also justifies the name:

Proposition 1 : *Let Γ be a basic connected isotypic R-order. Then Γ has the following form*

$$\Gamma = \begin{pmatrix} \Omega & \Omega & \cdots & \Omega & \Omega \\ \omega & \Omega & \cdots & \Omega & \Omega \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \omega & \omega & \cdots & \omega & \Omega \end{pmatrix}_q, \tag{11}$$

with

$$J = \begin{pmatrix} \omega & \Omega & \cdots & \Omega & \Omega \\ \omega & \omega & \cdots & \Omega & \Omega \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \omega & \omega & \cdots & \omega & \omega \end{pmatrix}_q, \tag{12}$$

which is generated by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \omega_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_q, \tag{13}$$

where ω is a principal ideal for the local order Ω, generated by ω_0. Conversely every such order is isotypic.

Remark 1 :

1. We point out, that $\Gamma/J \simeq \oplus_q \Omega/\omega \cdot \Omega$, and so all the local algebras in the definition are the **same** .

2. In case $\omega = rad(\Omega)$ the corresponding order Γ is a hereditary order.

We now return to the group ring RG from above. From Equation 10 and the Diagram 9 it follows, that Γ is an isotypic order and thus has the structure given in Equation 11. But in our case we have $\Omega/\omega \simeq \mathbb{Z}/p \cdot \mathbb{Z}$ is uniserial local. To describe the group ring RG we have to give a description of Ω and ω_0. We recall, that RC_n is the projective indecomposable RG-module which has trivial top, so it can be described as a pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_{p^n}) & \longrightarrow & RC_{p^n} & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I(C_{p^n}) & \longrightarrow & \Lambda_{p^n} & \longrightarrow & R/p^nR & \longrightarrow & 0\,. \end{array} \tag{14}$$

Taking fixed points is again an exact functor, and we get the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(C_{p^n})^{C_q} & \longrightarrow & \mathcal{E}_{p^n} & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_0 \cdot \Omega_{p^n} & \longrightarrow & \Omega_{p^n} & \longrightarrow & R/p^nR & \longrightarrow & 0 \end{array}, \tag{15}$$

where $\omega_0 = \prod_{i=1}^{q} ({}^{c_q{}^i} c_{p^n} - 1)$ is the norm of $(c_{p^n} - 1)$ under C_q. Moreover, also $\mathcal{E}_{p^n}$ can be described inductively. We illustrate this only in case $n = 2$. Ω_{p^2} is a pullback

$$\begin{array}{ccc} \Omega_{p^2} & \longrightarrow & R_p^{C_q} \\ \downarrow & & \downarrow \phi \\ R_{p^2}^{C_q} & \xrightarrow{\psi} & \mathbb{F}_p[X]/X^{(p-1)/q}\,, \end{array} \tag{16}$$

where both ϕ and ψ are reduction modulo $rad^{(p-1)/q}$.

So RG is the pullback of a $q \times q$ isotypic order with q copies of R, identified along the diagonal modulo ω_0 and p^n respectively. So with such a block we associate the tree, which is a star with q edges. With the central vertex we associate the local order $\{\Omega, \omega\}$ with the outside vertices the orders $\{R, p^m \cdot R\}$.

3 Green orders

My definition of a Green order as given in [7] for R a finite extension of $\hat{\mathbb{Z}}_p$ is quite involved.

Below I shall give a definition suggested by the referee.

Let K be the field of fractions of R, which is assumed to be big enough for all algebras under considerations (a finite set) for simplicity [4]. Let Λ be an R-order in a semisimple K-algebra A. Let I be a set of pairwise orthogonal primitive idempotents of Λ, such that $\sum_{i \in I} i = 1$, and let E be a set of idempotents [5] of $Z(K \otimes_R \Lambda)$, the centre of $K \otimes_R \Lambda$ [6], such that $\sum_{e \in E} e = 1$. We put

$$T := \{(i, s) \,|\, i \in I \,,\, e \in E \,:\, i \cdot e \neq 0\}\,,$$

and we identify the pair $(i, e) \in T$ with $i \cdot e$. Let

$$\pi : T \longrightarrow I \text{ and } \theta : T \longrightarrow E \text{ be the canonical maps.}$$

Definition 2 *Λ is a* **Green order** *with respect to E, if Λ and E fulfill the following conditions:*

1. *T is a* **tree***: that is to say, T defines a connected relation on E and we have $|E| = |I| + 1$ and $|\pi^{-1}(i)| = 2$ for any $i \in I$.*

2. *The family $\{\Lambda \cdot t\}_{t \in T}$[7] of (isomorphismhism classes of) Λ-modules is a* **cycle** *for the action of the* **Heller translator** *Ω: that is to*

[4]This assumption is not made in [7], since there we were mainly interested in blocks over the p-adic integers.

[5]This does not necessarily mean primitive idempotents.

[6]We identify Λ with $1 \otimes_R \Lambda$ in $K \otimes_R \Lambda$.

[7]Recall that we have identified $(i, e) \in T$ with $i \cdot e$.

say, there is a transitive permutation w *of* T *and, for any* $t \in T$, *a* Λ*-module homomorphism*

$$f_t : \Lambda\cdot\pi(t) \longrightarrow \Lambda\cdot\pi(w(t)) \text{ such that } Im(f_t) = Ker(f_{w(t)}) \simeq \Lambda\cdot t\,.$$

Remark 2 1. Of course, w is the well-known 'walk around the Brauer tree' introduced by Green (but it deserves an explicit formal definition).

2. By Green's results, if Λ is the basic algebra of $RG \cdot b$, and b is a block of G with cyclic defect groups, then Λ is a Green order with respect to a suitable E (i.e.

$$E = \{e_\Delta\} \cup \{e_\mu\}_{\mu\in M} \text{ , where } e_\Delta = \sum_{\delta\in\Delta} e_\delta\,,$$

Δ and M being respectively the sets of exceptional and non-exceptional irreducible characters of G in b, respectively in the basic algebra).

Let me give an **example**: Assume we are given local R-orders Ω_i and principal Ω_i-ideals $\omega_i\,, 1 \le i \le 5$ such that for $1 \le i,j \le 5$ we have $\Omega_i/\omega_i \simeq \Omega_j/\omega_j$.

Assume now that we are given the tree

$$\begin{array}{ccccccc} (\Omega_1,\omega_1) & \overset{P_1}{—} & (\Omega_2,\omega_2) & \overset{P_2}{—} & (\Omega_3,\omega_3) & \overset{P_3}{—} & (\Omega_4,\omega_4) \\ & & & & \Big| P_4 & & \\ & & & & (\Omega_5,\omega_5) & & \end{array} \tag{17}$$

1. The vertex 1 has one edge, so we associate with it an isotypic order for (Ω_1,ω_1) of size (1×1), call this Λ_1.

2. The vertex 2 has two edges, so we associate with it a basic isotypic order for (Ω_2,ω_2) of size (2×2), call this Λ_2.

3. The vertex 3 has three edges, so we associate with it a basic isotypic order for (Ω_3,ω_3) of size (3×3), call this Λ_3.

4. The vertex 4 has one edge, so we associate with it an isotypic order for (Ω_4, ω_4) of size (1×1), call this Λ_4.

5. The vertex 5 has one edge, so we associate with it an isotypic order for (Ω_5, ω_5) of size (1×1), call this Λ_5.

These are the projections of the order Λ we are going to construct. Up to now we have defined some projections of Λ, namely the orders Λ_i. Now we shall describe the pullbacks:

1. There is an edge labelled $'P_1'$ between the vertices one and two, and so we identify the $(1,1)$-position of Λ_1 with the $(1,1)$-position of Λ_2 via the given isomorphism $\Omega_1/\omega_1 \simeq \Omega_2/\omega_2$.

2. There is an edge labelled $'P_2'$ between the vertices two and three, and so we identify the $(2,2)$-position of Λ_2 with the $(1,1)$-position of Λ_3 via the given isomorphism $\Omega_2/\omega_2 \simeq \Omega_3/\omega_3$.

3. There is an edge labelled $'P_3$ between the vertices three and four, and so we identify the $(2,2)$-position of Λ_3 with the $(1,1)$-position of Λ_4 via the given isomorphism $\Omega_3/\omega_3 \simeq \Omega_4/\omega_4$.

4. There is an edge labelled $'P_4'$ between the vertices three and five, and so we identify the $(3,3)$-position of Λ_3 with the $(1,1)$-position of Λ_5 via the given isomorphism $\Omega_3/\omega_3 \simeq \Omega_5/\omega_5$.

We now denote by M_i^j the non isomorphic indecomposable projective left Λ_i-lattices in the j^{th} column of Λ_i. A projective resolution of M_1^1 is given by "Green's walk around the Brauer tree" in our case counter clockswise, since we have defined Λ by walking around the tree clockwise.

This way we have constructed from the given tree our order Λ, and it is obvious, that such a construction can be done with every tree T, where there are associated to the vertices given local R orders Ω_i and principal Ω_i-ideals ω_i such that for i, j we have a given isomorphism $\Omega_i/\omega_i \simeq \Omega_j/\omega_j$.

The order Λ constructed in this way is then the Green order associated to the tree T and the orders Ω_i with ideals ω_i (cf. [7]).

A projective resolution of the projective indecomposable lattice corresponding to an end of the tree is given by the walk around the tree.

We are now in the position to state the main result:

Theorem 1 *Let R with field of fractions be a finite extension of $\hat{\mathbb{Z}}_p$ and let B be a block with cyclic defect of RG for a finite group G. Then*

1. *B is a Green order with associated tree the Brauer tree of B (over KG), and exceptional vertex v_0.*

2. *To the vertices there correspond the isotypic R-orders*

$$B \cdot \eta_i := \begin{pmatrix} \Omega_i & \Omega_i & \cdots & \Omega_i & \Omega_i \\ \omega_i & \Omega_i & \cdots & \Omega_i & \Omega_i \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \omega_i & \omega_i & \cdots & \omega_i & \Omega_i \end{pmatrix}_{n_i}, \tag{18}$$

   where n_i is the valency of the vertex v_i.

3. *For $i \neq 0$; i.e. for the non exceptional vertices, the algebras $K \otimes_R B \cdot \eta_i$ are simple.*

4. *If K is a splitting field for G, then $\Omega_i = R$ for every $i \neq 0$, and for $i \neq 0$ a generator for ω_i is π^ν, where ν is independent of i.*

5. *In general Ω_i for $i \neq 0$ is an unramified extension of R.*

6. *The local algebras Ω_i/ω_i are uniserial for every i.*

This and the results in [7] have the following more elegant formulation, suggested by the referee:

Theorem 2 *1. If Λ is a Green order with respect to E in the sense of Definition 2, then there exists a family of R-orders $\{\Omega_e\}_{e\in E}$, a torsion R-algebra $\overline{\Omega}$ and a family*

$$\{\Omega_e \longrightarrow \overline{\Omega}\}_{e\in E}$$

of surjective R-homomorphisms such that Λ is isomorphic to the R-algebra of matrices

$$(a_{t,t'})_{t,t'\in T} \text{ in } M_T(\otimes_{e\in E}\Omega_e)\,,$$

where T is totally ordered by w and where the choice of a first element satisfies, for any $t,t'\in T$, the following four conditions:

(a) If $\theta(t)\neq\theta(t')$ then $a_{t,t'}=0$.

(b) If $\theta(t)=\theta(t')$ then $a_{t,t'}\in\Omega_{\theta(t)}$.

(c) If $\theta(t)=\theta(t')$ and $t<t'$ then $\overline{a_{t,t'}}=0$ in $\overline{\Omega}$.

(d) If $\pi(t)0\pi(t')$ then $\overline{a_{t,t}}=\overline{a_{t',t'}}$ in $\overline{\Omega}$.

2. If Λ is the basic order of $RG\cdot b$, then

$$\overline{\Omega}=R/|P|\cdot R \text{ and } B_{e_\mu}\simeq R \text{ for any } \mu\in M\,,$$

(a fact already proved by Plesken [5]) and it follows from the note by Linkelmann [4] that

$$\Omega_{e_\Delta}\simeq (RP)^{\mathcal{T}}/R\cdot(\sum_{u\in P}u)$$

and that the R-algebra homomorphism from this R-order onto $R/|P|\cdot\ R$ is determined by the augmentation of RP, where P is a defect group of b and $\mathcal{T}$ is an inertial quotient.

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Rigid and Exceptional Sheaves on a Del Pezzo Surface

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1 Introduction

It is quite clear nowdays that certain developments in algebraic geometry go along lines quite similar to those in representation theory. The goal of the paper is to present a series of such a results about algebraic vector bundles and sheaves. So the text is aimed at non-specialists and we will permit ourselves to repeat some factes quite trivial to those who are in the field and to discuss general philosophy a little more than usual in mathematical papers. The most technical proofs are omitted.

The paper is written during my visit to MIT, I would like to thank the Institute and the people there.

In the following we will work with vector bundles over an algebraic surface S. Let it always be smooth , complete and projective and sometimes S will be assumed to be a Del Pezzo surface.

It is well known that an algebraic vector bundle is determined by its sheaf of sections which would be a locally free coherent sheaf. And vice versa a locally free coherent sheaf is the sheaf of section of a vector bundle. Usually this is used to identify vector bundles with a full subcategory in the coherent sheaves category $Sh\ S$, because homomorphisms between vector bundles and their sheaves of sections are the same.

All the sheaves that we will talk about are supposed to be algebraic and coherent over an algebraic surface S

Let us fix standard notations:

$\mathbf{k}$ – the ground field,

$r(A)$ – rank of a sheaf A (which, if A is a sheaf of sections of a vector bundle, is equal to the dimension of a fiber of the bundle),

$\mathcal{K}_S$, K_S – canonical sheaf on S and corresponding element in the Neron-Severy group of divisor classes or algebraic cycles in S (canonical class of S),

$c_1(A)$ – the first Chern class of A which we will consider as element of the Neron-Severy group,

$c_2(A)$ – the second Chern class of A which here will be an integer number,

${}^0\langle A, B\rangle$ or Hom(A, B) – the ($\mathbf{k}$- vector) space of (global) morphisms from A to B,

V. Dlab and L. L. Scott (eds.), Finite Dimensional Algebras and Related Topics, 277–291.

${}^i\langle A,B\rangle$ or $\mathrm{Ext}^i(A,B)$ – the i-th (global) ext between A and B,

$\chi(A,B)=\sum_i(-1)^i \dim {}^i\langle A,B\rangle$ – Euler characteristic form.

And there are two important extensively used theorems.

Theorem 1.1 (Riemann-Roch theorem)

$$\chi(A,B)=\frac{1}{2}\left[r(A)c_1(B)-r(B)c_1(A)\right]\cdot(-K_S)+r(A)r(B)+$$

$$+r(A)(\frac{1}{2}c_1(B)^2-c_2(B))-r(B)(\frac{1}{2}c_1(A)^2-c_2(A))-c_1(A)\cdot c_1(B)\,.$$

Theorem 1.2 (Serre duality theorem)

$${}^i\langle A,B\rangle={}^{(\dim S-i)}\langle B,A\otimes K_S\rangle^*\,.$$

2 Filtrations and stability

We work with coherent sheaves on an algebraic variety X, and I first remind you that a sheaf has a support. The support supp A is the minimal closed subset in X such that a sheaf A is zero outside of it. And the dimension of supp A is an important characteristic of a sheaf. We will say that A is an m-supported sheaf if $\dim \operatorname{supp} A=m$, and that A has pure m-support if it is m-supported one and has no subsheaves with support of smaller dimension. If $\dim\operatorname{supp} A=\dim X$ then A has pure support iff A has no torsion, because $\dim\operatorname{supp} B<\dim X$ is equivalent to B being a torsion sheaf.

The support of a subsheaf or a factor sheaf belongs to the support of the sheaf, so there is no nonzero morphism from A to B, if $\dim\operatorname{supp} A<\dim\operatorname{supp} B$ and B has pure support.

There is a filtration in a sheaf which is described by the following Support Filtration Theorem.

Theorem 2.1 *Let A be a coherent sheaf on a variety X and* $\dim X=n$. *Then there is a filtration in A*

$$A=F_0A\supset F_1A\supset\cdots\supset F_nA\supset F_{n+1}A=0$$

such that $\dim\operatorname{supp} F_iA\le(n-i)$ *and that factors Gr_iA are of pure $(n-i)$-support if non zero.*

So $\mathrm{Hom}(Gr_jA,Gr_iA)=0$ for $i<j$. We will call this filtration a support filtration. The result is standart and it is among basic properties of sheaves but rarely it is stated in this form. See [Hart] for the proof.

There is another filtration that works for pure support sheaves and which we will use for sheaves without torsion on Del Pezzo surfaces. As it requires some definitions that are quite technical for the general case, we will restrict ourselves to sheaves on a Del Pezzo surface.

Definition 2.1 *Let A be a sheaf on a Del Pezzo surface S and assume A has no torsion (then* supp $A = S$ *and rank A is non zero). Let $r(A)$ be rank A and $c_1(A)$ the first Chern class of A, and let the slopes $\mu(A)$ be defined by*

$$\mu(A) = \frac{c_1(A) \cdot (-K_S)}{r(A)}$$

where $\cdot$ means intersection pairing and K_S is the canonical class (the image of the canonical sheaf $\mathcal{K}_S$) in the Neron-Severi group of S.

Slopes belong to the rational number field $\mathbf{Q}$. The slope of a canonical sheaf $\mathcal{K}_S$

$$\mu(\mathcal{K}_S) = -K_S \cdot K_S = -K_S^2$$

is an important characteristic of a surface S. It goes from 9 (for $S = \mathbf{P}^2$) to 8,7, ...,1 for Del Pezzo surfaces [Man].

Definition 2.2 *A sheaf A is called semistable if for any non zero subsheaf B in A*

$$\mu(B) \leq \mu(A),$$

and it is called stable if for such a subsheaf B whose rank is smaller than rank A

$$\mu(B) < \mu(A).$$

The definition is meaningful only for sheaves without torsion and*we will presume that saying a sheaf is stable or semistable implies that it has no torsion.* Also we will recall that we are using sheaves on Del Pezzo surfaces at moments when the property in question really depends on this assumption.

We will state below most useful properties of stable and semistable sheaves.

Proposition 2.1 (i) *If A is stable sheaf and sheaves B and C have non zero rank and there is an exact sequence*

$$0 \to B \to A \to C \to 0,$$

then $\mu(B) < \mu(A) < \mu(C)$.

(ii) *If A is semistable and the rest is the same as above, then $\mu(B) \leq \mu(A) \leq \mu(C)$.*

(iii) *If A and B are semistable and $\mu(A) < \mu(B)$, then* $\mathrm{Hom}(B, A) = 0$.

(iv) *If A and B are stable and $\mu(A) \leq \mu(B)$, then either* $\mathrm{Hom}(B, A) = 0$ *or $A = B$ and* $\mathrm{Hom}(B, A) = \mathbf{k}$.

(v) *If A and B are semistable sheaves on Del Pezzo surface and $\mu(A) \leq \mu(B)$ then ${}^2\langle A, B\rangle = 0$.*

One could prove (i) and (ii) by straight calculation of slopes based on additivity of rank and Chern classes. Next statement follows from this and the definition of semistability. Statement (iv) is quite an analog of Schur's lemma and the proof goes the same way. The last one follows from (iii) having in mind the Serre duality

$$^2\langle A, B\rangle = {}^0\langle B, A \otimes \mathcal{K}_S\rangle^*$$

and the fact that $\mu(A \otimes \mathcal{K}_S) = \mu(A) + \mu(\mathcal{K}_S) \leq \mu(A) - 1 < \mu(B)$.

For more detailed exposition of stability see [Ocon].

Now we are ready for theorems.

Theorem 2.2 *Let A be a coherent sheaf on a Del Pezzo surface S and A has no torsion. Then there is a filtration in A*

$$A = F_1 A \supset F_2 A \supset \cdots \supset F_k A \supset F_{k+1} A = 0$$

such that factors $Gr_i A$ are semistable and $\mu(Gr_i A) < \mu(Gr_{i+1} A)$

Usually the filtration of the theorem is called *the Harder-Narasimhan filtration.*

Remarks.

1. The filtration is canonical and unique for given A.
2. Length k of the filtration varies from 1 for semisimple A to $+\infty$ but *factors are always non zero.*
3. From the proposition 2.1(iii),(v) we have $^0\langle Gr_j A, Gr_i A\rangle = 0$, $^2\langle Gr_i A, Gr_j A\rangle = 0$ for $i < j$.
4. It is general opinion among specialists that there is such a theorem for any variety when "good enough" stability is defined, but proofs are written only for special cases [H-N, D-L]. The proof for Del Pezzo surfaces is outlined in [Gor, Kul].

The third filtration theorem we are dealing with establishes relation between semistable and stable sheaves.

Theorem 2.3 *If A is a semistable sheaf then there is a filtration*

$$A = F_1 A \supset F_2 A \supset \cdots \supset F_m A \supset F_{m+1} A = 0$$

such that factors $Gr_i A$ are stable and $\mu(Gr_i A) = \mu(Gr_{i+1} A) = \mu(A)$.

Remark. Here the filtration is not unique and one can say that it is a Jordan-Holder type filtration. It is usually called the Jordan-Holder filtration and it was studied in relation with a completion of moduli space for stable sheaves [Mar].

We will also use the following theorem.

Theorem 2.4 *Let A be a sheaf with a filtration*

$$A = F_1A \supset F_2A \supset \cdots \supset F_{n+1}A = 0$$

and G_i denoting factors of the filtration. Then there is a spectral sequence which abutts to $\mathrm{Hom}(A, A)$ *such that*

$$E_1^{p,q} = \oplus_i \ {}^{p+q}\langle G_i, G_{i+p}\rangle$$

The outlines of the proof goes as follows. Let $A \to \mathbf{U}$ be an injective resolution of A such that $\mathbf{U}$ has a filtration with a property that its factorization gives injective resolutions $Gr_iA \to Gr_i\mathbf{U}$. Then there is a filtration in the complex $\mathrm{Hom}(A, \mathbf{U})$

$$F_i\mathrm{Hom}(A, \mathbf{U}) = \{\varphi \mid \varphi(F_kA) \subset F_{k+i}\mathbf{U}\}$$

There is a standart spectral sequence for computing homology of filtered complex and it is what is needed. We apply it to the filtered complex $\mathrm{Hom}(A, \mathbf{U})$ and we need only to interpret E_1-term and it is quite a standard calculation.

3 Basic properties of exceptional sheaves

Our goal here is to state and prove some basic properties of exceptional sheaves on a Del Pezzo surface.

Definition 3.1 *Let E be a coherent sheaf on a Del Pezzo surface S. We will call E exceptional if:*
(1) ${}^0\langle E, E\rangle = \mathrm{Hom}(E, E) = \mathbf{k}$,
(2) ${}^1\langle E, E\rangle = 0$,
(3) ${}^2\langle E, E\rangle = 0$.

Remarks.

1. As we consider sheaves on a surface, higher ext-s would be zero, so ${}^0\langle E, E\rangle = \mathbf{k}$ and ${}^i\langle E, E\rangle = 0$ for $i > 0$.
2. Really one could prove that for sheaves on Del Pezzo (1),(2) $\Rightarrow$ (3). But we omit this because it involves too technical details.
3. Condition (1) implies that E is not a direct some of sheaves and (2) implies that E has no deformations because ${}^1\langle A, A\rangle$ is tangent space to a moduli scheme (which represented parameters for deformations) for E.

The folowing is the central theorem of the section.

Theorem 3.1 *Let E be an exceptional sheaf on a Del Pezzo surface S.*

Then either E is locally free stable sheaf (and corresponds to vector bundle) or E is a purely 1-supported torsion sheaf.

If E is a torsion sheaf then supp E *is a smooth irreducible exceptional curve in S (which is a projective line) and the restriction E to the curve is a line bundle on it. Any such a sheaf is exceptional.*

We will go though several propositions and lemmas to get the proof. The next proposition is the key point of the proof. It was proved by several authors under more and more relaxed conditions on S. First it was proved for projective plane [Rul, Dr] then for the most of Del Pezzo surfaces [Gor], then the last restrictions were dropped off by D Orlov in his thesis [Orl].

Proposition 3.1 *If E has no torsion, then E is semistable.*

To prove this let us apply the Spectral sequence theorem 2.4 to the Harder-Narasimhan filtration in E. Let us denote by G_i factors of the filtration. They are non zero and have no torsion also. We ought to prove that there is actually only one factor.

So we have got a spectral sequence where $\mathrm{E}_1^{p,q}$ could be different from zero only for $p+q = 0, 1, 2$. But

$$\mathrm{E}_1^{p,q} = 0 \text{ for } \begin{cases} p+q=0,\ p<0, \\ p+q=2,\ p\geq 0. \end{cases}$$

This follows from proposition 2.1.(iii),(v) because factors of the filtration are semisimple and their slopes decrease. Thus all the differentials related to $\mathrm{E}^{0,1}$ vanish, so

$$\mathrm{E}_1^{0,1} \ = \ \mathrm{E}_\infty^{0,1} \quad \text{but} \quad \mathrm{E}_\infty^{0,1} \ = \ 0 \quad \text{because} \quad {}^1\langle E, E\rangle = 0.$$

And also this vanishing of differentials permits us to say that there are two parts of the spectral sequence that above $q = 1$ and that below $q = 1$ which do not interact with each other. So we can apply the Euler characteristic rule to each of those parts. Hence

$$\sum_{q>1}(-1)^{p+q} \dim \mathrm{E}_1^{p,q} \ = \sum_{q>1}(-1)^{p+q} \dim \mathrm{E}_\infty^{p,q} \ = \ 0$$

and

$$\sum_{q<1}(-1)^{p+q} \dim \mathrm{E}_1^{p,q} \ = \sum_{q<1}(-1)^{p+q} \dim \mathrm{E}_\infty^{p,q} \ = \ 1.$$

But, from the construction of the spectral sequence, we have

$$\sum_{q>1}(-1)^{p+q} \dim \mathrm{E}_1^{p,q} \ = \sum_i \ \dim {}^2\langle G_i, G_i\rangle + \sum_{p<0,\, i} \ \dim \chi(G_i,\, G_{i+p}).$$

And similarly

$$\sum_{q<1}(-1)^{p+q} \dim \mathrm{E}_1^{p,q} \ = \sum_i \ \dim {}^0\langle G_i, G_i\rangle \ + \sum_{p>0,\, i} \ \dim \chi(G_i,\, G_{i+p}).$$

Then we substract one line from another, and we get

$$1 \ = \ 1 \ - \ 0 \ = \sum_i [\dim {}^0\langle G_i, G_i\rangle - \dim {}^2\langle G_i, G_i\rangle] \ + \sum_{p>0,\, i} [\dim \chi(G_i,\, G_{i+p}) - \dim \chi(G_{i+p},\, G_i)].$$

All the summands on the right side are integers and the following lemmas show that they are non negative ones. Hence there is really only one summand and that means that the filtration has only one factor. This finishes the proof.

To use them later the lemmas are stated in more generality that in is needed at the moment.

Lemma 3.1 *If a sheaf F has no 0-supported torsion, then*

$$dim\ {}^0\langle F, F\rangle \ \geq\ dim\ {}^2\langle F, F\rangle + 1.$$

Lemma 3.2 *If $r(A) \neq 0$, $r(B) \neq 0$ and $\mu(A) < \mu(B)$, then*

$$\chi(A, B) - \chi(B, A) > 0.$$

Proof of lemma 3.2. Straightforward calculation from the Riemann-Roch theorem shows that

$$\chi(A, B) - \chi(B, A) \ = r(A)\, r(B)\, (\mu(B) - \mu(A))$$

and this proves the lemma.

Proof of lemma 3.1. By the Serre duality theorem

$$\dim\ {}^2\langle F, F\rangle \ = \ \dim\ {}^0\langle F, F \otimes \mathcal{K}_S\rangle.$$

And any morphism of sheaves $\mathcal{K}_S \to \mathcal{O}_S$ (or their corresponding line bundles) give us a morphism

$${}^0\langle F, F \otimes \mathcal{K}_S\rangle \ \to \ {}^0\langle F, F\rangle.$$

Morphisms $\mathcal{K}_S \to \mathcal{O}_S$ could be constructed as multiplication by a global section of the line bunlde $\mathcal{K}_S^{-1}$ (anticanonical bundle which is ample by the very definition of Del Pezzo surface and which has at least two dimensional space of global sections for any Del Pezzo surface). And as supp$F$ is at least one dimensional, there exists a section that is non zero on an open dense subset in supp$F$. The resulting morphism

$${}^0\langle F, F \otimes \mathcal{K}_S\rangle \ \to \ {}^0\langle F, F\rangle$$

will be a monomorphism, and its image will not intersect the one dimensional subspace generated by the identity in ${}^0\langle F, F\rangle$, because zeros of any section of $\mathcal{K}_S^{-1}$ has will have non trivial intersection with supp$F$. Hence we have proved the lemma.

Proposition 3.2 *An exceptional sheaf E is either purely 1-supported sheaf or has no torsion.*

Proof goes along the same lines. Let us take the support filtration in E

$$E = F_1 \supset F_2 \supset F_3 \supset F_4 = 0$$

with factors G_0, G_1, G_2 which could also be equal to zero. Applying Spectral sequence theorem 2.4 we get a spectral sequence with

$$\mathrm{E}_1^{p,q} = 0 \ \text{ for } p + q = 0,\ p < 0$$

because there are no morphisms between purely supported sheaves from smaller supported sheaf to larger supported one. And by the same reason for $p + q = 2$, $p > 0$ there is

$$\mathrm{E}_1^{p,q} \ = \oplus_i\ {}^2\langle G_i, G_{i+p}\rangle \ = \oplus_i\ {}^0\langle G_{i+p}, G_i \otimes \mathcal{K}_S\rangle^* \ = \ 0,$$

because tensoring on an invertible sheaf does not change the support. So again all the differentials related to $\mathrm{E}^{0,1}$ vanish hence

$$\mathrm{E}_1^{0,1} \ = \ \mathrm{E}_\infty^{0,1} \ \text{ but } \mathrm{E}_\infty^{0,1} \ = 0.$$

Thus

$${}^1\langle G_0, G_0\rangle = {}^1\langle G_1, G_1\rangle = {}^1\langle G_2, G_2\rangle = 0.$$

And the following lemma shows that this implies $G_2 = 0$, as G_2 is a purely 0-supported sheaf by the construction of the spectral sequence.

Lemma 3.3 *For a non zero purely 0-supported sheaf F*

$$^1\langle F, F\rangle \neq 0$$

Going further by the same Euler charactiristic reasoning as in the proof of the previous proposition, we come to an equality

$$1 = (\dim\ {}^0\langle G_0, G_0\rangle - \dim\ {}^2\langle G_0, G_0\rangle) + (\dim\ {}^0\langle G_1, G_1\rangle - \dim\ {}^2\langle G_1, G_1\rangle) + [\,\chi(G_0, G_1) - \chi(G_1, G_0)\,]$$

whose right side summands we will look at providing that both G_1 and G_0 are non zero.

First the lemma 3.1 shows that both the first two summands are positive integers. The last summand we can evaluate by the Riemann-Roch theorem

$$\chi(G_0, G_1) - \chi(G_1, G_0) = r(G_0)\,(\,c_1(G_1)\cdot(-K_S)\,).$$

And $c_1(G_1)$ is an effective element of the Neron-Severy group, thus its intersection with an ample element $(-K_S)$ is positive. Hence

$$\chi(G_0, G_1) - \chi(G_1, G_0) > 0,$$

and we have come to a contradiction. So, to prove lemma 3.3 is the last remaining step in our reasoning:

Proof of lemma 3.3. For such a sheaf B one has $r(B) = 0$ and $c_1(B) = 0$ as follows from basic properties of Chern classes [Hart]. And then the Riemann-Roch theorem implies that

$$\chi(B, B) = 0.$$

So $^1\langle B, B\rangle = 0$ is impossible.

Proposition 3.3 *If an exceptional sheaf E has no torsion, then it is stable.*

We have proved that E is semistable. To prove the proposition we will use the same way of working with a filtration in E. All we need is to prove triviality of the Jordan-Holder filtration in E. But we have not enough properties for this filtration, so we will make a more special one.

If F is a stable subsheaf of E then its slope is equal to $\mu(E) = \mu$, so it is not difficult to show that there is a filtration in E

$$E = F_0 \supset F_1 \supset F_2 \cdots \supset F_m = F \supset F_{m+1} = 0$$

such that all the factors G_i except $E/F_1 = G_0$ are isomorphic to F and that $\mathrm{Hom(F, G_0)} = 0$. One need to use proposition 2.1 and proceed by induction on rank E. And we will have $\mathrm{Hom}(F_1, G_0) = 0$ also.

Now let us apply the Spectral sequence theorem 2.4 to a shortened filtration

$$E \supset F_1 \supset 0$$

with two factors $G_0 = E/F_1$ and $G_1 = F_1$. For this filtration we have

$$\mathrm{Hom(G_1, G_0)} = 0.$$

By the proposition 2.1 we could have non zero $E_1^{p,q}$ only for $p+q = 0, 1$ because $\mu(G_0) = \mu(F_1) = \mu$. And as we have only two factors we have zeros outside $p = -1, 0, 1$.

Also

$$E_1^{-1,1} = {}^0\langle G_1, G_0\rangle = 0$$

by the mentioned above property of the filtration. From this we conclude that

$$E_1^{-1,2} = E_\infty^{-1,2} = 0, \qquad E_1^{1,-1} = E_\infty^{1,-1} = 0, \qquad E_1^{0,1} = E_\infty^{0,1} = 0.$$

Hence $\chi(G_1, G_0) = 0$. But we know that

$$\chi(G_1, G_0) - \chi(G_0, G_1) = r(G_1)\, r(G_0)\, (\mu - \mu) = 0.$$

Thus $\chi(G_1, G_0) = 0$, so $\dim {}^0\langle G_1, G_0\rangle = \dim {}^1\langle G_1, G_0\rangle$. This shows us that

$$E_1^{1,0} = {}^0\langle G_1, G_0\rangle = 0.$$

As a net result all $E_1^{p,q}$ are zero except $E_1^{0,0}$ hence

$$E_1^{0,0} = E_\infty^{0,0} = \mathbf{k}.$$

This implies that ${}^0\langle G_0, G_0\rangle = 0$ so $E = F_1$.

Now we have a filtration

$$E = F_1 \supset F_2 \cdots \supset F_m = F \supset F_{m+1} = 0\,,$$

and let us again apply the Spectral sequence theorem 2.4 to it. We have got

$$E_1^{p,q} = 0 \text{ for } p + q \neq 0,\ 1, \text{ or } p \geq m,\ p \leq -m$$

So $E_1^{-m,-m+1} = E_\infty^{-m,-m+1} = 0$ and this means that

$${}^1\langle F_m, F_1/F_2\rangle = {}^1\langle F, F\rangle = 0$$

Thus E is isomorphic to a direct sum

$$E = F \oplus \cdots \oplus F$$

but as ${}^0\langle E, E\rangle = \mathbf{k}$, hence $m = 1$, only one summand is possible and $E = F$ as desired.

We left to the readers to prove themselves or to consult in literature for those detailes in the theorem 3.1 which are not proved yet. The fact that E is locally free was proved by S Mukai [Muk] in a quite general setting and you can find the proof in [Gor] also. The description of torsion exceptional sheaves on Del Pezzo surfaces where made by D Orlov [Orl].

Theorem 3.2 *Exceptional sheaves on a Del Pezzo surface are uniquelly determined by their K_0-image in a sense that:*
if sheaves E_1 and E_2 are exceptional, then

$$[E_1] = [E_2] \text{ in } K_0(S) \text{ implies } E_1 = E_2;$$

if E is a non torsion exceptional sheaf and F is a stable sheaf, then

$$[F] = [E] \text{ in } K_0(S) \text{ implies } F = E.$$

To prove the second statement we notice first that $[F] = [E]$ implies $r(F) = r(E) \neq 0$ and $c_1(F) = c_1(E)$, $c_2(F) = c_2(E)$, so $\mu(F) = \mu(E)$. And we get

$$\chi(E,F) = \chi(E,E) = 1$$

by the Riemann-Roch theorem. Hence

$$\dim {}^0\langle E,F\rangle + \dim {}^2\langle E,F\rangle \geq 1.$$

But by proposition 2.1(v) we have ${}^2\langle E,F\rangle = 0$, so ${}^0\langle E,F\rangle$ would be non zero. And now by proposition 2.1(iv) any non trivial morphism of stable sheaves of equal slope is isomorphism.

The first statement for non torsion sheaves follows from the second one, so we need only to investigate torsion exceptional sheaves. But they are of the type $i_*\mathcal{O}_l(m)$ where $i : l \to S$ is an inclusion of an exceptional curve l into S and $\mathcal{O}_l(m)$ is a line bundle of a degree m on l. For such a sheaf we have $c_1 = l$, $c_2 = -(m+1)$, hence it is uniquelly determined by its K_0-image. This proves the statement.

4 Admissible pairs and mutations

We will present here some ways to construct exceptional sheaves. Their significance will be stressed by a deep and fundamental theorem at the end on the section. They are also used in the next section to define a braid group action on systems of exceptional sheaves which gives way to classify exceptional sheaves on a Del Pezzo surface.

Definition 4.1 *A pair A, B of exceptional sheaves is called an exceptional pair if*

$${}^k\langle B,A\rangle = 0 \text{ for all } k.$$

Lemma 4.1 *Let A, B be an exceptional pair.*

If one of the following occur:

(La): ${}^k\langle A,B\rangle = 0$ for $i \neq 0$ and the canonical morphism $\varphi : {}^0\langle A,B\rangle \otimes A \to B$ is a part of an exact sequence

$$0 \to L \to {}^0\langle A,B\rangle \otimes A \to B \to 0,$$

(Lb): ${}^k\langle A,B\rangle = 0$ for $i \neq 0$ and the canonical morphism $\varphi : {}^0\langle A,B\rangle \otimes A \to B$ is a part of an exact sequence

$$0 \to {}^0\langle A,B\rangle \otimes A \to B \to L \to 0,$$

(Lc): ${}^k\langle A,B\rangle = 0$ for $i \neq 1$ and L is defined as universal extention of the type

$$0 \to B \to L \to {}^1\langle A,B\rangle \otimes A \to 0,$$

then the pair L, A is exceptional.

Definition 4.2 *If the condition of lemma 4.1 holds then the pair A, B is called left admissible and the pair L, A is called a left mutation of A, B.*

Remark. A pair $\mathcal{O}(i), \mathcal{O}(i+p)$ of sheaves on $\mathbf{P}^n$ is a left admisible exceptional pair for $0 < p \leq n$.

Proof of lemma 4.1. In some sense it is just a calculation but let us make it. Let us begin with (La) case and look at long exact sequences for Ext-functors, first at one related to functor Hom(A,):

$$\begin{aligned} 0 \to {}^0\langle A, L\rangle \to {}^0\langle A, B\rangle \otimes {}^0\langle A, A\rangle \to {}^0\langle A, B\rangle \to \\ \to {}^1\langle A, L\rangle \to {}^0\langle A, B\rangle \otimes {}^1\langle A, A\rangle \to {}^1\langle A, B\rangle \to \\ \to {}^2\langle A, L\rangle \to {}^0\langle A, B\rangle \otimes {}^2\langle A, A\rangle \to {}^2\langle A, B\rangle \to \cdots \end{aligned}$$

It is important to notice that the canonical morphism induces an isomorphism

$${}^0\langle A, B\rangle \otimes {}^0\langle A, A\rangle \to {}^0\langle A, B\rangle \otimes \mathbf{k} \to {}^0\langle A, B\rangle$$

in the first row, so taking our condition into account, we get

$${}^k\langle A, L\rangle = 0 \text{ for any } k. \tag{1}$$

Now let us use functor Hom(, L). We have got one more exact sequence:

$$\begin{aligned} 0 \to {}^0\langle B, L\rangle \to {}^0\langle A, B\rangle^* \otimes {}^0\langle A, L\rangle \to {}^0\langle L, L\rangle \to \\ \to {}^1\langle B, L\rangle \to {}^0\langle A, B\rangle^* \otimes {}^1\langle A, L\rangle \to {}^1\langle L, L\rangle \to \\ \to {}^2\langle B, L\rangle \to {}^0\langle A, B\rangle^* \otimes {}^2\langle A, L\rangle \to {}^2\langle L, L\rangle \to \cdots \end{aligned}$$

and the middle column is all zeros, so there are isomorphisms

$${}^i\langle L, L\rangle \to {}^{i+1}\langle B, L\rangle\,. \tag{2}$$

And taking functor Hom(B,) we get another one

$$\begin{aligned} 0 \to {}^0\langle B, L\rangle \to {}^0\langle A, B\rangle \otimes {}^0\langle B, A\rangle \to {}^0\langle B, B\rangle \to \\ \to {}^1\langle B, L\rangle \to {}^0\langle A, B\rangle \otimes {}^1\langle B, A\rangle \to {}^1\langle B, B\rangle \to \\ \to {}^2\langle B, L\rangle \to {}^0\langle A, B\rangle \otimes {}^2\langle B, A\rangle \to {}^2\langle B, B\rangle \to \cdots \end{aligned}$$

from which we get isomorphisms

$${}^i\langle B, B\rangle \to {}^{i+1}\langle B, L\rangle\,, \tag{3}$$

because the middle column is again all zeros.

Combining isomorphisms (2) and (3) we see that L is exceptional, and (1) shows that L, A is an exceptional pair.

Remark. We use the exceptionality of B at the very last moment in the reasoning. One could generalize the lemma to make a kind of functor $B \mapsto L$ preserving ext-s. Something of this kind was made in [Tju].

Proof for (Lb) and (Lc) cases goes surprisingly along the same lines. One might apply the same functors in the same order to have the same conclusions up to minor changes in indices. We leave this to the interested reader.

Lemma 4.2 *Let A, B be an exceptional pair.*
If one of the following occur:
(Ra): ${}^k\langle A,B\rangle = 0$ for $i \neq 0$ and the canonical morphism $\psi : A \to {}^0\langle A,B\rangle^* \otimes B$ is a part of an exact sequence

$$0 \to A \to {}^0\langle A,B\rangle^* \otimes B \to R \to 0,$$

(Rb): ${}^k\langle A,B\rangle = 0$ for $i \neq 0$ and the canonical morphism $\psi : A \to {}^0\langle A,B\rangle^* \otimes B$ is a part of an exact sequence

$$0 \to R \to A \to {}^0\langle A,B\rangle^* \otimes B \to 0,$$

(Rc): ${}^k\langle A,B\rangle = 0$ for $i \neq 1$ and R is defined as universal extention of the type

$$0 \to {}^1\langle A,B\rangle^* \otimes B \to R \to A \to 0,$$

then the pair B, R is exceptional.

We leave the proof of this lemma to the reader.

Definition 4.3 *If the condition of lemma 4.2 holds then the pair A, B is called right admissible and the pair B, R is called a right mutation of A, B.*

Remark. A pair $\mathcal{O}(i), \mathcal{O}(i+p)$ of sheaves on $\mathbf{P}^n$ is also a right admisible pair for $0 < p \leq n$.

Theorem 4.1 *For sheaves on a Del Pezzo surface any exceptional pair is left and right admissible.*

This is a wonderful result and an important one for all the following. It was proved by A Gorodentsev [Gor] under minor restriction on the surface in question, and the restriction was deleted by D Orlov [Orl]. To prove it one ought to work with complexes and to move to a derived category setting. We will not discuss it here.

5 Braid group action theorems

In this section we will give an overview of properties of exceptional systems for sheaves on a Del Pezzo surface.

Definition 5.1 *A system of exceptional sheaves $E_0, E_1, \ldots, E_m$ is called an exceptional system if any pair E_i, E_j for $i < j$ is exceptional.*

Proposition 5.1 *If $\mathbf{e} = (E_0, \ldots, E_{i-1}, E_i, E_{i+1}, E_{i+2}, \ldots, E_m)$ is an exceptional system of sheaves on a Del Pezzo surface S and L, E_i is a left mutation of a pair E_i, E_{i+1} then the system*

$$\mathcal{L}_i(\mathbf{e}) = (E_0, \ldots, E_{i-1},\ L, E_i,\ E_{i+2} \ldots, E_m)$$

is an exceptional system.

Theorem 5.1 *The operations $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{m-1}$ define Braid group action on the set of exceptional systems of length m of sheaves on S.*

To prove the proposition we need first to show that

$$^k\langle E_j, L\rangle = 0 \quad \text{for all } \; k,\; j > (i+1)$$

and

$$^k\langle L, E_j\rangle = 0 \quad \text{for all } \; k,\; j < i$$

which is just simple calculation with long exact sequences of ext-spaces.

To prove the theorem we need to check the braid group relations

$$\mathcal{L}_i\mathcal{L}_j = \mathcal{L}_j\mathcal{L}_i \quad \text{for } i \neq j-1\,,\, j+1,$$

$$\mathcal{L}_i\mathcal{L}_{i+1}\mathcal{L}_i = \mathcal{L}_{i+1}\mathcal{L}_i\mathcal{L}_{i+1}.$$

The plan here is that we will check the relations on a K_0-level which means showing that images in $K_0(S)$ of the left and right side are the same. Then actual equality will follow from theorem 3.2. To do this we can look at images in a $\mathbf{Q}$-vector space $K_0(S)\otimes\mathbf{Q}$, because $K_0(S)$ is a free $\mathbf{Z}$-module for Del Pezzo surface S. But let us proceed using more general setting.

Let V be a $\mathbf{Q}$-vector space with a bilinear form $(\ ,\ )$.

Definition 5.2 *A system $e_0, e_1, \dots, e_m$ is called triangular if*

$$(e_i, e_i) = 1 \quad \textit{for } 0 \le i \le m,$$

$$(e_j, e_i) = 0 \qquad \textit{for } i < j.$$

Proposition 5.2 *Let L_i be a transformation on the set of triangular systems of length m in V such that it moves a system $e_0, \dots, e_{i-1}, e_i, e_{i+1}, e_{i+2}, \dots, e_m$ to the system*

$$e_0, \dots, e_{i-1}, e_{i+1} - (e_i, e_{i+1})\, e_i,\; e_{i+2}, \dots, e_m$$

then the operations $L_0, L_1, \dots, L_{m-1}$ define Braid group action on the set of triangular systems of length m in V.

It required a series of elementary computations to prove all the statments of the proposition. We leave it to the reader.

To finish the proof of the theorem we mention first that it is sufficient to prove the equality of $K_0(S)$-images of exceptional sheaves up to signs: if the sheaves are non torsion, then positivity of ranks on both sides will imply an equality, and if they are torsion sheaves, then positivity of first Chern classes will do the same.

Now we could take as V a $\mathbf{Q}$-vector space $K_0(S)\otimes\mathbf{Q}$ with a naturally induced form.

Lemma 5.1 *Let $\mathbf{e} = (E_0, \dots, E_m)$ be an exceptional system of sheaves on S. The image $[\mathbf{e}]$ in V of an exceptional system $\mathbf{e}$ is a triangular system and the system $[\mathcal{L}_i\mathbf{e}]$ coincides with $L_i[\mathbf{e}]$ up to signs of its elements.*

It is important to mention that if we change some signs in

$$e_0, e_1, \ldots, e_m$$

then it will always result only in some sign change in

$$L_i(e_0, e_1, \ldots, e_m).$$

Again we leave this and the lemma to check to the reader, and that completes the proof of the theorem.

Now I would like to state without proof some recent results.

Definition 5.3 *An exceptional system $E_0, E_1, \ldots, E_m$ of sheaves on S is called full if its image in $K_0(S) \otimes \mathbf{Q}$ is a basis of this vector space.*

Theorem 5.2 *Let the ground field* $\mathbf{k}$ *be algebraically closed.*
Then there exist a full exceptional system of sheaves on a Del Pezzo surface.

It was proved in full by D Orlov in [Orl] and before by different people for some types of the surfaces. The proof works for algebraically closed field because it uses a construction of Del Pezzo surfaces via blowing up which gives all the Del Pezzo surfaces over such a field. Quite probably the assumption is redundant.

Theorem 5.3 *Let the ground field* $\mathbf{k}$ *be algebraically closed.*
Then any exceptional system on a Del Pezzo surface can be enlarged to a full exceptional system.

As a particular case this means that any exceptional sheave is in some full exceptional system.

This is a recent result proved by A Kuleshov and D Orlov, and will be published somewhere. It also depends on blowing up construction and on previous results for the projective plane and quadrics that there proved in [Ru1],[Dr],[Ru2],[Zu].

Theorem 5.4 *Let the ground field* $\mathbf{k}$ *be algebraically closed.*
Then the action of Braid group on full exceptional systems of sheaves on a Del Pezzo surface is transitive.

This is the most recent result of the same A Kuleshov and D Orlov and it uses similar methods as the previous one and probably will be published in the same paper. For minimal Del Pezzo surfaces it was proved in [Ru1],[Dr], [Rud2] and for some other cases in [Nog].

The last three theorems together permit us to find all exceptional sheaves on a Del Pezzo surface via a kind of an algorithmic procedure.

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Quasihereditary algebras and Kazhdan-Lusztig theory

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A major unsolved problem in finite group theory centers on determining the characters and degrees of the irreducible modular representations of finite groups of Lie type in their defining characteristic. Lusztig took a significant step toward a solution in 1979 by formulating his celebrated conjecture [22] for the characters of simple modules for semisimple algebraic groups. Since that time, mathematicians have devoted considerable effort to establishing this conjecture, which would completely solve the above problem for primes p of modest size relative to the root system ($p \geq h$, the Coxeter number, which is n for SL_n). In addition, recent work by Richard Dipper and Gordon James suggest parallels between the describing and nondescribing characteristic representation theory, so that a proof of the conjecture might provide insight into the irreducible representations of these finite groups in all characteristics, cf. [14], [13], [18].

A similar conjecture, by Kazhdan and Lusztig [20], for the composition factor multiplicities of Verma modules for semisimple complex Lie algebras, has already been settled [4], [6], by reduction to a formula of Kazhdan-Lusztig for perverse sheaves [21]. For some time, CPS has worked to develop algebraic techniques capturing some of the geometric methods used in the proof of this similar conjecture. The common ground for the algebra and geometry is the theory of finite dimensional algebras. Quasihereditary algebras were introduced because we could prove [7] that all of the 'highest weight categories'

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in Lie theory relevant to either conjecture could be described as module categories for quasihereditary algebras, and we hoped the same was true of the perverse sheaf categories instrumental in the characteristic zero proof. This was ultimately proved in [24, §5] (and observed there to be formally equivalent to an independent result of Morollo-Villonen).

This laid the groundwork for the deeper project, still in progress, of applying the insight gained from the geometry to inspire common algebraic approaches to all of these areas: characteristic p and characteristic 0 Lie theory, and perverse sheaves. Our first success in this direction is the notion of a Kazhdan-Lusztig theory [9], [10], [11], which provides conditions on an arbitrary quasihereditary algebra which are necessary and sufficient for the validity of the Lusztig or Kazhdan-Lusztig conjecture in a Lie-theoretic context, as well as new consequences of the conjecture. I have outlined this theory below, and have included new material explaining the algebraic and ℓ-adic reasons that relevant perverse sheaf categories fit the theory.

It should be mentioned that there is another approach to the Lusztig conjecture, proposed by Lusztig [23], which uses quantum groups. Quasihereditary algebras are relevant to the quantum approach as well, cf. [16]. For recent progress in the quantum direction, the reader is referred to [2].

Highest weight categories: notation, conventions, and examples. Fix a field k, and let $\mathcal{C}$ be an abelian $k-$category (that is, $\mathcal{C}$ is abelian, all Hom sets are $k-$modules, and multiplication of morphisms is $k-$linear). We suppose the nonisomorphic irreducible objects $L(\lambda)$ to be indexed by the elements λ of a poset Λ, called *weights.* For simplicity we will assume Λ is finite here; for a more general notion (requiring only that the *intervals* of Λ be finite), the reader is referred to [7]. We will also assume for simplicity that all objects of $\mathcal{C}$ have finite length, that Hom sets between objects are finite-dimensional over k, and, moreover, that $\mathrm{End}(L(\lambda) \cong k$ for each $\lambda \in \Lambda$. We assume that $\mathcal{C}$ has enough projectives, and let $P(\lambda)$ denote the projective cover of $L(\lambda)$. We say that $\mathcal{C}$ is a *highest weight* category if there are objects $V(\lambda), \lambda \in \Lambda$, such that

1) $V(\lambda)$ has head $L(\lambda)$, and all other composition factors of $V(\lambda)$ have smaller weight than λ.

2) There is an epimorphism $P(\lambda) \to V(\lambda)$ with kernel filtered by objects $V(\mu)$ with μ greater than λ.

These conditions imply that $V(\lambda)$ is the largest epimorphic image of $P(\lambda)$ with λ maximal among the weights of its composition factors. Such objects arise naturally in Lie-theoretic contexts. We call $V(\lambda)$ a *Weyl object*, since it is a Weyl module in our favorite context of characteristic p algebraic group representations. Other good names are *Verma object*, or simply *Standard object*. Typically, these objects are well understood, and the main object of research is to write the irreducible objects in terms of them in the Grothendieck group (that is, to obtain their "Weyl character formula"). This is precisely what the Lusztig conjecture purports to do.

Three examples in Lie theory

1) The example which first motivated us is the following: Let G be a semisimple, simply connected algebraic group over an algebraically closed field k of positive characteristic p. Let T be a fixed maximal torus, and denote the root system of T acting on the Lie algebra of G by Φ. We choose a set Φ^+ of positive roots, and let B denote the Borel subgroup corresponding to the associated set Φ^- of *negative* roots. The set $X(T)$ of characters (weights) on T is partially ordered by the rule: $\lambda \leq \mu \Leftrightarrow \mu - \lambda = \sum_{\alpha\in\Phi^+} n_\alpha \alpha$ for non-negative integers n_α. We also have an induced poset structure on the set $X(T)^+$ of dominant weights (relative to Φ^+). Fix any finite set Λ_0 of dominant weights, let Λ be the (finite) set of dominant weights λ for which $\lambda \leq \lambda_0$ for some $\lambda_0 \in \Lambda_0$. Then the category $\mathcal{C}$ of finite-dimensional $G-$modules (in the sense of algebraic groups) which have composition factors each with maximal $T-$weight in Λ is a highest weight category with weight poset Λ. The Weyl modules are obtained as duals of modules induced to G, in the sense of algebraic groups, from dominant weights in $X(T)^+$ extended to B. (These induced modules are all finite-dimensional!)

Projecting $\mathcal{C}$ onto any block of $G-$modules also gives a highest weight category. If $p \geq h$, the Coxeter number of the root system, it is well-known [19] that the character formulas for all irreducible modules are deducible from those in the principal block. The weights for the latter are the dominant weights in the orbit $W_p \cdot 0$ of 0 under the 'dot' action of the affine Weyl group W_p (defined by $w \cdot \mu = w(\mu + \rho) - \rho$, where ρ is the sum of all the fundamental dominant weights, for $w \in W_p$ and $\mu \in X(T)$.) Also, Steinberg's tensor product theorem allows us to restrict attention to *restricted* weights, those with coefficients less than p when expressed in terms of fundamental weights. Let us redefine Λ to be the set of dominant weights which are in the orbit $W_p.0$ and bounded above by a restricted weight in that orbit. Lusztig's conjecture [22] may then be written

$$\mathrm{ch}L(w \cdot 0) = \sum_{y.0 \in \Lambda} (-1)^{\ell(w)-\ell(y)} P_{yw_0, ww_0}(1) \mathrm{ch}(V(y.0),$$

for any weight $w.0$ in Λ. Here y, w are in W_p and w_0 denotes the long word in the ordinary Weyl group W. The terms $P_{yw_0,ww_0}(1)$, are values at 1 of *Kazhdan-Lusztig polynomials*, which are defined in a purely combinatorial way for any pair of elements in a Coxeter group. Finally $\ell(w)$ denotes the length of w in the sense of Coxeter groups (the number of fundamental reflections in a minimal expression), and the function ch(-) just assigns an object of $\mathcal{C}$ to the associated element in the Grothendieck group of $\mathcal{C}$.

To repeat: the conjectured formula would give character formulas for all irreducible $G-$modules, so long as $p \geq h$, and these would in turn give corresponding character formulas for any finite group $G(\mathbf{F_q})$ of Lie type associated to G, with q a power of p.

Lusztig obtained his conjecture by analogy with his conjecture with Kazhdan for complex Lie algebras, which we describe next.

2) Let g be a complex semisimple Lie algebra, and fix a Cartan subalgebra h and Borel subalgebra b containing h. Consider the corresponding category $\mathcal{O}$ of BGG. The objects are the g-modules which are h-diagonalizable with finite-dimensional weight spaces., and with the set of nonzero weights bounded above by some finite set of weights. We will also restrict attention to the case where all weights are *integral*; equivalently, they belong to the set $X(T)$ of characters for a torus T associated to h. It is again true that any block of such modules forms a highest weight category, and all character formulas for irreducible modules are obtainable from the principal block case. The standard objects this time are the Verma modules $M_\lambda, \lambda \in X(T)$, obtained by tensor induction of λ at the universal enveloping algebra level from b to g. We write $V(\lambda) = M_\lambda$, and let $L(\lambda)$ denote the irreducible head of $V(\lambda)$. The weights Λ indexing irreducible modules in the principal block are just those in the orbit $\Lambda = W \cdot -2\rho$. (This is also the orbit of 0 under the 'dot' action, since $0 = w_0 \cdot -2\rho$.) They correspond bijectively to elements of the Weyl group. The Kazhdan-Lusztig conjecture (now a theorem, as we mentioned) reads

$$\mathrm{ch}(L(w \cdot -2\rho) = \sum_{y \in W} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \mathrm{ch}(V(y. - 2\rho),$$

where again $P_{y,w}(1)$ is the value at 1 of a Kazhdan-Lusztig polynomial.

The similarity of this formula and the previous one is remarkable, and all the more so when one considers that the standard modules in the first case are finite-dimensional, but infinite-dimensional here. The next case is even more remarkable, in that we obtain precisely the same character formula for standard objects which are not modules at all, but complexes of sheaves.

3) A key ingredient in the proof of the a Kazhdan-Lusztig conjecture was the Kazhdan-Lusztig formula for the stalk dimensions of the cohomology of perverse sheaves. It can be written as a character formula in the Grothendieck group sense we are using here, and we describe it below.

Let $X = G/B$ denote the flag variety obtained from the simply connected semisimple complex Lie group G associated to the Lie algebra g above, and consider the category C of perverse sheaves on X with respect to the Schubert stratification and the middle perversity. (Thus a stratum is a Schubert cell $S(w) = BwB/B, w \in W$, and a perverse sheaf is a complex of sheaves of complex vector spaces with cohomology locally constant (thus constant) and finite-dimensional on Schubert cells, with certain support conditions satisfied.) The poset is W, with its Bruhat-Chevalley order, and the Weyl objects $V(w), w \in W$ are quite easy to describe: $V(w) = i_{S(w)!}\mathbf{C}[\ell(w)]$, the extension by 0 of the constant sheaf, shifted downward as a complex in the derived category by degree $\ell(w)$. Every Weyl object $V(w)$ has a unique irreducible quotient $L(w)$, and the axioms for a highest weight category are satisfied [24, §5]. Though unnecessary in our discussion, it is a remarkable fact that $L(w)$ is the downward shift by $\ell(w)$ of the complex (extended by zero to X) defining Goreski-MacPherson intersection cohomology on $\overline{S(w)}$.

The Grothendieck group formula of Kazhdan-Lusztig [21] reads

$$\mathrm{ch}(L(w \cdot -2\rho) = \sum_{y \in W} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\mathrm{ch}(V(y. - 2\rho),$$

which is identical to the form of the Verma module Kazhdan-Lusztig conjecture above. Essentially, the latter conjecture was proved through an equivalence of categories reducing it to the above formula.

Quasi-hereditary algebras

Every highest weight category with finite weight poset and all objects of finite length is the category of finite-dimensional modules for a quasihereditary algebra S. Indeed, CPS introduced quasihereditary algebras for this reason, and proved that, conversely, the category of modules for a quasihereditary algebra could be viewed as a highest weight category [26], [24], [7]. (In our context here, one should consider only finite-dimensional modules, and assume that k is algebraically closed or that endomorphism rings of irreducible S-modules are 1-dimensional.) We will not reproduce the axioms for a quasihereditary algebra here, but note that examples include hereditary algebras and poset algebras [24], as well as all finite-dimensional algebras of global dimension two [15]. All quotient algebras of hereditary algebras are quasihereditary. Further Lie-theoretic examples of quasihereditary algebras include Schur algebras and $q-$Schur algebras, and their generalizations, cf. [8], [16].

Every quasihereditary algebra S has finite global dimension. The opposite algebra S^{op} is quasihereditary with the same weight poset, and the S-module $A(\lambda)$ dual to the Weyl module $V^{op}(\lambda)$ for S^{op} has the following remarkable property [7, p.98,bottom]

$$\mathrm{Ext}^n(V(\mu), A(\lambda)) = \begin{cases} k, \text{ if } \lambda = \mu, \\ 0 \text{ otherwise.} \end{cases}$$

Using this, we can give a very general character formula for $L(\lambda)$, due in its first Lie-theoretic context to Delorme:

If $X \in \mathrm{Ob}(D^b(\mathcal{C}))$, we define, for each $\mu \in \Lambda$, associated left and right Poincaré "polynomials" $p^L_{\mu,X}$, $p^R_{\mu,X} \in \mathbf{Z}[t, t^{-1}]$ by the formulas:

$$p^L_{\mu,X} = \sum_n \dim_k \mathrm{Hom}^n(X, A(\mu))t^n$$

and

$$p^R_{\mu,X} = \sum_n \dim_k \mathrm{Hom}^n(V(\mu), X)t^n.$$

It is clear that $p^L_{\mu,X}$ and $p^R_{\mu,X}$ belong to $\mathbf{Z}[t, t^{-1}]$. Of course, if $X \in \mathrm{Ob}(\mathcal{C})$, then, for all $\mu \in \Lambda$, $p^L_{\mu,X}$, $p^R_{\mu,X} \in \mathbf{Z}[t]$.

The polynomials $p_{\mu,\lambda} \equiv p^L_{\mu,\lambda} \equiv p^L_{\mu,L(\lambda)}$ and $p^R_{\mu,\lambda} \equiv p^R_{\mu,L(\lambda)}$ are the *Poincaré polynomials* of the simple module $L(\lambda)$. If $\ell : \Lambda \to \mathbf{Z}$. is any function (called a "length" function) the *Kazhdan-Lusztig polynomials* $P_{\mu,\lambda}$ are given by the formula

$$P_{\mu,\lambda} \equiv P^L_{\mu,\lambda} = t^{\ell(\lambda)-\ell(\mu)}\bar{p}^L_{\mu,\lambda}$$

These polynomials are indexed by pairs of weights in Λ, while the classical Kazhdan-Lusztig polynomials KL1 are indexed by pairs of elements in a Coxeter group.[1] The classical Kazhdan-Lusztig polynomials are determined by the Coxeter group, and may be described combinatorially by a recursion. These classical Kazhdan-Lusztig polynomials may be regarded as known, whereas ours are generally unknown. Nevertheless, we have the following general formula:

Proposition 1 *For $X \in Ob(\mathcal{C})$, let $ch(X)$ denote the corresponding object in the Grothendieck group of $\mathcal{C}$. Then*

$$ch(L(\lambda)) = \sum_{\mu \in \Lambda} (-1)^{\ell(\lambda)-\ell(\mu)} P_{\mu,\lambda}(-1) ch(V(\mu)).$$

Proof. This is a general formula for the character of any object X : The expression $(-1)^{\ell(\lambda)-\ell(\mu)} P_{\mu,\lambda}(-1)$ is just the Euler characteristic $p_{\mu,\lambda}(-1) = p^L_{\mu,L(\lambda)}(-1)$, which makes sense when $L(\lambda)$ is replaced by X. From this point of view, the formula may be proved by checking it on any basis of the Grothendieck group. For $X = V(\lambda)$, we have polynomial $p^L_{\mu,V(\lambda)} \equiv 1$, by the displayed Ext formula, so the formula reduces to the tautology $\mathrm{ch}(V(\lambda)) = \mathrm{ch}(V(\lambda))$. ∎

Kazhdan-Lusztig theory

Though the above proposition is only formal, the similarity with the Lusztig conjecture, Kazhdan-Lusztig conjecture, and Kazhdan-Lusztig formula for perverse sheaves is striking, and demonstrates an underlying unity in a very general setting. There is one difference, however: the minus sign that appears in the general formula of the proposition is replaced by a plus sign in each of the more special formulas. This is explained by the fact that the Kazhdan-Lusztig polynomials associated with Coxeter groups are polynomials in $q = t^2$.

Definition Let $\mathcal{C}$ be a highest weight category as above, with weight poset Λ and length function $\ell : \Lambda \to \mathbf{Z}$. We say that $\mathcal{C}$ has a *Kazhdan-Lusztig* theory

[1] In the presence of a valid Lusztig conjecture (or Kazhdan-Lusztig conjecture) our Kazhdan-Lusztig polynomials identify with the Kazhdan-Lusztig polynomials for the affine Weyl group or W_p (or Weyl group W) as follows:

$$P_{y\cdot 0, w\cdot 0}(t^2) = P_{yw_0, ww_0}(q) = P_{w_0 y, w_0 w}(q) \ .$$

relative to ℓ if each polynomial $P_{\mu,\lambda}$ and its dual $P^R_{\mu,\lambda}$ are polynomials in t^2, for $\lambda, \mu \in \Lambda$. Equivalently,

$$\begin{aligned}&\mathrm{Ext}^n(V(\mu), L(\lambda)) \neq 0 \Rightarrow n \equiv \ell(\lambda) - \ell(\mu) \pmod 2, \quad \text{and}\\ &\mathrm{Ext}^n(L(\lambda), A(\mu)) \neq 0 \Rightarrow n \equiv \ell(\lambda) - \ell(\mu) \pmod 2, \text{ for each } \lambda, \mu \in \Lambda.\end{aligned}$$

Kazhdan and Lusztig proved the polynomial version of this condition for perverse sheaves in [21]. David Vogan [28] proved the Ext version for Verma modules in the category $\mathcal{O}$, under the assumption of the Kazhdan-Lusztig conjecture (not yet known to be true at the time), and work of Andersen [1] gave a corresponding result regarding the Lusztig conjecture. Vogan left open in [29] the question of whether or not these parity conditions implied the validity of the Kazhdan-Lusztig conjecture. He formulated in [28] and [29] another conjecture, now called the Vogan conjecture, on the complete reducibility of certain modules (not described here), which would imply the Kazhdan-Lusztig conjecture, and which was, indeed, equivalent to it.

CPS first encountered these parity conditions studying work of MacPherson, as viewed by Springer [27]. Implicit in this article is the device of keeping track of degree information in a derived category of sheaves in the presence of certain even-odd vanishing conditions on stalk cohomology. Those conditions translate precisely into the above requirements for a Kazhdan-Lusztig theory in an abstract highest weight category

In [9] we were able to answer Vogan's question affirmatively (see also [10] and [11]): We may summarize our results as follows:

Theorem 1 *In each of the three examples above, the Lusztig conjecture or its analog (the Kazhdan-Lusztig conjecture, or Kazhdan-Lusztig formula) is equivalent to the validity of Kazhdan-Lusztig theory relative to the length function furnished by affine Weyl group (or Weyl group). Also equivalent is each of the following statements:*

1) For each $\lambda, \mu \in \Lambda$, $Ext^1(V(\mu), L(\lambda)) \neq 0 \Longrightarrow \ell(\lambda) - \ell(\mu) \equiv 1 \ (mod\, 2)$.

2) For each $\lambda \in \Lambda$, and each weight λ' adjacent to λ (in the sense that the affine Weyl group or Weyl group element associated to λ' is obtained from that associated to λ by right multiplication by a simple reflection), we have $Ext^1(L(\lambda'), L(\lambda)) \neq 0$. (By a duality principle, one may take here $\lambda' < \lambda$. or $\lambda' > \lambda$).

3)For each $\lambda, \mu \in \Lambda$, the natural map $Ext^1(L(\mu), L(\lambda)) \rightarrow Ext^1(V(\mu), L(\lambda))$ is surjective.

By a duality principle, one can replace $V(\mu), L(\lambda)$ in condition 1) by $L(\lambda), A(\mu)$, and a similar replacement may be made in 3), for any given λ, μ. The same duality implies the order of λ, λ' does not matter in 2); if $\lambda' < \lambda$ are adjacent weights, then $\mathrm{Ext}^1(V(\lambda'), L(\lambda))$ is known to be nonzero (1-dimensional), which explains why 3)$\Longrightarrow$ 2). One can also assume both weights in 2) are *restricted* when trying to prove the Lusztig conjecture, cf. [10].

Vogan knew that condition 2) implied inductively the strong form of 1) which is the Kazhdan-Lusztig theory property. We proved in [9, (4.1)] with an elementary but delicately formulated result, stated below, showing that the form of 1) above (with its dual) implies the complete reducibility conjectured by Vogan. We called the result the "parity theorem", where "parity" here is something of a parody on "purity" (The "purity" theorem is a much more sophisticated result of Gabber [3] which implies a related complete reducibility in the perverse sheaf case.)

Theorem 2 *Let $\mathcal{C}$ be a highest weight category having weight poset Λ, and fix $X \in Ob(\mathcal{C})$. For $\lambda, \nu \in \Lambda$, assume that*

$$Hom(X, A(\lambda)) \neq 0 \neq Hom(X, A(\nu)) \Rightarrow Ext^1(V(\lambda), L(\nu)) = 0,$$

and

$$Hom(V(\lambda), X) \neq 0 \neq Hom(V(\nu), X) \Rightarrow Ext^1(L(\nu), A(\lambda)) = 0.$$

Then X is completely reducible.

As per our conventions here, the objects of the highest weight category are required to have finite length, but there are no further requirements, not even a length function. The result is very general, applying, for example, to (the finite-dimensional module categories of) any quasihereditary algebra. Our formulation of the above result was extracted from a first, much more sophisticated, derived category argument, found trying to understand the perverse sheaf situation. The original derived category argument (which we omit here) remains a principal ingredient in the following theorem, from [9] and [12]. Note that the "length function" as we use it here is used only to define a parity.

Theorem 3 *Let $\mathcal{C}$ be an abstract highest weight category as above, with finite weight poset Λ, and suppose $\mathcal{C}$ has a Kazhdan-Lusztig theory relative to a length function $\ell : \Lambda \to \mathbf{Z}$. Then*

1) For each $\lambda, \mu \in \Lambda$ *and any integer* $n \geq 0$*, there is a formula*

$$\dim_k Ext^n(L(\mu), L(\lambda)) = \sum_{\substack{r+s=n \\ \nu \in \Lambda}} \dim_k Ext^r(V(\nu), L(\lambda)) \dim_k Ext^s(L(\mu), A(\nu)).$$

2) For each $\lambda, \mu \in \Lambda$ *and any integer* $n \geq 0$*, the natural maps*

$$\begin{aligned} &Ext^n(L(\mu), L(\lambda)) \to Ext^n(V(\mu), L(\lambda)), \text{ and} \\ &Ext^n(L(\lambda), L(\mu)) \to Ext^n(L(\lambda), A(\mu)) \end{aligned}$$

are surjective.

3) Let L *be a finite direct sum of copies of the simple modules* $L(\lambda)$*, each of the latter appearing with nonzero multiplicity. Then the Ext algebra*

$$Ext^*(L, L)$$

is quasihereditary. Its category of finite-dimensional modules may be viewed as a highest weight category with weight poset Λ^{op} *and Weyl objects* $Ext^*(L, A(\mu))$*, for* $\mu \in \Lambda$.

The derived category enters as follows: The Kazhdan-Lusztig property allows us (after a lemma) to consider each simple object $L(\mu)$ as "filtered" in the derived category by objects $V(\nu)[\ell(v) + 2m]$, $m \in \mathbf{Z}$, where the bracket denotes degree shift in the derived category, and there is, similarly, a filtration for $L(\lambda)$ using objects $A(\nu)[\ell(v) + 2m]$. These two filtrations can be used to calculate $\mathrm{Hom}^n_{D^b(\mathrm{C})}(L(\mu), L(\lambda))$ $=\mathrm{Ext}^n(L(\mu), L(\lambda))$, and this gives assertion 1). Also, it turns out the first filtration must begin with the distinguished triangle $V(\mu) \to L(\mu) \to V(\mu)/L(\mu)[1] \to$. This implies that $\mathrm{Hom}^n(-, L(\lambda))$ applied to this triangle yields either 0 or a short exact sequence. This gives the first surjectivity in 2), and the second is obtained similarly.

For a discussion of 3), see Brian Parshall's article in this volume. We remark that the surjection

$$\mathrm{Ext}^*(L, L(\mu)) \to \mathrm{Ext}^*(L, A(\mu))$$

guaranteed by 2) is the required map of a projective cover of a simple object onto the corresponding Weyl object which appears in the definition of a highest weight category.

CPS has found that it is quite easy to construct a lot of quasihereditary algebras with (a finite-dimensional module category that has) a Kazhdan-Lusztig theory, using the recursive construction of [24]. The following questions and remarks concerning familiar classes of algebras are taken from [12].

Questions and remarks. (a) Let A be a finite dimensional algebra of global dimension ≤ 2 such that $A/\mathrm{rad}(A)$ is k-split. Then A is quasi-hereditary [15], so that $\mathcal{C} = \text{mod}-A$ is a highest weight category as explained in (1.1.3). It would be interesting to determine when $\mathcal{C}$ admits a Kazhdan-Lusztig theory relative to some length function.

Suppose the stronger condition that A is hereditary holds (i. e., $\mathrm{gl.dim}(A) \leq 1$) and, for simplicity, that the weight poset Λ is such that, for each weight λ, we have that $A(\lambda) = L(\lambda)$ and that $V(\lambda)$ is the projective cover of $L(\lambda)$. (This is always possible.) It is then easy to determine when $\mathcal{C}$ has a Kazhdan-Lusztig theory. In fact, we can replace A by a Morita equivalent algebra to assume that it is basic. Then A is isomorphic to the path algebra $k\vec{\Delta}$ for a directed graph $\vec{\Delta}$ having no oriented cycles (see [17]). We say that two simple modules $L(\lambda)$ and $L(\nu)$ have "opposite parity" provided $\mathrm{Ext}^1(L(\lambda), L(\nu)) \neq 0$. Clearly, this relation is consistent if and only if each (necessarily unoriented) cycle in $\vec{\Delta}$ has an even number of edges. If this holds, we can define a length function ℓ compatible with the parity, and $\mathcal{C}$ has a Kazhdan-Lusztig theory.

(b) Consider a finite poset Λ, and let A be the associated poset algebra, as discussed, for example, in [24, §6]. Then A is a quasi-hereditary algebra with weight poset Λ. In the highest weight category $\mathcal{C}$ =mod-A the Weyl module $V(\lambda)$ identifies with the projective indecomposable cover $P(\lambda)$ of the simple module $L(\lambda)$. Also, we have $A(\lambda) = L(\lambda)$ for all weights λ. Suppose that ℓ is a length function defined on Λ. We use ℓ to assign a "parity" to each simple module $L(\lambda)$, according to whether $\ell(\lambda)$ is even or odd. A necessary condition that $\mathcal{C}$ admit a Kazhdan-Lusztig theory is that $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\nu)) \neq 0$ imply that $L(\lambda)$ and $L(\nu)$ have opposite parity. It is very easy to write down simple examples of posets in which such an assignment of parity is not possible. (For example, the "pentagonal" poset $\Lambda = \{a, b, c, d, e\}$ defined by $a > b > e$ and $a > c > d > e$ has this property.) It would be interesting to have a combinatorial characterization of those posets Λ for which the associated highest weight category does admit a Kazhdan-Lusztig theory (relative to an appropriate length function).

Attacking the Ext^1 conditions. The main outstanding question is how to use one of the Ext^1 conditions in Theorem 1, especially condition 2), to prove the Lusztig conjecture (or even to provide an algebraic proof of the Kazhdan-Lusztig conjecture).

Perverse sheaves. A reasonable question to ask is, how do the perverse sheaves happen to satisfy such a condition? The answer involves sophisticated mathematics, but is not difficult to discover:

There is a natural 1−dimensional constant sheaf on any two adjacent Schubert cells. (Recall that a Schubert cell is the image in the flag variety G/B of a B, B double coset. Thus Schubert cells are indexed by the Weyl group W.) Extending by zero, we obtain a sheaf on the entire flag variety. This provides an extension in the sense of objects in an abelian category of extensions by zero of the 1-dimensional constant sheaves on the cells themselves. In the language of highest weight categories, we have an extension of a shifted Weyl object $V(w)[-\ell(w)]$ by a shifted Weyl object $V(ws)[-\ell(ws)]$. Here w and ws are elements of W, the element s is a fundamental reflection, and the lengths of w, ws satisfy $\ell(w) < \ell(ws)$. (Thus $\ell(ws) = \ell(w) + 1$.) Viewing W as a poset, the elements w, ws are adjacent in the sense of part 2) of Theorem 1, and all adjacent weights in the sense of that theorem arise this way. The extension gives a morphism in the derived category $V(w)[-\ell(w)] \to V(ws)[-\ell(ws)][1] = V(ws)[-\ell(w)]$. Shifting downward by $\ell\{w\}$ we obtain a morphism

$$V(w) \to V(ws).$$

In the parallel case of the principal block of the category $\mathcal{O}$, such morphisms also exist; in fact $\mathrm{Hom}_{\mathcal{O}}(V(w), V(ws))$ is 1−dimensional. (A similar statement can be made in the characteristic p situation of our first example.) It is not the existence of this morphism in itself which is so special to the perverse sheaf situation, but its interaction with Deligne's theory of weights. In the perverse sheaf situation, the extension of sheaves we have described is defined over the prime field, as is the above morphism. Sheaves defined over the prime field (and their shifts) come equipped with a natural "action" of the Frobenius morphism, and one may begin to apply the Deligne theory of weights to them after checking they are "mixed". This requires checking certain conditions on stalk eigenvalues for this "action", trivial here.. See [3, Chpt. 5] for a discussion and the formalism of weights. In the following, we will take the notion of an object being "mixed" and "having weight" in some interval as a primitive notion satisfying certain properties, which we will describe, with specific references, as we go along. (Note: We prefer to apply these adjectives at all times to objects in the derived category of sheaves over the flag variety of the algebraic closure of $\mathbf{F}_p$, with the understanding that these objects are arising from objects over the underlying $\mathbf{F}_p$–scheme.)

The sheaves $V(w)[-\ell(w)]$ and $V(ws)[-\ell(ws)]$ we have described both are mixed and "have weights ≤ 0, but not ≤ -1" in the sense of [3, 5.1.8], by virtue of the trivial action of the Frobenius morphism and its powers on stalks at rational points over the corresponding finite fields. Derived category shifts also shift weights, according to [3, 5.1.8], so $V(w)$ "has weights $\leq \ell(w)$, but not $\leq \ell(w) - 1$". A similar statement holds for $V(ws)$, using $\ell(ws)$.

The theory of weights also tells us [3, 5.3.1] that bounds on weights (above or below) for a perverse sheaf defined over the prime field also hold for its subobjects and quotient objects defined over the prime field. Conversely, it is easy to show, using [3, 5.1.8, 5.1.14(iii)], that any upper or lower bound which holds for some subobject and the corresponding quotient must hold for the original perverse sheaf. An object with upper bound $\leq m$ and lower bound $\geq m$ is said to be *pure* of weight m (an integer).

According to [3, Thm. 5.3.5, Thm. 5.3.8], there is an increasing filtration W_*, defined over the prime field, of $V(w)$, of $V(ws)$, or of any perverse sheaf X defined over the prime field, with quotients $W_m X / W_{m-1} X$ of successive terms completely reducible and pure of weight m. All morphisms defined over the prime field preserve these filtrations. Obviously, the top quotient of $V(ws)$ must be the simple object $L(ws)$, and must be pure of weight $\ell(ws)$. The remaining quotients are forced to have smaller weights. Also, the top quotient of $V(w)$ must be pure of weight $\ell(w) = \ell(ws) - 1$, and the same is true of the image of $V(w)$ in $V(ws)$. It follows that the image of $V(w)$ is not contained in $W_{\ell(ws)-2}V(ws)$, and so $V(ws)/W_{\ell(ws)-2}V(ws)$ has a quotient which is an extension (nontrivial) of $L(ws)$ by $L(w)$, as required by condition 2) of Theorem 1.

Graded quasihereditary algebras. The above argument simplifies considerably if we make use of gradings, which the algebras associated to perverse sheaves on flag varieties are known to possess. In fact, it is shown in [5] and [25] that, at least with suitable choice of p – –likely unnecessary—that these algebras are graded Koszul algebras. We will not require the Koszul property for our arguments, only that we are dealing with a graded quasihereditary algebra $S = \bigoplus_{i \geq 0} S_i$ for which S_0 is semisimple.

It will then follow as in [8] that all Weyl modules have a graded structure, with the top degree taken by their heads. As remarked above, there is a nonzero morphism $V(w) \to V(ws)$. It follows that there is a nonzero graded homomorphism to some shift in degree $V(ws)(m)$ of $V(ws)$. Thus, to make the above argument work, one has to show it is possible to take $V(ws)(m)$ to be generated in degree one higher than the generating degree for $V(w)$.

Automorphisms. Actually, we took the point of view in [25] that the perverse sheaf situation might inspire an algebraic approach to gradings. The idea was that the Frobenius morphism induces a self-equivalence on the perverse sheaf category fixing the simple objects, and thus an automorphism on the underlying quasihereditary algebra with the same property. We found purely algebraic eigenvalue conditions on such an automorphism, satisfied in the perverse sheaf case, which can guarantee an algebra (especially, a quasihereditary algebra) to be Koszul.

The above discussion of perverse sheaves now suggests that, to verify the Ext^1 condition 2) of Theorem 1, one should first try to demonstrate a grading on the algebra, perhaps making use of an automorphism (or family of automorphisms) in the spirit of [25], then exhibit a graded homomorphism between certain of its Weyl modules.

The process of exhibiting a grading on an algebra that does not have an obvious grading is not well understood by algebraists, so it is a helpful insight to see that the key grading in the perverse sheaf case arises from an algebra automorphism, as [25] demonstrates. Perhaps this may make the problem of exhibiting such gradings more tractable, or at least make the prospect of finding them less intimidating. But presently we will have to leave this issue to future research.

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CYCLES IN MODULE CATEGORIES

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ABSTRACT. Let A be an artin algebra over a commutative artin ring R and mod$A$ be the category of finitely generated right A-modules. A cycle in mod$A$ is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \cdots \to M_n = M_0$ between indecomposable modules from mod$A$. The main aim of this survey article is to show that study of cycles in mod$A$ leads to interesting information on indecomposable A-modules, the Auslander-Reiten quiver of A, and the ring structure of A. We present recent advances in some areas of the representation theory of artin algebras which should be of interest to a wider audience. In the paper, we also pose a number of open problems and indicate some new research directions.

0. Introduction

The present notes are an extended version of a series of three lectures given during the CMS Annual Seminar/NATO Advanced Research Workshop on "Representations of Algebras and Related Topics" held at Carleton University in August 1992. We present an outlook on the representation theory of artin algebras through the cycles of finitely generated indecomposable modules and show that study of cycles in module categories leads to interesting information on indecomposable modules, algebras and Auslander-Reiten components. We report on recent advances in some areas of the representation theory of artin algebras, pose a number of open problems which seem to be worthwhile to study, and indicate some new research directions. For the convenience of the reader we define here all needed concepts. Moreover, in order to show new ideas, we include some proofs of the presented results, or indicate their main steps. It is our hope that this survey article on the role of cycles in module categories will encourage a further investigation of cycles and their applications in the representation theory of artin algebras and related topics.

Let A be an artin algebra over a commutative artin ring R. We denote by mod$A$ the category of finitely generated A-modules, by rad(mod$A$) the Jacobson radical of mod$A$, by $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ the intersection of all powers $\mathrm{rad}^{i}(\mathrm{mod}A)$, $i \geq 1$, of rad(mod$A$), and by Γ_A the Auslander-Reiten quiver of A. A cycle in the category mod$A$ is a sequence $M_0 \to M_1 \to \cdots \to M_n = M_0$ of non-zero non-isomorphisms between indecomposable

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modules from mod$A$. Such a cycle is said to be finite if the morphisms forming this cycle do not belong to $\mathrm{rad}^{\infty}(\mathrm{mod}A)$.

It is known that A is of finite representation type if and only if $\mathrm{rad}^{\infty}(\mathrm{mod}A) = 0$. If this is the case, we may recover the morphisms in mod$A$ from the quiver Γ_A. In general, Γ_A describes "only" the quotient category $\mathrm{mod}A/\mathrm{rad}^{\infty}(\mathrm{mod}A)$. Moreover, if A is not of finite representation type, then mod$A$ always admits cycles. In fact, it was recently shown that all but finitely many $D\mathrm{Tr}$-orbits in Γ_A consists entirely of modules lying on cycles. We note also that the indecomposable modules which don't lie on a cycle in mod$A$ (directing modules) are rather well understood because their supports are tilted algebras. Hence, in order to obtain information on unknown indecomposable modules, we may try to study properties of cycles containing such modules. We show in these notes that the indecomposable modules which lie only on large cycles or finite cycles are better behaved that the other indecomposable modules. An another important research direction is to study the behaviour of components of Γ_A in the category mod$A$. We present many results showing how the shape of components depends on the properties of cycles passing through the modules lying in these components. We devote our special attention to the components $\mathcal{C}$ in Γ_A such that $\mathrm{rad}^{\infty}(X,Y) = 0$ for all modules X and Y from $\mathcal{C}$, which we call generalized standard components. Observe that such components consist entirely of modules which do not lie on infinite loops. We are also very much interested in the problem how to distinguish the representation types of algebras A (finite type, domestic, polynomial growth, tame, wild) by properties of cycles in mod$A$, properties of $\mathrm{rad}^{\infty}(\mathrm{mod}A)$, and the behaviour of Auslander-Reiten components in mod$A$. We consider the presented notes as a starting point of more systematic studies in the raised above directions.

We divide the notes into the following parts:
1. Preliminaries.
2. The Auslander-Reiten quiver.
3. Tilting configurations of modules.
4. Generalized standard components.
5. Regular components and cycles of modules.
6. Selfinjective algebras and generalized standard components.
7. Composition factors of indecomposable modules.
8. Module categories without infinite loops.
9. Simply connected algebras of polynomial growth.

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1. Preliminaries

In this section we recall some terminology and facts on artin algebras and their module categories which will be used in the article.

Throughout the article A will denote a fixed artin algebra over a commutative artin ring R, that is, A is an R-algebra which is finitely generated as an R-module. We may assume that A is connected (has no non-trivial central idempotents), and hence R is a local ring. We denote by n the rank of the Grothendieck group $K_0(A)$ of A. By a module is always meant a finitely generated right module. We shall denote by $\operatorname{mod} A$ the category of all (finitely generated right) A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by all indecomposable modules. Then $\operatorname{rad}(\operatorname{mod} A)$ denotes the **Jacobson radical** of $\operatorname{mod} A$, that is, the ideal in $\operatorname{mod} A$ generated by all non-invertible morphisms between indecomposable modules. The **infinite radical** $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ of $\operatorname{mod} A$ is the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Recall that A is called **representation-finite** if $\operatorname{ind} A$ admits only finitely many pairwise non-isomorphic objects. The following theorem characterizes the representation-finite artin algebras in terms of the infinite radical.

1.1. THEOREM. *A is representation-finite if and only if* $\operatorname{rad}^{\infty}(\operatorname{mod} A) = 0$.

PROOF. Assume that A is representation-finite. Then there is a common bound on the length of indecomposable A-modules, and $\operatorname{rad}^{\infty}(\operatorname{mod} A) = 0$ by the Harada-Sai lemma [38]. Assume now that A is representation-infinite. Then, by a result of Auslander [9], there exists an infinite sequence in $\operatorname{mod} A$

$$\cdots \longrightarrow M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0$$

where f_i are epimorphisms and M_i are pairwise non-isomorphic indecomposable A-modules. Let $g : A \to M_0$ be a non-zero map in $\operatorname{mod} A$. Then $g = g_i f_i \dots f_1$ for some $g_i \in \operatorname{Hom}_A(A, M_i)$, $i \geq 1$. Hence $g \in \operatorname{rad}^{\infty}(\operatorname{mod} A)$ and $\operatorname{rad}^{\infty}(\operatorname{mod} A) \neq 0$.

We denote by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the **standard duality** $D = \operatorname{Hom}_R(-, \overline{I})$, where $\overline{I}$ is the injective envelope of $R/\operatorname{rad} R$ over R.

A **path** of length t in $\operatorname{mod} A$ is a sequence of non-zero non-isomorphisms

$$(*) \qquad M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{t-1} \xrightarrow{f_t} M_t$$

where $t \geq 1$ and all M_i belong to $\operatorname{ind} A$. If $M_0 \simeq M_t$ then $(*)$ is called a **cycle of length** t. A cycle of length at most 2 is said to be **short** [50]. A **loop** in $\operatorname{mod} A$ is a cycle of length 1. Finally, a cycle $(*)$ is said to be **infinite** if $f_i \in \operatorname{rad}^{\infty}(\operatorname{mod} A)$ for some $1 \leq i \leq t$. An indecomposable A-module M is called **directing** [56] if it does not lie on any cycle in $\operatorname{mod} A$.

A module M from $\operatorname{mod} A$ is said to be **sincere** provided every simple A-module occurs as a composition factor of M. Further, an A-module M is called **faithful** if its annihilator $\operatorname{ann} M = \{a \in A;\ Ma = 0\}$ is zero. It is well-known that $M \in \operatorname{mod} A$ is faithful if and only if A_A is isomorphic to a submodule of a direct sum of copies of M. Dually, $M \in \operatorname{mod} A$ is faithful if and only if the injective cogenerator $D(A)_A$ is isomorphic to a factor module of a direct sum of copies of M. Clearly, a faithful module is always sincere.

In the remaining part of this section we shall recall some basic concepts from the Auslander-Reiten theory [11], [12]. Recall that if X is an A-module and $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$ is its minimal projective presentation, then the **transpose** $\operatorname{Tr} X$ of X is the cokernel of the map $\operatorname{Hom}_A(f, A)$ in $\operatorname{mod} A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A. We denote by τ_A and τ_A^- the **Auslander-Reiten translations** $D\operatorname{Tr}$ and $\operatorname{Tr} D$ on $\operatorname{mod} A$, respectively. A morphism $f : X \to Y$ in $\operatorname{mod} A$ is called **irreducible** if f is neither a split epimorphism nor a split monomorphism and, if there is a factorisation $f = gh$, then either g is a split monomorphism or h is a split epimorphism. It is known that a morphism $f : X \to Y$ with

X and Y from ind$A$ is irreducible if and only if f belongs to rad(mod$A$) $\setminus$ rad^2(mod$A$). Let Z be a module in mod$A$. Then a **minimal right almost split morphism** for Z is a morphism $g : E \to Z$ in mod$A$ such that (1) g is not a split epimorphism, (2) if $g = hg$ then h is an automorphism of E and (3) if $h : E' \to Z$ is not a split epimorphism, then there exists $h' : E' \to E$ such that $h = h'g$. A **minimal left almost split morphism** for Z is defined dually. An **Auslander-Reiten sequence (almost split sequence)** in mod$A$ is a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} E \xrightarrow{g} Z \longrightarrow 0$$

such that f is a minimal left almost split morphism for X and g is a minimal right almost split morphism for Z. The following theorem proved by M. Auslander and I. Reiten plays a crucial role in the representation theory of artin algebras.

1.2. THEOREM. (i) *For every non-projective indecomposable A-module Z there exists a unique (up to isomorphism) Auslander-Reiten sequence $0 \to X \to E \to Z \to 0$. Moreover, then X is indecomposable and isomorphic to $\tau_A Z$.*

(ii) *For every non-injective indecomposable A-module X there exists a unique Auslander-Reiten sequence $0 \to X \to E \to Z \to 0$. Moreover, then Z is indecomposable and isomorphic to $\tau_A^- X$.*

(iii) *If P is an indecomposable projective A-module then the inclusion* rad$P \hookrightarrow P$ *is a minimal right almost split morphism for P.*

(iv) *If I is an indecomposable injective A-module then the projection $I \to I/$soc$I$ is a minimal left almost split morphism for I.*

Given Z from ind$A$ and a minimal right almost split morphism $g : E \to Z$ for Z the irreducible morphisms ending at Z are described as follows: a morphism $u : Y \to Z$ in mod$A$ is irreducible if and only if $u = vg$ for some split monomorphism $v : Y \to E$. Dually, if X is from ind$A$ and $f : X \to E$ is a minimal left almost split morphism for X, then a morphism $h : X \to Y$ in mod$A$ is irreducible if and only if $h = fp$ for some split epimorphism $p : E \to Y$.

We end this section with some useful formulas involving the translations τ_A and τ_A^-. Namely, the translations τ_A and τ_A^- reduce the computation of Ext-groups to that of Hom-groups. For modules X, Y from mod$A$, let $\underline{\mathrm{Hom}}_A(X,Y)$ be the factor group of $\mathrm{Hom}_A(X,Y)$ modulo the subgroup of all maps $X \to Y$ which factor through a projective module. Similarly, let $\overline{\mathrm{Hom}}_A(X,Y)$ be the factor group of $\mathrm{Hom}_A(X,Y)$modulo the subgroup of all maps $X \to Y$ which factor through an injective module. Then we have the following facts.

1.3. THEOREM. *Let X and Y be A-modules. Then there are isomorphisms*

$$D\overline{\mathrm{Hom}}_A(Y, \tau_A X) \simeq \mathrm{Ext}_A^1(X,Y) \simeq D\underline{\mathrm{Hom}}_A(\tau_A^- Y, X).$$

1.4. THEOREM. *Let M be an A-module. Then*

(i) $\mathrm{pd}_A M \leq 1$ *if and only if* $\mathrm{Hom}_A(D(_AA), \tau_A M) = 0$.

(ii) $\mathrm{id}_A M \leq 1$ *if and only if* $\mathrm{Hom}_A(\tau_A^- M, A_A) = 0$.

1.5. COROLLARY. *Let X and Y be A-modules. Then*
(i) *If* $\mathrm{pd}_A X \leq 1$, *then* $\mathrm{Ext}_A^1(X,Y) \simeq D\mathrm{Hom}_A(Y,\tau_A X)$.
(ii) *If* $\mathrm{id}_A Y \leq 1$, *then* $\mathrm{Ext}_A^1(X,Y) \simeq D\mathrm{Hom}_A(\tau_A^- Y, X)$.

For proofs of the above facts we refer to [11] and [56], respectively.
We shall need also the following lemma.

1.6. LEMMA. *Let M be a faithful A-module. Then*
(i) *If* $\mathrm{Hom}_A(M,\tau_A M) = 0$, *then* $\mathrm{pd}_A M \leq 1$.
(ii) *If* $\mathrm{Hom}_A(\tau_A^- M, M) = 0$, *then* $\mathrm{id}_A M \leq 1$.

PROOF. Since M is faithful, there are an epimorphism $M^r \to D(A)_A$ and a monomorphism $A_A \to M^s$, for some $r,s \geq 1$. Then $\mathrm{Hom}_A(M,\tau_A M) = 0$ implies $\mathrm{Hom}_A(D(A),\tau_A M) = 0$, and $\mathrm{pd}_A M \leq 1$, by Theorem 1.4. Similarly, if $\mathrm{Hom}_A(\tau_A^- M, M) = 0$ we obtain that $\mathrm{Hom}_A(\tau_A^- M, A_A) = 0$, and so $\mathrm{id}_A M \leq 1$.

2. The Auslander-Reiten quiver

The aim of this section is to present some results on the shapes of components of an Auslander-Reiten quiver. Recall that the **Auslander-Reiten quiver** Γ_A of A is a valued translation quiver defined as follows: the vertices of Γ_A are the isoclasses $\{X\}$ of indecomposable modules in $\mathrm{mod}A$; $\{X\} \to \{Y\}$ is an arrow in Γ_A if there is an irreducible morphism from X to Y; the translation in Γ_A is the Auslander-Reiten translation τ_A; the valuation (d_{XY}, d'_{XY}) for an arrow $\{X\} \to \{Y\}$ in Γ_A is defined such that d_{XY} is the multiplicity of Y in the codomain of the minimal left almost split morphism for X and d'_{XY} is the multiplicity of X in the domain of the minimal right almost split morphism for Y. We shall not distinguish between an indecomposable A-module X and the corresponding vertex $\{X\}$ in Γ_A. If X is a module in Γ_A, then the τ_A**-orbit** of X is the set of all possible modules $\tau_A^m X$, $m \in \mathbb{Z}$. The module X is called τ_A**-periodic** if $X \simeq \tau_A^m X$ for some $m \geq 1$. By a **component** of Γ_A we mean a connected component in Γ_A.

It is known (see [10]) that the Auslander-Reiten quiver Γ_A describes the quotient category $\mathrm{mod}A/\mathrm{rad}^\infty(\mathrm{mod}A)$. Namely, two indecomposable A-modules X and Y belong to the same component of Γ_A if and only if there exists a sequence of indecomposable A-modules $X = Z_0, Z_1, \ldots, Z_s = Y$, $s \geq 1$, such that for each $1 \leq i \leq s$, either $\mathrm{Hom}_A(Z_{i-1}, Z_i)/\mathrm{rad}^\infty(Z_{i-1}, Z_i) \neq 0$ or $\mathrm{Hom}_A(Z_i, Z_{i-1})/\mathrm{rad}^\infty(Z_i, Z_{i-1}) \neq 0$. Therefore, the shapes of components in Γ_A are the first basic invariants of the category $\mathrm{mod}A$. Moreover, it seems to be worthwhile to study the **component quiver** Σ_A of A whose vertices are the components of Γ_A, and two components $\mathcal{C}$ and $\mathcal{D}$ are connected in Σ_A by an arrow $\mathcal{C} \to \mathcal{D}$ if $\mathrm{rad}^\infty(X,Y) \neq 0$ for some modules X from $\mathcal{C}$ and Y from $\mathcal{D}$ (see Theorem 9.4). We shall distinguish now some important types of Auslander-Reiten components.

Let $\mathcal{C}$ be a component in Γ_A. Then $\mathcal{C}$ is said to be **preprojective** if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module. Dually, $\mathcal{C}$ is said to be **preinjective** if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module. Further, $\mathcal{C}$ is called **regular** if $\mathcal{C}$ contains neither a projective module nor an injective module. Finally, $\mathcal{C}$ is called **semi-regular** if $\mathcal{C}$ does not contain both a projective module and an injective module. It is known that the Auslander-Reiten quiver Γ_A of a representation-infinite (connected) hereditary algebra H consists of a preprojective component containing all projective modules, a preinjective component containing all injective module and regular components which are either stable tubes, if H is of Euclidean type, or are of the form $\mathbb{Z}\mathbb{A}_\infty$, if H is wild (see [25], [26], [53], [54], [56], [21]). Recall that, if $\Delta = (\Delta_0, \Delta_1, d, d')$ is a locally finite valued quiver without multiple arrows and loops, then $\mathbb{Z}\Delta = ((\mathbb{Z}\Delta)_0, (\mathbb{Z}\Delta)_1, d, d', \tau)$ is the valued translation quiver defined as follows: $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$, given a valued arrow $\alpha : x \xrightarrow{(dxy,d'xy)} y$ from Δ_1, there are valued arrows $(i, \alpha) : (i, x) \xrightarrow{(dxy,d'xy)} (i, y)$, $(i, \alpha)' : (i, y) \xrightarrow{(d'xy,dxy)} (i+1, x)$, and the translation τ of $\mathbb{Z}\Delta$ is defined by $\tau(i, x) = (i-1, x)$ for $i \in \mathbb{Z}$, $x \in \Delta_0$. In particular, if Δ is the quiver $\mathbb{A}_\infty$

$$\cdots \to 2 \to 1 \to 0,$$

with trivial valuations $(1, 1)$, then $\mathbb{Z}A_\infty$ is of the form

$$\begin{array}{ccccccccccccccc}
 & & & & (i-1,0) & & & & (i,0) & & & & (i+1,0) & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\cdots & & (i-1,1) & & & & (i,1) & & & & (i+1,1) & & & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & & & (i,2) & & & & (i+1,2) & & & & (i+2,2) & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\
 & & & & & & & & \vdots & & & & & &
\end{array}$$

with $\tau(i, j) = (i-1, j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$.

For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i, j) in $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i, j)$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$, and call it the **stable tube of rank** r.

The following two theorems proved by Happel-Preiser-Ringel and Zhang, respectively, describe the shapes of regular components od Γ_A. For less combinatorial proofs of these results we refer to [42] and [43].

2.1. THEOREM [34]. *A regular component C of Γ_A contains a periodic module if and only if C is a stable tube.*

2.2. THEOREM [74]. *If C is a regular component of Γ_A containing no periodic module, then C is of the form $\mathbb{Z}\Delta$ for some valued locally finite quiver Δ without oriented cycles.*

But the following problem still remains open.

PROBLEM 1. Which translation quivers $\mathbb{Z}\Delta$ occur as regular components of Γ_A?

Some results in this direction have been obtained by Ringel [58] and by Crawley-Boevey and Ringel [24]. Ringel proved that, if Δ is a connected finite valued quiver with at least three vertices and no oriented cycle, then $\mathbb{Z}\Delta$ occurs as a regular component of some Γ_A if and only if Δ is neither a Dynkin nor a Euclidean quiver. The result of Crawley-Boevey and Ringel says that if Δ is a locally finite, connected, symmetrizable, valued quiver without oriented cycles and such that after deleting finitely many vertices and arrows we obtain a disjoint union of quivers of type $\mathbb{A}_\infty$, then $\mathbb{Z}\Delta$ occurs as a component in some Γ_A.

We pose the following problem concerning the rank of stable tubes.

PROBLEM 2. Let $\mathcal{T}$ be a regular component of Γ_A containing a periodic module. Is it true that the rank of $\mathcal{T}$ is equal or less than the length $|A|$ (over R) of A? Find the best estimate for the rank of $\mathcal{T}$.

By the corresponding results due to J.F. Carlson [18], [19] it is true in case A is a group algebra KG of a finite group G over an algebraically closed field K. It is also true for arbitrary generalized standard stable tubes (see 4.7).

Recently, S. Liu described the shapes of semi-regular components of Auslander-Reiten quivers.

2.3. THEOREM [43]. *Let $\mathcal{C}$ be a semi-regular component of Γ_A containing an oriented cycle. Then $\mathcal{C}$ is either a ray tube or a coray tube (in the sense below).*

2.4. THEOREM [43]. *Let $\mathcal{C}$ be a semi-regular component of Γ_A containing no oriented cycle. Then there is a valued locally finite quiver Δ containing no oriented cycle such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under predecessors or closed under successors.*

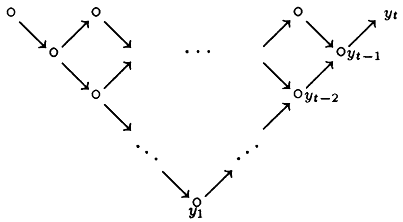
It is easy to see that if a semi-regular component $\mathcal{C}$ in Γ_A contains a periodic module then it is regular, and then the Happel-Preiser-Ringel theorem applies. One way to study the shape of an arbitrary component $\mathcal{C}$ in Γ_A is to delete the τ_A-orbits which contain projective or injective modules. We then obtain better behaved subquivers of $\mathcal{C}$: the stable part $\mathcal{C}_s$, the left stable part $\mathcal{C}_l$ and the right stable part $\mathcal{C}_r$ (see [43]). Applying then the corresponding analogues of Theorems 2.1 – 2.4 for the connected components of $\mathcal{C}_s$, $\mathcal{C}_l$, and $\mathcal{C}_r$ (see [34], [43], [74]) we may try to recover $\mathcal{C}$ from these parts. Observe that if $\mathcal{C}$ is infinite then one of the semi-stable parts $\mathcal{C}_l$ or $\mathcal{C}_r$ is not empty. In general, the knowledge of components containing both a projective module and an injective module is rather small. In the remaining part of this section we shall present some components containing many oriented cycles, and playing an important role in some recent investigations. These components can be obtained from stable tubes by a sequence of admissible operations (see [5], [6]).

Let (Γ, τ) be a connected translation quiver with trivial valuations. Recall that a path $z_0 \to z_1 \to \cdots \to z_r$ in Γ is called **sectional** if $z_{i-2} \neq \tau z_i$ for each i, $2 \leq i \leq r$. For a vertex x in Γ, called the **pivot**, we shall define three operations modifying (Γ, τ) to a new translation quiver (Γ', τ') depending on the shape of paths in Γ starting from x.

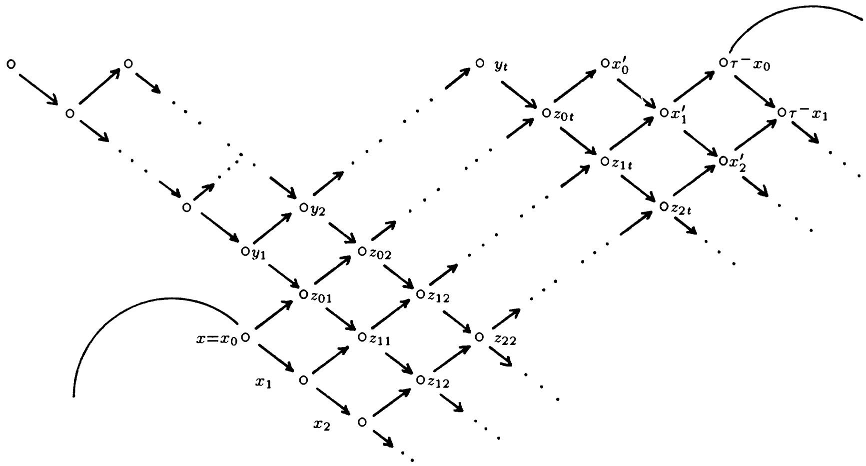
1) Suppose that Γ admits an infinite sectional path

$$x = x_0 \to x_1 \to x_2 \to \cdots$$

starting at x, and assume that every sectional path in Γ starting at x is a subpath of the above path. For $t \geq 1$, let Γ_t be the following translation quiver, isomorphic to the Auslander-Reiten quiver of the full $t \times t$ lower triangular matrix algebra,



We then let Γ' be the translation quiver having as vertices those of Γ, those of Γ_t, additional vertices z_{ij} and x'_i (where $i \geq 0$, $1 \leq j \leq t$) and having arrows as in the figure below

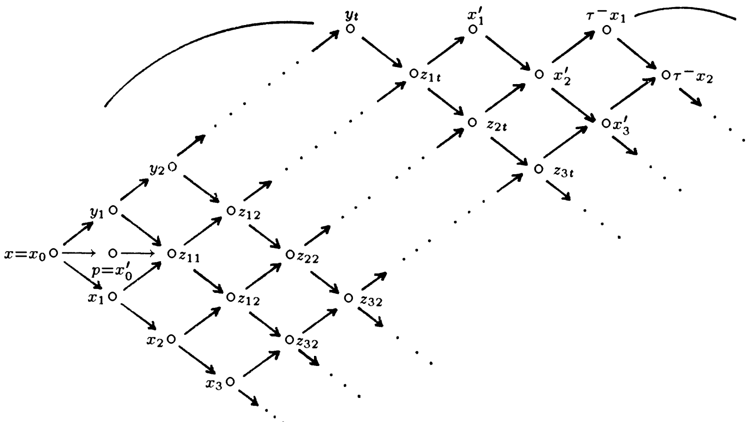


The translation τ' is defined as follows: $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geq 2, j \geq 2$, $\tau' z_{i1} = x_{i-1}$ if $i \geq 1$, $\tau' z_{0j} = y_{j-1}$ if $j \geq 2$, $p = z_{01}$ is projective, $\tau' x'_0 = y_t$, $\tau' x'_i = z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} x_i) = x'_i$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_t, respectively. If $t = 0$, the new translation quiver Γ' is obtained from Γ by inserting only the sectional path consisting of the x'_i.

2) Suppose that Γ admits two sectional paths starting at x, one infinite and the other finite with at least one arrow

$$y_t \leftarrow \cdots \leftarrow y_1 \leftarrow x = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$$

such that any sectional path starting at x is a subpath of one of these paths and x_0 is injective. Then Γ' is the translation quiver having as vertices those of Γ, additional vertices denoted by $p = x'_0$, z_{ij}, x'_i (where $i \geq 1$, $1 \leq j \leq t$), and having arrows as in the figure below

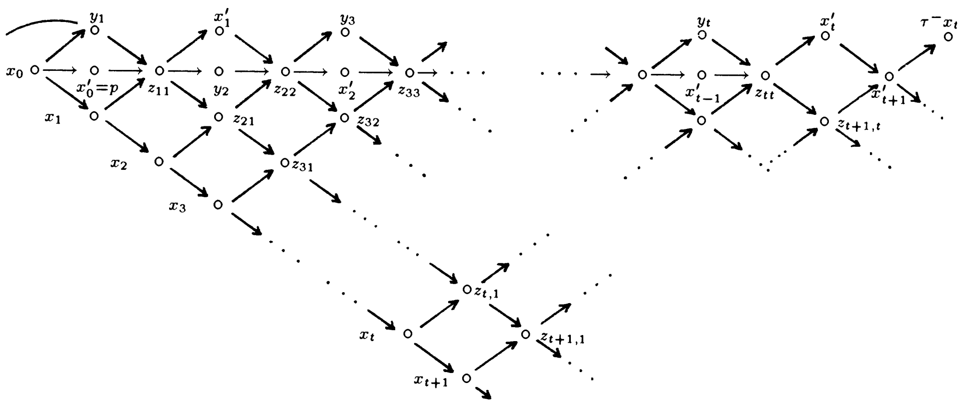


The translation τ' is defined as follows: $p = x'_0$ is projective-injective, $\tau' z_{ij} = z_{i-1,j-1}$ $(i \geq 2,\ j \geq 2)$, $\tau' z_{i1} = x_{i-1}$ $(i \geq 1)$, $\tau' z_{1j} = y_{j-1}$ $(j \geq 2)$, $\tau' x'_i = z_{i-1,t}$ $(i \geq 2)$, $\tau' x'_1 = y_t$, $\tau'(\tau^- x_i) = x'_i$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

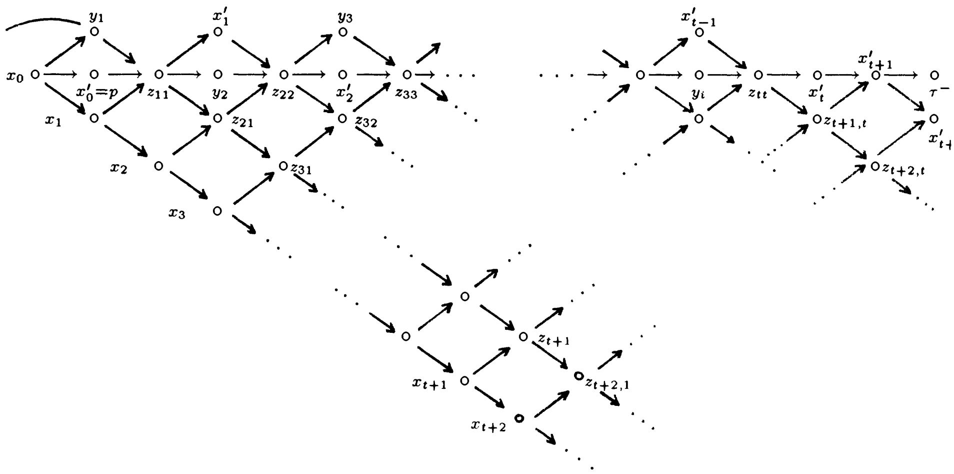
3) Suppose that Γ admits a full translation subquiver

$$\begin{array}{ccccccccccccc}
 & & y_1 & \to & y_2 & \to & \cdots & \to & y_t & & & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & & & \\
x & = & x_0 & \to & x_1 & \to & \cdots & \to & x_{t-1} & \to & x_t & \to & \cdots
\end{array}$$

with $t \geq 2$, x_{t-1} injective, the paths $y_1 \to y_2 \to \cdots \to y_t$, $x_0 \to x_1 \to x_2 \to \cdots$ sectional and such every sectional path in Γ starting at x_0 (respectively, at y_1) is a subpath of one of the paths $x_0 \to y_1$ or $x_0 \to x_1 \to x_2 \to \cdots$ (respectively, of $y_1 \to \cdots \to y_t$). Then Γ' is the translation quiver having as vertices those of Γ, additional vertices denoted by $p = x'_0$, z_{ij}, x'_i (where $i \geq 1, 1 \leq j \leq t$), and having arrows as in the figures below
- If t is odd



- If t is even



The translation τ' is defined as follows: $p = x'_0$ is projective, $\tau' z_{ij} = z_{i-1,j-1}$ $(i \geq 2, 2 \leq j \leq t)$, $\tau' z_{i1} = x_{i-1}$ $(i \geq 1)$, $\tau' x'_i = y_i$ $(1 \leq i \leq t)$, $\tau' x'_i = z_{i-1,t}$ $(i > t)$, $\tau' y_j = x'_{j-2}$ $(2 \leq j \leq t)$, $\tau'(\tau^- x_i) = x'_i$ $(i \geq t)$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. In both cases, x'_{t-1} is injective. For the remaining vertices of Γ', τ' coincides with τ.

We shall denote by 1*), 2*), 3*) the respective duals of 1), 2), 3) and these six operations will be called **admissible**. Following [5] a translation quiver is called a **coil** if there is a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ such that Γ_0 is a stable tube and, for each $0 \leq i \leq m$, Γ_{i+1} is obtained from Γ_i by an admissible operation. We refer also to [6] for an axiomatic definition of a coil and examples. Hence any stable tube is trivially a coil. A tube (in the sense of [56]) is a coil having the property that each admissible operation in the sequence defining it is of the form 1) or 1*). If we apply only operations of type 1) (respectively of type 1*)) then such a coil is called a **ray tube** (respectively, **coray tube**). Observe that a coil without projective (respectively, injective) vertices is a coray tube (respectively, a ray tube). A **quasi-tube** (in the sense of [61]) is a coil having the property that each admissible operation in the sequence defining it is of the form 1), 1*), 2) or 2*). The quasi-tubes occur frequently in the Auslander-Reiten quivers of the selfinjective algebras (see [2], [47], [60]). We note also that almost all vertices of any coil lie on oriented cycles. Moreover, we have the following proposition (see [6]).

2.5. PROPOSITION. *Let Γ be a coil. Then there exists an artin algebra A of finite global dimension such that Γ_A admits a component isomorphic to Γ.*

For an application of coils in studying the simply connected algebras of polynomial growth we refer to Section 9.

3. Tilting configurations of modules

Following [15] and [35], we shall call a module T from mod$A$ a **tilting module** if it satisfies the following conditions:

(T1) $\mathrm{pd}_A T \leq 1$
(T2) $\mathrm{Ext}^1_A(T,T) = 0$
(T3) The number of isomorphism classes of indecomposable direct summands of T is equal to n (= rank of $K_0(A)$).

It is known that under conditions (T1) and (T2), the condition (T3) is equivalent to

(T3') There exists a short exact sequence $0 \to A_A \to T' \to T'' \to 0$ with T' and T'' direct sums of direct summands of T.

Without loss of generality we may only consider the tilting modules which are direct sums of indecomposable pairwise non-isomorphic modules. A module T satisfying the conditions (T1) and (T2) is called a **partial tilting module**.

It was shown by Brenner-Butler and Happel-Ringel that there exists a close connection between the representation theories of the algebras A and $B = \mathrm{End}_A(T)$. This connection is known as a tilting theory. We refer to [1] for a good introduction to this theory. The aim of this section is to present some results from the tilting theory, including some recent ones, which we shall apply in our article.

Let us recall first the following lemma proved by Bongartz [15].

3.1. LEMMA. *Let M be a partial tilting A-module. Then there is a tilting A-module $T = M \oplus N$.*

The following proposition gives a useful criterion for a module to be a tilting module (see [51]).

3.2. PROPOSITION. *Let M be a faithful A-module having the properties:* $\mathrm{Hom}_A(M, \tau_A M) = 0$, $\mathrm{Hom}_A(\tau_A^- M, M) = 0$ *and, if* $\mathrm{Hom}_A(M, X) \neq 0$ *for some indecomposable A-module X which is not direct summand of M, then $Hom_A(\tau_A^- M, X) \neq 0$. Then M is a tilting A-module.*

PROOF. By Lemma 1.6 we get that $\mathrm{pd}_A M \leq 1$ and $\mathrm{id}_A M \leq 1$. Moreover, by Theorem 1.3, we have also that $\mathrm{Ext}^1_A(M,M) \simeq D\overline{\mathrm{Hom}}_A(M, \tau_A M) = 0$. Hence, M is a partial tilting module. We shall prove now that it is in fact a tilting module. Since M is finitely generated, there is a homomorphism $f : A_A \to M^r$, for some $r \geq 1$, such that the induced map $\mathrm{Hom}_A(M^r, M) \to \mathrm{Hom}_A(A, M)$ is an epimorphism. Then f is a monomorphism because M is faithful. Hence we have an exact sequence

$$0 \longrightarrow A_A \xrightarrow{f} M^r \longrightarrow N \longrightarrow 0.$$

We shall show that $\mathrm{pd}_A(M \oplus N) \leq 1$ and $\mathrm{Ext}^1_A(M \oplus N, M \oplus N) = 0$. Since $\mathrm{Ext}^2_A(N, -) \simeq \mathrm{Ext}^2_A(M^r, -)$, we have $\mathrm{pd}_A N \leq 1$, because $\mathrm{pd}_A M \leq 1$. Applying $\mathrm{Hom}_A(-, M)$ to the above exact sequence gives the exact sequence

$$\mathrm{Hom}_A(M^r, M) \to \mathrm{Hom}_A(A, M) \to \mathrm{Ext}_A^1(N, M) \to \mathrm{Ext}_A^1(M^r, M),$$

and hence $\mathrm{Ext}_A^1(N,M) = 0$. Further, applying $\mathrm{Hom}_A(N,-)$ we get an epimorphism $\mathrm{Ext}_A^1(N,M^r) \to \mathrm{Ext}_A^1(N,N)$, so that $\mathrm{Ext}_A^1(N,N) = 0$. Finally, applying $\mathrm{Hom}_A(M,-)$ gives the exact sequence $0 = \mathrm{Ext}_A^1(M,M^r) \to \mathrm{Ext}_A^1(M,N) \to \mathrm{Ext}_A^2(M,A) = 0$, so that $\mathrm{Ext}_A^1(M,N) = 0$. Therefore, we proved that $M \oplus N$ satisfies the conditions (T1), (T2) and (T3'), and hence is a tilting module. Suppose that there is an indecomposable direct summand X of N which is not a direct summand of M. From the exact sequence $0 \to A \to M^r \to N \to 0$ we have $\mathrm{Hom}_A(M,X) \neq 0$, and therefore it follows from our assumption that $\mathrm{Hom}_A(\tau_A^- M, X) \neq 0$. Since $\mathrm{id}_A M \leq 1$, we have, by Corollary 1.5, that $\mathrm{Ext}_A^1(X,M) \simeq D\mathrm{Hom}_A(\tau_A^- M, X) \neq 0$. This gives the desired contradiction, and hence M is a tilting A-module.

The following lemma proved in [62] will play a crucial role in our article.

3.3. LEMMA. *Let $M_1, \ldots, M_r$ be pairwise non-isomorphic indecomposable A-modules and $M = M_1 \oplus \cdots \oplus M_r$. Assume that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Then $r \leq n$.*

PROOF. Let I be the annihilator of M and $B = A/I$. Since $\mathrm{Hom}_A(M,\tau_A M) = 0$ and $\tau_B M$ is a submodule of $\tau_A M$, we get $\mathrm{Hom}_B(M,\tau_B M) = 0$. Then $\mathrm{Ext}_B^1(M,M) \simeq D\overline{\mathrm{Hom}}_B(M,\tau_B M) = 0$. Moreover, M is a faithful B-module with $\mathrm{Hom}_B(M,\tau_B M) = 0$, and so $\mathrm{pd}_B M \leq 1$, by Lemma 1.6. Therefore M is a partial tilting B-module which, by Lemma 3.1, may be extended to a tilting B-module $M \oplus X$. Then r is less than or equal to the rank at $K_0(B)$, which is clearly less than or equal to n.

We shall need also the following lemma.

3.4. LEMMA. *Let $\mathcal{C}$ be a component of Γ_A and Δ be a family of pairwise non-isomorphic modules from $\mathcal{C}$ satisfying the following conditions:*

(i) $\mathrm{Hom}_A(X,\tau_A Y) = 0$ *for all X, Y from Δ.*

(ii) $\mathrm{rad}^\infty(X,Y) = 0$ *for all X, Y from Δ.*

(iii) *Every path in $\mathcal{C}$ with source and target in Δ consists entirely of modules from Δ. Let M be the direct sum of modules from Δ and $H = \mathrm{End}_A(M)$. Then H is a hereditary artin algebra.*

PROOF. First observe that, by (i) and Lemma 3.3, Δ is finite, and hence H is an artin algebra. Let Z be a module from Δ. We shall show that the radical of the projective H-module $\mathrm{Hom}_A(M,Z)$ is also projective. Consider the exact sequence $0 \to \tau_A Z \to X \oplus Y \xrightarrow{f} Z$, where f is a minimal right almost split morphism for Z, X is a direct sum of modules from Δ and Y has no direct summands from Δ. It induces the following exact sequence of H-modules $0 \to \mathrm{Hom}_A(M,\tau_A Z) \to \mathrm{Hom}_A(M, X \oplus Y) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}_A(M,Z) \to \mathrm{Hom}_A(M,Z)/\mathrm{rad}(M,Z) \to 0$. From (i) we get $\mathrm{Hom}_A(M,\tau_A Z) = 0$, and so $\mathrm{Hom}_A(M,f)$ is a monomorphism. Then (ii) implies $\mathrm{rad}^\infty(M,Y) = 0$, because if $0 \neq g \in \mathrm{rad}^\infty(M,Y)$ then $0 \neq gf \in \mathrm{rad}^\infty(M,Z)$. We claim that $\mathrm{Hom}_A(M,Y) = 0$. Indeed, if this is not the case, then there exists a path in $\mathcal{C}$ from an indecomposable direct summand U of M to an indecomposable direct summand V of Y. Since there is an arrow in $\mathcal{C}$ from V to Z, there is in $\mathcal{C}$ a path from U to Z. But then, by (iii), V belongs to Δ, a contradiction. Observe also that $\mathrm{Hom}_A(M, Z)/\mathrm{rad}(M, Z)$ is a simple H-module. Therefore $\mathrm{Hom}_A(M, X)$ is the radical of $\mathrm{Hom}_A(M, Z)$ in $\mathrm{mod}H$, and obviously is projective. This proves that H is a hereditary algebra.

We devote the remaining part of this section to tilted algebras. In particular, we shall present a new characterization of such algebras. Recall that a **tilted algebra** B is an algebra of the form $\mathrm{End}_H(T)$, where H is a hereditary artin algebra and T is a tilting H-module. Then T determines a torsion theory $(\mathcal{F}(T), \mathcal{G}(T))$ in $\mathrm{mod}H$, where $\mathcal{F}(T) = \{X \in \mathrm{mod}H;\ \mathrm{Hom}_A(T, X) = 0\}$, $\mathcal{G}(T) = \{X \in \mathrm{mod}H;\ \mathrm{Ext}^1_H(T, X) = 0\}$, and a splitting torsion theory $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\mathrm{mod}B$, where $\mathcal{Y}(T) = \{N \in \mathrm{mod}B;\ \mathrm{Tor}^B_1(N, T) = 0\}$, $\mathcal{X}(T) = \{N \in \mathrm{mod}B; N \otimes_B T = 0\}$. By the Brenner-Butler theorem the functor $\mathrm{Hom}_H(T, -)$ induces an equivalence of $\mathcal{G}(T)$ and $\mathcal{Y}(T)$, and the functor $\mathrm{Ext}^1_A(T, -)$ induces an equivalence of $\mathcal{F}(T)$ and $\mathcal{X}(T)$. The images $\mathrm{Hom}_H(T, I)$ of all indecomposable injective H-modules I via $\mathrm{Hom}_H(T, -)$ belong to one component $\mathcal{C}_T$ of Γ_B, called the **connecting component** of Γ_B determined by T. This component connects the torsion-free part $\mathcal{Y}(T)$ of $\mathrm{mod}B$ with the torsion part $\mathcal{X}(T)$ of $\mathrm{mod}B$. Moreover, the component $\mathcal{C}_T$ admits a faithful section of the same type as the type of H. Recall that a full connected valued subquiver Δ of a component $\mathcal{C}$ in Γ_A is called a **section** if the following conditions are satisfied: (1) Δ has no oriented cycle, (2) Δ intersects each τ_A-orbit of $\mathcal{C}$ exactly once, (3) Each path in $\mathcal{C}$ with source and target from Δ lies entirely in Δ, (4) If $X \to Y$ is an arrow in $\mathcal{C}$ with X from Δ (respectively, Y from Δ), then either Y or $\tau_A Y$ is in Δ (respectively, either X or $\tau_A^- X$ is in Δ). The section Δ is called **faithful** provided the direct sum of modules from Δ is a faithful A-module.

The following theorem characterizes the semi-regular connecting components of tilted algebras.

3.5. Theorem [57]. *Let H be a hereditary artin algebra, T a tilting H-module, $B = \mathrm{End}_H(T)$, and $\mathcal{C}_T$ the connecting component of Γ_B determined by T. Then*

(i) *$\mathcal{C}_T$ does not contain projective modules if and only if T has no preinjective direct summands.*

(ii) *$\mathcal{C}_T$ does not contain injective modules if and only if T has no preprojective direct summands.*

(iii) *$\mathcal{C}_T$ is regular if and only if T is regular.*

Moreover we have the following theorem proved by Ringel.

3.6. Theorem [58]. *Let H be a connected hereditary artin algebra of type Δ. Then there exists a regular tilting H-module if and only if Δ is neither of Dynkin nor of Euclidean type, and has more than two vertices.*

In [56] Ringel characterized the tilted algebras by existence of a slice in their module categories. But one of his conditions requires knowledge of paths in the whole module category, and hence is rather difficult for checking. Recently, S. Liu [45] and the author [64] obtained independently the following characterization of tilted algebras which seems to be more suitable for practical reasons.

3.7. THEOREM. *An artin algebra A is a tilted algebra if and only if Γ_A admits a component $\mathcal{C}$ with a faithful section Δ such that $\mathrm{Hom}_A(X, \tau_A Y) = 0$ for all X, Y in Δ. Moreover, in this case $\mathcal{C}$ is a connecting component of Γ_A determined by a tilting module over a hereditary algebra.*

The proof of the above theorem is a direct consequence of Proposition 3.2, Lemma 3.3, Lemma 3.4, and the following lemma

3.8. LEMMA [45] [64]. *Let $\mathcal{C}$ be a component of Γ_A with a finite section Δ, and M be the direct sum of all modules in Δ. Then the following conditions are equivalent:*

(i) $\mathrm{Hom}_A(M, \tau_A M) = 0$.
(ii) $\mathrm{Hom}_A(\tau_A^- M, M) = 0$.
(iii) $\mathrm{rad}^\infty(M, M) = 0$.

In the representation theory of artin algebras an important role is played by a special class of tilted algebras, called concealed algebras. A **concealed algebra** is an algebra of the form $B = \mathrm{End}_H(T)$, where H is a connected, representation-infinite hereditary artin algebra and T is a preprojective (equivalently, preinjective) tilting H-module. If H is of Euclidean type then B is called **tame concealed**, otherwise B is called **wild concealed**.

For a description of Auslander-Reiten components of arbitrary tilted algebras we refer to [39], [40], [44] and [56].

4. Generalized standard components

In the representation theory of finite dimensional algebras over an algebraically closed field, an important role is played by the standard Auslander-Reiten components (see [17], [56]). Recall that, if R is an algebraically closed field K, then a component $\mathcal{C}$ of Γ_A is called **standard** if the full subcategory of $\mathrm{mod} A$ formed by modules from $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C}) = K\mathcal{C}/I_{\mathcal{C}}$ of $\mathcal{C}$, where $K\mathcal{C}$ is the path category of $\mathcal{C}$ and $I_{\mathcal{C}}$ is the ideal of $K\mathcal{C}$ generated by the elements $\sum \alpha_i \beta_i$ for all meshes

$$\begin{tikzcd}
 & Y_1 \arrow[ddr, "\beta_1"] & \\
 & Y_2 \arrow[dr, "\beta_2"] & \\
\tau_A Z \arrow[uur, "\alpha_1"] \arrow[ur, "\alpha'"] \arrow[dr, "\alpha_r"'] & \vdots & Z \\
 & Y_r \arrow[ur, "\beta_r"'] &
\end{tikzcd}$$

(where some Y_i may coincide) in $\mathcal{C}$.

We proposed in [63] a natural generalization of the notion of standard component, called generalized standard component, which is simpler and makes sense for any artin algebra. A component $\mathcal{C}$ of Γ_A is called **generalized standard** if $\mathrm{rad}^\infty(X, Y) = 0$ for all modules X and Y from $\mathcal{C}$. Observe that, if $\mathcal{C}$ is a generalized standard component in Γ_A, then $\mathrm{rad}^\infty(X, X) = 0$ for any module X in $\mathcal{C}$, or equivalently, $\mathcal{C}$ consists entirely of modules which don't lie on infinite loops. Moreover, the component quiver Σ_A of A has no loop if and only if every component in Γ_A is generalized standard.

In this section we shall present some results on the structure of generalized standard components proved in [63] and [64]. First we shall exhibit some important classes of such components.

4.1. EXAMPLES. (1) If A is representation-finite, then Γ_A is generalized standard, by Theorem 1.1. Note also that, if R is an algebraically closed field K of characteristic 2, then there are representation-finite K-algebras Λ with Γ_Λ non-standard (Riedtmann's examples [52]).

(2) Every preprojective (respectively, preinjective) component is generalized standard. Indeed, let $\mathcal{P}$ be a preprojective component of Γ_A. Then $\mathcal{P}$ consists of τ_A-orbits of projective modules and has no oriented cycle. Hence, for any path $M_0 \to M_1 \to \cdots \to M_t = Y$ in $\operatorname{mod} A$ with target Y from $\mathcal{P}$, the modules $M_0, \ldots, M_{t-1}$ also belong to $\mathcal{P}$, and moreover there is a common bound on the length of such paths. This implies that $\operatorname{rad}^\infty(X,Y) = 0$ for all X and Y from $\mathcal{P}$, and so $\mathcal{P}$ is generalized standard. The proof for preinjective components is dual.

(3) The connecting components of tilted algebras are generalized standard. Indeed, let H be a hereditary artin algebra, T a tilting H-module, $B = \operatorname{End}_H(T)$, and $\mathcal{C}_T$ the connecting component of Γ_B determined by T. Then $\mathcal{C}_T$ admits a finite section Δ such that all predecessors of Δ in $\mathcal{C}_T$ belong to the torsion-free part $\mathcal{Y}(T)$, and all proper successors of Δ belong to the torsion part $\mathcal{X}(T)$ of $\operatorname{mod} B$. Clearly, $\mathcal{C}_T$ has no oriented cycle. Suppose that $\operatorname{rad}^\infty(X,Y) \neq 0$ for some X and Y from $\mathcal{C}_T$. Then there are in $\mathcal{C}_T$ paths $X = Z_0 \to Z_1 \to \cdots \to Z_s$ and $V_r \to \cdots \to V_1 \to V_0 = Y$ such that $Z_s \in \mathcal{X}(T)$, $V_r \in \mathcal{Y}(T)$ and $\operatorname{rad}^\infty(Z_s, V_r) \neq 0$. On the other hand, $\operatorname{Hom}_B(Z_s, V_r) = 0$ because there are no non-zero maps from torsion modules to torsion-free modules. This gives the required contradiction, and so $\mathcal{C}_T$ is generalized standard.

(4) All components of tame tilted algebras and tubular algebras are generalized standard (see [39], [56]).

(5) Recently , S. Liu proved in [46] that, if A is a finite dimensional algebra over an algebraically closed field and $\mathcal{C}$ is a standard component in Γ_A, then $\mathcal{C}$ is generalized standard.

(6) Let Γ be a coil in the sense of Section 2. It was shown that in [6] that, for any algebraically closed field K, there is a finite dimensional algebra Λ such that Γ_Λ admits a standard, hence generalized standard, component $\mathcal{C}$ which is isomorphic to Γ.

(7) It is shown in [67] (see (9.4)) that a finite dimensional strongly simply connected algebra Λ over an algebraically closed field is of polynomial growth if and only if every component in Γ_Λ is (generalized) standard. If this is the case, then every component of Γ_Λ containing a cycle is a multicoil (in the sense of [6]), that is, a glueing of finitely many coils by directed parts.

The following theorem describes the shape of regular generalized standard components.

4.2. THEOREM [63]. *Let $\mathcal{C}$ be a regular generalized standard component of Γ_A. Then $\mathcal{C}$ is either a stable tube or is of the form $Z\Delta$ for some finite valued quiver Δ without oriented cycles.*

PROOF. Assume that $\mathcal{C}$ is not a stable tube. Then, by Theorems 2.1 and 2.2, $\mathcal{C} \simeq Z\Delta$ for some valued quiver Δ without oriented cycles. Fix a copy of Δ in $\mathcal{C}$. Since $\mathcal{C}$ is generalized standard without oriented cycles, we get $\mathrm{Hom}_A(X, \tau_A Y) = 0$ for all X, Y from Δ. Then, by Lemma 3.3, Δ is finite.

In particular, we get the following corollary which solves the Problem 3 in [57].

4.3. COROLLARY. *Let R be an algebraically closed field K and $\mathcal{C}$ be a regular standard component of Γ_A. Then $\mathcal{C}$ is either a stable tube or is of the form $Z\Delta$ for some finite quiver Δ without oriented cycles.*

PROOF. This follows from the above theorem and the fact that every standard component is generalized standard. But a direct copy of the proof of Theorem 4.2 gives a much simpler proof. Indeed, if $\mathcal{C}$ is not a stable tube , then $\mathcal{C} \simeq Z\Delta$ for some quiver Δ without oriented cycles, again by Theorems 2.1 and 2.2. Fix a copy of Δ in $\mathcal{C}$. Since the full subcategory of mod$A$ formed by modules from $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$, we infer that $\mathrm{Hom}_A(X, \tau_A Y) = 0$ for all modules X, Y from Δ. Then, by Lemma 3.3, Δ is finite. In fact, this also proves that $\mathcal{C}$ is generalized standard because then for any modules U and V in $\mathcal{C}$ there is a bound on the length of path in $\mathcal{C}$ from U to V, and so $\mathrm{rad}^\infty(U, V) = 0$.

Generalizing the above arguments we may prove the following fact on the shape of an arbitrary generalized standard component.

4.4. THEOREM [63]. *Let $\mathcal{C}$ be a generalized standard component of Γ_A. Then almost all τ_A-orbits in $\mathcal{C}$ are periodic.*

The following result shows that the Ringel's Problem 1 in [57] has a positive solution for generalized standard components.

4.5. THEOREM [63]. *Let $\mathcal{C}$ be a component in Γ_A such that almost all τ_A-orbits in $\mathcal{C}$ are periodic. Then, for each positive integer d, there are at most finitely many modules of length d in $\mathcal{C}$.*

The following lemma gives a characterization of generalized standard stable tubes.

4.6. LEMMA [63]. *Let $\mathcal{T}$ be a stable tube of Γ_A. Then the following conditions are equivalent:*

(i) *$\mathcal{T}$ is generalized standard.*

(ii) $\mathrm{rad}^\infty(M, M) = 0$ *for any module M in $\mathcal{T}$.*

(iii) *The mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks (the endomorphism rings are division rings).*

Moreover, we have the following consequence of Lemmas 3.3 and 4.6.

4.7. LEMMA [63]. *Let $\mathcal{T}$ be a generalized standard stable tube of rank r in Γ_A. Then $r \leq n$. Moreover, if $\mathcal{T}$ consists of modules which don't lie on infinite short cycles, then $r \leq n - 1$.*

We conjecture that the rank of any generalized standard stable tube in Γ_A is less than or equal to $n - 1$. For some partial solution of this problem we refer to (7.13).

For a component $\mathcal{C}$ of Γ_A, we denote by $\operatorname{ann}\mathcal{C}$ the **annihilator** of $\mathcal{C}$ in A, that is, the intersection of the annihilators $\operatorname{ann} M$ of all modules M in $\mathcal{C}$. If $\operatorname{ann}\mathcal{C} = 0$, $\mathcal{C}$ is said to be **faithful** . Observe that, if $\mathcal{C}$ is a component in Γ_A and $B = A/\operatorname{ann}\mathcal{C}$, then $\mathcal{C}$ is generalized standard in $\operatorname{mod}A$ if and only if $\mathcal{C}$ is so in $\operatorname{mod}B$. The following theorem describes the structure of generalized standard semi-regular components without oriented cycles.

4.8. THEOREM [63]. *Let $\mathcal{C}$ be a component of Γ_A without projective (repectively, injective) modules and $B = A/\operatorname{ann}\mathcal{C}$. Then the following conditions are equivalent*

(i) *$\mathcal{C}$ is generalized standard and contains no oriented cycle.*

(ii) *B is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary artin algebra and T is a tilting H-module without preinjective (respectively, preprojective) direct summands, and $\mathcal{C}$ is the connecting component of Γ_B determined by T.*

PROOF. The implication (ii) $\Rightarrow$ (i) follows from Theorem 3.5 and Example 4.1 (3). In order to prove (i) $\Rightarrow$ (ii), assume that $\mathcal{C}$ is generalized standard without projective modules and oriented cycles. Then, by Theorem 2.4, there is a valued quiver Δ without oriented cycles such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under predecessors. Fix a copy of Δ in $\mathcal{C}$. Since $\mathcal{C}$ is generalized standard and Δ is a section of $\mathcal{C}$, we infer that $\operatorname{Hom}_A(X, \tau_A Y) = 0$ and $\operatorname{Hom}_A(\tau_A^- Y, X) = 0$ for all X and Y from Δ. In particular, Δ is finite, by Lemma 3.3. Let M be the direct sum of modules from Δ. Since Δ is a finite section of $\mathcal{C}$, we easily infer that $\operatorname{ann}\mathcal{C} = \operatorname{ann} M$. Therefore, M is a faithful B-module with $\operatorname{Hom}_B(M, \tau_B M) = 0$, $\operatorname{Hom}_B(\tau_B^- M, M) = 0$ and such that, if $\operatorname{Hom}_B(M, X) \neq 0$ for an indecomposable B-module X which is not a direct summand of M, then $\operatorname{Hom}_B(\tau_B^- M, X) \neq 0$. Thus, by Proposition 3.2, M is a tilting B-module. Moreover, the family of modules from Δ satisfies the conditions of Lemma 3.4, and so $H = \operatorname{End}_B(M)$ is a hereditary algebra. Then, by [35], B is a tilted algebra of the form $B = \operatorname{End}_H(T)$ for the tilting H-module $T = D({}_H M)$, and $\mathcal{C}$ is the connecting component of Γ_B determined by T. Since $\mathcal{C}$ has no projective modules, the module T has no preinjective direct summands, by Theorem 3.5. The proof for $\mathcal{C}$ without injective modules is similar.

4.9. COROLLARY [63]. *Let $\mathcal{C}$ be a regular component in Γ_A without periodic modules, and $B = A/\operatorname{ann}\mathcal{C}$. Then $\mathcal{C}$ is generalized standard if and only if B is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra and T is a regular tilting H-module, and $\mathcal{C}$ is the connecting component of Γ_B determined by T.*

Note that Theorem 3.6 implies existence of many regular tilting modules, and hence generalized standard regular components without periodic modules. But the following theorem shows that a given artin algebra may have only finitely many such components.

4.10. THEOREM [63]. *All but finitely many generalized standard components in Γ_A are stable tubes.*

We pose the following problem concerning the structure of generalized standard stable tubes.

PROBLEM 3. Describe artin algebras A such that Γ_A admits a faithful generalized standard stable tube.

Some partial solutions of this problem will be presented in the next sections.

We end this section with some facts on generalized standard components without oriented cycles. Such components are described completely in [64]. In particular, we have proved there the following result.

4.11. THEOREM [64]. *There are at most two faithful generalized standard components without oriented cycles in* Γ_A, *and if two, then* A *is a concealed algebra.*

We note that in general there exist more than two sincere generalized standard components without oriented cycles (see [63, Section 4]). Finally, we may state the following characterization of tilted algebras.

4.12. THEOREM [45] [64]. *An artin algebra* A *is a tilted algebra if and only if* Γ_A *admits a faithful generalized standard component with a section.*

PROOF. Repeat the arguments from the proof of Theorem 4.8.

5. Regular components and cycles of modules

In this section we shall discuss the relationship between the shape of regular components and the directedness of indecomposable modules. We start with the following result proved in [62].

5.1. THEOREM. *Let* $\mathcal{C}$ *be a regular component in* Γ_A *with infinitely many* τ_A*-orbits. Then for every module* M *in* $\mathcal{C}$ *there is a cycle* $M = M_0 \to M_1 \to \cdots \to M_s = M$ *in* mod$A$ *with* $M_1, \ldots, M_{s-1}$ *from* $\mathcal{C}$.

PROOF. The statement is clear if $\mathcal{C}$ is a stable tube. Hence assume that $\mathcal{C} \simeq Z\Delta$ for some infinite valued quiver Δ without oriented cycles (compare Theorems 2.1 and 2.2). Let M be a module in $\mathcal{C}$. It is not hard to prove (see [62, Lemma 4]) that there is a sectional path $M = M_0 \to M_1 \to \cdots \to M_n$ in $\mathcal{C}$ (n is the rank of $K_0(A)$) such that $\mathcal{C}$ has no paths of the form $M_i = Z_s \to Z_{s-1} \to \cdots \to Z_1 = \tau_A M_j$, $0 \leq i,j \leq n$. Since $\mathcal{C}$ has no oriented cycle, the modules $M_0, M_1, \ldots, M_n$ are pairwise non-isomorphic. Then, by Lemma 3.3, there are M_p and M_q, $0 \leq p,q \leq n$, such that $\mathrm{Hom}_A(M_p, \tau_A M_q) \neq 0$. In particular, we have in mod$A$ a path $M_p \to \tau_A M_q \to M_{q-1}$, if $q \geq 1$, and a path $M_p \to \tau_A M_0 \to \tau_A M_1 \to M_0$, if $q = 0$. Let r be the least number with $1 \leq r \leq n$ and such that mod$A$ admits a path

$$(*) \qquad M_i = Y_t \xrightarrow{f_t} Y_{t-1} \xrightarrow{f_{t-1}} \cdots \longrightarrow Y_1 = \tau_A M_r \xrightarrow{f_1} Y_0 = M_{r-1}$$

with $Y_1, \ldots Y_t$ from $\mathcal{C}$ and $0 \leq i \leq n$. We shall show that $r = 1$. In this case we have a required cycle $M = M_0 \to \cdots \to M_i = Y_t \to \cdots \to Y_1 \to Y_0 = M_0 = M$. Suppose $r \geq 2$. Since $\mathcal{C}$ has no path from M_i to $\tau_A M_j$, $0 \leq i,j \leq n$, there exists u such that $f_u \in \mathrm{rad}^\infty(\mathrm{mod}A)$. Then there is a path $(*)$ with $t > n$, $f_1 \ldots, f_n$ irreducible, and $Y_1, \ldots, Y_t$ from $\mathcal{C}$. Since $Y_0, Y_1, \ldots, Y_n$ are pairwise non-isomorphic, by Lemma 3.3, we get that $\mathrm{Hom}_A(Y_l, \tau_A Y_m) \neq 0$ for some $0 \leq l, m \leq n$. Therefore we have in mod$A$ a path

$$M_i = Y_t \to \cdots \to Y_l \to \tau_A Y_m \to \cdots \tau_A Y_1 \to \tau_A Y_0 \to M_{r-2},$$

contradicting our choice of r. This finishes our proof.

As an application of the above theorem we get the following characterization of regular components containing directing modules.

5.2. THEOREM [62]. *Let $\mathcal{C}$ be a regular component of Γ_A containing a directing module. Write $A_A = P \oplus Q$, where the simple summands of $P/\mathrm{rad}P$ are exactly the composition factors of modules in $\mathcal{C}$, and denote by $t_Q(A)$ the ideal of A generated by the images of all homomorphisms from Q to A. Then*

(i) *$\mathcal{C}$ is generalized standard.*

(ii) *All modules from $\mathcal{C}$ are directing.*

(iii) $\mathrm{ann}\,\mathcal{C} = t_Q(A)$.

(iv) *$B = A/t_Q(A)$ is a tilted algebra given by a regular tilting module and $\mathcal{C}$ is a connecting component of Γ_B.*

We get then the following fact announced in the introduction, proved also independently by L. Peng and J. Xiao.

5.3. THEOREM [49] [62]. *Γ_A admits at most finitely many τ_A-orbits containing directing modules.*

PROOF. By an argument as in the proof of Theorem 5.1, we may prove that a component $\mathcal{C}$ of Γ_A admits at most finitely many τ_A-orbits containing directing modules. On the other hand, in the notation of Theorem 5.2, there are only finitely many ideals of the form $t_Q(A)$. Therefore, there are at most finitely many components in Γ_A containing directing modules. This proves our claim.

We pose the following problem.

PROBLEM 4. Let A be an artin algebra such that every τ_A-orbit in Γ_A contains a directing module. Is it true that then A is a representation-finite and Γ_A has no oriented cycles?

In Section 4 we characterized generalized standard stable tubes as those stable tubes for which their modules don't lie on infinite loops. It would be interesting to solve the following problems.

PROBLEM 5. Let $\mathcal{C}$ be a regular component in Γ_A containing no periodic module. Assume that $\mathrm{rad}^{\infty}(M, M) = 0$ (equivalently, $\mathrm{rad}(M, M) = 0$) for any module M in $\mathcal{C}$. Is it true that $\mathcal{C}$ is generalized standard?

PROBLEM 6. Let $\mathcal{C}$ be a regular component in Γ_A containing no periodic module. Assume that $\mathcal{C}$ admits a module M which doesn't lie on a short cycle in mod$A$. Is it true that $\mathcal{C}$ is generalized standard?

We end this section with the following theorem proved by the author and M. Wenderlich in [71], showing that frequently projective and injective modules lie on cycles.

5.4. THEOREM. *Assume that Γ_A admits a regular component which is neither a stable tube nor of the form $\mathbb{Z}A_\infty$. Then there are a non-directing indecomposable projective module and a non-directing indecomposable injective module.*

6. Selfinjective algebras and generalized standard components

In this section we shall present some joint results proved with K. Yamagata on the structure of selfinjective algebras whose Auslander-Reiten quiver admits non-periodic generalized standard subquivers. For the sake of simplicity we assume here that R is an algebraically closed field K. We refer the reader to [72] and [73] for the corresponding results on arbitrary selfinjective artin algebras. Recall that A is called **selfinjective** if A_A is injective. Moreover, A is called **symmetric** if $A \simeq D(A)$ as $A-A$-bimodules.

For an algebra B, we denote by $\hat{B}$ its (locally finite dimensional) **repetitive algebra**

$$\hat{B} = \begin{bmatrix} \ddots & \ddots & & & 0 \\ & B_1 & M_1 & & \\ & & B_0 & M_0 & \\ & & & B_{-1} & \ddots \\ 0 & & & & \ddots \end{bmatrix}$$

where $B_i = B$ and $M_i = {}_B(DB)_B$ for all $i \in \mathbb{Z}$, all the remaining entries are zero, addition is the usual addition of matrices, and multiplication is induced from the canonical maps $B \otimes_B DB \to DB$, $DB \otimes_B B \to DB$, and the zero map $DB \otimes_B DB \to 0$. Denote by $\nu_{\hat{B}} : \hat{B} \to \hat{B}$ the **Nakayama automorphism** of $\hat{B}$ such that $\nu_{\hat{B}}(B_i) = B_{i+1}$ and $\nu_{\hat{B}}(M_i) = M_{i+1}$ for all $i \in \mathbb{Z}$. Recall that if B is of finite global dimension then the stable module category $\underline{\mathrm{mod}}\,\hat{B}$ is equivalent, as a triangulated category, with the derived category $D^b(\mathrm{mod}B)$ of bounded complexes over $\mathrm{mod}B$ [32]. This equivalence has been used in [4] to obtain a characterization of algebras B for which every cycle in the derived category $D^b(\mathrm{mod}B)$ is finite.

If B is a tilted algebra of type Δ which is not Dynkin quiver, then the Auslander-Reiten quiver of $\Gamma_{\hat{B}}$ is of the form

$$\Gamma_{\hat{B}} = \bigvee_{p \in \mathbb{Z}} (\mathcal{C}_p \vee \mathcal{X}_p),$$

where $\mathcal{C}_p$ are components whose stable parts are of the form $\mathbb{Z}\Delta$, and $\mathcal{X}_p$ are unions of components whose stable parts are either tubes (if Δ is Euclidean) or of the form $\mathbb{Z}A_\infty$ (if Δ is wild). Moreover, $\nu_{\hat{B}}(\mathcal{C}_p) = \mathcal{C}_{p+2}$ and $\nu_{\hat{B}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$, for any $p \in \mathbb{Z}$. A K-linear automorphism $\varphi : \hat{B} \to \hat{B}$ is called **positive** (respectively, **strictly positive**) if, for the induced automorphism $\varphi : \mathrm{mod}\hat{B} \to \mathrm{mod}\hat{B}$, we have $\varphi(\mathcal{C}_0) = \mathcal{C}_i$ for some $i \geq 0$ (respectively, $i > 0$). Clearly, the Nakayama automorphism $\nu_{\hat{B}}$ is strictly positive. Every strictly positive K-linear automorphism $g : \hat{B} \to \hat{B}$ generates an infinite cyclic group $G = (g)$, the quotient $\hat{B}/G$ is a finite dimensional selfinjective K-algebra, and there is a

Galois covering $F^B : \widehat{B} \to \widehat{B}/G$. Moreover, since $\widehat{B}$ is locally support-finite (in the sense of [27]), the push-down functor $F_\lambda^B : \mathrm{mod}\widehat{B} \to \mathrm{mod}\widehat{B}/G$, induced by F^B, is dense. Hence the Auslander-Reiten quiver $\Gamma_{\widehat{B}/G}$ of $\widehat{B}/G$ is isomorphic to the quotient $\Gamma_{\widehat{B}}/G$ of $\Gamma_{\widehat{B}}$ by G (see [30]). Observe that if $G = (\nu_{\widehat{B}})$ then $\widehat{B}/G$ is the trivial extension $B \ltimes DB$ of B by DB.

A subquiver Γ of Γ_A is said to be **generalized standard** if $\mathrm{rad}^\infty(X,Y) = 0$ for all X and Y from Γ. Moreover, if Γ has no periodic module, it is said to be **non-periodic**.

We shall show now that, for a selfinjective algebra A, existence of a non-periodic generalized standard subquiver in Γ_A, determines completely the structure of A.

6.1. THEOREM [73]. *The following conditions are equivalent:*

(i) Γ_A *admits a non-periodic generalized standard right stable (respectively, left stable) full translation subquiver which is closed under successors (respectively, predecessors).*

(ii) $A \simeq \widehat{B}/(\nu_{\widehat{B}}\varphi)$, *where B is a tilted algebra not of Dynkin type and φ is a positive automorphism of $\widehat{B}$.*

(iii) $A \simeq \widehat{B}/(\nu_{\widehat{B}}\varphi)$, *where B is a tilted algebra of the form* $\mathrm{End}_H(T)$, *for some hereditary algebra H and a tilting H-module T without preprojective (respectively, preinjective) direct summand, and φ is a positive automorphism of $\widehat{B}$.*

We shall indicate the main steps in the proof of (i) $\Rightarrow$ (iii). Let $\mathcal{D}$ be a non-periodic generalized standard right stable subquiver of Γ_A which is closed under successors. Then $\mathcal{D} \simeq (-\mathbb{N})\Delta$ for some quiver Δ without oriented cycles which is moreover finite, by Lemma 3.3. We may assume that Δ does not contain socle factors of projective modules. Let $I = \mathrm{ann}\,\mathcal{D}$, $B = A/I$, and e be an idempotent of A such that the simple direct summands of $eA/e(\mathrm{rad}A)$ are exactly the simple composition factors of modules in $\mathcal{D}$. Then $B = eAe/eIe$. Let $e = e_1 + \cdots + e_r$ be a decomposition of e into a sum of primitive orthogonal idempotents. We prove the following facts:

(a) B is a tilted algebra of the form $\mathrm{End}_H(T)$, where H is a hereditary algebra, T is a tilting H-module without preprojective direct summands, and $\mathcal{D}$ is the torsion part of the connecting component of Γ_B determined by T.

(b) $IeI = 0$, and hence Ie is a right B-module and eI is a left B-module.

(c) The Hochschild cohomology $H^2(B, eIe)$ vanishes, and hence the extension

$$0 \to eIe \to eAe \to B \to 0$$

of B by the 2-nilpotent ideal eIe splits.

(d) The set $\{\nu_A^m(e_i);\ m \in \mathbb{Z}, 1 \le i \le r\}$, where ν_A is the Nakayama automorphism of A, contains a complete set of primitive orthogonal idempotents of A.

Next we define a functor $F : \widehat{B} \to A$ such that $F(e_{m,i}\widehat{B}) = \nu_A^m(e_i)A$, where $e_{m,i}$ are the standard idempotents in $\widehat{B}$ corresponding to the idempotents e_i in B, $m \in \mathbb{Z}$, $1 \le i \le r$, and show that this is a required Galois covering $F : \widehat{B} \to \widehat{B}/(\nu_{\widehat{B}}\varphi) = A$, for some positive automorphism $\varphi : \widehat{B} \to \widehat{B}$.

We get the following consequences of the above theorem.

6.2. COROLLARY [73]. Γ_A *admits a non-periodic non-regular generalized standard component if and only if $A \simeq \widehat{B}/(\nu_{\widehat{B}}\varphi)$, where B is a tilted algebra of the form* $\mathrm{End}_H(T)$, *for some hereditary algebra H and a non-regular tilting H-module T without preprojective (respectively, preinjective) direct summands and φ is a strictly positive automorphism of $\widehat{B}$.*

6.3. COROLLARY [73]. Γ_A *admits a non-periodic regular generalized standard component if and only if $A \simeq \widehat{B}/(\nu_{\widehat{B}}\varphi)$, where B is a tilted algebra given by a regular tilting module and φ is a positive automorphism of $\widehat{B}$.*

6.4. COROLLARY [73]. *Let A be a symmetric algebra. Then Γ_A admits a generalized standard non-periodic component if and only if $A \simeq B \ltimes DB$, where B is a tilted algebra given by a regular tilting module.*

6.5. COROLLARY [73]. *The following two conditions are equivalent:*

(i) *There is a non-periodic non-projective indecomposable A-module and every component in Γ_A is generalized standard.*

(ii) *$A \simeq \widehat{B}/(\nu_{\widehat{B}}\varphi)$, where B is a representation-infinite tilted algebra of Euclidean type and φ is a strictly positive automorphism of $\widehat{B}$.*

Moreover, we have the following interesting fact.

6.6. PROPOSITION [73]. *Let A be a symmetric algebra, and assume that Γ_A admits a generalized standard stable tube. Then the Cartan matrix of A is singular.*

It is well-known that the Cartan matrices of blocks of group algebras of finite groups are non-singular. Hence, we get the following consequence of the above results.

6.7. COROLLARY. *Let Λ be a block of a group algebra. Then Γ_Λ admits a generalized standard component if and only if Λ is representation-finite.*

On the other hand, it is easy to see that the trivial extention $C \ltimes DC$ of an arbitrary canonical algebra C (in the sense of [56]) is a symmetric algebra and its Auslander-Reiten quiver admits at least two one-parameter families of generalized standard stable tubes. We pose the following related problem.

PROBLEM 7. Describe selfinjective algebras A for which Γ_A admits a generalized standard stable tube.

It would be also interesting to describe selfinjective algebras whose Auslander-Reiten quivers admit components with finitely many orbits. We expect the following solution of this question.

PROBLEM 8. Let A be a selfinjective, basic, connected algebra. It is true that Γ_A admits a component with finitely many τ_A-orbits if and only if there is a tilted algebra B and a (not necessarily Galois) covering $\widehat{B} \to A$?

7. Composition factors of indecomposable modules

The aim this section is to present some recent results on the composition factors of indecomposable modules. For a module M in mod$A$ we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same composition factors including the multiplicities. We may ask when two indecomposable modules M and N have the same composition factors. In particular, it would be interesting to find sufficient conditions for an indecomposable module M to be uniquely determined (up to isomorphism) by its composition factors. This is known to be the case if M is directing (see [56]). Modules determined by their composition factors have been investigated by Auslander and Reiten in [13], where they proved the following important formulas.

7.1. THEOREM [13]. *Let X and Y be arbitrary modules in* mod$A$. *Suppose that $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation of X and $0 \to X \to I_0 \to I_1$ is a minimal injective copresentation of X. Then*

(i) $|\mathrm{Hom}_A(X,Y)| - |\mathrm{Hom}_A(Y,\tau_A X)| = |\mathrm{Hom}_A(P_0,Y)| - |\mathrm{Hom}_A(P_1,Y)|$.

(ii) $|\mathrm{Hom}_A(Y,X)| - |\mathrm{Hom}_A(\tau_A^- X,Y)| = |\mathrm{Hom}_A(Y,I_0)| - |\mathrm{Hom}_A(Y,I_1)|$.

Here, for an R-module V, we denote by $|V|$ its length over R.

We shall prove now the following theorem showing a close relation between the considered above problems and the existence of short cycles in mod$A$.

7.2. THEOREM [50]. *Let M and N be two non-isomorphic indecomposable A-modules such that $[M] = [N]$. Then M and N lie on short cycles.*

PROOF. Suppose that M does not lie on short cycle. Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Since $[M] = [N]$, we have the following equalities $|\mathrm{Hom}_A(P_0,M)| = |\mathrm{Hom}_A(P_0,N)|$ and $|\mathrm{Hom}_A(P_1,M)| = |\mathrm{Hom}_A(P_1,N)|$. Then, by Theorem 7.1, we get $|\mathrm{Hom}_A(M,M)| - |\mathrm{Hom}_A(M,\tau_A M)| = |\mathrm{Hom}_A(M,N)| - |\mathrm{Hom}_A(N,\tau_A M)|$. Observe that $\mathrm{Hom}_A(M,\tau_A M) = 0$, because otherwise there is a short cycle $M \to U \to M$ for some indecomposable direct summand U of the middle term of an Auslander-Reiten sequence $0 \to \tau_A M \to E \to M \to 0$. Then $|\mathrm{Hom}_A(M,M)| > 0$ implies $\mathrm{Hom}_A(M,N) \neq 0$, and so we have a non-zero non-isomorphism $M \to N$. Dually, since DM does not lie on short cycle in modA^{op}, and DM and DN have the same composition factors, we get $\mathrm{Hom}_{A^{\mathrm{op}}}(DM,DN) \neq 0$, and hence $\mathrm{Hom}_A(N,M) \neq 0$. Therefore, there is a non-zero non-isomorphism $N \to M$, and hence a short cycle $M \to N \to M$, a contradiction.

We have the following obvious consequence.

7.3. COROLLARY. *Let M be an indecomposable A-module which does not lie on short cycle in modA. Then M is uniquely determined by its composition factors.*

An important role in the representation theory is played by periodic modules. It was shown in [20] that, if R is an algebraically closed field and A is of tame representation type, then, in each dimension d, almost all indecomposable modules lie in stable tubes, and hence are periodic. The knowledge of periodic modules over arbitrary artin algebras is rather poor, and it is an important task to understand the structure of such modules. In particular, it would be interesting to develop criteria for when two indecomposable periodic modules are isomorphic, at least for some special types of algebras. Unfortunately, the above corollary cannot be applied to periodic modules lying in stable tubes, because such modules always lie on short cycles. But often the short cycles containing modules from stable tubes are finite. We shall present now some results on the composition factors of periodic modules lying in generalized standard stable tubes of Γ_A, proved by the author in [65].

Recall that if $\mathcal{T}$ is a stable tube in Γ_A then the τ_A-orbit in $\mathcal{T}$ formed by the modules having exactly one predecessor is called the **mouth** of $\mathcal{T}$. Then, for every module M in $\mathcal{T}$, there exists a unique sectional path $X_1 \to X_2 \to \cdots \to X_r = M$ in $\mathcal{T}$ with X_1 lying on the mouth of $\mathcal{T}$, and r is called the **quasi-length** of M which we shall denote by $\mathrm{ql}\,(M)$. Hence, $\mathrm{ql}\,(M)$ measures the distance of M from the mouth of $\mathcal{T}$. Recall also (Lemma 4.6) that the tube $\mathcal{T}$ is generalized standard if and only if $\mathcal{T}$ consists entirely of modules which don't lie on infinite loops. Hence, all stable tubes consisting of modules which don't lie on infinite short cycles are generalized standard.

We have the following useful simple lemma.

7.4. Lemma. *Let $\mathcal{T}$ be a generalized standard stable tube in Γ_A, and M, N be two modules from $\mathcal{T}$. Then*
$$|\mathrm{Hom}_A(N, \tau_A M)| = |\mathrm{Ext}^1_A(M, N)| = |\mathrm{Hom}_A(\tau_A^- N, M)|.$$

Combining this with Theorem 7.1 we get the following formulas.

7.5. Proposition. *Let M and N be indecomposable A-modules such that $[M] = [N]$, and assume that M lies in a generalized standard stable tube of Γ_A. Then*

(i) $|\mathrm{Hom}_A(M, M)| - |\mathrm{Ext}^1_A(M, M)| = |\mathrm{Hom}_A(M, N) - |\mathrm{Hom}_A(N, \tau_A M)|$.

(ii) $|\mathrm{Hom}_A(M, M)| - |\mathrm{Ext}^1_A(M, M)| = |\mathrm{Hom}_A(N, M)| - |\mathrm{Hom}_A(\tau_A^- M, N)|$.

The following technical result is crucial in our investigations.

7.6. Proposition. *Let $\mathcal{T}$ be a generalized standard stable tube of rank r in Γ_A and M be an indecomposable module from $\mathcal{T}$. Then*

(i) $|\mathrm{Hom}_A(M, M)| = (k + 1)|\mathrm{Hom}_A(M, M)/\mathrm{rad}(M, M)|$, *where $k \geq 0$ is such that $kr < \mathrm{ql}\,(M) \leq (k + 1)r$.*

(ii) $|\mathrm{Ext}^1_A(M, M)| = k|\mathrm{Hom}_A(M, M)/\mathrm{rad}(M, M)|$, *where $k \geq 0$ is such that $kr \leq \mathrm{ql}\,(M) < (k + 1)r$.*

Then we get the following corollary.

7.7. Corollary. *Let $\mathcal{T}$ be a generalized standard stable tube of rank r in Γ_A and M be a module from $\mathcal{T}$. Then*

(i) $|\mathrm{Hom}_A(M, M)| \geq |\mathrm{Ext}^1_A(M, M)|$.

(ii) $|\mathrm{Hom}_A(M, M)| = |\mathrm{Ext}^1_A(M, M)|$ *if and only if r divides* $\mathrm{ql}\,(M)$.

Applying the above results we proved the following facts on the composition factors of modules lying in generalized standard stable tubes.

7.8. THEOREM. *Let $\mathcal{T}$ be a generalized standard stable tube of rank $r > 1$ in Γ_A. Assume that M and N are non-isomorphic indecomposable modules from $\mathcal{T}$. Then $[M] = [N]$ if and only if* $\mathrm{ql}\,(M) = \mathrm{ql}\,(N) = cr$ *for some $c \geq 1$.*

7.9. COROLLARY. *Let $\mathcal{T}$ be a stable tube of rank $r > 1$ in Γ_A consisting of modules which don't lie on infinite short cycles, and M be a module from $\mathcal{T}$. Then M is uniquely determined by the composition factors if and only if r does not divide* $\mathrm{ql}\,(M)$.

7.10. COROLLARY. *Let $\mathcal{T}$ and $\mathcal{T}'$ be different stable tubes in Γ_A consisting of modules which don't lie on infinite short cycles. Let r be the rank of $\mathcal{T}$ and r' be the rank of $\mathcal{T}'$. Assume that $[M] = [N]$ for M from $\mathcal{T}$ and M' from $\mathcal{T}'$. Then r divides* $\mathrm{ql}\,(M)$, *r' divides* $\mathrm{ql}\,(M')$, *and the tubes $\mathcal{T}$ and $\mathcal{T}'$ are orthogonal.*

7.11. COROLLARY. *Let I be a linearly ordered set and $\mathcal{T}_i$, $i \in I$, be a family of pairwise different stable tubes in Γ_A consisting of modules which don't lie on infinite short cycles. Assume that* $\mathrm{Hom}_A(\mathcal{T}_i, \mathcal{T}_j) \neq 0$ *if and only if $i \leq j$. Then I is countable.*

PROOF. For each $i \in I$, choose a module $X_i \in \mathcal{T}_i$. Consider the function $f : I \to K_0(A)$ defined by $f(i) = [X_i]$. Then, by the above corollary and our assumption, f is an injection. Since $K_0(A) \simeq \mathbb{Z}^n$, we get that I is countable.

7.12. PROPOSITION. *Let $\mathcal{T} = (\mathcal{T}_\alpha)_{\alpha\in\Omega}$ and $\mathcal{T}' = (\mathcal{T}'_\lambda)_{\lambda\in\Sigma}$ be two disjoint families of stable tubes in Γ_A consisting of modules which don't lie on infinite short cycles. Assume that the following conditions are satisfied*

(a) *There are $M_\alpha \in \mathcal{T}_\alpha$, $\alpha \in \Omega$ such that $[M_\alpha] = [M_\beta]$ for all $\alpha, \beta \in \Omega$.*

(b) *There are $M'_\lambda \in \mathcal{T}'_\lambda$, $\lambda \in \Sigma$, such that $[M'_\lambda] = [M'_\rho]$ for all $\lambda, \rho \in \Sigma$.*

(c) $\mathrm{Hom}_A(\mathcal{T}_\beta, \mathcal{T}'_\rho) \neq 0$ *for some $\beta \in \Omega$ and $\rho \in \Sigma$.*

Then $\mathrm{Hom}_A(\mathcal{T}_\alpha, \mathcal{T}'_\lambda) \neq 0$ *for all $\alpha \in \Omega$ and $\lambda \in \Sigma$.*

Applying the above results, Lemma 3.3, and tilting theory we get the following theorem (see [65]).

7.13. THEOREM. *Let $\mathcal{T}_i$, $i \in I$, be a family of pairwise different stable tubes in Γ_A consisting of modules which don't lie on infinite short cycles. Assume that each $\mathcal{T}_i$ contains a module M_i such that $[M_i] = [M_j]$ for all $i, j \in I$. Denote by r_i the rank of $\mathcal{T}_i$. Then*

$$\sum_{i\in I}(r_i - 1) \leq n - 2.$$

It is easy to see that the above bound is the best possible.

It is well-known that, if A is a tame tilted algebra or a tubular algebra, then any cycle of indecomposable A-modules lies entirely in a tube of Γ_A, and hence is finite (see [56], [39]). The results presented above give a general explanation of the known properties of stable tubes over these important classes of tame algebras.

Generalized standard stable tubes occur also naturally in the module categories of algebras of infinite global dimension. Let R be an algebraically closed field K and Λ be the path algebra $K\Delta$ of the quiver

$$\Delta: \quad \begin{tikzcd} 1 \arrow[drr] & 2 \arrow[dr] & & 3 \arrow[dl] & 4 \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

of type $\widetilde{\mathbf{D}}_4$, and A be the trivial extension $\Lambda \ltimes D\Lambda$. Then Γ_A consists of the two non-periodic components $\mathcal{C}$ and $\mathcal{C}'$, whose stable parts $\mathcal{C}_s$ and $\mathcal{C}'_s$ are of the form $Z\Delta$, and two families $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ and $\mathcal{T}' = (\mathcal{T}'_\rho)_{\rho \in \mathbf{P}_1(k)}$ of (generalized) standard stable tubes, where three tubes of $\mathcal{T}$, respectively of $\mathcal{T}'$, have rank 2 and the remaining ones have rank 1. Then for any $\lambda \in \mathbf{P}_1(k)$ and indecomposable module M_λ in $\mathcal{T}_\lambda$ there are $\rho \in \mathbf{P}_1(k)$ and M'_ρ in $\mathcal{T}'_\rho$ such that $[M_\lambda] = [M'_\rho]$, and there is an infinite short cycle $M_\lambda \to M'_\rho \to M_\lambda$. This shows that our assumptions on non-existence of infinite short cycles passing through modules lying in stable tubes are essential for the validity of Corollaries 7.9 and 7.10. Moreover, the following example due to J. F. Carlson [18] shows that Theorems 7.8 and 7.13 are not true for arbitrary stable tubes. Let K be an algebraically closed field of characteristic $p > 0$, and $\mathbf{F}_p$ be its simple subfield. Let G be the following subgroup of $\mathrm{Gl}(3, \mathbf{F}_p)$

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; \quad a, b, c \in \mathbf{F}_p \right\}$$

Then the group algebra $B = KG$ is local, symmetric with non-singular Cartan matrix, and hence Γ_B does not admit generalized stable tubes, by Proposition 6.6. On the other hand, by [18], for any $\lambda \in K \setminus \mathbf{F}_p$ there exists a stable tube $\mathcal{T}_\lambda$ of rank p in Γ_B such that one of the mouth modules in $\mathcal{T}_\lambda$ has dimension p, and the remaining $p-1$ mouth modules in $\mathcal{T}_\lambda$ have dimension $p^3 + p$.

8. Module categories without infinite loops

We are interested now in artin algebras A such that $\mathrm{rad}^\infty(M, M) = 0$ for any M from $\mathrm{ind}A$. We shall call them briefly **loop-finite** algebras. Observe that if R is an algebraically closed field K then any loop-finite finite dimensional K-algebra is of tame representation type, because the module categories of wild algebras have many infinite loops. The class of loop-finite artin algebras contains all representation-finite algebras, all algebras for which every Auslander-Reiten component is generalized standard, and all **cycle-finite** algebras (every cycle in the module category is finite). In particular, all tame tilted algebras, tubular algebras and multicoil algebras (in the sense of [5] [6]) are loop-finite. We expect that the loop-finite algebras share many properties of the best understood examples of algebras of polynomial growth (see [61]). We believe that the following problems have positive answers.

PROBLEM 9. Let $\mathcal{C}$ be a component in Γ_A consisting of modules which don't lie on infinite loops. Is it true that all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic?

PROBLEM 10. Let A be a loop-finite artin algebra. Is it true that, for each positive integer d, all but finitely many isomorphism classes of indecomposable modules of length d lie in stable tubes of rank 1?

Observe that, by Theorem 4.5, the validity of Problem 9 implies that, if $\mathcal{C}$ is a component in Γ_A formed by modules which don't lie on infinite loops, then $\mathcal{C}$ contains at most finitely many modules of any given length (see Ringel's Problem 1 in [57]). Applying results of Section 7 and [43], we were able to prove in [65] the following related results.

8.1. THEOREM. *Let $\mathcal{C}$ be a component in Γ_A consisting of modules which don't lie on infinite short cycles. Then, for each positive integer d, there are at most finitely many modules of length d in $\mathcal{C}$. Moreover, if $\mathcal{C}$ has no oriented cycle, then every module in $\mathcal{C}$ is uniquely determined by the composition factors.*

8.2. THEOREM. *Assume that every short cycle in* mod$A$ *is finite. Then, for each positive d, all but finitely many isomorphism classes of indecomposable A-modules of length d lie in stable tubes of rank 1.*

PROOF. Let d be a positive integer. Suppose that there are infinitely many isomorphism classes of indecomposable A-modules of length d which don't lie in stable tubes of rank 1. Then there is a sequence $X_1, X_2, X_3, \ldots$ of pairwise non-isomorphic indecomposable A-modules such that $[X_i] = [X_j]$ and $\tau_A X_s \not\simeq X_s$ for all $i, j, s \geq 1$. From Theorems 2.1, 2.2 and 8.1, we infer that there are stable tubes $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$ of ranks > 1 in Γ_A and modules $M_i \in \mathcal{T}_i$, $i \geq 1$, such that $[M_i] = [M_j]$ for all $i, j \geq 1$. Our assumption on A implies that the tubes $\mathcal{T}_i$, $i \geq 1$, consist of modules which don't lie on infinite short cycles. But then we have a contradiction with Theorem 7.13.

We get the following direct consequence of the above theorem.

8.3. COROLLARY. *Assume that every short cycle in* mod$A$ *is finite. Then all but countably many components of Γ_A are stable tubes of rank one.*

Our next aim is to describe the loop-finite artin algebras which are minimal representation-infinite. This will lead to a criterion for a loop-finite algebra to be representation-infinite. Moreover, we show that the second Brauer-Thrall conjecture is valid for all loop-finite artin algebras over the rings R with $R/\text{rad}R$ infinite. A representation-infinite artin algebra A is said to be **minimal representation-infinite** if, for every non-zero two-sided ideal I in A, the algebra A/I is representation-finite. It is well known (see [21], [23], [25], [26] and [56]) that if B is a tame concealed artin algebra then: (1) B is minimal representation-infinite, (2) every cycle in mod$A$ is finite, (3) almost all indecomposable-modules have projective dimension one, and (4) almost all indecomposable B-modules have injective dimension one. If R is an algebraically closed field then the tame concealed algebras are completely described in [16], [37].

We have the following characterization of minimal representation-infinite loop-finite artin algebras.

8.4. THEOREM [70]. *Let A be a connected representation-infinite artin algebra. Then the following conditions are equivalent:*

(i) *A is a tame concealed algebra.*

(ii) *A is loop-finite and minimal representation-infinite.*

(iii) mod$A$ *has no infinite short cycle and, for every non-zero idempotent e of A,*

A/AeA is representation-finite.

(iv) *A is not wild concealed, almost all indecomposable A-modules have projective dimension one, and , for every non-zero idempotent e of A, A/AeA is representation-finite.*

(v) *A is not wild concealed, almost all indecomposable A-modules have injective dimension one, and, for every non-zero idempotent e of A, A/AeA is representation-finite.*

If R is an algebraically closed field and A has at least three non-isomorphic simple A-modules, then by [36], we may drop in (v) and (vi) the assumption that A is not wild concealed.

We have the following direct consequences of the above theorem.

8.5. COROLLARY. *Assume that every short cycle in* mod$A$ *is finite. Then A is representation-infinite if and only if there is an idempotent e in A such that A/AeA is tame concealed.*

8.6. COROLLARY. *Assume that A is loop-finite. Then the following conditions are equivalent*

(i) *A is representation-infinite.*

(ii) *There is an ideal I in A such that A/I is tame concealed.*

(iii) *There is a generic A-module.*

(iv) *Γ_A admits infinitely many stable tubes of rank* 1.

If in addition, the simple A-modules are infinite, then the above conditions are equivalent to

(v) *A is of strongly unbounded representation type.*

Recall that following [22], [23] an indecomposable A-module M is called **generic** if it is of infinite length over A but of finite length over $\mathrm{End}_A(M)$. It was shown in [22] that, if R is an algebraically closed field, then A is representation-infinite if and only if there is a generic A-module. The above corollary shows that such equivalence is also true for arbitrary loop-finite artin algebras. Recall also that A is of **strongly unbounded representation type** if there is an infinite sequence of positive integers $d_1 < d_2 < d_3 < \ldots$ such that, for each $i \geq 1$, there are infinitely many pairwise non-isomorphic indecomposable A-modules of length d_i. The remarkable second Brauer-Thrall conjecture says that, if $R/\mathrm{rad}R$ is an infinite field, then A is representation-infinite if and only if A is of strongly unbounded representation type. This is known to be the case if R is an algebraically closed field (for a proof we refer to [29]). The above corollary shows that the second Brauer-Thrall conjecture is valid for loop-finite artin algebras.

It was shown in [56] that if R is an algebraically closed field and every module in ind$A$ is directing then A is representation-finite. Recently, Happel and Liu proved in [33] that if A is an artin algebra such that mod$A$ has no short cycle then A is representation-finite. It is known that a tame concealed algebra admits indecomposable finitely generated modules whose endomorphism rings are not division algebras, because its Auslander-Reiten quiver admits a stable tube (see [21], [25], [26], [56]). Hence we get from Corollary 8.6 the following generalization of the above results.

8.7. Corollary [70]. *Assume that, for every module M in* $\operatorname{ind} A$, $\operatorname{End}_A(M)$ *is a division ring. Then A is representation-finite.*

We end this section with the following problems on possible characterizations of representation-finite algebras.

Problem 11. Assume that there is a positive integer m such that $\operatorname{rad}^m(\operatorname{End}_A(M)) = 0$ for any module M in $\operatorname{ind} A$. Is then A representation-finite?

Problem 12. Assume that Γ_A admits only finitely many components. Is then A representation-finite?

Problem 13. Assume that Γ_A admits only finitely many τ_A-orbits. Is then A representation-finite?

If R is an algebraically closed field, then the Problem 13 is solved in [42].

9. Simply connected algebras of polynomial growth

Throughout this section we assume that R is an algebraically closed field K. Recall that following Drozd [28] an algebra A is called **tame** if, for any dimension d, there is a finite number of $K[X]-A$-bimodules $M_1, \dots, M_r$ which are finitely generated and free as left $K[X]$-modules, and satisfy the following condition: all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $K[X]/(x-\lambda)\otimes_{K[X]} M_i$ for some $\lambda \in K$ and some i. We denote by $\mu_A(d)$ the least number of bimodules M_i satisfying the above condition. Then A is called of **polynomial growth** [59] (respectively, **domestic**) if there is a natural number m such that $\mu_A(d) \leq d^m$ (respectively, $\mu_A(d) \leq m$) for all $d \geq 1$. By [22] this concept of domestic algebra coincides with that introduced in [55]. Moreover, by the validity of the second Brauer-Thrall conjecture, we get that an algebra A is representation-finite if and only if $\mu_A(d) = 0$ for all $d \geq 1$. On the other hand, by the Drozd's remarkable "Tame and Wild Theorem" [28] we know that if A is not of tame type then the classification of indecomposable A-modules contains the classical unsolved wild problem of reducing pair of matrices under simultaneous conjugations. Examples of domestic algebras are provided by tilted algebras of Euclidean type. The Ringel's tubular algebras [56] are non-domestic of polynomial growth (see [61, 3.6]). Finally, the algebras $K[x,y]/(xy, x^n, y^n)$, $n \geq 3$, investigated by Gelfand and Ponomarev [31], are tame but not of polynomial growth (see [27], [59]).

We would like to characterize the domestic algebras, algebras of polynomial growth, and tame algebras by properties of cycles of indecomposable modules and the infinite radical of their module categories. The following problem was raised by O. Kerner and the author in [41].

Problem 14. Assume that $\operatorname{rad}^\infty(\operatorname{mod} A)$ is nilpotent. Is then A domestic?

We pose here the following problem.

Problem 15. Assume that every loop in $\operatorname{mod} A$ is finite. Is then A of polynomial growth?

It is easy to see that, if $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ is nilpotent or A is loop-finite, then A is tame (see [41], [5]). Moreover, it was shown in [20] that, if A is a tame algebra, then for every positive integer d, almost all indecomposable A-modules of dimension d lie in stable tubes of rank 1. Hence, in order to solve the above problems, we must describe the supports of periodic modules lying in the stable tubes of the considered classes of tame algebras. We have the following partial results.

9.1. THEOREM [69]. *For a finite-dimensional, basic, connected K-algebra A the following conditions are equivalent:*

(i) *A is cycle-finite and admits a sincere indecomposable module lying in a stable tube of Γ_A.*

(ii) *A is either tame concealed or tubular.*

Then we get the following immediate consequence

9.2. COROLLARY [69]. *Let A be a cycle-finite algebra. Then A is of polynomial growth.*

The following theorem gives some characterization of domestic cycle-finite algebras.

9.3. THEOREM [69]. *Let A be a cycle-finite algebra. The following conditions are equivalent:*

(i) *A is domestic.*

(ii) *A does not contain a tubular algebra as a full convex subalgebra.*

(iii) $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ *is nilpotent.*

(iv) *All but finitely many components of Γ_A are stable tubes of rank one.*

In the representation theory of finite dimensional algebras over an algebraically closed field an important role is played by simply connected algebras. The importance of simply connected algebras follows from the fact that often we may reduce, with the help of coverings, study of the module category of an algebra to that for the corresponding simply connected algebras. This is known to be the case for all representation-finite algebras (see [17]) and many important classes of tame algebras. Recall that following [3] an algebra A is called **simply connected** if A is triangular (the ordinary quiver of A has no oriented cycle) and for any presentation $A \simeq KQ/I$ of A as a bound quiver algebra, the fundamental group $\pi_1(Q,I)$ is trivial. It was shown in [61] that a triangular algebra A is simply connected if and only if A does not admit a proper Galois covering. Following [66] an algebra A is called **strongly simply connected** if every full convex subalgebra of A is simply connected. An important class of simply connected algebras is formed by algebras having the separation property of Bautista, Larrion and Salmeron [14]. Recall that, if A is a triangular algebra and Q is the ordinary quiver of A, then A has the **separation property** if the radical of any indecomposable projective module $P_A(x)$ is a direct sum of pairwise non-isomorphic indecomposable modules whose supports are contained in different connected components of the subquiver $Q(x)$ of Q obtained by deleting all those vertices of Q being a source of a path with target x (including the trivial path from x to x). Clearly, if the ordinary quiver of an algebra A is a tree, then A has the separation property. It is known that a representation-finite algebra A is simply connected if and only if A has the separation property [14]. On the other hand, there are representation-infinite simply connected algebras without separation property. But it was shown in [66] that an algebra is strongly simply connected if and only if every full convex subalgebra of A has the separation property. Hence, if A is strongly simply connected then A has the separation property.

In the study of simply connected algebras of polynomial growth an important role is played by multicoils and multicoil algebras introduced by I. Assem and the author in [5], [6]. Recall that a component $\mathcal{C}$ of Γ_A is called a **multicoil** if it contains a full translation subquiver $\mathcal{C}'$ such that $\mathcal{C}'$ is a disjoint union of coils (see Section 2) and no module from $\mathcal{C} \setminus \mathcal{C}'$ lies on an oriented cycle in $\mathcal{C}$. Then an algebra A is said to be a **multicoil algebra** if for any cycle $M_0 \to M_1 \to \cdots \to M_t = M_0$ in mod$A$, the indecomposable modules M_i belong to one standard coil of a multicoil in Γ_A. Clearly, every multicoil algebra is cycle-finite, and hence is of polynomial growth.

We shall now describe the module categories of loop-finite algebras with the separation property. In particular, we characterize the strongly simply connected algebras of polynomial growth by properties of their module categories.

9.4. THEOREM [67]. *Let A be an algebra with the separation property. Then the following conditions are equivalent.*

(i) mod$A$ *has no infinite loop.*
(ii) mod$A$ *has no infinite cycle.*
(iii) *Every component of* Γ_A *is generalized standard.*
(iv) *Every component of* Γ_A *is standard.*
(v) *The component quiver* Σ_A *has no oriente cycle.*
(vi) *A is a multicoil algebra.*

If A is strongly simply connected then the above conditions are equivalent to

(vii) *A is polynomial growth.*

Observe that (v) gives a partial order of Auslander-Reiten components in the category mod$A$. Further, (vi) gives a complete control of cycles in mod$A$, and hence a possibility for classification of non-directing indecomposable modules. In a joint research with I. Assem we investigate the sincere non-directing indecomposable modules over arbitrary multicoil algebras. If such a module lies in a stable tube then the algebra is either tame concealed or tubular [5] (see also Theorem 9.1). A description of the remaining indecomposable non-directing modules will be presented in [7] and [8]. We note also that if an indecomposable module is directing then its support is a tilted algebra (see [56]). The class of representation-infinite tame algebras with sincere directing modules is investigated by de la Peña (see [48]). The Theorem 9.4 is applied in [68] to obtain a different characterization of strongly simply connected algebras of polynomial growth. Namely, we distinguish in [68] a class of so called p-critical algebras (suitable non-polynomial growth enlargements of tame concealed algebras of type $\widetilde{\mathbb{D}}_m$) and prove that a strongly simply connected algebra A is of polynomial growth if and only if A does not contain p-critical full subalgebra and the Tits form of A is weakly non-negative. The Theorem 9.4 leads also to a characterization of polynomial growth algebras having strongly simply connected Galois coverings (see [68]).

Applying Theorem 9.4 one can prove also the following characterization of domestic strongly simply connected algebras.

9.5. THEOREM. *Let A be a strongly simply connected algebra. Then A is domestic if and only if* $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ *is nilpotent.*

Let us mention that there are polynomial growth algebras with the separation property which are not loop-finite. Therefore the conditions (i) - (v) in Theorem 9.4 don't characterize arbitrary simply connected algebras of polynomial growth. Hence we pose the following problem:

PROBLEM 16. Characterize the simply connected algebras of polynomial growth by properties of cycles in their module categories.

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RELATIVE HOMOLOGY

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Introduction

Throughout this paper all rings are assumed to be artin R-algebras over a fixed commutative artin ring R, i.e. R-algebras which are finitely generated R-modules. All modules are assumed to be finitely generated and we denote the category of finitely generated left modules over an artin algebra Λ by $\operatorname{mod}\Lambda$. Our purpose in this paper is to explain some applications of relative homological algebra to the study of the modules over an artin R-algebra Λ. We start by describing two applications of relative homological algebra, one ring theoretic and the other module theoretic. We hope this will encourage the reader to read the rest of the paper, which consists of a brief summary of relative homological algebra for modules over artin algebras and an explanation of how the two results given in the first section follow from the general theory.

Throughout the paper all subcategories are assumed to be additive subcategories. Recall that an *additive* subcategory of $\operatorname{mod}\Lambda$ is a full subcategory

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$\mathcal{X}$ closed under isomorphisms and all finite direct sums of modules in $\mathcal{X}$ and all direct summands of modules in $\mathcal{X}$ again are in $\mathcal{X}$. Moreover, for any module A in $\operatorname{mod}\Lambda$ we let $\operatorname{add} A$ denote the additive subcategory of $\operatorname{mod}\Lambda$ generated by A, i.e. $\operatorname{add} A$ consists of the Λ-modules isomorphic to direct summands of finite direct sums of A. Also, the symbol $\oplus$ denotes the direct sum in $\operatorname{mod}\Lambda$.

1. Results

Before stating our first result it is convenient to make the following definition. We say an artin algebra Λ is *$D\,\mathrm{Tr}$-selfinjective* if the subcategory $\operatorname{add}\{(\mathrm{Tr}\,D)^i\Lambda\}_{i=0}^{\infty}$ is of finite type, i.e. there is a module C in $\operatorname{mod}\Lambda$ such that $\operatorname{add} C = \operatorname{add}\{(\mathrm{Tr}\,D)^i\Lambda\}_{i=0}^{\infty}$. Here $\mathrm{Tr}\,D^0$ is defined to be the identity map. Clearly Λ being $D\,\mathrm{Tr}$-selfinjective is equivalent to either of the following equivalent conditions: (a) the $D\,\mathrm{Tr}$-orbit of all the indecomposable injective modules are finite or (b) the union of the $\mathrm{Tr}\,D$-orbits of all the indecomposable projective modules coincide with the union of the $D\,\mathrm{Tr}$-orbits of all the indecomposable injective modules. It is obvious that selfinjective algebras as well as algebras of finite representation type are $D\,\mathrm{Tr}$-selfinjective. The somewhat less obvious result that the Auslander algebras of selfinjective algebras of finite representation type are $D\,\mathrm{Tr}$-selfinjective gives other examples of $D\,\mathrm{Tr}$-selfinjective algebras. In this connection it is worth noting that not all Auslander algebras are $D\,\mathrm{Tr}$-selfinjective.

Our reason for introducing $D\,\mathrm{Tr}$-selfinjective algebras is that these algebras are intimately related to the better known class of algebras of injective dimension 2 which are also of dominant dimension 2. However, before stating the precise relationship between these type of algebras we recall that an artin algebra Γ is of *dominant dimension* at least $r \geq 1$ (notation: $\operatorname{dom.dim}\Gamma \geq r$), if in a minimal injective resolution

$$0 \to \Gamma \to I_0 \to I_1 \to \cdots$$

the injective Γ-modules I_j are also projective for $j = 0, 1, \ldots, r-1$.

THEOREM 1.1. *An artin algebra Γ has $\operatorname{dom.dim}\Gamma = 2$ and $\operatorname{id}_\Gamma \Gamma = 2$ if and only if there is a module T over a $D\,\mathrm{Tr}$-selfinjective algebra Λ satisfying the following:*

(a) $\Gamma = \operatorname{End}_\Lambda(T)$

(b) $T = C \oplus M$, *where* $\operatorname{add} C = \operatorname{add}\{(\mathrm{Tr}\,D)^i\Lambda\}_{i=0}^{\infty}$ *and* $M \simeq D\,\mathrm{Tr}\,M$.

The following is a special case of this result which seems of particular interest.

COROLLARY 1.2. *Suppose Λ is a symmetric algebra (e.g. Λ is a modular group ring) and A a Λ-module of finite Ω-period n. Let $M = \oplus_{i=0}^{n-1}\Omega_\Lambda^i(A)$ and let $T = \Lambda \oplus M$. Then $\Gamma = \operatorname{End}_\Lambda(T)$ has the property* $\operatorname{dom.dim}\Gamma = 2 = \operatorname{id}_\Gamma\Gamma$.

For those readers unfamiliar with the notation $\Omega^i_\Lambda(M)$ used above, we recall that $\Omega_\Lambda(M)$ denotes the kernel of a projective cover of a module M and the $\Omega^i_\Lambda(M)$ for all $i = 0, 1, \ldots$ are defined inductively as follows: $\Omega^0_\Lambda(M) = M$ and $\Omega^{i+1}_\Lambda(M) = \Omega_\Lambda(\Omega^i_\Lambda(M))$ for all $i \geq 0$. The $\Omega^i_\Lambda(M)$ is called the *i-th syzygy* of M.

Before giving our other application we recall the notion of homologically finite subcategories introduced by M. Auslander and S. O. Smalø in (4). Let Λ be an artin algebra. An additive subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is *contravariantly finite* in $\operatorname{mod}\Lambda$, if given any module C in $\operatorname{mod}\Lambda$ there is a morphism $f: X_C \to C$ with X_C in $\mathcal{X}$ such that $\operatorname{Hom}_\Lambda(X, X_C) \to \operatorname{Hom}_\Lambda(X, C) \to 0$ is exact for all X in $\mathcal{X}$. For a given module C such a morphism $f: X_C \to C$ is called a *right $\mathcal{X}$-approximation* of C. The approximation is called *minimal* if in addition a morphism $g: X_C \to X_C$ is an isomorphism whenever $f \circ g = f$. It is easy to see that if C has a right $\mathcal{X}$-approximation, then there is a minimal right $\mathcal{X}$-approximation which is unique up to isomorphism. Dually, an additive subcategory $\mathcal{Y}$ of $\operatorname{mod}\Lambda$ is *covariantly finite* in $\operatorname{mod}\Lambda$, if given any module A in $\operatorname{mod}\Lambda$ there is a morphism $g: A \to Y^A$ with Y in $\mathcal{Y}$ such that $\operatorname{Hom}_\Lambda(Y^A, Y) \to \operatorname{Hom}_\Lambda(A, Y) \to 0$ is exact for all Y in $\mathcal{Y}$. Dually one also defines left $\mathcal{Y}$-approximations and minimal left $\mathcal{Y}$-approximations. For example, if $\mathcal{X} = \operatorname{add} A$ for some module A in $\operatorname{mod}\Lambda$, then $\mathcal{X}$ is both contravariantly and covariantly finite in $\operatorname{mod}\Lambda$. A subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ which is both contravariantly and covariantly finite in $\operatorname{mod}\Lambda$ is called *functorially finite* in $\operatorname{mod}\Lambda$. Moreover, a subcategory of $\operatorname{mod}\Lambda$ satisfying one of these notions of finiteness is called a *homologically finite* subcategory of $\operatorname{mod}\Lambda$.

Now we have the necessary background to explain our second application. In (3) M. Auslander and I. Reiten showed that the additive subcategory generated by all n-th syzygy modules, $\Omega^n_\Lambda(\operatorname{mod}\Lambda)$, is functorially finite for all positive n. We generalize this result as follows. Let $\mathcal{X}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ containing all the projective Λ-modules. For each module C in $\operatorname{mod}\Lambda$ let

$$\cdots \to X_1(C) \xrightarrow{f_1(C)} X_0(C) \xrightarrow{f_0(C)} C \to 0$$

be exact with each $f_i(C): X_i(C) \to \operatorname{Im} f_i(C)$ being a minimal right $\mathcal{X}$-approximation. For each module C in $\operatorname{mod}\Lambda$ denote $\operatorname{Im} f_n(C)$ by $\Omega^n_{\mathcal{X}}(C)$. Let $\Omega^n_{\mathcal{X}}(\operatorname{mod}\Lambda)$ denote the additive subcategory of $\operatorname{mod}\Lambda$ generated by the modules $\Omega^n_{\mathcal{X}}(C)$ for all modules C in $\operatorname{mod}\Lambda$. Then we have the following result.

THEOREM 1.3. *The subcategory $\mathcal{X} \cup \Omega^n_{\mathcal{X}}(\operatorname{mod}\Lambda)$ is functorially finite in* $\operatorname{mod}\Lambda$ *for all positive n.*

A special case of this result is the following. Let $\mathcal{X} = \operatorname{add} G$ for a generator G of $\operatorname{mod} \Lambda$. Then $\mathcal{X}$ is functorially finite in $\operatorname{mod} \Lambda$ and contains all the projective Λ-modules. In this case the above result says that the union of $\mathcal{X}$ and the additive subcategory of $\operatorname{mod} \Lambda$ generated by all the kernels of the minimal right $\mathcal{X}$-approximations is functorially finite in $\operatorname{mod} \Lambda$. By (4, Proposition 3.13 (a)) removing a subcategory of finite type from a homologically finite subcategory does not change the homological finiteness. Hence it follows that the additive subcategory of $\operatorname{mod} \Lambda$ generated by all the kernels of the minimal right $\mathcal{X}$-approximations is functorially finite in $\operatorname{mod} \Lambda$.

2. Relative Homological Algebra

In this section we give a brief summary of relative homological algebra for modules over artin algebras.

Let Λ be an artin algebra. A *relative homology theory* for the category $\operatorname{mod} \Lambda$ consists of a class $\mathcal{E}$ of short exact sequences $0 \to A \to B \to C \to 0$ in $\operatorname{mod} \Lambda$ which is closed under isomorphisms of exact sequences, pushouts, pullbacks and finite sums, i.e. if $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$ and $0 \to A_1' \xrightarrow{f_1'} A_2' \xrightarrow{f_2'} A_3' \to 0$ are in $\mathcal{E}$, then $0 \to A_1 \oplus A_1' \xrightarrow{(f_1, f_1')} A_2 \oplus \xrightarrow{(f_2, f_2')} A_3 \oplus A_3' \to 0$ is in $\mathcal{E}$. Of particular concern to us are the relative homological theories one obtains as follows.

Let $\mathcal{X}$ be an arbitrary additive subcategory of $\operatorname{mod} \Lambda$. Then associated with $\mathcal{X}$ is the relative homology theory $\mathcal{E}_{\mathcal{X}}$ consisting of all exact sequences $0 \to A \to B \to C \to 0$ such that $0 \to \operatorname{Hom}_\Lambda(X, A) \to \operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C) \to 0$ is exact for all X in $\mathcal{X}$ and also the relative homological theory $\mathcal{E}^{\mathcal{X}}$ consisting all exact sequences $0 \to A \to B \to C \to 0$ in $\operatorname{mod} \Lambda$ such that $0 \to \operatorname{Hom}_\Lambda(C, X) \to \operatorname{Hom}_\Lambda(B, X) \to \operatorname{Hom}_\Lambda(A, X) \to 0$ is exact for all X in $\mathcal{X}$.

Now it is not difficult to see that associated with a relative homological theory $\mathcal{E}$ of $\operatorname{mod} \Lambda$ is the additive subfunctor F of the bifunctor $\operatorname{Ext}^1_\Lambda(\ ,\ ): (\operatorname{mod} \Lambda)^{\mathrm{op}} \times \operatorname{mod} \Lambda \to \mathrm{Ab}$ given by $F(C, A)$ consisting of all exact sequences $0 \to A \to B \to C \to 0$ in $\operatorname{Ext}^1_\Lambda(C, A)$ which are in $\mathcal{E}$, where Ab denotes the category of abelian groups and $(\operatorname{mod} \Lambda)^{\mathrm{op}}$ denotes the opposite category of $\operatorname{mod} \Lambda$. Recall that a functor $F: (\operatorname{mod} \Lambda)^{\mathrm{op}} \times \operatorname{mod} \Lambda \to \mathrm{Ab}$ is called a *bifunctor*. A bifunctor $F: (\operatorname{mod} \Lambda)^{\mathrm{op}} \times \operatorname{mod} \Lambda \to \mathrm{Ab}$ is said to be *additive* if for each C in $(\operatorname{mod} \Lambda)^{\mathrm{op}}$ and A in $\operatorname{mod} \Lambda$ the functors $F(C,\ ): \operatorname{mod} \Lambda \to \mathrm{Ab}$ and $F(\ , A): (\operatorname{mod} \Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ are additive functors. Moreover, a subfunctor F of $\operatorname{Ext}^1_\Lambda(\ ,\ ): (\operatorname{mod} \Lambda)^{\mathrm{op}} \times \operatorname{mod} \Lambda \to \mathrm{Ab}$ is called an *additive* subfunctor if F is an additive bifunctor. In fact the map which assigns to each relative homological theory $\mathcal{E}$ the additive subfunctor F of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ gives a bijection between the set of relative homological theories and the set of additive subfunctors of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. Since the only subfunctors of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ which are of interest to us are those associated with a relative homological theory $\mathcal{E}$ of $\operatorname{mod}\Lambda$, we assume from now on that when we say that F is a subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ we mean that F is an additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$.

Suppose now that $\mathcal{E}$ is a relative homological theory with corresponding subfunctor F of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. We say that an exact sequence $\cdots \to C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} 0$ is *F-exact* if the exact sequences $0 \to \operatorname{Ker} f_i \to C_i \to \operatorname{Ker} f_{i-1} \to 0$ are in $\mathcal{E}$ for all $i = 1, 2, \ldots$. We will often use the notation $\operatorname{Ext}^1_F(C, A)$ for $F(C, A)$ when we want to emphasize that $F(C, A)$ consists of the F-exact sequences $0 \to A \to B \to C \to 0$. As with standard homological algebra, if $0 \to A \to B \to C \to 0$ is F-exact, then for all X in $\operatorname{mod}\Lambda$ there are exact sequences

$$(*)\quad 0 \to \operatorname{Hom}_\Lambda(X, A) \to \operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C) \to \operatorname{Ext}^1_F(X, A),$$

which are functorial in X and the F-exact sequence $0 \to A \to B \to C \to 0$. As in standard homological algebra we would like to define additive bifunctors $\operatorname{Ext}^i_F(\ ,\ ) \colon (\operatorname{mod}\Lambda)^{\mathrm{op}} \times \operatorname{mod}\Lambda \to \mathrm{Ab}$ for $i = 0, 1, \ldots$, such that for an F-exact sequence $0 \to A \to B \to C \to 0$ we get the usual type of long exact sequences of additive functors

$$\begin{array}{lllllll} 0 & \to & \operatorname{Hom}_\Lambda(X, A) & \to & \operatorname{Hom}_\Lambda(X, B) & \to & \operatorname{Hom}_\Lambda(X, C) \\ & \to & \operatorname{Ext}^1_F(X, A) & \to & \operatorname{Ext}^1_F(X, B) & \to & \operatorname{Ext}^1_F(X, C) \\ & \to & \operatorname{Ext}^2_F(X, A) & \to & \operatorname{Ext}^2_F(X, B) & \to & \operatorname{Ext}^2_F(X, C) \\ & \to & \ldots, & & & & \end{array}$$

which are functorial in X and the F-exact sequence $0 \to A \to B \to C \to 0$. For this purpose it is convenient to introduce the notion of an F-projective module for a corresponding relative homological theory $\mathcal{E}$.

We say that a module P is *F-projective* if $0 \to \operatorname{Hom}_\Lambda(P, A) \to \operatorname{Hom}_\Lambda(P, B) \to \operatorname{Hom}_\Lambda(P, C) \to 0$ is exact for all F-exact sequences $0 \to A \to B \to C \to 0$ in $\operatorname{mod}\Lambda$. We denote by $\mathcal{P}(F)$ the subcategory of $\operatorname{mod}\Lambda$ consisting of all F-projective modules. Using the exact sequence $(*)$, it is easy to see as in the standard homological algebra, that a module P is F-projective if and only if $\operatorname{Ext}^1_F(P,\ ) = 0$. It is clear that $\mathcal{P}(F)$ contains $\mathcal{P}(\Lambda)$, where $\mathcal{P}(\Lambda)$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the projective Λ-modules. In particular, from the existence of almost split sequences, it follows that $\mathcal{P}(F_\mathcal{X}) = \mathcal{P}(\Lambda) \cup \mathcal{X}$, where $F_\mathcal{X}$ is the subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ corresponding to $\mathcal{E}_\mathcal{X}$. Using this one can show that a relative homological theory $\mathcal{E}$ has the property that $F = F_{\mathcal{P}(F)}$ if and only if $F = F_\mathcal{X}$ for some additive subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$.

Suppose now that $F = F_{\mathcal{P}(F)}$. Then for each F-exact sequence $0 \to A \to B \to C \to 0$ and X in $\operatorname{mod}\Lambda$ we have an exact sequence

$$\begin{array}{lllllll} 0 & \to & \operatorname{Hom}_\Lambda(X, A) & \to & \operatorname{Hom}_\Lambda(X, B) & \to & \operatorname{Hom}_\Lambda(X, C) \\ & \to & \operatorname{Ext}^1_F(X, A) & \to & \operatorname{Ext}^1_F(X, B) & \to & \operatorname{Ext}^1_F(X, C), \end{array}$$

which is functorial in X and F-exact sequences $0 \to A \to B \to C \to 0$. Thus we see that the hypothesis that $F = F_{\mathcal{P}(F)}$ gives us part of our desired long exact sequence. However in order to get a fully satisfactory theory, it seems necessary to make further assumptions on F.

We say that a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ has *enough projectives* if and only if given any Λ-module C there is an F-exact sequence $0 \to K \to P \to C \to 0$ with P an F-projective module. From this it follows that each Λ-module C has an F-projective resolution, i.e. there is an F-exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to C \to 0$$

with the P_i in $\mathcal{P}(F)$ for all i. In analogy with standard homological algebra we define $\mathrm{Ext}^i_F(C, X)$ to be the i-th homology of the complex

$$0 \to \mathrm{Hom}_\Lambda(P_0, X) \to \mathrm{Hom}_\Lambda(P_1, X) \to \mathrm{Hom}_\Lambda(P_2, X) \to \cdots.$$

If one substitutes F-exact sequences and F-projective modules for exact sequences and projective modules, then most of the basic results and concepts of standard homological algebra translate verbatim to the relative context. For instance, it is clear what one means by $\mathrm{pd}_F C$, the F-projective dimension of a module C as well as $\mathrm{gl.dim}_F \Lambda$, the F-global dimension of Λ. Moreover, for each F-exact sequence $0 \to A \to B \to C \to 0$ and X in $\mathrm{mod}\,\Lambda$ we get our desired long exact sequence

$$\begin{array}{lllllll} 0 & \to & \mathrm{Hom}_\Lambda(X,A) & \to & \mathrm{Hom}_\Lambda(X,B) & \to & \mathrm{Hom}_\Lambda(X,C) \\ & \to & \mathrm{Ext}^1_F(X,A) & \to & \mathrm{Ext}^1_F(X,B) & \to & \mathrm{Ext}^1_F(X,C) \\ & \to & \mathrm{Ext}^2_F(X,A) & \to & \mathrm{Ext}^2_F(X,B) & \to & \mathrm{Ext}^2_F(X,C) \\ & \to \ldots & & & & & \end{array}$$

functorial in the F-exact sequence $0 \to A \to B \to C \to 0$ and X in $\mathrm{mod}\,\Lambda$. In view of these observations it is of particular interest to know when a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ has enough projectives. Here we use the notion of homologically finite subcategories defined in section 1.

It is relatively straightforward to see that a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ has enough projective modules if and only if $\mathcal{P}(F)$ is contravariantly finite in $\mathrm{mod}\,\Lambda$ and $F = F_{\mathcal{P}(F)}$. In particular, if $\mathcal{X} = \mathrm{add}\, A$ for some module A, then $F_\mathcal{X}$ has enough projectives and $\mathcal{P}(F_\mathcal{X}) = \mathcal{P}(\Lambda) \cup \mathcal{X}$.

We also have the dual notion of F-injective modules for a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$. A module I in $\mathrm{mod}\,\Lambda$ is *F-injective* if and only if $0 \to \mathrm{Hom}_\Lambda(C, I) \to \mathrm{Hom}_\Lambda(B, I) \to \mathrm{Hom}_\Lambda(A, I) \to 0$ is exact for all F-exact sequences $0 \to A \to B \to C \to 0$ in $\mathrm{mod}\,\Lambda$. The subcategory of all F-injective modules is denoted by $\mathcal{I}(F)$, which clearly contains the subcategory $\mathcal{I}(\Lambda)$ of all injective Λ-modules. The subfunctor F has *enough injectives* if and only if given any Λ-module A there is an F-exact sequence $0 \to A \to$

$I \to K \to 0$ with I in $\mathcal{I}(F)$. Let $F^{\mathcal{X}}$ denote the subfunctor of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ corresponding to the relative homological theory $\mathcal{E}^{\mathcal{X}}$ for a subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$. Then, dual to the characterization of subfunctors of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ with enough projectives, a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ has enough injectives if and only if $\mathcal{I}(F)$ is covariantly finite in $\mathrm{mod}\,\Lambda$ and $F = F^{\mathcal{I}(F)}$.

In general the F-projective and F-injective modules are related in the following way. Using that for an exact sequence $0 \to A \to B \to C \to 0$ the sequence $\mathrm{Hom}_\Lambda(X, B) \to \mathrm{Hom}_\Lambda(X, C) \to 0$ is exact if and only if $\mathrm{Hom}_\Lambda(B, D\,\mathrm{Tr}\,X) \to \mathrm{Hom}_\Lambda(A, D\,\mathrm{Tr}\,X) \to 0$ is exact by (1, III, Corollary 4.2), it follows that $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup D\,\mathrm{Tr}\,\mathcal{P}(F)$ and $\mathcal{P}(F) = \mathcal{P}(\Lambda) \cup \mathrm{Tr}\,D\mathcal{I}(F)$. This implies that for an additive subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ the equality $\mathcal{I}(F_{\mathcal{X}}) = \mathcal{I}(\Lambda) \cup D\,\mathrm{Tr}\,\mathcal{X}$ holds. The duality D and the transpose Tr map contravariantly finite subcategories to covariantly finite subcategories and vice versa. By (4, Proposition 3.13) adding a subcategory of finite type to a homologically finite subcategory does not change the type of homologically finiteness. Using these observations a subfunctor F of $\mathrm{Ext}^1_\Lambda(\ ,\ )$ has enough projective and injective modules if and only if $\mathcal{P}(F)$ is functorially finite in $\mathrm{mod}\,\Lambda$ and $F = F_{\mathcal{P}(F)}$. In particular, if $\mathcal{X} = \mathrm{add}\,A$ for some module A in $\mathrm{mod}\,\Lambda$, then (a) $F_{\mathcal{X}}$ has enough projective nodules and enough injective modules and (b) $\mathcal{P}(F) = \mathcal{P}(\Lambda) \cup \mathcal{X}$ and $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup D\,\mathrm{Tr}\,\mathcal{X}$.

The main feature of relative homological algebra is that the class of extensions of modules is restricted while the homomorphisms remain the same. This has the effect of making some Λ-modules which are not projective or injective Λ-modules behave like projective or injective objects in the relative homological setting. For example, in the relative homological theory of $F = F^{\mathcal{P}(\Lambda)}$, the projective Λ-modules also behave like injective objects since an exact sequence $0 \to A \to B \to C \to 0$ is F-exact if and only if $0 \to \mathrm{Hom}_\Lambda(C, \Lambda) \to \mathrm{Hom}_\Lambda(B, \Lambda) \to \mathrm{Hom}_\Lambda(A, \Lambda) \to 0$ is exact. In fact, $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup \mathcal{P}(\Lambda)$.

But this is not the only change that occurs when one goes from standard to relative homology. Some subcategories of $\mathrm{mod}\,\Lambda$ may be closed under extensions in a suitable relative theory without being extension closed in the standard theory. For example, the functorially finite subcategory $\mathrm{Sub}(\Lambda)$ consisting of all modules isomorphic to submodules of free Λ-modules is extension closed in the relative theory $F = F^{\mathcal{P}(\Lambda)}$ but not in general extension closed in the standard theory. In fact $\mathrm{Sub}(\Lambda)$ is extension closed in the standard theory if and only if Λ satisfies the rather restrictive condition that the projective dimension of the Λ^{op}-injective envelope of Λ^{op} is at most 1 (2, Proposition 3.5).

Many of the results in the theory of homologically finite subcategories require that subcategories are closed under extensions. As we saw above not all homologically finite subcategories are closed under extensions, hence many of the results for homologically finite subcategories do not apply. One example of such a result is the Auslander-Reiten Lemma, which says the following. If $\mathcal{Y}$ is an extension closed covariantly finite subcategory of $\operatorname{mod}\Lambda$, then the subcategory $\{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(X, \ )|_{\mathcal{Y}} = 0\}$ is an extension closed contravariantly finite subcategory of $\operatorname{mod}\Lambda$. But it turns out that most of the results in the standard theory for homologically finite subcategories generalize to relative homological settings when reformulated properly. For example, the generalization of the Auslander-Reiten Lemma to the relative setting is the following.

THEOREM 2.1. *Let F be a subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ with enough projectives. If $\mathcal{Y}$ is an F-extension closed covariantly finite subcategory of $\operatorname{mod}\Lambda$, then the subcategory $\{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_F(X, \ )|_{\mathcal{Y}} = 0\}$ is an F-extension closed contravariantly finite subcategory of $\operatorname{mod}\Lambda$.*

We observed above that the subcategory $\operatorname{Sub}(\Lambda)$ is covariantly finite in $\operatorname{mod}\Lambda$ and closed under the relative extensions given by $F = F^{\mathcal{P}(\Lambda)}$, which has enough projectives. Then Theorem 2.1 implies that the subcategory $\mathcal{X} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_F(X, \ )|_{\operatorname{Sub}(\Lambda)} = 0\}$ is an F-extension closed contravariantly finite subcategory of $\operatorname{mod}\Lambda$. Even though $\mathcal{X}$ is given by expressions involving relative homology functors, we have the following description of $\mathcal{X}$ not involving any relative homological algebra. The subcategory $\mathcal{X}$ is equal to the subcategory $\{X \in \operatorname{mod}\Lambda \mid \underline{\operatorname{Hom}}_\Lambda(X, \ )|_{\mathcal{A}} = 0\}$, where $\mathcal{A} = \{A \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(A, \Lambda) = 0\}$ and $\underline{\operatorname{Hom}}_\Lambda(X, A)$ is $\operatorname{Hom}_\Lambda(X, A)$ modulo the morphisms from X to A which factor through a projective Λ-module. For more information on this and additional topics the reader is referred to the papers (5; 6; 7; 8).

The first examples of relative homology theories for categories of modules over a ring were given and developed by G. Hochshild in (13, 1956). Relative homological algebra was generalized and extended to abstract categories by A. Heller in (12, 1958) and by D. A. Buchsbaum in (9, 1959). M. C. R. Butler and G. Horrocks showed in (10, 1961) that the relative homology theories in a module category over a ring Λ are in one-one correspondence with the category of additive subfunctors of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. It is their approach to relative homological algebra we have taken in this paper and also in the papers (5; 6; 7; 8). As a background reference for relative homological algebra we refer to (11).

3. Outline of proofs

This section is devoted to describing some of the theoretical background for the proofs of the theorems stated in section 1.

We start with the ring theoretic application given in Theorem 1.1, which is proven using relative cotilting theory. We begin with a brief review of this theory.

Let Λ be an artin algebra. The object of cotilting theory is to compare Λ and $\mathrm{End}_\Lambda(T)$ for a cotilting module T over Λ. A cotilting module T is a Λ-module with (i) $\mathrm{Ext}^i_\Lambda(T,T) = 0$, for $i > 0$, (ii) $\mathrm{id}_\Lambda T < \infty$ and (iii) for any injective Λ-module I there is an exact sequence $0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to I \to 0$ with T_i in $\mathrm{add}\, T$ for all i. During the discussion of the theoretical background for Theorem 1.1 we fix F to be $F_{\mathrm{add}\, G}$ for some generator G of $\mathrm{mod}\,\Lambda$. Recall that $F(C,A) = \{0 \to A \to B \to C \to 0 \mid \mathrm{Hom}_\Lambda(G,B) \to \mathrm{Hom}_\Lambda(G,C) \to 0$ is exact$\}$. Then $\mathcal{P}(F) = \mathrm{add}\, G$ and F has enough projectives and injectives. A relative cotilting module is given by the following. An *F-cotilting module* is a Λ-module T with

(i) $\mathrm{Ext}^i_F(T,T) = 0$ for all $i > 0$,

(ii) $\mathrm{id}_F T < \infty$

(iii) for all modules I in $\mathcal{I}(F)$ there is an F-exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to I \to 0$$

with T_i is in $\mathrm{add}\, T$ for all i.

An example of an F-cotilting module that always exists is a module T such that $\mathrm{add}\, T = \mathcal{I}(F)$, which trivially satisfies (i)–(iii). It is this type of relative cotilting modules that we use in proving Theorem 1.1. The standard cotilting modules are obtained by letting $G = \Lambda$ so $F = \mathrm{Ext}^1_\Lambda(\ ,\ )$.

The relative extension groups may be zero without the standard extension groups being zero. This enables us to construct more relative cotilting modules than standard ones. For instance, for standard cotilting theory self-injective algebras have only the trivial cotilting module Λ, while using various relative homological theories one obtains relative cotilting modules other than Λ which give nontrivial results of which Theorem 1.1 is an example.

Let T be a Λ-module which is an F-cotilting module. We want to compare Λ and Γ. Since T is a module over Γ, we have the functor $\mathrm{Hom}_\Lambda(\ ,T)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ and $\mathrm{Hom}_\Lambda(\Lambda, {}_\Lambda T) \simeq {}_\Gamma T$. As in the standard case the naturally induced homomorphism $\Lambda \to \mathrm{End}_\Gamma({}_\Gamma T_{\Lambda^{\mathrm{op}}})$ is an isomorphism. Moreover, one can show that the number of nonisomorphic simple Γ-modules is the same as the number of nonisomorphic indecomposable modules in $\mathcal{P}(F) = \mathrm{add}\, G$. So, if the subcategory $\mathcal{P}(F)$ properly contains $\mathcal{P}(\Lambda)$, then Γ has more nonisomorphic simple modules than Λ. Furthermore, we have the following connections between Λ and Γ.

THEOREM 3.1. *Let $F = F_{\mathrm{add}\, G}$ for a generator G of* $\mathrm{mod}\,\Lambda$ *and let T be a Λ-module which is an F-cotilting module. Denote* $\mathrm{End}_\Lambda(T)$ *by Γ. Then the subcategory* $\mathrm{Hom}_\Lambda(\mathcal{P}(F),T)$ *in* $\mathrm{mod}\,\Gamma$ *contains* $\mathrm{add}\,{}_\Gamma T$ *and is* $\mathrm{add}\, T_0$ *for a standard cotilting module T_0 over Γ with* $\mathrm{id}_\Gamma T_0 \leq \mathrm{id}_F T + 2$. *Moreover,*

$\mathrm{id}_\Gamma\, {}_\Gamma T \leq \mathrm{id}_F\, {}_\Lambda T$ *and the modules* ${}_\Lambda T$ *and* ${}_\Gamma T_0$ *have the property that the natural homomorphism* ${}_\Gamma T_0 \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma({}_\Gamma T_0, {}_\Gamma T), {}_\Lambda T)$ *is an isomorphism.*

The last property of the modules T and T_0 mentioned in Theorem 3.1 turns out to be important. In this connection it is convenient to make the following definition. Let M be a Λ-module and $\Sigma = \mathrm{End}_\Lambda(M)$. If A is a Λ-module such that the natural homomorphism $A \to \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(A, M), M)$ is an isomorphism, then M is said to *dualize* A. If A is a direct summand of M, then A is called a *dualizing summand* of M. Since $\mathrm{add}\, {}_\Gamma T$ is contained in $\mathrm{Hom}_\Lambda(\mathcal{P}(F), T) = \mathrm{add}\, T_0$, we can choose T_0 such that ${}_\Gamma T$ is a direct summand of ${}_\Gamma T_0$. Then Theorem 3.1 says that ${}_\Gamma T$ is a dualizing summand of the standard cotilting module ${}_\Gamma T_0$. The converse of this result is also true, namely all dualizing summands of standard cotilting modules are obtained from relative cotilting modules. More precisely, we have the following result.

THEOREM 3.2. *Let* $T = T_1 \oplus T_2$ *be a standard cotilting module in* $\mathrm{mod}\,\Gamma$ *for an artin algebra* Γ. *Let* $\Lambda = \mathrm{End}_\Gamma(T_1)$. *Define* F *to be the subfunctor of* $\mathrm{Ext}^1_\Lambda(\ ,\ )$ *given by* $F_{\mathcal{P}(F)}$, *where* $\mathcal{P}(F) = \mathrm{Hom}_\Gamma(\mathrm{add}\, {}_\Gamma T, T_1)$. *Then* T_1 *is a dualizing summand of the* Γ*-module* T *if and only if the* Λ*-module* ${}_\Lambda T_1$ *is an* F*-cotilting module.*

This shows that relative cotilting modules and dualizing summands of standard cotilting modules are intimately connected and give a way of comparing algebras with a different number of nonisomorphic simple modules.

We saw that a module T such that $\mathrm{add}\, T = \mathcal{I}(F)$ always is an F-cotilting module. In this case we have even more information than in the general case.

PROPOSITION 3.3. *Let* $F = F_{\mathrm{add}\, G}$ *for a generator* G *of* $\mathrm{mod}\,\Lambda$, *where* $\mathrm{add}\, G$ *properly contains* $\mathcal{P}(\Lambda)$. *Let* T *be a* Λ*-module such that* $\mathrm{add}\, T = \mathcal{I}(F)$. *Denote* $\mathrm{End}_\Lambda(T)$ *by* Γ. *Then the* Γ*-module* $\mathrm{Hom}_\Lambda(G, T)$ *is a standard cotilting module over* Γ *with* $\mathrm{id}_\Gamma \mathrm{Hom}_\Lambda(G, T) = 2$.

Recall that Theorem 1.1 from section 1 says the following.

THEOREM 1.1. *An artin algebra* Γ *has* $\mathrm{dom.dim}\,\Gamma = 2$ *and* $\mathrm{id}_\Gamma\,\Gamma = 2$ *if and only if there is a module* T *over a* $D\,\mathrm{Tr}$*-selfinjective algebra* Λ *satisfying the following:*

(a) $\Gamma = \mathrm{End}_\Lambda(T)$

(b) $T = C \oplus M$, *where* $\mathrm{add}\, C = \mathrm{add}\{(\mathrm{Tr}\, D)^i \Lambda\}_{i=0}^\infty$ *and* $M \simeq D\,\mathrm{Tr}\, M$.

Next we explain part of the proof of this result using what we have discussed so far.

Let Λ be an artin algebra. Assume that $\mathcal{O}_\Lambda = \mathrm{add}\{(\mathrm{Tr}\, D)^i \Lambda\}_{i=0}^\infty$ is finite. This implies that all the injective Λ-modules are in $\mathcal{O}_\Lambda$. Let C be a Λ-module such that $\mathrm{add}\, C = \mathcal{O}_\Lambda$ and M a Λ-module such that $M \simeq D\,\mathrm{Tr}\, M$.

Denote $C \oplus M$ by T. Let $F = F_{\operatorname{add} T}$. Using that $\mathcal{I}(F) = \mathcal{I}(\Lambda) \cup \operatorname{Tr} D\mathcal{P}(F)$ and $\mathcal{P}(F) = \operatorname{add} T$, it follows that $\mathcal{P}(F) = \mathcal{I}(F) = \operatorname{add} T$. Hence, T is an F-cotilting module. By the above result we have that $\operatorname{id}_\Gamma \Gamma = 2$. By Theorem 3.1 we have that $\operatorname{id}_\Gamma {}_\Gamma T = 0$. Since ${}_\Lambda T$ is a generator, it follows that ${}_\Gamma T$ is a projective injective Γ-module. Let $P_1 \to P_0 \to T \to 0$ be a projective presentation of T as a Λ-module. Apply the functor $\operatorname{Hom}_\Lambda(\ ,T)$ to this sequence and we obtain the following exact sequence

$$0 \to \Gamma \to \operatorname{Hom}_\Lambda(P_0,T) \to \operatorname{Hom}_\Lambda(P_1,T) \to I_2 \to 0.$$

The modules $\operatorname{Hom}_\Lambda(P_i,T)$ for $i = 0,1$ are in $\operatorname{add} {}_\Gamma T$, so that the dominant dimension of Γ is at least 2. Since $\operatorname{id}_\Gamma \Gamma = 2$, the dominant dimension of Γ must be 2. This proves one direction of Theorem 1.1 we mentioned above. The other direction is more involved and the reader is referred to (8) for further details of the proof.

We end this paper by giving some details of the proof of the second result, which we recall next.

Let $\mathcal{X}$ be a functorially finite subcategory of $\operatorname{mod} \Lambda$ containing the projective Λ-modules. For each module C in $\operatorname{mod} \Lambda$ let $\cdots \to X_1(C) \xrightarrow{f_1(C)} X_0(C) \xrightarrow{f_0(C)} C \to 0$ be exact with each $f_i(C)\colon X_i(C) \to \operatorname{Im} f_i(C)$ being a minimal right $\mathcal{X}$-approximation. Denote $\operatorname{Im} f_n(C)$ by $\Omega^n_{\mathcal{X}}(C)$ for each C in $\operatorname{mod} \Lambda$ and $n > 0$. Let $\Omega^n_{\mathcal{X}}(\operatorname{mod} \Lambda)$ denote the additive subcategory of $\operatorname{mod} \Lambda$ generated by the modules $\Omega^n_{\mathcal{X}}(C)$ for all modules C in $\operatorname{mod} \Lambda$.

THEOREM 1.3. *The subcategory* $\mathcal{X} \cup \Omega^n_{\mathcal{X}}(\operatorname{mod} \Lambda)$ *is functorially finite in* $\operatorname{mod} \Lambda$ *for all positive* n.

In (3) M. Auslander and I. Reiten proved that $\Omega^n_\Lambda(\operatorname{mod} \Lambda)$ is functorially finite in $\operatorname{mod} \Lambda$ for all positive n. We claimed in the first section that the above result is an generalization of this result. To see this let F be the subfunctor $F_{\mathcal{X}}$ of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. Then $\mathcal{P}(F) = \mathcal{X}$ and F has enough projectives and injectives. Since a relative projective cover of a module C in $\operatorname{mod} \Lambda$ is the same as a minimal right $\mathcal{X}$-approximation, we have that relative syzygies $\Omega^n_F(C)$ coincide with $\Omega^n_{\mathcal{X}}(C)$ for every module C in $\operatorname{mod} \Lambda$. This gives the analogy with the standard result.

To prove the theorem we have to show that for each n the subcategory $\mathcal{X} \cup \Omega^n_{\mathcal{X}}(\operatorname{mod} \Lambda)$ is functorially finite. We first prove that it is a contravariantly finite subcategory of $\operatorname{mod} \Lambda$. The proof goes by induction on n, where we only indicate the steps for $n = 1$. Since F has enough injectives, we also have negative relative syzygies $\Omega^{-n}_F(C)$ using a minimal relative injective resolution of C in $\operatorname{mod} \Lambda$. Let C be an arbitrary module in $\operatorname{mod} \Lambda$. Then construct the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_F(I) & = & \Omega_F(I) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Z & \to & P_I & \to & \Omega_F^{-1}(C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C & \to & I & \to & \Omega_F^{-1}(C) & \to & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the diagram is induced by starting with the minimal right $\mathcal{X}$-approximat $0 \to \Omega_F(I) \to P_I \to I \to 0$. Since $0 \to Z \to P_I \to \Omega_F^{-1}(C) \to 0$ is a right $\mathcal{X}$-approximation, $Z \simeq \Omega_F(\Omega_F^{-1}(C)) \oplus X$ for some module X in $\mathcal{X}$. Then it is relatively straightforward to see that $Z \to C$ is a right $\mathcal{X} \cup \Omega^1_{\mathcal{X}}(\operatorname{mod}\Lambda)$-approximation.

To prove that $\mathcal{X} \cup \Omega^n_{\mathcal{X}}(\operatorname{mod}\Lambda)$ is covariantly finite in $\operatorname{mod}\Lambda$ we make use of the following result by M. Auslander and I. Reiten.

PROPOSITION 3.4 ((2, Proposition 1.2 (b))). *Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories and $G: \mathcal{C} \to \mathcal{D}$ and $H: \mathcal{D} \to \mathcal{C}$ is an adjoint pair of functors with G a left adjoint and H a right adjoint. Then* $\operatorname{Im} H$ *is covariantly finite in $\mathcal{C}$ and the homomorphism $I \to HG$ given by the adjointness gives a left* $\operatorname{Im} H$*-approximation $C \to HG(C)$ for each object C in $\mathcal{C}$.*

Let $\underline{\operatorname{mod}}_F\,\Lambda$ denote the category $\operatorname{mod}\Lambda$ modulo the subcategory $\mathcal{P}(F) = \mathcal{X}$. Then we show that $\Omega^n_{\mathcal{X}}: \underline{\operatorname{mod}}_F\,\Lambda \to \underline{\operatorname{mod}}_F\,\Lambda$ has a right adjoint, hence $\Omega^n_{\mathcal{X}}(\operatorname{mod}\Lambda)$ is covariantly finite in $\underline{\operatorname{mod}}_F\,\Lambda$. Since $\mathcal{X}$ is covariantly finite in $\operatorname{mod}\Lambda$, it follows easily that $\mathcal{X} \cup \Omega^n_{\mathcal{X}}(\operatorname{mod}\Lambda)$ is covariantly finite in $\operatorname{mod}\Lambda$.

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Tilting Theory and Selfinjective Algebras

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1 *Introduction*

Tilting theory is known as a powerful tool for studying representation theory of artin algebras. Though there are no non-trivial tilting modules over selfinjective algebras, we have a way of using tilting theory for trivial extension selfinjective algebras, starting from a tilting module, which is given by constructing equivalent functors between corresponding stable module categories over trivial extension selfinjective algebras of original algebras.

Since trivial extension selfinjective algebras are not general enough in the class of all selfinjective algebras, it is natural to ask a generalization of the above method of using tilting theory for trivial extension selfinjective algebras to more general selfinjective algebras. In order to do so, we need some description of general selfinjective algebras.

On the other hand, there is a natural generalization of usual tilting modules; and, we can construct stably equivalent functors by using such generalized tilting modules, similarly to the case of usual tilting modules.

Further, some homological problems, such as Finitistic Dimension Conjecture, Generalized Nakayama Conjecture and original Nakayama Conjecture are naturally related to the foundamental properties of generalized tilting modules.

Hence it is necessary to develope the theory of generalized tilting modules and selfinjective algebras, so that we can consider stable equivalences and some homological problems in a unified way.

V. Dlab and L. L. Scott (eds.), Finite Dimensional Algebras and Related Topics, 361–390.

In this paper, we will examine some foundamental properties of generalized tilting modules and consider a general structure of selfinjective algebras first, and next, by using those results, we will construct some equivalent functors between stable module categories of suitable selfinjective algebras which are not of the form of trivial extension algebras generally.

Throughout this paper all algebras and modules are assumed to be finite dimensional over an algebraically closed field, for simplicity. Homomorphisms operate from the opposite side of the scalar. The ordinary duality functor is denoted by D.

2 *Generalized Tilting Modules*

For a module T_A over an algebra A, we call an exact sequence

$$0 \to X_A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots$$

a left dominant T_A-resolution of X_A if it has the following two properties:

- All terms T_i are in add(T_A).
- The isomorphisms $Ext^1_A(Imf_i, T) = 0$ hold for all $i \geq 0$.

By $\mathcal{C}og^*(T_A)$ we denote the class of all modules X_A for which there are left dominante T_A-resolutions. As the dual of this, we define right dominant T_A-resolutions and the class $\mathcal{G}en^*(T_A)$. Usually we consider module classes as full subcategories of corresponding whole module categories.

We call a module T_A a generalized tilting module if it has the following two properties:

- The regular module A_A is in $\mathcal{C}og^*(T_A)$.
- The isomorphisms $Ext^i_A(T, T) = 0$ hold for all $i \geq 1$.

Throughout this paper, we call generalizes tilting modules just tilting modules for short.

In this section, we recall the basic properties and results on tilting modules without any proofs. See [1],[24],[32],[44]-[47],[49-52] for the proofs.

For a tilting module T_A, putting $B = End(T_A)$, it is proved that the left module ${}_BT$ is again a tilting module with $End({}_BT) = A$. So we may call the bimodule ${}_BT_A$ a tilting bimodule. Further, it should be noted that ${}_BT_A$ is a tilting bimodule if and only if so is its dual ${}_AD(T)_B$ ([49]).

For a tilting bimodule ${}_BT_A$, it is proved that the adjoint pair of functors $Hom_A(T,?)$ and $(? \otimes_B T)$ induces category equivalences

$$\mathcal{G}en^*(T_A) \approx Fix(\eta^T) \cap \bigcap_{i \geq 1} KerTor_i^B(?,T)$$

and

$$Fix(\epsilon^T) \cap \bigcap_{i \geq 1} KerExt_A^i(T,?) \approx \mathcal{C}og^*(D(T)_B),$$

where ϵ^T and η^T stand for the counit and the unit of the adunction.

Let us put

$$\mathcal{C}(T_A) = \mathcal{G}en^*(T_A) \cap \bigcap_{i \geq 1} KerExt_A^i(T,?)$$

and

$$\mathcal{D}(T_A) = \mathcal{C}og^*(T_A) \cap \bigcap_{i \geq 1} KerTor_i^A(T,?).$$

Then, we have category equivalences

$$\mathcal{C}(T_A) \approx \mathcal{D}(D(T)_B)$$

and

$$\mathcal{D}(T_A) \approx \mathcal{C}(D(T)_B),$$

which are induced by the adjoint pairs of functors

$$(Hom_A(T,?),(? \otimes_B T))$$

and

$$((? \otimes_A D(T)), Hom_B(D(T),?)),$$

respectively.

For a module class $\mathcal{M}$, we define the classes $\mathcal{I}\mathcal{M}$ and $\mathcal{P}\mathcal{M}$as follows:

$$\mathcal{I}\mathcal{M} = \{X \mid Ext^i(\mathcal{M}, X) = 0 \text{ } for \text{ } all \text{ } i > 0\}$$

and

$$\mathcal{P}\mathcal{M} = \{X \mid Ext^i(X, \mathcal{M}) = 0 \text{ } for \text{ } all \text{ } i > 0\}.$$

We have the following relations of our classes ([50,51]):

1. $\mathcal{C}(T_A) = \mathcal{G}en^*(T_A) \cap \mathcal{I}(T_A)$ and $\mathcal{D}(T_A) = \mathcal{C}og^*(T_A) \cap \mathcal{P}(T_A)$
2. $\mathcal{I}\mathcal{D}(T_A) \subseteq \mathcal{C}(T_A)$ and $\mathcal{P}\mathcal{C}(T_A) \subseteq \mathcal{D}(T_A)$.

3. $\mathcal{IPC}(T_A) = \mathcal{C}(T_A)$ and $\mathcal{PID}(T_A) = \mathcal{D}(T_A)$.

4. $\mathcal{C}(T_A) \cap \mathcal{PC}(T_A) = add(T_A)$ and $\mathcal{D}(T_A) \cap \mathcal{ID}(T_A) = add(T_A)$.

5. The restrictins of the category eqivalences $\mathcal{C}(T_A) \approx \mathcal{D}(D(T)_B)$ and $\mathcal{D}(T_A) \approx \mathcal{C}(D(T)_B)$ induce the corresponding category equivalences $\mathcal{ID}(T_A) \approx \mathcal{PC}(D(T)_B)$ and $\mathcal{PC}(T_A) \approx \mathcal{ID}(D(T)_B)$, respectively.

6. If the minimal projective resolution of T_A is ultimately closed, then

$$\mathcal{C}(T_A) = \mathcal{G}en^*(T_A)$$

and

$$\mathcal{D}(D(T)_B) = \mathcal{P}(D(T)_B) = \bigcap_{i \geq 1} KerTor_i^B(?,T)$$

hold.

7. If both of the minimal projective resolutions of T_A and ${}_BT$ are ultimately closed, then

$$\mathcal{C}(T_A) = \mathcal{G}en^*(T_A) = \mathcal{I}(T_A) = \bigcap_{i \geq 1} KerExt_A^i(T,?)$$

and

$$\mathcal{D}(D(T)_B) = \mathcal{C}og^*(D(T)_B) = \mathcal{P}(D(T)_B) = \bigcap_{i \geq 1} KerTor_i^B(?,T)$$

hold.

Remark 1 *The last relation above can be seen as a generalization of the classical result on Nakayama Conjecture given by B. J. Müller[31]. If the minimal injective resolution of the regular module A_A are ultimately closed and all the terms in the resolution are projective, then, denoting by T_A the direct sum of all indecomposable injective-projective modules which appear in the resolution and putting $B = End(T_A)$, ${}_BT_A$ is a tilting module with the property stated in the above relation, so ${}_BT_A$ becomes a progenerator since $\mathcal{G}en^*(T_A) = \mathcal{I}(T_A) = mod\,A$, because T_A is projective. Therefore, A must be self-injective.*

As a generalization of the theorem of Brenner-Butler for genuine tilting modules, we have the following long exact sequences for generalized tilting modules ${}_BT_A$ ([49]):

$$\cdots \to Tor_2^B(Hom_A(T,?),T) \to Tor_3^B(Hom_A(T,\Omega^{-1}(?)),T) \to$$

$$\begin{gathered}\to Tor_3^B(Ext_A^1(T,?),T) \to\to Tor_1^B(Hom_A(T,?),T) \to \\ \to Tor_2^B(Hom_A(T,\Omega^{-1}(?)),T) \to Tor_2^B(Ext_A^1(T,?),T) \to \\ \to Ker\,\epsilon^T \to Tor_1^B(Hom_A(T,\Omega^{-1}(?)),T) \to Tor_1^B(Hom_A(T,?),T) \to \\ \to Cok\,\epsilon^T \to Ker\,\epsilon^T\Omega^{-1} \to Ext_A^1(T,?)\otimes_B T \to 0\end{gathered}$$

and

$$\begin{gathered}0 \to Hom_A(T,Tor_B^1(?,T)) \to Cok\,\eta^T\Omega \to Ker\,\eta^T \to \\ \to Ext_A^1(T,Tor_1^B(?,T)) \to Ext_A^1(T,\Omega(?)\otimes_B T) \to Cok\,\eta^T \to \\ \to Ext_A^2(T,Tor_1^B(?,T)) \to Ext_A^2(T,\Omega(?)\otimes_B T) \to Ext_A^1(T,(?\otimes_B T)) \to \\ \to Ext_A^3(T,Tor_1^B(?,T)) \to Ext_A^3(T,\Omega(?)\otimes_B T) \to Ext_A^2(T,(?\otimes_B T)) \to \cdots.\end{gathered}$$

Applying the above result to a tilting module ${}_BT_A$ with $pd\,{}_BT \leq 1$ and $pd\,T_A \leq 1$, we have the original Brenner-Butler's theorem. Similarly, if we start from the assumptions $pd\,{}_BT \leq 2$ and $pd\,T_A \leq 2$, we obtain the following exact sequences and relations of functors:

$$0 \to Tor_2^B(Ext_A^1(T,?),T) \to Hom_A(T,?)\otimes_B T \to 1_{mod-A} \to Cok\,\epsilon^T \to 0,$$

$$\begin{gathered}0 \to Tor_1^B(Ext_A^1(T,?),T) \to Cok\,\epsilon^T \to Tor_2^B(Ext_A^2(T,?),T) \to \\ \to Ext_A^1(T,?)\otimes_B T \to 0,\end{gathered}$$

$$0 \to Ker\,\eta^T \to 1_{mod-B} \to Hom_A(T,(?\otimes_B T)) \to Ext_A^2(T,Tor_1^B(?,T)) \to 0,$$

$$\begin{gathered}0 \to Hom_A(T,Tor_1^B(?,T)) \to Ext_A^2(T,Tor_2^B(?,T)) \to Ker\,\eta^T \to \\ \to Ext_A^1(T,Tor_1^B(?,T)) \to 0,\end{gathered}$$

and

$$\begin{aligned}Tor_2^B(Hom_A(T,?),T) = Tor_1^B(Hom_A(T,?),T) &= Tor_1^B(Ext_A^2(T,?),T) \\ &= Ext_A^2(T,?)\otimes_B T = 0, \\ Ext_A^2(T,(?\otimes_B T)) = Ext_A^1(T,(?\otimes_B T)) &= Ext_A^1(T,Tor_2^B(?,T)) \\ &= Hom_A(T,Tor_2^B(?,T)) = 0.\end{aligned}$$

In this way, almost all results on genuine tilting modules are naturally generalized to generalized tilting modules.

Now we give the definition of the classes $\mathcal{C}_{fin}(T_A)$ and $\mathcal{D}_{fin}(T_A)$, where ${}_BT_A$ is a generalized tilting module. It is easy to see that any module X_A with a finite right dominant T_A-resolution

$$0 \to T_n \xrightarrow{f_{n-1}} T_{n-1} \to \cdots \to T_1 \xrightarrow{f_0} T_0 \to X_A \to 0$$

belongs to the class $\mathcal{ID}(T_A) \subseteq \mathcal{C}(T_A)$. We denote by $\mathcal{C}_{fin}(T_A)$ the class of all modules with finite right domonant T_A-resolutions. Similarly, we define $\mathcal{D}_{fin}(T_A)$ as the class of all modules with finite left dominant T_A-resolutions.

We have the following relations:

$$add(T_A) \subseteq \mathcal{C}_{fin}(T_A) \subseteq \mathcal{ID}(T_A) \subseteq \mathcal{C}(T_A)$$

and

$$add(T_A) \subseteq \mathcal{D}_{fin}(T_A) \subseteq \mathcal{PC}(T_A) \subseteq \mathcal{D}(T_A).$$

By restricting the category equivalent functors $\mathcal{C}(T_A) \approx \mathcal{D}(D(T)_B)$ and $\mathcal{D}(T_A) \approx \mathcal{C}(D(T)_B)$, we have the following equivalences:

$$\mathcal{C}_{fin}(T_A) \approx \{Y_B \mid pd\, Y_B < \infty\} \cap \mathcal{D}(D(T)_B)$$

and

$$\mathcal{D}_{fin}(T_A) \approx \{Y_B \mid id\, Y_B < \infty\} \cap \mathcal{C}(D(T)_B).$$

Cocerning Finititic Dimension Conjecture (FDC for short), we make the following problem which may be consider as a tilting version of FDC :

$FDC^{(tilting)}$ Are the lengthes of finite minimal right dominant resolutios for a tilting modules bounded?

By the above category equivalences, it is easy to see that the validity of $FDC^{(tilting)}$ follows from the validity of FDC. Conversely, the varidity of FDC follows from that of $FDC^{(tilting)}$. For, if we take the cogenerator ${}_AD(A)_A$ as a tilting module ${}_BT_A$, then

$$\{Y_B \mid pd\, Y_B < \infty\} \cap \mathcal{D}(D(T)_B) = \{X_A \mid pd\, X_A < \infty\}$$

and finitistic projective dimensions are bounded by the bound of lengthes of finite minimal right dominant resolutions for $D(A)_A$.

We can consider the above problem for each given tilting module and it may be not so hard to check the validity.

For a pair of module classes $\mathcal{I}$ and $\mathcal{P}$, we call an exact sequence

$$0 \to X_A \to I \to P \to 0$$

with $I \in \mathcal{I}$ and $P \in \mathcal{P}$ a kernel expression of X_A with respect to $(\mathcal{I}, \mathcal{P})$. By $Ker(\mathcal{I}, \mathcal{P})$ we denote the class of all modules for which there are kernel expressions with respect to a pair $(\mathcal{I}, \mathcal{P})$. As this dual, we define cokernel expressions of modules with respect to $(\mathcal{I}, \mathcal{P})$ and the class $Cok(\mathcal{I}, \mathcal{P})$.

It is proved that the relations

$$Ker(\mathcal{C}(T_A), \mathcal{PC}(T_A)) = mod\,A$$

and

$$Cok(\mathcal{ID}(D(T)_B), \mathcal{D}(D(T)_B)) = mod\,B$$

hold in case $\mathcal{C}(T_A)$ or $\mathcal{PC}(T_A)$ being functorially finite in $mod\,A$, or $pd\,_BT$, $pd\,T_A < \infty$.

Actually, in case of $pd\,_BT$ and $pd\,T_A$ being finite, $\mathcal{PC}(T_A) = \mathcal{D}_{fin}(T_A)$ and $\mathcal{ID}(D(T)_B) = \mathcal{C}_{fin}(D(T)_B)$ hold, so we have

$$Ker(\mathcal{C}(T_A), \mathcal{D}_{fin}(T_A)) = mod\,A$$

and

$$Cok(\mathcal{C}_{fin}(D(T)_B), \mathcal{D}(D(T)_B)) = mod\,B.$$

Those are proved in [50] and [51]. Further, in this case, $FDC^{(tilting)}$ is affirmative ([50]).

3 *Construction of Tilting Modules*

We call a module X_A selforthogonal if it satisfies $Ext^i_A(X, X) = 0$ for all $i > 0$.

Starting from the trivial tilting module A_A, a selforthogonal module U_A was called a tilting module if $A_A \in \mathcal{C}og^*(U_A)$. Similarly to this, starting from any tilting module T_A, we may ask whether a selforthogonal module U_A is a tilting module if $T_A \in \mathcal{C}og^*(U_A)$. Remind that, in case we start from the regular module, $U_A \in \mathcal{C}(A_A)$ holds.

Theorem 3.1 *Assume T_A is a tilting module. If $T_A \in \mathcal{C}og^*(U_A)$ holds for a selforthogonal module U_A in $\mathcal{C}(T_A)$, then U_A itself is a tilting module and $T_A \in \mathcal{D}(U_A)$.*

Proof. It is enough to show $A_A \in \mathcal{C}og^*(U_A)$. We will construct a left dominant U_A-resolution of A_A.

Let

$$0 \to A \to T_0 \to T_1 \to \cdots$$

and

$$0 \to T_i \to U_{i0} \to U_{i1} \to \cdots$$

be left dominant T_A-resolution of A_A and left dominant U_A -resolution of T_i, respectively. Put $T_i' = Im(T_i \to T_{i+1})$ and $U_{ij}' = Im(U_{ij} \to U_{i,j+1})$.

By the following diagram, we define the module C_0 first:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & T_0 & \to & T_0' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & U_{00} & \to & C_0 & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & U_{00}' & = & U_{00}' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

It is easy to see that all modules T_i' and U_{ij}' are in the class $\mathcal{P}(add(U_A))$. So, $C_0 \in \mathcal{P}(add(U_A))$ follows.

Next, we define the module K_0 by the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & T_0' & \to & T_1 & \to & T_1' & \to & 0 \\ & & \downarrow & PO & \downarrow & & \| & & \\ 0 & \to & C_0 & \to & K_0 & \to & T_1' & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & U_{00}' & = & U_{00}' & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

The module K_0 is again in $\mathcal{P}(add(U_A))$.

From the left dominant U_A-resolutions

$$0 \to T_1 \to U_{10} \to U_{11} \to \cdots$$

and

$$0 \to U_{00}' \to U_{01} \to U_{02} \to \cdots,$$

we get a left dominant U_A-resolution of K_0

$$0 \to K_0 \to U_{10} \oplus U_{01} \to U_{11} \oplus U_{02} \to \cdots.$$

We put $C_1 = Cok(C_0 \to U_{10} \oplus U_{01})$ and $K_0' = Cok(K_0 \to U_{10} \oplus U_{01})$. Then, K_0' belongs to $\mathcal{P}(add(U_A))$ and $0 \to T_1' \to C_1 \to K_0' \to 0$ is exact.

Continuing this process, we have a sequence $C_0, K_0, C_1, K_1, C_2, K_2, \ldots$ and obtain a left dominant U_A-resolution of A_A

$$0 \to A \to U_{00} \to U_{10} \oplus U_{01} \to U_{20} \oplus U_{11} \oplus U_{02} \to \cdots.$$

So we have proved $A_A \in \mathcal{C}og^*(U_A)$. ∎

Proposition 3.2 (Reflection) *Assume $T_A = S_A \oplus X_A$ is a tilting module and S_A is cogenerated by X_A. Let $S_A \xrightarrow{f} \bar{X}$ be the minimal left X_A-approximation of S_A and $U_A = Cok(f)$. Then, $T'_A = U_A \oplus X_A$ becomes a tilting module.*

Proof. By the above result, it is enough to prove that the module T'_A is selforthogonal.

From the sequence $0 \to S_A \xrightarrow{f} \bar{X} \to U \to 0$, applying $Hom_A(?, X)$, we know $Ext^i_A(U, X) = 0$ for $i > 0$, since $Ext^i_A(S \oplus \bar{X}, X) = 0$ for all $i > 0$. Then, by applying $Hom_A(U, ?)$ to the sequence, we have the isomorphisms $Ext^i_A(U, U) \approx Ext^{i+1}_A(U, S)$ for all $i > 0$. On the other hand,since $Ext^i_A(S \oplus \bar{X}, S) = 0$ hold for all $i > 0$, applying $Hom_A(?, S)$, we obtain the isomorphisms $Ext^i_A(U, S) = 0$ for all $i > 1$. Therefore, U_A is selforthogonal.

From $U_A \in \mathcal{C}(T_A)$,the isomorphisms $Ext^i_A(X, U) = 0$ for $i > 0$ follow. ∎

Proposition 3.3 (Bongartz Construction) *Let T_A be a tilting module and $U_A \in \mathcal{C}(T_A)$ a selforthogonal module. Suppose $Ext^i_A(U, T) = 0$ for all $i > 1$ and*

$$0 \to T_A \to E_A \to \oplus U_A \to 0$$

an exact sequence with the induced map $Hom_A(U, \bigoplus U) \to Ext^1_A(U, T)$ being surjective. Then, $E_A \oplus U_A$ is a tilting module.

Proof. By the similar way as above, it is proved that $E_A \oplus U_A$ is selforthogonal. Then, the assertion follows from the theorem, since $E_A \oplus U_A \in \mathcal{C}(T_A)$ and $T_A \in \mathcal{C}og^*(E_A \oplus U_A)$ hold. ∎

Proposition 3.4 *Let $\mathcal{I}$ and $\mathcal{P}$ be subcategories of mod A such that $\mathcal{I} = \mathcal{I}\mathcal{P}$, $\mathcal{P} = \mathcal{P}\mathcal{I}$ and $\mathcal{P} \subseteq Ker(\mathcal{I}, \mathcal{P})$. If there is a module T_A with $add(T_A) = \mathcal{I} \cap \mathcal{P}$, then this module T_A is a tilting module.*

Proof. By $0 \to X \to I(X) \to P(X) \to 0$, we denote the minimal kernel expression of X_A in $Ker(\mathcal{I}, \mathcal{P})$. Since projective modules are in $\mathcal{P}$, we have an exact sequence

$$0 \to A \to I(A) \to IP(A) \to IP^2(A) \to \cdots$$

with $Im(IP^{i-1}(A) \to IP^i(A)) = P^i(A)$ for any $i > 0$. Since $IP^i(A) \in \mathcal{I} \cap \mathcal{P} = add(T_A)$, this sequence is a left dominant T_A-resolution of A_A. So, by the theorem, T_A is a tilting module. ▮

In the above, it arises a natural question whether any indecomposable direct summand of T_A appears as a direct summand of one of terms $IP^i(A)$. In order to consider this question, we make the following definition:

Let ${}_BT_A$ be a tilting bimodule and

$$0 \to A_A \to T_0 \to T_1 \to \cdots$$

and

$$0 \to {}_B B \to T_0' \to T_1' \to \cdots$$

the minimal left dominant T-resolutions of A_A and ${}_BB$, respectively. By $T(A)_A$ and ${}_BT(B)$, we denote the unique basic direct summands of T_A and ${}_BT$ such that

$$add(T(A)_A) = add(\bigoplus_{i \geq 0} T_i)$$

and

$$add({}_BT(B)) = add(\bigoplus_{i \geq 0} T_i').$$

Concerning Generalized Nakayama Conjecture (GNC for short), we consider the following problem:

$GNC^{(tilting)}$ $T(B) = T = T(A)$?

It is not hard to see that GNC and $GNC^{(tilting)}$ are equivalent.

Theorem 3.5 *Let T_A be a tilting module and*

$$0 \to A_A \to T_0 \to T_1 \to \cdots$$

the minimal left dominant T_A-resolution. Denote by $\mathcal{C}$ the class of all modules X_A such that the functors $Hom_A(?, X)$ preserve the exactness of the above T_A-resolution. Then we have $\mathcal{C} = \mathcal{G}en^*(T(A)_A)$.

Proof. First,we prove $\mathcal{C} \subseteq \mathcal{G}en^*(T(A)_A)$. Put $A_i = Im(T_{i-1} \to T_i)$ and $T_{i-1} \xrightarrow{q_{i-1}} A_i \xrightarrow{v_i} T_i$. Let $X_A \in \mathcal{C}$. Since

$$Hom_A(T_0, X) \to Hom_A(A, X) \approx X_A$$

is surjective, the module X_A is generated by T_0 and so by $T(A)_A$. Denote by $T_0' \xrightarrow{p_0} X$ the minimal $T(A)_A$-approximation of X_A. Let $X_1 \xrightarrow{u_0} T_0'$ be the kernel of the map p_0. We want to prove that the module X_1 is generated by $T(A)_A$.

For any map $A_A \xrightarrow{f} X$, there are maps $T_0 \xrightarrow{f_0} T_0'$ and $A_1 \xrightarrow{f_1} X$ such that $f_0 \cdot v_0 = u_0 \cdot f$ and $f_1 \cdot q_0 = p_0 \cdot f_0$. By the property of X, the map f_1 has an extension $T_1 \xrightarrow{f_1'} X$ through v_1. Further, the map f_1' has a lift $T_1 \xrightarrow{g} T_0'$ through p_0. Then, there is a map h from T_0 to X_1 such that $f_0 = g \cdot v_1 \cdot q_0 + u_0 \cdot h$. This map satisfies $f = h \cdot v_0$. Thus we have proved that the map $Hom_A(T_0, X_1) \rightarrow Hom_A(A, X_1) \approx X_1$ is surjective and, hence, X_1 is generated by T_0 and also by $T(A)_A$.

Continuing this process, we can obtain the minimal right dominant $T(A)_A$-resolution of X_A and get $X_A \in \mathcal{G}en^*(T(A)_A)$.

Next, we prove $\mathcal{G}en^*(T(A)_A) \subseteq \mathcal{C}$.

Suppose $X_A \in \mathcal{G}en^*(T(A)_A)$ and

$$\cdots \rightarrow T_1' \rightarrow T_0' \rightarrow X_A \rightarrow 0$$

be a right dominant $T(A)_A$-resolution of X_A. Let us put $X_i = Im(T_{i+1}' \rightarrow T_i')$ and $X_{i+1} \xrightarrow{u_i} T_i' \xrightarrow{p_i} X_i$.

For any map f from A to X, there are maps $A \xrightarrow{g} T_0$ and $T_0 \xrightarrow{g'} T_0'$ such that $f = p_0 \cdot g$ and $g = g' \cdot v_0$. So, the map $p_0 \cdot g'$ is an extension of f through v_0. Thus, $Hom_A(v_0, X)$ is surjective.

Similarly, for any map $A_{i+1} \xrightarrow{f} X$, there are maps $T_{i-j} \xrightarrow{f_{i-j}'} T_j'$ and $A_{i-j} \xrightarrow{f_{i-j}} X_{j+1}$ for $0 \leq j \leq i$ such that $f_{i-j} \cdot q_{i-j} = p_j \cdot f_{i-j-1}'$ and $f_{i-j}' \cdot v_{i-j} = u_j \cdot f_{i-j}$, where we consider f as f_{i+1}.

Since $X_{i+1} \in \mathcal{G}en^*(T(A)_A)$, there is a map $T_0 \xrightarrow{\delta_0} X_{i+1}$ such that $\delta_0 \cdot v_0 = f_0$. Starting from δ_0, we obtain the maps $T_{i-j} \xrightarrow{\delta_{i-j}} X_{j+1}$ and $A_{i-j+1} \xrightarrow{\gamma_{i-j}} T_j'$ such that $f_{i-j} = \delta_{i-j} \cdot v_{i-j} + p_{j+1} \cdot \gamma_{i-j-1}$ and $f_{i-j}' = \gamma_{i-j} \cdot q_{i-j} + u_j \cdot \delta_{i-j}$.

Since the map $A_{i+1} \xrightarrow{\gamma_i} T_0'$ has an extension $T_{i+1} \xrightarrow{g} T_0'$ through v_{i+1}, $f = f_{i+1}$ also has an extension $p_0 \cdot g$. Thus, it has been proved that the map $Hom_A(v_{i+1}, X)$ is surjective for any i. Therefore, X_A belongs to $\mathcal{C}$. ∎

Corollary 3.6 $\mathcal{C}(T_A) \subseteq \mathcal{C}(T(A)_A)$.

Proof. It is obvious that $\bigcap_{i>0} KerExt_A^i(T, ?) \subseteq \bigcap_{i>0} KerExt_A^i(T(A)_A, ?)$.

On the other hand, from the proof of the previous theorem, it is clear that $\mathcal{G}en^*(T_A) \subseteq \mathcal{C} = \mathcal{G}en^*(T(A)_A)$.

Therefore, $\mathcal{C}(T_A) \subseteq \mathcal{C}(T(A)_A)$ holds. ∎

Proposition 3.7 *Put $T_A = T(A)_A \oplus S_A$ and assume that $pd\, S_A < \infty$, then S_A belongs to $add(T(A)_A)$.*

Proof. Assume $\cdots \to T_1' \to T_0' \to X_A \to 0$ is a right dominant $T(A)_A$-resolution of a module X_A in the class $\mathcal{C}(T(A)_A)$. Putting $X_i = Im(T_i' \to T_{i-1}')$, we have the following isomorphisms $Ext_A^i(S, X) \approx Ext_A^{i+j}(S, X_j)$ for all i, $j > 0$. From this isomorphisms, we have $S_A \in \mathcal{C}(T(A)_A) \cap \mathcal{PC}(T(A)_A) = add(T(A)_A)$. ∎

Corollary 3.8 (GNC) *For any algebra A, if $id_A A < \infty$ holds, then all indecomposable injective modules appear in the minimal injective resolution of A_A as direct summands of some terms.*

Remark 2 *Suppose $T_A = T(A)_A \oplus S_A$ and*

$$\cdots \to T_1' \to T_0' \to S_A \to 0$$

be a right dominant $T(A)_A$-resolution of S_A with $S_i = Im(T_i' \to T_{i-1}')$. Then it is not hard to see that all the modules $T(A)_A \oplus S_i$ are tilting modules for $i > 0$.

Remark 3 *For a tilting module ${}_BT_A$, by using idempotent elements $f \in B$ and $e \in A$, we can denote as $T(A) = f \cdot T$ and $T(B) = T \cdot e$.*

In this case, it is checked that ${}_BBf_{fBf}$, ${}_{fBf}fTe_{eAe}$ and ${}_{eAe}eA_A$ are tilting modules and

$$T(B) = {}_B Te_{eAe} \overset{\approx}{\leftarrow} {}_B Bf \otimes_{fBf} fTe_{eAe}$$

$$T(A) = {}_{fBf} fT_A \overset{\approx}{\leftarrow} {}_{fBf} fTe \otimes_{eAe} eA_A$$

$${}_BT_A \overset{\approx}{\leftarrow} {}_B Bf \otimes_{fBf} fTe \otimes_{eAe} eA_A,$$

canonically.

Using this description of tilting bimodules, we can consider relationship of foundamental properties of tilting modules and GNC easily.

4 *Selfinjective Algebras*

In this section, we consider the structure of selfinjective algebras over an algebraically closed field K. Let us start by recalling the definitions of some classes of selfinjective algebras. See Curtis-Reiner[15] for details.

An algebra Λ is said to be selfinjective if Λ_Λ is injective. It is well-known that Λ_Λ is injective if and only if ${}_\Lambda\Lambda$ is injective. In this paper, we call selfinjective algebras QF-algebras (Quasi-Frobenius algebras) for short.

If there exists an isomorphism of right modules

$$\Lambda_\Lambda \xrightarrow{\theta} D(\Lambda)_\Lambda,$$

Λ is called a Frobenius algebra. If Λ is Frobenius, by applying the duality functor D, we have an isomorphism of left modules

$${}_\Lambda\Lambda \xrightarrow{\pi_\Lambda}{}_\Lambda\, D^2(\Lambda) \xrightarrow{D(\theta)}{}_\Lambda\, D(\Lambda),$$

where π stands for the canonical natural transformation $id \to D^2$.

It is obvious that any Frobenius algebra is QF. Conversely, if the algebra is basic, QF means Frobenius.

Generally, to give an isomorphism θ is equivalent to giving a non-degenerate K-bilinear map

$$\psi_\theta : \ \Lambda \approx \Lambda \otimes_\Lambda \Lambda \to K \ (\lambda_1 \otimes \lambda_2 \mapsto \theta(\lambda_1)(\lambda_2)).$$

It is checked that the non-degenerate bilinear map ψ_θ is symmetric, i.e., $\psi_\theta(\lambda_1 \otimes \lambda_2) = \psi_\theta(\lambda_2 \otimes \lambda_1)$, if and only if $\theta = D(\theta) \cdot \pi_\Lambda$. Further,the condition $\theta = D(\theta) \cdot \pi_\Lambda$ is eqivalent to saying that θ is Λ-bilinear.

A Frobenius algebra is said to be symmetric if there is an isomorphism θ which satisfies the above condition.

Since $End(\Lambda_\Lambda) = End(D(\Lambda)_\Lambda) = \Lambda$, for a Frobenius algebra Λ with an isomorphism $\Lambda_\Lambda \xrightarrow{\theta} D(\Lambda)_\Lambda$, there is an algebra automorphism ν_θ of Λ such that

$${}_\Lambda\Lambda_\Lambda \xrightarrow{\theta} {}_{\nu_\theta}D(\Lambda)_\Lambda,$$

i.e, $\theta(\lambda_1 \cdot \lambda_2) = \nu_\theta(\lambda_1) \cdot \theta(\lambda_2)$. This is called a Nakayama automorphism of Λ and Λ is symmetric if and only if ν_θ is the identity map for some θ.

We consider next a general construction of algebras.

Let A be an algebra and ${}_AN_A$ a bimodule. Any bilinear map ϕ from ${}_AN \otimes_A N_A$ to ${}_AN_A$ is said to be associative if

$$\phi(n_1 \otimes \phi(n_2 \otimes n_3)) = \phi(\phi(n_1 \otimes n_2) \otimes n_3) \text{ for all } n_1,\, n_2,\, n_3 \in N.$$

From any associative map ϕ, we can construct an algebra $\Lambda(\phi) = A \oplus N$ by defining its multiplication as

$$(a_1, n_1) \cdot (a_2, n_2) = (a_1 a_2, a_1 n_2 + n_1 a_2 + \phi(n_1 \otimes n_2)).$$

In the algebra $\Lambda(\phi)$, N is always an ideal. We call the associative map ϕ nilpotent if N is nilpotent in $\Lambda(\phi)$. In this case, we call the map ϕ a nilpotent algebra.

When the underlying module ${}_AN_A$ is decomposed as ${}_AM_A \oplus {}_AS_A$ and the map ϕ is given by

$$\phi((m_1, s_1) \otimes (m_2, s_2)) = (\varphi(m_1 \otimes m_2), \psi(m_1 \otimes m_2))$$

for ${}_AM \otimes_A M_A \xrightarrow{\varphi} {}_AM_A$ and ${}_AM \otimes_A M_A \xrightarrow{\psi} {}_AS_A$, we write as $\phi = (\varphi, \psi)$ and $\Lambda(\phi) = \Lambda(\varphi, \psi)$.

The pair (φ, ψ) is a nilpotent algebra if and only if φ is nilpotent algebra and ψ satisfies $\psi(m_1 \otimes \varphi(m_2 \otimes m_3)) = \psi(\varphi(m_1 \otimes m_2) \otimes m_3)$ for $m_1, m_2, m_3 \in M$.

In the algebra $\Lambda(\varphi, \psi) = A \oplus M \oplus S$, the multiplication is given by

$$(a_1, m_1, s_1) \cdot (a_2, m_2, s_2) =$$

$$(a_1 a_2, a_1 m_2 + m_1 a_2 + \varphi(m_1 \otimes m_2), a_1 s_2 + s_1 a_2 + \psi(m_1 \otimes m_2))$$

and its Jacobson radical is denoted as

$$J(\Lambda(\varphi, \psi)) = J(A) \oplus M \oplus S.$$

In this algebra, the ideal S satisfies $S^2 = 0$.

For a progenerator ${}_BP_A$, we put $P^* = {}_AHom_A(P, A)_B$. From any nilpotent algebra $({}_AN_A, \phi)$, we can define a nilpotent algebra over B as follows:

Set ${}_BN^*_B = {}_BP \otimes_A N \otimes_A P^*_B$ and define the map ${}_BN^* \otimes_B N^*_B \xrightarrow{\phi^*}{}_B N^*_B$ by

$$\phi^*((p_1 \otimes n_1 \otimes f_1) \otimes (p_2 \otimes n_2 \otimes f_2)) = p_1 \otimes \phi(n_1 \otimes f_1(p_2) \cdot n_2) \otimes f_2,$$

where $p_1, p_2 \in P$, $n_1, n_2 \in N$ and $f_1, f_2 \in P^* = Hom_A(P, A)$.

The following result is checked easily.

Proposition 4.1 *The isomorphism $\Lambda(\phi^*) \approx End(P \otimes_A \Lambda(\phi)_{\Lambda(\phi)})$ holds generally.*

Similarly, for a nilpotent algebra $\phi = (\varphi, \psi)$, we can define $\phi^* = (\varphi^*, \psi^*)$ and get the isomorphism $\Lambda(\varphi^*, \psi^*) \approx End(P \otimes_A \Lambda(\varphi, \psi)_{\Lambda(\varphi,\psi)})$.

We call a nilpotent algebra (φ, ψ) a nilpotent QF-algebra if the following two coditions are satisfied:

1. The module ${}_AS_A$ is an injective cogenerator, i.e., both ${}_AS$ and S_A are injective cogenerators and $A = End(S_A)$, $End({}_AS) = A$.

2. The map ψ is non-degenerate, i.e., each condition $\psi(m \otimes M) = 0$ and $\psi(M \otimes m) = 0$ implies $m = 0$ for $m \in M$.

Remark 4 *Fixing a progenerator ${}_BP_A$ with B being the basic algebra of A, it is checked that there is a one to one correspondence between the set of all injective cogenerators ${}_AS_A$ and $Aut_K(B)/InnAut_K(B)$ the factor group of the group of all K-algebra automorphisms of B by the subgroup of inner automorphisms. For an automorphism $\tau \in Aut_K(B)$, the corresponding injective cogenerator is defined as ${}_AD(P_\tau^* \otimes_B P)_A$.*

Remark 5 *Since there are isomorphisms*

$$Hom_{A-A}(M, Hom_{mod\,A}(M,S)) \approx Hom_{A-A}(M \otimes_A M, S) \approx \\ Hom_{A-A}(M, Hom_{A\,mod}(M,S)),$$

denoting by $\ell_\psi \leftrightarrow \psi \leftrightarrow r_\psi$ the corresponding maps$(\ell_\psi(m_1)(m_2) = \psi(m_1 \otimes m_2) = r_\psi(m_2)(m_1))$, we have the relation $r_\psi = Hom(\ell_\psi, S) \cdot \pi_M^S$, where π^S denotes the canonical natural transformation $id \to (Hom_A(?, S))^2$, and we know that ℓ_ψ is bijective if and only if so is r_ψ and those conditions are equivalent to the condition of ψ being non-degenerate.

We call a nilpotent QF-algebra (φ, ψ) Frobenius or symmetric if ${}_AS_A \approx {}_\tau D(A)_A$ for some $\tau \in Aut_K(A)$ or ${}_AS_A \approx {}_AD(A)_A$ and $\psi(m_1 \otimes m_2)(1) = \psi(m_2 \otimes m_1)(1)$ for m_1, $m_2 \in M$, respectively.

Remark 6 *Let ${}_BP_A$ be a progenerator. From a nilpotent algebra (φ, ψ) with the undelying module ${}_AM_A \oplus_A S_A$, we constructed a nilpotent algebra (φ^*, ψ^*) over B with the underlying module ${}_BM_B^* \oplus {}_BS_B^*$, before. Though (φ^*, ψ^*) is not Frobenius for a nilpotent Frobenius algebra (φ, ψ) generally, it is symmetric if so is (φ, ψ).*

Remark 7 *For a nilpotent Frobenius algebra (φ, ψ) with underlying module ${}_AM_A \oplus {}_\tau D(A)_A$, we can define the isomorphisms*

$${}_AM_A \xrightarrow{\chi_\psi} {}_\tau D(M)_A$$

and

$$_A M_A \overset{\chi'_\psi}{\to} {}_A D(M)_{\tau^{-1}}$$

by $\psi(m_1 \otimes m_2)(1) = \chi_\psi(m_1)(m_2) = \chi'_\psi(m_2)(m_1)$. *In stead of* (φ, ψ), *we consider* (φ, χ_ψ) *as a nilpotent Frobenius algebra sometimes.*

Theorem 4.2 *Suppose a nilpotent algebra* (φ, ψ) *is QF. Then* $\Lambda(\varphi, \psi)$ *is a QF-algebra.*

Proof. Let us put $\Lambda = \Lambda(\varphi, \psi)$. Denote by $\tilde{\psi}$ the map

$$_A\Lambda \otimes_\Lambda \Lambda_A \overset{can.}{\to} {}_A \Lambda_A \overset{\kappa}{\to} {}_A S_A,$$

where the map κ is the projection.

Obviously, the map $\tilde{\psi}$ is non-degenerate and we have the isomorphisms $_A\Lambda_\Lambda \overset{\ell_{\tilde{\psi}}}{\to} {}_A\, Hom_A(\Lambda, S)_\Lambda$ and $_\Lambda\Lambda_A \overset{r_{\tilde{\psi}}}{\to} {}_\Lambda\, Hom_A(\Lambda, S)_A$.

For any right Λ-module X_Λ, the map $\ell_{\tilde{\psi}}$ induces an isomorphism

$$_A Hom_\Lambda(X, \Lambda) \overset{\alpha_X}{\to} {}_A Hom_A(X, S)$$

which is defined by $\alpha_X(f) = \kappa \cdot f$ for $X_\Lambda \overset{f}{\to} \Lambda_\Lambda$.

Similarly, for any left Λ-module $_\Lambda Y$, the isomorphism $Hom_\Lambda(Y, \Lambda)_A \overset{\beta_Y}{\to} Hom_A(Y, S)_A$ is defined by using $r_{\tilde{\psi}}$.

We have a relation

$$\beta_{Hom(X,\Lambda)} \cdot \pi_X^\Lambda = Hom(\alpha_X, S) \cdot \pi_X^S$$

for any module X_Λ.

Since the maps $\beta_{Hom(X,\Lambda)}$, π_X^S and $Hom(\alpha_X, X)$ are bijective, so is the map

$$X_\Lambda \overset{\pi_X^\Lambda}{\to} Hom_\Lambda(Hom_\Lambda(X, \Lambda), \Lambda)_\Lambda.$$

Therefore, $_\Lambda\Lambda_\Lambda$ is an injective cogenerator. ∎

Theorem 4.3 *The algebra* $\Lambda(\varphi, \psi)$ *is Frobenius(resp. symmetric) whenever nilpotent algebra* (φ, ψ) *with underlying module* $_A M_A \oplus {}_\tau D(A)_A$ *is Frobenius(resp. symmetric).*

Proof. Assume that (φ, ψ) is Frobenius. Define the map

$$\Lambda \xrightarrow{\tilde{\chi}} D(\Lambda)$$

as $\tilde{\chi}\,((a_1, m_1, f_1))(a_2, m_2, f_2) = f_1(a_2) + \chi_\psi(m_1)(m_2) + f_2(\tau(a_1))$. It is easy to see that this is bijective and a right Λ-map, where $\Lambda = \Lambda(\varphi, \psi)$.

In the case (φ, ψ) is symmetric, since $\chi_\psi = \chi'_\psi$, $\tilde{\chi}$ is a left Λ-map also. ∎

Theorem 4.4 *Let Λ be a QF-algebra without semi-simple part. Then, there is an isomorphism*

$$\Lambda \approx \Lambda(\varphi, \psi)$$

for some nilpotent QF-algebra (φ, ψ).

Proof. By proposition 4.1, we may suppose that Λ is indecomposable and basic.

By our assumption, there is a subalgebra A in Λ which is isomorphic to a direct product of the field K and Λ is decomposed as an A-module

$$\Lambda = A \oplus J(\Lambda),$$

where $J(*)$ means Jacobson radical. Since, Λ is not semi-simple, its socle S is included in $J(\Lambda)$ and, as an A-module, $J(\Lambda)$ is also decomposed as

$$J(\Lambda) = M \oplus S$$

for some A-submodule M. So, we have a decomposition

$$\Lambda = A \oplus M \oplus S.$$

The multiplication in Λ is given by

$$(a_1, m_1, s_1) \cdot (a_2, m_2, s_2) = (a_1 a_2, a_1 m_2 + m_1 a_2, a_1 s_2 + s_1 a_2) + m_1 \cdot m_2.$$

Since $M^2 \subseteq J(\Lambda) = M \oplus S$, $m_1 \cdot m_2$ can be written as

$$m_1 \cdot m_2 = (0, \varphi(m_1 \otimes m_2), \psi(m_1 \otimes m_2))$$

for some maps ${}_AM \otimes_A M_A \xrightarrow{\varphi} {}_AM_A$ and ${}_AM \otimes_A M_A \xrightarrow{\psi} {}_AS_A$.

It is easy to see that (φ, ψ) is a nilpotent QF-algebra over A. ∎

The following result is obtained by routine work.

Lemma 4.5 *Let (φ,ψ) be a nilpotent algebra with underlying module ${}_AM_A \oplus {}_AS_A$ and put $\Lambda = \Lambda(\varphi,\psi)$.*

Then any map χ from $\Lambda_\Lambda = (A \oplus M \oplus S)_\Lambda$ to $D(\Lambda)_\Lambda = (D(S) \oplus D(M) \oplus D(A))_\Lambda$ has a matrix expression

$$\chi = \begin{pmatrix} \chi_A & 0 & 0 \\ \alpha & \chi_M & 0 \\ \epsilon & \beta & \chi_S \end{pmatrix}$$

Those maps satisfy the following conditions:

1. *All maps are right A-maps.*
2. *$\beta(m)(a) = \alpha(1)(ma)$ for $a \epsilon A$ and $m \epsilon M$.*
3. *$\chi_S(s)(a) = \chi_A(1)(sa)$ for $a \epsilon A$ and $s \epsilon S$.*
4. *$\chi_M(m_1)(m_2) = \alpha(1)(\varphi(m_1 \otimes m_2)) + \chi_A(1)(\psi(m_1 \otimes m_2))$ for m_1, $m_2 \epsilon M$.*

Conversely, any map χ which has the above matrix expression with the components satisfying the above conditions is a right Λ-map.

Proposition 4.6 *Let (φ,ψ) be a nilpotent QF-algebra. Then the algebra $\Lambda(\varphi,\psi)$ is Frobenius if and only if so is (φ,ψ).*

Proof. If part. This is already proved in theorem 4.3.

Only if part. Let us put $\Lambda = \Lambda(\varphi,\psi)$. We will use the notation in the above lemma.

It is easy to see $Ann_\Lambda(M \oplus S) = S$, $Ann_{D(\Lambda)}(M \oplus S) = D(A)$, $Ann_\Lambda(S) = M \oplus S$ and $Ann_{D(\Lambda)}(S) = D(M) \oplus D(A)$. From those relations, we know that χ is bijective if and only if all the maps χ_A,χ_M and χ_S are bijective. The isomorphism $S_A \overset{\chi_S}{\rightarrow} D(A)_A$ becomes an A-bimodule isomorphism ${}_AS_A \rightarrow {}_\tau D(A)_A$ for some $\tau \,\epsilon\, Aut_K(A)$, because $A = End(D(A)_A)$. ∎

Theorem 4.7 *Let Λ be a Frobenius algebra without semi-simple part. Then, there is an isomorphism*

$$\Lambda \approx \Lambda(\varphi,\psi)$$

for some nilpotent Frobenius algebra (φ,ψ).

Proof. This follows from Theorem 4.4 and Proposition 4.6. ∎

Theorem 4.8 *Let Λ be a symmetric algebra without semi-simple part. Then there is an isomorphism*

$$\Lambda \approx \Lambda(\varphi, \psi)$$

for some nilpotent symmetric algebra (φ, ψ).

Proof. The proof is similar to that of Theorem 4.4. We may suppose that Λ is indecomposable and basic. We will use the notation of Lemma 4.5.

Let A be a semi-simple subalgebra of Λ which is isomorphic to the factor algebra $\Lambda/J(\Lambda)$ and $\Lambda = A \oplus J(\Lambda)$. Put $S = Soc(\Lambda)$ and $J(\Lambda) = M \oplus S$. Since Λ is symmetric, we have ${}_AS_A \approx {}_AD(A)_A$.

By Lemma 4.5, any isomorphism

$${}_\Lambda\Lambda_\Lambda \xrightarrow{\chi} {}_\Lambda D(\Lambda)_\Lambda$$

is written as

$$\chi = \begin{pmatrix} \chi_A & 0 & 0 \\ \alpha & \chi_M & 0 \\ \epsilon & \beta & \chi_S \end{pmatrix}$$

Since χ is a Λ-bimodule map, by Lemma 4.5, we have the following relations:

1. $\beta(m)(a) = \alpha(1)(ma) = \alpha(1)(am)$.

2. $\chi_S(s)(a) = \chi_A(1)(sa) = \chi_A(1)(as)$.

3. $\chi_M(m_1)(m2) = \alpha(1)(\varphi(m_1 \otimes m_2)) + \chi_A(\psi(m_1 \otimes m_2)) = \alpha(1)(\varphi(m_2 \otimes m_1)) + \chi_A(\psi(m_2 \otimes m_1)) = \chi_M(m_2)(m_1)$.

4. χ_A, χ_M, χ_S *are bijective.*

We denote by ψ^* the composition

$${}_AM \otimes_A M_A \xrightarrow{\psi} {}_AS_A = {}_AD(A)_A \xrightarrow{\chi_S} {}_AD(A)_A.$$

It is obvious that (φ, ψ^*) is also a nilpotent QF-algebra, further, by the above relations, (φ, ψ^*) is symmetric actually. On the other hand, we have an algebra isomorphism from $\Lambda = \Lambda(\varphi, \psi)$ to $\Lambda(\varphi, \psi^*)$ defined by $(a, m, s) \mapsto (a, m, \chi_S(s))$. ∎

5 *Construction of Symmetric Algebras*

In this section, we will show some constructions of symmetric algebras. We would like to mention here that many symmetric algebras including DJK-algebras of Brauer quivers can be reconstructed by our way. We will use one of our constructions to define some stably equivalent functors, in the next section.

Once we get a symmetric algebra by our way, there is a general method to construct non-symmetric Frobenius algebras from the given algebra. We will explain this at the end of the section.

We start from a trivial construction. Let (φ_i, ψ_i) be nilpotent algebras over A with underlying modules ${}_A(M_i)_A \oplus {}_AD(A)_A$ for $i = 1, 2$. Denote by ${}_AM_A$, the direct sum ${}_A(M_1 \oplus M_2)_A$. We can define the maps $\varphi_1 \oplus \varphi_2 : {}_AM \otimes_A M_A \to {}_AM_A$ and $\psi_1 \oplus \psi_2 : {}_AM \otimes_A M_A \to {}_AD(A)_A$ by

$$(\varphi_1 \oplus \varphi_2)((m_1, m_2) \otimes (m_1', m_2')) = (\varphi_1(m_1 \otimes m_1'), \varphi_2(m_2 \otimes m_2'))$$

$$(\psi_1 \oplus \psi_2)((m_1, m_2) \otimes (m_1', m_2')) = \psi_1(m_1 \otimes m_1') + \psi_2(m_2 \otimes m_2')$$

for m_1, $m_1' \in M_1$ and m_2, $m_2' \in M_2$.

It is easy to prove that $(\varphi_1 \oplus \varphi_2, \psi_1 \oplus \psi_2)$ is again a nilpotent algebra with underlying module ${}_AM_A \oplus {}_AD(A)_A$. We denote this by $(\varphi_1, \psi_1) \oplus (\varphi_2, \psi_2)$.

Proposition 5.1 *For nilpotent symmetric algebras (φ_1, ψ_1) and (φ_2, ψ_2), their direct sum $(\varphi_1, \psi_1) \oplus (\varphi_2, \psi_2)$ is again a nilpotent symmetric algebra.*

Next, we show a construction which starts from any nilpotent ideal of A. We describe this in a more general fashion.

Let ${}_AI \otimes_A I_A \xrightarrow{\phi} {}_AI_A$ be a nilpotent algebra. When we consider an ideal I in A, the map ϕ is just the map induced from the multiplication in A. Then, over the module ${}_A(I \oplus D(I))_A \oplus {}_AD(A)_A$, we can define a nilpotent algebra $(\varphi_\phi, \psi_\phi)$ by

$$\varphi_\phi((a_1, f_1) \otimes (a_2, f_2)) = (\phi(a_1 \otimes a_2), f_1(\phi(a_2 \otimes ?) + f_2(\phi(? \otimes a_1))$$

and

$$\psi_\phi((a_1, f_1) \otimes (a_2, f_2)) = f_1(?a_2) + f_2(a_1?).$$

Proposition 5.2 *For a nilpotent algebra ϕ, $(\varphi_\phi, \psi_\phi)$ becomes a nilpotent symmetric algebra.*

Generally, for a given symmetric algebra Λ, the decomposition $\Lambda = A \oplus M \oplus D(A)$ is not uniquely determined. In some cases, we need a subalgebra A to be bigger. In this sense, the following cunstruction is usefull.

Proposition 5.3 *Let (φ_Y, χ_Y) be a nilpotent symmetric algebra over ${}_AY_A$, where $\chi_Y = \chi_{\psi_Y}$. Put $B = \Lambda(\varphi_Y)$ and suppose (φ_X, χ_X) is a nilpotent symmetric algebra over ${}_BX_B$, where $\chi_X = \chi_{\psi_X}$. Then, over the module ${}_AM_A = {}_AX_A \oplus {}_AY_A$, by defining φ and χ as*

$$\varphi((x,y)\otimes(x',y')) = (xy' + yx' + \varphi_X(x\otimes x'), \varphi_Y(y\otimes y') + \chi_Y^{-1}(\chi_X(x)(x'?)))$$

and

$$\chi((x,y))((x',y')) = \chi_X(x)(x') + \chi_Y(y)(y')$$

we have a nilpotent symmetric algebra (φ, χ).

Starting from a polynomial algebra $K[X_1, X_2, \ldots, X_n]$, we can obtain many symmetric factor algebras Γ easily. In this case, the algebras Γ are local and $V = J(\Gamma)/soc(\Gamma)$ are nilpotent symmetric algebras over our field K. If E is a symmetric algebra over K, by taking the tensor product $V \otimes_K E$, we get a new nilpotent symmetric algebra.

Further, if we have an algebra map $A \xrightarrow{\rho} E$, we can consider $V \otimes_K E$ as a nilpotent symmetric algebra over A. In this way, we obtain a lot of symmetric algebras $\Lambda = A \oplus V \otimes_K E \oplus D(A)$. For example, from any module M_A, by putting $E = End({}_KM)$, we have a full matrix algebra and hence a symmetric algebra and a canonical algebra map $A \xrightarrow{\rho_M} E$ which is called a matrix representation.

It is not so hard to prove that any DJK-algebra corresponding to a Brauer quiver or, more generally, any Brauer quiver algebra can be described in this way.

The next construction is a formalization of the above remark.

Proposition 5.4 *Let (φ_X, χ_X) and (φ_Y, χ_Y) be nilpotent symmetric algebras with underlying modules ${}_AX_A$ and ${}_BY_B$. Then, on the underlying module ${}_{A\otimes B}(X \otimes_K Y)_{A\otimes B}$, we have a nilpotent symmetric algebra $(\varphi_X \otimes \varphi_Y, \chi_X \otimes \chi_Y)$, where $\varphi_X \otimes \varphi_Y$ and $\chi_X \otimes \chi_Y$ are defined as*

$$(\varphi_X \otimes \varphi_Y)((x\otimes y)\otimes(x'\otimes y')) = \varphi_X(x\otimes x') \otimes \varphi_Y(y \otimes y')$$

and

$$(\chi_X \otimes \chi_Y)(x\otimes y)(x'\otimes y') = \chi_X(x)(x')\cdot\chi_Y(y)(y').$$

Finally in this section, we explain how we can get non-symmetric Frobenius algebras.

Let A be an algebra and $\tau \in Aut_K(A)$ a K-automorphism of A. Suppose we have a module X_A and A-isomorphism $X_\tau \xrightarrow{\kappa} X_A$. Let us put $E = D(X) \otimes_K X \approx End({}_K X)$. The multiplication in E is given by

$$(f \otimes x) \cdot (g \otimes y) = g(x) \cdot f \otimes y.$$

Let (φ, ψ) be a nilpotent symmetric algebra with underlying module ${}_K V_K$, where we consider $V \otimes_K V \xrightarrow{\psi} K$.

From those data, we get two algebras:

$$\Lambda = A \ \oplus V \otimes_K E \ \oplus \ {}_A D(A)_A$$

and

$$\Lambda_\kappa = A \ \oplus V \otimes_K E \ \oplus \ {}_\tau D(A)_A.$$

The algebra Λ is a usual one, so this is symmetric. In the algebra Λ_κ, its multiplication is given by

$(a, v \otimes f \otimes x, q) \cdot (b, u \otimes g \otimes y, p) =$

$(ab, u \otimes ag \otimes y + v \otimes f \otimes xb + g(x)\varphi(v \otimes u) \otimes f \otimes y, \tau(a)p + qb + g(x)\psi(v \otimes u)f(\kappa(y?)))$.

It is checked that the algebra Λ_κ is Frobenius.

In the case τ induces a non-identity permutation on the complete set of non-isomorphic primitive idempotents of A, the algebra Λ_κ never be symmetric.

6 *Construction of Stable Equivalences*

In the papers [44,45],[50,51], we constructed stably equivalent functors between trivial extension selfinjective algebras, by making use of tilting modules. In this section, we try to generalize such a construction to a wider class of symmetric algebras. Our construction is very similar to that of trivial extension algebras, but, since our algebra $\Lambda = A \oplus M \oplus D(A)$ has non-zero middle part M generally, our generalization is not trivial.

Let A and B be algebras and ${}_B T_A$ a tilting module. Similarly to the original situation, we assume that our tilting module ${}_B T_A$ satisfies the following conditions:

$$Ker(\mathcal{C}(T_A), \mathcal{PC}(T_A)) = mod\, A$$

and

$$Cok(\mathcal{ID}(D(T)_B), \mathcal{D}(D(T)_B)) = mod\, B.$$

Let ${}_B U_A$ be another module and we assume the following hypotheses, throughout this section.

H1 $Hom_A(U, \mathcal{C}(T_A)) \subseteq add(D(T)_B)$ and $\mathcal{D}(D(T)_B) \otimes {}_BU \subseteq add(T_A)$.

H2 There is a map ${}_BU \otimes_A D(T) \otimes_B U_A \xrightarrow{\varphi} {}_BU_A$ such that;

1. $D(T) \otimes \varphi$ and $\varphi \otimes D(T)$ are nilpotent algebras.
2. $\varphi(\varphi(u \otimes f \otimes v) \otimes g \otimes w) = \varphi(u \otimes f \otimes \varphi(v \otimes g \otimes w))$.

H3 There is a map ${}_BU \otimes_A D(T) \otimes_B U_A \xrightarrow{\psi} {}_BT_A$ such that the maps

$$\chi_A^* = Hom_{B\,mod}(U, \psi) \cdot \eta^U_{D(T)\otimes U}$$

and

$$\chi_B^* = Hom_{mod\,A}(U, \psi) \cdot \eta^U_{U\otimes D(T)}$$

are bijective.

H4
1. $f(\psi(u \otimes g \otimes v)) = g(\psi(v \otimes f \otimes u))$.
2. $\psi(\varphi(u \otimes f \otimes v) \otimes g \otimes w) = \psi(u \otimes f \otimes \varphi(v \otimes g \otimes w))$.

Remark 8 *Defining the maps χ_A and χ_B by*

$${}_AD(T) \otimes_B U_A \xrightarrow{\chi_A^*} {}_AHom_B(U,T)_A \approx {}_AD(D(T) \otimes_B U)_A$$

and

$${}_BU \otimes_A D(T)_B \xrightarrow{\chi_B^*} {}_BHom_A(U,T)_B \approx {}_BD(U \otimes_A D(T))_B,$$

it is checked that χ_A and χ_B correspond to the non-degenerate maps $D(T)\otimes\psi$ and $\psi \otimes D(T)$, respectively.

Here, we have used the fact that ${}_AD(A)_A \approx {}_AD(T)\otimes_B T_A$ and ${}_BD(B)_B \approx {}_BT \otimes_A D(T)_B$. This is equivalent to saying that the module ${}_BT_A$ is faithfully balanced,i.e., $B = End(T_A)$ and $End({}_BT) = A$.

Hypotheses imply the fact that

$$(D(T) \otimes \varphi, D(T) \otimes \psi)$$

and

$$(\varphi \otimes D(T), \psi \otimes D(T))$$

are nilpotent symmetric algebras with the underlying modules

$${}_AD(T) \otimes_B U_A \oplus {}_AD(A)_A$$

and

$${}_BU \otimes_A D(T)_B \oplus {}_BD(B)_B.$$

Therefore, we have two symmetric algebras $\Lambda = \Lambda(D(T) \otimes \varphi, D(T) \otimes \psi)$ and $\Gamma = \Lambda(\varphi \otimes D(T), \psi \otimes D(T))$.

Our purpose in this section is to construct stable functors between the stable module categories of Λ and Γ.

Let us start by considering the structure of right Λ-modules.

Since, A is a subalgebra of Λ, any Λ-module X_Λ can be seen as an A-module. We call this module X_A the underlying module. The operations of $D(T)\otimes U$ and $D(A)\approx D(T)\otimes T$ to X_Λ define the maps $X\otimes {}_AD(T)\otimes {}_BU_A \overset{\alpha_X}{\to} X_A$ and $X\otimes {}_AD(T)\otimes {}_BT_A \overset{\beta_X}{\to} X_A$. Those maps satisfy the following conditions:

1. $\beta_X\cdot\beta_X\otimes D(T)\otimes T=0$.
2. $\alpha_X\cdot\beta_X\otimes D(T)\otimes U=0$.
3. $\beta_X\cdot\alpha_X\otimes D(T)\otimes T=0$.
4. $\alpha_X\cdot\alpha_X\otimes D(T)\otimes U=\alpha_X\cdot X\otimes D(T)\otimes\varphi+\beta_X\cdot X\otimes D(T)\otimes\psi$.

Conversely, if the triple (X_A,α_X,β_X) satisfies the above conditions, X has a unique Λ-module structure. So, we may consider Λ-modules as triples (X_A,α_X,β_X) satisfying the above conditions.

We have the similar description for right Γ-modules.

Lemma 6.1 *There is an isomorphism $\Lambda\otimes_A D(T)\approx D(T)\otimes_B\Gamma$ of additive groups. By using this map, we can identify $\Lambda\otimes_A D(T)$ with $D(T)\otimes_B\Gamma$. Then, this define a Λ-Γ bimodule.*

Proof. Routine verification. ▮

We denote this module by ${}_\Lambda\Theta_\Gamma$ and consider the adjoint pair of $(?\otimes_\Lambda\Theta_\Gamma)$ and $Hom_\Gamma(\Theta,?)$.

The next one is also proved by routine work.

Lemma 6.2 *For a module $X_\Lambda=(X_A,\alpha_X,\beta_X)$, we have*

$$X\otimes_\Lambda\Theta_\Gamma=(X\otimes D(T),\alpha_X\otimes D(T),\beta_X\otimes D(T)).$$

From $X_\Lambda=(X_A,\alpha,\beta)$, we obtain two Γ-modules

$$X\otimes_\Lambda\Theta_\Gamma$$

and

$$Hom_A(T,V(X))\otimes_B\Gamma_\Gamma,$$

where
$$0 \to X \overset{u_X}{\to} V(X) \to W(X) \to 0$$
is the minimal kernel expression of X_A with respect to $(\mathcal{C}(T_A), \mathcal{PC}(T_A))$.

From $X \otimes_\Lambda \Theta_B = X \otimes_A D(T)_B$ to $Hom_A(T, V(X)) \otimes_B \Gamma_B$ which is isomorphic to the direct sum of $Hom_A(T, V(X))_B$, $Hom_A(T, V(X)) \otimes_B U \otimes_A D(T)_B$ and $Hom_A(T, V(X)) \otimes_B T \otimes_A D(T)_B$, we can define a B-morphism $\ell_{(X:u_X)}$ by the following matrix:

$$\begin{pmatrix} Hom(T, u_X \cdot \beta) \cdot \eta^T_{X \otimes D(T)} \\ (c^T_{U,V(X)} \cdot Hom(T, V(X)) \otimes \chi^*_B)^{-1} \cdot Hom(U, u_X \cdot \alpha) \cdot \eta^U_{X \otimes D(T)} \\ (\epsilon^T_{V(X)} \otimes D(T))^{-1} \cdot u_X \otimes D(T) \end{pmatrix},$$

where $c^T_{U,V(X)}$ means the composition map in the category $mod\, A$. It is easy to see that $\ell_{(X:u_X)}$ is injective. We denote the cokernel of $\ell_{(X:u_X)}$ by
$$\mathcal{S}(X : u_X).$$

Lemma 6.3 *The map $\ell_{(X:u_X)}$ is actually a Γ-morphism.*

Proof. Routine verification. ∎

For a morphism f from $X_\Lambda = (X_A, \alpha_X, \beta_X)$ to $Y_\Lambda = (Y_A, \alpha_Y, \beta_X)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & X_A & \xrightarrow{u_X} & V(X)_A & \xrightarrow{p_X} & W(X)_A & \to & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \to & Y_A & \xrightarrow{u_Y} & V(Y)_A & \xrightarrow{p_Y} & W(Y)_A & \to & 0 \end{array}$$

Then using those maps g and h, we get Γ-modules $\mathcal{S}(X : u_X)$,$\mathcal{S}(Y : u_Y)$ and a Γ -morphism $\mathcal{S}(f : g)$ by the following diagram:

$$\begin{array}{ccccc} X \otimes_\Lambda \Theta_\Gamma & \xrightarrow{\ell_{(X:u_X)}} & Hom_A(T, V(X)) \otimes_B \Gamma_\Gamma & \to & \mathcal{S}(X : u_X)_\Gamma \\ {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle f^*}\downarrow \\ Y \otimes_\Lambda \Theta_\Gamma & \xrightarrow{\ell_{(Y:u_Y)}} & Hom_A(T, V(Y)) \otimes_B \Gamma_\Gamma & \to & \mathcal{S}(Y : u_Y)_\Gamma \end{array}$$

where $f' = f \otimes \Theta$,$g' = Hom(T, g) \otimes \Gamma$ and $f^* = \mathcal{S}(f : g)$.

Generally, though the morphism g is not uniquely determined from f, but, in the similar way of the case for trivial extension selfinjective algebras, it is proved that the morphism $\underline{\mathcal{S}(f:g)}$ in the stable category $\underline{mod}\,\Gamma$ is uniquely determined.

So we have a functor $\mathcal{S} : mod\,\Lambda \rightarrow \underline{mod}\,\Gamma$, i.e.,$\mathcal{S}(X) = \underline{\mathcal{S}(X : u_X)}$ and $\mathcal{S}(f) = \underline{\mathcal{S}(f:g)}$.

Further, it is proved that the module $\mathcal{S}(\Lambda : u_\Lambda)_\Gamma$ is projective. Hence, we have

Proposition 6.4 *The functor $\mathcal{S} : mod\,\Lambda \rightarrow \underline{mod}\,\Gamma$ induces a stable functor $\underline{mod}\,\Lambda \rightarrow \underline{mod}\,\Gamma$.*

We denote the induced stable functor by $\mathcal{S}$ also. By left-right symmetry, we have a stable functor $\mathcal{S}'$ from $\Gamma\,\underline{mod}$ to $\Lambda\,\underline{mod}$. Let us put

$$\mathcal{Q} = D \cdot \mathcal{S}' \cdot D : \underline{mod}\,\Gamma \rightarrow \underline{mod}\,\Lambda.$$

Now, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \rightarrow & X^* & \xrightarrow{u_X^*} & P & \rightarrow & W(X) & \rightarrow & 0 \\ & & {\scriptstyle q^*}\downarrow & & {\scriptstyle q}\downarrow & & \| & & \\ 0 & \rightarrow & X & \xrightarrow{u_X} & V(X) & \rightarrow & W(X) & \rightarrow & 0 \end{array}$$

where q is the projective cover of $V(X)_A$.

By using above maps, it is proved that there is an epimorphism λ from the direct sum L of $Hom_A(T, V(X))_B$, $Hom_A(U, V(X))_B$ and $P \otimes_A D(T)_B$ to our module $\mathcal{S}(X : u_X)_B$ with the kernel $X^* \otimes_A D(T)_B$:

$$0 \rightarrow X^* \otimes_A D(T)_B \xrightarrow{\mu} L_B \xrightarrow{\lambda} \mathcal{S}(X : u_X)_B \rightarrow 0$$

The map μ has the following components μ_1, μ_2 and μ_3:

$$\mu_1 = Hom(T, u_X) \cdot \epsilon^{D(T)}_{Hom(T,X)} \cdot \beta_X \otimes D(T) \cdot q^* \otimes D(T)$$

$$\mu_2 = Hom(U, u_X) \cdot \epsilon^{D(T)}_{Hom(U,X)} \cdot \alpha_X \otimes D(T) \cdot q^* \otimes D(T)$$

$$\mu_3 = u_X^* \otimes D(T)$$

By hypothesis H1, $Hom_A(U, V(X))_B$ is a module in the class $add(D(T)_B)$, and $X^* \otimes_A D(T)_B$ is in $\mathcal{ID}(D(T)_B)$, so we have $L_B \epsilon \mathcal{D}(D(T)_B)$ and $X^* \otimes_A D(T)_B \epsilon \mathcal{ID}(D(T)_B)$.

Therefore, in this way, we get a cokernel expression of $\mathcal{S}(X : u_X)_B$ with respect to $(\mathcal{ID}(D(T)_B), \mathcal{D}(D(T)_B))$.

Using the expression λ, routine verification shows

$$\mathcal{Q}(\mathcal{S}(X : u_X) : \lambda) \approx P \otimes_A \Lambda \oplus Hom_B(D(T), Hom_A(U, V(X))) \otimes_A \Lambda \oplus X$$

as Λ-modules. Hence, we have the following

Theorem 6.5 *The functors $\mathcal{S}$ and $\mathcal{Q}$ induce a stable equivalence*

$$\underline{mod}\,\Lambda \approx \underline{mod}\,\Gamma$$

Though our situation in this section looks very special, some familiar symmetric algebras such as DJK-algebras corresponding to Brauer quiver with one exceptional cycle can be described. But, it is clear that ours are not enough to study some well known classes of symmetric algebras. So it is desired to develope more general theory.

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