ORTHOGONAL RATIONAL TWISTED GENERALIZED WEYL ALGEBRA (RTGWA)

Abstract. This paper is concerned with explicitly producing generating sets and defining relations for the rational TGW algebra and with the foundation of a constructive homomorphism from the universal enveloping of complex orthogonal Lie algebra to the invariant subalgebra of constructed one over the commutative invariant subalgebra which is realized as rational extension of Gelfand-Tsetlin subalgebra.

Contents

1. Introduction

We follow the Gelfand-Tsetlin formal constructions concerning the orthogonal Lie algebra ([GZ1]; [GZ2]; [Zh]; [BR], Section 10.1.B, p. 286), the construction of TGWA ([Ba], [FuHa2]), the construction of skew monoid rings ([FO-GO]), Gelfand-Tsetlin bases for orthogonal Lie algebra ([Mol]), the description of orthogonal Lie algebras [Maz-ort]. In this paper we used the ideas of introducing involutions on generalized Weyl algebras from the papers [MT], and of rational Galois order [Hart-R]. and so on so on so on so on so on

In this paper we construct an algebra $\mathcal{A}$ called rational twisted generalized Weyl algebra, we investigate some of its properties, in particular, we consider an invariant subalgebra $\mathcal{A}^G$, and build a homomorphism homomorphism from the universal enveloping of complex orthogonal Lie algebra $\mathrm{U}(\mathcal{O}_n)$ to the invariant subalgebra $\mathcal{A}^G$, Theorem 1.

2. Notation and preliminaries

The ground field will be $\mathbb{C}$ denoted as $\Bbbk$, $\mathtt{i}$ denotes the imaginary unit, $*$ or $^-$ denotes the complex conjugation. We denote by $\mathbb{N}$ the set of all natural (non negative) integers, and let $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\}$. The set of integers k with $a \leqslant k \leqslant b$ (resp., with $a \leqslant k < b$) is denoted by $[a;b]$ (resp., by $[a;b)$).

Associative rings and algebras are understood to be over $\Bbbk$, to be unital, and homomorphisms of rings and algebras are unital.

2.1. **Orthogonal Lie algebra.** We first review some notations and definitions related to orthogonal (complex) Lie algebra. Throughout the paper we fix an integer $n > 1$. For any $m \in \mathbb{N}$, $m > 1$, we denote by $\mathcal{O}_m = \mathcal{O}(m, \Bbbk)$ the Lie algebra of skew symmetric $m \times m$-matrices over $\Bbbk$ and call it the *orthogonal Lie algebra*. It is a matrix algebra realized as the alternating matrices $\mathcal{O}_m = \{X \in \mathrm{Mat}_{m\times m}(\Bbbk) : {}^tX + X = 0\}$. We denote by $e_{ij} \in \mathrm{Mat}_{m\times m}(\Bbbk)$ the matrix unit whose i, j-component is one and all others are zero. The matrices $E_{ji} = e_{ji} - e_{ij}$, $1 \leq i < j \leq m$ form a standard generators of $\mathcal{O}_m$, and the set of matrices $E_{j+1j} = e_{j+1j} - e_{j\,j+1}$, $j = 1, \ldots, m-1$ is a minimal system of generators. (See definition 2 in Appendix.) The standard involution $* : \mathcal{O}_m \to \mathcal{O}_m$ provides $E_{ji} = -E_{ij}$.

The *universal enveloping algebra* $\mathrm{U}_m = \mathrm{U}(\mathcal{O}_m)$ is the associative algebra with generators E_{ij} and the defining relations

$$E_{kl}E_{rt} - E_{rt}E_{kl} = \kappa_{kr}E_{lt} + \kappa_{lt}E_{kr} - \kappa_{kt}E_{lr} - \kappa_{lr}E_{kt}, \tag{1}$$

with κ_{ij} be a Kronecker delta.

We identify $\mathcal{O}_m$ for $m \leqslant n$ with a Lie subalgebra of $\mathcal{O}_n$ spanned by $\{E_{ji} \,|\, 1 \leq i < j \leq m\}$, with E_{ji} to be understanding as $n \times n$-matrices, so we have natural embeddings on the left upper corner

$$\mathcal{O}_1 \subset \mathcal{O}_2 \subset \ldots \subset \mathcal{O}_n \text{ and induced embeddings } \mathrm{U}_1 \subset \mathrm{U}_2 \subset \ldots \subset \mathrm{U}_n.$$

Clearly, $\mathcal{O}_1 = 0$ and $\mathrm{U}_1 = 0$.

Following the Harish-Chandra theory, the Gelfand-Zetlin subalgebra Γ_m can be obtained by the isomorphism from the center $Z_m = Z(\mathrm{U}(\mathcal{O}_m))$ of the universal enveloping algebra $\mathrm{U}(\mathcal{O}_m)$ to the Cartan subalgebra that is invariant under the Weyl group W_m. By agreement, the *Gelfand-Zetlin subalgebra* Γ in U_n is generated by $\{\Gamma_m \,|\, m = 1, \ldots, n\}$, which is a polynomial algebra in $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ variables (see [Maz-ort]).

2.2. **Polynomial ring Λ.**

For any $m \in \mathbb{N}$, let $\mathtt{p}(m) = [\frac{m+1}{2}]$, and

$$N = \sum_{m\in[1;n-1]} \mathtt{p}(m) = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = \begin{cases} p^2, & n = 2p, \\ p^2 - p, & n = 2p-1; \end{cases} \qquad N_1 = N + \mathtt{p}(n). \tag{2}$$

We denote by $\mathfrak{I}_m = [1; \mathtt{p}(m)]$ an interval. We consider the triangular double indexed sets

$$\mathbb{J}_{[1;n]} = \{ki \mid k \in [1;n], i \in \mathfrak{I}_k\}, \qquad \mathbb{J}_{[1;n)} = \{ki \mid k \in [1;n), i \in \mathfrak{I}_k\},$$

then $|\mathbb{J}_{[1;n]}| = N_1$, $|\mathbb{J}_{[1;n)}| = N$. The record $ki, mj \in \mathbb{J}_{[1;n]}$ always means either $k \neq m$ or $i \neq j$.

For any $m \in [1;n]$, we denote by Λ_m a polynomial $\Bbbk$-algebra $\Lambda_m = \Bbbk\Big[\big\{\lambda_{mj}\big\}_{j\in\mathfrak{I}_m}\Big]$ in $\mathtt{p}(m)$ variables, and we denote by Λ a polynomial $\Bbbk$-algebra $\Lambda = \Bbbk\big[\big\{\lambda_{mj}\big\}_{m\in[1;n],j\in\mathfrak{I}_m}\big]$ in N_1 variables.

We use both notation λ_{mj} and $\lambda_{m,j}$ equally, but we use a comma as a rule if the number of symbols in the lower index is greater than two.

For any $m \in \mathbb{N}$, we put $\zeta_m = 0$ if m is odd, and $\zeta_m = 1$ if m is even. So ζ_m is the parity of integer $m+1$ modulo 2. For any λ_{mj}, we introduce the *conjugate* element $\lambda^*_{mj} = \bar{\lambda}_{mj}$ in Λ such that

$$\bar{\lambda}_{mj} = \zeta_m - \lambda_{mj} = \begin{cases} -\lambda_{mj}, & \text{if } m \text{ is odd}, \\ 1 - \lambda_{mj} & \text{if } m \text{ is even}. \end{cases} \tag{3}$$

Extending $*$ to the product of elements, we obtain an involuted automorphism $* \in \mathrm{Aut}_{\Bbbk}(\Lambda)$.

The following obviously hold for $m, k \in [1;n]$ under the condition $|m-k| = 1$:

$$\lambda_{mj} + \bar{\lambda}_{mj} + \lambda_{ki} + \bar{\lambda}_{ki} = 1 \ (= \zeta_m + \zeta_k), \qquad j \in \mathfrak{I}_m,\ i \in \mathfrak{I}_k. \tag{4}$$

2.3. **Group action.**

For any $m \in [1;n]$, we denote by $\mathfrak{S}_{(m)} \simeq S_{\mathtt{p}(m)}$ a symmetric group acting by the permutation of second indexes at the variable set $\{\lambda_{m1}, \dots, \lambda_{m,\mathtt{p}(m)}\}$ (at the m-th level). We assume that $\mathfrak{S}_{(m)}$ acts trivially on Λ_k for $k \neq m$. We denote by $\mathfrak{S} = \mathfrak{S}_{[1;n]} = \prod_{m=1}^{n} \mathfrak{S}_{(m)}$ the group generated by all $\mathfrak{S}_{(m)}$. So we assume, the group $\mathfrak{S}$ acts naturally by the permutations on the variables having the same first index such that for any $\pi \in \mathfrak{S}_{(m)}$, and for any $j \in \mathfrak{I}_m$, it holds $\pi : \lambda_{mj} \mapsto \lambda_{m\pi(j)}$ and $\pi : \bar{\lambda}_{mj} \mapsto \bar{\lambda}_{m\pi(j)}$, all other actions are identical.

Clearly, the involution $*$ commutes with $\mathfrak{S}_{[1;n]}$. This allows us to define the groups $G_{(m)} = \mathfrak{S}_{(m)} \times \mathcal{E}$, $m \in [1;n)$, and $G = G_{(n)} = \mathfrak{S} \times \mathcal{E}$ with $\mathcal{E} = <\varepsilon> \simeq \mathbb{Z}_2$, such that $f^{\varepsilon} = f^*$, $f \in \Lambda$.

For every $ki \in \mathbb{J}_{[1;n)}$, we denote by $\sigma_{ki}^{\pm 1} \in \mathrm{Aut}_{\Bbbk}\Lambda$ the shift automorphism such that

$$\sigma_{ki}^{\pm 1} := \begin{cases} \lambda_{ki} & \mapsto \quad \lambda_{ki} \mp 1, \\ \lambda_{mj} & \mapsto \quad \lambda_{mj}, \qquad mj \neq ki. \end{cases}$$

Let $\Sigma = \prod_{ki \in \mathbb{J}_{[1;n)}} \langle \sigma_{ki} \rangle$ be the commutative group, $\Sigma \simeq \mathbb{Z}^{\otimes N}$ with $N = \frac{n(n-1)}{2}$. Those group Σ is called *the group of shiftings* of polynomial ring Λ.

By agreement, for $f \in \Lambda$ and $\sigma \in \Sigma$, we will use the both denotations $\sigma(f)$ and f^{σ} equivalently.

2.4. **Invariant subalgebra Γ.**

Let $\vec{x} := \{x_1, \dots, x_q\}$ be a set of free indeterminates, $\Bbbk[\vec{x}\,]$ be a polynomial ring, S_q be a symmetric group in q variables acting on $\vec{x}$. We denote by

$$\mathtt{e}_r^q(\vec{x}), \quad r \in \mathbb{N},$$

the r-th elementary polynomial, with $\mathtt{e}_0^q = 1$, and $\mathtt{e}_r^q = 0$ if $r > q$. We denote by

$$\mathrm{Sym}(\vec{x}) = \Bbbk[\vec{x}\,]^{S_q} = \Bbbk\big[\{\mathtt{e}_k^q\}_{k=1}^q\big]$$

the $\Bbbk$-algebra of symmetric polynomials in q variables $\vec{x}$.

Assume, $*$ acts on the $\Bbbk[\vec{x}\,]$ by the rule $x_i^* = -x_i$. Then the invariant subring in $\Bbbk[\vec{x}\,]$ of the group $S_q \times \mathcal{E}$ is generated by the polynomials

$$\mathtt{e}_{2k}^q(\vec{x}),\ \mathtt{ie}_{2k+1}^q(\vec{x}), \ \ \forall\ k \in \mathbb{N}.$$

Then, in particular, $\sum_{i=1}^{q} x_i^2 = -\left(\mathtt{ie}_1^q(\vec{x})\right)^2 - 2\mathtt{e}_2^q(\vec{x}) \in \Bbbk[\vec{x}\,]^{S_q \times \mathcal{E}}$ is an invariant.

Now we introduce the new variables in Λ which are invariant under the involution $*$. For any $m \in [1;n]$, $j \in \mathfrak{I}_m$, we put $\ell_{mj} = \lambda_{mj}\bar{\lambda}_{mj}$. The action of the group $\mathfrak{S}_{(m)}$ is transferred to this variables in the natural way while $*$ acts identically.

The above gives grounds for characterizing of the invariant subring $\Gamma := \Lambda^G$ in Λ as an integral domain, which is the subring of G-fixed points of Λ.

For $m \in [1;n]$ with $q = \mathtt{p}(m)$, let $\vec{\lambda}_{(m)} = \{\lambda_{m1}, \dots, \lambda_{mq}\}$ and $\vec{\ell}_{(m)} = \{\ell_{m1}, \dots, \ell_{mq}\}$. Each group $\mathfrak{S}_{(m)}$ acts independently at $\Bbbk[\vec{\lambda}_{(m)}]$ and trivially at $\Bbbk[\vec{\lambda}_{(k)}]$, $k \neq m$. We consider the elementary symmetric polynomials

$$\mathtt{s}_r^{(m)}\big(\vec{\lambda}_{(m)}\big) := \mathtt{e}_r^q\big(\vec{\lambda}_{(m)}\big), \quad \mathtt{s}_r^{(m)}\big(\vec{\ell}_{(m)}\big) := \mathtt{e}_r^q\big(\vec{\ell}_{(m)}\big), \quad 0 \leqslant r \leqslant q,$$

Remark 1. *For $m = 2q-1 \in [1;n]$ odd, the invariant subalgebra $\Gamma_{(m)} := \Bbbk[\vec{\lambda}_{(m)}]^{G_{(m)}} = \Bbbk[\vec{\lambda}_{(m)}]^{G}$ is generated by the elementary symmetric polynomials*

$$\mathtt{s}_r^{(m)}\big(\vec{\lambda}_{(m)}\big),\ r \in [1;q],\ r \equiv 0(\bmod 2), \qquad \mathtt{is}_r^{(m)}\big(\vec{\lambda}_{(m)}\big),\ r \in [1;q],\ r \equiv 1(\bmod 2).$$

Then, in particular, $\mathrm{Sym}\big(\vec{\ell}_{(m)}\big) \subset \Gamma_{(m)}$, *where* $\ell_{mj} = \lambda_{mj}\bar{\lambda}_{mj} = -\lambda_{mj}^2$.

Remark 2. *For $m = 2q \in [1;n]$ even, we consider the variables $\lambda'_{mj} = \lambda_{mj} - 1/2$, $j \in [1;q]$. Then $(\lambda'_{mj})^* = -\lambda'_{mj}$. In this case, the invariant subalgebra $\Gamma_{(m)} := \Bbbk[\vec{\lambda'}_{(m)}]^G$ is generated by the polynomials*

$$\mathtt{s}_r^{(m)}(\vec{\lambda'}_{(m)}),\ r \in [1;q],\ r \equiv 0(\bmod 2), \qquad \mathtt{is}_r^{(m)}(\vec{\lambda'}_{(m)}),\ r \in [1;q],\ r \equiv 1(\bmod 2),$$

and, $\mathrm{Sym}\big(\vec{\ell}_{(m)}\big) \subset \Gamma_{(m)}$, *where* $\ell_{mj} = \lambda_{mj}\bar{\lambda}_{mj} = 1/4 - (\lambda'_{mj})^2$.

Note that, for the arbitrary $m \in [1;n]$, there holds $\mathrm{Gal}(\Lambda_{(m)}/\Gamma_{(m)}) \simeq G_{(m)}$.

Finally, we denote by $\Gamma = \bigoplus_{m\in[1;n]} \Gamma_{(m)}$ the $\Bbbk$-ring generated by all $\Gamma_{(m)}$, $m \in [1;n]$. It is easy to see that $\Gamma = \Lambda^G$ is a ring of function which are symmetric with respect to each subset of variables $\lambda_{m1}, \ldots, \lambda_{mq}$, $m \in [1;n]$ independently. Then Γ is generated by elementary polynomials on all levels that may be multiplied by the imaginary unit $\mathtt{i}$. We call such functions *polysymmetric* concerning a set of variables $\vec{\lambda}$.

Remark 3. *One can observe that $\Gamma = \Lambda^G$ and $Gal(\Lambda/\Gamma) = G$.*

Let $\mathrm{L} = \mathrm{Frac}(\Lambda)$ be a field of fractions of Λ over $\Bbbk$ (or of rational functions in variables λ_{ki}, $ki \in \mathbb{J}_{[1;n]}$). We denote $\mathrm{L}^\times = \mathrm{L} \setminus \{0\}$.

The involution map $* : \Lambda \to \Lambda$ defined by (3), is extended to the map $* \in \mathrm{Aut}_\Bbbk(\mathrm{L})$ with $(f/g)^* = f^*/g^*$, $g \neq 0$, as well as the action of the group $\mathfrak{S}$. Similarly, any automorphism from Σ is extended to the mapping from $\mathrm{Aut}_\Bbbk(\mathrm{L})$ with $(\mathtt{f}/\mathtt{g})^\sigma = \mathtt{f}^\sigma/\mathtt{g}^\sigma$. Therefore, we assume $\Sigma \subset \mathrm{Aut}_\Bbbk(\mathrm{L})$.

Denote $\mathrm{K} = \mathrm{Frac}(\Gamma)$ to be the field of rational fractions of Γ, then $\mathrm{K} = \mathrm{L}^G$.

3. The rational skew twisted algebra (RTGWA) $\mathcal{A}$

3.1. The indexed parameters from L.

3.1.1. *The vectors* $\mathtt{t}$, $\bar{\mathtt{t}}$ *from* $\mathrm{L}^{\otimes N}$.

For any $ki \in \mathbb{J}_{[1;n)}$, we denote

$$\mathtt{t}_{ki} = -\frac{\prod\limits_{m=k\pm1, j\in\mathfrak{I}_m} R_{ki,mj}}{h_{ki}H_{ki}\,H_{ki}^{\sigma_{ki}^{-1}}}; \quad \bar{\mathtt{t}}_{ki} = -\frac{\prod\limits_{m=k\pm1, j\in\mathfrak{I}_m} \bar{R}_{ki,mj}}{\bar{h}_{ki}H_{ki}\,H_{ki}^{\sigma_{ki}}}; \tag{5}$$

where $H_{ki} = \prod\limits_{j\in\mathfrak{I}_k\setminus\{i\}} H_{ki,kj}$, $H_{ki}^{\sigma_{ki}^{\pm1}} = \sigma_{ki}^{\pm1}(H_{ki})$, with

$$H_{ki,kj} := \ell_{kj} - \ell_{ki} = (\lambda_{ki} - \lambda_{kj})(\lambda_{ki} - \bar{\lambda}_{kj}) = \begin{cases} (\lambda_{ki}^2 - \lambda_{kj}^2), & \text{if } \zeta_k = 0; \\ (\lambda_{ki} - \lambda_{kj})(\lambda_{ki} + \lambda_{kj} - 1), & \text{if } \zeta_k = 1; \end{cases} \tag{6}$$

and for any $ki, mj \in \mathbb{J}_{[1;n)}$ with $|k-m| = 1$, there are

$$\begin{array}{rcll} R_{ki,mj} &=& (\lambda_{ki} + \lambda_{mj})(\lambda_{ki} + \bar{\lambda}_{mj}); & \\ \bar{R}_{ki,mj} &=& (\bar{\lambda}_{ki} + \lambda_{mj})(\bar{\lambda}_{ki} + \bar{\lambda}_{mj}) & = R_{ki,mj}^* = \sigma_{ki}(R_{ki,mj}); \end{array} \tag{7}$$

and

$$\begin{array}{llll} h_{ki} & := & \lambda_{ki}^2\,(\lambda_{ki}-\bar\lambda_{ki})\,\sigma_{ki}^{-1}(\lambda_{ki}-\bar\lambda_{ki}) & = \lambda_{ki}^2\,(4\lambda_{ki}^2-1); \\ \bar h_{ki} & := & \bar\lambda_{ki}^2\,(\lambda_{ki}-\bar\lambda_{ki})\,\sigma_{ki}(\lambda_{ki}-\bar\lambda_{ki}) & = \bar\lambda_{ki}^2\,(4\bar\lambda_{ki}^2-1) \quad = h_{ki}^* = \sigma_{ki}(h_{ki}); \end{array} \tag{8}$$

for k even, and $h_{ki}=\bar h_{ki}=4$ for k odd.

By the definition, $\mathtt{t}_{ki}^\sigma, \bar{\mathtt{t}}_{ki}^\sigma$ and their inverses belong to $\mathrm{L}^\times$ for all $\sigma\in\Sigma$, and there hold $\bar{\mathtt{t}}_{ki} = \mathtt{t}_{ki}^* = \sigma_{ki}(\mathtt{t}_{ki}), \quad \forall\, ki\in\mathbb{J}_{[1;n)}$.

3.1.2. *Matrices $\boldsymbol{\mu}^{xx}$, $\boldsymbol{\mu}^{xy}$.*

We consider the following matrices from $\mathrm{Mat}_{N\times N}(\mathrm{L}^\times)$:

$$\boldsymbol{\mu}^{xx} = (\!(\mu^{xx}_{ki,mj})\!), \qquad \boldsymbol{\mu}^{xy} = (\!(\mu^{xy}_{ki,mj})\!)$$

with the entries to be the invertible elements from $\mathrm{L}^\times$, defined as follows.

Let $k,m\in[1;n)$. If $|k-m|>1$ then for all appropriate $i,j,\quad \mu^{xx}_{ki,mj}=\mu^{xy}_{ki,mj}=1$.

Let $|k-m|=1$. We denote

$$\mu^{xx}_{ki,mj} = \frac{\sigma_{ki}(\lambda_{ki}+\bar\lambda_{mj})}{\lambda_{ki}+\bar\lambda_{mj}} = -\frac{\bar\lambda_{ki}+\lambda_{mj}}{\lambda_{ki}+\bar\lambda_{mj}}; \qquad \mu^{xy}_{ki,mj} = \frac{\sigma_{ki}(\lambda_{ki}+\lambda_{mj})}{\lambda_{ki}+\lambda_{mj}} = -\frac{\bar\lambda_{ki}+\bar\lambda_{mj}}{\lambda_{ki}+\lambda_{mj}}. \tag{9}$$

For the case of equal first indexes and $i\neq j$, we denote

$$\mu^{xx}_{ki,kj} = \frac{\sigma_{ki}^{-1}(\lambda_{ki}-\lambda_{kj})}{\sigma_{ki}(\lambda_{ki}-\lambda_{kj})} = \frac{\lambda_{ki}-\lambda_{kj}+1}{\lambda_{ki}-\lambda_{kj}-1}; \qquad \mu^{xy}_{ki,kj} = \frac{\sigma_{ki}^{-1}(\lambda_{ki}-\bar\lambda_{kj})}{\sigma_{ki}(\lambda_{ki}-\bar\lambda_{kj})} = \frac{\lambda_{ki}-\bar\lambda_{kj}+1}{\lambda_{ki}-\bar\lambda_{kj}-1}. \tag{10}$$

We define the diagonal elements as follows: $\mu^{xx}_{ki,ki}=1$, and

$$\mu^{xy}_{ki,ki} := \frac{\bar{\mathtt{t}}_{ki}}{\mathtt{t}_{ki}} = \left(\frac{\lambda_{ki}^2}{\bar\lambda_{ki}^2}\,\frac{\sigma_{ki}^{-1}(\lambda_{ki}-\bar\lambda_{ki})}{\sigma_{ki}(\lambda_{ki}-\bar\lambda_{ki})}\right)^{\zeta_k} \cdot \prod_{m=k\pm1,\, j\in\mathfrak{I}_m} \mu^{xx}_{ki,mj}\,\mu^{xy}_{ki,mj} \cdot \prod_{j\in\mathfrak{I}_k\setminus\{i\}} \mu^{xx}_{ki,kj}\,\mu^{xy}_{ki,kj}. \tag{11}$$

This definition is consistent with the previous one, because by (8), for k even, there is

$$\frac{h_{ki}}{\bar h_{ki}} = \frac{\lambda_{ki}^2\,(4\lambda_{ki}^2-1)}{\bar\lambda_{ki}^2\,(4\bar\lambda_{ki}^2-1)} = \frac{\lambda_{ki}^2\,(2\lambda_{ki}-1)(2\lambda_{ki}+1)}{\bar\lambda_{ki}^2\,(2\bar\lambda_{ki}-1)(2\bar\lambda_{ki}+1)} = \frac{\lambda_{ki}^2\,(2\lambda_{ki}+1)}{\bar\lambda_{ki}^2\,(2\lambda_{ki}-3)} = \frac{\lambda_{ki}^2}{\bar\lambda_{ki}^2}\,\frac{\sigma_{ki}^{-1}(\lambda_{ki}-\bar\lambda_{ki})}{\sigma_{ki}(\lambda_{ki}-\bar\lambda_{ki})};$$

and

$$\frac{\sigma_{ki}^{-1}(H_{ki,kj})}{\sigma_{ki}(H_{ki,kj})} = \frac{\sigma_{ki}^{-1}(\ell_{ki}-\ell_{kj})}{\sigma_{ki}(\ell_{ki}-\ell_{kj})} = \frac{\sigma_{ki}^{-1}(\lambda_{ki}-\lambda_{kj})\,\sigma_{ki}^{-1}(\lambda_{ki}-\bar\lambda_{kj})}{\sigma_{ki}(\lambda_{ki}-\lambda_{kj})\,\sigma_{ki}(\lambda_{ki}-\bar\lambda_{kj})} = \mu^{xx}_{ki,kj}\mu^{xy}_{ki,kj}.$$

We note the obvious properties of matrices:

(1) matrix $\boldsymbol{\mu}^{xy}$ is symmetric : $\mu^{xy}_{ki,mj}=\mu^{xy}_{mj,ki}$, and $(\mu^{xy}_{ki,mj})^* = (\mu^{xy}_{ki,mj})^{-1}$;

(2) matrix $\boldsymbol{\mu}^{xx}$ is $*$-symmetric : $\mu^{xx}_{ki,mj} = (\mu^{xx}_{mj,ki})^* = (\mu^{xx}_{mj,ki})^{-1}$.

3.1.3. *Free member C_{2q}.*

For any $q\in[1,\left[\frac{n}{2}\right]]$, we consider the following variables:

$$\vec\lambda^\times_{(2q-1)} := \lambda_{2q-1,1}\dots\lambda_{2q-1,q} = \mathtt{s}_q^{(2q-1)}\big(\vec\lambda_{(2q-1)}\big),$$

$$\vec\ell^\times_{(2q)} := \ell_{2q,1}\dots\ell_{2q,q} = \mathtt{s}_q^{(2q)}\big(\vec\ell_{(2q)}\big).$$

Then $\vec\ell^\times_{(2q)}$ and $\mathtt{i}^\alpha\,\vec\lambda^\times_{(2q-1)}$ are invariants of G where $\alpha\in\{0,1\}$, $\alpha\equiv q \bmod 2$, because $(\vec\lambda^\times_{(2q-1)})^* = (-1)^q\vec\lambda^\times_{(2q-1)}$, and $\mathtt{i}^*=-\mathtt{i}$.

We put $C_m=0$ for m odd, and, for $m=2q\in[1;n)$, we set

$$C_{2q} = (-1)^q\,\mathtt{i}\,\frac{\prod_{r=1}^{q}\lambda_{2q\text{-}1,r}\,\prod_{r=1}^{q+1}\lambda_{2q+1,r}}{\prod_{r=1}^{q}\lambda_{2q,r}\,\bar\lambda_{2q,r}} = (-1)^q\,\mathtt{i}\,\frac{\vec\lambda^\times_{(2q-1)}\cdot\vec\lambda^\times_{(2q+1)}}{\vec\ell^\times_{(2q)}} \in \mathrm{K}. \tag{12}$$

For m even, C_m is an invariant of $*$, while $q=|\mathfrak{I}_{2q-1}|$ and $q+1=|\mathfrak{I}_{2q+1}|$ have the distinct parities.

3.2. Definition of rational twisted generalized Weyl algebra (RTGWA) $\mathcal{A}$.

Definition 1. *Given a polynomial ring Λ together with the rings $\mathrm{L} = \mathrm{Frac}(\Lambda)$, $\Gamma = \Lambda^G$, $\mathrm{K} = \mathrm{L}^G$, above defined, we assume that*

- *there exist an involution automorphism $* \in \mathrm{Aut}_{\Bbbk}(\mathrm{L})$;*
- *there exist the $*$-symmetric matrix $\boldsymbol{\mu}^{xx} = ((\mu^{xx}_{ki,mj}))$ and symmetric matrix $\boldsymbol{\mu}^{xy} = ((\mu^{xy}_{ki,mj}))$ from $\mathrm{Mat}_{N\times N}(\mathrm{L}^\times)$, defined in (9), (10), (11);*
- *there exist the N-vectors $\mathtt{t} = \big(\mathtt{t}_{ki}\big)_{ki\in\mathbb{J}_{[1;n)}}$, $\bar{\mathtt{t}} = \big(\bar{\mathtt{t}}_{ki}\big)_{ki\in\mathbb{J}_{[1;n)}}$ with coefficients in $\mathrm{L}^\times$ defined in (5), such that $\bar{\mathtt{t}}_{ki} := \mathtt{t}^*_{ki} = \sigma_{ki}(\mathtt{t}_{ki})$.*

We consider the twisted skew anti involuted $\Bbbk$-algebra (RTGWA)

$$\mathcal{A} = \mathcal{A}(n, \mathbb{C}) = \mathcal{A}lg_n(\mathbb{C}, \Lambda, G, \Sigma, *, \boldsymbol{\mu}, \mathtt{t})$$

generated over the ring L, the field of rational fractions of Λ, by the variables $\mathcal{Z} = \big\{ X_{ki}, Y_{ki} \mid ki \in \mathbb{J}_{[1;n)} \big\}$, under the commutation relations such that

$$\begin{array}{rcl} X_{ki} f = \sigma_{ki}(f) X_{ki}, & Y_{ki} f = \sigma^{-1}_{ki}(f) Y_{ki}, & f \in \mathrm{L}, \\ & Y_{ki}X_{ki} = \mathtt{t}_{ki}, \quad X_{ki}Y_{ki} = \bar{\mathtt{t}}_{ki}, & \\ & X_{ki}Y_{mj} = \mu^{xy}_{ki,mj} Y_{mj} X_{ki}, & \\ & X_{ki}X_{mj} = \mu^{xx}_{ki,mj} X_{mj} X_{ki}, & \end{array} \qquad ki, mj \in \mathbb{J}_{[1;n)},\ ki \neq mj. \tag{13}$$

Moreover, we assume there exist an anti involution $* : \mathcal{A} \to \mathcal{A}$, that continues an involuted automorphism of the ring L, such that $X^*_{ki} = -Y_{ki}$, $Y^*_{ki} = -X_{ki}$.

We assume, the algebra $\mathcal{A}$ is equipped with the canonical embedding of L-rings $\rho : \mathrm{L} \to \mathcal{A}$.

The action of the group $\mathfrak{S}$ on L, above defined, has a prolongation to the action on $\mathcal{A}$ by the rule: if $ki \in \mathbb{J}_{[1;n)}$, then for any transposition π, $X^\pi_{ki} = X_{k\pi(i)}$ if $\pi \in \mathfrak{S}_k$, $X^\pi_{ki} = X_{ki}$ otherwise.

Here L is a maximal commutative subalgebra in $\mathcal{A}$.

By the definition, the anti involution $* : \mathcal{A} \to \mathcal{A}$ keep all defining relations true.

For $ki \neq mj$, we will use also the variables $\mu^{yx}_{ki,mj} = (\mu^{xy}_{ki,mj})^*$ and $\mu^{yy}_{ki,mj} = (\mu^{xx}_{ki,mj})^*$ such that

$$Y_{ki}X_{mj} = \mu^{yx}_{ki,mj} \cdot X_{mj}Y_{ki}, \qquad Y_{ki}Y_{mj} = \mu^{yy}_{ki,mj} \cdot Y_{mj}Y_{ki}.$$

Remark 4. *Consider the countable set $\Omega = \{\lambda_{ki} \pm \lambda_{mj} + c,\ \lambda_{ki} + c,\ 2\lambda_{ki} + c \mid ki \neq mj,\ c \in \mathbb{Z}\}$. It is easy to see that the numerators and denominators of all fractions in the formulae (5), (9), (10), and (11) belong to the multiplicative set S generated by Ω. The additional observation is that the set S is invariant under the action of the group G since $s^g \in S$ for each $s \in S$ and each $g \in G$. Therefore, for our purposes in this text, it is enough to consider as L a localization $S^{-1}\Lambda$ instead of* $\mathrm{Frac}\,\Lambda$.

The following remark fixes some properties of the algebra $\mathcal{A}$. It is due the [[FuHa2], Theorem A] which gives a sufficient condition for a TGWA to be a nontrivial ring. This condition consists in the implementation of equalities (14) and (15) below.

Remark 5. *The following two set of relations are satisfied in $\mathcal{A}$:*

$$\sigma_{ki}\sigma_{mj}(\mathtt{t}_{ki}\mathtt{t}_{mj}) = \sigma_{ki}(\mu^{xy}_{mj,ki})\,\sigma_{mj}(\mu^{xy}_{ki,mj})\,\sigma_{ki}(\mathtt{t}_{ki})\,\sigma_{mj}(\mathtt{t}_{mj}) \quad \forall\ ki, mj \in \mathbb{J}_{[1;n)}. \tag{14}$$

$$\mathtt{t}_{mj}\,\sigma_{ki}\sigma_{rl}(\mathtt{t}_{mj}) = \sigma_{ki}(\mathtt{t}_{mj})\,\sigma_{rl}(\mathtt{t}_{mj}), \quad \forall\ ki, mj, kl \ \text{ pairwise different.} \tag{15}$$

Proof of (14). It is enough to compare two calculations of $Y_{ki}Y_{mj}X_{ki}X_{mj}Y_{ki}X_{ki}$, namely

$$Y_{ki}\ (Y_{mj}X_{ki})\ X_{mj}Y_{ki}X_{ki} = \frac{1}{\sigma^{-1}_{ki}(\mu^{xy}_{ki,mj})} Y_{ki}X_{ki}Y_{mj}X_{mj}Y_{ki}X_{ki} = \frac{\mathtt{t}^2_{ki}\mathtt{t}_{mj}}{\sigma^{-1}_{ki}(\mu^{xy}_{ki,mj})};$$

$$Y_{ki}Y_{mj}X_{ki}\ (X_{mj}Y_{ki})\ X_{ki} =$$

$$= \sigma^{-1}_{mj}(\mu^{xy}_{mj,ki}) Y_{ki}Y_{mj}X_{ki}Y_{ki}X_{mj}X_{ki} = \sigma^{-1}_{mj}(\mu^{xy}_{mj,ki}) Y_{ki}Y_{mj}\sigma_{ki}(\mathtt{t}_{ki})X_{mj}X_{ki} =$$

$$= \sigma^{-1}_{mj}(\mu^{xy}_{mj,ki})\sigma^{-1}_{mj}(\mathtt{t}_{ki})\, Y_{ki}Y_{mj}X_{mj}X_{ki} = \sigma^{-1}_{mj}(\mu^{xy}_{mj,ki})\sigma^{-1}_{mj}(\mathtt{t}_{ki})\sigma^{-1}_{ki}(\mathtt{t}_{mj})\mathtt{t}_{ki}.$$

Then the equality $\mathtt{t}_{ki}\mathtt{t}_{mj} = \sigma^{-1}_{ki}(\mu^{xy}_{ki,mj}\,\mathtt{t}_{mj})\,\sigma^{-1}_{mj}(\mu^{xy}_{mj,ki}\,\mathtt{t}_{ki})$ yields (14). □

Proof of (15). First, we calculate

$$(Y_{mj}X_{mj})\ X_{ki}X_{rl}\ (Y_{mj}X_{mj})\ = \mathtt{t}_{mj}\sigma_{ki}\sigma_{rl}(\mathtt{t}_{mj})\cdot X_{ki}X_{rl}.$$

Next, we have

$$Y_{mj}X_{mj}X_{ki} = \sigma_{mj}^{-1}(\mu_{mj,ki}^{xx})\mu_{mj,ki}^{yx}X_{ki}Y_{mj}X_{mj} = \mu_{mj,ki}^{yx}\sigma_{mj}^{-1}(\mu_{mj,ki}^{xx})\sigma_{ki}(\mathtt{t}_{mj})X_{ki} = T_{mj,ki}\sigma_{ki}(\mathtt{t}_{mj})X_{ki}$$
$$X_{ki}Y_{mj}X_{mj} = \mu_{ki,mj}^{xy}\sigma_{mj}^{-1}(\mu_{ki,mj}^{xx})Y_{mj}X_{mj}X_{ki} = \mu_{ki,mj}^{xy}\sigma_{mj}^{-1}(\mu_{ki,mj}^{xx})\mathtt{t}_{mj}\,X_{ki} = T_{ki,mj}\,\mathtt{t}_{mj}X_{ki}$$

Then

$$\begin{aligned}(Y_{mj}X_{mj}X_{ki})\ X_{rl}\,Y_{mj}X_{mj} &= T_{mj,ki}\sigma_{ki}(\mathtt{t}_{mj})X_{ki}\,X_{rl}\,Y_{mj}X_{mj} = \\ &= \sigma_{ki}(\mathtt{t}_{mj})\,T_{mj,ki}\,\mu_{ki,rl}^{xx}\,X_{rl}\,(X_{ki}Y_{mj}X_{mj}) = \sigma_{ki}(\mathtt{t}_{mj})\,T_{mj,ki}\,\mu_{ki,rl}^{xx}\,X_{rl}\,\mathtt{t}_{mj}\,T_{ki,mj}X_{ki} = \\ &= \sigma_{ki}(\mathtt{t}_{mj})\sigma_{rl}(\mathtt{t}_{mj})\,T_{mj,ki}T_{ki,mj}\,\mu_{ki,rl}^{xx}\mu_{rl,ki}^{xx}\,X_{ki}X_{rl} = \sigma_{ki}(\mathtt{t}_{mj})\sigma_{rl}(\mathtt{t}_{mj})\ X_{ki}X_{rl}\end{aligned}$$

because $\sigma_{rl}(T_{ki,mj}) = T_{ki,mj}$ and $\mu_{\alpha,\beta}^{xy}\mu_{\beta,\alpha}^{yx} = 1$, $\mu_{\alpha,\beta}^{xx}\mu_{\beta,\alpha}^{xx} = 1$ for all distinct $\alpha,\beta \in \{ki, mj, kl\}$. Then, using the cancellation property in $\mathcal{A}$, we obtain (15). □

4. Proposition 1 : the proof of the second defining equality

4.1. The defining Lie bracket relations in $\mathcal{A}$.

Now we establish some properties of RTGWA $\mathcal{A}$.

Lemma 1. *For any $k \in [1;n)$, and any $i \in \mathfrak{I}_k$,, the record*

$$\mathcal{U}_k = \sum_{i\in\mathfrak{I}_k} X_{ki} - \sum_{i\in\mathfrak{I}_k} Y_{ki} + C_k \in \mathcal{A}^G \tag{16}$$

is an invariant of the group G in $\mathcal{A}$ where C_k is defined by (12)*.*

Indeed, by the construction, $X_{ki} \overset{*}{\rightleftarrows} -Y_{ki}$, and C_{2q} is invariant of G.

We denote by $\mathfrak{A} = \mathrm{K}\left\{\mathcal{U}_k\right\}_{k\in[1;n)}\mathrm{K}$ the subalgebra generated over K by all $\mathcal{U}_k$, $k \in [1;n)$. By Lemma 1, $\mathfrak{A} \subset \mathcal{A}^G$ is an invariant $\Bbbk$-algebra.

Proposition 1. *Let $k, m \in [1;n)$. The following equations are satisfied in $\mathcal{A}$:*

$$[\mathcal{U}_k, \mathcal{U}_m] = 0, \qquad |k-m| > 1; \tag{17}$$
$$[[\mathcal{U}_k, \mathcal{U}_m], \mathcal{U}_m] = -\mathcal{U}_k, \qquad |k-m| = 1; \tag{18}$$
$$[[[\mathcal{U}_k, \mathcal{U}_{k+1}], \mathcal{U}_{k+2}], \mathcal{U}_{k+1}] = 0 \qquad k \in [1;n-2]. \tag{19}$$

Here, the expression $\mathcal{U}_k$ does not depends on variables from the m-th level if $|m-k| > 1$, then $\mathcal{U}_k$ and $\mathcal{U}_m$ commute. The proof of this proposition is the issue of the next chapter considerations.

The proof of the equation (18) of Proposition 1 consists from the following lemmas.

4.2. General calculations.

In this section, we assume $k, m \in [1;n)$, $|k-m| = 1$. Unless otherwise stated, the notation $i, j \in \mathfrak{I}_m$ implies $i \neq j$.

Remark 6. *Note that the Lie bracket identities considered in the above steps state true under the action of anti involution $*$.*

Step 4.1. *For $k, m \in [1,n)$ with $|k-m| = 1$, and $i, j \in \mathfrak{I}_m$, $i \neq j$, $l \in \mathfrak{I}_k$, for $Z_{kl} = X_{kl}$ or $Z_{kl} = Y_{kl}$, there holds*

$$\begin{gathered}[[Z_{kl}, X_{mi}], X_{mj}] + [[Z_{kl}, X_{mj}], X_{mi}] = 0;\\ [[Z_{kl}, Y_{mi}], Y_{mj}] + [[Z_{kl}, Y_{mj}], Y_{mi}] = 0.\end{gathered} \tag{20}$$

Proof. We consider the case $Z_{kl} = X_{kl}$. The left part of (20) equals $T_{ij} + T_{ji}$ with

$$T_{ij} = X_{mi}X_{mj}X_{kl} + X_{kl}X_{mi}X_{mj} - 2X_{mi}X_{kl}X_{mj} =$$
$$= X_{mi}X_{mj}X_{kl} + \mu^{xx}_{kl,mi}X_{mi}X_{kl}X_{mj} - 2X_{mi}\mu^{xx}_{kl,mj}X_{mj}X_{kl} =$$
$$= X_{mi}X_{mj}X_{kl} + \mu^{xx}_{kl,mi}X_{mi}\mu^{xx}_{kl,mj}X_{mj}X_{kl} - 2(\mu^{xx}_{kl,mj})^{\sigma_{mi}}X_{mi}X_{mj}X_{kl} =$$
$$= \big(1 + \mu^{xx}_{kl,mi}\sigma_{mi}(\mu^{xx}_{kl,mj}) - 2\sigma_{mi}(\mu^{xx}_{kl,mj})\big) \cdot X_{mi}X_{mj}X_{kl} =$$
$$= \big(1 + \mu^{xx}_{kl,mi}\mu^{xx}_{kl,mj} - 2\mu^{xx}_{kl,mj}\big) \cdot X_{mi}X_{mj}X_{kl} = \frac{\lambda_{mj} - \lambda_{mi} + 1}{(\lambda_{kl} + \bar{\lambda}_{mi})(\lambda_{kl} + \bar{\lambda}_{mj})} \cdot X_{mi}X_{mj}X_{kl}.$$

This yields $T_{ij}+T_{ji}=\dfrac{1}{(\lambda_{kl}+\bar{\lambda}_{mi})(\lambda_{kl}+\bar{\lambda}_{mj})}\big((\lambda_{mj}-\lambda_{mi}+1)X_{mi}X_{mj}+(\lambda_{mi}-\lambda_{mj}+1)X_{mj}X_{mi}\big)X_{kl}=0$ because $\mu^{xx}_{mi,mj} = \dfrac{\lambda_{mi} - \lambda_{mj} + 1}{\lambda_{mi} - \lambda_{mj} - 1}$.

The case $Z_{kl} = Y_{kl}$, and the second equation are proved similarly. □

Step 4.2. *For $m, k \in [1, n)$ with $|m - k| = 1$, and $j \in \mathfrak{I}_m$, $l \in \mathfrak{I}_k$, and for $Z_{kl} = X_{kl}$ or $Z_{kl} = Y_{kl}$, there holds*

$$[[Z_{kl}, X_{mi}], X_{mi}] = 0, \qquad [[Z_{kl}, Y_{mi}], Y_{mi}] = 0. \tag{21}$$

Proof. Assume $Z_{kl} = X_{kl}$. We calculate

$$[[X_{kl}, X_{mi}], X_{mi}] = X_{mi}\, X_{mi}\, X_{kl} + X_{kl}\, X_{mi}\, X_{mi} - 2X_{mi}\, X_{kl}\, X_{mi} =$$
$$= \big(1 + \mu^{xx}_{kl,mi}(\mu^{xx}_{kl,mi})^{\sigma_{mi}} + 2(\mu^{xx}_{kl,mi})^{\sigma_{mi}}\big) \cdot X_{mi}\, X_{mi}\, X_{kl} =$$
$$= \left(1 + \frac{\bar{\lambda}_{kl} + \lambda_{mi}}{\lambda_{kl} + \bar{\lambda}_{mi}} \cdot \frac{\bar{\lambda}_{kl} + \lambda_{mi} - 1}{\lambda_{kl} + \bar{\lambda}_{mi} + 1} + 2\frac{\bar{\lambda}_{kl} + \lambda_{mi} - 1}{\lambda_{kl} + \bar{\lambda}_{mi} + 1}\right) \cdot X_{mi}\, X_{mi}\, X_{kl} =$$
$$= \left(1 - \frac{\bar{\lambda}_{kl} + \lambda_{mi}}{\lambda_{kl} + \bar{\lambda}_{mi} + 1} + 2\frac{\bar{\lambda}_{kl} + \lambda_{mi} - 1}{\lambda_{kl} + \bar{\lambda}_{mi} + 1}\right) \cdot X_{mi}\, X_{mi}\, X_{kl} = \left(\frac{\zeta_k + \zeta_m - 1}{\lambda_{kl} + \bar{\lambda}_{mi} + 1}\right) \cdot X_{mi}\, X_{mi}\, X_{kl} = 0.$$

Other cases are considered similarly. □

Step 4.3. *For $k, m \in [1, n)$ with $|k - m| = 1$, and $i, j \in \mathfrak{I}_k$, $i \neq j$, $l \in \mathfrak{I}_k$, for $Z_{kl} = X_{kl}$ or $Z_{kl} = Y_{kl}$, there holds*

$$[[Z_{kl}, X_{mi}], Y_{mj}] + [[Z_{kl}, Y_{mj}], X_{mi}] = 0. \tag{22}$$

Proof. Let $Z_{kl} = X_{kl}$. The left part of (22) equals $T_{ij} + T_{ji}$ with

$$T_{ij} = X_{mi}Y_{mj}X_{kl} + X_{kl}X_{mi}Y_{mj} - 2X_{mi}X_{kl}Y_{mj} =$$
$$= X_{mi}Y_{mj}X_{kl} + \mu^{xx}_{kl,mi}X_{mi}X_{kl}Y_{mj} - 2X_{mi}\mu^{xy}_{kl,mj}Y_{mj}X_{kl} =$$
$$= X_{mi}Y_{mj}X_{kl} + \mu^{xx}_{kl,mi}X_{mi}\mu^{xy}_{kl,mj}Y_{mj}X_{kl} - 2(\mu^{xy}_{kl,mj})^{\sigma_{mi}}X_{mi}Y_{mj}X_{kl} =$$
$$= \Big(1 + \mu^{xx}_{kl,mi}\sigma_{mi}(\mu^{xy}_{kl,mj}) - 2\sigma_{mi}(\mu^{xy}_{kl,mj})\Big) \cdot X_{mi}Y_{mj}X_{kl} =$$
$$= \Big(1 + \mu^{xx}_{kl,mi}\mu^{xy}_{kl,mj} - 2\mu^{xy}_{kl,mj}\Big) \cdot X_{mi}Y_{mj}X_{kl} = -\frac{\lambda_{mj} + \lambda_{mi} - 2}{(\lambda_{kl} + \bar{\lambda}_{mi})(\lambda_{kl} + \lambda_{mj})} \cdot X_{mi}Y_{mj}X_{kl}.$$

$$T_{ji} = Y_{mj}X_{mi}X_{kl} + X_{kl}Y_{mj}X_{mi} - 2Y_{mj}X_{kl}X_{mi} =$$
$$= X_{mi}Y_{mj}X_{kl} + \mu^{xy}_{kl,mj}Y_{mj}X_{kl}X_{mi} - 2Y_{mj}\mu^{xx}_{kl,mi}X_{mi}X_{kl} =$$
$$= X_{mi}Y_{mj}X_{kl} + \mu^{xy}_{kl,mj}\mu^{xx}_{kl,mi}Y_{mj}X_{mi}X_{kl} - 2\mu^{xx}_{kl,mi}Y_{mj}X_{mi}X_{kl} =$$
$$= (1 + \mu^{xy}_{kl,mj}\mu^{xx}_{kl,mi} - 2\mu^{xx}_{kl,mi})Y_{mj}X_{mi}X_{kl} = -\frac{\lambda_{mj} + \lambda_{mi}}{(\lambda_{kl} + \bar{\lambda}_{mi})(\lambda_{kl} + \lambda_{mj})} \cdot Y_{mj}X_{mi}X_{kl}.$$

This yields $T_{ij}+T_{ji}=\dfrac{1}{(\lambda_{kl}+\bar\lambda_{mi})(\lambda_{kl}+\bar\lambda_{mj})}\big((\lambda_{mj}-\lambda_{mi}+1)X_{mi}X_{mj}+(\lambda_{mi}-\lambda_{mj}+1)X_{mj}X_{mi}\big)X_{kl}=0$ because $\mu^{xy}_{mi,mj}=\dfrac{\lambda_{mi}-\bar\lambda_{mj}+1}{\lambda_{mi}-\bar\lambda_{mj}-1}=\dfrac{\lambda_{mi}+\lambda_{mj}}{\lambda_{mi}+\lambda_{mj}-2}$. □

Step 4.4. *For $m,k\in[1,n)$ with $|m-k|=1$, and for any $l\in\mathfrak{I}_k$, $i\in\mathfrak{I}_m$, there holds*

$$[[Z_{kl},X_{mi}],Y_{mi}]+[[Z_{kl},Y_{mi}],X_{mi}]=-\left(\frac{(2\lambda_{mi}+1+\zeta_k)\,\mathtt{t}_{mi}}{R_{mi,kl}}+\frac{(2\bar\lambda_{mi}+1+\zeta_k)\,\bar{\mathtt{t}}_{mi}}{\bar R_{mi,kl}}\right)Z_{kl},$$

where either $Z_{kl}=X_{kl}$ or $Z_{kl}=Y_{kl}$, and $R_{mi,kl},\bar R_{mi,kl}$ are defined in (7).

Proof. Using (7), we denote $\mathtt{t}_{mi}=R_{mi,kl}\cdot\mathtt{t}'_{mi}$, $\bar{\mathtt{t}}_{mi}=\bar R_{mi,kl}\cdot\bar{\mathtt{t}}'_{mi}$, where by (7), $R_{mi,kl}=(\lambda_{mi}+\lambda_{kl})(\lambda_{mi}+\bar\lambda_{kl})$, $\bar R_{mi,kl}=(\bar\lambda_{mi}+\lambda_{kl})(\bar\lambda_{mi}+\bar\lambda_{kl})$. We have $[[X_{kl},X_{mi}],Y_{mi}]+[[X_{kl},Y_{mi}],X_{mi}]=$

$$\begin{aligned}&=(X_{kl}X_{mi}Y_{mi}+X_{mi}Y_{mi}X_{kl}-2X_{mi}X_{kl}Y_{mi})+(X_{kl}Y_{mi}X_{mi}+Y_{mi}X_{mi}X_{kl}-2Y_{mi}X_{kl}X_{mi})=\\&=(X_{kl}\bar{\mathtt{t}}_{mi}+\bar{\mathtt{t}}_{mi}X_{kl}-2\mu^{xx}_{mi,kl}X_{kl}\bar{\mathtt{t}}_{mi})+(X_{kl}\mathtt{t}_{mi}+\mathtt{t}_{mi}X_{kl}-2\mu^{yx}_{mi,kl}X_{kl}\mathtt{t}_{mi})=\\&=(\bar{\mathtt{t}}_{mi}+\sigma_{kl}(\bar{\mathtt{t}}_{mi})-2\mu^{xx}_{mi,kl}\sigma_{kl}(\bar{\mathtt{t}}_{mi}))X_{kl}+(\mathtt{t}_{mi}+\sigma_{kl}(\mathtt{t}_{mi})-2\mu^{yx}_{mi,kl}\sigma_{kl}(\mathtt{t}_{mi}))X_{kl}=\\&=-\big(\bar{\mathtt{t}}'_{mi}(2\bar\lambda_{mi}+1+\zeta_k)+\mathtt{t}'_{mi}(2\lambda_{mi}+1+\zeta_k)\big)X_{kl}=-\left(\frac{(2\lambda_{mi}+1+\zeta_k)\,\mathtt{t}_{mi}}{R_{mi,kl}}+\frac{(2\bar\lambda_{mi}+1+\zeta_k)\,\bar{\mathtt{t}}_{mi}}{\bar R_{mi,kl}}\right)X_{kl},\end{aligned}$$

because $\mathtt{t}_{mi}+\sigma_{kl}(\mathtt{t}_{mi})-2\mu^{yx}_{mi,kl}\sigma_{kl}(\mathtt{t}_{mi})=\mathtt{t}'_{mi}\big((\lambda_{mi}+\lambda_{kl})(\lambda_{mi}+\bar\lambda_{kl})+\sigma_{kl}(\lambda_{mi}+\lambda_{kl})\sigma_{kl}(\lambda_{mi}+\bar\lambda_{kl})$ $-2\mu^{yx}_{mi,kl}\sigma_{kl}(\lambda_{mi}+\lambda_{kl})\sigma_{kl}(\lambda_{mi}+\bar\lambda_{kl})\big)=\mathtt{t}'_{mi}\big((\lambda_{mi}+\lambda_{kl})(\lambda_{mi}+\bar\lambda_{kl})+(\lambda_{mi}+\lambda_{kl}-1)(\lambda_{mi}+\bar\lambda_{kl}+1)-$ $2(\lambda_{mi}+\lambda_{kl})(\lambda_{mi}+\bar\lambda_{kl}+1)\big)=-(2\lambda_{mi}+1+\zeta_k)\mathtt{t}'_{mi}$. □

4.3. **The case of an odd parameter m.**

Remark 7. *Let $m=2q-1$. Then*

$$\Phi(m,\alpha)=\sum_{i\in\mathfrak{I}_m}\left(\frac{\lambda_{mi}^{\alpha-1}(\lambda_{mi}+1)^\alpha}{H_{mi}H_{mi}^{\sigma_{mi}^{-1}}}-\frac{\lambda_{mi}^{\alpha-1}(\lambda_{mi}-1)^\alpha}{H_{mi}H_{mi}^{\sigma_{mi}}}\right)=\begin{cases}0, & \alpha\in[0,2q)\\ 2, & \alpha=2q-1.\end{cases}\tag{23}$$

Proof. By (6), $H_{mi}=\prod\limits_{j\in\mathfrak{I}_m\setminus\{i\}}R_{mi,mj}=\prod\limits_{j\in\mathfrak{I}_m\setminus\{i\}}(\lambda_{mj}^2-\lambda_{mi}^2)$; and $H_{mi}^{\sigma_{mi}^{\pm1}}=\prod\limits_{j\in\mathfrak{I}_m\setminus\{i\}}\big(\lambda_{mj}^2-(\lambda_{mi}\mp1)^2\big)$. Then (23) is a direct consequence of Proposition 2 from Appendix. □

Step 4.5. *Let $\zeta_m=0$, $\zeta_k=1$, $|k-m|=1$, $l\in\mathfrak{I}_k$, then there holds*

$$\sum_{i\in\mathfrak{I}_m}\left(\frac{\lambda_{mi}+1}{R_{mi,kl}}\mathtt{t}_{mi}+\frac{\bar\lambda_{mi}+1}{\bar R_{mi,kl}}\bar{\mathtt{t}}_{mi}\right)=-\frac{1}{2}.\tag{24}$$

Proof. In this case, $R_{mi,kl}=(\lambda_{mi}(\lambda_{mi}+1)+\ell_{kl})$, $\bar R_{mi,kl}=(\lambda_{mi}(\lambda_{mi}-1)+\ell_{kl})$ (see (7)).

Since $m=2q-1$, then $k=2q-2$ or $k=2q$. We consider the index set $\mathbb{J}=\mathfrak{I}_{m-1}\cup\mathfrak{I}_{m+1}$, then $|\mathbb{J}|=\mathtt{p}(m-1)+\mathtt{p}(m+1)=2q-1=m$.

Denote by $\mathcal{F}^{(kl)}=\mathrm{Sym}\left(\{\ell_{m-1,r}\}_{r=1}^{q-1}\cup\{\ell_{m+1,r}\}_{r=1}^{q}\setminus\{\ell_{kl}\}\right)$ the ring of symmetric polynomials on $2q-2$ variables. There exists a polynomial $F(t)$ with coefficients in $\mathcal{F}^{(kl)}$ of degree $2q-2$, with leading coefficient 1 such that the following hold (see (5)):

$$\frac{\mathtt{t}_{mi}}{R_{mi,kl}}=-\frac{F\left(\lambda_{mi}(\lambda_{mi}+1)\right)}{h_{mi}H_{mi}H_{mi}^{\sigma_{mi}^{-1}}},\qquad\frac{\bar{\mathtt{t}}_{mi}}{\bar R_{mi,kl}}=-\frac{F\left(\lambda_{mi}(\lambda_{mi}-1)\right)}{\bar h_{mi}H_{mi}H_{mi}^{\sigma_{mi}}},$$

where $F\left(\lambda_{mi}(\lambda_{mi}+1)\right)=\prod\limits_{t\alpha\in\mathbb{J}\setminus\{kl\}}R_{mi,t\alpha}$, $\quad F\left(\lambda_{mi}(\lambda_{mi}-1)\right)=\prod\limits_{t\alpha\in\mathbb{J}\setminus\{kl\}}\bar R_{mi,t\alpha}$ and $R_{mi,t\alpha}=(\lambda_{mi}(\lambda_{mi}+1)+\ell_{t\alpha})$, $\bar R_{mi,t\alpha}=(\lambda_{mi}(\lambda_{mi}-1)+\ell_{t\alpha})$.

By (8), for m odd, $h_{mi}=\bar{h}_{mi}=4$. Then

$$\frac{\lambda_{mi}+1}{R_{mi,kl}}\mathtt{t}_{mi}+\frac{\bar{\lambda}_{mi}+1}{\bar{R}_{mi,kl}}\bar{\mathtt{t}}_{mi}=\frac{-(\lambda_{mi}+1)\,F\left(\lambda_{mi}(\lambda_{mi}+1)\right)}{4\prod\limits_{l\neq i}\left(\lambda_{mj}^2-\lambda_{mi}^2\right)\left((\lambda_{mj}+1)^2-\lambda_{mi}^2\right)}-\frac{-(\lambda_{mi}-1)\,F\left(\lambda_{mi}(\lambda_{mi}-1)\right)}{4\prod\limits_{j\neq i}\left(\lambda_{mj}^2-\lambda_{mi}^2\right)\left((\lambda_{mj}-1)^2-\lambda_{mi}^2\right)}.$$

Then the sum from (24) is a linear combination over $\mathrm{Sym}(\mathcal{F}^{(kl)})$ of $\Phi(m,\alpha)$ for $\alpha\in[0,2q-1]$, besides, the leading coefficient (for $\alpha=2q-1$)) equals $-1/4$. By (23), this sum equals 2, and therefore the whole sum equals $-1/2$.

The proof is complete. □

Step 4.6. *For $k,m\in[1,n)$ with $|k-m|=1$ and $\zeta_k=1$, $\zeta_m=0$, there holds*

$$\sum_{i\in\mathcal{I}_k}\frac{2}{\lambda_{mi}}\left(\mathtt{t}_{mi}-\bar{\mathtt{t}}_{mi}\right)=-1. \tag{25}$$

Proof.

$$\sum_{i\in\mathcal{I}_k}\frac{2}{\lambda_{mi}}\left(\mathtt{t}_{mi}-\bar{\mathtt{t}}_{mi}\right)\overset{(5)}{=}-\sum_{i\in\mathcal{I}_m}\left(\frac{\prod\limits_{r=k\pm1,l\in\mathcal{I}_r}(\lambda_{mi}(\lambda_{mi}+1)+\ell_{kl})}{2\lambda_{mi}\,H_{mi}\,H_{mi}^{\sigma_{mi}^{-1}}}-\frac{\prod\limits_{r=k\pm1,l\in\mathcal{I}_r}(\lambda_{mi}(\lambda_{mi}-1)+\ell_{kl})}{2\lambda_{mi}\,H_{mi}\,H_{mi}^{\sigma_{mi}^{+1}}}\right).$$

where H_{mi} are defined in (6). This sum is a linear combination over $\mathrm{Sym}(\mathcal{F}^{(kl)})$ of $\Phi(m,\alpha)$ for $\alpha\in[0,2q-1]$, besides, the leading coefficient (for $\alpha=2q-1$)) equals -1. By 23, the whole sum equals -1, which yield (25). □

Step 4.7. *For $k,m\in[1,n)$ with $\zeta_k=1$, $\zeta_m=0$ and any $l\in\mathcal{I}_k$, for $Z_{kl}\in\{X_{kl},Y_{kl}\}$, there holds*

$$\sum_{i\in\mathcal{I}_m}\left[\left[Z_{kl},X_{mi}\right],Y_{mi}\right]+\left[\left[Z_{kl},Y_{mi}\right],X_{mi}\right]=Z_{kl}; \tag{26}$$

By the step 4.4 and (24), we have

$$\left[\left[X_{kl},X_{mi}\right],Y_{mi}\right]+\left[\left[X_{kl},Y_{mi}\right],X_{mi}\right]=-2\left(\frac{\lambda_{mi}+1}{R_{mi,kl}}\mathtt{t}_{mi}+\frac{\bar{\lambda}_{mi}+1}{\bar{R}_{mi,kl}}\bar{\mathtt{t}}_{mi}\right)X_{kl}=X_{kl}.$$

The equality for Y_{kl} is obtained by applying of the involution $\star$ for the both sides.

Step 4.8. *For $k,m\in[1,n)$ with $|k-m|=1$ and $\zeta_k=1$, $\zeta_m=0$, and $i,j\in\mathcal{I}_k$, $i\neq j$, there holds*

$$\begin{gathered}\left[\left[C_k,X_{mi}\right],X_{mj}\right]+\left[\left[C_k,X_{mj}\right],X_{mi}\right]=0;\\ \left[\left[C_k,Y_{mi}\right],Y_{mj}\right]\;+\;\left[\left[C_k,Y_{mj}\right],Y_{mi}\right]=0;\\ \left[\left[C_k,X_{mi}\right],X_{mi}\right]=0,\quad\left[\left[C_k,Y_{mi}\right],Y_{mi}\right]=0.\end{gathered} \tag{27}$$

Proof. We have $T_{ij}=X_{mi}\,X_{mj}\,C_k+C_k\,X_{mi}\,X_{mj}-2X_{mi}\,C_k\,X_{mj}=\left(C_k^{\sigma_{mi}\sigma_{mj}}+C_k-2C_k^{\sigma_{mi}}\right)\cdot X_{mi}\,X_{mj}=C_k\,\dfrac{\lambda_{mj}-\lambda_{mi}+1}{\lambda_{mi}\lambda_{mj}}\cdot X_{mi}\,X_{mj}$ because

$$C_k+C_k^{\sigma_{mi}\sigma_{mj}}-2C_k^{\sigma_{mi}}=\frac{C_k}{\lambda_{mi}\lambda_{mj}}\left(\lambda_{mi}\lambda_{mj}+(\lambda_{mi}-1)(\lambda_{mj}-1)-2(\lambda_{mi}-1)\lambda_{mj}\right)=C_k\,\frac{\lambda_{mj}-\lambda_{mi}+1}{\lambda_{mi}\lambda_{mj}}.$$

Then the left part of equality above equals $T_{ij}+T_{ji}=\dfrac{C_k}{\lambda_{mi}\lambda_{mj}}\left((\lambda_{mj}-\lambda_{mi}+1)\mu^{xx}_{mi,mj}+(\lambda_{mi}-\lambda_{mj}+1)\right)\cdot X_{mj}\,X_{mi}=0$.

Similarly, we have $\left[\left[C_k,X_{mi}\right],X_{mi}\right]=\left(C_k+C_k^{\sigma_{mi}\sigma_{mi}}-2C_k^{\sigma_{mi}}\right)\cdot X_{mi}X_{mi}=0$. □

Step 4.9. *For $k,m\in[1,n)$ with $|k-m|=1$ and $\zeta_k=1$, $\zeta_m=0$, and $i,j\in\mathcal{I}_k$, $i\neq j$, there holds*

$$\left[\left[C_k,X_{mi}\right],Y_{mi}\right]+\left[\left[C_k,Y_{mi}\right],X_{mi}\right]=C_k\,\frac{2}{\lambda_{mi}}\left(\mathtt{t}_{mi}-\bar{\mathtt{t}}_{mi}\right)=C_k. \tag{28}$$

Proof. The left part of equality equals

$$(Y_{mi}X_{mi}C_k + C_kY_{mi}X_{mi} - 2Y_{mi}C_kX_{mi}) + (X_{mi}Y_{mi}\,C_k + C_k\,X_{mi}Y_{mi} - 2X_{mi}C_kY_{mi}) =$$
$$= 2(C_k - C_k^{\sigma_{mi}^{-1}})\,\mathtt{t}_{mi} + 2(C_k - C_k^{\sigma_{mi}^{+1}})\,\bar{\mathtt{t}}_{mi} = C_k\,\frac{2}{\lambda_{mi}}\,(-\mathtt{t}_{mi} + \bar{\mathtt{t}}_{mi}) \overset{(25)}{=} C_k.$$

□

Step 4.10. *Let $\zeta_m = 0$, $\zeta_k = 1$, $|k-m| = 1$. By (16), $\mathcal{U}_k = \sum_{i\in\mathcal{I}_k} X_{ki} - \sum_{i\in\mathcal{I}_k} Y_{ki} + C_k$, and $\mathcal{U}_m = \sum_{j\in\mathcal{I}_m} X_{mj} - \sum_{j\in\mathcal{I}_m} Y_{mj}$. Then the identity $[[\mathcal{U}_k,\mathcal{U}_m],\mathcal{U}_m] = -\mathcal{U}_k$ is satisfied.*

Proof. Using (20), (21), (22) and (26), we obtain

$$[[X_{kl},\mathcal{U}_m],\mathcal{U}_m] = -\sum_{i\in\mathcal{I}_m}\Big([[X_{kl},X_{mi}],Y_{mi}] + [[X_{kl},Y_{mi}],X_{mi}]\Big) \overset{(26)}{=} -X_{kl}$$

and similarly $[[Y_{kl},\mathcal{U}_m],\mathcal{U}_m] = -Y_{kl}$. Further, by (27) and (28), we obtain $[[C_k,\mathcal{U}_m],\mathcal{U}_m] = -C_k$. Then $[[\mathcal{U}_k,\mathcal{U}_m],\mathcal{U}_m] = \mathcal{U}_k$. □

4.4. **The case of an even parameter m.**

Remark 8. *For $m,k \in [1,n)$ with $|m-k| = 1$ and $\zeta_m = 1$, $\zeta_k = 0$, and for some $l \in \mathcal{I}_k$, there holds*

$$\sum_{i\in\mathcal{I}_m}\left(\frac{2\lambda_{mi}+1}{R_{mi,kl}}\mathtt{t}_{mi} + \frac{2\bar{\lambda}_{mi}+1}{\bar{R}_{mi,kl}}\bar{\mathtt{t}}_{mi}\right) = 1 + C_m^2\frac{1}{\lambda_{kl}^2}, \tag{29}$$

where by (12), $C_m^2 = -\dfrac{(\vec{\lambda}^{\times}_{(m-1)})^2\cdot(\vec{\lambda}^{\times}_{(m+1)})^2}{(\vec{\ell}^{\times}_{(m)})^2}$.

Proof. Let $m = 2q$. We consider the index set $\mathbb{J} = \{tr \mid t = m\pm 1, r\in\mathcal{I}_t\}$, then $|\mathbb{J}| = |\mathcal{I}_{2q-1}| + |\mathcal{I}_{2q+1}| = 2q+1$. For $t = 2q\pm 1$, we have $R_{mi,tr} = -(\lambda_{mi}^2 - \lambda_{tr}^2)$, $\bar{R}_{mi,tr} = -(\lambda_{mi}^2 - \lambda_{tr}^2)$ (see (7)).

As before in Step (4.7), we denote by $\mathcal{F}^{(kl)} = \mathrm{Sym}\left(\{\ell_{m-1,r}\}_{r=1}^{q} \cup \{\ell_{m+1,r}\}_{r=1}^{q+1} \setminus \{\ell_{kl}\}\right)$ the ring of symmetric polynomials on $2q$ variables. Then $\prod_{tr\in\mathbb{J}\setminus\{kl\}} R_{mi,tr}$ and $\prod_{tr\in\mathbb{J}\setminus\{kl\}} \bar{R}_{mi,tr}$ are both the polynomials in λ_{mi}^2 and $\bar{\lambda}_{mi}^2$ correspondingly of degree $2q$ with coefficients from $\mathcal{F}^{(kl)}$, with leading coefficient 1, and with the free member equals $C^{(kl)} = \prod_{tr\in\mathbb{J}\setminus\{kl\}}(-\lambda_{tr}^2) = \dfrac{(\vec{\lambda}^{\times}_{(m-1)})^2\cdot(\vec{\lambda}^{\times}_{(m+1)})^2}{\lambda_{kl}^2}$. We can write

$$\prod_{tr\in\mathbb{J}\setminus\{kl\}} R_{mi,tr} = \lambda_{mi}^2\,F\left(\lambda_{mi}^2\right) + C^{(kl)}, \qquad \prod_{tr\in\mathbb{J}\setminus\{kl\}} \bar{R}_{mi,tr} = \bar{\lambda}_{mi}^2\,F\left(\bar{\lambda}_{mi}^2\right) + C^{(kl)}$$

where $F(t)$ is a polynomial over $\mathcal{F}^{(kl)}$ of degree $2q-1$, with leading coefficient 1.

Then we represent one summand from (29) as follows

$$\frac{2\lambda_{mi}+1}{\bar{\lambda}_{mi}^2 - \lambda_{kl}^2}\mathtt{t}_{mi} + \frac{2\bar{\lambda}_{mi}+1}{\lambda_{mi}^2 - \lambda_{kl}^2}\bar{\mathtt{t}}_{mi} \overset{(5)}{=} \frac{(2\lambda_{mi}+1)\prod_{tr\in\mathbb{J}\setminus\{kl\}} R_{mi,tr}}{h_{mi}\,H_{mi}\,H_{mi}^{\sigma_{mi}^{-1}}} + \frac{(2\bar{\lambda}_{mi}+1)\prod_{tr\in\mathbb{J}\setminus\{kl\}} \bar{R}_{mi,tr}}{\bar{h}_{mi}\,H_{mi}\,H_{mi}^{\sigma_{mi}^{+1}}} =$$
$$\overset{(8)}{=} \frac{\lambda_{mi}^2\,F\left(\lambda_{mi}^2\right) + C^{(kl)}}{\lambda_{mi}^2(2\lambda_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{-1}}} + \frac{\bar{\lambda}_{mi}^2\,F\left(\bar{\lambda}_{mi}^2\right) + C^{(kl)}}{\bar{\lambda}_{mi}^2(2\bar{\lambda}_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{+1}}} = S_{mi}^{(1)} + C^{(kl)}\,S_{mi}^{(2)}$$

with $S_{mi}^{(1)} = \dfrac{F(\lambda_{mi}^2)}{(2\lambda_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{-1}}} + \dfrac{F(\bar{\lambda}_{mi}^2)}{(2\bar{\lambda}_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{+1}}}$, $\deg(F(t)) = 2q-1$, and

$$S_{mi}^{(2)} = \sum_{j\in\mathcal{I}_m}\left(\frac{1}{\lambda_{mi}^2(2\lambda_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{-1}}} + \frac{1}{\bar{\lambda}_{mi}^2(2\bar{\lambda}_{mi}-1)\,H_{mi}\,H_{mi}^{\sigma_{mi}^{+1}}}\right).$$

For any $j \in [1;q]$, we substitute $\lambda_{mi} = \dfrac{1}{2} + x_i$, $\bar\lambda_{mi} = \dfrac{1}{2} - x_i$, then $\ell_{mi} = \dfrac{1}{4} - x_i^2$, so $(\ell_{mi} - \ell_{mj}) = x_j^2 - x_i^2$, $\quad \sigma_{mi}^{\mp 1}(\ell_{mi} - \ell_{mj}) = ((\lambda_{mi} \pm 1)(\bar\lambda_{mi} \mp 1) - \ell_{mj}) = x_j^2 - (x_i \pm 1)^2$, and

$$H_{mi} = \prod_{j\in \mathfrak{I}_m\setminus\{i\}} (\ell_{mj} - \ell_{mi}) = (-1)^{2q-1} \prod_{j\in \mathfrak{I}_m\setminus\{i\}} (x_i^2 - x_j^2);$$

$$H_{mi}^{\sigma_{mi}^{\pm 1}} = \prod_{j\in \mathfrak{I}_m\setminus\{i\}} \big(\ell_{mj} - (\lambda_{mi} \mp 1)(\bar\lambda_{mi} \pm 1)\big) = (-1)^{2q-1} \prod_{j\in \mathfrak{I}_m\setminus\{i\}} ((x_i \pm 1)^2 - x_j^2).$$

We have

$$S_{mi}^{(1)} = \frac{F(x_i(x_i+1)+1/4)}{2x_i \prod\limits_{j\neq i}(x_i^2 - x_j^2)((x_i+1)^2 - x_j^2)} + \frac{F(x_i(x_i-1)+1/4)}{-2x_i \prod\limits_{j\neq i}(x_i^2 - x_j^2)((x_i-1)^2 - x_j^2)} = 1$$

by Proposition 2 because the leading term of F equals $1 \cdot (x_i(x_i+1))^{2q-1}$.

It remains to calculate the sum $S_{mj}^{(2)}$ in terms x_i. By Proposition 3, we obtain

$$S_{mi}^{(2)} = \sum_{i=1}^{q} \left(\frac{1}{2x_i\,(x_i+1/2)^2 \mathcal{H}_{(q,i)} \mathcal{H}_{(q,i)}^-} - \frac{1}{2x_i\,(x_i-1/2)^2 \mathcal{H}_{(q,i)} \mathcal{H}_{(q,i)}^+} \right) = \frac{-1}{\prod_{i=1}^q (x_i^2 - 1/4)^2} = \frac{-1}{\prod_{i=1}^q \ell_{mi}^2},$$

because $1/4 - x_i^2 = (1/2 - x_i)(1/2 + x_i) = \lambda_{mi}\bar\lambda_{mi} = \ell_{mi}$.

Finally, $C^{(kl)} \sum\limits_{i=1}^q S_{mi}^{(2)} = -\dfrac{1}{\lambda_{kl}^2} \cdot \dfrac{(\vec\lambda^\times_{(m-1)})^2\,(\vec\lambda^\times_{(m+1)})^2}{(\vec\ell^\times)^2_{(m)}} \overset{(12)}{=} C_m^2 \dfrac{1}{\lambda_{kl}^2}$. The proof of 29 is over. □

Step 4.11. *For $k, m \in [1, n)$ with $\zeta_k = 0$, $\zeta_m = 1$ with $|k - m| = 1$, and any $i \in \mathfrak{I}_m$, $l \in \mathfrak{I}_k$, there holds*

$$\sum_{j\in\mathfrak{I}_m} [[X_{kl}, X_{mi}], Y_{mi}] + \sum_{j\in\mathfrak{I}_m} [[X_{kl}, Y_{mi}], X_{mi}] = \big(1 + C_m^2 \frac{1}{\lambda_{mj}^2}\big) X_{kl};$$
$$\sum_{j\in\mathfrak{I}_m} [[Y_{kl}, X_{mi}], Y_{mi}] + \sum_{j\in\mathfrak{I}_m} [[Y_{kl}, Y_{mi}], X_{mi}] = \big(1 + C_m^2 \frac{1}{\lambda_{mj}^2}\big) Y_{kl}. \tag{30}$$

Proof. By the step 4.4, for $k, m \in [1, n)$ with $\zeta_k = 0$, $\zeta_m = 1$ and any $i \in \mathfrak{I}_m$, $l \in \mathfrak{I}_k$, there holds

$$[[X_{kl}, X_{mi}], Y_{mi}] + [[X_{kl}, Y_{mi}], X_{mi}] = \left(\frac{2\lambda_{mi}+1}{R_{mi,kl}^2} \mathtt{t}_{mi} - \frac{2\lambda_{mi}-1}{\bar R_{mi,kl}^2} \bar{\mathtt{t}}_{mi} \right) X_{kl}.$$

In this case, $R_{mi,kl} = -(\lambda_{mi}^2 + \ell_{kl}) = -(\lambda_{mi}^2 - \lambda_{kl}^2)$, $\bar R_{kl,mi} = -(\bar\lambda_{mi}^2 + \ell_{kl}) = -(\bar\lambda_{mi}^2 - \lambda_{kl}^2)$. Then the proof follows from the remark 8. The second equality is equivalent. □

Step 4.12. *For $k, m \in [1, n)$ with $|k - m| = 1$ and $\zeta_m = 1$, $\zeta_k = 0$, $l \in \mathfrak{I}_k$ and $Z_{kl} \in \{X_{kl}, Y_{kl}\}$, there holds*

$$[[X_{kl}, C_m], X_{mi}] + [[X_{kl}, X_{mi}], C_m] = 0;$$
$$[[X_{kl}, C_m], Y_{mi}] + [[X_{kl}, Y_{mi}], C_m] = 0. \tag{31}$$

Proof. We proof the first equation of (31) for the case $Z_{kl} = X_{kl}$, other cases are similar.

We denote $C_m = C_m' \dfrac{\lambda_{kl}}{\lambda_{mi}\bar\lambda_{mi}}$. We calculate $\quad [[X_{kl}, C_m], X_{mi}] + [[X_{kl}, X_{mi}], C_m] =$

$$= (X_{mi} C_m X_{kl} + C_m X_{mi} X_{kl} - 2X_{mi} X_{kl} C_m) + (X_{kl} C_m X_{mi} + X_{kl} X_{mi} C_m - 2C_m X_{kl} X_{mi}) =$$
$$= (C_m + C_m^{\sigma_{mi}} - 2C_m^{\sigma_{kl}\sigma_{mi}}) X_{mi} X_{kl} + (C_m^{\sigma_{kl}\sigma_{mi}} + C_m^{\sigma_{kl}} - 2C_m) X_{kl} X_{mi} =$$
$$= 2C_m' 2 \frac{\bar\lambda_{mi}}{\lambda_{mi}\bar\lambda_{mi}(\lambda_{mi}-1)(\bar\lambda_{mi}+1)} \big((\lambda_{kl} + \bar\lambda_{mi}) X_{mi} X_{kl} + (\bar\lambda_{kl} + \lambda_{mi}) X_{kl} X_{mi}\big) = 0$$

because $C_m + C_m^{\sigma_{kl}} - 2C_m^{\sigma_{kl}\sigma_{mi}} = C'_m\left(\dfrac{\lambda_{kl}-1}{(\lambda_{mi}-1)(\bar\lambda_{mi}+1)} - \dfrac{\lambda_{kl}+1}{\lambda_{mi}\bar\lambda_{mi}}\right) = 2\bar\lambda_{mi}(\lambda_{kl}+\bar\lambda_{mi})$, and $C_m + C_m^{\sigma_{mi}} - 2C_m^{\sigma_{kl}\sigma_{mi}} = 2\bar\lambda_{mi}(\bar\lambda_{kl}+\lambda_{mi})$. The assertion of step 4.12 follows. □

Step 4.13. *For $m, k \in [1, n)$ with $|m-k| = 1$ and $\zeta_k = 0$, $\zeta_m = 1$, there hold*

$$[[X_{kl}, C_m], C_m] = C_m^2\frac{1}{\lambda_{kl}^2}X_{kl}, \qquad [[Y_{kl}, C_m], C_m] = C_m^2\frac{1}{\lambda_{kl}^2}Y_{kl}. \tag{32}$$

Proof. $[[X_{kl}, C_m], C_m] = \left(C_m^2 + (C_m^{\sigma_{kl}})^2 - 2C_mC_m^{\sigma_{kl}}\right)X_{kl} = \left(1 + \dfrac{(\lambda_{kl}-1)^2}{\lambda_{kl}^2} - 2\dfrac{\lambda_{kl}-1}{\lambda_{kl}}\right)C_m^2\cdot X_{kl} = C_m^2\dfrac{1}{\lambda_{kl}^2}X_{kl}$. □

Now we are ready to finish the proof of equality (18).

Step 4.14. *Let $\zeta_m = 1$, $\zeta_k = 0$, $|k-m| = 1$. By (16), $\mathcal{U}_k = \sum_{i\in\mathfrak{I}_k} X_{ki} - \sum_{i\in\mathfrak{I}_k} Y_{ki}$, and $\mathcal{U}_m = \sum_{j\in\mathfrak{I}_m} X_{mj} - \sum_{j\in\mathfrak{I}_m} Y_{mj} + C_m$. Then the identity $[[\mathcal{U}_k, \mathcal{U}_m], \mathcal{U}_m] = -\mathcal{U}_k$ is satisfied.*

Proof. The proof follows from the following observations:

$$\begin{aligned}
\sum_{i,j\in\mathfrak{I}_m, i\neq j}\left([[X_{kl}, X_{mi}], X_{mj}] + [[X_{kl}, X_{mj}], X_{mi}]\right) &\overset{(20)}{=} 0\\
\sum_{i,j\in\mathfrak{I}_m, i\neq j}\left([[X_{kl}, Y_{mi}], Y_{mj}] + [[X_{kl}, Y_{mj}], Y_{mi}]\right) &\overset{(20)}{=} 0\\
\sum_{i\in\mathfrak{I}_m}\left([[X_{kl}, X_{mi}], X_{mi}] + [[X_{kl}, Y_{mi}], Y_{mi}]\right) &\overset{(21)}{=} 0\\
-\sum_{i,j\in\mathfrak{I}_m, i\neq j}\left([[X_{kl}, X_{mi}], Y_{mj}] + [[X_{kl}, Y_{mj}], X_{mi}]\right) &\overset{(22)}{=} 0\\
-\sum_{i\in\mathfrak{I}_m}\left([[X_{kl}, X_{mi}], Y_{mi}] + [[X_{kl}, Y_{mi}], X_{mi}]\right) &\overset{(30)}{=} -(1 + C_m^2\frac{1}{\lambda_{mj}^2})X_{kl}\\
\sum_{i\in\mathfrak{I}_m}\left([[X_{kl}, X_{mi}], C_m] + [[X_{kl}, C_m], X_{mi}]\right) &\overset{(31)}{=} 0\\
-\sum_{i\in\mathfrak{I}_m}\left([[X_{kl}, Y_{mi}], C_m] + [[X_{kl}, C_m], Y_{mi}]\right) &\overset{(31)}{=} 0\\
\sum_{i\in\mathfrak{I}_m}[[X_{kl}, C_m], C_m] &\overset{(32)}{=} C_m^2\frac{1}{\lambda_{kl}^2}X_{kl}
\end{aligned}$$

The cases when we accept Y_{kl} instead of X_{kl} are true. We summarize this equalities and obtain the above equation. □

Thus, the proof of the equation (18) of Proposition 1 is complete.

5. Proposition 1 : the proof of the third defining equality

The proof of this proposition consists from the following lemmas.

Remark 9. *We will check the equality (19) for separate summands of $[\mathcal{U}_k, [\mathcal{U}_{k+1}, [\mathcal{U}_{k+2}, \mathcal{U}_{k+1}]]]$. We assume that the coefficient for each monomial is equal to zero. In addition, these intermediate formulas remain true by the action of the group G, and even by the action of the group G if they do not contain the summand of a type C_m. In particular, they are invariant with respect to the applying of inversions ε_{tl} for every suitable index.*

We give some preliminary observations. We assume $q = \left[\frac{k}{2}\right] + 1$, we have the following two cases:

(i) $\zeta_{k+1} = 0$, $k = 2q-2$, then $\zeta_k = \zeta_{k+2} = 1$, and $\mathfrak{I}_k = [1; q-1]$, $\mathfrak{I}_{k+1} = \mathfrak{I}_{k+2} = [1; q]$;

(ii) $\zeta_{k+1}=1$, $k=2q-1$, then $\zeta_k=\zeta_{k+2}=0$, and $\mathcal{I}_k=\mathcal{I}_{k+1}=[1;q]$, $\mathcal{I}_{k+2}=[1;q{+}1]$.

For brevity and the simplicity of the denotations, we will indicate the considered indexes in the short form in such a way. Let Z_α be one of the variables $\{X_{kl},Y_{kl}\mid l\in\mathcal{I}_k\}$, and let Z_β be one of the variables $\{X_{k+2,l},Y_{k+2,l}\mid l\in\mathcal{I}_{k+2}\}$. Respectively, λ_α, λ_β be the variables with correspondent indexes. Since $[\mathcal{U}_k,\mathcal{U}_{k+2}]=0$, hence $[Z_\alpha,Z_\beta]=0$.

Besides, we will write index i or j instead of ${}_{k+1,i}$ or ${}_{k+1,j}$.

By (9), we have

$$\mu^{xx}_{i,\alpha}=-\frac{\lambda_\alpha+\bar\lambda_i}{\bar\lambda_\alpha+\lambda_i},\quad \mu^{xx}_{\alpha,i}=(\mu^{xx}_{i,\alpha})^{-1},\quad \mu^{xy}_{\alpha,i}=-\frac{\bar\lambda_\alpha+\bar\lambda_i}{\lambda_\alpha+\lambda_i},\quad \mu^{yx}_{i,\alpha}=(\mu^{xy}_{\alpha,i})^{-1}. \tag{33}$$

The definitions of the parameters μ for j instead of i and β instead of α is analogous.

Moreover, we have

$$\mu^{xx}_{i,j}=\frac{\lambda_i-\lambda_j+1}{\lambda_i-\lambda_j-1},\qquad X_iX_j=\mu^{xx}_{i,j}X_jX_i. \tag{34}$$

5.1. **General calculations.**

If $\zeta_{k+1}=1$ then $C_k=C_{k+2}=0$, and $C_{k+1}=\dfrac{\lambda_\alpha\lambda_\beta}{\ell_i\ell_j}\cdot C'_{k+1}$ where C'_{k+1} do not depend on the $\lambda_\alpha,\lambda_\beta,\lambda_i,\lambda_j$.

If $\zeta_{k+1}=0$ then $C_{k+1}=0$, $C'_k=\dfrac{\lambda_i\lambda_j}{\ell_\alpha}\cdot C'_k$, $C'_{k+2}=\dfrac{\lambda_i\lambda_j}{\ell_\beta}\cdot C'_{k+2}$, where C'_k, C'_{k+2} do not depend on the $\lambda_\alpha,\lambda_\beta,\lambda_i,\lambda_j$.

We shall use such notations:

$$\theta_i=(1-\mu^{xx}_{i,\alpha})(1-\mu^{xx}_{\beta,i})=\frac{1}{(\bar\lambda_\alpha+\lambda_i)(\bar\lambda_i+\lambda_\beta)},\quad \bar\theta_i=(1-\mu^{yx}_{i,\alpha})(1-\mu^{xy}_{\beta,i})=\frac{1}{(\bar\lambda_\alpha+\bar\lambda_i)(\lambda_i+\lambda_\beta)}. \tag{35}$$

There are $Y_iX_i=\mathtt{t}_i$, $X_iY_i=\bar{\mathtt{t}}_i$, where $\mathtt{t}_i$, $\bar{\mathtt{t}}_i$ are defined in (5) ((5)).

We assume, the formulae obtained from above ones by action of the group G are also true.

Remark 10. *There hold* $[X_\alpha,X_\beta]=0$, $[X_t,\lambda_r]=0$, $r\neq t$, $[X_\alpha,X_i]=(1-\mu^{xx}_{i,\alpha})X_\alpha X_i$, *and*

$$[[X_\alpha,X_i],X_\beta]=\theta_iX_\alpha X_iX_\beta;\qquad [[X_\alpha,Y_i],X_\beta]=\bar\theta_iX_\alpha Y_iX_\beta; \tag{36}$$

By direct calculation.

Lemma 2. *The following hold*

$$\mathcal{S}=[[[X_\alpha,X_i],X_\beta],X_j]+[[[X_\alpha,X_j],X_\beta],X_i]=0,\quad i\neq j, \tag{37}$$

$$[[[X_\alpha,X_i],X_\beta],X_i]=0. \tag{38}$$

Proof of (37). By (9), $\mu^{xx}_{i,\alpha}=-\dfrac{\lambda_\alpha+\bar\lambda_i}{\bar\lambda_\alpha+\lambda_i}$ and $\mu^{xx}_{\beta,i}=-\dfrac{\bar\lambda_\beta+\lambda_i}{\lambda_\beta+\bar\lambda_i}$. Then by (36), $[[[X_\alpha,X_i],X_\beta],X_j]=\theta_i\big(\mu^{xx}_{\beta,j}X_\alpha X_iX_jX_\beta-\mu^{xx}_{j,\alpha}X_\alpha X_jX_iX_\beta\big)$. We obtain

$$\begin{aligned}\mathcal{S}&=\theta_i\big(\mu^{xx}_{\beta,j}X_\alpha X_iX_jX_\beta-\mu^{xx}_{j,\alpha}X_\alpha X_jX_iX_\beta\big)+\theta_j\big(\mu^{xx}_{\beta,i}X_\alpha X_jX_iX_\beta-\mu^{xx}_{i,\alpha}X_\alpha X_iX_jX_\beta\big)=\\&=\big(\theta_i\mu^{xx}_{\beta,j}-\theta_j\mu^{xx}_{i,\alpha}\big)X_\alpha X_iX_jX_\beta+\big(\theta_j\mu^{xx}_{\beta,i}-\theta_i\mu^{xx}_{j,\alpha}\big)X_\alpha X_jX_iX_\beta=\\&=\theta_i\theta_j(\bar\lambda_\alpha+\bar\lambda_\beta+\lambda_i+\lambda_j-1)\big((\lambda_i-\lambda_j-1)X_\alpha X_iX_jX_\beta+(\lambda_j-\lambda_i-1)X_\alpha X_jX_iX_\beta\big)=0\end{aligned}$$

because $\mu^{xx}_{ij}=\dfrac{\lambda_i-\lambda_j+1}{\lambda_i-\lambda_j-1}$. □

Proof of (38). Using (36), we obtain $[[[X_\alpha,X_i],X_\beta],X_i]=\theta_i\,X_\alpha X_i\,\mu^{xx}_{\beta,i}X_iX_\beta-X_i\theta_iX_\alpha X_iX_\beta=$ $=\big(\theta_i\sigma_i(\mu^{xx}_{\beta,i})-\sigma_i(\theta_i)\mu^{xx}_{i,\alpha}\big)\,X_\alpha X_iX_iX_\beta=0.$ □

We denote $R_{i,\alpha} = (\lambda_i{+}\lambda_\alpha)(\lambda_i{+}\bar\lambda_\alpha)$, $\bar R_{i,\alpha} = (\bar\lambda_i{+}\lambda_\alpha)(\bar\lambda_i{+}\bar\lambda_\alpha)$, and $\mathtt{t}'_i = \dfrac{\mathtt{t}_i}{R_{i,\alpha}R_{i,\beta}}$, $\bar{\mathtt{t}}'_i = \dfrac{\bar{\mathtt{t}}_i}{\bar R_{i,\alpha}\bar R_{i,\beta}}$. Then the following lemma holds.

Lemma 3. *There are*

$$[[[X_\alpha, X_i], X_\beta], Y_i]{+}[[[X_\alpha, Y_i], X_\beta], X_i] = \tag{39}$$
$$= \begin{cases} (\lambda_\alpha{+}\lambda_\beta{-}1)\big((2\lambda_i{+}1)\mathtt{t}'_i + (2\bar\lambda_i{+}1)\bar{\mathtt{t}}'_i\big)\cdot X_\alpha X_\beta, & \text{if } \zeta_{k+1}=1;\\ (\bar\lambda_\alpha{+}\bar\lambda_\beta)\big(2(\bar\lambda_i{-}1)\mathtt{t}'_i + 2(\lambda_i{-}1)\bar{\mathtt{t}}'_i\big)\cdot X_\alpha X_\beta, & \text{if } \zeta_{k+1}=0.\end{cases}$$

and

$$[[[X_\alpha, X_i], Y_\beta], Y_i]{+}[[[X_\alpha, Y_i], Y_\beta], X_i] = \tag{40}$$
$$= \begin{cases} (\lambda_\alpha{+}\bar\lambda_\beta{-}1)\big((2\lambda_i{+}1)\mathtt{t}'_i + (2\bar\lambda_i{+}1)\bar{\mathtt{t}}'_i\big)\cdot X_\alpha Y_\beta, & \text{if } \zeta_{k+1}=1;\\ (\bar\lambda_\alpha{+}\lambda_\beta)\big(2(\bar\lambda_i{-}1)\mathtt{t}'_i + 2(\lambda_i{-}1)\bar{\mathtt{t}}'_i\big)\cdot X_\alpha Y_\beta, & \text{if } \zeta_{k+1}=0.\end{cases}$$

Proof. By (36), we have We calculate

$$T = [[[X_\alpha, X_i], X_\beta], Y_i] = [\theta_i X_\alpha X_i X_\beta, Y_i] = \theta_i X_\alpha X_i \mu^{xy}_{\beta,i} Y_i X_\beta - Y_i\theta_i X_\alpha X_i X_\beta =$$
$$= \theta_i X_\alpha \sigma_i(\mu^{xy}_{\beta,i}) X_i Y_i X_\beta - \sigma_i^{-1}(\theta_i) Y_i X_\alpha X_i X_\beta = \theta_i\sigma_i(\mu^{xy}_{\beta,i}) X_\alpha \bar{\mathtt{t}}_i X_\beta - \sigma_i^{-1}(\theta_i)\mu^{yx}_{i,\alpha} X_\alpha \mathtt{t}_i X_\beta.$$

Similarly, $\bar T = [[[X_\alpha, Y_i], X_\beta], X_i] = \bar\theta_i\sigma_i^{-1}(\mu^{xx}_{\beta,i}) X_\alpha \mathtt{t}_i X_\beta - \sigma_i(\bar\theta_i)\mu^{xx}_{i,\alpha} X_\alpha \bar{\mathtt{t}}_i X_\beta$.
Then there are

$$T + \bar T = \big(\theta_i\sigma_i(\mu^{xy}_{\beta,i}) - \sigma_i(\bar\theta_i)\mu^{xx}_{i,\alpha}\big) X_\alpha\bar{\mathtt{t}}_i X_\beta - \big(\sigma_i^{-1}(\theta_i)\mu^{yx}_{i,\alpha} - \bar\theta_i\sigma_i^{-1}(\mu^{xx}_{\beta,i})\big) X_\alpha \mathtt{t}_i X_\beta =$$
$$= \big(\theta_i\sigma_i(\mu^{xy}_{\beta,i}) - \sigma_i(\bar\theta_i)\mu^{xx}_{i,\alpha}\big)\ \sigma_\alpha(\bar\psi_\alpha)\bar\psi_\beta \bar{\mathtt{t}}'_i X_\alpha X_\beta$$
$$+ \big(\bar\theta_i\sigma_i^{-1}(\mu^{xx}_{\beta,i}) - \sigma_i^{-1}(\theta)\mu^{yx}_{i,\alpha}\big)\sigma_\alpha(\psi_\alpha)\psi_\beta \mathtt{t}'_i X_\alpha X_\beta$$

where $\mathtt{t}_i = \psi_\alpha\psi_\beta\mathtt{t}'_i$, $\bar{\mathtt{t}}_i = \bar\psi_\alpha\bar\psi_\beta\bar{\mathtt{t}}'_i$ with $\psi_\alpha = (\lambda_i{+}\lambda_\alpha)(\lambda_i{+}\bar\lambda_\alpha)$, $\bar\psi_\alpha = (\bar\lambda_i{+}\lambda_\alpha)(\bar\lambda_i{+}\bar\lambda_\alpha)$, $\psi_\beta = (\lambda_i{+}\lambda_\beta)(\lambda_i{+}\bar\lambda_\beta)$, $\bar\psi_\beta = (\bar\lambda_i{+}\lambda_\beta)(\bar\lambda_i{+}\bar\lambda_\beta)$. We calculate

$$\theta_i\sigma_i(\mu^{xy}_{\beta,i}) - \sigma_i(\bar\theta_i)\mu^{xx}_{i,\alpha} = \frac{1}{(\bar\lambda_\alpha{+}\lambda_i)(\bar\lambda_i{+}\lambda_\beta)}\frac{-(\bar\lambda_i{+}\bar\lambda_\beta{+}1)}{\lambda_i{+}\lambda_\beta{-}1} - \frac{1}{(\bar\lambda_\alpha{+}\bar\lambda_i{+}1)(\lambda_i{+}\lambda_\beta{-}1)}\frac{-(\lambda_\alpha{+}\bar\lambda_i)}{\bar\lambda_\alpha{+}\lambda_i} =$$
$$= \frac{1}{(\bar\lambda_\alpha{+}\lambda_i)(\lambda_i{+}\lambda_\beta{-}1)}\left(\frac{(\bar\lambda_i{+}\lambda_\alpha)}{(\bar\lambda_\alpha{+}\bar\lambda_i{+}1)} - \frac{(\bar\lambda_i{+}\bar\lambda_\beta{+}1)}{(\bar\lambda_i{+}\lambda_\beta)}\right),$$

then

$$\big(\theta_i\sigma_i(\mu^{xy}_{\beta,i}) - \sigma_i(\bar\theta_i)\mu^{xx}_{i,\alpha}\big)\ \sigma_\alpha(\bar\psi_\alpha)\bar\psi_\beta = (\bar\lambda_i{+}\lambda_\alpha)(\bar\lambda_i{+}\lambda_\beta) - (\bar\lambda_i{+}\bar\lambda_\beta{+}1)(\bar\lambda_i{+}\bar\lambda_\alpha{+}1);$$
$$\big(\bar\theta_i\sigma_i^{-1}(\mu^{xx}_{\beta,i}) - \sigma_i^{-1}(\theta)\mu^{yx}_{i,\alpha}\big)\ \sigma_\alpha(\psi_\alpha)\psi_\beta = (\lambda_i{+}\lambda_\alpha)(\lambda_i{+}\lambda_\beta) - (\lambda_i{+}\bar\lambda_\beta{+}1)(\lambda_i{+}\bar\lambda_\alpha{+}1).$$

Finally, we substitute $\bar\lambda_i = \zeta_{k+1} - \lambda_i$ to obtain (39); the equation (40) can be proved similarly. □

5.2. **The case of an even parameter $k+1$.**

Lemma 4. *Let $\zeta_{k+1} = 1$ and $\zeta_k = \zeta_{k+2} = 0$. Then the following hold*

$$\big[[[Z_\alpha, Z_i], Z_\beta], C_{k+1}\big] + \big[[[Z_\alpha, C_{k+1}], Z_\beta], Z_i\big] = 0. \tag{41}$$

$$[[[X_\alpha, C_{k+1}], X_\beta], C_{k+1}] = (\lambda_\alpha + \lambda_\beta - 1)\frac{C^2_{k+1}}{\lambda^2_\alpha\lambda^2_\beta}\, X_\alpha X_\beta, \tag{42}$$

$$[[[X_\alpha, C_{k+1}], Y_\beta], C_{k+1}] = (\lambda_\alpha + \bar\lambda_\beta - 1)\frac{C^2_{k+1}}{\lambda^2_\alpha\lambda^2_\beta}\, X_\alpha Y_\beta. \tag{43}$$

Proof. We proof (41) for the variables X_α, X_β, X_i, for other variables the proof is similar.

In this case, X_α, X_β commutes. Then $\bar\lambda_i = 1-\lambda_i$, $\bar\lambda_\alpha = -\lambda_\alpha$, $\bar\lambda_\beta = -\lambda_\beta$. We denote $C_{k+1} = \frac{\lambda_\alpha\lambda_\beta}{\lambda_i\bar\lambda_i}C'$ where C' do not depend on the $\lambda_\alpha, \lambda_i, \lambda_\beta$.

First, using (36), we obtain $\big[[[X_\alpha, X_i], X_\beta], C_{k+1}\big] = C'\big[(1-\mu_{i,\alpha})(1-\mu_{\beta,i})X_\alpha X_i X_\beta, \frac{\lambda_\alpha\lambda_\beta}{\lambda_i\bar\lambda_i}\big] =$
$= C'\theta_i\left(\frac{(\lambda_\alpha-1)(\lambda_\beta-1)}{(\lambda_i-1)(\bar\lambda_i+1)} - \frac{\lambda_\alpha\lambda_\beta}{\lambda_i\bar\lambda_i}\right)X_\alpha X_i X_\beta$, where $\theta_i = (1-\mu_{i,\alpha})(1-\mu_{\beta,i}) = \frac{1}{(\bar\lambda_\alpha+\lambda_i)(\lambda_\beta+\bar\lambda_i)}$.

On the other hand, $[[X_\alpha, C_{k+1}], X_\beta] = -\frac{1}{\lambda_i\bar\lambda_i}C'X_\alpha X_\beta$, and then

$$\big[[[X_\alpha, C_{k+1}], X_\beta], X_i\big] = C'\left(\frac{\mu^{xx}_{i,\alpha}}{(\lambda_i-1)(\bar\lambda_i+1)} - \frac{\mu^{xx}_{\beta,i}}{\lambda_i\bar\lambda_i}\right)X_\alpha X_i X_\beta.$$

Then the multiplier for $C'X_\alpha X_i X_\beta$ in the sum (41) equals
$$\left(\frac{\mu^{xx}_{i,\alpha}}{(\lambda_i-1)(\bar\lambda_i+1)} + \theta_i\,\frac{(\lambda_\alpha-1)(\lambda_\beta-1)}{(\lambda_i-1)(\bar\lambda_i+1)}\right) - \left(\frac{\mu^{xx}_{\beta,i}}{\lambda_i\bar\lambda_i} + \theta_i\,\frac{\lambda_\alpha\lambda_\beta}{\lambda_i\bar\lambda_i}\right) =$$
$$\frac{\mu^{xx}_{i,\alpha}+\theta_i(\lambda_\alpha-1)(\lambda_\beta-1)}{(\lambda_i-1)(\bar\lambda_i+1)} - \frac{\mu^{xx}_{\beta,i}+\theta_i\lambda_\alpha\lambda_\beta}{\lambda_i\bar\lambda_i} = \theta_i\,(\lambda_\alpha+\lambda_\beta-\lambda_i)\cdot\left(\frac{(\lambda_i-2)}{(\lambda_i-1)(\bar\lambda_i+1)} - \frac{1}{\bar\lambda_i}\right) = 0. \quad \square$$

Proof of (42). We denote $C_{k+1} = \lambda_\alpha\lambda_\beta C$ where C do not depend on the variables λ_α and λ_β. While $[X_\alpha, X_\beta] = 0$, we have

$$\begin{aligned}[[[X_\alpha, C_{k+1}], X_\beta], C_{k+1}] &= C^2[[[X_\alpha, \lambda_\alpha\lambda_\beta], X_\beta], \lambda_\alpha\lambda_\beta] = C^2[-\lambda_\beta X_\alpha X_\beta + (\lambda_\beta-1)X_\beta X_\alpha, \lambda_\alpha\lambda_\beta]\\ &= C^2(\lambda_\alpha+\lambda_\beta-1)\big(\lambda_\beta X_\alpha X_\beta - (\lambda_\beta-1)X_\beta X_\alpha\big) = C^2(\lambda_\alpha+\lambda_\beta-1)\,X_\alpha X_\beta.\end{aligned}$$

The proof of (43) is analogous. $\square$

Remark 11. *Let $\zeta_{k+1} = 1$, $k = 2q-1$, and $\zeta_k = \zeta_{k+2} = 0$. Then for some $\alpha \in \mathfrak{I}_k$ and some $\beta \in \mathfrak{I}_{k+2}$, there holds*

$$\sum_{i\in\mathfrak{I}_{k+1}}\left(\frac{2\lambda_i+1}{R_{i,\alpha}\,R_{i,\beta}}\mathtt{t}_i + \frac{2\bar\lambda_i+1}{\bar R_{i,\alpha}\,\bar R_{i,\beta}}\bar{\mathtt{t}}_i\right) = C^2_{k+1}\frac{1}{\lambda^2_\alpha\lambda^2_\beta}, \tag{44}$$

where by (12), $C^2_{k+1} = -\frac{(\vec\lambda^{\times}_{(k)})^2\cdot(\vec\lambda^{\times}_{(k+2)})^2}{(\vec\ell^{\times}_{(k+1)})^2}$.

The proof of remark 11 is absolutely similar to the proof of remark 8. In particular, we write

$$\prod_{\gamma\neq\alpha,\beta} R_{i,\gamma} = \lambda_i^2\,F\left(\lambda_i^2\right) + C^2_{k+1}\frac{1}{\lambda^2_\alpha\lambda^2_\beta},\qquad \prod_{\gamma\neq\alpha,\beta}\bar R_{i,\gamma} = \bar\lambda_i^2\,F\left(\bar\lambda_i^2\right) + C^2_{k+1}\frac{1}{\lambda^2_\alpha\lambda^2_\beta}$$

where $F(t)$ is a polynomial of degree $2q-2$, with leading coefficient 1.

Lemma 5. *Let $\zeta_{k+1} = 1$, $k = 2q-1$, and $\zeta_k = \zeta_{k+2} = 0$. Then the following hold*

$$\sum_{i\in\mathfrak{I}_{k+1}}\left([[[X_\alpha, X_i], X_\beta], Y_i] + [[[X_\alpha, Y_i], X_\beta], X_i]\right) = -(\lambda_\alpha+\lambda_\beta-1)\,C^2_{k+1}\frac{1}{\lambda^2_\alpha\lambda^2_\beta}\cdot X_\alpha X_\beta \tag{45}$$

Proof. Using (39), we represent the i-th summand of (45) in the form

$$[[[X_\alpha, X_i], X_\beta], Y_i] + [[[X_\alpha, Y_i], X_\beta], X_i] = -(\lambda_\alpha+\lambda_\beta-1)\left(\frac{2\lambda_i+1}{R_{i,\alpha}\,R_{i,\beta}}\mathtt{t}_i + \frac{2\bar\lambda_i+1}{\bar R_{i,\alpha}\,\bar R_{i,\beta}}\bar{\mathtt{t}}_i\right)\cdot X_\alpha X_\beta.$$

Then the proof of (45) follows from the remark 11. $\square$

Corollary 1. *Let $\zeta_k = \zeta_{k+2} = 0$, and $\zeta_{k+1} = 1$, then for all $Z_\alpha \in \{X_\alpha, Y_\alpha\}$ and $Z_\beta \in \{X_\beta, Y_\beta\}$*

$$-\sum_{i=1}^{p}\Big([[[Z_\alpha, X_i], Z_\beta], Y_i] + [[[Z_\alpha, Y_i], Z_\beta], X_i]\Big) + [[[Z_\alpha, C_{k+1}], Z_\beta], C_{k+1}] = 0.$$

Proof. By (42) and (45), this holds for $Z_\alpha = X_\alpha$ and $Z_\beta = X_\beta$. For other cases the proof is similar. □

Finally, applying (37), (38), (41) and the similar equalities obtained with an involution, we finish the proof of (19) from Proposition 1 for the case of an even $k+1$.

5.3. The case of an odd parameter $k+1$.

Remark 12. *There holds*

$$[[C_k, X_i], X_\beta] = \frac{C_k}{\lambda_i}[X_i, X_\beta], \qquad [[C_k, Y_i], X_\beta] = \frac{C_k}{\bar\lambda_i}[Y_i, X_\beta]. \tag{46}$$

Direct verification.

Lemma 6. *Let $\zeta_k = \zeta_{k+2} = 1$, and $\zeta_{k+1} = 0$. Then*

$$[[[C_k, X_{k+1,i}], X_{k+2,\beta}], X_{k+1,j}] + [[[C_k, X_{k+1,j}], X_{k+2,\beta}], X_{k+1,i}] = 0. \tag{47}$$

$$[[[C_k, X_{k+1,i}], X_{k+2,\beta}], X_{k+1,i}] = 0. \tag{48}$$

$$[[[C_k, X_{k+1,i}], C_{k+2}], X_{k+1,j}] + [[[C_k, X_{k+1,j}], C_{k+2}], X_{k+1,i}] = 0. \tag{49}$$

$$[[[C_k, X_{k+1,i}], C_{k+2}], Y_{k+1,i}] + [[[C_k, Y_{k+1,i}], C_{k+2}], X_{k+1,i}] = 0. \tag{50}$$

$$[[[C_k, X_{k+1,i}], C_{k+2}], X_{k+1,i}] = 0. \tag{51}$$

Proof of (47). We denote $C_k = C\cdot\lambda_i\lambda_j$. We consider

$$\frac{1}{C}[[[C_k, X_i], X_\beta], X_j] + \frac{1}{C}[[[C_k, X_j], X_\beta], X_i] = [[[\lambda_i\lambda_j, X_i], X_\beta], X_j] + [[[\lambda_i\lambda_j, X_j], X_\beta], X_i] =$$
$$= [\lambda_j[X_i, X_\beta], X_j] + [\lambda_i[X_j, X_\beta], X_i] = \lambda_j\mu^{xx}_{\beta,j}(1-\mu^{xx}_{\beta,i})X_iX_jX_\beta - (\lambda_i-1)(1-\mu^{xx}_{\beta,j})X_iX_jX_\beta +$$
$$+\lambda_i\mu^{xx}_{\beta,i}(1-\mu^{xx}_{\beta,j})X_jX_iX_\beta - (\lambda_j-1)(1-\mu^{xx}_{\beta,i})X_jX_iX_\beta = (\Phi_{ij} + \Phi_{ji}\mu^{xx}_{ji})X_iX_jX_\beta = 0$$

because

$$\Phi_{ij}/\Phi_{ji} = \frac{(\lambda_i+\bar\lambda_j-1)(\lambda_i+\lambda_j-\lambda_\beta)}{(\bar\lambda_i+\lambda_\beta)(\bar\lambda_j+\lambda_\beta)} : \frac{(\lambda_j+\bar\lambda_i-1)(\lambda_i+\lambda_j-\lambda_\beta)}{(\bar\lambda_i+\lambda_\beta)(\bar\lambda_j+\lambda_\beta)} = \frac{(\lambda_i+\bar\lambda_j-1)}{(\lambda_j+\bar\lambda_i-1)} \overset{(34)}{=} -\mu^{xx}_{ji}.$$

The proof is over. □

Proof of (48). In this case, $[[[C_k, X_i], X_\beta], X_i] \overset{(46)}{=} \frac{C_k}{\lambda_i}[[X_i, X_\beta], X_i] = \frac{C_k}{\lambda_i}\Phi\cdot X_iX_iX_\beta$, where

$$\Phi = 2\sigma_i(\mu^{xx}_{\beta,i}) - \mu^{xx}_{\beta,i}\sigma_i(\mu^{xx}_{\beta,i}) - 1 = -2\frac{\lambda_i+\bar\lambda_\beta-1}{\bar\lambda_i+\lambda_\beta+1} - \frac{(\lambda_i+\bar\lambda_\beta)(\lambda_i+\bar\lambda_\beta-1)}{(\bar\lambda_i+\lambda_\beta)(\bar\lambda_i+\lambda_\beta+1)} - 1 = 0.$$ □

Proof of (49). We present $C_k = \lambda_i\lambda_jC'_k$ and $C_{k+2} = \lambda_i\lambda_jC'_{k+2}$. Then

$$[[[C_k, X_i], C_{k+2}], X_j] = C'_kC'_{k+2}[[[\lambda_i\lambda_j, X_i], \lambda_i\lambda_j], X_j].$$

We calculate $[[[\lambda_i\lambda_j, X_i], \lambda_i\lambda_j], X_j] = [[\lambda_jX_i, \lambda_i\lambda_j], X_j] = -[\lambda_j^2X_i, X_j] = -\lambda_j^2X_iX_j + (\lambda_j-1)^2X_jX_i$. Then the sum (49) equals $-\lambda_j^2X_iX_j + (\lambda_j-1)^2X_jX_i - \lambda_i^2X_jX_i + (\lambda_i-1)^2X_iX_j =$
$= (\lambda_i+\lambda_j-1)\big((\lambda_i-1-\lambda_j) + (\lambda_j-1-\lambda_i)\mu^{xx}_{ji}\big)X_iX_j \overset{(34)}{=} 0.$ □

Proof of (50). We present $C_k = \lambda_iC'_k$ and $C_{k+2} = \lambda_iC'_{k+2}$. Then the sum equals:

$$C'_kC'_{k+2}\Big([[[\lambda_i, X_i], \lambda_i], Y_i] + [[[\lambda_i, Y_i], \lambda_i], X_i]\Big) = -C'_kC'_{k+2}\Big([X_i, Y_i] + [Y_i, X_i]\Big) = 0.$$ □

Proof of (51). We present $C_k = \lambda_iC'_k$ and $C_{k+2} = \lambda_iC'_{k+2}$. Then $[[[C_k, X_i], C_{k+2}], X_i] =$
$= C'_kC'_{k+2}[[[\lambda_i, X_i], \lambda_i], X_i] = C'_kC'_{k+2}[[X_i, \lambda_i], X_i] = -C'_kC'_{k+2}[X_i, X_i] = 0.$ □

Lemma 7. *Let $\zeta_{k+1} = 0$, $k = 2p$, and $\zeta_k = \zeta_{k+2} = 1$. Then the following hold*

$$[[[X_\alpha, X_i], X_\beta], Y_i] + [[[X_\alpha, Y_i], X_\beta], X_i] = 0.$$

Proof. Let Υ denotes the set of all indexes

$$\Upsilon = \{kr \mid r \in \mathfrak{I}_k \setminus \{\alpha\}\} \bigcup \{k{+}2, r \mid r \in \mathfrak{I}_{k+2} \setminus \{\beta\}\},$$

then $|\Upsilon| = |\mathfrak{I}_k| + |\mathfrak{I}_{k+2}| - 2 = 2p - 3$. Let $\prod_{v}$ denotes the product $\prod_{v\in\Upsilon}$. By (39), and (5), we obtain

$$\begin{aligned}
[[[X_\alpha, X_i], X_\beta], Y_i] + [[[X_\alpha, Y_i], X_\beta], X_i] &= -(\lambda_\alpha + \lambda_\beta)\big(2(\bar\lambda_i - 1)\mathtt{t}'_i + 2(\lambda_i - 1)\bar{\mathtt{t}}'_i\big) \cdot X_\alpha X_\beta = \\
&= 2(\lambda_\alpha + \lambda_\beta)\big((\lambda_i + 1)\mathtt{t}'_i + (\bar\lambda_i + 1)\bar{\mathtt{t}}'_i\big) \cdot X_\alpha X_\beta = \\
= 2(\lambda_\alpha + \lambda_\beta)\left(\frac{(\lambda_i + 1)\prod_v(\lambda_i^2 + \lambda_i + \ell_v)}{4H_i H_i^{\sigma_i^{-1}}} + \frac{(\bar\lambda_i + 1)\prod_v(\bar\lambda_i^2 + \bar\lambda_i + \ell_v)}{4H_i H_i^{\sigma_i}}\right) &\cdot X_\alpha X_\beta
\end{aligned}$$

with h_i, H_i to be defined in (8) and (6).

We substitute $\lambda_i = x_i$, then by (6), $H_{i,j} := \ell_i - \ell_j = -(\lambda_i - \lambda_j)(\lambda_i - \bar\lambda_j) = -(x_i^2 - x_j^2)$, and

$$(\lambda_i + 1)\mathtt{t}'_i + (\bar\lambda_i + 1)\bar{\mathtt{t}}'_i = \frac{F(x_i(x_i + 1))}{2x_i\,\mathcal{H}_{(p,i)}\mathcal{H}^-_{(p,i)}} - \frac{F(x_i(x_i - 1))}{2x_i\,\mathcal{H}_{(p,i)}\mathcal{H}^+_{(p,i)}}.$$

The polynomial $F(r) = r\prod_v(r + \ell_v)$ has degree $|\Upsilon| = 2p - 3$ with free member equals zero. The applying of Proposition 2 finishes the proof. $\square$

Lemma 8. *Let $\zeta_{k+1} = 0$ and $\zeta_k = \zeta_{k+2} = 1$. Then*

$$\begin{aligned}
&\sum_{i\in\mathfrak{I}_{k+1}} \big([[[C_k, X_{k+1,i}], X_{k+2,\beta}], Y_{k+1,i}] + [[[C_k, Y_{k+1,i}], X_{k+2,\beta}], X_{k+1,i}]\big) = 0, \\
&\sum_{i\in\mathfrak{I}_{k+1}} \big([[[X_\alpha, X_{k+1,i}], C_{k+2}], Y_{k+1,i}] + [[[Y_\alpha, Y_{k+1,i}], C_{k+2}], X_{k+1,i}]\big) = 0.
\end{aligned} \tag{52}$$

Proof of (52). In this case, $[C_k, X_\beta] = 0$. We prove the first equality. Using (46), we calculate

$$\begin{aligned}
[[[C_k, X_i], X_\beta], Y_i] + [[[C_k, Y_i], X_\beta], X_i] &= \frac{C_k}{\lambda_i}\big[[X_i, X_\beta], Y_i\big] + \frac{C_k}{\bar\lambda_i}\big[[Y_i, X_\beta], X_i\big] = \\
= \frac{C_k}{\lambda_i}\Big((X_iX_\beta Y_i - X_\beta X_i Y_i - Y_i X_i X_\beta + Y_i X_\beta X_i) &- (Y_i X_\beta X_i - X_\beta Y_i X_i - X_i Y_i X_\beta + X_i X_\beta Y_i))\Big) = \\
&= -\frac{C_k}{\lambda_i}\big(X_\beta Y_i X_i - X_\beta X_i Y_i + X_i Y_i X_\beta - Y_i X_i X_\beta\big) = \\
&= -2C_k(\lambda_\beta - 1)\left(\frac{\mathtt{t}'_i}{\lambda_i} + \frac{\bar{\mathtt{t}}'_i}{\bar\lambda_i}\right) \cdot X_\beta = \\
= -2C_k(\lambda_\beta - 1)\sum_{i\in\mathfrak{I}_{k+1}}\left(\frac{\mathtt{t}_i}{\lambda_i(\lambda_i + \lambda_\beta)(\lambda_i + \bar\lambda_\beta)}\right. &+ \left.\frac{\bar{\mathtt{t}}_i}{\bar\lambda_i(\bar\lambda_i + \lambda_\beta)(\bar\lambda_i + \bar\lambda_\beta)}\right)X_\beta.
\end{aligned}$$

The rest of the proof is analogous with the proof of Lemma 7. $\square$

Finally, applying Lemmas 6, 7, and 8, and the similar equalities obtained with an involution, the proof of (19) from Proposition 1 for the case of an even k is complete.

Therefore, we finish the proof of Proposition 1 in all.

5.4. The homomorphism of $\mathrm{U}(\mathcal{O}_n)$ to RTGWA $\mathcal{A}$.

In [Zh], Ch. X.70, Theorem 7, the Gelfand-Tsetlin formulae (in Zhelobenko form) are given for the action of generators of the orthogonal algebra $\mathcal{O}_n$ on a Gelfand-Tsetlin basis of a finite dimensional irreducible representation. These formulas provide a homomorphism $\mathrm{U}(\mathcal{O}_n) \to \mathcal{A}^G$.

Now we formulate Theorem which is a direct consequence of Proposition 1.

Let $\mathfrak{A}$ be an associative algebra generated over K by $\{\mathcal{U}_m \mid m \in [1;n)\}$, where $\mathcal{U}_m \in \mathcal{A}^G$ are defined in (16), then $\mathfrak{A} \subset \mathcal{A}^G$. Remember that the elements $E_{ji} = e_{j\,i} - e_{i\,j}$, $i, j \in [1, n]$, $i \neq j$, together with K subject to the defining relations (57), generate the universal enveloping of the orthogonal Lie algebra $\mathrm{U}(\mathcal{O}_n)$ as $\Bbbk$-algebra.

The following theorem holds.

Theorem 1. *The mapping $\tau : U(\mathcal{O}_n) \to \mathfrak{A}$ such that $\tau : \Bbbk \xrightarrow{\sim} \Bbbk$ and*

$$\tau\left(E_{m+1,m}\right) = \mathcal{U}_m = \sum_{i\in\mathfrak{I}_m} X_{mi} - \sum_{i\in\mathfrak{I}_m} Y_{mi} + C_m, \quad m \in [1;n),$$

is a homomorphism of $\Bbbk$-algebras.

Proof. For any $m \in [1;n)$, let us put $\Theta_{m,m+1} = \mathcal{U}_m$, $\Theta_{m+1,m} = -\mathcal{U}_m$, and $\Theta_{m,m} = 0$. By recursive definition, for $m, l \in [1;n)$, $l > m+1$, we denote

$$\Theta_{m,l} = \left[\Theta_{m,m+1}, \Theta_{m+1,l}\right], \quad \Theta_{l,m} = \left[\Theta_{l,l-1}, \Theta_{l-1,m}\right].$$

Then $\mathfrak{A}$ is generated by $\{\Theta_{m,l} \mid m, l \in [1;n)\}$.

Using Proposition 1, we can apply Proposition 4 (and Lemma 13) from Appendix to obtain that the equalities (61) hold on $\mathfrak{A}$. Then the map τ such that $\tau(E_{k,l}) = \Theta_{k,l}$, $k, l \in [1;n]$ defines a homomorphism of $\Bbbk$-algebras because the defining relations between generators of $\mathrm{U}(\mathcal{O}_n)$ (57) are annihilated by τ. Therefore τ is a homomorphism of $\Bbbk$-algebras. □

6. Appendix

6.1. **Ring of symmetric polynomials: formulae.** For integer $q > 1$, we consider the polynomial ring $\Bbbk[x_1, \ldots, x_q]$ in indeterminates $x_1, \ldots, x_q$ endowed by the shiftings $\Sigma = \{\sigma_i \mid i \in [1;n]\}$ such as

$$f(x_k)^{\sigma_k} = f(x_k - 1), \quad f(x_k)^{\sigma_k^{-1}} = f(x_k + 1)$$

for any $f(t) \in \Bbbk[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r]$. We denote by $R := \mathrm{Sym}_{\Bbbk}(x_1, \ldots, x_r) \subset \Bbbk[x_1, \ldots, x_r]$ the ring of symmetric polynomials.

For $q > 1$, we consider the sums $\mathcal{H}_{(q,i)} = \prod\limits_{j\neq i} \left(x_i^2 - x_j^2\right)$ and

$$\mathcal{H}^-_{(q,i)} = \prod_{j\neq i} \left((x_i+1)^2 - x_j^2\right) = \mathcal{H}^{\sigma_i^{-1}}_{(q,i)}, \quad \mathcal{H}^+_{(q,i)} = \prod_{j\neq i} \left((x_i-1)^2 - x_j^2\right) = \mathcal{H}^{\sigma_i}_{(q,i)}.$$

We put $\mathcal{H}_{(1,1)} = \mathcal{H}^-_{(1,1)} = \mathcal{H}^+_{(1,1)} = 1$. Then the Least Common Multiple equals

$$D_q = \mathrm{lcm}\left\{\mathcal{H}_{(q,i)}, \mathcal{H}^-_{(q,i)}, \mathcal{H}^+_{(q,i)}\right\}_{i\in[1;q]} = \mathcal{H}_{(q,i)}\mathcal{H}^-_{(q,i)}\mathcal{H}^+_{(q,i)} \in \mathrm{Sym}(x_1, \ldots, x_q)[x_i].$$

For any $i, j \in [1, q]$, let

$$b^-(i,j) = \left(x_i^2 - x_j^2\right)\left((x_i+1)^2 - x_j^2\right), \quad b^+(i,j) = \left(x_i^2 - x_j^2\right)\left((x_i-1)^2 - x_j^2\right)$$

and $b(i,j) = \mathrm{lcm}\left(b^-(i,j), b^+(i,j)\right) = \left(x_i^2 - x_j^2\right)\left((x_i+1)^2 - x_j^2\right)\left((x_i-1)^2 - x_j^2\right)$. Observe that $\mathcal{H}_{(q,i)}\mathcal{H}^-_{(q,i)}\mathcal{H}^+_{(q,i)} = \mathcal{H}_{(q-1,i)}\mathcal{H}^-_{(q-1,i)}\mathcal{H}^+_{(q-1,i)} \cdot b(i,q)$. The following formulae obviously hold

$$\begin{aligned} b^-(i,j) &= x_i^2(x_i+1)^2 - 2x_i(x_i+1)x_j^2 + x_j^2(x_j^2-1), \\ b^+(i,j) &= x_i^2(x_i-1)^2 - 2x_i(x_i-1)x_j^2 + x_j^2(x_j^2-1). \end{aligned} \tag{53}$$

Denotation 1. *We use the following denotations*

For $q = 0$, we denote $G^-_{(1,1,0)} = \dfrac{1}{2x_1}$, $G^+_{(1,1,0)} = \dfrac{1}{2x_1}$, and $G_{(1,0)} = 0$.

For $q = 1$, we have $G^-_{(1,1,1)} = \dfrac{x_1+1}{2}$, $G^+_{(1,1,1)} = \dfrac{x_1-1}{2}$, and $G_{(1,1)} = 1$, *For $q>1$, and $l\geqslant 0$, we use the recurrence formulae:*

$$G_{(q,l)}=\sum_{i=1}^{q}G^{-}_{(q,i,l)}-\sum_{i=1}^{q}G^{+}_{(q,i,l)},\quad G^{-}_{(q,i,l)}=\frac{x_i^{l-1}(x_i+1)^l}{2\,\mathcal{H}_{(q,i)}\mathcal{H}^{-}_{(q,i)}},\quad G^{+}_{(q,i,l)}=\frac{x_i^{l-1}(x_i-1)^l}{2\,\mathcal{H}_{(q,i)}\mathcal{H}^{+}_{(q,i)}}. \tag{54}$$

Here, $G_{(q,l)}\in\mathrm{Sym}(x_1,\dots,x_q)$ because $G^{*}_{(q,l)}=G_{(q,l)}$.

Lemma 9. *For $q>1$, $1\leqslant i<q$, and $l\geqslant 2$, there hold:*

$$\begin{aligned} G^{-}_{(q,i,l)}-2G^{-}_{(q,i,l-1)}\,x_q^2+G^{-}_{(q,i,l-2)}\,x_q^2(x_q^2-1) &= G^{-}_{(q-1,i,l-2)};\\ G^{-}_{(q,q,l)}-2G^{-}_{(q,q,l-1)}\,x_q^2+G^{-}_{(q,q,l-1)}\,x_q^2(x_q^2-1) &= 0. \end{aligned}$$

The same formulae hold for the case G^{+} in view of symmetry.

Proof. It follows from (53), because for any $l\geqslant 2$,

$$\begin{aligned} b^{-}(i,j)\,x_i^{l-2}(x_i+1)^{l-2} &= x_i^{l}(x_i+1)^{l}-2x_i^{l-1}(x_i+1)^{l-1}x_j^2+x_i^{l-2}(x_i+1)^{l-2}x_j^2(x_j^2-1),\\ b^{+}(i,j)\,x_i^{l-2}(x_i-1)^{l-2} &= x_i^{l}(x_i-1)^{l}-2x_i^{l-1}(x_i-1)^{l-1}x_j^2+x_i^{l-2}(x_i-1)^{l-2}x_j^2(x_j^2-1). \end{aligned}$$

□

Corollary 2. *For $q>1$, $1\leqslant i<q$, there hold:*

$$G_{(q,l)}-2G_{(q,l-1)}\,x_q^2+G_{(q,l-2)}\,x_q^2(x_q^2-1) = G_{(q-1,l-2)}.$$

Lemma 10. *For any $q,l\in\mathbb{Z}$, $q\geqslant 1$, the following hold: $G_{(q,2q-1)}=1$, $G_{(q,l)}=0$, for $0\leqslant l<2q{-}1$.*

Proof. It can be easy to check this assertion for $q=2$. Let $q\geqslant 3$. By the induction assumption, we assume Proposition to be true for all integer less then q, in particular, $G_{(q-1,2q-3)}=1$. For any l integer, $G_{(q,l)}$ is a symmetric polynomial. Therefore we obtain the following equation

$$G_{(q,2q-1)}-1=2G_{(q,2q-2)}x_q^2-G_{(q,2q-3)}x_q^2(x_q^2-1).$$

Note that the left part of this equality is either zero or a symmetric polynomial, but the right part is symmetric polynomial if and only if $G_{(q,2q-2)}=G_{(q,2q-3)}=0$. We conclude that $G_{(q,2q-1)}=1$ and $2G_{(q,2q-2)}=G_{(q,2q-3)}(x_q^2-1)$, but, again, this is possible only if $G_{(q,2q-2)}=G_{(q,2q-3)}=0$.

In such a way, consistently applying Corollary 2 for $k=2q-2,\dots,2$, we complete the proof of Lemma 10. □

Proposition 2. *In $\Bbbk[x_1,\dots,x_q]$, for any $q,\mathtt{d}\in\mathbb{N}$, $q>1$, the following hold:*

$$\sum_{i=1}^{q}\frac{x_i^{\mathtt{d}-1}(x_i+1)^{\mathtt{d}}}{2\prod\limits_{j\neq i}\mathcal{H}_{(q,i)}\mathcal{H}^{-}_{(q,i)}}-\sum_{i=1}^{q}\frac{x_i^{\mathtt{d}-1}(x_i-1)^{\mathtt{d}}}{2\prod\limits_{j\neq i}\mathcal{H}_{(q,i)}\mathcal{H}^{+}_{(q,i)}}=\begin{cases}0, & 0\leqslant\mathtt{d}<2q-1;\\ 1, & \mathtt{d}=2q-1.\end{cases}$$

It is a consequence of Lemma 10 in denotations (54).

Denotation 2. *We use the following denotations*

For any $q,l\in\mathbb{Z}$, $q>1$, using the denotations 1, we consider the symmetric functions

$$G'_{(q,l)}=\sum_{i=1}^{q}\frac{1}{x_i(x_i+1)+1/4}\,G^{-}_{(q,i,l)}-\sum_{i=1}^{q}\frac{1}{x_i(x_i-1)+1/4}\,G^{+}_{(q,i,l)}. \tag{55}$$

Then, in particular, for $q=1$ we have

$$G'_{(1,0)}=-\frac{1}{(x_1^2-1/4)^2},\quad G'_{(1,1)}=-\frac{1}{4(x_1^2-1/4)^2}. \tag{56}$$

Corollary 3 (from Lemma 9)**.** *For $q>1$, $1\leqslant i<q$, there hold:*

$$G'_{(q,l)}-2G'_{(q,l-1)}\,x_q^2+G'_{(q,l-2)}\,x_q^2(x_q^2-1) = G'_{(q-1,l-2)}.$$

Lemma 11. *For any* $0 \leqslant l \leqslant 2q$*, the following hold:* $G'_{(q,l)} = \dfrac{(-1)^{l+1}}{4^l \prod\limits_{i=1}^{q} (x_i^2 - 1/4)^2}$.

Proof. By the definition, for any l, $G_{(q,l)} = G'_{(q,l+1)} + \dfrac{1}{4} G'_{(q,l)}$. Therefore, due to Lemma 10, $G'_{(q,l+1)} = -\dfrac{1}{4} G'_{(q,l)}$ for any $l = 0, \ldots, 2q-2$, and consequently $G'_{(q,l)} = \dfrac{(-1)^l}{4^l} G'_{(q,0)}$.

Now, we proof by induction on q the assertion $G'_{(q,0)} = -\dfrac{1}{4 \prod\limits_{i=1}^{q} (x_i^2 - 1/4)^2}$. By (56), it is true for $q = 0, 1$.

For $q = 2$ it can be checked directly. By Corollary 3, for $q > 1$, $1 \leqslant i < q$, there hold:

$$G'_{(q-1,0)} = G'_{(q,2)} - 2G'_{(q,1)}\, x_q^2 + G'_{(q,0)}\, x_q^2 (x_q^2 - 1) = G'_{(q,0)} \left(\frac{1}{16} + \frac{1}{2} x_q^2 + x_q^2 (x_q^2 - 1) \right) = G'_{(q,0)} \left(x_q^2 - 1/4 \right)^2.$$

Then $G'_{(q,0)} = G'_{(q-1,0)} \left(x_q^2 - 1/4 \right)^{-2} = -\prod\limits_{i=1}^{q} (x_i^2 - 1/4)^{-2}$. $\square$

Proposition 3. *In* $\Bbbk[x_1, \ldots, x_q]$*, there holds*

$$\sum_{i=1}^{q} \left(\frac{1}{2x_i \left(x_i + 1/2 \right)^2 \mathcal{H}_{(q,i)} \mathcal{H}^-_{(q,i)}} - \frac{1}{2x_i \left(x_i - 1/2 \right)^2 \mathcal{H}_{(q,i)} \mathcal{H}^+_{(q,i)}} \right) = -\frac{1}{\prod_{i=1}^{q} (x_i^2 - 1/4)^2}.$$

It is a consequence of Lemma 11, because the right part of above equality equals $G'_{(q,0)}$ (see denotation (55)).

6.2. Generating systems of complex orthogonal Lie algebra.

The most direct approach uses the entries defines the Lie algebra by generators and relations. Taking into account the realization of orthogonal Lie algebras forming complex orthogonal matrices, we propose the following definition.

Definition 2. *We consider the complex orthogonal Lie algebra* $\mathcal{O}(n)$ *as an algebra given by such a system of standard generators*

$$\{\, E_{ij}, \quad i, j \in [1; n],\ i \neq j \,\}$$

under the defining relations $E_{ji} = -E_{ij}$*, and*

$$[\, E_{kl}, E_{rt} \,] = \kappa_{kr} E_{lt} + \kappa_{lt} E_{kr} - \kappa_{kt} E_{lr} - \kappa_{lr} E_{kt}, \tag{57}$$

with κ_{ij} *be a Kronecker delta. The set of generators* $E_{j,j+1}$*,* $j = 1, \ldots, n-1$ *is a minimal system of generators.*

In matrix realization, we have $E_{ij} = e_{ij} - e_{ji}$, $i, j \in [1; n]$, $i \neq j$.

Further, we presents the equivalent definitions of the complex orthogonal Lie algebra $\mathcal{O}_n$.

Definition 3. *We consider the Lie algebra* $\Psi(n)$ *given by a system of generators*

$$\{\, A_k \mid k \in [1; n) \,\}$$

and defining relations:

$$\left[A_k,\, A_l \right] = 0, \quad k, l \in [1; n), \quad |k - l| > 1; \tag{58}$$

$$\left[\left[A_k,\, A_l \right], A_l \right] = -A_k, \quad k, l \in [1; n), \quad |k - l| = 1. \tag{59}$$

$$\left[\left[\left[A_k, A_{k+1} \right], A_{k+2} \right], A_{k+1} \right] = 0, \quad k \in [1; n - 2). \tag{60}$$

Note, that this system of relations (58)-(60) is self dual (up to the reverse of multipliers). It is not quite obvious for (60). The last group can be described in generally as follows:
every normalized Lie product of length 4 *where entering* 3 *consequent generators and nothing more, equals zero.*

Proposition 4. *The systems of generators and defining relations given in definitions 2 and 3 are equivalent. Therefore, the Lie algebras $\mathcal{O}(n)$ and $\Psi(n)$ are isomorphic to each other.*

The proof follows from Lemmas 12 and 13.

Lemma 12. *In $\mathcal{O}(n)$, we denote $A_k = E_{k,k+1}$, $k \in [1;n)$. Then for the elements $\{A_k \mid k \in [1;n)\}$, the properties (58)-(60) hold:*

Moreover, if the elements $\{A_k\}$ satisfy (58) and (59), then for any $k \in [1;n)$, each of the following equations is equivalent to the equation (60):

$$\big[\big[\big[A_{k+2}, A_{k+1}\big], A_k\big], A_{k+1}\big] \;=\; 0; \quad \big[\big[A_k, A_{k+1}\big], \big[A_{k+1}, A_{k+2}\big]\big] \;=\; 0.$$

Proof. We use the equalities $\big[A_k, A_{k+1}\big] = -E_{k,k+2}$ and $\big[\ldots\big[A_k, A_{k+1}\big], \ldots, A_{k+t}\big] = (-1)^t E_{k,k+t+1}$, $t > 0$, and $\big[\ldots\big[A_{k+t}, A_{k+t-1}\big], \ldots, A_k\big] = E_{k,k+t+1}$, $t > 0$. The (58) obviously hold. We verify (59): $[[A_k, A_{k+1}], A_{k+1}]] = -\big[E_{k,k+2},\, E_{k+1,k+2}\big] = -A_k$.

Finally, we obtain (60) by the way: $\big[\big[\big[A_k, A_{k+1}\big], A_{k+2}\big], A_{k+1}\big] = \big[E_{k,k+3}, E_{k+1,k+2}\big] \;=\; 0$.

$\big[\big[\big[A_{k+2}, A_{k+1}\big], A_k\big], A_{k+1}\big] \overset{(Ji)}{=} -\big[\big[\big[A_{k+1}, A_k\big], A_{k+2}\big], A_{k+1}\big] - \big[\big[\big[A_k, A_{k+2}\big], A_{k+1}\big], A_{k+1}\big] \overset{(58)}{=}$ $\big[\big[\big[A_k, A_{k+1}\big], A_{k+2}\big], A_{k+1}\big]$, and

$$\begin{aligned}\big[\big[A_k, A_{k+1}\big], \big[A_{k+1}, A_{k+2}\big]\big] &\overset{(Ji)}{=} -\big[\big[\big[A_{k+1}, A_{k+2}\big], A_k\big], A_{k+1}\big] - \big[\big[A_{k+1}, \big[A_{k+1}, A_{k+2}\big]\big], A_k\big] = \\ &\overset{(59)}{=} \big[\big[\big[A_k, A_{k+1}\big], A_{k+2}\big], A_{k+1}\big] + \big[\big[\big[A_{k+2}, A_k\big], A_{k+1}\big], A_{k+1}\big] + \big[A_{k+2}, A_k\big] = \\ &\overset{(58)}{=} \big[\big[\big[A_k, A_{k+1}\big], A_{k+2}\big], A_{k+1}\big].\end{aligned}$$ □

Lemma 13. *Let $\mathfrak{A}$ be a $\Bbbk$-algebra generated by the set of elements $\{A_k \mid k \in [1;n)\}$, $n \in \mathbb{N}$ such that the properties (58)-(60) are satisfying.*

For any $k \in [1;n)$, let us put $\Theta_{k,k+1} = A_k$, $\Theta_{k+1,k} = -A_k$, and $\Theta_{k,k} = 0$. By recursive definition, for $k, l \in [1;n)$, $l > k+1$, we denote

$$\Theta_{k,l} = -\big[\Theta_{k,k+1}, \Theta_{k+1,l}\big], \quad \Theta_{l,k} = -\big[\Theta_{l,l-1}, \Theta_{l-1,k}\big].$$

Then for any $k, l, r, t \in [1;n)$, such that $k \leqslant l$ and $r \leqslant t$, there hold

$$[\Theta_{k,l}, \Theta_{r,t}] = \kappa_{kr}\Theta_{l,t} + \kappa_{lt}\Theta_{k,r} - \kappa_{kt}\Theta_{l,r} - \kappa_{lr}\Theta_{k,t}. \tag{61}$$

Proof. By the definition,

$$\Theta_{k,l} = (-1)^{l-k}\big[\ldots\big[A_k, A_{k+1}\big], \ldots, A_{l-1}\big], \quad \Theta_{l,k} = \big[\ldots\big[A_{l-1}, A_{l-2}\big], \ldots, A_k\big].$$

For the further proof, we will make extensive use of Jacobi identity in two possible forms: $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$, $\quad [[x,y],z] + [[y,z],x] + [[z,x],y] = 0$. We will prove Proposition 13 in several steps.

Step 6.1. $\Theta_{k,l} = -\Theta_{l,k}$.

Proof. Induction on $|l-k|$. If $|l-k| = 1$ then it follows from definition. Let $l-k > 1$, then, by the definition and induction assumption, we have $\Theta_{k,l} = -\big[\Theta_{k,k+1}, \Theta_{k+1,l}\big] = -\big[\Theta_{l,k+1}, \Theta_{k+1,k}\big] = -\Theta_{l,k}$. □

Step 6.2. *For any $k, r \in [1;n)$ s.t. $|r-k| > 1$ it holds* $\big[\Theta_{k,k+1},\, \Theta_{r,r+1}\big] \;=\; \big[A_k,\, A_r\big] \overset{(58)}{=} 0$.

Step 6.3. *For any $k \in (1; n-1)$ there are*

$$\big[\big[\Theta_{k-1,k}, \Theta_{k,k+1}\big], \big[\Theta_{k,k+1}, \Theta_{k+1,k+2}\big]\big] \;=\; \big[\big[A_{k-1}, A_k\big], \big[A_k, A_{k+1}\big]\big] \;=\; 0, \tag{62}$$

by Lemma 12, or, equivalently, $\big[\Theta_{k-1,k+1}, \Theta_{k,k+2}\big] \;=\; 0$.

Step 6.4. *If $k, l, r, t \in [1; n)$ and $k < r < t < l$, then $\big[\Theta_{k,r}, \Theta_{t,l}\big] = 0$.*

Proof. First we prove the following assertion: For any $k, r, t \in [1; n]$ s. t. $k + 1 < r < t$ it holds $\big[\Theta_{k,k+1},\, \Theta_{r,t}\big] = 0$.

We use the induction on $t - r$. The case $t = r + 1$ follows from Step 6.2. We assume $t > r + 1$, then, by the induction assumption, we have $\big[\Theta_{k,k+1},\, \Theta_{r,t}\big] = -\big[A_k,\, \big[A_r,\, \Theta_{r+1,t}\big]\big] = = \big[\Theta_{r+1,t},\, \big[A_k, A_r\big]\big] + \big[A_r,\, \big[\Theta_{r+1,t}, A_k\big]\big] = -\big[A_r,\, \big[A_k, \Theta_{r+1,t}\big]\big] = 0$.

Similarly, for any $k, r, t \in [1; n]$ s. t. $r < t < k$ it holds $\big[\Theta_{k,k+1},\, \Theta_{r,t}\big] = 0$.

In order to check the general case, we use the induction on $r - k + l - t$. The cases $r - k = 1$ or (and) $l - t = 1$ are already considered. Assume $r > k + 1$, then

$$\big[\Theta_{k,r}, \Theta_{t,l}\big] = -\big[\big[\Theta_{k,k+1}, \Theta_{k+1,r}\big], \Theta_{t,l}\big] = \big[\big[\Theta_{t,l}, \Theta_{k,k+1}\big], \Theta_{k+1,r}\big] + \big[\big[\Theta_{k+1,r}, \Theta_{t,l}\big], \Theta_{k,k+1}\big] = 0$$

by the induction assumption. The case $l > t + 1$ is similar to the previous one. □

Step 6.5. *If $k, l, r, t \in [1; n)$ and $k < r < t < l$, then $\big[\Theta_{k,l}, \Theta_{r,t}\big] = 0$.*

Proof. We use the induction on $(r - k) + (t - r) + (l - t)$. If $(r - k) + (t - r) + (l - t) = 3$ then

$$\begin{aligned}\big[\Theta_{k,l}, \Theta_{r,t}\big] &= \big[\Theta_{k,k+3}, \Theta_{k+1,k+2}\big] = \big[\big[\Theta_{k,k+1}, \Theta_{k+1,k+3}\big], \Theta_{k+1,k+2}\big] = \\ &= -\big[\big[\Theta_{k+1,k+2}, \Theta_{k,k+1}\big], \Theta_{k+1,k+3}\big] - \big[\big[\Theta_{k+1,k+3}, \Theta_{k+1,k+2}\big], \Theta_{k,k+1}\big] = \\ &= \big[\Theta_{k,k+2}, \Theta_{k+1,k+3}\big] - \big[\Theta_{k+2,k+3}, \Theta_{k,k+1}\big] = 0\end{aligned}$$

by Steps 6.4 and 6.3. Otherwise assume $r - k > 1$, then

$$\big[\Theta_{k,l}, \Theta_{r,t}\big] = \big[\big[\Theta_{k,k+1}, \Theta_{k+1,l}\big], \Theta_{r,t}\big] = -\big[\big[\Theta_{r,t}, \Theta_{k,k+1}\big], \Theta_{k+1,l}\big] - \big[\big[\Theta_{k+1,l}, \Theta_{r,t}\big], \Theta_{k,k+1}\big] = 0$$

by the induction assumption. The case $t - r > 1$ can be considered similarly. Let $l - t > 1$, then $r < t < l - 1 < l$, and we obtain

$$\big[\Theta_{k,l}, \Theta_{r,t}\big] = \big[\big[\Theta_{k,l-1}, \Theta_{l-1,l}\big], \Theta_{r,t}\big] = -\big[\big[\Theta_{r,t}, \Theta_{k,l-1}\big], \Theta_{l-1,l}\big] - \big[\big[\Theta_{l-1,l}, \Theta_{r,t}\big], \Theta_{k,l-1}\big]$$

by the induction assumption and Step 6.4. □

Step 6.6. *If $\{k, l\} \cap \{r, t\} = \varnothing$, and $k < r < l < t$, then $\big[\Theta_{k,l}, \Theta_{r,t}\big] = 0$.*

Proof. We use the induction on $t - k$. The case $t - k = 3$ means $\big[\Theta_{k,l}, \Theta_{r,t}\big] = \big[\Theta_{k,k+2}, \Theta_{k+1,k+3}\big]$, it was considered in Step 6.3.

Consider the case $r - k > 1$.

$$\big[\Theta_{k,l}, \Theta_{r,t}\big] = \big[\big[\Theta_{k,k+1}, \Theta_{k+1,l}\big], \Theta_{r,t}\big] = -\big[\big[\Theta_{r,t}, \Theta_{k,k+1}\big], \Theta_{k+1,l}\big] - \big[\big[\Theta_{k+1,l}, \Theta_{r,t}\big], \Theta_{k,k+1}\big]$$

Here $\big[\Theta_{r,t}, \Theta_{k,k+1}\big] = -\big[\Theta_{k,k+1}, \Theta_{r,t}\big] = 0$ by Step 6.4, and $\big[\Theta_{k+1,l}, \Theta_{r,t}\big] = 0$, by the induction assumption. Therefore, we get zero. The case $t - l > 1$ is analogous.

For the case $l - r > 1$, due to Step 6.5 we have $\big[\Theta_{k,l}, \Theta_{r,t}\big] = \big[\Theta_{k,l}, \big[\Theta_{r,r+1}, \Theta_{r+1,t}\big]\big] = -\big[\Theta_{r+1,t}, \big[\Theta_{k,l}, \Theta_{r,r+1}\big]\big] - \big[\Theta_{r,r+1}, \big[\Theta_{r+1,t}, \Theta_{k,l}\big]\big] = 0$. □

Step 6.7. *For any $k, l \in [1; n)$, $l - k > 1$, it holds*

$$\big[\Theta_{k,k+1},\, \Theta_{k,l}\big] \;=\; \Theta_{k+1,l}. \tag{63}$$

Proof. We use induction on $l - k$. If $l = k + 2$ then $\big[\Theta_{k,k+1},\, \Theta_{k,k+2}\big] = -\big[\Theta_{k,k+1},\, \big[\Theta_{k,k+1},\, \Theta_{k+1,k+2}\big]\big] = -\big[A_k,\, \big[A_k, A_{k+1}\big]\big] = A_{k+1} = \Theta_{k+1,k+2}$.

If $l > k + 2$, then using Step 6.4, we obtain $\big[\Theta_{k,k+1},\, \Theta_{k,l}\big] = -\big[\Theta_{k,k+1},\, \big[\Theta_{k,k+2}, \Theta_{k+2,l}\big]\big] = \big[\Theta_{k+2,l},\, \big[\Theta_{k,k+1}, \Theta_{k,k+2}\big]\big] + \big[\Theta_{k,k+2},\, \big[\Theta_{k+2,l}, \Theta_{k,k+1}\big]\big] = \big[\Theta_{k+1,k+2}, \Theta_{k+2,l}\big] = \Theta_{k+1,l}$. □

Step 6.8. *For any $k, l, r \in [1; n)$, $l, r > k$, $r \neq l$, it holds*

$$\big[\Theta_{k,r},\, \Theta_{k,l}\big] \;=\; \Theta_{r,l}. \tag{64}$$

Proof. We use induction on $r+l-k$. If $r=k+1$ or (and) $l=k+1$ then the equality follows from Steps 6.1, 6.7. Otherwise $k+1<l$ and $k+1<r$. Without loss of generality, we can assume $r<l$. We have $k<k+1<r<l$ and using Step 6.5, we obtain

$$\big[\Theta_{k,r},\,\Theta_{k,l}\big]=\big[\big[\Theta_{k,k+1},\Theta_{k+1,r}\big],\Theta_{k,l}\big]=-\big[\big[\Theta_{k,l},\Theta_{k,k+1}\big],\Theta_{k+1,r}\big]-\big[\big[\Theta_{k+1,r},\Theta_{k,l}\big],\Theta_{k,k+1}\big]$$

$$\stackrel{(63)}{=}\big[\Theta_{k+1,l},\Theta_{k+1,r}\big]+\big[\big[\Theta_{k,l},\Theta_{k+1,r}\big],\Theta_{k,k+1}\big]=\Theta_{l,r}.$$ □

Step 6.9. *$\Theta_{k,r}=-\big[\Theta_{k,l},\Theta_{l,r}\big]$ and $\Theta_{r,k}=\big[\Theta_{r,l},\Theta_{l,k}\big]$ for $k,l,r\in[1;n)$ such that $k<l<r$.*

Proof. We prove the first equality by induction on $l-k$. If $l=k+1$ then the assertion follows from definition. Let $l-k>1$, then by Jakobi identity and induction assumption, we obtain

$$\big[\Theta_{k,l},\Theta_{l,r}\big]=-\big[\big[\Theta_{k,k+1},\Theta_{k+1,l}\big],\Theta_{l,r}\big]=\big[\big[\Theta_{k+1,l},\Theta_{l,r}\big],\Theta_{k,k+1}\big]+\big[\big[\Theta_{l,r},\Theta_{k,k+1}\big],\Theta_{k+1,l}\big]=$$

$=-\big[\Theta_{k+1,r},\Theta_{k,k+1}\big]=-\Theta_{k,r}$, because $\big[\Theta_{k,k+1},\Theta_{l,r}\big]=0$ by Step 6.4. □

Step 6.10.
For $l\neq r$, $\big[\Theta_{k,l},\Theta_{k,r}\big]=\Theta_{l,r}$.

Proof. If $l,r>k$, then it follows from Step 6.8. The case $l,r<k$ is similar. And the case $l<k<r$ follows from Step 6.9. □

Step 6.11. *If $\{k,l\}\cap\{r,t\}=\varnothing$, then $\big[\Theta_{k,l},\Theta_{r,t}\big]=0$.*

Proof. If $\{k,l\}\cap\{r,t\}=\varnothing$, then the assertion $|\{k,l\}\cap\{r,t\}|=1$ follows from Steps 6.4, 6.5, 6.6 □

The considered cases provide the full proof of Lemma 13. □

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