JOURNAL OF ALGEBRA 133, 249–271 (1990)

Twisted Homogeneous Coordinate Rings

M. ARTIN* AND M. VAN DEN BERGH†

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139; and
†Departement Wiskunde en Informatica, Universitaire Instelling Antwerpen, B-2610 Wilrijk, Belgium

Communicated by Susan Montgomery

Received March 2, 1990

DEDICATED TO THE MEMORY OF I. N. HERSTEIN

1. INTRODUCTION

Let X be a projective scheme over a field k, and let $\mathscr{L}$ be the invertible sheaf $\mathcal{O}_X(1)$ of linear forms. The homogeneous coordinate ring of X is a graded k-algebra which, in high degree, is isomorphic to the algebra

$$B = \bigoplus_{n \geqslant 0} H^0(X, \mathscr{L}^{\otimes n}).$$

The following theorem of Serre [S] is one of the basic results about the graded ring B. Denote by ($\mathcal{O}_X$-mod) the category of quasi-coherent sheaves on a scheme X, and by (B-gr) the category of graded left modules over a graded ring B. A graded module $M = \oplus M_n$ is called *right bounded* if $M_n = 0$ for $n \gg 0$, and *torsion* if it is a direct limit $M = \varinjlim_\alpha M_{(\alpha)}$, in which each $M_{(\alpha)}$ is right bounded. Let (tors) denote the full subcategory of (B-gr) of torsion modules.

THEOREM (Serre). *The quotient category* (B-gr)/(tors) *is equivalent with the category* ($\mathcal{O}_X$-mod).

Let σ be an automorphism of X. A standard construction provides a twisted version $B = B(X, \sigma, \mathscr{L})$ of the homogeneous coordinate ring [ATV]. Denoting the pullback $\sigma^*\mathscr{L}$ by $\mathscr{L}^\sigma$, we set

$$B_n = H^0(X, \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}) \tag{1.1}$$

for $n \geqslant 0$, and $B = \oplus B_n$. Multiplication of sections is defined by the rule that if $a \in B_m$ and $b \in B_n$, then

$$a \cdot b = a \otimes b^{\sigma^m}. \tag{1.2}$$

0021-8693/90 $3.00

The main purpose of this note is to prove an extension of Serre's theorem to some of these graded rings.

Given a pair (X, σ) consisting of a noetherian scheme X and an automorphism σ, we call an invertible sheaf $\mathscr{L}$ on X *σ-ample* if for every coherent sheaf $\mathscr{F}$ on X, the cohomology group $H^q(X, \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}} \otimes \mathscr{F})$ vanishes for $q > 0$ and $n \gg 0$. A Cartier divisor D is *σ-ample* if $\mathcal{O}_X(D)$ is a σ-ample sheaf. When σ is the identity, these definitions are equivalent with the usual definitions of ample invertible sheaf and ample divisor.

Our version of Serre's theorem is as follows:

THEOREM (1.3). *Let σ be an automorphism of a projective scheme X over k, and let $\mathscr{L}$ be a σ-ample invertible sheaf on X. Let $B = B(X, \sigma, \mathscr{L})$ be the ring defined as above. Then the categories* ($\mathcal{O}_X$-mod) *and* (B-gr)/(tors) *are naturally equivalent.*

We also prove:

THEOREM (1.4). *With the assumptions of the previous theorem, B is a finitely generated noetherian k-algebra.*

The question of which invertible sheaves are σ-ample is fairly subtle, and we do not have a characterization of the automorphisms σ for which such an invertible sheaf exists. However, it is a relatively simple matter to prove the following:

PROPOSITION (1.5). *Let (X, σ) be as above, and let $\mathscr{L}$ be an ample invertible sheaf on X. Assume that a positive power of the sheaf $\mathscr{L}^\sigma \otimes \mathscr{L}^{-1}$ is in the connected component* $\mathrm{Pic}^0 X$ *of the Picard scheme of X. Then $\mathscr{L}$ is σ-ample, and moreover*

$$gk\text{-dim } B = \dim X + 1.$$

COROLLARY (1.6). *If a positive power σ^n of σ is algebraically equivalent to the identity automorphism, then an invertible sheaf $\mathscr{L}$ is σ-ample if and only if $\mathscr{L} \otimes \mathscr{L}^\sigma \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}$ is ample.*

Note that the proposition and the corollary apply to an arbitrary automorphism of a scheme of dimension 1.

Theorem (1.4) provides an alternative proof of the fact that the rings constructed in [ATV] are noetherian. The proof given in [ATV] was based on reduction to the case of a finite field.

Another example: Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let σ be the automorphism which interchanges the two factors. The corollary asserts that the sheaf $\mathscr{L} = pr_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ is σ-ample, though it is not ample.

The proofs of these results are in the next three sections of the paper. In the last section, we analyze the case that X is a smooth surface. Recall that a linear operator P is called *quasi-unipotent* if its eigenvalues are roots of unity, or equivalently, if some power of P is unipotent. We prove the following theorem, in which the last assertion may be somewhat surprising at first glance.

THEOREM (1.7). *Let σ be an automorphism of a smooth proper algebraic surface X, and let P denote the action of σ on the Neron–Severi lattice $NS(X)/(torsion)$.*

(i) *If there exists a σ-ample divisor on X, then P is quasi-unipotent.*

(ii) *If P is quasi-unipotent, then a divisor D is σ-ample if and only if there is an integer n such that $D+\sigma D+\cdots+\sigma^{n-1}D$ is ample.*

(iii) *Suppose that P^k is unipotent, and let D be a σ-ample divisor on X. The gk-dimension of the ring B is 3 if $P^k=I$, and is 5 otherwise.*

Translation along the fibres of an elliptic pencil provides examples in which P is quasi-unipotent and the gk-dimension of B is 5 (see Example (5.19)).

2. TWISTING THE SHEAF OF GRADED ALGEBRAS

In this section we discuss sheaves of graded algebras on a scheme X which are twisted by an automorphism σ of X. A slight complication arises because open sets may not be σ-invariant; in fact X may not have any proper σ-invariant open sets at all. But aside from this, things are similar to the untwisted case (the case $\sigma=$ identity).

For the moment, let σ be an automorphism of an arbitrary noetherian scheme X. We denote the structure sheaf of X by $\mathcal{O}$. To establish notation, let us say that σ operates on the left on X and on the right on $\mathcal{O}$. So by definition,

$$\text{if } f\in\mathcal{O}(\sigma U) \text{ then } f^{\sigma}=f\circ\sigma\in\mathcal{O}(U). \tag{2.1}$$

If $\mathcal{F}$ is a quasi-coherent sheaf on X, we denote by $\mathcal{F}^{\sigma}$ the pullback $\sigma^*\mathcal{F}$ via σ. Thus by definition,

$$\mathcal{F}^{\sigma}(U)=\mathcal{F}(\sigma U). \tag{2.2}$$

The complication which arises is illustrated by the skew polynomial construction. Suppose that $X=\operatorname{Spec} R$, so that σ operates on the right on R. The ring $R[t, t^{-1}; \sigma]$ of skew Laurent polynomials is defined by the commutation rule $ta = a^{\sigma}t$, or

$$at^i bt^j = ab^{\sigma^i} t^{i+j} \tag{2.3}$$

for all $a, b \in R$ and $i, j \in \mathbb{Z}$.

Let us extend this definition to schemes, by defining a graded quasi-coherent sheaf $\mathcal{O}[t, t^{-1}; \sigma] = \mathcal{B}$ on X. We set

$$\mathcal{O}[t, t^{-1}, \sigma] = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}t^n, \tag{2.4}$$

where $\mathcal{O}t^n = \mathcal{B}_n$ is the free left $\mathcal{O}$-module of rank one with basis $\{t^n\}$. According to (2.1), a multiplication rule satisfying (2.3) sends

$$\mathcal{B}_i(U) \times \mathcal{B}_j(\sigma^i U) \to \mathcal{B}_{i+j}(U). \tag{2.5}$$

This does not define a sheaf of rings in the usual sense, because a game of musical chairs is being played on the open sets.

The multiplication rule (2.5) can be interpreted as a tensor product of bimodules. By a *coherent $\mathcal{O}$-bimodule $\mathcal{M}$* on a scheme X we mean a coherent sheaf on $X \times X$ whose support Z has this property: The two projections

$$\pi_i \colon Z \to X \qquad (i = 1, 2) \tag{2.6}$$

to X are finite morphisms. We may view $\mathcal{M}$ as a left $\mathcal{O}$-module via the first projection pr_{1*} and as a right module via pr_{2*}. A section of $\mathcal{M}$ will be defined on an open subset of $X \times X$ or of Z. We also adopt the asymmetrical convention that sections of $\mathcal{M}$ on an open subset U of X are to be interpreted as sections on $U \times X$:

$$\mathcal{M}(U) := \mathcal{M}(U \times X) = \mathrm{pr}_{1*}\mathcal{M}(U), \qquad \text{if} \quad U \subset X. \tag{2.7}$$

The tensor product of two coherent bimodules $\mathcal{M}, \mathcal{N}$ is defined to be the bimodule

$$\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N} = \mathrm{pr}_{13*}[(\mathrm{pr}_{12}^*(\mathcal{M})) \otimes_{\mathcal{O}_{X^3}} (\mathrm{pr}_{23}^*(\mathcal{N}))]. \tag{2.8}$$

If $\mathcal{L}$ is a coherent left $\mathcal{O}$-module and σ is an automorphism of X, then a coherent bimodule can be formed canonically as the pullback $(\mathrm{pr}_1^* \mathcal{L}) \otimes \mathcal{O}_\Gamma = \pi_1^* \mathcal{L}$, where $\pi_1 \colon \Gamma \to X$ is the first projection of the graph Γ of σ to X. We will denote this bimodule by $\mathcal{L}_\sigma$. The subscript σ indicates that the right action on $\mathcal{L}$ is twisted by σ, i.e., that the product $sa \in \mathcal{L}(U)$ of sections $s \in \mathcal{L}(U)$ and $a \in \mathcal{O}(\sigma U)$ is defined by the formula $sa = a^\sigma s$.

Thus the right module structure on $\mathscr{L}_\sigma$ is

$$\mathrm{pr}_{2*}\mathscr{L}_\sigma = \sigma_*\mathscr{L} = \mathscr{L}^{\sigma^{-1}}. \tag{2.9}$$

The module structure is the given one, and with the convention (2.7), we have $\mathscr{L}(U) = \mathscr{L}_\sigma(U)$. Thus $\mathscr{L}_\sigma(U)$ has been made into an $(\mathcal{O}(U), \mathcal{O}(\sigma U))$-bimodule:

$$_{\mathcal{O}(U)}[\mathscr{L}_\sigma(U)]_{\mathcal{O}(\sigma U)}. \tag{2.10}$$

The support of $\mathscr{L}_\sigma$ is of course contained in the graph Γ. In particular, $\mathscr{L}_1$ will denote the bimodule formed in the trivial way: $sa = as$, with support on the diagonal Δ of $X \times X$. We often write $\mathscr{L} = \mathscr{L}_1$.

We omit the proof of the following lemma.

LEMMA (2.11). *A coherent bimodule $\mathscr{M}$ on X has the form $\mathscr{M} \approx \mathscr{L}_\sigma$, where $\mathscr{L}$ is the left $\mathcal{O}$-module* $\mathrm{pr}_{1*}\mathscr{M}$, *if and only if its support is contained in the graph of σ.*

We can also define the tensor product of a left or right $\mathcal{O}$-module with a bimodule. If $\mathscr{M}$ is a right module and $\mathscr{N}$ is a coherent bimodule, then by definition $\mathscr{M} \otimes \mathscr{N}$ is the right module $\mathrm{pr}_2^*((\mathrm{pr}_{1*}\mathscr{M}) \otimes_{\mathcal{O}_{X^2}} \mathscr{N})$. Similarly, if $\mathscr{M}'$ is a left module, then $\mathscr{N} \otimes \mathscr{M}'$ is the left module $\mathrm{pr}_1^*(\mathscr{N} \otimes_{\mathcal{O}_{X^2}} \mathrm{pr}_2^*\mathscr{M}')$. If U is an affine open in X and $\mathscr{N} = \mathscr{L}_\sigma$, where $\mathscr{L}$ is a locally free left $\mathcal{O}$-module, then

$$[\mathscr{M} \otimes \mathscr{L}_\sigma](\sigma U) = \mathscr{M}(U) \otimes_{\mathcal{O}(U)} \mathscr{L}_\sigma(U) \tag{2.12}$$

and

$$[\mathscr{L}_\sigma \otimes \mathscr{M}'](U) = \mathscr{L}_\sigma(U) \otimes_{\mathcal{O}(\sigma U)} \mathscr{M}'(\sigma U). \tag{2.13}$$

The proof of the following lemma is just a matter of sorting out the definitions.

LEMMA (2.14). *Let σ, τ be automorphisms of a scheme X, and let $\mathscr{L}$, $\mathscr{M}$ be invertible left $\mathcal{O}$-modules. Then*

$$\mathscr{L}_\sigma \otimes \mathscr{M}_\tau \approx (\mathscr{L} \otimes \mathscr{M}^\sigma)_{\tau\sigma}.$$

Proof. Let Γ, Γ', and Γ'' denote the graphs of σ, τ, and $\tau\sigma$, respectively. Note that according to (2.9), $(\mathrm{pr}_1^*\mathscr{M}) \otimes \mathcal{O}_{\Gamma'} \approx (\mathrm{pr}_2^*\tau_*\mathscr{M}) \otimes \mathcal{O}_{\Gamma'}$. Thus

$$\begin{aligned}
\mathscr{L}_\sigma \otimes \mathscr{M}_\tau &= \mathrm{pr}_{13*}[(\mathrm{pr}_1^* \mathscr{L}) \otimes \mathscr{O}_{\Gamma\times X}) \otimes (\mathrm{pr}_2^* \mathscr{M}) \otimes \mathscr{O}_{X\times\Gamma'})] \\
&\approx \mathrm{pr}_{13*}[(\mathrm{pr}_1^* \mathscr{L}) \otimes \mathscr{O}_{\Gamma\times X}) \otimes (\mathrm{pr}_3^* \tau_* \mathscr{M}) \otimes \mathscr{O}_{X\times\Gamma'})] \\
&\approx (\mathrm{pr}_1^* \mathscr{L}) \otimes (\mathrm{pr}_2^* \tau_* \mathscr{M}) \otimes \mathscr{O}_{\Gamma''} \\
&\approx \mathrm{pr}_1^*(\mathscr{L} \otimes (\tau\sigma)^* \tau_* \mathscr{M}) \otimes \mathscr{O}_{\Gamma''} \\
&\approx \mathrm{pr}_1^*(\mathscr{L} \otimes \sigma^* \mathscr{M}) \otimes \mathscr{O}_{\Gamma''} \approx (\mathscr{L}_1 \otimes \mathscr{M}^\sigma)_{\tau\sigma}.
\end{aligned}$$

Note that $\mathscr{L}_\sigma \otimes \mathscr{M}_\tau(U)$ is an $(\mathscr{O}(U), \mathscr{O}(\tau\sigma(U)))$-bimodule.

A coherent bimodule $\mathscr{M}$ will be called *invertible* if there exists an "inverse" bimodule $\mathscr{M}'$ such that $\mathscr{M} \otimes \mathscr{M}' \approx \mathscr{O}_1 \approx \mathscr{M}' \otimes \mathscr{M}$.

PROPOSITION (2.15). *A coherent bimodule $\mathscr{M}$ on X is invertible if and only if it is isomorphic to a module of the form $\mathscr{L}_\sigma$, where $\mathscr{L}$ is an invertible left $\mathscr{O}$-module and σ is an automorphism of X.*

Proof. Formula (2.14) shows that if $\mathscr{L}$ is an invertible left $\mathscr{O}$-module, then $((\mathscr{L}^*)^{\sigma^{-1}})_{\sigma^{-1}}$ is the inverse of $\mathscr{L}_\sigma$. Conversely, let $\mathscr{M}$ be an invertible bimodule, with inverse $\mathscr{M}'$. Denote the supports of these two bimodules by Γ, Γ', and let W be the support on X^3 of the module $\mathscr{F} = (\mathrm{pr}_{12}^* \mathscr{M}) \otimes_{\mathscr{O}_{E^3}} (\mathrm{pr}_{23}^* \mathscr{M}')$. Thus $W = \Gamma \times X \cap X \times \Gamma'$. The map $\mathrm{pr}_{13}\colon W \to X \times X$ is a finite map whose image is the composed correspondence $\Gamma' \circ \Gamma$. By assumption, $\mathscr{M} \otimes \mathscr{M}' = \mathrm{pr}_{13}^* \mathscr{F} \approx \mathscr{O}_1$. Therefore the projection pr_{13} defines an isomorphism of W with the diagonal $\Delta \subset X \times X$, and $\mathscr{F} \approx \mathscr{O}_W$. So $\Gamma' \circ \Gamma = \Delta$, and there is a map $\sigma\colon X \to X$ such that W is, scheme-theoretically, the locus of points $\{(p, \sigma(p), p) \mid p \in X\}$. A consideration of the isomorphism $\mathscr{M}' \otimes \mathscr{M} \approx \mathscr{O}_1$ shows that σ is an isomorphism. At this point, set-theoretic considerations show that Γ is the graph of σ. So by Lemma (2.11), $\mathscr{M} \approx \mathscr{L}_\sigma$ and similarly $\mathscr{M}' \approx \mathscr{L}'_{\sigma^{-1}}$ for some left modules $\mathscr{L}, \mathscr{L}'$. By (2.14), $\mathscr{O}_1 \approx \mathscr{M} \otimes \mathscr{M}' \approx \mathscr{L}_\sigma \otimes \mathscr{L}'_{\sigma^{-1}} = (\mathscr{L}_1 \otimes \mathscr{L}'^\sigma)_1$. Therefore $\mathscr{L}$ is invertible.

When an invertible left $\mathscr{O}$-module $\mathscr{L}$ and an automorphism σ are given, we can construct a sheaf of graded *skew algebras* $\mathscr{B} = \mathscr{O}[\mathscr{L}; \sigma]$ as follows. We set

$$\mathscr{B} = \bigoplus_{n \in \mathbb{Z}} \mathscr{B}_n, \qquad \text{where} \quad \mathscr{B}_n = \mathscr{L}_\sigma^{\otimes n}. \tag{2.16}$$

To make this explicit, we note that for an open set U of X and for $n > 0$, we have

$$\begin{aligned}
\mathscr{B}_n(U) = \mathscr{L}_\sigma^{\otimes n}(U) &\approx \mathscr{L}_\sigma(U) \otimes_{\mathscr{O}(\sigma U)} \mathscr{L}_\sigma(\sigma U) \otimes \cdots \otimes_{\mathscr{O}(\sigma^{n-1} U)} \mathscr{L}_\sigma(\sigma^{n-1} U) \\
&\approx \mathscr{L}(U) \otimes_{\mathscr{O}(U)} \mathscr{L}^\sigma(U) \otimes \cdots \otimes_{\mathscr{O}(U)} \mathscr{L}^{\sigma^{n-1}}(U).
\end{aligned}$$

The multiplication law $\mathscr{B}_i \otimes \mathscr{B}_j \to \mathscr{B}_{i+j}$ is given by the natural isomorphism

$\mathscr{L}_\sigma^{\otimes i} \otimes \mathscr{L}_\sigma^{\otimes j} \approx \mathscr{L}_\sigma^{\otimes (i+j)}$, and to multiply sections, the open sets must be as in (2.5). When $\mathscr{L}$ is the trivial invertible sheaf $\mathcal{O}$, we recover the algebra of skew Laurent polynomials $\mathcal{O}[t, t^{-1}; \sigma]$ defined above (2.4).

Let $\mathscr{B} = \mathcal{O}[\mathscr{L}; \sigma]$ be a skew algebra constructed as above. By graded left $\mathscr{B}$-module $\mathscr{M}$, we mean a graded quasi-coherent sheaf of left $\mathcal{O}$-modules together with a multiplication rule

$$\mathscr{B}_i \otimes \mathscr{M}_j \to \mathscr{M}_{i+j}, \tag{2.17}$$

or on sections,

$$\mathscr{B}_i(U) \otimes_{\mathcal{O}(\sigma^i U)} \mathscr{M}_j(\sigma^i U) \to \mathscr{M}_{i+j}(U),$$

which satisfies the axioms for a module.

PROPOSITIONS (2.18). *Let $\mathscr{L}$ be an invertible $\mathcal{O}$-module and let $\mathscr{B} = \mathcal{O}[\mathscr{L}; \sigma]$ be the skew algebra constructed as above. The functors*

$$\mathscr{F} \rightsquigarrow \mathscr{B} \otimes \mathscr{F} \qquad \text{and} \qquad \mathscr{M}_0 \leftsquigarrow \mathscr{M}$$

are quasi-inverses, which define equivalences between the categories of quasi-coherent left $\mathcal{O}$-modules and of quasi-coherent graded left $\mathscr{B}$-modules.

The proof of this proposition is routine and we omit it. (See [NV] for the affine analogue.)

We call a graded left $\mathscr{B}$-module $\mathscr{M}$ *coherent* if the corresponding $\mathcal{O}$-module $\mathscr{M}_0$ is coherent. Since X is noetherian, coherent graded $\mathscr{B}$-modules have the ascending chain condition on graded submodules.

3. PROOF OF THE THEOREMS (1.3) AND (1.4)

Before turning to the proof of Theorem (1.3), we need to extend some simple facts about ample invertible sheaves to sequences. As before, $\mathcal{O}$ denotes the structure sheaf of the noetherian scheme X. Let $\mathscr{L}_{(1)}, \mathscr{L}_{(2)}, \ldots$ be a sequence of invertible sheaves on X, and let $\mathscr{B}_n = \mathscr{L}_{(1)} \otimes \cdots \otimes \mathscr{L}_{(n)}$. The sequence will be called *ample* if it satisfies the following condition:

$$\text{For every coherent left } \mathcal{O}\text{-module } \mathscr{F}, \text{ there is an integer } n_0 \text{ such that } H^q(X, \mathscr{B}_n \otimes \mathscr{F}) = 0 \text{ for } q > 0 \text{ and } n \geqslant n_0. \tag{3.1}$$

Clearly, the constant sequence $\mathscr{L} = \mathscr{L}_{(1)} = \mathscr{L}_{(2)} = \cdots$ is ample if and only if $\mathscr{L}$ is ample in the usual sense.

An invertible bimodule $\mathscr{L}_\sigma$ is called ample if the sequence $\mathscr{L}, \mathscr{L}^\sigma, \mathscr{L}^{\sigma^2}, \ldots$ is ample. This means that (3.1) holds when $\mathscr{B}_n$ is defined by (2.16).

PROPOSITION (3.2). *Let $\mathscr{L}_{(1)}, \mathscr{L}_{(2)}, \ldots$ be an ample sequence of invertible sheaves on a projective scheme X over k.*

(i) *$[\mathscr{F}] = [\mathscr{F} \to \mathscr{F}' \to \mathscr{F}'']$ be a complex of coherent left $\mathcal{O}$-modules. Then $[\mathscr{F}]$ is exact if and only if there is an integer n_0 such that $H^0(\mathscr{B}_n \otimes [\mathscr{F}])$ is exact for all $n \geqslant n_0$.*

(ii) *Let $\mathscr{F}$ be a coherent left $\mathcal{O}$-module. Then there is an integer n_0 such that $\mathscr{B}_n \otimes \mathscr{F}$ is generated by its sections if $n \geqslant n_0$.*

(iii) *For sufficiently large n, $\mathscr{B}_n$ is a very ample line bundle on X.*

Proof. We first prove the "only if" part of (i). Suppose that $[\mathscr{F}]$ is exact. To show that the complex $H^0(\mathscr{B}_n \otimes [\mathscr{F}])$ is exact for $n \gg 0$, it suffices to treat the case that the complex extends to an exact sequence $0 \to \mathscr{F} \to \mathscr{F}' \to \mathscr{F}'' \to 0$. In this case, the assertion follows by applying the definition of ample bimodule to the sheaf $\mathscr{F}$.

To prove (ii), we first verify the lemma:

LEMMA (3.3). *$\mathscr{B}_n$ is generated by its sections for large n.*

Proof. We may assume that the ground field k is algebraically closed. Let $d = \dim X$, and let p be a point at which X is Cohen–Macaulay and of dimension d. We choose a regular sequence $s_1, \ldots, s_d$ of sections of $\mathcal{O}_X(1)$ which vanish at p. (To do this, suppose that $s_1, \ldots, s_i$ have been found. Let $Z_1, \ldots, Z_r$ be the associated primes of $V_i := V(s_1, \ldots, s_i)$, and choose a section s_{i+1} which does not vanish identically on any Z_ν. Then s_{i+1} is not a zero divisor on V_i.) Then $S = V_d$ is a scheme of dimension zero which contains p, and $\mathcal{O}_S$ has a Koszul resolution of the form

$$0 \to \mathscr{K}_d \to \cdots \to \mathscr{K}_1 \to \mathscr{K}_0 \to \mathcal{O}_S \to 0,$$

where $\mathscr{K}_0 = \mathcal{O}_X$, and where each term $\mathscr{K}_i$ is a direct sum of line bundles $\mathcal{O}_X(-\nu)$, these line bundles being independent of the regular sequence (hence of the point p). We choose n_1 large enough so that if $n \geqslant n_1$, then $H^q(\mathscr{B}_n \otimes \mathscr{K}_i) = 0$ for all i and all $q > 0$. Let $\mathscr{N}$ be the kernel of the map $\mathcal{O}_X \to \mathcal{O}_S$. Working from the left of the Koszul resolution, we find that $H^q(\mathscr{B}_n \otimes N) = 0$ for $q > 0$ as well. Therefore the sections of $\mathcal{O}_S \otimes \mathscr{B}_n$ lift to $\mathscr{B}_n$ if $n \geqslant n_1$. Since S is zero-dimensional, $\mathcal{O}_S \otimes \mathscr{B}_n$ has a section which does not vanish at p, and so $\mathscr{B}_n$ does too. This shows that the sections of $\mathscr{B}_n$ generate at every Cohen–Macaulay point p, if $n \geqslant n_1$.

Let $Y \subset X$ be the proper closed subscheme of points at which X is not maximally Cohen–Macaulay. We note that the restriction $\mathcal{O}_Y \otimes \mathscr{L}_{(i)}$ of the sequence $\mathscr{L}_{(i)}$ to a closed subscheme is an ample sequence on the subscheme. By noetherian induction on X, we may assume that $\mathcal{O}_Y \otimes \mathscr{B}_n$ is generated by its sections for $n \gg n_2$. Let $\mathscr{I}$ denote the ideal sheaf defining

Y. The definition of ample sequence, applied to this ideal sheaf, shows that sections of $\mathcal{O}_Y \otimes \mathcal{B}_n$ lift to $\mathcal{B}_n = \mathcal{O}_X \otimes \mathcal{B}_n$ for $n \geqslant n_3$. Thus the sections of $\mathcal{B}_n$ generate the sheaf along Y if $n \geqslant n_3$. Taking $n_0 = \max(n_1, n_2, n_3)$, we are done.

We now return to the proof of part (ii) of Proposition (3.2). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We first note that if $\mathcal{F}$ is generated by its sections, then so is $\mathcal{B}_n \otimes \mathcal{F}$, provided that $n \gg 0$. This follows from Lemma (3.3). Next, we may assume that $\mathcal{F} \neq 0$. Then there is a point p such that $k(p) \otimes \mathcal{F} \neq 0$. Applying (i) to the exact sequence $\mathcal{F} \to k(p) \otimes \mathcal{F} \to 0$ and using the fact that $k(p) \otimes \mathcal{B}_n \otimes \mathcal{F}$ is generated by its sections for every n, we find that $\mathcal{B}_n \otimes \mathcal{F}$ is generated by its sections at the point p, if $n \gg 0$. It is permissible to replace $\mathcal{F}$ by $\mathcal{B}_n \otimes \mathcal{F}$ and reindex, and we do this. Let $Y \subset X$ be the subscheme of points at which the global sections fail to generate $\mathcal{F}$. This is a proper closed subscheme. By noetherian induction on X, we may assume that the assertion (ii) is true for the sheaf $\mathcal{O}_Y \otimes \mathcal{F}$. Then one more application of the lifting argument completes the proof. The other implication of (i) now follows easily.

It remains to prove that $\mathcal{B}_n$ is very ample if $n \gg 0$. We have seen that $\mathcal{B}_n(-1)$ is generated by its sections for large n. Together with the fact that $\mathcal{O}(1)$ is very ample, this implies that $\mathcal{B}_n$ is very ample.

PROPOSITION (3.4). *Suppose that X is projective. A sequence $\mathcal{L}_{(1)}, \mathcal{L}_{(2)}, \ldots$ is ample if for each $k > 0$ there is an integer n_0 such that if $n \geqslant n_0$ and $q > 0$, then $H^q(\mathcal{B}_n \otimes \mathcal{O}(-k)) = 0$.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ admits a resolution

$$\cdots \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{P}_i$ is a sum of invertible sheaves of the form $\mathcal{O}(-k)$. If the condition of the proposition is satisfied, there exists an integer n_0 such that for all $n \geqslant n_0$, $H^q(\mathcal{B}_n \otimes \mathcal{P}_i) = 0$ if $q > 0$ and if $i \leqslant \dim X + 1$. The spectral sequence associated to this resolution shows that $H^q(\mathcal{B}_n \otimes \mathcal{F}) = 0$ as well.

For the rest of this section, we let $\mathcal{B} = \mathcal{O}[\mathcal{L}; \sigma]$ be the skew algebra on a projective scheme X, which is defined by an ample invertible bimodule $\mathcal{L}_\sigma$. We define a graded k-algebra B by

$$B := H^0(X, \mathcal{B})_{\geqslant 0} = \bigoplus_{n \geqslant 0} H^0(X, \mathcal{B}_n). \tag{3.5}$$

Since X is a proper, B is a graded ring with the property that $\dim_k B_n < \infty$ for all n.

As before, a *torsion* B-module M is defined to be a graded module which is a direct limit of right bounded submodules N, i.e., ones such that $N_r = 0$ if $r \gg 0$. The category of torsion modules is a localizing subcategory of the category of all graded B-modules; i.e., it is closed under submodules, quotients, extensions, and direct limits.

With notation as in the introduction, we are going to show that the quotient category (B-gr)/(tors) is equivalent to the category ($\mathcal{O}$-mod) of quasi-coherent sheaves on X, as is true for commutative graded rings. By Proposition (2.18), relating (B-gr)/(tors) to the category ($\mathcal{B}$-gr) of graded left $\mathcal{B}$-modules amounts to the same thing.

We define adjoint functors between the two categories (B-gr)/(tors) and ($\mathcal{B}$-gr) as follows: If $\mathcal{M}$ is a sheaf of graded $\mathcal{B}$-modules, then

$$\Gamma_*(\mathcal{M}) := H^0(X, \mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{M}_n). \tag{3.6}$$

LEMMA (3.7). (i) *For any graded left $\mathcal{B}$-module $\mathcal{M}$, $H^q(X, \mathcal{M})$ is a graded left B-module; hence $\Gamma_*(\mathcal{M})$ is a graded left B-module. If $q > 0$, then $H^q(X, \mathcal{M})$ is a torsion module. If $\mathcal{M}$ is coherent and $q > 0$, then $H^q(X, \mathcal{M})$ is right bounded.*

(ii) *The functor Γ_* is exact modulo* (tors).

(iii) *For any $\mathcal{B}$-module $\mathcal{M}$, $\Gamma_*(\mathcal{M})$ is torsion free, and if $\mathcal{M} \neq 0$, then $\Gamma_*(\mathcal{M}) \neq 0$.*

Proof. The first assertion of (i) is clear, because multiplication by $b \in H^0(X, \mathcal{B})$ is an $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{M}$. If $\mathcal{M}$ is coherent, then Proposition (3.3) implies that $H^1(X, \mathcal{M}_n) = 0$ for $n \gg 0$, hence that this module is right bounded, hence torsion. In general, we may write $\mathcal{M}_0$ as a limit of coherent sheaves. Then $\mathcal{M}$ is the corresponding limit. Since cohomology commutes with direct limits, $H^1(X, \mathcal{M})$ is a torsion module. Part (ii) follows.

To prove (iii), we use the fact that $\mathcal{B}_n$ and $\mathcal{B}_n \otimes \mathcal{M}$ are generated by global sections for large n. Let $b_1, ..., b_r$ be sections which generate $\mathcal{B}_n$. Since $\mathcal{M}_{k+n} = \mathcal{B}_n \otimes \mathcal{M}_k$, the map $\mathcal{M}_k \to \bigoplus_1^r \mathcal{M}_{k+n}$ defined by multiplication by $(b_1, ..., b_r)$ is injective. Therefore this map does not annihiliate any nonzero global sections of $\mathcal{M}_k$; i.e., B_n does not annihilate any element of $\Gamma_*(\mathcal{M})$.

To go in the other direction, let M be a graded left B-module. We regard B and M as constant sheaves on X. With the obvious conventions, the canonical map $B \to \mathcal{B}$ is a homomorphism which allows us to construct a sheaf of graded $\mathcal{B}$-modules

$$\tilde{M} = \mathcal{B} \otimes_B M. \tag{3.8}$$

By definition, this is the graded sheaf associated to the graded presheaf

$$U \rightsquigarrow \mathscr{B}(U) \otimes_B M. \tag{3.9}$$

Note that if $V \subset U$ are affine open sets in X, then

$$\mathscr{B}(V) = \mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathscr{B}(U),$$

so

$$\mathscr{B}(V) \otimes_B M = \mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \otimes_B M.$$

This shows that Formula (3.9) defines the sections of the tensor product sheaf on affine open subsets, i.e., that taking the associated sheaf is not necessary on affine opens.

LEMMA (3.10). *If M is a finitely generated graded B-module, then $\tilde{M}$ is coherent.*

Proof. If M is finitely generated, then it is a quotient of a projective module P which is a finite sum of shifts of B. Since tensor product is a right exact operation, $\tilde{M}$ is a quotient of $\tilde{P}$, and on the other hand, $\tilde{P}$ is a sum of shifts of $\mathscr{B}$. Therefore $\tilde{M}$ is a quotient of a coherent graded $\mathscr{B}$-module, which implies that it is coherent itself.

It is clear that the functors Γ_* and $\sim$ are adjoint, i.e., that

$$\mathrm{Hom}_B(M, \Gamma_* \mathscr{N}) \approx \mathrm{Hom}_{\mathscr{B}}(\tilde{M}, \mathscr{N}) \tag{3.11}$$

for all $M \in (B\text{-gr})$ and $\mathscr{N} \in (\mathscr{B}\text{-gr})$.

We are now ready to prove Theorem (1.3), which we restate more precisely here.

THEOREM (3.12). *Let σ be an automorphism of a projective scheme X over k, and let $\mathscr{L} = \mathscr{L}_\sigma$ be an ample invertible bimodule on X. Let $\mathscr{B} = \mathcal{O}[\mathscr{L}; \sigma]$, and let $B = \bigoplus_{n \geqslant 0} H^0(X, \mathscr{B}_n)$. Let* (tors) *denote the full subcategory of* (B-gr) *of torsion modules. The functors Γ_* and $(\mathscr{B} \otimes_B \cdot)_0$ induce inverse equivalences*

$$(B\text{-gr})/(\text{tors}) \underset{\Gamma_*}{\overset{(\mathscr{B} \otimes_B \cdot)_0}{\rightleftarrows}} (\mathcal{O}\text{-mod}).$$

Let $\bar{M}$ denote the module $\Gamma_*(\tilde{M})$. The adjointness property of the two functors provides a functorial map $M \to \bar{M}$. Since $\sim$ and Γ_* both commute with direct limits, the functor $M \to \bar{M}$ does too. The proof of Theorem (3.12) is obtained by combining assertions (i), (iii), and (v) of the next lemma.

LEMMA (3.13). *Let M be a graded B-module.*

(i) *The cokernel of the map $M \to \bar{M}$ is a torsion module. If M is finitely generated, the cokernel is right bounded.*

(ii) *If M is a torsion module, then $\tilde{M} = 0$.*

(iii) *The kernel of the map $M \to \bar{M}$ is the torsion submodule of M.*

(iv) *The functor $\sim$ is exact.*

(v) *For any graded $\mathscr{B}$-module $\mathscr{M}$, the map $\mathscr{M} \to \Gamma_*(\mathscr{M})^\sim$ is an isomorphism.*

Proof. (i) Since $\sim$ and Γ_* both commute with direct limits, it suffices to prove the assertion for a finitely generated graded module. We choose a surjective map $P \to M \to 0$, where P is a finite sum of shifts of B. Direct inspection shows that $P \to \bar{P}$ is injective, and that its cokernel is right bounded. Let $\mathscr{K}$ be the kernel of the map $\tilde{P} \to \tilde{M}$. Since tensor product is a right exact functor, we obtain an exact sequence

$$0 \to \mathscr{K} \to \tilde{P} \to \tilde{M} \to 0$$

of coherent graded $\mathscr{B}$-modules. The bottom row of the diagram

$$\begin{tikzcd} & & P \arrow[r] \arrow[d] & M \arrow[d] & \\ 0 \arrow[r] & \Gamma_*(\mathscr{K}) \arrow[r] & \bar{P} \arrow[r] & \bar{M} \arrow[r] & H^1(\mathscr{K}) \end{tikzcd}$$

is exact. Also, since $\mathscr{L}_\sigma$ is ample, $H^1(\mathscr{K})$ is right bounded. Since the cokernel of the map $P \to \bar{P}$ is right bounded, this shows that the cokernel of $M \to \bar{M}$ is right bounded too.

(ii) Since $\sim$ commutes with direct limits, it suffices to show that $\tilde{M} = 0$ if M has finite length. Let M be a finite length graded module. Then (3.10) $\tilde{M}$ is coherent; so if it is not zero, then $H^0(\tilde{M}_n) \neq 0$ for large n (3.2ii). This implies that $\bar{M} \neq 0$ for large n. Therefore $\bar{M}/\mathrm{im}\, M$ is not right bounded, which contradicts (i).

(iii) The torsion submodule and the map $M \to \bar{M}$ are both compatible with direct limits. So it suffices to prove that the kernel of the map is the torsion submodule when M is finitely generated. We choose a surjective map $P \to M \to 0$, where P is a finite sum of shifts of B, as before. Let R be the module of relations, so that

$$0 \to R \to P \to M \to 0$$

is exact, and let $\mathscr{K}$ be the kernel of the map $\tilde{R} \to \tilde{P}$, so that the sequence

$$0 \to \mathscr{K} \to \tilde{R} \to \tilde{P} \to \tilde{M} \to 0$$

is exact. Applying Γ_*, we obtain a diagram

$$\begin{tikzcd} & 0 \arrow[r] & R \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma_*\mathscr{K} \arrow[r] & \bar{R} \arrow[r] & \bar{P} \arrow[r] & \bar{M} \arrow[r] & 0 \end{tikzcd}$$

We work modulo torsion modules. Then the middle vertical arrow in this diagram is injective, and the rows are exact (3.7ii). Also, all vertical arrows are surjective by (i). It follows that $\Gamma_*\mathscr{K}=0$ and that all of the vertical arrows are bijective, modulo torsion. Thus the kernel of $M\to\bar{M}$ is a torsion module.

Now let T be the torsion submodule of M, and let $M'=M/T$. Then $\tilde{T}=0$ by (ii); hence, since $\sim$ is right exact, $\tilde{M}\approx\tilde{M}'$ and $\bar{M}\approx\bar{M}'$. Moreover, the map $M'\to\bar{M}'$ is injective because M' is torsion-free. Therefore T is the kernel of the map $M\to\bar{M}$, as required.

(iv) Since $\sim$ is right exact, it suffices to show that if $N\subset M$ then $\tilde{N}\subset\tilde{M}$. Let $\mathscr{K}$ be the kernel of the map $\tilde{N}\to\tilde{M}$. Then modulo torsion, the sequence $0\to\Gamma_*(\mathscr{K})\to N\to M$ is exact. So $\Gamma_*(\mathscr{K})$ is a torsion module. By (3.7iii), $\mathscr{K}=0$.

(v) Since $\sim$ and Γ_* are compatible with direct limits, it suffices to prove this when $\mathscr{M}$ is a coherent graded module. We may choose a presentation $\mathscr{P}_2\to\mathscr{P}_1\to\mathscr{M}\to 0$, where $\mathscr{P}_i$ is a finite sum of shifts of $\mathscr{B}$. This follows from the fact that $\mathscr{M}_n$ is generated by its sections for large n. The fact that the map $\mathscr{P}\to\Gamma_*(\mathscr{P})^\sim$ is bijective when $\mathscr{P}$ is a sum of shifts of $\mathscr{B}$ is seen by inspection. Since $\sim$ and Γ_* are right exact, the map is bijective for $\mathscr{M}$ as well.

The statement of Theorem (1.4) in terms of bimodules is as follows:

THEOREM (3.14). *Let σ be an automorphism of a scheme X which is projective over k, and let $\mathscr{L}_\sigma$ be an ample invertible bimodule on X. Let $\mathscr{B}=\mathscr{O}[\mathscr{L};\sigma]$, and let $B=\bigoplus_{n\geqslant 0}H^0(X,\mathscr{B}_n)=(\Gamma_*(\mathscr{B}))_{\geqslant 0}$. Then B is a finitely generated, noetherian k-algebra.*

Proof. We first show that B is finitely generated. Let r be a positive integer such that $\mathscr{B}_r$ is generated by its sections, and let $B_r=H^0(\mathscr{B}_r)$, as before. We form an exact sequence

$$0\to\mathscr{F}\to B_r\otimes_k\mathscr{O}\to\mathscr{B}_r\to 0.$$

Since $\mathscr{L}_\sigma$ is ample, there is an integer n such that $H^1(\mathscr{B}_m \otimes \mathscr{F}) = 0$ for all $m \geqslant n$. The cohomology sequence associated to the exact sequence

$$0 \to \mathscr{B}_m \otimes \mathscr{F} \to \mathscr{B}_m \otimes_k B_r \to \mathscr{B}_{m+r} \to 0$$

provides a surjective map $B_r \otimes_k B_m \to B_{m+r}$. So the global sections of degree $m+r$ are generated by products of sections of degrees m and r, respectively. It follows by induction that B is generated by $B_0 \oplus \cdots \oplus B_{n+r-1}$, and since this is a finite-dimensional vector space, finitely many generators suffice.

We now prove that B is noetherian. Note first that a set of homogeneous elements generates B as B_0-algebra if and only if it generates the augmentation ideal $\mathfrak{m} = B_{\geqslant 1}$ as left or right ideal. Thus $\mathfrak{m}$ is finitely generated.

We omit the proof of the following lemma.

LEMMA (3.15). *Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of graded left B-modules. If M' is finitely generated and M'' has finite length, then M is finitely generated. If M and M'' are finitely generated, so is M'.*

COROLLARY (3.16). *Let $\mathscr{B}(\nu)$ be a shift of $\mathscr{B}$. Then $\Gamma_*(\mathscr{B}(\nu))_{\geqslant 0}$ is a finitely generated B-module.*

Proof. By inspection, $\Gamma_*(\mathscr{B}(\nu))_{\geqslant 0}$ is a shift of the tail $B_{\geqslant \nu}$ if $\nu \geqslant 0$, and is a shift of B if $\nu < 0$. The lemma implies that these tails are finitely generated.

LEMMA (3.17). *With the notation as in the proof of Theorem* (3.14), *let M be a graded left B-module such that $\tilde{M}$ is a coherent $\mathscr{B}$-module. Then $\bar{M}_{\geqslant 0}$ is a finitely generated graded B-module.*

Proof. We choose a surjection $\mathscr{P} \to \tilde{M}$, where $\mathscr{P}$ is a finite sum of shifts of $\mathscr{B}$. The kernel $\mathscr{K}$ of this map is a coherent $\mathscr{B}$-module, and so $H^1(\mathscr{K})$ is right bounded. Also, $\Gamma_*(\mathscr{P})_{\geqslant 0}$ is finitely generated by Corollary (3.16). The lemma follows by applying Lemma (3.15) to the exact sequence

$$0 \to \Gamma_*(\mathscr{K})_{\geqslant 0} \to \Gamma_*(\mathscr{P})_{\geqslant 0} \to \bar{M}_{\geqslant 0} \to H^1(\mathscr{K})_{\geqslant 0}.$$

Now let M be a graded left ideal of B, and consider the diagram

$$\begin{array}{ccc} M & \subset & B \\ \downarrow & & \downarrow \\ \bar{M} & \longrightarrow & \bar{B} \end{array}$$

The right vertical arrow is a bijection. Therefore the left vertical arrow is injective. Moreover, its cokernel has finite length (3.13i). Lemmas (3.15) and (3.17) show that M is finitely generated. This proves that B is noetherian.

4. Proof of Proposition (1.5)

LEMMA (4.1). *Let σ be an automorphism of a noetherian scheme X, and let m be a positive integer. An invertible sheaf $\mathscr{L}$ on X is σ-ample if and only if $\mathscr{B}_m := \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \cdots \otimes \mathscr{L}^{\sigma^{m-1}} = \mathscr{L}_\sigma^{\otimes m}$ is σ^m-ample.*

Proof. The implication $\Rightarrow$ is immediate from the definition of a σ-ample sheaf. Conversely, suppose that $\mathscr{L}_\sigma^{\otimes m}$ is ample, and let us show that $\mathscr{L}_\sigma$ is ample. Let $\mathscr{F}$ be a coherent sheaf on X and let i be an integer. We apply the definition of ampleness to the sheaf $\mathscr{L}_\sigma^{\otimes i} \otimes \mathscr{F}$, concluding that there is an integer k_i such that $H^q(\mathscr{L}_\sigma^{\otimes mk} \otimes \mathscr{L}_\sigma^{\otimes i} \otimes \mathscr{F}) = 0$ for all $k \geqslant k_i$. Let $n_0 = \max\{mk_0, mk_1 + 1, ..., mk_{m-1} + m - 1\}$. Then $H^q(\mathscr{L}_\sigma^{\otimes n} \otimes \mathscr{F}) = 0$ if $n \geqslant n_0$; hence $\mathscr{L}$ is σ-ample.

To prove Proposition (1.5), we recall that according to a theorem of Murre [M], the Picard functor is representable. Its connected component $\operatorname{Pic}^0 X$ is a group scheme of finite type over k. Suppose as in the statement of the proposition that $\mathscr{L}$ is ample and that the rth power of the sheaf $\mathscr{N} := \mathscr{L}^\sigma \otimes \mathscr{L}^{-1}$ is in $\operatorname{Pic}^0 X$. Set $\mathscr{B}_n = \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}$ as above. If we write

$$\mathscr{B}_n = \mathscr{L}_1^{\otimes n} \otimes \mathscr{N}_n, \tag{4.2}$$

then the rth power of $\mathscr{N}_n$ is in $\operatorname{Pic}^0 X$ too.

Let $\mathscr{F}$ be a coherent sheaf on X. We are to prove that the sheaf $\mathscr{B}_n \otimes \mathscr{F}$ has trivial cohomology if $n \gg 0$, which is a consequence of this lemma:

LEMMA (4.3). *Let r be a positive integer, let $\mathscr{F}$ be a coherent sheaf on X, and let $\mathscr{L}$ be an ample line bundle. There is an integer n_0 such that for every invertible sheaf $\mathscr{N}$ whose rth power is in* $\operatorname{Pic}^0 X$, $H^q(X, \mathscr{L}_1^{\otimes n} \otimes \mathscr{N} \otimes \mathscr{F}) = 0$ *if $q > 0$ and $n \geqslant n_0$.*

Proof. The invertible sheaves $\mathscr{N}$ in question form a limited family. For any particular sheaf $\mathscr{N}$, $H^q(X, \mathscr{L}_1^{\otimes n} \otimes \mathscr{N} \otimes \mathscr{F}) = 0$ for all $q > 0$ if $n \gg 0$, because $\mathscr{L}$ is ample. So the lemma follows from the fact that the dimensions of cohomology groups are semicontinuous with respect to a parameter.

It remains to determine the gk-dimension of B. Let $B\langle k \rangle$ denote the Veronese subring $\bigoplus_{n \geqslant 0} B_{nk}$ of B. Then in the notation of (1.1), $B\langle k \rangle = B(X, \sigma^n, \mathscr{B}_n)$. By what has been shown, $\mathscr{B}_n$ is σ^n-ample; therefore $B\langle k \rangle$ is a noetherian ring, and B is a finitely generated $B\langle k\rangle$-module, generated by $B_0 + \cdots + B_{k-1}$. Thus gk-dim $B \leqslant$ gk-dim $B\langle k\rangle$. The other inequality is trivial, so B and $B\langle k\rangle$ have the same gk-dimension. We may therefore replace σ by σ^n and $\mathscr{L}$ by $\mathscr{B}_n$. This reduces us to the case that $\mathscr{N} \in \operatorname{Pic}^0 X$. In this case, the sheaves $\mathscr{N}_n$ defined in (4.2) are in $\operatorname{Pic}^0 X$ too. Since $\operatorname{Pic}^0 X$ is connected, formula (4.2) shows that $\chi(\mathscr{B}_n) = \chi(\mathscr{L}_1^{\otimes n})$. Moreover, $H^q(\mathscr{B}_n) = 0$ if $q > 0$ and $n \gg 0$. Therefore $H^0(\mathscr{B}_n) = H^0(\mathscr{L}_1^{\otimes n})$ if $n \gg 0$. This shows that the gk-dimension of B is the same as that of the homogeneous coordinate ring of X, which is $\dim X + 1$. ∎

Corollary (1.6) follows by combining Proposition (1.5) and Lemma (4.1).

5. The Case of an Algebraic Surface

The object of this section is to prove Theorem (1.7). Let σ be an automorphism of a smooth projective surface X. Denote the Neron–Severi group of X by NS. We set $V_{\mathbb{Z}} = NS/(\text{torsion})$, $V = NS \otimes \mathbb{R}$, and $V_{\mathbb{C}} = V \otimes \mathbb{C}$, and we denote the operation of σ on V by P. So $P \in GL(V_{\mathbb{Z}})$, and P is orthogonal with respect to the intersection form on V. If D is a divisor on X, we set $D_i = \sigma^i D$, and $\Delta_n = D_0 + \cdots + D_n$.

By the Hodge Index Theorem [G, GH], the signature of the intersection form on V is $(1, l)$. Before proving Theorem (1.7), we derive some elementary consequences of this fact for the operator P. So for the present we consider a linear operator on a real vector space V which preserves a nondegenerate symmetric form of signature $(1, l)$.

If λ is a complex eigenvalue of P, we denote the generalized eigenspace $\ker(P-\lambda)^k$ $(k \gg 0)$ in $V_{\mathbb{C}}$ by V_λ. In case λ is real, we denote the corresponding subspace of V by the same symbol.

Lemma (5.1). (i) *A subspace W of V which contains a vector D such that $(D \cdot D) > 0$ is nondegenerate with respect to the form. In other words, the form is negative semidefinite on every degenerate subspace.*

(ii) *Let λ, μ be eigenvalues of P. Then $V_\lambda \perp V_\mu$ unless $\lambda\mu = 1$.*

(iii) *If λ is an eigenvalue of P, so is λ^{-1}, and the subspace $W_\lambda := V_\lambda + V_{\lambda^{-1}}$ is nondegenerate.*

Proof. (i) Let Y be any nonzero vector in W. If $(D \cdot D) > 0$ and $(D \cdot Y) = 0$, the signature of the form shows that $(Y \cdot Y) < 0$. So either $(D \cdot Y) \neq 0$ or $(Y \cdot Y) \neq 0$. Thus Y is not in $W^\perp$.

(ii) Assume that $\lambda\mu \neq 1$. We use induction on the order of nilpotence of $P-\lambda$ on $Y\in V_\lambda$ and $P-\mu$ on $Z\in V_\mu$, writing $P=\lambda+(P-\lambda)=\mu+(P-\mu)$. Since P is orthogonal,

$$\begin{aligned}(Y\cdot Z)=\lambda\mu(Y\cdot Z)&+((P-\lambda)Y\cdot\mu Z)\\&+(\lambda Y\cdot(P-\mu)Z)+((P-\lambda)Y\cdot(P-\mu)Z).\end{aligned}$$

All terms on the right side except for $\lambda\mu(Y\cdot Z)$ vanish by induction, and we are left with $(Y\cdot Z)=\lambda\mu(Y\cdot Z)$. Hence $(Y\cdot Z)=0$.

(iii) Let $Y\in V_\lambda$. Then by (ii), $Y\perp V_\mu$ for all $\mu\neq\lambda^{-1}$. Since Y is not a null vector, $V_{\lambda^{-1}}\neq 0$. So λ^{-1} is an eigenvalue, and there is a vector $Z\in V_{\lambda^{-1}}$ such that $(Y\cdot Z)\neq 0$. It follows that W_λ is nondegenerate.

LEMMA (5.2). *If P has an eigenvalue λ of absolute value >1, then λ is real and* $\dim V_\lambda=1$. *Moreover there is at most one such eigenvalue.*

Proof. Suppose that $|\lambda|>1$. Let $\bar\lambda$ be the complex conjugate of λ, and let U_λ denote the real vector space $(V_\lambda+V_{\bar\lambda})\cap V$. This space is isotropic by (5.1ii). The form is nondegenerate on $T=U_\lambda+U_{\lambda^{-1}}$, because $T_{\mathbb{C}}$ is the orthogonal sum $W_\lambda+W_{\bar\lambda}$ if $\lambda\neq\bar\lambda$, or is W_λ if $\lambda=\bar\lambda$. Thus T is a sum of the isotropic subspaces U_λ and $U_{\lambda^{-1}}$. This implies that the dimensions of these two spaces are equal, say to d, and that the signature of the form on T is (d, d). Thus $d=1$ and $\lambda=\bar\lambda$. Since the form is negative definite on $T^\perp$, there is no eigenvalue of absolute value >1 other than λ.

LEMMA (5.3). *If all eigenvalues of a linear operator P have absolute value* 1, *and if P fixes a $\mathbb{Z}$-lattice in V, then P is quasi-unipotent.*

Proof. In this case the eigenvalues are algebraic integers all of whose complex conjugates have absolute value 1. This implies that they are roots of unity.

LEMMA (5.4). *Suppose that P is unipotent and that $P\neq I$. Let $D\in V$ be a vector with $(D\cdot D)>0$, and let W be the subspace spanned by $\{P^nD\}$. Then* $\dim W=3$, *W is a nondegenerate subspace, and the restriction of P to $W^\perp$ is the identity. Thus the Jordan form of P is*

$$\left[\begin{array}{ccc|ccc}1&&&&&\\1&1&&&&\\&1&1&&&\\\hline&&&1&&\\&&&&\ddots&\\&&&&&1\end{array}\right].\tag{5.5}$$

Proof. The subspace W is obviously P-invariant, and it is nondegenerate by Lemma (5.1i). Since W contains D and $(D \cdot D) > 0$, the form is negative definite on $W^{\perp}$. So since P is unipotent, its restriction to $W^{\perp}$ is the identity.

We write $P = 1 + N$. Expansion of the equation $(Y \cdot Z) = (PY \cdot PZ)$ shows that

$$(Y \cdot NZ) = -(NY \cdot PZ). \tag{5.6}$$

The proper invariant subspaces of W are $W_i = N^i W$, $i = 1, ..., k-1$, where $\dim W = k$ and $\dim W_i = k - i$. It follows from (5.6) that W_1 is orthogonal to $W_{k-1} = \ker N$. Since the dimensions are correct, $W_{k-1} = W_1^{\perp}$, so the form induces a nondegenerate form on $\tilde{W} := W_1/W_{k-1}$. Since the form is degenerate on W_1, it is negative semidefinite on this subspace (5.1i). So the induced form on $\tilde{W}$ is negative definite. Since the operator $\tilde{P}$ induced by P on $\tilde{W}$ is orthogonal and unipotent, $\tilde{P} = I$ and $\tilde{N} = 0$. On the other hand, $\tilde{N}\tilde{W} = \tilde{W}_2$, and hence $\tilde{W}_2 = 0$. This implies that $k = \dim W \leqslant 3$.

It remains to eliminate the possibility that $\dim W < 3$, and we know that $\dim W > 1$ because $P \neq I$. Suppose that $\dim W = 2$, and let $Z = ND$. Then $Z \neq 0$, but $NZ = 0$. By (5.6), $(Z \cdot Z) = 0$. Applying (5.6) once more shows that $(D \cdot Z) = 0$, which contradicts the Hodge index theorem. Thus $\dim W = 3$, as required.

Proof of Theorem (1.7i). For the rest of the section, P will denote the operator on $V = NS \otimes \mathbb{R}$ induced by an automorphism σ of the smooth projective surface X.

Assume that P has an eigenvalue λ which is not a root of unity. We must show that no divisor D is σ-ample. By Lemma (4.1) and Proposition (3.2iii), we may assume that D is ample, and we do so. According to Lemma (5.3), there is an eigenvalue λ with $|\lambda| \neq 1$, hence (5.2) a unique one with $|\lambda| = r > 1$, and this eigenvalue is real, so $\lambda = \pm r$. Applying Lemma (4.1), we may replace σ by σ^2, and λ by λ^2. Hence we may assume that $\lambda = r > 1$.

LEMMA (5.7). *Let C be an eigenvector with eigenvalue λ. Then* $(D \cdot C) \neq 0$.

Proof. Since λ is real and $\dim V_{\lambda} = 1$, we may assume that C is a real vector. Also, we have $(C \cdot C) = 0$, by (5.1ii). Since D is ample, $(D \cdot D) > 0$. The Hodge Index Theorem asserts that the orthogonal space to D is negative definite. Thus C is not orthogonal to D.

LEMMA (5.8). (i) *For any divisor Y, there is a constant c_1 such that $(D \cdot P^n Y) \leqslant c_1 r^n$ for all $n \geqslant 0$.*

(ii) *If Y is a positive divisor and if the projection Y_λ of Y to the eigenspace V_λ is not zero, then there is a constant $c_2 > 0$ such that for sufficiently large n, $c_2 r^n < (D \cdot P^n Y)$.*

Proof of (ii). Since $\lambda = r$ is the eigenvalue of largest absolute value, $Y_\lambda = \lim_{n\to\infty} r^{-n} P^n Y$; hence $(D \cdot Y_\lambda) = \lim_{n\to\infty} r^{-n}(D \cdot P^n Y)$. Since Y is positive and D is ample, $(D \cdot P^n Y) > 0$ for all n, and so by Lemma (5.7), $(D \cdot Y_\lambda) > 0$. The assertion follows.

LEMMA (5.9). *Let D be an ample irreducible effective divisor on X, and let $\Delta_n = D_0 + \cdots + D_n$. For any divisor Y, $H^2(\mathcal{O}(\Delta_n - Y)) = 0$ if n is sufficiently large.*

Proof. By duality, it suffices to show that $H^0(\mathcal{O}(K + Y - \Delta_n)) = 0$ for large n. Since D is ample, some multiple rD is effective. Then rD_i is effective too. It follows that for large n, $r(K + Y - \Delta_n)$ is not effective. Then $K + Y - \Delta_n$ is not effective either, which shows that $H^0(\mathcal{O}(K + Y - \Delta_n)) = 0$ if n is large enough.

LEMMA (5.10). *With the notation of the previous lemma, there is a divisor Y such that $H^1(\mathcal{O}_{D_n}(\Delta_n - Y)) \neq 0$ for sufficiently large n.*

Proof. We have $(D_n \cdot Z) = (D \cdot P^{-n} Z)$, so

$$\begin{aligned}(D_n \cdot \Delta_n) &= (D_n \cdot D_n) + (D_n \cdot D_{n-1}) + \cdots + (D_n \cdot D_0)\\ &= (D \cdot (1 + P^{-1} + \cdots + P^{-n}) D).\end{aligned}$$

Since P has an eigenvalue > 1, P^{-1} does too. We replace σ by σ^{-1} and P by P^{-1} in Lemma (5.8). The lemma tells us that for a suitable divisor Y, for suitable constants k, c, c', c'' and for $n \gg 0$, the inequalities

$$(D_n \cdot \Delta_n) \leqslant k + \sum_{i=0}^{n} c_1 r^i \leqslant c r^n,$$

$$(D_n \cdot Y) > c_2 r^n$$

hold. Replacing Y by a multiple, we may assume that $c_2 > c$. Then $(D_n \cdot Y - \Delta_n) \to \infty$ as $n \to \infty$, and so for large n, $(D_n \cdot (Y - \Delta_n)) > 2g - 2$, where g is the arithmetic genus of D. This inequality implies that $H^1(\mathcal{O}_{D_n}(\Delta_n - Y)) \neq 0$.

It follows from Lemmas (5.9) and (5.10) that X contains no σ-ample divisor D. For, by Lemma (4.1), we may assume that D is very ample, hence that it is an irreducible curve. Choose Y as in Lemma (5.10). The exact sequence

$$0 \to \mathcal{O}(\Delta_{n-1} - Y) \to \mathcal{O}(\Delta_n - Y) \to \mathcal{O}_{D_n}(\Delta_n - Y) \to 0,$$

together with Lemma (5.9), shows that $H^1(\mathcal{O}(\Delta_n - Y)) \neq 0$ for large n, as required.

Proof of Theorem (1.7ii). We suppose that P is quasi-unipotent, and we let D be a divisor such that Δ_m is ample for some m. To show that D is σ-ample, we may apply Lemma (4.1) to reduce to the case that D is ample and that P is unipotent, which we assume from now on. We may also assume that P is not the identity.

The Jordan form of P is determined by Proposition (5.4). We write $P = 1 + N$, where $N^2 \neq 0$ but $N^3 = 0$. Then for any divisors Y, Z,

$$(Z \cdot P^n Y) = (Z \cdot Y) + n(Z \cdot NY) + \tfrac{1}{2} n(n-1)(Z \cdot N^2 Y). \qquad (5.11)$$

Also, it follows from (5.4) that $N^2 D \neq 0$.

LEMMA (5.12). *Assume that $N^2 Y \neq 0$. Then $(D \cdot N^2 Y) \neq 0$. If in addition $Y > 0$, then $(D \cdot N^2 Y) > 0$.*

Proof. Formula (5.6) shows that $(N^2 Y \cdot N^2 Y) = 0$, hence by the Hodge Index Theorem that $(D \cdot N^2 Y) \neq 0$. Suppose that $Y > 0$. Then $P^n Y > 0$ too; hence $(D \cdot P^n Y) > 0$. But the dominant term of (5.11), with $Z = D$, is $\frac{1}{2} n(n-1)(D \cdot N^2 Y)$. Hence $(D \cdot N^2 Y) > 0$.

LEMMA (5.13). (i) *$(D \cdot P^n D) = cn^2 + O(n)$, $(D \cdot \Delta_n) = c'n^3 + O(n^2)$, and $(\Delta_n \cdot \Delta_n) = c''n^4 + O(n^3)$, with $c, c', c'' > 0$.*

(ii) *For any Y, $(D \cdot P^n Y) \leqslant c(Y) n^2$.*

This lemma follows from (5.11) and (5.12).

LEMMA (5.14). *For any divisor Y, $H^0(\mathcal{O}(\Delta_n - Y)) \neq 0$ if n is sufficiently large.*

Proof. The Riemann–Roch formula is

$$\chi(\mathcal{O}(\Delta_n - Y)) = \chi(\mathcal{O}) + \tfrac{1}{2}((\Delta_n - Y)^2 - ((\Delta_n - Y) \cdot K)),$$

and Lemma (5.13) shows that $\chi(\mathcal{O}(\Delta_n - Y)) > 0$ for large n. Also, $H^2(\mathcal{O}(\Delta_n - Y)) = 0$ for large n, by Lemma (5.9). It follows that $H^0(\mathcal{O}(\Delta_n - Y)) > 0$.

Combined with Lemma (4.1), this lemma allows us to assume that $D > 0$, which we do from now on.

LEMMA (5.15). *Let Z be a proper one-dimensional scheme which is isomorphic to a divisor on a smooth algebraic surface, and let $\mathcal{L}$ be an invertible sheaf on Z. There is an integer n_0 with this property: For every embedding of Z as a divisor into a smooth surface X, and for every set $D_1, ..., D_n$ of* *ample divisors on X with $n \geqslant n_0$, the sheaf $\mathscr{F} = \mathcal{O}(D_1 + \cdots + D_n) \otimes \mathscr{L}$ is generated by its sections, and $H^q(\mathscr{F}) = 0$ if $q > 0$.*

Proof. Choose a reduced irreducible curve C which is a component of Z, and let Z' be the divisor $Z - C$. Let $\mathscr{I}$ be the ideal sheaf of Z' in Z, which is an invertible sheaf on C, so that the sequence

$$0 \to \mathscr{I} \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$$

is exact. Since D_i is ample, $(D_i \cdot C) \geqslant 1$ for each i. Hence $\mathscr{I} \otimes \mathscr{F}$ has degree at least n. If n is sufficiently large, then $\mathscr{I} \otimes \mathscr{F}$ is generated by its sections and $H^q(\mathscr{I} \otimes \mathscr{F}) = 0$. The lemma follows by induction, by tensoring the above exact sequence with $\mathscr{F}$.

The following lemma, combined with Proposition (3.4), completes the proof of Theorem (1.7ii).

LEMMA (5.16). *Let C be any divisor on X. Then $H^q(\mathcal{O}(\Delta_n - C)) = 0$ for $q > 0$ and for sufficiently large n.*

Proof. We first treat the case that $C = 0$. We choose a divisor Y which is the sum of sufficiently many ample divisors so that the previous lemma applies, with $Z = D$ and $\mathscr{L} = \mathcal{O}_Z$. We may also assume that $H^q(\mathcal{O}(Y)) = 0$ for $q > 0$. We apply the lemma to the right-hand terms of the exact sequences

$$0 \to \mathcal{O}(\Delta_{n-1} + Y) \to \mathcal{O}(\Delta_n + Y) \to \mathcal{O}_{D_n}(\Delta_n + Y) \to 0,$$

concluding by induction that $H^q(\mathcal{O}(\Delta_n + Y)) = 0$ for $q > 0$ and for all $n \geqslant 0$. The same is true when Y is replaced by $\sigma^s Y$, for any $s \in \mathbb{Z}$.

Let $\approx$ stand for linear equivalence. Applying Lemma (5.14), we may choose an integer k so that $\Delta_k \approx Y + Z$, with $Z > 0$. Then

$$\Delta_n = \sigma^k \Delta_{n-k} + \Delta_k \approx \sigma^k \Delta_{n-k} + Y + Z.$$

So there is an exact sequence

$$0 \to \mathcal{O}(\sigma^k \Delta_{n-k} + Y) \to \mathcal{O}(\Delta_n) \to \mathcal{O}_Z(\Delta_n) \to 0.$$

The first term is $\sigma^k(\mathcal{O}(\Delta_{n-k} + \sigma^{-k} Y))$. Hence $H^q(\mathcal{O}(\sigma^k \Delta_{n-k} + Y)) = 0$ for $q > 0$ and for all $n \geqslant 0$. Also, $H^q(\mathcal{O}_Z(\Delta_n)) = 0$ for $q > 0$ and for large n, by Lemma (5.15). Hence $H^q(\mathcal{O}(\Delta_n)) = 0$ for $q > 0$ and for large n.

Now let C be arbitrary. By Lemma (5.14), there is a k such that $\Delta_k \approx C + Z$, with $Z > 0$. Then $\Delta_n - C \approx \sigma^k \Delta_{n-k} + Z$, and the exact sequence

$$0 \to \mathcal{O}(\sigma^k \Delta_{n-k}) \to \mathcal{O}(\sigma^k \Delta_{n-k} + Z) \to \mathcal{O}_Z(\sigma^k \Delta_{n-k} + Z) \to 0,$$

together with Lemma (5.15), completes the proof.

Part (iii) of Theorem (1.7) is essentially proved. The case $P^k = I$ is included in Proposition (1.5). If $P^k \neq I$, we note that the dominant term in the Riemann–Roch formula for $\chi(\mathcal{O}(\Delta_n))$ is $\frac{1}{2}(\Delta_n \cdot \Delta_n)$. According to Lemma (5.13) this term has the order of n^4. Hence dim $B_n = O(n^4)$, which means that the gk-dimension of B is 5.

COROLLARY (5.17). *Let σ be an automorphism of a surface X and let D be an ample divisor on X. The ring B is noetherian if it has polynomial growth.*

Proof. If P is quasi-unipotent, then B has polynomial growth and is noetherian, by Theorems (1.7) and (1.4). If P is not quasi-unipotent, then P has an eigenvalue of absolute value >1. In this case Riemann–Roch on X, together with Lemmas (5.8) and (5.9) shows that B has exponential growth.

EXAMPLE (5.18). Translation along a pencil of elliptic curves: Let $\pi: S \to Z$ be a fibration of a smooth surface by a pencil of elliptic curves with two sections $Z_0 \neq Z_1$ such that $Z_1 - Z_0 \neq 0$ in $NS \otimes \mathbb{Q}$. Then taking Z_0 as zero section, we can consider the automorphism σ which represents translation along the fibres by Z_1. Assume for simplicity that Z_1 passes through the connected component of each fibre F. We choose the $\mathbb{Q}$-basis for NS

$$F; Z_0; Y = Z_1 - Z_0; Z_2, ..., Z_r; \theta_1, ..., \theta_s,$$

where $Z_1, ..., Z_r$ is a $\mathbb{Q}$-basis for the group of sections, and where θ_i are the components of the reducible fibres, with the connected components omitted. We may assume that the sections Z_i pass through the connected components of each fibre. Then P acts as

$$PF = F; PY = Y + nF; PZ_0 = Z_0 + Y;$$

$$PZ_i = Z_i + Y + n_i F\,(i > 1); P\theta_j = \theta_j,$$

for suitable n, n_i. Thus P is unipotent.

The coefficients m, n_i can be computed using the relations

$$(PZ_i \cdot PZ_i) = (Z_i \cdot Z_i) = (Z_0 \cdot Z_0).$$

For instance, $PZ_1 = 2Z_1 - Z_0 + nF$. To determine the coefficient n, we compute

$$(Z_0 \cdot Z_0) = (PZ_1 \cdot PZ_1) = 5(Z_0 \cdot Z_0) - 4(Z_1 \cdot Z_0) + 2n.$$

Hence $n = 2((Z_1 \cdot Z_0) - (Z_0 \cdot Z_0)) = -(Z_1 - Z_0)^2$ is the canonical height of the section Z_1. Since Z_1 is not of finite order, $n > 0$, P is not the identity, and by Theorem (1.7), the associated ring B has gk-dimension 5.

Questions (5.19). (i) What is the extension of Theorem (1.7) to higher dimension?

(ii) Does the existence of a σ-ample divisor imply that B has polynomial growth?

References

[ATV] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, Grothendieck, Vol. 1, Birkhäuser, Boston, 1990, (to appear).

[G] A. Grothendieck, Sur une note de Mattuck–Tate, *J. Reine Angew. Math.* **200** (1958), 208–215.

[GH] P. Griffiths and J. Harris, "Principles of Algebraic Geometry," p. 741, Wiley, New York, 1978.

[M] J. P. Murre, On contravariant functors from the category of preschemes over a field into the category of Abelian groups, *Publ. Inst. Math. Hautes Etudes Sci.*, No. 23, 1964.

[NV] C. Năstăsescu and F. Van Oystaeyen, "Graded Ring Theory," p. 16, North-Holland, Amsterdam, 1982.

[S] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. of Math.* **61** (1955), 197.