CANONICAL BASIS IN IRREDUCIBLE REPRESENTATIONS OF gl_3 AND ITS APPLICATIONS .

I.Gelfand, A.Zelevinsky
Moscow State University, USSR

In this paper we define and study the new basis in irreducible finite dimensional representations of the Lie algebra gl_3 , which we call canonical. This basis has some advantages against the commonly used Gelfand-Tsetlin basis. The Gelfand-Tsetlin basis is defined by means of the restriction of representations of gl_3 to the chain of subalgebras $gl_1 \subset gl_2 \subset gl_3$. But this chain depends on the arbitrary choice of ordering of the basis vectors e_1 , e_2, e_3 in the standard representation of gl_3 ; another choice leads to another Gelfand-Tsetlin basis whose relation to the first basis is rather complicated (see [6]). Meantime, the canonical basis does not depend on this arbitrary choice. (In more invariant terms, the canonical basis depends only on the choice of Cartan and Borel subalgebras in gl_3 whereas to define the Gelfand-Tsetlin basis we need in addition an ordering of simple roots). We will see that the canonical basis is related to the standard Gelfand-Tsetlin basis as well as to the opposite Gelfand-Tsetlin basis by a simple triangular transformation; so in some sense it lies "in the middle" between these two bases.

Another advantage of the canonical basis is that the action of the generators of gl_3 in it is given by integral matrices, and the matrix elements have no denominators. Besides, the canonical basis has a beautiful polynomial realization.

the classical Racah and Clebsch-Gordan coefficients (related to sl_2). Namely, as indicated above the canonical basis gives the factorization of the transition matrix between two Gelfand-Tsetlin bases for gl_3 into the product of two triangular matrices. But it is shown in [6], [7] that elements of this matrix are closely related to Racah coefficients (or Wigner's 6j-symbols). As a consequence we get a simple triangular factorization of the matrix of 6j-symbols and of the Clebsch-Gordan matrix. We are grateful to V.N.Tolstoy who called our attention to the papers [6], [7].

The paper is organized as follows. In § 1 the canonical basis is defined. The transition matrix between the canonical basis and the Gelfand-Tsetlin basis and the action of generators of gl_3 in the canonical basis are computed in §2. In §3 we evaluate the canonical basis in the realization of irreducible representations in the space of polynomials on the base affine space of the group GL_3.

In §4 the action of the subgroup of permutations $S_3 \subset GL_3$ in the canonical basis is computed. Since a permutation sends the Gelfand-Tsetlin basis into another Gelfand-Tsetlin basis, this enables one to compute the transition matrices between the canonical basis and all Gelfand-Tsetlin bases.

Finally, the applications of the results of §4 to Clebsch-Gordan coefficients and 6j-symbols are given in §5.

The main results of this paper were announced in [3]. A possibility of generalization of the results to gl_n is discussed in [2].

1. Definition of the canonical basis. We recall that a weight for gl_n is an integral vector $\gamma = (\gamma_1, \dots, \gamma_n)$; the weight γ is called highest if $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n$. Let V_λ denote the irreducible finite dimensional representation of gl_n with highest weight λ, and

$V_\lambda(\gamma)$ be the weight subspace of weight γ in V_λ; thus, $V_\lambda(\gamma) = \{ v \in V_\lambda : E_{ii} v = \gamma_i v$ for $1 \leq i \leq n \}$. If $\nu = (\nu_1, \dots, \nu_n)$ is one more highest weight for gl_n then we put $V_\lambda(\gamma, \nu) = \{ v \in V_\lambda(\gamma) : E_{i,i+1}^{\nu_i - \nu_{i+1} + 1} v = 0$ for $1 \leq i \leq n-1 \}$.

A basis in V_λ will be called proper if each of the subspaces $V_\lambda(\gamma, \nu)$ (for all possible γ and ν) is spanned by its subset.

Theorem 1. Each irreducible finite dimensional representation of gl_n has a proper basis.

Theorem is proved in [2].

For general n and λ a proper basis in V_λ is not unique. It turns out, however that it is essentially unique for $n=3$.

Theorem 2. In each irreducible finite dimensional representation of gl_3 there is only one proper basis up to scalar multiples. We call this basis (normalized as indicated in §2 below) canonical.

This theorem is also proved in [2]. Another proof is outlined in §2.

Hereafter we deal only with gl_3. For each vector f belonging to an irreducible representation of gl_3 we define the exponents $d_1 = d_1(f)$ and $d_2 = d_2(f)$ to be the least nonnegative integers such that $E_{12}^{d_1+1} f = E_{23}^{d_2+1} f = 0$.

Proposition 1. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a highest weight for gl_3 and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ a fixed weight in V_λ. Then each vector f of the canonical basis in $V_\lambda(\gamma)$ is uniquely determined by $d_1(f)$ (as well as by $d_2(f)$). When f varies, each of the exponents of f runs some interval in Z; the sum $d_1(f) + d_2(f)$ is independent of f and equals $\max(\lambda_1 - \gamma_1, \gamma_3 - \lambda_3)$.

This Proposition follows from the results in [2]; another proof is outlined in § 2.

By Proposition 1, a vector f of the canonical basis in V_λ may be parametrized by its weight and any one of the exponents. However, another parametrization will be more convenient for our purposes viz., the parametrization by Gelfand-Tsetlin patterns. It will be introduced in the next section.

2. Canonical basis and Gelfand-Tsetlin basis. In this section we compute the transition matrix between the canonical basis and the Gelfand-Tsetlin basis, and the action of the generators of gl_3 on the canonical basis. We begin with some well-known facts about Gelfand-Tsetlin patterns.

Let V_λ be an irreducible representation of gl_3 with highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Recall that a Gelfand-Tsetlin pattern (G-pattern) of type λ is a triangular array

$$\Lambda = \begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \alpha_1 & & \alpha_2 & \\ & & \beta & & \end{array}$$

of integers satisfying the betweenness conditions: $\lambda_1 \geqslant \alpha_1 \geqslant \lambda_2 \geqslant \alpha_2 \geqslant \lambda_3$, $\alpha_1 \geqslant \beta \geqslant \alpha_2$. The weight $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of a G-pattern Λ is defined from the equations

$$\gamma_1 = \beta, \quad \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2, \quad \gamma_1 + \gamma_2 + \gamma_3 = \lambda_1 + \lambda_2 + \lambda_3 \tag{1}$$

It is known that the Gelfand-Tsetlin basis (G-basis) in $V_\lambda(\gamma)$ is parametrized by G-patterns of type λ and weight γ; let $|\Lambda\rangle$ denote the normalized vector of the basis corresponding to Λ. It will be more convenient for us to consider $e_\Lambda = d(\Lambda)\,|\Lambda\rangle$ instead of $|\Lambda\rangle$, where the normalization constant $d(\Lambda)$ is defined to be

$$d(\Lambda) = \{(\lambda_1-\alpha_1)!\,(\lambda_1-\alpha_2+1)!\,(\lambda_2-\alpha_2)!\,(\alpha_1-\beta)!\,(\alpha_1-\lambda_2)! \cdot (\alpha_1-\lambda_3+1)!\,(\alpha_2-\lambda_3)!\,(\beta-\alpha_2)!/(\alpha_1-\alpha_2+1)!\,(\alpha_1-\alpha_2)!\}^{\frac{1}{2}} \tag{2}$$

When the type λ of Λ is fixed we write simply $\Lambda = \begin{smallmatrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{smallmatrix}$ and $e_\Lambda = e^{\alpha_1\ \alpha_2}_{\ \ \beta}$. Rewriting the well-known Gelfand-Tsetlin formulas [4], [5] with the account of the normalization (2) we see that the action of the generators of gl_3 on the basis (e_Λ) is given by

$$E_{12}e^{\alpha_1\ \alpha_2}_{\ \ \beta} = (\alpha_1-\beta)e^{\alpha_1\ \alpha_2}_{\ \ \beta+1}, \quad E_{21}e^{\alpha_1\ \alpha_2}_{\ \ \beta} = (\beta-\alpha_2)\,e^{\alpha_1\ \alpha_2}_{\ \ \beta-1},$$

$$E_{23}e^{\alpha_1\ \alpha_2}_{\ \ \beta} = (\lambda_1-\alpha_1)e^{\alpha_1+1\ \alpha_2}_{\ \ \ \beta} + \frac{(\lambda_2-\alpha_2)(\beta-\alpha_2)(\lambda_1-\alpha_2+1)}{(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2)} \cdot e^{\alpha_1\ \alpha_2+1}_{\ \ \beta}, \tag{3}$$

$$E_{32}e^{\alpha_1\ \alpha_2}_{\ \ \beta} = \frac{(\alpha_1-\beta)(\alpha_1-\lambda_2)(\alpha_1-\lambda_3+1)}{(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2)}\,e^{\alpha_1-1\ \alpha_2}_{\ \ \ \beta} + (\alpha_2-\lambda_3)\,e^{\alpha_1\ \alpha_2-1}_{\ \ \ \beta}.$$

Now we return to the canonical basis. We reformulate the definition in terms of the exponents $d_1(f)$ and $d_2(f)$ introduced in §1. Namely, the canonical basis in V_λ is determined up to normalization by the condition that for any $d_1, d_2 \geqslant 0$ and any weight γ the subspace of $f \in V_\lambda(\gamma)$ such that $d_1(f) \leqslant d_1$, $d_2(f) \leqslant d_2$ is spanned by a subset of this basis. Our next goal is to define the normalization of the vectors of the canonical basis and their parametrization by G-patterns. We need some more definitions.

Let $\Lambda = \begin{smallmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \alpha_1 & & \alpha_2 & \\ & & \beta & & \end{smallmatrix}$ and $\Lambda' = \begin{smallmatrix} \lambda_1' & & \lambda_2' & & \lambda_3' \\ & \alpha_1' & & \alpha_2' & \\ & & \beta' & & \end{smallmatrix}$ be two G-patterns.

We write $\Lambda' < \Lambda$ if Λ and Λ' have the same type and

weight and $\alpha_1' < \alpha_1$. By (1), a G-pattern of fixed type and weight is determined by one parameter, e.g. by α_1; therefore, the relation $\Lambda' < \Lambda$ is a linear order on such patterns. For each integer k we put $\Lambda[k] = \begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \alpha_1+k & & \alpha_2-k & \\ & & \beta & & \end{matrix}$; thus $\Lambda' < \Lambda$ if and only if $\Lambda' = \Lambda[-k]$ for some $k > 0$. Let $V(\Lambda)$ denote the subspace of V_λ (where λ is the type of a G-pattern Λ) spanned by $e_{\Lambda'}$'s with $\Lambda' \leqslant \Lambda$.

Proposition 2. Let Λ be a G-pattern of type λ and weight γ. Then $V(\Lambda) = \{f \in V_\lambda(\gamma) : d_1(f) \leqslant d_1(e_\Lambda)\}$.

Proof. By (3) we have

$$d_1(e^{\alpha_1 \alpha_2}_{\beta}) = \alpha_1 - \beta \tag{4}$$

Hence $d_1(e_{\Lambda'}) < d_1(e_\Lambda)$ if $\Lambda' < \Lambda$. This implies our proposition.

Proposition 3. For any G-pattern Λ there is the unique vector of the canonical basis belonging to $V(\Lambda) - \sum_{\Lambda'<\Lambda} V(\Lambda')$. We normalize it so that the coefficient of e_Λ in its expansion in the G-basis were 1; denote this vector by f_Λ.

Thus, this proposition gives the normalization of the canonical basis and its parametrization by G-patterns.

Proof. By Proposition 2, each of the subspaces $V(\Lambda)$ is spanned by a subset of the canonical basis. This implies our proposition by induction in Λ.

The vector f_Λ may be written in the form

$$f_\Lambda = \sum_{k \geqslant 0} (-1)^k b_k(\Lambda)\, e_{\Lambda[-k]}\,; \tag{5}$$

it is also clear that e_Λ is of the form

$$e_\Lambda = \sum_{k \geqslant 0} a_k(\Lambda)\, f_{\Lambda[-k]} \tag{6}$$

We compute $a_k(\Lambda)$ and $b_k(\Lambda)$, and the action of generators of gl_3 on the (f_Λ). It turns out that all formulas are piecewise polynomial with singularities on a certain hyperplane in the space of G-patterns. This hyperplane is defined by the equation $\lambda_2 = \gamma_2$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the type of a G-pattern and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is its weight.

A G-pattern Λ of type λ and weight γ is of class I if $\gamma_2 > \lambda_2$, of class II if $\gamma_2 < \lambda_2$ and of class O if $\gamma_2 = \lambda_2$.

Theorem 3. Let $\Lambda = \begin{matrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{matrix}$ be a G-pattern of type λ. Then

$$a_k(\Lambda) = \binom{\beta-\alpha_2}{k}\binom{\lambda_2-\alpha_2}{k}\Big/\binom{\alpha_1-\alpha_2}{k}, \tag{7^{I}}$$

$$b_k(\Lambda) = \binom{\beta-\alpha_2}{k}\binom{\lambda_2-\alpha_2}{k}\Big/\binom{\alpha_1-\alpha_2+1-k}{k} \tag{8^{I}}$$

if Λ is of class I or O;

$$a_k(\Lambda) = \binom{\alpha_1-\beta}{k}\binom{\alpha_1-\lambda_2}{k}\Big/\binom{\alpha_1-\alpha_2}{k}, \tag{7^{II}}$$

$$b_k(\Lambda) = \binom{\alpha_1-\beta}{k}\binom{\alpha_1-\lambda_2}{k}\Big/\binom{\alpha_1-\alpha_2+1-k}{k} \tag{8^{II}}$$

if Λ is of class II or O.

Theorem 4. The action of generators of gl_3 on the (f_Λ) is given by the formulas

$$E_{12} f^{\alpha_1 \alpha_2}_{\beta} = (\alpha_1-\beta) f^{\alpha_1 \alpha_2}_{\beta+1} + (\lambda_2-\alpha_2)\chi(\mathrm{I}) f^{\alpha_1-1\,\alpha_2+1}_{\beta+1},$$

$$E_{21} f^{\alpha_1 \alpha_2}_{\beta} = (\beta-\alpha_2) f^{\alpha_1 \alpha_2}_{\beta-1} + (\alpha_1-\lambda_2)\chi(\mathrm{II}) f^{\alpha_1-1\,\alpha_2+1}_{\beta-1},$$

$$E_{23} f^{\alpha_1 \alpha_2}_{\beta} = (\lambda_1-\alpha_1) f^{\alpha_1+1\,\alpha_2}_{\beta} + (\lambda_1+\lambda_2+\beta-2\alpha_1-\alpha_2)\cdot \chi(\mathrm{II}) f^{\alpha_1 \alpha_2+1}_{\beta}, \tag{9}$$

$$E_{32} f^{\alpha_1 \alpha_2}_{\beta} = (\alpha_2-\lambda_3) f^{\alpha_1 \alpha_2-1}_{\beta} + (\alpha_1+2\alpha_2-\lambda_2-\lambda_3-\beta)\cdot \chi(\mathrm{I}) f^{\alpha_1-1\,\alpha_2}_{\beta}.$$

where the symbol $\chi(J)$ equals 1 if $\begin{matrix}\lambda_1 & & \lambda_2 & & \lambda_3\\ & \alpha_1 & & \alpha_2 & \\ & & \beta & & \end{matrix}$ is of class J, and equals 0, otherwise.

Now we give a sketch of the proof of Theorems 2,3,4 and Proposition 1. Denote temporarily the right-hand side of (5) by $\tilde{f}_\Lambda$, where the $b_k(\Lambda)$ are defined by (8^{I}) (8^{II}). The formulas (6) and (9) with f replaced by $\tilde{f}$ and $a_k(\Lambda)$ defined by (7^{I}), (7^{II}) are directly verified (to verify (9) one only needs to use (5) and (3)). It easily follows from (9) that

$$d_1(\tilde{f}\begin{matrix}\alpha_1 & & \alpha_2\\ & \beta & \end{matrix}) = d_1(e\begin{matrix}\alpha_1 & & \alpha_2\\ & \beta & \end{matrix}) = \alpha_1 - \beta \quad \text{, and that}$$

$$d_2(\tilde{f}\begin{matrix}\alpha_1 & & \alpha_2\\ & \beta & \end{matrix}) = \lambda_1 - \alpha_1 + \max(0,\lambda_2-\gamma_2) \text{ . Hence for}$$

$\Lambda'<\Lambda$ we have $d_1(\tilde{f}_{\Lambda'})<d_1(\tilde{f}_\Lambda)$ and $d_2(\tilde{f}_{\Lambda'})>d_2(\tilde{f}_\Lambda)$; moreover, $d_1(\tilde{f}_\Lambda)+d_2(\tilde{f}_\Lambda)$ depends only on the type and weight of Λ, and has the value given by Proposition 1.

Let λ be a highest weight for gl_3, γ a weight, and d_1, d_2 nonnegative integers. The just proven inequalities between the exponents of $\tilde{f}_\Lambda$ easily imply that the subspace $\{f\in V_\lambda(\gamma): d_1(f)\leq d_1,\ d_2(f)\leq d_2\}$ is spanned by the vectors $\tilde{f}_\Lambda$ contained in it. So $\tilde{f}_\Lambda$ form a proper basis. Furthermore, if Λ is a G-pattern of type λ and weight γ, and $d_1=d_1(\tilde{f}_\Lambda)$, $d_2=d_2(\tilde{f}_\Lambda)$ then the corresponding subspace is spanned by $\tilde{f}_\Lambda$. This proves uniqueness of a proper basis for gl_3. Therefore, the basis $(\tilde{f}_\Lambda)$ is canonical, which proves all our assertions.

The formulas (7)-(9) have interesting symmetry properties. For each G-pattern

$$\Lambda = \begin{matrix}\lambda_1 & & \lambda_2 & & \lambda_3\\ & \alpha_1 & & \alpha_2 & \\ & & \beta & & \end{matrix} \qquad \text{we put} \qquad \tilde{\Lambda} = \begin{matrix}-\lambda_3 & & -\lambda_2 & & -\lambda_1\\ & -\alpha_2 & & -\alpha_1 & \\ & & -\beta & & \end{matrix}$$

The following properties of the mapping $\Lambda\longrightarrow\tilde{\Lambda}$ follow at once from definitions.

Proposition 4. The mapping $\Lambda\longrightarrow\tilde{\Lambda}$ is an involution, i.e. $\tilde{\tilde{\Lambda}}=\Lambda$. If Λ is of type λ and weight γ then $\tilde{\Lambda}$ is of type $\tilde{\lambda}=(-\lambda_3,-\lambda_2,-\lambda_1)$ and weight $-\gamma$. If Λ is of class I (resp. II, 0) then $\tilde{\Lambda}$ is of class II (resp. I, 0). We have $\Lambda[k]=\tilde{\Lambda}[k]$; in particular, if $\Lambda'<\Lambda$ then $\tilde{\Lambda}'<\tilde{\Lambda}$.

Let $T:V_\lambda\longrightarrow V_{\tilde{\lambda}}$ denote the linear operator such that $T(e_\Lambda)=e_{\tilde{\Lambda}}$ for any G-pattern Λ of type λ. Let ω be the automorphism of gl_3 such that $\omega(E_{ii})=-E_{ii}$, $\omega(E_{i,i\pm1})=E_{i\pm1,i}$.

Proposition 5. (a) For any G-pattern Λ of type λ we have $T(f_\Lambda)=f_{\tilde{\Lambda}}$.

(b) We have $T\circ\omega(X)=X\circ T$ for any $X\in gl_3$.

Proof. By (7), (8) and Proposition 4, for each G-pattern Λ we have

$$a_k(\tilde{\Lambda})=a_k(\Lambda),\ b_k(\tilde{\Lambda})=b_k(\Lambda) \tag{10}$$

This proves (a). Part (b) follows at once from (3) or (9).

3. Geometric realization of the canonical basis. In this section we compute the canonical basis in the realization of irreducible finite dimensional representations of gl_3 in the space of functions on the base affine space A of GL_3. By definition, $A=GL_3/N$, where N is the subgroup of unipotent upper triangular matrices in GL_3. We realize A as a cone in the seven-dimensional affine space. Namely, let $V=V_{(1,0,0)}$ be the standard three-dimensional representation of gl_3 and (e_1,e_2,e_3) the standard basis in V. We let $W=V\oplus\Lambda^2V\oplus\Lambda^3V$; then the vectors e_i, $e_i\wedge e_j$ for $i<j$, and $e_1\wedge e_2\wedge e_3$ form a basis in W. Let (x_i,x_{ij},x_{123}) be the corresponding coordinates in W. Then A may be realized as the subset of W determined by the inequalities

$(x_1,x_2,x_3) \neq 0$, $(x_{12},x_{13},x_{23}) \neq 0$, $x_{123} \neq 0$, and the equation

$$x_1x_{23} - x_2x_{13} + x_3x_{12} = 0 \qquad (11)$$

It is known that the natural representation of GL_3 (or gl_3) in the space $C[A]$ of regular (algebraic) functions on A is the sum of all the irreducible representations V_λ each occuring exactly once.

A monomial $\prod_i x_i^{m_i} \cdot \prod_{i<j} x_{ij}^{m_{ij}} \cdot x_{123}^{m_{123}}$ is said to be proper if all its exponents are integers, all except m_{123} are nonnegative, and $m_2m_{13} = 0$. It follows easily from (11) that proper monomials form a basis in $C[A]$. It turns out that union of canonical bases in all V_λ is just this basis. To be more precise we assign to each G-pattern $\Lambda = \begin{matrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{matrix}$ of type λ and weight γ the proper monomial $x^{m(\Lambda)}$ whose exponents are defined as follows: if Λ is of class I or O then

$$m_1 = \beta - \alpha_2,\ m_3 = \lambda_1 - \alpha_1,\ m_{12} = \alpha_2 - \lambda_3,\ m_{23} = \lambda_2 - \alpha_2,$$
$$m_{123} = \lambda_3,\ m_2 = \gamma_2 - \lambda_2,\ m_{13} = 0\ ; \qquad (12^{I})$$

if Λ is of class II or O then

$$m_1 = \alpha_1 - \lambda_2,\ m_3 = \lambda_1 - \alpha_1,\ m_{12} = \alpha_2 - \lambda_3,\ m_{23} = \alpha_1 - \beta,$$
$$m_{123} = \lambda_3,\ m_2 = 0,\ m_{13} = \lambda_2 - \gamma_2\ . \qquad (12^{II})$$

Theorem 5. The linear mapping $\bigoplus_\lambda V_\lambda \longrightarrow C[A]$ sending each f_Λ to the proper monomial $x^{m(\Lambda)}$ is an isomorphism of representations of GL_3 (and gl_3).

Proof. It suffices to compute the action of generators of gl_3 in the basis of proper monomials (this is done in a straightforward way with the help of (11)) and then to note that the replacement of monomials by G-patterns according to (12) transforms this action into the action given by (9).

Remarks. 1) Theorem 5 and the formulas (6) and (7) give the realization of the G-basis in $C[A]$. Similar formulas were obtained in [5], p. 303.

2) The basis of proper monomials in $C[A]$ was mentioned in [5], p. 341 but its canonical definition was not given.

4. The action of permutations in GL_3 on the canonical basis. In this section we compute the action of the permutation subgroup $S_3 \subset GL_3$ on the canonical basis in each V_λ. It is convenient for us to choose two permutations

$$s_1 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \text{and} \quad s = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

as generators of S_3.

Proposition 6. The action of s_1 in the canonical basis in V_λ is given by

$$s_1 f^{\begin{matrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{matrix}} = (-1)^{\alpha_2} \sum_{\alpha_1' \leq \alpha_1} \binom{d}{\alpha_1 - \alpha_1'} f^{\begin{matrix} \alpha_1' & & \alpha_1+\alpha_2-\alpha_1' \\ & \alpha_1+\alpha_2-\beta & \end{matrix}},$$

where $d = \min(\alpha_1-\lambda_2, \alpha_1-\beta, \lambda_2-\alpha_2, \beta-\alpha_2)$.

Sketch of the proof. We realize f_Λ's as proper monomials on the base affine space A (see Theorem 5). Now s_1 acts on $C[A]$ by an automorphism, and we have $s_1x_1 = x_2$, $s_1x_2 = x_1$, $s_1x_3 = x_3$, $s_1x_{12} = -x_{12}$, $s_1x_{13} = x_{23}$, $s_1x_{23} = x_{13}$, $s_1x_{123} = -x_{123}$. Acting by s_1 on the proper monomial $x^{m(\Lambda)}$ corresponding to f_Λ we obtain a monomial not proper in general. But it is decomposed in a linear combination of proper monomials with the help of (11). It remains to use (12) for translating the re-

sults back into the language of G-patterns.

To describe the action of s we introduce some notation. For each weight $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ we let $s\gamma = (\gamma_3, \gamma_2, \gamma_1)$. Now let $\Lambda = \begin{matrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{matrix}$ be a G-pattern of type λ and weight γ. Define the G-pattern $s\Lambda$ of type λ and weight $s\gamma$ to be $s\Lambda = \begin{matrix} \lambda_1+\alpha_2-\beta, & & \lambda_2+\lambda_3-\alpha_2 \\ & \gamma_3 & \end{matrix}$ if Λ is of class I or 0, and $s\Lambda = \begin{matrix} \lambda_1+\lambda_2-\alpha_1, & & \lambda_3+\alpha_1-\beta \\ & \gamma_3 & \end{matrix}$ if Λ is of class II. It is directly verified that the mapping $\Lambda \longrightarrow s\Lambda$ is an involution, i.e. $s(s\Lambda) = \Lambda$ and that $s(\Lambda[k]) = (s\Lambda)[-k]$ for all integer k; in particular, if $\Lambda' < \Lambda$ then $s\Lambda' > s\Lambda$.

Theorem 6. The action of $s \in GL_3$ on the canonical basis in V_λ is given by $sf_\Lambda = (-1)^{\lambda_2} f_{s\Lambda}$.

Proof is similar to that of Proposition 6.

Theorems 3 and 6 allow us to decompose the matrix of the operator $s : V_\lambda \longrightarrow V_\lambda$ in the G-basis (e_Λ) into the product of two triangular matrices. Put $g_\Lambda = (-1)^{\lambda_2} s\, e_{s\Lambda}$. By (5), (6) and Theorem 6 we have:

$$f_\Lambda = \sum_{k \geq 0} (-1)^k b_k(s\Lambda)\, g_{\Lambda[k]} \tag{13}$$

$$g_\Lambda = \sum_{k \geq 0} a_k(s\Lambda)\, f_{\Lambda[k]} \tag{14}$$

(the coefficients in (13) and (14) are given by (7) and (8)).

The elements g_Λ form one more basis in $V_\lambda(\gamma)$ parametrized by G-patterns of type λ and weight γ. Up to parametrization (2), this is the G-basis corrresponding to the chain $gl_1 \subset gl_2 \subset gl_3$ which is obtained from the standard chain by conjugation by $s \in GL_3$. Define the transition matrix $(c(\Lambda, \Lambda'))$ by

$$g_\Lambda = \sum_{\Lambda'} c(\Lambda, \Lambda')\, e_{\Lambda'}\ ; \tag{15}$$

by (5) and (14) $c(\Lambda, \Lambda') = 0$ unless $\Lambda' = \Lambda[r]$ for some $r \in Z$, and

$$c(\Lambda, \Lambda[r]) = \sum_{k, \ell \geq 0,\, k-\ell = r} (-1)^{\ell} a_k(s\Lambda)\, b_\ell(\Lambda[k]) \tag{16}$$

5. Triangular factorization for 6j-symbols and Clebsch-Gordan coefficients. It is shown in [6], [7] that the transition matrix between two G-bases for gl_3 may be expressed in terms of Racah coefficients (or Wigner's 6j-symbols) corresponding to sl_2. So there is an interpretation of (16) in terms of 6j-symbols. Recall the definitions of the Clebsch-Gordan coefficients and 6j-symbols.

As is common in physical papers we parametrize irreducible (finite dimensional) representations of sl_2 by a half-integer $j \geq 0$; the representation V_j corresponding to j has dimension $2j+1$. Choose an invariant scalar product on V_j and an orthonormal basis of weight vectors; these vectors are denoted by $|jm\rangle$ ($m = -j, -j+1, \ldots, j-1, j$). Sometimes another basis (e^j_m) in V_j is more convenient, namely

$$e^j_m = \{(j+m)!\,(j-m)!\}^{\frac{1}{2}}\, |j\, m\rangle \tag{17}$$

Let E_+, E_- and H be standard generators of sl_2 satisfying the commutation relations $[H, E_\pm] = \pm 2E_\pm$, $[E_+, E_-] = H$; then their action on the e^j_m is given by

$$E_+ e^j_m = (j-m)\, e^j_{m+1}\ ,\quad E_- e^j_m = (j+m)\, e^j_{m-1}\ ,\quad H e^j_m = 2m\, e^j_m \tag{18}$$

It is known that the tensor product $V_{j_1} \otimes V_{j_2}$ is multiplicity free. Furthermore, V_j occurs in $V_{j_1} \otimes V_{j_2}$ if and only if j_1+j_2+j is an integer and each of the numbers j_1, j_2 and j does not exceed the sum of two others (these two conditions will be referred to as the

triangle condition). Thus there is the unique (up to scalar multiple) operator $T(j_1 j_2 j) : V_j \longrightarrow V_{j_1} \otimes V_{j_2}$ commuting with the action of sl_2. Its matrix elements, $C^{j_1 j_2 j}_{m_1 m_2 m}$, defined by

$$T(j_1 j_2 j)\ |jm\rangle = \sum_{m_1, m_2} C^{j_1 j_2 j}_{m_1 m_2 m}\ |j_1 m_1\rangle \otimes |\ j_2 m_2\rangle \qquad (19)$$

are called the Clebsch-Gordan coefficients; therefore, $C^{j_1 j_2 j}_{m_1 m_2 m} = 0$ unless j_1, j_2 and j satisfy the triangle condition and $m_1 + m_2 = m$. The normalization of $T(j_1 j_2 j)$ is uniquely determined by the condition that this is an isometry and the phase condition $C^{j_1 j_2 j}_{j_1, j-j_1, j} > 0$.

Now consider the tensor product of three irreducible representations $V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$. We may decompose it into irreducible components in two different ways. The first is to decompose $V_{j_1} \otimes V_{j_2}$ into $\oplus V_{j_{12}}$ and then to decompose each $V_{j_{12}} \otimes V_{j_3}$; alternatively, we may decompose $V_{j_2} \otimes V_{j_3}$ into $\oplus V_{j_{23}}$ and then decompose each $V_{j_1} \otimes V_{j_{23}}$. Hence for any V_j there are two different bases in the space of operators $V_j \longrightarrow V_{j_1} \otimes \otimes V_{j_2} \otimes V_{j_3}$ commuting with the action of sl_2. The first is parametrized by j_{12}'s such that each of the triples (j_1, j_2, j_{12}) and (j_{12}, j_3, j) satisfies the triangle condition; the corresponding operator is the composition of $T(j_{12} j_3 j) : V_j \longrightarrow V_{j_{12}} \otimes V_{j_3}$ and $T(j_1 j_2 j_{12}) \otimes \mathbb{1} : V_{j_{12}} \otimes V_{j_3} \longrightarrow V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$. Denote this operator by $L(j_1 j_2 j_3 j j_{12})$. Similarly, for each j_{23} such that (j_2, j_3, j_{23}) and (j_1, j_{23}, j) satisfy the triangle condition define $R(j_1 j_2 j_3 j j_{23})$ to be the composition of $T(j_1 j_{23} j) : V_j \longrightarrow V_{j_1} \otimes V_{j_{23}}$ and $\mathbb{1} \otimes T(j_2 j_3 j_{23}) : V_{j_1} \otimes V_{j_{23}} \longrightarrow V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$; then the operators $R(j_1 j_2 j_3 j j_{23})$ form another basis in the same vector space. Let $\left(\left\langle \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\rangle\right)$ denote the transition matrix between these bases, i.e.

$$R(j_1 j_2 j_3 j j_{23}) = \sum_{j_{12}} \left\langle \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\rangle L(j_1 j_2 j_3 j j_{12}) \qquad (20)$$

This matrix is expressed in terms of 6j-symbols as follows:

$$\left\langle \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\rangle = (-1)^{j_1+j_2+j_3+j} \{(2j_{12}+1)(2j_{23}+ + 1)\}^{\frac{1}{2}} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\} \qquad (21)$$

(see e.g. [6], (1.18b)).

Now we return to gl_3. For any G-pattern $\Lambda = \begin{matrix} \alpha_1 & & \alpha_2 \\ & \beta & \end{matrix}$ we let $j(\Lambda) = (\alpha_1 - \alpha_2)/2$; if sl_2 is embedded in gl_3 by $E_+ = E_{12}$, $E_- = E_{21}$, $H = E_{11} - E_{22}$ then e_Λ generates under the action of this subalgebra an irreducible representation isomorphic to $V_{j(\Lambda)}$.

Theorem 7. Let Λ and Λ' be G-patterns of type λ and weight γ. Then the coefficient $c(\Lambda, \Lambda')$ from (15) is

$$c(\Lambda, \Lambda') = \frac{d(s\Lambda)}{d(\Lambda')} \left\langle \begin{matrix} \frac{\lambda_1-\gamma_2}{2} & \frac{\lambda_1-\gamma_1}{2} & j(\Lambda') \\ \frac{\lambda_2-\lambda_3}{2} & \frac{\lambda_1-\gamma_3}{2} & j(s\Lambda) \end{matrix} \right\rangle \qquad (22)$$

where $d(\Lambda)$ is defined by (2) and $s\Lambda$ is defined in §4.

This Theorem is a reformulation of (2.4b) from [6] (see also [7]).

Using (21) and (22) we see that (16) gives the "triangular factorization" for 6j-symbols.

It is well-known that Clebsch-Gordan coefficients may be considered as limits of Racah coefficients (see e.g. [1], (3.300)). In our notation this result has the form

$$C^{j_1 j_2 j}_{m_1, m-m_1, m} = \lim_{j_3 \to \infty} \left\langle \begin{matrix} j_3 & j_1 & j_3+m_1 \\ j_2 & j_3+m & j \end{matrix} \right\rangle \tag{23}$$

By (23), Theorem 7 implies the triangular factorization of the Clebsch-Gordan coefficients. For this we assign to each j_1, j_2, m_1, m_2 as above and any integer $t \geqslant 0$ the G-pattern

$$\Lambda_t \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} = \begin{matrix} 2j_1 + 2j_2 + t, & 2j_2, & 0 \\ m_1 + j_1 + 2j_2 + t, & m_2+j_2 & \\ & 2j_2 + t & \end{matrix} \tag{24}$$

Theorem 8. Fix j_1, j_2, m_1, m_2, and j. For any integer $t \geqslant 0$ put $\Lambda'_t = \Lambda_t \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}$, $r = j(s\Lambda'_t) - j$ and $\Lambda_t = \Lambda'_t\,[r]$. Then

$$C^{j_1 j_2 j}_{m_1, m_2, m_1+m_2} = \lim_{t \to \infty} \frac{c(\Lambda_t, \Lambda'_t)\, d(\Lambda'_t)}{d(s\,\Lambda_t)} \tag{25}$$

Proof follows at once from (22)-(24).

Theorem 8 has an interpretation and proof not using 6j-symbols. Fix j_1 and j_2 and for each integer $t \geqslant 0$ consider the subspace $V(t) = \bigoplus_{\gamma} V_\lambda(\gamma) \subset V_\lambda$, where

$\lambda = (2j_1 + 2j_2 + t, 2j_2, 0)$ and γ runs all the weights such that $\gamma_1 = 2j_2 + t$. If $\Lambda = \Lambda_t \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}$ we write $e_{m_1 m_2}(t)$, $f_{m_1 m_2}(t)$ and $g_{m_1 m_2}(t)$ instead of e_Λ, f_Λ, and g_Λ. Clearly, each of the families $(e_{m_1 m_2}(t))$, $(f_{m_1 m_2}(t))$ and $(g_{m_1 m_2}(t))$, where $m_i = -j_i, -j_i+1, \ldots, j_i$ $(i = 1,2)$, is a basis in $V(t)$. We consider sl_2 as the subalgebra in gl_3 generated by $E_+ = E_{23}$, $E_- = E_{32}$, $H = E_{22} - E_{33}$; clearly, $V(t)$ is invariant under this subalgebra.

Assign to each pair (m_1, m_2) as above the pair (j,m) as follows: $m = m_1+m_2$, and $j = j_1+m_2$ (resp. $j = j_2 - m_1$) if $m \geqslant j_2-j_1$ (resp. $m \leqslant j_2-j_1$). Geometrically, $2j$ is the length of the hook shown on Fig. 1, which contains the point (m_1, m_2). It is easy to see that this construction gives a one-to-one correspondence between the pairs (m_1, m_2) and pairs (j,m) such that j_1, j_2 and j satisfy the triangle condition and $m \in \{-j, -j+1, \ldots, j\}$. In all subsequent formulas we suppose that (m_1, m_2) and (j,m) are related by this correspondence. In terms of G-patterns this means the following: if $\Lambda = \Lambda_t \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}$ then $j = j(s\Lambda)$; furthermore, Λ is of class I (resp. II, 0) if $m > j_2-j_1$ (resp. $m < j_2-j_1$, $m = j_2-j_1$), see Fig. 1.

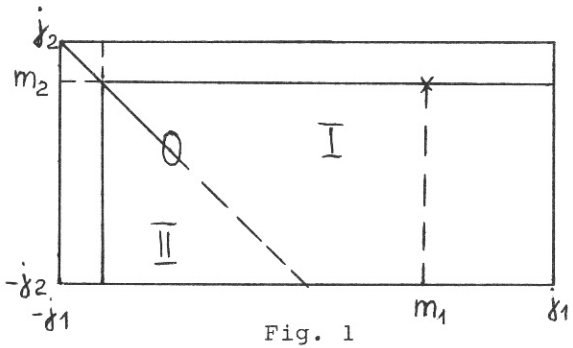

Fig. 1

The action of sl_2 on each of the bases $(e_{m_1m_2}(t))$, $(f_{m_1m_2}(t))$ and $(g_{m_1m_2}(t))$ is computed easily, using (3) and (9). The result for $g_{m_1m_2}(t)$ may be formulated as follows:

Proposition 7. For each integer $t \geqslant 0$ the linear mapping $V_j \longrightarrow V(t)$ sending e_m^j to $g_{m_1m_2}(t)$ (where (m_1,m_2) and (j,m) are related as above) commutes with the sl_2- action (see (17), (18)).

By Proposition 7, we may identify each $V(t)$ as the representation of sl_2 with $V_{j_1} \otimes V_{j_2}$ by means of the isomorphism sending $g_{m_1m_2}(t)$ to $T(j_1j_2j)\, e_m^j$ (see (19)). Using this identification we obtain the basis $(g_{m_1m_2})$ in $V_{j_1} \otimes V_{j_2}$ and two one-parametric families of bases $(e_{m_1m_2}(t))$ and $(f_{m_1m_2}(t))$.

Proposition 8.

$$g_{m_1m_2} = \sum_{k \geqslant 0} \binom{j_1-m_1}{k}\binom{j_2+m_2}{k} \Big/ \binom{2j}{k} f_{m_1+k,m_2-k}(t) \tag{26}$$

$$f_{m_1m_2}(t) = \sum_{k \geqslant 0} (-1)^k \frac{\binom{j_1+j_2-j}{k}\binom{j_1+j_2-j+t}{k}}{\binom{j_1+j_2+m_1-m_2+1-k+t}{k}} e_{m_1-k,m_2+k}(t) \tag{27}$$

$$e_{m_1m_2}(t) = \sum_{k \geqslant 0} \frac{\binom{j_1+j_2-j}{k}\binom{j_1+j_2-j+t}{k}}{\binom{j_1+j_2+m_1-m_2+t}{k}} f_{m_1-k,m_2+k}(t) \tag{28}$$

$$f_{m_1m_2}(t) = \sum_{k \geqslant 0} (-1)^k \binom{j_1-m_1}{k}\binom{j_2+m_2}{k} \Big/ \binom{2j+1-k}{k} g_{m_1+k,m_2-k} \tag{29}$$

This follows at once from (5)-(8), (13) and (14).

In particular, we see that the basis $(f_{m_1m_2}(t))$ is independent on t (so we may write simply $(f_{m_1m_2})$) and is connected with each of the bases $(g_{m_1m_2})$ and $(e_{m_1m_2}(t))$ by a triangular transformation. Now let $t \longrightarrow \infty$ in (27) and (28); clearly, the limit vectors $e_{m_1m_2} = \lim_{t \to \infty} e_{m_1m_2}(t)$ exist and form a basis in $V_{j_1} \otimes V_{j_2}$, and we have

$$f_{m_1m_2} = \sum_{k \geqslant 0} (-1)^k \binom{j_1+j_2-j}{k} e_{m_1-k,m_2+k} \tag{27'}$$

$$e_{m_1m_2} = \sum_{k \geqslant 0} \binom{j_1+j_2-j}{k} f_{m_1-k,m_2+k} \tag{28'}$$

Proposition 9. The vector $e_{m_1m_2}$ in $V_{j_1} \otimes V_{j_2}$ equals the tensor product $e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}$. So up to normalization factors of type (17) the transition matrix between the bases $(g_{m_1m_2})$ and $(e_{m_1m_2})$ is the Clebsch-Gordan matrix.

Proof. Letting $t \longrightarrow \infty$ in the formulas expressing the action of sl_2 on the $e_{m_1m_2}(t)$ we obtain that the limit basis $(e_{m_1m_2})$ is transformaed under the action of sl_2 in the same way as the basis $(e_{m_1}^{j_1} \otimes e_{m_2}^{j_2})$. It follows that these two bases differ only by normalization factors; the fact that all the factors are equal to 1 is proved directly.

Now we see that (26) and (27') give the triangular factorization of the Clebsch-Gordan matrix. This is our interpretation of Theorem 8.

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MULTIPLICITIES AND PROPER BASES FOR gl_n

I.Gelfand, A.Zelevinsky
Moscow State University

1. Formulations of results. The first main result of this paper is a new expression for multiplicities of irreducible components in the tensor product of two irreducible finite-dimensional representations of the Lie algebra gl_n. The answer is the number of integral points in a certain convex polytope in the space of Gelfand-Tsetlin patterns (G-patterns). The Kostant function and the weight multiplicity for gl_n may also acquire a similar interpretation. Thus for gl_n all these three fundamental functions of the representation theory may be treated in the same geometric way: there is a convex polytope in the space of large dimension with a linear projection on the domain of our function and the function value at each point x equals the number of integral points in the fiber of the projection over x (clearly this fiber is also a convex polytope). This approach gives a natural explanation of the piece-wise polynomial behaviour of these functions.

To formulate the result we need some notations. Recall that the weight lattice for gl_n consists of integral vectors $\gamma = (\gamma_1,\dots,\gamma_n)$. The weight $\lambda = (\lambda_1,\dots,\lambda_n)$ is called highest if $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$. Irreducible finite-dimensional representations of gl_n are parametrized by highest weights; denote the representation with highest weight λ by V_λ.

A G-pattern for gl_n is defined as a sequence $(\lambda^{(1)}, \lambda^{(2)},\dots, \lambda^{(n)})$, where $\lambda^{(i)}=(\lambda_1^{(i)}, \lambda_2^{(i)},\dots,$