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Formal analytic continuation of Gel'fand's finite dimensional representations of gl(n,C)[a]

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The article contains three results: I. It is shown that among the $2^{-n}\, n!(n+1)!$ discrete series of representations of the Lie algebra gl(n,C) of complex $n\times n$ matrices described in the literature, the majority are not representations at all. Thus for $n=3$ and 4 one has respectively 12 and 45 series of representations instead of 18 and 180. II. In addition to the $p+1$ discrete unitary series of representations of u(p,q) [the Lie algebra of the group U(p,q), $p\geq q$, and $p+q=n$] there exist other discrete series of gl(n,C) which become unitary when restricted to its real subalgebra u(p,q). For $n=3$ there are four such series all corresponding to the chain u(2,1) ⊃ u(1,1) ⊃ u(1); for $n=4$ there exist six such series for u(3,1) and four series for u(2,2). Furthermore, some of the gl(n,C) series whose restriction to the real case do not provide unitary representations in general, do contain (infinitely many) particular representations which are unitary. Such unitary representations are contained inside of two of the four series for $n=3$ and inside of seven of the 27 series for $n=4$. III. Some properties of indecomposable representations of the Lie algebras for the groups of inhomogeneous transformations are shown using the discrete series of gl(n,C).

I. INTRODUCTION

The explicit representation theory as given by Gel'fand and Tseitlin for the groups U(n) and O(n) and later extended by Gel'fand and Graev to some representation of Gl(n,C) is undoubtedly the most suitable form of the theory for extensive computations. It is therefore important to investigate the limits of its validity. In the present paper we are concerned with this question.

In a paper by Gel'fand and Tseitlin[1] every finite dimensional irreducible representation of gl(n,C) is described by labeling the basis vectors and giving explicit formulas for the representatives of a generating set of gl(n,C). In a supplement to a later paper[2] on this subject Gel'fand and Graev present a systematic study of formal analytic continuations of both the labeling and the generating operators of these representations. The goal of this paper is threefold. First we shall show that, contrary to the claim of Gel'fand and Graev, a sizeable proportion of these analytic continuations are not representations of gl(n,C). In fact we give a necessary condition to these operators to satisfy the commutation relations of gl(n,C). Secondly, we consider the restrictions of these representations to the real forms u(p,q) of gl(n,C), determining in particular all these restrictions which are discrete unitary irreducible representations of u(p,q). Finally, we shall indicate by an example some further applications of these formal Gel'fand representations to determine interesting classes of explicitly defined representations of certain subalgebras of gl(n,C). In order to make this paper as self-contained as possible, we begin with a brief outline of the material presented in the supplement to Ref. 2.

As is well known, every finite-dimensional irreducible representation of the Lie algebra gl(n,C) of the group GL(n,C) of all nonsingular $n\times n$ complex matrices is specified by a set of n integers $m_{1n}\geqslant\cdots\geqslant m_{nn}$. The representation space H has an orthonormal basis labeled by all possible triangular arrays (patterns) of integers,

$$m=\begin{pmatrix} m_{1n} & & m_{2n} & & \cdots & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\ & & & \cdots & & & \\ & & & m_{11} & & & \end{pmatrix}, \tag{1}$$

where the components m_{ij} satisfy the inequalities

$$m_{ij}\geqslant m_{i,j-1}\geqslant m_{i+1,j}\quad \text{for } i<j\leqslant n.$$

The algebra gl(n,C) consists of all complex $n\times n$ matrices and has the standard basis $\{e_{ij}|i,j=1,2,\cdots,n\}$, where e_{ij} denotes the $n\times n$ complex matrix with 1 at the intersection of the ith row and jth column and zeroes elsewhere. The commutation product of this algebra is then given by

$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{il}e_{kj}, \tag{2}$$

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where δ is the Kronecker delta function. Thus in order to specify a representation of $\mathrm{gl}(n,C)$ on H, it suffices to define linear operators E_{ij} (representative of e_{ij}) on H satisfying the commutation relations (2). In fact, it actually suffices to define only the representatives E_{kk}, $E_{k,k-1}$ and $E_{k-1,k}$, since the other operators can be obtained from these.

Gel'fand and Tseitlin give explicit formulas for these generating operators as follows: For any basis vector $\xi(m)$, where m is the label given in (1), we have

$$E_{kk}\xi(m) = (r_k - r_{k-1})\xi(m), \tag{3}$$

where $r_k = m_{1,k} + \cdots + m_{kk}$ for $k = 1,2,\cdots,n$ and $r_0 = 0$,

$$E_{k,k-1}\xi(m) = a^1_{k-1}\xi(m^1_{k-1}) + \cdots + a^{k-1}_{k-1}\xi(m^{k-1}_{k-1}), \tag{4}$$

where m^j_{k-1} denotes the array obtained from m replacing $m_{j,k-1}$ by $m_{j,k-1} - 1$,

$$a^j_{k-1} = \left[- \frac{\Pi^k_{i=1}(m_{ik} - m_{j,k-1} - i + j + 1)\Pi^{k-2}_{i=1}(m_{k,k-2} - m_{j,k-1} - i + j)}{\Pi_{i\neq j}(m_{i,k-1} - m_{j,k-1} - i + j + 1)(m_{i,k-1} - m_{j,k-1} - i + j)} \right]^{1/2},$$

$$E_{k-1,k}\xi(m) = b^1_{k-1}\xi(m^1_{k-1}) + \cdots + b^{k-1}_{k-1}\xi(m^{k-1}_{k-1}), \tag{5}$$

where m^j_{k-1} denotes the array obtained from m replacing $m_{j,k-1}$ by $m_{j,k-1} + 1$, and

$$b^j_{k-1} = \left[- \frac{\Pi^k_{i=1}(m_{ik} - m_{j,k-1} - i + j)\Pi^{k-2}_{i=1}(m_{i,k-2} - m_{j,k-1} - i + j - 1)}{\Pi_{i\neq j}(m_{i,k-1} - m_{j,k-1} - i + j)(m_{i,k-1} - m_{j,k-1} - i + j - 1)} \right]^{1/2}.$$

A detailed derivation of these formulas as well as an effective description of the operators of irreducible finite dimensional representations of the groups $\mathrm{GL}(n,C)$ and $\mathrm{U}(n)$ is given in the paper of Gel'fand and Graev.[2] (Another derivation of these results is given by Baird and Biedenharn.[3])

In a supplement to the Gel'fand and Graev paper the labeling patterns given in (1) are slightly altered and the generating operators E_{kk}, $E_{k,k-1}$, and $E_{k-1,k}$ defined to operate on the new basis vectors in the following way: To each $k = 1,2,\cdots,n-1$ we assign a pair of integers $\{i_k,i'_k\}$ where

$$i_k\in\{0,1,\cdots,k\},\; i'_k\in\{1,2,\cdots,k+1\}, \text{and } i_k < i'_k. \tag{6}$$

For each such set of indices one defines a Hilbert space $H\{i_k,i'_k\}$ having an orthonormal basis labeled by the set of all possible triangular arrays of integers where the top row is fixed and the other components satisfy the following set of inequalities,

$$\begin{array}{lll}
(1) & m_{jk} \geqslant m_{j+1,k}, & \text{for } j < k \leqslant n,\\
(2) & m_{j-1,k+1} + 1 \geqslant m_{jk} \geqslant m_{j,k+1} + 1, & \text{for } j \leqslant i_k,\\
(3) & m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}, & \text{for } i_k < j < i'_k,\\
(4) & m_{j+1,k+1} - 1 \geqslant m_{jk} \geqslant m_{j+2,k+1} - 1, & \text{for } j \geqslant i'_k
\end{array} \tag{7}$$

(by convention we set $m_{0,k+1} = +\infty$ and $m_{k+2,k+1} = -\infty$). The original finite dimensional space corresponds to the case where $i_k = 0$ and $i'_k = k+1$ for $k = 1,2,\ldots,n-1$. All other spaces defined above are infinite dimensional. The operators E_{kk}, $E_{k,k-1}$, and $E_{k-1,k}$ are then defined on $H\{i_k,i'_k\}$ by the same formulas (3)–(5), as in the case of the finite dimensional representation, on noting that the argument of the coefficients a^j_{k-1} and b^j_{k-1} are taken to be $(\pi/2)N$, where N is the common number of negative factors under the radial signs in the expressions for a^j_{k-1} and b^j_{k-1}. It is clear that these operators map any basis vector of $H\{i_k,i'_k\}$ into a finite linear combination of basis elements of the same space $H\{i_k,i'_k\}$. In a formal sense these operators on $H\{i_k,i'_k\}$ represent analytic continuations of the finite dimensional operators. It is claimed that in this manner, for each set of indices $\{i_k,i'_k\}$, one obtains a series of irreducible representations of the algebra $\mathrm{gl}(n,C)$ on the Hilbert space $H\{i_k,i'_k\}$. In Sec. II we show that this claim is false for a sizeable proportion of these sets of indices.

Leaving this problem for the moment, we now outline a second set of results in the supplement to the Gel'fand and Graev paper which we wish to expand upon. For any fixed nonnegative integers p and q with $p \geqslant q$ and $p + q = n$, we denote by $\mathrm{U}(p,q)$ the group of all $n\times n$ complex matrices which preserve the Hermitian form

$$|x_1|^2 + |x_2|^2 + \cdots + |x_p|^2 - |x_{p+1}|^2 - \cdots - |x_n|^2.$$

Since $\mathrm{U}(p,q)$ is one of the real forms of the group $\mathrm{GL}(n,C)$, its Lie algebra $\mathrm{u}(p,q)$ is a real form of $\mathrm{gl}(n,C)$. A representation of the algebra $\mathrm{u}(p,q)$ is then said to be *unitary* iff the generators of the representation are all skew-Hermitian. Gel'fand and Graev show that among the representations of $\mathrm{gl}(n,C)$ defined above, there exist $p+1$ series of irreducible unitary representations of $\mathrm{u}(p,q)$. This result follows by selecting a basis of $\mathrm{u}(p,q)$ consisting of

$$\begin{array}{ll}
ie_{kk}, & \text{for } k = 1,2,\cdots,n,\\
e_{kl} - e_{lk},\; i(e_{kl} + e_{lk}), & \text{for } k,l \leqslant p \text{ or } k,l > p,\\
e_{kl} + e_{lk},\; i(e_{kl} - e_{lk}) & \text{for } k > p \text{ and } l \leqslant p,
\end{array} \tag{8}$$

and observing that the representatives of these basis elements on the space $H\{i_k,i'_k\}$ are skew-Hermitian iff we have

(1) $E^+_{kk} = E_{kk}$, for $k = 1,2,\cdots,n$,

and

$$(2)\ E^{+}_{k,k-1} = \begin{cases} E_{k-1,k}, & \text{for } k \neq p, \\ -E_{p-1,p}, & \text{for } k = p. \end{cases} \tag{9}$$

These conditions are equivalent to requiring that the coefficients a^j_k are real for all $k \neq p$ and all $j = 1,2,...,k$, and that a^j_p are purely imaginary for $j = 1,2,...,p$. By inspection the only sets of indices $\{i_k, i'_k\}$ satisfying these conditions are the following:

$$\begin{aligned} &i_k = 0, \quad i'_k = k+1, \quad \text{for } k<p, \\ &i_p = l \in \{0,1,2,...,p\}, \ i'_p = l+1, \\ &i_k = l, \quad i'_k = l+k-p+1, \quad \text{for } k>p. \end{aligned} \tag{10}$$

In each of these $p+1$ cases one observes that the labeling formalism yields an explicit description of the branching rule for the canonical chain of subalgebras u(p,q) $\supset$u($p,q-1$)$\supset\cdots\supset$u(p,0)$\supset\cdots\supset$u(1). Note that the operators E_{ij} for $i,j \leqslant \mu$ leave the top $n-\mu+1$ rows of the labels invariant, and hence the subalgebra of u(p,q) consisting of linear combinations of operators from $\{E_{ij} | i,j \leqslant \mu\}$ can be viewed as operating only on the bottom $\mu - 1$ rows of the labeling arrays. It is in this sense that we associate a chain of subalgebras of u(p,q) with the given basis labels.

In Sec. III we show that the restriction of other representations of gl(n,C) to u(p,q) also provides discrete series of unitary irreducible representations of the Lie algebra u(p,q), which in some cases correspond to different chains of subalgebras.

II. A NECESSARY CONDITION FOR EXISTENCE OF A DISCRETE SERIES OF gl(*n,C*)

In this section we show that for certain sets of indices $\{i_k, i'_k\}$, allowed by condition (6), the generators E_{kk}, $E_{k,k-1}$, and $E_{k-1,k}$ defined on $H\{i_k,i'_k\}$ do *not* provide representations of the algebra gl(n,C). In fact, we claim that a necessary condition for these operators to satisfy the commutation relations (2) and hence provide a representation of gl(n,C) on $H\{i_k,i'_k\}$ is that for each $k = 2,3,...,n-1$ we have

$$i_{k-1}, i'_{k-1} \notin [1, i_k - 1] \cup [i_k + 1, i'_k - 2] \cup [i'_k, k-1]. \tag{11}$$

Here $[a,b]$ denotes the set of integers $\{a, a+1, \cdots, b\}$; by convention $[a,b] = \emptyset$ if $b < a$.

To illustrate this condition consider the case of $n = 3$. Gel'fand and Graev claim that for each fixed set of integers $m_{13} \geqslant m_{23} \geqslant m_{33}$ their construction yields 18 inequivalent irreducible representations of gl($3,C$); one representation for each of the 18 different sets of indices (see Table I). Of these 18 sets of indices, six do not satisfy condition (11). To be specific, consider the particular case $(i_2,i'_2) = (0,3)$ and $(i_1,i'_1) = (0,1)$. These values are clearly within the range allowed by (6), but forbidden by (11) because $i'_1 = 1 \in [i_2+1, i'_2-2] = [1,1]$. From Table I one reads off the inequalities imposed on the elements of an array belonging to the space $H\{(0,3),(0,1)\}$. Namely, for any fixed integers $m_{13} \geqslant m_{23} \geqslant m_{33}$ one has

$$m_{13} \geqslant m_{12} \geqslant m_{23}, \quad m_{23} \geqslant m_{22} \geqslant m_{33}, \quad m_{22} - 1 \geqslant m_{11}. \tag{12}$$

Consider the vector

$$m = \begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{23} & \\ & & m_{23} - 1 & & \end{pmatrix} \in H\{(0,3),(0,1)\},$$

TABLE I. Gel'fand representations of gl($3,C$) and their restrictions.

(i_2,i'_2) \ (i_1,i'_1)	(0,2) $m_{12} \geqslant m_{11} \geqslant m_{22}$	(0,1) $m_{22} - 1 \geqslant m_{11}$	(1,2) $m_{11} \geqslant m_{12} + 1$
(0,3)$\equiv$ $m_{13} \geqslant m_{12} \geqslant m_{23}$ $m_{23} \geqslant m_{22} \geqslant m_{33}$	Unitary su(3)$\supset$su(2)	Not a representation	Not a representation
(0,2)$\equiv$ $m_{13} \geqslant m_{12} \geqslant m_{23}$ $m_{33} - 1 \geqslant m_{22}$	Unitary if $m_{13} = m_{23}$ su(2,1)$\supset$su(2)	Unitary su(2,1)$\supset$su(1,1)	Unitary su(2,1)$\supset$su(1,1)
(0,1)$\equiv$ $m_{23} - 1 \geqslant m_{12} \geqslant m_{33} - 1$ $m_{33} - 1 \geqslant m_{22}$	Unitary su(2,1)$\supset$su(2)	Not a representation	Not a representation
(1,3)$\equiv$ $m_{12} \geqslant m_{13} + 1$ $m_{23} \geqslant m_{22} \geqslant m_{33}$	Unitary if $m_{23} = m_{33}$ su(2,1)$\supset$su(2)	Unitary su(2,1)$\supset$su(1,1)	Unitary su(2,1)$\supset$su(1,1)
(1,2)$\equiv$ $m_{12} \geqslant m_{13} + 1$ $m_{33} - 1 \geqslant m_{22}$	Unitary su(2,1)$\supset$su(2)	Never unitary	Never unitary
(2,3)$\equiv$ $m_{12} \geqslant m_{13} + 1$ $m_{13} + 1 \geqslant m_{22} \geqslant m_{23} + 1$	Unitary su(2,1)$\supset$su(2)	Not a representation	Not a representation

where the elements m_{22} and m_{11} have reached their highest values compatible with (12). Then

$$E_{32}m = a_2^1 m_2^1 = a_2^1\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12}-1 & & m_{23} & \\ & & m_{23-1} & & \end{pmatrix}, \tag{13}$$

which differs from zero as long as $m_{12} > m_{23}$. If, however, $m_{12} = m_{23}$, then necessarily $a_2^1 = 0$; otherwise the array m_2^1 on the right of (13) would not satisfy the inequalities (12). Substituting the values $m_{12} = m_{23}$, $m_{22} = m_{23}$, and $m_{11} = m_{23} - 1$ into the expression for a_2^1, one verifies that both the numerator and denominator of a_2^1 contain a factor equal to zero. In order to avoid a contradiction one is forced to define that $a_2^1 = 0$ whenever the numerator contains zero, regardless of the denominator. Indeed, that convention is tacitly adopted in Ref. 2. The point we want to make here is that even then a contradiction is not avoided. It can be shown as follows:

$$(E_{22} - E_{33})\xi(m) = (2m_{12} - m_{13} - m_{33} + 1)\xi(m), \tag{14}$$

and

$$\begin{aligned}[E_{23},E_{32}]\xi(m) &= [(a_2^1)^2 - (b_2^1)^2]\xi(m) \\ &= \Big((m_{13} - m_{12} + 1)(m_{12} - m_{33} + 1)\frac{(m_{12} - m_{23})(m_{12} - m_{23} + 1)}{(m_{12} - m_{23})(m_{12} - m_{23} + 1)} \\ &\quad - (m_{13} - m_{12})(m_{12} - m_{33} + 2)\frac{(m_{12} - m_{23} + 1)(m_{12} - m_{23} + 2)}{(m_{12} - m_{23} + 1)(m_{12} - m_{23} + 2)}\Big)\xi(m).\end{aligned} \tag{15}$$

Algebraic simplification of the coefficients verifies that $[E_{23},E_{32}]\xi(m) = (E_{22} - E_{33})\xi(m)$. Now, however, if we assume that $m_{12} = m_{23}$, we have

$$(E_{22} - E_{33})\xi(m) = (2m_{23} - m_{13} - m_{33} + 1)\xi(m), \tag{16}$$

and

$$\begin{aligned}[E_{23},E_{32}]\xi(m) &= -(b_2^1)^2\xi(m) \\ &= \Big(-(m_{13} - m_{23})(m_{23} - m_{33} + 2)\frac{(m_{23} - m_{23} + 1)(m_{23} - m_{23} + 2)}{(m_{23} - m_{23} + 1)(m_{23} - m_{23} + 2)}\Big)\xi(m).\end{aligned} \tag{17}$$

[Note that the term $(a_2^1)^2$ of Eq. (15) does not occur in Eq. (17).] Equating the coefficients in (16) and (17), we obtain $(m_{23} - m_{13} - 1)(m_{23} - m_{33} + 1) = 0$, which is impossible, since $m_{13} \geqslant m_{23} \geqslant m_{33}$. We may thus conclude that the operators E_{ij} defined as above on $H\{(0,1),(0,3)\}$ do not provide a representation of the algebra gl$(3,C)$. The problem arises here in the passage from Eq. (15) to Eq. (17) as $m_{12} \to m_{23}$. In particular the coefficient $(a_2^1)^2$ tends to $(m_{13} - m_{23} + 1)(m_{23} - m_{33} + 1) \neq 0$ as $m_{12} \to m_{23}$ whereas in the limiting case of $m_{12} = m_{23}$ this term must vanish due to the constraints on the arrays of integers belonging to $H\{(0,3),(0,1)\}$. This problem can be resolved in the context of this formalism only by insuring that whenever $m_{12} = m_{22}$ we also have $m_{11} = m_{12} = m_{22}$; this condition, translated in terms of the indices, implies that if $(i_2,i_2') = (0,3)$ we must have $(i_1,i_1') = (0,2)$.

In the general case, although the coefficients are much more complicated, we arrive at essentially the same problem. Whenever the set of indices $\{i_k,i_k'\}$ allows an array in which $m_{jk} = m_{j+1,k}$, we must also have that $m_{j-1,k-1} - 1; m_{j,k-1}$ or $m_{j+1,k-1} + 1 = m_{jk} = m_{j+1,k}$ in order to preserve the continuity of the coefficients a_k^i and b_k^j at the boundary values of the arrays belonging to $H\{i_k,i_k'\}$. This condition can easily be translated into condition (11) on the set of indices.

For $n = 3$ one can directly verify that if the set of indices $\{i_k,i_k'\}$ satisfies condition (11) the operators E_{ij} defined on $H\{i_k,i_k'\}$ do provide a representation of the algebra gl$(3,C)$. For n arbitrary, however, we have been able to give a proof of the sufficiency of condition (11) only in certain special cases.

III. DISCRETE SERIES OF UNITARY REPRESENTATIONS OF u(p,q)

As noted earlier, Gel'fand and Graev have shown that the restrictions of certain of these series of irreducible representations of gl(n,C) to its real form u(p,q) yield discrete unitary irreducible representations with the Gel'fand bases corresponding to the chain of subalgebras u(p,q) $\supset$ u$(p,q-1) \supset \cdots \supset$ u$(p,0)$. In this section we show that it is possible to obtain additional unitary irreducible representations of u(p,q) corresponding to this same chain of subalgebras as well as other chains of subalgebras.

In terms of the Gel'fand basis of the space $H\{(i_k,i_k')\}$, we have $E_{ij}^T = E_{ji}$, and moreover, $E_{jj}^+ = E_{jj}^T = E_{jj}$, where T

and + denote transposition and Hermitian conjugation, respectively. We now wish to investigate under what conditions we have $E^{+}_{\mu,\mu-1} = \pm E_{\mu-1,\mu}$. By formula (4) we have $E_{\mu,\mu-1}\xi(m) = a^{1}_{\mu-1}\xi(m^{1}_{\mu-1}) + \cdots + a^{\mu-1}_{\mu-1}\xi(m^{\mu-1}_{\mu-1})$, where the coefficient $a^{j}_{\mu-1}$ is either real or purely imaginary, depending on the set of indices $\{(i_k,i'_k)\}$. (Note this is independent of the particular array.) Thus, if $a^{j}_{\mu-1}$ is real for $j = 1,2,\ldots,\mu-1$ we have $E^{+}_{\mu,\mu-1} = E_{\mu-1,\mu}$ and if $a^{j}_{\mu-1}$ is purely imaginary for $j = 1,2,\ldots,\mu-1$ we have $E^{+}_{\mu,\mu-1} = -E_{\mu-1,\mu}$. By simply counting the number of negative factors in the expression for $a^{j}_{\mu-1}$ we can determine whether $a^{j}_{\mu-1}$ is real or purely imaginary. This determination is then displayed in the following scheme:

	$j\leqslant i_{\mu-1}$	$i_{\mu-1}<j<i'_{\mu-1}$	$i'_{\mu-1}\leqslant j$
$j\geqslant i_{\mu-2}$	real	imaginary	real
$i_{\mu-2}<j\leqslant i'_{\mu-2}$	imaginary	real	imaginary
$i'_{\mu-2}<j$	real	imaginary	real

Using this scheme we can conclude that

(1) If $(i_{\mu-2},i_{\mu-2}) = (l,l')$ and $(i_{\mu-1},i'_{\mu-1}) = (l,l'+1)$, then $E^{+}_{\mu,\mu-1} = E_{\mu-1,\mu}$.

(2) If (a) $(i_{\mu-2},i_{\mu-2}) = (0,l)$ and $(i_{\mu-1},i'_{\mu-1}) = (l,\mu)$, or (b) $(i_{\mu-2},i_{\mu-2}) = (l,\mu-1)$ and $(i_{\mu-1},i'_{\mu-1}) = (0,l+1)$, or (c) $(i_{\mu-2},i_{\mu-2}) = (0,\mu-1)$ and $(i_{\mu-1},i'_{\mu-1}) = (l,l+1)$, then $E^{+}_{\mu,\mu-1} = -E_{\mu-1,\mu}$.

In all other cases one finds that in general, the coefficients of $E_{\mu,\mu-1}\xi(m)$ contain both real and purely imaginary terms and hence $E^{+}_{\mu,\mu-1} \neq \pm E_{\mu-1,\mu}$. If, however, we place particular restrictions on the values of the defining constants $m_{1n},\cdots,m_{nn}$ of the representation it is still possible to have $E^{+}_{\mu,\mu-1} = \pm E_{\mu-1,\mu}$. Later we describe this situation for $n = 3$ and 4.

Consider now any sequence $\epsilon = \{\epsilon_1,\epsilon_2,\ldots,\epsilon_n\}$ where $\epsilon_1 = 1$ and $\epsilon_i = \pm 1$ for $i = 2,3,\ldots,n$. [This sequence can be understood to be the signature of the u(p,q) invariant form.] We shall say that a set of indices $\{(i_k,i'_k)\}$ such that $E^{+}_{\mu,\mu-1} = \epsilon_{\mu-1}\epsilon_\mu E_{\mu-1,\mu}$ for $\mu = 2,3,\ldots,n$ is *compatible* with the sequence $\{\epsilon_1,\ldots,\epsilon_n\}$. If $\epsilon_i = +1$ for all $i = 1,2,\ldots,n$ there is exactly one compatible set of indices, namely $i_k = 0$ and $i'_k = k+1$ for $k = 1,2,\ldots,n-1$. If $\nu < n$ then there are $\nu + 1$ distinct sets of indices compatible with the sequence $\{\epsilon_1,\ldots,\epsilon_n\}$ where $\epsilon_1 = \cdots = \epsilon_\nu = +1$ and $\epsilon_{\nu+1} = \cdots = \epsilon_n = -1$. These are precisely the sets of indices (9) considered by Gel'fand and Graev, where one takes $p = \nu$. Finally for each other sequence there are two compatible sets of indices. Note that the sets of indices compatible with any sequence satisfy condition (11).

Choose some $p < n$ and restrict attention now to those sequences which contain either p or $q = n - p$ terms equal to $+1$. To any such sequence we associate a set of generators of a representation of the algebra u(p,q) as follow:

$$\{iE_{\mu\mu}\,|\,\mu = 1,2,\ldots,n\}$$
$$\cup\{i(E_{\mu,\mu-1} - E_{\mu-1,\mu}),E_{\mu,\mu-1} + E_{\mu-1,\mu}$$
$$|\ \text{for all } \mu \text{ with } \epsilon_{\mu-1}\epsilon_\mu = -1\},$$
$$\cup\{i(E_{\mu,\mu-1} + E_{\mu-1,\mu}),E_{\mu,\mu-1} - E_{\mu-1,\mu}$$
$$|\ \text{for all } \mu \text{ with } \epsilon_{\mu-1}\epsilon_\mu = +1\}.$$

For any set of indices $\{i_k,i'_k\}$ compatible with the given sequence each of these generators is a skew-Hermitian operator on the space $H\{(i_k,i'_k)\}$. Therefore, assuming the sufficiency of condition (11), these operators provide a discrete irreducible unitary representation of the algebra u(p,q) on the space $H\{(i_k,i'_k)\}$. If we set p_j (resp. q_j) equal to the number of $+$'s (resp. -1's) in the truncated sequence $\{\epsilon_1,\cdots,\epsilon_j\}$, then the Gel'fand basis of this representation space corresponds to the chain of subalgebras u$(p,q)\supset$u$(p_{n-1},q_{n-1})\supset\cdots\supset$u(1).

ble representations of u(p,q) the unitary condition depends solely on the set of indices $\{(i_k,i'_k)\}$ and is valid for all possible choices of the defining integral parameters $m_{1n}\geqslant\cdots\geqslant m_{nn}$, i.e., for all irreducible representations of the series. We now observe that some other sets of indices also yield unitary representations, but only for certain values of the parameters $m_{1n}\geqslant\cdots\geqslant m_{nn}$. For the cases of $n = 3$ and $n = 4$ we have tabulated these additional series of unitary representations by specifying for each such set of indices the restrictions on the values of $m_{1n}\geqslant\cdots\geqslant m_{nn}$ and the sequence to which it is compatible.

To illustrate the results of this section let us again consider the case of $n = 3$. In this case there are four possible sequences (1,1,1), (1,1, $-$1), (1, $-$1,1) and (1, $-$1, $-$1), each of which will be treated separately.

The sequence (1,1,1) gives rise to the real subalgebra u(3) of gl$(3,C)$ and the only compatible set of indices is $\{(0,2),(0,3)\}$. Not surprisingly, this set of indices provides us with the unique finite dimensional unitary irreducible representation of u(3).

The sequence (1,1, $-$1) gives rise to the real subalgebra u(2,1) of gl$(3,C)$ and there are three compatible sets of indices, namely $\{(0,2),(0,1)\}$, $\{(0,2),(1,2)\}$, and $\{(0,2),(2,3)\}$. The associated representations of these sets of indices are precisely the three discrete unitary irreducible representations of u(2,1) described by Gel'fand and Graev. In the decomposition of each of these representations with respect to the subalgebra u(2,0) generated by $\{iE_{11},iE_{22},i(E_{12}+E_{21}),E_{12}-E_{21}\}$ we observe that the second row components (m_{12},m_{22}) label the infinite number of finite dimensional irreducible unitary u(2) subrepresentations. Thus the Gel'fand basis formalism corresponds to the chain of subalgebras u(2,1)$\supset$u(2).

The sequence (1, $-$1,1) also gives rise to the real subalgebra u(2,1) of gl$(3,C)$, and there are two compatible sets of indices $\{(0,1),(1,3)\}$ and $\{(1,2),(0,2)\}$. The Gel'fand bases for the two associated representations corresponds in this case to the chain of subalgebras u(2,1)$\supset$u(1,1). In fact, if we decompose these representations into their irreducible components with respect to the subalgebra u(1,1) generated by $\{iE_{11}, iE_{22}, i(E_{12}-E_{21}), E_{12}+E_{21}\}$, we find that the components (m_{12}, m_{22}) of the second row label the infinite dimensional unitary irreducible u(1,1) subrepresentations.

Finally the sequence $(1,-1,-1)$ again gives rise to the real subalgebra u(2,1) and there are two compatible sets of indices $\{(0,1),(0,2)\}$ and $\{(1,1),(1,3)\}$. As in the previous case the Gel'fand labeling corresponds to the chain of subalgebra $\mathrm{u}(2,1)\supset\mathrm{u}(1,1)$.

A summary of these results for the case $n=3$ is displayed in Table I. a similar analysis of the case $n=4$ has also been carried out with the results given in Tables II(a), II(b), and II(c).

IV. EXAMPLES

In this section we illustrate the results of Secs. II and III by considering the following specific examples where for simplicity we assume that $m_{13}=m_{23}=m_{33}=0$.

A. Space in which operators do not provide a representation of gl(3,*C*)

The space $H\{(0,1),(0,3)\}$ has a basis consisting of

$$\left\{\xi\begin{pmatrix}0 & & 0\\ & z & \end{pmatrix}\middle|\, z\leqslant -1\right\}.$$

Using the formulas (3)–(5) we have

$$(E_{22}-E_{33})\xi\begin{pmatrix}0 & & 0\\ & z & \end{pmatrix} = -z\xi\begin{pmatrix}0 & & 0\\ & z & \end{pmatrix},$$

whereas

$$[E_{23},E_{32}]\xi\begin{pmatrix}0 & & 0\\ & z & \end{pmatrix}=0.$$

Thus on this space $[E_{23},E_{32}]\neq E_{22}-E_{33}$, i.e., these operators do *not* yield a representation of gl(3,*C*).

B. Representation of gl(3,*C*) whose restriction to the real forms u(3) or u(2,1) are not unitary

The space $H\{(1,2),(1,2)\}$ has a basis consiting of

TABLE IIA. Gel'fand representations of gl(4,*C*) and their restrictions. Assume $(i_1,i_1')=(0,1)$, i.e., $m_{22}-1\geqslant m_{11}$.

(i_2,i_2') / (i_3,i_3')	(0,3) $m_{13}\geqslant m_{12}\geqslant m_{23}$ $m_{23}\geqslant m_{22}\geqslant m_{33}$	(0,2) $m_{13}\geqslant m_{12}\geqslant m_{23}$ $m_{11}-1\geqslant m_{22}$	(0,1) $m_{23}-1\geqslant m_{12}\geqslant m_{33}-1$ $m_{11}-1\geqslant m_{22}$	(1,3) $m_{12}\geqslant m_{13}+1$ $m_{23}\geqslant m_{22}\geqslant m_{33}$	(1,2) $m_{12}\geqslant m_{13}+1$ $m_{11}-1\geqslant m_{22}$	(2,3) $m_{12}\geqslant m_{13}+1$ $m_{13}+1\geqslant m_{22}\geqslant m_{23}+1$
(0,4) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(0,3) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{44}-1\geqslant m_{33}$	Not a representation	Unitary on $\mathrm{su}(3,1)\supset\mathrm{su}(2,1)$ $\supset\mathrm{su}(1,1)$	Not a representation	Not a representation	Not a representation	Not a representation
(0,2) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{34}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Unitary on $\mathrm{su}(2,2)\supset\mathrm{su}(2,1)$ $\supset\mathrm{su}(1,1)$	Not a representation	Not a representation
(0,1) $m_{24}-1\geqslant m_{13}\geqslant m_{34}-1$ $m_{34}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(1,4) $m_{13}\geqslant m_{14}+1$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	Not a representation	Not a representation	Not a representation	Unitary on $\mathrm{su}(3,1)\supset\mathrm{su}(2,1)$ su(1,1)	Not a representation	Not a representation
(1,3) $m_{13}\geqslant m_{14}+1$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{44}-1\geqslant m_{33}$	Not a representation	Never Unitary	Not a representation	Never unitary	Never unitary	Not a representation
(1,2) $m_{13}\geqslant m_{14}+1$ $m_{34}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Never unitary	Not a representation	Not a representation
(2,4) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	Not a representation	Unitary on $\mathrm{su}(2,2)\supset\mathrm{su}(2,1)$ $\supset\mathrm{su}(1,1)$	Not a representation	Not a representation	Not a representation	Not a representation
(2,3) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{44}-1\geqslant m_{33}$	Not a representation	Never unitary	Not a representation	Not a representation	Not a representation	Not a representation
(3,4) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{24}+1\geqslant m_{33}\geqslant m_{34}+1$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation

TABLE IIB. Gel'fand representations of gl(4,C) and their restrictions. Assume $(i_1,i_1')=(0,2)$, i.e., $m_{12}\geqslant m_{11}\geqslant m_{22}$.

(i_2,i_2') / (i_3,i_3')	(0,3) $m_{13}\geqslant m_{12}\geqslant m_{23}$ $m_{23}\geqslant m_{22}\geqslant m_{33}$	(0,2) $m_{13}\geqslant m_{12}\geqslant m_{23}$ $m_{11}-1\geqslant m_{22}$	(0,1) $m_{23}-1\geqslant m_{12}\geqslant m_{33}-1$ $m_{11}-1\geqslant m_{22}$	(1,3) $m_{12}\geqslant m_{13}+1$ $m_{23}\geqslant m_{22}\geqslant m_{33}$	(1,2) $m_{12}\geqslant m_{13}+1$ $m_{11}-1\geqslant m_{22}$	(2,3) $m_{12}\geqslant m_{13}+1$ $m_{13}+1\geqslant m_{22}\geqslant m_{23}+1$
(0,4) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	Unitary on su(4)$\supset$su(3) $\supset$su(2)	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(0,3) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{44}-1\geqslant m_{33}$	If $m_{14}=m_{24}=m_{34}$ unitary on su(3,1)$\supset$su(3) $\supset$su(2)	If $m_{14}=m_{24}=m_{34}$ unitary on su(2,2)$\supset$su(2,1) $\supset$su(2)	Not a representation	Not a representation	Not a representation	Unitary on su(3,1)$\supset$su(2,1) $\supset$su(2)
(0,2) $m_{14}\geqslant m_{13}\geqslant m_{24}$ $m_{14}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	If $m_{14}=m_{24}$ unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Not a representation	Unitary on su(2,2)$\supset$su(2,1) $\supset$su(2)	Never unitary	Not a representation	Not a representation
(0,1) $m_{24}-1\geqslant m_{13}\geqslant m_{14}-1$ $m_{14}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	Unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(1,4) $m_{13}\geqslant m_{14}+1$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	If $m_{24}=m_{34}=m_{44}$ unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Not a representation	unitary on su(3,1)$\supset$su(2,1) $\supset$su(2)	If $m_{24}=m_{34}=m_{44}$ unitary on su(2,2)$\supset$su(2,1) $\supset$su(2)	Not a representation	Not a representation
(1,3) $m_{13}\geqslant m_{14}+1$ $m_{24}\geqslant m_{23}\geqslant m_{34}$ $m_{44}-1\geqslant m_{33}$	If $m_{24}=m_{34}$ unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Never unitary	Never unitary	Never unitary	Unitary on su(2,2)$\supset$su(2,1) $\supset$su(2)	Never unitary
(1,2) $m_{13}\geqslant m_{14}+1$ $m_{34}-1\geqslant m_{23}\geqslant m_{44}-1$ $m_{44}-1\geqslant m_{33}$	Unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Not a representation	Never unitary	Never unitary	Not a representation	Not a representation
(2,4) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{34}\geqslant m_{33}\geqslant m_{44}$	If $m_{34}=m_{44}$ unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Never unitary	Not a representation	Not a representation	Not a representation	Unitary on su(2,2)$\supset$su(2,1) $\supset$su(2)
(2,3) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{44}-1\geqslant m_{33}$	Unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Never unitary	Not a representation	Not a representation	Not a representation	Never unitary
(3,4) $m_{13}\geqslant m_{14}+1$ $m_{14}+1\geqslant m_{23}\geqslant m_{24}+1$ $m_{24}+1\geqslant m_{33}\geqslant m_{34}+1$	Unitary on su(3,1)$\supset$su(3) $\supset$su(2)	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation

$$\left\{\xi\begin{pmatrix} x & & y \\ & z & \end{pmatrix}\middle|\quad x\geqslant 1; y\leqslant -1 \text{ and } z\geqslant x+1\right\}.$$

By direct calculation one can verify that the operators defined by formulas (3)–(5) provide an irreducible representation of gl(3,C) on this space. However we also observe that

$$\left\langle\xi\begin{pmatrix} x+1 & & y \\ & z & \end{pmatrix}\middle|E_{23}^{+}\middle|\xi\begin{pmatrix} x & & y \\ & z & \end{pmatrix}\right\rangle$$
$$=-\left(\frac{x(x+1)(x+2)(z-x-1)}{(x-y+1)(x-y+2)}\right)^{1/2}$$
$$=\left\langle\xi\begin{pmatrix} x+1 & & y \\ & z & \end{pmatrix}\middle|E_{32}\middle|\xi\begin{pmatrix} x & & y \\ & z & \end{pmatrix}\right\rangle$$

and

$$\left\langle\xi\begin{pmatrix} x & & y+1 \\ & z & \end{pmatrix}\middle|E_{23}^{+}\middle|\xi\begin{pmatrix} x & & y \\ & z & \end{pmatrix}\right\rangle$$
$$=-i\left(\frac{(1-y)(-y)(-y-1)(z-y)}{(x-y+1)(x-y)}\right)^{1/2}$$
$$=-\left\langle\xi\begin{pmatrix} x & & y+1 \\ & z & \end{pmatrix}\middle|E_{32}\middle|\xi\begin{pmatrix} x & & y \\ & z & \end{pmatrix}\right\rangle.$$

(all factors under the radical signs have been made nonnegative). Thus $E_{32}^{+}\neq\pm E_{32}$, and consequently the restriction of this representation to either u(3) or u(2,1) does not consist solely of skew-Hermitian matrices, i.e., these restrictions are not unitary.

C. Representation of gl(3,*C*) whose restriction to u(2,1) is unitary but not equivalent to any of those given by Gel'fand and Graev

The space $H\{(0,1),(1,3)\}$ has a basis consisting of

$$\left\{\xi\begin{pmatrix} x & & 0 \\ & z & \end{pmatrix}\middle|\quad x\geqslant 1 \text{ and } z\leqslant -1\right\}.$$

By direct calculation one can see that the linear operators defined by formulas (3)–(5) provide an irreducible representation of gl(3,C) and moreover that the restriction of this representation to u(2,1) is unitary. This representation is however not equivalent to any of the unitary representations of u(2,1) listed by Gel'fand and Graev. This follows immedi-

TABLE IIC. Gel'fand representations of gl(4,C) and their restrictions. Assume $(i_1,i_1') = (1,2)$, i.e., $m_{11} \geqslant m_{12} + 1$.

(i_1,i_1') \ (i_2,i_2')	(0,3) $m_{13} \geqslant m_{12} \geqslant m_{23}$, $m_{23} \geqslant m_{22} \geqslant m_{33}$	(0,2) $m_{13} \geqslant m_{12} \geqslant m_{23}$, $m_{33} - 1 \geqslant m_{22}$	(0,1) $m_{23} - 1 \geqslant m_{12} \geqslant m_{33} - 1$, $m_{33} - 1 \geqslant m_{22}$	(1,3) $m_{12} \geqslant m_{13} + 1$, $m_{23} \geqslant m_{22} \geqslant m_{33}$	(1,2) $m_{12} \geqslant m_{13} + 1$, $m_{33} - 1 \geqslant m_{22}$	(2,3) $m_{12} \geqslant m_{13} + 1$, $m_{13} + 1 \geqslant m_{22} \geqslant m_{23} + 1$
(0,4) $m_{14} \geqslant m_{13} \geqslant m_{24}$, $m_{24} \geqslant m_{23} \geqslant m_{34}$, $m_{34} \geqslant m_{33} \geqslant m_{44}$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(0,3) $m_{14} \geqslant m_{13} \geqslant m_{24}$, $m_{24} \geqslant m_{23} \geqslant m_{34}$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Unitary on su(3,1)⊃su(2,1) ⊃su(1,1)	Not a representation	Not a representation	Not a representation	Not a representation
(0,2) $m_{14} \geqslant m_{13} \geqslant m_{24}$, $m_{34} - 1 \geqslant m_{23} \geqslant m_{44} - 1$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Unitary on su(2,2)⊃su(2,1) ⊃su(1,1)	Not a representation	Not a representation
(0,1) $m_{24} - 1 \geqslant m_{13} \geqslant m_{34} - 1$, $m_{34} - 1 \geqslant m_{23} \geqslant m_{44} - 1$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation
(1,4) $m_{13} \geqslant m_{14} + 1$, $m_{24} \geqslant m_{23} \geqslant m_{34}$, $m_{34} \geqslant m_{33} \geqslant m_{44}$	Not a representation	Not a representation	Not a representation	Unitary on su(3,1)⊃su(2,1) ⊃su(1,1)	Not a representation	Not a representation
(1,3) $m_{13} \geqslant m_{14} + 1$, $m_{24} \geqslant m_{23} \geqslant m_{34}$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Never unitary	Not a representation	Never unitary	Never unitary	Not a representation
(1,2) $m_{13} \geqslant m_{14} + 1$, $m_{34} - 1 \geqslant m_{23} \geqslant m_{44} - 1$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Not a representation	Not a representation	Never unitary	Not a representation	Not a representation
(2,4) $m_{13} \geqslant m_{14} + 1$, $m_{14} + 1 \geqslant m_{23} \geqslant m_{24} + 1$, $m_{34} \geqslant m_{33} \geqslant m_{44}$	Not a representation	Unitary on su(3,1)⊃su(2,1) ⊃su(1,1)	Not a representation	Not a representation	Not a representation	Not a representation
(2,3) $m_{13} \geqslant m_{14} + 1$, $m_{14} + 1 \geqslant m_{23} \geqslant m_{24} + 1$, $m_{44} - 1 \geqslant m_{33}$	Not a representation	Never unitary	Not a representation	Not a representation	Not a representation	Not a representation
(3,4) $m_{13} \geqslant m_{14} + 1$, $m_{14} + 1 \geqslant m_{23} \geqslant m_{24} + 1$, $m_{24} + 1 \geqslant m_{33} \geqslant m_{34} + 1$	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation	Not a representation

ately on comparing the weight space decompositions of these representations.

D. Unitary representation from a series which is in general nonunitary

In general, the representations of gl(3,C) associated with the set of indices $\{(0,2),(0,2)\}$ do not restrict to unitary representations of u(3) or u(2,1), however if we take $m_{13} = m_{23} = m_{33} = 0$, then the space $H\{(0,2),(0,2)\}$ has a basis consisting of

$$\left\{ \xi \begin{pmatrix} 0 & & y \\ & z & \end{pmatrix} \middle| \quad y \leqslant -1;\ z \leqslant y - 1 \right\}$$

and the matrix element a_2^1 and b_2^1 are identically zero. Therefore, by writing out the linear operators on this space one can directly verify that the generators of the representation are skew-Hermitian when restricted to u(2,1) and hence the representation is unitary. Moreover, the Gel'fand basis is associated with the chain of subalgebras u(2,1)⊃u(2) and by considering the weight space decomposition, this representation is not equivalent to any of those specified in Ref. 2.

V. REPRESENTATIONS OF NONSEMISIMPLE SUBALGEBRAS OF gl(n,C)

The explicitly defined Gel'fand representations of gl(n,C) offer possible avenues to study the representations of other subalgebras of gl(n,C) in a detailed manner. As an example consider the group G of inhomogeneous transformations consisting of all 3×3 complex matrices

$$g = \begin{bmatrix} \alpha & 0 \\ z & 1 \end{bmatrix},$$

where $\alpha \in \mathrm{GL}(2,C)$ and $z \in C^2$. The elements of G can be viewed as operating on the two dimensional affine space

$\{(x,y,1) \mid x,y \in C\}$ where $(x,y,1)g = ((x,y)\alpha + z, 1)$. The Lie algebra L of G, considered as a subalgebra of $\mathrm{gl}(3,C)$ has a basis given by $\{e_{ij} \mid i = 1,2,3;\ j = 1,2\}$. Consider an arbitrary, finite dimensional representation (ρ, V) of L. When viewed as a representation of the subalgebra $\mathrm{gl}(2,C)$, the space V decomposes into a finite direct sum of irreducible representations, say $V = W_1 \oplus \cdots \oplus W_n$. Define two operators $N = \rho(e_{32})$ and $M = \rho((e_{11} - e_{22} + 1)e_{31} + e_{32}e_{21})$ on the space V and note the following properties:

(1) By an extreme vector of V we mean any vector $v \in V$ such that $\rho(e_{12})v = 0$. Since $[\rho(e_{12}), N] = [\rho(e_{12}), M] = 0$, we find that N and M map extreme vectors to extreme vectors.

(2) Denote by $\langle S \rangle$ the $\mathrm{gl}(2,C)$ subrepresentation of V generated by a set $S \subseteq V$. Then for any nonzero extreme vector $v \in V$ we have

$$\mathrm{Dim}\langle Nv \rangle = \mathrm{Dim}\langle v \rangle + 1 \quad \text{or } Nv = 0,$$

and

$$\mathrm{Dim}(Mv\rangle = \mathrm{Dim}\langle v \rangle - 1 \quad \text{or } Mv = 0.$$

(3) Combining remarks 1 and 2 with the fact that $[N,M] = 0$ we find that N and M are commuting nilpotent linear operators on V. This in turn implies that for any fixed nonzero extreme vector $v \in V$ the set of all nonzero vectors of the form $N^{\mu}M^{\nu}v$ is linearly independent. {The operators N and M are clearly simple modifications of the Nagel–Moshinsky operators [cf.(4)].}

Consider now $V_0 = \langle \{N^{\mu}M^{\nu}v_1 \mid v_1 \text{ is an extreme vector of } W_1 \text{ and } \mu, \nu \text{ are nonnegative integers}\}\rangle$. It is clear that $W_1 \subseteq V_0$ and V_0 is an L subrepresentation of V. We shall now explicitly construct a representation of L equivalent to V_0 using the Gel'fand representations of $\mathrm{gl}(3,C)$.

Let $d+1$ denote the dimension of the space W_1 and let k be the smallest nonnegative integer such that $N^{k}v_1 \neq 0$ and $N^{k+1}v_1 = 0$. Denote by (ρ_1, U) the restriction to L of the unique finite dimensional irreducible representation of $\mathrm{gl}(3,C)$ labeled by the sequence $m_{13} = k+d$, $m_{23} = k$ and $m_{33} = 0$. Let U' equal to the L subrepresentation of U generated by

$$\left\{ \xi(m) \,\middle|\, m = \begin{pmatrix} k+d & & k & & 0 \\ & k+d-\nu & & k-\mu & \\ & & k+d-\nu & & \end{pmatrix}, \text{where } N^{\mu}M^{\nu}v_1 = 0 \right\}.$$

We claim that V_0 and U/U' are equivalent L representations. In fact, by the properties of N and M listed above we find that as a $\mathrm{gl}(2,C)$ representation

$$V_0 = \sum_{\substack{\mu,\nu \text{ nonnegatice integer} \\ \text{s.t. } N^{\mu}M^{\nu}v_1 \neq 0}} \oplus \langle N^{\mu}M^{\nu}v_1 \rangle,$$

where $\langle N^{\mu}M^{\nu}v_1 \rangle$ is a $(d+1+\mu-\nu)$-dimensional irreducible $\mathrm{gl}(2,C)$ subrepresentation. The corresponding decomposition of U/U' as a $\mathrm{gl}(2,C)$ representation yields

$$U/U' = \sum_{\substack{\mu,\nu \text{ nonnegatice integer} \\ \text{s.t. } N^{\mu}M^{\nu}v_1 \neq 0}} \oplus \left\langle \xi \begin{pmatrix} k+d & & k & & 0 \\ & k+d-\nu & & k-\mu & \\ & & k+d-\nu & & \end{pmatrix} + U' \right\rangle,$$

where

$$\left\langle \xi \begin{pmatrix} k+d & & k & & 0 \\ & k+d-\nu & & k-\mu & \\ & & k+d-\nu & & \end{pmatrix} + U' \right\rangle$$

is a $(d+1+\mu-\nu)$-dimensional irreducible $\mathrm{gl}(2,C)$ subrepresentation. Thus $V_0 \cong U/U'$ as $\mathrm{gl}(2,C)$ representations.

To complete the verification of the equivalence of V_0 and U/U' as L representation it suffices to observe that the map ϕ, given by setting

$$\phi(N^{\mu}M^{\nu}v_1) = K_{\mu\nu}\xi \begin{pmatrix} k+d & & k & & 0 \\ & k+d-\nu & & k-\mu & \\ & & k+d-\nu & & \end{pmatrix} + U',$$

where the coefficients $K_{\mu\nu}$ are constants defined by the equation

$$\rho_1(e_{32})^{\mu}\rho_1((e_{11} - e_{22} + 1)e_{31} + e_{32}e_{21}))^{\nu}\xi \begin{pmatrix} k+d & & k & & 0 \\ & k+d & & k & \\ & & k+d & & \end{pmatrix} + U'$$

$$= K_{\mu\nu}\xi\begin{pmatrix} k+d & & & k & & 0 \\ & k+d-\nu & & & k-\mu & \\ & & k+d-\nu & & & \end{pmatrix} + U',$$

can be extended in a natural fashion to an L representation equivalence between V_0 and U/U'.

It is easily seen that the L representation U/U' and hence V_0 is indecomposable and one can explicitly write out the matrix elements of the representations. Thus we have a very explicit presentation for a wide class of finite dimensional representations of L.

This technique has a wide range of possible applications which would appear to be worthy of further investigation.

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