Quantised coordinate rings of semisimple groups are unique factorisation domains

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Abstract

We show that the quantum coordinate ring of a semisimple group is a unique factorisation domain in the sense of Chatters and Jordan in the case where the deformation parameter q is a transcendental element.

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Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{C}^*$ is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the present authors, together with L Rigal, [11], have shown that many quantum algebras are noetherian UFD. In particular, we have shown that the quantum group $O_q(SL_n)$ is a noetherian UFD.

Let G be a connected simply connected complex semisimple algebraic group. Since in the classical setting it was shown by Popov, [12], that the ring of regular functions on G is a unique factorisation domain, one can ask if a similar result holds for the quantisation

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$O_q(G)$ of the coordinate ring of G. The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of $O_q(G)$ that was constructed by Joseph, [8].

1 Quantised enveloping algebras and quantum coordinate rings

1.1 Quantised enveloping algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank n. We denote by $\pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that π is a basis of a euclidean vector space E over $\mathbb{R}$, whose inner product is denoted by $(\ ,\ )$ (E is usually denoted by $\mathfrak{h}^*_{\mathbb{R}}$ in Bourbaki). We denote by W the Weyl group of $\mathfrak{g}$; that is, the subgroup of the orthogonal group of E generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \dots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, for $i \in \{1, \dots, n\}$. If $w \in W$, we denote by $l(w)$ its length. Further, we denote by w_0 the longest element of W. Throughout this paper, the Coxeter group W will be endowed with the Bruhat order that we denote by $\leq$. We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by R^+ the set of positive roots and by R the set of roots. We set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$. We denote by $\varpi_1, \dots, \varpi_n$ the fundamental weights, by P the $\mathbb{Z}$-lattice generated by $\varpi_1, \dots, \varpi_n$, and by P^+ the set of dominant weights. In the sequel, P will always be endowed with the following partial order:

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$

Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data.

Recall that the scalar product of two roots (α, β) is always an integer. As in [1], we assume that the short roots have length $\sqrt{2}$.

For each $i \in \{1, \dots, n\}$, set $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and

$$\left[\begin{array}{c} m \\ k \end{array} \right]_i := \frac{(q_i - q_i^{-1}) \dots (q_i^{m-1} - q_i^{1-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \dots (q_i^k - q_i^{-k})(q_i - q_i^{-1}) \dots (q_i^{m-k} - q_i^{k-m})}$$

for all integers $0 \leq k \leq m$. By convention, we have

$$\left[\begin{array}{c} m \\ 0 \end{array} \right]_i := 1.$$

We will use the definition of the quantised enveloping algebra given in [1, I.6.3, I.6.4]. The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by indeterminates $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ subject to the following relations:

$$K_iK_j = K_jK_i \qquad K_iK_i^{-1} = 1$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j \qquad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_i E_i^{1-a_{ij}-k}E_jE_i^k = 0 \ (i \neq j)$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_i F_i^{1-a_{ij}-k}F_jF_i^k = 0 \ (i \neq j).$$

Note that $U_q(\mathfrak{g})$ is a Hopf algebra; its comultiplication is defined by

$$\Delta(K_i) = K_i \otimes K_i \qquad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

its counit by

$$\varepsilon(K_i) = 1 \qquad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and its antipode by

$$S(K_i) = K_i^{-1} \qquad S(E_i) = -K_i^{-1}E_i \qquad S(F_i) = -F_iK_i.$$

We refer the reader to [1, 7, 8] for more details on this algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \dots, E_n$ and by $U_q(\mathfrak{b}^+)$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \dots, E_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$. In a similar manner, $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \dots, F_n$ and $U_q(\mathfrak{b}^-)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$.

1.2 Representation theory of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra $U_q(\mathfrak{g})$ is analogous to the representation theory of the classical enveloping algebra $U(\mathfrak{g})$. In this section, we collect the properties that will be needed in the rest of the paper.

As usual, if M is a left $U_q(\mathfrak{g})$-module, we denote its dual by M^*. Observe that M^* is a right $U_q(\mathfrak{g})$-module in a natural way. However, by using the antipode of $U_q(\mathfrak{g})$, this right action of $U_q(\mathfrak{g})$ on M^* can be twisted to a left action, so that M^* can be viewed as a left $U_q(\mathfrak{g})$-module.

Let M be a $U_q(\mathfrak{g})$-module and $m \in M$. The element m is said to have weight $\lambda \in P$ if $K_i.m = q^{(\lambda,\alpha_i)}m$ for all $i \in \{1,\dots,n\}$. For each $\lambda \in P$, set

$$M_\lambda := \{m \in M \mid K_i.m = q^{(\lambda,\alpha_i)}m \text{ for all } i \in \{1,\dots,n\}\}.$$

If $M_\lambda \neq 0$ then M_λ is said to be a weight space of M and λ is a weight of M.

It is well-known, see, for example [1, 7], that, for each dominant weight $\lambda \in P^+$, there exists a unique (up to isomorphism) simple finite dimensional $U_q(\mathfrak{g})$-module of highest weight λ that we denote by $V(\lambda)$. In the following proposition, we collect some well-known properties of the $V(\lambda)$, for $\lambda \in P^+$. We refer the reader to [1, especially I.6.12], [6] and [7] for details and proofs.

Proposition 1.1 *Denote by $\Omega(\lambda)$ the set of those weights $\mu \in P$ such that $V(\lambda)_\mu \neq 0$.*

1. $V(\lambda) = \oplus_{\mu\in\Omega(\lambda)} V(\lambda)_\mu$

2. *The weights of $V(\lambda)$ are given by Weyl's character formula. In particular, if $\mu \in \Omega(\lambda)$, then $w\mu \in \Omega(\lambda)$ for all $w \in W$.*

3. *For all $w \in W$, one has* $\dim_{\mathbb{C}} V(\lambda)_{w\lambda} = 1$.

4. $V(\lambda)^* \simeq V(-w_0\lambda)$.

5. *The weight $w_0\lambda$ is the unique lowest weight of $V(\lambda)$.*
   In particular, for all $\mu \in \Omega(\lambda)$, one has $w_0\lambda \leq \mu \leq \lambda$.

6. $\Omega(\lambda) = \{\lambda - w\mu \mid w \in W$ *and* $\mu \in P^+$ *such that* $\mu \leq \lambda\}$.

For all $w \in W$ and $\lambda \in P^+$, let $u_{w\lambda}$ denote a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Then we denote by $V_w^+(\lambda)$ the Demazure module associated to the pair λ, w, that is:

$$V_w^+(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{b}^+)u_{w\lambda}.$$

We also set

$$V_w^-(\lambda) := U_q^-(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{b}^-)u_{w\lambda}.$$

(Observe that these definitions are independent of the choice of $u_{w\lambda}$ because of Proposition 1.1 (3).)

The following result may be well-known; however, we have been unable to locate a precise statement.

Proposition 1.2 *1.* $V_{w_0}^+(\lambda) = V(\lambda) = V_{id}^-(\lambda)$.

2. For all $i, j \in \{1, \ldots, n\}$, one has

$$V_{w_0 s_i}^+(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0 \varpi_j\}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise,} \end{cases}$$

and

$$V_{s_i}^-(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{\varpi_j\}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise.} \end{cases}$$

Proof. We only prove the assertions corresponding to "positive" Demazure modules, the proof for "negative" Demazure modules is similar.

Since $w_0\lambda$ is the lowest weight of $V(\lambda)$, we have $U_q^+(\mathfrak{g})u_{w_0\lambda} = V(\lambda)$; that is, $V_{w_0}^+(\lambda) = V(\lambda)$. This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then $s_i(\varpi_j) = \varpi_j$. Hence, in this case, one has: $V_{w_0 s_i}^+(\varpi_j) = U_q^+(\mathfrak{g})u_{w_0 s_i \varpi_j} = U_q^+(\mathfrak{g})u_{w_0 \varpi_j} = V_{w_0}^+(\varpi_j) = V(\varpi_j)$.

Next, let $j \in \{1, \ldots, n\}$. Then $s_j(\varpi_j) = \varpi_j - \alpha_j$. Let $\mu \in \Omega(\varpi_j)$ with $\mu \neq w_0\varpi_j$, and let $m \in V(\varpi_j)_\mu$ be any nonzero element. It follows from the first assertion that there exists $x \in U_q^+(\mathfrak{g})$ such that $m = x.u_{w_0\varpi_j}$. The element x can be written as a linear combination of products $E_{i_1} \ldots E_{i_k}$, with $k \in \mathbb{N}^*$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Naturally, one can assume that $E_{i_1} \ldots E_{i_k}.u_{w_0\varpi_j} \neq 0$ for each such product. Let $E_{i_1} \ldots E_{i_k}$ be one of these products. Since $w_0\pi = -\pi$, there exists $l \in \{1, \ldots, n\}$ such that $w_0\alpha_{i_k} = -\alpha_l$. We will prove that $l = j$. Indeed, assume that $l \neq j$. Since $E_{i_k}.u_{w_0\varpi_j}$ is a nonzero vector of $V(\varpi_j)$ of weight $w_0\varpi_j + \alpha_{i_k}$, we get that

$$w_0\varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).$$

Then, we deduce from Proposition 1.1 that

$$s_l w_0 \left(w_0\varpi_j + \alpha_{i_k}\right) \in \Omega(\varpi_j),$$

that is,

$$s_l\varpi_j + \alpha_l \in \Omega(\varpi_j).$$

Further, since we have assumed that $l \neq j$, we get $s_l\varpi_j = \varpi_j$, so that

$$\varpi_j + \alpha_l \in \Omega(\varpi_j).$$

This contradicts the fact that ϖ_j is the highest weight of $V(\varpi_j)$.

Thus, we have just proved that $w_0\alpha_{i_k} = -\alpha_j$ for all products $E_{i_1}\dots E_{i_k}$ that appear in x. Now, observe that $E_{i_k}.u_{w_0\varpi_j}$ is a nonzero vector of $V(\varpi_j)$ of weight $w_0\varpi_j + \alpha_{i_k} = w_0(\varpi_j + w_0\alpha_{i_k}) = w_0(\varpi_j - \alpha_j) = w_0s_j\varpi_j$. Since $\dim_{\mathbb{C}} V(\varpi_j)_{w_0s_j\varpi_j} = 1$, we get that $E_{i_k}.u_{w_0\varpi_j} = a u_{w_0s_j\varpi_j}$ for a certain nonzero complex number a. Hence we get that

$$m = x.u_{w_0\varpi_j} = \sum \bullet E_{i_1}\dots E_{i_k}.u_{w_0\varpi_j} = y.u_{w_0s_j\varpi_j},$$

where $\bullet$ denote some nonzero complex numbers and $y \in U_q^+(\mathfrak{g})$. Thus $m \in V^+_{w_0s_j}(\varpi_j)$. This shows that

$$\bigoplus_{\mu\in\Omega(\varpi_j)\setminus\{w_0\varpi_j\}} V(\varpi_j)_\mu \subseteq V^+_{w_0s_j}(\varpi_j).$$

As the reverse inclusion is trivial, this finishes the proof. $\square$

1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let G be a connected, simply connected, semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. Since $U_q(\mathfrak{g})$ is a Hopf algebra, one can define its Hopf dual $U_q(\mathfrak{g})^*$ (see [8, 1.4]) via

$$U_q(\mathfrak{g})^* := \{f \in \mathrm{Hom}_{\mathbb{C}}(U_q(\mathfrak{g}), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension}\}.$$

The quantised coordinate ring $O_q(G)$ of G is the subalgebra of $U_q(\mathfrak{g})^*$ generated by the coordinate functions $c^\lambda_{\xi,v}$ for all $\lambda \in P^+$, $\xi \in V(\lambda)^*$ and $v \in V(\lambda)$, where $c^\lambda_{\xi,v}$ is the element of $U_q(\mathfrak{g})^*$ defined by

$$c^\lambda_{\xi,v}(u) := \xi(uv) \text{ for all } u \in U_q(\mathfrak{g}),$$

see, for example, [8, Chapter 9]. As usual, if $\xi \in V(\lambda)^*_\eta$ and $v \in V(\lambda)_\mu$, we write $c^\lambda_{\eta,\mu}$ instead of $c^\lambda_{\xi,v}$. Naturally, this leads to some ambiguity. However, when $\mu \in W.\lambda$ and $\eta \in W.(-w_0\lambda)$, then $\dim(V(\lambda)_\mu) = 1 = \dim(V(\lambda)^*_\eta)$, so that this ambiguity is very minor.

It is well-known that $O_q(G)$ is a noetherian domain and a Hopf-subalgebra of $U_q(\mathfrak{g})^*$, see [1, 8]. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, [1, 1.9.25] or [8, 1.3.5]), and then to obtain an action of the torus $\mathcal{H} := (\mathbb{C}^*)^{2n}$ on $O_q(G)$ (see [2, 5.2]). More precisely, observe that the torus $H := (\mathbb{C}^*)^n$ can be identified with $\mathrm{Hom}(P, \mathbb{C}^*)$ via:

$$h(\lambda) = h_1^{\lambda_1}\dots h_n^{\lambda_n},$$

where $h = (h_1, \ldots, h_n) \in H$ and $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_n\varpi_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. Then, it is known (see [5, 3.3] or [1, I.1.18]) that the torus $\mathcal{H}$ acts rationally by $\mathbb{C}$-algebra automorphisms on $O_q(G)$ via:

$$g.c^{\lambda}_{\xi,v} = g_1(\mu)g_2(\eta)c^{\lambda}_{\xi,v},$$

for all $g = (g_1, g_2) \in \mathcal{H} = H \times H$, $\lambda \in P^+$, $\xi \in V(\lambda)^*_\mu$ and $v \in V(\lambda)_\eta$.
(We refer the reader to [1, II.2.6] for the definition of a rational action.)

As usual, we denote by $\mathrm{Spec}(O_q(G))$ the set of prime ideals in $O_q(G)$. Recall that Joseph has proved [9] that every prime in $O_q(G)$ is completely prime.

Since $\mathcal{H}$ acts by automorphisms on $O_q(G)$, this induces an action of $\mathcal{H}$ on the prime spectrum of $O_q(G)$. As usual, we denote by $\mathcal{H}$-$\mathrm{Spec}(O_q(G))$ the set of those primes ideals of $O_q(G)$ that are $\mathcal{H}$-invariant. This is a finite set since Brown and Goodearl [2, Section 5] (see also [1, II.4]) have shown using previous results of Joseph that

$$\mathcal{H}\text{-Spec}(O_q(G)) = \left\{ Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W \right\},$$

where

$$Q^+_{w_+} := \langle c^{\lambda}_{\xi,v} \mid \lambda \in P^+,\ v \in V(\lambda)_\lambda \text{ and } \xi \in (V^+_{w_+}(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$
$$Q^-_{w_-} := \langle c^{\lambda}_{\xi,v} \mid \lambda \in P^+,\ v \in V(\lambda)_{w_0\lambda} \text{ and } \xi \in (V^-_{w_-w_0}(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$

and

$$Q_{w_+,w_-} := Q^+_{w_+} + Q^-_{w_-}.$$

Since q is transcendental, it follows from [10, Théorème 3] that it is enough to consider the fundamental weights in the definition of $Q^+_{w_+}$ and $Q^-_{w_-}$. More precisely, we deduce from [10, Théorème 3] the following result.

Theorem 1.3 (Joseph)

$$\mathcal{H}\text{-Spec}(O_q(G)) = \left\{ Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W \right\},$$

where

$$Q^+_{w_+} := \langle c^{\varpi_j}_{\xi,v} \mid j \in \{1, \ldots, n\},\ v \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V^+_{w_+}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

$$Q^-_{w_-} := \langle c^{\varpi_j}_{\xi,v} \mid j \in \{1, \ldots, n\},\ v \in V(\varpi_j)_{w_0\varpi_j} \text{ and } \xi \in (V^-_{w_-w_0}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

and

$$Q_{w_+,w_-} := Q^+_{w_+} + Q^-_{w_-}.$$

Moreover the prime ideals Q_{w_+,w_-}, for $(w_+, w_-) \in W \times W$, are pairwise distinct.

2 $O_q(G)$ is a noetherian UFD.

In this section, we prove that $O_q(G)$ is a noetherian UFD (We refer the reader to [11, Section 1] for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal $\mathcal{H}$-eigenvectors $r_1, \dots, r_k$ of $O_q(G)$ such that each $\langle r_i \rangle$ is (completely) prime, and that each nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains one of the r_i. This property may be thought of as a "weak factoriality" result: $O_q(G)$ is an $\mathcal{H}$-UFD in the terminology of [11].

2. Secondly, by using the H-stratification theory of Goodearl and Letzter (see [1, II]), we show that the localisation of $O_q(G)$ with respect to the multiplicative system generated by the r_i is a noetherian UFD.

3. Finally, we use a noncommutative analogue of Nagata's Lemma (see [11, Proposition 1.6]) to prove that $O_q(G)$ itself is a noetherian UFD.

2.1 $O_q(G)$ is an $\mathcal{H}$-UFD

This aim of this section is two-fold. First, we show that for each $i \in \{1, \dots n\}$, the ideal generated by the normal element $c^{\varpi_i}_{-\varpi_i, w_0\varpi_i}$ or $c^{\varpi_i}_{-w_0\varpi_i, \varpi_i}$ is (completely) prime and then we prove that every nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains either one of the $c^{\varpi_i}_{-\varpi_i, w_0\varpi_i}$ or one of the $c^{\varpi_i}_{-w_0\varpi_i, \varpi_i}$.

Lemma 2.1 *Let $i \in \{1, \dots n\}$. Then $Q_{w_0, s_i w_0} = \langle c^{\varpi_i}_{-\varpi_i, w_0\varpi_i} \rangle$ and $Q_{w_0 s_i, w_0} = \langle c^{\varpi_i}_{-w_0\varpi_i, \varpi_i} \rangle$.*

Proof. Recall that

$$Q_{w_0, s_i w_0} = Q^+_{w_0} + Q^-_{s_i w_0},$$

where

$$Q^+_{w_0} = \langle c^{\varpi_j}_{\xi, v} \mid j \in \{1, \dots, n\},\, v \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V^+_{w_0}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

$$Q^-_{s_i w_0} = \langle c^{\varpi_j}_{\xi, v} \mid j \in \{1, \dots, n\},\, v \in V(\varpi_j)_{w_0\varpi_j} \text{ and } \xi \in (V^-_{s_i}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle.$$

Next, it follows from Proposition 1.2(1) that $V^+_{w_0}(\varpi_j) = V(\varpi_j)$ for all j, so that $Q^+_{w_0} = (0)$. Also, we deduce from Proposition 1.2(2) that $V^-_{s_i}(\varpi_j) = V(\varpi_j)$ if $j \neq i$, and $V^-_{s_i}(\varpi_i) = \oplus_{\mu \in \Omega(\varpi_i) \setminus \{\varpi_i\}} V(\varpi_i)_\mu$. Hence,

$$Q^-_{s_i w_0} = \langle c^{\varpi_i}_{\xi, v} \mid v \in V(\varpi_i)_{w_0\varpi_i} \text{ and } \xi \in V(\varpi_i)^*_{-\varpi_i} \rangle,$$

that is, $Q^-_{s_iw_0} = \langle c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}\rangle$. Therefore $Q_{w_0,s_iw_0} = Q^+_{w_0} + Q^-_{s_iw_0} = \langle c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}\rangle$, as desired.

The second claim of the lemma is obtained in the same way. $\square$

Now observe that, in [8], Joseph uses slighty different conventions for the dual M^* of a left $U_q(\mathfrak{g})$-module. Indeed, it is mentioned in [8, 9.1] that the dual M^* is viewed with its natural right $U_q(\mathfrak{g})$-module structure. As a consequence, Joseph's convention for the weights of the dual $L(\lambda)^*$ of $L(\lambda)$, for $\lambda \in P^+$, is not exactly the same as our convention. In particular, the elements $c^{\varpi_i}_{\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{w_0\varpi_i,\varpi_i}$, $i \in \{1,\dots,n\}$, that appear in [8, Corollary 9.1.4], correspond to the elements $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$ in our notation. With this in mind, it follows from [8, Corollary 9.1.4] that the elements $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$, for $i \in \{1,\dots,n\}$, are normal in $O_q(G)$. Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata's Lemma in order to prove that $O_q(G)$ is a noetherian UFD.

Corollary 2.2 *The $2n$ elements $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$, for $i \in \{1,\dots,n\}$, are nonzero normal elements of $O_q(G)$ and they generate pairwise distinct completely prime ideals of $O_q(G)$.*

Since the $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$, for $i \in \{1,\dots,n\}$, are $\mathcal{H}$-eigenvectors of $O_q(G)$, in order to prove that $O_q(G)$ is an $\mathcal{H}$-UFD in the sense of [11, Definition 2.7], it only remains to prove that every nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains either one of the $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ or one of the $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$. This is what we do next.

Lemma 2.3 *Let $\mathbf{w} = (w_+,w_-) \in W \times W$, with $\mathbf{w} \neq (w_0,w_0)$. Then $Q_{\mathbf{w}}$ contains either one of the $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$, or one of the $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$.*

Proof. Since $\mathbf{w} \neq (w_0,w_0)$, either $w_+ \neq w_0$, or $w_- \neq w_0$. Assume, for instance, that $w_+ \neq w_0$, so that there exists $i \in \{1,\dots,n\}$ such that $w_+ \leq w_0 s_i$. One can easily check from the definition of $Q_{\mathbf{w}}$ that this forces $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i} \in Q^+_{w_+}$, so that

$$c^{\varpi_i}_{-w_0\varpi_i,\varpi_i} \in Q^+_{w_+} \subseteq Q_{\mathbf{w}},$$

as required. $\square$

As a consequence of Corollary 2.2 and Lemma 2.3, we get the following result.

Corollary 2.4 *$O_q(G)$ is an $\mathcal{H}$-UFD.*

Proof. Theorem 1.3 establishes that $\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+,w_-} \mid (w_+,w_-) \in W \times W\}$. Note that $Q_{w_+,w_-} = 0$ precisely when $w_+ = w_- = w_0$. Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero $\mathcal{H}$-prime ideal of $O_q(G)$ contains a nonzero $\mathcal{H}$-prime of height one that is generated by a normal $\mathcal{H}$-eigenvector. Thus, $O_q(G)$ is an $\mathcal{H}$-UFD. $\square$

2.2 $O_q(G)$ is a noetherian UFD.

Set T to be the localisation of $O_q(G)$ with respect to the multiplicatively closed set generated by the normal $\mathcal{H}$-eigenvectors $c^{\varpi_i}_{-\varpi_i,w_0\varpi_i}$ and $c^{\varpi_i}_{-w_0\varpi_i,\varpi_i}$, for $i \in \{1,\ldots,n\}$. Then the rational action of $\mathcal{H}$ on $O_q(G)$ extends to an action of $\mathcal{H}$ on the localisation T by $\mathbb{C}$-algebra automorphisms, since we are localising with respect to $\mathcal{H}$-eigenvectors, and this action of $\mathcal{H}$ on T is also rational, by using [1, II.2.7]. The following result is a consequence of Corollary 2.4 and [11, Proposition 3.5].

Proposition 2.5 *The ring T is $\mathcal{H}$-simple; that is, the only $\mathcal{H}$-ideals of T are 0 and T.*

We are now in position to show that $O_q(G)$ is a noetherian UFD.

Theorem 2.6 *$O_q(G)$ is a noetherian UFD.*

Proof. By [11, Proposition 1.6], it is enough to prove that the localisation T is a noetherian UFD. Now, as proved in Proposition 2.5, T is an $\mathcal{H}$-simple ring. Thus, using [1, II.3.9], T is a noetherian UFD, as required. $\square$

As a consequence, we deduce from Theorem 2.6 and [4, Theorem 2.4] the following result.

Corollary 2.7 *$O_q(G)$ is a maximal order.*

The fact that $O_q(G)$ is a maximal order can also be proved directly by using a suitable localisation of $O_q(G)$, [8, Corollary 9.3.10], which is itself a maximal order.

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