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"Classical" Flag Varieties for Quantum Groups: The Standard Quantum $\mathrm{SL}(n, \mathbf{C})$

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We suggest a possible programme to associate geometric "flag-like" data to an arbitrary simple quantum group, in the spirit of the noncommutative algebraic geometry developed by Artin, Tate, and Van den Bergh. We then carry out this programme for the standard quantum $\mathrm{SL}(n)$ of Drinfel'd and Jimbo, where the varieties involved are certain T-stable subvarieties of the (ordinary) flag variety.

0. INTRODUCTION

The study of quantum analogues of flag varieties, first suggested by Manin [31], has been undertaken during the past decade by several authors, from various points of view; see e.g. [1, 8, 13, 16, 17, 23, 26, 29, 35, 38, 39, 40]. Around the same time, an approach to noncommutative projective algebraic geometry was initiated by Artin *et al.* [4, 5] and Artin and Van den Bergh [6], and considerably developed since (see e.g. [3, 7, 9, 24, 28, 37, 41, 42, 44]). One attractive feature of their approach is the association of actual *geometric* data to certain classes of graded *non*commutative algebras.

The present work is an attempt to study quantum flag varieties from this point of view. As a consequence, our "quantum flag varieties" will be actual varieties (with some bells and whistles).

Recall the original idea of [4–6]. If A is the homogeneous coordinate ring of a projective scheme E, then the points of E are in one-to-one correspondence with the isomorphism classes of the so-called *point modules* of A, i.e. $\mathbf{N}$-graded cyclic A-modules P such that $\dim P_n = 1$ for all n. Now if A is an $\mathbf{N}$-graded *non*commutative algebra, one may still try to parametrize

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its point modules by the points of some projective scheme E. Of course, one cannot hope to reconstruct A from E alone, but there is now an additional ingredient: the *shift* operation $\sigma : P \mapsto P[1]$, where $P[1]$ is the $\mathbf{N}$-graded A-module defined by $P[1]_n := P_{n+1}$. (When A is commutative, this shift is trivial: $P[1] \simeq P$ for every point module P.) Assume that σ may be viewed as an automorphism of E: one may then hope, at least in "good" cases, to recover A from the triple $(E, \sigma, \mathscr{L})$, where $\mathscr{L}$ is the line bundle over E defined by its embedding into a projective space. The first step of this recovery is the construction of the *twisted homogeneous coordinate ring* $B(E, \sigma, \mathscr{L})$ of a triple $(E, \sigma, \mathscr{L})$, defined in [6] as follows:

$$B(E, \sigma, \mathscr{L}) = \bigoplus_{n \in \mathbf{N}} B_n, \qquad B_n := \mathrm{H}^0(E, \mathscr{L} \otimes \mathscr{L}^{\sigma} \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}})$$

(where $\mathscr{L}^{\sigma}$ denotes the pullback of $\mathscr{L}$ along σ), the multiplication being given by $\alpha\beta := \alpha \otimes \beta^{\sigma^m}$ for all $\alpha \in B_m$, $\beta \in B_n$. (When σ is the identity, this algebra coincides, in high degree, with the (commutative) homogeneous coordinate ring of E w.r.t. the polarization $\mathscr{L}$.) If the triple $(E, \sigma, \mathscr{L})$ comes from an algebra A as above, the second step then consists in analysing the canonical morphism $A \to B(E, \sigma, \mathscr{L})$. The initial success of this method has been a complete study of all regular algebras of dimension three [4] (where the kernel of $A \to B(E, \sigma, \mathscr{L})$ turns out to be generated by a single element of degree three).

The present paper is organized as follows.

In Part I, we give a general outline of a possible theory of flag varieties for quantum groups, using a multigraded version of the ideas of [4–6] recalled above, some of which have already been introduced by Chan [10]. This part is largely conjectural and contains no (significant) new results; its purpose is rather to set up a framework that will be used in Parts II and III.

More specifically, we proceed as follows. Let G be a simple complex group; our interest in flag varieties allows us to assume without harm that G is simply connected. Let P^+ be the monoid of dominant integral weights of G (w.r.t. some Borel subgroup $B \subset G$): the *shape algebra* M of G is a P^+-graded G-algebra whose term of degree λ is the irreducible representation of G of highest weight λ. Now consider the definition of a point module (see above), but with $\mathbf{N}$-gradings replaced by P^+-gradings: we obtain the notion of a *flag module* of M (Definition 2.1); this terminology is justified by the fact that the isomorphism classes of such modules are indeed parametrized by the points of the flag variety G/B (Proposition 2.2).

If a quantum group has the "same" representation theory as G (in the sense of Definition 1.1), then the we may still define a (P^+-graded) shape algebra. We then discuss the possibility to parametrize the latter's flag modules (up to isomorphism) by the points of some scheme E, and to realize shifts $F \mapsto F[\lambda]$ ($\lambda \in P^+$) as automorphisms σ_λ of E. It will of course be sufficient to know the automorphisms $\sigma_1, \dots, \sigma_\ell$ associated to the fundamental weights $\varpi_1, \dots, \varpi_\ell$, which freely generate P^+. Moreover, since we are in a *multi*graded situation, it will be more natural to view E as a subscheme of a *product* of ℓ projective spaces, corresponding to ℓ line bundles $\mathscr{L}_1, \dots, \mathscr{L}_\ell$ over E.

We then consider the converse problem of reconstructing the shape algebra from E, the σ_i, and the $\mathscr{L}_i$, using Chan's construction [10] of a twisted *multi*homogeneous coordinate ring: this is the P^+-graded algebra

$$B(E, \sigma_1, \dots, \sigma_\ell, \mathscr{L}_1, \dots, \mathscr{L}_\ell) := \bigoplus_{\lambda \in P^+} \mathrm{H}^0(E, \mathscr{L}_\lambda),$$

where the line bundles $\mathscr{L}_\lambda$ are constructed inductively from the rules $\mathscr{L}_{\varpi_i} = \mathscr{L}_i$, $\mathscr{L}_{\varpi_i+\lambda} = \mathscr{L}_i \otimes \mathscr{L}_\lambda^{\sigma_i}$. (Again, if $E = G/B$, if each σ_i is the identity, and if the $\mathscr{L}_i$ are the line bundles associated with the fundamental G-modules $V^1, \dots, V^\ell$, then this algebra is the (commutative) multihomogeneous coordinate ring of $G/B \subset \mathbb{P}(V^1) \times \cdots \times \mathbb{P}(V^\ell)$, which in turn is equal to the shape algebra $\mathcal{O}(\overline{G/U})$, U the unipotent radical of B.)

We stress that the ideas developed in this Part are *not* restricted to the standard quantum groups of Drinfel'd [14] and Jimbo [22], but could, in principle, be applied to other quantum groups as well, as long as they have the "same" representation theory as a given simple complex group. (Potential other examples include the multiparameter quantum groups of Artin *et al.* [2, 19], the quantum $\mathrm{SL}(n)$ of Cremmer and Gervais [11, 18], or the quantum $\mathrm{SL}(3)$'s classified in [32].)

In Parts II and III, we *do* restrict ourselves to a standard Drinfel'd–Jimbo quantum group $\mathcal{O}_q^{\mathrm{DJ}}(G)$, with q not a root of unity. Thanks to the results of Lusztig [30] and Rosso [34], $\mathcal{O}_q^{\mathrm{DJ}}(G)$ has the "same" representation theory as the group G, so one can define a shape algebra M^{DJ}.

In Part II, we construct geometric data E^{DJ}, σ_i, and $\mathscr{L}_i$, and we conjecture that these data indeed correspond to M^{DJ} as described above (Conjectures 9.1 and 9.2). The scheme E^{DJ} will actually be a union of certain T-stable subvarieties of the (ordinary) flag variety G/B. Since the latter may be of independent interest to algebraic geometers, we have decided to describe them in a separate note [33] (but we recall their construction here, without proofs).

In Part III, we prove Conjecture 9.1 for $G = \mathrm{SL}(n)$, thus obtaining a "flag variety" for the standard Drinfel'd–Jimbo quantum $\mathrm{SL}(n)$. The proof uses special features of the group $\mathrm{SL}(n)$ (the Weyl group is the symmetric group, all fundamental representations are minuscule, etc.) and is essentially combinatorial; it is therefore not likely to be extendable to an arbitrary G.

Conventions. All vector spaces, dimensions, algebras, tensor products, varieties, schemes, etc. will be over the field $\mathbf{C}$ of complex numbers. If G is a linear algebraic group, we denote by $\mathcal{O}(G)$ the Hopf algebra of polynomial functions on G. If A is a (co)algebra, then the dual of a left A-(co)module is a right A-(co)module, and vice versa; morphisms of A-(co)modules will simply be called *A-morphisms*. When V is a vector space and $v \in V$ is nonzero, we will sometimes still denote by v the corresponding point in the projective space $\mathbb{P}(V)$.

Contents.

PART I. AN APPROACH TO QUANTUM FLAG VARIETIES: GENERAL OUTLINE

This part contains no (significant) new results. It rather discusses a possible theory of flag varieties for simple quantum groups, asking several questions along the way (as well as two ambitious problems at the end).

Most of what we will say here is a multigraded version of some of the main ideas of [4–6], applied in a Lie-theoretic setting.

1. SIMPLE QUANTUM GROUPS AND THEIR SHAPE ALGEBRAS

Let G be a simply connected simple complex group, $B \subset G$ a Borel subgroup, and P^+ the set of dominant integral weights of G w.r.t. B. For each $\lambda, \mu, \nu \in P^+$, denote by

- d^λ the dimension of the simple G-module of highest weight λ, and by

- $c_\nu^{\lambda\mu}$ the multiplicity of the simple G-module of highest weight ν inside the tensor product of those of highest weights λ and μ.

Bearing in mind that the algebra $\mathcal{O}(G)$ of polynomial functions on G is a commutative Hopf algebra, and that (finite-dimensional) left G-modules correspond to right $\mathcal{O}(G)$-comodules, recall the following definition from [32].

DEFINITION 1.1. We call a *quantum* G any (not necessarily commutative) Hopf algebra A (over $\mathbf{C}$) such that

(a) there is a family $\{V^\lambda \mid \lambda \in P^+\}$ of simple and pairwise nonisomorphic (right) A-comodules, with $\dim V^\lambda = d^\lambda$,

(b) every A-comodule is isomorphic to a direct sum of these,

(c) for every $\lambda, \mu \in P^+$, $V^\lambda \otimes V^\mu$ is isomorphic to $\oplus_\nu c_\nu^{\lambda\mu} V^\nu$.

For convenience, we will write

$$V_\lambda \coloneqq (V^\lambda)^*.$$

For every $\lambda, \mu \in P^+$, Definition 1.1(c) yields an injective A-morphism $V^{\lambda+\mu} \to V^\lambda \otimes V^\mu$ that is unique up to scalars. Denote by

$$m_{\lambda\mu} : V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$$

the corresponding projection. Gluing these together on

$$M_A \coloneqq \bigoplus_\lambda V_\lambda,$$

we get a (not necessarily associative) multiplication $m \colon M_A \otimes M_A \to M_A$.

DEFINITION 1.2. The algebra M_A is called the *shape algebra* of A.

QUESTION A. Is it possible to renormalize the $m_{\lambda\mu}$ in such a way that the multiplication m becomes associative?

Recall that this question has a positive answer in the commutative case $A = \mathcal{O}(G)$: if U is a maximal unipotent subgroup, then by the Borel–Weil theorem, we may set

$$M_{\mathcal{O}(G)} = \mathcal{O}(\overline{G/U}) \coloneqq \{f \in \mathcal{O}(G) \mid f(gu) = f(g)\ \forall g \in G,\ \forall u \in U\}.$$

The next proposition provides a criterion for a positive answer to Question A. We first introduce some more notation: let ℓ be the rank of G, denote by $\varpi_1, \ldots, \varpi_\ell$ the fundamental weights, and let us use the shorthand notation

$$V_i := V_{\varpi_i}, \qquad V^i := V^{\varpi_i}, \qquad 1 \leqslant i \leqslant \ell.$$

For every $1 \leqslant i, j, k \leqslant \ell$, Definition 1.1(c) implies that $V_i \otimes V_j \otimes V_k$ contains a unique subcomodule isomorphic to $V_{\varpi_i + \varpi_j + \varpi_k}$; denote this subcomodule by W_{ijk}.

PROPOSITION 1.3. *Question A has a positive answer* (*for a given* A) *if and only if there exist* A*-isomorphisms* $R_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ *for all* $i > j$, *such that the braid relation*

$$(R_{jk} \otimes \mathrm{id})(\mathrm{id} \otimes R_{ik})(R_{ij} \otimes \mathrm{id})|_{W_{ijk}} = (\mathrm{id} \otimes R_{ij})(R_{ik} \otimes \mathrm{id})(\mathrm{id} \otimes R_{jk})|_{W_{ijk}} \tag{1.1}$$

holds for all $i > j > k$.

We defer the proof to Appendix A.

COROLLARY 1.4. *Question A has a positive answer in each of the following situations*:

- *when* G *is of rank* 2,
- *when* A *is dual quasitriangular*,
- *when* $G = \mathrm{SL}(n)$ (*by the main result of* [25]).

Since $\varpi_1, \ldots, \varpi_\ell$ generate the monoid P^+, and since the $m_{\lambda\mu}$ are surjective, the algebra M_A is generated by

$$M_1 := V_1 \oplus \cdots \oplus V_\ell.$$

In this way, M_A may be viewed as an $\mathbf{N}$-graded algebra. More explicitly, if $\lambda \in P^+$ decomposes as $\sum_i a_i \varpi_i$ (with each $a_i \in \mathbf{N}$), and if we write $h(\lambda) := \sum a_i$ for the *height* of λ, then the $\mathbf{N}$-grading on M_A is given by $M_k := \oplus_{h(\lambda)=k} V_\lambda$.

QUESTION B. Is the shape algebra M_A quadratic (as an $\mathbf{N}$-graded algebra)?

In the commutative case $A = \mathcal{O}(G)$, the shape algebra $\mathcal{O}(\overline{G/U})$ is indeed quadratic by a well-known theorem of Kostant (see [27, Theorem 1.1] for a proof). This remains true for the standard Drinfel'd–Jimbo quantum $\mathrm{SL}(n)$:

a presentation of the corresponding shape algebra by generators and (quadratic) relations has been given by Taft and Towber [40].

QUESTION C. Is the shape algebra M_A a Koszul algebra?

To finish this section, let us take a closer look at the quadratic relations in M_A. For every $1 \leqslant i,j \leqslant \ell$, let K_{ij} be the kernel of the multiplication $V_i \otimes V_j \to V_{\varpi_i+\varpi_j}$. By Definition 1.1(c), the A-comodules $V_i \otimes V_j$ and $V_j \otimes V_i$ are isomorphic, and, rescaling the A-isomorphism $R_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ of Proposition 1.3 if necessary, we may assume that the diagram (A.1) (in Appendix A) commutes. Using Definition 1.1(c), we see that the quadratic relations in M_A of degree $\varpi_i + \varpi_j$ are of two kinds:

$(\mathrm{I})_{ij}$ $\xi = 0$, for $\xi \in K_{ij}$;

$(\mathrm{II})_{ij}$ $\xi = R_{ij}(\xi)$, for $\xi \in V_i \otimes V_j$.

Remark 1.5. By Definition 1.1(c), relations $(\mathrm{I})_{ij}$ and $(\mathrm{II})_{ij}$ for arbitrary i,j are consequences of relations $(\mathrm{I})_{ij}$ for $i \geqslant j$ only and relations $(\mathrm{II})_{ij}$ for $i > j$ only.

2. THE SCHEME OF FLAG MODULES

Assume that Question A has a positive answer. The following definition is a multigraded analogue of the point modules introduced in [5].

DEFINITION 2.1. A *flag module* is a P^+-graded right M_A-module F such that

(a) $\dim F_\lambda = 1$ for each $\lambda \in P^+$,

(b) F is cyclic.

The terminology is justified by the commutative case. Indeed, let $B \subset G$ be a Borel subgroup and U the unipotent radical of B. Then we have the following

PROPOSITION 2.2. *The isomorphism classes of flag modules of $M_{\mathcal{O}(G)} = \mathcal{O}(\overline{G/U})$ are parametrized by the points of the flag variety G/B.*

Proof. First, recall from the Borel–Weil theorem that the decomposition $\mathcal{O}(\overline{G/U}) = \oplus_{\lambda \in P^+} V_\lambda$ is given by

$$V_\lambda = \{f \in \mathcal{O}(G) \mid f(gb) = \lambda(b)f(g) \ \forall g \in G, \ \forall b \in B\}.$$

Now fix $g \in G$ and endow a vector space $F = \oplus_{\lambda \in P^+} \mathbf{C} e_\lambda$ with the flag module structure defined by

$$e_\lambda . f = f(g) e_{\lambda+\mu} \qquad \text{for all } f \in V_\mu.$$

If we replace g by gb for some $b \in B$, the expression for $e_\lambda . f$ is just multiplied by $\mu(b)$, so up to isomorphism (of graded modules), the flag module thus obtained only depends on the class $gB \in G/B$.

Conversely, assume that F is a flag module of $\mathcal{O}(\overline{G/U})$, and choose a graded basis $\{e_\lambda \mid \lambda \in P^+\}$ of F. For each $\lambda, \mu \in P^+$, let $v_\lambda^\mu \in V^\mu$ be defined by

$$e_\lambda . f = \langle f, v_\lambda^\mu \rangle \, e_{\lambda+\mu} \qquad \text{for all } f \in V_\mu.$$

Since the algebra $\mathcal{O}(\overline{G/U})$ is commutative, we have $(e_0 . f).f' = (e_0 . f').f$ for every $f \in V_\lambda$, $f' \in V_\mu$, hence

$$v_0^\lambda \otimes v_\lambda^\mu = v_\mu^\lambda \otimes v_0^\mu. \tag{2.1}$$

It follows that v_λ^μ is a multiple of $v_0^\mu =: v^\mu$, say $v_\lambda^\mu = a_\lambda v^\mu$. Inserting back into (2.1) yields $a_\lambda = a_\mu$, for all $\lambda, \mu \in P^+$. Since $a_0 = 1$, we get $a_\lambda = 1$ for all $\lambda \in P^+$. Therefore,

$$e_0 . f = \langle f, v^\mu \rangle \, e_\lambda \qquad \text{for all } f \in V_\lambda.$$

The collection $\{v^\lambda \mid \lambda \in P^+\}$ defines a linear form v on $\mathcal{O}(\overline{G/U})$. Furthermore, we have $e_0.(ff') = (e_0 . f).f'$ for all $f \in V_\lambda$, $f' \in V_\mu$, so $\langle ff', v^{\lambda+\mu} \rangle = \langle f, v^\lambda \rangle \langle f', v^\mu \rangle$, which shows that the linear form v is a character on $\mathcal{O}(\overline{G/U})$, corresponding to a point x of the affine variety $\overline{G/U}$. Moreover, since F is cyclic, each v^λ must be nonzero, so x actually lies in G/U, say $x = gU$. This yields an element $gB \in G/B$.

It is clear that these two constructions are inverse to each other. ∎

We will now discuss a possible picture of this kind in the noncommutative situation: if A is a quantum G, we would like to parametrize the isomorphism classes of flag modules over the shape algebra M_A by the (closed) points of some scheme E.

Moreover, given a flag module F and a weight $\lambda \in P^+$, consider the *shifted* flag module $F[\lambda]$, defined as the P^+-graded module such that $F[\lambda]_\mu := F_{\lambda+\mu}$. We would then like that, for each λ, the shift operation $F \mapsto F[\lambda]$ corresponds to an automorphism of schemes $\sigma_\lambda : E \to E$.

To achieve this, let us encode the structure of a flag module more geometrically, as follows. If F is a flag module with basis $\{e_\lambda \mid \lambda \in P^+\}$, then for each $\lambda \in P^+$ and each $1 \leqslant i \leqslant \ell$, let $v_\lambda^i \in V^i$ be defined by

$$e_\lambda . f = \langle f, v_\lambda^i \rangle \, e_{\lambda+\varpi_i} \qquad \text{for all } f \in V_i. \tag{2.2}$$

Replacing F by an isomorphic flag module (i.e. rescaling the e_λ) only multiplies each v_λ^i by a scalar, so let p_λ^i be the corresponding point in $\mathbb{P}(V^i)$. To simplify notation, let us write

$$\mathbb{P}^{1\ldots\ell} := \mathbb{P}(V^1) \times \cdots \times \mathbb{P}(V^\ell)$$

and denote by $\mathrm{pr}^i : \mathbb{P}^{1\ldots\ell} \to \mathbb{P}(V^i)$ the natural projection. For any point $p \in \mathbb{P}^{1\ldots\ell}$, we use the shorthand notation $p^i := \mathrm{pr}^i(p)$. Thus, to an isomorphism class of flag modules, we associate a collection of points $\{p_\lambda \mid \lambda \in P^+\}$ in $\mathbb{P}^{1\ldots\ell}$.

From now on, we assume that Question B has a positive answer. The quadratic relations (I) and (II) in M_A (see the end of Section 1) impose some conditions on this collection of points, which we now analyse.

For relations of type (I), identify $\mathbb{P}(V^i) \times \mathbb{P}(V^j)$ with its image in $\mathbb{P}(V^i \otimes V^j)$ under the Segre embedding. Relations $(\mathrm{I})_{ij}$ may be viewed as equations defining a subscheme Γ^{ij} of $\mathbb{P}(V^i) \times \mathbb{P}(V^j)$. We then have

$$(p_\lambda^i, p_{\lambda+\varpi_i}^j) \in \Gamma^{ij} \tag{2.3}$$

for all $\lambda \in P^+$ and all $1 \leqslant i,j \leqslant \ell$.

Similarly, for relations of type (II), we consider the map $\mathbb{P}(R^{ji}) : \mathbb{P}(V^j \otimes V^i) \to \mathbb{P}(V^i \otimes V^j)$, where R^{ji} denotes the transpose of R_{ij}. Then we must have

$$(p_\lambda^i, p_{\lambda+\varpi_i}^j) = \mathbb{P}(R^{ji})(p_\lambda^j, p_{\lambda+\varpi_j}^i) \tag{2.4}$$

(again identifying $\mathbb{P}(V^i) \times \mathbb{P}(V^j)$ with its image under the Segre embedding).

Gluing together conditions (2.3) and (2.4) for all i,j, we are led to consider the subscheme $\Gamma \subset (\mathbb{P}^{1\ldots\ell})^{\ell+1}$ of all $(\ell+1)$-tuples $(p_0, p_1, \ldots, p_\ell)$ satisfying

$$\begin{gathered}(p_0^i, p_i^j) \in \Gamma^{ij}, \\ (p_0^i, p_i^j) = \mathbb{P}(R^{ji})(p_0^j, p_j^i)\end{gathered}$$

for all $1 \leqslant i,j \leqslant \ell$. We may now rephrase conditions (2.3) and (2.4) by saying that the collection $\{p_\lambda \mid \lambda \in P^+\}$ satisfies

$$(p_\lambda, p_{\lambda+\varpi_1}, \ldots, p_{\lambda+\varpi_\ell}) \in \Gamma \qquad \text{for all } \lambda \in P^+. \tag{2.5}$$

Proposition 2.3. *Assume that M_A is quadratic (as an* **N**-*graded algebra). Then there is a one-to-one correspondence between isomorphism* *classes of flag modules over* M_A *and families* $\{p_\lambda \mid \lambda \in P^+\}$ *of points of* $\mathbb{P}^{1\ldots\ell}$ *satisfying* (2.5).

Proof. It remains to show that the above construction can be reversed, so assume that $\{p_\lambda \mid \lambda \in P^+\}$ is a collection of points in $\mathbb{P}^{1\ldots\ell}$ satisfying (2.5). Choose a (nonzero) representative $v^i_\lambda \in V^i$ for each p^i_λ, and endow a vector space $\oplus_{\lambda \in P^+} \mathbf{C} e_\lambda$ with the flag module structure defined by rule (2.2). By (2.3), this rule is compatible with relations of type (I) in M_A. By (2.4), it is also compatible with relations of type (II), *provided* that, for each $\lambda \in P^+$ and each $1 \leqslant i,j \leqslant \ell$, we suitably rescale one of v^i_λ, $v^j_{\lambda+\varpi_i}$, v^j_λ, $v^i_{\lambda+\varpi_j}$. Proceeding by induction over the height $h(\lambda)$, we may perform this rescaling in a consistent way.

It is clear that the two constructions are inverse to each other. ∎

Remark 2.4. Rescaling the $m_{\lambda\mu}$ only multiplies the R_{ij} by scalars. Therefore, the scheme Γ does not depend on the normalizations of the multiplication in M_A, but only on A itself.

The following question is inspired by the description given in the Introduction of [4].

QUESTION D. Do there exist a subscheme E of $\mathbb{P}^{1\ldots\ell}$ and ℓ pairwise commuting automorphisms $\sigma_1, \ldots, \sigma_\ell : E \to E$ such that the scheme Γ is given by

$$\Gamma = \{(p, \sigma_1(p), \ldots, \sigma_\ell(p)) \mid p \in E\}? \tag{2.6}$$

A positive answer to this question would fulfill the aim of parametrizing flag modules, as expressed at the beginning of this section. Indeed, assume that E and $\sigma_1, \ldots, \sigma_\ell$ as in Question D do exist. For each weight $\lambda = \sum a_i \varpi_i$, define $\sigma_\lambda := \sigma_1^{a_1} \ldots \sigma_\ell^{a_\ell}$; since the σ_i commute, we have $\sigma_{\lambda+\mu} = \sigma_\lambda \sigma_\mu$. Then for every family $\{p_\lambda \mid \lambda \in P^+\}$ of points in $\mathbb{P}^{1\ldots\ell}$ satisfying (2.5), the realization (2.6) shows that $p_\lambda = \sigma_\lambda(p_0)$ for all $\lambda \in P^+$, with $p_0 \in E$. Conversely, for every $p \in E$, the family $\{\sigma_\lambda(p) \mid \lambda \in P^+\}$ satisfies (2.5) and thus defines an isomorphism class of flag modules by Proposition 2.3. Therefore, if Question D had a positive answer, flag modules (up to isomorphism) would be parametrized by the points of E, with σ_λ corresponding to the shift operation $F \mapsto F[\lambda]$.

Finally, for future reference, we define, for each $1 \leqslant i \leqslant \ell$, the line bundle $\mathscr{L}_i$ over E as the pullback of $\mathcal{O}_{\mathbb{P}(V^i)}(1)$ along pr^i (restricted to E), and we call the tuple

$$\mathrm{T}(M_A) \coloneqq (E, \sigma_1, \ldots, \sigma_\ell, \mathscr{L}_1, \ldots, \mathscr{L}_\ell)$$

the *flag tuple* associated to A.

3. BRAIDED TUPLES AND RECONSTRUCTION OF SHAPE ALGEBRAS

Chan [10] has given a construction in the opposite direction, starting from a scheme E, automorphisms $\sigma_1, \ldots, \sigma_\ell$ of E, and line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_\ell$ over E (satisfying some compatibility conditions; see Definition 3.1), and building a P^+-graded algebra from these data. Let us briefly recall his construction. To improve legibility, we will write $\mathscr{L}^\sigma$ for the pullback of a line bundle $\mathscr{L}$ along a map σ.

DEFINITION 3.1. We call a tuple $T = (E, \sigma_1, \ldots, \sigma_\ell, \mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ as above a *braided tuple* if

(a) the σ_i pairwise commute,

(b) for every $i > j$, there exists an equivalence $R_{ij} : \mathscr{L}_i \otimes \mathscr{L}_j^{\sigma_i} \xrightarrow{\sim} \mathscr{L}_j \otimes \mathscr{L}_i^{\sigma_j}$ of line bundles such that the braid relation

$$(R_{jk} \otimes \mathrm{id})(\mathrm{id} \otimes R_{ik}^{\sigma_j})(R_{ij} \otimes \mathrm{id}) = (\mathrm{id} \otimes R_{ij}^{\sigma_k})(R_{ik} \otimes \mathrm{id})(\mathrm{id} \otimes R_{jk}^{\sigma_i}) \tag{3.1}$$

holds for every $i > j > k$ (both sides being equivalences $\mathscr{L}_i \otimes \mathscr{L}_j^{\sigma_i} \otimes \mathscr{L}_k^{\sigma_i \sigma_j} \xrightarrow{\sim} \mathscr{L}_k \otimes \mathscr{L}_j^{\sigma_k} \otimes \mathscr{L}_i^{\sigma_k \sigma_j}$).

Note that if we set $R_{ii} \coloneqq \mathrm{id}$ for all i and $R_{ji} \coloneqq R_{ij}^{-1}$ for all $i > j$, then (3.1) becomes true for all i, j, k.

If $\lambda \in P^+$ decomposes as $\lambda = \sum a_i \varpi_i$, then define $\sigma_\lambda \coloneqq \sigma_1^{a_1} \ldots \sigma_\ell^{a_\ell}$, as before (so $\sigma_{\lambda+\mu} = \sigma_\lambda \sigma_\mu$). Define a line bundle $\mathscr{L}_\lambda$ over E by the following inductive rules (with $\mathscr{L}_0$ the trivial bundle):

$$\mathscr{L}_{\varpi_i} = \mathscr{L}_i, \qquad 1 \leqslant i \leqslant \ell,$$

$$\mathscr{L}_{\lambda+\mu} = \mathscr{L}_\lambda \otimes \mathscr{L}_\mu^{\sigma_\lambda}. \tag{3.2}$$

As is shown in [10], this procedure is, thanks to (3.1), well defined up to unique equivalences of line bundles built from the R_{ij} (cf. also the proof of Proposition 1.3). Now define the product of two section, $\alpha \in \mathrm{H}^0(E, \mathscr{L}_\lambda)$ and $\beta \in \mathrm{H}^0(E, \mathscr{L}_\mu)$ by

$$\alpha\beta \coloneqq \alpha \otimes \beta^{\sigma_\lambda} \in \mathrm{H}^0(E, \mathscr{L}_{\lambda+\mu}). \tag{3.3}$$

THEOREM 3.2 (Chan [10]). *The product rule* (3.3) *turns the direct sum*

$$B(T) := \bigoplus_{\lambda \in P^+} \mathrm{H}^0(E, \mathscr{L}_\lambda)$$

into an associative P^+-graded algebra.

The algebra $B(T)$ is not quadratic in general, so we consider its quadratic cover

$$M(T) := B(T)^{(2)}.$$

(If B is any $\mathbf{N}$-graded algebra, we define its *quadratic cover* $B^{(2)}$ as follows: consider the canonical homomorphism $\mathrm{T}(B_1) \to B$ and its kernel $J = \oplus_{k \geqslant 2} J_k$, then set $B^{(2)} := \mathrm{T}(B_1)/(J_2)$. Here we view $B(T)$ as an $\mathbf{N}$-graded algebra via the height function $h(\lambda)$.)

The quadratic algebra $M(T)$ may also be described more directly in terms of the braided tuple T, as follows. For each $1 \leqslant i \leqslant \ell$, set $V_i := \mathrm{H}^0(E, \mathscr{L}_i)$, denote by V^i the dual of V_i, and consider the map $\mathrm{Pl}^i : E \to \mathbb{P}(V^i)$ corresponding to the line bundle $\mathscr{L}_i$. For every $1 \leqslant i, j \leqslant \ell$, the map $\mathrm{Pl}^i \boxtimes (\mathrm{Pl}^j)^{\sigma_i}$ corresponding to the line bundle $\mathscr{L}_i \otimes \mathscr{L}_j^{\sigma_i}$ is then given by the composite

$$E \overset{\text{diag.}}{\to} E \times E \overset{\mathrm{id} \times \sigma_i}{\to} E \times E \overset{\mathrm{Pl}^i \times \mathrm{Pl}^j}{\to} \mathbb{P}(V^i) \times \mathbb{P}(V^j) \overset{\text{Segre}}{\to} \mathbb{P}(V^i \otimes V^j). \quad (3.4)$$

Denote by Γ^{ij} the image of this map and by $K_{ij} \subset V_i \otimes V_j$ the subspace of linear forms on $V^i \otimes V^j$ vanishing on Γ^{ij}.

For every $1 \leqslant i, j \leqslant \ell$, Definition 3.1 implies that there exists a linear isomorphism $R^{ji} : V^j \otimes V^i \to V^i \otimes V^j$ such that the following diagram commutes:

$$\begin{tikzcd} & E \times E \arrow[r, "\mathrm{id} \times \sigma_i"] & E \times E \arrow[r, "\mathrm{Pl}^i \times \mathrm{Pl}^j"] & \mathbb{P}(V^i) \times \mathbb{P}(V^j) \arrow[r, "\text{Segre}"] & \mathbb{P}(V^i \otimes V^j) \\ E \arrow[ur, "\text{diag.}"] \arrow[dr, "\text{diag.}"'] & & & & \\ & E \times E \arrow[r, "\mathrm{id} \times \sigma_j"'] & E \times E \arrow[r, "\mathrm{Pl}^j \times \mathrm{Pl}^i"'] & \mathbb{P}(V^j) \times \mathbb{P}(V^i) \arrow[r, "\text{Segre}"'] & \mathbb{P}(V^j \otimes V^i). \arrow[uu, "\mathbb{P}(R^{ji})"'] \end{tikzcd} \quad (3.5)$$

Let $R_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ be the transpose of R^{ji}. It is clear that modulo K_{ij} and K_{ji}, the map R_{ij} is unique up to a scalar.

The algebra $M(T)$ is then generated by $V_1 \oplus \cdots \oplus V_\ell$, with relations given by $(\mathrm{I})_{ij}$ and $(\mathrm{II})_{ij}$ for all $1 \leqslant i, j \leqslant \ell$ (see the end of Section 1; Remark 1.5 still applies).

QUESTION E. What is the kernel of the canonical morphism $M(T) \to B(T)$?

Having constructed the algebra $M(T)$ from a braided tuple T, we may formulate a converse to Question D:

QUESTION F. Assume that A is a quantum G such that the shape algebra M_A is quadratic. Does there exist a braided tuple T such that $M_A = M(T)$?

This Question is a priori weaker than Question D, for the following reason. If M_A is quadratic and does admit a flag tuple T as in Question D, then the reconstructed algebra $M(T)$ is canonically isomorphic to M_A. However, we might also have $M(T') = M_A$ for some subtuple T' of T (i.e. a subscheme E' of E stabilized by each σ_i, with σ_i' and $\mathscr{L}_i'$ the obvious restrictions).

PROBLEM G. Given a simple complex group G, characterize the flag tuples of all quantum G's intrinsically (i.e. as braided tuples).

For $G = \mathrm{SL}(2)$, this is elementary: E must be the projective line $\mathbb{P}^1$, σ can be an arbitrary automorphism of infinite order, and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(1)$. The three possible forms of σ correspond to three different quantum SL(2)'s, namely, $\mathscr{O}(\mathrm{SL}(2))$ (when $\sigma = \mathrm{id}$), the standard Drinfel'd–Jimbo quantum SL(2) for q not a root of unity (when σ has two fixed points), and the Jordanian quantum SL(2) [12] (when σ has one fixed point). These are known [43] to be the only quantum SL(2)'s (in the sense of Definition 1.1). The associated shape algebras are $\mathbf{C}\langle x, y\rangle/(xy - yx)$, $\mathbf{C}\langle x, y\rangle/(xy - q\,yx)$, and $\mathbf{C}\langle x, y\rangle/(xy - yx - y^2)$, respectively.

PROBLEM H. Reconstruct not only a shape algebra, but a quantum G itself from a braided tuple satisfying the conditions found in Problem G.

PART II. A CONJECTURAL FLAG TUPLE FOR THE STANDARD DRINFEL'D–JIMBO QUANTUM GROUPS

In this part, we describe ingredients for a potential braided tuple, and we conjecture that these geometric data provide positive answers to Questions F and D for the standard quantum groups of Drinfel'd and Jimbo. (The conjecture concerning Question F will be proved for SL(n) in Part III.)

Again, G will denote a simply-connected simple complex group.

4. RECOLLECTIONS ON $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{g})$ AND $\mathcal{O}_q^{\mathrm{DJ}}(G)$

Let $\mathfrak{g}$ be the Lie algebra of G. Drinfel'd [14] and Jimbo [22] have defined (independently) a Hopf algebra $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{g})$ that depends on a parameter $q \in \mathbf{C}^*$ and that is a "quantum analogue" of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ (in the sense that its comultiplication is no longer cocommutative). When q is not a root of unity, finite-dimensional $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{g})$-modules have been studied (independently) by Lusztig [30] and by Rosso [34]: in particular, discarding unwanted nontrivial one-dimensional modules, there still exists a family $\{V_\lambda \mid \lambda \in P^+\}$ of $\mathrm{U}_q(\mathfrak{g})$-modules satisfying conditions (a) and (c) of Definition 1.1 (condition (c) follows e.g. from Theorem 4.12(b) of [30]).

Therefore, if $\mathcal{O}_q^{\mathrm{DJ}}(G)$ denotes the subspace of $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{g})^*$ spanned by the matrix coefficients of the modules V_λ, then $\mathcal{O}_q^{\mathrm{DJ}}(G)$ (for q not a root of unity) is a quantum G in the sense of Definition 1.1. (If G is not assumed to be simply connected, then $\mathcal{O}_q^{\mathrm{DJ}}(G)$ may still be defined in this way, provided P^+ is replaced by the appropriate submonoid.) We call $\mathcal{O}_q^{\mathrm{DJ}}(G)$ the *standard* quantum G. When $G = \mathrm{SL}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(n)$, a presentation of $\mathcal{O}_q^{\mathrm{DJ}}(G)$ by generators and relations has been given by Faddeev *et al.* [15].

5. RECOLLECTIONS FROM [33]

Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and let $W := \mathrm{N}_G(T)/T$ be the associated Weyl group. Denote by Φ and Φ^+ the root system and the set of positive roots, respectively. To each $\alpha \in \Phi$ are associated a reflection $s_\alpha \in W$, a root group U_α, and a copy $L_\alpha = \langle U_\alpha, U_{-\alpha}\rangle$ of (P)SL(2) in G.

Recall the following construction from [33]: an *orthocell* (of rank d) is a left coset in W of the form

$$C = C(w; \alpha_1, \ldots, \alpha_d) := w\langle s_{\alpha_1}, \ldots, s_{\alpha_d}\rangle,$$

where $w \in W$ and $\alpha_1, \ldots, \alpha_d$ are positive and pairwise orthogonal roots.

Warning: the α_k are *not* assumed to be *strongly* orthogonal, i.e. the sum of two of them may well be a root.

By orthogonality, the reflections $s_{\alpha_1}, \ldots, s_{\alpha_d}$ pairwise commute. Therefore, the following notation makes sense, and we will use it frequently:

$$s_L := \prod_{k \in L} s_{\alpha_k}, \qquad L \subset \{1, \ldots, d\}.$$

(Note that the elements of C are those of the form ws_L.)

Reordering the sequence $\alpha_1, \ldots, \alpha_d$ if necessary, assume that it is *nonincreasing*, in the sense that $\alpha_k \not< \alpha_{k'}$ for all $k < k'$; then define

$$E(C) := \{\dot{w} g_1 \ldots g_d B \mid g_k \in L_{\alpha_k} \ \forall k\} \subset G/B,$$

where $\dot{w} \in \mathrm{N}_G(T)$ is some representative of w. In [33], we show that $E(C)$ only depends on C as a coset (and not on the choice of w in C, nor of its representative $\dot{w}$, nor on the chosen nonincreasing ordering of the α_k). Furthermore, we show that $E(C)$ is a T-stable subvariety of G/B, isomorphic to the product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of d projective lines.

Remark 5.1. Orthocells may also be defined in terms of *right* cosets: if we set

$$C(\alpha_1, \ldots, \alpha_d; w) := \langle s_{\alpha_1}, \ldots, s_{\alpha_d} \rangle w,$$

then $C(\alpha_1, \ldots, \alpha_d; w) = C(w; w^{-1}\alpha_1, \ldots, w^{-1}\alpha_d)$. Moreover, recall (see [36], end of Section 9.2.1) that for each $w \in W$ and each root α, we have $wU_\alpha w^{-1} = U_{w\alpha}$, and hence $wL_\alpha w^{-1} = L_{w\alpha}$. It follows that for $C = C(\alpha_1, \ldots, \alpha_d; w)$, we have

$$E(C) = \{g_1 \ldots g_d w B \mid g_k \in L_{\alpha_k} \ \forall k\}.$$

6. MONOGRESSIVE ORTHOCELLS AND THE VARIETY E^{DJ}

Denote by $<$ the Bruhat order on W and by $\lessdot$ the associated cover relation (i.e. $w \lessdot w'$ if $w < w'$ and if no element of W lies between w and w'). Denote also by $\ell(w)$ the length of an element $w \in W$. Recall the following combinatorial characterization (see e.g. [20, Sect. 5.9, 5.11]):

$$w \lessdot w' \ \Leftrightarrow \ell(w') = \ell(w) + 1 \text{ and } w' = ws \text{ for some reflection } s.$$

Assume that $w \in C$ has been chosen of minimal length.

DEFINITION 6.1. An orthocell $C = C(w; \alpha_1, \ldots, \alpha_d)$ will be called *monogressive* if

$$ws_L \lessdot ws_L s_{\alpha_k} \qquad \forall L \subset \{1, \ldots, d\}, \quad \forall k \notin L,$$

or, equivalently, if

$$\ell(ws_L) = \ell(w) + |L| \qquad \forall L \subset \{1, \ldots, d\}.$$

We then define the variety $E^{\mathrm{DJ}} \subset G/B$ by

$$E^{\mathrm{DJ}} := \bigcup_{C \text{ monogressive}} E(C).$$

7. THE AUTOMORPHISMS $\sigma_1, \ldots, \sigma_\ell$

Let $\beta_1, \ldots, \beta_\ell$ be the simple roots. Then the morphism

$$T \to (\mathbf{C}^*)^\ell : t \mapsto (\beta_1(t), \ldots, \beta_\ell(t))$$

is surjective, so we may choose, for each $1 \leqslant i \leqslant \ell$, an element $t_i \in T$ such that

$$\beta_j(t_i) = q^{-(\varpi_i|\beta_j)} = \begin{cases} q^{-(\beta_j|\beta_j)/2} & \text{if } j = i, \\ 1 & \text{if } j \neq i. \end{cases}$$

By (multiplicative) linearity, it then follows that $\alpha(t_i) = q^{-(\varpi_i|\alpha)}$ for every root $\alpha \in \Phi$.

Now let $C = C(w; \alpha_1, \ldots, \alpha_d)$ be a monogressive orthocell. Since $E(C)$ is T-stable in G/B, the automorphism

$$\sigma_{i,C} : E(C) \to E(C) : gB \mapsto wt_iw^{-1}gB$$

is well defined, and it is independent of the choice of t_i because the kernel of the above morphism $T \to (\mathbf{C}^*)^\ell$ is equal to the centre of G (see e.g. [36, Proposition 8.1.1]).

PROPOSITION 7.1. *For each i, the automorphisms $\sigma_{i,C}$ glue together to form a well defined automorphism σ_i of E^{DJ}.*

We defer the proof to Appendix B.

8. THE LINE BUNDLES $\mathscr{L}_1, \ldots, \mathscr{L}_\ell$

Recall that for each $\lambda \in P^+$, the highest weight point in $\mathbb{P}(V^\lambda)$ is fixed by B, hence we get a well-defined *Plücker map*

$$\mathrm{Pl}^\lambda : G/B \to \mathbb{P}(V^\lambda).$$

Let $\varpi_1, \ldots, \varpi_\ell$ be the fundamental weights and write $\mathrm{Pl}^i := \mathrm{Pl}^{\varpi_i}$ for each i.

We then define the a line bundle $\mathscr{L}_i$ as the pullback of $\mathcal{O}_{\mathbb{P}(V^i)}(1)$ along Pl^i, restricted to E^{DJ}.

Warning. We may not define $\mathscr{L}_\lambda$ to be the pullback of $\mathcal{O}_{\mathbb{P}(V^\lambda)}(1)$ for all $\lambda \in P^+$: this would cause a conflict with the recursion rule (3.2).

9. MAIN CONJECTURES AND RESULT

Conjecture 9.1 (Positive Answer to Question F). *Assume that $q \in \mathbf{C}^*$ is not a root of unity. The tuple $T^{\mathrm{DJ}} = (E^{\mathrm{DJ}}, \sigma_1, \dots, \sigma_\ell, \mathscr{L}_1, \dots, \mathscr{L}_\ell)$ defined in Section* 6–8 *is a braided tuple* (*see Definition* 3.1), *and the associated quadratic algebra $M(T^{\mathrm{DJ}})$ is the shape algebra of the standard quantum group $\mathcal{O}_q^{\mathrm{DJ}}(G)$.*

Conjecture 9.2 (Positive Answer to Question D). *Moreover, the same tuple T^{DJ} is the flag tuple associated to $\mathcal{O}_q^{\mathrm{DJ}}(G)$ (i.e. E^{DJ} parametrizes* all *flag modules of the shape algebra of $\mathcal{O}_q^{\mathrm{DJ}}(G)$).*

Theorem 9.3. *Conjecture* 9.1 *is true for $G = \mathrm{SL}(n)$.*

PART III. THE STANDARD QUANTUM SL(n)

In this part, we will describe the objects of Sections 5–7 more explicitly when $G = \mathrm{SL}(n)$, and we prove Conjecture 9.1 in that case.

10. THE VARIETIES $E(C)$

From now on, it will be more convenient to view orthocells as *right* cosets (see Remark 5.1).

Let us first recall the usual realization of the flag variety $\mathrm{SL}(n)/B$, of the Plücker maps Pl^i, and of the subgroups L_α.

We let $B \subset \mathrm{SL}(n)$ be the subgroup of all upper triangular matrices, i.e. the stabilizer of the flag

$$\mathbf{C}e_1 \subset \mathbf{C}e_1 \oplus \mathbf{C}e_2 \subset \cdots \subset \mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_{n-1},$$

where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbf{C}^n$. This identifies $\mathrm{SL}(n)/B$ with the set of all (full) flags in $\mathbf{C}^n$ (or in $\mathbb{P}^{n-1} := \mathbb{P}(\mathbf{C}^n)$).

We also let $T \subset B$ be the subgroup of all diagonal matrices: the Weyl group W then identifies with the symmetric group S_n, and the reflections correspond exactly to the transpositions.

For each $1 \leqslant i \leqslant n-1$, recall that the fundamental representation $V^i := V^{\varpi_i}$ is given by the exterior power $\Lambda^i \mathbf{C}^n$, and that the map $\mathrm{Pl}^i : \mathrm{SL}(n)/B \to \mathbb{P}(\Lambda^i \mathbf{C}^n)$ may be described as follows: given a flag $F \in \mathrm{SL}(n)/B$, choose a basis $f_1, \ldots, f_i$ of its component F_i of dimension i, then send F to the point $f_1 \wedge \cdots \wedge f_i \in \mathbb{P}(\Lambda^i \mathbf{C}^n)$ (which is independent of the choice of the basis). Moreover, the elements

$$e^i_w := e_{w(1)} \wedge \cdots \wedge e_{w(i)}, \qquad w \in S_n,$$

form a basis of $\Lambda^i \mathbf{C}^n$ (up to obvious redundancies).

Let $\alpha \in \Phi^+$ and write $s_\alpha = (a\,b)$, $1 \leqslant a < b \leqslant n$. Then the subgroup $L_\alpha \subset \mathrm{SL}(n)$ is the group $\mathrm{SL}(2)$ acting naturally on $\mathbf{C}e_a \oplus \mathbf{C}e_b$ and trivially on all other e_c. Clearly, if s_α, s_β commute (i.e. if α, β are orthogonal), then so do L_α and L_β. (This is not true for arbitrary G.)

Now fix an orthocell $C = C(\alpha_1, \ldots, \alpha_d; w)$ and let us describe the variety $E(C)$, or rather, its images under the maps Pl^i, $1 \leqslant i \leqslant n-1$.

Remark 10.1. For each $1 \leqslant k \leqslant d$, the following conditions are equivalent:

- $s_{\alpha_k} w \varpi_i = w \varpi_i$,
- the transposition s_{α_k} leaves the set $\{w(1), \ldots, w(i)\}$ invariant,
- $e^i_{s_{\alpha_k} w} = \pm e^i_w$.

Number the α_k in such a way that for some $1 \leqslant a \leqslant d$, the above conditions hold for $1 \leqslant k \leqslant a$ and do not hold for $a+1 \leqslant k \leqslant d$. For each k, write $s_{\alpha_k} = (a_k\, b_k)$, $a_k < b_k$, and pick an element $g_k \in L_{\alpha_k}$ acting as $\begin{pmatrix} x_k & * \\ y_k & * \end{pmatrix}$ on $\mathbf{C}e_{a_k} + \mathbf{C}e_{b_k}$ (and trivially on the other e_c). For any subset $L \subset \{1, \ldots, a\}$, write

$$\bar{L} := \{1, \ldots, a\} \backslash L, \qquad x_L := \prod_{k \in L} x_k, \qquad y_L := \prod_{k \in L} y_k.$$

The above description of the map Pl^i and of the subgroups L_{α_k} now imply that

$$\mathrm{Pl}^i(g_1 \ldots g_d w B) = \sum_{L \subset \{1, \ldots, a\}} x_{\bar{L}} y_L \, e^i_{s_L w} \in \mathbb{P}(\Lambda^i \mathbf{C}^n). \tag{10.1}$$

Remark 10.2. The variety $E(C)$ being a product of d projective lines, we may view $(x_1 : y_1), \ldots, (x_d : y_d)$ as homogeneous coordinates on these lines.

A more geometric description of the varieties $E(C)$ (not needed here) can be found in [33, Example 5.1].

11. MONOGRESSIVITY

Let us first recall a more explicit description of the Bruhat cover relation in S_n. Write a permutation $w \in S_n$ as an array $[w(1) \ldots w(n)]$, and write e.g. $w = [\cdot a \cdot b \cdot c \cdot]$ to signify that in the array w, a appears to the left of b and b appears to the left of c.

If $s \in S_n$ is a transposition, say $s = (a\, b)$ with $a < b$, then $w \lessdot sw$ if and only if (i) $w = [\cdot a \cdot b \cdot]$ and (ii) whenever $w = [\cdot a \cdot c \cdot b \cdot]$, c is outside of the (numerical) interval $[a, b]$. For example, if $n = 7$ and $s = (4\, 6)$, then we have $[3472651] \lessdot [3672451]$, but $[7415623] \not\lessdot [7615423]$ because the subarray $[4156]$ contains 5.

Now let $C = C(\alpha_1, \ldots, \alpha_d; w)$ be an orthocell, and write again $s_{\alpha_k} = (a_k\, b_k)$, $a_k < b_k$, for all k. The above description of the Bruhat cover relation shows the following

CRITERION 11.1. *With the above notation, the orthocell C is monogressive if and only if the following conditions hold for all k:*

- $w = [\cdot a_k \cdot b_k \cdot]$,
- *whenever* $w = [\cdot a_k \cdot c \cdot b_k \cdot]$, *$c$ is outside of the interval* $[a_k, b_k]$,
- *whenever* $w = [\cdot a_k \cdot a_{k'} \cdot b_k \cdot]$ *for some* $k' \neq k$, *both* $a_{k'}$ *and* $b_{k'}$ *are outside of the interval* $[a_k, b_k]$, *and similarly whenever* $w = [\cdot a_k \cdot b_{k'} \cdot b_k \cdot]$.

EXAMPLE 11.2 ($n = 4$). There are 58 monogressive orthocells of rank 1, viz. those of one of the following forms:

- $\{[ijkl], [jikl]\}$, $\{[kijl], [kjil]\}$, or $\{[klij], [klji]\}$ $(i < j)$,
- $\{[ikjl], [jkil]\}$ or $\{[likj], [ljki]\}$ $(i < j;\ k \notin [i, j])$,
- $\{[iklj], [jkli]\}$ $(i < j;\ k, l \notin [i, j])$.

There are 11 monogressive orthocells $C(\alpha_1, \alpha_2; w)$ of rank 2, given by

- $s_{\alpha_1} = (1\, 2)$, $s_{\alpha_2} = (3\, 4)$, $w = [1234]$, $[3412]$, $[1324]$, $[3142]$, $[1342]$, or $[3124]$;
- $s_{\alpha_1} = (1\, 3)$, $s_{\alpha_2} = (2\, 4)$, $w = [1324]$ or $[2413]$;
- $s_{\alpha_1} = (1\, 4)$, $s_{\alpha_2} = (2\, 3)$, $w = [1423]$, $[2314]$, or $[2143]$.

(Pictures for the corresponding varieties $E(C)$ may be found in [33, Example 5.1].)

EXAMPLE 11.3 ($n = 3$). View $\mathrm{SL}(3)/B$ as the set of flags (p, l) in $\mathbb{P}^2 = \mathbb{P}(\mathbf{C}^3)$. Consider e_1, e_2, e_3 as points in $\mathbb{P}^2$ and let $e_{ab} \subset \mathbb{P}^2$ be the line through

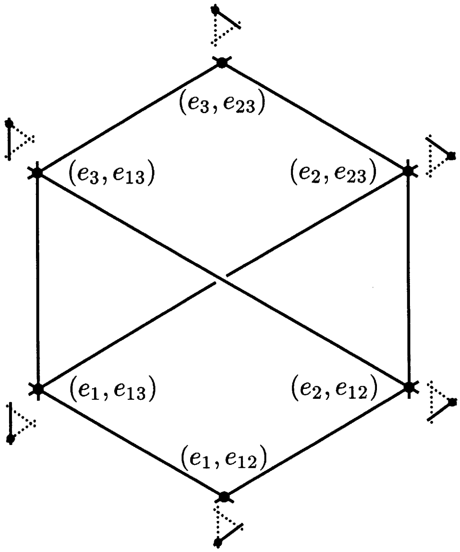

FIG. 1. The subvariety E^{DJ} in $\mathrm{SL}(3)/B$. Its eight irreducible components intersect in six points. Next to each point is a small picture, viewing it as a flag in $\mathbb{P}^2$. The "missing" diagonal corresponds to the orthocell $\{[123],[321]\}$, which is not monogressive.

e_a and e_b. Then E^{DJ} is the union of the following eight curves in $\mathrm{SL}(3)/B$:

$$\begin{aligned}
&\{(e_a, l) \mid e_a \in l\}, \quad a = 1,2,3,\\
&\{(p, e_{ab}) \mid p \in e_{ab}\}, \quad ab = 12, 13, 23,\\
&\{(p, l) \mid e_{12} \ni p \in l \ni e_3\},\\
&\{(p, l) \mid e_{23} \ni p \in l \ni e_1\}.
\end{aligned}$$

See Fig. 1.

12. THE AUTOMORPHISMS σ_i

Denote again the simple roots by $\beta_1, \ldots, \beta_{n-1}$. If $t = \mathrm{diag}(x_1, \ldots, x_n) \in T$ (with $\prod_j x_j = 1$), then recall that $\beta_i(t) = x_i x_{i+1}^{-1}$. Therefore, $t_i \in T$ is equal, up to a factor, to the matrix $\mathrm{diag}(1, \ldots, 1, q, \ldots, q)$ (i times 1 and $n-i$ times q).

If $C = C(\alpha_1, \ldots, \alpha_d; w)$, with $s_{\alpha_k} = (a_k\, b_k)$ as before, then the action of the associated automorphism $\sigma_i : gB \mapsto w t_i w^{-1} gB$ on $E(C)$ may be described more explicitly using the homogeneous coordinates $(x_1 : y_1), \ldots, (x_d : y_d)$ of Remark 10.2:

$$\sigma_i : (x_k : y_k) \mapsto \begin{cases} (x_k : q y_k) & \text{if } s_{\alpha_k} w \varpi_i \neq w \varpi_i, \\ (x_k : y_k) & \text{if } s_{\alpha_k} w \varpi_i = w \varpi_i. \end{cases}$$

EXAMPLE 12.1 ($n = 3$). On each of the eight components of E^{DJ} (see Example 11.3), σ_1, σ_2 act as homotheties (viewing the two T-stable points on this component as 0 and ∞). The ratios for σ_1 are, respectively, $1, 1, 1, q, q, q, q, q$, and those for σ_2 are $q, q, q, 1, 1, 1, q, q$.

13. PROOF OF CONJECTURE 9.1 FOR $G = \mathrm{SL}(n)$

For each $1 \leqslant i \leqslant n-1$, consider the vector space $V^i := \Lambda^i \mathbf{C}^n$. On one hand, $\mathrm{SL}(n)$ acts on it naturally, and the corresponding map $\mathrm{SL}(n)/B \to \mathbb{P}(V^i)$ induces a line bundle $\mathscr{L}_i$ on $E^{\mathrm{DJ}} \subset \mathrm{SL}(n)/B$ (see Section 8). On the other hand, we will make $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$ act on V^i (see below, before Lemma 13.4), turning V^i into the simple $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-module of highest weight ϖ_i.

Both the algebra $M(T^{\mathrm{DJ}})$ and the shape algebra M^{DJ} thus become quotients of the tensor algebra $T(V_1 \oplus \cdots \oplus V_{n-1})$ (where $V_i := (V^i)^*$). Note also that $M(T^{\mathrm{DJ}})$ is quadratic by definition, and M^{DJ} is quadratic by Taft and Towber [40]. So we need to show that relations of types (I) and (II) (see end of Section 1) agree for both algebras (and, of course, that the tuple T^{DJ} is braided in the first place).

We will break down the proof into several lemmas.

DEFINITION 13.1. Let $1 \leqslant i, j \leqslant n-1$. An orthocell $C = C(\alpha_1, \ldots, \alpha_d; w)$ will be called *ij-effective* if, for every $1 \leqslant k \leqslant d$, we have both $s_{\alpha_k} w\varpi_i \neq w\varpi_i$ and $s_{\alpha_k} w\varpi_j \neq w\varpi_j$. In this case, we define the following element of $V^i \otimes V^j$:

$$e_C^{ij} := \sum_{L \subset \{1,\ldots,d\}} q^{|L|} e^i_{s_{\bar{L}}w} \otimes e^j_{s_L w},$$

where, as before, $s_L := \prod_{k \in L} s_{\alpha_k}$ and $\bar{L} := \{1, \ldots, d\} \backslash L$.

We denote by $V^{ij} \subset V^i \otimes V^j$ the linear span of the image of the map $\mathrm{Pl}^i \boxtimes (\mathrm{Pl}^j)^{\sigma_i}$ (see (3.4)).

LEMMA 13.2. *The subspace V^{ij} is linearly spanned by the e_C^{ij} for C monogressive and ij-effective.*

Proof. First, let $C = C(\alpha_1, \ldots, \alpha_d; w)$ be monogressive and ij-effective. If a point $p \in E(C)$ has homogeneous coordinates $(x_1 : y_1), \ldots, (x_d : y_d)$ (see Remark 10.2), then by ij-effectiveness, the coordinates of $\sigma_i(p)$ are $(x_1 : qy_1), \ldots, (x_d : qy_d)$. Using (10.1), we therefore see that $\mathrm{Pl}^i \boxtimes (\mathrm{Pl}^j)^{\sigma_i}$ sends p to the following point in $\mathbb{P}(V^i \otimes V^j)$:

$$\sum_{L, M \subset \{1,\ldots,d\}} q^{|M|} x_{\bar{L}} x_{\bar{M}} y_L y_M \, e^i_{s_L w} \otimes e^j_{s_M w}.$$

Using the change of "variables" $I := L \triangle M$ (symmetric difference), $J := L \cap M$, and $N := M \backslash L = M \backslash J \subset I$, this expression may be rewritten as

$$\begin{aligned}&\sum_{\substack{I,J\subset\{1,\ldots,d\}\\ I\cap J=\emptyset}} q^{|J|}(x_{\overline{I\cup J}})^2 x_I y_I (y_J)^2 \left(\sum_{N\subset I} q^{|N|} e^i_{s_{\bar{N}} s_J w} \otimes e^j_{s_N s_J w} \right)\\ &= \sum_{\substack{I,J\subset\{1,\ldots,d\}\\ I\cap J=\emptyset}} q^{|J|}(x_{\overline{I\cup J}})^2 x_I y_I (y_J)^2 \, e^{ij}_{C(\alpha_I; s_J w)},\end{aligned}$$

where α_I is shorthand for the set $\{\alpha_k \mid k \in I\}$. By induction over d, we may assume that $e^{ij}_{C'} \in V^{ij}$ for all monogressive ij-effective orthocells C' of rank smaller than d (the case $d = 0$ being trivial). Since the above sum is in V^{ij} by definition, the only remaining term, namely e^{ij}_C, is in V^{ij} as well.

We still need to show that the image of a point $p \in E(C)$ is in the span of the $e^{ij}_{C'}$ (for C' monogressive and ij-effective) even if C is not ij-effective (but still monogressive). Reordering the α_k if necessary, we may assume that, for some $1 \leqslant a \leqslant b \leqslant c \leqslant d$, they satisfy

$$\begin{array}{lll} s_{\alpha_k} w\varpi_i \neq w\varpi_i, & s_{\alpha_k} w\varpi_j \neq w\varpi_j & \text{if } 1 \leqslant k \leqslant a; \\ s_{\alpha_k} w\varpi_i \neq w\varpi_i, & s_{\alpha_k} w\varpi_j = w\varpi_j & \text{if } a+1 \leqslant k \leqslant b; \\ s_{\alpha_k} w\varpi_i = w\varpi_i, & s_{\alpha_k} w\varpi_j \neq w\varpi_j & \text{if } b+1 \leqslant k \leqslant c; \\ s_{\alpha_k} w\varpi_i = w\varpi_i, & s_{\alpha_k} w\varpi_j = w\varpi_j & \text{if } c+1 \leqslant k \leqslant d. \end{array}$$

Let again $(x_1 : y_1), \ldots, (x_d : y_d)$ be homogeneous coordinates for a point $p \in E(C)$. This time, the coordinates for $\sigma_i(p)$ are obtained by multiplying y_k by q only for $1 \leqslant k \leqslant b$. Furthermore, we have

$$\mathrm{Pl}^i(p) = \sum_{\substack{L\subset\{1,\ldots,a\}\\ L'\subset\{a+1,\ldots,b\}}} x_{\bar{L}} x_{\bar{L}'} y_L y_{L'} \, e^i_{s_L s_{L'} w},$$

and a similar expression for $\mathrm{Pl}^j(p)$, with $\{a+1, \ldots, b\}$ replaced by $\{b+1, \ldots, c\}$. A computation similar to the one above shows that $\mathrm{Pl}^i \boxtimes (\mathrm{Pl}^j)^{\sigma_i}$ now sends p to

$$\sum_{\substack{I,J\subset\{1,\ldots,a\}\\ I\cap J=\emptyset\\ L'\subset\{a+1,\ldots,b\}\\ M'\subset\{b+1,\ldots,c\}}} q^{|J|}(x_{\overline{I\cup J}})^2 x_I x_{\bar{L}'} x_{\bar{M}'} y_I (y_J)^2 y_{L'} y_{M'} \, e^{ij}_{C(\alpha_I; s_J s_{L'} s_{M'} w)} \tag{13.1}$$

(where we have used the fact that $e^i_w = e^i_{s_{M'} w}$ and $e^j_w = e^j_{s_{L'} w}$). ∎

LEMMA 13.3. *Let* $1 \leqslant i,j \leqslant n-1$, *with, say,* $i<j$. *Then*

$$\dim V^{ij} \leqslant D_{n;i,j} := \binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j+1}.$$

Proof. Consider a monogressive ij-effective cell $C = C(\alpha_1, \dots, \alpha_d; w)$. For each $1 \leqslant k \leqslant d$, write again $s_{\alpha_k} = (a_k\, b_k)$, $a_k < b_k$ (hence $w = [\cdot a_k \cdot b_k \cdot]$ by monogressivity). Reorder the α_k in such a way that $b_1 < \cdots < b_d$. By ij-effectiveness, each a_k must appear in the subarray $[w(1) \dots w(i)]$, and each b_k in the subarray $[w(j+1) \dots w(n)]$. Now let $S_{ij} := S_i \times S_{j-i} \times S_{n-(i+j)} \subset S_n$, and note that for each $\pi \in S_{ij}$, replacing w by $w\pi$ in C leaves e_C^{ij} invariant up to a sign. Choosing π appropriately, we may assume that w takes the following form:

$$[w(1) \dots w(i-d)\, a_1 \dots a_d\, w(i+1) \dots w(j)\, b_d \dots b_1\, w(j+d+1) \dots w(n)],$$

with, say, the following orderings:

$$w(1) > \cdots > w(i-d), \qquad w(i+1) > \cdots > w(j),$$

$$w(j+d+1) > \cdots > w(n).$$

This rearrangement does not affect the monogressivity of C (nor, for that matter, its ij-effectiveness). Indeed, the only nonobvious point here is the relative ordering of the a_k and the b_k: by monogressivity, we have $w \lessdot s_{\alpha_k} w \lessdot s_{\alpha_k} s_{\alpha_{k'}} w$ and $w \lessdot s_{\alpha_{k'}} w \lessdot s_{\alpha_k} s_{\alpha_{k'}} w$, so if $k < k'$, then, whatever the order in which $a_k, a_{k'}, b_k, b_{k'}$ appear in the original array $[w(1) \dots w(n)]$, we must have either $a_k < b_k < a_{k'} < b_{k'}$, or $a_{k'} < a_k < b_k < b_{k'}$. In both cases, $a_{k'}$ and $b_{k'}$ are outside of the (numerical) interval $[a_k, b_k]$, so they may indeed appear between a_k and b_k in the new array without affecting monogressivity.

An orthocell thus modified will be called *ij-normal*. The proof will be finished if we show that there are $D_{n;i,j}$ ij-normal orthocells in S_n. Since we have the recursion rule

$$D_{n+1;i,j} = D_{n;i-1,j-1} + D_{n;i-1,j} + D_{n;i,j-1} + D_{n;i,j},$$

it is enough to show that the number of ij-normal orthocells in S_{n+1} satisfies the same recursion rule. If C is such an orthocell, there are two possibilities.

- Either each s_{α_k} fixes $n+1$. Removing $n+1$ from the array $[w(1) \dots w(n+1)]$, we then obtain an orthocell in S_n, which is $(i-1)(j-1)$-normal, $i(j-1)$-normal, or ij-normal, according to the position of $n+1$ in the array, relative to $w(i)$ and $w(j)$.

- Or some s_{α_k} involves $n+1$: necessarily, $k=d$ and $b_d=n+1$. Removing again $n+1$ from the array, and discarding α_d from C, we then obtain an $(i-1)j$-normal orthocell in S_n.

Clearly, this procedure may be reversed, starting from a normal orthocell in S_n and inserting $n+1$ at all possible places in the corresponding array. Hence the desired recursion rule. ∎

Now let us recall a presentation for the quantized enveloping algebra $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$, following e.g. [21]: it is generated by $4(n-1)$ elements K_β, K_β^{-1}, X_β, Y_β (β a simple root), subject to the commutation relations

$$\begin{aligned} K_\beta K_\beta^{-1} &= 1 = K_\beta^{-1}K_\beta, \qquad K_\beta K_\gamma = K_\gamma K_\beta, \\ K_\beta X_\gamma K_\beta^{-1} &= q^{(\beta|\gamma)}X_\gamma, \\ K_\beta Y_\gamma K_\beta^{-1} &= q^{-(\beta|\gamma)}Y_\gamma, \\ X_\beta Y_\gamma - Y_\gamma X_\beta &= \delta_{\beta\gamma}\frac{K_\beta - K_\beta^{-1}}{q-q^{-1}}, \end{aligned}$$

as well as the quantized Serre relations

$$\begin{aligned} X_\beta^2 X_\gamma - (q+q^{-1})X_\beta X_\gamma X_\beta + X_\gamma X_\beta^2 &= 0 \qquad \text{if } \beta,\gamma \text{ adjacent}, \\ X_\beta X_\gamma - X_\gamma X_\beta &= 0 \qquad \text{if } \beta,\gamma \text{ not adjacent}, \\ Y_\beta^2 Y_\gamma - (q+q^{-1})Y_\beta Y_\gamma Y_\beta + Y_\gamma Y_\beta^2 &= 0 \qquad \text{if } \beta,\gamma \text{ adjacent}, \\ Y_\beta Y_\gamma - Y_\gamma Y_\beta &= 0 \qquad \text{if } \beta,\gamma \text{ not adjacent}. \end{aligned}$$

Moreover, $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$ is a Hopf algebra whose comultiplication is given on the generators by

$$\begin{aligned} \Delta K_\beta^{\pm 1} &= K_\beta^{\pm 1} \otimes K_\beta^{\pm 1}, \\ \Delta X_\beta &= X_\beta \otimes 1 + K_\beta \otimes X_\beta, \\ \Delta Y_\beta &= Y_\beta \otimes K_\beta^{-1} + 1 \otimes Y_\beta. \end{aligned} \tag{13.2}$$

We then define a $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-module structure on V^i as follows. For every $w \in W$ and every simple root β, we set

$$K_\beta e_w^i = q^{(w\varpi_i|\beta)}e_w^i, \qquad K_\beta^{-1}e_w^i = q^{-(w\varpi_i|\beta)}e_w^i,$$

and

$$\begin{aligned}
&X_\beta e^i_w = 0, && Y_\beta e^i_w = e^i_{s_\beta w} && \text{if } (w\varpi_i|\beta) = 1;\\
&X_\beta e^i_w = 0, && Y_\beta e^i_w = 0 && \text{if } (w\varpi_i|\beta) = 0;\\
&X_\beta e^i_w = e^i_{s_\beta w}, && Y_\beta e^i_w = 0 && \text{if } (w\varpi_i|\beta) = -1.
\end{aligned}$$

(These are the only possible values for $(w\varpi_i|\beta)$, because ϖ_i is minuscule.) It is straightforward to check that this module structure is well defined, and that it is the simple $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-module of highest weight ϖ_i.

Lemma 13.4. *The subspace V^{ij} is a $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-submodule of $V^i \otimes V^j$.*

By Lemma 13.2, the statement means that the action of a generator of $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$ on a vector e^{ij}_C, C monogressive and ij-effective, must again be a linear combination of such vectors. We postpone these rather tedious computations to Appendix C.

Corollary 13.5. *The subspace V^{ij} is equal to the (unique) $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-submodule of $V^i \otimes V^j$ of highest weight $\varpi_i + \varpi_j$, and the e^{ij}_C (for C monogressive and ij-effective) are linearly independent.*

Proof. The vector $e^{ij}_{C(-;1)} = e^i_1 \otimes e^j_1$ is a highest weight vector, of weight $\varpi_i + \varpi_j$. Now apply Lemmas 13.2 and 13.3, noting that the dimension of the simple module of highest weight $\varpi_i + \varpi_j$ is precisely $D_{n;i,j}$. ∎

Lemma 13.6. *The linear map $R^{ji} : V^{ji} \to V^{ij}$ defined by*

$$R^{ji}(e^{ji}_C) = e^{ij}_C \qquad \textit{for all monogressive ij-effective } C \tag{13.3}$$

is an isomorphism of $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-modules.

Proof. This is immediate from the action of the generators of $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$ on the basis elements of V^{ij} and V^{ji}, as described in Appendix C: the formulas obtained there are symmetric in i and j. ∎

Extend R^{ji} to an isomorphism $V^j \otimes V^i \xrightarrow{\sim} V^i \otimes V^j$ of $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-modules (in an arbitrary way).

Lemma 13.7. *The maps R^{ji} induce isomorphisms $R_{ij} : \mathscr{L}_i \otimes \mathscr{L}_j^{\sigma_i} \xrightarrow{\sim} \mathscr{L}_j \otimes \mathscr{L}_i^{\sigma_j}$ of line bundles over E^{DJ}, and the latter satisfy* (3.1) *for all i, j, k.*

Proof. The first statement amounts to the commutativity of the diagram (3.5), which immediately follows from (13.1) and (13.3).

For the second statement, consider the composite map

$$E^{\mathrm{DJ}} \overset{\text{diag.}}{\to} E^{\mathrm{DJ}} \times E^{\mathrm{DJ}} \times E^{\mathrm{DJ}} \overset{\mathrm{id}\times\sigma_i\times\sigma_i\sigma_j}{\to} E^{\mathrm{DJ}} \times E^{\mathrm{DJ}} \times E^{\mathrm{DJ}}$$

$$\overset{\mathrm{Pl}^i\times\mathrm{Pl}^j\times\mathrm{Pl}^k}{\to} \mathbb{P}(V^i) \times \mathbb{P}(V^j) \times \mathbb{P}(V^k) \overset{\text{Segre}}{\to} \mathbb{P}(V^i \otimes V^j \otimes V^k) \tag{13.4}$$

corresponding to the line bundle $\mathscr{L}_i \otimes \mathscr{L}_j^{\sigma_i} \otimes \mathscr{L}_k^{\sigma_i\sigma_j}$, and denote by $V^{ijk} \subset V^i \otimes V^j \otimes V^k$ the linear span of the image of this map.

Claim A. *The subspace V^{ijk} is contained in the unique simple $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-submodule W^{ijk} of $V^i \otimes V^j \otimes V^k$ of highest weight $\varpi_i + \varpi_j + \varpi_k$.*

Indeed, let K_{ij} be the kernel of a projection $V_i \otimes V_j \to V_{\varpi_i+\varpi_j}$, and define similarly K_{jk}, K_{ijk}. Since M^{DJ} is quadratic (cf. [40]), we have $K_{ijk} = K_{ij} \otimes V_k + V_i \otimes K_{jk}$, so dually, $W^{ijk} = V^{ij} \otimes V^k \cap V^i \otimes V^{jk}$, and V^{ijk} is clearly contained on the right-hand side. This shows Claim A.

The proof will be finished if we show the following claim (from which (3.1) follows):

Claim B. *Consider the maps $(R^{ji} \otimes \mathrm{id})(\mathrm{id} \otimes R^{ki})(R^{kj} \otimes \mathrm{id})$ and $(\mathrm{id} \otimes R^{kj})(R^{ki} \otimes \mathrm{id})(\mathrm{id} \otimes R^{ji})$ from $V^k \otimes V^j \otimes V^i$ to $V^i \otimes V^j \otimes V^k$. Their restrictions to V^{kji} agree.*

By Claim A, it will be enough to show that the restrictions to W^{kji} agree. Since both maps are morphisms between the simple $\mathrm{U}_q^{\mathrm{DJ}}(\mathfrak{sl}(n))$-modules W^{kji} and W^{ijk}, they must be equal up to a constant. But they both send the (highest weight) vector $e_1^k \otimes e_1^j \otimes e_1^i$ to $e_1^i \otimes e_1^j \otimes e_1^k$, so this constant is equal to 1. This shows Claim B. ∎

It now follows from Lemma 13.7 that the tuple $T^{\mathrm{DJ}} = (E^{\mathrm{DJ}}, \sigma_1, \ldots, \sigma_\ell, \mathscr{L}_1, \ldots, \mathscr{L}_\ell)$ is braided. It also follows from Corollary 13.5 and Lemma 13.6 that the quadratic algebras $M(T^{\mathrm{DJ}})$ and M^{DJ} agree (as quotients of $T(V_1 \oplus \cdots \oplus V_{n-1})$). Conjecture 9.1 is thus proved for $G = \mathrm{SL}(n)$.

APPENDIX A. PROOF OF PROPOSITION 1.3

Assume that M_A is associative. By Definition 1.1(c), there exists an A-isomorphism $R_{ij} : V_i \otimes V_j \to V_j \otimes V_i$. Rescaling R_{ij} if necessary, we may assume that the diagram

$$\begin{tikzcd}
V_i \otimes V_j \arrow[rr, "R_{ij}"] \arrow[dr, "m"'] & & V_j \otimes V_i \arrow[dl, "m"] \\
& V_{\varpi_i+\varpi_j} &
\end{tikzcd} \tag{A.1}$$

commutes. Now consider the following diagram:

$$\begin{tikzcd}
& & & V_iV_jV_k \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
V_jV_iV_k \arrow[rr] \arrow[dd] \arrow[dr] & & V_{i+j}V_k \arrow[dr] & & V_iV_{j+k} \arrow[dl] & & V_iV_kV_j \arrow[ll] \arrow[dl] \arrow[dd] \\
& V_jV_{i+k} \arrow[rr] & & V_{i+j+k} & & V_{i+k}V_j \arrow[ll] & \\
V_jV_kV_i \arrow[ur] \arrow[rr] \arrow[drrr] & & V_{j+k}V_i \arrow[ur] & & V_kV_{i+j} \arrow[ul] & & V_kV_iV_j \arrow[ll] \arrow[ul] \arrow[dlll] \\
& & & V_kV_jV_i, \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

where we have omitted all tensor product symbols and written V_{i+j} instead of $V_{\varpi_i+\varpi_j}$, etc. (The arrows are the obvious ones, coming either from the multiplication m or from the R_{ij}.) All diamonds commute by associativity, and all triangles commute, being instances of (A.1). Moreover, each object in the diagram contains a unique copy of $V_{\varpi_i+\varpi_j+\varpi_k}$, and when all arrows are restricted to these subcomodules, they become isomorphisms. Therefore, the outer rim commutes, i.e. (1.1) holds.

Conversely, assume that (1.1) holds for all $i > j > k$. We first extend the definition of the R_{ij} by setting $R_{ii} := \mathrm{id}$ for all i and $R_{ji} := R_{ij}^{-1}$ for all $i > j$. Relation (1.1) then holds for all i, j, k.

We will realize M_A as a quotient of the tensor algebra $\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell)$. Let Γ be the free monoid on $\{1, \ldots, \ell\}$. For every $I = i_1 \ldots i_r \in \Gamma$, let

$$\varpi_I := \varpi_{i_1} + \cdots + \varpi_{i_r}, \qquad V^{\otimes I} := V_{i_1} \otimes \cdots \otimes V_{i_r},$$

so $\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell) = \oplus_{I \in \Gamma} V_I$. By Definition 1.1(c), $V^{\otimes I}$ contains a unique copy of V_{ϖ_I}; let $K_I \subset V^{\otimes I}$ be its unique invariant supplement. Then the direct sum $K = \oplus_{I \in \Gamma} K_I$ is a (two-sided) ideal in $\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell)$ (again by Definition 1.1(c)), so we get a quotient algebra

$$\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell)/K =: \bigoplus_{I \in \Gamma} V_I.$$

We still need to identify V_I with V_J whenever $\varpi_I = \varpi_J$. The argument will be fairly standard: use the R_{ij} to exchange generators from different V_i's, and check that this is consistent, using (1.1). More explicitly, denote by S_r the symmetric group. If $\pi \in S_r$ and $I = i_1 \dots i_r \in \Gamma$, define $\pi I := i_{\pi(1)} \dots i_{\pi(r)}$. Denote the usual generators of S_r by $s_j := (j, j+1)$, $1 \leqslant j \leqslant r-1$, and define an A-isomorphism $R_{I,j} : V^{\otimes I} \to V^{\otimes s_j I}$ by $R_{i_j i_{j+1}}$ on $V_{i_j} \otimes V_{i_{j+1}}$ and by id on all other V_{i_k}. If $\pi \in S_r$ decomposes as $\pi = s_{j_1} \dots s_{j_t}$ (*not* necessarily in a reduced way), define $R_{I,\pi} : V^{\otimes I} \to V^{\otimes \pi I}$ by $R_{I,\pi} := R_{I,j_1} \dots R_{I,j_t}$. Since this is an A-isomorphism, it restricts to $R_{I,\pi} : V_I \to V_{\pi I}$. Note that this restriction does not depend on the chosen decomposition of π, thanks to $R_{ii} = \mathrm{id}$, to $R_{ij} R_{ji} = \mathrm{id}$, and to (1.1).

Now consider $\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell)/K$ as a P^+-graded algebra, the term of degree $\lambda \in P^+$ being $U_\lambda := \oplus_{I\,:\,\varpi_I = \lambda} V_I$. Let $S_\lambda \subset U_\lambda$ be the span of all elements $x - R_{I,\pi}(x)$, $x \in V_I$, where I runs over all elements of Γ such that $\varpi_I = \lambda$. Then U_λ / S_λ consists of just one copy of V_λ: indeed, on one hand, $\varpi_I = \varpi_J$ if and only if $J = \pi I$ for some $\pi \in S_r$, and on the other hand, the construction of the $R_{I,\pi}$ implies that

$$R_{I,\pi\pi'} = R_{\pi' I,\pi} R_{I,\pi'}.$$

Moreover, the direct sum $S := \oplus_{\lambda \in P^+} S_\lambda$ is an ideal in $\mathrm{T}(V_1 \oplus \cdots \oplus V_\ell)/K$, so the corresponding quotient yields the desired associative realization of the shape algebra M_A.

APPENDIX B. PROOF OF PROPOSITION 7.1

We need to show that for any monogressive orthocells C_1, C_2, the automorphisms σ_{i,C_1} and σ_{i,C_2} agree on $E(C_1) \cap E(C_2)$. Since this intersection is clearly T-stable and closed, it is a union of T-orbit closures in each of $E(C_1)$ and $E(C_2)$, hence [33, Corollary 6.2] a union of $E(C')$ with C' a common subcell of C_1 and C_2.

It is therefore enough to show that the $\sigma_{i,C}$ are compatible with restriction to subcells. Consider a monogressive orthocell $C = C(w; \alpha_1, \dots, \alpha_d)$ and a subcell, say, $C' = C(ws_L; \alpha'_1, \dots, \alpha'_e)$, with $L \subset \{1, \dots, d\}$, $\{\alpha'_1, \dots, \alpha'_e\} \subset \{\alpha_1, \dots, \alpha_d\}$, and $\alpha_k \notin \{\alpha'_1, \dots, \alpha'_e\}$ for all $k \in L$ (so that ws_L is again of minimal length in C'). For each $\alpha \in \Phi$, fix an isomorphism $u_\alpha : (\mathbf{C}, +) \to U_\alpha$ such that $t u_\alpha(z) t^{-1} = u_\alpha(\alpha(t) z)$ for all $t \in T$ and all $z \in \mathbf{C}$ [36, Proposition 8.1.1(i)]. Then the set of all $\dot{w}\dot{s}_L u_{-\alpha'_1}(z_1) \dots u_{-\alpha'_e}(z_e) B$, $(z_1, \dots, z_e) \in \mathbf{C}^e$, is an open dense subset of $E(C')$ (cf. [33, Proof of Theorem 4.1]), and the action of $\sigma_{i,C'}$ on such an element is given by

$$w s_L t_i s_L^{-1} w^{-1} \dot{w}\dot{s}_L u_{-\alpha_1}(z_1) \dots u_{-\alpha_d}(z_d) B$$

$$= \dot{w}\dot{s}_L t_i u_{-\alpha_1}(z_1) \dots u_{-\alpha_d}(z_d) B$$

$$= \dot{w}\dot{s}_L u_{-\alpha_1}(\alpha_1(t_i)^{-1} z_1) \dots u_{-\alpha_d}(\alpha_d(t_i)^{-1} z_d) B,$$

whereas the action of $\sigma_{i,C}$ (i.e. multiplication by wt_iw^{-1} instead of $ws_Lt_is_L^{-1}w^{-1}$) is given by the same expression, with t_i replaced by $s_L^{-1}t_is_L$. But since s_L is a product of reflections w.r.t. roots orthogonal to each α'_k, we have $\alpha'_k(s_L^{-1}ts_L) = (s_L\alpha'_k)(t) = \alpha'_k(t)$. Thus, the restriction of $\sigma_{i,C}$ to C' coincides with $\sigma_{i,C'}$, and the result follows.

APPENDIX C. PROOF OF LEMMA 13.4

We begin by collecting some more explicit information on the root system of $\mathrm{SL}(n)$. First, $(\alpha|\alpha) = 2$ for every root α, so in particular,

$$s_\alpha(\lambda) = \lambda - (\lambda|\alpha)\,\alpha$$

for any weight λ. Recall also that all fundamental weights $\varpi_1, \dots, \varpi_{n-1}$ are minuscule, so for any $w \in S_n$ and any root α, $(w\varpi_i|\alpha) = 0,\ 1,$ or -1. Moreover, if $\alpha > 0$, then

$$w < s_\alpha w \Rightarrow (w\varpi_i|\alpha) = 0 \text{ or } 1,$$

$$w > s_\alpha w \Rightarrow (w\varpi_i|\alpha) = 0 \text{ or } -1.$$

Now let $\alpha \neq \alpha'$ be two positive roots and $s_\alpha = (a\,b)$, $s_{\alpha'} = (a'\,b')$, with $a < b$ and $a' < b'$. Then

$$(\alpha|\alpha') = \begin{cases} 1 & \text{if } a = a' \text{ or } b = b' \text{ (but not both)}, \\ 0 & \text{if } \{a, b\} \cap \{a', b'\} = \emptyset, \\ -1 & \text{if } a = b' \text{ or } b = a'. \end{cases}$$

The preceding information will be used freely in the sequel, without explicit reference.

We fix a simple root β. Consider first the action of the generator K_β on a vector e_C^{ij}, where $C = C(\alpha_1, \dots, \alpha_d; w)$ is monogressive and ij-effective. Recalling expression (13.2) for ΔK_β, we get

$$K_\beta e_C^{ij} = \sum_{L \subset \{1,\dots,d\}} q^{(s_{\bar{L}}w\varpi_i|\beta) + (s_Lw\varpi_j|\beta)} q^{|L|}\, e^i_{s_{\bar{L}}w} \otimes e^j_{s_Lw}.$$

For each $1 \leqslant k \leqslant d$, we have $s_{\alpha_k} w\varpi_j = w\varpi_j - \alpha_k$. More generally, $s_L w\varpi_j = w\varpi_j - \sum_{k \in L} \alpha_k$, and similarly for $s_{\bar{L}} w\varpi_i$; therefore,

$$K_\beta e_C^{ij} = q^{(w(\varpi_i + \varpi_j)|\beta) - \sum_{k=1}^{d} (\alpha_k|\beta)} e_C^{ij}.$$

A similar formula holds for $K_\beta^{-1} e_C^{ij}$.

Now we study the action of X_β and of Y_β on a vector e_C^{ij}. Note that the root β will be orthogonal to all defining roots of the orthocell C, except at most two. Thus, there are four cases to consider:

Case I. $C = C(\alpha, \alpha', \alpha_1, \ldots, \alpha_d; w)$.

Case II. $C = C(\alpha, \alpha_1, \ldots, \alpha_d; w)$.

Case III. $C = C(\alpha_1, \ldots, \alpha_d; w)$.

Case IV. $C = C(\beta, \alpha_1, \ldots, \alpha_d; w)$, where, in all cases, $\alpha, \alpha', \alpha_1, \ldots, \alpha_d$ are pairwise orthogonal, $(\beta|\alpha) = \pm 1$, $(\beta|\alpha') = \pm 1$, and $(\beta|\alpha_k) = 0$ for all k.

We will first treat these four cases when $d = 0$, and then describe how to deduce results for arbitrary d from this particular case.

Let us use the notation $c \lessdot c'$ even when c, c' are integers, meaning that $c' = c + 1$. We will also use the following notation throughout:

$$s := s_\alpha =: (a\, b), \qquad s' := s_{\alpha'} =: (a'\, b'), \qquad t := s_\beta,$$

with $a < b$ and $a' < b'$. Note that monogressivity and ij-effectiveness *exclude* the orderings $a < a' < b < b'$ and $a' < a < b' < b$.

Case I. $\quad C = C(\alpha, \alpha'; w)$.

Subcase I.1: $\quad (\beta|\alpha) = (\beta|\alpha') = -1$. Exchanging α, α' if necessary, we may assume that $a < a'$. Then we must have $a < b \lessdot a' < b'$ and $t = (b\, a')$. Furthermore, $(sw\varpi_i|\beta) = (s'w\varpi_i|\beta) = (w\varpi_i|\beta) - 1$ and $(ss'w\varpi_i|\beta) = (w\varpi_i|\beta) -2$. Since all these inner products must be equal to 0, 1, or -1, we get

$$(w\varpi_i|\beta) = -1, \qquad (sw\varpi_i|\beta) = (s'w\varpi_i|\beta) = 0, \qquad (ss'w\varpi_i|\beta) = 1,$$

and similarly for ϖ_i replaced by ϖ_j. Recalling expression (13.2) for ΔX_β and ΔY_β, we obtain

$$\begin{aligned} X_\beta e_C^{ij} &= X_\beta (q^2 e_w^i \otimes e_{ss'w}^j + q\, e_{sw}^i \otimes e_{s'w}^j + q\, e_{s'w}^i \otimes e_{sw}^j + e_{ss'w}^i \otimes e_w^j) \\ &= q^2 X_\beta e_w^i \otimes e_{ss'w}^j + K_\beta e_{ss'w}^i \otimes X_\beta e_w^j \\ &= q^2 e_{tw}^i \otimes e_{ss'w}^j + q\, e_{ss'w}^i \otimes e_{tw}^j, \end{aligned}$$

and, by a similar computation,

$$Y_\beta e_C^{ij} = q^2 e_w^i \otimes e_{tss'w}^j + q\, e_{tss'w}^i \otimes e_w^j.$$

The vanishing inner products obtained above imply that $tsw\varpi_i = sw\varpi_i$ and $ts'w\varpi_i = s'w\varpi_i$. Using the Coxeter relations $(ts)^3 = (ts')^3 = 1$, we then also have the equalities

$$\begin{aligned} tw\varpi_i &= stw\varpi_i = s'tw\varpi_i = ss'tw\varpi_i, \\ ss'w\varpi_i &= sts'w\varpi_i = s'tsw\varpi_i = ss'tss'tw\varpi_i, \\ w\varpi_i &= tstw\varpi_i = ts'tw\varpi_i = tss'tw\varpi_i, \\ tss'w\varpi_i &= stss'w\varpi_i = s'tss'w\varpi_i = ss'tss'w\varpi_i \end{aligned}$$

(and similarly for ϖ_i replaced by ϖ_j), which may be used in the above expressions for $X_\beta e_C^{ij}$ and $Y_\beta e_C^{ij}$; cf. Remark 10.1. Since $a<b<a'<b'$, monogressivity implies that there are four possible relative positions of a, b, a', b' inside the array w, yielding the following expressions for $X_\beta e_C^{ij}$ and $Y_\beta e_C^{ij}$:

w	$X_\beta e_C^{ij}$	$Y_\beta e_C^{ij}$:
$[\cdot a \cdot a' \cdot b \cdot b' \cdot]$	$-q\, e^{ij}_{C(ss'(\beta);ss'tw)}$	$-q\, e^{ij}_{C(ss'(\beta);tss'tw)}$,
$[\cdot a' \cdot a \cdot b' \cdot b \cdot]$	$-q\, e^{ij}_{C(ss'(\beta);tw)}$	$-q\, e^{ij}_{C(ss'(\beta);w)}$,
$[\cdot a' \cdot a \cdot b \cdot b' \cdot]$	$-q\, e^{ij}_{C(ss'(\beta);s'tw)}$	$-q\, e^{ij}_{C(ss'(\beta);ts'tw)}$,
$[\cdot a \cdot a' \cdot b' \cdot b \cdot]$	$-q\, e^{ij}_{C(ss'(\beta);stw)}$	$-q\, e^{ij}_{C(ss'(\beta);tstw)}$.

These results are valid provided all orthocells involved are monogressive (their ij-effectiveness being clear). In each case, this may easily be checked using Criterion 11.1. For example, in the first line, we have $w = [\cdot a \cdot a' \cdot b \cdot b' \cdot]$, $ss'tw = [\cdot b \cdot a \cdot b' \cdot a' \cdot]$, and $s_{ss'(\beta)} = (a\, b')$: since $C = C(\alpha, \alpha'; w)$ is monogressive by assumption, the subarray $[a' \dots b]$ of w contains no numbers in the (numerical) interval $[a, b]$, nor in $[a', b']$, and therefore not in $[a, b']$ (because $a<b\lessdot a'<b'$); thus, the orthocell $C(ss'(\beta); ss'tw)$ is again monogressive.

Subcase I.2: $(\beta|\alpha) = 1$ and $(\beta|\alpha') = -1$. Here we must have either $a\lessdot a' < b' < b$ and $t = (a\, a')$, or $a<a'<b'\lessdot b$ and $t = (b'\, b)$. Furthermore,

$$(w\varpi_i|\beta) = (ss'w\varpi_i|\beta) = 0, \qquad (sw\varpi_i|\beta) = -1, \qquad (s'w\varpi_i|\beta) = 1,$$

and similarly for ϖ_i replaced by ϖ_j. It follows that

$$X_\beta e_C^{ij} = q\, e_{tsw}^i \otimes e_{s'w}^j + q^2 e_{s'w}^i \otimes e_{tsw}^j,$$

$$Y_\beta e_C^{ij} = q\, e_{sw}^i \otimes e_{ts'w}^j + q^2 e_{ts'w}^i \otimes e_{sw}^j.$$

Arguments similar to those of Subcase I.1 then show (whether $t = (a\,a')$ or $t = (b'\,b)$) that the orthocells $C(ss'(\beta); s'tw)$ and $C(ss'(\beta); ts'tw)$ are monogressive and that

$$X_\beta e_C^{ij} = -q\, e^{ij}_{C(ss'(\beta);s'tw)}, \qquad Y_\beta e_C^{ij} = -q\, e^{ij}_{C(ss'(\beta);ts'tw)}.$$

(We omit the details.)

Subcase I.3: $(\beta|\alpha) = (\beta|\alpha') = 1$. These inner products force $a<a'<b<b'$ or $a'<a<b'<b$, contradicting the fact that $C(\alpha, \alpha'; w)$ is monogressive and ij-effective. Therefore, this Subcase is impossible.

Case II. $C = C(\alpha; w)$.

Subcase II.1: $(\beta|\alpha) = -1$. We must have either $c \lessdot a < b$ and $t = (c\,a)$, or $a<b\lessdot c$ and $t = (b\,c)$. Moreover, since $(sw\varpi_i|\beta) = (w\varpi_i|\beta) + 1$, and similarly for ϖ_j, we obtain the following cases.

- If $(w\varpi_i|\beta) = (w\varpi_j|\beta) = -1$, then arguments similar to those of Case I show that

$$Y_\beta e_C^{ij} = 0$$

and that $X_\beta e_C^{ij} = q\, e_{tw}^i \otimes e_{sw}^j + e_{sw}^i \otimes e_{tw}^j$ is given by the following table:

t	w	$X_\beta e_C^{ij}$:
$(c\,a)$	$[\cdot a \cdot b \cdot c \cdot]$	$e^{ij}_{C(s(\beta);tw)}$,
$(c\,a)$	$[\cdot a \cdot c \cdot b \cdot]$	$e^{ij}_{C(s(\beta);stw)}$,
$(b\,c)$	$[\cdot a \cdot c \cdot b \cdot]$	$-e^{ij}_{C(s(\beta);stw)}$,
$(b\,c)$	$[\cdot c \cdot a \cdot b \cdot]$	$-e^{ij}_{C(s(\beta);tw)}$.

- If $(w\varpi_i|\beta) = (w\varpi_j|\beta) = 0$, then

$$X_\beta e_C^{ij} = 0$$

and $Y_\beta e^{ij}_C = q\, e^i_w \otimes e^j_{tsw} + e^i_{tsw} \otimes e^j_w$ is given by the following table:

t	w	$Y_\beta e^{ij}_C$:
$(c\,a)$	$[\cdot c \cdot a \cdot b \cdot]$	$-e^{ij}_{C(s(\beta);tw)}$,
$(c\,a)$	$[\cdot a \cdot c \cdot b \cdot]$	$-e^{ij}_{C(s(\beta);w)}$,
$(b\,c)$	$[\cdot a \cdot c \cdot b \cdot]$	$e^{ij}_{C(s(\beta);w)}$,
$(b\,c)$	$[\cdot a \cdot b \cdot c \cdot]$	$e^{ij}_{C(s(\beta);tw)}$.

- If $(w\varpi_i|\beta) = -1$ and $(w\varpi_j|\beta) = 0$, or vice versa, then

$$X_\beta e^{ij}_C = -q\, e^{ij}_{C(-;stw)}, \qquad Y_\beta e^{ij}_C = -q\, e^{ij}_{C(-;stsw)}.$$

Subcase II.2: $(\beta|\alpha) = 1$. Here we have either $a \lessdot c < b$ and $t = (a\,c)$, or $a < c \lessdot b$ and $t = (c\,b)$. Moreover, since $(sw\varpi_i|\beta) = (w\varpi_i|\beta) - 1$, and similarly for ϖ_j, we obtain the following cases.

- If $(w\varpi_i|\beta) = (w\varpi_j|\beta) = 1$, then

$$X_\beta e^{ij}_C = 0, \qquad Y_\beta e^{ij}_C = \pm e^{ij}_{C(s(\beta);tw)}.$$

- If $(w\varpi_i|\beta) = (w\varpi_j|\beta) = 0$, then

$$X_\beta e^{ij}_C = \pm e^{ij}_{C(s(\beta);tw)}, \qquad Y_\beta e^{ij}_C = 0.$$

- The case $(w\varpi_i|\beta) = 1$ and $(w\varpi_j|\beta) = 0$, or vice versa, contradicts the ij-effectiveness of C and is therefore impossible.

Case III: $C = C(-;w)$. We obtain the following table:

$(w\varpi_i|\beta)$	$(w\varpi_j|\beta)$	$X_\beta e^{ij}_C$	$Y_\beta e^{ij}_C$:
1	1	0	$q^{-1}\, e^{ij}_{C(\beta;w)}$,
1	0	0	$e^{ij}_{C(-;tw)}$,
0	1	0	$e^{ij}_{C(-;tw)}$,
0	0	0	0,
0	-1	$e^{ij}_{C(-;tw)}$	0,
-1	0	$e^{ij}_{C(-;tw)}$	0,
-1	-1	$q^{-1}\, e^{ij}_{C(\beta;tw)}$	0.

Case IV: $C = C(\beta; w)$. Since C is monogressive and *ij*-effective, we must have $(w\varpi_i|\beta) = (w\varpi_j|\beta) = 1$, hence

$$X_\beta e_C^{ij} = (q^2+1)e_{C(-;w)}^{ij}, \qquad Y_\beta e_C^{ij} = (q^2+1)e_{C(-;tw)}^{ij}.$$

Finally, we show how, in the preceding four cases, one can deduce the action of X_β and Y_β for arbitrary d from that for $d=0$. The idea is that $\alpha_1, \ldots, \alpha_d$, being orthogonal to β, α, α', do not "interfere" with the computations done above. To make this idea precise, we will restrict ourselves to the very first case treated above (all other cases being similar), namely, the action of X_β on e_C^{ij} when $C = C(\alpha, \alpha', \alpha_1, \ldots, \alpha_d; w)$, $(\beta|\alpha) = (\beta|\alpha') = -1$, $(\beta|\alpha_k) = 0$ for all k, $s_\alpha = (a\, b)$ and $s_{\alpha'} = (a'\, b')$ with $a < b \lessdot a' < b'$ (so $t := s_\beta = (b\, a')$), and $w = [\cdot a \cdot a' \cdot b \cdot b' \cdot]$.

Since β is orthogonal to each α_k, we have $(s_L\lambda|\beta) = (\lambda|\beta)$ for any weight λ and any $L \subset \{1, \ldots, d\}$, so we still get

$$(s_L w\varpi_i|\beta) = -1, \qquad (ss_L w\varpi_i|\beta) = (s's_L w\varpi_i|\beta) = 0, \qquad (ss's_L w\varpi_i|\beta) = 1.$$

Therefore, the action of X_β on each term of

$$e^{ij}_{C(\alpha,\alpha',\alpha_1,\ldots,\alpha_d;w)} = \sum_{L\subset\{1,\ldots,d\}} q^{|L|}(q^2 e^i_{s_{\bar{L}}w} \otimes e^j_{ss's_L w} + q\, e^i_{ss_{\bar{L}}w} \otimes e^j_{s's_L w} + q\, e^i_{s's_{\bar{L}}w} \otimes e^j_{ss_L w} + e^i_{ss's_{\bar{L}}w} \otimes e^j_{s_L w})$$

is still computed in a similar way to that on $e^{ij}_{C(\alpha,\alpha';w)}$, viz.,

$$X_\beta e^{ij}_{C(\alpha,\alpha',\alpha_1,\ldots,\alpha_d;w)} = \sum_{L\subset\{1,\ldots,d\}} q^{|L|}(q^2 e^i_{ts_{\bar{L}}w} \otimes e^j_{ss's_L w} + q\, e^i_{ss's_{\bar{L}}w} \otimes e^j_{ts_L w}).$$

It follows that

$$X_\beta e^{ij}_{C(\alpha,\alpha',\alpha_1,\ldots,\alpha_d;w)} = -q\, e^{ij}_{C(ss'(\beta),\alpha_1,\ldots,\alpha_d;ss'tw)},$$

provided the orthocell $C(ss'(\beta), \alpha_1, \ldots, \alpha_d; ss'tw)$ is monogressive. But it is easy to see that the analysis of the monogressivity of $C(ss'(\beta); ss'tw)$ made earlier, using Criterion 11.1, remains valid if $\alpha_1, \ldots, \alpha_d$ are added to the orthocells $C(\alpha, \alpha'; w)$ and $C(ss'(\beta); ss'tw)$.

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