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MULTIPLICITIES AND PROPER BASES FOR gl_n

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1. Formulations of results. The first main result of this paper is a new expression for multiplicities of irreducible components in the tensor product of two irreducible finite-dimensional representations of the Lie algebra gl_n. The answer is the number of integral points in a certain convex polytope in the space of Gelfand-Tsetlin patterns (G-patterns). The Kostant function and the weight multiplicity for gl_n may also acquire a similar interpretation. Thus for gl_n all these three fundamental functions of the representation theory may be treated in the same geometric way: there is a convex polytope in the space of large dimension with a linear projection on the domain of our function and the function value at each point x equals the number of integral points in the fiber of the projection over x (clearly this fiber is also a convex polytope). This approach gives a natural explanation of the piece-wise polynomial behaviour of these functions.

To formulate the result we need some notations. Recall that the weight lattice for gl_n consists of integral vectors $\gamma = (\gamma_1,\dots,\gamma_n)$. The weight $\lambda = (\lambda_1,\dots,\lambda_n)$ is called highest if $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$. Irreducible finite-dimensional representations of gl_n are parametrized by highest weights; denote the representation with highest weight λ by V_λ.

A G-pattern for gl_n is defined as a sequence $(\lambda^{(1)}, \lambda^{(2)},\dots, \lambda^{(n)})$, where $\lambda^{(i)}=(\lambda_1^{(i)}, \lambda_2^{(i)},\dots,$

$\lambda_i^{(i)})$ is a highest weight for $\mathfrak{gl}_i$ such that

$$\lambda_j^{(i+1)} \geqslant \lambda_j^{(i)} \geqslant \lambda_{j+1}^{(i+1)} \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant n-1 \qquad (1)$$

A G-pattern is usually represented as a triangular array

$$\Lambda = \begin{array}{ccccc} \lambda_1^{(n)} & \lambda_2^{(n)} & \dots & \lambda_n^{(n)} \\ & \lambda_1^{(n-1)} & \dots & \lambda_{n-1}^{(n-1)} \\ & & \lambda_1^{(1)} & \end{array}$$

We call $\lambda^{(n)}$ the type of Λ and define the weight $\gamma = (\gamma_1, \dots, \gamma_n)$ of Λ by

$$\gamma_1 + \dots + \gamma_i = \lambda_1^{(i)} + \lambda_2^{(i)} + \dots + \lambda_i^{(i)} \qquad (1 \leqslant i \leqslant n) \qquad (2)$$

For $1 \leqslant j \leqslant i \leqslant n-1$ we put

$$d_j^{(i)}(\Lambda) = \sum_{1 \leqslant h < j} (\lambda_h^{(i+1)} - 2\lambda_h^{(i)} + \lambda_h^{(i-1)}) + (\lambda_j^{(i+1)} - \lambda_j^{(i)}) \qquad (3)$$

We call the $d_j^{(i)}(\Lambda)$ the exponents of a G-pattern Λ.

Theorem 1. For any three highest weights λ, μ, ν for $\mathfrak{gl}_n$ the multiplicity of the irreducible representation V_μ in the tensor product $V_\lambda \otimes V_\nu$ equals the number of G-patterns Λ of type λ and weight $\mu - \nu$ such that the exponents of Λ satisfy

$$d_j^{(i)}(\Lambda) \leqslant \nu_i - \nu_{i+1} \qquad (1 \leqslant j \leqslant i \leqslant n-1) \qquad (4)$$

This theorem as well as all the other results will be proved in § 2.

Now we give a similar expression for the weight multiplicities for $\mathfrak{gl}_n$. Let $V_\lambda(\gamma)$ denote the weight subspace of weight γ in V_λ, i.e. $V_\lambda(\gamma) = \{ v \in V_\lambda : E_{ii} v = \gamma_i v$ for $1 \leqslant i \leqslant n \}$; by definition the multiplicity of γ in V_λ is $\dim V_\lambda(\gamma)$.

Proposition 1. The multiplicity of a weight γ in V_λ equals the number of G-patterns of type λ and weight γ.

This proposition in an equivalent form with Young tableaux instead of G-patterns goes back to A.Young (cf. e.g. [6], Ch. 1, (6.4)).

Let us give a geometric interpretation of Theorem 1 and Proposition 1. Consider the vector space $R^{n(n+1)/2}$ with coordinates $(\lambda_j^{(i)})$ $(1 \leqslant j \leqslant i \leqslant n)$ and the (unbounded) convex polytope Γ in it defined by inequalities (1); then G-patterns for $\mathfrak{gl}_n$ may be viewed as integral points in Γ. The mapping assigning to each G-pattern Λ its type and weight extends to the linear mapping $p: \Gamma \longrightarrow R^{2n}$. Proposition 1 means that the multiplicity of γ in V_λ equals the number of integral points in $p^{-1}(\lambda, \gamma)$. Furthermore, for any highest weight ν let Γ_ν denote the subset of Γ determined by inequalities (4); clearly Γ_ν is also a convex polytope. Let $p_\nu : \Gamma_\nu \to R^{2n}$ be the restriction of p to Γ_ν. Theorem 1 asserts that the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ equals the number of untegral points in $p_\nu^{-1}(\lambda, \mu - \nu)$.

Proposition 1 has natural representation-theoretic interpretation. Namely there is the Gelfand-Tsetlin basis in V_λ parametrized by G-patterns of type λ, and each weight subspace $V_\lambda(\gamma)$ is spanned by its subset corresponding to G-patterns of weight γ. It turns out that Theorem 1 may be similarly interpreted.

Definition. (a) Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ be two highest weights for $\mathfrak{gl}_n$, and γ a weight. Set

$$V_\lambda(\gamma, \nu) = \{ v \in V_\lambda(\gamma) : E_{i,i+1}^{\nu_i - \nu_{i+1} + 1} v = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n-1 \}$$

(b) For any $v \in V_\lambda$ and $i = 1, 2, \dots, n-1$ define the exponent $d_i = d_i(v)$ to be the smallest nonnegative integer such that $E_{i,i+1}^{d_i+1} v = 0$.

(c) A basis in V_λ is proper if each of the subspaces

$V_\lambda(\gamma,\nu)$ for all possible γ and ν is spanned by its subset.

(d) Let B be a basis in V_λ. By a proper parametrization of B we mean its parametrization by G-patterns of type λ $(\Lambda \longrightarrow f_\Lambda)$ such that:

(1) if the weight of Λ is γ then $f_\Lambda \in V_\lambda(\gamma)$;

(2) $d_i(f_\Lambda) = \max\limits_{1\leq j\leq i} d_j^{(i)}(\Lambda)$, where $d_j^{(i)}(\Lambda)$ is defined by (3), for each $i = 1,\dots, n-1$.

Proposition 2. For any three highest weights λ, μ, ν for gl_n the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ equals $\dim V_\lambda(\mu-\nu,\nu)$.

This fact is well-known (see e.g. [5]).

Proposition 3. (a) The subspace $V_\lambda(\gamma,\nu)$ consists of vectors $v \in V_\lambda(\gamma)$ such that $d_i(v) \leq \nu_i - \nu_{i+1}$ for $1 \leq i \leq n-1$.

(b) A properly parametrized basis in V_λ is proper. Conversely, each proper basis in V_λ has a proper parametrization.

Theorem 2. In each irreducible finite dimensional representation V_λ of gl_n it is possible to choose a properly parametrized basis.

Now let (f_Λ) be a properly parametrized basis in V_λ. By Proposition 3, each of the subspaces $V_\lambda(\gamma,\nu)$ in V_λ has as a basis the subset of (f_Λ) corresponding to G-patterns Λ of type λ and weight γ satisfying (4); in particular, $\dim V_\lambda(\gamma,\nu)$ is the number of such patterns. By Proposition 2 this implies Theorem 1. So, a properly parametrized basis "explains" Theorem 1 in the same way as the Gelfand-Tsetlin basis explains Proposition 1.

Theorem 3. In the case of gl_3 a properly parametrized basis in each irreducible finite dimensional representation is unique up to scalar factors.

The proper basis for gl_3 will be called canonical. Its properties are studied in [1].

Note that our proof of Theorem 2 uses the result of the very interesting paper [7]. Namely, in [7] for each irreducible finite dimensional representation of gl_n there is constructed a special basis with remarkable branching properties; we show that these properties imply that the basis is proper.

Proper bases may be used for explicit decomposition into irreducible components of the tensor product of two irreducible finite dimensional representation of gl_n. Namely, a choice of a basis in $V_\lambda(\mu-\nu,\nu)$ allows one to construct the complete family of irreducible components of $V_\lambda \otimes V_\nu$ which are isomorphic to V_μ. More precisely, note that Proposition 2 may be refined as follows: for each λ, μ, ν there is a natural isomorphism between $V_\lambda(\mu-\nu,\nu)$ and the space $\mathrm{Hom}(V_\mu, V_\lambda \otimes V_\nu)$ of linear operators $T: V_\mu \longrightarrow V_\lambda \otimes V_\nu$ commuting with the gl_n-action. A proper basis in V_λ by definition gives the basis in each $V_\lambda(\mu-\nu,\nu)$ and so makes it possible to choose the basis $T_1,\dots,T_N$ in $\mathrm{Hom}(V_\mu, V_\lambda \otimes V_\nu)$. The subspaces $T_i(V_\mu)$ are irreducible components of $V_\lambda \otimes V_\nu$ isomorphic to V_μ; taking all these subspaces for all possible μ we obtain a decomposition of $V_\lambda \otimes V_\nu$ into irreducible components. This application of proper bases stimulates their investigation.

The main results of this paper are announced in [2]. Another approach to the decomposition problem of tensor products based on the study of representation models is worked out by the authors in [3].

2. Proofs. Proof of Theorem 1. We call a weight $\gamma = (\gamma_1,\dots,\gamma_n)$ for gl_n polynomial if each γ_i is nonnegative. The first step in the proof is reduction to the case of polynomial λ, μ, ν. For this we put $\delta = (1,1,\dots,1)$ and note that the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ as well as the number of G-patterns from Theorem 1 do not change under the replacement $(\lambda \longrightarrow \lambda + k\delta, \nu \longrightarrow \nu + \ell\delta, \mu \longrightarrow \mu + (k+\ell)\delta)$ for arbitrary in-

tegers k and ℓ; for k and ℓ large enough λ, μ and ν are polynomial. In this case the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ is given by the classical Littlewood-Richardson rule (see e.g. [6], Ch. 1, §9). Recall its formulation.

First of all, the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ may be nonzero only when $\mu - \nu$ is polynomial, i.e., $\mu_i \geqslant \nu_i$ for $1 \leqslant i \leqslant n$. Let $D(\mu - \nu)$ denote the set $\{(i,j) \in Z^2 : 1 \leqslant i \leqslant n,\ \nu_i < j \leqslant \mu_i\}$; this set is called a skew diagram. By a Young tableau of shape $\mu - \nu$ we mean the mapping $T : D(\mu - \nu) \longrightarrow \{1,2,\dots,n\}$ satisfying the following conditions:

$$T(i,j+1) \geqslant T(i,j) \tag{5}$$

$$T(i+1,j) > T(i,j) \tag{6}$$

A vector $\lambda = (\lambda_1,\dots,\lambda_n)$, where λ_k is the number of points $(i,j) \in D(\mu - \nu)$ such that $T(i,j) = k$, is called the weight of T.

In drawing, the nodes of a skew diagram are usually replaced by squares as in Fig. 1. A Young tableau T is described graphically by numbering each square (i,j)

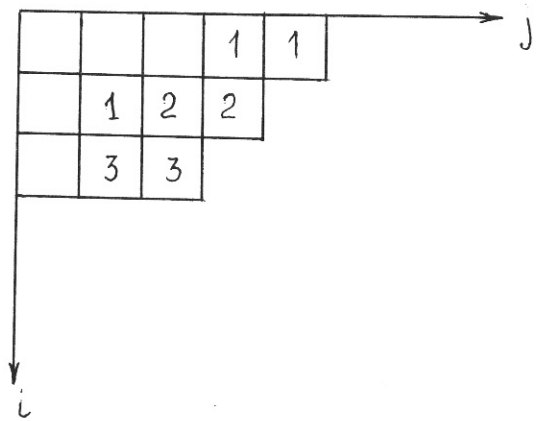

Fig. 1

with the number $T(i,j)$; the conditions (5) and (6) mean that the numbers must increase strictly down each column, and weakly from left to right along each row. The component λ_k of the weight of T is the number of squares numbered with k. For example, the Young tableau in Fig.1 is of shape (5,4,3) - (3,1,1) and weight (3,2,2). Let T be a Young tableau. Let $w(T)$ denote the word obtained by reading the symbols in T from right to left in successive rows, starting with the top row; for example, for T in Fig. 1 $w(T) = 1122133$. A word $w = a_1a_2\dots a_N$ in the symbols $1, 2,\dots, n$ is said to be a lattice permutation if for $1 \leqslant r \leqslant N$ and $1 \leqslant i \leqslant n-1$ the number of occurences of the symbol i in $a_1a_2\dots a_r$ is not less than the number of occurences of $i+1$.

Littlewood-Richardson rule. Let λ, μ, ν be polynomial highest weights for gl_n. Then the multiplicity of V_μ in $V_\lambda \otimes V_\nu$ equals the number of tableaux T of shape $\mu - \nu$ and weight λ such that $w(T)$ is a lattice permutation.

For the proof of Theorem 1 it remains to give a one-to-one correspondence between G-patterns described in Theorem 1 and Young tableaux described in the Littlewood-Richardson rule. Let T be a Young tableau of shape $\mu-\nu$ and $1 \leqslant i, j \leqslant n$; denote by $\lambda_j^{(i)}$ the number of occurences of j in the first i rows of T (to be more formal, $\lambda_j^{(i)}$ is the number of points $(i',j') \in D(\mu - \nu)$ such that $i' \leqslant i$ and $T(i',j') = j$). Let $\Lambda = \Lambda(T)$ denote the pattern formed by the numbers $\lambda_j^{(i)}$ for $1 \leqslant j \leqslant i \leqslant n$.

Proposition 4. The mapping $T \longrightarrow \Lambda(T)$ gives a one-to-one correspondence between Young tableaux T of shape $\mu - \nu$ and weight λ such that $w(T)$ is a lattice permutation, and G-patterns Λ of type λ and weight $\mu - \nu$ satisfying (4).

Proof. It is verified directly that $w(T)$ is a lattice permutation if and only if $\lambda_j^{(i)} = 0$ for $j > i$ and

$\lambda_j^{(i)} \geqslant \lambda_{j+1}^{(i+1)}$ for $1 \leqslant j \leqslant i \leqslant n-1$; it follows that the pattern $\Lambda(T)$ is triangular and satisfies half of the conditions (1). Another half, namely, $\lambda_j^{(i)} \geqslant \lambda_j^{(i-1)}$, follows from the fact that $\lambda_j^{(i)} - \lambda_j^{(i-1)}$ is the number of squares in the i-th row of T occupied by j. By (5) T is uniquely recovered from $\Lambda = \Lambda(T)$ i.e. the i-th row of T consists of $(\lambda_1^{(i)} - \lambda_1^{(i-1)})$ 1's, $(\lambda_2^{(i)} - \lambda_2^{(i-1)})$ 2's,..., $\lambda_i^{(i)}$ i's. This description of T readily implies that the conditions (6) for T are equivalent to the inequalities (4) for $\Lambda(T)$. Finally, the fact that $\Lambda(T)$ is of type λ and weight $\mu - \nu$ follows at once from definitions.

Proposition 4 and Theorem 1 are proved.

Proof of Proposition 3. Part (a) follows at once from definitions. To prove (b) we use the following.

Proposition 5. Let λ, ν be highest weights for gl_n and γ a weight. Then $\dim V_\lambda(\gamma, \nu)$ equals the number of G-patterns Λ of type λ and weight γ satisfying (4).

Proof. If $\gamma+\nu$ is a highest weight then our statement follows from Theorem 1 and Proposition 2. We show that if it is not the case then $V_\lambda(\gamma, \nu) = 0$ and there are no G-patterns of weight γ satisfying (4).

Lemma 1. If v is a nonzero vector of V_λ of weight $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $1 \leqslant i \leqslant n-1$ then $d_i(v) \geqslant \gamma_{i+1} - \gamma_i$.

Proof. Restrict V_λ to the subalgebra with the basis $\{E_- = E_{i+1,i},\ H = E_{ii} - E_{i+1,i+1},\ E_+ = E_{i,i+1}\}$; this subalgebra is isomorphic to sl_2, and $Hv = (\gamma_i - \gamma_{i+1})v$. It remains to use the following well-known property of representations of sl_2: if v is a nonzero vector in a finite dimensional representation of sl_2 such that $Hv = -hv$ for some integer $h \geqslant 0$ then $E_+^h v \neq 0$.

By this lemma and part (a) of proposition 3, if $V_\lambda(\gamma, \nu) \neq 0$ then $\gamma_{i+1} - \gamma_i \leqslant \nu_i - \nu_{i+1}$ for $1 \leqslant i \leqslant n-1$ and hence $\gamma + \nu$ is a highest weight.

Now let Λ be a G-pattern of weight γ satisfying (4).

Using (1) to (4) we deduce

$$\gamma_{i+1} - \gamma_i = \sum_{j \geqslant 1} (\lambda_j^{(i+1)} - 2\lambda_j^{(i)} + \lambda_j^{(i-1)}) = d_i^{(i)}(\Lambda) - \lambda_i^{(i)} + \lambda_{i+1}^{(i+1)} \leqslant d_i^{(i)}(\Lambda) \leqslant \nu_i - \nu_{i+1},$$

which again implies that $\gamma+\nu$ is a highest weight. This proves Proposition 5.

Now let (f_Λ) be a properly parametrized basis in V_λ. By definition and part (a) of Proposition 3 $f_\Lambda \in V_\lambda(\gamma, \nu)$ if and only if a G-pattern Λ is of type λ, weight γ and satisfies (4). Since by Proposition 5 $\dim V_\lambda(\gamma, \nu)$ is the number of such patterns, $V_\lambda(\gamma, \nu)$ is spanned by the vectors f_Λ contained in it. Therefore, the basis (f_Λ) is proper.

Conversely, let B be a proper basis in V_λ. We fix a weight γ. For each family $(d_1, \ldots, d_{n-1})$ of nonnegative integers let $\varphi_1(d_1, \ldots, d_{n-1})$ denote the number of vectors $f \in B$ with weight γ and such that $d_i(f) = d_i$ for $1 \leqslant i \leqslant n-1$; denote also by $\varphi_2(d_1, \ldots, d_{n-1})$ the number of G-patterns Λ of type λ and weight γ such that $\max_j d_j^{(i)}(\Lambda) = d_i$ for $1 \leqslant i \leqslant n-1$. By Proposition 3 (a) and Proposition 5, for all $d_1, \ldots, d_{n-1}$ we have

$$\sum_{0 \leqslant d_i' \leqslant d_i} \varphi_1(d_1', \ldots, d_{n-1}') = \sum_{0 \leqslant d_i' \leqslant d_i} \varphi_2(d_1', \ldots, d_{n-1}');$$

indeed, both parts equal $\dim V_\lambda(\gamma, \nu)$, where ν is such that $\nu_i - \nu_{i+1} = d_i$ for $1 \leqslant i \leqslant n-1$. It follows that $\varphi_1(d_1, \ldots, d_{n-1}) = \varphi_2(d_1, \ldots, d_{n-1})$ for all $d_1, \ldots, d_{n-1}$. Therefore, B admits a proper parametrization. Proposition 3 is proved.

Proof of Theorem 2. We fix an irreducible finite dimensional representation V of gl_n. By Proposition 3 it suffices to prove that V has a proper basis. In [7] a special basis in V is constructed satisfying some branching properties; we show that these properties im-

ply that the basis is proper. We give the result of [7] in a convenient form.

For each family $\underline{s} = (s_1,\dots,s_k)$ of positive integers with sum n denote by $gl_{\underline{s}}$ the product $gl_{s_1} \times \dots \times gl_{s_k}$; $gl_{\underline{s}}$ is viewed as a Lie subalgebra of gl_n. Irreducible finite dimensional representations of $gl_{\underline{s}}$ are parametrized by families $\underline{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)})$, where $\lambda^{(i)}$ is a highest weight for gl_{s_i}; the corresponding representation of $gl_{\underline{s}}$ is $V_\lambda = V_{\lambda^{(1)}} \otimes \dots \otimes V_{\lambda^{(k)}}$. Let $\lambda = (\lambda_1,\dots,\lambda_m)$ and $\lambda' = (\lambda'_1,\dots,\lambda'_m)$ be highest weights for gl_m; we shall write $\lambda \geqslant \lambda'$ if

$$\lambda_1 \geqslant \lambda'_1,\ \lambda_1+\lambda_2 \geqslant \lambda'_1+\lambda'_2,\ \dots,\ \lambda_1+\dots+\lambda_{m-1} \geqslant \lambda'_1+\dots+\lambda'_{m-1},\ \lambda_1+\dots+\lambda_m = \lambda'_1+\dots+\lambda'_m .$$

If $\underline{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)})$ and $\underline{\lambda}' = (\lambda^{(1)\prime},\dots,\lambda^{(k)\prime})$ are two families of highest weights for the same $gl_{\underline{s}}$ then we shall write $\underline{\lambda} \geqslant \underline{\lambda}'$ if $\lambda^{(i)} \geqslant \lambda^{(i)\prime}$ for $1 \leqslant i \leqslant k$.

Consider the restriction of V to $gl_{\underline{s}}$. For each $\underline{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)})$ corresponding to $gl_{\underline{s}}$ let $V(\underline{s},\underline{\lambda})$ denote the sum of all irreducible components of this restriction which are isomorphic to $V_{\underline{\lambda}}$. We put

$$\overline{V(\underline{s},\underline{\lambda})} = \bigoplus_{\underline{\lambda}' \leqslant \underline{\lambda}} V(\underline{s},\underline{\lambda}'),\quad V^{o}(\underline{s},\underline{\lambda}) = \bigoplus_{\underline{\lambda}' < \underline{\lambda}} V(\underline{s},\underline{\lambda}')$$

so that $V(\underline{s},\underline{\lambda})$ is identified with $\overline{V(\underline{s},\underline{\lambda})}/V^{o}(\underline{s},\underline{\lambda})$.

Theorem 4. [7]. In each irreducible finite dimensional representation V of gl_n there is a basis B such that for all $\underline{s}$ and $\underline{\lambda}$ the following conditions hold:

(a) The subspace $V(\underline{s},\underline{\lambda})$ is spanned by a subset of B;

(b) The vectors of B belonging to $\overline{V(\underline{s},\underline{\lambda})} - V^{o}(\underline{s},\underline{\lambda})$ are separated into some groups so that the images of vectors of each group in $\overline{V(\underline{s},\underline{\lambda})}/V^{o}(\underline{s},\underline{\lambda})$ form a basis in some irreducible component under the action of $gl_{\underline{s}}$.

The conditions (a) and (b) are very strong; it is even possible that B is uniquely determined by them up to scalar factors. The fact that B is proper turns out to be equivalent to a small part of these conditions.

Proposition 6. A basis B in irreducible representation V of gl_n is proper if and only if it satisfies the condition (a) of Theorem 4 for all $\underline{s} = (s_1,\dots,s_k)$ such that $s_i \leqslant 2$ for all i (and all $\underline{\lambda}$).

Proof. Recall that a family of subspaces in V is called a lattice if it is closed under sums and intersections. Let B be a basis in V; it is clear that the subspaces spanned by the subsets of B form a lattice. Therefore, to prove Proposition it suffices to show that the families of subspaces $V(\gamma,\nu)$ (for all possible γ and ν) and $V(\underline{s},\underline{\lambda})$ (for $\underline{s} = (s_1,\dots,s_k)$ such that $s_i \leqslant 2$ and all possible $\underline{\lambda}$) generate the same lattice. We prove this statement for an arbitrary finite dimensional representation of gl_n (not nesseccarily irreducible).

If $\underline{s} = (1,1,\dots,1)$ then the subspaces $V(\underline{s},\underline{\lambda})$ are just the weight subspaces in V. Now we use a lemma about representations of gl_2.

Lemma 2. Let V be a finite dimensional representation of gl_2, $\gamma = (\gamma_1,\gamma_2)$ a weight and $d \geqslant \max(0,\gamma_2-\gamma_1)$ an integer, Let v be a nonzero vector in V of weight γ. Then the following conditions are equivalent: (1) $d_1(v) \leqslant d$; (2) $v \in \overline{V(\underline{s},\underline{\lambda})}$, where $\underline{s} = (2)$, $\underline{\lambda} = (\gamma_1+d,\ \gamma_2-d)$.

This lemma follows at once from definitions and the well-known description of irreducible representation of gl_2.

Now for each $i = 1,\dots,n-1$ consider the subalgebra of gl_n isomorphic to gl_2 with the basis $(E_{ii}, E_{i,i+1}, E_{i+1,i}, E_{i+1,i+1})$. Using Proposition 3 (a) and Lemma 1 it is easy to see that our statement that two lattices coincide follows from Lemma 2 applied to the restrictions of V to these subalgebras. This proves

Proposition 6 and Theorem 2.

Proof of Theorem 3. We use the following lemma.

Lemma 3. Let Λ be a G-pattern for gl_3 of type $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and weight $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Then $d_1^{(1)}(\Lambda) + \max(d_1^{(2)}(\Lambda), d_2^{(2)}(\Lambda))$ depends only on λ and γ and equals $\max(\lambda_1 - \gamma_1, \gamma_3 - \lambda_3)$.

Proof. Follows at once from (2) and (3) in §1.

Now let (f_Λ) be a properly parametrized basis in the irreducible representation V_λ of gl_3. Let Λ be a G-pattern of type λ. Let γ denote the weight of Λ, and $\nu = (\nu_1, \nu_2, \nu_3)$ be a highest weight for gl_3 such that $\nu_1 - \nu_2 = d_1^{(1)}(\Lambda)$, $\nu_2 - \nu_3 = \max(d_1^{(2)}(\Lambda), d_2^{(2)}(\Lambda))$.

Using Lemma 3, Proposition 5 and the definition of a proper parametrization we see that the subspace $V_\lambda(\gamma,\nu)$ is one dimensional and is spanned by f_Λ. Thus, f_Λ is uniquely determined by Λ up to a multiple. This proves Theorem 3.

Remark. Let B be a basis in the irreducible finite dimensional representation V_λ of gl_n satisfying (a) and (b) from Theorem 4. For each $i = 1,\dots,n$ consider the restriction of V_λ to the subalgebra gl_i embedded in gl_n in a standard way (i.e. gl_i has a basis (E_{jk}) with $1 \leqslant j, k \leqslant i$). Let $f \in B$; it is easy to see that there is the unique highest weight $\lambda^{(i)}$ for gl_i such that $f \in V_\lambda((i), \lambda^{(i)}) - V^0((i), \lambda^{(i)})$. Furthermore, the highest weights $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)}$ form a G-pattern of type λ. Therefore, we obtain a parametrization of B by G-patterns of type λ. Apparently, for the basis constructed in [7] this parametrization is proper.

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