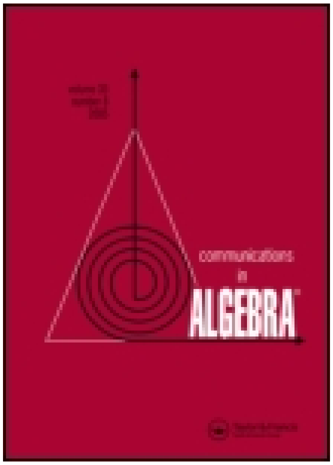

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The wedderburn-malcev theorem for comodule algebras

D. Ştefan [a] & F. Van Oystaeyen [b]

[a] Department of Mathematics , University of Bucharest , Str. Academiei 14, Bucharest, RO-70109, Romania

[b] Department of Mathematics , University of Antwerp (UIA) , Universiteitsplein 1, Wilrijk, B-2610, Belgium

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THE WEDDERBURN-MALCEV THEOREM FOR COMODULE ALGEBRAS

D. Ştefan
University of Bucharest,
Department of Mathematics
Str. Academiei 14, RO-70109
Bucharest, Romania

F. Van Oystaeyen
University of Antwerp (UIA),
Department of Mathematics
Universiteitsplein 1, B-2610
Wilrijk, Belgium

Abstract

Let H be a Hopf algebra over a field k. Under some assumptions on H we state and prove a generalization of the Wedderburn-Malcev theorem for H-comodule algebras. We show that our version of this theorem holds for a large enough class of Hopf algebras, such as coordinate rings of completely reducible affine algebraic groups, finite dimensional Hopf algebras over fields of characteristic 0 and group algebras. Some dual results are also included.

Introduction

Let k be an arbitrary field and suppose that A is a finite dimensional k-algebra of Jacobson radical J such that A/J is separable. The Wedderburn-Malcev theorem asserts that the canonical projection $\pi : A \to A/J$ has a section $u : A/J \to A$ which is an algebra map (equivalently, there is a subalgebra B in A such that $A \simeq B \oplus J$, direct sum of vector spaces). In the case when $J^2 = 0$, the idea of the construction of u (due to Hochschild) is to pick up an arbitrary k-linear section σ of π and then to modify σ by a certain Hochschild 1-chain. More precisely, since $J^2 = 0$ then σ defines an $A/J - A/J$ bimodule structure on J by $\widehat{a}.x = \sigma(\widehat{a})x$ and $x.\widehat{a} = x\sigma(\widehat{a})$, where $x \in J$ and $a \in A$. Moreover, the map $\omega : A/J \otimes A/J \to J$, $\omega(\widehat{x} \otimes \widehat{y}) = \sigma(\widehat{xy}) - \sigma(\widehat{x})\sigma(\widehat{y})$ is an Hochschild 2-cocycle of A/J with coefficients in J. Since the Hochschild cohomology of a separable algebra is trivial, there is an 1-chain $\theta : A/J \to J$ such that $d_1(\theta) = \omega$. Now it is easy to see that $u := \sigma + \theta$ is an algebra section of π. The general case, when J^2 is not necessarily zero, may be immediately settled by induction.

In this paper we prove the following generalization of the Wedderburn-Malcev theorem (for definitions see page 4):

Theorem 2.4. *Let H be a cosemisimple Hopf algebra such that there is a left integral $\lambda \in H^*_{\mathrm{ad}}$. Let A be a finite dimensional comodule algebra over H such that J, the Jacobson radical of A, is a subcomodule of A and A/J is separable. Then there is a subcomodule algebra B in A such that $A = B \oplus J$ (direct sum of H-comodules).*

We adopt the same strategy for proving this theorem, namely we construct a morphism of comodule algebras $u : A/J \to A$ which is a section of the canonical projection $\pi : A \to A/J$. Let us suppose that $J^2 = 0$. Since H is cosemisimple we can pick up an *H-colinear* section σ of π that affords an *H-colinear* Hochschild 2-cocycle ω. Furthermore, the H-colinear Hochschild cochains define a subcomplex of the standard Hochschild complex. By using a Maschke type theorem for Hopf bimodules (see Theorem 2.1), we prove that this subcomplex is exact under the assumptions of the Theorem 2.4. This property will allow us to construct an *H-colinear* map $\theta : A/J \to J$ such that $d_1(\theta) = \omega$. We end the proof by remarking that $u := \sigma + \theta$ is a comodule algebra section of π.

In the first section of the paper we prove some preliminary results which we need in the proof of Theorem 2.1. For instance, if M and N are two right H-comodules we recall the definition of $\mathrm{HOM}_k(M, N)$ and we show that it is the rational part of a certain H^*-module structure on $\mathrm{Hom}_k(M, N)$.

The main result, the generalization of the Wedderburn-Malcev theorem, is proved in the second part of the article. Next we show that the Theorem 2.4 holds for a large enough class of Hopf algebras, including cosemisimple finite dimensional Hopf algebras over fields of characteristic zero (Corollary 2.7), group algebras of arbitrary groups (Corollary 2.8) and coordinate rings of completely reducible affine algebraic groups (Corollary 2.10). Finally, following referee's suggestion, some dual results are studied: Theorem 2.12 and Corollary 2.13.

1 The functor $\mathrm{HOM}_k(-,-) : \mathcal{M}^H \times \mathcal{M}^H \to \mathcal{M}^H$

Let H be a Hopf algebra. If A is a right H-comodule algebra and M, N are two right A-Hopf modules then the right H-comodule $\mathrm{HOM}_A(M, N)$ was defined by Ulbrich in [8]. In particular, by taking $A = k$, one associates to each pair of right H-comodules $M, N \in \mathcal{M}^H$ a right H-comodule $\mathrm{HOM}_k(M, N)$. In this paragraph we study some properties of the functor $\mathrm{HOM}_k(-,-) : \mathcal{M}^H \times \mathcal{M}^H \to \mathcal{M}^H$.

We start by providing an equivalent description of this functor. Namely, we shall prove that $\mathrm{HOM}_k(M, N)$ is the rational part of a certain H^*-module structure on $\mathrm{Hom}_k(M, N)$, where M, N are H-comodules, as above.

Proposition 1.1. *Let $M, N \in \mathcal{M}^H$. For any $\alpha \in H^*$ and $f \in \mathrm{Hom}_k(M, N)$ define $\alpha.f \in \mathrm{Hom}_k(M, N)$ by $(\alpha.f)(m) = \sum \alpha[f(m_0)_1 S(m_1)] f(m_0)_0$. Then "." defines a left H^*-module structure on $\mathrm{Hom}_k(M, N)$.*

Proof. Let α, β be arbitrary elements in H^* and $f \in \mathrm{Hom}_k(M,N)$. Then by the definition of the convolution product and of the action "." we have:

$$\begin{aligned}[(\alpha * \beta).f](m) &= \sum (\alpha * \beta)[f(m_0)_1 S(m_1)] f(m_0)_0 \\ &= \sum \alpha[f(m_0)_1 S(m_2)] \beta[f(m_0)_2 S(m_1)] f(m_0)_0.\end{aligned}$$

On the other hand

$$(\beta.f)(m) = \sum \beta[f(m_0)_1 S(m_1)] f(m_0)_0,$$

so

$$\sum (\beta.f)(m_0)_1 \otimes (\beta.f)(m_0)_1 = \sum \beta[f(m_0)_2 S(m_1)] f(m_0)_0 \otimes f(m_0)_1.$$

Thus

$$\begin{aligned}[\alpha.(\beta.f)](m) &= \sum \alpha[(\beta.f)(m_0)_1 S(m_1)](\beta.f)(m_0)_0 \\ &= \sum \alpha[f(m_0)_1 S(m_2)] \beta[f(m_0)_2 S(m_1)] f(m_0)_0,\end{aligned}$$

so the proposition is proved. ∎

Definition 1.2. *If M, N are two right H-comodules then we define $\mathrm{HOM}_k(M,N)$ to be the rational part of $\mathrm{Hom}_k(M,N)$, i.e. the unique maximal rational H^*-submodule of $\mathrm{Hom}_k(M,N)$.*

Remark 1.3. By the definition of the rational part of an H^*-module there is an unique right H-comodule structure $\rho : \mathrm{HOM}_k(M,N) \to \mathrm{HOM}_k(M,N) \otimes H$ such that the corresponding H^* action is ".". Then, for any $f \in \mathrm{HOM}_k(M,N)$ we have $\rho(f) = \sum f_0 \otimes f_1$ if and only if

$$\sum \alpha(f_1) f_0(m) = \sum \alpha[f(m_0)_1 S(m_1)] f(m_0)_0, \tag{1}$$

for all $\alpha \in H^*$ and $m \in M$.

Proposition 1.4. *If $M, N \in \mathcal{M}^H$ then $\mathrm{HOM}_k(M,N)$ defined in 1.2 and Ulbrich's construction are identical.*

Proof. Let us briefly recall Ulbrich's construction. If we put

$$\begin{aligned}\omega &: \mathrm{Hom}_k(M,N) \to \mathrm{Hom}_k(M, N \otimes H),\ \omega(f)(m) = \sum f(m_0)_0 \otimes f(m_0)_1 S(m), \\ \varphi &: \mathrm{Hom}_k(M,N) \otimes H \to \mathrm{Hom}_k(M, N \otimes H),\ \varphi(f \otimes h) = f(m) \otimes h.\end{aligned}$$

then it is easy to see that φ is an injective map. Let $\overline{\mathrm{HOM}}_k(M,N)$ be the set $\{f \in \mathrm{Hom}_k(M,N) \mid \omega(f) \in \mathrm{Im}(\varphi)\}$. In [8] Ulbrich proved that the k-linear function $\overline{\rho} : \overline{\mathrm{HOM}}_k(M,N) \to \overline{\mathrm{HOM}}_k(M,N) \otimes H$, $\overline{\rho} = \varphi^{-1} \circ \omega$ defines a comodule structure on $\overline{\mathrm{HOM}}_k(M,N)$, such that $\overline{\mathrm{HOM}}_k(M,N)$ with the induced left H^*-module structure is a submodule of $\mathrm{Hom}_k(M,N)$. It follows that $\overline{\mathrm{HOM}}_k(M,N) \subseteq \mathrm{HOM}_k(M,N)$, since $\mathrm{HOM}_k(M,N)$ is by definition the largest rational submodule of $\mathrm{Hom}_k(M,N)$. For the other inclusion, let us take $f \in \mathrm{HOM}_k(M,N)$ and write

$$\rho(f) = \sum f_0 \otimes f_1.$$

Then by (1) we get

$$\sum \alpha(f_1) f_0(m) = \sum \alpha[f(m_0)_1 S(m_1)] f(m_0)_0.$$

Therefore we have:

$$(\mathrm{id} \otimes \alpha)(z) = 0,$$

where $z = \varphi(\sum f_0 \otimes f_1)(m) - \omega(f)(m)$. If $B = \{e_i \mid i \in I\}$ is a basis on N and $z = \sum_{i \in J} e_i \otimes h_i$, for a certain finite set $J \subseteq I$, then

$$\textstyle\sum_{i \in J} \alpha(h_i) e_i = 0. \quad \forall \alpha \in H^*.$$

It results that $\alpha(h_i) = 0$, for all $\alpha \in H^*$, so $h_i = 0$ for every $i \in J$. In conclusion $z = 0$, that implies $\omega(f) = \varphi(\sum f_0 \otimes f_1)$. Furthermore, the above equality shows that $\rho(f) = \overline{\rho}(f)$, since $\rho(f) = \sum f_0 \otimes f_1$ and $\overline{\rho}(f) = (\varphi^{-1} \circ \omega)(f)$, whenever $\omega(f) \in \mathrm{Im}(\varphi)$. Hence $\mathrm{HOM}_k(M, N)$ and $\overline{\mathrm{HOM}}_k(M, N)$ are identical not only as sets but as right H-comodules. ∎

We now study the case when A is a right H-comodule algebra and M, N are right A-Hopf modules. We recall that A is a comodule algebra over H if the algebra A is endowed with a right H-comodule structure via an *algebra* map $\rho_A : A \to A \otimes H$. A right A-module M is called Hopf module if $\rho_M : M \to M \otimes H$ defines a comodule on M such that

$$\rho_M(ma) = \sum m_0 a_0 \otimes m_1 a_1, \quad \forall m \in M, \ \forall a \in A.$$

A morphism of Hopf modules $f : M \to N$ is by definition a map which is right A-linear and H-colinear. The set of all morphism of Hopf modules from M to N will be denoted by $\mathrm{Hom}_A^H(M, N)$ and the category of right A-Hopf modules will be denoted by $\mathcal{M}_A^H$.

Proposition 1.5. *If A is a right H-comodule algebra and M, N are right A-Hopf modules, then $\mathrm{Hom}_A(M, N)$ is a left H^*-submodule of $\mathrm{Hom}_k(M, N)$.*

Proof. Let $f \in \mathrm{Hom}_k(M, N)$ and $\alpha \in H^*$. The action of H^* on $\mathrm{Hom}_k(M, N)$ is defined in (1.1), so for $m \in M$ and $a \in A$ we have:

$$\begin{aligned}
(\alpha.f)(ma) &= \sum \alpha[f((ma)_0)_1 S((ma)_1)] f((ma)_0)_0 \\
&= \sum \alpha[f(m_0)_1 a_1 S(a_2) S(m_1)] f(m_0)_0 a_0 \\
&= \sum \alpha[f(m_0)_1 S(m_1)] f(m_0)_0 a \\
&= (\alpha.f)(m) a,
\end{aligned}$$

since M, N are Hopf modules and f is right A-linear. ∎

Definition 1.6. *For any right H-comodule algebra and any right A-Hopf modules M, N we define* $\mathrm{HOM}_A(M, N) := \mathrm{Hom}_A(M, N) \bigcap \mathrm{HOM}_k(M, N)$.

Remark 1.7. a) Since $\mathrm{Hom}_A(M, N)$ is an H^*-submodule of $\mathrm{Hom}_k(M, N)$, it follows that $\mathrm{HOM}_A(M, N)$ is an H^*-submodule too. Actually, $\mathrm{HOM}_A(M, N)$ is a rational submodule since any submodule of a rational module is rational, [7, Theorem 2.1.3].

b) Clearly the functions ω and φ defined in the proof of Proposition 1.4 map $\mathrm{Hom}_A(M,N)$ and $\mathrm{Hom}_A(M,N)\otimes H$ respectively to $\mathrm{Hom}_A(M,N\otimes H)$. Hence the restrictions of ρ and $\overline{\rho}$ to $\mathrm{HOM}_A(M,N)$ define the same comodule structure on $\mathrm{HOM}_A(M,N)$, namely Ulbrich's one.

Lemma 1.8. *Let M, $N\in\mathcal{M}^H$ be two H-comodules. If M is finite dimensional then* $\mathrm{HOM}_k(M,N)=\mathrm{Hom}_k(M,N)$.

Proof. First, let us study the case $N=k$. If $\{e_1,\dots,e_s\}$ is a basis on M and $\{e_1^*,\dots,e_s^*\}$ is its dual basis then it is easy to see that

$$\alpha.e_i^*=\sum\alpha(S(h_{ip}))e_p^*,$$

where the set $\{h_{ip}\mid 1\le i,p\le s\}$ is such that $\rho(e_i)=\sum_{p=1}^s e_p\otimes h_{pi}$. This proves that e_i^* belongs to the rational part of M^*, so M^* is rational. Now the general case can be settled by remarking that $\mathrm{Hom}_k(M,N)\simeq N\otimes M^*$ as left H^*-modules. ■

Corollary 1.9. *Let A be a right H-comodule algebra and M, N be Hopf modules. If M is finite dimensional then* $\mathrm{HOM}_A(M,N)=\mathrm{Hom}_A(M,N)$.

Proof. Obviously, since $\mathrm{HOM}_A(M,N)=\mathrm{Hom}_A(M,N)\bigcap\mathrm{HOM}_k(M,N)$. ■

Remark 1.10. For any $M\in\mathcal{M}^H$ define $M^{\mathrm{co}(H)}:=\{m\in M\mid\rho_M(m)=m\otimes 1\}$. If M, $N\in\mathcal{M}^H$ then $\mathrm{HOM}_k(M,N)^{\mathrm{co}(H)}=\mathrm{Hom}^H(M,N)$, the space of all H-colinear maps from M to N. Furthermore, if A is a right H-comodule algebra and M, N are in $\mathcal{M}_A^H$ then $\mathrm{HOM}_A(M,N)^{\mathrm{co}(H)}=\mathrm{Hom}_A^H(M,N)$, the space of all morphisms of Hopf modules, see [8].

We end this section by studying some properties of the class of *ad-invariant* elements in H^*, the analogue of central functions of a group.

Definition 1.11. *Let H be a Hopf algebra. An element $\alpha\in H^*$ is called* ad-invariant *if $\alpha(\sum x_1yS(x_2))=\varepsilon(x)\alpha(y)$, for all x, $y\in H$. The set of all ad-invariant elements in H^* will be denoted by H^*_{ad}.*

Some elementary properties of ad-invariant elements are collected in the following proposition.

Proposition 1.12. *Let H be a Hopf algebra.*

a) The set H^*_{ad} is a subalgebra of H^*.

b) If λ is a left integral of a finite dimensional cosemisimple Hopf algebra H over a field of characteristic 0 then $\lambda\in H^*_{\mathrm{ad}}$.

Proof. 1) Let α, $\beta\in H^*_{\mathrm{ad}}$. Then:

$$\begin{aligned}(\alpha*\beta)(\sum x_1yS(x_2)) &= \sum\alpha[(x_1)_1y_1S(x_2)_1]\beta[(x_1)_2y_2S(x_2)_2]\\ &= \sum\alpha[(x_1)y_1S(x_4)]\beta[(x_2)y_2S(x_3)]\end{aligned}$$

$$\begin{aligned}
&= \sum \varepsilon(x_2)\alpha[x_1 y_1 S(x_3)]\beta(y_2) \\
&= \sum \alpha[x_1 y_1 S(x_2)]\beta(y_2) \\
&= \varepsilon(x)\sum \alpha(y_1)\beta(y_2) \\
&= \varepsilon(x)(\alpha * \beta)(y).
\end{aligned}$$

b) Since H is cosemisimple and $\mathrm{char}(k) = 0$ then H is semisimple, unimodular and moreover $S^2 = \mathrm{id}_H$, see [4] and [3]. By [6, Lemma 3] it follows that λ is cocommutative (i.e. $\lambda(xy) = \lambda(yx)$, $\forall x, y \in H$). We conclude this part by remarking that

$$\lambda(\sum x_1 y S(x_2)) = \lambda(\sum y S(x_2) x_1) = \varepsilon(x)\lambda(y).$$

Note that $\sum S(x_2)x_1 = \varepsilon(x)1_H$, since $S^2 = \mathrm{id}_H$. ∎

We recall that M is an object in the category ${}_A\mathcal{M}_A^H$ if M is an $A - A$ bimodule and a right H-comodule via $\rho : M \to M \otimes H$ such that

$$\rho(am) = \sum a_0 m_0 \otimes a_1 m_1, \tag{2}$$
$$\rho(ma) = \sum m_0 a_0 \otimes m_1 a_1. \tag{3}$$

The map $f : M \to N$ is a morphism in ${}_A\mathcal{M}_A^H$ if f is a morphism of $A - A$ bimodules and of H-comodules. Define $\mathrm{HOM}_{A-A}(M, N) := \mathrm{Hom}_{A-A}(M, N) \bigcap \mathrm{HOM}_k(M, N)$.

Proposition 1.13. *Let A be an H-comodule algebra and let M, N be two Hopf bimodules. If $f \in \mathrm{HOM}_{A-A}(M, N)$ and $\alpha \in H^*_{\mathrm{ad}}$ then $\alpha.f \in \mathrm{HOM}_{A-A}(M, N)$.*

Proof. Since $\mathrm{Hom}_A(M_A, N_A) \bigcap \mathrm{HOM}_k(M, N)$ is a left H^*-module it follows that $\alpha.f$ belongs to this intersection for *any* $\alpha \in H^*$. We just have to prove that $\alpha.f$ is left A-linear too, provided $\alpha \in H^*_{\mathrm{ad}}$. Indeed, we have:

$$\begin{aligned}
(\alpha.f)(am) &= \sum \alpha[f((am)_0)_1 S((am)_1)] f((am)_0)_0 \\
&= \sum \alpha[f(a_0 m_0)_1 S(a_1 m_1)] f(a_0 m_0)_0 \quad \text{(by 2)} \\
&= \sum \alpha[(a_0)_1 f(m_0)_1 S(m_1) S(a_1)](a_0 f(m_0))_0 \quad (f \text{ is left } A\text{-linear}) \\
&= \sum \alpha[(a_0)_1 f(m_0)_1 S(m_1) S(a_1)](a_0)_0 f(m_0)_0 \quad \text{(by 2)} \\
&= \sum \alpha[a_1 f(m_0)_1 S(m_1) S(a_2)] a_0 f(m_0)_0 \\
&= \sum \alpha[f(m_0)_1 S(m_1)] a f(m_0)_0 \quad (\alpha \in H^*_{\mathrm{ad}}) \\
&= a(\alpha.f)(m),
\end{aligned}$$

so the proposition follows. ∎

Remark 1.14. Note that, in general, $\mathrm{HOM}_{A-A}(M, N)$ is not a right H-comodule since $\mathrm{HOM}_{A-A}(M, N)$ is just an H^*_{ad}-module, not an H^*-module.

2 The main results

The main aim of this section is to prove a generalization of Wedderburn-Malcev theorem for finite dimensional H-comodule algebras, where H is a cosemisimple Hopf algebra which has an ad-invariant left integral.

Recall that a Hopf algebra is cosemisimple if and only if there is a left integral $\lambda \in H^*$ such that $\lambda(1) = 1$.

Theorem 2.1. *Let H be a cosemisimple Hopf algebra such that there is a nonzero ad-invariant left integral $\lambda \in H^*_{\mathrm{ad}}$. Suppose that A is an H-comodule algebra, M and N are in ${}_A\mathcal{M}^H_A$ and $f : M \to N$ is a morphism of Hopf bimodules. If f has a section in $\mathrm{HOM}_{A-A}(N, M)$ then f has a section in $\mathrm{Hom}^H_{A-A}(N, M)$.*

Proof. Let $\sigma \in \mathrm{HOM}_{A-A}(N, M)$ be an $A-A$ bilinear map such that $f \circ \sigma = I_N$. We want to construct a morphism of Hopf bimodules $\pi : N \to M$ such that $f \circ \pi = I_N$.

We take by definition $\pi = \lambda.\sigma$. We know that $\mathrm{HOM}_{A-A}(M, N)$ is an H^*_{ad}-module, so π is in $\mathrm{HOM}_{A-A}(M, N)$. In particular it follows that π is $A-A$ bilinear, so it remains to show that π is an H-colinear section of f.

Obviously, π is H-colinear if and only if π is H^*-linear, so let $\alpha \in H^*$ and $n \in N$. Since λ is an integral we have $\sum \lambda(x_2)x_1 = \lambda(x)1_H$, for all $x \in H$. Therefore

$$\begin{aligned} \alpha.\pi(n) &= \alpha. \sum \lambda[\sigma(n_0)_1 S(n_1)]\sigma(n_0)_0 \\ &= \sum \lambda[\sigma(n_0)_2 S(n_1)]\alpha(\sigma(n_0)_1)\sigma(n_0)_0 \\ &= \sum \alpha\{\lambda[\sigma(n_0)_2 S(n_1)]\sigma(n_0)_1 S(n_2)n_3\}\sigma(n_0)_0 \\ &= \sum \alpha\{\lambda[\sigma(n_0)_1 S(n_1)]n_2\}\sigma(n_0)_0 \\ &= \sum \alpha(n_2)\lambda[\sigma(n_0)_1 S(n_1)]\sigma(n_0)_0 \\ &= \pi(\alpha.n). \end{aligned}$$

It remains to prove that π is a section for f. We have:

$$\begin{aligned} (f \circ \pi)(n) &= f[\sum \lambda(\sigma(n_0)_1 S(n_1))\sigma(n_0)_0] \\ &= \sum \lambda[\sigma(n_0)_1 S(n_1)] f(\sigma(n_0)_0). \end{aligned} \tag{4}$$

Since f is H-colinear and σ is a section of f we get

$$\sum f(\sigma(n)_0) \otimes \sigma(n)_1 = \rho(f(\sigma(n))) = \rho(n) = \sum n_0 \otimes n_1.$$

By (4) we obtain $(f \circ \pi)(n) = \sum \lambda(n_1 S(n_2))n_0 = \sum \varepsilon(n_1)\lambda(1)n_0 = n$, where in the last equality we have used that $\lambda(1) = 1$, since H is cosemisimple. ∎

The Hochschild cohomology of an algebra A (over an field k) with coefficients in the $A-A$ bimodule M is the homology of the complex $(\mathbf{C}^n(A, M), d_n)_{n\in\mathbb{N}}$:

$$0 \to M \xrightarrow{d_0} \mathrm{Hom}_k(A, M) \xrightarrow{d_1} \mathrm{Hom}_k(A \otimes A, M) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d_n} \cdots. \tag{5}$$

The differentials $d_\bullet$ are defined as follows: $d_0(m)(a) = am - ma$ and for $n > 1$ we set $d_{n-1}(f) := \sum_{i=0}^n (-1)^i d_{n-1,i}(f)$, where:

$$d_{n-1,i}(f)(a^1 \otimes \cdots \otimes a^n) = \begin{cases} a^1 f(a^2 \otimes \cdots \otimes a^n), & \text{if } i = 0; \\ f(a^1 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^n), & \text{if } 1 \le i \le n-1; \\ f(a^1 \otimes \cdots \otimes a^{n-1})a^n, & \text{if } i = n. \end{cases}$$

It is well-known that A is a separable algebra (that is A is a projective $A-A$ bimodule) if and only if $\mathbf{H}^1(A, M) = 0$ for any $M \in {}_A\mathcal{M}_A$. Here $\mathbf{H}^\bullet(A, M)$ denotes the Hochschild cohomology of A with coefficients in M. We are interested in studying some properties of the Hochschild cohomology in the case when H is a Hopf algebra, A is an H-comodule algebra and $M \in {}_A\mathcal{M}_A^H$.

Recall that the tensor product $M \otimes N$ of two right H-comodules (M, ρ_M) and (N, ρ_N) is an H-comodule via $\rho_{M\otimes N} : M \otimes N \to M \otimes N \otimes H$, given by:

$$\rho_{M\otimes N}(m \otimes n) = \sum m_0 \otimes n_0 \otimes m_1 n_1. \tag{6}$$

Similarly one can define a comodule structure on the tensor product of an arbitrary but finite number of H-comodules. In particular, $A^{\otimes n}$ is a right H-comodule, for any H-comodule algebra A. We shall denote the set of H-colinear maps from $A^{\otimes n}$ to M by $\mathrm{Hom}^H(A^{\otimes n}, M)$. Let us prove that d_0 maps $M^{\mathrm{co}(H)}$ to $\mathrm{Hom}^H(A, M)$ and d_n maps $\mathrm{Hom}^H(A^{\otimes n}, M)$ to $\mathrm{Hom}^H(A^{\otimes n+1}, M)$ for any $n > 0$.

Let $m \in M^{\mathrm{co}(H)}$ and $a \in A$. Then we have to prove that

$$\rho[d_0(m)(a)] = \sum d_0(m)(a_0) \otimes a_1.$$

But $d_0(m)(a) = am - ma$, so $\rho[d_0(m)(a)] = \sum (a_0 m - m a_0) \otimes a_1$, since m is in $M^{\mathrm{co}(H)}$ and $M \in {}_A\mathcal{M}_A^H$. For $n > 0$ we show that each $d_{n,i}$ maps $\mathrm{Hom}^H(A^{\otimes n}, M)$ to $\mathrm{Hom}^H(A^{\otimes n+1}, M)$. Indeed, we have:

$$\begin{aligned}
\rho[d_{n,0}(f)(a^1 \otimes \cdots \otimes a^n)] &= \rho[a^1 f(a^2 \otimes \cdots \otimes a^n)] \\
&= \sum a_0^1 f(a_0^2 \otimes \cdots \otimes a_0^n) \otimes a_1^1 a_1^2 \cdots a_1^n \\
&= \sum d_{n,0}(f)(a_0^1 \otimes \cdots \otimes a_0^n) \otimes a_1^1 a_1^2 \cdots a_1^n,
\end{aligned}$$

since $f \in \mathrm{Hom}^H(A^{\otimes n}, M)$ and $M \in {}_A\mathcal{M}_A^H$. Similarly one can show that $d_{n,i}$ satisfies the required property, for $i \geq 1$. In conclusion, we have proved:

Proposition 2.2. *If H is a Hopf algebra, A is a right H-comodule algebra and $M \in {}_A\mathcal{M}_A^H$ then*

$$0 \to M^{\mathrm{co}(H)} \xrightarrow{d_0} \mathrm{Hom}^H(A, M) \xrightarrow{d_1} \mathrm{Hom}^H(A \otimes A, M) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \mathrm{Hom}^H(A^{\otimes n}, M) \xrightarrow{d_n} \cdots$$

is a subcomplex of (5). It will be denoted by $(\widetilde{\mathbf{C}}^n(A, M), d_n)_{n\in\mathbb{N}}$ and we shall write $\widetilde{\mathbf{H}}^\bullet(A, M)$ for its cohomology.

We have already noticed that $A \in {}_A\mathcal{M}_A$ is projective if and only if $\mathbf{H}^1(A, M) = 0$. In the next proposition we shall prove a similar results for $A \in {}_A\mathcal{M}_A^H$.

Proposition 2.3. *Let A be a right comodule-algebra over the Hopf algebra H. Then the following conditions are equivalent.*

a) The product map $A \otimes A \to A$ has a section in the category ${}_A\mathcal{M}_A^H$.

b) For each $M \in {}_A\mathcal{M}_A^H$ and $n > 0$ we have $\widetilde{\mathbf{H}}^n(A, M) = 0$.

c) For each $M \in {}_A\mathcal{M}_A^H$ we have $\widetilde{\mathbf{H}}^1(A, M) = 0$.

d) If $0 \to K \to M \to Q \to 0$ is exact in ${}_A\mathcal{M}_A^H$ and splits H-colinearly then $0 \to \mathrm{Hom}_{A-A}^H(A, K) \to \mathrm{Hom}_{A-A}^H(A, M) \to \mathrm{Hom}_{A-A}^H(A, Q) \to 0$ is exact too.

If in addition A is separable, H is cosemisimple and there is an integral $\lambda \in H_{\mathrm{ad}}^*$ then these four conditions hold true.

Proof. (a$\Longrightarrow$b) Let $\sigma : A \to A \otimes A$ be a section in ${}_A\mathcal{M}_A^H$ of the multiplication map of A. Write $\sigma(1) = \sum_{i=1}^d l_i \otimes r_i$. Since σ is a morphism in ${}_A\mathcal{M}_A^H$ we get

$$\begin{aligned} \textstyle\sum_{i=1}^d al_i \otimes r_i &= \textstyle\sum_{i=1}^d l_i \otimes r_i a, &(7)\\ \textstyle\sum_{i=1}^d l_i r_i &= 1, &(8)\\ \textstyle\sum_{i=1}^d (l_i)_0 \otimes (r_i)_0 \otimes (l_i)_1 (r_i)_1 &= \textstyle\sum_{i=1}^d l_i \otimes r_i \otimes 1. &(9) \end{aligned}$$

For $n > 0$ we have to prove that each n-cocycle in $\widetilde{\mathbf{C}}^n(A, M)$ is an n-coboundary. Suppose that $f \in \widetilde{\mathbf{C}}^n(A, M)$ is such a cocycle and define $g \in \mathbf{C}^{n-1}(A, M)$ by

$$g(a^1 \otimes \cdots \otimes a^{n-1}) = \begin{cases} \sum_{i=1}^d f(l_i) r_i & \text{if } n = 1; \\ \sum_{i=1}^d f(a^1 \otimes \cdots a^{n-1} \otimes l_i) r_i & \text{if } n > 1. \end{cases}$$

The relation (9) implies that g is an n-cochain in $\widetilde{\mathbf{C}}^{n-1}(A, M)$. On the other hand if $m = d_{n-1}(g)(a^1 \otimes \cdots \otimes a^n)$ then

$$\begin{aligned} m &= a^1 g(a^2 \otimes \cdots \otimes a^n) - g(a^1 a^2 \otimes \cdots \otimes a^n) + \cdots + \\ &\quad + (-1)^{n-1} g(a^1 \otimes \cdots \otimes a^{n-1} a^n) + (-1)^n g(a^1 \otimes \cdots \otimes a^{n-1}) a^n \\ &= \textstyle\sum_{i=1}^d a^1 f(a^2 \otimes \cdots \otimes a^n \otimes l_i) r_i - \sum_{i=1}^d f(a^1 a^2 \otimes \cdots \otimes a^n \otimes l_i) r_i + \cdots + \\ &\quad + (-1)^{n-1} \textstyle\sum_{i=1}^d f(a^1 \otimes \cdots \otimes a^{n-1} \otimes l_i) r_i + \\ &\quad + (-1)^n \textstyle\sum_{i=1}^d f(a^1 \otimes \cdots \otimes a^{n-1} \otimes l_i) r_i a^n. \end{aligned}$$

By (7) we have $\sum_{i=1}^d f(a^1 \otimes a^2 \otimes \cdots \otimes l_i) r_i a^n = \sum_{i=1}^d f(a^1 \otimes a^2 \otimes \cdots \otimes a^n l_i) r_i$ so

$$\begin{aligned} m &= d_n(f)(a^1 \otimes \cdots \otimes a^n \otimes l_i) r_i + (-1)^n \textstyle\sum_{i=1}^d f(a^1 \otimes a^2 \otimes \cdots \otimes a^n) l_i r_i \\ &= (-1)^n f(a^1 \otimes a^2 \otimes \cdots \otimes a^n). \end{aligned}$$

The last equality comes by (8) and by the fact that f is a cocycle, therefore f is an n-coboundary in $\widetilde{\mathbf{C}}^n(A, M)$.

(b$\Longrightarrow$c) Obviously.

(c$\Longrightarrow$d) Since $0 \to K \to M \to Q \to 0$ is exact and splits H-colinearly it follows that $0 \to \widetilde{\mathbf{C}}^\bullet(A, K) \to \widetilde{\mathbf{C}}^\bullet(A, M) \to \widetilde{\mathbf{C}}^\bullet(A, Q) \to 0$ is an exact sequence of complexes. The implication now follows by the long exact sequence for cohomology and by remarking that the natural isomorphism $\mathbf{H}^0(A, M) \simeq \mathrm{Hom}_{A-A}(A, M)$ induces an isomorphism $\widetilde{\mathbf{H}}^0(A, M) \simeq \mathrm{Hom}_{A-A}^H(A, M)$, $\forall M \in {}_A\mathcal{M}_A^H$.

(d$\Longrightarrow$a) The multiplication map $\omega : A \otimes A \to A$ is a morphism in ${}_A\mathcal{M}_A^H$ that splits H-colinearly (the map $\sigma : A \to A \otimes A$, $\sigma(a) = a \otimes 1$ is an H-colinear section). By hypothesis the map obtained by applying the functor $\mathrm{Hom}_{A-A}^H(A, -)$ to ω is surjective, so there is $i : A \to A \otimes A$, a morphism in ${}_A\mathcal{M}_A^H$, that splits ω.

Finally, if A is separable then the multiplication map ω has an $A - A$ linear section and A is finite dimensional. So $\mathrm{HOM}_{A-A}(A, A \otimes A) = \mathrm{Hom}_{A-A}(A, A \otimes A)$ and by the Theorem 2.1 it results that ω has a section in $\mathrm{Hom}_{A-A}^H(A, A \otimes A)$ too. ∎

Theorem 2.4. *Let H be a cosemisimple Hopf algebra such that there is a left integral $\lambda \in H_{\mathrm{ad}}^*$. Let A be a finite dimensional comodule algebra over H such that J,* *the Jacobson radical of A, is a subcomodule of A and A/J is separable. Then there is a subcomodule algebra B in A such that $A = B \oplus J$ (direct sum of H-comodules).*

Proof. We divide the proof into two cases.

The case: $J^2 = 0$. Let $\pi : A \to A/J$ be the canonical projection. Since H is cosemisimple there is $\sigma : A/J \to A$, an H-colinear section of π. By assumption $J^2 = 0$, so the section σ defines an A/J-bimodule structure on J as follows:

$$\widehat{a}.x = \sigma(\widehat{a})x \quad \text{and} \quad x.\widehat{a} = x\sigma(\widehat{a}), \quad \forall x \in J, \forall \widehat{a} \in A/J.$$

Actually, we may check easily that $J \in {}_{A/J}\mathcal{M}^H_{A/J}$, since σ is H-colinear. Moreover, $\omega : A/J \otimes A/J \to J$, $\omega(\widehat{x} \otimes \widehat{y}) = \sigma(\widehat{xy}) - \sigma(\widehat{x})\sigma(\widehat{y})$ belongs to $\widetilde{\mathbf{Z}}^2(A/J, J)$, the set of 2-cocycles of $\widetilde{\mathbf{C}}^\bullet(A/J, J)$. By the preceding proposition it follows that $\widetilde{\mathbf{H}}^2(A/J, J) = 0$, so we find $\theta \in \mathrm{Hom}^H(A/J, J)$ such that $d_1(\theta) = \omega$. We now may check by a straightforward computation that $u : A/J \to A$, $u := \sigma + \theta$ is a morphism of algebras. Since σ and θ are H-colinear it follows that u is so. Obviously, u is a section of π as σ is a section of this map and $\mathrm{Im}(\theta) \subseteq \mathrm{Ker}(\pi)$. Thus we can take in this case $B := \mathrm{Im}(u)$.

The case: $J^2 \neq 0$. We shall proceed by induction on the dimension of A. If $\dim(A) = 1$ then $A = k$, so we have nothing to prove. Let J be the Jacobson radical of A. If $J^2 = 0$ then the theorem follows by the first step. Suppose that $J^2 \neq 0$. Then $A' := A/J^2$ is an H-comodule algebra (J^2 is an H-subcomodule) and obviously $\dim(A') < \dim(A)$. The Jacobson radical of A/J^2 is J/J^2, so by the induction hypothesis there is B' an H-subcomodule algebra of A' such that $A' = B' \oplus (J/J^2)$, direct sum of right comodules. Let B_0 be the unique subcomodule algebra of A such that $B_0/J^2 \simeq B'$. The Jacobson radical of A is nilpotent so $J^2 \subset J$ and $J^2 \neq J$. It follows $\dim(B_0) < \dim(A)$ and the Jacobson radical of B_0 is J^2. By applying the induction hypothesis once again, it results that there is a subcomodule algebra B of B_0 such that $B \oplus J^2 = B_0$. Thus $B \oplus J = A$. ■

Finally we consider some applications of the preceding theorem. We start by proving the classical Wedderburn-Malcev theorem.

Corollary 2.5. *Let A be a finite dimensional algebra such that A/J is separable, where J is the Jacobson radical of A. Then there is a subalgebra B of A such that $B \oplus J = A$ (direct sum of vector spaces).*

Proof. Straightforward, by taking the trivial coaction of $H := k$ on A. ■

Corollary 2.6. *Suppose that A is a finite dimensional Hopf algebra over a field of characteristic 0. If the Jacobson radical J of A is a Hopf ideal (for example take A to be the dual of a pointed Hopf algebra) then the algebra A is the smash product of $A^{\mathrm{co}(A/J)}$ by A/J.*

Proof. Since $\mathrm{char}(k) = 0$ it follows that A/J is separable and coseparable. Since J is a right A/J-comodule, by the Theorem 2.4, there is a morphism $u : A/J \to A$ of A/J-comodule algebras which is a section of the canonical projection. Now we can apply [5, Lemma 2.2]. ■

Corollary 2.7. *Let H be a finite dimensional cosemisimple Hopf algebra over a field of characteristic* 0. *Let A be a finite dimensional right H-comodule algebra such that J, the Jacobson radical of A, is an H-subcomodule. Then there is a subcomodule algebra B of A such that $B \oplus J = A$ (direct sum of comodules).*

Proof. By Proposition 1.12(b) there is a nonzero left integral $\lambda \in H^*_{\text{ad}}$ so we may apply Theorem 2.4. Note that A/J is separable since the base field is of characteristic zero. ■

Corollary 2.8. *Let G be an arbitrary group and A a finite dimensional G-graded algebra over a field k. If the Jacobson radical of A is a homogeneous ideal, i.e. $\sum_{g\in G}(J \bigcap A_g) = A$, and A/J is separable then there is a graded subalgebra B in A such that $B \oplus J = A$ (direct sum of G-graded vector spaces).*

Proof. The group algebra of G, denoted by $k[G]$, is a cosemisimple Hopf algebra and $\lambda \in k[G]^*$ defined by $\lambda(g) = \delta_{e,g}$ (the Kronecker symbol) is an ad-invariant left integral. It is well-known that an algebra A is a $k[G]$-comodule algebra if and only if A is G-graded. For instance, if $\rho : A \to A \otimes H$ is the comodule structure of A define $A_g := \{a \in A \mid \rho(a) = a \otimes 1\}$. Then $A = \oplus_{g\in G} A_g$ and $A_g A_h = A_{gh}$, so A is G-graded. Conversely, if A is G-graded with homogeneous components $(A_g)_{g\in G}$ then we can take $\rho : A \to A \otimes H$ to be the k-linear map given by $\rho(a) = a \otimes g$, $\forall a \in A_g$. Hence, the corollary follows by applying Theorem 2.4 once again. ■

Now let us consider the case of completely reducible affine algebraic groups. Let G be such a group over an algebraically closed field k. The algebra $\mathcal{O}(G)$, the coordinate ring of G, is a Hopf algebra via $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, the algebra map induced by the multiplication in G. The counit $\varepsilon : \mathcal{O}(G) \to k$ is the evaluation map $f \longmapsto f(1)$, where 1 is the neutral element of G. Moreover, for $f \in \mathcal{O}(G)$ we have

$$\Delta(f) = \textstyle\sum_i f_i' \otimes f_i''$$

if and only if $f(xy) = \sum_i f_i'(x) f_i''(y)$, $\forall x, y \in G$.

Furthermore, if V is a finite dimensional rational representation of G, that is a morphism of algebraic groups $\omega : G \to GL(V)$, then V may be viewed as an $\mathcal{O}(G)$-comodule as follows. Take $\{e_1, \ldots, e_n\}$ be a basis of V and write

$$\omega(g)(e_i) = \textstyle\sum_{j=1}^n \omega_{ji}(g) e_j.$$

Then $\omega_{ij} \in \mathcal{O}(G)$ and we define $\rho : V \to V \otimes \mathcal{O}(G)$ by

$$\rho(e_i) = \textstyle\sum_{j=1}^n e_j \otimes \omega_{ji}.$$

It is easy to see that ρ defines a comodule structure on V.

Lemma 2.9. *Let A be a finite dimensional algebra over a field k. If A is a rational representation of an affine algebraic group G via the map $\omega : G \to GL(A)$ and $\omega(g)$ is an algebra automorphism for any $g \in G$ then A is an $\mathcal{O}(G)$-comodule algebra.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of A and write $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$. We know that $\omega(g)$ is an algebra automorphism, so $\omega(g)(e_i e_j) = \omega(g)(e_i)\omega(g)(e_j)$. Therefore

$$\sum_{k=1}^n c_{ij}^k \omega(g)(e_k) = \sum_{p=1}^n \omega_{pi}(g) e_p \sum_{q=1}^n \omega_{qi}(g) e_q.$$

Hence

$$\sum_{r,k=1}^n \omega_{rk}(g) c_{ij}^k e_r = \sum_{p,q,r=1}^n \omega_{pi}(g)\omega_{qj}(g) c_{pq}^r e_r.$$

Thus

$$\sum_{k=1}^n \omega_{rk}(g) c_{ij}^k = \sum_{p,q=1}^n \omega_{pi}(g)\omega_{qj}(g) c_{pq}^r, \tag{10}$$

for all $1 \leq i, j, r \leq n$ and $g \in G$. On the other hand

$$\begin{aligned} \rho(e_i)\rho(e_j) &= \sum_{p=1}^n e_p \otimes \omega_{pi} \sum_{q=1}^n e_q \otimes \omega_{qj} = \sum_{p,q=1}^n e_p e_q \otimes \omega_{pi}\omega_{qj} \\ &= \sum_{p,q,r=1}^n c_{pq}^r e_r \otimes \omega_{pi}\omega_{qj} = \sum_{r=1}^n e_r \otimes \sum_{p,q=1}^n c_{pq}^r \omega_{pi}\omega_{qj} \end{aligned} \tag{11}$$

and

$$\rho(e_i e_j) = \sum_{k=1}^n c_{ij}^k \rho(e_k) = \sum_{k,r=1}^n c_{ij}^k e_r \otimes \omega_{rk} = \sum_{r=1}^n e_r \otimes \sum_{k=1}^n c_{ij}^k \omega_{rk}. \tag{12}$$

Thus by (10), (11) and (12) we obtain $\rho(e_i e_j) = \rho(e_i)\rho(e_j)$, for all $1 \leq i, j \leq n$. ∎

Corollary 2.10. *Let A be a finite dimensional algebra over an algebraically closed field k. Suppose that G is a completely reducible affine algebraic group and any $g \in G$ is an algebra automorphisms of A. If J is the Jacobson radical of A then there is a subalgebra $B \subseteq A$ such that $B \oplus J = A$ and $g(B) \subseteq B$ for all $g \in G$.*

Proof. By the above lemma A is an $\mathcal{O}(G)$-comodule algebra. Since any element of G is an algebra automorphism it follows that J is an $\mathcal{O}(G)$-subcomodule of A. It is well-known that G is completely reducible if and only if the Hopf algebra $\mathcal{O}(G)$ is cosemisimple if and only if there is a nonzero left integral (since $\mathcal{O}(G)$ is commutative $\lambda(1) \neq 0$, for any integral $\lambda \neq 0$). For these properties of $\mathcal{O}(G)$ the reader is referred to [1]. Obviously, if λ is such an integral then λ is ad-invariant, so we may apply the theorem in order to obtain the required subalgebra. Note that $g(B) = B$, $\forall g \in G$, since B is an $\mathcal{O}(G)$-subcomodule of A. ∎

Remark 2.11. Let G be a finite group such that the characteristic of k does not divide $|G|$. Then G may be regarded as an affine algebraic group and $\mathcal{O}(G) = k[G]^*$. The assumption on the order of G implies that $\mathcal{O}(G)$ is cosemisimple, so Corollary 2.10 holds for any such finite groups.

We end the paper by studying the dual result of our main theorem.

Theorem 2.12. *Let H be a finite dimensional semisimple Hopf algebra. Suppose that a nonzero integral t of H satisfies $\Delta(t) = \sum t_2 \otimes S^2(t_1)$. Let C be a left H-module coalgebra. If the coradical C_0 of C is coseparable and H-stable, then there is an H-linear coalgebra projection $C \to C_0$.*

Proof. If C is finite dimensional then the theorem follows by duality from (2.4). In the general case we can proceed as in the proof of [1, Theorem 2.3.11]. Indeed, it is enough to apply Zorn's lemma to the set of pairs (F, π), where F is an H-stable subcoalgebra of C and π is an H-linear projection from F to $F \bigcap C_0$, and to note that any finite dimensional subcoalgebra D is contained in an H-stable one (since H is finite dimensional then the H submodule generated by D is so). ∎

Corollary 2.13. *Suppose that* $\mathrm{char}(k) = 0$. *Let A be a Hopf algebra such that $H = A_0$, the coradical of A, is a finite dimensional Hopf subalgebra (for example take A to be a pointed Hopf algebra with finite dimensional coradical). Then the coalgebra A is a smash co-product of A/H^+A by H.*

Proof. By the above theorem there is an H-linear coalgebra projection $A \to A_0$, so we conclude by using [2, Corollary 2.5]. ∎

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