Several Complex Variables with Connections to Algebraic Geometry and Lie Groups

Joseph L. Taylor

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ABSTRACT. A graduate text with an integrated treatment of several complex variables and complex algebraic geometry, with applications to the structure theory and representation theory of Lie groups.

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Preface

This text evolved from notes I developed for use in a course on several complex variables at the University of Utah. The eclectic nature of the topics presented in the text reflects the interests and motivation of the graduate students who tended to enroll for this course. These students were almost all planning to specialize in either algebraic geometry or representation theory of semisimple Lie groups. The algebraic geometry students were primarily interested in several complex variables because of its connections with algebraic geometry, while the group representations students were primarily interested in applications of complex analysis – both algebraic and analytic – to group representations.

The course I designed to serve this mix of students involved a simultaneous development of basic complex algebraic geometry and basic several complex variables, which emphasized and capitalized on the similarities in technique of much of the foundational material in the two subjects. The course began with an exposition of the algebraic properties of the local rings of regular and holomorphic functions, first on $\mathbb{C}^n$ and then on varieties. This was followed by a development of abstract sheaf theory and sheaf cohomology and then by the introduction of coherent sheaves in both the algebraic and analytic settings. The fundamental vanishing theorems for both kinds of coherent sheaves were proved and then exploited. Typically the course ended with a proof and applications of Serre's GAGA theorems, which show the equivalence of the algebraic and analytic theories in the case of projective varieties. The notes for this course were corrected and refined, with the help of the students, each time the course was taught. This text is the result of that process.

There were instances where the course continued through the summer as a reading course for students in group representations. One summer, the objective was to prove the Borel-Weil-Bott theorem; another time, it was to explore a complex analysis approach to the study of representations of real semisimple Lie groups. Material from these summer courses was expanded and then included in the text as the final three chapters.

The material on several complex variables in the text owes a great debt to the text of Gunning and Rossi [**GR**], and the recent rewriting of that text by Gunning [**Gu**]. It was from Gunning and Rossi that I learned the subject, and the approach to the material that is used in Gunning and Rossi is also the approach used in this text. This means a thorough treatment of the local theory using the tools of commutative algebra, an extensive development of sheaf theory and the theory of coherent analytic sheaves, proofs of the main vanishing theorem for such sheaves (Cartan's Theorem B) in full generality, and a complete proof of the finite dimensionality of the cohomologies of coherent sheaves on compact varieties (the Cartan-Serre theorem). This does not mean that I have included treatments of all the topics covered in Gunning and Rossi. There is no discussion of pseudoconvexity, for example, or global embeddings, or the proper mapping theorem, or envelopes of holomorphy. I have included, however, a more extensive list of applications of the main results of the subject – particularly if one includes in this category Serre's GAGA theorems and the material on complex semisimple Lie groups and the proof of the Borel-Weil-Bott theorem.

Several complex variables is a very rich subject, which can be approached from a variety of points of view. The serious student of several complex variables should consult, not only Gunning's rewriting of Gunning and Rossi, but also the many excellent texts which approach the subject from other points of view. These include [**D**], [**Fi**], [**GRe**], [**GRe2**], [**Ho**], [**K**], and [**N**], to name just a few.

Interwoven with the material on several complex variables in this text is a simultaneous treatment of basic complex algebraic geometry. This includes the structure theory of local rings of regular functions and germs of varieties, dimension theory, the vanishing theorems for coherent and quasi-coherent algebraic sheaves, structure of regular maps between varieties, and the main theorems on the cohomology of coherent sheaves on projective spaces.

There are real advantages to this simultaneous development of algebraic and analytic geometry. Results in the two subjects often have essentially the same proofs; they both rely heavily on the same background material – commutative algebra for the local theory and homological algebra and sheaf theory for the global theory; and often a difficult proof in several complex variables can be motivated and clarified by an understanding of the often similar but technically simpler proof of the analogous result in algebraic geometry.

Several complex variables and complex algebraic geometry are not just similar; they are equivalent when done in the context of projective varieties. This is the content of Serre's GAGA theorems. We give complete proofs of these results in Chapter 13, after first studying the cohomology of coherent sheaves on projective spaces in Chapter 12.

The text could easily have ended with Chapter 13. This is where the course typically ends. The material in Chapters 14 through 16 is on quite a different subject – Lie groups and their representations – albeit one that involves the extensive use of several complex variables and algebraic geometry. Chapter 16 is devoted to a proof of the Borel-Weil-Bott theorem. This is the theorem which pinpoints the relationship between finite dimensional holomorphic representations of a complex semisimple Lie group G and the cohomologies of G-equivariant holomorphic line bundles on a projective variety, called the *flag variety*, constructed from G. Chapter 15 is a brief treatment of the subject of complex algebraic groups. This is included in order to provide proofs of some of the basic structure results for complex semisimple Lie groups that are needed in the formulation and proof of the Borel-Weil-Bott theorem. Chapter 14 is a survey of the background material needed if one is to understand Chapters 15 and 16. It includes material on topological groups and their representations, compact groups, Lie groups and Lie algebras, and finite dimensional representations of semisimple Lie algebras. These last three chapters are included primarily for the benefit of the student of Lie theory and group representations. This material illustrates that both several complex variables and complex algebraic geometry are essential tools in the modern study of group representations. The chapter on algebraic groups (Chapter 15) provides particularly compelling examples of the utility of algebraic geometry applied in the context of the structure theory of Lie groups. The proof of the Borel-Weil-Bott theorem in Chapter 16 involves applications of a wide range of material from several complex variables and algebraic geometry. In particular, it provides nice applications of the sheaf theory of Chapter 7, the Cartan-Serre theorem from Chapter 11, the material on projective varieties in Chapter 12, Serre's theorems in Chapter 13, and of course, the background material on algebraic groups and general Lie theory from Chapters 14 and 15.

I have tried to make the text as self-contained as possible. However, students who attempt to use it will need some background. This should include knowledge of the material from typical first year graduate courses in real and complex analysis, modern algebra, and topology. Also, students who wishes to confront the material in Chapters 14 through 16 will be helped greatly if they have had a basic introduction to Lie theory. Though the background material in Chapter 14 is reasonably self-contained, it is intended as a survey, and so some of the more technical proofs have been left out. For example, the basic theorems relating Lie algebras and Lie groups are stated without proof, as is the existence of compact real forms for complex semisimple groups and the classification of finite dimensional representations of semisimple Lie algebras.

Each chapter ends with an exercise set. Many exercises involve filling in details of proofs in the text or proving results that are needed elsewhere in the text, while others supplement the text by exploring examples or additional material. Cross-references in the text to exercises indicate both the chapter and the exercise number; that is, Exercise 2.5 refers to Exercise 5 of Chapter 2.

There are many individuals who contributed to the completion of this text. Edward Dunne, Editor for the AMS book program, noticed an early version of the course notes on my website and suggested that I consider turning them into a textbook. Without this suggestion and Ed's further advice and encouragement, the text would not exist. Several of my colleagues provided valuable ideas and suggestions. I received encouragement and much useful advice on issues in several complex variables from Hugo Rossi. Aaron Bertram, Herb Clemens, Dragan Miličić, Paul Roberts, and Angelo Vistoli gave me valuable advice on algebraic geometry and commutative algebra, making up, in part, for my lack of expertise in these areas. Henryk Hecht, Dragan Miličić, and Peter Trombi provided help on Lie theory and group representations. Without Dragan's help and advice, the chapters on Lie theory, algebraic groups, and the Borel-Weil-Bott theorem would not exist. The proof of the Borel-Weil-Bott theorem presented in Chapter 16 is due to Dragan, and he was the one who insisted that I approach structure theorems for semisimple Lie groups from the point of view of algebraic groups. The students who took the course the three times it was offered while the notes were being developed caught many errors and offered many useful suggestions. One of these students, Laura Smithies, after leaving Utah with a Ph.D. and taking a position at Kent State, volunteered to proofread the entire manuscript. I gratefully accepted this offer, and the result was numerous corrections and improvements. My sincere thanks goes out to all of these individuals and to my wife, Ulla, who showed great patience and understanding while this seemingly endless project was underway.

Joseph L. Taylor

Chapter 1

Selected Problems in One Complex Variable

The study of holomorphic functions of several complex variables involves the use of powerful tools from many areas of modern mathematics, areas such as commutative algebra, functional analysis, homological algebra, sheaf theory, and algebraic topology. For this reason, a course in several complex variables is a great opportunity to teach students how the seemingly separate fields of pure mathematics can be used in concert to solve difficult problems and produce striking results. However, this fact also makes the study of holomorphic functions in several variables much more difficult and sophisticated than the study of other classes of functions - continuous functions, differentiable functions, holomorphic functions of a single variable – that students encounter in their early graduate work. When they begin to realize this, students tend to ask questions such as: Why are we developing all this machinery? Where is this headed? What is this good for? It is difficult to answer these questions until much of the language and machinery of several complex variables has been developed. However, in this introductory section we will attempt to give some indication of where we are headed and why we are headed there, by discussing several problems from the theory of a single complex variable that illustrate some of the issues that will be central in the several variable theory.

While the main purpose of this chapter is to illustrate and motivate what is to come, that is not its only purpose. Some of the results developed in this chapter will be needed later. This is true, for example, of the results on partitions of unity in section 1.3 and those on the inhomogeneous Cauchy-Riemann equation in section 1.4.

1.1 Preliminaries

The complex plane will be denoted by $\mathbb{C}$, while complex n-space, the Cartesian product of n copies of $\mathbb{C}$, will be denoted by $\mathbb{C}^n$. The open disc of radius $r \geq 0$ centered at $a \in \mathbb{C}$ will be denoted $\Delta(a,r)$, while the closed disc with this radius and center will be denoted $\bar{\Delta}(a,r)$. If U is an open set in $\mathbb{C}$ and f a complex valued function defined on U, then f is *holomorphic* if its complex derivative

$$f'(z) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w}$$

exists for each $z \in U$. We will denote the space of all holomorphic functions on U by $\mathcal{H}(U)$.

We assume that the reader is familiar with the basic properties of holomorphic functions of a single variable as presented in standard texts (e.g. [**R**]) in the subject: A holomorphic function on U has a convergent power series expansion in a neighborhood of each point of U; a differentiable function is holomorphic if and only if it satisfies the Cauchy-Riemann equations; the space $\mathcal{H}(U)$ is an algebra over the complex field under the operations of pointwise addition, multiplication, and scalar multiplication; if a sequence $\{f_n\}$ in $\mathcal{H}(U)$ converges uniformly on each compact subset of U, then the limit function is also holomorphic on U; holomorphic functions satisfy the Cauchy integral theorem and formula, the identity theorem, and the maximum modulus theorem.

A function on an open set U in $\mathbb{C}^n$ is holomorphic if it is holomorphic in each variable separately (in the next chapter we shall prove that this is equivalent to the existence of local multi-variable power series expansions of the function). In the n variable case, we shall also denote the space of holomorphic functions on U by $\mathcal{H}(U)$. The space of continuous functions on a topological space U will be denoted by $\mathcal{C}(U)$, the space of n times continuously differentiable functions on an open set U in a Euclidean space by $\mathcal{C}^n(U)$, and the space of infinitely differentiable functions on U by $\mathcal{C}^\infty(U)$. As usual, the Euclidean norm of a point $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ is defined by $||z|| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}$.

1.2 A Simple Problem

Here we introduce a problem which is easy to state, but not so easy to solve. Its solution illustrates, in a relatively simple setting, many of the issues we will have to deal with later in this text.

Let U be an open set in $\mathbb{C}$. Since $\mathcal{H}(U)$ is an algebra, it is natural to try to find all of its maximal ideals. It is easy to see that each point $\lambda \in U$

determines a maximal ideal

$$M_\lambda = \mathrm{Ker}\{f \to f(\lambda) : \mathcal{H}(U) \to \mathbb{C}\}$$

and, in fact, every maximal ideal with $\mathbb{C}$ as quotient field has this form. To see this, observe that if M is a maximal ideal with quotient field $\mathbb{C}$, $\phi : \mathcal{H}(U) \to \mathbb{C}$ is the quotient homomorphism modulo M, and $\lambda = \phi(z)$, then $z - \lambda$ belongs to M. But, for each $f \in \mathcal{H}(U)$, it follows from the power series expansion of f at λ that $f(z) - f(\lambda)$ is divisible by $z - \lambda$. That is, $\dfrac{f(z) - f(\lambda)}{z - \lambda}$ has a removable singularity at λ and so it defines an $h \in \mathcal{H}(U)$ such that $f(z) - f(\lambda) = (z - \lambda)h(z)$. Thus, $f - f(\lambda)$ also belongs to M and $\phi(f) - f(\lambda) = \phi(f - f(\lambda)) = 0$. This means that ϕ agrees with the evaluation homomorphism $f \to f(\lambda)$ and $M = M_\lambda$.

Is every maximal ideal of $\mathcal{H}(U)$ of the form M_λ? No. Let $\{a_n\} \subset U$ be a sequence of points which has no limit point in U. Let I be the set of functions in $\mathcal{H}(U)$ which vanish at all but finitely many points of $\{a_n\}$. Then I is a proper ideal of $\mathcal{H}(U)$. However, there is no single point at which all the functions in I vanish. This follows from the fact that, given any discrete set S of points of U, there is a function in $\mathcal{H}(U)$ which vanishes exactly on S (a corollary of the Weierstrass theorem, which we will prove later in the chapter). Thus, no maximal ideal that contains I can be of the form M_λ. The following is true, however:

1.2.1 Theorem. *Each finitely generated maximal ideal of $\mathcal{H}(U)$ is of the form M_λ for some $\lambda \in U$.*

If a maximal ideal M of $\mathcal{H}(U)$ is finitely generated, say by $\{g_1, \dots, g_n\}$, and if these generators all vanish at some point $\lambda \in U$, then all the functions in M vanish at λ and we have $M \subset M_\lambda$. This, of course, implies that $M = M_\lambda$, since M is a maximal ideal. Thus, Theorem 1.2.1 will be proved if we can show that: if a finite set of functions $\{g_1, \dots, g_n\}$ does not have a common zero, then it does not generate a proper ideal. That is the content of the following proposition.

1.2.2 Proposition. *If a finite set of functions $\{g_1, \dots, g_n\} \subset \mathcal{H}(U)$ has no common zero in U, then the equation*

$$f_1 g_1 + \cdots + f_n g_n = 1 \tag{1.2.1}$$

has a solution for $f_1, \dots, f_n \in \mathcal{H}(U)$.

The proof of this result will occupy most of the chapter. Along the way, we shall prove a number of important results from the theory of holomorphic functions of one complex variable that are often not covered in a first year graduate course on the subject. Although the ideal theory problem posed here provides motivation for these results, they have much broader applicability.

1.3 Partitions of Unity

Proving Proposition 1.2.2 is a typical example of what we will call a *local to global* problem. That is, we know that the equation (1.2.1) has local solutions in the sense that for each $w \in U$ there is a neighborhood V of w and a solution to (1.2.1) consisting of functions f_i holomorphic on V. In fact, for some j, g_j does not vanish at w. Then $V_j = \{z \in U : g_j(z) \neq 0\}$ is a neighborhood of w on which equation (1.2.1) has a solution, given by setting $f_j = g_j^{-1}$ and $f_i = 0$ for $i \neq j$. Thus, we will have proved Proposition 1.2.2 if we can show that: if equation (1.2.1) has a solution locally in a neighborhood of each point of U, then it has a global solution.

We will encounter many of these local to global problems in the course of our study. Proposition 1.2.2 is a special case of a more general result concerning a system of linear equations

$$GF = H, \tag{1.3.1}$$

where U is an open set in $\mathbb{C}^n$, G is a given $p \times q$ matrix with entries from $\mathcal{H}(U)$, H is a given p vector of functions from $\mathcal{H}(U)$, and a solution F is sought which is a q vector of functions from $\mathcal{H}(U)$. Is it true that, if this system of equations has a solution locally in a neighborhood of each point of U, then it has a global solution on U? The answer is "yes", provided U is what is called a *domain of holomorphy*. To prove this result requires much of the machinery that we shall develop in this text. Every open set in $\mathbb{C}$ is a domain of holomorphy and so the answer is always "yes" for functions of a single variable. We won't prove that in this chapter, although we will prove it in the special case of equation (1.2.1).

While the local to global problem posed by (1.3.1) is quite formidable for holomorphic functions on an open set U in $\mathbb{C}^n$, the same problem for the classes of continuous or infinitely differentiable functions is actually quite easy. This is due to the fact that the classes of continuous and infinitely differentiable functions have a strong separation property – Urysohn's lemma. Urysohn's lemma for continuous functions on a locally compact Hausdorff space should be familiar to the reader. A similar result holds for $\mathcal{C}^\infty$ functions on Euclidean space.

1.3.1 Lemma. *If $K \subset U \subset \mathbb{R}^n$, with K compact and U open, then there exists $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $0 \leq f(x) \leq 1$ for all x, $f(x) = 1$ for $x \in K$, and $f(x) = 0$ for $x \notin U$.*

Proof. Let $a < b$ be positive numbers. The function $\lambda(t)$ which is $\exp(-t^{-1})$ for positive t and 0 at all other points on the line is infinitely differentiable (Exercise 1.1). Thus, the function $\psi(t) = \lambda(t-a)\lambda(b-t)$ is also in $\mathcal{C}^\infty(\mathbb{R})$. Note that

$$\psi(t) = \exp((a-t)^{-1} + (t-b)^{-1}) \quad \text{for} \quad a < t < b$$

and $\psi(t) = 0$ for all other values of t. We use ψ to define a function

$$\phi(t) = \int_t^b \psi(s)\, ds \left(\int_a^b \psi(s)\, ds \right)^{-1}$$

which has the properties that $0 \le \phi(t) \le 1$ for all real t, $\phi(t) = 1$ for $t \le a$, and $\phi(t) = 0$ for $t \ge b$.

We can now prove the lemma in the case where K is the closed ball $\{x \in \mathbb{R}^n : ||x-y|| \le a\}$ of radius a centered at y and U is the open ball $\{x \in \mathbb{C}^n : ||x-y|| < b\}$ of radius b centered at y. In fact, the function $f(x) = \phi(||x-y||)$ satisfies the conditions of the lemma in this case.

To prove the lemma in general, we cover K with finitely many closed balls B_i contained in U. Each such ball is contained in an open ball $V_i \subset U$ with the same center. We then choose functions f_i, as above, for each pair $B_i \subset V_i$. The function

$$f(x) = 1 - \prod_i (1 - f_i(x))$$

then has the required properties.

Given an open cover $\mathcal{V}$ of an open set U in $\mathbb{R}^n$, a $\mathcal{C}^\infty$ *partition of unity* subordinate to $\mathcal{V}$ is a collection $\{\phi_i\} \subset \mathcal{C}^\infty(U)$ such that: $0 \le \phi_i(x) \le 1$ for each $x \in U$ and each i; $\phi_i(x) = 0$ except on a compact subset of some member of $\mathcal{V}$; on any compact subset $K \subset U$, all but finitely many ϕ_i vanish identically; and

$$\sum_i \phi_i(x) = 1$$

for every $x \in U$.

1.3.2 Lemma. *Given an open cover $\mathcal{V}$ of an open set U in $\mathbb{R}^n$, there exists a $\mathcal{C}^\infty$ partition of unity subordinate to $\mathcal{V}$.*

Proof. Let $\{B_i\}$ be an enumeration of the set of all open balls in $\mathbb{R}^n$ with rational radii, centered at points with rational coordinates, and with the property that the closure of the ball is contained in some member of $\mathcal{V}$. Let

A_i be the ball with the same center as B_i but with half the radius. Clearly, $\cup_i A_i = U$.

By Lemma 1.3.1, there are functions f_i with $0 \le f_i(x) \le 1$ for all x, $f_i(x) = 1$ on $\bar{A}_i$, and $f_i(x) = 0$ on the complement of B_i. We define $\phi_1 = f_1$ and

$$\phi_i = (1 - f_1)(1 - f_2)\dots(1 - f_{i-1})f_i \quad \text{for} \quad i > 1.$$

Note that $\phi_i = 0$ on the complement of B_i. Also, an induction argument shows that

$$1 - (\phi_1 + \phi_2 + \cdots + \phi_n) = (1 - f_1)(1 - f_2)\dots(1 - f_n)$$

and so

$$\phi_1 + \phi_2 + \cdots + \phi_n = 1 \quad \text{on} \quad A_1 \cup A_2 \cup \cdots \cup A_n.$$

Given a compact subset K of U, there is an n so that $K \subset A_1 \cup A_2 \cup \cdots \cup A_n$. It follows that, for $i > n$, ϕ_i vanishes identically on K. Thus, the infinite sum $\sum_i \phi_i$ makes sense and is identically 1 on U.

In what follows, $\mathcal{C}^\infty(U)^p$ will denote the space of vectors of length p with entries from $\mathcal{C}^\infty(U)$. We consider $\mathcal{C}^\infty(U)^p$ a module over the algebra $\mathcal{C}^\infty(U)$ through the usual coordinate-wise operations of addition and scalar multiplication.

1.3.3 Theorem. *If $U \subset \mathbb{R}^n$ is an open set, G is a $p \times q$ matrix with entries from $\mathcal{C}^\infty(U)$, and $H \in \mathcal{C}^\infty(U)^p$, then the equation*

$$GF = H$$

has a solution $F \in \mathcal{C}^\infty(U)^q$, provided it has a solution in $\mathcal{C}^\infty(V)^q$ for some neighborhood V of each point of U.

Proof. We choose an open cover $\mathcal{V}$ of U by sets on which the equation has a solution. We then choose a partition of unity $\{\phi_i\}$ subordinate to this cover. Thus, each ϕ_i vanishes off a compact subset of a set V_i in $\mathcal{V}$ and there is an $F_i \in \mathcal{C}^\infty(V_i)^q$ such that $GF_i = H$ on V_i. If we define $\phi_i F_i$ to be 0 in the complement of V_i, then the result is a vector which is in $\mathcal{C}^\infty(U)^q$, as is the sum $F = \sum_i \phi_i F_i$. Furthermore,

$$GF = G\sum_i \phi_i F_i = \sum_i \phi_i G F_i = \sum_i \phi_i H = H.$$

Thus, F is a global solution to our equation.

Of course, this argument won't work in the case of systems of equations involving holomorphic functions, because we do not have anything like Urysohn's lemma or partitions of unity in the class of holomorphic functions – even in the case of one variable. How do we know this? The identity theorem says that a holomorphic function on a connected open set in $\mathbb{C}$ is identically 0, if it is 0 on an open subset. In fact, the set on which a non-constant holomorphic function of one variable on a connected open set is 0 (or any other constant value) is always a discrete set. Thus, there certainly are no holomorphic functions with properties like those of the ϕ_i in the above argument.

So how do we proceed to prove a result like Proposition 1.2.2? We reduce the problem to the one we just solved by using the inhomogeneous Cauchy-Riemann equations to relate holomorphic functions to $\mathcal{C}^\infty$ functions.

1.4 The Cauchy-Riemann Equations

In this section we will make use of the notation of differential forms in the plane. Thus, if f is a $\mathcal{C}^\infty$ function on an open set in the plane, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1.4.1}$$

An expression of the form $g\,dx + h\,dy$ is called a *differential 1-form*. Equation (1.4.1) defines an operator d from $\mathcal{C}^\infty$ functions to $\mathcal{C}^\infty$ differential 1-forms.

A $\mathcal{C}^\infty$ *differential 2-form* is an expression of the form $f\,dx \wedge dy$ with f a $\mathcal{C}^\infty$ function. The exterior product $u \wedge v$ of two 1-forms, $u = u_1\,dx + u_2\,dy$ and $v = v_1\,dx + v_2\,dy$, is a 2-form defined by

$$u \wedge v = (u_1 v_2 - u_2 v_1) dx \wedge dy.$$

This product is distributive and anti-commutative ($u \wedge v = -v \wedge u$). Given a $\mathcal{C}^\infty$ 1-form $u = g\,dx + h\,dy$ we set

$$du = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx \wedge dy.$$

This defines an operator from 1-forms to 2-forms. Note that $d(df) = 0$, for any $\mathcal{C}^\infty$ function f.

The coefficient functions of 1-forms and 2-forms are allowed to be complex valued. In particular there are 1-forms

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy.$$

For a complex C^∞ function f, we can write df in terms of dz and $d\bar{z}$ as follows:

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z},$$

where the complex differential operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ are defined by

$$\frac{\partial}{\partial z} = 1/2\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \bar{z}} = 1/2\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right).$$

Using the above notation, the homogeneous Cauchy-Riemann equations can be written as the single complex differential equation

$$\frac{\partial}{\partial \bar{z}} f(z) = 0.$$

A complex differentiable function f on an open set U is holomorphic on U if and only if it satisfies this equation.

It is also important to consider the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial}{\partial \bar{z}} g(z) = f(z).$$

We shall show that this has a solution $g \in C^\infty(U)$ for every open set U and every $f \in C^\infty(U)$. The first step is to prove the generalized Cauchy integral formula:

1.4.1 Theorem. *Let U be an open subset of $\mathbb{C}$ bounded by a simple closed rectifiable curve γ. If f is a C^∞ function on a neighborhood of $\overline{U}$ and $z \in U$, then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\iint\limits_U \frac{\partial f(\zeta)}{\partial \bar{\zeta}}\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Proof. Note that, as forms in ζ for fixed z,

$$d\left(f(\zeta)\frac{d\zeta}{\zeta - z}\right) = \frac{\partial}{\partial \bar{\zeta}}\left(\frac{f(\zeta)}{\zeta - z}\right) d\bar{\zeta} \wedge d\zeta = \frac{\partial f(\zeta)}{\partial \bar{\zeta}}\frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Thus, if γ_r is the boundary of the disc $\Delta(z, r)$ and if r is chosen small enough that this disc is contained in U, then Stokes' theorem implies that

$$\iint\limits_{U_r} \frac{\partial f(\zeta)}{\partial \bar{\zeta}}\frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_\gamma f(\zeta)\frac{d\zeta}{\zeta - z} - \int_{\gamma_r} f(\zeta)\frac{d\zeta}{\zeta - z}$$

where $U_r = U - \Delta(z, r)$. Now $(\zeta - z)^{-1}$ is a function of ζ on any bounded region of the plane, as is easily seen by integrating its absolute value using polar coordinates centered at z. Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{r\to 0} \iint\limits_{U_r} \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \iint\limits_{U} \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Also,

$$\lim_{r\to 0} \int_{\gamma_r} f(\zeta) \frac{d\zeta}{\zeta - z} = \lim_{r\to 0} \int_0^{2\pi} f(z + re^{it})\, i dt = 2\pi i f(z).$$

The result follows.

1.4.2 Proposition. *If $f \in \mathcal{C}^\infty(U)$, for an open set $U \subset \mathbb{C}$ containing a compact set K, then there exists an open set V, with $K \subset V \subset U$, and a $g \in \mathcal{C}^\infty(V)$, such that $\partial g/\partial \bar{z} = f$ in V.*

Proof. We modify f so that it is actually $\mathcal{C}^\infty$ on all of $\mathbb{C}$, with compact support in U, by multiplying it by a $\mathcal{C}^\infty$ function which is 1 in a neighborhood V of K and has compact support in U, and then extending the resulting function to be 0 on the complement of U. With f so modified, the integral

$$g(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

is defined for all $z \in \mathbb{C}$ and defines a function $g \in \mathcal{C}^\infty(\mathbb{C})$. We calculate the derivative $\partial g/\partial \bar{z}$ of g using the change of variables $\zeta \to \zeta + z$:

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}} g(z) &= \frac{1}{2\pi i} \frac{\partial}{\partial \bar{z}} \iint f(\zeta + z) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} \\
&= \frac{1}{2\pi i} \iint \frac{\partial f(\zeta + z)}{\partial \bar{z}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} = \frac{1}{2\pi i} \iint \frac{\partial f(\zeta + z)}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} \\
&= \frac{1}{2\pi i} \iint \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} = f(z),
\end{aligned}$$

where the last two identities follow from reversing the change of variables and then using the generalized Cauchy integral theorem on U (recall that f vanishes on the complement of a compact subset of U and so the line integral in Theorem 1.4.1 vanishes). Thus, $\partial g/\partial \bar{z} = f$ on all of $\mathbb{C}$. Of course, we modified f on the complement of V and so this equation holds for our original f only on V, but this is what was to be shown.

1.4.3 Theorem. *If U is any open set in the plane and $f \in \mathcal{C}^\infty(U)$, then there exists $g \in \mathcal{C}^\infty(U)$ such that $\partial g/\partial \bar{z} = f$.*

Proof. Let $\{K_n\}$ be a sequence of compact sets with the following properties: (a) K_n is contained in the interior of K_{n+1} for each n; (b) the union of the interiors of the K_n's is U; and (c) each bounded component of the complement of K_n meets the complement of U. This can be done (Exercise 1.2 or [**R**], 13.3) and, by Runge's theorem, it ensures that each function holomorphic in a neighborhood of K_n can be uniformly approximated on K_n by rational functions which have poles only in the complement of U ([**R**], 13.6).

We will prove by induction that there is a sequence $\{g_n\}$, with g_n a $\mathcal{C}^\infty$ function satisfying $\partial g_n/\partial \bar{z} = f$ on a neighborhood of K_n, and with $|g_n(z) - g_{n-1}(z)| < 2^{-n}$ for all $z \in K_{n-1}$ if $n > 1$. We may choose g_1 satisfying these conditions by Proposition 1.4.2. Suppose a sequence $\{g_n\}$ satisfying these conditions has been chosen for $n \leq m$. We apply Proposition 1.4.2 to choose a function h which is $\mathcal{C}^\infty$ on a neighborhood of K_{m+1} and satisfies $\partial h/\partial \bar{z} = f$ on this neighborhood. On a neighborhood of K_m we then have $\partial(h - g_m)/\partial \bar{z} = 0$, and so $h - g_m$ is holomorphic on this neighborhood. By Runge's theorem, we may choose a rational function r, with poles in the complement of U, such that $|h(z) - g_m(z) - r(z)| < 2^{-m}$ for $z \in K_m$. If we set $g_{m+1} = h - r$, then $\partial g_{m+1}/\partial \bar{z} = f$ on a neighborhood of K_{m+1} and $|g_{m+1}(z) - g_m(z)| < 2^{-m}$ on K_m. By induction, a sequence $\{g_n\}$ with the required properties exists.

Clearly, the sequence $\{g_m\}$ converges uniformly on each K_n to a function g defined on U. Furthermore, $g_m - g_n$ is holomorphic on a neighborhood of K_n for each $m > n$. Thus, for each fixed n, $\{g_m - g_n\}$ is a sequence of holomorphic functions on a neighborhood of K_n which is uniformly convergent on K_n. It follows that the limit function $g - g_n$ is holomorphic on the interior of K_n. Hence, g is $\mathcal{C}^\infty$ on the interior of K_n. Since this is true of each n, g is $\mathcal{C}^\infty$ on all of U. Clearly, $\partial g/\partial \bar{z} = f$.

1.5 The Proof of Proposition 1.2.2.

We think of the n-tuple of functions $\{g_1, g_2, \ldots, g_n\}$ as defining, for each open set $V \subset U$, an $\mathcal{H}(V)$-module homomorphism

$$\phi_0 : \mathcal{H}(V)^n \to \mathcal{H}(V)$$

by

$$\phi_0(f_1, f_2, \ldots, f_n) = g_1 f_1 + g_2 f_2 + \cdots + g_n f_n = FG^t,$$

where G and F are the vectors $G = (g_1, g_2, \ldots, g_n)$ and $F = (f_1, f_2, \ldots, f_n)$. The hypothesis of Proposition 1.2.2 is that each point of U has a neighborhood V for which this map is surjective. We want to conclude that it is surjective when $V = U$. To do this, we need to know something about the kernel of the map ϕ_0.

We define another $\mathcal{H}(V)$-module homomorphism $\phi_1 : \mathcal{H}(V)^m \to \mathcal{H}(V)^n$, where $m = \dfrac{n(n-1)}{2}$. Here we represent $\mathcal{H}(V)^m$ as the space of $n \times n$ skew-symmetric matrices with entries from $\mathcal{H}(V)$ and, for such a matrix $A = \{a_{ij}\}$, define $\phi_1(A) = GA$, where G is the vector $(g_1, g_2, \ldots, g_n)$. Then $\phi_0 \circ \phi_1(A) = GAG^t = 0$, since A is skew-symmetric and GAG^t, being a scalar function, is symmetric. Thus, ϕ_1 has image contained in the kernel of ϕ_0. In fact, locally, ϕ_1 maps onto the kernel of ϕ_0. To see this, let V be a neighborhood on which ϕ_0 is surjective. We choose $F = (f_1, f_2, \ldots, f_n)$ in $\mathcal{H}(V)^n$, so that $FG^t = \phi_0(F) = 1$ on V, and let H be any vector in the kernel of ϕ_0. Then $A = F^tH - H^tF$ is a skew-symmetric $n \times n$ matrix, with the property that

$$\phi_1(A) = GF^tH - GH^tF = H.$$

Thus, we have a sequence of $\mathcal{H}(V)$-module homomorphisms

$$\mathcal{H}(V)^m \xrightarrow{\phi_1} \mathcal{H}(V)^n \xrightarrow{\phi_0} \mathcal{H}(V) \longrightarrow 0$$

for which the composition of any two succeeding maps is 0 and which is locally exact – that is, each point of U has a neighborhood V on which the kernel of each map in the sequence is the image of the preceding map.

Note that the maps ϕ_0 and ϕ_1 are defined for tuples of $\mathcal{C}^\infty$ functions as well and have the same exactness properties. In fact, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}(U)^m & \xrightarrow{\phi_1} & \mathcal{H}(U)^n & \xrightarrow{\phi_0} & \mathcal{H}(U) & \longrightarrow & 0 \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & \\
\mathcal{C}^\infty(U)^m & \xrightarrow{\phi_1} & \mathcal{C}^\infty(U)^n & \xrightarrow{\phi_0} & \mathcal{C}^\infty(U) & \longrightarrow & 0 \\
{\scriptstyle \frac{\partial}{\partial \bar z}}\downarrow & & {\scriptstyle \frac{\partial}{\partial \bar z}}\downarrow & & {\scriptstyle \frac{\partial}{\partial \bar z}}\downarrow & & \\
\mathcal{C}^\infty(U)^m & \xrightarrow{\phi_1} & \mathcal{C}^\infty(U)^n & \xrightarrow{\phi_0} & \mathcal{C}^\infty(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0, & &
\end{array}$$

where i is the inclusion of holomorphic vectors into $\mathcal{C}^\infty$ vectors and $\partial/\partial\bar{z}$ acts on $\mathcal{C}^\infty$ vectors by acting on each component. The fact that $\partial/\partial\bar{z}$ commutes with ϕ_0 and ϕ_1 follows from the fact that these maps are defined by matrices with holomorphic entries – entries which are killed by $\partial/\partial\bar{z}$ (Exercise 1.6).

The rows and columns of the above diagram are complexes (the composition of successive maps is 0), the columns are exact (the kernel of each map is the image of the preceding map) by Theorem 1.4.3, and the second and third rows are exact by Theorem 1.3.3.

Now the proof of Proposition 1.2.2 is a simple diagram chase. We choose $\hat{F} \in \mathcal{C}^\infty(U)^n$ so that $\phi_0(\hat{F}) = 1$. Then $\phi_0(\partial\hat{F}/\partial\bar{z}) = \partial 1/\partial\bar{z} = 0$. Since the bottom row of the diagram is exact, there exists $A \in \mathcal{C}^\infty(U)^m$ such that $\phi_1(A) = \partial\hat{F}/\partial\bar{z}$. Since the columns are exact, there exists $B \in \mathcal{C}^\infty(U)^m$ such that $\partial B/\partial\bar{z} = A$. Then $F = \hat{F} - \phi_1(B)$ is holomorphic on U, because

$$\frac{\partial}{\partial\bar{z}}F = \frac{\partial}{\partial\bar{z}}(\hat{F} - \phi_1(B)) = \frac{\partial}{\partial\bar{z}}\hat{F} - \phi_1(A) = 0.$$

Furthermore, $\phi_0(F) = \phi_0(\hat{F}) - \phi_0(\phi_1(B)) = 1$. This completes the proof of Proposition 1.2.2.

1.6 The Mittag-Leffler and Weierstrass Theorems

The solvability of the inhomogeneous Cauchy-Riemann equation was the key to the above proof of Proposition 1.2.2. It has other important applications as well, among them proofs of the Mittag-Leffler and Weierstrass theorems. We will conclude this chapter with a development of these results. They both follow easily from the following proposition, which is a solution to a single variable version of a problem which was posed in several variables by Cousin. Thus, we will call it the single variable Cousin problem.

1.6.1 Proposition. *Let $\{U_i\}$ be an indexed collection of open sets in $\mathbb{C}$ and $\{g_{ij}\}$ an indexed set of holomorphic functions satisfying:*
(i) $g_{ij} \in \mathcal{H}(U_i \cap U_j)$ *for each pair* (i,j)*;*
(ii) $g_{ij} = -g_{ji}$ *for each pair* (i,j)*; and*
(iii) $g_{ij} + g_{jk} + g_{ki} = 0$ *for each triple* (i,j,k)*.*

Then there exists an indexed set $\{h_i\}$ of functions, with $h_i \in \mathcal{H}(U_i)$ for each i and $g_{ij} = h_j - h_i$ for each pair (i,j).

Proof. Let $\{\phi_p\}$ be a $\mathcal{C}^\infty$ partition of unity subordinate to the cover $\{U_i\}$. Then, for each p, there is a k_p so that ϕ_p has compact support in U_{k_p}; on any given compact subset of $U = \cup_i U_i$, all but finitely many of the ϕ_p vanish identically; and $\sum_p \phi_p = 1$. Note that $\phi_p g_{k_p i}$ may be defined to be a function in $\mathcal{C}^\infty(U_i)$ by defining it to be 0 on $U_i - (U_i \cap U_{k_p})$. For each i, we set

$$f_i = \sum_p \phi_p g_{k_p i} \quad \text{on} \quad U_i.$$

Then, on $U_i \cap U_j$,

$$f_j - f_i = \sum_p \phi_p (g_{k_p j} - g_{k_p i}) = \sum_p \phi_p g_{ij} = g_{ij}.$$

This gives us a $\mathcal{C}^\infty$ solution to the Cousin problem.

Note that $\partial f_i/\partial \bar{z} = \partial f_j/\partial \bar{z}$ on $U_i \cap U_j$, since $f_j - f_i = g_{ij}$ is holomorphic. It follows that there is a function $u \in \mathcal{C}^\infty(U)$ such that $u = \partial f_i/\partial \bar{z}$ on U_i for each i. If $v \in \mathcal{C}^\infty(U)$ is chosen so that $\partial v/\partial \bar{z} = u$, then $h_i = f_i - v$ is holomorphic on U_i and $h_j - h_i = g_{ij}$. The set $\{h_i\}$ is the required holomorphic solution to the Cousin problem.

An indexed set of holomorphic functions $\{g_{ij}\}$ satisfying conditions (i), (ii), (iii) of the above proposition is called a set of *Cousin data* for the cover $\{U_i\}$.

The Mittag-Leffler theorem is a simple application of the above result. It concerns the existence of meromorphic functions with given principal parts. A *meromorphic* function on an open set $U \subset \mathbb{C}$ is a function which is defined and holomorphic on U, except at a discrete set of points where it has poles. If λ is a point at which a meromorphic function f has a pole of order k, then $(z-\lambda)^k f(z)$ has a removable singularity at λ. Thus, $f(z) = g(z)(z-\lambda)^{-k}$, where $g(z)$ is holomorphic in a neighborhood of λ and, hence, has a power series expansion which converges in some disc centered at λ. In this disc, we can use the power series expansion of g to write f in the form

$$f(z) = a_{-k}(z-\lambda)^{-k} + \cdots + a_{-1}(z-\lambda)^{-1} + a_0 + a_1(z-\lambda) + a_2(z-\lambda)^2 + \cdots$$

The rational function $a_{-k}(z-\lambda)^{-k} + \cdots + a_{-1}(z-\lambda)^{-1}$ is then called the *principal part* of f at λ.

1.6.2 Mittag-Leffler Theorem. *Let U be an open set in $\mathbb{C}$, let S be a discrete subset of U, and, for each point $\lambda \in S$, let P_λ be a polynomial in $(z-\lambda)^{-1}$ with no constant term. Then there is a meromorphic function f on U which has a pole with principal part P_λ at each $\lambda \in S$ and which has no other poles.*

Proof. We choose an open cover $\{U_i\}$ of U with the property that each U_i contains at most one point of S. We assign a meromorphic function f_i on

U_i to each index i in the following way: If U_i contains a point $w_i \in S$, then $f_i = P_{w_i}$; otherwise, $f_i = 0$. We then define Cousin data for the cover $\{U_i\}$ by setting

$$g_{ij} = f_j - f_i \quad \text{on} \quad U_i \cap U_j$$

and noting that g_{ij} is holomorphic on $U_i \cap U_j$. It follows easily that the collection $\{g_{ij}\}$ satisfies (i), (ii), and (iii) of Proposition 1.6.1. Hence, by that proposition, there exists a collection $\{h_i\}$, with h_i holomorphic on U_i and with $h_j - h_i = g_{ij}$ on $U_i \cap U_j$. Then,

$$f_i - h_i = f_j - h_j \quad \text{on} \quad U_i \cap U_j$$

for each pair i, j, and so these functions fit together to define a meromorphic function f on U. Because adding a holomorphic function to f_i doesn't affect its poles and principal parts, f has poles exactly at the points of S and, at each such pole λ, the principal part of f is P_λ.

With a little more work, we can prove the Weierstrass theorem concerning the existence of holomorphic functions with prescribed zeroes and poles.

1.6.3 Weierstrass Theorem. *Given an open set $U \subset \mathbb{C}$, a discrete set $S \subset U$, and a non-zero integer k_λ assigned to each point λ of S, there exists a meromorphic function f which has a zero of order k_λ at each $\lambda \in S$ with $k_\lambda > 0$ and a pole of order $-k_\lambda$ at each $\lambda \in S$ with $k_\lambda < 0$ and which has no other zeroes or poles.*

Proof. We may choose a sequence $\{U_n\}$ of open sets with compact closures such that $\overline{U}_n \subset U_{n+1}$, the union of the U_n's is U, and each bounded component of the complement of $\overline{U}_n$ contains a point of the complement of U (see Exercise 1.2 or [**R**], 13.3). It is easy to see that this can be done in such a way that no points of S lie on the boundary of any set U_n. We set $K_n = \overline{U}_n$ and $S_n = S \cap K_n = S \cap U_n$. We will inductively construct a sequence $\{f_n\}$ with the following properties:

(1) f_n is a rational function on $\mathbb{C}$;
(2) f_n has zeroes and poles of the required orders at the points of S_n and no other zeroes or poles on K_n; and
(3) the zeroes and poles of $f_n^{-1} f_{n+1}$ occur in pairs, with each pair consisting of a zero and a pole of the same order occurring at points which lie in the same component of the complement of K_n.

We set $f_1(z) = \prod_{\lambda \in S_1} (z - \lambda)^{k_\lambda}$. Then f_1 is a rational function with the required zeroes and poles on K_1. Assume that f_i with the above properties have been chosen for $i \le n$. We will then show how to construct f_{n+1}. We choose a rational function v on $\mathbb{C}$ so that v has the required zeroes and poles at points of S_{n+1} and no other zeroes or poles on K_{n+1}. Now the function $f_n^{-1}v$ is rational and has its poles and zeroes in the complement of K_n. Thus, this function has the form

$$f_n^{-1}(z)v(z) = \prod (z - a_i)^{m_i},$$

with each a_i in the complement of K_n. Now each a_i is in the same component of the complement of K_n as some point b_i in the complement of K_{n+1}. If we set

$$f_{n+1}(z) = v(z) \prod_i (z - b_i)^{-m_i},$$

then f_{n+1} has the required zeroes and poles on K_{n+1}, and the zeroes and poles of $f_n^{-1} f_{n+1}$ occur in pairs (a_i, b_i) as in (3) above. Thus, by induction, a sequence with properties (1), (2), and (3) exists.

Now if h is any rational function and γ is a simple closed curve in $\mathbb{C}$ which doesn't meet a zero or pole of h, then

$$(2\pi i)^{-1} \int_\gamma \frac{h'(\zeta)}{h(\zeta)}\, d\zeta$$

counts the number of zeroes minus the number of poles (counted according to multiplicity) in the bounded component of the complement of γ ([**R**], 10.43). It follows that if V is an open set with the property that the poles and zeroes of h occur in pairs, each consisting of a zero and a pole with the same multiplicity, located in the same component of the complement of V, then any such integral must vanish if γ lies in V. This, in turn, implies that there is a well-defined holomorphic logarithm of h defined on V – that is, a function $g \in \mathcal{H}(V)$ such that $h(z) = \exp(g(z))$ on V. In fact, on any given component of V, such a function is given by

$$g(z) = \log h(z_0) + \int_{\gamma_z} \frac{h'(\zeta)}{h(\zeta)}\, d\zeta$$

where z_0 is a fixed point of the component and γ_z is any smooth curve in V joining z_0 to z. If we apply this in the case where $h = f_n^{-1} f_{n+1}$ and $V = U_n$, we conclude that there is a function g_n, holomorphic on U_n, so that

$$\exp(g_n) = f_n^{-1} f_{n+1}$$

on U_n.

We now define Cousin data $\{g_{ij}\}$ for the cover $\{U_n\}$ by requiring that $g_{ij} = -g_{ji}$ for all i, j and, for $i < j$, setting

$$g_{ij} = g_i + g_{i+1} + \cdots + g_{j-1} \quad \text{on} \quad U_i = U_i \cap U_j.$$

It follows easily that $\exp(g_{ij}) = f_i^{-1} f_j$ on $U_i \cap U_j$ for each pair of indices (i, j). Given a triple of indices $i < j < k$, we have

$$g_{ij} + g_{jk} + g_{ki} = g_i + \cdots + g_{j-1} + g_j + \cdots + g_{k-1} - g_i - \cdots - g_{k-1} = 0$$

on $U_i \cap U_j \cap U_k = U_i$.

By Proposition 1.6.1, there exist $h_i \in \mathcal{H}(U_i)$ such that $g_{ij} = h_j - h_i$ on $U_i \cap U_j$. Then

$$f_j \exp(-h_j) = f_i \exp(g_{ij}) \exp(-h_j) = f_i \exp(-h_i) \quad \text{on} \quad U_i \cap U_j,$$

and so there is a holomorphic function f, defined on U, with $f = f_i \exp(-h_i)$ on U_i. Since f_i has zeroes and poles of the required orders at the required points in U_i, the same thing is true of f. Thus, f has the required zeroes and poles of the required orders on all of U.

This is not the most efficient proof of the Weierstrass theorem but it is a proof which illustrates the methods we have been developing.

Several nice applications of the Mittag-Leffler and Weierstrass theorems appear in the exercises.

1.7 Conclusions and Comments

We have seen how the local to global problem posed by Proposition 1.2.2 can be solved by using the solvability of the inhomogeneous Cauchy-Riemann equation to pass from C^∞ solutions of a linear equation to holomorphic solutions. This will be a major theme later in the several variable theory. There, the local to global problem is formulated as the problem of showing that certain sheaf cohomology groups vanish. The first theorem of this type, and the one on which all later ones are based, is Dolbeault's lemma which asserts the solvability of certain systems of differential equations which are several variable analogues of the inhomogeneous Cauchy-Riemann equation.

The Cousin problem of section 1.6 makes sense for holomorphic functions of several variables. In later chapters, we will formulate it as the problem of showing the vanishing of Čech cohomology groups for sheaves of holomorphic functions. The Mittag-Leffler and Weierstrass theorems also have several variable analogues and they are related to Čech cohomology in the same way the single variable versions were related to the Cousin problem in section 1.6. The strongest theorem we will prove on the vanishing of cohomology of sheaves related to holomorphic functions is Cartan's Theorem B. Much of the machinery developed in the text is aimed at proving this result.

The special nature of the zero set of a holomorphic function played a big role in the preceding pages, as did detailed knowledge of the local structure of holomorphic and meromorphic functions. Specifically, a non-trivial holomorphic function f has a discrete zero set; in a neighborhood V of each point λ in the domain of f, there is a factorization $f(z) = (z-\lambda)^k g(z)$, where g is a non-vanishing holomorphic function on V and k is a non-negative integer; a meromorphic function locally has a similar factorization with k an arbitrary integer. A detailed knowledge of the zero sets of holomorphic functions and the local structure of holomorphic functions will be just as essential in the several variable theory. However, it will not be so easily obtained. Much of the material in the next four chapters is devoted to these subjects. The zero sets of holomorphic functions are no longer discrete in several variables. They are, however, sets with very special structure (holomorphic varieties) and will be studied in detail. The relationship between holomorphic varieties and ideals in the local algebra of holomorphic functions will be a major story, culminating in Chapter 4 in the Nullstellensatz, which is one of the main theorems of the subject.

As we develop the machinery of several complex variables, we will also give a parallel introductory development of complex algebraic geometry. The general strategies in the two subjects, as treated in this text, are the same: Develop detailed knowledge of the local structure and then use techniques of sheaf cohomology to pass from local information to global information. We feel it is quite instructive to see the two subjects developed in parallel. We can point out the way in which the subjects differ as well as ways in which they are similar. This also lays the foundation for proofs of Serre's GAGA theorems in Chapter 13. These essentially show that the two subjects are equivalent if one is working on a projective variety. Serre's proofs use both the major results of the theory of several complex variables (including Cartan's Theorem B) and the basics of complex algebraic geometry.

Although the approach we have adopted for this text is based heavily on commutative algebra, we will also use a substantial amount of analysis. There are several arguments in Chapter 10 which use properties of Banach spaces. In Chapter 11 we will need information about more general topological vector spaces. The approximation argument using Runge's theorem in the proof of Theorem 1.4.3 will recur in Chapter 10 and in Chapter 11 in a more abstract context, where we will need to use properties of topological vector spaces (specifically, Fréchet spaces). We will use a certain amount of Hilbert space theory in Chapter 14. There will be arguments which involve integrating or differentiating Banach space valued functions. The definitions and basic properties of these calculus concepts, in the setting of Banach space valued functions, are not in any substantial way different from the versions for real or complex valued functions and so we will use them without much comment. We will, in particular, have occasion to use holomorphic functions which are Banach space valued in Chapter 11. The elementary facts we will need concerning such functions in plane regions are developed in Exercises 1.16 – 1.20. Most of the analysis background that will be needed in the coming pages is contained in Walter Rudin's first year graduate text [**R**], although there will be an occasional reference to more advanced texts in functional analysis, such as [**R2**]

Exercises

1. Show that the function which is $\exp(-t^{-1})$ for $t \geq 0$ and 0 for $t < 0$ is $\mathcal{C}^\infty$ on the real line.
2. Show that if U is an open set in $\mathbb{C}$, then there exists a sequence $\{K_n\}$ of compact subsets of U such that
   (i) $K_n \subset \operatorname{int}(K_{n+1})$ for each n;
   (ii) $\cup_n K_n = U$;
   (iii) each bounded component of the complement of K_n contains a point of the complement of U.
3. Verify that the function

$$g(z) = \log h(z_0) + \int_{\gamma_z} \frac{h'(\zeta)}{h(\zeta)}\, d\zeta,$$

   which appears in the proof of Theorem 1.6.3, is indeed a logarithm for h. That is, verify that $\exp g(z) = h(z)$ on V.
4. Show that if U is an open subset of $\mathbb{C}$, every finitely generated ideal of $\mathcal{H}(U)$ is a principal ideal.
5. If K is a compact subset of $\mathbb{C}$, let $\mathcal{H}(K)$ denote the algebra of functions holomorphic in a neighborhood of K, where two functions are identified in $\mathcal{H}(K)$ if they agree on some neighborhood of K. Prove that each maximal ideal of $\mathcal{H}(K)$ is of the form $M_\lambda = \{f \in \mathcal{H}(K) : f(\lambda) = 0\}$ for some point $\lambda \in K$.
6. Let U be an open set in $\mathbb{C}$. Prove that each $\mathcal{C}^\infty(U)$-module homomorphism $\phi : \mathcal{C}^\infty(U)^m \to \mathcal{C}^\infty(U)^n$ is given by an $n \times m$ matrix with entries

in $\mathcal{C}^\infty(U)$. Prove that ϕ commutes with $\partial/\partial\bar{z}$ if and only if this matrix has holomorphic entries.

7. Show that every meromorphic function on an open set $U \subset \mathbb{C}$ is of the form $f(z)/g(z)$, where f and g are holomorphic functions on U.
8. Let U be an open subset of $\mathbb{C}$. Prove there is a function $f \in \mathcal{H}(U)$ which cannot be analytically continued to any larger open set.
9. Use the Mittag-Leffler and Weierstrass theorems to prove the interpolation theorem: If $\{\lambda_i\}$ is a sequence of points in the open set U, with no limit point in U, and if $\{\alpha_i\}$ a sequence of complex numbers, then there is a holomorphic function $f \in \mathcal{H}(U)$ such that $f(\lambda_i) = \alpha_i$ for each i.
10. Prove the following strong version of the interpolation theorem: Given a discrete sequence $\{a_i\}$ in an open set $U \subset \mathbb{C}$, a sequence of positive integers $\{n_i\}$, and a sequence of functions $\{f_i\}$, where f_i is holomorphic in a neighborhood of a_i for each i, show that there exists an $f \in \mathcal{H}(U)$ such that $f(z)$ agrees with f_i to order n_i (that is, $f - f_i$ has a zero of order at least n_i) for each i.
11. Let K be a compact connected subset of $\mathbb{C}$ and consider the ring $\mathcal{H}(K)$, defined in Exercise 1.5. Show that this ring is a principle ideal domain (an integral domain such that every ideal is a principle ideal).
12. Let U be an open subset of $\mathbb{C}$ such that $\mathbb{C} - U$ has only finitely many connected components. Let $\{\lambda_j\}$ be a set consisting of one point from each of these components. Prove that each non-vanishing function f in $\mathcal{H}(U)$ has the form
$$f(z) = \prod_j (z - \lambda_j)^{n_j} \exp g(z),$$
for some $g \in \mathcal{H}(U)$.
13. Let $g(z)$ be a meromorphic function on $\mathbb{C}$ which is holomorphic except at a discrete set of points $\{\lambda_j\} \subset \mathbb{C}$, where it has poles of order one. Suppose the residue of f at λ_j is an integer n_j for each j. Let $U = \mathbb{C} - \{\lambda_j\}$. Prove that if γ_z is a smooth simple curve in U joining a fixed point $z_0 \in U$ to a variable point $z \in U$, then the function
$$f(z) = \exp\left(\int_{\gamma_z} g(\zeta)d\zeta\right)$$
is a well-defined holomorphic function on U and has a zero of order n_j at λ_j for each j. Show that the Weierstrass theorem for the domain $\mathbb{C}$ follows from this result and the Mittag-Leffler theorem.
14. Give a simpler proof of the existence of solutions of the inhomogeneous Cauchy-Riemann equation (Theorem 1.4.3) in the case where $U = \mathbb{C}$, one that does not involve Runge's theorem.

15. Here is another type of inhomogeneous Cauchy-Riemann problem: Let U be an open set in $\mathbb{C}$ and f a function which is C^∞ on U and has compact support in U. By Theorem 1.4.3, $f = \frac{\partial g}{\partial \bar{z}}$ for some C^∞ function on U, but g may not have compact support. Prove that a solution g exists with compact support in U if and only if $\int_U f(z)h(z)dz \wedge d\bar{z} = 0$ for every $h \in \mathcal{H}(U)$.
16. Let X be a complex Banach space (see [**R**]) and U an open subset of $\mathbb{C}$. A function $f : U \to X$ is said to be *holomorphic* in U if its complex derivative
$$f'(z) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w}$$
exists at each point of U, in the sense that the indicated limit exists in the norm of X. Contour integrals of continuous X-valued functions in plane regions are defined in the same way as for complex valued functions, with convergence of the integral taking place in the norm of X. Prove the Cauchy theorem for Banach space valued functions; that is, prove that if $f : U \to X$ is holomorphic, Γ is a piecewise continuously differentiable simple closed curve in U, and the bounded component of the complement of Γ is contained in U, then
$$\int_\Gamma f(\zeta)d\zeta = 0 \quad \text{and} \quad f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$
for each z in the bounded component of the complement of Γ.
17. If X is a Banach space and f is an X-valued holomorphic function in the open disc $\{z \in \mathbb{C} : |z| < R\}$, then prove that f has a power series expansion
$$f(z) = x_0 + x_1 z + \frac{x_2}{2}z^2 + \cdots + \frac{x_k}{k!}z^k + \cdots,$$
where
$$x_k = \frac{1}{2\pi i}\int \frac{f(z)}{z^{k+1}}dz = \frac{f^{(k)}(0)}{k!}.$$
Prove this power series converges uniformly in the norm of X in every disc $\{z \in \mathbb{C} : |z| \leq r\}$ with $r < R$. Hint: Use the result of the previous exercise.
18. Prove that if X is a Banach space, f is a holomorphic X-valued function in a neighborhood of the closed disc $\{z \in \mathbb{C} : |z| \leq R\}$, and if $|f(z)| \leq M$ on this disc, then
$$||f^{(k)}(0)|| \leq \frac{k!M}{R^k}$$
for $k = 1, 2, \ldots$.

19. Let X be a Banach space and let f be an X-valued holomorphic function in some neighborhood of 0 in $\mathbb{C}$. Prove that if R is the largest positive number r such that f can be extended to be holomorphic on the disc $\{z \in \mathbb{C} : |z| < r\}$, then $\limsup ||x_k||^{1/k} = R^{-1}$, where the elements x_k are the power series coefficients of f.
20. Prove Liouville's theorem for Banach space valued functions: If X is a Banach space and f is a holomorphic X-valued function on $\mathbb{C}$ such that $||f(z)||$ is bounded, then f is constant.

Chapter 2

Holomorphic Functions of Several Variables

There are a number of possible ways to define what it means for a complex valued function on an open set in $\mathbb{C}^n$ to be holomorphic. One could simply insist that the function be holomorphic in each variable separately. Or one could insist the function be continuous (as a function of several variables) in addition to being holomorphic in each variable separately. The a priori strongest condition would be to insist that a holomorphic function have a convergent expansion as a multi-variable power series in a neighborhood of each point of the domain. The main object of this chapter is to show that these possible definitions are all equivalent. We will also develop some of the elementary properties of holomorphic functions of several variables and introduce a concept that will play a central role in later chapters – the concept of domain of holomorphy. All open sets in $\mathbb{C}$ are domains of holomorphy. The fact that this is not true in several variables is one of the things that makes the several variable theory strikingly different from the single variable theory.

2.1 Cauchy's Formula and Power Series Expansions

For holomorphic functions of one complex variable, the existence of local power series expansions is usually deduced from Cauchy's integral formula. The same technique works in several variables.

In what follows, $\Delta(a,r)$ will denote the open polydisc centered at the point $a = (a_1, a_2, \ldots, a_n)$ and with polyradius $r = (r_1, r_2, \ldots, r_n)$. That is,

$$\Delta(a,r) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_j - a_j| < r_j, \quad j = 1, 2, \ldots, n\}.$$

The corresponding closed polydisc will be denoted $\bar{\Delta}(a,r)$. Note that $\Delta(a,r)$ is just the Cartesian product of the open discs $\Delta(a_i, r_i) \subset \mathbb{C}$ and $\bar{\Delta}(a,r)$ is the Cartesian product of the closed discs $\bar{\Delta}(a_i, r_i)$.

2.1.1 Proposition (Cauchy's Integral Formula). *If f is a function which is defined in a neighborhood U of the closed polydisc $\bar{\Delta}(a,r)$ and holomorphic in each variable z_i at each point of U, then*

$$(2.1.1) \quad f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}$$

for each $z = (z_1, z_2, \ldots, z_n) \in \Delta(a,r)$.

This follows immediately from repeated application of the one variable Cauchy formula.

Note that, at this point, we do not assume that the function f of the above lemma is continuous or bounded on $\bar{\Delta}(a,r)$ or even integrable as a function on the product of the circles $|\zeta_i - a_i| = r_i$. It is measurable, since a separately continuous function on $\mathbb{R}^k$ is a Borel function (Exercise 8.8 in [**R**]). However, the integral of the lemma makes perfectly good sense, as an iterated integral, though, as far as we know at this point, not necessarily as an integral relative to the corresponding product measure.

2.1.2 Osgood's Lemma. *If f is a function on an open set $U \subset \mathbb{C}^n$, which is bounded on each compact subset of U, and holomorphic in each variable separately on U, then, for each compact polydisc $\bar{\Delta}(a,s) \subset U$, there is a power series of the form*

$$\sum_{i_1, \ldots, i_n = 0}^{\infty} c_{i_1 \ldots i_n} (z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i_n}$$

which converges uniformly to f on $\bar{\Delta}(a,s)$.

Proof. Since $\bar{\Delta}(a,s)$ is compact in U, we may choose numbers $r_i > s_i$ such that the compact polydisc $\bar{\Delta}(a,r)$ with polyradius $r = (r_1, \ldots, r_n)$ is also contained in U. Then f is bounded on $\bar{\Delta}(a,r)$. We substitute into the integrand of (2.1.1) the series expansion:

$$\frac{1}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} = \sum_{i_1, \ldots, i_n = 0}^{\infty} \frac{(z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i_n}}{(\zeta_1 - a_1)^{i_1 + 1} \ldots (\zeta_n - a_n)^{i_n + 1}}.$$

This series converges uniformly in ζ and z for $|z_i - a_i| \leq s_i < r_i = |\zeta_i - a_i|$ and, thus, can be integrated termwise. The resulting series in z is uniformly convergent for $z \in \bar{\Delta}(a, s)$ and yields a series expansion for f of the form

$$f(z) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1\ldots i_n}(z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i_n},$$

uniformly convergent on $\bar{\Delta}(a, s)$, with

$$c_{i_1\ldots i_n} = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - z_n| = r_n} \cdots \int_{|\zeta_1 - z_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - a_1)^{i_1+1} \ldots (\zeta_n - a_n)^{i_n+1}}.$$

This completes the proof.

A function f, defined on an open set $U \subset \mathbb{C}^n$, is called *holomorphic* at a if it has a power series expansion, as in Osgood's lemma, convergent to f in some open polydisc centered at a. If f is holomorphic at each point of U, then it is called holomorphic on U. Thus, Osgood's lemma says that a function on U, which is holomorphic in each variable separately at each point of U, is holomorphic on U, provided it is bounded on each compact subset of U. In particular, it is holomorphic if it is holomorphic in each variable separately and is continuous. We will later prove that the boundedness or continuity hypothesis is redundant.

Cauchy's formula can also be used, as in the single variable case, to obtain bounds on the derivatives of a holomorphic function:

2.1.3 Proposition (Cauchy's Inequalities). *If f is a holomorphic function in a neighborhood of the closed polydisc $\bar{\Delta}(a, r)$, and $|f|$ is bounded by M in this polydisc, then for each multi-index $(i_1, \ldots, i_n)$,*

$$\left| \frac{\partial^{i_1+\cdots+i_n}}{\partial {z_1}^{i_1} \ldots \partial {z_n}^{i_n}} f(a) \right| \leq M(i_1!) \ldots (i_n!) {r_1}^{-i_1} \ldots {r_n}^{-i_n}.$$

Proof. If f is expressed as a power series, convergent on $\bar{\Delta}(a, r)$, as in Lemma 2.1.2, then repeated differentiation yields

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial {z_1}^{i_1} \ldots \partial {z_n}^{i_n}} f(a) = (i_1!) \ldots (i_n!) c_{i_1\ldots i_n},$$

where, according to the proof of Lemma 2.1.2,

$$c_{i_1\ldots i_n} = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - z_n| = r_n} \cdots \int_{|\zeta_1 - z_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - a_1)^{i_1+1} \ldots (\zeta_n - a_n)^{i_n+1}}.$$

From the obvious bounds on the integrand of this integral, it follows that

$$|c_{i_1\ldots i_n}| \leq M {r_1}^{-i_1} \ldots {r_n}^{-i_n},$$

and the Cauchy inequalities follow from this.

2.2 Hartog's Theorem

In this section we will remove the boundedness hypothesis from Lemma 2.1.2. The result is Hartog's theorem.

In what follows, the notation $|U|$ will be used to denote the volume of a set $U \subset \mathbb{C}^n$ and dV will denote volume measure on $\mathbb{C}^n$.

2.2.1 Theorem (Jensen's Inequality). *If f is holomorphic in a neighborhood of $\bar{\Delta} = \bar{\Delta}(a, r)$, then*

$$\log|f(a)| \leq \frac{1}{|\Delta|}\int_\Delta \log|f(z)|dV(z).$$

Proof. We recall the single variable version of Jensen's inequality([**R**], 3.3):

$$\log|g(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|g(a+\rho e^{i\theta})|d\theta,$$

and apply it to f, considered as a function of only z_1, with the other variables fixed at $a_2, \ldots, a_n$. For $0 \leq \rho_1 \leq r_1$, this yields:

$$\log|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(a_1+\rho_1 e^{i\theta_1}, a_2, \ldots, a_n)|d\theta_1.$$

We now apply Jensen's inequality to the integrand of this expression, where f is considered as a function of z_2 with z_1 fixed at $a_1 + \rho_1 e^{i\theta_1}$ and the remaining variables fixed at $a_3, \ldots, a_n$ to obtain, for $0 \leq \rho_2 \leq r_2$:

$$\log|f(a)| \leq \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|f(a_1+\rho_1 e^{i\theta_1}, a_2+\rho_2 e^{i\theta_2}, a_3, \ldots, a_n)|d\theta_1 d\theta_2.$$

Continuing in this way, we obtain:

$$\log|f(a)| \leq \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} \log|f(a_1+\rho_1\, e^{i\theta_1}, \ldots, a_n+\rho_n\, e^{i\theta_n})|d\theta_1 \ldots d\theta_n.$$

Finally, we multiply both sides of this by $\rho_1 \ldots \rho_n$ and integrate with respect to $d\rho_1 \ldots d\rho_n$ over the set $\{0 \leq \rho_i \leq r_i; \quad i = 1, \ldots, n\}$ to obtain the inequality of the theorem.

In the next lemma we will write a point $z \in \mathbb{C}^n$ as $z = (z', z_n)$, with $z' \in \mathbb{C}^{n-1}$. Similarly we will write a polyradius r as $r = (r', r_n)$, so that a polydisc $\Delta(a, r)$ can be written as $\Delta(a', r') \times \Delta(a_n, r_n)$.

2.2.2 Hartogs' Lemma. *Let f be holomorphic in $\Delta(0,r)$ and let*

$$f(z) = \sum_k f_k(z')z_n^k$$

be the power series expansion of f in the variable z_n, where the f_k are holomorphic in $\Delta(0,r')$. If there is a number $c > r_n$ such that this series converges in $\bar{\Delta}(0,c)$, for each $z' \in \Delta(0,r')$, then it converges uniformly on each compact subset of $\Delta(0,r') \times \Delta(0,c)$. Thus, f extends to be holomorphic in this larger polydisc.

Proof. Choose a point $a' \in \Delta(0,r')$, a closed polydisc $\bar{\Delta}(a',s') \subset \Delta(0,r')$, and some positive $b < r_n$. Since f is holomorphic in $\Delta(0,r)$, there is an upper bound $M > 1$ for $|f|$ on the compact polydisc $\bar{\Delta}(a',s') \times \bar{\Delta}(0,b) \subset \Delta(0,r)$. It follows from Cauchy's inequalities that $|f_k(z')| \le Mb^{-k}$, for all $z' \in \bar{\Delta}(a',s')$. Hence,

$$k^{-1}\log|f_k(z')| \le k^{-1}\log M - \log b \le \log M - \log b = M_0,$$

for all $z' \in \bar{\Delta}(a',s')$ and all k. If the above series converges at (z',c), then $|f_k(z')|c^k$ converges to 0. Hence,

$$\limsup_k k^{-1}\log|f_k(z')| \le -\log c$$

for all $z' \in \bar{\Delta}(a',s')$.

The functions $k^{-1}\log|f_k(z')|$ are bounded and measurable on $\bar{\Delta}(a',s')$, and so, with $\Delta = \Delta(a',s')$, Fatou's lemma implies

$$\begin{aligned}\limsup_k \int_\Delta k^{-1}\log|f_k(z')|dV(z') &\le \int_\Delta \limsup_k k^{-1}\log|f_k(z')|dV(z') \\ &\le -|\Delta|\log c.\end{aligned}$$

We conclude that, if $0 < c_0 < c$, then there is a k_0 and a $\delta > 0$ such that

$$\int_\Delta k^{-1}\log|f_k(z')|dV(z') \le -|\Delta|\log c_0 - \delta,$$

for all $k \ge k_0$. Now, since $k^{-1}\log|f_k(z')|$ is bounded by M_0 for all k and all $z' \in \Delta$, we may choose ϵ sufficiently small that, if $\Delta = \Delta(a',s')$ is replaced by $\Delta_\epsilon = \Delta(w',s'-\hat{\epsilon})$, where $w' \in \Delta(a',\hat{\epsilon})$ and $\hat{\epsilon} = (\epsilon,\ldots,\epsilon)$, we will have

$$\int_{\Delta_\epsilon} k^{-1}\log|f_k(z')|dV(z') \le -|\Delta_\epsilon|\log c_0.$$

Through Jensen's inequality, this yields

$$k^{-1}\log|f_k(w')| \le -\log c_0,$$

or

$$|f_k(w')|c_0^k \le 1,$$

for all $k \ge k_0$ and $w' \in \Delta_\epsilon$. This implies that our series is uniformly convergent on compact subsets of $\Delta_\epsilon \times \Delta(0, c_0)$. Since a' is an arbitrary point of $\Delta(0, r')$ and c_0 is an arbitrary positive number less than c, our power series serves to extend f to a function which is continuous on $\Delta(0, r') \times \Delta(0, c)$ and holomorphic in each variable. It follows from Osgood's lemma that this extension of f is, in fact, holomorphic.

2.2.3 Hartog's Theorem. *If a complex valued function f is holomorphic in each variable separately in a domain $U \subset \mathbb{C}^n$, then it is holomorphic in U.*

Proof. We prove this by induction on the dimension n. There is nothing to prove if $n = 1$. Thus, we suppose that $n > 1$ and that the theorem is true for dimension $n - 1$.

Let a be a point in U and let $\bar{\Delta}(a, r)$ be a closed polydisc contained in U. As in the previous lemma, we write $z = (z', z_n)$ for points of $\mathbb{C}^n$ and write $\bar{\Delta}(a, r) = \bar{\Delta}(a', r') \times \bar{\Delta}(a_n, r_n)$.

Consider the sets

$$X_k = \{z_n \in \bar{\Delta}(a_n, r_n/2) : \quad |f(z', z_n)| \le k \quad \forall z' \in \bar{\Delta}(a', r')\}.$$

For each k, this set is closed, since $f(z', z_n)$ is continuous in z_n for each fixed z'. On the other hand, by the induction assumption, $f(z', z_n)$ is also continuous in z' and, hence, bounded on $\bar{\Delta}(a', r')$, for each fixed $z_n \in \bar{\Delta}(a_n, r_n/2)$. We conclude that $\bar{\Delta}(a_n, r_n/2) \subset \bigcup_k X_k$. It follows from the Baire category theorem that, for some k, the set X_k contains a neighborhood $\Delta(b_n, \delta)$ of some point $b_n \in \Delta(a_n, r_n/2)$.

We now know that f is separately holomorphic and uniformly bounded in the polydisc $\Delta(a', r') \times \bar{\Delta}(b_n, \delta)$. We conclude from Osgood's lemma that f is holomorphic on $\Delta(a', r') \times \Delta(b_n, \delta)$ and, in fact, has a power series expansion about (a', b_n) which converges uniformly on compact subsets of this polydisc.

We choose $s_n > r_n/2$ so that $\Delta(b_n, s_n) \subset \Delta(a_n, r_n)$. Then $f(z', z_n)$ is holomorphic in z_n on $\Delta(b_n, s_n)$ for each $z' \in \Delta(a', r')$ and, hence, its power series expansion about (a', b_n) converges as a power series in z_n on $\Delta(b_n, s_n)$ for each fixed point $z' \in \Delta(a', r')$. It follows from Hartog's lemma that f is actually holomorphic on all of $\Delta(a', r') \times \Delta(b_n, s_n)$. Since a is in this set and a was an arbitrary element of U, the proof is complete.

2.3 The Cauchy-Riemann Equations

As we did in the single variable case in Chapter 1, we introduce the first order partial differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then a function is holomorphic in each variable $z_j = x_j + iy_j$ separately in a domain U if and only if, for each j, it is differentiable as a function of the pair of real variables x_j, y_j and satisfies the equation $\partial f/\partial \bar{z}_j = 0$ in U. Thus, by Hartogs theorem, we have:

2.3.1 Corollary. *A function defined in an open set $U \subset \mathbb{C}^n$ is holomorphic in U provided, for $j = 1, \ldots, n$, it is differentiable in x_j, y_j and $\partial f/\partial \bar{z}_j = 0$ in U.*

A much stronger result is true: A distribution ψ, defined in a domain in $\mathbb{C}^n$, which satisfies the Cauchy-Riemann equations, $\partial \psi/\partial \bar{z}_j = 0, j = 1, \ldots, n$ in the distribution sense, is actually a holomorphic function. We will not prove this here, but it follows from regularity theorems for elliptic PDE's.

2.4 Convergence Theorems

If U is an open subset of $\mathbb{C}^n$, then the space of all holomorphic functions on U will be denoted by $\mathcal{H}(U)$. This is obviously a complex algebra under the operations of pointwise addition and multiplication of functions. It also has a natural topology – the topology of uniform convergence on compact subsets of U. This section is devoted to proving a couple of important theorems about this topology.

Recall that a *seminorm* on a real or complex vector space X is a function $\rho : X \to \mathbb{R}^+$ such that $\rho(x + y) \leq \rho(x) + \rho(y)$ and $\rho(\lambda x) = |\lambda|\rho(x)$ for all $x, y \in X$ and all scalars λ. A seminorm has all the properties of a norm, except it is not required that $\rho(x) = 0$ implies $x = 0$.

The topology of uniform convergence on compacta is defined as follows. For each compact set $K \subset U$, we define a seminorm $||\cdot||_K$ on $\mathcal{H}(U)$ by

$$||f||_K = \sup_{z \in K} |f(z)|.$$

Then a neighborhood basis at $g \in \mathcal{H}(U)$ consists of all sets of the form

$$\{f \in \mathcal{H}(U) : ||f - g||_K < \epsilon\}$$

for K a compact subset of U and $\epsilon > 0$. If $\{K_n\}$ is an increasing sequence of compact subsets of U, with the property that each compact subset of

U is contained in some K_n, then a basis for this topology is also obtained using just the sets of the above form with K one of the sets K_n and ϵ one of the numbers m^{-1}, for m a positive integer. Thus, there is a countable basis for the topology at each point and, in fact, $\mathcal{H}(U)$ is a metric space in this topology. Clearly, a sequence converges in this topology if and only if it converges uniformly on each compact subset of U. A nice application of Osgoods's lemma is the proof that $\mathcal{H}(U)$ is complete in this topology. Since a sequence $\{f_j\}$ in $\mathcal{H}(U)$, which is Cauchy in each seminorm $||\cdot||_K$, converges uniformly to a continuous function on each K, the completeness of $\mathcal{H}(U)$ amounts to the following:

2.4.1 Theorem. *If $\{f_n\}$ is a sequence of holomorphic functions on U, which is uniformly convergent on each compact subset of U, then the limit is also holomorphic on U.*

Proof. For holomorphic functions of one variable, this is a standard result. Its proof is a simple application of Morera's theorem. In the several variable case we simply apply this result in each variable separately (with the other variables fixed) to conclude that the limit of a sequence of holomorphic functions, converging uniformly on compacta, is holomorphic in each variable separately. Such a limit is also obviously continuous, and so is holomorphic, by Osgood's lemma (we could appeal to Hartog's theorem but we really only need the more elementary Osgood's lemma).

A topological vector space, with a topology defined by a sequence of seminorms, as above, and which is complete in this topology, is called a *Fréchet space.* Thus, $\mathcal{H}(U)$ is a Fréchet space. It is, in fact, a Montel space. The content of this statement is that it is a Fréchet space, with the following additional property:

2.4.2 Theorem. *Every closed bounded subset of $\mathcal{H}(U)$ is compact, where $A \subset \mathcal{H}(U)$ is bounded if $\{||f||_K : f \in A\}$ is bounded for each compact subset K of U.*

Proof. Since $H(U)$ is a metric space we need only prove that every bounded sequence in $\mathcal{H}(U)$ has a convergent subsequence. Thus, let $\{f_n\}$ be a bounded sequence in $\mathcal{H}(U)$. By the previous theorem, we need only show that it has a subsequence that converges uniformly on compacta to a continuous function – the limit will then automatically be holomorphic as well. By the Ascoli-Arzela theorem, a bounded, equicontinuous sequence of functions on a compact set has a uniformly convergent subsequence. It follows from Cauchy's inequalities (Proposition 2.1.3) that the sequence $\{f_n\}$ has uniformly bounded sequences of complex partial derivatives $\{\partial f_n/\partial z_j\}$ on each compact set. This, together with the Cauchy-Riemann equations,

$\partial f_n/\partial \bar{z}_j = 0$, implies that the real and imaginary parts of $\{f_n\}$ have uniformly bounded sequences of real partial derivatives on each compact set. Then the mean value theorem implies that $\{f_n\}$ is equicontinuous on each compact set and, hence, has a uniformly convergent subsequence on each compact set (Exercise 2.7).

We choose a sequence $\{K_m\}$ of compact subsets of U, with the property that each compact subset of U is contained in some K_m. By the previous paragraph, we may choose inductively a sequence of subsequences $\{f_{n_{m,i}}\}_i$ of $\{f_n\}$ with the property that $\{f_{n_{m+1,i}}\}_i$ is a subsequence of $\{f_{n_{m,i}}\}_i$ and $\{f_{n_{m,i}}\}_i$ is uniformly convergent on K_m. The diagonal of the resulting array of functions converges uniformly on each K_m and, hence, on each compact subset of U. This completes the proof.

2.5 Domains of Holomorphy

If U is an open subset of $\mathbb{C}$, then there is a function $f \in \mathcal{H}(U)$ such that f cannot be extended to be holomorphic on any larger open set (Exercise 1.8). The situation in several variables is quite different. In fact there are open sets $U \subset V$ with U a proper subset of V and with the property that *every* holomorphic function on U extends to be holomorphic on V. This is demonstrated by the following proposition:

2.5.1 Proposition. *Let $\Delta(0, r)$ be an open polydisc in $\mathbb{C}^n$ with $n > 1$. Set $r = (r_1, \dots, r_n)$ and $r' = (r_1, \dots, r_{n-1})$. Let U be a connected open subset of $\Delta(0, r)$ and for each $z' \in \Delta(0, r')$ set $U_{z'} = \{z_n \in \mathbb{C} : (z', z_n) \in U\}$. Assume that U has the following properties:*

(i) *there is a fixed $s < r_n$ such that $\bar{\Delta}(0, s)$ contains the complement of $U_{z'}$ in $\Delta(0, r_n)$ for each $z' \in \Delta(0, r')$;*

(ii) *the equality $U_{z'} = \Delta(0, r_n)$ holds for all z' in some open subset of $\Delta(0, r')$.*

Then every holomorphic function on U has a holomorphic extension to $\Delta(0, r)$.

Proof. Given $z_n \in \Delta(0, r_n)$, we may choose a smooth simple closed curve $\gamma \subset \Delta(0, r_n) - \bar{\Delta}(0, s)$ such that z_n and $\bar{\Delta}(0, s)$ are both contained in the bounded component of the complement of γ. For $f \in \mathcal{H}(U)$, $z' \in \Delta(0, r')$, we set

$$\hat{f}(z', z_n) = (2\pi i)^{-1} \int_\gamma \frac{f(z', \zeta)}{\zeta - z_n}\, d\zeta.$$

By the Cauchy integral theorem, this function is independent of the choice of γ. It is clearly holomorphic in z_n. It is also holomorphic in z', since, for a given z_n, the curve γ is independent of z'. Thus, $\hat{f}$ is a holomorphic function on $\Delta(0,r)$. If z' is a point for which $\Delta(0,r_n) = U_{z'}$, then $f(z',z_n)$ is holomorphic in z_n on all of $\Delta(0,r_n)$ and the Cauchy integral formula tells us that $\hat{f}(z',z_n) = f(z',z_n)$, for all $z_n \in \Delta(0,r_n)$. Since there is a non-empty open set of such points z' and since U is connected, the identity theorem (Exercise 2.2) implies that $\hat{f} = f$ on U.

It is easy to construct examples of the above situation:

2.5.2 Example. Let $\Delta(0,r)$ be an open polydisc and $U = \Delta(0,r) - K$, where K is any compact subset of $\Delta(0,r)$ which does not separate $\Delta(0,r)$. It is obvious that U and $\Delta(0,r)$ satisfy the conditions of Proposition 2.5.1. Thus, any function holomorphic on $\Delta(0,r) - K$ extends to be holomorphic on all of $\Delta(0,r)$. In Exercise 10.21 this result is extended to the case where U is an arbitrary open set, rather than a polydisc.

In the above example, U is not topologically trivial. In fact, it has non-vanishing 2-dimensional homology. It might be expected that this would cause trouble. However, in the next example U is actually contractible.

2.5.3 Example. Let A be the open annulus $\Delta(0,1) - \bar{\Delta}(0,1/2)$ and set $U = (\Delta(0,1) \times A) \cup (\Delta(0,1/2) \times \Delta(0,1)) \subset \Delta(0,(1,1))$. Here, $z' = z_1$ and U_{z_1} is all of $\Delta(0,1)$ if $|z_1| < 1/2$ and is the annulus A if $|z_1| \geq 1/2$. Thus, U and $\Delta(0,(1,1))$ satisfy the conditions of Proposition 2.5.1. Some idea of what U looks like can be obtained by looking at its three-dimensional projection under the map $(z_1,z_2) \to (|z_1|, z_2)$. The result is a solid cylinder of length 1 and radius 1 with a hole of radius 1/2 drilled half way through from one end.

The phenomenon expressed by Proposition 2.5.1 is of fundamental importance. It is a phenomenon that does not occur in one variable. It turns out that, for sets U like the one in Proposition 2.5.1, there are obstructions to the solution of local to global problems like those of Chapter 1. On the other hand, on sets which are completely free of this phenomena, these local to global problems always have solutions. What does it mean for an open set to be completely free of the phenomena of Proposition 2.5.1? It means that the set is a *domain of holomorphy* as expressed in the following definition.

2.5.4 Definition. An open set $U \subset \mathbb{C}^n$ is called a domain of holomorphy if there is a function $f \in \mathcal{H}(U)$, with the following property: For each point w on the boundary of U, and each polyradius r, there is no holomorphic function on $\Delta(w,r)$ which is equal to f on a component of $\Delta(w,r) \cap U$.

In other words, U is a domain of holomorphy if there is a holomorphic function on U which has no local holomorphic extension across part of the boundary of U.

Note that there is a difference between saying that a function f doesn't have local extensions across the boundary in the sense of Definition 2.5.4 and saying that f doesn't have a global extension to any larger domain. For example, the log function on $U = \mathbb{C} - \{x \in \mathbb{R}; x \leq 0\}$ has no global holomorphic extension to a larger open set, but it does have local extensions across the boundary. That is, if $x < 0$ and $0 < r < -x$, then f has holomorphic extensions to $\Delta(x, r)$ from each of the two components of $\Delta(x, r) \cap U$ (they don't agree, of course).

We conclude this chapter by giving a useful characterization of domains of holomorphy. That they are free of obstructions to solving local to global problems, like those of Chapter 1, is a special case of our main vanishing theorem for sheaf cohomology – Cartan's Theorem B – which will be proved in Chapter 11.

Suppose an open set U is fixed. If K is a compact subset of U, then we set

$$\hat{K} = \{w \in U : |f(w)| \leq ||f||_K \text{ for all } f \in \mathcal{H}(U)\},$$

where $||f||_K = \sup\{|f(z)| : z \in K\}$. The set $\hat{K}$ is the *holomorphically convex hull* of K in U. Note that, if $\hat{K}$ is also compact, then it is its own holomorphically convex hull.

2.5.5 Definition. An open set $U \subset \mathbb{C}^n$ is said to be holomorphically convex if $\hat{K}$ is compact for each compact set $K \subset U$.

2.5.6 Proposition. *If U is a holomorphically convex open set, then U is a domain of holomorphy.*

Proof. Let $\{w_i\}$ be an enumeration of a dense subset of U in which each element of the set appears infinitely often. For each i, let S_i be the open sphere of largest radius that is centered at w_i and is contained in U. Let $\{K_j\}$ be an increasing sequence of non-empty, compact subsets of U such that every compact subset of U is contained in some K_j.

Since U is holomorphically convex, $\hat{K}_j$ is compact for each j. Since S_j is either all of $\mathbb{C}^n$ or contains points arbitrarily near the boundary of U, there is a point $z_j \in S_j$ which is not in $\hat{K}_j$. Thus, we may choose a holomorphic function $f_j \in \mathcal{H}(U)$ such that $f_j(z_j) = 1$ and $||f_j||_{K_j} < 1$. By replacing f_j by a high enough power of itself we may assume that $||f_j||_{K_j} < 2^{-j}$. We may also assume that f_j is not identically 1 on any component of U. Then $\prod_j (1 - f_j)^j$ converges uniformly on each K_j to a function $f \in \mathcal{H}(U)$ which does not vanish identically on any component of U (Exercise 2.5). Furthermore, f has a zero of total order at least j at z_j (this means that the lowest order non-vanishing term in the power series expansion of f at z_j has order at least j). If Δ is a polydisc centered at a boundary point of U, then every component of $U \cap \Delta$ contains infinitely many of the spheres S_j. Thus, f has zeroes of arbitrarily high total order in each component of $\Delta \cap U$. If f could be continued from one of these components to a holomorphic function $\hat{f}$ on Δ, then $\hat{f}$ and all of its partial derivatives would have to vanish at each point of Δ which is a boundary point of the component. Then $\hat{f}$ would vanish identically on Δ and, consequently, on every component of U which meets Δ. This is impossible, since f does not vanish identically on any component of U. It follows that U is a domain of holomorphy.

2.5.7 Theorem. *If U is an open set in $\mathbb{C}^n$, then U is a domain of holomorphy if and only if U is holomorphically convex.*

Proof. In view of Proposition 2.5.6, we need only prove that, if U is a domain of holomorphy, then it is holomorphically convex.

Let K be a compact subset of U. There is a number $s > 0$ such that the polydisc $\Delta(a, \hat{s})$ is contained in U for every $a \in K$. Here, we will use the convention that, for $r > 0$, the polyradius $\hat{r}$ is $(r, r, \dots, r)$. If f is a function holomorphic on U, and $0 < r < s$, then it follows from Cauchy's estimates (Proposition 2.1.3) that

$$\left| \frac{\partial^{i_1+\cdots+i_n}}{\partial {z_1}^{i_1} \dots \partial {z_n}^{i_n}} f(a) \right| \leq M(i_1!) \dots (i_n!) r^{-i_1-\cdots-i_n}$$

whenever $a \in K$. Here, M is the supremum of $|f|$ on the compact set consisting of the union of all closed polydiscs of radius r centered at points of K. By the definition of $\hat{K}$, the same inequalities will hold for all points $a \in \hat{K}$.

Now $\hat{K}$ is always a bounded set, since each coordinate function z_i is holomorphic on U and, hence, is bounded on $\hat{K}$ by the same bound it satisfies on K. So, if $\hat{K}$ is not compact, it must share a boundary point with U. But then, given such a point b, we can find a point $a \in \hat{K}$ close enough to b that the polydisc $\Delta(a, \hat{s})$ contains b. However, the above estimates imply that the power series expansion of f about a converges in $\Delta(a, \hat{s})$. If U is a domain of holomorphy, then there is a function f for which this is impossible. Thus, $\hat{K}$ must be compact for every compact subset K of U and U is holomorphically convex.

With this result in hand, it is possible to show that a wide variety of open sets are domains of holomorphy. Exercises 2.8 through 2.10 provide several examples of results of this type.

Exercises

1. If U is a domain in $\mathbb{C}^n$ and $F : U \to \mathbb{C}^m$ is a map, then F is called holomorphic if each of its coordinate functions is holomorphic. Prove that the composition of two holomorphic mappings is holomorphic.
2. Prove that a holomorphic function on a connected open set $U \subset \mathbb{C}^n$, which vanishes to infinite order at some point of U, vanishes identically. Here, a holomorphic function is said to vanish to infinite order at a point if it and its partial derivatives of all orders vanish at the point.
3. Formulate and prove the maximum modulus theorem for holomorphic functions of several variables.
4. Prove Schwarz's lemma for holomorphic functions of several variables: If f is holomorphic on $\Delta(0, \hat{r})$, $\hat{r} = (r, r, \dots, r)$, $|f(z)| \leq 1$ for $z \in \Delta(0, \hat{r})$, and k is the degree of the lowest order non-vanishing homogeneous polynomial in the power series expansion of f, then $|f(z)| \leq ||z||^k r^{-k}$ for all $z \in \Delta(0, \hat{r})$.
5. Prove the claims about the convergence and non-vanishing of the infinite product in the proof of Proposition 2.5.6. Note that an infinite product is defined to be the limit of its sequence of finite partial products, if that limit exists. (hint: Use logarithms.)
6. Prove that if a holomorphic function on a connected open set is not identically 0, then the set where it vanishes has $2n$-dimensional Lebesgue measure 0 (hint: Use Jensen's inequality.)
7. Prove the claim used in the proof of Theorem 2.4.2: A sequence of real valued functions on an open set $U \subset \mathbb{R}^n$, with first order partial derivatives which are uniformly bounded on each compact subset of U, is equicontinuous on each compact subset of U.
8. Prove that the Cartesian product of any collection of n open subsets of $\mathbb{C}$ is a domain of holomorphy in $\mathbb{C}^n$. In particular, $\mathbb{C}^n$ itself is a domain of holomorphy.
9. Prove that if $U \subset \mathbb{C}^n$ is a domain of holomorphy, $f_1, \dots, f_k \in \mathcal{H}(U)$, and $V = \{z \in U : |f_i(z)| < 1, i = 1, \dots, k\}$, then V is also a domain of holomorphy.
10. Prove that each convex open subset of $\mathbb{C}^n$ is a domain of holomorphy.
11. Prove that an open set in $\mathbb{C}^n$ is a domain of holomorphy if and only if each of its connected components is a domain of holomorphy.
12. Prove that if Δ is an open polydisc in $\mathbb{C}^n$, $n > 1$, and K is a compact subset of Δ, then $\Delta - K$ is not a domain of holomorphy.

13. Prove that an open subset U of $\mathbb{C}^n$ is a domain of holomorphy if and only if, for every sequence $\{z_i\} \subset U$, which converges to a point on the boundary of U, there is an $f \in \mathcal{H}(U)$ such that $\{f(z_i)\}$ is unbounded.
14. Prove that a holomorphic function is an open map (the image of every open subset of the domain is open).
15. Let U be a connected open subset of $\mathbb{C}^n$ and $f \in \mathcal{H}(U)$. Prove that, if f vanishes on a non-empty set of the form $U \cap (z_0 + \mathbb{R}^n)$, then f is identically 0. What must be true of a real linear subspace $L \subset \mathbb{C}^n$ if this result is to hold with $\mathbb{R}^n$ replaced by L?
16. Let $U \subset \mathbb{C}^n$ be a connected open set and set $U^* = \{\bar{z} : z \in U\}$. Prove that, if $f \in \mathcal{H}(U \times U^*)$ and $f(z, \bar{z}) \equiv 0$ on U, then $f(z, w) \equiv 0$ on $U \times U^*$.
17. Suppose U is an open subset of $\mathbb{C}^n$. Prove that a sequence $\{f_i\} \subset \mathcal{H}(U)$, which is bounded on each compact subset of U, and converges pointwise on U, actually converges uniformly on compact subsets of U.
18. If $n > 1$ and $U \subset \mathbb{C}^n$ is an open set, prove that a holomorphic function on U cannot have an isolated zero in U.
19. If $n > 1$, prove that a function that is holomorphic on all of $\mathbb{C}^n$ cannot have a non-empty bounded set as its set of zeroes.
20. Prove that, if a power series
$$\sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1\ldots i_n} z_1^{i_1} \ldots z_n^{i_n}$$
converges at a given point $z = (z_1, \ldots, z_n)$, then there is a positive constant K such that
$$|c_{i_1\ldots i_n}| \le K r_1^{-i_1} \ldots r_n^{-i_1}$$
where $r_i = |z_i|$. Then prove that the power series converges uniformly and absolutely on every compact polydisc contained in $\Delta(0, r)$, where $r = (r_1, \ldots, r_n)$.
21. For the power series in the previous problem, let C be the subset of $\mathbb{C}^n$ consisting of all z such that the series converges in a neighborhood of z. Prove that (i) if $(z_1, \ldots, z_n) \in C$, then $(w_1, \ldots, w_n) \in C$ for all $(w_1, \ldots, w_n)$ with $|w_i| \le |z_i|$ for each i; and (ii) C is *logarithmically convex*, meaning that
$$\log|C| = \{(\log|z_1|, \ldots, \log|z_n|) : (z_1, \ldots, z_n) \in C, z_i \neq 0, i = 1, \ldots, n\}$$
is a convex subset of $\mathbb{R}^n$. The set C is called the *domain of convergence* of the power series.
22. Show that the domain of convergence (see the previous problem) of a power series need not be a polydisc, by finding the domain of convergence for the power series $\sum_{k=0}^{\infty} z_1^k z_2^k$.

Chapter 3

Local Rings and Varieties

In this chapter we study the behavior of holomorphic functions in small neighborhoods of a point. This leads to the notion of the *germ* of a function at a point. The set of all germs of holomorphic functions at a point $\lambda_0 \in \mathbb{C}^n$ forms a local ring which we call the ring of germs of holomorphic functions at λ. This ring has many things in common with the local ring of rational functions which are regular at a point in $\mathbb{C}^n$. Both are Noetherian rings and both have the property that their ideals are closely related to the underlying geometry of $\mathbb{C}^n$ through the theory of germs of varieties.

The basic tools for the study of the local ring of germs of holomorphic functions are the Weierstrass preparation theorem and Weierstrass division theorem. These allow us to reduce problems involving germs of holomorphic functions in n variables to problems involving polynomials with coefficients which are germs of holomorphic functions in $n - 1$ variables. These results lead not only to the fact that the local ring of germs of holomorphic functions is Noetherian, but also to results, such as the implicit and inverse mapping theorems, which do not have analogues for the local rings of rational functions.

There are deeper results concerning the local structure of holomorphic functions that will have to wait until the next chapter. These include detailed descriptions of the local structure of holomorphic subvarieties of $\mathbb{C}^n$ and the exact relationship between these geometric objects and the ideals of the local ring of germs of holomorphic functions.

3.1 Rings of Germs of Holomorphic Functions

Let X be a topological space and x a point of X. If f and g are functions defined in neighborhoods U and V of x and if $f(y) = g(y)$ for all y in some third neighborhood W of x with $W \subset U \cap V$, then we say that f and g are *equivalent* at x. The equivalence class consisting of all functions equivalent to f at x is called the *germ* of f at x.

The set of germs of complex valued functions at x is clearly an algebra over the complex field, with the algebra operations defined in the obvious way. In fact, this algebra can be described as the inductive limit $\varinjlim F(U)$, where $F(U)$ is the algebra of complex valued functions on U, and the limit is taken over the directed set consisting of neighborhoods of x. The germs of continuous functions at x obviously form a subalgebra of the germs of all complex valued functions at x and, in the case where $X = \mathbb{C}^n$, the germs of holomorphic functions at x form a subalgebra of the germs of $\mathcal{C}^\infty$ functions at x which, in turn, form a subalgebra of the germs of continuous functions at X.

We shall denote the algebra of holomorphic functions in a neighborhood $U \subset \mathbb{C}^n$ by $\mathcal{H}(U)$ and the algebra of germs of holomorphic functions at $\lambda \in \mathbb{C}^n$ by $\mathcal{H}_\lambda$, or by ${}_n\mathcal{H}_\lambda$ in case it is important to stress the dimension n. We have that $\mathcal{H}_\lambda = \varinjlim\{\mathcal{H}(U) : \lambda \in U\}$. We also have as an immediate consequence of the definition of holomorphic function that:

3.1.1 Proposition. *The algebra ${}_n\mathcal{H}_0$ may be described as $\mathbb{C}\{z_1, ..., z_n\}$, the algebra of convergent power series in n variables.*

Here, by a convergent power series in the n variables $z_1, \dots, z_n$, we mean one which converges in some polydisc $\Delta(0, r)$ with $r_i > 0, i = 1, \dots, n$.

Another important algebra of germs of functions is the algebra ${}_n\mathcal{O}_\lambda$ of germs of regular functions at $\lambda \in \mathbb{C}^n$. Here we give $\mathbb{C}^n$ the *Zariski topology.* This is the topology in which a set is closed if and only if it is the set of common zeroes of some set of polynomials in $\mathbb{C}[z_1, ..., z_n]$. A *regular function* on a Zariski open set U is a rational function (ratio of two polynomials) with a denominator which does not vanish on U. The algebra of regular functions on a Zariski open set U will be denoted $\mathcal{O}(U)$. The algebra of germs of regular functions at $\lambda \in \mathbb{C}^n$ is then ${}_n\mathcal{O}_\lambda = \varinjlim\{\mathcal{O}(U) : \lambda \in U\}$. It follows that ${}_n\mathcal{O}_\lambda$ is the algebra of functions of the form f/g, where f and g are polynomials in $\mathbb{C}[z_1, ..., z_n]$ and g does not vanish at λ. Thus,

3.1.2 Proposition. *The algebra ${}_n\mathcal{O}_\lambda$ is just the algebra of fractions of the algebra $\mathbb{C}[z_1, ..., z_n]$ with respect to the multiplicative set consisting of polynomials which do not vanish at λ.*

By a *local ring* we will mean a commutative ring with a unique maximal ideal.

3.1.3 Proposition. *The algebras ${}_n\mathcal{O}_\lambda$ and ${}_n\mathcal{H}_\lambda$ are local rings and, in each case, the maximal ideal consists of the elements which are represented by functions which vanish at λ.*

Proof. In each of these algebras, a germ represented by a function which does not vanish at λ will be invertible. This implies that every proper ideal of the algebra is contained in the ideal consisting of germs represented by functions which vanish at λ.

The obvious translation operator defines an isomorphism between ${}_n\mathcal{H}_\lambda$ and ${}_n\mathcal{H}_0$ and between ${}_n\mathcal{O}_\lambda$ and ${}_n\mathcal{O}_0$. Thus, in what follows, we will restrict our attention to ${}_n\mathcal{H}_0$ and ${}_n\mathcal{O}_0$.

3.2 Hilbert's Basis Theorem

The algebras ${}_n\mathcal{O}_0$ and ${}_n\mathcal{H}_0$ are, in fact, Noetherian rings (every ideal is finitely generated). For ${}_n\mathcal{O}_0$ this is a well-known and elementary fact from commutative algebra. We will give the proof here because the main ingredient (Hilbert's basis theorem) will also be needed in the proof that ${}_n\mathcal{H}_0$ is Noetherian.

We will use the elementary fact that if M is a finitely generated module over a Noetherian ring A, then every submodule and every quotient module of M is also finitely generated.

3.2.1 Hilbert's Basis Theorem. *If A is a Noetherian ring, then the polynomial ring $A[x]$ is also Noetherian.*

Proof. Let I be an ideal in $A[x]$ and let J be the ideal of A consisting of all leading coefficients of elements of I. Since A is Noetherian, J has a finite set of generators $\{a_1, ..., a_n\}$. For each i, there is an $f_i \in I$ such that $f_i = a_i x^{r_i} + g_i$ where g_i has degree less than r_i.

Let $r = \max_i r_i$ and let $f = ax^m + g$ (deg $g < m$) be an element of I of degree $m \geq r$. We may choose $b_i, ..., b_n \in A$ such that $a = \sum_i b_i a_i$. Then $f - \sum_i b_i f_i x^{m-r_i}$ belongs to I and has degree less than m. By iterating this process, we conclude that every element of I may be written as the sum of an element of the ideal generated by $\{f_1, ..., f_n\}$ and a polynomial of degree less than r belonging to I. However, the polynomials of degree less than r form a finitely generated A-module and, hence, the submodule consisting of its intersection with I is also finitely generated. A generating set for this

A-module together with $\{f_1, ..., f_n\}$ provides a set of generators for I as an $A[x]$-module. This completes the proof.

An induction argument, based on the above result, shows that the polynomial ring $\mathbb{C}[z_1, ..., z_n]$ is Noetherian. This implies that ${}_n\mathcal{O}_0$ is Noetherian as follows: If I is an ideal of ${}_n\mathcal{O}_0$ and

$$J = I \cap \mathbb{C}[z_1, ..., z_n],$$

then J generates I as an ${}_n\mathcal{O}_0$-module but J is finitely generated as a $\mathbb{C}[z_1, ..., z_n]$-module. It follows that I is finitely generated as an ${}_n\mathcal{O}_0$-module. In summary:

3.2.2 Theorem. *The polynomial algebra $\mathbb{C}[z_1, ..., z_n]$ and the local algebra ${}_n\mathcal{O}_0$ are Noetherian rings.*

3.3 The Weierstrass Theorems

We now proceed to develop the tools needed to prove that ${}_n\mathcal{H}_0$ is Noetherian. In what follows, we shall use bold letters to denote germs of functions, in order to distinguish them from ordinary functions.

A holomorphic function f, defined in a neighborhood of 0, is said to have *vanishing order* k in z_n at 0, provided $f(0, ..., 0, z_n)$ has a zero of order k at 0, with $0 \le k < \infty$. We will say that f has *finite vanishing order* in z_n at 0 if $f(0, ..., 0, z_n)$ has vanishing order k for some k, that is, if it does not vanish identically in any neighborhood of $z_n = 0$. Thus, if f is not identically 0 in a neighborhood of $z = 0$, then coordinates can be chosen for $\mathbb{C}^n$ so that f has finite vanishing order in z_n at 0. A germ $\mathbf{f} \in \mathcal{H}_0$ is said to have *vanishing order* k in z_n if it has a representative in some neighborhood of 0 which has vanishing order k in z_n at 0.

3.3.1 Proposition. *If f is holomorphic in a neighborhood U of 0 in $\mathbb{C}^n$ and has vanishing order k in z_n at 0, then there is a polydisc $\Delta(0, r') \times \Delta(0, r_n)$ such that, for each $z' \in \Delta(0, r')$, as a function of z_n, $f(z', z_n)$ has exactly k zeroes in $\Delta(0, r_n)$, counting multiplicity, and no zeroes on the boundary of $\Delta(0, r_n)$.*

Proof. Choose r_n small enough that the only zeroes of $f(0, z_n)$ on $\bar{\Delta}(0, r_n)$ occur at $z_n = 0$. Set

$$\delta = \inf\{|f(0, z_n)| : \quad |z_n| = r_n\}$$

and choose r' small enough that

$$|f(z', z_n) - f(0, z_n)| < \delta \text{ whenever } z' \in \Delta(0, r'), \quad |z_n| = r_n.$$

Then, for each $z' \in \Delta(0, r')$, $f(z', z_n)$ has no zeroes on the boundary of $\Delta(0, r_n)$ and, by Rouche's theorem ([**R**], Theorem 10.43), the functions $f(0, z_n)$ and $f(z', z_n)$ of z_n have the same number of zeroes in the disc $\Delta(0, r_n)$. This completes the proof.

A *thin subset* of an open set $U \subset \mathbb{C}^n$ is a subset which, locally at each point of U, is contained in the zero set of a non-trivial holomorphic function. That is, T is thin in U if, for each $z \in U$, there exists a neighborhood V of z and $f \in \mathcal{H}(V)$ such that $f = 0$ on $V \cap T$, but f does not vanish identically on any neighborhood of z.

3.3.2 Removable Singularity Theorem. *If f is bounded and holomorphic on an open set of the form $U - T$, where U is open and T is a thin subset of U, then f has a unique holomorphic extension to all of U.*

Proof. This is a local result and needs only to be proved for a neighborhood of each point $a \in U$. We may assume that the intersection of T with some neighborhood V of a is contained in the zero set of a non-trivial holomorphic function g on V, which we may assume has vanishing order k in z_n, for some k (otherwise we simply perform a coordinate change). By the preceding result, there is a neighborhood of the form $\Delta(a, r) = \Delta(a', r') \times \Delta(a_n, r_n)$ with the property that, for each $z' \in \Delta(a', r')$, the set T meets $\{z'\} \times \bar{\Delta}(a_n, r_n)$ in at most k points, all of which lie in $\{z'\} \times \Delta(a_n, r_n)$. Then the function

$$h(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n - a_n| = r_n} \frac{f(z', \zeta_n)}{\zeta_n - z_n} d\zeta_n$$

is holomorphic in all of $\Delta(a, r)$ and agrees with f off T, by the Cauchy integral theorem and the removable singularity theorem for holomorphic functions of one complex variable.

We next prove a classical result about the elementary symmetric functions which will be useful in the proof of the Weierstrass preparation theorem. The elementary symmetric functions of $z = (z_1, z_2, \ldots, z_n)$ are the functions $\phi_j(z)$, $j = 1, \ldots, n$ which appear in the expansion

$$\prod_i (\lambda - z_i) = \lambda^n - \phi_1(z)\lambda^{n-1} + \cdots + (-1)^n \phi_n(z).$$

Thus, $\phi_1(z) = \sum\limits_i z_i$, $\phi_2(z) = \sum\limits_{i<j} z_i z_j$, etc. On the other hand, the power sum functions of z are the functions

$$s_k(z) = z_1^k + z_2^k + \cdots + z_n^k, \quad k = 1, \ldots, n.$$

3.3.3 Lemma. *Each of the elementary symmetric functions ϕ_j may be written as a polynomial in the power sum functions $s_1, \ldots, s_n$.*

Proof. For a fixed value of $z = (z_1, \ldots, z_n)$, we define a polynomial

$$\psi(\lambda) = \prod_{i=1}^{n}(1 - \lambda z_i) = 1 - \phi_1\lambda + \phi_2\lambda^2 + \cdots + (-1)^n\phi_n\lambda^n$$

and note that, since $\psi(0) = 1$, $\psi(\lambda)$ has a holomorphic logarithm $\log\psi(\lambda)$ in some neighborhood of $\lambda = 0$. The derivative of $\log\psi(\lambda)$ is

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = \sum_{i=1}^{n} \frac{-z_i}{1 - \lambda z_i} = -s_1 - s_2\lambda - \cdots - s_n\lambda^{n-1} - \cdots,$$

where the sum on the right converges in a neighborhood of $\lambda = 0$. This leads to the relation

$$-\psi'(\lambda) = \psi(\lambda) \cdot \sum_{k=1}^{\infty} s_k\lambda^{k-1}.$$

On equating powers of λ, we obtain the equations

$$\begin{aligned} \phi_1 &= s_1, \\ -2\phi_2 &= -\phi_1 s_1 + s_2, \\ 3\phi_3 &= \phi_2 s_1 - \phi_1 s_2 + s_3, \\ &\;\;\vdots \end{aligned}$$

For $1 \le j \le n$, the jth equation expresses ϕ_j as a polynomial in the s_k with $k \le j$ and the ϕ_i with $i < j$. Note that the coefficients of these polynomials are independent of z. A simple induction argument then shows that each ϕ_j is a polynomial in the s_k with coefficients independent of z.

A *Weierstrass polynomial* of degree k in z_n is a polynomial $\mathbf{h} \in {}_{n-1}\mathcal{H}_0[z_n]$ of the form

$$\mathbf{h}(z) = z_n^k + \mathbf{a}_1(z')z_n^{k-1} + \cdots + \mathbf{a}_{k-1}(z')z_n + \mathbf{a}_k(z'),$$

where $z = (z', z_n)$ and each $\mathbf{a}_i$ is a non-unit in ${}_{n-1}\mathcal{H}_0$.

3.3.4 Weierstrass Preparation Theorem. *If $\mathbf{f} \in {}_n\mathcal{H}_0$ has vanishing order k in z_n, then $\mathbf{f}$ has a unique factorization as $\mathbf{f} = \mathbf{uh}$, where $\mathbf{h}$ is a Weierstrass polynomial of degree k in z_n and $\mathbf{u}$ is a unit in ${}_n\mathcal{H}_0$.*

Proof. We fix some representative f of $\mathbf{f}$ and use Proposition 3.3.1 to choose a polydisc $\Delta(0,r)$ in which $f(z',z_n)$ has exactly k zeroes (none on the boundary) as a function of z_n for each $z' \in \Delta(0,r')$. We label the zeroes $b_1(z'),\dots,b_k(z')$. The polynomial we are seeking is the germ of

$$h(z) = \prod_{j=1}^{k}(z_n - b_j(z')) = z_n^k - a_1(z')z_n^{k-1} + \cdots + (-1)^k a_k(z').$$

Of course, the functions $b_j(z')$ need not even be continuous, because of the arbitrary choices made in labeling the zeroes of f. However, the functions $a_j(z')$ are, in fact, holomorphic. To see this, note that these functions are the elementary symmetric functions of the b_j's and these, by Lemma 3.3.3, may be written as polynomials in the power sums s_m where

$$s_m = \sum_{j=1}^{k} b_j^m.$$

Also, it follows from residue theory that

$$s_m(z') = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \zeta^m \frac{\partial f/\partial\zeta(z',\zeta)}{f(z',\zeta)}\, d\zeta.$$

These functions and, consequently, the $a_j(z')$ are holomorphic in $\Delta(0,r')$. Note that the b_j's all vanish at $z'=0$ and, thus, so do the a_j's. We conclude that the germ $\mathbf{h}$ of h at 0 is a Weierstrass polynomial.

The germs of h and u will provide the required factorization if we can show that $u = f/h$ is holomorphic and non-vanishing in $\Delta(0,r)$. For each fixed $z' \in \Delta(0,r')$, the function $z_n \to f(z',z_n)/h(z',z_n)$ has a holomorphic extension to $\Delta(0,r_n)$, since numerator and denominator have exactly the same zeroes (counting order) in this polydisc. Furthermore, h is bounded away from 0 on $\Delta(0,r') \times \partial\Delta(0,r_n)$. This and the maximum modulus principal imply that f/h is bounded on $\Delta(0,r)$. Since it is holomorphic in this set except where h vanishes, Theorem 3.3.2 implies that it extends to be holomorphic and non-vanishing in the entire polydisc. Furthermore, it is clear from the construction that h is the only Weierstrass polynomial of degree k which has the same zeroes as f for each $z' \in \Delta(0,r')$. It follows that the factorization is unique. This completes the proof.

3.3.5 Weierstrass Division Theorem. *If* $\mathbf{h} \in {}_{n-1}\mathcal{H}_0[z_n]$ *is a Weierstrass polynomial of degree* k *and* $\mathbf{f} \in {}_n\mathcal{H}_0$*, then* $\mathbf{f}$ *can be written uniquely in the form* $\mathbf{f} = \mathbf{g}\mathbf{h} + \mathbf{q}$*, where* $\mathbf{g} \in {}_n\mathcal{H}_0$ *and* $\mathbf{q} \in {}_{n-1}\mathcal{H}_0[z_n]$ *is a polynomial in* z_n *of degree less than* k*. Furthermore, if* $\mathbf{f}$ *is a polynomial in* z_n*, then so is* $\mathbf{g}$.

Proof. We choose representatives f and h of $\mathbf{f}$ and $\mathbf{h}$ which are defined in a neighborhood of a polydisc $\bar{\Delta}(0,r)$, chosen small enough that $h(z', z_n)$ has exactly k zeroes in $\bar{\Delta}(0,r_n)$, as a function of z_n, for each $z' \in \bar{\Delta}(0,r')$, with none occurring on $|z_n| = r_n$. Then the function

$$g(z) = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{f(z',\zeta)}{h(z',\zeta)}\frac{d\zeta}{\zeta - z_n}$$

is holomorphic in $\Delta(0,r)$, as is the function $q = f - gh$. The function q may be written as

$$q(z) = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{f(z',\zeta)}{h(z',\zeta)}\frac{h(z',\zeta) - h(z',z_n)}{\zeta - z_n} d\zeta.$$

But the function

$$\frac{h(z',\zeta) - h(z',z_n)}{\zeta - z_n}$$

is a polynomial in z_n of degree less than k (with coefficients which are functions of (z',ζ)). This implies that q is a polynomial in z_n of degree less than k.

To show that this representation is unique, suppose we have two representations

$$f = gh + q = g_1 h + q_1,$$

where q and q_1 are both polynomials of degree less than k in z_n. Then $q - q_1 = h(g_1 - g)$ is a polynomial of degree less than k in z_n, with at least k zeroes for each fixed value of $z' \in \Delta(0,r')$. This is possible only if it is identically 0.

Now if f itself is a polynomial in z_n, then the usual division algorithm for polynomials over a commutative ring gives a representation of f, as above, with g a polynomial in z_n. The uniqueness implies that this must coincide with the representation given above. This completes the proof.

3.4 The Local Ring of Holomorphic Functions Is Noetherian

The Weierstrass preparation theorem and Weierstrass division theorem provide the tools needed for a large number of induction arguments involving germs of holomorphic functions. In particular, we can now prove the following:

3.4.1 Theorem. *The ring ${}_n\mathcal{H}_0$ is a Noetherian ring.*

Proof. We proceed by induction on the dimension n. When $n = 0$, the algebra ${}_n\mathcal{H}_0$ is just the base field $\mathbb{C}$, which has no non-trivial ideals, and is trivially Noetherian. Suppose that ${}_{n-1}\mathcal{H}_0$ is Noetherian. Then we know, from Hilbert's basis theorem, that ${}_{n-1}\mathcal{H}_0[z_n]$ is Noetherian. Suppose that $\mathcal{I}$ is a proper ideal in ${}_n\mathcal{H}_0$. Let $\mathbf{h}$ be any non-zero element of $\mathcal{I}$. By performing a coordinate change, if necessary, we may assume that $\mathbf{h}$ has finite vanishing order in z_n and, from the Weierstrass preparation theorem, we may (after multiplying by a unit if necessary) assume that $\mathbf{h}$ is a Weierstrass polynomial in z_n. In other words, $\mathbf{h}$ belongs to $\mathcal{I} \cap {}_{n-1}\mathcal{H}_0[z_n]$, which is an ideal in the Noetherian ring ${}_{n-1}\mathcal{H}_0[z_n]$ and is, thus, generated by a finite set $\{\mathbf{g}_1, ..., \mathbf{g}_m\}$. Now, by the Weierstrass division theorem, each $\mathbf{f} \in \mathcal{I}$ may be written as $\mathbf{f} = \mathbf{gh} + \mathbf{q}$, with $\mathbf{g} \in {}_n\mathcal{H}_0$ and $\mathbf{q} \in {}_{n-1}\mathcal{H}_0[z_n]$. But this means that $\mathbf{h}$ and $\mathbf{q}$ both belong to $\mathcal{I} \cap {}_{n-1}\mathcal{H}_0[z_n]$ and, hence, to the ideal generated by $\mathbf{g}_1, ..., \mathbf{g}_m$. Therefore, $\mathbf{f}$ belongs to the ideal generated by $\mathbf{g}_1, ..., \mathbf{g}_m$, and we conclude that this set of elements generates $\mathcal{I}$. This completes the proof.

3.5 Varieties

If S is a finite subset of $\mathcal{H}(U)$, for an open set $U \subset \mathbb{C}^n$, then we set

$$V(S) = \{z \in U : f(z) = 0 \ \text{ for every } \ f \in S\}.$$

A subset $V \subset U$ is a *holomorphic subvariety* of U if, for each point $\lambda \in U$, there is a neighborhood W_λ of λ and a finite set $S \subset \mathcal{H}(W_\lambda)$ such that

$$V \cap W_\lambda = V(S).$$

In other words, a holomorphic subvariety of U is a subset V which, locally at each point of U, is the set of common zeroes of a finite set of holomorphic functions. Note that a holomorphic subvariety V of U is necessarily a closed subset of U.

Algebraic subvarieties are defined in the same way, except that Zariski open sets are used and regular functions replace holomorphic functions. Thus, a subvariety of a Zariski open set $U \subset \mathbb{C}^n$ is a subset V of U which is locally (in the Zariski topology) the set of common zeroes of a finite set of regular functions.

Let U be a Zariski open subset of $\mathbb{C}^n$ and V an algebraic subvariety of U. Since a rational function which is regular on an open set W vanishes at the same points of W as does its numerator, we can assume that each point λ of U has a neighborhood W_λ such that $V \cap W_\lambda$ is the set of common zeroes in W_λ of a finite set of polynomials. This means that $W_\lambda \cap V$ agrees with the intersection of W_λ with a Zariski closed subset of $\mathbb{C}^n$. It follows that V is a Zariski closed subset of U. Thus, the algebraic subvarieties of U are Zariski closed subsets of U. Conversely, a Zariski closed subset is the set of common zeroes in U of some set of polynomials. Since the polynomial ring is Noetherian, there is a finite set of polynomials with the same set of common zeroes. Thus, the algebraic subvarieties of U are the common zero sets of finite sets of polynomials and these are the closed subsets of U in the Zariski topology.

It is obvious that finite unions and intersections of holomorphic subvarieties of an open set U are also holomorphic subvarieties of U. We may also define the *germ* of a holomorphic subvariety V of U at a point $\lambda \in V$. We define two holomorphic subvarieties V and W of U to be *equivalent at* $\lambda \in V \cap W$ if there is a neighborhood U_λ of λ such that $V \cap U_\lambda = W \cap U_\lambda$. The germ of a holomorphic subvariety V at $\lambda \in V$ is then the equivalence class containing V. We will say $\mathbf{V}$ is the *germ of a holomorphic variety* at $\lambda \in \mathbb{C}^n$ if $\mathbf{V}$ is the germ of a holomorphic subvariety V of some neighborhood of λ, where necessarily $\lambda \in V$.

Germs of algebraic varieties are defined in the same way. In much of the coming discussion, the statements made are true whether we are talking about holomorphic varieties or algebraic varieties. When this is the case, we shall simply use the term "variety". We shall use bold letters to denote germs of varieties.

3.5.1 Example. For $k = 1, \dots, n$ let

$$\begin{aligned} L_k &= \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_{k+1} = \cdots = z_n = 0\}, \\ M_k &= \{z \in L_k : z_k \neq 0\}, \\ N_k &= \{z \in L_k : \operatorname{Re} z_k > 0\}. \end{aligned}$$

Each L_k is an algebraic as well as holomorphic subvariety of $\mathbb{C}^n$. Note that L_k is a linear subspace of dimension k. Each set M_k is a holomorphic and algebraic subvariety of the open set $U_k = \{z \in \mathbb{C}^n : z_k \neq 0\}$, but is not a subvariety of $\mathbb{C}^n$, since it is not closed in $\mathbb{C}^n$. Each set N_k is a holomorphic subvariety of the open set $V_k = \{z \in \mathbb{C}^n : \operatorname{Re} z_k > 0\}$, but not of $\mathbb{C}^n$. The set V_k is not a Zariski open set and so it does not make sense to say that N_k is an algebraic subvariety of V_k.

Note that, for $n > 1$, it is no longer the case that the zero set of a holomorphic function must be discrete. The set L_k above is the set of common zeroes of $n - k$ holomorphic functions and is a set of complex codimension $n - k$ in $\mathbb{C}^n$. This illustrates the typical situation: Generically, adding another holomorphic function to a finite set of such functions reduces the complex dimension of the set of common zeros by 1. Of course, to formulate precisely when this statement is true, and to prove it, will take a lot of work. In particular, we will need a definition of dimension for a variety. All of this must wait until Chapter 5.

3.5.2 Example. Let V_1 be the subvariety of $U = \{(z,w) \in \mathbb{C}^2 : w \neq 0\}$ defined by the vanishing of the holomorphic function $z - e^{1/w}$, and let V_2 be the subvariety of $\mathbb{C}^2$ defined by $V_2 = \{(z,w) \in \mathbb{C}^2 : w = 0\}$. Then V_1 is a holomorphic subvariety of U but not an algebraic subvariety. The closure of V_1 in $\mathbb{C}^2$ is $V_1 \cup V_2$. However, any holomorphic function on $\mathbb{C}^2$ which vanishes on $V_1 \cup V_2$ vanishes identically (Exercise 3.15). Therefore, in $V_1 \cup V_2$ we have a union of two subvarieties which is not a subvariety. The problem, of course, is that V_1 and V_2 are not subvarieties of the same open set.

Given finitely many germs of varieties $\mathbf{V}_1, \dots, \mathbf{V}_k$ at λ, we may choose (by taking intersections, if necessary) a common neighborhood U of λ in which these germs have representatives. The germ of the intersection of these varieties is then well defined independent of the choice of U and the representatives of the $\mathbf{V}_i$. We will call this the intersection, $\mathbf{V}_1 \cap \cdots \cap \mathbf{V}_k$, of the germs $\mathbf{V}_1, \dots, \mathbf{V}_k$. The union of finitely many germs of varieties is defined in a similar manner, as is the containment relation "$\subset$". Under this relation, the set of germs of varieties at λ is a partially ordered set. We will say that the germ $\mathbf{f}$ of a holomorphic (regular) function at λ vanishes on the germ of a variety $\mathbf{V}$ at λ if there is a neighborhood U of λ and representatives f for $\mathbf{f}$ and V for $\mathbf{V}$ in U so that f vanishes on V. Each of these relationships is independent of the choice of the representatives chosen for the germs involved.

As with germs of functions, in studying germs of varieties at λ, we may as well assume $\lambda = 0$, since we may always move any point to 0 with a translation.

3.5.3 Definition. If $\mathbf{V}$ is the germ of a holomorphic variety at 0, then the ideal of $\mathbf{V}$, denoted $\operatorname{id}\mathbf{V}$, is the ideal of ${}_n\mathcal{H}_0$ consisting of all germs which vanish on $\mathbf{V}$. If $\mathcal{I}$ is an ideal of ${}_n\mathcal{H}_0$, then the locus of $\mathcal{I}$, denoted $\operatorname{loc}\mathcal{I}$, is the germ of $V(S)$, where S is a finite set of holomorphic functions, defined in a neighborhood U of 0, such that the germs of the members of S form a set of generators for $\mathcal{I}$. The ideal of a germ of an algebraic variety and the locus of a germ of an algebraic variety are defined analogously.

It is necessary to check that $\operatorname{loc}\mathcal{I}$ is well defined; that is, we must show the germ defined in Definition 3.5.3 is independent of the choice of the neighborhood U and the set $S \subset \mathcal{H}(U)$. This is a consequence of the following proposition, which gives a characterization of $\operatorname{loc}\mathcal{I}$ which is independent of these choices:

3.5.4 Proposition. *If $\mathcal{I}$ is an ideal of ${}_n\mathcal{H}_0$ (${}_n\mathcal{O}_0$), then* $\operatorname{loc}\mathcal{I}$ *is a germ of a variety on which each element of $\mathcal{I}$ vanishes, and it contains every germ of a variety with this property.*

Proof. We prove this in the case of an ideal in ${}_n\mathcal{H}_0$. The proof for ${}_n\mathcal{O}_0$ is the same. If S is a finite subset of $\mathcal{H}(U)$ whose germs at $0 \in U$ generate the ideal $\mathcal{I}$, then each $\mathbf{f} \in \mathcal{I}$ vanishes on the germ $\mathbf{V}$ of $V = V(S)$ at 0. Suppose $\mathbf{W}$ is a germ of a subvariety at 0 which also has this property. Then $\mathbf{W}$ has a representative W in some 0-neighborhood $U_1 \subset U$. Since the germ of each member of S belongs to $\mathcal{I}$, we may choose, for each $f \in S$, a 0-neighborhood $U_f \subset U_1$ so that f vanishes on $W \cap U_f$. Then each element of S vanishes on $W \cap U_2$, where $U_2 = \cap\{U_f : f \in S\}$. Thus, $W \cap U_2 \subset V$ and the germ of W, which is the same as the germ of $W \cap U_2$, is contained in the germ of V.

The next proposition lists elementary properties of id and loc that follow easily from the definitions:

3.5.5 Proposition. *The following relationships hold between ideals of ${}_n\mathcal{H}_0$ (${}_n\mathcal{O}_0$) and germs of holomorphic (algebraic) subvarieties at* 0.

(i) $\mathbf{V}_1 \subset \mathbf{V}_2$ *implies* $\operatorname{id}\mathbf{V}_1 \supset \operatorname{id}\mathbf{V}_2$;

(ii) $\mathcal{I}_1 \subset \mathcal{I}_2$ *implies* $\operatorname{loc}\mathcal{I}_1 \supset \operatorname{loc}\mathcal{I}_2$;

(iii) $\mathbf{V} = \operatorname{loc}\operatorname{id}\mathbf{V}$;

(iv) $\mathcal{I} \subset \operatorname{id}\operatorname{loc}\mathcal{I}$, *but they are not generally equal;*

(v) $\operatorname{id}(\mathbf{V}_1 \cup \mathbf{V}_2) = (\operatorname{id}\mathbf{V}_1) \cap (\operatorname{id}\mathbf{V}_2) \supset (\operatorname{id}\mathbf{V}_1) \cdot (\operatorname{id}\mathbf{V}_2)$;

(vi) $\operatorname{id}(\mathbf{V}_1 \cap \mathbf{V}_2) \supset (\operatorname{id}\mathbf{V}_1) + (\operatorname{id}\mathbf{V}_2)$;

(vii) $\operatorname{loc}(\mathcal{I}_1 \cdot \mathcal{I}_2) = \operatorname{loc}(\mathcal{I}_1 \cap \mathcal{I}_2) = \operatorname{loc}(\mathcal{I}_1) \cup \operatorname{loc}(\mathcal{I}_2)$;

(viii) $\operatorname{loc}(\mathcal{I}_1 + \mathcal{I}_2) = \operatorname{loc}(\mathcal{I}_1) \cap \operatorname{loc}(\mathcal{I}_2)$.

Proof. Parts (i) and (ii) are obvious. We will prove parts (iii), (v) and (vii). The others will be left as exercises. Again, the proofs are the same for ${}_n\mathcal{H}_0$ and ${}_n\mathcal{O}_0$, so we give just the proof for ${}_n\mathcal{H}_0$.

(iii) We use Proposition 3.5.4. Since each element of $\operatorname{id}\mathbf{V}$ vanishes on $\mathbf{V}$, we have that $\mathbf{V} \subset \operatorname{loc}\operatorname{id}\mathbf{V}$. On the other hand, since $\mathbf{V}$ is a germ of a variety, we may choose a neighborhood U and a representative V of $\mathbf{V}$ so that V is given as the set of common zeroes of a finite subset of $\mathcal{H}(U)$. Since the germs of the functions in this subset clearly belong to $\operatorname{id}\mathbf{V}$, we have that $\operatorname{loc}\operatorname{id}\mathbf{V} \subset \mathbf{V}$.

(v) A germ $\mathbf{f}$ vanishes on $\mathbf{V}_1 \cup \mathbf{V}_2$ if and only if it vanishes on $\mathbf{V}_1$ and on $\mathbf{V}_2$ and this is certainly the case if it is the product $\mathbf{gh}$ of a germ $\mathbf{g}$ which vanishes on V_1 and a germ $\mathbf{h}$ which vanishes on $\mathbf{V}_2$.

(vii) We have containments

$$\operatorname{loc}(\mathcal{I}_1) \cup \operatorname{loc}(\mathcal{I}_2) \subset \operatorname{loc}(\mathcal{I}_1 \cap \mathcal{I}_2) \subset \operatorname{loc}(\mathcal{I}_1 \cdot \mathcal{I}_2),$$

the first due to the fact that a germ which is in both $\mathcal{I}_1$ and $\mathcal{I}_2$ must vanish on both $\operatorname{loc}(\mathcal{I}_1)$ and $\operatorname{loc}(\mathcal{I}_2)$ and the second due to (ii) and the fact that $\mathcal{I}_1 \cdot \mathcal{I}_2 \subset \mathcal{I}_1 \cap \mathcal{I}_2$. Now let U be a neighborhood of 0 on which there are finite subsets S_1 and S_2 of ${}_n\mathcal{H}(U)$ whose germs form sets of generators for $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Then $S_1 \cdot S_2$ is a subset of ${}_n\mathcal{H}(U)$ whose germs form a generating set for $\mathcal{I}_1 \cdot \mathcal{I}_2$. We clearly have that $V(S_1 \cdot S_2)$ is equal to $V(S_1) \cup V(S_2)$. On passing to germs at 0, this implies that $\operatorname{loc}(\mathcal{I}_1 \cdot \mathcal{I}_2)$ is equal to $\operatorname{loc}(\mathcal{I}_1) \cup \operatorname{loc}(\mathcal{I}_2)$. This, together with the above containments, establishes (vii).

Part (iv) of the above proposition raises some questions: What is $\operatorname{id}\operatorname{loc}\mathcal{I}$ when it is not equal to $\mathcal{I}$? When are they equal? In other words, when can we recover $\mathcal{I}$ as the ideal of all germs which vanish on some germ of a subvariety? The answers are very simple:

$$\operatorname{id}\operatorname{loc}\mathcal{I} = \sqrt{\mathcal{I}} = \{\mathbf{f} \in {}_n\mathcal{H}_0 : \mathbf{f}^k \in \mathcal{I} \text{ for some } k\},$$

and so $\operatorname{id}\operatorname{loc}\mathcal{I} = \mathcal{I}$ if and only if $\mathcal{I}$ is its own radical $\sqrt{\mathcal{I}}$. Although it is easy to state, this is a deep result. The result is called the *Nullstellensatz* and its proof will occupy most of the next chapter.

3.6 Irreducible Varieties

In this section, all germs are germs at the origin.

A subvariety V of an open set U is said to be *reducible* if $V = V_1 \cup V_2$, where V_1 and V_2 are both subvarieties contained properly in V. If V is not reducible, then it is called irreducible. Similarly, a germ $\mathbf{V}$ of a variety is called *reducible* if $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are both germs of varieties contained in but not equal to $\mathbf{V}$. Again, if $\mathbf{V}$ is not reducible, then it is called irreducible.

An elementary result which follows easily from Proposition 3.5.5 is:

3.6.1 Proposition. *A germ of a variety* $\mathbf{V}$ *is irreducible if and only if* $\operatorname{id}\mathbf{V}$ *is a prime ideal.*

Proof. If $\mathbf{V}$ is reducible, then it is $\mathbf{V}_1 \cup \mathbf{V}_2$ with $\mathbf{V}$ not contained in $\mathbf{V}_1$ or in $\mathbf{V}_2$. Then $\operatorname{id}\mathbf{V}_1 \cdot \operatorname{id}\mathbf{V}_2 \subset \operatorname{id}\mathbf{V}$, by (v) and, for each i, $\operatorname{id}\mathbf{V}_i$ is not contained in $\operatorname{id}\mathbf{V}$, by (ii) and (iii). Thus, $\operatorname{id}\mathbf{V}$ is not a prime ideal in this case.

On the other hand, if $\operatorname{id}\mathbf{V}$ is not prime, then it contains $\mathcal{I}_1 \cdot \mathcal{I}_2$ for ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, neither of which is contained in $\operatorname{id}\mathbf{V}$. Then $\mathbf{V} \subset \operatorname{loc}\mathcal{I}_1 \cup \operatorname{loc}\mathcal{I}_2$ with $\mathbf{V}$ not contained in $\operatorname{loc}\mathcal{I}_1$ or in $\operatorname{loc}\mathcal{I}_2$ by (i), (iv) and (vii). If we set $\mathbf{V}_i = \mathbf{V} \cap \operatorname{loc}\mathcal{I}_i$ for $i = 1, 2$, then $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$. Consequently, $\mathbf{V}$ is reducible.

The next proposition shows that each germ of a variety can be uniquely decomposed into irreducible varieties.

3.6.2 Proposition. *A germ of a variety can be uniquely (up to order) written as* $\mathbf{V}_1 \cup \cdots \cup \mathbf{V}_k$*, where each* $\mathbf{V}_i$ *is an irreducible germ of a variety, each* $\mathbf{V}_i$ *is a proper subgerm of* $\mathbf{V}$*, and* $\mathbf{V}_i$ *is not contained in* $\mathbf{V}_j$ *if* $i \neq j$.

Proof. Suppose $\mathbf{V}$ is a germ of a variety which cannot be written as a finite union of irreducible germs of varieties. Then $\mathbf{V}$ itself is not irreducible and so it can be written as $\mathbf{V}_1 \cup \mathbf{V}_2$ for two germs of varieties which are properly contained in $\mathbf{V}$. Then at least one of $\mathbf{V}_1$ and $\mathbf{V}_2$ also fails to be a finite union of irreducibles. We assume, without loss of generality, that this is $\mathbf{V}_1$, and then write $\mathbf{V}_1$ as the union of two proper subgerms. Continuing in this way will produce an infinite decreasing sequence of germs of varieties, each properly containing the next. By Proposition 3.5.5(ii) and (iii), this means that the corresponding sequence of ideals of these germs will form an infinite ascending chain of ideals of our Noetherian local ring (${}_n\mathcal{H}$ or ${}_n\mathcal{O}$), which is impossible. Thus we have proved by contradiction that every germ of a variety is a finite union of irreducibles. By deleting redundant germs of varieties, we get the other condition satisfied.

If

$$\mathbf{V} = \mathbf{V}_1 \cup \cdots \cup \mathbf{V}_k = \mathbf{V}'_1 \cup \cdots \cup \mathbf{V}'_m$$

are two non-redundant ways of writing $\mathbf{V}$ as a union of irreducibles, then, for each i, $\mathbf{V}_i = (\mathbf{V}_i \cap \mathbf{V}'_1) \cup \cdots \cup (\mathbf{V}_i \cap \mathbf{V}'_m)$, which implies that $\mathbf{V}_i$ is contained in one of the $\mathbf{V}'_j$. Likewise, each $\mathbf{V}'_j$ is contained in one of the $\mathbf{V}_i$'s. Since the decompositions are non-redundant we conclude that each $\mathbf{V}_i$ is equal to some $\mathbf{V}'_j$ and viceversa. In other words, the decomposition is unique up to order. This completes the proof.

Of particular interest are non-singular subvarieties. We shall describe these in the holomorphic case first, where we have the implicit and inverse mapping theorems available as tools. The algebraic case requires a different approach and will be discussed later.

3.7 Implicit and Inverse Mapping Theorems

If U and U' are open subsets of $\mathbb{C}^n$, then a *biholomorphic mapping* from U to U' is a holomorphic map with a holomorphic inverse. A *holomorphic submanifold* of an open set U is a closed subset V such that for each point $\lambda \in V$ there is a neighborhood U_λ and a biholomorphic map F of U_λ onto an open polydisc $\Delta(0,r)$ in $\mathbb{C}^n$ such that $F(\lambda) = 0$ and

$$F(U_\lambda \cap V) = \{z \in \Delta(0,r) : z_{k+1}, \ldots, z_n = 0\},$$

for some integer k. The integer k is called the dimension of the submanifold at λ.

Thus, a submanifold is a subvariety that locally, up to biholomorphic equivalence, looks like a complex linear subspace. A germ of a holomorphic subvariety is called *non-singular* or *regular* if it is the germ of a holomorphic submanifold; otherwise, it is called *singular*. If V is a holomorphic subvariety of an open set in $\mathbb{C}^n$ and $z \in V$, then z is called a singular point of V if the germ of V at z is singular and a *non-singular* or *regular* point of V if the germ of V at z is non-singular. In order to characterize non-singular points of varieties, we need the complex version of the familiar implicit mapping theorem from advanced calculus. We first prove a special case, the implicit function theorem:

3.7.1 Implicit Function Theorem. *Suppose U is an open subset of $\mathbb{C}^n$, $f \in \mathcal{H}(U)$, and $\lambda \in U$ with $f(\lambda) = 0$. Suppose also that $\partial f/\partial z_n(\lambda) \neq 0$. Then there is a polydisc*

$$\Delta(\lambda, r) = \Delta(\lambda', r') \times \Delta(\lambda_n, r_n) \subset \mathbb{C}^{n-1} \times \mathbb{C}$$

and a holomorphic function $g : \Delta(\lambda', r') \to \Delta(\lambda_n, r_n)$ such that $f(z) = 0$ for $z = (z', z_n) \in \Delta(\lambda, r)$ if and only if $g(z') = z_n$.

Proof. The hypotheses that $f(\lambda) = 0$ and $\partial f/\partial z_n(\lambda) \neq 0$ simply mean that f has vanishing order 1 in z_n at λ. Thus, we may apply the Weierstrass preparation theorem to conclude that there is a polydisc $\Delta(\lambda, r)$, a non-vanishing holomorphic function u on $\Delta(\lambda, r)$, and a holomorphic function g on $\Delta(\lambda', r')$ such that $g(\lambda') = 0$ and $f(z', z_n) = u(z', z_n)(z_n - g(z'))$ on $\Delta(\lambda, r)$. Since $f(z', z_n)$ and $z_n - g(z')$ have the same zero set on $\Delta(\lambda, r)$, this proves the theorem.

If U is a domain in $\mathbb{C}^n$ and $F : U \to \mathbb{C}^m$ is a holomorphic map with coordinate functions $f_1, \dots, f_m$, then the *Jacobian* of F is the $m \times n$ matrix $J_F(z) = \left(\dfrac{\partial f_i}{\partial z_j}(z)\right)$.

3.7.2 Implicit Mapping Theorem. *Let $F : U \to \mathbb{C}^m$ be a holomorphic mapping, as above, and suppose $\lambda \in U$ with $F(\lambda) = 0$. Suppose also that the last m columns of $J_F(\lambda)$ form a non-singular $m \times m$ matrix. Then there is a polydisc*

$$\Delta(\lambda, r) = \Delta(\lambda', r') \times \Delta(\lambda'', r'') \subset \mathbb{C}^{n-m} \times \mathbb{C}^m$$

and a holomorphic map $G : \Delta(\lambda', r') \to \Delta(\lambda'', r'')$ such that $F(z) = 0$ for $z = (z', z'') \in \Delta(\lambda, r)$ if and only if $G(z') = z''$.

Proof. The case $m = 1$ of this theorem is the implicit function theorem. We prove the general case by induction on m. Thus, we assume that the result is true for $m-1$ and proceed to prove it for m. Let $J_F(\lambda) = (J'_F(\lambda), J''_F(\lambda))$ be the separation of $J_F(\lambda)$ into its first $n-m$ columns and its last m columns. The hypothesis is that $J''_F(\lambda)$ is non-singular. By a linear change of variables in the range space $\mathbb{C}^m$, we may assume that $J''_F(\lambda)$ is the $m \times m$ identity matrix. Then, since $\partial f_m/\partial z_n = 1$ at λ, it follows from the implicit function theorem that there is a polydisc $\Delta(\lambda, r)$ and a holomorphic function

$$h : \Delta((\lambda_1, \ldots, \lambda_{n-1}), (r_1, \ldots, r_{n-1})) \to \Delta(\lambda_n, r_n)$$

such that $f_m(z) = 0$ for $z \in \Delta(\lambda, r)$ exactly when $z_n = h(z_1, \ldots, z_{n-1})$. We define a holomorphic mapping

$$F' : \Delta((\lambda_1, \ldots, \lambda_{n-1}), (r_1, \ldots, r_{n-1})) \to \mathbb{C}^{m-1}$$

by defining its coordinate functions $f'_1, \ldots, f'_{m-1}$ to be

$$f'_i(z_1, \ldots, z_{n-1}) = f_i(z_1, \ldots, z_{n-1}, h(z_1, \ldots, z_{n-1})).$$

Then $F'(\lambda_1, \ldots, \lambda_{n-1}) = 0$, and the Jacobian $J_{F'}(\lambda)$ has the property that its last $m-1$ columns form an $(m-1) \times (m-1)$ identity matrix. It follows from the induction hypothesis that, after possibly shrinking the polydisc, there is a holomorphic mapping

$$G' : \Delta(\lambda', r') \to \Delta((\lambda_{n-m+1}, \ldots, \lambda_{n-1}), (r_{n-m+1}, \ldots, r_{n-1}))$$

so that, for $(z_1, \ldots, z_{n-1}) \in \Delta((\lambda_1, \ldots, \lambda_{n-1}), (r_1, \ldots, r_{n-1}))$, the equation $F'(z_1, \ldots, z_{n-1}) = 0$ holds precisely when $G'(z') = (z_{n-m+1}, \ldots, z_{n-1})$. Since $F(z) = 0$ for $z \in \Delta(\lambda, r)$ precisely when both $z_n = h(z_1, \ldots, z_{n-1})$ and $F'(z_1, \ldots, z_{n-1}) = 0$, the mapping

$$G(z') = (G'(z'), h(z', G'(z')))$$

has the required properties. This completes the proof.

3.7.3 Inverse Mapping Theorem. *If H is a holomorphic mapping from a neighborhood U of $\lambda \in \mathbb{C}^n$ into $\mathbb{C}^n$, and if $J_H(\lambda)$ is non-singular, then, on some possibly smaller neighborhood U' of λ, H is a biholomorphic mapping onto some neighborhood of $H(\lambda)$.*

Proof. This follows immediately from the implicit mapping theorem applied to the mapping $F : \mathbb{C}^n \times U \to \mathbb{C}^n$ defined by $F(z', z'') = H(z'') - z'$. The mapping G of that theorem is an inverse for H on an appropriate polydisc.

3.7.4 Theorem. *If F is a holomorphic mapping from a domain U in $\mathbb{C}^n$ into $\mathbb{C}^m$ and if J_F has constant rank k in U, then, for each point $\lambda \in U$, there is a neighborhood U_λ of λ in which F is biholomorphically equivalent to the mapping $(z_1, \ldots, z_n) \to (z_1, \ldots, z_k, 0, \ldots, 0)$ from a neighborhood of 0 in $\mathbb{C}^n$ to a neighborhood of 0 in $\mathbb{C}^m$. Thus, $\{z \in U : F(z) = 0\}$ is a submanifold of dimension $n - k$ in U_λ and, for each $\lambda \in U$, $F(U_\lambda)$ is a submanifold of dimension k in a neighborhood of $F(\lambda)$.*

Proof. We may assume that λ and $F(\lambda)$ are both the origin. After a linear change of coordinates, we may assume that the upper left $k \times k$ submatrix of $J_F(z)$ is nonsingular in a neighborhood U' of 0. We then define a new mapping G from U' into $\mathbb{C}^n$ by

$$G(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_k(z_1, \ldots, z_n), z_{k+1}, \ldots, z_n).$$

Then J_G is non-singular in U', and G is a biholomorphic mapping of one neighborhood U'' of $0 \in \mathbb{C}^n$ to another. Also, $F \circ G^{-1}$ has the form $(z_1, \ldots, z_k, f'_{k+1}, \ldots, f'_m)$. Since the Jacobian of $F \circ G^{-1}$ also has rank k throughout U'', it follows that the functions $\partial f'_j / \partial z_i$ vanish identically for $i > k$ and, thus, that the functions f'_j are functions of $z_1, \ldots, z_k$ alone. If we set

$$H(z_1, \ldots, z_m) = (z_1, \ldots, z_k, z_{k+1} - f'_{k+1}(z_1, \ldots, z_k), \ldots, z_m - f'_m(z_1, \ldots, z_k)),$$

then H is a biholomorphic map on some neighborhood of $0 \in \mathbb{C}^m$, and

$$H \circ F \circ G^{-1}(z_1, \ldots, z_n) = (z_1, \ldots, z_k, 0, \ldots, 0)$$

on a neighborhood U of $0 \in \mathbb{C}^n$, as required. This completes the proof.

3.7.5 Corollary. *If $F : U \to C^m$ is a holomorphic map defining a subvariety $V = \{z \in U : F(z) = 0\}$, then V is non-singular at each point $z \in V$ such that* rank J_F *is constant in a neighborhood of z in U.*

Note that, for any holomorphic mapping F on a domain U, the set on which J_F has rank less than or equal to k is a subvariety of U, since it is defined by the condition that the determinants of certain submatrices of J_F vanish. Let

$$V = \{z \in U : F(z) = 0\} \text{ and } W_k = \{z \in U : \text{rank} J_F(z) \leq k\}.$$

If j is the largest integer k for which W_k is a proper subvariety of U, then $U - W_j$ is an open dense set on which rank J_F is equal to $j + 1$. The set $V_0 = V \cap (U - W_j)$ consists of non-singular points of V and is a submanifold of dimension $n-j$ of $U - W_j$. It would be nice to know that V_0 is an open dense subset of V, since this would show that most points of a variety are non-singular. However, it is possible for the varieties V and W_j of the above discussion to coincide (see the paragraph following the examples below), in which case $V_0 = \emptyset$ and the Jacobian of F tells us nothing about the regular points of V. In this case, we have made a bad choice of a mapping F to define our subvariety V. We need to be able to describe a variety as the set of common zeroes of a set of functions which are chosen in a way that gives us much more detailed information about the local structure of our subvariety at a point. We obtain such an *optimal choice of functions defining a subvariety* in the next chapter and use it to prove the Nullstellensatz and several other important facts concerning varieties, including the one about regular points alluded to above.

3.7.6 Example. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by $F(z,w) = z^2 - w$, and let $V = \{(z,w) \in \mathbb{C}^2 : F(z,w) = 0\}$. Here, $J_F(z,w) = (2z, -1)$, and so J_F has rank 1 everywhere. It follows from Theorem 3.7.4 that V is non-singular everywhere.

3.7.7 Example. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by $F(z,w) = z^2 - w^2$ and let $V = \{(z,w) \in \mathbb{C}^2 : F(z,w) = 0\}$. Here, $J_F(z,w) = (2z, -2w)$ and so J_F has rank 1 everywhere except at the origin where it has rank 0. Theorem 3.7.4 implies V is non-singular everywhere except possibly the origin. In fact, $F(z,w) = (z-w)(z+w)$ and so V is just the union of the two planes defined by $z - w = 0$ and $z + w = 0$. Thus, V is reducible. Clearly, V is singular at the origin, where the two planes cross.

3.7.8 Example. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by $F(z,w) = z^2 - w^3$ and let $V = \{(z,w) \in \mathbb{C}^2 : F(z,w) = 0\}$. Here, $J_F(z,w) = (2z, -3w^2)$, which has rank 1 everywhere except at the origin where it has rank 0. Once again, V is non-singular everywhere except at the origin where it is singular (Exercise 3.12). In this case, however, V is irreducible (Exercise 3.9).

Each of the preceding examples is an algebraic as well as a holomorphic subvariety of $\mathbb{C}^2$. It is easy to construct holomorphic subvarieties of $\mathbb{C}^n$ which are not algebraic subvarieties.

3.7.9 Example. Let $F : \mathbb{C}^2 \to C$ be defined by $F(z,w) = z - e^w$ and let $V = \{(z,w) \in C^2 : F(z,w) = 0\}$. No non-trivial polynomial in z, w vanishes identically on V (Exercise 3.13) and so V is not an algebraic variety. As a holomorphic variety, it is non-singular everywhere, since its Jacobian is $(1, e^w)$.

Note that, in each of these examples, if F is replaced by F^2, then the Jacobian becomes $J_{F^2} = 2FJ_F$, which has rank 0 on all of V. Thus, although

F^2, as well as F, is a function which defines V, we can tell nothing about the set of non-singular points of V by looking at the Jacobian of F^2. Thus, F^2 is a poor choice for a defining function for the variety V.

3.8 Holomorphic Functions on a Subvariety

3.8.1 Definition. If V is a holomorphic (algebraic) subvariety of an open set in $\mathbb{C}^n$, then a holomorphic (regular) function f on V is a complex valued function on V with the property that, for each point $\lambda \in V$, there is a neighborhood U_λ of λ in $\mathbb{C}^n$ such that f extends to be holomorphic (regular) in U_λ. The algebra of functions holomorphic (regular) on V will be denoted $\mathcal{H}(V)$ ($\mathcal{O}(V)$) while the algebra of germs at $\lambda \in V$ of functions holomorphic (regular) on neighborhoods in V of λ will be denoted ${}_V\mathcal{H}_\lambda$ (${}_V\mathcal{O}_\lambda$). The latter is called the *local ring* of V at λ.

The following is immediate from the definition:

3.8.2 Proposition. *If V is a holomorphic (algebraic) subvariety of an open set in $\mathbb{C}^n$ and $\mathbf{V}_\lambda$ is its germ at λ, then ${}_V\mathcal{H}_\lambda = {}_n\mathcal{H}_\lambda / \operatorname{id} \mathbf{V}_\lambda$ and this ring is a Noetherian local ring. The analogous statement holds for ${}_V\mathcal{O}_\lambda$.*

We can also define holomorphic mappings between subvarieties:

3.8.3 Definition. If V and W are holomorphic (algebraic) subvarieties of open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$ and $F : V \to W$ is a mapping, then F is called holomorphic (regular) if each of its coordinate functions is a holomorphic (regular) complex valued function on V. A holomorphic (regular) function with a holomorphic (regular) inverse is called biholomorphic (biregular).

Again, it is immediate from the definition that:

3.8.4 Proposition. *If V and W are holomorphic subvarieties of open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$, $F : V \to W$ is a mapping, and $\lambda \in V$, then F is a holomorphic mapping on some neighborhood of λ if and only if $F^*(g) = g \circ F$ defines an algebra homomorphism F^* from ${}_W\mathcal{H}_{F(\lambda)}$ to ${}_V\mathcal{H}_\lambda$. The analogous result holds for a mapping between algebraic subvarieties.*

A somewhat deeper result is the following:

3.8.5 Theorem. *If V and W are holomorphic subvarieties of open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$, $\lambda \in V$, $\mu \in W$, and $\phi : {}_W\mathcal{H}_\mu \to {}_V\mathcal{H}_\lambda$ is any algebra homomorphism, then ϕ is induced, as in Proposition 3.8.4, by a holomorphic mapping from a neighborhood of λ in V to a neighborhood of μ in W. The analogous result also holds for algebraic subvarieties.*

Proof. The algebra homomorphism ϕ maps units to units. However, each element $\mathbf{f} \in {}_W\mathcal{H}_\mu$ has the property that $\mathbf{f}(\mu)$ is the unique complex number c such that $\mathbf{f} - \mathrm{c}$ is a non-unit. The analogous statement is true of ${}_V\mathcal{H}_\lambda$. It follows that ϕ also maps non-units to non-units. That is, $\mathbf{f}$ is in the maximal ideal of ${}_W\mathcal{H}_\mu$ if and only if $\phi(\mathbf{f})$ is in the maximal ideal of ${}_V\mathcal{H}_\lambda$. If $\mathbf{w}_1, \dots, \mathbf{w}_m$ are the germs in ${}_W\mathcal{H}_\mu$ of the restrictions $w_1, \dots, w_m$ of the coordinate functions in $\mathbb{C}^m$ to W, then $\phi(\mathbf{w}_1), \dots, \phi(\mathbf{w}_m)$ are germs in ${}_V\mathcal{H}_\lambda$ and thus are represented by functions which extend to be holomorphic in some neighborhood of λ in $\mathbb{C}^n$. Let $f_1, \dots, f_m$ be such holomorphic extensions. Then they are the coordinate functions of a holomorphic map F from a neighborhood of λ in $\mathbb{C}^n$ into $\mathbb{C}^m$. Since, for each j, $\phi(\mathbf{w}_j - \mathbf{w}_j(\mu))$ is the germ of $f_j|_V - w_j(\mu)$ at λ, which belongs to the maximal ideal of ${}_V\mathcal{H}_\lambda$, it follows that $f_j(\lambda) = w_j(\mu)$ and, hence, that $F(\lambda) = \mu$. Thus, the mapping F induces an algebra homomorphism $F^* : {}_m\mathcal{H}_\mu \to {}_n\mathcal{H}_\lambda$. If we follow this by restriction to V, then we have a homomorphism $\tilde{F}^* : {}_m\mathcal{H}_\mu \to {}_V\mathcal{H}_\lambda$. On the other hand, if we precede ϕ by restriction from $\mathbb{C}^m$ to W, then it also determines a homomorphism $\tilde{\phi} : {}_m\mathcal{H}_\mu \to {}_V\mathcal{H}_\lambda$. The proof will be complete if we can show that $\tilde{\phi} = \tilde{F}^*$ and that F maps some neighborhood of λ in V into W.

By the construction of F, $\tilde{F}^*(\mathbf{w}_j) = \tilde{\phi}(\mathbf{w}_j)$ for each j. This implies that the two homomorphisms agree on polynomials. Since, for each k, every element of ${}_m\mathcal{H}_\mu$ is a polynomial of degree k plus an element of the kth power of the maximal ideal of ${}_m\mathcal{H}_\mu$, and since both $\tilde{F}^*$ and $\tilde{\phi}$ map the kth power of the maximal ideal of ${}_m\mathcal{H}_\mu$ to the kth power of the maximal ideal of ${}_V\mathcal{H}_\lambda$, we conclude that $\tilde{F}^*(f) - \tilde{\phi}(f)$ belongs to the kth power of the maximal ideal of ${}_V\mathcal{H}_\lambda$, for every $f \in {}_m\mathcal{H}_\mu$ and every positive integer k. However, by Nakayama's lemma (Exercise 3.5), the intersection of all powers of the maximal ideal in a Noetherian local ring is 0. Therefore, $\tilde{F}^* = \tilde{\phi}$.

It remains to prove that F maps a neighborhood of λ in V into W. However, $\operatorname{id} W$ is in the kernel of $\tilde{\phi} = \tilde{F}^*$, by the definition of $\tilde{\phi}$. Hence, for every $f \in \operatorname{id} W$ we have $(f \circ F)|_V = \tilde{F}^*(f) = 0$, i.e. f vanishes on $F(V)$. If we apply this fact to a finite set of generators of $\operatorname{id} W$, we conclude that a suitably small neighborhood of λ in V is mapped by F into W.

Two germs of varieties $\mathbf{V}$ and $\mathbf{W}$ are said to be *biholomorphically equivalent* if there is a biholomorphic map $F : V \to W$ between suitable representative varieties V and W. We have the following two corollaries of Theorem 3.8.5, the first of which is an immediate consequence of the theorem.

3.8.6 Corollary. *Two germs of holomorphic varieties are biholomorphically equivalent if and only if their local rings are isomorphic as algebras.*

3.8.7 Corollary. *A holomorphic variety V is non-singular at $\lambda \in V$ if and only if its local ring ${}_V\mathcal{H}_\lambda$ is isomorphic to ${}_k\mathcal{H}_0$ for some k.*

Proof. A holomorphic subvariety V of an open set in $\mathbb{C}^n$ is non-singular at $\lambda \in V$ if there is a biholomorphic map F from a neighborhood U of λ in $\mathbb{C}^n$ to a neighborhood W of 0 in $\mathbb{C}^n$ such that F maps $V \cap U$ onto

$$\{z \in W : z_{k+1} = \cdots = z_n = 0\}.$$

If $G : V \to \mathbb{C}^k$ is the map whose coordinate functions are the first k coordinate functions of F, then G is a biholomorphic map between $V \cap U$ and $\mathbb{C}^k \cap W$. Furthermore, every germ of a biholomorphic map G between a neighborhood of λ in V and a neighborhood of 0 in $\mathbb{C}^k$ arises in this way, as the restriction to $V \cap U$ of a biholomorphic map $F : U \to W$ between neighborhoods in $\mathbb{C}^n$. In fact, given G, its coordinate functions g_i extend to yield holomorphic functions f_i, $i = 1, \dots, k$, in some neighborhood U of λ in $\mathbb{C}^n$. If we set $f_i(z_1, \dots, z_n) = z_i$ for $i = k+1, \dots, z_n$, then the f_i are the coordinate functions of a holomorphic map $F : U \to \mathbb{C}^n$. This map, after shrinking U if necessary, has a holomorphic inverse defined, on some neighborhood of 0, to be the inverse of G in the first k coordinates and the identity in the remaining coordinates. Thus, $\lambda \in V$ is a non-singular point if and only if the germ of V at λ is biholomorphically equivalent to the germ of $\mathbb{C}^k$ at 0. The corollary then follows immediately from the preceding corollary.

Exercises

1. A unique factorization domain is an integral domain in which each element has a unique (up to units) factorization as a finite product of irreducible factors. Prove that if A is a unique factorization domain, then so is $A[x]$.
2. Prove that $\mathbb{C}[z_1, ..., z_n]$ and ${}_n\mathcal{O}_0$ are unique factorization domains.
3. Prove that a Weierstrass polynomial $\mathbf{p} \in {}_{n-1}\mathcal{H}_0[z_n]$ is irreducible in ${}_{n-1}\mathcal{H}_0[z_n]$ if and only if it is irreducible in ${}_n\mathcal{H}_0$.
4. Prove that ${}_n\mathcal{H}_0$ is a unique factorization domain. Hint: Use the results of Exercises 3.1 and 3.3.
5. Prove Nakayama's lemma: If M is a finitely generated module over a local ring A with maximal ideal $\mathfrak{m}$ and if $\mathfrak{m}M = M$, then $M = 0$. Hint: Prove that if M has a generating set with k elements, with $k > 0$, then it also has a generating set with $k - 1$ elements.

6. Prove parts (iv), (vi), and (viii) of Proposition 3.5.5.
7. Give an example which shows that the implicit function theorem, the inverse mapping theorem, and the Weierstrass preparation theorem fail in the algebraic case.
8. Prove that if f is holomorphic in a neighborhood of 0 in $\mathbb{C}^n$ and there is an i such that $\dfrac{\partial f}{\partial z_i}$ does not vanish at 0, then the germ of f at 0 is an irreducible element of ${}_n\mathcal{H}_0$.
9. Consider the polynomial on $\mathbb{C}^2$ defined by $p(z,w) = z^2 - w^3$. Prove that the germ $\mathbf{p}$ of p is irreducible in both ${}_2\mathcal{H}_0$ and ${}_2\mathcal{O}_0$. Show that this implies that $\mathbf{p}$ generates a prime ideal in each algebra.
10. Show that, in either ${}_2\mathcal{H}_0$ or ${}_2\mathcal{O}_0$, the ideal generated by the polynomial $\mathbf{p}$ of Exercise 3.9 is $\operatorname{id}\mathbf{V}$, where $\mathbf{V}$ is the germ at 0 of the subvariety $V = \{(z,w) \in \mathbb{C}^2 : p(z,w) = 0\}$.
11. With V as in Exercise 3.10, show that there are irreducible elements $\mathbf{f}, \mathbf{g} \in {}_V\mathcal{O}_0$ such that $\mathbf{f}^2 = \mathbf{g}^3$. Conclude that ${}_V\mathcal{O}$ and ${}_V\mathcal{H}$ are not unique factorization domains.
12. Show that the holomorphic variety V of Exercise 3.10 is singular at the origin.
13. Verify that no non-trivial polynomial in (z,w) vanishes identically on the holomorphic variety V of Example 3.7.9.
14. Use Proposition 3.3.1 to prove that if U is a connected open subset of $\mathbb{C}^n$ and V is a subvariety of U, then $U - V$ is dense in U.
15. For the subvarieties V_1 and V_2 of Example 3.5.2, show that if Δ is a polydisc centered at a point of V_2 and f is a holomorphic function in $\mathcal{H}(\Delta)$ that vanishes on $V_1 \cap \Delta$, then f is identically 0. Hint: Use the Picard theorem.
16. Give an example of an algebraic subvariety V of $\mathbb{C}^2$, containing the origin, such that ${}_V\mathcal{O}_0$ is not isomorphic to ${}_1\mathcal{O}_0$ but, when V is considered as a holomorphic subvariety, ${}_V\mathcal{H}_0 \simeq {}_1\mathcal{H}_0$ (so that the origin is a non-singular point).
17. Show that the polynomial $p(z,w) = w^2 - z^2(1-z)$ is irreducible in ${}_2\mathcal{O}_0$ but reducible in ${}_2\mathcal{H}_0$.
18. A continuous map $f : X \to Y$ between topological spaces is said to be *proper* if the inverse image under f of each compact set in Y is compact in X. Prove that a continuous map from a bounded open set $U \subset \mathbb{R}^n$ to a bounded open set $V \subset \mathbb{R}^m$ is proper if and only if each sequence in U which converges to a boundary point of U has image in V which converges to a boundary point of V.

19. Use the result of the preceding exercise to prove that if $U \subset \mathbb{C}^p$, $V \subset \mathbb{C}^q$, and $W \subset \mathbb{C}^n$ are bounded open sets, and there exists a proper holomorphic map $f : U \times V \to W$, then there exists a non-constant holomorphic map $f : V \to \mathbb{C}^n$ with image contained in the boundary of W.
20. Prove that there is no non-constant holomorphic map of a connected open set $U \subset \mathbb{C}$ into $\mathbb{C}^n$ which has its image in the unit sphere

$$\{z = (z_1, \dots, z_n) \in \mathbb{C}^n : ||z||^2 = |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

Hint: Let $f : U \to \mathbb{C}^n$ be a holomorphic function which satisfies the equation $||f(z)||^2 = 1$ on U. Apply $\dfrac{\partial^2}{\partial z \partial \bar{z}}$ to this equation and conclude that $\dfrac{\partial f}{\partial z} = 0$.
21. Use the results of the preceding two exercises to prove that, if $n \geq 2$, there is no proper holomorphic map from a bounded open polydisc in $\mathbb{C}^n$ onto the open unit ball in $\mathbb{C}^n$. In particular, there is no biholomorphic map from a bounded polydisc to the ball.

Chapter 4

The Nullstellensatz

The Nullstellensatz states the exact connection between germs of holomorphic varieties and ideals in the ring of germs of holomorphic functions: That $\operatorname{id}\operatorname{loc} I$ is the radical of the ideal I. The same statement makes sense and is true for the polynomial ring and the ring of germs of regular functions. In fact, for these rings the proof of the Nullstellensatz is relatively simple. We will prove it for the polynomial ring in section 4.3 and then proceed to the much more difficult case of the ring of germs of holomorphic functions.

The proof of the Nullstellensatz for ${}_n\mathcal{H}_0$ depends on a fairly precise description of the locus of a prime ideal as the germ of a finite branched holomorphic cover of a neighborhood in $\mathbb{C}^m$ for some $m < n$. This description is developed in sections 4.4 and 4.5 using properties of algebraic ring and field extensions. In section 4.2, we give a summary of results concerning such extensions that will be needed in this development.

4.1 Reduction to the Case of Prime Ideals

In rings A for which it is true, the Nullstellensatz says that

$$\operatorname{id}\operatorname{loc} I = \sqrt{I} = \{x \in A : x^n \in I \text{ for some } n\},$$

for each ideal I of A. This is true for the rings ${}_n\mathcal{H}_0$ and ${}_n\mathcal{O}_0$ as well as for the ring of polynomials $\mathbb{C}[z_1, \dots, z_n]$. Here, id and loc, defined for the local rings $\mathcal{H}_0$ and $\mathcal{O}_0$ in Definition 3.5.3, have the obvious meanings for globally defined rings of functions such as $\mathbb{C}[z_1, \dots, z_n]$, and the analogue of Proposition 3.5.5 holds. In each case, it is trivial from the definition that $\sqrt{I} \subset \operatorname{id}\operatorname{loc} I$. Also, in each case, the theorem can easily be reduced to the case of prime ideals. In a Noetherian ring, each ideal I has a primary decomposition $I = \bigcap_{j=1}^m I_j$ and primary ideals have radicals which are prime (see [**AM**]). Thus, if we assume the Nullstellensatz for prime ideals, then we have

$$\begin{aligned}\operatorname{id}\operatorname{loc} I &= \operatorname{id}\left(\bigcup_{j=1}^m \operatorname{loc} I_j\right) = \bigcap_{j=1}^m \operatorname{id}\operatorname{loc} I_j \subset \bigcap_{j=1}^m \operatorname{id}\operatorname{loc}\sqrt{I_j}\\ &= \bigcap_{j=1}^m \sqrt{I_j} = \sqrt{I}.\end{aligned}$$

This gives the Nullstellensatz for general ideals. Thus, we have proved:

4.1.1 Proposition. *If A is ${}_n\mathcal{H}_0$, ${}_n\mathcal{O}_0$ or $\mathbb{C}[z_1,\dots,z_n]$, and* $\operatorname{id}\operatorname{loc} I = \sqrt{I}$ *is true whenever I is a prime ideal of A, then it is true for all ideals of A.*

4.2 Survey of Results on Ring and Field Extensions

In proving the Nullstellensatz for both $\mathbb{C}[z_1,\dots,z_n]$ and for ${}_n\mathcal{H}_0$, we will make heavy use of the theory of finite ring and field extensions. Since this discussion will not make much sense to someone who doesn't know the requisite algebra, we present below a list of facts from this subject that we will use implicitly or explicitly. This is presented as a study guide for those who need to brush up on the subject. Most of these results are quite standard and proofs can be found in the standard texts on rings and fields (e.g.: [**Hun**], [**E**], [**AM**], [**Mat**]). We will include proofs of some of the more specialized results which occur in the second half of the section.

If $K \subset F$ are fields and $x \in F$, then x is called *algebraic* over K if it is a root of a polynomial with coefficients in K. The field F is called an *algebraic field extension* of K if each of its elements is algebraic over K.

4.2.1 Theorem of the Primitive Element. *If K is an infinite field and F is a finitely generated algebraic field extension of K, then F is generated by a single element which may be chosen to be a linear combination of any given set of generators.*

The subfield of F generated by $x_1,\dots,x_n$ over K will be denoted by $K(x_1,\dots,x_n)$.

4.2.2 Theorem. *If $x \in F$ is algebraic over K, then x is a root of a unique irreducible monic polynomial with coefficients in K. If the degree of that polynomial is n, then $\{x^{n-1},\dots,x,1\}$ is a basis for $K(x)$ as a vector space over K.*

The unique irreducible monic polynomial having x as a root is called the *minimal polynomial* of the element x.

If p is a polynomial with coefficients in K, then a *splitting field* F for p is an extension field of K which is generated over K by the roots of p, and, in which, p factors as a product of linear factors (so that all possible roots of p are included in F).

4.2.3 Theorem. *If p is a polynomial with coefficients in K, then there is a splitting field for p, and it is unique up to isomorphism. If p is the minimal polynomial of an algebraic element x over K, then a splitting field for p may be chosen which is a field extension of $K(x)$.*

The *Galois group* of a field extension $K \subset F$ is the group of automorphisms of F which leave all elements of K fixed. The extension is called a Galois extension if K is exactly the set of elements fixed by the Galois group.

4.2.4 Theorem. *If F is a splitting field for some polynomial p with coefficients in K, and if K has characteristic 0, then F is a Galois extension of K. Every element of the Galois group of such an extension is uniquely determined by a permutation of the roots of p. If p is irreducible, then the Galois group acts transitively on the roots.*

Now let B be a commutative ring and A a commutative B-algebra. This just means that there is an identity preserving ring homomorphism $B \to A$. An element of A is said to be *integral* over B if it is a root of a monic polynomial with coefficients in B. If every element of A is integral over B, then A is said to be integral over B. If, in addition, B is actually a subalgebra of A, then A is called an *integral extension* of B. We have the following equivalence:

4.2.5 Theorem. *The following are equivalent for a commutative ring B and a B-algebra A:*

(i) *A is integral over B and is finitely generated as a B-algebra;*
(ii) *A is generated as a B-algebra by finitely many elements which are integral over A;*
(iii) *A is finitely generated as a B-module.*

If the equivalent conditions of Theorem 4.2.5 are satisfied, then A is said to be *finite* over B and A is called a *finite B-algebra.* If, in addition, B is a subalgebra of A, then A is called a *finite extension* of B.

The following two results are simple consequences of Theorem 4.2.5.

4.2.6 Corollary. *If B is a commutative ring and A is a B-algebra, then the set of elements in A which are integral over B is a subalgebra of A.*

The next result says that integral dependence is transitive.

4.2.7 Corollary. *If $C \subset B \subset A$ are commutative rings, with B integral over C, and A integral over B, then A is integral over C.*

An integral domain A is said to be a *normal domain* if it is *integrally closed* in its quotient field K, that is, if each element of K which is integral over A belongs to A.

4.2.8 Theorem. *A unique factorization domain is a normal domain.*

If $B \subset A$ are rings and B is a normal domain with quotient field K, then the minimal polynomial for an element $a \in A$, integral over B, has an unambiguous meaning. That is, there is a monic polynomial $p \in K[x]$ of minimal degree with the property that $p(a) = 0$. In fact, p actually belongs to $B[x]$ and, hence, is also the monic polynomial of minimal degree in $B[x]$ with a as a root. This is due to the fact that the coefficients of p are elementary symmetric polynomials in the roots of p in some splitting field for p containing $B[a]$. Since the Galois group acts transitively on the roots, each of them is also integral over B and, hence, the coefficients of p are integral over B. They belong to the quotient field K of B and, hence, to B, since B is normal.

If p is a polynomial with coefficients in a field K, then the *discriminant* of p is the product

$$d = \prod_{i \neq j} (x_i - x_j),$$

where $x_1, \dots, x_n$ are the roots of p in some splitting field. A priori, this is just an element of the chosen splitting field. However, it actually belongs to K. In fact, we have the following:

4.2.9 Theorem. *If A is a normal domain with quotient field K, and p is a polynomial with coefficients in A, then the discriminant of p is a well-defined element of A, which is non-zero if and only if p has no multiple roots.*

Proof. The discriminant d is left fixed by each element of the Galois group, since such elements just permute the roots $\{x_i\}$. Thus, $d \in K$. However, each x_i is integral over A and, hence, d is as well, by Corollary 4.2.6. Since A is normal and K is its quotient field, d belongs to A. It is obviously 0 if and only if $x_i = x_j$ for some $i \neq j$, that is, if and only if p has a multiple root.

4.2.10 Theorem. *If p is an irreducible polynomial with coefficients in a field K, and if K has characteristic 0, then p has no multiple roots and, thus, has non-vanishing discriminant.*

Proof. Let p' be the formal derivative of p. Since p is irreducible and p' has degree less than p, the polynomials p and p' are relatively prime in $K[x]$. However, $K[x]$ is a principal ideal domain. Thus, if p and p' generate a proper ideal, then this ideal must be generated by a single polynomial, which necessarily divides both p and p' and, hence, must be a unit. We conclude that p and p' do not generate a proper ideal. This means there are polynomials r and s so that $rp + sp' = 1$. Thus, there is no multiple root of p, since p and p' must both vanish at any multiple root.

4.2.11 Theorem. *If p is a polynomial with coefficients in K and if its roots in some splitting field are $x_1, \ldots, x_n$, then the discriminant d of p is the square of the Vandermonde determinant*

$$\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}.$$

Proof. For a set of indeterminants $\{x_1, \ldots, x_n\}$, let

$$\delta = \prod_{i<j} (x_i - x_j) \quad \text{and} \quad \delta_\sigma = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)}),$$

for each permutation σ of $\{1, 2, \ldots, n\}$. Then we have that $\delta_\sigma = \operatorname{sgn}(\sigma)\delta$ and $\delta^2 = d$. We shall prove by induction on n that the Vandermonde determinant is equal to δ. This identity is clearly true for $n = 1$ (the product of an empty set of factors is understood to be 1). Assuming it is true for $n - 1$ indeterminants, we will prove that it is true for n indeterminants.

Given n indeterminants $x_1, \ldots, x_n$, each monomial in the expansion of the product defining δ has the form

$$\prod_i x_i^{q_i} \text{ with } 0 \le q_i \le n-1,\ i = 1, \ldots, n \text{ and } \sum_i q_i = \frac{n(n-1)}{2}.$$

Collecting terms involving the $n - 1$ power of each x_i yields

$$\delta = x_1^{n-1}a_1 - x_2^{n-1}a_2 + \cdots + (-1)^{n-1}x_n^{n-1}a_n + b,$$

where b is the sum of all monomials in the expansion of δ which do not involve an indeterminant raised to the $n - 1$ power. The powers q_i that occur in each monomial that occurs in b are integers less than $n - 1$, which add up to $n(n - 1)/2$. These cannot all be distinct integers. Hence, each monomial in b must involve two indeterminants, x_i and x_j, raised to the same power. Each such monomial is fixed by the permutation which interchanges i and j. However, this is an odd permutation and, hence, each such monomial must have coefficient 0 after like terms are combined. Thus, $b = 0$. Furthermore, for each k, the coefficient a_k is given by

$$a_k = \prod\{(x_i - x_j) : i < j,\ i, j \neq k\},$$

which by the induction assumption, is the Vandermonde for indeterminants $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. So $\delta = x_1^{n-1}a_1 - x_2^{n-1}a_2 + \cdots + (-1)^n x_n^{n-1}a_n$ is the expansion by minors of the first column of the Vandermonde for $x_1, \ldots, x_n$. This completes the induction.

4.2.12 Theorem. *If A is a normal domain with quotient field K of characteristic 0 and if $F = K(x)$ is the field extension generated by an element x, integral over A, with minimal polynomial p of degree n, then every element of F which is integral over A belongs to the A-submodule of F generated by the elements $\frac{x^{n-1}}{d}, \ldots, \frac{x}{d}, \frac{1}{d}$, where d is the discriminant of p.*

Proof. For an element $f(x) \in K(x)$, integral over A, we wish to find coefficients $a_0, \ldots, a_{n-1}$ such that

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = d \cdot f(x).$$

Let $x_1 = x$ and let $x_2, \ldots, x_n$ be the other roots of p in a splitting field for p. We may then write down a system of n equations, each of which is a copy of the one above, but with x replaced by x_j in the jth equation. If we consider this as a system of equations in which the unknowns are the elements $a_1, \ldots, a_n$, Kramer's rule gives as solution

$$a_j = d \begin{vmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^{n-1} & \cdots & x_2 & 1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{n-1} & \cdots & x_n & 1 \end{vmatrix}^{-1} \begin{vmatrix} x_1^{n-1} & \cdots & f(x_1) & \cdots & x_1 & 1 \\ x_2^{n-1} & \cdots & f(x_2) & \cdots & x_2 & 1 \\ \cdot & \cdots & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot & \cdot \\ x_n^{n-1} & \cdots & f(x_n) & \cdots & x_n & 1 \end{vmatrix},$$

where in the second determinant the $f(x_i)$ replace the jth column of the first determinant. Now, of course, the first determinant is the Vandermonde determinant, which has square equal to d, by Theorem 4.2.11. Thus, a_j is the product of the Vandermonde and the determinant obtained from the Vandermonde by replacing its jth column with the column formed by the $f(x_i)$. Clearly this product is left fixed by any permutation of the roots $x_1, \ldots, x_n$, since this just amounts to applying the same permutation to the rows in both matrices. Thus, the elements a_j, so determined, are fixed by the Galois group of the splitting field of p, and belong to K, by Theorem 4.2.4. However, since x and $f(x)$ are integral over A, so are all the x_i and $f(x_i)$, again by Theorem 4.2.4. It follows that the a_j are also integral over A, since they lie in the ring generated by the x_i and $f(x_i)$. However, A is integrally closed in its quotient field K and, hence, $a_j \in A$, for $j = 1, \ldots, n$. This completes the proof.

We end this section with one more result concerning finite ring extensions. This result will play a crucial role in the next chapter where we develop the properties of dimension for varieties.

Let $B \subset A$ be a finite extension of commutative rings, where A and B are integral domains. Let K and L be the quotient fields of B and A, respectively. Suppose a basis $e_1, e_2, \ldots, e_k$ is chosen for L as a vector space over K. Then each element x of L determines a matrix (α_{ij}), with entries from K, by the equations

$$xe_i = \alpha_{i1}e_1 + \alpha_{i2}e_2 + \cdots + \alpha_{ik}e_k \tag{4.2.1}$$

If z is an indeterminant, then we define a polynomial $p_x(z)$ by

$$p_x(z) = \det(\alpha_{ij} - z\delta_{ij}),$$

where (δ_{ij}) is the $k \times k$ identity matrix. This is the *characteristic polynomial* of x for the extension $B \to A$. It is a polynomial which has x as a root. Note that the polynomial p_x is independent of the choice of basis. The coefficient of degree 0 in this polynomial is $N(x) = \det(\alpha_{ij})$. The element $N(x)$ is called the *norm* of the element x for the extension $A \to B$. The next theorem summarizes the properties of N.

4.2.13 Theorem. *Let $B \subset A$ be a finite extension of commutative rings with A and B integral domains with fields of quotients K and L. Then,*

(i) *$N(xy) = N(x)N(y)$ for each pair $x, y \in L$;*

(ii) *$N(x) = (-1)^k c_0^m$, where c_0 is the coefficient of degree 0 of the minimal polynomial of x over K and m is the dimension of L as a vector space over $K(x)$;*

(iii) *if B is normal, then $N(a) \in B$ for each $a \in A$;*

(iv) *$N(b) = b^k$ for each $b \in B$.*

Proof. The fact that the determinant is multiplicative implies (i). Part (iii) follows from part (ii), since if B is normal, the minimal polynomial for a over K has coefficients in B. Part (vi) is obvious, since the matrix (α_{ij}) of the system (4.2.1) is just $(b\delta_{ij})$ in this case. Thus, we just need to prove (ii). We will do this first in the case where x generates L over K.

Suppose $L = K(x)$ and the minimal polynomial q for x over K is

$$q(z) = z^k + c_{k-1}z^{k-1} + \cdots + c_1 z + c_0.$$

Then the set $\{x^{k-1}, \dots, x, 1\}$ is a basis for L as a K vector space. Relative to this basis, the determinant of the matrix (α_{ij}) of the system (4.2.1) is

$$\begin{vmatrix} -c_{k-1} & -c_{k-2} & \cdots & -c_1 & -c_0 \\ 1 & 0 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{vmatrix} = (-1)^k c_0.$$

This proves (ii) in the case where $L = K(x)$.

To finish the argument, we let x be any element of L and set $F = K(x)$. If $\{f_1, \dots, f_n\}$ is a basis for $K(x)$ as a K vector space and $\{g_1, \dots, g_m\}$ is a basis for L as a $K(x)$ vector space, then $e_{\mu\nu} = f_\mu g_\nu$ defines a basis $\{e_{\mu\nu}\}$ for L as a K vector space. Evidently the matrix (α_{ij}) of the system (4.2.1), in this case, is a block diagonal matrix consisting of m copies, down the diagonal, of the matrix (β_{ij}) of the system

$$xf_i = \beta_{i1}f_1 + \beta_{i2}f_2 + \cdots + \beta_{ik}f_k$$

and zeroes everywhere else. The determinant of (β_{ij}) is the norm of x as an element of the field extension $K(x)$ of K. Hence, by the previous paragraph, it is equal to $(-1)^n c_0$, where c_0 is the constant term of the minimal polynomial of x over K. Hence, the determinant of (α_{ij}) is $((-1)^n c_0)^m = (-1)^k c_0^m$.

Note that the above argument can also be used to show that the characteristic polynomial of x for the extension L of K is the mth power of the characteristic polynomial of x for the extension $K(x)$ of K.

This completes our review of the theory of ring and field extensions. We now return to our development of the machinery necessary to prove the Nullstellensatz. We first prove Hilbert's Nullstellensatz for polynomial algebras and then the more complicated Rückert's Nullstellensatz for ${}_n\mathcal{H}_0$.

4.3 Hilbert's Nullstellensatz

The proof we give here of Hilbert's Nullstellensatz is not the most efficient possible, but it has the advantage of following the same strategy that we will use later to prove Rückert's Nullstellensatz, without the technical complications of the latter result. The main ingredients of the proof are Noether's normalization theorem and the going up theorem.

4.3.1 Noether's Normalization Theorem. *If A is a finitely generated commutative algebra over an infinite field K, then there is an algebraically independent set of elements $\{x_1, \dots, x_m\} \subset A$, consisting of linear combinations of the generators of A, such that if $B = K[x_1, \dots, x_m]$, then A is a finite extension of B.*

Proof. The proof is by induction on the number of generators of A. It is trivial if A is generated by zero generators – that is, if $A = K$. Thus, we suppose the theorem is true for algebras generated by $n-1$ generators and let A be an algebra generated by n generators $y_1, \dots, y_n$. If these generators are algebraically independent over K, then A itself is a polynomial algebra, we may choose $B = A$, and there is nothing more to prove. If they are not independent, they satisfy a non-trivial polynomial equation $p(y_1, \dots, y_n) = 0$ over K. Because the field K is infinite, there are non-zero elements $\lambda_1, \dots, \lambda_{n-1} \in K$ such that $q(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$, where q is the sum of the homogeneous terms of highest degree in p. If we make the change of variables $y_i' = y_i - \lambda_i y_n$, for $1 \le i < n$ and $y_n' = y_n$, then

$$\begin{aligned} 0 &= p(y_1' + \lambda_1 y_n, \dots, y_{n-1}' + \lambda_{n-1} y_n, y_n) \\ &= q(\lambda_1, \dots, \lambda_{n-1}, 1) p'(y_1', \dots, y_{n-1}', y_n), \end{aligned}$$

where p' is a polynomial in $y_1', \dots, y_{n-1}', y_n$ which is monic as a polynomial in y_n. Thus, y_n is integral over A', where A' is the subalgebra of A generated by $y_1', \dots, y_{n-1}'$. Since A' is generated by $n-1$ elements, the induction hypothesis implies it has a subalgebra $B \subset A'$ such that B is isomorphic to a polynomial algebra $K[x_1, \dots, x_m]$, with each x_i a linear combination of the y_j', and A' is integral over B. By the transitivity of integral dependence, A is also integral over B. This completes the induction.

4.3.2 Proposition. *If A and B are integral domains and A is an integral extension of B, then A is a field if and only if B is a field.*

Proof. Suppose A is a field. Then each non-zero element $b \in B$ has an inverse $a \in A$. Since a is integral over B, there are coefficients $c_{n-1}, \dots, c_0$ in B such that

$$a^n + c_{n-1} a^{n-1} + \cdots + c_0 = 0. \tag{4.3.1}$$

If we multiply this equation by b^{n-1} and use the fact that $ab = 1$, we have

$$a = -(c_{n-1} + \cdots + c_0 b^{n-1}) \in B.$$

Thus, B is a field if A is a field.

Suppose B is a field and a is a non-zero element of A. Let (4.3.1) be the minimal polynomial for a. This polynomial is irreducible, which implies that $c_0 \neq 0$. Then the element $-c_0^{-1}(a^{n-1}+c_{n-1}a^{n-2}+\cdots+c_1)$ is an inverse for a. Thus, A is a field if B is a field.

4.3.3 Corollary. *If A is an integral extension of B and if P is a prime ideal of A, then P is maximal if and only if $P \cap B$ is a maximal ideal of B.*

Proof. If A is an integral extension of B, then the integral domain A/P is clearly an integral extension of the integral domain $B/(P \cap B)$. By the preceding proposition, A/P is a field if and only if $B/(P \cap B)$ is a field.

4.3.4 Going Up Theorem. *If A is an integral extension of B and if P is a prime ideal of B, then $P = Q \cap B$ for some prime ideal Q of A.*

Proof. Let S be the set of all elements of B that do not belong to P. Then S is a multiplicative system in both B and A. We denote the corresponding rings of fractions $S^{-1}B$ and $S^{-1}A$ by B_P and A_P, respectively. We claim that A_P is integral over B_P. In fact, if $a/s \in A_P$, with $a \in A$ and $s \in S$, and a satisfies a polynomial equation $a^n + b_{n-1}a^{n-1} + \cdots + b_0 = 0$, with coefficients in B, then a/s satisfies the polynomial equation

$$\left(\frac{a}{s}\right)^n + \frac{b_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \cdots + \frac{b_0}{s^n} = 0,$$

with coefficients in B_P. This establishes the claim.

Suppose J is a maximal ideal of A_P. Then, by the above corollary, $J \cap B_P$ is the unique maximal ideal $S^{-1}P$ of the local ring B_P. Furthermore, $Q = J \cap A$ is a prime ideal of A and $Q \cap B = J \cap B = P$. This completes the proof.

4.3.5 Theorem (Hilbert's Nullstellensatz). *If I is an ideal of the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$, then* $\operatorname{id}\operatorname{loc} I = \sqrt{I}$.

Proof. By Proposition 4.1.1, we may assume that the ideal I is prime. Suppose $f \in \mathbb{C}[z_1, \ldots, z_n]$ and $f \notin I$. By the Noether normalization theorem, we may assume that coordinates in $\mathbb{C}^n$ and an integer m have been chosen so that $A = \mathbb{C}[z_1, \ldots, z_n]/I$ is a finite extension of $B = \mathbb{C}[z_1, \ldots, z_m]$. Then the image of f in A/I satisfies a minimal monic polynomial equation over B. That is, there exist $b_i \in B, i = 1, \ldots, k-1$, so that

$$f^k + b_{k-1}f^{k-1} + \cdots + b_0 \in I, \tag{4.3.2}$$

and k is the minimal degree of such an equation. The minimality of k implies that $b_0 \neq 0$, and so there is a point $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, where $b_0(\lambda) \neq 0$.

Let M be the maximal ideal $\operatorname{id}\{\lambda\}$ of B. By Theorem 4.3.4 and Corollary 4.3.3, there is a maximal ideal N of A such that $N \cap B = M$. The residue field A/N is a finite extension of $B/M = \mathbb{C}$, and since $\mathbb{C}$ is algebraically closed, it is equal to $\mathbb{C}$. Let ζ be the point of $\mathbb{C}^n$ whose coordinates are the images in A/N of the generators $z_1, \dots, z_n$. It follows that N is the maximal ideal of A determined by the maximal ideal of $\mathbb{C}[z_1, \dots, z_n]$ consisting of polynomials which vanish at ζ. Polynomials in I vanish at ζ, since $A = \mathbb{C}[z_1, \dots, z_n]/I$. Thus, $\zeta \in \operatorname{loc} I$. Also, $\zeta_i = \lambda_i$, for $i = 1, \dots, m$, since $N \cap B = M$. It follows from (4.3.2) that

$$f^n(\zeta) + b_{n-1}(\lambda) f^{n-1}(\zeta) + \cdots + b_0(\lambda) = 0.$$

Since $b_0(\lambda) \neq 0$, this implies that $f(\zeta) \neq 0$, and hence, that $f \notin \operatorname{id} \operatorname{loc} I$. This completes the proof.

The proof of the Nullstellensatz for ${}_n\mathcal{H}_0$ follows almost the same strategy as the proof given above. That is, there is an analogue of the Noether normalization theorem, which says that if $\mathcal{P}$ is a prime ideal of $A = {}_n\mathcal{H}_0$, then there is a subalgebra B of A such that B is isomorphic to ${}_m\mathcal{H}_0$ for some $m \leq n$, and A is a finite extension of B (Theorem 4.5.2). The proof of Hilbert's Nullstellensatz followed from showing that each maximal ideal of $B = \mathbb{C}[z_1, \dots, z_m]$, determined by a point of $\mathbb{C}^m$, lifts to a maximal ideal of $A = \mathbb{C}[z_1, \dots, z_n]/I$, determined by a point of $\operatorname{loc} I$. This argument works only because there are enough maximal ideals of B determined by points of $\mathbb{C}^m$ to separate points in B. This is certainly not true in the case where B is the local ring ${}_m\mathcal{H}_0$. The solution to this problem is to show that the fact that ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0$ implies that a certain lifting property holds throughout a polydisc centered at 0. Specifically, there is a polydisc $\Delta \subset \mathbb{C}^m$, centered at 0, and a representative V of $\operatorname{loc} \mathcal{P}$ in $\pi^{-1}(\Delta)$, where $\pi : \mathbb{C}^n \to \mathbb{C}^m$ is the projection, so that each point of Δ lifts to a point of V under π. All of the technical difficulties in the proof lie in establishing this lifting theorem. Once the lifting theorem is established, the proof of the Nullstellensatz for ${}_n\mathcal{H}_0$ is identical to the proof given above for Hilbert's Nullstellensatz.

The lifting theorem that we need is Theorem 4.5.4, which characterizes the underlying geometric structure associated with a finite ring extension of the form ${}_m\mathcal{H}_0 \to {}_n\mathcal{H}_0/\mathcal{P}$. This is the structure of the germ of a finite branched holomorphic cover. Thus, our next task is to develop the essential properties of finite branched holomorphic covers.

4.4 Finite Branched Holomorphic Covers

A *proper map* $\pi : V \to W$, between two topological spaces, is a continuous map with the property that $\pi^{-1}(K)$ is compact in V whenever K is a compact subset of W.

4.4.1 Definition. *Let V and W be holomorphic subvarieties of open subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let $\pi : V \to W$ be a finite to one proper holomorphic map. Then π is said to be a finite branched holomorphic cover if there are dense open subsets $W_0 \subset W$ and $V_0 = \pi^{-1}(W_0) \subset V$ such that $W - W_0$ is a subvariety of W and $\pi : V_0 \to W_0$ is a locally biholomorphic mapping of V_0 onto W_0. We will call $\pi : V_0 \to W_0$ a dense regular subcover of $\pi : V \to W$.*

Here, by a *locally biholomorphic mapping* $\pi : V_0 \to W_0$, between holomorphic subvarieties, we mean a holomorphic map with the property that, for each $\lambda \in V_0$, there is a neighborhood U of λ on which π is a biholomorphic map from U to a neighborhood of $\pi(\lambda)$ in W_0. If each point $w \in W_0$ has a neighborhood A such that $\pi^{-1}(A)$ is a finite disjoint union of open sets, on each of which π is a biholomorphic map onto A, then we say that $\pi : V_0 \to W_0$ is a *finite holomorphic covering map.* Note that a locally biholomorphic map need not be a finite holomorphic covering map.

4.4.2 Proposition. *If $\pi : V \to W$ is a finite branched holomorphic cover with $\pi : V_0 \to W_0$ as a dense regular subcover, then $\pi : V_0 \to W_0$ is a finite holomorphic covering map.*

Proof. Let w be a point of W. Let $\pi^{-1}(w) = \{\lambda_1, \dots, \lambda_k\}$ and suppose we have chosen, for each i, a neighborhood U_i of λ_i in V, in such a way that $U_i \cap U_j = \emptyset$ for $i \neq j$. If A is a neighborhood of w with compact closure in W_0, then $\pi^{-1}(\overline{A}) - \bigcup U_i$ is a compact subset of V_0. The collection of sets of this form is closed under finite intersection and so, if they are all non-empty, there is a point λ in their intersection. Then, necessarily, $\pi(\lambda) = w$ and, hence, λ is one of the λ_i. This is not possible, since λ is in the complement of each U_i. It follows that, for some choice of A, the set $\pi^{-1}(A)$ is the disjoint union of the open sets $\pi^{-1}(A) \cap U_i$.

Now suppose that $w \in W_0$ and the U_i are chosen to be open subsets of V_0 such that $\pi|_{U_i}$ is a biholomorphic map onto a neighborhood of w for each i. If A is chosen so that $A \subset \bigcap \pi(U_i)$ and so that $\pi^{-1}(A) = \bigcup(\pi^{-1}(A) \cap U_i)$, then, for each i, π is a biholomorphic map of $\pi^{-1}(A) \cap U_i$ onto A. Thus, each point of W_0 has a neighborhood which, under π, is covered by a finite number of biholomorphic copies of itself. That is, π is a finite holomorphic covering map.

Note that the number of points in the inverse image of a point of W_0 is locally constant on W_0. If W_0 is connected, then this number is a constant r and, in this case, we say that $\pi : V \to W$ is a finite branched holomorphic cover of *pure order* r and $\pi : V_0 \to W_0$ is a finite holomorphic covering map of *pure order* r.

4.4.3 Proposition. *With $\pi : V \to W$ and W_0 as above, if W_0 is locally connected in W, $w \in W$, and $\lambda \in \pi^{-1}(w)$, then there are arbitrarily small neighborhoods U of λ and $A = \pi(U)$ of w such that $\pi : U \to A$ is a finite branched holomorphic cover of pure order.*

Proof. Let $\pi^{-1}(w) = \{\lambda_1, \dots, \lambda_k\}$. Suppose $\{U_1, \dots, U_k\}$ is a pairwise disjoint collection of open subsets of V with $\lambda_i \in U_i$. As in the proof of Proposition 4.4.2, there is a neighborhood A of w so that $\pi^{-1}(A) \subset \bigcup U_i$. By replacing each U_i with $U_i \cap \pi^{-1}(A)$, we may assume that

$$\pi^{-1}(A) = \bigcup U_i.$$

We may also choose A so that $A_0 = W_0 \cap A$ is connected. Then $\pi^{-1}(A_0)$ is a finite holomorphic cover of A_0 of pure order. If we set $U_i' = \pi^{-1}(A_0) \cap U_i = U_i \cap V_0$, then U_i' is dense in U_i and the restriction of π to U_i' is also a finite holomorphic cover of pure order. It follows that the restriction of π to U_i is a finite branched holomorphic cover of A of pure order.

With $\pi : V \to W$ as in the above proposition and $\lambda \in V$, we know that there are arbitrarily small neighborhoods of λ on which π is a finite branched holomorphic cover of pure order. For small enough neighborhoods the order must stabilize at some positive integer $o_\pi(\lambda)$. We call this integer the *branching order* of π at λ.

In what follows, we will make extensive use of the properties of a monic polynomial p with coefficients in $\mathcal{H}(W)$, where W is a connected open subset of $\mathbb{C}^m$. The ring $\mathcal{H}(W)$ is a normal domain (Exercise 4.15) so, by Theorem 4.2.9, the discriminant of p is an element of $\mathcal{H}(W)$. Also by Theorem 4.2.9, the subset of W on which p has distinct roots is exactly the set on which its discriminant is non-zero.

The next result will be our main tool for constructing finite branched holomorphic covers. In it, and throughout the remainder of this chapter, we will use the following conventions: A choice of coordinates for $\mathbb{C}^n$ and a non-negative integer $m \le n$ determine a decomposition of $\mathbb{C}^n$ as $\mathbb{C}^m \times \mathbb{C}^{n-m}$ and a corresponding projection $\pi : \mathbb{C}^n \to \mathbb{C}^m$. For a point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$, where $z' = (z_1, \dots, z_m)$ and $z'' = (z_{m+1}, \dots, z_n)$. If W is an open subset of $\mathbb{C}^m$ and, for some $j > m$,

$$p = a_0 + a_1 z_j + \cdots + a_n z_j^n$$

is a polynomial in $\mathcal{H}(W)[z_j]$ (so that the coefficients $a_0, \dots, a_n$ are functions in $\mathcal{H}(W)$), then we will use the notation

$$p(z', z_j) = a_0(z') + a_1(z')z_j + \cdots + a_n(z')z_j^n.$$

The resulting function of $z' = (z_1, \dots, z_m)$ and z_j gives rise to a holomorphic function $z \to p(\pi(z), z_j)$ on $\pi^{-1}(W) \subset \mathbb{C}^n$, where z_j, of course, stands for the jth coordinate of z.

4.4.4 Proposition. *Let W be a connected open set in $\mathbb{C}^m$, let $\pi : \mathbb{C}^n \to \mathbb{C}^m$ be the projection, and set $U = \pi^{-1}(W)$. For $j = m+1, \dots, n$ let p_j be a monic polynomial in z_j with coefficients in $\mathcal{H}(W)$ and degree at least* 1. *If*

$$V = \{z = (z_1, \dots, z_n) \in U : p_j(\pi(z), z_j) = 0, \quad j = m+1, \dots, n\},$$

then $\pi : V \to W$ is a finite branched holomorphic cover.

Proof. Let $d_j \in \mathcal{H}(W)$ be the discriminant of p_j, for $j = m+1, \dots, n$, and let D be the union of the zero sets of the d_j. Then $W_0 = W - D$ is an open subset of W. We set $V_0 = \pi^{-1}(W_0)$, where $\pi : V \to W$ is the restriction of the projection $U \to W$ to V. It follows from Exercise 3.14 that W_0 is dense in W. We need to show that V_0 is dense in V and that π is proper and finite to one on V and locally biholomorphic on V_0.

Let $K \subset W$ be compact and set $L = \pi^{-1}(K)$. We claim that L is a bounded subset of $\mathbb{C}^n$. Clearly the coordinate function z_j is bounded on L if $j \leq m$, since it is bounded on K. For $j > m$, z_j is bounded on L because points $z \in L$ satisfy the equation $p_j(\pi(z), z_j) = 0$, which is a polynomial equation in z_j with coefficients which are bounded functions on K. If we divide this equation by $z_j^{n_j-1}$, where n_j is the order of p_j, we may use the resulting equation to estimate $|z_j|$, on the set where $|z_j| \geq 1$, in terms of the coefficients of p_j . We conclude that $|z_j|$ is bounded by the maximum of 1 and the sum of the suprema of the absolute values of the coefficients of p_j on K. Thus, the coordinate functions z_j are bounded on L for all j and, therefore, L is a bounded set. The set L is also closed in $\mathbb{C}^n$, since it is just the set of points in $\mathbb{C}^n$ which map to K under the projection $\mathbb{C}^n \to \mathbb{C}^m$ and, at which, each p_j vanishes. Hence, L is compact and π is a proper holomorphic map.

For $j = m+1, \dots, n$ and each fixed value of $z' \in W$, there is a non-empty, finite set of values of z_j for which $p_j(z', z_j) = 0$. This implies that $\pi : V \to W$ is finite to one and surjective.

Let $a = (a_1, \dots, a_n)$ be a point of V with $a' = \pi(a)$ the corresponding point of W. Each polynomial $p_j(a', z_j)$ is monic in z_j and vanishes to some order greater than 0 at $z_j = a_j$. Proposition 3.3.1 implies that, for each

$\epsilon > 0$, there is a neighborhood Δ' of a' so that, for $z' \in \Delta'$, each polynomial $p_j(z', z_j)$ has a root that lies within ϵ of a_j. This implies, in particular, that for each point of V, there are arbitrarily nearby points of V that lie over points of the open dense set W_0 of W. In other words, V_0 is dense in V.

Since W_0 is exactly the set of $z' \in W$ at which the roots of all the p_j are distinct, the inverse image V_0 of this set under π is the subset of U on which each p_j vanishes, but its derivative with respect to z_j does not vanish. Thus, the map $F : \mathbb{C}^n \to \mathbb{C}^{n-m}$, defined by

$$F(z) = (p_{m+1}(\pi(z), z_{m+1}), \dots, p_n(\pi(z), z_n)),$$

has Jacobian J_F in which the last $n - m$ columns form a diagonal matrix with entries that do not vanish on V_0. Thus, J_F has rank $n - m$ in an open set containing V_0. It follows from the implicit mapping theorem that, for each $\lambda \in V_0$, there is a neighborhood A_λ of λ in $\mathbb{C}^n$, a neighborhood B_λ of $\pi(\lambda)$ in $\mathbb{C}^m$, and a holomorphic map $G : B_\lambda \to A_\lambda$ such that the points of A_λ where F vanishes (i.e. the points of $V_0 \cap A_\lambda$) are exactly the points in the image of G. It follows that G is a holomorphic inverse for the restriction of π to $V_0 \cap A_\lambda$. Thus, π locally has a holomorphic inverse on V_0. In other words, π is locally biholomorphic on V_0. Consequently $V \to W$ as a finite branched holomorphic cover. This completes the proof.

Our further study of finite branched holomorphic covers depends on the following technical lemma.

4.4.5 Lemma. *For each pair of positive integers n and r, there exists a finite set $\{f_1, \dots, f_q\}$ of linear functionals on $\mathbb{C}^n$ such that, for any set of r distinct points $\{z_1, \dots, z_r\} \subset \mathbb{C}^n$, there is some i for which the numbers $f_i(z_1), \dots, f_i(z_r)$ are distinct.*

Proof. We may assume $r \geq 2$. We choose an integer $q > \frac{1}{2}r(r-1)(n-1)$ and a set of linear functionals $\{f_1, \dots, f_q\}$ on $\mathbb{C}^n$, such that every subset with n elements of this set is linearly independent. If the functionals f_i are interpreted as the rows of a $q \times n$ matrix, then this is just the condition that each $n \times n$ submatrix has non-vanishing determinant. Such a choice is clearly possible, since the union of the zero sets in $\mathbb{C}^{nq}$ of these determinants is a proper subvariety of $\mathbb{C}^{nq}$.

Given distinct integers j and k between 1 and r and $\{z_1, \dots, z_r\} \subset \mathbb{C}^n$, the set of linear functionals f on $\mathbb{C}^n$ for which $f(z_j - z_k) = 0$ is a linear subspace of dimension $n - 1$ and, hence, it may contain at most $n - 1$ of the functionals $f_1, \dots, f_q$. There are $\frac{1}{2}r(r-1)$ unordered pairs of distinct elements of the set $\{z_1, \dots, z_r\}$ and so there are at most $\frac{1}{2}r(r-1)(n-1)$ integers i for which an equation of the form

$$f_i(z_j - z_k) = 0, \quad \text{with} \quad j \neq k,$$

can be satisfied. By the choice of q, there must be at least one index i between 1 and q such that no such equation holds. For this i, the functional f_i separates the points $z_1, \dots, z_r$.

4.4.6 Proposition. *Let W be a connected open subset of $\mathbb{C}^m$, let D be a proper subvariety of W, and set $W_0 = W - D$. Denote the projection $\mathbb{C}^n \to \mathbb{C}^m$ by π. If V_0 is a subvariety of $\pi^{-1}(W_0)$, with closure $\overline{V}_0$ in $\pi^{-1}(W)$, and if $\pi : V_0 \to W_0$ is a holomorphic covering map of order r and $\pi : \overline{V}_0 \to W$ a proper map, then*

(i) *$\overline{V}_0$ is a subvariety of $\pi^{-1}(W)$;*
(ii) *$\pi : \overline{V}_0 \to W$ is a finite branched holomorphic cover;*
(iii) *for each $w \in W$, there are at most r elements in $\pi^{-1}(w) \cap \overline{V}_0$;*
(iv) *each $f \in \mathcal{H}(\overline{V}_0)$ is a root of a monic polynomial of degree r with coefficients in $\mathcal{H}(W)$.*

Proof. We first show that $\pi : \overline{V}_0 \to W$ is surjective. We know that W_0 is dense in W (Exercise 3.14). Thus, if $w \in W$, we can choose a sequence $\{w_i\}$ in W_0 which converges to w. The inverse image in $\overline{V}_0$ of the set consisting of the points of this sequence, together with its limit point w, is a compact set, since $\pi : \overline{V}_0 \to W$ is a proper map. It follows that there is a sequence $\{v_i\}$ in $\overline{V}_0$ with $\pi(v_i) = w_i$ and with a subsequence converging in $\overline{V}_0$ to a point v with $\pi(v) = w$. Thus, $\pi : \overline{V}_0 \to W$ is surjective. If we can show that $\overline{V}_0$ is a subvariety of $\pi^{-1}(W)$, then $\pi : \overline{V}_0 :\to W$ is a finite branched holomorphic cover, by Definition 4.4.1.

Since W_0 is connected, π is a cover of pure order r for some r. Let the points of V_0 over a point $w \in W_0$ be labeled $\lambda_1(w), \dots, \lambda_r(w)$. For a function f, continuous on $\overline{V}_0$ and holomorphic on V_0, let p be the polynomial in the indeterminant x defined by

$$p(w, x) = \prod_{j=1}^{r} (x - f(\lambda_j(w))).$$

Since $\pi : V_0 \to W_0$ is a holomorphic covering map, each $w \in W_0$ has a neighborhood in which it is possible to choose the $\lambda_j(w)$ in such a way that these functions are holomorphic. Of course, it may not be possible to do this globally. However, since the coefficients of the polynomial p are independent of the labeling of the roots, they are well defined and holomorphic in all of W_0. In fact, since $\pi : \overline{V}_0 \to W$ is a proper map, f is locally bounded on $\overline{V}_0$, which implies that, for each point $w \in W_0$, there is a neighborhood Δ of w such that the coefficients of p are bounded on $\Delta \cap W_0$. The removable singularity theorem (Theorem 3.3.2) then implies that these coefficients extend to be holomorphic in all of W. Thus, p is a polynomial with coefficients holomorphic in W and with the property that, whenever $w \in W_0$, the roots of $p(w, x)$ are exactly the values assumed by f on the set $\pi^{-1}(w) \cap V_0$. In particular, $p(\pi(z), f(z))$ vanishes on V_0. By continuity, this function also vanishes on $\overline{V}_0$. If we can establish that $\overline{V}_0$ is a subvariety, then this argument shows that every function holomorphic on $\overline{V}_0$ is a root of a monic polynomial of degree r over $\mathcal{H}(W)$, which proves (iv).

We apply the previous lemma to obtain linear functionals $f_1, \ldots, f_q$ such that any set of $r+1$ distinct points in $\mathbb{C}^n$ can be separated by some one of the functionals f_i. We then let $p_1, \ldots, p_q$ be the polynomials constructed, as above, for the functions $f_1, \ldots, f_q$. Each of the functions $p_j(\pi(z), f_j(z))$ vanishes on $\overline{V}_0$, and so $\overline{V}_0 \subset V$, where V is the subvariety of U on which they all vanish. We endeavor to prove that $\overline{V}_0 = V$. To this end, let a_1 be a point of V, and let $a_1, a_2, \ldots, a_k$ be a distinct set of points of $\pi^{-1}(\pi(a_1)) \cap V$, where $k \leq r+1$. Then there is a j for which the numbers $f_j(a_1), \ldots, f_j(a_k)$ are all distinct. These are all roots of the polynomial $p_j(\pi(a_1), x)$. Since this is a polynomial of degree r, we conclude that $k \leq r$ and, hence, that the inverse image in V of a point of W contains no more than r points. This proves (iii).

Now that we know there are at most r points of $\pi^{-1}(\pi(a_1)) \cap V$, we may choose j so that f_j separates them. Then, for any $w \in W_0$ sufficiently near $\pi(a_1)$, there must be a root of $p_j(w, x)$ near $f_j(a_1)$. But, by the construction of p_j, this root must be a value assumed by f_j on $\pi^{-1}(w) \cap V_0$. It follows that, for any sequence $\{w_i\} \subset W_0$, converging to $\pi(a_1)$, there is a sequence $\{v_i\} \subset V_0$ with $\pi(v_i) = w_i$ and with $f_j(v_i)$ converging to $f_j(a_1)$. Since $\pi : \overline{V}_1 \to W$ is a proper map, the sequence $\{v_i\}$ must have a limit point $v \in V$. Necessarily, $\pi(v) = \pi(a_1)$ and $f_j(v) = f_j(a_1)$. Hence, $v = a_1$, since f_j separates the points of $\pi^{-1}(\pi(a_1)) \cap V$. It follows that $a_1 \in \overline{V}_0$ and $V = \overline{V}_0$. This establishes (i). Since (ii) follows from (i), by the first paragraph of the proof, the proof is complete.

This is an important result. It and its corollaries will play key roles in the remainder of this chapter.

4.4.7 Corollary. *Let W be a connected open set in $\mathbb{C}^m$, $\pi : \mathbb{C}^n \to \mathbb{C}^m$ the projection, and V a subvariety of $\pi^{-1}(W)$. If $\pi : V \to W$ is a finite branched holomorphic cover, and if $\pi : V_0 \to W_0$ is a dense regular subcover, then*

(i) *there are at most r points of V lying over a given point of W, where r is the order of the cover $\pi : V_0 \to W_0$;*
(ii) *the ring $\mathcal{H}(V)$ is an integral extension of the ring $\mathcal{H}(W)$;*
(iii) *if A_0 is a component of V_0, then its closure $\overline{A}_0$ in V is a subvariety of V, and $\pi : \overline{A}_0 :\to W$ is also a finite branched holomorphic cover.*

Proof. If $\pi : V \to W$ is a finite branched holomorphic cover, then π is a proper map and $\overline{V}_0 = V$. Thus, V_0 and $\pi : V_0 \to W_0$ satisfy the conditions of Proposition 4.4.6. Parts (i) and (ii) of the corollary then follow immediately from (iii) and (iv) of Proposition 4.4.6. If A_0 is a component of V_0, then $\pi : A_0 \to W_0$ is a finite holomorphic covering map and $\pi : \overline{A}_0 \to W$ is a proper map. Thus, (iii) of the corollary follows from (i) and (ii) of Proposition 4.4.6.

The *branch locus* of a finite branched holomorphic cover $\pi : V \to W$ is the subset of V consisting of all points where the branching order is at least 2.

4.4.8 Corollary. *Let $\pi : V \to W$ be a finite branched holomorphic cover, with W and V as in Corollary 4.4.7. If k is any positive integer, then the subset of V on which π has branching order at least k is a subvariety of V. In particular, the branch locus of π is a subvariety of V. Furthermore, the image under π of the branch locus is a subvariety of W.*

Proof. We choose linear functionals $f_1, \dots, f_q$ and polynomials $p_1, \dots, p_q$ as in the proof of Proposition 4.4.6. Then each $p_j(w, x)$ is a polynomial of degree r in the indeterminant x with coefficients in $\mathcal{H}(W)$, and, for each $w \in W_0$, the roots of $p_j(w, x)$ are exactly the values that f_j assumes on $\pi^{-1}(w)$.

Let $o_\pi(\lambda)$ denote the branching order of $\pi : V \to W$ at λ. Then there are arbitrarily small neighborhoods Ω of λ and $\Delta = \pi(\Omega)$ of $\pi(\lambda)$ such that Δ contains points w of W_0 for which $\pi^{-1}(w) \cap \Omega$ contains exactly $k = o_\pi(\lambda)$ points, $a_1, a_2, \dots, a_k$. By the construction of the p_j in Proposition 4.4.6, the numbers $f_j(a_1), \dots, f_j(a_k)$ are k of the r roots of $p_j(w, x)$ (they may not all be distinct). This clearly implies that $m \geq o_\pi(\lambda)$, where m is the multiplicity of the root $f_j(\lambda)$ of $p_j(\pi(\lambda), x)$.

Now choose j so that f_j separates the points of the set $\pi^{-1}(\pi(\lambda)) \cap V$ (this set has at most r elements). Let $\lambda_1, \dots, \lambda_\nu$ be the distinct points of this set, and let m_i be the multiplicity of the root $f_j(\lambda_i)$ of $p_j(\pi(\lambda), x)$. Then $\sum m_i = r$. We also have $\sum o_\pi(\lambda_i) = r$, where $o_\pi(\lambda_i)$ is the branching order of π at λ_i. However, by the previous paragraph, $m_i \geq o_\pi(\lambda_i)$ for each i. It follows that $m_i = o_\pi(\lambda_i)$ for each i. Thus, we have proved that $o_\pi(\lambda)$ is less than or equal to the multiplicity of the root $f_j(\lambda)$ of $p_j(\pi(\lambda), x)$, for each j, and there is a j for which they are equal. In other words, $o_\pi(\lambda)$ is the minimum of the multiplicities of the roots $f_j(\lambda)$ of the polynomials $p_j(\pi(\lambda), x)$.

Let $p_j^{(s)}$ denote the s-fold derivative of the polynomial $p_j(w, x)$ with respect to x. A root is a multiple root of $p_j(w, x)$ of multiplicity k if and only if the polynomials $p_j^{(s)}(w,x)$ vanish there for $s=0,\dots,k$, but not for $j=k+1$. Thus, it follows from the above paragraph that $\lambda \in V$ is a point of branching order at least k if and only if $p_j^{(s)}(\pi(\lambda), f_j(\lambda)) = 0$, for $s=0,\dots,k$ and for all j. The set of $\lambda \in V$ for which this happens is a subvariety of V. This proves the first part of the corollary. The second part follows from the fact that the set where the branching order is at least 2 is the set where all the p_j have multiple roots, and this is the set where the discriminants of all the polynomials p_j vanish. These discriminants are functions in $\mathcal{H}(W)$, and the set on which they vanish simultaneously is a subvariety of W. This completes the proof.

4.4.9 Example. We let V be the variety of Example 3.7.8. That is

$$V = \{(z,w) \in \mathbb{C}^2 : z^2 - w^3 = 0\}.$$

We set $W = \mathbb{C}$ and define $\pi_i : V \to W$, for $i = 1,2$, by $\pi_1(z,w) = z$ and $\pi_2(z,w) = w$. Finally, we let $V_0 = V - \{(0,0)\}$ and $W_0 = W - \{0\}$. Here, in each case, $\pi_i : V \to W$ is a finite branched holomorphic cover with branch locus $\{(0,0)\}$ and $\pi_i : V_0 \to W_0$ is a finite holomorphic covering map. Note, however, that π_1 has branching order 3 at $(0,0)$, while π_2 has branching order 2 at $(0,0)$. Thus, the branching order at a point of V depends, not just on the variety V, but on the specific branched holomorphic cover.

4.4.10 Example. We set

$$p(u,v,z) = z^3 - 3uz + 2v, \quad V = \{(u,v,z) \in \mathbb{C}^3 : p(u,v,z) = 0\}$$

and let $\pi : V \to W = \mathbb{C}^2$ be the projection $\pi(u,v,z) = (u,v)$. Then π is a finite branched holomorphic cover by Proposition 4.4.4. It has branching order at least 2 at points where p and $\partial p/\partial z$ vanish simultaneously. Since $\partial p/\partial z(u,v,z) = 3z^2 - 3u$, we conclude that V has branching order at least 2 at points where $u = z^2$ and $v = z^3$. This can happen only if $u^3 = v^2$. The branching order is 3 at points where p, $\partial p/\partial z$, and $\partial^2 p/\partial z^2$ vanish simultaneously. Since $\partial^2 p/\partial z^2(u,v,z) = 6z$, we conclude that the origin is the only point of branching order 3. Let $D = \{(u,v) \in W : u^3 = v^2\}$. Above each point of $D - \{0\}$, there is one point of V where the branching order is 2 (the point (u,v,z) with $u = z^2, v = z^3$) and one point where the branching order is 1 (the point (u,v,z) with $u = z^2/4, v = -z^3/8$). Above each point of $W_0 = W - D$, there are three points of V and they each have branching order 1.

4.5 The Nullstellensatz

Recall that a B-algebra A is finite over B if it is finitely generated as a B-module (section 4.2). By Theorem 4.2.5, this is equivalent to the condition that A is finitely generated as a B-algebra by finitely many elements which are integral over B. This, in turn, is equivalent to the condition that A is a finitely generated B-algebra, and every element of A is integral over B. If A is finite over B and the natural map $B \to A$ is injective, then A is called a finite extension of B.

In what follows, once a coordinate system of $\mathbb{C}^n$ is fixed, for $j < n$ we consider ${}_j\mathcal{H}_0$ to be the subalgebra of ${}_n\mathcal{H}_0$ consisting of those germs of holomorphic functions that depend only on the variables $z_1, \dots, z_j$. Then ${}_n\mathcal{H}_0$ is a ${}_j\mathcal{H}_0$-algebra, as is each of its quotients ${}_n\mathcal{H}_0/\mathcal{I}$ by an ideal $\mathcal{I}$. The natural map ${}_j\mathcal{H}_0 \to {}_n\mathcal{H}_0/\mathcal{I}$ is injective if and only if $\mathcal{I} \cap {}_j\mathcal{H}_0 = (0)$. In this case, we may consider ${}_j\mathcal{H}_0$ to be a subring of ${}_n\mathcal{H}_0/\mathcal{I}$. Recall that we will distinguish germs from their representatives by the use of boldface type for germs. Also recall the definition of vanishing order in section 3.3.

4.5.1 Proposition. *If $\mathcal{I} \subset {}_n\mathcal{H}_0$ is an ideal, then the following two conditions on a linear coordinate system for $\mathbb{C}^n$ and an integer $m < n$ are equivalent:*

(i) *there exists $\mathbf{f}_j \in {}_j\mathcal{H}_0 \cap \mathcal{I}$, for $j = m+1, \dots, n$, such that $\mathbf{f}_j$ has finite vanishing order in $\mathbf{z}_j$;*

(ii) *${}_n\mathcal{H}_0/\mathcal{I}$ is a finite ${}_m\mathcal{H}_0$ algebra, which is generated by the images of $\mathbf{z}_{m+1}, \dots, \mathbf{z}_n$ in ${}_n\mathcal{H}_0/\mathcal{I}$.*

Proof. For $j \le n$ let ${}_j\tilde{\mathcal{H}}_0$ denote ${}_j\mathcal{H}_0/(\mathcal{I} \cap {}_j\mathcal{H}_0)$. Suppose (i) holds. In view of the Weierstrass preparation theorem, we may choose the $\mathbf{f}_j$ to be Weierstrass polynomials. Since these polynomials belong to $\mathcal{I}$, this implies that, for $j > m$, the image $\tilde{\mathbf{z}}_j$ of $\mathbf{z}_j$ in ${}_n\tilde{\mathcal{H}}_0$ is integral over ${}_{j-1}\tilde{\mathcal{H}}_0$, for $j = m+1, \dots, n$. The Weierstrass division theorem implies that, for each $\mathbf{f} \in {}_j\mathcal{H}_0$, there are elements $\mathbf{g} \in {}_j\mathcal{H}_0$ and $\mathbf{r} \in {}_{j-1}\mathcal{H}_0[\mathbf{z}_j]$ such that $\mathbf{f} = \mathbf{f}_j\mathbf{g} + \mathbf{r}$. On passing to residue classes mod $\mathcal{I}$ and remembering that $\mathbf{f}_j \in \mathcal{I}$, we conclude that every element of ${}_j\tilde{\mathcal{H}}_0$ is a polynomial in $\tilde{\mathbf{z}}_j$ over the ring ${}_{j-1}\tilde{\mathcal{H}}_0$. Thus, ${}_j\tilde{\mathcal{H}}_0$ is the integral extension of the ring ${}_{j-1}\tilde{\mathcal{H}}_0$ by the element $\tilde{\mathbf{z}}_j$. This means that ${}_n\tilde{\mathcal{H}}_0$ is obtained from ${}_m\tilde{\mathcal{H}}_0$ through successive integral extensions by the elements $\tilde{\mathbf{z}}_j$ for $j = m+1, \dots, n$. Corollary 4.2.7 now implies that ${}_n\tilde{\mathcal{H}}_0 = {}_m\tilde{\mathcal{H}}_0[\tilde{\mathbf{z}}_{m+1}, \dots, \tilde{\mathbf{z}}_n]$, where each of the elements $\tilde{\mathbf{z}}_j$ is integral over ${}_m\tilde{\mathcal{H}}_0$. Thus, (i) implies (ii).

Now suppose that (ii) holds. Then for $m < j \le n$, the image of $\mathbf{z}_j$ in ${}_n\tilde{\mathcal{H}}_0$ is a root of a monic polynomial with coefficients in ${}_m\tilde{\mathcal{H}}_0$. In other words, there is a monic polynomial $\mathbf{f}_j \in {}_m\mathcal{H}_0[\mathbf{z}_j] \cap \mathcal{I}$. Then $\mathbf{f}_j$ is clearly an element of ${}_j\mathcal{H}_0$ which has finite vanishing order in $\mathbf{z}_j$. This completes the proof.

4.5.2 Theorem. *If $\mathcal{P}$ is a non-zero prime ideal of ${}_n\mathcal{H}_0$, then linear coordinates for $\mathbb{C}^n$ and an integer $m < n$ may be chosen so that ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0$, generated by the images of $\mathbf{z}_{m+1}, \dots, \mathbf{z}_n$. Furthermore, given such coordinates, an additional change of coordinates involving only $\mathbf{z}_{m+1}, \dots, \mathbf{z}_n$ will ensure that the image of $\mathbf{z}_{m+1}$ generates the quotient field ${}_n\tilde{\mathcal{M}}$ of ${}_n\mathcal{H}_0/\mathcal{P}$ over the quotient field ${}_m\mathcal{M}$ of ${}_m\mathcal{H}_0$.*

Proof. As in the previous proposition, we set ${}_j\tilde{\mathcal{H}}_0 = {}_j\mathcal{H}_0/(\mathcal{P} \cap {}_j\mathcal{H}_0)$. Since $\mathcal{P} \neq 0$, we may choose a non-zero $\mathbf{f}_n \in \mathcal{P}$, and then, by a suitable linear change of coordinates, arrange that $\mathbf{f}_n$ has finite vanishing order in $\mathbf{z}_n$. Suppose we have chosen $\mathbf{f}_j \in {}_j\mathcal{H}_0 \cap \mathcal{P}$, for $j = k+1, \dots, n$, such that $\mathbf{f}_j$ has finite vanishing order in $\mathbf{z}_j$. Then either ${}_k\mathcal{H}_0 \cap \mathcal{P} = 0$, or there is a non-zero $\mathbf{f}_k \in {}_k\mathcal{H}_0 \cap \mathcal{P}$. If it exists, $\mathbf{f}_k$ can be made to have finite vanishing order in $\mathbf{z}_k$ by a linear change of coordinates that involves only the first k coordinates and, hence, does not affect the vanishing order of $\mathbf{f}_j$ or the definition of ${}_j\mathcal{H}_0$ for $j > k$. We continue this process until it terminates at some integer $k = m \geq 0$. Then ${}_m\mathcal{H}_0 \cap \mathcal{P} = 0$ and, for $j > m$, we have a function $\mathbf{f}_j \in {}_j\mathcal{H}_0 \cap \mathcal{P}$ such that $\mathbf{f}_j$ has finite vanishing order in $\mathbf{z}_j$. It follows that the natural map ${}_m\mathcal{H}_0 \to {}_n\tilde{\mathcal{H}}_0 = {}_n\mathcal{H}_0/\mathcal{P}$ is injective and, by Proposition 4.5.1, ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0$.

With coordinates for $\mathbb{C}^n$ and $m \leq n$ chosen as above, ${}_n\tilde{\mathcal{M}}$ is an algebraic field extension of ${}_m\mathcal{M}$ generated by the images $\tilde{\mathbf{z}}_j$ of the elements $\mathbf{z}_j$, for $j = m+1, \dots, n$. By the theorem of the primitive element (Theorem 4.2.1), ${}_n\tilde{\mathcal{M}}$ is actually generated over ${}_m\tilde{\mathcal{M}}$ by a single element which may be chosen to be a linear combination of the $\tilde{\mathbf{z}}_j$. Another linear change of variables, affecting only these coordinates, can be used to transform this element into $\tilde{\mathbf{z}}_{m+1}$. Such a change of variables does not change the embedding of ${}_m\mathcal{H}_0$ in ${}_n\tilde{\mathcal{H}}_0$. This completes the proof.

Let $\mathcal{P}$ be a non-zero prime ideal of ${}_n\mathcal{H}_0$. We will choose a finite set of elements of $\mathcal{P}$ which will lead directly to our description of $\operatorname{loc} P$ as the germ of a finite branched holomorphic cover of a neighborhood in $\mathbb{C}^m$. We assume that linear coordinates and an integer m have been chosen as in Theorem 4.5.2. As before, we let $\tilde{\mathbf{z}}_j$ denote the image of $\mathbf{z}_j$ in ${}_n\mathcal{H}_0/\mathcal{P}$. Thus, ${}_m\mathcal{H}_0 \to {}_n\mathcal{H}_0/\mathcal{P}$ is a finite ring extension and the corresponding field extension ${}_m\mathcal{M} \to {}_n\tilde{\mathcal{M}}$ is generated by $\tilde{\mathbf{z}}_{m+1}$.

Suppose $m < j \leq n$. Then $\tilde{\mathbf{z}}_j$ is integral over ${}_m\mathcal{H}_0$. Let $\mathbf{p}_j$ be its minimal polynomial; that is, $\mathbf{p}_j$ is the polynomial of minimal degree in ${}_m\mathcal{H}_0[\mathbf{z}_j] \subset {}_n\mathcal{H}_0$ which belongs to $\mathcal{P}$. Being monic, p_j has finite vanishing order in z_j, as a function of the variables $z_1, \dots, z_m, z_j$, and, hence, by the Weierstrass preparation theorem, factors as a unit times a Weierstrass polynomial in z_j. Therefore, we may as well assume that p_j is a Weierstrass polynomial in $\mathbf{z}_j$. The polynomials $\mathbf{p}_j$, for $j = m+1, \ldots, n$, are some of the elements of $\mathcal{P}$ that we are seeking.

We next use Theorem 4.2.12 to choose the remaining elements. For $j = m+2, \ldots, n$, we have $\tilde{\mathbf{z}}_j \in {}_n\tilde{\mathcal{H}}_0$ is integral over ${}_m\mathcal{H}_0$, $\tilde{\mathbf{z}}_{m+1}$ generates the quotient field of ${}_n\tilde{\mathcal{H}}_0$ over the quotient field of ${}_m\mathcal{H}_0$, and $\mathbf{p}_{m+1}$ is the minimal polynomial of $\tilde{\mathbf{z}}_{m+1}$. Thus, by Theorem 4.2.12,

$$\mathbf{d} \cdot \tilde{\mathbf{z}}_j = \mathbf{s}_j(\tilde{\mathbf{z}}_{m+1}),$$

where $\mathbf{d}$ is the discriminant of $\mathbf{p}_{m+1}$ and $\mathbf{s}_j$ is a unique polynomial of degree less than the degree of $\mathbf{p}_{m+1}$ with coefficients in ${}_m\mathcal{H}_0$. Then

$$\mathbf{q}_j(\mathbf{z}_{m+1}, \mathbf{z}_j) = \mathbf{d} \cdot \mathbf{z}_j - \mathbf{s}_j(\mathbf{z}_{m+1})$$

belongs to ${}_m\mathcal{H}_0[\mathbf{z}_{m+1}, \mathbf{z}_j] \cap \mathcal{P}$. The $\mathbf{q}_j$ together with the $\mathbf{p}_j$ are the elements we need to adequately describe $\mathcal{P}$ and $\operatorname{loc}\mathcal{P}$ in the coming arguments.

We also introduce a subideal of $\mathcal{P}$ that will play a role in what follows. This is the ideal $\mathcal{I}$ of the following lemma.

4.5.3 Lemma. *Let $\mathcal{I}$ be the ideal in ${}_n\mathcal{H}_0$ generated by $\mathbf{p}_{m+1}, \mathbf{q}_{m+2}, \ldots, \mathbf{q}_n$. Let $\mathbf{d}$ be the discriminant of $\mathbf{p}_{m+1}$, and let $\mathbf{D}$ be the locus of the ideal in ${}_n\mathcal{H}_0$ generated by $\mathbf{d}$. Then there is an integer μ such that*

$$\mathbf{d}^\mu \cdot \mathcal{P} \subset \mathcal{I} \subset \mathcal{P} \quad \textit{and} \quad \operatorname{loc} P \subset \operatorname{loc}\mathcal{I} \subset \operatorname{loc}\mathcal{P} \cup \mathbf{D}.$$

Proof. The equation $\mathbf{d} \cdot \mathbf{z}_j = \mathbf{q}_j + \mathbf{s}_j$ implies that $\mathbf{d} \cdot \mathbf{z}_j = \mathbf{s}_j \bmod \mathcal{I}$, with $\mathbf{s}_j \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$. Then, for every polynomial $\mathbf{p} \in {}_m\mathcal{H}_0[\mathbf{z}_j]$, we have $\mathbf{d}^\nu \cdot \mathbf{p} = \mathbf{r} \bmod \mathcal{I}$ for some $\mathbf{r} \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$, where $\nu = \deg \mathbf{p}$. If $\mathbf{p} \in \mathcal{P}$, then $\mathbf{r} \in \mathcal{P}$. By the Weierstrass division theorem, $\mathbf{r} = \mathbf{h}\mathbf{p}_{m+1} + \mathbf{r}_0$ for some $\mathbf{h} \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$ and $\mathbf{r}_0 \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$ of degree less than $\deg \mathbf{p}_{m+1}$. Since $\mathbf{r}_0 \in \mathcal{P} \cap {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$, and $\mathbf{p}_{m+1}$ is the minimal polynomial in this ideal, we conclude that $\mathbf{r}_0 = 0$ and, hence, that $\mathbf{r}$ and $\mathbf{d}^\nu \cdot \mathbf{p}$ belong to $\mathcal{I}$. If we apply this result to each of the polynomials $\mathbf{p}_j$, we conclude that $\mathbf{d}^{\nu_j} \cdot \mathbf{p}_j \in \mathcal{I}$, for $j = m+1, \ldots, n$, where $\nu_j = \deg \mathbf{p}_j$.

If $n = m$, then $\mathcal{P}$ is the zero ideal and there is nothing to prove. Thus, we assume $n \geq m+1$. We will prove by induction on $j \geq m+1$ that, if $\mathbf{f} \in \mathcal{P} \cap {}_j\mathcal{H}_0$, then $\mathbf{d}^{\rho_j} \cdot \mathbf{f} \in \mathcal{I}$, where

$$\rho_j = \sum_{i=m+2}^{j} \nu_i.$$

If $j = m+1$, then the Weierstrass division theorem, with $\mathbf{p}_{m+1}$ as divisor, implies that $\mathbf{f} = \mathbf{g}_{m+1}\mathbf{p}_{m+1} + \mathbf{r}_{m+1}$, with $\mathbf{r}_{m+1} \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$ of degree less than the degree of $\mathbf{p}_{m+1}$. Since $\mathbf{f}$ and $\mathbf{p}_{m+1}$ are both in $\mathcal{P}$, the polynomial $\mathbf{r}_{m+1}$ is as well. Since $\mathbf{p}_{m+1}$ is the minimal polynomial in $\mathcal{P} \cap {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$, we conclude that $\mathbf{r}_{m+1} = 0$. Hence, $\mathbf{f} \in \mathcal{I}$ and the claim is true for $j = m+1$. Suppose $k > m+1$ and the claim is true for $j < k$. The polynomial $\mathbf{p}_k$ is a Weierstrass polynomial, and so the Weierstrass division theorem allows us to write $\mathbf{f} = \mathbf{p}_k \mathbf{g}_k + \mathbf{r}_k$, for some $\mathbf{g}_k \in {}_k\mathcal{H}_0$ and some $\mathbf{r}_k \in {}_{k-1}\mathcal{H}_0[\mathbf{z}_k]$ of degree less than $\nu_k = \deg \mathbf{p}_k$. Thus, $\mathbf{d}^{\nu_k}\mathbf{f} = \mathbf{d}^{\nu_k}\mathbf{r}_k \bmod \mathcal{I}$. Furthermore, $\mathbf{d} \cdot \mathbf{z}_k = \mathbf{s}_k \bmod \mathcal{I}$ with $\mathbf{s}_k \in {}_m\mathcal{H}_0[\mathbf{z}_{m+1}]$. It follows that $\mathbf{d}^{\nu_k} \cdot \mathbf{r}_k$ is equal to an element of $\mathcal{P} \cap {}_{k-1}\mathcal{H}_0 \bmod \mathcal{I}$. Then the same thing is true of $\mathbf{d}^{\nu_k} \cdot \mathbf{f}$. By the induction assumption, $\mathbf{d}^{\rho_k} \cdot \mathbf{f} = \mathbf{d}^{\rho_{k-1}} \cdot \mathbf{d}^{\nu_k} \cdot \mathbf{f} \in \mathcal{I}$. This completes the induction.

Thus, we have proved that $\mathbf{d}^{\mu} \cdot \mathcal{P} \subset \mathcal{I} \subset \mathcal{P}$, where $\mu = \sum \deg \mathbf{p}_j$. It follows immediately that $\operatorname{loc}\mathcal{P} \subset \operatorname{loc}\mathcal{I} \subset \operatorname{loc}\mathcal{P} \cup \mathbf{D}$.

4.5.4 Theorem. *Let $\mathcal{P}$ be a prime ideal of $\mathcal{H}_0$. Suppose coordinates for $\mathbb{C}^n$ and an integer m have been chosen so that ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0$. Then $\mathbf{V} = \operatorname{loc}\mathcal{P}$ is a union $\mathbf{V} = \mathbf{V}' \cup \mathbf{V}''$ of germs of subvarieties such that $\pi : \mathbb{C}^n \to \mathbb{C}^m$ exhibits $\mathbf{V}'$ as the germ of a finite branched holomorphic cover of pure order of a neighborhood of 0 in $\mathbb{C}^m$.*

Proof. According to Theorem 4.5.2, by making a linear change of variables involving only $z_{m+1}, \dots, z_n$, we may arrange that the image of z_{m+1} in ${}_n\mathcal{H}_0/\mathcal{P}$ generates the quotient field of this algebra over the quotient field of ${}_m\mathcal{H}_0$. We may then choose polynomials $\mathbf{p}_{m+1}, \dots, \mathbf{p}_n, \mathbf{q}_{m+2}, \dots, \mathbf{q}_n$, as in Lemma 4.5.3. Recall that $\mathbf{d}$ denotes the discriminant of $\mathbf{p}_{m+1}$. The polynomials $\mathbf{p}_j$ and $\mathbf{q}_j$ have coefficients which are germs in ${}_m\mathcal{H}_0$. We choose a polydisc $\Delta \subset \mathbb{C}^m$, centered at 0, on which these germs have representatives and let $p_{m+1}, \dots, p_n, q_{m+2}, \dots, q_n$ be the corresponding polynomials with coefficients in $\mathcal{H}(\Delta)$. Then the discriminant d of p_{m+1} is a representative in $\mathcal{H}(\Delta)$ of $\mathbf{d}$.

Let $U = \pi^{-1}(\Delta)$, where $\pi : \mathbb{C}^n \to \mathbb{C}^m$ is the projection. We obtain a finite branched holomorphic cover by applying Proposition 4.4.4 to the set

$$V_1 = \{z = (z_1, \dots, z_n) \in U : p_{m+1}(\pi(z), z_{m+1}) = \cdots = p_n(\pi(z), z_n) = 0\}$$

and the map $\pi : V_1 \to \Delta$ obtained by restricting $\pi : U \to \Delta$ to V_1. Since each p_j is a Weierstrass polynomial in z_j with coefficients in $\mathcal{H}(\Delta)$, Proposition 3.3.1 implies that, for each $\epsilon > 0$, we may choose Δ small enough that, for each $z' \in \Delta$, all roots of $p_j(z', z_j)$ are in the disc on which $|z_j| < \epsilon$. It follows that we can choose Δ small enough to ensure that V_1 is contained in a given neighborhood of 0 in $\mathbb{C}^n$. In particular, we may choose Δ small enough that V_1 is contained in a neighborhood in which each element of a finite set of generators for $\mathcal{P}$, containing $\mathbf{p}_{m+1}, \dots, \mathbf{p}_n, \mathbf{q}_{m+2}, \dots, \mathbf{q}_n$, has a representative. If V is the variety of common zeroes of this set, then V is a subvariety of V_1 and the germ of V at 0 is $\mathbf{V} = \operatorname{loc}\mathcal{P}$.

We let $E \subset \Delta$ be the zero set of the discriminant d of $p_{m+1}, \dots, p_n$ and set $\Delta_0 = \Delta - E$, $U_0 = \pi^{-1}(\Delta_0)$, and $V_0 = V \cap U_0$.

It is clear that $\pi : V \to \Delta$ is a finite to one proper holomorphic map, because these things are true of $\pi : V_1 \to \Delta$, and V is a subvariety of V_1. We next show that $\pi : V_0 \to \Delta_0$ is a holomorphic covering map of pure order r. To this end, we define a subvariety W of $U \cap \mathbb{C}^{m+1}$ by setting

$$W = \{(z', z_{m+1}) \in U \cap \mathbb{C}^{m+1} : p_{m+1}(z', z_{m+1}) = 0\}.$$

By Proposition 4.4.4, the projection $\mathbb{C}^{m+1} \to \mathbb{C}^m$ induces a finite branched holomorphic cover $\rho : W \to \Delta$. Since p_{m+1} is a Weierstrass polynomial of order r, this cover is necessarily of pure order r, by Proposition 3.3.1. Also, if $W_0 = W \cap U_0$, then W_0 is an open subset of W and $\rho : W_0 \to \Delta_0$ is a holomorphic covering map, since the discriminant d of p_{m+1} is non-vanishing on Δ_0.

Lemma 4.5.3 implies that $\operatorname{loc}\mathcal{P} \subset \operatorname{loc}\mathcal{I} \subset \operatorname{loc}\mathcal{P} \cup \mathbf{D}$, where $\mathcal{I}$ is the ideal generated by $\mathbf{p}_{m+1}, \mathbf{q}_{m+2}, \dots, \mathbf{q}_n$ and $\mathbf{D}$ is the germ of $D = \pi^{-1}(E)$. Consequently,

$$\mathbf{V} \subset \mathbf{V}_1 \subset \mathbf{V} \cup \mathbf{D}.$$

Thus, Δ may be chosen so that $(z', z_{m+1}, \dots, z_n) \in U$ is in V_0 exactly when

$$\begin{aligned} &z' \in \Delta_0, \\ &p_{m+1}(z', z_{m+1}) = 0, \text{ and for } j = m+2, \dots, n, \\ &q_j(z', z_{m+1}, z_j) = d(z')z_j - s_j(z', z_{m+1}) = 0. \end{aligned} \tag{4.5.1}$$

Since d is non-vanishing on Δ_0, the projection $\mathbb{C}^n \to \mathbb{C}^{m+1}$ yields a biholomorphic mapping $V_0 \to W_0$, with inverse given by the equations

$$z_j = d(z')^{-1} s_j(z', z_{m+1}), \ j = m+2, \dots, n.$$

Thus, $\pi : V_0 \to \Delta_0$ is a finite holomorphic covering map of degree r, since it factors as $V_0 \to W_0$ followed by $W_0 \to \Delta_0$.

Now π is a proper map on $\overline{V}_0$, since it is proper on V_1. Since V_0 is a subvariety of $U_0 = \pi^{-1}(\Delta_0)$ and $\pi : V_0 \to \Delta_0$ is a finite holomorphic covering map, it follows from Proposition 4.4.6 that $\overline{V}_0$ is a subvariety of U. Since V is closed in U, $V' = \overline{V}_0$ is also a subvariety of V. Since Δ_0 is dense in Δ and V_0 is dense in V', $\pi : V' \to \Delta$ is a finite branched holomorphic cover with $\pi : V_0 \to \Delta_0$ as a dense regular subcover. Clearly V is the union of the two subvarieties V' and $V'' = V \cap D$. This completes the proof.

With Theorem 4.5.4 in hand, the proof of the Nullstellensatz for $\mathcal{H}_0$ proceeds in much the same way as the proof of Hilbert's Nullstellensatz (Theorem 4.3.5).

4.5.5 Theorem (Rückert's Nullstellensatz). *If $\mathcal{I}$ is an ideal of ${}_n\mathcal{H}_0$, then* $\operatorname{id}\operatorname{loc}\mathcal{I} = \sqrt{\mathcal{I}}$.

Proof. By Proposition 4.1.1, we need only prove that $\operatorname{id}\operatorname{loc}\mathcal{P} \subset \mathcal{P}$ if $\mathcal{P}$ is a prime ideal of ${}_n\mathcal{H}_0$. Suppose $\mathbf{f} \in {}_n\mathcal{H}_0$, $\mathbf{f} \notin \mathcal{P}$, and coordinates have been chosen so that ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0$. Then the image of $\mathbf{f}$ in ${}_n\mathcal{H}_0/\mathcal{P}$ satisfies a minimal polynomial equation over ${}_m\mathcal{H}_0$. That is, there are elements $\mathbf{a}_0, \dots, \mathbf{a}_{k-1}$ in ${}_m\mathcal{H}_0$ such that

$$\mathbf{p}(\mathbf{f}) = \mathbf{f}^k + \mathbf{a}_{k-1}\mathbf{f}^{k-1} + \cdots + \mathbf{a}_1\mathbf{f} + \mathbf{a}_0 \in \mathcal{P},$$

and no polynomial in $\mathbf{f}$ of degree less than k has this property. Since $\mathcal{P}$ is prime, $\mathbf{a}_0 \neq 0$; otherwise, $\mathbf{p}(\mathbf{f})$ would factor as $\mathbf{f}$ times a lower degree polynomial and neither factor would belong to $\mathcal{P}$.

We choose a polydisc $\Delta \subset \mathbb{C}^m$ on which each $\mathbf{a}_i$ has a representative a_i. If $\pi : \mathbb{C}^n \to \mathbb{C}^m$ is the usual projection, Theorem 4.5.4 implies that we may choose Δ small enough that $\operatorname{loc}\mathcal{P}$ has a representative $V \subset \pi^{-1}(\Delta)$, and V is the union of two subvarieties V' and V'', with $\pi : V' \to \Delta$ a finite branched holomorphic cover. Let f be a representative of $\mathbf{f}$ in some neighborhood $U \subset \pi^{-1}(\Delta)$ of 0 in $\mathbb{C}^n$. Then U contains points $z \in V'$, arbitrarily close to 0, at which

$$f^k + a_{k-1}f^{k-1} + \cdots + a_1 f + a_0 = 0$$

and $a_0(\pi(z)) \neq 0$. Let z be such a point. Clearly f cannot vanish at z, since this would force a_0 to vanish at $\pi(z)$. It follows that $\mathbf{f}$ does not vanish on the germ of V, which is $\operatorname{loc}\mathcal{P}$, and so $\mathbf{f} \notin \operatorname{id}\operatorname{loc}\mathcal{P}$. This completes the proof.

In view of Proposition 3.6.1, the following corollary is a direct result of Rückert's Nullstellensatz:

4.5.6 Corollary. *Let $\mathcal{I}$ be an ideal of $\mathcal{H}_0$. Then $\sqrt{\mathrm{I}}$ is prime if and only if* $\operatorname{loc}\mathcal{P}$ *is an irreducible germ of a variety.*

The next theorem summarizes information on the local structure of irreducible varieties, contained in Theorems 4.5.2 and 4.5.4, in the form we will be using in later sections.

4.5.7 Local Paramaterization Theorem. *If $\mathbf{V}$ is a germ of an irreducible variety at 0 in $\mathbb{C}^n$, then there is a choice of coordinates in $\mathbb{C}^n$ and an $m \leq n$ such that ${}_V\mathcal{H}_0$ is a finite extension of ${}_m\mathcal{H}_0$. With this choice of coordinates:*

(i) *there is an irreducible representative V of $\mathbf{V}$, in a neighborhood of 0, such that the projection $\pi : \mathbb{C}^n \to \mathbb{C}^m$ expresses V as a finite branched holomorphic cover of a polydisc $\Delta \subset \mathbb{C}^m$;*

(ii) *the induced morphism $\mathcal{H}(\Delta) \to \mathcal{H}(V)$ expresses the ring $\mathcal{H}(V)$ as a finite extension of the ring $\mathcal{H}(\Delta)$;*

(iii) *after a linear change of coordinates involving only $z_{m+1}, \ldots, z_n$, the coordinate z_{m+1} generates the quotient field of $\mathcal{H}(V)$ over that of $\mathcal{H}(\Delta)$;*

(iv) *with coordinates chosen as in (iii), $\pi : V \to \Delta$ has a dense regular subcover $\pi : V_0 \to \Delta_0$, where V_0 is the subset of $\pi^{-1}(\Delta)$ defined by (4.5.1).*

Proof. The first statement is Theorem 4.5.2 with $\mathcal{P} = \mathrm{id}\,\mathbf{V}$. Since $\mathbf{V}$ is $\mathrm{loc}\,\mathcal{P}$, (i) is a result of Theorem 4.5.4 and the observation that the $\mathbf{V}'$ of that theorem must, in fact, be $\mathbf{V}$, since $\mathbf{V}$ is irreducible and $\mathbf{V}''$ cannot be V. That the representative V can be chosen to be irreducible is proved as follows: If V_0 is the dense subset of V on which π is a holomorphic covering map, then only one connected component of V_0 can have 0 in its closure, otherwise $\mathbf{V}$ would fail to be irreducible, since the closure of each component is a subvariety, by Corollary 4.4.7(iii). The closure of that one component of V_0 is then a representative of $\mathbf{V}$. It is irreducible, by the result of Exercise 4.9.

Part (ii) follows from Corollary 4.4.7(ii) while part (iii) follows from an application of the theorem of the primitive element (Theorem 4.2.1), as in the proof of Theorem 4.5.2. Part (iv) follows from the proof of Theorem 4.5.4.

A *thin subset* W, of a subvariety V, is a set which is contained in a subvariety of V with dense complement. Clearly, the union of finitely many thin sets is thin.

4.5.8 Corollary. *For any subvariety V of a domain in $\mathbb{C}^n$, the set of singular points is a thin set.*

Proof. This is a local result; it suffices to prove it for the intersection of V with a sufficiently small neighborhood of each point of V. Given a point $p \in V$, we decompose the germ at p of the variety V into a union of irreducible germs of subvarieties. Then, if we replace V by its intersection with a sufficiently small neighborhood Δ of p, we will have $V = \bigcup_i V_i$, where each V_i is a subvariety with irreducible germ at p. Each singular point of V is clearly either in the union of the singular sets of the V_i or the union of the intersection sets $V_i \cap V_j$ for $i \neq j$.

For each i, if Δ is chosen small enough, Theorem 4.5.7 implies that V_i will be a finite branched holomorphic cover of a polydisc in $\mathbb{C}^m$ for some m. If V_{i0} is the regular part of V_i for the covering map (the complement of the branch locus), then V_{i0} is dense in V_i, and the singular set of V_i is contained in the branch locus of the cover. This is a subvariety, by Corollary 4.4.8. Thus, the singular sets of the V_i are all thin subsets of V.

It remains to prove that each intersection set $V_i \cap V_j$ for $i \neq j$ is a thin subset of V. For a fixed i and each $j \neq i$, the set $V_i \cap V_j$ is a proper subvariety of V_i which, necessarily, meets V_{i0} in a proper subvariety of V_{i0}. However, V_{i0} is locally biholomorphic to a polydisc in $\mathbb{C}^m$, and so the complement of a proper subvariety is dense. Since V_{i0} is a dense open subset of V_i, it follows that the complement of $V_i \cap V_j$ in V_i is also dense in V_i. We conclude that the complement of $V_i \cap V_j$ in V is dense in V. Thus, each intersection set is thin, and the proof is complete.

4.6 Morphisms of Germs of Varieties

Using the Nullstellensatz and Theorem 4.5.7, we can now state and prove some very useful results concerning morphisms between germs of holomorphic varieties. For the purposes of this discussion, we may assume that a germ $\mathbf{V}$ of a variety is, for some n, the germ at 0 of a subvariety V of a neighborhood of 0 in $\mathbb{C}^n$. The local ring of germs of holomorphic functions associated with a germ $\mathbf{V}$ of a variety will be denoted ${}_V\mathcal{H}_0$. By a *morphism* $\mathbf{V} \to \mathbf{W}$, between germs of holomorphic varieties, we mean the germ of a holomorphic map $V \to W$, between representatives of $\mathbf{V}$ and $\mathbf{W}$, which maps 0 to 0. Recall that, by Proposition 3.8.4 and Theorem 3.8.5, each morphism of germs of varieties $\pi : \mathbf{V} \to \mathbf{W}$ induces a ring homomorphism $\phi : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$, and, conversely, each ring homomorphism $\phi : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$ is induced by an underlying morphism of germs of varieties.

We will say that a morphism $\pi : \mathbf{V} \to \mathbf{W}$ is a *surjective morphism of germs of varieties* if there are representatives V, W, and $\pi : V \to W$ of $\mathbf{V}$, $\mathbf{W}$, and π so that the image of each neighborhood of 0 in V, under π, contains a neighborhood of 0 in W. Similarly, a morphism of varieties will be called *injective* if there are representatives, as above, so that π is injective on some neighborhood of 0 in V.

If $\pi : \mathbf{V} \to \mathbf{W}$ is a morphism of germs of varieties and $\mathbf{A}$ is the germ of a subvariety of $\mathbf{W}$, then $\pi^{-1}(\mathbf{A})$ is the germ of a subvariety $\pi^{-1}(A)$, where A is a subvariety of W, A and W are representatives of $\mathbf{A}$ and $\mathbf{W}$, and $\pi : V \to W$ is a representative of $\pi : \mathbf{V} \to \mathbf{W}$. Since $f \circ \pi$ is a holomorphic function on V, vanishing on $\pi^{-1}(A)$, whenever f is a holomorphic function on W, vanishing on A, it is clear that the inverse image of a germ of a subvariety under a holomorphic map is, indeed, a germ of a subvariety.

A *finite morphism* of germs of varieties is a morphism $\pi : \mathbf{V} \to \mathbf{W}$ such that $\pi^{-1}(\mathbf{0}) = \mathbf{0}$. The main objective of this section is to show that a morphism $\pi : \mathbf{V} \to \mathbf{W}$ is finite if and only if the induced ring homomorphism

$\pi^* : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$ is finite. First, we note the following simple consequences of the definitions.

4.6.1 Proposition. *Let $\pi : \mathbf{V} \to \mathbf{W}$ be a morphism of germs of varieties and let $\mathbf{V} = \mathbf{V}_i \cup \cdots \cup \mathbf{V}_k$ be the decomposition of $\mathbf{V}$ into irreducible subvarieties. Then*
(i) *π is finite on $\mathbf{V}$ if and only if it is finite on each $\mathbf{V}_i$;*
(ii) *π is surjective if its restriction to some $\mathbf{V}_i$ is surjective.*

Clearly the classes of injective, surjective, and finite morphisms of germs of varieties are each closed under composition.

If $\pi : \mathbf{V} \to \mathbf{W}$ is a morphism of varieties, we have no reason to believe that the image $\pi(\mathbf{V})$ makes sense. That is, there is no reason, in general, to believe that there is a germ of a subvariety $\mathbf{Y} \subset \mathbf{W}$ so that π factors as a surjective morphism $\mathbf{V} \to \mathbf{Y}$, followed by the inclusion $\mathbf{Y} \to \mathbf{W}$. If this does happen, however, then we shall say that the image $\pi(\mathbf{V})$ exists as a subvariety of $\mathbf{W}$. The next proposition gives an important case in which the image of a morphism does exist. We use the symbol $\mathbf{C^m}$ to denote the germ of $\mathbb{C}^m$ at 0.

4.6.2 Proposition. *Let $m \le n$ be non-negative integers and $\mathcal{P} \subset {}_n\mathcal{H}_0$ a prime ideal such that ${}_n\mathcal{H}_0/\mathcal{P}$ is finite over ${}_m\mathcal{H}_0$. Then there is a germ of a holomorphic subvariety $\mathbf{W} \subset \mathbf{C^m}$ such that $\pi : \operatorname{loc}\mathcal{P} \to \mathbf{C^m}$ factors as a germ of a finite branched holomorphic cover $\operatorname{loc}\mathcal{P} \to \mathbf{W}$ followed by the inclusion $\mathbf{W} \to \mathbf{C^m}$.*

Proof. By assumption, ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_m\mathcal{H}_0/(\mathcal{P} \cap {}_m\mathcal{H}_0)$. Since $\mathcal{P} \cap {}_m\mathcal{H}_0$ is a prime ideal of ${}_m\mathcal{H}_0$, Theorem 4.5.2 implies that we may choose coordinates in $\mathbb{C}^m$ and an integer k so that ${}_m\mathcal{H}_0/(\mathcal{P} \cap {}_m\mathcal{H}_0)$ is a finite extension of ${}_k\mathcal{H}_0$. Then ${}_n\mathcal{H}_0/\mathcal{P}$ is also a finite extension of ${}_k\mathcal{H}_0$.

Let $\mathbf{V} = \operatorname{loc} P$ and $\mathbf{Y} = \operatorname{loc}(\mathcal{P} \cap {}_m\mathcal{H}_0)$. The projections $\mathbb{C}^n \to \mathbb{C}^k$, $\mathbb{C}^n \to \mathbb{C}^m$, and $\mathbb{C}^m \to \mathbb{C}^k$ induce germs $\pi' : \mathbf{V} \to \mathbf{C^k}$, $\pi : \mathbf{V} \to \mathbf{Y}$, and $\pi'' : \mathbf{Y} \to \mathbf{C^k}$ of holomorphic mappings. By Theorem 4.5.7, π' and π'' are the germs of finite branched holomorphic covers of a neighborhood of 0 in $\mathbb{C}^k$. Clearly, $\pi' = \pi'' \circ \pi$.

Choose representatives U, V, and Y of $\mathbf{C^k}$, $\mathbf{V}$, and $\mathbf{Y}$ so that $\pi' : V \to U$ and $\pi'' : Y \to U$ are finite branched holomorphic covers of the connected open set $U \subset \mathbb{C}^k$, $\pi(V) \subset Y$ and $\pi' = \pi'' \circ \pi$ holds on V. For a suitable proper subvariety $D \subset U$ the sets $V_0 = \pi'^{-1}(U - D)$ and $Y_0 = \pi''^{-1}(U - D)$ are sets on which π' and π'' are finite holomorphic covering maps. Set $W_0 = \pi(V_0)$. Since $\pi' = \pi'' \circ \pi$, it is clear that $\pi : V \to Y$ is a finite to one proper holomorphic mapping and $\pi : V_0 \to W_0$ is a finite holomorphic covering map. The set W_0 is, therefore, the union of some connected components of

Y_0. Its closure W in Y is a subvariety by Proposition 4.4.6. Since V_0 is dense in V, and π is a proper, continuous mapping, it follows that the closure W of W_0 in Y is $\pi(V)$. Thus, W is a subvariety, and $\pi : V \to W$ is a finite branched holomorphic cover. This completes the proof.

4.6.3 Theorem. *Let $\mathcal{I}$ be an ideal of ${}_n\mathcal{H}_0$ and set $\mathbf{V} = \operatorname{loc}\mathcal{I}$. Then, for a choice of coordinates in $\mathbb{C}^n$ and an integer $m \leq n$, the following three conditions are equivalent:*

(i) *if $\mathbf{L}$ is the germ of $\{z \in \mathbb{C}^n : z_1 = \cdots = z_m = 0\}$, then $\mathbf{L} \cap \mathbf{V} = \{\mathbf{0}\}$;*
(ii) *${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_m\mathcal{H}_0/(\mathcal{I} \cap {}_m\mathcal{H}_0)$;*
(iii) *for each irreducible component $\mathbf{V}_i$ of $\mathbf{V}$, the image $\pi(\mathbf{V}_i)$ exists as a germ of a subvariety contained in $\mathbf{C}^\mathbf{m}$ and the projection $\mathbb{C}^n \to \mathbb{C}^m$ induces the germ $\pi_i : \mathbf{V}_i \to \pi(\mathbf{V}_i)$ of a finite branched holomorphic cover.*

Proof. We first show (ii) implies (iii). Assume (ii) holds, let $\mathbf{V}_1, \dots, \mathbf{V}_r$ be the irreducible components of $\mathbf{V}$, and set $\mathcal{P}_i = \operatorname{id}\mathbf{V}_i$. Then each $\mathcal{P}_i$ is a prime ideal containing $\mathcal{I}$. Since ${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_m\mathcal{H}_0/(\mathcal{I} \cap {}_m\mathcal{H}_0)$, it follows that ${}_n\mathcal{H}_0/\mathcal{P}_i$ is a finite extension of ${}_m\mathcal{H}_0/(\mathcal{P}_i \cap {}_m\mathcal{H}_0)$ for each i. Then (iii) follows from the previous proposition applied to each $\mathcal{P}_i$.

Now suppose condition (iii) is satisfied. Let $\mathbf{V}_1, \dots, \mathbf{V}_r$ be the irreducible components of $\mathbf{V}$, as before. The projection $\mathbb{C}^n \to \mathbb{C}^m$ induces the germ $\pi_i : \mathbf{V}_i \to \mathbf{W}_i$ of a finite branched holomorphic cover, with $\mathbf{W}_i = \pi(\mathbf{V}_i)$ the germ of a subvariety of a neighborhood of 0 in $\mathbb{C}^m$. If $\mathbf{L}$ is the germ of $\{z \in \mathbb{C}^n : z_1 = \cdots = z_m = 0\}$, then $\mathbf{L} \cap \mathbf{V}_i = \pi_i^{-1}(\mathbf{0})$, which must be the germ of a finite set containing 0, since π_i is finite to one. But the germ at 0 of a finite set containing 0 is $\{\mathbf{0}\}$. Thus, (iii) implies (i).

Finally, we show that (i) implies (ii). Suppose $\mathcal{I}$ satisfies condition (i) for some $m < n$. This means that the ideal $\mathcal{J}$ generated by $\mathcal{I}$ and $\mathbf{z}_1, \dots, \mathbf{z}_m$ satisfies $\operatorname{loc}\mathcal{J} = \{\mathbf{0}\}$. Since $\mathbf{z}_n$ vanishes at 0, it belongs to $\operatorname{id}\operatorname{loc}\mathcal{J}$, which is $\sqrt{\mathcal{J}}$, by the Nullstellensatz. This implies that $\mathbf{z}_n^\nu \in \mathcal{J}$, for some ν. In other words,

$$\mathbf{z}_n^\nu = \mathbf{f}_n + \mathbf{z}_1\mathbf{g}_1 + \cdots + \mathbf{z}_m\mathbf{g}_m,$$

for some $\mathbf{f}_n \in \mathcal{I}$ and $\mathbf{g}_1, \dots, \mathbf{g}_m \in {}_n\mathcal{H}_0$. Thus, there is an $\mathbf{f}_n \in \mathcal{I}$ which has finite vanishing order in $\mathbf{z}_n$. It follows from Proposition 4.5.1 that ${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_{n-1}\mathcal{H}_0/(\mathcal{I} \cap {}_{n-1}\mathcal{H}_0)$. The rest of the proof amounts to iterating this procedure as long as we can.

In order to carry out the next step, we must prove that if $\mathcal{I} \subset {}_n\mathcal{H}_0$ satisfies (i), for some $m < n$, then the ideal $\mathcal{I}_{n-1} = \mathcal{I} \cap {}_{n-1}\mathcal{H}_0$ satisfies (i) in $\mathbb{C}^{n-1}$, for the same m. Since we have just showed that ${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_{n-1}\mathcal{H}_0/\mathcal{I}_{n-1}$ and we know that (ii) implies (iii), we conclude that the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ maps $\mathbf{V}$ to the germ of a variety $\mathbf{V}_{n-1} \subset \mathbf{C}^{\mathbf{n-1}}$. Condition (i) for $\mathcal{I}$ implies that $\mathbf{L} \cap \mathbf{V}_{n-1} = \mathbf{0}$. Clearly $\mathcal{I}_{n-1} \subset \operatorname{id} \mathbf{V}_{n-1}$. But if $\mathbf{f} \in \operatorname{id} \mathbf{V}_{n-1}$, then, when $\mathbf{f}$ is considered as an element of ${}_n\mathcal{H}_0$, constant in $\mathbf{z}_n$, it vanishes on $\mathbf{V}$ and, by the Nullstellensatz, some power of it belongs to $\mathcal{I}$ and, hence, to $\mathcal{I}_{n-1}$. It follows that $\mathcal{I}_{n-1} \subset \operatorname{id} \mathbf{V}_{n-1} \subset \sqrt{\mathcal{I}_{n-1}}$ and from this that $\operatorname{loc} \mathcal{I}_{n-1} = \mathbf{V}_{n-1}$. Thus, $\mathcal{I}_{n-1}$ satisfies condition (i), as desired. Either $m = n-1$ or we may now conclude, as above, that ${}_{n-1}\mathcal{H}_0/\mathcal{I}_{n-1}$ is a finite extension of ${}_{n-2}\mathcal{H}_0/\mathcal{I}_{n-2}$ and, hence, that ${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_{n-2}\mathcal{H}_0/\mathcal{I}_{n-2}$. Clearly we can repeat this procedure $n-m$ times and finally show that ${}_n\mathcal{H}_0/\mathcal{I}$ is a finite extension of ${}_m\mathcal{H}_0/\mathcal{I}_m = {}_m\mathcal{H}_0/(\mathcal{I} \cap {}_m\mathcal{H}_0)$. This completes the proof.

4.6.4 Corollary. *For the germ $\pi : \mathbf{V} \to \mathbf{W}$ of a holomorphic mapping between two germs of holomorphic varieties, the following are equivalent:*

(i) *$\pi : \mathbf{V} \to \mathbf{W}$ is finite;*

(ii) *${}_V\mathcal{H}_0$ is finite over ${}_W\mathcal{H}_0$ under $\pi^* : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$;*

(iii) *for each irreducible component $\mathbf{V}_i$ of $\mathbf{V}$, the image $\pi(\mathbf{V}_i)$ of $\mathbf{V}_i$ exists as the germ of a subvariety of $\mathbf{W}$, and $\pi : \mathbf{V}_i \to \pi(\mathbf{V}_i)$ is the germ of a finite branched holomorphic cover.*

Proof. We may represent $\mathbf{V}$ and $\mathbf{W}$ by subvarieties of neighborhoods in $\mathbb{C}^n$ and $\mathbb{C}^m$ $(m \le n)$ in such a way that the projection $\mathbb{C}^n \to \mathbb{C}^m$ induces π, by Exercise 4.8. Then $\pi^{-1}(\mathbf{0}) = \mathbf{L} \cap \mathbf{V}$, where $\mathbf{L}$ is the germ of the subvariety $\{z \in \mathbb{C}^n : z_1 = \cdots = z_m = 0\}$. Thus, π is finite if and only if $\mathbf{L} \cap \mathbf{V} = \mathbf{0}$. The equivalences of the corollary are then just the equivalences of Theorem 4.6.3, with $\mathcal{I} = \operatorname{id} \mathbf{V}$.

4.6.5 Corollary. *Let $\pi : \mathbf{V} \to \mathbf{W}$ be a morphism of germs of varieties. If π is surjective, then the induced ring homomorphism $\pi^* : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$ is injective. Conversely, if π is finite and π^* is injective, then π is surjective.*

Proof. If π is surjective, then $\pi^*(\mathbf{f}) = \mathbf{f} \circ \pi \ne 0$ if $0 \ne \mathbf{f} \in {}_W\mathcal{H}_0$ and, thus, π^* is injective. On the other hand, if π is finite, then its image exists as a germ of a subvariety of $\mathbf{W}$, by Corollary 4.6.4. If this germ of a subvariety is not $\mathbf{W}$ itself, then some non-zero $\mathbf{f} \in {}_W\mathcal{H}_0$ vanishes on it. Then $\pi^*(\mathbf{f}) = 0$ and π^* is not injective.

The next corollary is an immediate consequence of Corollary 4.6.4,

4.6.6 Corollary. *If $\mathbf{V}$ is an irreducible germ of a variety, then $\pi : \mathbf{V} \to \mathbf{W}$ is the germ of a finite branched holomorphic cover if and only if π is finite and surjective.*

4.6.7 Corollary. *If* $\mathbf{V}$ *is the germ of an irreducible holomorphic variety, then, for a unique integer* m, *there exists a finite surjective morphism of germs of varieties* $\pi : \mathbf{V} \to \mathbf{C^m}$.

Proof. This follows from Theorem 4.5.2, applied to the prime ideal $\operatorname{id}\mathbf{V}$, and Corollary 4.6.6.

A bijective morphism between germs of varieties is one which is both injective and surjective. One might hope that a bijective morphism is an isomorphism. This is not true in general (Exercise 4.22), but it is true if the target variety is normal, where a germ of a holomorphic variety $\mathbf{W}$ is called *normal* if its local ring ${}_W\mathcal{H}_0$ is a normal domain. A normal germ of a variety is necessarily irreducible, since its local ring is an integral domain. A point on a subvariety is called a *normal point* if the germ of the variety at that point is normal.

4.6.8 Theorem. *If* $\pi : \mathbf{V} \to \mathbf{W}$ *is a bijective morphism of germs of varieties and* $\mathbf{W}$ *is normal, then* π *is an isomorphism.*

Proof. The induced morphism $\pi^* : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$ is injective. To prove the theorem, we must prove that it is also surjective. Thus, let $\mathbf{f}$ be a germ in ${}_V\mathcal{H}_0$. The morphism π is finite, and so $\mathbf{f}$ is integral over ${}_W\mathcal{H}_0$. Let $\mathbf{p} \in {}_W\mathcal{H}_0[z]$ be the minimal polynomial for f. Note that, since ${}_W\mathcal{H}_0$ is normal, $\mathbf{p}$ is a monic polynomial which is of minimal degree among all polynomials in ${}_W\mathcal{H}_0[z]$, monic or not, for which $\mathbf{f}$ is a root (see the comment following Theorem 4.2.8). We choose representatives V, W, and $\pi : V \to W$ for $\mathbf{V}$, $\mathbf{W}$, and $\pi : \mathbf{V} \to \mathbf{W}$ such that π is bijective. We may assume W is a subvariety of a neighborhood U of 0 in $\mathbb{C}^n$ on which $\mathbf{p}$ has a representative $p \in \mathcal{H}(W)[z]$ which extends to an element $\tilde{p} \in \mathcal{H}(U)[z]$.

The germ $\tilde{\mathbf{p}}$ of $\tilde{p}$ at 0 is monic and has only 0 as a root at the origin, so it is a Weierstrass polynomial. The Weierstrass division theorem implies that, mod $\tilde{\mathbf{p}}$, every element of ${}_{n+1}\mathcal{H}_0$ is a polynomial in ${}_n\mathcal{H}_0[z]$ of degree less than $k = \deg \mathbf{p}$. On restricting to $\mathbf{W}$, this implies that every function holomorphic in a neighborhood of 0 in $W \times \mathbb{C}$ has germ at 0 equal, mod $\mathbf{p}$, to an element of ${}_W\mathcal{H}_0[z]$ of degree less than k.

Let A be the zero set of p in $W \times \mathbb{C}$. Then π factors as $\pi = \rho \circ \psi$, where $\rho : A \to W$ is the projection $W \times \mathbb{C} \to W$, restricted to A, and $\psi : V \to A$ is defined by $\psi(v) = (\pi(v), f(v))$. Then the germ of ψ is a finite morphism and so it determines a surjective morphism onto a subvariety of $\mathbf{A}$. If this is a proper subvariety, there is a non-zero element $\mathbf{g}$ of ${}_A\mathcal{H}_0$ which vanishes on it. Then $\mathbf{g}$ has a representative g so that the equation $0 = g \circ \psi(v) = g(\pi(v), f(v))$ holds in a neighborhood of 0 in V. However, by the previous paragraph, $\mathbf{g}$ is represented by a polynomial $\mathbf{q} \in {}_W\mathcal{H}_0[z]$ of degree less than $k = \deg \mathbf{p}$. The identity $g(\pi(v), f(v)) = 0$, for $v \in V$, means that the equation $\mathbf{q}(f) = 0$ holds in ${}_V\mathcal{H}_0$. This violates the minimality of $\mathbf{p}$. We conclude that ψ is surjective. Since π is bijective, we conclude that ρ must be injective.

Since $\mathbf{W}$ is normal, $\mathbf{p}$ has a discriminant $\mathbf{d} \in {}_W\mathcal{H}_0$, by Theorem 4.2.9. If $k > 1$, then $\mathbf{d}$ is non-zero and the zero set of a representative defines a proper subvariety $\mathbf{D}$ of $\mathbf{W}$. On passing to appropriate representatives p, D, and W, the polynomial p has k roots over points of $W - D$. However, ρ is injective and so there is only one root of p over each point of some neighborhood of 0 in W. It follows that $k = 1$, which means that $f \in {}_W\mathcal{H}_0$.

Exercises

1. Assuming the Nullstellensatz for $\mathbb{C}[z_1, \dots, z_n]$, prove it for ${}_n\mathcal{O}_0$.
2. Prove that if $U \subset \mathbb{C}^n$ is a connected open set, and D is a subvariety of U, then $U - D$ is also connected.
3. Prove that if V is a connected holomorphic submanifold of an open set in $\mathbb{C}^n$, and if V_1 is a subvariety of V with non-empty interior as a subset of V, then $V_1 = V$.
4. Let W be an open set in $\mathbb{C}^m$ and $\pi : V \to W$ a finite branched holomorphic cover. Prove that each point where the branching order is 1 is a regular point of the variety V – that is, a point which has a neighborhood in V biholomorphic to a polydisc in $\mathbb{C}^m$ for some m.
5. Prove that if $\mathbf{V}$ is an irreducible germ of a variety at $0 \in \mathbb{C}^n$, then there are arbitrarily small polydiscs $\Delta \subset \mathbb{C}^n$, centered at 0, such that $\mathbf{V}$ has a representative V in Δ which is irreducible.
6. Formulate and prove the Nullstellensatz for the local ring ${}_V\mathcal{H}_0$ of the germ of a holomorphic variety V at 0.
7. Use Corollary 4.6.4 to prove that if U is an open subset of $\mathbb{C}^n$, and a holomorphic map $f : U \to \mathbb{C}^n$ is injective, then its image $f(U)$ is open, and $f : U \to f(U)$ is biholomorphic.
8. Prove that if $\pi : \mathbf{V} \to \mathbf{W}$ is the germ of a holomorphic map between two varieties, then we may represent $\mathbf{V}$ and $\mathbf{W}$ by subvarieties of neighborhoods in $\mathbb{C}^n$ and $\mathbb{C}^m$ ($m \le n$) in such a way that the projection $\mathbb{C}^n \to \mathbb{C}^m$ induces π.
9. Prove that if W is a connected open subset of $\mathbb{C}^m$, and $\pi : V \to W$ is a finite branched holomorphic cover, then the set V_0 of Definition 4.4.1

is connected if and only if V is an irreducible subvariety. Hint: use the result of Exercise 4.3.

10. Strengthen Corollary 4.6.6 by proving that if $\pi : \mathbf{V} \to \mathbf{W}$ is any germ of a finite branched holomorphic cover, then π is finite and surjective.
11. Show that the converse of the result of Exercise 4.10 is not true. That is, show that there is a finite surjective morphism of germs of varieties $\pi : \mathbf{V} \to \mathbf{W}$ which is not a finite branched holomorphic cover (necessarily $\mathbf{V}$ will be reducible).
12. Let $\pi : V_0 \to W_0$ be a finite covering map, with W_0 a connected open subset of $\mathbb{C}^m$. Show that if A_0 is a component of V_0, then $\pi : A_0 \to W_0$ is also a finite covering map. In particular, it is surjective.
13. Let W be an open subset of $\mathbb{C}^m$. Prove that if $\pi : V \to W$ is a finite branched holomorphic cover with branching order 1 at every point, then π is actually a holomorphic covering map.
14. Let A and B be finitely generated commutative algebras over an infinite field K and suppose $B \subset A$. Prove the following generalization of the Noether normalization theorem: There is a non-zero element $y \in B$ such that if B_y and A_y are the localizations of B and A relative to the multiplicative system consisting of the powers of y, then A_y is a finite extension of a polynomial algebra $B_y[x_1, \ldots, x_k]$ over B_y.
15. Prove that if W is a connected open set in $\mathbb{C}^n$, then $\mathcal{H}(W)$ is a normal domain. Hint: By Exercise 3.4 and Theorem 4.2.8, each local ring ${}_n\mathcal{H}_w$, for $w \in W$, is a normal domain.
16. Let $\Delta = \Delta(0, r) \subset \mathbb{C}^m$, and let $p(z, w) = a_0(z) + \cdots + a_{k-1}(z)w^{k-1} + w^k$ be a monic polynomial in w with coefficients in $\mathcal{H}(\Delta)$. Prove that if $p(0, w)$ has a single root λ_0 (of multiplicity k) which dominates all roots of $p(z, w)$ for $z \in \Delta$ (that is, $|\lambda| \leq |\lambda_0|$ if $z \in \Delta$, $\lambda \in \mathbb{C}$ and $p(z, \lambda) = 0$), then the coefficients $a_i(z)$ of p are all constants.
17. Prove the maximal modulus theorem for varieties: If V is a connected subvariety of an open set in $\mathbb{C}^n$, $f \in \mathcal{H}(V)$, and $|f|$ has a local maximum at some point z_0 of V, then f is constant on V. Hint: Apply Theorem 4.5.7 and the result of the previous exercise.
18. Let $\mathbf{V}$ be an irreducible germ of a subvariety at 0 in $\mathbb{C}^n$, and let coordinates in $\mathbb{C}^n$, a representative V of $\mathbf{V}$, and a polydisc $\Delta \in \mathbb{C}^m$ be chosen, as in Theorem 4.5.7. Prove that there is a non-zero element $d \in \mathcal{H}(\Delta)$ and a positive integer k such that, for each $f \in \mathcal{H}(V)$, $d(\pi(z))f(z)$ is equal on V to a unique polynomial $p \in \mathcal{H}(\Delta)[z_{m+1}]$ of degree less than k. Hint: Use Theorem 4.2.12.
19. Let V, Δ, d, and k be as in the previous exercise. Let $\phi : \mathcal{H}(V) \to \mathcal{H}^k(\Delta)$ be the $\mathcal{H}(\Delta)$-module morphism which sends each $f \in \mathcal{H}(V)$ to the k-tuple of coefficients of p of the previous exercise. Prove ϕ is continuous if $\mathcal{H}(V)$ and $\mathcal{H}(\Delta)$ are given the topology of uniform convergence on compacta.

Hint: Theorem 4.2.12 gives a formula for each coefficient $a_i(z')$ of p in terms of the values of z_{m+1} and f at points $z \in V$ where $\pi(z) = z'$.

20. Prove that the image of the morphism ϕ of the previous two exercises is a finitely generated $\mathcal{H}(\Delta)$-submodule of $\mathcal{H}^k(\Delta)$.
21. In Chapter 11 we will prove that if Δ is a polydisc in $\mathbb{C}^m$, and k is a positive integer, then every finitely generated submodule of $\mathcal{H}^k(\Delta)$ is closed in the topology of uniform convergence on compacta. Assuming this is true, and using the results of the preceding three exercises, prove that $\mathcal{H}(V)$ is complete in the topology of uniform convergence on compacta if V is a subvariety of an open set in $\mathbb{C}^n$ which has an irreducible germ at each of its points.
22. Let V be the variety of Example 4.4.9 and define a morphism $\phi : C \to V$ by $\phi(\lambda) = (\lambda^3, \lambda^2)$. Show that the germ of this morphism is bijective, but is not an isomorphism. Conclude that the germ $\mathbf{V}$ of V at the origin is not normal.

Chapter 5

Dimension

We continue our study of the local properties of regular and holomorphic functions and algebraic and holomorphic varieties. In this section, we will be concerned with a germ $\mathbf{V}$ of an algebraic or holomorphic variety V and with its corresponding local ring. Unless otherwise specified, we will assume that $0 \in V$ and $\mathbf{V}$ is the germ at 0 of V. Then the corresponding local ring is ${}_V\mathcal{O}_0$ or ${}_V\mathcal{H}_0$. In each case, there are three notions of dimension of the local ring: topological dimension, geometric dimension, and tangential dimension. We show that the first two agree and that they agree with the third if and only if the variety is regular at the point in question. It follows from this that the singular locus of a variety is a subvariety. Here the *regular locus* of a holomorphic variety is the set of points at which it is locally a complex submanifold, and the *singular locus* is the complement of the regular locus. The regular and singular locus of an algebraic variety have not yet been defined. In fact, the usual definition is that the regular locus is the set where the second and third of the three dimensions, referred to above, are equal.

Through most of this chapter, we will focus our attention on holomorphic varieties, turning to the study of dimension for algebraic varieties only in the later sections.

5.1 Topological Dimension

We initially take as our definition of the dimension of a holomorphic variety the topological definition. Each component of the regular locus of a subvariety V is a complex submanifold of some dimension. However, the various components may have different dimensions.

5.1.1 Definition. *If V is a holomorphic variety then the dimension of V is the maximum dimension of a component of the regular locus of V.*

Note that if $\lambda \in V$, then the dimensions of smaller and smaller neighborhoods of λ in V eventually stabilize, and so it makes sense to talk about the dimension of the germ of V at λ. Also note that if V is not irreducible, and

$$\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_k$$

is the decomposition of $\mathbf{V}$ into its irreducible components, then the dimension of $\mathbf{V}$ is the maximum of the dimensions of the $\mathbf{V}_j$.

The results of section 4.6 imply the following (Exercise 5.1):

5.1.2 Lemma. *Let $\pi : \mathbf{V} \to \mathbf{W}$ be a morphism of germs of varieties. Then*
(i) *if π is finite, then* $\dim \mathbf{V} \leq \dim \mathbf{W}$;
(ii) *if π is finite and surjective, then* $\dim \mathbf{V} = \dim \mathbf{W}$.

5.1.3 Proposition. *If $\mathbf{V}$ and $\mathbf{W}$ are germs of holomorphic varieties, with $\mathbf{V} \subset \mathbf{W}$, then $\dim \mathbf{V} \leq \dim \mathbf{W}$, and the two are equal exactly when $\mathbf{V}$ and $\mathbf{W}$ have a common irreducible component of dimension $\dim \mathbf{W}$.*

Proof. Since $\mathbf{V} \subset \mathbf{W}$, each irreducible component of $\mathbf{V}$ is contained in some irreducible component of $\mathbf{W}$. Thus, the theorem can be reduced to the case where $\mathbf{V}$ and $\mathbf{W}$ are irreducible. We make that assumption for the remainder of the proof.

That $\dim \mathbf{V} \leq \dim \mathbf{W}$ is obvious. Suppose they are equal. Corollary 4.6.7 implies that there is a finite surjection $\pi : \mathbf{W} \to \mathbf{C^m}$ for some m, where $\mathbf{C^m}$ denotes the germ of $\mathbb{C}^m$ at 0. Lemma 5.1.2 implies that m is the dimension of $\mathbf{W}$. Clearly, $\pi : \mathbf{V} \to \mathbf{C^m}$ is finite as well. By Theorem 4.6.3, π maps $\mathbf{V}$ to a germ of a subvariety $\mathbf{A}$ of $\mathbf{C^m}$. If this is not $\mathbf{C^m}$ itself, then there is a finite surjective morphism $\mathbf{A} \to \mathbf{C^k}$ for some $k < m$, by Corollary 4.6.7 again. In this case, $\mathbf{A}$ has dimension less than m. Since $\mathbf{V}$ and $\mathbf{A}$ have the same dimension, by Lemma 5.1.2, this is impossible. Thus, $\pi : \mathbf{V} \to \mathbf{C^m}$ is also a finite surjection.

We may choose a polydisc $\Delta \subset \mathbb{C}^m$ and a representative $W \subset \mathbb{C}^n$ of $\mathbf{W}$ so that π is the germ of a finite branched holomorphic cover $W \to \Delta$, induced by the projection $\mathbb{C}^n \to \mathbb{C}^m$. By shrinking Δ, if necessary, we may assume that the projection $\mathbb{C}^n \to \mathbb{C}^m$ also induces a finite branched holomorphic cover $V \to \Delta$ of a representative V of $\mathbf{V}$. There is a proper subvariety D of Δ, such that if $\Delta_0 = \Delta - D$, $V_0 = \pi^{-1}(D) \cap V$, and $W_0 = \pi^{-1}(\Delta_0) \cap W$, then V_0 and W_0 are dense in V and W, respectively, and each is a finite holomorphic cover of Δ_0. We may assume that both V and W are irreducible (Exercise 4.5). Then both V_0 and W_0 are connected

(Exercise 4.9). Since $V_0 \subset W_0$, and they are complex manifolds of the same dimension m, it follows that V_0 and W_0 must coincide. Then their closures V and W also coincide.

5.1.4 Theorem. *If* $\mathbf{V}$ *is the germ of a holomorphic subvariety at the origin in* $\mathbb{C}^n$, *then* $\dim \mathbf{V}$ *is the smallest integer* k *so that there is a germ* $\mathbf{L}$ *of a linear subspace* L, *of dimension* $n-k$, *such that* $\mathbf{L} \cap \mathbf{V} = \mathbf{0}$.

Proof. Suppose there is a $n-k$-dimensional subspace L with $\mathbf{V} \cap \mathbf{L} = \mathbf{0}$. We choose coordinates for $\mathbb{C}^n$ such that $L = \{z \in \mathbb{C}^n : z_1 = \cdots = z_k = 0\}$. By Theorem 4.6.3, the projection $\pi : \mathbf{V} \to \mathbf{C^k}$ is a finite map and its image is a subvariety $\mathbf{W}$ of $\mathbf{C^k}$. Then there is a finite surjective map $\rho : \mathbf{W} \to \mathbf{C^m}$ with $m \leq k$. The composition $\rho \circ \pi$ is also a finite surjective map and, thus, the dimension of $\mathbf{V}$ is $m \leq k$.

On the other hand, if $\dim \mathbf{V} = m$, then we have a finite surjective map $\pi : \mathbf{V} \to \mathbf{C^m}$, which we may assume is induced by the projection $\pi : \mathbb{C}^n \to \mathbb{C}^m$. Since $\pi : \mathbf{V} \to \mathbf{C^m}$ is finite, $\mathbf{L} \cap \mathbf{V} = \mathbf{0}$, where $\mathbf{L}$ is the germ of $L = \pi^{-1}(0)$. This completes the proof.

5.2 Subvarieties of Codimension 1

A germ of a variety has *pure dimension* r if each of its irreducible components has dimension r.

5.2.1 Proposition. *Let* $\mathbf{V} \subset \mathbf{C^n}$ *be a germ of a holomorphic subvariety. Then* $\mathbf{V}$ *has pure dimension* $n-1$ *if and only if* $\operatorname{id} \mathbf{V}$ *is a principal ideal of* ${}_n\mathcal{H}_0$.

Proof. It is easy to see that this is true in general if it is true for irreducible germs of varieties (Exercise 5.2). Thus, let $\mathbf{V}$ be an irreducible germ of a variety such that the prime ideal $\mathcal{P} = \operatorname{id} \mathbf{V}$ is generated by a single element $\mathbf{f}$. Then $\mathbf{f}$ must be irreducible. After a change of variables, we may also assume that $\mathbf{f}$ has finite vanishing order in $\mathbf{z}_n$ and, after multiplying by a unit, we may assume it is a Weierstrass polynomial in ${}_{n-1}\mathcal{H}_0[z_n]$. We claim that $\mathcal{P} \cap {}_{n-1}\mathcal{H}_0 = 0$. Otherwise, there is an $\mathbf{h} \in {}_n\mathcal{H}_0$ such that $\mathbf{g} = \mathbf{fh} \in {}_{n-1}\mathcal{H}_0$. If we choose representatives f, g, h for $\mathbf{f}, \mathbf{g}, \mathbf{h}$, then, for each $z' \in \mathbb{C}^{n-1}$, sufficiently near the origin, the polynomial $f(z', z_n)$ has at least one root. This implies that $g(z') = 0$ for all z' near the origin, i.e. that $\mathbf{g}$ is the zero germ. This establishes the claim. It follows that ${}_n\mathcal{H}_0/\mathcal{P}$ is a finite extension of ${}_{n-1}\mathcal{H}_0$ and, hence, that $\pi : \mathbf{V} \to \mathbf{C^{n-1}}$ is a finite surjection. By Lemma 5.1.2, the dimension of $\mathbf{V}$ is $n-1$.

Conversely, suppose that $\mathbf{V}$ is an irreducible germ of a variety of dimension $n-1$. By Corollary 4.6.7, there is a finite surjection $\pi : \mathbf{V} \to \mathbf{C^m}$, for some m. However, by Lemma 5.1.2, m is the dimension of $\mathbf{V}$ and, thus, $m = n - 1$. Then ${}_{\mathbf{V}}\mathcal{H}_0 = {}_n\mathcal{H}_0/(\mathrm{id}\,\mathbf{V})$ is a finite extension of ${}_{n-1}\mathcal{H}_0$, by Theorem 4.6.3. Let p be the minimal polynomial for $\mathbf{z}_n$ mod $\mathrm{id}\,\mathbf{V}$. If $\mathbf{f} \in \mathrm{id}\,V$, then $\mathbf{f} = \mathbf{pg} + \mathbf{r}$, for some $\mathbf{g} \in {}_n\mathcal{H}$ and some polynomial $\mathbf{r} \in {}_{n-1}\mathcal{H}[z_n]$ of degree less than the degree of $\mathbf{p}$. However, since $\mathbf{f}$ and $\mathbf{p}$ both belong to $\mathrm{id}\,\mathbf{V}$, so does $\mathbf{r}$. We may choose a neighborhood W of 0 in $\mathbb{C}^{n-1}$, a representative $r \in \mathcal{H}_0(W)[z_n]$ for $\mathbf{r}$, and a representative $\pi : V \to W$ of $\pi : \mathbf{V} \to \mathbf{C}^{\mathbf{n-1}}$ which is a finite branched holomorphic cover. Then, for z' in a dense open subset W_0 of W, the nth coordinates of points in $\pi^{-1}(z')$ are exactly the roots of $p(z', z_n)$. These must also be roots of $r(z', z_n)$, but there are too many of them unless $r = 0$. Thus, $\mathbf{r} = 0$, and $\mathbf{p}$ generates $\mathrm{id}\,\mathbf{V}$. This completes the proof.

The following lemma will be used both here and later (when we study dimension for algebraic varieties), to pass from a codimension 1 result, like the one for $\mathbb{C}^n$ in the preceding proposition, to a codimension 1 result on general varieties.

5.2.2 Lemma. *Let B be a normal domain and A an integral domain which is a finite extension of B. If I is a principal ideal of A, then there is a principal ideal J of B such that $A/\sqrt{I}$ is a finite extension of $B/\sqrt{J}$.*

Proof. Let I be generated by $a \in A$. Let $p(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0$ be the minimal polynomial of a over B. Let J be the ideal generated by b_0 in B. We claim that J has the required properties.

Since the equation

$$p(a) = a^k + b_{k-1}a^{k-1} + \cdots + b_0 = 0$$

holds in A, we have $J \subset I \cap B$, and so $\sqrt{J} \subset \sqrt{I} \cap B$. On the other hand, if $b \in \sqrt{I} \cap B$, then $b^\omega = ca$ for some $c \in A$ and some integer ω. At this point, we make us of the norm function N for the extension $B \to A$, introduced in section 4.2. By Theorem 4.2.13, $N(b)^\omega = N(c)N(a)$, $N(a) = (-1)^\mu b_0^\nu$ for integers μ and ν, $N(c) \in B$, and, since $b \in B$, $N(b) = b^\mu$. It follows that $b^{\mu\omega} = (-1)^\mu N(c) b_0^\nu$, and so $b \in \sqrt{J}$. Thus, $\sqrt{J} = \sqrt{I} \cap B$ and $B/\sqrt{J}$ is naturally a subring of $A/\sqrt{I}$. Since A is a finitely generated, integral extension of B, it follows that $A/\sqrt{I}$ is a finitely generated, integral extension of $B/\sqrt{J}$. That is, $A/\sqrt{I}$ is a finite extension of $B/\sqrt{J}$, as required.

The above lemma is the key to the following extension of Proposition 5.2.1.

5.2.3 Proposition. *Let $\mathbf{V}$ be a germ of a holomorphic variety of pure dimension m, and let $\mathbf{f} \in {}_{\mathbf{V}}\mathcal{H}_0$ be neither a unit nor a zero divisor. Then* *the locus of the ideal generated by* $\mathbf{f}$ *in* ${}_V\mathcal{H}_0$ *is a subvariety of* $\mathbf{V}$ *of pure dimension* $m-1$.

Proof. If $\mathbf{V}_1, \dots, \mathbf{V}_k$ are the irreducible components of $\mathbf{V}$, then each $\mathbf{V}_i$ has dimension m. Also, the fact that $\mathbf{f}$ is not a zero divisor implies that $\mathbf{f}$ does not vanish identically on any $\mathbf{V}_i$. That is, the restriction of $\mathbf{f}$ to $\mathbf{V}_i$ is a non-zero non-unit in ${}_{\mathbf{V}_i}\mathcal{H}_0$ for each i. Thus, it suffices to prove the theorem in the case where $\mathbf{V}$ is irreducible.

Assuming $\mathbf{V}$ is irreducible, we set $A = {}_V\mathcal{H}_0$ and $B = {}_m\mathcal{H}_0$. Since the dimension of $\mathbf{V}$ is m, there is a finite surjection $\mathbf{V} \to \mathbf{C}^\mathbf{m}$ which induces a morphism $B \to A$, relative to which A is a finite extension of B. Let I be the principal ideal of A generated by f. By the previous lemma, there is a principal ideal J of B such that $A/\sqrt{I}$ is a finite extension of $B/\sqrt{J}$. By the Nullstellensatz, $B/\sqrt{J} = {}_Y\mathcal{H}_0$ and $A/\sqrt{I} = {}_W\mathcal{H}_0$, where $\mathbf{W} = \operatorname{loc} I$ in $\mathbf{V}$, and $\mathbf{Y} = \operatorname{loc} J$ in $\mathbf{C}^\mathbf{m}$. It follows that there is a finite surjective morphism $\mathbf{W} \to \mathbf{Y}$ and, by Lemma 5.1.2, $\mathbf{W}$ and $\mathbf{Y}$ have the same dimension. Since J is principal, Proposition 5.2.1 implies that $\dim \mathbf{Y} = m-1$. It follows that $\dim \mathbf{W} = m-1$.

5.3 Krull Dimension

5.3.1 Definition. *Let A be a local ring. The Krull dimension of A is the supremum of the set of integers k for which there exists a strict chain $P_0 \subset P_1 \subset \cdots \subset P_k \subset A$ of prime ideals of A – that is, a chain in which all the containments are proper.*

This is the second notion of dimension we referred to earlier – the geometric one.

5.3.2 Theorem. *If* $\mathbf{V}$ *is the germ of a holomorphic variety, and* ${}_V\mathcal{H}_0$ *is its local ring, then the dimension of* $\mathbf{V}$ *is equal to the Krull dimension of* ${}_V\mathcal{H}_0$.

Proof. By the Nullstellensatz, there is a one to one correspondence between strict chains of prime ideals in ${}_V\mathcal{H}_0$ and strict chains of irreducible subvarieties of $\mathbf{V}$. Thus, the Krull dimension of ${}_V\mathcal{H}_0$ is the supremum of the lengths of strict chains of irreducible subvarieties of $\mathbf{V}$.

We claim that if $\mathbf{W} \subset \mathbf{W}'$ are germs of irreducible subvarieties of $\mathbf{V}$, and $\dim \mathbf{W}' - \dim \mathbf{W} \geq 2$, then there is another irreducible subvariety $\mathbf{W}''$ which is properly contained in $\mathbf{W}'$ and properly contains $\mathbf{W}$. Indeed, there must be a germ $\mathbf{f} \in {}_V\mathcal{H}_0$ which belongs to $\operatorname{id} \mathbf{W}$ but not to $\operatorname{id} \mathbf{W}'$. The zero locus in $\mathbf{W}'$ of such a germ contains $\mathbf{W}$ and, by Proposition 5.2.3, is a subvariety of $\mathbf{W}'$ of pure dimension $\dim \mathbf{W}' - 1$. Thus, some irreducible component of this variety properly contains $\mathbf{W}$ and is properly contained in $\mathbf{W}'$. This will do as our $\mathbf{W}''$.

It is clear from the above paragraph that if $0 = \mathbf{W}_k \subset \mathbf{W}_{k-1} \subset \cdots \subset \mathbf{W}_0$ is a maximal chain of irreducible subvarieties of $\mathbf{V}$, then successive varieties in the chain differ in dimension by exactly 1. It follows that $k = \dim \mathbf{W}_0$ for such a chain. Clearly k will be largest possible when $\mathbf{W}_0$ is an irreducible subvariety of $\mathbf{V}$ of largest dimension. That is, when $\mathbf{W}_0$ is an irreducible component of $\mathbf{V}$ with the same dimension as $\mathbf{V}$. This completes the proof.

5.4 Tangential Dimension

The third notion of dimension for a germ of a variety is tangential dimension. It is defined in terms of the dimension of a formal tangent space to the variety.

5.4.1 Definition. *If $\mathbf{V}$ is a germ at 0 of a holomorphic or algebraic variety, then a tangent vector to $\mathbf{V}$ is a derivation at 0 – that is, a linear map $t : {}_V\mathcal{H} \to \mathbb{C}$ ($t : {}_V\mathcal{O} \to \mathbb{C}$) such that $t(\mathbf{fg}) = f(\mathbf{0})t(\mathbf{g}) + g(\mathbf{0})t(\mathbf{f})$. The vector space of all tangent vectors is called the tangent space to $\mathbf{V}$ and is denoted $T(\mathbf{V})$. Its dimension is the tangential dimension of $\mathbf{V}$ and is denoted* $\operatorname{tdim} \mathbf{V}$.

5.4.2 Proposition. *The vector space $T(\mathbf{V})$ is naturally isomorphic to the dual of $\mathcal{M}/\mathcal{M}^2$, where $\mathcal{M}$ is the maximal ideal of ${}_V\mathcal{H}_0$ (${}_V\mathcal{O}_0$).*

Proof. If $t \in T(\mathbf{V})$, then $t(\mathbf{1}) = t(\mathbf{1}^2) = 2t(\mathbf{1})$. Consequently, t kills constants and is, thus, determined by its restriction to $\mathcal{M}$. However, if t is any linear functional on ${}_V\mathcal{H}$ (${}_V\mathcal{O}$) which kills constants, and if we set $\mathbf{f} = \mathbf{1} + \mathbf{f}_1, \mathbf{g} = \mathbf{1} + \mathbf{g}_1$, with $\mathbf{f}_1, \mathbf{g}_1 \in \mathcal{M}$, then

$$t(\mathbf{fg}) = t(\mathbf{f}_1) + t(\mathbf{g}_1) + t(\mathbf{f}_1\mathbf{g}_1) = \mathbf{g}(\mathbf{0})t(\mathbf{f}) + \mathbf{f}(\mathbf{0})t(\mathbf{g}) + t(\mathbf{f}_1\mathbf{g}_1).$$

We conclude from this that t is a tangent vector if and only if t vanishes on $\mathcal{M}^2$. Thus, restriction to $\mathcal{M}$ defines an isomorphism between $T(\mathbf{V})$ and the dual of $\mathcal{M}/\mathcal{M}^2$. This completes the proof.

It is clear that if $\mathbb{C}^n$ is considered as either a holomorphic or an algebraic variety, and $\mathbf{C}^n$ is its germ at 0, then $T(\mathbf{C}^n)$ is naturally isomorphic to $\mathbb{C}^n$, where a given point $t = (t_1, \dots, t_n)$ of $\mathbb{C}^n$ is associated to the derivation on ${}_n\mathcal{H}_0$ or ${}_n\mathcal{O}_0$ defined by $t(\mathbf{f}) = \sum t_i \dfrac{\partial f}{\partial z_i}(0)$, for any representative f of $\mathbf{f}$.

If $\mathbf{F} : \mathbf{V} \to \mathbf{W}$ is a germ of a morphism between two holomorphic varieties or two algebraic varieties, then $\mathbf{F}$ induces an algebra homomorphism

$\mathbf{F}^* : {}_W\mathcal{H}_0 \to {}_V\mathcal{H}_0$ ($\mathbf{F}^* : {}_W\mathcal{O}_0 \to {}_V\mathcal{O}_0$) which, in turn, induces a linear map $d\mathbf{F} : T(\mathbf{V}) \to T(\mathbf{W})$ by $d\mathbf{F}(t)(\mathbf{f}) = t(\mathbf{F}^*(\mathbf{f}))$. The linear map $d\mathbf{F}$ is called the *differential of* $\mathbf{F}$. If $F : V \to W$ is a morphism between varieties, then, for each $z \in V$, $d\mathbf{F}_z$ will denote the differential of the germ $\mathbf{F}_z : \mathbf{V}_z \to \mathbf{W}_{F(z)}$.

5.4.3 Proposition. *If* $\mathbf{V} \subset \mathbf{C}^n$ *is a germ of a subvariety, and* $\mathbf{F} : \mathbf{V} \to \mathbf{C}^n$ *is the inclusion, then* $d\mathbf{F} : T(\mathbf{V}) \to T(\mathbf{C}^n)$ *is injective and its image is the set of all elements of* $T(\mathbf{C}^n)$ *which vanish on* $\operatorname{id}\mathbf{V}$.

Proof. For $g \in {}_n\mathcal{H}_0$ and $t \in T(\mathbf{V})$, we have that

$$d\mathbf{F}(t)(\mathbf{g}) = t(\mathbf{F}^*(\mathbf{g})) = t(\mathbf{g} \circ \mathbf{F}) = t(\mathbf{g}|_\mathbf{V}).$$

Since every germ in ${}_V\mathcal{H}_0$ is the restriction to $\mathbf{V}$ of a germ in ${}_n\mathcal{H}_0$, we have $d\mathbf{F}(t) = 0$ if and only if $t = 0$. Furthermore, each derivation on ${}_n\mathcal{H}_0$ of the form $d\mathbf{F}(t)$ clearly vanishes on $\operatorname{id}\mathbf{V} = \{\mathbf{g} \in {}_n\mathcal{H} : \mathbf{g}|_\mathbf{V} = 0\}$. Conversely, if s is a derivation on ${}_n\mathcal{H}_0$ which vanishes on $\operatorname{id}\mathbf{V}$, then s determines a well-defined linear functional t on ${}_V\mathcal{H}_0 = {}_n\mathcal{H}_0/\operatorname{id}\mathbf{V}$ such that $t(\mathbf{g}|_\mathbf{V}) = s(\mathbf{g})$. Since s is a derivation, t clearly is as well. This completes the proof for germs of holomorphic varieties. The proof is the same for germs of algebraic varieties.

In the above proposition, suppose that $\operatorname{id}\mathbf{V}$ is generated by $\mathbf{g}_1, \dots, \mathbf{g}_m$. Then a derivation in $T(\mathbf{C}^n)$ vanishes on $\operatorname{id}\mathbf{V}$ if and only if it vanishes at each $\mathbf{g}_i$. Thus, if we identify $T(\mathbf{V})$ with its image under $d\mathbf{F}$ and choose representatives for the $\mathbf{g}_j$, then

$$T(\mathbf{V}) = \{t \in \mathbb{C}^n : \sum_i t_i \frac{\partial g_j}{\partial z_i}(0) = 0 \quad \text{for} \quad j = 1, \dots, m\}.$$

The dimension of the space of solutions $t \in \mathbb{C}^n$ of this system of equations is $n - \operatorname{rank} J_G(0)$, where $G = (g_1, g_2, \dots, g_m)$. Thus,

5.4.4 Corollary. *If* $\mathbf{V} \subset \mathbf{C}^n$ *is a germ of a holomorphic (algebraic) subvariety, and if* $\mathbf{g}_1, \dots, \mathbf{g}_m$ *generate* $\operatorname{id}\mathbf{V}$, *then*

$$\operatorname{tdim}\mathbf{V} = n - \operatorname{rank} J_G(0),$$

where $G : U \to \mathbb{C}^m$ *is a holomorphic (algebraic) map with representatives of the germs* $\mathbf{g}_j$ *on the* 0*-neighborhood* U *as coordinate functions, and* J_G *is its Jacobian matrix.*

Note that the Jacobian matrix $J_G(0)$, in the above corollary, is just the matrix representing the linear transformation $d\mathbf{G}_0$, where $\mathbf{G}_0$ is the germ of the map G at 0.

The tangential dimension $\operatorname{tdim}\mathbf{V}$ determines when a holomorphic variety $\mathbf{V}$ can be realized as a germ of a subvariety of a neighborhood in $\mathbb{C}^n$:

5.4.5 Proposition. *A germ $\mathbf{V}$ of a holomorphic variety can be represented as a germ of a subvariety of a neighborhood in $\mathbb{C}^n$ if and only if $n \geq \operatorname{tdim} \mathbf{V}$.*

Proof. Certainly $\operatorname{tdim} \mathbf{V} \leq n$ if $\mathbf{V}$ is represented as a germ of a subvariety of a neighborhood in $\mathbb{C}^n$. On the other hand, if $\mathbf{V}$ is a subvariety of $\mathbf{C}^\mathbf{m}$, $g_1, \dots, g_k$ are holomorphic functions on a neighborhood of 0 in $\mathbb{C}^m$, whose germs at 0 form a set of generators for $\operatorname{id} \mathbf{V}$, and if G is the map with these functions as coordinate functions, then $\operatorname{rank} J_G(0) = m - q$, where $q = \operatorname{tdim} \mathbf{V}$. Without loss of generality, we may assume that the map $G' : U \to \mathbb{C}^{m-q}$, whose coordinate functions are the first $m - q$ coordinate functions of G, also has rank $m - q$. It follows from the implicit mapping theorem that the subvariety W of U, on which G' vanishes, has 0 as a regular point, and so its germ $\mathbf{W}$ at 0 is the germ of a submanifold of dimension q. Since $\mathbf{V} \subset \mathbf{W}$, it follows that $\mathbf{V}$ is biholomorphically equivalent to the germ of a subvariety of $\mathbb{C}^n$, provided $q \leq n$.

A germ of a variety $\mathbf{V} \subset \mathbf{C}^\mathbf{n}$ is said to be *neatly embedded* in $\mathbf{C}^\mathbf{n}$ if $n = \operatorname{tdim} V$. The above theorem says that every germ of a holomorphic variety can be neatly embedded.

Since its proof uses the implicit function theorem, one might suspect that the algebraic version of Proposition 5.4.5 is not true. This is the case, as the following example shows. Note that if V is an irreducible algebraic subvariety of $\mathbb{C}^n$, and the germ of V at some point $\lambda \in V$ is embeddable in $\mathbb{C}^m$, then the quotient field of $\mathcal{O}(V) = \mathbb{C}[z_1, \dots, z_n]/\operatorname{id}(V)$ is the same as the quotient field of the local ring ${}_V\mathcal{O}_\lambda$, and this is generated by m elements over $\mathbb{C}$.

5.4.6 Example. Consider the polynomial $p(z, w) = z^3 + w^3 + 1$ and the algebraic subvariety V of $\mathbb{C}^2$ defined by the equation $p(z, w) = 0$. It is easy to see that p is irreducible (Exercise 5.5), and so V is irreducible. This implies that the ideal generated by p is prime and, hence, that p generates $\operatorname{id}(V)$, by the Nullstellensatz. The Jacobian matrix of p is $(3z^2, 3w^2)$, which has rank 1 at each point of V. It follows from Corollary 5.4.4 that $\operatorname{tdim} V_\lambda = 1$ at each point of V. Suppose the germ of V at such a point is embeddable in $\mathbb{C}^1$. Then there is a single element x which generates the quotient field of $\mathcal{O}(V)$ over $\mathbb{C}$. The images of z and w in the quotient field of $\mathcal{O}(V)$ must be rational functions r and s of this generator. This implies $r^3 + s^3 + 1 = 0$. On clearing denominators, it follows that there exist polynomials $q, v, u \in \mathbb{C}[x]$, not all of degree 0, such that

$$q^3 + u^3 + v^3 = 0.$$

This is impossible (Exercise 5.6). Thus, no germ of V is embedded in $\mathbb{C}^1$, even though the tangential dimension of V is 1 at all points of V.

5.5 Dimension and Regularity

The following is a form of the inverse mapping theorem that holds for holomorphic varieties.

5.5.1 Theorem. *If* $\mathbf{F} : \mathbf{V} \to \mathbf{W}$ *is a morphism between germs of holomorphic varieties, and if* $d\mathbf{F} : T(\mathbf{V}) \to T(\mathbf{W})$ *is injective, then* $\mathbf{F}$ *is an isomorphism from* $\mathbf{V}$ *to a holomorphic subvariety of* $\mathbf{W}$.

Proof. We may assume that $\mathbf{V}$ and $\mathbf{W}$ are neatly embedded germs of subvarieties at the origin in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then $\mathbf{F}$ may be regarded as a germ of a holomorphic map from $\mathbf{V}$ into $\mathbf{C^n}$. Furthermore, $\mathbf{F}$ extends to a germ of a holomorphic mapping $\mathbf{G} : \mathbf{C^m} \to \mathbf{C^n}$, since each of its coordinate functions has a holomorphic extension to a neighborhood of 0 in $\mathbb{C}^m$. Since $\mathbf{V}$ and $\mathbf{W}$ are neatly embedded, we have that $T(\mathbf{V}) = \mathbb{C}^m$ and $T(\mathbf{W}) = \mathbb{C}^n$, with the natural identifications given by Proposition 5.4.3. It follows that the linear maps $d\mathbf{F} : T(\mathbf{V}) \to T(\mathbf{W})$ and $d\mathbf{G} : \mathbb{C}^m \to \mathbb{C}^n$ agree after this identification, and, hence, that $d\mathbf{G}$ is injective. We now extend $\mathbf{G}$ to a germ of a holomorphic map $\mathbf{H} : \mathbf{C^n} = \mathbf{C^n} \oplus \mathbf{C^{n-m}} \to \mathbf{C^n}$, by letting $\mathbf{H}$ be $\mathbf{G}$ on $\mathbf{C^m}$, and defining it on $\mathbf{C^{n-m}}$ to be the germ of a linear isomorphism onto a complement in $\mathbb{C}^n$ of the image of $d\mathbf{G}$. Then $d\mathbf{H}$ is an isomorphism. The matrix representing $d\mathbf{H}$ is the Jacobian matrix $J_H(0)$ for some representative H of $\mathbf{H}$, and so the usual inverse mapping theorem implies that H is a biholomorphic map of one neighborhood of 0 in $\mathbb{C}^n$ onto another. We conclude that G is a biholomorphic map of a neighborhood of 0 in $\mathbb{C}^m$ onto an m-dimensional submanifold of a neighborhood in $\mathbb{C}^n$. Its germ $\mathbf{G}$, restricted to $\mathbf{V}$, is then an isomorphism of $\mathbf{V}$ onto a subvariety of $\mathbf{W}$, as required.

The next proposition is useful for identifying cases where the previous theorem applies.

5.5.2 Proposition. *If* $\mathbf{F} : \mathbf{V} \to \mathbf{C^n}$ *is a morphism between germs of holomorphic varieties, and if the coordinate functions of* $\mathbf{F}$ *generate the maximal ideal of* ${}_V\mathcal{H}_0$, *then* $d\mathbf{F} : T(\mathbf{V}) \to T(\mathbf{C^n}) = \mathbb{C}^n$ *is injective.*

Proof. This follows from Corollary 5.4.4 in the following fashion: Assume $\mathbf{V}$ is neatly embedded as a germ at the origin of a subvariety of $\mathbb{C}^m$. Extend $\mathbf{F}$ to a morphism $\mathbf{G} : \mathbb{C}^m \to \mathbb{C}^n$, as in the previous theorem. Let $\{\mathbf{k}_1, \dots, \mathbf{k}_p\}$ be a set of germs in ${}_m\mathcal{H}_0$ which generates $\operatorname{id}\mathbf{V}$, and let $\mathbf{K} : \mathbb{C}^m \to \mathbb{C}^p$ be the morphism with these as coordinate functions. If $\mathbf{H} = \mathbf{G} \oplus \mathbf{K} : \mathbb{C}^m \to \mathbb{C}^{n+p}$, then the set of coordinate functions of $\mathbf{H}$ is the union of the set of coordinate functions of $\mathbf{G}$ and those of $\mathbf{K}$, and so it clearly generates the maximal ideal of ${}_m\mathcal{H}_0$. Since the variety $\{0\}$ has tangential dimension 0, it follows from Corollary 5.4.4 that rank $J_{\mathbf{H}}(0) = m$ and, hence, that $d\mathbf{H}$ is injective. However, $\mathbf{K}$ has coordinate functions which generate $\operatorname{id} \mathbf{V}$. Since $\mathbf{V}$ is neatly embedded, $\operatorname{tdim} \mathbf{V} = m$, and so it follows, also by Corollary 5.4.4, that rank $J_{\mathbf{K}}(0) = 0$. Thus, $d\mathbf{K} = 0$, and we conclude that $d\mathbf{G}$ is injective. Since $d\mathbf{F}$ and $d\mathbf{G}$ agree after the appropriate identification, the proof is complete.

5.5.3 Theorem. *If $\mathbf{V}$ is the germ of a holomorphic variety, then*

$$\dim \mathbf{V} \le \operatorname{tdim} \mathbf{V}$$

and equality holds if and only if $\mathbf{V}$ is regular.

Proof. That $\dim \mathbf{V} \le \operatorname{tdim} \mathbf{V}$ clearly follows from Proposition 5.4.5. A germ of a variety is regular if and only if it is isomorphic to the germ of a neighborhood of 0 in $\mathbb{C}^n$, where $n = \dim \mathbf{V}$. By Proposition 5.1.3 and Proposition 5.4.5 again, this is equivalent to $\operatorname{tdim} \mathbf{V} = n$.

5.6 Dimension of Algebraic Varieties

If V is an irreducible algebraic subvariety of $\mathbb{C}^n$, then $P = \operatorname{loc} V$ is a prime ideal of $\mathbb{C}[z_1, \ldots, z_n]$. By Noether's normalization theorem (Theorem 4.3.1), coordinates may be chosen for $\mathbb{C}^n$ so that $\mathcal{O}(V) = \mathbb{C}[z_1, \ldots, z_n]/P$ is a finite extension of $\mathbb{C}[z_1, \ldots, z_m]$. The integer m is well defined and, in fact, is the transcendence degree of the quotient field of $\mathcal{O}(V)$ over $\mathbb{C}$. We define the dimension of V to be this integer m. The dimension of a general algebraic subvariety of $\mathbb{C}^n$ is the maximal dimension of its irreducible components. If all the irreducible components of V have the same dimension r, we say V has pure dimension r.

Now each germ $\mathbf{V}$ of a subvariety at $0 \in \mathbb{C}^n$ has a representative which is an algebraic subvariety of $\mathbb{C}^n$. The dimension of $\mathbf{V}$ is defined to be the minimum of the dimensions of such representatives. If the germ $\mathbf{V}$ is irreducible, a representative V may be chosen which is also irreducible. Note that if two irreducible subvarieties of $\mathbb{C}^n$ have the same germ at 0, then they are identical. Thus, an irreducible germ of a variety has a unique representative which is an irreducible subvariety of $\mathbb{C}^n$. Clearly, the dimension of the germ is the dimension of this representative. If V is an irreducible subvariety of $\mathbb{C}^n$, then germs $\mathbf{V}_\lambda$ have the same dimension at all points $\lambda \in V$. It follows from this discussion that if a germ $\mathbf{V}$ of a subvariety is not irreducible, then its dimension is the maximum of the dimensions of its irreducible components. As with varieties, a germ of a variety has pure dimension r if each of its irreducible components has dimension r.

Because there is no difference between the dimension of an irreducible subvariety of $\mathbb{C}^n$ and the dimension of each of its germs, it is convenient to develop the basic properties of dimension in the context of subvarieties of $\mathbb{C}^n$ rather than in the context of germs of varieties.

Propositions 5.2.1 and 5.2.3 and Theorem 5.3.2 have analogues in the algebraic case with proofs that are almost the same. We will state these but leave the proofs as exercises.

5.6.1 Proposition. *A an algebraic subvariety $V \subset \mathbb{C}^n$ has pure dimension $n-1$ if and only if* $\operatorname{id} V$ *is a principal ideal of* $\mathbb{C}[z_1, \dots, z_n]$.

5.6.2 Proposition. *Let V be an algebraic subvariety of $\mathbb{C}^n$ of pure dimension m, and let* $\mathrm{f} \in \mathcal{O}(V)$ *be neither a unit nor a zero divisor. Then the subset of V on which f vanishes is a subvariety of V of pure dimension $m-1$.*

5.6.3 Theorem. *If V is an algebraic subvariety of $\mathbb{C}^n$, then the dimension of V is equal to the Krull dimension of $\mathcal{O}(V)$.*

Proposition 5.6.2 implies that if V is any subvariety of $\mathbb{C}^n$, $f \in \mathcal{O}(V)$, and V_1 is an irreducible component of V, then the subvariety of V on which f vanishes either contains V_1 or meets V_1 in a set of dimension $\dim V_1 - 1$. Thus, its intersection with each irreducible component of V has dimension the same as, or 1 less than, the dimension of the component. This is the basis for an induction argument which proves the following proposition.

5.6.4 Proposition. *Let V be a subvariety of $\mathbb{C}^n$ of pure dimension m, and let W be a subvariety of V defined by the vanishing of k regular functions on V. Then each irreducible component of W has dimension at least $m-k$.*

Our next goal is to develop information about the tangential dimension of a germ of an algebraic variety. We first prove that $\operatorname{tdim}(\mathbf{V}_\lambda)$ is an upper semicontinuous function of λ.

5.6.5 Proposition. *If V is an algebraic subvariety of $\mathbb{C}^n$ and k is an integer, then $\{\lambda \in V : \operatorname{tdim}(\mathbf{V}_\lambda) < k\}$ is an open set.*

Proof. Since $\mathbb{C}[z_1, \dots, z_n]$ is a Noetherian ring, the ideal of polynomials which vanish on V is finitely generated. Let $\{f_1, \dots, f_k\}$ be a set of generators for this ideal. We claim that this same set of polynomials generates the ideal $\operatorname{id} \mathbf{V}_\lambda$ in ${}_n\mathcal{O}_\lambda$, for each λ. In fact, an element of $\operatorname{id} \mathbf{V}_\lambda$ has the form g/h, where h is a polynomial which does not vanish at λ, and g is a polynomial which vanishes on $V \cap U$, where U is some Zariski neighborhood of λ in $\mathbb{C}^n$. Then g vanishes on every irreducible component of V which meets U and, hence, on every irreducible component of V which contains λ. If u is a polynomial which vanishes on all the irreducible components of V which don't contain λ, but which does not vanish at λ, then u is a unit in ${}_n\mathcal{O}_\lambda$

and ug vanishes on V. Hence, ug is in the ideal generated by $\{f_1, \dots, f_k\}$ in $\mathbb{C}[z_1, \dots, z_n]$, and $f = ug/uh$ is in the ideal generated by $\{f_1, \dots, f_k\}$ in ${}_n\mathcal{O}_\lambda$.

If $F : \mathbb{C}^n \to \mathbb{C}^k$ is the morphism with coordinate functions $f_1, \dots, f_k$, then $\operatorname{tdim} \mathbf{V}_\lambda = n - \operatorname{rank} J_F(\lambda)$, by Corollary 5.4.4. Thus, $\operatorname{tdim} \mathbf{V}_\lambda < k$ if and only if $\operatorname{rank} J_F(\lambda) > n - k$, and this happens if and only if some square submatrix of $J_F(\lambda)$, of size $n - k + 1$, has non-vanishing determinant. There are finitely many such submatrices, and so this condition defines a Zariski open subset of V.

The inequality $\dim V \leq \operatorname{tdim} V$, proved in Theorem 5.5.3 for holomorphic varieties, also holds for algebraic varieties. The proof of this inequality in the algebraic case relies on Noether's normalization theorem rather than the implicit mapping theorem.

5.6.6 Proposition. *Let V be an irreducible algebraic subvariety of $\mathbb{C}^n$. Then, for each $\lambda \in V$,*

$$\dim \mathbf{V}_\lambda \leq \operatorname{tdim} \mathbf{V}_\lambda,$$

and equality holds on a non-empty Zariski open subset of V.

Proof. We use essentially the same technique used in the proof of the Nullstellensatz in section 4.5. By Noether's normalization theorem (Theorem 4.3.1), we may choose coordinates for $\mathbb{C}^n$ and an integer $m \leq n$ such that the following is true: If $\mathbb{C}[z_1, \dots, z_m]$ is considered a subalgebra of $\mathbb{C}[z_1, \dots, z_n]$, then, under restriction to V, $\mathbb{C}[z_1, \dots, z_m]$ is isomorphic to a subalgebra B of the algebra $A = \mathcal{O}(V)$, and A is a finite extension of B. If $m = n$, then $A = B$, $V = \mathbb{C}^n$, and the proposition is clearly true. If $m < n$, by the theorem of the primitive element (Theorem 4.2.1), the coordinates for $\mathbb{C}^n$ may be chosen so that z_{m+1} generates the field of quotients of A over the field of quotients of B. Then if p is the minimal polynomial of z_{m+1} in $B[z_{m+1}]$, and $d \in B$ is its discriminant, Theorem 4.2.12 implies that each element f of A satisfies an equation $d \cdot f = q$ on V, where q is a polynomial in $B[z_{m+1}] = \mathbb{C}[z_1, \dots, z_m, z_{m+1}]$. In particular, there are polynomials $q_{m+2}, \dots, q_n$ in $\mathbb{C}[z_1, \dots, z_{m+1}]$ such that the equations

$$d(z_1, \dots, z_m) \cdot z_j = q_j(z_1, \dots, z_{m+1}), \text{ for } j = m+2, \dots, n,$$

hold at all points of V.

Let W be the subset of $\mathbb{C}^{m+1}$ on which p vanishes, and let $\pi : V \to W$ be the restriction to V of the projection $\mathbb{C}^n \to \mathbb{C}^{m+1}$. Then π is a morphism of algebraic varieties. If W_0 and V_0 are the open subsets of W and V on which d does not vanish, then $\pi|_{V_0}$ has an inverse $\phi : W_0 \to V_0 \subset \mathbb{C}^n$, where the jth coordinate function ϕ_j of ϕ is given by

$$\phi_j(z_1, \ldots, z_n) = \begin{cases} z_j & \text{if } 1 \leq j \leq m+1 \\ d^{-1}(z_1, \ldots, z_m)q_j(z_1, \ldots, z_{m+1}) & \text{if } m+2 \leq j \leq n. \end{cases}$$

We conclude that V_0 and W_0 are isomorphic as algebraic varieties and, hence, have the same dimension and tangential dimension at each point.

Now W is the irreducible subvariety of $\mathbb{C}^{m+1}$ defined by the vanishing of the irreducible polynomial $p \in \mathbb{C}[z_1, \ldots, z_{m+1}]$. Hence, W has dimension m, by Proposition 5.6.1. It also has tangential dimension m at each point where the differential of p is non-vanishing, by Corollary 5.4.4. This is true on a Zariski open subset of W and, hence, on a Zariski open subset of W_0. It follows that V has dimension m and has tangential dimension m on a Zariski open subset. However, by Proposition 5.6.5, the set on which it has dimension at least m is Zariski closed. Since V is irreducible, this implies that V has tangential dimension at least m at all points of V.

We have yet to define regular and singular points for an algebraic variety. The lack of an implicit function theorem suggests that we cannot often expect an algebraic variety to be locally isomorphic to a Zariski open subset of $\mathbb{C}^n$. However, Theorem 5.5.3 suggests the following definition:

5.6.7 Definition. *We say a germ of an algebraic variety* $\mathbf{V}$ *is regular if* $\dim \mathbf{V} = \operatorname{tdim} \mathbf{V}$ *and singular if* $\dim \mathbf{V} < \operatorname{tdim} V$. *A point of an algebraic variety is a regular point (singular point) if the germ of the variety is regular (singular) at the point. The singular locus of an algebraic variety* V *is the set of all of its singular points. An algebraic variety which contains only regular points is called non-singular or smooth.*

5.6.8 Theorem. *If* V *is an algebraic subvariety of* $\mathbb{C}^n$*, then the singular locus of* V *is a subvariety of* V *which does not contain any irreducible component of* V*. That is, the set of regular points of* V *is an open dense subset of* V*.*

Proof. Since $\dim V_\lambda$ is constant on each irreducible component of V, it follows from Proposition 5.6.5 that the singular locus of V meets every irreducible component of V in a subvariety – a proper subvariety by Proposition 5.6.6. The theorem follows from this.

The regular points on an algebraic subvariety of $\mathbb{C}^n$ may be characterized as follows:

5.6.9 Proposition. *Let V be an algebraic subvariety of $\mathbb{C}^n$. Then $\lambda \in V$ is a regular point of V if and only if there is a Zariski neighborhood U of λ in V and a regular map $F : U \to C^k$ such that $dF_z : T(\mathbf{V}_z) \to C^k$ is bijective for each $\mathbf{z} \in U$. In this case, k is* $\dim V$.

Proof. If such a U and F exist, then the tangential dimension of $\mathbf{V}_z$ is k at every $z \in U$. However, by the previous theorem, the set of regular points is dense in V and, hence, in U. It follows that k is the dimension of $\mathbf{V}_z$ at every point of U. Therefore, every point of U is a regular point and, in particular, λ is a regular point.

Now suppose that λ is a regular point of V and $\dim \mathbf{V}_\lambda = k$. Let $\mathcal{M}$ denote the maximal ideal of ${}_V\mathcal{O}_\lambda$. We choose a Zariski neighborhood W of λ in $\mathbb{C}^n$ and regular functions $g_1, \dots, g_k$ on W, vanishing at λ, whose restrictions to $V \cap W$ have germs at λ which span $\mathcal{M}/\mathcal{M}^2$. Then the map $G : W \to \mathbb{C}^k$, with these functions as coordinate functions, has differential at λ, $dG_\lambda : \mathbb{C}^n \to \mathbb{C}^k$, which is bijective when restricted to the subspace $T(\mathbf{V}_\lambda)$ of $\mathbb{C}^n$. Now let $h_1, \dots, h_m$ be polynomials in $\mathbb{C}[z_1, \dots, z_n]$ which generate the ideal of functions vanishing on V. As in the proof of Proposition 5.6.5, these polynomials also generate the ideal $\operatorname{id} \mathbf{V}_z$ of ${}_V\mathcal{O}_z$ at every point $z \in V$. It follows from Corollary 5.4.4 that the differential dH_z at z of the map $H : \mathbb{C}^n \to \mathbb{C}^n$, with coordinate functions $h_1, \dots, h_m$, has rank $n - \operatorname{tdim} \mathbf{V}_z$ at every point $z \in V$. Furthermore, dH_z vanishes on $T(\mathbf{V}_z)$ at every $z \in V$. Then the regular map $G \oplus H : \mathbb{C}^n \to \mathbb{C}^{k+m}$ has differential of rank $n + k - \operatorname{tdim} \mathbf{V}_z$ at every $z \in V$. At λ this rank is n, since $\operatorname{tdim} \mathbf{V}_\lambda = k$. It follows that it must be n in some Zariski neighborhood U of λ in V. However, for $z \in V$, the rank of dH_z is $n - \operatorname{tdim} \mathbf{V}_z$, and its kernel contains $T(\mathbf{V}_z)$. Thus, the only way the rank of the differential $dG_z \oplus dH_z$ can be n is if dG_z is bijective on $T(\mathbf{V}_z)$. This, of course, forces $\operatorname{tdim} \mathbf{V}_z$ to be k. Thus, if F is the restriction of G to V, then F and U have the required properties.

5.7 Algebraic vs. Holomorphic Dimension

To each algebraic subvariety V of $\mathbb{C}^n$, there is associated a holomorphic subvariety V^h. This is the same subset of $\mathbb{C}^n$, but with the Euclidean rather than the Zariski topology, and with ${}_V\mathcal{H}_\lambda$ rather than ${}_V\mathcal{O}_\lambda$ as the local ring associated to a point $\lambda \in V$. It makes sense to ask whether or not the germ of an algebraic variety at a point has the same dimension (or tangential dimension), as the germ of the corresponding holomorphic variety. It also makes sense to ask if the singular locus of an algebraic variety V agrees with the singular locus of V^h. We can give partial answers to these questions now.

A complete answer must wait until we develop additional machinery in a later chapter.

5.7.1 Theorem. *If $\mathbf{V}$ is a germ of an algebraic subvariety of $\mathbb{C}^n$ and $\mathbf{V}^h$ the corresponding germ of a holomorphic subvariety, then*

$$\dim \mathbf{V} = \dim \mathbf{V}^h \leq \operatorname{tdim} \mathbf{V}^h \leq \operatorname{tdim} \mathbf{V}.$$

Proof. Given a generating set $\mathbf{g}_1, \dots, \mathbf{g}_k$ for $\operatorname{id} \mathbf{V}$, we can expand it to a generating set for $\operatorname{id} \mathbf{V}^h$ by adjoining finitely many germs $\mathbf{g}_{k+1}, \dots, \mathbf{g}_m$ from $\operatorname{id} \mathbf{V}^h$. If $G(z) = (g_1(z), \dots, g_k(z))$, and $\tilde{G}(z) = (g_1(z), \dots, g_m(z))$, where the g_i are representatives of the $\mathbf{g}_i$ on some neighborhood of 0, then clearly $\operatorname{rank} J_G(0) \leq \operatorname{rank} J_{\tilde{G}}(0)$. Thus, the inequality $\operatorname{tdim} \mathbf{V}^h \leq \operatorname{tdim} \mathbf{V}$ follows from Corollary 5.4.4.

Suppose $\mathbf{V}$ is irreducible of dimension m and V is an irreducible subvariety of $\mathbb{C}^n$ which is a representative of $\mathbf{V}$. Then, as in the proof of Proposition 5.6.6, there is a Zariski open subset V_0 of V which is isomorphic as an algebraic variety to a Zariski open subset W_0 of the zero set of a polynomial in $\mathbb{C}[z_1, \dots, z_{m+1}]$. By Proposition 5.2.1, W_0 has dimension m as a holomorphic variety. It follows that V^h has dimension m as well. Hence, for an arbitrary germ $\mathbf{V}$ of an algebraic subvariety, $\dim \mathbf{V} = \dim \mathbf{V}^h$, since this is true of each irreducible component of $\mathbf{V}$.

5.7.2 Corollary. *If V is an algebraic subvariety of $\mathbb{C}^n$, and V^h the corresponding holomorphic subvariety, then the singular set of V^h is contained in the singular set of V.*

Proof. By Theorem 5.7.1, at each $\lambda \in V$ we have

$$0 \leq \operatorname{tdim} \mathbf{V}^h_\lambda - \dim \mathbf{V}^h_\lambda \leq \operatorname{tdim} \mathbf{V}_\lambda - \dim \mathbf{V}_\lambda.$$

By Theorem 5.5.3 and Definition 5.6.7, if λ is a regular point of V, it is also a regular point of V^h.

Actually, much stronger results than Theorem 5.7.1 and Corollary 5.7.2 are true. We will prove in Chapter 13 that a generating set for an ideal $\operatorname{id} \mathbf{V}$ is also a generating set for the ideal $\operatorname{id} \mathbf{V}^h$. That the tangential dimensions of the germs $\mathbf{V}^h$ and $\mathbf{V}$ are actually the same then follows from Corollary 5.4.4. This, in turn, implies that the singular sets of an algebraic variety V and the corresponding holomorphic variety V^h are the same. We cannot prove these things now because the proof that a generating set for $\operatorname{id} \mathbf{V}$ is also a generating set for the ideal $\operatorname{id} \mathbf{V}^h$ requires machinery from the theory of completions of local rings. We have opted to postpone the treatment of this machinery until it is needed for the proofs of Serre's theorems, in order to more quickly complete our development of the fundamentals of function theory in several complex variables.

Another result closely related to the ideas of this chapter is that the singular set of a holomorphic variety V is itself a subvariety of V. The analogous result for algebraic subvarieties of $\mathbb{C}^n$ was proved in Theorem 5.6.8. A proof of the result for holomorphic varieties must wait until we prove, in Chapter 9, that, in some neighborhood of each point $\lambda \in V \subset \mathbb{C}^n$, a finite set of functions $\{g_1, \dots, g_k\}$ can be chosen so that their germs generate the ideal of the variety at each point of the neighborhood (not just at the point λ). The proof that this suffices to prove the singular set of a variety is a subvariety is left as an exercise (Exercise 5.9).

Exercises

1. Prove Lemma 5.1.2.
2. Let $\mathbf{V} \subset \mathbf{C}^n$ be a germ of a subvariety. Prove that $\operatorname{id}\mathbf{V}$ is a principal ideal in ${}_n\mathcal{H}_0$ if and only if $\operatorname{id}\mathbf{V}_i$ is a principal ideal for every irreducible component $\mathbf{V}_i$ of $\mathbf{V}$.
3. Prove that a compact subvariety of an open set in $\mathbb{C}^n$ must be a finite set.
4. Prove that if $\mathbf{V}$ is a germ of a variety, then $\operatorname{tdim}\mathbf{V}$ is the minimal number of generators for the maximal ideal of ${}_V\mathcal{H}_0$.
5. Prove that the polynomial $p(z_1, z_2) = z_1^3 + z_2^3 + 1$ is irreducible in $\mathbb{C}[z_1, z_2]$.
6. Let q, u, and v be polynomials in $\mathbb{C}[z]$ such that $q^3 + u^3 + v^3 = 0$. Prove that q, u, and v all have degree 0. Hint: Assume the contrary and choose solutions q, u, and v of this equation for which the integer $\max\{\deg q, \deg u, \deg v\}$ is positive and is the minimal such integer. Assume $\deg q$ is this integer. Show that $q^3 = -(u+v)(u+jv)(u+j^2v)$, where j is a cube root of unity. Then show that this implies that q factors as fgh, where $f^3 + g^3 + h^3 = 0$.
7. Prove that if A is a Noetherian ring and P a prime ideal of A, then $\operatorname{height}(P) + \operatorname{depth}(P) = \dim A$, where $\operatorname{height}(P)$ is the maximal length of a strict chain of prime ideals contained in P and $\operatorname{depth}(P)$ is the maximal length of a strict chain of prime ideals containing P.
8. Let V be an irreducible holomorphic subvariety of an open set in $\mathbb{C}^n$. Prove that $\dim \mathbf{V}_\lambda$ is constant for $\lambda \in V$.

9. Let V be a holomorphic subvariety of an open set U in $\mathbb{C}^n$. Suppose there is a finite subset of $\mathcal{H}(U)$ which generates the ideal of $\mathbf{V}_\lambda$ at each $\lambda \in V$. Prove that the singular locus of V is a subvariety of V. Hint: Use Corollary 5.4.4.
10. Prove the algebraic analogue of Lemma 5.1.2. That is, suppose that V is an algebraic subvariety of $\mathbb{C}^n$, W an algebraic subvariety of $\mathbb{C}^m$, and $f : V \to W$ is a regular map which is *finite* in the sense that $\mathcal{O}(V)$ is finite over $\mathcal{O}(W)$ under the induced ring homomorphism $f^* : \mathcal{O}(W) \to \mathcal{O}(V)$. Prove that $\dim V \leq \dim W$. If $f(V)$ is contained in no proper subvariety of W, prove that $\mathcal{O}(V)$ is a finite extension of $\mathcal{O}(W)$ and $\dim V = \dim W$.
11. Prove Proposition 5.6.1. Hint: Proceed as in the proof of Proposition 5.2.1.
12. Prove Proposition 5.6.2. Hint: Proceed as in the proof of Proposition 5.2.3, using the results of the preceding two exercises.
13. Prove Theorem 5.6.3. Hint: Proceed as in the proof of Theorem 5.3.2.
14. Prove that if $\mathbf{V}$ and $\mathbf{W}$ are irreducible germs at 0 of holomorphic subvarieties of a neighborhood of 0 in $\mathbb{C}^n$, with $\dim \mathbf{V} = n-1$ and $\dim \mathbf{W} = r$, then either $\mathbf{W} \subset \mathbf{V}$ or $\dim \mathbf{V} \cap \mathbf{W} = r - 1$.
15. For an irreducible algebraic subvariety V of $\mathbb{C}^n$, let $K(V)$ denote the quotient field of $\mathcal{O}(V)$. Show that irreducible algebraic subvarieties V of $\mathbb{C}^n$ and W of $\mathbb{C}^m$ have the property that $K(V) \simeq K(W)$ if and only if there are Zariski open subsets $V_0 \subset V$ and $W_0 \subset W$ and a biregular map $V_0 \to W_0$. Is there an analogue of this result for irreducible holomorphic subvarieties of open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$?
16. Prove that every subvariety of $\mathbb{C}^2$ defined by an irreducible quadratic polynomial is isomorphic to either $\mathbb{C}$ or $\mathbb{C}^* = \mathbb{C} - \{0\}$.
17. If V is a holomorphic subvariety of an open set in $\mathbb{C}^n$, and $\lambda \in V$ is a regular point of V, then ${}_V\mathcal{H}_\lambda$ is isomorphic to ${}_m\mathcal{H}_0$ for some m. Show, by example, that the analogous statement for algebraic subvarieties and their local rings is not true.
18. Let $\mathbf{G} = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$ be a finite morphism from $\mathbf{C}^n$ to itself. If $\mathbf{g}_1, \ldots, \mathbf{g}_n$ generate the maximal ideal of ${}_n\mathcal{H}_0$, show that Corollary 5.4.4 implies that $\mathbf{G}$ is the germ of a biholomorphic map.
19. Let $\mathbf{G}$ be a finite morphism, as in the previous exercise, but assume that $\mathbf{g}_1, \ldots, \mathbf{g}_n$ generate an ideal $\mathcal{I}$ which may not be the maximal ideal of ${}_n\mathcal{H}_0$. Prove that $\mathcal{I}$ has finite codimension in ${}_n\mathcal{H}_0$.
20. With $\mathbf{G}$ as in the previous exercise, the codimension in ${}_n\mathcal{H}_0$ of the ideal generated by $\mathbf{g}_1, \ldots, \mathbf{g}_n$ is an important invariant of the map $\mathbf{G}$. It is equal to the branching order at 0 of a finite branched holomorphic cover with germ $\mathbf{G}$. This is not easy to prove in general (see [**D**], 2.2.1). Prove it when $n = 1$.

Chapter 6

Homological Algebra

At this point, we leave the study of local properties of rings of holomorphic and regular functions and focus our attention on developing techniques for finding global solutions to problems which we know how to solve locally. The techniques we need are those of sheaf theory and sheaf cohomology. These are the techniques of homological algebra applied to the category of sheaves on a topological space X. In this chapter, we give an introduction to homological algebra. We then illustrate its use by applying it to the category of modules over a commutative ring. This material is quite standard and can be found in any number of graduate texts (e.g. [**CE**], [**GM**], [**Go**], [**Gro**], [**KS**], [**Mac**], [**Mit**], [**Ro**]).

In order to avoid set theoretic difficulties and to be able to use the axiom of choice whenever it is needed, we will assume that all of our constructions in this chapter take place inside one large set that satisfies the conditions of a Grothendieck universe [**Gr**]. With this convention, collections that are not normally sets become sets. For example, the set of all modules over a fixed ring is a set, because, by convention, we allow only modules that belong to the given universe. This does not really restrict the generality of our results, because every module, even if not in the universe, is isomorphic to a module that is in the universe.

6.1 Abelian Categories

Much of the machinery of mathematics, including that of homological algebra, works in great generality. It is wasteful to develop this machinery over and over again, in different contexts, but using basically the same constructions and proofs. Category theory provides a language with which to state and prove very general theorems, which can then be applied in a wide variety of contexts. We give a very brief introduction to this language. We assume that most students taking courses at the level of this text are already familiar with the language of abelian categories, but some may benefit from a brief refresher course.

6.1.1 Definition. *A category $\mathcal{A}$ consists of the following:*

(i) *a set of objects* $\mathrm{Ob}(\mathcal{A})$*;*

(ii) *for each ordered pair of objects A, B in* $\mathrm{Ob}(\mathcal{A})$*, a set* $\mathrm{Hom}_{\mathcal{A}}(A, B)$ *of morphisms* $\alpha : A \to B$*;*

(iii) *for each ordered triple of objects A, B, C, a composition law which assigns to morphisms* $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ *and* $\beta \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ *a morphism* $\beta \circ \alpha$ *in* $\mathrm{Hom}_{\mathcal{A}}(A, C)$*; this composition law is required to be associative;*

(iv) *for each object in* $\mathrm{Ob}(\mathcal{A})$*, an identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ *which leaves other morphisms fixed under composition on either side.*

Given a category $\mathcal{A}$ and objects A and B, the notation $\alpha : A \to B$ will indicate that α is an element of $\mathrm{Hom}_{\mathcal{A}}(A, B)$. A morphism $\alpha : A \to B$ will be called an *isomorphism* if there is a morphism $\beta : B \to A$ such that $\alpha \circ \beta = \mathrm{id}_B$ and $\beta \circ \alpha = \mathrm{id}_A$. The element β is then called the *inverse* of α and denoted α^{-1}.

A *monomorphism* is a morphism $\alpha : A \to B$ such that, for any pair of morphisms $\gamma_1, \gamma_2 \in \mathrm{Hom}_{\mathcal{A}}(C, A)$, the identity $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$ implies $\gamma_1 = \gamma_2$. A morphism α is called an *epimorphism*, if it satisfies such a cancellation rule on the other side, that is, if $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$ implies $\gamma_1 = \gamma_2$, for morphisms $\gamma_1, \gamma_2 \in \mathrm{Hom}_{\mathcal{A}}(B, C)$.

A category $\mathcal{B}$ is called a *subcategory* of a category $\mathcal{A}$ if $\mathrm{Ob}(\mathcal{B}) \subset \mathrm{Ob}(\mathcal{A})$, for each ordered pair $A, B \in \mathrm{Ob}(\mathcal{B})$, $\mathrm{Hom}_{\mathcal{B}}(A, B) \subset \mathrm{Hom}_{\mathcal{A}}(A, B)$, and for each $A \in \mathrm{Ob}(\mathcal{B})$, the identity morphisms of $\mathrm{Hom}_{\mathcal{B}}(A, A)$ and $\mathrm{Hom}_{\mathcal{A}}(A, A)$ agree. Of course, the composition law for morphisms of $\mathcal{B}$ should be induced by the one for $\mathcal{A}$. A *full subcategory* $\mathcal{B}$ of $\mathcal{A}$ is a subcategory such that $\mathrm{Hom}_{\mathcal{B}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$, for each ordered pair $A, B \in \mathcal{B}$.

Examples abound. There is the category whose objects are sets and whose morphisms are functions between sets. There is the category whose objects are topological spaces and whose morphisms are continuous maps. There is the category whose objects are groups and whose morphisms are group homomorphisms. This has, as a full subcategory, the category of abelian groups and homomorphisms of abelian groups. Rings and ring homomorphisms form a category with commutative rings and ring homomorphisms as a full subcategory. Given a fixed ring A, there is the category of A-modules and A-module homomorphisms. Germs of holomorphic varieties and morphisms of germs, as discussed in section 4.6, form a category.

Given categories $\mathcal{A}$ and $\mathcal{X}$, a *functor* F from $\mathcal{A}$ to $\mathcal{X}$ is a correspondence which assigns to each object A of $\mathcal{A}$ an object $F(A)$ of $\mathcal{X}$, and to each morphism $\alpha : A \to B$ a morphism $F(\alpha) : F(A) \to F(B)$. This correspondence is required to respect composition of morphisms and identity morphisms. That is, when $\beta \circ \alpha$ is defined, we should have

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha),$$

and, for each $A \in \mathcal{A}$, we should have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} .$$

Given a category $\mathcal{A}$, there is another category $\mathcal{A}^{op}$, which has the same objects as $\mathcal{A}$, but $\mathrm{Hom}_{\mathcal{A}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(B, A)$, for each ordered pair of objects A, B. In other words, we simply reverse all arrows. A functor from A^{op} to another category $\mathcal{X}$ is what is commonly called a *contravariant* functor from $\mathcal{A}$ to $\mathcal{X}$. That is, it reverses arrows.

Given a category $\mathcal{A}$, the set of all morphisms between objects of $\mathcal{A}$ forms the set of objects of a category Morph($\mathcal{A}$) in which the morphisms are commutative diagrams of morphisms. That is, a morphism from the object $\{\alpha : A_1 \to A_2\}$ to the object $\{\beta : B_1 \to B_2\}$ is a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_2} \\ B_1 & \xrightarrow{\beta} & B_2. \end{array}$$

The set of functors from a category $\mathcal{A}$ to a category $\mathcal{X}$ forms the set of objects of another category Funct($\mathcal{A}, \mathcal{X}$). The morphisms of this category are defined as follows: Given functors $F, G : \mathcal{A} \to \mathcal{X}$, we define a morphism ϕ from F to G to be an assignment of a morphism $\phi(A) : F(A) \to G(A)$ to each object $A \in \mathrm{Ob}\,\mathcal{A}$, in such a way that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\alpha)} & F(B) \\ {\scriptstyle \phi(A)}\downarrow & & {\scriptstyle \phi(B)}\downarrow \\ G(A) & \xrightarrow{G(\alpha)} & G(B) \end{array}$$

is commutative for every morphism $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$.

An *equivalence* between categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ with a quasi-inverse $G : \mathcal{B} \to \mathcal{A}$ – that is, a functor G such that $F \circ G$ is isomorphic to the identity functor in Funct($\mathcal{A}$), and $G \circ F$ is isomorphic to the identity functor in Funct($\mathcal{B}$).

Abelian categories are the categories in which kernels, cokernels, and images of morphisms exist and have the right properties – so that we may define exact sequences. The category of left modules over a (generally non-commutative) ring forms an abelian category. Conversely, any abelian category is equivalent to one which is a full abelian subcategory of the category of modules over some ring [**Mit**]. Thus, one will not be led astray if one thinks of objects in an abelian category as modules and morphisms as module homomorphisms. Before defining abelian categories precisely, we first define additive categories.

6.1.2 Definition. *An additive category $\mathcal{A}$ is a category in which*

(i) *for each ordered pair of objects A, B, the set $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is given the structure of an (additive) abelian group, in such a way that the composition law is bilinear;*

(ii) *there is an object 0 in $\mathcal{A}$ so that $\mathrm{Hom}_{\mathcal{A}}(0,0) = \{0\}$;*

(iii) *to each ordered pair of objects A, B, there is assigned a third object $A \oplus B$ and an isomorphism of functors $Hom_{\mathcal{A}}(A, \cdot\,) \oplus Hom_{\mathcal{A}}(B, \cdot\,) \simeq Hom_{\mathcal{A}}(A \oplus B, \cdot\,)$.*

(iv) *for each ordered pair of objects A, B, there is also an isomorphism of functors $Hom_{\mathcal{A}}(\,\cdot\,, A) \oplus Hom_{\mathcal{A}}(\,\cdot\,, B) \simeq Hom_{\mathcal{A}}(\,\cdot\,, A \oplus B)$.*

Actually, if conditions (i), (ii) and (iii) of the above definition hold, then (iv) does as well (Exercise 6.1).

Condition (iii) of Definition 6.1.2 implies there is an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(A \oplus B, A \oplus B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, A \oplus B) \oplus \mathrm{Hom}_{\mathcal{A}}(B, A \oplus B).$$

The image of $\mathrm{id} \in \mathrm{Hom}_{\mathcal{A}}(A \oplus B, A \oplus B)$ under this isomorphism is an element $i_1 \oplus i_2$ where $i_1 \in \mathrm{Hom}_{\mathcal{A}}(A, A \oplus B)$ and $i_2 \in \mathrm{Hom}_{\mathcal{A}}(B, A \oplus B)$. These are the inclusion morphisms. They have the following universal property: Given any pair of morphisms $\alpha : A \to C$ and $\beta : B \to C$, there is a morphism $\gamma : A \oplus B \to C$ such that $\alpha = \gamma \circ i_1$ and $\beta = \gamma \circ i_2$. This follows from another application of (iii) of Definition 6.1.2, this time to $\alpha \oplus \beta$. Using (iv) in the same way, we obtain projection morphisms $\pi_1 : A \oplus B \to A$ and $\pi_2 : A \oplus B \to B$ with the analogous universal property (just reverse the arrows).

If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, then an *additive functor* from $\mathcal{A}$ to $\mathcal{X}$ is a functor F such that $F : \mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{X}}(F(A), F(B))$ is a homomorphism of abelian groups.

Throughout the remainder of this section, $\mathcal{A}$ will be an additive category and objects and morphisms will be objects and morphisms of $\mathcal{A}$.

6.1.3 Definition. *Let $\alpha : A \to B$ be a morphism. Suppose there exists an object K and a morphism $\kappa : K \to A$ such that every morphism $\gamma : C \to A$*

with $\alpha \circ \gamma = 0$ *factors as* $\kappa \circ \gamma'$*, where* $\gamma' \in \mathrm{Hom}_{\mathcal{A}}(C, K)$*. Suppose, further, that the correspondence* $\gamma \to \gamma'$ *from* $\{\gamma \in \mathrm{Hom}_{\mathcal{A}}(C, A) : \alpha \circ \gamma = 0\}$ *to* $\mathrm{Hom}_{\mathcal{A}}(C, K)$ *is an isomorphism of functors of* C*. Then* K *is called the kernel of* α *and is denoted* $\mathrm{Ker}(\alpha)$*.*

Suppose we have a commutative diagram of morphisms

$$\begin{array}{ccccc} K_1 & \xrightarrow{\kappa_1} & A_1 & \xrightarrow{\alpha_1} & B_1 \\ & & \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \nu} \\ K_2 & \xrightarrow{\kappa_2} & A_2 & \xrightarrow{\alpha_2} & B_2, \end{array}$$

where K_1 and K_2 are kernels for α_1 and α_2. Then $\alpha_2 \circ \mu \circ \kappa_1 = \nu \circ \alpha_1 \circ \kappa_1 = 0$ implies that there is a morphism $\gamma : K_1 \to K_2$ such that $\mu \circ \kappa_1 = \kappa_2 \circ \gamma$. In the case where $A_1 = A_2$, $B_1 = B_2$, and $\alpha_1 = \alpha_2$, this can be used to show that $\mathrm{Ker}(\alpha)$ and the morphism $\kappa : \mathrm{Ker}(\alpha) \to A$, if they exist, are unique up to isomorphism. It also implies that $\mathrm{Ker}(\alpha)$ is a functor of the morphism α.

If $\alpha : A \to B$ is a morphism in $\mathcal{A}$, and the corresponding morphism $B \to A$ in $\mathcal{A}^{op}$ has a kernel L, then the corresponding object in $\mathcal{A}$ is called the *cokernel* of α and is denoted $\mathrm{Coker}(\alpha)$. It has the property that it defines an isomorphism from $\{\gamma \in \mathrm{Hom}_{\mathcal{A}}(B, C) : \gamma \circ \alpha = 0\}$ to $\mathrm{Hom}(L, C)$, which is a functor of C.

With $\alpha : A \to B$ as above, if $\mathrm{Ker}(\alpha) \to A$ has a cokernel, we call it the *coimage* of α and denote it $\mathrm{Coim}(\alpha)$. Similarly, if $B \to \mathrm{Coker}(\alpha)$ has a kernel, then we call it the *image* of α and denote it by $\mathrm{Im}(\alpha)$.

Note that α is a monomorphism if and only if $0 = \mathrm{Ker}(\alpha)$; in this case, $A = \mathrm{Coim}(\alpha)$. Similarly, α is an epimorphism if and only if $0 = \mathrm{Coker}(\alpha)$; in this case, $B = \mathrm{Im}(\alpha)$.

Since the composition $A \to B \to \mathrm{Coker}(\alpha)$ is 0, there is a morphism $A \to \mathrm{Im}(\alpha)$ so that $\alpha : A \to B$ factors as $A \to \mathrm{Im}(\alpha) \to B$. The composition $\mathrm{Ker}(\alpha) \to A \to \mathrm{Im}(\alpha)$ is 0 and so there is a morphism $\mathrm{Coim}(\alpha) \to \mathrm{Im}(\alpha)$ so that $A \to \mathrm{Im}(\alpha)$ factors as $A \to \mathrm{Coim}(\alpha) \to \mathrm{Im}(\alpha)$.

6.1.4 Definition. *An additive category* $\mathcal{A}$ *is called an abelian category if* $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Coker}(\alpha)$ *exist for every morphism* α *and if the natural morphism* $\mathrm{Coim}(\alpha) \to \mathrm{Im}(\alpha)$ *is an isomorphism for every morphism* α*.*

Note that, in an abelian category, a morphism $\alpha : A \to B$ is an isomorphism if and only if $\mathrm{Ker}(\alpha) = 0$ and $\mathrm{Coker}(\alpha) = 0$. That an isomorphism has vanishing kernel and cokernel is clear. On the other hand, if the kernel and cokernel both vanish, then $A = \mathrm{Coim}(\alpha)$ and $B = \mathrm{Im}(\alpha)$, so the fact that α is an isomorphism, in this case, is part of the definition of abelian category.

6.1.5 Definition. *Let $\mathcal{A}$ be an abelian category. A sequence of two morphisms*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is called an exact sequence if $\beta \circ \alpha = 0$, and if the resulting induced morphism $\mathrm{Im}(\alpha) \to \mathrm{Ker}(\beta)$ *is an isomorphism. A general sequence of morphisms is called exact if each pair of successive morphisms in the sequence is exact.*

A functor between abelian categories is called an *exact functor* if it takes exact sequences to exact sequences.

Note that every morphism $\alpha : A \to B$ in an abelian category can be resolved into two exact sequences

$$0 \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow A \longrightarrow \mathrm{Im}(\alpha) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Im}(\alpha) \longrightarrow B \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow 0.$$

A subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$ is called an *abelian subcategory* if $\mathcal{A}$ is an abelian category and if the inclusion of $\mathcal{A}$ into $\mathcal{B}$ is an exact functor.

6.1.6 Example. Let A be any ring (possibly non-commutative) with identity. Then the category of left modules and left module homomorphisms over A is an abelian category. It turns out that every abelian category is equivalent to a full abelian subcategory of the category of left modules over some ring [**Mit**].

6.1.7 Example. Let A be a Noetherian commutative ring. The category of all A-modules is, of course, an abelian category. It has the category of finitely generated A-modules as a full abelian subcategory.

6.1.8 Example. The category of Banach spaces and continuous linear maps is an additive category. The kernel and cokernel of every morphism exist, but the condition that $\mathrm{Coim}(\alpha) \to \mathrm{Im}(\alpha)$ be an isomorphism fails (Exercise 6.2). Hence, this is not an abelian category.

6.1.9 Example. Let A be an algebra over a field K. The category whose objects are A-modules and whose morphisms are K-linear maps is not an abelian category. It is, however, a full subcategory of the abelian category of vector spaces over K. Also, it has the abelian category of A-modules and A-module homomorphisms as a subcategory – not a full subcategory, of course.

6.2 Complexes

Let $\mathcal{A}$ be an abelian category. A *complex* $A = \{A^n, \delta^n\}$ in $\mathcal{A}$ is a sequence of objects and morphisms of $\mathcal{A}$ of the form

$$\cdots \xrightarrow{\delta^{n-2}} A^{n-1} \xrightarrow{\delta^{n-1}} A^n \xrightarrow{\delta^n} A^{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

such that $\delta^{n+1} \circ \delta^n = 0$ for each n. The morphisms δ^n are the *differentials* of the complex.

A complex A, as above, is said to be *bounded below* if there is an integer j so that $A^n = 0$ for $n \leq j$, and *bounded above* if there is an integer k so that $A^n = 0$ for $n > k$. If it is both bounded above and bounded below, then we say the complex is *bounded.*

If $A = \{A^n, \delta^n_A\}$ and $B = \{B^n, \delta^n_B\}$ are complexes in $\mathcal{A}$, then a *morphism of complexes* $\alpha : A \to B$ is a sequence of morphisms $\alpha^n : A^n \to B^n$ so that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_A^{n-2}"] & A^{n-1} \arrow[r, "\delta_A^{n-1}"] \arrow[d, "\alpha^{n-1}"] & A^n \arrow[r, "\delta_A^n"] \arrow[d, "\alpha^n"] & A^{n+1} \cdots \arrow[r, "\delta_A^{n+2}"] \arrow[d, "\alpha^{n+1}"] & {} \\
\cdots \arrow[r, "\delta_B^{n-2}"] & B^{n-1} \arrow[r, "\delta_B^{n-1}"] & B^n \arrow[r, "\delta_B^n"] & B^{n+1} \arrow[r, "\delta_B^{n+2}"] & \cdots
\end{tikzcd} \tag{6.2.1}$$

is commutative. With this definition of morphism, the complexes in $\mathcal{A}$ and their morphisms form a category $\mathcal{C}(\mathcal{A})$.

Since $\delta^n \circ \delta^{n-1} = 0$, for a complex $A = \{A^n, \delta^n\}$, it follows that there is a well-defined morphism $\operatorname{Im} \delta^{n-1} \to \operatorname{Ker} \delta^n$.

6.2.1 Definition. *If A is a complex, then its cohomology is the graded group $H(A) = \{H^n(A)\}$, where $H^n(A) = \operatorname{Coker}\{\operatorname{Im} \delta^{n-1} \to \operatorname{Ker} \delta^n\}$.*

Note that a complex is exact (is an exact sequence in the sense of Definition 6.1.5), if and only if $H^n(A) = 0$ for every n.

If $\alpha : A \to B$ is a morphism of complexes, then the commutativity of (6.2.1) and the fact that Ker and Im are functors implies that $\alpha^n : A^n \to B^n$ induces morphisms $\operatorname{Ker} \delta^n_A \to \operatorname{Ker} \delta^n_B$ and $\operatorname{Im} \delta^{n-1}_A \to \operatorname{Im} \delta^{n-1}_B$ such that the diagram

$$\begin{tikzcd}
\operatorname{Im} \delta_A^{n-1} \arrow[r] \arrow[d] & \operatorname{Ker} \delta_A^n \arrow[d] \\
\operatorname{Im} \delta_B^{n-1} \arrow[r] & \operatorname{Ker} \delta_B^n
\end{tikzcd}$$

is commutative. The fact that Coker is a functor now implies that α induces a morphism $\tilde{\alpha} : H^n(A) \to H^n(B)$ for each n.

6.2.2 Definition. *A morphism $\alpha : A \to B$ of complexes is homotopic to* 0 *if there are morphisms $h^n : A^n \to B^{n-1}$ such that*

$$h^{n+1} \circ \delta_A^n + \delta_B^{n-1} \circ h^n = \alpha^n$$

for each n. Here, $\delta_A^n : A^n \to A^{n+1}$ and $\delta_B^{n-1} : B^{n-1} \to B^n$ are the differentials in the complexes A and B. Two morphisms $\alpha, \beta : A \to B$ of complexes are said to be homotopic if their difference, $\alpha - \beta$, is homotopic to 0.

For complexes A and B in $\mathcal{C}(\mathcal{A})$, the morphisms in $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A, B)$ which are homotopic to 0 form a subgroup. If a morphism homotopic to 0 is composed on either side with another morphism, then the result is also homotopic to 0. It follows that we may define a new category, whose objects are the objects of $\mathcal{C}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms of complexes, where two morphisms are equivalent if they are homotopic. The resulting category is called the *homotopy category* for $\mathcal{A}$ and is denoted $\mathcal{K}(\mathcal{A})$.

6.2.3 Proposition. *If A and B are complexes in $\mathcal{C}(\mathcal{A})$ and α and β are homotopic morphisms in $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A, B)$, then α and β induce the same morphism $H^n(A) \to H^n(B)$ on cohomology.*

Proof. We prove the equivalent statement: If $\alpha \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A, B)$ is homotopic to 0, then it induces the zero morphism on cohomology. Let $\{h^n\}$ be a homotopy for α and consider the equation

$$h^{n+1} \circ \delta_A^n + \delta_B^{n-1} \circ h^n = \alpha^n.$$

If we compose this with the natural morphism $\kappa : \mathrm{Ker}(\delta_A^n) \to A^n$, we obtain

$$\delta_B^{n-1} \circ h^n \circ \kappa = \alpha^n \circ \kappa,$$

which implies that $\alpha^n \circ \kappa$ is killed by composition with $B^n \to \mathrm{Coker}(\delta_B^{n-1})$. This implies that $\alpha^n \circ \kappa$ factors through $\mathrm{Im}(\delta_B^{n-1})$ and, hence, its composition with $\mathrm{Ker}(\delta_B^n) \to H^n(B)$ is 0. This implies that the induced morphism $\tilde{\alpha} : H^n(A) \to H^n(B)$ is 0.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

The following is a standard, much used fact. Most students will have seen a proof of this in some context. Here we will give an outline of the proof, leaving some of the details as an exercise (Exercise 6.3).

6.2.4 Theorem. *Given a short exact sequence of complexes in $\mathcal{C}(\mathcal{A})$,*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0, \tag{6.2.2}$$

there is a morphism $\tilde{\delta}^n : H^n(C) \to H^{n+1}(A)$, for each n, so that

$$\cdots \longrightarrow H^n(A) \xrightarrow{\tilde{\alpha}} H^n(B) \xrightarrow{\tilde{\beta}} H^n(C) \xrightarrow{\tilde{\delta}} H^{n+1}(A) \longrightarrow \cdots$$

is an exact sequence in $\mathcal{A}$. Furthermore, this assignment of a long exact sequence to each short exact sequence defines a functor from the category of short exact sequences in $\mathcal{C}(\mathcal{A})$ to the category of exact sequences in $\mathcal{C}(\mathcal{A})$.

Proof. Let $A = \{A^n, \delta^n\}$ be a complex. It follows immediately from the definition of H^n that the differential $\delta^n_A : A^n \to A^{n+1}$ induces a morphism $\hat{\delta}^n_A : \mathrm{Coker}(\delta^{n-1}_A) \to \mathrm{Ker}(\delta^{n+1}_A)$ so that the sequence

$$0 \longrightarrow H^n(A) \xrightarrow{i_A} \mathrm{Coker}(\delta^{n-1}_A) \xrightarrow{\hat{\delta}^n_A} \mathrm{Ker}(\delta^{n+1}_A) \xrightarrow{p_A} H^{n+1} \longrightarrow 0$$

is exact. When applied to each complex in the short exact sequence of the theorem, this leads to a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^n(A) & \xrightarrow{\tilde{\alpha}} & H^n(B) & \xrightarrow{\tilde{\beta}} & H^n(C) & & \\
 & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} & & \downarrow{\scriptstyle i_C} & & \\
 & & \mathrm{Coker}(\delta^{n-1}_A) & \xrightarrow{\alpha} & \mathrm{Coker}(\delta^{n-1}_B) & \xrightarrow{\beta} & \mathrm{Coker}(\delta^{n-1}_C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\delta}^n_A} & & \downarrow{\scriptstyle \hat{\delta}^n_B} & & \downarrow{\scriptstyle \hat{\delta}^n_C} & & \\
0 & \longrightarrow & \mathrm{Ker}(\delta^{n+1}_A) & \xrightarrow{\alpha} & \mathrm{Ker}(\delta^{n+1}_B) & \xrightarrow{\beta} & \mathrm{Ker}(\delta^{n+1}_C) & & \\
 & & \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_C} & & \\
 & & H^{n+1}(A) & \xrightarrow{\tilde{\alpha}} & H^{n+1}(B) & \xrightarrow{\tilde{\beta}} & H^{n+1}(C) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. If $K = \mathrm{Ker}(\hat{\delta}^n_C \circ \beta) = \mathrm{Ker}(\beta \circ \hat{\delta}^n_B)$ and

$$L = \mathrm{Coker}\{\alpha : \mathrm{Coker}(\delta^{n-1}_A) \to K)\},$$

then β induces a monomorphism $L \to \mathrm{Coker}(\delta_A^{n-1})$, with image equal to $\mathrm{Ker}(\hat{\delta}_C^n) = \mathrm{Im}(i_C)$. This defines an isomorphism $L \to H^n(C)$. Also, $\hat{\delta}_B^n$ induces a morphism $L \to \mathrm{Ker}(\delta_A^{n+1})$, which may be composed with p_A to yield a morphism $L \to H^{n+1}(A)$. Putting this together yields the morphism $\tilde{\delta}^n : H^n(C) \to H^{n+1}(A)$. That this fits together with $\tilde{\alpha}$ and $\tilde{\beta}$ to form a long exact sequence, we leave as an exercise (Exercise 6.3). Each morphism constructed in the process of defining $\tilde{\delta}^n$ is a morphism of functors of the short exact sequence (6.2.2). Thus, δ^n is as well.

Note that if

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\mu"] & B \arrow[r, "\beta"] \arrow[d, "\nu"] & C \arrow[r] \arrow[d, "\omega"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r] & 0
\end{tikzcd}$$

is a morphism of short exact sequences of complexes, then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(A) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\mu}"] & H^n(B) \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{\nu}"] & H^n(C) \arrow[r, "\tilde{\delta}"] \arrow[d, "\tilde{\omega}"] & H^{n+1}(A) \arrow[r] \arrow[d, "\tilde{\mu}"] & \cdots \\
\cdots \arrow[r] & H^n(A') \arrow[r, "\tilde{\alpha}'"] & H^n(B') \arrow[r, "\tilde{\beta}'"] & H^n(C') \arrow[r, "\tilde{\delta}'"] & H^{n+1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

induced on cohomology, is commutative and, thus, is a morphism of complexes. The commutativity of the squares involving $\tilde{\alpha}$, $\tilde{\alpha}'$, $\tilde{\beta}$, and $\tilde{\beta}'$ is due to the fact that the H^n are functors. The commutativity of the squares involving $\tilde{\delta}$ and $\tilde{\delta}'$ is due to the fact that δ^n is a morphism of functors of the short exact sequence (6.2.2). This proves the last statement of the theorem.

6.3 Injective and Projective Resolutions

An additive functor $F : \mathcal{A} \to \mathcal{X}$, between abelian categories, is called *exact* if, for each short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}(\mathcal{A})$, the complex

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

in $\mathcal{C}(\mathcal{X})$ is also exact. Since any exact complex may be decomposed into a series of short exact sequences, it follows that an additive functor is exact if and only if it preserves the exactness of arbitrary exact complexes.

The functor F is called *left exact* if, whenever

$$0 \to A \to B \to C$$

is exact, so is

$$0 \to F(A) \to F(B) \to F(C)$$

(exactness is preserved at the first two places). It is called *right exact* if , whenever

$$A \to B \to C \to 0$$

is exact, so is

$$F(A) \to F(B) \to F(C) \to 0$$

(exactness is preserved at the last two places). An additive functor is exact if and only if it is both left and right exact.

6.3.1 Proposition. *For an abelian category $\mathcal{A}$ and an object $A \in \mathcal{A}$,*
(i) *the functor $B \to \mathrm{Hom}_{\mathcal{A}}(A, B)$ is left exact;*
(ii) *the functor $B \to \mathrm{Hom}_{\mathcal{A}}(B, A)$ is left exact as a functor on $\mathcal{A}^{op}$.*

Proof. Let

$$0 \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3$$

be an exact sequence of morphisms in $\mathcal{A}$ and let $\alpha : A \to B_1$ be a morphism. If $\beta_1 \circ \alpha = 0$, then α factors through $\mathrm{Ker}\,\beta_1$. But $\mathrm{Ker}\,\beta_1 = 0$. Thus, $\alpha = 0$. This establishes the exactness of the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, B_1) \to \mathrm{Hom}_{\mathcal{A}}(A, B_2) \to \mathrm{Hom}_{\mathcal{A}}(A, B_3) \tag{6.3.1}$$

at the first stage.

Now suppose $\alpha : A \to B_2$ is a morphism with $\beta_2 \circ \alpha = 0$. Then α factors through $\mathrm{Ker}\,\beta_2$. Since $\mathrm{Im}\,\beta_1 \to \mathrm{Ker}\,\beta_2$ and $B_1 \to \mathrm{Im}\,\beta_1$ are both isomorphisms, it follows that α factors through B_1. This establishes exactness of (6.3.1) at the second stage and completes the proof of (i).

For part (ii), recall that $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, A) = \mathrm{Hom}_{\mathcal{A}^{op}}(A, \,\cdot\,)$. That this is left exact as a functor on $\mathcal{A}^{op}$ now follows from part (i).

In an abelian category $\mathcal{A}$, an object A is called *injective* if $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, A)$ is an exact functor and *projective* if $\mathrm{Hom}_{\mathcal{A}}(A, \,\cdot\,)$ is exact. In this section, we shall focus our attention on injective objects. Of course, every statement we make about injectives has an analogous statement for projectives, since the projectives of $\mathcal{A}$ are just the injectives of $\mathcal{A}^{op}$. The following summarizes the most elementary properties of injectives:

6.3.2 Proposition. *In an abelian category $\mathcal{A}$:*

(i) *an object A of $\mathcal{A}$ is injective if and only if, given any monomorphism $i : B \to C$, each morphism $\beta : B \to A$ extends to a morphism $\gamma : C \to A$ such that $\beta = \gamma \circ i$;*

(ii) *each monomorphism $\alpha : A \to B$, with A injective, splits (has a left inverse);*

(iii) *if $0 \to A \to B \to C \to 0$ is an exact sequence, and A is injective, then B is injective if and only if C is injective.*

Proof. Since $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, A)$ is left exact on $\mathcal{A}^{op}$, it is exact if and only if, for every monomorphism $i : B \to C$, the morphism $\gamma \to \gamma \circ i : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ is surjective. Thus, part (i) holds.

Part (ii) follows from applying (i) to the case of the monomorphism $\alpha : A \to B$ and the morphism $\mathrm{id} : A \to A$. An extension of $\mathrm{id} : A \to A$ to a morphism $\gamma : B \to A$ yields a left inverse for α.

Part (iii) follows from part (ii), since a short exact sequence, as in (iii), with A injective, splits. Then $B \simeq A \oplus C$. If A is injective, then clearly $A \oplus C$ is injective if and only if C is injective.

The corresponding proposition for projective objects is the following. It is proved by applying Proposition 6.3.2 to $\mathcal{A}^{op}$.

6.3.3 Proposition. *In an abelian category $\mathcal{A}$:*

(i) *an object A of $\mathcal{A}$ is projective if and only if, for every epimorphism $\pi : C \to B$, each morphism $\beta : A \to B$ lifts to a morphism $\gamma : A \to C$ such that $\beta = \pi \circ \gamma$;*

(ii) *each epimorphism $\pi : B \to A$, with A projective, splits (has a right inverse);*

(iii) *if $0 \to C \to B \to A \to 0$ is an exact sequence, and A is projective, then B is projective if and only if C is projective.*

An abelian category $\mathcal{A}$ is said to have *enough injectives* if, for every object A, there is an injective object I^0 and a monomorphism $A \to I^0$. This, and the fact that every morphism has a cokernel, allows one to construct, for each object A, an exact sequence of the form

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^n \to \cdots,$$

where each I^n is injective. Such an exact sequence is called an *injective resolution* of A. There is another way of thinking about such resolutions which yields both economy of notation and additional insight and so is worth introducing. We will let I denote the complex

$$0 \to I^0 \to I^1 \to \cdots \to I^n \to \cdots$$

and identify A with the complex

$$0 \to A \to 0,$$

where A appears in degree 0. Then an injective resolution of A may be thought of as a morphism of complexes

$$i : A \to I,$$

where I is a complex of injective objects (0 in negative degrees) and i induces an isomorphism on cohomology (both complexes have vanishing cohomology in all degrees except 0, where the cohomology is A). A morphism of complexes which induces an isomorphism of cohomology is called a *quasi-isomorphism.* Thus, an injective resolution of an object A is a quasi-isomorphism $A \to I$, where I is a complex of injectives which vanishes in negative degrees. Actually, insisting that I vanish in negative degrees is equivalent, for the purposes of this theory, to insisting that it be bounded on the left – that is, vanish for sufficiently high negative degrees.

Similarly, we say $\mathcal{A}$ has *enough projectives* if $\mathcal{A}^{op}$ has enough injectives – that is, if, for every object A of $\mathcal{A}$, there is a projective object P^0 of $\mathcal{A}$ and an epimorphism $P^0 \to A$. If $\mathcal{A}$ has enough projectives, then every object A has a projective resolution

$$\cdots \to P^n \to \cdots \to P^1 \to P^0 \to A \to 0.$$

As before, this may be thought of as a quasi-isomorphism of complexes $P \to A$, with P a complex of projectives, if A is identified with the complex $0 \to A \to 0$.

A key result is the following:

6.3.4 Proposition. *If $\alpha : A \to B$ is a morphism in an abelian category and $A \to I$ and $B \to J$ are injective resolutions of A and B, then there is a morphism of complexes $\tilde{\alpha} : I \to J$ such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & I \arrow[d, "\tilde{\alpha}"] \\ B \arrow[r] & J \end{tikzcd}$$

is commutative. Furthermore, any two morphisms $I \to J$, with this property, are homotopic.

Proof. We may construct a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & I^0 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\tilde{\alpha}^0} \\ B & \longrightarrow & J^0 \end{array}$$

by choosing $\tilde{\alpha}^0$ to be an extension to I^0 of the morphism $A \to J^0$, obtained by composing $\alpha : A \to B$ with $B \to J^0$. This extension exists, because J_0 is injective. Next, we repeat this argument with A replaced by $\mathrm{Coker}\{A \to I^0\}$, B replaced by $\mathrm{Coker}\{B \to J^0\}$, and I^0 and J^0 replaced by I^1 and J^1. The result is a morphism $\tilde{\alpha}^1$ which makes commutative the diagram

$$\begin{array}{ccc} I^0 & \longrightarrow & I^1 \\ {\scriptstyle\tilde{\alpha}^0}\downarrow & & \downarrow{\scriptstyle\tilde{\alpha}^1} \\ J^0 & \longrightarrow & J^1. \end{array}$$

This is the basis for an induction argument which proves the first statement of the proposition. The second statement is proved with a similar induction argument, which we leave as an exercise (Exercise 6.4).

When Proposition 6.3.4 is applied to two different injective resolutions of A and the identity morphism from A to A, it implies the following:

6.3.5 Corollary. *If $i : A \to I$ and $j : A \to J$ are two injective resolutions of A, then there are morphisms $\alpha : I \to J$ and $\beta : J \to I$ such that*

$$\begin{array}{ccccc} I & \longleftarrow & A & \longrightarrow & I \\ \downarrow{\scriptstyle\alpha} & & \| & & \uparrow{\scriptstyle\beta} \\ J & \longleftarrow & A & \longrightarrow & J \end{array}$$

is a commutative diagram, and both $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identity.

Of course, there are analogous results for projective resolutions.

6.4 Higher Derived Functors

If α and β in $\mathcal{A}$ are homotopic morphisms between complexes in $\mathcal{C}(\mathcal{A})$, and $F : \mathcal{A} \to \mathcal{X}$ is an additive functor into another abelian category $\mathcal{X}$, then $F(\alpha)$ and $F(\beta)$ are also homotopic. This fact and Proposition 6.3.4 imply the following:

6.4.1 Proposition. *If $\alpha : A \to B$ is a morphism, $A \to I$ and $B \to J$ are injective resolutions of A and B, and F is an additive functor from $\mathcal{A}$ to another abelian category $\mathcal{X}$, then the morphism $\tilde{\alpha} : I \to J$ of Proposition 6.3.4 induces a morphism of cohomology $\alpha^* : H(F(I)) \to H(F(J))$, which is independent of the choice of $\tilde{\alpha}$. This is an isomorphism if α is an isomorphism.*

Because of the assumption that our categories, their objects, and their morphisms all belong to a Grothendieck universe, we may use the axiom of choice to assign to each object $A \in \mathrm{Ob}(\mathcal{A})$ a particular injective resolution $I(A)$. In many cases, this can be done in such a fashion that $A \to I(A)$ is a functor – ideally an exact functor. This is nice when it can be done (and it can be for sheaves), but it is not necessary for the development of the theory. In what follows, we will assume we are working in a category $\mathcal{A}$ with enough injectives and that a choice of injective resolution $A \to I(A)$ has been assigned to each object A. We may then construct the higher derived functors of a left exact functor F as follows: For an object $A \in \mathrm{Ob}(\mathcal{A})$ we set

$$R^n F(A) = H^n(F(I(A))),$$

the nth cohomology of the complex obtained by applying the functor F to the complex of injectives $I(A)$. Of course, by Proposition 6.4.1 different choices of resolutions $I(A)$ will yield isomorphic objects $R^n F(A)$. It also follows from Proposition 6.4.1 that R^n is a functor.

6.4.2 Theorem. *If F is a left exact functor from an abelian category with enough injectives to an abelian category, then there is an isomorphism of functors $F \to R^0 F$.*

Proof. The fact that F is left exact means that, if $0 \to A \to I^0 \to I^1 \to \cdots$ is an injective resolution, then exactness is preserved by F at the first two terms; that is,

$$0 \to F(A) \to F(I^0) \to F(I^1)$$

is exact. The theorem follows.

6.4.3 Theorem. *If $\mathcal{A}$ is an abelian category with enough injectives, F is a left exact functor from $\mathcal{A}$ to an abelian category $\mathcal{X}$, and*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{A}$, then there exists, for each p, a morphism $\delta^n : R^p F(C) \to R^{p+1} F(A)$ so that the sequence

$$\begin{gathered} 0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to \cdots \\ \cdots \to R^n F(A) \to R^n F(B) \to R^n F(C) \to R^{n+1} F(A) \to \cdots \end{gathered}$$

is exact. Furthermore, this defines a functor from the category of short exact sequences in $\mathcal{C}(\mathcal{A})$ to the category of exact sequences in $\mathcal{C}(\mathcal{X})$.

Proof. Given injective resolutions $A \to I(A)$ and $C \to I(C)$, we may construct an injective resolution $B \to J$, along with morphisms $I(A) \to J$ and $J \to I(C)$, such that

$$0 \longrightarrow I(A) \longrightarrow J \longrightarrow I(C) \longrightarrow 0$$

is an exact sequence of complexes. To do this, we set $J^n = I^n(A) \oplus I^n(C)$ and define morphisms $j : B \to J^0$ and $\delta^n : J^n \to J^{n+1}$, as follows: Using the fact that $I^0(A)$ is injective, we extend $i_A : A \to I^0(A)$ to a morphism $j_1 : B \to I^0(A)$. We let $j_2 : B \to I^0(C)$ be the composition of $B \to C$ with $i_C : C \to I^0(C)$. Then $j = j_1 \oplus j_2$. Clearly j is a monomorphism of B into J^0 and the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "i_A"'] & B \arrow[r] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "i_C"'] & 0 \\
0 \arrow[r] & I^0(A) \arrow[r] & J^0 \arrow[r] & I^0(C) \arrow[r] & 0
\end{tikzcd}$$

is commutative. We repeat this argument, with $i_A : A \to I^0(A)$ and $i_C : C \to I^0(C)$ replaced by $\operatorname{Coker} i_A \to I^1(A)$ and $\operatorname{Coker} i_C :\to I^1(C)$, respectively, and obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "i_A"'] & B \arrow[r] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "i_C"'] & 0 \\
0 \arrow[r] & I^0(A) \arrow[r] \arrow[d, "\delta^0_A"'] & J^0 \arrow[r] \arrow[d, "\delta^0"'] & I^0(C) \arrow[r] \arrow[d, "\delta^0_C"'] & 0 \\
0 \arrow[r] & I^1(A) \arrow[r] & J^1 \arrow[r] & I^1(C) \arrow[r] & 0
\end{tikzcd}$$

Continuing in this way, we construct an injective resolution $B \to J$ of B and morphisms of complexes $I(A) \to J$ and $J \to I(C)$ for which the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "i_A"'] & B \arrow[r] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "i_C"'] & 0 \\
0 \arrow[r] & I(A) \arrow[r] & J \arrow[r] & I(C) \arrow[r] & 0
\end{tikzcd}$$

is commutative, with exact rows. On applying any left exact functor F, we obtain the diagram

$$(6.4.1) \qquad 0 \longrightarrow F(I(A)) \longrightarrow F(J) \longrightarrow F(I(C)) \longrightarrow 0.$$

By construction, each sequence $0 \to I^p(A) \to J^p \to I^p(C) \to 0$ is a split short exact sequence. Any additive functor takes a split short exact sequence to a split short exact sequence. Thus, (6.4.1) is a short exact sequence of complexes. The existence of the long exact sequence of the theorem now follows from Theorem 6.2.4.

The fact that a morphism of short exact sequences induces a morphism of the corresponding long exact sequences of cohomology follows from Theorem 6.2.4, Proposition 6.3.4 , and a diagram chase, involving the construction of the resolution J, to show that a morphism of short exact sequences induces a morphism between the corresponding short exact sequences of injective complexes constructed above. We will skip the details. Note that the morphisms $R^nF(A) \to R^nF(B)$ and $R^nF(B) \to R^nF(C)$ are just those induced by $A \to B$ and $B \to C$; i.e. they are the images of these morphisms under the functor R^nF. The connecting morphisms $R^nF(C) \to R^{n+1}F(A)$, a priori, depend on the choices made in the construction of J. In fact, they do not depend on these choices. They are well defined and depend functorially on the short exact sequence $0 \to A \to B \to C \to 0$. The proof of this is one of the details being skipped.

As indicated in the above theorem, we regard the sequence $\{R^nF, \delta^n\}$ as a functor from the category of short exact sequences of objects in $\mathcal{A}$ to the category of long exact sequences of objects in $\mathcal{X}$. It has a special form, in that each R^nF takes a short exact sequence $0 \to A \to B \to C \to 0$ to a complex with three terms, $R^nF(A) \to R^nF(B) \to R^nF(C)$, and the connecting morphisms $\delta^n : R^nF(C) \to R^{n+1}F(A)$ interlace these to form a long exact sequence. We will call a functor from short exact sequences in $\mathcal{A}$ to long exact sequences in $\mathcal{X}$, which has this special form, a *δ-functor*. The set of δ-functors itself forms a category in the obvious way.

6.4.4 Proposition. *Let $F : \mathcal{A} \to \mathcal{X}$ be a left exact functor between abelian categories, and let I be an injective object of $\mathcal{A}$. Then $R^nF(I) = 0$, for all $n \neq 0$.*

Proof. If I is injective, then $0 \to I \to I \to 0$ is an injective resolution of I. It follows from Corollary 6.3.5 that $R^nF(I)$ may be computed by applying F to this resolution and taking cohomology. Thus, $R^nF(I) = 0$, for $n \neq 0$.

The δ-functor constructed in Theorem 6.4.3 is unique in the sense of the following proposition.

6.4.5 Proposition. *If $\{G^n, \gamma^n\}$ is a δ-functor with the properties listed for $\{R^nF, \delta^n\}$ in Theorems 6.4.2 and 6.4.3 and Proposition 6.4.4, then there is an isomorphism of δ-functors $\{R^nF, \delta^n\} \to \{G^n, \gamma^n\}$.*

Proof. The isomorphism $R^0F \to G^0$ is determined by the isomorphisms $F \to R^0F$ and $F \to G^0$ of Theorem 6.4.2.

If A is an object of $\mathcal{A}$, and $A \to I$ is an injective resolution, then we have a short exact sequence

$$0 \to A \to I^0 \to K^1 \to 0, \tag{6.4.2}$$

where $K^1 = \operatorname{Ker}\{I^1 \to I^2\}$. Since $R^nF(I^1) = G^n(I^1) = 0$, for $n > 1$, by Proposition 6.4.4, the long exact sequences for F and G give us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A) \arrow[r] \arrow[d] & R^0F(I^0) \arrow[r] \arrow[d] & R^0F(K^1) \arrow[r] \arrow[d] & R^1F(A) \arrow[r] & 0 \\
0 \arrow[r] & G^0(A) \arrow[r] & G^0(I^0) \arrow[r] & G^0(K^1) \arrow[r] & G^1(A) \arrow[r] & 0
\end{tikzcd}$$

with exact rows and with the three vertical arrows isomorphisms. It follow that the isomorphism $R^0F(K^1) \to G^0(K^1)$ induces an isomorphism $R^1F(A) \to G^1(A)$. This is true for every A. The result is an isomorphism of functors of A, since this is true of the other vertical arrows in the diagram.

If we assume that an isomorphism of functors $R^nF \to G^n$ has been constructed for some $n > 0$, then higher degree portions of the long exact sequences associated to (6.4.2) yield the diagram of isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & R^nF(K^1) \arrow[r] \arrow[d] & R^{n+1}F(A) \arrow[r] & 0 \\
0 \arrow[r] & G^n(K^1) \arrow[r] & G^{n+1}(A) \arrow[r] & 0.
\end{tikzcd}$$

This defines an isomorphism $R^{n+1}F(A) \to G^{n+1}(A)$, which fills in the diagram to be commutative. This is clearly an isomorphism of functors of A. By induction, the proof is complete.

In practice, one does not really compute the objects $R^nF(A)$ using injective resolutions. Typically, one uses the long exact sequence to reduce the computation of $R^nF(A)$ for complicated objects A to that for simpler objects for which the answer is known, or one uses Theorem 6.4.6 below which often allows one to compute $R^nF(A)$ using much simpler resolutions of A.

An object C is said to be F-*acyclic* if $R^nF(C) = 0$ for $n > 0$. Proposition 6.4.4 implies that injective objects are F-acyclic for every left exact functor F.

6.4.6 Theorem. *Let F be as in Theorem 6.4.3, and suppose that $A \to J$ is a resolution of A by a complex of F-acyclic objects. Then there is an isomorphism $R^nF(A) \to H^n(F(J))$ for each n. Furthermore, these isomorphisms are functors of the pair consisting of A and the resolution $A \to J$.*

Proof. Let $K^n = \mathrm{Ker}\{J^n \to J^{n+1}\}$, and consider the long exact sequences for $\{R^nF\}$ determined by the short exact sequences

$$0 \to K^p \to J^p \to K^{p+1} \to 0, \quad p \geq 0,$$

where $K^0 = A$. Using the fact that $R^nF(J^p) = 0$ for $n > 0$, we conclude from these long exact sequences that

$$R^qF(K^p) \simeq R^{q-1}F(K^{p+1}), \quad p \geq 0,\ q > 1,$$

and

$$R^1F(K^{n-1}) \simeq \mathrm{Coker}\{F(J^{n-1}) \to F(K^n)\} \simeq H^n(J).$$

An induction argument, using these two isomorphisms, then shows that $R^nF(A) = R^nF(K^0) \simeq H^n(J)$, as required. One checks that each of the isomorphisms used in this argument depends functorially on A and the resolution $A \to J$.

6.5 Ext

Let $\mathcal{A}$ be an abelian category and A an object of $\mathcal{A}$. Then, by Proposition 6.3.1, the correspondence $B \to \mathrm{Hom}_{\mathcal{A}}(A, B)$ is a left exact functor from $\mathcal{A}$ to the category of abelian groups. We next apply the machinery developed in the previous section to this functor.

If $\mathcal{A}$ has enough injectives, then we have a sequence of derived functors for $\mathrm{Hom}_{\mathcal{A}}(A, \cdot\,)$.

6.5.1 Definition. *If $\mathcal{A}$ has enough injectives, we define* $\mathrm{Ext}^n_{\mathcal{A}}(A, \cdot\,)$ *to be the nth derived functor of* $\mathrm{Hom}_{\mathcal{A}}(A, \cdot\,)$.

By construction, $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ is a functor of $B \in \mathcal{A}$. It is also a functor of its first argument A. To see this, recall that $\mathrm{Ext}^n_{\mathcal{A}}(A, B) = H^n(\mathrm{Hom}_{\mathcal{A}}(A, I))$, where $B \to I$ is an injective resolution of B. With I fixed, this is clearly a contravariant functor of A. Thus, $\mathrm{Ext}_{\mathcal{A}}(\cdot\,, \cdot\,)$ is a *bifunctor* from $\mathcal{A}^{op} \times \mathcal{A}$ to abelian groups.

6.5.2 Theorem. *The sequence of bifunctors* $\{\mathrm{Ext}^n_{\mathcal{A}}\}$ *has the following properties:*

(i) $\mathrm{Ext}^0_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$;

(ii) $\mathrm{Ext}^n_{\mathcal{A}}(A,B)=0$ *if* B *is injective and* $n>0$;
(iii) $\mathrm{Ext}^n_{\mathcal{A}}(A,B)=0$ *if* A *is projective and* $n>0$;
(iv) *for each short exact sequence* $0\to B_1\to B_2\to B_3\to 0$, *there is a corresponding long exact sequence*

$$0\to \mathrm{Hom}_{\mathcal{A}}(A,B_1)\to \mathrm{Hom}_{\mathcal{A}}(A,B_2)\to \mathrm{Hom}_{\mathcal{A}}(A,B_3)\to \mathrm{Ext}^1_{\mathcal{A}}(A,B_1)\to\cdots$$

$$\to \mathrm{Ext}^n_{\mathcal{A}}(A,B_1)\to \mathrm{Ext}^n_{\mathcal{A}}(A,B_2)\to \mathrm{Ext}^n_{\mathcal{A}}(A,B_3)\to \mathrm{Ext}^{n+1}_{\mathcal{A}}(A,B_1)\to\cdots;$$

(v) *for each short exact sequence* $0\to A_1\to A_2\to A_3\to 0$, *there is a corresponding long exact sequence*

$$0\to \mathrm{Hom}_{\mathcal{A}}(A_3,B)\to \mathrm{Hom}_{\mathcal{A}}(A_2,B)\to \mathrm{Hom}_{\mathcal{A}}(A_1,B)\to \mathrm{Ext}^1_{\mathcal{A}}(A_3,B)\to\cdots$$

$$\to \mathrm{Ext}^n_{\mathcal{A}}(A_3,B)\to \mathrm{Ext}^n_{\mathcal{A}}(A_2,B)\to \mathrm{Ext}^n_{\mathcal{A}}(A_1,B)\to \mathrm{Ext}^{n+1}_{\mathcal{A}}(A_3,B)\to\cdots.$$

Proof. Parts (i), (ii), and (iv) follow directly from Theorems 6.4.2 and 6.4.3 and Proposition 6.4.4. If A is projective, then $\mathrm{Hom}_{\mathcal{A}}(A,\,\cdot\,)$ is an exact functor and, hence, preserves quasi-isomorphisms. Thus, if $B\to I$ is an injective resolution, then $\mathrm{Hom}_{\mathcal{A}}(A,B)\to\mathrm{Hom}_{\mathcal{A}}(A,I)$ is a quasi-isomorphism. This means it is a morphism of complexes which induces an isomorphism on cohomology. Since $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is a complex which is 0 except in degree 0, this means that $\mathrm{Hom}_{\mathcal{A}}(A,I)$ has vanishing cohomology except in degree 0. This establishes (iii).

If $0\to A_1\to A_2\to A_3\to 0$ is a short exact sequence and $B\to I$ is an injective resolution, then

$$0\to \mathrm{Hom}_{\mathcal{A}}(A_3,I)\to \mathrm{Hom}_{\mathcal{A}}(A_2,I)\to \mathrm{Hom}_{\mathcal{A}}(A_1,I)\to 0$$

is also exact, since each term in the complex I is injective. Thus, the existence of the long exact sequence of (v) follows from Theorem 6.2.4.

Now suppose that $\mathcal{A}$ has enough projectives. Then $\mathcal{A}^{op}$ has enough injectives and so the sequence $\{\mathrm{Ext}^n_{\mathcal{A}^{op}}\}$ is defined. If $\mathcal{A}$ also has enough injectives, then $\{\mathrm{Ext}^n_{\mathcal{A}}\}$ is defined. How are they related? There is an isomorphism of bifunctors

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B)\simeq \mathrm{Ext}^n_{\mathcal{A}^{op}}(B,A). \tag{6.5.1}$$

This follows immediately from Proposition 6.4.5, since Theorem 6.5.2 implies that each of the functors $\{\mathrm{Ext}^n_{\mathcal{A}}(A,\,\cdot\,)\}$ and $\{\mathrm{Ext}^n_{\mathcal{A}^{op}}(\,\cdot\,,A)\}$ satisfies the axioms defining the sequence of derived functors for $\mathrm{Hom}_{\mathcal{A}}(A,\,\cdot\,)$. In the case where the category $\mathcal{A}$ has enough projectives, but not enough injectives, the opposite category $\mathcal{A}^{op}$ has enough injectives, and we use (6.5.1) as the definition of $\mathrm{Ext}^n_{\mathcal{A}}(A,B)$.

6.6 The Category of Modules, Tor

The category of modules over a commutative ring A is an abelian category. We will show that it has enough injectives and enough projectives. Thus, all of the preceding machinery applies. The morphisms in this category are module homomorphisms. Given A-modules M and N, we denote the abelian group of all homomorphisms from M to N by $\mathrm{Hom}_A(M, N)$.

A free A-module is a module which is the direct sum of (possibly infinitely many) copies of A. Equivalently, a module F is free if it has a set S of generators which are independent over A, in the sense that no non-trivial equation of the form $a_1s_1 + \cdots + a_ns_n = 0$, $s_i \in S, a_i \in A$, holds in F. Such a set of generators is called a *basis* for F over A. A free module F, with a basis S, has the property that there is a natural identification between $\mathrm{Hom}_A(F, M)$ and the set of functions from S to M. If ϕ is such a function, then the homomorphism it defines is γ, where $\gamma(a_1s_1 + \cdots + a_ns_n) = a_1\phi(s_1) + \cdots + a_n\phi(s_n)$.

Given a set S, there is a free A-module $F(S)$ which contains the set S as a basis. The module $F(S)$ can be described as the set of all formal sums $a_1s_1 + \cdots + a_ks_k$ with $a_i \in A$ and $s_i \in S$, with addition and multiplication by elements of A defined in the obvious way.

6.6.1 Proposition. *For every A-module M, there is a free A-module F and an epimorphism $F \to M$.*

Proof. Let S be a set of generators of M over A, and let $F(S)$ be the free A-module generated by S. Then F is a free module, with S as a basis. The module homomorphism $F \to M$ induced by the inclusion $S \to M$ is the required epimorphism.

6.6.2 Proposition. *An A-module is projective if and only if it is a direct summand of a free module.*

Proof. If F is a free module over a set of generators S, and $\alpha : M \to N$ and $\beta : F \to N$ are module homomorphisms, with α an epimorphism, then for each $s \in S$, we may choose $\phi(s) \in M$ such that $\beta(s) = \alpha(\phi(s))$. Then ϕ determines a homomorphism $\gamma : F \to M$ such that $\beta = \alpha \circ \gamma$. Thus, a free module is projective. It is easy to see that a direct summand of a projective module is also projective, and so direct summands of free modules are projective.

Conversely, suppose P is a projective module. By the previous proposition, there is a free module F and an epimorphism $F \to P$. Since P is projective, the identity homomorphism $P \to P$ must lift to a homomorphism $P \to F$. That is, $F \to P$ has a right inverse. Thus, P is a direct summand of F.

The preceding two results show that there are enough projectives in the category of A-modules.

The category of modules over a ring also has enough injectives and, in fact, there is a functor which assigns to each module a monomorphism into an injective module. We give a brief description of the construction:

An abelian group is injective if and only if it is divisible (for every element g, and every integer n, there is an element h so that $nh = g$) (Exercise 6.5). If G is an abelian group, let $F(G)$ be the free abelian group generated by G as a set. The identity map $G \to G$ induces a surjective group homomorphism $F(G) \to G$. Let $K(G)$ be its kernel. Then let $Q(G)$ be the free module over the rationals generated by G. Note that $Q(G)$ contains $F(G)$ and $K(G)$ as subgroups. Finally, we set $D(G) = Q(G)/K(G)$. Then $D(G)$ is a divisible group containing a copy of G $(F(G)/K(G))$ as a subgroup. If A is any ring and M is a module over A, then we consider M as an abelian group and construct $D(M)$. Then the A-module $\mathrm{Hom}_Z(A, D(M))$ is injective (Exercise 6.6), and there is a monomorphism $M \to \mathrm{Hom}_Z(A, D(M))$ defined by

$$m \to (a \to am),$$

where M is regarded as a subgroup of $D(M)$. Thus, we have defined a functorial way to assign to each A-module M a monomorphism of M into an injective module.

If A happens to be an algebra over a field K, then $M \to \mathrm{Hom}_K(A, M)$ is a simpler way of embedding each A-module into an injective A-module (Exercise 6.7).

As a result of the above discussion, we have:

6.6.3 Proposition. *If A is a commutative ring, then the category of A-modules has enough injectives and enough projectives.*

Thus, the category of A-modules is a category for which Ext is defined. That is, we have a sequence of bifunctors $\{\mathrm{Ext}^n_A\}$, with the properties specified in Theorem 6.5.2. It may be defined using either injective resolutions of its second argument or projective resolutions of its first argument. For the category of A-modules, Ext has additional structure. In fact, here $\mathrm{Hom}_A(\cdot\,,\,\cdot)$ is a bifunctor to the category of A-modules, not just the category of abelian groups. It follows that the derived functors $\{\mathrm{Ext}^n_A\}$ are also bifunctors to the category of A-modules. Thus, for each pair M, N of A-modules and each n, $\{\mathrm{Ext}^n_A(M, N)\}$ has a natural A-module structure.

Given two modules M and N over a commutative ring A, their *tensor product* $N \otimes_A N$ over A is defined to be the abelian group generated by the symbols $m \otimes n$, for $m \in M, n \in N$, and the relations

$$
\begin{aligned}
&(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \\
&m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \\
&am \otimes n = m \otimes an,
\end{aligned}
\tag{6.6.1}
$$

for all $a \in A$, $m, m_1, m_2 \in M$, and $n, n_1, n_2 \in N$. This is a new A-module, where the module action is defined by letting A act on either M or N in $M \otimes N$ (the result is the same). The relations imposed in (6.6.1) are the minimal relations needed to ensure that the map $(m, n) \to m \otimes n$ is an A-bilinear morphism from $M \times N$ to $M \otimes_A N$.

It is clear from the construction that the tensor product is characterized among A-modules by the following:

6.6.4 Proposition. *Given any A-bilinear morphism $\phi : M \times N \to L$, there exists a unique A-module homomorphism $\lambda : M \otimes_A N \to L$ such that $\phi(m, n) = \lambda(m \otimes n)$.*

It follows from the above that $M \otimes_A (\,\cdot\,)$ is a functor from A-modules to A-modules for each fixed A-module M. In fact, a morphism $\alpha : N_1 \to N_2$ induces an A-bilinear map $(m, n) \to m \otimes \alpha(n)$ from $M \times N_1$ to $M \otimes_A N_2$, and this, in turn, induces a homomorphism $\mathrm{id} \otimes \alpha : M \otimes_A N_1 \to M \otimes_A N_2$, by the above proposition.

Note that the map $(m, n) \to n \otimes m$ induces a module isomorphism from $M \otimes_A N$ to $N \otimes_A M$. From this and the previous paragraph, it follows that $(\,\cdot\,) \otimes_A N$ is a functor from A-modules to A-modules, for each N.

6.6.5 Proposition. *For each fixed A-module M, the functor $N \to M \otimes N$ is right exact.*

Proof. Consider an exact sequence of A-modules,

$$
N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \longrightarrow 0.
$$

Let L be the Cokernel of $\mathrm{id} \otimes \alpha : M \otimes_A N_1 \to M \otimes_A N_2$. Then we have a commutative diagram

$$
\begin{tikzcd}
M \otimes_A N_1 \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d, equal] & M \otimes_A N_2 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\sigma"'] & 0 \\
M \otimes_A N_1 \arrow[r, "\mathrm{id}\otimes\alpha"] & M \otimes_A N_2 \arrow[r, "\mathrm{id}\otimes\beta"] & M \otimes_A N_3 \arrow[r] & 0,
\end{tikzcd}
$$

with the top row exact. For each pair $(m, n_3) \in M \times N_3$, there is an element $\phi(m, n_3) \in L$ defined to be the image of $m \otimes n_2$ in L, where n_2 is chosen so that $\beta(n_2) = n_3$. Note that $\phi(m, n_3)$ does not depend on the choice of n_2, since the morphism $M \otimes_A N_2 \to L$ kills each element in the image of $\mathrm{id} \otimes \alpha$. It follows that ϕ is an A-bilinear morphism from $M \times N_3$ to L and, hence, it induces a homomorphism $\tau : M \otimes_A N_3 \to L$. By the construction, this is an inverse for σ. Thus, the two rows of our diagram are isomorphic, and the bottom row must also be exact.

6.6.6 Proposition. *Let N and $\{M_\alpha\}_\alpha$ be A-modules. Then*
(i) *there is a natural isomorphism $A \otimes_A N \to N$;*
(ii) *if $M = \oplus_\alpha M_\alpha$ is the direct sum, then $M \otimes_A N \simeq \oplus_\alpha (M_\alpha \otimes_A N)$.*

Proof. The defining relations for $A \otimes_A N$ imply that the product map $a \otimes n \to an : A \otimes_A N \to N$ has $n \to 1 \times n$ as inverse. This proves (i).

The A-bilinear morphism $(\oplus m_\alpha) \times n \to \oplus(m_\alpha \otimes n)$ induces a homomorphism $\phi : M \otimes_A N \to \oplus_\alpha (M_\alpha \otimes_A N)$. Also, for each α, there is a homomorphism $M_\alpha \otimes_A N \to M \otimes_A N$, induced by the inclusion $M_\alpha \to M$. The sum of these homomorphisms is an inverse for ϕ.

A trivial consequence of this proposition is that the functor $M \otimes_A (\,\cdot\,)$ is exact in the case where $M = A$. This is also clearly true if M is an arbitrary direct sum of copies of A. Thus, the functor $M \otimes_A (\,\cdot\,)$ is exact when M is any free A-module. Finally, it is an easy consequence of the definition of direct summand that, if M is a direct summand of a free A-module, then $M \otimes_A (\,\cdot\,)$ is also an exact functor. A module M for which $M \otimes_A (\,\cdot\,)$ is an exact functor is called a *flat* A-module. Thus, we have just proved:

6.6.7 Proposition. *Every projective A-module is flat.*

We may regard $M \otimes_A (\,\cdot\,)$ as a functor from $\mathcal{A}^{op}$ to $\mathcal{A}^{op}$, where $\mathcal{A}$ denotes the category of A-modules. As such, it is a left exact functor. Furthermore, projectives in $\mathcal{A}$ are injectives in $\mathcal{A}^{op}$. Thus, the higher derived functors of $M \otimes_A (\,\cdot\,)$ as a functor from $\mathcal{A}^{op}$ to $\mathcal{A}^{op}$ are defined and are the functors

$$N \to H^n(P(N)),$$

where $P(N) \to N$ is a choice of a projective resolution of N for each A-module N.

6.6.8 Definition. *Given an A-module M, we define $\mathrm{Tor}^n_A(M, \,\cdot\,)$ to be the nth derived functor of the functor $M \otimes_A (\,\cdot\,)$, in the above sense.*

Note that, since $M \otimes_A (\,\cdot\,)$ is a functor from the category of A-modules to itself, the same thing is true of $\mathrm{Tor}^n_A(M, \,\cdot\,)$.

Thus, by Theorems 6.4.2 and 6.4.3, we have:

6.6.9 Theorem. *Given A-modules M and N,*
(i) $\mathrm{Tor}_0^A(M,N)$ *is naturally isomorphic to* $M \otimes_A N$;
(ii) *there is a functor from short exact sequences*

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

of A-modules to long exact sequences

$$\cdots \to \mathrm{Tor}_{n+1}^A(M,N_3) \to \mathrm{Tor}_n^A(M,N_1) \to \mathrm{Tor}_n^A(M,N_2) \to \mathrm{Tor}_n^A(M,N_3) \to$$

$$\cdots \to \mathrm{Tor}_1(M,N_3) \to M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3 \to 0$$

(iii) *the same thing holds with the roles of M and N reversed.*

6.6.10 Theorem. *For an A-module M, the following statements are equivalent:*
(i) *M is flat;*
(ii) $\mathrm{Tor}_n^A(M,N) = 0$, *for all $n > 0$ and all A-modules N;*
(iii) $\mathrm{Tor}_1^A(M,N) = 0$, *for all A-modules N.*
Of course, the same equivalence holds with M and N reversed.

Proof. If M is flat, then tensoring with M preserves the exactness of a projective resolution $0 \to N \to P$ of N. Hence (i) implies (ii). Trivially (ii) implies (iii). That (iii) implies (i) follows immediately from the long exact sequence for $\mathrm{Tor}^A(M, \cdot\,)$.

6.7 Hilbert's Syzygy Theorem

In this section, we will investigate when modules over a commutative ring have projective resolutions of finite length. This leads to the study of homological dimension. We will be interested, specifically, in Noetherian local rings and, ultimately, in the local rings associated to a holomorphic or algebraic variety.

If A is a commutative ring, M is an A-module, and

$$0 \to P_n \to P_z n - 1 \to \cdots \to P^0 \to M \to 0$$

is a projective resolution of M, then n is called the length of the resolution. If a projective resolution does not terminate, then we say it has infinite length. The length of an injective resolution is defined similarly.

The next proposition follows easily from the definitions and the long exact sequence for Ext. We leave its proof as an exercise (Exercise 6.15).

6.7.1 Proposition. *For a commutative ring A, an A-module M, and an integer n, the following statements are equivalent:*
(i) $\mathrm{Ext}_A^{n+1}(M,N)=0$, *for all A-modules N;*
(ii) *if there is an exact sequence of A-modules*

$$0\to N\to P_{n-1}\to\cdots\to P_0\to M\to 0,$$

with each P_i projective, then N is also projective;
(iii) *M has a projective resolution of length n.*

Of course, there is an analogous result for injective modules and injective resolutions (Exercise 6.16).

6.7.2 Definition. *Let A be a commutative ring and M an A-module. Then*
(i) *the projective dimension of M is the minimum of the lengths of projective resolutions of M;*
(ii) *the injective dimension of M is the minimum of the lengths of injective resolutions of M;*
(iii) *the global dimension of A is the supremum of the projective dimensions of all A-modules.*

In fact, the global dimension of A is also the supremum of the injective dimensions of all A-modules and the supremum of integers n for which $\mathrm{Ext}_A^n(M,N)$ is non-zero, for some pair of A-modules M,N. This follows immediately from the following proposition.

6.7.3 Proposition. *Let A be a commutative ring and n a non-negative integer. Then the following statements are equivalent:*
(i) *every A-module has projective dimension at most n;*
(ii) *every finitely generated A-module has projective dimension at most n;*
(iii) *every A-module has injective dimension at most n;*
(iv) $\mathrm{Ext}_A^{n+1}(M,N)=0$, *for every pair of A-modules M,N.*

Proof. That (i) implies (ii) and (iii) implies (iv) are trivial. We will prove (ii) implies (iii) and (iv) implies (i).

Assume (ii) is true for n. Given an A-module M, there is an exact sequence

$$0\to M\to I_0\to\cdots\to I_{n-1}\to J\to 0,$$

with each I^p injective. There is an induction argument (Exercise 6.14) which shows that $\mathrm{Ext}_A^1(N,J)\simeq\mathrm{Ext}_A^{n+1}(N,M)$. This is 0, by (ii), if N is any finitely generated A-module. Consequently, for any short exact sequence

$$0\to K\to Q\to N\to 0$$

of A-modules, with N finitely generated, any morphism from $K \to J$ extends to a morphism from Q to J. This implies that J is injective (Exercise 6.12) and completes the proof that (ii) implies (iii).

Now suppose (iv) is true, and M is an A-module. Then there is an exact sequence

$$0 \longrightarrow Q \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

with each P_i projective. As in the previous paragraph, an induction argument shows that $\mathrm{Ext}^1_A(Q, N) \simeq \mathrm{Ext}^{n+1}_A(M, N) = 0$, for any A-module N. Thus, Q is projective and (i) holds.

For Noetherian local rings, the situation is particularly nice, due to the following lemma.

6.7.4 Lemma. *Let A be a Noetherian local ring with maximal ideal M and residue field $k = A/M$. If N is a finitely generated A-module for which $\mathrm{Tor}^A_1(k, N) = 0$, then N is free.*

Proof. Given a finitely generated A-module N, we choose a set of elements $\{n_1, \dots, n_p\} \subset N$ whose images in the k vector space $k \otimes_A N$ form a basis. Then the morphism $\phi : A^p \to N$ defined by

$$\phi(a_1, \dots, a_n) = a_1 n_1 + \cdots + a_p n_p$$

has the property that $1 \otimes \phi : k \otimes_A A^p \to k \otimes_A N$ is an isomorphism. If $C = \mathrm{Coker}\,\phi$, then this implies that $k \otimes_A C = 0$. It follows from Nakayama's lemma (Exercise 3.5) that $C = 0$ and, hence, ϕ is surjective. If $K = \mathrm{Ker}\,\phi$, then the exact sequence

$$\cdots \mathrm{Tor}^A_1(k, N) \longrightarrow k \otimes_A K \xrightarrow{1\otimes\phi} k \otimes_A A^p \longrightarrow k \otimes_A N,$$

along with the assumption that $\mathrm{Tor}^A_1(k, N) = 0$, implies that $k \otimes_A K = 0$. That $K = 0$ follows from another application of Nakayama's lemma and the fact that, since A is Noetherian, K is also finitely generated. Thus, ϕ is an isomorphism, and so N is a free module.

6.7.5 Corollary. *If A is a Noetherian local ring, then every finitely generated projective A-module is free.*

6.7.6 Proposition. *Let A be a Noetherian local ring with maximal ideal M. If $k = A/M$, and N is a non-negative integer, then*

(i) *if N is any finite dimensional A-module, N has projective dimension at most n if and only if $\mathrm{Tor}^A_{n+1}(N, k) = 0$;*

(ii) *A has global dimension at most n if and only if $\mathrm{Tor}^A_{n+1}(k, k) = 0$.*

Proof. If N has a projective resolution of length n, then $\mathrm{Tor}^{n+1}(N,k)=0$. Conversely, if $\mathrm{Tor}^{n+1}(N,k)=0$, and

$$0 \to Q \to P_{n-1} \to \cdots \to P_0 \to N \to 0$$

is an exact sequence, with each P_k projective, then $\mathrm{Tor}_1^A(Q,k)=0$, by the same induction argument used to do Exercise 6.14 . By Lemma 6.7.4, this implies that Q is free and, hence, that N has a projective resolution of length n. This completes the proof of (i).

If A has global dimension at most n, then certainly $\mathrm{Tor}_{n+1}^A(k,k)=0$. On the other hand, if $\mathrm{Tor}_{n+1}^A(k,k)=0$, then part (i) implies that k has a projective resolution of length n. Then $\mathrm{Tor}_{n+1}^A(N,k)=0$ for every finitely generated A-module N. Applying part (i) again, we conclude that every finitely generated A-module has a projective resolution of length n and, by Proposition 6.7.3, that the global dimension of A is a most n. This completes the proof of part (ii).

The following proposition provides an easy way to compute the global dimension of a class of local rings that includes those of greatest interest to us.

6.7.7 Proposition. *Let A be a local ring with maximal ideal M, and suppose there exist a sequence $\{a_1,a_2,\dots,a_m\}\subset M$ such that*

(i) *$\{a_1,\dots,a_m\}$ is a set of generators of M;*

(ii) *for $j=1,\dots,n$, the image of a_j in $A/(a_1,\dots,a_{j-1})$ is not a zero divisor.*

Then m is the global dimension of A.

Proof. We will show that $\mathrm{Tor}_{m+1}^A(k,k)=0$, and $\mathrm{Tor}_m^A(k,k)\neq 0$, where $k=A/M$. The theorem then follows from Proposition 6.7.6(ii).

Set $A_0=A$, and for $j=1,\dots m$, set $A_j=A/(a_1,\dots,a_j)$. Then $A_m=k$. We will prove by induction that

$$\mathrm{Tor}_p^A(A_j,k)=0 \quad \text{for } p>j \text{ and } \quad \mathrm{Tor}_j^A(A_j,k)\neq 0. \tag{6.7.1}$$

Clearly (6.7.1) is true for $j=0$. Suppose it is true for a given $j<m$. Since a_{j+1} is not a zero divisor in its action on A_j, we have a short exact sequence

$$0 \longrightarrow A_j \xrightarrow{a_{j+1}} A_j \longrightarrow A_{j+1} \longrightarrow 0.$$

The long exact sequence for Tor shows that $\mathrm{Tor}_p^A(A_{j+1},k)=0$, for $p>j+1$. It also gives us an exact sequence

$$0 \longrightarrow \mathrm{Tor}_{j+1}^A(A_{j+1},k) \longrightarrow \mathrm{Tor}_j^A(A_j,k) \xrightarrow{a_{j+1}} \mathrm{Tor}_j^A(A_j,k).$$

Since a_{j+1} acts as 0 on the A-module k, it also acts as 0 on the A-module $\mathrm{Tor}_j^A(A_j,k)$ (Exercise 6.12). Hence, $\mathrm{Tor}_{j+1}^A(A_{j+1},k) \simeq \mathrm{Tor}_j^A(A_j,k)$. Since the later module is non-zero, this completes the induction and establishes (6.7.1) for all j. The case $j = m$ completes the proof of the theorem.

We say that a Noetherian local ring A is a *regular* local ring if its Krull dimension $\dim A$ is the same as its tangential dimension $\operatorname{tdim} A$. The latter is defined to be $\dim_k\{M/M^2\}$, where M is the maximal ideal of A, and $k = A/M$ is its residue field. A regular local ring contains a set of $m = \dim A$ elements $\{a_1, \dots, a_m\}$ which form a basis for M mod M^2. Such a set is called a *regular system of parameters* for A. It turns out that a regular system of parameters for a regular local ring always satisfies (i) and (ii) of the above proposition. That (i) is satisfied is a simple consequence of Nakayama's lemma (Exercise 6.18). Part (ii) is another matter – we have not developed enough dimension theory for abstract Noetherian local rings to prove this here (the proof involves the theory of Hilbert polynomials) – a proof can be found in Matsumura ([**Mat**], Theorem 36). Thus, a regular local ring has finite global dimension and, in fact, its global dimension, Krull dimension, and tangential dimension are all the same. Conversely, by a theorem of Serre, every Noetherian local ring of finite global dimension is regular ([**Mat**], Theorem 45). By definition, ${}_V\mathcal{O}_\lambda$ is a regular local ring if and only if λ is a regular point of the algebraic subvariety V. By Theorem 5.5.3, the same thing is true of ${}_V\mathcal{H}_\lambda$ if V is a holomorphic subvariety.

For ${}_n\mathcal{O}_0$ and ${}_n\mathcal{H}_0$ it is easy to see that the set of germs of coordinate functions $\{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ satisfies (i) and (ii) of Proposition 6.7.7 (Exercise 6.19). Thus, these rings have global dimension, dimension, and tangential dimension all equal to n. Clearly, the same thing is true of ${}_V\mathcal{H}_\lambda$ if V is a holomorphic subvariety of dimension n, and λ is a regular point of V. The analogous result for ${}_V\mathcal{O}_\lambda$ is also true, by the general theorem for regular local rings cited in the previous paragraph, but we are not in a position to prove it here.

With the machinery developed in this section, it is now a simple matter to prove a form of the Hilbert syzygy theorem. We first note that, if A is any commutative ring, then a set of k_0 generators of a finitely generated A-module M determines a surjection $A^{k_0} \to M$. If A is Noetherian, then the kernel of this surjection is also finitely generated and, hence, is the image of a morphism $\delta_1 : A^{k_1} \to A^{k_0}$. Continuing in this fashion, we conclude that a finitely generated module M over a Noetherian ring A has a resolution by free finitely generated modules, that is, a resolution of the form

$$(6.7.2) \qquad \dots \xrightarrow{\delta_{n+1}} A^{k_n} \xrightarrow{\delta_n} \dots \xrightarrow{\delta_1} A^{k_0} \longrightarrow M \longrightarrow 0.$$

Such a resolution is called a *chain of syzygies for M*. A chain of syzygies, as above, is said to *terminate at the nth stage* if the kernel of δ_{n-1} is free. In this case, the kernel of δ_{n-1} may be used to replace A^{k_n}, resulting in a chain in which the terms beyond the nth one are all 0.

6.7.8 Hilbert's Syzygy Theorem. *If A is ${}_n\mathcal{H}_0$ or ${}_n\mathcal{O}_0$, and M is a finitely generated A-module, then any chain of syzygies for M terminates at the nth stage.*

Proof. Given a chain of syzygies (6.7.2), let K_n denote the kernel of δ_{n-1}. By Proposition 6.7.7, A has global dimension n. By Proposition 6.7.1, K_n is projective, and by Corollary 6.7.5, K_n is free.

6.7.9 Corollary. *If A is ${}_n\mathcal{H}_0$ or ${}_n\mathcal{O}_0$, then every finitely generated A-module M has a free resolution of the form:*

$$0 \to A^{k_n} \to A^{k_{n-1}} \to \cdots \to A^{k_0} \to M \to 0.$$

Of course, the same results hold for regular local rings in general (see [**Mat**]).

Exercises

1. Prove that (i), (ii), and (iii) of Definition 6.1.2 imply (iv).
2. If $\mathcal{B}$ is the category of Banach spaces and bounded linear maps, prove that every morphism α in $\mathcal{B}$ has a kernel and a cokernel, but the condition $\operatorname{Coim}(\alpha) \simeq \operatorname{Im}(\alpha)$ fails for some morphisms α.
3. Fill in the details in the proof of Theorem 6.2.4.
4. Prove the second statement of Proposition 6.3.4.
5. Prove that an abelian group is injective if and only if it is divisible (for every element g and every integer n, there is an element h so that $nh = g$).
6. Prove that $\operatorname{Hom}_Z(A, D)$ is an injective A-module if D is a divisible abelian group, and A is a commutative ring.
7. Prove that, if A is an algebra over a field K, X is a vector space over K, and $\operatorname{Hom}_K(A, X)$ is given the obvious A-module structure, then $\operatorname{Hom}_K(A, X)$ is an injective A-module. Show that every A-module is a submodule of a module of this form. Then prove that an A-module is injective if and only if it is a direct summand of a module of the form $\operatorname{Hom}_K(A, X)$.

8. Fix $\lambda \in \mathbb{C}$. Find a resolution of the form given in Corollary 6.7.9 for the 1-dimensional $\mathbb{C}[z]$-module, $\mathbb{C}_\lambda$ on which each $p \in \mathbb{C}[z]$ acts as multiplication by the scalar $p(\lambda)$.
9. If V is any vector space and $L : V \to V$ is any linear transformation, then we can make V into a $\mathbb{C}[z]$-module V_L by letting $p \in \mathbb{C}[z]$ act on V as the linear transformation $p(L)$. Show that $\operatorname{Tor}_1^A(\mathbb{C}_\lambda, V_L) = \operatorname{Ker}(\lambda - L)$, and $\operatorname{Tor}_0^A(\mathbb{C}_\lambda, V_L) = \operatorname{Coker}(\lambda - L)$. Thus, $\lambda - L$ is invertible if and only if both of these Tor groups vanish.
10. Do the 2-dimensional analogue of Exercise 6.8: If $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $\mathbb{C}_\lambda$ is the 1-dimensional $\mathbb{C}[z_1, z_2]$-module on which each $p \in \mathbb{C}[z_1, z_2]$ acts as multiplication by the scalar $p(\lambda)$, then find a resolution of $\mathbb{C}_\lambda$ of the form given in Corollary 6.7.9. Then do the 2-dimensional analogue of Exercise 6.9.
11. Show that each non-zero element of $\operatorname{Ext}_A^1(M, N)$ corresponds to a non-trivial extension of M by N, that is, to a short exact sequence
$$0 \to N \to Q \to M \to 0$$
which does not split.
12. If A is a commutative ring, and M and N are A-modules, then for each p, $\operatorname{Tor}_p^A(M, N)$ and $\operatorname{Ext}_A^p(M, N)$ have natural A-module structures. Show that, if $a \in A$ kills either M or N, then it kills $\operatorname{Tor}_p^A(M, N)$ and $\operatorname{Ext}_A^p(M, N)$ for each p.
13. If A is a commutative ring, prove that an A-module J is injective if and only if, whenever Q is an A-module and K an A-submodule of Q such that it and one other element of A generate Q, then every morphism from K to J extends to a morphism from Q to J.
14. Let A be a commutative ring. Prove by induction on n that, if there is an exact sequence of A-modules
$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to J,$$
with I_k injective for $k = 0, \ldots, n-1$, then there is a natural isomorphism from $\operatorname{Ext}_A^1(N, J)$ to $\operatorname{Ext}_A^{n+1}(N, M)$.
15. Prove Proposition 6.7.1.
16. Formulate and prove the analogue of Proposition 6.7.1 for injective modules and injective resolutions.
17. Prove that the global dimension of a Noetherian ring A is the supremum of the projective dimensions of finitely generated A-modules.
18. Let A be a local ring with maximal ideal M and residue field $k = A/M$. Use Nakayama's lemma to prove that, if a set $\{a_1, \ldots, a_n\} \subset M$ generates M/M^2 as a k vector space, then it also generates M as an ideal.
19. If A is ${}_n\mathcal{H}_0$ or ${}_n\mathcal{O}_0$, prove that the sequence $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ satisfies (i) and (ii) of Proposition 6.7.7.

20. If ${}_V\mathcal{H}_0$ is a regular ring, M is its maximal ideal, and $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\} \subset {}_V\mathcal{H}_0$ is a regular system of parameters (see the paragraph following the proof of Proposition 6.7.7), then prove that $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ satisfies (i) and (ii) of Proposition 6.7.7.
21. With V as in Exercise 3.9, let M be the maximal ideal of ${}_V\mathcal{H}$. Show that M is generated by two of its elements in such a way that the resulting morphism ${}_V\mathcal{H} \to M$ has kernel isomorphic to M as a ${}_V\mathcal{H}$-module. Conclude that the Hilbert syzygy theorem fails to hold for ${}_V\mathcal{H}$. In fact, show that ${}_V\mathcal{H}$ does not have finite global dimension.

Chapter 7

Sheaves and Sheaf Cohomology

Sheaf cohomology provides the formal machinery for passing from local to global solutions for a wide variety of problems, as well as for classifying the obstructions to doing so when local solutions do not give rise to global solutions. Later chapters will make heavy use of this machinery. Here we will define sheaves and develop sheaf cohomology as an application of homological algebra to the category of sheaves on a topological space. We will then use these results to give brief developments of two classical cohomology theories (de Rham and Čech).

The language of sheaves leads to the concept of a ringed space. This is a topological space equipped with a sheaf of algebras – its structure sheaf – which defines additional structure on the space. This concept provides an economical way to define manifolds of various types as well as the morphisms between them.

Sheaf theory is a very rich subject that has come to be routinely used in a wide variety of fields of mathematics. Because of its importance in modern pure mathematics we have chosen to include a somewhat more extensive treatment of sheaf theory than is usually found in texts on several complex variables. Nevertheless, our treatment is far from comprehensive. A more comprehensive treatment may be found in [**KS**].

7.1 Sheaves

In this section, we define sheaves and morphism between sheaves. Before doing so, however, we give several examples of local to global problems that can be posed in the language of sheaves and solved using sheaf theory techniques.

7.1.1 Examples. The following are typical examples of local to global problems:

1. If X is a compact Hausdorff space, and f is a continuous complex valued function on X which never vanishes, then f locally has a continuous logarithm. Does it have a logarithm globally? In other words, is there a continuous function g on X such that $f = \exp g$?
2. If U is a domain in $\mathbb{C}$, and g is a C^∞ function on U, then the equation $\dfrac{\partial f}{\partial \bar{z}} = g$ has a solution locally in a neighborhood of each point. Does it have a global solution on U? We answered this in the affirmative in Chapter 1.
3. If U is a domain in $\mathbb{C}^n$, $V \subset U$ is a holomorphic subvariety, and f is holomorphic on V, then for each $\lambda \in V$ there is a holomorphic function defined in a neighborhood U_λ of λ in $\mathbb{C}^n$ whose restriction to $U_\lambda \cap V$ agrees with that of f. Is there a holomorphic function defined on all of U whose restriction to V is f?
4. If U is a domain in $\mathbb{C}^n$ and $V \subset U$ a holomorphic subvariety, then V is locally defined as the set of common zeroes of some set of holomorphic functions. Is there a set of holomorphic functions defined on all of U so that V is its set of common zeroes?

Generally these problems involve classes of functions – continuous, holomorphic, C^∞, etc. – which make sense on any open set in the underlying space. The notion of sheaf simply abstracts this idea:

7.1.2 Definition. *Let X be a topological space. We consider the collection of open subsets of X to be a category, where the morphisms are the inclusions $U \subset V$. Then a presheaf on X is a contravariant functor from this category to the category of abelian groups. A morphism between two presheaves on X is a morphism of functors.*

Thus, a presheaf S on X assigns to each open set $U \subset X$ an abelian group $S(U)$ and to each inclusion of open sets $U \subset V$ a homomorphism $\rho_{U,V} : S(V) \to S(U)$, called the *restriction map*, with $\rho_{U,U} = \text{id}$, for each open set U, and $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$, for each triple $U \subset V \subset W$.

A morphism $\phi : S \to T$, between two presheaves on X, assigns a morphism $\phi_U : S(U) \to T(U)$ to each open set U in a way which commutes with restriction. Unless the context dictates otherwise, we shall usually drop the subscript from ϕ_U and write simply ϕ.

7.1.3 Examples. The following are examples of presheaves:

1. The presheaf $\mathcal{C}$ of continuous functions is the functor which assigns to each open subset $U \subset X$ the algebra of continuous complex valued functions $\mathcal{C}(U)$ and to each inclusion of open sets $U \subset V$ the usual restriction map of functions $f \to f|_U : \mathcal{C}(V) \to \mathcal{C}(U)$.
2. The presheaf $\mathcal{C}^\infty$ of $\mathcal{C}^\infty$ functions on a $\mathcal{C}^\infty$ manifold is defined similarly.
3. On $\mathbb{C}^n$, we have the presheaf ${}_n\mathcal{H}$ of holomorphic functions, with restriction again defined as ordinary restriction of functions.
4. If $\mathbb{C}^n$ is given the Zariski topology, then we may define on it the presheaf ${}_n\mathcal{O}$ of regular functions.
5. If X is any topological space and G is a fixed abelian group, then we may define a presheaf called the *constant presheaf* by assigning G to each non-empty open set and 0 to the empty set.
6. If X is a topological space, G an abelian group, and $p \in X$, then the presheaf which assigns G to each open set containing p and 0 to each open set not containing p is called the *skyscraper presheaf* at p with stalk G.

The first four of the above examples are actually presheaves of algebras not just of abelian groups. As we shall see, the existence of additional structure on the objects $S(U)$ for a presheaf S is the typical situation, although the abelian group structure is all that is needed in much of the theory.

If $U \subset V$, then the image of $s \in S(V)$ under $\rho_{U,V} : S(V) \to S(U)$ will often be denoted $s|_U$ and called the *restriction* of s to U.

A sheaf is a presheaf which is locally defined in a sense made precise in the following definition:

7.1.4 Definition. *If S is a presheaf on X, then S is called a sheaf if the following conditions are satisfied for each open subset $U \subset X$ and each open cover $\mathcal{V}$ of U:*

(i) *if $s \in S(U)$ and $s|_V = 0$ for all $V \in \mathcal{V}$, then $s = 0$;*

(ii) *if $\{s_V \in S(V)\}_{V \in \mathcal{V}}$ is a collection of elements with the property that $s_V|_{V \cap W} = s_W|_{V \cap W}$ for each pair $V, W \in \mathcal{V}$, then there is an $s \in S(U)$ such that $s|_V = s_V$ for all $V \in \mathcal{V}$.*

Note that since the empty cover is an open cover of the empty set, it follows from Definition 7.1.4(i) that $S(\emptyset) = 0$ if S is a sheaf.

The presheaves of continuous, $\mathcal{C}^\infty$, holomorphic, and regular functions described in Examples 7.1.3 are obviously sheaves, as is the skyscraper presheaf. However, the presheaf which assigns a fixed group G to each non-empty open set is not a sheaf unless the underlying space has the property that every open set is connected. There is, however, a closely related sheaf: the sheaf of locally constant functions with values in G (see Example 7.1.9). In fact, for every presheaf there is an associated sheaf, as we shall show in Theorem 7.1.8.

If S is a presheaf on X, then the *stalk* of S at $x \in X$ is the group

$$S_x = \varinjlim\{S(U) : x \in U\}.$$

Given an element $s \in S(U)$ and an $x \in U$, the image of s in S_x is denoted s_x and is called the *germ* of s at x. The stalks contain the local information in a presheaf. If $\phi : S \to T$ is a morphism of presheaves, then clearly ϕ induces a morphism $\phi_x : S_x \to T_x$ for each x. The fact that sheaves are presheaves that are locally defined is illustrated by the following two results:

7.1.5 Proposition. *If S is a sheaf on X, U is an open subset of X, and $s \in S(U)$, then*

(i) *if $s_x = 0$ for some $x \in U$, then there is a neighborhood V_x of x such that $s_y = 0$ for all $y \in V_x$;*

(ii) *$s_x = 0$ for each $x \in U$ if and only if $s = 0$.*

Proof. If $s_x = 0$, then there is a neighborhood V_x of x such that the restriction of s to V_x is 0. Then $s_y = 0$ for all $y \in V_x$. If $s_x = 0$ for all $x \in U$, then U is covered by neighborhoods V_x such that the restriction of s to V_x is 0. By Definition 7.1.4(i), this implies that $s = 0$.

7.1.6 Proposition. *If $\phi : S \to T$ is a morphism of sheaves on X, then ϕ_U is injective for each open set U if and only if ϕ_x is injective for each $x \in X$. Furthermore, ϕ_U is an isomorphism for each open set U if and only if ϕ_x is an isomorphism for each $x \in X$.*

Proof. It is obvious from the definition of direct limit that ϕ_x is injective (surjective) for each $x \in X$ if ϕ_U is injective (surjective) for each open set $U \subset X$.

If each ϕ_x is injective, then an $s \in S(U)$, in the kernel of ϕ_U, has vanishing germ at each $x \in U$. By the previous proposition, this implies that $s = 0$. Thus, ϕ_U is injective.

If ϕ_x is an isomorphism for each $x \in X$, then ϕ_U is injective for each open set U, by the previous paragraph. So to complete the proof, we must show it is also surjective. Given $t \in T(U)$, there is an $s_x \in S_x$ such that $\phi_x(s_x) = t_x$ for each $x \in U$. But this means that for each $x \in U$ there is a neighborhood U_x of x and a section $s_{U_x} \in S(U_x)$ such that $\phi(s_{U_x}) = t|_{U_x}$. Then for $x, y \in U$,

$$\phi(s_{U_x}|_{U_x \cap U_y} - s_{U_y}|_{U_x \cap U_y}) = 0.$$

Since $\phi_{U_x \cap U_y}$ is injective, this implies that $s_{U_x}|_{U_x \cap U_y} = s_{U_y}|_{U_x \cap U_y}$ and, by Definition 7.1.4(ii), that there exists $s \in S(U)$ such that $s|_{U_x} = s_{U_x}$ for each $x \in U$. It follows that $\phi(s)_x = t_x$ for each $x \in U$ and, hence, that $\phi(s) = t$. Thus, ϕ_U is surjective and the proof is complete.

Note that Proposition 7.1.6 does not say that $\phi_U : S(U) \to T(U)$ is necessarily surjective for each open set U if $\phi_x : S_x \to T_x$ is surjective for each $x \in X$. This is, in fact, not true (Example 7.2.3 or Exercise 7.4).

7.1.7 Definition. *Let S be a presheaf on X, and let Y be a subset of X. By a section of S over Y, we will mean a function f which assigns to each point $y \in Y$ an element $f(y) \in S_y$ in such a way that the following continuity condition is satisfied: For each $y_0 \in Y$ there is a neighborhood V of y_0 and an element $s \in S(V)$ such that $f(y) = s_y$ for every $y \in V \cap Y$. We denote the group of sections of S over Y by $\Gamma(Y, S)$.*

If S is a presheaf, then $\mathcal{S}(U) = \Gamma(U, S)$ clearly defines another presheaf $\mathcal{S}$. Furthermore, each element $s \in S(U)$ defines a section $x \to s_x$ over U. Thus, there is a natural morphism of presheaves $S \to \mathcal{S}$. By the following theorem, $\mathcal{S}$ is actually a sheaf. We shall call it the *sheaf of sections* of the presheaf S.

7.1.8 Theorem. *If S is a presheaf, and $\mathcal{S}$ is defined as above, then*
- (i) *$\mathcal{S}$ is a sheaf;*
- (ii) *the natural morphism $S \to \mathcal{S}$ induces an isomorphism $S_x \to \mathcal{S}_x$ for each $x \in X$;*
- (iii) *$S \to \mathcal{S}$ is an isomorphism of presheaves if and only if S is a sheaf;*
- (iv) *every morphism of presheaves $S \to \mathcal{T}$, where $\mathcal{T}$ is a sheaf, factors uniquely through $S \to \mathcal{S}$.*

Proof. Part (i) is obvious since the continuity condition defining sections is a local condition.

If $s \in S(U)$ has germ at x which is sent to 0 by $S_x \to \mathcal{S}_x$, then $s_y = 0$ for all y in some neighborhood $V \subset U$ of x. In particular, $s_x = 0$. Thus, $S_x \to \mathcal{S}_x$ is injective. That it is surjective follows immediately from the definition of section. This proves part (ii).

Part (iii) follows from (ii) and Proposition 7.1.6.

Suppose $\mathcal{T}$ is a sheaf, and $\phi : S \to \mathcal{T}$ is a morphism of presheaves. Then ϕ induces a morphism $\tilde{\phi}$ from the sheaf $\mathcal{S}$ of sections of S to the sheaf of sections of $\mathcal{T}$ by $\tilde{\phi}(f)(x) = \phi_x(f(x))$, where $f \in \mathcal{S}(U)$ and $x \in U$. In fact, if $s \in S(V)$ for some neighborhood V with $x \in V \subset U$ and $s_y = f(y)$ for all $y \in V$, then $\tilde{\phi}(f)(y) = \phi_y(f(y)) = \phi_y(s_y) = \phi(s)_y$ for all $y \in V$. Thus, $\tilde{\phi}(f)$ is a section of $\mathcal{T}$ on U, and $\tilde{\phi}$ is a morphism from $\mathcal{S}$ to the sheaf of sections of $\mathcal{T}$, as claimed. Then ϕ factors as the natural map $S \to \mathcal{S}$, followed by $\tilde{\phi}$, followed by the inverse of the isomorphism between $\mathcal{T}$ and its sheaf of sections given by (iii). It is easy to see that any factorization of ϕ through $\mathcal{S}$ has this form. This proves (iv).

The content of (iv) of the above proposition is that $S \to \mathcal{S}$ is the left adjoint of the forgetful functor which assigns to a sheaf its associated presheaf. This just means that $\operatorname{Hom}(G(S), \mathcal{T}) \simeq \operatorname{Hom}(S, F(\mathcal{T}))$, where F is the functor which forgets that a sheaf is a sheaf and just considers it a presheaf, and G is the functor which assigns to a presheaf S its sheaf of sections $G(S) = \mathcal{S}$.

7.1.9 Example. For a topological space X and an abelian group G, consider the constant presheaf which assigns G to each non-empty open set $U \subset X$. The stalk of this sheaf at each point $x \in X$ is just G. A section of this sheaf over an open set U is a G-valued function on U which is locally constant. Thus, the corresponding sheaf of sections is the sheaf of locally constant G-valued functions on X. This is called the *constant sheaf* on X with stalk G.

7.2 Morphisms of Sheaves

Suppose $\phi : \mathcal{S} \to \mathcal{T}$ is a morphism of sheaves. It is easy to see that the functor $U \to \operatorname{Ker} \phi_U$ is, in fact, a sheaf and is the category theoretic kernel of ϕ as a sheaf morphism (Exercise 7.1). However, the analogous statement is not true in general for the cokernel (Exercise 7.2). It is clearly true that the presheaf $U \to \operatorname{Coker} \phi_U$ is a cokernel for ϕ as a morphism of presheaves. Unfortunately, $U \to \operatorname{Coker} \phi_U$ need not be a sheaf, so it is not generally the cokernel of ϕ in the category of sheaves. However, we have the following:

7.2.1 Proposition. *Suppose $\phi : \mathcal{S} \to \mathcal{T}$ is a morphism of sheaves. Then the sheaf of sections of the presheaf $U \to \operatorname{Coker} \phi_U$ is a cokernel for ϕ in the category of sheaves.*

Proof. If C is the presheaf $U \to \operatorname{Coker} \phi_U$ and $\mathcal{C}$ is its sheaf of sections, then the composition of the presheaf morphisms $\mathcal{T} \to C$ and $C \to \mathcal{C}$ is a sheaf morphism γ such that $\gamma \circ \phi = 0$. Any sheaf morphism $\delta : \mathcal{T} \to \mathcal{D}$ such that $\delta \circ \phi = 0$ must factor through $\mathcal{T} \to C$ as a presheaf morphism, but since it is a sheaf morphism, it must actually factor through γ, by Theorem 7.1.8(iv). This completes the proof.

We shall denote the sheaf of sections of the presheaf $U \to \operatorname{Coker} \phi_U$ by $\operatorname{Coker} \phi$. This may seem ambiguous, since one might use the same notation for $U \to \operatorname{Coker} \phi_U$ itself, it being the cokernel of ϕ as a morphism of presheaves. However, we will never do this, since our focus will be on the category of sheaves.

We emphasize that if U is an open set, then the space of sections $(\operatorname{Coker}\phi)(U)$ is not the obvious candidate, $\operatorname{Coker}\phi_U$, since $U \to \operatorname{Coker}\phi_U$ is not generally a sheaf; however, it is true that we get the obvious thing at the level of stalks: that is, $(\operatorname{Coker}\phi)_x = \operatorname{Coker}(\phi_x)$. This follows from the fact that a presheaf and its sheaf of sections have the same stalks (Theorem 7.1.8(ii)). It follows that $\operatorname{Im}\phi = \operatorname{Ker}(\mathcal{T} \to \operatorname{Coker}\phi)$ is characterized by

$$(\operatorname{Im}\phi)(U) = \{t \in \mathcal{T}(U) : t_x \in \operatorname{Im}\phi_x \ \forall\, x \in U\}.$$

In particular, ϕ is an epimorphism ($\operatorname{Coker}\phi = 0$) if and only if each ϕ_x is surjective.

Note that we have defined things in such a way that, for a morphism of sheaves, the notions of kernel, image, and cokernel, as well as epimorphism and monomorphism, are all local, that is, are all defined stalkwise. In particular, a sequence $A \to B \to C$ is exact ($\operatorname{Im}(A \to B) = \operatorname{Ker}(B \to C)$) if and only if the corresponding sequence of stalks $A_x \to B_x \to C_x$ is exact for each $x \in X$. Using this fact, it is easy to see that the category of sheaves on X is an abelian category (Exercise 7.3).

What we have gained in passing from presheaves to sheaves is that sheaves and their morphisms are defined locally. What we have lost in passing from presheaves to sheaves is that the functor which assigns to a sheaf $\mathcal{S}$ its group of sections $\mathcal{S}(U)$ over an open set U may no longer be exact (Example 7.2.3 or Exercise 7.4). Studying this possible loss of exactness is the central theme of sheaf theory.

Recall that $\Gamma(Y, \cdot\,)$ is the functor which assigns to a sheaf its group of sections over a set Y. If $\mathcal{S}$ is a sheaf, then it may be identified with its sheaf of sections, by Theorem 7.1.8(ii). Under this identification, $\Gamma(U, \mathcal{S}) = \mathcal{S}(U)$ for each open set U. In case $U = X$, we will sometimes write simply $\Gamma(\mathcal{S})$ for $\Gamma(X, \mathcal{S})$. It may seem clumsy to have two different notations, $\Gamma(U, \mathcal{S})$ and $\mathcal{S}(U)$, for the same object. We retain the Γ notation for two reasons: (1) it is the historical notation for sections of a sheaf, and (2) $\Gamma(Y, \mathcal{S})$ makes sense for any subset Y of X, not just for open subsets.

7.2.2 Proposition. *For each subset $Y \subset X$, the functor $\Gamma(Y, \cdot\,)$ is left exact.*

Proof. Given an exact sequence of sheaves on X,

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C},$$

the corresponding sequence of morphisms of stalks at x,

$$0 \longrightarrow \mathcal{A}_x \xrightarrow{\alpha_x} \mathcal{B}_x \xrightarrow{\beta_x} \mathcal{C},$$

is also exact, for each $x \in X$. Obviously, this implies the morphism induced by α on sections over Y is injective.

If b is a section of $\mathcal{B}$ over Y, and $\beta(b) = 0$, then $\beta(b(y)) = 0$ for each $y \in Y$. This implies that for each $y \in Y$ there is an $a(y) \in \mathcal{A}_y$ such that $\alpha(a(y)) = b(y)$. This element is necessarily unique, because $\alpha_y : \mathcal{A}_y \to \mathcal{B}_y$ is injective. Given $y_0 \in Y$, there is a neighborhood V of y_0, and an element $t \in \mathcal{A}(V)$ such that $a(y_0) = t_{y_0}$. But then $\alpha(t)_{y_0} = \alpha(a(y_0)) = b_{y_0}$. Now since b is a section of $\mathcal{B}$ over Y, there is a neighborhood of y_0 (which we may assume is V), and an element $s \in \mathcal{B}(V)$ so that $s_y = b(y)$ for all $y \in V \cap Y$. Then $\alpha(t)_{y_0} = s_{y_0}$. But if $\alpha(t)$ and s have the same germ at y_0, they must have the same germ at y for all y in a neighborhood of y_0. It follows that the neighborhood V may be chosen so that $\alpha(t)_y = s_y = b(y)$ for all $y \in V \cap Y$. Since α is injective, we have $t_y = a(y)$ for all $y \in V$. We conclude that the function $y \to a(y)$ defines a section a of $\mathcal{A}$ over Y. By construction, $\alpha(a) = b$.

7.2.3 Example. Let $\mathcal{Z}$ denote the constant sheaf with stalk $\mathbb{Z}$ on the interval $[0, 1]$ (see Example 7.1.9). Let $\mathcal{Z}_0$ denote the subsheaf of $\mathcal{Z}$ consisting of sections which vanish at 0 and 1, and let $\mathcal{Q}$ be the quotient sheaf. Then $\mathcal{Q}$ is a direct sum of two skyscraper sheaves – one with stalk $\mathbb{Z}$ at 0 and one with stalk $\mathbb{Z}$ at 1. The sequence

$$0 \to \mathcal{Z}_0 \to \mathcal{Z} \to \mathcal{Q} \to 0$$

is an exact sequence of sheaves, but the corresponding sequence of global sections is not exact on the right. In fact, $\Gamma(\mathcal{Q})$ is a direct sum of a copy of $\mathbb{Z}$ corresponding to 0 and a copy of $\mathbb{Z}$ corresponding to 1. On passing to global sections, the map $\mathcal{Z} \to \mathcal{Q}$ becomes the diagonal map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, which is not surjective.

We shall show that the category of sheaves has enough injectives. From this it follows that we have naturally defined right derived functors for every left exact functor – in particular, for the functors $\Gamma(Y, \cdot)$. The resulting functors are those of sheaf cohomology and are the subject of most of the remainder of this chapter.

7.3 Operations on Sheaves

In much of this chapter, $\mathcal{R}$ will be a sheaf of commutative rings and $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules. This means that $\mathcal{R}(U)$ is a commutative ring, and $\mathcal{M}(U)$ is a module over $\mathcal{R}(U)$ for each open set $U \subset X$. Also, for each inclusion $V \subset U$, the restriction map $\mathcal{R}(U) \to \mathcal{R}(V)$ is a ring homomorphism, and the restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ is an $\mathcal{R}(U)$-module homomorphism, where $\mathcal{M}(V)$ is an $\mathcal{R}(U)$-module via the morphism $\mathcal{R}(U) \to \mathcal{R}(V)$. A morphism $\alpha : \mathcal{M} \to \mathcal{N}$ between sheaves of $\mathcal{R}$-modules is a sheaf morphism for which $\alpha_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is an $\mathcal{R}(U)$-module homomorphism for each open set U. The group of all morphisms from $\mathcal{M}$ to $\mathcal{N}$ will be denoted $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$.

There are a number of standard operations on sheaves which produce new sheaves. Among the most important are four functors, f^{-1}, f_*, $f^!$, and $f_!$, associated with a continuous map $f : X \to Y$, and two bifunctors, sheaf tensor product and sheaf Hom. We will discuss f^{-1} and f_* in this section. Sheaf tensor product and sheaf Hom will be defined, but any additional development of these functors will be postponed until they are needed. We will not discuss $f^!$ and $f_!$. For a detailed development of all the key sheaf operations and the relationships between them see [**KS**].

7.3.1 Definition. *Let $f : Y \to X$ be a continuous map of topological spaces.*

(i) *If $\mathcal{T}$ is a sheaf on Y, then the direct image $f_*\mathcal{T}$ of $\mathcal{T}$ under f is the sheaf on X defined by $U \to \mathcal{T}(f^{-1}(U))$;*

(ii) *if $\mathcal{S}$ is a sheaf on X, then the inverse image $f^{-1}(\mathcal{S})$ of $\mathcal{S}$ under f is the sheaf of sections associated to the presheaf on Y defined by*

$$U \to \varinjlim\{\mathcal{S}(V) : V \text{ open}, \ f(U) \subset V\}.$$

The definition of $f^{-1}\mathcal{S}$ looks complicated, but f^{-1} is actually a relatively simple functor with very nice properties.

7.3.2 Proposition. *Let $f : Y \to X$ be a continuous map and $\mathcal{S}$ a sheaf on X. Then,*

(i) *for each $y \in Y$, the stalk of $f^{-1}\mathcal{S}$ at y is isomorphic to $\mathcal{S}_{f(y)}$;*

(ii) *if U is open in Y, then an element of $f^{-1}\mathcal{S}(U)$ is a function g which assigns an element of $\mathcal{S}_{f(y)}$ to each $y \in U$ and satisfies the following condition: For each point $p \in U$, there is a neighborhood U_p of p in U, an open set $V_p \subset X$ containing $f(U_p)$, and an $s \in \mathcal{S}(V_p)$ such that $g(y) = s_{f(y)}$ for each $y \in U_p$;*

(iii) *the inverse image functor f^{-1} is an exact functor from sheaves on X to sheaves on Y;*

(iv) *if $\mathcal{S}$ is a constant sheaf on X, then $f^{-1}\mathcal{S}$ is a constant sheaf on Y.*

Proof. To prove (i), note that for $y \in Y$

$$\begin{aligned}(f^{-1}\mathcal{S})_y &= \varinjlim\{f^{-1}\mathcal{S}(U) : y \in U, U \text{ open}\} \\ &= \varinjlim\{\varinjlim\{\mathcal{S}(V) : f(U) \subset V, V \text{ open}\} : y \in U, U \text{ open}\} \\ &= \varinjlim\{\mathcal{S}(V) : f(y) \in V, V \text{ open}\} = \mathcal{S}_{f(y)}.\end{aligned}$$

Part (ii) follows directly from (i) and the definition of the sheaf of sections of a presheaf.

Part (iii) is an immediate consequence of (i), since a sequence of sheaves is exact if it is exact stalkwise.

For part (iv), recall that the constant sheaf with stalk G is the sheaf of locally constant G-valued functions. Parts (i) and (ii) imply that if $\mathcal{S}$ is the sheaf of locally constant G-valued functions on X, then $f^{-1}\mathcal{S}$ is the sheaf of locally constant G-valued functions on Y.

Part (ii) of the above proposition may be paraphrased as follows: An element of $f^{-1}\mathcal{S}(U)$ is a function on U which is locally the composition with f of a local section of $\mathcal{S}$.

A special case of the inverse image functor is the restriction functor. Here, $Y \subset X$ is a subset and $i : Y \to X$ is the inclusion. For a sheaf $\mathcal{S}$ on X, $i^{-1}\mathcal{S}$ is denoted $\mathcal{S}|_Y$ and is called the *restriction* of $\mathcal{S}$ to Y. By Proposition 7.3.2, $i^{-1}\mathcal{S}$ has stalks $(i^{-1}\mathcal{S})_y = \mathcal{S}_y$ at $y \in Y$, and a section of $i^{-1}\mathcal{S}$ over an open subset U of Y is a function on Y which, in some Y-neighborhood of each point of Y, agrees with a section of $\mathcal{S}$ defined in an open subset of X. However, this is how sections of $\mathcal{S}$ over U are defined. Thus, we have proved:

7.3.3 Proposition. *If $i : Y \to X$ is the inclusion of a subset Y into X, and $\mathcal{S}$ is a sheaf on X, then*

$$\Gamma(U, \mathcal{S}|_Y) = \Gamma(U, i^{-1}\mathcal{S}) = \Gamma(U, \mathcal{S})$$

for each relatively open subset U of Y. In particular, $\Gamma(Y, \mathcal{S}|_Y) = \Gamma(Y, \mathcal{S})$.

The direct image functor f_* from sheaves on Y to sheaves on X is not exact in general. To see this, consider the map f which collapses Y to a point p; in this case, $f_*\mathcal{T}$ is the sheaf which assigns to p the group $\Gamma(Y, \mathcal{T})$, and we know that Γ is not always exact (Example 7.2.3 or Exercise 7.4). On the other hand, an argument like the one in Proposition 7.2.2 shows that f_* is always left exact. Thus, it is another functor for which we expect to be able to construct right derived functors.

Suppose $\mathcal{R}$ is a sheaf of rings on X. Then $f^{-1}\mathcal{R}$ is a sheaf of rings on Y. If $\mathcal{N}$ is a sheaf of $\mathcal{R}$-modules on X, then it is easy to see that $f^{-1}\mathcal{N}$ is naturally a sheaf of $f^{-1}\mathcal{R}$-modules on Y. Also, if $\mathcal{M}$ is a sheaf of $f^{-1}\mathcal{R}$-modules on Y, then $f_*\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules on X.

The inverse image and direct image functors are adjoint functors in the sense of the following proposition.

7.3.4 Proposition. *Let $f : X \to Y$ be a continuous map, and let $\mathcal{R}$ be a sheaf of rings on Y. If $\mathcal{N}$ is a sheaf of $\mathcal{R}$-modules on Y, and $\mathcal{M}$ is a sheaf of $f^{-1}\mathcal{R}$-modules on X, then there is a natural isomorphism* $\mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}\mathcal{N}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, f_*\mathcal{M})$.

Proof. Let $\alpha : f^{-1}\mathcal{N} \to \mathcal{M}$ be a morphism. Since $f^{-1}\mathcal{N}$ is generated by the presheaf

$$U \to L(U) = \varinjlim\{\mathcal{N}(V) : f(U) \subset V\} = \varinjlim\{\mathcal{N}(V) : U \subset f^{-1}(V)\},$$

there is a morphism of presheaves $L \to f^{-1}\mathcal{N}$. The composition of the resulting morphism $\mathcal{N}(V) = L(f^{-1}(V)) \to f^{-1}\mathcal{N}(f^{-1}(V))$ with $\alpha_{f^{-1}(V)}$ defines a morphism $\tilde{\alpha}_V : \mathcal{N}(V) \to \mathcal{M}(f^{-1}(V)) = f_*\mathcal{M}(V)$ for each V. This commutes with restriction and, hence, defines a morphism $\tilde{\alpha} : \mathcal{N} \to f_*\mathcal{M}$.

On the other hand, if $\beta : \mathcal{N} \to f_*\mathcal{M}$ is a morphism, then each β_V, for V open in Y, defines a morphism $\mathcal{N}(V) \to \mathcal{M}(f^{-1}(V))$. Thus, with L as above, β defines a morphism $L(U) \to \varinjlim\{\mathcal{M}(f^{-1}(V)) : U \subset f^{-1}(V)\}$. By composing this morphism with restriction to U, we obtain a morphism $L(U) \to \mathcal{M}(U)$. This commutes with restriction and, hence, defines a morphism from the presheaf L to $\mathcal{M}$. Since $f^{-1}\mathcal{N}$ is the sheaf of sections of L, this defines a morphism $\hat{\beta} : f^{-1}\mathcal{N} \to \mathcal{M}$, by Theorem 7.1.8(iv). It is clear from the construction that $\alpha \to \tilde{\alpha}$ and $\beta \to \hat{\beta}$ are inverses of one another. This completes the proof.

If we apply the above result in the case where $\mathcal{M} = f^{-1}\mathcal{N}$, then it tells us that the identity morphism in $\mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}\mathcal{N}, f^{-1}\mathcal{N})$ induces a canonical morphism $\mathcal{N} \to f_*f^{-1}\mathcal{N}$ of sheaves on X. Similarly, in the case where $\mathcal{N} = f_*\mathcal{M}$, the identify in $\mathrm{Hom}_{\mathcal{R}}(f_*\mathcal{M}, f_*\mathcal{M})$ induces a canonical morphism $f^{-1}f_*\mathcal{M} \to \mathcal{M}$ of sheaves on Y. These are the *adjunction morphisms.*

The next proposition will often be useful in identifying the stalks of the direct image sheaf $f_*\mathcal{S}$.

If $\mathcal{S}$ is a sheaf on X, Y is a subspace of X, and U is an open set containing Y, then restriction defines a morphism $\Gamma(U, \mathcal{S}) \to \Gamma(Y, \mathcal{S})$. Thus, restriction defines a morphism $\psi : \varinjlim\{\Gamma(U, \mathcal{S}) : Y \subset U\} \to \Gamma(Y, \mathcal{S})$.

7.3.5 Proposition. *The morphism*

$$\psi : \varinjlim\{\Gamma(U, \mathcal{S}) : Y \subset U\} \to \Gamma(Y, \mathcal{S})$$

is injective. It is an isomorphism if X is Hausdorff and Y is compact, or if X is paracompact and Y is closed.

Proof. If $Y \subset U$, $s \in \Gamma(U, \mathcal{S})$, and s has image 0 in $\Gamma(Y, \mathcal{S})$, then $s_x = 0$ for each $x \in Y$. This implies that s has restriction 0 in an open set V containing Y, and hence, that s determines the 0 element of the group $\varinjlim\{\Gamma(U, \mathcal{S}) : Y \subset U\}$. Thus, ψ is injective.

Now suppose $s \in \Gamma(Y, \mathcal{S})$. Then there is an open cover $\mathcal{W}$ of X, and on each $W \in \mathcal{W}$, a section in $\Gamma(W, \mathcal{S})$ which agrees with s on $W \cap Y$. If X is Hausdorff and Y compact, or if X is paracompact and Y closed, then there is a locally finite open cover $\mathcal{V}$ of Y such that each $V \in \mathcal{V}$ has closure $\overline{V}$ contained in some member W_V of $\mathcal{W}$. For each such V, we let s_V denote the section in $\Gamma(W_V, \mathcal{S})$ which agrees with s on $W_V \cap Y$. The important thing here is that each s_V is a section defined in an open set containing $\overline{V}$.

Now for each $x \in X$ we set

$$\mathcal{V}(x) = \{V \in \mathcal{V} : x \in \overline{V}\} \quad \text{and}$$

$$U = \{x \in X : (s_V)_x = (s_{V'})_x \;\forall\, V, V' \in \mathcal{V}(x)\}.$$

Note that $\mathcal{V}(x)$ is a finite set, since $\mathcal{V}$ is locally finite. We shall show that U is an open set. For a given x, let Δ be a neighborhood of x which meets $\overline{V}$ for only finitely many $V \in \mathcal{V}$. Then the set

$$\Delta_x = \Delta - \bigcup\{\overline{V} : V \in \mathcal{V}, V \notin \mathcal{V}(x)\}$$

is an open set containing x. In fact, Δ_x is the set of $y \in \Delta$ such that $\mathcal{V}(y) \subset \mathcal{V}(x)$. Now if $x \in U$, choose N_x to be a neighborhood of x, contained in Δ_x, on which $s_V|_{N_x} = s_{V'}|_{N_x}$ for $V, V' \in \mathcal{V}(x)$. Then the conditions for membership in U will be satisfied by each $y \in N_x$. Thus, U is an open subset of X. The fact that the s_V fit together to define a section s on Y means that $Y \subset U$. By construction, the s_V fit together on U to define a section in $\Gamma(U, S)$ which restricts to s on Y.

One application of the above result is a characterization, under appropriate conditions, of the group of sections of $f_*\mathcal{S}$ over a compact set. By definition, if $f : Y \to X$ is a continuous map, then $\Gamma(f^{-1}(U), \mathcal{S}) = \Gamma(U, f_*\mathcal{S})$ for each open set $U \subset X$. In particular, $\Gamma(Y, \mathcal{S}) = \Gamma(X, f_*\mathcal{S})$. Now suppose that X and Y are locally compact, and f is a proper map (i.e. $f^{-1}(K)$ is compact whenever K is compact). Then a simple compactness argument shows that each neighborhood of $f^{-1}(K)$, for K compact, contains a neighborhood of the form $f^{-1}(V)$, with V a neighborhood of K in X. In view of this, the following is a consequence of Proposition 7.3.5.

7.3.6 Corollary. *If $f : Y \to X$ is a continuous, proper map between locally compact spaces, and $\mathcal{S}$ is a sheaf on Y, then*

(i) *if $K \subset X$ is compact, then $\Gamma(f^{-1}K, \mathcal{S})$ is naturally isomorphic to $\Gamma(K, f_*\mathcal{S})$;*

(ii) *if* $x \in X$, *then the stalk of* $f_*\mathcal{S}$ *at* x *is naturally isomorphic to* $\Gamma(f^{-1}(x), \mathcal{S})$.

If the hypothesis that f is proper is dropped, the above result fails. There is another type of direct image functor – direct image with proper supports ($f_!$) which is sometimes more appropriate when one is dealing with maps which are not proper (see [**KS**]).

If $\mathcal{R}$ is a sheaf of commutative rings on X, and $\mathcal{S}$ and $\mathcal{T}$ are sheaves of $\mathcal{R}$-modules, then we may construct an $\mathcal{R}(U)$-module $\mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U)$ for each open set $U \subset X$. If $V \subset U$ are two open sets, then the restriction maps $\mathcal{S}(U) \to \mathcal{S}(V)$ and $\mathcal{T}(U) \to \mathcal{T}(V)$ are $\mathcal{R}(U)$-module morphisms and, hence, define a morphism $\mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U) \to \mathcal{S}(V) \otimes_{\mathcal{R}(U)} \mathcal{T}(V)$. The passage from $\mathcal{S}(V) \otimes_{\mathcal{R}(U)} \mathcal{T}(V)$ to $\mathcal{S}(V) \otimes_{\mathcal{R}(V)} \mathcal{T}(V)$ is then just the quotient map defined by imposing the additional relations determined by elements of $\mathcal{R}(V)$ not in the image of $\mathcal{R}(U)$. It follows that there is a natural $\mathcal{R}(U)$-module morphism

$$\mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U) \to \mathcal{S}(V) \otimes_{\mathcal{R}(V)} \mathcal{T}(V)$$

and, hence, that the correspondence $U \to \mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U)$ is a presheaf of $\mathcal{R}$-modules.

7.3.7 Definition. *The* $\mathcal{R}$*-module tensor product of sheaves of* $\mathcal{R}$*-modules* $\mathcal{S}$ *and* $\mathcal{T}$ *is denoted* $\mathcal{S} \otimes_{\mathcal{R}} \mathcal{T}$ *and is defined to be the sheaf of sections of the presheaf* $U \to \mathcal{S}(U) \otimes_{\mathcal{R}(U)} \mathcal{T}(U)$.

With $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ as above, we may likewise consider the correspondence which assigns to each open set U the $\mathcal{R}(U)$-module $\mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{S}|_U, \mathcal{T}|_U)$. It is easy to see that this is actually a sheaf as it stands.

7.3.8 Definition. *Given two sheaves of* $\mathcal{R}$*-modules* $\mathcal{S}$ *and* $\mathcal{T}$*, the sheaf of* $\mathcal{R}$*-modules* $U \to \mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{S}|_U, \mathcal{T}|_U)$ *will be called the sheaf of homomorphisms from* $\mathcal{S}$ *to* $\mathcal{T}$ *and will be denoted* $\mathcal{H}om_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$.

Note that, by definition, the module of global sections of $\mathcal{H}om_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$ is $\mathrm{Hom}_{\mathcal{R}}(\mathcal{S}, \mathcal{T})$.

7.4 Sheaf Cohomology

Given a sheaf of modules $\mathcal{M}$ over a sheaf of rings $\mathcal{R}$ on X, there is a standard method for embedding $\mathcal{M}$ in an injective sheaf of $\mathcal{R}$-modules. For each stalk $\mathcal{M}_x$ of $\mathcal{M}$, we let $\tilde{\mathcal{M}}_x$ be the injective $\mathcal{R}_x$-module containing $\mathcal{M}_x$, constructed using the method of Proposition 6.6.3. Then $\mathcal{I}^0(\mathcal{M})$ is the sheaf defined as follows: $\mathcal{I}^0(\mathcal{M})(U)$ is the $\mathcal{R}(U)$-module consisting of all functions which assign to each $x \in U$ an element of $\tilde{\mathcal{M}}_x$. Note that there is no requirement that these functions satisfy the continuity condition of Definition 7.1.7. Clearly $\mathcal{M}$ is embedded in $\mathcal{I}^0(\mathcal{M})$ as those functions which have values in the modules $\mathcal{M}_x$ and do satisfy this continuity condition, so that they are sections of $\mathcal{M}$.

7.4.1 Proposition. *The sheaf $\mathcal{I}^0(\mathcal{M})$ is an injective object in the category of sheaves of $\mathcal{R}$-modules.*

Proof. Suppose $\alpha : \mathcal{S} \to \mathcal{T}$ is a monomorphism of sheaves of $\mathcal{R}$-modules, and $\beta : \mathcal{S} \to \mathcal{I}^0(\mathcal{M})$ is a morphism of sheaves of $\mathcal{R}$-modules. Then β determines a morphism $\phi_x \in \mathrm{Hom}_{\mathcal{R}_x}(\mathcal{S}_x, \tilde{\mathcal{M}}_x)$, for each x, by $\phi_x(s_x) = \beta(s_x)(x)$ (note $\beta(s_x)$ is a germ in $\mathcal{I}_0(\mathcal{M})_x$, and $\beta(s_x)(x)$ is this germ evaluated at x). By the injectivity of $\tilde{\mathcal{M}}_x$, the morphism $\phi_x : \mathcal{S}_x \to \tilde{\mathcal{M}}_x$ has an extension to a morphism $\psi_x : \mathcal{T}_x \to \tilde{\mathcal{M}}_x$ such that $\psi_x(\alpha(s_x)) = \phi_x(s_x)$ (we fix one such extension for each x, using the axiom of choice). If we set $\gamma(t)(x) = \psi_x(t_x)$ for U open and $t \in \mathcal{T}(U)$, then we have defined a morphism $\gamma : \mathcal{T} \to \mathcal{I}^0(\mathcal{M})$. Note that

$$\gamma(\alpha(s_x))(x) = \psi_x(\alpha(s_x)) = \phi_x(s_x) = \beta(s_x)(x)$$

for each germ s_x of a section of $\mathcal{S}$. That is, $\gamma \circ \alpha = \beta$. Thus, $\mathcal{I}^0(\mathcal{M})$ is injective in the category of sheaves of $\mathcal{R}$-modules.

7.4.2 Corollary. *If $\mathcal{R}$ is a sheaf of rings on X, then the category of sheaves of $\mathcal{R}$-modules on X has enough injectives.*

In fact, it is clear that the above construction can be used recursively to define a functor which assigns to each $\mathcal{R}$-module $\mathcal{M}$ an injective resolution $\mathcal{M} \to \mathcal{I}(\mathcal{M})$.

Since we have enough injectives, we know that for any left exact functor from the category of sheaves of $\mathcal{R}$-modules to an abelian category there are corresponding right derived functors for which Theorems 6.4.2, 6.4.3 and 6.4.6 and Propositions 6.4.4 and 6.4.5 of the previous chapter hold. This applies, in particular, to the sections functor $\Gamma(Y, \cdot)$ over a subset Y of X and to the direct image functor f_* associated with a continuous map $f : Y \to X$. There are a number other left exact functors associated with sheaves. We describe two of these in the next paragraph.

If S is a sheaf on X, and $s \in \Gamma(X, S)$ is a section, then the support of s is the (necessarily closed) set $K = \{x \in X : s_x \neq 0\}$. If $Y \subset X$ is any subset of X, then $\Gamma_Y(X, S)$ is the group of sections $s \in \Gamma(X, S)$ with support contained in Y. If ϕ is a family of subsets of X which is closed under finite unions, then $\Gamma_\phi(X, S)$ is the group of sections $s \in \Gamma(X, S)$ with support contained in some member of ϕ. A family ϕ, as above, is called a family of supports. A common situation in which $\Gamma_\phi(X, S)$ is useful is when

X is locally compact and the family of supports ϕ is the family of compact subsets of X. It is easy to see that both $\Gamma_Y(X, \cdot)$ and $\Gamma_\phi(X, \cdot)$ are also left exact functors on the category of sheaves.

Thus, we have right derived functors for the left exact functors $\Gamma(Y, \cdot)$, f_*, $\Gamma_Y(X, \cdot)$ and $\Gamma_\phi(X, \cdot)$.

7.4.3 Definition. *If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules on X, $Y \subset X$ is a subset of X, and ϕ a family of closed subsets of X, closed under finite union, then for each p we set*

(i) $H^p(Y, \mathcal{M}) = R^p\Gamma(Y, \mathcal{M})$,
(ii) $H^p_Y(X, \mathcal{M}) = R^p\Gamma_Y(X, \mathcal{M})$, *and*
(iii) $H^p_\phi(X, \mathcal{M}) = R^p\Gamma_\phi(X, \mathcal{M})$.

These groups are called the sheaf cohomology groups of $\mathcal{M}$ on Y, of $\mathcal{M}$ on X with support in Y, and of $\mathcal{M}$ on X with supports in ϕ, respectively.

If $f : Y \to X$ is a continuous map, then there are also higher derived functors $R^p f_*$ for the direct image functor f_*, though we won't bother to give them special names.

A sheaf $\mathcal{M}$ on X is said to be *acyclic* if $H^p(X, \mathcal{M}) = 0$ for all $p > 0$. If Y is a subset of X, then $\mathcal{M}$ is said to be *acyclic on Y* if $H^p(Y, \mathcal{M}) = 0$ for all $p > 0$. That is, a sheaf is acyclic on Y if it is a $\Gamma(Y, \cdot)$-acyclic object in the category of sheaves (as defined in section 6.4).

Sheaf cohomology is actually a special case of Ext. In fact, if $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules on X, then $\Gamma(X, \mathcal{M})$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M})$ (Exercise 7.5). It follows that $H^p(X, \mathcal{M})$ is naturally isomorphic to $\mathrm{Ext}^p_{\mathcal{R}}(\mathcal{R}, \mathcal{M})$.

We will need to know that certain operations preserve injectives. Among them are restriction to an open set, direct image, and direct product:

7.4.4 Proposition.

(i) *If $\mathcal{I}$ is an injective sheaf of $\mathcal{R}$-modules on X, and $U \subset X$ is an open set, then $\mathcal{I}|_U$ is an injective sheaf of $\mathcal{R}|_U$-modules;*
(ii) *if $f : X \to Y$ is a continuous map, $\mathcal{R}$ a sheaf of rings on Y, and $\mathcal{I}$ an injective sheaf of $f^{-1}\mathcal{R}$-modules, then $f_*\mathcal{I}$ is an injective sheaf of $\mathcal{R}$-modules;*
(iii) *the direct product of any collection of injective sheaves of $\mathcal{R}$-modules on X is an injective sheaf of $\mathcal{R}$-modules.*

Proof. If $\mathcal{M}$ is a sheaf on U, let $\mathcal{M}^X$ be the sheaf $\mathcal{M}$ extended by zero to X; that is, $\mathcal{M}^X$ is the sheaf of sections of the presheaf which assigns $\mathcal{M}(V)$ to an open set $V \subset U$ and 0 to any open set V which is not contained in U. This sheaf has stalks $\mathcal{M}^X_x = \mathcal{M}_x$ at points $x \in U$ and $\mathcal{M}^X_x = 0$ at points x not in U. It is easily verified that $\mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{M}, \mathcal{I}|_U) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^X, \mathcal{I})$. The functor

$\mathcal{M} \to \mathcal{M}^X$ is clearly exact. Thus, if $\mathcal{I}$ is injective, then $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^X, \mathcal{I})$ is an exact functor of $\mathcal{M}$. It follows that $\mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{M}, \mathcal{I}|_U)$ is an exact functor of $\mathcal{M}$ and, hence, that $\mathcal{I}|_U$ is injective. This proves (i).

By Proposition 7.3.4, $\mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}\mathcal{M}, \mathcal{I}) \simeq \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, f_*\mathcal{I})$, for a sheaf $\mathcal{M}$ of $\mathcal{R}$-modules on Y and a sheaf $\mathcal{I}$ of $f^{-1}R$-modules on X. Furthermore, $\mathcal{M} \to f^{-1}\mathcal{M}$ is exact. If $\mathcal{I}$ is injective, it follows that $\mathcal{M} \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, f_*\mathcal{I})$ is exact and, hence, that $f_*\mathcal{I}$ is injective. This proves (ii).

Let $\{\mathcal{I}_\alpha\}$ be any set of injective sheaves on X, and suppose $i : \mathcal{M} \to \mathcal{N}$ is a monomorphism of sheaves on X. Each morphism $f : \mathcal{M} \to \prod_\alpha \mathcal{I}_\alpha$ is just the product $\prod_\alpha f_\alpha$ of a set of morphisms $f_\alpha : \mathcal{M} \to \mathcal{I}_\alpha$. Each of these extends to a morphism $g_\alpha : \mathcal{N} \to \mathcal{I}_\alpha$. The product $\prod g_\alpha : \mathcal{N} \to \prod_\alpha \mathcal{I}_\alpha$ is then an extension of f. Thus, $\prod_\alpha \mathcal{I}_\alpha$ is also injective. This completes the proof.

Let $f : Y \to X$ be a continuous map between topological spaces, and let $\mathcal{S}$ be a sheaf on Y. As an application of the above result, we show how to define a canonical morphism

$$H^p(X, f_*\mathcal{S}) \to H^p(Y, \mathcal{S}). \tag{7.4.1}$$

Let $\mathcal{S} \to \mathcal{I}$ be an injective resolution of $\mathcal{S}$. Then by Proposition 7.4.4(ii), the complex $f_*\mathcal{I}$ is a complex of injective sheaves, although it may no longer be a resolution of $\mathcal{S}$, since f_* is not an exact functor. However, let $f_*\mathcal{S} \to \mathcal{J}$ be an injective resolution of $f_*\mathcal{S}$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{S} & \longrightarrow & \mathcal{J}^0 & \longrightarrow & \mathcal{J}^1 & \longrightarrow & \cdots \\
 & & {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & f_*\mathcal{S} & \longrightarrow & f_*\mathcal{I}^0 & \longrightarrow & f_*\mathcal{I}^1 & \longrightarrow & \cdots
\end{array}$$

which is unique up to homotopy. The vertical morphisms in this diagram are constructed, as in Proposition 6.3.4, using the injectivity of the sheaves $f_*\mathcal{I}^p$. Since $\Gamma(X, f_*\mathcal{I}^p) \simeq \Gamma(Y, \mathcal{I}^p)$, on passing to global sections and taking cohomology, this diagram gives us the morphisms (7.4.1).

Note that if f_* does preserve the exactness of the resolution $\mathcal{S} \to \mathcal{I}$, so that $f_*\mathcal{S} \to f_*\mathcal{I}$ is an injective resolution of $f_*\mathcal{S}$, then the morphisms (7.4.1) are isomorphisms. This happens, in particular, when Proposition 7.3.5 applies, so that each stalk $(f_*\mathcal{I}^P)_x$ is $\Gamma(f^{-1}(x), \mathcal{I}^p)$, and the sheaf $\mathcal{S}$ is acyclic on each set $f^{-1}(x)$. Thus, we have proved:

7.4.5 Proposition. *Let $f : Y \to X$ be a continuous map between two topological spaces, and let $\mathcal{S}$ be a sheaf on Y. Then f induces a natural morphism of cohomology*

$$H^p(X, f_*\mathcal{S}) \to H^p(Y, \mathcal{S}).$$

If f is a proper map between locally compact spaces, and $\mathcal{S}$ is acyclic on each fiber $f^{-1}(x)$ for $x \in X$, then this morphism is an isomorphism.

This result is a special case of a very general relationship (the Leray spectral sequence) between the cohomology groups of $\mathcal{S}$ on Y and the cohomology groups of the sheaves $R^p f_* \mathcal{S}$ on X (see [**Br**], [**KS**]).

If we apply Proposition 7.4.5 to a sheaf $\mathcal{S}$ which is of the form $f^{-1}\mathcal{T}$, where $\mathcal{T}$ is a sheaf on Y, then we obtain a morphism

$$H^p(X, f_* f^{-1}\mathcal{T}) \to H^p(Y, f^{-1}\mathcal{T}).$$

The adjunction morphism $\mathcal{T} \to f_* f^{-1}\mathcal{T}$, introduced following Proposition 7.3.4, induces a morphism $H^p(X, \mathcal{T}) \to H^p(Y, f_* f^{-1}\mathcal{T})$ which, when composed with the above, yields a morphism

$$H^p(X, \mathcal{T}) \to H^p(Y, f^{-1}\mathcal{T}).$$

The adjunction morphism $\mathcal{T} \to f_* f^{-1}\mathcal{T}$ is an isomorphism when f is a proper map between locally compact spaces, and the fibers of f are connected (Exercise 7.9). Furthermore, on each fiber of f the sheaf $f^{-1}\mathcal{S}$ is a constant sheaf. Thus, we have:

7.4.6 Vietoris-Begle Theorem. *Let $f : Y \to X$ be a continuous map between two topological spaces, and let $\mathcal{T}$ be a sheaf on X. Then f induces a natural morphism of cohomology*

$$H^p(X, \mathcal{T}) \to H^p(Y, f^{-1}\mathcal{T}).$$

If f is a proper map between locally compact spaces, and if each fiber $f^{-1}(x)$, $x \in X$, is connected and is a space on which every constant sheaf is acyclic, then this morphism is an isomorphism.

If X is a topological space, and G is an abelian group, then the pth cohomology of X with coefficients in G is the group $H^p(X, G) = H^p(X, \mathcal{G})$, where $\mathcal{G}$ is the constant sheaf on X with stalk G. If $f : Y \to X$ is a continuous map, and $\mathcal{G}$ is the constant sheaf with stalk G on X, then $f^{-1}\mathcal{G}$ is the constant sheaf on Y with stalk G. Thus, the Vietoris-Begle theorem implies that f induces a map

$$H^p(X, G) \to H^p(Y, G),$$

which is an isomorphism if f has connected fibers on which every constant sheaf is acyclic.

One of the common uses of the long exact sequence theorem (Theorem 6.4.3) is to derive relationships between the cohomology of a sheaf on X

and its cohomology on certain subsets of X. We will end this section with a brief discussion of these matters.

Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules on X, let $i : Y \to X$ be the inclusion of a closed subset, and let $U = X - Y$. Consider the sheaf $\mathcal{M}_Y = i_*i^{-1}\mathcal{M}$ on X. This is the sheaf $V \to \mathcal{M}|_Y(Y \cap V) = \Gamma(Y \cap V, \mathcal{M})$, by Proposition 7.3.3. An intuitive description of the construction of $\mathcal{M}_Y$ is "restrict $\mathcal{M}$ to Y and then extend by zero to X". There is an epimorphism from $\mathcal{M}$ to $\mathcal{M}_Y$ given by $s \to s|_{Y\cap V} : \mathcal{M}(V) \to \mathcal{M}_Y(V) = \Gamma(Y \cap V, \mathcal{M})$. In fact, $\mathcal{M}_Y$ is the unique quotient of $\mathcal{M}$ whose stalks at points of Y agree with those of $\mathcal{M}$ and whose stalks at points not in Y are 0. We will denote by $\mathcal{M}_U$ the kernel of the quotient map $\mathcal{M} \to \mathcal{M}_Y$. This is the unique subsheaf of $\mathcal{M}$ which has stalks equal to those of $\mathcal{M}$ at points of U and equal to zero at points not in U. Thus, its intuitive description is also "restrict $\mathcal{M}$ to U and then extend by zero to all of X". However, its precise description is quite different from that of $\mathcal{M}_Y$. It is not $j_*j^{-1}\mathcal{M}$ for the inclusion map $j : U \to X$. It is the sheaf that assigns to an open set V the subspace of $\mathcal{M}(V)$ consisting of sections which vanish on $V \cap Y$. By combining these two constructions, one may define $\mathcal{M}_Z$ when Z is any locally closed subset of X. A locally closed set Z is relatively open in its closure $\overline{Z}$, so that $\overline{Z} - Z$ is a closed set. Thus we may define $\mathcal{M}_Z$ to be $(\mathcal{M}_{\overline{Z}})_U$ where U is the complement in X of the closed set $\overline{Z} - Z$. This is a sheaf which has stalks equal to those of $\mathcal{M}$ at points of Z and 0 stalks at points of the complement of Z.

If Y is closed in X, and $U = X - Y$, then we have a short exact sequence

$$0 \longrightarrow \mathcal{M}_U \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_Y \longrightarrow 0 \tag{7.4.2}$$

associated to any sheaf on $\mathcal{M}$. Given two closed subsets Y_1 and Y_2 of X, there is also a short exact sequence

$$0 \longrightarrow \mathcal{M}_{Y_1\cup Y_2} \longrightarrow \mathcal{M}_{Y_1} \oplus \mathcal{M}_{Y_2} \longrightarrow \mathcal{M}_{Y_1\cap Y_2} \longrightarrow 0 \tag{7.4.3}$$

associated to a sheaf $\mathcal{M}$ on X. Here, the first map in this sequence is defined by $f \to \alpha_1(f) \oplus \alpha_2(f)$ and the second by $g \oplus h \to \beta_1(g) - \beta_2(h)$ where $\alpha_i : \mathcal{M}_{Y_1\cup Y_2} \to \mathcal{M}_{Y_i}$ and $\beta_i : \mathcal{M}_{Y_i} \to \mathcal{M}_{Y_1\cap Y_2}$, $i = 1, 2$, are the restriction maps.

By Theorem 6.4.3, corresponding to the short exact sequences (7.4.2) and (7.4.3), we have long exact sequences of cohomology:

7.4.7 Proposition. *Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules on X,*

(i) *if Y is a closed subset of X and $U = X - Y$, then there is an exact sequence:*

$$\cdots \to H^p(X, \mathcal{M}_U) \to H^p(X, \mathcal{M}) \to H^p(X, \mathcal{M}_Y) \to H^{p+1}(X, \mathcal{M}_U) \to \cdots ;$$

(ii) *if Y_1 and Y_2 are closed subsets of X, then there is an exact sequence*

$$\cdots \to H^p(X, \mathcal{M}_{Y_1 \cup Y_2}) \to H^p(X, \mathcal{M}_{Y_1}) \oplus H^p(X, \mathcal{M}_{Y_2}) \\ \to H^p(X, \mathcal{M}_{Y_1 \cap Y_2}) \to H^{p+1}(X, \mathcal{M}_{Y_1 \cup Y_2}) \to \cdots .$$

It remains to interpret the cohomology groups of sheaves like $\mathcal{M}_Y$ and $\mathcal{M}_U$, that appear in this proposition, in terms of cohomology groups of the original sheaf $\mathcal{M}$. This will be done in the next section.

7.5 Classes of Acyclic Sheaves

From Theorem 6.4.6 we know that we can compute the cohomology groups of a sheaf $\mathcal{M}$ from any resolution of $\mathcal{M}$ by sheaves that are acyclic for the functor Γ. In this section we will describe various classes of acyclic sheaves that are particularly useful for the computation of sheaf cohomology.

7.5.1 Definition. *If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules on X, then*

(i) *$\mathcal{M}$ is called flabby if the restriction map $\Gamma(X, \mathcal{M}) \to \Gamma(U, \mathcal{M})$ is surjective for every open set $U \subset X$;*

(ii) *$\mathcal{M}$ is called soft if the restriction map $\Gamma(X, \mathcal{M}) \to \Gamma(Y, \mathcal{M})$ is surjective for every closed set $Y \subset X$;*

(iii) *$\mathcal{M}$ is called fine if, for every open cover $\mathcal{V}$ of X, there is a family of morphisms $\{\phi_\alpha : \mathcal{M} \to \mathcal{M}\}$ such that each ϕ_α is supported in some $V \in \mathcal{V}$, for each point $x \in X$ there is a neighborhood of x on which all but finitely many ϕ_α vanish, and $\sum_\alpha \phi_\alpha = \mathrm{id}$. The family $\{\phi_\alpha\}$ is called a partition of unity for the sheaf $\mathcal{M}$ subordinate to the cover $\mathcal{V}$.*

The definition of fine sheaf needs some comment. The statement that ϕ_α is supported in V means that there is a closed subset K of X, with $K \subset V$, such that $(\phi_\alpha)_x = 0$ for all $x \in X - K$. The sum $\sum_\alpha \phi_\alpha$ makes sense as a morphism from $\mathcal{M}$ to $\mathcal{M}$, because all but finitely many ϕ_α vanish in a neighborhood of a given point x. Hence, the sum makes sense and defines a morphism in such a neighborhood; but then these local morphisms fit together to define a morphism of sheaves globally.

7.5.2 Theorem. *Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Then in the category of sheaves of $\mathcal{R}$-modules on X,*

(i) *if the first two terms of a short exact sequence are flabby, then so is the third;*

(ii) *injective sheaves are flabby, and flabby sheaves are acyclic.*

Proof. For U open and $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules on X, let $\mathcal{R}_U$ be the restriction of $\mathcal{R}$ to U followed by extension by zero to all of X. Then we have an inclusion $\mathcal{R}_U \to \mathcal{R}$. Thus, if $\mathcal{M}$ is injective, then each morphism from $\mathcal{R}_U$ to $\mathcal{M}$ extends to a morphism from $\mathcal{R}$ to $\mathcal{M}$. But a morphism $\mathcal{R}_U \to \mathcal{M}$ is just a section of $\mathcal{M}$ over U, and a morphism $\mathcal{R} \to \mathcal{M}$ is just a section of $\mathcal{M}$ over X (Exercise 7.5). Thus, sections of $\mathcal{M}$ over U extend to sections of $\mathcal{M}$ over X. This proves injective sheaves are flabby.

Now suppose that $\mathcal{L}$ is a flabby sheaf and

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N} \longrightarrow 0$$

is an exact sequence of sheaves of $\mathcal{R}$-modules. We wish to prove that

$$0 \longrightarrow \Gamma(X,\mathcal{L}) \xrightarrow{\alpha} \Gamma(X,\mathcal{M}) \xrightarrow{\beta} \Gamma(X,\mathcal{N}) \longrightarrow 0 \tag{7.5.1}$$

is also exact. Of course, we need only prove that $\beta : \Gamma(X,\mathcal{M}) \to \Gamma(X,\mathcal{N})$ is surjective. To this end, we let f be an element of $\Gamma(X,\mathcal{N})$ and consider the class of pairs (U, g) where U is an open subset of X, $g \in \Gamma(U,\mathcal{M})$ and $\beta(g) = f|_U$. This class is non-empty, since f is locally in the image of β. It is partially ordered under the relation: $(U_1, g_1) < (U_2, g_2)$ if $U_1 \subset U_2$ and $g_1 = g_2|_{U_1}$. It also has the property that a maximal totally ordered subset has a maximal element (by taking union). Thus, it follows from Zorn's lemma that there is a maximal element (U, g) in this class. If $U = X$ we are done. If not, then there is an $x \in X - U$. We may choose a neighborhood V of x and an element $g_0 \in \Gamma(V,\mathcal{M})$ such that $\beta(g_0) = f|_V$. We then have $\beta(g|_{U\cap V} - g_0|_{U\cap V}) = 0$, and so there is an element $h_0 \in \Gamma(U \cap V, \mathcal{L})$ such that $\alpha(h_0) = g|_{U\cap V} - g_0|_{U\cap V}$. The fact that $\mathcal{L}$ is flabby allows us to extend h_0 to a section h of $\mathcal{L}$ on all of X. Then g and $g_0 + \alpha(h)$ agree when restricted to $U \cap V$ and, hence, define a section $g' \in \Gamma(U'; \mathcal{M})$ where $U' = U \cup V$. But then (U', g') is in our class and is larger than (U, g), contradicting the maximality. This proves the exactness of (7.5.1).

We may now prove part (i) of the theorem. If $\mathcal{L}$ and $\mathcal{M}$ are flabby in the above short exact sequence, then their restrictions to an open set U are also flabby. It follows from the above paragraph and the fact that $\mathcal{L}|_U$ is flabby that any section f of $\mathcal{N}$ over U is $\beta(g)$ for some section of $\mathcal{M}$ over U. But, since $\mathcal{M}$ is flabby, the section g is the restriction to U of a global section g' of $\mathcal{M}$. Then $\beta(g')$ provides an extension of f to a global section of $\mathcal{N}$.

Now to prove that a flabby sheaf $\mathcal{L}$ is acyclic, we embed it in an injective sheaf of modules $\mathcal{I}$ and use the long exact sequence of cohomology for the short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{I} \xrightarrow{\beta} \mathcal{N} \longrightarrow 0,$$

where α is the inclusion and $\mathcal{N}$ is its cokernel. Since $\mathcal{I}$ is acyclic, we conclude that

$$H^p(X,\mathcal{L}) \simeq H^{p-1}(X,\mathcal{N}), \quad p > 1,$$

and

$$H^1(X,\mathcal{L}) \simeq \operatorname{Coker}\{\beta : \Gamma(X,\mathcal{I}) \to \Gamma(X,\mathcal{N})\}.$$

The exactness of (7.5.1), with $\mathcal{M} = \mathcal{I}$, implies $H^1(X,\mathcal{L}) = 0$. Now $\mathcal{N}$ is flabby, because $\mathcal{L}$ and $\mathcal{I}$ are flabby. Thus, $H^2(X,\mathcal{L}) \simeq H^1(X,\mathcal{N}) = 0$. By iterating this argument, we conclude that $H^p(X,\mathcal{L}) = 0$ for all $p > 0$. Thus, we have proved that flabby sheaves are acyclic.

There is a similar result with a similar proof for soft sheaves. It has a slightly more complicated proof and requires that the space be paracompact.

7.5.3 Theorem. *If X is paracompact, then for sheaves of $\mathcal{R}$-modules on X,*

(i) *if the first two terms of a short exact sequence are soft, then so is the third;*
(ii) *flabby sheaves are soft, and soft sheaves are acyclic;*
(iii) *fine sheaves are soft.*

Proof. Let $U \subset X$ be open and let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules on X. Since X is paracompact, Proposition 7.3.5 implies that every section of a sheaf on a closed set Y extends to an open set containing Y and, hence, extends to all of X if the sheaf is flabby. Thus, flabby sheaves are soft.

Now suppose that $\mathcal{L}$ is a soft sheaf and

$$0 \longrightarrow \mathcal{L} \xrightarrow{\ \alpha\ } \mathcal{M} \xrightarrow{\ \beta\ } \mathcal{N} \longrightarrow 0$$

is an exact sequence of sheaves of $\mathcal{R}$-modules. We wish to prove that

$$(7.5.2) \qquad 0 \longrightarrow \Gamma(X,\mathcal{L}) \xrightarrow{\ \alpha\ } \Gamma(X,\mathcal{M}) \xrightarrow{\ \beta\ } \Gamma(X,\mathcal{N}) \longrightarrow 0$$

is also exact. Of course, we need only prove that $\beta : \Gamma(X,\mathcal{M}) \to \Gamma(X,\mathcal{N})$ is surjective. To this end, let f be a section of $\mathcal{N}$ over X, and note that f is locally in the image of β. Thus, for each $x \in X$, there is a neighborhood U of x so that $f|_U$ is in the image of β_U. Since X is paracompact, it is normal. Hence, there exists a neighborhood V of x with $\overline{V} \subset U$. We conclude that there is an open cover $\mathcal{V}$ of X, which we may assume locally finite, since X is paracompact, and for each $V \in \mathcal{V}$, a section g_V of $\mathcal{M}$, defined in a neighborhood of $\overline{V}$, such that $\beta(g_V)$ agrees with f in a neighborhood of $\overline{V}$. We then consider the class of pairs (Y, g), where Y is a subset of X which is a union of some of the sets $\overline{V}$ for $V \in \mathcal{V}$, $g \in \Gamma(Y,\mathcal{M})$, and $\beta(g) = f|_Y$.

Any set Y from such a pair is closed, due to the fact that the collection $\mathcal{V}$ is locally finite. This class is non-empty, since it contains each pair $(\overline{V}, g_V)$, and it is partially ordered under the relation: $(Y_1, g_1) < (Y_2, g_2)$ if $Y_1 \subset Y_2$ and $g_1 = g_2|_{Y_1}$. It also has the property that a maximal totally ordered subset has a maximal element (by taking union). Thus, it follows from Zorn's lemma that there is a maximal element (Y, g) in this class. If $Y = X$, we are done. If not, then there is a $V \in \mathcal{V}$ so that $\overline{V}$ is not contained in Y. We then have $\beta(g|_{Y\cap\overline{V}} - g_V|_{Y\cap\overline{V}}) = 0$ and so there is an element $h_0 \in \Gamma(Y \cap \overline{V}, \mathcal{L})$ such that $\alpha(h_0) = g|_{Y\cap\overline{V}} - g_V|_{Y\cap\overline{V}}$. We use the fact that $\mathcal{L}$ is soft to extend h_0 to a section h of $\mathcal{L}$ on all of $\overline{V}$. Then g and $g_V - \alpha(h)$ agree when restricted to $Y \cap \overline{V}$ and, hence, define a section $g' \in \Gamma(Y', \mathcal{M})$ where $Y' = Y \cup \overline{V}$. But then (Y', b') is in our class and is larger than (Y, b), contradicting the maximality. This proves the exactness of (7.5.2). This result may be used to prove part (i) and that soft sheaves are acyclic, exactly as in the proof of Theorem 7.5.2.

It remains to prove that fine sheaves are soft. Thus, let $Y \subset X$ be closed, and let $\mathcal{M}$ be a fine sheaf on X. If $s \in \Gamma(Y, \mathcal{M})$, then for each $x \in Y$, the germ of s at x is represented by a section defined in a neighborhood of x, which agrees with s when restricted to that neighborhood intersected with Y. Thus, we may choose an open cover $\mathcal{V}$ of X and elements $s_V \in \Gamma(V, \mathcal{M})$ such that $s|_{V\cap Y} = s_V|_{V\cap Y}$ for each $V \in \mathcal{V}$ – one of these open sets will be the complement of Y and will have the zero section assigned to it, and the others will be neighborhoods of points of Y. Because X is paracompact, we may assume that $\mathcal{V}$ is locally finite. Now because $\mathcal{M}$ is fine, there is a family of morphisms $\{\phi_\alpha : \mathcal{M} \to \mathcal{M}\}$ such that each ϕ_α is supported in some $V_\alpha \in \mathcal{V}$, for each point $x \in X$ there is a neighborhood of x on which all but finitely many ϕ_α vanish, and $\sum_\alpha \phi_\alpha = \mathrm{id}$. For each α, we interpret $\phi_\alpha s_{V_\alpha}$ to be a section on all of X by extending it to be 0 on the complement of V_α. We then set $s' = \sum \phi_\alpha s_{V_\alpha} \in \Gamma(X, \mathcal{M})$. That this sum makes sense follows from the local finiteness of the open cover, which means that in a neighborhood of any point we are summing only finitely many non-zero terms. We also have that $s'_x = s_x$ at each point $x \in Y$, so that s' is an extension of s to all of X. It follows that $\mathcal{M}$ is soft. Thus, we have proved that fine sheaves are soft. This completes the proof of the theorem.

As an application of the above theorem, we finish the discussion of cohomology, subspaces, and the functor $\mathcal{M} \to \mathcal{M}_Z$ that ended the previous section. The functor $\mathcal{M} \to \mathcal{M}_Z$ is exact, but it may not take injective sheaves to injective sheaves. However, we do have:

7.5.4 Proposition. *If $\mathcal{M}$ is a soft sheaf on a paracompact space X, and Z is a locally closed subset of X, then $\mathcal{M}_Z$ is also soft.*

Proof. First, suppose Z is an open set U. If $Y \subset X$ is closed, and m is a section of $\mathcal{M}_U$ on Y, then m vanishes on $Y \cap (X - U)$. Thus, we obtain a section m_1 of $\mathcal{M}$ on the closed set $Y \cup (X - U)$, by defining m_1 to be m on Y and 0 on $X - U$. Since $\mathcal{M}_U \subset \mathcal{M}$ and $\mathcal{M}$ is soft, this section extends to a section m_2 of $\mathcal{M}$ which vanishes on $X - U$ and, hence, is actually a section of $\mathcal{M}_U$. Since m_2 restricted to Y is m, we have proved that $\mathcal{M}_U$ is soft.

Now suppose that Z is a closed set Y and set $U = X - Y$. We have a short exact sequence

$$0 \to \mathcal{M}_U \to \mathcal{M} \to \mathcal{M}_Y \to 0.$$

Since $\mathcal{M}$ and $\mathcal{M}_U$ are soft, so is $\mathcal{M}_Y$, by Theorem 7.5.3.

For a general locally closed set Z, the proposition follows from the fact that $\mathcal{M}_Z = (\mathcal{M}_Y)_U$, where $Z = Y \cap U$ with Y closed and U open.

Now let $\mathcal{M} \to \mathcal{I}$ be an injective resolution of $\mathcal{M}$ on X. By Proposition 7.5.4 and the fact that $\mathcal{M} \to \mathcal{M}_Z$ is exact, we have that $\mathcal{M}_Z \to \mathcal{I}_Z$ is a resolution of $\mathcal{M}_Z$ by soft sheaves, which are acyclic, by Theorem 7.5.3. Thus, the cohomology groups $H^p(X, \mathcal{M}_Z)$ are isomorphic to the cohomology groups of the complex $\Gamma(X, \mathcal{I}_Z)$. Let $\mathcal{I}^p$ be the pth term of the complex $\mathcal{I}$. Each section of any sheaf on X has support which is closed in X, but the support of a section of $\mathcal{I}^p_Z$ lies in Z. So such a section is supported on a subset of Z which is closed in X. Conversely, a section of a sheaf in $\mathcal{I}^p$, supported on a subset of Z which is closed in X, uniquely defines a section of $\mathcal{I}_Z$. It follows that

$$H^p(X, \mathcal{M}_Z) \simeq H^p_\phi(Z, \mathcal{M}) \simeq H^p_\phi(Z, \mathcal{M}|_Z),$$

where ϕ is the family of subsets of Z which are closed in X. Note that if Z is closed, then every section of a sheaf on Z is automatically supported on a member of ϕ – Z itself. Thus, for Y a closed subset of X, we have

$$H^p(X, \mathcal{M}_Y) \simeq H^p(Y, \mathcal{M}) \simeq H^p(Y, \mathcal{M}|_Y).$$

This yields the following reinterpretation of Proposition 7.4.7.

7.5.5 Proposition. *Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules on a paracompact space X. Then*

(i) *if Y is a closed subset of X, $U = X - Y$, and ϕ is the family of subsets of U which are closed in X, then there is an exact sequence*

$$\cdots \to H^p_\phi(U, \mathcal{M}) \to H^p(X, \mathcal{M}) \to H^p(Y, \mathcal{M}) \to H^{p+1}_\phi(U, \mathcal{M}) \to \cdots;$$

(ii) *if Y_1 and Y_2 are closed subsets of X, then there is an exact sequence*

$$\cdots \to H^p(Y_1 \cup Y_2, \mathcal{M}) \to H^p(Y_1, \mathcal{M}) \oplus H^p(Y_2, \mathcal{M})$$
$$\to H^p(Y_1 \cap Y_2, \mathcal{M}) \to H^{p+1}(Y_1 \cup Y_2, \mathcal{M}) \to \cdots.$$

The sequence in part (ii) above is called the Meyer-Vietoris sequence. The long exact sequences of this proposition are particularly useful in algebraic topology. There, the sheaves of greatest interest are the constant sheaves and the sheaves derived from constant sheaves through the standard sheaf theoretic operations.

7.5.6 Example. Recall Example 7.2.3. There X is the unit interval $[0,1]$ and $\mathcal{Z}$ is the constant sheaf with stalks $\mathbb{Z}$. If $U=(0,1)$, and $Y=\{0\}\cup\{1\}$, then the sheaves $\mathcal{Z}_0$ and $\mathcal{Q}$ of that example are $\mathcal{Z}_U$ and $\mathcal{Z}_Y$, respectively. By Proposition 7.5.5(i), the short exact sequence $0\to\mathcal{Z}_U\to\mathcal{Z}\to\mathcal{Z}_Y\to 0$ induces a long exact sequence

$$(7.5.3)\quad \cdots\to H^p_\phi(U,\mathcal{Z})\to H^p(X,\mathcal{Z})\to H^p(Y,\mathcal{Z})\to H^{p+1}_\phi(U,\mathcal{Z})\to\cdots.$$

The constant sheaf $\mathcal{Z}$ has vanishing cohomology on $X=[0,1]$ in degrees greater than 0 and cohomology $\mathbb{Z}$ in degree 0 by Exercise 7.12 . The sheaf $\mathcal{Z}_U$ has no non-zero global sections. Also, $H^p(Y,\mathcal{Z})$ is $\mathbb{Z}\oplus\mathbb{Z}$ for $p=0$ and is 0 for $p>0$, since Y consists of two points. Thus, the long exact sequence (7.5.3) reduces to

$$0\to\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\to H^1_\phi(U;\mathcal{Z})\to 0\to\cdots\to 0\to H^p_\phi(U,\mathcal{Z})\to 0\to\cdots.$$

It follows that $H^1_\phi(U,\mathcal{Z})=\mathbb{Z}$ and $H^p_\phi(U,\mathcal{Z})=0$ for $p\neq 1$.

7.6 Ringed Spaces

Sheaves provide an economical language with which to describe the structure of a wide variety of spaces that occur in analysis, topology and geometry.

7.6.1 Definition. *A ringed space $(X,\mathcal{R})$ is a topological space X, together with a sheaf of commutative rings $\mathcal{R}$ on X called the structure sheaf. A morphism $\phi:(X,\mathcal{R})\to(Y,\mathcal{S})$ of ringed spaces is a continuous mapping $\phi:X\to Y$, together with a morphism of sheaves of rings $\alpha:\phi^{-1}\mathcal{S}\to\mathcal{R}$.*

Note that a sheaf morphism $\alpha:\phi^{-1}\mathcal{S}\to\mathcal{R}$ is determined by a morphism from the presheaf $U\to\varinjlim\{\mathcal{S}(V):V \text{ open}, \phi(U)\subset V\}$ to $\mathcal{R}$. Note also that, for $x\in X$, the stalk $(\phi^{-1}\mathcal{S})_x$ is naturally isomorphic to the stalk $\mathcal{S}_{\phi(x)}$, and so the morphism α induces a ring homomorphism $\mathcal{S}_{\phi(x)}\to\mathcal{R}_x$ for each $x\in X$.

The ringed spaces we will be using all have structure sheaves which are sheaves of complex subalgebras of the sheaf of continuous complex valued functions. They all also have the property that their stalks are local rings. Morphisms between such ringed spaces have a simple characterization:

7.6.2 Proposition. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be ringed spaces with the property that the structure sheaves $\mathcal{R}$ and $\mathcal{S}$ are sheaves of complex subalgebras of the sheaf of continuous functions, with stalks that are local rings. Then a continuous map $\phi : X \to Y$ is the map associated to a morphism of ringed spaces from $(X, \mathcal{R})$ to $(Y, \mathcal{S})$ if and only if $f \circ \phi \in \mathcal{R}(\phi^{-1}(V))$, for each open set $V \subset Y$ and each $f \in \mathcal{S}(V)$. In this case, $f \to f \circ \phi$ defines the unique morphism of sheaves of rings $\alpha : \phi^{-1}\mathcal{S} \to \mathcal{R}$.*

Proof. Suppose $f \to f \circ \phi$ maps $\mathcal{S}(V)$ into $\mathcal{R}(\phi^{-1}(V))$, for each open set $V \subset Y$. Then, for U open in X, and V open in Y and containing $\phi(U)$, the map $f \to (f \circ \phi)|_U$ defines a ring homomorphism from $\mathcal{S}(V) \to \mathcal{R}(U)$. On passing to the limit over all open sets V containing $\phi(U)$, we obtain a ring homomorphism $f \to f \circ \phi$ from $\varinjlim\{\mathcal{S}(V) : V \text{ open}, \ \phi(U) \subset V\}$ to $\mathcal{R}(U)$. This homomorphism commutes with restriction and, hence, defines a morphism of sheaves of rings from $\phi^{-1}\mathcal{S}$ to $\mathcal{R}$. Thus, ϕ determines a morphism of ringed spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$.

Now suppose there is some morphism of sheaves of rings $\alpha : \phi^{-1}\mathcal{S} \to \mathcal{R}$. If U is an open subset of X, then for each open set $V \subset Y$, containing $\phi(U)$, each $f \in \mathcal{S}(V)$ and each $x \in U$, the number $f(\phi(x))$ is the unique complex number λ such that the germ of $\lambda - f$ fails to be invertible in the local ring $\mathcal{S}_{\phi(x)}$. But then $f(\phi(x))$ is also the unique complex number λ so that the germ of $\alpha(\lambda - f) = \lambda - \alpha(f)$ fails to be invertible in the local ring $\mathcal{R}_x$. It follows that $\alpha(f)(x) = f(\phi(x))$ for all $x \in U$ and, hence, $\alpha(f) = f \circ \phi$. This implies that $f \circ \phi \in \mathcal{R}(\phi^{-1}(V))$ for each open set $V \subset Y$ and each $f \in \mathcal{S}(V)$. It also proves that $f \to f \circ \phi$ is the unique morphism of sheaves of rings from $\phi^{-1}\mathcal{S}$ to $\mathcal{R}$. This completes the proof.

7.6.3 Examples. Each of the following is a ringed space:

(i) $(X, {}_X\mathcal{C})$, where X is a topological space, and ${}_X\mathcal{C}$ is the sheaf of continuous functions on X. Any continuous map $\phi : X \to Y$, between topological spaces, determines a morphism of ringed spaces. By Proposition 7.6.2, the unique ring homomorphism $\phi^{-1}{}_Y\mathcal{C} \to {}_X\mathcal{C}$ is defined by $f \to f \circ \phi$.

(ii) $(U, {}_U\mathcal{C}^\infty)$, where U is an open subset of $\mathbb{R}^n$, and ${}_U\mathcal{C}^\infty$ is the sheaf of functions on U with continuous partial derivatives of all orders. A continuous map $\phi : U \to V$, between open sets in Euclidean spaces, defines a morphism of ringed spaces $(U, {}_U\mathcal{C}^\infty) \to (V, {}_V\mathcal{C}^\infty)$ if and only if the coordinate functions of ϕ are $\mathcal{C}^\infty$ functions. Here again, the ring homomorphism $\phi^{-1}{}_V\mathcal{C}^\infty \to {}_U\mathcal{C}^\infty$ is unique and is defined by $f \to f \circ \phi$.

(iii) $(V, {}_V\mathcal{H})$, where V is a holomorphic subvariety of an open set in $\mathbb{C}^n$, and ${}_V\mathcal{H}$ is the sheaf of holomorphic functions on V. A continuous map $\phi : V \to W$, between two holomorphic subvarieties, induces a morphism of ringed spaces if and only if ϕ is a holomorphic map. Once again, the unique morphism $\phi^{-1}{}_W\mathcal{H} \to {}_V\mathcal{H}$ is defined by $f \to f \circ \phi$.

(iv) $(V, {}_V\mathcal{O})$, where V is an algebraic subvariety of an open set in $\mathbb{C}^n$ and ${}_V\mathcal{O}$ is the sheaf of regular functions on V. A map $\phi : V \to W$, between two algebraic subvarieties, induces a morphism of ringed spaces if and only if ϕ is a regular map. Again, the morphism $\phi^{-1}{}_W\mathcal{O} \to {}_V\mathcal{O}$ is unique and is defined by $f \to f \circ \phi$.

The concept of ringed space allows us to give quick and easy definitions of manifolds and varieties of various types. A topological manifold is a Hausdorff topological space X which is locally homeomorphic to a neighborhood in $\mathbb{R}^n$. That is, each point of X has a neighborhood V which is homeomorphic to a neighborhood in $\mathbb{R}^n$. Other types of manifolds are defined by insisting that X be a ringed space and these local homeomorphisms be ringed space isomorphisms for one of the standard structure sheaves on $\mathbb{R}^n$. This same idea leads to the definitions of abstract holomorphic and algebraic varieties.

7.6.4 Definition. *Let $(X, \mathcal{R})$ be a ringed space, with X Hausdorff and second countable as a topological space.*

(i) *If each point of X has a neighborhood U such that $(U, \mathcal{R}|_U)$ is isomorphic to $(V, {}_VC^\infty)$, where V is a neighborhood in $\mathbb{R}^n$ for some n, then $(X, \mathcal{R})$ is called a C^∞ manifold; in this case, the structure sheaf $\mathcal{R}$ is denoted ${}_XC^\infty$ (or just C^∞) and is called the sheaf of C^∞ functions on X.*

(ii) *If each point of X has a neighborhood U such that $(U, \mathcal{R}|_U)$ is isomorphic to $(V, {}_V\mathcal{H})$, where V is a neighborhood in $\mathbb{R}^n$ for some n, then $(X, \mathcal{R})$ is called a complex analytic manifold; in this case, the structure sheaf $\mathcal{R}$ is denoted ${}_X\mathcal{H}$ (or just $\mathcal{H}$) and is called the sheaf of holomorphic functions on X.*

(iii) *if each point of X has a neighborhood U such that $(U, \mathcal{R}|_U)$ is isomorphic to a ringed space of the form $(V, {}_V\mathcal{H})$, where V is a holomorphic subvariety of an open set in $\mathbb{C}^n$ for some n, then $(X, \mathcal{R})$ is called a holomorphic variety; in this case, the structure sheaf $\mathcal{R}$ is denoted ${}_X\mathcal{H}$ (or just $\mathcal{H}$) and is called the sheaf of holomorphic functions on X.*

Algebraic varieties are defined similarly – as ringed spaces locally isomorphic to algebraic subvarieties of $\mathbb{C}^n$. However, the Hausdorff condition doesn't make sense and needs to be replaced by another condition. For this reason, we put off the precise definition of algebraic varieties until the next chapter.

The language of sheaves also provides an economical way to think of vector bundles over a space X. We give a brief discussion of $\mathcal{C}^\infty$ vector bundles in preparation for the next section. Topological, holomorphic, and algebraic vector bundles are defined the same way but using ringed spaces with structure sheaves the sheaves of continuous, holomorphic, or regular functions.

A $\mathcal{C}^\infty$ vector bundle over a $\mathcal{C}^\infty$ manifold X is a $\mathcal{C}^\infty$ manifold E, together with a surjective morphism $\pi : E \to X$, and a finite dimensional complex vector space structure on each fiber $\pi^{-1}(x)$, $x \in X$ (of course, by morphism here we mean morphism of ringed spaces). This structure is required to be locally trivial in the sense that each point of X has a neighborhood U such that $\pi : \pi^{-1}(U) \to U$ is equivalent, via a morphism which is linear on the fibers, to a projection $U \times F \to U$, where F is a finite dimensional complex vector space. A morphism $\phi : (E_1, \pi_1) \to (E_2, \pi_2)$ between $\mathcal{C}^\infty$ vector bundles is a morphism of ringed spaces $\phi : E_1 \to E_2$, with $\pi_2 \circ \phi = \pi_1$, which is linear on each fiber $\pi_1^{-1}(x)$.

A $\mathcal{C}^\infty$ *section* of a $\mathcal{C}^\infty$ vector bundle $\pi : E \to X$, over an open set $U \subset X$, is a morphism of ringed spaces $f : U \to E$ such that $\pi \circ f = \mathrm{id} : U \to U$. Note, this says a section is a $\mathcal{C}^\infty$ map. The sections of a $\mathcal{C}^\infty$ vector bundle form a sheaf of modules over the sheaf of rings $\mathcal{C}^\infty$, where addition and multiplication by $\mathcal{C}^\infty$ functions are defined pointwise.

Note that if (E, π) is the trivial bundle ($E = X \times F$, with $\pi : X \times F \to X$ the projection, for some finite dimensional vector space F), then the space of sections of (E, π) is simply the space of $\mathcal{C}^\infty$ functions from X to F (here a function $f : X \to F$ is identified with the section $x \to (x, f(x))$). Fixing a basis for F then determines an isomorphism from this module to the direct sum of n copies of $\mathcal{C}^\infty$, where n is the dimension of F. A module of this form is called a *free module* of rank n over $\mathcal{C}^\infty$. Since every $\mathcal{C}^\infty$ vector bundle is locally trivial, we are led to the following equivalence:

7.6.5 Proposition. *If X is a $\mathcal{C}^\infty$ manifold, then the category of $\mathcal{C}^\infty$ vector bundles is equivalent to the category of locally free finite rank modules over the sheaf of rings $\mathcal{C}^\infty$. The analogous result holds for holomorphic varieties and holomorphic vector bundles and for topological spaces and topological vector bundles.*

Proof. The preceding discussion shows that the module of sections of a locally free $\mathcal{C}^\infty$ vector bundle is a locally free finite rank module over $\mathcal{C}^\infty$. Obviously, a morphism of vector bundles induces a morphism of the corresponding modules of sections. Thus, the assignment to a vector bundle of its sheaf of sections is a functor from $\mathcal{C}^\infty$ vector bundles to locally free finite rank sheaves of $\mathcal{C}^\infty$-modules. It remains to describe the inverse functor.

Let $\mathcal{M}$ be a locally free finite rank module over $\mathcal{C}^\infty$. If $x \in X$, we set $F_x = \mathcal{M}/\mathcal{I}_x\mathcal{M}$, where $\mathcal{I}_x$ is the sheaf of ideals of $\mathcal{C}^\infty$ consisting of functions which vanish at x ($f \in \mathcal{C}^\infty(U)$ belongs to $\mathcal{I}_x(U)$ if $f(x) = 0$ when $x \in U$). Then F_x is a finite dimensional vector space. We set $E = \cup_x F_x$ and define $\pi : E \to X$ by $\pi(F_x) = x$. If U is an open subset of X on which $\mathcal{M}$ is free, then, for some integer k, there is an isomorphism of ${}_U\mathcal{C}^\infty$-modules $\mathcal{M}|_U \to ({}_U\mathcal{C}^\infty)^k$. This determines a bijection $E|_U \to U \times \mathbb{C}^k$ which is a linear isomorphism of the fiber F_x over x to $x \times C^k$, for each $x \in U$. We use this bijection to give $E|_U$ the structure of a $\mathcal{C}^\infty$ vector bundle over U. This structure is independent of the isomorphism $\mathcal{M}|_U \to ({}_U\mathcal{C}^\infty)^k$, since any two such isomorphisms are related by an invertible $k \times k$ matrix with entries from $\mathcal{C}^\infty(U)$. By covering X by open sets on which $\mathcal{M}$ is free, we determine a well-defined structure of a $\mathcal{C}^\infty$ vector bundle on X. It is easy to see that $\mathcal{M}$ is isomorphic, as a sheaf of $\mathcal{C}^\infty$-modules, to the sheaf of sections of (E, π). The correspondence $\mathcal{M} \to (E, \pi)$ is the inverse of the global sections functor. Clearly the same argument works for topological bundles on topological spaces and holomorphic bundles on holomorphic varieties.

7.7 De Rham Cohomology

Fine sheaves of rings have a particularly nice property: Any sheaf of modules over such a sheaf of rings is also fine (Exercise 7.7).

7.7.1 Proposition. *The sheaf $\mathcal{C}$ of continuous functions on a paracompact space is a fine sheaf of rings, as is the sheaf $\mathcal{C}^\infty$ of infinitely differentiable functions on a $\mathcal{C}^\infty$ manifold.*

Proof. Paracompact implies normal, which implies that Urysohn's lemma holds. Urysohn's lemma can be used to construct continuous partitions of unity subordinate to any locally finite open cover (as in Lemma 1.3.2). A continuous function with support inside a given open set defines, by multiplication, an endomorphism of $\mathcal{C}$ with support in the open set. Thus, a partition of unity in the algebra of continuous functions on X defines a partition of unity for the sheaf $\mathcal{C}$, in the sense of Definition 7.5.1(iii). This proves that $\mathcal{C}$ is a fine sheaf. The same result for $\mathcal{C}^\infty$ follows from the fact that, on a $\mathcal{C}^\infty$ manifold, partitions of unity consisting of $\mathcal{C}^\infty$ functions may be constructed using Lemma 1.3.1 and an argument like that in Lemma 1.3.2.

Thus, a sheaf of modules over $\mathcal{C}$ or $\mathcal{C}^\infty$ is fine, hence, acyclic. In particular, the sheaf of $\mathcal{C}^\infty$ sections of a $\mathcal{C}^\infty$ vector bundle on a $\mathcal{C}^\infty$ manifold is a fine sheaf. The sheaves of differential forms that appear in the de Rham complex are of this form. The de Rham complex is a particular fine resolution of the constant sheaf $\mathbb{C}$. We give a brief description of it in this section.

Let X be a $\mathcal{C}^\infty$ manifold. If U is an open subset of X, then a *vector field* on U is a linear map $v : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$ which obeys the product rule

$$v(fg) = fv(g) + gv(f) \quad \text{for all} \quad f, g \in \mathcal{C}^\infty(U).$$

If $h \in \mathcal{C}^\infty(U)$, and v is a vector field on U, then $(hv)(f) = hv(f)$ defines a new vector field hv. Clearly the sum of two vector fields is a vector field, and so the vector fields on U form a module $\mathcal{D}^1(U)$ over $\mathcal{C}^\infty(U)$. In fact, the correspondence $U \to \mathcal{D}^1(U)$ is a sheaf of $\mathcal{C}^\infty$-modules on X.

If U is an open set in $\mathbb{R}^n$, then it is easy to see that the vector fields on U are the differential operators of the form

$$\sum g_i \frac{\partial}{\partial x_i} \quad \text{for} \quad g_i \in \mathcal{C}^\infty(U).$$

Thus, $\mathcal{D}^1$ is a free $\mathcal{C}^\infty$-module of rank n on any open subset of $\mathbb{R}^n$. If X is a $\mathcal{C}^\infty$ manifold, then X is locally isomorphic as a ringed space to a neighborhood in $\mathbb{R}^n$. It follows that the sheaf $\mathcal{D}^1$ on X is locally free of finite rank and, hence, is the sheaf of sections of a vector bundle. This vector bundle is the *complexified tangent bundle* $T(X)$ of X. Its fiber $T(X)_x$ at x is called the *complexified tangent space* of X at x and is the vector space of derivations on the local ring $\mathcal{C}_x$ (see Definition 5.4.1). The complex dual of $T(X)_x$ is the *complexified cotangent space* of X at x and is denoted $T^*(X)_x$.

A complex differential p-form on an open set $U \subset X$ is an alternating $\mathcal{C}^\infty$ p-multilinear form on $\mathcal{D}^1(U)$ – that is, a map $\omega : \mathcal{D}^1(U)^n \to \mathcal{C}^\infty(U)$ which is linear over $\mathcal{C}^\infty(U)$ in each variable and is skew-symmetric in each pair of variables. On an open set U in $\mathbb{R}^n$, where we may choose bases $\{\frac{\partial}{\partial x_i}\}$ for the tangent space and $\{dx_i\}$ for the cotangent space, corresponding to a basis $\{x_i\}$ for $\mathbb{R}^n$, the typical differential p-form may be written as

$$\phi = \sum \phi_{i_1 \dots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $\phi_{i_1 \dots i_p} \in \mathcal{C}^\infty(U)$. Here $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ stands for the p-form which is 1 (1) on an even (odd) permutation of the p-tuple $(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_p}})$ and is 0 on all other p-tuples of the basis vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.

We denote the $\mathcal{C}^\infty$-module of differential p-forms on U by $\mathcal{E}^p(U)$. The correspondence $U \to \mathcal{E}^p(U)$ is a sheaf of $\mathcal{C}^\infty$ -modules, denoted $\mathcal{E}^p$, which is also locally free of finite rank. It is, in fact, the sheaf of sections of the vector bundle whose fiber at x is the vector space of complex alternating p-forms on $T(X)_x$.

Exterior differentiation $d^p : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is defined, in any local coordinate system, by

$$d^p(\phi_{i_1 \dots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_i \frac{\partial \phi_{i_1 \dots i_p}}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The operator d^p also has a coordinate free description and, hence, is defined on any $\mathcal{C}^\infty$ manifold. A simple calculation shows that $d^{p+1} \circ d^p = 0$. We refer the reader to [**GP**] for these and other basic facts regarding differential forms.

The de Rham complex of sheaves is the complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{p-1}} \mathcal{E}^p \xrightarrow{d^p} \cdots, \tag{7.7.1}$$

where d^p is exterior differentiation, and $\mathbb{C}$ here stands for the constant sheaf with stalks $\mathbb{C}$.

The Poincaré lemma ([**GP**]) says that if U is any convex open set in $\mathbb{R}^n$, then the complex of sections of (7.7.1) over U is exact (this is proved by constructing an explicit homotopy between the identity and 0, using integration along lines from a fixed point in the convex set). Since a $\mathcal{C}^\infty$ manifold looks locally like a ball in $\mathbb{R}^n$, it follows that the de Rham complex is exact, as a complex of sheaves. Thus, it defines a resolution $\mathbb{C} \to \mathcal{E}$ of the constant sheaf $\mathbb{C}$, by a complex $\mathcal{E}$ of fine sheaves. On passing to global sections of $\mathcal{E}$, we obtain the classical de Rham complex $\mathcal{E}(X)$ of differential forms on X. The cohomology of this complex is called the de Rham cohomology of X (with coefficients in $\mathbb{C}$). By Theorem 6.4.6 we have:

7.7.2 Theorem. *There is a natural isomorphism between the de Rham cohomology of a $\mathcal{C}^\infty$ manifold X and the sheaf cohomology $H(X, \mathbb{C})$ for the constant sheaf $\mathbb{C}$ on X.*

7.8 Čech Cohomology

We next discuss Čech cohomology and its relation to sheaf cohomology. Let $\mathcal{S}$ be a sheaf of $\mathcal{R}$-modules on X, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, indexed by some index set I. If $\alpha = (i_0, \dots, i_p) \in I^{p+1}$ is a multi-index, then we set $U_\alpha = U_{i_0} \cap \cdots \cap U_{i_p}$, and let $i_\alpha : U_\alpha \to X$ be the inclusion map. Then the sheaf $(i_\alpha)_* i_\alpha^{-1}\mathcal{S}$ is just the sheaf $U \to \mathcal{S}(U \cap U_\alpha)$. Its module of global sections is $\mathcal{S}(U_\alpha)$. The sheaf of p-cochains is the direct product

$$\prod_{\alpha \in I^{p+1}} (i_\alpha)_* i_\alpha^{-1}\mathcal{S}$$

of these sheaves. Thus, it is the sheaf of $\mathcal{R}$-modules which assigns to each open set U the $\mathcal{R}(U)$-module

$$\prod_{\alpha \in I^{p+1}} \mathcal{S}(U_\alpha \cap U).$$

An element of this module is a function f which assigns to each $\alpha \in I^{p+1}$ an element $f(\alpha) \in \mathcal{S}(U_\alpha \cap U)$. If this is an alternating function of α (so that $f(\sigma(\alpha)) = \operatorname{sgn}(\sigma) f(\alpha)$ for each permutation σ of $\alpha = (i_0, \dots, i_p)$), then we say that f is an *alternating p-cochain*. The alternating p-cochains form a subsheaf of $\mathcal{R}$-modules of the sheaf of p-cochains. We denote the sheaf of alternating p-cochains by $\mathcal{C}^p(\mathcal{U}, \mathcal{S})$. The module of global sections of this sheaf is $\mathcal{C}^p(\mathcal{U}, \mathcal{S})(X)$. This is the classical space of alternating Čech cochains for $\mathcal{S}$ and the cover $\mathcal{U}$, and it will be denoted $C^p(\mathcal{U}, \mathcal{S})$.

We define a coboundary morphism

$$\delta^p : \mathcal{C}^p(\mathcal{U}, \mathcal{S}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{S})$$

by

$$\delta^p f(\alpha) = \sum_{j=0}^{p+1} (-1)^j f(\alpha_j)|_{U_\alpha \cap U},$$

where $f \in \mathcal{C}^p(\mathcal{U}, \mathcal{S})(U)$, and $\alpha_j \in I^{p+1}$ is obtained from $\alpha \in I^{p+2}$ by deleting its jth entry.

7.8.1 Proposition. *With δ^p as defined above, $\delta^{p+1} \circ \delta^p = 0$, so that*

$$0 \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta^0} \mathcal{C}^1(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} \mathcal{C}^p(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta^p} \cdots$$

is a complex of sheaves of $\mathcal{R}$-modules.

Proof. For $\alpha \in I^{p+3}$, let $\alpha_{j,k}$ denote the result of deleting both the jth and the kth entries from α. Then

$$(\alpha_j)_k = \alpha_{j,k} \quad \text{if} \quad 0 \le k < j \le p+2,$$

and

$$(\alpha_j)_k = \alpha_{j,k+1} \quad \text{if} \quad 0 \le j < k \le p+1.$$

It follows that, for $f \in \mathbb{C}^p(\mathcal{U}, \mathcal{S})(U)$, we have

$$\begin{aligned}
\delta^{p+1} \circ \delta^p f(\alpha) &= \sum_{j=0}^{p+2} (-1)^j \left[\sum_{k=0}^{p+1} (-1)^k f((\alpha_j)_k)|_{U_{\alpha_j} \cap U} \right] |_{U_\alpha \cap U} \\
&= \sum_{k<j} (-1)^{j+k} f(\alpha_{j,k})|_{U_\alpha \cap U} + \sum_{k \ge j} (-1)^{j+k} f(\alpha_{j,k+1})|_{U_\alpha \cap U}.
\end{aligned}$$

This vanishes, due to the fact that, in the last line above, the second term is equal to the negative of the first, which is evident from the change of variables $j \to k$, $k+1 \to j$, applied to the second term, and the observation that $\alpha_{j,k} = \alpha_{k,j}$.

The restrictions $\mathcal{S}(U) \to \mathcal{S}(U_i \cap U)$ define a morphism $\epsilon : \mathcal{S} \to \mathcal{C}^0(\mathcal{U}, \mathcal{S})$, whose composition with δ^0 is 0. In fact, much more is true:

7.8.2 Proposition. *If $\mathcal{U}$ is an open cover of X, and $\mathcal{S}$ a sheaf on X, then the complex*

$$0 \longrightarrow \mathcal{S} \xrightarrow{\ \epsilon\ } \mathcal{C}^0(\mathcal{U},\mathcal{S}) \xrightarrow{\ \delta^0\ } \cdots \xrightarrow{\ \delta^{p-1}\ } \mathcal{C}^p(\mathcal{U},\mathcal{S}) \xrightarrow{\ \delta^p\ } \cdots$$

is an exact sequence of sheaves.

Proof. Fix an $x \in X$. Let U be any neighborhood of x which is contained in a member of $\mathcal{U}$, say $U \subset U_k$. Now suppose $f \in \mathcal{C}^p(\mathcal{U},\mathcal{S})(U)$, and $\delta^p f = 0$. Define $g \in \mathcal{C}^{p-1}(\mathcal{U},\mathcal{S})(U)$ by

$$g(\beta) = f((k,\beta)),$$

where we set $(k,\beta) = (k, i_0, \dots, i_{p-1}) \in I^{p+1}$ for $\beta = (i_0, \dots, i_{p-1}) \in I^p$. Note that $g \in \mathcal{C}^{p-1}(\mathcal{U},\mathcal{S})(U)$, due to the fact that $U \subset U_k$, which implies that $U_k \cap U_\beta \cap U = U_\beta \cap U$. Then, for $\alpha \in I^{p+1}$,

$$\begin{aligned} 0 = \delta^p f((k,\alpha)) &= f(\alpha)|_{U_k \cap U_\alpha \cap U} - \sum_{j=0}^{p} (-1)^j f((k,\alpha_j))|_{U_k \cap U_\alpha \cap U} \\ &= f(\alpha) - \delta^{p-1} g(\alpha), \end{aligned}$$

again due to the fact that $U_k \cap U_\alpha \cap U = U_\alpha \cap U$. This shows that, locally, the kernel of δ^p is the image of δ^{p-1} for $p > 0$. The same argument also works for $p = 0$, with δ^{-1} replaced by ϵ. This proves the exactness of the sequence of sheaves of the proposition.

The complex $\{\mathcal{C}^p(\mathcal{U},\mathcal{S}), \delta^p\}$ is called the *Čech complex of sheaves* for the cover $\mathcal{U}$ and is denoted $\mathcal{C}(\mathcal{U},\mathcal{S})$. Its complex of global sections, $\Gamma(\mathcal{C}(\mathcal{U},\mathcal{S})) = C(\mathcal{U},\mathcal{S})$, is what is classically called the Čech complex for the sheaf $\mathcal{S}$ and the open cover $\mathcal{U}$. We shall call it the *global Čech complex.*

7.8.3 Definition. *The Čech cohomology $\{\check{H}^p(\mathcal{U},\mathcal{S})\}_{p\geq 0}$ of the sheaf $\mathcal{S}$ for the cover $\mathcal{U}$ is the cohomology of the global Čech complex $C(\mathcal{U},\mathcal{S})$.*

7.8.4 Proposition. *For any sheaf $\mathcal{S}$ and any open cover $\mathcal{U}$, we have*

(i) *$\check{H}^0(\mathcal{U},\mathcal{S}) \simeq \Gamma(X,\mathcal{S})$; and*

(ii) *if $\mathcal{S}$ is injective, then $\check{H}^p(\mathcal{U},\mathcal{S}) = 0$ for $p > 0$.*

Proof. By Proposition 7.8.2, we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{S} \xrightarrow{\ \epsilon\ } \mathcal{C}^0(\mathcal{U},\mathcal{S}) \xrightarrow{\ \delta^0\ } \mathcal{C}^1(\mathcal{U},\mathcal{S}),$$

which implies that

$$0 \longrightarrow \Gamma(X,\mathcal{S}) \xrightarrow{\ \epsilon\ } C^0(\mathcal{U},\mathcal{S}) \xrightarrow{\ \delta^0\ } C^1(\mathcal{U},\mathcal{S})$$

is exact, since the global sections functor is left exact. This implies statement (i).

Note that the sheaf $\mathcal{C}^p(\mathcal{U},\mathcal{S})$ is a direct product of sheaves $(i_\alpha)_* i_\alpha^{-1}\mathcal{S}$, one for each equivalence class of multi-indices α under permutation, where the map $i_\alpha : U_\alpha \to X$ is the inclusion. If $\mathcal{S}$ is injective then so is its restriction $i_\alpha^{-1}\mathcal{S}$ to U, by Proposition 7.4.4(i). By Proposition 7.4.4(ii) the direct image of an injective sheaf is injective It follows that $(i_\alpha)_* i_\alpha^{-1}\mathcal{S}$ is injective for each α. Then Proposition 7.4.4(iii) implies that $\mathcal{C}^p(\mathcal{U},\mathcal{S})$ is injective for each p. Thus, the complex in Proposition 7.8.2 is an exact sequence of injective sheaves. By Proposition 6.4.4, the complex obtained from it by applying $\Gamma(X, \cdot\,)$ is also exact. Part (ii) follows.

Let $\mathcal{S}$ be a sheaf on X, $\mathcal{U}$ an open cover of X, and $\mathcal{S} \to \mathcal{C}(\mathcal{U},\mathcal{S})$ the corresponding Čech resolution, as in Proposition 7.8.2. Let $\mathcal{S} \to \mathcal{I}$ be an injective resolution of $\mathcal{S}$. Then the injectivity of the terms in $\mathcal{I}$ can be used, as in Proposition 6.3.4, to inductively construct a morphism of complexes $\mathcal{C}(\mathcal{U},\mathcal{S}) \to \mathcal{I}$ for which the diagram

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{C}(\mathcal{U},\mathcal{S}) \\ \| & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{I} \end{array}$$

is commutative. This morphism is unique up to homotopy and so, after we apply $\Gamma(X, \cdot\,)$, it determines a well-defined morphism $\check{H}^p(\mathcal{U},\mathcal{S}) \to H^p(X,\mathcal{S})$.

The open cover $\mathcal{U}$ is called a *Leray cover* for the sheaf $\mathcal{S}$ if, for each multi-index α, the sheaf $\mathcal{S}$ is acyclic on U_α. For example, it follows from the Poincaré lemma that the de Rham cohomology of any convex open set in $\mathbb{R}^n$ vanishes in positive degrees. From this and Theorem 7.7.2 we conclude that the constant sheaf $\mathbb{C}$ is acyclic on any convex open set in $\mathbb{R}^n$. This implies that any open cover of $\mathbb{R}^n$ by open convex sets is a Leray cover for the constant sheaf $\mathbb{C}$.

7.8.5 Theorem. *If $\mathcal{U}$ is a Leray cover for the sheaf $\mathcal{S}$, then the natural morphism*

$$\check{H}^p(\mathcal{U},\mathcal{S}) \to H^p(X,\mathcal{S})$$

is an isomorphism.

Proof. Choose an embedding $\mathcal{S} \to \mathcal{F}$ of $\mathcal{S}$ in an injective sheaf. Let $\mathcal{G}$ be the cokernel, and consider the short exact sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then from the long exact sequence and the fact that $\mathcal{S}$ is acyclic on $\mathcal{U}_\alpha$, we conclude that

$$0 \longrightarrow \mathcal{S}(U_\alpha) \longrightarrow \mathcal{F}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha) \longrightarrow 0$$

is exact for each multi-index α and, from this, that we have an exact sequence

$$0 \longrightarrow C(\mathcal{U},\mathcal{S}) \longrightarrow C(\mathcal{U},\mathcal{F}) \longrightarrow C(\mathcal{U},\mathcal{G}) \longrightarrow 0$$

of global Čech complexes. This, in turn, gives us a long exact sequence of Čech cohomology, which along with the long exact sequence for sheaf cohomology, the induced morphisms from Čech to sheaf cohomology discussed above, and the fact that $\mathcal{F}$ is injective, gives us the following commutative diagrams with exact rows:

(7.8.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{H}^0(\mathcal{U},\mathcal{S}) & \longrightarrow & \check{H}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{U},\mathcal{G}) & \longrightarrow & \check{H}^1(\mathcal{U},\mathcal{S}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X,\mathcal{S}) & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(X,\mathcal{G}) & \longrightarrow & H^1(X,\mathcal{S}) & \longrightarrow & 0, \end{array}$$

and for $p > 0$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^p(\mathcal{U},\mathcal{G}) & \longrightarrow & \check{H}^{p+1}(\mathcal{U},\mathcal{S}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^p(X,\mathcal{G}) & \longrightarrow & H^{p+1}(X,\mathcal{S}) & \longrightarrow & 0. \end{array} \tag{7.8.2}$$

The first three vertical arrows of (7.8.1) are isomorphisms and, hence, so is the fourth. This establishes our result in the case $p = 1$. In (7.8.2), if the first vertical arrow is an isomorphism, so is the second. Consider the class of all sheaves $\mathcal{T}$ such that the open cover $\mathcal{U}$ is a Leray cover for $\mathcal{T}$. From the long exact sequence for sheaf cohomology, it follows that if the first two sheaves in a short exact sequence belong to this class, then so does the third. By hypothesis, $\mathcal{S}$ belongs to this class, and $\mathcal{F}$ does, since it is injective. It follows that $\mathcal{G}$ belongs as well. Thus, if we have proved the theorem for all members of this class and for all degrees less than or equal to p, then (7.8.2) shows that it is also true for degree $p+1$. By induction, the proof is complete.

What we have been discussing so far is Čech cohomology for an open cover and a sheaf. Čech cohomology for a sheaf, without reference to a cover, is obtained by passing to the limit over covers. More precisely, if $\mathcal{V}$ is an open cover which is a refinement of the open cover $\mathcal{U}$, then the restriction maps define a morphism of complexes of sheaves $\mathcal{C}(\mathcal{U},\mathcal{S}) \to \mathcal{C}(\mathcal{V},\mathcal{S})$. By passing to the limit over the directed set of open covers of X, we obtain a complex of sheaves $\mathcal{C}(\mathcal{S})$, which we shall call the limit Čech complex of sheaves, and a quasi-isomorphism $\mathcal{S} \to \mathcal{C}(\mathcal{S})$. The complex of global sections of $\mathcal{C}(\mathcal{S})$ is $C(\mathcal{S}) = \mathcal{C}(\mathcal{S})(X) = \varinjlim C(\mathcal{U},\mathcal{S})$ and is called the limit global Čech complex. Its cohomology is the Čech cohomology of $\mathcal{S}$ on X and is denoted $\{\check{H}^p(X,\mathcal{S})\}$. Clearly the morphisms $\check{H}^p(\mathcal{U},\mathcal{S}) \to H^p(X,\mathcal{S})$ induce a morphism $\check{H}^p(X,\mathcal{S}) \to H^p(X,\mathcal{S})$.

7.8.6 Theorem. *If X is paracompact, then the natural morphism*

$$\check{H}^p(X,\mathcal{S}) \to H^p(X,\mathcal{S})$$

is an isomorphism.

Proof. We shall prove that $\mathcal{S} \to \mathcal{C}(\mathcal{S})$ is a resolution of $\mathcal{S}$ by soft sheaves. Since soft sheaves are acyclic, the result will then follow from Theorem 6.4.6.

We have already remarked that $\mathcal{S} \to \mathcal{C}(\mathcal{S})$ is a quasi-isomorphism – in other words, that the exactness of the sequences in Proposition 7.8.2 is preserved on passing to the direct limit – since direct limits generally preserve exactness. It remains to prove that each $\mathcal{C}^p(\mathcal{S})$ is a soft sheaf. Thus, let $Y \subset X$ be closed, and suppose f is a section of $\mathcal{C}^p(\mathcal{S})$ over Y. Since X is paracompact, and Y is closed, Proposition 7.3.5 implies that f may be represented by a section in a neighborhood U of Y. This section may, in turn, be represented by an element $f' \in \mathcal{C}^p(\mathcal{U},\mathcal{S})(U)$, for some open cover $\mathcal{U}$ of X. We may choose a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$, with the property that each set in $\mathcal{V}$ is either contained in U, or is contained in $X - Y$. We then define a section g' of $\mathcal{C}^p(\mathcal{V},\mathcal{S})$ on X by setting g' equal to the image of f' under the refinement map on those multi-indices for which all the corresponding sets in $\mathcal{V}$ are contained in U, and setting it equal to 0 otherwise. On passing to the image g of g' in the space of limit Čech p-cochains, we obtain a global section of $\mathcal{C}^p(\mathcal{S})$ which has f as its restriction to Y. Thus, $\mathcal{C}^p(\mathcal{S})$ is soft, and the proof is complete.

When $\mathcal{S}$ is a constant sheaf with stalk G, the Čech cohomology in the sense of this chapter is just classical Čech cohomology with coefficients in G.

We end this section with an example which shows how to solve one of the local to global problems posed at the beginning of the chapter. This is the problem of finding a global logarithm for a non-vanishing continuous function on X. We assume X is paracompact.

7.8.7 Example. Let $\mathcal{C}$ denote the sheaf of continuous functions on X, with addition as group operation, and $\mathcal{C}^*$ the sheaf of invertible (non-vanishing) continuous functions, with multiplication as group operation. Then, due to the fact that a non-vanishing continuous function has a logarithm locally in a neighborhood of each point, the sequence of sheaves:

$$0 \longrightarrow 2\pi i\mathcal{Z} \longrightarrow \mathcal{C} \xrightarrow{\exp} \mathcal{C}^* \longrightarrow 0$$

is exact. Since $\mathcal{C}$ is a fine sheaf and, hence, is acyclic, we conclude from the long exact sequence of cohomology for this short exact sequence that

$$\mathcal{C}^*(X)/\exp(\mathcal{C}(X)) \simeq H^1(X,\mathbb{Z}) \simeq \check{H}^1(X,\mathbb{Z})$$

where, of course, $\mathcal{Z}$ stands for the constant sheaf with stalk $\mathbb{Z}$, and we use the fact that $2\pi i\mathcal{Z} \simeq \mathcal{Z}$. Thus, every non-vanishing continuous function on X has a global logarithm if and only if first Čech cohomology of X, with integral coefficients, vanishes. More generally, there is an epimorphism $f \to [f]$ from the group of non-vanishing continuous functions on X to the Čech cohomology group $\check{H}^1(X,\mathbb{Z})$, with the property that f has a global logarithm if and only if $[f]$ vanishes. This is an elementary but very instructive example of the use of sheaf theory to analyze a problem involving passing from local to global solutions.

7.9 Line Bundles and Čech Cohomology

Recall from section 7.6 that the category of n-dimensional topological vector bundles on X is equivalent to the category of locally free $\mathcal{C}$-modules of rank n on X. The case $n = 1$ has an interesting connection with Čech cohomology.

A 1-dimensional vector bundle is called a *line bundle.* The corresponding sheaf of local sections is a locally free $\mathcal{C}$-module of rank 1 – that is, a sheaf of modules which is locally isomorphic to $\mathcal{C}$. This concept makes sense, not just for $\mathcal{C}$, but for any sheaf of commutative rings with identity on X – that is, it makes sense for any ringed space.

Thus, let X be a ringed space with structure sheaf $\mathcal{A}$. If $\mathcal{L}$ and $\mathcal{K}$ are two locally free rank 1 $\mathcal{A}$-modules, then their tensor product $\mathcal{L} \otimes_{\mathcal{A}} \mathcal{K}$ is another one (Definition 7.3.7). This follows from the fact that, locally, $\mathcal{L}$ and $\mathcal{K}$ are isomorphic to $\mathcal{A}$, and $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}$ is isomorphic to $\mathcal{A}$ (Exercise 7.16). Thus, tensor product defines an operation on rank one locally free sheaves of $\mathcal{A}$-modules. This corresponds to the operation of vector bundle tensor product.

Consider the set of isomorphism classes of rank 1 locally free $\mathcal{A}$-modules. The tensor product operation on this set is clearly associative and commutative. The module $\mathcal{A}$ itself acts as an identity. In fact, this set is an abelian group under tensor product. The inverse of a module $\mathcal{L}$ is given by

$$\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{A}}(\mathcal{L},\mathcal{A})$$

(Definition 7.3.8). In fact, there is a morphism $\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^{-1} \to \mathcal{A}$ defined by $g \otimes f \to f(g)$, and it is easy to see that this is an isomorphism by looking at what happens on a neighborhood where $\mathcal{L}$ is isomorphic to $\mathcal{A}$. It turns out that the locally free rank 1 modules are exactly those locally free $\mathcal{A}$-modules which have an inverse under tensor product. For this reason, locally free rank 1 sheaves of $\mathcal{A}$-modules are called *invertible sheaves* of $\mathcal{A}$-modules.

7.9.1 Definition. *Given a ringed space $(X, \mathcal{A})$, the abelian group of isomorphism classes of invertible sheaves of $\mathcal{A}$-modules on X is called the Picard group of X and is denoted* $\mathrm{Pic}(X)$.

Given an invertible $\mathcal{A}$-module $\mathcal{L}$, let $\{U_i\}$ be an open cover of X such that $\mathcal{L}$ is free on U_i for each i. Then there is an isomorphism $f_i : \mathcal{A}|_{U_i} \to \mathcal{L}|_{U_i}$ for each i. If we set

$$\phi_{ij} = f_i^{-1} \circ f_j \text{ on } U_i \cap U_j, \tag{7.9.1}$$

then ϕ_{ij} is an automorphism of $\mathcal{A}|_{U_i \cap U_j}$, as a module over itself – that is, it is multiplication by a function in $\mathcal{A}^*(U_i \cap U_j)$, where $\mathcal{A}^*$ denotes the sheaf of invertible elements of $\mathcal{A}$, with multiplication as group operation. The correspondence $\{i, j\} \to \phi_{ij}$ defines a Čech 1-cochain for the cover $\{U_i\}$ and the sheaf $\mathcal{A}^*$. Clearly,

$$\phi_{ji} = \phi_{ij}^{-1}$$

and

$$\delta\phi_{ijk} = \phi_{jk}\phi_{ik}^{-1}\phi_{ij} = \mathrm{id}\,.$$

Thus, we have an alternating 1-cocycle for the cover $\{U_i\}$ and the sheaf $\mathcal{A}^*$.

Suppose $\mathcal{L}$ and $\mathcal{L}'$ are locally free rank n sheaves of $\mathcal{A}$-modules, which determine cocycles ϕ and ϕ', through local isomorphisms $\{f_i\}$ and $\{f_i'\}$ for some open cover $\{U_i\}$, as in (7.9.1). If $h : \mathcal{L} \to \mathcal{L}'$ is an isomorphism between these two sheaves, then we may define a 0-cochain g by

$$g_i = f_i^{-1} h f_i'.$$

Then

$$\phi_{ij} = g_i \phi_{ij}' g_j^{-1}.$$

In other words, the cocycles ϕ and ϕ' are equivalent and, hence, define the same element of $H^1(X, \mathcal{A}^*)$. It follows that there is a well-defined map from $\mathrm{Pic}(X)$ to the set $H^1(X, \mathcal{A}^*)$. The image of an invertible sheaf $\mathcal{L}$ under this map is called the *Chern class* of $\mathcal{L}$ and is denoted $\mathrm{ch}(\mathcal{L})$. This map takes the tensor product of two modules to the product of the corresponding cocycles and is bijective (Exercise 7.17). Thus, we have:

7.9.2 Proposition. *The Chern class map* ch : $\mathrm{Pic}(X) \to H^1(X, \mathcal{A}^*)$ *is an isomorphism of abelian groups.*

This result applies, in particular, in the cases where the ringed space X is a topological space, a $\mathcal{C}^\infty$ manifold, a holomorphic variety, and an algebraic variety. This covers the cases where $\mathrm{Pic}(X)$ consists of isomorphism classes of continuous line bundles, $\mathcal{C}^\infty$ line bundles, holomorphic line bundles and algebraic line bundles.

Finally, note that if X is paracompact, then $H^1(X, \mathcal{C}^*) \simeq H^2(X, \mathbb{Z})$ (Exercise 7.18), and so $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$.

Exercises

1. Prove that if ϕ is a morphism of sheaves, then the presheaf $U \to \mathrm{Ker}\, \phi_U$ is a sheaf and is a kernel for ϕ in the category of sheaves.
2. Give an example of a morphism of sheaves ϕ such that the presheaf $U \to \mathrm{Coker}\, \phi_U$ is not a sheaf.
3. Prove that the category of sheaves is an abelian category.
4. Let $\mathcal{H}$ be the sheaf of holomorphic functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and let $\mathcal{K}$ be the subsheaf consisting of sections which vanish in a neighborhood of 0. Show that the quotient $\mathcal{S} = \mathcal{H}/\mathcal{K}$ is a skyscraper sheaf at 0, with stalk $\mathcal{H}_0$. Thus, there is a short exact sequence
$$0 \to \mathcal{K} \to \mathcal{H} \to \mathcal{S} \to 0.$$
Show that the corresponding sequence of global sections is not exact.
5. Prove that if U is an open subset of X, $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules on X, and $\mathcal{R}_U$ is the extension by zero of the restriction of $\mathcal{R}$ to U, then $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}_U, \mathcal{M}) \simeq \Gamma(U, \mathcal{M})$.
6. Finish the proof of Theorem 7.5.3 by proving that if the first two terms of a short exact sequence of sheaves are soft, then so is the third. Hint: You may use the fact, proved in 7.5.3, that if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of sheaves on X, and $\mathcal{A}$ is soft, then $\Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$ is surjective.
7. Prove that if a sheaf of rings is fine, then so is every sheaf of modules over this sheaf of rings.
8. Prove that if $f : Y \to X$ is any continuous map between topological spaces, then $f_*\mathcal{S}$ is a flabby sheaf on X if $\mathcal{S}$ is a flabby sheaf on Y.
9. Prove that the adjunction morphism $\mathcal{T} \to f_* f^{-1}\mathcal{T}$ is an isomorphism, for every sheaf $\mathcal{T}$ on X, if $f : Y \to X$ is a proper surjective map between locally compact spaces, and the fibers $f^{-1}(x)$ are all connected.

10. Show that the Čech cohomology $\check{H}^p(\mathcal{V}, \mathcal{S})$ of a cover $\mathcal{V}$ vanishes, for all $p > n$, if the open cover $\mathcal{V}$ has the property that the intersection of any $n + 2$ distinct sets in $\mathcal{V}$ is empty.
11. Prove that if $I = [0, 1]$ is the unit interval, then each open cover of I has a refinement $\mathcal{V}$ with the property that the intersection of any 3 distinct sets in $\mathcal{V}$ is empty. Conclude that $H^p(I, \mathcal{S}) = 0$ for $p > 1$ and for any sheaf $\mathcal{S}$ on I. Can you generalize this to a result about n-cubes? How about general n-dimensional manifolds?
12. Prove that if I is the interval $[0, 1]$, and $\mathcal{S}$ is a constant sheaf on I, then $H^p(I; \mathcal{S}) = 0$ for all $p > 0$. Hint: By Exercise 7.11 you only need to prove that $H^1(I; \mathcal{S}) = 0$. Do this by using Exercise 7.11 to directly compute the Čech cohomology of I in degree 1.
13. Let X be a locally compact space, and $\pi : X \times I \to X$ the projection, where $I = [0, 1]$. If $\mathcal{S}$ is a constant sheaf on X, use Theorem 7.4.6 and the previous exercise to show that π induces an isomorphism $H^p(X, \mathcal{S}) \to H^p(X \times I, \pi^{-1}\mathcal{S})$. Also show that the inverse of this isomorphism is the map induced, as in Theorem 7.4.6, by the inclusion $i_t : X \to X \times I$, defined by $i_t(x) = (x, t)$. Conclude that this map is independent of t.
14. Maps $f : Y \to X$ and $g : Y \to X$, between locally compact spaces, are said to be *homotopic* if there is a continuous map $h : Y \times I \to X$ such that $f(y) = h(y, 0)$, and $g(y) = h(y, 1)$. Prove that if f and g are homotopic maps, then they induce the same morphism $H^p(X, G) \to H^p(Y, G)$, where G is any abelian group (see the remark following Theorem 7.4.6). Hint: Use the result of the previous exercise.
15. A space X is *contractible* if the identity map $X \to X$ and the map which collapses X to some point $x_0 \in X$ are homotopic. Use the result of the previous exercise to prove that a contractible locally compact space satisfies $H^p(X, G) = 0$ for $p > 0$ and any group G. Use this result, induction on n, and Proposition 7.4.7 to prove that $H^p(S^n, G) \simeq G$ if $p = 0$ or n, and that it vanishes otherwise. Here S^n denotes the real n-dimensional sphere (the unit sphere in $\mathbb{R}^{n+1}$).
16. If $\mathcal{A}$ is a sheaf of commutative rings (with identity) on a topological space X, prove that $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}$ is naturally isomorphic to $\mathcal{A}$. Use this to prove that if $\mathcal{L}$ and $\mathcal{K}$ are locally free rank 1 sheaves of $\mathcal{A}$-modules on X, then $\mathcal{L} \otimes_{\mathcal{A}} \mathcal{K}$ is also.
17. Complete the proof of Proposition 7.9.2 by proving that the map

$$\mathrm{ch} : \mathrm{Pic}(X) \to H^1(X, \mathcal{A}^*)$$

is a group homomorphism and is bijective.
18. Let X be paracompact. Prove that $H^1(X, \mathcal{C}^*) \simeq H^2(X, \mathbb{Z})$ (see section 7.8).

19. Prove that S^2 is the only real n-sphere on which there is a non-trivial complex line bundle.
20. Prove that if $i : Y \to X$ is an embedding of Y as a closed subspace of a topological space X, and if $\mathcal{S}$ is any sheaf on Y, then $H^p(X, i_*\mathcal{S})$ is naturally isomorphic to $H^p(Y, \mathcal{S})$. If X were a locally compact Hausdorff space, this would follow from Proposition 7.4.5, but here we do not assume X and Y are even Hausdorff. Hint: Use the result of Exercise 7.8.

Chapter 8

Coherent Algebraic Sheaves

Abstract holomorphic varieties were defined in the last chapter. In this chapter, we will define abstract algebraic varieties and introduce and study the classes of quasi-coherent and coherent algebraic sheaves on such varieties. In the next chapter, we take up the study of coherent analytic sheaves on holomorphic varieties.

Quasi-coherent algebraic sheaves are the sheaves which arise from the theory of localization of modules. In fact, on an affine variety V, the category of quasi-coherent algebraic sheaves is equivalent to the category of $\mathcal{O}(V)$-modules, where the sheaf corresponding to a module is obtained by localizing the module, and the module corresponding to a sheaf is obtained by taking global sections of the sheaf. Under this equivalence, the coherent sheaves correspond to the finitely generated modules. In general, the category of quasi-coherent algebraic sheaves on an algebraic variety is an abelian full subcategory of the category of sheaves of $\mathcal{O}$-modules. It contains the locally free $\mathcal{O}$-modules and has good homological properties. For example, each quasi-coherent algebraic sheaf on an affine variety is acyclic. In fact, this property characterizes affine varieties among all algebraic varieties.

The sheaf of ideals corresponding to a subvariety is a coherent algebraic sheaf. Once we prove this, we can prove that the singular set of an algebraic variety is a subvariety. The last two sections are devoted to results on preservation of coherence or quasi-coherence under direct or inverse image and to the structure and properties of morphisms between algebraic varieties. These results will play an essential role in Chapter 13 on Serre's theorems and Chapter 15 on algebraic groups.

8.1 Abstract Varieties

As before, if V is an algebraic subvariety of a domain in $\mathbb{C}^n$, then ${}_V\mathcal{O}$ will denote the sheaf of regular functions on V. That is, for each open subset $U \subset V$, we let ${}_V\mathcal{O}(U)$ be the space of regular functions on U. When V is understood, we will usually drop the prefix V and simply write $\mathcal{O}(U)$ for the space of regular functions on U.

Recall the definition of ringed space from section 7.6. A holomorphic variety is a ringed space in which each point has a neighborhood isomorphic to a holomorphic subvariety of a Euclidean open set in $\mathbb{C}^n$, and whose topological space is Hausdorff and second countable. If a holomorphic variety is locally isomorphic, as a ringed space, to an open set in $\mathbb{C}^n$, then it is called a complex manifold.

An algebraic subvariety V of a Zariski open set in $\mathbb{C}^n$ is a ringed space with structure sheaf ${}_V\mathcal{O}$. Thus, as with holomorphic varieties, it should be possible to define an abstract algebraic variety to be a ringed space which is locally isomorphic to a subvariety of a Zariski open set in $\mathbb{C}^n$. However, the definition of abstract algebraic variety is somewhat more complicated, due to the need for a condition which replaces Hausdorff. We begin by defining the notion of affine variety. Recall the discussion of algebraic subvarieties of $\mathbb{C}^n$ in section 3.5.

8.1.1 Definition. *An affine variety is a ringed space which is isomorphic to an algebraic subvariety of* $\mathbb{C}^n$.

In much of the algebraic geometry literature an affine variety is defined to be an *irreducible* subvariety of $\mathbb{C}^n$, but this is too restrictive for our purposes.

The next proposition shows that the topology of an affine variety has a basis of neighborhoods which are also affine varieties.

8.1.2 Proposition. *If V is an affine variety and $f \in \mathcal{O}(V)$, then the open set $V_f = \{z \in V : f(z) \neq 0\}$ is also an affine variety.*

Proof. Suppose V is an algebraic subvariety of $\mathbb{C}^n$, and consider the algebraic subvariety W of $\mathbb{C}^{n+1}$ defined by

$$W = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} : 1 - f(z) \cdot z_{n+1} = 0,\ z \in V\}.$$

Then the map $z \to (z, f(z)^{-1}) : V_f \to W$ is an isomorphism of ringed spaces (Exercise 8.4), with inverse $(z, z_{n+1}) \to z : W \to V_f$. Thus, V_f is affine, since it is isomorphic to W.

An *algebraic prevariety* is a ringed space which has a finite cover by open sets which are affine varieties. Note that, by Proposition 8.1.2, each open subset of an affine variety V is, itself, a finite union of open sets which are affine varieties (sets of the form V_f). Thus, an open subset of a prevariety is a prevariety. A closed subset of an affine variety is also, clearly, an affine variety, and so a closed subset of an algebraic prevariety is also an algebraic prevariety.

We need one more condition to define the class of algebraic varieties. This is to eliminate pathological examples like the following:

8.1.3 Example. Consider the space X which is two copies of the complex plane glued together by the identity map everywhere except at 0. Under the Zariski topology (the topology in which open sets are complements of finite sets), this is a union of two open sets, each of which is a copy of $\mathbb{C}$ with the Zariski topology. Thus, it is an algebraic prevariety. However, it is a strange space. It looks like a copy of the complex plane with the origin split into two points.

We don't want pathological spaces, like the one in this example, to be algebraic varieties. Note that if $f, g : \mathbb{C} \to X$ are the embeddings of the two copies of $\mathbb{C}$ into X, then each is a morphism of ringed spaces but $\{z \in \mathbb{C} : f(z) = g(z)\} = \mathbb{C} - 0$ is not a closed subset of $\mathbb{C}$. Thus, this space X will not be an algebraic variety if we use the following definition:

8.1.4 Definition. *An algebraic variety X is an algebraic prevariety with the property that, for any algebraic prevariety W, and any pair of morphisms $f : W \to X$ and $g : W \to X$, the set $\{w \in W : f(w) = g(w)\}$ is closed in W.*

It turns out that the above condition is equivalent to the condition that the diagonal in $X \times X$ is a closed set (Exercise 8.2). However, in order to make sense of this, we must first define the Cartesian product of two prevarieties. As a pointset, this is the ordinary Cartesian product, but it does not have the Cartesian product topology. In order that the product turn out to be a prevariety, we must proceed as follows: We first define the product of two affine varieties V and W to be the pointset $V \times W$, with the Zariski topology determined by the tensor product $\mathcal{O}(V) \otimes \mathcal{O}(W)$ of the corresponding rings of regular functions. This ring can be regarded as a ring of functions on $V \times W$ via the map $f \otimes g \to \{(z, w) \to f(z)g(w)\}$. The localizations of this ring to Zariski open sets of $V \times W$ define a presheaf whose sheaf of germs is the structure sheaf for $V \times W$ as a ringed space. It is easy to see that the product in this sense of two affine varieties is isomorphic, as a ringed space, to an affine variety. In fact, if V is realized as a subvariety of $\mathbb{C}^n$, and W as a subvariety of $\mathbb{C}^m$, then $V \times W$ is isomorphic to the corresponding subvariety of $\mathbb{C}^{n+m}$. Once the topology and structure sheaf on a product of affine varieties is defined, then one defines the topology on the product of general algebraic prevarieties by declaring a set to be open if its intersection with each Cartesian product of affine subvarieties is open. Similarly, a complex valued function on an open subset U of the product is called regular if its restriction to the intersection of U with each product of affines is regular. The sheaf of regular functions is then the structure sheaf of the product prevariety.

It is easy to see that affine varieties are algebraic varieties. In fact, the following theorem shows this and much more.

8.1.5 Proposition. *If X is an algebraic prevariety with the property that, for any two points x and y in X, there is an open affine subvariety of X containing both x and y, then X is an algebraic variety.*

Proof. Let $f, g : W \to X$ be two morphisms from a prevariety W to X, and let $Z = \{w \in W : f(w) = g(w)\}$. For $w \in W - Z$, set $x = f(w)$, $y = g(w)$, and let U be an open affine subvariety of X containing both x and y. Then $Q = f^{-1}(U) \cap g^{-1}(U)$ is an open set in W containing w. However, f and g both map Q into the affine variety U. We may regard U as a subvariety of $\mathbb{C}^n$ for some n and, using subtraction in $\mathbb{C}^n$, write

$$Z \cap Q = \{w' \in Q : f(w') - g(w') = 0\}.$$

This implies that $Z \cap Q$ is closed in Q and, hence, that its complement in Q is an open subset of W containing w and missing Z. Hence, Z is closed in W.

8.1.6 Example. Let S denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with the topology in which the open sets are the sets with finite complements in S. For such an open set U, we define $\mathcal{O}(U)$ to be the algebra of rational functions with poles in the complement of U. If we give it the structure sheaf $\mathcal{O}$, then S is a ringed space. The complement of each point $\lambda \in S$ is isomorphic, as a ringed space, to $\mathbb{C}$ with the Zariski topology and, hence, is an affine variety. Thus, S is a prevariety. In fact, by the previous proposition, it is an algebraic variety. It is clearly not affine itself, because the only global sections of $\mathcal{O}$ are the constant functions.

Note that this example has some similarities with Example 8.1.3. It may be regarded as two copies of $\mathbb{C}$ glued together along the complement of 0. However, in this case, the gluing is via the map $z \to 1/z$ rather than the identity map.

The familiar concepts we discussed in earlier chapters for algebraic subvarieties of $\mathbb{C}^n$ all make sense for abstract algebraic varieties: A *subvariety* of an abstract algebraic variety X is a subset of X which is closed. An *irreducible variety* is one which is not the union of two proper subvarieties. Every abstract variety X is the union of a finite set of irreducible subvarieties, each of which has non-empty interior. These are called the *irreducible components* of X. The *dimension* of an irreducible algebraic variety X is the dimension of any of its affine open subsets and this is the same as the dimension of the germ of X at any of its points. The dimension of a reducible variety is the maximum of the dimensions of its irreducible components. The *singular* and *regular* points of X are defined just as in Chapter 5, and X is called *smooth* if each of its points is a regular point.

For more on abstract algebraic varieties see [**H**] or [**Mum**].

8.2 Localization

We will denote the structure sheaf of an abstract algebraic variety X by ${}_X\mathcal{O}$ and of an abstract holomorphic variety X by ${}_X\mathcal{H}$. The corresponding algebras of global sections will be denoted $\mathcal{O}(X)$ and $\mathcal{H}(X)$.

In what follows, we will make heavy use of localization in the context of a ring which is not necessarily an integral domain. If A is such a ring, and S is a multiplicative system (a subset of A, closed under multiplication), then the localization A_S of A relative to S is the set of equivalence classes of fractions a/s, $a \in A$, $s \in S$, where fractions a/s and b/t are equivalent if $(at - bs)u = 0$ for some $u \in S$. Addition and multiplication are then defined in the obvious way. Note that an element a/s is 0 in A_S if and only if $au = 0$ for some $u \in S$.

We will need a precise description of the global regular functions on an affine variety. Recall that, by definition, a regular function f on a subvariety V of an open set in $\mathbb{C}^n$ is a function with the property that, for each point $z \in V$, there is a neighborhood U of z in $\mathbb{C}^n$, and a rational function on $\mathbb{C}^n$, with denominator non-vanishing on U, such that f agrees with this rational function on $U \cap V$. However, it happens that regular functions on a subvariety V of $\mathbb{C}^n$ are actually restrictions to V of polynomials on $\mathbb{C}^n$.

Proposition 8.2.1. *If V is a subvariety of $\mathbb{C}^n$, and A is the algebra of restrictions to V of polynomials on $\mathbb{C}^n$, then*

(i) *as a topological space, V is the space of maximal ideals of A, with its Zariski topology;*
(ii) *if $f \in A$ and $V_f = \{\lambda \in V : f(\lambda) \neq 0\}$, then the algebra $\mathcal{O}(V_f)$ of regular functions on V_f is the localization A_f of A relative to the multiplicative set $\{f^n : n \geq 0\}$;*
(iii) *in particular, $\mathcal{O}(V) = A$.*

Proof. Each point λ of V determines a maximal ideal

$$M_\lambda = \{f \in A : f(\lambda) = 0\}.$$

On the other hand, if M is a maximal ideal of A, then M is generated by a finite set $\{f_1, \dots, f_m\} \subset A$. If there is a point $\lambda \in V$ where the functions f_i all vanish, then $M \subset M_\lambda$ and, hence, $M = M_\lambda$, since M is maximal. Thus, to prove that the maximal ideal M has the form M_λ, it suffices to prove that the set $\{f_1, \dots, f_m\}$ has a common zero on V.

Suppose the set of common zeroes of $\{f_1, \dots, f_m\}$ is empty. We choose polynomials $h_1, \dots, h_k \in \mathbb{C}[z_1, \dots, z_n]$ such that $h_j|_V = f_j$ for $j = 1, \dots, m$ and such that $\{h_{m+1}, \dots, h_k\}$ is a set of polynomials for which V is exactly its set of common zeroes. Then the set $\{h_1, \dots, h_k\}$ has no common zeroes on $\mathbb{C}^n$; in other words, the ideal I which it generates has $\operatorname{loc}(I) = \emptyset$. It follows from the Nullstellensatz that $\sqrt{I} = \mathbb{C}[z_1, \dots, z_n]$. But this implies that the identity is in $\sqrt{I}$ and, hence, in I. However, the restriction to V of a function in I belongs to M, and the identity cannot be an element of the maximal ideal M. The resulting contradiction shows that the set $\{f_1, \dots, f_m\}$ does have a common zero. This implies that $\lambda \to M_\lambda$ is a bijection from points of V to maximal ideals of A. The topology of V is the relative topology inherited from $\mathbb{C}^n$. This is the topology in which closed sets are common zero sets of families of polynomials. Thus, the closed subsets of V are common zero sets of families of polynomials restricted to V. This is the Zariski topology induced on V by A. Thus, we have proved part (i).

One direction of (ii) is trivial: Restriction to V_f defines a natural map $A_f \to \mathcal{O}(V_f)$. Suppose $g/f^k \in A_f$ ($g \in A$) determines the zero section of ${}_V\mathcal{O}$ over V_f. This means that $g = 0$ on V_f, which implies that $fg = 0$ in A. This, in turn, implies that $g/f^k = 0$ in A_f. Thus, $A_f \to \mathcal{O}(V_f)$ is injective. The surjectivity will be proved after we prove (iii).

We now prove (iii). Certainly the restriction to V of a polynomial on $\mathbb{C}^n$ is a regular function on V, and so $A \subset \mathcal{O}(V)$. Thus, we must show that the restriction map is surjective. Suppose that $h \in \mathcal{O}(V)$. We can cover V with a collection $\{U_i\}$ of open sets such that $h|_{U_i}$ has the form p_i/q_i, where p_i and q_i are elements of A, and q_i does not vanish on U_i. Since A is Noetherian, we may assume the collection $\{U_i\}$ is finite. Furthermore, we may assume each set U_i is of the form V_g for some $g \in A$, since these sets form a base for the topology. In fact we may actually assume that $U_i = V_{q_i} = \{v \in V : q_i(v) \neq 0\}$, since if this is not true, it can be achieved by replacing p_i and q_i by rp_i and rq_i for an appropriate $r \in A$. Then the condition that the p_i/q_i define a global section of ${}_V\mathcal{O}$ is that $p_i/q_i = p_j/q_j$ on $V_{q_i} \cap V_{q_j}$ for each pair i, j. This means that $p_iq_j - p_jq_i = 0$ on $V_{q_i} \cap V_{q_j}$, which means, as in the previous paragraph, that its product with q_iq_j is the zero element of A. Thus, if we set $p'_i = p_i q_i$ and $q'_i = q_i^2$ for each i, then $p'_i/q'_i = p_i/q_i$ in A_{q_i} for each i, and for each pair i, j, the equation $p'_i q'_j = p'_j q'_i$ holds in A. Since the q'_j have no common zeroes on V, part (i) implies we may choose a set $\{g_i\} \subset A$ such that $\sum g_i q'_i = 1$. Then

$$q'_j \sum g_i p'_i = p'_j \sum g_i q'_i = p'_j,$$

or $q'_j h' = p'_j$, where $h' = \sum g_i p'_i \in A$. Since $h' = p'_j/q'_j = p_j/q_j = h$ on U_j for each j, the image of h' in $\mathcal{O}(V)$ is h. Thus, $A \to \mathcal{O}(V)$ is surjective and (iii) is proved.

We showed above that $A_f \to \mathcal{O}(V_f)$ is injective. To prove the surjectivity in (ii) we simply apply (iii) to the image of V_f under the map $z \to (z, f(z)^{-1}) : V_f \to \mathbb{C}^{n+1}$. As we showed in Proposition 8.1.2, this map is an isomorphism of V_f onto a subvariety W of $\mathbb{C}^{n+1}$. The composition of this map with a polynomial on $\mathbb{C}^{n+1}$ is the restriction to V_f of a function of the form g/f^k, where g is a polynomial on $\mathbb{C}^n$. This shows that $A_f \to \mathcal{O}(V_f)$ is surjective and finishes the proof of (ii).

In section 7.3, we defined the tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ of two sheaves of modules over a sheaf of rings $\mathcal{A}$. One may also define, for a fixed ring A, the tensor product $\mathcal{M} \otimes_A \mathcal{N}$ of two sheaves of A-modules, or the tensor product of an A-module M with a sheaf of A-modules $\mathcal{N}$. These sheaves are defined as the sheaves of sections of the obvious presheaves. That is, $\mathcal{M} \otimes_A \mathcal{N}$ is the sheaf of sections of the presheaf $U \to \mathcal{M}(U) \otimes_A \mathcal{N}(U)$, while $M \otimes_A \mathcal{N}$ is the sheaf of sections of the presheaf $U \to M \otimes_A \mathcal{N}(U)$.

One such construction, localization, lies at the foundation of algebraic geometry. We shall describe it in some detail. Suppose that $\mathcal{A}$ is a sheaf of rings on a space X, $A = \mathcal{A}(X) = \Gamma(X, \mathcal{A})$, and M is an A-module. Then one can *localize* M on X by constructing the sheaf $\mathcal{A} \otimes_A M$. This is the sheaf of germs of the presheaf $U \to \mathcal{A}(U) \otimes_A M$. Clearly, $M \to \mathcal{A} \otimes_A M$ is a functor from the category of A-modules to the category of sheaves of $\mathcal{A}$-modules. While there is no reason to expect this functor to have nice properties, in general, there are many situations where it is very nice indeed. What properties would we like it to have? We would like to be able to recover M from $\mathcal{A} \otimes_A M$ as $\Gamma(X, \mathcal{A} \otimes_A M)$. It would be nice if $M \to \mathcal{A} \otimes_A M$ were an exact functor or, even better, an equivalence of categories with $\Gamma(X, \cdot)$ as inverse. This is exactly what happens for the localization functor of algebraic geometry. The version that we describe below is a special case of a much more general theory.

Suppose V is an affine variety, and M is a module over the algebra $\mathcal{O}(V)$. Then we set

$$\tilde{M} = {}_V\mathcal{O} \otimes_{\mathcal{O}(V)} M.$$

As noted above, this is the sheaf of germs of the presheaf

$$U \to M(U) = {}_V\mathcal{O}(U) \otimes_{\mathcal{O}(V)} M.$$

Now, a basis of neighborhoods for the topology of V is given by the sets of the form $V_f = \{v \in V : f(v) \neq 0\}$ for $f \in \mathcal{O}(V)$. Thus, the sheaf $\tilde{M}$ is determined by the restriction of the presheaf $U \to M(U)$ to sets of the form $U = V_f$.

8.2.2 Proposition. *Let V be an affine variety, M an $\mathcal{O}(V)$-module, and f an element of $\mathcal{O}(V)$. Then there is a natural isomorphism $M(V_f) \to M_f$, where M_f is the localization of the module M relative to the multiplicative set $\{f^k : k \geq 0\}$.*

Proof. The localization of M relative to the multiplicative set $\{f^k : k \geq 0\}$ consists of equivalence classes of expressions of the form m/f^k where $m \in M$, with the obvious module operations. Two such expressions m/f^k and n/f^j are equivalent if $f^p(f^j m - f^k n) = 0$ for some $p \geq 0$. Note that this is the same as the localization of M relative to the multiplicative set consisting of functions in $\mathcal{O}(V)$ which are non-vanishing on V_f. This is due to the fact that if $g \in \mathcal{O}(V)$ is non-vanishing on V_f, then by the Nullstellensatz, f is in the radical of the ideal generated by g in $\mathcal{O}(V)$, from which it follows that $f^k = hg$ for some $k \geq 0$ and some $h \in \mathcal{O}(V)$. This implies that the two multiplicative sets $\{f^k : k \geq 0\} \subset \{g \in \mathcal{O}(V) : g(v) \neq 0, \ \forall v \in V_f\}$ yield the same localization.

As noted in Proposition 8.2.1, the ring $\mathcal{O}(V_f)$ is the localization of $\mathcal{O}(V)$ relative to $\{f^k : k \geq 0\}$. We define the morphism

$$M(V_f) = \mathcal{O}(V_f) \otimes_{\mathcal{O}(V)} M \to M_f$$

by

$$g/f^k \otimes m \to gm/f^k.$$

Its inverse is the morphism determined by

$$m/f^k \to 1/f^k \otimes m : M_f \to M(V_f).$$

It is clear that both morphisms are well defined and they are inverses of one another.

If we apply the above result, along with Proposition 8.2.1, in the case where $M = \mathcal{O}(V)$ itself, we obtain the following:

8.2.3 Corollary. *If V is an affine variety, and $M = \mathcal{O}(V)$, then $\tilde{M} = {}_V\mathcal{O}$.*

8.2.4 Proposition. *Let V be an affine algebraic variety. Then*

(i) *for each open set $U \subset V$, of the form $U = V_f$, the functor $M \to M(U)$ is exact; that is, $\mathcal{O}(U)$ is a flat $\mathcal{O}(V)$-module;*

(ii) *the localization functor $M \to \tilde{M}$ is exact.*

Proof. The definition of $M(U)$ as $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} M$ makes it clear that this functor is right exact. Thus, we need only prove that if $N \subset M$ is a submodule, then $N(U) \to M(U)$ is injective. To prove this, we use Proposition 8.2.2, which tells us that $M(U) = M_f$ and $N(U) = N_f$ if $U = V_f$. Thus, let n/f^k represent an element of N_f, and suppose that it determines the zero element of M_f. This means that $f^p \cdot n = 0$ for some p. But if this equation holds in M, it also holds in N, and hence, n/f^k represents the zero element of N_f as well. This proves (i). However, (ii) is an immediate consequence of (i) and the fact that sets of the form V_f form a basis for the topology of V.

The following proposition shows that localization commutes with restriction from an affine variety to an affine open subset.

8.2.5 Proposition. *Suppose W is an affine open subset of the affine variety V, M is an $\mathcal{O}(V)$-module, and $\tilde{M}$ is its localization on V. Then $\tilde{M}|_W$ is the localization on W of the $\mathcal{O}(W)$-module $M(W)$.*

Proof. For each open subset U of W,

$$\begin{aligned}\mathcal{O}(U) \otimes_{\mathcal{O}(W)} M(W) &= \mathcal{O}(U) \otimes_{\mathcal{O}(W)} \mathcal{O}(W) \otimes_{\mathcal{O}(V)} M \\ &\simeq \mathcal{O}(U) \otimes_{\mathcal{O}(V)} M = M(U),\end{aligned}$$

which implies that $\tilde{M}|_W$ and the localization of $M(W)$ on W are sheaves of sections of isomorphic presheaves on W.

8.2.6 Proposition. *If V is an affine variety, and M is an $\mathcal{O}(V)$-module, then*

(i) *on an open set of the form V_f, the morphism $M_f = M(V_f) \to \tilde{M}(V_f)$ is an isomorphism;*

(ii) *in particular, $M \to \tilde{M}(V)$ is an isomorphism.*

Proof. If V is affine, so is each V_f. Furthermore, it follows from Propositions 8.2.2 and 8.2.5 that $\tilde{M}|_{V_f}$ is isomorphic to the localization $\tilde{M}_f$ of M_f on V_f. Thus, (i) follows from (ii), and so we just need to prove (ii).

This is almost the same as the proof of Proposition 8.2.1(ii) and (iii), but there are a few differences. Suppose $m \in M$ determines the zero section of $\tilde{M}$ over V. This means that we may cover V with finitely many open sets U_i such that the image of m in $M(U_i)$ is 0 for each i. Without loss of generality, we may assume that these sets are of the form $U_i = V_{q_i}$ where $q_i \in \mathcal{O}(V)$. Then for each i, there is an integer n_i such that $q_i^{n_i} m = 0$. Since the sets U_i cover V, the collection $\{q_i\}$ has no common zero on V. Since V is affine, Proposition 8.2.1(i) implies there exists a set $\{g_i\} \subset \mathcal{O}(V)$ such that $\sum g_i q_i^{n_i} = 1$ in $\mathcal{O}(V)$. However, this implies that $m = 0$, since m is killed by $q_i^{n_i}$ for every i. We conclude that $M \to \tilde{M}(V)$ is injective.

Now suppose that $s \in \tilde{M}(V)$. We can cover V with a finite collection $\{U_i = V_{q_i}\}$ of basic open sets such that $s|_{U_i}$ is the image in $\tilde{M}(U_i)$ of an element $m_i/q_i^{n_i} \in M(U_i)$, where $m_i \in M$. In fact, by relabeling each $q_i^{n_i}$ as q_i, we may assume that $s|_{U_i}$ is the image in $\tilde{M}(U_i)$ of an element of the form m_i/q_i. Since these elements fit together to form a section over V, we may assume (after replacing the cover by a refinement, if necessary) that $(m_i/q_i)|_{U_i \cap U_j} - (m_j/q_j)|_{U_i \cap U_j} = 0$. However, $U_i \cap U_j = V_{q_i q_j}$, and so this equality means that there is a positive integer n so that

$$(q_i q_j)^n (q_j m_i - q_i m_j) = 0$$

in M for each pair i, j. If we simply relabel $q_i^n m_i$ by m_i and q_i^{n+1} by q_i, then the fractions m_i/q_i don't change, but the above equality becomes simply

$$q_j m_i - q_i m_j = 0.$$

Since V is affine and the sets U_i cover V, there exist elements $g_i \in \mathcal{O}(V_f)$ such that $\sum g_i q_i = 1$ on V. Then the equation

$$q_j \left(\sum g_i m_i\right) = \left(\sum g_i q_i\right) m_j = m_j$$

holds in M. It says that $m_j = q_j m$ in M, where $m = \sum g_i m_i$. Then m is an element of M with image m_j/q_j in $M(U_j)$ for each j. It follows that s is the image of m under the map $M \to \tilde{M}(V)$. Thus, $M \to \tilde{M}(V)$ is surjective.

8.3 Coherent and Quasi-coherent Algebraic Sheaves

By an *algebraic sheaf* on an algebraic variety X, we will mean a sheaf of ${}_X\mathcal{O}$-modules.

8.3.1 Definition. *An algebraic sheaf $\mathcal{S}$ on X is said to be quasi-coherent if each point of X is contained in an affine neighborhood V such that $\mathcal{S}|_V$ is isomorphic to $\tilde{M}$, for some $\mathcal{O}(V)$-module M. A quasi-coherent sheaf is called coherent if, for each point of X, this can be achieved with a module M which is finitely generated over $\mathcal{O}(V)$.*

Note that the structure sheaf ${}_X\mathcal{O}$ of an algebraic variety X is coherent, since on any affine open set V it is the localization to V of the ring $\mathcal{O}(V)$, by Corollary 8.2.3. It follows that direct sums of copies of the structure sheaf are quasi-coherent and finite direct sums are coherent.

8.3.2 Lemma. *If V is an affine algebraic variety, $f \in \mathcal{O}(V)$, and $\mathcal{S}$ a quasi-coherent algebraic sheaf on V, then*

(i) *if $s \in \mathcal{S}(V)$ and $s|_{V_f} = 0$, then there exists a positive integer n so that $f^n s = 0$;*

(ii) *if $t \in \mathcal{S}(V_f)$, then there exist a positive integer n such that $f^n t$ is the restriction to V_f of a section in $\mathcal{S}(V)$;*

(iii) *restriction defines a natural isomorphism $\mathcal{S}(V)_f \to \mathcal{S}(V_f)$.*

Proof. We may cover V by affine opens sets on which $\mathcal{S}$ is the localization of a module. By Proposition 8.2.5, if $\mathcal{S}$ is the localization of a module on one affine open set, then it is the localization of a module on any of its affine open subsets. It follows that we may choose a finite collection of basic open sets $\{V_{g_i}\}$ such that $\mathcal{S}|_{V_{g_i}} = \tilde{M}_i$ for an $\mathcal{O}(V_{g_i})$-module M_i for each i. If $s \in \mathcal{S}(V)$, then for each i, we have that $s_i = s|_{V_{g_i}} \in \tilde{M}_i(V_{g_i})$ may be regarded as as element of M_i, by Proposition 8.2.6. If $s|_{V_f} = 0$, then $(s_i)|_{V_f \cap V_{g_i}} = 0$ for each i. Since $V_f \cap V_{g_i} = V_{fg_i}$, this implies that the image of s_i in $(M_i)_f$ is 0, by Proposition 8.2.2. It follows that $f^n s_i = 0$ in M_i for some n and each i. We may choose n independent of i, since the open cover $\{V_{g_i}\}$ is finite. Then $f^n s$ is a global section of $\mathcal{S}$ which restricts to 0 on each set in this cover and, hence, is the zero section. This proves part (i).

Now suppose that $t \in \mathcal{S}(V_f)$. Then for each i, $t|_{V_f \cap V_{g_i}}$ may be regarded as an element of $(M_i)_f$, using Propositions 8.2.6 and 8.2.2 again. Thus, for each i, $t|_{V_{fg_i}}$ is a fraction with the numerator the restriction of an element $t_i \in M_i$ and the denominator a power of f. That is, there is an n so that $t_i \in M_i$ and $f^n t \in \mathcal{S}(V)$ agree when restricted to V_{fg_i}. The integer n may be chosen independent of i. Now t_i and t_j are sections of $\mathcal{S}$ which agree on $V_{fg_ig_j}$, since they both agree with the restriction of $f^n t$. By part(i), there is an integer m such that $f^m(t_i - t_j) = 0$ on $V_{g_ig_j}$. Again, we may choose m large enough to work for all i, j. But this means that the sections $f^m t_i$ on the sets V_{g_i} agree on intersections and, thus, define a global section s of $\mathcal{S}$. Clearly, the restriction of s to V_f is $f^{n+m}t$. This completes the proof of (ii).

Part(iii) means exactly that (i) and (ii) hold.

8.3.3 Proposition. *If V is an affine variety, $\mathcal{S}$ is a quasi-coherent sheaf on V, and $M = \mathcal{S}(V)$, then there is a natural isomorphism*

$$\tilde{M} \to \mathcal{S}$$

of sheaves of ${}_V\mathcal{O}$-modules. Furthermore, $\mathcal{S}$ is coherent if and only if M is finitely generated.

Proof. Let Q denote the collection of basic open sets V_f for which $\mathcal{S}|_{V_f}$ is the localization of an $\mathcal{O}(V_f)$-module. By Proposition 8.2.5, if $V_g \subset V_f$ and

V_f belongs to Q, then V_g also belongs to Q. Thus, the sets in Q form a basis for the topology of V.

On a set $V_f \in Q$, necessarily $\mathcal{S}|_{V_f}$ is the localization of $\mathcal{S}(V_f)$, by Proposition 8.2.6. Also, we have an isomorphism $M_f = \mathcal{S}(V)_f \to \mathcal{S}(V_f)$, by Lemma 8.3.2. Thus, $\mathcal{S}|_{V_f}$ is the localization of M_f on V_f.

It also follows from Propositions 8.2.6 and 8.2.5 that the localization $\tilde{M}_f$ of M_f on V_f is $\tilde{M}|_{V_f}$. Thus, on each open set $V_f \in Q$, we have an isomorphism $\tilde{M}|_{V_f} \to \mathcal{S}|_{V_f}$, defined by the composition of the isomorphisms $\tilde{M}|_{V_f} \to \tilde{M}_f$ and $\tilde{M}_f \to \mathcal{S}|_{V_f}$. This clearly commutes with restriction from V_f to V_g in case $V_g \subset V_f$. Since open sets in Q form a basis for the topology of V, this defines an isomorphism of sheaves of ${}_V\mathcal{O}$-modules $\tilde{M} \to \mathcal{S}$.

If M is finitely generated, then $\mathcal{S}$ is coherent, by definition, since it is then the localization of a finitely generated module. On the other hand, if $\mathcal{S}$ is coherent, then it is locally the localization of a finitely generated module. That is, we may cover V with basic open sets V_{f_i} such that, for each i, $\mathcal{S}|_{V_{f_i}}$ is the localization of a finitely generated module, necessarily isomorphic to M_{f_i}. In other words, there is a finite set $\{f_i\} \subset \mathcal{O}(V)$, with no common zeroes on V, and with the property that M_{f_i} is finitely generated for each i. This implies that M is finitely generated (Exercise 8.5). This completes the proof.

8.3.4 Corollary. *On an algebraic variety X, a sheaf $\mathcal{S}$ of ${}_X\mathcal{O}$-modules is quasi-coherent if and only if, for every affine open subset $V \subset X$, $\mathcal{S}|_V$ is the localization of the $\mathcal{O}(V)$-module $\mathcal{S}(V)$.*

Proof. Let V be an affine open subset of X. If $\mathcal{S}$ is quasi-coherent, it follows from Proposition 8.2.5 that the restriction of $\mathcal{S}$ to V is also quasi-coherent. Then $\mathcal{S}|_V$ is the localization of $\mathcal{S}(V)$ by Proposition 8.3.3.

Much of what we have shown in this section and the previous section is summarized in the following theorem.

8.3.5 Theorem. *If V is an affine variety, then the functor $M \to \tilde{M}$ is an equivalence of categories from the category of $\mathcal{O}(V)$-modules to the category of quasi-coherent sheaves of ${}_V\mathcal{O}$-modules on V with $\Gamma(V, \cdot)$ as a quasi-inverse functor. When restricted to the subcategory of finitely generated modules, the functor $M \to \tilde{M}$ is an equivalence from this subcategory to the category of coherent sheaves of $\mathcal{O}(V)$-modules.*

Proof. By Propositions 8.2.6 and 8.3.3, the composition of $M \to \tilde{M}$ and $\mathcal{S} \to \mathcal{S}(V) = \Gamma(V, \mathcal{S})$, in either order, is naturally isomorphic to the identity. Thus, each is an equivalence of categories and they are quasi-inverses of one another. That the equivalence $M \to \tilde{M}$ restricts to an equivalence from the category of finitely generated modules to the category of coherent sheaves follows from the second statement of Proposition 8.3.3.

If $\mathcal{A}$ is a sheaf of rings, and $\mathcal{M}$ a sheaf of $\mathcal{A}$-modules on X, then a set $\{g_1, \ldots, g_k\}$ of sections in $\mathcal{M}(X)$ is said to *generate* $\mathcal{M}$ over $\mathcal{A}$ if the sheaf morphism

$$(f_1, \ldots, f_k) \to \sum f_i g_i : \mathcal{A}^n \to \mathcal{M}$$

is surjective.

8.3.6 Corollary. *If $\mathcal{M}$ is a coherent sheaf of ${}_X\mathcal{O}$-modules on an algebraic variety X, and if U is an affine open subset of X, then $\mathcal{M}|_U$ is generated over ${}_U\mathcal{O}$ by a finite set of sections in $\mathcal{M}(U)$.*

Proof. Given an affine open set $U \subset X$, the sheaf $\mathcal{M}|_U$ is the localization of a finitely generated $\mathcal{O}(U)$-module M. Let $\{m_1, m_2, \ldots, m_k\}$ be a set of generators of M. Then the module homomorphism

$$(f_1, f_2, \ldots, f_k) \to f_1 m_1 + f_2 m_2 + \cdots + f_k m_k : \mathcal{O}(U)^k \to M$$

is surjective. Since localization is an exact functor, by Proposition 8.2.4, the image of this morphism under the localization functor remains surjective. If g_i is the section of $\tilde{M} = \mathcal{M}|_U$ determined by m_i for each i, then it follows that the set $\{g_1, g_2, \ldots, g_k\}$ forms a set of generators for $\mathcal{M}$ on U.

8.4 Theorems of Artin-Rees and Krull

Our next major result is that quasi-coherent sheaves on an affine variety are acyclic. To prove this, we need some additional results from commutative algebra. Specifically, we need Krull's theorem, which is a consequence of the Artin-Rees theorem. For completeness, we will prove both these results in this section before moving on to the vanishing theorem for quasi-coherent sheaves. The student who has been through a text on commutative algebra such as [**AM**] will already be familiar with these results and may wish to skip this section.

A graded ring is a ring $A = \oplus_{n=0}^{\infty} A_n$ which is the direct sum of subspaces A_n in such a way that $A_n \cdot A_m \subset A_{n+m}$ for all n, m. The elements of A_n are said to be *homogeneous of degree n*. A graded Noetherian ring is a graded ring which is also Noetherian as a ring.

8.4.1 Artin-Rees Theorem. *Let A be a Noetherian ring, and K an ideal of A. Let M be a finitely generated A-module, and N a submodule of M.*

Then there exists $m_0 \in \mathbb{Z}^+$ such that

$$N \cap K^{p+m_0}M = K^p(N \cap K^{m_0}M)$$

for all $p \in \mathbb{Z}^+$.

Proof. Put $A^* = \oplus_{n=0}^{\infty} K^n$, where $K^0 = A$. Then A^* has a natural structure of a graded ring. Let $\{a_1, \dots, a_s\}$ be a set of generators for the ideal K. For each n, K^n is generated as an A-module by the monomials of degree n in the a_i. Thus, we have a surjective morphism $A[x_1, \dots, x_s] \to A^*$ determined by $x_{i_1} \dots x_{i_n} \to a_{i_1} \dots a_{i_n} \in K^n$. Since $A[x_1, \dots, x_s]$ is Noetherian, this implies that A^* is Noetherian and, hence, is a graded Noetherian ring.

Let $M^* = \oplus_{n=0}^{\infty} K^n M$. Then M^* is a graded A^*-module. It is clearly generated by $M_0^* = M$ as an A^*-module. Since M is a finitely generated A-module, we conclude that M^* is a finitely generated A^*-module.

Now set $N^* = \oplus_{n=0}^{\infty}(N \cap K^n M) \subset M^*$. Then the containments

$$K^p(N \cap K^n M) \subset K^p N \cap K^{n+p}M \subset N \cap K^{n+p}M$$

imply that N^* is an A^*-submodule of M^*. Since A^* is a Noetherian ring, N^* is finitely generated. Hence, there exists $m_0 \in \mathbb{Z}_+$ such that $\oplus_{n=0}^{m_0}(N \cap K^n M)$ generates N^*. Then for $p \in \mathbb{Z}_+$,

$$N \cap K^{p+m_0}M = \sum_{s=0}^{m_0} K^{p+m_0-s}(N \cap K^s M) \subset K^p(N \cap K^{m_0}M) \subset N \cap K^{p+m_0}M.$$

Therefore, the inclusions are equalities and the proof is complete.

8.4.2 Krull's Theorem. *If A is a Noetherian ring, K an ideal of A, M a finitely generated A-module, and $N \subset M$ a submodule, then for each positive integer n, there is a positive integer m such that $N \cap K^m M \subset K^n N$.*

Proof. For each positive integer n, let $m = n + m_0$, where m_0 is the integer given by the Artin-Rees theorem. Then

$$N \cap K^m M = K^n(N \cap K^{m_0}M) \subset K^n N.$$

8.4.3 Proposition. *If A is a Noetherian ring, I an injective A-module, and K an ideal of A, then the submodule $J \subset I$ defined by*

$$J = \{x \in I : K^n x = 0, \text{ for some } n\}$$

is also injective.

Proof. To prove that J is injective, it suffices to prove that, if $N \subset M$ are A-modules with M finitely generated, then every morphism $N \to J$ extends to a morphism $M \to J$ (by Exercise 8.6). Thus, suppose that $\phi : N \to J$ is such a morphism. Then, since $\phi(N) \subset J$, and N is also finitely generated, we may choose a fixed n such that $\phi(K^n N) = K^n\phi(N) = 0$. By Theorem 8.4.2, there is an integer m such that $K^m M \cap N \subset K^n N$. Thus, ϕ factors through the morphism $N \to N/(K^m M \cap N)$. Since I is injective, the morphism $N/(K^m M \cap N) \to J \subset I$, induced by ϕ, extends to a morphism $\psi : M/K^m M \to I$. However, the image of ψ lies in J, since it is killed by K^m. Then the composition of ψ with $M \to M/K^m M$ is the required extension of ϕ. This completes the proof.

8.5 The Vanishing Theorem for Quasi-coherent Sheaves

The fact that quasi-coherent sheaves on an affine variety are acyclic (Theorem 8.5.2) will follow easily from the next result.

8.5.1 Proposition. *Let V be an affine variety. If I is an injective module over the ring $\mathcal{O}(V)$, then $\tilde{I}$ is a flabby sheaf on V.*

Proof. We first show that for $f \in \mathcal{O}(V)$ the natural map $I \to I_f$ is surjective. To this end, let x/f^n be an element of I_f, with $x \in I$ and n a non-negative integer. We consider the morphism $f^{n+1}g \to fgx : f^{n+1}\mathcal{O}(V) \to I$. This is well defined, since if $f^{n+1}g$ is 0, then so is fg, and hence, fgx. Since I is injective, this morphism extends to a morphism $\phi : \mathcal{O}(V) \to I$ such that $\phi(f^{n+1}g) = fgx$. Then $f^{n+1}y = fx$ if $y = \phi(1)$. However, this implies that x/f^n is the image of y under the localization map $I \to I_f$. Thus, $I \to I_f$ is surjective. This proves that the restriction map $\Gamma(V, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective in the case where $U \subset V$ is an open subset of the form V_f.

To complete the proof, we must show that $\Gamma(V, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective if U is an arbitrary open subset of V. Let Y be the subvariety of V which is the support of $\tilde{I}$ (see Exercise 8.12). If $Y \cap U = \emptyset$, then we are through, since the only section of $\tilde{I}$ over U is then 0. Suppose $Y \cap U \neq \emptyset$. Then there is an open set of the form $V_f \subset U$ such that $Y \cap V_f \neq \emptyset$. If $s \in \Gamma(U, \tilde{I})$, then, by the first paragraph, the restriction of s to V_f is also the restriction to V_f of a global section t. Then $s = t|_U + r$ where $r \in \Gamma(U, \tilde{I})$ has its support in $Z = V - V_f$, which is the zero set of f. That is, $r \in \Gamma_Z(U, \tilde{I})$. Suppose we can show that $\Gamma_Z(V, \tilde{I}) \to \Gamma_Z(U, \tilde{I})$ is surjective. Then $r = r'|_U$ for a section $r' \in \Gamma_Z(V, \tilde{I})$, and $s = (t + r')|_U$ for the global section $t + r' \in \Gamma(V, \tilde{I})$.

Now if $J = \{x \in I : f^n x = 0 \text{ for some } n\}$, then $\tilde{J}$ is the subsheaf of $\tilde{I}$ consisting of sections killed by some power of f, and this is exactly the subsheaf of sections with support in Z. Thus, $\Gamma_Z(V, \tilde{I}) = \Gamma(V, \tilde{J})$ and

$\Gamma_Z(U, \tilde{I}) = \Gamma(U, \tilde{J})$. The support of $\tilde{J}$ is contained in $Y \cap Z = Y - V_f$, which is a proper subvariety of Y. Furthermore, J is also an injective $\mathcal{O}(V)$-module, by Proposition 8.4.3. Thus we have reduced our problem to the same problem for a different injective module – one with smaller support. Since subvarieties of an affine variety satisfy the descending chain condition, we may reduce the problem, by induction, to the case where the support is the empty set. The module is the zero module, in this case, and its localization is the zero sheaf, which is flabby. This completes the proof.

The above result raises an interesting question: Don't we know, on the basis of Theorem 8.3.5, that if I is an injective module, then $\tilde{I}$ is an injective sheaf and, hence, flabby? Why do we need to prove this? The answer is as follows: On the basis of Theorem 8.3.5, we know that if I is an injective module, then $\tilde{I}$ is an injective object in the category of quasi-coherent sheaves of ${}_V\mathcal{O}$-modules. We do not know it is an injective object in the category of all sheaves of ${}_V\mathcal{O}$-modules. We would need to know it was injective in the latter sense in order to be able to conclude from this line of reasoning that $\tilde{I}$ is flabby.

8.5.2 Theorem. *Let V be an affine algebraic variety. Then $H^p(V, \mathcal{S}) = 0$ for $p > 0$ and for all quasi-coherent sheaves $\mathcal{S}$ on V.*

Proof. We set $M = \Gamma(V, \mathcal{S})$, and choose an injective resolution $M \to I$ of M. On localizing this, and using Theorem 8.3.5 and Proposition 8.5.1, we obtain a resolution $\mathcal{S} = \tilde{M} \to \tilde{I}$ of $\mathcal{S}$ by flabby sheaves. Thus, we obtain the cohomology of $\mathcal{S}$ by taking global sections of $\tilde{I}$ and then taking cohomology of the resulting complex. However, by Theorem 8.3.5, we simply get back the resolution $M \to I$ when we apply the global sections functor to $\mathcal{S} \to \tilde{I}$. Thus, $H^p(V, \mathcal{S}) = 0$ for all $p > 0$.

8.6 Cohomological Characterization of Affine Varieties

In this section, we characterize affine varieties as those algebraic varieties on which all coherent algebraic sheaves are acyclic. As a first step, we define ideal sheaves and show that the ideal sheaf of a subvariety is coherent.

8.6.1 Definition. *On an algebraic variety X, a sheaf of submodules (ideals) of the structure sheaf ${}_X\mathcal{O}$ is called an ideal sheaf. If an ideal sheaf is coherent as a sheaf of modules, then it is called a coherent ideal sheaf. Any subvariety $Y \subset X$ determines an ideal sheaf – the sheaf of sections of ${}_X\mathcal{O}$ which vanish on Y. This is called the ideal sheaf of Y and is often denoted $\mathcal{I}_Y$.*

8.6.2 Theorem. *Let X be an algebraic variety, and $Y \subset X$ a subvariety. Then the ideal sheaf $\mathcal{I}_Y$ is a coherent algebraic sheaf.*

Proof. We may cover X with affine open sets V. For each such V, the algebra $\mathcal{O}(V)$ is a quotient of a polynomial algebra and, hence, is Noetherian. Thus, the ideal

$$I = \mathcal{I}_Y(V) = \{g \in \mathcal{O}(V) : g(z) = 0, \ \forall z \in Y \cap V\}$$

is finitely generated. We claim that $\mathcal{I}_Y(V_f) = I_f$, for each basic open set $V_f \subset V$ for $f \in \mathcal{O}(V)$. In fact, an element of $\mathcal{I}_Y(V_f)$ is a function on V_f of the form g/f^n, with $g \in \mathcal{O}(V)$, which vanishes on $Y \cap V_f$. Then $fg \in \mathcal{O}(V)$ vanishes on $Y \cap V$ and, hence, belongs to I. Thus, on V_f we have

$$g/f^n = gf/f^{n+1} \in I_f.$$

This proves that, on V, the ideal sheaf $\mathcal{I}$ is the localization $\tilde{I}$ of the finitely generated ideal I. Since X may be covered by such sets V, we have proved that the ideal sheaf is coherent.

Suppose $\mathcal{I}_Y$ is the ideal sheaf of a subvariety Y of an algebraic variety X. Then, by Proposition 8.2.1, on an affine neighborhood U of X, the quotient $\mathcal{O}(U)/\mathcal{I}_Y(U)$ is isomorphic to ${}_Y\mathcal{O}(U \cap Y)$ via the restriction map $f \to f|_{U \cap Y}$. It follows that the quotient sheaf ${}_X\mathcal{O}/\mathcal{I}_Y$ is isomorphic to $i_{*Y}\mathcal{O}$, where $i : Y \to X$ is the inclusion. Thus, we have a short exact sequence

$$0 \to \mathcal{I}_Y \to {}_X\mathcal{O} \to i_{*Y}\mathcal{O} \to 0.$$

It follows from Theorem 8.6.2 and Exercise 8.10 that $i_{*Y}\mathcal{O}$ is also a coherent sheaf of ${}_X\mathcal{O}$-modules. Thus, we have:

8.6.3 Corollary. *If X is an algebraic variety, Y is an algebraic subvariety of X, and $i : Y \to X$ is the inclusion, then $i_{*Y}\mathcal{O}$ is a coherent sheaf of ${}_X\mathcal{O}$-modules.*

In particular, this implies that the skyscraper sheaf with stalk $\mathbb{C}$ at a point $\lambda \in X$ has the structure of a coherent sheaf of ${}_X\mathcal{O}$-modules.

Obviously, Theorem 8.3.5 implies that, on an affine variety V, a coherent sheaf of ideals $\mathcal{I}$ is the localization of an ideal $\Gamma(V, \mathcal{I})$ of $\mathcal{O}(V)$ and, conversely, an ideal $I \subset \mathcal{O}(V)$ is the ideal of global sections of a coherent sheaf of ideals $\tilde{I}$.

The next few lemmas lead up to our characterization of affine varieties. They show that the vanishing of first cohomology for all coherent sheaves is a very powerful condition.

8.6.4 Lemma. *Let V be an algebraic variety for which $H^1(V, \mathcal{S}) = 0$ for all coherent algebraic sheaves $\mathcal{S}$ on V. Then*

(i) *each maximal ideal of $\mathcal{O}(V)$ is $M_\lambda = \{f \in \mathcal{O}(V) : f(\lambda) = 0\}$ for some $\lambda \in V$;*

(ii) *the functions in $\mathcal{O}(V)$ separate points in V;*

(iii) *each neighborhood of a point $\lambda \in V$ contains an affine neighborhood of λ of the form V_f for some $f \in \mathcal{O}(V)$.*

Proof. For $\lambda \in V$, let U be an affine neighborhood of λ, and set $Y = V - U$. Consider the exact sequence of sheaves

$$0 \to \mathcal{I}_{Y \cup \{\lambda\}} \to \mathcal{I}_Y \to \mathbb{C}_{\{\lambda\}} \to 0,$$

where $\mathcal{I}_{Y \cup \{\lambda\}}$ and $\mathcal{I}_Y$ are the ideal sheaves in ${}_V\mathcal{O}$ of the subvarieties $Y \cup \{\lambda\}$ and Y, respectively, and $\mathbb{C}_{\{\lambda\}}$ is the skyscraper sheaf which is $\mathbb{C}$ at λ and 0 at all other points. Here the first map is the inclusion and the second is the quotient map. Since the ideal sheaf $\mathcal{I}_{Y \cup \{\lambda\}}$ is coherent, $H^1(V, \mathcal{I}_{Y \cup \{\lambda\}}) = 0$, and the long exact sequence of cohomology implies that

$$0 \to \Gamma(V, \mathcal{I}_{Y \cup \{\lambda\}}) \to \Gamma(V, \mathcal{I}_Y) \to \Gamma(V, \mathbb{C}_{\{\lambda\}}) \to 0$$

is also exact. This just means that there is a function $f \in \mathcal{O}(V)$ which vanishes on Y and does not vanish at λ. In other words, every affine neighborhood U of λ contains a neighborhood of λ of the form V_f, where f is a global section in $\mathcal{O}(V)$. This proves (iii). Since, (ii) follows trivially from (iii), it just remains to prove (i).

Suppose M is a maximal ideal of $\mathcal{O}(V)$ which does not have the form M_λ. Then there is no point of V at which all functions in M vanish. This means that the collection of open sets $\{V_f : f \in M\}$ covers V, and this, in turn, implies that some finite subcollection covers V, by Exercise 8.3. In other words, there is a finite set $\{f_i\} \subset M$, with no common zero on V. Given any such set $\{f_i\}$, consider the sheaf morphism

$$\oplus^n {}_V\mathcal{O} \to {}_V\mathcal{O},$$

defined by $(g_1, \dots, g_n) \to \sum_i g_i f_i$. This is an epimorphism, since at each λ in V some f_i is non-vanishing and, hence, invertible in some neighborhood of λ. The kernel of this map is coherent (Exercise 8.10) and, thus, has vanishing first cohomology. It follows from the long exact sequence of cohomology that the map induced on global sections is also surjective, and hence, that we may solve for $g_1, \dots, g_n \in \mathcal{O}(V)$ such that $\sum g_i f_i = 1$. This contradicts the assumption that $\{f_i\}$ is contained in a maximal ideal and establishes (i).

8.6.5 Lemma. *Let V satisfy the hypothesis of Lemma 8.6.4. Then for each affine open set of the form V_f, the restriction map $\mathcal{O}(V)_f \to \mathcal{O}(V_f)$ is an isomorphism.*

Proof. Let $\{V_i\}$ be a cover of V by affine open sets. Suppose the restriction of $g/f^n \in \mathcal{O}(V)_f$ to V_f vanishes. Then $g|_{V_f} = 0$, which implies that g restricts to 0 in each of the open sets $V_f \cap V_i$. Since the V_i are affine, it follows from Proposition 8.2.1 that there is an integer k such that $f^k g$ restricts to 0 in V_i for each i. However, this means that $f^k g = 0$ in $\mathcal{O}(V)$ and, hence, that g/f^n is the zero element of $\mathcal{O}(V)_f$. Thus, $\mathcal{O}(V)_f \to \mathcal{O}(V_f)$ is injective.

Now suppose that $h \in \mathcal{O}(V_f)$ and h_i is its restriction to $V_f \cap V_i$. Then, also by Proposition 8.2.1, there is an integer n so that, for each i, $f^n h_i = g_i$ for some section $g_i \in \mathcal{O}(V_i)$. Then g_i and g_j agree on the intersection $V_i \cap V_j$ for each pair i, j, and so the g_i define a global section g. Clearly $h = g/f^n$ on V_f. Thus, $\mathcal{O}(V)_f \to \mathcal{O}(V_f)$ is surjective and $\mathcal{O}(V_f)$ is the localized algebra $\mathcal{O}(V)_f$.

Lemmas 8.6.4(iii) and 8.6.5 imply that, under the hypothesis of Lemma 8.6.4, the structure sheaf ${}_V\mathcal{O}$ of V is just the sheaf obtained by localizing the algebra $\mathcal{O}(V)$.

8.6.6 Lemma. *If V satisfies the hypothesis of Lemma 8.6.4, then the algebra $\mathcal{O}(V)$ is finitely generated.*

Proof. By Lemma 8.6.4, we may choose a finite affine open cover of V of the form $\{V_{f_i}\}$, with $f_i \in \mathcal{O}(V)$ for each i. We know that each of the algebras $\mathcal{O}(V_{f_i}) = \mathcal{O}(V)_{f_i}$ is finitely generated, since it is a quotient of a polynomial algebra, by Proposition 8.2.1. Thus, we may choose an integer k and elements $h_{ij} \in \mathcal{O}(V)$ such that, for each i, the set $\{h_{ij}/f_i^k\}_j$ generates $\mathcal{O}(V_{f_i})$. We may also choose elements $g_i \in \mathcal{O}(V)$ such that $\sum f_i g_i = 1$. Let A be the subalgebra of $\mathcal{O}(V)$ generated by $\{f_i\} \cup \{g_i\} \cup \{h_{ij}\}$. Then, for each i, we have $A_{f_i} = \mathcal{O}(V_{f_i})$. Thus, if $g \in \mathcal{O}(V)$, then we may choose an integer n and elements $p_i \in A$ such that the equation $f_i^n g = p_i$ holds in $\mathcal{O}(V)$. However, the fact that the equation $\sum f_i g_i = 1$ holds in A implies that the set $\{f_i\}$ is contained in no maximal ideal of A, and this implies that the set $\{f_i^n\}$ is contained in no maximal ideal of A. Hence, we may solve the equation $\sum g_i' f_i^n = 1$ for $g_i' \in A$. Then

$$g = g \sum g_i' f_i^n = \sum g_i' p_i \in A.$$

Thus, $A = \mathcal{O}(V)$, and $\mathcal{O}(V)$ is finitely generated.

8.6.7 Theorem. *Let V be an algebraic variety. Then the following statements are equivalent:*

(i) *V is affine;*

(ii) *$H^p(V, \mathcal{S}) = 0$ for $p > 0$ and for all quasi-coherent sheaves $\mathcal{S}$ on V;*

(iii) *$H^1(V, \mathcal{S}) = 0$ for all coherent $\mathcal{S}$ sheaves on V.*

Proof. That (i) implies (ii) is Theorem 8.5.2. The implication (ii) implies (iii) is trivial. Thus, to complete the proof, we must prove that (iii) implies (i).

Assume (iii) holds. By Lemma 8.6.4(i) and (ii), V is the space of maximal ideals of the algebra $\mathcal{O}(V)$. It has the Zariski topology induced by $\mathcal{O}(V)$, by Lemma 8.6.4(iii). The structure sheaf of V is given by localization of $\mathcal{O}(V)$, by Lemma 8.6.5. By Lemma 8.6.6, $\mathcal{O}(V)$ is finitely generated and, hence, is a quotient of the polynomial algebra $\mathbb{C}[z_1, \dots, z_n]$ by an ideal I for some n. Since $\mathcal{O}(V)$ has vanishing Jacobson radical, $\sqrt{I} = I$, and by Hilbert's Nullstellensatz, $\mathcal{O}(V)$ must be isomorphic to the algebra $\mathcal{O}(W)$, where $W = \operatorname{loc} I$ is a subvariety of $\mathbb{C}^n$. By Proposition 8.2.1, W is the space of maximal ideals of $\mathcal{O}(W)$ with the Zariski topology and with structure sheaf given by localizing $\mathcal{O}(W)$. Necessarily then, $(V, {}_V\mathcal{O})$ and $(W, {}_W\mathcal{O})$ are isomorphic as ringed spaces. Thus, V is an affine variety.

8.7 Morphisms – Direct and Inverse Image

A morphism $\phi : X \to Y$ of algebraic varieties is, of course, a ringed space morphism – a map of sets with the property that $f \circ \phi \in {}_X\mathcal{O}(U)$ whenever U is an open subset of X and $f \in {}_Y\mathcal{O}(V)$ for some neighborhood V of $f(U)$. This means that ϕ induces a morphism $\phi^{-1}{}_Y\mathcal{O} \to {}_X\mathcal{O}$.

Of particular interest to us will be morphisms which are embeddings. A morphism $\phi : X \to Y$, between algebraic varieties, is said to be an *embedding* if it is an isomorphism of X onto a subvariety of an open subset of Y. An embedding with closed image – that is, with image a subvariety of Y itself – is called a *closed embedding.* A morphism ϕ is an embedding if and only if it is a homeomorphism of X onto a subvariety of an open subset of Y, and the induced map $(\phi^{-1}{}_Y\mathcal{O})_x \to {}_X\mathcal{O}_x$ is surjective for each $x \in X$. This last condition guarantees that ϕ induces an isomorphism between the structure sheaf of X and the structure sheaf of its image in Y (Exercise 8.17). Exercise 8.18 gives a useful characterization of embeddings and closed embeddings.

If $\phi : Y \to X$ is a morphism of algebraic varieties, and $\mathcal{M}$ is a sheaf of ${}_X\mathcal{O}$-modules on X, then the inverse image sheaf $\phi^{-1}\mathcal{M}$ is a sheaf of $\phi^{-1}{}_X\mathcal{O}$-modules on Y (see section 7.3). It is not naturally a sheaf of ${}_Y\mathcal{O}$-modules. However, there is a morphism of sheaves of algebras $\phi^{-1}{}_X\mathcal{O} \to {}_Y\mathcal{O}$, since

ϕ is a morphism of ringed spaces. This makes ${}_Y\mathcal{O}$ a sheaf of modules over $\phi^{-1}{}_X\mathcal{O}$ and allows us to form the tensor product sheaf ${}_Y\mathcal{O} \otimes_{\phi^{-1}{}_X\mathcal{O}} \phi^{-1}\mathcal{M}$. Recall that this is the sheaf of sections on Y of the presheaf defined by $U \to {}_Y\mathcal{O}(U) \otimes_{\phi^{-1}{}_X\mathcal{O}(U)} \phi^{-1}\mathcal{M}(U)$. It is clearly a sheaf of ${}_Y\mathcal{O}$-modules.

8.7.1 Definition. *With $\phi : Y \to X$ and $\mathcal{M}$ as above, we define the algebraic inverse image of $\mathcal{M}$ under ϕ to be $\phi^*\mathcal{M} = {}_Y\mathcal{O} \otimes_{\phi^{-1}{}_X\mathcal{O}} \phi^{-1}\mathcal{M}$.*

Algebraic inverse image is a reasonably nice functor. In particular, it preserves quasi-coherence and coherence.

8.7.2 Proposition. *If $\phi : Y \to X$ is a morphism of algebraic varieties, and $\mathcal{M}$ is a quasi-coherent (coherent) sheaf of ${}_X\mathcal{O}$-modules, then $\phi^*\mathcal{M}$ is a quasi-coherent (coherent) sheaf of ${}_Y\mathcal{O}$-modules.*

Proof. For $y \in Y$, let V be an affine open subset of X containing $f(y)$, and let $W \subset \phi^{-1}(V)$ be an affine neighborhood of y. If $\mathcal{M}$ is quasi-coherent, then $\mathcal{M}$ is the localization on V of the $\mathcal{O}(V)$-module $M = \mathcal{M}(V)$. That is, $\mathcal{M}|_V = {}_V\mathcal{O} \otimes_{\mathcal{O}(V)} M$. Consider the sheaf of algebras $\mathcal{A} = (\phi^{-1}{}_V\mathcal{O})|_W$ on W. There is a natural algebra homomorphism $\mathcal{O}(V) \to \mathcal{A}(W)$ which makes $\mathcal{A}$ a sheaf of $\mathcal{O}(V)$-modules on W. It follows easily from the definitions that

$$\phi^{-1}\mathcal{M}|_W = \phi^{-1}({}_V\mathcal{O} \otimes_{\mathcal{O}(V)} M)|_W = (\phi^{-1}{}_V\mathcal{O})|_W \otimes_{\mathcal{O}(V)} M = \mathcal{A} \otimes_{\mathcal{O}(V)} M$$

and, hence, that

$$\phi^*\mathcal{M}|_W = {}_W\mathcal{O} \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}(V)} M) = {}_W\mathcal{O} \otimes_{\mathcal{O}(V)} M = {}_W\mathcal{O} \otimes_{\mathcal{O}(W)} N,$$

where N is the $\mathcal{O}(W)$-module $\mathcal{O}(W) \otimes_{\mathcal{O}(V)} M$. That is, if $\mathcal{M}$ is the localization of M on V, then $\phi^*\mathcal{M}$ is the localization on W of the module $N = \mathcal{O}(W) \otimes_{\mathcal{O}(V)} M$. It follows that $\phi^*\mathcal{M}$ is quasi-coherent if $\mathcal{M}$ is quasi-coherent. Since $\mathcal{O}(W) \otimes_{\mathcal{O}(V)} M$ is a finitely generated $\mathcal{O}(W)$-module if M is a finitely generated $\mathcal{O}(V)$-module, it also follows that $\phi^*\mathcal{M}$ is coherent if $\mathcal{M}$ is coherent.

Note that the functor ϕ^* takes the structure sheaf on X to the structure sheaf on Y. That is,

$$\phi^*{}_X\mathcal{O} = {}_Y\mathcal{O} \otimes_{\phi^{-1}{}_X\mathcal{O}} \phi^{-1}{}_X\mathcal{O} = {}_Y\mathcal{O}.$$

The algebraic inverse image functor is not always exact. The functor ϕ^{-1} is exact, but ϕ^* is ϕ^{-1} followed by the tensor functor ${}_Y\mathcal{O} \otimes \phi^{-1}{}_X\mathcal{O}(\,\cdot\,)$. This is right exact but not always exact. In fact, it is exact precisely when the stalk ${}_Y\mathcal{O}_y$ is a flat module over the stalk ${}_X\mathcal{O}_{\phi(y)}$ of $\phi^{-1}{}_X\mathcal{O}$ at y for each $y \in Y$. A morphism ϕ with this property is called a *flat morphism.* Thus,

8.7.3 Proposition. *Let $\phi : Y \to X$ be a morphism of algebraic varieties. Then ϕ^* is an exact functor if and only if ϕ is a flat morphism.*

Now suppose $\mathcal{N}$ is a sheaf of ${}_Y\mathcal{O}$-modules on Y, and $\phi : Y \to X$ is a morphism of algebraic varieties. Then $\mathcal{N}$ is also a sheaf of $\phi^{-1}{}_X\mathcal{O}$-modules via the map $\phi^{-1}{}_X\mathcal{O} \to {}_Y\mathcal{O}$ induced by ϕ. Hence, $\phi_*\mathcal{N}$ has a natural structure of a sheaf of ${}_X\mathcal{O}$-modules on X (see section 7.3). Thus, ϕ_* may be regarded as a functor from the category of sheaves of ${}_Y\mathcal{O}$-modules on Y to the category of sheaves of ${}_X\mathcal{O}$-modules on X. Does it also preserve coherence and quasi-coherence? It turns out that quasi-coherence is preserved (Proposition 8.7.4 below and Exercise 8.16), but coherence is not preserved in general (Exercise 8.13). There are reasonable hypotheses under which coherence is preserved. We will include a couple of results of this kind, but we refer the reader to [**H**] for a more complete discussion of this issue.

8.7.4 Proposition. *If $\phi : Y \to X$ is a morphism between affine algebraic varieties, then the functor ϕ_* takes quasi-coherent sheaves of ${}_Y\mathcal{O}$-modules to quasi-coherent sheaves of ${}_X\mathcal{O}$-modules.*

Proof. Let $\mathcal{M}$ be a quasi-coherent sheaf on Y. Since Y is affine, $\mathcal{M}$ is the localization on Y of the $\mathcal{O}(Y)$-module $M = \mathcal{M}(Y)$. However the $\mathcal{O}(X)$-module $N = \phi_*\mathcal{M}(X)$ is just M considered as an $\mathcal{O}(X)$-module via the algebra homomorphism $f \to f \circ \phi : \mathcal{O}(X) \to \mathcal{O}(Y)$. We claim that $\phi_*\mathcal{M}$ is just the localization of N on X and, hence, is quasi-coherent. In fact, given an affine neighborhood in X of the form X_f, we have $\tilde{N}(X_f) = N_f = M_{f\circ\phi} = \mathcal{M}(Y_{f\circ\phi}) = \phi_*\mathcal{M}(X_f)$.

The analogous statement for coherent algebraic sheaves is not true (Exercise 8.13). The problem is that the module N, in the above argument, may fail to be a finitely generated $\mathcal{O}(X)$-module even though M is a finitely generated $\mathcal{O}(Y)$-module. We can fix this by assuming that ϕ is a *finite morphism*, as described in the following definition (see Exercise 5.10).

8.7.5 Definition. *Let $\phi : Y \to X$ be a morphism between algebraic varieties. Then ϕ is said to be a finite morphism if X can be covered by affine open sets V with the property that $\phi^{-1}(V)$ is also affine, and the ring $\mathcal{O}(\phi^{-1}(V))$ is finite over $\mathcal{O}(V)$.*

Note that a closed embedding $i : Y \to X$ is a finite morphism. It follows from the going up theorem (Theorem 4.3.4) that a finite morphism has closed image (Exercise 8.19). Thus, an embedding which is not closed is not a finite morphism.

8.7.6 Proposition. *If $\phi : Y \to X$ is a finite morphism of algebraic varieties, then ϕ_* takes coherent algebraic sheaves on Y to coherent algebraic sheaves on X.*

Proof. The statement is local over X, and so we may assume that X and Y are both affine and that $\mathcal{O}(Y)$ is finite over $\mathcal{O}(X)$. Then $\mathcal{O}(Y)$ is a finitely generated $\mathcal{O}(X)$-module, and so any finitely generated $\mathcal{O}(Y)$-module will also be finitely generated as an $\mathcal{O}(X)$-module. Thus, the module N of Proposition 8.7.4 will be finitely generated if M is finitely generated. It follows that $\phi_*\mathcal{M}$ is coherent if $\mathcal{M}$ is coherent.

8.8 An Open Mapping Theorem

There are situations where it is important to know that a morphism is an open map or at least that its image contains an open set. Results of this kind will play a key role in the proofs of Serre's theorems in Chapter 13 and in the study of algebraic groups in Chapter 15. For holomorphic mappings between holomorphic manifolds, the inverse mapping theorem yields a result of this type: If a holomorphic map from a holomorphic manifold of dimension m to a holomorphic manifold of dimension n has a differential of rank n at a point, then it maps a neighborhood of that point onto a neighborhood of its image (see Theorem 3.7.4). Thus, if the differential has rank n at every point, then the map is an open map. Unfortunately, there is no inverse mapping theorem for morphisms of algebraic varieties. Still, we can prove an analogous open mapping theorem for algebraic varieties. Instead of the inverse mapping theorem, we use the Noether normalization theorem and the properties of dimension.

In this section, if $f : X \to Y$ is a morphism of algebraic varieties, and $y \in Y$, then X_y will denote the *fiber* of f over y – that is, the set $f^{-1}(y)$.

8.8.1 Proposition. *Let $f : X \to Y$ be a morphism between irreducible algebraic varieties, and suppose that f has dense image. Then $f(X)$ contains a set U such that:*

(i) *U is open and dense in Y, and*

(ii) *X_y has pure dimension* $\dim X - \dim Y$ *for each $y \in U$.*

Proof. Since Y is irreducible, any non-empty open subset V of Y is dense in Y and is an irreducible variety. Furthermore, $f(X) \cap V = f(f^{-1}(V))$ is dense in V, and $f^{-1}(V)$ is an irreducible variety. Thus, if Y and X are replaced by V and $f^{-1}(V)$, the hypotheses of the proposition still hold. Since there is a non-empty open subset of Y which is affine, we may assume, without loss of generality, that Y is affine.

Also, X is covered by a finite collection $\{X_1, \dots, X_k\}$ of affine open sets, each of which is dense in X. Hence, for each i, the restriction f_i of f to X_i satisfies the hypothesis of the lemma. Suppose the conclusion holds for each of these maps f_i. Let $U_i \subset f_i(X_i)$ be the open set in Y for which (i) and (ii) hold for f_i. Then (i) and (ii) also hold for f and the open set $U = \bigcap_i U_i$. Thus, we have reduced the proof to the case where X and Y are both affine.

We now let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$. The fact that $f(X)$ is dense in Y implies the map $h \to h \circ f : B \to A$ is injective. Thus, A may be considered a finitely generated extension of B. By a generalization of the Noether normalization theorem (Exercise 4.14), there is a non-zero element $b \in B$ and a set of algebraically independent elements $\{a_1, \dots, a_k\}$ in A such that if B_b and A_b are the localizations of B and A relative to the multiplicative system consisting of powers of b, then A_b is a finite extension of $B_b[a_1, \dots, a_k]$. This means that, if U is the open subset of Y on which b is non-vanishing, and $V = f^{-1}(U)$, then the map $f : V \to U$ factors as a finite morphism $V \to \mathbb{C}^k \times U$ followed by the projection $\mathbb{C}^k \times U \to U$. The finite morphism $V \to \mathbb{C}^k \times U$ has dense image and, hence, is surjective (Exercise 8.19). Thus, U is a non-empty open subset of $f(X)$. It is dense in Y, since Y is irreducible. This proves that (i) holds for f and U.

It remains to show that X_y has pure dimension equal to $\dim X - \dim Y$. However, a finite morphism is dimension preserving (Exercise 8.21), and so

$$\dim X = \dim V = k + \dim U = k + \dim Y.$$

Since k is clearly the dimension of the fiber X_y of f over any point $y \in U$, the proof of (ii) is complete.

8.8.2 Corollary. *If $f : X \to Y$ is a morphism between algebraic varieties, then $f(X)$ contains a set which is open and dense in the closure of $f(X)$.*

Proof. We may as well assume that Y is the closure of $f(X)$. Then $f(X)$ is dense in Y and, for each open subset $U \subset Y$, $f(X) \cap U$ is dense in U. If $Y = Y_1 \cup \cdots \cup Y_k$ is the decomposition of Y into irreducible components, then the complement of $Y_2 \cup \cdots \cup Y_k$ is a dense subset of Y_1 and is an open subset of Y. It follows that it meets $f(X)$ in a dense subset of itself and, hence, in a dense subset of Y_1. Thus, $f : f^{-1}(Y_1) \to Y_1$ is a morphism with dense image. The same thing must be true of $f : X_1 \to Y_1$ for some irreducible component X_1 of $f^{-1}(Y_1)$. By part (i) of the previous proposition, $f(X_1)$ contains an open dense subset of Y_1, and so $f(X)$ does as well. The same thing is true of each of the other components of Y, and hence, $f(X)$ contains an open dense subset of Y.

Proposition 5.6.4 gives a lower bound on the dimension of the inverse image of a point under a morphism from a variety X to $\mathbb{C}^n$. That is, the dimension of the fiber of a morphism $f : X \to \mathbb{C}^n$ is at least $\dim X - n$. The key to our version of the open mapping theorem is a pair of similar results concerning the dimension of the inverse image of a subvariety of Y under a morphism of varieties $f : X \to Y$. These give the same bound, but under different sets of hypotheses.

8.8.3 Proposition. *Let $f : X \to Y$ be a morphism between irreducible algebraic varieties. If V is an irreducible subvariety of Y, and W an irreducible component of $f^{-1}(V)$ such that $f(W)$ is dense in V, then*

$$\dim W \geq \dim X - \dim Y + \dim V.$$

In particular, if $y \in f(X)$, then each irreducible component of $X_y = f^{-1}(y)$ has dimension at least $\dim X - \dim Y$.

Proof. Note that it is enough to prove, for each point q of Y, that the proposition is true with Y replaced by some neighborhood U of q and X replaced by $f^{-1}(U)$. Thus, without loss of generality, we may assume that Y is affine. The proof of the proposition in this form will be by induction on $k = \dim Y - \dim V$.

If $k = 0$, then $V = Y$, $W = X$, and there is nothing to prove. Thus, for $k > 0$, we suppose the proposition is true whenever $\dim Y - \dim V < k$, and we consider a situation where $\dim Y - \dim V = k$. Since $k > 0$, V is a proper subvariety of Y, and we may choose a non-zero function $h \in \mathcal{O}(Y)$ which vanishes on V. Then the zero set $V(h)$ of h is a subvariety of Y, containing V, with irreducible components of dimension $\dim Y - 1$. Also, the subvariety $f^{-1}(V(h)) = (h \circ f)^{-1}(0)$ of X has irreducible components, each of dimension at least $\dim X - 1$ ($h \circ f$ could be identically 0 on X).

Let $Y_1, \dots, Y_m$ be the irreducible components of $V(h)$. Since $f(W)$ is dense in V, if Y_i does not contain V, then it does not contain $f(W)$, and so $f^{-1}(Y_i)$ does not contain W. Thus, a component Y_i contains V if and only if $f^{-1}(Y_i)$ contains W. Let X_1 be an irreducible component of $f^{-1}(V(h))$ which contains W. Then X_1 must be contained in $f^{-1}(Y_i)$ for some i and, necessarily, this will be an i for which Y_i contains V. By renumbering if necessary, we may assume $i = 1$ – that is, we may assume that $V \subset Y_1$ and $W \subset X_1 \subset f^{-1}(Y_1)$. If f_1 is the restriction of f to X_1, then $f_1 : X_1 \to Y_1$ is a morphism of irreducible varieties with Y_1 affine, V is an irreducible subvariety of Y_1, and W is an irreducible component of $f^{-1}(V)$ such that $f(W)$ is dense in V. Since $\dim Y_1 - \dim V = k - 1$, the induction assumption tell us that

$$\dim W \geq \dim X_1 - \dim Y_1 + \dim V \geq \dim X - \dim Y + \dim V.$$

This completes the induction step and finishes the proof.

We obtain a stronger result along the same lines if we assume the varieties X and Y are smooth.

8.8.4 Proposition. *Let $f : X \to Y$ be a morphism between smooth irreducible algebraic varieties. If V is an irreducible subvariety of Y and $V \cap f(X) \neq \emptyset$, then the dimension of each irreducible component of $f^{-1}(V)$ is at least* $\dim X - \dim Y + \dim V$.

Proof. Again, it is enough to prove, for each point q of Y, that the proposition is true with Y replaced by some neighborhood U of q and X replaced by $f^{-1}(U)$.

Let G be the graph of f. That is, $G = \{(x, y) \in X \times Y : f(x) = y\}$. This is a subvariety of $X \times Y$ which is isomorphic to X. Let (p, q) be a point of G. If $n = \dim Y$, then the fact that Y is smooth allows us to choose a system of parameters $\{z_1, \dots, z_n\}$ defined over an affine neighborhood of q in Y – that is, a set of functions with the property that, if z is the morphism of this neighborhood to $\mathbb{C}^n$ with these functions as coordinates, then dz is bijective at each point of the neighborhood (Proposition 5.6.9). Without loss of generality, we may assume this affine neighborhood is Y itself. Let H be defined by

$$H = (z \circ f - z)^{-1}(0) = \{(x, y) \in X \times Y : z \circ f(x) - z(y) = 0\}.$$

Then H is the inverse image of the graph of $z \circ f : X \to \mathbb{C}^n$ under the map $\mathrm{id} \times z : X \times Y \to X \times \mathbb{C}^n$. Since z has bijective differential at each point, $\mathrm{id} \times z$ has also, and so H has the same tangential dimension as the graph of $z \circ f$ – that is, it has tangential dimension $\dim X$ at each point. On the other hand, since $X \times Y$ has dimension $\dim X + n$, and H is defined by the vanishing of n regular functions on this variety, Proposition 5.6.4 implies that it has dimension at least $\dim X$. Since $\dim H \leq \operatorname{tdim} H$, we conclude that these two dimensions are both equal to $\dim X$ at every point of H. Hence, H is a smooth algebraic variety of the same dimension as its subvariety G. It follows that G is a connected component of H.

Now $f^{-1}(V)$ is isomorphic to $W = G \cap (X \times V)$ under the projection of $X \times Y$ on X. Hence, W is a union of connected components of the set $H \cap (X \times V)$. This is the inverse image of 0, in the variety $X \times V$, of the map $z \circ f - z : X \times V \to \mathbb{C}^n$. It follows from Proposition 5.6.4 that each irreducible component of W has dimension at least $\dim X + \dim V - n = \dim X + \dim V - \dim Y$. This completes the proof.

8.8.5 Proposition. *Let $f : X \to Y$ be a morphism of irreducible algebraic varieties, and for $y \in f(X)$ set $X_y = f^{-1}(y)$. Suppose*

(i) *for each irreducible subvariety* V *of* Y *containing* y*, the dimension of each irreducible component of the subvariety* $f^{-1}(V) \subset X$ *is at least* $\dim X - \dim Y + \dim V$*; and*
(ii) *each irreducible component of* X_y *has dimension* $\dim X - \dim Y$.

Then y *is an interior point of* $f(X)$.

Proof. Let V be any irreducible subvariety of X containing y, and let A be an irreducible component of $f^{-1}(V)$ which contains an irreducible component of X_y. By (i), we have

$$\dim X - \dim Y \leq \dim A - \dim V.$$

Since $y \in f(A)$, it follows from (ii) and Proposition 8.8.3, applied to the morphism $f|_A : A \to \overline{A}$, that

$$\dim A - \dim V \leq \dim A - \dim \overline{f(A)} \leq \dim X_y = \dim X - \dim Y.$$

We conclude that these inequalities are all equalities, and so

$$\dim X - \dim Y = \dim A - \dim V, \tag{8.8.1}$$

and

$$\dim V = \dim \overline{f(A)}.$$

This last equality implies that $f(A)$ is dense in V, since V is irreducible. This holds, in particular, when $V = Y$ and $A = X$. Hence, f has dense image in Y.

We now prove the proposition using induction on $\dim Y$. If $\dim Y = 0$, there is nothing to prove. Suppose the proposition is true whenever Y has dimension less than n, and consider the case where Y has dimension n. Since f has dense image, Proposition 8.8.1 implies there is a non-empty open set U contained in $f(X)$. If $y \in U$, then there is nothing more to prove. If $y \notin U$, then y belongs to the proper subvariety $V = Y - U$ of U. The subvariety V has dimension less than n and, if A is any irreducible component of $f^{-1}(V)$ containing an irreducible component of X_y, (8.8.1) implies that conditions (i) and (ii) of the proposition hold with f replaced by $f|_A : A \to V$. Thus, the induction hypothesis, applied to $f|_A$, implies that y is an interior point of $V \cap f(X)$ in V. That is, there is a subvariety W of V whose complement in V is contained in $V \cap f(X)$ and contains y. Since the complement of V is contained in $f(X)$, the complement of W in Y is also contained in $f(X)$ and, hence, is an open subset of $f(X)$ containing y. This completes the induction and the proof.

8.8.6 Theorem. *If $f : X \to Y$ is a morphism with dense image, between smooth algebraic varieties, and if X_y has pure dimension which is constant for $y \in f(X)$, then f is an open map.*

Proof. By Proposition 8.8.1, there is an open set contained in $f(X)$ on which $\dim X_y$ has pure dimension $\dim X - \dim Y$. Since $\dim X_y$ is constant, $\dim X_y = \dim X - \dim Y$ for all $y \in f(X)$. By Proposition 8.8.4, f satisfies the hypotheses of Proposition 8.8.5 at each point of $f(X)$. Thus, $f(X)$ is open in Y.

Now let $U \subset X$ be any open set and consider $f|_U : U \to Y$. Each fiber of this morphism is a union of open subsets of the irreducible components of the corresponding fiber of f. Thus, each fiber of $f|_U$ is either empty or has pure dimension $\dim X - \dim Y$. It follows that $f(U)$ is also dense in Y, for if not then its closure is an algebraic subvariety V of Y of lower dimension. Then the map $f|_U : U \to V$ violates Proposition 8.8.4, since U has the same dimension as X, and each non-empty fiber of $f|_U$ has the same dimension as the corresponding fiber of f. Thus, $f|_U$ also satisfies the hypotheses of the theorem, as a morphism from U to Y. By the result of the preceding paragraph, $f(U)$ is open in Y.

The above theorem is true under weaker hypotheses. It is not necessary that X be smooth and the condition that Y be smooth can be replaced by the condition that each point of Y has local ring $\mathcal{O}_y$ which is a normal domain (see [**Ch**]).

A continuous map between topological spaces which is bijective and open is an isomorphism of topological spaces (a homeomorphism). However, for morphisms between algebraic varieties the situation is not so simple. It is quite possible for a morphism to be bijective and open and still not be an isomorphism of algebraic varieties (Exercise 8.22).

Exercises

1. Prove that the product $X \times Y$ of two prevarieties and the projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ have the following universal property: For each prevariety Q and each pair of morphisms $f : Q \to X$ and $g : Q \to Y$, there is a morphism $h : Q \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.
2. Prove that an algebraic prevariety X is an algebraic variety if and only if the diagonal is closed in $X \times X$.

3. Prove that every open cover of an algebraic variety has a finite subcover.
4. Prove that if V is a subvariety of an open set in $\mathbb{C}^n$, W is a subvariety of an open set in $\mathbb{C}^m$, and $f : V \to W$ is algebraic (has coordinate functions which are regular), then f is a morphism of ringed spaces.
5. Prove that if V is an affine variety, $\{f_i\}$ a finite set of elements of $\mathcal{O}(V)$ which have no common zero on V, and M is an $\mathcal{O}(V)$-module such that M_{f_i} is finitely generated for each i, then M is finitely generated.
6. Let A be an algebra, and I an A-module. Use a transcendental induction argument to prove that I is injective if and only if for every singly generated A-module M and every submodule $N \subset M$, each morphism $N \to I$ has an extension to M.
7. If $A = \oplus_n A_n$ is a graded Noetherian ring, prove A_0 is Noetherian.
8. If $A = \oplus_n A_n$ is a graded Noetherian ring, prove A is a finitely generated A_0 algebra.
9. A sheaf supported on a single point is called a skyscraper sheaf. Prove that every skyscraper sheaf is flabby.
10. Prove that the kernel, image, and cokernel of a morphism between quasi-coherent (coherent) sheaves on an algebraic variety is also quasi-coherent (coherent). Hint: Use Theorem 8.3.5.
11. Prove that if V is a subvariety of $\mathbb{C}^n$, and $V = V_1 \cup V_2 \cup \dots V_k$ is its decomposition into irreducible subvarieties, then the singular set of V is the union of the intersection sets $V_i \cap V_j$ and the singular sets of the V_i.
12. Prove that if $\mathcal{S}$ is a coherent algebraic sheaf on an algebraic variety X, and if $Y = \text{Support}(\mathcal{S}) = \{x \in X : \mathcal{S}_x \neq 0\}$, then Y is a subvariety of X.
13. Give an example of a morphism $f : Y \to X$ between two affine algebraic varieties and a coherent algebraic sheaf $\mathcal{M}$ on Y for which $f_*\mathcal{M}$ is not coherent.
14. Prove that an algebraic variety is affine if and only if each of its irreducible components is affine. Hint: Use Theorem 8.6.7 and results of section 8.7.
15. A coherent sheaf of algebras on an algebraic variety X is a sheaf of ${}_X\mathcal{O}$-algebras (with identity) which is coherent as a sheaf of ${}_X\mathcal{O}$-modules. Prove that $\mathcal{B}$ is a coherent sheaf of ${}_X\mathcal{O}$-algebras if and only if there is an algebraic variety Y, and a finite morphism $\phi : Y \to X$ such that $\mathcal{B} \simeq \phi_{*Y}\mathcal{O}$.
16. Prove that if $f : Y \to X$ is morphism between two algebraic varieties, and $\mathcal{M}$ is a quasi-coherent algebraic sheaf on Y, then $f_*\mathcal{M}$ is a quasi-coherent algebraic sheaf on X. Hint: This is a local statement over X and so you may assume that X is affine. Show that if $\{U_i\}$ is a finite cover of Y by affine open sets, then there is an exact sequence

$$0 \to f_*\mathcal{M} \to \oplus_i f_*\mathcal{M}|_{U_i} \to \oplus_{ij} f_*\mathcal{M}|_{U_i \cap U_j}.$$

Then use Proposition 8.7.4.

17. Prove that if $\phi : X \to Y$ is a morphism of algebraic varieties, then ϕ is an embedding if and only if it is a homeomorphism onto a subvariety of an open subset of Y and the induced map $\phi^{-1}{}_Y\mathcal{O} \to {}_X\mathcal{O}$ is an epimorphism of sheaves.
18. Let X and Y be algebraic varieties. Prove that a morphism $\phi : X \to Y$ is an embedding if and only if there is a finite collection $\{U_i\}$ of open subsets of Y such that $\phi(X) \subset \bigcup_i U_i$, each set $V_i = \phi^{-1}(U_i)$ is affine, and the morphism $f \to f \circ \phi : \mathcal{O}(U_i) \to \mathcal{O}(V_i)$ is surjective. Also, prove that the morphism ϕ is a closed embedding if and only if there is an affine open cover $\{U_i\}$ of Y with these properties.
19. Prove that every finite morphism has closed image (Hint: Use Theorem 4.3.4).
20. Let X and Y be algebraic varieties, and let $p : X \times Y \to Y$ be the projection. Prove that if $\mathcal{M}$ is a quasi-coherent algebraic sheaf on Y, and U and V are affine neighborhoods in X and Y, respectively, then there is an $\mathcal{O}(U \times V)$-module isomorphism $\mathcal{O}(U) \otimes \mathcal{M}(V) \to p^*\mathcal{M}(U \times V)$. You will need to recall the definition of the structure sheaf for $X \times Y$, given in section 8.1, and it will be helpful to review the proof of Proposition 8.7.2.
21. Prove that if $f : X \to Y$ is a finite morphism between irreducible algebraic varieties, then X and Y have the same dimension.
22. Consider the morphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f(z, w) = (z^2, w^3)$. Prove that f is bijective and open, but it is not an isomorphism of algebraic varieties.

Coherent Analytic Sheaves

In this chapter we will define the category of coherent analytic sheaves on a holomorphic variety X and show that this category includes the sheaves of greatest interest in holomorphic function theory. We will also show that the category of coherent analytic sheaves on X is a full abelian subcategory of the category of sheaves of ${}_X\mathcal{H}$-modules. Later chapters will deal with vanishing theorems for the cohomology of coherent analytic sheaves and applications to holomorphic function theory.

9.1 Coherence in the Analytic Case

A coherent algebraic sheaf $\mathcal{S}$ on an algebraic variety X is a sheaf of ${}_X\mathcal{O}$-modules such that each point of X has an affine neighborhood V on which $\mathcal{S}$ can be realized as the localization $\tilde{M}$ of a finitely generated $\mathcal{O}(V)$-module M. Since M is finitely generated, there is a surjection $\mathcal{O}(V)^n \to M$, and since $\mathcal{O}(V)$ is Noetherian, the kernel of this map is also finitely generated. Thus, there is an exact sequence

$$\mathcal{O}(V)^m \to \mathcal{O}(V)^n \to M \to 0.$$

If we localize this sequence over V, we obtain an exact sequence of sheaves

$${}_V\mathcal{O}^m \to {}_V\mathcal{O}^n \to \mathcal{S}|_V \to 0.$$

Thus, a coherent algebraic sheaf is locally the cokernel of a morphism ${}_V\mathcal{O}^m \to {}_V\mathcal{O}^n$ between free finite rank sheaves of $\mathcal{O}$-modules. The converse is also true, by Exercise 8.10. Thus, this condition characterizes coherent algebraic sheaves. We will use the analogous condition to define coherent analytic sheaves.

9.1.1 Definition. *Let X be a holomorphic variety. An analytic sheaf on X is a sheaf of ${}_X\mathcal{H}$-modules, and a morphism of analytic sheaves is a morphism of sheaves of ${}_X\mathcal{H}$-modules. An analytic sheaf $\mathcal{S}$ will be called a coherent analytic sheaf if each point of X is contained in a neighborhood V such that $\mathcal{S}|_V$ is the cokernel of a morphism ${}_V\mathcal{H}^m \to {}_V\mathcal{H}^n$ between free finite rank analytic sheaves.*

There are reasons why a definition of coherent sheaves in terms of localization, like that of Chapter 8, does not work well in the analytic case. To begin with, $\mathcal{H}(V)$, for V open in $\mathbb{C}^n$, is never Noetherian (Exercise 9.1). For certain well-behaved compact sets K, the algebra $\mathcal{H}(K) = \Gamma(K, \mathcal{H})$ is Noetherian (see [**Siu**]) and a localization theory for modules over $\mathcal{H}(K)$, for such sets K, is possible (see [**Bj**], section 1.4). However, to show that this localization theory has the right properties (is an exact functor and yields an equivalence between the category of $\mathcal{H}(K)$-modules and the category of coherent sheaves defined in a neighborhood of K) requires the use of powerful results (Cartan's theorems) that can only be proved after extensive development of the properties of coherent sheaves. For this reason, we cannot proceed as in the previous chapter, but must use a different definition and different methods in our discussion of coherence in the analytic case. Once we have proved Cartan's theorems, we will give a brief treatment of localization in the analytic case as one application of these theorems (section 11.9).

If X is a holomorphic variety, then each morphism of analytic sheaves $\phi : {}_X\mathcal{H} \to {}_X\mathcal{H}$ is given by multiplication by a global holomorphic function f. In fact, f is just $\phi(1)$ – the image of the identity section of ${}_X\mathcal{H}$. Similarly, each morphism of analytic sheaves $\phi : {}_X\mathcal{H}^k \to {}_X\mathcal{H}^m$ is given by an $m \times k$ matrix with entries which are holomorphic functions on X.

9.1.2 Definition. *An analytic sheaf $\mathcal{S}$ is said to be locally finitely generated if for each $x \in X$, there is a neighborhood U of x and finitely many sections $s_1, \dots, s_k$ of $\mathcal{S}$ such that the germs of these sections at y generate $\mathcal{S}_y$ for every $y \in U$.*

In other words, $\mathcal{S}$ is locally finitely generated if for each $x \in X$, there is a neighborhood U of x and an epimorphism ${}_U\mathcal{H}^k \to \mathcal{S}|_U$ of analytic sheaves. Thus, $\mathcal{S}$ is coherent if it is locally finitely generated in such a way that the resulting morphisms ${}_U\mathcal{H}^k \to \mathcal{S}|_U$ have kernels which are also locally finitely generated.

9.1.3 Example. Consider the morphism of sheaves of $\mathcal{H}$-modules on $\mathbb{C}$ defined by $f \to gf : \mathcal{H} \to \mathcal{H}$, where $g(z) = z$. The cokernel $\mathcal{C}$ of this morphism is a coherent analytic sheaf on $\mathbb{C}$, by Definition 9.1.1. It is, in fact, just the skyscraper sheaf at 0 with stalk $\mathbb{C}$, considered as an $\mathcal{H}_0$-module via the evaluation map $f \to f(0) : \mathcal{H}_0 \to \mathbb{C}$.

9.1.4 Example. Let $\mathcal{K}$ be the sheaf of ideals in $\mathcal{H}$ on $\mathbb{C}$ consisting of functions which have vanishing germs at 0. This sheaf has stalk $\mathcal{K}_\lambda = \mathcal{H}_\lambda$ if $\lambda \neq 0$ and has stalk 0 at $\lambda = 0$. Thus, the stalks of $\mathcal{K}$ are finitely generated $\mathcal{H}$-modules. However, $\mathcal{K}$ is not finitely generated in any neighborhood of 0, since every section of $\mathcal{K}$ in a connected neighborhood of 0 vanishes identically. Thus, $\mathcal{K}$ is not coherent. The quotient $\mathcal{S} = \mathcal{H}/\mathcal{K}$ is the skyscraper sheaf with stalk $\mathcal{H}_0$ at 0. It also fails to be coherent, even though it is a locally finitely generated sheaf of $\mathcal{H}$-modules. This follows from Theorem 9.5.4.

9.1.5 Example. Let X be a holomorphic variety, and let $\mathcal{M}$ be the sheaf of local sections of a holomorphic vector bundle over X, as discussed in section 7.6. By Proposition 7.6.5, $\mathcal{M}$ is a locally free finite rank sheaf of ${}_X\mathcal{H}$-modules. That is, it locally has the form ${}_X\mathcal{H}^k$. It is then coherent, since it is locally the cokernel of the zero morphism ${}_X\mathcal{H}^m \to {}_X\mathcal{H}^k$, for any m.

It will take some work to show that coherence, as defined above, is a reasonable condition. The key result is Oka's theorem, which serves as a substitute for the Noetherian property.

9.2 Oka's Theorem

Oka's theorem states that if U is an open subset of $\mathbb{C}^n$, then the kernel of any morphism $\alpha : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$ is locally finitely generated. This is the key ingredient in showing that the category of coherent analytic sheaves is a full abelian subcategory of the category of analytic sheaves. The proof will consist of a series of lemmas. The first of these reduces the result to the case where $k = 1$.

9.2.1 Lemma. *Let k be a positive integer. If for every open set $U \subset \mathbb{C}^n$ and every positive integer m, every morphism ${}_U\mathcal{H}^m \to {}_U\mathcal{H}$ has locally finitely generated kernel, then the same thing is true of every open set $U \subset \mathbb{C}^n$ and every morphism ${}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$.*

Proof. We prove this using induction on k. The case $k = 1$ is the hypothesis of the lemma. Thus, we fix $k > 1$ and suppose that, for every U, every morphism ${}_U\mathcal{H}^m \to {}_U\mathcal{H}^j$ with $j < k$ has locally finitely generated kernel. We then fix an open set U and consider a morphism $\alpha : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$.

We may write $\alpha = (\beta, \gamma)$, where $\beta : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^{k-1}$ is α followed by the projection $(f_1, \dots, f_k) \to (f_1, \dots, f_{k-1})$, and γ is α followed by the projection $(f_1, \dots, f_k) \to f_k$. Now, by the induction hypothesis, $\operatorname{Ker}\beta$ is locally finitely generated. Hence, if $x \in U$, then there is a neighborhood V of x, contained in U, a $p > 0$, and a morphism $\phi : {}_V\mathcal{H}^p \to {}_V\mathcal{H}^m$ which maps onto $\operatorname{Ker}\beta$. Thus, we have a diagram

$$\begin{tikzcd} {}_V\mathcal{H}^p \arrow[r, "\phi"] & {}_V\mathcal{H}^m \arrow[r, "\beta"] \arrow[d, "\gamma"'] & {}_V\mathcal{H}^{k-1} \\ & {}_V\mathcal{H} & \end{tikzcd}$$

with the top row exact. We also have, by assumption, that the kernel of $\gamma \circ \phi : {}_V\mathcal{H}^p \to {}_V\mathcal{H}$ is locally finitely generated. This means, after shrinking V if necessary, we can find $q > 0$ and a morphism $\psi : {}_V\mathcal{H}^q \to {}_V\mathcal{H}^p$ which maps onto $\operatorname{Ker}\gamma \circ \phi$. But $\operatorname{Ker}\alpha = \operatorname{Ker}\beta \cap \operatorname{Ker}\gamma = \phi(\operatorname{Ker}\gamma \circ \phi) = \operatorname{Im}\phi \circ \psi$. Thus, the kernel of α is also locally finitely generated, as was to be shown.

The proof of Oka's theorem will be by induction on the dimension n of the underlying complex Euclidean space. The key to the induction step is to use the Weierstrass theorems to reduce the problem in dimension n to an analogous one involving polynomials in z_n, of degree bounded by a fixed degree d, with coefficients which are holomorphic in $z_1, \dots, z_{n-1}$. The essential step in this reduction is stated in the following lemma concerning modules over the local ring ${}_n\mathcal{H}_0$.

9.2.2 Lemma. *Let* $(\mathbf{p}_1, \dots, \mathbf{p}_m)$ *be an* m*-tuple of monic polynomials belonging to* ${}_{n-1}\mathcal{H}_0[\mathbf{z}_n]$*, and let* d *be the maximum of the degrees of the* $\mathbf{p}_j$*. Let* $\rho : {}_n\mathcal{H}_0^m \to {}_n\mathcal{H}_0$ *be the morphism determined by*

$$\rho(\mathbf{f}_1, \dots, \mathbf{f}_m) = \mathbf{p}_1\mathbf{f}_1 + \cdots + \mathbf{p}_m\mathbf{f}_m.$$

If K_d *is the subset of* $\operatorname{Ker}\rho$ *consisting of* m*-tuples whose entries are polynomials of degree at most* d *in* ${}_{n-1}\mathcal{H}_0[\mathbf{z}_n]$*, then* K_d *generates* $\operatorname{Ker}\rho$ *as an* ${}_n\mathcal{H}_0$*-module.*

Proof. Since $\mathbf{p}_1$ is a monic polynomial in $\mathbf{z}_n$, it has finite vanishing order in $\mathbf{z}_n$ (see section 3.3). The Weierstrass preparation theorem implies that we may factor $\mathbf{p}_1$ as $\mathbf{p}_1 = \mathbf{p}_1'\mathbf{p}_1''$, where $\mathbf{p}_1'$ is a Weierstrass polynomial, and $\mathbf{p}_1''$ is a unit. By the Weierstrass division theorem, $\mathbf{p}_1''$ is also a polynomial in $\mathbf{z}_n$. We set

$$d_1' = \deg \mathbf{p}_1', \quad d_1'' = \deg \mathbf{p}_1'',$$

and note that

$$d_1' + d_1'' = \deg \mathbf{p}_1 \le d.$$

If $\mathbf{h} = (\mathbf{h}_1, \dots, \mathbf{h}_m) \in {}_n\mathcal{H}_0^m$, we use the Weierstrass division theorem on each component of $\mathbf{h}$ to write $\mathbf{h} = \mathbf{p}_1'\mathbf{h}'' + \mathbf{r}'$, where $\mathbf{r}', \mathbf{h}'' \in {}_n\mathcal{H}_0^m$, and $\mathbf{r}'$ has components which are polynomials in $\mathbf{z}_n$ of degree less than d_1'. If we set $\mathbf{h}' = (\mathbf{p}_1'')^{-1}\mathbf{h}''$ then

$$\mathbf{h} = \mathbf{p}_1\mathbf{h}' + \mathbf{r}'.$$

Also,

$$\mathbf{p}_1\mathbf{h}' = \mathbf{q} + \sum_{j=2}^{m} \mathbf{h}_j'\mathbf{e}_j,$$

where

$$\mathbf{q} = \left(\sum \mathbf{p}_i\mathbf{h}_i', 0, \dots, 0\right) \quad \text{and} \quad \mathbf{e}_j = (-\mathbf{p}_j, 0, \dots, \mathbf{p}_1, \dots, 0),$$

with $\mathbf{p}_1$ occurring in the jth position of $\mathbf{e}_j$. Note that each $\mathbf{e}_j$ belongs to K_d. Thus, if we set $\mathbf{r} = \mathbf{r}' + \mathbf{q}$, then

$$\mathbf{h} = \mathbf{r} + \sum_{j=2}^{m} \mathbf{h}_j'\mathbf{e}_j,$$

where $\mathbf{e}_j \in K_d$, and $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_m)$ is an element of ${}_n\mathcal{H}_0^m$, with $\mathbf{r}_2, \dots, \mathbf{r}_m$ polynomials in $\mathbf{z}_n$ of degree less than d_1'. Now suppose $\mathbf{h}$ belongs to $\operatorname{Ker}\rho$. Since each $\mathbf{e}_j$ belongs to $\operatorname{Ker}\rho$, so does $\mathbf{r}$. This means that

$$\mathbf{p}_1\mathbf{r}_1 = -(\mathbf{p}_2\mathbf{r}_2 + \cdots + \mathbf{p}_m\mathbf{r}_m),$$

which implies that $\mathbf{p}_1'\mathbf{p}_1''\mathbf{r}_1 = \mathbf{p}_1\mathbf{r}_1$ is a polynomial in $\mathbf{z}_n$ of degree less than $d + d_1'$, since this is true of each term on the right above. However, since $\mathbf{p}_1'$ is a Weierstrass polynomial, the Weierstrass division theorem implies that $\mathbf{p}_1''\mathbf{r}_1$ is also a polynomial. The degree of $\mathbf{p}_1''\mathbf{r}_1$ is then at most d. Thus, the entries of $\mathbf{p}_1''\mathbf{r}$ are all polynomials in z_n of degree at most d, and so $\mathbf{p}_1''\mathbf{r} \in K_d$. Since $\mathbf{p}_1''$ is a unit we conclude that $\mathbf{r}$ and, thus, $\mathbf{h}$ belong to the submodule of ${}_n\mathcal{H}_0$ generated by K_d. Therefore, K_d generates the kernel of ρ as an ${}_n\mathcal{H}_0$-module.

9.2.3 Oka's Theorem. *Let $U \subset \mathbb{C}^n$ be an open set. Then each ${}_U\mathcal{H}$-module morphism ${}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$ has locally finitely generated kernel.*

Proof. We prove this by induction on n. It is trivial for $n = 0$, since $\mathbb{C}^0$ is a point, and ${}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$, in this case, is a linear map between finite dimensional vector spaces over $\mathbb{C}$. Thus, we assume that $n > 0$ and that the theorem is true in dimension less than n.

Lemma 9.2.1 reduces the proof to showing, in dimension n, that the kernel of a morphism of the form $\alpha : {}_U\mathcal{H}^m \to {}_U\mathcal{H}$ is locally finitely generated, under the assumption that the theorem holds for all m and k in dimensions less than n. The strategy of the proof is to use the Weierstrass theorems to reduce the problem to an analogous one involving polynomials of a fixed degree in z_n and then to apply the induction assumption to the coefficients of these polynomials.

Given a point $\lambda \in U$, we must show that $\operatorname{Ker} \alpha$ is finitely generated in some neighborhood of λ. Without loss of generality, we may assume that the point λ is the origin. The map α has the form $\alpha(g_1, \dots, g_m) = \sum f_i g_i$, for an m-tuple of functions $\{f_i\} \subset {}_n\mathcal{H}(U)$. By appropriate choice of coordinates, we may assume that the germ at 0 of each f_i has finite vanishing order at 0 and, hence, by the Weierstrass preparation theorem, has the form $\mathbf{u}_i \mathbf{p}_i$, where $\mathbf{p}_i$ is a Weierstrass polynomial, and $\mathbf{u}_i$ is a unit. We may replace these germs by their representatives p_i and u_i in some neighborhood V of 0 and, by choosing V small enough, we may assume the u_i are non-vanishing in V. Then the map $(g_1, \dots, g_m) \to (u_1 g_1, \dots, u_m g_m)$ is an automorphism of ${}_V\mathcal{H}^m$ which maps the kernel of α to the kernel of the morphism determined by the m-tuple $(p_1, \dots, p_m)$. Thus, without loss of generality, we may replace the f_i's with the p_i's and assume that α has the form $\alpha(g_1, \dots, g_m) = \sum p_i g_i$. Let d be the maximum of the degrees of the p_i's. We may assume that V has the form $V' \times V''$ for open sets $V' \subset \mathbb{C}^{n-1}$ and $V'' \subset \mathbb{C}$.

Let $\mathcal{K}_d$ denote the sheaf on V' defined as follows: For each open subset $W \subset V'$,

$$\mathcal{K}_d(W) \subset \operatorname{Ker}\{\alpha : {}_V\mathcal{H}^m(W \times V'') \to {}_V\mathcal{H}(W \times V'')\}$$

is the subspace consisting of m-tuples whose entries are polynomials of degree less than or equal to d in ${}_{n-1}\mathcal{H}(W)[z_n]$. This is a sheaf of ${}_{V'}\mathcal{H}$-modules. We shall show that it is locally finitely generated on V'. For each neighborhood $W \subset V'$, the space of m-tuples $(q_1, \dots, q_m)$, where each q_i is a polynomial of degree at most d in ${}_{n-1}\mathcal{H}(W)[z_n]$, is a free module of rank $m(d+1)$ over ${}_{n-1}\mathcal{H}(W)$. The map α determines an ${}_{n-1}\mathcal{H}(W)$-module morphism of this free module into the free ${}_{n-1}\mathcal{H}(W)$-module of rank $2d + 1$ consisting of polynomials of degree at most $2d$ in ${}_{n-1}\mathcal{H}(W)[z_n]$. Furthermore, $\mathcal{K}_d(W)$ is the kernel of this morphism. In other words, we may regard α as determining a morphism of sheaves of ${}_{V'}\mathcal{H}$-modules, ${}_{V'}\mathcal{H}^{m(d+1)} \to {}_{V'}\mathcal{H}^{2d+1}$, and our sheaf $\mathcal{K}_d$ is its kernel. By the induction hypothesis, $\mathcal{K}_d$ is locally finitely generated as a sheaf of ${}_{V'}\mathcal{H}$-modules.

To complete the proof, we will show that $\mathcal{K}_d$ generates $\operatorname{Ker} \alpha$ as a sheaf of ${}_V\mathcal{H}$-modules. This is a statement regarding the equality of a sheaf and one of its subsheaves. Thus, it holds provided it holds stalkwise, that is, provided, at each point $\lambda \in V$, the stalk $(\mathcal{K}_d)_\lambda$ generates the stalk $(\operatorname{Ker} \alpha)_\lambda$ over ${}_n\mathcal{H}_\lambda$. By translating, we may assume that λ is the origin. The polynomials p_i

may no longer have germs at the origin which are Weierstrass polynomials, but they will still be monic polynomials in $\mathbf{z}_n$. Then Lemma 9.2.2 gives the desired result. This completes the induction and the proof.

9.2.4 Corollary. *If U is an open subset of $\mathbb{C}^n$, and $\mathcal{M}$ is a locally finitely generated sheaf of submodules of ${}_U\mathcal{H}^k$, then $\mathcal{M}$ is coherent. In particular, if $\phi : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$ is a morphism of analytic sheaves, then* $\operatorname{Ker}\phi$ *and* $\operatorname{Im}\phi$ *are coherent analytic sheaves on U.*

Proof. Since $\mathcal{M}$ is locally finitely generated, each point of U has a neighborhood V on which $\mathcal{M}$ is the image of a morphism $\theta : {}_V\mathcal{H}^m \to {}_V\mathcal{H}^k$ of sheaves of ${}_V\mathcal{H}$-modules. By Oka's theorem, we know that $\operatorname{Ker}\theta$ is locally finitely generated. Thus, by shrinking V if necessary, we may assume $\operatorname{Ker}\theta$ is the image of a morphism $\psi : {}_V\mathcal{H}^p \to {}_V\mathcal{H}^m$. Then on V, $\mathcal{M}$ is the cokernel of ψ. Thus, $\mathcal{M}$ is coherent.

Since the image of a morphism $\phi : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$ of analytic sheaves is finitely generated, by definition, and the kernel is locally finitely generated, by Oka's theorem, they are both coherent by the result of the previous paragraph.

9.2.5 Corollary. *If U is an open subset of $\mathbb{C}^n$, and $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves of submodules of ${}_U\mathcal{H}^m$, then so is $\mathcal{M} \cap \mathcal{N}$.*

Proof. Every point of U has a neighborhood V on which there are morphisms $\phi : {}_V\mathcal{H}^p \to {}_V\mathcal{H}^m$, with image $\mathcal{M}|_V$, and $\psi : {}_V\mathcal{H}^q \to {}_V\mathcal{H}^m$, with image $\mathcal{N}|_V$. Consider the map $\theta : {}_V\mathcal{H}^{p+q} \to {}_V\mathcal{H}^m$ defined by writing ${}_V\mathcal{H}^{p+q}$ as ${}_V\mathcal{H}^p \oplus {}_V\mathcal{H}^q$ and setting $\theta(f \oplus g) = \phi(f) - \psi(g)$. The kernel of θ is coherent, by Oka's theorem, and hence, is locally finitely generated. Furthermore, on V, $\mathcal{M} \cap \mathcal{N}$ is the image of the kernel of θ under ϕ. Thus, after shrinking V if necessary, we may choose a finite set of generators for $\operatorname{Ker}\theta$. Then the image of this set under ϕ will generate $\mathcal{M} \cap \mathcal{N}$ on V. Thus, $\mathcal{M} \cap \mathcal{N}$ is locally finitely generated. In view of the previous corollary, this proves that it is coherent.

9.3 Ideal Sheaves

If X is a holomorphic variety, then an *ideal sheaf* of ${}_X\mathcal{H}$ is a subsheaf $\mathcal{I}$ of ${}_X\mathcal{H}$ such that $\mathcal{I}(U)$ is an ideal of $\mathcal{H}(U)$ for each open set $U \subset X$. If $Y \subset X$ is a subvariety, then the ideal sheaf of Y, denoted $\mathcal{I}_Y$, is the sheaf defined by

$$\mathcal{I}_Y(U) = \{f \in \mathcal{H}(U) : f(y) = 0,\ \forall\, y \in U \cap Y\}.$$

A major objective of this section and the next section is to prove that the ideal sheaf of a subvariety is coherent.

If $U \subset \mathbb{C}^n$ is an open set, and $\mathcal{I}$ and $\mathcal{J}$ are ideal sheaves of ${}_U\mathcal{H}$, then $\mathcal{I} : \mathcal{J}$ will denote the presheaf which assigns to the open set $V \subset U$ the ideal

$$\mathcal{I}(V) : \mathcal{J}(V) = \{f \in \mathcal{H}(V) : f\mathcal{J}(V) \subset \mathcal{I}(V)\}.$$

One easily checks that this is actually a sheaf (Exercise 9.2).

In the case where $\mathcal{J}$ is singly generated, by say $f \in \mathcal{H}(U)$, then we will write $\mathcal{I} : f$ for $\mathcal{I} : \mathcal{J}$. That is $\mathcal{I} : f$ is the sheaf which assigns to $V \subset U$ the ideal

$$\mathcal{I}(V) : f = \{g \in \mathcal{H}(V) : fg \in \mathcal{I}(V)\}.$$

Note that $f \in \mathcal{I}(V)$ if and only if $\mathcal{I} : f = \mathcal{H}$.

9.3.1 Proposition. *If U is an open subset of $\mathbb{C}^n$, and $\mathcal{I}$ and $\mathcal{J}$ are coherent ideal sheaves in ${}_U\mathcal{H}$, then so is $\mathcal{I} : \mathcal{J}$.*

Proof. First, suppose that $\mathcal{J}$ is generated by a single element $h \in {}_n\mathcal{H}(U)$ and $\mathcal{I}$ is generated by the elements $g_1, \dots, g_k \in {}_n\mathcal{H}(U)$. Consider the map $\phi : {}_U\mathcal{H}^{k+1} \to {}_U\mathcal{H}$ defined by

$$\phi(f_0, \dots, f_k) = hf_0 - g_1 f_1 - \cdots - g_k f_k.$$

Then $\mathcal{I} : \mathcal{J} = \mathcal{I} : h$ is the image of $\operatorname{Ker}\phi$ under the projection of ${}_U\mathcal{H}^{k+1}$ onto its first coordinate. The kernel of ϕ is locally finitely generated, by Oka's theorem, and thus, so is $\mathcal{I} : \mathcal{J}$. It is then coherent, by Corollary 9.2.4.

In the general case, for any point of U we may choose a neighborhood V in which $\mathcal{J}$ is finitely generated, say by $h_1, \dots, h_m$. On V, we have $\mathcal{I} : \mathcal{J} = \bigcap_i (\mathcal{I} : h_i)$. Thus, it follows from Corollary 9.2.5 that $\mathcal{I} : \mathcal{J}$ is coherent.

The strategy of the proof that an ideal sheaf $\mathcal{I}_Y$ is coherent is to use the local description of an irreducible variety as a finite branched holomorphic cover, as in section 4.5. This produces an explicit set of functions which generate $\mathcal{I}_Y$ at a point and in a neighborhood of the point, except possibly on an exceptional subvariety Z. The next lemma is used to conclude that, in this situation, the set actually generates $\mathcal{I}_Y$ in a full neighborhood of the point.

9.3.2 Lemma. *Let U be an open set in $\mathbb{C}^n$ containing the origin. Let $f_1, \dots, f_p$ be a set of holomorphic functions on U, with Y as its set of common zeroes and $\mathcal{I}$ as the ideal sheaf it generates. Suppose that $\mathcal{I}_0$ is a prime ideal of ${}_n\mathcal{H}_0$, and $\mathcal{I}_\lambda = \operatorname{id}\mathbf{Y}_\lambda$ at all points $\lambda \in Y - Z$, where Z is a holomorphic subvariety of Y with $\mathbf{Z}_0 \neq \mathbf{Y}_0$. Then there is a 0-neighborhood $W \subset U$ such that $\mathcal{I}_\lambda = \operatorname{id}\mathbf{Y}_\lambda$ at all points $\lambda \in W$.*

Proof. Since the functions $f_1, \dots, f_p$ determine Y, it follows from the Nullstellensatz that $\mathcal{I}_0 \subset \operatorname{id} \mathbf{Y}_0 \subset \sqrt{\mathcal{I}_0}$. However, by assumption, $\mathcal{I}_0$ is prime, and so $\mathcal{I}_0 = \operatorname{id} Y_0 = \sqrt{\mathcal{I}_0}$.

Let d be a function, holomorphic in a 0-neighborhood $W \subset U$, with germ $\mathbf{d}_0$ which belongs to $\operatorname{id} \mathbf{Z}_0$, but not to $\operatorname{id} \mathbf{Y}_0$. We may assume (by shrinking W if necessary) that d vanishes on $Z \cap W$, but does not vanish identically on $Y \cap W$. By Proposition 9.3.1, the sheaf $\mathcal{I} : d$ on W is locally finitely generated, and hence, by shrinking W, we may assume it is finitely generated. Let $g_1, \dots, g_q \in {}_n\mathcal{H}(W)$ be a set of generators for this sheaf. Then for each j, $dg_j \in \mathcal{I}(W)$ and, in particular, its germ at 0 belongs to $\mathcal{I}_0$. However, $\mathcal{I}_0$ is a prime ideal and it does not contain the germ of d. We conclude that the germ of g_j at 0 belongs to $\mathcal{I}_0$ for each j. That is, the germ at 0 of each g_j belongs to the ideal at 0 generated by the f_i. If this is true at 0, it is true in a neighborhood of 0, and hence, we may choose W small enough that it is true at every point of W. This implies that

$$\mathcal{I} : d = \mathcal{I} \quad \text{on } W. \tag{9.3.1}$$

For a point $\lambda \in Y \cap W$ and a germ $\mathbf{f}_\lambda \in \operatorname{id} Y_\lambda$, we choose a representative f of $\mathbf{f}_\lambda$ in a neighborhood V_λ of λ. We note that, if the neighborhood is sufficiently small, then $\mathcal{I} : f$ will be finitely generated on V_λ, by Proposition 9.3.1. The hypothesis that $\mathcal{I} = \operatorname{id} Y$ at each point of $Y - Z$ implies that $\mathcal{I} : f = {}_n\mathcal{H}$ at each point of $Y - Z$. Thus, if $h_1, \dots, h_m \in {}_n\mathcal{H}(V_\lambda)$ are generators of $\mathcal{I} : f$ on V_λ, then the set of common zeroes of the h_i must lie in $Z \cap V_\lambda$. The function d vanishes on Z, and hence, by the Nullstellensatz, $d^r_\lambda \in (\mathcal{I} : f)_\lambda$ for some r. This means that $(d^r f)_\lambda \in \mathcal{I}_\lambda$. It then follows from (9.3.1) that $f \in \mathcal{I}_\lambda$. Therefore, $\mathcal{I}_\lambda = \operatorname{id} \mathbf{Y}_\lambda$ at all points of $W \cap Y$.

Finally, we can prove:

9.3.3 Theorem. *If Y is a holomorphic subvariety of an open set $U \subset \mathbb{C}^n$, then its ideal sheaf $\mathcal{I}_Y$ is coherent.*

Proof. This is a local result, and so we only need to prove it in some neighborhood of each point of Y. The proof uses the machinery of section 4.5. We fix a point of Y, which we may assume is 0. We assume at first that the germ $\mathbf{Y}_0$ is irreducible. We choose coordinates in $\mathbb{C}^n$ and an integer m, as in Theorem 4.5.7(iii). Then $\pi : \mathbb{C}^n \to \mathbb{C}^m$ expresses $\mathbf{Y}_0$ as a germ of a finite branched holomorphic cover of $\mathbf{C^m}$. By Theorem 4.5.7, we may choose a polydisc $\Delta \subset \mathbb{C}^n$, centered at 0 and a representative (which we may assume is Y) in $\pi^{-1}(\Delta)$ of $\mathbf{Y}_0$ such that $\pi : Y \to \Delta$ is a finite branched holomorphic cover. Furthermore, this cover has a dense regular subcover

$\pi : Y_0 \to \Delta_0$, with Y_0 the subset of $\pi^{-1}(\Delta_0)$ defined by the equations in (4.5.1).

We may assume the neighborhood U of 0 in $\mathbb{C}^n$ is chosen so that there are representatives $f_1, \dots, f_p \in \mathcal{H}(U)$ for a set of generators of $\operatorname{id}\mathbf{Y}_0$. We assume this set of generators includes the polynomials $\mathbf{p}_{m+1} \in {}_m\mathcal{H}_0[z_{m+1}]$ and $\mathbf{q}_j \in {}_m\mathcal{H}_0[z_{m+1}, z_j]$, for $m+2 \le j \le n$, that appear in the proof of Theorem 4.5.4.

Let E be the complement of Δ_0 in Δ, and set $Z = \pi^{-1}(E) \cap U$. Then Z is a subvariety of U with germ at 0 properly contained in $\mathbf{Y}_0$. Consider the holomorphic map $\phi : U \to \mathbb{C}^n$ with coordinate functions $z_1, \dots, z_m, p_{m+1}, q_{m+2}, \dots, q_n$. The Jacobian of ϕ is non-singular on $U - Z$, due to the form of the polynomials q_j in (4.5.1) and the fact that, on Δ_0, the polynomial p_{m+1} has distinct roots and, therefore, non-vanishing derivative with respect to z_{m+1}. By the inverse function theorem, each $\lambda \in Y - Z$ has a neighborhood $U_\lambda \subset U$, on which these functions form a coordinate system – a coordinate system in which $Y \cap U_\lambda$ is expressed as the set where the last $n - m$ coordinates vanish. It follows that the functions $f_1, \dots, f_p$ generate $\operatorname{id}\mathbf{Y}_\lambda$ at each $\lambda \in Y - Z$. By the previous lemma, these functions generate $(\mathcal{I}_Y)_\lambda = \operatorname{id}\mathbf{Y}_\lambda$ at all points λ of some 0-neighborhood contained in U. Thus, $\mathcal{I}_Y$ is finitely generated in a neighborhood of 0.

If the germ $\mathbf{Y}_0$ is not irreducible, then we write $Y = Y_1 \cup \cdots \cup Y_q$ in some neighborhood W of 0, where the varieties Y_j have irreducible germs at 0. Then $\mathcal{I}_{Y_j}$ is finitely generated in some neighborhood of 0 for each i, by the previous paragraph. We may choose W to be a neighborhood in which this is true for all j. Then each ideal sheaf $\mathcal{I}_{Y_j}$ is coherent on W, and hence, so is the intersection $\mathcal{I}_{Y_j} \cap \cdots \cap \mathcal{I}_{Y_q}$, by Corollary 9.2.5. But this intersection is just $\mathcal{I}_Y$. This completes the proof.

In section 5.6, we proved that the singular set of an algebraic variety is an algebraic subvariety. Now that we know the ideal sheaf of a holomorphic subvariety of $\mathbb{C}^n$ is coherent, we can prove the analogous result in the holomorphic case.

9.3.4 Theorem. *If Y is a holomorphic variety, then the set of singular points of Y is a holomorphic subvariety of Y.*

Proof. This is a local result and so it suffices to prove the following: If Y is a subvariety of an open set U in $\mathbb{C}^n$ and $0 \in Y$, then there is a neighborhood V of 0 such that the singular points of $V \cap Y$ form a subvariety of $V \cap Y$.

The ideal sheaf $\mathcal{I}_Y \subset {}_U\mathcal{H}$ of the subvariety Y is coherent by, Theorem 9.3.3. Thus, $\mathcal{I}_Y$ is locally finitely generated, and we may choose V such that $\mathcal{I}_Y$ is generated on V by finitely many sections $g_1, g_2, \dots, g_k \in \mathcal{I}_Y(V)$.

It then follows from Corollary 5.4.4 that the singular locus of $Y \cap V$ is a subvariety of $Y \cap V$ (see Exercise 5.8).

9.4 Coherent Sheaves on Varieties

We are now in a position to prove the strong form of Oka's theorem.

9.4.1 Oka's Theorem on Varieties. *Let $\phi : {}_X\mathcal{H}^m \to {}_X\mathcal{H}^k$ be a morphism of analytic sheaves on a holomorphic variety X. Then* $\operatorname{Ker}\phi$ *is coherent.*

Proof. This is a local result, and so we may assume that X is a subvariety of an open set U contained in $\mathbb{C}^n$. The morphism ϕ is determined by a $k \times m$ matrix with entries which are holomorphic functions on X, and hence, extend locally to be holomorphic in neighborhoods in U. Again, since we are proving a local result, we may as well assume that the entries of this matrix extend to be holomorphic in U. Then ϕ extends to a morphism $\tilde{\phi} : {}_U\mathcal{H}^m \to {}_U\mathcal{H}^k$. Furthermore, by Theorem 9.3.3, we know that the ideal sheaf $\mathcal{I}_X$ is locally finitely generated, and so we may as well assume that it is finitely generated on U. Then we may represent it as the image of a ${}_U\mathcal{H}$-module morphism $\psi : {}_U\mathcal{H}^p \to {}_U\mathcal{H}$. Let $\psi^k : {}_U\mathcal{H}^{kp} \to {}_U\mathcal{H}^k$ denote the morphism that is just the direct sum of k copies of ψ. Then consider the morphism

$$\theta : {}_U\mathcal{H}^m \oplus {}_U\mathcal{H}^{kp} \to {}_U\mathcal{H}^k, \text{ where } \theta(f,g) = \tilde{\phi}(f) - \psi^k(g).$$

Note that $f \in {}_U\mathcal{H}^m(V)$ is the first element of a pair $(f,g) \in \operatorname{Ker}\theta$, over a neighborhood V, if and only if $\tilde{\phi}(f) \in \mathcal{I}_X^k(V)$ – that is, if and only if the restriction $\phi(f)$ of $\tilde{\phi}(f)$ to $X \cap V$ is 0. Thus, the kernel of ϕ on V can be characterized as those functions which are restrictions to $X \cap V$ of first elements of pairs (f,g) in the kernel of θ on V. It is then clear that $\operatorname{Ker}\phi$ is locally finitely generated if $\operatorname{Ker}\theta$ is locally finitely generated. But $\operatorname{Ker}\theta$ is locally finitely generated by Oka's theorem. Thus, $\operatorname{Ker}\phi$ is locally finitely generated. However, knowing this for all such morphisms ϕ allows us to prove, in exactly the same manner as in Corollary 9.2.4, that a locally finitely generated subsheaf of a free analytic sheaf ${}_X\mathcal{H}^m$ is coherent. It follows that $\operatorname{Ker}\phi$ is coherent.

As noted in the preceding proof, once we know that the kernel of any morphism ${}_X\mathcal{H}^m \to {}_X\mathcal{H}^k$ is locally finitely generated, then we may prove, as in Corollary 9.2.4, that any locally finitely generated subsheaf of a free finite rank analytic sheaf ${}_X\mathcal{H}^m$ is coherent. It follows that the analogs, for holomorphic varieties, of Corollaries 9.2.4, 9.2.5, and Proposition 9.3.1 are all true and have exactly the same proofs. We state this fact as a corollary of the above theorem.

9.4.2 Corollary. *If X is a holomorphic variety, then*

(i) *each locally finitely generated analytic subsheaf of ${}_X\mathcal{H}^m$ is coherent;*

(ii) *the image of each morphism of analytic sheaves $\phi : {}_X\mathcal{H}^m \to {}_X\mathcal{H}^k$ is coherent;*

(iii) *the intersection of two coherent subsheaves of ${}_X\mathcal{H}^m$ is coherent;*

(iv) *if $\mathcal{I}$ and $\mathcal{J}$ are two coherent ideal sheaves in ${}_X\mathcal{H}$, then $\mathcal{I} : \mathcal{J}$ is also coherent.*

Using Corollary 9.4.2 and Theorem 9.3.3, it is a simple matter to prove the following:

9.4.3 Cartan's Theorem. *If X is any holomorphic variety and $Y \subset X$ is a holomorphic subvariety, then the ideal sheaf $\mathcal{I}_Y \subset {}_X\mathcal{H}$ is coherent.*

Proof. Again, this is a local result and so we may assume that X is a subvariety of an open set U in $\mathbb{C}^n$. Then Y is also a subvariety of U and, as such, its ideal sheaf is a coherent sheaf of ${}_U\mathcal{H}$-modules, by Theorem 9.3.3. In particular, it is locally finitely generated as a ${}_U\mathcal{H}$-module and so we may as well assume that it is finitely generated on U. But then the restriction to X of a set of generators of this ideal sheaf will be a set of generators over ${}_X\mathcal{H}$ of its ideal sheaf $\mathcal{I}_Y$. Thus, $\mathcal{I}_Y$ is locally finitely generated and, hence, coherent by Corollary 9.4.2.

9.5 Morphisms between Coherent Sheaves

The main result remaining to be proved in this chapter is that the kernel, image, and cokernel of a morphism between coherent analytic sheaves are also coherent. The next three results lead up to this theorem.

9.5.1 Proposition. *If X is a holomorphic variety, and $\phi : {}_X\mathcal{H}^m \to \mathcal{S}$ is an epimorphism of analytic sheaves, then for each morphism of analytic sheaves $\psi : {}_X\mathcal{H}^k \to \mathcal{S}$ and each point $x \in X$, there is a neighborhood U of x in which ψ lifts to a morphism of analytic sheaves $\rho : {}_U\mathcal{H}^k \to {}_U\mathcal{H}^m$ such that $\phi \circ \rho = \psi$ on U.*

Proof. The stalk ${}_U\mathcal{H}^k_x$ is a free, hence projective, ${}_U\mathcal{H}_x$-module, and so the morphism $\psi : {}_U\mathcal{H}^k_x \to \mathcal{S}_x$ lifts to a morphism $\rho : {}_U\mathcal{H}^k_x \to {}_U\mathcal{H}^m_x$ such that $\phi \circ \rho = \psi$. The morphism ρ is represented by a matrix with entries from ${}_U\mathcal{H}_x$, and we may assume that U is chosen small enough that each of these entries is represented by a holomorphic function on U. The resulting matrix defines a morphism of analytic sheaves $\rho : {}_U\mathcal{H}^k \to {}_U\mathcal{H}^m$. The identity $\phi \circ \rho - \psi = 0$ holds at x. That is, $(\phi \circ \rho f - \psi f)_x = 0$ for each $f \in {}_U\mathcal{H}^k$. Since ${}_U\mathcal{H}^k$ is finitely generated, the identity $\phi \circ \rho - \psi = 0$ holds in a neighborhood of x. This completes the proof.

9.5.2 Proposition. *If X is a holomorphic variety, then each locally finitely generated analytic subsheaf $\mathcal{M}$ of a coherent analytic sheaf $\mathcal{S}$ is also coherent.*

Proof. Since $\mathcal{S}$ is coherent, each point x of X has a neighborhood U for which there is an epimorphism of analytic sheaves $\phi : {}_U\mathcal{H}^m \to \mathcal{S}|_U$, with a locally finitely generated kernel. Also, since $\mathcal{M}$ is locally finitely generated, we may assume U is chosen small enough that $\mathcal{M}|_U$ is finitely generated. Thus, there is a morphism of analytic sheaves $\psi : {}_U\mathcal{H}^k \to \mathcal{S}|_U$ which has $\mathcal{M}|_U$ as image. In order to prove that $\mathcal{M}$ is coherent, we need to prove that $\operatorname{Ker}\psi$ is locally finitely generated or, since our chosen point x is quite general, that $\operatorname{Ker}\psi$ is finitely generated if U is chosen small enough.

By Proposition 9.5.1, if U is chosen small enough, we may lift ψ to a morphism of analytic sheaves $\rho : {}_U\mathcal{H}^k \to {}_U\mathcal{H}^m$ such that $\phi \circ \rho = \psi$ on U. Now the image of ρ is a finitely generated subsheaf of ${}_U\mathcal{H}^m$, as is the kernel of ϕ. Thus, both sheaves are coherent, as is their intersection, by Corollary 9.4.2. Since, $\operatorname{Ker}\phi \cap \operatorname{Im}\rho$ is coherent, it is, after shrinking U if necessary, the image of a morphism of analytic sheaves $\sigma : {}_U\mathcal{H}^p \to {}_U\mathcal{H}^m$. Using Proposition 9.5.1 again, we conclude that if U is chosen small enough, there is a morphism of analytic sheaves $\lambda : {}_U\mathcal{H}^p \to {}_U\mathcal{H}^k$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
{}_U\mathcal{H}^m & \xrightarrow{\text{id}} & {}_U\mathcal{H}^m & \xrightarrow{\phi} & \mathcal{S} \\
{\scriptstyle\sigma}\big\uparrow & & {\scriptstyle\rho}\big\uparrow & & {\scriptstyle\psi}\big\uparrow \\
{}_U\mathcal{H}^p & \xrightarrow{\lambda} & {}_U\mathcal{H}^k & \xrightarrow{\text{id}} & {}_U\mathcal{H}^k.
\end{array}$$

Now given $y \in U$, a germ $\mathbf{f} \in {}_U\mathcal{H}^k_y$ is in the kernel of ψ if and only if $\rho(\mathbf{f}) \in \operatorname{Ker}\phi \cap \operatorname{Im}\rho = \operatorname{Im}\sigma$, that is, if and only if there exists $\mathbf{g} \in {}_U\mathcal{H}^p_y$ such that $\rho(\mathbf{f}) = \sigma(\mathbf{g}) = \rho \circ \lambda(\mathbf{g})$. The latter is equivalent to $\mathbf{f} - \lambda(\mathbf{g}) \in \operatorname{Ker}\rho$. Since $\operatorname{Ker}\rho$ is coherent, we may, after shrinking U again if necessary, assume that $\operatorname{Ker}\rho$ is the image of a morphism $\tau : {}_U\mathcal{H}^q \to {}_U\mathcal{H}^k$. It follows that $\operatorname{Ker}\psi$ is equal to $\operatorname{Im}\lambda + \operatorname{Im}\tau$. This implies that $\operatorname{Ker}\psi$ is locally finitely generated, and hence, that $\mathcal{M}$ is coherent.

The following has a proof much like that of Corollary 9.2.5 (Exercise 9.3):

9.5.3 Corollary. *If $\mathcal{M}$ and $\mathcal{N}$ are coherent subsheaves of a coherent analytic sheaf on a holomorphic variety X, then $\mathcal{M} \cap \mathcal{N}$ is also coherent.*

9.5.4 Theorem. *If $\mathcal{M}$ and $\mathcal{N}$ are coherent analytic sheaves over a holomorphic variety X, then the image, kernel, and cokernel of any morphism of analytic sheaves $\phi : \mathcal{M} \to \mathcal{N}$ are also coherent.*

Proof. Since $\mathcal{M}$ is locally finitely generated, so is $\operatorname{Im}\phi$. Hence, $\operatorname{Im}\phi$ is coherent, by Proposition 9.5.2.

Fix a point $x \in X$. Since $\mathcal{N}$ is coherent, there is a neighborhood U of x and an epimorphism of analytic sheaves $\psi : {}_U\mathcal{H}^n \to \mathcal{N}|_U$ with finitely generated kernel. Since $\operatorname{Im}\phi$ is locally finitely generated, we may choose U so that there is also an epimorphism $\rho : {}_U\mathcal{H}^p \to \operatorname{Im}\phi|_U$. In fact, by choosing U small enough and using Proposition 9.5.1 to construct λ, we may construct the following commutative diagram of morphisms of analytic sheaves:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\phi|_U \arrow[r] & \mathcal{N}|_U \arrow[r] & \operatorname{Coker}\phi|_U \arrow[r] & 0 \\
& {}_U\mathcal{H}^p \arrow[u, "\rho"] \arrow[r, "\lambda"] & {}_U\mathcal{H}^n \arrow[u, "\psi"] \arrow[r, "\mathrm{id}"] & {}_U\mathcal{H}^n \arrow[u, "\sigma"] &
\end{tikzcd}$$

The top row of this diagram is exact, the vertical maps are all epimorphisms, and $\operatorname{Ker}\psi$ is finitely generated. It is evident from the diagram that the kernel of σ is $\psi^{-1}(\operatorname{Im}\phi) = \operatorname{Im}\lambda + \operatorname{Ker}\psi$. This is finitely generated, and so $\operatorname{Coker}\phi$ is coherent.

It remains to prove that $\operatorname{Ker}\phi$ is coherent. To this end, we fix $x \in X$ and choose a neighborhood U of x small enough that we may find epimorphisms of analytic sheaves $\sigma : {}_U\mathcal{H}^m \to \mathcal{M}$ and $\rho : {}_U\mathcal{H}^n \to \mathcal{N}$ such that $\operatorname{Ker}\rho$ and $\operatorname{Ker}\sigma$ are finitely generated and, thus, coherent. We then construct the following commutative diagram, after shrinking U appropriately:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\phi|_U \arrow[r] & \mathcal{M}|_U \arrow[r, "\phi"] & \mathcal{N}|_U \\
& {}_U\mathcal{H}^p \arrow[r, "\psi"] & {}_U\mathcal{H}^m \arrow[u, "\sigma"] \arrow[r, "\theta"] & {}_U\mathcal{H}^n \arrow[u, "\rho"] \\
& & {}_U\mathcal{H}^q \arrow[u, "\eta"] \arrow[r, "\mathrm{id}"] & {}_U\mathcal{H}^q . \arrow[u, "\lambda"]
\end{tikzcd}$$

Here we use Proposition 9.5.1 to construct the map θ. Its image is finitely generated and, hence, coherent. Then $\operatorname{Ker}\rho \cap \operatorname{Im}\theta$ is coherent, and hence, is the image of a morphism λ as above. Then η is obtained as another application of Proposition 9.5.1. The map ψ is a morphism of analytic sheaves which maps onto the kernel of θ, and it exists, for small enough U, by Corollary 9.4.2(ii). A diagram chase shows $\sigma \circ (\psi + \eta) : {}_U\mathcal{H}^p \oplus {}_U\mathcal{H}^q \to \mathcal{M}|_U$ has $\operatorname{Ker}\phi|_U$ as its image. This completes the proof that $\operatorname{Ker}\phi$ is locally finitely generated, and hence, coherent.

The above theorem implies that the category of coherent analytic sheaves is an abelian subcategory of the category of analytic sheaves.

Finally, we have the following result:

9.5.5 Theorem. *If*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N} \longrightarrow 0$$

is an exact sequence of analytic sheaves, and if any two of the three are coherent, then the third is also coherent.

Proof. Two of the three cases have already been proved in the preceding theorem. It remains to prove that if $\mathcal{K}$ and $\mathcal{N}$ are coherent, then $\mathcal{M}$ is coherent. To this end, let x be a point of X and U a neighborhood of x for which there are epimorphisms of analytic sheaves $\rho : {}_U\mathcal{H}^k \to \mathcal{K}|_U$ and $\tau : {}_U\mathcal{H}^n \to \mathcal{N}|_U$, with finitely generated kernels. We may then use Proposition 9.5.1 to construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}|_U \arrow[r, "\alpha"] & \mathcal{M}|_U \arrow[r, "\beta"] & \mathcal{N}|_U \arrow[r] & 0 \\
0 \arrow[r] & {}_U\mathcal{H}^k \arrow[r] \arrow[u, "\rho"] & {}_U\mathcal{H}^{k+n} \arrow[r] \arrow[u, "\sigma"] & {}_U\mathcal{H}^n \arrow[r] \arrow[u, "\tau"] & 0.
\end{tikzcd}$$

Here the morphisms on the bottom row are just the canonical inclusion and projection associated with writing ${}_U\mathcal{H}^{k+n}$ as ${}_U\mathcal{H}^k \oplus {}_U\mathcal{H}^n$. The morphism σ is constructed, for sufficiently small U, by using Proposition 9.5.1 to lift τ to a morphism $\tau' : {}_U\mathcal{H}^n \to \mathcal{M}|_U$ with $\beta \circ \tau' = \tau$, then writing ${}_U\mathcal{H}^{k+n}$ as ${}_U\mathcal{H}^k \oplus {}_U\mathcal{H}^n$, and defining σ by

$$\sigma(f \oplus g) = \alpha \circ \rho(f) + \tau'(g).$$

Now the fact that ρ and τ are epimorphisms implies that σ is an epimorphism. Also, the kernels of the vertical maps form a short exact sequence

$$0 \longrightarrow \operatorname{Ker} \rho|_U \longrightarrow \operatorname{Ker} \sigma|_U \longrightarrow \operatorname{Ker} \tau|_U \longrightarrow 0$$

in which the first and third terms are locally finitely generated and, hence, coherent. Thus, we may repeat the above argument for this sequence and conclude that, for sufficiently small U, $\operatorname{Ker} \sigma|_U$ is also finitely generated. It follows that $\mathcal{M}$ is coherent.

9.6 Direct and Inverse Image

If $\phi : Y \to X$ is a holomorphic map between holomorphic varieties, we may define an analytic inverse image functor ϕ^* using the same construction that was used in section 8.7. That is, if $\mathcal{M}$ is a sheaf of ${}_X\mathcal{H}$-modules on X, then $\phi^{-1}\mathcal{M}$ is a sheaf of $\phi^{-1}{}_X\mathcal{H}$-modules on Y, as is ${}_Y\mathcal{H}$ under the morphism $\phi^{-1}{}_X\mathcal{H} \to {}_Y\mathcal{H}$, which is part of the structure of ϕ as a morphism of ringed spaces. We set

$$\phi^*\mathcal{M} = {}_Y\mathcal{H} \otimes_{\phi^{-1}{}_X\mathcal{H}} \phi^{-1}\mathcal{M}.$$

Clearly, $\phi^*\mathcal{M}$ is a sheaf of ${}_Y\mathcal{H}$-modules and ϕ^* is a functor from the category of sheaves of ${}_X\mathcal{H}$-modules to the category of sheaves of ${}_Y\mathcal{H}$-modules. There is the following analogue of Proposition 8.7.2. Its proof is somewhat different from that of Proposition 8.7.2, since here coherence is not defined in terms of localization. We leave the proof as an exercise (Exercise 9.12).

9.6.1 Proposition. *If $\phi : Y \to X$ is a holomorphic map between holomorphic varieties, and $\mathcal{M}$ is a coherent sheaf of ${}_X\mathcal{H}$-modules, then $\phi^*\mathcal{M}$ is a coherent sheaf of ${}_Y\mathcal{H}$-modules.*

As with the algebraic inverse image functor, the analytic inverse image functor is not exact in general. The stalk of $\phi^*\mathcal{M}$ at $y \in Y$ is

$$(\phi^*\mathcal{M})_y = {}_Y\mathcal{H}_y \otimes_{{}_X\mathcal{H}_{\phi(y)}} \mathcal{M}_{\phi(y)}$$

and so ϕ^* is right exact. It is exact if and only if ${}_Y\mathcal{H}_y$ is a flat ${}_X\mathcal{H}_{\phi(y)}$-module for each $y \in Y$. A holomorphic map with this property is called a *flat holomorphic map.* A discussion of flat holomorphic maps may be found in [**GPR**], Chapter II, Section 2.

The direct image functor ϕ_*, for a holomorphic map $\phi : Y \to X$, takes sheaves of ${}_Y\mathcal{H}$-modules to sheaves of ${}_X\mathcal{H}$-modules (Exercise 9.9). It sometimes preserves coherence and sometimes does not (Exercises 9.8 through 9.11). As our final result in this chapter, we will prove the finite mapping theorem, which asserts that finite holomorphic maps preserve coherence. A *finite holomorphic map* between holomorphic varieties is a proper holomorphic map for which the inverse image of each point in the image is finite. Note that the germ of such a map at each point of its domain is a finite morphism of germs of varieties, in the sense of section 4.6. The first step in the proof of the finite mapping theorem is to prove the following lemma, which is a generalization of the Weierstrass factorization theorem.

Note that a polynomial $\mathbf{p} \in {}_{n-1}\mathcal{H}_0[z_n]$ is a polynomial in z_n whose coefficients are germs at 0 of holomorphic functions of $z' \in U$, for some neighborhood U of 0 in $\mathbb{C}^{n-1}$. If p is the polynomial in $\mathcal{H}(U)[z_n]$ with these holomorphic functions as coefficients, then we say $\mathbf{p}$ is the germ at $0 \in \mathbb{C}^{n-1}$ of p. However, p itself is a function of $z = (z', z_n) \in U \times \mathbb{C} \subset \mathbb{C}^n$ and, hence, we may evaluate it at $z' = 0$ to obtain a polynomial $p(0, z_n)$ with complex coefficients. Also, we may consider its germ at any point of the form $w = (0, \mu)$ with $\mu \in \mathbb{C}$. Since these depend only on $\mathbf{p}$ and not on the representative p, we will denote them by $\mathbf{p}(0, z_n)$ and $\mathbf{p}_w$, respectively.

Recall that a polynomial $\mathbf{p} \in {}_{n-1}\mathcal{H}_0[z_n]$ is a Weierstrass polynomial in the variable $z_n - \lambda$, for some $\lambda \in \mathbb{C}$, provided it is the germ at $z' = 0 \in \mathbb{C}^{n-1}$ of a polynomial of the the form

$$p(z', z_n) = (z_n - \lambda)^m + a_{m-1}(z')(z_n - \lambda)^{m-1} + \cdots + a_0(z'),$$

where each $a_j(z')$ is a holomorphic function in a neighborhood of 0 in $\mathbb{C}^{n-1}$, which vanishes at 0. In other words, $\mathbf{p} \in {}_{n-1}\mathcal{H}_0[z_n]$ is a Weierstrass polynomial in $z_n - \lambda$ if $\mathbf{p}(0, z_n) = (z_n - \lambda)^m$ for some m – that is, if $\mathbf{p}(0, z_n)$ is a monic polynomial with λ as its only root.

9.6.2 Lemma. *Let $\mathbf{p}$ be a monic polynomial in ${}_{n-1}\mathcal{H}_0[z_n]$. Let $\lambda_1, \dots, \lambda_k$ be the distinct roots of $\mathbf{p}(0, z_n)$, and set $w_j = (0, \lambda_j)$ for $j = 1, \dots, k$. Then*

(i) *$\mathbf{p}$ factors uniquely as $\mathbf{p} = \mathbf{q}_1 \dots \mathbf{q}_k$, where $\mathbf{q}_j$ is a Weierstrass polynomial in $z_n - \lambda_j$ for each j;*

(ii) *if $\mathbf{f}_1, \dots, \mathbf{f}_k$ are germs of holomorphic functions at $w_1, \dots, w_k$, respectively, then there exist unique germs $\mathbf{g}_1, \dots, \mathbf{g}_k$ at these points and a unique polynomial $\mathbf{r} \in {}_{n-1}\mathcal{H}_0[z_n]$, of degree less than $\deg \mathbf{p}$, such that $\mathbf{f}_j = \mathbf{g}_j \mathbf{p}_{w_j} + \mathbf{r}_{w_j}$ for each j.*

Proof. The proof of (i) is by induction on k. If $k = 1$, then λ_1 is the unique root of $\mathbf{p}(0, z_n)$, and $\mathbf{p}$ is already a Weierstrass polynomial in $z_n - \lambda_1$. Thus (i) is true in this case. Suppose (i) is true for polynomials with less than k roots and let $\mathbf{p}$ be a polynomial with k roots. Then $\mathbf{p}$ has finite vanishing order in $z_n - \lambda_k$ at $z' = 0$ (see section 3.3). The Weierstrass preparation theorem implies that $\mathbf{p} = \mathbf{q}_k \mathbf{h}$, where $\mathbf{q}_k$ is a Weierstrass polynomial in $z_n - \lambda_k$ at $z' = 0$, and $\mathbf{h}$ is a unit in $\mathcal{H}_{w_k}$. Furthermore, since $\mathbf{p}$ is a polynomial in $z_n - \lambda_k$, so is $\mathbf{h}$, and since $\mathbf{p}$ and $\mathbf{q}_k$ are monic, so is $\mathbf{h}$. Thus $\mathbf{h}$ is a monic polynomial in ${}_{n-1}\mathcal{H}_0[z_n]$ which has roots only at $\lambda_1, \dots, \lambda_{k-1}$. By the induction assumption, $\mathbf{h}$ factors as $\mathbf{q}_1 \dots \mathbf{q}_{k-1}$ with $\mathbf{q}_j$ a Weierstrass polynomial in $z_j - \lambda_j$ at $z' = 0$. Then $\mathbf{p}$ has the required factorization $\mathbf{q}_1 \dots \mathbf{q}_{k-1}\mathbf{q}_k$. The uniqueness follows from the uniqueness of the Weierstrass factorization. The proof of (i) is complete.

To prove (ii), we let $\mathbf{p}$ be factored as above, set

$$\mathbf{e}_i = \prod_{\nu \neq i} \mathbf{q}_\nu, \quad \mathbf{e}_{ij} = \prod_{\nu \neq i,j} \mathbf{q}_\nu,$$

and note that $\mathbf{e}_j = \mathbf{q}_i \mathbf{e}_{ij}$. Since $\mathbf{q}_\nu$ is a Weierstrass polynomial in $z_n - \lambda_\nu$, its germ at w_j for $j \neq \nu$ is a unit. Thus, $\mathbf{e}_j$ is a unit at w_j. It follows from the Weierstrass division theorem that $\mathbf{f}'_j = \mathbf{f}_j \mathbf{e}_j^{-1}$ has, as a germ at w_j, a representation as

$$\mathbf{f}'_j = \mathbf{g}'_j \mathbf{q}_j + \mathbf{r}_j,$$

where $\mathbf{g}'_j \in {}_n\mathcal{H}_{w_j}$, $\mathbf{r}_j \in {}_{n-1}\mathcal{H}_0[z_n]$, and $\deg \mathbf{r}_j < \deg \mathbf{q}_j$. Then, as germs at w_j,

$$\mathbf{f}_j = \mathbf{e}_j(\mathbf{g}'_j\mathbf{q}_j + \mathbf{r}_j) = \mathbf{e}_{ij}\mathbf{q}_i(\mathbf{g}'_j\mathbf{q}_j + \mathbf{r}_j) = \mathbf{q}_j(\mathbf{e}_j\mathbf{g}'_j - \sum_{i\neq j} \mathbf{e}_{ij}\mathbf{r}_i) + \sum_i \mathbf{e}_i\mathbf{r}_i.$$

Thus, if we define a germ $\mathbf{g}_j$ at w_j by

$$\mathbf{g}_j = \mathbf{e}_j\mathbf{g}'_j - \sum_{i\neq j} \mathbf{e}_{ij}\mathbf{r}_i,$$

for each j, and polynomial $\mathbf{r} \in {}_{n-1}\mathcal{H}_0[z_n]$ by

$$\mathbf{r} = \sum_i \mathbf{e}_i\mathbf{r}_i,$$

then $\mathbf{f}_j = \mathbf{g}_j\mathbf{q}_j + \mathbf{r}_{w_j}$, and $\deg \mathbf{r} < \deg \mathbf{p}$. This gives the existence in (ii).

To prove uniqueness in (ii), it suffices to show that if $0 = \mathbf{g}_j\mathbf{p}_{w_j} + \mathbf{r}_{w_j}$ for each j, with $\mathbf{r} \in {}_{n-1}\mathcal{H}_0[z_n]$ a polynomial of degree less than $\deg \mathbf{p}$, then $\mathbf{r} = 0$. However, this follows easily from counting zeroes of representatives of $\mathbf{p}$ and $\mathbf{r}$, as functions of z_n, for z' in a neighborhood of 0 (Exercise 9.12). This completes the proof of (ii).

9.6.3 Lemma. *Let U be a neighborhood of 0 in $\mathbb{C}^{n-1}$, $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection, and X a subvariety of $\pi^{-1}(U)$. Set $\phi = \pi|_X$ and suppose that $\phi : X \to U$ has finite germ at 0. Then, after replacing U by a smaller neighborhood if necessary, we have $\phi_*\mathcal{M}$ is coherent on U for each coherent sheaf $\mathcal{M}$ on X.*

Proof. We argue as in the proof of Theorem 4.6.3. If J is the ideal in ${}_n\mathcal{H}_0$ generated by $\operatorname{id} \mathbf{X}$ and $\{\mathbf{z}_1, \dots, \mathbf{z}_{n-1}\}$, then the fact that the germ of $\phi^{-1}(0) = \pi^{-1}(0) \cap X$ is 0 implies that $\operatorname{loc} J = 0$. Since $\mathbf{z}_n$ vanishes at 0, the Nullstellensatz implies that some power of $\mathbf{z}_n$ belongs to J. Hence, there exist a positive integer k, elements $\mathbf{g}_1, \dots, \mathbf{g}_{n-1} \in {}_n\mathcal{H}_0$, and $\mathbf{h} \in \operatorname{id} \mathbf{X}$ such that

$$\mathbf{h}(\mathbf{z}) = \mathbf{z}_n^k - \mathbf{z}_1\mathbf{g}_1(\mathbf{z}) - \cdots - \mathbf{z}_{n-1}\mathbf{g}_{n-1}(\mathbf{z}).$$

Then $\mathbf{h}$ is an element of $\operatorname{id} \mathbf{X}$ which has vanishing order k in $\mathbf{z}_n$ at 0. It follows from the Weierstrass preparation theorem that there is a Weierstrass polynomial $\mathbf{p} \in {}_{n-1}\mathcal{H}_0[\mathbf{z}_n]$ which vanishes on $\mathbf{X}$. After shrinking U, if necessary, we may choose a representative $p \in \mathcal{H}(U)[z_n]$ of $\mathbf{p}$ which vanishes on X. Thus, if Y is the zero set of p in $\pi^{-1}(U)$, then X is a subvariety of Y.

Now $\phi : X \to U$ is the composition of the inclusion $i : X \to Y$ and $\psi : Y \to U$, where $\psi = \pi|_Y$. Since i_* preserves coherence (Exercise 9.10), to prove the lemma, we need only show that ψ_* preserves coherence. First we show that $\psi_{*Y}\mathcal{H}$ is free of finite rank as a sheaf of ${}_U\mathcal{H}$-modules.

Let m be the degree of p. We define a ${}_U\mathcal{H}$-module homomorphism $\theta : {}_U\mathcal{H}^m \to \psi_{*Y}\mathcal{H}$ by

$$\theta(h_1, \dots, h_m)(z) = h_1(z')z_n^{m-1} + \cdots + h_{m-1}(z')z_n + h_m(z'),$$

where $z = (z', z_n)$, $(h_1, \dots, h_m) \in \mathcal{H}(V)^m$ for some neighborhood V in U, and the function $z \to h_1(z')z_n^{m-1} + \cdots + h_{m-1}(z')z_n + h_m(z')$ is considered an element of $\psi_{*Y}\mathcal{H}(V) = \mathcal{H}(\psi^{-1}(V))$. If u is a point of U, we claim that $\theta_u : {}_U\mathcal{H}_u^m \to \psi_{*Y}\mathcal{H}_u$ is an isomorphism. This amounts to showing that each element of $\psi_{*Y}\mathcal{H}_u$ is the germ at u of a unique polynomial of degree less than m in z_n. This is exactly the content of Lemma 9.6.2(ii). Thus, θ is a stalkwise isomorphism and, hence, an isomorphism of sheaves of ${}_U\mathcal{H}$-modules. This proves that $\psi_{*Y}\mathcal{H}$ is free of finite rank as a sheaf of ${}_U\mathcal{H}$-modules. It follows that ψ_* takes free finite rank sheaves of ${}_Y\mathcal{H}$-modules to free finite rank sheaves of ${}_U\mathcal{H}$-modules.

Now let $\mathcal{M}$ be a coherent analytic sheaf on Y. On some neighborhood of 0, $\mathcal{M}$ may be represented as the cokernel of a morphism between free finite rank sheaves. Thus, we may as well assume that on Y there is an exact sequence

$${}_Y\mathcal{H}^s \to {}_Y\mathcal{H}^t \to \mathcal{M} \to 0.$$

Since ψ is finite, ψ_* is an exact functor (Exercise 9.14). Hence, we have an exact sequence

$$\psi_{*Y}\mathcal{H}^s \to \psi_{*Y}\mathcal{H}^t \to \psi_*\mathcal{M} \to 0.$$

Since ψ_* sends free finite rank sheaves to free finite rank sheaves, we conclude that $\psi_*\mathcal{M}$ is coherent on U. This completes the proof.

9.6.4 Theorem. *If $\phi : Y \to X$ is a finite holomorphic map, then ϕ_* takes coherent analytic sheaves on Y to coherent analytic sheaves on X.*

Proof. Let x be a point of X, and let $y_1, \dots, y_k$ be the points of $\phi^{-1}(x)$. Then for some neighborhood U of x, $\phi^{-1}(U)$ is the union of disjoint subvarieties $V_1, \dots, V_k$ with $y_i \in V_i$. Let $\phi_i : V_i \to U$ be the restriction of ϕ to V_i. Then on U, $\phi_*\mathcal{M}$ is the direct sum of the sheaves $\phi_{i*}\mathcal{M}|_{V_i}$ and, hence, is coherent if each of these is coherent. Since the theorem is local over X, it suffices to prove that if $x \in X$ is a point for which $\phi^{-1}(x)$ is a single point and if $\mathcal{M}$ is a coherent analytic sheaf on Y, then $\phi_*\mathcal{M}$ is coherent in some neighborhood of x.

Without loss of generality, we may assume that X is a subvariety of a neighborhood U of $x = 0$ in $\mathbb{C}^m$. We may also assume that, for some $n \geq m$, Y is a subvariety of $\pi^{-1}(U)$, and $\phi = \pi|_Y$, where $\pi : \mathbb{C}^n \to \mathbb{C}^m$ is projection on the first m coordinates, and $y = 0$ is the only point in $\phi^{-1}(0)$ (Exercise 4.8). The proof is then by induction on $n - m$ beginning with $n - m = 0$.

The result is trivial when $n - m = 0$. Suppose that $n > m$ and assume that the theorem is true whenever ϕ can be realized, as in the previous paragraph, but with n replaced by $n - 1$. Denote by $\pi' : \mathbb{C}^{n-1} \to \mathbb{C}^m$ and $\pi'' : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projections. Then the germ at 0 of the restriction of π'' to Y is a finite morphism $\mathbf{Y} \to \mathbf{C}^{n-1}$ of germs of varieties. By Corollary 4.6.4, the image of $\mathbf{Y}$ under π'' exists as a germ of a holomorphic variety. Thus, by shrinking U if necessary, we may assume that the image of Y under π'' is a subvariety Z of $\pi'^{-1}(X)$. The map $\phi : Y \to X$ factors as $\phi = \phi' \circ \phi''$ with $\phi' = \pi'|_Z : Z \to X$ and $\phi'' = \pi''|_Y : Y \to Z$. Clearly, each of ϕ' and ϕ'' is also a finite holomorphic map. The functor ϕ'_* preserves coherence by the induction assumption, and ϕ''_* preserves coherence by Lemma 9.6.3 and Exercise 9.13. It follows that the composition $\phi_* = \phi'_* \circ \phi''_*$ also preserves coherence. This completes the induction and finishes the proof.

A deep theorem along these lines, which we shall not prove here, is the proper mapping theorem for coherent sheaves. It states that ϕ_* preserves coherence if ϕ is a proper holomorphic map. We refer the interested reader to [**GRe2**], Chapter 10.

Exercises

1. Show that if U is a non-empty open subset of $\mathbb{C}^n$ for $n > 0$, then $\mathcal{H}(U)$ is not Noetherian.
2. Prove that the presheaf $\mathcal{I} : \mathcal{J}$, as defined in section 9.3, is a sheaf.
3. Prove Corollary 9.5.3.
4. Prove that if $\phi : {}_X\mathcal{H}^k \to {}_X\mathcal{H}^m$ is a morphism of analytic sheaves, then the map $\phi_x : {}_X\mathcal{H}^k_x \to {}_X\mathcal{H}^m_x$ is surjective if and only if the matrix of holomorphic functions on X, defining ϕ, has rank m at x.
5. Prove that if $\mathcal{M}$ is a coherent analytic sheaf on a holomorphic variety X, then $\operatorname{Support}(\mathcal{M}) = \{x \in X : \mathcal{M}_x \neq 0\}$ is a holomorphic subvariety of X. Hint: Use the result of the preceding exercise.
6. Use the result of the preceding exercise to prove that if $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism between two coherent analytic sheaves, then the set of $x \in X$ for which ϕ_x is not injective and the set of $x \in X$ for which

ϕ_x is not surjective are holomorphic subvarieties of X. Prove that the corresponding statements for coherent algebraic sheaves are also true.

7. Let X be a holomorphic variety of dimension n. That is, n is the maximal dimension of the germs $\mathbf{X}_\lambda$ at points $\lambda \in X$. Prove that there is a finite filtration of X by closed subvarieties
$$X_0 \subset X_1 \subset \cdots \subset X_n = X,$$
such that X_0 is a discrete set and $X_k - X_{k-1}$ is a complex manifold for $k = 1, \ldots, n$.
8. If Y is a subvariety of a holomorphic variety X, and $i : Y \to X$ is the inclusion, then prove that the sheaf $i_{*Y}\mathcal{H}$ on X is a coherent sheaf of ${}_X\mathcal{H}$-modules.
9. Show that if $\phi : Y \to X$ is any holomorphic map between holomorphic varieties and $\mathcal{M}$ is a sheaf of ${}_Y\mathcal{H}$-modules, then $\phi_*\mathcal{M}$ has a natural structure of a sheaf of ${}_X\mathcal{H}$-modules.
10. Let Y be a subvariety of the holomorphic variety X, and let $i : Y \to X$ be the inclusion. Prove that if $\mathcal{M}$ is a coherent analytic sheaf on Y, then $i_*\mathcal{M}$ is a coherent analytic sheaf on X. Hint: Use the result of Exercise 9.8.
11. Show by example that if $\phi : Y \to X$ is a holomorphic map between holomorphic varieties, and $\mathcal{M}$ is a coherent sheaf of ${}_Y\mathcal{H}$-modules, then it is not necessarily true that $\phi_*\mathcal{M}$ is a coherent sheaf of ${}_X\mathcal{H}$-modules.
12. Let $\mathbf{p}$ be a monic polynomial in ${}_{n-1}\mathcal{H}_0[z_n]$ such that the distinct zeroes of $\mathbf{p}(0, z_n)$ are $\lambda_1, \ldots, \lambda_k$, and let $\mathbf{r} \in {}_{n-1}\mathcal{H}_0[z_n]$ be a polynomial with $\deg \mathbf{r} < \deg \mathbf{p}$. Set $w_j = (0, \lambda_j) \in \mathbb{C}^n$ for each j. If, for each j, there is a germ $\mathbf{g}_j \in {}_n\mathcal{H}_{w_j}$ such that $\mathbf{g}_j\mathbf{p}_{w_j} = \mathbf{r}_{w_j}$, then prove that $\mathbf{r} = 0$ and that $\mathbf{g}_j = 0$ for each j. Hint: Let p and r be representatives of $\mathbf{p}$ and $\mathbf{r}$ in a neighborhood and count zeroes of the polynomials $p(z', z_n)$ and $r(z', z_n)$ for z' in this neighborhood.
13. Let $\mathcal{M}$ be a coherent analytic sheaf on a holomorphic variety X. Suppose the support of $\mathcal{M}$ is contained in a proper subvariety Y of X, and $\mathcal{M}|_Y$ is naturally a sheaf of ${}_Y\mathcal{H}$-modules (that is, the ideal sheaf of Y acts trivially on $\mathcal{M}$, so that the action of ${}_X\mathcal{H}$ on $\mathcal{M}$ induces an action of ${}_Y\mathcal{H}$ on $\mathcal{M}$). Then show that $\mathcal{M}|_Y$ is coherent as an analytic sheaf on Y.
14. Let $\phi : Y \to X$ be a finite holomorphic map. Prove that ϕ_* is an exact functor from the category of sheaves on Y to the category of sheaves on X.
15. Prove that if X is a compact connected holomorphic variety, then every holomorphic function on X is constant.
16. A holomorphic variety Y is said to be *normal* if each of its local rings ${}_Y\mathcal{H}_y$ is a normal domain. Use Theorem 4.6.8 to prove that, if $f : Y \to X$

is a bijective holomorphic map between holomorphic varieties and Y is normal, then f is a biholomorphic map.

17. Let $\mathcal{M}$ and $\mathcal{N}$ be analytic sheaves on a holomorphic variety X. Prove that if $\mathcal{M}$ is coherent, and $x \in X$, then every morphism $\psi \in \mathrm{Hom}_{\mathcal{H}_x}(\mathcal{M}_x, \mathcal{N}_x)$ is ϕ_x for some $\phi \in \mathrm{Hom}_{\mathcal{H}|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$ and some neighborhood U of x.
18. Consider the sheaf $\mathcal{H}om_{\mathcal{H}}(\mathcal{M}, \mathcal{N})$ of Definition 7.3.8, with $\mathcal{M}$ and $\mathcal{N}$ as in the previous exercise. Given $x \in X$, the morphisms

$$\phi \to \phi_x : \mathrm{Hom}_{\mathcal{H}|_U}(\mathcal{M}|_U, \mathcal{N}|_U) \to \mathrm{Hom}_{\mathcal{H}_x}(\mathcal{M}_x, \mathcal{N}_x), \quad x \in U,$$

determine a morphism $\mathcal{H}om_{\mathcal{H}}(\mathcal{M}, \mathcal{N})_x \to \mathrm{Hom}_{\mathcal{H}_x}(\mathcal{M}_x, \mathcal{N}_x)$. Prove that this is an isomorphism for each x.
19. Suppose $\mathcal{M}$ and $\mathcal{N}$ are coherent analytic sheaves on a holomorphic variety X. Using the result of the previous exercise, prove that $\mathcal{H}om_{\mathcal{H}}(\mathcal{M}, \mathcal{N})$ is also coherent.
20. Prove that if $\mathcal{M}$ and $\mathcal{N}$ are coherent analytic sheaves on a holomorphic variety X, then $\mathcal{M} \otimes_{\mathcal{H}} \mathcal{N}$ is also coherent (see Definition 7.3.7).

Chapter 10

Stein Spaces

Stein spaces play the role in the study of holomorphic functions and varieties that is played by affine varieties in the algebraic theory. On a Stein space, a coherent analytic sheaf has a rich supply of global sections. In fact, it is generated over $\mathcal{H}$ by its set of global sections (Cartan's Theorem A). Furthermore, on a Stein space, every coherent analytic sheaf is acyclic (Cartan's Theorem B). In this chapter we define the category of Stein spaces and lay the ground work for proving Cartan's theorems. The proofs of the theorems themselves will be carried out in the next chapter, where we develop and employ some approximation results for coherent analytic sheaves.

The key result in this chapter is a vanishing theorem which states that a coherent analytic sheaf defined in a neighborhood of a compact polydisc has vanishing higher cohomology on the polydisc. We first prove this in the case where the sheaf is the structure sheaf $\mathcal{H}$ itself using a differential equations argument based on the solution of the inhomogeneous Cauchy-Riemann equations (Dolbeault's lemma). This also establishes the result in the case of a free sheaf of $\mathcal{H}$-modules (a direct sum of copies of $\mathcal{H}$). The strategy for proving the result for general coherent analytic sheaves is to show that if a coherent sheaf has a terminating free resolution (a chain of syzygies), then its higher cohomologies vanish (section 10.2) and then to prove that every coherent sheaf on a polydisc has such a resolution.

10.1 Dolbeault Cohomology

In sheaf theory, a *vanishing theorem* is a theorem which asserts that cohomology vanishes for some class of sheaves in some range of degrees p (usually $p > 0$). Vanishing theorems generally insure the existence of global solutions to certain locally solvable problems. The Poincaré lemma, referred to in section 7.7, is an example of a vanishing theorem. It asserts that the cohomology of the constant sheaf $\mathbb{C}$ vanishes on any convex open set in $\mathbb{R}^n$.

In this section we prove the one vanishing theorem for analytic sheaves that can be proved without a great deal of work – Dolbeault's lemma. This is a vanishing theorem for the structure sheaf ${}_n\mathcal{H}$ of $\mathbb{C}^n$. Its proof involves constructing a resolution of ${}_n\mathcal{H}$ by a complex of sheaves of differential forms which is analogous to the de Rham complex. Hence, we will make heavy use of the language of differential forms as discussed in section 7.7.

If U is an open set in $\mathbb{C}^n$, then we consider it as an open subset of the real vector space $\mathbb{R}^{2n}$ with basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. As in section 1.4, instead of the usual bases for the complexified tangent and cotangent spaces, we use the basis consisting of

$$dz_i = dx_i + i\,dy_i, \quad d\bar{z}_i = dx_i - i\,dy_i, \quad i = 1, \ldots, n$$

for the complexified cotangent space and

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right), \quad i = 1, \ldots, n$$

for the complexified tangent space. Note that these are dual bases to one another. That is,

$$dz_i\left(\frac{\partial}{\partial z_j}\right) = \delta_{ij} = d\bar{z}_i\left(\frac{\partial}{\partial \bar{z}_j}\right)$$

and

$$dz_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = 0 = d\bar{z}_i\left(\frac{\partial}{\partial z_j}\right) \text{ for all } i, j.$$

In terms of the above basis for the cotangent space, a differential form in $\mathcal{E}^r(U)$ may be written as

$$\sum_{p+q=r} \phi_{j_i \ldots j_p k_1 \ldots k_q} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q},$$

with coefficients $\phi_{j_i \ldots j_p k_1 \ldots k_q} \in C^\infty(U)$. If we let $\mathcal{E}^{p,q}(U)$ denote the subspace of $\mathcal{E}^{p+q}(U)$ consisting of forms of degree p in the dz_i and of degree q in the $d\bar{z}_i$, then we have a direct sum decomposition:

$$\mathcal{E}^r(U) = \sum_{p+q=r} \mathcal{E}^{p,q}(U).$$

Forms in the space $\mathcal{E}^{p,q}(U)$ are said to have *bidegree* (p, q) and *total degree* $p + q$.

When restricted to $\mathcal{E}^{p,q}$, exterior differentiation defines a map

$$d : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q} + \mathcal{E}^{p,q+1}.$$

If we define $\partial : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q}$ and $\bar{\partial} : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}$ to be d followed by projection on $\mathcal{E}^{p+1,q}$ and $\mathcal{E}^{p,q+1}$, respectively, then

$$d = \partial + \bar{\partial}.$$

Note that $0 = d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$. After sorting out terms of different bidegree, it follows that

$$\partial^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \quad \bar{\partial}^2 = 0.$$

Also, since

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^r \phi \wedge d\psi,$$

it follows that

$$\partial(\phi \wedge \psi) = \partial\phi \wedge \psi + (-1)^r \phi \wedge \partial\psi, \quad \bar{\partial}(\phi \wedge \psi) = \bar{\partial}\phi \wedge \psi + (-1)^r \phi \wedge \bar{\partial}\psi$$

for $\phi \in \mathcal{E}^r(U)$ and ψ an arbitrary form.

The operator $\bar{\partial}$ is given explicitly by

$$\begin{aligned} &\bar{\partial}(\phi_{j_i \dots j_p k_1 \dots k_q} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}) \\ &= \sum_i \frac{\partial \phi_{j_i \dots j_p k_1 \dots k_q}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}. \end{aligned}$$

In particular, for $f \in \mathcal{E}^{0,0}(U) = \mathcal{C}^\infty(U)$,

$$\bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i,$$

and so the Cauchy-Riemann equations assert that $\bar{\partial} f = 0$ if and only if f is holomorphic in U.

10.1.1 Definition. *For an open set $U \subset \mathbb{C}^n$ the pth Dolbeault complex is the complex*

$$0 \longrightarrow \mathcal{E}^{p,0}(U) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}(U) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,n}(U) \longrightarrow 0.$$

Its qth cohomology group is called the (p,q)-Dolbeault cohomology of U and is denoted $\mathfrak{H}^{p,q}(U)$.

Note that the $(p,0)$-Dolbeault cohomology is just the space of p-forms in $dz_1, \dots, dz_n$ with holomorphic coefficients. This is called the space of *holomorphic p-forms*, and it will be denoted $\mathcal{H}^{(p)}(U)$. The correspondence $U \to \mathcal{H}^{(p)}(U)$ is clearly a sheaf, and it will be denoted $\mathcal{H}^{(p)}$. Note that $\mathcal{H}^{(0)}$ is the sheaf of holomorphic functions $\mathcal{H}$.

Our main objective in this section is to prove that Dolbeault cohomology vanishes for $q > 0$ if U is a polydisc. Among other things, this will imply that for each p, as a complex of sheaves, the Dolbeault complex is exact except in degree 0 and provides a fine resolution of the sheaf of holomorphic p-forms. In particular, when $p = 0$ the Dolbeault complex provides a fine resolution of the sheaf of holomorphic functions and, hence, can be used to compute its sheaf cohomology.

If $K \subset \mathbb{C}^n$ is compact, we denote by $\mathcal{E}^{p,q}(K)$ the space $\Gamma(K, \mathcal{E}^{p,q})$ of $\mathcal{C}^\infty$ forms of bidegree (p, q) defined in a neighborhood of K. The Dolbeault cohomology $\mathfrak{H}^{p,q}(K)$ for K is then the cohomology of the complex $\{\mathcal{E}^{p,q}(K), \bar\partial\}$.

10.1.2 Lemma. *If $\overline{\Delta}$ is a compact polydisc in $\mathbb{C}^n$, then $\mathfrak{H}^{p,q}(\overline{\Delta}) = 0$ for $q > 0$ and for each p.*

Proof. Suppose $q > 0$, and let $\phi \in \mathcal{E}^{p,q}(\overline{\Delta})$ be a form such that $\bar\partial\phi = 0$. We must show that $\phi = \bar\partial\psi$ for some form $\psi \in \mathcal{E}^{p,q-1}(\overline{\Delta})$. Let k be the least integer such that the expression for ϕ involves no conjugate differential $d\bar z_j$ with $j > k$; that is, ϕ can be written in terms of the conjugate differentials $d\bar z_1, \dots, d\bar z_k$ and the differentials $dz_1, \dots, dz_n$. We proceed by induction on k. If $k = 0$, then $\phi = 0$, since $q > 0$, and hence, there is nothing to prove. Thus, we assume that $k > 0$ and the assertion is true for integers less than k. We write ϕ as

$$\phi = d\bar z_k \wedge \alpha + \beta,$$

where α and β involve only the conjugate differentials $d\bar z_1, \dots, d\bar z_{k-1}$. Then

$$0 = \bar\partial\phi = -d\bar z_k \wedge \bar\partial\alpha + \bar\partial\beta.$$

If $j > k$, then no cancellation can occur between terms of $\bar\partial\alpha \wedge d\bar z_k$ involving $d\bar z_j$ and terms of $\bar\partial\beta$ involving $d\bar z_j$. It follows that such terms individually vanish and, hence, the coefficients of α and β are holomorphic in the variables $z_{k+1}, \dots, z_n$.

Now recall Proposition 1.4.2. It states that if f is a $\mathcal{C}^\infty$ function in a neighborhood of a compact set in $\mathcal{C}$, then $f = \partial g/\partial \bar z$ for some g which is also $\mathcal{C}^\infty$ in a neighborhood of the compact set. It is clear from the explicit integral formula that gives the solution g in Proposition 1.4.2 that if f depends in a holomorphic or $\mathcal{C}^\infty$ fashion on other parameters, then g does as well. Thus, if f is a coefficient of α, then $f = \partial g/\partial \bar z_k$ for some $g \in \mathcal{C}^\infty(\overline{\Delta})$ which is also holomorphic in the variables $z_{k+1}, \dots, z_n$. By replacing each coefficient f of α by the corresponding g, as above, we obtain a $(p, q-1)$-form γ with the property that

$$\bar\partial\gamma = \delta + d\bar z_k \wedge \alpha$$

where δ is a form involving only the conjugate differentials $d\bar z_1, \dots, d\bar z_{k-1}$. Then $\phi - \bar\partial\gamma = \beta - \delta$ involves only the conjugate differentials $d\bar z_1, \dots, d\bar z_{k-1}$. Furthermore, $\bar\partial(\phi - \bar\partial\gamma) = 0$. Thus, by the induction hypothesis, we conclude that $\bar\partial\eta = \phi - \bar\partial\gamma$ for some $\eta \in \mathcal{E}^{p,q-1}(\overline{\Delta})$, from which it follows that $\phi = \bar\partial\psi$ with $\psi = \eta + \gamma \in \mathcal{E}^{p,q-1}(\overline{\Delta})$. This completes the proof.

Note that it was not really important in the above argument that $\bar\Delta$ be a polydisc. It was important that it be a Cartesian product – that is, a set of the form $K_1 \times K_2 \times \cdots \times K_n$ for some collection of compact sets $K_i \subset \mathbb{C}$. This is due to the fact that the solution was obtained by applying Proposition 1.4.2 in each variable separately, while treating the other variables as parameters.

In section 1.4, we used an approximation argument, along with Proposition 1.4.2 , to establish the solvability of the inhomogeneous Cauchy-Riemann equation on any open set. In the next theorem we use a similar approximation argument, along with Lemma 10.1.2, to establish a vanishing result for Dolbeault cohomology on any open polydisc. Note that we allow some or all of the radii r_i to be infinite. Thus, $\mathbb{C}^n$ itself is an open polydisc.

10.1.3 Dolbeault's Lemma. *Let Δ be an open polydisc in $\mathbb{C}^n$ for some n. Then $\mathcal{H}^{p,q}(\Delta) = 0$ for $q > 0$ and for all p.*

Proof. Let $\{\Delta_j\}$ be a sequence of open polydiscs with compact closure such that $\overline{\Delta}_j \subset \Delta_{j+1}$ and $\bigcup_j \Delta_j = \Delta$. If $\phi \in \mathcal{E}^{p,q}(\Delta)$ and $\bar\partial\phi = 0$, then we will construct $\psi \in \mathcal{E}^{p,q-1}(\Delta)$ such that $\bar\partial\psi = \phi$ inductively, using Lemma 10.1.2 on the sets $\overline{\Delta}_j$. We first take care of the case $q > 1$, which is different and considerably easier than the case $q = 1$.

If $q > 1$, we inductively construct a sequence of forms $\{\psi_j\}$ such that $\psi_j \in \mathcal{E}^{p,q-1}(\overline{\Delta}_i)$, $\bar\partial\psi_j = \phi$ on Δ_j, and $\psi_{j+1}|_{\Delta_j} = \psi_j$. Clearly this will give the desired result, since we can then define a solution $\psi \in \mathcal{E}^{p,q-1}(\Delta)$ by $\psi|_{\Delta_j} = \psi_j$. Suppose the sequence $\{\psi_j\}$ has been constructed, with the above properties, for $j < k$. Then we use Lemma 10.1.2 to find $\theta \in \mathcal{E}^{p,q-1}(\overline{\Delta}_k)$ such that $\bar\partial\theta = \phi$ in a neighborhood of $\overline{\Delta}_k$. We then have

$$\bar\partial(\theta - \psi_{k-1}) = 0$$

in a neighborhood of $\overline{\Delta}_{k-1}$, and since $q > 1$, we may apply Lemma 10.1.2 again to find $\eta \in \mathcal{E}^{p,q-2}(\overline{\Delta}_{k-1})$ such that $\bar\partial\eta = \theta - \psi_{k-1}$ in a neighborhood of $\overline{\Delta}_{k-1}$. By multiplying by a $\mathcal{C}^\infty$ function which is 1 in a neighborhood of $\overline{\Delta}_{k-1}$ and has compact support in a slightly larger neighborhood of $\overline{\Delta}_{k-1}$, we may assume that η is in $\mathcal{E}^{p,q-2}(\mathbb{C}^n)$. Then $\psi_k = \theta - \bar{\partial}\eta \in \mathcal{E}^{p,q-1}(\overline{\Delta}_k)$ gives the required next function in our sequence, since

$$\bar{\partial}\psi_k = \bar{\partial}(\theta - \bar{\partial}\eta) = \phi$$

on Δ_k, and $\psi_k = \psi_{k-1}$ on Δ_{k-1}. This completes the proof in the case $q > 1$.

In the case $q = 1$, we use the sequence $\{\Delta_j\}$, as before, but this time we inductively construct a sequence $\{\psi_j\} \subset \mathcal{E}^{p,q-1}(\overline{\Delta}_j)$ such that $\bar{\partial}\psi_j = \phi$ on Δ_j, and $||\psi_j - \psi_{j+1}||_j < 2^{-j}$ for each j. Here, $||\theta||_j$ is the sum of the supremum norms on $\overline{\Delta}_j$ of the coefficients of the form θ. Suppose such a sequence $\{\psi_j\}$ has been constructed for all indices $j < k$. We use Lemma 10.1.2 to find $\theta \in \mathcal{E}^{p,0}(\overline{\Delta}_k)$ such that $\bar{\partial}\theta = \phi$ in a neighborhood of $\overline{\Delta}_k$. As before,

$$\bar{\partial}(\theta - \psi_{k-1}) = 0$$

in a neighborhood of $\overline{\Delta}_{k-1}$. This means that $\theta - \psi_{k-1}$ has coefficients which are holomorphic in a neighborhood of $\overline{\Delta}_{k-1}$. If we represent these coefficients as convergent power series about the point which is the center of $\overline{\Delta}_{k-1}$, then it is clear that we may choose a form η, with polynomial coefficients, such that $||\theta - \psi_{k-1} - \eta||_{k-1} < 2^{-k+1}$. Then $\psi_k = \theta - \eta$ satisfies $\bar{\partial}\psi_k = \phi$ in a neighborhood of Δ_k, and $||\psi_k - \psi_{k-1}||_{k-1} < 2^{-k+1}$. Thus, by induction, we may construct the sequence $\{\psi_j\}$ as claimed.

On a given $\overline{\Delta}_k$, consider the sequence $\{(\psi_j)|_{\overline{\Delta}_k}\}_{j=k}^{\infty}$. This is a Cauchy sequence in the norm $||\cdot||_k$, since $||\psi_{j+1} - \psi_j||_k \le ||\psi_{j+1} - \psi_j||_j < 2^{-j}$ for $j \ge k$. Furthermore, for each $j \ge k$, the form $\psi_j - \psi_k$ on $\overline{\Delta}_k$ has coefficients holomorphic on Δ_k (since it is killed by $\bar{\partial}$). The sequence $\{\psi_j - \psi_k\}_j$ is also convergent in the norm $||\cdot||_k$ on $\overline{\Delta}_k$, and hence, it converges to a form θ_k with coefficients holomorphic on Δ_k. Furthermore, $\lim_{j\to\infty} \psi_j = \theta_k + \psi_k$ on Δ_k. Since this is true for each k, $\psi = \lim_{j\to\infty} \psi_j$ is a form in $\mathcal{E}^{p,q-1}(\Delta)$ which satisfies $\bar{\partial}\psi = \phi$. This completes the proof.

It is not hard to see that the division of $\mathcal{E}^r$ into bigraded terms $\mathcal{E}^{p,q}$ is independent of the choice of complex coordinate system. It follows that the Dolbeault complex and Dolbeault cohomology can be defined on any complex manifold,

10.1.4 Corollary. *If X is any complex manifold, then for each p there is a natural isomorphism $\mathfrak{H}^{p,q}(X) \to H^q(X, \mathcal{H}^{(p)})$ between Dolbeault cohomology on X and sheaf cohomology of $\mathcal{H}^{(p)}$ on X. In particular, $\mathfrak{H}^{0,q}(X)$ is isomorphic to the sheaf cohomology $H^q(X, \mathcal{H})$ of the sheaf of holomorphic functions.*

Proof. Since a complex manifold has a neighborhood base consisting of sets which are biholomorphically equivalent to open polydiscs, it follows from Lemma 10.1.3 that the complex

$$0 \longrightarrow \mathcal{H}^{(p)} \longrightarrow \mathcal{E}^{p,0} \longrightarrow \mathcal{E}^{p,1} \longrightarrow \dots \longrightarrow \mathcal{E}^{p,n} \longrightarrow 0$$

is exact. Since each $\mathcal{E}^{p,q}$ is a sheaf of $\mathcal{C}^\infty$-modules, this sequence is a fine resolution of $\mathcal{H}^{(p)}$, and hence, may be used to compute its sheaf cohomology.

10.1.5 Corollary. *If $\Delta \subset \mathbb{C}^n$ is an open polydisc, then for each p, the sheaf cohomology $H^q(\Delta, \mathcal{H}^{(p)})$ vanishes for $q > 0$.*

Since the Dolbeault complex terminates at degree $q = n$, Dolbeault cohomology vanishes in degree greater than n. Thus:

10.1.6 Corollary. *If X is a complex manifold of dimension n, then for each p, the sheaf cohomology $H^q(X, \mathcal{H}^{(p)})$ vanishes for $q > n$.*

10.2 Chains of Syzygies

Our strategy for proving that every coherent analytic sheaf on a polydisc is acyclic is based on the following lemma.

10.2.1 Lemma. *Let Δ be an open polydisc in $\mathbb{C}^n$. Suppose there is an exact sequence of sheaves of $\mathcal{H}$-modules on Δ of the form*

$$0 \longrightarrow \mathcal{H}^{p_m} \longrightarrow \dots \longrightarrow \mathcal{H}^{p_1} \longrightarrow \mathcal{H}^{p_0} \longrightarrow \mathcal{S} \longrightarrow 0.$$

Then $H^q(\Delta, \mathcal{S}) = 0$ for $q > 0$, and the sequence of global sections

$$0 \longrightarrow \mathcal{H}^{p_m}(\Delta) \longrightarrow \dots \longrightarrow \mathcal{H}^{p_0}(\Delta) \longrightarrow \mathcal{S}(\Delta) \longrightarrow 0$$

is also exact.

Proof. The exact sequence of sheaves in the hypothesis can be decomposed into a collection of short exact sequences of coherent analytic sheaves

$$0 \longrightarrow \mathcal{L}_k \longrightarrow \mathcal{H}^{p_k} \longrightarrow \mathcal{L}_{k-1} \longrightarrow 0,$$

where $\mathcal{L}_k = \operatorname{Ker}\{\mathcal{H}^{p_k} \to \mathcal{H}^{p_{k-1}}\}$ for $k > 0$, $\mathcal{L}_0 = \operatorname{Ker}\{\mathcal{H}^{p_0} \to \mathcal{S}\}$, and $\mathcal{L}_{-1} = \mathcal{S}$. For such a sequence, the long exact sequence of cohomology and the fact that $\mathcal{H}^{p_k}$ has vanishing qth cohomology for $q > 0$ imply that, if $\mathcal{L}_k$ is also a sheaf with vanishing qth cohomology for all $q > 0$, then $\mathcal{L}_{k-1}$ is as well, and the sequence

$$0 \longrightarrow \mathcal{L}_k(\Delta) \longrightarrow \mathcal{H}^{p^k}(\Delta) \longrightarrow \mathcal{L}_{k-1}(\Delta) \longrightarrow 0$$

is exact. The theorem follows from descending induction using this fact, beginning on the left at $k = m$, and noting that $\mathcal{L}_m = 0$.

In a corollary to Hilbert's syzygy theorem (Corollary 6.7.9), we proved that every finitely generated module over the local ring ${}_n\mathcal{H}_0$ has a free finite rank resolution of length n – a syzygy of length n. Thus, given a coherent sheaf $\mathcal{S}$ on an open set $U \subset \mathbb{C}^n$, a terminating free resolution can be constructed for the stalk $\mathcal{S}_\lambda$ at each point $\lambda \in U$. Using the results on coherence of Chapter 9, we are able to do this in a coherent fashion so as to construct a terminating resolution by free sheaves for the sheaf $\mathcal{S}$ in a neighborhood of any point.

10.2.2 Proposition. *If $\mathcal{S}$ is a coherent analytic sheaf defined on an open set U in $\mathbb{C}^n$, then for each point $\lambda \in U$, there is a neighborhood W of λ in U and an exact sequence:*

$$0 \longrightarrow {}_W\mathcal{H}^{p_n} \xrightarrow{\phi_n} \dots \longrightarrow {}_W\mathcal{H}^{p_1} \xrightarrow{\phi_1} {}_W\mathcal{H}^{p_0} \xrightarrow{\phi_0} \mathcal{S}|_W \longrightarrow 0$$

In other words, $\mathcal{S}$ locally has a free finite rank resolution of length n.

Proof. Let λ be any point of U. By the definition of analytic coherence, there is a neighborhood W of λ and an exact sequence

$${}_W\mathcal{H}^{p_1} \xrightarrow{\phi_1} {}_W\mathcal{H}^{p_0} \xrightarrow{\phi_0} \mathcal{S}|_W \longrightarrow 0.$$

We know, by Corollary 9.2.4, that $\operatorname{Ker}\phi_1$ is also coherent. Thus, after shrinking W if necessary, we may express $\operatorname{Ker}\phi_1$ as the image of a morphism $\phi_2 : {}_W\mathcal{H}^{p_2} \to {}_W\mathcal{H}^{p_1}$. Continuing in this manner, we construct the sequence of the proposition, up to stage $n-1$. At this stage we have a morphism $\phi_{n-1} : {}_W\mathcal{H}^{p_{n-1}} \to {}_W\mathcal{H}^{p_{n-2}}$.

By Hilbert's syzygy theorem (Theorem 6.7.8), the stalk $(\operatorname{Ker}\phi_{n-1})_\lambda$ is a free finite rank $\mathcal{H}_\lambda$-module. We choose germs which form a basis for $(\operatorname{Ker}\phi_{n-1})_\lambda$ and then choose representatives in a neighborhood (which we may as well assume is W) for these germs. The resulting finite set of sections of $\operatorname{Ker}\phi_{n-1}$ defines a morphism $\phi_n : {}_W\mathcal{H}^{p_n} \to \operatorname{Ker}\phi_{n-1}$, which is an isomorphism between the stalks of these two sheaves at λ. By Exercise 9.6, ϕ is an isomorphism in a neighborhood of λ. Therefore, after again shrinking W if necessary, the morphism ϕ_n completes the construction of our resolution.

The final step in proving a vanishing theorem for the cohomology of coherent sheaves on a polydisc will be to piece together the local resolutions given by the above proposition to obtain such a resolution on the entire polydisc. This is not so easy to do. The construction is based on a factorization lemma (Cartan's factorization lemma) for holomorphic matrix valued functions.

10.3 Functional Analysis Preliminaries

The proof of Cartan's factorization lemma we will give here uses several results from functional analysis – specifically, an open mapping theorem for non-linear maps between Banach spaces and some elementary results concerning the group of invertible elements of a Banach algebra. In this section, we give a brief development of these functional analysis preliminaries.

A Banach algebra is a complex algebra A with a norm which makes A a Banach space and which is submultiplicative – that is, it satisfies $||ab|| \leq ||a|| ||b||$ for all $a, b \in A$. We will consider only Banach algebras with an identity, and we will assume the identity has norm 1. The set of invertible elements in a Banach algebra A forms a group which we will sometimes denote A^{-1}. We will show in the next theorem that A^{-1} is a *topological group*. That is, it is a group with a Hausdorff topology (given by the norm in A) such that the multiplication map $(a, b) \to ab : A^{-1} \times A^{-1} \to A^{-1}$ is continuous, and inversion $a \to a^{-1} : A^{-1} \to A^{-1}$ is continuous. The submultiplicative property of the norm implies that multiplication is continuous in A and, hence, in A^{-1}. The fact that Banach algebras are Banach spaces and, hence, are complete, allows us to use power series arguments. These yield the following elementary results concerning A^{-1}:

10.3.1 Theorem. *If A is a Banach algebra with identity, then*

(i) *if $a \in A$, and $||a|| < 1$, then $1 - a$ has an inverse* $(1-a)^{-1} = \sum_{k=0}^{\infty} a^k$;

(ii) *A^{-1} is open in A, inversion is continuous in A^{-1}, and A^{-1} is a topological group;*

(iii) *there is a map* $a \to \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ *from A to A^{-1} which is a homeomorphism from some neighborhood of* 0 *in A to the neighborhood* $\{b : ||1-b|| < 1\}$ *in* A^{-1};

(iv) *on* $\{b \in A : ||1 - b|| < 1\}$ *the map* $b \to \log(b) = -\sum_{n=1}^{\infty} \frac{(1-b)^n}{n}$ *is an inverse for* exp;

(v) *the subgroup of A^{-1} generated by the image of* exp *is open and is equal to the connected component of the identity in* A^{-1}.

Proof. We prove statements (i), (ii), and (v) and leave (iii) and (iv) as an exercise (Exercise 10.1). Since the norm in a Banach algebra is subadditive and submultiplicative, we have

$$||\sum_{k=n}^{m} a^k|| \leq \sum_{k=n}^{m} ||a||^k$$

for $n \leq m$. Thus, if $||a|| < 1$, the sequence of partial sums of the series $\sum_{k=0}^{\infty} a^k$ is Cauchy in norm and, hence, converges in norm to an element of the Banach algebra A. This element is clearly an inverse for $1 - a$. Thus, the open unit ball in A, centered at 1, consists of invertible elements. If we write $(1-a)^{-1}$ in the form $1 - b$, then $b = -\sum_{k=1}^{\infty} a^k$ and

$$||b|| \leq \sum_{k=1}^{\infty} ||a||^k = ||a||(1 - ||a||)^{-1}.$$

This proves that inversion is continuous at 1.

It follows from the submultiplicitivity of the norm that multiplication in A is continuous, and in particular, multiplication by an invertible element is a homeomorphism of A to itself. It follows that every element of A^{-1} has a neighborhood in A which is also contained in A^{-1}, and so inversion is continuous at every point of A^{-1}. This proves (i) and (ii).

In a topological group, a subgroup with an interior point is necessarily both open and closed (Exercise 10.2). Therefore, if we assume (iii) and (iv), the subgroup $E \subset A^{-1}$ generated by the image of exp is open and closed, because it contains $\{b \in A : ||1-b|| < 1\}$. A typical element of this subgroup has the form

$$a = \exp(b_1) \cdot \exp(b_2) \ldots \exp(b_k)$$

with $b_1, \ldots, b_k \in A$. This element is connected to the identity in A^{-1} by the arc

$$t \to \exp(tb_1) \cdot \exp(tb_2) \ldots \exp(tb_k) : [0,1] \to A^{-1}.$$

Hence, E is connected. Since E disconnects A^{-1} and is connected itself, it is the connected component of the identity in A^{-1}. This proves (v).

Let $\Phi : X \to Y$ be a surjective bounded linear map between two Banach spaces. By the open mapping theorem ([**R**]), Φ is an open map. This implies that there is a constant K with the property that, for each $y \in Y$, there is an $x \in X$ such that $\Phi x = y$ and $||x|| \leq K||y||$. In this case, K will be called an *inversion constant* for Φ.

10.3.2 Lemma. *Let $\Phi : X \to Y$ be a surjective bounded linear map between Banach spaces. Then there is a $\delta > 0$ and a $K > 0$ such that Ψ is surjective with inversion constant K, whenever $\Psi : X \to Y$ is a bounded linear map with $||\Phi - \Psi|| < \delta$.*

Proof. Let K_0 be an inversion constant for Φ and choose $\delta = (2K_0)^{-1}$. For a bounded linear map $\Psi : X \to Y$ with $||\Phi - \Psi|| < \delta$ and a $y \in Y$, we seek an x such that $\Psi x = y$. We choose u_0 such that $\Phi u_0 = y$, and $||u_0|| \le K_0||y||$. We then choose inductively a sequence $\{u_n\}$ of elements of X such that

$$\Phi u_n = (\Phi - \Psi)u_{n-1}, \quad ||u_n|| \le 2^{-n}K_0||y||.$$

The choice of δ clearly makes this possible. If $x_n = u_0 + u_1 + \cdots + u_n$, then the sequence x_n converges in X, and

$$\begin{aligned} y - \Psi x_n &= \Phi u_0 - \Psi x_n = \Phi u_1 - \Psi(x_n - u_0) \\ &= \Phi u_2 - \Psi(x_n - u_0 - u_1) = \cdots = \Phi u_{n+1}. \end{aligned}$$

This implies that $\Psi x = y$ if $x = \lim x_n$. It also follows from our estimate on the norms of the u_n that $||x|| \le 2K_0||y||$. This establishes the lemma with $K = 2K_0$.

If f is a function from an open subset U of a Banach space X to a Banach space Y, then the *differential* (if it exists) of f at $x \in U$ is a bounded linear map $L : X \to Y$ with the property that

$$\lim_{u \to 0} ||u||^{-1}||f(x+u) - f(x) - Lu|| = 0.$$

If the differential of f at x exists, we will denote it by $df(x)$. It then can be computed as follows:

$$df(x)u = \frac{d}{dt}f(x+tu)|_{t=0} = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t},$$

where the limit is in the norm in the Banach space Y.

The next result is a version of the open mapping theorem for non-linear maps between Banach spaces.

10.3.3 Theorem. *Let X and Y be Banach spaces, and let $f : U \to Y$ be a (non-linear) function from a neighborhood of 0 in X into Y. If $df(x)$ exists at each $x \in U$, is a continuous operator valued function of x, and is surjective at $x = 0$, then the image of f contains a neighborhood of $f(0)$.*

Proof. By the previous lemma and the continuity of df, we may assume that U is small enough that there is a $K > 0$ such that $df(x)$ is surjective with inversion constant K for all $x \in U$. We may also assume that U is convex, so that if x and $x+u$ lie in U, then so does the line segment joining them. It follows from the fundamental theorem of calculus (Exercise 10.5) and the chain rule (Exercise 10.4) that

$$f(x+u) - f(x) = \int_0^1 \frac{d}{dt}f(x+tu)\,dt = \int_0^1 df(x+tu)u\,dt.$$

Thus,

$$f(x+u) - f(x) - df(x)u = \int_0^1 [df(x+tu) - df(x)]u\,dt. \tag{10.3.1}$$

By shrinking U if necessary and using the continuity of df again, we may assume that $||df(x+tu) - df(x)|| \le (2K)^{-1}$ when $x \in U$ and $x+u \in U$. This and (10.3.1) imply that

$$||f(x+u) - f(x) - df(x)u|| \le (2K)^{-1}||u|| \tag{10.3.2}$$

for $x, x+u \in U$.

The remainder of the proof is just an application of Newton's method. We choose $\delta > 0$ so that $||u|| < 2\delta$ implies that $u \in U$. We will show that, whenever $||y - f(0)|| < K^{-1}\delta$, we can solve the equation $f(x) = y$. We proceed as in Newton's method, using $x_0 = 0$ as our initial guess. We then choose $x_1 \in X$ so that $df(0)x_1 = y - f(0)$, and $||x_1|| \le K||y - f(0)||$. Note that $||x_1|| < \delta$. We then inductively choose x_n so that $x_n = x_{n-1} + u_n$, where

$$df(x_{n-1})u_n = y - f(x_{n-1}),$$

and

$$||u_n|| \le K||y - f(x_{n-1})||.$$

Then by (10.3.2), we have

$$\begin{aligned} ||y - f(x_n)|| &= ||y - f(x_{n-1}) - (f(x_n) - f(x_{n-1}))|| \\ &= ||df(x_{n-1})u_n - (f(x_{n-1} + u_n) - f(x_{n-1}))|| \\ &< (2K)^{-1}||u_n||. \end{aligned}$$

This implies

$$||u_{n+1}|| < 2^{-1}||u_n||.$$

Hence, $||u_1|| < \delta$ implies that $||u_2|| < 2^{-1}\delta$, and that $||u_n|| < 2^{-n+1}\delta$ in general. It follows that the sequence $\{x_n\}$ is contained in U and converges to an element $x \in U$. The estimate above on $||y - f(x_n)||$ shows that $f(x) = y$. This completes the proof.

10.4 Cartan's Factorization Lemma

Roughly speaking, Theorem 10.3.3 says that a certain non-linear problem has a solution if the linearized version has a solution. In our application of this result, the solvability of the linearized problem is given by the next lemma. In this lemma and in what follows we will use the following geometric setup.

By an open (compact) *box* in $\mathbb{C}^n$ we will mean an open (compact) set U which is the Cartesian product of intervals – one from each of the $2n$ real and imaginary coordinate axes. An *aligned pair* of open (compact) boxes will be a pair (U_1, U_2) of boxes determined by two ordered sets of intervals which are identical except in one (real or imaginary) coordinate, and in that coordinate the two intervals are overlapping. The coordinate in which the defining intervals are allowed to be different will be called the *exceptional coordinate.* It is clear from the definition that if (U_1, U_2) is an aligned pair of boxes, then $U_1 \cap U_2$ and $U_1 \cup U_2$ are also boxes, and they are obtained from U_1 and U_2 by taking intersection or union of the two defining intervals in the exceptional coordinate, and leaving the defining intervals in all other coordinates the same.

10.4.1 Lemma. *Let (U_1, U_2) be an aligned pair of open boxes in $\mathbb{C}^n$. Then each bounded holomorphic function f on $U_1 \cap U_2$ is the difference $f_1 - f_2$ of a bounded holomorphic function f_1 on U_1 and a bounded holomorphic function f_2 on U_2.*

Proof. Without loss of generality, we may assume that the exceptional coordinate for (U_1, U_2) is x_1, so that the pair (U_1, U_2) has the form

$$U_1 = I_1 \times iK \times W,$$
$$U_2 = I_2 \times iK \times W,$$

where I_1 and I_2 are overlapping open intervals on the line, K is an open interval on the line, and W is an open box in $\mathbb{C}^{n-1}$.

We choose a bounded C^∞ function ϕ on $\mathbb{R}$ that is 1 in a neighborhood of $I_1 - I_1 \cap I_2$ and is 0 in a neighborhood of $I_2 - I_1 \cap I_2$. We then consider ϕ to be a function defined on $\mathbb{C}^n$ which is constant in all variables except x_1. Then $(1-\phi)f$ extends by zero to be a bounded C^∞ function g_1 in U_1, while $-\phi f$ extends by zero to be a bounded C^∞ function g_2 in U_2. Furthermore, on $U_1 \cap U_2$, $g_1 - g_2 = f$. In other words, we have solved our problem in the class of bounded functions which are C^∞ in the variable z_1 and holomorphic in the remaining variables. Now we need to modify this solution to arrive at one which is holomorphic in z_1 as well.

The fact that $g_1 - g_2 = f$ is holomorphic on $U_1 \cap U_2$ implies that

$$\frac{\partial g_1}{\partial \bar{z}_1} = \frac{\partial g_2}{\partial \bar{z}_1}$$

on $U_1 \cap U_2$, and this implies that these functions define a bounded $\mathbb{C}^\infty$ function q on $U_1 \cup U_2$, which is $\dfrac{\partial g_i}{\partial \bar{z}_1}$ on U_i. Let V be an open set with compact closure in $(I_1 \cup I_2) \times K$, and let λ_1 be a C^∞ function of z_1 which is 1 on V and which has compact support in $(I_1 \cup I_2) \times K$. Set $\lambda_2 = 1 - \lambda_1$. We then proceed as in the proof of Proposition 1.4.2 (the solvability of the inhomogeneous Cauchy-Riemann equation). We set $D = U_1 \cup U_2$ and

$$h_i(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_D \frac{\lambda_i(\zeta_1) q(\zeta_1, z_2, \ldots, z_n)\, d\zeta_1 \wedge d\bar{\zeta}_1}{\zeta_1 - z_1}$$

and let $h = h_1 + h_2$. As in the proof of Proposition 1.4.2, $(\zeta_1 - z_1)^{-1}$ is integrable on any bounded subset of $\mathbb{C}$, so that the functions h_i are bounded C^∞ functions on $U_1 \cup U_2$. They are also holomorphic in $(z_2, \ldots, z_n)$.

Since λ_1 is compactly supported in $(I_1 \cup I_2) \times K$, so that the line integral term in the generalized Cauchy theorem (Theorem 1.4.1) can be made to vanish, we conclude, as in the proof of Proposition 1.4.2, that $\dfrac{\partial h_1}{\partial \bar{z}_1} = \lambda_1 q$ in $U_1 \cup U_2$. This means that $\dfrac{\partial h_1}{\partial \bar{z}_1} = q$ on $V \times W$. On the other hand, because λ_2 vanishes on V, the function h_2 is holomorphic on $V \times W$. Thus, $\dfrac{\partial h}{\partial \bar{z}_1} = q$ in $V \times W$. However, h is independent of the choice of V and λ and, hence, $\dfrac{\partial h}{\partial \bar{z}_1} = q$ in all of $U_1 \cup U_2$.

We now have a bounded C^∞ function h on $U_1 \cup U_2$ which is holomorphic in $(z_2, \ldots, z_n)$ and which satisfies

$$\frac{\partial h}{\partial \bar{z}_1} = \frac{\partial g_i}{\partial \bar{z}_1}, \quad \text{in} \quad U_i.$$

Then $f_i = g_i - h$ is bounded and holomorphic in U_i and $f = f_1 - f_2$ on the intersection $U_1 \cap U_2$. This completes the proof.

In what follows, $Gl_n(\mathbb{C})$ will denote the group of invertible $n \times n$ complex matrices. This is the group of invertible elements of the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices. The latter is a Banach algebra under the standard matrix norm

$$||a|| = \sup\{||ax|| : x \in \mathbb{C}^n,\ ||x|| \le 1\},$$

and so, with the topology determined by this norm, $Gl_n(\mathbb{C})$ is a topological group. If U is a domain in $\mathbb{C}^n$, we will also be concerned with the Banach algebra $\mathcal{H}_b(U, M_n)$ of bounded M_n-valued holomorphic functions on U. Here the norm is given by $||f|| = \sup\{||f(x)|| : x \in U\}$, where $||f(x)||$ is the matrix norm of the matrix $f(x) \in M_n(\mathbb{C})$. The invertible group of this Banach algebra is the group of holomorphic $Gl_n(\mathbb{C})$-valued functions which are bounded and have bounded inverse.

10.4.2 Proposition. *If $U \subset \mathbb{C}^n$ is the Cartesian product of simply connected open subsets of $\mathbb{C}$, and K is a compact subset of U, then each holomorphic mapping $f : U \to Gl_n(\mathbb{C})$ may be uniformly approximated on K by holomorphic mappings from $\mathbb{C}^n$ to $Gl_n(\mathbb{C})$.*

Proof. By the Riemann mapping theorem, we may, without loss of generality, assume that U is a polydisc centered at the origin. We let V be an open polydisc centered at the origin, containing K, and with compact closure in U. Then on V, f and f^{-1} are bounded holomorphic functions with values in $Gl_n(\mathbb{C})$ – that is, f is an element of the invertible group A^{-1} of the Banach algebra $A = \mathcal{H}_b(V, M_n(\mathbb{C}))$.

We may construct a curve in A^{-1}, joining f to the constant matrix $f(0)$, by setting $f_t(z) = f(tz)$ for each $t \in [0,1]$. Since $Gl_n(\mathbb{C})$ itself is connected (Exercise 10.3), this proves that A^{-1} is connected. By Theorem 10.3.1(v), this implies that f is a product of elements in the range of the exponential function. Hence, on V,

$$f = \exp(g_1)\exp(g_2)\dots\exp(g_k) \quad \text{with} \quad g_1, g_2, \dots, g_k \in A.$$

Now each g_i may be regarded as a matrix with entries which are bounded holomorphic functions on V. By truncating the power series of each entry of each g_i, we may approximate each g_i by matrices h_i with polynomial entries as closely as we like in the uniform topology on K. Then

$$\tilde{f} = \exp(h_1)\exp(h_2)\dots\exp(h_k)$$

will be a holomorphic $Gl_n(\mathbb{C})$-valued function on all of $\mathbb{C}^n$. Clearly, the h_i can be chosen so that $\tilde{f}$ approximates f arbitrarily closely on K.

The next lemma is the key to the vanishing theorem we are seeking:

10.4.3 Cartan's Factorization Lemma. *Let (K_1, K_2) be an aligned pair of compact boxes in $\mathbb{C}^n$. Then each holomorphic $Gl_n(\mathbb{C})$-valued function f, defined in a neighborhood of $K_1 \cap K_2$, may be factored as $f = f_2^{-1} f_1$, where f_i is a holomorphic $Gl_n(\mathbb{C})$-valued function in a neighborhood of K_i, for $i = 1, 2$.*

Proof. We may construct an aligned pair (U_1, U_2) of open boxes such that $K_i \subset U_i$, and f and f^{-1} are holomorphic and bounded in $U_1 \cap U_2$. Let A_i be the Banach algebra $\mathcal{H}_b(U_i, M_n(\mathbb{C}))$ for $i = 1, 2$, and let B be the Banach algebra $\mathcal{H}_b(U_1 \cap U_2, M_n(\mathbb{C}))$. Consider the non-linear map ϕ, from a neighborhood of 0 in $A_1 \oplus A_2$ to B, defined by

$$\phi(g_1 \oplus g_2) = \log((\exp g_2)^{-1} \exp g_1).$$

Now it is easy to see that a function from a Banach algebra to itself which is defined by a convergent power series is infinitely differentiable. It is also easy to see that the analogues of the chain rule (Exercise 10.4) and the product rule hold for functions between Banach algebras. It follows that ϕ is infinitely differentiable in a neighborhood of 0. We have that $\phi(0) = 0$, and a simple calculation (Exercise 10.6) shows that

$$d\phi(0)(h_1 \oplus h_2) = h_1 - h_2.$$

By Lemma 10.4.1, $d\phi(0)$ is surjective. By Theorem 10.3.3, the image of ϕ contains a neighborhood of 0 in B. After composing ϕ with exp, we conclude that the map

$$g_1 \oplus g_2 \to (\exp g_2)^{-1} \exp g_1$$

has image which contains a neighborhood of the identity in B. This means that the theorem is true for f sufficiently close to the identity in B. However, by the previous theorem, we may approximate f arbitrarily closely on $U_1 \cap U_2$ by a $Gl_n(\mathbb{C})$-valued function h which is holomorphic on all of $\mathbb{C}^n$. Thus, for an appropriate choice of h we will have fh^{-1} sufficiently close to the identity in B that we may write $fh^{-1} = g_2^{-1}g_1$ with g_i a holomorphic $Gl_n(\mathbb{C})$-valued function on U_i. Then the desired solution is $f = f_2^{-1}f_1$, with $f_1 = g_1 h$ and $f_2 = g_2$.

10.5 Amalgamation of Syzygies

We next use Cartan's factorization lemma to piece together two syzygies defined over an aligned pair of boxes.

10.5.1 Proposition. *Let (K, L) be an aligned pair of compact boxes in $\mathbb{C}^n$. Let $\mathcal{S}$ be a coherent analytic sheaf defined in a neighborhood of $K \cup L$, and suppose that there are exact sequences of analytic sheaves*

$$0 \longrightarrow \mathcal{H}^{p_m} \xrightarrow{\alpha_m} \dots \xrightarrow{\alpha_2} \mathcal{H}^{p_1} \xrightarrow{\alpha_1} \mathcal{H}^{p_0} \xrightarrow{\alpha_0} \mathcal{S} \longrightarrow 0$$

over a neighborhood of K and

$$0 \longrightarrow \mathcal{H}^{q_m} \xrightarrow{\beta_m} \dots \xrightarrow{\beta_2} \mathcal{H}^{q_1} \xrightarrow{\beta_1} \mathcal{H}^{q_0} \xrightarrow{\beta_0} \mathcal{S} \longrightarrow 0$$

over a neighborhood of L. Then there exists an exact sequence of analytic sheaves

$$0 \longrightarrow \mathcal{H}^{r_m} \xrightarrow{\gamma_m} \dots \xrightarrow{\gamma_2} \mathcal{H}^{r_1} \xrightarrow{\gamma_1} \mathcal{H}^{r_0} \xrightarrow{\gamma_0} \mathcal{S} \longrightarrow 0$$

defined in a neighborhood of $K \cup L$.

Proof. The proof is by induction on the length m of the syzygies. If $m = 0$, then we have a pair of isomorphisms:

$$0 \longrightarrow \mathcal{H}^{p_0} \xrightarrow{\alpha_0} \mathcal{S} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{H}^{q_0} \xrightarrow{\beta_0} \mathcal{S} \longrightarrow 0,$$

with the first defined over a neighborhood of K and the second over a neighborhood of L. The composition $\phi = \beta_0^{-1} \circ \alpha_0 : \mathcal{H}^{p_0} \to \mathcal{H}^{q_0}$ is an isomorphism defined over a neighborhood of $K \cap L$. This implies that p_0 and q_0 are the same integer k, and that ϕ is determined by a holomorphic $Gl_k(\mathbb{C})$-valued function in a neighborhood of $K \cap L$. By Cartan's factorization lemma, this function may be factored as $\phi = \mu^{-1} \circ \lambda$ where λ (resp. μ) is a holomorphic $Gl_k(\mathbb{C})$-valued function in a neighborhood of K (resp. L). Then $\beta_0 \circ \mu^{-1} = \alpha_0 \circ \lambda^{-1}$ in a neighborhood of $K \cap L$ and so these two morphisms fit together to define an isomorphism

$$0 \longrightarrow \mathcal{H}^{k} \xrightarrow{\gamma_0} \mathcal{S} \longrightarrow 0$$

over a neighborhood of $K \cup L$, as required.

For the induction step, we assume that the theorem is true of all pairs of sequences, as above, of length less than m and we suppose we are given a pair of length m. By applying the Riemann mapping theorem in each variable we see that $K \cap L$ has arbitrarily small neighborhoods U which are biholomorphically equivalent to open polydiscs. On a sufficiently small such neighborhood, Lemma 10.2.1 implies that the sequences

$$\dots \xrightarrow{\alpha_2} \mathcal{H}^{p_1}(U) \xrightarrow{\alpha_1} \mathcal{H}^{p_0}(U) \xrightarrow{\alpha_0} \mathcal{S}(U) \longrightarrow 0,$$

$$\dots \xrightarrow{\beta_2} \mathcal{H}^{q_1}(U) \xrightarrow{\beta_1} \mathcal{H}^{q_0}(U) \xrightarrow{\beta_0} \mathcal{S}(U) \longrightarrow 0$$

are exact. We can use the fact that $\mathcal{H}^{p_0}(U)$ is a free $\mathcal{H}(U)$-module to construct the morphism λ which makes commutative the diagram

$$\begin{array}{ccccc} \mathcal{H}^{p_0}(U) & \xrightarrow{\alpha_0} & \mathcal{S}(U) & \longrightarrow & 0 \\ {\scriptstyle\lambda}\downarrow & & \| & & \\ \mathcal{H}^{q_0}(U) & \xrightarrow{\beta_0} & \mathcal{S}(U) & \longrightarrow & 0. \end{array}$$

Now λ is a matrix with entries which are holomorphic functions on U and, as such, it defines a morphism of analytic sheaves $\lambda : \mathcal{H}^{p_0} \to \mathcal{H}^{q_0}$ over U

such that $\beta_0 \circ \lambda = \alpha_0$. A similar argument shows that we may construct a morphism of analytic sheaves $\mu : \mathcal{H}^{q_0} \to \mathcal{H}^{p_0}$ over U with $\alpha_0 \circ \mu = \beta_0$.

We next modify each of the sequences so that the free modules that appear in degree 0 will be identical. Thus, the first sequence is modified by taking its direct sum with the exact sequence

$$0 \longrightarrow \mathcal{H}^{q_0} \xrightarrow{\mathrm{id}} \mathcal{H}^{q_0} \longrightarrow 0$$

to obtain

$$\ldots \longrightarrow \mathcal{H}^{p_1} \oplus \mathcal{H}^{q_0} \xrightarrow{\tilde{\alpha}_1} \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0} \xrightarrow{\tilde{\alpha}_0} \mathcal{S} \longrightarrow 0$$

where $\tilde{\alpha}_1 = \alpha_1 \oplus \mathrm{id}$ and $\tilde{\alpha}_0(f \oplus g) = \alpha_0(f)$. Similarly, we modify the second sequence by taking its direct sum (on the other side) with the exact sequence

$$0 \longrightarrow \mathcal{H}^{p_0} \xrightarrow{\mathrm{id}} \mathcal{H}^{p_0} \longrightarrow 0$$

to obtain

$$\ldots \longrightarrow \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_1} \xrightarrow{\tilde{\beta}_1} \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0} \xrightarrow{\tilde{\beta}_0} \mathcal{S} \longrightarrow 0$$

where $\tilde{\beta}_1 = \mathrm{id} \oplus \beta_1$ and $\tilde{\beta}_0(f \oplus g) = \beta_0(g)$.

We define two endomorphisms ϕ and ψ of $\mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0}$ on U to be the maps which have the following matrix representations relative to the direct sum decomposition:

$$\phi = \begin{bmatrix} 1 & -\mu \\ \lambda & 1 - \lambda\mu \end{bmatrix}, \quad \psi = \begin{bmatrix} 1 - \mu\lambda & \mu \\ -\lambda & 1 \end{bmatrix}.$$

A calculation shows that $\tilde{\beta}_0 \circ \phi = \tilde{\alpha}_0$, $\tilde{\alpha}_0 \circ \psi = \tilde{\beta}_0$, and $\psi = \phi^{-1}$. Thus, ϕ is an isomorphism, and we have the following commutative diagram over U

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{H}^{p_1} \oplus \mathcal{H}^{q_0} \arrow[r, "\tilde{\alpha}_1"] & \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0} \arrow[r, "\tilde{\alpha}_0"] \arrow[d, "\phi"'] & \mathcal{S} \arrow[r] \arrow[d, equal] & 0 \\
\ldots \arrow[r] & \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_1} \arrow[r, "\tilde{\beta}_1"] & \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0} \arrow[r, "\tilde{\beta}_0"] & \mathcal{S} \arrow[r] & 0
\end{tikzcd}$$

By Cartan's lemma, ϕ can be factored over some neighborhood of $K \cap L$ as $\delta \circ \theta^{-1}$, where θ is an automorphism of $\mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0}$ over some neighborhood of K, and δ is an automorphism of $\mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0}$ over some neighborhood of L. Then in a neighborhood of $K \cap L$ we have

$$\tilde{\beta}_0 \circ \delta = \tilde{\alpha}_0 \circ \theta,$$

which means that there is a single morphism

$$\gamma_0 : \mathcal{H}^{p_0} \oplus \mathcal{H}^{q_0} \to \mathcal{S},$$

over a neighborhood of $K \cup L$, such that $\gamma_0 = \tilde{\alpha}_0 \circ \theta$ on a neighborhood of K and $\gamma_0 = \tilde{\beta}_0 \circ \delta$ on a neighborhood of L.

If $\mathcal{K}$ is the kernel of γ_0, then we have exact sequences

$$0 \longrightarrow \mathcal{H}^{p_m} \xrightarrow{\alpha_m} \dots \xrightarrow{\alpha_2} \mathcal{H}^{p_1'} \xrightarrow{\alpha_1'} \mathcal{K} \longrightarrow 0$$

over a neighborhood of K and

$$0 \longrightarrow \mathcal{H}^{q_m} \xrightarrow{\beta_m} \dots \xrightarrow{\beta_2} \mathcal{H}^{q_1'} \xrightarrow{\beta_1'} \mathcal{K} \longrightarrow 0$$

over a neighborhood of L, where $p_1' = p_1 + q_0$, $q_1' = q_1 + p_0$, $\alpha_1' = \theta^{-1} \circ \alpha_1$, and $\beta_1' = \delta^{-1} \circ \alpha$. Since these sequences have length $m - 1$, the induction hypothesis implies that there exists an exact sequence

$$0 \longrightarrow \mathcal{H}^{r_m} \xrightarrow{\gamma_m} \dots \xrightarrow{\gamma_2} \mathcal{H}^{r_1} \xrightarrow{\gamma_1} \mathcal{K} \longrightarrow 0$$

in a neighborhood of $K \cup L$. Combining this with the morphism γ_0 gives us the required sequence for $\mathcal{S}$ over a neighborhood of $K \cup L$. This completes the proof of the proposition.

10.5.2 Proposition. *Let U be an open polydisc in $\mathbb{C}^n$, and $\mathcal{S}$ a coherent analytic sheaf on U. For any compact subset $K \subset U$, there is an exact sequence of the form*

$$0 \longrightarrow \mathcal{H}^{p_n} \longrightarrow \dots \longrightarrow \mathcal{H}^{p_1} \longrightarrow \mathcal{H}^{p_0} \longrightarrow \mathcal{S} \longrightarrow 0,$$

defined in some neighborhood of K. In other words, $\mathcal{S}$ has a free finite rank resolution of length n in a neighborhood of each compact subset of U.

Proof. By applying the Riemann mapping theorem in each coordinate, we may reduce the problem to one in which U is an open box in $\mathbb{C}^n$. Also, without loss of generality, we may assume that K is a compact box contained in U. By Proposition 10.2.2, we know that $\mathcal{S}$ has a resolution, like the one above, in a neighborhood of each point of U. If K is a Cartesian product $I_1 \times \cdots \times I_{2n}$ of closed intervals, we partition each of these intervals into m subintervals of equal length. This results in a partition of K into m^n compact boxes. By choosing m large enough, we may assume that $\mathcal{S}$ has a free finite rank resolution of length n in a neighborhood of each of these boxes. Fix a row of these boxes in which only the coordinate x_1 changes. Clearly, adjacent pairs of boxes in this row are aligned pairs. In fact, the pair consisting of the kth box in the row and the union of the first $k-1$ boxes is an aligned pair of boxes. Hence, if we have constructed a free finite rank resolution of length n in a neighborhood of the union of the first $k-1$ boxes in the row, then we may use Proposition 10.5.1 to construct such a resolution in a neighborhood of the union of the first k boxes. Thus, by induction, we can construct a free finite rank resolution of length n for $\mathcal{S}$ in a neighborhood of the union of the boxes in any such row. These unions provide a new partition of K into m^{n-1} compact boxes in which adjacent boxes are aligned. Repeating this argument one variable at a time, we eventually end up with a free finite rank resolution of $\mathcal{S}$ in a neighborhood of K, as required.

10.5.3 Theorem. *Let U be an open set in $\mathbb{C}^n$ and let $\mathcal{S}$ be a coherent analytic sheaf on U. Then $H^p(\Delta, \mathcal{S}) = 0$ for $p > 0$ and for any open polydisc Δ with compact closure in U.*

Proof. This follows directly from Lemma 10.2.1 and Proposition 10.5.2.

Theorem 10.5.3 has the following corollaries, which will be important tools in the next chapter.

10.5.4 Corollary. *Suppose Y is a subvariety of an open polydisc U, and Δ is an open polydisc with compact closure in U. Then every holomorphic function on $\Delta \cap Y$ is the restriction of a holomorphic function on Δ.*

Proof. On U we have an exact sequence of coherent analytic sheaves:

$$0 \to \mathcal{I}_Y \to {}_U\mathcal{H} \to {}_Y\mathcal{H} \to 0.$$

By Theorem 10.5.3, the sheaf $\mathcal{I}_Y$ has vanishing pth cohomology on Δ for each $p > 0$. Thus, the long exact sequence of cohomology implies that the morphism $\mathcal{H}(\Delta) \to \mathcal{H}(\Delta \cap Y)$ is surjective.

10.5.5 Corollary. *If X is an analytic variety, then X has a neighborhood base for its topology consisting of sets W, with compact closure, and with the properties that*

(i) *W may be embedded as a subvariety of a polydisc $\Delta \subset \mathbb{C}^n$ in such a way that the restriction map $\mathcal{H}(\Delta) \to \mathcal{H}(W)$ is surjective;*
(ii) *each coherent sheaf on a neighborhood of $\overline{W}$ is acyclic on W.*

The proof of Corollary 10.5.5 is left as an exercise (Exercise 10.19).

10.6 Stein Spaces

We now have a vanishing theorem for coherent sheaves on polydiscs. Our goal is to prove a vanishing theorem (Cartan's Theorem B) for coherent sheaves on a very general class of holomorphic varieties – the class of Stein spaces. Once we define Stein spaces, our strategy for proving Cartan's Theorem B is roughly as follows: We next prove a vanishing theorem for coherent sheaves on domains which can be realized as subvarieties of polydiscs (Oka-Weil domains). We then show that Stein spaces can be approximated in a strong sense by Oka-Weil domains. Finally, we use an approximation argument, like the one used in the proof of Dolbeault's lemma, to prove our vanishing theorem on Stein spaces. This last step is carried out in the next chapter.

10.6.1 Definition. *Let X be a holomorphic variety.*

(i) *If K is a compact subset of X, then the holomorphically convex hull of K in X is the set*
$$\widehat{K} = \{x \in X : |f(x)| \le \sup_{y \in K} |f(y)|, \ \forall\, f \in \mathcal{H}(X)\};$$
(ii) *a compact subset K of X is said to be holomorphically convex in X if $\widehat{K} = K$;*
(iii) *X is said to be holomorphically convex if $\widehat{K}$ is compact for every compact subset $K \subset X$.*

Note that $\widehat{K}$ is always a closed subset of X, and so it will be compact if it is contained in a compact subset of X. Note also that if X is a holomorphic variety, U an open subset of X, and K a compact subset of U, then it makes sense to talk about the holomorphically convex hull of K in X and in U. These are not necessarily the same. The open set U is said to be holomorphically convex if the holomorphically convex hull of K in U is compact for every compact subset K of U – that is, if U is holomorphically convex as a holomorphic variety in its own right. It is easy to see that the intersection of any finite set of holomorphically convex open subsets of a holomorphic variety is also holomorphically convex (Exercise 10.7).

10.6.2 Definition. *A holomorphic variety X is said to be a Stein space if*

(i) *X is holomorphically convex;*
(ii) *for each $x \in X$, the maximal ideal of $\mathcal{H}_x$ is generated by a set of global sections of $\mathcal{H}$;*
(iii) *the global sections of $\mathcal{H}$ separate points in X.*

Note that conditions (ii) and (iii) are automatically inherited by open subsets, and so an open subset of a Stein space is also a Stein space, provided it is holomorphically convex. In particular, $\mathbb{C}^n$ is clearly a Stein space

(Exercise 10.8) and so each of its holomorphically convex open subsets is also a Stein space. It follows from Theorem 2.5.7 that an open subset of $\mathbb{C}^n$ is a Stein space if and only if it is a domain of holomorphy. It is also easy to see that a subvariety of a Stein space is a Stein space (Exercise 10.9).

Actually condition (ii) of the above definition is redundant. That is, a holomorphic variety X is a Stein space if and only if it is holomorphically convex and has enough global holomorphic functions to separate points (Exercise 11.13). However, to prove this requires several results from the next chapter.

10.6.3 Definition. *An open subset W of a holomorphic variety X is said to be an Oka-Weil subdomain of X if the following conditions are satisfied:*

(i) *W has compact closure $\overline{W}$ in X;*

(ii) *there is a holomorphic map $\phi : X \to \mathbb{C}^n$ which maps a neighborhood W_1 of $\overline{W}$ biholomorphically onto a subvariety of a neighborhood of a closed polydisc $\overline{\Delta}(0,r)$ in $\mathbb{C}^n$;*

(iii) *the image of W under ϕ is a subvariety of $\Delta(0,r)$.*

Thus, an Oka-Weil subdomain $W \subset X$ is biholomorphically equivalent to a subvariety of an open polydisc. However, we also require that the map which embeds W in the polydisc extends to a globally defined holomorphic map of X into $\mathbb{C}^n$, which also embeds a neighborhood of $\overline{W}$ as a subvariety of a polydisc. We will prove a vanishing theorem for coherent sheaves on Oka-Weil domains and then attack the desired vanishing theorem on Stein manifolds by writing a Stein manifold as an increasing union of Oka-Weil domains.

10.6.4 Theorem. *If W is an Oka-Weil subdomain of a holomorphic variety X, then every coherent sheaf defined in a neighborhood of the closure $\overline{W}$ of W is acyclic on W.*

Proof. By Definition 10.6.3, there is a neighborhood W_1 of $\overline{W}$ which may be identified with a subvariety of an open set $U_1 \subset \mathbb{C}^n$ in such a way that $\overline{\Delta}(0,r) \subset U_1$ and $W = W_1 \cap \Delta(0,r)$. If a coherent sheaf $\mathcal{S}$ is defined in a neighborhood W_2 of $\overline{W}$, then we may assume that $W_2 \subset W_1$. There is an open set $U_2 \subset U$, containing $\overline{\Delta}(0,r)$, such that $W_2 = W_1 \cap U_2$. If $i : W_2 \to U_2$ is the inclusion, then $i_*\mathcal{S}$ is a coherent analytic sheaf on U_2 (Exercise 9.10). It follows from Theorem 10.5.3 that such a sheaf is acyclic on $\Delta(0,r)$. By Proposition 7.4.5, $\mathcal{S}$ is acyclic on W. This completes the proof.

That a Stein space may be strongly approximated by Oka-Weil subdomains is the content of the next theorem.

10.6.5 Theorem. *If X is a Stein space, K is a compact holomorphically convex subset of X, and U is an open subset of X, containing K, then there exists an Oka-Weil subdomain W of X such that $K \subset W \subset \overline{W} \subset U$.*

Proof. We may as well assume that U has compact closure $\overline{U}$. If $x \in \partial U$, we may choose a function $f \in \mathcal{H}(X)$ such that $|f(x)| > 1$ and $||f||_K < 1$, where $||f||_K$ denotes the supremum norm of f on K. Since the inequality $|f(y)| > 1$ will also hold for y in some neighborhood of x, and since ∂U is compact, we may choose a finite set of functions $\{f_i\}_{i=1}^k$ in $\mathcal{H}(X)$ so that

$$K \subset W_1 = \{x \in \overline{U} : |f_i(x)| < 1, \quad i = 1, \dots k\}$$

and

$$\overline{W}_1 \subset \{x \in \overline{U} : |f_i(x)| \leq 1, \quad i = 1, \dots k\} \subset U.$$

Then the functions $\{f_i\}_{i=1}^k$ are the coordinate functions of a holomorphic map $\phi_0 : X \to \mathbb{C}^k$ which maps W_1 into $\Delta(0,1)$. In fact, ϕ_0 is a proper holomorphic map of W_1 into $\Delta(0,1)$, since the inverse image in W_1 of any compact subset of $\Delta(0,1)$ will be closed not only in W_1, but also in the compact set $\overline{U}$.

Now, by (ii) of Definition 10.6.2, for each point x of $\overline{W}_1$ there is a finite set of global sections of $\mathcal{H}$ which vanish at x and generate the maximal ideal of $\mathcal{H}_x$. Without loss of generality, we may assume that these functions all have modulus less than 1 at each point of W_1. By Theorem 5.5.1 and Proposition 5.5.2, the map $X \to \mathbb{C}^m$, with these functions as coordinate functions, is a biholomorphic map of some neighborhood of x onto a subvariety of some neighborhood of $f(x)$. Since $\overline{W}_1$ is compact, finitely many such neighborhoods will cover $\overline{W}_1$. By adjoining all the corresponding functions to the set $\{f_i\}_{i=1}^k$, we obtain a set $\{f_i\}_{i=1}^q$ of functions defining a holomorphic map $\phi_1 : X \to \mathbb{C}^q$, which is proper on W_1, and also has the property that, in a neighborhood of each point $x \in \overline{W}_1$, it is a biholomorphic map onto a subvariety of a neighborhood of $f(x)$.

In particular, ϕ_1 is locally one to one. This means that the diagonal in $\overline{W}_1 \times \overline{W}_1$ has a neighborhood V in which $\phi_1(x) - \phi_1(y)$ is non-vanishing except on the diagonal itself. Another compactness argument, along with (iii) of Definition 10.6.2, shows that we can find another finite set of global sections of $\mathcal{H}$ (again with modulus less than 1 at points of W_1) such that, whenever $(x, y) \in \overline{W}_1 \times \overline{W}_1 - V$, there is some function f in this set such that $f(x) \neq f(y)$. By adjoining this finite set to the set $\{f_i\}_{i=1}^q$, we obtain an ordered set $\{f_i\}_{i=1}^m$ of functions defining a holomorphic map $\phi : X \to \mathbb{C}^m$ which is proper and injective on $\overline{W}_1$, and locally maps neighborhoods in W_1 onto subvarieties of neighborhoods in $\Delta(0,1)$.

A proper, injective, continuous map from one locally compact space into another has closed image and is a homeomorphism onto its image (Exercise

10.10). Thus, the image Y of ϕ on W_1 is a closed subset of $\Delta(0,1)$, and ϕ is a holomorphic homeomorphism onto Y. Furthermore, for each $x \in W_1$, ϕ maps some neighborhood of x biholomorphically onto an open set in Y which is a holomorphic subvariety of an open set in $\Delta(0,1)$. It follows that Y is a subvariety of $\Delta(0,1)$ and ϕ maps W_1 biholomorphically onto Y.

For each polyradius r which has all coordinates less than 1, we define $W_r = \phi^{-1}(\Delta(0,r))$. Then each W_r is obviously an Oka-Weil subdomain of X. If we choose r such that $K \subset W_r$, then $W = W_r$ is an Oka-Weil subdomain containing K and contained in U.

10.6.6 Corollary. *If X is a Stein space, then X is the union of a sequence $\{W_n\}$ of Oka-Weil subdomains such that $\overline{W}_n \subset W_{n+1}$ for each n.*

Proof. Since X is countable at infinity, it is the union of an increasing sequence $\{K_n\}$ of compact sets. Suppose we have managed to find, for $j < n$, Oka-Weil subdomains W_j such that $K_j \subset W_j$, and $\overline{W}_{j-1} \subset W_j$. Then $C_n = \overline{W}_{n-1} \cup K_n$ is compact. Since X is a Stein space, $\widehat{C}_n$ is also compact. By Theorem 10.6.5, $\widehat{C}_n$ is contained in an Oka-Weil subdomain W_n. By construction, $\overline{W}_{n-1} \subset W_n$, and $K_n \subset W_n$. Thus, the corollary follows by induction.

In view of Theorems 10.6.4 and 10.6.5, all that remains for us to be able to prove that all coherent sheaves on a Stein space X are acyclic (Cartan's Theorem B) is to express X as the union of an increasing sequence of Oka-Weil subdomains and then do an approximation argument like that used in the proof of the vanishing lemma for Dolbeault cohomology (Lemma 10.1.3). However, to carry out such a program, we need a topology on the space of sections of a coherent sheaf. Specifically, we need to establish that the space of sections of a coherent analytic sheaf has a canonical Fréchet space topology. This project is left to the next chapter.

Exercises

1. Prove parts (iii) and (iv) of Theorem 10.3.1.
2. Prove that if G is a topological group, and H is a subgroup with an interior point, then H is both open and closed in G.
3. Prove that $GL_n(\mathbb{C})$ is connected.
4. Prove that the chain rule holds for differentials of functions between Banach spaces, as defined in section 10.3. That is, prove that if X, Y, and

Z are Banach spaces, $U \subset X$ and $V \subset Y$ are open sets, $x \in X$, and $g : U \to V$ and $f : V \to Z$ are functions such that $dg(x)$ exists and $df(g(x))$ exists, then $d(f \circ g)(x) = df(g(x)) \circ dg(x)$.

5. Prove that the fundamental theorem of calculus holds for Banach space valued functions on the real line. That is, prove that
$$g(b) - g(a) = \int_a^b \frac{d}{dt} g(t)\, dt$$
if g is a function from $[a, b]$ to a Banach space X which has a continuous derivative on $[a, b]$. Here the derivative is defined as a limit of a difference quotient and the integral is defined as a limit of Riemann sums, just as in ordinary calculus, except that g is an X-valued function and so the limits must be taken in X.
6. Calculate $d\phi(0)$ for the function ϕ in the proof of Lemma 10.4.3.
7. Prove that the intersection of any finite collection of holomorphically convex open subsets of a holomorphic variety is also holomorphically convex. Then prove that a holomorphic variety has a locally finite open cover $\mathcal{W}$ with the property that any finite intersection of sets from $\mathcal{W}$ is a Stein space.
8. Prove that $\mathbb{C}^n$ is a Stein space and that an open subset U of $\mathbb{C}^n$ is a Stein space if and only if it is holomorphically convex.
9. Prove that a subvariety of a Stein space is a Stein space.
10. Prove that a proper, injective, continuous map from one locally compact space into another has closed image and is a homeomorphism onto its image.
11. If X is a Stein space, then an *analytic polyhedron* in X is a set of the form
$$\{x \in X : |f_i(x)| < r_i,\ i = 1, \ldots, k\},$$
where $f_1, \ldots, f_k \in \mathcal{H}(X)$ and $r_1, \ldots, r_k \in \mathbb{R}^+$. Prove that an analytic polyhedron in a Stein space is also a Stein space. In particular, open polydiscs are Stein spaces.
12. Prove that $Gl_n(\mathbb{C})$ and $Sl_n(\mathbb{C})$ are Stein spaces, where $Gl_n(\mathbb{C})$ is the space of invertible $n \times n$ complex matrices, considered as an open subset of $\mathbb{C}^{2n}$, and $Sl_n(\mathbb{C})$ is the subgroup of $Gl_n(\mathbb{C})$ consisting of matrices with determinant 1.
13. Let X be a Stein space and $Y \subset X$ a holomorphic subvariety. Prove that, on each open set U with compact closure in X, there exist a finite set of holomorphic functions with $Y \cap U$ as its set of common zeroes on U.
14. Let X be a Stein space and $K \subset X$ a compact holomorphically convex set. Prove that each function which is holomorphic in a neighborhood of K may be uniformly approximated on K by functions in $\mathcal{H}(X)$.

15. Let X be a holomorphic variety and let Y and Z be subvarieties of X which are Stein spaces. Prove that $X \cap Y$ is also a Stein space.
16. Prove that the Cartesian product of two Stein spaces is a Stein space.
17. Let $U \subset \mathbb{C}^n$ be an open set, Δ an open polydisc with compact closure in U, and $\mathcal{M}$ a coherent analytic sheaf on U. Prove that if the sections $g_1, \dots, g_k \in \mathcal{M}(U)$ generate $\mathcal{M}$ over $\mathcal{H}$ on U, then the restrictions of these sections to Δ also generate $\mathcal{M}(\Delta)$ over $\mathcal{H}(\Delta)$.
18. Let W be an Oka-Weil subdomain of a Stein space X and let $\mathcal{M}$ be a coherent analytic sheaf defined on a neighborhood of $\overline{W}$. Prove that, for some k, there is a surjective morphism $\mathcal{H}^k \to \mathcal{M}$ defined in a neighborhood of $\overline{W}$. Also prove that the morphism of sections $\mathcal{H}^k(W) \to \mathcal{M}(W)$ is surjective.
19. Prove Corollary 10.5.5.
20. Let U and V be open subsets of $\mathbb{C}^n$ such that V has compact closure contained in U. Set $X = U - V$. Use the exact sequence in Proposition 7.5.5(i) to prove that there is an exact sequence
$$0 \longrightarrow \mathcal{H}(U) \longrightarrow \mathcal{H}(X) \xrightarrow{\delta} H^1_c(V, \mathcal{H}).$$
Show that δ is surjective if U is a polydisc. Here, c is the family of compact subsets of V, and $\mathcal{H}(X) = \Gamma(X, \mathcal{H})$ is the space of functions holomorphic in a neighborhood of X.
21. Use Proposition 2.5.1 to prove that, if $n > 1$, then the space $H^1_c(V, \mathcal{H})$; of the previous exercise, is (0) for any open $V \subset \mathbb{C}^n$ with compact closure and with connected complement in $\mathbb{C}^n$. Use this to prove the following theorem of Hartogs: If U is any open set in $\mathbb{C}^n$ and K is any compact subset of U, with connected complement in $\mathbb{C}^n$, then every function holomorphic on $U - K$ has a unique extension to a holomorphic function on U.

Fréchet Sheaves – Cartan's Theorems

The approximation argument we will use to finally prove Cartan's Theorems A and B requires knowing that a coherent analytic sheaf has the structure of a Fréchet sheaf – that is, a sheaf with a Fréchet topological vector space structure on the space of sections over each open set with the property that the restriction maps are continuous. In this section we will show that there always is such a structure and that it is unique subject to certain conditions. We will also show that morphisms between coherent sheaves are automatically continuous for this structure. We then prove Cartan's theorems. These theorems have numerous powerful consequences and we will develop a number of them in this chapter.

We will also prove that the cohomology modules of a coherent algebraic sheaf on a compact holomorphic variety are finite dimensional. This is the Cartan-Serre theorem. It will play a key role, along with Cartan's theorems, in the proof of Serre's theorems in Chapter 13. The proof of the Cartan-Serre theorem is based on the fact that the topology on the space of sections of a coherent sheaf is actually a Montel topology, and on a theorem of Schwartz on compact perturbations of surjective morphisms between Fréchet spaces. We will give a proof of Schwarz's theorem in the last section.

We will need a number of results from topological vector space theory in this chapter. Those which we regard as elementary, to be found in any basic functional analysis text, we will simply list in section 11.1. Those which we regard as beyond the scope of a basic functional analysis course we will prove, for the sake of completeness, in the final section of the chapter.

11.1 Topological Vector Spaces

In this section, we give a brief summary, without proofs, of some introductory material from the theory of topological vector spaces. Proofs and further results can be found in [**Sc**] or [**R**].

Unless otherwise specified, vector spaces are vector spaces over the complex field $\mathbb{C}$. A *topological vector space* is a vector space with a topology under which the operations of addition and scalar multiplication are continuous. A topological vector space is *locally convex* if it has a neighborhood base at 0 consisting of convex balanced sets, where a set is balanced if it is closed under multiplication by complex scalars of modulus one. A topological vector space is locally convex if and only if its topology is given by a family $\{\rho_\alpha\}$ of seminorms, where a seminorm on a vector space X is a function ρ from X to the non-negative reals which satisfies

$$\rho(x+y) \le \rho(x) + \rho(y) \quad \text{and} \quad \rho(\lambda x) = |\lambda|\rho(x)$$

for $x, y \in X$ and $\lambda \in \mathbb{C}$. The topology determined by a family $\{\rho_\alpha\}$ of seminorms is the topology in which a basis of neighborhoods at 0 is given by the collection of finite intersections of sets of the form

$$\{x \in X : \rho_\alpha(x) < \epsilon\},$$

where ϵ is a positive number.

A continuous linear functional on a topological vector space is a complex valued function which is linear and continuous. The space of all continuous linear functionals is called the dual of X and is denoted X^*. There are a number of topologies that can be put on X^* which make it a locally convex topological vector space. They have different properties and are used in different circumstances.

A *Fréchet space* is a locally convex topological vector space F which is complete and which has its topology defined by a sequence $\{\rho_n\}$ of seminorms. Without loss of generality, the sequence $\{\rho_n\}$ may be chosen to be increasing. Equivalently, a Fréchet space is a topological vector space which is the inverse limit of a sequence of Banach spaces and bounded linear maps. Equivalently, a Fréchet space is a complete locally convex topological vector space with a topology defined by a translation invariant metric (Exercise 11.20). A bounded subset of a Fréchet space is a set B with the property that each of the defining seminorms ρ_n is bounded on B; equivalently, B is bounded if for every neighborhood U of 0, there is a positive number k such that $B \subset kU$. A Fréchet space F is called a *Montel space* if every closed bounded subset of F is compact.

The quotient of a topological vector space X by a subspace Y is the vector space quotient X/Y, with the topology in which a set is open in X/Y

if and only its inverse image in X is open. This is a Hausdorff topology if and only if Y is a closed subspace. In this case, we say that X/Y is a *separated quotient* of X.

We will assume knowledge of the following elementary facts concerning locally convex topological vector spaces. The proofs can be found in any text on functional analysis or topological vector space theory – for example, [**Sc**], [**R2**].

11.1.1 Proposition. *Closed subspaces, separated quotients, and countable direct products of Fréchet spaces are Fréchet spaces.*

11.1.2 Proposition. *Closed subspaces, separated quotients, and countable direct products of Montel spaces are Montel spaces.*

11.1.3 Open Mapping Theorem. *A surjective continuous linear map from a Fréchet space to a Fréchet space is an open map.*

The graph of a linear map $\phi : X \to Y$ between vector spaces is the subspace $\{(x, \phi(x)) : x \in X\}$ of the direct sum $X \oplus Y$.

11.1.4 Closed Graph Theorem. *A linear map from a Fréchet space to a Fréchet space is continuous if and only if its graph is closed.*

The strong form of the Hahn-Banach theorem says that a real linear functional on a subspace of a real vector space, dominated by a convex functional defined on the whole space, extends to the whole space with preservation of the dominance. We will need the following two consequences of this result:

11.1.5 Hahn-Banach Theorem. *Every continuous linear functional on a linear subspace of a locally convex topological vector space extends to a continuous linear functional on the whole space.*

11.1.6 Convex Separation Theorem. *If B is a closed convex balanced set in a locally convex topological vector space X, and $x_0 \in X$ is a point not in B, then there exists a continuous linear functional f on X such that $|f(x)| \leq 1$ for all $x \in B$ but $|f(x_0)| > 1$.*

Finally, we have:

11.1.7 Proposition. *Every locally compact topological vector space is finite dimensional.*

This completes the list of basic facts from topological vector space theory that we will assume in the following pages.

11.2 The Topology of $\mathcal{H}(X)$

By the results of section 2.4, if U is a domain in $\mathbb{C}^n$, then $\mathcal{H}(U)$ is a Fréchet space in the topology of uniform convergence on compact sets. In fact, by Theorem 2.4.2, $\mathcal{H}(U)$ is a Montel space. In this section we will prove that the same things are true of $\mathcal{H}(X)$ if X is a holomorphic variety. The proof of this result is quite delicate and requires several very technical preliminary results. The difficult part is showing that $\mathcal{H}(X)$ is complete in the topology of uniform convergence on compacta. That is, it is quite non-trivial to show that the limit of a sequence of holomorphic functions in this topology is itself holomorphic. In Exercises 4.18 through 4.21 we outlined a strategy for proving this (at least in the case of an irreducible subvariety of an open set in $\mathbb{C}^n$). We are now in a position to develop the machinery needed to successfully carry out this strategy.

The preliminary results leading to the proof that $\mathcal{H}(X)$ is complete also lay the groundwork for the proof of the closure of modules theorem at the end of the section. We will use this result in the next section to define a canonical Fréchet space topology on the space of sections of a coherent sheaf on an open subset of a holomorphic variety.

In what follows, for any open subset $U \subset \mathbb{C}^n$ and any positive integer k, we consider $\mathcal{H}(U)^k$ to be an $\mathcal{H}(U)$-module and a Fréchet topological vector space with the topology of uniform convergence on compact subsets of U.

We will need the following lemma, which is a strengthened form of the Weierstrass division theorem.

11.2.1 Lemma. *Let $\mathbf{h} \in {}_{n-1}\mathcal{H}_0[z_n]$ be a Weierstrass polynomial of degree k. For each sufficiently small open disc $\Delta'' \subset \mathbb{C}$, centered at 0, there is a 0-neighborhood $U' \subset \mathbb{C}^{n-1}$ such that for every open polydisc $\Delta' \subset U'$, centered at 0, the polydisc $\Delta = \Delta' \times \Delta''$ has the following property: There is a representative h of $\mathbf{h}$ on Δ and a continuous linear map $\gamma : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ such that $f - \gamma(f)h$ is a polynomial of degree less than k for each $f \in \mathcal{H}(\Delta)$.*

Proof. By the Weierstrass division theorem, every germ $\mathbf{f} \in \mathcal{H}_0$ has a unique representation as $\mathbf{f} = \mathbf{gh} + \mathbf{q}$, where $\mathbf{g}$ is the germ of a function g, holomorphic in a neighborhood of 0, and $\mathbf{q}$ is a polynomial in $\mathbf{z}_n$ of degree less than k, with coefficients which are germs at 0 of holomorphic functions in the variables $z_1, \dots, z_{n-1}$. The proof of the lemma simply involves revisiting the proof of the Weierstrass theorem (Theorem 3.3.5), and checking that g depends linearly and continuously on f if Δ is chosen properly. The following construction of g is equivalent to the one given in the proof of Theorem 3.3.5.

Let a polyradius $s = (s', s_n)$ be chosen so that $\mathbf{h}$ has a representative h in a neighborhood of the closed polydisc $\overline{\Delta}(0, s)$. If $0 < r_n \le s_n$, we may choose a polyradius $r' \le s'$ such that $h(z', z_n)$ has exactly $k = \deg \mathbf{h}$ zeroes, as a function of z_n, for each $z' \in \Delta(0, r')$, all of which occur on the open disc where $|z_n| < r_n$. We set $\Delta'' = \Delta(0, r_n)$ and $U' = \Delta(0, r')$. If Δ' is any open polydisc, centered at 0 and contained in U', and $\Delta = \Delta' \times \Delta''$, then we claim that Δ has the required property.

If $f \in \mathcal{H}(\Delta)$, then g is defined by the integral formula

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta|=t} \frac{f(z', \zeta)\, d\zeta}{h(z', \zeta)(\zeta - z_n)},$$

where t is chosen so that $|z_n| < t < r_n$, and so that the k zeroes of $h(z', \cdot)$ are all inside the open disk of radius t. By the Cauchy integral theorem, the resulting function is independent of the choice of t. Note that if z is restricted to lie in a given compact subset of Δ, then t may be chosen independent of z. It follows that the function g belongs to $\mathcal{H}(\Delta)$ and depends linearly and continuously on $f \in \mathcal{H}(\Delta)$. We set $\gamma(f) = g$. Then γ is a continuous linear map $\gamma : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$. The same argument used in the proof of Theorem 3.3.5 shows that $f - \gamma(f)h$ is a polynomial of degree less than k in $\mathbf{z}_n$, for all $f \in \mathcal{H}(\Delta)$.

The next result is a preliminary version of the theorem on closure of modules as well as a key step in the proof that $\mathcal{H}(X)$ is complete if X is a holomorphic variety.

Given an open set $U \subset \mathbb{C}^n$, a point $\lambda \in U$, and an $\mathcal{H}_\lambda$-submodule $M \subset \mathcal{H}^k_\lambda$, let $\mathcal{H}^k(U)_M$ denote the set of sections in $\mathcal{H}^k(U)$ whose germs at λ belong to M.

11.2.2 Proposition. *Let U be an open subset of $\mathbb{C}^n$ and k a positive integer.*

(i) *If $\mathcal{M}$ is a coherent analytic subsheaf of $\mathcal{H}^k$ on U, and λ is a point of U, then coordinates in $\mathbb{C}^n$ may be chosen so that there is a polydisc Δ, centered at λ, with $\Delta \subset U$, such that each element f in the closure of $\mathcal{H}^k(\Delta)_{\mathcal{M}_\lambda}$ in $\mathcal{H}^k(\Delta)$ actually belongs to $\mathcal{M}(\Delta)$. Furthermore, given finitely many coherent sheaves $\mathcal{M}_1, \dots, \mathcal{M}_j$ on U, the polydisc Δ may be chosen so that this result holds simultaneously on Δ for each of the $\mathcal{M}_i$.*

(ii) *If $\lambda \in U$, and M is a submodule of $\mathcal{H}^k_\lambda$, then $\mathcal{H}^k(U)_M$ is closed in $\mathcal{H}^k(U)$.*

(iii) *If $\phi : \mathcal{H}^m \to \mathcal{H}^k$ is a morphism of analytic sheaves on U, and Δ is an open polydisc with compact closure in U, then the induced map $\phi : \mathcal{H}^m(\Delta) \to \mathcal{H}^k(\Delta)$ has closed image.*

Proof. If (i) is true for a given n and k, then (ii) is also true for the same n and k. In fact, a submodule $M \subset \mathcal{H}^k_\lambda$ is finitely generated by elements of M which have representatives in any sufficiently small neighborhood $V \subset U$ of λ. These representatives generate a coherent subsheaf $\mathcal{M}$ of $\mathcal{H}^k$ on V with $\mathcal{M}_\lambda = M$. A sequence in $\mathcal{H}^k(U)_M$ which converges in $\mathcal{H}^k(U)$, when restricted to V, yields a sequence in $\mathcal{H}^k(V)_M$ which converges in $\mathcal{H}^k(V)$. If (i) holds, then λ has a neighborhood $\Delta \subset V$ so that the limit of this sequence has restriction to Δ belonging to $\mathcal{M}(\Delta)$. Then its germ at λ belongs to M and, hence, the limit of the original sequence belongs to $\mathcal{H}^k(U)_M$.

Thus, we need only prove (i) and (iii). We first show that if (i) (and hence (ii)) hold for given integers n and k, then part (iii) holds for the same n and k and arbitrary m. Given ϕ as in (iii), let $\mathcal{L} = \operatorname{Im}\{\phi : \mathcal{H}^m \to \mathcal{H}^k\}$. Let Δ be an open polydisc with compact closure in U. Then $\mathcal{L}(\Delta)$ consists of those sections of $\mathcal{H}^k(\Delta)$ with germ at λ belonging to the submodule $\mathcal{L}_\lambda$ at every point $\lambda \in \Delta$. By part (ii), this is a closed set. If $\mathcal{K}$ is the kernel of $\phi : \mathcal{H}^m \to \mathcal{H}^k$, then we have a short exact sequence of coherent analytic sheaves on U:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H}^m \xrightarrow{\phi} \mathcal{L} \longrightarrow 0.$$

From the long exact sequence of cohomology and the vanishing theorem of the previous chapter (Theorem 10.5.3), we conclude that the corresponding sequence of sections over Δ is also exact. Thus, $\phi : \mathcal{H}^m(\Delta) \to \mathcal{L}(\Delta)$ is surjective. Since $\mathcal{L}(\Delta)$ is closed in $\mathcal{H}^k(\Delta)$, the proof that (i) implies (iii) is complete.

The proof of part (i) will be by induction on n. Part (i) is trivially true in the case $n = 0$, and so we assume that $n > 0$ and that the proposition is true whenever U is an open subset of $\mathbb{C}^{n-1}$. Under this assumption, we next prove part (i) when U is an open subset of $\mathbb{C}^n$ and $k = 1$.

Thus, let $\mathcal{M}$ be a coherent sheaf of submodules (ideals) of $\mathcal{H}$ on U, and let λ be a point of U. We may as well assume that $\lambda = 0$. Let $\{p_1, \dots, p_\nu\}$ be a set of sections of $\mathcal{M}$ which generates $\mathcal{M}$ on a neighborhood of 0, which we may as well assume is U. We may choose coordinates for $\mathbb{C}^n$ so that each germ $\mathbf{p}_i$ at 0 has finite vanishing order in z_n. By the Weierstrass preparation theorem we may, after shrinking U if necessary, choose the p_i to be Weierstrass polynomials in z_n. We assume the p_i are indexed so that p_1 has the largest degree and we set $d = \deg p_1$. We may also assume that U has the form $U = U' \times U''$ for 0 neighborhoods $U' \subset \mathbb{C}^{n-1}$ and $U'' \subset \mathbb{C}$.

Let Δ'' be a disc centered at 0, with compact closure contained in U''. By the previous lemma, there is a 0-neighborhood in $\mathbb{C}^{n-1}$, which we may assume is U', such that for any open polydisc $\Delta' \subset U'$ (in any coordinate system), centered at 0, the polydisc $\Delta = \Delta' \times \Delta''$ has the property that there is a continuous linear map $\gamma : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ such that $f - \gamma(f)p_1$ is always a polynomial in z_n of degree less than $d = \deg p_1$.

We next consider the free ${}_{n-1}\mathcal{H}$-module $\mathcal{F}$ of rank $2d$ on U' consisting of polynomials in z_n of degree less than $2d$, with coefficients from ${}_{n-1}\mathcal{H}$. This has as a coherent subsheaf the sheaf $\mathcal{N}$ generated by the polynomials of the form $z_n^j p_i$ for $j = 0, 1, \dots, d-1$ and $i = 1, \dots, \nu$. By the induction assumption, after a linear change of coordinates in the variables $z_1, \dots, z_{n-1}$, we may choose a polydisc Δ', centered at 0, with compact closure in U' and with the property of part (i) for the coherent sheaf $\mathcal{N}$. Note that a linear change of variables in $z_1, \dots, z_{n-1}$ does not change the fact that the p_i are Weierstrass polynomials in z_n. Let $\Delta = \Delta' \times \Delta''$. We claim Δ has the property required in part (i) for the sheaf $\mathcal{M}$.

Let h be a section in $\mathcal{M}(\Delta)$ which is a polynomial in z_n of degree less than $2d$. Then, since the p_i generate $\mathcal{M}$ over a neighborhood of the closure of Δ, it follows from Exercise 10.17 that h may be written as

$$h = g_1p_1 + \cdots + g_\nu p_\nu$$

for a set of functions $g_1, \dots, g_\nu \in \mathcal{H}(\Delta)$. We then write $g_i = \gamma(g_i)p_1 + r_i$ for $i = 2, \dots, \nu$, where each r_i is a polynomial of degree less than d in z_n which also belongs to $\mathcal{H}(\Delta)$. This yields

$$h = sp_1 + r_2p_2 + \cdots + r_\nu p_\nu \quad \text{where} \quad s = g_1 + \sum_{i=2}^{\nu} \gamma(g_i)p_i.$$

Since h and each r_ip_i is a polynomial in z_n of degree less than $2d$, and since p_1 is a Weierstrass polynomial, it follows from the Weierstrass preparation theorem that s is a polynomial of degree less than d. We conclude that every $h \in \mathcal{M}(\Delta)$, which is a polynomial in z_n of degree less than $2d$, is in the $\mathcal{H}(\Delta')$-module generated by the elements $z_n^j p_i$ for $i = 1, \dots, \nu$ and $j = 0, \dots, d-1$. That is, each such h is a section over Δ' of the sheaf $\mathcal{N}$.

By the choice of Δ', every h in the closure in $\mathcal{F}(\Delta') \simeq \mathcal{H}^{2d}(\Delta')$ of the set of sections with germs in $\mathcal{N}_0$ at 0 belongs to $\mathcal{N}(\Delta')$. Now if f is in the closure of the set of functions in $\mathcal{H}(\Delta)$ with germs in $\mathcal{M}_0$ at 0, then it follows from the continuity of γ that $f - \gamma(f)p_1$ is in the closure of the set of functions in $\mathcal{H}(\Delta)$ which are polynomials of degree less than d with germs at 0 belonging to $\mathcal{M}_0$. Hence, it is in the closure in $\mathcal{F}(\Delta')$ of the set of sections with germs at 0 belonging to $\mathcal{N}_0$. This implies that the restriction of $f - \gamma(f)p_1$ to Δ' actually belongs to $\mathcal{N}(\Delta')$ and, hence, to $\mathcal{M}(\Delta)$. Since p_1 also belongs to $\mathcal{M}(\Delta)$, it follows that f does as well. This completes the induction step under the assumption that $k = 1$.

Note that to choose Δ so that the above argument works simultaneously for finitely many coherent sheaves $\mathcal{M}_1, \dots, \mathcal{M}_j$ just requires that we choose coordinates so that all the generators chosen for the modules $(\mathcal{M}_i)_0$ have finite vanishing order in z_n, and that we then choose U' small enough to make Lemma 11.2.1 work for each of the polynomials that occur in the role of p_1.

To complete the induction on n for general k, we use induction on k. We assume that part (i) is true with k replaced by $k-1$ and prove that it is also true for k. Thus, we let $\mathcal{M}$ be a coherent analytic subsheaf of ${}_n\mathcal{H}^k$ on U and define coherent subsheaves $\mathcal{L} \subset {}_n\mathcal{H}^{k-1}$ and $\mathcal{K} \subset {}_n\mathcal{H}$ on U by

$$\mathcal{L}(V) = \{(f_1, \ldots, f_k) \in \mathcal{M}(V) : f_k = 0\},$$
$$\mathcal{K}(V) = \{f_k : (f_1, \ldots, f_k) \in \mathcal{M}(V)\}.$$

Then $\mathcal{K}$ is isomorphic to $\mathcal{M}/\mathcal{L}$ under the projection map which sends a k-tuple of functions to its kth element. By shrinking U if necessary, we may choose a finite set of elements of $\mathcal{M}(U)$ which generates $\mathcal{M}$ over U. This set of elements then determines a morphism of analytic sheaves over U

$$\psi : \mathcal{H}^m \to \mathcal{H}^k,$$

with image equal to $\mathcal{M}$. Over U, we define $\phi : \mathcal{H}^m \to \mathcal{H}$ to be ψ followed by the projection of $\mathcal{H}^k$ on its last component. Then the image of ϕ is $\mathcal{K}$.

Now suppose that a polydisc Δ is chosen, with compact closure in U, and satisfying part (i) of the proposition for both the sheaves $\mathcal{L} \subset \mathcal{H}^{k-1}$ and $\mathcal{K} \subset \mathcal{H}$. Since part (i) of the proposition holds for $k = 1$, part (iii) also holds for $k = 1$. This means that $\phi : \mathcal{H}^m(\Delta) \to \mathcal{H}(\Delta)$ has closed image. By the open mapping theorem for Fréchet spaces, this implies that ϕ is an open mapping onto its image.

Let $\{f_j\}$ be a sequence in $\mathcal{H}^k(\Delta)$, with $(f_j)_0 \in \mathcal{M}_0$ for every j, and suppose that this sequence converges to $f \in \mathcal{H}^k(\Delta)$. If g_j and g are the last components of f_j and f, then $g_j \to g$ and, by the choice of Δ, $g \in \mathcal{K}(\Delta)$. Since $\phi : \mathcal{H}^m(\Delta) \to \mathcal{H}(\Delta)$ is an open map onto its image, there exists a convergent sequence $h_j \to h$ in $\mathcal{H}^m(\Delta)$ such that $g_j = \phi(h_j)$ (Exercise 11.1). Then $f_j - \psi(h_j)$ is a convergent sequence in $\mathcal{H}^{k-1}(\Delta)$ consisting of elements whose germs at 0 belong to $\mathcal{L}_0$. Again, by our choice of Δ, the limit $f - \psi(h)$ belongs to $\mathcal{L}(\Delta)$. However, $\psi(h)$ is in $\mathcal{M}(\Delta)$ and $\mathcal{L}(\Delta) \subset \mathcal{M}(\Delta)$. Hence, f belongs to $\mathcal{M}(\Delta)$. With the additional observation that this part of the argument can also be carried out simultaneously for any finite set of coherent sheaves on U, this completes the induction on k and also the induction on n. Hence, parts (i), (ii), and (iii) are proved in general.

The final technical result leading to our proof that $\mathcal{H}(X)$ is complete is the following proposition.

11.2.3 Proposition. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be germs of varieties at* 0 *in* $\mathbb{C}^n$ *and assume* $\mathbf{V}$ *is irreducible. There is a neighborhood* U *of* 0 *in* $\mathbb{C}^n$*, in which* $\mathbf{V}$ *and* $\mathbf{W}$ *have representatives* V *and* W*, with the following property: If* $\{f_n\} \subset \mathcal{H}(V \cup W)$ *is a sequence of functions whose germs vanish on* $\mathbf{W}$*, and if the sequence converges uniformly on compact subsets of* $V \sup W$ *to a function* f*, then* f *is the restriction to* $V \cup W$ *of a holomorphic function on* U *which vanishes on* W*.*

Proof. If we choose coordinates in $\mathbb{C}^n$, an integer $m \leq n$, an open polydisc $\Delta \subset \mathbb{C}^m$, an open set $U \subset \mathbb{C}^n$, and a representative V of $\mathbf{V}$ in U in accordance with Theorem 4.5.7, then V is an irreducible variety which the projection $\pi : \mathbb{C}^n \to \mathbb{C}^m$ expresses as a finite branched holomorphic cover of Δ, $\mathcal{H}(V)$ is a finite extension of $\mathcal{H}(\Delta)$, and z_{m+1} generates the quotient field of $\mathcal{H}(V)$ over the quotient field of $\mathcal{H}(\Delta)$. We may choose U to be the Cartesian product of Δ with a polydisc in $\mathbb{C}^{n-m}$ and choose it small enough that $\mathbf{W}$ also has a representative W in U. Given one choice of these sets, if we replace Δ by a smaller polydisc centered at 0, and we also replace U, V, and W by their intersections with the inverse image under π of the new Δ, then we will have a new choice of these sets with the same properties. Thus, we will regard Δ as a variable and U, V, and W as functions of Δ.

By Exercise 4.18, there is a function $d \in \mathcal{H}(\Delta)$, an integer k, and an $\mathcal{H}(\Delta)$-module morphism

$$\phi : \mathcal{H}(V) \to \mathcal{H}^k(\Delta),$$

where ϕ is defined by the condition that, if $\phi(f) = (a_0, \dots, a_{k-1})$, then $p(z', z_{m+1}) = a_0(z') + a_1(z')z_{m+1} + \cdots + a_{k-1}(z')z_{m+1}^{k-1}$ is the unique polynomial of degree less than k in $\mathcal{H}(\Delta)[z_{m+1}]$ such that

$$d \cdot f = p \quad \text{on} \quad V.$$

Here $z' = (z_1, \dots, z_m)$, and both d and p are regarded as functions on U which happen to be constant in some of the variables.

We may define ϕ for arbitrarily small 0-neighborhoods Δ, and by the uniqueness, it commutes with restriction. Hence there is also an ${}_m\mathcal{H}_0$-module morphism, which we will also call ϕ, from ${}_V\mathcal{H}_0$ to ${}_m\mathcal{H}_0^k$ such that $\mathbf{d} \cdot \mathbf{f} = \mathbf{p}$ on $\mathbf{V}$ if $\mathbf{p}$ is the polynomial in ${}_m\mathcal{H}_0[z_{m+1}]$ with coefficients given by the entries of $\phi(\mathbf{f})$.

Let M be the submodule of ${}_m\mathcal{H}_0^k$ consisting of all germs of elements of the form $\phi(f|_V)$, where for some choice of Δ, $f \in \mathcal{H}(U)$ and f vanishes on the corresponding W. Since ${}_m\mathcal{H}_0$ is Noetherian, M is finitely generated. Let $\{\mathbf{h}_i = (\mathbf{h}_{i0}, \dots, \mathbf{h}_{ik-1})\}_{i=1}^{\nu}$ be a set of generators of M, and $\{h_i\}_{i=1}^{\nu}$ a set of representatives of these generators, defined in a neighborhood of 0, which we may as well assume is Δ. By shrinking Δ, if necessary, we may choose functions $g_1, \ldots, g_\nu \in \mathcal{H}(U)$ so that $g_i|_W = 0$ and $\phi(g_i|_V) = h_i$. This means that

$$d \cdot g_i = \sum_{j=0}^{k-1} h_{ij} z_{m+1}^j \quad \text{on} \quad V.$$

Now let $\mathcal{M}$ be the coherent subsheaf of ${}_m\mathcal{H}^k$ on Δ generated by the sections $h_1, \ldots, h_\nu$. Then $\mathcal{M}_0 = M$. By shrinking Δ, if necessary, we may assume that $\mathcal{M}$ is defined in a neighborhood of $\overline{\Delta}$ and that condition (i) of Proposition 11.2.2 holds for the sheaf $\mathcal{M}$ on Δ and the point $\lambda = 0$.

By the result of Exercise 4.19, the morphism $\phi : \mathcal{H}(V) \to \mathcal{H}(\Delta)$ is continuous if $\mathcal{H}(V)$ and $\mathcal{H}(\Delta)$ are each given the topology of uniform convergence on compacta. Thus, if $\{f_i\}$ is a Cauchy sequence in $\mathcal{H}(V \cup W)$, then $\{\phi(f_i|_V)\}$ is a Cauchy, hence convergent, sequence in $\mathcal{H}^k(\Delta)$. If each f_i vanishes on W, then each $\phi(f_i|_V)$ has germ at 0 belonging to $\mathcal{M}_0$. Since Δ satisfies (i) of Proposition 11.2.2 for $\mathcal{M}$ and $\lambda = 0$, it follows that the section $u = \lim \phi(f_i|_V) \in \mathcal{H}^k(\Delta)$ is actually a section of $\mathcal{M}$. Furthermore, by Exercise 10.17, the set $\{\phi(g_j|_V)\}$ generates the space of sections of $\mathcal{M}$ over Δ, since it generates the sheaf $\mathcal{M}$ over a neighborhood of the closure of Δ. Thus,

$$u = \sum_j b_j \phi(g_j), \text{ where } b_j \in \mathcal{H}(\Delta).$$

Then

$$u = \phi(f|_V), \text{ where } f = \sum_j b_j g_j \in \mathcal{H}(U).$$

Now $\{f_i\}$ converges to a continuous function $\tilde{f}$ on $V \cup W$, and by the definition of ϕ, we have $d \cdot f = d \cdot \tilde{f}$ on V. Since both f and $\tilde{f}$ vanish on W and the complement of the zero set of d in V is dense, we conclude that $\tilde{f} = f$ on $V \cup W$. Thus, f is the required extension of $\tilde{f}$. This completes the proof.

The next proposition is essentially the completeness result we have been aiming for.

11.2.4 Proposition. *Let* $\mathbf{V}$ *be a germ of a holomorphic variety at* 0 *in* $\mathbb{C}^n$. *Let* $U \subset \mathbb{C}^n$ *be a neighborhood of* 0 *in which* $\mathbf{V}$ *has a representative* V. *If* $\{f_j\} \subset \mathcal{H}(V)$ *is a sequence which converges uniformly on compact subset of* V *to a function* f, *then* f *is holomorphic in a neighborhood of* 0.

Proof. If $\mathbf{V}$ is irreducible, this is just a special case of Proposition 11.2.3. We will prove the proposition in general by induction on the number of irreducible branches of $\mathbf{V}$. Thus, we assume the proposition is true for germs of varieties with fewer than k irreducible components. Let $\mathbf{V}$ be a germ of a variety with k irreducible components, say $\mathbf{V} = \mathbf{V}_1 \cup \cdots \cup \mathbf{V}_k$ and let U be a neighborhood of 0 in $\mathbb{C}^n$ in which each $\mathbf{V}_i$ has a representative V_i.

If $\{f_j\}$ is a sequence of holomorphic functions on V, converging to f in the topology of uniform convergence on compacta, then, by the induction assumption, the restriction of f to $W = V_2 \cup \cdots \cup V_k$ is holomorphic in a neighborhood of 0 in W. Hence, there is a 0-neighborhood $U_1 \subset U$, so that $f|_{W \cap U_1}$ extends to a holomorphic function g on U_1. We set $h_i = f_i - g$ on $V \cap U_1$. Then $\{h_i\}$ is a sequence of holomorphic functions on $V \cap U_1$, vanishing on $W \cap U_1$, and converging to $f - g$ in the uniform topology of $\mathcal{H}(V \cap U_1)$. It follows from Proposition 11.2.3, applied to the pair of germs $\mathbf{V}_1$ and $\mathbf{W}$, that there is some 0-neighborhood $U_2 \subset U_1$ such that the restriction of $f - g$ to $V \cap U_2 = (V_1 \cup W) \cap U_2$ extends to a holomorphic function on U_2. It follows that $f - g$, and hence, f are holomorphic on a neighborhood of 0 in V. This completes the induction.

The above result allows us to finally prove that $\mathcal{H}(X)$ is Fréchet and, in fact, Montel.

11.2.5 Theorem. *If X is a holomorphic variety, then $\mathcal{H}(X)$ is a Montel space in the topology of uniform convergence on compact sets.*

Proof. We first prove that $\mathcal{H}(X)$ is a Fréchet space. If $\{K_j\}$ is an increasing sequence of compact subsets of X such that every compact subset of X is contained in some K_j, then the sequence of seminorms $\{||\cdot||_{K_j}\}$ determines the topology of uniform convergence on compacta. That $\mathcal{H}(X)$ is complete in this topology follows immediately from the preceding proposition, since each point of X has a neighborhood which embeds as a subvariety of an open set in $\mathbb{C}^n$ for some n. That is, a Cauchy sequence in $\mathcal{H}(X)$, with the topology of uniform convergence on compacta, converges to a continuous function on X, and the previous proposition shows that the limit is actually holomorphic in a neighborhood of each point and, hence, holomorphic on X. Thus, $\mathcal{H}(X)$ is a Fréchet space.

We proceed to show that $\mathcal{H}(X)$ is a Montel space. We know by Corollary 10.5.4 that X has a neighborhood base consisting of open sets V with the property that V may be embedded as a subvariety of a polydisc Δ in $\mathbb{C}^n$ in such a way that the restriction map $\mathcal{H}(\Delta) \to \mathcal{H}(V)$ is surjective. This map is also continuous since both spaces have the topology of uniform convergence on compacta. It follows from the open mapping theorem for Fréchet spaces that $\mathcal{H}(V)$ has the quotient topology it inherits from $\mathcal{H}(\Delta)$. By Proposition 11.1.2, since $\mathcal{H}(\Delta)$ is a Montel space, so is $\mathcal{H}(V)$.

Now let $\{V_i\}$ be a countable cover of X by neighborhoods with the properties of the neighborhood V of the previous paragraph. Then

$$f \to \prod_i f|_{V_i}$$

embeds $\mathcal{H}(X)$ as a linear subspace of $\prod_i \mathcal{H}(V_i)$. It is a closed subspace, because it may be characterized as the subspace consisting of all $\{f_i\}$ such that $f_i = f_j$ on $V_i \cap V_j$ for all i, j. Thus, $\mathcal{H}(X)$ is a closed subspace of a countable product of Montel spaces and is itself Montel, by Proposition 11.1.2. This completes the proof.

Part (ii) of Proposition 11.2.2 extends to arbitrary holomorphic varieties. The result is the closure of modules theorem.

11.2.6 Closure of Modules Theorem. *Let X be a holomorphic variety, and $\mathcal{S}$ a coherent subsheaf of ${}_X\mathcal{H}^k$ for some k. Then*

(i) *if $x \in X$, and M is a submodule of $\mathcal{H}_x$, then $\{f \in \mathcal{H}^k(X) : \mathrm{f}_x \in M\}$ is closed in $\mathcal{H}^k(X)$;*

(ii) *$\mathcal{S}(X)$ is a closed subspace of the Fréchet space $\mathcal{H}^k(X)$.*

Proof. Corollary 10.5.4 implies that there is a neighborhood W of x which may be identified with a subvariety of an open polydisc $\Delta \subset \mathbb{C}^n$, in such a way that the restriction map $\mathcal{H}(\Delta) \to \mathcal{H}(W)$ is surjective. If a sequence $\{f_j\}$ converges to f in $\mathcal{H}^k(X)$, then the open mapping theorem implies there is a convergent sequence $\{g_j\} \subset \mathcal{H}^k(\Delta)$ such that $g_j|_W = f_j|_W$ for each j (Exercise 11.1). Furthermore, if the functions f_j all have germs at x belonging to a submodule $M \subset {}_X\mathcal{H}_x$, then the g_j will have germs at x belonging to the inverse image $N \subset {}_n\mathcal{H}^k_x$ of this submodule under the quotient map ${}_n\mathcal{H}^k_x \to {}_X\mathcal{H}^k_x$. It follows from Proposition 11.2.2 that if g is the limit of the sequence $\{g_j\}$, then g has germ at x belonging to N and, hence, its restriction f to W has germ at x belonging to M. This proves part (i).

Now part (ii) is an immediate consequence, since a global section of $\mathcal{S}$ is just a global section of ${}_X\mathcal{H}^k$ which has germ at x belonging to $\mathcal{S}_x$ for each $x \in X$.

11.3 Fréchet Sheaves

A Fréchet sheaf on a space X is just a sheaf of Fréchet spaces on X. Of course, the restriction maps are required to be morphisms in the category of Fréchet spaces – that is, continuous linear maps. In this section we will show that a coherent analytic sheaf on a holomorphic variety carries a canonical structure of a Fréchet sheaf.

Let X be a holomorphic variety, and $\mathcal{S}$ a coherent analytic sheaf on X. By Corollary 10.5.5, we may choose a neighborhood base $\mathcal{U}$ for the topology of X consisting of sets U which have compact closure and have the property that coherent sheaves defined in a neighborhood of $\overline{U}$ are acyclic on U. Furthermore, we may choose $\mathcal{U}$ so that, for each $U \in \mathcal{U}$, the sheaf $\mathcal{S}$ is the cokernel of a morphism between free finite rank sheaves on a neighborhood of $\overline{U}$.

Fix $U \in \mathcal{U}$. Then there is an exact sequence of analytic sheaves

$$\mathcal{H}^m \xrightarrow{\phi} \mathcal{H}^k \longrightarrow \mathcal{S} \longrightarrow 0$$

defined in a neighborhood of $\overline{U}$. The fact that every coherent sheaf defined in a neighborhood of $\overline{U}$ is acyclic on U implies that the induced sequence on sections over U:

$$\mathcal{H}^m(U) \xrightarrow{\phi} \mathcal{H}^k(U) \longrightarrow \mathcal{S}(U) \longrightarrow 0$$

is also exact. Furthermore, by Theorem 11.2.6, the image of the morphism $\phi : \mathcal{H}^m(U) \to \mathcal{H}^k(U)$ is closed. Hence, $\mathcal{S}(U)$, as a separated quotient of $\mathcal{H}^k(U)$, inherits a Fréchet space topology, in fact, a Montel space topology. Now suppose that $V \subset U$ is another set in our basis $\mathcal{U}$, and suppose for this set we have an exact sequence

$$\mathcal{H}^p \xrightarrow{\psi} \mathcal{H}^q \longrightarrow \mathcal{S} \longrightarrow 0$$

defined in a neighborhood of $\overline{V}$. Then we may construct the following commutative diagram of $\mathcal{H}(U)$-module homomorphisms with exact rows:

$$\begin{tikzcd}
\mathcal{H}^m(U) \arrow[r, "\phi"] \arrow[d, "\beta"'] & \mathcal{H}^k(U) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{S}(U) \arrow[r] \arrow[d, "r_{V,U}"'] & 0 \\
\mathcal{H}^p(V) \arrow[r, "\psi"] & \mathcal{H}^q(V) \arrow[r] & \mathcal{S}(V) \arrow[r] & 0,
\end{tikzcd}$$

where $r_{V,U}$ is the restriction map, and α and β are constructed by using the familiar lifting argument (the projectivity of free modules). The maps α and β are given by matrices of holomorphic functions on V, through vector-matrix multiplication. This clearly implies that they are continuous. If $\mathcal{S}(U)$ has the topology it inherits from being a quotient of $\mathcal{H}^k(U)$, then $\mathcal{H}^k(U) \to \mathcal{S}(U)$ is an open map. Also, the map $\mathcal{H}^q(V) \to \mathcal{S}(V)$ is continuous if $\mathcal{S}(V)$ is given the quotient topology. These facts combine to force the map $r_{V,U}$ to be continuous. We draw the following conclusions from this:

11.3.1 Lemma. *The topology, defined above, on $\mathcal{S}(U)$ is independent of the way in which it is expressed as a quotient of a free finite rank $\mathcal{H}(U)$-module. Furthermore, if $V \subset U$ are two sets in our basis, then the restriction map $r_{V,U} : \mathcal{S}(U) \to \mathcal{S}(V)$ is continuous.*

To define the topology on $\mathcal{S}(U)$ for a general open set U, we cover U by a countable collection $\{W_i\}$ of sets from our basis. Then

$$f \to \{f|_{W_i}\} : \mathcal{S}(U) \to \prod_i \mathcal{S}(W_i)$$

is an injective continuous linear map of $\mathcal{S}(U)$ onto a closed subspace of the Montel space $\prod_i \mathcal{S}(W_i)$. The image is closed, because it is the subspace of $\prod_i \mathcal{S}(W_i)$ consisting of $\{g_i\}$ such that $g_i = g_j$ on $W_i \cap W_j$ for all i, j. Since a closed subspace of a Montel space is Montel, this serves to put a Fréchet space structure on $\mathcal{S}(U)$ under which it is actually a Montel space.

Note that, by construction, the above topology on $\mathcal{S}(U)$ has the property that, for each $x \in U$, there is a basic neighborhood $W \subset U$, containing x, such that the restriction map $\mathcal{S}(U) \to \mathcal{S}(W)$ is continuous. Now suppose that $V \subset U$ is another open set and $\mathcal{S}(V)$ is given a Fréchet space topology with this same property. Then we claim that the restriction morphism $r_{V,U} : \mathcal{S}(U) \to \mathcal{S}(V)$ has closed graph, and hence, is continuous. To see this, let $\{(f_n, r_{V,U}(f_n))\}$ be a sequence in the graph which converges to the point (f, g). Then $f_n \to f$, and $r_{V,U}(f_n) \to g$. Now, for each point x of V, we can choose a basic neighborhood W of x such that the restriction morphisms $\mathcal{S}(U) \to \mathcal{S}(W)$ and $\mathcal{S}(V) \to \mathcal{S}(W)$ are both continuous. This clearly implies that $r_{V,U}(f)|_W = g|_W$. But since this is true for a neighborhood of each point of V, we conclude that $r_{V,U}(f) = g$, and hence, that the graph of $r_{V,U}$ is closed, as required. This proves that:

11.3.2 Lemma. *The Fréchet topology defined above on $\mathcal{S}(U)$ is uniquely determined by the property that, for each basic open set $W \subset U$, the restriction map $\mathcal{S}(U) \to \mathcal{S}(W)$ is continuous. Furthermore, if $V \subset U$ are two open subsets of X, then the restriction map $r_{V,U} : \mathcal{S}(U) \to \mathcal{S}(V)$ is continuous.*

Thus, we have proved the existence of a Fréchet sheaf structure on $\mathcal{S}$, under which it is a Montel sheaf. Furthermore, it is clear from the closed graph theorem argument used in the proof, that a morphism $\mathcal{T} \to \mathcal{S}$, between any two Fréchet sheaves on a topological space, is continuous if and only if it is continuous locally – that is, if and only if $\mathcal{T}(W) \to \mathcal{S}(W)$ is continuous for each W in a neighborhood basis of X. This, and the construction of the topology on basic open sets shows that the topology, defined above, has the property that if $\mathcal{H}^k \to \mathcal{S}$ is a surjective morphism over an open set U, then $\mathcal{H}^k(U) \to \mathcal{S}(U)$ is continuous.

11.3.3 Theorem. *Let X be a holomorphic variety.*

(i) *If $\mathcal{S}$ is a coherent analytic sheaf on X, then there is a unique structure of a Fréchet sheaf on $\mathcal{S}$, with the property that if U is any open set and ${}_U\mathcal{H}^k \to \mathcal{S}|_U$ is a surjective morphism of analytic sheaves for some k, then $\mathcal{H}^k(U) \to \mathcal{S}(U)$ is continuous. Furthermore, $\mathcal{S}$ is a Montel sheaf with this structure.*

(ii) *A morphism of sheaves of $\mathcal{H}$-modules, between two coherent analytic sheaves, is automatically continuous.*

Proof. We have already proved everything in part (i) but the uniqueness. Let $\phi : \mathcal{S} \to \mathcal{T}$ be a morphism between coherent analytic sheaves on X, and assume that each of $\mathcal{S}$ and $\mathcal{T}$ is equipped with a Fréchet sheaf structure such that the condition in part (i) is satisfied. As noted above, the morphism ϕ will be continuous if and only if it is continuous locally. If x is a point of X, then we may choose a neighborhood W for x such that there are surjective morphisms $\alpha : \mathcal{H}^k \to \mathcal{S}$ and $\beta : \mathcal{H}^m \to \mathcal{T}$ defined over W. By Corollary 10.5.5, we may choose W so that the corresponding morphisms of sections over W are also surjective. The usual lifting argument gives us the map λ in the following commutative diagram

$$\begin{tikzcd} \mathcal{H}^k(W) \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & \mathcal{S}(W) \arrow[r] \arrow[d, "\phi"'] & 0 \\ \mathcal{H}^m(W) \arrow[r, "\beta"] & \mathcal{T}(W) \arrow[r] & 0. \end{tikzcd}$$

The maps α, β, and λ are continuous. Also, α is open, by the open mapping theorem for Fréchet spaces. It follows that ϕ is continuous for each such neighborhood W. Thus, ϕ is locally continuous and, hence, continuous. Applied in the case where $\mathcal{S} = \mathcal{T}$, this proves (i). Applied in general, it proves (ii).

In view of the above theorem, we may, henceforth, assume that every coherent analytic sheaf comes equipped with a canonical structure of a Montel sheaf and that morphisms between coherent analytic sheaves are automatically continuous.

11.4 Cartan's Theorems

We are now in a position to prove the main vanishing theorem for coherent analytic sheaves – Cartan's Theorem B. The first step is an approximation result.

11.4.1 Proposition. *If X is a Stein space, $\mathcal{S}$ a coherent analytic sheaf on X, and $W \subset X$ an Oka-Weil subdomain, then the space of restrictions to W of global sections of $\mathcal{S}$ is dense in $\mathcal{S}(W)$.*

Proof. We first prove that this is true in the case where $\mathcal{S}$ is the structure sheaf $\mathcal{H}$. Since W is an Oka-Weil subdomain, there is a holomorphic map $\phi : X \to \mathbb{C}^n$ such that ϕ maps W biholomorphically onto a subvariety of the unit polydisc $\Delta(0,1)$. If f is a holomorphic function on W and K is a compact subset of W, then it follows from Corollary 10.5.4 that f has the form $g \circ \phi$ in a neighborhood of K, where g is a holomorphic function on an open polydisc Δ with closure contained in $\Delta(0,1)$. If $\{h_j\}$ is a sequence of polynomials converging to g in the topology of uniform convergence on compact subsets of Δ, then $f_j = h_j \circ \phi$ defines a sequence of holomorphic functions on X which converge uniformly to f on K.

In order to prove the proposition for a general coherent analytic sheaf $\mathcal{S}$, we choose a sequence $\{W_i\}$ of Oka-Weil subdomains with $W = W_0$ and $\overline{W}_i \subset W_{i+1}$ for each i. We claim that the image of $\mathcal{S}(W_j)$, under restriction, is dense in $\mathcal{S}(W_i)$ if $i < j$. By Exercise 10.18, there is a surjective morphism $\mathcal{H}^k \to \mathcal{S}$ defined on a neighborhood of $\overline{W}_j$ in X. Then it follows from Theorem 10.6.4 that both $\mathcal{H}^k(W_j) \to \mathcal{S}(W_j)$ and $\mathcal{H}^k(W_i) \to \mathcal{S}(W_i)$ are surjective. Thus, an element f of $\mathcal{S}(W_i)$ can be lifted to an element g of $\mathcal{H}^k(W_i)$, and this can be expressed as the limit of a sequence of restrictions of elements $h_p \in \mathcal{H}^k(W_j)$, by the result of the above paragraph. The image of the sequence $\{h_p\}$ in $\mathcal{S}(W_j)$ will then have the property that its restriction to $\mathcal{S}(W_i)$ converges to f. This establishes the claim.

To finish the proof, we choose a translation invariant metric σ_j defining the topology of $\mathcal{S}(W_j)$ for each j (Exercise 11.20). We set $\rho_j(f) = \sigma_j(f, 0)$, so that $\sigma_j(f,g) = \rho_j(f - g)$. Since the restriction maps $\mathcal{S}(W_j) \to \mathcal{S}(W_i)$, $i < j$, are continuous, the metric σ_j may be replaced by the metric $\sum_{i=0}^{j} \sigma_i$, without changing the topology it defines on $\mathcal{S}(W_j)$. Thus, we may assume, without loss of generality, that the sequence of metrics is increasing, in the sense that $\rho_i(f|_{W_i}) \leq \rho_j(f)$ if $i < j$ and $f \in \mathcal{S}(W_j)$. Then for each $\epsilon > 0$, and $f \in \mathcal{S}(W) = \mathcal{S}(W_0)$, we choose $g_1 \in \mathcal{S}(W_1)$ with $\rho_0(f - g_1) < \epsilon/2$. We then inductively choose elements $g_i \in \mathcal{S}(W_i)$ with $\rho_i(g_i - g_{i+1}|_{W_i}) < \epsilon 2^{-i-1}$. Clearly, for each i, the sequence $\{(g_j)|_{W_i} : j > i\}$ converges in the metric ρ_i to an element $h_i \in \mathcal{S}(W_i)$. Furthermore, $h_i = (h_j)|_{W_i}$ for $i < j$ and $\rho_0(f - h_0) < \epsilon$. Hence, the h_j define a global section $h \in \mathcal{S}(X)$ such that $\rho_0(f - h) < \epsilon$. This completes the proof.

11.4.2 Cartan's Theorem A. *If X is a Stein space, and $\mathcal{S}$ is a coherent analytic sheaf on X, then $\mathcal{S}(X)$ generates $\mathcal{S}_x$ at every point $x \in X$.*

Proof. Let M_x be the submodule of $\mathcal{S}_x$ generated by the global sections of $\mathcal{S}$. There exists a neighborhood W of x on which $\mathcal{S}$ is finitely generated. That is, there is a positive integer k and a surjective morphism $\phi : \mathcal{H}^k \to \mathcal{S}$

over W. In particular, this implies that $\mathcal{S}(W)$ generates $\mathcal{S}_x$. We may assume that W is an Oka-Weil subdomain, and hence, that $\phi : \mathcal{H}^k(W) \to \mathcal{S}(W)$ is also surjective. Then the space of sections $f \in \mathcal{H}^k(W)$ such that $\phi(f)_x \in M_x$ is closed, by Theorem 11.2.6. It follows from the open mapping theorem that the set N, of all $g \in \mathcal{S}(W)$ such that $g_x \in M_x$, is also closed. However, N contains the set of restrictions to W of all elements of $\mathcal{S}(X)$, which by Proposition 11.4.1 is dense in $\mathcal{S}(W)$. Thus, $N = \mathcal{S}(W)$. Since $\mathcal{S}(W)$ generates $\mathcal{S}_x$, it follows that $M_x = \mathcal{S}_x$ and the proof is complete.

11.4.3 Cartan's Theorem B. *If X is a Stein space, and $\mathcal{S}$ is a coherent analytic sheaf on X, then $\mathcal{S}$ is acyclic on X.*

Proof. Given the machinery we have developed so far, the remainder of the proof is similar to the proof of Dolbeault's Lemma (Theorem 10.1.3). We use Corollary 10.6.6 to express X as the union of a sequence of Oka-Weil subdomains $\{W_n\}$ such that $\overline{W}_n \subset W_{n+1}$. Then $\mathcal{S}$ is acyclic on each W_n, by Theorem 10.6.4. Also, since the W_n's form a nested sequence, $\{W_n\}$ is a Leray cover of X for $\mathcal{S}$. We also have that the space of global sections of $\mathcal{S}$ is dense in $\mathcal{S}(W_n)$ for each n. That $\mathcal{S}$ is acyclic on X now follows from a principle that is quite general and which we state in the next lemma.

11.4.4 Lemma. *Suppose $\mathcal{S}$ is a sheaf on a topological space X. If X is the union of an increasing sequence $\{W_i\}$ of open subsets such that $\mathcal{S}$ is acyclic on each W_i, then*

(i) *$H^p(X, \mathcal{S}) = 0$ for $p > 1$;*

(ii) *if $\mathcal{S}$ is a Fréchet sheaf, and the restriction map $\mathcal{S}(W_{i+1}) \to \mathcal{S}(W_i)$ has dense image for each i, then $H^1(X, \mathcal{S}) = 0$.*

Proof. Let

$$0 \longrightarrow \mathcal{S} \xrightarrow{\eta} \mathcal{F}^0 \xrightarrow{\delta^0} \mathcal{F}^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} \mathcal{F}^p \xrightarrow{\delta^p} \cdots$$

be a flabby resolution of $\mathcal{S}$. Suppose $p > 1$, and $f \in \mathcal{F}^p(X)$, $\delta f = 0$. Then we will prove by induction that there is a sequence $\{g_n\}$ with $g_n \in \mathcal{F}^{p-1}(W_n)$, $g_n = g_{n-1}$ on W_{n-1}, and $\delta g_n = f_n$ on W_n. Clearly, if we can show this, then part (i) of the lemma will be established, since such a sequence determines a global section $g \in \mathcal{F}^{p-1}(X)$ such that $g = g_n$ on W_n and, consequently, $\delta g = f$ on all of X.

Suppose we have managed to construct the sequence $\{g_n\}$ for all $n \leq m$. Because $\mathcal{S}$ is acyclic on each W_n, there exists a section $\tilde{g}_{m+1} \in \mathcal{F}^{p-1}(W_{m+1})$ such that $\delta \tilde{g}_{m+1} = f$. Of course, $\tilde{g}_{m+1} - g_m$ may not be 0 on W_m. However, it does belong to the kernel of δ on W_m, and hence, there exists a section $h_{m+1} \in \mathcal{F}^{p-2}(W_m)$ such that $\delta h_{m+1} = \tilde{g}_{m+1} - g_m$ on W_m. Since $\mathcal{F}^{p-2}$ is flabby, we may assume that h_{m+1} is actually a section defined on all of X. We then set $g_{m+1} = \tilde{g}_{m+1} - \delta h_{m+1}$ on W_{m+1}. This serves to extend our sequence $\{W_n\}$ to $n = m+1$ and completes the induction. This completes the proof of part (i). Note that this proof does not work when $p = 1$, since in this case, there is no flabby $\mathcal{F}^{p-2}$.

We now proceed with the proof of part (ii). By Leray's theorem (Theorem 7.8.5) we may compute $H^1(X, \mathcal{S})$ using Čech cohomology for the Leray cover $\{W_n\}$. Our argument will involve an approximation argument for 0-cochains. The space of 0-cochains for $\mathcal{S}$ and the cover $\{W_n\}$ on an open set U is just $\prod_n \mathcal{S}(W_n \cap U)$ and, as a countable product of Fréchet spaces, is itself a Fréchet space. For each j, we let σ_j be a translation invariant metric on the Fréchet space of 0-Čech cochains on W_j. As in the proof of Proposition 11.4.1, we set $\rho_j(f) = \sigma_j(f, 0)$, and we assume that the sequence $\{\rho_j\}$ is increasing, in the sense that $\rho_j(g|_{W_j}) \le \rho_k(g)$ if $j < k$, and g is a 0-cochain on W_k. Suppose f is a 1-cocycle for the cover $\{W_n\}$. We inductively construct a sequence $\{g_j\}$, where g_j is a 0-cochain on W_j such that $\delta g_j = f$ on W_j, and $\rho_j(g_j - g_{j+1}|_{W_j}) < 2^{-j}$ for each j. Suppose such a sequence $\{g_j\}$ has been constructed for all indices $j < k$. We use the fact that $H^1(W_k, \mathcal{S}) = 0$ to find a 0-cochain t on W_k such that $\delta t = f$ on W_k. Then,

$$\delta(t - g_{k-1}) = 0$$

in W_{k-1}. This means that $t - g_{k-1}$ is the 0-cochain on W_{k-1} determined by a section $r \in \mathcal{S}(W_{k-1})$. Using the density hypotheses, there is a section $s \in \mathcal{S}(W_k)$ such that $\rho_{k-1}(t - g_{k-1} - s) < 2^{-k+1}$. Then $g_k = t - s$ has the properties that $\delta g_k = f$ on W_k and $\rho_{k-1}(g_k - g_{k-1}) < 2^{-k+1}$. Thus, by induction, we may construct the sequence $\{g_j\}$ as claimed.

Now on a given W_k, consider the sequence $\{g_j|_{W_k}\}_{j=k}^{\infty}$. This is a Cauchy sequence in the metric ρ_k, since $\rho_k(g_{j+1}|_{W_k} - g_j|_{W_k}) < \rho_j(g_{j+1}|_{W_j} - g_j) < 2^{-j}$ for $j \ge k$. Furthermore, the terms of this sequence differ from the first term by cocycles (since the difference is killed by δ). Thus, the sequence may be regarded as a fixed cochain plus a uniformly convergent sequence of cocycles. It follows that this sequence actually converges in the topology of 0-cochains on W_k, and the limit h_k satisfies $\delta h_k = f|_{W_k}$. Furthermore, $h_{k+1} = h_k$ on W_k, and hence, the h_k determine a 0-cochain h on X. Clearly, $\delta h = f$. Thus, every 1-cochain is a coboundary and the proof is complete.

As is the case for affine algebraic varieties, Stein spaces are actually characterized among all holomorphic varieties as those for which cohomology in degree 1 vanishes for all coherent sheaves.

11.4.5 Proposition. *Let X be a holomorphic variety. Then X is a Stein space if and only if $H^1(X, \mathcal{S}) = 0$ for every coherent analytic sheaf $\mathcal{S}$ on X.*

Proof. Suppose $H^1(X, \mathcal{S}) = 0$ for every coherent analytic sheaf $\mathcal{S}$ on X. If D is any discrete subset of X, then D is a subvariety, and hence, we have a short exact sequence of sheaves

$$0 \to \mathcal{I}_D \to {}_X\mathcal{H} \to {}_D\mathcal{H} \to 0,$$

where $\mathcal{I}_D$ is the ideal sheaf of D. Since $\mathcal{I}_D$ is coherent, Theorem 11.4.3 implies that $H^1(X, \mathcal{I}_D) = 0$. Consequently, the sequence of global sections

$$0 \to \mathcal{I}_D(X) \to \mathcal{H}(X) \to \mathcal{H}(D) \to 0$$

is also exact. However, since D is discrete, $\mathcal{H}(D)$ is just the space of all complex valued functions on D. We conclude that every complex valued function on D is the restriction to D of a holomorphic function on X. This implies immediately that the holomorphic functions on X separate the points of X. We will also show that it implies X is holomorphically convex.

Suppose K is a compact subset of X. Let D be a discrete subset of its holomorphically convex hull $\hat{K}$. Then every complex function on D is the restriction to D of a holomorphic function on X. However, such a function f is bounded in modulus on $\hat{K}$ by $||f||_K$. It follows that every complex function on D is bounded. This implies that D is finite. Thus, the only discrete subsets of $\hat{K}$ are finite sets. This implies that $\hat{K}$ is compact.

Given $x \in X$, let $\mathcal{M}_x$ be the maximal ideal of $\mathcal{H}_x$ at x, and let $\mathcal{I}$ be the ideal sheaf defined by

$$\mathcal{I}(U) = \{f \in \mathcal{H}(U) : \mathbf{f}_x \in \mathcal{M}_x^2 \text{ if } x \in U\}.$$

Then we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{H} \to \mathcal{H}/\mathcal{I} \to 0,$$

where $\mathcal{H}/\mathcal{I}$ is a skyscraper sheaf at x, with $\mathcal{H}_x/\mathcal{M}_x^2$ as stalk at x. Clearly $\mathcal{I}$ is coherent, and so the corresponding sequence of global sections is also exact. This means that every element of $\mathcal{H}_x/\mathcal{M}_x^2$ is the image of a global holomorphic function. In particular, there are global holomorphic functions $f_1, \ldots, f_k$ whose germs at x generate $\mathcal{M}_x$ modulo $\mathcal{M}_x^2$. It follows from Nakayama's lemma that they generate the ideal $\mathcal{M}_x$ (Exercise 6.18).

11.5 Applications of Cartan's Theorems

Cartan's Theorem B has a host of applications. We list some of these in the next few pages. The first five are proved using sheaf theory techniques which should be familiar by now, and so we leave their proofs as exercises.

The first corollary is the ultimate generalization of Proposition 1.2.2 from Chapter 1.

11.5.1 Corollary. *If X is a Stein space, then every epimorphism $\mathcal{S} \to \mathcal{T}$ of coherent analytic sheaves induces an epimorphism $\mathcal{S}(X) \to \mathcal{T}(X)$ on global sections.*

The next corollary is also a generalization of a single variable theorem from Chapter 1 – the interpolation theorem (Exercise 1.9).

11.5.2 Corollary. *If Y is a subvariety of a Stein space X, then every holomorphic function on Y is the restriction to Y of a holomorphic function on X.*

Recall that we began this text in section 1.2 with the problem of characterizing the maximal ideals of $\mathcal{H}(U)$ for U a domain in $\mathbb{C}$. We succeeded in characterizing the finitely generated maximal ideals as those which are given by points of U in Theorem 1.2.1. We can now generalize Theorem 1.2.1 to the ring of holomorphic functions on a Stein space.

11.5.3 Corollary. *Let X be a Stein space.*

(i) *If $\{f_i\}$ is a finite set of holomorphic functions on X which do not vanish simultaneously at any point of X, then there is a set of holomorphic functions $\{g_i\}$ on X such that $\sum g_i f_i = 1$;*

(ii) *each finitely generated maximal ideal of $\mathcal{H}(X)$ has the form M_x, where $M_x = \{f \in \mathcal{H}(X) : f(x) = 0\}$ for some $x \in X$.*

If K is a compact subset of a holomorphic variety, then by $\mathcal{H}(K)$ we mean the algebra of functions holomorphic in a neighborhood of K, where two such functions are identified in $\mathcal{H}(K)$ if they agree in some neighborhood of K. It follows from Proposition 7.3.5 that this is the same as the space $\Gamma(K, \mathcal{H})$ of sections of the sheaf $\mathcal{H}$ on K.

A *Stein compact set* is a ringed space which is isomorphic to a holomorphically convex compact subset of a Stein space. A coherent analytic sheaf on a Stein compact set K is a sheaf which is the restriction to K of a coherent analytic sheaf, defined in a neighborhood of K, in some Stein space X in which K is embedded as a holomorphically convex subset. It follows from Theorem 10.6.5, that a compact subset of a Stein space X is holomorphically convex if and only if it is the intersection of a nested sequence of open subsets of X which are themselves Stein spaces. This characterization is useful in proving the following corollary.

11.5.4 Corollary. *Let K be a Stein compact set. Then every coherent analytic sheaf on K is acyclic and is generated as a sheaf of $\mathcal{H}$-modules by a finite set of sections over K.*

On a Stein compact set we have a stronger version of Corollary 11.5.3 – one that characterizes all maximal ideals.

11.5.5 Corollary. *Let K be a Stein compact set. Then every maximal ideal of $\mathcal{H}(K)$ is of the form $M_x = \{f \in \mathcal{H}(X) : f(x) = 0\}$, for some $x \in K$.*

11.6 Invertible Groups and Line Bundles

Let $\mathcal{H}^*$ denote the sheaf of invertible holomorphic functions, with multiplication as the group operation. We have the short exact sequence

$$0 \longrightarrow \mathcal{Z} \xrightarrow{2\pi i} \mathcal{H} \xrightarrow{\exp} \mathcal{H}^* \longrightarrow 0,$$

where $\mathcal{Z}$ denotes the constant sheaf with the integers as stalk, and the map $\mathcal{Z} \xrightarrow{2\pi i} \mathcal{H}$ is multiplication by $2\pi i$, followed by inclusion. We pass to global sections and use the long exact sequence of cohomology and the fact that $\mathcal{H}$ is acyclic on a Stein space to conclude:

11.6.1 Proposition. *If X is a Stein space then*
(i) $H^1(X, \mathbb{Z}) \simeq \mathcal{H}^*(X)/\exp(\mathcal{H}(X))$;
(ii) $H^2(X, \mathbb{Z}) \simeq H^1(X, \mathcal{H}^*)$.

Recall that we developed the analogous result for the algebra of continuous functions on a paracompact space in Example 7.8.7 and Exercise 7.17. If X is a Stein space, then the inclusion $\mathcal{H} \to \mathcal{C}$ gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(X) \arrow[r, "\exp"] \arrow[d] & \mathcal{H}^*(X) \arrow[r] \arrow[d] & H^1(X, Z) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathcal{C}(X) \arrow[r, "\exp"] & \mathcal{C}^*(X) \arrow[r] & H^1(X, Z) \arrow[r] & 0,
\end{tikzcd}$$

with rows exact, by Proposition 11.6.1 and Example 7.8.7. We conclude:

11.6.2 Corollary. *The natural map*

$$\mathcal{H}^*(X)/\exp(\mathcal{H}(X)) \to \mathcal{C}^*(X)/\exp(\mathcal{C}(X))$$

is an isomorphism.

This means that every invertible continuous function f has the form $h\exp(g)$, where h is an invertible holomorphic function and g a continuous function on X. Furthermore, h is unique up to factors of the form $\exp(k)$, with $k \in \mathcal{H}(X)$.

Recall from section 7.9 that if X is any ringed space with structure sheaf $\mathcal{A}$, then the *Picard group* of X, denoted $\mathrm{Pic}(X)$, is the group of isomorphism classes of invertible sheaves of $\mathcal{A}$-modules on X. Here the group operation is tensor product. We will be interested in this notion in the cases where X is a paracompact topological space ($\mathcal{A} = \mathcal{C}$), a complex variety ($\mathcal{A} = \mathcal{H}$), or an algebraic variety ($\mathcal{A} = \mathcal{O}$). In each case, a sheaf of modules is an invertible sheaf if and only if it is the sheaf of sections of a line bundle.

Recall from section 7.9 that if $\mathcal{L}$ is an invertible sheaf of $\mathcal{A}$-modules on the ringed space $(X, \mathcal{A})$, then its Chern class $\mathrm{ch}(\mathcal{L})$ is an element of the cohomology group $H^1(X, \mathcal{A}^*)$ and, in fact, ch defines an isomorphism from the group $\mathrm{Pic}(X)$ to $H^1(X, \mathcal{A}^*)$ (Proposition 7.9.2). In particular, if X is a holomorphic variety, then $\mathrm{ch} : \mathrm{Pic}(X) \to H^1(X, \mathcal{H}^*)$ is an isomorphism, where in this case, $\mathrm{Pic}(X)$ is the group of isomorphism classes of holomorphic line bundles. Let X^t denote the ringed space obtained by forgetting the holomorphic structure on X. That is, X^t is the topological space X with $\mathcal{C}$ as structure sheaf. Then we have two Picard groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X^t)$. There are natural morphisms $\mathrm{Pic}(X) \to \mathrm{Pic}(X^t)$, defined by forgetting that a holomorphic line bundle is holomorphic, and $H^1(X, \mathcal{H}^*) \to H^1(X, \mathcal{C}^*)$, defined by the inclusion $\mathcal{H} \to \mathcal{C}$. It is clear that the diagram

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{\mathrm{ch}} & H^1(X, \mathcal{H}^*) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X^t) & \xrightarrow{\mathrm{ch}} & H^1(X, \mathcal{C}^*) \end{array}$$

is commutative. By Proposition 11.6.1 and Exercise 7.18, the vertical arrow on the right is an isomorphism, with both groups naturally isomorphic to $H^2(X, \mathbb{Z})$. This proves:

11.6.3 Corollary. *Let X be a Stein space. Then* $\mathrm{Pic}(X) \to \mathrm{Pic}(X^t)$ *is an isomorphism. Thus, both groups are isomorphic to $H^2(X, \mathbb{Z})$ via the Chern class.*

This means that every continuous line bundle on a Stein space X is isomorphic to a holomorphic line bundle, which is unique up to isomorphism. The generalization of this result to vector bundles is also true and is due to Grauert [**Gr**]. We will not attempt to prove it here.

11.7 Meromorphic Functions

Now suppose X is a connected complex manifold. Then for each connected open set $U \subset X$, the ring $\mathcal{H}(U)$ is an integral domain. Let $\mathcal{M}$ denote the sheaf on X generated by the presheaf which assigns to each connected open set U the quotient field of $\mathcal{H}(U)$. The sheaf $\mathcal{M}$ is called the *sheaf of* *meromorphic functions* on U. It is easy to see that $\mathcal{H}$ is a subsheaf of $\mathcal{M}$, and at each $x \in X$, the stalk $\mathcal{M}_x$ is the quotient field of the stalk $\mathcal{H}_x$.

We let $\mathcal{M}^*$ be the sheaf of non-zero elements of $\mathcal{M}$ under multiplication. Clearly, $\mathcal{H}^* \subset \mathcal{M}^*$. The quotient sheaf $\mathcal{M}^*/\mathcal{H}^*$ is denoted $\mathcal{D}$ and called the *sheaf of divisors* on X.

A global section of $\mathcal{M}$ is called a *meromorphic function* on X, while a global section of $\mathcal{D}$ is called a *divisor* on X.

Divisors are related to line bundles. In fact, if D is a divisor, then for each $x \in X$, there is a neighborhood U of x in which D can be represented by a function $f \in \mathcal{M}^*(U)$. Any other such representative has the form hf, with $h \in \mathcal{H}^*(U)$. Thus, the sheaf of $\mathcal{H}|_U$-submodules of $\mathcal{M}$ generated by f^{-1} is independent of the choice of representative f. In this way, D uniquely defines a sheaf of submodules of $\mathcal{M}|_U$. The uniqueness implies that these sheaves agree on overlapping neighborhoods, and hence, define a sheaf of $\mathcal{H}$-submodules of $\mathcal{M}$. We denote this sheaf by $\mathcal{H}(D)$. It follows easily from the construction that $\mathcal{H}(D_1) \otimes_{\mathcal{H}} \mathcal{H}(D_2) = \mathcal{H}(D_1 D_2)$, and $\mathcal{H}(1) = \mathcal{H}$. Thus, we have:

11.7.1 Proposition. *For each divisor D on X, $\mathcal{H}(D)$ is an invertible sheaf of $\mathcal{H}$-modules, and the correspondence $D \to \mathcal{H}(D)$ is a group homomorphism from $\mathcal{D}(X)$ to* $\operatorname{Pic}(X)$.

The long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathcal{H}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$$

yields an exact sequence

$$0 \to \mathcal{H}^*(X) \to \mathcal{M}^*(X) \to \mathcal{D}(X) \to H^1(X, \mathcal{H}^*) \to H^1(X, \mathcal{M}^*).$$

Since $H^1(X, \mathcal{H}^*)$ is isomorphic to $\operatorname{Pic}(X)$, this gives another morphism of $\mathcal{D}(X)$ to $\operatorname{Pic}(X)$. One easily checks that it is the same as the one given in the previous proposition. Now if ξ is a holomorphic line bundle representing an element of $\operatorname{Pic}(X)$, and if ξ has a section that is not identically 0, then the Chern class of ξ is sent to 0 by the map $H^1(X, \mathcal{H}^*) \to H^1(X, \mathcal{M}^*)$ (Exercise 11.8). Now suppose X is a *Stein manifold* – that is, a Stein space which is also a complex manifold. By Cartan's Theorem A, every coherent sheaf on a Stein space is generated by its global sections. We conclude that every holomorphic line bundle on X has a section which is not identically 0. Hence, $H^1(X, \mathcal{H}^*) \to H^1(X, \mathcal{M}^*)$ is the zero map, and we conclude from the above exact sequence and Proposition 11.6.1 that:

11.7.2 Theorem. *If X is a Stein manifold, then there is an exact sequence*

$$0 \to \mathcal{H}^*(X) \to \mathcal{M}^*(X) \to \mathcal{D}(X) \to H^2(X, \mathbb{Z}) \to 0.$$

Thus, every element of $H^2(X, \mathbb{Z})$ *is the Chern class of some divisor, and a divisor has Chern class* 0 *if and only if it is the divisor of a global meromorphic function.*

A meromorphic function is locally of the form f/g, for holomorphic functions f and g. When is a meromorphic function of this form globally? With a little more work, the previous corollary leads to an answer in the case where X is a Stein manifold. First we need to develop a little more information about the local structure of meromorphic functions on a Stein manifold X. For one thing, note that $\mathcal{H}_x$ is a unique factorization domain for each $x \in X$, and so every element of the quotient field $\mathcal{M}_x$ has a unique representation as $\mathbf{f}/\mathbf{g}$, with $\mathbf{f}, \mathbf{g}$ relatively prime elements of $\mathcal{H}_x$. We have the following characterization of relatively prime pairs of elements of $\mathcal{H}_x$.

We denote the locus of the principle ideal generated by $\mathbf{f} \in \mathcal{H}_x$ by $\mathrm{loc}(\mathbf{f})$. This is the germ at x of the zero set of a representative of $\mathbf{f}$.

11.7.3 Proposition. *If* X *is a complex manifold of dimension* n, $x \in X$, *and* $\mathbf{f}$ *and* $\mathbf{g}$ *are non-units in* $\mathcal{H}_x$, *then* $\mathbf{f}$ *and* $\mathbf{g}$ *are relatively prime if and only if* $\mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$ *has pure dimension* $n-2$.

Proof. If $\mathbf{f}$ and $\mathbf{g}$ are not relatively prime, there is a non-unit $\mathbf{h} \in \mathcal{H}_x$ which is a greatest common divisor for $\mathbf{f}$ and $\mathbf{g}$. Then $\mathrm{loc}(\mathbf{h}) \subset \mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$. Since $\dim \mathrm{loc}(\mathbf{h}) = n-1$, by Proposition 5.2.1, the dimension of $\mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$ is at least $n-1$.

On the other hand, the irreducible branches of $\mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$ all have dimension $n-1$ or $n-2$, by Proposition 5.2.3. Thus, if $\mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$ does not have pure dimension $n-2$, then we may choose an irreducible subvariety $\mathbf{V}$ of $\mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$ which has dimension $n-1$. But a germ of a subvariety of pure dimension $n-1$ is the locus of a principal ideal, by Proposition 5.2.1. Hence, there is an irreducible $\mathbf{h} \in \mathcal{H}_x$, with $\mathrm{loc}(\mathbf{h}) \subset \mathrm{loc}(\mathbf{f}) \cap \mathrm{loc}(\mathbf{g})$. By the Nullstellensatz, $\mathbf{h}$ divides some power of $\mathbf{f}$ and some power of $\mathbf{g}$. Since $\mathbf{h}$ is irreducible and $\mathcal{H}_x$ is a unique factorization domain, this implies that $\mathbf{h}$ divides both $\mathbf{f}$ and $\mathbf{g}$. Thus, $\mathbf{f}$ and $\mathbf{g}$ are not relatively prime.

Now let m be a meromorphic function on the complex manifold X. Then its germ $\mathbf{m}_x \in \mathcal{M}_x$ may or may not belong to $\mathcal{H}_x$. We call the set of $x \in X$ such that $\mathbf{m}_x \notin \mathcal{H}_x$ the *pole set* of m and denote it by $P(m)$.

11.7.4 Proposition. *If* m *is a meromorphic function on a complex manifold* X, *and* m *is not holomorphic, then the pole set of* m *is a subvariety of pure dimension* $n-1$.

Proof. Given $x \in P(m)$, let $\mathbf{m}_x = \mathbf{f}_x/\mathbf{g}_x$, with $\mathbf{f}_x$ and $\mathbf{g}_x$ relatively prime elements of $\mathcal{H}_x$. Let f and g be representatives of $\mathbf{f}_x$ and $\mathbf{g}_x$ in some neighborhood U of x, and suppose U is chosen small enough that $\mathbf{m}_y = \mathbf{f}_y/\mathbf{g}_y$, for all $y \in U$. Since $\mathbf{f}_x$ and $\mathbf{g}_x$ are relatively prime, we have that the germ of $\operatorname{loc}(f) \cap \operatorname{loc}(g)$ has dimension $n-2$ at x. Therefore, $\operatorname{loc}(f) \cap \operatorname{loc}(g)$ has pure dimension $n-2$ as a subvariety of U if U is chosen small enough. It follows that f and g have relatively prime germs at every point of such a neighborhood U. Then the pole set $P(m)$ has, as intersection with U, the zero set $\operatorname{loc}(g)$ of g. This proves that $P(m)$ is a subvariety of pure dimension $n-1$.

11.7.5 Theorem. *If X is a Stein manifold such that $H^2(X, \mathbb{Z}) = 0$, then every meromorphic function m on X has the form f/g, with $f, g \in \mathcal{H}(X)$.*

Proof. If m is not holomorphic, the pole set $P = P(m)$ is a subvariety of X of pure dimension $n-1$. It follows that its ideal sheaf $\mathcal{I}_P$ has stalks which are principle ideals. However, if g is a holomorphic function defined in a neighborhood of $x \in X$, and its germ $\mathbf{g}_x$ generates $(\mathcal{I}_P)_x$, then g actually generates $\mathcal{I}_P$ over a neighborhood of x (Exercise 9.6). Furthermore, $\mathbf{g}$ is unique with this property, up to factors which are units. Thus, $\mathcal{I}_P$ uniquely determines a divisor D on X. If $H^2(X, \mathbb{Z}) = 0$, then Theorem 11.7.2 implies that D is determined by a global meromorphic function g. This function has the property that if $x \in X$ and $\mathbf{m}_x = \mathbf{k}_x/\mathbf{h}_x$ with $\mathbf{k}_x, \mathbf{h}_x$ relatively prime elements of $\mathcal{H}_x$, then $\mathbf{h}_x = \mathbf{u}_x\mathbf{g}_x$ for some unit $\mathbf{u}_x \in \mathcal{H}_x^*$. It follows that g and $f = gm$ are holomorphic. Then $m = f/g$ is the required global representation of m.

11.7.6 Example. If X is a Riemann surface (a connected complex manifold of pure dimension 1), and U is an open subset of X, then two elements of $\mathcal{M}^*(U)$ are equivalent mod $\mathcal{H}^*(U)$ if and only if they have zeroes and poles of the same orders at the same points of X. Thus, a divisor on U is determined by assigning an integer to each point – the order of the pole of a representative meromorphic function at the point (zeroes are poles of negative order), in such a way that only a discrete set of points are assigned non-zero integers. Thus, the sheaf of divisors $\mathcal{D}$ is the sheaf of locally finitely non-zero integer valued functions.

It turns out that every non-compact Riemann surface X is a Stein space and has vanishing $H^2(X, \mathbb{Z})$ (see [**Fo**] and Exercise 11.14). Thus, both Theorem 11.7.2 and Theorem 11.7.5 apply. It follows that, on a non-compact Riemann surface, every divisor is the divisor of a global meromorphic function, and every global meromorphic function has the form f/g for a pair of holomorphic functions f and g. This result generalizes the Weierstrass theorem from Chapter 1 (Theorem 1.6.3).

Every compact Riemann surface X has a non-constant global meromorphic function (Exercise 11.17), and this fact can be used to show that there is a finite holomorphic map $f : X \to P^1$, where P^1 is the Riemann sphere (Exercise 12.7). The existence of a non-constant meromorphic function is a nice application of the finiteness of cohomology for coherent sheaves on a compact holomorphic variety – a result we shall prove later in this chapter (Theorem 11.10.2).

11.8 Holomorphic Functional Calculus

There is a very nice application of several complex variables techniques to functional analysis. This is the proof of the existence of the holomorphic functional calculus for commutative Banach algebras. Banach algebras were introduced and discussed briefly in section 10.3.

In this section, by the term *algebra*, we will mean *complex algebra with identity.* In a Banach algebra the identity is required to have norm 1. Scalar multiplication of the identity by an element of $\mathbb{C}$ determines an injection of $\mathbb{C}$ into A. We will denote the image of $\lambda \in \mathbb{C}$ under this injection simply by λ. Thus, "$\lambda - a$" for $\lambda \in \mathbb{C}$ and $a \in A$ means "λ times the identity minus a".

Let A be a commutative algebra over $\mathbb{C}$. For an n-tuple $a = (a_1, \dots, a_n)$ of elements of A and a complex polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$, there is a well-defined element $p(a) = p(a_1, \dots, a_n) \in A$ – that is, the polynomial p acts on the n-tuple a. More precisely, a determines a unique algebra homomorphism

$$p \to p(a) : \mathbb{C}[z_1, \dots, z_n] \to A$$

taking the identity to the identity and z_i to a_i for each i. The holomorphic functional calculus for commutative Banach algebras states that, if A is a commutative Banach algebra, then this homomorphism extends to a homomorphism of the algebra $\mathcal{H}(U)$ into A, where U is any open set containing a certain compact set $\sigma(a) \subset \mathbb{C}^n$, determined by a. The set $\sigma(a)$ is called the *joint spectrum* of a and is defined below.

There is a relatively simple proof of the existence of the holomorphic functional calculus, based on the vanishing of cohomology for coherent analytic sheaves on a polydisc. The purpose of this section is to present that proof. We do not use the full power of Cartan's Theorem B, since the coherent sheaves involved are defined on all of $\mathbb{C}^n$ and so Theorem 10.5.3 applies. We first need a few preliminaries on commutative Banach algebras.

Let A be a Banach algebra. The spectrum of an element $a \in A$ is the set of all $\lambda \in \mathbb{C}$ such that the element $\lambda - a$ is not invertible in A. If A is commutative and if $a = (a_1, \dots, a_n)$ is an n-tuple of elements of A, then the *joint spectrum* $\sigma(a)$ of a is the set of n-tuples $\lambda \in \mathbb{C}^n$ for which the equation

$$(\lambda_1 - a_1)b_1 + \cdots + (\lambda_n - a_n)b_n = 1 \tag{11.8.1}$$

fails to have a solution $(b_1, \dots, b_n) \in A^n$. The complement in $\mathbb{C}^n$ of the spectrum of a is called the *resolvent set* of a.

The resolvent set is always open, and hence, the spectrum is closed. This is due to the fact that, if λ is in the resolvent set of a, so that (11.8.1) has a solution, then a small perturbation of the n-tuple a will leave the left side of (11.8.1) sufficiently close to 1 that it will be an invertible element c of A, by Theorem 10.3.1. Then multiplying each b_i by c^{-1} will yield a solution to (11.8.1) for the perturbed a.

The joint spectrum of $a \in A^n$ is also a bounded set, and hence, is compact. In fact,

$$\sigma(a) \subset \overline{\Delta}(0, r), \text{ if } r = (||a_1||, \dots, ||a_n||).$$

This is due to the fact that, if $\lambda > ||a_i||$, for some i, then $1 - \lambda^{-1}a_i$ has an inverse, by Theorem 10.3.1, and hence, so does $\lambda - a_i$. Then $b_i = (\lambda - a_i)^{-1}$ and $b_j = 0$ for $j \neq i$ provides a solution to (11.8.1).

In the case of a single element a of a Banach algebra, $\sigma(a)$ is a compact subset of the disc $\overline{\Delta}(0, ||a||)$. The smallest number r such that $\sigma(a) \subset \overline{\Delta}(0, r)$ is called the *spectra radius* of a. It is, in fact, given by (Exercises 11.21 and 11.22):

$$r = \lim_n ||a^n||^{1/n}.$$

The spectrum of an element of A is also non-empty. This follows easily from Liouville's theorem:

11.8.1 Proposition. *If A is a Banach algebra, and $a \in A$, then $\sigma(a)$ is non-empty.*

proof. Given $\lambda, \mu \in \mathbb{C}$, we have the identity

$$\frac{1}{\lambda - a} - \frac{1}{\mu - a} = \frac{\mu - \lambda}{(\lambda - a)(\mu - a)}.$$

Since inversion is a continuous operation on A^{-1} (Theorem 10.3.1(ii)), we conclude that

$$\lim_{\mu \to \lambda} \frac{1}{\lambda - \mu}\left[\frac{1}{\lambda - a} - \frac{1}{\mu - a}\right] = \frac{-1}{(\lambda - a)^2}.$$

Thus $\lambda \to (\lambda - a)^{-1}$ is a holomorphic A-valued function on the resolvent set of a. If the resolvent set of a is all of $\mathbb{C}$, then this is an entire function. It is also bounded, in fact

$$\lim_{\lambda \to \infty} (\lambda - a)^{-1} = \lim_{\lambda \to \infty} \lambda^{-1}(1 - \lambda^{-1}a)^{-1} = 0.$$

It follows from Liouville's theorem for Banach space valued entire functions (Exercise 1.20) that $(\lambda - a)^{-1}$ is identically 0. This is clearly impossible, and so we conclude that the resolvent set of a cannot be all of $\mathbb{C}$. Thus, the spectrum is non-empty.

11.8.2 Corollary. *A Banach algebra which is a field is isomorphic to* $\mathbb{C}$.

Proof. If A is a Banach algebra which is a field, then every non-zero element of A is invertible. However, for every $a \in A$ there is a $\lambda \in \sigma(a)$, that is, a $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. Then $\lambda - a$ must be 0. In other words, every element of A is a scalar multiple of the identity. This means that the natural map $\mathbb{C} \to A$ is an algebra isomorphism. It is norm preserving, since the identity has norm 1 in both $\mathbb{C}$ and A.

A complex homomorphism of a complex algebra is a homomorphism onto the field of complex numbers. Obviously the kernel of a complex homomorphism is a maximal ideal. In a commutative Banach algebra, the converse is also true.

11.8.3 Proposition. *If A is a commutative Banach algebra, then every maximal ideal of A is the kernel of a complex homomorphism. Furthermore, every complex homomorphism is continuous.*

Proof. The closure of an ideal of A is also an ideal of A, if it is a proper subset of A. However, by Theorem 10.3.1(i), there is a neighborhood of the identity in A, consisting of invertible elements, and an ideal contains no invertible elements. It follows that the closure of an ideal is an ideal, and hence, that maximal ideals are closed.

If M is a maximal ideal of A, then A/M is also a Banach algebra under the induced norm defined by $||a + M|| = \inf\{||a + m|| : m \in M\}$. Since M is maximal, A/M is also a field. By the previous corollary, A/M is isomorphic as a Banach algebra to $\mathbb{C}$. Then the composition of $A \to A/M$ and the isomorphism $A/M \to \mathbb{C}$ is a continuous complex homomorphism with kernel M. Thus, every maximal ideal is the kernel of a continuous complex homomorphism.

It remains to show that every complex homomorphism is continuous. If h is such a homomorphism, then h has a maximal ideal as kernel. It follows that h has the same kernel as some continuous complex homomorphism h'. Since $h(1) = 1 = h'(1)$, if h and h' have the same kernel, then they are the same homomorphism. Thus, $h = h'$, and h is continuous.

Let X denote the set of all complex homomorphisms of the commutative Banach algebra A. Note that each complex homomorphism is, in particular, a bounded linear functional on A, and so $X \subset A^*$.

If h is a complex homomorphism of A and if $a \in A$ with $||a|| \leq 1$, then

$$|h(a)|^2 = |h(a^2)| \leq ||h|| ||a^2|| \leq ||h||.$$

On taking the supremum over all a with $||a|| \leq 1$, this yields $||h||^2 \leq ||h||$, and hence, $||h|| \leq 1$. Since $h(1) = 1$, we have that $||h|| = 1$.

Recall that the weak-$*$ topology on the dual B^* of a Banach space B is the weakest topology for which each of the functions $f \to f(b) : B^* \to \mathbb{C}$, for $b \in B$, is continuous. The Banach-Alaoglu Theorem states that the closed unit ball in B^* is compact in the weak-$*$ topology (see [**R2**], 3.15). In view of this and the preceding paragraph, we conclude the following:

11.8.4 Proposition. *The set X is a weak-$*$ closed subset of the unit ball in A^*, and is, therefore, a weak-$*$ compact subset of A^*.*

The space X, with the compact topology it inherits from the weak-$*$ topology of A^*, is called the *maximal ideal space of A*.

If $a \in A$, then the *Gelfand transform* of a is the continuous function $\hat{a}$ on X defined by $\hat{a}(h) = h(a)$. Then $a \to \hat{a}$ is an algebra homomorphism from A into $C(X)$. It is injective if and only if 0 is the only element killed by all complex homomorphisms $h \in X$, that is, if and only if 0 is the intersection of all maximal ideals of A. A commutative Banach algebra with this property is called *semisimple.*

If $a = (a_1, \ldots, a_n)$ is an n-tuple of elements of A, then its Gelfand transform is the continuous function $\hat{a} : X \to \mathbb{C}^n$ defined by $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)$.

11.8.5 Proposition. *If a is an n-tuple in A^n, then its joint spectrum $\sigma(a)$ is the image of $\hat{a}$ in $\mathbb{C}^n$.*

Proof. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the equation (11.8.1) fails to have a solution if and only if the ideal generated by the elements $\lambda_1 - a_1, \ldots, \lambda_n - a_n$ is proper – that is, if and only if these elements are contained in some maximal ideal. This happens if and only if there is some complex homomorphism $h \in X$ such that $0 = h(\lambda_i - a_i) = \lambda_i - \hat{a}_i(h)$ for $i = 1, \ldots, n$ – that is, if and only if there is an $h \in X$ such that $\hat{a}(h) = \lambda$. This completes the proof.

The proof of the holomorphic functional calculus we will present here is based on the following lemma.

11.8.6 Lemma. *Let $a = (a_1, \dots, a_n)$ be an n-tuple in A^n and U an open set in $\mathbb{C}^n$ containing $\sigma(a)$. Then there is an integer $m \geq n$, an m-tuple $b = (b_1, \dots, b_m)$ in A^m, a finite set $\mathcal{P}$ of polynomials in $\mathbb{C}[z_1, \dots, z_m]$, and a polydisc $\Delta \subset \mathbb{C}^m$, centered at 0, such that*

(i) *$b_i = a_i$ for all $i \leq n$;*

(ii) *the set $\mathcal{P}$ has a non-singular subvariety V of $\mathbb{C}^m$ as its set of common zeroes, and it generates the ideal sheaf in ${}_m\mathcal{H}$ of this subvariety;*

(iii) *the projection $\pi : \mathbb{C}^m \to \mathbb{C}^n$ on the first n coordinates maps $\Delta \cap V$ into U;*

(iv) *$p(b) = 0$ for every $p \in \mathcal{P}$;*

(v) *Δ contains the closed polydisc with polyradius given by the norms of the entries of b.*

Proof. Denote by $\overline{\Delta}_1$ the closed polydisc $\overline{\Delta}(0, r)$, where $r_i = ||a_i||$ for $i = 1, \dots, n$. Recall that $\sigma(a) \subset \overline{\Delta}_1$. Since $\sigma(a) \subset U$, by hypothesis, $\sigma(a) \subset \overline{\Delta}_1 \cap U$. Let $K = \overline{\Delta}_1 - \overline{\Delta}_1 \cap U$. Each $\lambda = (\lambda_1, \dots, \lambda_n) \in K$ is not in $\sigma(a)$, and so there is a solution $(c_1, \dots, c_n) \in A^n$ to the equation

$$(\lambda_1 - a_1)c_1 + \cdots + (\lambda_n - a_n)c_n = 1. \tag{11.8.2}$$

We define a polynomial $p \in \mathbb{C}[z_1, \dots, z_{2n}]$ by

$$p(z_1, \dots, z_{2n}) = (\lambda_1 - z_1)z_{n+1} + \cdots + (\lambda_n - z_n)z_{2n} - 1.$$

Then (11.8.2) says that $p(a_1, \dots, a_n, c_1, \dots, c_n) = 0$. Note that, at each point of the zero set of p, at least one of the partial derivatives $\dfrac{\partial p}{\partial z_{n+i}}$, $i = 1, \dots, n$, is non-vanishing.

Consider the closed polydisc $\overline{\Delta}(0, \rho)$ in $\mathbb{C}^{2n}$ with polyradius

$$\rho = (||a_1||, \dots, ||a_n||, ||c_1||, \dots, ||c_n||).$$

Let $\pi : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the projection. Since $\pi^{-1}(\lambda)$ does not meet the zero set $V(p)$ of p, the set $K_\lambda = \pi(V(p) \cap \overline{\Delta}(0, \rho))$ is a compact subset of $\overline{\Delta}_1$ which does not contain λ. If we carry out this procedure for each $\lambda \in K$, then K and the resulting sets K_λ constitute a family of compact sets with empty intersection. It follows that some finite subfamily $\{K, K_{\lambda^1}, \dots, K_{\lambda^{q-1}}\}$ also has empty intersection. Each of the corresponding points $\lambda^j \in K$ determines an equation of the form (11.8.2). We denote the solution of this equation by $c^j = (c^j_1, \dots, c^j_n)$. We then form a qn-tuple $b = (b_1, \dots, b_{qn}) \in A^{qn}$ by setting

$$b_i = a_i \text{ for } i = 1, \dots, n,$$

$$b_{jn+i} = c^j_i \text{ for } j = 1, \dots, q-1,\ i = 1, \dots, n.$$

Thus, (i) is satisfied for this tuple. For $j = 1, \dots, q-1$, we define polynomials $p_j \in \mathbb{C}[z_1, \dots, z_{qn}]$ by

$$p_j(z_1, \dots, z_{qn}) = (\lambda_1^j - z_1)z_{jn+1} + \cdots + (\lambda_n^j - z_n)z_{jn+n} - 1,$$

where $\lambda^j = (\lambda_1^j, \dots, \lambda_n^j)$. Then (11.8.2), with λ replaced by λ^j and c by c^j, becomes the equation $p_j(b) = 0$. Thus, (iv) is satisfied.

Let $V \subset \mathbb{C}^{qn}$ denote the set of common zeroes of the polynomials p_j. Note that, at each point of V and for each $j = 1, \dots, q-1$, at least one of the partial derivatives $\dfrac{\partial p_j}{\partial z_{jn+i}}$, $i = 1, \dots, n$, is non-vanishing. This implies that the Jacobian of the map $\mathbb{C}^{qn} \to \mathbb{C}^{q-1}$, with the p_j as coordinate functions, has rank $q-1$ at each point of V. In view of the implicit mapping theorem (Theorem 3.7.4), this implies that V is a non-singular variety, and its ideal sheaf is generated by the set of polynomials $\{p_j\}$. Thus, (ii) is satisfied.

The fact that the family $\{K, K_{\lambda^1}, \dots, K_{\lambda^{q-1}}\}$ has empty intersection implies that the intersection of V with the polydisc $\overline{\Delta}(0, (||b_1||, \dots, ||b_{qn}||))$ projects into U under $\pi : \mathbb{C}^{qn} \to \mathbb{C}^n$. The same thing will be true of the intersection of V with a slightly larger open polydisc Δ. Then (iii) and (v) are satisfied. Thus, with $m = qn$, the m-tuple b, set of polynomials $\mathcal{P} = \{p_1, \dots, p_{q-1}\}$, and polydisc Δ satisfy (i) through (v) of the lemma.

We are now in a position to prove the existence of the holomorphic functional calculus.

11.8.7 Theorem. *Let $a = (a_1, \dots, a_n)$ be an n-tuple of elements of the commutative Banach algebra A, and let $U \subset \mathbb{C}^n$ be an open set containing $\sigma(a)$. Then there is an algebra homomorphism $f \to f(a) : \mathcal{H}(U) \to A$ which takes z_i to a_i for each i. Furthermore, $\widehat{f(a)} = f \circ \hat{a}$, and if A is semisimple, $f(a)$ is uniquely determined by this property.*

Proof. We choose m, $b \in A^m$, $\Delta \subset \mathbb{C}^m$, and $V \subset \mathbb{C}^m$, as in the previous lemma. Let $\Delta = \Delta(0, r)$. Then (v) of the lemma says that $||b_i|| < r_i$ for $i = 1, \dots, m$. This implies that, if $g \in \mathcal{H}(\Delta)$, then the power series expansion for g in Δ yields a convergent series in A when the variables $z_1, \dots, z_m$ are replaced by the Banach algebra elements $b_1, \dots, b_m$. Hence, there is a homomorphism

$$g \to g(b) : \mathcal{H}(\Delta) \to A$$

which takes z_i to b_i for each i. We will use this to define a homomorphism of $\mathcal{H}(U)$ into A.

Since $\pi : \mathbb{C}^m \to \mathbb{C}^n$ maps $V \cap \Delta$ into U, each $f \in \mathcal{H}(U)$ yields an element $f \circ \pi \in \mathcal{H}(V \cap \Delta)$. This has an extension $\tilde{f}$ to an element $f \in \mathcal{H}(\Delta)$, by Corollary 10.5.4 (this is one place a vanishing theorem for cohomology of coherent sheaves enters into the proof). We then define $f(a)$ to be the element $\tilde{f}(b) \in A$, given by substitution in the power series for $\tilde{f}$, as in the previous paragraph. This appears to depend on the choice of the extension $\tilde{f}$, but we will show that it does not.

Since the set of polynomials $\mathcal{P}$ generates the ideal sheaf of V, it follows from Theorem 10.5.3 (see Exercise 10.17) that it also generates its ideal of sections in $\mathcal{H}(\Delta)$ – that is, $\mathcal{P}$ generates the ideal in $\mathcal{H}(\Delta)$ consisting of functions which vanish on $V \cap \Delta$ (this is the other place a vanishing theorem for cohomology of coherent sheaves enters into the proof). Each element of $\mathcal{P}$ is sent to 0 by the homomorphism $g \to g(b)$ by (iv) of Lemma 11.8.6. Consequently, each function which vanishes on $V \cap \Delta$ is also sent to 0 by $g \to g(b)$. Thus, for $f \in \mathcal{H}(U)$, any two extensions of $f \circ \pi \in \mathcal{H}(V \cap \Delta)$ to functions in $\mathcal{H}(\Delta)$ are sent to the same element of A by the homomorphism $g \to g(b)$. In other words, the map $f \to f(a) : \mathcal{H}(U) \to A$ is well defined. It is clearly an algebra homomorphism which takes z_i to a_i for each i.

If h is a complex homomorphism in X, then

$$\widehat{f(a)}(h) = h(\tilde{f}(b)) = \tilde{f}(h(b)) = f \circ \pi(h(b)) = f(h(a)) = \hat{a}(h), \tag{11.8.3}$$

since $\pi(h(b)) = h(a)$. If A is semisimple, so that complex homomorphisms separate points in A, then the element $f(a)$ is uniquely defined by (11.8.3) and does not depend on the choices made in Lemma 11.8.6. This completes the proof.

In some applications of the holomorphic functional calculus, it is important to know that we can arrange for the set U in the above theorem to be a Stein space – that is, to be holomorphically convex (see Exercise 10.8). This will be true if U is a polynomial polyhedron, that is, a set of the form

$$U = \{z \in \mathbb{C}^n : |p_i(z)| < r_i, \ i = 1, \dots, k\},$$

for a finite set $\{p_1, \dots, p_k\}$ of polynomials, and numbers $r_i > 0$, $i = 1, \dots, k$. The next proposition shows how to reduce the general case of the holomorphic functional calculus to one in which the open set is a polynomial polyhedron.

11.8.8 Proposition. *Given an n-tuple $a \in A^n$ and an open set $U \subset \mathbb{C}^n$, with $\sigma(a) \subset U$, there exists an $m > n$, an m-tuple $b \in A^m$, and a polynomial polyhedron $W \subset \mathbb{C}^m$, such that $b_i = a_i$ for $i = 1, \dots, n$, $\sigma(b) \subset W$, and the projection $\pi : \mathbb{C}^m \to \mathbb{C}^n$ maps W into U.*

Proof. Let b, $\mathcal{P}$, V, and $\overline{\Delta}_1$ be chosen as in the proof of Lemma 11.8.6. We then have $\sigma(b) \subset \overline{\Delta}_1 \cap V$, and $\pi(\overline{\Delta}_1 \cap V) \subset U$. Since $\overline{\Delta}_1 \cap V$ is compact, it is contained in neighborhoods W which also satisfy $\pi(W) \subset U$. We need to show that we can choose one which is a polynomial polyhedron. By definition,

$$\overline{\Delta}_1 \cap V = \{z \in \mathbb{C}^m : |z_i| \leq ||a_i||,\ |p(z)| = 0, \text{ for } i = 1, \dots, n,\ p \in \mathcal{P}\}.$$

It follows that each neighborhood of $\overline{\Delta}_1 \cap V$ contains one of the form

$$W = \{z \in \mathbb{C}^m : |z_i| \leq r_i,\ |p(z)| < \epsilon, \text{ for } i = 1, \dots, n,\ p \in \mathcal{P}\}$$

for appropriate choices of $r_i > ||a_i||$ and $\epsilon > 0$. A set W of this form is a polynomial polyhedron. This completes the proof.

The holomorphic functional calculus seems particularly suited to proving results which connect the topology of the maximal ideal space X with the algebraic structure of the algebra A. We finish this section with three examples of this type of result.

11.8.9 Shilov's Idempotent Theorem. *If A is a commutative Banach algebra with maximal ideal space X, then the additive subgroup of A generated by the idempotents of A is isomorphic to $H^0(X, \mathbb{Z})$.*

Proof. The additive subgroup of $C(X)$ generated by the idempotents of $C(X)$ is the additive group of integer valued continuous functions on X, and this is the group of sections of the constant sheaf with stalk $\mathbb{Z}$. Thus, the proof will be complete if we can show that every idempotent in $C(X)$ is the Gelfand transform of an idempotent in A, equivalently, that every open-compact subset of X is the support of $\hat{p}$ for some idempotent $p \in A$.

Let $K \subset X$ be both open and compact. Since the set of functions of the form $\hat{a}$, $a \in A$, separates points in X, we can choose an n-tuple $a = (a_1, \dots, a_n)$, such that $\hat{a} : X \to \mathbb{C}^n$ takes K and its complement to disjoint compact sets. There are then open sets U_1 and U_2 in $\mathbb{C}^n$ with $U_1 \cap U_2 = \emptyset$, $\hat{a}(K) \subset U_1$ and $\hat{a}(X - K) \subset U_2$. Then $U = U_1 \cup U_2$ is an open set containing $\sigma(a)$ and the function f, which is 1 on U_1 and 0 on U_2, is an idempotent holomorphic function on U. Thus, the element $p = f(a)$, given by the holomorphic functional calculus, is an idempotent of A with $\hat{p}$ equal to the characteristic function of K.

The next result involves the invertible group A^{-1} of A and its subgroup $\exp(A)$, defined by $\exp(A) = \{\exp(a) : a \in A\}$. Recall from Theorem 10.3.1(v) that this subgroup is open and, in fact, is the identity component of A^{-1}. Thus the group $A^{-1}/\exp(A)$ is the group of components of A^{-1}. A word about notation is in order: In sections 11.6 and 11.7, we used $\mathcal{H}^*$, $\mathcal{C}^*$, and $\mathcal{M}^*$ to denote the sheaves of invertible sections, under multiplication, for the indicated sheaves of algebras. However, we denote the invertible group of a Banach algebra A by A^{-1} rather than by A^*. This is done both because the notation A^{-1} is the one typically used in Banach algebra theory and because we use A^* here to denote the Banach space dual of A.

11.8.10 Arens-Royden Theorem. *The group $A^{-1}/\exp(A)$ is naturally isomorphic to $H^1(X, \mathbb{Z})$.*

Proof. We know that this is true for the Banach algebra $C(X)$, by Example 7.8.7. The analogous statement is also true for the algebra $\mathcal{H}(X)$ if X is any Stein space, by Proposition 11.6.1. The proof of the Arens-Royden theorem is just a matter of putting these facts together with the holomorphic functional calculus in the right way.

The Gelfand transform $a \to \hat{a} : A \to C(X)$, being an algebra homomorphism, maps A^{-1} into $C(X)^{-1}$. Since it is also continuous, it commutes with the exponential function, and hence, sends $\exp(A)$ into $\exp(C(X))$. Consequently, the Gelfand transform defines a group homomorphism

$$A^{-1}/\exp(A) \to C(X)/\exp(C(X)) \simeq H^1(X, \mathbb{Z}).$$

To complete the proof, we need to show that this is surjective and injective.

The subalgebra of $C(X)$ generated by the functions $\hat{a}$ and their conjugates, for $a \in A$, is dense in $C(X)$, by the Stone-Weierstrass theorem. Thus, if $f \in C(X)^{-1}$, then there is an n-tuple $a \in A^n$ and a function $g(z) \in C(\mathbb{C}^n)$, which is a polynomial in $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$, such that $|g \circ \hat{a} - f| < |f|$ on X. Hence, $|f^{-1} g \circ \hat{a} - 1| < 1$ on X, which implies that $f^{-1} g \circ \hat{a}$ has a continuous logarithm ϕ. Then $f = g \circ \hat{a} \exp \phi$.

The restriction of g to $\sigma(a) \subset \mathbb{C}^n$ is a non-vanishing continuous function. Hence, its restriction to some open set U containing $\sigma(a)$ is also non-vanishing. Proposition 11.8.8 implies that, by adjoining additional elements of A to the n-tuple a, if necessary, we may assume, without loss of generality, that U is a polynomial polyhedron and, hence, a Stein space.

Since U is a Stein space, it follows from Corollary 11.6.2 that there is a holomorphic function $h \in \mathcal{H}(U)$ and a continuous function $\psi \in C(U)$ such that $g = h \exp \psi$ on U. We apply the holomorphic functional calculus (Theorem 11.8.7) to h to obtain an element $b = h(a) \in A$ with $\hat{b} = h \circ \hat{a}$. By construction,

$$f = g \circ \hat{a} \exp(\phi) = h \circ \hat{a} \exp(\psi \circ \hat{a}) \exp(\phi) = \hat{b} \exp(\psi \circ \hat{a} + \phi).$$

This proves that the map $A^{-1}/\exp(A) \to C(X)/\exp(C(X))$ is surjective.

The injectivity is similar. Suppose $a_1 \in A^{-1}$ and $\hat{a}_1 = \exp(\phi)$ for some $\phi \in C(X)$. We must show that $a_1 \in \exp(A)$. We first construct an n-tuple $a = (a_1, \ldots, a_n) \in A^n$, with a_1 as first entry, and with the property that there exists $\psi \in C(\mathbb{C}^n)$, which is a polynomial in $z_1, \ldots, z_n$ and $\bar{z}_1, \ldots, \bar{z}_n$, such that ϕ and $\psi \circ \hat{a}$ are sufficiently close that $|\hat{a}_1 \exp(-\psi \circ \hat{a}) - 1| < 1$ on X. It follows that $|z_1 \exp(-\psi) - 1| < 1$ on a neighborhood U of $\sigma(a)$. This implies that $z_1 \exp(-\psi)$ has a continuous logarithm on U which, in turn, implies that z_1 has a continuous logarithm on U. Because of Proposition 11.8.8, we may assume, without loss of generality, that U is a polynomial polyhedron. Then Corollary 11.6.2 implies that z_1 has a holomorphic logarithm ω on U. If we apply the holomorphic functional calculus to ω, we obtain an element $b = \omega(a) \in A$ such that $\exp(\hat{b}) = \hat{a}_1$. Then $c = 1 - a_1 \exp(-b)$ has Gelfand transform 0 and, hence, has spectrum $\{0\}$. This implies that $\lim_k ||c^k||^{1/k} = 0$ (Exercise 11.21 and 11.22) and, hence, that the power series for $\log(1-c)$ converges in A. That is, $c = \exp(u)$ for some $u \in A$. We conclude that $a_1 = \exp(b + u)$. This completes the proof.

We mention one more result along these lines, but without proof. The Picard group $\mathrm{Pic}(A)$ of an algebra A is the group of isomorphism classes of invertible projective modules over A, where a projective A-module M is called *invertible* if there is a projective A-module N such that $M \otimes_A N \simeq A$ (compare with the corresponding notion for ringed spaces, as discussed in sections 7.9 and 11.6).

11.8.11 Forster's Theorem. *There is a natural isomorphism from* $\mathrm{Pic}(A)$ *to* $H^2(X, \mathbb{Z})$.

This follows from Corollary 11.6.3 and an appropriate use of the holomorphic functional calculus, but the proof involves additional technical issues we don't wish to get into here. There are many more results of this kind. For example, there is a result relating the algebraic K-theory of A to the topological K-theory of X. See [**T2**] for an expository article which includes proofs of these results as well as Forster's theorem.

There are more sophisticated approaches to the holomorphic functional calculus that yield somewhat more information. We will briefly mention one of these. It involves defining a notion of spectrum, not just for a commuting n-tuple in a commutative Banach algebra, but for a commuting n-tuple $a = (a_1, \ldots, a_n)$ of operators on a Banach space M. The n-tuple of operators a is thought of as defining on M the structure of a P_n-module M_a, where $P_n = \mathbb{C}[z_1, \ldots, z_n]$, and the module action is $(p, m) \to p(a)m$. In the same way, a number $\lambda \in \mathbb{C}^n$ defines the structure of a P_n-module $\mathbb{C}_\lambda$ on the space $\mathbb{C}$. Then λ is in the resolvent set of a if $\mathrm{Tor}_q^{P_n}(M_a, \mathbb{C}_\lambda) = 0$ for all q, including $q = 0$; otherwise, λ is said to be in the spectrum of a. The resulting spectrum $\sigma(a)$ is compact and non-empty. In this context, the holomorphic functional calculus asserts that there is a natural extension of the P_n-module structure on M to an $\mathcal{H}(\sigma(a))$-module structure. The case of an n-tuple of elements in a commutative Banach algebra A is covered as the special case in which the Banach space M is A and the elements of the n-tuple act as operators via multiplication. The holomorphic functional calculus, defined in this way, is canonical and does not involve the kinds of arbitrary choices made in Theorem 11.8.7. For details see [**T**].

11.9 Localization

We promised earlier to discuss localization in the context of modules over algebras of holomorphic functions. This type of localization is used in the theory of analytic $\mathcal{D}$-modules as expounded in [**Bj**]. As we pointed out in Chapter 9, the ring $\mathcal{H}(U)$ is typically not Noetherian and this poses a problem for any localization theory. However, for a broad class of Stein compact sets K, the ring $\mathcal{H}(K)$ is Noetherian. Frisch proved in [**F**] that this is true of semianalytic Stein compact sets. Siu [**Siu**] proved the following characterization of Stein compact sets K for which $\mathcal{H}(K)$ is Noetherian.

11.9.1 Theorem. *Let K be a Stein compact subset of a holomorphic variety. Then $\mathcal{H}(K)$ is Noetherian if and only if, for each subvariety V of a neighborhood of K, $K \cap V$ has finitely many connected components.*

We will say that a Stein compact set K has the *Noether property* if $\mathcal{H}(K)$ is Noetherian.

As mentioned above, Stein compact semianalytic sets have the Noether property (see [**F**] for the definition of semianalytic set). Every compact analytic polyhedron is a semianalytic Stein compact set, and so every compact analytic polyhedron has the Noether property. Here, a compact analytic polyhedron is a compact set defined by finitely many inequalities of the form $|f(x)| \le 1$ for $f \in \mathcal{H}(X)$, X a Stein space.

Let K be a set which has the Noether property. We will show how localization can be carried out for modules over $\mathcal{H}(K)$. Let M be such a module. Then, as in Chapter 8, we define the *localization* of M on K to be the sheaf $\tilde{M}$ defined by

$$\tilde{M} = \mathcal{H} \otimes_{\mathcal{H}(K)} M.$$

To show that $M \to \tilde{M}$ is a reasonable functor, we must show that it is exact and is an equivalence of categories, with $\Gamma(K, \cdot\,)$ as quasi-inverse. It is not hard to prove the later, at least for finitely generated modules and coherent sheaves.

11.9.2 Proposition. *Let K be a Stein compact set with the Noether property. Then localization is an equivalence of categories from the category of finitely generated $\mathcal{H}(K)$-modules to the category of coherent sheaves on K.*

Proof. We realize K as a holomorphically convex compact subset of a Stein space X. Then , for a free finite rank module $M = \mathcal{H}^k(K)$, $\tilde{M} = \mathcal{H}^k|_K$, and $\Gamma(K, \tilde{M}) = \Gamma(K, \mathcal{H}^k) = \mathcal{H}^k(K)$. On the other hand, if $\mathcal{M} = \mathcal{H}^k|_K$ is a free finite rank $\mathcal{H}|_K$-module, then $\Gamma(K, \mathcal{M}) = \mathcal{H}^k(K)$, which has localization $\mathcal{M}$. Thus, when restricted to free finite rank objects, the functors of localization on K and taking sections on K are quasi-inverses of one another.

Let M be a finitely generated $\mathcal{H}(K)$-module. Then, since $\mathcal{H}(K)$ is Noetherian, there are morphisms α and β of $\mathcal{H}(X)$-modules, such that the sequence

$$\mathcal{H}^k(K) \xrightarrow{\beta} \mathcal{H}^m(K) \xrightarrow{\alpha} M \longrightarrow 0$$

is exact. Since tensor product is right exact, on localizing, we have an exact sequence

$$\mathcal{H}^k|_K \xrightarrow{\tilde{\beta}} \mathcal{H}^m|_K \xrightarrow{\tilde{\alpha}} \tilde{M} \longrightarrow 0. \tag{11.9.1}$$

The matrix determining β has entries which are holomorphic in a neighborhood of K. Thus, β determines a sheaf morphism $\tilde{\beta} : \mathcal{H}^m \to \mathcal{H}^n$ over a neighborhood of K, and its kernel and cokernel are coherent sheaves over this neighborhood. This implies that the sheaf $\tilde{M}$ is the restriction to K of a coherent sheaf $\mathcal{M}$ defined on a neighborhood of K. It also implies that we have the following two short exact sequences of coherent sheaves on a neighborhood of K:

$$0 \longrightarrow \operatorname{Ker}(\tilde{\alpha}) \longrightarrow \mathcal{H}^m \xrightarrow{\tilde{\alpha}} \mathcal{M} \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ker}(\tilde{\beta}) \longrightarrow \mathcal{H}^k \xrightarrow{\tilde{\beta}} \operatorname{Ker}(\tilde{\alpha}) \longrightarrow 0.$$

Since K is a Stein compact set, these sequences remain exact when we pass to sections over K. It follows that (11.9.1) remains exact when we pass to global sections, which implies that the sequence

$$\mathcal{H}^k(K) \xrightarrow{\beta} \mathcal{H}^m(K) \xrightarrow{\alpha} \Gamma(K, \tilde{M}) \longrightarrow 0$$

is exact. It follows that the natural map $M \to \Gamma(K, \tilde{M})$ is an isomorphism.

Now let $\mathcal{M}$ be a coherent sheaf defined in a neighborhood of K. By Cartan's Theorem A, for each $x \in K$, we can find finitely many global sections of $\mathcal{M}$ which generate $\mathcal{M}_x$ as an $\mathcal{H}_x$-module. This same set of sections will then generate the stalks of $\mathcal{M}$ in some neighborhood of x, by Exercise 9.6. Since K is compact, we can choose finitely many sections of $\mathcal{M}$ which generate the stalks of $\mathcal{M}$ at all points of a neighborhood U of K. Hence, there is an epimorphism $\mathcal{H}^m \to \mathcal{M}$ on U. Since the kernel of this morphism is also coherent, we may choose a finite generating set for it over a neighborhood of K as well. Thus, we have an exact sequence

$$\mathcal{H}^k \longrightarrow \mathcal{H}^m \longrightarrow \mathcal{M} \longrightarrow 0 \tag{11.9.2}$$

defined over a neighborhood of K. Since we are dealing with coherent sheaves, this remains exact when we pass to sections over K and yields an exact sequence

$$\mathcal{H}^k(K) \longrightarrow \mathcal{H}^m(K) \longrightarrow \Gamma(K, \mathcal{M}) \longrightarrow 0. \tag{11.9.3}$$

Thus, $M = \Gamma(K, \mathcal{M})$ is finitely generated. Furthermore, again using the fact that tensor product is right exact, we see that localizing (11.9.3), yields the sequence

$$\mathcal{H}^k \longrightarrow \mathcal{H}^m \longrightarrow \tilde{M} \longrightarrow 0.$$

On comparing this with (11.9.2) we conclude that the natural map $\tilde{M} \to \mathcal{M}$ is an isomorphism.

11.9.3 Proposition. *Let K be a Stein compact set with the Noether property. Then*

(i) *for each $x \in K$, $\mathcal{H}_x$ is a flat $\mathcal{H}(K)$-module;*

(ii) *the localization functor $M \to \tilde{M}$, for general $\mathcal{H}(K)$-modules, is exact.*

Proof. On the category of finitely generated $\mathcal{H}(K)$-modules, the localization functor is exact, because it is an equivalence from this abelian category to the abelian category of coherent analytic sheaves on K.

To prove that $\mathcal{H}_x$ is a flat $\mathcal{H}(K)$-module, it is sufficient to prove that if N is a submodule of a finitely generated $\mathcal{H}(K)$-module M, then the morphism $\mathcal{H}_x \otimes_{\mathcal{H}(K)} N \to \mathcal{H}_x \otimes_{\mathcal{H}(K)} M$ is injective. But this follows from the preceding paragraph. This proves (i). Obviously, (ii) follows from (i).

At this point, one could define a quasi-coherent analytic sheaf on a Stein compact set K to be an analytic sheaf which is obtained by localizing an $\mathcal{H}(K)$-module. We won't pursue this idea any further here.

11.10 Coherent Sheaves on Compact Varieties

We now turn to the final topic of this chapter – cohomology of coherent sheaves on compact holomorphic varieties. We cannot expect a compact variety to be a Stein space, since the only global holomorphic functions on a compact connected variety are constants (Exercise 9.15). Thus, we cannot expect every coherent analytic sheaf on a compact variety to be acyclic. However, we will prove that every coherent analytic sheaf on a compact holomorphic variety has finite dimensional cohomology (the Cartan-Serre theorem). The proof depends on a theorem of Schwartz which states that a compact perturbation of a surjective continuous linear map between Fréchet spaces has closed image and finite dimensional cokernel (Theorem 11.11.8).

A continuous linear map $\phi : X \to Y$ between two topological vector spaces is said to be *compact* if there exists a neighborhood U of 0 in X such that $\phi(U)$ has compact closure in Y.

11.10.1 Proposition. *If $\mathcal{S}$ is a coherent analytic sheaf on a Stein space X and U is an open set with compact closure in X, then the restriction map $\mathcal{S}(X) \to \mathcal{S}(U)$ is a compact linear map.*

Proof. We first note that this is true if $\mathcal{S}$ is the structure sheaf $\mathcal{H}$. In fact, $\{f \in \mathcal{H}(X) : ||f(x)||_{\overline{U}} < 1\}$ is a neighborhood of 0 in $\mathcal{H}(X)$, and its image in $\mathcal{H}(U)$ is bounded and, hence, has compact closure, since $\mathcal{H}(U)$ is a Montel space. Thus, the proposition is true if $\mathcal{S} = \mathcal{H}$. Clearly, this implies that it is also true if $\mathcal{S} = \mathcal{H}^k$ for some k.

Now suppose $\mathcal{S}$ is any coherent sheaf on X. We choose an Oka-Weil subdomain W such that $\overline{U} \subset W$. By Exercise 10.18 (or by Cartan's theorems), for some k, there is a surjective morphism $\mathcal{H}^k \to \mathcal{S}$, defined over a neighborhood of $\overline{W}$, with the corresponding map on sections over W also surjective. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{H}^k(W) & \longrightarrow & \mathcal{S}(W) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{H}^k(U) & \longrightarrow & \mathcal{S}(U) & \longrightarrow & 0 \end{array}$$

where the vertical maps are restrictions. There is a neighborhood of 0 in $\mathcal{H}^k(W)$ whose image in $\mathcal{H}^k(U)$ has compact closure, by the result of the previous paragraph. The image of this neighborhood in $\mathcal{S}(W)$ is a neighborhood of 0 in $\mathcal{S}(W)$, by the open mapping theorem, and its image in $\mathcal{S}(U)$ will have compact closure, by the commutativity of the diagram. Thus, $\mathcal{S}(W) \to \mathcal{S}(U)$ is a compact map. However, $\mathcal{S}(X) \to \mathcal{S}(U)$ is the composition of this map with the restriction map $\mathcal{S}(X) \to \mathcal{S}(W)$. Since the composition of a continuous linear map with a compact map is clearly compact, the proof is complete.

We now prove the Cartan-Serre theorem assuming Schwartz's theorem. We will end the chapter with a proof of Schwartz's theorem.

11.10.2 Cartan-Serre Theorem. *If X is a compact holomorphic variety, and $\mathcal{S}$ is a coherent analytic sheaf, then $H^p(X, \mathcal{S})$ is finite dimensional for all p.*

Proof. We choose a finite open cover $\mathcal{W} = \{W_i\}$ of X consisting of sets which are Stein spaces. Finite intersections of sets in this cover are also Stein spaces (Exercise 10.7). We then choose another such cover $\mathcal{U} = \{U_j\}$ which is a refinement of the first cover and, in fact, has the property that, for each j, there is an integer $\iota(j)$ such that $\overline{U}_j \subset W_{\iota(j)}$. Then for each multi-index $\alpha = (j_0, \ldots, j_p)$, the multi-index $\iota(\alpha) = (\iota(j_0), \ldots, \iota(j_p))$ has the property that $\overline{U}_\alpha \subset W_{\iota(\alpha)}$, so that the restriction map $\mathcal{S}(W_{\iota(\alpha)}) \to \mathcal{S}(U_\alpha)$ is a compact map. It follows that the refinement morphism $\iota^* : \mathcal{C}^p(\mathcal{W}) \to \mathcal{C}^p(\mathcal{U})$ from the space of alternating Čech p-cochains for $\mathcal{W}$ to the space of alternating Čech p-cochains for $\mathcal{U}$ is a compact map between Fréchet spaces. Since both covers $\mathcal{W}$ and $\mathcal{U}$ are Leray covers for $\mathcal{S}$, the map ι^* induces an isomorphism of cohomology (section 7.8). Hence, if $\mathcal{Z}^p(\mathcal{W})$ and $\mathcal{Z}^p(\mathcal{U})$ are the spaces of Čech p-cocycles for $\mathcal{W}$ and $\mathcal{U}$, then the map

$$f \oplus g \to \delta^{p-1}(f) + \iota^*(g) : \mathcal{C}^{p-1}(\mathcal{U}) \oplus \mathcal{Z}^p(\mathcal{W}) \to \mathcal{Z}^p(\mathcal{U})$$

is surjective. Since ι^* is compact, δ^{p-1} is a compact perturbation of a surjective morphism of Fréchet spaces. It follows from Schwartz's theorem (Theorem 11.11.8) that δ^{p-1} has closed image and finite dimensional cokernel. Hence, $H^p(X, \mathcal{S})$ is finite dimensional.

11.11 Schwartz's Theorem

It remains to prove Schwartz's theorem. We first prove a dual version of Schwartz's theorem. The theorem itself will follow from this and duality theory for locally convex topological vector spaces.

11.11.1 Theorem. *Let X and Y be locally convex topological vector spaces, and let $A : X \to Y$ be a continuous linear map which has closed image and is a topological isomorphism onto its image. Let $C : X \to Y$ be a compact continuous linear map. Then $B = A + C$ has finite dimensional kernel K and closed image I, and the map $X/K \to I$, induced by B, is a topological isomorphism.*

Proof. A and $-C$ agree on the kernel K of B. Thus, $A|_K$ is a topological isomorphism of K onto a closed subspace of Y. But it is also a compact map. This implies that the image of K in Y, under A, has a neighborhood of 0 with compact closure. However, a topological vector space which is locally compact is necessarily finite dimensional (Proposition 11.1.7). It follows that K is finite dimensional.

There is a closed subspace $L \subset X$, which is complementary to K in X. This follows from the Hahn-Banach theorem. In fact, if $\{x_i\}$ is a basis for the vector space K, then the Hahn-Banach theorem implies that we can find, for each i, a continuous linear functional f_i on X with $f_i(x_j) = \delta_{ij}$. The intersection of the kernels of the f_i will then be a closed complement for K. If L is such a complement, then $A|_L$ and $C|_L$ are continuous linear maps of L into Y, the first a topological isomorphism onto its image and the second a compact map. Furthermore, $A|_L + C|_L$ has the same image as $A + C$. Thus, to complete the proof it is enough to prove the theorem in the case where $B = A + C$ is injective.

We need to prove that B has closed image and is a topological isomorphism onto its image. To prove this, it is sufficient to prove that the image under B of each closed subset of X is closed in Y. To this end, we factor the map $A : X \to Y$ as $E \circ D$ where $D : X \to Y \times Y$ is given by $D(x) = (B(x), C(x))$ and $E : Y \times Y \to Y$ is given by $E(y, z) = y - z$. Then D and E are continuous, D is injective, and A is a topological isomorphism onto its image, which is closed. It follows that E is injective on the image of D and, from this, it follows that D is also a topological isomorphism onto its image.

Now let J be a closed subset of X, and let y be a point in the closure of $B(J)$. Since C is compact, there is a continuous seminorm ρ on X such that $U_\rho = \{x \in X : \rho(x) < 1\}$ is a neighborhood which C maps to a set with compact closure in Y. Suppose there is a subset S of J such that $\rho(S)$ is bounded, and y is in the closure of $B(S)$. Then $C(S)$ has compact closure T, and $D(S) \subset Y \times T$. Since T is compact, the projection $Y \times T \to Y$ is a closed map, and hence, there is a point in the closure of $D(S)$ with y as first coordinate. Let (y, z) be such a point. Since A has closed image, there is a unique $x \in S$ such that $A(x) = E(y, z) = y - z$. Since D is a topological isomorphism onto its image, $D(x) = (y, z)$. Thus, $B(x) = y$.

Now suppose that there is no set $S \subset J$ with $\rho(S)$ bounded and y in the closure of $B(S)$. Then if $J_n = \{x \in J : \rho(x) \geq n\}$, necessarily y is in the closure of each set $B(J_n)$. Then the set $S = \{x/\rho(x) : x \in J_1\}$ consists of elements x' with $\rho(x') = 1$ and $B(S)$ has 0 in its closure. Thus, we may argue as in the preceding paragraph, with $y = 0$. We conclude that there is a point x in the closure of S such that $Bx = 0$. Since $\rho(x) = 1$, we have $x \neq 0$. But this contradicts the assumption that B is injective. Thus, the previous case is the only one possible. We have, therefore, proved that $B(J)$ is closed in Y for every closed set $J \subset X$. Hence B has closed image and is a topological isomorphism onto its image. This completes the proof of the theorem.

The previous theorem is the dual of the one we want. In order to use it to prove Schwartz's theorem, we must first define a topology on the dual of a Fréchet space and prove a number of results about duality for locally convex topological vector spaces. Among these are the Mackey-Arens theorem and the fact that the dual of a compact linear map is compact.

Recall that the dual of a locally convex topological vector space X is the vector space X^* of all continuous linear functionals on X. It follows from the Hahn-Banach theorem (11.1.5) that the functionals in X^* separate points in X. It also follows from the Hahn-Banach theorem that, if Y is a closed subspace of X, and $Y^\perp = \{f \in X^* : f(y) = 0, \; \forall\, y \in Y\}$, then

$$Y^* = X^*/Y^\perp,$$

and

$$(X/Y)^* = Y^\perp.$$

There are many ways to topologize the dual of a locally convex topological vector space. The useful ways are of the following type:

A set S in a topological vector space is said to be *balanced* if $T \cdot S = S$, where T is the unit circle in $\mathbb{C}$. The closed convex balanced hull of a set S is the smallest closed convex balanced set containing S. *A saturated family* of bounded subsets of a topological vector space X is a family which is closed under subsets, multiplication by scalars, finite unions, and closed convex balanced hulls. Such a family is said to *cover* X if X is the union of the sets in the family.

11.11.2 Definition. *If X is a locally convex topological vector space, and κ is a saturated family of bounded subsets of X, which covers X, then X^*_κ will denote the space of continuous linear functionals on X, with the topology of uniform convergence on sets in κ.*

Clearly X^*_κ is a locally convex topological vector space. A family of seminorms defining the topology is the family of all seminorms of the form ρ_K where K is a set in κ and $\rho_K(f) = \sup\{|f(x)| : x \in K\}$. The family of sets we want to use is the family c of all sets with compact closure. This is not always a saturated family since it is not always true that the closed convex balanced hull of a compact set is compact. However, for Fréchet spaces we have:

11.11.3 Proposition. *If X is a Fréchet space, then the closed convex balanced hull of every compact subset of X is also compact.*

Proof. If D is the closed unit disc in $\mathbb{C}$, and $K \subset X$ is compact, then $D \cdot K$ is the image of the compact set $D \times K$ under the scalar multiplication map and is, hence, compact. The closed convex hull of $D \cdot K$ will be a closed convex balanced set containing K. Hence, to prove the proposition, it suffices to prove that the closed convex hull of a compact set in a Fréchet space is compact.

Recall that a subset of a complete metric space is compact if and only if it is closed and totally bounded. A subset S of a metric space is *totally bounded* if, for each $\epsilon > 0$, there is a finite set F in S so that each point of S is within ϵ of some point of F.

Choose a translation invariant metric ρ defining the topology of X. If K is a compact subset of the Fréchet space X, then K is totally bounded. Hence, given $\epsilon > 0$, there exists a finite set of points $\{y_i\}_{i=1}^n \subset K$ such that each $x \in K$ is within $\epsilon/2$ of some y_i. Then the convex hull L of the set $\{y_i\}$ is the image of the map

$$(s_1, \dots, s_n) \to \sum s_i y_i : S \to X,$$

where S is the simplex $\{(s_1, \dots, s_n) \in (R^+)^n : \sum s_i = 1\}$. Thus, L is compact. Furthermore, every element of the convex hull of K is within $\epsilon/2$ of a point of L. Since L is compact, it is also totally bounded and we may find a finite set of points $\{x_j\}$ such that every point of L is within $\epsilon/2$ of some x_j. It follows that every point of the convex hull of K is within ϵ of some x_j. Thus, we have proved that the convex hull of K is totally bounded. The closure of a totally bounded set is clearly totally bounded as well. Thus, the closed convex hull of a compact subset of a Fréchet space is totally bounded and, hence, compact.

The above theorem implies that the family c of sets with compact closure in a Fréchet space is a saturated family. The topology it determines on X^* is the topology of uniform convergence on compact subsets of X. The space X^*, with this topology, will be denoted X_c^*.

Now let X and Y be Fréchet spaces and $A : X \to Y$ a continuous linear map. We consider the duals X_c^* and Y_c^* of X and Y in the topology of uniform convergence on compact subsets. Then the continuous linear map $A : X \to Y$ has a dual $A^* : Y_c^* \to X_c^*$, defined by

$$A^*(f)(x) = f(A(x)).$$

Since A is continuous, it maps compact sets to compact sets, from which it follows that A^* is continuous.

11.11.4 Proposition. *If $C : X \to Y$ is a compact linear map between Fréchet spaces, then $C^* : Y_c^* \to X_c^*$ is also compact.*

Proof. Since C is compact, there is a 0-neighborhood U in X such that $C(U)$ has compact closure K in Y. Then the 0-neighborhood

$$V_K = \{f \in Y_c^* : \sup_{x \in K} |f(x)| < 1\}$$

in Y^* has the property that $C^*(V_K)$ is a family of continuous functions which is uniformly bounded by 1, in modulus, on U. If L is any compact subset of X, then $C^*(V_K)$ is uniformly bounded on L, since $L \subset kU$ for some $k > 0$. Also, $C^*(V_K)$ is equicontinuous on L, since $|f(x) - f(x')| < \epsilon$ if $x - x' \in \epsilon U$ and $f \in C^*(V_K)$. It follows from the Ascoli-Arzela theorem that $C^*(V_K)$ has compact closure in the space of all functions on X, in the topology of uniform convergence on compact subsets of X. However, the space of continuous linear functionals is closed in this topology, since X is a metric space. Thus, $C^*(V_K)$ has compact closure in X_c^*, and C^* is a compact operator.

11.11.5 Proposition. *If X and Y are Fréchet spaces, and $A : X \to Y$ is a surjective continuous linear map, then $A^* : Y_c^* \to X_c^*$ has closed image and is a topological isomorphism onto its image.*

Proof. The fact that A is surjective obviously implies that A^* is injective. Let S be a closed set in Y_c^*, and let $g \in X_c^*$ be a point of the closure of $A^*(S)$. Let K be a compact subset of Y. The fact that A is an open map (Theorem 11.1.3) implies that K is the image under A of a compact subset L of X (Exercise 11.18). Then $g|_L$ is in the uniform closure on L of $\{A^*(f) = f \circ A : f \in S\}$, and this implies that there is an f_K in the uniform closure of S on K with $f_K \circ A = g|_L$ on L. Since this is true for every compact set $K \subset Y$, and A is surjective, the functions f_K fit together to define a function f on Y with $f|_K = f_K$ for each compact subset K of Y. The resulting function is clearly linear on Y and continuous on compact subsets of Y. Since Y is a metric space, this implies it is continuous. Furthermore, f is in S, since S is closed in the topology of uniform convergence on compact sets. Also, $A^*(f) = f \circ A = g$. Hence, the image of each closed subset of A^* is closed. This implies that A^* has closed image and is a topological isomorphism onto its image.

The converse of the above proposition is also true (Proposition 11.11.7). To prove it, we must first prove the Mackey-Arens theorem.

Note that, for any saturated family κ covering X, each element $x \in X$ determines a continuous linear functional on X_κ^* by $f \to f(x) : X_\kappa^* \to \mathbb{C}$. Thus, X embeds in the second dual $(X_\kappa^*)^*$. In the case where the sets in κ are precompact, every continuous linear functional on X^* has this form.

11.11.6 Mackey-Arens Theorem. *If X is a locally convex topological vector space, and κ is a saturated family of precompact subsets of X which covers X, then every continuous linear functional on X^*_κ is determined by an element of X. Thus, the map $X \to (X^*_\kappa)^*$ is bijective.*

Proof. If $\lambda \in (X^*_\kappa)^*$, then there is a 0-neighborhood V in X^*_κ such that $|\lambda(f)| < 1$ for all $f \in V$. We may assume that the neighborhood V has the form

$$V = \{f \in X^*_c : \sup_{x \in K} |f(x)| < 1\}$$

for some $K \in \kappa$. Without loss of generality, we may assume K is compact, convex, and bounded, since κ is saturated and consists of precompact sets. We regard X as embedded in $(X^*_\kappa)^*$, and we give the latter space the weak-$*$ topology – that is, the topology of pointwise convergence of functions on X^*_κ. This may also be described as the topology of uniform convergence on the family σ of sets which are convex balanced hulls of finite sets. On the image of X in $(X^*_c)^*$, the σ topology is weaker than the original topology on X, and hence, K is also compact in this topology.

Let us assume, for the moment, that the theorem is true in the case of the space X^*_c and the σ topology on its dual $(X^*_c)^*$. Then every continuous linear functional on $(X^*_c)^*_\sigma$ is given by an element of X^*_c. If λ is not an element of K, then it follows from the convex separation theorem (Proposition 11.1.6) that there exists a continuous linear functional f on $(X^*_c)^*_\sigma$ such that $|f(\lambda)| > 1$ and $|f(x)| < r < 1$ for all $x \in K$. However, under our assumption about the σ topology, we must have that $f \in X^*_c$, and it then follows that $f \in V$. This is a contradiction, since $|f(\lambda)| = |\lambda(f)| < 1$ if $f \in V$. We conclude that λ does belong to K, and in particular, λ belongs to X. Thus, the theorem is true, in general, if it is true whenever $\kappa = \sigma$.

Thus, let κ be the family σ generated by finite sets. Let λ, V, and K be as above. Since $\kappa = \sigma$, we have that K is the convex balanced hull of a finite set $\{x_i\}_{i=1}^n$. Then the set of $f \in X^*$ such that $f(x_i) = 0$ for $i = 1, \dots, n$ may be described as the intersection of the sets $k^{-1}V$ for $k = 1, 2, \dots$. We conclude that λ vanishes on this set, since $|\lambda|$ is less than 1 on V. It then follows from elementary linear algebra (Exercise 11.19) that λ must be a linear combination of the x_i and, hence, belongs to X.

11.11.7 Proposition. *Let X and Y be Fréchet spaces and $B : X \to Y$ a continuous linear map. If $B^* : Y^*_c \to X^*_c$ has closed image and is a topological isomorphism onto its image, then B is surjective.*

Proof. The map B induces a continuous linear map $B_1 : X/\mathrm{Ker}\, B \to Y$ which is injective. If this map is surjective, then so is B. The map B_1 has as dual the map B^* considered as a map from Y_c^* to

$$(\operatorname{Ker} B)^\perp = \{f \in X_c^* : f(x) = 0, \ \forall\, x \in \operatorname{Ker} B\}.$$

This map has closed image, and so, if it is not surjective, then the Hahn-Banach theorem implies that there is a continuous linear functional F on X^* which vanishes identically on $B^*(Y_c^*)$ and does not vanish identically on $(\operatorname{Ker} B)^\perp$. However, by the Mackey-Arens theorem, the functional F has to have the form $F(f) = f(x)$ for some $x \in X$. In other words, there is an $x \in X$ such that $g(B(x)) = B^*(g)(x) = 0$ for all $g \in Y_c^*$, but $f(x) \neq 0$ for some $f \in X_c^*$ which vanishes on $\operatorname{Ker} B$. This is impossible, since such an x would necessarily be in $\operatorname{Ker} B$. Thus, B^* is surjective as a map from Y_c^* to $(\operatorname{Ker} B)^\perp$. It follows that the proposition is true in general if it is true in the case where B is injective and B^* is surjective. Thus, in the remainder of the proof, we assume B has these properties. We are assuming, in particular, that B^* is a topological isomorphism of Y_c^* onto X_c^*.

Suppose that $\{x_n\}$ is a sequence in X such that $\{B(x_n)\}$ converges to $y_0 \in Y$. Then the set $S = \{B(x_n)\} \cup \{y_0\}$ is compact. Hence, the set

$$U = \{f \in Y^* : |f(y)| < 1, \ \forall\, y \in S\} \subset \{f \in Y^* : |f(B(x_n))| < 1, \ \forall\, n\}$$

is a neighborhood of 0 in Y_c^*, from which it follows that its image $B^*(U)$ is a neighborhood of 0 in X_c^*. This implies that there is a compact set $K \subset X$ and a $\delta > 0$ such that

$$\{g \in X^* : |g(x)| < \delta, \ \forall\, x \in K\} \subset B^*(U) \subset \{g \in X_c^* : |g(x_n)| < 1, \ \forall\, n\},$$

or

$$\{g \in X^* : |g(x)| < 1, \ \forall\, x \in \delta^{-1}K\} \subset B^*(U) \subset \{g \in X_c^* : |g(x_n)| < 1, \ \forall\, n\}.$$

From the convex separation theorem (Theorem 11.1.6), it follows that the sequence $\{x_n\}$ lies in the closed convex balanced hull of $\delta^{-1}K$, which is a compact subset of X by Proposition 11.11.3. Then it has a cluster point x and, clearly, $B(x) = y$. Thus, B is surjective.

11.11.8 Schwartz's Theorem. *Let X and Y be Fréchet spaces and suppose that $A : X \to Y$ is a surjective continuous linear map and $C : X \to Y$ is a compact continuous linear map. Then $B = A + C$ has closed image and finite dimensional cokernel.*

Proof. The dual map $A^* : Y_c^* \to X_c^*$ has closed image and is a topological isomorphism onto its image by Proposition 11.11.5, while the dual map $C^* : Y_c^* \to X_c^*$ is compact by Proposition 11.11.4.

We now have the hypotheses of Theorem 11.11.1 satisfied for the pair of operators A^* and C^*. We conclude that $B^* = A^* + C^*$ has finite dimensional kernel Z and closed image and induces a topological isomorphism from Y_c^*/Z to its image in X_c^*. Now the map $Y_c^*/Z \to X_c^*$, induced by B^*, is the dual of the continuous linear map B' which is B considered as a map from X to $\{y \in Y : f(y) = 0, \forall f \in Z\}$. The latter space is closed, with finite codimension in Y. By Proposition 11.11.7, B' is surjective. This completes the proof.

Exercises

1. Prove that a bounded linear map $\phi : X \to Y$ between two Fréchet spaces is an open map if and only if every convergent sequence in Y lifts under ϕ to a convergent sequence in X.
2. If $\mathcal{S}$ is a Fréchet sheaf, we give each stalk $\mathcal{S}_x = \varinjlim\{\mathcal{S}(U) : x \in U\}$ the direct limit topology. This is the topology in which a neighborhood basis at the origin is the set of convex balanced sets with open inverse image under each restriction map $\mathcal{S}(U) \to \mathcal{S}_x$, $x \in U$. Prove that a sheaf morphism $\mathcal{S} \to \mathcal{T}$ is continuous if and only if the induced map on each stalk is continuous.
3. Prove Corollary 11.5.1 .
4. Prove Corollary 11.5.2 .
5. Prove Corollary 11.5.3 .
6. Prove Corollary 11.5.4 .
7. Prove Corollary 11.5.5 .
8. Let X be a complex manifold. Prove that if the line bundle corresponding to an element of $H^1(X, \mathcal{H}^*)$ has a non-zero section, then the element has image 0 in $H^1(X, \mathcal{M}^*)$ (see sections 11.6 and 11.7).
9. Use the finite mapping theorem (Theorem 9.6.4) to prove that if X is a Stein space, Y is a holomorphic variety, and there exists a finite holomorphic map $f : Y \to X$, then Y is also a Stein space.
10. Let X be a holomorphic variety. We say an open set U is holomorphically convex in X if for each compact subset K of U the holomorphically convex hull of K in X is contained in U. Prove that if an open set U is holomorphically convex in X, and X is a Stein space, then the restriction map $\mathcal{H}(X) \to \mathcal{H}(U)$ has dense image.
11. Let X be a holomorphic variety which has enough global holomorphic functions to separate points. Prove that if K is a compact subset of X

with $\hat{K} = K$, then there is an open set U, containing K, such that U is a Stein space, U has compact closure, and U is holomorphically convex in X. Hint: Use the techniques in the proof of Theorem 10.6.5 to show that there is an open set U, containing K, with compact closure, and a finite holomorphic map of U onto a subvariety of a polydisc.

12. Let X be a holomorphic variety which is holomorphically convex and has enough global holomorphic functions to separate points. Prove that X is the union of an increasing sequence $\{U_i\}$ of open subsets such that U_i is a Stein space, and the restriction map $\mathcal{M}(U_{i+1}) \to \mathcal{M}(U_i)$ has dense image for each i. Hint: Use the results of the preceding two exercises.
13. Prove that a holomorphic variety is a Stein space if and only if it is holomorphically convex, and it has enough global holomorphic functions to separate points. Hint: See the preceding exercise.
14. One proof that a non-compact Riemann surface X is a Stein manifold first establishes, using methods similar to those of Chapter 1 (see Exercise 1.9), that X has the interpolation property – that is, for every discrete sequence $\{x_i\} \subset X$, and every sequence $\{\lambda_i\} \subset \mathbb{C}$, there is a holomorphic function $f \in \mathcal{H}(X)$ such that $f(x_i) = \lambda_i$. Assuming that a non-compact Riemann surface X has this property, prove that X is a Stein space.
15. Let X be a compact Riemann surface. If a divisor D on X is considered a finitely non-zero integer valued function, as in Example 11.7.6, then we can put a partial order on the set of divisors by declaring that $D_1 \leq D_2$ if $D_1(x) \leq D_2(x)$ for all $x \in X$. Prove that $D_1 \leq D_2$ if and only if $\mathcal{H}(D_1) \subset \mathcal{H}(D_2)$ (see section 11.7). In this case, show that the quotient sheaf $\mathcal{H}(D_2)/\mathcal{H}(D_1)$ is a direct sum of skyscraper sheaves supported on the set $\{x \in X : D_1(x) < D_2(x)\}$ and with stalk $\mathbb{C}^{m_x}$ at each point of this set, where $m_x = D_2(x) - D_1(x)$.
16. If D is a divisor on a compact Riemann surface, then $\deg(D)$ is defined to be the sum of all the values of D considered as an integer valued function. Prove that
$$\dim H^0(X, \mathcal{H}(D)) - \dim H^1(X, \mathcal{H}(D)) = \deg(D) + 1.$$
Note that the dimensions involved are finite, by Theorem 11.10.2. Hint: First prove that $\dim H^0(X, \mathcal{H}(D)) - \dim H^1(X, \mathcal{H}(D)) - \deg(D)$ is constant by using the result of the previous exercise to prove that it is the same for D_1 and D_2 whenever $D_1 \leq D_2$.
17. Use the result of the previous exercise to prove that there exists a non-constant meromorphic function on every compact Riemann surface.
18. Prove that if $X \to Y$ is a surjective continuous linear map between Fréchet spaces, then each compact subset of Y is the image of a compact subset of X. Hint: Use the fact that in a complete metric space the compact sets are the closed totally bounded sets.

19. Let X be a vector space and let Y be a vector space of linear functionals on X which separates points in X. If $\{x_1, \ldots, x_k\}$ is a finite subset of X and $x \in X$ has the property that $f(x) = 0$ whenever $f \in Y$ and $f(x_i) = 0$ for $i = 1, \ldots, k$, then prove that x is a linear combination of $x_1, \ldots, x_k$.
20. If X is a Fréchet space, with topology determined by a sequence $\{\rho_n\}$ of seminorms, prove that

$$\sigma(x, y) = \sum_n \frac{1}{2^n} \frac{\rho_n(x-y)}{1+\rho_n(x-y)}$$

defines a translation invariant metric σ on X, which determines the same topology.
21. Let A be a Banach algebra and a an element of A with spectrum $\sigma(a)$. Prove that the spectral radius (section 11.8) of a is $\rho = \limsup ||a^n||^{1/n}$. Hint: Let r be the spectral radius of a. Prove that $r \leq \rho$ by constructing an inverse for $\lambda - a$ if $|\lambda| > \rho$. Then prove that $\rho \leq r$ using the result of Exercise 1.19.
22. Prove that the sequence $||a^n||^{1/n}$ of the previous exercise actually converges.

Projective Varieties

Affine varieties are algebraic varieties on which every coherent algebraic sheaf is acyclic, while Stein spaces are holomorphic varieties on which every coherent analytic sheaf is acyclic. An important class of varieties with quite different properties is the class of projective varieties – varieties which can be realized as subvarieties of complex projective space. This concept makes sense both for algebraic varieties and for holomorphic varieties. Projective varieties are compact in the Euclidean topology, and hence, projective algebraic (holomorphic) varieties have no non-constant global regular (holomorphic) functions. Projective varieties support coherent sheaves with non-vanishing higher cohomologies, yet these cohomologies have reasonable properties and are amenable to study. Furthermore, there is a stunning fact concerning the categories of projective algebraic and projective holomorphic varieties: They are equivalent. Furthermore, given a projective algebraic variety X and its corresponding holomorphic variety $\tilde{X}$, the category of coherent algebraic sheaves on X is equivalent to the category of coherent analytic sheaves on $\tilde{X}$. These are Serre's theorems from the famous GAGA paper [**S**].

In this chapter we define complex projective space and projective varieties and develop some of the basic properties of coherent algebraic and analytic sheaves on projective varieties. We do this in preparation for proving Serre's theorems in the next chapter. The results of this chapter will also be used in the proof of the Borel-Weil-Bott theorem in Chapter 16.

12.1 Complex Projective Space

Complex algebraic projective space of dimension n is the algebraic variety defined in the following fashion: We consider the point set P^n which is

$\mathbb{C}^{n+1} - \{0\}$ modulo the equivalence relation defined by

$$(\lambda z_0, \dots, \lambda z_n) \sim (z_0, \dots, z_n), \ \forall \, \lambda \in \mathbb{C} - \{0\}.$$

We shall define a topology and a ringed space structure on P^n and show that the resulting ringed space is an algebraic variety.

Let $p \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial of some degree – say k. Then p does not define a function on P^n unless $k = 0$, but the relation

$$p(\lambda z_0, \dots, \lambda z_n) = \lambda^k p(z_0, \dots, z_n)$$

means the zero set of p is invariant under multiplication by $\lambda \in \mathbb{C}^{n+1} - \{0\}$ and, hence, defines a subset of P^n. Similarly, the set of common zeroes of any set of homogeneous polynomials defines a subset of P^n.

12.1.1 Definition. *An algebraic subset of P^n is a subset which is the set of common zeroes of some family of homogeneous polynomials.*

Of course, because subvarieties of $\mathbb{C}^{n+1}$ satisfy the descending chain condition, an algebraic subset of P^n is actually the zero set of a finite set of homogeneous polynomials.

It is easy to see that the collection of algebraic subsets of P^n is closed under finite union and arbitrary intersection. Thus, this collection may be taken as the collection of closed sets in a topology for P^n.

12.1.2 Definition. *The Zariski topology on P^n is the topology in which the open sets are the complements of algebraic sets.*

12.2 Projective Space as an Algebraic and a Holomorphic Variety

We next associate to each integer k a sheaf $\mathcal{O}(k)$ on P^n. Let

$$\pi : \mathbb{C}^{n+1} - \{0\} \to P^n$$

be the projection. If U is open in P^n in the Zariski topology, then $\pi^{-1}(U)$ is Zariski open in $\mathbb{C}^{n+1} - \{0\}$ and, in fact, is the complement of the set of common zeroes of a finite set of homogeneous polynomials.

Throughout the discussion of the sheaves $\mathcal{O}(k)$ in this section, the topology on P^n and $\mathbb{C}^{n+1}$ will be the Zariski topology.

12.2.1 Definition. *If U is an open subset of P^n and k is an integer, we define $\mathcal{O}(k)(U)$ to be the space of regular functions on $\pi^{-1}(U)$ which are homogeneous of degree k.*

Clearly $\mathcal{O}(k)$ forms a sheaf for each k. Furthermore, if $f \in \mathcal{O}(j)(U)$ and $g \in \mathcal{O}(k)(U)$, then $fg \in \mathcal{O}(j+k)(U)$. Thus, $\bigoplus_{k=-\infty}^{\infty} \mathcal{O}(k)$ is a sheaf of graded rings. In particular, $\mathcal{O}(0)$ is a sheaf of rings and each $\mathcal{O}(k)$ is a sheaf of modules over $\mathcal{O}(0)$. Note that, for $U \subset P^n$, a function on $\pi^{-1}(U)$ which is homogeneous of degree 0 is constant on equivalence classes and, hence, determines a well-defined function on U. Thus, $\mathcal{O}(0)$ may be considered a sheaf of rings of continuous functions on P^n.

12.2.2 Definition. *We make P^n into a ringed space by defining the structure sheaf to be the sheaf of rings $\mathcal{O} = \mathcal{O}(0)$.*

Throughout this chapter, let U_i denote the open set in P^n which is the complement of the algebraic set defined by the zero set of the ith coordinate function z_i. We define a morphism $\phi_i : U_i \to \mathbb{C}^n$ by $\phi_i \circ \pi = \psi_i$, where $\psi_i : \pi^{-1}(U_i) \to \mathbb{C}^n$ is defined by

$$\psi_i(z_0, \dots, z_n) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right).$$

The map ϕ_i is well defined, since the coordinate functions of ψ_i are homogeneous of degree 0. In fact:

12.2.3 Proposition. *The map $\phi_i : U_i \to \mathbb{C}^n$ is an isomorphism of ringed spaces.*

Proof. We may as well assume that $i = 0$. We first prove that ϕ_0 is a homeomorphism. Note that ϕ_0 is bijective, with inverse given by

$$\phi_0^{-1}(z_1, \dots, z_n) = \pi(1, z_1, \dots, z_n).$$

To prove that it is a homeomorphism, we must show that a set is closed in U_0 if and only if its image is closed in $\mathbb{C}^n$.

Every closed subset of U_0 is a finite intersection of sets of the form $U_0 \cap Z(p)$ where p is a homogeneous polynomial and $Z(p) \subset P^n$ is its zero set. But if $z_0 \neq 0$, then

$$p(z_0, z_1, \dots, z_n) = 0 \quad \text{if and only if} \quad p(1, z_1/z_0, \dots, z_n/z_0) = 0.$$

Since the latter function is a polynomial composed with ψ_0, the image of $U_0 \cap Z(p)$ under ϕ_0 is a subvariety of $\mathbb{C}^n$ – that is, a closed set. It follows that the image of every closed subset of U_0 is closed in $\mathbb{C}^n$.

The closed subsets of $\mathbb{C}^n$ are finite intersections of zero sets of polynomials. Let q be a polynomial in $z_1, \dots, z_n$ of degree k. Then we may define a homogeneous polynomial of degree k in $z_0, \dots, z_n$ by

$$p(z_0, z_1, \dots, z_n) = z_0^k q(z_1/z_0, \dots, z_n/z_0),$$

and p is 0 exactly at points which ϕ_0 maps to zeros of q. That is, $Z(p) \cap U_0$ is the inverse image under ϕ_0 of the zero set of q. It follows that the inverse image under ϕ_0 of a closed set in $\mathbb{C}^n$ is a closed set in U_0.

To finish the proof, we must show that ϕ_0 induces an isomorphism between the structure sheaves of U_0 and $\mathbb{C}^n$. That is, for each open set $W \subset \mathbb{C}^n$, we must show that $g \to g \circ \phi_0$ is an isomorphism from the ring of regular functions on W to the ring of regular functions on $\phi_0^{-1}(W)$. In terms of ψ_0, this means we must show that, for each open set $W \subset \mathbb{C}^n$, $g \to g \circ \psi_0$ is an isomorphism from the ring of regular functions on W to the ring of homogeneous regular functions on $\pi^{-1}(W)$. Since ψ_0 is algebraic and homogeneous, it is clear that $g \to g \circ \psi_0$ is a ring homomorphism from $\mathcal{O}(W)$ to regular homogeneous functions on $\pi^{-1}(W)$. To see that it is an isomorphism, we simply note that its inverse is given by $f \to \tilde{f}$ where $\tilde{f}(z_1, \dots, z_n) = f(1, z_1, \dots, z_n)$. This completes the proof.

12.2.4 Proposition. *The ringed space P^n is an algebraic variety.*

Proof. We have that $\{U_i\}$ is a cover of P^n by open subsets which are isomorphic as ringed spaces to $\mathbb{C}^n$. Thus, P^n is an algebraic prevariety. It is also clear that, given any two points p and q of P^n, we may choose our coordinate system in $\mathbb{C}^{n+1}$ in such a way that one of the U_i contains both p and q. In other words, given any two points, there is an affine open subset containing both. By Proposition 8.1.5 this implies that P^n is actually an algebraic variety.

12.2.5 Definition. *A projective variety is an algebraic variety which is isomorphic to a subvariety of P^n, for some n. A quasi-projective variety is an algebraic variety which is isomorphic to an open subset of a projective variety.*

Since $\mathbb{C}^n$ is isomorphic to an open subset of P^n (any one of the U_i), it is quasi-projective. Clearly a subvariety of a quasi-projective variety is also quasi-projective, and hence, every affine variety is quasi-projective.

To any algebraic variety we may associate, in a canonical way, a holomorphic variety. An algebraic variety is locally isomorphic, as a ringed space, to an algebraic subvariety of $\mathbb{C}^n$. There is a canonical way to associate to an algebraic subvariety V of $\mathbb{C}^n$ a holomorphic subvariety V^h – we simply give V the Euclidean topology instead of the Zariski topology and let its structure sheaf be the sheaf of holomorphic functions rather than the sheaf of regular functions. This means that, for any algebraic variety, we have a canonical way of changing the topology and ringed space structure on any affine open subset in such a way as to make it a holomorphic variety. This is canonical, because any two ways of representing an affine open subset as a subvariety of complex Euclidean space are related by a biregular map between the two subvarieties. This will necessarily be a biholomorphic map between the associated holomorphic subvarieties. In particular, this implies that the ringed space structures, defined in this way, on affine subsets of an algebraic variety will agree on intersections and, thus, define a global structure of a holomorphic variety. One thing that does need to be checked is that the resulting topological space is Hausdorff (Exercise 12.4).

From the above discussion, we conclude that projective space P^n may also be considered as a holomorphic variety – in fact, as a complex manifold, since the maps $\phi_i : U_i \to \mathbb{C}^n$ give local biholomorphic maps onto $\mathbb{C}^n$. We initially defined regular functions on $U \subset P^n$ as regular functions on $\pi^{-1}(U)$ which are homogeneous of degree 0. It turns out that the holomorphic functions on a Euclidean open set U have a similar description:

12.2.6 Definition. *For each integer k we define a sheaf $\mathcal{H}(k)$ on P^n with the Euclidean topology as follows: If $U \subset P^n$ is open in the Euclidean topology, then $\mathcal{H}(k)(U)$ is the space of functions in $\mathcal{H}(\pi^{-1}(U))$ which are homogeneous of degree k.*

12.2.7 Proposition. *The sheaf $\mathcal{H}(0)$ is a sheaf of rings, canonically isomorphic to the structure sheaf $\mathcal{H}$ of P^n, and the sheaves $\mathcal{H}(k)$ are sheaves of $\mathcal{H}$-modules.*

Proof. Clearly $\mathcal{H}(0)$ is a sheaf of rings, and each $\mathcal{H}(k)$ is a sheaf of modules over $\mathcal{H}(0)$. Thus, we need only show that $\mathcal{H}(0)$ is canonically isomorphic to $\mathcal{H}$. The isomorphism is obviously the one which sends $f \in \mathcal{H}(U)$ to $f \circ \pi$. It is easy to see, using the definition of the holomorphic structure on P^n, that $\pi : \mathbb{C}^{n+1} - 0 \to P^n$ is holomorphic, so that $f \to f \circ \pi$ is a homomorphism of $\mathcal{H}$ to $\mathcal{H}(0)$. To show that it is an isomorphism, it is enough to show it is an isomorphism on each U_i and, without loss of generality, we may assume $i = 0$. However, on U_0, the morphism $g \to g \circ \rho_0$ is an inverse for $f \to f \circ \pi$, where $\rho_0 : U_0 \to \mathbb{C}^{n+1} - 0$ is the composition of $\phi_0 : U_0 \to \mathbb{C}^n$ with $(z_1, \ldots, z_n) \to (1, z_1, \ldots, z_n) : \mathbb{C}^n \to \mathbb{C}^{n+1} - 0$.

12.3 The Sheaves $\mathcal{O}(k)$ and $\mathcal{H}(k)$

We now turn to the study of the sheaves $\mathcal{O}(k)$ and $\mathcal{H}(k)$, introduced above. The sheaves $\mathcal{O}(k)$ are sheaves of $\mathcal{O} = \mathcal{O}(0)$-modules. If V is an open subset of U_i, then it is easy to see that $f \in \mathcal{O}(V)$ if and only if $z_i^k f \in \mathcal{O}(k)(V)$. Thus, $f \to z_i^k f$ defines an $\mathcal{O}|_{U_i}$-module isomorphism from $\mathcal{O}|_{U_i}$ to $\mathcal{O}(k)|_{U_i}$. In other words, $\mathcal{O}(k)$ is locally free of rank 1 as a sheaf of $\mathcal{O}$-modules. Exactly the same thing is obviously true of $\mathcal{H}(k)$ as a sheaf of $\mathcal{H}$-modules. Recall from section 7.9 that a sheaf of modules with this property is called an *invertible sheaf*, since such a sheaf always has an inverse under tensor product relative to the structure sheaf. In this case, the inverse of $\mathcal{O}(k)(\mathcal{H}(k))$ is obviously $\mathcal{O}(-k)(\mathcal{H}(-k))$, in view of the following proposition:

12.3.1 Proposition. *If j and k are integers, then multiplication defines an isomorphism*

$$\mathcal{O}(j) \otimes_{\mathcal{O}} \mathcal{O}(k) \to \mathcal{O}(j+k).$$

The analogous statement is true for the sheaves $\mathcal{H}(k)$.

Proof. The multiplication map $f \otimes g \to fg$ defines a morphism of algebraic sheaves from $\mathcal{O}(j) \otimes_{\mathcal{O}} \mathcal{O}(k)$ to $\mathcal{O}(j+k)$. The only question is whether or not it is an isomorphism. However, a morphism of sheaves is an isomorphism if and only if it is locally an isomorphism. Thus, it suffices to show that the morphism is an isomorphism on U_i for each i. However, on U_i, $\mathcal{O}(j)$ and $\mathcal{O}(k)$ are free of rank 1 with generators z_i^j and z_i^k while $\mathcal{O}(j) \otimes_{\mathcal{O}} \mathcal{O}(k)$ is free of rank 1 with generator $z_i^j \otimes z_i^k$. The multiplication map sends $z_i^j \otimes z_i^k$ to z_i^{j+k}, which is the generator of $\mathcal{O}(j+k)$ on U_i. This proves the theorem in the algebraic case. The proof in the analytic case is the same.

The sheaf of sections of a (finite dimensional) holomorphic vector bundle is a locally free finite rank sheaf of $\mathcal{H}$-modules and vice-verse, by Proposition 7.6.5. Sheaves of modules which are locally free of rank 1 (invertible sheaves), such as our sheaves $\mathcal{H}(k)$, can be realized as sheaves of sections of holomorphic line bundles – vector bundles with 1-dimensional fiber. Note that locally free sheaves of finite rank are, of course, coherent, since coherence is a local property, and locally free finite rank sheaves locally have the form $\mathcal{H}^m$ (or $\mathcal{O}^m$ in the algebraic case).

Our next main objective is to compute the sections and cohomology of the sheaves $\mathcal{O}(k)$ and $\mathcal{H}(k)$. It will be convenient to introduce three additional sheaves on P^n:

12.3.2 Definition. *Let $\mathcal{S}$ denote the sheaf on P^n which assigns to an open set $U \subset P^n$ the ring $\mathcal{O}(\pi^{-1}(U))$, and let $\mathcal{T}$ denote the sheaf which assigns to U the ring $\mathcal{H}(\pi^{-1}(U))$. We also let $\mathcal{T}_0$ be the subsheaf of $\mathcal{T}$ spanned by the subsheaves $\mathcal{H}(k)$.*

Thus, $\mathcal{O}(k)$ is the subsheaf of $\mathcal{S}$ consisting of elements homogeneous of degree k, while $\mathcal{H}(k)$ is the subsheaf of $\mathcal{T}$ consisting of elements homogeneous of degree k, and

$$\mathcal{S} = \bigoplus_{k=-\infty}^{\infty} \mathcal{O}(k), \quad \text{and} \quad \mathcal{T}_0 = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}(k).$$

Also note that, if $\mathcal{O}$ and $\mathcal{H}$ denote the sheaves of regular and holomorphic functions on $\mathbb{C}^{n+1} - \{0\}$, then $\mathcal{S} = \pi_*\mathcal{O}$, and $\mathcal{T} = \pi_*\mathcal{H}$.

We will compute cohomology using the Čech complex for the open cover $\{U_i\}$. Note if $\alpha = (i_0, \ldots, i_p)$ is a multi-index, then $U_\alpha = U_{i_0} \cap \cdots \cap U_{i_p} = U_{z_\alpha}$, where $z_\alpha = z_{i_0} \ldots z_{i_p}$, and U_{z_α} is the complement of the algebraic set in P^n determined by the vanishing of z_α. Thus, U_α is the subset of $U_{i_0} \simeq \mathbb{C}^n$ on which the regular function $z_{i_0}^{-p} z_{i_1} \ldots z_{i_p}$ does not vanish. This implies that U_α is an affine variety, for each multi-index α, from which it follows that $\mathcal{O}(k)$ is acyclic on U_α, for each α and each k. Hence, $\{U_i\}$ is a Leray cover for $\mathcal{O}(k)$, for each k. Thus, Leray's theorem (Theorem 7.8.5) applies and we may, in fact, compute the sheaf cohomology of $\mathcal{O}(k)$ or $\mathcal{S}$ using the Čech complex for the cover $\{U_i\}$.

We also have that each U_i is biholomorphically equivalent to $\mathbb{C}^n$, and $\mathcal{H}(k)$ is a free $\mathcal{H}$-module on each U_i. Furthermore, since each U_α is affine, as an algebraic variety, it is a Stein space (in fact, a Stein manifold), as a holomorphic variety. It follows from Cartan's Theorem B (Theorem 11.4.3) that $\mathcal{H}(k)$ is also acyclic on each U_α, and hence, that $\{U_i\}$ is also a Leray cover for the sheaf $\mathcal{H}(k)$. Again, Leray's theorem implies that we may compute the cohomology of $\mathcal{H}(k)$ or $\mathcal{T}_0$ using Čech cohomology for the cover $\{U_i\}$.

We will compute Čech cohomology for $\mathcal{O}(k)$ and $\mathcal{H}(k)$ by computing it for $\mathcal{S}$ and $\mathcal{T}_0$ and then projecting out the part which is homogeneous of degree k. We begin by expressing each element of $\mathcal{S}(U_\alpha)$ and $\mathcal{T}(U_\alpha)$ in terms of its *Fourier coefficients.* That is, if $\alpha = (i_0, \ldots, i_p)$, then U_α is the set of all $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $z_{i_j} \neq 0$ for $j = 0, \ldots, p$. This is a Cartesian product of copies of $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$, a copy of $\mathbb{C}$ for each $i = 0, \ldots, n$ which is not one of the i_j appearing in α, and a copy of $\mathbb{C}^*$ for each $i = 0, \ldots, n$ which is one of the i_j appearing in α. A function $f \in \mathcal{T}(U_\alpha)$ has a multiple Laurent series expansion

$$\sum a_{m_0 \ldots m_n} z_0^{m_0} \ldots z_n^{m_n}$$

in which $a_{m_0 \ldots m_n}$ is non-zero only for terms such that $(m_0, \ldots, m_n) \in W_\alpha$, where

$$W_\alpha = \{m = (m_0, \ldots, m_n) : m_i \geq 0 \quad \text{if} \quad i \notin \{i_0, \ldots, i_p\}\}.$$

For such an expansion to converge on U_α, the coefficients $a_{m_0 \ldots m_n}$ must decay faster than $r^{|m_0| + \cdots + |m_n|}$, for every positive number r. Thus, we may identify $\mathcal{T}(U_\alpha)$ with the space of functions $m \to a_m : \mathbb{Z}^{n+1} \to \mathbb{C}$ which decay at infinity faster than any geometric series, and which are supported on W_α.

Of course, $\mathcal{S}(U_\alpha)$ is the subspace of $\mathcal{T}(U_\alpha)$ consisting of sums, as above, with coefficients which are non-vanishing for only finitely many indices.

The spaces of global sections $\mathcal{S}(P^n)$ and $\mathcal{T}(P^n)$ may be described, as above, if we simply use for α the empty index. Thus, $\mathcal{T}(P^n)$ is the set of sums, as above, for which the coefficient function $m \to a_m$ satisfies the decay condition and is supported on K^+, where

$$K^+ = \{m = (m_0, \dots, m_n) \in \mathbb{Z}^k : m_i \geq 0 \quad \text{for} \quad i = 0, \dots, n\}.$$

Similarly, $\mathcal{S}(P^n)$ is the set of sums, as above, with coefficient function which is finitely non-zero and supported on K^+.

In $\mathcal{S}(U_\alpha)$ and $\mathcal{T}(U_\alpha)$, the elements which are homogeneous of degree k are those whose Fourier coefficient functions are supported on the set L_k, where

$$L_k = \{m = (m_0, \dots, m_n) \in \mathbb{Z}^k : m_0 + m_1 + \cdots + m_n = k\}.$$

Thus, the coefficient functions of global sections of $\mathcal{O}(k)$ and $\mathcal{H}(k)$ are supported on $K^+ \cap L_k$. Since there are only finitely many tuples $(m_0, \dots, m_n)$ of non-negative integers with $m_0 + m_1 + \cdots + m_n = k$, we conclude that $\mathcal{O}(k)(P^n)$ and $\mathcal{H}(k)(P^n)$ are finite dimensional, are equal, and in fact:

12.3.3 Theorem. *For each integer k, $H^0(P^n, \mathcal{O}(k))$ and $H^0(P^n, \mathcal{H}(k))$ both equal the space of homogeneous polynomials of degree k in $\mathbb{C}[z_0, \dots, z_n]$. This means, in particular, that $H^0(P^n, \mathcal{O}) = H^0(P^n, \mathcal{H}) = \mathbb{C}$, and for $k < 0$, $H^0(P^n, \mathcal{O}(k)) = H^0(P^n, \mathcal{H}(k)) = 0$.*

It turns out that the cohomology of the sheaves $\mathcal{O}(k)$ and $\mathcal{H}(k)$ in degrees other than 0 is quite simple. It vanishes in all degrees except n and in degree n it is the dual of degree 0 cohomology, but for k replaced by $-n-k-1$. The proof uses the notion of the support in $\mathbb{Z}^n$ of an element of pth cohomology and a lemma concerning this support.

Via the Fourier representation described above, a p-cochain f for the sheaf $\mathcal{S}$ and the cover $\{U_i\}$ is described by a series whose coefficients $\{a_m^\alpha\}$ comprise an alternating function which assigns to each $\alpha = (i_0, \dots, i_p)$ a finitely non-zero function $m \to a_m^\alpha : Z^{n+1} \to \mathbb{C}$ with support in the set W_α. We say a cochain is supported in a set $A \subset \mathbb{Z}^n$ if its coefficient function $m \to a_m^\alpha$ is supported in A for every α. Now multiplication by the characteristic function of a given subset A of Z^{n+1} is a projection operator P_A on cochains, which commutes with the coboundary operator, and has as range the cochains with support in A. Thus, P_A defines a projection operator on Čech cohomology as well. In particular, the cohomology of $\mathcal{O}(k)$ can be obtained from that of $\mathcal{S}$ by applying the projection P_{L_k} determined by the set L_k. Also, it makes sense to talk about an element of cohomology of $\mathcal{S}$ being supported in a set A – that is, $\xi \in H^p(P^n, \mathcal{S})$ is supported in A if $P_A\xi = \xi$. The same things are true for the sheaf $\mathcal{T}_0$.

12.3.4 Lemma. *For $p > 0$, every element of $H^p(P^n, \mathcal{S})$ or $H^p(P^n, \mathcal{T}_0)$ is supported in the set $K^- = \{m = (m_0, \ldots, m_n) : m_i < 0,\ for\, i = 0, \ldots, n\}$.*

Proof. We know that $H^p(U_j, \mathcal{S}) = 0$, for $p > 0$, since U_j is affine, and $\mathcal{S}$ is quasi-coherent (it is an infinite direct sum of coherent sheaves). This means that, for a fixed j, and $p > 0$, any p-cocycle f for $\mathcal{S}$ and the cover $\{U_i\}$ of P^n is, when restricted to U_j, the coboundary of some $(p-1)$-cochain g for $\mathcal{S}$ and the open cover $\{U_j \cap U_i\}_i$ of U_j. Such a cochain assigns to each multi-index $\alpha = (i_0, \ldots, i_{p-1})$ an element $g(\alpha) \in \mathcal{S}(U_\alpha \cap U_j)$. The Fourier coefficient function of such an element is supported in $W_{\alpha\vee j}$, where $\alpha \vee j = (i_0, \ldots, i_{p-1}, j)$. The only thing that keeps g from being a cochain for the cover $\{U_i\}$ of P^n is that $g(\alpha)$ has coefficient function supported in $W_{\alpha\vee j}$ rather than W_α. Since $W_\alpha = W_{\alpha\vee j} \cap A_j$, where $A_j = \{m \in Z^{n+1} : m_j \geq 0\}$, the projection $P_{A_j} g$ is a cochain for $\{U_i\}$ which is mapped by the coboundary onto $P_{A_j} f$. This implies that P_{A_j} kills elements of $H^p(P^n, \mathcal{S})$ for $p > 0$, and hence, that $H^p(P^n, \mathcal{S})$ is supported on the complement of A_j. Since this is true for every j, we conclude that $H^p(P^n, \mathcal{S})$ is supported on K^-, as claimed. The same proof works for $H^p(P^n, \mathcal{T}_0)$.

This leads directly to:

12.3.5 Theorem. *For each integer k we have*

(i) *$H^p(P^n, \mathcal{O}(k)) = 0$ if p is not 0 or n;*

(ii) *$H^n(P^n, \mathcal{O}(-n-1)) \simeq \mathbb{C}$, and $H^n(P^n, \mathcal{O}(k)) = 0$ for $k > -n-1$;*

(iii) *$H^n(P^n, \mathcal{O}(-n-1-k))$ is the vector space dual of $H^0(P^n, \mathcal{O}(k))$ under the pairing induced by multiplication:*

$$H^0(P^n, \mathcal{O}(k)) \times H^n(P^n, \mathcal{O}(-n-k-1)) \to H^n(P^n, \mathcal{O}(-n-1)) \simeq \mathbb{C}.$$

Also, the same statements are true with $\mathcal{O}(k)$ replaced by $\mathcal{H}(k)$.

Proof. An alternating Čech cochain vanishes on any multi-index with a repeated entry. Thus, n is the largest value of p for which there are non-zero alternating p-cochains for $\{U_i\}$, from which it follows that $H^p(P^n, S) = 0$ for $p > n$.

If f is a p-cochain, then the coefficient function of $f(\alpha)$ is supported on W_α. It follows that an element of $H^p(P^n, \mathcal{S})$ is supported on the union of the W_α for $\alpha \in \mathbb{Z}^{p+1}$. However, $W_\alpha \cap K^- = \emptyset$ for all $\alpha \in \mathbb{Z}^{p+1}$, if $0 < p < n$. It follows from Lemma 12.3.4 that $H^p(P^n, \mathcal{S}) = 0$ for $0 < p < n$. This proves (i).

For $p = n$, the only multi-index we need consider is $\alpha = (0, 1, \ldots, n)$, since all others without repeated entries are permutations of this one. For this index α, the set W_α is all of Z^{n+1}. Thus, an n-cochain for $\mathcal{S}$ and

$\{U_i\}$ is described by just a single coefficient function $a = \{a_m\}$, with no restriction, other than only finitely many coefficients may be non-zero. By Lemma 12.3.4, $H^n(P^n, S)$ is supported on K^-. However, every n-cochain is a cocycle and the space of $(n-1)$-cochains is supported in the complement of K^-. Thus, it follows that $H^n(P^n, S)$ is actually isomorphic to the space of finitely non-zero coefficient functions supported in K^-. Since $K^- \cap L_{-n-1}$ is the single point consisting of the $(n+1)$-tuple $(-1, -1, \ldots, -1)$, we conclude that $H^n(P^n, \mathcal{O}(-n-1))$ consists of functions with a single non-vanishing coefficient corresponding to the index $(-1, \ldots, -1)$. If $k > -n-1$, then $L_k \cap K^- = \emptyset$ and so $H^n(P^n, \mathcal{O}(k)) = 0$. This proves part (ii).

We have, from the previous theorem, that $H^0(P^n, \mathcal{S})$ may be regarded as the space of finitely non-zero coefficient functions which are supported in K^+ and, from the previous paragraph, that $H^n(P^n, \mathcal{S})$ may be regarded as the space of finitely non-zero coefficient functions which are supported on K^-. Clearly, reflection through the hyperplane $\sum m_i = (-n-1)/2$ is a bijection between K^+ and K^-. This pairs basis elements for $H^0(P^n, \mathcal{S})$ and $H^n(P^n, \mathcal{S})$. It pairs functions supported on L_k with functions supported on L_{-n-k-1}, and thus, it pairs basis elements for $H^0(P^n, \mathcal{O}(k))$ and $H^n(P^n, \mathcal{O}(-n-k-1))$, as required in the theorem. This completes the proof of part (iii).

The argument is almost the same in the analytic case, where $\mathcal{O}(k)$ is replaced by $\mathcal{H}(k)$. The difference is that we work with $\mathcal{T}_0$ rather than $\mathcal{S}$. However, $\mathcal{T}_0$ is just the direct sum of the $\mathcal{H}(k)$ and, hence, also has vanishing cohomology on each U_α. The rest of the argument is the same if one observes that the projection operators P_A preserve the faster than exponential decay at infinity required of coefficient functions in $\mathcal{T}(U_\alpha)$.

Since we use the same cover by Zariski open sets to compute both, there is a map $H^p(P^n, \mathcal{O}(k)) \to H^p(P^n, \mathcal{H}(k))$, defined by inclusion. Clearly, the above theorem has as a consequence:

12.3.6 Corollary. *The map $H^p(P^n, \mathcal{O}(k)) \to H^p(P^n, \mathcal{H}(k))$ is an isomorphism for every p and every k.*

12.3.7 Example. The space P^1 is covered by two open sets, U_0 and U_1, which are copies of $\mathbb{C}$, via the maps $\phi_i : U_i \to \mathbb{C}$, $i = 0, 1$. Note that ϕ_0^{-1} and ϕ_1^{-1} are the maps $w \to (1, w)$ and $z \to (z, 1)$, followed by $\pi : \mathbb{C}^2 \to P^1$. The map $\phi_0 \circ \phi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ identifies these two copies of $\mathbb{C}$ along $\mathbb{C}^*$ to form P^1. Note that $\phi_0 \circ \phi_1^{-1}(z) = 1/z$. Thus, we may think of U_0 as a complex plane with coordinate w and U_1 as a complex plane with coordinate z. The overlap $U_0 \cap U_1$ is a copy of the punctured plane $\mathbb{C}^*$ and the two coordinate functions are related by $w = 1/z$ on the overlap.

On U_0, the sheaf $\mathcal{O}(k)$ is generated over $\mathcal{O}$ by the section determined by the k-homogeneous function z^k on $\pi^{-1}(U_0)$, while on U_1, it is generated by the section determined by the k-homogeneous function w^k on $\pi^{-1}(U_1)$. Thus, $\mathcal{O}(k)$ is a copy of $\mathcal{O}$ on each of U_0 and U_1 and these two copies of $\mathcal{O}$ are identified on $U_0 \cap U_1$ by multiplication by the transition function determined by the homogeneous function z^k/w^k on $\pi^{-1}(U_0 \cap U_1)$. In the z coordinate for U_1 this is the function z^k. In the w coordinate for U_0 it is the function w^{-k}. Thus, in terms of the z coordinate, a global section of $\mathcal{O}(k)$ consists of a polynomial $f(z)$ in z and a polynomial $g(1/z)$ in $1/z$ such that $f(z) = z^k g(1/z)$. If f and g are not identically 0, this forces k to be non-negative and f and g to have degree less than or equal to k. Thus, there are non-zero global sections of $\mathcal{O}(k)$ only for $k \geq 0$, as asserted in Theorem 12.3.3. A similar analysis for the sheaves $\mathcal{H}(k)$ on P^1 leads to similar conclusions.

Note that the sheaf $\mathcal{H}^{(1)}$ of holomorphic 1-forms on P^1 (section 10.1) is an invertible sheaf and it is natural to ask if it is isomorphic to one of the sheaves $\mathcal{H}(k)$. In fact, it is easy to see that it is isomorphic to $\mathcal{H}(-2)$. This follows from the fact that dz generates $\mathcal{H}^{(1)}$ on U_1, $-dw$ generates $\mathcal{H}^{(1)}$ on U_0 and $-dw = -d(1/z) = z^{-2}dz$ on $U_0 \cap U_1$. By Theorem 12.3.4, we expect that $\mathcal{H}^{(1)}$ has cohomology $\mathbb{C}$ in degree 1 and vanishing cohomology in all other degrees. This is the conclusion of Exercise 7.18.

12.4 Applications of the Sheaves $\mathcal{O}(k)$

In this section, we show how to make use of the sheaves $\mathcal{O}(k)$ in the study of coherent sheaves on projective varieties. We show that a coherent sheaf on P^n may be *twisted* by tensoring it with $\mathcal{O}(k)$, and if k is large enough, the resulting twisted sheaf is acyclic and is generated by finitely many of its global sections. This implies that every coherent sheaf on projective space has finite dimensional cohomology. Since projective varieties have closed embeddings in projective space, information about coherent sheaves on P^n leads to information on coherent sheaves on projective varieties in general.

12.4.1 Definition. *If $\mathcal{F}$ is a coherent algebraic sheaf on P^n, then we set $\mathcal{F}(k) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(k)$. The sheaf $\mathcal{F}(k)$ is said to be obtained from $\mathcal{F}$ through twisting it by $\mathcal{O}(k)$.*

Note that the direct sum $\oplus_k \mathcal{F}(k)$ is a sheaf of modules over the sheaf of rings $\mathcal{S} = \oplus_k \mathcal{O}(k)$, where the product hf of $h \in \mathcal{O}(k)$ and $f \in \mathcal{F}(m)$ is $h \otimes f \in \mathcal{F}(k+m)$.

Note also, since $\mathcal{O}(k)$ is locally free of rank 1, $\mathcal{F}(k)$ is locally isomorphic to $\mathcal{F}$, but it is not generally globally isomorphic to $\mathcal{F}$, because of the twist introduced by tensoring with $\mathcal{O}(k)$.

12.4.2 Proposition. *If $\mathcal{F}$ is a coherent algebraic sheaf on P^n, then for some $k \geq 0$, the sheaf $\mathcal{F}(k)$ is generated by finitely many of its global sections.*

Proof. For each i, the open set U_i is affine, and hence, the coherent sheaf $\mathcal{F}|_{U_i}$ is the image under localization of $\mathcal{F}(U_i)$, which is finitely generated. Let $\{f_{ij}\}_j$ be a finite generating set of sections for $\mathcal{F}(U_i)$. Now z_i is a global section of $\mathcal{O}(1)$ which vanishes exactly on the complement of U_i. For some integer m, which may be chosen large enough to work for all i, j, the product $z_i^m f_{ij}$ extends to be a global section of $\mathcal{F}(m)$ (Exercise 12.5). The resulting collection of global sections $\{z_i^m f_{ij}\}_{ij}$ clearly generates $\mathcal{F}(m)$ on each U_i and, hence, on all of P^n.

12.4.3 Proposition. *If $\mathcal{F}$ is a coherent algebraic sheaf on P^n, then $\mathcal{F}$ is a quotient of a sheaf of the form $\mathcal{O}(k)^r$, for integers $k \leq 0$ and $r > 0$.*

Proof. By the previous proposition, there exists a non-negative integer m such that $\mathcal{F}(m)$ is generated by finitely many global sections. If there are r of these sections, then they determine a surjection $\mathcal{O}^r \to \mathcal{F}(m)$. On tensoring this with $\mathcal{O}(-m)$, we get a surjection $\mathcal{O}(-m)^r \to \mathcal{F}$, as required.

12.4.4 Theorem. *If $\mathcal{F}$ is a coherent algebraic sheaf on P^n, then*

(i) *$H^p(P^n, \mathcal{F})$ is finite dimensional for each p and vanishes for $p > n$;*

(ii) *there exists an integer m_0 such that $H^p(P^n, \mathcal{F}(m)) = 0$, for all $p > 0$ and all $m > m_0$.*

Proof. That $H^p(P^n, \mathcal{F})$ vanishes for $p > n$ follows from the fact that we can use alternating Čech cochains for the cover $\{U_i\}$ to compute it. Note that this implies that the m_0 in part (ii) can be chosen independent of p, if it can be chosen at all, since there are only finitely many p to worry about. With these things in mind, the proof is by reduction on p. We assume both statements are true for some $p+1 \leq n+1$ and then prove they are also true for p. Using Proposition 12.4.3, we express $\mathcal{F}$ as a quotient of a sheaf $\mathcal{O}(k)^r$ so that we have a short exact sequence of coherent sheaves

$$0 \to \mathcal{K} \to \mathcal{O}(k)^r \to \mathcal{F} \to 0.$$

Then part (i) for p follows from the long exact sequence for cohomology associated to this short exact sequence, the assumption that the theorem is true for $p+1$, and Theorems 12.3.3 and 12.3.5. Part(ii) follows in a similar fashion from Theorem 12.3.5 and the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{K}(m) \to \mathcal{O}(k+m)^r \to \mathcal{F}(m) \to 0,$$

obtained through tensoring by the sheaf $\mathcal{O}(m)$. This completes the proof.

The next result shows that part (i) of the above theorem holds for projective varieties in general. There is also an analogue of part (ii) for projective varieties which is developed in the next section (Theorem 12.5.8(ii)).

12.4.5 Theorem. *Let X be a projective variety and $\mathcal{M}$ a coherent algebraic sheaf on X. Then $H^p(X, \mathcal{M})$ is finite dimensional for all p and vanishes for $p > n$, where n is the dimension of a projective space P^n into which X embeds.*

Proof. Let $i : X \to P^n$ be a closed embedding. Then $i_*\mathcal{M}$ is a coherent algebraic sheaf on P^n, by Proposition 8.7.6, and $H^p(X, \mathcal{M}) \simeq H^p(P^n, i_*\mathcal{M})$, by Exercise 7.20. The result now follows from part (i) of the preceding theorem.

12.5 Embeddings in Projective Space

In this section we develop criteria for a variety to be projective or quasi-projective (Definition 12.2.5). This material is not really needed in the remainder of the text, but we include it for completeness. Most of the proofs will be left as exercises.

Consider any morphism ϕ of an algebraic variety X into projective space P^n. The invertible sheaf $\mathcal{O}(1)$ on P^n is generated by its global sections determined by the 1-homogeneous functions $z_0, \dots, z_n$ on $\mathbb{C}^{n+1} - 0$. It follows that $\phi^*\mathcal{O}(1)$ is an invertible sheaf on X which is generated by sections $g_0, \dots, g_n$ corresponding to $z_0, \dots, z_n$. Thus, the morphism $\phi : X \to P^n$ determines an invertible sheaf on X and a set of $n + 1$ global sections which generate this sheaf.

On the other hand, suppose X is an algebraic variety and $\mathcal{L}$ an invertible sheaf on X which is generated by a set $\{g_0, g_1, \dots, g_n\}$ of global sections. Let V be an open subset of X on which $\mathcal{L}$ is free. This means that there is a section $h \in \mathcal{L}(V)$ which generates $\mathcal{L}$ on V, and this section determines an isomorphism $f \to fh : \mathcal{O}|_V \to \mathcal{L}|_V$. Then $g_i = f_i h$, $i = 0, \dots, n$ for a set $\{f_0, \dots, f_n\}$ of regular functions on V. The fact that the g_i generate the sheaf $\mathcal{L}$ at each point of V just means that at least one of the functions f_i is non-vanishing at each point of V. Hence, the f_i are the coordinate functions of a morphism $\tilde{\phi} : V \to \mathbb{C}^{n+1} - \{0\}$. If we follow this by the projection $\pi : \mathbb{C}^{n+1} - \{0\} \to P^n$, we have a morphism $\phi : V \to P^n$. Now the morphism $\tilde{\phi}$ depends on the choice of the generating section h on V, but the morphism ϕ does not, since any two such sections differ by a factor which is a non-vanishing regular function and this factor disappears when we apply π. Thus, the invertible sheaf $\mathcal{L}$ and sections $g_0, \dots, g_n$ determine a well-defined morphism from V to P^n. The morphisms so determined on two overlapping neighborhoods V_1 and V_2 will agree on the intersection $V_1 \cap V_2$. It follows that there is a well-defined morphism $\phi : X \to P^n$, determined by $\mathcal{L}$ and $g_0, \dots, g_n$, with the property that $\phi^*\mathcal{O}(1) \simeq \mathcal{L}$, and the sections $g_0, \dots, g_n$ of $\mathcal{L}$ are those determined by the sections $z_0, \dots, z_n$ of $\mathcal{O}(1)$. In summary:

12.5.1 Proposition. *Let X be an algebraic variety. Each morphism of algebraic varieties $\phi : X \to P^n$ determines an invertible sheaf $\mathcal{L} = \phi^*\mathcal{O}(1)$ on X and a set $\{g_0, \dots, g_n\}$ of global sections which generates $\mathcal{L}$.*

Conversely, each pair consisting of an invertible sheaf $\mathcal{L}$ on X and a set of $n+1$ global sections, which generates $\mathcal{L}$, uniquely determines a morphism $\phi : X \to P^n$ such that $\mathcal{L} = \phi^*\mathcal{O}(1)$, and the set of sections is that determined by the coordinate sections in $\mathcal{O}(1)$.

Of particular interest is the case where the morphism $\phi : X \to P^n$ is an embedding. Recall that a morphism $\phi : X \to Y$ between algebraic varieties is said to be an *embedding* if it is an isomorphism of X onto a subvariety of an open subset of Y. An embedding with closed image – that is, with image a subvariety of Y itself – is called a *closed embedding.*

12.5.2 Definition. *Let X be an algebraic variety. An invertible sheaf $\mathcal{L}$ on X is said to be very ample if it is generated by a set of global sections which determines an embedding of X into P^n for some n.*

If X is a quasi-projective variety, then, by definition, there is an embedding $i : X \to P^n$. This embedding is determined, as in Proposition 12.5.1, by the sheaf $i^*\mathcal{O}(1)$ and the sections determined by the coordinate functions $z_0, \dots, z_n$. Thus, $i^*\mathcal{O}(1)$ is a very ample sheaf on X, in this case. In particular, there is a very ample sheaf on X if X is either projective or affine.

12.5.3 Definition. *An invertible sheaf $\mathcal{L}$ on an algebraic variety X is said to be ample if, for every coherent algebraic sheaf $\mathcal{M}$ on X, there is a positive integer k such that $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated by its global sections.*

As an easy consequence of Proposition 12.4.2 we have:

12.5.4 Proposition. *Every very ample invertible sheaf is ample.*

The proof is left as an exercise (Exercise 12.8).

The next result shows that if a variety has an ample sheaf, then it has a very ample sheaf. Its proof is not so easy, but is carried out in a series of exercises (Exercises 12.8 – 12.11).

12.5.5 Theorem. *An invertible sheaf $\mathcal{L}$, on an algebraic variety X, is ample if and only if $\mathcal{L}^k$ is very ample for some positive integer k.*

If a variety X has an ample sheaf, then Theorem 12.5.5 implies that there is an embedding of X into projective space – that is, X is quasi-projective. It need not be projective, of course, because its image need not be closed. However, there is a condition, analogous to compactness for Hausdorff topologies, which guarantees that the image will be closed. This is the completeness condition.

12.5.6 Definition. *An algebraic variety X is said to be complete if, for every algebraic variety Y, the projection $X \times Y \to Y$ is a closed map.*

We will show in the next chapter that every projective variety is complete (Exercise 13.17). Of course, every projective variety has an ample sheaf. The next result shows that these two conditions characterize projective varieties. It follows immediately from the fact that a morphism $f : X \to Y$, with X complete, has closed image (Exercise 12.12).

12.5.7 Proposition. *A complete variety is projective if and only if it has an ample sheaf.*

The next theorem gives a cohomological criterion for a sheaf to be ample.

12.5.8 Theorem. *Let X be an algebraic variety and $\mathcal{L}$ an invertible sheaf on X. Then*

(i) *$\mathcal{L}$ is ample if, for each coherent algebraic sheaf $\mathcal{M}$ on X, there is a positive integer k_0 such that $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is acyclic for all $k > k_0$;*

(ii) *the converse is true provided X is complete.*

Proof. We will prove part (i) and leave part (ii) as an exercise (Exercise 12.14).

Let $\mathcal{M}$ be a coherent algebraic sheaf on X, x a point of X, $\mathcal{I}$ the ideal sheaf of $\{x\}$, and $\mathcal{S}$ the quotient sheaf $\mathcal{M}/\mathcal{I}\mathcal{M}$. Note that $\mathcal{S}$ is a skyscraper sheaf at x, with stalk the geometric fiber $\mathcal{M}_x/\mathcal{I}_x$ of $\mathcal{M}$ at x. We have a short exact sequence

$$0 \to \mathcal{I}\mathcal{M} \to \mathcal{M} \to \mathcal{S} \to 0.$$

We twist this by a power of $\mathcal{L}$ and obtain the sequence

$$0 \to \mathcal{I}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k \to \mathcal{S} \otimes_{\mathcal{O}} \mathcal{L}^k \to 0.$$

This is exact, because $\mathcal{L}^k$ is locally free and, hence, has flat stalks. The sheaf $\mathcal{I}\mathcal{M}$ is coherent, and so, by hypothesis, there is an integer k_0 so that

$\mathcal{IM} \otimes_{\mathcal{O}} \mathcal{L}^k$ is acyclic for $k \geq k_0$. Then $\Gamma(X, \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k) \to \Gamma(X, \mathcal{S} \otimes_{\mathcal{O}} \mathcal{L}^k)$ is surjective. In other words, the global sections of the sheaf $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ generate its geometric fiber at x.

Now it follows from Nakayama's lemma that the global sections of the sheaf $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ generate its stalk at x as an $\mathcal{O}_x$-module (Exercise 12.15). By the algebraic analogue of Exercise 9.6, $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated by its global sections throughout a neighborhood of x. A priori this neighborhood depends on k, but we will show that we may choose it independent of $k > k_0$.

The sheaf $\mathcal{L}$ is coherent, and so, by the result of the previous paragraph, there is an integer j and a neighborhood of x in which $\mathcal{L}^j$ is generated by global sections. Let U be a neighborhood of x in which this is true and in which it is also true that $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated by global sections for $k_0 \leq k \leq k_0 + j$. Then each positive power $\mathcal{L}^{mj}$ of $\mathcal{L}^j$ is generated by global sections in U, as is each sheaf of the form $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^{k+mj} = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k \otimes_{\mathcal{O}} \mathcal{L}^{mj}$ for $k_0 \leq k < k_0 + j$ and $m \geq 0$. We conclude that each sheaf $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated by global sections on U for $k > k_0$.

It follows from the above that there is an open cover $U_1, \ldots, U_n$ of X and a set of positive integers $k_1, \ldots, k_n$ such that $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated by global sections on U_i if $k \geq k_i$. Then $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}^k$ is generated on all of X by its global sections, provided $k \geq \max\{k_1, \ldots, k_n\}$. This shows that $\mathcal{L}$ is ample and completes the proof of (i).

Exercises

1. A homogeneous ideal of $\mathbb{C}[z_0, \ldots, z_n]$ is an ideal of the form $I = \oplus_k I_k$, where I_k consists of the polynomials in I which are homogeneous of degree k. Prove that an ideal is homogeneous if and only if it is generated by a finite set of homogeneous polynomials.
2. Prove that a homogeneous ideal I of $\mathbb{C}[z_0, \ldots, z_n]$ has, as its set of common zeroes in $\mathbb{C}^{n+1} - \{0\}$, a set of the form $\pi^{-1}(V)$, where V is a unique subvariety of P^n, and $\pi : \mathbb{C}^{n+1} - \{0\} \to P^n$ is the canonical projection. The subvariety V is called the *zero set* of the homogeneous ideal I and is denoted $Z(I)$.
3. Prove that every subvariety of P^n is the zero set of some homogeneous ideal.
4. Prove that if V is an algebraic variety, then the topological space V^h, constructed by giving each affine open subset of V its Euclidean topology, is Hausdorff and countable at infinity.

5. Model the proof of Lemma 8.3.2(ii) to prove that, if $\mathcal{F}$ is a coherent sheaf on P^n, and $f \in \mathcal{F}(U_i)$, then for some positive integer m the section $z_i^m f \in \Gamma(U_i, \mathcal{F}(m))$ extends to a global section of $\mathcal{F}(m)$.
6. Prove that $P^n \times P^m$ may be embedded as a subvariety of P^k for some k. Thus, if X and Y are projective or quasi-projective varieties, then so is $X \times Y$.
7. Prove that, for every compact Riemann surface X, there is a finite holomorphic map $\phi : X \to P^1$, where P^1 is considered a complex manifold with the Euclidean topology (see section 9.6). Hint: Use Exercise 11.17.
8. Prove that an invertible sheaf $\mathcal{L}$, on an algebraic variety X, is ample if $\mathcal{L}^k$ is very ample for some positive integer k. This proves Proposition 12.5.4 and one half of Theorem 12.5.5.
9. Let $\mathcal{L}$ be an ample sheaf on an algebraic variety X. Let U be an affine open subset of X, and suppose $x \in U$. Prove there is a positive integer p, and a section s of $\mathcal{L}^p$ such that $s_y \in M_y\mathcal{L}_y$ at each point y of $X - U$, but $s_x \notin M_x\mathcal{L}_x$. Here, M_y is the maximal ideal of $\mathcal{O}_y$.
10. With X, $\mathcal{L}$, and s as in the previous exercise, let U_s be the affine neighborhood of x consisting of all points $y \in X$ for which $s_y \notin M_y\mathcal{L}_y$. Let $f_1, \ldots, f_n$ be a set of generators of $\mathcal{O}(U_s)$ as an algebra over $\mathbb{C}$. Prove that there is an integer m such that $s^m f_j$ extends to be a global section of $\mathcal{L}^{mp}$. Conclude that $\mathcal{L}^{mp}$ has global sections $g_0, g_1, \ldots, g_n$ such that $g_0 = s^m$ and $g_j = f_j g_0$ on U for $j = 1, \ldots, n$.
11. Complete the proof of Theorem 12.5.5, by proving that if $\mathcal{L}$ is an ample invertible sheaf on an algebraic variety X, then $\mathcal{L}^k$ is very ample for some positive integer k. Hint: Cover X by finitely many sets of the form U_s, and construct the sections $g_0, \ldots, g_n$ of the sheaf $\mathcal{L}^{pm}$, as in the previous exercise, for each of these open sets. Note that the integers p and m may be chosen so that they work for each set in the open cover. Show that the union of the resulting sets of sections of $\mathcal{L}^{pm}$ determines an embedding of X into projective space of some dimension. Hint: See Exercise 8.18.
12. Prove that if $f : X \to Y$ is a morphism of algebraic varieties, and X is complete, then $f(X)$ is closed in Y.
13. Prove that there is an ample invertible sheaf on every variety of the form $X \times Y$, where X is affine, and Y is projective.
14. Prove (ii) of Theorem 12.5.8 by using the fact that, for some m, $\mathcal{L}^m$ determines a closed embedding of X into P^n.
15. Use Nakayama's lemma to prove that if S is a finitely generated module over a Noetherian local ring A, M is the maximal ideal of A, and F is a subset of S whose image in S/MS generates S/MS, then F generates S.
16. Prove that the discrete union of a finite set of projective varieties is also a projective variety. In other words, show that if $V_1, \ldots, V_k$ are projective

varieties, then for some n, there are closed embeddings $\phi_i : V_i \to P^n$, $i = 1, \dots, k$, with disjoint images.

17. Let Y be an affine algebraic variety, set $X = P^n \times Y$, and let $p : X \to P^n$ be the projection. If ${}_X\mathcal{O}(k) = p^*\mathcal{O}(k)$, then prove the following analogue of Theorem 12.3.5: $H^j(X, {}_X\mathcal{O}(k))$ is a free finite rank $\mathcal{O}(Y)$-module for each k, and it is 0 unless $j = 0$ or n.
18. Let X and Y be as in the previous exercise. Use the result of that exercise and the strategy of section 12.4 to prove that if $\mathcal{M}$ is a coherent algebraic sheaf on X, then $H^p(X, \mathcal{M})$ is a finitely generated $\mathcal{O}(Y)$-module for each p and is 0 for $p > n$.
19. Let X and Y be as in the previous exercise. Use the result of that exercise and Exercise 8.16 to prove that if $\pi : X \to Y$ is the projection, and $\mathcal{M}$ is a coherent algebraic sheaf on X, then $\pi_*\mathcal{M}$ is a coherent algebraic sheaf on Y.
20. A morphism $\phi : X \to Y$, between algebraic varieties, is called a *projective morphism* if it factors as an embedding of X as a closed subvariety of $P^n \times Y$, followed by the projection $P^n \times Y \to Y$. Prove that if ϕ is a projective morphism, and $\mathcal{S}$ is a coherent algebraic sheaf on X, then $\phi_*\mathcal{S}$ is a coherent algebraic sheaf on Y.

Chapter 13

Algebraic vs. Analytic – Serre's Theorems

In this chapter, we prove that, for all practical purposes, analytic projective varieties and algebraic projective varieties and their coherent sheaves are the same. These are the results of Serre's famous GAGA paper *Géométrie Algébrique et Géométrie Analytique* [**S**]. The strategy is to first establish that the local ring of holomorphic functions at a point of an algebraic variety is a faithfully flat ring extension of the local ring of regular functions at this point, by proving that the two local rings have the same M-adic completion. This allows us to define an exact functor from coherent algebraic sheaves on an algebraic variety to coherent analytic sheaves on the corresponding holomorphic variety. We then use the cohomological results of the preceding chapters to show that this is a cohomology preserving equivalence of categories in the case where the algebraic variety is projective. This theorem has a number of applications, a few of which will be discussed here.

13.1 Faithfully Flat Ring Extensions

In this section, $A \subset B$ will be a pair consisting of a commutative ring with identity and a subring containing the identity. In this situation, there is a functor $X \to X_B$ from the category of A-modules to the category of B-modules, defined by

$$X_B = B \otimes_A X.$$

The situation is particularly nice when this functor is both exact and faithful ($X \neq 0$ implies $X_B \neq 0$). This leads to the following definition.

13.1.1 Definition. *The ring B is said to be faithfully flat over its subring A if B is a flat A-module, and for each non-zero A-module X, the B-module $X_B = B \otimes_A X$ is non-zero.*

There are several equivalent ways of saying that B is faithfully flat over A.

13.1.2 Lemma. *The following statements are equivalent:*
(i) *B is faithfully flat over A;*
(ii) *B is flat over A, and $x \to 1 \otimes x : X \to X_B$ is injective for every A-module X;*
(iii) *B is flat over A, and $BI \cap A = I$ for every ideal I of A;*
(iv) *B/A is a flat A-module.*

Proof. If B is faithfully flat over A, and X is an A-module, let K be the kernel of $x \to 1 \otimes x : X \to X_B$. Then the flatness of B implies that $B \otimes_A K \to B \otimes_A X$ is injective. On the other hand, the composition of the maps

$$K \to B \otimes_A K \to B \otimes_A X$$

is the zero map. This implies that $k \to 1 \otimes k : K \to B \otimes_A K$ is the zero map; but the image of this map generates $B \otimes_A K$ over B and, hence, $B \otimes_A K = 0$. It follows from (i) that $K = 0$. This proves that (i) implies (ii). That (ii) implies (i) is obvious.

Note that $I \subset BI \cap A$ for any ideal $I \subset A$. Also, if B is flat over A, and $I \subset A$ is an ideal, then applying $B \otimes_A (\cdot)$ to the short exact sequence

$$0 \to I \to A \to A/I \to 0,$$

and noting that $B \otimes_A I = BI$, yields the isomorphism $B \otimes_A (A/I) \simeq B/BI$. Thus, the condition $BI \cap A = I$ is equivalent to the injectivity of the map $A/I \to B \otimes_A (A/I)$. This proves that (ii) implies (iii). It also proves (iii) implies (ii), since for any A-module X and any $x \in X$, the submodule $Ax \subset X$ has the form A/I for the ideal $I = \{a \in A : ax = 0\}$.

For an A-module X, the short exact sequence

$$0 \to A \to B \to B/A \to 0$$

gives rise to an exact sequence

$$\mathrm{Tor}_1^A(A, X) \to \mathrm{Tor}_1^A(B, X) \to \mathrm{Tor}_1^A(B/A, X) \to A \otimes_A X \to B \otimes_A X.$$

Since $\mathrm{Tor}_1^A(A, X) = 0$, and $A \otimes_A X = X$, this becomes

$$0 \to \mathrm{Tor}_1^A(B, X) \to \mathrm{Tor}_1^A(B/A, X) \to X \to B \otimes_A X.$$

We conclude that $\operatorname{Tor}_1^A(B/A, X) = 0$ if and only if $\operatorname{Tor}_1^A(B, X) = 0$, and $X \to B \otimes_A X$ is injective. Since an A-module Y is flat if and only if $\operatorname{Tor}_1^A(Y, X)$ vanishes for every A-module X (Theorem 6.6.10), it follows that (ii) is equivalent to (iv). This completes the proof.

13.1.3 Proposition. *If $A \subset B \subset C$ are rings, with C faithfully flat over A, and C faithfully flat over B, then B is faithfully flat over A.*

Proof. Suppose $X \to Y$ is an injective morphism of A-modules, and let N be the kernel of $B \otimes_A X \to B \otimes_A Y$. Then, since C is B-flat, we have an exact sequence

$$0 \to C \otimes_B N \to C \otimes_B (B \otimes_A X) \to C \otimes_B (B \otimes_A Y).$$

By the associativity of tensor product, we have that $C \otimes_B (B \otimes_A X) = C \otimes_A X$ and $C \otimes_B (B \otimes_A Y) = C \otimes_A Y$. But, since C is A-flat, it follows that $C \otimes_A X \to C \otimes_A Y$ is injective and, hence, $C \otimes_A N = 0$. This implies that $N = 0$, since C is faithfully flat over A. Thus, we have proved that B is A-flat.

Now suppose that X is an A-module, and consider the pair of morphisms $X \to B \otimes_A X \to C \otimes_A X$. Since the composition is an injection, due to the fact that C is faithfully flat over A, it follows that the first map is an injection as well, and hence, that B is faithfully flat over A. This completes the proof.

The next lemma says that the functor $X \to X_B = X \otimes_A B$ behaves well with respect to Hom if B is a faithfully flat extension of A.

If X and Y are A-modules, then the functor $X \to X_B$ defines a natural morphism of A-modules

$$\operatorname{Hom}_A(X, Y) \to \operatorname{Hom}_B(X_B, Y_B)$$

which, since the right side is a B-module, induces a B-module morphism

$$\iota : \operatorname{Hom}_A(X, Y)_B \to \operatorname{Hom}_B(X_B, Y_B).$$

13.1.4 Lemma. *The morphism ι, defined above, is an isomorphism if A is Noetherian, X is finitely generated over A, and B is faithfully flat over A.*

Proof. For a fixed module Y, consider the two functors T and T' from A-modules to B-modules defined by

$$T(X) = \operatorname{Hom}_A(X, Y)_B$$

and

$$T'(X) = \mathrm{Hom}_B(X_B, Y_B) = \mathrm{Hom}_A(X, Y_B).$$

Then $\iota : T \to T'$ is a morphism of functors, and we are to show that it is an isomorphism on finitely generated modules. Clearly, ι is an isomorphism if $X = A$, since $T(A)$ and $T'(A)$ are both equal to Y_B in this case. Similarly, ι is an isomorphism if $X = A^n$ for some n. Note that T and T' are left exact, since Hom is left exact in its first variable, and B is faithfully flat over A. Since A is Noetherian, for each finitely generated A-module X, we can construct an exact sequence of the form

$$A^n \longrightarrow A^m \longrightarrow X \longrightarrow 0.$$

On applying T and T', we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] \arrow[d, "\iota"'] & T(A^m) \arrow[r] \arrow[d, "\iota"'] & T(A^n) \arrow[d, "\iota"'] \\
0 \arrow[r] & T'(X) \arrow[r] & T'(A^m) \arrow[r] & T'(A^n),
\end{tikzcd}$$

with exact rows and with the last two vertical maps isomorphisms. It follows that the first vertical map is also an isomorphism and the proof is complete.

13.2 Completion of Local Rings

Let $\mathbf{V}$ be the germ of an algebraic variety V at $0 \in \mathbb{C}^n$. We will prove that ${}_V\mathcal{H}_0$ is faithfully flat over ${}_V\mathcal{O}_0$. Our strategy will be to inject both rings into a third ring – the M-adic completion of ${}_V\mathcal{O}_0$ with respect to its maximal ideal M – and to show that this algebra is faithfully flat over both ${}_V\mathcal{O}_0$ and ${}_V\mathcal{H}_0$. Then Proposition 13.1.3 will give us the desired result. To this end, we need to study the completion $\hat{A}$ of a local ring with respect to its maximal ideal.

Let A be a Noetherian local ring with maximal ideal M. Given any A-module X, we define a topology on X by declaring a neighborhood base for the topology at $x \in X$ to consist of the sets $x + M^n X$ for $n \in \mathbb{Z}^+$. This is a uniform topology for the additive group structure of X, and so we may define a completion $\hat{X}$ of X relative to this topology. The completion consists of equivalence classes of Cauchy sequences from X, where a sequence $\{x_k\}$ is Cauchy if, for each $n \in \mathbb{Z}_+$, there is a $K \in \mathbb{Z}^+$ such that $x_j - x_k \in M^n X$ whenever $j, k > K$. In other words, a sequence is Cauchy if and only if it is eventually constant mod $M^n X$, for each $n \in \mathbb{Z}^+$. Two Cauchy sequences $\{x_k\}$ and $\{y_k\}$ are equivalent if, for each $n \in \mathbb{Z}^+$, there is a K for which

$x_k - y_k \in N^n$ for $k > K$. This description makes it clear that $\hat{X}$ may also be described as the inverse limit

$$\hat{X} = \varprojlim X/M^n X.$$

When $X = A$, we obtain a completion $\hat{A}$ for A itself. It is easy to see that $\hat{A}$ is also a ring, and for each module X over A, the completion $\hat{X}$ is a module over $\hat{A}$. In fact, $X \to \hat{X}$ is a covariant functor from A-modules to $\hat{A}$-modules. This is an exact functor when restricted to finitely generated modules, as is shown below.

13.2.1 Proposition. *Let A be a Noetherian local ring, and let*

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence of finitely generated A-modules. Then the sequence

$$0 \to \hat{X} \to \hat{Y} \to \hat{Z} \to 0$$

is also exact.

Proof. By the Artin-Rees theorem (Theorem 8.4.1), there exists an m_0 such that

$$M^{p+m_0}X \subset M^{p+m_0}Y \cap X \subset M^p X.$$

This implies that the completion of X with respect to the topology determined by the filtration $\{M^p X\}$ agrees with that determined by the filtration $\{X \cap M^p Y\}$. In other words, if $X_p = X/(X \cap M^p Y)$, then $\hat{X} = \varprojlim X_p$. But we have, for each p, a short exact sequence

$$0 \to X_p \to Y_p \to Z_p \to 0,$$

where $Y_p = Y/M^p Y$, and $Z_p = Z/M^p Z$. Now limits of inverse sequences preserve left exactness, but do not always preserve right exactness. Right exactness is, however, preserved in the case where the left sequence $\{X_p\}$ is *surjective* in the sense that each map $X_{p+1} \to X_p$ is surjective (Exercise 13.1), as is true in our situation. It follows that

$$0 \to \hat{X} \to \hat{Y} \to \hat{Z} \to 0$$

is exact, as required.

13.2.2 Proposition. *If A is a Noetherian local ring, and X is a finitely generated A-module, then $\hat{A} \otimes_A X \to \hat{X}$ is an isomorphism.*

Proof. It is clear that $X \to \hat{X}$ commutes with taking finite direct sums. Thus, since $\hat{A} \otimes_A A \to \hat{A}$ is an isomorphism, we conclude that $\hat{A} \otimes_A Y \to \hat{Y}$ is an isomorphism whenever Y is a finitely generated free module. Since X is finitely generated, there is a short exact sequence of the form

$$0 \to K \to A^n \to X \to 0$$

for some n. This yields a diagram

$$\begin{array}{ccccccccc} & & \hat{A} \otimes_A K & \longrightarrow & \hat{A} \otimes_A A^n & \longrightarrow & \hat{A} \otimes_A X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{K} & \longrightarrow & \hat{A}^n & \longrightarrow & \hat{X} & \longrightarrow & 0, \end{array}$$

in which the bottom row is exact by Proposition 13.2.1, the top row is right exact, and the middle vertical map is an isomorphism. A simple diagram chase shows that $\hat{A} \otimes_A X \to \hat{X}$ is surjective. However, since A is Noetherian, we also have that K is a finitely generated A-module, and by what we just proved, the map $\hat{A} \otimes_A K \to \hat{K}$ is also surjective. It then follows from another diagram chase that $\hat{A} \otimes_A X \to \hat{X}$ is injective. This completes the proof.

The main theorem of this section is the following:

13.2.3 Theorem. *If A is a Noetherian local ring, then $\hat{A}$ is faithfully flat over A.*

Proof. It is easy to see (Exercise 13.2) that an A-module Y is flat if and only if, whenever X_1 and X_2 are finitely generated A-modules, and $X_1 \to X_2$ is injective, then $Y \otimes_A X_1 \to Y \otimes_A X_2$ is also injective. Since Propositions 13.2.1 and 13.2.2 prove that $\hat{A} \otimes_A (\cdot)$ preserves exactness of short exact sequences of finitely generated A-modules, we conclude that $\hat{A}$ is flat as an A-module.

Now suppose that X is a finitely generated A-module. Then the kernel of the morphism $X \to \hat{X}$ is $E = \bigcap_n M^n X$. It follows from the Artin-Rees theorem, applied to $E \subset X$, that $ME = E$. Then Nakayama's lemma implies that $E = 0$. By Proposition 13.2.2, we conclude that the map $X \to \hat{A} \otimes_A X$ is injective whenever X is finitely generated. But this clearly implies that this map is injective in general and, hence, that $\hat{A}$ is faithfully flat over A.

We conclude this section with a couple of results concerning the structure of the ring $\hat{A}$ when A is a Noetherian local ring. In particular, we show that $\hat{A}$ is also a Noetherian local ring.

13.2.4 Proposition. *If A is a Noetherian local ring, then*

(i) *the unique maximal ideal of $\hat{A}$ is $\hat{M} = \hat{A}M$;*
(ii) *$M^n = A \cap \hat{M}^n$ for all $n \in \mathbb{Z}^+$; and*
(iii) *$A/M^n \to \hat{A}/\hat{M}^n$ is an isomorphism for all n.*
(iv) *$\hat{A}$ is complete in the $\hat{M}$-adic topology.*

Proof. Since $M^p(A/M) = 0$ for $p \in \mathbb{Z}^+$, we have that A/M is complete in the M-adic topology. We apply the completion functor to the exact sequence

$$0 \to M \to A \to A/M \to 0,$$

and use the fact that this functor is exact, to conclude that the sequence

$$0 \to \hat{M} \to \hat{A} \to A/M \to 0$$

is exact. This implies that $\hat{M}$ is a maximal ideal of $\hat{A}$, since A/M is a field. By Proposition 13.2.2, $\hat{M} = \hat{A} \otimes_A M = \hat{A}M$, which implies that $\hat{M}^n = \hat{A}M^n$. Since $\hat{A}$ is faithfully flat over A, Lemma 13.1.2(iii) implies that $\hat{A}M^n \cap A = M^n$, and we conclude that $\hat{M}^n \cap A = M^n$. This proves (ii).

That $A/M^n \to \hat{A}/\hat{M}^n$ is surjective follows from the fact that a Cauchy sequence in the M-adic topology is eventually constant modulo M^n. That this map is injective follows from (ii). This completes the proof of (iii).

Part (iv) follows immediately from (iii), which shows that the $\hat{M}$-adic completion of $\hat{A}$ is $\hat{A}$.

To complete the proof of (i), we must show that $\hat{M}$ is the only maximal ideal of $\hat{A}$. To do this, we need only show that $1 - a$ is a unit in $\hat{A}$ for every $a \in M$. In fact, the inverse of $1 - a$ for $a \in M$ is $1 + a + a^2 + \cdots + a^n + \cdots$, which converges in the $\hat{M}$-adic topology of $\hat{A}$.

13.2.5 Theorem. *If A is a Noetherian local ring, then $\hat{A}$ is also a Noetherian local ring.*

Proof. By the previous proposition, $\hat{A}$ is a local ring. Thus, we need only show that $\hat{A}$ is Noetherian. The graded ring $G(A) = \oplus_{n=0}^{\infty} M^n/M^{n+1}$ associated to A is a finitely generated algebra over the field A/M and is, therefore, Noetherian by the Hilbert basis theorem. It follows that $G(\hat{A})$ is also Noetherian, since it is isomorphic to $G(A)$, by Proposition 13.2.4(iii).

Suppose I is an ideal of $\hat{A}$. If we give I the filtration $\{\hat{M}^n \cap I\}$, then $G(I)$ embeds as an ideal of $G(\hat{A})$ and, as such, it is finitely generated. Let $\{\bar{a}_i\,;\, i = 1, \dots, n\}$ be a set of homogeneous generators of $G(I)$, set $r_i = \deg(\bar{a}_i)$, and let $a_i \in I \cap M^{r_i}$ be a representative of $\bar{a}_i$ for each i. Let J be the ideal in $\hat{A}$ generated by $a_1, \dots, a_n$.

We will prove that $J = I$. Clearly $G(I) = G(J)$. Suppose $u \in I$. Since $\hat{A}$ is Hausdorff, there exists p such that $u \in \hat{M}^p - \hat{M}^{p+1}$. Then there exist $v_{0i} \in \hat{M}^{p-r_i}$ such that $u - \sum v_{0i}a_i \in I \cap \hat{M}^{p+1}$. By continuing this construction we obtain sequences $\{v_{ji} : j \in \mathbb{Z}_+, i = 1, \ldots, n\}$ such that $v_{ji} \in \hat{M}^{p+j-r_i}$ and

$$u - \sum_{i=1}^{n} \sum_{j=0}^{s} v_{ji}a_i \in I \cap \hat{M}^{p+s+1}.$$

Since $\hat{A}$ is complete, the series $\sum_{j=0}^{\infty} v_{ji}$ converges to some $v_i \in \hat{A}$ for each i and, clearly, $u = \sum_{i=1}^{n} v_i a_i$. Thus, $u \in J$ and the proof is complete.

13.3 Local Rings of Algebraic vs. Holomorphic Functions

We now return to the study of the algebras ${}_V\mathcal{O}_\lambda$ and ${}_V\mathcal{H}_\lambda$. We may assume that V is an algebraic subvariety of $\mathbb{C}^n$, and $\lambda = 0$. We shall do so throughout this section. We note first that, if A is ${}_n\mathcal{O}_0$ or ${}_n\mathcal{H}_0$, and M is the maximal ideal of A, then A/M^p is just the quotient of the ring of polynomials in n variables modulo the ideal consisting of polynomials all of whose terms are of degree at least p. Thus, the following result is obvious from the definitions:

13.3.1 Lemma. *The algebras ${}_n\mathcal{O}_0$ and ${}_n\mathcal{H}_0$ both have as completion the algebra of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$.*

The following technical lemma due to Chevalley ([**ZS**], Ch. VIII, section 13) is the key to showing that ${}_V\mathcal{O}_0$ and ${}_V\mathcal{H}_0$ also have the same completion.

13.3.2 Lemma. *Let V be an algebraic subvariety of $\mathbb{C}^n$. Then there are no non-zero nilpotent elements of ${}_V\hat{\mathcal{O}}_0$. That is, ${}_V\hat{\mathcal{O}}_0$ is reduced.*

Proof. We first reduce to the case where the germ $\mathbf{V}$ of V at 0 is irreducible. If $\mathbf{V}$ is not irreducible, and $\mathbf{V} = \mathbf{V}_1 \cup \cdots \cup \mathbf{V}_k$ is its irreducible decomposition, then consider the map

$${}_V\mathcal{O}_0 \to \bigoplus_i {}_{V_i}\mathcal{O}_0$$

induced by restricting to each irreducible component. This is an injection, and hence, it remains an injection if we apply the ${}_V\mathcal{O}_0$-module completion functor to obtain

$${}_V\hat{\mathcal{O}}_0 \to \bigoplus_i {}_{V_i}\hat{\mathcal{O}}_0.$$

However, the completion of ${}_{V_i}\mathcal{O}_0$ as a ${}_V\mathcal{O}_0$-module is the same as its completion as a local ring. Therefore, if the theorem is true for each of the irreducible varieties V_i, then a nilpotent element of ${}_V\hat{\mathcal{O}}_0$ must map to 0 in each ${}_{V_i}\hat{\mathcal{O}}_0$ and, therefore, must be 0. It follows that the theorem is also true for V.

Thus, we will assume that the germ $\mathbf{V}$ of V at 0 is irreducible. By the Noether normalization theorem (Theorem 4.3.1), there is an integer m and a subalgebra $A \subset {}_V\mathcal{O}_0$, isomorphic to ${}_m\mathcal{O}_0$, such that ${}_V\mathcal{O}_0$ is a finite ring extension of A. If K is the field of fractions of A and L is the field of fractions of ${}_V\mathcal{O}_0$, then the theorem of the primitive element (Theorem 4.2.1) implies that L is generated over K by a single element $g \in {}_V\mathcal{O}_0$. Then Theorem 4.2.12 implies that every element of L which is integral over ${}_V\mathcal{O}_0$ is in the A-submodule of L generated by $\{d^{-1}, d^{-1}g, \dots, d^{-1}g^{k-1}\}$, where d is the discriminant, and k the degree of the minimal polynomial of g. Thus, every element of L, integral over ${}_V\mathcal{O}_0$, belongs to $d^{-1}{}_V\mathcal{O}$.

At this point we pass to the completion $\hat{A}$, which is just the ring of formal power series, by Lemma 13.3.1, and hence, is an integral domain. We denote its field of fractions by $\tilde{K}$ and remark that this is an extension field of K. The completion ${}_V\hat{\mathcal{O}}_0$ of ${}_V\mathcal{O}_0$ is $\hat{A} \otimes_A {}_V\mathcal{O}_0$, by Exercise 13.3. We define a $\tilde{K}$ algebra $\tilde{L} = \tilde{K} \otimes_K L$. This may also be described as the result of passing from L to $\hat{A} \otimes_A L$, and then localizing relative to the multiplicative set consisting of the non-zero elements of $\hat{A}$. By Theorem 13.2.3, $\hat{A}$ is faithfully flat over A, and so ${}_V\hat{\mathcal{O}}_0 = \hat{A} \otimes_A {}_V\mathcal{O}_0$ is embedded as a subalgebra of $\hat{A} \otimes_A L$. We claim that this algebra is, in turn, embedded as a subalgebra of $\tilde{L}$. That is, we must show that nothing is killed when we localize. This means we must show that $am = 0$ for $0 \neq a \in \hat{A}$, and $m \in \hat{A} \otimes_A L$ implies that $m = 0$. However, L is a finite dimensional vector space over K so, as an A-module, it is a finite direct sum of copies of K. Thus, it suffices to show that if $am = 0$ for some $m \in \hat{A} \otimes_A K$ and non-zero $a \in \hat{A}$, then $m = 0$. But this follows immediately from the fact that $\hat{A}$ is an integral domain and $\hat{A} \otimes_A K$ is just the localization of $\hat{A}$ relative to the multiplicative system consisting of non-zero elements of A (Exercise 13.4). Thus, we have shown that ${}_V\hat{\mathcal{O}}_0$ is embedded as a subalgebra of $\tilde{L}$.

Since L is generated over K by the element $g \in {}_V\mathcal{O}_0$, the algebra $\tilde{L}$ is generated over $\tilde{K}$ by g. Thus, we may again use Theorem 4.2.12 to conclude that any element of $\tilde{L}$ which is integral over $\hat{A}$ is actually in the $\hat{A}$-submodule generated by the elements $d^{-1}g^j$, for $j = 0, 1, \dots, k-1$. In particular, such an element belongs to $d^{-1}{}_V\hat{\mathcal{O}}_0$.

Now let $x \in {}_V\hat{\mathcal{O}}_0$ be nilpotent. If $0 \neq a \in A$, then x/a is a nilpotent element of $\tilde{L}$. Since a nilpotent element is integral, it follows, as above, that $x/a \in d^{-1}{}_V\hat{\mathcal{O}}_0$. This holds, in particular, when $a = y^p$ for any non-zero element y of M and any positive integer p. Thus, $dx \in M^p{}_V\hat{\mathcal{O}}_0 \subset \hat{M}^p$ for each p. Since ${}_V\hat{\mathcal{O}}_0$ is a Noetherian local ring, by Theorem 13.2.5, it follows as in the proof of Theorem 13.2.3, that $\bigcap_{p=1}^{\infty} \hat{M}^p = 0$, and so $dx = 0$. Since multiplication by d is injective on ${}_V\mathcal{O}$ and completion is an exact functor, we conclude that $x = 0$. This completes the proof.

13.3.3 Proposition. *Let* $\mathbf{V}$ *be the germ at* 0 *of an algebraic variety in* $\mathbb{C}^n$, *and let* $\mathbf{V}^h$ *be the corresponding germ of a holomorphic variety. Then* $\operatorname{id}\mathbf{V}^h = {}_n\mathcal{H}_0 \cdot \operatorname{id}\mathbf{V}$.

Proof. Let $I = \operatorname{id}\mathbf{V} \subset {}_n\mathcal{O}_0$. If $J = {}_n\mathcal{H}_0 I$, then $\operatorname{loc} J = \mathbf{V}^h$, and so it follows from the Nullstellensatz that $\operatorname{id}\mathbf{V}^h = \sqrt{J}$. Thus, if $f \in \operatorname{id}\mathbf{V}^h$, then $f^m \in J$ for some positive integer m. If we consider f to be a formal power series, then we have an element $f \in {}_n\hat{\mathcal{O}}_0$ such that f^m belongs to the ideal ${}_n\hat{\mathcal{O}}_0 I$ in the completion ${}_n\hat{\mathcal{O}}_0 = \mathbb{C}[[z_1, \dots, z_n]]$ of ${}_n\mathcal{O}_0$. By the exactness property of completion for finitely generated modules (Proposition 13.2.1), we have that the quotient of ${}_n\hat{\mathcal{O}}_0$ modulo this ideal is just ${}_V\hat{\mathcal{O}}_0$. We conclude that the image of f in ${}_V\hat{\mathcal{O}}_0$ is nilpotent and, hence, 0 by the previous lemma. Thus, $f \in {}_n\hat{\mathcal{O}}_0 I = {}_n\hat{\mathcal{O}}_0 J$. Since $f \in {}_n\mathcal{H}_0$, it follows from Lemma 13.1.2(iii) that $f \in J = {}_n\mathcal{H}_0 I$. We conclude that $\operatorname{id}\mathbf{V}^h = {}_n\mathcal{H}_0 \cdot \operatorname{id}\mathbf{V}$, as claimed.

13.3.4 Proposition. *If* $\mathbf{V}$ *is a germ of an algebraic variety at* $0 \in \mathbb{C}^n$, *then* ${}_{V^h}\hat{\mathcal{H}}_0 = {}_V\hat{\mathcal{O}}_0$.

Proof. Let $\{p_1, \dots, p_r\}$ be a set of generators for $\operatorname{id}\mathbf{V} \subset {}_n\mathcal{O}_0$. Then by the previous theorem, it is also a set of generators for $\operatorname{id}\mathbf{V}^h \subset {}_n\mathcal{H}_0$. Thus, we have a commutative diagram

$$\begin{array}{ccccccc}
{}_n\mathcal{O}_0^r & \longrightarrow & {}_n\mathcal{O}_0 & \longrightarrow & {}_V\mathcal{O}_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
{}_n\mathcal{H}_0^r & \longrightarrow & {}_n\mathcal{H}_0 & \longrightarrow & {}_{V^h}\mathcal{H}_0 & \longrightarrow & 0
\end{array}$$

with exact rows, where $(f_1, \dots, f_r) \to p_1 f_1 + \cdots + p_r f_r$ defines the maps ${}_n\mathcal{O}_0^r \to {}_n\mathcal{O}_0$ and ${}_n\mathcal{H}_0^r \to {}_n\mathcal{H}_0$. On applying the completion functor to this diagram, the first two vertical maps become isomorphisms and, hence, the third must be one also. This completes the proof.

We now have ${}_V\mathcal{O}_0 \subset {}_{V^h}\mathcal{H}_0 \subset {}_V\hat{\mathcal{O}}_0 = {}_{V^h}\hat{\mathcal{H}}_0$. In view of Proposition 13.1.3 and Theorem 13.2.3, we have proved the following key result:

13.3.5 Theorem. *If* V *is a germ of an algebraic variety, then* ${}_{V^h}\mathcal{H}_0$ *is faithfully flat over* ${}_V\mathcal{O}_0$.

Proposition 13.3.3 allows us to finish the proof of a result which was mentioned, but not proved, in Chapter 5:

13.3.6 Proposition. *If V is an algebraic variety, and V^h is the corresponding holomorphic variety, then* $\operatorname{tdim} \mathbf{V}_\lambda = \operatorname{tdim} \mathbf{V}^h_\lambda$ *at each point $\lambda \in V$. Thus, the singular set of V agrees with the singular set of V^h, as a point set.*

Proof. This is a local statement, and so we may assume without loss of generality that V is an algebraic subvariety of $\mathbb{C}^n$. By Proposition 13.3.3, a set of generators for $\operatorname{id} \mathbf{V}_\lambda$ over ${}_n\mathcal{O}_\lambda$ is also a set of generators for $\operatorname{id} \mathbf{V}^h_\lambda$ over ${}_n\mathcal{H}_\lambda$. By Corollary 5.4.4, this implies that $\operatorname{tdim} \mathbf{V}^h_\lambda = \operatorname{tdim} \mathbf{V}_\lambda$.

A point λ of V is a singular point if and only if $\dim \mathbf{V}_\lambda \neq \operatorname{tdim} \mathbf{V}_\lambda$. The same is true of V^h. Since $\dim \mathbf{V}^h_\lambda = \dim \mathbf{V}_\lambda$, and $\operatorname{tdim} \mathbf{V}_\lambda = \operatorname{tdim} \mathbf{V}^h_\lambda$, the singular points of V are the same as the singular points of V^h.

13.4 The Algebraic to Holomorphic Functor

We pointed out in section 12.2 that an algebraic variety V also has the structure of a holomorphic subvariety V^h. This structure is defined on each affine neighborhood by embedding the neighborhood in $\mathbb{C}^n$ and giving it the Euclidean topology and structure sheaf it inherits from $\mathbb{C}^n$. The resulting ringed space structure is independent of the embedding. The Euclidean open subsets of V are then defined to be the sets whose intersection with each affine neighborhood is a Euclidean open set. The holomorphic functions on such a set are those which are holomorphic on each affine neighborhood. The resulting topology on V is Hausdorff and countable at infinity (Exercise 12.4). The resulting ringed space structure is unique subject to the condition that, on each affine neighborhood, it agrees with the natural Euclidean structure. We summarize this in the following proposition.

13.4.1 Proposition. *If X is an algebraic variety, then there is a unique holomorphic variety X^h, which is X as a point set, for which every open subset of X is open in X^h, and for which every affine open subset has its natural holomorphic structure.*

We will make use of the notation X^h only when it is necessary for clarity. More often, we will simply drop the superscript and use X to denote either the algebraic variety or its associated holomorphic variety, with the context dictating which is meant.

Our next task will be to show that a coherent algebraic sheaf $\mathcal{M}$ on an algebraic variety X gives rise to a coherent analytic sheaf $\mathcal{M}^h$ on X^h. First, if $\mathcal{M}$ is any sheaf on X, let $\mathcal{M}'$ be the sheaf on X^h which is the inverse image of $\mathcal{M}$ under the continuous map $X^h \to X$. Thus, for each open set

U in X^h,

$$\mathcal{M}'(U) = \varinjlim\{\mathcal{M}(W) : U \subset W, \quad W \text{ open in } X\}.$$

Note that at each point $x \in X$, the stalks $\mathcal{M}_x$ and $\mathcal{M}'_x$ are the same.

13.4.2 Definition. *If $\mathcal{M}$ is any sheaf of $\mathcal{O}$-modules on an algebraic variety X, then we define a corresponding analytic sheaf $\mathcal{M}^h$ on X^h by setting $\mathcal{M}^h = \mathcal{H} \otimes_{\mathcal{O}'} \mathcal{M}'$.*

By definition, the sheaf $\mathcal{M}^h = \mathcal{H} \otimes_{\mathcal{O}'} \mathcal{M}'$ is the sheaf of germs of the presheaf $U \to \mathcal{H}(U) \otimes_{\mathcal{O}'(U)} \mathcal{M}'(U)$, and so its stalk at $x \in X$ is

$$\mathcal{M}^h_x = \varinjlim\{\mathcal{H}(U) \otimes_{\mathcal{O}'(U)} \mathcal{M}'(U) : x \in U\} = \mathcal{H}_x \otimes_{\mathcal{O}'_x} \mathcal{M}'_x = \mathcal{H}_x \otimes_{\mathcal{O}_x} \mathcal{M}_x.$$

The second equality above follows easily from the definitions of direct limit and tensor product (Exercise 13.6).

The next theorem is a direct application of the results of the preceding section.

13.4.3 Theorem. *If X is an algebraic variety, then $\mathcal{M} \to \mathcal{M}^h$ is a faithful exact functor from the category of sheaves of $\mathcal{O}$-modules on X to the category of sheaves of $\mathcal{H}$-modules on X^h.*

Proof. It is clear that $\mathcal{M} \to \mathcal{M}^h$ is a functor from sheaves of $\mathcal{O}$-modules to sheaves of $\mathcal{H}$-modules. We have that $\mathcal{M}^h_x = \mathcal{H}_x \otimes_{\mathcal{O}_x} \mathcal{M}_x$, and by Theorem 12.3.5, $\mathcal{H}_x$ is faithfully flat over $\mathcal{O}_x$. It follows that $\mathcal{M} \to \mathcal{M}^h$ is a faithful exact functor.

13.4.4 Proposition. *For an algebraic variety X, the functor $\mathcal{M} \to \mathcal{M}^h$ takes*

(i) *$\mathcal{O}^m$ to $\mathcal{H}^m$ for each m,*
(ii) *coherent algebraic sheaves to coherent analytic sheaves,*
(iii) *the ideal sheaf in $\mathcal{O}$ of an algebraic subvariety V to the ideal sheaf in $\mathcal{H}$ of the corresponding holomorphic subvariety V^h.*

Proof. Part (i) is trivial since $\mathcal{H}(U) \otimes_{\mathcal{O}'(U)} \mathcal{O}'(U) = \mathcal{H}(U)$ for any Euclidean open set U.

If $\mathcal{M}$ is a coherent algebraic sheaf on X, then for each affine open subset $U \subset X$, there is a finitely generated $\mathcal{O}(U)$-module M with $\mathcal{M}|_U$ isomorphic to $\mathcal{O} \otimes_{\mathcal{O}(U)} M$. Since M is finitely generated, there is a surjective morphism $\mathcal{O}(U)^m \to M$ and, since $\mathcal{O}(U)$ is Noetherian, the kernel of this morphism is also finitely generated. Thus, there is a morphism $\mathcal{O}(U)^k \to \mathcal{O}(U)^m$ so that the sequence

$$\mathcal{O}(U)^k \longrightarrow \mathcal{O}(U)^m \longrightarrow M \longrightarrow 0$$

is exact. If we apply the localization functor $\mathcal{O} \otimes_{\mathcal{O}(U)} (\,\cdot\,)$ to this sequence, we obtain an exact sequence of sheaves

$$\mathcal{O}^k|_U \longrightarrow \mathcal{O}^m|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0.$$

On applying the functor $\mathcal{M} \to \mathcal{M}^h$ to this sequence and then using the previous theorem and part (i) above, we conclude there is an exact sequence

$$\mathcal{H}^k|_U \longrightarrow \mathcal{H}^m|_U \longrightarrow \mathcal{M}^h|_U \longrightarrow 0.$$

Thus, by definition, $\mathcal{M}^h$ is coherent. This completes the proof of part (ii).

If V is an algebraic subvariety of X, then the inclusion $0 \to \mathcal{I}_V \to \mathcal{O}$ yields an inclusion $0 \to \mathcal{I}_V^h \to \mathcal{H}$ by Theorem 13.4.3. The image consists of the sheaf of ideals generated in $\mathcal{H}$ by the image of $\mathcal{I}'_V$. By Proposition 13.3.3, this is the ideal sheaf of V^h in $\mathcal{H}$. This completes the proof of (iii).

Now suppose that $i : Y \to X$ is the inclusion of a holomorphic subvariety in a holomorphic variety X, and $\mathcal{M}$ is a sheaf of ${}_Y\mathcal{H}$-modules on Y. Then $i_*\mathcal{M}$ – the extension of $\mathcal{M}$ by zero – is a sheaf on X which is supported on Y. It is certainly a sheaf of $i_*{}_Y\mathcal{H}$-modules. However, as discussed in section 9.6, $i_*\mathcal{M}$ is also a sheaf of ${}_X\mathcal{H}$-modules. Exactly the same thing is true for a closed embedding $i : Y \to X$ of algebraic varieties and a sheaf $\mathcal{M}$ of ${}_Y\mathcal{O}$-modules – the sheaf $i_*\mathcal{M}$ is then a sheaf of ${}_X\mathcal{O}$-modules (section 8.7).

13.4.5 Proposition. *If $i : Y \to X$ is a closed embedding of algebraic varieties and $\mathcal{M}$ is a coherent algebraic sheaf on Y, then $(i_*\mathcal{M})^h = i_*(\mathcal{M}^h)$.*

Proof. There is an inclusion $\mathcal{M}' \to \mathcal{M}^h$ of sheaves of ${}_Y\mathcal{O}'$-modules, which yields a morphism $i_*(\mathcal{M}') \to i_*(\mathcal{M}^h)$ of sheaves of ${}_X\mathcal{O}'$-modules. Clearly, we have $i_*(\mathcal{M}') = (i_*\mathcal{M})'$, and so there is a morphism $(i_*\mathcal{M})' \to i_*(\mathcal{M}^h)$ of sheaves of ${}_X\mathcal{O}'$-modules. On tensoring with ${}_X\mathcal{H}$ we obtain a morphism

$$(i_*\mathcal{M})^h \to {}_X\mathcal{H} \otimes_{{}_X\mathcal{O}'} i_*(\mathcal{M}^h).$$

Since $i_*(\mathcal{M}^h)$ is already an ${}_X\mathcal{H}$-module, there is a morphism

$${}_X\mathcal{H} \otimes_{{}_X\mathcal{O}'} i_*(\mathcal{M}^h) \to i_*(\mathcal{M}^h)$$

given by the module action. Thus, we have a morphism

$$(i_*\mathcal{M})^h \to i_*(\mathcal{M}^h).$$

To complete the argument, we need to show that this is an isomorphism on each stalk. For this, we only need to consider stalks at points $x \in Y$, since the stalks of both sheaves are 0 off Y. At such a point x, this amounts to showing that the natural map

$$ {}_X\mathcal{H}_x \otimes_{{}_X\mathcal{O}_x} \mathcal{M}_x \to {}_Y\mathcal{H}_x \otimes_{{}_Y\mathcal{O}_x} \mathcal{M}_x $$

is an isomorphism. The second module can be written as

$$ ({}_X\mathcal{H}_x \otimes_{{}_X\mathcal{O}_x} {}_Y\mathcal{O}_x) \otimes_{{}_Y\mathcal{O}_x} \mathcal{M}_x $$

by using the fact that ${}_X\mathcal{H}_x \otimes_{{}_X\mathcal{O}_x} {}_Y\mathcal{O}_x \simeq {}_Y\mathcal{H}_x$, which is deduced from tensoring ${}_X\mathcal{H}_x$ with the exact sequence $0 \to (\mathcal{I}_Y)_x \to {}_X\mathcal{O}_x \to {}_Y\mathcal{O}_x \to 0$, and using flatness and Proposition 13.4.4(iii). The theorem then follows from the associativity of tensor product and the fact that ${}_Y\mathcal{O}_x \otimes_{{}_Y\mathcal{O}_x} \mathcal{M}_x \simeq \mathcal{M}_x$.

We end this section with a result relating the topologies of an algebraic variety X and its corresponding holomorphic variety X^h.

13.4.6 Proposition. *If X is an algebraic variety and U is an open subset of X, then the closures of U in X and in X^h coincide as sets.*

Proof. We may assume that U is non-empty and not equal to X, since the result is trivially true otherwise. Let $\overline{U}$ denote the closure of U in X (i.e. in the Zariski topology). Then $\overline{U}$ is a subvariety of X. Also, it contains the closure of U in X^h (i.e. in the Euclidean topology), since the Euclidean topology is stronger than the Zariski topology. To prove the proposition, we must show that U is dense in $\overline{U}$ in the Euclidean topology.

Without loss of generality, we may assume that $\overline{U} = X$. If $Y = X - U$, then Y is a proper algebraic subvariety of X. In fact, if $x \in Y$, then since every neighborhood of x meets U, the germ of Y at x is properly contained in the germ of X at x. Thus, the restriction map ${}_X\mathcal{O}_x \to {}_Y\mathcal{O}_x$ has non-zero kernel. On the other hand, if x is not in the closure of U in the Euclidean topology, then there is a neighborhood of x in X^h which does not meet U and, hence, is contained in Y. This means that the restriction map ${}_X\mathcal{H}_x \to {}_Y\mathcal{H}_x$ is an isomorphism. Since the functor $\mathcal{M} \to \mathcal{M}^h$ is a faithful exact functor taking ${}_X\mathcal{O}$ to ${}_X\mathcal{H}$ and ${}_Y\mathcal{O}$ to ${}_Y\mathcal{H}$ (Theorem 13.4.3 and Proposition 13.4.4), this is impossible. Hence, each point of Y is in the closure of U in the Euclidean topology. This completes the proof.

13.5 Serre's Theorems

We can now prove the three main theorems of Serre's GAGA paper.

If X is an algebraic variety, and $\mathcal{M}$ is a sheaf on X, then $\mathcal{M}$ and $\mathcal{M}'$ have the same global sections and, in fact, the same cohomology. The latter is due to the fact that a flabby resolution of $\mathcal{M}'$ is also a flabby resolution of $\mathcal{M}$, when restricted to the Zariski open sets. So $H^p(X, \mathcal{M}) = H^p(X^h, \mathcal{M}')$. Furthermore, the morphism $m \to 1 \otimes m : \mathcal{M}' \to \mathcal{M}^h$ induces a morphism $H^p(X^h, \mathcal{M}') \to H^p(X^h, \mathcal{M}^h)$ for each p. Combining these facts gives us a morphism $H^p(X, \mathcal{M}) \to H^p(X^h, \mathcal{M}^h)$. The following is the first of Serre's three GAGA theorems:

13.5.1 Theorem. *If X is a projective algebraic variety, and $\mathcal{M}$ is a coherent algebraic sheaf on X, then the natural map $H^p(X, \mathcal{M}) \to H^p(X^h, \mathcal{M}^h)$ is an isomorphism for every p.*

Proof. Since X is projective, we may assume that it is embedded as a subvariety of P^n. If $i : X \to P^n$ is the embedding, then $i_*\mathcal{M}$ is a coherent algebraic sheaf on P^n, by Proposition 8.7.6. Furthermore, the cohomology groups of $\mathcal{M}$ and $i_*\mathcal{M}$ are the same (Exercise 7.20). Likewise, $i_*\mathcal{M}^h$ is a coherent analytic sheaf on P^n, by Exercise 9.10, and the cohomology groups of $\mathcal{M}^h$ and $i_*\mathcal{M}^h$ are the same. By Proposition 13.4.5, $(i_*\mathcal{M})^h = i_*(\mathcal{M}^h)$. It follows that the theorem is true in general if it is true for coherent algebraic sheaves on P^n. Thus, we may assume that $X = P^n$.

By Proposition 12.4.3, $\mathcal{M}$ is the quotient of a sheaf $\mathcal{F}$ which is a finite direct sum of sheaves of the form $\mathcal{O}(k)$. If $\mathcal{K}$ is the kernel of $\mathcal{F} \to \mathcal{M}$, then we have a short exact sequence of coherent algebraic sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$$

If we apply $(\,\cdot\,)^h$ to this sequence, and use Theorem 13.4.3 and Proposition 13.4.4, then we obtain an exact sequence

$$0 \longrightarrow \mathcal{K}^h \longrightarrow \mathcal{F}^h \longrightarrow \mathcal{M}^h \longrightarrow 0$$

of coherent analytic sheaves, with $\mathcal{F}^h$ a finite direct sum of sheaves of the form $\mathcal{H}(k)$. If we apply the morphism of cohomology induced by $(\,\cdot\,)^h$ to the long exact sequences of cohomology corresponding to these short exact sequences, we obtain a commutative diagram

$$\begin{tikzcd}
H^p(\mathcal{K}) \arrow[r] \arrow[d, "\epsilon_1"'] & H^p(\mathcal{F}) \arrow[r] \arrow[d, "\epsilon_2"'] & H^p(\mathcal{M}) \arrow[r] \arrow[d, "\epsilon_3"'] & H^{p+1}(\mathcal{K}) \arrow[r] \arrow[d, "\epsilon_4"'] & H^{p+1}(\mathcal{F}) \arrow[d, "\epsilon_5"'] \\
H^p(\mathcal{K}^h) \arrow[r] & H^p(\mathcal{F}^h) \arrow[r] & H^p(\mathcal{M}^h) \arrow[r] & H^{p+1}(\mathcal{K}^h) \arrow[r] & H^{p+1}(\mathcal{F}^h),
\end{tikzcd}$$

where, to save space, we have suppressed the P^n in each cohomology group, and written, for example, $H^p(\mathcal{M})$ rather than $H^p(P^n, \mathcal{M})$. In this diagram, ϵ_2 and ϵ_5 are always isomorphisms by Corollary 12.3.6 and the fact that $\mathcal{F}$ is a direct sum of sheaves $\mathcal{O}(k)$. Suppose that $H^{p+1}(\mathcal{M}) \to H^{p+1}(\mathcal{M}^h)$

is an isomorphism for every coherent algebraic sheaf $\mathcal{M}$. Then ϵ_4 is an isomorphism. This implies that ϵ_3 is surjective for every $\mathcal{M}$ and, hence, that ϵ_1 is surjective. This, along with the fact that ϵ_2 and ϵ_4 are isomorphisms, implies that ϵ_3 is also injective and, thus, is an isomorphism. Thus, the proof will be complete if we can show that there is a p_0 so that $H^p(\mathcal{M}) \to H^p(\mathcal{M}^h)$ is an isomorphism for all $\mathcal{M}$ for $p > p_0$. However, $H^p(\mathcal{M}) = 0$ for $p > n$, by Theorem 12.4.4(i). The same argument used in Theorem 12.4.4(i) also works in the holomorphic case and shows that $H^p(\mathcal{M}^h) = 0$ for $p > n$. Thus, $H^p(\mathcal{M}) \to H^p(\mathcal{M}^h)$ is trivially an isomorphism for $p > n$. This completes the proof.

The next theorem is the second of Serre's GAGA theorems.

13.5.2 Theorem. *If $\mathcal{M}$ and $\mathcal{N}$ are two coherent algebraic sheaves on a projective algebraic variety X, then every morphism of analytic sheaves $\mathcal{M}^h \to \mathcal{N}^h$ is induced by a morphism $\mathcal{M} \to \mathcal{N}$ of algebraic sheaves.*

Proof. Let $\mathcal{A}$ denote the sheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$. Recall that this is the sheaf of $\mathcal{O}$-modules which assigns to an open $U \subset X$ the $\mathcal{O}(U)$-module consisting of all morphisms $\mathcal{M}|_U \to \mathcal{N}|_U$ in the category of sheaves of ${}_U\mathcal{O}$-modules (Definition 7.3.7). Let $\mathcal{B} = \mathcal{H}om_{\mathcal{H}}(\mathcal{M}^h, \mathcal{N}^h)$ be the analogous sheaf for the sheaves of $\mathcal{H}$-modules $\mathcal{M}^h$ and $\mathcal{N}^h$. The functor $(\,\cdot\,)^h$ clearly defines a sheaf morphism $\mathcal{A}' \to \mathcal{B}$, and since $\mathcal{B}$ is an $\mathcal{H}$-module, this induces a morphism $\mathcal{A}^h = \mathcal{H} \otimes_{\mathcal{O}'} \mathcal{A}' \to \mathcal{B}$.

We claim that the morphism $\mathcal{A}^h \to \mathcal{B}$ is an isomorphism. As usual, it suffices to check this for the stalk at each point of X. The fact that $\mathcal{M}$ is coherent and, hence, locally finitely generated, implies that each $\mathcal{O}_x$-module homomorphism from $\mathcal{M}_x$ to $\mathcal{N}_x$ extends to a morphism from $\mathcal{M}|_U$ to $\mathcal{N}|_U$ for some neighborhood U of x, and that a morphism from $\mathcal{M}|_U$ to $\mathcal{N}|_U$, which vanishes at x, also vanishes in a neighborhood of x. These two statements, taken together, and their analogues for coherent analytic sheaves mean that

$$\mathcal{A}_x = \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{M}_x, \mathcal{N}_x) \quad \text{and} \quad \mathcal{B}_x = \mathrm{Hom}_{\mathcal{H}_x}(\mathcal{M}^h_x, \mathcal{N}^h_x).$$

We also have that

$$\mathcal{A}^h_x = \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{M}_x, \mathcal{N}_x) \otimes_{\mathcal{O}_x} \mathcal{H}_x, \quad \mathcal{M}^h_x = \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x, \quad \mathcal{N}^h_x = \mathcal{N}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x.$$

Thus, our claim will be established if we can show that the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{M}_x, \mathcal{N}_x) \otimes_{\mathcal{O}_x} \mathcal{H}_x \to \mathrm{Hom}_{\mathcal{H}_x}(\mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x, \mathcal{N}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x)$$

is bijective. But this follows from Lemma 13.1.4, since $\mathcal{O}_x$ is Noetherian, $\mathcal{M}_x$ is finitely generated, and $\mathcal{H}_x$ is faithfully flat over $\mathcal{O}_x$.

To finish the proof, we consider the morphisms

$$H^0(X, \mathcal{A}) \to H^0(X^h, \mathcal{A}^h) \to H^0(X^h, \mathcal{B}).$$

The first of these morphisms is an isomorphism, by Theorem 13.5.1, provided we can show that $\mathcal{A}$ is coherent. This is left as Exercise 13.8. The second morphism is an isomorphism, by the claim proved in the previous paragraph. Thus, the composition, $H^0(X, \mathcal{A}) \to H^0(X^h, \mathcal{B})$, is an isomorphism. This completes the proof of the theorem, since the global sections of $\mathcal{B}$ are the morphisms from $\mathcal{M}^h$ to $\mathcal{N}^h$, while the global sections of $\mathcal{A}$ are the morphisms from $\mathcal{M}$ to $\mathcal{N}$.

We will say that a coherent analytic sheaf on X is *algebraically induced* if it is of the form $\mathcal{N}^h$ for some coherent algebraic sheaf $\mathcal{N}$ on X. The third and most difficult of Serre's GAGA theorems states that every coherent analytic sheaf on a projective variety is algebraically induced. We will prove this first for $X = P^n$, using induction and a series of lemmas. We will then prove it for a general projective variety X by embedding X in P^n for some n.

The *geometric fiber* of a coherent analytic sheaf $\mathcal{S}$ at a point x is the $\mathcal{H}_x$-module $\mathcal{S}_x/M_x\mathcal{S}_x$, where M_x is the maximal ideal of $\mathcal{H}_x$. The geometric fiber of a coherent algebraic sheaf is defined analogously.

13.5.3 Lemma. *If $\mathcal{S}$ is a coherent analytic sheaf on a holomorphic variety X, $x \in X$, and $F \subset H^0(X, \mathcal{S})$ is a set of sections which generates the geometric fiber of $\mathcal{S}$ at x, then F generates $\mathcal{S}|_U$ for some neighborhood U of x. The analogous statement is true for coherent algebraic sheaves.*

Proof. It follows from Nakayama's lemma that, if F generates $\mathcal{S}_x/M_x\mathcal{S}_x$ as a vector space, then it generates $\mathcal{S}_x$ as an $\mathcal{H}_x$-module (Exercise 12.15). By Exercise 9.6, the set of y at which F fails to generate $\mathcal{S}_y$ is a subvariety. Hence, there is a neighborhood U of x such that F generates $\mathcal{S}|_U$.

If $\mathcal{M}$ is a coherent analytic sheaf on P^n, we may twist $\mathcal{M}$ by $\mathcal{H}(k)$ to construct a coherent analytic sheaf $\mathcal{M}(k) = \mathcal{M} \otimes_{\mathcal{H}} \mathcal{H}(k)$ for each k, just as we did for coherent algebraic sheaves in Chapter 12. Furthermore, $\mathcal{N}(k)^h \simeq (\mathcal{N}^h)(k)$, due to the fact that $\mathcal{O}(k)^h = \mathcal{H}(k)$.

By a *hyperplane* in P^n, we mean a subvariety of P^n determined by the zero set of a linear (homogeneous of degree 1) function on $\mathbb{C}^{n+1}$. Note that each hyperplane in P^n is isomorphic to P^{n-1}.

13.5.4 Lemma. *Assume that every coherent analytic sheaf on P^{n-1} is algebraically induced. Let E be a hyperplane in P^n, and $\mathcal{M}$ a coherent analytic* *sheaf on E. Then there is a k_0 such that $H^p(E, \mathcal{M}(k)) = 0$ for all $p > 0$, $k > k_0$.*

Proof. Since E is a copy of P^{n-1}, our assumption means that $\mathcal{M} = \mathcal{N}^h$ for some coherent algebraic sheaf $\mathcal{N}$ on E. That $H^p(E, \mathcal{M}(k)) = 0$ for $p > 0$ and large enough k then follows from Theorems 13.5.1 and 12.4.4.

13.5.5 Lemma. *Assume that every coherent analytic sheaf on P^{n-1} is algebraically induced. Then, for every coherent analytic sheaf $\mathcal{M}$ on P^n, there is an integer k_0 such that, for every $k > k_0$, the sheaf $\mathcal{M}(k)$ is generated over $\mathcal{H}$ by its space of global sections $H^0(P^n, \mathcal{M}(k))$.*

Proof. If $\mathcal{M}(k)_x$ is generated by $H^0(P^n, \mathcal{M}(k))$, then $\mathcal{M}(k+p)_x$ is generated by $H^0(P^n, \mathcal{M}(k+p))$ for all $p > 0$. This is due to the fact that $\mathcal{M}(k+p)_x = \mathcal{M}(k)_x \otimes_{\mathcal{H}_x} \mathcal{H}(p)_x$, and $\mathcal{H}(p)_x$ is generated by its global sections if $p \geq 0$ (since $\mathcal{H}(p)$ is the sheaf of holomorphic sections of a line bundle, one only needs to have one global section which is non-vanishing at x in order to have the global sections generate $\mathcal{H}(p)_x$ and the existence of such a section follows from Theorem 12.3.3). In view of these remarks, the compactness of P^n, and Lemma 13.5.3, to prove the lemma it suffices to prove that, for each $x \in P^n$, there is a k for which the geometric fiber of the module $\mathcal{M}(k)_x$ is generated by $H^0(P^n, \mathcal{M}(k))$.

Let E be a hyperplane in P^n, let $i : E \to P^n$ be the inclusion, and let $\mathcal{I}_E$ be the ideal sheaf of E in $\mathcal{H}$. We may as well assume that E is the hyperplane on which the coordinate function z_0 vanishes. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{H} \longrightarrow i_{*E}\mathcal{H} \longrightarrow 0.$$

Multiplication by z_0 defines a monomorphism from $\mathcal{H}(-1)$ to $\mathcal{H}$. The image of this morphism is the ideal sheaf $\mathcal{I}_E$. Thus, we have an isomorphism $\mathcal{H}(-1) \to \mathcal{I}_E$. Using this, the above exact sequence becomes

$$0 \longrightarrow \mathcal{H}(-1) \longrightarrow \mathcal{H} \longrightarrow i_{*E}\mathcal{H} \longrightarrow 0,$$

which, on tensoring with $\mathcal{M}$ relative to $\mathcal{H}$, yields an exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{M}(-1) \longrightarrow \mathcal{M} \longrightarrow \mathcal{B} \longrightarrow 0,$$

where $\mathcal{B} = i_{*E}\mathcal{H} \otimes_{\mathcal{H}} \mathcal{M}$ and $\mathcal{C} = \operatorname{Tor}_1^{\mathcal{H}}(i_{*E}\mathcal{H}, \mathcal{M})$. If we tensor this with $\mathcal{H}(k)$, we get the sequence of coherent analytic sheaves

$$0 \longrightarrow \mathcal{C}(k) \longrightarrow \mathcal{M}(k-1) \longrightarrow \mathcal{M}(k) \longrightarrow \mathcal{B}(k) \longrightarrow 0.$$

If we set $\mathcal{L}_k = \mathrm{Ker}(\mathcal{M}(k) \to \mathcal{B}(k))$, then this sequence breaks up into two short exact sequences of analytic sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}(k) & \longrightarrow & \mathcal{M}(k-1) & \longrightarrow & \mathcal{L}_k & \longrightarrow & 0, \\ 0 & \longrightarrow & \mathcal{L}_k & \longrightarrow & \mathcal{M}(k) & \longrightarrow & \mathcal{B}(k) & \longrightarrow & 0. \end{array}$$

We next apply the long exact sequences of cohomology to these two short exact sequences. The relevant parts for us are

$$(13.5.1)\qquad \begin{array}{ccccc} H^1(P^n, \mathcal{M}(k-1)) & \longrightarrow & H^1(P^n, \mathcal{L}_k) & \longrightarrow & H^2(P^n, \mathcal{C}(k)), \\ H^1(P^n, \mathcal{L}_k) & \longrightarrow & H^1(P^n, \mathcal{M}(k)) & \longrightarrow & H^1(P^n, \mathcal{B}(k)). \end{array}$$

Since $\mathcal{B}$ and $\mathcal{C}$ are kernel and cokernel, respectively, of a morphism between coherent analytic sheaves, they are coherent sheaves of $\mathcal{H}$-modules. However, in each case, the action of $\mathcal{H}$ factors through $i_{*E}\mathcal{H}$ and, hence, they may be realized as images under i_* of coherent analytic sheaves on E (Exercise 13.9). It follows that $\mathcal{B}(k)$ and $\mathcal{C}(k)$ are also images under i_* of coherent analytic sheaves on E. By assumption, coherent analytic sheaves on E are algebraically induced. It follows from this and Proposition 13.4.5 that $\mathcal{B}(k)$ and $\mathcal{C}(k)$ are images under the functor $(\,\cdot\,)^h$ of coherent algebraic sheaves on P^n. It then follows from Theorems 13.5.1 and 12.4.4 that $\mathcal{B}(k)$ and $\mathcal{C}(k)$ have vanishing pth cohomology for $p > 0$ and k sufficiently large.

Thus, the exact sequences (13.5.1) imply that, for large k, we have surjective morphisms

$$H^1(P^n, \mathcal{M}(k-1)) \to H^1(P^n, \mathcal{L}_k), \quad \text{and} \quad H^1(P^n, \mathcal{L}_k) \to H^1(P^n, \mathcal{M}(k)),$$

which implies that

$$\dim H^1(P^n, \mathcal{M}(k-1)) \geq \dim H^1(P^n, \mathcal{L}_k) \geq \dim H^1(P^n, \mathcal{M}(k)).$$

At this point, we appeal to the Cartan-Serre theorem (Theorem 11.10.2), which states that all cohomology spaces of a coherent analytic sheaf on a compact holomorphic variety are finite dimensional vector spaces. In particular, the spaces $H^1(P^n, \mathcal{M}(k))$ are finite dimensional and, by the above, the dimension is a non-increasing function of k, for sufficiently large k. This implies that eventually the dimension must become constant as k increases. Thus, for sufficiently large k, $H^1(P^n, \mathcal{L}_k) \to H^1(P^n, \mathcal{M}(k))$ is a surjective morphism between finite dimensional vector spaces of the same dimension. This implies that it is injective as well. If we use this fact on the long exact sequence of cohomology

$$H^0(P^n, \mathcal{M}(k)) \to H^0(P^n, \mathcal{B}(k)) \to H^1(P^n, \mathcal{L}_k) \to H^1(P^n, \mathcal{M}(k)),$$

we conclude that $H^0(P^n, \mathcal{M}(k)) \to H^0(P^n, \mathcal{B}(k))$ is surjective for k sufficiently large. Also, since $\mathcal{B}$ is the image under $(\cdot)^h$ of a coherent algebraic sheaf, $\mathcal{B}(k)_x$ is generated by $H^0(P^n, \mathcal{B}(k))$ if k is sufficiently large (since the analogous thing is true of coherent algebraic sheaves, by Proposition 12.4.2). We conclude that $H^0(P^n, \mathcal{M}(k))$ generates $\mathcal{B}(k)_x = \mathcal{M}(k)_x \otimes_{\mathcal{H}_x} i_{*E}\mathcal{H}_x$. The latter module is just the quotient module $\mathcal{M}(k)_x/(\mathcal{I}_E)_x\mathcal{M}(k)_x$. Since this quotient module of $\mathcal{M}(k)_x$ has the same geometric fiber as $\mathcal{M}(k)_x$, we conclude that $H^0(P^n, \mathcal{M}(k))$ generates the geometric fiber of $\mathcal{M}(k)_x$. This completes the proof.

13.5.6 Lemma. *Every coherent analytic sheaf on P^n is algebraically induced.*

Proof. We will prove this by induction on n. It is trivial when $n = 0$ since P^0 is a point and coherent algebraic and analytic sheaves are just finite dimensional vector spaces. Thus, we will assume that $n > 0$ and that the theorem is true in dimensions less than n.

By Lemma 13.5.5, we know that if $\mathcal{M}$ is a coherent analytic sheaf, then there is an integer k such that $\mathcal{M}(k)$ is generated by $H^0(P^n, \mathcal{M}(k))$. Since $H^0(P^n, \mathcal{M}(k))$ is finite dimensional, by Theorem 11.10.2, there is a surjection $\mathcal{H}^p \to \mathcal{M}(k)$ for some p. If we twist this morphism by the sheaf $\mathcal{H}(-k)$, we obtain a surjection $\mathcal{H}^p(-k) \to \mathcal{M}$. Now by applying the same analysis to the kernel of this map, which is also a coherent analytic sheaf, we obtain an exact sequence of the form

$$\mathcal{H}^q(-j) \xrightarrow{\alpha} \mathcal{H}^p(-k) \longrightarrow \mathcal{M} \longrightarrow 0.$$

By the results of section 12.3, $\mathcal{H}^q(-j) = \mathcal{O}^q(-j)^h$ and $\mathcal{H}^p(-k) = \mathcal{O}^p(-k)^h$, and so, by Theorem 13.5.2, the morphism α is induced by a morphism of coherent algebraic sheaves $\beta : \mathcal{O}^q(-j) \to \mathcal{O}^p(-k)$. If $\mathcal{N}$ is the cokernel of β, then the exact functor

$$(\cdot)^h$$

, applied to the exact sequence

$$\mathcal{O}^q(-j) \xrightarrow{\beta} \mathcal{O}^p(-k) \longrightarrow \mathcal{N} \longrightarrow 0,$$

yields an exact sequence

$$\mathcal{H}^q(-j) \xrightarrow{\alpha} \mathcal{H}^p(-k) \longrightarrow \mathcal{N}^h \longrightarrow 0.$$

But this implies that $\mathcal{M} \simeq \mathcal{N}^h$, which completes the induction and the proof of the lemma.

We are now in a position to prove Serre's third theorem.

13.5.7 Theorem. *If X is a projective algebraic variety, and $\mathcal{M}$ is a coherent analytic sheaf on X^h, then there is a coherent algebraic sheaf $\mathcal{N}$ on X such that $\mathcal{N}^h \simeq \mathcal{M}$. Furthermore, $\mathcal{N}$ is unique up to isomorphism.*

Proof. The uniqueness is an immediate consequence of Theorem 13.5.2.

If $i : X \to P^n$ is a closed embedding of X in P^n, then $i_*\mathcal{M}$ is a coherent analytic sheaf on P^n, by Exercise 9.10. By Lemma 13.5.6, there is a coherent algebraic sheaf $\mathcal{S}$ on P^n, with $\mathcal{S}^h \simeq i_*\mathcal{M}$. We claim that $\mathcal{S}$ is $i_*\mathcal{N}$ for a coherent algebraic sheaf $\mathcal{N}$ on X. By Exercise 13.9, this will follow if we can show that $\mathcal{I}\mathcal{S} = 0$, where $\mathcal{I} \subset {}_{P^n}\mathcal{O}$ is the ideal sheaf of $i(X)$.

If $f \in \mathcal{I}(U)$ for some neighborhood $U \subset P^n$, then multiplication by f determines an endomorphism $\phi : \mathcal{S}|_U \to \mathcal{S}|_U$. The corresponding morphism $\phi^h : \mathcal{S}^h|_U \to \mathcal{S}^h|_U$ is still multiplication by f. It is the zero morphism, because $\mathcal{S}^h \simeq i_*\mathcal{M}$, and the $\mathcal{H}$-module action on $i_*\mathcal{M}$ factors through the quotient map $\mathcal{H} \to i_{*X}\mathcal{H}$. But if ϕ^h vanishes in a neighborhood, then so does ϕ, by Theorem 13.4.3. Thus, we have proved that $\mathcal{I}\mathcal{S} = 0$, and hence, that $\mathcal{S} = i_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on X.

Now Proposition 13.4.5 implies that $i_*\mathcal{N}^h \simeq (i_*\mathcal{N})^h = \mathcal{S}^h \simeq i_*\mathcal{M}$, which implies that $\mathcal{N}^h \simeq \mathcal{M}$. This completes the proof.

The results of Theorems 13.5.1, 13.5.2, and 13.5.7 (Serre's Theorems 1, 2, and 3) can be summarized as follows:

13.5.8 Theorem. *If X is a projective algebraic variety, then the functor $\mathcal{M} \to \mathcal{M}^h$ is a cohomology preserving equivalence of categories from the category of coherent algebraic sheaves on X to the category of coherent analytic sheaves on X^h.*

13.6 Applications

The above results have a wide variety of applications. We state and prove some of these below. For a more complete discussion of applications, we refer the reader to Serre's paper [**S**].

At this point, we will drop the use of the X^h notation, except in situations in which it is needed to avoid confusion. If X is an algebraic variety, then we will generally also use X to denote the corresponding holomorphic variety – that is, we will think of an algebraic variety as having two ringed space structures – one algebraic and one holomorphic. We will call a holomorphic variety X *algebraic* if it is the holomorphic variety associated to some algebraic variety.

Our first application is the following theorem of Chow:

13.6.1 Corollary. *If X is a projective variety, then every holomorphic subvariety of X is an algebraic subvariety.*

Proof. Let Y be a holomorphic subvariety of X, and consider the ideal sheaf $\mathcal{I}_Y$. Then the quotient $\mathcal{H}/\mathcal{I}_Y$ is a coherent analytic sheaf on X which is isomorphic to $i_{*Y}\mathcal{H}$, where $i : Y \to X$ is the inclusion. The support of the sheaf $\mathcal{H}/\mathcal{I}_Y$ is clearly the subvariety Y. Now by Theorem 13.5.8, there is a coherent algebraic sheaf $\mathcal{N}$ with the property that $\mathcal{N}^h \simeq \mathcal{H}/\mathcal{I}_Y$. The support of $\mathcal{N}$ is the same set as the support of $\mathcal{N}^h$, due to the fact that $\mathcal{H}_x$ is faithfully flat over $\mathcal{O}_x$ for each x. Thus, the support of $\mathcal{N}$ is Y. However, the support of a coherent algebraic sheaf is an algebraic subvariety (Exercise 9.6). This completes the proof.

In order to prove our next application of Serre's theorems, we need another result of Chow:

13.6.2 Chow's Lemma. *If X is an algebraic variety, then there is a projective variety Y, a dense open subset $U \subset Y$, and a surjective morphism $f : U \to X$ which has a closed graph in $Y \times X$.*

Proof. The lemma is clearly true if X is affine, since every affine variety is an open subset of a projective variety Y, and so we can choose $U = X$ and $f : U \to X$ the identity in this case. The graph of f is closed in $Y \times X$ because it is just the intersection of $Y \times X$ with the diagonal in $Y \times Y$, which is closed in $Y \times Y$.

Suppose that X is irreducible, and $X = V_1 \cup V_2 \cup \cdots \cup V_k$, where each V_i is an affine open subset of X. Then there are projective varieties Y_i, dense open subsets $U_i \subset Y_i$, and surjective morphisms $f_i : U_i \to V_i$ with graphs closed in $Y_i \times V_i$. Since X is irreducible, $V = \bigcap_i V_i$ is an open dense subset of X. Let $W \subset U_1 \times \cdots \times U_k$ be defined by

$$W = \{(y_1, \dots, y_k) : f_1(y_1) = \cdots = f_k(y_k)\}.$$

On W, we set $g(y_1, \dots, y_k) = f_i(y_i)$, for $i = 1, \dots, k$. Then we have a well-defined surjective morphism $g : W \to V$. Let Y be the closure of W in $Y_1 \times \cdots \times Y_k$. Then we can extend the morphism g from W to the open subset

$$\tilde{U}_i = \{(y_1, \dots, y_k) \in Y : y_i \in U_i\}$$

of Y by sending $(y_1, \dots, y_k)$ to $f_i(y_i)$ for $(y_1, \dots, y_k) \in \tilde{U}_i$. The extensions to $\tilde{U}_i$ and $\tilde{U}_j$ agree on their intersection, because W is dense in both. Thus, if $U = \bigcup_i \tilde{U}_i$, then these extensions define a morphism $f : U \to X$.

It remains to show that f has closed graph in $Y \times X$. Let $\overline{\Gamma}$ be the closure in $Y \times X$ of the graph Γ of f. This is the same as the closure in

$Y \times X$ of the graph of g. If $y = (y_1, \ldots, y_k, x) \in \overline{\Gamma}$, and $x \in V_i$, then (y_i, x) belongs to the closure in $Y_i \times V_i$ of the graph of f_i. But this graph is closed. Hence, $y_i \in \tilde{U}_i$, and $y \in \Gamma$. This completes the proof in the case where X is irreducible.

In the case where X is not irreducible, say $X = X_1 \cup \cdots \cup X_p$ is its irreducible decomposition, we simply choose, for each i, a projective variety Y_i, an open dense subset U_i, and a morphism $f_i : U_i \to X_i$ as above. Then the discrete union of the sets Y_i is a projective variety Y (Exercise 12.16), containing the discrete union of the sets U_i as a dense open subset U of Y and on this set, the f_i collectively define a surjective morphism $f : U \to X$ with closed graph in $Y \times X$. This completes the proof.

13.6.3 Corollary. *If X is an algebraic variety, then every compact holomorphic subvariety of X is an algebraic subvariety.*

Proof. Let K be a compact holomorphic subvariety of X. We choose Y, U, and $f : U \to X$ as in Chow's lemma. Then the graph Γ of f is closed in $Y \times X$. Since both Y and K are compact in the Euclidean topology, the set $\Gamma' = \Gamma \cap (Y \times K)$ is also compact in the Euclidean topology. Thus, the same thing is true of its projection L onto the factor Y. But L is $f^{-1}(K)$ and, hence, is a holomorphic subvariety of U. Since it is compact, it is also a holomorphic subvariety of the projective variety Y. By Corollary 13.6.1, L is an algebraic subvariety of Y. The image K of L under f is compact, hence, closed in the Euclidean topology.

Let $\overline{K}$ be the closure of K in the Zariski topology. Then $f : L \to \overline{K}$ is a regular morphism of algebraic varieties, with dense image. It follows from Proposition 8.8.1, that the image of f contains a dense open subset of $\overline{K}$; that is, the Zariski interior of K is dense in $\overline{K}$. However, the Zariski and Euclidean closures of a Zariski open set coincide (Proposition 13.4.6). Since K is closed in the Euclidean topology, it follows that $K = \overline{K}$ – that is, K is also closed in the Zariski topology. Thus, K is an algebraic subvariety of X.

From this, we easily deduce the following:

13.6.4 Corollary. *If X is an algebraic variety which is compact in its Euclidean topology, then every holomorphic map from X to an algebraic variety Y is regular.*

Proof. Let $f : X \to Y$ be holomorphic, where X and Y are algebraic and X is compact in the Euclidean topology. Then the graph Γ of f is a compact holomorphic subvariety of the algebraic variety $X \times Y$. By the previous corollary, it is algebraic. This implies that f is regular (Exercise 13.10).

This has the obvious consequence that:

13.6.5 Corollary. *A compact holomorphic variety has at most one structure of an algebraic variety (up to isomorphism).*

It is natural to ask whether or not every compact holomorphic variety (or at least every compact holomorphic manifold) is algebraic. The answer is affirmative for manifolds in dimension 1 – that is, every compact Riemann surface is algebraic (Exercise 13.14). However, in dimensions greater than 1, there are holomorphic manifolds which are not algebraic (see [**H**], appendix B).

Recall from Proposition 7.6.5 that the category of holomorphic vector bundles on a holomorphic variety may be identified with the category of locally free finite rank sheaves of $\mathcal{H}$-modules – a vector bundle is identified with its sheaf of sections. In the same way, the category of algebraic vector bundles on an algebraic variety may be identified with the category of locally free finite rank sheaves of $\mathcal{O}$-modules. Here, in both the algebraic and holomorphic cases, the free modules correspond to the trivial vector bundles. The functor $\mathcal{M} \to \mathcal{M}^h$ takes free finite rank sheaves of $\mathcal{O}$-modules to free finite rank sheaves of $\mathcal{H}$-modules. Since Zariski open sets are also Euclidean open sets, it is also true that this functor takes locally free finite rank sheaves of $\mathcal{O}$-modules to locally free finite rank sheaves of $\mathcal{H}$-modules. In the case where X is a projective variety, if $\mathcal{M}$ is a coherent algebraic sheaf, then $\mathcal{M}^h$ is free of finite rank if and only if $\mathcal{M}$ is free of finite rank (Exercise 13.18). Also, it is true, but not at all obvious, that every locally free finite rank $\mathcal{H}$-module is the image under $\mathcal{M} \to \mathcal{M}^h$ of a locally free finite rank $\mathcal{O}$-module (Exercise 13.20). The functor on vector bundles corresponding to $\mathcal{M} \to \mathcal{M}^h$ is just the functor which assigns to an algebraic vector bundle $\pi : E \to X$, over an algebraic variety, the holomorphic bundle $\pi^h : E^h \to X^h$ obtained by putting the canonical analytic structure on both total space and base. Thus, Theorem 13.5.8 has the following corollary.

13.6.6 Corollary. *If X is a projective algebraic variety, then the category of algebraic vector bundles on X is equivalent to the category of holomorphic vector bundles on X^h under the natural correspondence. An algebraic vector bundle is trivial (isomorphic to a trivial bundle $X \times C^n \to X$) if and only if the corresponding holomorphic vector bundle is trivial.*

Serre's paper contains a more general result of the above type which concerns bundles with structure groups other than $Gl_n(\mathbb{C})$. We will not attempt to describe or prove this result here.

Exercises

1. Prove that the inverse limit of an inverse sequence of short exact sequences is exact, provided the sequence on the left is surjective (each morphism in the sequence is surjective).
2. Prove that an A-module Y is flat if and only if, whenever X_1 and X_2 are finitely generated A-modules, and $X_1 \to X_2$ is an injective morphism, then $Y \otimes_A X_1 \to Y \otimes_A X_2$ is also injective.
3. Let A be a Noetherian local ring with maximal ideal M, and let B be a local ring which is a finite extension of A. Prove that if N is the maximal ideal of B, then the completion of B in the N-adic topology is the same as its completion as an A-module with the M-adic topology.
4. Suppose $A \subset B$ are integral domains, and K is the field of fractions of A. Prove that $K \otimes_A B$ is the localization of B relative to the multiplicative set consisting of the non-zero elements of A.
5. Prove that the formal power series ring $\mathbb{C}[[z_1, \dots, z_n]]$ is a unique factorization domain. Hint: Use induction on the number of variables, Gauss's theorem (A is a UFD implies $A[z]$ is a UFD), and an extension of the Weierstrass preparation theorem to formal power series.
6. Prove that if $\mathcal{M}$ and $\mathcal{N}$ are sheaves of modules over a sheaf of rings $\mathcal{R}$, then
$$(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})_x \simeq \mathcal{M}_x \otimes_{\mathcal{R}_x} \mathcal{N}_x.$$
Recall that $(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N})_x = \varinjlim\{\mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{N}(U) : x \in U\}$.
7. Prove that if $\mathcal{M}$ and $\mathcal{N}$ are coherent algebraic sheaves on an algebraic variety X, then the sheaf $\mathcal{M} \otimes_{\mathcal{O}} N$ is also a coherent algebraic sheaf. Also, verify the analogous statement for coherent analytic sheaves on a holomorphic variety.
8. Prove that if $\mathcal{M}$ and $\mathcal{N}$ are coherent algebraic sheaves on an algebraic variety X, then the sheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ is also a coherent algebraic sheaf. The proof of the analogous result for coherent analytic sheaves was outlined in Exercises 9.17 – 9.19. Show that the same approach works in the algebraic case.
9. Suppose Y is a subvariety of X, $i : Y \to X$ is the inclusion, $\mathcal{I}$ is the ideal sheaf of Y, and $\mathcal{S}$ is a coherent sheaf on X which satisfies $\mathcal{I}\mathcal{S} = 0$. Prove that $\mathcal{S}$ is supported on Y, its restriction $i^{-1}\mathcal{S}$ to Y is a coherent sheaf on Y, and $i_* i^{-1}\mathcal{S} = \mathcal{S}$. It doesn't matter whether the sheaves and spaces are algebraic or analytic.
10. Prove that if X and Y are algebraic varieties, and $f : X \to Y$ is a holomorphic map with an algebraic graph, then f is regular.

11. Let $\mathcal{M}$ and $\mathcal{N}$ be coherent algebraic sheaves on an algebraic variety X. Prove that $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})^h \simeq \mathcal{M}^h \otimes_{\mathcal{H}} \mathcal{N}^h$.
12. Suppose X is a projective variety, and $\mathcal{A}$ is a coherent analytic sheaf of $\mathcal{H}$-algebras on X (a sheaf of $\mathcal{H}$-algebras with identity which is coherent as a sheaf of $\mathcal{H}$-modules). Prove that $\mathcal{A} \simeq \mathcal{B}^h$ for a coherent sheaf of $\mathcal{O}$-algebras $\mathcal{B}$ on X. Hint: Use Serre's theorems and the result of the preceding exercise.
13. Use the results of the previous exercise and Exercises 8.15 and 9.16 to prove that if Y is a normal holomorphic variety, X is a projective variety and $\phi : Y \to X$ is a finite holomorphic map (section 9.6), then Y is algebraic.
14. Use the results of the previous exercise and Exercise 12.7 to prove that every compact Riemann surface is algebraic.
15. Let $A \to B$ be a faithfully flat ring extension. Prove that if B is a normal domain, then A is also.
16. Prove that if V is an algebraic variety, and $x \in V$ is a regular point, then ${}_V\mathcal{O}_x$ is a normal domain.
17. Prove that a projective variety is complete (Definition 12.5.6). Hint: Use the fact that a projective variety is compact in the Euclidean topology.
18. Prove that if X is a projective variety and $\mathcal{M}$ is a coherent algebraic sheaf on X for which $\mathcal{M}^h$ is a free sheaf of $\mathcal{H}$-modules, then $\mathcal{M}$ is a free sheaf of $\mathcal{O}$-modules.
19. Prove that if A and B are Noetherian local rings, with B a faithfully flat ring extension of A, and if M is an A-module, then the B-module $M_B = B \otimes_A M$ is free of finite rank n if and only if the A-module M is free of finite rank n.
20. Use the result of the preceding exercise to prove that if X is an algebraic variety and $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}$-modules on X such that $\mathcal{M}^h$ is locally free of finite rank as a sheaf of $\mathcal{H}$-modules, then $\mathcal{M}$ is also locally free of finite rank. Conclude that, on a projective variety, every locally free finite rank sheaf of $\mathcal{H}$-modules is $\mathcal{M}^h$ for some locally free finite rank sheaf $\mathcal{M}$ of $\mathcal{O}$-modules.

Chapter 14

Lie Groups and Their Representations

The final three chapters of this text are devoted to the study of complex semisimple Lie groups and their finite dimensional representations. This is a significant departure from the main thrust of the text, and it may not be of interest to every student who uses the text. However, the subject does provide significant insight into how the results of the preceding chapters are used in practice – the results on complex algebraic geometry, as well as the results on several complex variables. It also makes effective use of the Serre theorems of the previous chapter, and it continues the theme of that chapter – that, under certain conditions, the holomorphic and algebraic theories coincide. For example, we will prove in the next chapter that a complex Lie group (a complex manifold with a group structure which is holomorphic), which is connected and semisimple, is actually an algebraic group (an algebraic variety with an algebraic group structure).

This chapter is a survey of basic results from harmonic analysis – specifically, topological groups, Lie groups, Lie algebras and group representations, with an emphasis on compact groups and the Peter-Weyl theorem, the structure of complex semisimple Lie groups and Lie algebras, and the finite dimensional representations of semisimple Lie algebras. Even though this is intended primarily as a survey, we have included proofs of most of the results. However, we have left out some proofs which would require major diversions into other subjects and some proofs which are long and tedious, without being particularly instructive. Where proofs are not included, we give references to texts where they can be found. The material is quite standard and detailed developments can be found in a number of texts.

After an introductory section on topological groups, we present a development of representation theory for compact groups, culminating in the Peter-Weyl theorem. The remainder of the chapter is devoted to a survey of general Lie theory, the structure of Lie algebras, particularly semisimple Lie algebras, and classification of the finite dimensional representations of semisimple Lie algebras.

This chapter is followed by a chapter on algebraic groups which presents the general theory of such groups and culminates in proofs of the main structure theorems for semisimple algebraic groups, as well as a proof that every connected complex semisimple Lie group is algebraic. The final chapter consists of a proof of the Borel-Weil-Bott theorem. This gives a geometric picture of the finite dimensional representations of a semisimple Lie group G in terms of the cohomologies of certain holomorphic line bundles on a G-homogeneous projective variety – the flag variety – associated with G.

14.1 Topological Groups

A *topological group* is a group G with a Hausdorff topology for which the group operations $(g_1, g_2) \to g_1 g_2 : G \times G \to G$ and $g \to g^{-1} : G \to G$ are continuous. We normally restrict attention to locally compact topological groups – groups for which the underlying topological space is locally compact. If the underlying topological space is a topological manifold, then G is called a *Lie group*. If this manifold is a complex manifold, and the group operations are holomorphic, then G is a *complex Lie group*. We will be particularly interested here in complex Lie groups for which the underlying complex manifold is algebraic, and the group operations are also algebraic. Groups of this type are called *algebraic groups*.

Let V be a locally convex topological vector space, and denote by $GL(V)$ the group of continuous linear transformations of V with continuous inverse. A *representation* of a topological group G on V is a group homomorphism $\pi : G \to GL(V)$ such that $g \to \pi(g)v : G \to V$ is continuous for every $v \in V$. In this situation, we will sometimes refer to V as a *G-module*, where it is understood that the module action of G on V is given by the representation π. An *invariant subspace* for the representation π is a subspace $W \subset V$ with $\pi(g)W \subset W$ for each $g \in G$ – that is, an invariant subspace is a *submodule.* A representation with no proper closed invariant subspaces is said to be *irreducible.*

The representations of a topological group G constitute the objects in a category in which the morphisms are described as follows: If (π_i, V_i) are representations of G, then a morphism from (π_1, V_1) to (π_2, V_2) is a continuous linear map $\phi : V_1 \to V_2$, such that $\phi(\pi_1(g)v) = \pi_2(g)\phi(v)$ for all $\mathbf{g} \in G$

and $v \in V_1$. Traditionally, such a map is called an *intertwining operator* from (π_1, V_1) to (π_2, V_2). Using module terminology, one would say that ϕ is a G-module homomorphism.

One cannot say very much about completely general representations of locally compact topological groups. More structure is needed before reasonable results become possible. For example, one could restrict attention to representations on finite dimensional spaces V. Another common restriction is to study *unitary representations.* These are representations (π, V), on a Hilbert space V, such that $\pi(g)$ is unitary (inner product preserving) for each $g \in G$. In the study of representations of real semisimple Lie groups, one usually restricts attention to a class of representations introduced by Harish-Chandra – the *admissible* representations. For complex Lie groups and algebraic groups, one often restricts attention to representations which are holomorphic or algebraic in some sense.

On each locally compact group G, there is a regular Borel measure μ which is left invariant – that is, $\mu(gE) = \mu(E)$ for every Borel set E and every $g \in G$. This measure is unique up to a multiplicative constant and is called *left Haar measure.* Here we use the term "regular" to mean that it satisfies the conditions of the measures produced in the Riesz representation theorem, as described in [**R**] – that is, it is outer regular at all Borel sets and inner regular at open sets and sets of finite measure. There is, of course, also a right Haar measure – a regular Borel measure μ which satisfies $\mu(Eg) = \mu(E)$. It is not always true that a left Haar measure is also a right Haar measure, although, for large classes of groups, this is true. It is, of course, true for abelian groups, and it is also true for compact groups, as we shall see in the next section.

The left invariance of a left Haar measure is used in various arguments in the following way: If μ is such a measure, and f is integrable with respect to μ, then a change of variables of the form $g \to g_1 g$ does not affect the integral of f with respect to μ. That is,

$$\int f(g)\, d\mu(g) = \int f(g_1 g)\, d\mu(g).$$

That this is true when f is the characteristic function of a Borel set is the definition of left invariance. That it is true in general follows from the fact that finite linear combinations of such functions form a dense subspace of $L^1(\mu)$. The analogous statement is true for right Haar measures – that is, a change of variables of the form $g \to g g_1$ does not effect the integral.

The proof that Haar measure exists in general is very tedious ([**Nai**], 27.5). However, for Lie groups, there is a relatively simple proof involving invariant differential forms ([**Wa**], 4.11), and for compact groups, there is a simple and elegant proof which we will present in the next section.

Because of the importance of unitary representations, particularly in the study of compact or abelian topological groups, we shall make extensive use of elementary Hilbert space theory in this chapter. Almost any treatment of the subject should suffice as background (e.g. [**R**], [**R2**], [**Nai**]). Most of the facts we shall use from the subject are contained in the following paragraph.

If H is a Hilbert space, we denote by (u, v) the inner product of the vectors $u, v \in H$. This is linear in u, conjugate linear in v, and positive definite ($(u, u) > 0$ if $0 \neq u \in H$). A Hilbert space is a Banach space under the norm defined by $||u|| = \sqrt{(u,u)}$. The Schwartz inequality

$$|(u, v)| \leq ||u|| \, ||v||$$

ensures that the inner product is a continuous function from $H \times H$ to $\mathbb{C}$. Two vectors $u, v \in h$ are *orthogonal* if $(u, v) = 0$. In this case, we write $u \perp v$. If M is a closed subspace of H, its *orthogonal complement* is the space

$$M^{\perp} = \{v \in H : u \perp v, \; \forall\, u \in M\}.$$

Then H decomposes as $M \oplus M^{\perp}$, and the projection of H onto M determined by this decomposition is called the *orthogonal projection* of H onto M. Let H^* denote the Hilbert space dual of H – that is, the Banach space of all bounded linear functionals on H. The Riesz representation theorem for Hilbert space says that the inner product on H determines a conjugate linear isometry $v \to v^*$ of H onto H^*, where $v^*(u) = (u, v)$. If $a : H_1 \to H_2$ is a bounded linear operator between two Hilbert spaces, then its Hilbert space adjoint $a^* : H_2 \to H_1$ is a bounded linear operator defined by the equation

$$(a^* u, v) = (u, av), \; \forall\, u \in H_2, v \in H_1.$$

The space of all bounded linear operators from a Hilbert space H to itself will be denoted $\mathcal{B}(H)$. The map $a \to a^*$ is an *involution* on $\mathcal{B}(H)$ – that is, it is conjugate linear and satisfies $(ab)^* = b^* a^*$. An operator $a \in \mathcal{B}(H)$ is called *self-adjoint*, or *hermitian*, if $a^* = a$ and *unitary* if $a^* = a^{-1}$. More generally, an operator $a : H_1 \to H_2$, between two Hilbert spaces, will be called *unitary* if it is invertible, and $a^* = a^{-1}$. Thus, an operator $a : H_1 \to H_2$ is unitary if and only if it is surjective, and $(au, av) = (u, v)$ for all $u, v \in H_1$. The unitary operators in $\mathcal{B}(H)$ form a subgroup of the group of all bounded invertible operators on H called the *unitary group* of H.

In addition to the elementary facts listed above, we will need one not so elementary result from Hilbert space operator theory. This is the Hilbert space version of Schur's lemma:

14.1.1 Schur's Lemma. *A unitary representation (π, H) of a locally compact group G is irreducible if and only if the only bounded linear operators on H which commute with $\pi(g)$, for every $g \in G$, are multiples of the identity.*

Proof. If the representation is not irreducible, then there is a proper closed invariant subspace $E \subset H$. It follows from the unitarity of the operators $\pi(g)$ that $E^{\perp}$ is also invariant. This implies that the orthogonal projection onto E commutes with each $\pi(g)$. This operator is not a multiple of the identity if E is a non-zero proper subspace of H.

On the other hand, suppose (π, H) is irreducible. If an operator a on H commutes with all $\pi(g)$, then

$$a^*\pi(g) = a^*\pi(g^{-1})^* = (\pi(g^{-1})a)^* = (a\pi(g^{-1}))^* = \pi(g)a^*,$$

that is, a^* also commutes with each $\pi(g)$. Thus, each $\pi(g)$ commutes with each of the self-adjoint operators $a + a^*$ and $i(a - a^*)$. Consequently, it suffices to prove that each self-adjoint operator which commutes with all $\pi(g)$ is a multiple of the identity. However, the spectral theorem for self-adjoint operators ([**R2**]) tells us that such an operator a has the form

$$a = \int \lambda dp(\lambda),$$

where p is a projection valued measure on $\mathbb{R}$ with values in the set of orthogonal projections on H which commute with every operator which commutes with a. Since there are no non-trivial projections which commute with all $\pi(g)$, the only values possible for p are the identity and 0. It follows that a is a multiple of the identity.

Schur's lemma allows us to easily characterize the irreducible unitary representations of an abelian locally compact group.

14.1.2 Proposition. *A unitary representation (π, H) of an abelian locally compact group is irreducible if and only if H is 1-dimensional.*

Proof. Suppose (π, H) is irreducible. By Shur's lemma, the only operators on H which commute with all $\pi(g)$ are multiples of the identity. This implies that each $\pi(g)$ is a multiple of the identity. Since H has no non-trivial invariant subspaces, it must be 1-dimensional. The converse is obvious.

Let G be an abelian locally compact group, and π an irreducible unitary representation of G. Then, for each $g \in G$, $\pi(g)$ is a multiple $\gamma(g)$ of the identity. The function γ is necessarily a homomorphism of G into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. The fact that π is unitary implies that $\gamma(g)^{-1} = \overline{\gamma(g)}$ and, hence, that γ is a homomorphism from G to the circle group T. The fact that the representation π is continuous implies that γ is continuous. We will call a continuous homomorphism from G to $\mathbb{C}^*$ a *character* and, if it takes values in T, a *unitary character.* Thus, every irreducible unitary representation of an abelian locally compact group G is determined by a unitary character of G. In fact, it is obvious that this gives us a bijective correspondence between isomorphism classes of irreducible unitary representations of G and unitary characters.

The unitary characters on G form an abelian group under pointwise multiplication. In fact, this is a locally compact group, if it is given the topology of uniform convergence on compact sets (see [**R3**]). The resulting locally compact abelian group is called the *dual group* of G and is usually denoted $\widehat{G}$.

We will not pursue the topic of locally compact abelian groups here, other than to point out that the Fourier transform can be defined in this context and has essentially the same properties as it has in the classical case, where $G = \mathbb{R}$.

Let μ be a Haar measure on the locally compact abelian group G. Then $L^P(G)$, for $1 \le p \le \infty$, is defined to be $L^p(\mu)$. The *Fourier transform* of $f \in L^1(G)$ is a continuous function $\widehat{f}$ on $\widehat{G}$ defined by

$$\widehat{f}(\gamma) = \int \gamma(g) f(g)\, d\mu(g).$$

The following results are standard. Proofs can be found in [**R3**].

Each $g \in G$ determines a character χ_g of $\widehat{G}$ by $\chi_g(\gamma) = \gamma(g)$. This defines a natural map $g \to \chi_g : G \to \widehat{\widehat{G}}$.

14.1.3 Pontryagin Duality Theorem. *The natural map $G \to \widehat{\widehat{G}}$, of G to its second dual, is an isomorphism of topological groups.*

14.1.4 Plancherel Theorem. *There is a choice of Haar measure ν on $\widehat{G}$ such that the Fourier transform on $L^1(G) \cap L^2(G)$ extends to an inner product preserving isomorphism of $L^2(G)$ onto $L^2(\widehat{G})$.*

With this same choice ν of Haar measure on $\widehat{G}$, we define the inverse Fourier transform $f \to f^\vee$ for $f \in L^1(\widehat{G})$ by

$$f^\vee(g) = \int \overline{\gamma(g)}\, d\nu(\gamma).$$

14.1.5 Fourier Inversion Theorem. *If $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then $f = (\widehat{f})^\vee$.*

The regular representation of the abelian locally compact group G is the unitary representation $g \to R_g$ on $L^2(G)$ defined by

$$R_g f(g_1) = f(g_1 g).$$

The fact that this representation is unitary follows from the translation invariance of Haar measure. A simple calculation shows that

$$\widehat{(R_g f)}(\gamma) = \gamma(g)\widehat{f}(\gamma) = \chi_g(\gamma)\widehat{f}(\gamma)$$

for $f \in L^2(G)$. Thus, the Fourier transform is an isomorphism from the regular representation of G on $L^2(G)$ to the representation $g \to \chi_g$ of G as a group of multiplication operators on $L^2(\widehat{G})$.

If G is a compact abelian group, then Haar measure on G is finite and may be chosen to have total mass 1. In this case, $\widehat{G}$ is discrete, and its corresponding Haar measure is counting measure. The elements of $\widehat{G}$ then belong to $L^2(G)$, and in fact, form an orthogonal basis for $L^2(G)$. In this case, the isomorphism of the Plancherel theorem is just the Fourier series decomposition of $L^2(G)$ as $\ell^2(\widehat{G})$. Note that this expresses the regular representation of G on $L^2(G)$ as a Hilbert space direct sum of the irreducible representations determined by the characters in $\widehat{G}$. We will generalize this result to non-abelian compact groups in the next section. Note that, for non-compact abelian groups, the Plancherel theorem can be regarded as expressing the regular representation of G as a "direct integral" of irreducible representations. This idea also generalizes to non-abelian groups, but we will not discuss it further here.

14.2 Compact Topological Groups

Compact topological groups have very special properties. Most notably: Every continuous representation of a compact topological group on a Hilbert space decomposes as a direct sum of irreducible representations, and every irreducible representation is finite dimensional. Since every complex semisimple Lie group has a compact real form, these facts play a key role in the study of complex semisimple Lie groups and their representations. This section is devoted to proving them. The key to these results is the existence of a finite Haar measure on every compact group. We begin with a proof of this fact.

Let G be any locally compact topological group. The *convolution product* $\mu * \nu$ of two complex regular Borel measures on G is the measure defined by the condition that

$$\int f\, d\mu * \nu = \int\int f(gg_1)d\mu(g)\, d\nu(g_1), \tag{14.2.1}$$

for every continuous function f which vanishes at infinity. The left side of this equation defines a bounded linear functional on $C_0(G)$, and the Riesz representation theorem ([**R**], 6.19) tells us it is given by a measure $\mu * \nu$, as above. It is easy to see that this operation is associative and bilinear relative to the natural vector space structure on the space $\mathfrak{M}(G)$ of all complex regular Borel measures. Furthermore, since the Riesz representation theorem says that the norm of a complex measure is the same as the norm of the bounded linear functional on $C_0(X)$ that it determines, it follows from the definition of convolution that

$$||\mu * \nu|| \leq ||\mu|| ||\nu||.$$

Thus, with convolution as multiplication, $\mathfrak{M}(G)$ is a Banach algebra.

It follows from a limit argument that (14.2.1) also holds when f is a bounded Borel function. If f is the characteristic function of a Borel set E, this implies that

$$\mu * \nu(E) = \int \mu(Eg^{-1})\, d\nu(g) = \int \nu(g^{-1}E)\, d\mu(g).$$

It follows easily from this that

$$\delta_g * \mu(E) = \mu(g^{-1}E) \quad \text{and} \quad \mu * \delta_g(E) = \mu(Eg^{-1}), \tag{14.2.2}$$

where δ_g is the unit point mass at $g \in G$.

A regular Borel measure on G which is positive and satisfies $\mu(G) = 1$ is called a *probability measure*. Clearly the convolution product of two probability measures is again a probability measure. The proof of the existence of Haar measure for compact groups makes use of this fact and a compactness argument to show the existence of a probability measure which is a fixed point for convolution by any probability measure.

14.2.1 Theorem. *Let K be a compact topological group. There is a probability measure μ on K such that $\mu * \nu = \mu = \nu * \mu$ for every probability measure ν on K. Such a measure is both a left and a right Haar measure for K and is the unique left (right) Haar measure which is a probability measure.*

Proof. The Riesz representation theorem says that $\mathfrak{M}(K)$ is the Banach space dual of the space of continuous functions $C(K)$. If P denotes the set of probability measures on K, then P is a closed convex subset of the unit ball in $\mathfrak{M}(K)$, in the weak-$*$ topology. Since the unit ball in the dual of a Banach space is compact in the weak-$*$ topology, we conclude that P is compact in this topology.

We claim that convolution on P is a continuous operation in the weak-$*$ topology. Since the weak-$*$ topology is the weakest topology under which each of the functionals $\nu \to \int f\,d\nu$, $f \in C(K)$, is continuous, it suffices to prove that the functional

$$(\nu,\omega) \to \int\int f(kk_1)\,d\nu(k)d\omega(k_1)$$

is continuous on $P \times P$ for every $f \in C(K)$. However, this follows from the Stone-Weierstrass theorem, which implies that the function $f(kk_1)$ may be uniformly approximated on $K \times K$ by finite sums of functions of the form $g(k)h(k_1)$ with $g, h \in C(K)$.

For each $\nu \in P$, the map $\omega \to \nu * \omega$ is a continuous affine transformation of the compact convex set P into itself. It, therefore, has a fixed point, by an easy special case of the Leray-Schauder fixed point theorem (Exercise 14.1). That is, for each $\nu \in P$, there is an $\omega \in P$ such that $\nu * \omega = \omega$. We define a partial order "$<<$" on P by declaring that $\omega << \nu$ if $\nu * \omega = \omega$. By Zorn's lemma, there is a maximal totally ordered set $Q \subset P$. For each $\nu \in Q$, let $Q_\nu = \{\omega \in Q : \omega << \nu\}$. Then $\{Q_\nu\}$ is a nested family of closed sets, so the compactness of P ensures there is a point $\mu \in \bigcap_\nu Q_\nu$. It follows that $\mu << \nu$ for every $\nu \in Q$. Because of the maximality of Q, $\mu \in Q$, and no other element ω of P satisfies $\omega << \mu$. That is, μ is a minimal element of P. Since some element of P is fixed by μ, we conclude that it must be μ – that is, $\mu * \mu = \mu$. Then, for every $\nu \in P$, we have $\mu * (\mu * \nu) = \mu * \nu$ – that is, $\mu * \nu << \mu$, which implies $\mu * \nu = \mu$. Now reverse the roles of left and right multiplication in the above argument to conclude that there is an element $\mu' \in P$ such that $\nu * \mu' = \mu'$ for every $\nu \in P$. Then $\mu = \mu * \mu' = \mu'$.

We now have an element $\mu \in P$ with the property that $\nu * \mu = \mu = \mu * \nu$, for every $\nu \in P$. If we set $\nu = \delta_k$, we conclude from (14.2.2) that μ is both left and right invariant. From the definition of convolution, it is obvious that any measure $\omega \in P$, which is left invariant, will satisfy $\nu * \omega = \omega$ for every $\nu \in P$. By setting $\nu = \mu$ in this identity and using the properties of μ, we conclude that the only such ω is μ. Thus, μ is the only left Haar measure in P. Similarly, it is the only right Haar measure in P.

Throughout the remainder of this section, K will denote a compact topological group, and μ will denote its normalized Haar measure. Our objective will be to study the unitary representations of K.

Unitary representations have a very important property that is not shared by more general representations. If (σ, H) is a unitary representation of a locally compact group G, and M is a closed invariant subspace of H, then $M^\perp$ is also an invariant subspace. In fact, if $v \in M^\perp$, then for every $u \in M$, we have $\sigma(g^{-1})u \in M$, and so

$$(u, \sigma(g)v) = (\sigma(g)^*u, v) = (\sigma(g^{-1})u, v) = 0,$$

which implies $\sigma(g)v \in M^\perp$. Thus, every closed invariant subspace M of a unitary representation results in a decomposition $H = M \oplus M^\perp$ of the representation as the orthogonal direct sum of two subrepresentations.

For non-compact locally compact groups, there are Hilbert space representations which have closed invariant subspaces with no complementary invariant subspaces, and so, not only are they not unitary, they cannot be made to be unitary by a change of inner product. For compact groups the reverse is true as the following proposition shows.

14.2.2 Proposition. *Let σ be a representation of a compact topological group K on a Hilbert space H. Then the inner product on H may be replaced with an equivalent inner product, with respect to which, σ is unitary.*

Proof. We simply set

$$(u,v)_1 = \int (\sigma(k)u, \sigma(k)v)\, d\mu(k).$$

The integrand is continuous, hence bounded on K, and so the integral exists. Clearly, the bilinear form $(\,\cdot\,,\,\cdot\,)_1$ is an inner product on H. Furthermore, since $k \to (\sigma(k)u, \sigma(k)u) = ||\sigma(k)u||^2$ is bounded on K for each $u \in H$, it follows from the Banach-Steinhaus theorem ([**R**]) that $k \to ||\sigma(k)||$ is also bounded on K. Let C be a bound for this function. Then for $k \in K$, $u \in H$,

$$(\sigma(k)u, \sigma(k)u) \le C^2(u,u).$$

Also, this inequality, with k and u replaced by k^{-1} and $\sigma(k)u$, implies that

$$(u,u) \le C^2(\sigma(k)u, \sigma(k)u),$$

for all $k \in K$ and all $u \in H$. If we integrate these two inequalities over K, we conclude that

$$C^{-1}||u|| \le ||u||_1 \le C||u||,$$

for all $u \in H$, where $||\cdot||_1 = \sqrt{(\,\cdot\,,\,\cdot\,)_1}$ is the norm on H determined by the new inner product. In other words, the two inner products determine equivalent norms on H.

Since the measure μ is right invariant, we have

$$\begin{aligned}(\sigma(k_1)u, \sigma(k_1)v)_1 &= \int (\sigma(kk_1)u, \sigma(kk_1)v)\, d\mu(k)\\ &= \int (\sigma(k)u, \sigma(k)v)\, d\mu(k) = (u,v)_1\end{aligned}$$

and so the representation σ is unitary under the new inner product on H.

The above result illustrates a technique that will be used often in this section, and which is the key to the elegance of the theory of representations of compact groups: Because a compact group K has a finite Haar measure, quantities which transform under K can be averaged over K. The next few results all use this technique.

Let (σ_1, H_1) and (σ_2, H_2) be unitary representations of K and suppose $a : H_1 \to H_2$ is a bounded linear operator. Recall that a is called an *intertwining operator* between the two representations if it is a morphism of representations – that is, if $\sigma_2(k)a = a\sigma_1(k)$ for all $k \in K$. We will next show how the averaging over K technique can be used to construct an intertwining operator starting with any bounded operator a from H_1 to H_2.

Given such an operator a, the function $k \to \sigma_2(k^{-1})a\sigma_1(k)$ need not be a continuous function in the operator norm. However, for each $u \in H_1$ and $v \in H_2$, the function

$$k \to (\sigma_2(k^{-1})a\sigma_1(k)u, v) = (a\sigma_1(k)u, \sigma_2(k)v)$$

is continuous and can be integrated over K. If we set

$$\psi(u, v) = \int (\sigma_2(k^{-1})a\sigma_1(k)u, v)\, d\mu(k),$$

then ψ is linear in u, conjugate linear in v, and satisfies

$$|\psi(u, v)| \leq ||a||||u||||v||.$$

It follows from the Riesz representation theorem for Hilbert space ([**R**], 4.12) that, for each $u \in H_1$, there is an element $bu \in H_2$ such that $(bu, v) = \psi(u, v)$ for all $v \in H_2$. The element bu clearly depends linearly on u and satisfies $||bu|| \leq ||a||||u||$. Thus, there is a bounded linear operator $b : H_1 \to H_2$ such that

$$(bu, v) = \int (\sigma_2(k^{-1})a\sigma_1(k)u, v)\, d\mu(k).$$

We will denote this operator by $\displaystyle\int \sigma_2(k^{-1})a\sigma_1(k)\, d\mu(k)$.

14.2.3 Proposition. *Given any bounded linear operator $a : H_1 \to H_2$, the operator $\int \sigma_2(k^{-1})a\sigma_1(k)\, d\mu(k)$, defined above, is an intertwining operator from the representation (σ_1, H_1) to the representation (σ_2, H_2).*

Proof. If $b = \int \sigma_2(k^{-1}) a \sigma_1(k)\, d\mu(k)$, then for $u \in H_1, v \in H_2$,

$$\begin{aligned}
(b\sigma_1(k_1)u, v) &= \int (\sigma_2(k^{-1}) a \sigma_1(k)\sigma_1(k_1)u, v)\, d\mu(k) \\
&= \int (a\sigma_1(kk_1)u, \sigma_2(k)v)\, d\mu(k) \\
&= \int (a\sigma_1(k)u, \sigma_2(kk_1^{-1})v)\, d\mu(k) \\
&= \int (\sigma_2(k^{-1}) a\sigma_1(k)u, \sigma_2(k_1^{-1})v)\, d\mu(k) \\
&= (bu, \sigma_2(k_1^{-1})v) = (\sigma_2(k_1)bu, v),
\end{aligned}$$

where the third equality uses the change of variables $k \to kk_1^{-1}$ and the right invariance of μ. This clearly implies that $b\sigma_1(k_1) = \sigma_2(k_1)b$.

The above proposition becomes a powerful tool when used in conjunction with the theory of compact operators. The next lemma is the key to this approach.

14.2.4 Lemma. *If there is a non-zero compact self-intertwining operator for a unitary representation (σ, H) of K, then there is a finite dimensional invariant subspace for the representation. If (σ, H) is irreducible, then the only compact self-intertwining operators for the representation are multiples of the identity.*

Proof. Recall, from section 11.8, that the spectrum of an element a of a Banach algebra is the set of complex numbers λ such that $\lambda - a$ is not invertible (here λ is identified with the corresponding multiple of the identity in the algebra). If a is a self-adjoint operator on a Hilbert space H, then the spectrum of a is contained in the real numbers and contains a non-zero element, unless $a = 0$ (Exercises 14.2 and 14.4).

Let a be a non-zero compact self-adjoint operator on H, and let λ be a non-zero element of its spectrum. Then $\lambda - a$ is a compact perturbation of an invertible operator, and hence, by Theorem 11.11.1, it has finite dimensional kernel and closed image. However, a simple calculation, using the fact that $\lambda - a$ is also self-adjoint, shows that the image of $\lambda - a$ is the orthogonal complement of the kernel of $\lambda - a$. Thus, either the kernel of $\lambda - a$ is a non-zero finite dimensional subspace or $\lambda - a$ is injective and surjective. The latter is impossible, since the open mapping theorem says that a continuous bijective operator must have a bounded inverse. Thus, we have shown that a non-zero compact self-adjoint operator has a non-zero eigenvalue, and the corresponding eigenspace is finite dimensional.

Now suppose a is any compact self-intertwining operator for a unitary representation (σ, H) of K. Then a^* is also an intertwining operator, since for $k \in K$,

$$\sigma(k)a^* = (a\sigma(k)^*)^* = (a\sigma(k)^{-1})^* = (\sigma(k)^{-1}a)^* = a^*\sigma(k).$$

It follows that the self-adjoint operators $a + a^*$ and $ia - ia^*$ are also compact self-intertwining operators for σ. If a is not 0, then at least one of these operators is not 0. However, if either of these is non-zero, then it has a non-zero eigenvalue with a finite dimensional eigenspace, and this must be an invariant subspace for the representation. If the representation is irreducible, any such eigenspace must be the whole space H. It follows that a is a multiple of the identity operator.

Note that the above result is a variant of Schur's lemma (Lemma 14.1.1), but because it deals only with compact intertwining operators, it has an elementary proof which does not use the full force of the spectral theorem. We will need only this weaker form of Schur's lemma in this section.

Lemma 14.2.4 implies, in particular, that an irreducible unitary representation is finite dimensional if it has a non-zero compact self-intertwining operator. This easily leads to the first of our main theorems concerning representations of compact groups.

If H is a Hilbert space, and $\{H_i\}$ is a family of closed subspaces, with $H_i \perp H_j$ for $i \neq j$, and if the sum of the subspaces H_i is dense in H, then we will say that H is the orthogonal direct sum of the H_i and write $H = \oplus_i H_i$. If σ is a representation of K on H, and each H_i is an invariant subspace, then we will say that (σ, H) is the orthogonal direct sum of the representations (σ_i, H_i), where σ_i is σ restricted to H_i.

14.2.5 Theorem. *Each unitary representation of K decomposes as an orthogonal direct sum of irreducible representations, and each irreducible unitary representation of K is finite dimensional.*

Proof. Let (σ, H) be a unitary representation of k, and let p be the orthogonal projection onto any non-zero finite dimensional subspace of H. Then p has the form

$$px = \sum_{i=1}^{n} (x, y_i) y_i,$$

where $\{y_1, \dots, y_n\}$ is a finite orthonormal set of vectors in H. It follows that p is a compact operator, and the function $k \to \sigma(k)^{-1} p \sigma(k)$ is continuous from K to the ideal of compact operators on H, with the operator norm topology. This ideal is closed in $\mathcal{B}(H)$. It follows that the operator

$$a = \int \sigma(k)^{-1} p \sigma(k)\, d\mu(k)$$

is also a compact operator. By Proposition 14.2.3, a is a self-intertwining operator for σ. It cannot be the zero operator because

$$(au, u) = \int (p\sigma(k)u, \sigma(k)u)\, d\mu(k)$$

and the integrand of this integral is non-negative, continuous, and positive at $k = e$, provided u is chosen to be a non-zero element of the space on which p projects. We conclude, from the previous lemma, that there is a non-zero finite dimensional invariant subspace for σ. This implies immediately that every irreducible unitary representation is finite dimensional.

Since every unitary representation of K has a non-zero finite dimensional invariant subspace, and any descending chain of such subspaces must terminate, we conclude that every unitary representation of K contains a finite dimensional, irreducible subrepresentation. Now suppose that H_1 is a proper closed invariant subspace of H which is an orthogonal direct sum of irreducible subrepresentations. Then the orthogonal complement of H_1 is also an invariant subspace, and hence, it too contains an irreducible subrepresentation. By taking the direct sum of this with H_1, we obtain a larger subspace of H which is an orthogonal direct sum of irreducible subrepresentations. Now a Zorn's lemma argument on the partially ordered set consisting of all such subspaces shows that H itself must be such an orthogonal direct sum.

Recall from section 14.1 that, if the compact group K is abelian, and σ is an irreducible representation of K, then Schur's lemma implies that σ is a 1-dimensional representation. Hence, we have the following corollary of Theorem 14.2.5.

14.2.6 Corollary. *If K is a compact abelian group, then every unitary representation of K decomposes as an orthogonal direct sum of* 1*-dimensional invariant subspaces.*

Schur's lemma also has the following consequence:

14.2.7 Proposition. *Each intertwining operator between two irreducible unitary representations of a compact group is a non-negative constant times a unitary operator.*

Proof. Suppose (σ_1, H_1) and (σ_2, H_2) are irreducible unitary representations of K, and $a : H_1 \to H_2$ is an intertwining operator. Then $a^* : H_2 \to H_1$ is also an intertwining operator, as is the operator $a^*a : H_1 \to H_1$. Thus, a^*a is a multiple of the identity, by Schur's lemma. Since a^*a is positive definite (i.e. $(a^*au, u) = (au, au) \geq 0$ for all $u \in H_1$), it must be a non-negative number times the identity. If this number is 0, then $a = 0$. If it is non-zero, we may divide a by its square root, obtaining an operator b such that $b^*b = \mathrm{id}$. Then $(u, v) = (b^*bu, v) = (bu, bv)$ for all $u, v \in H_1$, and so b is an isometry of H_1 onto a closed subspace of H_2. Since b is an intertwining operator, this subspace is an invariant subspace for σ_2, and hence, is all of H_2, by the irreducibility of σ_2. Because it is surjective and inner product preserving, b is unitary.

We are now in a position to determine the structure of $L^2(K)$ as a Hilbert space with two (left and right) unitary representations of K.

The two representations of K on $L^2(K)$ are the left regular representation $k \to L_k$, defined by

$$L_k f(k_1) = f(k^{-1}k_1),$$

and the right regular representation $k \to R_k$, defined by

$$R_k f(k_1) = f(k_1 k).$$

The continuity of these representation is proved as follows. If f is a continuous function on k, then the uniform continuity of f implies that $k \to L_k f$ is continuous from K to $\mathcal{C}(K)$ with the uniform topology, and hence, it is also continuous as a map from K to $L^2(K)$. That $k \to L_k f : K \to L^2(K)$ is also continuous when $f \in L^2(K)$ follows from an approximation argument using the fact that $\mathcal{C}(K)$ is dense in $L^2(K)$. The same argument works for $k \to R_k$. That these representations are, indeed, unitary follows directly from the translation invariance of μ.

Note that each L_{k_1} commutes with each R_{k_2}, and so, together, they determine a unitary representation $(k_1, k_2) \to L_{k_1}R_{k_2}$ of $K \times K$ on $L^2(K)$. This representation decomposes as an orthogonal direct sum of irreducibles, which are necessarily finite dimensional. We will characterize the constituents in this decomposition.

Let (σ, H_σ) be any irreducible unitary representation of K. If $u, v \in H_\sigma$, then the continuous function on K defined by

$$k \to (u, \sigma(k)v)$$

is called the *matrix coefficient* defined by the pair u, v. The correspondence which assigns the matrix coefficient to the pair u, v is linear in u and conjugate linear in v – that is, it is linear in the element $v^* \in H_\sigma^*$ defined by $v^*(\,\cdot\,) = (\,\cdot\,, v)$. Thus, the matrix coefficient correspondence is a bilinear map from $H_\sigma \times H_\sigma^*$ to $\mathcal{C}(K)$ and, hence, determines a linear map $\phi_\sigma : H_\sigma \otimes H_\sigma^* \to \mathcal{C}(K)$. There is a natural unitary representation σ^* of K on the space H_σ^* defined by $\sigma^*(k)v^*(u) = v^*(\sigma(k^{-1})u)$ (where the inner product on H_σ^* is defined by $(u^*, v^*) = (v, u)$). This representation is conjugate isomorphic to σ under the map $v \to v^*$ and, hence, is irreducible. The tensor product representation $\sigma \otimes \sigma^*$ of $K \times K$ on $H_\sigma \otimes H_\sigma^*$ is then defined by

$$(k_1, k_2) \to \sigma(k_1) \otimes \sigma^*(k_2).$$

We make $H_\sigma \otimes H_\sigma^*$ into a Hilbert space by giving it the inner product defined on elementary tensors by

$$(u_1 \otimes v_1^*, u_2 \otimes v_2^*) = (u_1, u_2)(v_1^*, v_2^*) = (u_1, u_2)(v_2, v_1).$$

Then the tensor product representation $\sigma \otimes \sigma^*$ is unitary and irreducible (Exercise 14.5).

The space $H_\sigma \otimes H_\sigma^*$ also has an algebra structure. In fact, it may be identified with the algebra of all linear transformations of H_σ, where an element $\sum u_i \otimes v_i^*$ of $H_\sigma \otimes H_\sigma^*$ is identified with the linear transformation $w \to \sum v_i^*(w)u_i$.

Similarly, $L^2(K)$ has an algebra structure. In fact, if we define the convolution product $f * h$ of two elements of $L^2(K)$ by

$$f * h(k) = \int f(k_1)h(k_1^{-1}k)\,d\mu(k_1),$$

then the resulting function $f * h$ is continuous (Exercise 14.6) and, hence, lies in $L^2(K)$. It is easy to see that convolution is associative and distributive, and hence, that $L^2(K)$ is an algebra, with convolution as multiplication. Also, it follows from the Schwartz inequality that $|f * h(k)| \le ||f||_2||h||_2$ for each $k \in K$ and, hence, that $||f * h||_2 \le ||f||_2||h||_2$. Here $||\cdot||_2$ is the norm in the Hilbert space $L^2(K)$. This shows that $L^2(K)$ is a Banach algebra under convolution, and that convolution is a continuous operation in the norm of $L^2(K)$.

14.2.8 Proposition. *Let (σ, H_σ) be an irreducible representation of K, of dimension n, and let ϕ_σ be the matrix coefficient map described above. Then,*

(i) *ϕ_σ is a $K \times K$ intertwining operator from $H_\sigma \otimes H_\sigma^*$ to $L^2(K)$;*

(ii) *$n^{1/2}\phi_\sigma$ is an isometry;*

(iii) *$n\phi_\sigma$ is an algebra isomorphism of the matrix algebra $H_\sigma \otimes H_\sigma^*$ onto a 2-sided ideal of the convolution algebra $L^2(K)$.*

Proof. If $u, v \in H_\sigma$ and $k, k_1 \in K$, then

$$\begin{aligned} L_k\phi_\sigma(u \otimes v^*)(k_1) &= (u, \sigma(k^{-1}k_1)v) = (\sigma(k)u, \sigma(k_1)v) \\ &= \phi_\sigma(\sigma(k)u \otimes v^*)(k_1), \text{ and} \\ R_k\phi_\sigma(u \otimes v^*)(k_1) &= (u, \sigma(k_1 k)v) = (u, \sigma(k_1)\sigma(k)v) \\ &= \phi_\sigma(u \otimes \sigma^*(k)v^*)(k_1). \end{aligned}$$

This shows that ϕ_σ is an intertwining operator from $H_\sigma \otimes H_\sigma^*$ to $L^2(K)$. Since the representation $\sigma \otimes \sigma^*$ is irreducible, ϕ_σ is an isomorphism of this representation of $K \times K$ onto an irreducible subrepresentation of $L^2(K)$. This proves (i).

Since ϕ_σ is an intertwining operator from $H_\sigma \otimes H_\sigma^*$ to $L^2(K)$, it follows that $\phi_\sigma^* : L^2(K) \to H_\sigma \otimes H_\sigma^*$ is also an intertwining operator, as is the composition $\phi_\sigma^*\phi_\sigma : H_\sigma \otimes H_\sigma^* \to H_\sigma \otimes H_\sigma^*$. By Lemma 14.2.4, this operator must be a positive multiple of the identity. Thus, ϕ_σ itself is a positive constant times an isometry. That is, there is a constant c such that

$$(14.2.3) \qquad \int (u_1, \sigma(k)v_1)\overline{(u_2, \sigma(k)v_2)}\, d\mu(k) = \mathrm{c}(u_1, u_2)(v_2, v_1)$$

for all $u_1, u_2, v_1, v_2 \in H_\sigma$. The constant c may be evaluated by choosing u_1 and u_2 both equal to an arbitrary unit vector u, setting $v_1 = v_2 = w_i$, and summing the above formula as w_i runs through an orthonormal basis for H_σ (that is, a set $\{w_i\}$ which spans H_σ and satisfies $(w_i, w_j) = \delta_{ij}$). Since, for each k, $\{\sigma(k)w_i\}$ is also an orthonormal basis for H_σ, we have

$$\sum_{i=1}^{n} (u, \sigma(k)w_i)\overline{(u, \sigma(k)w_i)} = (u, u) = 1$$

for each $k \in K$. Thus,

$$1 = \sum_{i=1}^{n} \int (u, \sigma(k)w_i)\overline{(u, \sigma(k)w_i)}\, d\mu(k) = \sum_{i=1}^{n} \mathrm{c}(u, u)(w_i, w_i) = n\mathrm{c},$$

which implies that $\mathrm{c} = n^{-1}$. Thus, (14.2.3) becomes a well-known formula for the inner product of two matrix coefficients:

$$(14.2.4) \qquad \int (u_1, \sigma(k)v_1)\overline{((u_2, \sigma(k)v_2)}\, d\mu(k) = n^{-1}(u_1, u_2)(v_2, v_1).$$

We conclude that the intertwining operator

$$n^{1/2}\phi_\sigma : H_\sigma \otimes H_\sigma^* \to L^2(K)$$

is an isometry onto an n^2-dimensional $K \times K$-invariant subspace of $L^2(K)$. This completes the proof of (ii).

Finally, consider two elementary tensors $u_1 \otimes v_1^*$ and $u_2 \otimes v_2^*$ in $H_\sigma \otimes H_\sigma^*$. Their product as operators on H_σ is $(u_2, v_1)u_1 \otimes v_2$. If we apply $n\phi$ to this element, we obtain the function

$$k \to n(u_2, v_1)(u_1, \sigma(k)v_2).$$

On the other hand, the convolution product of $n\phi(u_1 \otimes v_1^*)$ and $n\phi(u_2 \otimes v_2^*)$ is

$$n^2 \int (u_1, \sigma(k_1)v_1)(u_2, \sigma(k_1^{-1}k)v_2)\, d\mu(k_1)$$
$$= n^2 \int (u_1, \sigma(k_1)v_1)\overline{(\sigma(k)v_2, \sigma(k_1)u_2)}\, d\mu(k_1).$$

By (14.2.4) this is $n(u_2, v_1)(u_1, \sigma(k)v_2)$. Thus, $n\phi : H_\sigma \otimes H_\sigma^* \to L^2(K)$ is an algebra isomorphism onto a subalgebra of $L^2(K)$. The image of $n\phi$ is both left and right invariant under K. That it is a 2-sided ideal of $L^2(K)$, then follows from the definition of convolution. This proves (iii).

If (σ, H_σ) is a unitary representation of a compact group K, and $Y \subset H_\sigma$ is a closed invariant subspace, then $Y^\perp$ is also invariant. It follows that the orthogonal projection p onto the subspace Y is an intertwining operator for σ. If Z is also a closed invariant subspace of H_σ, then the restriction of p to Z is an intertwining operator from Z to Y. Hence, if the restrictions of σ to Y and to Z are inequivalent irreducible representations of the group, then the restriction of p to Z must be 0. In other words, $Z \perp Y$. We conclude that inequivalent irreducible subrepresentations of a unitary representation are orthogonal.

Let $\widehat{K}$ denote the set of equivalence classes of irreducible unitary representations of K. For each $\sigma \in \widehat{K}$, set

$$M_\sigma = \phi_\sigma(H_\sigma \otimes H_\sigma^*).$$

If we apply the observation of the previous paragraph in the case of the representation of $K \times K$ on $L^2(K)$, we conclude that $M_\sigma \perp M_{\sigma'}$ if σ and σ' are distinct elements of $\widehat{K}$.

Let $W = \oplus_{\sigma \in \widehat{K}} M_\sigma$ be the Hilbert space direct sum of the subspaces M_σ – that is, W is the closure in $L^2(K)$ of the sum of the subspaces M_σ. We will show that W is equal to $L^2(K)$. To do this, it suffices to show that $W^\perp = (0)$. Note that W and $W^\perp$ are both invariant subspaces for the left and right regular representations. Suppose H is an irreducible invariant subspace of $W^\perp$ for the left regular representation $k \longrightarrow L_k$. For $f, h \in H$, the matrix coefficient $k \longrightarrow (f, L_k h)$ is the function

$$f * h^\#(k) = \int f(k_1)\overline{h(k^{-1}k_1)}\, d\mu(k_1),$$

where $h^\#(k) = \overline{h(k^{-1})}$. If g belongs to W, then a simple calculation shows that

$$(f * h^\#, g) = (f, g * h) = 0,$$

since W is a 2-sided ideal for convolution. However, one of the summands M_σ in W corresponds to the irreducible representation obtained by restricting $k \longrightarrow L_k$ to H. Thus, one choice for g is the matrix coefficient $f * h^\#$ itself, and this implies that $f * h^\# = 0$. On the other hand, if we choose $h = f$, then this matrix coefficient is continuous (Exercise 14.6) and equal to $||f||_2$ at the identity of K. We conclude that $f = 0$, and hence, that $H = 0$. Consequently, W is all of $L^2(K)$.

In summary, we have proved:

14.2.9 The Peter-Weyl Theorem. *If K is a compact group, and $\widehat{K}$ is the set of equivalence classes of irreducible representations of K, then there is an orthogonal direct sum decomposition*

$$L^2(K) = \bigoplus_{\sigma \in \widehat{K}} M_\sigma,$$

where each M_σ is a 2-sided ideal and an irreducible $K \times K$-submodule of $L^2(K)$, isomorphic to $H_\sigma \otimes H_\sigma^*$ both as an $K \times K$-module and as an algebra.

For $\sigma \in \widehat{K}$, the identity element of the matrix algebra $H_\sigma \otimes H_\sigma^*$ is the element $\sum w_i \otimes w_i^*$, where $\{w_i\}$ is an orthonormal basis for H_σ. The image of this element under the matrix coefficient map ϕ_σ plays a special role in the subject. This is the function

$$\chi_\sigma(k) = \sum (w_i, \sigma(k) w_i) = \operatorname{tr}(\sigma(k^{-1}))$$

and is called the *character* of the representation. Note that, if n is the dimension of H_σ, then $n\chi_\sigma$ is the identity element of the subalgebra M_σ of $L^2(K)$.

14.2.10 Corollary. *For each $\sigma \in \widehat{K}$, the character χ_σ is a central element of the convolution algebra $L^2(K)$. If n is the dimension of the representation σ, then $n\chi_\sigma$ acts, under convolution, as the orthogonal projection of $L^2(K)$ onto M_σ.*

Proof. The map $n\phi_\sigma$ is an algebra isomorphism onto M_σ, which is a 2-sided ideal of $L^2(K)$. It follows that the image of the identity under this map acts (by convolution on either side) on M_σ as the identity. It acts on $M_{\sigma'}$, for $\sigma' \neq \sigma$ as the zero operator, since its product with an element of $M_{\sigma'}$ lies in both $M_{\sigma'}$ and M_σ. The corollary follows directly from Theorem 14.2.9 and this observation.

14.3 Lie Groups and Lie Algebras

A *Lie group* is a topological group such that the underlying topological space is a manifold and the group operations are morphisms for the manifold structure. There are several categories of Lie groups, each corresponding to a category of manifolds.

Here we will deal with three categories of Lie groups. A *real Lie group* G is a topological group for which the underlying topological space is a topological manifold. It is a theorem (the solution to Hilbert's fifth problem [**MZ**]) that G is then, necessarily, a real analytic manifold, and the group operations are real analytic. If the underlying space has the structure of a complex manifold, in such a way that the group operations are holomorphic, then G is called a *complex Lie group.* An *algebraic group* is a group for which the underlying space is an algebraic variety, and the group operations are regular maps. In this case, since $g \to hg : G \to G$ is a biregular map, each neighborhood of a point $h \in G$ is equivalent to a neighborhood of the identity $e \in G$. It follows that G must be a non-singular variety. By the same argument, we need only know of a group G that its underlying space is a holomorphic variety, for which the group operations are holomorphic, to conclude that it is a complex Lie group. Note that, for algebraic groups, we drop the requirement that the underlying topological space be Hausdorff. Of course, each algebraic group has an associated complex Lie group, obtained by applying the functor $X \to X^h$ of Chapter 13, which associates to an algebraic variety its corresponding holomorphic variety, and this complex Lie group does have a Hausdorff topology.

Algebraic groups will be discussed in the next chapter. In this chapter we will focus on real and complex Lie groups.

In any topological group, the connected component containing the identity is a closed normal subgroup. Since a Lie group is locally connected, it follows that the connected component of the identity is also open, and this implies that each of its cosets is both open and closed. Thus, a Lie group is the discrete union of the cosets of its identity component.

A morphism $G \to H$, between Lie groups, is a group homomorphism which is also a morphism of the underlying ringed spaces (real analytic manifold, complex manifold, or algebraic variety). Thus, a morphism of complex Lie groups is a holomorphic map, in addition to being a group homomorphism.

To each complex Lie group G is associated a real Lie group, obtained by forgetting the complex structure on G and just regarding it as a real analytic manifold.

14.3.1 Example. Let V be a complex vector space. Then the group $GL(V)$ of all invertible linear transformations of V is the subset of the vector space of all linear transformations of V defined by the non-vanishing of the determinant. Thus, it is an open subset of a complex vector space and, therefore, a complex manifold. Furthermore, given any coordinate system for V, the group operations are holomorphic functions in these coordinates. Hence, $GL(V)$ is a complex Lie group. Similarly, the group $SL(V)$ of all linear transformations of determinant 1 on V is a complex Lie group, since it is both a holomorphic subvariety of $GL(V)$ and a subgroup.

Of course, if V is a real vector space, then the groups $GL(V)$ and $SL(V)$ may also be defined as above and are real Lie groups.

14.3.2 Example. Let V be a complex vector space, and let $\Theta : V \times V \to \mathbb{C}$ be a bilinear form. Then the subgroup of $GL(V)$ consisting of elements g such that

$$\Theta(gv, gv) = \Theta(v, v)$$

is a complex Lie group, since the set of $g \in GL(V)$ that satisfy the above equation is both a subgroup and a holomorphic subvariety of $GL(V)$. In the case where the bilinear form Θ is the standard inner product

$$\Theta(z, w) = z_1 w_1 + \cdots + z_n w_n, \quad z = (z_1, \ldots, z_n), \quad w = (w_1, \ldots, w_n),$$

this is just the complex orthogonal group $O_n(\mathbb{C})$.

Similarly, the set of invertible transformations which preserve a real bilinear form on a real vector space is a real Lie group. The real orthogonal groups and the unitary groups arise in this way.

A *Lie subgroup* of a complex Lie group is a subgroup which is also a holomorphic submanifold. Lie subgroups of real Lie groups are defined analogously. Recall that submanifolds are closed, so a Lie subgroup is, in particular, a closed subgroup.

An *action* of a group G on a space X is a map $(g, x) \to gx : G \times X \to X$, which is associative: $g_1(g_2 x) = (g_1 g_2)x$, for $g_1, g_2 \in G, x \in X$, and also satisfies $ex = x$, for $x \in X$ (these conditions just mean that the map ϕ, defined by $\phi(g)x = gx$, is a group homomorphism from G to the group of transformations of X). If G is a complex Lie group, and X is a complex manifold, then an action of G on X is said to be a *holomorphic action* if the action map $(g, x) \to xg : G \times X \to X$ is holomorphic.

If H is a Lie subgroup of a complex Lie group, then the space of left cosets G/H has a G-action defined by $(g, g_1H) \to gg_1H : G \times G/H \to G/H$. We define a topology on G/H by declaring a set to be open if its inverse image under the quotient map $\rho : G \to G/H$ is open. Then ρ is a continuous open map. We make G/H into a ringed space by defining the structure sheaf to be the sheaf which assigns, to an open set $U \subset G/H$, the ring of complex valued functions f on U, such that $f \circ \rho$ is a section of the structure sheaf of G on $\rho^{-1}(U)$. Then the quotient map $G \to G/H$ and the action map $G \times G/H \to G/H$ are both morphisms of ringed spaces. This follows easily from the definition of G/H and the fact that the multiplication map $G \times G \to G$ is a morphism of ringed spaces. We will show that the ringed space G/H is a holomorphic manifold on which there is a holomorphic G-action. The analogous result for real Lie groups also holds and has essentially the same proof.

14.3.3 Lemma. *If G is a complex Lie group, and H is a complex Lie subgroup, then there is a complex submanifold Y of a neighborhood of the identity e in G, with $e \in Y$, such that the map $(y, h) \to yh : Y \times H \to G$ is a biholomorphic map of $Y \times H$ onto an open subset of G. The analogous statement is true for a real Lie group G and a real Lie subgroup H.*

Proof. We give the proof in the complex case. The proof in the real case is the same.

Since H is a submanifold of G, we can find a submanifold Y_1 of some neighborhood of the identity e in G, with $e \in Y_1$, so that the tangent spaces of H and Y_1 at e are complementary linear subspaces of the tangent space of G at e . It follows that the map $(y, h) \to yh : H \times Y_1 \to G$ has a non-singular differential at (e, e), and hence, by the inverse mapping theorem, maps some neighborhood of (e, e), which we may assume has the form $Y_1 \times V_1$ for a neighborhood V_1 of e in H, biholomorphically onto a neighborhood W_1 of e in G.

Since V_1 is open in $W_1 \cap H$, we can find a neighborhood W of e, contained in W_1, with the property that $W \cap H = V_1$. We may then choose a neighborhood V of e in V_1 and a neighborhood Y of e in Y_1 so that, if $U = Y \cdot V$, then $U^{-1} \cdot U \subset W$. We then claim that the map $\phi : Y \times H \to G$, defined by $\phi(y, h) = yh$, is a biholomorphic map onto an open set in G.

Since ϕ maps $Y \times V$ biholomorphically to an open subset of W, it also maps $Y \times Vh$ biholomorphically to an open subset of Wh for each $h \in H$. Thus, to establish the claim, we need only show that ϕ is injective. Suppose

$y_1 h_1 = y_2 h_2$ for some pairs $y_1, y_2 \in Y$ and $h_1, h_2 \in H$. Then $h_2 = y_2^{-1} y_1 h_1$ is in $W \cap H = V_1$. By the same argument, $h_1 \in V_1$. However, on $Y_1 \times V_1$ the map $(y, h) \to yh$ is biholomorphic and, hence, injective. It follows that $h_1 = h_2$, and $y_1 = y_2$. We conclude that ϕ is injective. This establishes the claim and finishes the proof in the complex case. The proof in the real case is the same.

14.3.4 Proposition. *If G is a complex Lie group, H is a complex Lie subgroup, and $\rho : G \to G/H$ is the quotient map, then*

(i) *G/H is a holomorphic manifold;*
(ii) *the action of G on G/H is holomorphic;*
(iii) *there is a neighborhood U of $\rho(e)$ in G/H and a holomorphic map $\alpha : U \to G$ such that $\rho \circ \alpha = \mathrm{id}$, and the map $\phi : U \times H \to \rho^{-1}(U)$, defined by $\phi(x, h) = \alpha(x)h$, is biholomorphic.*

The analogous statements are true for a real Lie group G and a real Lie subgroup H.

proof. We choose Y as in the previous lemma and set $U = \rho(Y) = \rho(Y \cdot H)$ and $\alpha = (\rho|_Y)^{-1}$. Since $Y \cdot H$ is open in G, and $\rho^{-1}(U) = Y \cdot H$, U is open in G/H in the quotient topology. Furthermore, if $V \subset U$, then $\rho^{-1}(V) = \alpha(V) \cdot H$ and $\alpha(V) = \rho^{-1}(V) \cap Y$. It follows that V is open in U if and only if $\alpha(V)$ is open in Y. Thus, $\alpha : U \to Y$ is a homeomorphism. It is also a ringed space isomorphism since, if $V \subset U$ is open, then a function f on V is a section of the structure sheaf for G/H if and only if $f \circ \rho$ is holomorphic on the set $\rho^{-1}(V) = \alpha(V) \cdot H$, and this is true if and only if $f(\rho(yh)) = f(\rho(y)) = f(\alpha^{-1}(y))$ is a holomorphic function of y. It follows that, as a ringed space, G/H is locally isomorphic to a holomorphic manifold. Hence, it is a holomorphic manifold. That the action of G on G/H is holomorphic is automatic from the definition of the ringed space structure on G/H. Thus, (i) and (ii) are proved.

Then ϕ is the biholomorphic map $(x, h) \to (\alpha(x), h) : U \times H \to Y \times H$ followed by the biholomorphic map $(y, h) \to yh : Y \times H \to \rho^{-1}(U)$. Hence, ϕ is biholomorphic, and the proof of (iii) is complete.

The tangent space at e, for a Lie group G, has a special structure – a Lie algebra structure – which can be viewed as the infinitesimal form of its Lie group structure. A *Lie algebra* over a field k is a vector space $\mathfrak{g}$ over k, with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *bracket*, which satisfies

$$(14.3.1) \qquad [\xi, \eta] = -[\eta, \xi] \quad \text{and} \quad [\xi, [\eta, \nu]] + [\eta, [\nu, \xi]] + [\nu, [\xi, \eta]] = 0$$

for $\xi, \eta, \nu \in \mathfrak{g}$. The latter condition is called the *Jacobi identity*.

Unless otherwise specified, each Lie algebra discussed here will be a finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, a *Lie algebra morphism* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map such that $\phi([\xi, \eta]) = [\phi(\xi), \phi(\eta)]$ for all $\xi, \eta \in \mathfrak{g}_1$. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a linear subspace which is closed under bracket. An *ideal* of $\mathfrak{g}$ is a linear subspace $\mathfrak{h}$ such that $[\xi, \eta] \in \mathfrak{h}$ whenever $\eta \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$.

If G is a complex Lie group, then a holomorphic vector field on G is a holomorphic section of the complex tangent bundle of G. Equivalently, a holomorphic vector field on G is a derivation ξ on the structure sheaf $\mathcal{H}$ of G – in other words, a linear sheaf endomorphism of $\mathcal{H}$ which satisfies $\xi(fg) = g\xi(f) + f\xi(g)$ (see section 7.7). In its action on stalks of $\mathcal{H}$, a vector field ξ assigns to each $x \in G$ a derivation $\xi_x : \mathcal{H}_x \to \mathcal{H}_x$. A vector field is *right invariant* if it commutes with each of the right translation operators R_y, where, on stalks, $R_y : \mathcal{H}_x \to \mathcal{H}_{xy^{-1}}$ is defined by $R_y f(z) = f(zy)$. The space of right invariant vector fields forms a complex Lie algebra under commutator bracket:

$$[\xi, \eta] = \xi\eta - \eta\xi.$$

The resulting complex Lie algebra $\mathfrak{g}$ is called the *Lie algebra of* G. As a complex vector space, it may be identified with the tangent space to G at e, since each right invariant vector field is determined by its value at the identity e. Thus, $\mathfrak{g}$ may be thought of as the tangent space to G at e, with the additional structure determined by the bracket operation.

The same discussion applies in the case of a real Lie group G, except that $\mathcal{H}$ is replaced by the sheaf $\mathcal{C}^\infty_{\mathbb{R}}$ of real valued $\mathcal{C}^\infty$ functions on G, and holomorphic vector fields are replaced by real $\mathcal{C}^\infty$ vector fields – that is, derivations from $\mathcal{C}^\infty_{\mathbb{R}}$ to itself. The right invariant real $\mathcal{C}^\infty$ vector fields form a real Lie algebra $\mathfrak{g}$ of the same real dimension as G. The complexification of this Lie algebra may be identified with the Lie algebra of all complex right invariant vector fields on G – that is, all right invariant derivations on the sheaf $\mathcal{C}^\infty$.

If $\phi : G_1 \to G_2$ is a Lie group morphism, then its differential $d\phi$, defined by $d\phi(\xi)(f) = \xi(f \circ \phi)$, maps vector fields to vector fields and right invariant vector fields to right invariant vector fields. In fact, it is easy to see that it defines a Lie algebra morphism from the Lie algebra $\mathfrak{g}_1$ of G_1 to the Lie algebra $\mathfrak{g}_2$ of G_2. It follows that the *Lie correspondence*, which assigns to each Lie group its Lie algebra and to each morphism of Lie groups its differential, is a functor from the category of Lie groups to the category of Lie algebras. The following theorem, which we will not prove here, summarizes the fundamental results of Lie theory concerning this functor ([**V**], 2.7.5 and 3.15.1):

14.3.5 Theorem. *The Lie correspondence $G \to \mathfrak{g}$ has the following properties:*

(i) *each Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected, simply connected Lie group;*

(ii) *if H and G are Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, and if H is connected and simply connected, then each morphism of Lie algebras $\mathfrak{h} \to \mathfrak{g}$ is the differential of a morphism of Lie groups $H \to G$.*

The above theorem says that the functor which assigns to a Lie group its Lie algebra determines an equivalence of categories between the category of connected, simply connected Lie groups and the category of finite dimensional Lie algebras.

The *simply connected* condition requires some comment. Hence, we digress briefly to discuss the universal covering space and fundamental group of a manifold. For more details see [**V**], section 2.6. Let X be a connected topological manifold and $e \in X$ a base point. Consider the space of continuous curves $\gamma : [0,1] \to X$ with $\gamma(0) = e$. We define an equivalence relation on this space by declaring γ_1 and γ_2 equivalent if $\gamma_1(1) = \gamma_2(1)$, and γ_1 and γ_2 are homotopic – that is, there is a continuous map $h : [0,1] \times [0,1] \to X$ with $h(0,t) = \gamma_1(t), h(1,t) = \gamma_2(t), h(s,0) = e$, and $h(s,1) = \gamma_1(1)$. The quotient mod this equivalence relation has a natural topology under which it is also a manifold. The resulting space is the *universal covering space* $\tilde{X}$ of X. The map $\gamma \to \gamma(1) : \tilde{X} \to X$ is a covering map, due to the fact that X is locally Euclidean. The inverse image of e under this covering map has a natural group structure. The resulting group is called the *fundamental group of* X and is denoted $\pi_1(X)$. The space X is called *simply connected* if $\pi_1(X)$ is the trivial group, that is, if $\tilde{X} \to X$ is a homeomorphism. If G is a Lie group, then it is not hard to see that $\tilde{G}$ also has the structure of a Lie group, and the covering map $\tilde{G} \to G$ is a morphism of Lie groups with kernel equal to $\pi_1(G)$. This is true for both real and complex Lie groups – that is, in the complex case, $\tilde{G}$ inherits the structure of a complex Lie group from G, and the covering map $\tilde{G} \to G$ is a morphism of complex Lie groups. The Lie group $\tilde{G}$ is called the *universal covering group* of the Lie group G.

Since the map $\tilde{G} \to G$ is a covering map, it follows that $\tilde{G}$ and G have isomorphic neighborhoods of the identity and, hence, they have the same Lie algebra. Thus, every connected Lie group is the quotient of a simply connected Lie group with the same Lie algebra. The kernel of this quotient map is a discrete central subgroup of $\tilde{G}$ and is isomorphic to $\pi_1(G)$.

If H is not simply connected, a morphism of Lie algebras $\phi : \mathfrak{h} \to \mathfrak{g}$ may fail to be the differential of a morphism of Lie groups $H \to G$. Also, there are many distinct isomorphism classes of Lie groups with a given Lie algebra, but only one of these consists of connected, simply connected groups.

14.3.6 Example. The real Lie algebra consisting of $\mathbb{R}$ with trivial bracket is the Lie algebra of both the Lie group $\mathbb{R}$ of real numbers under addition and the circle group $T = \{z \in \mathbb{C} : |z| = 1\}$. Thus, these groups have isomorphic Lie algebras but are not isomorphic as Lie groups. There is a morphism $t \to e^{it} : \mathbb{R} \to T$, but no morphism in the other direction. Of course, $\mathbb{R}$ is simply connected, and T is not.

An *integral subgroup* of a Lie group G is a Lie group H together with an injective morphism $i : H \to G$ which is an *immersion* (has differential $di : \mathfrak{h} \to \mathfrak{g}$ which is injective). An integral subgroup H is not necessarily a submanifold, in the sense we have defined the term in this text, because H need not be closed and the topology on H may not be the relative topology it inherits as a subset of G (see Exercise 14.11). Theorem 14.3.5 implies that the Lie correspondence determines a bijection between the connected integral subgroups of G and the Lie subalgebras of its Lie algebra $\mathfrak{g}$. A connected integral subgroup is normal if and only if the corresponding Lie subalgebra is an ideal ([**V**], 2.13.4).

If G is a Lie group over $k = \mathbb{R}$ or $\mathbb{C}$, with Lie algebra $\mathfrak{g}$, then each element $\xi \in \mathfrak{g}$ determines a 1-dimensional Lie subalgebra $\{t\xi\}_{t\in k}$. This determines a Lie group homomorphism of k into G. The image of the identity under this homomorphism is called $\exp(\xi)$, and the homomorphism itself is then the map $t \to \exp(t\xi) : k \to G$. Since the differential of this map at the origin is supposed to be $t \to t\xi : k \to \mathfrak{g}$, we conclude that, for $\xi \in \mathfrak{g}$ and f a section of the structure sheaf of G in a neighborhood of $x \in G$,

$$\xi(f)(x) = \frac{d}{dt}|_{t=0} f(x \exp(t\xi)).$$

It can be shown that exp is, in fact, a holomorphic map in the complex case and a real analytic map in the real case ([**V**], Theorem 2.10.1). Then the above identity implies that the differential at the origin of $\exp : \mathfrak{g} \to G$ is the identity map from $\mathfrak{g}$ to $\mathfrak{g}$. The inverse mapping theorem then implies that there is a neighborhood of the origin in $\mathfrak{g}$ on which exp is a map, onto a neighborhood of the identity in G, which is biholomorphic in the complex case and bianalytic in the real case.

Let $\mathfrak{g}_1$ be a Lie subalgebra of the Lie algebra $\mathfrak{g}$ and let $G_1 \to G$ be the corresponding morphism from the connected, simply connected Lie group with Lie algebra $\mathfrak{g}_1$ into G. That is, the image of $G_1 \to G$ is the integral subgroup of G corresponding to $\mathfrak{g}_1$. Since the exponential map is the unique analytic $\phi : \mathfrak{g} \to G$ which is a homomorphism on each 1-dimensional subspace of $\mathfrak{g}$ and which has differential at the origin equal to the identity (Exercise 14.12), the restriction to $\mathfrak{g}_1$ of the exponential map for $\mathfrak{g}$ must be the exponential map of $\mathfrak{g}_1$ followed by $G_1 \to G$. It follows that the image of G_1 in G is the subgroup of G generated by the image of exp restricted to $\mathfrak{g}_1$.

The preceding discussion of exp is summarized in the following proposition.

14.3.7 Proposition. *If G is a complex Lie group with Lie algebra $\mathfrak{g}$, then there is a holomorphic map* $\exp : \mathfrak{g} \to G$ *with the following properties:*

(i) *there is a neighborhood of* 0 *in* $\mathfrak{g}$ *which* exp *maps biholomorphically onto a neighborhood of the identity in G;*

(ii) exp *is a group homomorphism on each* 1*-dimensional subspace of* $\mathfrak{g}$*;*

(iii) *the differential of* exp *at the origin is the identity map;*

(iv) *if G_1 is an integral subgroup of G, with Lie algebra $\mathfrak{g}_1 \subset \mathfrak{g}$, then the restriction of the exponential map for G to $\mathfrak{g}_1$ is the exponential map for G_1, followed by the immersion $G_1 \to G$.*

The same statements are true in the real case with "holomorphic" and "biholomorphic" replaced by "analytic" and "real analytic".

Note that (i) of the above proposition implies that the image of exp generates the identity component of the group G. Also, the exponential map is the unique holomorphic (analytic) map satisfying (ii) and (iii) (Exercise 14.12).

14.3.8 Example. Let V be a vector space over $k = \mathbb{R}$ or $\mathbb{C}$. As noted in Example 14.3.1, the groups $GL(V)$ and $SL(V)$ are Lie groups over k. Let $\mathfrak{gl}(V)$ denote the Lie algebra consisting of all linear transformations of V, with commutator bracket as bracket operation, and let $\mathfrak{sl}(V)$ denote its Lie subalgebra consisting of operators of trace 0. If $a \in \mathfrak{gl}(V)$, then a determines a right invariant vector field ξ_a on $GL(V)$ by

$$\xi_a(f)(b) = \frac{d}{dt}|_{t=0} f(b + tab),$$

where f is a section of the structure sheaf in a neighborhood of $b \in GL(V)$. It is easy to see that the correspondence $a \to \xi_a$ is a Lie algebra isomorphism from $\mathfrak{gl}(V)$ to the Lie algebra of $GL(V)$.

The exponential map in this case is the map

$$a \to \mathrm{e}^a = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

discussed in Theorem 10.3.1. In fact, this map is a homomorphism on each 1-dimensional subspace of $\mathfrak{gl}(V)$ and has differential at the origin given by $a \to \frac{d}{dt}|_{t=0}\, \mathrm{e}^{ta} = a$. Hence, by Exercise 14.12, $a \to \mathrm{e}^a$ is the exponential map.

Since a matrix has trace 0 if and only if its exponential has trace 1, the integral subgroup corresponding to $\mathfrak{sl}(V)$ is $SL(V)$.

If a basis for V is chosen, determining an isomorphism of V with $\mathbb{C}^n$, then $GL(V), SL(V), \mathfrak{gl}(V)$, and $\mathfrak{sl}(V)$ are usually denoted $GL_n(k), SL_n(k), \mathfrak{gl}_n(k)$, and $\mathfrak{sl}_n(k)$, respectively.

A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra morphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. We will denote such a representation by (π, V) when it is important to name the vector space involved, and simply by π, otherwise. We will also sometimes use module terminology and refer to V as a $\mathfrak{g}$-module.

In studying representations of Lie groups, we will be particularly interested in those which are holomorphic or analytic in an appropriate sense. A *holomorphic representation* of a complex Lie group is a representation (π, V) for which $\pi(g)v$ is a holomorphic V-valued function for each $v \in V$. For a real Lie group, we define the notion of *analytic* representation analogously. If V is finite dimensional, then a representation of G on V is holomorphic (resp. analytic) if and only if the map $\pi : G \to GL(V)$ is a morphism of complex (resp. real) Lie groups. The differential of such a representation is a complex (resp. real) Lie algebra morphism from the Lie algebra $\mathfrak{g}$ of G to $\mathfrak{gl}(V)$ – that is, it is a *representation* of the Lie algebra $\mathfrak{g}$. Every representation of the Lie algebra $\mathfrak{g}$ is the differential of a representation of G if G is connected and simply connected. If G is not simply connected, a given representation of $\mathfrak{g}$ may or may not be the differential of a representation of G. It is traditional to use the same symbol (e.g. π) to denote a representation of G and the corresponding representation of $\mathfrak{g}$.

Note that if (π, V) is a representation of G, and $\xi \in \mathfrak{g}$, then

$$\pi(\exp(\xi)) = e^{\pi(\xi)}.$$

This is due to the fact that $\pi(\exp(t\xi))$ and $e^{\pi(t\xi)}$ are both solutions to the matrix valued differential equation

$$u' = \pi(\xi)u,$$

satisfying the initial condition $u(0) = I$. If G is connected, it follows that a subspace of V is invariant under each $\pi(\xi)$, for $\xi \in \mathfrak{g}$, if and only if it is invariant under each $\pi(x)$, for $x \in G$. Thus, a finite dimensional representation of a connected group G is irreducible if and only if the corresponding representation of its Lie algebra $\mathfrak{g}$ is irreducible.

The *adjoint representation* of a Lie algebra $\mathfrak{g}$ on itself is the representation $\xi \to \mathrm{ad}_\xi$, where ad_ξ is defined by $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. That ad : $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is, in fact, a Lie algebra morphism follows from equations (14.3.1).

If G is a Lie group with Lie algebra $\mathfrak{g}$, and $g \in G$, then Ad_g is the automorhism of $\mathfrak{g}$ which is the differential of the automorphism $g_1 \to g g_1 g^{-1}$ of G. The *adjoint representation* of G is then $g \to \mathrm{Ad}_g : G \to GL(\mathfrak{g})$. Note that this is a holomorphic representation if G is a complex Lie group. A calculation shows that the representation ad : $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the differential of the representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$.

14.4 Lie Algebras

This section is devoted to a brief introduction to the foundations of Lie algebra structure theory. Included are the theorems of Engels and Lie and Cartan's two criteria. These results concern characterizations of the three types of Lie algebras – nilpotent, solvable, and semisimple.

Our main focus in this chapter will be on complex Lie algebras. However, many of the results discussed in this section hold in the real case as well. Thus, except where the field is specified, Lie algebras will be over a fixed field k, which may be either $\mathbb{R}$ or $\mathbb{C}$. Also, the Lie algebras discussed in this chapter will be assumed to be finite dimensional. Note that every complex Lie algebra may be considered a real Lie algebra if we simply forget the complex structure. Also, every real Lie algebra gives rise to a complex Lie algebra – its complexification. This is the complex vector space $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, with the bracket operation defined by $[\lambda \otimes x, \mu \otimes y] = \lambda\mu \otimes [x, y]$ for $\lambda, \mu \in \mathbb{C}$, and $x, y \in \mathfrak{g}$.

Recall that an ideal $\mathfrak{h}$, of a Lie algebra $\mathfrak{g}$, is a linear subspace such that $[x, y] \in \mathfrak{h}$ whenever $x \in \mathfrak{g}$, and $y \in \mathfrak{h}$. The set $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \{[x, y] : x, y \in \mathfrak{g}\}$ is called the *derived algebra* of $\mathfrak{g}$. It follows from the Jacobi identity (14.3.1) that it is an ideal of $\mathfrak{g}$. In fact, if $\mathfrak{h}$ is any ideal of $\mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{h}]$ is also an ideal of $\mathfrak{g}$. Note that the quotient $\mathfrak{g}/\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, by an ideal $\mathfrak{h}$, inherits a natural Lie algebra structure such that the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a morphism of Lie algebras.

14.4.1 Definition. *If $\mathfrak{g}$ is a Lie algebra, then we say*

(i) *$\mathfrak{g}$ is solvable if it has a filtration by ideals, with successive subquotients all 1-dimensional;*

(ii) *$\mathfrak{g}$ is nilpotent if the sequence of ideals $\mathfrak{g}^0, \mathfrak{g}^1, \dots, \mathfrak{g}^n, \dots$, defined by $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$, is eventually 0;*

(iii) *$\mathfrak{g}$ is semisimple if it is a direct product of simple Lie algebras – that is, non-abelian Lie algebras with no non-trivial ideals.*

It is clear that if

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{p} \to 0$$

is a short exact sequence of Lie algebras, with $\mathfrak{h}$ and $\mathfrak{p}$ solvable, then $\mathfrak{g}$ is also solvable.

Note that every nilpotent Lie algebra is solvable, since the subquotients of the filtration $\{\mathfrak{g}^n\}$ defined in Definition 14.4.1(ii) are abelian, and every abelian Lie algebra has a filtration by ideals (linear subspaces) with successive subquotients 1-dimensional.

To show that a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is nilpotent, it suffices to show that V has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$, such that $\mathfrak{g}V_i \subset V_{i-1}$, for $i = 1, \ldots, k$. For then, $\mathfrak{g}^n V \subset V^{k-n}$, and so $\mathfrak{g}^k = 0$. Thus, a Lie subalgebra of the Lie algebra of strictly upper triangular matrices in $\mathfrak{gl}_n(k)$ is nilpotent.

An element $a \in \mathrm{End}(V)$ is said to be *nilpotent* if $a^n = 0$ for some n.

14.4.2 Engel's Theorem. *If V is a finite dimensional vector space, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, and every element of $\mathfrak{g}$ is nilpotent as an element of* $\mathrm{End}(V)$, *then $\mathfrak{g}$ is a nilpotent Lie algebra.*

Proof. We claim that if a Lie algebra $\mathfrak{g}$ has a representation on a vector space in which every element of $\mathfrak{g}$ acts as a nilpotent operator, then the vector space has a non-zero subspace killed by $\mathfrak{g}$. Assuming this for the moment, the hypotheses of the theorem implies that V has a non-zero subspace V_1 which is killed by every element of $\mathfrak{g}$. Then $\mathfrak{g}$ acts on V/V_1 by nilpotent operators, and so there is a subspace V_2 of V so that V_2/V_1 is killed by $\mathfrak{g}$ acting on V/V_1. That is, $\mathfrak{g}V_2 \subset V_1$. Continuing in this way, we produce a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$, with $\mathfrak{g}V_i \subset V_{i-1}$ for $i = 1, \ldots, k$. As noted above, this proves that $\mathfrak{g}$ is nilpotent.

We prove the claim by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 0$, the claim is trivially true. Suppose the claim is true of all representations of Lie algebras of dimension less than n, and let $\mathfrak{g}$ be a Lie algebra of dimension n acting as an algebra of nilpotent operators on a vector space V. We may assume $\mathfrak{g}$ acts faithfully, otherwise, its image in $\mathfrak{gl}(V)$ is of dimension less than n. Thus, we will regard $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}(V)$.

If $x, y \in \mathrm{End}(V)$, then $\mathrm{ad}_x^n(y) = x^n y + n x^{n-1} y(-x) + \cdots + y(-x)^n$, with each term involving either x or $-x$ raised to a power at least $n/2$. From this, it follows that if x is nilpotent, then so is ad_x. Thus, $\mathfrak{g}$ acts on itself, under the adjoint representation, as a Lie algebra of nilpotent operators.

Let $\mathfrak{m}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$. Then $0 < \dim \mathfrak{m} < n$, and $\mathfrak{m}$ acts on $\mathfrak{g}/\mathfrak{m}$ as a Lie algebra of nilpotent operators. It follows from the induction assumption that there is a non-zero element of $\mathfrak{g}/\mathfrak{m}$ which is killed by $\mathfrak{m}$. This means there is an element $y \in \mathfrak{g}$, $y \notin \mathfrak{m}$ such that $[x, y] \in \mathfrak{m}$ for all $x \in \mathfrak{m}$. This, in turn, implies that the subspace spanned by y and $\mathfrak{m}$ is a subalgebra which, by the maximality of $\mathfrak{m}$, must be $\mathfrak{g}$. We conclude that $\mathfrak{m}$ is an ideal of codimension 1 in $\mathfrak{g}$.

By the induction assumption, the subspace $W = \{v \in V : \mathfrak{m}v = 0\}$ is non-trivial. Since $\mathfrak{m}$ is an ideal of $\mathfrak{g}$, the subspace W is invariant for $\mathfrak{g}$. Furthermore, $\mathfrak{g}$ is spanned by y and $\mathfrak{m}$. Since y is nilpotent, there is a non-zero vector in W that is killed by y. Such a vector is killed by all of $\mathfrak{g}$. This completes the induction and the proof.

The next result holds for both real and complex solvable Lie algebras, but it concerns a representation of such an algebra on a complex vector space.

14.4.3 Lie's Theorem. *If $\mathfrak{g}$ is a real or complex solvable Lie algebra, then every finite dimensional complex $\mathfrak{g}$ module V has a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

by complex submodules, such that V_{i-1} has complex codimension 1 *in V_i for $i = 1, \dots, k$.*

Proof. It is enough to prove that every finite dimensional complex module over a solvable Lie algebra has a 1-dimensional invariant subspace, since a simple induction argument then produces a filtration, as above.

As in the previous theorem, the proof is by induction on the dimension (real or complex) of $\mathfrak{g}$. There is nothing to prove if $\dim \mathfrak{g} = 0$. Let n be a positive integer. We assume that every complex module over a solvable Lie algebra of dimension less than n has a complex 1-dimensional invariant subspace, and we let V be a module over $\mathfrak{g}$, where $\mathfrak{g}$ is a solvable Lie algebra of dimension n.

Since $\mathfrak{g}$ is solvable, it has an ideal $\mathfrak{p}$ of codimension 1. By the induction assumption, there is a complex 1-dimensional invariant subspace of V for the action of $\mathfrak{p}$. This means there is a linear functional $\lambda \in \mathfrak{p}^*$ and a non-zero $v \in V$ such that

$$pv = \lambda(p)v, \ \forall\, p \in \mathfrak{p}. \tag{14.4.1}$$

Let W_λ be the subspace of V consisting of all $v \in V$ which satisfy (14.4.1).

We claim that W_λ is invariant under $\mathfrak{g}$. Let x be an element of $\mathfrak{g}$ not in $\mathfrak{p}$, so that x and $\mathfrak{p}$ span $\mathfrak{g}$. Let v be a non-zero element of W_λ. We set $v_j = x^j v$ and let A_j be the complex linear span of $\{v_0, \dots, v_j\}$. Note that there is a smallest integer k so that $A_j = A_k$ for all $j \geq k$. If we set $A = A_k$, then $\{v_0, \ldots, v_k\}$ forms a basis for A, and A is invariant under x. Then for $p \in \mathfrak{p}$,

$$pv_j = pxv_{j-1} = xpv_{j-1} + [p, x]v_{j-1}.$$

Since $[p, x] \in \mathfrak{p}$, an induction argument shows that each A_j is invariant under $\mathfrak{p}$ and that

$$pv_j = \lambda(p)v_j \bmod A_{j-1}$$

for $j \geq 1$. In other words, relative to the basis $\{v_0, \ldots, v_k\}$ for A, the restriction of each $p \in \mathfrak{p}$ to A is an upper triangular matrix, with each diagonal entry equal to $\lambda(p)$.

If tr_A denotes the trace for operators on A, then for $p \in \mathfrak{p}$,

$$(k+1)\lambda([p, x]) = \mathrm{tr}_A([p, x]) = \mathrm{tr}_A(px) - \mathrm{tr}_A(xp) = 0.$$

Thus, $\lambda([p, x]) = 0$, and so

$$pxv = xpv + [p, x]v = \lambda(p)xv + \lambda([p, x])v = \lambda(p)xv.$$

This implies that $xv \in W_\lambda$ and, hence, that W_λ is invariant under $\mathfrak{g}$, as claimed.

Since x acts on W_λ, it has a non-zero eigenvector $w \in W_\lambda$. Then w is a common eigenvector for all elements of $\mathfrak{g}$. It therefore spans a 1-dimensional invariant subspace for $\mathfrak{g}$. This completes the induction and finishes the proof.

The *Killing form* for a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form

$$\langle x, y \rangle = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y).$$

Much of elementary Lie algebra theory follows from the use of the Killing form and some basic tools from linear algebra. One of these tools is the Jordan decomposition, as discussed below.

If V is a complex vector space, and $a \in \mathrm{End}(V)$, we say a is *semisimple* if it is diagonalizable – that is, if V has a basis consisting of eigenvectors for the transformation a. The Jordan decomposition allows us to write each element a of $\mathrm{End}(V)$ as $a = s + n$, where s is semisimple, and n is nilpotent. We will need the strong version of the Jordan decomposition. This is given to us by the Jordan-Chevalley decomposition lemma:

14.4.4 Lemma. *If V is a finite dimensional complex vector space, and a is an element of* $\mathrm{End}(V)$, *then $a = s + n$, where*

(i) *s and n are commuting elements of* $\mathrm{End}(V)$, *with s semisimple, and n nilpotent;*

(ii) *each of s and n is a polynomial in a without constant term;*
(iii) *s and n commute with every element of* $\mathrm{End}(V)$ *that commutes with a;*
(iv) *s and n leave invariant any subspace of V which is invariant under a;*
(v) *the decomposition $a = s + n$ is the unique decomposition of this form with property (i).*

Proof. Note that (iii) and (iv) follow immediately from (ii) and so it is enough to prove (i), (ii), and (v).

Let the distinct eigenvalues of a be $\lambda_1, \ldots, \lambda_k$, and let the multiplicity of λ_i be m_i for each i. Let $V_i \subset V$ be the generalized eigenspace of a for the eigenvalue λ_i. That is, V_i is the subspace on which $(\lambda_i - a)^{m_i}$ vanishes. Then V is the direct sum of the subspaces V_i.

Let x be an indeterminate. Note that the polynomials $(\lambda_i - x)^{m_i}$, for $i = 1, \ldots, k$, are pairwise relatively prime elements of the polynomial algebra $\mathbb{C}[x]$, and if 0 is not one of the λ_i's, then each of these elements is relatively prime to x. Thus, by the Chinese remainder theorem ([**Hun**], III. 2.25), there is a polynomial $p(x) \in \mathbb{C}[x]$ such that $p(x) = \lambda_i \bmod (\lambda_i - x)^{m_i}$ for $i = 1, \ldots, k$, and $p(x) = 0 \bmod x$ (whether 0 is one of the λ_i or not). This implies that $p(a) = \lambda_i$ on V_i for $i = 1, \ldots, k$, and that $p(a)$ is a polynomial in a with no constant term, as is $q(a) = a - p(a)$. Thus, $p(a)$ is a semisimple element of $\mathrm{End}(V)$. Since $q(a)$ is $a - \lambda_i$ on V_i, it is nilpotent on V_i for each i, and is, therefore, nilpotent. Clearly, setting $s = p(a)$ and $n = q(a)$ gives us a decomposition $a = s + n$ which satisfies (i), (ii), (iii), and (iv).

Suppose $a = s' + n'$ is another decomposition of a which satisfies (i). Then $s - s' = n' - n$. Furthermore, s' commutes with s, since it commutes with a, and s is a polynomial in a. Consequently, V has a basis consisting of common eigenvectors for s and s' (Exercise 14.10). This implies that their difference is also semisimple. Likewise, $n' - n$ is nilpotent because it is the difference of commuting nilpotent elements of $\mathrm{End}(V)$. However, the only element of $\mathrm{End}(V)$ which is both semisimple and nilpotent is the zero element. Hence, $s' = s$ and $n' = n$.

If n is a nilpotent element of $\mathrm{End}(V)$, then $\mathrm{tr}(n) = 0$. This follows from the fact that there is a basis for V relative to which n is strictly upper triangular. On the other hand, in the next proposition, vanishing conditions involving trace are used to show that certain Lie subalgebras of $\mathfrak{gl}(V)$ are nilpotent or solvable.

In the proof of the next proposition, we will use the following simple fact: The algebra of complex valued functions on a finite set Q is generated by any function f which separates the points of Q. In fact, for each $p \in Q$, the function g_p defined by

$$g_p(q) = \prod_{p' \neq p} (f(q) - f(p'))$$

is non-zero at $q = p$, zero at all other points of Q and is a polynomial in f. Clearly every complex valued function on Q is a linear combination of the functions g_p and, hence, is also a polynomial in f. Note that the polynomials in f with no constant term form the ideal in the algebra of functions on Q consisting of functions which vanish where f vanishes.

14.4.5 Proposition. *Let V be a finite dimensional complex vector space, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\operatorname{tr}(ab) = 0$ for all $a \in [\mathfrak{g}, \mathfrak{g}]$ and $b \in \mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, and $\mathfrak{g}$ is solvable.*

Proof. For $x, y \in \mathfrak{g}$, let $a = s + n$ be the Jordan decomposition in $\operatorname{End}(V)$ of $a = [x, y]$. Then ad_s is a semisimple element (Exercise 14.14), and ad_n a nilpotent element of $\operatorname{End}(\mathfrak{gl}(V))$. Because ad_s and ad_n commute, the Jordan decomposition of ad_a in $\operatorname{End}(\mathfrak{gl}(V))$ is $\operatorname{ad}_a = \operatorname{ad}_s + \operatorname{ad}_n$. It follows from Lemma 14.4.4 that ad_s and ad_n are polynomials in ad_a with no constant term, and hence, that each of them carries $\mathfrak{g}$ into $[\mathfrak{g}, \mathfrak{g}]$.

Let $V = V_1 \oplus \cdots \oplus V_k$ be the decomposition of V into eigenspaces for the distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of s, and let $s^* \in \operatorname{End}(V)$ be the element which is $\bar{\lambda}_i$ times the identity on each V_i. Then, by the remark preceding this proposition, s^* is a polynomial in s. The eigenvalues of ad_s are the numbers $\lambda_j - \lambda_i$, with the corresponding eigenspaces being the subspaces $A_{ij} \subset \operatorname{End}(V)$ consisting of transformations mapping V_i into V_j and mapping all $V_{i'}$ for $i \neq i'$ to 0 (Exercise 14.14). The analogous statement is true for ad_{s^*}. We conclude that ad_{s^*} is $\bar{\lambda}_j - \bar{\lambda}_i$ times the identity on the $\lambda_j - \lambda_i$ eigenspace for ad_s. Hence, ad_{s^*} is a polynomial in ad_s with no constant term. This implies that it also carries $\mathfrak{g}$ into $[\mathfrak{g}, \mathfrak{g}]$.

Since s^* commutes with n, ns^* is nilpotent. It follows that

$$\operatorname{tr}(ss^*) = \operatorname{tr}(ss^*) + \operatorname{tr}(ns^*) = \operatorname{tr}(as^*).$$

Since $a = [x, y]$, we have

$$\begin{aligned} \operatorname{tr}(as^*) &= \operatorname{tr}([x, y]s^*) = \operatorname{tr}(xys^*) - \operatorname{tr}(yxs^*) \\ &= \operatorname{tr}(s^*xy) - \operatorname{tr}(xs^*y) = \operatorname{tr}(\operatorname{ad}_{s^*}(x)y) = 0, \end{aligned}$$

since $\operatorname{ad}_{s^*}(x) \in [\mathfrak{g}, \mathfrak{g}]$, and $y \in \mathfrak{g}$. We conclude that $\operatorname{tr}(ss^*) = 0$. On the other hand, $\operatorname{tr}(ss^*) = \sum_i |\lambda_i|^2$. It follows that $s = 0$ and, hence, that a is nilpotent.

We now have that $[\mathfrak{g},\mathfrak{g}]$ consists entirely of nilpotent elements. Engel's theorem implies that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. Since $[\mathfrak{g},\mathfrak{g}]$ is an ideal of $\mathfrak{g}$, and the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian, it follows that $\mathfrak{g}$ is solvable.

With the above result in hand, Cartan's theorems classifying Lie algebra types in terms of the Killing form quickly fall into place.

14.4.6 Theorem (Cartan's First Criterion). *A Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing form vanishes identically on the derived algebra $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$.*

Proof. Without loss of generality, we may assume that $\mathfrak{g}$ is a complex Lie algebra, since the theorem is clearly true of a real Lie algebra if it is true of its complexification.

Let $\mathfrak{q}$ be the image of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ under ad and let $\mathfrak{a}$ be the kernel of ad. Then $\mathfrak{a}$ is abelian, and there is a short exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{q} \to 0.$$

Thus, $\mathfrak{g}$ is solvable if $\mathfrak{q}$ is solvable. If the Killing form vanishes on $\mathfrak{g}^1$, then $\operatorname{tr}(ab) = 0$ for all $a, b \in \mathfrak{q}^1$. By the previous proposition, this implies that $[\mathfrak{q}^1,\mathfrak{q}^1]$ is nilpotent and $\mathfrak{q}^1$ is solvable. Since $\mathfrak{q}^1$ is an ideal of $\mathfrak{q}$ such that the quotient $\mathfrak{q}/\mathfrak{q}^1$ is abelian, we conclude that $\mathfrak{q}$ is solvable. Thus, $\mathfrak{g}$ is solvable if its Killing form vanishes identically on $\mathfrak{g}^1$.

On the other hand, if $\mathfrak{g}$ is solvable, then Lie's theorem, applied to its adjoint representation, implies that $\mathfrak{g}$ has a basis relative to which $\mathfrak{q}$ is a Lie algebra of upper triangular matrices. Then the elements of $\mathfrak{q}^1$ are all strictly upper triangular and, hence, have trace 0. It follows that the Killing form vanishes identically on $\mathfrak{g}^1$.

14.4.7 Theorem (Cartan's Second Criterion). *The following statements are equivalent for a Lie algebra $\mathfrak{g}$:*

(i) *$\mathfrak{g}$ is semisimple;*
(ii) *the Killing form for $\mathfrak{g}$ is non-singular;*
(iii) *$\mathfrak{g}$ has no solvable ideals.*

Proof. As with the previous theorem, we may assume, without loss of generality, that $\mathfrak{g}$ is complex.

Let $\mathfrak{p}$ be an ideal of $\mathfrak{g}$, and consider the annihilator $\mathfrak{p}^\perp$ of $\mathfrak{p}$ under the Killing form. That is:

$$\mathfrak{p}^\perp = \{x \in \mathfrak{g} : \langle x, y\rangle = 0,\ \forall\, y \in \mathfrak{p}\}.$$

Then the identity

$$\mathrm{tr}([a,b]c) = \mathrm{tr}([b,c]a), \quad \text{for } a,b,c \in \mathrm{End}(\mathfrak{g}),$$

applied to $a = \mathrm{ad}_x$, $b = \mathrm{ad}_y$, $c = \mathrm{ad}_z$, implies that

$$\langle [x,y], z\rangle = \langle [y,z], x\rangle \quad \text{for } x, y \in \mathfrak{g},\ z \in \mathfrak{p}.$$

It follows that $\mathfrak{p}^\perp$ is also an ideal, as is $\mathfrak{p} \cap \mathfrak{p}^\perp$. Furthermore, the Killing form for $\mathfrak{g}$ vanishes on $\mathfrak{p} \cap \mathfrak{p}^\perp$, and by Exercise 14.15, this means that the Killing form for $\mathfrak{p} \cap \mathfrak{p}^\perp$ vanishes identically. By the previous theorem, $\mathfrak{p} \cap \mathfrak{p}^\perp$ is a solvable ideal.

If $\mathfrak{g}$ has no non-zero solvable ideals, we conclude that $\mathfrak{p} \cap \mathfrak{p}^\perp = 0$ for every ideal $\mathfrak{p}$. A linear algebra argument shows that $\dim \mathfrak{p} + \dim \mathfrak{p}^\perp \geq \dim \mathfrak{g}$. Hence, if $\mathfrak{p} \cap \mathfrak{p}^\perp = 0$, then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{p}^\perp$. Consequently, if (iii) holds, then every ideal of $\mathfrak{g}$ has a complementary ideal. It follows that $\mathfrak{g}$ then decomposes into a direct sum of minimal ideals. Since none of these can be abelian if (iii) holds, we conclude that $\mathfrak{g}$ is a direct sum of simple ideals. Thus, (iii) implies (i).

Since $\mathfrak{g}^\perp$ is an ideal of $\mathfrak{g}$, we must have $\mathfrak{g}^\perp = 0$ if $\mathfrak{g}$ is simple. This means that the Killing form is non-singular for a simple Lie algebra. It follows from Exercise 14.15 that the Killing form is also non-singular for a semisimple algebra. Thus, (i) implies (ii).

If $\mathfrak{g}$ contains a solvable ideal, then it contains an abelian ideal $\mathfrak{a}$. In this case, let a be a non-zero element of $\mathfrak{a}$. Then $\mathrm{ad}_a \circ \mathrm{ad}_x \circ \mathrm{ad}_a = 0$ for every $x \in \mathfrak{g}$. In particular $\mathrm{ad}_x \circ \mathrm{ad}_a$ is nilpotent (its square is 0), and so it has trace 0. This means that $\langle x, a\rangle = 0$ for every $x \in \mathfrak{g}$. This is impossible if the Killing form is non-singular. Thus, (ii) implies (iii).

If $\mathfrak{p}$ and $\mathfrak{q}$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{p}+\mathfrak{q}$ is an ideal. It is also solvable, since $\mathfrak{p}$ and $(\mathfrak{p}+\mathfrak{q})/\mathfrak{p} \simeq \mathfrak{q}/(\mathfrak{p}\cap\mathfrak{q})$ are both solvable. Thus, the sum of two solvable ideals of a Lie algebra $\mathfrak{g}$ is again a solvable ideal. It follows that every Lie algebra $\mathfrak{g}$ has a maximal solvable ideal $\mathfrak{r}$ (called the *radical* of $\mathfrak{g}$) and that the quotient Lie algebra $\mathfrak{g}/\mathfrak{r}$ has no non-zero solvable ideals. In view of the previous theorem, we have:

14.4.8 Proposition. *If $\mathfrak{r}$ is the radical of the Lie algebra $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{r}$ is semisimple.*

14.5 Structure of Semisimple Lie Algebras

This section is devoted to a summary of the results we will need from the structure theory of semisimple Lie algebras.

Since the Killing form is non-singular for a semisimple Lie algebra $\mathfrak{g}$, the adjoint representation ad : $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful. Thus, $\mathfrak{g}$ is isomorphic to a certain subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$. It follows from the Jacobi identity that each element in the image of ad is a *derivation* of $\mathfrak{g}$ – that is, a linear transformation δ of $\mathfrak{g}$ which satisfies

$$\delta[x,y] = [\delta x, y] + [x, \delta y], \quad \forall\, x, y \in \mathfrak{g}. \tag{14.5.1}$$

If we denote the Lie algebra of all derivations of $\mathfrak{g}$ by $\mathrm{Der}(\mathfrak{g})$, then in fact:

14.5.1 Proposition. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then its adjoint map* ad *is a Lie algebra isomorphism of* $\mathfrak{g}$ *onto* $\mathrm{Der}(\mathfrak{g})$.

Proof. We need to show that ad : $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is surjective. We first note that the identity (14.5.1) may be rewritten as

$$[\delta, \mathrm{ad}_x] = \mathrm{ad}_{\delta x}, \quad \forall\, \delta \in \mathrm{Der}(\mathfrak{g}),\ x \in \mathfrak{g}. \tag{14.5.2}$$

Thus, if $\tilde{\mathfrak{g}}$ denotes the image of $\mathfrak{g}$ under ad, then $\tilde{\mathfrak{g}}$ is an ideal of the Lie algebra $\mathrm{Der}(\mathfrak{g})$. As in the proof of Theorem 14.4.7, if we set

$$\tilde{\mathfrak{g}}^{\perp} = \{\delta \in \mathrm{Der}(\mathfrak{g}) : \langle \delta, \mathrm{ad}_x \rangle = 0,\ \forall\, x \in \mathfrak{g}\},$$

then $\tilde{\mathfrak{g}}^{\perp}$ is also an ideal of $\mathrm{Der}(\mathfrak{g})$, and its intersection with $\tilde{\mathfrak{g}}$ is an ideal of $\tilde{\mathfrak{g}}$ with Killing form identically 0. Since such an ideal must be solvable, and $\mathfrak{g}$ is semisimple, we conclude that $\tilde{\mathfrak{g}}^{\perp} \cap \tilde{\mathfrak{g}} = 0$. This implies that $[\delta, \mathrm{ad}_x] = 0$ if $\delta \in \tilde{\mathfrak{g}}^{\perp}$ and $x \in \mathfrak{g}$. In view of (14.5.2) and the fact that ad is a faithful representation of $\mathfrak{g}$, we conclude that $\tilde{\mathfrak{g}}^{\perp} = 0$.

For every $\delta \in \mathrm{Der}(\mathfrak{g})$ the linear functional $a \to \langle \delta, a \rangle$, on $\tilde{\mathfrak{g}}$, is realized by some $b \in \tilde{\mathfrak{g}}$. That is, there is a $b \in \tilde{\mathfrak{g}}$ such that

$$\langle \delta, a \rangle = \langle b, a \rangle, \quad \forall\, a \in \tilde{\mathfrak{g}}.$$

Then $\langle \delta - b, a \rangle = 0$ for every $a \in \tilde{\mathfrak{g}}$, and so $\delta = b$. This completes the proof.

The previous result, along with the Jordan-Chevalley decomposition (Lemma 14.4.4), allows us to prove an abstract Jordan decomposition theorem for complex semisimple Lie algebras.

14.5.2 Theorem. *If* $\mathfrak{g}$ *is a complex semisimple Lie algebra, then each* $x \in \mathfrak{g}$ *has a unique decomposition* $x = s + n$*, with* $s, n \in \mathfrak{g}$*, such that the Jordan decomposition of* ad_x *is* $\mathrm{ad}_x = \mathrm{ad}_s + \mathrm{ad}_n$.

Proof. Set $a = \mathrm{ad}_x$, and let $a = t + m$ be the Jordan decomposition of a in $\mathrm{End}(\mathfrak{g})$. We claim that t and m are derivations of $\mathfrak{g}$. To prove this, we consider an eigenvalue λ of a, and let $\mathfrak{g}_\lambda$ be the generalized eigenspace of a for λ – that is, $\mathfrak{g}_\lambda$ is the set of all elements $y \in \mathfrak{g}$ such that some power of $a - \lambda$ kills y. Then $\mathfrak{g}$ is the direct sum of the subspaces $\mathfrak{g}_\lambda$. A calculation, using the fact that a is a derivation, shows that, for two eigenvalues λ and μ,

$$(a - (\lambda + \mu))^k[y, z] = \sum_j \frac{k!}{(k-j)!j!}[(a-\lambda)^j y, (a-\mu)^{k-j} z].$$

If $y \in \mathfrak{g}_\lambda$ and $z \in \mathfrak{g}_\mu$, then the right side of this expression is 0 for sufficiently large k. This implies that either $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = 0$, or $\lambda + \mu$ is also an eigenvalue of a, and $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. This, along with the fact that $t = \lambda$ on $\mathfrak{g}_\lambda$, implies that t is a derivation of $\mathfrak{g}$. Since a and t are derivations, m is as well. By Proposition 14.5.1, we have that t and m are in the image of ad. Hence, there exist s and n in $\mathfrak{g}$ such that $x = s + n$, and ad takes s and n to the semisimple and nilpotent parts of $a = \mathrm{ad}_x$ in $\mathrm{End}(\mathfrak{g})$. Since ad is a faithful representation of $\mathfrak{g}$, s and n are unique with this property.

We will call s and n the *semisimple* and *nilpotent* parts of x, respectively, and $x = s+n$ will be called the *abstract Jordan decomposition* of the element $x \in \mathfrak{g}$.

The abstract Jordan decomposition implies that a semisimple Lie algebra must contain non-zero semisimple elements, otherwise, it would contain only nilpotent elements, and such a Lie algebra is necessarily a nilpotent Lie algebra, by Engel's theorem.

14.5.3 Proposition. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{h}$ is a subalgebra consisting entirely of semisimple elements, then $\mathfrak{h}$ is abelian.*

Proof. Let x be an element of $\mathfrak{h}$. Since x is a semisimple element, $\mathfrak{g}$ is a direct sum of the eigenspaces for the distinct eigenvalues of ad_x. The projection on a given one of these eigenspaces is a polynomial in ad_x. Since $\mathfrak{h}$ is invariant under ad_x, it is also invariant under this projection. It follows that $\mathfrak{h}$ is also a direct sum of eigenspaces for distinct eigenvalues of ad_x – that is, ad_x also acts semisimply on $\mathfrak{h}$. We will prove that there is only one eigenspace for ad_x acting on $\mathfrak{h}$ – the one for eigenvalue 0.

Suppose $y \in \mathfrak{h}$ is an eigenvector of ad_x with eigenvalue $\lambda \neq 0$. Then $\mathrm{ad}_x(y) = \lambda y$ and $\mathrm{ad}_y(x) = -\lambda y$. Now ad_y also acts semisimply on $\mathfrak{h}$, and so we may write $x = e_1 + \cdots + e_k$, where $\{e_1, \ldots, e_k\}$ is a linearly independent set of eigenvectors of ad_y in $\mathfrak{h}$. Then $-\lambda y = \mathrm{ad}_y(x) = \mu_1 e_1 + \cdots + \mu_k e_k$, where μ_i is the eigenvalue of ad_y for the eigenvector e_i. Thus,

$$0 = \mathrm{ad}_y(\mu_1 e_1 + \cdots + \mu_k e_k) = \mu_1^2 e_1 + \cdots + \mu_k^2 e_k.$$

Since the e_i are linearly independent, we conclude that $e_i = 0$ unless $\mu_i = 0$. This implies that $\mathrm{ad}_y(x) = 0$, which contradicts the choice of y. We conclude that $[x, y] = 0$ for all $x, y \in \mathfrak{h}$ and, hence, that $\mathfrak{h}$ is abelian.

A *Cartan subalgebra* $\mathfrak{h}$, of a Lie algebra $\mathfrak{g}$, is a nilpotent subalgebra which is its own normalizer. For complex semisimple algebras $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ is a Cartan subalgebra if and only if it is a maximal toral subalgebra, where a subalgebra is *toral* if all of its elements are semisimple ([**Hum**]). Here we will only be concerned with Cartan subalgebras of complex semisimple algebras and so we may adopt *maximal toral* as the definition of such a subalgebra.

Throughout the remainder of this section, $\mathfrak{g}$ will denote a complex semisimple Lie algebra. It is clear from the discussion preceding Proposition 14.5.3 that non-trivial Cartan subalgebras of $\mathfrak{g}$ exist. Furthermore, the proposition implies that every Cartan subalgebra $\mathfrak{h}$ is actually an abelian subalgebra. We will prove in Proposition 14.5.5 below that, in fact, $\mathfrak{h}$ is maximal abelian.

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $h \in \mathfrak{h}$, then ad_h is semisimple, and so $\mathfrak{g}$ has a basis consisting of eigenvectors of ad_h. In fact, since $\mathfrak{h}$ is abelian, the operators ad_h, for $h \in \mathfrak{h}$, commute with one another, and this implies that there is a basis for $\mathfrak{h}$ consisting of vectors which are simultaneously eigenvectors for all ad_h with $h \in \mathfrak{h}$ (Exercise 14.10). The eigenvalue of ad_h corresponding to a given one of these eigenvectors clearly depends linearly on h. It follows that $\mathfrak{g}$ decomposes as a direct sum of subspaces $\mathfrak{g}_\alpha$, where $\alpha \in \mathfrak{h}^*$ – the vector space dual of $\mathfrak{h}$ – and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \mathrm{ad}_h(x) = \alpha(h)x, \ \forall\, h \in \mathfrak{h}\}.$$

The non-zero elements $\alpha \in \mathfrak{h}^*$, for which $\mathfrak{g}_\alpha$ is non-empty, are called *roots* and the subspaces $\mathfrak{g}_\alpha$ are called *root spaces.* We denote the set of all roots by Δ. Thus,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

This is called the *root space decomposition* of $\mathfrak{g}$ determined by the Cartan subalgebra $\mathfrak{h}$. A number of properties of this decomposition follow almost immediately from the definitions:

14.5.4 Proposition. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then*
(i) *Δ generates $\mathfrak{h}^*$ as a complex vector space;*
(ii) *$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \Delta$;*
(iii) *$\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ for $\alpha, \beta \in \Delta \cup \{0\}$, unless $\alpha = -\beta$.*

Proof. If $h \in \mathfrak{h}$ has the property that $\alpha(h) = 0$ for all $\alpha \in \Delta$, then ad_h vanishes on each $\mathfrak{g}_\alpha$, for $\alpha \in \Delta$, and on $\mathfrak{g}_0$. Then $\mathrm{ad}_x = 0$ and, hence, $x = 0$. Part (i) follows.

Part (ii) is a simple calculation.

Part (ii) implies that ad_x for $x \in \mathfrak{g}_\alpha$ acts on $\mathfrak{g}$ by shifting each eigenspace $\mathfrak{g}_\sigma$, for $\sigma \in \Delta \cup \{0\}$, into the eigenspace $\mathfrak{g}_{\alpha+\sigma}$. Similarly, $\mathrm{ad}_x \mathrm{ad}_y$, for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, maps $\mathfrak{g}_\sigma$ into $\mathfrak{g}_{\alpha+\beta+\sigma}$. Since only finitely many of these eigenspaces are non-zero, $\mathrm{ad}_x \mathrm{ad}_y$ is nilpotent unless $\alpha + \beta = 0$. Since nilpotent matrices have trace 0, this proves (iii).

The subspace $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ and is a Lie subalgebra of $\mathfrak{g}$. In fact, it is equal to $\mathfrak{h}$:

14.5.5 Proposition. *With $\mathfrak{h}$ and $\mathfrak{g}_0$ as above, $\mathfrak{h} = \mathfrak{g}_0$. Furthermore, the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is also non-singular.*

Proof. For $x \in \mathfrak{g}_0$, let $x = s + n$ be the abstract Jordan decomposition of x. Then ad_s commutes with every element of $\mathrm{End}(\mathfrak{g})$ that commutes with ad_x. In particular, it commutes with ad_h for all $h \in \mathfrak{h}$. This implies that $[s, h] = 0$ for all $h \in \mathfrak{h}$ and, hence, that $s \in \mathfrak{h}$, since by definition, $\mathfrak{h}$ is maximal toral. It follows that the semisimple elements of $\mathfrak{g}_0$ all belong to $\mathfrak{h}$, and each element of $\mathfrak{g}_0$ is the sum of an element of $\mathfrak{h}$ and a nilpotent element of $\mathfrak{g}_0$.

If $n \in \mathfrak{g}_0$ is nilpotent, and $s \in \mathfrak{h}$, then ad_n and ad_h act as commuting nilpotent operators on $\mathfrak{g}_0$ under the adjoint action. Since they commute, their sum is also nilpotent. By Engel's theorem, the derived algebra of $\mathfrak{g}_0$ is nilpotent, and hence, so is $\mathfrak{g}_0$.

It follows from (iii) of Proposition 14.5.4 that the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}_0$ is non-singular. This implies that there is no non-zero nilpotent element n in the center of $\mathfrak{g}_0$. In fact, if $n \in \mathfrak{g}_0$, and $y \in \mathfrak{g}_0$, with $[n, y] = 0$, then $\mathrm{ad}_y \mathrm{ad}_n$ is a nilpotent operator and, hence, has trace 0. This means that $\langle y, n \rangle = 0$. If n belongs to the center of $\mathfrak{g}_0$, then this is true of every $y \in \mathfrak{g}_0$, which forces $n = 0$. This argument also shows that $\langle h, n \rangle = 0$ for every $h \in \mathfrak{h}$ and every nilpotent element n of $\mathfrak{g}_0$.

Since $\langle [x, y], h \rangle = \langle [h, x], y] \rangle = 0$, if $h \in \mathfrak{h}$ and $x, y \in \mathfrak{g}_0$, each element $h \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ satisfies $\langle h, x \rangle = 0$ if x is either the semisimple or the nilpotent part of an element of $\mathfrak{g}_0$. Since the Killing form is non-singular on $\mathfrak{g}_0$, we conclude that $h = 0$. Thus, $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$.

Since, $\mathfrak{g}_0$ is a nilpotent Lie algebra, there is a largest integer k for which the ideal $\mathfrak{g}_0^k$ is non-zero. Then this ideal is contained in the center of $\mathfrak{g}_0$. Let x be a non-zero element of $\mathfrak{g}_0^k$. Then the nilpotent part n of x is also in the center of $\mathfrak{g}_0$ and, hence, is 0. Thus, x lies in $\mathfrak{h}$, by the first paragraph. If

$k > 0$, this means that $x \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$, and hence, that $x = 0$. The resulting contradiction shows that $k = 0$, which means that $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$. Thus, $\mathfrak{g}_0$ is abelian, every element of $\mathfrak{g}_0$ is semisimple, and thus, $\mathfrak{g}_0 = \mathfrak{h}$. This proves the first statement of the proposition.

Since the Killing form restricted to $\mathfrak{g}_0$ is non-singular, the second statement of the proposition follows immediately.

Note that the above proposition says that every Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian subalgebra.

14.5.6 Example. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of all 2×2 matrices with 0 trace. It is semisimple, by Exercise 14.16. It has a basis consisting of matrices h, x, and y, where h is diagonal with $1, -1$ down the diagonal, and x (resp. y) is strictly upper triangular (resp. lower triangular) with 1 as sole non-zero entry. We have

$$[h, x] = 2x, \ [h, y] = -2y, \text{ and } [x, y] = h.$$

Thus, h spans a Cartan subalgebra $\mathfrak{h}$, with root vectors x and y corresponding to the roots α and $-\alpha$, where $\alpha(h) = 2$. The set $\{\alpha, -\alpha\} \subset \mathfrak{h}^*$ is the root system for $\mathfrak{sl}_2(\mathbb{C})$ and the Cartan $\mathfrak{h}$.

Since the restriction of the bilinear form $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}$ remains non-singular, it defines a linear isomorphism $\alpha \to t_\alpha : \mathfrak{h}^* \to \mathfrak{h}$ with the property that

$$\alpha(h) = \langle t_\alpha, h \rangle, \quad \text{for } \alpha \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

This allows us to define a bilinear form on $\mathfrak{h}^*$, which we will also denote $\langle \cdot, \cdot \rangle$, by setting

$$\langle \alpha, \beta \rangle = \langle t_\alpha, t_\beta \rangle = \alpha(t_\beta), \quad \text{for } \alpha, \beta \in \mathfrak{h}^*.$$

14.5.7 Theorem. *Let α and β be roots for the complex semisimple Lie algebra $\mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$. Then*

(i) *$-\alpha$ is also a root;*
(ii) *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional and is spanned by t_α;*
(iii) *$\langle \alpha, \alpha \rangle = \alpha(t_\alpha) \neq 0$;*
(iv) *$\mathfrak{g}_\alpha$ is 1-dimensional;*
(v) *$n\alpha$ is not a root for $n = 2, 3, \dots$;*
(vi) *$\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, and t_α span a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$;*

Proof. If $x \in \mathfrak{g}_\alpha$, then Proposition 14.5.4(iii) implies that $\langle x, y \rangle = 0$ if $y \in \mathfrak{g}_\beta$, $\beta \in \Delta \cup \{0\}$, and $\beta \neq -\alpha$. Since the Killing form is non-singular on $\mathfrak{g}$, we must have $\mathfrak{g}_{-\alpha} \neq 0$. This proves (i).

If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, and $h \in \mathfrak{h}$, then

$$\langle [x,y],h\rangle = \langle [h,x],y\rangle = \alpha(h)\langle x,y\rangle = \langle t_\alpha,h\rangle\langle x,y\rangle.$$

Thus, $\langle [x,y] - \langle x,y\rangle t_\alpha, h\rangle = 0$ for every $h \in \mathfrak{h}$. Since, $[x,y] \in \mathfrak{g}_0 = \mathfrak{h}$, and the Killing form is non-singular on $\mathfrak{h}$, we conclude that $[x,y] = \langle x,y\rangle t_\alpha$. To finish the proof of (ii), it suffices to prove that $\langle x,y\rangle$ is not 0 for some choice of $x \in \mathfrak{g}_\alpha$ and $y \in g_{-\alpha}$. However, as in the previous paragraph, if x is a non-zero element of $\mathfrak{g}_\alpha$ then $\mathfrak{g}_{-\alpha}$ must contain an element y with $\langle x,y\rangle \neq 0$. This proves (ii).

Suppose $x \in \mathfrak{g}_\alpha$, and $y \in \mathfrak{g}_{-\alpha}$. Since $[x,y] = \langle x,y\rangle t_\alpha$, we have

$$[[x,y],x] = \langle x,y\rangle[t_\alpha,x] = \langle x,y\rangle\alpha(t_\alpha)x$$

and

$$[[x,y],y] = \langle x,y\rangle[t_\alpha,y] = -\langle x,y\rangle\alpha(t_\alpha)y.$$

Since x and y may be chosen so that $\langle x,y\rangle \neq 0$, we conclude that $\alpha(t_\alpha) = 0$ if and only if $[x,y]$ commutes with both x and y. In this case, x and y generate a 3-dimensional solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ with center $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] = [\mathfrak{s},\mathfrak{s}]$. Lie's theorem, applied to the representation of $\mathfrak{s}$ on $\mathfrak{g}$ given by ad, implies there is a basis for $\mathfrak{g}$, relative to which each operator ad_s for $s \in \mathfrak{s}$ is upper triangular. Then ad_s for $s \in [\mathfrak{s},\mathfrak{s}]$ is strictly upper triangular and, hence, nilpotent. However $t_\alpha \in [\mathfrak{s},\mathfrak{s}]$, and t_α is semisimple (being an element of $\mathfrak{h}$). Since the only operator which is both nilpotent and semisimple is the zero operator, we conclude that $t_\alpha = 0$. This contradicts our choice of t_α. Thus, it must be the case that $\langle\alpha,\alpha\rangle = \alpha(t_\alpha) \neq 0$. This proves (iii).

Choose a non-zero $y \in g_{-\alpha}$, and consider the linear subspace M of $\mathfrak{g}$ spanned by y and the spaces $\mathfrak{g}_{n\alpha}$ for $n = 0,1,2,\dots$. If x is a non-zero element of $\mathfrak{g}_\alpha$, then M is invariant under ad_x, ad_y, and $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x,\mathrm{ad}_y]$. The trace of $\mathrm{ad}_{[x,y]}$ on M is 0, since it is a commutator of two operators. On the other hand, $[x,y] = \langle x,y\rangle t_\alpha \in \mathfrak{h}$, by the previous paragraph, and so $\mathrm{ad}_{[x,y]}$ is a diagonal operator on M with trace $\langle x,y\rangle\alpha(t_\alpha)(-1+k_1+2k_2+\cdots)$, where k_n is the dimension of $\mathfrak{g}_{n\alpha}$. Since $\langle x,y\rangle\alpha(t_\alpha) \neq 0$, we conclude that $k_1 = 1$, and $k_n = 0$ for $n > 1$. This proves (iv) and (v).

If we set $h_\alpha = 2t_\alpha\langle\alpha,\alpha\rangle^{-1}$, and choose generators $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha,y_\alpha] = h_\alpha$, then x_α, y_α, and h_α satisfy the same commutation relations as the standard basis elements of $\mathfrak{sl}_2(\mathbb{C})$ described in Example 14.5.6. Hence, they span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. This proves (vi).

A *system of positive roots* for a Cartan subalgebra $\mathfrak{h}$ is a subset $\Delta^+ \subset \Delta$ such that: (i) Δ^+ is closed under addition, in the sense that if $\alpha,\beta \in \Delta^+$, then $\alpha + \beta \in \Delta^+$, provided $\alpha + \beta$ is a root; and (ii) for every root $\alpha \in \Delta$, exactly one of α, $-\alpha$ belongs to Δ^+. Such a system may be constructed by choosing a real hyperplane through 0 in $\mathfrak{h}^*$, which does not meet Δ, and then letting Δ^+ be the set of all roots on one side of this hyperplane. If Δ^+ is a system of positive roots, then so is its complement in Δ, and this is usually denoted Δ^-.

A subalgebra of a Lie algebra is called a *Borel subalgebra* if it is maximal solvable. In semisimple Lie algebras, positive root systems determine Borel subalgebras as follows:

Let Δ^+ be a system of positive roots, and set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-.$$

14.5.8 Proposition. *The subalgebra $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) as a nilpotent ideal.*

Proof. It follow from the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ that $\mathfrak{n}^+$ is nilpotent and is an ideal of $\mathfrak{b}^+$. Since $\mathfrak{h}$ is abelian, $[\mathfrak{b}^+, \mathfrak{b}^+] \subset \mathfrak{n}^+$. It follows that $\mathfrak{b}^+$ is solvable.

Suppose $\mathfrak{p}$ is any subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}^+$. If $\mathfrak{p} \neq \mathfrak{b}^+$, there exists a non-zero $x \in \mathfrak{p}$ which belongs to the direct sum of the subspaces $\mathfrak{g}_\alpha$ for $\alpha \in \Delta^-$. For any pair $\alpha, \beta \in \Delta^-$ with $\alpha \neq \beta$, there exists an $h \in \mathfrak{h}$ such that $\alpha(h) \neq \beta(h)$. This means that the operator $\mathrm{ad}_h - \beta(h)$ on $\mathfrak{g}$ kills $\mathfrak{g}_\beta$ but not $\mathfrak{g}_\alpha$. Each such operator leaves $\mathfrak{p}$ invariant, since $\mathfrak{p}$ is a subalgebra containing $\mathfrak{h}$. Using these operators, we may kill off each summand of x except the one in $\mathfrak{g}_\alpha$. We conclude that each of the summands of x belongs to $\mathfrak{p}$. Thus, if $\mathfrak{p} \neq \mathfrak{b}^+$, then there exists an $\alpha \in \Delta^-$ such that $\mathfrak{g}_\alpha \subset \mathfrak{p}$. Since $-\alpha \in \Delta^+$, $\mathfrak{g}_{-\alpha} \subset \mathfrak{p}$ also. However, by the previous proposition, the Lie subalgebra generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is a copy of $\mathfrak{sl}_2(\mathbb{C})$, which is semisimple (Exercise 14.16) and, hence, not solvable. It follows that $\mathfrak{p}$ also fails to be solvable. Consequently, $\mathfrak{b}^+$ is maximal solvable.

In Chapter 15 we will show that every Borel subalgebra of a complex semisimple Lie algebra arises in the above fashion from a Cartan subalgebra $\mathfrak{h}$ and system of positive roots Δ^+.

14.5.9 Example. Recall that $\mathfrak{gl}_n(\mathbb{C})$ is the Lie algebra of all $n \times n$ matrices and $\mathfrak{sl}_n(\mathbb{C})$ the Lie algebra of all $n \times n$ matrices with trace 0. Note that $\mathfrak{gl}_n(\mathbb{C})$ is not semisimple, because it has a non-trivial center (the constant matrices). However, $\mathfrak{sl}_n(\mathbb{C})$ is semisimple (Exercise 14.16). It has the space of diagonal matrices as a Cartan subalgebra. Each off diagonal elementary matrix spans a root space for this Cartan. Those which are upper triangular determine a system of positive roots Δ^+. The system of positive roots corresponding to the lower triangular elementary matrices is then Δ^-. The Borel subalgebras corresponding to this Cartan and these positive root systems are the subalgebras of upper and lower triangular matrices.

14.6 Representations of $\mathfrak{sl}_2(\mathbb{C})$

The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are easy to characterize, and they play a special role in the study of representations of general semisimple Lie algebras. We describe them below.

We first note that the group $SL_2(\mathbb{C})$ is simply connected. This follows from the polar decomposition for complex matrices. That is, each matrix $a \in GL_n(\mathbb{C})$ can be uniquely written in the form $a = ub$, where u is a unitary matrix and b is an invertible positive definite matrix. In fact, b is the positive definite square root of a^*a, and $u = ab^{-1}$. Here, a^* denotes the conjugate transpose of a. If a has determinant 1, then both u and b do as well. Furthermore, the set of invertible positive definite matrices of determinant 1 is the homeomorphic image under exp of the real vector subspace of $\mathfrak{sl}_n(\mathbb{C})$ consisting of hermitian matrices. Thus, as a topological space, $SL_n(\mathbb{C})$, for any n, is a Cartesian product of a vector space with the group SU_n of unitary matrices of determinant 1. In the case where $n = 2$, a simple calculation shows that SU_2 consists of matrices of the form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$$

with $|z|^2 + |w|^2 = 1$. Thus, SU_2 is a homeomorphic to a three sphere and is, therefore, simply connected. We conclude that $Sl_2(\mathbb{C})$ is also simply connected.

Since $SL_2(\mathbb{C})$ is simply connected and has $\mathfrak{sl}_2(\mathbb{C})$ as Lie algebra, every finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is the differential of a holomorphic representation of $SL_2(\mathbb{C})$. This has the following consequence:

14.6.1 Theorem. *Let $h \in sl_2(\mathbb{C})$ be the diagonal matrix of Example 14.5.6, and let (π, V) be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Then*

(i) *(π, V) decomposes as a direct sum of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$;*
(ii) *V has a basis consisting of eigenvectors for $\pi(h)$;*
(iii) *the eigenvalues of $\pi(h)$ are real.*

Proof. To prove (i), we will show that the corresponding representation of $SL_2(\mathbb{C})$ (also denoted by π) decomposes as a direct sum of irreducible representations. The key here is to use the fact that $SL_2(\mathbb{C})$ has the compact group SU_2 as a real form, and then use the results of section 14.2 on compact groups.

That SU_2 is a *real form* of $SL_2(\mathbb{C})$ means that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of the Lie algebra $\mathfrak{su}_2$ of SU_2. In fact, $\mathfrak{su}_2$ is the Lie algebra of skew-hermitian matrices in $\mathfrak{sl}_2(\mathbb{C})$ (matrices a with $a^* = -a$), and so clearly, $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2 \oplus i\mathfrak{su}_2$. The exponential map defines a holomorphic map of $\mathfrak{sl}_2(\mathbb{C})$ onto $SL_2(\mathbb{C})$ which takes $\mathfrak{su}_2$ onto SU_2. Suppose f is a holomorphic function on $SL_2(\mathbb{C})$ which vanishes on SU_2. Then $f \circ \exp$ is a holomorphic function on $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathbb{C}^3$ which vanishes on $\mathfrak{su}_2 \simeq \mathbb{R}^3$, and hence, vanishes identically. From this, it follows that any holomorphic function on $SL_2(\mathbb{C})$, which vanishes on SU_2, vanishes identically.

Using Proposition 14.2.2, we may choose an inner product on V for which the representation π is unitary when restricted to SU_2. Then Theorem 14.2.5 implies that V decomposes as a direct sum of irreducible invariant subspaces for the restriction of π to SU_2. However, a subspace $W \subset V$ is an invariant subspace for this representation if and only if $(\pi(u)w, v) = 0$ for every $w \in W$, $v \in W^\perp$, and $u \in SU_2$. Since $g \to (\pi(g)w, v)$ is a holomorphic function of $g \in SL_2(\mathbb{C})$, we conclude from the previous paragraph that, if it vanishes on SU_2, then it vanishes on $SL_2(\mathbb{C})$. It follows that every invariant subspace for π restricted to SU_2 is also an invariant subspace for π. Thus, V decomposes as a direct sum of invariant subspaces for the representation π. This proves (i).

To prove (ii), we restrict π to the subgroup T of SU_2 consisting of the diagonal unitary matrices of determinant 1. This is the group of matrices of the form $\exp(ith)$ for $t \in \mathbb{R}$ and is a copy of the unit circle. Since T is compact and abelian, the representation π, restricted to T, decomposes as a direct sum of 1-dimensional invariant subspaces. These subspaces are all invariant under $\pi(h)$ – that is, they are eigenspaces for $\pi(h)$. This proves (ii).

Since $\pi(\exp(ith)) = \exp(it\pi(h))$ is unitary for each real number t, the operator $\pi(h)$ is self-adjoint (hermitian) and, hence, has real eigenvalues. This proves (iii).

14.6.2 Corollary. *If $\mathfrak{h}$ is a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, and (π, V) is a finite dimensional representation of $\mathfrak{g}$, then V has a basis consisting of common eigenvectors for the elements of $\mathfrak{h}$.*

Proof. By Theorem 14.5.7, $\mathfrak{h}$ is spanned by elements h_α, each of which is the h in a standard basis for a copy of $\mathfrak{sl}_2(\mathbb{C})$ which is a subalgebra of $\mathfrak{g}$. It follows from Theorem 14.6.1 that each $\pi(h_\alpha)$ acts semisimply on V. Since the elements $\pi(h_\alpha)$ are mutually commuting, there is a basis of V consisting of common eigenvectors of the $\pi(h_\alpha)$. These are then common eigenvectors for all elements of $\mathfrak{h}$.

14.6.3 Corollary. *If s is a semisimple element of a complex semisimple Lie algebra $\mathfrak{g}$, and (π, V) is any finite dimensional representation of $\mathfrak{g}$, then $\pi(s)$ is also semisimple.*

Proof. Since s is semisimple, it belongs to a maximal subalgebra of $\mathfrak{g}$ which consists of semisimple elements – that is, it belongs to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Thus, the result follows from the previous corollary.

Our next objective is to construct a class of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ – a class that, it turns out, contains an isomorphic copy of each irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

Consider the differential operators

$$\begin{aligned}\eta &= z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w},\\ \xi^+ &= z\frac{\partial}{\partial w},\\ \xi^- &= w\frac{\partial}{\partial z}\end{aligned}$$

acting on the space of polynomials in z and w. It is easy to check that these three operators satisfy the same commutation relations as the generators of $\mathfrak{sl}_2(\mathbb{C})$ in Example 14.5.6. Thus, they span a copy of $\mathfrak{sl}_2(\mathbb{C})$ and determine a representation σ of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space $\mathbb{C}[z,w]$. The homogeneous polynomials of a given degree form an invariant subspace for this representation. Thus, $(\sigma, \mathbb{C}[z,w])$ decomposes as a direct sum of representations σ_n, where σ_n is σ restricted to the space $V(n)$ of homogeneous polynomials of degree n in $\mathbb{C}[z,w]$.

We continue to denote the standard basis elements for $\mathfrak{sl}_2(\mathbb{C})$ by h, x, and y, as in Example 14.5.6. Then for the representation σ on $\mathbb{C}[z,w]$, described above, we have $\sigma(h) = \eta$, $\sigma(x) = \xi^+$, and $\sigma(y) = \xi^-$.

In keeping with terminology that will be introduced in the next section, if (σ, V) is any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, the eigenvalues of $\sigma(h)$ will be called the *weights* of the representation σ. The corresponding eigenspaces of $\sigma(h)$ in V will be called the *weight spaces* of $\sigma(h)$. Note that, by Theorem 14.6.1(iii), the weights are real numbers. Hence, there is a largest one, and we will call it the *highest weight*.

14.6.4 Proposition. *With the representations $(\sigma_n, V(n))$ of $\mathfrak{sl}_2(\mathbb{C})$ as defined above,*

(i) *the weight spaces of $\sigma_n(h)$ are the 1-dimensional spaces spanned by the vectors $z^{n-k}w^k$ for $k = 0, \ldots, n$;*

(ii) *the weight corresponding to the weight space spanned by* $z^{n-k}w^k$ *is* $n-2k$, *and so the highest weight is* n;

(iii) *each* $(\sigma_n, V(n))$ *is irreducible;*

(iv) *each finite dimensional irreducible representation of* $\mathfrak{sl}_2(\mathbb{R})$ *is isomorphic to one of the representations* σ_n.

Proof. Parts (i) and (ii) follow by direct computation. Note also that any non-zero linear combination of the vectors $z^{n-k}w^k$ can be reduced to a non-zero multiple of z^n, by applying some power of ξ^+, and each of the vectors $z^{n-k}w^k$ is a scalar multiple of a power of ξ^- applied to z^n. It follows that we can get from any non-zero element of $V(n)$ to any other by applying an operator in the algebra generated by η, ξ^+, and ξ^-. This implies that σ_n is irreducible and proves (iii).

Now let (π, V) be any irreducible finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. By Theorem 14.6.1, V has a basis consisting of weight vectors. Let the highest weight for this representation be λ. Let v_0 be a weight vector of weight λ, and set $v_k = \pi(y)^k v_0$. Then we claim

$$
\begin{aligned}
\pi(y)v_k &= v_{k+1}, \\
\pi(h)v_k &= (\lambda - 2k)v_k, \qquad (14.6.1)\\
\pi(x)v_k &= k(\lambda - k + 1)v_{k-1}.
\end{aligned}
$$

The first of these equations is immediate from the definition of v_k. The second follows by induction, using the commutation relation $[h, y] = -2y$. To prove the third, we use the relation $[x, y] = h$, which implies that

$$
\pi(x)v_k = \pi(x)\pi(y)v_{k-1} = (\pi(h) + \pi(y)\pi(x))v_{k-1}. \qquad (14.6.2)
$$

If we set $a_0 = 0$ and define a_k, for $k \geq 1$, so that $\pi(x)v_k = a_k v_{k-1}$, then (14.6.2) and the first two equations of (14.6.1) imply that, for $k \geq 1$,

$$
a_k = \lambda - 2(k-1) + a_{k-1}.
$$

Since $a_0 = 0$, an induction argument, using this recursion formula, establishes the third equation of (14.6.1).

Since the representation (π, V) is finite dimensional, only finitely many of the v_k can be non-zero. This means there is a largest non-negative integer n for which $v_n \neq 0$. Then $0 = \pi(x)v_{n+1} = (n+1)(\lambda - n)v_n$, by (14.6.1), and this implies that $\lambda = n$. However, with $\lambda = n$, equations (14.6.1) also hold in $V(n)$ with π replaced by σ_n and each vector v_k replaced by the vector $\dfrac{n!}{(n-k)!} z^{n-k}w^k$. It follows that this correspondence between basis vectors defines an isomorphism from the representation (π, V) to the representation $(\sigma_n, V(n))$. This proves (iv).

14.7 Representations of Semisimple Lie Algebras

In this section, $\mathfrak{g}$ will denote a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Δ its system of roots. We will study the structure of finite dimensional representations of $\mathfrak{g}$ and characterize the irreducible finite dimensional representations.

From Corollary 14.6.2 it follows that if (π, V) is a finite dimensional representation of $\mathfrak{g}$, then V is a direct sum of subspaces V_λ, where $\lambda \in \mathfrak{h}^*$, and

$$V_\lambda = \{v \in V : \pi(h)v = \lambda(h)v, \ \forall\, h \in \mathfrak{h}\}.$$

The elements $\lambda \in \mathfrak{h}^*$ for which $V_\lambda \neq 0$ are called the *weights* of the representation π. For each weight λ, the space V_λ is called the *weight space* with weight λ. Since the representation is finite dimensional, there can be only finitely many weights. How does the rest of the Lie algebra $\mathfrak{g}$ act on V? We get an idea by using the root space decomposition of $\mathfrak{g}$ described in the previous section. In fact, a trivial computation yields the following:

14.7.1 Proposition. *Let (π, V) be a finite dimensional representation of $\mathfrak{g}$, and for each weight λ of (π, V), let V_λ be the corresponding weight space. Then*

$$\pi(x)V_\lambda \subset V_{\lambda+\alpha}$$

for each $x \in \mathfrak{g}_\alpha$ and each root α.

If the representation (π, V) is irreducible, then each weight space V_λ must generate V under the action of $\mathfrak{g}$. In view of the preceding proposition, this implies that one must be able to obtain all the weights from a given one by successively adding roots.

14.7.2 Definition. *For a finite dimensional representation (π, V), a positive root system $\Delta^+ \subset \Delta$, and weights λ and μ, we say $\mu < \lambda$ if $\lambda - \mu$ is a sum of positive roots. A weight is called a highest weight if it is maximal relative to this relation. If a weight λ has the property that, for each root α, either $\lambda + \alpha$ or $\lambda - \alpha$ is not a weight for (π, V), then λ will be called an extremal weight.*

If λ is a highest weight, then λ is an extremal weight, since $\lambda + \alpha$ fails to be a weight for every positive root α. We also have $\pi(\mathfrak{g}_\alpha)V_\lambda = 0$ for every $\alpha \in \Delta^+$, since $V_{\lambda+\alpha} = 0$.

Let (π, V) be an irreducible representation of $\mathfrak{g}$ with λ as a highest weight, and let W be the subspace of V consisting of all vectors killed by $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta^+$. Then W contains V_λ and is invariant under $\mathfrak{h}$. Hence, it decomposes as a direct sum of weight spaces of the form $W \cap V_\mu$. Let μ be a minimal weight for which $W \cap V_\mu \neq 0$. It follows from the commutation relations in $\mathfrak{g}$ that the $\mathfrak{g}$-invariant subspace of V generated by any non-zero vector $v \in W \cap V_\mu$ is contained in the span of v and the spaces V_ν for $\nu < \mu$. Since the representation is irreducible, we conclude that $\mu = \lambda$, $V_\lambda = W$, and V_λ is 1-dimensional. Thus, we have proved the following theorem:

14.7.3 Theorem. *Given an irreducible finite dimensional representation (π, V) of $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, and a system of positive roots Δ^+, there is a unique highest weight $\lambda \in \mathfrak{h}^*$. Furthermore, the corresponding weight space V_λ is 1-dimensional and is equal to the set of vectors in V that are killed by $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta^+$.*

By Theorem 14.5.7(vi), the Lie algebra $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2(\mathbb{C})$ for each root α. We gain considerable information about a representation of $\mathfrak{g}$ by restricting it to each of these copies of $\mathfrak{sl}_2(\mathbb{C})$ and exploiting what we know about representations of $\mathfrak{sl}_2(\mathbb{C})$. This is the strategy in the next proposition. Recall that, for a root α, $t_\alpha \in \mathfrak{h}$ is chosen so that

$$\beta(t_\alpha) = \langle t_\beta, t_\alpha \rangle = \langle \beta, \alpha \rangle$$

for each root β. Also,

$$h_\alpha = \frac{2t_\alpha}{\langle \alpha, \alpha \rangle} \tag{14.7.1}$$

is the basis element in a Cartan in a standard basis for the copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

14.7.4 Proposition. *Let (π, V) be a finite dimensional representation of $\mathfrak{g}$. Let α be a root of $\mathfrak{g}$ and λ a weight of (π, V). Then*

$$\lambda(h_\alpha) = \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer.

Proof. By restricting the representation π to the copy of $\mathrm{sl}_2(\mathbb{C})$ in $\mathfrak{g}$ spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, we express V as a module over $\mathfrak{sl}_2(\mathbb{C})$. If λ is a weight for the representation π, then its restriction to $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is also a weight for this $\mathfrak{sl}_2(\mathbb{C})$-module. By Theorem 14.6.1, as a $\mathfrak{sl}_2(\mathbb{C})$ module, V is a direct sum of irreducibles, and by Proposition 14.6.4, the element h from a standard basis for $\mathfrak{sl}_2(\mathbb{C})$ acts as an integer on each weight space of each irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. It follows that $\lambda(h_\alpha)$ is an integer. That this integer is also equal to $\dfrac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ follows from (14.7.1).

14.7.5 Corollary. *For roots $\alpha, \beta \in \Delta$,*

(i) $\beta(h_\alpha) = \dfrac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ *is an integer;*

(ii) $\langle \alpha, \alpha \rangle$ *is a positive rational number;*

(iii) $\langle \alpha, \beta \rangle$ *is a rational number.*

Proof. Part (i) is just the previous proposition in the case where π is the adjoint representation.

There is a basis for $\mathfrak{g}$ consisting of an element from each $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$, together with a basis for $\mathfrak{h}$. Relative to this basis, ad_h for $h \in \mathfrak{h}$ acts on $\mathfrak{g}$ as a diagonal matrix with eigenvalues the numbers 0 and $\beta(h)$ for $\beta \in \Delta$. Thus,

$$\langle h, h \rangle = \mathrm{tr}(\mathrm{ad}_h \, \mathrm{ad}_h) = \sum_{\beta \in \Delta} \beta(h)^2.$$

If $h = h_\alpha$, this, along with (i), implies that $\langle h_\alpha, h_\alpha \rangle$ is a positive integer. However, it follows from (14.7.1) that $\langle \alpha, \alpha \rangle = 4\langle h_\alpha, h_\alpha \rangle^{-1}$. Hence, $\langle \alpha, \alpha \rangle$ is a positive rational number. This proves (ii). Part (iii) follows from (i) and (ii).

Let $\mathfrak{h}'$ be the real subspace of $\mathfrak{h}^*$ spanned by the roots. It follows from Corollary 14.7.5 that $\mathfrak{h}'$ is the real subspace of $\mathfrak{h}^*$ consisting of those elements λ for which $\langle \lambda, \alpha \rangle$ is real for every $\alpha \in \Delta^+$. It also follows from Corollary 14.7.5 that the Killing form is positive definite on $\mathfrak{h}'$, and hence, defines an inner product on this real vector space. Thus, we may define a Euclidean norm on $\mathfrak{h}'$ by setting

$$||\lambda|| = \sqrt{\langle \lambda, \lambda \rangle}.$$

Each $\alpha \in \Delta^+$ defines a hyperplane $\{\mu \in \mathfrak{h}' : \langle \mu, \alpha \rangle = 0\}$ in $\mathfrak{h}'$. The complement in $\mathfrak{h}'$ of the union of these hyperplanes has finitely many components. These are open sets called *Weyl chambers.* That is, a Weyl chamber is a non-empty subset of $\mathfrak{h}'$ of the form

$$\{\lambda \in \mathfrak{h}' : \epsilon(\alpha)\langle \lambda, \alpha \rangle > 0, \ \forall \, \alpha \in \Delta^+\},$$

where ϵ is a function from Δ^+ to $\{1, -1\}$ (not every such function defines a non-empty set). The positive Weyl chamber is the one defined by choosing $\epsilon(\alpha) = 1$ for every α.

For $\alpha \in \Delta^+$, the operator s_α on $\mathfrak{h}'$, defined by

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

is reflection through the hyperplane determined by α. We may also consider s_α to be a complex linear transformation of $\mathfrak{h}^*$, defined by the same formula as above, but with $\lambda \in \mathfrak{h}^*$.

14.7.6 Definition. *The Weyl group W for $\mathfrak{g}$ is the group of transformations of $\mathfrak{h}'$ generated by the set of reflections of the form s_α.*

Since it is reflection through a hyperplane, each s_α is an isometry for the metric determined by the Killing form. Hence, the Weyl group is a group of isometries of $\mathfrak{h}'$.

It is clear that the Weyl chambers are in one to one correspondence with the possible choices of positive root systems in Δ. That is, for each Weyl chamber C there is exactly one positive root system $\Delta^+ \subset \Delta$ for which C is the positive Weyl chamber. It is also easy to see that the Weyl group acts transitively on the set of Weyl chambers.

We set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

This element plays a special role throughout the theory of semisimple Lie algebras.

We now return to the study of irreducible representations of $\mathfrak{g}$ with a theorem which lists the key properties satisfied by the system of weights for such a representation.

14.7.7 Theorem. *Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, Δ^+ a positive root system, $\Lambda \subset \mathfrak{h}^*$ the set of weights for (π, V), and $\lambda \in \Lambda$ the highest weight. Then*

(i) *Λ is closed under the action of the Weyl group;*
(ii) *if $\alpha \in \Delta^+$, then $\langle \lambda, \alpha \rangle \geq 0$, and $\langle \rho, \alpha \rangle > 0$;*
(iii) $||\mu|| \leq ||\lambda|| \quad \forall\, \mu \in \Lambda$;
(iv) *$\mu \in \Lambda$ is extremal if and only if $||\mu|| = ||\lambda||$, and the Weyl group acts transitively on the set of extremal weights;*
(v) *$||\mu + \rho|| < ||\lambda + \rho||$ for every $\mu \in \Lambda$ distinct from λ.*

Proof. If $\mu \in \Lambda$, $\alpha \in \Delta$, then the weight spaces $V_{\mu - n\alpha}$, for n an integer, span a subspace of V that is invariant under the action of the copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. The set of weights of a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is symmetric about the origin, since this is true of irreducible representations, by Proposition 14.6.4, and every finite dimensional representation is a sum of irreducibles, by Theorem 14.6.1. It follows that there is an integer m so that $\mu - m\alpha \in \Lambda$ and $\mu(h_\alpha) - m\alpha(h_\alpha) = -\mu(h_\alpha)$. Since $\alpha(h_\alpha) = 2$, this implies that

$$m = \mu(h_\alpha) = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle},$$

which implies that $\mu - m\alpha = s_\alpha(\mu)$. Hence, $s_\alpha(\mu) \in \Lambda$. This proves (i).

The first inequality of part (ii) also follows from the above argument, applied to the case where $\mu = \lambda$, since then only non-negative integers n yield non-zero subspaces $V_{\lambda - n\alpha}$. Thus, the integer m above must be non-negative in this case, which implies that $\langle \lambda, \alpha \rangle \geq 0$. The proof of the second inequality in part (ii) is more complicated. It requires a development of the properties of systems of simple roots. We leave this development and the resulting proof of the fact that ρ is in the positive chamber to the exercises (Exercises 14.17 – 14.22).

If μ is any weight in Λ, it has the form $\mu = \lambda - \nu$, where ν is a sum of positive roots. If μ also satisfies $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$, then we have

$$\langle \mu, \mu \rangle = \langle \lambda, \lambda \rangle - \langle \nu, \lambda \rangle - \langle \mu, \nu \rangle \leq \langle \lambda, \lambda \rangle.$$

Thus, $||\mu|| \leq ||\lambda||$ for every weight μ in the closure of the positive chamber. However, this implies $||\mu|| \leq ||\lambda||$ for all $\mu \in \Lambda$, since every such μ may be brought into the closure of the positive chamber by applying a Weyl group transformation, and Weyl group transformations are isometries. This completes the proof of part (iii);

Suppose that $\mu \in \Lambda$ is not an extremal weight. Then there is a root α such that $\mu + \alpha$ and $\mu - \alpha$ are both roots. Then

$$||\mu||^2 + ||\alpha||^2 = \frac{1}{2}(||\mu + \alpha||^2 + ||\mu - \alpha||^2) \leq ||\lambda||^2,$$

by part (iii), and this implies that $||\mu|| < ||\lambda||$. Thus, only the extremal weights can have norm equal to $||\lambda||$. The proof of part (iv) will be complete if we can show that the Weyl group acts transitively on the set of extremal weights, since this will imply that they all have norm equal to that of λ.

Using a Weyl group transformation, we can move any extremal weight into the closure of the positive chamber and it will still be an extremal weight. Thus, we will have completed the proof of (iv) if we show that λ is the only extremal weight in the closure of the positive chamber. Suppose that μ is extremal and lies in the closure of the positive chamber. Then, for each $\alpha \in \Delta^+$, we have $\langle \mu, \alpha \rangle \geq 0$, and either $\mu + \alpha$ or $\mu - \alpha$ is not a weight. However, the $\mathfrak{sl}_2(\mathbb{C})$ argument of the first paragraph shows that it must be $\mu + \alpha$ that fails to be a weight if $\langle \mu, \alpha \rangle \geq 0$. Hence, μ is the highest weight λ, and the proof of part(iv) is complete.

If μ is any weight in Λ, then

$$\begin{aligned}\langle \mu + \rho, \mu + \rho \rangle &= \langle \mu, \mu \rangle + 2\langle \mu, \rho \rangle + \langle \rho, \rho \rangle \leq \langle \lambda, \lambda \rangle + 2\langle \mu, \rho \rangle + \langle \rho, \rho \rangle \\ &= \langle \lambda + \rho, \lambda + \rho \rangle - 2\langle \lambda - \mu, \rho \rangle < \langle \lambda + \rho, \lambda + \rho \rangle\end{aligned}$$

by part (ii) and the fact that $\lambda - \mu$ is a sum of positive roots. This proves part (v).

The elements of $\mathfrak{h}^*$ that satisfy the integrality condition of Proposition 14.7.4 are called *integral weights.* Thus, only the integral weights of $\mathfrak{h}^*$ can be weights of finite dimensional representations of $\mathfrak{g}$.

14.7.8 Definition. *We say that a weight $\lambda \in \mathfrak{h}'$ is dominant relative to a system of positive roots if $\langle \alpha, \lambda + \rho \rangle > 0$ for every positive root α. Thus, a weight λ is dominant if and only if $\lambda + \rho$ belongs to the positive Weyl chamber.*

We know that, for every finite dimensional irreducible representation, there is a unique highest weight, and it is easy to see from part (ii) of Theorem 14.7.7 that highest weights are dominant. In fact, the finite dimensional irreducible representations of a complex semisimple Lie algebra are classified by their highest weights. This statement is made precise by the following theorem which we will not prove here (see [**Hum**], section 21.2).

14.7.9 Theorem. *If $\mathfrak{g}$ is a semisimple Lie algebra, then each of its dominant integral weights λ is the highest weight for a unique (up to isomorphism) finite dimensional irreducible representation of $\mathfrak{g}$.*

14.8 Compact Semisimple Groups

A Lie group is called semisimple if it has a semisimple Lie algebra. Each connected complex semisimple Lie group G has a compact real form K and the category of finite dimensional holomorphic representations of G is equivalent to the category of all finite dimensional representations of K. This powerful result, noted and used extensively by Weyl, allows one to apply the Peter-Weyl theorem to the study of finite dimensional holomorphic representations of complex semisimple groups. In this section, we discuss the existence of the compact real form K, although we will not give a complete proof of it. For this we refer the reader to [**V**]. In the next chapter we will establish the equivalence between finite dimensional holomorphic representations of G and finite dimensional representations of K.

We begin with a Lie algebra characterization of compact semisimple Lie groups. A real Lie algebra $\mathfrak{k}$ is called *compact* if its Killing form is negative definite, that is, if $\langle k, k \rangle < 0$ for every non-zero $k \in \mathfrak{k}$. Since a negative definite form is non-singular, every compact Lie algebra is semisimple. The term *compact* comes from the fact that a connected Lie group with a compact Lie algebra is necessarily a compact group. We will prove this below. However, first we need a result which characterizes the image of a semisimple group under the adjoint representation.

Given a semisimple Lie algebra $\mathfrak{g}$, it is easy to describe a Lie group with $\mathfrak{g}$ as Lie algebra. Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of all automorphisms of $\mathfrak{g}$ –

that is, all operators a on $\mathfrak{g}$ such that

$$a[x, y] = [ax, ay], \ \forall\, x, y \in \mathfrak{g}.$$

Let $\mathrm{Aut}^0(\mathfrak{g})$ denote the identity component of $\mathrm{Aut}(\mathfrak{g})$. If G is any Lie group with Lie algebra $\mathfrak{g}$, then the image of G under Ad clearly lies in $\mathrm{Aut}(\mathfrak{g})$. If G is connected, then this image lies in $\mathrm{Aut}^0(\mathfrak{g})$. Furthermore we have:

14.8.1 Proposition. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*
(i) $\mathrm{Aut}(\mathfrak{g})$ *is a Lie group with Lie algebra* $\mathrm{Der}(\mathfrak{g}) \simeq \mathfrak{g}$;
(ii) *if G is any connected Lie group with Lie algebra $\mathfrak{g}$, then the adjoint map* $\mathrm{Ad} : G \to \mathrm{Aut}^0(\mathfrak{g})$ *is a covering map.*

Proof. In view of Propositions 14.3.7 and 14.5.1, to prove (i) we need to show that, for a linear transformation δ of $\mathfrak{g}$, $\mathrm{e}^{t\delta}$ is an automorphism for each real number t if and only if δ is a derivation. First, suppose that $\mathrm{e}^{t\delta}$ is an automorphism for all t. Then

$$\begin{aligned}\delta[x, y] &= \frac{d}{dt}\Big|_{t=0} \mathrm{e}^{t\delta}[x, y] \\ &= \frac{d}{dt}\Big|_{t=0}[\mathrm{e}^{t\delta}\, x, \mathrm{e}^{t\delta}\, y] \\ &= [\delta x, y] + [x, \delta y],\end{aligned}$$

where the last equality follows from the chain rule. Thus, $\delta \in \mathrm{Der}(\mathfrak{g})$.

On the other hand, if we assume that $\delta \in \mathrm{Der}(\mathfrak{g})$, then

$$\begin{aligned}\frac{d}{dt}[\mathrm{e}^{t\delta}\, x, \mathrm{e}^{t\delta}\, y] &= [\delta\, \mathrm{e}^{t\delta}\, x, \mathrm{e}^{t\delta}\, y] + [\mathrm{e}^{t\delta}\, x, \delta\, \mathrm{e}^{t\delta}\, y] \\ &= \delta[\mathrm{e}^{t\delta}\, x, \mathrm{e}^{t\delta}\, y]\end{aligned}$$

for $t \in \mathbb{R}$ and $x, y \in \mathfrak{g}$. That is, the $\mathfrak{g}$-valued function $\phi(t) = [\mathrm{e}^{t\delta}\, x, \mathrm{e}^{t\delta}\, y]$ satisfies the differential equation $\phi'(t) = \delta\phi(t)$. Since this is also satisfied by the function $t \to \mathrm{e}^{t\delta}[x, y]$, and since these two functions have the same initial value $[x, y]$, we conclude that they are the same function. Hence, $\mathrm{e}^{t\delta} \in \mathrm{Aut}(\mathfrak{g})$ for all t. This completes the proof that the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$. That this is isomorphic to $\mathfrak{g}$ is Proposition 14.5.1. Recall that the isomorphism is given by $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$.

Part (ii) follows immediately, since (i) implies that the differential ad of the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}^0(\mathfrak{g})$ is an isomorphism.

14.8.2 Proposition. *The real semisimple Lie algebra $\mathfrak{k}$ is compact if and only if the Lie group* $\mathrm{Aut}^0(\mathfrak{k})$*, described above, is compact.*

Proof. The group $\mathrm{Aut}^0(\mathfrak{k})$ is clearly a closed subgroup of $GL(\mathfrak{k})$. If the Killing form is negative definite on $\mathfrak{k}$, then its negative is a positive definite form on $\mathfrak{k}$. The identity

$$\langle \mathrm{ad}_x(y), z\rangle = -\langle y, \mathrm{ad}_x(z)\rangle$$

implies that each ad_x, for $x \in \mathfrak{k}$, is a skew-adjoint operator relative to this positive definite form, and hence, its exponential belongs to the corresponding orthogonal group. It follows that $\mathrm{Aut}^0(\mathfrak{k})$ is contained in the orthogonal group for the vector space $\mathfrak{k}$ and the positive definite form $-\langle\cdot,\cdot\rangle$. Since it is a closed subgroup of the orthogonal group, it is compact.

On the other hand, if $\mathrm{Aut}^0(\mathfrak{k})$ is compact, then by Proposition 14.2.2, there is an inner product on the complexification of $\mathfrak{k}$ such that $\mathrm{Aut}^0(\mathfrak{k})$ consists of unitary operators. Thus, for each $x \in \mathfrak{k}$, the 1-parameter group $t \to \exp(t\,\mathrm{ad}_x)$ lies in the unitary group. This can happen only if the operator ad_x is semisimple with purely imaginary eigenvalues. If $x \neq 0$ these eigenvalues are not all 0. Then $\langle x, x\rangle = \mathrm{tr}(\mathrm{ad}_x^2) < 0$. Thus, the Killing form is negative definite in this case.

It remains to prove that every connected Lie group, with compact Lie algebra $\mathfrak{k}$, is a compact group. To show this, we will prove that the simply connected covering group of a compact semisimple Lie group is compact. The proof is based on the following lemma, which uses Haar measure to show that a certain kind of cocycle on a compact group is also a coboundary.

14.8.3 Lemma. *Let K be a compact group, and let f be a continuous real valued function on $K \times K$ which satisfies the cocycle condition*

$$f(k_1, k_2) - f(k_1, k_2k_3) + f(k_1k_2, k_3) - f(k_2, k_3) = 0,$$

for all $k_1, k_2, k_3 \in K$. Then f is a coboundary, in the sense that there exists a continuous real valued function h such that

$$f(k_1, k_2) = h(k_1) - h(k_1k_2) + h(k_2),$$

for all $k_1, k_2 \in K$.

Proof. This is another result that follows from the "average over K" trick. We simply set

$$h(k) = \int f(k, k_1)\, d\mu(k_1),$$

where μ is normalized Haar measure on K. Then

$$\begin{aligned} f(k_1, k_2) &= \int f(k_1, k_2)\, d\mu(k_3) \\ &= \int [f(k_1, k_2k_3) - f(k_1k_2, k_3) + f(k_2, k_3)]\, d\mu(k_3) \\ &= h(k_1) - h(k_1k_2) + h(k_2), \end{aligned}$$

where the last equality uses the left invariance of μ and the definition of h.

14.8.4 Proposition. *Let G be a locally compact group for which there is no non-trivial continuous homomorphism $G \to \mathbb{R}$. If C is a discrete, finitely generated central subgroup of G, for which G/C is compact, then C is finite, and G is compact.*

Proof. Since C is a finitely generated abelian group, it is the product of a finite group and $\mathbb{Z}^k$ for some non-negative integer k. If C is not compact, then $k \neq 0$, and there exists a non-trivial homomorphism of C to $\mathbb{Z}$, and, hence a non-trivial homomorphism $\phi : C \to \mathbb{R}$. To complete the proof, we will show that any such ϕ extends to a non-trivial homomorphism of G to $\mathbb{R}$.

Let G be covered by open sets with compact closure. The images of these sets in G/C form an open cover of G/C, and so finitely many of them cover G/C. If U is the union of the corresponding finitely many open subsets of G, then U is an open set with compact closure such that $C \cdot U = G$. With $\phi : C \to \mathbb{Z}$ as above, let ψ be a continuous non-negative real valued function, with compact support on G, which is positive on U. Since C is discrete, and ψ has compact support K, $C \cap K$ is finite and the sum

$$u(g) = \sum_{c \in C} \psi(cg)\, e^{\phi(c)}$$

is a finite sum in a neighborhood of each point of G, and hence, defines a continuous function on G. This function is everywhere positive, since ψ is positive on U, and $C \cdot U = G$.

Note that

$$u(cg) = e^{\phi(c)}\, u(g), \ \forall\, c \in C,\ g \in G, \tag{14.8.1}$$

and this implies that the function $\log(u(g_1 g_2)) - \log(u(g_1)) - \log(u(g_2))$ on $G \times G$ is constant on cosets of C in both variables. It therefore defines a function f on $G/C \times G/C$. Also, the function f satisfies the cocycle condition of the previous lemma. Since G/C is compact, we may apply the conclusion of that lemma to assert there is a continuous function h on G/C such that

$$f(\bar{g}_1, \bar{g}_2) = h(\bar{g}_1) - h(\bar{g}_1 \bar{g}_2) + h(\bar{g}_2)$$

for all $g_1, g_2 \in G$, where $\bar{g}$ denotes the image in G/C of an element $g \in G$. Then the function $\tilde{\phi}$, defined by $\tilde{\phi}(g) = \log(u(g)) + h(\bar{g})$, is a homomorphism from G to $\mathbb{R}$. On C, it is ϕ plus a constant by (14.8.1). However, since both ϕ and $\tilde{\phi}$ are homomorphisms, they must agree on C. This completes the proof.

14.8.5 Theorem. *If K is a connected real semisimple Lie group, then K is compact if and only if its Lie algebra is compact.*

Proof. If K has a compact Lie algebra $\mathfrak{k}$, then $\mathrm{Aut}^0(\mathfrak{k})$ is compact and has K as a covering group, by Propositions 14.8.1 and 14.8.2. Then the kernel C of $K \to \mathrm{Aut}^0(\mathfrak{k})$ is central, discrete, and finitely generated (Exercises 14.8 and 14.9). Also, $\mathrm{Aut}^0(\mathfrak{k})$ is a Lie group with the same Lie algebra as K. If it has a non-trivial continuous homomorphism to $\mathbb{R}$, then this homomorphism is analytic and its differential is a morphism of the Lie algebra $\mathfrak{k}$ to the Lie algebra $\mathbb{R}$. However. a semisimple Lie algebra has no such homomorphisms. Hence, $\mathrm{Aut}^0(\mathfrak{k})$ has no non-trivial homomorphism to $\mathbb{R}$. By the previous proposition, C is finite and K is also compact.

On the other hand, if K is compact, then $\mathrm{Aut}^0(\mathfrak{k})$, as a quotient of K, is also compact, and thus, $\mathfrak{k}$ is compact by Proposition 14.8.2.

If $\mathfrak{g}$ is a complex Lie algebra, and $\mathfrak{k}$ is a real Lie subalgebra such that $\mathfrak{k} \cap i\mathfrak{k} = (0)$, and $\mathfrak{k} + i\mathfrak{k} = \mathfrak{g}$, then it is easy to see that $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{k}$. In this case, we say that $\mathfrak{k}$ is a *real form* of $\mathfrak{g}$. If G is a connected complex Lie group and $K \subset G$ is a real Lie subgroup, then we say K is a real form of G if the Lie algebra of K is a real form of the Lie algebra of G. Every complex semisimple Lie group has a compact real form – that is, a real form which is a compact subgroup. The first step in proving this is to prove that every complex semisimple Lie algebra has a compact real form. We will not give a complete proof of this result, but we will give a brief discussion of the strategy of the proof. For a complete proof, see [**V**], section 4.11.

The proof is based on the construction of a *Weyl system* for a complex semisimple Lie algebra $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ and a system of positive roots Δ^+. For each $\alpha \in \Delta^+$, we choose non-zero elements $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ in such a way that

$$\langle x_\alpha, x_{-\alpha} \rangle = 1. \tag{14.8.2}$$

We then define numbers $N_{\alpha,\beta}$, for $\alpha, \beta \in \Delta$ by the equations

$$\begin{aligned} [x_\alpha, x_\beta] &= N_{\alpha,\beta} x_{\alpha+\beta} \quad \text{if} \quad \alpha+\beta \in \Delta, \\ N_{\alpha,\beta} &= 0 \quad \text{if} \quad \alpha+\beta \notin \Delta. \end{aligned} \tag{14.8.3}$$

There are a number of identities that are satisfied by the numbers $N_{\alpha,\beta}$, and these identities make it possible to prove the following: Let $\tilde{\mathfrak{g}}$ be another semisimple Lie algebra, $\tilde{\mathfrak{h}}$ a Cartan subalgebra of $\tilde{\mathfrak{g}}$, and $\tilde{\Delta} \subset \tilde{h}^*$ the corresponding system of roots. Suppose, $h \to \tilde{h}$ is a linear isomorphism of $\mathfrak{h}$ onto $\tilde{\mathfrak{h}}$ such that its adjoint maps $\tilde{\Delta}$ onto Δ. Let $\alpha \to \tilde{\alpha} : \Delta \to \tilde{\Delta}$ be the inverse of this bijection. Then the adjoint of $h \to \tilde{h}$ preserves the Killing form. Furthermore, it is possible to choose elements $\tilde{x}_{\tilde{\alpha}} \in \tilde{g}_{\tilde{\alpha}}$ for each $\tilde{\alpha} \in \tilde{\Delta}$ in such a way that

$$[x_{\tilde{\alpha}}, x_{\tilde{\beta}}] = N_{\alpha,\beta} x_{\tilde{\alpha}+\tilde{\beta}}.$$

This, in turn, implies that the isomorphism $h \to \tilde{h}$, and the map $x_\alpha \to x_{\tilde{\alpha}}$, define a Lie algebra isomorphism of $\mathfrak{g}$ onto $\tilde{\mathfrak{g}}$. This leads to the following theorem (for details see [**V**]):

14.8.6 Theorem. *Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be complex semisimple Lie algebras with Cartan subgroups $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$. Then each linear isomorphism $\mathfrak{h} \to \tilde{\mathfrak{h}}$, whose adjoint maps the set of roots for $\tilde{\mathfrak{g}}$ onto the set of roots for $\mathfrak{g}$, extends to a Lie algebra isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$. In particular, each automorphism of $\mathfrak{h}$, whose adjoint permutes the set of roots Δ, extends to a Lie algebra automorphism of $\mathfrak{g}$.*

If ϕ and ψ are two isomorphisms from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$, which extend the isomorphism $h \to \tilde{h}$, as above, then $\psi^{-1} \circ \phi$ is an automorphism of $\mathfrak{g}$ which fixes each element of $\mathfrak{h}$. These automorphisms are characterized in Exercise 14.24 as the automorphisms of the form $\mathrm{Ad}_{\exp(h)}$ for $h \in \mathfrak{h}$. Thus, the isomorphism ϕ of Theorem 14.8.6 is unique up to composition with $\mathrm{Ad}_{\exp(h)}$ for some $h \in \mathfrak{h}$.

Since each Weyl group element permutes the roots and (considered as a transformation of $\mathfrak{h}^*$) is the adjoint of a linear isomorphism of $\mathfrak{h}$, we have:

14.8.7 Corollary. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then every element of the Weyl group of $\mathfrak{g}$ is the adjoint of an automorphism of $\mathfrak{h}$ which extends to be an automorphism of $\mathfrak{g}$.*

The map $h \to -h$ is an automorphism of $\mathfrak{h}$ whose adjoint $\alpha \to -\alpha$ permutes the roots (Theorem 14.5.7(i)). Hence, there is an automorphism ϕ of $\mathfrak{g}$ such that $\phi(h) = -h$ for $h \in \mathfrak{h}$. Since ϕ maps $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and preserves the Killing form, we have, for $x \in \mathfrak{g}_\alpha$,

$$\langle x, \phi(x) \rangle = \langle \phi(x), \phi^2(x) \rangle = c\langle \phi(x), x \rangle = c\langle x, \phi(x) \rangle$$

where $\phi^2(x) = cx$. It follows that $c = 1$ and $\phi^2 = \mathrm{id}$. That is, ϕ is an *involutive* automorphism. For each $\alpha \in \Delta^+$, it is clearly possible to choose an element $x_\alpha \in \mathfrak{g}_\alpha$ such that $\langle x_\alpha, \phi(x_\alpha) \rangle = -1$. If we then set $x_{-\alpha} = -\phi(x_\alpha)$, we have a set $\{x_\alpha\}_{\alpha \in \Delta}$ which satisfies (14.8.2). The fact that ϕ is an automorphism implies that the numbers $N_{\alpha\beta}$ of (14.8.3) also satisfy

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}.$$

A choice of a set $\{x_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ with this property and the property expressed in (14.8.2) is called a *Weyl system.* Thus, Theorem 14.8.6 implies that Weyl systems exist.

Let $\mathfrak{h}_0$ be the real subspace of $\mathfrak{h}$ consisting of elements h for which $\alpha(h)$ is real for every root α. The real subspace $\mathfrak{g}_0$ of $\mathfrak{g}$ is then defined to be the real linear span of $\mathfrak{h}_0$ and the elements of a Weyl system constructed, as above, using the automorphism ϕ. Clearly, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. We define a conjugation (an automorphism which is conjugate linear and involutive) σ on $\mathfrak{g}$ by setting

$$\sigma(x+iy) = x - iy \text{ for } x, y \in \mathfrak{g}_0.$$

Then $\tau = \sigma \circ \phi$ is also a conjugation.

Since τ is involutive, $\mathfrak{g}$ decomposes as the direct sum of the eigenspaces of τ with eigenvalues $+1$ and -1. The eigenspace with eigenvalue $+1$ is the set of elements fixed by τ. We denote it by $\mathfrak{k}$. Then the eigenspace with eigenvalue -1 is clearly $i\mathfrak{k}$. Since τ is an automorphism of real Lie algebras, $\mathfrak{k}$ is a real Lie subalgebra of $\mathfrak{g}$. Hence, it is a real form of $\mathfrak{g}$.

It not difficult to see that $\mathfrak{k}$ is spanned by $i\mathfrak{h}_0$ and the elements of the form $x_\alpha - x_{-\alpha}$ and $i(x_\alpha + x_{-\alpha})$ for $\alpha \in \Delta^+$. We know that the Killing form is positive definite on $\mathfrak{h}_0$ so that it is negative definite on $i\mathfrak{h}_0$. By Proposition 14.5.4, $\langle h, x_\alpha\rangle = 0$ for $h \in \mathfrak{h}$ and $\alpha \in \Delta$, and $\langle x_\alpha, x_\beta\rangle = 0$ if $\beta \neq -\alpha$. Furthermore, $\langle x_\alpha, x_{-\alpha}\rangle = 1$ for each $\alpha \in \Delta$, by (14.8.2). A simple calculation using these facts shows that $\langle\cdot,\cdot\rangle$ is negative definite on $\mathfrak{t}$ and so $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$. Thus, we have:

14.8.8 Theorem. *Every complex semisimple Lie algebra has a compact real form.*

The analogous result for groups follows easily:

14.8.9 Theorem. *Each connected complex semisimple Lie group G has a connected compact real form K. Furthermore, G and K have the same fundamental group.*

Proof. If G is a complex semisimple Lie group, then its Lie algebra $\mathfrak{g}$ has a compact real form $\mathfrak{k}$. Hence, there is a simply connected Lie group $\tilde{K}$ with $\mathfrak{k}$ as Lie algebra. By Theorem 14.8.5, the Lie group $\tilde{K}$ is compact. Furthermore, the inclusion $\mathfrak{k} \to \mathfrak{g}$ is the differential of a morphism of Lie groups $\tilde{K} \to G$. The image K of this morphism is compact, hence closed in G, and is, therefore, a Lie subgroup of G with Lie algebra $\mathfrak{k}$. That is, K is a compact real form of G.

That G and K have the same fundamental group follows from a polar decomposition result which we will describe but not prove. Let $\mathfrak{p}$ denote the real linear subspace $i\mathfrak{k}$ of $\mathfrak{g}$. Then $\mathfrak{g} \simeq \mathfrak{p} \oplus \mathfrak{k}$ and the Killing form is positive definite on $\mathfrak{p}$. Using this representation of $\mathfrak{g}$, we define a conjugation operation on $\mathfrak{g}$, by setting $\bar{x} = p - k$ if $x = p + k$ for $p \in \mathfrak{p}$ and $k \in \mathfrak{k}$, and a hermitian inner product on $\mathfrak{g}$ by setting

$$(x, y) = \langle x, \bar{y} \rangle \text{ for } x, y \in \mathfrak{g}.$$

A simple calculation shows that, relative to this inner product, each operator ad_p for $p \in \mathfrak{p}$ is self-adjoint (hermitian), while each ad_k for $k \in \mathfrak{k}$ is skew-adjoint (skew-hermitian). The polar decomposition result we referred to is the following: Each element of G has the form $k \exp(p)$ for some $k \in K$ and some $p \in \mathfrak{p}$. In fact, the map

$$k \times p \to k \exp(p) : K \times \mathfrak{p} \to G$$

is a diffeomorphism. This is proved by showing that the differential of this map is an isomorphism at each point, that the map is injective (this follows from the uniqueness of the usual polar decomposition in $GL_n(\mathbb{C})$), and that the map has closed image. It then follows from the inverse function theorem that it is actually a diffeomorphism of $K \times \mathfrak{p}$ onto G (see [**V**], Chapter 4, Exercise 33). Since G is homeomorphic to the Cartesian product of K with a Euclidean space, it has the same fundamental group as K.

14.8.10 Corollary. *If G is a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, then the adjoint map* $\mathrm{Ad} : G \to \mathrm{Aut}^0(\mathfrak{g})$ *has a finite kernel. That is, the center of G is finite.*

Proof. By the previous theorem, G has a compact real form K, and the kernel of $\mathrm{Ad} : G \to \mathrm{Aut}^0(\mathfrak{g})$ agrees with the kernel of $\mathrm{Ad} : K \to \mathrm{Aut}^0(\mathfrak{k})$. This is a closed discrete subgroup of a compact group, and hence, is finite.

Exercises

1. The Leray-Schauder fixed point theorem says that, if C is a non-empty compact convex subset of a locally convex topological vector space, and if $F : C \to C$ is a continuous map, then F has a fixed point. Prove this in the special case where F is affine (a function F on a convex set C is affine if $F(tx + (1-t)y) = tF(x) + (1-t)F(y)$ for all $x, y \in C$ and $t \in [0, 1]$). Hint: Choose $x \in C$, and consider the sequence $s_n = n^{-1} \sum\limits_{k=1}^{n} F^k(x)$.

2. Prove that the spectrum of a self-adjoint operator on a Hilbert space H is contained in the set of real numbers. Hint: If $b = a - i$, and a is self-adjoint, show that $||h|| \leq ||bh||$ for every $h \in H$.
3. Prove that $||a^2|| = ||a||^2$, if a is a self-adjoint operator on a Hilbert space. Hint: Use Schwartz's inequality.
4. Prove that there is a real number λ with $|\lambda| = ||a||$ in the spectrum of any self-adjoint Hilbert space operator a.
5. For $i = 1, 2$, let K_i be a compact group, and (σ_i, H_i) an irreducible unitary representation of K_i. Prove that $\sigma_1 \otimes \sigma_2$ is an irreducible unitary representation of $K_1 \times K_2$ on $H_1 \otimes H_2$.
6. If K is a compact topological group, prove that the convolution product $f * h$ is a continuous function for all pairs $f, h \in L^2(K)$.
7. Verify the identity $(f * h^{\#}, g) = (f, g * h)$ for $f, h, g \in L^2(K)$, which was used in the proof of the Peter-Weyl theorem (Theorem 14.2.9).
8. If $\tilde{G}$ is the universal covering group of a Lie group G, then prove that the kernel of the covering map $\tilde{G} \to G$ is a discrete central subgroup of $\tilde{G}$.
9. Prove that if G is a connected locally compact group, K a connected compact group, and $\rho : G \to K$ a group homomorphism and a covering map, then $C = \operatorname{Ker} \rho$ is finitely generated. Hint: Show there is an open set U, with compact closure in G, such that $U^{-1} = U$, and $\rho(U) = K$. Then show there is a finite subset F of C such that $U^2 \subset F \cdot U$. Prove that the union of F with the finite set $U \cap C$ generates C.
10. Prove that if V is a finite dimensional vector space, and A is a set of mutually commuting semisimple linear transformations of V, then V has a basis consisting of vectors each of which is an eigenvector for each $a \in A$.
11. Let T be the circle group ($T = \{z \in \mathbb{C} : |z| = 1\}$). Show that there is an injective group homomorphism $\phi : R \to T \times T$, which is an immersion, and which has dense image. Conclude that $\phi(R)$ is an integral subgroup of $T \times T$, but not a Lie subgroup.
12. If G is a real Lie group with Lie algebra $\mathfrak{g}$, prove that the exponential map is the unique analytic map $\phi : \mathfrak{g} \to G$ which is a homomorphism on each 1-dimensional subspace of $\mathfrak{g}$, and which has differential at the origin equal to the identity.
13. Prove that if G is a topological group, and B a subgroup of G, then G/B is Hausdorff under the quotient topology if and only if B is closed in G.
14. Prove that if V is a vector space, and s is a semisimple element of $\operatorname{End}(V)$, with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then ad_s is a semisimple element of $\operatorname{End}(\mathfrak{gl}(V))$ with eigenvalues $\{\lambda_i - \lambda_j\}_{i,j=1}^n$. Identify the eigenspaces of ad_s in terms of those for s.
15. Suppose that $\mathfrak{l}$ is an ideal of the Lie algebra $\mathfrak{g}$. Prove that the restriction to $\mathfrak{l}$ of the Killing form of $\mathfrak{g}$ is the Killing form of $\mathfrak{l}$.
16. Prove that $\mathfrak{sl}_n(\mathbb{C})$ is semisimple.

17. This and the following 5 exercises refer to the set of roots for a Cartan subalgebra of a complex semisimple Lie algebra. Let α and β be roots. Use the fact that $\dfrac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ and $\dfrac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$ are integers (Corollary 14.7.5) to prove that if $||\beta|| \le ||\alpha||$, then $\dfrac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is $0, 1$, or -1.
18. Use the result of the previous exercise to prove that if α and β are distinct roots and $\langle\beta,\alpha\rangle > 0$, then $\alpha - \beta$ is a root.
19. Let Δ^+ be a system of positive roots. A simple root in Δ^+ is one which is not a sum of two roots in Δ^+. Use the result of Exercise 18 to prove that if α and β are distinct simple roots in Δ^+, then $\langle\beta,\alpha\rangle < 0$.
20. Prove that the set $\{\alpha_i\}$ of simple roots in Δ^+ is a basis for the real vector space spanned by the roots. Furthermore, each element of Δ^+ has an expansion in this basis with coefficients which are all positive. This basis is called the *system of simple roots* generating Δ^+.
21. Let $\{\alpha_i\}$ be the system of simple roots generating Δ^+, as in Exercise 20. Choose a basis $\{\alpha_i'\}$ for $\mathfrak{h}^*$ so that $\langle\alpha_i,\alpha_j'\rangle = \delta_{ij}$. Prove that a weight is in the positive Weyl chamber if and only if its expansion in terms of the basis $\{\alpha_i'\}$ has all positive coefficients.
22. Prove that $\langle\rho,\alpha\rangle > 0$ for every positive root α, where ρ is one-half the sum of the positive roots.
23. Let (π, V) be a finite dimensional representation of a Lie group G, with Lie algebra $\mathfrak{g}$, and let Y and W be subspaces of V. Then,

(a) if $B = \{g \in G : \pi(g)Y \subset W\}$, prove that B is a Lie subgroup with Lie algebra $\mathfrak{b} = \{x \in \mathfrak{g} : \pi(x)Y \subset W\}$;

(b) if $H = \{g \in G : \pi(g)y = y,\ \forall\, y \in Y\}$, prove that H is a Lie subgroup of G with Lie algebra $\mathfrak{h} = \{x \in \mathfrak{g} : \pi(x)y = 0,\ \forall\, y \in Y\}$.

24. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Prove that every automorphism ϕ of $\mathfrak{g}$ that fixes each element of $\mathfrak{h}$ has the form $\exp(\operatorname{ad} h) = \operatorname{Ad}_{\exp(h)}$ for some element $h \in \mathfrak{h}$. Hint: Prove that ϕ acts as multiplication by a scalar c_α on each $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$, and that these scalars satisfy $c_\alpha c_\beta = c_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in \Delta$. Using the result of Exercise 20, prove that there is an $h \in \mathfrak{h}$ such that $c_\alpha = \exp(\alpha(h))$ for each $\alpha \in \Delta$. Show that this element h has the property that $\phi = \exp(\operatorname{ad}_h)$.
25. Prove every real Lie algebra has a maximal compact Lie subalgebra and every real Lie group has a maximal connected compact Lie subgroup.
26. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ consisting of the diagonal matrices of trace 0. Calculate the restriction to $\mathfrak{h}$ of the Killing form on $\mathfrak{sl}_3(\mathbb{C})$. Choose an orthonormal basis for $\mathfrak{h}$ relative to this form (choose real matrices), and calculate the roots in $\mathfrak{h}' = \mathbb{R}^2$ for $\mathfrak{h}$ relative to this basis. You should get six vectors of equal length, with equal angular spacing.

Algebraic Groups

The complex Lie groups that arise most commonly in practice are actually algebraic varieties with group operations which are morphisms of algebraic varieties. Such an object is called an *algebraic group.* In some arguments involving such groups, there are strong advantages to be gained by exploiting this algebraic group structure. In other arguments, there are advantages to sticking with the holomorphic structure on the Lie group and using analytic techniques. Thus, the theory of complex Lie groups fits nicely with the general philosophy of this text, which is that complex algebraic geometry and holomorphic function theory are companion subjects that illuminate, enrich, and complement one another.

In this chapter, we will give a brief introduction to the theory of algebraic groups. We will develop just enough of this subject to allow us to prove the key structure theorems for complex semisimple Lie groups using algebraic group methods. The last section of the chapter is devoted to proving that every complex semisimple Lie group does, in fact, have the structure of an algebraic group.

15.1 Algebraic Groups and Their Representations

Algebraic groups can be defined over any field. However, for us the only field of interest is the complex field. For this reason, we will discuss only complex algebraic groups, and the term "algebraic group" will be used to mean "complex algebraic group".

Thus, for us, an *algebraic group* is a group for which the underlying space is a complex algebraic variety, and the group operations are regular maps. If G is such a group, since $g \to g_1 g : G \to G$ is a biregular map, each neighborhood of a point $g_1 \in G$ is equivalent to a neighborhood of the identity $e \in G$. It follows that G must be a smooth variety (a variety with no singular points). Of course, unlike the usual situation for topological groups, the topology on an algebraic group is not Hausdorff.

As is the case for Hausdorff topological groups, the connected component of the identity in an algebraic group is a normal subgroup which is both open and closed. Thus, the quotient of an algebraic group modulo its identity component is a discrete group.

A morphism $G \to H$, between algebraic groups, is a group homomorphism which is also a morphism of the underlying algebraic varieties. An *algebraic subgroup* of an algebraic group is a subgroup which is also an algebraic subvariety – that is, a subgroup which is closed in the Zariski topology. Note that the closure of a subgroup is again a subgroup (Exercise 15.1), and so every subgroup has an algebraic subgroup as its closure.

To each algebraic group G is associated a complex Lie group, obtained by replacing the algebraic variety G with its associated holomorphic variety. Since complex Lie groups have complex Lie algebras, there is a complex Lie algebra $\mathfrak{g}$ associated to each algebraic group G. This may be described intrinsically, without reference to the complex Lie algebra associated to G, as the Lie algebra of morphisms $\xi : \mathcal{O} \to \mathcal{O}$ which are right invariant derivations – commute with all right translation operators and satisfy

$$\xi(fh) = h\xi(f) + f\xi(h), \ \forall\, f, h \in \mathcal{O}(U).$$

We leave as exercises that the Lie algebra defined this way is isomorphic, as a vector space, to the tangent space to G at e, and is isomorphic, as a Lie algebra, to the Lie algebra of the complex Lie group associated to G (Exercises 15.2 and 15.3).

An *affine algebraic group* is an algebraic group which is affine as an algebraic variety. In this text we will restrict our attention to affine algebraic groups.

15.1.1 Examples. Let V be a complex vector space. Then the general linear group $GL(V)$ is the subset of the vector space of all linear transformations of V defined by the non-vanishing of the determinant. If a basis for V is fixed, then each linear transformation of V is represented by a matrix and the determinant is a polynomial in the entries of this matrix. It follows that $GL(V)$ is an affine algebraic variety. Furthermore, the group operations in $GL(V)$ are regular functions of the matrix entries. Hence, $GL(V)$ is an affine algebraic group. Suppose a basis for V is fixed so that elements of $GL(V)$ are represented as matrices. Then each of the following is a subgroup of $GL(V)$ defined by the vanishing of finitely many regular functions in the matrix entries and, hence, is an algebraic subgroup of $GL(V)$:

(i) the special linear group $SL_n(\mathbb{C})$, consisting of matrices of determinant one;
(ii) the diagonal group $D_n(\mathbb{C})$, consisting of invertible diagonal matrices;
(iii) the triangular group $T_n(\mathbb{C})$, consisting of upper triangular matrices;
(iv) the subgroup of $GL(V)$ consisting of elements g such that

$$\Lambda(gv, gw) = \Lambda(v, w), \ \forall \, v, w \in V,$$

where $\Lambda : V \times V \to \mathbb{C}$ is a bilinear form; as special cases of this we have:
(v) the complex orthogonal group $O_n(\mathbb{C})$, determined by the usual dot product

$$\Lambda(z, w) = z_1 w_1 + \cdots + z_n w_n;$$

(vi) the group $O_{n,m}(\mathbb{C})$ – here the dimension of V is $n + m$, and

$$\lambda(z, w) = z_1 w_1 + \cdots + z_n w_n - z_{n+1} w_{n+1} - \cdots - z_{n+m} w_{n+m};$$

(vii) the symplectic group $Sp_{2n}(\mathbb{C})$ – here the dimension of V is $2n$, and

$$\Lambda(z, w) = z_1 w_{n+1} + \cdots + z_n w_{2n} - z_{n+1} w_1 - \cdots - z_{2n} w_n.$$

15.1.2 Example. A complex vector space V, with addition as operation, is an abelian affine algebraic group.

There are projective varieties which have the structure of an algebraic group. These are called *abelian varieties.* Obviously such an algebraic group is not affine, since it has only the constants as global regular functions. It turns out that every algebraic group G has a normal subgroup H, which is affine, such that G/H is an abelian variety ([**Ch2**]). The proof of this is fairly difficult, and we will not attempt to give it here. We will be concerned only with affine algebraic groups.

We will have occasion to study representations of algebraic groups on infinite dimensional vector spaces, although our primary interest will be in representations on finite dimensional spaces. For this reason, we need to make sense of the notion of a regular function with values in an infinite dimensional vector space V. This is easy to do on an affine space X. We realize X as a subvariety of $\mathbb{C}^n$, and then define a regular V-valued function to be the restriction to X of a polynomial in $z_1, \ldots, z_n$ with coefficients in V. One easily checks that the resulting class of V-valued functions on X is independent of the realization of X as a subvariety of Euclidean space. If X is any algebraic variety, we define a V-valued function on X to be regular if its restriction to each affine open subset of X is regular.

Having defined regular V-valued functions as above, we immediately note that there is an equivalent and somewhat simpler definition. That is, a regular V-valued function f on an affine space X has all its values in a finite dimensional subspace of V – the span of the coefficients of a polynomial representing f. Since every algebraic variety has a finite cover by affine open sets, every regular V-valued function on a variety has its values in a finite dimensional subspace of V. Thus, we could, just as well, define a V-valued regular function on a variety X to be a regular map from X to a finite dimensional subspace of V.

15.1.3 Definition. *Let G be an algebraic group, and V a complex vector space (possibly infinite dimensional). An algebraic representation of G on V is a homomorphism π from G into the group of linear transformations of V such that the function $g \to \pi(g)v : G \to V$ is regular for each $v \in V$.*

From the remarks preceding the above definition, it is clear that if (π, V) is an algebraic representation, and $v \in V$, then the linear span of the orbit $\{\pi(g)v : g \in G\}$ of v is finite dimensional and is invariant under π. Hence, we have:

15.1.4 Proposition. *If (π, V) is an algebraic representation of an algebraic group G, then V is the union of its finite dimensional invariant subspaces.*

Our main use of this result will be in connection with the left and right regular representations of an affine algebraic group G on $\mathcal{O}(G)$. These are defined, as they were in Chapter 14, by

$$L_g f(g_1) = f(g^{-1}g_1) \quad \text{and} \quad R_g f(g_1) = f(g_1 g)$$

for $f \in \mathcal{O}(G)$ and $g, g_1 \in G$. The fact that $(g, g_1) \to g^{-1}g_1$ and f are regular implies that, if G is realized as a subvariety of $\mathbb{C}^n$, then $(g, g_1) \to f(g^{-1}g_1)$ is the restriction to $G \times G$ of a polynomial on $\mathbb{C}^n \times \mathbb{C}^n$. It follows that, for each $f \in \mathcal{O}(G)$, the function $g \to L_g f$ is the restriction to G of a polynomial on $\mathbb{C}^n$ with coefficients in $\mathcal{O}(G)$. Thus, the left regular representation is algebraic. Similarly, the right regular representation is algebraic, as is the representation $LR = \{(g, g') \to L_g R_{g'}\}$ of $G \times G$ on $\mathcal{O}(G)$. If we define left, right, and 2-sided invariant subspaces of $\mathcal{O}(G)$ to be subspaces which are invariant under L, R, and LR, respectively, then Proposition 15.1.4 implies the following:

15.1.5 Proposition. *Let G be an affine algebraic group. Then $\mathcal{O}(G)$ is the union of its finite dimensional 2-sided invariant subspaces.*

This is a simple but important result, and it will be used often. The next proposition is a typical application of this result.

15.1.6 Proposition. *If G is an affine algebraic group, then there is a faithful finite dimensional algebraic representation of G.*

Proof. Since G is an affine variety, there is a finite subset $S \subset \mathcal{O}(G)$ which separates points in G. This set is contained in a finite dimensional right invariant subspace M of $\mathcal{O}(G)$. The right regular representation of G, restricted to M, is a faithful finite dimensional algebraic representation of G.

This means that every affine algebraic group G has an injective morphism into $GL(V)$, for some finite dimensional vector space V.

We use the above result to show that the adjoint representation of G on its Lie algebra $\mathfrak{g}$ is an algebraic representation. Here the adjoint representation $g \to \mathrm{Ad}_g : G \to GL(\mathfrak{g})$ is defined, as in the previous chapter, for G considered as a complex Lie algebra. We know it is a holomorphic representation of the complex Lie group G. The only question is whether or not this map is algebraic if G is given its algebraic structure. However, by the previous proposition, G has a faithful finite dimensional algebraic representation (π, V). The differential of π embeds the Lie algebra $\mathfrak{g}$ of G as a subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{gl}(V)$, and the adjoint representation for G is obtained by composing π with the adjoint representation for $GL(V)$, and then restricting the resulting representation of G to the invariant subspace $\tilde{\mathfrak{g}} \subset \mathfrak{gl}(V)$. Thus, the adjoint representation for G is algebraic if the adjoint representation for $GL(V)$ is algebraic. However, if a basis for V is fixed, so that $GL(V)$ is represented as the group of invertible $n \times n$ matrices, and $\mathfrak{gl}(V)$ as the vector space of all $n \times n$ matrices, the adjoint representation for $GL(V)$ is given by matrix multiplication as

$$\mathrm{Ad}_a(b) = a^{-1}ba.$$

This is clearly algebraic. Thus, we have:

15.1.7 Proposition. *For an affine algebraic group, the adjoint representation is algebraic.*

15.2 Quotients and Group Actions

Recall that if G is a topological group, and H a closed subgroup, then the quotient G/H is given the unique topology which makes the quotient map $\rho : G \to G/H$ a continuous open map; that is, a set $U \subset G/H$ is open if and only if $\rho^{-1}(U)$ is open in G. If G is an algebraic group, then G/H is given a ringed space structure by defining the structure sheaf to be the sheaf which assigns to an open set U the ring $\mathcal{O}(\rho^{-1}(U))$. With this definition, the map

$\rho : G \to G/H$ is a morphism of ringed spaces. Recall from the previous chapter that the ringed space structure on G/H is induced from that of G, in a similar fashion, in the case of a complex or real Lie group.

We proved in Proposition 14.3.4 that the quotient G/H, of a complex Lie group G by a Lie subgroup, is a holomorphic manifold on which G acts holomorphically. This followed from the fact that, locally over G/H, G looks like a product of H with a neighborhood in G/H. This, in turn, was proved using the implicit mapping theorem which, of course, does not hold for maps between algebraic varieties. Thus, to prove that the quotient of an algebraic group by an algebraic subgroup exists as an algebraic variety requires entirely different techniques – it exists as a ringed space, but is this ringed space an algebraic variety? The purpose of this section is to develop these techniques. In the next section, we finally prove the existence of quotients for affine algebraic groups.

We begin by studying algebraic group actions. Recall that an action of a group G on a set X is a map $(g,x) \to gx : G \times X \to X$, which satisfies $ex = x$, and the associative law $g(g_1x) = (gg_1)x$, so that it defines a group homomorphism of G into the group of invertible transformations of X. Given an action of a group G on a space X, and a point $x \in X$, the *orbit* of x is the set $\{gx : g \in G\}$, while the *isotropy group* or *stabilizer* of x is $\{g \in G : gx = x\}$. If G is an algebraic group, and X is an algebraic variety, then an action of G on X is *algebraic* if the action map $G \times X \to X$ is a morphism of algebraic varieties. Note that, for an algebraic group action, the isotropy group of a point is an algebraic subgroup. It is also true that orbits of algebraic group actions have nice properties. This is illustrated in the next three propositions. The first of these tells us that orbits of an algebraic group action are locally closed subsets.

15.2.1 Proposition. *If G is an algebraic group, X an algebraic variety, and $(g,x) \to gx : G \times X \to X$ an algebraic action of G on X, then each orbit of this action is open in its closure.*

Proof. Here, of course, we are using the Zariski topology. If x_0 is a point of X, and Y is the orbit containing x_0, then the map $g \to gx_0 : G \to X$ is a regular map with Y as its image. By Proposition 8.8.1, Y contains a non-empty set V which is open in the closure $\overline{Y}$ of Y. Then, since Y is the union of the sets gV for $g \in G$, it is also open in $\overline{Y}$.

The above proposition implies that each orbit of an algebraic group action is itself an algebraic variety. Furthermore, given any two points of this variety, each neighborhood of the first is isomorphic, via the group action, to a neighborhood of the second. It follows that each orbit is a smooth variety, and each connected orbit is irreducible.

If an algebraic group acts algebraically on an algebraic variety X, and Y is an orbit of this action, then it is clear that the closure $\overline{Y}$ of Y is invariant under the G-action, and hence, $G \times \overline{Y} \to \overline{Y}$ is also an algebraic group action. It follows that the boundary of Y is a union of orbits of dimension less than the dimension of Y. The next proposition is an immediate consequence of this fact:

15.2.2 Proposition. *For an algebraic action of an algebraic group, there exist orbits of minimal dimension, and each such orbit is closed.*

This has the following simple but quite surprising consequence:

15.2.3 Theorem. *Let $\phi : H \to G$ be a morphism of algebraic groups. Then $\phi(H)$ is an algebraic subgroup of G.*

Proof. Consider the action of H on G given by $(h, g) \to \phi(h)g : H \times G \to G$. This is an algebraic action, and so, by the previous proposition, it has a closed orbit. However, the orbits of this action are just the right cosets of $\phi(H)$, and each of them is isomorphic to $\phi(H)$ under right translation by some element of G. Hence, $\phi(H)$ is closed.

The next proposition proves an important special case of the fact that the quotient G/H, of an algebraic group by an algebraic subgroup, is an algebraic variety. The general case will eventually be reduced to this case.

15.2.4 Proposition. *Given an algebraic group action of G on X, and an element $x \in X$, let H be the isotropy group of x, and Y the orbit containing x. Then the natural map $G/H \to Y$, induced by $g \to gx : G \to X$, is an isomorphism from the ringed space G/H to the algebraic variety Y. Thus, G/H is an algebraic variety in this case.*

Proof. Without loss of generality, we may assume that G is connected, and hence, that both G and Y are irreducible smooth varieties. It follows that each open subset of either space is dense. Let the map $\alpha : G \to Y$ be defined by $\alpha(g) = gx$. The fibers of the map α are the left cosets of H, and so they are all isomorphic and, in particular, of the same dimension. By Theorem 8.8.6, α is an open map, and this implies that the induced map $G/H \to Y$ is a homeomorphism.

To show that the induced map $G/H \to Y$ is an isomorphism of ringed spaces, we must show that, given an affine open subset $V \subset Y$ and its inverse image $U = \alpha^{-1}(V)$, the map $f \to f \circ \alpha : \mathcal{O}(V) \to \mathcal{O}(U)$ is an isomorphism of the ring $\mathcal{O}(V)$ onto the subring of $\mathcal{O}(U)$ consisting of functions which are constant on left cosets of H. The only thing in question is the surjectivity of $f \to f \circ \alpha$. Thus, if B is the image of this map, then we need to show that every function $f \in \mathcal{O}(U)$ which is constant on the left cosets of H is actually in B. Let f be such a function and let A be the subalgebra of $\mathcal{O}(U)$ generated by f and B.

Since the ring $B \simeq \mathcal{O}(V)$ is finitely generated over $\mathbb{C}$, so is the ring A. Hence, there is an affine variety W such that $A \simeq \mathcal{O}(W)$. The inclusion $B \to A$ induces a regular map $W \to V$. On the other hand, the inclusion $A \to \mathcal{O}(U)$ induces a regular map $U \to W$, which is constant on cosets of H, and whose composition with $W \to V$ is $\alpha|_U$. Since no element of A vanishes on U, the map $U \to W$ has dense image, and in fact, contains a dense open subset of W, by Proposition 8.8.1. Since this map is constant on cosets of H, it induces a map $\tilde{U} \to W$, where $\tilde{U}$ is the open set in G/H which is the image of U. We then have morphisms of ringed spaces $\tilde{U} \to W \to V$, with the composition a homeomorphism, and the first map has image containing an open subset of W. It follows that the dimension of W at a point of this open set is the same as the dimension of V at the corresponding point in V. If f is not algebraic over B, then A is isomorphic to the polynomial ring in one variable over B, which is $\mathcal{O}(V \times \mathbb{C})$. Since $V \times \mathbb{C}$ has dimension at each point greater than the dimension of V at the corresponding point, this is impossible. Thus, f is algebraic over B.

Since each point of V is regular, each local ring ${}_V\mathcal{O}_x$ is a normal domain, by Exercise 13.16. It follows that the minimal polynomial of f has a non-zero discriminant which is an element of $B = \mathcal{O}(V)$ (Theorem 4.2.9). Over points $x \in V$ where this discriminant doesn't vanish, the minimal polynomial has as many distinct roots as its degree, and these roots must be values assumed by f on points of W which lie over x. Since $W \to V$ is injective on an open subset, we conclude that the minimal polynomial for f has degree 1. In other words, $f \in B$.

15.2.5 Corollary. *If $\phi : G \to G'$ is a morphism of algebraic groups and $H = \operatorname{Ker}\phi$, then G/H is an algebraic group, and ϕ factors as the quotient morphism $G \to G/H$ followed by an inclusion of G/H as an algebraic subgroup of G'.*

Proof. The morphism ϕ determines an algebraic action $(g, g') \to \phi(g)g'$ of G on G'. The isotropy group of $e \in G'$ under this action is the kernel H of ϕ, and the orbit of e is the image of ϕ, which is closed by Theorem 15.2.3. It follows from the preceding proposition that the quotient G/H is an algebraic variety, and the induced map $G/H \to G'$ is an isomorphism onto the image of ϕ. Since H is a normal subgroup of G, G/H is a group. Since the induced map $G/H \to \operatorname{Im}(\phi)$ is an isomorphism of varieties and an isomorphism of groups, and since $\operatorname{Im}(\phi)$ is an algebraic group, G/H is also an algebraic group.

Note that each of the preceding five results is false in the cases of real or complex Lie groups, and analytic or holomorphic group actions, as the following example shows.

15.2.6 Example. Let α be an irrational real number, and consider the complex Lie groups $X = \mathbb{C}^* \times \mathbb{C}^*$ and $G = \mathbb{C}$ and the holomorphic action of G on X given by $(z, (u, v)) \to (e^z u, e^{\alpha z} v)$. Under this action, the isotropy group H of any point is 0 and so G/H is just G. The orbits of this action are exactly the cosets of the subgroup of X consisting of all elements of the form $(e^z, e^{\alpha z})$. The closure of this subgroup consists of all elements $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$ with $|u|^\alpha = |v|$. This is a real three-dimensional subgroup of X, while G has real dimension 2. Thus, the orbits of this action of G are not open in their closures; orbits of minimal dimension are not closed; and G is not even homeomorphic to its image in X under an orbit map. Thus, Propositions 15.2.1, 15.2.2, and 15.2.4, Theorem 15.2.3, and Corollary 15.2.5 all fail in this example. This is true whether the groups involved are considered as real or complex Lie groups.

15.3 Existence of the Quotient

In this section, we prove that the quotient G/H of an affine algebraic group by an algebraic subgroup H is an algebraic variety. The key to the proof is Proposition 15.2.4. In order to apply it, we need to show that H is the isotropy group of a point under some algebraic action of G on a variety. This is accomplished in the next two lemmas. We will also show that, if H is a normal subgroup, then G/H is affine and, hence, is itself an affine algebraic group.

Given a representation (π, V) of a group G, and a linear subspace $N \subset V$, the subgroup $H = \{g \in G : \pi(g)N = N\}$ will be called the *stabilizer* of N under π, or simply the *stabilizer* of N if π is understood. We shall also say that a subset of G *stabilizes* a subspace N of V if each element of the set maps N onto N.

15.3.1 Lemma. *Let H be an algebraic subgroup of an affine algebraic group G. Then there exists a finite dimensional algebraic representation (π, M) of G and a subspace $N \subset M$ such that H is the stabilizer of N.*

Proof. Let I be the ideal of functions in $\mathcal{O}(G)$ which vanish on H. Since H is a subvariety of an affine variety G, it follows from Corollary 8.3.6 and Theorem 8.6.2 that the ideal sheaf of H is generated by a finite set $F \subset I$. It follows that H is exactly the set of points where the functions in F vanish simultaneously. By Proposition 15.1.5, there is a finite dimensional subspace M of $\mathcal{O}(G)$ which is invariant under the right regular representation R of

G on $\mathcal{O}(G)$, and which contains F. If π is R restricted to M, then we claim that the representation (π, M) and the subspace $N = I \cap M$ have the required properties.

Note that $F \subset N$. Clearly, under R, H leaves N invariant. Suppose $g \in G$, and $\pi(g)N \subset N$. Then, for each $f \in F$ we have $\pi(g)f \in I$, and so $f(g) = \pi(g)f(e) = 0$. We conclude that $g \in H$. This completes the proof.

Thus, we can always realize an algebraic subgroup H of an affine algebraic group G as the stabilizer of a certain subspace of a finite dimensional algebraic representation of G. The next lemma says we can achieve this with a subspace which is 1-dimensional.

15.3.2 Lemma. *Let H be an algebraic subgroup of an affine algebraic group G. Then there is a finite dimensional algebraic representation (σ, V) of G and a 1-dimensional subspace $W \subset V$ such that H is the stabilizer of W.*

Proof. Let (π, M) and N be chosen as in Lemma 15.3.1. If the dimension of N is k, then we set $V = \bigwedge^k M$, $W = \bigwedge^k N$, and $\sigma = \wedge^k \pi$. Then σ is an algebraic representation of G on V, and W is naturally a 1-dimensional subspace of V, which is invariant under $\sigma(H)$. We need to show that every element of G which leaves W invariant belongs to H.

To finish the proof, we will show that an invertible linear transformation $\phi : M \to M$ leaves N invariant if and only if its k fold exterior product $\wedge^k \phi$ leaves $\bigwedge^k N$ invariant. We may choose a basis $\{e_1, \dots, e_n\}$ for M and an integer m such that $\{e_1, \dots, e_k\}$ is a basis for N and $\{e_m, \dots, e_{m+k-1}\}$ is a basis for $\phi(N)$. This can be done by choosing a basis for $N \cap \phi(N)$ first, labeling it starting with the index $m = k - \dim N \cap \phi(N)$, then filling this out to a basis for $N + \phi(N)$, indexed from 1 to $m + k - 1$, and then filling this out to a basis for M by choosing elements $e_{m+k}, \dots, e_n$. With this choice of basis, $\wedge^k \phi(e_1 \wedge \cdots \wedge e_k) = \phi(e_1) \wedge \cdots \wedge \phi(e_k)$ is a scalar times $e_m \wedge \cdots \wedge e_{m+k-1}$. Hence, $\wedge^k \phi$ maps $\bigwedge^k N$ into itself if and only if $m = 1$ – that is, if and only if ϕ maps N into itself. This completes the proof.

The above lemma leads directly to the existence theorem for quotients.

15.3.3 Theorem. *If G is an affine algebraic group, and $H \subset G$ an algebraic subgroup, then G/H is an algebraic variety.*

Proof. Let $P = P(V)$ be the projective space of the vector space V in the previous lemma. That is, P is the set of all 1-dimensional subspaces of V, with the structure of an algebraic variety described in Chapter 12. Then the algebraic representation σ determines an algebraic action of G on P with the property that H is the isotropy group of the point in P determined by

W. We conclude from Proposition 15.2.4 that G/H is an algebraic variety isomorphic to the orbit of this point.

In the next theorem we make use of the notion of a *character* of an algebraic group G. This is a morphism of G to the multiplicative group $\mathbb{C}^* = D_1(\mathbb{C})$. If (σ, V) is a representation of G, and γ is a character of G, then the space V_γ of all vectors $v \in V$ for which $\sigma(g)v = \gamma(g)v$ for $g \in G$ is called the *weight space* of (σ, V) for the character γ. A non-zero element of V_γ is called a *weight vector* with weight γ. A set $\{v_1, \ldots, v_k\}$ of weight vectors with distinct weights $\gamma_1, \ldots, \gamma_k$ is necessarily linearly independent (Exercise 15.4). We will study characters of algebraic groups in more detail in section 15.5.

15.3.4 Theorem. *If G is an affine algebraic group, and $H \subset G$ is a normal algebraic subgroup, then G/H is an affine algebraic group.*

Proof. Let V and W be as in Lemma 15.3.2. Since W is invariant under H, and W is 1-dimensional, there is a character $\gamma^0 : H \to \mathbb{C}^*$ such that

$$\sigma(h)w = \gamma^0(h)w, \ \forall\, w \in W, h \in H.$$

For each character γ of H, we set

$$V_\gamma = \{v \in V : \sigma(h)v = \gamma(h)v, \ \forall\, h \in H\}.$$

Then $W \subset V_{\gamma^0}$.

If $g \in G$, then $ghg^{-1} \in H$, since H is normal. Thus for any character γ of H, we may define a character γ_g of H by

$$\gamma_g(h) = \gamma(g^{-1}hg).$$

If $v \in V_\gamma$, and $g \in G$, then

$$\sigma(h)\sigma(g)v = \sigma(g)\sigma(g^{-1}hg)v = \sigma(g)\gamma_g(h)v = \gamma_g(h)\sigma(g)v,$$

and hence, $\sigma(g)v \in V_{\gamma_g}$. It follows that the linear span of the spaces V_γ is invariant under the action of G. Without loss of generality, we may assume that this span is V itself.

Since a set of weight vectors with distinct weights is linearly independent (Exercise 15.4), it follows that we have a direct sum decomposition

$$V = \bigoplus_\gamma V_\gamma. \tag{15.3.1}$$

Of course, since V is finite dimensional, only finitely many of the V_γ are non-zero. By definition, the subgroup H of G consists of elements which, under σ, act as scalars on each V_γ. We set

$$B = \{b \in \mathfrak{gl}(V) : bV_\gamma \subset V_\gamma,\ \forall\, \gamma\}.$$

That is, B is the set of linear transformations of V which have block diagonal form relative to the decomposition (15.3.1). Then, for each $h \in H$, the operator $\sigma(h)$ commutes with every element of B. We claim H is exactly the set of elements of G with this property. In fact, if g is such an element, then $\sigma(g)$ acts as a scalar on each V_γ, by Schur's lemma. In particular, it acts as a scalar on V_{γ^0} and, hence, stabilizes every subspace of V_{γ^0}, including the subspace W. This implies g belongs to H.

If we follow the representation $\sigma : G \to GL(V)$ with the adjoint representation $\mathrm{Ad} : GL(V) \to GL(\mathfrak{gl}(V))$, we obtain an algebraic representation of G on $\mathfrak{gl}(V)$. The subspace $B \subset \mathfrak{gl}(V)$ is an invariant subspace for this representation, since $\sigma(g)V_\gamma = V_{\gamma_g}$ for each $g \in G$. Let $\pi(g) = \mathrm{Ad} \circ \sigma(g)|_B$ for $g \in G$. Then π is an algebraic representation of G, and since H is the subgroup of G consisting of elements h such that $\sigma(h)$ commutes with every element of B, the kernel of π is H. By Theorem 15.2.3, $\pi(G)$ is a subvariety of $GL(B)$ and, hence, is affine. It follows from Proposition 15.2.4 that G/H is isomorphic to $\pi(G)$, and so it is affine as well.

15.4 Jordan Decomposition

A linear operator t, on a possibly infinite dimensional space V, is called *semisimple* if V has a basis consisting of eigenvectors for t. A linear operator u on V is called *unipotent* if $u = \mathrm{id} + n$, where each $v \in V$ is killed by some power of n. Note that if V is finite dimensional, then u is unipotent if and only if $u = \mathrm{id} + n$, where n is nilpotent.

If V is a finite dimensional vector space, and $t \in GL(V)$, then t has a unique decomposition $t = s + n$ where $s, n \in \mathrm{End}(V)$, s is semisimple, n is nilpotent, and s and n are polynomials in t (Lemma 14.4.4). It is evident from the proof of this lemma that s has the same eigenvalues as t and, hence, it also belongs to $GL(V)$. If we set $u = \mathrm{id} + s^{-1}n$, then $t = su$, u is unipotent, s and u commute with each other and with every element of $GL(V)$ which commutes with t. It is also true that s and u stabilize every subspace of V which is stabilized by t. This is true of s and n, by Lemma 14.4.4. It is also true of s^{-1}, since if s stabilizes a subspace $M \subset V$, then s^{-1} does as well. It follows that $u = \mathrm{id} + s^{-1}n$ also stabilizes every subspace of V stabilized by t. It also follows from Lemma 14.4.4 that this decomposition is the unique way of writing t as a product of commuting elements of $GL(V)$, one of which is semisimple and the other unipotent. We will call this decomposition the *multiplicative Jordan decomposition* or, when no confusion will result, simply the *Jordan decomposition* of t. Our objective in this section is to prove the existence of an *abstract multiplicative Jordan decomposition* for elements of arbitrary affine algebraic groups. The first step in this direction is the following lemma.

15.4.1 Lemma. *Let V be a finite dimensional complex vector space, and set $G = GL(V)$. Then*

(i) *if $s \in G$ is semisimple as a transformation of V, then R_s is semisimple on $\mathcal{O}(G)$;*

(ii) *if $u \in G$ is unipotent as a transformation of V, then R_u is unipotent on $\mathcal{O}(G)$.*

Proof. The group $GL(V)$ is the open subset of $E = \mathrm{End}(V)$ defined by the non-vanishing of the function $\delta(g) = \det(g)$. Thus, $\mathcal{O}(G)$ is obtained from the algebra $P(E)$ of polynomials on the vector space E by localizing relative to the multiplicative system consisting of powers of δ. That is, $\mathcal{O}(G)$ is the union of the subspaces $\delta^{-n}P(E)$ for $n = 0, 1, 2, \dots$. Since $\delta(g_1 g) = \delta(g_1)\delta(g)$, each of these subspaces is right invariant. Furthermore, R_s is semisimple (R_u is unipotent) on $\delta^{-n}P(E)$ if R_s is semisimple (R_u is unipotent) on $P(E)$. Thus, we need only show that R_s (resp. R_u) is semisimple (resp. unipotent) on $P(E)$.

The algebra $P(E)$ decomposes as

$$P(E) = \mathbb{C} \oplus E^* \oplus S^2(E^*) \oplus \cdots \oplus S^n(E^*) \oplus \cdots,$$

where $S^n(E^*)$ is the space of homogeneous polynomials of degree n. Each of the spaces in this decomposition is a 2-sided invariant subspace of $\mathcal{O}(G)$. We need to analyze the action of R_g on each of them.

Of course, each R_g acts as the identity on the constant factor $\mathbb{C}$. If we represent $E^* = \mathrm{End}(V)^*$ as $V \otimes V^*$, where $v \otimes v^* \in V \otimes V^*$ acts on $\mathrm{End}(V)$ as the linear functional f_{v,v^*}, defined by $f_{v,v^*}(b) = v^*(bv)$, then the right regular action of G on E^* is given by $R_g f_{v,v^*} = f_{gv,v^*}$. Thus, E^* is just the G-module V tensored with the vector space V^*. In other words, as a G-module under the right regular representation, it is just a direct sum of finitely many copies of V. Thus, if s acts semisimply on V, then it also acts semisimply on E^* via the right regular representation. If $h_1, \dots, h_k$ is a basis for E^* consisting of eigenvectors for R_s, then $\{h_i h_j\}_{i,j=1}^k$ is a basis for $S^2(E^*)$ consisting of eigenvectors for R_s, and so R_s acts semisimply on $S^2(E^*)$ as well. A similar analysis shows that R_s acts semisimply on each of the subspaces $S^n(E^*)$ and, hence, acts semisimply on $P(E)$. It is equally clear that if u is unipotent on V, then it is unipotent on E^* and on each $S^n(E^*)$. This completes the proof.

We will say that an element s of an affine algebraic group G is *semisimple* if R_s is semisimple as a linear transformation on $\mathcal{O}(G)$. Similarly, $u \in G$ is *unipotent* if R_u is unipotent. The following is the multiplicative abstract Jordan decomposition:

15.4.2 Theorem. *If G is an affine algebraic group, then each element g of G has a unique decomposition $g = su$, where s is semisimple, u is unipotent, and s and u commute.*

Proof. Suppose first that $G = GL(V)$ for some finite dimensional vector space V. Then we know from the usual Jordan decomposition that each $g \in G$ has a unique decomposition $g = su$ such that s and u commute, s is semisimple on V, and u is unipotent on V. By the previous lemma, R_s is semisimple on $\mathcal{O}(G)$, and R_u is unipotent on $\mathcal{O}(G)$. That the decomposition $R_g = R_sR_u$ is unique with these properties follows from the uniqueness of the ordinary Jordan decomposition, applied to each finite dimensional right invariant subspace of $\mathcal{O}(G)$. This then implies the uniqueness of the decomposition $g = su$. Thus, the theorem is true for groups of the form $GL(V)$.

Now any affine algebraic group has a faithful finite dimensional algebraic representation, by Proposition 15.1.6, and this implies that it is isomorphic to an algebraic subgroup of some $GL(V)$, by Corollary 15.2.5. Thus, the proof will be complete if we can show that the theorem is true of each algebraic subgroup G of G_1 whenever it is true of a given affine algebraic group G_1.

Thus, suppose the theorem is true of G_1, and let G be an algebraic subgroup of G_1. If $g \in G$, let $g = su$ be its Jordan decomposition in G_1, which exists by assumption. Then $R_g = R_sR_u$ is the Jordan decomposition of R_g on $\mathcal{O}(G_1)$. The ideal I, consisting of functions which vanish on G, is invariant under R_g, as is each subspace of the form $I \cap J$, where J is any finite dimensional right G_1-invariant subspace of $\mathcal{O}(G_1)$. It follows from the properties of the Jordan decomposition that each of these spaces $I \cap J$ is also invariant under R_s and R_u. Since I is the union of such spaces, this implies that I is also invariant under R_s and R_u. Then $f(s) = R_sf(e) = 0$, and $f(u) = R_uf(e) = 0$ for every $f \in I$. It follows, as in the proof of Lemma 15.3.1, that $s, u \in G$, and hence, that the Jordan decomposition exists for each element of G. The uniqueness follows, as above, from the uniqueness of the ordinary Jordan decomposition.

For simplicity in what follows, we will refer to the above decomposition as simply the *Jordan decomposition* of $g \in G$. The next proposition says that it is preserved by morphisms of algebraic groups.

15.4.3 Proposition. *Let $\phi : G \to G_1$ be a morphism of affine algebraic groups. If $g = su$ is the Jordan decomposition of an element $g \in G$, then $\phi(g) = \phi(s)\phi(u)$ is the Jordan decomposition of $\phi(g) \in G_1$.*

Proof. If G_1 is a quotient of G, say $G_1 = G/H$, then $\mathcal{O}(G_1)$ may be regarded as the subspace of $\mathcal{O}(G)$ consisting of functions constant on cosets of H. This subspace is right invariant under G, and so we may argue, as in the previous theorem, that it is invariant under R_s and R_u, where $g = su$ is the Jordan decomposition of $g \in G$. The restriction of R_g to $\mathcal{O}(G_1)$ is $R_{\phi(g)}$, with similar statements for s and u. Since the restriction of R_s to $\mathcal{O}(G_1)$ is semisimple and the restriction of R_u to $\mathcal{O}(G_1)$ is unipotent, it follows that $R_{\phi(g)} = R_{\phi(s)}R_{\phi(u)}$ is the unique Jordan decomposition of $R_{\phi(g)}$. Hence, $\phi(g) = \phi(s)\phi(u)$ is the Jordan decomposition of $\phi(g)$.

Now consider the case where $\phi : G \to G_1$ is the inclusion of G as an algebraic subgroup of G_1. We argue, as in the proof of the previous theorem, that the ideal I of functions in $\mathcal{O}(G_1)$ which vanish on $\phi(G)$ is stabilized by $\phi(g)$, for $g \in G$, and is, therefore, also stabilized by the components of the Jordan decomposition of $\phi(g)$ on $\mathcal{O}(G_1)$. It follows, as in the previous theorem, that these components must then belong to $\phi(G)$, and hence, must be $\phi(s)$ and $\phi(u)$, where $g = su$ is the Jordan decomposition of g in G. This proves the proposition in the case of an inclusion.

By Corollary 15.2.5, every morphism of algebraic groups factors as a quotient map followed by an inclusion. Thus, combining the two previous paragraphs gives the proof in general.

15.4.4 Corollary. *If G is an affine algebraic group, $g \in G$, and $g = su$ is the Jordan decomposition of g as an element of G, then for any finite dimensional algebraic representation (π, V) of G, $\pi(g) = \pi(s)\pi(u)$ is the Jordan decomposition of $\pi(g)$ as an invertible linear transformation of V.*

Proof. It follows from Lemma 15.4.1 that the ordinary Jordan decomposition of an element of $GL(V)$ agrees with its abstract Jordan decomposition in the sense of this section. Hence, the corollary follows directly from Proposition 15.4.3 applied to the morphism $\pi : G \to GL(V)$.

15.5 Tori

Recall from section 15.3 that a character of an algebraic group G is a morphism $\gamma : G \to C^*$. Thus, the characters of G are the functions in $\mathcal{O}(G)$ which are non-vanishing and multiplicative ($\gamma(gg_1) = \gamma(g)\gamma(g_1)$). The characters of an algebraic group themselves form a group under multiplication. Observe that an element $\gamma \in \mathcal{O}(G)$ is a character of G if and only if $\gamma(e) = 1$, and γ is an eigenvector of R_g for each $g \in G$. Then γ is a weight vector for $(\mathcal{O}(G), R)$ with weight γ.

Since characters of G are homomorphisms from G to an abelian group, a group with enough characters to separate points must be abelian. However, not every abelian affine algebraic group has non-trivial characters. In fact, a complex vector space, as an algebraic group under addition, fails to have any characters other than the identically 1 character (Exercise 15.6) – it has characters as a complex Lie group, but not as an algebraic group. The diagonal group $D_n(\mathbb{C})$ is a group which does have enough characters to separate points – the projection on each diagonal entry is a character.

Consider $D_1(\mathbb{C}) = \mathbb{C}^*$, the group of non-vanishing complex numbers under multiplication. Clearly, its characters are the functions of the form z^n for $n = 0, \pm 1, \pm 2, \dots$, and $\mathcal{O}(\mathbb{C}^*)$ is the linear span of these functions. It is easy to see that the same thing is true of $D_n(\mathbb{C}) \simeq \mathbb{C}^{*n}$ – that is, its ring of regular functions is the linear span of its group of characters.

If G is an abelian algebraic group, then we will denote its group of characters by $\widehat{G}$. Note that $\widehat{G}$ is always a linearly independent set of functions on G (Exercise 15.5), and so if it spans $\mathcal{O}(G)$, it forms a basis of $\mathcal{O}(G)$ as a vector space. We will call an abelian algebraic group G for which $\widehat{G}$ is a basis for $\mathcal{O}(G)$ a *d-group*.

15.5.1 Proposition. *If G is a d-group, and H is an algebraic subgroup of G, then*

(i) *H is also a d-group;*
(ii) *H is defined by finitely many equations of the form $\gamma(g) = 1$ for $\gamma \in \widehat{G}$; and*
(iii) *every element of $\widehat{H}$ is the restriction to H of an element of $\widehat{G}$.*

Proof. Since G is affine, and H is a subvariety, $\mathcal{O}(H)$ is the set of restrictions to H of elements of $\mathcal{O}(G)$. Since the latter space is the span of the characters on G, and since the restriction to H of a character of G is a character of H, we conclude that $\mathcal{O}(H)$ is the span of those characters of H which are restrictions to H of characters of G. Since the set of characters of H is linearly independent, by Exercise 15.5, it agrees with the set of restrictions of characters of G, and it forms a basis for $\mathcal{O}(H)$. This proves (i) and (iii).

Since H is an algebraic subvariety, it is defined by finitely many equations of the form $f(g) = 0$ for $f \in \mathcal{O}(G)$. But each such f is a linear combination $f = \sum a_i \gamma_i$ of characters of G, and the restrictions of these characters to H are characters of H. The set of such restrictions is linearly independent. Hence, the only way an equation $\sum a_i \gamma_i = 0$ can hold on H is if the γ_i can be grouped into sets, each of which consists of characters which have the same restriction to H, with the coefficients for characters in a given set summing to 0. This means an equation of this form on H is equivalent to a set of equations of the form $\gamma_i(g) = \gamma_j(g)$ or, setting $\gamma = \gamma_i^{-1}\gamma_j$, a set of equations of the form $\gamma(g) = 1$ with $\gamma \in \widehat{G}$. This proves (ii) and completes the proof of the proposition.

The abelian algebraic groups G which have enough characters to separate points are characterized in the following proposition.

15.5.2 Proposition. *For an abelian algebraic group G, the following are equivalent:*

(i) *the elements of $\widehat{G}$ separate the points of G;*
(ii) *G is isomorphic to an algebraic subgroup of $D_n(\mathbb{C})$ for some n;*
(iii) *G is a d-group;*
(iv) *every element of G is semisimple.*

Proof. If (i) is true, then there are characters $\gamma_1, \dots, \gamma_n$ such that the map $g \to (\gamma_1(g) \dots, \gamma_n(g)) : G \to \mathbb{C}^{*n} \simeq D_n(\mathbb{C})$ is injective. It is clearly a morphism of algebraic groups, and hence, it is an isomorphism of G onto an algebraic subgroup of $D_n(\mathbb{C})$, by Corollary 15.2.5. Thus, (i) implies (ii).

That (ii) implies (iii) follows from the preceding proposition and the fact that $D_n(\mathbb{C})$ is a d-group.

If (iii) is true, then $\widehat{G}$ is a basis for $\mathcal{O}(G)$. However, each character of G is an eigenvector for R_g for every $g \in G$. Thus, each element of G is semisimple.

A commuting family of semisimple transformations of a finite dimensional vector space has a set of common eigenvectors which forms a basis for the vector space (Exercise 14.10). Since $\mathcal{O}(V)$ is a union of finite dimensional subspaces which are invariant under the right regular representation, (iv) implies that $\mathcal{O}(G)$ has a basis consisting of elements which are eigenvectors for each R_g. Such an element γ, if normalized so that $\gamma(e) = 1$, is necessarily a character. We conclude that $\mathcal{O}(G)$ is spanned by characters, in this case, and hence, there are enough characters to separate points. This proves that (iv) implies (i)

A *complex torus* is an algebraic group which is isomorphic to $D_n(\mathbb{C})$. Obviously, every complex torus is a d-group. Conversely, we have:

15.5.3 Theorem. *Every connected d-group is a complex torus.*

Proof. Let G be a connected d-group. Since G is a d-group, it is isomorphic to an algebraic subgroup of $D_n(\mathbb{C})$ for some n. By Proposition 15.5.1(iii), $\widehat{G}$ is a quotient of the character group of $D_n(\mathbb{C})$. Since $D_n(\mathbb{C})$ is a product of n copies of $\mathbb{C}^*$, its group of characters is $\mathbb{Z}^n$. This implies $\widehat{G}$ is a finitely generated abelian group and, hence, is isomorphic to a product of $\mathbb{Z}^k$ with a finite abelian group. However, $\widehat{G}$ contains no non-trivial elements of finite order; for if $\gamma \in \widehat{G}$ and $\gamma^m = 1$, then γ maps G to the group of mth roots of unity. Since G is connected, this implies that $\gamma = 1$. We conclude that $\widehat{G} \simeq \mathbb{Z}^k$.

Let $\gamma_1, \dots, \gamma_k$ be an independent set of generators for $\widehat{G}$. Then

$$\phi(g) = (\gamma_1(g), \dots, \gamma_k(g))$$

defines an injective morphism $\phi : G \to D_n(\mathbb{C})$. The image is an algebraic subgroup of $D_n(\mathbb{C})$, and if it is proper, Proposition 15.5.1(ii) implies it is defined by equations of the form $\lambda(g) = 1$, for some non-empty set of characters $\lambda \neq 1$ on $D_n(\mathbb{C})$. However, if such an equation holds, it means that $\lambda \circ \phi = 1$, and this is a non-trivial relation among the γ_i. Since the γ_i were independent generators of the free abelian group $\widehat{G}$, there are no such relations. Hence, $\phi(G)$ is not a proper subgroup of $D_n(\mathbb{C})$, and ϕ is an isomorphism of G onto $D_n(\mathbb{C})$.

Let G be a connected, affine, algebraic group, and $H \subset G$ an algebraic subgroup of G which is a complex torus. Let (π, V) be any finite dimensional algebraic representation of G. Since each element of H is semisimple, by Proposition 15.5.2, V decomposes as a direct sum of subspaces V_i, each of which is the weight space for some character γ_i of H. We will call the corresponding elements $\gamma_i \in \widehat{H}$ the *weights* of H for the representation π.

We claim that there is always an element $h \in H$ which separates the elements γ_i. This is because each of the equations $\gamma_i(h)\gamma_j^{-1}(h) = 1$ defines a codimension 1 subgroup of H, and there are only finitely many such equations. If h does separate the elements γ_i, then for each i, V_i is exactly the subspace of eigenvectors for $\pi(h)$ with eigenvalue $\gamma_i(h)$. An element $g \in G$ for which $\pi(g)$ commutes with h must then stabilize each V_i. This implies that $\pi(g)$ commutes with every element of $\pi(H)$. If the representation π is faithful, then this implies that the centralizer of h in G is the same as the centralizer of H in G. Since every affine algebraic group has a faithful finite dimensional representation, we have proved the following proposition.

15.5.4 Proposition. *If G is an affine algebraic group, and H is an algebraic subgroup of G which is a complex torus, then there exists an element $h \in H$ such that the centralizer of h in G is equal to the centralizer of H in G.*

We will call an element $h \in H$ with the property described in Proposition 15.5.4 a *regular* element of H relative to G. Note that this notion depends, not just on H, but also on the embedding of H in G.

15.6 Solvable Algebraic Groups

In this section we study the structure of solvable algebraic groups. A key result is a fixed point theorem for actions of solvable algebraic groups. This result has a number of useful consequences, some of which will be discussed here.

A *solvable algebraic group* is an algebraic group G for which there is a filtration $e = G_n \subset \cdots \subset G_1 \subset G_0 = G$ by normal algebraic subgroups such that G_i/G_{i+1} is abelian for $i = 0, \ldots, n-1$. Note that since each normal algebraic subgroup of G corresponds to an ideal of the Lie algebra $\mathfrak{g}$, and since a connected Lie group is abelian if and only if its Lie algebra is abelian, a filtration, as above, implies the existence of a filtration $\{\mathfrak{g}_i\}$ of $\mathfrak{g}$ by ideals such that the subquotients $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ are abelian. Thus, a solvable algebraic group has a solvable Lie algebra. The converse is also true and can be proved easily using the results of this section. We leave the proof as an exercise (Exercise 15.11).

15.6.1 Theorem. *If G is a connected solvable affine algebraic group, and X is a projective variety with an algebraic G-action $(g, x) \to gx$, then there is a point $x_0 \in X$ such that $gx_0 = x_0$ for every $g \in G$.*

Proof. We prove this by induction on the dimension of G. The theorem is clearly true if G is 0-dimensional. Thus, we assume it is true for all groups of dimension less than n and let G be a group of dimension n.

Let $e = G_n \subset \cdots \subset G_1 \subset G_0 = G$ be a filtration of G with abelian subquotients G_i/G_{i+1}. The commutator subgroup C of G is the subgroup generated by the image Y of the morphism of algebraic varieties

$$(g, h) \to ghg^{-1}h^{-1} : G \times G \to G. \tag{15.6.1}$$

Note that $C \subset G_1$, since G/G_1 is abelian. Thus, we still have a filtration of G by closed subgroups with abelian subquotients if we replace G_1 by the closure $\overline{C}$ of C and each remaining G_i by its intersection with $\overline{C}$. Since G is connected, Y is connected, as is each of the sets Y^k. Since C is the union of these sets, and since they form an increasing sequence of sets (due to the fact that $e \in Y$), it follows that C is connected, as is its closure G_1.

We now have that G_1 is a group of dimension less than $\dim G$ which also satisfies the hypotheses of the theorem. Hence, there is a fixed point for its action on X. Let X_1 be the set of all fixed points for the action of G_1. Note that X_1 is defined by a set of algebraic equations and, hence, is a subvariety of X. Note also that $GX_1 = X_1$, since G_1 is normal in G. By Proposition 15.2.2, there is a closed orbit Q for the action of G on X_1. The orbit Q is connected and projective, since X is projective, and Q is closed in X_1 and, hence, in X. However, the isotropy group of any point of Q contains G_1, and so Q is isomorphic to a quotient of the abelian group G/G_1, by Proposition 15.2.4. Since G is affine, G/G_1 and Q are also affine, by Theorem 15.3.4. The only projective varieties that are also affine are finite sets. Since Q is connected, we conclude that Q is a point. Hence, it is a fixed point for the action of G on X.

One consequence of the above theorem is that a finite dimensional representation π of a solvable affine algebraic group can be put in triangular form. That is, there is a basis for the representation space relative to which each operator $\pi(g)$ is a lower triangular matrix. This is the content of the following proposition.

15.6.2 Proposition. *If (π, V) is a finite dimensional algebraic representation of a solvable affine algebraic group G, then there exists a filtration*

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of V, by invariant subspaces for π, such that each subquotient V_{i+1}/V_i is 1-dimensional.

Proof. The argument is by induction on the dimension of V. Let n be $\dim V$, and suppose the corollary is true of representations on spaces of dimension less than n. By Theorem 15.6.1 there is a fixed point for the action of G on the projective space $P(V)$ induced by π. This means there is a 1-dimensional subspace V_1 of V which is invariant for the representation π. Applying the induction assumption to the representation that π induces on $V' = V/V_1$ yields a filtration $(0) = V_1' \subset V_2' \subset \cdots \subset V_n' = V'$ by invariant subspaces, with each of codimension 1 in the next. Letting $V_2, \ldots, V_n$ be the inverse images of $V_2', \ldots, V_n'$ under the projection $V \to V/V_1$ gives us the required filtration of V.

An algebraic group is called *unipotent* if each of its elements is a unipotent element.

15.6.3 Proposition. *A connected abelian affine algebraic group is isomorphic to the product of a torus and an abelian unipotent group.*

Proof. Let G be a connected abelian affine algebraic group. By Proposition 15.1.6 there is a faithful finite dimensional algebraic representation (π, V) of G, and by Corollary 15.2.5, the morphism $\pi : G \to \pi(G)$ is an isomorphism onto an algebraic subgroup of $GL(V)$. By the previous proposition, there exists a filtration $(0) = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V by invariant subspaces for π, with each V_i of codimension 1 in the next.

Let H be the set of all elements $h \in G$ such that $\pi(h)$ is semisimple. Since G is abelian, each of the invariant subspaces V_i has a basis consisting of common eigenvectors for all the elements of $\pi(H)$ (Exercise 14.10). In fact, a basis $v_1, \dots, v_n$ for V can be chosen so that each v_i is a common eigenvector for the elements of $\pi(H)$, and for each $k \le n$, the set $\{v_1, \dots, v_k\}$ is a basis for V_k. Relative to this basis, each element of $\pi(G)$ is lower triangular, and H is exactly the set of elements $h \in H$ for which $\pi(h) \in D_n(\mathbb{C})$. It follows that H is an algebraic subgroup of G. It is a d-group, since it is isomorphic to a subgroup of $D_n(\mathbb{C})$.

Let N be the subgroup of $GL(V)$ consisting of elements which, relative to the above basis, are lower triangular matrices with all 1's on the diagonal. This is an algebraic subgroup of $GL(V)$, and so $M = \pi^{-1}(N)$ is an algebraic subgroup of G. Every element of N is unipotent, and since π is faithful, M is a unipotent group. Clearly, $M \cap H = \{e\}$, and $H \cdot M = G$ (by the Jordan decomposition). Thus, the map $(h, m) \to hm : H \times M \to G$ is a bijective morphism of algebraic groups. It is an isomorphism, by Corollary 15.2.5.

Each proper algebraic subgroup of a connected algebraic group must have lower dimension, since it is a proper subvariety. It follows that any strictly increasing sequence of connected algebraic subgroups must be finite and, hence, must have a maximal element. This implies that each torus in an algebraic group is contained in a maximal torus.

15.6.4 Proposition. *Let G be a connected solvable affine algebraic group. Then the set of all unipotent elements of G is a normal algebraic subgroup N of G, and G/N is a complex torus.*

Proof. Propositions 15.1.6 and 15.6.2 imply that there is a faithful representation (π, V) of G and a basis for V, relative to which π maps G into the group of lower triangular matrices. The unipotent elements of the group of lower triangular matrices are those with all 1's on the diagonal. It follows from the properties of the Jordan decomposition, that an element of G is unipotent if and only if π maps it into this group. This implies that the set of unipotent elements of G is a normal algebraic subgroup, and the quotient by this subgroup is isomorphic to a connected subgroup of $D_n(\mathbb{C})$. Such a subgroup is a complex torus by Theorem 15.5.3.

The group N of the above proposition is obviously the unique maximal unipotent subgroup of G.

15.6.5 Proposition. *If G is a connected solvable affine algebraic group, H is a maximal torus in G, and N is the maximal unipotent subgroup of G, then the quotient map $\phi : G \to G/N$ maps H isomorphically onto G/N.*

Proof. We must show that $\phi|_H : H \to G/N$ is bijective. It is obviously injective, since H consists of semisimple elements, and $N = \operatorname{Ker}\phi$ consists of unipotent elements.

To finish the proof, we must show that ϕ maps H onto G/N. Consider the set of connected algebraic subgroups K of G such that $H \subset K$, and ϕ maps K onto G/N. This set is non-empty, since it contains G. Any strictly decreasing sequence of subgroups in this set must be finite, and hence, the set contains a minimal element. That is, we may choose a minimal connected algebraic subgroup K, containing H, and mapping onto G/N under ϕ. The proof will be complete if we can show that $K = H$.

Let C be the centralizer of H in K – that is, the set of elements of K which commute with every element of H. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{c}$, and $\mathfrak{k}$ be the Lie algebras of G, N, C, and K, respectively. Then $\mathfrak{c}$ is the subalgebra of $\mathfrak{k}$ consisting of elements fixed by $\operatorname{Ad}(h)$ for every $h \in H$. Since H is a torus, $\mathfrak{k}$ decomposes into weight spaces for $\operatorname{Ad}|_H$ and $\mathfrak{c}$ is the weight space for the identically 1 weight. However, every element of $\mathfrak{k}$ is fixed by $\operatorname{Ad}(h)$ mod $\mathfrak{n} \cap \mathfrak{k}$, since G/N is abelian. It follows that the weight spaces for weights other than the identically 1 weight are all contained in $\mathfrak{n} \cap \mathfrak{k}$. In other words, the differential of ϕ maps $\mathfrak{c}$ onto $\mathfrak{g}/\mathfrak{n}$. It follows that ϕ maps the connected component of the identity in C onto G/N, since G/N is connected. From the minimality of K, we conclude that K is the identity component of C. In other words, every element of K commutes with every element of H. Thus, if $k \in K$ and $k = su$ is its Jordan decomposition, then s commutes with H and, hence, belongs to H, since H is a maximal torus. Since, $u \in N$, we have that H and K are equal mod N, and by the definition of K, this means they are equal.

15.6.6 Lemma. *If A is any abelian subgroup of an algebraic group G, and if $g \in G$ normalizes A, then the map $\phi : A \to A$ defined by*

$$\phi(a) = gag^{-1}a^{-1}$$

is a morphism of algebraic groups. If A is connected and $\operatorname{Ker}\phi \cap \operatorname{Im}\phi = (e)$, *then* $\operatorname{Ker}\phi \cdot \operatorname{Im}\phi = A$.

Proof. Since g normalizes A, ϕ maps A to itself . Since A is abelian,

$$\begin{aligned}\phi(ab) &= abgb^{-1}a^{-1}g^{-1} = aga^{-1}b^{-1}g^{-1}b \\ &= aga^{-1}g^{-1}gb^{-1}g^{-1}b = aga^{-1}g^{-1}bgb^{-1}g^{-1} \\ &= \phi(a)\phi(b).\end{aligned}$$

Thus, ϕ is a homomorphism.

If $\operatorname{Ker}\phi \cap \operatorname{Im}\phi = (e)$, then $\operatorname{Ker}\phi \cdot \operatorname{Im}\phi$ is an algebraic subgroup of A of the same dimension as A. If A is connected, it follows that $\operatorname{Ker}\phi \cdot \operatorname{Im}\phi = A$.

15.6.7 Theorem. *If G is a solvable algebraic group, then any two maximal complex tori in G are conjugate.*

Proof. Suppose S and T are maximal tori in G. Let $s \in S$ be a regular element of S relative to G (see Proposition 15.5.4 and the comment which follows it). Then the centralizers of s and S in G agree. Since S is a maximal torus, it contains any torus in its centralizer and, hence, any torus in the centralizer of s. This implies it is the unique maximal torus containing s. Similarly, each conjugate of s is contained in a unique maximal torus. Hence, if we can show that s has a conjugate $x^{-1}sx$ in T, it will follow that $x^{-1}Sx = T$.

To complete the proof, we will show that, given a connected solvable group G and a maximal torus T, every semisimple element s of G has a conjugate in T. We prove this by induction on the dimension of the maximal unipotent subgroup N. If $\dim N = 0$, so that $N = (e)$, there is nothing to prove, since Proposition 15.6.5 implies that G itself is the unique maximal torus in G.

We suppose $N \neq (e)$ and the assertion is true of connected solvable groups with maximal unipotent subgroup of dimension less than $\dim N$. Then there is a non-trivial connected normal abelian algebraic subgroup N_0 of N. Let $\pi : G \to G/N_0$ be the quotient map. Then $\pi(T)$ is a maximal torus in G/N_0. Therefore $\pi(s)$ is conjugate in G/N_0 to an element of $\pi(T)$. This means that there are elements $g \in G$, $t \in T$, and $n \in N_0$ such that $g^{-1}sg = nt$. Thus, our problem is reduced to showing that a semisimple element of the form nt is conjugate to an element of T.

Consider the map $\phi : N_0 \to N_0$ defined by

$$\phi(x) = xtx^{-1}t^{-1}.$$

By Lemma 15.6.6, ϕ is a morphism of algebraic groups. Its kernel is the subgroup C of N_0 consisting of those elements which commute with t. If $\mathrm{c} = xtx^{-1}t^{-1} \in \phi(N_0) \cap C$, then $\mathrm{c}t = xtx^{-1}$, which implies that $\mathrm{c}t$ is semisimple. Since c is unipotent and c and t commute, the uniqueness of the Jordan decomposition implies that $\mathrm{c} = e$. Thus, $C \cap \phi(N_0) = (e)$. We conclude from Lemma 15.6.6, that $C\phi(N_0) = N_0$.

We may now write our element $n \in N_0$ in the form $\mathrm{c}xtx^{-1}t^{-1}$, so that our semisimple element nt becomes $nt = \mathrm{c}xtx^{-1}$, where $\mathrm{c}, x \in N_0$, and c commutes with t. Since N_0 is abelian, c also commutes with x. Thus, $nt = x^{-1}\mathrm{c}tx$. This implies that $\mathrm{c}t$ is semisimple, and again by the uniqueness of the Jordan decomposition, that $\mathrm{c} = e$. We conclude that our semisimple element nt is conjugate to an element of T. This completes the proof.

Finally, we prove a connectedness result that will be useful in the next section. If G is an algebraic group and A and B are subgroups of G, then the *centralizer* of A in B is the group

$$C_B(A) = \{b \in B : ab = ba, \ \forall \ a \in A\}.$$

This is clearly an algebraic subgroup of B if B is an algebraic subgroup of G.

15.6.8 Proposition. *Let G be a solvable Lie group, and let T be any torus in G. Then $C_G(T)$ is connected.*

Proof. Let H be any maximal torus in G containing T. Then $H \subset C_G(T)$. By Proposition 15.6.5, $G = H \cdot N$, where N is the maximal unipotent subgroup of G. Then $C_G(T) = H \cdot C_N(T)$. However, every closed subgroup of a unipotent group is connected (Exercise 15.13). Thus, $C_N(T)$ is connected, from which it follows that $C_G(T)$ is connected.

15.7 Semisimple Groups and Borel Subgroups

We will say that an algebraic group is *semisimple* if it is affine and contains no non-trivial connected abelian normal algebraic subgroup. If an affine algebraic group G has a semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ contains no non-trivial abelian ideals, and so it follows that G is semisimple as an algebraic group.

If G is any affine algebraic group, and $\mathfrak{g}$ is its Lie algebra, then $\mathfrak{g}$ has a unique maximal solvable ideal $\mathfrak{r}$, called the radical of $\mathfrak{r}$, and $\mathfrak{g}/\mathfrak{r}$ is semisimple (Proposition 14.4.8). Because $\mathfrak{r}$ is a maximal solvable ideal, it is exactly the set of elements $x \in \mathfrak{g}$ such that ad_x maps $\mathfrak{g}$ into $\mathfrak{r}$ – that is, the set of x such that ad_x induces the zero map on $\mathfrak{g}/\mathfrak{r}$. It follows that the group consisting of elements $g \in G$ such that Ad_g induces the identity map on $\mathfrak{g}/\mathfrak{r}$ is a normal algebraic subgroup of G, with Lie algebra $\mathfrak{r}$ (Exercise 14.23). The identity component of this group is called the *radical* of G and will be denoted $R(G)$. The quotient $G/R(G)$ has Lie algebra $\mathfrak{g}/\mathfrak{r}$ and, hence, is semisimple. Since $\mathfrak{r}$ is solvable, $R(G)$ is solvable, by Exercise 15.11. The fact that $G/R(G)$ is semisimple implies that $R(G)$ contains every connected solvable normal algebraic subgroup of G, and so it is the unique maximal such subgroup. Note that $\mathfrak{r} = \{0\}$ if and only if $R(G) = \{e\}$, and this happens if and only if G is semisimple. In summary, we have:

15.7.1 Proposition. *Let G be an affine algebraic group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ and $R(G)$ the radical of G. Then*

(i) *$\mathfrak{r}$ is the Lie algebra of $R(G)$;*

(ii) *$R(G)$ is the unique maximal connected solvable normal algebraic subgroup of G; and*
(iii) *G is semisimple if and only if $R(G) = \{e\}$, and this is true if and only if $\mathfrak{g}$ is semisimple.*

Borel subalgebras play a big role in the structure theory of semisimple Lie algebras, and it should be expected that their analogues for algebraic groups would be equally important. Recall that a Borel subalgebra of a Lie algebra is a maximal solvable subalgebra. Similarly, by a *Borel subgroup* of an algebraic group G we will mean a maximal connected solvable algebraic subgroup of G. The connection between Borel subgroups of an algebraic group G and Borel subalgebras of its Lie algebra $\mathfrak{g}$ is the obvious one, as the next proposition shows.

15.7.2 Proposition. *Let G be an affine algebraic group. Then*
(i) *every Borel subalgebra of $\mathfrak{g}$ is the Lie algebra of a Borel subgroup of G;*
(ii) *every Borel subgroup of G has a Borel subalgebra of $\mathfrak{g}$ as its Lie algebra;*
(iii) *if G is semisimple, every Cartan subalgebra of $\mathfrak{g}$ is the Lie algebra of a maximal torus in G;*
(iv) *if G is semisimple, and H is a maximal torus of G, then the Lie algebra $\mathfrak{h}$ of H is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. If $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, let B be the identity component of the normalizer of $\mathfrak{b}$ in G (the algebraic subgroup of G consisting of those elements $b \in G$ such that $\mathrm{Ad}_b(\mathfrak{b}) \subset \mathfrak{b}$). Since $\mathfrak{b}$ is maximal solvable, it is its own normalizer in $\mathfrak{g}$ (that is, $\mathfrak{b}$ contains every $x \in \mathfrak{g}$ such that $\mathrm{ad}_x(\mathfrak{b}) \subset \mathfrak{b}$). It follows that B has $\mathfrak{b}$ as its Lie algebra (Exercise 14.23). If B_1 is any connected solvable algebraic subgroup of G containing B, then its Lie algebra is a solvable Lie algebra containing $\mathfrak{b}$ and, hence, equal to $\mathfrak{b}$. It follows that $B_1 = B$ and, from this, that B is maximal solvable. This proves (i).

If B is a Borel subgroup, then its Lie algebra $\mathfrak{b}$ is solvable, and hence, is contained in a maximal solvable Lie subalgebra $\mathfrak{b}_1$ of $\mathfrak{g}$. Then (i) implies that $\mathfrak{b}_1$ is the Lie algebra of a Borel subgroup B_1 containing B. Necessarily, $B = B_1$ and $\mathfrak{b} = \mathfrak{b}_1$. This proves (ii).

The proofs of (iii) and (vi) are similar. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, let H be the identity component of the centralizer of $\mathfrak{h}$ in G (the set of $g \in G$ such that Ad_g fixes each element of $\mathfrak{h}$). Notice that $\mathfrak{h}$ is its own centralizer in $\mathfrak{g}$ because it is maximal abelian. It follows that H has $\mathfrak{h}$ as its Lie algebra (Exercise 14.23). Since each element of $\mathfrak{h}$ is semisimple, the same thing is true of H. Since H is connected, it is a torus, by Proposition 15.5.2 and Theorem 15.5.3. Conversely, every torus has a Lie algebra consisting of semisimple elements. Thus, the fact that $\mathfrak{h}$ is maximal with this property implies that H is a maximal connected, algebraic subgroup of G consisting of semisimple elements. Hence, it is a maximal torus. This proves (iii).

If H is a maximal torus, then its Lie algebra $\mathfrak{h}$ consists of semisimple elements, and it follows from (iii) that it is maximal with this property, since H is maximal. Thus, $\mathfrak{h}$ is a Cartan subalgebra. This proves (iv).

A maximal torus in a semisimple group G will be called a *Cartan subgroup*. Then, for semisimple groups, (iii) and (iv) of the above proposition say that the Lie correspondence is a one to one correspondence between Cartan subgroups of G and Cartan subalgebras of $\mathfrak{g}$. Parts (i) and (ii) say the analogous thing for Borel subgroups and Borel subalgebras.

In this section, we will prove a number of key results about Borel subgroups, among them: that any two Borel subgroups are conjugate, that the quotient of G by a Borel subgroup is a projective variety, and that each Borel subgroup is its own normalizer. The proof of the first two of these results involves knowing that the set of full flags in a vector space forms a projective variety. To prove this, we must digress into a discussion of Grassman varieties and flag varieties.

The *Grassman variety* $\mathfrak{G}_k(V)$ is the set consisting of the k-dimensional subspaces of the complex vector space V. This is a projective variety in a natural way. To prove this, we embed $\mathfrak{G}_k(V)$ as a subvariety of the projective space $P(\bigwedge^k V)$ of the k-fold exterior product $\bigwedge^k V$. The map ϕ that does this is described as follows: If W is a k-dimensional subspace of V, then $\bigwedge^k W$ is a well-defined 1-dimensional subspace of $\bigwedge^k V$. We define $\phi(W)$ to be the point of projective space $P(\bigwedge^k V)$ determined by this 1-dimensional subspace.

We claim that ϕ embeds $\mathfrak{G}_k(V)$ as a Zariski closed subset of $P(\bigwedge^k V)$. To prove this, it is enough to prove that its intersection with U is closed in U, for each set U in an affine open cover of $P(\bigwedge^k V)$. The sets U of the following type form such a cover. We fix a k-dimensional subspace A of V, and then choose a basis $v_1, \dots, v_n$ for V such that $\{v_1, \dots, v_k\}$ is a basis for A. Let B denote the subspace spanned by the remaining basis vectors, and let $\rho : V \to A$ be the projection with kernel B. The vectors of the form $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}$, for $\sigma : \{1, \dots, k\} \to \{1, \dots, n\}$ an increasing function, form a basis for $\bigwedge^k V$. Then the set of vectors in $\bigwedge^k V$ of the form $v_1 \wedge \cdots \wedge v_k + y$, where y is an arbitrary linear combination of the other vectors from this basis, is a hyperplane H in $\bigwedge^k V$ which projects onto an affine open subset U of $P(\bigwedge^k V)$.

The inverse image of U under ϕ consists of those k-dimensional subspaces of V which the projection ρ maps isomorphically onto A. These are the subspaces of V spanned by bases of the form $\{v_1 + x_1, \ldots, v_k + x_k\}$, where each x_i is an arbitrary member of B. We set

$$\psi(x_1, \ldots, x_k) = (v_1 + x_1) \wedge \cdots \wedge (v_k + x_k) = \sum_j p_j(x_1, \ldots, x_k).$$

Here, p_j is the sum of the homogeneous terms of degree j in the entries of $(x_1, \ldots, x_k) \in B^k$ in the expansion of $(v_1+x_1)\wedge\cdots\wedge(v_k+x_k)$. In particular,

$$p_1(x_1, \ldots, x_k) = \sum_i v_1 \wedge \ldots v_{i-1} \wedge x_i \wedge v_{i+1} \cdots \wedge v_k$$

is linear in $(x_1, \ldots, x_k)$. It is also clearly bijective as a linear map from B^k to the linear subspace C of $\bigwedge^k V$ spanned by the basis vectors formed from the wedge product of $k - 1$ members of $\{v_1, \ldots, v_k\}$ and one member of $\{v_{k+1}, \ldots, v_n\}$.

Let $\eta : \bigwedge^k V \to B^k$ be the composition of the linear projection of $\bigwedge^k V$ onto C and the inverse of p_1. Then the image of ψ is the intersection of H with the subset of $\bigwedge^k V$ defined by the equation

$$u - \psi \circ \eta(u) = 0. \tag{15.7.1}$$

This is a system of polynomial equations in the coefficients of $u \in \bigwedge^k V$. Hence, the image of ψ is a subvariety of H, and its projection into $P(\bigwedge^k V)$ is a subvariety of U. In other words, the image of ϕ meets U in a subvariety of U. Since $P(\bigwedge^k V)$ is covered by neighborhoods U of this form, we conclude that the image of ϕ is a subvariety of $P(\bigwedge^k V)$. This gives the Grassman variety $\mathfrak{G}_k(V)$ the structure of a projective variety.

Next, for $k \leq m$, we consider the subspace of $\mathfrak{G}_k(V) \times \mathfrak{G}_m(V)$ consisting of pairs (W_1, W_2) of subspaces of V, of dimensions k and m, respectively, with $W_1 \subset W_2$. We claim that this is a subvariety of $\mathfrak{G}_k(V) \times \mathfrak{G}_m(V)$. To prove this, we consider a neighborhood U in $P(\bigwedge^k V)$ of the form described above, so that U is the projection in $P(\bigwedge^k V)$ of a hyperplane $H \subset \bigwedge^k V$. We construct the map $\eta : \bigwedge^k V \to B^k$ so that $H \cap \psi(B^k)$ is defined by the equation (15.7.1). Then, for each $u \in H$, $\eta(u) = (x_1, \ldots, x_k)$ determines the k-dimensional subspace $W(\eta(u))$ with basis $\{v_1 + x_1, \ldots, v_k + x_k)$. This subspace is contained in an m-dimensional subspace W' if and only if $\bigwedge^m W' \wedge (v_i + x_i) = 0$ for $i = 1, \ldots, k$. Since η is a polynomial map, this condition defines an algebraic subvariety of $U \times \mathfrak{G}_m(V)$. Since neighborhoods of the form U cover $G_k(V)$, the claim is proved.

A *flag* for V is a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V by subspaces. If V_i/V_{i-1} is 1-dimensional for each i, then the flag is a *full flag*. The *flag variety* of V is the set of all full flags. This can be described as the set $\mathfrak{F}(V)$ of all points $(V_0, V_1, \ldots, V_n) \in \mathfrak{G}_0(V) \times \cdots \times \mathfrak{G}_n(V)$ such that $V_{i-1} \subset V_i$ for $i = 1, \ldots, n$. In view of the previous paragraph, $\mathfrak{F}(V)$ is a subvariety of $\mathfrak{G}_0(V) \times \cdots \times \mathfrak{G}_n(V)$. Since the Cartesian product of projective varieties is a projective variety (Exercise 12.6), we conclude that the flag variety $\mathfrak{F}(V)$ has a natural structure of a projective variety. This fact leads to the following result concerning Borel subgroups.

15.7.3 Theorem. *If G is a connected affine algebraic group, and B is a Borel subgroup of B, then*
(i) *G/B is a projective variety; and*
(ii) *every Borel subgroup of G is conjugate to B.*

Proof. If we can establish (i) and (ii) for one Borel subgroup, then they will be true of all Borel subgroups. Thus, we may assume that B has maximal dimension among all Borel subgroups. Let (π, V) be a representation of G with the property that there is a 1-dimensional subspace V_1 such that B is the stabilizer of V_1 (Lemma 15.3.2). Since B is solvable, Proposition 15.6.2 implies that there is a full flag for V/V_1 which is stabilized by B – that is, each subspace in the flag is invariant for B. Pulling this back to V yields a full flag $V_0 \subset V_1 \subset \cdots \subset V_n$ for V such that B is exactly its stabilizer. Then G/B is isomorphic to the orbit X of this flag in $\mathfrak{F}(V)$.

We claim X is an orbit of minimal dimension and, hence, is closed. If not, let $y \in \mathfrak{F}(V)$ be a point in an orbit of lower dimension. Let K be the kernel of the representation π. The stabilizer in G/K of a full flag is necessarily solvable (Exercise 15.9), and K is a normal subgroup of G which is also solvable, since it is a subgroup of B. It follows that the isotropy group of y is solvable, its identity component is a subgroup of a Borel subgroup, and so its dimension is less than or equal to the dimension of B. This is a contradiction, since the quotient of G by this subgroup is an orbit of dimension less than that of X. Thus, G/B is isomorphic to a closed orbit X in $\mathfrak{F}(V)$ and is, therefore, a projective variety. This completes the proof of (i).

Let B' be any other Borel subgroup of G and consider the action of B' on X. By Theorem 15.6.1, there is a fixed point x_0 in X for this action. If $x \in X$ is the point with B as its isotropy group and if $gx = x_0$, then $B'gx = gx$ and so $g^{-1}B'g \subset B$. Since B' is a Borel subgroup, it is maximal connected solvable and, hence, must be equal to B. This proves (ii).

The next proposition says, among other things, that most elements of a semisimple algebraic group belong to a Cartan subgroup.

15.7.4 Proposition. *Let G be a semisimple algebraic group. If H is a Cartan subgroup of G, then the union of all conjugates of H contains a dense open subset of G. If B is a Borel subgroup of G, then the union of all conjugates of B is G.*

Proof. Consider the map $\phi : G \times H \to G$ defined by $\phi(g,h) = ghg^{-1}$. Note that ϕ is constant on each coset $gH \times h$ of the subgroup $H \times e$ of $G \times H$. It, therefore, defines a morphism of algebraic varieties $\psi : G/H \times H \to G$.

The fiber of ψ over $x \in G$ is the set

$$\{(gH, h) : \ g \in G, \ h \in H, \ ghg^{-1} = x\}.$$

If $x = \psi(H, h_0) = h_0 \in H$, then this fiber consists of all points of the form $(gH, g^{-1}h_0g)$ with $g^{-1}h_0g \in H$.

Note that $g^{-1}h_0g \in H$ implies that $g^{-1}h_0g$ commutes with every element of H, and this implies that gHg^{-1} is contained in the centralizer of h_0. If we choose h_0 to be a regular element of H, it follows that gHg^{-1} is contained in the centralizer of H (Proposition 15.5.4). Since H is a maximal torus, this implies that $gHg^{-1} = H$. In this case, Ad_g is an automorphism of $\mathfrak{g}$ which leaves the Lie algebra $\mathfrak{h}$ of H invariant. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, by Proposition 15.7.2, the restriction of such an automorphism to $\mathfrak{h}$ determines and is determined by a permutation of the roots in $\mathfrak{h}^*$. The set of such automorphisms of $\mathfrak{h}$ is finite. On the other hand, if Ad_g is the identity automorphism on $\mathfrak{h}$, then g belongs to the centralizer C of $\mathfrak{h}$ in G. Since $\mathfrak{h}$ is maximal abelian, C and H have the same Lie algebra, and hence, H is the identity component of C. Then C/H is finite (Exercise 15.7). It follows that, for regular h_0, the set of cosets gH, for which $g^{-1}h_0g \in H$, is finite. We conclude that the fiber of ψ which lies over $x = h_0$ is finite, in this case. This implies that the dimension of the closure of the image of ψ is $\dim G/H + \dim H = \dim G$ (Proposition 8.8.3), and hence, that the image of ψ contains an open dense set (Proposition 8.8.1).

Now let B be a Borel subgroup. Let Y be the subset of $G/B \times G$ consisting of elements of the form (gB, gbg^{-1}) for $g \in G$ and $b \in B$. The inverse image of Y under the projection $G \times G \to G/B \times G$ is

$$\{(g, gbg^{-1}) : g \in G, b \in B\} = \{(g, g_1) : g, g_1 \in G, g^{-1}g_1g \in B\},$$

and this is clearly closed in $G \times G$. Since the projection $G \times G \to G/B \times G$ is an open map, it follows that Y is closed in $G/B \times G$. The union of the conjugates of B is the image of Y under the projection of $G/B \times G$ on its second factor. Since G/B is projective, by the previous theorem and, hence, complete (Definition 12.5.6 and Exercise 13.17), it follows that this image is closed in G. Thus, the union of the conjugates of B is closed in G. However,

B contains a Cartan subgroup H and the union of the conjugates of H is dense. We conclude that the union of the conjugates of B is all of G.

15.7.5 Corollary. *If G is a connected semisimple algebraic group and H a Cartan subgroup of G, then the centralizer of H in G is contained in every Borel subgroup of G which contains H. In particular, the center of G is contained in every Borel subgroup of B.*

Proof. Given a Borel subgroup B, containing H, and an element c of the centralizer of H, the preceding proposition implies there is $g \in G$ such that $c \in gBg^{-1}$. That is, gB is a fixed point for c acting on G/B. Let Y be the set of all fixed points of c acting on G/B. Then Y is a subvariety of G/B and, hence, a projective variety. Also, it is invariant under the action of B on G/B, since each element of B commutes with c. It follows from Theorem 15.6.1 that there is a fixed point $y \in Y$ for the action of B on Y. Then y has the form gB, where g belongs to the normalizer of B in G. Since y is also a fixed point of c, $cgB = gB$. This implies that $cB = cgBg^{-1} = gBg^{-1} = B$ and, hence, that $c \in B$.

Proposition 15.7.4 also allows us to prove the following extension of Proposition 15.6.8 to arbitrary connected groups.

15.7.6 Proposition. *If G is a connected affine algebraic group, then each torus T in G has a connected centralizer $C_G(T)$.*

Proof. This is true if G is solvable, by Proposition 15.6.8. To prove it in general, we just need to show that if $x \in C_G(T)$, then there is a Borel subgroup of G containing both x and T, for this implies there is a connected subgroup of $C_G(T)$ containing both x and T, by Proposition 15.6.8.

We know there is a Borel subgroup B containing x, by Proposition 15.7.4. This means that B is a fixed point for the action of x on G/B. Let $Y \subset G/B$ be the set of all fixed points for x. This is a subvariety of G/B and is, therefore, a projective variety. Since the elements of T commute with x, the variety Y is invariant under the action of T on G/B. Since T is solvable, there is a fixed point yB in Y for this action. Since yB is a fixed point for both x and T, the Borel subgroup yBy^{-1} contains both x and T. This completes the proof.

15.7.7 Proposition. *If B is a Borel subgroup of the connected affine algebraic group G, then B is its own normalizer.*

Proof. We first note that $B/R(G)$ is a Borel subgroup of $G/R(G)$. Furthermore, if $B/R(G)$ is its own normalizer in $G/R(G)$, then B is its own normalizer in G. Thus, if the proposition is true for Borel subgroups of the semisimple group $G/R(G)$, then it is true for Borel subgroups of the group G.

We prove the proposition by induction on the dimension of the group G. If the dimension of G is 0, there is nothing to prove. Thus, we assume it is true of all connected algebraic groups of dimension less than n, and we let G be an algebraic group of dimension $n > 0$. By the above remark, we may assume G is semisimple. Let g be an element of G such that $gBg^{-1} \subset B$. We must show that $g \in B$. Let H be a maximal torus of B. Then gHg^{-1} is also a maximal torus of B. By Theorem 15.6.7, there is an element $b \in B$ such that $H = bgHg^{-1}b^{-1}$. If we can show that $bg \in B$, then $g \in B$ also. Hence, without loss of generality, we may assume that $g = bg$ – in other words, that g also normalizes H.

By Lemma 15.6.6, the map $\phi : H \to H$ defined by $\phi(h) = hgh^{-1}g^{-1}$ is a morphism of algebraic groups. Assume that $\operatorname{Ker}\phi$ is discrete (hence, finite). Then $\phi(H)$ has the same dimension as H and, since H is connected, is equal to H. If M is the algebraic subgroup of G generated by g and B, this implies that H is contained in the subgroup (M, M) generated by all commutators $\{xyx^{-1}y^{-1}\}$ for $x, y \in M$. Let (π, V) be a finite dimensional representation of G such that M is the subgroup which stabilizes a certain 1-dimensional subspace W of V (Lemma 15.3.2). Let w be a non-zero element of W. Then there is a character γ of M such that $\pi(x)w = \gamma(x)w$ for $x \in M$. The character γ is trivial on all commutators $\{xyx^{-1}y^{-1}\}$ and, hence, on H. It is also trivial on the unipotent radical N of B, since representations preserve unipotency, and $\mathbb{C}^*$ has no unipotent elements other than 1. It follows that $x \to \pi(x)w$ is constant on B and, hence, induces a morphism of algebraic varieties $G/B \to V$. However, since G/B is projective and V is affine, the image of such a morphism is a single point (necessarily w). This follows from the fact that elements of $\mathcal{O}(V)$ separate points, while $\mathcal{O}(G/B)$ consists of just constant functions. Since M is the stabilizer of W, we conclude that $M = G$. However, G is connected and semisimple and B is normal in M. Thus, if $G = M$, then Proposition 15.7.4 implies that $G = B$, which is impossible, since a group cannot be both solvable and semisimple. We conclude that $\operatorname{Ker}\phi$ cannot be discrete, and hence, it has a non-trivial identity component T.

By its definition, T is a subtorus of H consisting of elements which commute with g. Let $C = C_G(T)$. Then C is a proper connected algebraic subgroup of G containing g, by the previous proposition. We claim that $C \cap B$ is a Borel subgroup of C. If this is true, then the induction hypothesis implies that $g \in C \cap B$. Thus, the proof will be complete once we establish this claim.

Clearly $C \cap B$ is solvable. If we can show that $C/(C \cap B)$ is projective, then it follows from Exercise 15.24 that $C \cap B$ is a Borel subgroup of C. Since $C/(C \cap B)$ is isomorphic to the image of C under the canonical map $\phi : G \to G/B$, we must show that this image is closed in G/B. This is equivalent to showing that $\phi^{-1}(C) = CB$ is closed in G. Let Y be the closure of CB in G. Then for each $y \in Y$ we have a morphism $Y \times T \to G$ defined by

$$(y, t) \to y^{-1}ty.$$

This morphism maps $CB \times T$ into B and so it also maps $Y \times T$ into B. Let ψ be this morphism, followed by the quotient map $B \to B/N$. Then ψ is a morphism of varieties from $Y \times T$ to a torus B/N. Furthermore, if we define $\psi_y : T \to B/N$ by $\psi_y(t) = \psi(y, t)$, then ψ_y is a morphism of algebraic groups for each $y \in Y$. However, morphisms between tori are subject to a kind of rigidity (Exercise 15.15), which in this context, implies that ψ_y is independent of y. We conclude that $\psi_y = \psi_e$ for each $y \in Y$. That is, $y^{-1}ty = t \mod N$ for each $y \in Y, t \in T$. In particular, $y^{-1}Ty \subset TN$.

Now TN is a solvable group with both T and yTy^{-1} as maximal tori. By Theorem 15.6.7, $y^{-1}Ty$ is conjugate to T in TN – that is, there exists $n \in N$ such that $n^{-1}y^{-1}Tyn = T$. Thus, yn belongs to the normalizer Q of T in G. This means $y \in QN \subset QB$. Thus, we have $CB \subset Y \subset QB$. By Exercise 15.16, the normalizer Q of T has the centralizer C of T as its identity component (another consequence of the rigidity of tori). Thus, QB is a finite union of sets of the form CxB with $x \in Q$. Any two of these sets are either equal or disjoint (they are inverse images in B of orbits of C acting on G/B). Hence, Y is contained in a finite disjoint union of sets of the form CxB. Since Y is closed under the action of C on the left and B on the right, it contains any set of this form which it meets. Thus, Y is equal to the disjoint union of finitely many sets of the form CxB. Furthermore, since an orbit is open in its closure, CB is open in Y. Thus, the other sets CxB which lie in Y lie in the boundary of Y. However, since a set of this form has the same dimension as CB, it cannot lie in the boundary of CB. We conclude that $Y = CB$. This completes the proof.

15.7.8 Theorem. *Let G be a connected, affine, algebraic group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. Then, under the adjoint representation of G on $\mathfrak{g}$,*

(i) *the stabilizer of $\mathfrak{b}$ is a Borel subgroup B with Lie algebra $\mathfrak{b}$;*

(ii) *G acts transitively on the set of all Borel subalgebras of $\mathfrak{g}$.*

Proof. The stabilizer B of $\mathfrak{b}$ under Ad is an algebraic subgroup of G with Lie algebra equal to the stabilizer of $\mathfrak{b}$ under ad. Since $\mathfrak{b}$ is maximal solvable, it is its own stabilizer under ad. Hence, B is an algebraic subgroup of G with Lie algebra $\mathfrak{b}$.

Since the Lie algebra $\mathfrak{b}$ of B is solvable, B is a solvable algebraic group by Exercise 15.11. Furthermore, its identity component B^0 is a maximal connected solvable algebraic subgroup, since any larger such subgroup would have to have as Lie algebra a solvable Lie subalgebra of $\mathfrak{g}$ larger than $\mathfrak{b}$.

Thus, we have that B^0 is a Borel subgroup of G. However, B normalizes B^0 and each Borel subgroup is its own normalizer, by Proposition 15.7.7. Thus, $B = B^0$, and we conclude that B is a Borel subgroup of G. This completes the proof of (i).

Part (ii) now follows from part (i) and Theorem 15.7.3(ii).

Together, parts (i) and (ii) of the above theorem allow us to identify the set X of all Borel subalgebras of G with the projective variety G/B, where B is any Borel subgroup of G. This is a key result in the study of complex semisimple Lie groups. To know that it applies to every complex semisimple Lie group, we need to know that each such group has the structure of an affine algebraic group. Proving this is the goal of the next section.

15.8 Complex Semisimple Lie Groups

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If we fix a basis $\{x_i\}$ for $\mathfrak{g}$ as a complex vector space, then the condition that a linear transformation a of $\mathfrak{g}$ be an automorphism can be written as $a[x_i, x_j] = [ax_i, ax_j]$ for all i, j. This is a finite set of algebraic equations in the matrix entries of a. Thus, $\mathrm{Aut}(\mathfrak{g})$ is an algebraic subvariety of $GL(\mathfrak{g})$, and as such, it has the structure of an affine algebraic group. If $\mathrm{Aut}^0(\mathfrak{g})$ denotes the connected component of the identity in $\mathrm{Aut}(\mathfrak{g})$, then $\mathrm{Aut}^0(\mathfrak{g})$ is a connected affine algebraic group with Lie algebra $\mathfrak{g}$. Hence, a complex semisimple Lie algebra is the Lie algebra of at least one complex Lie group which has the structure of an affine algebraic group. Actually, every connected complex semisimple Lie group G has the structure of an affine algebraic group. Furthermore, every finite dimensional holomorphic representation of G is an algebraic representation of the corresponding algebraic group. We will prove both of these statements below. The proofs are beautiful applications of the Peter-Weyl theorem (Theorem 14.2.9).

Theorem 14.8.9 tells us that every connected complex semisimple Lie group G has a compact real form – that is, a compact connected subgroup K, such that the Lie algebra of G is the complexification of the Lie algebra of K. This means that each point of K has a neighborhood U in G which is biholomorphic to a neighborhood of 0 in $\mathbb{C}^n$, in such a way that $U \cap K$

is mapped to $V \cap \mathbb{R}^n$. Consequently, any holomorphic function on G, which vanishes on K, vanishes identically on G.

15.8.1 Proposition. *If G is a connected complex semisimple Lie group, and K is a compact real form of G, then every finite dimensional continuous representation of K is the restriction to K of a unique holomorphic representation of G.*

Proof. Let (σ, V) be a continuous, finite dimensional representation of K. The corresponding representation of the Lie algebra $\mathfrak{k}$ of K, also denoted σ, has an extension to a complex linear representation (π, V) of the complexification $\mathfrak{g}$ of $\mathfrak{k}$. This, in turn, induces a holomorphic representation, also denoted (π, V), of the simply connected covering group $\tilde{G}$ of G. By Corollary 14.8.10, the covering map $\tilde{G} \to G$ has a finite kernel. Thus, if $\tilde{K}$ is the inverse image of K under $\tilde{G} \to G$, then $\tilde{K}$ is a compact real form of $\tilde{G}$, and $\tilde{K} \to K$ is a covering map with the same kernel C as $\tilde{G} \to G$. It follows from Theorem 14.8.9 that $\tilde{K}$ is connected.

The representation $\pi|_{\tilde{K}}$ has the same differential as σ and, hence, agrees with the pullback of σ to $\tilde{K}$. This implies that π is the identity on C, and hence, that it descends to a representation of $G = \tilde{G}/C$. The resulting representation of G is clearly an extension of σ. It is the unique extension, since any holomorphic function on G, which vanishes on K, must vanish identically.

15.8.2 Proposition. *Every complex semisimple Lie group G has a faithful finite dimensional holomorphic representation.*

Proof. Let K be a compact real form of G. We know of one holomorphic representation of G – its adjoint representation on $\mathfrak{g}$. This has a finite kernel C, which is a subgroup of K. It follows from the Peter-Weyl theorem (Theorem 14.2.9) that there are enough finite dimensional irreducible representations of a compact group to separate points. Thus, we can find a finite dimensional representation of K which separates the points of C. If we extend this to a holomorphic representation of G and take its direct sum with the adjoint representation of G, we obtain the required faithful representation.

The complete reducibility of unitary representations of compact groups (Theorem 14.2.5) implies the corresponding result for finite dimensional holomorphic representations of complex semisimple groups:

15.8.3 Proposition. *If (π, V) is a finite dimensional holomorphic representation of a complex semisimple Lie group G, then every invariant subspace of V has a complementary invariant subspace. Consequently, (π, V) has a direct sum decomposition into irreducible subrepresentations.*

Proof. If σ is the restriction of π to a compact real form K of G, then there is an inner product $(\cdot,\cdot)$ on V for which (σ, V) is a unitary representation (Proposition 14.2.2). If W is a subspace of V which is invariant under π, then it and, hence, its orthogonal complement $W^\perp$ are invariant under σ. It follows that $W^\perp$ is also invariant under π. This is due to the fact that a holomorphic function of the form $(\pi(g)u, v)$, for $u \in W^\perp, v \in W$ vanishes for $g \in K$ if and only if it vanishes identically on G.

We can now prove the main results of this section – that complex semisimple Lie groups and their finite dimensional representations are, in fact, algebraic.

15.8.4 Theorem. *Every complex semisimple Lie group G is the complex Lie group associated to an algebraic group.*

Proof. Let (π, V) be a faithful finite dimensional holomorphic representation of G. Let $\overline{G}$ be the Zariski closure of $\pi(G)$ in the algebraic group $GL(V)$. Then $\overline{G}$ is an algebraic subgroup of $GL(V)$ (Exercise 15.1). We will prove that $\overline{G} = \pi(G)$.

We consider the adjoint representation of $GL(V)$ on its Lie algebra $\mathfrak{gl}(V)$. This is an algebraic representation of $GL(V)$. Let $\bar{\mathfrak{g}}$ be the Lie algebra of $\overline{G}$. Via the differential of π and the inclusion $\bar{\mathfrak{g}} \to \mathfrak{gl}(V)$, we may consider $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ to be linear subspaces of $\mathfrak{gl}(V)$. They are, in fact, invariant subspaces for the action of G on $\mathfrak{gl}(V)$ given by the composition of π with Ad. By Proposition 15.8.3, there is a subspace $\mathfrak{z}$ of $\bar{\mathfrak{g}}$ which is complementary to $\mathfrak{g}$ and is also invariant under the action of G. Since $\overline{G}$ is the Zariski closure of $\pi(G)$, and since the adjoint action is algebraic, any subspace of $\mathfrak{gl}(V)$, invariant under G, is also invariant under $\overline{G}$. In particular, $\mathfrak{g}$ and $\mathfrak{z}$ are invariant under $\overline{G}$. Thus, we have an algebraic action of $\overline{G}$ on $\mathfrak{g}$ which extends the adjoint action of G on $\mathfrak{g}$. Then

$$\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{z},$$

where both $\mathfrak{g}$ and $\mathfrak{z}$ are invariant subspaces for the adjoint action of $\overline{G}$. It follows that this is a decomposition of $\bar{\mathfrak{g}}$ as the direct sum of two ideals. This, in turn, implies that $[\mathfrak{g}, \mathfrak{z}] = 0$, and hence, that G fixes each element of $\mathfrak{z}$. Then $\overline{G}$ also fixes each element of $\mathfrak{z}$, from which it follows that $\mathfrak{z}$ is the center of $\bar{\mathfrak{g}}$.

Suppose W is an irreducible invariant subspace of V for the representation π. Again, since $\overline{G}$ is the Zariski closure of $\pi(G)$, and the representation of $\overline{G}$ on V is algebraic, W must also be invariant under $\overline{G}$. It follows from Schur's lemma that each element of the center $\mathfrak{z}$ of $\bar{\mathfrak{g}}$ must act as a multiple of the identity on W. Since $\mathfrak{g}$ is semisimple, it has no non-trivial abelian quotients, and hence, its representation on W must lie in $\mathfrak{sl}(W)$, and the corresponding representation of G must lie in $SL(W)$. It follows that the same is true of $\overline{G}$. Hence, each element of $\mathfrak{z}$ acts as a multiple of the identity with trace 0 – that is, as 0 itself. Since this is true of every irreducible subspace of V for π, and since V decomposes into a direct sum of such subspaces, we conclude that $\mathfrak{z} = 0$. Hence, $\bar{\mathfrak{g}} = \mathfrak{g}$, from which we conclude that $\pi(G) = \overline{G}$, since $\overline{G}$ is connected. Thus, $\pi(G)$ is an algebraic subgroup of $GL(V)$, and G is its associated complex Lie group.

Let G be a complex semisimple Lie group. If (σ, V_σ) is a finite dimensional holomorphic representation of G, $u \in V_\sigma, v^* \in V_\sigma^*$, then the holomorphic function

$$g \to v^*(\sigma(g^{-1})u)$$

is called the *matrix coefficient* determined by σ, u, and v^*. These are the holomorphic versions of the matrix coefficients defined for finite dimensional representations of compact groups in Chapter 14.

The assignment of the matrix coefficient function in $\mathcal{H}(G)$ to the pair u, v^* is bilinear, and hence, determines a linear map $\theta_\sigma : V_\sigma \otimes V_\sigma^* \to \mathcal{H}(G)$ such that

$$\theta_\sigma(u \otimes v^*)(g) = v^*(\sigma(g^{-1})u).$$

Note that

$$\theta_\sigma(\sigma(g_1)u, v^*)(g) = \theta_\sigma(u \otimes v^*)(g_1^{-1}g),$$
$$\theta_\sigma(u, \sigma^*(g_1)v^*)(g) = \theta_\sigma(u \otimes v^*)(gg_1).$$

Thus, θ_σ is an intertwining operator from the representation $\sigma \otimes \sigma^*$ of $G \times G$ on $V_\sigma \otimes V_\sigma^*$ to the representation of $G \times G$ on $\mathcal{H}(G)$ determined by left and right translation.

Now suppose σ is irreducible. Then $\sigma \otimes \sigma^*$ is an irreducible representation of $G \times G$ on $V_\sigma \otimes V_\sigma^*$. It follows that θ_σ is injective in this case. Let $A_\sigma = \mathrm{Im}(\theta_\sigma)$. Then θ_σ is an isomorphism from $V_\sigma \otimes V_\sigma^*$ to A_σ which is an intertwining operator for the corresponding representation of $G \times G$.

The set of matrix coefficients for the direct sum of two representations is clearly the sum of the sets of matrix coefficients of the two representations. Thus, the set of all matrix coefficients is a linear subspace of $\mathcal{H}(G)$. Since every finite dimensional holomorphic representation of G is the direct sum of irreducible holomorphic representations, every matrix coefficient is a sum of matrix coefficients of irreducible holomorphic representations. Thus, the set of all matrix coefficients is $A = \sum_{\sigma} A_\sigma$.

If (σ, V_σ) and (ρ, V_ρ) are irreducible holomorphic representations of G, then $A_\sigma \cdot A_\rho$ is the set of matrix coefficients for the tensor product representation $\sigma \otimes \rho$ on $V_\sigma \otimes V_\rho$. It follows that the set A of matrix coefficients is a subalgebra of $\mathcal{H}(G)$. The next theorem identifies this subalgebra.

15.8.5 Theorem. *Let G be a connected complex semisimple Lie group, and suppose G is equipped with a compatible structure of an affine algebraic group. Then $\mathcal{O}(G)$ is the set A of holomorphic matrix coefficients of G. As a 2-sided G-module, $\mathcal{O}(G)$ decomposes as the direct sum of the modules $A_\sigma \simeq V_\sigma \otimes V_\sigma^*$ for σ an irreducible holomorphic representation of G.*

Proof. The algebra $\mathcal{O}(G)$ is the union of finite dimensional subspaces which are invariant for the left regular representation, by Proposition 15.1.5 . Let V be such a subspace and let π denote the restriction of the left regular representation to V. Let K be a compact real form of G. We may choose an inner product on the vector space V which makes the representation π unitary on K (Proposition 14.2.2). Since the evaluation map $u \to u(e)$ is a linear functional on V, there is an element $v \in V$ such that $u(e) = (u, v)$ for every $u \in V$. Then u is the matrix coefficient

$$u(g) = \pi(g^{-1})u(e) = (\pi(g^{-1})u, v) = v^*(\pi^{-1}(g)u).$$

This implies that $u \in A$. Also, on K the conjugate of u is the function

$$\bar{u}(k) = (\pi(k)v, u) = u^*(\pi(k)v).$$

This is the restriction to K of the regular function $g \to u^*(\pi(g)v)$. We conclude that $\mathcal{O}(G)$ is a subalgebra of A, and its set of restrictions to K is closed under conjugation.

Since $\mathcal{O}(G)|_K$ is a subalgebra of $C(K)$ which is closed under conjugation and separates the points of K, it is dense in $C(K)$, by the Stone-Weierstrass theorem. It follows from the Peter-Weyl theorem, that, for each irreducible holomorphic representation σ of G, the orthogonal projection P_σ of $L^2(G)$ onto $M_\sigma = A_\sigma|_K$ is an intertwining operator for both the left and right regular representations and is given by

$$P_\sigma f(k) = h_\sigma * f(k) = \int h(k_1) f(k_1^{-1}k)\, d\mu(k_1),$$

where h_σ is a central element of A_σ. It follows that the equation

$$P_\sigma f(g) = \int h(k_1) f(k_1^{-1}g)\, d\mu(k_1),$$

for $f \in \mathcal{H}(G)$ and $g \in G$, defines a projection of A onto A_σ which is an intertwining operator for both left and right regular representations of G on A. Note also that P_σ maps each subspace of A which is finite dimensional and invariant under the left regular representation into itself. In particular, $\mathcal{O}(G)$ is mapped into itself by P_σ, since it is a union of finite dimensional left invariant subspaces.

Now we have $\mathcal{O}(G)|_K \subset A|_K \subset C(K)$ and $\mathcal{O}(G)|_K$ dense in $C(K)$. It follows that $P_\sigma \mathcal{O}(G) \subset A_\sigma$, and $P_\sigma \mathcal{O}(G)|_K$ is dense in $M_\sigma = A_\sigma|_K$. Since M_σ is finite dimensional, and the restriction map $f \to f|_K : A_\sigma \to M_\sigma$ is injective, we conclude that $P_\sigma \mathcal{O}(G) = A_\sigma$, and hence, that $A_\sigma \subset \mathcal{O}(G)$. This shows that $\mathcal{O}(G) = \sum A_\sigma$, as required. Since the restrictions of the A_σ to K are orthogonal subspaces of $L^2(G)$, the sum is a direct sum.

15.8.6 Corollary. *Let G be a complex semisimple Lie group. Then G has a unique structure of an affine algebraic group which is compatible with its given structure as a complex Lie group.*

Proof. Any two such structures yield the same algebra of regular functions $\mathcal{O}(G) = \bigoplus_\sigma A_\sigma$.

15.8.7 Corollary. *Let G be a complex semisimple Lie group. Then every finite dimensional holomorphic representation of G is algebraic for the unique algebraic group structure on G.*

Proof. By Theorem 15.8.5 the matrix coefficients of any finite dimensional holomorphic representation of G are regular functions. It follows that the representation is algebraic.

Exercises

1. Let G be an algebraic group, and let H be an arbitrary subgroup of G. Prove that the Zariski closure of H is also a subgroup of G.
2. Recall that the tangent space to an algebraic variety at a point x is the space of point derivations at x of the stalk $\mathcal{O}_x$ of the structure sheaf. Prove that if G is an algebraic group, then each element of the tangent space at $e \in G$ is the value at e of a unique left invariant derivation on $\mathcal{O}$. Here, by the value at x of a derivation $\delta : \mathcal{O} \to \mathcal{O}$ we mean the composition of $\delta_x : \mathcal{O}_x \to \mathcal{O}_x$ with evaluation $\mathcal{O}_x \to \mathbb{C}$ at x.

3. Prove that there is a natural isomorphism between the Lie algebra of an algebraic group G, as defined in section 15.1, and the Lie algebra of the associated complex Lie group, as defined in section 14.3.
4. If (σ, V) is an algebraic representation of an algebraic group G, prove that a set of weight vectors for (σ, V), with weights given by distinct characters, is linearly independent (see section 15.3).
5. Prove that the set of characters of an affine algebraic group G is a linearly independent subset of $\mathcal{O}(G)$.
6. Prove that the additive group of a complex vector space is an abelian affine algebraic group with no characters other than the identically 1 character.
7. Prove that the identity component G^0 of an algebraic group G is a subgroup of finite index (i.e. G/G^0 is finite).
8. Describe the structure of an abelian affine algebraic group.
9. Prove that if (π, V) is a faithful algebraic representation of an affine algebraic group G, then the stabilizer of any full flag for V is a solvable subgroup of G.
10. Prove that an algebraic subgroup of a solvable algebraic group is a solvable algebraic group.
11. Prove that an algebraic group is a solvable algebraic group if its Lie algebra is solvable. This is the converse of the result derived in the second paragraph of section 15.6. (Hint: Use the adjoint representation and the results of the previous two exercises.)
12. Prove that an element of finite order in an affine algebraic group is necessarily semisimple.
13. Let G be an affine algebraic group. Prove that a closed subgroup of G consisting of unipotent elements must be connected.
14. Let T be a complex torus. Prove that the union of the set of elements of finite order is a dense subgroup of T.
15. Prove the rigidity of tori theorem: Let V be a connected algebraic variety, S and T complex tori, and $\psi : V \times T \to S$ a morphism of algebraic varieties. Suppose that $\psi_v : T \to S$ is a morphism of algebraic groups for each $v \in V$, where $\psi_v(t) = \psi(v, t)$. Then ψ_v is constant as a function of v. Hint: First use the result of the previous exercise to prove this in the case where $S = \mathbb{C}^*$, and V contains a point v_0 where $\psi_{v_0} = \mathrm{id}$. Then reduce the general case to this case by using characters of S.
16. Prove that if G is an affine algebraic group, and T is a torus in G, then the normalizer $N_G(T)$ of T in G has the same identity component as the centralizer $C_G(T)$. Hint: Use the previous exercise.
17. Prove that if G is a connected, solvable algebraic group, and T is a torus in G, then the normalizer $N_G(T)$ and centralizer $C_G(T)$ of T in G agree. Hint: Use Exercise 15.13 and Exercise 15.16.

18. Use the result of the previous exercise and Proposition 15.7.7 to prove that if G is a connected affine algebraic group, and H is a maximal torus in G, then $N_G(H)/C_G(H)$ acts simply and transitively, under Ad, on the set of Borel subgroups of G which contain H.
19. Let G be a connected complex semisimple Lie group and let H be a maximal torus in G. Use the result of Exercise 14.24 to prove that every automorphism of G which fixes each point of H is of the form $g \to h^{-1}gh$ for $h \in H$. Then prove that every automorphism of G which stabilizes H is inner.
20. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let G be any connected Lie group with Lie algebra $\mathfrak{g}$. Prove that the Weyl group of $\mathfrak{g}$ is isomorphic to $N_G(H)/C_G(H)$. Hint: Use the results of the preceding two exercises and results from section 14.7.
21. If G is a complex semisimple Lie group, and H a maximal torus in G, prove that $C_G(H) = HZ$, where Z is the center of G.
22. Prove that the sets of upper triangular matrices and lower triangular matrices in $SL_n(\mathbb{C})$ are Borel subgroups. Hint: See Example 14.5.9.
23. Prove that if P is an algebraic subgroup of an algebraic group G, and if G/P is projective, then P contains a Borel subgroup. Hint: Use Theorem 15.6.1.
24. Use the result of the previous exercise to prove that if B is an algebraic subgroup of an algebraic group G, then B is a Borel subgroup if and only if B is solvable and G/B is projective.

Chapter 16

The Borel-Weil-Bott Theorem

In this chapter we give an application to group representation theory of some of the machinery developed in previous chapters. The application is to the study of finite dimensional representations of a complex semisimple Lie group and, specifically, to the proof of the Borel-Weil-Bott theorem.

Theorems 15.7.8 and 15.8.4 imply that, if $\mathfrak{g}$ is a semisimple Lie algebra, then the set of Borel subalgebras of $\mathfrak{g}$ has the structure of a projective algebraic variety X. Furthermore, if G is a connected, complex Lie group with Lie algebra $\mathfrak{g}$, then G has the structure of an algebraic group which acts algebraically and transitively on X. The variety X is called the *flag variety* of $\mathfrak{g}$. The Borel-Weil-Bott theorem calculates the cohomology in each degree of each G-equivariant holomorphic line bundle on X. Specifically: this cohomology is non-vanishing in only one degree, and the G-action on the resulting cohomology space is an irreducible finite dimensional holomorphic representation of G. Every irreducible finite dimensional holomorphic representation of G arises in this way, and the theorem specifies the representation and the degree in which it occurs in terms of a weight in $\mathfrak{h}^*$ determined by the line bundle. A key step in the proof is to relate G-equivariant line bundles on X to the invertible sheaves $\mathcal{O}(k)$ of Chapter 12 and to use the vanishing theorem proved in Chapter 12 for coherent sheaves on projective space.

We will find it useful in this development that we may regard a semisimple Lie group G as either a complex Lie group or as an algebraic group (by Theorem 15.8.4) and that we may work with either the algebraic or the holomorphic theory in dealing with coherent sheaves on X (by the Serre theorems of Chapter 13).

The proof of the Borel-Weil-Bott theorem we give here is due to Miličić [M]. It is not terribly difficult, but it does require detailed knowledge of the structure theory for complex semisimple Lie algebras and their finite dimensional representations, as presented in Chapter 14.

16.1 Vector Bundles and Induced Representations

If G is a Lie group and B a Lie subgroup, then vector bundles on G/B which are G-equivariant (have a G-action compatible with the action of G on G/B) arise from finite dimensional representations of the subgroup B through a procedure called *induction.* In the case where G is complex semisimple, B is a Borel subgroup, and the bundles are holomorphic line bundles, these are the objects of study in the Borel-Weil-Bott theorem. In this section we describe the induction procedure and establish some of its properties.

Although, we are primarily interested in induction in the case where G is a complex semisimple Lie group, and B is a Borel subgroup, we will describe it and develop its basic properties in the more general context where G is any complex Lie group, and B any Lie subgroup. The induction procedure involves using a representation of the subgroup B to construct a locally free G-equivariant sheaf (i.e. a G-equivariant vector bundle) on G/B. Then G acts on the vector space of sections of this sheaf, as well as on each of its cohomology spaces. Thus, each non-vanishing cohomology space of this *induced sheaf* yields a representation of G. In this way, representations of B *induce* representations of G. The same development may be carried out, essentially without change, for real Lie groups G and real Lie subgroups B. This construction also makes sense in the algebraic case, but it is not so obvious that the resulting induced sheaf is locally free.

A holomorphic G-space is a holomorphic variety with a holomorphic action $G \times X \to X$ of G on X. Given a holomorphic G-space X, we define a *G-equivariant holomorphic vector bundle* on X to be a holomorphic vector bundle $\gamma : M \to X$, with a holomorphic G-action $G \times M \to M$, such that the projection $\gamma : M \to X$ is a morphism of G-spaces ($\gamma(gm) = g\gamma(m)$), and such that $m \to gm : M_x \to M_{gx}$ is linear for each $g \in G$ and $x \in X$, where $M_x = \gamma^{-1}(x)$ (see section 7.6 for a discussion of vector bundles). The induction procedure, described below, assigns a vector bundle of this type on $X = G/B$ to each finite dimensional, holomorphic representation of B.

Let (σ, Q) be a finite dimensional, holomorphic representation of B. The space $G \times_B Q$ is constructed from $G \times Q$ by identifying points which lie in the same orbit of the B-action described by $(b, (g, q)) \to (gb^{-1}, \sigma(b)q)$. We give $G \times_B X$ the quotient topology and quotient structure sheaf it inherits from the quotient map $G \times X \to G \times_B X$. It then becomes a ringed space. Projection on the first factor of $G \times Q$, followed by the projection $\rho : G \to X = G/B$, induces a well-defined morphism $\gamma : G \times_B Q \to X$ of ringed spaces. The inverse image of the typical point under this projection is $gB \times_B Q$, which has a well-defined vector space structure under which it is isomorphic to Q.

16.1.1 Proposition. *With the ringed space structure on $G \times_B Q$, the morphism $\gamma : G \times_B Q \to X$, and the vector space structure on each $\gamma^{-1}(x)$, described above, $G \times_B Q$ is a G-equivariant holomorphic vector bundle.*

Proof. Each point of X has a neighborhood V such that $\rho^{-1}(V)$ is isomorphic as a right holomorphic B-space to a product $V \times B$ (Proposition 14.3.4(iii)). For such a neighborhood V, $\gamma^{-1}(V)$ is isomorphic as a ringed space to $V \times B \times_B Q = V \times Q$. It follows that $G \times_B Q$ is a holomorphic manifold. Furthermore, the isomorphism $\gamma^{-1}(V) \simeq V \times Q$ is linear on the fiber over each point of V. Thus, $G \times_B Q$ is a holomorphic vector bundle over X, with fiber Q (see section 7.6). Furthermore, G acts holomorphically on $G \times_B Q$ through its action on the left in the first factor. Clearly, $\gamma : G \times_B Q \to X$ is a morphism of holomorphic G-spaces. Hence, $G \times_B Q$ is a G-equivariant holomorphic vector bundle on X.

Note that the above argument uses the holomorphic version of the implicit function theorem, since it rests on Proposition 14.3.4(iii). For this reason, other methods (along the lines of sections 15.2 and 15.3) are needed to prove the existence of the induced bundle $G \times_B Q$ in the algebraic case. We won't try to do this here. For our purposes, it is enough to develop induction in the holomorphic case. On the other hand, the real case poses no additional difficulties, and the induced bundle is defined exactly as above when G and B are real Lie groups, except that we use the sheaves of real analytic or $\mathbb{C}^\infty$ functions as structure sheaves.

16.1.2 Definition. *Let (σ, Q) be a finite dimensional holomorphic representation of a Lie subgroup $B \subset G$. Then we will call $G \times_B Q$ the induced bundle over X determined by (σ, Q), and denote it by $I(\sigma)$.*

If (π, V) is a finite dimensional holomorphic representation of G, then we may speak of the trivial G-equivariant holomorphic vector bundle over X with fiber V. This is the complex manifold $X \times V$, with projection $X \times V \to X$ just the projection on the first factor, and G-action given by $(g, (x, v)) \to (gx, \pi(g)v)$. The restriction σ of π to B is a representation of B on V, and so we may consider its induced bundle $I(\sigma) = X \times_B V$. In fact, the map

$$(g, v) \to (\rho(g), \pi(g)v) : G \times V \to X \times V$$

induces a G-equivariant vector bundle isomorphism $G \times_B V \to X \times V$. Thus, the trivial G-equivariant bundle $X \times V$ and the induced bundle $I(\sigma)$ are isomorphic. This is part of what is to be shown in the next theorem.

16.1.3 Theorem. *The induction functor $(\sigma, Q) \to I(\sigma)$ is an equivalence of categories from the category of finite dimensional holomorphic representations of B to the category of G-equivariant holomorphic vector bundles over X. Furthermore, the bundle $I(\sigma)$ is trivial if and only if the representation σ is the restriction to B of a holomorphic representation of G.*

Proof. Let $x_0 = \rho(e)$. Then B is the isotropy group of x_0. If $M \to X$ is any G-equivariant vector bundle over X, then the fiber M_{x_0} over x_0 is invariant under B, and hence, is a finite dimensional representation of B. The resulting correspondence $M \to M_{x_0}$ is a functor from G-equivariant holomorphic vector bundles over X to finite dimensional holomorphic representations of B.

Given a finite dimensional holomorphic representation (σ, Q) of B, the fiber $\gamma^{-1}(x_0)$ of $I(\sigma)$ over x_0 is $B \times_B Q$, with B-action given by multiplication on the left factor. There is a B-module isomorphism from Q to this fiber, given by $q \to (e, q) : Q \to B \times Q$, followed by the projection $B \times Q \to B \times_B Q$. Thus, the functor $(\sigma, Q) \to I(\sigma)$ composed with $M \to M_{x_0}$ is isomorphic to the identity.

On the other hand, if M is a G-equivariant holomorphic vector bundle, then we define a map $G \times M_{x_0} \to M$ by $(g, m) \to gm$. This induces a map $G \times_B M_{x_0} \to M$, which is an isomorphism of G-equivariant vector bundles. Thus, the composition $M \to M_{x_0}$ followed by $(\sigma, Q) \to I(\sigma)$ is also isomorphic to the identity. This completes the proof of the first statement of the theorem. We have already proved the second statement in one direction. We leave the other direction as as exercise (Exercise 16.1).

The next proposition gives a useful description of the sheaf of sections of the vector bundle $I(\sigma)$.

16.1.4 Proposition. *The sheaf of sections of $I(\sigma)$ is isomorphic, as an analytic sheaf, to the sheaf $\mathcal{I}(\sigma)$ defined as follows:*

$$\mathcal{I}(\sigma)(U) = \{f \in \mathcal{H}(\rho^{-1}(U), Q) : f(gb^{-1}) = \sigma(b)f(g),\ \forall\ b \in B\}$$

for each open set $U \subset X$.

Proof. Here $\mathcal{H}(\rho^{-1}(U), Q)$ denotes the space of holomorphic Q-valued functions on $\rho^{-1}(U)$.

Let $x_0 = \rho(e)$. As noted in the proof of the previous theorem, there is a B-module isomorphism from Q to the fiber $\gamma^{-1}(x_0) \simeq B \times_B Q$. We use this to identify Q and $\gamma^{-1}(x_0)$. A section of $I(\sigma)$ over an open set $U \subset X$ is a holomorphic function $\tilde{f} : U \to G \times_B Q$ with $\gamma \circ \tilde{f} = \text{id}$. Given such a section, we construct a Q-valued holomorphic function f on $\rho^{-1}(U) \subset G$ as follows:

$$f(g) = g^{-1}\tilde{f}(gx_0).$$

Note, this has its values in $\gamma^{-1}(x_0) \simeq Q$. Then

$$f(gb^{-1}) = bg^{-1}\tilde{f}(gx_0) = \sigma(b)f(g).$$

Thus, a section of the bundle $I(\sigma)$ over U gives rise to a section of the sheaf $\mathcal{I}(\sigma)$ over U.

Conversely, each section $f \in \mathcal{I}(\sigma)(U)$ defines a section $\tilde{f}$ of $I(\sigma)$ by

$$\tilde{f}(x) = gf(g), \text{ if } x = gx_0,$$

where $f(g)$ is interpreted as an element of the fiber $I(\sigma)_{x_0}$, and $gf(g)$ as an element of the fiber $I(\sigma)_{gx_0}$. The resulting section $\tilde{f}$ is well define, because $gbf(gb) = gf(g)$ if $b \in B$. Clearly, the correspondence $f \to \tilde{f}$ defines an isomorphism between the sheaf of sections of the bundle $I(\sigma)$ and the sheaf $\mathcal{I}(\sigma)$.

Note that $\mathcal{I}(\sigma)$ is locally free of finite rank, since it is the sheaf of sections of a vector bundle. It is, therefore, a coherent analytic sheaf.

16.1.5 Example. The group $GL_{n+1}(\mathbb{C})$ acts linearly and algebraically on $\mathbb{C}^{n+1}$, and hence, it acts algebraically on complex projective space P^n. If $\{x_0\}$ is a point of P^n, let G_{x_0} be the isotropy group of this action. The action is transitive, and so $GL_{n+1}(\mathbb{C})/G_{x_0}$ is isomorphic to P^n. Let (σ, Q) be any representation of G_{x_0} on a 1-dimensional vector space Q. We consider $GL_{n+1}(\mathbb{C})$ as a complex Lie group acting holomorphically on P^n as complex manifold. Then Proposition 16.1.4 says that the induced sheaf $\mathcal{I}(\sigma)$ is a $GL_{n+1}(\mathbb{C})$-equivariant locally free rank 1 analytic sheaf – that is, it is the sheaf of sections of a $GL_{n+1}(\mathbb{C})$-equivariant line bundle on P^n. Theorem 16.1.3 says that every $GL_{n+1}(\mathbb{C})$-equivariant line bundle on P^n has this form.

To see what these bundles look like, we let x_0 be the point of P^n corresponding to the 1-dimensional subspace $Q_0 = \{(z, 0, \dots, 0) : z \in \mathbb{C}\}$. Then G_{x_0} consists of all matrices $\{a_{ij}\} \in GL_{n+1}(\mathbb{C})$ with first column entries 0 except for a_{11}. Now Q_0 is invariant under G_{x_0}, and in fact, $\{a_{ij}\} \in G_{x_0}$ acts on Q_0 by multiplication by a_{11}. This means that $\{a_{ij}\} \to a_{11}$ is a homomorphism of G_{x_0} to $\mathbb{C}^*$. Then, for each integer k, the map $\sigma_k : G_{x_0} \to \mathbb{C}^*$ defined by $\sigma_k(\{a_{ij}\}) = a_{11}^k$ is also a homomorphism and, hence, determines a 1-dimensional representation of G_{x_0}. In fact, it is easy to see that every 1-dimensional representation of G_{x_0} has this form. We conclude that the sheaves $\mathcal{I}(\sigma_k)$ are exactly the sheaves corresponding to $GL_{n+1}(\mathbb{C})$-equivariant line bundles on P^k. Of course, we have encountered these sheaves before – they are just the sheaves $\mathcal{H}(k)$ of Chapter 12 (Exercise 16.2). In this case, it is easy to see that the corresponding algebraically induced sheaves are just the sheaves $\mathcal{O}(k)$.

Let (σ, Q) be any finite dimensional holomorphic representation of B, and suppose that $X = G/B$ is compact. Since $\mathcal{I}(\sigma)$ is a coherent analytic sheaf, it follows from the Cartan-Serre theorem (Theorem 11.10.2) that its cohomology is finite dimensional in each degree. We claim that each cohomology space $H^p(X, \mathcal{I}(\sigma))$ has a natural $\mathfrak{g}$-module structure. In fact, if $\xi \in \mathfrak{g}$ and we realize ξ as a right invariant differential operator on ${}_G\mathcal{H}$, then ξ maps $\mathcal{I}(\sigma)(U)$ into itself for each open set $U \subset X$, and hence, it defines an endomorphism of the sheaf $\mathcal{I}(\sigma)$. This shows that there is a natural action of $\mathfrak{g}$ as a Lie algebra of endomorphisms of $\mathcal{I}(\sigma)$. Since $H^p(X, \cdot\,)$ is a functor, it follows that there is a corresponding representation of $\mathfrak{g}$ on $H^p(X, \mathcal{I}(\sigma))$. If G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, then there is a corresponding holomorphic representation of G on each cohomology space. For $p = 0$, this action has the obvious description as the left regular representation of G on $\mathcal{H}(G)$, followed by restriction to $\mathcal{I}(\sigma)(X)$. Even when the group G is not simply connected, it is true that the action of $\mathfrak{g}$ on $\mathcal{I}(\sigma)$ is the differential of a holomorphic representation of G. This is obvious when $p = 0$, since we have an explicit description of the action. For $p \neq 0$, things are not so simple, and the only proof we know of requires developing a substantial amount of machinery. We won't need this result here and so we won't attempt to prove it.

16.2 Equivariant Line Bundles on the Flag Variety

We now turn to the special case of the induction construction which is of interest in the Borel-Weil-Bott theorem. This is induction of 1-dimensional representations of a Borel subgroup of a semisimple group.

Let $\mathfrak{g}$ denote a complex semisimple Lie algebra, and suppose G is a connected Lie group with Lie algebra $\mathfrak{g}$. Corollary 15.8.6 implies that G has a unique structure of an algebraic group. We will use this structure whenever it is convenient, otherwise we will continue to regard G as a complex Lie group.

Recall that a Borel subalgebra of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$, while a *Borel subgroup* of G is a maximal connected solvable algebraic subgroup of G. By Exercise 15.11, a connected affine algebraic group is solvable if and only if its Lie algebra is solvable. Corresponding to each

Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ there is a Borel subgroup B of G with $\mathfrak{b}$ as Lie algebra. This is the stabilizer $B = \{x \in G : \mathrm{Ad}_x(\mathfrak{b}) = \mathfrak{b}\}$ of $\mathfrak{b}$ in G (Theorem 15.7.8). Since any maximal connected solvable subgroup of G as a complex Lie group also has a maximal solvable Lie subalgebra of $\mathfrak{g}$ as Lie algebra, we see that the Borel subalgebras of G may also be describe as the maximal connected solvable Lie subalgebras of G as a complex Lie group.

By Theorem 15.7.8, the adjoint action of G on the set X of Borel subalgebras of $\mathfrak{g}$ is transitive and the stabilizer of a given Borel subalgebra $\mathfrak{b}$ is the corresponding Borel subgroup B. Thus, X is in one to one correspondence with the variety G/B, which is projective by Theorem 15.7.3. This correspondence gives X the structure of a projective algebraic variety on which the group G acts algebraically and transitively.

16.2.1 Proposition. *Up to isomorphism, the algebraic variety X is independent of the choice of Lie group G with Lie algebra $\mathfrak{g}$ and the choice of Borel subgroup $B \subset G$.*

Proof. Let G be the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}$, and let B be a Borel subalgebra of G with corresponding Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Let G_1 be any other connected complex Lie group with Lie algebra $\mathfrak{g}$, and let B_1 be any Borel subgroup of G_1. Then G_1 is a quotient of G by a finite central subgroup. Let $\phi : G \to G_1$ be the quotient map. Then $\phi(B)$ is a connected algebraic subgroup of G_1, by Theorem 15.2.3, and it has $\mathfrak{b}$ as its corresponding Lie subalgebra of $\mathfrak{g}$. It follows from Proposition 15.7.2 that $\phi(B)$ is a Borel subgroup of G_1. By Theorem 15.7.3, there is an element $g_1 \in G_1$ such that $B_1 = g_1\phi(B)g_1^{-1}$. If we define a morphism of algebraic groups $\psi : G \to G_1$ by

$$\psi(g) = g_1\phi(g)g_1^{-1},$$

then ψ is a surjective morphism with the same kernel as ϕ, and $\psi(B) = B_1$. By Corollary 15.7.5, B contains the center of G and, hence, the kernel of ϕ. Consequently, $\psi^{-1}(B_1) = B$. Then $(g, x) \to \psi(g)x : G \times G_1/B_1 \to G_1/B_1$ is a transitive algebraic group action of G on G_1/B_1, with the isotropy group of the identity coset equal to B. By Proposition 15.2.4, this action induces an isomorphism $G/B \to G_1/B_1$ of algebraic varieties. This completes the proof.

The set of Borel subalgebras of $\mathfrak{g}$, with its unique structure of a projective algebraic variety, is called the *flag variety* for $\mathfrak{g}$. We will denote it by X. Every connected complex Lie group with Lie algebra $\mathfrak{g}$ acts algebraically on X.

In the remainder of this chapter, we will assume that G is a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, B is a Borel subgroup of G, H a

Cartan subgroup contained in B, and N the maximal unipotent subgroup of B. The corresponding Lie subalgebras of $\mathfrak{g}$ will be denoted $\mathfrak{b}$, $\mathfrak{h}$, and $\mathfrak{n}$.

The Cartan subgroup H is not simply connected, since it is a complex torus of some dimension. Thus, if an element $\lambda \in \mathfrak{h}^*$ is considered as a Lie algebra homomorphism $\lambda : \mathfrak{h} \to \mathbb{C} = \mathfrak{gl}_1(\mathbb{C})$, it will not necessarily be the differential of a Lie group homomorphism $H \to \mathbb{C}^* = GL_1(\mathbb{C})$. That is, not every $\lambda \in \mathfrak{h}^*$ is the differential of a character of H. Those that are form a discrete additive subgroup of $\mathfrak{h}^*$ (Exercise 16.3).

16.2.2 Proposition. *If G is simply connected, a weight $\lambda \in \mathfrak{h}^*$ is the differential of a character of H if and only if it is an integral weight.*

Proof. Integral weights were defined in section 14.7. The important things for us are that the integral weights form an additive subgroup of $\mathfrak{h}^*$, and every dominant integral weight is the highest weight of an irreducible finite dimensional representation of $\mathfrak{g}$ (Theorem 14.7.9). Recall that an integral weight λ is called dominant if it lies in the cone defined by the condition $\langle \lambda, \alpha \rangle \geq 0$ for every positive root α.

Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight, and choose an irreducible representation (π, V) of $\mathfrak{g}$ with highest weight λ. Since λ is a weight of (π, V), there is a vector $v_\lambda \in V$ such that $\pi(\xi)v_\lambda = \lambda(\xi)v_\lambda$ for all $\xi \in \mathfrak{h}$. Since G is simply connected, the representation (π, V) is the differential of a holomorphic (hence, algebraic) representation of G on V (which we also denote by π). The fact that $\pi(\xi)$ has v_λ as an eigenvector for each $\xi \in \mathfrak{h}$ implies that $\pi(h)$ has v_λ as an eigenvector for each $h \in H$. That is, there is a character γ_λ of H such that $\pi(h)v_\lambda = \gamma_\lambda(h)v_\lambda$ for every $h \in H$. Note that λ is the differential of γ_λ.

The set of $\lambda \in \mathfrak{h}^*$ such that λ is the differential of a character of H is a subgroup of the additive group $\mathfrak{h}^*$. We have just shown that this subgroup contains all dominant integral weights. However, any integral weight can be transformed to a dominant integral weight by adding a multiple of ρ to it. Since ρ itself is dominant integral, it follows that the subgroup of $\mathfrak{h}^*$ generated by the dominant integral weights is the group of all integral weights. Thus, every integral weight is the differential of a character of H.

On the other hand, let (π, V) be a faithful holomorphic (hence, algebraic) representation of G. Then, we may choose a basis $\{v_i\}_{i=1}^n$ for V such that $\pi(H)$ is contained in the corresponding diagonal subgroup $D_n(\mathbb{C})$ of $GL(V)$. Each vector v_i determines a character γ_i of H such that $\pi(h)v_i = \gamma_i(h)v_i$ for $h \in H$, and this character has differential equal to the weight corresponding to the weight vector v_i. Since $\pi : H \to GL(V)$ is faithful, H is isomorphic to $\pi(H)$. Every character of $\pi(H)$ is the restriction to $\pi(H)$ of a character of $D_n(\mathbb{C})$, by Proposition 15.5.1. Since γ_i is $\pi|_H$, followed by evaluation of the ith entry in $D_n(\mathbb{C})$, it follows that the set $\{\gamma_i\}$ generates the group of characters of H (see Exercise 16.3). Consequently, every $\lambda \in \mathfrak{h}^*$ which is the differential of a character of H is an integral weight.

If G is not simply connected, then the weights corresponding to characters of H will form a proper subgroup of the group of integral weights. In fact, it can be shown that the quotient of the group of integral weights by this subgroup is the fundamental group of G.

Let σ_λ denote the character of H determined by an integral weight λ. We can extend this to a character of B (also denoted σ_λ) by defining it to be identically 1 on the maximal unipotent subgroup N of G (by Propositions 15.6.4 and 15.6.5, N is a normal subgroup of B, $B = HN$, and $H \cap N = \{e\}$). Each 1-dimensional holomorphic representation of B arises in this way (Exercise 16.4). If we apply the induction functor to this representation of B, we obtain a G-equivariant holomorphic line bundle $I(\sigma_\lambda)$. In view of Theorem 16.1.3 we have:

16.2.3 Proposition. *For integral weight λ, the line bundle $I(\sigma_\lambda)$ is a G-equivariant holomorphic line bundle on X. Every G-equivariant holomorphic line bundle on X has this form.*

We denote the sheaf of holomorphic sections of $I(\sigma_\lambda)$ by $\mathcal{H}(\lambda)$. Thus, if U is an open subset of X, and $\rho : G \to G/B = X$ is the projection, then

$$\mathcal{H}(\lambda)(U) = \{f \in \mathcal{H}(\rho^{-1}(U)) : f(gb^{-1}) = \sigma_\lambda(b)f(g)\}.$$

The pth cohomology, $H^p(X, \mathcal{H}(\lambda))$ of this sheaf is a finite dimensional vector space with a natural $\mathfrak{g}$-action. In the case $p = 0$, or when G is simply connected, this action is the differential of a holomorphic representation of G.

There is a slightly different but equivalent description of the sheaves $\mathcal{H}(\lambda)$, for integral λ, which is sometimes useful. It just amounts to noticing that, since the representations σ_λ are trivial on the maximal unipotent subgroup N of B, we might as well factor N out of the picture right from the beginning.

Let B be the Borel subgroup of G, and let N be its maximal unipotent subgroup, then $H = B/N$ is a complex torus, isomorphic to a Cartan subgroup of B, which acts holomorphically on the right on the complex manifold $Y = G/N$. Then the quotient map $G \to X = G/B$ factors as $G \to Y$, followed by a quotient map $\nu : Y \to X$, with fibers which are the orbits of the action of this right action of H on Y. Of course G acts holomorphically and transitively on Y on the left, and ν is a morphism of G-spaces from Y

to the flag variety X. We then describe $\mathcal{H}(\lambda)$ by

$$\mathcal{H}(\lambda)(U) = \{f \in \mathcal{H}(\nu^{-1}(U)) : f(yh^{-1}) = \sigma_\lambda(h)f(y)\},$$

where σ_λ now stands for the character of H determined by the weight $\lambda \in \mathfrak{h}^*$. This description of $\mathcal{H}(\lambda)$ is equivalent to the one given above. The difference is that sections of $\mathcal{H}(\lambda)$ are described in terms of holomorphic functions on Y which transform in accordance with σ_λ under H rather than functions on G which transform in accordance with σ_λ under B.

16.2.4 Example. Consider the group $G = SL_2(\mathbb{C})$ acting on $\mathbb{C}^2$. Let B be the Borel subgroup of upper triangular matrices. Then N is the unipotent subgroup consisting of upper triangular matrices with 1's on the diagonal. Under the natural action of $G = SL_2(\mathbb{C})$ on $\mathbb{C}^2$, N is the isotropy group of the point $(1, 0) \in \mathbb{C}^2$. It follows that $C^2 - \{0\}$ is isomorphic to the quotient space $Y = G/N$, where the map $G/N \to C^2 - \{0\}$ is induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a, c) : G \to \mathbb{C}^2 - \{0\}.$$

When restricted to the Borel subgroup B, this is the map which sends an upper triangular matrix of the above form to $(a, 0)$. This demonstrates that $H = B/N$ is isomorphic to the 1-dimensional torus $\mathbb{C}^*$. Under this identification, the action of $z \in H = C^*$ on the right on $G/N = \mathbb{C}^2 - \{0\}$ is simply scalar multiplication by z. The quotient of Y by this action is $X = P^1$.

The characters of the torus H are the functions

$$z \longrightarrow z^k.$$

Note that there is one for each of the integral elements $k \in \mathfrak{h}^*$. The G-equivariant sheaf on P^1 induced, as above, from the character of H determined by the integer k is the sheaf whose sections over an open set $U \subset P^1$ consist of the k-homogeneous functions on the inverse image of U in $\mathbb{C}^2 - \{0\}$ – that is, it is the sheaf $\mathcal{H}(k)$ of Chapter 12.

Returning to the case of a general complex semisimple Lie group G and a weight λ which is the differential of a character of H, there are several natural questions concerning the sheaves $\mathcal{H}(\lambda)$ obtained by inducing σ_λ from B to G:

1. Is the representation of $\mathfrak{g}$ on $H^p(X, \mathcal{H}(\lambda))$ the differential of a holomorphic G-action for all p and all G?
2. For which integral weights λ and integers p is $H^p(X, \mathcal{H}(\lambda))$ non-zero?

3. For which integral weights λ and for which integers p is $H^p(X, \mathcal{H}(\lambda))$ an irreducible representation of $\mathfrak{g}$?
4. Which finite dimensional representations of $\mathfrak{g}$ arise this way?
5. If $H^p(X, \mathcal{H}(\lambda)) \neq (0)$, how is the parameter λ related to the weights which occur in this representation of $\mathfrak{g}$.

The Borel-Weil-Bott theorem (Theorem 16.5.3) answers all these questions. The proof we present here is based on the results of Chapter 12, concerning coherent sheaves on projective space, but it also makes strong use of Lie algebra structure theory, as outlined in Chapter 14, and information about the action of the Casimir operator for a semisimple Lie algebra. The next section is devoted to a description of this operator.

16.3 The Casimir Operator

The *enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the quotient of the free associative algebra generated by $\mathfrak{g}$, as a vector space, modulo the 2-sided ideal generated by all elements of the form $xy - yx - [x, y]$, for $x, y \in \mathfrak{g}$. It is an associative algebra, characterized up to isomorphism by the following properties (Exercise 16.5):

(1) There is a Lie algebra morphism $\iota : \mathfrak{g} \to U(\mathfrak{g})$; and
(2) every Lie algebra morphism $\psi : \mathfrak{g} \to A$, of $\mathfrak{g}$ into an associative algebra, uniquely determines a morphism $\phi : U(\mathfrak{g}) \to A$ of associative algebras such that $\psi = \phi \circ \iota$.

Note that every associative algebra also has a Lie algebra structure, where commutator bracket is the bracket operation. When we refer to a Lie algebra morphism from a Lie algebra to an associative algebra, we really mean that the target is the Lie algebra associated to the given associative algebra.

If $\mathfrak{g}$ is the Lie algebra of a complex Lie group G, and the elements of $\mathfrak{g}$ are realized as right invariant holomorphic vector fields on G, then there is also a description of $U(\mathfrak{g})$ in terms of differential operators on G. It is isomorphic to the algebra of all right invariant holomorphic differential operators on G – that is, all differential operators $\xi : {}_G\mathcal{H} \to {}_G\mathcal{H}$ which commute with right translation. Since G is algebraic, each of these operators is actually algebraic (maps ${}_G\mathcal{O}$ to itself) (Exercise 16.6).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The *Casimir operator* is a particular element of the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$. Its description is given in the following proposition. Recall that if $\{\xi_i\}$ is a basis for a vector space, then its *dual basis* $\{\eta_i\}$, relative to a given non-singular symmetric bilinear form $\langle \cdot, \cdot \rangle$, is the unique basis $\{\eta_i\}$ such that $\langle \xi_i, \eta_j \rangle = \delta_{ij}$.

16.3.1 Proposition. *Let $\{\xi_i\}$ be a basis for $\mathfrak{g}$, and let $\{\eta_i\}$ be its dual basis relative to the Killing form. Then the element*

$$\Omega = \sum_i \xi_i \eta_i$$

belongs to $\mathcal{Z}(\mathfrak{g})$ and is independent of the choice of basis.

Proof. We use the Killing form $\langle \cdot, \cdot \rangle$ to define an isomorphism from $\mathfrak{g} \otimes \mathfrak{g}$ to $\operatorname{End}(\mathfrak{g})$ in the following way: We send an elementary tensor $\phi \otimes \psi \in \mathfrak{g} \otimes \mathfrak{g}$ to the linear transformation $L_{\phi\otimes\psi}$ defined by

$$L_{\phi\otimes\psi}(\xi) = \langle \psi, \xi \rangle \phi.$$

The resulting bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\operatorname{End}(\mathfrak{g})$ then extends to a linear map $\Phi \to L_\Phi : \mathfrak{g} \otimes \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$. This is an isomorphism, because the Killing form is non-singular.

We also have

$$\operatorname{Ad}_g \circ L_{\phi\otimes\psi} = L_{\operatorname{Ad}_g(\phi)\otimes\psi}$$

and

$$L_{\phi\otimes\psi} \circ \operatorname{Ad}_{g^{-1}} = L_{\phi\otimes\operatorname{Ad}_g(\psi)}.$$

The latter identity follows from the fact that the Killing form is invariant under Ad_g for each $g \in G$, which implies that $\langle \operatorname{Ad}_g(\psi), \xi \rangle = \langle \psi, \operatorname{Ad}_{g^{-1}}(\xi) \rangle$. These two identities imply that, if Φ is a tensor such that L_Φ commutes with Ad_g, then $\operatorname{Ad}_g \otimes \operatorname{Ad}_g$ fixes Φ. If $\Phi = \sum \xi_i \otimes \eta_i$, for our dual pair of bases $\{\xi_i\}$ and $\{\eta_i\}$, then L_Φ is the identity operator, and hence, it commutes with Ad_g for each $g \in G$. It follows that Φ is fixed by $\operatorname{Ad}_g \otimes \operatorname{Ad}_g$ for all $g \in G$, and this clearly implies that the element $\Omega = \sum \xi_i \eta_i$ of $U(\mathfrak{g})$ is fixed by Ad_g. On differentiating, we conclude that Ω is killed by ad_ξ for every $\xi \in \mathfrak{g}$. In other words, Ω commutes with each $\xi \in \mathfrak{g}$ and, hence, belongs to the center of $U(\mathfrak{g})$.

That Ω does not depend on the choice of basis follows from the fact that the linear transformation L_Φ is the identity no matter what basis is used in its definition. This implies that the element $\Phi \in \mathfrak{g} \otimes \mathfrak{g}$ is independent of the choice of basis. Since Ω is just the image of Φ under the multiplication map from $\mathfrak{g} \otimes \mathfrak{g}$ to $U(\mathfrak{g})$, we conclude that Ω is independent of the choice of basis as well.

By choosing a particular basis for $\mathfrak{g}$, we express the Casimir operator in a form which allows us to easily compute its action on each irreducible representation of $\mathfrak{g}$. We let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{b}$. As in Chapter 14, Δ denotes the set of roots for $\mathfrak{h}$ and

Δ^+ a set of positive roots, chosen so that the Borel subalgebra $\mathfrak{b}$ is the corresponding algebra $\mathfrak{b}^-$. We then choose for our basis of $\mathfrak{g}$ the elements of a self-dual basis $\{\eta_i\}$ for $\mathfrak{h}$, together with a set $\{\xi_\alpha\}$ consisting of an element $\xi_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$, chosen so that $\langle \xi_\alpha, \xi_{-\alpha}\rangle = 1$. Then this basis is dual to the one consisting of the same set of elements but with the pairing described as follows: Each η_i is paired with itself, while each ξ_α is paired with $\xi_{-\alpha}$. Using this dual pair of bases, the Casimir operator becomes

$$\Omega = \sum_i \eta_i^2 + \sum_{\alpha\in\Delta^+} (\xi_\alpha\xi_{-\alpha} + \xi_{-\alpha}\xi_\alpha) = \sum_i \eta_i^2 + \sum_{\alpha\in\Delta^+} (2\xi_{-\alpha}\xi_\alpha + [\xi_\alpha, \xi_{-\alpha}]).$$

For each α, the element $\theta_\alpha = [\xi_\alpha, \xi_{-\alpha}]$ belongs to $\mathfrak{h}$. Also, for $\eta \in \mathfrak{h}$,

$$\langle \theta_\alpha, \eta\rangle = \langle [\xi_\alpha, \xi_{-\alpha}], \eta\rangle = \langle [\eta, \xi_\alpha], \xi_{-\alpha}\rangle = \alpha(\eta)\langle \xi_\alpha, \xi_{-\alpha}\rangle = \alpha(\eta).$$

Thus, θ_α is the element of $\mathfrak{h}$ corresponding to $\alpha \in \mathfrak{h}^*$ under the pairing determined by the Killing form (in section 14.5, this element was called t_α). It follows that $\lambda(\theta_\alpha) = \langle \lambda, \alpha\rangle$ for each $\lambda \in \mathfrak{h}^*$, where $\langle \cdot, \cdot\rangle$ also denotes the form on $\mathfrak{h}^*$ which is dual to the Killing form. Thus, we have

$$\Omega = \sum_i \eta_i^2 + \sum_{\alpha\in\Delta^+} \theta_\alpha + 2\sum_{\alpha\in\Delta^+} \xi_{-\alpha}\xi_\alpha.$$

Let $\rho \in \mathfrak{h}^*$ be one half the sum of the positive roots.

16.3.2 Proposition. *On a finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ, the Casimir operator acts as the scalar $\langle \lambda, \lambda + 2\rho\rangle$.*

Proof. Since π is irreducible, Schur's lemma implies that $\pi(\Omega)$ must be a scalar times the identity operator. We calculate the scalar by applying $\pi(\Omega)$ to a highest weight vector. Thus, let v_λ be a highest weight vector in V. Then $\pi(\xi_\alpha)v_\lambda = 0$ for each $\alpha \in \Delta^+$. It follows that

$$\pi(\Omega)v_\lambda = \sum_i \lambda(\eta_i)^2 v_\lambda + \sum_{\alpha\in\Delta^+} \lambda(\theta_\alpha)v_\lambda.$$

From the definition of the dual form on $\mathfrak{h}^*$, it follows that $\sum \lambda(\eta_i)^2 = \langle \lambda, \lambda\rangle$. Since $\lambda(\theta_\alpha) = \langle \lambda, \alpha\rangle$, we conclude that

$$\pi(\Omega) = \langle \lambda, \lambda\rangle + \sum_{\alpha\in\Delta^+} \langle \lambda, \alpha\rangle = \langle \lambda, \lambda\rangle + \langle \lambda, 2\rho\rangle = \langle \lambda, \lambda + 2\rho\rangle.$$

This completes the proof.

We will also need to know how the Casimir operator acts on each of the induced sheaves $\mathcal{H}(\lambda)$, defined in the previous section. The representation of $\mathfrak{g}$ on $\mathcal{H}(\lambda)$ generates a corresponding action of the enveloping algebra $U(\mathfrak{g})$ on $\mathcal{H}(\lambda)$. In particular, the Casimir operator acts on this sheaf.

16.3.3 Proposition. *The Casimir operator acts as the scalar* $\langle \lambda, \lambda + 2\rho \rangle$ *on the sheaf* $\mathcal{H}(\lambda)$.

Proof. Let x be a point of X, B the Borel subgroup which is the isotropy group of x, and $\mathfrak{b}$ the Borel subalgebra corresponding to B. Then we may represent X as G/B and x as the identity coset of B. We choose a Cartan $\mathfrak{h} \subset \mathfrak{b}$ and a positive root system such that the Borel subalgebra $\mathfrak{b}$ is $\mathfrak{b}^-$. Let $\rho : G \to G/B$ be the projection. A local section of $\mathcal{H}(\lambda)$ on a neighborhood U of x is a function $f \in \mathcal{H}(\rho^{-1}(U))$ which transforms on the right under B in accordance with the representation σ_λ of B. The action of an element of $U(\mathfrak{g})$ on such a section is given by the realization of $U(\mathfrak{g})$ as the algebra of holomorphic right invariant differential operators on G.

Let f be a section of $\mathcal{H}(\lambda)$ on a neighborhood U of x, realized as a function in $\mathcal{H}(\rho^{-1}(U))$, as above. Then, on the identity coset B, we have

$$f(b^{-1}b_1) = \sigma_\lambda(b_1^{-1})\sigma_\lambda(b)f(e) = \sigma_\lambda(b)\sigma_\lambda(b_1^{-1})f(e) = \sigma_\lambda(b)f(b_1).$$

If we set $b = \exp(t\xi)$, for $\xi \in \mathfrak{b}$, and differentiate with respect to t, we conclude that the function $\xi f - \sigma_\lambda(\xi)f$ vanishes on B. In particular, on B,

$$\xi_{-\alpha} f = 0, \quad \text{and} \quad \eta f = \lambda(\eta) f,$$

for each $\alpha \in \Delta^+$ and each $\eta \in \mathfrak{h}$. Since ξ_α maps $\mathcal{H}(\lambda)(U)$ to itself, it follows that $\xi_{-\alpha}\xi_\alpha f$ also vanishes on B, for each $\alpha \in \Delta^+$.

That $\Omega f = \langle \lambda, \lambda+2\rho \rangle f$ on B now follows, as in the previous proposition, from the identity

$$\Omega = \sum \eta_i^2 + \sum_{\alpha \in \Delta^+} \theta_\alpha + 2 \sum_{\alpha \in \Delta^+} \xi_{-\alpha}\xi_\alpha.$$

This says that the sections Ωf and $\langle \lambda, \lambda + 2\rho \rangle f$, of the line bundle corresponding to $\mathcal{H}(\lambda)$, agree at the point x. However, since x was arbitrary, it follows that they agree at all points of X. Note that here we are using the fact that the Casimir operator is independent of the choice of basis, since the basis we used to expressed it in the above argument does depend on x. This completes the proof.

Note that

$$(16.3.1) \qquad \langle \lambda, \lambda + 2\rho \rangle = \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle = ||\lambda + \rho||^2 - ||\rho||^2.$$

If μ is a weight of the form $\mu = w\rho - \rho$ for some Weyl group element w, then

$$||\mu + \rho||^2 - ||\rho||^2 = ||w\rho||^2 - ||\rho||^2 = 0,$$

since the Weyl group is a group of isometries of $\mathfrak{h}^*$. Thus:

16.3.4 Corollary. *Let W be the Weyl group of G. Then the Casimir acts as the zero operator on $\mathcal{H}(\mu)$ for all $\mu \in \{w\rho - \rho : w \in W\}$.*

Finally, we need to analyze the action of the Casimir operator on the sheaf induced from a general finite dimensional representation of B. Let (θ, Q) be such a representation. Then, unlike the situation described in Proposition 16.3.3, where Q is 1-dimensional, the Casimir operator no longer acts on $\mathcal{I}(\theta)$ as a scalar or even as a scalar matrix. It acts as a differential operator. However, we will introduce a filtration of $\mathcal{I}(\theta)$, by G-equivariant subsheaves, such that Ω acts as a scalar operator on each subquotient for this filtration. This leads to a decomposition of $\mathcal{I}(\theta)$ as a direct sum of "generalized eigenspaces" for Ω. This decomposition will play a key role in the next two sections.

We begin by constructing a filtration $\{Q_j\}_{j=0}^n$ of Q by B-submodules such that the subquotients Q_j/Q_{j-1} are all 1-dimensional. We can do this, by Lie's theorem (Theorem 14.4.3). Since, on each 1-dimensional B-module, B necessarily acts by a character σ_ν corresponding to some weight $\nu \in \mathfrak{h}^*$, we have a sequence $\{\nu_j\}$ of elements of $\mathfrak{h}^*$ such that B acts as σ_{ν_j} on Q_j/Q_{j-1}. The elements ν_j are necessarily weights which occur in the representation (θ, Q).

It follows from Theorem 16.1.3 that the sheaf $\mathcal{I}(\theta)$ is filtered by a sequence of G-equivariant subsheaves $\{\mathcal{I}(\theta_j)\}$, where θ_j is θ restricted to Q_j, and $\mathcal{I}(\theta_j)/\mathcal{I}(\theta_{j-1})$ is isomorphic to $\mathcal{H}(\nu_j) = \mathcal{I}(\sigma_{\nu_j})$. Each $\mathcal{I}(\theta_j)$ is a $\mathfrak{g}$-submodule as well as an $\mathcal{H}$-submodule of $\mathcal{I}(\theta)$. In particular, each $\mathcal{I}(\theta_j)$ is invariant under the action of the Casimir. By Proposition 16.3.2, the Casimir Ω acts on $\mathcal{H}(\nu_j)$ as $\langle \nu_j, \nu_j + 2\rho \rangle$. From this, it follows that $\prod_j(\Omega - \langle \nu_j, \nu_j + 2\rho \rangle)$ acts as 0 on $\mathcal{I}(\theta)$. We rewrite this product as $\prod_t(\Omega - t)^{k_t}$, where t ranges over the set of distinct eigenvalues $\langle \nu_j, \nu_j + 2\rho \rangle$, and k_t is the multiplicity of the eigenvalue t. Then $\mathcal{I}(\theta)$ is a sum of subsheaves $\mathcal{S}_t$, where $\mathcal{S}_t$ is the subsheaf on which $(\Omega - t)^{k_t}$ vanishes. This is a direct sum, since on each non-zero subquotient of the filtration $\{\mathcal{I}(\theta_j) \cap \mathcal{S}_t\}$ of $\mathcal{S}_t$ the Casimir Ω acts as the scalar t. Also note that each $\mathcal{S}_t$ is a sheaf of $U(\mathfrak{g})$-modules, since Ω belongs to the center of $U(\mathfrak{g})$. However, $\mathcal{S}_t$ is not necessarily a subsheaf of $\mathcal{H}$-modules, since Ω does not commute with all elements of $\mathcal{H}$.

Observe that the numbers $t = \langle \nu_j, \nu_j + 2\rho \rangle$ that parameterize the above decomposition really do deserve to be called the eigenvalues of Ω acting on $\mathcal{I}(\theta)$. They are exactly the numbers t for which there exists a non-zero section f of $\mathcal{I}(\theta)$, over some open set, such that $\Omega f = tf$ (Exercise 16.7).

Finally, note that if one of the eigenvalues t occurs with multiplicity 1 – that is, $t = \langle \nu_j, \nu_j + 2\rho \rangle$ for exactly one index j – then the subsheaf $\mathcal{S}_t$ is isomorphic to $\mathcal{H}(\sigma_{\nu_j})$. In summary, we have proved:

16.3.5 Proposition. *Let (θ, Q) be a finite dimensional representation of B. Then there is a direct sum decomposition*

$$\mathcal{I}(\theta) = \oplus_t \mathcal{S}_t,$$

where each $\mathcal{S}_t$ is a sheaf of $U(\mathfrak{g})$-modules, t ranges over the distinct eigenvalues for the action of Ω on $\mathcal{I}(\theta)$, and t is the sole eigenvalue for the action of Ω on $\mathcal{S}_t$. If $t = \langle \nu, \nu + 2\rho \rangle$ is an eigenvalue of multiplicity 1, then $\mathcal{S}_t$ is isomorphic to $\mathcal{H}(\nu)$.

16.4 The Borel-Weil Theorem

We can now begin the proof of the Borel-Weil-Bott theorem. The first result in this direction is a lemma which says that, if we can compute $H^p(X, \mathcal{H}(\nu))$ for a weight ν, then we can compute it for all weights ω for which $\nu + \rho$ and $\omega + \rho$ belong to the same Weyl chamber. Note that the set of all integral weights ν for which $\nu + \rho$ belongs to a given Weyl chamber can be written as

$$\{w\lambda + \mu = w(\lambda + \rho) - \rho : \lambda \text{ dominant integral}\},$$

where $\mu = w\rho - \rho$, and $w \in W$ is a Weyl group element which maps the dominant chamber to the given chamber.

16.4.1 Lemma. *Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ. Let w be an element of W, and set $\mu = w\rho - \rho$. Then for all p,*

(i) *$H^p(X, \mathcal{H}(\mu))$ is a trivial $\mathfrak{g}$-module; and*

(ii) *$H^p(X, \mathcal{H}(w\lambda + \mu)) \simeq H^p(X, \mathcal{H}(\mu)) \otimes V$, as $\mathfrak{g}$-modules.*

Proof. By Corollary 16.3.4, we know that the Casimir acts on $\mathcal{H}(\mu)$ as the zero operator. It follows that the Casimir acts on $H^p(X, \mathcal{H}(\mu))$ as the zero operator as well. By Proposition 16.3.2, the Casimir operator acts on any irreducible submodule of $H^p(X, \mathcal{H}(\mu))$ as $\langle \lambda, \lambda + \rho \rangle$, where λ is the highest weight of the submodule. Since $\langle \lambda, \rho \rangle \geq 0$, we conclude that $\lambda = 0$. However, this implies that all the weights of the representation of $\mathfrak{g}$ on $H^p(X, \mathcal{H}(\mu))$ are 0, and hence, that it is a trivial representation – that is, a representation on which every element of $\mathfrak{g}$ acts as the zero operator. This proves part (i). Note that $H^p(X, \mathcal{H}(\mu))$ may be zero or non-zero. Which it is depends on the relationship between w and p, as we shall see in the next section.

We let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, and consider the representation of G corresponding to (π, V). We denote by θ its restriction to B. Then the induced bundle $I(\theta)$ is trivial, by Theorem 16.1.3, and in fact, is just the trivial G-equivariant bundle $X \times V$. Thus, its sheaf of sections $\mathcal{I}(\theta)$ is just $\mathcal{H} \otimes V$. On the other hand, we may *twist* θ by tensoring it with the 1-dimensional representation σ_μ, determined by the weight μ, to obtain a representation θ_μ of B. Then the corresponding G-equivariant bundle is $I(\theta_\mu) = I(\sigma_\mu) \otimes I(\theta)$. The corresponding sheaf of sections is

$$\mathcal{I}(\theta_\mu) = \mathcal{H}(\mu) \otimes_{\mathcal{H}} \mathcal{I}(\theta) = \mathcal{H}(\mu) \otimes_{\mathcal{H}} (\mathcal{H} \otimes V) \simeq \mathcal{H}(\mu) \otimes V.$$

Here the $\mathfrak{g}$-action is the tensor product of the natural action on $\mathcal{H}(\mu)$ with the action on V given by the representation θ. It follows that, as $\mathfrak{g}$-modules,

$$H^p(X, \mathcal{I}(\theta_\mu)) \simeq H^p(X, \mathcal{H}(\mu)) \otimes V.$$

Thus, as a $\mathfrak{g}$-module, $H^p(X, \mathcal{I}(\theta_\mu))$ is the tensor product of a trivial $\mathfrak{g}$-module, $H^p(X, \mathcal{H}(\mu))$ and the module (π, V). In particular, this implies that the Casimir acts on $H^p(X, \mathcal{I}(\theta_\mu))$ as the scalar $\langle \lambda, \lambda + 2\rho \rangle$, by Proposition 16.3.2.

By Proposition 16.3.5, $\mathcal{I}(\theta_\mu)$ decomposes as a direct sum of subsheaves $\mathcal{S}_t$, where t runs through the set of distinct eigenvalues for Ω on $\mathcal{I}(\theta_\mu)$, each of which is of the form $\langle \omega, \omega + 2\rho \rangle$ for ω a weight of the representation θ_μ, k_t is the multiplicity of t, and $\mathcal{S}_t$ is the subsheaf on which $(\Omega - t)^{k_t}$ vanishes. The weights of θ_μ are the weights of the form $\omega = \nu + \mu$, where ν is a weight of θ.

Since θ is the restriction to B of a representation of G, its set of weights Λ is invariant under the Weyl group. Addition of ρ to each element of this set shifts it in the direction of the positive chamber and leaves us with a set of weights, $\Lambda + \rho$, with a unique element $\lambda + \rho$ of largest norm (Theorem 14.7.7(v)). Since the norm induced by the Killing form is invariant under the Weyl group, $w(\lambda + \rho)$ is the unique element of largest norm in the set $w(\Lambda + \rho) = \Lambda + w\rho = \Lambda + \mu + \rho$. It follows from this and (16.3.1) that $w\lambda + \mu$ is the unique weight ω of θ_μ for which $\langle \omega, \omega + 2\rho \rangle = \langle \lambda, \lambda + 2\rho \rangle$. Hence,

$$t = \langle \lambda, \lambda + 2\rho \rangle = \langle w\lambda + \mu, w\lambda + \mu + 2\rho \rangle$$

is an eigenvalue of multiplicity 1 for Ω acting on $\mathcal{I}(\theta_\mu)$. By Proposition 16.3.5, the corresponding summand of $\mathcal{I}(\theta_\mu)$ must be a copy of $\mathcal{H}(w\lambda + \mu)$. Then

$$H^p(X, \mathcal{I}(\theta_\mu)) \simeq H^p(X, \mathcal{H}(w\lambda + \mu)) \oplus H^p(X, \mathcal{J})$$

where $\mathcal{J}$ is a summand of $\mathcal{I}(\theta_\mu)$ on which $\Omega - \langle \lambda, \lambda + 2\rho \rangle$ is injective. However, we proved above that $\Omega - \langle \lambda, \lambda + 2\rho \rangle$ acts as the zero operator on the left side of this equality. It also acts as 0 on the first term on the right side, but is injective on the second term. It follows that $H^p(X, \mathcal{J}) = 0$ for all p and

$$H^p(X, \mathcal{H}(\mu)) \otimes V \simeq H^p(X, \mathcal{I}(\theta_\mu)) \simeq H^p(X, \mathcal{H}(w\lambda + \mu))$$

for all p. This completes the proof of part (ii).

The above is a strong result. In particular, it implies the following:

16.4.2 Corollary. *The set of integers p, such that $H^p(X,\mathcal{H}(\nu))$ is non-vanishing, is constant as $\nu+\rho$ varies over the set of integral weights in a given Weyl chamber.*

Proof. If $\nu+\rho$ belongs to the image of the positive Weyl chamber under $w \in W$, then there is a dominant root λ such that $\nu+\rho = w(\lambda+\rho)$. Thus, $\nu = w\lambda+\mu$, where $\mu = w\rho-\rho$. It then follows from the previous lemma that $H^p(X,\mathcal{H}(\nu))$ is non-vanishing if and only if $H^p(X,\mathcal{H}(\mu))$ is non-vanishing. Since μ depends only on the chamber determined by w, the proof is complete.

We will use the above corollary to prove that $H^p(X,\mathcal{H}(\nu))$ vanishes for $p>0$ for all ν with $\nu+\rho$ in the positive chamber by showing it is true for at least one such ν. We show this by appealing to Theorem 12.4.4, which says that any coherent analytic sheaf on projective space can be twisted, by tensoring it with $\mathcal{H}(k)$ for sufficiently large k, so that the resulting sheaf is acyclic. In order to use this result, we need to embed X in projective n-space in just the right way.

Let (π, V) be a finite dimensional representation of G. Recall from Theorem 14.7.3 that there is a unique highest weight λ for the representation π, and the corresponding weight space is 1-dimensional. Let Q be the 1-dimensional subspace of the dual space V^* consisting of functionals which vanish on all weight spaces of V for weights other than λ. Then Q is exactly the subspace of V^* consisting of vectors killed by $\mathfrak{g}_\alpha$ for each negative root α. It is also the weight space of V^* for the weight $-\lambda$, which is a lowest weight for π^*. The representation π^* determines an action of G on the projective space $P(V^*)$. If q denotes the point of $P(V^*)$ determined by the subspace Q, then B fixes q, and hence, the map $g \to \pi^*(g)q : G \to P(V^*)$ induces an algebraic morphism ϕ of $X = G/B$ onto a subvariety of $P(V^*)$.

A weight λ is said to be *regular* if $\langle\lambda,\alpha\rangle \neq 0$ for every root α.

16.4.3 Lemma. *For a regular weight λ, the morphism $\phi : X \to P(V^*)$, described above, is an isomorphism of X onto a subvariety of $P(V^*)$. Furthermore, $\mathcal{H}(\lambda) \simeq \phi^*\mathcal{H}(1)$, where $\mathcal{H}(1)$ is the sheaf of 1-homogeneous functions on $P(V^*)$, as described in Chapter 12.*

Proof. The group G is an affine algebraic group, by Corollary 15.8.6, and the representation π is algebraic, by Corollary 15.8.7. Hence, the action of G on $P(V)$ is algebraic. By Proposition 15.2.1, each orbit is Zariski open in its Zariski closure. Since X is compact in the Euclidean topology, the orbit $\phi(X)$ is compact as well. This implies it is closed in the Zariski topology, by Proposition 13.4.6. Thus, $\phi(X)$ is an algebraic subvariety of $P(V^*)$.

If λ is a regular weight, then $\langle\lambda,\alpha\rangle \neq 0$ for every root α. This implies that $\lambda(\tau_\alpha) \neq 0$ for every root α, where τ_α is a non-zero element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

It follows that there is no root α for which $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ both kill Q. Suppose ξ is an element of $\mathfrak{g}$ which takes Q into itself. Then each of the components of ξ, relative to the decomposition $\mathfrak{g} = \mathfrak{h} + \sum \mathfrak{g}_\alpha$, must do the same – the component in $\mathfrak{h}$ acts as a scalar on Q, and the component in $\mathfrak{g}_\alpha$ sends Q into the weight space for weight $\alpha - \lambda$, and since the weight spaces are linearly independent, this means it kills Q. Hence, either $\xi \in \mathfrak{b}^-$ or there must be an $\alpha \in \Delta^+$ for which $\mathfrak{g}_\alpha$ kills Q. Since $\mathfrak{g}_{-\alpha}$ also kills Q, this contradicts the assumption that λ is regular. We conclude that $\mathfrak{b}^-$ is exactly the subalgebra of $\mathfrak{g}$ which leaves Q invariant. The fact that B is the normalizer of $\mathfrak{b}$ in G then implies that B is the stabilizer of Q – i.e. the isotropy subgroup of the point q in $P(V^*)$.

By Proposition 15.2.4, the morphism $\phi : X \to P(V^*)$, which is induced by $g \to \pi(g)q : G \to P(V^*)$, is an isomorphism of algebraic varieties from X to $\phi(X)$. If these spaces are given their natural holomorphic structures, the results of Chapter 13 imply that $\phi : X \to \phi(X)$ is a biholomorphic map.

Let $\mathcal{H}(1)$ be the sheaf on $P(V^*)$ determined by 1-homogeneous functions on $V^* - \{0\}$, as in Chapter 12. To complete the proof, we must show that $\mathcal{H}(\lambda) = \phi^*\mathcal{H}(1)$. Let U be an open subset of X and A an open subset of $P(V^*)$ containing $\phi(U)$. Let $\gamma : V^* - \{0\} \to P(V^*)$ be the projection. Then $\mathcal{H}(1)(A)$ is the space of 1-homogeneous functions in $\mathcal{H}(\gamma^{-1}(A))$. Given such a function f and a non-zero vector $v^* \in V^*$, the function ψf defined by $\psi f(g) = f(\pi^*(g)v^*)$ is defined on the inverse image of U under $G \to X$ and satisfies

$$\begin{aligned}\psi f(gb) &= f(\pi^*(gb)v^*) = f(\sigma_\lambda(b^{-1})\pi^*(g)v^*) \\ &= \sigma_\lambda(b^{-1})f(\pi^*(g)v^*) = \sigma_\lambda(b^{-1})\psi f(g).\end{aligned}$$

Thus, ψf defines a section of $\mathcal{H}(\lambda)$ over U. The correspondence $f \to \psi f$ defines a morphism of sheaves $\psi : \phi^{-1}\mathcal{H}(1) \to \mathcal{H}(\lambda)$ and, on tensoring with ${}_X\mathcal{H}$ relative to $\phi^{-1}{}_{P(V^*)}\mathcal{H}$, a morphism of sheaves of ${}_X\mathcal{H}$-modules $\psi : \phi^*\mathcal{H}(1) \to \mathcal{H}(\lambda)$. Since both sheaves involved here are locally free of rank 1, to show that ψ is an isomorphism, we just need to show that, locally, a generator is sent to a generator. This, however, is obvious, since if a local section of $\phi^*\mathcal{H}(1)$ is non-vanishing, then so is its image under ψ. This completes the proof.

The Borel-Weil theorem follows easily from Lemma 16.4.1, Corollary 16.4.2, and Lemma 16.4.3.

16.4.4 Borel-Weil Theorem. *If λ is a dominant integral weight, then*
(i) *$H^0(X, \mathcal{H}(\lambda))$ is the irreducible finite dimensional $\mathfrak{g}$-module with λ as highest weight; and*
(ii) *$H^p(X, \mathcal{H}(\lambda)) = 0$ for $p \neq 0$.*

Proof. Let (π, V) be the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ. By Lemma 16.4.1, with $w = \mathrm{id}$, we have

$$H^p(X, \mathcal{H}(\lambda)) = H^p(X, \mathcal{H}) \otimes V.$$

This immediately implies part (i), since $H^0(X, \mathcal{H})$ is the space $\mathbb{C}$ of constants, by virtue of the fact that X is compact and connected. Also, the set of p for which $H^p(X, \mathcal{H}(\lambda))$ is non-vanishing is independent of λ as long as λ is dominant, by Corollary 16.4.2. Thus, part (ii) will be proved if we can show that there exists a dominant weight ν such that $H^p(X, \mathcal{H}(\nu)) = 0$ for all $p > 0$.

The weight ρ, which is half the sum of the positive roots, is a regular dominant integral weight (Exercise 14.22). Let (π, V) be a finite dimensional irreducible representation of G with highest weight ρ. Let $\phi : X \to P(V^*)$ be the morphism of Lemma 16.4.3. Since $\phi^*\mathcal{H}(1) = \mathcal{H}(\rho)$, it follows that

$$\phi^*\mathcal{H}(k) = \phi^*(\otimes_{\mathcal{H}}^k \mathcal{H}(1)) = \otimes_{\mathcal{H}}^k \mathcal{H}(\rho) = \mathcal{H}(k\rho).$$

Thus, by Proposition 7.4.5, the cohomology of $\mathcal{H}(k\rho)$ on X is the cohomology of

$$\phi_*\mathcal{H}(k\rho) = \phi_*\phi^*\mathcal{H}(k) \simeq (\phi_*\phi^*\mathcal{H}) \otimes_{\mathcal{H}} \mathcal{H}(k)$$

on $P(V^*)$. Note that $\phi_*\phi^*\mathcal{H}$ is just the structure sheaf of the subvariety $\phi(X)$, extended by zero to all of $P(V^*)$. Since $\phi_*\phi^*\mathcal{H}$ is coherent, its tensor product with $\mathcal{H}(k)$ is acyclic for sufficiently large k, by Theorem 12.4.4. It follows that $\mathcal{H}(k\rho)$ is acyclic on X for sufficiently large k. As noted above, this implies $\mathcal{H}(\lambda)$ is acyclic for all dominant integral weights λ. This completes the proof.

16.5 The Borel-Weil-Bott Theorem

With the Borel-Weil theorem, we have computed $H^p(X, \mathcal{H}(\lambda))$ for all integral weights λ with $\lambda+\rho$ in the positive Weyl chamber. The Borel-Weil-Bott theorem completes the analysis by computing $H^p(X, \mathcal{H}(\lambda))$ for all integral weights. The next step toward this goal is to consider the case where $\lambda+\rho$ is not in a Weyl chamber but lies in a wall. This just means that $\langle \lambda+\rho, \alpha \rangle = 0$ for some $\alpha \in \Delta^+$. An integral weight with this property is said to be *singular.*

16.5.1 Proposition. *If λ is an integral weight for which $\lambda + \rho$ lies in a wall, then*

$$H^p(X, \mathcal{H}(\lambda)) = 0$$

for all p, including $p = 0$.

Proof. If $H^p(X, \mathcal{H}(\lambda)) \neq 0$ for some p, then it is a $\mathfrak{g}$-module with highest weight μ for some dominant weight μ. Furthermore, the Casimir operator acts on $H^p(X, \mathcal{H}(\lambda))$ as $\langle \mu, \mu+2\rho \rangle = ||\mu+\rho||^2 - ||\rho||^2$, by Proposition 16.3.2. On the other hand, the Casimir acts as $\langle \lambda, \lambda+2\rho \rangle = ||\lambda+\rho||^2 - ||\rho||^2$ on $\mathcal{H}(\lambda)$ and, hence, on $H^p(X, \mathcal{H}(\lambda))$, by Proposition 16.3.3. Thus, $||\lambda+\rho|| = ||\mu+\rho||$. This implies that $\lambda + \rho$ and $\mu + \rho$ are in the same Weyl group orbit, by Theorem 14.7.7(iv). Thus, if $\lambda + \rho$ belongs to a wall, then $\mu + \rho$ does as well. This is impossible, since μ is dominant. Thus, if $H^p(X, \mathcal{H}(\lambda)) \neq 0$, then $\lambda + \rho$ does not belong to a wall.

We need one more lemma before proving the Borel-Weil-Bott theorem:

16.5.2 Lemma. *Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{g}$, and let Λ be its set of weights. Let α be an element of Δ, and let Δ^+ be a system of positive roots. If $\delta \in \mathfrak{h}'$ satisfies $\langle \delta, \alpha \rangle = 0$, and $\langle \delta, \beta \rangle > 0$ for all $\beta \in \Delta^+$ distinct from α and $-\alpha$, then the maximal value of $||\delta + \nu||$ for $\nu \in \Lambda$ is achieved at exactly two points, $\nu = \mu$ and $\nu = s_\alpha(\mu)$, where μ is the highest weight in Λ.*

Proof. Since $||\nu||^2 = \langle \nu, \nu \rangle$ is a convex function of ν in $\mathfrak{h}'$, the maximum clearly can occur only at weights $\delta + \mu$ for which μ is an extremal weight in Λ. Given two extremal weights μ and ν, we have $||\mu|| = ||\nu||$ by Theorem 14.7.7, and hence,

$$||\delta + \mu||^2 - ||\delta + \nu||^2 = 2\langle \delta, \mu - \nu \rangle.$$

If μ is the highest weight in Λ, then $\mu - \nu$ is a sum of positive roots, and so

$$||\delta + \mu||^2 - ||\delta + \nu||^2 > 0,$$

except in the case where $\mu - \nu$ involves only the root α, i.e. has the form $n\alpha$ for some n. The only extremal weight of the form $\nu = \mu - n\alpha$ is $s_\alpha(\mu)$.

Two Weyl chambers C and C' are *adjacent* if their closures meet in a *wall*, that is, a set of dimension $k-1$ in $\mathfrak{h}'$, where $\dim \mathfrak{h}' = k$. This happens if and only if there is a root $\alpha \in \Delta^+$ such that, for $\lambda \in C$, $\lambda' \in C'$, $\langle \lambda, \alpha \rangle$ and $\langle \lambda', \alpha \rangle$ have opposite signs , while $\langle \lambda, \beta \rangle$ and $\langle \lambda', \beta \rangle$ have the same sign for all positive roots $\beta \neq \alpha$. Thus, one passes from C to C' by crossing the hyperplane determined by α, while not crossing any hyperplanes determined by other positive roots. In this case, $s_\alpha(C) = C'$. The distance from a Weyl chamber (or one of its elements) to the positive chamber is defined to be the minimal number of such wall crossings needed to pass from the positive chamber to the given chamber. Thus, it is the number of negative numbers in the set $\{\langle\lambda,\beta\rangle : \beta \in \Delta^+\}$ for λ in the chamber. The *length* of a Weyl group element w is the distance from $w\rho$ to the positive chamber.

Note that the condition on the weight δ in the previous lemma can be satisfied only if the positive chamber C^+ is adjacent to $s_\alpha(C^+)$; then it is equivalent to the condition that δ belong to the wall between C^+ and $s_\alpha(C^+)$ and not to any other walls. An example of such a weight is $\delta = \rho + s_\alpha(\rho)$.

16.5.3 Borel-Weil-Bott Theorem. *Let λ be an integral weight. Then*
(i) *if $\lambda+\rho = w(\nu+\rho)$ for a dominant weight ν and a $w \in W$ of length d, then $H^d(X,\mathcal{H}(\lambda))$ is isomorphic to the irreducible $\mathfrak{g}$-module of highest weight ν;*
(ii) *$H^p(X,\mathcal{H}(\lambda))$ vanishes in all other cases.*

Proof. We already know from Proposition 16.5.1 that $H^p(X,\mathcal{H}(\lambda)) = 0$ if $\lambda+\rho$ lies in a wall. Thus, we may assume that $\lambda+\rho$ lies in a Weyl chamber – i.e. that $\lambda = w\nu + \mu$ for a dominant integral weight ν, where $\mu = w\rho - \rho$. By Lemma 16.4.1, we have

$$H^p(X,\mathcal{H}(\lambda)) = H^p(X,\mathcal{H}(\mu)) \otimes V,$$

where V is the irreducible finite dimensional representation of highest weight ν. Thus, the theorem is true for a given λ, with $\lambda+\rho$ in the chamber determined by w, if and only if it is true for the weight $\mu = w\rho - \rho$ – that is, if and only if $H^p(X,\mathcal{H}(\mu)) = 0$ unless p is the length of w, in which case $H^p(X,\mathcal{H}(\mu)) = \mathbb{C}$. Of course, this means that the theorem is true for all weights λ with $\lambda+\rho$ in a given chamber if and only if it is true for one such weight.

We prove the theorem by induction on the distance d from our chamber to the positive chamber. The case $d=0$ is the Borel-Weil theorem. Thus, we suppose the theorem is true for Weyl chambers at distance $d-1$ from the positive chamber and consider a chamber C at a distance d, obtained by applying w of length d to the positive chamber. We choose an integral weight ν within the positive chamber. Then ν is the highest weight of an irreducible finite dimensional representation (π, V). Also $w(\nu)$ is in our chamber C. Now C is adjacent to a chamber C' at distance $d-1$ from the positive chamber. Let α be the positive root defining the wall separating the two chambers. Then $w' = s_\alpha \circ w$ is a Weyl group element of length $d-1$ which maps the positive chamber to C'. We set $\eta = w'(\nu) \in C$. Since $\langle\alpha,\eta\rangle$ and $\langle\alpha, s_\alpha(\eta)\rangle$ have opposite signs and $w = s_\alpha \circ w'$ has length 1 greater than that of w', it must be that $\langle\alpha,\eta\rangle > 0$ and $\langle\alpha, s_\alpha(\eta)\rangle < 0$.

Let δ be an integral weight which satisfies the condition of Lemma 16.5.2 for the system of positive roots $w\Delta^+$. This means that δ is in the wall separating C and C' and no other walls. For example, $\delta = w\rho + s_\alpha(w\rho)$

would do. Since $s_\alpha(\eta) = w(\nu)$ is a highest weight for π relative to the system of positive roots $w\Delta^+$, Lemma 16.5.2 implies that $||\omega + \delta||$ achieves its maximum value, as ω ranges over the weights of π, at the points $\omega = \eta$ and $\omega = s_\alpha(\eta)$.

As in the proof of Lemma 16.4.1, let θ be the representation π restricted to the Borel subgroup B. Set $\tau = \delta - \rho$, and let θ_τ be the tensor product of θ with the 1-dimensional representation σ_τ of B determined by the integral weight τ. We then consider the induced bundle $I(\theta_\tau)$ and its sheaf of sections $\mathcal{I}(\theta_\tau)$. As in the proof of Lemma 16.4.1, $I(\theta_\tau) = I(\sigma_\tau) \otimes I(\theta)$ as G-equivariant bundles, $\mathcal{I}(\theta_\tau) = \mathcal{H}(\tau) \otimes V$ as sheaves of $\mathfrak{g}$-modules, and for each p

$$H^p(X, \mathcal{I}(\theta_\tau)) = H^p(X, \mathcal{H}(\tau)) \otimes V,$$

as $\mathfrak{g}$-modules. Since $\tau + \rho = \delta$ is in a wall, it follows from Proposition 16.5.1 that

$$H^p(X, \mathcal{H}(\tau)) = 0, \text{ and so } H^p(X, \mathcal{I}(\theta_\tau)) = 0,$$

for all p.

Since $\langle \alpha, \eta \rangle > 0$, and

$$s_\alpha(\eta) = \eta - 2\frac{\langle \alpha, \eta \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

we conclude that $s_\alpha(\eta) < \eta$ in the ordering of integral weights determined by Δ^+. We may define a B-submodule V' of V to be the span of all weight spaces for weights $\omega < \eta$. If V'' is the quotient V/V', then we have a short exact sequence of B-modules

$$0 \to V' \to V \to V'' \to 0,$$

with V' containing the weight space for weight $s_\alpha(\eta)$, and V'' containing the weight space for weight η. If we tensor this sequence with the 1- dimensional representation of B with character σ_τ and then induce, we are led to a corresponding short exact sequence of sheaves of $\mathfrak{g}$-modules:

$$0 \to \mathcal{I}(\theta'_\tau) \to \mathcal{I}(\theta_\tau) \to \mathcal{I}(\theta''_\tau) \to 0, \tag{16.5.1}$$

where θ'_τ and θ''_τ will denote the subrepresentation and quotient representation of θ_τ corresponding to V' and V''.

By Proposition 16.3.5, the sheaf $\mathcal{I}(\theta_\tau)$ is a direct sum of subsheaves which are generalized eigenspaces for the action of the Casimir Ω. The possible eigenvalues are of the form

$$\langle \omega + \tau, \omega + \tau + 2\rho \rangle = ||\omega + \tau + \rho||^2 - ||\rho||^2,$$

where ω is a weight of the representation (π, V). The particular eigenvalue $t = ||\eta+\tau+\rho||^2 - ||\rho||^2 = ||s_\alpha(\eta)+\tau+\rho||^2 - ||\rho||^2$ occurs with multiplicity 2 for the module $\mathcal{I}(\theta_\tau)$, since η and $s_\alpha\eta$ are the two weights ω where $||\omega+\tau+\rho|| = ||\omega + \delta||$ achieves its maximum. If we apply the projection onto the Ω-eigenspace for eigenvalue t to each term in (16.5.1), we obtain a short exact sequence:

$$(16.5.2) \qquad 0 \to \mathcal{I}(\theta'_\tau)_t \to \mathcal{I}(\theta_\tau)_t \to \mathcal{I}(\theta''_\tau)_t \to 0.$$

Since $s_\alpha\eta + \tau$ is a weight of θ'_τ, and $\eta + \tau$ is a weight of θ''_τ, it follows that t is an eigenvalue of multiplicity 1 for Ω on each of $\mathcal{I}(\theta'_\tau)$ and $\mathcal{I}(\theta''_\tau)$. We conclude from Proposition 16.3.5 that

$$\mathcal{I}(\theta'_\tau)_t = \mathcal{H}(s_\alpha\eta + \tau), \text{ and } \mathcal{I}(\theta''_\tau)_t = \mathcal{H}(\eta + \tau).$$

Since $\mathcal{I}(\theta_\tau)$ has vanishing cohomology in all degrees, the same thing is true of its direct summand $\mathcal{I}(\theta_\tau)_t$. From the long exact sequence of cohomology associated to (16.5.2), we conclude that

$$(16.5.3) \qquad H^{p+1}(X, \mathcal{H}(s_\alpha(\eta) + \tau)) \simeq H^p(X, \mathcal{H}(\eta + \tau))$$

for all p. Since $\tau + \rho$ is in the wall separating C and C', it is fixed by s_α, and since η is in the chamber C', it follows that $\eta+\tau+\rho$ is also in C'. Then

$$s_\alpha(\eta) + \tau + \rho = s_\alpha(\eta + \tau + \rho)$$

is the corresponding element of the chamber C. Since we have assumed the theorem true for all $\lambda + \rho$ in C', and hence, for $\lambda + \rho = \eta + \tau + \rho$, (16.5.3) shows that the theorem is also true when $\lambda+\rho$ is the element $s_\alpha(\eta+\tau+\rho)$ of the chamber C. As noted above, the theorem is true for all integral weights $\lambda + \rho$ in a chamber if it is true for one. This completes the proof.

16.5.4 Example. As in Example 16.2.4, we consider $G = SL_2(\mathbb{C})$. We let B be the Borel subgroup of upper triangular matrices and N the unipotent subgroup consisting of upper triangular matrices with 1's on the diagonal. Then N is the isotropy group of the point $(1, 0) \in \mathbb{C}^2$ and $\mathbb{C}^2 - \{0\}$ is isomorphic to the quotient space $Y = G/N$. The flag variety is $P^1 = P^1(\mathbb{C})$, obtained by taking the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $H = \mathbb{C}^*$ under scalar multiplication.

We identify $\mathfrak{h}$ with $\mathbb{C}$. We also identify $\mathfrak{h}^*$ with $\mathbb{C}$ through the usual pairing of $\mathbb{C}$ with itself. If h, x and y are the standard basis elements for $\mathfrak{sl}_2(\mathbb{C})$ as in Example 14.5.6, then h is identified with 1 under the above identification of $\mathfrak{h}$ with $\mathbb{C}$. Also, the commutation relation $[h, x] = 2x$ implies that the root $\alpha \in \mathfrak{h}^*$ corresponding to the root vector x is identified with the number 2. Since α and $-\alpha$ are the only roots, we may choose $\{\alpha\}$ as a system of positive roots. The element ρ which is half the sum of the positive roots is then identified with the number 1.

The characters of the Cartan subgroup H are the functions $z \to z^k$. Note that there is one for each of the integral elements $k \in \mathfrak{h}^*$. The sheaves of sections of G-equivariant line bundles on X are the sheaves $\mathcal{H}(k)$ described in Chapter 12. According to Theorem 12.3.3 and Theorem 12.3.5,

1. for $k \geq 0$, $H^0(X, \mathcal{H}(k))$ is the space of k-homogeneous polynomials on $\mathbb{C}^2$;
2. for $k = 1$, $H^p(X, \mathcal{H}(k)) = 0$ for all p; and
3. for $k \leq 2$, $H^1(X, \mathcal{H}(k)) \simeq H^0(X, \mathcal{H}(-2-n))^*$ and $H^p(X, \mathcal{H}(k)) = 0$ if $p \neq 1$.

These are exactly the results predicted by the Borel-Weil-Bott theorem, for the following reasons: There are two Weyl chambers $\mathbb{R}^+$ and $\mathbb{R}^-$ with $\mathbb{R}^+$ being the positive chamber, of course. We have $k + \rho = k + 1 \in \mathbb{R}^+$ if and only if $k \geq 0$, while $k + \rho \in \mathbb{R}^-$ if and only if $k \leq 2$. The case $k = -1$ is the case where $k + \rho$ lies in a wall.

16.6 Consequences for Real Semisimple Lie Groups

With the results of this chapter and the classification theorem (Theorem 14.7.9), we have a fairly good understanding of the irreducible holomorphic representations of complex semisimple Lie groups. We know they correspond to integral weights which are differentials of characters of a Cartan subgroup, and the Borel-Weil-Bott theorem gives us a variety of geometric realizations of each of them (one for each Weyl chamber). It is natural to ask what this tells us about the irreducible finite dimensional representations of a real semisimple Lie group.

If G_0 is a real semisimple Lie group, and (π, V) is an irreducible finite dimensional representation of G_0, one might try to relate (π, V) to a representation of a complex semisimple Lie group in the following way: Embed G_0 as a real form of a complex semisimple Lie group G, and then extend the representation π to a holomorphic representation of G. Unfortunately, this does not work, since a real semisimple Lie group may not be realizable as a real form of a complex Lie group (Exercise 16.12). However, something very similar to this does work.

If G_0 is a connected real semisimple Lie group, then its Lie algebra $\mathfrak{g}_0$ has a complexification $\mathfrak{g}$ which is a complex semisimple Lie algebra. If $\tilde{G}_0$ is the universal covering group of G_0, and $\tilde{G}$ is the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}$, then the inclusion $\mathfrak{g}_0 \to \mathfrak{g}$ induces a morphism of real Lie groups $\phi : \tilde{G}_0 \to \tilde{G}$. Let C be the kernel of the natural map $\tilde{G}_0 \to G_0$. Since C is in the center of $\tilde{G}_0$, $\phi(C)$ is contained in the center of $\tilde{G}$, which is finite. If we set $G = \tilde{G}/\phi(C)$, then ϕ induces a morphism $\psi : G_0 \to G$ with $\operatorname{Ker}\psi = \operatorname{Ker}\phi/(C \cap \operatorname{Ker}\phi)$.

Each finite dimensional representation (π, V) of G_0 arises from a representation of $\tilde{G}_0$ which is trivial on C and which has as differential a finite dimensional representation of $\mathfrak{g}_0$. This, in turn, uniquely extends by linearity to a representation of the complex Lie algebra $\mathfrak{g}$ which is the differential of a holomorphic representation $\tilde{\pi}$ of $\tilde{G}$ on V. Necessarily, π and $\tilde{\pi} \circ \phi$ agree. Thus, $\tilde{\pi}$ is trivial on $\phi(C)$, and therefore, induces a holomorphic representation π' of G on V such that $\pi = \pi' \circ \psi$. In summary:

Proposition 16.6.1. *Let G_0 be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_0$, and let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Then there is a complex Lie group G, with Lie algebra $\mathfrak{g}$, and a morphism of real Lie groups $\psi : G_0 \to G$ such that each finite dimensional representation (π, V) of G_0 has the form $\pi = \pi' \circ \psi$ for a unique finite dimensional holomorphic representation (π', V) of G.*

By the result of Exercise 16.12, the group G in the above proposition need not be a group which has G_0 as a real form. In other words, the morphism ψ need not be injective.

Since the irreducible finite dimensional holomorphic representations of the complex group G have geometric descriptions in terms of G-equivariant line bundles on X, this gives us a similar description of the irreducible finite dimensional representations of the real group G_0. That is, for each such representation and each Weyl chamber, there is a G_0-equivariant holomorphic line bundle on X (which is actually G-equivariant), which has non-vanishing cohomology in one degree (the distance to the positive chamber), and this cohomology is isomorphic as a G_0-module to the given representation.

The action of G_0 on X gives rise to representations of G_0 other than those described above, representations also determined by G_0-equivariant line bundles, but constructed not on all of X, but on G_0 orbits which are proper subsets of X, and they give rise to infinite dimensional representations of G_0. This is the topic of the next and final section of the chapter.

16.7 Infinite Dimensional Representations

So far our focus in this chapter has been on finite dimensional representations of semisimple Lie groups. However, perhaps the most important thing about the Borel-Weil-Bott theorem is that it points the way to a geometric description of a large and important class of infinite dimensional representations of real semisimple Lie groups. In this final section we will give a brief survey of this topic; a full development is beyond the scope of this text, involving as it does a large body of technical machinery not touched on here (see **[HM]**, **[HT]**, **[KSc]**, **[Sm]**, **[Vo]** for various approaches to the subject).

In the following discussion, by a *representation* (π, V) of a locally compact topological group G_0 we will mean a continuous representation π on a complete locally convex topological vector space V – continuous in the sense that the function

$$g \to \pi(g)v : G \to V$$

is continuous for each $v \in V$. Such a representation is called *irreducible* if V contains no non-trivial closed invariant subspaces for π.

Unless the group is compact, one does not get very far in studying the harmonic analysis of a locally compact group if the only irreducible representations that are understood are those which are finite dimensional. Non-compact real semisimple Lie groups have a wealth of irreducible infinite dimensional representations on Banach spaces or more general locally compact topological vector spaces. It is hopeless to attempt to classify all of them for the following reasons: (1) a given representation can be modified in countless ways by simply changing the topology on the representation space, and then completing in the new topology; (2) there are pathological representations which are badly behaved, and are unlikely to fit into any reasonable classification scheme. Consequently, one usually focuses the study of the representations of a real Lie group on some restricted class, which contains the representations which are known to arise in interesting problems, and for which there is some hope of a reasonable classification scheme. Harish-Chandra introduced such a class for real semisimple Lie groups – the class of admissible representations. A brief description of this class is given below.

Let G_0 be a connected real semisimple Lie group, and let K_0 be a maximal connected compact subgroup of G_0. Such a subgroup exists, because an increasing chain of connected Lie subgroups of a Lie group must terminate, due to dimension considerations. Let (π, V) be a representation of G_0 on a Banach space V. A vector $v \in V$ is called K_0*-finite* if $\{\pi(k)v : k \in K_0\}$ spans a finite dimensional subspace of V. The set V_0 of K_0-finite vectors is spanned by finite dimensional K_0-invariant subspaces of V, and hence, is a direct sum of irreducible representations of K_0. If the irreducible representations of K_0 which occur in this decomposition of V_0 occur with finite multiplicity, then the representation (π, V) is called *admissible.* In this case, each K_0-finite vector is a $\mathcal{C}^\infty$ vector – that is, a vector v for which the function $g \to \pi(g)v : G_0 \to V$ is an infinitely differentiable vector valued function (Exercise 16.15). Furthermore, V_0 is a dense subspace of V (Exercise 16.16) which has an action of the Lie algebra $\mathfrak{g}$ (given by differentiation of $g \to \pi(g)v$) and an action of the compact group K_0. The resulting space V_0 together with its $\mathfrak{g}$-module and K_0-module structures is called the *Harish-Chandra module* of V. Harish-Chandra's strategy for studying infinite dimensional representations was to restrict attention to admissible representations and to study them by studying their Harish-Chandra modules. The Harish-Chandra module depends on the choice of a maximal compact subgroup K_0 of G_0, but this is not a serious problem, since the maximal compact subgroups of G_0 are all conjugate ([**He**], Chapter VI, Theorem 2.2).

The Harish-Chandra module of an admissible representation is a dense subspace which determines the essential features of the representation, but is not affected by tampering with the topology of the representation and completing. This makes the category of Harish-Chandra modules a reasonable object of study. In fact, the irreducible Harish-Chandra modules have been classified. One way of classifying them is through a theory developed by Beilinson and Bernstein [**BB**] which defines an equivalence of categories between the category of Harish-Chandra modules and a certain category of quasi-coherent sheaves of $\mathcal{O}$-modules on the flag variety of the complexification of the Lie algebra of G_0. This equivalence is a powerful generalization of the Borel-Weil theorem. There are other equivalences of this nature, corresponding to Weyl chambers other than the positive chamber which, taken together, constitute a vast generalization of the Borel-Weil-Bott theorem.

We will not attempt to describe the Beilinson-Bernstein theory here. It would require the development of a significant amount of machinery just to give a non-trivial example of the kind of sheaf which occurs. There is another, more recent, approach to the problem of understanding infinite dimensional representations of G_0 that is similar to that of Beilinson-Bernstein, in that it also describes representations in terms of sheaves on the flag variety, and it is also a generalization of the Borel-Weil-Bott theorem, but it is somewhat easier to describe the sheaves which occur, and we will attempt to do so below.

If (π, V) is an admissible representation of G_0, then an *analytic vector* for π is a vector $v \in V$ for which the vector valued function $g \to \pi(g)v : G_0 \to V$ is analytic, in the sense that $g \to f(\pi(g)v)$ is real analytic for every $f \in V^*$. If V_a denotes the space of analytic vectors, then V_a contains the Harish-Chandra module V_0 of V (for any choice of maximal compact K_0) and is invariant under the action of G_0 (unlike the Harish-Chandra module). The space V_a may be given a natural topology, stronger than the topology it inherits from V, which makes it a very special kind of complete topological vector space (a dual nuclear Fréchet space, or DNF space). For each $v \in V_a$, the map $g \to \pi(g)v$ is not only continuous, it is also analytic (in the DNF topology of V_a, not just in the topology inherited from V). Also, V_0 is dense in V_a with the DNF topology. Thus, V_a is a kind of completion of V_0. In fact, there is a sense in which V_a is the minimal completion of the Harish-Chandra module V_0 on which there is a G_0-action compatible with the $\mathfrak{g}$-action on V_0. For that reason V_a is called the *minimal globalization* of V_0 (see [**Sch**], [**KSc**]).

We will call the G_0-modules which occur as minimal globalizations of Harish-Chandra modules (equivalently, as the modules of analytic vectors in admissible representations) *analytic modules of finite type.* As with Harish-Chandra modules, they may all be described in terms of a certain abelian category of sheaves on the flag variety X. These are analytic sheaves which have an analytic G_0-action and satisfy certain finiteness conditions. Rather than giving the technical definition of this category, we will describe its irreducible objects – the sheaves we call *standard analytic sheaves.* For a full development of this subject see [**HT**] and [**Sm**].

For simplicity of exposition in what follows, we will assume that the real connected semisimple group G_0 is a real form of a connected complex semisimple Lie group G. The flag variety for G will be denoted X.

Through its inclusion in G, the real group G_0 acts on X. It turns out that there are only finitely many orbits for this action, and each of them is a real analytic submanifold of X. There is always a unique closed (hence compact) orbit, which is a real form of the complex manifold X, and hence, has real dimension equal to the complex dimension n of X. There are also open orbits, and these have real dimension $2n$. The other orbits have real dimension between n and $2n$. Each orbit Q is given a ringed space structure by letting the structure sheaf be $\mathcal{H}$ restricted to Q. This is the ordinary complex structure on the open orbits, is the real analytic structure on the closed orbit, and on a general orbit, is a mixed real and complex analytic structure called a CR structure.

Suppose Q is a G_0 orbit of X. Let $\mathcal{H}(\lambda)$ be the sheaf of holomorphic sections of the line bundle on X determined by an integral weight $\lambda \in \mathfrak{h}^*$. We let $\mathcal{H}_Q(\lambda)$ denote the sheaf which is the extension by zero to X of the restriction of $\mathcal{H}(\lambda)$ to Q. Recall from section 7.4 that this is the unique sheaf on X which has 0 stalk at every point of the complement of Q, and which has the same restriction to Q as does $\mathcal{H}(\lambda)$. Since Q is a G_0 orbit, there is a natural action of G_0 on the sheaf $\mathcal{H}_Q(\lambda)$, inherited from the action of G on $\mathcal{H}(\lambda)$. This, in turn, induces a representation of G_0 on each of the cohomology spaces of $\mathcal{H}_Q(\lambda)$. Now $\mathcal{H}_Q(\lambda)$ is an analytic sheaf, but is not coherent. Hence, there is no reason to expect its cohomologies to be finite dimensional. In fact, they are not. They do, however, possess natural DNF topologies for which the resulting representations of G_0 are analytic of finite type. This, then, is one way to construct a fairly large family of well-behaved infinite dimensional representations of G_0.

One can enlarge the class of representations of G_0, obtained as above, by observing that the induction procedure used to construct the sheaves $\mathcal{H}(\lambda)$ for integral λ can be generalized to produce, on some orbits Q, sheaves which correspond to non-integral weights λ.

Recall the discussion of induction in the paragraph preceding Example 16.2.4. Let Q be a G_0 orbit in X, and let q be a point of Q. Let B be the Borel subgroup of G which is the isotropy group of q, let N be its maximal unipotent subgroup, and set $Y = G/N$, and $H = B/N$. The right action of B on G induces a right action of H on Y, with orbits equal to the fibers of the natural projection $\nu : Y \to X$. The identity coset of N determines a point $p \in Y = G/N$ with $\nu(p) = q$. If C is the G_0 orbit $G_0 p$ in Y, then $\nu(C) = Q$. The isotropy group of q in the real group G_0 is $P = G_0 \cap B$. Its quotient $A = P/(P \cap N)$ is a real subgroup of the complex torus H (it is, in fact, a real form of H) which acts on the right on C and has orbits equal to the fibers of the map $\nu : C \to Q$. If σ is a character of A, we define a sheaf $\mathcal{I}(Q, \sigma)$ on Q by setting

$$\mathcal{I}(Q,\sigma)(U) = \{f \in i^{-1}\mathcal{H}(\nu^{-1}(U)) : f(yh^{-1}) = \sigma(h)f(y)\},$$

where $i : C \to Y$ is the inclusion. This is exactly what was done to construct $\mathcal{H}(\lambda)$ in the paragraph preceding Example 16.2.4, except that here we are doing it on the space C and its quotient Q by A instead of on Y and its quotient X by H, and the character σ is a character of the real group A, rather than a holomorphic character of the complex group H.

16.7.1 Definition. *With Q, σ, and $\mathcal{I}(Q, \sigma)$ as above, the standard analytic sheaf $\mathcal{H}_Q(\sigma)$ is the extension of $\mathcal{I}(Q, \sigma)$ by zero to X.*

The standard analytic sheaf $\mathcal{H}_Q(\sigma)$ has a natural G_0-action determined by the action of G_0 by left translation on Y, and this induces a representation of G_0 on each of its cohomologies. Each of these cohomologies has a natural DNF topology, and the G_0-action is analytic with respect to this topology. In fact, each non-trivial cohomology module of a standard analytic sheaf yields an analytic G_0-module of finite type.

How are the sheaves $\mathcal{H}_Q(\sigma)$ related to the sheaves $\mathcal{H}_Q(\lambda)$, defined above for integral λ? The restriction of σ to the identity component of A has differential which agrees on the Lie algebra of A with some weight $\lambda \in \mathfrak{h}^*$; however, this weight may not be integral. Furthermore, if A is not connected, then there will be several different representations σ corresponding to the same weight λ. Only in the case where λ is integral, and the corresponding holomorphic character σ_λ on H agrees with σ on each component of A, do we expect $\mathcal{H}_Q(\sigma)$ to be isomorphic to $\mathcal{H}_Q(\lambda)$. In fact, this is exactly what happens.

The class of standard analytic sheaves is sufficiently rich that its cohomologies generate the category of analytic modules of finite type which, as mentioned above, is equivalent to the category of Harish-Chandra modules. Here the closed orbit plays a special role. The standard analytic sheaves for this orbit are acyclic and their modules of sections are the so-called *principle series* representations of G_0.

To give some idea of how all of this works, we will describe explicitly what happens in the case of $SL_2(\mathbb{R})$.

16.7.2 Example. Recall the discussion of $SL_2(\mathbb{C})$ in Examples 16.2.4 and 16.5.4. The group of upper triangular matrices in $SL_2(\mathbb{C})$ is a Borel subgroup and the flag variety X is the Riemann sphere $P^1 = P^1(\mathbb{C})$. The sheaves $\mathcal{H}(\lambda)$ of the Borel-Weil-Bott theorem are defined for integral weights λ which, in this case, are represented by actual integers. For each integer n, the sheaf $\mathcal{H}(n)$ is the sheaf of n-homogeneous functions on $\mathbb{C}^2$, as defined in Chapter 12.

We let G_0 be the subgroup of $SL_2(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{16.7.1}$$

Then G_0 is a copy of $SL_2(\mathbb{R})$ (Exercise 16.17). The G_0 orbits of P^1 are the circle T, determined by the set of $(z, w) \in \mathbb{C}^2$ with $|z| = |w|$, and the hemispheres D^+ and D^-, determined by the inequalities $|z| < |w|$ and $|w| < |z|$.

Let p^+ be the point of D^+ determined by $\{(z, 0) : z \in \mathbb{C}^*\}$, and let p^- be the point of D^- determined by $\{(0, w) : w \in \mathbb{C}^*\}$. The points p^+ and p^- have the same isotropy group in G_0 – the group K_0 consisting of matrices of the form (16.7.1) with $b = 0$ and $|a| = 1$. This is a copy of the circle group. It is both the P and the A of the discussion preceding Definition 16.7.1, since $N \cap P = 0$ in this case. Only integral weights give rise to characters (irreducible representations) of K_0, and so each character of K_0 is the restriction to K_0 of a holomorphic character of H. This implies that the standard analytic sheaves $\mathcal{H}_{D^\pm}(n)$, for these orbits, are obtained from $\mathcal{H}(n)$ by restricting to $D^\pm$, and then extending by 0. For each i and n, the resulting sheaf has no non-zero sections, but it does have non-vanishing first cohomology, and this yields a representation of G_0.

If we choose for the orbit T the point determined by $\{(z,z) : z \in \mathbb{C}^*\}$, then its isotropy group B in G is the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $a+b = c+d$, while N consists of the elements of B with $a+b=1$. Then $P = B \cap G_0$ is the set of matrices in G_0 with $a+b$ real. Furthermore, $H = B/N$ is $\mathbb{C}^*$, as before, and $A = P/(P \cap N)$ is identified with $\mathbb{R}^*$ via the projection map $P \to A$ which sends a matrix of the above form in P to the number $a+b$. Each weight (complex number) λ defines two characters σ_λ^+ and σ_λ^- of $\mathbb{R}^*$, where

$$\sigma_\lambda^+(x) = |x|^\lambda,$$

and

$$\sigma_\lambda^-(x) = \operatorname{sgn}(x)|x|^\lambda.$$

Then the construction of Definition 16.7.1, applied to these characters, yields standard sheaves $\mathcal{H}_T(\sigma_\lambda^\pm)$. The resulting sheaves are acyclic and have non-trivial global sections. The representations of G_0 on the modules of global sections are the *principle series* representations.

Notice that some of the sheaves $\mathcal{H}_T(\sigma_\lambda^\pm)$ come from restricting the globally defined sheaves $\mathcal{H}(n)$ to T – namely those for which $\sigma_\lambda^\pm$ is the restriction to $\mathbb{R}^*$ of a holomorphic character $z \to z^n$ of $\mathbb{C}^*$. This will be true if and only if λ is an integer n and the sign in $\sigma_\lambda^\pm$ is determined by the parity of n – positive for even n and negative for odd n. If this is the case, and if we denote by $\mathcal{H}_T(n)$ the resulting sheaf, then we have an exact sequence of analytic sheaves on P^1:

$$0 \to \mathcal{H}_{D^+}(n) \oplus \mathcal{H}_{D^-}(n) \to \mathcal{H}(n) \to \mathcal{H}_T(n) \to 0. \tag{16.7.2}$$

If n is non-negative, then $\mathcal{H}(n)$ is acyclic and has module of global sections equal to the space of n-homogeneous polynomials on $\mathbb{C}^2$. If $n \le -2$, then $\mathcal{H}(n)$ has no non-trivial global sections, but $H^1(\mathcal{H}(n))$ is non-trivial, and in fact, isomorphic to $H^0(\mathcal{H}(-n-2))$ (Example 16.5.4) (all cohomologies in this discussion will be over P^1 and so we will suppress the symbol P^1 in the notation $H^p(P^1, \mathcal{H}(n))$). Thus, we have short exact sequences
(16.7.3)

$$0 \to H^0(\mathcal{H}(n)) \to H^0(\mathcal{H}_T(n)) \to H^1(\mathcal{H}_{D^+}(n)) \oplus H^1(\mathcal{H}_{D^-}(n)) \to 0,$$

in the case $n \ge 0$, and

$$0 \to H^0(\mathcal{H}_T(n)) \to H^1(\mathcal{H}_{D^+}(n)) \oplus H^1(\mathcal{H}_{D^+}(n)) \to H^1(\mathcal{H}(n)) \to 0,$$

in the case where $n \le -2$. For $n = -1$, $\mathcal{H}(n)$ has vanishing cohomology in all degrees, including 0, and so

$$H^0(\mathcal{H}_T(-1)) \simeq H^1(\mathcal{H}_{D^+}(-1)) \oplus H^1(\mathcal{H}_{D^-}(-1)).$$

This shows that the principle series representations $H^0(\mathcal{H}_T(n))$ are reducible and indicates how they decompose. In all other cases, the principle series representations $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$ are irreducible (Exercise 16.21).

It is instructive to look at a Laurent series description of $H^0(\mathcal{H}_T(n))$. By Exercise 16.18, $H^0(\mathcal{H}_T(n))$ is isomorphic as a G_0-module to the space of series

$$\sum_{k=-\infty}^{\infty} a_k z^{n-k} w^k \tag{16.7.4}$$

where the sequence of coefficients $\{a_k\}$ satisfies

$$|a_k| \le M t^{|k|}, \tag{16.7.5}$$

for some $M > 0$, and some t with $0 < t < 1$. This ensures that (16.7.4) converges uniformly in a neighborhood of the set $C \subset \mathbb{C}^2$ defined by $w = \bar{z}$ – that is, the set of points in $\mathbb{C}^2$ of the form $(re^{i\theta}, re^{-i\theta})$, $r > 0$, $0 \le \theta < 2\pi$. This is the set C of the discussion preceding Definition 16.7.1. It is the G_0 orbit of the point $(1,1) \in \mathbb{C}^2$. It projects onto T under the quotient map $\nu : \mathbb{C}^2 - \{0\} \to P^1$, and the fibers of this projection $C \to T$ are the orbits of the action of R^* on C via scalar multiplication. Here, the group G_0 acts on the functions defined by (16.7.4) via its action on the orbit C. It follows that, in the corresponding action of the Lie algebra $\mathfrak{g}$, the standard basis for $\mathfrak{sl}_2(\mathbb{C})$ acts by the differential operators ξ^+, ξ^-, and η of section 14.6. One can see explicitly from this description that, for $n \ge 0$, $H^0(\mathcal{H}_T(n))$ has a finite dimensional irreducible submodule consisting of those series which are polynomials in z and w and that the quotient is a direct sum of two irreducible $\mathfrak{g}$ modules (Exercise 16.19). This is the decomposition expressed in (16.7.3).

Finally, there is also a Laurent series description of the other principle series representations $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$. We begin by noticing that, on the G_0 orbit C, zw is a positive real valued function, and so the monomial $(zw)^{\lambda/2}$ is defined. In fact, it may be defined on the neighborhood U of C defined by the condition $\mathrm{Re}(zw) > 0$ by using the principle branch of the log function and setting $(zw)^{\lambda/2} = \mathrm{e}^{(\lambda/2)\log(zw)}$. With this definition, each series of the form

$$\sum_{k=-\infty}^{\infty} a_k z^{-k} w^k (zw)^{\lambda/2}, \tag{16.7.6}$$

with coefficients satisfying (16.7.5), makes sense and converges uniformly in a neighborhood of C. We conclude that it determines a well-defined holomorphic function in a neighborhood of C. Note that each monomial in this series is an even function. The resulting space of functions is a G_0-module, with G_0-action given by the action of G_0 on C. The corresponding action of the Lie algebra $\mathfrak{g}$ is again by the differential operators ξ^+, ξ^-, and η of section 14.6. The functions f defined by (16.7.6) satisfy

$$f(sz, sw) = |s|^\lambda f(z, w), \ \forall\ s \in \mathbb{R}^*,$$

which means they define elements of $H^0(\mathcal{H}_T(\sigma_\lambda^+))$. It is not hard to see that every element of $H^0(\mathcal{H}_T(\sigma_\lambda^+))$ can be put in this form (Exercise 16.20). This describes $H^0(\mathcal{H}_T(\sigma_\lambda^+))$. The description of $H^0(\mathcal{H}_T(\sigma_\lambda^-))$ is very similar. It can be described as the space of series of the form

$$\sum_{k=-\infty}^{\infty} a_k z^{1-k} w^k (zw)^{(\lambda-1)/2}, \tag{16.7.7}$$

again with coefficients satisfying (16.7.5). Note that the monomials in this series are odd functions. The functions represented by such series satisfy

$$f(sz, sw) = \operatorname{sgn}(s)|s|^\lambda f(z, w), \ \forall\ s \in \mathbb{R}^*,$$

and hence, belong to $H^0(\mathcal{H}_T(\sigma_\lambda^-))$.

If λ is an even integer n, then the space $H^0(\mathcal{H}_T(\sigma_\lambda^+))$ of functions represented by (16.7.6) is actually the same as the space $H^0(\mathcal{H}_T(n))$ of functions represented by (16.7.4). To see this, simply make the change of variables $k \to k + n/2$ in (16.7.4). If λ is an odd integer n, then the change of variables $k \to k + (n-1)/2$ shows that the description of $H^0(\mathcal{H}_T(\sigma_n^-))$ given by (16.7.7) is equivalent to the description of $H^0(\mathcal{H}_T(n))$ given by (16.7.4).

The monomials that appear in the above Laurent series expansions are weight vectors for the action of the compact group K_0, and this makes it easy to identify the underlying Harish-Chandra modules (modules of K_0-finite vectors) for the analytic modules $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$ and $H^0(\mathcal{H}_{D^\pm}(n))$. In each case, they consist of the series with only finitely many non-vanishing coefficients (Exercise 16.22).

One final comment: There are clearly many ways to impose a norm topology on any one of the standard modules, described above, in such a way that the G_0-action is continuous. The completion of the standard module in the resulting topology will always be an admissible representation of G_0, with the same underlying standard module as its module of analytic vectors, and with the same underlying Harish-Chandra module as its module of K_0-finite vectors. For example, for the principle series representations (those corresponding to the orbit T), such norms may be obtained by restricting the functions in the standard module to the circle $\tilde{T} = \{(e^{i\theta}, e^{-i\theta}) : 0 \le \theta < 2\pi\}$ and imposing the supremum norm, or one of the L^p norms for Lebesgue measure on $\tilde{T}$. This illustrates the point made earlier: The category of Harish-Chandra modules, or equivalently, the category of finite type analytic modules is a much more tractable category to deal with than is the category of all Banach space representations of G_0.

Exercises

1. Let (σ, W) be a finite dimensional holomorphic representation of a Lie subgroup B of a complex Lie group G. Finish the proof of Theorem 16.1.3 by proving that if the induced bundle $I(\sigma)$ is isomorphic to a trivial G-equivariant vector bundle, then σ is the restriction to B of a holomorphic representation of G.
2. Prove that the induced sheaves of Example 16.1.5 are the sheaves $\mathcal{H}(k)$ introduced in Chapter 12.
3. For the complex torus $H = D_n(\mathbb{C}) \simeq (\mathbb{C}^*)^n$, give an explicit description of the group of characters of H. Identify the subset of $\mathfrak{h} = \mathbb{C}^n$ consisting of all differentials of characters of H.
4. Let B be a connected solvable algebraic group, and let N be the maximal unipotent subgroup of B. Prove that every character of B is the composition of the quotient map $B \to B/N$ with a character of the torus B/N.
5. Prove that the enveloping algebra $U(\mathfrak{g})$ has properties (1) and (2), as claimed in the first paragraph of section 16.3, and prove that any associative algebra with these properties is isomorphic to $U(\mathfrak{g})$.
6. Prove that the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of a complex Lie group G is isomorphic to the algebra of right invariant holomorphic differential operators on ${}_G\mathcal{H}$. Also prove that if G is algebraic, then each right invariant holomorphic differential operator is algebraic (maps ${}_G\mathcal{O}$ to itself).
7. Let (θ, W) be a finite dimensional representation of B, as in the proof of Proposition 16.3.5, and let t be a complex number. Prove that there exists a non-empty open set $U \subset X$ and a section $f \in \mathcal{I}(\theta)(U)$, such that $\Omega f = tf$ on U, if and only if $t = \sigma_\nu$ for some weight ν of θ.
8. For $G = SL_2(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the flag variety X is $P^1(\mathbb{C})$. Verify the claim made in Example 16.5.4 that, in this case, the G-equivariant holomorphic line bundles are the sheaves $\mathcal{H}(k)$ of Chapter 12.

9. Let $G = SL_3(\mathbb{C})$, let B be the Borel subgroup of G consisting of upper triangular matrices, and let A be the subgroup of $SL_3(\mathbb{C})$ consisting of matrices in which every element of the first column is 0 except the first one. Let X be the flag variety of G. Show that A is the isotropy group of an action of G on $P^2(\mathbb{C})$. Then show that the containments $B \subset A \subset G$ lead to a description of X as a fiber bundle with base $P^2(\mathbb{C})$ and fiber $P^1(C)$ such that the action of G on X is equivariant (in the sense that the projection $X \to P^2(\mathbb{C})$ is a morphism of G-spaces).
10. With G, B, and A as in the previous exercise, show that each irreducible representation of G can be obtained from a finite dimensional irreducible representation of A by inducing to a G-equivariant holomorphic vector bundle on $G/A \simeq P^2(\mathbb{C})$ and then passing to global sections.
11. With Exercise 16.9 as a guide, show how the flag variety of $SL_n(\mathbb{C})$ can be built from the projective spaces $P^k(\mathbb{C})$ for $k < n$ by successive fibrations.
12. Prove that the universal covering group of $SL_2(R)$ is not a real form of some complex semisimple Lie group.
13. Let G_0 be any real Lie group and let (π, V) be a representation of G_0 on a Banach space V. If ϕ is a $\mathcal{C}^\infty$ function with compact support, m is a left Haar measure on G_0, and $v \in V$, prove that the vector $u = \pi(\phi)v = \int \phi(g)\pi(g)v\, dm(g)$ is a $\mathcal{C}^\infty$ vector – that is, prove that $g \to \pi(g)u$ is a $\mathcal{C}^\infty$ vector valued function. This exercise and Exercises 16.14 and 16.15 will require that you know a little bit about integration and differentiation of Banach space valued functions.
14. With G_0 and (π, V) as in the previous exercise, prove that the set of $\mathcal{C}^\infty$ vectors in V is dense.
15. Let G_0 be a real semisimple Lie group with maximal compact subgroup K_0, and let (π, V) be an admissible representation of G_0 on a Banach space (see section 16.7). Prove that each K_0-finite vector V is a $\mathcal{C}^\infty$ vector. Hint: Let W_σ be the sum of all irreducible K_0-invariant subspaces of V equivalent to a given irreducible representation σ of K_0. By the Peter-Weyl theorem, there is a function $h_\sigma \in \mathcal{C}^\infty(K)$ such that the map $v \to \int h(k)\pi(k)v\, dm(k)$ is a projection operator P_σ mapping V onto W_σ. Show that this operator preserves the space of $\mathcal{C}^\infty$ vectors, and use this result, along with the previous exercise, to prove that each element of W_σ must be a $\mathcal{C}^\infty$ vector. Conclude that each element of V_0 is a $\mathcal{C}^\infty$ vector.
16. With G_0, K_0, and (π, V) as in the previous exercise, use the Peter-Weyl theorem to prove that the set V_0 of K_0-finite vectors is dense in V.
17. Prove that the set of matrices in $SL_2(\mathbb{C})$ of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

is a real Lie subgroup conjugate to $SL_2(\mathbb{R})$ in $SL_2(\mathbb{C})$.

18. Let n be an integer, and let $\mathcal{H}_T(n)$ be the standard analytic sheaf of Example 16.7.2. Show that $H^0(\mathcal{H}_T(n))$ is isomorphic as a $\mathfrak{g}$-module to the space of Laurent series (16.7.4).
19. Using the previous exercise, show directly that the $\mathfrak{g}$-module $H^0(\mathcal{H}_T(n))$, for $n \geq 0$, contains the n-homogeneous polynomials in z and w as an irreducible submodule and that the quotient decomposes as a direct sum of two submodules. Describe them as spaces of Laurent series and show that they are both irreducible. These must be the modules $H^1(\mathcal{H}_{D^\pm}(n))$ of (16.7.3).
20. In Example 16.7.2, verify that every element of $H^0(\mathcal{H}_T(\sigma_\lambda^+))$ has a Laurent series expansion (16.7.6) convergent in a neighborhood of the set C. In the same way, show that every element of $H^0(\mathcal{H}_T(\sigma_\lambda^-))$ has a Laurent series (16.7.7) convergent in a neighborhood of C.
21. In Example 16.7.2, use their descriptions in terms of Laurent series to prove that the $\mathfrak{g}$-modules $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$ are irreducible, except in the cases where λ is an integer n, and $\sigma_\lambda^\pm$ agrees with $r \to r^n$ on $\mathbb{R}^*$. Recall a module is said to be irreducible if it has no closed invariant subspaces. You may assume that the topology of $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$ is such that a subspace is closed if and only if it contains the limit of each of its sequences defined and converging uniformly on compact subsets of a neighborhood in $\mathbb{C}^2$ of the set C. (Hint: Prove first that every non-zero closed invariant subspace contains a monomial of the type that appears in (16.7.6) or (16.7.7) and then show that it must contain every such monomial).
22. Let K_0 be the subgroup of G_0 in Example 16.7.2 consisting of diagonal matrices with entries of modulus 1. When each of the standard modules $H^0(\mathcal{H}_T(\sigma_\lambda^\pm))$, $H^1(\mathcal{H}_{D^+}(n))$, and $H^1(\mathcal{H}_{D^-}(n))$ is expressed in terms of Laurent series, as in Exercises 16.18, 16.19, and 16.20, show that the corresponding Harish-Chandra module (module of K_0-finite vectors) is, in each case, the space of series with only finitely many non-zero terms.

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