Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum grassmannians.

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Abstract

We study quantum analogues of quotient varieties, namely quantum grassmannians and quantum determinantal rings, from the point of view of regularity conditions. More precisely, we show that these rings are AS-Cohen-Macaulay and determine which of them are AS-Gorenstein. Our method is inspired by the one developed by De Concini, Eisenbud and Procesi in the commutative case. Thus, we introduce and study the notion of a *quantum graded algebra with a staightening law* on a partially ordered set, showing in particular that, among such algebras, those whose underlying poset is wonderful are AS-Cohen-Macaulay. Then, we prove that both quantum grassmannians and quantum determinantal rings are quantum graded algebras with a staightening law on a wonderful poset, hence showing that they are AS-Cohen-Macaulay. In this last step, we are lead to introduce and study (to some extent) natural quantum analogues of Schubert varieties.

2000 Mathematics subject classification: 16W35, 16P40, 16S38, 17B37, 20G42.

Key words: Quantum matrices, quantum grassmannian, quantum Schubert variety, quantum determinantal ring, straightening laws, Cohen-Macaulay, Gorenstein.

Introduction.

Let $\Bbbk$ be an arbitrary field, m,n be positive integers such that $m \leq n$ and q be a non-zero element of $\Bbbk$. We denote by $\mathcal{O}_q(M_{m,n}(\Bbbk))$ the quantum deformation of the coordinate ring of the variety $M_{m,n}(\Bbbk)$ of $m \times n$ matrices with entries in $\Bbbk$ and by $\mathcal{O}_q(G_{m,n}(\Bbbk))$ the quantum deformation of the homogeneous coordinate ring of the Grassmannian $G_{m,n}(\Bbbk)$ of m-dimensional subspaces of $\Bbbk^n$ (see Definitions 3.1.1 and 3.1.4). In addition, if $1 \leq t \leq m$, we denote by $\mathcal{I}_t$ the ideal of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the $t \times t$ quantum minors (see the beginning of Subsection 3.1). As is well known, the algebras $\mathcal{O}_q(M_{m,n}(\Bbbk))$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are noncommutative analogues of $\mathcal{O}(M_{m,n}(\Bbbk))$ (the coordinate ring of the affine variety $M_{m,n}(\Bbbk))$ and of $\mathcal{O}(G_{m,n}(\Bbbk))$ (the homogeneous coordinate ring of the projective variety $G_{m,n}(\Bbbk))$ in the sense that the usual coordinate rings of these varieties are recovered when the parameter q is taken to be 1 (and the base field is algebraically closed). In the same way, $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ is a quantum deformation of the coordinate ring of the determinantal variety $M_{m,n}^{\leq t-1}(\Bbbk)$ of those matrices in $M_{m,n}(\Bbbk)$ whose rank is at most $t-1$.

*This research was supported by the Leverhulme Research Interchange Grant F/00158/X. Part of the work was done while the authors were visiting Antwerp supported by the European Science Foundation programme 'Noncommutative Geometry', and while the second author was visiting the University of Edinburgh supported by a grant from the Edinburgh Mathematical Society.

The commutative rings $\mathcal{O}(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}(G_{m,n}(\Bbbk))$ have been extensively studied in the past decades; an extensive account of their interest and properties can be found, for example, in [BV]. They are rings of invariants for natural actions of the general and special linear groups on a suitable polynomial ring. It follows that they are normal domains. In addition, if $\Bbbk$ is an algebraically closed field of characteristic zero, it follows by the Hochster-Roberts theorem (see [BH; Theorem 6.5.1]) that these rings are Cohen-Macaulay. In fact, the Cohen-Macaulay property of these rings still holds over an arbitrary field. One way to prove this is to make use of the notion of *graded algebras with a staightening law*, initiated in works by De Concini, Eisenbud and Procesi (see [E] and [DEP]). For a detailed description of this method and extensive comments on the historical background, the reader is referred to [BV] and [BH].

Hence, $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ can be thought of as quantum analogues of quotient varieties under the action of reductive groups. In support of this point of view, note that K.R. Goodearl, A.C. Kelly and the authors have proved that $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ is the ring of co-invariants for a natural co-action of the Hopf algebra $\mathcal{O}_q(GL_t(\Bbbk))$ on $\mathcal{O}_q(M_{m,t}(\Bbbk))\otimes\mathcal{O}_q(M_{t,n}(\Bbbk))$, see [GLR], and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is the ring of co-invariants for a natural co-action of the Hopf algebra $\mathcal{O}_q(SL_m(\Bbbk))$ on $\mathcal{O}_q(M_{m,n}(\Bbbk))$, see [KLR]. In this context, it is natural to study $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ from the point of view of regularity properties, where by regularity properties we mean the maximal order property, the AS-Cohen-Macaulay property and the AS-Gorenstein property, which are noncommutative analogues of classical regularity properties in commutative algebra and algebraic geometry. In [KLR] and [LR], the maximal order property for the above rings has been investigated. Recall that the maximal order property is a non commutative analogue for normality. Here, our aim is to study the $\Bbbk$-algebras $\mathcal{O}_q(G_{m,n}(\Bbbk))$ and $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ from the point of view of the AS-Cohen-Macaulay and AS-Gorenstein properties, as defined in recent works on noncommutative algebraic geometry (see, for example, [JZ]). These properties are noncommutative analogues of the notions of Cohen-Macaulay and Gorenstein rings.

There are many problems associated with obtaining quantum analogues of these famous commutative results. Of course, the most obvious problems arise due to the lack of commutativity; this forces us to study in detail specific kinds of commutation relations. Note that, although the algebras we consider are known to be algebras of co-invariants of suitable Hopf algebra coactions, at the moment we have no known methods for exploiting this fact.

In order to prove that $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are AS-Cohen-Macaulay $\Bbbk$-algebras, we introduce and study the notion of *quantum graded algebras with a staightening law* (quantum graded A.S.L. for short). Roughly speaking, a quantum graded A.S.L. is an $\mathbb{N}$-graded algebra with a partially ordered finite set Π of homogeneous generators, satisfying the following properties: standard monomials (that is products of elements of Π in increasing order) form a free family; the product of any two incomparable elements of Π can be written as a linear combination of standard monomials in a way compatible with the partial order on Π (these are the so-called *straightening relations*); given two elements $\alpha,\beta\in\Pi$, there exists a skew commutator of α and β which can be written as a linear combination of standard monomials in a way compatible with the partial order on Π (these are the so-called *commutation relations*). We then show that quantum graded A.S.L. are AS-Cohen-Macaulay (under a mild assumption on Π). The main point then is to show that $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are examples of such quantum graded A.S.L. We expect that the notion of a quantum graded A.S.L. will prove useful in many other contexts.

The paper is organised as follows. In Section 1, we introduce and study the notion of quantum graded A.S.L. on a partially ordered finite set Π. In particular, we show that such algebras are noetherian and we compute their Gelfand-Kirillov dimension. We also study in detail certain ideals of a quantum graded A.S.L., namely those which are generated by Π-ideals (see the beginning of Subsection 1.2). In particular, we show that these ideals are generated by normalising sequences. The Π-ideals are essential for our arguments because they allow us to study quantum graded A.S.L. by inductive arguments. In Section 2, we first introduce the homological background necessary to define and study the AS-Cohen-Macaulay and AS-Gorenstein properties and then recall the results concerning these notions that we will need. Then we prove that a quantum graded A.S.L. is AS-Cohen-Macaulay, provided that the poset Π is *wonderful*. The aim of Section 3 is then to show that the algebras $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are quantum graded A.S.L. For this, we first study $\mathcal{O}_q(G_{m,n}(\Bbbk))$ in detail: this essentially consists of proving the existence of straightening relations and specific kinds of commutation relations between maximal minors of $\mathcal{O}_q(M_{m,n}(\Bbbk))$. These two types of relations are established by introducing and studying natural quantum analogues of (coordinate rings on) Schubert varieties over arbitrary integral domains. In this way, we prove that $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is a quantum graded A.S.L. and deduce that $\mathcal{O}_q(M_{m,n}(\Bbbk))$ enjoys the same property by means of a dehomogenisation map that relates these two algebras. It is then easy to show that the $\Bbbk$-algebras $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ are also quantum graded A.S.L. Finally, in Section 4, we deduce that the $\Bbbk$-algebras $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ and $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are AS-Cohen-Macaulay and determine which of them are AS-Gorenstein by means of a criterion involving their Hilbert series.

If S is a finite set, we denote its cardinality by $|S|$.

We would like to thank the many colleagues with whom we have discussed this work, especially J. Alev, G. Cauchon, K.R. Goodearl, P. Jørgensen, S. Launois, S.P. Smith and J.J. Zhang.

1 Quantum graded algebras with a straightening law.

Throughout this section, $\Bbbk$ is a field. An $\mathbb{N}$-graded $\Bbbk$-algebra is a $\Bbbk$-algebra together with a family of $\Bbbk$-subspaces A_i, for $i \in \mathbb{N}$, such that $A = \oplus_{i\in\mathbb{N}} A_i$ and $A_iA_j \subseteq A_{i+j}$, for each pair $i, j \in \mathbb{N}$. An $\mathbb{N}$-graded $\Bbbk$-algebra is said to be *connected* if $A_0 = \Bbbk.1$. An $\mathbb{N}$-graded $\Bbbk$-algebra is connected if and only if it is generated as a $\Bbbk$-algebra by $\oplus_{i\geq 1} A_i$.

1.1 Definition and elementary properties.

Let A be an $\mathbb{N}$-graded algebra and Π a finite subset of elements of A with a partial order $<_{\rm st}$. A *standard monomial* on Π is an element of A which is either 1 or of the form $\alpha_1 \dots \alpha_s$, for some $s \geq 1$, where $\alpha_1, \dots, \alpha_s \in \Pi$ and $\alpha_1 \leq_{\rm st} \cdots \leq_{\rm st} \alpha_s$.

Definition 1.1.1 – *Let A be an $\mathbb{N}$-graded $\Bbbk$-algebra and Π a finite subset of A equipped with a partial order $<_{\rm st}$. We say that A is a* quantum graded algebra with a straightening law *(*quantum graded A.S.L. *for short) on the poset $(\Pi, <_{\rm st})$ if the following conditions are satisfied.*

(1) The elements of Π are homogeneous with positive degree.
(2) The elements of Π generate A as a $\Bbbk$-algebra.
(3) The set of standard monomials on Π is a free family.
(4) If $\alpha, \beta \in \Pi$ are not comparable for $<_{\rm st}$, then $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\rm st} \mu$ and $\lambda <_{\rm st} \alpha, \beta$.
(5) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \Bbbk^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\rm st} \mu$ and $\lambda <_{\rm st} \alpha, \beta$.

Notice that, in the above definition, the case $\Pi = \emptyset$ is not excluded. Hence, $\Bbbk$ is a quantum graded A.S.L. on $\emptyset$ and, of course, it is the only one.

Remark 1.1.2 – Let A be a quantum graded A.S.L. on the set Π. If ω is a standard monomial on Π and $\omega \neq 1$, then condition (3) of Definition 1.1.1 implies that it can be written in a unique way as a product $\alpha_1 \dots \alpha_s$, for some $s \geq 1$, where $\alpha_1, \dots, \alpha_s \in \Pi$ and $\alpha_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} \alpha_s$. In this case, we say that ω is a standard monomial of *length* s. For convenience, we say that $1 \in A$ is the (unique) standard monomial of length 0.

Let π be an element of a finite poset Π with partial order $<_{\mathrm{st}}$; we define the *rank* of π, denoted $\mathrm{rk}\pi$, as in [BV; Chap. 5.C, p. 55]. Thus, $\mathrm{rk}\pi = k$ if and only if there is a chain $\pi_1 <_{\mathrm{st}} \cdots <_{\mathrm{st}} \pi_{k-1} <_{\mathrm{st}} \pi_k = \pi$, with $\pi_i \in \Pi$ for $1 \leq i \leq k$ and no such chain of greater length exists. If Ω is a subset of Π, we define its rank by $\mathrm{rk}\Omega = \max\{\mathrm{rk}\pi,\, \pi \in \Omega\}$. The rank of the empty set is taken to be 0.

Let A be a quantum graded A.S.L. on the set Π partially ordered by $<_{\mathrm{st}}$. For $s \in \mathbb{N}^*$, we denote by Π^s the set of elements of A which either equal 1 or can be written as a product of t elements of Π, with $1 \leq t \leq s$. In addition, we let V^s be the $\Bbbk$-subspace of A generated by Π^s. On the other hand, we denote by Π^s_{st} the set of those elements of Π^s which are standard monomials of length at most equal to s and by V^s_{st} the $\Bbbk$-subspace of A generated by Π^s_{st}. For convenience, we put $\Pi^0 = \Pi^0_{\mathrm{st}} = \{1\}$ and $V^0 = V^0_{\mathrm{st}} = \Bbbk$. Clearly, $V^s_{\mathrm{st}} \subseteq V^s$, for $s \in \mathbb{N}$. Of course, since $\Pi^1_{\mathrm{st}} = \Pi^1 = \Pi \cup \{1\}$, we have $V^1_{\mathrm{st}} = V^1$. Also, as an easy consequence of conditions (4) and (5) in the definition of a quantum graded A.S.L., we have $V^2_{\mathrm{st}} = V^2$.

Lemma 1.1.3 – *Let $s \in \mathbb{N}$, $s \geq 2$ and assume that $V^s_{\mathrm{st}} = V^s$. If $\pi_1, \dots, \pi_{s+1}$ are elements of Π such that $\pi_2 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} \pi_{s+1}$, then $\pi = \pi_1 \dots \pi_{s+1} \in V^{s+1}_{\mathrm{st}}$.*

Proof. We proceed by induction on $\mathrm{rk}\pi_1$. If $\mathrm{rk}\pi_1 = 1$, then π_1 is minimal in Π with respect to $<_{\mathrm{st}}$. Two cases may occur. If π_1 and π_2 are not comparable with respect to $<_{\mathrm{st}}$, then the minimality of π_1 and condition (4) in the definition of a quantum graded A.S.L. imply that $\pi_1\pi_2 = 0$ and thus $\pi = 0$. Otherwise, π_1, π_2 are comparable and, since π_1 is minimal, we have $\pi_1 \leq_{\mathrm{st}} \pi_2$. It follows that π is a standard monomial. In both cases, $\pi \in V^{s+1}_{\mathrm{st}}$.

Now, assume that the result is true when $1 \leq \mathrm{rk}\pi_1 \leq r$, where $r < \mathrm{rk}\Pi$ and consider an element π as above, for which $\mathrm{rk}\pi_1 = r + 1$. Thus, $\pi = \pi_1 \dots \pi_{s+1}$, where $\pi_1, \pi_2, \dots, \pi_{s+1}$ are elements of Π such that $\pi_2 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} \pi_{s+1}$.

First case: π_1 and π_2 are not comparable. Condition (4) in the definition of a quantum graded A.S.L. shows that π is a linear combination of terms of $V^s = V^s_{\mathrm{st}}$ and of terms $\pi''_1 \dots \pi''_{s+1}$ such that $\pi''_1 <_{\mathrm{st}} \pi_1$.

Second case: π_1 and π_2 are comparable. If $\pi_1 \leq_{\mathrm{st}} \pi_2$, π is standard and thus in V^{s+1}_{st}. If $\pi_1 >_{\mathrm{st}} \pi_2$, condition (5) in the definition of a quantum graded A.S.L. shows that π is the sum of $\omega \in V^s_{\mathrm{st}}$ and of a linear combination of $\pi_2\pi_1\pi_3 \dots \pi_{s+1}$ together with terms of the form $\pi'_1 \dots \pi'_{s+1}$ such that $\pi'_1 <_{\mathrm{st}} \pi_1$. Hence, all the terms in this linear combination are elements of the form $\pi''_1 \dots \pi''_{s+1}$ with $\pi''_1 <_{\mathrm{st}} \pi_1$.

Thus, in all cases, $\pi_1 \dots \pi_{s+1}$ is the sum of an element of V^{s+1}_{st} and of a linear combination of terms $\pi''_1 \dots \pi''_{s+1}$ such that $\pi''_1 <_{\mathrm{st}} \pi_1$. In order to conclude, it suffices to show that any product $\pi''_1 \dots \pi''_{s+1}$ such that $\pi''_1 <_{\mathrm{st}} \pi_1$ is in V^{s+1}_{st}. In such a product, $\pi''_2 \dots \pi''_{s+1}$ may be rewritten (using $V^s_{\mathrm{st}} = V^s$) as the sum of an element of $V^{s-1}_{\mathrm{st}} \subseteq V^s_{\mathrm{st}}$ and a linear combination of terms $\pi'''_2 \dots \pi'''_{s+1}$ with $\pi'''_2 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} \pi'''_{s+1}$. Thus, such a product $\pi''_1\pi''_2 \dots \pi''_{s+1}$ may be rewritten as the sum of an element of $V^s = V^s_{\mathrm{st}}$ and a linear combination of terms $\pi''_1\pi'''_2 \dots \pi'''_{s+1}$ with

$\pi_2''' \leq_{\rm st} \cdots \leq_{\rm st} \pi_{s+1}'''$. But, since $\pi_1'' <_{\rm st} \pi_1$, we have $\mathrm{rk}\pi_1'' < \mathrm{rk}\pi_1$ and, by the inductive hypothesis, $\pi_1''\pi_2''' \ldots \pi_{s+1}''' \in V_{\rm st}^{s+1}$. It follows that $\pi \in V_{\rm st}^{s+1}$ as required. This completes the proof. ∎

Lemma 1.1.4 – *In the notation above,* $V_{\rm st}^s = V^s$*, for each* $s \in \mathbb{N}$*.*

Proof. We proceed by induction on s, the cases $s = 0, 1, 2$ being already proved. Assume that the result is true for some integer $s \geq 2$. We must show that $V^{s+1} \subseteq V_{\rm st}^{s+1}$; that is, we must show that $\Pi^{s+1} \subseteq V_{\rm st}^{s+1}$. Let π be an element of Π^{s+1}. If $\pi \in \Pi^s \subseteq V^s$, we have $\pi \in V^s = V_{\rm st}^s \subseteq V_{\rm st}^{s+1}$, by the inductive hypothesis; and so it suffices to consider the case where $\pi = \pi_1 \ldots \pi_{s+1}$, where $\pi_1, \ldots, \pi_{s+1} \in \Pi$. By the inductive hypothesis, $\pi_2 \ldots \pi_{s+1} \in V_{\rm st}^s$ and so this product may be written as a sum of $\omega \in V_{\rm st}^{s-1}$ and a linear combination of standard monomials of length s. It follows that we may rewrite π as the sum of $\pi_1\omega \in V^s = V_{\rm st}^s \subseteq V_{\rm st}^{s+1}$ and a linear combination of elements of Π^{s+1} of the form $\pi_1\pi_2' \ldots \pi_{s+1}'$, where $\pi_2', \ldots, \pi_{s+1}'$ are elements of Π such that $\pi_2' \leq_{\rm st} \cdots \leq_{\rm st} \pi_{s+1}'$. By the inductive hypothesis and Lemma 1.1.3, such elements are in $V_{\rm st}^{s+1}$. This completes the proof. ∎

Proposition 1.1.5 – *Let* A *be a quantum graded A.S.L. on the set* Π*. The set of standard monomials on* Π *form a* $\Bbbk$*-basis of* A*.*

Proof. The set of standard monomials on Π is free over $\Bbbk$ by hypothesis; and so we need only prove that this set generates A as a $\Bbbk$-vector space. However, since Π generates A as a $\Bbbk$-algebra, it is enough to show that a product of elements of Π is a linear combination of standard monomials. This is a consequence of Lemma 1.1.4. ∎

Proposition 1.1.6 – *Let* A *be a quantum graded A.S.L. on the set* Π*; then* $\mathrm{GKdim}A = \mathrm{rk}\Pi$.

Proof. We use the notation introduced before Lemma 1.1.3. In this notation, $V := V^1$ is the span of $\Pi \cup \{1\}$ and V^s is the usual s-th power of the vector space V for $s \geq 2$. Since $1 \in V$ and since V is a finite dimensional vector space generating A as a $\Bbbk$-algebra, we have $\mathrm{GKdim}A = \overline{\lim}_{n\to\infty} \log_n(\dim V^n)$. Thus, by Lemma 1.1.4,

$$\mathrm{GKdim}A = \overline{\lim}_{n\to\infty} \log_n(\dim V_{\rm st}^n).$$

Let us denote by M the number of subsets of Π which are totaly ordered for $<_{\rm st}$ and maximal (with respect to inclusion) for this property. These sets all have rank less than or equal to that of Π and, clearly, for $n \in \mathbb{N}$:

$$\dim V_{\rm st}^n \leq M \begin{pmatrix} n + \mathrm{rk}\Pi \\ \mathrm{rk}\Pi \end{pmatrix},$$

so that $\mathrm{GKdim}A \leq \mathrm{rk}\Pi$. On the other hand, there is a totally ordered subset of Π of rank $\mathrm{rk}\Pi$. Hence,

$$\dim V_{\rm st}^n \geq \begin{pmatrix} n + \mathrm{rk}\Pi \\ \mathrm{rk}\Pi \end{pmatrix};$$

and so $\mathrm{GKdim}A \geq \mathrm{rk}\Pi$. ∎

To each element $\alpha \in \Pi$ we associate the ideal I_α of A generated by those γ in Π such that $\gamma <_{\rm st} \alpha$ (with the convention that an ideal generated by the empty set is $\langle 0 \rangle$).

Lemma 1.1.7 – *Let* A *be a quantum graded A.S.L. on the set* Π*. Any element* α *in* Π *is normal modulo the ideal* I_α*. In particular, any minimal element in* Π *is normal. In addition, if* Π *has a unique minimal element* α *for* $<_{\rm st}$*, then* α *is a regular normal element of* A*.*

Proof. Condition (5) in the definition of a quantum graded A.S.L. shows that, for all β in Π, there exists a scalar $c_{\alpha\beta} \in \mathbb{k}^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha \in I_\alpha$. Denoting by π_α the canonical projection of A onto A/I_α, this shows that, $\pi_\alpha(\alpha)$ commutes up to non zero scalar with all the elements $\pi(\beta)$, for $\beta \in \Pi$. However, the elements $\pi(\beta)$, with $\beta \in \Pi$, form a set of algebra generators of A/I_α, so α is normal modulo the ideal I_α. Now, assume that α is the unique minimal element of Π. In this case, $\alpha \leq_{\rm st} \beta$, for all $\beta \in \Pi$. Thus, if ω is a standard monomial, $\alpha\omega$ is still a standard monomial. Since the set standard monomials is a $\mathbb{k}$-basis of A, by Proposition 1.1.5, it follows that left multiplication by α in A is an injective map. However, as we already mentioned above, condition (5) in the definition of a quantum graded A.S.L. shows that α commutes up to non-zero scalar with any standard monomial. Thus, the injectivity of right multiplication by α in A follows from the injectivity of left multiplication. We have shown that α is a regular normal element of A. ∎

1.2 Properties of Π-ideals.

Let Π be a set with a partial order $<_{\rm st}$. If $<_{\rm tot}$ is a total order on Π, we say that $<_{\rm tot}$ *respects* $<_{\rm st}$ if, for $\alpha, \beta \in \Pi$, $\alpha <_{\rm st} \beta \Longrightarrow \alpha <_{\rm tot} \beta$. Notice that, for a finite poset $(\Pi, <_{\rm st})$, there always exists a total order on Π which respects $<_{\rm st}$.

Let Π be a set with a partial order $<_{\rm st}$. If Ω is a subset of Π, we say that Ω is a Π-*ideal* provided it satisfies the following condition: if $\alpha \in \Omega$ and if $\beta \in \Pi$, with $\beta \leq_{\rm st} \alpha$, then $\beta \in \Omega$. (Notice that what we call a Π-ideal is called an *ideal in* Π in [BV]; see [BV; Proposition 5.1.].)

Lemma 1.2.1 – *Let A be a quantum graded A.S.L. on the poset $(\Pi, <_{\rm st})$ and let Ω be a Π-ideal. In addition, let $<_{\rm tot}$ be any total order on Π which respects $<_{\rm st}$. Then the elements of Ω, ordered by $<_{\rm tot}$ form a normalising sequence of generators of the ideal $\langle\Omega\rangle$ generated by Ω in A.*

Proof. Let $\Omega = \{\omega_1, \dots, \omega_s\}$ with $\omega_1 <_{\rm tot} \cdots <_{\rm tot} \omega_s$. For $1 \leq i \leq s$, consider $\alpha \in \Pi$ such that $\alpha <_{\rm st} \omega_i$. Then, since Ω is a Π-ideal, $\alpha \in \Omega$. Moreover, since $<_{\rm tot}$ respects $<_{\rm st}$, we also have $\alpha <_{\rm tot} \omega_i$. Thus, $\alpha \in \{\omega_1, \dots, \omega_{i-1}\}$ (this set being empty if $i = 1$). Thus, $I_{\omega_i} \subseteq \langle\omega_1, \dots, \omega_{i-1}\rangle$. But then ω_i is normal modulo $\langle\omega_1, \dots, \omega_{i-1}\rangle$ since, by Lemma 1.1.7, it is normal modulo I_{ω_i}. ∎

Lemma 1.2.2 – *A quantum graded A.S.L. is noetherian and satisfies polynomial growth ((PG) for short) in the sense of [Lev; 5.4].*

Proof. Let A be a quantum graded A.S.L. on the poset $(\Pi, <_{\rm st})$ and let $<_{\rm tot}$ be any total order on Π which respects $<_{\rm st}$. Since Π is clearly a Π-ideal, by Lemma 1.2.1, the elements of Π ordered by $<_{\rm tot}$ form a nomalising sequence in A. Since $A/\langle\Pi\rangle \cong \mathbb{k}$ is noetherian, [ATV; Lemma 8.2] shows that A is noetherian. In the same way, since $A/\langle\Pi\rangle \cong \mathbb{k}$ satisfies (PG), [Lev; Proposition 5.6] shows that A satisfies (PG). ∎

A nice feature of Π-ideals is the fact that they can be described in a very simple way in terms of the standard monomials. This description is given in the next proposition. If $\omega \in \Pi$, we say that a standard monomial $\mu \neq 1$ *involves* ω if $\mu = \omega_1 \dots \omega_s$, where $\omega_1, \dots, \omega_s \in \Pi$, with $\omega_1 \leq_{\rm st} \cdots \leq_{\rm st} \omega_s$, and $\omega \in \{\omega_1, \dots, \omega_s\}$.

The following remark will be useful.

Remark 1.2.3 – Let A be a quantum graded A.S.L. on the poset $(\Pi, <_{\rm st})$ and let $<_{\rm tot}$ be any total order on Π which respects $<_{\rm st}$. Let $\Omega = \{\omega_1, \dots, \omega_s\}$, for some $s \geq 2$, be a Π-ideal where

$\omega_1, \dots, \omega_s$ are elements of Π such that $\omega_1 <_{\text{tot}} \cdots <_{\text{tot}} \omega_s$ and let $\Omega' = \{\omega_1, \dots, \omega_{s-1}\}$. Consider $\pi \in \Pi$ such that there exists $1 \le i \le s-1$ with $\pi \le_{\text{st}} \omega_i$. Then, $\pi \in \Omega$ since Ω is a Π-ideal. Now, suppose that $\pi = \omega_s$. Then we have $\omega_s = \pi \le_{\text{st}} \omega_i$. It follows that $\omega_s = \pi \le_{\text{tot}} \omega_i <_{\text{tot}} \omega_s$, a contradiction. Thus, $\pi \in \Omega'$. We have proved that Ω' is also a Π-ideal.

Proposition 1.2.4 – *Let A be a quantum graded A.S.L. on the set Π and let Ω be a Π-ideal. The set of standard monomials involving an element of Ω form a $\Bbbk$-basis of $\langle \Omega \rangle$.*

Proof. Let $<_{\text{tot}}$ be any total order on Π which respects $<_{\text{st}}$. Notice that the set of standard monomials involving an element of Ω is free by condition (3) of the definition of a quantum graded A.S.L. and that it is included in $\langle \Omega \rangle$. Thus it remains to prove that this set generates $\langle \Omega \rangle$ as a $\Bbbk$-vector space.

We proceed by induction on the cardinality of Ω. First, suppose that Ω is a Π-ideal with $|\Omega| = 1$, say $\Omega = \{\omega_1\}$. Then ω_1 is a minimal element of Π with respect to $\le_{\text{st}}$, and $\langle \Omega \rangle = \omega_1 A$, by Lemma 1.2.1. Let V be the subspace of A generated by those standard monomials involving ω_1. Clearly, $V \subseteq \langle \Omega \rangle$. Conversely, suppose that $x \in \langle \Omega \rangle$; and so there is an element $x_1 \in A$ such that $x = \omega_1 x_1$. Set $x_1 = c_1 y_1 + \cdots + c_t y_t$, where $c_1, \dots, c_t$ are scalars and $y_1, \dots, y_t$ standard monomials; so that $x = \omega_1 x_1 = c_1 \omega_1 y_1 + \cdots + c_t \omega_1 y_t$. Thus, it is enough to show that, if $\pi_1, \dots, \pi_s$ are elements in Π such that $\pi_1 \le_{\text{st}} \cdots \le_{\text{st}} \pi_s$, then $\omega_1 \pi_1 \dots \pi_s \in V$. Of course, if $\omega_1 \le_{\text{st}} \pi_1$, this is trivial. Thus, we assume that $\omega_1 \not\le_{\text{st}} \pi_1$. Since ω_1 is minimal in Π, it follows that ω_1 and π_1 are not comparable with respect to $\le_{\text{st}}$. Thus, condition (4) of the definition of a quantum graded A.S.L. entails $\omega_1 \pi_1 = 0$ and then $\omega_1 \pi_1 \dots \pi_s \in V$. We have shown that the result is true when $|\Omega| = 1$.

Next, suppose that $s > 1$, and assume that the result is true for all the Π-ideals of cardinality less than s. Set $\Omega = \{\omega_1, \dots, \omega_s\}$ where $\omega_1, \dots, \omega_s$ are elements of Π such that $\omega_1 <_{\text{tot}} \cdots <_{\text{tot}} \omega_s$. Then, $\langle \Omega \rangle = \omega_1 A + \cdots + \omega_s A$, by Lemma 1.2.1. Let V be the $\Bbbk$-subspace of A generated by those standard monomials involving an element of Ω. Again, it is clear that $V \subseteq \langle \Omega \rangle$. Now, let $\Omega' = \{\omega_1, \dots, \omega_{s-1}\}$ and denote by V' the $\Bbbk$-subspace of A spanned by those standard monomials involving an element of Ω'. In fact, Ω' is a Π-ideal as we noticed in Remark 1.2.3. Thus, the inductive hypothesis gives $\langle \Omega' \rangle = \omega_1 A + \cdots + \omega_{s-1} A = V'$. Now, let $x \in \langle \Omega \rangle$. Then there exist $x_1, \dots, x_s \in A$ such that $x = \omega_1 x_1 + \cdots + \omega_s x_s$. We want to show that $x \in V$. Since $\omega_1 x_1 + \cdots + \omega_{s-1} x_{s-1} \in \langle \Omega' \rangle = V' \subseteq V$, it is enough to show that $\omega_s x_s \in V$. Of course, since x_s is a linear combination of standard monomials, it is enough to show that, if $\pi_1, \dots, \pi_t$ are elements in Π such that $\pi_1 \le_{\text{st}} \cdots \le_{\text{st}} \pi_t$, then $\omega_s \pi_1 \dots \pi_t \in V$. First, assume that $\omega_s \not\ge_{\text{st}} \pi_j$ for each j with $1 \le j \le t$. In this case, if ω_s and π_1 are comparable with respect to $\le_{\text{st}}$ then $\omega_s <_{\text{st}} \pi_1$; and so $\omega_s \pi_1 \dots \pi_t$ is a standard monomial which is obviously in V. Next, consider the case where ω_s and π_1 are not comparable with respect to $\le_{\text{st}}$. In this case, by condition (4) of the definition of quantum graded A.S.L., $\omega_s \pi_1$ is a linear combination of standard monomials involving an element $\lambda \in \Pi$ such that $\lambda <_{\text{st}} \omega_s$. Any such λ is in Ω, since Ω is a Π-ideal; and so $\lambda \in \Omega'$, since $\lambda \ne \omega_s$. It follows that $\omega_s \pi_1 \in \langle \Omega' \rangle$ and then that $\omega_s \pi_1 \dots \pi_t \in \langle \Omega' \rangle \subseteq V$. It remains to consider the case where there exists an integer i, with $1 \le i \le t$, such that $\omega_s \ge_{\text{st}} \pi_i$. Let j denote the greatest such integer. Thus, we have $\pi_1 \le_{\text{st}} \cdots \le_{\text{st}} \pi_j \le_{\text{st}} \omega_s$. By condition (5) of the definition of quantum graded A.S.L., for $1 \le i \le j$, there exists a scalar $c_i \in \Bbbk^*$ such that $\omega_s \pi_i - c_i \pi_i \omega_s$ is a linear combination of standard monomials involving an element $\lambda \in \Pi$ such that $\lambda <_{\text{st}} \omega_s$. As we have already mentioned, this implies that each such λ is in Ω'. Thus, $\omega_s \pi_i - c_i \pi_i \omega_s \in \langle \Omega' \rangle$, for $1 \le i \le j$. Setting $c = c_1 \dots c_j$, it follows that $\omega_s \pi_1 \dots \pi_j - c \pi_1 \dots \pi_j \omega_s \in \langle \Omega' \rangle$ and, as a consequence, $\omega_s \pi_1 \dots \pi_t - c \pi_1 \dots \pi_j \omega_s \pi_{j+1} \dots \pi_t \in \langle \Omega' \rangle$. Since $\langle \Omega' \rangle = V' \subseteq V$, it remains to prove that $\pi_1 \dots \pi_j \omega_s \pi_{j+1} \dots \pi_t \in V$. Again, we isolate two cases. First, assume that ω_s and π_{j+1} are comparable. Then, by definition of j, we have $\omega_s <_{\text{st}} \pi_{j+1}$. In this case, $\pi_1 \dots \pi_j \omega_s \pi_{j+1} \dots \pi_t$ is a standard monomial that is obviously in V. Next, assume that ω_s and π_{j+1} are not comparable. Then, as we have already noticed, $\omega_s\pi_{j+1}$ is a linear combination of standard monomials involving an element $\lambda \in \Omega'$. Hence, $\pi_1 \dots \pi_j\omega_s\pi_{j+1}\dots\pi_t \in \langle\Omega'\rangle \subseteq V$. The proof is complete. ■

Corollary 1.2.5 – *Let A be a quantum graded A.S.L. on the set Π and let Ω be a Π-ideal. Suppose that $p : A \longrightarrow A/\langle\Omega\rangle$ is the canonical projection. Then the $\Bbbk$-algebra $A/\langle\Omega\rangle$ is a quantum graded A.S.L. on the set $p(\Pi \setminus \Omega)$.*

Proof. Since $\langle\Omega\rangle$ is generated by homogeneous elements, it is a graded ideal and $A/\langle\Omega\rangle$ inherits an $\mathbb{N}$-grading. By Proposition 1.2.4 and the linear independence of standard monomials, $\Pi\cap\langle\Omega\rangle = \Omega$. In fact $p(\Pi \setminus \Omega) \cong \Pi \setminus \Omega$ and we may equip $p(\Pi \setminus \Omega)$ with the partial order inherited from that of Π. Clearly, the elements of $p(\Pi \setminus \Omega)$ are homogeneous elements of positive degree generating $A/\langle\Omega\rangle$ as a $\Bbbk$-algebra. In addition, the family of standard monomials on $p(\Pi \setminus \Omega)$ is free. The fact that conditions (4) and (5) are satisfied for $A/\langle\Omega\rangle$ is then clear. ■

2 The AS-Cohen-Macaulay property.

Throughout this section, $\Bbbk$ is a field.

2.1 AS-Cohen-Macaulay algebras.

In this subsection, we recall the notions of AS-Cohen-Macaulay and AS-Gorenstein algebras as defined in [JZ]. We also recall some results about these notions that we will need latter.

Throughout this subsection, A stands for a noetherian $\mathbb{N}$-graded connected $\Bbbk$-algebra; that is, $A = \oplus_{i\geq 0}A_i$ is an $\mathbb{N}$-graded $\Bbbk$-algebra such that $A_0 = \Bbbk$. It follows that A is a finitely generated $\Bbbk$-algebra and that it is locally finite in the sense that A_n is finite dimensional as a $\Bbbk$-vector space for each $n \in \mathbb{N}$. For any integer n, let $A_{\geq n} = \oplus_{i\geq n}A_i$. We denote the unique maximal graded ideal, $A_{\geq 1}$, of A by $\mathfrak{m}$. The category of graded left A-modules and homogeneous homomorphisms of degree zero is denoted $\mathsf{GrMod}(A)$. If $M = \oplus_{i\in\mathbb{Z}}M_i$ is an object of $\mathsf{GrMod}(A)$, then for any integer n, the graded module $M(n)$ is the n-th shift of M; that is, $M(n) = \oplus_{i\in\mathbb{Z}}M(n)_i$ where $M(n)_i = M_{n+i}$ for $i \in \mathbb{Z}$. Let M, N be objects in $\mathsf{GrMod}(A)$, then $\mathrm{Hom}_{\mathsf{GrMod}(A)}(M, N)$ will stand for the abelian group of homogeneous homomorphisms of degree 0. The Ext groups associated to $\mathrm{Hom}_{\mathsf{GrMod}(A)}$ are then denoted $\mathrm{Ext}^i_{\mathsf{GrMod}(A)}$. In addition, we put

$$\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{GrMod}(A)}(M, N(m)).$$

The Ext groups associated to $\underline{\mathrm{Hom}}_A$ are then denoted $\underline{\mathrm{Ext}}^i_A$. Thus, if M, N are objects in $\mathsf{GrMod}(A)$,

$$\underline{\mathrm{Ext}}^i_A(M, N) \cong \bigoplus_{m\in\mathbb{Z}} \mathrm{Ext}^i_{\mathsf{GrMod}(A)}(M, N(m)).$$

We now define the local cohomology functors. If M is an object in $\mathsf{GrMod}(A)$, we define its *torsion submodule*, $\Gamma_{\mathfrak{m}}(M)$, by

$$\Gamma_{\mathfrak{m}}(M) := \{m \in M \,|\, \exists\, n \in \mathbb{N},\ A_{\geq n}m = 0\}.$$

It is easy to check that $\Gamma_{\mathfrak{m}}$ (and restriction of morphisms) defines a left exact functor. We denote the associated right derived functors by $H^i_{\mathfrak{m}}$. Then, if M is an object in $\mathsf{GrMod}(A)$,

$$H^i_{\mathfrak{m}}(M) \cong \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M).$$

We write A° for the opposite algebra of A. Of course, A° is again a noetherian connected $\mathbb{N}$-graded algebra and for any integer i, the i-th local cohomology functor associated to A° will be denoted $H^i_{\mathfrak{m}^\circ}$.

Definition 2.1.1 – [JZ; Definition 0.1, Definition 0.2].
The algebra A is said to be AS-Cohen-Macaulay *if there exists $n \in \mathbb{N}$ such that*

$$\forall\, i \in \mathbb{N},\, i \neq n, H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}^\circ}(A) = 0.$$

The algebra A is said to be AS-Gorenstein *if it satisfies the following conditions.*
(1) We have $\mathrm{injdim}_A A = \mathrm{injdim}_{A^\circ} A = n < \infty$ *(the injective dimension being measured in* $\mathsf{GrMod}(A)$ *and* $\mathsf{GrMod}(A^\circ)$ *respectively).*
(2) There exists an integer ℓ such that

$$\underline{\mathrm{Ext}}^i_A(\Bbbk, A) \cong \underline{\mathrm{Ext}}^i_{A^\circ}(\Bbbk, A) \cong \left\{ \begin{array}{cl} 0 & \textit{for } i \neq n, \\ \Bbbk(\ell) & \textit{for } i = n. \end{array} \right.$$

Remark 2.1.2 – Notice that, by [Lev; Lemma 3.3], in the definition of the AS-Gorenstein property, we might equally well use the usual injective dimension.

The following definition appeared in [Z; p. 392].

Definition 2.1.3 – *Let B be a noetherian connected $\mathbb{N}$-graded $\Bbbk$-algebra. If for every non-simple graded prime factor B/P of B there is a non-zero homogeneous normal element in $(B/P)_{\geq 1}$, then we say B has* enough normal elements.

As we will see in Remark 2.1.9, if A has enough normal elements, it behaves in a similar way to commutative noetherian local rings with respect to the AS-Gorenstein and AS-Cohen-Macaulay properties.

Remark 2.1.4 –
1. As mentioned in [Z; p. 392], if a noetherian connected $\mathbb{N}$-graded $\Bbbk$-algebra B has a normalising sequence $x_1, \ldots, x_n$ of elements in $B_{\geq 1}$ such that $B/(x_1, \ldots, x_n)$ is finite dimensional as a $\Bbbk$-vector space, then B as enough normal elements.
2. Clearly, a quantum graded A.S.L. is connected and it is noetherian by Lemma 1.2.2. Hence, from the previous point and Lemma 1.2.1, it follows at once that any quantum graded A.S.L. has enough normal elements.

The next result is due to Yekutieli and Zhang (see [YZ]). (Here, we take the convention that the Gelfand-Kirillov dimension of the zero module is $-\infty$.)

Theorem 2.1.5 – *Assume that A has enough normal elements. Then, for each finitely generated graded left A-module M,*

$$\sup\{i \in \mathbb{N} \,|\, H^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{GKdim}_A M < +\infty.$$

Proof. For unexplained terminology, the reader is referred to [YZ]. By the conjunction of [YZ; Theorem 5.13] and [YZ; Theorem 5.14], the algebra A is graded-Auslander (that is, A has a graded-Auslander balanced dualizing complex R) and for every finitely generated graded left A-module M, we have $\mathrm{Cdim} M = \mathrm{GKdim}_A M$ (see [YZ; 4.8] and [YZ; 3.10]). It remains to apply [YZ; 4.14] to conclude that $\sup\{i \in \mathbb{N} \,|\, H^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{GKdim}_A M < +\infty$, for every finitely generated graded left A-module M. ∎

Given a finitely generated graded left A-module M, the previous theorem provides an upper bound for the set of integers i such that $H^i_{\mathfrak{m}}(M) \neq 0$. To obtain a lower bound, we need the notion of depth.

Definition 2.1.6 – *Let M be a finitely generated graded left A-module, the depth of M is defined by*

$$\operatorname{depth}_A M := \inf\{i \in \mathbb{N} \mid \underline{\operatorname{Ext}}^i_A(\Bbbk, M) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

We list the results we need about depth in Lemma 2.1.7. We also give a sketch of proof for those well-known results since we have not been able to locate them in the literature.

Lemma 2.1.7 –
(i) Let M be a finitely generated graded left A-module. Then

$$\operatorname{depth}_A M = \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

(ii) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence in $\mathsf{GrMod}(A)$ of finitely generated graded left A-modules, and if ℓ, m, n denote the depth of L, M, N, respectively, then

$$\ell \geq \min\{m, n+1\},\ m \geq \min\{\ell, n\},\ n \geq \min\{\ell - 1, m\}.$$

(iii) If $x \in A$ is a regular normal homogeneous element of positive degree, then $\operatorname{depth}_{A/\langle x\rangle} A/\langle x\rangle = \operatorname{depth}_A A - 1$.

Proof. Point (i) is proved as in the commutative case (see [W; Theorem 4.6.3]).
(ii) This follows immediately from examination of the long exact sequence of local cohomology groups associated with the short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$.
(iii) By the graded analogue of Rees' Lemma, we have

$$\forall\, i \in \mathbb{N}, \qquad \underline{\operatorname{Ext}}^i_{A/\langle x\rangle}(\Bbbk, A/\langle x\rangle) \cong \underline{\operatorname{Ext}}^{i+1}_A(\Bbbk, A).$$

The result follows at once. ∎

Remark 2.1.8 – Assume that A has enough normal elements and let M be a non-zero finitely generated graded left A-module. By Theorem 2.1.5 and Lemma 2.1.7, $\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\}$ is non-empty, contains the integers $\operatorname{depth}_A M$ and $\operatorname{GKdim}_A M$ and any i in this set satisfies $\operatorname{depth}_A M \leq i \leq \operatorname{GKdim}_A M$. This result is similar to the classical vanishing theorem of Grothendieck for commutative noetherian local rings (see [BH; Theorem 3.5.7]).

Remark 2.1.9 – The noetherian connected $\mathbb{N}$-graded algebras with enough normal elements behave in a similar way to commutative noetherian local rings from the point of view of the AS-Cohen-Macaulay and the AS-Gorenstein properties. Indeed, if A has enough normal elements, then A is AS-Gorenstein if and only if $\operatorname{injdim}_A A = \operatorname{injdim}_{A^\circ} A < \infty$, see [Z; Proposition 2.3(2)], and, if A is AS-Gorenstein, then it is AS-Cohen-Macaulay (use [Z; Theorem 3.1(1)], [Z; Proposition 2.3(2)] and Remark 2.1.8).

The following result is [YZ ; Lemma 4.5, p. 29], in conjunction with [JZ ; Lemma 5.8]. It will be useful later.

Proposition 2.1.10 – *Assume that A has enough normal elements. Let M be a graded left A-module. Let I be a graded ideal in $\operatorname{Ann}_A M$ and set $\mathfrak{n} = \mathfrak{m}/I$. Then $H^i_{\mathfrak{m}}(M)$ and $H^i_{\mathfrak{n}}(M)$ are isomorphic as A/I-modules, for $i \in \mathbb{N}$.*

2.2 The AS-Cohen-Macaulay property for a Quantum graded A.S.L..

The aim of this section is to prove that, if A is a quantum graded A.S.L. on a so-called *wonderful* poset Π, then A is AS-Cohen-Macaulay.

First, we prove a lemma which will be of central importance.

Remark 2.2.1 – Let A be a noetherian $\mathbb{N}$-graded connected $\Bbbk$-algebra and assume that A has enough normal elements. Let K be an homogeneous ideal of A, with $K \neq A$. Then the ring A/K is AS-Cohen-Macaulay if and only if

$$\mathrm{depth}_A A/K = \mathrm{GKdim}_A A/K = \mathrm{GKdim}_{A^\circ} A/K = \mathrm{depth}_{A^\circ} A/K.$$

(Note that the central equality is always true, by [KL; 5.4].) This remark is an easy consequence of Remark 2.1.8 and Proposition 2.1.10.

Lemma 2.2.2 – *Let A be a noetherian $\mathbb{N}$-graded connected $\Bbbk$-algebra and assume that A has enough normal elements.*
(1) Assume that A has polynomial growth ((PG) for short) in the sense of [Lev; 5.4]. Let $x \in A$ be a normal homogeneous element of positive degree which is not a zero divisor. Then, the ring A/xA is AS-Cohen-Macaulay if and only if the ring A is AS-Cohen-Macaulay.
(2) Suppose that I and J are homogeneous ideals of A such that:
(i) $\mathrm{GKdim}_A A/I = \mathrm{GKdim}_A A/J = \mathrm{GKdim}_A A$,
(ii) $\mathrm{GKdim}_A A/(I+J) = \mathrm{GKdim}_A A - 1$,
(iii) $I \cap J = (0)$,
(iv) the rings A/I and A/J are AS-Cohen-Macaulay.
Then, the ring A is AS-Cohen-Macaulay if and only if the ring $A/(I+J)$ is AS-Cohen-Macaulay.

Proof. (1) According to [Lev; 5.7], $\mathrm{GKdim}_A A/(x) = \mathrm{GKdim}_A A - 1$ and $\mathrm{GKdim}_{A^\circ} A/(x) = \mathrm{GKdim}_{A^\circ} A - 1$. According to point (iii) in Lemma 2.1.7 and Proposition 2.1.10, $\mathrm{depth}_A A/(x) = \mathrm{depth}_{A/(x)} A/(x) = \mathrm{depth}_A A - 1$ and $\mathrm{depth}_{A^\circ} A/(x) = \mathrm{depth}_{A^\circ} A - 1$. Thus, the result follows at once from Remark 2.2.1.
(2) Because of the hypothesis (iii), there is a short exact sequence of left-A-modules (respectively right-A-modules)

$$0 \longrightarrow A \xrightarrow{f} A/I \oplus A/J \xrightarrow{g} A/(I+J) \longrightarrow 0, \tag{1}$$

where, for $a, b \in A$, $f(a) = (a+I, -a+J)$ and $g((a+I, b+J)) = a+b+I+J$. Indeed, clearly $\ker f = I \cap J = (0)$, while g is onto and $g \circ f = 0$. Moreover, for $a, b \in A$, if $0 = g((a+I, b+J)) = a+b+I+J$, then there exist $i \in I$ and $j \in J$ such that $a+b = i+j$. Thus, $(a+I, b+J) = (x+I, -x+J) = f(x)$ where $x = a-i = j-b$. From these short exact sequences, using point (ii) of Lemma 2.1.7, we derive

$$\begin{aligned} \mathrm{depth}_A A &\geq \min\{\mathrm{depth}_A A/I \oplus A/J, \mathrm{depth}_A A/(I+J)+1\}, \\ \mathrm{depth}_{A^\circ} A &\geq \min\{\mathrm{depth}_{A^\circ} A/I \oplus A/J, \mathrm{depth}_{A^\circ} A/(I+J)+1\}, \end{aligned} \tag{2}$$

$$\begin{aligned} \mathrm{depth}_A A/(I+J) &\geq \min\{\mathrm{depth}_A A - 1, \mathrm{depth}_A A/I \oplus A/J\}, \\ \mathrm{depth}_{A^\circ} A/(I+J) &\geq \min\{\mathrm{depth}_{A^\circ} A - 1, \mathrm{depth}_{A^\circ} A/I \oplus A/J\}. \end{aligned} \tag{3}$$

Using [KL; 5.4], hypotheses (i) and (ii) may be rewritten
(i) $\mathrm{GKdim}_A A/I = \mathrm{GKdim}_A A/J = \mathrm{GKdim}_A A = \mathrm{GKdim}_{A^\circ} A = \mathrm{GKdim}_{A^\circ} A/I = \mathrm{GKdim}_{A^\circ} A/J$,
(ii) $\mathrm{GKdim}_A A/(I+J) = \mathrm{GKdim}_A A - 1 = \mathrm{GKdim}_{A^\circ} A - 1 = \mathrm{GKdim}_{A^\circ} A/(I+J)$.

Because of Remark 2.2.1, hypothesis (iv) may be rewritten
(iv) $\mathrm{depth}_A A/K = \mathrm{GKdim}_A A/K = \mathrm{GKdim}_{A^\circ} A/K = \mathrm{depth}_{A^\circ} A/K$, where $K = I$ or J.
But, by [KL; 5.1], $\mathrm{GKdim}_A(A/I \oplus A/J) = \max\{\mathrm{GKdim}_A A/I, \mathrm{GKdim}_A A/J\}$. Thus, as a consequence of hypothesis (i), $\mathrm{GKdim}_A(A/I \oplus A/J) = \mathrm{GKdim}_A A$. In addition, point (ii) of Lemma 2.1.7 shows that $\mathrm{depth}_A(A/I \oplus A/J) \geq \min\{\mathrm{depth}_A A/I, \mathrm{depth}_A A/J\}$. Thus, hypotheses (i) and (iv) give $\mathrm{depth}_A(A/I \oplus A/J) \geq \mathrm{GKdim}_A A$. Since, by 2.1.8 the depth of a non-zero finitely generated graded module is bounded by its GK-dimension, it follows that $\mathrm{depth}_A(A/I \oplus A/J) = \mathrm{GKdim}_A(A/I \oplus A/J) = \mathrm{GKdim}_A A$. Clearly, the same holds with A° instead of A. Thus, we have proved

$$\begin{aligned} \mathrm{depth}_A(A/I \oplus A/J) = \mathrm{GKdim}_A(A/I \oplus A/J) = \mathrm{GKdim}_A A \\ \mathrm{depth}_{A^\circ}(A/I \oplus A/J) = \mathrm{GKdim}_{A^\circ}(A/I \oplus A/J) = \mathrm{GKdim}_{A^\circ} A. \end{aligned} \tag{4}$$

Now, assume that $A/(I+J)$ is AS-Cohen-Macaulay. By 2.2.1, this means that $\mathrm{depth}_A A/(I+J) = \mathrm{GKdim}_A A/(I+J) = \mathrm{GKdim}_{A^\circ} A/(I+J) = \mathrm{depth}_{A^\circ} A/(I+J)$. Thus, using hypothesis (ii), this is equivalent to $\mathrm{depth}_A A/(I+J) = \mathrm{GKdim}_A A - 1 = \mathrm{GKdim}_{A^\circ} A - 1 = \mathrm{depth}_{A^\circ} A/(I+J)$. Thus, from relations (2) and (4), it follows that $\mathrm{depth}_A A \geq \mathrm{GKdim}_A A$ and $\mathrm{depth}_{A^\circ} A \geq \mathrm{GKdim}_{A^\circ} A$. But, as we noticed before, the depth is bounded above by the GK-dimension, thus $\mathrm{depth}_A A = \mathrm{GKdim}_A A = \mathrm{GKdim}_{A^\circ} A = \mathrm{depth}_{A^\circ} A$. This means that the ring A is AS-Cohen-Macaulay, by Remark 2.2.1 .

A similar argument, using (3) instead of (2) shows that, if the ring A is AS-Cohen-Macaulay, then the ring $A/(I+J)$ is AS-Cohen-Macaulay. ∎

Now, we need to introduce combinatorial tools. For this we essentially follow [BV; Chapter 5.D]. An important notion, in what follows, is that of a *wonderful poset.* For the definition of such posets, the reader is referred to [BV; Chapter 5.D]. Note that any distributive lattice is a wonderful poset (see [BV; p. 58]). We will also use Lemma 5.13 of [BV] which contains all the properties of wonderful posets that we need. Finally, let Π be a set partially ordered by $<_{\mathrm{st}}$; following [BV; Chap. 5.A], to any subset Σ of Π we associate two Π-ideals as follows. The Π-*ideal generated by* Σ is the smallest Π-ideal containing Σ; that is, the set $\{\xi \in \Pi \,|\, \exists\, \sigma \in \Sigma,\ \xi \leq_{\mathrm{st}} \sigma\}$. The Π-*ideal cogenerated by* Σ is the greatest Π-ideal disjoint from Σ; that is, the set $\{\xi \in \Pi \,|\, \forall\, \sigma \in \Sigma,\ \xi \not\geq_{\mathrm{st}} \sigma\}$.

Theorem 2.2.3 – *If A is a quantum graded A.S.L. on a wonderful poset $(\Pi, <_{\mathrm{st}})$, then A is AS-Cohen-Macaulay.*

Proof. Recall that, by Remark 2.1.4, a quantum graded A.S.L. is a noetherian $\mathbb{N}$-graded connected $\Bbbk$-algebra with enough normal elements. The proof is by induction on $|\Pi|$. The result is true if $|\Pi| = 1$ since, in this case, A is a (commutative) polynomial ring in one indeterminate, which is well known to be AS-Cohen-Macaulay. Now, we suppose the result is true up to cardinality $n \geq 1$ and we consider a quantum graded A.S.L. A on a wonderful poset Π of cardinality $n+1$. Two cases may occur according to the number of minimal elements in Π.

First, assume that $(\Pi, <_{\mathrm{st}})$ has a single minimal element ξ. By Lemma 1.1.7, ξ is a regular normal element of A. In addition, Lemma 1.2.2 shows that A satisfies (PG). Moreover, Corollary 1.2.5, [BV; Lemma 5.13 (b)] and the inductive hypothesis show that, $A/\xi A$ is AS-Cohen-Macaulay. So, by part (1) of Lemma 2.2.2, A is AS-Cohen-Macaulay.

Now, assume that the minimal elements of Π are $\xi_1, \ldots, \xi_k$, with $k \geq 2$. Let Ω be the Π-ideal cogenerated by ξ_1 and Ψ be the Π-ideal cogenerated by $\xi_2, \ldots, \xi_k$. We let $I = \langle \Omega \rangle$ and $J = \langle \Psi \rangle$. Consider $x \in I \cap J$ and assume that x is non-zero. Then, by Proposition 1.1.5 we may write $x = \lambda_1 x_1 + \cdots + \lambda_s x_s$, with $s \geq 1$, where $\lambda_1, \ldots, \lambda_s$ are non-zero scalars and $x_1, \ldots, x_s$ are standard monomials. Using Proposition 1.2.4, we see that for each $i \in \{1, \dots, s\}$, the term x_i must involve an element $\omega_i \in \Omega$ and an element $\psi_i \in \Psi$. But, x_i being standard, ω_i and ψ_i are comparable for $<_{\mathrm{st}}$ and, Ω and Ψ being Π-ideals, the smaller element of ω_i and ψ_i must be in $\Omega \cap \Psi$. However, [BV; 5.13 (e)(iii)] shows that $\Omega \cap \Psi = \emptyset$; a contradiction. Thus, we have $I \cap J = \langle 0 \rangle$. We have $I + J = \langle \Omega \cup \Psi \rangle$ and clearly $\Omega \cup \Psi$ is a Π-ideal. By [BV; 5.13(e)], $\Pi \setminus \Omega$, $\Pi \setminus \Psi$ and $\Pi \setminus (\Omega \cup \Psi)$ are wonderful posets and we have $\mathrm{rk}(\Pi \setminus \Omega) = \mathrm{rk}(\Pi \setminus \Psi) = \mathrm{rk}\Pi$ and $\mathrm{rk}(\Pi \setminus (\Omega \cup \Psi)) = \mathrm{rk}\Pi - 1$. Now, from Corollary 1.2.5 and Proposition 1.1.6, we deduce that $\mathrm{GKdim}_A A/I = \mathrm{GKdim}_A A/J = \mathrm{GKdim}_A A$ and $\mathrm{GKdim}_A A/(I+J) = \mathrm{GKdim}_A A - 1$ (see [KL; 5.1(c)]). In addition, the induction hypothesis gives that the rings A/I, A/J and $A/(I+J)$ are AS-Cohen-Macaulay. So, we are in position to apply Lemma 2.2.2 which shows that the ring A is AS-Cohen-Macaulay. The proof is complete. ∎

3 Examples of quantum graded A.S.L.

The aim of this section is to show that quantum grassmannians and quantum determinantal rings are quantum graded A.S.L.

For our purposes, we need to define quantum grassmannians in a slightly more general setting than usual; that is, over an arbitrary commutative domain. We also need to introduce quantum analogues of coordinate rings of Schubert varieties. This is done in the first subsection. In the second subsection, we exhibit useful bases for quantum grassmannians, called *standard bases*, in this general setting, thus extending results from [KLR] where such bases were constructed in the case where the base ring is a field. Using this material, we show in Subsections 3 and 4 that quantum grassmannians over a field are examples of quantum graded A.S.L. Finally, in Subsection 5, we deduce that quantum determinantal rings over a field are quantum graded A.S.L.

Throughout the rest of this work, we fix two positive integers m, n.

3.1 Quantum grassmannians.

Recall the definition of the quantum analogue, $\mathcal{O}_q(M_{m,n}(\Bbbk))$, of the coordinate ring of $m \times n$ matrices with entries in the field $\Bbbk$.

Definition 3.1.1 – *Let $\Bbbk$ be a field and q be a non-zero element of $\Bbbk$. We denote by $\mathcal{O}_q(M_{m,n}(\Bbbk))$ the $\Bbbk$-algebra generated by mn indeterminates X_{ij}, for $1 \le i \le m$ and $1 \le j \le n$, subject to the following relations:*

$$\begin{array}{ll}
X_{ij}X_{il} = qX_{il}X_{ij}, & \text{for } 1 \le i \le m, \text{ and } 1 \le j < l \le n\,; \\
X_{ij}X_{kj} = qX_{kj}X_{ij}, & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j \le n\,; \\
X_{ij}X_{kl} = X_{kl}X_{ij}, & \text{for } 1 \le k < i \le m, \text{ and } 1 \le j < l \le n\,; \\
X_{ij}X_{kl} - X_{kl}X_{ij} = (q - q^{-1})X_{il}X_{kj}, & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j < l \le n.
\end{array}$$

More generally, we make the following definition.

Definition 3.1.2 – *Let A be a commutative integral domain and denote by $\Bbbk$ its field of fractions. For any unit $u \in A$, we define $\mathcal{O}_u(M_{m,n}(A))$ to be the A-subalgebra of $\mathcal{O}_u(M_{m,n}(\Bbbk))$ generated over A by $\{X_{ij}\}$, for $1 \le i \le m$ and $1 \le j \le n$.*

Remark 3.1.3 – Let A be a commutative domain whose field of fractions is denoted by $\mathbb{k}$ and let u be a unit of A.
1. It is well-known that $\mathcal{O}_u(M_{m,n}(\mathbb{k}))$ is an iterated skew polynomial extension of $\mathbb{k}$ where the indeterminates are added in the order $X_{11}, \dots, X_{1n}, X_{21}, \dots, X_{2n}, \dots, X_{m1}, \dots, X_{mn}$. As a well-known consequence, $\mathcal{O}_u(M_{m,n}(\mathbb{k}))$ is a noetherian domain.
2. It is not difficult to see, as a consequence of the previous point, that $\mathcal{O}_u(M_{m,n}(A))$ is an iterated Ore extension of the ring A where, at each step, the endomorphism of the skew-derivation that is used is in fact an automorphism.
3. It is well known that there is a $\mathbb{k}$-algebra isomorphism $\mathrm{Tr} : \mathcal{O}_u(M_{m,n}(\mathbb{k})) \longrightarrow \mathcal{O}_u(M_{n,m}(\mathbb{k}))$ such that $X_{ij} \mapsto X_{ji}$, which we call the *transpose isomorphism*. It follows at once that there is a similar transpose isomorphism of A-algebras $\mathrm{Tr} : \mathcal{O}_u(M_{m,n}(A)) \longrightarrow \mathcal{O}_u(M_{n,m}(A))$.

It follows from point 3 in Remark 3.1.3 that, in what follows, we may assume that $m \leq n$ without loss of generality. We do this for the rest of the section.

If A is a commutative domain and u a unit in A, we associate to $\mathcal{O}_u(M_{m,n}(A))$ the $m \times n$ matrix $X = (X_{ij})$ which we call the *generic matrix* of $\mathcal{O}_u(M_{m,n}(A))$.

Let $1 \leq t \leq m$. A pair (I, J) is an *index pair (of cardinality t)* provided that $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ with $|I| = |J| = t$. If $I = \{i_1, \dots, i_t\}$ and $J = \{j_1, \dots, j_t\}$ with $1 \leq i_1 < \cdots < i_t \leq m$ and $1 \leq j_1 < \cdots < j_t \leq n$, we will write $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$. The set of all index pairs of cardinality t, where t takes all possible values in $\{1, \dots, m\}$ is denoted $\Delta_{m,n}$ (or Δ if there is no possible confusion). To any index pair (I, J) of cardinality t with $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$ we associate the *quantum $t \times t$ minor* $[I|J]$ of X. This is the element of $\mathcal{O}_u(M_{m,n}(A))$ defined by

$$[I|J] := \sum_{\sigma \in \mathfrak{S}_t} (-u)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} \dots X_{i_t, j_{\sigma(t)}}$$

(here, $\mathfrak{S}_t$ is the symmetric group on $\{1, \dots, t\}$ and ℓ is the usual length function on $\mathfrak{S}_t$).

In the same way, an *index set* J is a subset J of m pairwise distinct elements of $\{1, \dots, n\}$. If $J = \{j_1, \dots, j_m\}$ with $1 \leq j_1 < \cdots < j_m \leq n$, we will denote $J = \{j_1 < \cdots < j_m\}$. The set of all index sets is denoted $\Pi_{m,n}$ (or Π if there is no possible confusion).
To any $J = \{j_1 < \cdots < j_m\} \in \Pi$, we associate the maximal quantum minor $[J] := [K|J]$ of X, where $K = \{1 < \cdots < m\}$, hence

$$[J] = \sum_{\sigma \in S_m} (-u)^{\ell(\sigma)} X_{1, j_{\sigma(1)}} \dots X_{m, j_{\sigma(m)}}.$$

The *standard partial order*, $\leq_{\mathrm{st}}$, on Π is defined as follows:

$$\{i_1 < \cdots < i_m\} \leq_{\mathrm{st}} \{j_1 < \cdots < j_m\} \qquad \text{iff} \qquad i_s \leq j_s \qquad \text{for each} \qquad 1 \leq s \leq m.$$

Clearly, Π can be identified with the subset of maximal quantum minors of $\mathcal{O}_u(M_{m,n}(A))$. Thus, in what follows, we also denote by Π the set of maximal quantum minors of $\mathcal{O}_u(M_{m,n}(A))$.

Definition 3.1.4 – *Let A be a commutative domain and u be a unit in A. The* quantum grassmannian $\mathcal{O}_u(G_{m,n}(A))$ *is the A-subalgebra of $\mathcal{O}_u(M_{m,n}(A))$ generated by $\{[I],\ I \in \Pi\}$.*

The following remark will be of constant use in what follows. We will often use it without explicit reference.

Remark 3.1.5 – Let A and B be commutative domains, let $u \in A$ and $v \in B$ be units and let $f : A \longrightarrow B$ be a morphism of rings such that $f(u) = v$. By using Remark 3.1.3 and the universal property of Ore extensions, it is easy to see that there is a unique morphism of rings $g : \mathcal{O}_u(M_{m,n}(A)) \longrightarrow \mathcal{O}_v(M_{m,n}(B))$ such that $g(X_{ij}) = X_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ which extends f. Clearly, for $I \in \Pi$, we then have $g([I]) = [I]$. Hence, g induces by restriction a morphism of rings $h : \mathcal{O}_u(G_{m,n}(A)) \longrightarrow \mathcal{O}_v(G_{m,n}(B))$ such that $h([I]) = [I]$ for any $I \in \Pi$. It is easy to see that, if f is injective (respectively, surjective), then both g and h are injective (respectively, surjective).

Remark 3.1.6 – In what follows, we will be mainly interested in the following cases.
1. The case where $A = \mathbb{Z}[t^{\pm 1}]$ and $u = t$. This case will be, in fact, of central importance, thus we fix the notation $\mathcal{A} = \mathbb{Z}[t^{\pm 1}]$ for the rest of this work.
2. The case where $A = \Bbbk$ is a field and q is any non-zero element in $\Bbbk$. Then, $\mathcal{O}_q(M_{m,n}(\Bbbk))$ is the usual quantum deformation of the coordinate ring on the variety $M_{m,n}(\Bbbk)$. Moreover, $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is the usual quantum deformation of the homogeneous coordinate ring of the grassmannian $G_{m,n}(\Bbbk)$.

In order to prove that quantum grassmannians have a straightening law, we will need to use certain quantum analogues of *Schubert varieties*. These are quotients of quantum grassmannians and the key to obtaining straightening relations in quantum grassmannians is to find good bases for these quantum Schubert varieties.

Definition 3.1.7 – *Let A be a commutative domain and u be a unit in A. To any $\gamma \in \Pi$ we associate the set $\Omega_\gamma := \{\pi \in \Pi \,|\, \pi \not\geq_{\mathrm{st}} \gamma\}$ and we set*

$$I_{\gamma,A} := \langle \Omega_\gamma \rangle \qquad \textit{and} \qquad \mathcal{O}_u(G_{m,n}(A))_\gamma := \mathcal{O}_u(G_{m,n}(A))/I_{\gamma,A}.$$

We denote by $\Theta_{\gamma,A}$ the canonical projection

$$\Theta_{\gamma,A} : \mathcal{O}_u(G_{m,n}(A)) \longrightarrow \mathcal{O}_u(G_{m,n}(A))_\gamma.$$

The quotient ring $\mathcal{O}_u(G_{m,n}(A))_\gamma$ will be called the quantum Schubert variety (over A) associated *with γ.*

3.2 Standard bases for quantum grassmannians.

Definition 3.2.1 – *Let A be any commutative domain and u be any invertible element of A. A* standard monomial *on Π is an element of $\mathcal{O}_u(G_{m,n}(A))$ which is either equal to 1 or is of the form $[I_1]\dots[I_s]$, with $I_1, \dots, I_s \in \Pi$ and $I_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} I_s$.*

Remark 3.2.2 – Recall from [KLR; Corollary 2.8] that, if A is any field and u is any nonzero element of A then the standard monomials on Π form an A-basis of $\mathcal{O}_u(G_{m,n}(A))$.

We want to extend Remark 3.2.2 to any commutative integral domain A.

Lemma 3.2.3 – *The set of standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ is a $\mathbb{Z}[t^{\pm 1}]$-generating set of $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$.*

Proof. Let x be a non zero element of $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$. We consider the natural embeddings $\mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{Q}(t)$ and $\mathbb{Z}[t^{\pm 1}] \hookrightarrow \mathbb{C}[t^{\pm 1}]$ and the associated injective morphisms of rings $h : \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}])) \hookrightarrow \mathcal{O}_t(G_{m,n}(\mathbb{Q}(t)))$ and $h' : \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}])) \hookrightarrow \mathcal{O}_t(G_{m,n}(\mathbb{C}[t^{\pm 1}]))$.

Remark 3.2.2 applied with $A = \mathbb{Q}(t)$ and $u = t$, shows that $h(x)$ may be written as a linear combination of standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{Q}(t)))$ with coefficients from $\mathbb{Q}(t)$. It follows at once that there are elements $k(t), k_1(t), \dots, k_s(t) \in \mathbb{Z}[t^{\pm 1}] \setminus \{0\}$ and standard monomials $M_1, \dots, M_s$ in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ such that $k(t)x = \sum_{i=1}^{s} k_i(t)M_i \in \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$. Of course, we may assume that $k(t), k_1(t), \dots, k_s(t)$ have no common irreducible factor in $\mathbb{Z}[t^{\pm 1}]$ and hence in $\mathbb{C}[t^{\pm 1}]$. By applying h' to this relation we obtain a relation $k(t)h'(x) = \sum_{i=1}^{s} k_i(t)M_i \in \mathcal{O}_t(G_{m,n}(\mathbb{C}[t^{\pm 1}])$. Now, consider $u \in \mathbb{C}^*$: to the natural map $ev_u : \mathbb{C}[t^{\pm 1}] \longrightarrow \mathbb{C}$, $t \mapsto u$ we may associate an epimorphism $\mathcal{O}_t(G_{m,n}(\mathbb{C}[t^{\pm 1}])) \longrightarrow \mathcal{O}_u(G_{m,n}(\mathbb{C}))$. Applying this map with u a non-zero root of $k(t)$ gives a nontrivial relation of linear dependence among standard monomials in $\mathcal{O}_u(G_{m,n}(\mathbb{C}))$ which violates Remark 3.2.2. Hence, there exists $c, d \in \mathbb{Z}$ such that $k(t) = ct^d$. As a consequence, we have the relation $cx = \sum_{i=1}^{s} t^{-d}k_i(t)M_i$ in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$. Here again, we may assume that in $\mathbb{Z}$ the greatest common divisor of c and the non zero coefficients of the $k_i(t)$, for $1 \leq i \leq s$, is one. Now, assume that p is a prime divisor of c in $\mathbb{Z}$. The natural map $\mathbb{Z}[t^{\pm 1}] \longrightarrow \mathbb{F}_p[t^{\pm 1}] \hookrightarrow \mathbb{F}_p(t)$ gives rise to a map $h'' : \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}])) \longrightarrow \mathcal{O}_t(G_{m,n}(\mathbb{F}_p(t)))$. Applying h'' to the above relation gives $0 = \sum_{i=1}^{s} t^{-d}(k_i(t) + p\mathbb{Z}[t^{\pm 1}])M_i$ in $\mathcal{O}_t(G_{m,n}(\mathbb{F}_p(t)))$ where the right hand side is a non trivial linear combination of standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{F}_p(t)))$. Again, this violates Remark 3.2.2. Thus, $c = \pm 1$ and we get the relation $x = \sum_{i=1}^{s} (\pm 1)t^{-d}k_i(t)M_i$ in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$, as required. ■

Proposition 3.2.4 – *Let A be a commutative domain and let u be an invertible element of A. The set of standard monomials in $\mathcal{O}_u(G_{m,n}(A))$ is a A-generating set of $\mathcal{O}_u(G_{m,n}(A))$.*

Proof. Since u is a unit in A, there is a natural ring homomorphism $\mathbb{Z}[t^{\pm 1}] \longrightarrow A$ such that $1 \mapsto 1_A$ and $t \mapsto u$. Let $f : \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}])) \longrightarrow \mathcal{O}_u(G_{m,n}(A))$ denote the induced morphism. Clearly, the set of products of the form $[I_1] \dots [I_s]$, with $I_1, \dots, I_s \in \Pi$ together with 1 form a generating set for the A-module $\mathcal{O}_u(G_{m,n}(A))$. Hence it is enough to prove that any such product is a linear combination with coefficients in A of standard monomials. Let $I_1, \dots, I_s \in \Pi$; we have $[I_1] \dots [I_s] = f([I_1] \dots [I_s])$. By Lemma 3.2.3, in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$, we have an identity $[I_1] \dots [I_s] = \sum_{i=1}^{d} k_i(t)M_i$, where $k_i(t) \in \mathbb{Z}[t^{\pm 1}]$ and M_i is a standard monomial, for $1 \leq i \leq d$. Applying f to this equality shows that in $\mathcal{O}_u(G_{m,n}(A))$, we have an identity $[I_1] \dots [I_s] = \sum_{i=1}^{d} f(k_i(t))M_i$, where $f(k_i(t)) \in A$ and M_i is a standard monomial, for $1 \leq i \leq d$. ■

Proposition 3.2.5 – *Let A be a commutative domain and let u be an invertible element of A. The set of standard monomials in $\mathcal{O}_u(G_{m,n}(A))$ is a A-basis of $\mathcal{O}_u(G_{m,n}(A))$.*

Proof. Let F be the field of fractions of A. The natural embedding $A \hookrightarrow F$ induces an embedding $\epsilon_A : \mathcal{O}_u(G_{m,n}(A)) \hookrightarrow \mathcal{O}_u(G_{m,n}(F))$ with $\epsilon_A([I_1] \dots [I_s]) = [I_1] \dots [I_s]$, for $I_1 \leq_{\text{st}} \dots \leq_{\text{st}} I_s$ in Π. The linear independence of standard monomials in $\mathcal{O}_u(G_{m,n}(A))$ thus follows from the linear independence of standard monomials in $\mathcal{O}_u(G_{m,n}(F))$ (see Remark 3.2.2). On the other hand, the standard monomials generate $\mathcal{O}_u(G_{m,n}(A))$ as an A-module, by Proposition 3.2.4. ■

3.3 Straightening relations in quantum grassmanians.

The aim of this subsection is to establish that quantum grassmannians have a straightening law. In order to complete this task, we will make use of the quantum Schubert varieties introduced earlier.

Remark 3.3.1 – Let A be a commutative domain and u a unit of A.
1. We say that a standard monomial $[I_1]\dots[I_s]$ in $\mathcal{O}_u(G_{m,n}(A))$ *involves* an element $[J]$, with $J \in \Pi$, if $J \in \{I_1, \dots, I_s\}$.
2. It is clear from the definition that the subset Ω_γ is a Π-ideal for any $\gamma \in \Pi$. As a consequence, a standard monomial $[I_1]\dots[I_s]$ in $\mathcal{O}_u(G_{m,n}(A))$ involves an element of Ω_γ if an only if $I_1 \in \Omega_\gamma$.

Recall that $\mathcal{A} = \mathbb{Z}[t^{\pm 1}]$. Our main aim in this section is to exhibit useful $\mathcal{A}$-bases for quantum Schubert varieties over $\mathcal{A}$ when $u = t$. In order to complete this task, we need some commutation relations established in the works [HL(1)] and [HL(2)] by Hodges and Levasseur. Since the point of view and the notation of these papers are quite different from those of this work, we recall some facts from them, for the convenience of the reader. Let $r \in \mathbb{N}^*$. Recall that, for any field $\Bbbk$ and any non-zero element $q \in \Bbbk$, the *quantum special linear group* is defined by $\mathcal{O}_q(SL_r(\Bbbk)) := \mathcal{O}_q(M_r(\Bbbk))/\langle D_q - 1\rangle$, where $\mathcal{O}_q(M_r(\Bbbk)) = \mathcal{O}_q(M_{r,r}(\Bbbk))$ and D_q is the quantum determinant of $\mathcal{O}_q(M_r(\Bbbk))$; that is, its unique $r \times r$ quantum minor.

The setting of [HL(1)] and [HL(2)] is the following. Fix an integer $r \geq 2$ and a complex number q which is not a root of unity. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sl}_r(\mathbb{C})$ and G the algebraic group $SL_r(\mathbb{C})$. Let $U_q(\mathfrak{g})$ be the quantum universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{C}$ as defined in [HL(1) ; §1.2] and let $\mathbb{C}_q[G]$ be the restricted dual associated with $U_q(\mathfrak{g})$. According to [HL(1); Theorem 1.4.1], the algebra $\mathbb{C}_q[G]$ is nothing but $\mathcal{O}_{q^2}(SL_r(\mathbb{C}))$ in our notation. Now, for any integer k, with $1 \leq k \leq r-1$, and any element w in the Weyl group $W \cong \mathfrak{S}_r$ associated with $\mathfrak{sl}_r(\mathbb{C})$, we consider the matrix coefficient $c^+_{k,w} \in \mathbb{C}_q[G]$ defined in [HL(1); 1.5]. As noted in [HL(1); 1.5], and making no distinction between a minor of $\mathcal{O}_{q^2}(M_r(\mathbb{C}))$ and its image in $\mathcal{O}_{q^2}(SL_r(\mathbb{C}))$, we have

$$c^+_{k,w} = [I|K] \in \mathbb{C}_q[G] = \mathcal{O}_{q^2}(SL_r(\mathbb{C})),$$

where $K = \{1, \dots, k\}$ and I is the index-set obtained from $\{w(1), \dots, w(k)\}$ after reordering. Then, for $x, y \in W$, applying [HL(2); Proposition 1.1] with $1 \leq i = j = k \leq r-1$ and $t = \mathrm{id} \in \mathfrak{S}_r$, we get an identity

$$c^+_{k,y}c^+_{k,x} - (q^2)^e c^+_{k,x}c^+_{k,y} = \sum_{\substack{u \in W \\ x <_k ux \\ uy <_k y}} g_u(q) c^+_{k,ux}c^+_{k,uy}, \tag{5}$$

where $e \in \mathbb{Z}$, and $g_u(q) \in \mathbb{C}$, for $u \in W$ such that $x <_k ux$ and $uy <_k y$. (For the definition of the relation $<_k$ in W, see [HL(2)].)

Now, set $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$ with $I, J \subseteq \{1, \dots, r\}$ and $k \leq r-1$. Let $K = \{1, \dots, k\}$. Once translated in to our notation for minors, the relation (5) yields a relation

$$[I|K][J|K] - (q^2)^e [J|K][I|K] = \sum_{\substack{J <_{\mathrm{st}} L \\ M <_{\mathrm{st}} I}} g_{L,M}[L|K][M|K], \tag{6}$$

in $\mathcal{O}_{q^2}(SL_r(\mathbb{C})$, where $g_{L,M} \in \mathbb{C}$, for $L, M \subseteq \{1, \dots, r\}$ such that $|L| = |M| = k$ while $J <_{\mathrm{st}} L$ and $M <_{\mathrm{st}} I$. (Here again, we use the same notation for a minor in $\mathcal{O}_{q^2}(M_r(\mathbb{C}))$ and its image in $\mathcal{O}_{q^2}(SL_r(\mathbb{C}))$.) On the other hand, by using the transpose automorphism of $\mathcal{O}_{q^2}(M_r(\mathbb{C}))$ (see [PW; 3.7.1(1)] and [PW; 4.1.1]) we can interchange rows and colums in (6). Finally, it is well-known that, for $1 \leq k < r$, the natural map $\mathcal{O}_{q^2}(G_{k,r}(\mathbb{C})) \hookrightarrow \mathcal{O}_{q^2}(M_r(\mathbb{C})) \longrightarrow \mathcal{O}_{q^2}(SL_r(\mathbb{C}))$ is an embedding. Hence, we end up with the following important proposition.

Proposition 3.3.2 – *We assume that $A = \mathbb{C}$ and $u = q \in \mathbb{C}$ is not a root of unity. Let $I, J \in \Pi$. Then there exists a relation*

$$[I][J] - q^{e_{I,J}}[J][I] = \sum_{\substack{J <_{\mathrm{st}} L \\ M <_{\mathrm{st}} I}} g_{L,M}[L][M],$$

in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$, where $e_{I,J} \in \mathbb{Z}$ and $g_{L,M} \in \mathbb{C}$, for $L, M \subseteq \{1, \dots, n\}$ such that $|L| = |M| = m$ while $J <_{\mathrm{st}} L$ and $M <_{\mathrm{st}} I$.

Theorem 3.3.3 – *Let $A = \mathbb{C}$ and suppose that $u = q \in \mathbb{C}$ is not a root of unity. Let $\gamma \in \Pi$. Then the ideal $I_{\gamma,\mathbb{C}}$ of $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ is generated by $\{\pi \in \Pi \,|\, \pi \not\geq_{\mathrm{st}} \gamma\}$ as a right ideal.*

Proof. The ideal $I_{\gamma,\mathbb{C}}$ is generated, as a two-sided ideal, by $\Omega_\gamma = \{\pi \in \Pi \,|\, \pi \not\geq_{\mathrm{st}} \gamma\}$. Now, consider any total order on Π that respects the partial order $\leq_{\mathrm{st}}$ on Π. Then the commutation relations in Proposition 3.3.2 together with the fact that Ω_γ is a Π-ideal show that the elements of Ω_γ ordered by this total order form a normalising sequence in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$. The result follows. ∎

We will also need the following adaptation of [GL; Corollary 1.8].

Proposition 3.3.4 – *Let $A = K$ be a field and let q be a nonzero element of K. Let $I_1, \dots, I_s$ be elements of Π. In $\mathcal{O}_q(G_{m,n}(K))$, the product $[I_1] \dots [I_s]$ can be expressed as a linear combination of standard monomials of the form $[J_1] \dots [J_s]$ such that $J_1 \leq_{\mathrm{st}} I_1$.*

Proof. In this proof, we will use notation from [GL]. Recall from Remark 3.2.2 that the standard monomials on Π form a K-basis of $\mathcal{O}_q(G_{m,n}(K))$. Hence, the product $[I_1] \dots [I_s]$ is a linear combination of standard monomials in $\mathcal{O}_q(G_{m,n}(K))$ which must have the form $[J_1] \dots [J_s]$ with $J_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} J_s \in \Pi$ as one may easily deduce using the standard grading of $\mathcal{O}_q(M_{m,n}(K))$. Now, consider the embedding $\mathcal{O}_q(M_{m,n}(K)) \hookrightarrow \mathcal{O}_q(M_n(K))$ defined by $X_{ij} \mapsto X_{ij}$. It induces an embedding $\iota : \mathcal{O}_q(G_{m,n}(K)) \hookrightarrow \mathcal{O}_q(M_n(K))$ such that $[I] \mapsto [L|I]$ where I is any element of Π and $L = \{1, \dots, m\}$. Now, it is clear that, if $I, J \in \Pi$, $I \leq_{\mathrm{st}} J$ implies that $(L, I) \leq (L, J)$ where $\leq$ stands for the order $(\leq_r, \leq_c)$ defined in [GL; §1.2]. Thus, applying ι to the above expression of $[I_1] \dots [I_s]$ as a linear combination of standard monomials $[J_1] \dots [J_s]$ leads to an expression of $[L|I_1] \dots [L|I_s]$ in $\mathcal{O}_q(M_n)$ as a linear combination of products $[L|J_1] \dots [L|J_s]$ whose associated bitableau

$$\begin{pmatrix} L & J_1 \\ \vdots & \vdots \\ L & J_s \end{pmatrix}$$

is preferred (in the sense of [GL; §1.3]). But, by [GL; Corollary 1.10], such products form a basis of $\mathcal{O}_q(M_n)$, hence we can compare this expression of $[L|I_1] \dots [L|I_s]$ with that given by [GL; Corollary 1.8] to conclude that $(L, J_1) \leq (L, I_1)$ which leads to $J_1 \leq_{\mathrm{st}} I_1$. ∎

Corollary 3.3.5 – *Let $A = \mathbb{C}$ and suppose that $u = q \in \mathbb{C}$ is not a root of unity. Let $\gamma \in \Pi$. The ideal $I_{\gamma,\mathbb{C}}$ of $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ is the $\mathbb{C}$-span of the standard monomials which involve an element of Ω_γ.*

Proof. We let $V_{\gamma,\mathbb{C}}$ denote the $\mathbb{C}$-span in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ of those standard monomials which involve an element of Ω_γ. Notice that, by Remark 3.3.1, a standard monomial involves an element of Ω_γ if and only if it starts with an element of Ω_γ; that is, it has the form $[I_1] \dots [I_s]$ where $I_1, \dots, I_s \in \Pi$ with $I_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} I_s$ and $I_1 \in \Omega_\gamma$. Clearly, $V_{\gamma,\mathbb{C}} \subseteq I_{\gamma,\mathbb{C}}$.

By Theorem 3.3.3 and Remark 3.2.2, any element in $I_{\gamma,\mathbb{C}}$ is a linear combination of terms of the form $[I]\mu$ where $I \in \Omega_\gamma$ and μ is a standard monomial. Hence, to show that $V_{\gamma,\mathbb{C}} \supseteq I_{\gamma,\mathbb{C}}$, it is enough to show that any product $[I]\mu$ where $I \in \Omega_\gamma$ and μ is a standard monomial lies in $V_{\gamma,\mathbb{C}}$. The case $\mu = 1$ is trivial; and so we just have to consider products $[I][I_1]\dots[I_s]$ with $I_1,\dots,I_s \in \Pi$ and $I_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} I_s$. By using Proposition 3.3.4, we see that $[I][I_1]\dots[I_s]$ is a linear combination of standard monomials $[J_1]\dots[J_{s+1}]$, where $J_1 \leq_{\mathrm{st}} I$. However, Ω_γ is a Π-ideal; and so from $J_1 \leq_{\mathrm{st}} I$ we deduce that $J_1 \in \Omega_\gamma$. Hence $I_{\gamma,\mathbb{C}} \subseteq V_{\gamma,\mathbb{C}}$. ∎

We now investigate the case of quantum Schubert varieties over the base ring $\mathcal{A} = \mathbb{Z}[t^{\pm 1}]$. Let q be an element of $\mathbb{C}$ that is transcendental over $\mathbb{Q}$. There is a canonical embedding $\mathbb{Z}[t^{\pm1}] \hookrightarrow \mathbb{C}$ sending t to q. This embedding induces an embedding

$$\varepsilon \,:\, \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}])) \hookrightarrow \mathcal{O}_q(G_{m,n}(\mathbb{C})),$$

such that, for $I \in \Pi$, $\varepsilon([I]) = [I]$. We denote by $I_{\gamma,\mathcal{A}}$ and $I_{\gamma,\mathbb{C}}$ the ideals of $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))$ and $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$, respectively, generated by the elements of the Π-ideal Ω_γ. In addition, we let $V_{\gamma,\mathcal{A}}$ denote the $\mathbb{Z}[t^{\pm1}]$-span of those standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))$ involving an element of Ω_γ. (Recall from the proof of Corollary 3.3.5 that $V_{\gamma,\mathbb{C}}$ is the $\mathbb{C}$-span of those standard monomials in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ involving an element of Ω_γ.)

Lemma 3.3.6 – *In the notation above, $I_{\gamma,\mathcal{A}} = \varepsilon^{-1}(I_{\gamma,\mathbb{C}})$ and $I_{\gamma,\mathcal{A}} = V_{\gamma,\mathcal{A}}$.*

Proof. We have $V_{\gamma,\mathcal{A}} \subseteq I_{\gamma,\mathcal{A}} \subseteq \varepsilon^{-1}(I_{\gamma,\mathbb{C}}) = \varepsilon^{-1}(V_{\gamma,\mathbb{C}}) = V_{\gamma,\mathcal{A}}$. Indeed, the first inclusion is obvious, the second is easy since ε is a ring homomorphism which sends $[I]$ to $[I]$ for any $I \in \Pi$, the first equality is Corollary 3.3.5 and the second equality is an easy consequence of the fact that the $\mathbb{Z}[t^{\pm1}]$-basis of standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}])$ is sent to the $\mathbb{C}$-basis of standard monomials in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ in the obvious way and of the fact (due to the transcendance of q over $\mathbb{Q}$) that a nonzero element in $\mathbb{Z}[t^{\pm1}]$ is sent by ε to a nonzero complex number. Hence, we have proved the desired equalities. ∎

Remark 3.3.7 – (i) It follows at once from Corollary 3.3.5 that, if $q \in \mathbb{C}$ is a non-root of unity, then the set of cosets of those standard monomials in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ of the form $[I_1]\dots[I_s]$ such that $\gamma \leq_{\mathrm{st}} I_1$ together with 1 form a $\mathbb{C}$-basis of $\mathcal{O}_q(G_{m,n}(\mathbb{C}))_\gamma$;
(ii) it then follows from Lemma 3.3.6 that the set of cosets of those standard monomials in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))$ of the form $[I_1]\dots[I_s]$ such that $\gamma \leq_{\mathrm{st}} I_1$ together with 1 form a $\mathbb{Z}[t^{\pm1}]$-basis of $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))_\gamma$.

We now prove the existence of straightening relations in $\mathcal{O}_q(G_{m,n}(A))$ for any commutative domain A.

Recall that, for any $\gamma \in \Pi$, we denote by $\Theta_{\gamma,\mathbb{Z}[t^{\pm1}]}$ the canonical projection

$$\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}])) \overset{\Theta_{\gamma,\mathbb{Z}[t^{\pm1}]}}{\longrightarrow} \mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))_\gamma \ .$$

Theorem 3.3.8 – *Let I, J be elements of Π which are not comparable with respect to $\leq_{\mathrm{st}}$. Then, in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm1}]))$, the product $[I][J]$ may be rewritten as*

$$[I][J] = \sum_{i=1}^{s} k_i [I_i][J_i],$$

with $k_i \in \mathbb{Z}[t^{\pm1}]$, while $I_i, J_i \in \Pi$ with $I_i <_{\mathrm{st}} I, J$ and $I_i \leq_{\mathrm{st}} J_i$.

Proof. From Proposition 3.2.5, we know that standard monomials form a $\mathbb{Z}[t^{\pm 1}]$-basis in the ring $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$. Thus, the product $[I][J]$ in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ may be written as a linear combination of standard monomials. In addition, $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ is a graded $\mathbb{Z}[t^{\pm 1}]$-algebra in which the degree of the canonical generators is one. It follows that we have a relation

$$[I][J] = \sum_{i=1}^{s} k_i [I_i][J_i],$$

with $k_i \in \mathbb{Z}[t^{\pm 1}] \setminus \{0\}$ while $I_i, J_i \in \Pi$ and $I_i \leq_{\mathrm{st}} J_i$. (Notice that the product $[I][J]$ is non zero, since $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ is a domain; and so the right hand side of the above relation contains at least one term.)

Consider a maximal element γ of $\{I_1, \ldots, I_s\} \subseteq \Pi$ with respect to $\leq_{\mathrm{st}}$. By applying $\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}$ to the above relation, we obtain a relation

$$\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([I])\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([J]) = \sum_{i=1}^{s} k_i \Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([I_i])\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([J_i]).$$

In the right hand side of this last relation, all the terms $\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([I_i])\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([J_i])$ such that $I_i \not\geq_{\mathrm{st}} \gamma$ are zero while the others form a non-empty family (it contains $\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([\gamma])\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([J_i])$ for some i) which is free by Remark 3.3.7. Thus, $\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([I])\Theta_{\gamma,\mathbb{Z}[t^{\pm 1}]}([J]) \neq 0$ and we deduce that $I, J \geq_{\mathrm{st}} \gamma$.

Thus, we have shown that, for any maximal element γ of $\{I_1, \ldots, I_s\}$, we have $I, J \geq_{\mathrm{st}} \gamma$. This entails that $I_1, \ldots, I_s \leq_{\mathrm{st}} I, J$. Now, if $I_i = I$ or $I_i = J$ for some $i \in \{1, \ldots, s\}$, since $I_i \leq_{\mathrm{st}} I, J$, it follows that I and J must be comparable, which is not the case. The proof is complete. ■

Theorem 3.3.9 – *Let A be a commutative domain, and let u be a unit in A. Suppose that I, J are elements of Π which are not comparable with respect to $\leq_{\mathrm{st}}$. Then, in $\mathcal{O}_u(G_{m,n}(A))$, the product $[I][J]$ may be rewritten as*

$$[I][J] = \sum_{i=1}^{s} k_i [I_i][J_i],$$

with $k_i \in A$ while $I_i, J_i \in \Pi$ with $I_i <_{\mathrm{st}} I, J$ and $I_i \leq_{\mathrm{st}} J_i$.

Proof. There is a morphism of rings $\mathbb{Z} \longrightarrow A$ such that $1 \mapsto 1_A$. Since u is a unit in A, it induces a morphism of rings $\mathbb{Z}[t^{\pm 1}] \longrightarrow A$ such that $t \mapsto u$. From this, we get a morphism of rings $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}])) \longrightarrow \mathcal{O}_u(G_{m,n}(A))$ sending $[I]$ to $[I]$. At this point, it is clear that the desired relations in $\mathcal{O}_u(G_{m,n}(A))$ follow from the corresponding ones in $\mathcal{O}_t(G_{m,n}(\mathbb{Z}[t^{\pm 1}]))$ given by Theorem 3.3.8. ■

3.4 Quantum grassmannians are quantum graded A.S.L.

In order to show that quantum grassmannians are quantum graded A.S.L., it still remains to obtain good commutation relations for them. This is the first goal of this section.

Proposition 3.4.1 – *Let $A = \mathbb{C}$ and suppose that $u = q \in \mathbb{C}$ is not a root of unity. Let $I, J \in \Pi$. Then there exists a relation*

$$[I][J] - q^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M],$$

in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$, where the sum is taken over pairs (L, M) of elements of Π such that $L <_{\mathrm{st}} I$ and where $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in \mathbb{C}$.

Proof. It is well-known that there exists an anti-isomorphism of algebras $\alpha : \mathcal{O}_q(M_n(\mathbb{C})) \longrightarrow \mathcal{O}_{q^{-1}}(M_n(\mathbb{C}))$ such that $X_{ij} \mapsto X_{ij}$, for $1 \leq i, j \leq n$ (see, for example, [PW; 3.7]). In addition, for $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$, with $1 \leq i_1 < \cdots < i_t \leq n$ and $1 \leq j_1 < \cdots < j_t \leq n$,

$$\begin{aligned} \alpha([I|J]) &= \alpha\left(\textstyle\sum_{\sigma\in\mathfrak{S}_t}(-q)^{\ell(\sigma)}X_{i_1,j_{\sigma(1)}}\dots X_{i_t,j_{\sigma(t)}}\right) \\ &= \textstyle\sum_{\sigma\in\mathfrak{S}_t}(-q)^{\ell(\sigma)}X_{i_t,j_{\sigma(t)}}\dots X_{i_1,j_{\sigma(1)}} \\ &= \textstyle\sum_{\sigma\in\mathfrak{S}_t}(-q)^{\ell(\sigma)}X_{i_{\omega(1)},j_{\sigma\omega(1)}}\dots X_{i_{\omega(t)},j_{\sigma\omega(t)}}, \end{aligned}$$

where ω is the longest element of the symmetric group $\mathfrak{S}_t$, that is the element which sends r to $t+1-r$, for $1 \leq r \leq t$. As is well known, $\ell(\sigma) = \ell(\omega) - \ell(\sigma\omega)$, for all $\sigma \in \mathfrak{S}_t$. Hence, we get

$$\begin{aligned} \alpha([I|J]) &= \textstyle\sum_{\sigma\in\mathfrak{S}_t}(-q)^{\ell(\omega)-\ell(\sigma\omega)}X_{i_{\omega(1)},j_{\sigma\omega(1)}}\dots X_{i_{\omega(t)},j_{\sigma\omega(t)}} \\ &= \textstyle\sum_{\sigma\in\mathfrak{S}_t}(-q)^{\ell(\omega)-\ell(\sigma)}X_{i_{\omega(1)},j_{\sigma(1)}}\dots X_{i_{\omega(t)},j_{\sigma(t)}} \\ &= [I|J]. \end{aligned}$$

(For the last equality, see [PW; 4.1.1].) It follows that α induces an anti-isomorphism $\beta : \mathcal{O}_q(G_{m,n}(\mathbb{C})) \longrightarrow \mathcal{O}_{q^{-1}}(G_{m,n}(\mathbb{C}))$ such that, for any $I \in \Pi$, $\beta([I]) = [I]$. Now, let $I, J \in \Pi$. By Proposition 3.3.2 we have a relation

$$[I][J] - q^{e_{I,J}}[J][I] = \sum_{\substack{J <_{\rm st} L \\ M <_{\rm st} I}} g_{L,M}[L][M],$$

in $\mathcal{O}_{q^{-1}}(G_{m,n}(\mathbb{C}))$, where $e_{I,J} \in \mathbb{Z}$ and $g_{L,M} \in \mathbb{C}$, for $L, M \subseteq \{1, \dots, n\}$ such that $|L| = |M| = m$ and $J <_{\rm st} L, M <_{\rm st} I$. Applying β^{-1} to the identity gives us a new identity

$$[I][J] - q^{-e_{I,J}}[J][I] = \sum_{\substack{J <_{\rm st} L \\ M <_{\rm st} I}} g'_{L,M}[M][L]. \tag{7}$$

This completes the proof. ∎

Proposition 3.4.2 – *Let $A = \mathbb{C}$ and suppose that $u = q \in \mathbb{C}$ is not a root of unity. Let $I, J \in \Pi$. Then there exists a relation*

$$[I][J] - q^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M],$$

in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$, where the sum is taken over pairs (L, M) of elements of Π such that $L <_{\rm st} I, J$ and either $L \leq_{\rm st} M$ or $M <_{\rm st} L$ and where $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in \mathbb{C}$.

Proof. Fix a pair $(I, J) \in \Pi \times \Pi$. If I and J are not comparable for $\leq_{\rm st}$, then applying Theorem 3.3.9 to both $[I][J]$ and $[J][I]$ shows that the result holds with, for example, $f_{I,J} = 0$. Now, suppose I and J are comparable. Of course, we may assume without loss of generality that $I \leq_{\rm st} J$. Now, Proposition 3.4.1 gives us a relation $[I][J] - q^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M]$, where the sum is taken over pairs (L, M) of elements of Π such that $L <_{\rm st} I$ (and so $L <_{\rm st} J$ also holds, since $I \leq_{\rm st} J$) and where $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in \mathbb{C}$. In addition, by applying Theorem 3.3.9 to each pair (L, M) of incomparable elements on the right hand side of this relation, we see that we can produce such a relation with the additional information that, for each pair (L, M) appearing, either $L \leq_{\rm st} M$ or $M <_{\rm st} L$. ∎

Theorem 3.4.3 – *Let $A = \mathbb{C}$ and suppose that $u = q \in \mathbb{C}$ is not a root of unity. Let $I, J \in \Pi$. Then there exists a relation*

$$[I][J] - q^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M],$$

in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$, where the sum is taken over pairs (L, M) of elements of Π such that $L <_{\rm st} I, J$ and $L \leq_{\rm st} M$ and where $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in \mathbb{C}$.

Proof. Let us introduce some notation. We let S be the subset of those pairs (L, M) in $\Pi \times \Pi$ such that $L \leq_{\rm st} M$ and let T the subset of those pairs (L, M) in $\Pi \times \Pi$ such that $M <_{\rm st} L$. In addition, for each pair $(I, J) \in \Pi \times \Pi$, we let $E_{I,J}$ be the subset of those pairs (L, M) in $\Pi \times \Pi$ such that $L <_{\rm st} I, J$. In this notation, Proposition 3.4.2 shows that, for each pair $(I, J) \in \Pi \times \Pi$, there exist an integer $f_{I,J}$ and scalars $k_{L,M} \in \mathbb{C}$ for each $(L, M) \in (S \cap E_{I,J}) \cup (T \cap E_{I,J})$ such that

$$[I][J] - q^{f_{I,J}}[J][I] = \sum_{(L,M)\in S\cap E_{I,J}} k_{L,M}[L][M] + \sum_{(L,M)\in T\cap E_{I,J}} k_{L,M}[L][M]. \qquad (8)$$

To each pair $(I, J) \in \Pi\times\Pi$ and each equation as in (8), we attach the integer $\max\{{\rm rk}L, (L, M) \in T\cap E_{I,J}, k_{L,M} \neq 0\}$ if $\{(L, M) \in T\cap E_{I,J}, k_{L,M} \neq 0\} \neq \emptyset$ and 0 otherwise. (The definition of the rank of an element in a poset was given just after Remark 1.1.2.) In addition, for $t \in \mathbb{N}$, we say that a pair (I, J) is of *type* t if there exists an equation as in (8) for which the attached integer is less than or equal to t. With this vocabulary, our aim is to show that each pair $(I, J) \in \Pi \times \Pi$ is of type 0. We will prove the following statement: for each $t \in \mathbb{N}$, if $(I, J) \in \Pi \times \Pi$ is of type t, then (I, J) is of type 0; for this, we proceed by induction on t. The result is trivial for $t = 0$. Now, assume that the statement is true for pairs of type s for some integer $s \geq 0$. Let $(I, J) \in \Pi \times \Pi$ be a pair of type $s + 1$. Then there exists $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in \mathbb{C}$ for $(L, M) \in (S \cap E_{I,J}) \cup (T \cap E_{I,J})$ such that $[I][J] - q^{f_{I,J}}[J][I] = \sum_{(L,M)\in S\cap E_{I,J}} k_{L,M}[L][M] + \sum_{(L,M)\in T\cap E_{I,J}} k_{L,M}[L][M]$, with in addition, ${\rm rk}L \leq s + 1$ for all $(L, M) \in T \cap E_{I,J}$ such that $k_{L,M} \neq 0$. By applying Proposition 3.4.2 to any pair $(L, M) \in T \cap E_{I,J}$ such that $k_{L,M} \neq 0$, we see that (L, M) has type s; and so by the inductive hypothesis, (L, M) has type 0. It follows that we have an equation $[L][M] = q^{f_{L,M}}[M][L] + \sum_{(L_1,M_1)\in S\cap E_{L,M}} k_{L_1,M_1}[L_1][M_1]$. Substituting this new expression in $\sum_{(L,M)\in T\cap E_{I,J}} k_{L,M}[L][M]$ gives us a new expression of $[I][J] - q^{f_{I,J}}[J][I]$ as in (8) but where the second summand in the right hand side term is zero, that is: (I, J) is of type zero. This finishes the proof by induction. ∎

Theorem 3.4.4 – *Let A be a commutative domain and let u be any invertible element of A. Let $I, J \in \Pi$; then there exists a relation*

$$[I][J] - u^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M],$$

in $\mathcal{O}_u(G_{m,n}(A))$, where the sum is taken over pairs (L, M) of elements of Π such that $L <_{\rm st} I, J$ and $L \leq_{\rm st} M$ and where $f_{I,J} \in \mathbb{Z}$ and $k_{L,M} \in A$.

Proof. Let $\mathcal{A} = \mathbb{Z}[t^{\pm 1}]$. Choose $q \in \mathbb{C}$ transcendental over $\mathbb{Q}$. There is an injective ring homomorphism $\mathcal{A} \longrightarrow \mathbb{C}$ such that $1 \mapsto 1$ and $t \mapsto q$. From Remark 3.1.5, it follows that there is an injective ring homomorphism $f : \mathcal{O}_t(G_{m,n}(\mathcal{A})) \longrightarrow \mathcal{O}_q(G_{m,n}(\mathbb{C}))$ such that $1 \mapsto 1$, $t \mapsto q$ and, for all $I \in \Pi$, $[I] \mapsto [I]$. Now, consider $I, J \in \Pi$; according to Theorem 3.4.3, there exists $f_{I,J} \in \mathbb{Z}$ and, for each $(L, M) \in \Pi$ such that $L \leq_{\rm st} M$, $L <_{\rm st} I, J$, a complex number $k_{L,M}$ such that

$$[I][J] - q^{f_{I,J}}[J][I] = \sum k_{L,M}[L][M], \qquad (9)$$

in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$. On the other hand, by Proposition 3.2.5, we can express the element $[I][J]-t^{f_{I,J}}[J][I]$ of $\mathcal{O}_t(G_{m,n}(\mathcal{A}))$ as a linear combination of standard monomials in $\mathcal{O}_t(G_{m,n}(\mathcal{A}))$: $[I][J]-t^{f_{I,J}}[J][I]=\sum z_{L,M}[L][M]$ where the sum is taken over all pairs (L,M) of elements of Π such that $L\leq_{\rm st} M$ and $z_{L,M}\in\mathcal{A}$ for each such pair. Now, applying f to this expression gives $[I][J]-q^{f_{I,J}}[J][I]=\sum f(z_{L,M})[L][M]$ in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ from which it follows, by using relation (9), the linear independence of standard monomials in $\mathcal{O}_q(G_{m,n}(\mathbb{C}))$ and the injectivity of f that $z_{L,M}$ is zero whenever $L\not<_{\rm st} I,J$. Hence, we end up with a relation

$$[I][J]-t^{f_{I,J}}[J][I]=\sum z_{L,M}[L][M], \tag{10}$$

where the sum is taken over all pairs (L,M) of elements of Π such that $L\leq_{\rm st} M$, $L<_{\rm st} I,J$ and $z_{L,M}\in\mathcal{A}$ for each such pair. Now, there is a natural ring homomorphism $\mathcal{A}\longrightarrow A$ such that $1\mapsto 1$ and $t\mapsto u$. Using Remark 3.1.5 again, we get a ring homomorphism $g:\mathcal{O}_t(G_{m,n}(\mathcal{A}))\longrightarrow\mathcal{O}_u(G_{m,n}(A))$ such that $1\mapsto 1$, $t\mapsto u$ and $[I]\mapsto[I]$ for all $I\in\Pi$. It is now enough to apply g to the relations (10) to finish the proof. ∎

We are now in position to prove the main theorem of this section.

Theorem 3.4.5 – *Let $\mathbb{k}$ be a field, q be a non-zero element of $\mathbb{k}$ and m,n be positive integers such that $m\leq n$. Then $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is a quantum graded A.S.L. on the poset $(\Pi_{m,n},<_{\rm st})$.*

Proof. Recall that $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is an $\mathbb{N}$-graded ring generated by the elements $[I]$, $I\in\Pi_{m,n}$ which have degree one. By Proposition 3.2.5, the set of standard monomials is a free family. It remains to use Theorems 3.3.9 and 3.4.4 to conclude. ∎

Corollary 3.4.6 – *Let $\mathbb{k}$ be a field, q be a non-zero element of $\mathbb{k}$ and m,n be positive integers such that $m\leq n$. For $\gamma\in\Pi_{m,n}$, the quantum Schubert variety $\mathcal{O}_q(G_{m,n}(\mathbb{k}))_\gamma$ is a quantum graded A.S.L. on the poset $(\Pi_{m,n}\setminus\Omega_\gamma,<_{\rm st})$, where $\Omega_\gamma=\{\pi\in\Pi_{m,n}\,|\,\pi\not\geq_{\rm st}\gamma\}$.*

Proof. For $\gamma\in\Pi_{m,n}$, $\mathcal{O}_q(G_{m,n}(\mathbb{k}))_\gamma=\mathcal{O}_q(G_{m,n}(\mathbb{k}))/\langle\Omega_\gamma\rangle$. Since Ω_γ is a $\Pi_{m,n}$-ideal, the result follows from Theorem 3.4.5 and Corollary 1.2.5. ∎

3.5 Quantum determinantal rings are quantum graded A.S.L.

Throughout this subsection, $\mathbb{k}$ denotes a field.

In this subsection, we want to derive the fact that quantum determinantal rings are quantum graded A.S.L. from Theorem 3.4.5. For this, we need to relate the ring of quantum matrices and quantum grassmannians, and this is done using a *dehomogenisation map* from [KLR] the construction of which we now recall.

From [KLR; Lemma 1.5], we know that $[n+1,\dots,n+m]$ is a normal element in $\mathcal{O}_q(G_{m,m+n}(\mathbb{k}))$. Hence, we may localise $\mathcal{O}_q(G_{m,m+n}(\mathbb{k}))$ with respect to the set of powers of $[n+1,\dots,n+m]$; this localisation is denoted $\mathcal{O}_q(G_{m,m+n}(\mathbb{k}))[[n+1,\dots,n+m]^{-1}]$. For convenience, if $I=\{i_1<\cdots<i_m\}$ with $1\leq i_1<\cdots<i_m\leq m+n$, we put $\{I\}=[I][n+1,\dots,n+m]^{-1}\in\mathcal{O}_q(G_{m,m+n}(\mathbb{k}))[[n+1,\dots,n+m]^{-1}]$. Now, let ϕ be the automorphism of $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$ defined by $\phi(X_{ij})=q^{-1}X_{ij}$ for $1\leq i\leq m$ and $1\leq j\leq n$. From [KLR; Corollary 4.4], we know that there is an isomorphism of $\mathbb{k}$-algebras

$$D_{m,n}\;:\;\mathcal{O}_q(M_{m,n}(\mathbb{k}))[y,y^{-1};\phi]\;\longrightarrow\;\mathcal{O}_q(G_{m,m+n}(\mathbb{k}))[[n+1,\dots,n+m]^{-1}]$$

defined by $D_{m,n}(X_{ij}) = \{\{j, n+1, \dots, n+\widehat{m+1}-i, \dots, n+m\}\}$ and $D_{m,n}(y) = [n+1, \dots, n+m]$. (Here $\{j, n+1, \dots, n+\widehat{m+1}-i, \dots, n+m\}$ stands for the (ordered) set obtained from $\{j, n+1, \dots, n+m\}$ by removing $n+m+1-i$.)

Recall from Subsection 3.1 the definition of $\Delta_{m,n}$. In addition, to any pair $(I,J) \in \Delta_{m,n}$, with $I = \{i_1 < \cdots < i_t\}$, $J = \{j_1 < \cdots < j_t\}$, $1 \le t \le m$, we associate the index set $K_{(I,J)} \in \Pi_{m,n+m}$ obtained by ordering the elements of the set $\{j_1, \dots, j_t, n+1, \dots, n+m\} \setminus \{n+m+1-i_1, \dots, n+m+1-i_t\}$.

Lemma 3.5.1 – *We keep the above notation. Let $(I,J) \in \Delta_{m,n}$ with $I = \{i_1 < \cdots < i_t\} \subseteq \{1, \dots, m\}$, $J = \{j_1 < \cdots < j_t\} \subseteq \{1, \dots, n\}$ and $1 \le t \le m$. Then*

$$D_{m,n}([I|J]) = \{K_{(I,J)}\} = [K_{(I,J)}][\{n+1, \dots, n+m\}]^{-1}.$$

Proof. The proof is by induction on t, the case $t = 1$ being nothing but the definition of $D_{m,n}$. Now, let t be an integer such that $1 \le t < m$ and assume that the result is true for $t \times t$ minors. We consider $(I,J) \in \Delta_{m,n}$ with $I = \{i_1 < \cdots < i_{t+1}\} \subseteq \{1, \dots, m\}$, $J = \{j_1 < \cdots < j_{t+1}\} \subseteq \{1, \dots, n\}$ and, for $1 \le k \le t+1$, we let I_k (respectively J_k) be the index set obtained from I (respectively J) by removing the k-th entry. Then, by [PW; Corollary 4.4.4], we have

$$[I|J] = \sum_{k=1}^{t+1} (-q)^{t+1-k} [I_{t+1}|J_k] X_{i_{t+1}, j_k}.$$

Hence, by using the inductive hypothesis and [KLR; Lemma 1.5], we get

$$\begin{aligned} D_{m,n}([I|J]) &= \sum_{k=1}^{t+1} (-q)^{t+1-k} \{K_{(I_{t+1}, J_k)}\}\{K_{(i_{t+1}, j_k)}\} \\ &= q \left(\sum_{k=1}^{t+1} (-q)^{t+1-k} [K_{(I_{t+1}, J_k)}][K_{(i_{t+1}, j_k)}] \right) [n+1, \dots, n+m]^{-2}. \end{aligned}$$

Now, by applying [KLR; Theorem 2.5] in $\mathcal{O}_q(G_{m,m+n}(\Bbbk))$ with $K = (J \cup \{n+1, \dots, n+m\}) \setminus \{m+n+1-i_1, \dots, m+n+1-i_t\}$, $J_1 = \emptyset$ and $J_2 = \{n+1, \dots, n+m\} \setminus \{n+m+1-i_{t+1}\}$ we obtain the quantum Plücker relation

$$\sum_{k=1}^{t+1} (-q)^{m-(k-1)} [K_{(I_{t+1}, J_k)}][K_{(i_{t+1}, j_k)}] + (-q)^{m-t-1} [K_{(I,J)}][n+1, \dots, n+m] = 0.$$

(Notice that the first summand corresponds to the case where $K'' \subseteq J$ while the second corresponds to the case where $K'' = \{m+n+1-i_{t+1}\}$.) We may rewrite this last relation as

$$q \sum_{k=1}^{t+1} (-q)^{t+1-k} [K_{(I_{t+1}, J_k)}][K_{(i_{t+1}, j_k)}] = [K_{(I,J)}][n+1, \dots, n+m].$$

Hence, we get

$$D_{m,n}([I|J]) = [K_{(I,J)}][n+1, \dots, n+m]^{-1} = \{K_{(I,J)}\}.$$

The proof is complete. ∎

On the set $\Delta_{m,n}$ introduced above, we put a partial order that we also denote by $\leq_{\mathrm{st}}$. Let u, v be integers such that $1 \leq u, v \leq m$ and let (I, J) and (K, L) be an index pairs with $I = \{i_1 < \cdots < i_u\} \subseteq \{1, \dots, m\}$, $J = \{j_1 < \cdots < j_u\} \subseteq \{1, \dots, n\}$, $K = \{k_1 < \cdots < k_v\} \subseteq \{1, \dots, m\}$, $L = \{l_1 < \cdots < l_v\} \subseteq \{1, \dots, n\}$. We define $\leq_{\mathrm{st}}$ as follows:

$$(I, J) \leq_{\mathrm{st}} (K, L) \Longleftrightarrow \begin{cases} u \geq v, \\ i_s \leq k_s \quad \text{for} \quad 1 \leq s \leq v, \\ j_s \leq l_s \quad \text{for} \quad 1 \leq s \leq v. \end{cases}$$

The following combinatorial lemma relates the orders on $\Delta_{m,n}$ and $\Pi_{m,n+m}$. Notice first that the map

$$\begin{array}{cccc} \delta_{m,n} & : & \Delta_{m,n} & \longrightarrow & \Pi_{m,n+m} \setminus \{\{n+1 < \cdots < n+m\}\} \\ & & (I, J) & \mapsto & K_{(I,J)} \end{array}$$

is bijective.

Lemma 3.5.2 – *For $(I, J), (K, L) \in \Delta_{m,n}$, one has $(I, J) \leq_{\mathrm{st}} (K, L)$ if and only if $\delta_{m,n}((I, J)) \leq_{\mathrm{st}} \delta_{m,n}((K, L))$. Hence, $\delta_{m,n}$ is a bijective map of posets between $(\Delta_{m,n}, \leq_{\mathrm{st}})$ and $(\Pi_{m,n+m} \setminus \{\{n+1 < \cdots < n+m\}\}, \leq_{\mathrm{st}})$.*

Proof. See the proof of [BV; Lemma 4.9]. ∎

Theorem 3.5.3 – *Let $\Bbbk$ be a field, m, n be positive integers such that $m \leq n$ and q be any element of $\Bbbk^*$. Then $\mathcal{O}_q(M_{m,n}(\Bbbk))$ is a quantum graded A.S.L. on the poset $(\Delta_{m,n}, \leq_{\mathrm{st}})$.*

Proof. Clearly, $\mathcal{O}_q(M_{m,n}(\Bbbk))$ is an $\mathbb{N}$-graded $\Bbbk$-algebra and the elements of $\Delta_{m,n}$ are homogeneous with positive degree and generate $\mathcal{O}_q(M_{m,n}(\Bbbk))$ as a $\Bbbk$-algebra. Recall, from Lemma 3.5.1, that $D_{m,n}([I|J]) = [\delta_{m,n}((I, J))][n+1, \dots, n+m]^{-1}$, for any index pair $(I, J) \in \Delta_{m,n}$. As a consequence, if $(I_1, J_1) \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} (I_t, J_t)$ is an increasing sequence of elements of $\Delta_{m,n}$, then there exists $\alpha \in \mathbb{Z}$ such that $D_{m,n}([I_1|J_1] \dots [I_t|J_t]) = q^{\alpha}[\delta_{m,n}((I_1, J_1))] \dots [\delta_{m,n}((I_t, J_t))][n+1, \dots, n+m]^{-t}$ (here we use [KLR; Lemma 1.5]). From this observation, and using Lemma 3.5.2 and the linear independence of standard monomials in $\mathcal{O}_q(G_{m,m+n}(\Bbbk))$, it follows easily that the set of standard monomials in $\mathcal{O}_q(M_{m,n}(\Bbbk))$ form a free family. Now, consider two incomparable elements (I, J) and (K, L) of $\Delta_{m,n}$. By Lemma 3.5.2, $\delta_{m,n}((I, J))$ and $\delta_{m,n}((K, L))$ are incomparable elements of $\Pi_{m,n+m}$. Then, by Theorem 3.3.9, there exist $K_i, L_i \in \Pi_{m,n+m}$ and $k_i \in \Bbbk$, for $1 \leq i \leq t$, such that

$$[\delta_{m,n}((I, J))][\delta_{m,n}((K, L))] = \sum_{i=1}^{t} k_i [K_i][L_i]$$

with, in addition $K_i \leq_{\mathrm{st}} L_i$ and $K_i <_{\mathrm{st}} \delta_{m,n}((I, J)), \delta_{m,n}((K, L))$, for $1 \leq i \leq t$. If we put $M = \{n+1, \dots, n+m\} \in \Pi_{m,n+m}$, by [KLR; Lemma 1.5], we then have

$$[\delta_{m,n}((I, J))][M]^{-1}[\delta_{m,n}((K, L))][M]^{-1} = \sum_{i=1}^{t} k'_i [K_i][M]^{-1}[L_i][M]^{-1},$$

where $k'_i \in \Bbbk$, for $1 \leq i \leq t$. At this point, it is worth mentioning that no K_i can equal M since $K_i <_{\mathrm{st}} \delta_{m,n}((I, J)), \delta_{m,n}((K, L))$ and M is maximal in $\Pi_{m,n+m}$. In contrast, it is possible for an L_i to equal M; so that in the above equation, a term $[L_i][M]^{-1}$ might very well equal 1. Now, by applying $D_{m,n}^{-1}$ to this last equation we obtain an expression for $[I|J][K|L]$ as a linear combination of terms of the form $[E|F][G|H]$ or $[E|F]$, where $(E,F),(G,H) \in \Delta_{m,n}$, $(E,F) \leq_{\rm st} (G,H)$ and $(E,F) <_{\rm st} (I,J),(K,L)$. This shows that condition (4) in the definition of a quantum graded A.S.L. is satisfied. Finally, consider two elements (I,J) and (K,L) in $\Delta_{m,n}$. By Theorem 3.4.4 there exist $f \in \mathbb{Z}$, elements $K_i, L_i \in \Pi_{m,n+m}$ and $k_i \in \Bbbk$, for $1 \leq i \leq t$, such that

$$[\delta_{m,n}((I,J))][\delta_{m,n}((K,L))] - q^f[\delta_{m,n}((K,L))][\delta_{m,n}((I,J))] = \sum_{i=1}^{t} k_i[K_i][L_i]$$

with, in addition, $K_i \leq_{\rm st} L_i$ and $K_i <_{\rm st} \delta_{m,n}((I,J)), \delta_{m,n}((K,L))$, for $1 \leq i \leq t$. Now, applying the same procedure as above, we end up with an expression for $[I|J][K|L] - q^g[K|L][I|J]$ (for some $g \in \mathbb{Z}$) as a linear combination of terms of the form $[E|F][G|H]$ or $[E|F]$, where $(E,F),(G,H) \in \Delta_{m,n}$ with $(E,F) \leq_{\rm st} (G,H)$ and $(E,F) <_{\rm st} (I,J),(K,L)$. This shows that condition (5) in the definition of a quantum graded A.S.L. is satisfied. ■

For $1 \leq t \leq m$, we denote by $\mathcal{I}_t$ the ideal of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the $t \times t$ quantum minors. On the other hand, for $t \geq 2$, we denote by Ω_t the $\Delta_{m,n}$-ideal defined by

$$\Omega_t = \{(I,J) \in \Delta_{m,n} \,|\, (I,J) \not\geq_{\rm st} (\{1,\ldots,t-1\},\{1,\ldots,t-1\})\},$$

and we put $\Omega_1 = \Delta_{m,n}$. For $1 \leq t \leq m$, it is easy to check that Ω_t is just the set of index pairs (I,J) such that $t \leq |I| = |J| \leq m$. Hence, $\mathcal{I}_t = \langle \Omega_t \rangle$, by using the well known Laplace expansions for quantum minors (see [PW; Corollary 4.4.4]).

Corollary 3.5.4 – *Let $\Bbbk$ be a field, m,n be positive integers such that $m \leq n$ and q be any element of $\Bbbk^*$. For $1 \leq t \leq m$, the quantum determinantal ring $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ is a quantum graded A.S.L. on the poset $\Delta_{m,n} \setminus \Omega_t$.*

Proof. According to the comments above, this is an immediate consequence of Theorem 3.5.3 and Corollary 1.2.5. ■

4 Quantum determinantal rings and quantum grassmannians are AS-Cohen-Macaulay.

In this section, we reach our main aim. We prove that quantum determinantal rings and quantum Grassmannians are AS-Cohen-Macaulay. Once this is done, we determine which of these rings are AS-Gorenstein, by means of their Hilbert series.

Theorem 4.1 – *Let $\Bbbk$ be a field, m,n be positive integers such that $m \leq n$ and q be any element of $\Bbbk^*$. For $1 \leq t \leq m$, the quantum determinantal ring $\mathcal{O}_q(M_{m,n}(\Bbbk))/\mathcal{I}_t$ is AS-Cohen-Macaulay.*

Proof. The case $t = 1$ is trivial, thus we assume that $t \geq 2$. First notice that $\Delta_{m,n} \setminus \Omega_t = \{(I,J) \in \Delta_{m,n} \,|\, (I,J) \geq_{\rm st} (\{1,\ldots,t-1\},\{1,\ldots,t-1\})\}$ which is a distributive lattice by [BV; Theorem 5.3] and hence a wonderful poset (see [BV; p. 58]). Thus, the result follows from Theorem 2.2.3 in conjunction with Corollary 3.5.4. ■

Theorem 4.2 – *Let $\Bbbk$ be a field, q be a non-zero element of $\Bbbk$ and m,n be positive integers such that $m \leq n$. For $\gamma \in \Pi_{m,n}$, the quantum Schubert variety $\mathcal{O}_q(G_{m,n}(\Bbbk))_\gamma$ is AS-Cohen-Macaulay. In particular, the quantum Grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is AS-Cohen-Macaulay.*

Proof. Recall that $\Pi_{m,n}$ has a single minimal element. Hence the quantum grassmannian is a special case of a quantum Schubert variety. Let $\gamma \in \Pi_{m,n}$. From Corollary 3.4.6, the quantum Schubert variety $\mathcal{O}_q(G_{m,n}(\mathbb{k}))_\gamma$ is a quantum graded A.S.L. on the poset $(\Pi_{m,n} \setminus \Omega_\gamma, <_{\rm st})$, where $\Omega_\gamma = \{\pi \in \Pi_{m,n} \,|\, \pi \not\geq_{\rm st} \gamma\}$. On the other hand, by [BV; Theorem 5.4], the poset $(\Pi_{m,n} \setminus \Omega_\gamma, <_{\rm st})$ is a distributive lattice and hence a wonderful poset (see [BV; p. 58]). Thus, the result follows from Theorem 2.2.3. ■

We now show that quantum grassmannians are AS-Gorenstein and determine which quantum determinantal rings are AS-Gorenstein. This can be easily deduced from the above results using the criterion given in [JZ; Theorem 6.2].

Recall that, if $A = \oplus_{i\in\mathbb{N}} A_i$ is a noetherian $\mathbb{N}$-graded connected $\mathbb{k}$-algebra, then it is locally finite and we can speak of its Hilbert series, namely

$$H_A(t) = \sum_{i\geq 0} (\dim_{\mathbb{k}} A_i) t^i \in \mathbb{Q}[[t]].$$

If, in addition, A is AS-Cohen-Macaulay, a domain and has enough normal elements, then [JZ; Theorem 6.2] shows that the fact that A is AS-Gorenstein or not can be read off from its Hilbert series. This is the criterion we will use to prove the following theorems.

Remark 4.3 – Let R be a noetherian commutative ring. Then, R is said to be Gorenstein if, for every maximal ideal $\mathfrak{p}$ of R, the local ring $R_\mathfrak{p}$ is of finite self-injective dimension. If, moreover, the Krull dimension of R is finite, then R is Gorenstein if and only if it has finite self-injective dimension (see [B; Theorem and definition §1]). Now, let $A = \oplus_{i\in\mathbb{N}} A_i$ be a commutative noetherian $\mathbb{N}$-graded connected $\mathbb{k}$-algebra. Since, clearly, A has finite Krull dimension, A is Gorenstein if and only if A has finite (self-)injective dimension. On the other hand it is clear that A has enough normal elements, since A is commutative. Thus, from Remark 2.1.9, it follows that A is AS-Gorenstein if and only if it is Gorenstein.

Theorem 4.4 – *Let $\mathbb{k}$ be a field, q be a non-zero element of $\mathbb{k}$ and m, n be positive integers such that $m \leq n$. Then the quantum Grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is AS-Gorenstein.*

Proof. The quantum grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ is a domain since it is a subalgebra of the domain $\mathcal{O}_q(M_{m,n}(\mathbb{k}))$. It is AS-Cohen-Macaulay by Theorem 4.2. And, it has enough normal elements since it is a quantum graded A.S.L. (see Theorem 3.4.5 and Remark 2.1.4).

On the other hand, it follows at once from Proposition 3.2.5 that for all $q \in \mathbb{k}^*$, the Hilbert series of $\mathcal{O}(G_{m,n}(\mathbb{k})) := \mathcal{O}_1(G_{m,n}(\mathbb{k}))$ and $\mathcal{O}_q(G_{m,n}(\mathbb{k}))$ coincide. Hence, using [JZ; Theorem 6.2], it is enough to prove that the (usual) homogeneous coordinate ring $\mathcal{O}(G_{m,n}(\mathbb{k}))$ is AS-Gorenstein. However, this follows at once from Remark 4.3 and [BV; Corollary 8.13]. ■

Theorem 4.5 – *Let $\mathbb{k}$ be a field, m, n be positive integers such that $m \leq n$ and q be any element of $\mathbb{k}^*$. For $1 \leq t \leq m$, the quantum determinantal ring $\mathcal{O}_q(M_{m,n}(\mathbb{k}))/\mathcal{I}_t$ is AS-Gorenstein if and only if $t = 1$ or $m = n$.*

Proof. The quantum determinantal rings $\mathcal{O}_q(M_{m,n}(\mathbb{k}))/\mathcal{I}_t$ are domains by [GL; Corollary 2.6]. They are AS-Cohen-Macaulay by Theorem 4.1, and they have enough normal elements since they are quantum graded A.S.L. (see Corollary 3.5.4 and Remark 2.1.4).

On the other hand, it follows at once from Theorem 3.5.4 and Proposition 1.1.5 that, for all $q \in \mathbb{k}^*$, the Hilbert series of $\mathcal{O}(M_{m,n}(\mathbb{k}))/\mathcal{I}_t := \mathcal{O}_1(M_{m,n}(\mathbb{k}))/\mathcal{I}_t$ and $\mathcal{O}_q(M_{m,n}(\mathbb{k}))/\mathcal{I}_t$ coincide. Hence, using [JZ; Theorem 6.2], it is enough to prove that the (usual) coordinate ring $\mathcal{O}(M_{m,n}(\Bbbk))/\mathcal{I}_t$ is AS-Gorenstein if and only if $t=1$ or $m=n$. However, this follows at once from Remark 4.3 and [BV; Corollary 8.9]. ∎

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