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An algorithm to compute bases and representation matrices for SL_{n+1}-representations

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Abstract

We construct a basis for irreducible representations of the complex Lie algebra sL_{n+1}. The basis is obtained by applying certain monomials in the enveloping algebra of SL_{n+1} to a highest weight vector. In addition we provide a straightening law which can be used to define an algorithm to compute the representation matrix of elements of sL_{n+1} with respect to this basis. The method can be generalized to all complex simple Lie algebras with a simply laced root system. © 1997 Elsevier Science B.V.

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1. Introduction

Most of the algorithms known in representation theory of complex simple Lie algebras deal with the combinatorial data of representations: weight space multiplicities, tensor product multiplicities, characters, etc. The algorithm presented here is a constructive procedure to calculate representation matrices for arbitrary irreducible representations of the group SL_{n+1}. There are different ways of approaching this problem. One possibility would be to use the classical theory of Young tableaux. The algorithm presented here has the advantage that it can be generalized directly to the groups $Spin_{2m}$, E_6, E_7 and E_8 (see Section 10). In fact, it is very likely that the algorithm can be generalized to all semisimple Lie algebras.

Let $\mathfrak{sl}_{n+1} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the usual decomposition in upper triangular, diagonal and lower triangular matrices, and set $N := \dim \mathfrak{n}^-$. We construct a basis of the enveloping algebra $U(\mathfrak{n}^-)$ as follows: Let $\alpha_1, \ldots, \alpha_n$ be the simple roots, and fix root vectors $Y_i \in \mathfrak{g}_{-\alpha_i}$ and $X_i \in \mathfrak{g}_{\alpha_i}$. For

$$(\boldsymbol{a}) = (a_1, b_2, b_1, c_3, c_2, c_1, \ldots, d_n, \ldots, d_1) \in \mathbb{N}^N$$

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denote by $Y^{(a)}$ the monomial

$$Y^{(a)} := Y_1^{(a_1)}\,(Y_2^{(b_2)}Y_1^{(b_1)})\,(\ldots)\,(Y_n^{(d_n)}\cdots Y_2^{(d_2)}Y_1^{(d_1)}).$$

We determine a subset $\mathscr{S} \subset \mathbb{N}^N$ such that the monomials $\mathbb{B} := \{Y^{(a)} \mid (a) \in \mathscr{S}\}$ form a basis of $U(\mathfrak{n}^-)$. The basis $\mathbb{B}$ has the following nice "universal" property: Let v_λ be a highest weight vector in $V(\lambda)$. We associate to λ a subset $\mathscr{S}(\lambda) \subseteq \mathscr{S}$ such that the following set $\mathbb{B}(\lambda)$ forms a basis of $V(\lambda)$:

$$\mathbb{B}(\lambda) := \{v^{(a)} \in V(\lambda) \mid v^{(a)} = Y^{(a)}v_\lambda,\ (a) \in \mathscr{S}(\lambda)\}.$$

We would like to point out that the construction holds over $\mathbb{Z}$, i.e. $\mathbb{B}(\lambda)$ is a basis of an admissible lattice $V_{\mathbb{Z}}(\lambda) \subset V(\lambda)$. As a consequence, the algorithm works over any field of characteristic zero, and, by reduction mod p, also for certain representations of $\mathfrak{sl}_{n+1}$ in positive characteristic.

The algorithm to compute the representation matrices is divided into two steps: Fix $(a) \in \mathscr{S}(\lambda)$. The first step consists of a procedure that expresses $X_i^{(m)}v^{(a)}$ respectively $Y_i^{(m)}v^{(a)}$ as integral linear combinations of vectors of the form $v^{(b)}$. The second step is a *straightening* procedure: It expresses a vector $v^{(b)} \notin \mathbb{B}(\lambda)$ as an integral linear combination:

$$v^{(b)} = \sum c_{(d)}v^{(d)} \quad \text{where } (d) < (b).$$

By repeating the procedure if necessary, this algorithm yields an expression of $v^{(b)}$ as an integral linear combination of the elements of $\mathbb{B}(\lambda)$, and it yields hence an expression for the matrix coefficients:

$$X_i^{(m)}v^{(a)} = \sum_{(c)\in\mathscr{S}(\lambda)} r_{(a,c)}v^{(c)} \quad \text{and} \quad Y_i^{(m)}v^{(a)} = \sum_{(d)\in\mathscr{S}(\lambda)} s_{(a,d)}v^{(d)}.$$

The indexing system is related to the reduced decomposition of the longest word in the Weyl group: $w_0 = s_1(s_2s_1)(s_3s_2s_1)(\ldots)(s_n\ldots s_2s_1)$. In order to give a nice combinatorial description of the set $\mathscr{S}(\lambda)$, we construct a natural bijection between $\mathscr{S}(\lambda)$ and the set of Gelfand–Tsetlin patterns of type λ.

Though quantum groups are never used in this article, the work has been influenced by the relationship between the crystal graph, good bases and the path model (see [4–12]). The proofs are in fact completely elementary, we need only standard results about P-B-W bases and the Gelfand–Tsetlin patterns. In [10], we construct generalizations of Gelfand–Tsetlin patterns (and associated bases) for arbitrary simple Lie algebras. These results strongly suggest that the algorithm can be reformulated for arbitrary semisimple Lie algebras.

2. P-B-W bases and admissible lattices

The aim of this section is to recall some basic facts and to fix the notation. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ into the direct sum of strictly upper triangular, diagonal, and strictly lower triangular matrices. Denote by U, U^+ and U^- the enveloping algebras of $\mathfrak{g}$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$. Let the decomposition of the root system $\Phi = \Phi^+ \cup -\Phi^+$ be such that

$$\mathfrak{n}^+ := \bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta.$$

We fix a Chevalley basis of $\mathfrak{g}: X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$ for $\beta \in \Phi^+$, and $H_\alpha \in \mathfrak{h}$ for α simple. Let $U_\mathbb{Z}, U_\mathbb{Z}^+$ and $U_\mathbb{Z}^-$ be the corresponding $\mathbb{Z}$-forms of the enveloping algebras. We use the following abbreviations:

$$Y_\beta^{(k)} := \frac{Y_\beta^k}{k!}, \quad X_\beta^{(k)} := \frac{X_\beta^k}{k!}, \quad \binom{H_\alpha}{k} := \frac{H_\alpha(H_\alpha - 1)\cdots(H_\alpha - k + 1)}{k!}.$$

Fix an ordering $\{\gamma_1, \ldots, \gamma_N\}$ of the positive roots. For $(\boldsymbol{n}) \in \mathbb{N}^N$ we set:

$$X^{(\boldsymbol{n})} := X_{\gamma_1}^{(n_1)} \cdots X_{\gamma_N}^{(n_N)}, \qquad Y^{(\boldsymbol{n})} := Y_{\gamma_1}^{(n_1)} \cdots Y_{\gamma_N}^{(n_N)}.$$

Fix an ordering $\{\alpha_1, \ldots, \alpha_n\}$ of the simple roots. For $(\boldsymbol{k}) \in \mathbb{N}^n$ we set:

$$H^{(\boldsymbol{k})} := \binom{H_{\alpha_1}}{k_1} \cdots \binom{H_{\alpha_n}}{k_n}.$$

Recall that the monomials $Y^{(\boldsymbol{m})} H^{(\boldsymbol{k})} X^{(\boldsymbol{n})}$ form a Poincaré–Birkhoff–Witt basis of $U_\mathbb{Z}$, and the monomials $X^{(\boldsymbol{n})}$ and $Y^{(\boldsymbol{m})}$ form a P-B-W basis of $U_\mathbb{Z}^+$ respectively $U_\mathbb{Z}^-$. For a dominant weight λ let $V(\lambda)$ be the corresponding irreducible representation of highest weight λ (over $\mathbb{C}$). Fix a highest weight vector v_λ in $V(\lambda)$. We denote by $V_\mathbb{Z}(\lambda) \subset V(\lambda)$ the admissible lattice

$$V_\mathbb{Z}(\lambda) := U_\mathbb{Z}^- v_\lambda = U_\mathbb{Z} v_\lambda.$$

3. Some lemmas on commutation

Suppose $\mathfrak{g} \subset M_n(\mathbb{C})$ is the Lie algebra of a real Lie group $G \subset GL_n(\mathbb{C})$, and let $e: \mathfrak{g} \to G$ be the exponential map. The connection between the Lie bracket in $\mathfrak{g}$ and the multiplication in G is given by the Campbell–Hausdorff series:

Theorem 1 (Campbell–Hausdorff). *If $X, Y \in \mathfrak{g}$, then there exists an $\varepsilon > 0$ such that for all $r, s \in \mathbb{R}$, $0 < |r|, |s| < \varepsilon$:*

$$e^{rX} e^{sY} = e^{rX + sY + (rs/2)[X,Y] + (r^2 s/12)[X,[X,Y]] + (rs^2/12)[Y,[Y,X]] + \cdots}$$

Consider the real Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{R})$. Note that (by the choice of the basis) one can assume $X_\beta, Y_\beta, H_\alpha \in \mathfrak{sl}_{n+1}(\mathbb{R})$. The relations which we will derive hold a priori only over $\mathbb{R}$. But since the coefficients are all integers, these relations hold also in $U_\mathbb{Z}(\mathfrak{sl}_{n+1})$. Let $\alpha, \gamma \in \Phi^+$ be positive roots which form a basis of a root subsystem of Φ of type

A_2, and set $\beta := \alpha + \gamma$. Since $[Y_\alpha, [Y_\alpha, Y_\gamma]] = [Y_\gamma, [Y_\gamma, Y_\alpha]] = 0$ and $[Y_\alpha, Y_\beta] = [Y_\gamma, Y_\beta] = 0$, the Campbell–Hausdorff series implies:

$$e^{rY_\alpha} e^{sY_\gamma} = e^{rY_\alpha + sY_\gamma + (rs/2)Y_\beta}, \qquad e^{sY_\gamma} e^{rY_\alpha} = e^{rY_\alpha + sY_\gamma - (rs/2)Y_\beta}.$$

By multiplying the second equation by e^{rsY_β} we get:

$$e^{rY_\alpha} e^{sY_\gamma} = e^{sY_\gamma} e^{rY_\alpha} e^{rsY_\beta} \quad \text{and} \quad e^{-sY_\gamma} e^{rY_\alpha} e^{sY_\gamma} = e^{rY_\alpha} e^{rsY_\beta}.$$

for small $r, s \in \mathbb{R}$. By comparing the coefficients of $r^m s^n$ (respectively $r^{m+n} s^n$ for the second equation) on both sides, we get the following useful lemma on commutation:

Lemma 2. *Set* $M := \min\{m, n\}$, *then*

$$Y_\alpha^{(m)} Y_\gamma^{(n)} = \sum_{l=0}^{M} Y_\gamma^{(n-l)} Y_\alpha^{(m-l)} Y_\beta^{(l)}, \qquad Y_\alpha^{(m)} Y_\beta^{(n)} = \sum_{l=0}^{n} (-1)^l Y_\gamma^{(l)} Y_\alpha^{(m+n)} Y_\gamma^{(n-l)}.$$

In the following we use some binomial coefficient identities, we refer to [3] for the proofs. To have binomial coefficients also available for negative integers, we use the definition:

$$\binom{a}{b} := \lim_{t \to 0} \frac{\Gamma(a + 1 + t)}{\Gamma(b - a + 1 + t)\Gamma(b + 1 + t)}.$$

Let $f(x, y, z) : \{(x, y, z) \in \mathbb{N}^3 \mid z \leq x, y\} \to \mathbb{Z}$ be the function defined by

$$f(x, y, z) := (-1)^{x-z} \binom{y - z - 1}{x - z}$$

Lemma 3. *If* $m < n$, *then*

$$Y_\alpha^{(m)} Y_\gamma^{(n)} = \sum_{q=0}^{m} f(m, n, q) Y_\gamma^{(n-q)} Y_\alpha^{(m)} Y_\gamma^{(q)}.$$

Proof. By Lemma 2 we have

$$\begin{aligned} Y_\alpha^{(m)} Y_\gamma^{(n)} &= \sum_{l=0}^{m} \sum_{k=0}^{l} (-1)^k \binom{n + k - l}{k} Y_\gamma^{(n-l+k)} Y_\alpha^{(m)} Y_\gamma^{(l-k)} \\ &= \sum_{q=0}^{m} Y_\gamma^{(n-q)} Y_\alpha^{(m)} Y_\gamma^{(q)} \left(\sum_{k=0}^{m-q} (-1)^k \binom{n - q}{k} \right) \\ &= \sum_{q=0}^{m} f(m, n, q) Y_\gamma^{(n-q)} Y_\alpha^{(m)} Y_\gamma^{(q)}. \qquad \square \end{aligned}$$

In the same way one gets from $e^{sY_\gamma} e^{rY_\alpha} = e^{rY_\alpha} e^{sY_\gamma} e^{-rsY_\beta}$:

Lemma 4.

$$Y_\gamma^{(q)} Y_\alpha^{(p)} = \sum_{l=0}^{\min\{p,q\}} (-1)^l Y_\alpha^{(p-l)} Y_\gamma^{(q-l)} Y_\beta^{(l)},$$

$$Y_\gamma^{(n)} Y_\beta^{(m)} = \sum_{l=0}^{m} (-1)^{l+m} Y_\alpha^{(l)} Y_\gamma^{(m+n)} Y_\alpha^{(m-l)}.$$

For nonnegative integers a,b,c,x with $x \le b$, set

$$p(a,b,c,x) := \binom{a+c-b}{a-x}, \qquad q(a,b,c,x) := \binom{a+c-b}{c-x}.$$

Lemma 5.

$$p(a,b,c,x) = \sum_{k=0}^{\min\{a,b\}-x} (-1)^k \binom{a+c-(x+k)}{c} \binom{b-x}{k},$$

$$q(a,b,c,x) = \sum_{l=x}^{\min\{b,c\}} (-1)^{x+l} \binom{a+c-l}{a} \binom{b-x}{b-l}.$$

Proof. Suppose first $a \ge b$. Then

$$\sum_{k=0}^{b-x} (-1)^k \binom{a+c-x-k}{c} \binom{b-x}{k} = \sum_{k=0}^{b-x} (-1)^{k+b-x} \binom{a+c-b+k}{c} \binom{b-x}{k}$$
$$= \binom{a+c-b}{x+c-b} = p(a,b,c,x).$$

And if $a \le b$, then

$$\sum_{k=0}^{a-x} (-1)^k \binom{a+c-x-k}{c} \binom{b-x}{k} = \sum_{k=0}^{a-x} (-1)^{k+a-x} \binom{c+k}{c} \binom{b-x}{b-a+k}$$
$$= \binom{a+c-b}{x+c-b} = q(a,b,c,x).$$

The proof for the function $q(a,b,c,x)$ is similar. $\square$

Proposition 6.

$$Y_\gamma^{(a)} Y_\alpha^{(b)} Y_\gamma^{(c)} = \sum_{x=0}^{\min\{a,b\}} p(a,b,c,x) Y_\alpha^{(b-x)} Y_\gamma^{(a+c)} Y_\alpha^{(x)},$$

$$Y_\gamma^{(a)} Y_\alpha^{(b)} Y_\gamma^{(c)} = \sum_{x=0}^{\min\{b,c\}} q(a,b,c,x) Y_\alpha^{(x)} Y_\gamma^{(a+c)} Y_\alpha^{(b-x)}.$$

Remark 7. The coefficient $p(a,b,c,x)\neq 0$ unless $a+c\geq b$, $b\geq c$ and $0\leq x<b-c$. We have a similar effect in the second sum: the coefficient $q(a,b,c,x)\neq 0$ unless $a+c\geq b$, $b\geq a$ and $0\leq x<b-a$.

Proof. By Lemma 2 and 4 we have

$$
\begin{aligned}
Y_\gamma^{(a)}Y_\alpha^{(b)}Y_\gamma^{(c)} &= \left(\sum_{l=0}^{\min\{a,b\}}(-1)^l Y_\alpha^{(b-l)}Y_\gamma^{(a-l)}Y_\beta^{(l)}\right)Y_\gamma^{(c)}\\
&= \sum_{l=0}^{\min\{a,b\}}(-1)^l\binom{a+c-l}{c}Y_\alpha^{(b-l)}Y_\gamma^{(a+c-l)}Y_\beta^{(l)}\\
&= \sum_{l=0}^{\min\{a,b\}}(-1)^l\binom{a+c-l}{c}Y_\alpha^{(b-l)}\left(\sum_{k=0}^{l}(-1)^{k+l}Y_\alpha^{(k)}Y_\gamma^{(a+c)}Y_\alpha^{(l-k)}\right)\\
&= \sum_{l=0}^{\min\{a,b\}}\sum_{k=0}^{l}(-1)^k\binom{a+c-l}{c}\binom{b+k-l}{k}Y_\alpha^{(b+k-l)}Y_\gamma^{(a+c)}Y_\alpha^{(l-k)}\\
&= \sum_{x=0}^{\min\{a,b\}}p(a,b,c,x)Y_\alpha^{(b-x)}Y_\gamma^{(a+c)}Y_\alpha^{(x)}.
\end{aligned}
$$

We get similarly:

$$
\begin{aligned}
Y_\gamma^{(a)}Y_\alpha^{(b)}Y_\gamma^{(c)} &= Y_\gamma^{(a)}\left(\sum_{l=0}^{\min\{b,c\}}Y_\gamma^{(c-l)}Y_\alpha^{(b-l)}Y_\beta^{(l)}\right)\\
&= \sum_{l=0}^{\min\{b,c\}}\binom{a+c-l}{a}Y_\gamma^{(a+c-l)}Y_\beta^{(l)}Y_\alpha^{(b-l)}\\
&\ \sum_{l=0}^{\min\{b,c\}}\binom{a+c-l}{a}\left(\sum_{k=0}^{l}(-1)^{k+l}Y_\alpha^{(k)}Y_\gamma^{(a+c)}Y_\alpha^{(l-k)}\right)Y_\alpha^{(b-l)}\\
&= \sum_{l=0}^{\min\{b,c\}}\sum_{k=0}^{l}(-1)^{k+l}\binom{a+c-l}{a}\binom{b-k}{b-l}Y_\alpha^{(k)}Y_\gamma^{(a+c)}Y_\alpha^{(b-k)}\\
&= \sum_{x=0}^{\min\{b,c\}}q(a,b,c,x)Y_\alpha^{(x)}Y_\gamma^{(a+c)}Y_\alpha^{(b-x)}. \qquad \square
\end{aligned}
$$

Suppose $a \geq 0$, $b \geq c > m \geq y \geq 0$ and $a+c > b$. We set

$$\begin{aligned} g(a,b,c,m,y) &:= \frac{(m+1-y)}{(y-c)}\binom{a+c-b-1}{c-m-1}\binom{b-a-c}{m+1-y}, \\ h(a,b,c,m,y) &:= \frac{(b-m-y)}{(y-a)}\binom{a+c-b-1}{c-m-1}\binom{b-a-c}{b-m-y}. \end{aligned} \tag{1}$$

Proposition 8. *Let $a,b,c,r \in \mathbb{N}$ be such that $b \geq c$, $r \geq c$ and $a > r+b-2c$. Set $m := r-a-c+b$, then*

$$\begin{aligned} Y_\alpha^{(a)}Y_\gamma^{(b)}Y_\alpha^{(c)} &= \sum_{y=0}^{m} g(a,b,c,m,y)Y_\alpha^{(a+c-y)}Y_\gamma^{(b)}Y_\alpha^{(y)} \\ &\quad + \sum_{y=0}^{b-m-1} h(a,b,c,m,y)Y_\alpha^{(y)}Y_\gamma^{(b)}Y_\alpha^{(a+c-y)}. \end{aligned} \tag{2}$$

Proof. Note that $a > r+b-2c$ and $r \geq c$ implies $a+c > r+b-c \geq b$. Further, $a > r+b-2c$ implies $c > m$. Since $b \geq c$ and $c > m$, Proposition 6 implies that $Y_\alpha^{(a)}Y_\gamma^{(b)}Y_\alpha^{(c)}$ is equal to

$$\sum_{x=0}^{m} q(a,b,c,x)Y_\gamma^{(x)}Y_\alpha^{(a+c)}Y_\gamma^{(b-x)} + \sum_{x=m+1}^{c} q(a,b,c,x)Y_\gamma^{(x)}Y_\alpha^{(a+c)}Y_\gamma^{(b-x)}.$$

By Proposition 6, the first sum is equal to (note that $x \leq m < a+c$):

$$\begin{aligned} &\sum_{x=0}^{m}\sum_{y=0}^{x} q(a,b,c,x)p(x,a+c,b-x,y)Y_\alpha^{(a+c-y)}Y_\gamma^{(b)}Y_\alpha^{(y)} \\ &= \sum_{y=0}^{m}\left(\sum_{x=y}^{m} q(a,b,c,x)p(x,a+c,b-x,y)\right)Y_\alpha^{(a+c-y)}Y_\gamma^{(b)}Y_\alpha^{(y)} \\ &= \sum_{y=0}^{m}\left(\sum_{k=0}^{m-y}\binom{a+c-b}{c-y-k}\binom{b-a-c}{k}\right)Y_\alpha^{(a+c-y)}Y_\gamma^{(b)}Y_\alpha^{(y)} \\ &= \sum_{y=0}^{m} g(a,b,c,m,y)Y_\alpha^{(a+c-y)}Y_\gamma^{(b)}Y_\alpha^{(y)}. \end{aligned}$$

The second sum is equal to (note that $b-x \leq b < a+c$ and $q(a,b,c,x)=0$ for $x > c$):

$$\begin{aligned} &\sum_{x=m+1}^{c}\sum_{y=0}^{b-x} q(a,b,c,x)q(x,a+c,b-x,y)Y_\alpha^{(y)}Y_\gamma^{(b)}Y_\alpha^{(a+c-y)} \\ &= \sum_{y=0}^{b-m-1}\left(\sum_{x=m+1}^{b-y} q(a,b,c,x)q(x,a+c,b-x,y)\right)Y_\alpha^{(y)}Y_\gamma^{(b)}Y_\alpha^{(a+c-y)} \end{aligned}$$

$$= \sum_{y=0}^{b-m-1} \left(\sum_{x=m+1}^{b-y} \binom{a+c-b}{c-x} \binom{b-a-c}{b-x-y} \right) Y_\alpha^{(y)} Y_\gamma^{(b)} Y_\alpha^{(a+c-y)}$$

$$= \sum_{y=0}^{b-m-1} h(a,b,c,m,y) Y_\alpha^{(y)} Y_\gamma^{(b)} Y_\alpha^{(a+c-y)}. \qquad \square$$

Example 9. Suppose $\mathfrak{g} = \mathfrak{sl}_3$, so α and γ are simple roots. Denote by ω_α and ω_γ the fundamental weights, and let $\lambda = r\omega_\alpha + s\omega_\gamma$ be a dominant weight. We fix a highest weight vector $v_\lambda \in V(\lambda)$. It follows easily by the relations proved above that $U_{\mathbb{Z}}^-$ is spanned by the monomials

$$\mathbb{B} = \{ Y_\alpha^{(a)} Y_\gamma^{(b)} Y_\alpha^{(c)} \mid b \geq c \},$$

and hence $V_{\mathbb{Z}}(\lambda)$ is spanned by $\tilde{\mathbb{B}}(\lambda) := \{ Y_\alpha^{(a)} Y_\gamma^{(b)} Y_\alpha^{(c)} v_\lambda \mid b \geq c \}$. Now $\mathfrak{sl}_2$-representation theory implies:

$$Y_\alpha^{(a)} Y_\gamma^{(b)} Y_\alpha^{(c)} v_\lambda = 0 \quad \text{if } c > r, \text{ or } b > s + c, \quad \text{or } c = 0 \text{ and } a > r + b.$$

Suppose now $r \geq c > 0$ and $b \leq s + c$, but $a > r + b - 2c$. The exponents in the second sum in Proposition 8 satisfy the following inequality:

$$a + c - y \geq a + c - (b - m - 1) \geq a + c - b + 1 + (r - a - c + b) > r.$$

Such a summand applied to v_λ gives 0. So Proposition 8 implies:

$$Y_\alpha^{(a)} Y_\gamma^{(b)} Y_\alpha^{(c)} v_\lambda = \sum_{y=0}^{m} g(a,b,c,m,y) Y_\alpha^{(a+c-y)} Y_\gamma^{(b)} Y_\alpha^{(y)} v_\lambda.$$

Note that $b \geq y$ and $c > y$ because $b \geq c > m \geq y$. Proceeding by induction on the last exponent, this shows that $V_{\mathbb{Z}}(\lambda)$ is spanned by:

$$\mathbb{B}(\lambda) = \{ Y_\alpha^{(a)} Y_\gamma^{(b)} Y_\alpha^{(c)} v_\lambda \mid c \leq b, \ c \leq r, \ b \leq s + c, \ a \leq r + b - 2c \}.$$

An easy dimension argument shows that $\mathbb{B}(\lambda)$ is in fact a basis of $V_{\mathbb{Z}}(\lambda)$.

4. Commutation rules for semi-standard blocks

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_n = \varepsilon_n - \varepsilon_{n+1}$ be the usual ordering of the simple roots, where $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ denotes the projection of a diagonal matrix onto its ith entry. We write in the following only Y_i for Y_{α_i}. For $(\boldsymbol{a}^j) := (a_j, \ldots, a_1) \in \mathbb{N}^j$ let $Y^{(\boldsymbol{a}^j)}$ be the *semi-standard block* of length j:

$$Y^{(\boldsymbol{a}^j)} := Y_j^{(a_j)} \cdots Y_3^{(a_3)} Y_2^{(a_2)} Y_1^{(a_1)}.$$

We call $Y^{(\boldsymbol{a}^j)}$ a *standard block* of length j if $a_j \geq \cdots \geq a_1$. The proof of the following lemma is obvious:

Lemma 10. *Let $Y^{(\boldsymbol{a}^j)}$ be a semi-standard block of length j.*

(i) *If $r > j+1$, then $Y^{(\boldsymbol{a}^j)}Y_r^{(a)} = Y_r^{(a)}Y^{(\boldsymbol{a}^j)}$.*

(ii) $Y_j^{(a)}Y^{(\boldsymbol{a}^j)} = \binom{a_j^j+a}{a} Y_j^{(a_j^j+a)} \cdots Y_2^{(a_2^j)} Y_1^{(a_1^j)}$.

(iii) $Y^{(\boldsymbol{a}^j)}Y_1^{(a)} = \binom{a_1^j+a}{a} Y_j^{(a_j^j)} \cdots Y_2^{(a_2^j)} Y_1^{(a_1^j+a)}$

Let p, q, f be the same functions as in Section 3. Lemma 3 implies

Lemma 11. *Let $Y^{(\boldsymbol{a}^j)}$ be a semi-standard block of length j such that $a_i < a_{i-1}$ for some $2 \le i \le j$. Then*

$$Y^{(\boldsymbol{a}^j)} = \sum_{q=0}^{a_i} f(a_i, a_{i-1}, q) Y_{i-1}^{(a_{i-1}-q)} (Y_j^{(a_j)} \cdots Y_i^{(a_i)} Y_{i-1}^{(q)} Y_{i-2}^{(a_{i-2})} \cdots Y_1^{(a_1)}).$$

Definition 12. We write $(\boldsymbol{a}^j) \ge (\boldsymbol{b}^j)$ for $(\boldsymbol{a}^j), (\boldsymbol{b}^j) \in \mathbb{N}^j$, if there exists an i such that $a_1 = b_1, a_2 = b_2, \ldots, a_{i-1} = b_{i-1}$, and $a_i > b_i$.

For example, $(\boldsymbol{b}^j) := (a_j, \ldots, a_i, q, a_{i-2}, \ldots, a_1)$ is such that $(\boldsymbol{b}^j) < (\boldsymbol{a}^j)$ in Lemma 11. We get by induction on "$<$":

Corollary 13. *A semi-standard block $Y^{(\boldsymbol{a}^j)}$ is a linear combination*

$$Y^{(\boldsymbol{a}^j)} = \sum_{(\boldsymbol{b}^j) \le (\boldsymbol{a}^j)} h_{(\boldsymbol{b}^j)} Y^{(\boldsymbol{b}^j)},$$

where the $(\boldsymbol{b}^j)$ are standard and the $h_{(\boldsymbol{b}^j)}$ are integral linear combinations of monomials in the $Y_i^{(k)}$, $i < j$.

Proposition 14. *Suppose $M := (Y_j^{(a_j)} \cdots Y_2^{(a_2)} Y_1^{(a_1)}) Y_r^{(a)}$ is such that $2 \le r \le j$. Then*

$$\begin{aligned} M &= \sum_{x=0}^{\min\{a_r, a_{r-1}\}} p(a_r, a_{r-1}, a, x) Y_{r-1}^{(a_{r-1}-x)} (Y_j^{(a_j)} \cdots Y_r^{(a_r+a)} Y_{r-1}^{(x)} \cdots Y_1^{(a_1)}) \\ &= \sum_{x=0}^{\min\{a_{r-1}, a\}} q(a_r, a_{r-1}, a, x) Y_{r-1}^{(x)} (Y_j^{(a_j)} \cdots Y_r^{(a+a_r)} Y_{r-1}^{(a_{r-1}-x)} \cdots Y_1^{(a_1)}). \end{aligned}$$

Proof. Note that Y_r commutes with Y_i for $i \ne r-1,\ r+1$, so

$$(Y_j^{(a_j)} \cdots Y_2^{(a_2)} Y_1^{(a_1)}) Y_r^{(a)} = Y_j^{(a_j)} \cdots Y_r^{(a_r)} Y_{r-1}^{(a_{r-1})} Y_r^{(a)} \cdots Y_1^{(a_1)}.$$

By Proposition 6, the monomial is equal to

$$\sum_{x=0}^{\min\{a_r, a_{r-1}\}} p(a_r, a_{r-1}, a, x) Y_j^{(a_j)} \cdots Y_{r-1}^{(a_{r-1}-x)} Y_r^{(a_r+a)} Y_{r-1}^{(x)} \cdots Y_1^{(a_1)}.$$

Since Y_{r-1} commutes with the Y_i, $i > r$, this implies the first part, the proof of the second part is similar. □

Proposition 15. *If* $1 \le i_1, \dots, i_r \le j$, *then*

$$Y_{i_1}^{(c_1)} \cdots Y_{i_r}^{(c_r)} = \sum_{(\boldsymbol{a}^j)\ \text{standard}} g_{(\boldsymbol{a}^j)} Y^{(\boldsymbol{a}^j)},$$

where the $g_{(\boldsymbol{a}^j)}$ *are integral linear combinations of monomials in the* $Y_i^{(k)}$, $i < j$.

Proof. Choose t minimal such that $i_t = j$. Set $(\boldsymbol{d}^j) := (c_t, 0, \dots, 0)$, then

$$Y_{i_1}^{(c_1)} \cdots Y_{i_r}^{(c_r)} = Y_{i_1}^{(c_1)} \cdots Y_{i_{t-1}}^{(c_{t-1})} Y^{(\boldsymbol{d}^j)} Y_{i_{t+1}}^{(c_{t+1})} \cdots Y_{i_r}^{(c_r)}.$$

By Lemma 10, Proposition 14, and Corollary 13, $Y^{(\boldsymbol{d}^j)} \cdots Y_{i_r}^{(c_r)}$ is a linear combination of standard blocks $Y^{(\boldsymbol{a}^j)}$, where the coefficients $h_{(\boldsymbol{a}^j)}$ are integral linear combinations of monomials in the $Y_i^{(k)}$, $i < j$. So

$$Y_{i_1}^{(c_1)} \cdots Y_{i_r}^{(c_r)} = \sum g_{(\boldsymbol{a}^j)} Y^{(\boldsymbol{a}^j)},$$

where $g_{(\boldsymbol{a}^j)} := Y_{i_1}^{(c_1)} \cdots Y_{i_{t-1}}^{(c_{t-1})} h_{(\boldsymbol{a}^j)}$. $\square$

Applying the proposition to the $g_{(\boldsymbol{a}^j)}$, we get by induction:

Corollary 16. *A monomial* $M = Y_{i_1}^{(c_1)} \cdots Y_{i_r}^{(c_r)}$, $1 \le i_1, \dots, i_r \le j$, *is an integral linear combination of products of standard blocks*:

$$M = \sum c(\boldsymbol{a}^1, \dots, \boldsymbol{a}^j) Y^{(\boldsymbol{a}^1)} Y^{(\boldsymbol{a}^2)} \cdots Y^{(\boldsymbol{a}^j)}, \quad c(\boldsymbol{a}^1, \dots, \boldsymbol{a}^j) \in \mathbb{Z}.$$

5. Standard monomials

We write s_i for the simple reflection s_{α_i} in the Weyl group W of SL_{n+1}. For the rest of the article we fix the following reduced decomposition of the longest word $w_0 \in W$:

$$w_0 = s_1(s_2 s_1)(s_3 s_2 s_1)(\cdots)(s_n s_{n-1} \cdots s_2 s_1). \tag{3}$$

A monomial in the Y_j is called *semi-standard* if it is of the form:

$$Y^{(\boldsymbol{a})} := Y_1^{(a_1^1)}(Y_2^{(a_2^2)} Y_1^{(a_1^2)})(Y_3^{(a_3^3)} Y_2^{(a_2^3)} Y_1^{(a_1^3)})(\cdots)(Y_n^{(a_n^n)} \cdots Y_2^{(a_2^n)} Y_1^{(a_1^n)}),$$

where $(\boldsymbol{a}) = (a_1^1, a_2^2, a_1^2, \dots, a_n^n, \dots, a_1^n) \in \mathbb{N}^N$. The tuple $(\boldsymbol{a})$ and the monomial $Y^{(\boldsymbol{a})}$ are called *standard* if:

$$(\boldsymbol{a}) \in \mathscr{S} := \{(\boldsymbol{a}) \in \mathbb{N}^N \mid a_2^2 \ge a_1^2,\ a_3^3 \ge a_2^3 \ge a_1^3, \dots, a_n^n \ge \cdots \ge a_2^n \ge a_1^n\}.$$

Theorem 17. *The set of standard monomials* $\mathbb{B}$ *is a basis of* $U_{\mathbb{Z}}^-$.

Proof. $U_{\mathbb{Z}}^-$ is spanned by monomials in the $Y_\beta^{(m)}$, $\beta\in\Phi^+$. If $\beta\in\Phi^+$ is not a simple root, then let $\alpha,\gamma\in\Phi^+$ be such that $\alpha+\gamma=\beta$. Since

$$Y_\beta^{(m)}=\sum_{l=0}^{m}(-1)^l Y_\gamma^{(l)}Y_\alpha^{(m)}Y_\gamma^{(m-l)}$$

by Lemma 2, it follows by induction on the height that any monomial in the $Y_\beta^{(m)}$ is an integral linear combination of monomials in the $Y_i^{(k)}$. By Corollary 16, such a monomial is an integral linear combination of standard monomials, so $U_{\mathbb{Z}}^-$ is spanned by $\mathbb{B}$. Fix the following ordering of Φ^+:

$$\begin{array}{llll}\beta_1:=\alpha_1, & \beta_2:=\alpha_1+\alpha_2, & \beta_4:=\alpha_1+\alpha_2+\alpha_3, & \dots\\ & \beta_3:=\alpha_2, & \beta_5:=\alpha_2+\alpha_3, & \dots\\ & & \beta_6:=\alpha_3, & \dots\end{array}$$

A simple counting argument shows that the number of elements of a given weight of the corresponding P-B-W basis is the same as the number of standard monomials of the same weight. Since the standard monomials span U^-, this proves that they are also linearly independent. $\square$

Let $Y^{(\boldsymbol{a})}$ be a semi-standard monomial. We refer to $Y^{(\boldsymbol{a}^j)}:=Y_j^{(a_j^j)}\dots Y_2^{(a_2^j)}Y_1^{(a_1^j)}$ as the jth (semi-standard) block of $Y^{(\boldsymbol{a})}$. We write $(\boldsymbol{a})>(\boldsymbol{b})$ if and only if there exists a j such that:

$$(\boldsymbol{a}^n)=(\boldsymbol{b}^n),\dots,(\boldsymbol{a}^{j+1})=(\boldsymbol{b}^{j+1})\text{ and }(\boldsymbol{a}^j)>(\boldsymbol{b}^j).$$

Theorem 18. *A semi-standard monomial $Y^{(\boldsymbol{b})}$ is an integral linear combination of standard monomials $Y^{(\boldsymbol{a})}$ such that $(\boldsymbol{a})\le(\boldsymbol{b})$:*

$$Y^{(\boldsymbol{b})}=\sum_{(\boldsymbol{a})\le(\boldsymbol{b})}c_{(\boldsymbol{a})}Y^{(\boldsymbol{a})}.$$

Proof. If $b_{i+1}^j<b_i^j$, then Lemma 11 implies

$$\begin{aligned}Y^{(\boldsymbol{b})}=\sum_{x=0}^{b_{i+1}^j} & f(b_{i+1}^j,b_i^j,x)Y^{(\boldsymbol{b}^1)}\cdots Y^{(\boldsymbol{b}^{j-1})}Y_i^{(b_i^j-x)}\\ & \times(Y_j^{(b_j^j)}\cdots Y_{i+1}^{(b_{i+1}^j)}Y_i^{(x)}\cdots Y_1^{(b_1^j)})Y^{(\boldsymbol{b}^{j+1})}\cdots Y^{(\boldsymbol{b}^n)}.\end{aligned}$$

By Corollary 16, $Y^{(\boldsymbol{b}^1)}\cdots Y^{(\boldsymbol{b}^{j-1})}Y_i^{(b_i^j-x)}$ is an integral linear combination of products of standard blocks $Y^{(\boldsymbol{d}^1)}\dots Y^{(\boldsymbol{d}^{j-1})}$. So $Y^{(\boldsymbol{b})}$ is an integral linear combination of semi-standard monomials:

$$Y^{(\boldsymbol{d})}:=Y^{(\boldsymbol{d}^1)}\dots Y^{(\boldsymbol{d}^{j-1})}(Y_j^{(b_j^j)}\cdots Y_{i+1}^{(b_{i+1}^j)}Y_i^{(x)}\cdots Y_1^{(b_1^j)})Y^{(\boldsymbol{b}^{j+1})}\cdots Y^{(\boldsymbol{b}^n)}.$$

Note that $(\boldsymbol{d})<(\boldsymbol{b})$: $(\boldsymbol{d}^k)=(\boldsymbol{b}^k)$ for $k>j$, $d_r^j=b_r^j$ for $r>i$, but $x<b_i^j$. It follows by induction on "$<$" that $Y^{(\boldsymbol{b})}$ is an integral linear combination of standard monomials $\sum c_{(\boldsymbol{a})}Y^{(\boldsymbol{a})}$ such that $(\boldsymbol{a})<(\boldsymbol{b})$. $\square$

The arguments above prove in fact the following more precise statement:

Corollary 19. *Let $Y^{(\boldsymbol{b})}$ be a semi-standard monomial. If j is maximal such that there exists an $i\le j$ with $b_{i+1}^j<b_i^j$, then $Y^{(\boldsymbol{b})}$ can be written as an integral linear combination of standard monomials*

$$Y^{(\boldsymbol{b})}=\sum_{\substack{(\boldsymbol{a})\le(\boldsymbol{b})\\ (\boldsymbol{a})\in\mathcal{S}}} c_{(\boldsymbol{a})}Y^{(\boldsymbol{a})},\quad c_{(\boldsymbol{a})}\in\mathbb{Z},$$

where the $(\boldsymbol{a})$ are such that $(\boldsymbol{a}^k)=(\boldsymbol{b}^k)$ for all $k>j$, and $a_i^j\le b_i^j$ $\forall i=1,\dots,j$.

6. Multiplication

The following rules are a first step towards a description of $Y_i^{(a)}Y^{(\boldsymbol{b})}$, $(\boldsymbol{b})$ standard, as a linear combination of standard monomials.

Proposition 20. *If $1\le r<j$, then $M:=Y_r^{(a)}(Y_j^{(b_j)}\cdots Y_2^{(b_2)}Y_1^{(b_1)})$ can be written as an integral linear combination*:

$$\begin{aligned}M&=\sum_{x=0}^{\min\{a,b_{r+1}\}}p(a,b_{r+1},b_r,x)(Y_j^{(b_j)}\cdots Y_{r+1}^{(b_{r+1}-x)}Y_r^{(a+b_r)}\cdots Y_1^{(b_1)})Y_{r+1}^{(x)}\\ &=\sum_{x=0}^{\min\{b_{r+1},b_r\}}q(a,b_{r+1},b_r,x)(Y_j^{(b_j)}\cdots Y_{r+1}^{(x)}Y_r^{(a+b_r)}\cdots Y_1^{(b_1)})Y_{r+1}^{(b_{r+1}-x)}.\end{aligned}$$

Proof. Since Y_r commutes with Y_i for $i\ne r-1,r+1$, we get

$$Y_r^{(a)}(Y_j^{(b_j)}\cdots Y_2^{(b_2)}Y_1^{(b_1)})=Y_j^{(b_j)}\cdots Y_r^{(a)}Y_{r+1}^{(b_{r+1})}Y_r^{(b_r)}\cdots Y_1^{(b_1)}.$$

By Proposition 6, this monomial is equal to

$$\sum_{x=0}^{\min\{a,b_{r+1}\}}p(a,b_{r+1},b_r,x)Y_j^{(b_j)}\cdots Y_{r+1}^{(b_{r+1}-x)}Y_r^{(b_r+a)}Y_{r+1}^{(x)}\cdots Y_1^{(b_1)}.$$

Since Y_{r+1} commutes with Y_i, $i<r$, this implies the first relation, the proof of the second relation is similar. $\square$

Set $M:=Y_{j+1}^{(x_{j+1})}(Y_j^{(a_j)}\cdots Y_2^{(a_2)}Y_1^{(a_1)})(Y_{j+1}^{(b_{j+1})}Y_j^{(b_j)}\cdots Y_2^{(b_2)}Y_1^{(b_1)})$.

Proposition 21. *M is an integral linear combination of semi-standard monomials*:

$$\sum_{x_j=0}^{\min\{a_j,x_{j+1}\}} \cdots \sum_{x_1=0}^{\min\{a_1,x_2\}} \left(\prod_{l=1}^{j} p(x_{l+1},a_l,b_{l+1},x_l) \binom{b_1+x_1}{b_1} \right)$$

$$\times (Y_j^{(a_j-x_j)} \cdots Y_2^{(a_2-x_2)} Y_1^{(a_1-x_1)})(Y_{j+1}^{(b_{j+1}+x_{j+1})} \cdots Y_2^{(b_2+x_2)} Y_1^{(b_1+x_1)})$$

or, with $z_l := a_l - x_l$ *for* $1 \le l \le j$ *and* $z_{j+1} := x_{j+1}$, *as*

$$\sum_{x_j=0}^{\min\{a_j,b_{j+1}\}} \cdots \sum_{x_1=0}^{\min\{a_1,b_2\}} \left(\prod_{l=2}^{j+1} q(z_l,a_{l-1},b_l,x_{l-1}) \binom{b_1+z_1}{b_1} \right)$$

$$\times (Y_j^{(x_j)} \cdots Y_2^{(x_2)} Y_1^{(x_1)})(Y_{j+1}^{(b_{j+1}+z_{j+1})} \cdots Y_2^{(b_2+z_2)} Y_1^{(b_1+z_1)}).$$

Proof. By Proposition 6 we know that M is equal to

$$\sum_{x_j=0}^{\min\{a_j,x_{j+1}\}} p(x_{j+1},a_j,b_{j+1},x_j)(Y_j^{(a_j-x_j)} Y_{j+1}^{(b_{j+1}+x_{j+1})})$$

$$\times Y_j^{(x_j)} (Y_{j-1}^{(a_{j-1})} \cdots Y_1^{(a_1)})(Y_j^{(b_j)} \cdots Y_1^{(b_1)}).$$

By reordering the factors and by applying Proposition 6 to monomials of the type $Y_k^{(x_k)}(Y_{k-1}^{(a_{j-1})} \cdots Y_1^{(a_1)})(Y_k^{(b_k)} \cdots Y_1^{(b_1)})$, we get inductively:

$$M = \sum_{x_j=0}^{\min\{a_j,x_{j+1}\}} \cdots \sum_{x_1=0}^{\min\{a_1,x_2\}} \left(\prod_{l=1}^{j} p(x_{l+1},a_l,b_{l+1},x_l) \right)$$

$$\times (Y_j^{(a_j-x_j)} \cdots Y_2^{(a_2-x_2)} Y_1^{(a_1-x_1)})(Y_{j+1}^{(b_{j+1}+x_{j+1})} \cdots Y_2^{(b_2+x_2)})(Y_1^{(x_1)} Y_1^{(b_1)}).$$

Since $Y_1^{(x_1)} Y_1^{(b_1)} = \binom{b_1+x_1}{b_1} Y_1^{(x_1+b_1)}$, this proves the first part of the proposition. The proof of the second part is similar. □

7. λ-Standard monomials and Gelfand–Tsetlin-patterns

Fix $(\boldsymbol{a}) = (a_1^1, \ldots, a_3^n, a_2^n, a_1^n) \in \mathbb{N}^N$. For a dominant weight λ let $v_\lambda \in V(\lambda)$ be a highest weight vector. Consider the following sequence of weight vectors:

$$v_\lambda,\ Y_1^{(a_1^n)} v_\lambda,\ Y_2^{(a_2^n)} Y_1^{(a_1^n)} v_\lambda,\ Y_3^{(a_3^n)} Y_2^{(a_2^n)} Y_1^{(a_1^n)} v_\lambda, \ldots, Y^{(\boldsymbol{a})} v_\lambda.$$

Denote by λ_i^j the weight of $(Y_i^{(a_i^j)} \cdots Y_1^{(a_1^j)})(\cdots)(Y_n^{(a_n^n)} \cdots Y_1^{(a_1^n)}) v_\lambda$, and set $\lambda_0^n := \lambda$, and $\lambda_0^{j-1} := \lambda_j^j$ for $1 \le j \le n$. We write H_i for H_{α_i}.

Definition 22. $\mathscr{S}(\lambda)$ is the set of all $(\boldsymbol{a})\in\mathscr{S}$ such that

$$\begin{array}{lllll}
\lambda_0^n(H_1)\ge a_1^n & \lambda_1^n(H_2)\ge a_2^n & \lambda_2^n(H_3)\ge a_3^n & \dots & \lambda_{n-1}^n(H_n)\ge a_n^n\\
\dots & \dots & \dots & \dots & \\
\lambda_0^3(H_1)\ge a_1^3 & \lambda_1^3(H_2)\ge a_2^3 & \lambda_2^3(H_3)\ge a_3^3 & & \\
\lambda_0^2(H_1)\ge a_1^2 & \lambda_1^2(H_2)\ge a_2^2 & & & \\
\lambda_0^1(H_1)\ge a_1^1 & & & &
\end{array}$$

Definition 23. A monomial $Y^{(\boldsymbol{a})}$ is called λ-standard if $(\boldsymbol{a})\in\mathscr{S}(\lambda)$.

Fix $p_1\ge\cdots\ge p_{n+1}\ge 0$ such that $\lambda=p_1\varepsilon_1+\cdots+p_{n+1}\varepsilon_{n+1}$. We write:

$$\lambda_j^j=g_{j,1}\varepsilon_1+\cdots+g_{j,n+1}\varepsilon_{n+1}$$

by subtracting the roots $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ from λ. We associate to $(\boldsymbol{a})$ the triangular scheme

$$\Delta(\boldsymbol{a}):=\begin{array}{c}
\boxed{p_1}\boxed{p_2}\boxed{\cdot}\boxed{\cdot}\boxed{p_n}\boxed{p_{n+1}}\\
\boxed{g_{n,1}}\boxed{g_{n,2}}\boxed{\cdot}\boxed{\cdot}\boxed{g_{n,n}}\\
\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\\
\boxed{g_{3,1}}\boxed{g_{3,2}}\boxed{g_{3,3}}\\
\boxed{g_{2,1}}\boxed{g_{2,1}}\\
\boxed{g_{1,1}}
\end{array}$$

Recall that such a scheme is called a Gelfand–Tsetlin pattern of shape λ if $p_i\ge g_{n,i}\ge p_{i+1}$ for $1\le i\le n$, and

$$g_{i,j}\ge\max\{g_{i-1,j},g_{i+1,j+1}\}\ge\min\{g_{i-1,j},g_{i+1,j+1}\}\ge g_{i,j+1}.$$

for all $1\le j\le i\le n$. A simple calculation shows for $(\boldsymbol{a})\in\mathscr{S}$:

$$(\boldsymbol{a})\in\mathscr{S}(\lambda)\Leftrightarrow\Delta(\boldsymbol{a})\text{ is a Gelfand–Tsetlin pattern of shape }\lambda,\tag{4}$$

and the correspondence is a bijection. By (4) we get

Lemma 24. *The dimension of $V(\lambda)$ is equal to the cardinality of $\mathscr{S}(\lambda)$.*

8. A basis of $V_{\mathbb{Z}}(\lambda)$

Fix a highest weight vector v_λ in the irreducible $\mathfrak{g}$-module $V(\lambda)$ of highest weight $\lambda\in X^+$, and let $V_{\mathbb{Z}}(\lambda)=U_{\mathbb{Z}}v_\lambda$ be the corresponding admissible lattice.

Theorem 25. $\mathbb{B}(\lambda):=\{Y^{(\boldsymbol{a})}v_\lambda\mid(\boldsymbol{a})\in\mathscr{S}(\lambda)\}$ *is a basis of $V_{\mathbb{Z}}(\lambda)$.*

Proof. We prove: If $Y^{(\boldsymbol{b})}$ is a semi-standard monomial such that either

$$b_i^j<b_{i-1}^j\text{ for some }2\le i\le j,\quad\text{or}\quad b_i^j>\lambda_{i-1}^j(H_i)\text{ for some }1\le i\le j,\tag{5}$$

then $Y^{(\boldsymbol{b})}v_\lambda$ is an integral linear combination:

$$Y^{(\boldsymbol{b})}v_\lambda = \sum c_{(\boldsymbol{a})}Y^{(\boldsymbol{a})}v_\lambda, \quad c_{(\boldsymbol{a})} \in \mathbb{Z} \tag{6}$$

where the $Y^{(\boldsymbol{a})}$ are semi-standard monomials such that $(\boldsymbol{a}) < (\boldsymbol{b})$.

We prove first that (6) implies the theorem: Note that a semi-standard monomial $Y^{(\boldsymbol{b})}$ is not λ-standard if and only if $Y^{(\boldsymbol{b})}$ satisfies (5). So if $Y^{(\boldsymbol{b})}$ is not λ-standard, then we can replace $Y^{(\boldsymbol{b})}v_\lambda$ by an integral linear combination of vectors $Y^{(\boldsymbol{a})}v_\lambda$, where $(\boldsymbol{a}) < (\boldsymbol{b})$.

If one of the $Y^{(\boldsymbol{a})}$ is not λ-standard, then we can replace $Y^{(\boldsymbol{a})}v_\lambda$ by (6) by an integral linear combination of vectors $Y^{(\boldsymbol{d})}v_\lambda$, where $(\boldsymbol{d})<(\boldsymbol{a})$, and hence $(\boldsymbol{d})<(\boldsymbol{b})$. This proves of course inductively that $Y^{(\boldsymbol{b})}v_\lambda$ is an integral linear combination of vectors of the form $Y^{(\boldsymbol{a})}v_\lambda$, $(\boldsymbol{a}) \in \mathscr{S}(\lambda)$. So $\mathbb{B}(\lambda)$ spans the lattice $V_{\mathbb{Z}}(\lambda)$. It follows now from Lemma 24 that $\mathbb{B}(\lambda)$ is a basis of $V_{\mathbb{Z}}(\lambda)$.

To prove (6), let $Y^{(\boldsymbol{b})}$ be a semi-standard monomial, and fix j maximal such that b_i^j satisfies (5) for some $i \le j$. Fix i minimal with this property, i.e. b_i^j is the right-most coefficient of $(\boldsymbol{b})$ satisfying (5).

If $b_i^j < b_{i-1}^j$, then (6) is a consequence of Theorem 18. So assume now that $b_i^j \ge \cdots \ge b_1^j$ and $b_i^j > \lambda_{i-1}^j(H_i)$. We prove the following stronger version of (6) by decreasing induction on j:

$$\text{Either } Y^{(\boldsymbol{b})}v_\lambda = 0, \quad \text{or} \quad Y^{(\boldsymbol{b})}v_\lambda = \sum c_{(\boldsymbol{a})}Y^{(\boldsymbol{a})}v_\lambda, \quad c_{(\boldsymbol{a})} \in \mathbb{Z} \tag{7}$$

where $(\boldsymbol{a}) < (\boldsymbol{b})$, and there exists an $k \ge j+1$ such that $(\boldsymbol{a}^k) < (\boldsymbol{b}^k)$.

If $j = n$, then $\mathfrak{sl}_2$-representation theory implies that $Y^{(\boldsymbol{b})}v_\lambda = 0$, which proves (7) in this case. Suppose now $j < n$. To simplify the notation, we write for $(\boldsymbol{b}^j)$ and $(\boldsymbol{b}^{j+1})$: $(\boldsymbol{b}^j) = (b_j, \ldots, b_1)$, $(\boldsymbol{b}^{j+1}) = (c_{j+1}, \ldots, c_1)$.

The second relation in Proposition 6 implies

$$\begin{aligned} Y^{(\boldsymbol{b})} = \sum_{x=0}^{(b_{i-1})} & q(0, b_i, b_{i-1}, x) Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} (Y_j^{(b_j)} \cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1})} \cdots Y_1^{(b_1)}) \\ & \times (Y_{j+1}^{(c_{j+1})} \cdots Y_i^{(b_i - x)} Y_{i+1}^{(c_{i+1})} Y_i^{(c_i)} \cdots Y_1^{(c_1)}) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)}. \end{aligned}$$

These monomials are not anymore semi-standard because of the term:

$$I(x) := Y_i^{(b_i - x)} Y_{i+1}^{(c_{i+1})} Y_i^{(c_i)}.$$

Set $r := \lambda_{i-1}^{j+1}(H_i)$ and $s := \lambda_{i-1}^{j+1}(H_{i+1})$. By the maximal choice of j we know

$$c_i \le r \quad \text{and} \quad c_{i+1} \le s + c_i = \lambda_i^{j+1}(H_{i+1}).$$

But $0 \le x \le b_{i-1}$ and $b_i > \lambda_{i-1}^j(H_i) = r - 2c_i + c_{i+1} + b_{i-1}$ imply

$$b_i - x > r - 2c_i + c_{i+1} + b_{i-1} - x \ge r + c_{i+1} - 2c_i.$$

Set $a := b_i - x$, $b := c_{i+1}$, $c := c_i$ and $m := r - a - c + b$, then the choice of j and the equations above imply:

$$b \geq c, \quad r \geq c, \quad a > r + b - 2c, \quad \text{and} \quad c > r + b - a - c = m.$$

By Proposition 8, we get

$$I(x) = \sum_{y=0}^{m} g(a,b,c,m,y) Y_i^{(a+c-y)} Y_{i+1}^{(b)} Y_i^{(y)} + \sum_{y=0}^{b-m-1} h(a,b,c,m,y) Y_i^{(y)} Y_{i+1}^{(b)} Y_i^{(a+c-y)}.$$

So if we replace $I(x)$ by the expression above, then we can write $Y^{(b)}$ as an integral linear combination of monomials of two types (according to the two sums in the expression for $I(x)$: After "moving" $Y_i^{(a+c-y)}$ to the left of the $(j+1)$st block, the monomials of the first type are of the form:

$$Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} (Y_j^{(b_j)} \cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1})} \cdots Y_1^{(b_1)}) Y_i^{(a+c-y)}$$
$$\times (Y_{j+1}^{(c_{j+1})} \cdots Y_{i+1}^{(b)} Y_i^{(y)} \cdots Y_1^{(c_1)}) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)},$$

where $y \leq m < c$. By Corollary 16, we can write the monomial

$$Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} (Y_j^{(b_j)} \cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1})} \cdots Y_1^{(b_1)}) Y_i^{(a+c-y)}$$

as an integral linear combination of products of standard blocks $Y^{(\boldsymbol{a}^1)} \cdots Y^{(\boldsymbol{a}^j)}$, so Y^b is an integral linear combination of monomials of the type:

$$Y^{(\boldsymbol{a})} = Y^{(\boldsymbol{a}^1)} \cdots Y^{(\boldsymbol{a}^j)} (Y_{j+1}^{(c_{j+1})} \cdots Y_{i+1}^{(b)} Y_i^{(y)} \cdots Y_1^{(c_1)}) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)}. \tag{8}$$

These monomials are semi-standard and $(\boldsymbol{a}) < (\boldsymbol{b})$: Note that $(\boldsymbol{a}^k) = (\boldsymbol{b}^k)$ for $k > j+1$, and $(\boldsymbol{a}^{j+1}) < (\boldsymbol{b}^{j+1})$ because $a_i^{j+1} = y < c = c_i = b_i^{j+1}$.

The summands coming from the first sum in (8) are hence integral linear combinations of semi-standard monomials $Y^{(\boldsymbol{a})}$ such that $(\boldsymbol{a}) < (\boldsymbol{b})$, and there exists an $k \geq j+1$ such that $(\boldsymbol{a}^k) < (\boldsymbol{b}^k)$. So the $Y^{(\boldsymbol{a})}$ satisfy (7).

It remains to consider the contributions coming from the second sum in (8). After "moving" $Y_i^{(y)}$ to the left of the $(j+1)$st block, the monomials in the second sum are of the form:

$$Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} (Y_j^{(b_j)} \cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1})} \cdots Y_1^{(b_1)}) Y_i^{(y)}$$
$$(Y_{j+1}^{(c_{j+1})} \cdots Y_{i+1}^{(b)} Y_i^{(a+c-y)} \cdots Y_1^{(c_1)}) Y^{\boldsymbol{b}^{(j+2)}} \cdots Y^{(\boldsymbol{b}^n)},$$

where $a + c - y \geq a + c + m + 1 - b = r + 1 > r$. As above, by Corollary 16, we can write the monomial

$$Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} (Y_j^{(b_j)} \cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1})} \cdots Y_1^{(b_1)}) Y_i^{(y)}$$

as an integral linear combination of products of standard blocks $Y^{(\boldsymbol{d}^1)} \cdots Y^{(\boldsymbol{d}^j)}$, so (9) is an integral linear combination of monomials of the form:

$$Y^{(\boldsymbol{d})} = Y^{(\boldsymbol{d}^1)} \cdots Y^{(\boldsymbol{d}^j)} (Y_{j+1}^{(c_{j+1})} \cdots Y_{i+1}^{(b)} Y_i^{(a+c-y)} \cdots Y_1^{(c_1)}) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)}.$$

The $Y^{(\boldsymbol{d})}$ are semi-standard, and $b_{i-1}^{j+1} = d_{i-1}^{j+1}, \dots, b_1^{j+1} = d_1^{j+1}$. But note that $d_i^{j+1} = a + c - y$ has the property:

$$d_i^{j+1} > r = \lambda_{i-1}^{j+1}(H_i). \tag{9}$$

By the maximal choice of j, this implies in addition that $d_i^{j+1} > b_i^{j+1}$. So $(\boldsymbol{d})$ satisfies also the second assumption of (7):

$$d_i^{j+1} \geq d_{i-1}^{j+1} \geq \cdots \geq d_1^{j+1}.$$

It follows by (decreasing) induction that either:

$$Y^{(\boldsymbol{d})} v_\lambda = 0, \quad \text{or} \quad Y^{(\boldsymbol{d})} v_\lambda = \sum_{(\boldsymbol{a}) < (\boldsymbol{d})} c_{(\boldsymbol{a})} Y^{(\boldsymbol{a})} v_\lambda, \quad c_{(\boldsymbol{a})} \in \mathbb{Z}.$$

Further, there exists an $k > j+1$ such that $(\boldsymbol{a}^k) < (\boldsymbol{d}^k)$. Since $(\boldsymbol{b}^l) = (\boldsymbol{d}^l)$ for $l > j+1$, this implies also: $(\boldsymbol{a}) < (\boldsymbol{b})$ and $(\boldsymbol{a}^k) < (\boldsymbol{b}^k)$.

Hence the summands coming from the second sum in (8) are integral linear combinations of semi-standard monomials $Y^{(\boldsymbol{a})}$ such that $(\boldsymbol{a}) < (\boldsymbol{b})$, and there exists an $k \geq j+1$ such that $(\boldsymbol{a}^k) < (\boldsymbol{b}^k)$. So the $Y^{(\boldsymbol{a})}$ satisfy also (7). $\square$

9. An algorithm to compute representation matrices

For a dominant weight $\lambda \in X^+$ let $V(\lambda)$ denote the corresponding irreducible representation. Fix a highest weight vector v_λ, and let $V_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}^- v_\lambda$ be the corresponding admissible lattice. By Theorem 25, we know that $V_{\mathbb{Z}}(\lambda)$ has as basis the vectors $\mathbb{B}(\lambda)$. The set of Gelfand–Tsetlin patterns of type λ provides a nice combinatorial tool to "encode" the set $\mathbb{B}(\lambda)$ (see (4)).

Suppose $(\boldsymbol{a}) \in \mathscr{S}(\lambda)$, so $v^{(\boldsymbol{a})} := Y^{(\boldsymbol{a})} v_\lambda \in \mathbb{B}(\lambda)$. It remains to describe an algorithm to compute the coefficients $r_{(\boldsymbol{c})}$ and $s_{(\boldsymbol{d})}$ in the expressions

$$X_i^{(m)} v^{(\boldsymbol{a})} = \sum_{(\boldsymbol{c}) \in \mathscr{S}(\lambda)} r_{(\boldsymbol{c})} v^{(\boldsymbol{c})}, \qquad Y_i^{(m)} v^{(\boldsymbol{a})} = \sum_{(\boldsymbol{d}) \in \mathscr{S}(\lambda)} s_{(\boldsymbol{d})} v^{(\boldsymbol{d})},$$

The algorithm is already implicitly given in the proofs of Theorems 17 and 25.

9.1. *The matrix coefficients of $Y_i^{(c)}$*

Step 1: In the first step we use Proposition 21 to express

$$Y_i^{(c)} Y^{(\boldsymbol{a})} = Y^{(\boldsymbol{a}^1)} \cdots Y^{(\boldsymbol{a}^{i-2})} (Y_i^{(c)} Y^{(\boldsymbol{a}^{i-1})} Y^{(\boldsymbol{a}^i)}) Y^{(\boldsymbol{a}^{i+1})} \cdots Y^{(\boldsymbol{a}^n)}$$

as a sum of semi-standard monomials. The coefficients occurring during the procedure can be computed using the functions p, q defined in Section 3.

Step 2: Suppose $Y^{(\boldsymbol{b})}$ is semi-standard. The second step is an algorithm to rewrite $Y^{(\boldsymbol{b})} v_\lambda$ as an integral linear combination of vectors $Y^{(\boldsymbol{a})} v_\lambda$, $(\boldsymbol{a}) \in \mathscr{S}(\lambda)$. The procedure is more or less the same as in the proof of Theorem 25.

Suppose $Y^{(\boldsymbol{b})}$ is a semi-standard monomial, but $Y^{(\boldsymbol{b})}$ is not λ-standard. Assume that $(\boldsymbol{b}^j)$ is the "right-most" block that contradicts the condition of being λ-standard. Fix i minimal such that either

$$b_i^j < b_{i-1}^j \text{ for some } 2 \le i \le j, \quad \text{or} \quad b_i^j > \lambda_{i-1}^j(H_i) \text{ for some } 1 \le i \le j. \tag{10}$$

(I) Suppose $b_i^j < b_{i-1}^j$. We "move" the Y_i to the left (Lemma 3):

$$\begin{aligned} &Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})}(Y_j^{(b_j^j)} \cdots Y_{i+1}^{(b_{i+1}^j)} Y_i^{(b_i^j)} \cdots Y_1^{(b_1^j)}) Y^{(\boldsymbol{b}^{j+1})} \cdots Y^{(\boldsymbol{b}^n)} \\ &= \sum_{q=0}^{(b_{i+1}^j)} f(b_{i+1}^j, b_i^j, q) Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} Y_i^{(b_i^j - q)} \\ &\quad \times (Y_j^{(b_j^j)} \cdots Y_{i+1}^{(b_{i+1}^j)} Y_i^{(q)} \cdots Y_1^{(b_1^j)}) Y^{(\boldsymbol{b}^{j+1})} \cdots Y^{(\boldsymbol{b}^n)}. \end{aligned}$$

Proposition 14 provides an inductive algorithm to express

$$Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})} Y_i^{(b_i^j - q)} = \sum c_{(\boldsymbol{d}^1, \ldots, \boldsymbol{d}^{j-1})} Y^{(\boldsymbol{d}^1)} \cdots Y^{(\boldsymbol{d}^{j-1})}, \quad c_{(\boldsymbol{d}^1, \ldots, \boldsymbol{d}^{j-1})} \in \mathbb{Z},$$

as a sum of products of semi-standard blocks. Hence $Y^{(\boldsymbol{b})}$ can be expressed as an integral linear combination of semi-standard monomials of the form:

$$Y^{(\boldsymbol{d})} = Y^{(\boldsymbol{d}^1)} \cdots Y^{(\boldsymbol{d}^{j-1})}(Y_j^{(b_j^j)} \cdots Y_{i+1}^{(b_{i+1}^j)} Y_i^{(q)} \cdots Y_1^{(b_1^j)}) Y^{(\boldsymbol{d}^{j+1})} \cdots Y^{(\boldsymbol{d}^n)},$$

where $Y^{(\boldsymbol{d}^k)} = Y^{(\boldsymbol{b}^k)}$ for $k \ge j+1$. Note that $(\boldsymbol{d}) < (\boldsymbol{b})$ because $d_i^j = q < b_i^j$ and $d_{i-1}^j = b_{i-1}^j, \ldots, d_1^j = b_1^j$. So $Y^{(\boldsymbol{b})} v_\lambda$ is an integral linear combination:

$$Y^{(\boldsymbol{b})} v_\lambda = \sum_{(\boldsymbol{d}) < (\boldsymbol{b})} c_{(\boldsymbol{d})} Y^{(\boldsymbol{d})} v_\lambda.$$

(IIa) Suppose now $b_i^j \ge b_{i-1}^j$ and $b_i^j > \lambda_{i-1}^j(H_i)$. If $j = n$, then $Y^{(\boldsymbol{b})} v_\lambda = 0$. Otherwise we use Proposition 6 to "move" the Y_i to next block to the right:

$$\begin{aligned} Y^{(\boldsymbol{b})} = &\sum_{x=0}^{b_{i-1}^j} q(0, b_i^j, b_{i-1}^j, x) Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})}(\cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1}^j)} \cdots) \\ &(\cdots Y_i^{(b_i^j - x)} Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(b_i^{j+1})} \cdots) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)}. \end{aligned}$$

Then we apply Proposition 8 to $Y_i^{(b_i^j - x)} Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(b_i^{j+1})}$. The resulting summands are of two types:

(IIb) The first type is of the form:

$$\begin{aligned} &Y^{(\boldsymbol{b}^1)} \cdots Y^{(\boldsymbol{b}^{j-1})}(\cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1}^j)} \cdots) Y_i^{(b_i^j - x + b_i^{j+1} - y)} \\ &(\cdots Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(y)} \cdots) Y^{(\boldsymbol{b}^{j+2})} \cdots Y^{(\boldsymbol{b}^n)}, \end{aligned}$$

where $y < b_i^{j+1}$. We apply the same inductive procedure (Proposition 14) as before to express

$$Y^{(b^1)} \cdots Y^{(b^{j-1})}(\cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1}^j)} \cdots) Y_i^{(b_i^j - x + b_i^{j+1} - y)}$$

as a sum of products of semi-standard blocks $Y^{(d^1)} \cdots Y^{(d^j)}$. So we have expressed the summands of the first type as an integral linear combination of semi-standard monomials of the form:

$$Y^{(d)} = Y^{(d^1)} \cdots Y^{(d^j)}(\cdots Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(y)} \cdots) Y^{(d^{j+2})} \cdots Y^{(d^n)},$$

where $(d^k) = (b^k)$ for $k \geq j+2$. We have proved in Section 8 that $(d) < (b)$.

(IIc) The summands of the second type are of the form:

$$Y^{(b^1)} \cdots Y^{(b^{j-1})}(\cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1}^j)} \cdots) Y_i^{(y)}(\cdots Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(b_i^j - x + b_i^{j+1} - y)} \cdots) Y^{(b^{j+2})} \cdots Y^{(b^n)}.$$

We apply the same inductive procedure (Proposition 14) as before to rewrite

$$Y^{(b^1)} \cdots Y^{(b^{j-1})}(\cdots Y_i^{(x)} Y_{i-1}^{(b_{i-1}^j)} \cdots) Y_i^{(y)}$$

as a sum of products of semi-standard blocks $Y^{(d^1)} \cdots Y^{(d^j)}$. The resulting summands are of the form:

$$Y^{(d)} := Y^{(d)^1} \cdots Y^{(d)^j}(\cdots Y_{i+1}^{(b_{i+1}^{j+1})} Y_i^{(b_i^j - x + b_i^{j+1} - y)} \cdots) Y^{(d^{j+2})} \cdots Y^{(d^n)}, \tag{11}$$

where $(d^k) = (b^k)$ for $k \geq j+2$. We have proved in (9) that

$$d_i^{j+1} = b_i^j - x + b_i^{j+1} - y > \lambda_{i-1}^{j+1}(H_i).$$

(IId) If $j+1 = n$, then $Y^{(d)} v_\lambda = 0$. Suppose now $j+1 < n$. We have proved in Section 8 that (d) satisfies the assumptions of (IIa). Apply (IIa) to $Y^{(d)} v_\lambda$, this procedure yields again summands of two different types:

The summands $Y^{(c)} v_\lambda$ of the first type are such that $(c) < (d)$. Note that we have proved in Section 8 that $(c) < (b)$. To the summands of second type we apply again (IIc) etc. We have proved in Section 8 that this algorithm yields:

$$Y^{(b)} v_\lambda = \sum_{(d) < (b)} c_{(d)} Y^{(d)} v_\lambda, \quad c_{(d)} \in \mathbb{Z}.$$

(III) So if (10) holds, then (I) and (II) provide algorithms to express $Y^{(b)} v_\lambda$ as an integral linear combination:

$$Y^{(b)} v_\lambda = \sum_{(d) < (b)} c_{(d)} Y^{(d)} v_\lambda, \quad c_{(d)} \in \mathbb{Z}. \tag{12}$$

Next we apply (I) respectively (II) to all summands such that $(d) \notin \mathscr{S}(\lambda)$, etc.

It follows by induction on the total order "$<$" that this is a finite algorithm. A (very rough) upper bound for the number of times this procedure has to repeated can be given as follows: For $(\boldsymbol{b}) \in \mathbb{N}^N$ denote by $|(\boldsymbol{b})|$ the sum:

$$|(\boldsymbol{b})| := \sum_{1 \le i \le j \le n} b_i^j.$$

Since $|(\boldsymbol{b})| = |(\boldsymbol{d})|$ for all summands in (12), all the $(\boldsymbol{d})$ are elements of the cube $\{(\boldsymbol{a}) \in \mathbb{N}^N \mid a_i^j \le |(\boldsymbol{b})|\}$. It follows that there are at most $|(\boldsymbol{b})|^N$ elements $(\boldsymbol{d}) \in \mathbb{N}^N$ such that $(\boldsymbol{d}) < (\boldsymbol{b})$ and $|(\boldsymbol{b})| = |(\boldsymbol{d})|$. So $|(\boldsymbol{b})|^N$ is an upper bound for the number of times one has to repeat (I) or (II) until we get an expression:

$$Y^{(\boldsymbol{b})} v_\lambda = \sum_{\substack{(\boldsymbol{d}) \in \mathscr{S}(\lambda) \\ (\boldsymbol{d}) < (\boldsymbol{b})}} c(\boldsymbol{b}) Y^{(\boldsymbol{d})} v_\lambda.$$

9.2. The matrix coefficients of $X_i^{(c)}$

Write H_i for the coroot $H_{\alpha_i} \in \mathfrak{h}$. Since X_i commutes with Y_j, $j \ne i$, we know $X_i^{(c)} Y^{(\boldsymbol{a})} = Y^{(\boldsymbol{a}^1)} \dots Y^{(\boldsymbol{a}^{i-2})} Y^{(\boldsymbol{a}^{i-1})} X_i^{(c)} Y^{(\boldsymbol{a}^i)} \dots Y^{(\boldsymbol{a}^n)}$. For the commutator of $X_i^{(c)}$ and $Y_i^{(a)}$ we have the well-known formula:

$$X_i^{(c)} Y_i^{(a)} = \sum_{k=0}^{\min\{a,c\}} Y_i^{(a-k)} \binom{H_i - a - c + 2k}{k} X_i^{(c-k)}.$$

Applying the rule to the vector $X_i^{(c)}(Y^{(\boldsymbol{a})} v_\lambda)$ yields:

$$\begin{aligned}
&\sum_{k_i=0}^{\min\{c, a_i^i\}} \cdots \sum_{k_n=0}^{\min\{c-k_i-\dots-k_{n-1}, a_i^n\}} Y^{(\boldsymbol{a}^1)} \dots Y^{(\boldsymbol{a}^{i-1})} \\
&\quad \left(Y_i^{(a_i^i - k_i)} \binom{H_i - c - a_i^i + 2k_i}{k_i} Y_{i-1}^{(a_{i-1}^i)} \dots Y_1^{(a_1^i)} \right) \\
&\quad \left(Y_{i+1}^{(a_{i+1}^{i+1})} Y_i^{(a_i^{i+1} - k_{i+1})} \binom{H_i - c - a_i^{i+1} + k_i + 2k_{i+1}}{k_i} Y_{i-1}^{(a_{i-1}^{i+1})} \dots Y_1^{(a_1^{i+1})} \right) \\
&\quad \dots \left(Y_n^{(a_n^n)} \dots Y_i^{(a_i^n - k_n)} \binom{H_i - c - a_i^n + k_i + \dots + k_{n-1} + 2k_n}{k_n} \dots Y_1^{(a_1^n)} \right) v_\lambda .
\end{aligned}$$

After the evaluation of the

$$\binom{H_i - c - a_i^j + k_i + \dots + k_{j-1} + 2k_j}{k_j},$$

the sum above becomes a sum of semi-standard monomials (with integral coefficients): $\sum c_{(\boldsymbol{d})} Y^{(\boldsymbol{d})} v_\lambda$. We apply now Step 2 above to the $Y^{(\boldsymbol{d})} v_\lambda$ to obtain an expression

$$X_i^{(c)}(Y^{(\boldsymbol{a})} v_\lambda) = \sum_{(\boldsymbol{b}) \in \mathscr{S}(\lambda)} c_{(\boldsymbol{b})} Y^{(\boldsymbol{b})} v_\lambda, \quad c_{(\boldsymbol{b})} \in \mathbb{Z}.$$

For $\lambda \in X^+$ let $\mathscr{S}(\lambda) = \{(\boldsymbol{b}_1), \ldots, (\boldsymbol{b}_m)\}$ be an enumeration such that $(\boldsymbol{b}_1) < (\boldsymbol{b}_2) < \cdots < (\boldsymbol{b}_m)$. Denote by V_λ^i the subspace spanned by the vectors $Y^{(\boldsymbol{b}_j)} v_\lambda$, $j \le i$. As an immediate consequence we get

Corollary 26. *The flag* $0 \subset V_\lambda^1 \subset V_\lambda^2 \subset \cdots \subset V_\lambda^m = V_\lambda$ *is B-stable.*

10. Generalization to the simply laced case

In this section we indicate roughly how to generalize the algorithm to the simple Lie algebras with simply laced Dynkin diagram. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the usual triangular decomposition. For a dominant weight $\lambda \in X^+$ let $V(\lambda)$ be the corresponding irreducible complex representation, and let v_λ be a highest weight vector. For the orthogonal algebra $\mathfrak{spin}_{2m}$, we fix the reduced decomposition (the numeration of the simple roots is in [1]):

$$w_0 = (s_{m-1}s_m)(s_{m-2}s_{m-1}s_m s_{m-2})(\cdots)(s_1 s_2 \cdots s_{m-1} s_m \cdots s_2 s_1),$$

for the exceptional groups we fix the decomposition of the longest of the Weyl group of E_8. As decompositions for the longest word of the Weyl group of E_6 and E_7 we take the induced ones.

$$\begin{aligned} w_0 = {} & s_3 s_2 (s_4 s_3 s_2 s_4)(s_5 s_4 s_3 s_2 s_4 s_5)(s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6) \\ & (s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6) \\ & (s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7)(s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6) s_7 \\ & s_8 (s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6 s_7) \\ & s_8 (s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7) s_8. \end{aligned}$$

As before, let N be the length of the longest word in the Weyl group. We fix a Chevalley basis $X_\beta, Y_\beta, H_\alpha$ of the Lie algebra, and for $(\boldsymbol{a}) \in \mathbb{N}^N$ let $Y^{(\boldsymbol{a})}$ be the monomial in the Y_{α_i} according to the fixed decomposition of w_0. In [10], we construct subsets $\mathscr{S}(\lambda)$ and $\mathscr{S} \subset \mathbb{N}^N$ such that $\mathbb{B} := \{Y^{(\boldsymbol{a})} \mid (\boldsymbol{a}) \in \mathscr{S}\}$ is a basis of $U_{\mathbb{Z}}(\mathfrak{n}^-)$ and $\mathbb{B}(\lambda) := \{Y^{(\boldsymbol{a})} v_\lambda \mid (\boldsymbol{a}) \in \mathscr{S}(\lambda)\}$ is a basis for the lattice $V_{\mathbb{Z}}(\lambda) := U(\mathfrak{n}^-) v_\lambda$. Since the Lie algebras are of simply laced type, it is now easy to check that, for the decompositions given above, the same algorithm applies: first express $X_i^{(m)} v^{(\boldsymbol{a})}$ respectively $Y_i^{(m)} v^{(\boldsymbol{a})}$ as integral linear combinations of vectors of the form $v^{(\boldsymbol{b})}$, $(\boldsymbol{b}) \in \mathscr{S}$, and then apply the straightening procedure. The proof is the same as in the $\mathfrak{sl}_{n+1}$-case. We recall below the definition of the set $\mathscr{S}$ for the Lie algebra $\mathfrak{spin}_{2m}$. We use the following notation:

$$\begin{aligned} (\boldsymbol{a}) := {} & (a_{m-1}^{m-1}, a_m^{m-1}, a_{m-2}^{m-2}, a_{m-1}^{m-2}, a_m^{m-2}, \bar{a}_{m-2}^{m-2}, \ldots, \\ & a_1^1, a_2^1, \ldots, a_{m-2}^1, a_{m-1}^1, a_m^1, \bar{a}_{m-2}^1, \ldots, \bar{a}_2^1, \bar{a}_1^1). \end{aligned}$$

Then

$$\mathcal{S} := \{(a) \in \mathbb{N}^N \mid a_i^i \geq \cdots \geq a_{m-2}^i \geq \left\{ \begin{matrix} a_{i,m-1} \\ \overline{a}_{i,m-1} \end{matrix} \right\} \geq \overline{a}_{m-2}^i \geq \cdots \geq \overline{a}_i^i \}.$$

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