# Some Algebras Associated to Automorphisms of Elliptic Curves

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## 1. Introduction

The main object of this paper is to relate a certain type of graded algebra, namely the regular algebras of dimension 3, to automorphisms of elliptic curves. Some of the results were announced in [V]. A graded algebra $A$ is called *regular* if it has finite global dimension, polynomial growth, and is Gorenstein. The precise definitions are reviewed in Section 2. As was shown in [A-S], there are two basic possibilities for a regular algebra $A$ of (global) dimension 3 which is generated in degree 1. Either $A$ can be presented by 3 generators and 3 quadratic relations, or else by 2 generators and 2 cubic relations. Throughout this paper, $A$ will denote an algebra so presented, over a ground field $k$. The number of generators will be denoted by $r$, and the degrees of the defining relations by $s$. Thus the possible values are

$$(1.1) \qquad (r,s) = \begin{cases} (3,2) \\ (2,3), \end{cases} \quad \text{and } r+s=5.$$

In ([A-S], (1.5)) it is shown that if $A$ is regular, then there are choices of generators $(x_i)$ and relations $(f_i = 0)$, $1 \leq i \leq r$, such that if we write $f_i = \sum_{j=1}^r m_{ij} x_j$ in the free associative algebra $T = k\langle x_1, \dots, x_r\rangle$ generated by the $x_i$, then the elements $g_j = \sum x_i m_{ij}$ are also defining relations, so that there is a non-singular matrix $Q = (q_{ij}) \in G\ell_r(k)$ with scalar entries such that $\sum_{i=1}^r x_i m_{ij} = \sum_{l=1}^r q_{jl} f_l$ for $j = l, \dots, r$. We call an algebra of this kind *standard* (see Section 2). In a standard algebra $A$, the linear transformation of $A_1 = \Sigma k x_i$ defined by $x_i \rightsquigarrow \sum_j x_j q_{ji}$ is canonical, independent of the choice of generators and relations. This "$Q$-transform" of $A_1$ is used in ([A-S], Section 3) to classify the standard algebras into "types," which are irreducible algebraic families; there are seven types for $r = 3$ and six types for $r = 2$. The generic algebra of each type is shown to be regular, with diagonalizable matrix $Q$, but in [A-S] no simple criterion is given for deciding whether a special standard algebra is regular.

In this paper we give such a criterion, and give a different sort of classification, in which the invariant $Q$ plays no role at all. The method involves multilinearization of homogeneous elements of the algebra $T = k\langle x_1, \dots, x_n\rangle$ (see Section 3). This $T$ is the tensor algebra of the space $A_1$. To a tensor $t \in A_1^{\otimes n} = T_n$ we associate the corresponding multilinear function $\tilde{t}$ on the product of $n$ copies of the dual space $A_1^*$. Since $\tilde{t}$ is multilinear, it is multihomogeneous and defines a locus of zeros in the product $(\mathbf{P}^{r-1})^n$ of $n$ copies of the $(r-1)$-dimensional projective space of lines in $A_1^*$.

Let $\Gamma$ be the locus of common zeros of the multilinearizations $\tilde{f}_i$ of the defining relations for $A$. Thus, $\Gamma \subset \mathbf{P}^2 \times \mathbf{P}^2$ if $r = 3$, and $\Gamma \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ if $r = 2$. Define the projections

$$\alpha\left(p^{(1)} \times \dots \times p^{(s)}\right) = \left(p^{(1)} \times \dots \times p^{(s-1)}\right) \quad \textit{(drop the last component)}$$

$$\beta\left(p^{(1)} \times \dots \times p^{(s)}\right) = \left(p^{(2)} \times \dots \times p^{(s)}\right) \quad \textit{(drop the first component).}$$

For a standard algebra $A$ the images of $\Gamma$ under these two projections are the same, both being given by the locus of zeros of $\det\widetilde{M}$, where $\widetilde{M}$ is the matrix $(\tilde{m}_{ij})$ with $m_{ij} \in T_{s-1}$ defined as above by $f_i = \Sigma m_{ij} x_j$. We call this locus $E : E = \alpha(\Gamma) = \beta(\Gamma)$, and we view $\Gamma$ as the graph of a correspondence $\alpha(p) \rightsquigarrow \beta(p)$ from $E$ to itself. If $r = 3$, then $E$ is a cubic divisor in $\mathbf{P}^2$ or is all of $\mathbf{P}^2$ if $\det\widetilde{M}$ is identically zero. If $r = 2$, then $E$ is a divisor of bidegree $(2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ or is all of $\mathbf{P}^1 \times \mathbf{P}^1$ if $\det\widetilde{M}$ is identically zero.

We call the algebra $A$ *nondegenerate* if $\Gamma$ is the graph of an automorphism $\sigma : E \to E$. For this to be so, it is necessary and sufficient that the $r \times r$ matrix $\widetilde{M}$ have rank at least $r - 1$ at every point of $\mathbf{P}^2(\bar{k})$, or of $\mathbf{P}^1(\bar{k}) \times \mathbf{P}^1(\bar{k})$, where $\bar{k}$ is an algebraic closure of $k$, i.e., that the 9 conics in $\mathbf{P}^2$, or the 4 curves of bidegree $(1,1)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the minors of $\widetilde{M}$ have no common zero. Our basic result is :

**Theorem 1.** *The regular algebras of global dimension 3 generated in degree 1 are exactly the nondegenerate standard algebras.*

To prove this theorem we first show (Section 5), by a brute force computation, that a degenerate standard algebra is not regular. Since regular algebras are standard, we can therefore limit our consideration to a nondegenerate standard $A$, and we must prove that it is regular. To such an $A$ we associate a *triple* $\mathcal{T} = (E, \sigma, L)$ consisting of (i) a scheme $E \subset (\mathbf{P}^{r-1})^{s-1}$ which is either a divisor of the type described above, or is the whole ambient space, (ii) an automorphism $\sigma$ of $E$ and (iii) the invertible $\mathcal{O}_E$-module $L = \pi^*\mathcal{O}(1)$, where $\pi$ is the inclusion of $E$ in $\mathbf{P}^2$ if $r = 3$, or is the projection of $E$ on the first factor $\mathbf{P}^1$ if $r = 2$. In the latter case, the projection of $E$ on the second factor is $\pi \circ \sigma$. In both cases the map $\pi : E \longrightarrow \mathbf{P}^{r-1}$ is the morphism defined by the sections of $L$. Hence we can recover the subscheme $\Gamma \subset (\mathbf{P}^{r-1})^s$ from the triple $\mathcal{T} = (E, \sigma, L)$; it is the image of $E$ under the closed immersion $p \rightsquigarrow (\pi(p) \cdot \pi(\sigma p), \ldots, \pi(\sigma^s p))$. Having $\Gamma$, we can try to recover $A$ as the quotient of the tensor algebra on $H^0(E, L) = A_1$ by the ideal generated by the tensors $f \in A_1^{\otimes S}$ whose multilinearizations $\tilde{f}$ vanish on $\Gamma$. Working this out, one is led to define a new graded algebra $B = \Sigma_n B_n$ by $B_n = H^0(E, L \otimes L^\sigma \otimes \cdots \otimes L^{\sigma^{n-1}})$, where $L^\sigma = \sigma^* L$, with multiplication defined by

$$bc = b \otimes c^{\sigma^n} \in B_{n+m},$$

for $b \in B_n$, $c \in B_m$. This $B = \mathcal{B}(\mathcal{T})$ is constructed so that there is a canonical isomorphism $A_1 \rightarrow B_1$ which extends to a homomorphism of graded algebras $A \rightarrow B$, and our approach to $A$ is via $B$ and this homomorphism. The point is that $A$, being defined only by generators and relations, is hard to get at, but $B$, whose homogeneous parts are spaces of sections of line bundles on $E$, is more amenable to study. In Section 6 we state (6.6) the required results about $B$, and show (6.8) how they not only imply that $A$ is regular, but also yield the following theorem about the structure of $A$ and $\mathcal{T}$.

**Theorem 2.** *Suppose $A$ is regular, let $\mathcal{T} = (E, \sigma, L)$ be the triple associated to $A$, and let $B = \mathcal{B}(\mathcal{T})$. Then (i) If $\dim E = 2$, then $A \xrightarrow{\sim} B$. If $\dim E = 1$, then $A/gA \xrightarrow{\sim} B$, where $g \in A_{s+1}$ is a (right and left) non-zero divisor such that $Ag = gA$. (ii) Let $\lambda \in \operatorname{Pic} E$ be the class of $L$. Then*

$$(1.2) \qquad (\sigma - 1)^2\lambda = 0 \text{ if } r = 3, \text{ and } (\sigma - 1)(\sigma^2 - 1)\lambda = 0 \text{ if } r = 2.$$

Section 7 is devoted to the proofs of the facts about $B$ which are stated and used in Section 6. In case $\dim E = 2$ this is elementary. When $\dim E = 1$, i.e., $E$ is a divisor on $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, then it involves Riemann-Roch and duality for $E$ and the surface. The proof is made by considering the cohomology of some relevant vector bundles on $E$. There are technical complications because $E$ need not be reduced or irreducible.

In fact, it is essential for Theorem 3 below that we study (Section 7) an arbitrary triple of the type described above, without assuming it comes from a nondegenerate standard algebra $A$. Such triples can be viewed as coming from nondegenerate algebras on $r$ generators with $r$ relations of degree $s$ of a kind more general than standard, which we call semi-standard (4.2). It turns out that the non-standard i.e., non-regular, among these give rise to triples $\mathcal{T}$ which do not satisfy (1.2). Hence, we call triples satisfying (1.2) *regular* triples. For the non-regular semi-standard algebras we have $A \xrightarrow{\sim} \mathcal{B}(\mathcal{T})$ except in one case, which we call *exceptional.* In that case, $B = A/gA$, where $g$ is normalizing of degree $s+1$, but is probably a zero divisor – we have not studied this case closely. In any case, the reader interested in regular algebras can ignore the discussion of non-standard algebras in Sections 5 and 6; it suffices to consider arbitrary triples in Section 7.

For regular algebras we obtain the following classification in terms of triples (6.8):

**Theorem 3.** *The isomorphism classes of regular graded algebras of dimension 3 generated in degree 1 are in bijective correspondence with isomorphism classes of triples $\mathcal{T}$ which are regular (i.e., satisfy condition (1.2)) and are such that, if $\dim E = 1$, then $(\sigma - 1)\lambda \neq 0$ if $r = 3$ and $(\sigma^2 - 1)\lambda \neq 0$ if $r = 2$.*

The triples excluded by the last condition are just those for which $E$ is a divisor, but the automorphism $\sigma$ can be extended from $E$ to the whole space $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. These triples give rise to the same $A$ as the triples with $E$ the whole space and $\sigma$ the extension. Incidentally, as is implicit in Theorem 2, the regular algebra $A$ is recovered from its triple by constructing $B = B(\mathcal{T})$ as above, and then keeping only the relations of degree $s$ defining $B$, i.e., ignoring the extra relation of degree $s+1$ needed to define $B$ when $\dim E = 1$.

Let us now consider some examples. The most familiar example of a regular algebra of dimension 3 with two generators is the enveloping algebra $A$ of the Heisenberg Lie algebra. It can be presented by the cubic defining relations

$$[x,[x,y]] = [y,[x,y]] = 0. \tag{1.3}$$

Calling these $f_1$ and $f_2$ we find

$$M = \begin{bmatrix} yx - 2xy & xx \\ -yy & 2yx - xy \end{bmatrix}$$

and, using $(x_1, y_1; x_2, y_2)$ as coordinates in $\mathbf{P}^1 \times \mathbf{P}^1$,

$$\widetilde{M} = \begin{bmatrix} y_1x_2 - 2x_1y_2 & x_1x_2 \\ -y_1y_2 & 2y_1x_2 - x_1y_2 \end{bmatrix}.$$

Thus, $\det \widetilde{M} = -2(x_1y_2 - x_2y_1)^2$, so that if $\mathrm{char}(k) \neq 2$, the divisor $E$ given by $\det \widetilde{M} = 0$ is the double diagonal $2\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$. Using the affine coordinate $t = y/x$ in $\mathbf{P}^1$, $E$ is given by $(t_2 - t_1)^2 = 0$, i.e., by the points $(t, t+\epsilon)$ such that $\epsilon^2 = 0$, and it is easy to check that the automorphism $\sigma$ is given by $\sigma(t, t+\epsilon) = (t+\epsilon, t+2\epsilon)$. Thus $\sigma$ is identity on the reduced curve $\Delta$ but is not trivial on $2\Delta$. If char $k = 2$, then $E = \mathbf{P}^1 \times \mathbf{P}^1$ and $\sigma$ is given by $(t_1 t_2) \rightsquigarrow (t_2, t_1)$.

One of the most interesting cases is that in which the automorphism $\sigma$ is a translation on elliptic curve $E$. It turns out that the translations correspond to algebras classified as type $A$ in ([A-S], (3.9, 11)). According to ([A-S], (10.14)), the equations defining a generic algebra of type $A$ with $r = 3$ can be put into the form

$$\begin{aligned} f_1 &= cx^2 + bzy + ayz = 0, \\ f_2 &= azx + cy^2 + bxz = 0, \\ f_3 &= byx + axy + cz^2 = 0, \end{aligned} \tag{1.4}$$

over an algebraically closed field. The multilinear equations which correspond to (1.4) are

$$\begin{aligned} \tilde{f}_1 &= cx_1x_2 + bz_1y_2 + ay_1z_2 = 0, \\ \tilde{f}_2 &= az_1x_2 + cy_1y_2 + bx_1z_2 = 0, \\ \tilde{f}_3 &= by_1x_2 + ax_1y_2 + cz_1z_2 = 0, \end{aligned} \tag{1.5}$$

where the coordinates in $\mathbf{P}^2 \times \mathbf{P}^2$ are labeled as $(x_1, y_1, z_1; x_2, y_2, z_2)$. Let $\Gamma \subset \mathbf{P}^2 \times \mathbf{P}^2$ denote the locus of common zeros of these equations. To compute the projection $\mathrm{pr}_1\Gamma = E$, we determine the values of $(x_1, y_1, z_1)$ for which there exist $(x_2, y_2, z_2)$ solving (1.5). Since the equations (1.5) are linear in $(x_2, y_2, z_2)$, the values of $(x_1, y_1, z_1)$ in question are those such that $\det \widetilde{M} = 0$, where, dropping the subscript 1,

$$\widetilde{M} = \begin{bmatrix} cx & bz & ay \\ az & cy & bx \\ by & ax & cz \end{bmatrix}.$$

Thus $E$ is the "level 3" cubic curve defined by the equation

$$(a^3+b^3+c^3)\,xyz = abc\,(x^3+y^3+z^3)\,. \tag{1.6}$$

Notice that the computation of $\mathrm{pr}_2\Gamma$ yields the same answer, in agreement with the fact that $\Gamma$ is the graph of an automorphism.

The automorphism $\sigma$ can also be computed from the equation (1.5). Given $(x,y,z)$ such that $\det \widetilde{M}=0$ we have $\sigma(x,y,z,)=(x',y',z')$ where $(x',y',z')$ is a vector orthogonal to the row vectors of $\widetilde{M}$. Such vectors are the "outer" or "cross" products of any two of those rows, and if $A$ is nondegenerate, i.e., if $\widetilde{M}$ has rank 2 at every point of $E$, then there will be a pair of rows whose outer product is not zero, so defines a point in $\mathbf{P}^2$. Taking the top two rows we find for example that in general

$$\sigma(x,y,z) = \left(acy^2-b^2xz,\, bcx^2-a^2yz,\, abz^2-c^2xy\right). \tag{1.7}$$

Let us choose the rational point $(1,-1,0)$ on $E$ as origin. Then $\sigma(1,-1,0)=(a,b,c)$. It follows that $\sigma$ is translation by the point $(a,b,c)$ in the group law on $E$. One way to see this is to recall that the only automorphisms of a generic elliptic curve which are not translations have order 2, and this is not the case for $\sigma$, provided that $(a,b,c)$ is generic.

Using Theorem 1 it is an easy exercise to check that, as conjectured in [A-S, 10.37i], the algebra (1.4) is regular unless either $a^3=b^3=c^3$, or two of the three quantities $(a,b,c)$ are zero. (But in the case $r=2$, 10.37i is not quite right; $A_{1,0,0}$ is regular, even linear.) If exactly one of them is zero, then $E$ is a "triangle" consisting of the lines $x=0$, $y=0$, and $z=0$, unless $a^3+b^3+c^3=0$, in which case $E=\mathbf{P}^2$. The automorphism $\sigma$ either carries the lines into themselves (if $c=0$) or permutes them cyclically, carrying the line $z=0$ to the line containing the point $(a,b,c)$. The non-regular case, when two of the $a,b,c$ are zero, is the case in which the point $(a,b,c)$ is a vertex of the triangle.

If $abc\neq 0$, then $E$ is smooth unless $(3abc)^3=(a^3+b^3+c^3)^3$, in which case, $E$ is again a triangle, unless char $k=3$, when $E$ is the triple line $(x+y+z)^3=0$. Again, the non-regular case $a^3=b^3=c^3$ occurs when the point $(a,b,c)$ is a vertex of the triangle, at least in char $\neq 3$.

In ([A-S], (6.11)), a cubic curve was introduced to describe the condition that $A$ is a skew polynomial ring. For algebras of type $A$, that cubic is the locus $x^3+y^3+z^3+6uxyz=0$, where $6u=(3a+3b)/c$. This curve is *not* isomorphic to our curve $E$ (1.6) for most choices of the parameters $(a,b,c)$. Thus there are two $j$-invariants associated to regular algebras of dimension 3. Each of them has significance for the algebra. They agree for types $E$, $H$ and $S$ in the classification [A-S], but not for types $A$ and $B$. The presence of two elliptic curves caused some confusion.

This work grew out of an attempt to understand the conjectural description of algebras of type $A$ in [A-S]. The main thrust of these conjectures is to describe the condition that an algebra defined by equation (1.4) be a finite module over its center, in terms of the point $p = (a, b, c) \in \mathbf{P}^2$. The condition is that $p$ must lie on one of a sequence of curves $C_n$ defined by certain recursive relations. It has turned out that these curves are modular curves, $C_n$ being the locus of points $p = (a, b, c) \in \mathbf{P}^2$ such that $p$ has order $n$ on the curve $E$ (1.5). In other words, the algebras which are finite modules over their centers are those such that the automorphism $\sigma$ has finite order. The same is true for all regular algebras of dimension 3.

It is quite easy to prove that, if $\sigma$ has finite order, then the algebra $B = \mathcal{B}(E, \sigma, L)$ defined above is finite over its center. This is done in Proposition (8.5), and we make use of that fact in the last section, to prove that regular algebras of dimension 3 are left and right noetherian. However, since our description of the algebra $A$ is indirect, our proof that $A$ is finite over its center if $\sigma$ is of finite order is considerably more complicated. It is based on a detailed study of $A$-modules. We will present this proof in a subsequent paper [A-T-V] (see also [V]).

The geometric approach to regular algebras $A$ of dimension 3 which we have used, relating them to triples, works beautifully and simply for regular algebras of dimension 2, which are those of the form $A = B(\mathbf{P}^1, \sigma, \mathcal{O}(1))$, where $\sigma$ is an arbitrary automorphism of $\mathbf{P}^1$. We learned only recently that the translations on elliptic curves have been used also to construct algebras in higher dimension. Already in 1983 Sklyanin [S] used an elliptic curve, translation by a point, and Theta functions $\theta_{ij}$ which are sections of a bundle of degree 4 to make an algebra with 4 generators and 6 quadratic relations which is presumably regular*, and recently Odessky and Feigin [O-F] generalized his method using sections of a bundle of degree $n$ to obtain algebras with $n$ generators and $\binom{n}{2}$ quadratic defining relations. The situation for $n > 3$ is more subtle than for $n = 3$, because the space of quadratic relations defining $A$ is *not* the space of *all* quadratic relations holding in $B$, as it is for us in dimension 3, but is, rather, an $\binom{n}{2}$-dimensional subspace of that $(n^2 - 2n)$-dimensional space, and the description of the subspace is not at all obvious.

## 2. Background and notational conventions

Let $k$ be a field. By *graded algebra $A$ over $k$*, we will always mean an algebra graded in positive degrees, such that the part of degree zero is $k$:

$$A = k + A_1 + A_2 + \dots . \tag{2.1}$$

*The regularity of these algebras was proved recently by P. Smith and T. Stafford.

Except for rings occurring incidentally the algebras we study will be generated by finitely many elements of degree 1. Such a ring is a quotient of the tensor algebra $T = T(A_1)$ on the vector space $A_1$, by an ideal $I$ which is generated by homogeneous elements: $A \approx T/I$.

By *module* $M$ over a graded ring $A$, we will always mean a graded left or right module, and homomorphisms are graded, i.e., degree-preserving, homomorphisms. To avoid confusion, we will sometimes indicate that we are dealing with a left or right module by a subscript $A$, viz. ${}_AM$ or $M_A$, depending on the case. The module $M$ is said to be *left bounded* if $M_n = 0$ for all $n \ll 0$. A key fact about a left bounded module $M$ is this version of the Nakayama lemma:

**Proposition 2.2.** *Let $I = A_1 + A_2 + \ldots$ be the augmentation ideal of $A$, let $M$ be a left bounded module, and let $\varphi : N \to M$ be a homomorphism. If the composed map $N \to M/IM$ is surjective, then $\varphi$ is surjective.* $\square$

A module $M$ is called *locally finite* if each graded piece $M_n$ is a finite dimensional $k$-vector space [An]. The *Hilbert function* of a locally finite module is the function

$$n \rightsquigarrow m_n = \dim_k M_n, \tag{2.3}$$

which takes finite values for all $n$. Since it is a quotient of the tensor algebra, an algebra $A$ of the form (2.1) is left bounded and locally finite, as an $A$-module, and so it has a Hilbert function.

The *shift* $M(r)$ of a left or right module $M$ is the module defined by

$$M(r)_n = M_{n+r}. \tag{2.4}$$

It is an elementary exercise to prove that a left bounded module $P$ is projective in the category of graded modules if and only if it is isomorphic to a direct sum of shifts of the module $A$:

$$P \approx \bigoplus_i A(-\ell_i), \tag{2.5}$$

in other words, if and only if $P$ is free, with basis $\{e_i\}$ consisting of homogeneous elements $e_i$ of varying degrees $\ell_i$ which are bounded below (see [C, Section 7, Proposition 9] or [$B_1$, Ch.10, Sec. 8, No. 7, Prop. 8]. Cartan's "gradué par une graduation $\geq 0$" can obviously be replaced by "left bounded"). Such a module $P$ is locally finite if and only if either the direct sum is finite or if $\ell_i \to \infty$ as $i \to \infty$. If we represent a direct sum as a row vector, then maps $f : P \to P'$ between two such projectives are given by right multiplication by row-finite matrices with homogeneous entries in $A$. Every left bounded module has a resolution by projective modules of this same form. In fact, it has a *minimal* resolution which is unique up to non-unique isomorphism and which is a direct summand of every resolution [C, Section 7, Definition 2]. If $M$ is locally finite, so are the projectives in its minimal resolution.

The *projective dimension* of a module $M$ is the minimal length of a projective resolution of $M$, and the algebra $A$ is said to have *global dimension* $d$ if every $A$-module $M$ has projective dimension $\leq d$ and $d$ is minimal with this property. A basic fact about graded algebras is that their global dimension is equal to the projective dimension of the left module ${}_Ak$. Indeed, let

$$0 \to P^d \longrightarrow \dots \xrightarrow{f_2} P^1 \xrightarrow{f_1} P^0 \longrightarrow {}_A k \to 0 \tag{2.6}$$

be a minimal resolution of ${}_Ak$ by left bounded, locally finite projectives of the form (2.5):

$$P^q = \bigoplus_j A\left(-\ell_{qj}\right), \text{ and } P^d \neq 0. \tag{2.7}$$

The shape of the minimal resolution of the right module $k_A$ is the same, as is shown by the formula

$$\mathrm{Tor}_q^A\left(k_A, {}_A k\right) \approx \oplus k\left(-\ell_{qj}\right), \tag{2.8}$$

and in particular, its length is also $d$. Hence $\mathrm{Tor}_i^A(k_A, M)$ and $\mathrm{Tor}_i^A(N, {}_A k)$ vanish for $i > d$ and for all left modules $M$ and right modules $N$. If $M$ or $N$ has a minimal resolution, it follows that its length is $\leq d$. Thus $\mathrm{Ext}_A^i(M, L) = 0$ for all $L$ and all $i > d$, if $M$ is left bounded. For arbitrary modules, see [Au, Theorem 1], where the ungraded case is treated. There is no difficulty in adapting the proof to the graded case.

In general, the projectives which appear in the resolution (2.6) need not be finitely generated. However, if they are finitely generated we may consider the transposed complex of right modules, obtained by applying the functor $M \rightsquigarrow M^* = \mathrm{Hom}_A(M, A)$. Note that $A(\nu)^*$ is the right module $A(-\nu)$. An algebra of finite global dimension $A$ will be called *Gorenstein* if

**(2.9)**. (i) the projectives appearing in a minimal resolution of ${}_Ak$ are finitely generated, and

(ii) the transposed complex is a resolution of a right module isomorphic to $k_A$, shifted to some degree $e$:

$$0 \leftarrow k_A(e) \leftarrow P^{d*} \leftarrow \ldots \leftarrow P^{1*} \leftarrow P^{0*} \leftarrow 0. \tag{2.10}$$

This last condition can also be expressed as follows:

$$\operatorname{Ext}^q_A({}_Ak, A) = 0 \text{ if } q \neq d, \text{ and } \operatorname{Ext}^d_A({}_Ak, A) \approx k_A(e). \tag{2.11}$$

Finally, a graded algebra $A$ is called a *regular* algebra of dimension $d$ if it satisfies these conditions:

(2.12)
(i) $A$ has global dimension $d$,
(ii) is Gorenstein, and
(iii) has polynomial growth,

i.e., has $\dim_k A_n \leq cn^\delta$ for some positive real numbers $c, \delta$. For every regular algebra that we know, the minimal such $\delta$ is equal to $d-1$.

Going back to the minimal resolution (2.6) of the module ${}_Ak$, it is clear that $P^0 \approx A$. The next two terms $P^1, P^2$ can be described in terms of the generators and relations of $A$. Suppose that $A$ is written as quotient of the free associative algebra $T = k\langle x_1, \ldots, x_n\rangle$, as $A \approx T/I$, where $(x_1, \ldots, x_n)$ is a minimal set of homogeneous generators. Let $\ell_{1j} = \deg x_j$. Then

$$P^1 \approx \bigoplus_{j=1}^{n} A(-\ell_{1j}),$$

and if we view the elements of $P^1$ as row vectors $(a_1, \ldots, a_n)$, with $a_j \in A(-\ell_{1j})$, the map $P^1 \to P^0$ is right multiplication by the column vector $x = (x_1, \ldots, x_n)^t$. This is rather trivial. The interpretation of $P^2$ in terms of defining relations is less obvious, but it is well known (cf. [W], p. 441). Let $\{f_j\}$ be a minimal set of homogeneous generators for the ideal $I$, and let $\ell_{2j} = \deg f_j$. Then $P^2 \approx \oplus_j A(-\ell_{2j})$, and the map $f_2 : P^2 \to P^1$ is right multiplication by the matrix $M$ defined as follows. In the non-commutative polynomial ring, we may write the defining relations uniquely in the form

$$f_i = \sum_j m_{ij} x_j, \tag{2.13}$$

where $m_{ij} \in T_{\ell_{2i} - \ell_{1j}}$. Then $M$ is the image in $A$ of the matrix $(m_{ij})$. Thus the start of a resolution of ${}_Ak$ has the form

$$\bigoplus_i A(-\ell_{2i}) \xrightarrow{M} \bigoplus_j A(-\ell_{1j}) \xrightarrow{x} A \longrightarrow {}_Ak \longrightarrow 0, \tag{2.14}$$

$x$ and $M$ being defined as above. Conversely, if (2.14) is an exact sequence and $(m_{ij})$ is a lifting of the matrix $M$ to the free algebra $T = k\langle x_1, \dots, x_n\rangle$, then the elements $f_i$ defined by (2.13) form a minimal set of defining relations for the algebra $A$ [W, loc cit].

It is difficult to interpret the remaining terms of a minimal resolution concretely, though canonical, non-minimal resolutions are known [C-E], [An]. However, if $A$ is a regular algebra of dimension 3, only one term in the minimal resolution remains to be found, and the Gorenstein condition together with the requirement of polynomial growth determine it completely. As is shown in ([AS], (1.6)), the minimal resolution of ${}_A k$ for a regular algebra $A$ of dimension 3 has the form

$$(2.15) \qquad 0 \to A(-s-1) \xrightarrow{x^t} A(-s)^r \xrightarrow{M} A(-1)^r \xrightarrow{x} A \longrightarrow {}_A k \to 0,$$

where $(r, s) = (3, 2)$ or $(2, 3)$. Thus such an algebra has $r$ generators and $r$ relations of degree $s$, and $r + s = 5$. The Gorenstein symmetry also tells us that the entries of the row vector $g = (x^t)M$ form a minimal generating set for the ideal $I$. Thus

$$(2.16) \qquad g^t = \left((x^t)\, M\right)^t = QMx = Qf,$$

for some $Q \in G\ell_r(k)$.

In each case, $r = 2$ and $r = 3$, the Hilbert function of a regular algebra of dimension 3 with $r$ generators is the same as that of a commutative polynomial algebra in three variables, the variables being of degrees $(1, 1, 1)$ if $r = 3$ and of degree $(1, 1, 2)$ if $r = 2$. Explicitly,

(2.17)
$$\begin{aligned} \dim_k A_n = &\ \tfrac{1}{2}\left(n^2 + 3n + 2\right) \text{ if r=3, and} \\ = &\begin{cases} \tfrac{1}{4}\left(n^2 + 4n + 4\right) & \text{if } r = 2 \text{ and } n \text{ is even,} \\ \tfrac{1}{4}\left(n^2 + 4n + 3\right) & \text{if } r = 2 \text{ and } n \text{ is odd.} \end{cases} \end{aligned}$$

We will call an algebra $A$ a *standard algebra* if $A$ can be presented by $r$ generators $x_j$ of degree 1 and $r$ relations $f_i$ of degree $s$, such that, with $M$ defined by (2.13), $(r, s) = (3, 2)$ or $(2, 3)$ as above, and there is an element $Q \in G\ell_r(k)$ such that (2.16) holds.

For any standard algebra $A$, the sequence (2.15) is a complex, which, by the discussion above, is exact at the first three terms on the right. We will speak of this complex as the *potential resolution* of ${}_A k$. This potential resolution is actually a resolution at all generic points of the variety which parametrizes such sequences, and if it is a resolution, then $A$ is a regular algebra of dimension 3.

## 3. Multilinearization

Let $T = k\langle x_0, \ldots, x_n\rangle$ denote the free associative $k$-algebra on generators $x_0, \ldots, x_n$ of degree 1, and let $V = T_1^*$ denote the dual space of $T_1$. A homogeneous element $f \in T_d = T_1^{\otimes d}$ of degree $d$ defines a linear map

$$\tilde{f} : V \otimes \ldots \otimes V = V^{\otimes d} \to k, \tag{3.1}$$

or equivalently, a multilinear form $\tilde{f} : V \times \ldots \times V = V^d \to k$. This form may be viewed as a section of the sheaf

$$\mathcal{O}(1, 1, \ldots, 1) = \mathrm{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \otimes \ldots \otimes pr_d^* \mathcal{O}_{\mathbf{P}}(1) \tag{3.2}$$

on the product $\mathbf{P} \times \ldots \times \mathbf{P}$ of $d$ copies of the projective space $\mathbf{P} \approx \mathbf{P}^n$ of lines in $V$ (or of hyperplanes in $T_1$). We will denote this product by $(\mathbf{P})^d$. Since $\tilde{f}$ is multilinear and hence multihomogeneous, its zeros define a locus in $(\mathbf{P})^d$.

Note that with natural conventions, we have

$$\widetilde{(fg)}(v_1, \ldots, v_p; w_1, \ldots, w_q) = \tilde{f}(v_1, \ldots, v_p)\, \tilde{g}(w_1, \ldots, w_q) \tag{3.3}$$

for $f \in T_p$, $g \in T_q$.

Let $A = T/I$ be a graded quotient ring of $T$, let $\tilde{I}_d$ be the space of multilinear forms $\tilde{f}$ such that $f \in I_d$. We will denote by $\Gamma_d$ the scheme of zeros of $\tilde{I}_d$ in the product $(\mathbf{P})^d$:

$$\Gamma_d = \mathcal{Z}\left(\tilde{I}_d\right) \subset (\mathbf{P})^d. \tag{3.4}$$

**Proposition 3.5.** (i) *For any $d$, $\Gamma_{d+1} \subset (\mathbf{P} \times \Gamma_d) \cap (\Gamma_d \times \mathbf{P})$, with equality if $I_{d+1} = T_1 I_d + I_d T_1$, for instance if $I$ is generated in degrees $\leq d$.*

(ii) *Let $\mathbf{P}_i$ denote the $i$-th factor of the product $(\mathbf{P})^d$, and for $1 \leq i < j \leq d$, let $\mathrm{pr}_{ij} = \mathrm{pr}_{ij}^{(d)}$ denote its projection to the product $\mathbf{P}_i \times \ldots \times \mathbf{P}_j = \prod_{\nu=i}^{j} \mathbf{P}_\nu$. Then $\mathrm{pr}_{ij}(\Gamma_d)$ is a closed subset of $\Gamma_{j-i+1}$.*

**Proof.** (i) With the obvious notation, the locus of zeros of $\widetilde{T}_1 \tilde{I}_d$ in $(\mathbf{P})^{d+1}$ is $\mathbf{P} \times \Gamma_d$, and similarly, the locus of zeros of $\tilde{I}_d \widetilde{T}_1$ is $\Gamma_d \times \mathbf{P}$. Since $T_1 I_d + I_d T_1 \subset I_{d+1}$, it follows that $(\mathbf{P} \times \Gamma_d) \cap (\Gamma_d \times \mathbf{P}) \supset \Gamma_{d+1}$. If $I$ is generated in degrees $\leq d$, then $T_1 I_d + I_d T_1 = I_{d+1}$. This implies that $(\mathbf{P} \times \Gamma_d) \cap (\Gamma_d \times \mathbf{P}) = \Gamma_{d+1}$.

(ii) The projection is closed because $\Gamma_d$ is proper. Also, $\mathrm{pr}_{ij}^{-1}(\Gamma_{j-i})$ is the locus of zeros in $(\mathbf{P})^d$ of $\widetilde{T}_{i-1} \tilde{I}_{j-i+1} \widetilde{T}_{d-j}$, and $T_{i-1} I_{j-i+1}\, T_{d-j} \subset I_d$. Therefore $\mathrm{pr}_{ij}^{-1}(\Gamma_{j-i+1}) \supset \Gamma^d$, and $\Gamma_{j-i+1} \supset \mathrm{pr}_{ij}(\Gamma_d)$. $\square$

**Proposition 3.6.** *Let $0 \le i \le d$, and let $\pi : \Gamma_{d+1} \to \Gamma_d$ denote the projection $(p_1, \dots, p_i, q, p_{i+1}, \dots, p_d) \rightsquigarrow (p_1, \dots, p_d)$.*
*(i) The fibres of $\pi$ are linear subspaces of $\mathbf{P}$.*
*(ii) Let $p \in \Gamma_d$, and let $L$ be the fibre of $\pi$ at $p$. If $\dim L \le 0$, then $\pi$ is a closed immersion locally in a neighborhood of $p \in \Gamma_d$.*

**Proof.** The first part follows from the fact that the defining equations of $\Gamma_{d+1}$ are multilinear. This implies that the scheme-theoretic fibre $L$ at $p \in \Gamma_d$ is defined by linear equations, and so it is a linear space. To prove the second part, note that the projection is a proper map. So if the fibre at $p$ has dimension zero, then $\pi$ is a finite map in some neighborhood of $p$, by Chevalley's Theorem [EGA III, 7.8]. This means that the localization of $\pi_* \mathcal{O}_{\Gamma_{d+1}}$ is a finite $\mathcal{O}_{\Gamma_{d,p}}$-module. The fibre is isomorphic to $p$ or else is empty. Hence, by the Nakayama Lemma, the map $\mathcal{O}_{\Gamma_d} \to \mathcal{O}_{\Gamma_{d+1}}$ is surjective locally at $p$. □

**Proposition 3.7.** (i) *Assume that for some $d$, $\mathrm{pr}_{1d}^{(d+1)}$ defines a closed immersion from $\Gamma_{d+1}$ to $\Gamma_d$, thus identifying $\Gamma_{d+1}$ with a closed subscheme $E \subset \Gamma_d$. Then $\Gamma_{d+1}$ defines a map $\sigma : E \to \Gamma_d$, by the rule*

$$\sigma(p_1, \dots, p_d) = (p_2, \dots, p_{d+1}),$$

*where $(p_1, \dots, p_{d+1})$ is the unique point of $\Gamma_{d+1}$ lying over $(p_1, \dots, p_d) \in \Gamma_d$.*
(ii) *If in addition $\sigma(E) \subset E$ and if $I$ is generated in degree $\le d$, then $\mathrm{pr}_{1d}^{(n)} : \Gamma_n \to E$ is an isomorphism for every $n \ge d+1$.*

**Proof.** The first assertion follows from the identification of $E$ with $\Gamma_{d+1}$. To prove the second one, we define an inverse function $\psi$ as follows: If $p = (p_1, \dots, p_d) \in E$, we set $\psi(p) = (p_1, \dots, p_n)$, where the coordinates $p_\nu$ are chosen so that $\mathrm{pr}_{1+i,d+1}^{(n)}(p) = (p_{1+i}, \dots, p_{d+i}) = \sigma^i(p)$ for $0 \le i \le n-d$. Since $\sigma^i(p) \in E$ for each $i$, it follows from the equality $\Gamma_{r+1} = (\Gamma_r \times \mathbf{P}) \cap (\mathbf{P} \times \Gamma_r)$ (3.5)(i) and induction that $q = \psi(p)$ is the unique point of $\Gamma_n$ such that $\mathrm{pr}_{1d}^{(n)}(q) = p$ for each $p \in E$. Hence the projection $\mathrm{pr}_{1d}^{(n)}$ maps $\Gamma_n$ isomorphically to $E$, by (3.6)(ii). □

We will now interpret the loci $\Gamma_n$ associated to a graded algebra $A$ in terms of certain $A$-modules which are analogous to points of projective varieties.

*Definition* **3.8**. A graded right $A$-module $M$ will be called a *point module* if it satisfies the following conditions:
(i) $M$ is generated in degree zero,
(ii) $M_0 = k$, and
(iii) $\dim M_i = 1$ for all $i \ge 0$.

By a *truncated point module of length* $d+1$, we mean a module $M$ satisfying (i) and (ii), and whose Hilbert function is

$$\dim M_i = \begin{cases} 1 & \text{if } 0 \le i \le d \\ 0 & \text{otherwise} \end{cases}.$$

We may also speak of a *family* of point modules parametrized by $S = \operatorname{Spec} R$, where $R$ is a commutative $k$-algebra. Such a family is, by definition a graded $R \otimes A$-module $M$, generated in degree zero, such that $M_0 = R$ and that $M_i$ is locally free of rank 1 for each $i$. Families of truncated point modules are defined similarly.

**Proposition 3.9.** *With the above notation, there is a one-to-one correspondence between points $(p_i, \ldots, p_d) \in \Gamma_d$ and truncated point modules of length $d+1$. More precisely, $\Gamma_d$ represents the functor of flat families of such modules: If $R$ is a commutative $k$-algebra and $S = \operatorname{Spec} R$, then points of $\Gamma_d$ with coordinates in $R$ correspond to isomorphism classes of families of truncated point modules, parametrized by $S$.*

**Proof.** Let $M$ be a family of truncated point modules. Since point modules have been rigidified by the requirement that $M_0 = R$, they are compatible with descent. So we may localize $R$, to reduce to the case that each graded piece $M_j$, $1 \le j \le d$, is a free $R$-module of rank 1. We choose a basis $m_j$ for each $M_j$. For $1 \le j \le d$ and $0 \le i \le n$, we write out the products of the bases $m_j$ by the generators $x_i$ of $A$:

$$m_{j-1} x_i = m_j a_{ij}, \tag{3.10}$$

for some $a_{ij} \in R$. In this way, we obtain a set of $d$ points

$$a_j = (a_{0j}, \ldots, a_{nj}) \in \mathbf{P} \tag{3.11}$$

with coordinates in $R$, i.e., an $R$-valued point of $(\mathbf{P})^d$. This point has the property that if $f(x)$ is any polynomial in $R \otimes T$ of degree $d$, then the formula

$$m_0 f(x) = m_d \tilde{f}\,(a_1, \ldots, a_d)$$

holds, where $M$ is given the structure of $T$-module via the map $T \to A$. If $f(x) \in I_d$, then $f(x)$ represents zero in $A$, hence $\tilde{f}(a_1, \ldots, a_d) = 0$. So by definition of the locus $\Gamma_d$, $(a_1, \ldots, a_d) \in \Gamma_d(R)$. This procedure is reversible: Starting with a point of $\Gamma_d(R)$, we localize $R$ so that the point can be represented by a $d$-tuple of homogeneous coordinates $(a_1, \ldots, a_d)$, with $a_{ij} \in R$. We let $M_j$ denote a free module of rank one with basis $m_j$, and we define a structure of module on the direct sum $M = M_0 \oplus \ldots \oplus M_d$

by the rule (3.10). It is easily checked that this definition produces a truncated point module, and that the two functors thus defined are quasi-inverses of each other. □

If $A = T$ is the free associative algebra itself, there are no equations, so $\Gamma_d = \Gamma_d(T)$ is just the product $(\mathbf{P})^d$. In this case, the universal family of truncated point modules corresponding to the functor we have described is a sheaf $M = M_0 \oplus \ldots \oplus M_d$ on $(\mathbf{P})^d$, where

$$(3.12) \qquad M_j = \operatorname{pr}_1^* \mathcal{O}_{\mathbf{P}}(1) \otimes \ldots \otimes \operatorname{pr}_j^* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}(1, \ldots, 1, 0, \ldots, 0).$$

Since $T_1 = H^0(\mathbf{P}, \mathcal{O}(1))$, multiplication of tensors provides us with a map $M_j \otimes_k T_1 \to M_{j+1}$, which defines the $T$-module structure on the sheaf $M$.

For an arbitrary quotient algebra $A$ of $T$, the universal truncated point module is obtained by restricting the sheaf $M$ defined above to $\Gamma_r(A)$.

Note that the functorial description of the schemes $\Gamma_d$ shows that they are intrinsic to the algebra $A$, and do not depend on its presentation as a quotient of $T$.

**Corollary 3.13.** *Let $\Gamma$ denote the inverse limit of the sets $\Gamma_d$. Then the points of $\Gamma$ are in one-to-one correspondence with point modules.*

We may also obtain an object which represents the functor of point modules analogous to (3.9), by considering $\Gamma$ as a pro-scheme instead of taking the inverse limit. Or, $\Gamma$ has the structure of an infinite-dimensional scheme, namely the subscheme of the infinite product $\mathbf{P} \times \mathbf{P} \times \ldots$ defined by the relations $\tilde{f} = 0$ for all $f \in I$.

Note that if one of the projections $\operatorname{pr}_{1d} : \Gamma_{d+1} \to \Gamma_d$ is an isomorphism, then the inverse system $\{\Gamma_i\}$ is constant for $i \geq d$, by Proposition (3.7)(ii), and hence $\Gamma \approx \Gamma_d$. This occurs in many cases, such as with $d = s = 5 - r$, when $A$ is regular algebra of dimension 3.

As a simple example, we may consider the commutative ring $A = k[x_0, \ldots, x_n]$. Here the defining equations are $x_i x_j = x_j x_i$, and the associated bilinear equations define the diagonal $\Gamma_2 = \Delta \subset \mathbf{P} \times \mathbf{P}$. Proposition (3.7) applies, $\sigma$ is the identity map $\mathbf{P} \to \mathbf{P}$, and $\Gamma_d \approx \mathbf{P}$ for all $d$. By the above Corollary, point modules are in bijective correspondence with points of the projective space $\mathbf{P}$.

As another example, we may consider the regular algebras of global dimension 2 which are generated in degree 1. They are defined by a single quadratic relation in two variables of the form

$$(3.14) \qquad (cx + dy)x - (ax + by)y = 0,$$

with $ad - bc \neq 0$. For these algebras, $\Gamma_2$ is the graph of an automorphism of $\mathbf{P}^1$, namely

$$(3.15) \qquad \sigma(x, y) = (ax + by, cx + dy),$$

and two such algebras are isomorphic if and only if the conjugacy classes of the corresponding automorphisms of $\mathbf{P}^1$ are the same. Proposition (3.7) shows that for these algebras as well, point modules are in bijective correspondence with points of $\mathbf{P}^1$. On the other hand, the relation $yx = 0$ defines an algebra which is of global dimension 2 but is not Gorenstein, hence is not a regular algebra. For this algebra, $\Gamma_2 = (p \times \mathbf{P}^1) \cup (\mathbf{P}^1 \times q)$, where $p = (1,0)$ and $q = (0,1)$, is the graph of a degenerate correspondence on $\mathbf{P}^1$. In this case, the schemes $\Gamma_n$ are all non-isomorphic.

*Question* **3.16**. Assume that $A$ is noetherian and finitely presented. Is it true that the inverse system $\{\Gamma_d\}$ is constant for large $d$?

**3.17**. *The algebra B*: The most naive attempt to recover an algebra $A$ from its multilinearization correspondences $\Gamma_d$ yields an algebra $B$ and a canonical homomorphism $A \to B$, which we will now describe. In fact, we can associate canonically a graded algebra $B$ to any sequence of subschemes $\{Z_d \subset (\mathbf{P})^d\}$ having the property

$$\text{(3.18)} \qquad \mathrm{pr}_{1,d-1}(Z_d) \subset Z_{d-1} \text{ and } \mathrm{pr}_{2,d}(Z_d) \subset Z_{d-1} \text{ for all } d.$$

It follows from this condition that $\mathrm{pr}_{ij}(Z_d) \subset Z_{j-i+1}$ for all $1 \le i < j \le d$, as in Proposition (3.5)(ii). Denote by $L_d$ the restriction of the invertible sheaf $\mathcal{O}(1,1,\ldots,1) = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}}(1) \otimes \ldots \otimes \mathrm{pr}_d^*\mathcal{O}_{\mathbf{P}}(1)$ to $Z_d$:

$$\text{(3.19)} \qquad L_d = \mathcal{O}(1,1,\ldots,1) \otimes_{\mathcal{O}_{(\mathbf{P})^d}} \mathcal{O}_{Z_d},$$

and let $B_d = H^0(Z_d, L_d)$. Since $\mathrm{pr}_{1,i}(Z_{i+j}) \subset Z_i$ and $\mathrm{pr}_{i+1,i+j}(Z_{i+j}) \subset Z_j$, we have $Z_i \times Z_j \supset Z_{i+j}$. Therefore we can define a multiplication map $B_i \times B_j \to B_{i+j}$ by applying $H^0$ to the obvious isomorphism $\mathrm{pr}_{1,i}^*(L_i) \otimes_{\mathcal{O}_{\Gamma_d}} \mathrm{pr}_{i+1,i+j}^*(L_j) \to L_{i+j}$. This multiplication makes $B = \oplus B_d$ into a graded associative algebra. We will refer to $B$ as the algebra *associated* to the sequence $\{Z_d\}$. We remark that $B$ need not be generated in degree 1, and if it is, then the sequence $\{Z_d\}$ may be properly contained in the sequence $\{\Gamma_d = \Gamma_d(B)\}$.

If $Z_d = (\mathbf{P})^d$ for all $d$, the associated algebra $B$ is the free algebra $T$. But if we start with an arbitrary algebra $A = T/I$ then the algebra $B$ associated to the sequence $\{Z_d = \Gamma_d(A)\}$ need not be isomorphic to $A$, though the two algebras are always related by a homomorphism:

**Proposition 3.20.** *Let $A = T/I$ be a quotient of the free ring, and let $\Gamma_d = \Gamma_d(A)$. Let $B$ be the algebra associated to the sequence $\{\Gamma_d\}$. There is a canonical homomorphism,*

$$\varphi : A \to B,$$

*which is bijective in degree* 1.

**Proof**. The functorial maps

$$H^0\left((\mathbf{P})^d, \mathcal{O}(1,1,\dots,1)\right) \to H^0\left(\Gamma_d, L_d\right)$$

define a homomorphism from the free algebra $T$ to $B$, which carries an element $f \in T_d$ to the restriction to $\Gamma_d$ of the corresponding section $\tilde{f}$ of $\mathcal{O}(1,\dots,1)$. Since $\Gamma_d$ is the scheme of zeros of $\tilde{I}_d = \{\tilde{f} | f \in I_d\}$, the ideal $I$ is in the kernel of the homomorphism $T \to B$. Therefore this homomorphism factors through $T/I = A$. In degree 1, $\Gamma_1$ is the projective space of hyperplanes in $T_1/I_1 = A_1$, and our homomorphism is the canonical isomorphism $A_1 \xrightarrow{\sim} H^0(\Gamma_1, \mathcal{O}(1)) = B_1$.

## 4. Multilinearization of semi-standard algebras

Throughout this section, $A$ will denote an algebra which is presented in the form $A = T/I$, where $T$ is a non-commutative polynomial ring with $r$ generators of degree 1 and $I$ is an ideal generated by $r$ linearly independent relations of degree $s$. We assume moreover that $(r,s) = (3,2)$ or $(2,3)$ as in the end of Section 2.

Let $f = (f_1, \dots, f_r)^t$ be a column vector of defining relations, i.e., of generators for $I$. As in Section 2, there are uniquely defined matrices $M, N$ with entries in the tensor algebra $T$ such that

$$f = Mx \text{ and } f^t = x^t N. \tag{4.1}$$

Let $\mathbf{P} = \mathbf{P}^{r-1}$. We consider the sequence of subschemes $\Gamma_d \subset (\mathbf{P})^d$ defined in the previous section as the zeros of the multilinearizations $\tilde{I}_d$. It follows from Proposition (3.5) that the whole sequence of schemes is determined by $\Gamma = \Gamma_s$, the locus of zeros of the multilinearized relations $\tilde{f}_i$ in $(\mathbf{P})^s$.

Let $E$ be the scheme-theoretic image of $\Gamma$ via the projection $\mathrm{pr}_{1,s-1}$. We denote the restriction of this projection to $\Gamma$ by $\pi_1 : \Gamma \to E$, and similarly, we denote by $\pi_2 : \Gamma \to E'$ the map from $\Gamma$ to its image under the map $\mathrm{pr}_{2,s}$. By definition, $E$ is the locus of points $p = (p_1, \dots, p_{s-1}) \in (\mathbf{P})^{s-1}$ for which there exists $p_s \in \mathbf{P}$ such that $(p_1, \dots, p_{s-1}, p_s) \in \Gamma$, i.e., for which the system of linear equations $\widetilde{M}(p_1, \dots, p_{s-1})x = 0$ has a non-trivial solution $x = p_s$. So $E$ is the locus of zeros in $(\mathbf{P})^{s-1}$ of the multihomogeneous polynomial $\det \widetilde{M} = 0$. Similarly, $E'$ is the locus $\det \widetilde{N} = 0$. Thus $E$ is either all of $(\mathbf{P})^{s-1}$ if $\det \widetilde{M} = 0$, or is a Cartier divisor in $(\mathbf{P})^{s-1}$, and similarly for $E'$.

We will call an algebra $A$ with $r$ generators and $r$ relations of degree $s$ as above *semi-standard* if the schemes $E$ and $E'$ are equal, or equivalently, if

$$\det \widetilde{M} = c \det \widetilde{N} \tag{4.2}$$

for some $c \in k^*$. The constant $c$ is independent of change of basis in $x$ and $f$, provided that $\det \widetilde{M}$ is not identically zero.

Suppose that our algebra $A$ is semi-standard, so that $E = E'$. We may view $\Gamma$ as the graph of a correspondence on $E$ via the closed immersion $(\pi_1, \pi_2) : \Gamma \to E \times E$, and we call $A$ *nondegenerate* if $\Gamma$ is the graph of an automorphism $\sigma$ of $E$, and *degenerate* otherwise. This terminology conflicts slightly with the terminology of [A-S, Section 2], where degenerate means not regular. However, we will show in the next section that regular algebras are nondegenerate, and ultimately, for the algebras considered in [A-S, Tables (3.9, 11)], that the nondegenerate ones are regular.

**Proposition 4.3.** *Let $A$ be a semi-standard algebra. With the above notation, the scheme $E$ is either a divisor in $(\mathbf{P})^{s-1}$, or else $E = (\mathbf{P})^{s-1}$. More precisely, we have the following cases.*

*Case $r = 3$:* Either

(a) *$E$ is a divisor of degree 3 in $\mathbf{P}^2$, and $\Gamma$ is the graph of an automorphism $\sigma$ of $E$, or*
(b) *$E = \mathbf{P}^2$, and $\Gamma$ is the graph of an automorphism $\sigma$ of $\mathbf{P}^2$, or*
(c) *the scheme $\Gamma$ contains a subset of the form $p \times \ell$, where $\ell$ is a line and $p$ is a point in $\mathbf{P}^2$.*

*Case $r = 2$:* Either

(a) *$E$ is a divisor in $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree (2,2), and $\Gamma$ defines an automorphism $\sigma$ of $E$, of the form $\sigma(p, q) = (q, f(p, q))$,*
(b) *$E = \mathbf{P}^1 \times \mathbf{P}^1$, and $\sigma$ defines an automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ which has the form $\sigma(p, q) = (q, \tau(p))$, where $\tau$ is an automorphism of $\mathbf{P}^1$, or*
(c) *the scheme $\Gamma$ contains a subset of the form $p \times q \times \mathbf{P}^1$, where $p, q$ are points of $\mathbf{P}^1$.*

*In both cases, $A$ is degenerate if and only if (c) holds.*

**Proof.** The main differences between the cases $r = 2$ and $r = 3$ are notational, so since there is a small additional point when $r = 2$, we will omit the proof of the case $r = 3$.

We have seen in Proposition (3.6) that $A$ is degenerate unless the projections $\pi_i : \Gamma \to E$ are isomorphisms. Assume that they are. Then $(\pi_1, \pi_2)\Gamma$ is the graph of the automorphism $\sigma = \pi_2 \pi_1{}^{-1}$ of $E$, and $E$, being the scheme of zeros of the "biquadratic" form $\det \widetilde{M}$, is either all of $(\mathbf{P})^2 \approx \mathbf{P}^1 \times \mathbf{P}^1$, if $\det \widetilde{M} = 0$, or is a divisor of bidegree $(2, 2)$.

*A priori*, the automorphism $\sigma$ has the form $\sigma(p,q) = (q, f(p,q))$ for some map $f : E \to \mathbf{P}^1$ (see (3.7)(i)). So to complete the proof, we need to show that if $E = \mathbf{P}^1 \times \mathbf{P}^1$, then $f(p,q)$ is independent of $q$. Now since the automorphism $\sigma$ has the above form, the map $p \rightsquigarrow f(p,q)$ defines an automorphism of $\mathbf{P}^1$ for each fixed point $q$. Varying $q$, we obtain a map $\mathbf{P}^1 \to PG\ell_2$. This map is constant because $\mathbf{P}^1$ is proper and $PG\ell_2$ is affine. Thus $f(p,q)$ is independent of $q$, as claimed. The fact that $\tau$ is an automorphism of $\mathbf{P}^1$ follows from the fact that $\sigma$ is an automorphism.

It remains to discuss the asymmetry in possibility (c) in case $A$ is degenerate. This is taken care of by the following lemma.

**Lemma 4.4.** *A semi-standard algebra $A$ is nondegenerate if and only if one of the projections $\pi_i : \Gamma \to E$ is an isomorphism, or equivalently, if either one of the matrices $\widetilde{M}$, $\widetilde{N}$ has rank at least $r-1$ at every point.*

**Proof**. As we have already remarked, it follows from Proposition (3.6) that $\pi_1$ is an isomorphism if and only if $\widetilde{M}$ has rank at least $r-1$ at every point. Now if $\pi_1$ is an isomorphism then $\Gamma$ is the graph of a surjective, dominant map $\sigma : E \to E$. Then since $E$ is either a curve or $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, the fibre of $\sigma$ can not contain a line, hence $\sigma$ is an isomorphism. If $E$ is a curve, this follows from the fact that $E$ contains finitely many irreducible components. If $E = \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ it follows from the fact that these surfaces do not contain curves which can be contracted to a point. The same reasoning works if $\pi_2$ is an isomorphism. □

The generic situation for a semi-standard algebra is Case (a). Case (b) also occurs; for example, when $r = 3$, it includes the commutative polynomial ring. As we have already remarked, we will show in Theorem (5.1) that regular algebras are nondegenerate, hence that Case (c) does not occur for them.

Proposition (4.3) tells us that we can associate to every nondegenerate semi-standard algebra $A$ the triple $\mathcal{T}(A) = (E, \sigma, L)$, where $E = \pi_1(\Gamma)$, $\sigma$ is the automorphism of $E$ defined by $\Gamma$, and $L$ is the invertible sheaf $pr_1^*\mathcal{O}(1)$. It will be convenient to consider triples given abstractly as well. So we make the following definition:

*Definition* 4.5. A *triple* $\mathcal{T}$ is a set $(E, \sigma, L)$, where $E$ is a scheme, $\sigma$ is an automorphism of $E$, and $L$ is an invertible sheaf on $E$ whose global sections define a morphism $\pi : E \to \mathbf{P} \approx \mathbf{P}^{r-1}$, and where one of the following holds:

*Case* $r = 3$:

(a) $E$ is a divisor of degree 3 in $\mathbf{P}^2$, and $L$ is the restriction of $\mathcal{O}_{\mathbf{P}^2}(1)$,
(b) $E = \mathbf{P}^2$, and $L = \mathcal{O}(1)$,

*Case* $r = 2$:

(a) $E$ is a divisor of bidegree $(2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, $\sigma$ has the form $\sigma(p,q) = (q, f(p,q))$, and $L = pr_1^*(\mathcal{O}_{\mathbf{P}^1}(1))$.

(b) $E = \mathbf{P}^1 \times \mathbf{P}^1$, $\sigma$ has the form $\sigma(p,q) = (q, \tau(p))$, where $\tau$ is an automorphism of $\mathbf{P}^1$, and $L = pr^*(\mathcal{O}_{\mathbf{P}^1}(1))$.

We will say that a triple of type (a) is *elliptic*. Similarly, we will say that a nondegenerate semi-standard algebra $A$ is *elliptic* if the associated triple $\mathcal{T}$ is elliptic, i.e., if we are in case (a) of (4.3). We have chosen this terminology because smooth divisors $E$ of degree 3 in $\mathbf{P}^2$ or of bidegree $(2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ are elliptic curves. However, we do not require our divisor $E$ to be smooth.

Triples of type (b) will be called *linear*.

**4.6**. *Standard algebras and regular triples.* Standard algebras are semi-standard. This follows from equation (2.16), which expresses the property that the entries of the row-vector $x^t M$ generate the defining ideal: $x^t M = f^t Q^t$ for some $Q \in Gl_r(k)$. Combining this with (4.1), we find for a regular algebra that

$$M = NQ^t, \quad Q \in Gl_r(k). \tag{4.7}$$

This condition obviously implies (4.2), with $c = \det Q$.

Next, let $\mathcal{T}(A) = (E, \sigma, L)$ be the triple associated to a regular algebra, and let $\lambda$ denote the class of $L$ in Pic $E$. It will turn out that the operation of the automorphism $\sigma$ on the class $\lambda$ satisfies the following relation:

$$\begin{cases} (\sigma - 1)^2 \lambda = 0 & \text{if } r = 3 \\ (\sigma - 1)(\sigma^2 - 1)\lambda = 0 & \text{if } r = 2 \end{cases}. \tag{4.8}$$

Recall that a triple $\mathcal{T}$ is said to be *regular* if it satisfies (4.8). These definitions of the words semi-standard, standard and regular form local terminology, introduced in this paper for convenience. Note that regularity of a triple has nothing to do with whether $E$ is a regular scheme.

Note that all linear triples are regular. A linear triple $\mathcal{T}$ satisfies the stronger condition

$$\begin{cases} \sigma\lambda = \lambda & \text{if } r = 3 \\ \sigma^2 \lambda = \lambda & \text{if } r = 2 \end{cases}, \tag{4.8$'$}$$

because Aut $\mathbf{P}$ acts trivially on Pic $\mathbf{P} = \mathbb{Z}$. In fact, this condition characterizes linear triples.

Our main results, Theorem (6.8) and Proposition (6.7) assert that the regular algebras of dimension 3 are precisely the nondegenerate standard algebras.

**4.9**. *Exceptional triples and exceptional algebras.* Triples of the following types play a special role:

(4.10) r=3: $E$ is the union of a line and a conic, and $\sigma$ interchanges these two components;
r=2: $E$ is the union of three components of bidegrees $(1,0)$, $(0,1)$, and $(1,1)$ respectively, and $\sigma$ permutees these components cyclically.

We will call a triple satisfying (4.10) *exceptional*, and similarly, we will call a semi-standard algebra $A$ *exceptional* if it is nondegenerate and the associated triple is exceptional.

The following proposition may be helpful for intuition, though it is not needed elsewhere.

**Proposition 4.11.** *Exceptional triples are not regular.*

**Proof**. Suppose that $r = 3$, and let $E = C_1 + C_2$, where $C_1$ is a line and $C_2$ is a conic. Denote by a subscript $i$ the restriction of a sheaf to $C_i$. Then deg $L_1$ = deg $(\sigma^2 L)_1 = 1$, while deg $(\sigma L)_1 = 2$. Thus deg $((\sigma - 1)^2 L)_1 = -2$, which shows that $(\sigma - 1)^2 L$ is not even zero numerically. The proof is similar for the case $r = 2$. □

*Remarks* 4.12. (i) It is not difficult to check that the only triples for which the condition (4.8) does not hold at least *numerically* are the exceptional triples.

(ii) It is possible to specialize a translation on a plane cubic curve to an automorphism which interchanges a line and a conic, but nevertheless the above proposition and Theorem (6.8ii) show that regular algebras are not exceptional.

For each type of regular algebra of dimension 3 in the classification of [A-S, (3.9, 11)], the generic algebras are elliptic. The divisors $E$ and automorphisms $\sigma$ which they determine are described below, when the ground field is algebraically closed. If $E$ is a smooth curve, the description involves the tacit choice of a suitable base point $p_0$ to make $E$ into an abelian variety. In each case, the automorphism $\sigma$ and line bundle $L$ are related by the condition that $\mathcal{T}$ be regular. The relation (4.8) follows from the theorem of the square for type $A$, for which $\sigma$ is a translation, and so it does not impose any condition on $L$ for these algebras. But it is not automatic for some of the other types.

**4.13**. *The divisors $E$ and automorphism $\sigma$ associated to generic regular algebras of dimension 3.*

*Case* $r = 3$:

*type* $A$ : $E$ is a smooth curve, and $\sigma$ is a translation by a point of $E$.

*type* $B$ : $E$ is a smooth curve, $\sigma$ is multiplication by $-1$, and $L \approx \mathcal{O}_E(2p_0 + q)$, where $q$ is a point of order 4 of $E$.

*type* $E$ : $E$ is a smooth curve, $j(E) = 0$, $\sigma$ is an automorphism of order 3, multiplication by a cube root of unity, and $L \approx \mathcal{O}_E(2(p_0) + (q))$, where $q$ is a point of order 3 on $E$ not fixed by $\sigma$.

*type* $H$ : $E$ is a smooth curve, $j(E) = 12^3$, $\sigma$ is an automorphism of order 4, multiplication by $i$, and $L \approx \mathcal{O}_E(2(p_0) + (q))$, where $q$ is a point of order 2 of $E$ which is not fixed by $\sigma$.

*type* $S_1$ : $E$ is a triangle, and $\sigma$ stabilizes the three components.

*type* $S_1'$ : $E$ is the union of a line and a conic meeting the line in two points, and $\sigma$ stabilizes the components.

*type* $S_2$ : $E$ is a triangle, and $\sigma$ interchanges two of its sides.

*Case* $r = 2$:

*type* $A$ : $E$ is a smooth curve, and $\sigma$ is a translation by a point of $E$.

*type* $E$ : $E$ is a smooth curve, $j(E) = 0$, $\sigma$ is a complex multiplication order 3, and $L \approx \mathcal{O}_E(p_0 + q)$, where $q$ is a point of order 3 of which is not fixed by $\sigma$.

*type* $H$ : $E$ is a smooth curve, $j(E) = 12^3$, $\sigma$ is a complex multiplication of order 4, and $L \approx \mathcal{O}_E(p_0 + q)$, where $q$ is a point of order 4 such that $2q$ is not fixed by $\sigma$.

*type* $S_1$ : $E$ is the union of two curves of bidegree $(1,1)$, and $\sigma$ stabilizes the components.

*type* $S_2$ : $E = C_1 + C_1 + C_1' + C_1'$, where the bidegrees of $C_i, C_i'$ are $(1,0), (0,1)$ respectively, and $\sigma$ interchanges $C_i$ and $C_i'$.

*type* $S_2'$ : $E = C_1 + C_1' + C_2$ is a union of curves of bidegree $(1,0), (0,1), (1,1)$ respectively, and $\sigma$ interchanges $C_1$ and $C_1'$.

## 5. Discussion of degenerate cases

We retain the notation of Section 4. In this section we show that regular algebras are nondegenerate, by showing that the degenerate standard algebras are the ones listed in [A-S, Lemma (4.2)].

**Theorem 5.1.** (1) *Regular algebras are neither degenerate nor exceptional.*
(2) *Let $A$ be a semi-standard algebra. Consider the following assertions:*
(i) *There is a basis for the defining ideal of $A$ having one of the forms of* [A-S, (4.2)], *viz:*

$$\begin{array}{lll} \textit{Case } r=3: & (\mathrm{a})\,(f_1,f_2,f_3)=\left(x^2,^*,^*\right), \\ & (\mathrm{b})\,(f_1,f_2,f_3)=\left(yz+cx^2,zx+dy^2,xy\right). \\ \textit{Case } r=2: & (\mathrm{a})\,(f_1,f_2) \quad =\left(x^3,^*\right), \\ & (\mathrm{b})\,(f_1,f_2) \quad =\left(yxy+cx^3,xyx\right), \end{array}$$

(ii) *There are non-zero elements $u \in T_{s-1}$ and $v \in T_1$ whose product $uv$ is in $I_s$.*

(iii) *$A$ is degenerate or exceptional.*

*The following implications hold:* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *Moreover if $A$ is either a degenerate standard algebra or an exceptional algebra, then* (i) *holds.*

**Proof.** (1): We know that regular algebras are standard. It is proved in [A-S (4.2)] that the standard algebras of the form (2i) are not regular, so the fact that neither degenerate nor exceptional algebras are regular follows from the last assertion of the theorem.

(2): It is trivial that (i) implies (ii). We prove that (ii) implies (iii) by showing that if (ii) holds and $A$ is nondegenerate, then $A$ is exceptional. So we assume, with our usual notation, that $\Gamma$ is the graph of an automorphism $\sigma$ of $E$. Suppose first that $r=3$. Then by hypothesis, $I_2$ contains an element $uv$ which is the product of two linear forms. We may take $uv=f_1$ as one of the three defining equations $f_1,f_2,f_3$. The locus of zeros of $\tilde{u}\tilde{v}$ in $\mathbf{P}\times\mathbf{P}$ ($\mathbf{P}=\mathbf{P}^2$) has the form $Z=(\ell_u\times\mathbf{P})\cup(\mathbf{P}\times\ell_v)$, where $\ell_u$ and $\ell_v$ are the zeros of $u$ and $v$ respectively. Since $\Gamma\subset Z$, we may decompose it accordingly: $\Gamma=\Gamma_u\cup\Gamma_v$, where $\Gamma_u=\{\tilde{f}_2=\tilde{f}_3=0\}\cap(\ell_u\times\mathbf{P})$ and $\Gamma_v=\{\tilde{f}_2=\tilde{f}_3=0\}\cap(\mathbf{P}\times\ell_v)$. Since $\Gamma$ is the graph of an automorphism, the triple $\mathcal{T}=\mathcal{T}(A)$ must be elliptic, and the divisor $E$ is reducible. Thus $\Gamma_u$ and $\Gamma_v$ are schemes of dimension 1 in $\mathbf{P}\times\mathbf{P}$, and $\Gamma_u$ is the graph of an isomorphism from $\ell_u$ to $C_u=\mathrm{pr}_2\Gamma_u$. Similarly, $\Gamma_v$ is the graph of an isomorphism from $C_v=pr_1\Gamma_v$ to $\ell_v$, and we have $E=\ell_u+C_v=\ell_v+C_u$. Since $E$ is of degree three, $C_u$ and $C_v$ are conics. Hence $\ell_u=\ell_v$, $C_u=C_v$, and the triple is exceptional.

Suppose that $r=2$. Then there is a defining equation $f_1=uv$, where $u$ is quadratic and $v$ is linear, and one other cubic defining relation $f_2$. As before, we have a decomposition $\Gamma=\Gamma_u\cup\Gamma_v$. Setting $\mathbf{P}=\mathbf{P}^1$, we have $\Gamma_v = \{\tilde{f}_2 = 0\} \cap (\mathbf{P} \times \mathbf{P} \times p_v)$, where $p_v$ is the zero of $\ell_v$ in $\mathbf{P}$ and $\Gamma_u = \{\tilde{f}_2 = 0\} \cap (D \times \mathbf{P})$, where $D$ is the locus of zeros of $\tilde{u}$, a divisor of bidegree $(1,1)$ in $\mathbf{P} \times \mathbf{P}$. Then $\Gamma_v$ is a curve of tridegree $(1,1,0)$ in $\mathbf{P} \times \mathbf{P} \times \mathbf{P}$, and $\Gamma_u$ has tridegree $(1,1,2)$. Since $\sigma$ is an automorphism, $\Gamma_v$ is the graph of an isomorphism $\mathrm{pr}_{12}\Gamma_v = C_0 \to C_1 = \mathrm{pr}_{23}\Gamma_v$, where $C_0$ and $C_1$ are components of $E$. So $E = C_0 \cup C_1 \cup C_2$, where $C_2$ has degree $(0,1)$. Then $\mathrm{pr}_{12}\Gamma_u = C_1 + C_2$, and $\mathrm{pr}_{23}\Gamma_u = C_0 + C_2$. A consideration of the degrees shows that $\sigma$ permutes the three components cyclically, hence that $A$ is exceptional. This completes the proof that (ii) implies (iii).

The following lemma shows that exceptional algebras have defining equations of the form (2i)(a).

**Lemma 5.2.** *Let $A$ be an exceptional algebra* (4.9). *There exists a non-zero element $x \in A_1$ such that $x^s = 0$ in $A_s$.*

This lemma will be proved in the course of proof of Proposition (6.7ii).

We now come to the main part of the proof, the verification that every degenerate standard algebra has defining equations of one of the forms (2i). This is done by an explicit computation.

We first treat the case $r = 3$. Let $A$ be a standard algebra which defines a degenerate correspondence. As we have seen (4.3), this implies that $\Gamma$ contains a subset of the form $p \times \tilde{\ell}$. Adjusting coordinates $(x, y, z)$ in $\mathbf{P}^2$ appropriately, we may assume that the pair $(p, \ell)$ is in one of two standard positions, according as $p$ is contained in $\ell$ or not:

*Case* 1: $p = (1,0,0), \ell = \{(0, y, z)\}$,

*Case* 2: $p = (1,0,0), \ell = \{(x, y, 0\}$.

We write the defining equations in the form

(5.3)
$$\begin{aligned} f_1 &= a_1x^2 + a_2xy + a_3xz + a_4yx + a_5y^2 + a_6yz + a_7zx + a_8zy + a_9z^2, \\ f_2 &= b_1x^2 + b_2xy + b_3xz + b_4yx + b_5y^2 + b_6yz + b_7zx + b_8zy + b_9z^2, \\ f_3 &= c_1x^2 + c_2xy + c_3xz + c_4yx + c_5y^2 + c_6yz + c_7zx + c_8zy + c_9z^2. \end{aligned}$$

Bilinearizing and substituting $(1,0,0)$ for the first variable yields

$$\tilde{f}_1((1,0,0),(x,y,z)) = a_1x + a_2y + a_3z,$$

etc. Therefore the coefficients $a_i, b_i, c_i$ vanish if $i = 2,3$ in Case 1, and if $i = 1,2$ in Case 2. We now analyze $(x,y,z)M = (g_1, g_2, g_3)$, which by hypothesis is also a set of generators for the ideal $(f_1, f_2, f_3)$ (cf. 2.16).

*Case* 1: The generators $f_i$ and $g_i$ have the form:

$$\begin{aligned}
f_1 &= a_1x^2 && + a_4yx + a_5y^2 + a_6yz + a_7zx + a_8zy + a_9z^2,\\
f_2 &= b_1x^2 && + b_4yx + b_5y^2 + b_6yz + b_7zx + b_8zy + b_9z^2,\\
f_3 &= c_1x^2 && + c_4yx + c_5y^2 + c_6yz + c_7zx + c_8zy + c_9z^2,\\
g_1 &= a_1x^2 + a_4xy + a_7xz && + b_1yx + b_4y^2 + b_7yz + c_1zx + c_4zy + c_7z^2,\\
g_2 &= \quad a_5xy + a_8xz && + b_5y^2 + b_8yz + c_5zy + c_8z^2,\\
g_3 &= \quad a_6xy + a_9xz && + b_6y^2 + b_9yz + c_6zy + c_9z^2.
\end{aligned}$$

Here the $xy$ and $xz$ coefficients of $g_i$ must vanish because they are not present in $f_i$. Hence $a_i = 0$ for all $i \geq 1$, and $f_1 = a_1x^2$. This algebra has the form (a).

*Case* 2: In this case, the generators have the form:

$$\begin{aligned}
f_1 &= && a_3xz + a_4yx + a_5y^2 + a_6yz + a_7zx + a_8zy + a_9z^2,\\
f_2 &= && b_3xz + b_4yx + b_5y^2 + b_6yz + b_7zx + b_8zy + b_9z^2,\\
f_3 &= && c_3xz + c_4yx + c_5y^2 + c_6yz + c_7zx + c_8zy + c_9z^2,\\
g_1 &= && a_4xy + a_7xz + b_4y^2 + b_7yz + c_4zy + c_7z^2,\\
g_2 &= && a_5xy + a_8xz + b_5y^2 + b_8yz + c_5zy + c_8z^2,\\
g_3 &= a_3x^2 && + a_6xy + a_9xz + b_3yx + b_6y^2 + b_9yz + c_3zx + c_6zy + c_9z^2.
\end{aligned}$$

As in the previous case, it follows that $a_3 = a_4 = a_5 = a_6 = 0$. This reduces $f_i$ and $g_i$ to

$$\begin{aligned}
f_1 &= a_7zx + a_8zy + a_9z^2,\\
f_2 &= b_3xz + b_4yx + b_5y^2 + b_6yz + b_7zx + b_8zy + b_9z^2,\\
f_3 &= c_3xz + c_4yx + c_5y^2 + c_6yz + c_7zx + c_8zy + c_9z^2,\\
g_1 &= a_7xz + b_4y^2 + b_7yz + c_4zy + c_7z^2,\\
g_2 &= a_8xz + b_5y^2 + b_8yz + c_5zy + c_8z^2,\\
g_3 &= a_9xz + b_3yx + b_6y^2 + b_9yz \quad c_3zx + c_6zy + c_9z^2.
\end{aligned}$$

*Subcase* (2a): $a_7 \neq 0$. Then it follows that $c_3 \neq 0$ as well, and that if we express $f_1$ as linear combination of the $g_i$, the coefficient of $g_3$ is not zero. Therefore $b_3 = 0$, which implies that $b_4 = c_4 = 0$ too.

We are still allowed a change of coordinates which preserves the flag $p \subset \ell$, i.e., one of the form

$$\begin{aligned} x &\rightsquigarrow d_{11}x+d_{12}y+d_{13}z, \\ y &\rightsquigarrow \phantom{d_{11}x+{}} d_{22}y+d_{23}z, \\ z &\rightsquigarrow \phantom{d_{11}x+d_{12}y+{}} d_{33}z. \end{aligned}$$

Using the substitution for $x$, we can eliminate the coefficients $a_8, a_9$. If also $b_5 = 0$, then we conclude in turn that $b_6 = c_5 = c_6 = b_7 = b_8 = b_9 = 0$, hence that $f_2 = 0$, which shows that $A$ can be defined by 2 relations, contradicting our assumption that $A$ is standard. Assume that $b_5 \neq 0$. Then we can use this coefficient to eliminate $b_6$ by a substitution $y \rightsquigarrow d_{22}y + d_{23}z$. Since $f_1$ is a linear combination of $g_2, g_3$, the vectors $(b_5, b_8, c_5, c_8)$ and $(b_6, b_9, c_6, c_9)$ are proportional. Since $b_6 = 0$, it follows that $b_9 = c_6 = c_9 = 0$, which forces $c_7 = c_8 = b_8 = c_5 = 0$. Hence $g_2 = b_5y^2$, and the algebra has the form (a).

*Subcase* (2b): $a_7 = 0$. If also $a_8 = 0$, the algebra has the form (a). If not, we eliminate $a_9$ by a change of variable $y$. Then if $b_3 = 0$, it follows that $c_3 \neq 0$, but $b_4 = c_4 = 0$. In this case, $f_1$ is a linear combination of $g_1$ and $g_3$. This forces $c_3 = 0$, which means that $f_3 = 0$. If $b_3 \neq 0$, we can eliminate $b_6, b_9$ by a change of variable $x$. When this is done, $f_1$ is a multiple of $g_1$, hence $b_4 = b_7 = c_7 = 0$, and $c_3 = 0$. Then $f_i$ is in the span of $\{g_1, g_i\}$, which implies that $b_8 = c_8 = c_5 = c_6 = 0$. This algebra has the form (b).

We now consider the case $r = 2$. Here $\Gamma \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and (4.3) we can assume that the projection $\mathrm{pr}_{12}$ is not an immersion at $p = (p_1, p_2) \in \mathbf{P}^1 \times \mathbf{P}^1$, hence that the fibre is all of $\mathbf{P}^1$. We write the equations as

(5.4)

$$\begin{aligned} f_1 &= a_1x^3 + a_2x^2y + a_3xyx + a_4xy^2 + a_5yx^2 + a_6yxy + a_7y^2x + a_8y^3, \\ f_2 &= b_1x^3 + b_2x^2y + b_3xyx + b_4xy^2 + b_5yx^2 + b_6yxy + b_7y^2x + b_8y^3, \end{aligned}$$

so that

$$\begin{aligned} g_1 &= a_1x^3 + a_3x^2y + a_5xyx + a_7xy^2 + b_1yx^2 + b_3yxy + b_5y^2x + b_7y^3, \\ g_2 &= a_2x^3 + a_4x^2y + a_6xyx + a_8xy^2 + b_2yx^2 + b_4yxy + b_6y^2x + b_8y^3. \end{aligned}$$

Adjusting coordinates, we have again two cases, according as $p_1 \neq p_2$ or $p_1 = p_2$:

*Case* 1: $p = ((1,0)(0,1))$. Substituting $p$ into $\tilde{f}_1$ gives $\tilde{f}_1(1,0;0,1;x,y) = a_3x + a_4y$. Thus $a_3 = a_4 = 0$, and similarly, $b_3 = b_4 = 0$. It follows in order that $a_2 = b_2 = a_6 = b_6 = a_5 = a_7 = a_8 = 0$,, hence that $f_1 = a_1x^3$, which is of the form (a).

*Case* 2: $p = ((1,0)(1,0))$. Substituting $p$ into $\tilde{f}_1$ gives $a_1x + a_2y$. Thus $a_1 = a_2 = 0$ and $b_1 = b_2 = 0$. It follows that $a_3 = a_4 = a_5 = b_5 = 0$, which reduces the relations to

$$
\begin{aligned}
f_1 &= a_6yxy + a_7y^2x + a_8y^3,\\
f_2 &= b_3xyx + b_4xy^2 + b_6yxy + b_7y^2x + b_8y^3,\\
g_1 &= a_7xy^2 + b_3yxy + b_7y^3,\\
g_2 &= a_6xyx + a_8xy^2 + b_4yxy + b_6y^2x + b_8y^3.
\end{aligned}
$$

*Subcase* (2a): $A_6 \neq 0$, hence $b_3 \neq 0$. In this case, $f_1$, is a multiple of $g_1$, which implies that $a_7 = 0$. We can make a change of variable fixing $p$, i.e., of the form $x \rightsquigarrow d_1x + d_2y$. Using such a change, we can eliminate $a_8$. Then it follows in order that $b_4 = b_7 = b_6 = 0$. This algebra has the form (b).

*Subcase* (2b): $a_6 = 0$, hence $b_3 = 0$. If $a_7 = 0$, then $A$ has the form (a). If not we eliminate $a_8$ by a change of variable. Then $f_1$ is a multiple of $g_2$, hence $b_4 = 0$, and this contradicts $a_7 \neq 0$. This completes the proof of Theorem 5.1. □

## 6. The algebras $A$ and $B$ defined by a triple

We have seen how multilinearization determines a triple $\mathcal{T}(A) = (E, \sigma, L)$ from a nondegenerate semi-standard algebra $A$. In this section we start with an arbitrary triple $\mathcal{T}$ and construct from it two algebras $\mathcal{A}(\mathcal{T})$ and $\mathcal{B}(\mathcal{T})$, and a homomorphism $\beta = \beta(\mathcal{T}) : \mathcal{A}(\mathcal{T}) \to \mathcal{B}(\mathcal{T})$. This construction has the property that if $A$ is a nondegenerate semi-standard algebra and if $\mathcal{T} = \mathcal{T}(A)$, then there is a canonical isomorphism $A \to \mathcal{A}(\mathcal{T})$ which allows us to identify the homomorphism $\beta$ with the map $A \to B$ constructed at the end of Section 3.

We will prove (6.8ii) that if $\mathcal{T}$ is a regular triple, then $\mathcal{A}(\mathcal{T})$ is a regular algebra of dimension 3, by using the homomorphism $\beta(\mathcal{T})$ together with the properties of $\mathcal{B}(\mathcal{T})$ stated in Theorem 6.6 and proved in the Section 7. We also show (6.7iii) that $\mathcal{T}(A)$ is regular if $A$ is nondegenerate standard. The fact that $A$ is isomorphic to $\mathcal{A}(\mathcal{T}(A))$ then shows that a nondegenerate standard algebra is regular.

We also show (6.8iii) that if $\mathcal{T}$ is regular then we recover $\mathcal{T} = \mathcal{T}(A)$ from the algebra $A = \mathcal{A}(\mathcal{T})$, unless $\mathcal{T}$ is elliptic and satisfies (4.8′). In that case, the automorphism $\sigma$ of $E$ extends uniquely to the ambient space $X = \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, and the triple attached to $\mathcal{A}(\mathcal{T})$ is the linear one obtained from that extension of $\sigma$.

*Construction of the algebras $\mathcal{A}(\mathcal{T})$ and $\mathcal{B}(\mathcal{T})$.* Let $\mathcal{T} = (E, \sigma, L)$ be a triple and $\pi : E \to \mathbf{P}^{r-1}$ ($r = 2$ or $3$) be the morphism determined by the global sections of $L$. With $\mathcal{T}$ we associate a graded ring $B = \mathcal{B}(\mathcal{T})$ as follows: For each integer $n \geq 0$, set

$$L_n = L \otimes L^{\sigma} \otimes \ldots \otimes L^{\sigma^{n-1}}. \tag{6.2}$$

Here $L^{\sigma} = \sigma^* L$ as before, $L_0 = \mathcal{O}_E$, and tensor products are taken over $\mathcal{O}_E$. As a graded vector space, $\mathcal{B}(\mathcal{T}) = B = \Sigma B_n$ is defined by the rule

$$B_n = H^0(E, L_n). \tag{6.3}$$

For every pair of integers $m, n \geq 0$, there is a canonical isomorphism

$$L_m \otimes L_n^{\sigma^m} \to L_{m+n}, \tag{6.4}$$

and hence a map

$$\mu_{m,n} : H^0(E, L_m) \otimes_k H^0(E, L_n) \to H^0(E, L_{m+n})$$

which defines the multiplication in $B$ : If $u \in H^0(E, L_m)$ and $v \in H^0(E, L_n)$, we define the product to be

$$uv = \mu_{m,n}\left(u \otimes v^{\sigma^m}\right) \in B_{m+n}, \tag{6.5}$$

where $v^{\sigma} = v \circ \sigma$. In this way we obtain an associative algebra $B = \mathcal{B}(\mathcal{T})$ from the triple $\mathcal{T}$.

To define $\mathcal{A}(\mathcal{T})$, let $T = \Sigma T_n$ be the tensor algebra over $k$ on $T_1 = H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$. The isomorphism

$$\pi^* : T_1 \xrightarrow{\sim} B_1$$

induces a homomorphism $T \to B$. Let $J = \Sigma J_n$ be its kernel, and let $I$ be the two-sided ideal of $T$ generated by $J_s$, where $s = 5 - r$. We then put $\mathcal{A}(\mathcal{T}) = T/I$. The composition of the natural homomorphisms

$$\mathcal{A}(\mathcal{T}) = T/I \to T/J \to B = \mathcal{B}(\mathcal{T})$$

gives us a canonical homomorphism $\mathcal{B}(\mathcal{T}) : \mathcal{A}(\mathcal{T}) \to \mathcal{B}(\mathcal{T})$ which is bijective in degree 1. Note that in fact $A = \mathcal{A}(\mathcal{T})$ depends only on $B = \mathcal{B}(\mathcal{T})$. Indeed, $A$ is the quotient of the tensor algebra on $B_1$ by the relations of degree $s$ defining $B$. Key properties of $B$ are listed in the following theorem.

**Theorem 6.6.** (Properties of $\mathcal{B}(\mathcal{T})$). *Let $\mathcal{T}$ be a triple, let $B = \mathcal{B}(\mathcal{T})$, and call $J$ the kernel of the map $T \to B$ defined above. Let $r = \dim B_1$ and $s = 5 - r$. Then*

(i) *$B = k[B_1]$, and the right and left socles of $B$ are zero (i.e., if $b \in B$ and if either $B_1 b = 0$ or $bB_1 = 0$, then $b = 0$).*

(ii) *If $\mathcal{T}$ is elliptic, then $\dim_k B_n = rn$ for $n > 0$. If $\mathcal{T}$ is linear, then the Hilbert function of $B$ is the same as that of a regular algebra of dimension 3 with $r$ generators* (cf. (2.17)).

(iii) *$J_n = 0$ for $n < s$,* and *$\dim_k J_s = r$.*

(iv) *For $n \geq 2$, $J_{s+n} = T_1 J_{s+n-1} + J_s T_n = T_n J_s + J_{s+n-1} T_1$.*

(v) *Let $U = J_{s+1}/(T_1 J_s + J_s T_1)$* and *$W = T_1 J_s \cap J_s T_1$. Then $\dim_k U = 1$ if $\mathcal{T}$ is elliptic and is either regular or exceptional, and $\dim_k U = 0$ otherwise. Moreover,*

$$\dim_k W = \begin{cases} \dim_k U & \text{if } \mathcal{T} \text{ is elliptic} \\ 1 & \text{if } \mathcal{T} \text{ is linear.} \end{cases}$$

(vi) *Suppose that $\mathcal{T}$ is regular, so that in particular, $\dim_k W = 1$. Then $W$ is a nondegenerate subspace of $T_1 J_s$ and of $J_s T_1$, in the following sense: If $w$ is a non-zero element of $W$, and if we write $w = \Sigma x_i f_i = \Sigma g_i x_i$, where $\{x_i\}$ is a basis for $T_1$, then both $\{f_i\}$ and $\{g_i\}$ are bases for $J_s$.*

A proof of this theorem will be given in the next section. It involves a study of various aspects of the cohomology of certain sheaves on $E$. For the rest of this section, we assume the theorem and derive the promised consequences for the algebra $\mathcal{A}(\mathcal{T}) = T/I$.

**Proposition 6.7.** *Let $\mathcal{T} = \mathcal{T}(\mathcal{A}') = (E, \sigma, L)$ be the triple associated to a nondegenerate semi-standard algebra $A'$ as in Section 4, and let $A = \mathcal{A}(\mathcal{T})$. Then*

(i) *The algebras $A'$ and $A$ are canonically isomorphic.*
(ii) *If the algebra $A'$ is exceptional, it is not standard.*
(iii) *If $A'$ is standard, then $\mathcal{T}$ is regular.*

**Proof.** (i) Both $A$ and $A'$ are quotients of the tensor algebra $T$ on $A'_1 = H^0(\mathbf{P}, \mathcal{O}(1))$ by ideals $I \supset I'$ generated in degree $s$. By (6.6iii), $\dim I_s = r$, and by definition of a semi-standard algebra, $\dim I'_s = r$ too (Section 4). Hence $I = I'$.

(ii) Suppose $A'$ is exceptional. We will show that there exists a non-zero element $x \in A'_1$ such that $x^s = 0$ in $A'_s$. The non-standardness of $A'$ follows, because a standard algebra with such an element $x$ is degenerate, as is shown in [A-S, (4.2)]. Let $B = \mathcal{B}(\mathcal{T})$. If $r = 3$, $E$ is the union of a line $C_1$ and a conic $C_2 = C_1^\sigma$, and we take $x$ to be a non-zero element of $T_1 = H^0(\mathbf{P}, \mathcal{O}(1)) = H^0(E, L) = B_1$ which vanishes on $C_1$. Then

$x \otimes x^\sigma \in H^0(E, L \otimes L^\sigma)$ vanishes on all of $E$, so $x^2 = 0$ in $B_2 = T_2/J_2 = T_2/I_2 = A_2 = A_2'$. Similarly, if $r = 2$, then $E$ is a union $C \cup \sigma C \cup \sigma^2 C$, where $C$ is a "line" of the form $p \times \mathbf{P}$, and we take $x$ to be a non-zero element of $T_1$ such that $x(p) = 0$. Then $x \otimes x^\sigma \otimes x^{\sigma^2}$ vanishes on all of $E$, so $x^3 = 0$ in $B_3 = T_3/J_3 = T_3/I_3 = A_3 = A_3'$.
(iii) Recall that if $A' = T/I$ is a standard algebra, there is a non-zero element $w = \Sigma x_i m_{ij} x_j \in T_1 I_s \cap I_s T_1$. Since $J_s = I_s$, it follows from (6.6v) that $\mathcal{T}$ is either regular or exceptional (recall that linear triples are regular). But by (ii), a standard algebra is not exceptional. □

**Theorem 6.8.** *Let $\mathcal{T}$ be a triple. Let $A = \mathcal{A}(\mathcal{T})$ and $B = \mathcal{B}(\mathcal{T})$. Then*
*(i) The canonical homomorphism $A \to B$ is always surjective. It is an isomorphism unless $\mathcal{T}$ is elliptic and either regular or exceptional, in which case its kernel has the form $gA = Ag$, where $g$ is a non-zero normalizing element of degree $s+1$.*
*(ii) If $\mathcal{T}$ is regular, then $A$ is a regular algebra of global dimension 3, and in the elliptic case the element $g$ of (i) is left and right regular.*
*(iii) Let $\mathcal{T}' = (E', \sigma', L')$ be the triple $\mathcal{T}(A)$ attached to $A$. Then $\mathcal{T} = \mathcal{T}'$ unless $\mathcal{T}$ is elliptic and satisfies the condition (4.8$'$). In that case $\mathcal{T}'$ is linear, $E'$ is the ambient space $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ in which $E$ is embedded, and $\sigma'$ is the unique extension of $\sigma$ from $E$ to an automorphism of $E'$.*

**Proof.** (i) We have $B = k[B_1]$ by (6.6i). Since $A_1 \approx B_1$, the homomorphism $A \to B$ is surjective. Denote its kernel by $K = J/I$, where $I$ is the ideal of $T$ generated by $J_s$. By (6.6iii), $K_n = 0$ for $n \le s$, and by (6.6iv), $K_n = T_1 K_{n-1} = K_{n-1} T_1$ for $n \ge s+2$. It follows by induction on $n$ that $K$ is generated as left ideal and as right ideal by $K_{s+1} = U$, and now (i) follows from (6.6v), on taking for $g$ a non-zero element of $U$ in case $\dim U = 1$. In that case, let $Z$ and $Z'$ be the right and left annihilators of $U$ in $A$. For each integer $n$, we have exact sequences

$$(6.9) \qquad \begin{gathered} 0 \to Z_{n-s-1} \to A_{n-s-1} \xrightarrow{\lambda(g)} A_n \to B_n \to 0 \\ 0 \to Z'_{n-s-1} \to A_{n-s-1} \xrightarrow{\rho(g)} A_n \to B_n \to 0, \end{gathered}$$

where $\lambda(g)$ and $\rho(g)$ denote left and right multiplication by $g$ respectively. Comparing these two sequences, we see that $\dim Z_n = \dim Z'_n$ for all $n$. Hence, to prove the statement about $g$ in (ii), it suffices to show that $Z = 0$ in case $\mathcal{T}$ is elliptic and regular.
(ii) We now suppose that $\mathcal{T}$ is regular and we prove that $A$ is regular, and if $\mathcal{T}$ is elliptic, that $Z = 0$. By (6.6v), $\dim W = 1$. Let $w$ be a non-zero element of $W$ and write $w = \Sigma x_i m_{ij} x_j$, where $(x_i)$ is a basis for $A_1 = T_1$, with $m_{ij} \in A_{s-1} = T_{s-1}$. By (6.6vi), $A$ is a standard algebra because the

elements $f_i = \Sigma_{ij} x_j$ span $J_s$, and so do the $g_j = \Sigma x_i m_{ij}$. The standard potential resolution of the left $A$-module $k$ (2.15) is the complex whose degree $n$ part is

$$(6.10) \qquad 0 \to A_{n-s-1} \xrightarrow{x^t} A^r_{n-s} \xrightarrow{M} A^r_{n-1} \xrightarrow{x} A_n \to k_n \to 0$$

(where $k_n = 0$ if $n \neq 0$, $k_0 = k$, $x$ is the column vector $(x_i)$ of length $r$, and $M = (m_{ij})$. To say that $A$ is a regular algebra means that this sequence is exact for each $n$. Since the elements $f_i$ span $I_s$, it is exact to the right of $M$. Let $P_{n-s-1}$ and $Q_{n-s}$ be the homology groups of the complex (6.10) at $A_{n-s-1}$ and $A^r_{n-s}$ respectively. Thus $P_n = \{f \in A_n | fA_1 = 0\}$ is the degree $n$ part of the left socle of $A$.

If a graded vector space is denoted by a capital letter, say $X$, then we will use the lower case letter to denote its dimension, viz. $x_n = \dim_k X_n$. With this notation, we need to show that $p_n = q_n = 0$ for all $n$, and also $z_n = 0$ in case $\mathcal{T}$ is elliptic. We also set

$$(6.11) \qquad \begin{aligned} e_n = & \text{ Hilbert function of a regular algebra} \\ & \text{ of dimension 3 on } r \text{ generators, and} \\ d_n = & \; e_n - a_n. \end{aligned}$$

From (6.10), the relation

$$a_{n-s-1} - ra_{n-s} + ra_{n-1} - a_n + \delta_{n,0} = p_{n-s-1} - q_{n-s}$$

holds for all $n \in \mathbb{Z}$, where $\delta$ is the Kronecker delta, and since (6.10) is exact for regular algebras, we also have

$$e_{n-s-1} - re_{n-s} + re_{n-1} - e_n + \delta_{n,0} = 0$$

for all $n$.

Subtracting these two equations, we find

$$(6.12) \qquad -d_{n-s-1} + rd_{n-s} - rd_{n-1} + d_n = p_{n-s-1} - q_{n-s}.$$

If $\mathcal{T}$ is linear, we have $b_n = e_n$ by (6.6ii), and since we have seen that $A \approx B$ in this case, we have $a_n = e_n$, i.e., $d_n = 0$ for all $n$. By (6.6i), $A$ has no socle, so $p_n = 0$ for all $n$. Then (6.12) shows that $q_n = 0$ for all $n$ as well, hence that $A$ is regular.

Suppose that $\mathcal{T}$ is elliptic. In this case we use induction on $n$ to prove simultaneously that $p_n, q_{n+1}$, and $z_n$ are zero, and thereby finish the proof of (ii). The key point is the following lemma:

**Lemma (6.13).** *Let $j$ be an integer. If $z_{j-s} = 0$ and $p_{j-s-1} = 0$, then $p_j = 0$.*

**Proof**. Let $u \in P_j$. Then since $B$ has trivial socle (6.6i), the image of $u$ in $B$ is zero. By (6.9), there is an element $h \in A_{j-s-1}$ such that $u = gh$. Then $ghA_1 = uA_1 = 0$, hence $hA_1 \in Z_{j-s}$. If $Z_{j-s} = 0$, then $h \in P_{j-s-1}$, and if $P_{j-s-1} = 0$ as well, then $h = 0$, and $u = gh = 0$, as was to be shown. $\square$

By (6.6ii), we have $b_n = rn$ if $n \geq 0$, and by (6.6i), $b_0 = 1$, and $b_n = 0$ for $n < 0$. From this fact, or by examining a particular regular algebra of dimension 3, it is easy to check that $b_n = e_n - e_{n-s-1}$. From (6.9), it follows that

$$z_{n-s-1} - a_{n-s-1} + a_n = e_n - e_{n-s-1},$$

i.e., that $z_{n-s-1} = d_n - d_{n-s-1}$ for all $n$. Together with (6.12), this gives

$$x_{n-s-1} + rd_{n-s} - rd_{n-1} = p_{n-s-1} - q_{n-s} \tag{6.14}$$

for all $n$.

We now show that $d_n = p_n = q_n = z_n = 0$ for all $n$ by induction on $n$. For each integer $m$, let $S_m$ denote the statement

$$S_m : \begin{cases} d_i = 0 & \text{for } i \leq m-1 \\ p_i = 0 & \text{for } i \leq m-1-s \\ q_i = 0 & \text{for } i \leq m-s \\ z_i = 0 & \text{for } i \leq m-1-s \end{cases}.$$

For $m \leq 0$, $S_m$ is trivially true. We show that $S_m$ implies $S_{m+1}$. Suppose that $S_m$ is true. By (6.12), it follows that $d_i = 0$ for $i \leq m$. By (6.14), with $n$ replaced by $m+1$

$$z_{m-s} = p_{m-s} - q_{m+1-s}. \tag{6.15}$$

Now the lemma, with $j = m - s$, shows that $p_{m-s} = 0$, and consequently, by (6.15), that $z_{m-s} = 0$ and $q_{m+1-s} = 0$. Thus $S_{m+1}$ is true. This completes the induction step and, therewith the proof of (ii).

(iii) It is tautological that $\mathcal{T} \subset \mathcal{T}'$, in the sense that $E \subset E'$ and that $\sigma$ and $L$ are the restrictions to $E$ of $\sigma'$ and $L'$. Thus $E = E'$, and hence $\mathcal{T} = \mathcal{T}'$, unless $E$ and $E'$ have different dimensions, i.e., unless $\mathcal{T}$ is elliptic and $\mathcal{T}'$ is linear. If $\mathcal{T}'$ is linear, then $\mathcal{T}'$ satisfies (4.8′), and so does $\mathcal{T}$ since $\mathcal{T} \subset \mathcal{T}'$. Conversely, suppose $\mathcal{T}$ is elliptic and satisfies (4.8′). Let $\pi : E \to \mathbf{P}^{r-1}$ be the map given by sections of $L$. If $r = 3$, then $L \approx L^\sigma$ and this means there is an automorphism $\sigma'$ of $\mathbf{P}^2$ such that $\pi\sigma = \sigma'\pi$, and $\Gamma'$ is the graph of $\sigma'$, so $E' = \mathbf{P}^2$. If $r = 2$, then $L \approx L^{\sigma^2}$, which means there is an automorphism $\tau$ of $\mathbf{P}^1$ such that $\pi\sigma^2 = \tau\pi$. Thus $\sigma$ extends to the automorphism $\sigma' : (p,q) \rightsquigarrow (q, \tau p)$ of $\mathbf{P}^1 \times \mathbf{P}^1$, $\Gamma'$ is the locus of points $(p,q,\tau p)$, and $E' = \mathbf{P}^1 \times \mathbf{P}^1$. $\square$

## 7. Proof of Theorem 6.6

Let $\mathcal{T} = (E, \sigma, L)$ be a triple, $B = \mathcal{B}(\mathcal{T})$ and let $r = \dim B_1$ and $s = 5 - r$ as usual. To prove Theorem 6.6, we will treat the cases $\mathcal{T}$ linear and $\mathcal{T}$ elliptic separately, but we begin with some remarks which apply in both cases.

**Lemma 7.1.** *To prove* (6.6i) *for $\mathcal{T}$ it suffices to show that*
(i) $H^0(E, \mathcal{O}_E) = k$, *and that*
(ii) *For each $n > 0$, the multiplication map*

$$H^0(E,L) \otimes H^0(E, L_n^\sigma) \to H^0(E, L \otimes L_n^\sigma) = H^0(E, L_{n+1})$$

*is surjective.*

**Proof.** These statements mean that $B_0 = k$ and that $B_1 \otimes B_n \to B_{n+1}$ is surjective. They imply that $B = k[B_1]$ by induction on the degree. The rest of (6.6i), that $B$ has trivial socle, follows from the fact that, by definition of a triple, $L$ is invertible and generated by its sections. Hence if a section of any sheaf is killed by multiplication with every element of $B_1 = H^0(E, L)$, it is zero. $\square$

**Lemma 7.2.** *The ideal $J$ is generated by $J_s$ if and only if* (6.6iii) *and* (6.6iv) *hold and the space $U$ in* (6.6v) *is* 0.

This is evident and is left to the reader. $\square$

**Lemma 7.3.** *The statement about* $\dim_k W$ *in* (6.6iv) *follows from the preceding parts of* (6.6).

**Proof.** We have

$$\begin{aligned}\dim J_{s+1} &= \dim U + \dim(T_1 J_s + J_s T_1) \\ &= \dim U + (T_1 J_s) + \dim(J_s T_1) - \dim W \\ &= \dim U + 2r^2 - \dim W.\end{aligned}$$

Combining this with $\dim J_n = r^n - \dim B_n$ gives the result in all cases, as we leave to the reader to check. $\square$

Suppose that $\mathcal{T}$ is a linear triple. Then $E$ is either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. Up to isomorphism, the invertible sheaves on $E$ are just $\mathcal{O}(n)$ or $\mathcal{O}(m,n)$

according to the case, and the effect of $\sigma$ on them is given by $\mathcal{O}(n)^\sigma = \mathcal{O}(n)$ in the first case and $\mathcal{O}(m,n)^\sigma = \mathcal{O}(n,m)$ in the second one. In the first case, $L = \mathcal{O}(1)$, hence $L_n \approx \mathcal{O}(n)$ for all $n \geq 0$. In the second case, $L = \mathcal{O}(1,0)$, hence $L_{2m} \approx \mathcal{O}(m,m)$ and $L_{2m+1} \approx \mathcal{O}(m+1,m)$ for all $m \geq 0$. Statements (6.6i) and (ii) follow easily from these facts and from Lemma (7.1).

Choose an isomorphism $\varphi : L \to L^\sigma$ if $E = \mathbf{P}^2$, or an isomorphism $\varphi : L \to L^{\sigma^2}$ if $E = \mathbf{P}^1 \times \mathbf{P}^1$, and let $\tau$ be the automorphism of $B_1 = H^0(E, L)$, such that, for $x \in B_1$, $\varphi(x) = (x^\tau)^\sigma$ or $(x^\tau)^{\sigma^2}$, according to the case. Let $\{x_i\}$, $1 \leq i \leq r$, be a basis for $B_1 = T_1$.

**Proposition 7.4.** *If $r = 3$, $J$ is generated by the three elements*

$$x_i x_j^\tau - x_j x_i^\tau, \quad 1 \leq i < j \leq 3. \tag{7.4'}$$

*If $r = 2$, $J$ is generated by the two elements*

$$x_1 x_j x_2^\tau - x_2 x_j x_1^\tau, \quad j = 1, 2. \tag{7.4''}$$

By lemmas (7.2) and (7.3), parts (iii-v) of Theorem 6.6 follow from this proposition, and part (vi) does too, using, in the two cases, the formulas

$$w = x_1 x_2^\tau x_3^{\tau^2} - x_1 x_3^\tau x_2^{\tau^2} + x_2 x_3^\tau x_1^{\tau^2} - x_2 x_1^\tau x_3^{\tau^2} + x_3 x_1^\tau x_2^{\tau^2} - x_3 x_2^\tau x_1^{\tau^2}$$

and

$$w = x_1^2 (x_2^\tau)^2 - x_1 x_2 x_2^\tau x_1^\tau - x_2 x_1 x_1^\tau x_2^\tau - x_2^2 (x_1^\tau)^2 ,$$

as we leave to the reader to check.

**Proof of 7.4**. If $r = 3$, the isomorphism

$$L \otimes L \xrightarrow{1 \otimes \varphi} L \otimes L^\sigma$$

carries the relation $xy = yx$ for $x, y \in H^0(E, L) = B_1$ into the relation $x(y^\tau)^\sigma = y(x^\tau)^\sigma$, which shows, by the definition (6.5) of multiplication in $B$, that we have $xy^\tau = yx^\tau$ in $B$ for any $x, y \in B_1$. If $r = 2$, the isomorphism

$$L \otimes L^\sigma \otimes L \xrightarrow{1 \otimes 1 \otimes \varphi} L \otimes L^\sigma \otimes L^{\sigma^2}$$

shows similarly that the relation

$$xyz^\tau = zyx^\tau$$

holds for any $x, y, z \in B_1$. Thus the elements (7.4′) and (7.4″) are in $J_s$.

Let $J' \subset J$ be the ideal generated by the relations (7.4′) or (7.4″), as the case may be, and let $B' = T/J'$.

Using those relations, it is easy to check that $B'_n$ is spanned by the monomials

$$x_{i_1} x_{i_2}^{\tau} \dots x_{i_n}^{\tau^{n-1}}, \quad \text{with} \quad i_1 \leq i_2 \leq \dots \leq i_n$$

in the first case, and by the monomials of degree $n$ of the form

$$x_{i_1} x_{j_1} x_{i_2}^{\tau} x_{j_2}^{\tau} x_{i_3}^{\tau^2} x_{i_3}^{\tau^2} \dots, \quad \text{with} \quad \begin{cases} i_1 \leq i_2 \leq \dots \\ j_1 \leq j_2 \leq \dots \end{cases}$$

in the second case. The number of such monomials is $\dim B_n$ in each case. Hence $B' = B$, $J' = J$, and incidentally, these monomials are linearly independent. This completes the proof of Theorem 6.6 in the case that $\mathcal{T}$ is linear.

For the rest of this section, we assume that our triple is elliptic. Thus if $r = 3$, then $E$ is a divisor of degree 3 in $X = \mathbf{P}^2$, and if $r = 2$, then $E$ is a divisor of bidegree $(2, 2)$ in $X = \mathbf{P}^1 \times \mathbf{P}^1$. In both cases the canonical sheaf $\omega_E$ is isomorphic to $\mathcal{O}_E$, and $E$ has arithmetic genus 1. This follows from the adjunction formula ([A-K Ch. I, 2.5], [M p.81, Theorem 3])).

We write the divisor $E$ in the form $E = \Sigma r_i C_i$, where $C_i$ are the irreducible components of its support. If $M$ is an invertible sheaf on $E$, we denote by $d_i(M)$ the degree of its restriction to $C_i$. By *degree* $\deg M$ of $M$ we mean its total degree: $\deg M = \Sigma r_i d_i(M)$.

**Lemma 7.5.** *Suppose that $\mathcal{T}$ is not exceptional, i.e., does not satisfy (4.9). Then for all $i$ we have $d_i(L^\sigma) = d_i(L)$ if $r = 3$, and $d_i(L^{\sigma^2}) = d_i(L)$ if $r = 2$.*

**Proof**. The assertion is trivial if $E$ has only one component. If $r = 3$, there are at most three components, and if there are three, then they are all lines. In that case $L$ has the same degree on each component, and so every automorphism preserves the degrees $d_i$. There remains the case of two components interchanged by $\sigma$, which is the exceptional configuration (4.10).

If $r = 2$, then we need to show that $\sigma^2$ preserves the degree $d_i(L)$, hence we need only consider cases in which the set of components has a $\sigma$-orbit of order at least 3. Now $E$ has at most four components, and if there are 4, then they consist of two pairs of rulings – a "quadrilateral." In that case a permutation of order 4 exists, but it satisfies the relation $d_i(L^{\sigma^2}) = d_i(L)$. The remaining case of three components is the exceptional configuration (4.10). □

If $M$ is a locally free sheaf on $E$, we will use the standard notation

$$(7.6)\qquad \begin{gathered} H^i(M) = H^i(E,M),\ \ h^i(M) = \dim_k H^i(M), \\ \det M = \Lambda^n M,\ \text{ if } n = \text{rank } M, \\ \deg M = \deg(\det M),\ \ M^* = \mathrm{Hom}_{\mathcal{O}_E}(M, \mathcal{O}_E)\,. \end{gathered}$$

The Riemann-Roch theorem and duality give

$$(7.7)\qquad \chi(M) := h^0(M) - h^1(M) = \deg M, \text{ and } h^1(M) = h^0(M^*).$$

(see [M] p. 79 Ths. 1,2 or [A-K]).

In order to fix attention on cases in which this formula can be used to compute $h^0$ and $h^1$ separately in terms of the degree, we introduce some more local terminology:

*Definition* 7.8: An invertible sheaf $M$ on $E$ is *tame* if either $h^0(M) = 0$ or $h^1(M) = 0$, or $M \approx \mathcal{O}_E$.

If $E$ is reduced and irreducible, e.g., a smooth elliptic curve, then all invertible sheaves are tame, because in that case an invertible $M$ of degree $\leq 0$ has no non-zero section unless $M \approx \mathcal{O}_E$, and dually, if $\deg M \geq 0$, then $h^1(M) = 0$ unless $M \approx \mathcal{O}_E$. However, on a reducible divisor $E$, not all invertible sheaves are so nice. For example, suppose that $E = C_1 + C_2$ consists of two curves $C_i$ of genus zero, meeting with multiplicity $(C_1 \cdot C_2) = 2$. If the first degree $d_1$ of an invertible sheaf $M$ on $E$ is at least 2, then $M$ will have non-zero sections which vanish on $C_2$ no matter how negative the total degree $d_1 + d_2$ is. In order to overcome this difficulty, we make a digression.

We recall the definition of numerically connected divisors [R]: Let $D$ be a positive divisor on a smooth surface $X$. A *decomposition* of $D$ is a pair of strictly positive divisors $A$, $B$ such that $D = A + B$. The divisor $D$ is said to be *numerically connected* if $(A \cdot B) > 0$ for every such decomposition. Ramanujam proves the following facts concerning numerically connected divisors:

**Proposition 7.9.** (i) *Suppose that the divisor $D$ moves in an algebraic family without fixed points on $X$, and that $(D^2) > 0$. Then $D$ is numerically connected.*
(ii) *Let $D$ be a numerically connected divisor. Then $H^0(D, \mathcal{O}_D) = k$.*

Ramanujam's method of proof of (ii) can be used to show the following:

**Proposition 7.10.** *Let $M$ be an invertible sheaf on a numerically connected divisor $D$, with the following properties:*
(i) $\deg(M) \leq 0$, *and*

(ii) *for every decomposition* $D = A + B$, $\deg(M_B) < (A \cdot B)$, *where* $M_B$ *denotes the restriction of* $M$ *to* $B$.

*Then either* $M \approx \mathcal{O}_D$, *or else* $H^0(D, M) = 0$.

**Proof**. Let $\alpha$ be a non-zero section of $M$. Assume first that $\alpha$ does not vanish identically on any component $C$ of $D$. Then $\deg(M_C) \geq 0$ for every $C$. Since the total degree of $M$ is non-positive, it follows that $\deg(M_C) = 0$ for every $C$. Then since $M_C$ has a non-zero section, it is the trivial sheaf and $\alpha$ does not vanish anywhere on $C$. This is true for each component, so $\alpha$ is a non-vanishing section on $D$, and defines an isomorphism $\mathcal{O}_D \xrightarrow{\sim} M$.

We want to rule out the possibility that $\alpha$ vanishes identically on some component of $D$. Assume that it does, and let $A$ be the largest divisor $< D$ on which $\alpha$ vanishes identically. Let $B = D - A$, and consider the exact sequence

$$0 \to M_B(-A) \to M \to M_A \to 0.$$

By definition of $A$, $\alpha$ is a section of $M_B(-A)$. Since the degree of that sheaf is $\deg(M_B) - (A \cdot B) < 0$, it follows that $\alpha$ vanishes on a component of $B$, contradicting maximality. □

We now return to our elliptic triple $\mathcal{T} = (E, \sigma, L)$.

**Proposition 7.11.** *The divisor* $E$ *is numerically connected, and* $H^0(E, \mathcal{O}_E) = k$.

This is an immediate consequence of Proposition 7.9, because every positive divisor on $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ moves in the family of divisors of the same degree or bidegree, which is without fixed points, and because $(E^2) = r^s > 0$. □

**Proposition 7.12.** *An invertible sheaf* $M$ *on* $E$ *is tame if it satisfies any one of the following conditions:*

(i) *The hypotheses of* (7.10) *hold with* $D$ *replaced by* $E$.
(ii) $M^*$ *is tame.*
(iii) $d_i(M) \leq 0$ *for all* $i$ *or* $d_i(M) \geq 0$ *for all* $i$.
(iv) $M$ *is generated by its sections.*

**Proof**. Proposition 7.10 shows that $M$ is tame in case (i). Case (ii) follows from the definition of tame and duality. If $d_i(M) \leq 0$ for all $i$, then $\deg L_B \leq 0 < (A \cdot B)$ for all decompositions $E = A + B$, so this is a subcase of (i). The case $d_i(M) \geq 0$ for all $i$ follows by duality, and it includes the case (iv). □

**Proposition 7.13.** *An invertible sheaf $M$ on $E$ such that $\deg M \geq 2$ and that $d_i(M) \geq 0$ for all $i$ is generated by its sections.*

**Proof.** We may suppose $k$ algebraically closed. Let $d = \deg M \geq 2$. For each $i$, we will show that $M$ is generated by its sections at every point of $C_i$.

*Case* 1: $d_i = d$. Then the irreducible component $C = C_i$ has multiplicity 1 in $E$, and the smooth points of $C$ are smooth points of $E$. Let $p$ be such a point. Then the sheaf $M(-(d-1)p)$ is of degree 1, and is tame by (7.12iii), so it has a non-zero section $x$. The divisor of zeros of $x$ is of degree 1, hence it consists of a single smooth point $q$. So $x$ generates $M$ everywhere except at $p$ and $q$. Since $M(-p)$ is tame of degree $d-1$, it has fewer sections than $M$, so the sections of $M$ generate $M$ at $p$. The same argument shows that they generate $M$ at $q$.

*Case* 2: $d_i < d$. Let $C = C_i$ and $B = E - C$. Then $C \approx \mathbf{P}^1$, because it is reduced and irreducible of genus 0. Moreover, $B$ is numerically connected. If $r = 3$, this is obvious. If $r = 2$, it follows from the fact that $B$ does not have bidegree $(2,0)$ or $(0,2)$ because, being reduced and irreducible, $C$ does not. Consider the exact sequence

$$0 \to M_B(-C) \to M \to M_C \to 0.$$

The sheaf $M_C$ is of degree $d_i \geq 0$ on $C$, and is therefore generated by its sections. To prove that $M$ is generated by its sections at points of $C$, it suffices to show that the map $H^0(M) \to H^0(M_C)$ is surjective. To do this we show that $h^1(M_B(-C)) = 0$ by duality on $B$. Since $\omega_E \approx \mathcal{O}_E$, we have $\omega_B \approx \mathcal{O}_B(-C)$, and therefore

$$\begin{aligned} h^1(M_B(-C)) = h^0\left(\omega_B \otimes (M_B(-C))^*\right) &= h^0\left(\mathcal{O}_B(-C) \otimes M_B^*(C)\right) \\ &= h^0\left(M_B^*\right) = 0 \end{aligned}$$

by (7.10), because $M_B^*$ is an invertible sheaf on the numerically connected divisor $B$ whose degree on each irreducible component is $\leq 0$, and whose total degree is $d_i - d < 0$. $\square$

*Notation* 7.14. Let $M$ be a locally free sheaf which is generated by its sections. We denote by [M] the canonical exact sequence

$$[\mathrm{M}] \qquad 0 \to M'' \to M' \xrightarrow{\varphi} m \to 0,$$

where $M' = \mathcal{O}_E \otimes_k H^0(M)$, and $M''$ is the kernel of the canonical map $\varphi : M' \to M$, which is surjective because $M$ is generated by its sections. Thus $M'$ is a free $\mathcal{O}_E$-module of rank $h^0(M)$, $M''$ is locally free, and since $H^0(\mathcal{O}_E) = k$,

$$H^0(M'') = 0. \tag{7.15}$$

Moreover,

(7.16)
$$\det M'' = (\det M)^*, \quad \deg M'' = -\deg M, \quad \text{and} \quad \chi(M'') = -\chi(M).$$

**Proposition 7.17.** *Let $M$ and $N$ be locally free sheaves on $E$ such that $M$ is generated by its sections and that $h^1(N) = 0$. Let*

$$\mu_{M,N} : H^0(M) \otimes H^0(N) \to H^0(M \otimes N)$$

*be the multiplication map. Then* $\ker \mu_{M,N} \approx H^0(M'' \otimes N)$, *coker* $\mu_{M,N} \approx H^1(M'' \otimes N)$ and $H^1(M \otimes N) = 0$.

This follows immediately from the exact sequence of cohomology of the exact sequence $[M] \otimes N$, which is

$$\begin{aligned} 0 \to H^0(M'' \otimes N) \to H^0(M) \otimes H^0(N) \xrightarrow{\mu_{M,N}} H^0(M \otimes N) \\ \to H^1(M'' \otimes N) \to 0 \to H^1(M \otimes N) \to 0. \end{aligned}$$

□

**Corollary.** *Let $M$ and $N$ be invertible sheaves on $E$ such that $M$ is generated by its sections and $N$ is tame of degree $\geq 0$. Then $M \otimes N$ is tame of degree $\geq 0$.*

**Proof**. If $N$ is isomorphic to $\mathcal{O}_E$, this follows from (7.12iv). If not, then $H^1(N) = 0$, and hence $H^1(M \otimes N) = 0$ by the proposition. □

**Lemma 7.18.** *The sheaf $L^* \otimes L_n^\sigma$ is tame for all $n \geq 1$.*

**Proof**. We have

$$L^* \otimes L_n^\sigma = (L^* \otimes L^\sigma) \otimes L^{\sigma^2} \otimes \ldots \otimes L^{\sigma^n}.$$

Since each $L^{\sigma^i}$ is generated by sections and since $\deg(L^* \otimes L^\sigma) = 0$, the above corollary and induction reduces us to showing that $L^* \otimes L^\sigma$ is tame. We verify the condition of Proposition 7.10. Let $E = A + B$ be a decomposition of the divisor $E$. If $r = 3$, we may assume that $\deg A = \deg L_A = 1$, and $\deg B = \deg L_B = 2$. There are two possibilities: Either $\deg L_A^\sigma = 1$ and $\deg L_B^\sigma = 2$, or else $\deg L_A^\sigma = 2$ and $\deg L_B^\sigma = 1$. In each case, $\deg(L^* \otimes L^\sigma)_A$ and $\deg(L^* \otimes L^\sigma)_B$ are less than $(A \cdot B) = 2$, as required.

Assume that $r = 2$, and say that the bidegree of $B$ is $(i, j)$. Then the bidegree of $A$ is $(2 - i, 2 - j)$, and $\deg L_B = i$, $\deg L_B^\sigma = j$. In each case, $\deg(L^* \otimes L^\sigma)_B = i - j < (A \cdot B) = 2(i + j - ij)$.

We can now prove parts (i-iii) of Theorem 6.6. Part (ii) is easy. We have

$$\dim B_n = h^0(L_n) = \deg L_n = rn,$$

for $n \geq 1$, because $L_n$ is the product of $n$ sheaves $L^{\sigma^i}$, each of which is of degree $r$ and is generated by its sections. Hence $L_n$ is generated by its sections, so it is a tame sheaf of degree $rn$.

Part (iii) of (6.6) follows from parts (i) and (ii), which imply that $\dim J_n = r^n - rn$ for all $n \geq 1$.

Part (i) is, by (7.1), a consequence of the fact that $H^0(L) \approx k$ (7.11), Proposition 7.17 (with $M = L$, $N = L_n^\sigma$) and

**Lemma 7.19.** *$H^1(L'' \otimes L_n^\sigma) = 0$ for $n \geq 1$.*

**Proof.** If $r = 2$, then $L''$ is of rank 1, so $L'' \approx \det L'' \approx L^*$, and $L'' \otimes L_n^\sigma \approx L^* \otimes L_n^\sigma$. This sheaf is tame by Lemma 7.18, and of degree $2(n-1) \geq 0$. Moreover, when the degree is zero, the sheaf in question is $L^* \otimes L^\sigma$, which is not isomorphic to $\mathcal{O}_E$. Indeed, $L$ is not isomorphic to $L^\sigma$, because $\pi_L : E \to \mathbf{P}^1$ is of degree 2, while $(\pi_L, \pi_{L^\sigma}) : E \to \mathbf{P}^1 \times \mathbf{P}^1$ is a closed immersion.

If $r = 3$, then $L''$ is of rank 2. The pairing $L'' \otimes L'' \to \det L''$ shows in this case that

$$(7.20) \qquad (L'')^* \approx L'' \otimes (\det L'')^* \approx L'' \otimes L.$$

We use Lemma 7.18 and put $M = L_n^\sigma$ in the following lemma to get our result. $\square$

**Lemma 7.21.** *Assume that $r = 3$. Let $M$ be an invertible sheaf on $E$ of degree at least 3 such that $L^* \otimes M$ is tame. Then $H^1(L'' \otimes M) = 0$.*

**Proof.** Dualizing the sequence [$L$] and tensoring it with $L^* \otimes M$ using (7.20) gives the exact sequence

$$0 \to (L^*)^{\otimes 2} \otimes M \to (L^* \otimes M)^3 \to L'' \otimes M \to 0.$$

If $L$ and $M$ are not isomorphic, then $H^1(L^* \otimes M) = 0$ because $\deg(L^* \otimes M) \geq 0$, and hence $H^1(L'' \otimes M) = 0$. To show that $H^1(L'' \otimes L) = 0$ too, we use (7.20), duality, and the fact that $H^0(L'') = 0$ (7.15). $\square$

In order to complete the proof of Theorem 6.6 in the case $r = 3$, we need one more result about generation by sections.

**Proposition 7.22.** *Assume that $r = 3$. Let $M$ be an invertible sheaf of degree 3 whose sections define an embedding of $E$ as a divisor of degree 3 in $\mathbf{P}^2$. Then $L'' \otimes M$ is generated by its sections.*

**Proof.** Let $N = L'' \otimes M$. This sheaf is of rank 2, and its determinant

$$\det N = \wedge^2 N = \wedge^2(L'' \otimes M) \approx \wedge^2(L'') \otimes M^{\otimes 2} \approx L^* \otimes M^{\otimes 2}$$

is of degree 3. Moreover, $d_i(\wedge^2 N) \geq 0$ for each $i$. For, if $E$ has more than one irreducible component, then the equation $3 = \Sigma d_i(L)$, together with the fact that $d_i(L) > 0$ for each $i$, imply that $1 \leq d_i(L) \leq 2$ for each $i$. The same holds for $M$, so that $2d_i(M) - d_i(L) \geq 0$ for each $i$, as claimed. Thus by (7.13), $\wedge^2 N$ is generated by its sections.

The same argument we used in the proof of Lemma 7.18 to prove that $L^* \otimes L^\sigma$ is tame when $r = 3$ shows that $L^* \otimes M$ is tame. Therefore $h^1(N) = 0$ by the preceding lemma, and consequently $h^0(N) = \deg N = 3$. Thus each of the three vector spaces

$$X = H^0(L), \; Y = H^0(M), \; \text{and} \; Z = H^0(N)$$

has dimension 3.

The sheaf exact sequence $[L] \otimes M$ gives an injection $i : N \to X \otimes_k M$. Applying the functor $\wedge^2$, we have another sheaf map

$$\wedge^2 N \xrightarrow{j} \wedge (X \otimes_k M) = \left(\wedge_k^2 X\right) \otimes_k M^{\otimes 2}. \tag{7.23}$$

Applying $H^0$ to these maps, we obtain an injection

$$Z \xrightarrow{i_*} X \otimes Y, \tag{7.24}$$

which identifies $Z$ with $\ker \mu_{L,M}$ (cf. (7.17)), and a commutative diagram

$$\begin{array}{ccc} \wedge_k^2 Z & \xrightarrow{\wedge^2 i_*} & \wedge_k^2 (X \otimes_k Y) \\ \downarrow \varphi & & \downarrow \psi \\ H^0\left(\wedge_k^2 N\right) & \xrightarrow{j_*} & \left(\wedge_k^2 X\right) \otimes \mathrm{Symm}_2(Y), \end{array}$$

in which the upper horizontal arrow is injective. The vertical arrows are the canonical maps $\wedge^2 H^0 \to H^0 \wedge^2$. At the bottom right we have used the isomorphism $\mathrm{Symm}_2(Y) = H^0(M^{\otimes 2})$, which follows from our hypothesis that the sections of $M$ embed $E$ as a plane cubic in $\mathbb{P}^2 = \mathbb{P}(Y)$, the projective space of planes in $Y$; the restriction map

$$\mathrm{Symm}_k(Y) = \bigoplus_{n \geq 0} H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)\right) \to \bigoplus_{n \geq 0} H^0\left(E, M^{\otimes n}\right)$$

is surjective, and its kernel is generated by a form of degree 3. So it is an isomorphism in degrees $\leq 2$.

We have seen in the first paragraph of this proof that $\wedge^2 N$ is generated by its sections. In order to conclude that $N$ itself is generated by sections, it suffices to prove that the map $\varphi$ is surjective. For if that is so, then for every point $p \in E$ there exist elements $z, z' \in Z$ such that $\varphi(z \wedge z')(p) = z(p) \wedge z'(p)$ is not zero, and such $z$ and $z'$ generate $N$ at $p$.

To show $\varphi$ is surjective, it suffices to show $\varphi$ injective, because $\varphi$ is a linear transformation from one three-dimensional vector space to another.

It is an elementary fact that every element of $\wedge^2 Z$ is a decomposable tensor, of the form $z^\wedge z'$. (In general, all elements of the $(n-1)$-st exterior power of an $n$-dimensional vector space are decomposable [B1], Ch. III, Sect. 11, No. 13, Cor. of Prop. 15.) We have therefore only to show that for $z, z' \in Z$, $\varphi(z \wedge z') = 0$ implies $z \wedge z' = 0$.

Let $\{x_i\}$, $i = 1, 2, 3$, be a basis for $X$, and write

$$i_* z = \sum_i x_i \otimes y_i, \quad i_* z' = \sum_i x_i \otimes y'_i,$$

with $y_i, y'_i \in Y$. Then if $\varphi(z \wedge z') = 0$, we have

$$\begin{aligned} 0 = j_* \varphi(z \wedge z') &= \psi\left(\wedge^2 i_*\right)(z \wedge z') = \psi(i_* z \wedge i_* z') \\ &= \psi((\sum_i x_i \otimes y_i) \wedge (\sum_i x_i \otimes y'_i)) = \sum_{i,j} (x_i \wedge x_j) \otimes y_i y'_j \\ &= \sum_{1 \le i,j \le 3} (x_i \wedge x_j)\left(y_i y'_j - y_j y'_i\right). \end{aligned}$$

Since the elements $x_i \wedge x_j$ for $i < j$ form a basis for $\wedge^2 X$, it follows that $y_i y'_j - y_j y'_i = 0$ in $\mathrm{Symm}_2 Y$ for all $i, j$, and from this we wish to conclude that $z \wedge z' = 0$.

We use unique factorization in the polynomial algebra Symm $Y$. If $y_i = 0$ for all $i$, then $i_* z = 0$, hence $z = 0$. So renumbering if necessary, we can suppose $y_1 \neq 0$. We have

$$y_1 y'_j = y_j y'_1 \tag{7.26}$$

for $j = 1, 2, 3$, and the $y$'s are linear forms, with $y_1 \neq 0$. Hence either $y_1$ divides $y'_1$, in which case $y'_1 = c y_1$ for some $c \in k$, or else $y_1$ divides $y_j$ for each $j$, in which case there are constants $c_j \in k$ such that $y_j = c_j y_1$. Substituting in (7.26) and canceling $y_1$, we conclude that for all $j$,

$$y'_j = \begin{cases} c y_j & \text{in the first case,} \\ c_j y'_1 & \text{in the second case} \end{cases}.$$

In the first case the conclusion is $i_* z' = c i_* z$, which means $z' = cz$ and $z^\wedge z' = 0$, as desired. In the second case, putting $x = \Sigma c_i x_i$, we have

$$i_* z = x \otimes y_1 \quad \text{and} \quad i_* z' = x \otimes y_1'.$$

We now use the fact that $i_* Z$ is the kernel of the multiplication map $\mu_{L,M}$, which implies that

$$xy_1 = 0 = xy_1'$$

in $H^0(L \otimes M)$. Since $c_1 = 1 \neq 0$, the section $x$ is not identically zero, and, since $E$ has no embedded components, it follows that the support of $x$ is not finite. Hence the linear forms $y_1$ and $y_1'$ have an infinite set of common zeros on $E$. This means that the lines $y_1 = 0$ and $y_1' = 0$ in $\mathbf{P}(Y) = \mathbf{P}^2$ have an infinite number of common points, which implies that $y_1' = c' y_1$ for some $c' \in k$. Then $i_* z' = c' i_* z$, $z' = c' z$, and $z \wedge z' = 0$, as before. This completes the proof of Lemma 7.22. $\square$

The lemma we have just proved is essentially equivalent to the following corollary, which may be of some interest in its own right.

**Corollary.** *Let $E$ be scheme, and let $L$ and $M$ be invertible sheaves on $E$ whose spaces of sections $X = H^0(E, L)$ and $Y = H^0(E, M)$ are of dimension 3 and give embeddings $\varphi_L : E \to \mathbf{P}(X)$ and $\varphi_M : E \to \mathbf{P}(Y)$ of $E$ as a cubic divisor in $\mathbf{P}^2$. Let $Z \subset X \otimes Y$ be the kernel of the multiplication map $\mu_{L,M} : X \otimes Y \to H^0(L \otimes M)$, and let $\Gamma \subset \mathbf{P}(X) \times \mathbf{P}(Y)$ be the locus of zeros of $Z$. Then $\Gamma \cap (\mathbf{P}(X) \times \varphi_M(E))$ is the graph of the isomorphism $\theta : \varphi_L(E) \to \varphi_M(E)$ such that $\varphi_M = \theta \circ \varphi_L$.*

**Proof.** With notation as in the proof just completed, let $(z^{(\nu)})$, $1 \leq \nu \leq 3$, be a basis for $Z$, and let $i_* z^{(\nu)} = \sum_j x_j y_j^{(\nu)}$. The lemma we have just proved amounts to the fact that the sections $\varphi(z^{(\mu)} \wedge z^{(\nu)})$ have no common zeros on $E$. The map $j$ (7.23) is an injection of sheaves and splits locally, because the same is true of $i : N \to X \otimes_k M$. It follows that the sections

$$j_* \varphi\left(z^{(\mu)} \wedge z^{(\nu)}\right) = \sum_{i<j} (x_i \wedge x_j) \otimes \left(y_i^{(\mu)} y_j^{(\nu)} - y_j^{(\mu)} y_i^{(\nu)}\right)$$

have no common zero, i.e., that the $3 \times 3$ matrix $(y_i^{(\mu)})$ has rank $\geq 2$ at every point of $E$. Thus there is at most one solution $p \in \mathbf{P}(X)$ to the equations

$$\sum_j x_j(p) y_j^{(\nu)}(q) = 0, \qquad \nu = 1, 2, 3,$$

for each point $q \in \varphi_M(E)$. And of course there is one solution, namely $p = \theta^{-1}(q)$. $\square$

We now return to the proof of Theorem 6.6, to establish the remaining properties (6.6iv)-(6.6vi) of the algebra $B = \mathcal{B}(\mathcal{T})$. For $\ell, m, n \geq 0$, let

$$K_{l,m} = \ker (B_l \otimes B_m \to B_{l+m}) \subset B_l \otimes B_m,$$

and

$$K_{l,m,n} = \ker (B_l \otimes B_m \otimes B_n \to B_{l+m+n}) \subset B_{l+m+n}.$$

Let $\theta_n$ denote the canonical map $T_n \to B_n$. The following elementary lemma has nothing to do with our special situation. It holds for any graded algebra $B$ such that $B = k[B_1]$.

**Lemma 7.27.** *For each triple of positive integers $\ell, m, n$, the map*

$$\theta_{l,m,n} = \theta_l \otimes \theta_m \otimes \theta_n : T_{l+m+n} \to B_l \otimes B_m \otimes B_n$$

*induces an isomorphism*

(*)
$$J_{l+m+n}/(T_l \otimes J_{m+n} + J_{l+m} \otimes T_n) \xrightarrow{\sim} K_{l,m,n}/(B_l \otimes K_{m,n} + K_{l,m} \otimes B_n).$$

**Proof**. The map $\theta = \theta_{l,m,n}$ is surjective because $B = k[B_1]$. We have immediately from the definitions

$$\begin{aligned}
\theta^{-1}(K_{l,m,n}) &= J_{l+m+n}, \\
\theta^{-1}(B_l \otimes K_{m,n}) &= T_l \otimes J_{m+n} + J_l \otimes T_{m+n}, \\
\theta^{-1}(K_{l,m} \otimes B_n) &= T_{l+m} \otimes J_n + J_{l+m} \otimes T_n.
\end{aligned}$$

Since $J_l \otimes T_{m+n} \subset J_{l+m} \otimes T_n$ and $T_{l+m} \otimes J_n \subset T_l \otimes J_{m+n}$, it follows that the inverse images, under $\theta$, of the top and bottom of the quotient on the right of (*) are, respectively, the top and bottom groups on the left. □

For $m \geq 1$ and $n \geq 1$, let $M = L_m^\sigma$, and $N = L_n^{\sigma^{n+1}}$, so that $L \otimes M = L_{m+1}$, $M \otimes N = L_{m+n}^\sigma$, and $L \otimes M \otimes N = L_{1+m+n}$.

**Lemma 7.28.** *For every $m, n \geq 1$, there is a commutative diagram*

$$\begin{array}{ccc}
K_{1,m} \otimes B_n & \xrightarrow{\alpha} & K_{1,m,n}/(B_1 \otimes K_{m,n}) \\
\downarrow & & \downarrow \\
H^0(L'' \otimes M) \otimes H^0(N) & \xrightarrow{\mu} & H^0(L'' \otimes M \otimes N)
\end{array}$$

*in which the vertical arrows are isomorphisms, $\alpha$ is the map induced by the inclusion $K_{1,m} \otimes B_n \subset K_{1,m,n}$, and $\mu = \mu_{L'' \otimes M, N}$ is the multiplication map.*

**Proof**. Applying the functor $H^0$ to the diagram

$$\begin{array}{c}[L]\otimes M\otimes N'\\ \downarrow\\ [L]\otimes M\otimes N\end{array}$$

and making such identifications as $B_m = H^0(M)$ via $\sigma$ so that $\mu_{L,M}$ becomes the multiplication map $B_1 \otimes B_m \to B_{1+m}$, etc., we obtain the exact commutative diagram

$$\begin{array}{ccccccc}0 \longrightarrow H^0(L'' \otimes M) \otimes H^0(N) & \xrightarrow{\alpha} & B_1 \otimes B_m \otimes B_n & \xrightarrow{\mu\otimes 1} & B_{1+m} \otimes B_n & \longrightarrow & 0\\ \Big\downarrow{\scriptstyle \mu_{L''\otimes M,N}} & & \Big\downarrow{\scriptstyle 1\otimes\mu} & & \Big\downarrow{\scriptstyle \mu} & & \\ H^0(L'' \otimes M \otimes N) & \longrightarrow & B_1 \otimes B_{m+n} & \xrightarrow{\mu} & B_{1+m+n} & \longrightarrow & 0\end{array}$$

in which the $\mu$'s without subscripts are multiplication in $B$. The top row shows that $H^0(L'' \otimes M) \otimes H^0(N) = K_{1,m} \otimes B_n$ (cf. (7.17)). The diagram shows that the image of $H^0(L'' \otimes M \otimes N)$ in $B_1 \otimes B_{m+n}$ is the same as the image of the subspace $K_{1,m,n}$ of $B_1 \otimes B_m \otimes B_n$, and this proves the lemma since the kernel of the vertical map of $B_1 \otimes B_m \otimes B_n$ is $B_1 \otimes K_{m,n}$. □

From now on we set $m = s - 1$, and we suppose $n \geq 1$. Combining the last two lemmas, we obtain

**Lemma 7.29.** *For $n \geq 1$, $J_{s+n}/(T_1 J_{s+n-1} + J_s T_n)$ is isomorphic to the cokernel of the multiplication map*

$$\mu_{L''\otimes M,N} : H^0(L'' \otimes M) \otimes H^0(N) \to H^0(L'' \otimes M \otimes N),$$

*where $M = L^{\sigma}_{s-1}$ and $N = L^{\sigma^s}_n$.*

**Lemma 7.30.** *The sheaf $L'' \otimes M$ is generated by its sections. It has rank $r-1$, its determinant is $L^* \otimes M^{\otimes(r-1)}$ and its degree is $r$. Moreover, $h^0(L'' \otimes M) = r$ and $h^1(L'' \otimes M) = 0$.*

**Proof**. The fact that $h^1 = 0$ is a consequence of (7.19). The rest is straightforward except the generation by sections. This follows from (7.22) if $r = 3$, because then $L'' \otimes M = L'' \otimes L^{\sigma}$. If $r = 2$, then $L'' \otimes M \approx L^* \otimes L^{\sigma} \otimes L^{\sigma^2}$, and it follows from (7.13) in that case, provided that we show $d_i(L'' \otimes M) \geq 0$ for all $i$. This is true. If the triple is not exceptional, (7.6) shows that $d_i(L'' \otimes M) = 0$ for each $i$. If the triple is exceptional and the components are ordered so that the multidegree of $L$ is $(d_1, d_2, d_3) = (1, 0, 1)$, it is easy to compute that the multidegree of $L^* \otimes L^{\sigma} \otimes L^{\sigma^2}$ is $(0, 2, 0)$. □

Since $L''\otimes M$ is generated by sections, we can analyze the multiplication map $\mu$ in (7.29) via (7.17) and the cohomology sequence of the sheaf exact sequence $[L''\otimes M]\otimes N$, instead of the sheaf exact sequence $(L''\otimes M)\otimes[N]$. The advantage of this is that $(L''\otimes M)''$ is of rank 1, whereas $N''$ is not, in general. Thus $(L''\otimes M)''$ is equal to its determinant, which is $L\otimes(M^*)^{\otimes r-1}$. By (7.17), $J_{s+n}/(T_1J_{s+n-1}+J_sT_n)$, which is the cokernel of the map $\mu$ in (7.29), is isomorphic to $H^1(L\otimes(M^*)^{\otimes r-1}\otimes N)$, i.e., to $H^1(F\otimes L_{n-1}^{\sigma^{s+1}})$, where

$$F=\begin{cases} L\otimes((L^*)^\sigma)^{\otimes^2}\otimes L^{\sigma^2} & \text{if } r=3,\\ L\otimes(L^*)^\sigma\otimes(L^*)^{\sigma^2}\otimes L^{\sigma^3} & \text{if } r=2.\end{cases}\tag{7.31}$$

Suppose that $E$ is not exceptional. Then, by (7.6), we have $d_i(F)=0$ for all $i$. Consequently, the sheaf $F\otimes L_{n-1}^{\sigma^{s+1}}$ is tame by (7.12iii)) for all $n\geq 1$; and since the total degree of that sheaf is $(n-1)r$, which is $>0$ for $n\geq 2$, statement (6.6iv) follows.

*Remark* 7.32. It is enough to prove one half of (6.6iv), as we have just done. The other half follows on replacing $\sigma$ by $\sigma^{-1}$ in our triple, because the algebra $\mathcal{B}(E,\sigma^{-1},L)$ is the opposite of $\mathcal{B}(E,\sigma,L)$. This is immediate from the definitions: Call the algebras $B'$ and $B$. Then the isomorphisms

$$\begin{aligned}\theta_n: B'_n = H^0\left(L\otimes L^{\sigma^{-1}}\otimes\ldots\otimes L^{\sigma^{-n+1}}\right)&\\ \xrightarrow{\sigma^{n-1}} H^0\left(L^{\sigma^{n-1}}\otimes\ldots\otimes L^{\sigma}\otimes L\right)&\\ \xrightarrow{\tau} H^0\left(L\otimes L^{\sigma^1}\otimes\ldots\otimes L^{\sigma^{n-1}}\right)&=B_n,\end{aligned}$$

where $\tau$ reverses the order of the tensors, satisfy the relation

$$\theta_{m+n}(fg)=\theta_m(g)\theta_n(f).$$

Taking $n=1$, we get (6.6v) in the non-exceptional case, because $F\approx\mathcal{O}_E$ if and only if $\mathcal{T}$ is regular. (Recall that, by (7.3), the assertion for $W$ in (6.6v) follows from the assertion for $U$, once (6.6i)-(6.6iv) are proven.)

If $E$ is exceptional, then $F$ is not tame, but we can check (6.6iv) and (6.6v) by explicit computation.

Assume that $E$ is exceptional and $r=3$. Then $E=C_1+C_2$, each $C_i$ is isomorphic to $\mathbf{P}^1$, and $(C_1\cdot C_2)=2$. Let $G$ be an invertible sheaf on $E$. If $d_1(G)\leq -1$, then $h^0(G)=\text{Max }(0,d_2(G)-1)$, because sections of $G$ are sections of $G_{C_2}(-C_1)$, a sheaf of degree $d_2(G)-2$ on a curve of genus zero. If we order the components so that $L$ has multidegree $(d_1,d_2)=(1,2)$, then $F$ has multidegree $(-2,2)$, so $h^0(F)=h^1(F)=1$. This proves (6.6v) in that case. For $n \geq 2$, the sheaf $F \otimes L_{n-1}^{\sigma^3}$ has degrees $d_1, d_2 \geq 0$, so it is tame and since its total degree is positive, $h^1 = 0$. This proves (6.6iv).

Assume that $E$ is exceptional and that $r = 2$. Then $E = C_1 + C_2 + C_3$ with components $C_i \approx \mathbf{P}^1$, and $(C_i \cdot C_j) = 1$ for all $i, j$. Let $G$ be an invertible sheaf on $E$. If $d_1(G) \leq -1$ and $d_2(G) \leq 0$, then $h^0(G) = \operatorname{Max}(0, d_3(G) - 1)$. (Restricting to $C_1$, then to $C_2$, one sees that sections of $G$ are the same as sections of $G_{C_3}(-C_1 - C_2)$.) Ordering the $C_i$ so that $L$ has multidegree $(1,1,0)$, we check that $F$ has multidegree $(1,1,-2)$, so the discussion above applies to $F^*$, giving $h^0(F^*) = h^1(F) = 1$ as before. The sheaf $F \otimes L^{\sigma^4}$ has multidegree $(2,-1,1)$ (or $(1,-1,2)$) and again, after changing $C_2$ and $C_3$, the discussion above applies to $F^*$ and gives $0 = h^0(F^*) = h^1(F)$. For $n \geq 3$, the sheaf $F \otimes L_{n-1}^{\sigma^4}$ has $d_i \geq 0$ for each $i$, so it is tame, of total degree $> 0$, and $h^1 = 0$.

There remains (6.6vi). We assume $\mathcal{T}$ is regular and must show that $W$ is a nondegenerate subspace of $J_s T_1$ and of $T_1 J_s$. By Remark 7.32, it suffices to treat only the inclusion $W \subset J_s T_1$. In the notation explained just before (7.27), we have $J_s = K_{1,s-1}$. Indeed if $r = 3$, $s = 2$, the two spaces coincide by definition. If $r = 2$, $s = 3$, then

$$J_3 = \ker(B_1 \otimes B_1 \otimes B_1 \to B_3) = \ker(B_1 \otimes B_2 \to B_3) = K_{1,2}$$

because $B_1 \otimes B_1 \to B_2$ is an isomorphism in this case. Similarly, $J_s = K_{s-1,1}$. Hence, taking $m = s-1$ and $n = 1$ in Lemma 7.28, we find that $W$, which by definition is the kernel of the map $\alpha$ in that lemma, fits into an exact sequence

$$0 \to W \xrightarrow{i} J_s \otimes T_1 \xrightarrow{\mu} H^0(L'' \otimes M \otimes N). \tag{7.33}$$

This is the sequence obtained by applying $H^0$ to the sheaf exact sequence $[L'' \otimes M] \otimes N$, which is of the form

$$0 \to F \xrightarrow{i} J_s \otimes N \xrightarrow{j} L'' \otimes M \otimes N \to 0, \tag{7.34}$$

and $F \approx \mathcal{O}_E$ because our triple is assumed to be regular.

Let $w$ be a basis for $W$ and let $\{x_i\}$, $1 \leq i \leq r$, be a basis for $T_1$. Letting

$$i(w) = \sum_{j=1}^{r} g_j x_j,$$

we wish to show that the $g_j$ span $J_s$. To do this, we let $\wedge : J_s \to k$ be a linear form vanishing on the $g_j$, and we proceed to show that $\wedge = 0$. Consider the homomorphism of $\mathcal{O}_E$-modules

$$\lambda \otimes 1 : J_s \otimes_k N \to k \otimes_k N = N.$$

By construction, the composed map $(\lambda \otimes 1) \circ i$ kills $W = H^0(F)$, and hence it kills $F$, because $F \approx \mathcal{O}_E$. Thus, by the exactness of (7.33), the sheaf map $\lambda \otimes 1$ is of the form $\gamma \circ j$ for some $\gamma : L'' \otimes M \otimes N \to N$. But $\mathrm{Hom}(L'' \otimes M \otimes N, N) = \mathrm{Hom}(L'' \otimes M, \mathcal{O}_E) = \mathrm{Hom}((L'' \otimes M)^*) = 0$ by duality and (7.30). Therefore $\lambda \otimes 1 = 0$ and this implies $\lambda = 0$, as was to be shown. This completes the proof of Theorem 6.6. $\square$.

## 8. Proof that regular algebras of dimension 3 are noetherian

The object of this section is to prove the following theorem:

**Theorem 8.1.** *Every nondegenerate semi-standard algebra, and in particular, every regular algebra of dimension 3, is left and right noetherian.*

Suppose that $A$ is such an algebra. By (6.7i) we have $A \approx \mathcal{A}(\mathcal{T})$, where $\mathcal{T}$ is the triple derived from $A$, and hence, by (6.8i), $A$ contains a normalizing element $g$, possibly zero, of degree $s+1$, such that $A/gA \approx B = \mathcal{B}(\mathcal{T})$. Thus the following lemma reduces Theorem 8.1 to Theorem 8.3, which is stated below.

**Lemma 8.2.** *Let $A$ be a graded $k$-algebra, and let $g \in A$ be a homogeneous normalizing element of positive degree. If $B = A/gA$ is left or right noetherian, so is $A$.*

**Proof**. Left multiplication by $g$ on a left $A$-module $M$:

$$\ell_g : M \to M$$

need not be $A$-linear, but nevertheless the kernel and image of $\ell_g$ are submodules of $M$, and $\ell_g$ induces a bijection between the $A$-submodules of $M/\ker \ell_g$ and the $A$-submodules of $\ell_g M$.

Assume that $A$ is not left noetherian. Then ([B], Ch. III, Sect. 2, no. 10, Lemme 1) $A$ has a *graded* left ideal which is not finitely generated. Using Zorn's Lemma, we may select a graded left ideal $L$ which is not finitely generated and is maximal with that property. Then $\bar{A} = A/L$ is a noetherian $A$-module. We consider left multiplication by $g$ on the short exact sequence

$$0 \to L \to A \to \bar{A} \to 0.$$

The snake lemma provides us with an exact sequence

$$K \xrightarrow{\delta} L/gL \xrightarrow{\epsilon} A/gA = B,$$

where $K = \ker(\ell_g : \bar{A} \to \bar{A})$. The map $\epsilon$ is $A$-linear, and so $\mathrm{im}\,\epsilon$ is a finitely generated submodule of the noetherian $A$-module $B$. Also, $K$ is a finitely generated module, because it is a submodule of the noetherian module $\bar{A}$.

The map $\delta$ is defined as follows: Let $\bar{y} \in K$ be represented by $y \in A$. Then $\delta(y)$ is represented by the element $gy$ of $L$. If $x_1, \ldots, x_n \in A$ represent generators $\bar{x}_i$ of $K$, and if $\bar{y} = \Sigma a_i \bar{x}_i$, then $\delta(\bar{y})$ is represented by $g(\Sigma a_i x_i)$, and this element can be rewritten in the form $\Sigma a_i' g x_i$. This shows that the elements $gx_i = \delta(\bar{x}_i)$ generate $\delta(K) = \ker \epsilon$, and hence that $L/gL$ is finitely generated. Since $g$ is of degree $> 0$, it follows that $L$ is finitely generated. (If M is a finitely generated graded submodule of $L$ such that $L = M + gL$, then, by induction on $n$, $L = M + g^n L$ for all $n > 0$. This implies $L_n = M_n$ for all $n$, hence $L = M$.) $\square$

**Theorem 8.3.** *Let $\mathcal{T}$ be a triple, and $B = \mathcal{B}(\mathcal{T})$. Then $B$ is (left and right) noetherian.*

Let us call a ring $A$ *essentially finite* if it is a finite module over its center $Z(A)$.

**Lemma 8.4.** *Let $A$ be a finitely generated graded algebra over $k$. Suppose $A$ is essentially finite. Then its center $Z(A)$ is also a finitely generated graded $k$- algebra, and $A$ is noetherian.*

For the proof, see [A-T]. The relevant theorem is stated there for commutative rings, but only the fact that the subring is central in the big one is used in the proof. $\square$

**Lemma 8.5.** *Let $\mathcal{T} = (E, \sigma, L)$ be a triple and let $n > 0$ be an integer such that $\sigma^n = 1$. Let $B = \mathcal{B}(\mathcal{T})$, and denote by $B\langle n\rangle$ the graded ring $\sum_\nu B_{\nu n}$. Then*

(i) *$B$ is essentially finite.*

(ii) *There is an automorphism $\tau$ of $B\langle n\rangle$, such that $(x_1 \ldots x_n)^\tau = x_n x_1 \ldots x_{n-1}$ for all $x_i \in B_1$.*

(iii) *For $b \in B_i$ and $f \in B\langle n\rangle$, we have $bf = f^{\tau^i} b$. In particular, $B\langle n\rangle$ is commutative and the subring of invariants*

$$C = \{f \in B\langle n\rangle | f^\tau = f\}$$

*is in the center of $B$.*

**Proof**. Let $i$ and $j$ be integers such that $i + j \equiv 0$ (modulo $n$). Then

$$fgh = hgf, \tag{8.6}$$

for all $f, h \in B_i$ and $g \in B_j$, because

$$fgh = f \otimes g^{\sigma^i} \otimes h^{\sigma^{i+j}} = f \otimes g^{\sigma^i} \otimes h = h \otimes g^{\sigma^i} \otimes f = h \otimes g^{\sigma^i} \otimes f^{\sigma^{i+j}} = hgf.$$

For $v \in B_{\nu n}$, write $v = \Sigma_\alpha g_\alpha y_\alpha$ with $g_\alpha \in B_{\nu n-1}$, $y_\alpha \in B_1$, and put $v^\tau = \Sigma_\alpha y_\alpha g_\alpha$. Then, by (8.6),

$$xv = v^\tau x, \tag{8.7}$$

for all $x \in B_1$. Since the socle of $B$ is zero, (8.7) shows that the element $v^\tau$ depends only on $v$, and is independent of the choice of the expression $\Sigma g_\alpha y_\alpha$. The map $v \rightsquigarrow v^\tau$ is clearly $k$-linear, and it is also multiplicative, since $xv_1v_2 = v_2^\tau x v_2 = v_1^\tau v_2^\tau x$, and $(v_1v_2)^\tau$ is the *unique* element such that $xv_1v_2 = (v_1v_2)^\tau x$ for all $x \in B_1$. This proves (ii), and (iii) follows immediately, by induction on $i$, from (8.7).

To prove (i), we note that since $B = k[B_1]$, we have $B\langle n\rangle = k[B_n]$, and so the commutative ring $B\langle n\rangle$ is a finitely generated $k$-algebra. Since $\tau$ is of finite order, $C$ is finitely generated and $B\langle n\rangle$ is a finite $C$-module ([B], Ch. V, Sect.1, No. 9, Th. 2). Also, $B$ is a finite left $B\langle n\rangle$-module generated by $B_0 + B_1 + \ldots + B_{n-1}$. Hence $B$ is a finite $C$-module. Since $C$ is noetherian, so is $B$. In fact, any ring $D$ between $C$ and $B$ is finitely generated over $k$, and $B$ is a finite left $D$-module. This applies in particular to the center $D = Z(B)$. □

To prove Theorem 8.3, we will use the fact that a triple consists of algebro-geometric data, and consequently that there is a reduction of $\mathcal{T}$ (modulo $p$) to a finite field. To be precise, we define a family $\mathcal{T}_R = (E_R, \sigma_R, L_R)$ of triples, parametrized by a commutative ring $R$, to be a flat and proper family of schemes $E_R$ over $R$, an automorphism $\sigma_R$ of this $R$-scheme, and an invertible sheaf $L_R$ on $E_R$, such that the conditions of (4.5) hold for each geometric fibre of $E_R$ over $R$.

**Lemma 8.8.** *Let $\mathcal{T}_R$ be a family of triples. Let $(L_R)_n$ be defined as in* (6.2). *Then $H^0(E_R, (L_R)_n)$ is a locally free $R$-module for every $n \geq 0$.*

**Proof**. This follows from ([EGA] III, 7.8, or [H], p. 288, Cor. 12.9), because $H^1(E_k, (L_k)_n) = 0$ for every geometric point $R \to k$ of Spec$R$ (7.12). □

Because of this lemma, the construction of the algebra $\mathcal{B}(\mathcal{T})$ from a triple $\mathcal{T}$ carries over to a family $\mathcal{T}_R$, and provides us with a flat graded $R$-algebra $B_R$ with the property that for every ring homomorphism $R \to S$, the algebra $S \otimes_R B_R$ is the one associated to the triple $\mathcal{T}_S = S \otimes_R \mathcal{T}_R$. Now if $\mathcal{T}_K$ is a triple defined over a field $K$, then there is a subring $R$ of $K$ which is finitely generated over $\mathbb{Z}$, and a family of triples $\mathcal{T}_R$ parametrized by Spec $R$, such that $K \otimes_R \mathcal{T}_R \approx \mathcal{T}_K$. Moreover, being a non-zero ring of finite type over $\mathbb{Z}$, $R$ contains a maximal ideal $M$, and the residue field $R/M = k'$ is finite. We may ([B], Ch. 6, Sect. 1, No. 2, Th. 2) choose a valuation ring $R$ of $K$ which dominates $R/M$ and with algebraic residue field extension. Then Lemma 8.9 below reduces us to the case that $K$ is algebraic over the prime field $\mathbf{F}_p$ for some prime integer $p$. In that case, we will show that $B$ is noetherian by proving (Lemma 8.10) that $B$ satisfies the hypothesis of Lemma 8.5 and is therefore a finite module over its center.

**Lemma 8.9.** *Let $R$ be a valuation ring with fraction field $K$ and residue field $k$, and let $A_R$ be a graded $R$-algebra with the property that $(A_R)_n$ is a free $R$-module of finite rank for each $n$. Let $A_K = A_R \otimes_R K$, and $A_k = A_R \otimes_R k$. If $A_k$ is noetherian, so is $A_K$, and if $R$ is a discrete valuation ring, then $A_R$ is noetherian too.*

**Proof**. As in the proof of Lemma 8.2, it suffices to prove *graded* ideals are finitely generated. Let $L$ be a graded left ideal of $A_K$. Then for each $n$, $L_n \cap (A_R)_n$ is a direct summand of $(A_R)_n$, because the quotient

$$(A_R)_n / (L_n \cap (A_R)_n) \approx (L_n + (A_R)_n) / L_n$$

is torsion-free as $R$-module, being a submodule of the $K$-vector space $(A_K)_n/L_n$, and is finitely generated as $R$-module because it is a homomorphic image of $(A_R)_n$. Such an $R$-module is free [B$_2$, Ch. 6, Sect. 3, No. 6, Lemme 1]. Thus

$$L_R := L \cap A_R$$

is a direct summand of $A_R$, as $R$-module. This implies that $L_k := L_R \otimes_R k$ is a direct summand of $A_k$. So the map $L_k \to A_k$ is injective, and maps $L_k$ isomorphically to a left ideal of $A_k$. Therefore by our hypothesis on $A_k$, $L_k$ is a finitely generated module over $A_k$. Let $S$ be a finite set of homogeneous elements of $L_R$ whose residues generate $L_k$. The ordinary Nakayama Lemma, applied to each degree separately, shows that $L_R$ is generated over $A_R$ by $S$, and therefore that $L$ is generated over $A_K$ by $S$ as well. This shows that $A_K$ is noetherian.

Assume that $R$ is a discrete valuation ring with maximal ideal $p$, and let $L$ be a left ideal of $A_R$. Let $L' = A_R \cap L_K$, where $L_K = K \otimes_R L$. It was shown above that $L'$ is finitely generated, from which it follows that $p^r L' \subset L$ for suitable $r$. The successive quotients of the filtration $p^\nu L' \cap L$ are submodules of the finitely generated $A_k$-modules $p^\nu L'/p^{\nu+1}L'$, and hence are finitely generated. Also, $p^r L'$ is finitely generated. It follows that $L$ is finitely generated. Thus $A_R$ is noetherian. □

Now the proof of Theorem 8.3 is completed by

**Lemma 8.10.** *An automorphism $\sigma$ of a one-dimensional projective scheme $E$ defined over a finite field $K_0$ has finite order.*

**Proof**. Let $L = \mathcal{O}_E(1)$. Replacing $\sigma$ by a power, we may assume that it stabilizes each irreducible component of $E$. Then the divisor class

$[L^\sigma]-[L]$ is in the connected component of Pic $E$, which has finitely many $K_0$-rational points. So replacing $\sigma$ by a power again, we may assume that $[L^\sigma]=[L]$. When this is so, $\sigma$ extends to an automorphism of the ambient projective space, and so it is an element of projective general linear group over $K_0$, which is finite. $\square$

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