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Retractability of set theoretic solutions of the Yang–Baxter equation ☆

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Abstract

It is shown that square free set theoretic involutive non-degenerate solutions of the Yang–Baxter equation whose associated permutation group (referred to as an involutive Yang–Baxter group) is abelian are retractable in the sense of Etingof, Schedler and Soloviev. This solves a problem of Gateva-Ivanova in the case of abelian IYB groups. It also implies that the corresponding finitely presented abelian-by-finite groups (called the structure groups) are poly-$\mathbb{Z}$ groups. Secondly, an example of a solution with an abelian involutive Yang–Baxter group which is not a generalized twisted union is constructed. This answers in the negative another problem of Gateva-Ivanova. The constructed solution is of multipermutation level 3. Retractability of solutions is also proved in the case where the natural generators of the IYB group are cyclic permutations. Moreover, it is shown that such solutions are generalized twisted unions.

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Keywords: Yang–Baxter equation; Set theoretic solution; Multipermutation solution; Permutation group; Group of I-type

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1. Introduction

In order to find new solutions of the Yang–Baxter equation, Drinfeld, in [2], posed the question of finding the simplest possible solutions, the so-called set theoretic solutions on a finite set X (see precise definition below). There are many papers in this area and with many links to different topics. For a detailed background and references we refer the reader to [3–8]. We mention a few highlights. Etingof, Schedler and Soloviev [3] and independently Gateva-Ivanova and Van den Bergh [8] gave a group theoretical interpretation of the set theoretic involutive non-degenerate solutions of the Yang–Baxter equation. In the latter paper it was then shown that the associated group and semigroup algebras share many homological properties with commutative polynomial algebras in finitely many variables, see also [9]. Rump in [12] proved that if such solutions are square free (on a set with more than one element) then they are decomposable, hence confirming a conjecture of Gateva-Ivanova. A detailed account on these aspects can be found in [10]. Very recent papers [5–7] focus on various specific constructions of set theoretic solutions, already introduced in [3]. The aim is to show that many solutions can be built recursively from solutions constructed on smaller sets.

We now first recall the precise definitions, notations and results needed to clearly state the problems tackled in this paper.

Let $X = \{x_1, x_2, \dots, x_n\}$, with $n > 1$. Recall that a set theoretic involutive non-degenerate solution of the Yang–Baxter equation on X is a pair (X, r), where r is a map $r : X^2 \longrightarrow X^2$ such that:

(1) $r^2 = \mathrm{id}_{X^2}$;
(2) for $i, j \in \{1, \dots, n\}$ there exist unique $k, l \in \{1, \dots, n\}$ such that $r(x_i, x_k) = (x_j, x_l)$;
(3) $r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}$, where $r_{ij} : X^3 \longrightarrow X^3$ is the map acting as r on the (i, j) components (in this order) and as the identity on the remaining component.

Such a solution (X, r) is called square free if $r(x_i, x_i) = (x_i, x_i)$ for every i. Condition (2) implies that the maps $\sigma_i, \gamma_i : X \longrightarrow X$ defined by $r(x_i, x_k) = (x_{\sigma_i(k)}, x_{\gamma_k(i)})$ are bijective. Denote by G_r the subgroup $\langle \sigma_1, \dots, \sigma_n \rangle$ of the symmetric group Sym_n. Following [1], we call the group G_r the involutive Yang–Baxter group (IYB group) associated to the solution (X, r). Note also that $G_r = \langle \gamma_1, \dots, \gamma_n \rangle$ (see for example [1]). It is known that G_r is solvable (see [3, Theorem 2.15]). Conversely, it remains an open problem to decide which solvable finite groups are IYB groups. In [1] this has been proved for several classes of groups, in particular, nilpotent finite groups of class 2 and thus for abelian finite groups.

Let $G(X, r)$ be the group defined by the presentation

$$\bigl\langle x_1, \dots, x_n \bigm| x_i x_j = x_k x_l \text{ if } r(x_i, x_j) = (x_k, x_l) \bigr\rangle.$$

This group is called the structure group of the solution (X, r), or the group of I-type associated to the solution (X, r) [8,10]. It is known that this group is isomorphic with the subgroup of the semidirect product $Fa_n \rtimes G_r$ of the free abelian group $Fa_n = \langle u_1, \dots, u_n \rangle$ of rank n, with G_r acting by $\sigma_i(u_j) = u_{\sigma_i(j)}$, generated by the set $\{(u_i, \sigma_i) \mid i = 1, \dots, n\}$. It is known that, identifying x_i with (u_i, σ_i),

$$G(X, r) = \bigl\{(a, \sigma_a) \bigm| a \in Fa_n\bigr\} \subseteq Fa_n \rtimes G_r,$$

where $a \mapsto \sigma_a$ is a mapping from Fa_n to G_r and $\sigma_i = \sigma_{u_i}$.

In particular, $G(X,r)$ is a solvable abelian-by-finite group. It is also a torsion free group ([8, Corollary 1.4] or [10, Corollary 8.2.7]). The earlier mentioned homological properties (see [8]) of the group algebra $K[G(X,r)]$ then yield that this algebra is a domain (this also follows from a result of Brown, see [11, Theorem 13.4.1]).

The main challenge is to construct new classes of solutions and to classify all possible solutions, which have proved to be very difficult problems. There are two approaches, both originating from the work of Etingof, Schedler and Soloviev [3]. The idea is to show that every solution can be built in a recursive way from certain solutions of smaller cardinality. The first approach is based on the retract relation $\sim$ on the set X, introduced in [3], and defined by $x_i \sim x_j$ if $\sigma_i = \sigma_j$. There is a natural induced solution $Ret(X,r) = (X/{\sim}, \tilde{r})$, and it is called the retraction of X. A solution (X,r) is called a multipermutation solution of level m if m is the smallest non-negative integer such that the solution $Ret^m(X,r)$ has cardinality 1. Here we define $Ret^k(X,r) = Ret(Ret^{k-1}(X,r))$ for $k > 1$. If such an m exists then one also says that the solution is retractable. In this case, the group $G(X,r)$ is a poly-$\mathbb{Z}$ group (see [10, Proposition 8.2.12]). For such groups, this of course then gives a direct proof for the fact that $K[G(X,r)]$ is a domain.

The second approach is based on the notion of generalized twisted union. In order to state the definition, first notice that there is a natural action of G_r on X defined by $\sigma(x_i) = x_{\sigma(i)}$. A set theoretic involutive non-degenerate solution (X,r) is called a generalized twisted union of solutions (Y, r_Y) and (Z, r_Z) if X is a disjoint union of two G_r-invariant non-empty subsets Y, Z such that for all $z, z' \in Z$, $y, y' \in Y$ we have

$$\sigma_{\gamma_y(z)|Y} = \sigma_{\gamma_{y'}(z)|Y}, \tag{1}$$

$$\gamma_{\sigma_z(y)|Z} = \gamma_{\sigma_{z'}(y)|Z}. \tag{2}$$

Here, to simplify notation, we write σ_x for σ_i if $x = x_i$, and similarly for all γ_i. If, moreover, (X,r) is a square free solution, then conditions (1) and (2) are equivalent to

$$\sigma_{\sigma_y(z)|Y} = \sigma_{z|Y}, \tag{3}$$

$$\sigma_{\sigma_z(y)|Z} = \sigma_{y|Z} \tag{4}$$

(see [4, Proposition 8.3] and its proof). Let $G_{r,Y}$ be the subgroup of G_r generated by the set $\{\sigma_y \mid y \in Y\}$ and let $G_{r,Z}$ be the subgroup of G_r defined in a similar way. Then (3) and (4) amount to saying that the elements of the same $G_{r,Y}$-orbit on Z determine the same permutation of Y and the elements of the same $G_{r,Z}$-orbit on Y determine the same permutation of Z. The simplest example (called a twisted union in [3]) motivating this definition is obtained by choosing any permutations $\sigma_1, \sigma_2 \in \mathrm{Sym}_n$, $n = |X|$, such that $\sigma_i(Y) = Y$ for $i = 1, 2$, and $\sigma_y = \sigma_1$ for every $y \in Y$ and $\sigma_z = \sigma_2$ for every $z \in Z$. An important step supporting this approach was made by Rump [12], who showed that the number of G_r-orbits on X always exceeds 1 if (X,r) is a non-degenerate involutive square free solution with $|X| > 1$.

The following conjectures were formulated by Gateva-Ivanova in [4].

(I) Every set theoretic involutive non-degenerate square free solution (X,r) of cardinality $n \geqslant 2$ is a multipermutation solution of level $m < n$.

(II) Every multipermutation square free solution of level m and of cardinality $n \geqslant 2$ is a generalized twisted union of multipermutation solutions of level less than m.

In Section 2 we show that Conjecture (I) is true for solutions with an abelian involutive Yang–Baxter group G_r. Actually, we prove more. Namely, that every such solution is retractable in a stronger sense, obtained by refining the relation $\sim$ on X by requesting additionally that the elements are in the same G_r-orbit on X. It follows that the corresponding structure groups $G(X,r)$ are poly-$\mathbb{Z}$ groups. Notice that this is not true in the case of non-square free solutions, as shown in [10, Example 8.2.14].

In Section 3 we give an example of a multipermutation solution of level 3 with an abelian involutive Yang–Baxter group that is not a generalized twisted union. Therefore Conjecture (II) does not hold.

Finally, in Section 4, we show that if every generator σ_i of the IYB group G_r is a cyclic permutation, then the corresponding solution (X,r) also is retractable. Moreover, we prove that such solutions are generalized twisted unions, provided that $|X| \geqslant 2$. As this assumption on G_r does not imply that the group G_r is abelian, this provides another class of solutions for which Conjecture (I) is confirmed.

2. Solutions with an abelian IYB group

In this section we confirm Conjecture (I) for set theoretic involutive non-degenerate square free solutions (X,r) with an associated abelian IYB group.

We will often use the following consequence of [10, Theorem 8.1.4, Corollary 8.2.4 and Theorem 9.3.10].

Lemma 2.1. *Assume that (X,r) is a set theoretic involutive non-degenerate square free solution. If $r(x_i,x_j)=(x_k,x_l)$ for some i,j,k,l then $\sigma_i\circ\sigma_j=\sigma_k\circ\sigma_l$ in Sym_n.*

By [12, Theorem 1], if $n=|X|>1$ then the number m of orbits in X under the action of G_r is greater than 1. Let $X_1,\dots,X_m$ denote these orbits.

Lemma 2.2. *Suppose that G_r is abelian. Let $i,j\in\{1,\dots,n\}$ be such that $\sigma_i(j)=j$. If $x_j\in X_k$, then $\sigma_i(l)=l$ for all $x_l\in X_k$.*

Proof. Let $x_l\in X_k$. Then there exist $i_1,\dots,i_t\in\{1,\dots,n\}$ such that $\sigma_{i_1}\dots\sigma_{i_t}(j)=l$. Since G_r is abelian, we have

$$\begin{aligned}\sigma_i(l)&=\sigma_i\sigma_{i_1}\dots\sigma_{i_t}(j)\\&=\sigma_{i_1}\dots\sigma_{i_t}\sigma_i(j)\\&=\sigma_{i_1}\dots\sigma_{i_t}(j)=l.\qquad\square\end{aligned}$$

Corollary 2.3. *Suppose that G_r is abelian. Then for all k and for all $x_i\in X_k$,*

$$\sigma_i|_{X_k}=\mathrm{id}_{X_k}.$$

Proof. Since r is square free, $\sigma_i(i)=i$. Thus the result follows by Lemma 2.2. $\square$

Lemma 2.4. *Let $k\in\{1,\dots,n\}$. Suppose that for all $x_i\in X_k$ we have that $\sigma_i|_{X_k}=\mathrm{id}_{X_k}$. Let $x_{i_1},\dots,x_{i_s}\in X_k$ and $x_{j_1}\in X$. If $j_2=\sigma_{i_1}\dots\sigma_{i_s}(j_1)$, then $\sigma_{j_1}|_{X_k}=\sigma_{j_2}|_{X_k}$.*

Proof. Clearly we may assume that $s = 1$. Thus $j_2 = \sigma_{i_1}(j_1)$. Then there exists $x_i \in X$ such that $r(x_{i_1}, x_{j_1}) = (x_{j_2}, x_i)$. By Lemma 2.1, we have that $\sigma_{i_1}\sigma_{j_1} = \sigma_{j_2}\sigma_i$. Let $x_j \in X_k$. Since $x_{\sigma_{j_1}(j)} \in X_k$, it follows that $\sigma_{i_1}(\sigma_{j_1}(j)) = \sigma_{j_1}(j)$. Since $x_i = x_{\sigma_{j_2}^{-1}(i_1)} \in X_k$, we have that $\sigma_{j_2}\sigma_i(j) = \sigma_{j_2}(j)$. Therefore $\sigma_{j_1}(j) = \sigma_{j_2}(j)$ and thus $\sigma_{j_1}|_{X_k} = \sigma_{j_2}|_{X_k}$. □

We say that the solution r is trivial if $r(x_i, x_j) = (x_j, x_i)$ for every i, j. This is equivalent to saying that σ_i is the identity map for every i, and also to the fact that $m = n$.

Theorem 2.5. *Assume that (X, r) is a set theoretic involutive non-degenerate square free solution and the group G_r is abelian. If r is not trivial then there exist $i, j \in \{1, \dots, n\}$ such that $\sigma_i = \sigma_j$, $i \neq j$ and $x_i, x_j \in X_k$ for some $k \in \{1, \dots, m\}$.*

Proof. Suppose the assertion does not hold. So, for every k, if $x_i, x_j \in X_k$ are distinct then $\sigma_i \neq \sigma_j$. Note that if $|X_q| = 1$ for all $q \in \{1, \dots, m\}$, then $\sigma_1 = \sigma_2 = \cdots = \sigma_n = \mathrm{id}$, a contradiction. Thus we may assume that $|X_1| > 1$. We shall prove by induction that for all $j \leqslant m$, there exists X_{k_j} such that $X_{k_1} = X_1$, the set $\{X_{k_1}, \dots, X_{k_j}\}$ has cardinality j and there exist $x_{i_{1,j}}, x_{i_{2,j}} \in X_{k_j}$ such that

(i) $x_{i_{1,j}} \neq x_{i_{2,j}}$,
(ii) for all $p < j$, there exist $x_{j_{p,1}}, \dots, x_{j_{p,t_p}} \in X_{k_p}$ such that

$$i_{2,j} = \sigma_{j_{p,1}} \dots \sigma_{j_{p,t_p}}(i_{1,j}).$$

For $j = 1$, we take $X_{k_1} = X_1$ and we choose any two different elements $x_{i_{1,1}}, x_{i_{2,1}} \in X_1$.

Suppose that $j > 1$ and there exist $X_{k_1}, \dots, X_{k_{j-1}}$ such that $X_{k_1} = X_1$, the set $\{X_{k_1}, \dots, X_{k_{j-1}}\}$ has cardinality $j - 1$ and there exist $x_{i_{1,j-1}}, x_{i_{2,j-1}} \in X_{k_{j-1}}$ such that

(i′) $x_{i_{1,j-1}} \neq x_{i_{2,j-1}}$,
(ii′) for all $p < j - 1$, there exist $x_{j_{p,1}}, \dots, x_{j_{p,t_p}} \in X_{k_p}$ such that

$$i_{2,j-1} = \sigma_{j_{p,1}} \dots \sigma_{j_{p,t_p}}(i_{1,j-1}).$$

By Lemma 2.4 and Corollary 2.3, we have that

$$\sigma_{i_{1,j-1}}|_{X_{k_l}} = \sigma_{i_{2,j-1}}|_{X_{k_l}}$$

for all $1 \leqslant l \leqslant j - 1$. Since $x_{i_{1,j-1}} \neq x_{i_{2,j-1}}$, we have that $\sigma_{i_{1,j-1}} \neq \sigma_{i_{2,j-1}}$. Thus there exists $i \in \{1, \dots, n\}$ such that $\sigma_{i_{1,j-1}}(i) \neq \sigma_{i_{2,j-1}}(i)$. Let k_j be the integer such that $x_i \in X_{k_j}$. Clearly the set $\{X_{k_1}, \dots, X_{k_j}\}$ has cardinality j. Let $i_{1,j} = \sigma_{i_{1,j-1}}(i)$ and $i_{2,j} = \sigma_{i_{2,j-1}}(i)$. Then $x_{i_{1,j}}, x_{i_{2,j}} \in X_{k_j}$. Let q be the order of the permutation $\sigma_{i_{1,j-1}}$. Then

$$i_{2,j} = \sigma_{i_{2,j-1}}(i) = \sigma_{i_{2,j-1}}\sigma_{i_{1,j-1}}^{-1}(i_{1,j}) = \sigma_{i_{2,j-1}}\sigma_{i_{1,j-1}}^{q-1}(i_{1,j}).$$

Note that $x_{i_{2,j-1}}, x_{i_{1,j-1}} \in X_{k_{j-1}}$. Thus, if $j = 2$, then (ii) is satisfied. Suppose that $j > 2$. Let $1 \leqslant p < j - 1$. By (ii′), there exist $x_{j_{p,1}}, \dots, x_{j_{p,t_p}} \in X_{k_p}$ such that

$$i_{2,j-1} = \sigma_{j_{p,1}} \dots \sigma_{j_{p,t_p}}(i_{1,j-1}).$$

Hence there exist $j'_1, \ldots, j'_{t_p} \in \{1, \ldots, n\}$ such that

$$r(x_{j_{p,t_p}}, x_{i_{1,j-1}}) = (x_{\sigma_{j_{p,t_p}}(i_{1,j-1})}, x_{j'_{t_p}})$$

and

$$r(x_{j_{p,t_p-v}}, x_{\sigma_{j_{p,t_p-v+1}} \cdots \sigma_{j_{p,t_p}}(i_{1,j-1})}) = (x_{\sigma_{j_{p,t_p-v}} \cdots \sigma_{j_{p,t_p}}(i_{1,j-1})}, x_{j'_{t_p-v}}),$$

for all $1 \leqslant v \leqslant t_p - 1$. Note that $x_{j'_1}, \ldots, x_{j'_{t_p}} \in X_{k_p}$. By Lemma 2.1 it follows that

$$\sigma_{j_{p,t_p}} \sigma_{i_{1,j-1}} = \sigma_{\sigma_{j_{p,t_p}}(i_{1,j-1})} \sigma_{j'_{t_p}}$$

and

$$\sigma_{j_{p,t_p-v}} \sigma_{\sigma_{j_{p,t_p-v+1}} \cdots \sigma_{j_{p,t_p}}(i_{1,j-1})} = \sigma_{\sigma_{j_{p,t_p-v}} \cdots \sigma_{j_{p,t_p}}(i_{1,j-1})} \sigma_{j'_{t_p-v}},$$

for all $1 \leqslant v \leqslant t_p - 1$. Hence

$$\begin{aligned}
\sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p}} \sigma_{i_{1,j-1}} &= \sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p-1}} \sigma_{\sigma_{j_{p,t_p}}(i_{1,j-1})} \sigma_{j'_{t_p}} \\
&= \sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p-2}} \sigma_{\sigma_{j_{p,t_p-1}} \sigma_{j_{p,t_p}}(i_{1,j-1})} \sigma_{j'_{t_p-1}} \sigma_{j'_{t_p}} \\
&= \cdots = \sigma_{\sigma_{j_{p,1}} \cdots \sigma_{j_{p,t_p}}(i_{1,j-1})} \sigma_{j'_1} \ldots \sigma_{j'_{t_p}} \\
&= \sigma_{i_{2,j-1}} \sigma_{j'_1} \ldots \sigma_{j'_{t_p}} \\
&= \sigma_{j'_1} \ldots \sigma_{j'_{t_p}} \sigma_{i_{2,j-1}}.
\end{aligned}$$

Therefore

$$\sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p}} \sigma_{i_{1,j-1}}(i) = \sigma_{j'_1} \ldots \sigma_{j'_{t_p}} \sigma_{i_{2,j-1}}(i),$$

that is

$$\sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p}}(i_{1,j}) = \sigma_{j'_1} \ldots \sigma_{j'_{t_p}}(i_{2,j}).$$

Hence, if q_z is the order of the permutation $\sigma_{j'_z}$, then we have that

$$i_{2,j} = \sigma_{j'_1}^{q_1-1} \ldots \sigma_{j'_{t_p}}^{q_{t_p}-1} \sigma_{j_{p,1}} \ldots \sigma_{j_{p,t_p}}(i_{1,j}),$$

and

$$x_{j'_1}, \ldots, x_{j'_{t_p}}, x_{j_{p,1}}, \ldots, x_{j_{p,t_p}} \in X_{k_p}.$$

Thus (ii) is satisfied. In particular, for $j = m$, there exist $X_{k_1}, \ldots, X_{k_m}$ such that $X_{k_1} = X_1$, the set

$$\{X_{k_1}, \ldots, X_{k_m}\} = \{X_1, \ldots, X_m\}$$

and there exist $x_{i_{1,m}}, x_{i_{2,m}} \in X_{k_m}$ such that

(i) $x_{i_{1,m}} \neq x_{i_{2,m}}$,
(ii) for all $p < m$, there exist $x_{j_{p,1}}, \dots, x_{j_{p,t_p}} \in X_{k_p}$ such that

$$i_{2,m} = \sigma_{j_{p,1}} \dots \sigma_{j_{p,t_p}}(i_{1,m}).$$

By Lemma 2.4 and Corollary 2.3, we have that

$$\sigma_{i_{1,m}}|_{X_{k_l}} = \sigma_{i_{2,m}}|_{X_{k_l}}$$

for all $1 \leqslant l \leqslant m$, that is $\sigma_{i_{1,m}} = \sigma_{i_{2,m}}$, a contradiction. Therefore the assertion follows. □

Motivated by Theorem 2.5, we define a relation ρ on X as follows:

$$(x_i, x_j) \in \rho \quad \text{if } x_i, x_j \in X_k \text{ for some } k \text{ and } \sigma_i = \sigma_j.$$

This can be used to define a stronger version of retractability of (X, r), based on the relation ρ. In order to make this idea work, we need some observations that are similar to those known for the retract relation $\sim$.

Recall that every set theoretic involutive non-degenerate square free solution satisfies the so-called cyclic condition. This says that if $r(x_w, x_{j_1}) = (x_{j_2}, x_{w'})$ then there exist $j_3, \dots, j_k$, such that

$$r(x_w, x_{j_2}) = (x_{j_3}, x_{w'}), \dots, r(x_w, x_{j_k}) = (x_{j_1}, x_{w'}),$$

see [10, Corollary 9.2.6].

Lemma 2.6. *The relation ρ is an equivalence relation that is compatible with r. That is, if $r(x_i, x_j) = (x_p, x_q)$ and $r(x_w, x_v) = (x_k, x_l)$ with $(x_i, x_w), (x_j, x_v) \in \rho$, then $(x_p, x_k), (x_q, x_l) \in \rho$.*

Proof. Since $(x_i, x_w) \in \rho$, we have that $\sigma_w(j) = \sigma_i(j) = p$. Thus there exists w' such that $r(x_w, x_j) = (x_p, x_{w'})$. Since $r^2 = \mathrm{id}_{X^2}$, it follows that

$$r(x_p, x_q) = (x_i, x_j) \quad \text{and} \quad r(x_p, x_{w'}) = (x_w, x_j).$$

Thus $\sigma_p(q) = i$ and $\sigma_p(w') = w$. Since $(x_i, x_w) \in \rho$, we thus obtain that $x_q, x_i, x_{w'}, x_w$ are in the same G_r-orbit on X. By Lemma 2.1, it follows that $\sigma_i \sigma_j = \sigma_p \sigma_q$ and $\sigma_w \sigma_j = \sigma_p \sigma_{w'}$. Hence, since $\sigma_i = \sigma_w$, we have that

$$\sigma_{w'} = \sigma_p^{-1} \sigma_w \sigma_j = \sigma_p^{-1} \sigma_i \sigma_j = \sigma_q.$$

Therefore $(x_q, x_{w'}) \in \rho$. Since $r(x_w, x_j) = (x_p, x_{w'})$, by the cyclic condition, there exists p' such that $r(x_w, x_{p'}) = (x_j, x_{w'})$. Thus $\sigma_j(w') = w$. Since $r(x_w, x_v) = (x_k, x_l)$, by the cyclic condition, there exists k' such that $r(x_w, x_{k'}) = (x_v, x_l)$. Thus $\sigma_v(l) = w$. Since $(x_j, x_v) \in \rho$, it follows that

$$l = \sigma_v^{-1}(w) = \sigma_j^{-1}(w) = w'.$$

Therefore $(x_q, x_l) \in \rho$. Since $\sigma_i(j) = p$, $\sigma_w(v) = k$ and $(x_j, x_v) \in \rho$, the elements x_j, x_p, x_v, x_k are in the same G_r-orbit on X. The assumption $r(x_w, x_v) = (x_k, x_l)$ implies, in view of Lemma 2.1, that $\sigma_w\sigma_v = \sigma_k\sigma_l$. Since $\sigma_l = \sigma_q$, $\sigma_w = \sigma_i$, $\sigma_v = \sigma_j$ and $\sigma_i\sigma_j = \sigma_p\sigma_q$, this yields that

$$\sigma_k = \sigma_w\sigma_v\sigma_l^{-1} = \sigma_i\sigma_j\sigma_q^{-1} = \sigma_p.$$

Therefore $(x_p, x_k) \in \rho$. $\square$

By Lemma 2.6, it is easy to see that the map $\bar{r} : (X/\rho)^2 \longrightarrow (X/\rho)^2$, defined by $\bar{r}(\overline{x_i}, \overline{x_j}) = (\overline{x_k}, \overline{x_l})$ if $r(x_i, x_j) = (x_k, x_l)$, yields a set theoretic involutive non-degenerate square free solution $(X/\rho, \bar{r})$. Let $|X/\rho| = n'$ and let $X/\rho = \{\overline{x_{i_1}}, \ldots, \overline{x_{i_{n'}}}\}$. We denote by y_j the element $y_j = \overline{x_{i_j}} \in X/\rho$. Let $\sigma'_j \in \mathrm{Sym}_{n'}$ be the permutation defined by $\bar{r}(y_j, y_k) = (y_{\sigma'_j(k)}, y_l)$. Let $G_{\bar{r}}$ denote the group $\langle \sigma'_1, \ldots, \sigma'_{n'} \rangle$.

Lemma 2.7. *The map $\phi : \{\sigma_1, \ldots, \sigma_n\} \longrightarrow \{\sigma'_1, \ldots, \sigma'_{n'}\}$, defined by $\phi(\sigma_i) = \sigma'_j$ if $\overline{x_i} = y_j$, extends to a group epimorphism $\widetilde{\phi} : G_r \longrightarrow G_{\bar{r}}$.*

Proof. Let $\sigma \in G_r$. We define $\widetilde{\phi}(\sigma)$ by

$$\widetilde{\phi}(\sigma)(k) = j,$$

where $\overline{x_{\sigma(i_k)}} = y_j$. Note that there exists l such that $r(x_i, x_{i_k}) = (x_{\sigma_i(i_k)}, x_l)$. Thus $\bar{r}(\overline{x_i}, \overline{x_{i_k}}) = (\overline{x_{\sigma_i(i_k)}}, \overline{x_l})$. Hence if $\overline{x_i} = y_p$ then $\overline{x_{\sigma_i(i_k)}} = y_{\sigma'_p(k)}$. Therefore

$$\widetilde{\phi}(\sigma_i) = \sigma'_p = \phi(\sigma_i).$$

So, $\widetilde{\phi}$ is an extension of ϕ.

Let $\sigma \in G_r$. Then $\widetilde{\phi}(\sigma)(k) = j$, where $\overline{x_{\sigma(i_k)}} = y_j$. Let $i \in \{1, \ldots, n\}$. Then there exists i' such that

$$r(x_i, x_{i_j}) = (x_{\sigma_i(i_j)}, x_{i'}).$$

Also there exists i'' such that

$$r(x_i, x_{\sigma(i_k)}) = (x_{\sigma_i\sigma(i_k)}, x_{i''}).$$

Since $\overline{x_{i_j}} = y_j = \overline{x_{\sigma(i_k)}}$, by Lemma 2.6, we have that $\overline{x_{\sigma_i(i_j)}} = \overline{x_{\sigma_i\sigma(i_k)}}$. Let $y_{j'} = \overline{x_{\sigma_i(i_j)}} = \overline{x_{\sigma_i\sigma(i_k)}}$. Then we have

$$\widetilde{\phi}(\sigma_i)\big(\widetilde{\phi}(\sigma)(k)\big) = \widetilde{\phi}(\sigma_i)(j) = j' = \widetilde{\phi}(\sigma_i\sigma)(k).$$

Hence

$$\widetilde{\phi}(\sigma_i)\widetilde{\phi}(\sigma) = \widetilde{\phi}(\sigma_i\sigma).$$

Thus, by induction on s, it is easy to see that

$$\widetilde{\phi}(\sigma_{k_1})\ldots\widetilde{\phi}(\sigma_{k_s})=\widetilde{\phi}(\sigma_{k_1}\ldots\sigma_{k_s}).$$

Therefore $\widetilde{\phi}$ is an epimorphism of groups. $\square$

Note that if $i=\sigma_{k_1}\ldots\sigma_{k_s}(i_k)$ and $\overline{x_i}=y_j$, then

$$j=\widetilde{\phi}(\sigma_{k_1})\ldots\widetilde{\phi}(\sigma_{k_s})(k).$$

Hence if x_i and x_u are in the same orbit under the action of G_r then $\overline{x_i}$ and $\overline{x_u}$ are in the same orbit under the action of $G_{\bar r}$. Conversely, if y_j and y_k are in the same orbit under the action of $G_{\bar r}$, then there exist $k_1,\ldots,k_s$ such that

$$j=\widetilde{\phi}(\sigma_{k_1})\ldots\widetilde{\phi}(\sigma_{k_s})(k)=\widetilde{\phi}(\sigma_{k_1}\ldots\sigma_{k_s})(k).$$

Thus $\overline{x_{i_j}}=\overline{x_{\sigma_{k_1}\ldots\sigma_{k_s}(i_k)}}$, and therefore x_{i_j} and $x_{\sigma_{k_1}\ldots\sigma_{k_s}(i_k)}$ are in the same orbit under the action of G_r. Hence x_{i_j} and x_{i_k} are in the same orbit under the action of G_r. So we have proved the following result.

Lemma 2.8. *The number of orbits of X under the action of G_r is the same as the number of orbits of X/ρ under the action of $G_{\bar r}$. Furthermore, if X_k is an orbit of X, then $\{\overline{x_i}\mid x_i\in X_k\}$ is an orbit of X/ρ.*

Now the notion of strong retractability of (X,r) may be defined as follows. First, let $Ret_\rho(X,r)=(X/\rho,\bar r)$ denote the induced solution. We say that (X,r) is strongly retractable if there exists $m\geqslant 1$ such that applying m times the operator Ret_ρ we get a trivial solution.

Since the IYB group corresponding to the solution $(X/\rho,\bar r)$ also is abelian by Lemma 2.7 if G_r is abelian, the following is a direct consequence of Theorem 2.5.

Corollary 2.9. *Assume that (X,r) is a set theoretic involutive non-degenerate square free solution and the group G_r is abelian. Then (X,r) is strongly retractable.*

Notice that, since Lemma 2.1 is true for infinite solutions (X,r), it is easy to see that the above proof remains valid for infinite solutions (X,r), provided that there are finitely many G_r-orbits on the set X.

3. A solution that is not a generalized twisted union

In this section we present a counterexample to Conjecture (II). Actually, such a construction can be given already in the case where the group G_r is abelian.

Theorem 3.1. *There exists a multipermutation square free solution of level* 3 *that it is not a generalized twisted union. Furthermore, the associated IYB group is abelian.*

Proof. Let $X=\{x_i\mid 1\leqslant i\leqslant 24\}$. Consider the following permutations in Sym_{24},

$$\sigma_1 = \sigma_2 = (9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24),$$
$$\sigma_3 = \sigma_4 = (9,11)(10,12)(13,15)(14,16)(17,18)(19,20)(21,22)(23,24),$$
$$\sigma_5 = \sigma_6 = (9,10)(11,12)(13,14)(15,16)(17,19)(18,20)(21,23)(22,24),$$
$$\sigma_7 = \sigma_8 = (9,11)(10,12)(13,15)(14,16)(17,19)(18,20)(21,23)(22,24),$$
$$\sigma_9 = \sigma_{12} = \sigma_{13} = \sigma_{16} = (1,5)(2,6)(3,7)(4,8)(17,21)(18,22)(19,23)(20,24),$$
$$\sigma_{10} = \sigma_{11} = \sigma_{14} = \sigma_{15} = (1,5)(2,6)(3,7)(4,8)(17,24)(18,23)(19,22)(20,21),$$
$$\sigma_{17} = \sigma_{20} = \sigma_{21} = \sigma_{24} = (9,13)(10,14)(11,15)(12,16)(1,3,2,4)(5,7,6,8),$$
$$\sigma_{18} = \sigma_{19} = \sigma_{22} = \sigma_{23} = (9,16)(10,15)(11,14)(12,13)(1,3,2,4)(5,7,6,8).$$

Consider the map $r : X^2 \longrightarrow X^2$ defined by $r(x_i, x_j) = (x_{\sigma_i(j)}, x_{\sigma^{-1}_{\sigma_i(j)}(i)})$ for all $i, j \in X$. Note that $r^2 = \mathrm{id}_X$ and $r(x_i, x_i) = (x_i, x_i)$ for all $x_i \in X$. It is well known that in order to prove that (X, r) is a square free solution it is sufficient to check that $\sigma_i \sigma_j = \sigma_k \sigma_l$ whenever $r(x_i, x_j) = (x_k, x_l)$ (see [10, Theorem 9.3.10]). This can be checked by a direct verification.

Consider the following permutations in Sym_{24},

$$\tau_1 = (9,10)(11,12)(13,14)(15,16),$$
$$\tau_2 = (9,11)(10,12)(13,15)(14,16),$$
$$\tau_3 = (9,13)(10,14)(11,15)(12,16),$$
$$\tau_4 = (9,16)(10,15)(11,14)(12,13),$$
$$\tau_5 = (17,18)(19,20)(21,22)(23,24),$$
$$\tau_6 = (17,19)(18,20)(21,23)(22,24),$$
$$\tau_7 = (17,21)(18,22)(19,23)(20,24),$$
$$\tau_8 = (17,24)(18,23)(19,22)(20,21),$$
$$\tau_9 = (1,5)(2,6)(3,7)(4,8),$$
$$\tau_{10} = (1,3,2,4)(5,7,6,8).$$

It is easy to check that the subgroups $\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$, $\langle \tau_5, \tau_6, \tau_7, \tau_8 \rangle$ and $\langle \tau_9, \tau_{10} \rangle$ are abelian. Hence, the group $\langle \tau_1, \dots, \tau_{10} \rangle$ is abelian. As $G_r = \langle \sigma_i \mid 1 \leqslant i \leqslant 24 \rangle \subseteq \langle \tau_1, \dots, \tau_{10} \rangle$, we thus obtain that G_r is abelian.

It is easy to see that there are three G_r-orbits on X:

$$X_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\},$$
$$X_2 = \{x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\},$$
$$X_3 = \{x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}\}.$$

Suppose (X, r) is a generalized twisted union $X = Y \cup Z$. Then Y or Z is equal to one of these orbits. Say, $Y = X_i$ for some i. Notice that $\sigma_{17}(1) = 3$ and $\sigma_{1|X_2 \cup X_3} \neq \sigma_{3|X_2 \cup X_3}$. This implies that

$Y \neq X_1$. Similarly, $\sigma_1(9) = 10$ and $\sigma_{9|X_1 \cup X_3} \neq \sigma_{10|X_1 \cup X_3}$ imply that $Y \neq X_2$. Finally, $\sigma_1(17) = 18$ and $\sigma_{17|X_1 \cup X_2} \neq \sigma_{18|X_{21} \cup X_2}$. Therefore $Y \neq X_3$. This contradiction shows that (X, r) is not a generalized twisted union.

It is easy to verify that applying twice the operator *Ret* we get a trivial solution of cardinality 3. Hence, (X, r) is a multipermutation solution of level 3. □

4. IYB groups generated by cyclic permutations

In this section (X, r) will stand for a set theoretic involutive non-degenerate square free solution with associated IYB group G_r, such that its generators σ_i, $i = 1, \dots, n$, are cyclic permutations.

Our aim is to prove that if $|X| > 1$ then (X, r) is a retractable solution, and moreover it is a generalized twisted union. Hence, we confirm Conjectures (I) and (II) in this case.

Recall that (see [10, Corollary 9.2.6 and Proposition 9.2.4]) every square free solution satisfies the so-called full cyclic condition. This says that for any distinct elements z and y in X there exist distinct elements $z = z_1, z_2, \dots, z_k$ and distinct $y = y_1, y_2, \dots, y_p$ in X such that

$$\begin{array}{cccc}
r(y_1, z_1) = (z_2, y_2), & r(y_1, z_2) = (z_3, y_2), & \dots, & r(y_1, z_k) = (z_1, y_2), \\
r(y_2, z_1) = (z_2, y_3), & r(y_2, z_2) = (z_3, y_3), & \dots, & r(y_2, z_k) = (z_1, y_3), \\
 & \vdots & & \\
r(y_p, z_1) = (z_2, y_1), & r(y_p, z_2) = (z_3, y_1), & \dots, & r(y_p, z_k) = (z_1, y_1).
\end{array}$$

Let $X_1, \dots, X_m$ be the distinct orbits of X under the action of G_r. We denote by r_j the restriction of r on X_j^2. Note that (X_j, r_j) also is a set theoretic involutive non-degenerate square free solution with associated IYB group G_{r_j}, such that its generators $\sigma_{i|X_j}$, for $x_i \in X_j$, are cyclic permutations.

Lemma 4.1. *Let $x_{i_1} \in X_i$. If $\sigma_{i_1|X_k} \neq \mathrm{id}_{X_k}$ for some k, then*

$$\sigma_{j|X_k} \neq \mathrm{id}_{X_k},$$

for all $x_j \in X_i$. Furthermore, σ_{i_1}, σ_j are conjugate elements in G_r for all $x_j \in X_i$.

Proof. Let $x_{i_2} \in X_i$ be an element such that $x_{i_1} \neq x_{i_2}$. Suppose that there exists $l \in \{1, \dots, n\}$ such that $\sigma_l(i_1) = i_2$. By the full cyclic condition there exist distinct elements $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ and distinct $x_l = x_{l_1}, x_{l_2}, \dots, x_{l_p}$ in X such that

$$\begin{array}{cccc}
r(x_{l_1}, x_{i_1}) = (x_{i_2}, x_{l_2}), & r(x_{l_1}, x_{i_2}) = (x_{i_3}, x_{l_2}), & \dots, & r(x_{l_1}, x_{i_k}) = (x_{i_1}, x_{l_2}), \\
r(x_{l_2}, x_{i_1}) = (x_{i_2}, x_{l_3}), & r(x_{l_2}, x_{i_2}) = (x_{i_3}, x_{l_3}), & \dots, & r(x_{l_2}, x_{i_k}) = (x_{i_1}, x_{l_3}), \\
 & \vdots & & \\
r(x_{l_p}, x_{i_1}) = (x_{i_2}, x_{l_1}), & r(x_{l_p}, x_{i_2}) = (x_{i_3}, x_{l_1}), & \dots, & r(x_{l_p}, x_{i_k}) = (x_{i_1}, x_{l_1}).
\end{array}$$

Since every σ_p is a cycle, it follows that $\sigma_l = \sigma_{l_1} = \sigma_{l_2} = (i_1, i_2, \ldots, i_k)$. From Lemma 2.1 we know that $\sigma_{l_1}\sigma_{i_1} = \sigma_{i_2}\sigma_{l_2}$, thus

$$\sigma_l \sigma_{i_1} \sigma_l^{-1} = \sigma_{i_2}.$$

Since $\sigma_{i_1|X_k} \neq \mathrm{id}_{X_k}$, clearly we have that $\sigma_{i_2|X_k} \neq \mathrm{id}_{X_k}$.

Let $x_j \in X_i$ be an element different from x_{i_1}. Since x_j, x_{i_1} are in the same orbit, there exist $j_1, \ldots, j_t \in \{1, \ldots, n\}$ such that

$$\sigma_{j_1} \ldots \sigma_{j_t}(i_1) = j.$$

Now it is easy to see by induction on t that σ_{i_1}, σ_j are conjugate elements in G_r, and the result follows. □

Theorem 4.2. *(X, r) is strongly retractable. Moreover, if $|X| > 1$ then (X, r) is a generalized twisted union.*

Proof. We shall prove both statements by induction on $|X| = n$. Clearly, we may assume that (X, r) is a non-trivial solution. In particular, $n > 2$. Also, we may assume that the result is true for all solutions of the same type as (X, r) with cardinality less than n.

Let $X_1, \ldots, X_m$ be the different orbits of X under the action of G_r. As mentioned before, by [12, Theorem 1], $m > 1$.

First assume that $|X_k| = 1$ for some k. Then, clearly, (X, r) is a generalized twisted union of X_k and $X \setminus X_k$. Moreover, if $X' = X \setminus X_k$ and $r' = r_{|X' \times X'}$ then the solution (X', r') inherits the assumptions on (X, r) and so it is strongly retractable by the induction hypothesis.

If the solution (X', r') is non-trivial, this means that there exist $x_i, x_j \in X'$, $i \neq j$, such that $\sigma_{i|X'} = \sigma_{j|X'}$ and x_i, x_j are in the same $G_{r'}$-orbit on X'. Then $\sigma_i = \sigma_j$ and x_i, x_j are in the same G_r-orbit on X, whence ρ is a non-trivial relation.

If (X', r') is a trivial solution then $\sigma_{i|X'} = \mathrm{id}_{|X'}$ for all $x_i \in X'$, so that all $\sigma_i, x_i \in X'$, are equal. Suppose that ρ is a trivial relation on X. This implies that every G_r-orbit contained in X' is of cardinality 1. But then (X, r) is a trivial solution, a contradiction. Therefore the relation ρ is non-trivial also in this case.

As the induced solution $(X/\rho, \bar{r})$ inherits the assumptions on (X, r), by the induction hypothesis, it follows that it is strongly retractable. Therefore (X, r) also is strongly retractable.

Hence, to complete the proof we may assume that $|X_k| > 1$ for all $k = 1, \ldots, n$.

By [12, Theorem 1], the number of G_{r_1}-orbits of X_1 is greater than 1. Therefore there exist $x_{i_1}, x_{i_2} \in X_1$ and $x_j \in X \setminus X_1$ such that $x_{i_1} \neq x_{i_2}$ and $\sigma_j(i_1) = i_2$. Let k be such that $x_j \in X_k$. By Lemma 4.1, $\sigma_{l|X_1} \neq \mathrm{id}_{X_1}$ for all $x_l \in X_k$. Thus $\sigma_{l|X_k} = \mathrm{id}_{X_k}$ for all $x_l \in X_k$. We claim that $\sigma_l = \sigma_j$ for all $x_l \in X_k$.

Let $x_{l_1} \in X_k$. Suppose first that $l_1 \neq j$ and $\sigma_u(j) = l_1$ for some $u \in \{1, \ldots, n\}$. In this case, there exists $x_v \in X$ such that $r(x_u, x_j) = (x_{l_1}, x_v)$. By the full cyclic condition, $\sigma_v(j) = l_1$. Thus $\sigma_{u|X_k} \neq \mathrm{id}_{X_k}$ and $\sigma_{v|X_k} \neq \mathrm{id}_{X_k}$. From Lemma 2.1 we know that $\sigma_u\sigma_j = \sigma_{l_1}\sigma_v$. Since

$$\sigma_{j|X_k} = \sigma_{l_1|X_k} = \mathrm{id}_{X_k}$$

and $\sigma_u, \sigma_v, \sigma_j, \sigma_{l_1}$ are cyclic permutations, this implies that $\sigma_u = \sigma_v$ and $\sigma_j = \sigma_{l_1}$, as desired. Now it is easy to see that $\sigma_l = \sigma_j$ for all $x_l \in X_k$, as claimed. In particular, the relation ρ is non-trivial on X.

On the other hand, by Lemma 2.4, for all $x_i \in X \setminus X_k$ and all $x_l \in X_k$, we have that $\sigma_{i|X_k} = \sigma_{\sigma_l(i)|X_k}$. Hence (X, r) is a generalized twisted union of X_k and $X \setminus X_k$.

Moreover, the induced solution $(X/\rho, \bar{r})$ satisfies all the assumptions on (X, r). Therefore it is strongly retractable by the induction hypothesis. It follows that (X, r) is strongly retractable as well. $\square$

Notice that there exist solutions of the above type with non-abelian groups G_r, see for example [7, Example 5.4]. Therefore, Theorem 4.2 confirms Conjecture (I) for a class of solutions not covered by Theorem 2.5.

Added in proof

In a recent preprint by P. Cameron and T. Gateva-Ivanova "Multipermutation solutions of the Yang–Baxter equation", arXiv:0907.4276, a further progress is made. In particular, the new notion of a strong twisted union is defined, which has an origin in [5]. This notion is weaker than that of a generalized twisted union. A new conjecture is considered in place of Conjecture (II), it says that every (involutive non-degenerate square-free) multipermutation solution of level m is a strong twisted union of (finitely many) multipermutation solutions of levels less than m. This conjecture is confirmed for every solution with an abelian IYB group, and also for every solution which is retractable of multipermutation level not exceeding 3. Notice that the proof of Corollary 8.13 in [4] actually yields exactly the latter statement, but this result is not correct as stated, as our example in Theorem 3.1 shows.

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