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Infinite-Dimensional Representations of the Lie Algebra $\mathfrak{gl}(n, \mathbb{C})$ Related to Complex Analogs of the Gelfand–Tsetlin Patterns and General Hypergeometric Functions on the Lie Group $\mathrm{GL}(n, \mathbb{C})$

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Abstract. Complex analogs of the Gelfand–Tsetlin patterns are introduced. Infinite-dimensional representations of $\mathfrak{gl}(n, \mathbb{C})$ in the vector spaces spanned on these patterns are constructed. Exponentials of these representations are described. These exponentials are operators $T(x)$, $x \in \mathrm{GL}(n, \mathbb{C})$, defined only in neighborhoods of the identity element of $\mathrm{GL}(n, \mathbb{C})$. A system of differential-difference equations for matrix elements of operators $T(x)$ is constructed. Explicit formulas for matrix elements are obtained for the case $x \in Z^{\pm}$, where Z^{+} and Z^{-} are the triangular unipotent subgroups. Representations of $\mathfrak{gl}(\infty, \mathbb{C})$ are also constructed; bases of these representations consist of Gelfand–Tsetlin patterns having infinitely many rows.

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1. Introduction

1. In [2], Gelfand–Tsetlin bases in spaces of irreducible finite-dimensional representations of the Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}(n, \mathbb{C})$ were introduced. Their construction is based on the fact that each irreducible representation of $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$ appears at most once in the restriction of an irreducible representation of $\mathfrak{gl}_n$ onto $\mathfrak{gl}_{n-1}$. Explicit formulas for matrix elements of irreducible representations with respect to the Gelfand–Tsetlin bases were obtained. In [3], explicit formulas for matrix elements of irreducible representations of the group $\mathrm{GL}(n, \mathbb{C})$ with respect to the same bases were obtained.

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Each vector of a Gelfand–Tsetlin basis is determined by a table of n integer rows (Gelfand–Tsetlin n-pattern):

$$m = \begin{pmatrix} m_{n1} & & \cdots & & \cdots & & m_{nn} \\ & m_{n-1,1} & & \cdots & & m_{n-1,n-1} & \\ & & & m_{1,1} & & & \end{pmatrix}.$$

The top row $m_n = (m_{n1}, \dots, m_{nn})$ of the table is the highest weight of the initial representation, the kth row from the bottom $m_k = (m_{k1}, \dots, m_{kk})$ is the highest weight of the subspace that is irreducible with respect to $\mathfrak{gl}(k, \mathbb{C})$ and contains this vector. The table determines each basis vector unambiguously up to a multiplier.

The basis of each irreducible representation consists of all patterns with a fixed top row and such that the elements of other rows satisfy the inequalities

$$m_{ij} \geqslant m_{i-1,j} \geqslant m_{i,j+1}.$$

2. It was proved in [3] that, for an appropriate normalization of the Gelfand–Tsetlin basis, matrix elements of the representations of $\mathfrak{gl}(n, \mathbb{C})$ are rational functions of the coordinates m_{ij} of the schema. Let us present formulas from [3] for operators of these representations in the Gelfand–Tsetlin basis.

It is convenient to replace patterns m by patterns λ, whose coordinates λ_{ij} are related to the coordinates m_{ij} of m by the relation:

$$\lambda_{ij} = m_{ij} - j.$$

Let us denote by $[\lambda]$ the basis element corresponding to the pattern λ.

By e_{ij} we shall denote the pattern λ with $\lambda_{ij} = 1$ and $\lambda_{i'j'} = 0$ for $(i'j') \neq (ij)$.

In this notation the operators E_{ij} corresponding to the standard basis of $\mathfrak{gl}(n, \mathbb{C})$ are given by the formulas:

$$E_{ii}[\lambda] = (k_i(\lambda) - k_{i-1}(\lambda) + i)[\lambda], \quad i = 1, \dots, n, \tag{1}$$

where

$$k_i(\lambda) = \sum_{j=1}^{i} \lambda_{ij} \quad \text{for } i > 0, \qquad k_0(\lambda) = 0; \tag{2}$$

$$E_{i+1,i}[\lambda] = \sum_{j=1}^{i} a_{ij}(\lambda)[\lambda - e_{ij}], \qquad E_{i,i+1}[\lambda] = \sum_{j=1}^{i} b_{ij}(\lambda)[\lambda + e_{ij}], \tag{3}$$

where

$$\begin{aligned} a_{ij}(\lambda) &= (-1)^i \frac{\prod_{k=j}^{i-1}(\lambda_{ij} - \lambda_{i-1,k}) \prod_{k=j+1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij} + 1)}{\prod_{k \neq j}(\lambda_{ij} - \lambda_{ik})}, \\ b_{ij}(\lambda) &= \frac{\prod_{k=1}^{j-1}(\lambda_{ij} - \lambda_{i-1,k} + 1) \prod_{k=1}^{j}(\lambda_{i+1,k} - \lambda_{ij})}{\prod_{k \neq j}(\lambda_{ij} - \lambda_{ik})} \end{aligned} \tag{4}$$

for $i > 1$; $a_{11} = -(\lambda_{22} - \lambda_{11} + 1)$, $b_{11}(\lambda) = \lambda_{21} - \lambda_{11}$.

For $|i-j|>1$, the operators E_{ij} are defined by induction on $|i-j|$:

$$E_{ij}=\begin{cases}[E_{i,i-1},E_{i-1,j}] & \text{for } i>j+1,\\ [E_{i,i+1},E_{i+1,j}] & \text{for } i<j-1.\end{cases}$$

3. The conditions that those operators really give a representation of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$, i.e., that

$$[E_{ij},E_{i'j'}]=\delta_{ji'}E_{ij'}-\delta_{j'i}E_{i'j}\quad \text{for all } i,i',j,j'=1,\dots,n, \tag{5}$$

can be presented as algebraic identities for the elements of Gelfand–Tsetlin patterns, see [3, 4, 6]. It is important that those identities are valid for any, not necessarily integer, value of the elements λ_{ij} of Gelfand–Tsetlin patterns. For instance, the relation

$$[E_{i+1,i},E_{i,i+1}]=E_{i+1,i+1}-E_{i,i}$$

is equivalent to the following identity, which is valid for any complex λ_{ij}:

$$\begin{aligned}\sum_{j=1}^{i}\Bigg(&\frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k}-\lambda_{ij}-1)\prod_{k=1}^{i-1}(\lambda_{i-1,k}-\lambda_{ij})}{\prod_{k\neq j}(\lambda_{ik}-\lambda_{ij}+1)(\lambda_{ik}-\lambda_{ij})}-\\ &-\frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k}-\lambda_{ij})\prod_{k=1}^{i-1}(\lambda_{i-1,k}-\lambda_{ij}-1)}{\prod_{k\neq j}(\lambda_{ik}-\lambda_{ij})(\lambda_{ik}-\lambda_{ij}-1)}\Bigg)\\ &=1+\sum_{j=1}^{i+1}\lambda_{i+1,j}-2\sum_{j=1}^{i}\lambda_{ij}+\sum_{j=1}^{i-1}\lambda_{i-1,j}.\end{aligned}$$

Thus the formulas from [3] give a representation of $\mathfrak{gl}(n,\mathbb{C})$ also when elements of the basis are patterns with complex coordinates. This paper is devoted to the description of the in such manner defined infinite-dimensional representations of the algebra $\mathfrak{gl}(n,\mathbb{C})$ and to the description of their exponentials, which are local representations of the group $\mathrm{GL}(n,\mathbb{C})$.

4. We consider the set of complex n-patterns λ such that $\lambda_{ij}-\lambda_{ij'}\notin\mathbb{Z}$ for all $i<n$ and $j\neq j'$. With each λ^0 we connect the infinite-dimensional linear space $L(\lambda^0)$ with the basis $\{\lambda=\lambda^0+l \mid l_{ij}\in\mathbb{Z};\ l_{nj}=0\}$. We construct a finite set of representations of $\mathfrak{gl}(n,\mathbb{C})$ in each space $L(\lambda^0)$ (the number of those representations equals $2^{n(n^2-1)/3}$). If $\lambda_{ij}-\lambda_{i-1,k}\notin\mathbb{Z}$ for any i, j, and k, then those representations are irreducible and mutually equivalent. That is not correct if $\lambda_{ij}-\lambda_{i-1,k}\in\mathbb{Z}$ for at least one triple i, j, k.

We construct also the exponentials of those representations. These exponentials are local representations T of the group $\mathrm{GL}(n,\mathbb{C})$, where the operators $T(x)$ are defined only for elements x from a neighborhood of the identity matrix. It is proved that the matrix elements of these operators are solutions of a system of differential-difference equations on the group $\mathrm{GL}(n,\mathbb{C})$. This system can be naturally regarded as general hypergeometric system of equations on the group.

The representations of $\mathfrak{gl}(n, \mathbb{C})$ are extendable to representations of the algebra $\mathfrak{gl}(\infty, \mathbb{C})$ in the spaces $L(\lambda)$, where each vector of a basis is determined by a pattern with infinite number of rows. An infinite (continuous) set of representations of $\mathfrak{gl}(\infty, \mathbb{C})$ is connected with every $L(\lambda)$. As for the case of $\mathfrak{gl}(n, \mathbb{C})$, those representations are irreducible and mutually equivalent if and only if $\lambda_{ij} - \lambda_{i-1,k} \notin \mathbb{Z}$ for all i, j, and k.

2. Infinite-Dimensional Representations of $\mathfrak{gl}(n, \mathbb{C})$, Associated with Gelfand–Tsetlin Patterns

2.1. INITIAL DEFINITIONS

Elements λ of the space $\mathbb{C} \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^n \simeq \mathbb{C}^N$, $N = n(n+1)/2$ are called Gelfand–Tsetlin n-patterns (or simply n-patterns); we shall denote them by

$$\lambda = \begin{pmatrix} \lambda_{n1} & & \cdots & & \cdots & & \lambda_{nn} \\ & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & \\ & & & \lambda_{1,1} & & & \end{pmatrix}.$$

By λ_i denote the ith row from the bottom of λ, i.e.,

$$\lambda_i = (\lambda_{i1}, \ldots, \lambda_{ii}).$$

Let us write $\lambda \geqslant \lambda'$ if all coordinates of $\lambda - \lambda'$ are nonnegative integers.

By $L = L_n$ denote the set of all n-patterns with integer coordinates and zero top row ($\lambda_n = 0$).

By e_{ij}, where $j \leqslant i$, $i = 1, \ldots, n$, denote the n-pattern λ such that $\lambda_{ij} = 1$ and $\lambda_{i'j'} = 0$ for $(i'j') \neq (ij)$.

DEFINITION. We say that a n-pattern λ is admissible if

$$\lambda_{ij} - \lambda_{ij'} \notin \mathbb{Z} \quad \text{for } i = 2, \ldots, n-1 \text{ and } j \neq j'. \tag{6}$$

We say that an admissible n-pattern λ^0 is strongly admissible, if it satisfies also the condition:

$$\lambda^0_{i+1,j} - \lambda^0_{ij'} \notin \mathbb{Z} \quad \text{for any } i,\ j,\ j'. \tag{7}$$

2.2. DESCRIPTION OF THE REPRESENTATIONS $U^{\epsilon}_{\lambda^0}$

Let us associate a finite set of representations of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ with any admissible n-pattern λ^0.

These representations act in the infinite-dimensional linear space $L(\lambda^0)$ with basis $\Lambda^0 = \{\lambda = \lambda^0 + l \mid l \in L\}$.

Each representation is defined by the set of numbers

$$\epsilon = \left\{\epsilon^i_{jk} \mid j \in [1, i],\ k \in [1, i-1], i \in [2, n]\right\},$$

where each ϵ^i_{jk} equals 0 or 1. Note that the number of different sets ϵ equals $2^{n(n^2-1)/3}$.

The operators $E_{ij} = E^{\epsilon}_{ij}$ of the representation, corresponding to the elements of the standard basis of $\mathfrak{gl}(n, \mathbb{C})$, are defined by the formulas:

$$E_{ii}[\lambda] = \big(k_i(\lambda) - k_{i-1}(\lambda) + i\big)[\lambda], \quad i = 1, \dots, n, \tag{8}$$

where $k_i(\lambda) = \lambda_{i1} + \cdots + \lambda_{ii}$ for $i > 0$, $k_0 = 0$;

$$E_{i+1,i}[\lambda] = \sum_{j=1}^{i} a^{\epsilon}_{ij}(\lambda)[\lambda - e_{ij}], \qquad E_{i,i+1}[\lambda] = \sum_{j=1}^{i} b^{\epsilon}_{ij}(\lambda)[\lambda + e_{ij}], \tag{9}$$

where

$$\begin{aligned} a^{\epsilon}_{ij}(\lambda) &= (-1)^i \frac{\prod_{k=1}^{i-1}(\lambda_{ij} - \lambda_{i-1,k})^{1-\epsilon^i_{jk}} \prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij} + 1)^{\epsilon^{i+1}_{kj}}}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})}, \\ b^{\epsilon}_{ij}(\lambda) &= \frac{\prod_{k=1}^{i-1}(\lambda_{ij} - \lambda_{i-1,k} + 1)^{\epsilon^i_{jk}} \prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij})^{1-\epsilon^{i+1}_{kj}}}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})}. \end{aligned} \tag{10}$$

For $|i - j| > 1$, the linear operators $E_{ij} = E^{\epsilon}_{ij}$ are defined by the induction on $|i - j|$:

$$E_{ij} = \begin{cases} [E_{i,i-1}, E_{i-1,j}] & \text{for } i > j + 1, \\ [E_{i,i+1}, E_{i+1,j}] & \text{for } i < j - 1. \end{cases}$$

It follows from the admissibility condition (6) that all the coefficients $a^{\epsilon}_{ij}(\lambda)$ and $b^{\epsilon}_{ij}(\lambda)$ are finite. Moreover, the product $a^{\epsilon}_{ij}(\lambda) b^{\epsilon}_{ij}(\lambda - e_{ij})$ does not depend on ϵ.

Let us present the expressions for $a^{\epsilon}_{ij}(\lambda)$ and $b^{\epsilon}_{ij}(\lambda)$ for some special choices of ϵ.

For $\epsilon^i_{jk} \equiv 0$ we have:

$$a^0_{ij}(\lambda) = (-1)^i \frac{\prod_{k=1}^{i-1}(\lambda_{ij} - \lambda_{i-1,k})}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})}, \qquad b^0_{ij}(\lambda) = \frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij})}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})} \tag{11}$$

for $i > 1$; $a^0_{11}(\lambda) = -1$, $b^0_{11}(\lambda) = (\lambda_{21} - \lambda_{11})(\lambda_{22} - \lambda_{11})$.

For $\epsilon^i_{jk} \equiv 1$ we have:

$$\begin{aligned} a^1_{ij}(\lambda) &= (-1)^i \frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij} + 1)}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})}, \\ b^1_{ij}(\lambda) &= \frac{\prod_{k=1}^{i-1}(\lambda_{ij} - \lambda_{i-1,k} + 1)}{\prod_{k\neq j}(\lambda_{ij} - \lambda_{ik})} \end{aligned} \tag{12}$$

for $i > 1$; $a^1_{11}(\lambda) = -(\lambda_{21} - \lambda_{11})(\lambda_{22} - \lambda_{11})$, $b^1_{11}(\lambda) = 1$.

In the case $\epsilon^i_{jk} = 1$ for $j > k$ and $\epsilon^i_{jk} = 0$ for $j \leqslant k$, the formulas for $a_{ij}(\lambda)$ and $b_{ij}(\lambda)$ have the form (4).

PROPOSITION 2.1. *Suppose an n-pattern λ^0 is strongly admissible. Then for any ϵ there exists a function $c_\epsilon(\lambda)$ on $\Lambda^0 = \lambda^0 + L$ with the following properties:*

(1) $c_\epsilon(\lambda) \neq 0$ *everywhere on* Λ^0;
(2) *the operators $E^\epsilon_{i+1,i}$ and $E^\epsilon_{i,i+1}$ are transfered onto $E^0_{i+1,i}$ and $E^0_{i,i+1}$, respectively, by the linear transform in $L(\lambda^0)$ that maps the basic vectors λ onto the vectors $\lambda/c_\epsilon(\lambda)$.*

This function is

$$c_\epsilon(\lambda) = \prod_{i=2}^{n}\prod_{k=1}^{i-1}\big(\Gamma(\lambda_{ij} - \lambda_{i-1,k} + 1)\big)^{\epsilon^i_{jk}}. \tag{13}$$

THEOREM 1. *For any admissible n-pattern λ^0 and any ϵ the operators $E_{ij} = E^\epsilon_{ij}$ satisfy the commutation relations*

$$[E_{ij}, E_{i'j'}] = \delta_{ji'}E_{ij'} - \delta_{j'i}E_{i'j}. \tag{14}$$

Thus they define a representation of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ in the space $L(\lambda^0)$. Denote this representation by $U^\epsilon_{\lambda^0}$.

Proof. Firstly, let us suppose that λ^0 is strongly admissible. It follows from Proposition 2.1 that in this case it is sufficient to verify the commutation relations for any fixed ϵ.

Such verification was done, for instance, in [3, 4, 6]. Note that only integer patterns were considered there; however the given proof remains valid without that assumption, too. For the completeness, checking the commutation relations is given in the Appendix to this paper.

The commutation relations are correct also for any admissible pattern λ^0, because any admissible pattern is a limit of strongly admissible patterns. □

Note. The definition of the operators E_{ij} and the proof of the commutation relations remain having sense under the replacement of $\mathbb{C}$ by any field K of zero characteristic. In that case elements of Gelfand–Tsetlin patterns belong to K and the above formulas define a representation of the algebra $\mathfrak{gl}(n, K)$ in a linear space over K.

2.3. PROPERTIES OF THE REPRESENTATIONS $U^\epsilon_{\lambda^0}$

Firstly, let us note that the representations $U^\epsilon_{\lambda^0}$ and $U^\epsilon_{\lambda^0+l}$ are equivalent for any $l \in L$.

Further, if λ^0 is an arbitrary strongly admissible pattern, then from the description of the operators of the representation and from Proposition 2.1 it follows that:

(1) The representations $U^\epsilon_{\lambda^0}$ for different ϵ are irreducible and mutually equivalent.

(2) Suppose a strongly admissible pattern λ^1 can be obtained from λ^0 by arbitrary transmutations of elements of each row; then U_{λ^1} is equivalent to U_{λ^0}.
(3) The representation $U^{\epsilon}_{\lambda^1}$ is equivalent to $U^{\epsilon}_{\lambda^0}$ if and only if $\lambda^1 = s\lambda^0 + l$, where $l \in L$ and $s\lambda^0$ can be obtained from λ^0 by permutations of elements in each row.

If a pattern λ^0 is admissible but not strongly admissible, then in general the representation U^{ϵ}_{λ} is semireducible and the representations $U^{\epsilon}_{\lambda^0}$, corresponding to different ϵ, are not equivalent.

3. Local Representations of the Group GL$(n, \mathbb{C})$ Related to Admissible Gelfand–Tsetlin Patterns

3.1. DEFINITION OF LOCAL REPRESENTATIONS OF GL$(n, \mathbb{C})$

With every representation $U^{\epsilon}_{\lambda^0}$ of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ in the space $L(\lambda^0)$, where λ^0 is an arbitrary admissible n-pattern, we shall connect a local representation $T = T^{\epsilon}_{\lambda^0}$ of the Lie group GL$(n, \mathbb{C})$ in the same space $L(\lambda^0)$.

Define operators $T(x)$ on $L(\lambda^0)$ by the equality

$$T(\exp X) = \exp(U^{\epsilon}_{\lambda^0}(X)), \tag{15}$$

where $X \mapsto \exp X$ is the exponential mapping from $\mathfrak{gl}(n, \mathbb{C})$ to GL$(n, \mathbb{C})$.

By the equality (15) the operators $T(x)$ are defined unambiguously for all elements of the subgroups $Z^{\pm}$ of lower and upper unipotent matrices. These operators form representations of these subgroups in the space $L(\lambda^0)$. It follows from the formulas for the operators E_{ij} that the matrix elements of the operators $T(x)$, where $x \in Z^{\pm}$, are polynomials in the elements of the matrices $x \in Z^{\pm}$.

For arbitrary elements $x \in$ GL$(n, \mathbb{C})$, the operators $T(x)$ are defined by the equality (15) only in a sufficiently small neighborhood of the identity matrix. In this sufficiently small neighborhood their matrix elements $F_{\lambda',\lambda}(x)$ are regular functions of elements of the matrix x and the relation $T(xy) = T(x)T(y)$ is satisfied. In this sense the operators $T(x)$ define a local representation of GL$(n, \mathbb{C})$.

Note. It is natural to consider matrix elements $F_{\lambda',\lambda}(x)$ of an operator $T(x)$ as generalizations of matrix elements $P_{\lambda',\lambda}(x)$ of irreducible finite-dimensional representations of GL$(n, \mathbb{C})$ in the Gelfand–Tsetlin basis. The latter matrix elements, up to the factor $(\det x)^{-k}$, are homogenous polynomials in elements of the matrix x. Their explicit description is known only for the simplest representations, see for instance [1, 5].

3.2. FORMULAS FOR THE OPERATORS $T(x) = T^{\epsilon}_{\lambda^0}(x)$

Let us obtain explicit formulas for the operators $\exp(tE_{ii})$, $\exp(tE_{i+1,i})$, and $\exp(tE_{i,i+1})$ of the local representation $T^{\epsilon}_{\lambda^0}(x)$ of GL$(n, \mathbb{C})$ in the space $L(\lambda^0)$;

these operators correspond to the matrices $\exp(te_{ii})$, $I+te_{i+1,i}=\exp(te_{i+1,i})$, and $I+te_{i,i+1}=\exp(te_{i,i+1})$, where I is the identity matrix.

For E_{ii} the following proposition follows from the formula (8).

PROPOSITION 3.1. *The operators* $\exp(tE_{ii})$ *are defined by the equalities:*

$$\exp(tE_{ii})[\lambda] = \mathrm{e}^{tr_i(\lambda)}[\lambda], \tag{16}$$

where $r_i(\lambda)=k_i(\lambda)-k_{i-1}(\lambda)+i$.

Obtaining formulas for the operators $\exp(tE_{i+1,i})$ and $\exp(tE_{i,i+1})$ we shall assume that the n-pattern λ^0 is strongly admissible. Then the representations $U^{\epsilon}_{\lambda^0}$ of $\mathfrak{gl}(n,\mathbb{C})$ are mutually equivalent for different ϵ. Therefore it is possible to restrict ourselves to the case $\epsilon=0$. For this case, the operators $E_{i+1,i}$ and $E_{i,i+1}$ are given by the formulas:

$$E_{i+1,i}[\lambda] \;=\; -\sum_{j=1}^{i}\frac{\prod_{k=1}^{i-1}(\lambda_{ij}-\lambda_{i-1,k})}{\prod_{k\neq j}(\lambda_{ik}-\lambda_{ij})}[\lambda-e_{ij}]; \tag{17}$$

$$E_{i,i+1}[\lambda] \;=\; \sum_{j=1}^{i}\frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k}-\lambda_{ij})}{\prod_{k\neq j}(\lambda_{ij}-\lambda_{ik})}[\lambda+e_{ij}]. \tag{18}$$

THEOREM 2. *The operators* $\exp(tE_{i+1,i})$ *and* $\exp(tE_{i,i+1})$ *corresponding to* $\epsilon=0$ *are defined by the formulas:*

$$\begin{aligned}\exp(tE_{i+1,i})[\lambda] &= \sum_{\lambda_i'\leqslant\lambda_i} A_i(\lambda',\lambda)t^{k_i(\lambda-\lambda')}[\lambda'],\\ \exp(tE_{i,i+1})[\lambda] &= \sum_{\lambda_i'\geqslant\lambda_i} B_i(\lambda',\lambda)t^{k_i(\lambda'-\lambda)}[\lambda'],\end{aligned} \tag{19}$$

where

$$A_i(\lambda',\lambda) \;=\; (-1)^{k_i(\lambda-\lambda')}\prod_{k,l}\frac{\Gamma(\lambda_{il}-\lambda_{i-1,k}+1)}{\Gamma(\lambda'_{il}-\lambda_{i-1,k}+1)}\prod_{k,l}\frac{\Gamma(\lambda_{ik}-\lambda_{il}+1)}{\Gamma(\lambda_{ik}-\lambda'_{il}+1)}\;\times$$
$$\times\prod_{k<l}\frac{\lambda'_{ik}-\lambda'_{il}}{\lambda_{ik}-\lambda_{il}}; \tag{20}$$

$$B_i(\lambda',\lambda) \;=\; \prod_{k,l}\frac{\Gamma(\lambda_{i+1,k}-\lambda_{il}+1)}{\Gamma(\lambda_{i+1,k}-\lambda'_{il}+11)}\prod_{k,l}\frac{\Gamma(\lambda_{il}-\lambda_{ik}+1)}{\Gamma(\lambda'_{il}-\lambda_{ik}+1)}\;\times$$
$$\times\prod_{k<l}\frac{\lambda'_{ik}-\lambda'_{il}}{\lambda_{ik}-\lambda_{il}}, \tag{21}$$

$\Gamma(s)$ *is the Euler Gamma-function. In particular:*

$$A_1(\lambda',\lambda) \;=\; (-1)^{\lambda_{11}-\lambda'_{11}}(\Gamma(\lambda_{11}-\lambda'_{11}+1))^{-1}, \tag{22}$$

$$B_1(\lambda',\lambda) \;=\; \frac{\Gamma(\lambda_{21}-\lambda_{11}+1)\Gamma(\lambda_{22}-\lambda_{11}+1)}{\Gamma(\lambda_{21}-\lambda'_{11}+1)\Gamma(\lambda_{22}-\lambda'_{11}+1)\Gamma(\lambda'_{11}-\lambda_{11}+1)}. \tag{23}$$

The summation in (19) *is over the set* $\lambda' \in \Lambda^0$ *of the form* $\lambda' = \lambda - \sum_{j=1}^{i} l_j e_{ij}$ *and* $\lambda' = \lambda + \sum_{j=1}^{i} l_j e_{ij}$*, respectively, here the* l_j *run over all nonnegative integers.*

To prove the theorem we need the following lemma:

LEMMA 1. *The following relation holds:*

$$\sum_{j=1}^{i} \frac{\prod_{k=1}^{i}(x_k - y_j)}{\prod_{k\neq j}^{i}(y_k - y_j)} = \sum_{j=1}^{i}(x_j - y_j). \tag{24}$$

Proof. The left-hand expression I is obviously symmetric with respect to x_j and y_j. Let us present it in the form:

$$I = \frac{P(x, y)}{\prod_{k<l}(y_k - y_l)},$$

where $P(x, y)$ is a polynomial in x and y. Because the denominator is antisymmetric with respect to y_j, we have that the numerator $P(x, y)$ is antisymmetric with respect to y_j, too. Hence $I = \alpha \sum_{j=1}^{i} x_j + \beta \sum_{j=1}^{i} y_j$. It is easy to check that $\alpha = -\beta = 1$. □

Proof of the theorem. By $T_i^-(t)$ and $T_i^+(t)$ denote the operators in the right-hand sides of (19). Let us check that

$$T_i^-(t) = \exp(tE_{i+1,i}), \qquad T_i^+(t) = \exp(tE_{i,i+i}). \tag{25}$$

From the expressions for $T_i^{\pm}(t)$ it immediately follows that $T_i^{\pm}(0)$ is the identity operator. Thus it is sufficient to prove that

$$\frac{\mathrm{d}T_i^-(t)}{\mathrm{d}t} = E_{i+1,i}T_i^-(t), \qquad \frac{\mathrm{d}T_i^+(t)}{\mathrm{d}t} = E_{i,i+1}T_i^+(t). \tag{26}$$

Let us prove the first of Equations (26). We have:

$$\begin{aligned} &E_{i+1,i}T_i^-(t)[\lambda] \\ &\quad = -c(\lambda)\sum_{\lambda',j}\Bigg(\frac{\prod_{k=1}^{i-1}(\lambda'_{ij} - \lambda_{i-1,k})}{\prod_{k,l}\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)}\prod_{k,l}\big(\Gamma(\lambda_{ik} - \lambda'_{il} + 1)\big)^{-1} \times \\ &\qquad\qquad \times \frac{\prod_{k<l}\lambda'_{ik} - \lambda'_{il}}{\prod_{k\neq j}(\lambda'_{ik} - \lambda'_{ij})}(-t)^{k_i(\lambda)-k_i(\lambda')}[\lambda' - e_{ij}]\Bigg), \end{aligned}$$

where

$$c(\lambda) = \prod_{k,l}\big(\Gamma(\lambda_{il} - \lambda_{i-1,k} + 1)\Gamma(\lambda_{ik} - \lambda_{il} + 1)\big)\prod_{k<l}(\lambda_{ik} - \lambda_{il})^{-1}.$$

Substitute $\lambda' \to \lambda' + e_{ij}$. By this substitution the factors

$$\frac{\prod_{k=1}^{i-1}(\lambda'_{ij} - \lambda_{i-1,k})}{\prod_{k,l}\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)} \quad \text{and} \quad \prod_{k,l}\big(\Gamma(\lambda_{ik} - \lambda'_{il} + 1)\big)^{-1}$$

are transformed, respectively, to

$$\prod_{k,l}\big(\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)\big)^{-1} \quad \text{and} \quad \frac{\prod_{k=1}^{i}(\lambda_{ik} - \lambda'_{ij})}{\prod_{k,l}\Gamma(\lambda_{ik} - \lambda'_{il} + 1)};$$

the factor $\frac{\prod_{k<l}(\lambda'_{ik}-\lambda'_{il})}{\prod_{k\neq j}(\lambda'_{ik}-\lambda'_{ij})}$ is preserved by the substitution because it does not depend on λ'_{ij}. Thus, using (24), we get:

$$\begin{aligned}
&E_{i+1,i}T_i^-(t)[\lambda]\\
&\quad = -\sum_{\lambda'}\left(A_i(\lambda',\lambda)\left(\sum_{j=1}^{i}\frac{\prod_{k=1}^{i}(\lambda_{ik} - \lambda'_{ij})}{\prod k \neq j(\lambda'_{ik} - \lambda'_{ij})}\right)(-t)^{k_i(\lambda)-k_i(\lambda')-1}\right)[\lambda']\\
&\quad = -\sum_{\lambda'}(k_i(\lambda) - k_i(\lambda'))A_i^-(\lambda',\lambda)(-t)^{k_i(\lambda)-k_i(\lambda')-1}[\lambda'] = \frac{\mathrm{d}T_i^-(t)}{\mathrm{d}t}.
\end{aligned}$$

The second equation (20) can be checked similarly. □

COROLLARY. *The operators* $\exp(tE^{\epsilon}_{i+1,i})$ *and* $\exp(tE^{\epsilon}_{i,i+1})$ *for arbitrary* ϵ *are given by the formulas:*

$$\begin{aligned}
\exp(tE^{\epsilon}_{i+1,i})[\lambda] &= \sum_{\lambda'_i \leqslant \lambda_i} A_i^{\epsilon}(\lambda',\lambda)t^{k_i(\lambda-\lambda')}[\lambda'],\\
\exp(tE^{\epsilon}_{i,i+1})[\lambda] &= \sum_{\lambda'_i \geqslant \lambda_i} B_i^{\epsilon}(\lambda',\lambda)t^{k_i(\lambda'-\lambda)}[\lambda'],
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
A_i^{\epsilon}(\lambda',\lambda) = (-1)^{k_i(\lambda-\lambda')}&\prod_{k,l}\left(\frac{\Gamma(\lambda_{il} - \lambda_{i-1,k} + 1)}{\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)}\right)^{1-\epsilon^i_{lk}} \times\\
&\times\prod_{k,l}\left(\frac{\Gamma(\lambda_{i+1,k} - \lambda'_{il} + 1)}{\Gamma(\lambda_{i+1,k} - \lambda_{il} + 1)}\right)^{\epsilon^{i+1}_{kl}} \times\\
&\times\prod_{k,l}\frac{\Gamma(\lambda_{ik} - \lambda_{il} + 1)}{\Gamma(\lambda_{ik} - \lambda'_{il} + 1)}\prod_{k<l}\frac{\lambda'_{ik} - \lambda'_{il}}{\lambda_{ik} - \lambda_{il}};
\end{aligned} \tag{28}$$

$$\begin{aligned}
B_i^{\epsilon}(\lambda',\lambda) = &\prod_{k,l}\left(\frac{\Gamma(\lambda_{i+1,k} - \lambda_{il} + 1)}{\Gamma(\lambda_{i+1,k} - \lambda'_{il} + 11)}\right)^{1-\epsilon^{i+1}_{kl}}\prod_{k,l}\left(\frac{\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)}{\Gamma(\lambda_{il} - \lambda_{i-1,k} + 1)}\right)^{\epsilon^i_{lk}} \times\\
&\times\prod_{k,l}\frac{\Gamma(\lambda_{il} - \lambda_{ik} + 1)}{\Gamma(\lambda'_{il} - \lambda_{ik} + 1)}\prod_{k<l}\frac{\lambda'_{ik} - \lambda'_{il}}{\lambda_{ik} - \lambda_{il}}.
\end{aligned} \tag{29}$$

The summation in (27) *is carried out over the same set of patterns as in* (19).

Remark. In the formulas for $A_i^\epsilon(\lambda', \lambda)$ and $B_i^\epsilon(\lambda', \lambda)$ the factor $\prod_{k<l} \frac{\lambda'_{ik}-\lambda'_{il}}{\lambda_{ik}-\lambda_{il}}$ can be removed. This can be done by renormalization of the basis vectors.

3.3. TURN TO ARBITRARY ADMISSIBLE PATTERNS

Let us present the expressions for the operators $\exp(tE_{i+1,i})$ and $\exp(tE_{i,i+1})$ in the form that preserves its sense for any admissible, not necessarily strongly admissible, n-pattern λ^0. For definiteness let us consider the case $\epsilon = 0$. Denote:

$$\widetilde{A}_i(l, \lambda) = A_i\left(\lambda - \sum_{j=1}^{i} l_j e_{ij}, \lambda\right),$$

$$\widetilde{B}_i(l, \lambda) = B_i\left(\lambda + \sum_{j=1}^{i} l_j e_{ij}\ , \lambda\right),$$

where l_j are nonnegative integers. In this notation $\lambda'_{ij} = \lambda_{ij} - l_j$ in the formula for the operator $\exp(tE_{i+1,i})$; therefore

$$\frac{\Gamma(\lambda_{il} - \lambda_{i-1,k} + 1)}{\Gamma(\lambda'_{il} - \lambda_{i-1,k} + 1)} = [\lambda_{il} - \lambda_{i-1,k}]_{l_j},$$

$$\frac{\Gamma(\lambda_{ik} - \lambda_{il} + 1)}{\Gamma(\lambda_{ik} - \lambda'_{il} + 1)} = \big((\lambda_{ik} - \lambda_{il} + 1)_{l_j}\big)^{-1},$$

where the following notation is used:

$$[a]_n = a(a-1)\cdots(a-n+1); \qquad (a)_n = a(a+1)\cdots(a+n-1);$$
$$[a]_0 = (a)_0 = 1.$$

Similarly, $\lambda'_{ij} = \lambda_{ij} + l_j$ in the formula for the operator $\exp(tE_{i+1,i})$; therefore

$$\frac{\Gamma(\lambda_{i+1,k} - \lambda_{il} + 1)}{\Gamma(\lambda_{i+1,k} - \lambda'_{il} + 1)} = [\lambda_{i+1,k} - \lambda_{il}]_{l_j},$$

$$\frac{\Gamma(\lambda_{il} - \lambda_{ik} + 1)}{\Gamma(\lambda'_{il} - \lambda_{ik} + 1)} = \big((\lambda_{ij} - \lambda_{ik} + 1)_{l_j}\big)^{-1}.$$

Thus the formulas for the operators $\exp(tE_{i+1,i})$ and $\exp(tE_{i,i+1})$ for $\epsilon = 0$ take the form:

$$\exp(tE_{i+1,i})[\lambda] = \sum_l \widetilde{A}_i(l, \lambda) t^{\sum_{j=1}^{i} l_j} \left[\lambda - \sum_{j=1}^{i} l_j e_{ij}\right], \tag{30}$$

$$\exp(tE_{i,i+1})[\lambda] = \sum_l \widetilde{B}_i(l, \lambda) t^{\sum_{j=1}^{i} l_j} \left[\lambda + \sum_{j=1}^{i} l_j e_{ij}\right], \tag{31}$$

where

$$\begin{aligned}\widetilde{A}_i(l,\lambda) &= (-1)^{\sum_{j=1}^{i} l_j} \frac{\prod_{k,l}[\lambda_{il}-\lambda_{i-1,k}]_{l_j}}{\prod_{k\neq l}(\lambda_{ik}-\lambda_{il}+1)_{l_j}\prod_{j=1}^{i} l_j!} \times \\ &\quad \times \prod_{k<l}\frac{\lambda_{ik}-\lambda_{il}-l_k+l_l}{\lambda_{ik}-\lambda_{il}}, \qquad (32)\\ \widetilde{B}_i(l,\lambda) &= \frac{\prod_{k,l}[\lambda_{i+1,k}-\lambda_{il}]_{l_j}}{\prod_{k\neq l}(\lambda_{il}-\lambda_{ik}+1)_{l_j}\prod_{j=1}^{i} l_j!}\prod_{k<l}\frac{\lambda_{ik}-\lambda_{il}+l_k-l_l}{\lambda_{ik}-\lambda_{il}}. \qquad (33)\end{aligned}$$

In a similar form one can represent the expressions for $\widetilde{A}_i^{\epsilon}(l,\lambda)$ and $\widetilde{B}_i^{\epsilon}(l,\lambda)$ for arbitrary $\epsilon = \{\epsilon_{jk}^{i}\}$. For instance, for $\epsilon_{jk}^{i} \equiv 1$ we have:

$$\widetilde{A}_i(l,\lambda) = (-1)^{\sum_{j=1}^{i} l_j} \frac{\prod_{k,l}(\lambda_{il}-\lambda_{i-1,k})_{l_j}}{\prod_{k\neq l}(\lambda_{ik}-\lambda_{il}+1)_{l_j}\prod_{j=1}^{i} l_j!}\prod_{k<l}\frac{\lambda_{ik}-\lambda_{il}-l_k+l_l}{\lambda_{ik}-\lambda_{il}},$$

$$\widetilde{B}_i(l,\lambda) = \frac{\prod_{k,l}(\lambda_{i+1,k}-\lambda_{il})_{l_j}}{\prod_{k\neq l}(\lambda_{il}-\lambda_{ik}+1)_{l_j}\prod_{j=1}^{i} l_j!}\prod_{k<l}\frac{\lambda_{ik}-\lambda_{il}+l_k-l_l}{\lambda_{ik}-\lambda_{il}}.$$

3.4. FUNCTIONS $F_{\lambda',\lambda}(x)$

Let $T(x) = T_{\lambda^0}^{\epsilon}(x)$ be a local representation of the group $\mathrm{GL}(n,\mathbb{C})$ in the space $L(\lambda^0)$, where λ^0 is an arbitrary admissible n-pattern. By definition, the basis of $L(\lambda^0)$ consists of the set Λ^0 of patterns of the form $\lambda = \lambda^0 + l$, where $l \in L$.

By $F_{\lambda',\lambda}(x) = F_{\lambda',\lambda}^{\epsilon}(x)$ denote the matrix elements of the operator $T(x)$ in the basis $\lambda \in \Lambda^0$, i.e.,

$$T(x)[\lambda] = \sum_{\lambda'\in\Lambda^0} F_{\lambda',\lambda}(x)[\lambda']. \qquad (34)$$

Thus the operator $T(x)$ is described by the following infinite matrix:

$$F(x) = \|F_{\lambda',\lambda}(x)\|_{\lambda,\lambda'\in\Lambda^0}.$$

It follows from the definition that

(1) For any pair $\lambda, \lambda' \in \Lambda^0$ the function $F_{\lambda',\lambda}(x)$ is defined and regular in a neighborhood of the identity matrix e, and also

$$F_{\lambda',\lambda}(e) = \delta_{\lambda'\lambda} \qquad (35)$$

(i.e., $F_{\lambda,\lambda}(e) = 1$, and $F_{\lambda',\lambda}(e) = 0$ for $\lambda' \neq \lambda$).

(2) For any elements x, y from a neighborhood of the identity matrix e the following condition holds:

$$\sum_{\lambda\in\Lambda^0} F_{\lambda^1,\lambda}(x)F_{\lambda,\lambda^2}(y) = F_{\lambda^1,\lambda^2}(xy) \qquad (36)$$

for any $\lambda^1, \lambda^2 \in \Lambda^0$.

(3) Restrictions of the functions $F_{\lambda',\lambda}(x)$ to the subgroups Z^- and Z^+ of lower and upper unipotent matrices are polynomials in elements of the matrices $x \in Z^{\pm}$, and the equality (36) holds for any elements of these subgroups.

(4) If the pattern λ^0 is strongly admissible, then the functions

$$\frac{c_\epsilon(\lambda')}{c_\epsilon(\lambda)} F^{\epsilon}_{\lambda',\lambda}(x)$$

do not depend on ϵ, where $c_\epsilon(\lambda)$ is defined by the equality (13).

Remark. By the initial definition, the functions $F_{\lambda',\lambda}(x)$ are defined for the subset of pairs (λ', λ) such that $\lambda - \lambda^0 \in L$ and $\lambda' - \lambda^0 \in L$, where λ^0 is an arbitrary admissible pattern. Because λ^0 is arbitrary, the functions $F_{\lambda',\lambda}(x)$ can be regarded as functions on the set of all pairs (λ', λ) of admissible patterns such that $\lambda' - \lambda \in L$.

4. General Hypergeometric Functions on the Group GL(n, $\mathbb{C}$)

4.1. RELATIONS FOR THE OPERATORS $T(x) = T^{\epsilon}_{\lambda^0}(x)$

THEOREM 3. *The operators $T(x) = T^{\epsilon}_{\lambda^0}(x)$ of the local representation of* $\mathrm{GL}(n, \mathbb{C})$ *in the space* $L(\lambda^0)$ *satisfy the following relations:*

$$\sum_{j=1}^{n} x_{i'j} \frac{\partial}{\partial x_{ij}} T(x) = E_{ii'} T(x), \qquad \sum_{i=1}^{n} x_{ij} \frac{\partial}{\partial x_{ij'}} T(x) = T(x) E_{jj'}, \tag{37}$$

where E_{ij} are Lie operators defined by the formulas (8) *and* (9).

Proof. Use the equality

$$T(\mathrm{e}^{te_{ii'}} x) = T(\mathrm{e}^{te_{ii'}}) T(x) = \exp(tE_{ii'}) T(x).$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(T(\mathrm{e}^{te_{ii'}} x)\big)\Big|_{t=0} = \sum_{j=1}^{n} x_{i'j} \frac{\partial}{\partial x_{ij}} T(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \exp(tE_{ii'})\Big|_{t=0} = E_{ii'},$$

this implies the first of Equations (37). The second equation can be proved similarly. □

COROLLARY. *The matrix elements $F_{\lambda',\lambda}(x)$ of the operators $T(x)$ satisfy the same equalities* (37).

THEOREM 4. *Suppose an operator $T_0(x)$ on $L(\lambda^0)$ is defined and regular in a neighborhood of an arbitrary fixed matrix $x_0 \in \mathrm{GL}(n, \mathbb{C})$ and satisfies the equation*

$$\sum_{j=1}^{n} x_{i'j} \frac{\partial}{\partial x_{ij}} T_0(x) = E_{ii'} T_0(x) \quad \textit{for } i' = i \textit{ and } i' = i \pm 1. \tag{38}$$

Then in a neighborhood of x_0 *it has the form:*

$$T_0(x) = T(xx_0^{-1})T_0(x_0), \quad \textit{where } T = T^{\epsilon}_{\lambda^0}. \tag{39}$$

In particular, if T_0 *is defined and regular in a neighborhood of the identity matrix* e *and* $T_0(e) = \mathrm{id}$, *then* $T_0(x) = T(x)$.

Proof. Set $y = \mathrm{e}^{te_{ii'}}$, where $i' = i$ or $i' = i \pm 1$. Because of the condition,

$$\sum_{j=1}^{n} y_{i'j} \frac{\partial}{\partial y_{ij}} T_0(y) = E_{ii'} T(y).$$

On the other hand,

$$\sum_{j=1}^{n} y_{i'j} \frac{\partial}{\partial y_{ij}} T_0(y) = \sum_{j=1}^{n} x_{i'j} \frac{\partial}{\partial x_{ij}} T_0(\mathrm{e}^{te_{ii'}} x) = \frac{\mathrm{d}}{\mathrm{d}t} T_1(t),$$

where $T_1(t) = T_0(\mathrm{e}^{te_{ii'}}x)$. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}T_1(t) = E_{ii'}T_1(t)$. Because $T_1(0) = T_0(x)$, we obtain:

$$T_0(\mathrm{e}^{te_{ii'}}x) = \exp(tE_{ii'})T_0(x) = T(\mathrm{e}^{te_{ii'}})T_0(x). \tag{40}$$

Because the elements $\mathrm{e}^{te_{ii'}}$ generate the group $\mathrm{GL}(n, \mathbb{C})$, it follows from the equality (40) for $x = e$ that:

$$T_0(gx) = T(g)T_0(x)$$

for matrices g from a neighborhood of the identity matrix. By substituting $g = xx_0^{-1}$ and $x = x_0$, we obtain the equality (39). □

4.2. GENERAL HYPERGEOMETRIC FUNCTIONS ON THE GROUP $\mathrm{GL}(n, \mathbb{C})$

By $\mathcal{L}$ denote the set of pairs (λ', λ) of admissible n-patterns such that $\lambda' - \lambda \in L$.

DEFINITION. The following system of differential-difference equations on the space of functions $f(\lambda', \lambda, x)$ on $\mathcal{L} \times \mathrm{GL}(n, \mathbb{C})$ is called the general hypergeometric system on the group $\mathrm{GL}(n, \mathbb{C})$:

$$\sum_{j=1}^{n} x_{ij} \frac{\partial}{\partial x_{ij}} f(\lambda', \lambda, x) = r_i(\lambda') f(\lambda', \lambda, x), \quad i = 1, \ldots, n; \tag{41}$$

$$\sum_{j=1}^{n} x_{ij} \frac{\partial}{\partial x_{i+1,j}} f(\lambda', \lambda, x) = \sum_{j=1}^{i} a^{\epsilon}_{ij}(\lambda' + e_{ij}) f(\lambda' + e_{ij}, \lambda, x),$$
$$i = 1, \ldots, n-1; \tag{42}$$

$$\sum_{j=1}^{n} x_{i+1,j} \frac{\partial}{\partial x_{ij}} f(\lambda', \lambda, x) = \sum_{j=1}^{i} b^{\epsilon}_{ij}(\lambda' - e_{ij}) f(\lambda' - e_{ij}, \lambda, x),$$
$$i = 1, \ldots, n-1; \tag{43}$$

here $a^{\epsilon}_{ij}(\lambda)$ and $b^{\epsilon}_{ij}(\lambda)$ are defined by the equalities (10).

Solutions of this system, analytic with respect to elements of the matrix x, are called general hypergeometric functions on $\mathrm{GL}(n, \mathbb{C})$.

Remark. The system (41)–(43) is redundant: it is sufficient to leave out any one of Equations (41).

It follows from Theorem 3 that the matrix elements $F^{\epsilon}_{\lambda',\lambda}(x)$ of the operators $T^{\epsilon}(x)$ of the local representation of $\mathrm{GL}(n, \mathbb{C})$ are general hypergeometric functions defined in a neighborhood of e.

THEOREM 5. *Suppose a general hypergeometric function $f(\lambda', \lambda, x)$ is regular with respect to x in a neighborhood of an arbitrary fixed matrix $x_0 \in \mathrm{GL}(n, \mathbb{C})$. Then $f(\lambda', \lambda, x)$ is uniquely determined by its values $f(\lambda', \lambda, x_0)$ in the point x_0 and has the following form in that neighborhood:*

$$f(\lambda', \lambda, x) = \sum_{l \in L} F^{\epsilon}_{\lambda',\lambda+l}(x x_0^{-1}) f(\lambda + l, \lambda, x_0), \tag{44}$$

where $F^{\epsilon}_{\lambda',\lambda}(x)$ are matrix elements of the operator $T^{\epsilon}(x)$ of the local representation of $\mathrm{GL}(n, \mathbb{C})$.

In particular, if a function $f(\lambda', \lambda, x)$ is regular in a neighborhood of e and $f(\lambda', \lambda, e) = \delta_{\lambda',\lambda}$, then $f(\lambda', \lambda, x) = F^{\epsilon}_{\lambda',\lambda}(x)$.

Proof. Let us restrict ourselves to regarding the subset Λ^0 of n-patterns λ such that $\lambda - \lambda^0 \in L$, where λ^0 is an arbitrary fixed admissible n-pattern.

Define the operator $T_0(x)$ in the space $L(\lambda^0)$, where x belongs to a neighborhood of x_0, by the equality

$$T_0(x)[\lambda] = \sum_{l \in L} f(\lambda + l, \lambda, x)[\lambda + l].$$

From the conditions of this theorem it follows that the operator T_0 satisfies the conditions of the Theorem 4. Hence because of that theorem $T_0(x) = T(x x_0^{-1}) T_0(x_0)$ for matrices x from a neighborhood of x_0, where $T(x) = T^{\epsilon}(x)$ is the operator of local representation of $\mathrm{GL}(n, \mathbb{C})$ in the space $L(\lambda^0)$. The obtained equality is equivalent to (44). □

5. Example: The Case $n = 2$

5.1. REPRESENTATIONS OF THE ALGEBRA $\mathfrak{gl}(2, \mathbb{C})$

All 2-patterns

$$\begin{pmatrix} \lambda_1 & & \lambda_2 \\ & \lambda & \end{pmatrix}$$

are admissible. The pattern is strongly admissible if $\lambda_1 - \lambda \notin \mathbb{Z}$ and $\lambda_2 - \lambda \notin \mathbb{Z}$.

The space $L(\lambda^0)$ has a basis of patterns such that $\lambda_1 = \lambda_1^0$, $\lambda_2 = \lambda_2^0$ and $\lambda - \lambda^0 \in \mathbb{Z}$. In the sequel for simplicity by $[\lambda]$ we denote the pattern from the basis of $L(\lambda^0)$ that has λ in the lowest row.

There are 4 (as many as sets ϵ) types of representations of $\mathfrak{gl}(2, \mathbb{C})$ in the space $L(\lambda^0)$.

The formulas for the operators E_{ii} are:

$$E_{11}[\lambda] = (\lambda + 1)[\lambda], \qquad E_{22}[\lambda] = (\lambda_1 + \lambda_2 - \lambda + 2)[\lambda].$$

The operators E_{21} and E_{12} for different ϵ are:

$$\begin{aligned}
&E_{21}^1[\lambda] = -[\lambda - 1], \qquad E_{12}^1[\lambda] = (\lambda_1 - \lambda)(\lambda_2 - \lambda)[\lambda + 1];\\
&E_{21}^2[\lambda] = -(\lambda_1 - \lambda + 1)[\lambda - 1], \qquad E_{12}^2[\lambda] = (\lambda_2 - \lambda)[\lambda + 1];\\
&E_{21}^3[\lambda] = -(\lambda_2 - \lambda + 1)[\lambda - 1], \qquad E_{12}^3[\lambda] = (\lambda_1 - \lambda)[\lambda + 1];\\
&E_{21}^4[\lambda] = (\lambda_1 - \lambda + 1)(\lambda_2 - \lambda + 1)[\lambda - 1], \qquad E_{12}^4[\lambda] = [\lambda + 1].
\end{aligned}$$

If the 2-pattern λ^0 is strongly admissible, then those 4 representations are irreducible and mutually equivalent. If λ^0 is not strongly admissible, then those representations are semi-reducible and mutually not equivalent. For example, if $\lambda_1^0 - \lambda^0 = k \in \mathbb{Z}$, then the subspace in $L(\lambda^0)$ generated by the patterns $[\lambda^0 + k - n]$, $n = 0, 1, \dots$ is invariant with respect to the operators of the first representation; the subspace generated by the patterns $[\lambda^0 + k + n]$, $n = 1, 2, \dots$ is invariant with respect to the operators of the second representation etc.

5.2. LOCAL REPRESENTATION OF THE GROUP GL(2, $\mathbb{C}$)

For definiteness let us restrict ourselves to the first of presented representations of the algebra $\mathfrak{gl}(2, \mathbb{C})$. The formulas for the operators $T(x)$ of the respective representation of the group GL(2, $\mathbb{C}$) are:

$$T\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}[\lambda] = t_1^{\lambda+1} t_2^{\lambda_1+\lambda_2-\lambda+2}[\lambda], \tag{45}$$

$$T\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}[\lambda] = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}[\lambda - n], \tag{46}$$

$$T\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}[\lambda] = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda_1 - \lambda + 1)\Gamma(\lambda_2 - \lambda + 1)}{\Gamma(\lambda_1 - \lambda - n + 1)\Gamma(\lambda_2 - \lambda - n + 1)} \frac{t^n}{n!}[\lambda + n]. \tag{47}$$

The expression for $T(x)$, where x is an arbitrary matrix from a neighborhood of the identity matrix, can be obtained from the Gauss decomposition:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix},$$

where $t_1 = \Delta/x_{22}$, $\Delta = \det x$; $t_2 = x_{22}$; $s_1 = x_{12}/x_{22}$; $s_2 = x_{21}x_{22}/\Delta$.

We have:

$$T(x)[\lambda] = \sum_{n=-\infty}^{\infty} F_{\lambda-n,\lambda}(x)[\lambda - n],$$

where

$$\begin{aligned} &F_{\lambda-n,\lambda}(x) \\ &\quad = t_1^{\lambda+1} t_2^{\lambda_1+\lambda_2-\lambda+2} \times \\ &\qquad \times \sum_{k=0}^{\infty} \frac{\Gamma(\lambda_1 - \lambda + k + 1)\Gamma(\lambda_2 - \lambda + k + 1)}{\Gamma(\lambda_1 - \lambda + n + 1)\Gamma(\lambda_2 - \lambda + n + 1)\Gamma(k - n + 1)k!} s_1^{k-n}(-s_2)^k. \end{aligned}$$

Up to a multiplication by a monomial from x_{ij} and $\det x$, the obtained series is the Gauss hyperegeometric function of $-s_1 s_2$; it converges in the domain $|s_1 s_2| < 1$.

Replacing t_1, t_2, s_1 and s_2 by their expressions in terms of elements of the matrix x, one can represent $F_{\lambda-n,\lambda}(x)$ in the following form:

$$\begin{aligned} F_{\lambda-n,\lambda}(x) = {} & x_{22}^{\lambda_1+\lambda_2-2\lambda+n+1} (\det x)^{\lambda+1} \times \\ & \times \sum_{k=0}^{\infty} \frac{\Gamma(\lambda_1 - \lambda + k + 1)\Gamma(\lambda_2 - \lambda + k + 1)}{\Gamma(\lambda_1 - \lambda + n + 1)\Gamma(\lambda_2 - \lambda + n + 1)} \frac{x_{12}^{k-n}}{\Gamma(k - n + 1)} \times \\ & \times \frac{(-x_{21})^k (\det x)^{-k}}{k!}. \end{aligned}$$

The function $F_{\lambda',\lambda}(x)$ satisfies the following differential-difference system of equations:

$$\begin{aligned} &\left(x_{i1} \frac{\partial}{\partial x_{i1}} + x_{i2} \frac{\partial}{\partial x_{i2}} \right) F_{\lambda',\lambda}(x) = r_i(\lambda') F_{\lambda',\lambda}(x), \quad i = 1, 2; \\ &\left(x_{11} \frac{\partial}{\partial x_{21}} + x_{12} \frac{\partial}{\partial x_{22}} \right) F_{\lambda',\lambda}(x) = -F_{\lambda'+1,\lambda}(x); \\ &\left(x_{21} \frac{\partial}{\partial x_{11}} + x_{22} \frac{\partial}{\partial x_{12}} \right) F_{\lambda',\lambda}(x) = (\lambda_1 - \lambda' + 1)(\lambda_2 - \lambda' + 1) F_{\lambda'-1,\lambda}(x). \end{aligned}$$

It is the only solution of this system that satisfies the two conditions: (1) $F_{\lambda',\lambda}(x)$ is regular with respect to x in a neighborhood of the identity matrix e and (2) $F_{\lambda',\lambda}(e) = \delta_{\lambda'\lambda}$.

6. Representations of Subgroups of Unipotent Matrices and General Beta-Functions

6.1. REPRESENTATIONS OF SUBGROUPS OF UNIPOTENT MATRICES

Consider the operators $T(x) = T^{\epsilon}(x)$ of the local representation of the group $\mathrm{GL}(n, \mathbb{C})$ and their matrix elements $F_{\lambda',\lambda}(x) = F^{\epsilon}_{\lambda',\lambda}(x)$. By $T^{\pm}(x)$ and $F^{\pm}_{\lambda',\lambda}(x)$

denote the restrictions of $T(x)$ and $F_{\lambda',\lambda}(x)$ to the subgroups Z^- and Z^+ of lower and upper unipotent matrices, respectively. Thus,

$$T^{\pm}(x)[\lambda] = \sum_{\lambda' \in \Lambda} F^{\pm}_{\lambda',\lambda}(x)[\lambda']. \tag{48}$$

The operators $T^{\pm}$ are defined on the set of all elements of the corresponding unipotent subgroups and define a representation of those subgroups in the space $L(\lambda^0)$, i.e.,

$$T^{\pm}(xy) = T^{\pm}(x)T^{\pm}(y) \quad \text{for any } x, y \in Z^{\pm}. \tag{49}$$

or equivalently

$$\begin{aligned} F^{-}_{\lambda^1,\lambda^2}(xy) &= \sum_{\lambda} F^{-}_{\lambda^1,\lambda}(x) F^{-}_{\lambda,\lambda^2}(y) \quad \text{for any } x, y \in Z^-; \\ F^{+}_{\lambda^1,\lambda^2}(xy) &= \sum_{\lambda} F^{+}_{\lambda^1,\lambda}(x) F^{+}_{\lambda,\lambda^2}(y) \quad \text{for any } x, y \in Z^+. \end{aligned} \tag{50}$$

It follows from the formulas for the operators $\exp(tE_{i+1,i})$ that the functions $F^{-}_{\lambda',\lambda}(x)$ are nonzero only for $\lambda' \leqslant \lambda$. Therefore in the first equality of (50) one can assume that $\lambda^1 \leqslant \lambda^2$ and summation is carried out only over the subset of n-patterns $\lambda \in \Lambda^0$ such that $\lambda^1 \leqslant \lambda \leqslant \lambda^2$. Similarly, the functions $F^{+}_{\lambda',\lambda}(x)$ are nonzero only for $\lambda' \geqslant \lambda$. Therefore in the second equality of (50) one can assume that $\lambda^1 \geqslant \lambda^2$ and summation is carried out only over the subset of n-patterns $\lambda \in \Lambda^0$ such that $\lambda^1 \geqslant \lambda \geqslant \lambda^2$. This implies that all sums (50) are finite.

6.2. THE OPERATORS $T_i^{\pm}(z_1, \ldots, z_i)$

Let us represent the operators $T^{\pm}(x)$ as products of operators of a simpler form. Namely, consider the commutative subgroups $Z_i^{\pm} \subset Z^{\pm}$ of elements $e + z_1 e_{i+1,1} + \cdots + z_i e_{i+1,i}$ and $e + z_1 e_{1,i+1} + \cdots + z_i e_{i,i+1}$, respectively. By $T_i^{\pm}(z) = T_i^{\pm}(z_1, .., z_i)$ denote the restrictions of the operators $T^{\pm}(x)$ to those subgroups, i.e.,

$$T_i^{-}(z_1, .., z_i) = T^{-}(e + z_1 e_{i+1,1} + \cdots + z_i e_{i+1,i}), \tag{51}$$

$$T_i^{+}(z_1, .., z_i) = T^{-}(e + z_1 e_{1,i+1} + \cdots + z_i e_{i,i+1}), \quad i = 1, \ldots, n-1. \tag{52}$$

In particular $T_1^{-}(z) = \exp(zE_{21})$, $T_1^{+}(z) = \exp(zE_{12})$. The following proposition is evident:

PROPOSITION 6.1. *The operators $T^{\pm}(x)$ on the subgroups $Z^{\pm}$ can be presented as products of the operators $T_i^{\pm}(z_1, .., z_i)$:*

$$T^{-}(x) = \prod_{i=2}^{n} T^{-}_{i-1}(x_{i1}, \ldots, x_{i,i-1}), \tag{53}$$

$$T^{+}(x) = \prod_{i=1}^{n} T^{+}_{i-1}(x_{1i}, \ldots, x_{i-1,i}). \tag{54}$$

Further, it is evident that

$$T_i^{\pm}(z_1 + z_1', \dots, z_i + z_i') = T_i^{\pm}(z_1, \dots, z_i) T_i^{\pm}(z_1', \dots, z_i') \tag{55}$$

and the recurrence relations hold:

$$\begin{aligned} &T_i^{-}(z_1, \dots, z_i) \\ &\quad = T_{i-1}^{-}(-z_1, \dots, -z_{i-1}) \exp(z_i E_{i+1,i}) T_{i-1}^{-}(z_1, \dots, z_{i-1}); \\ &T_i^{+}(z_1, \dots, z_i) \\ &\quad = T_{i-1}^{+}(-z_1, \dots, -z_{i-1}) \exp(z_i E_{i,i+1}) T_{i-1}^{+}(z_1, \dots, z_{i-1}). \end{aligned} \tag{56}$$

6.3. GENERAL BETA-FUNCTIONS

By $B_i^{\pm}(\lambda', \lambda | z_1, .., z_i)$ denote the matrix elements of the operators $T_i^{\pm}(z_1, \dots, z_i)$, i.e.,

$$T_i^{\pm}(z_1, \dots, z_i)[\lambda] = \sum_{\lambda' \in \Lambda} B_i^{\pm}(\lambda', \lambda | z_1, .., z_i)[\lambda']. \tag{57}$$

Because the operator $T_i^{-}(z_1, \dots, z_i)$ can be presented as a product of the operators $\exp(tE_{j+1,j})$, where $j \leqslant i$, summation in (57) for T^- is carried out over the subset of n-patterns λ' such that $\lambda' \leqslant \lambda$ and $\lambda_j' = \lambda_j$ for $j > i$. Similarly, summation in (57) for T^+ is carried out over the subset of n-patterns λ' such that $\lambda' \geqslant \lambda$ and $\lambda_j' = \lambda_j$ for $j > i$.

Therefore the equality (57) can be represented in the form:

$$T_i^{\pm}(z_1, \dots, z_i)[\lambda] = \sum_{l \in L^i} B_i^{\pm}(\lambda \pm l, \lambda | z_1, .., z_i)[\lambda \pm l],$$

where L^i is the set of integer-valued patterns $l \geqslant 0$ having $l_j = 0$ for $j > i$.

DEFINITION. The functions on the set of pairs $\lambda, \lambda' = \lambda \pm l$, where $l \in L^i$, defined by the equality

$$B_i^{\pm}(\lambda', \lambda) = B_i^{\pm}(\lambda', \lambda | 1, \dots, 1). \tag{58}$$

are called general beta-functions associated with the representations $T_i^{\pm}$.

Because $\lambda' = \lambda \pm l$, where $l \in L^i$, we shall write $B_i^{\pm}(l, \lambda | z_1, .., z_i)$ and $B_i^{\pm}(l, \lambda)$ instead of $B_i^{\pm}(\lambda \pm l, \lambda | z_1, .., z_i)$ and $B_i^{\pm}(\lambda \pm l^i, \lambda)$, respectively.

Consider the subset $L_+^i \subset L^i$ of n-patterns $l \in L^i$ having

$$k_j(l) \geqslant k_{j-1}(l) \quad \text{for } j = 1, \dots, i.$$

PROPOSITION 6.2. *The function $B_i^{\pm}(l, \lambda | z_1, \dots, z_i)$ is nonzero only for $l \in L_+^i$*

and

$$B_i^{\pm}(l,\lambda|z_1,\dots,z_i) = B_i^{\pm}(l,\lambda)\prod_{j=1}^{i} z_j^{\rho_j(l)} \quad \textit{for } l \in L_+^i, \tag{59}$$

where $\rho_j(l) = k_j(l) - k_{j-1}(l)$.

Proof. Because $B_i^-(\lambda', \lambda|z_1,\dots,z_i)$ is an analytic function of z_i, it is sufficient to prove (59) supposing that all z_i are sufficiently close to 1. Under that assumption the operator $T(\delta) = \prod_{j=1}^{i} \exp(\log z_j E_{jj})$ is defined, where $\delta = \mathrm{diag}(z_1,\dots,z_i, 1,\dots,1)$ and

$$T_i^-(z_1,\dots,z_i) = T(\delta^{-1})T_i^-(1,\dots,1)T(\delta). \tag{60}$$

It follows from the formulas (16) for $\exp(tE_{jj})$ that (60) is equivalent to the relation (59) for the matrix elements of the operator $T_i^-(z_1,\dots,z_i)$.

From the other hand, $B_i^-(l,\lambda|z_1,\dots,z_i)$ is an entire function of z_j. Hence $B_i^-(l,\lambda|z_1,\dots,z_i) = 0$ if $\rho_j(l) < 0$ for at least one j, i.e., if $l \notin L_+^i$. For B_i^+ the proof is similar. □

COROLLARY. *The matrix elements of the operators $T^{\pm}(x)$ of the representations of the unipotent subgroups Z^- and Z^+ have the form:*

$$\begin{aligned} F_{\lambda',\lambda}^-(x) &= \textstyle\sum_{l^1,\dots,l^{n-1}} \left(c_{\lambda',\lambda}^-(l^1,\dots,l^{n-1}) \prod\limits_{i>j} x_{ij}^{\rho_j(l^{i-1})} \right), \\ F_{\lambda',\lambda}^+(x) &= \textstyle\sum_{l^1,\dots,l^{n-1}} \left(c_{\lambda',\lambda}^+(l^1,\dots,l^{n-1}) \prod\limits_{i<j} x_{ij}^{\rho_i(l^{j-1})} \right), \end{aligned} \tag{61}$$

where

$$c_{\lambda',\lambda}^{\pm}(l^1,\dots,l^{n-1}) = \prod_{i=1}^{n-1} B_i^{\pm}\left(l^i, \lambda' \mp \sum_{k=1}^{i} l^k\right). \tag{62}$$

Summation in (61) *is carried out over all $l^i \in L_+^i$ such that $l^1 + \dots + l^{n-1} = \pm(\lambda' - \lambda)$.*

6.4. RELATIONS FOR BETA-FUNCTIONS

The relations (55) and (56) for the operators $T_i^{\pm}$ are equivalent to the following relations for general beta-functions:

(1) Analogs of the binomial formula

$$\begin{aligned} &B_i^{\pm}(l,\lambda)\prod_{j=1}^{i}(z_j + z_j')^{\rho_j(l)} \\ &\quad = \sum_{l'+l''=l} B_i^{\pm}(l'', \lambda \pm l')B_i^{\pm}(l',\lambda)\prod_{j=1}^{i} z_j^{\rho_j(l'')} z_j'^{\rho_j(l')}. \end{aligned} \tag{63}$$

(Summation over the set of n-patterns $l', l'' \in L^i$ such that $l' + l'' = l$.)

(2) Recurrence relations:

$$B_i^{\pm}(\lambda^2, \lambda^1) = \sum_{\lambda', \lambda''} (-1)^{\lambda^1 - \lambda''} A^{\pm}(\lambda'', \lambda') B_{i-1}^{\pm}(\lambda^1, \lambda'') B_{i-1}^{\pm}(\lambda', \lambda^2), \tag{64}$$

where $A^{-}(\lambda'', \lambda')$ and $A^{+}(\lambda'', \lambda')$ are matrix elements of the group operators $\exp E_{i+1,i}$ and $\exp E_{i,i+1}$, respectively.

Note that the sums (63) and (64) are finite.

7. Representations of the Algebra $\mathfrak{gl}(\infty)$ and of the Group GL(∞) Associated with Infinite Gelfand–Tsetlin Patterns

Let us define the Lie algebra $\mathfrak{gl}(\infty)$ and the group GL(∞) as inductive limits of the sequences:

$$\mathfrak{gl}(1, \mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n, \mathbb{C}) \subset \cdots$$

and

$$\mathrm{GL}(1, \mathbb{C}) \subset \cdots \subset \mathrm{GL}(n, \mathbb{C}) \subset \cdots,$$

where the embeddings $\mathfrak{gl}(n, \mathbb{C}) \hookrightarrow \mathfrak{gl}(n+1, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}(n+1, \mathbb{C})$ are defined as follows:

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

respectively.

The above constructed representations of the algebras $\mathfrak{gl}(n, \mathbb{C})$ and local representations of the groups $\mathrm{GL}(n, \mathbb{C})$ can be naturally extended to representations of $\mathfrak{gl}(\infty)$ and local representations of GL(∞).

DEFINITION. An infinite sequence of rows $\lambda_1, \ldots, \lambda_n, \ldots$, numerated from the bottom, where $\lambda_n = (\lambda_{n1}, \ldots, \lambda_{nn})$, $\lambda_{ij} \in \mathbb{C}$ is called an infinite Gelfand–Tsetlin pattern.

The definitions of admissible and strongly admissible patterns are extended to the case of infinite patterns automatically.

By L denote the set of integer-valued infinite patterns with a finite number of nonzero rows.

With any admissible pattern λ^0 let us associate an infinite (continuous) family of representations of the algebra $\mathfrak{gl}(\infty)$. The representations are defined in the infinite-dimensional space $L(\lambda^0)$ with the basis consisting of all admissible patterns $\lambda = \lambda^0 + l$, $l \in L$.

Every representation of $\mathfrak{gl}(\infty)$ in the space $L(\lambda^0)$ is specified by an infinite set of numbers

$$\epsilon = \left\{\epsilon^i_{jk} \mid j \leqslant i, \quad k < i, \ 1 < i < \infty\right\},$$

where every ϵ^i_{jk} is equal to 0 or 1.

The operators of the representation $U = U^{\epsilon}_{\lambda^0}$, associated with an arbitrary fixed set ϵ, are defined by the formulas (8) and (9).

In Subsection 2.3 the properties of the representations of $\mathfrak{gl}(n, \mathbb{C})$, associated with strongly admissible patterns, were formulated. Those properties are correct for the algebra $\mathfrak{gl}(\infty)$, too. In particular, these representations are irreducible.

PROPOSITION 7.1. *If a pattern λ^0 is strongly admissible, then for different sets ϵ the representations $U^{\epsilon}_{\lambda^0}$ of $\mathfrak{gl}(\infty)$ in the space $L(\lambda^0)$ are mutually equivalent.*

Proof. Denote, see Subsection 2.2:

$$c^n_\epsilon(\lambda) = \prod_{i=2}^{n} \prod_{k=1}^{i-1} (\Gamma(\lambda_{ij} - \lambda_{i-1,k} + 1))^{\epsilon^i_{jk}}.$$

Set $\lambda = \lambda^0 + l$. If $l_i = 0$ for $i > n$, then set:

$$c_\epsilon(\lambda) = \frac{c^n_\epsilon(\lambda)}{c^n_\epsilon(\lambda^0)}.$$

$c_\epsilon(\lambda)$ is well defined, because

$$\frac{c^m_\epsilon(\lambda)}{c^m_\epsilon(\lambda^0)} = \frac{c^n_\epsilon(\lambda)}{c^n_\epsilon(\lambda^0)}$$

for any $m > n$.

By the condition of strong admissibility, the function $c_\epsilon(\lambda)$ is everywhere finite and nonzero. Consider the linear transform in $L(\lambda^0)$ mapping the basic vectors λ to the vectors $\lambda / c_\epsilon(\lambda)$.

Similarly to the case of the algebra $\mathfrak{gl}(n, \mathbb{C})$, it is easy to check that under this linear transform the operators $E^{\epsilon}_{i+1,i}$ and $E^{\epsilon}_{i,i+1}$, for arbitrary ϵ are transformed into the operators $E^0_{i+1,i}$ and $E^0_{i,i+1}$ associated with the set $\epsilon = 0$. $\square$

Let us show that if λ^0 is admissible but not strongly admissible, then for any set ϵ there exists a proper subspace in $L(\lambda^0)$ that is invariant with respect to the representation U^ϵ of $\mathfrak{gl}(\infty)$.

Let λ^0 be an arbitrary admissible pattern. If λ^0 is not strongly admissible, then there exists at least one pair of coordinates λ^0_{ij} and $\lambda^0_{i-1,k}$ such that $\lambda^0_{ij} - \lambda^0_{i-1,k} \in \mathbb{Z}$. Then $\lambda_{ij} - \lambda_{i-1,k} \in \mathbb{Z}$ for any pattern $\lambda \in \Lambda$, too.

Represent Λ as the disjunct union:

$$\Lambda = \Lambda^0_{ijk} \sqcup \Lambda^1_{ijk},$$

where $\Lambda^0_{ijk} = \{\lambda \in \Lambda \mid \lambda_{ij} - \lambda_{i-1,k} \geqslant 0\}$, $\Lambda^1_{ijk} = \{\lambda \in \Lambda \mid \lambda_{i-1,k} - \lambda_{ij} \geqslant 1\}$. By L^0_{ijk} and L^1_{ijk} denote the linear subspaces in $L(\lambda^0)$ generated by the patterns $\lambda \in \Lambda^0_{ijk}$ and $\lambda \in \Lambda^1_{ijk}$, respectively. We have $L(\lambda^0) = L^0_{ijk} \oplus L^1_{ijk}$.

PROPOSITION 7.2. *The subspaces L^0_{ijk} and L^1_{ijk} are invariant with respect to the representations U^ϵ of $\mathfrak{gl}(\infty)$ with $\epsilon^i_{jk} = 0$ and $\epsilon^i_{jk} = 1$, respectively.*

Proof. Note first of all that ith and $(i-1)$th rows are transformed only by the operators $E^\epsilon_{i+1,i}$, $E^\epsilon_{i,i-1}$, $E^\epsilon_{i,i+1}$, and $E^\epsilon_{i-1,i}$. Thus it is sufficient to check invariance with respect to these operators only.

First regard the case $\epsilon^i_{jk} = 0$. The space L^0_{ijk} is invariant with respect to the operators $E^\epsilon_{i,i-1}$ and $E^\epsilon_{i,i+1}$, because under the action of these operators on λ, the difference $\lambda_{ij} - \lambda_{i-1,k}$ can increase only. Let us prove that L^0_{ijk} is invariant with respect to the operators $E^\epsilon_{i+1,i}$ and $E^\epsilon_{i-1,i}$, too.

Suppose $\lambda \in L^0_{ijk}$. If $\lambda_{ij} - \lambda_{i-1,k} > 0$, then it is obvious that $E^\epsilon_{i+1,i}[\lambda]$ and $E^\epsilon_{i-1,i}[\lambda]$ belong to L^0_{ijk}. If $\lambda_{ij} = \lambda_{i-1,k}$, then it follows from the formulas (10) for the matrix elements of the operators $E^\epsilon_{i+1,i}$ and $E^\epsilon_{i-1,i}$ that $a^\epsilon_{ij}(\lambda) = b^\epsilon_{i-1,k}(\lambda) = 0$ for $\epsilon^i_{jk} = 0$. Thus the decompositions of $E^\epsilon_{i+1,i}[\lambda]$ and $E^\epsilon_{i-1,i}[\lambda]$ include patterns $\lambda \in \Lambda^0_{ijk}$ only.

Now let us regard the case $\epsilon^i_{jk} = 1$. The space L^1_{ijk} is invariant under the operators $E^\epsilon_{i+1,i}$ and $E^\epsilon_{i-1,i}$, because under the action of these operators on the patterns λ, the difference $\lambda_{i-1,k} - \lambda_{i,j}$ can increase only. Let us prove that L^1_{ijk} is invariant under $E^\epsilon_{i,i-1}$ and $E^\epsilon_{i,i+1}$, too.

Suppose $\lambda \in L^1_{ijk}$. If $\lambda_{i-1,k} - \lambda_{ij} > 1$, then obviously $E^\epsilon_{i,i-1}[\lambda]$ and $E^\epsilon_{i,i+1}[\lambda]$ belong to L^1_{ijk}. If $\lambda_{i-1,k} - \lambda_{i-1,k} = 1$, then it follows from the formulas (10) for the matrix elements of $E^\epsilon_{i,i-1}$ and $E^\epsilon_{i,i+1}$ that $a^\epsilon_{i-1,k}(\lambda) = b^\epsilon_{ij}(\lambda) = 0$ for $\epsilon^i_{jk} = 1$. Thus the decompositions of $E^\epsilon_{i,i-1}[\lambda]$ and $E^\epsilon_{i,i+1}[\lambda]$ include patterns $\lambda \in L^1_{ijk}$ only. □

Appendix. Verification of the Commutation Relations for the Operators E_{ij}

Let us verify that the operators E_{ij} introduced in 2.2 satisfy the commutation relations

$$[E_{ij}, E_{i'j'}] = \delta_{ji'} E_{ij'} - \delta_{j'i} E_{i'j} \quad \text{for all } i, i', j, j' = 1, \dots, n. \tag{65}$$

It is sufficient to restrict ourselves to the case when the matrix elements a_{ij} and b_{ij} of the operators $E_{i+1,i}$ and $E_{i,i+1}$, respectively, are given by the formulas:

$$\begin{aligned} a_{ij}(\lambda) &= (-1)^i \frac{\prod_{k=1}^{i-1} (\lambda_{ij} - \lambda_{i-1,k})}{\prod_{k \neq j} (\lambda_{ij} - \lambda_{ik})}, \\ b_{ij}(\lambda) &= (-1)^{i+1} \frac{\prod_{k=1}^{i+1} (\lambda_{ij} - \lambda_{i+1,k})}{\prod_{k \neq j} (\lambda_{ij} - \lambda_{ik})} \end{aligned} \tag{66}$$

for $i > 1$; $a_{11}(\lambda) = 1$, $b_{11}(\lambda) = -(\lambda_{11} - \lambda_{21})(\lambda_{11} - \lambda_{22})$.

A.1. RELATIONS (65) FOR $i' = j' = k$

In the case $|i - j| = 1$ these relations follow immediately from the definition of the operators $E_{i+1,i}$, $E_{i,i+1}$, and E_{kk}. For arbitrary i and j they can be obtained by induction on $|i - j|$.

A.2. RELATIONS (65) FOR $i > j$ AND $i' > j'$

Because the matrix elements of the operators $E_{i+1,i}$ depend only on the rows λ_i and λ_{i-1}, we have

$$[E_{i+1,i}, E_{j+1,j}] = 0 \quad \text{for } |i - j| > 1.$$

The equality

$$[E_{i+1,i}, E_{i,i-1}] = E_{i+1,i-1}$$

follows from the definition of the operator $E_{i+1,i-1}$.

By induction on $|i - j|$ and $|i' - j'|$, the relations (65) for arbitrary $i > j$ and $i' > j'$ are reduced to the above relations and to the relations

$$\begin{aligned} &[E_{i+1,i-1}, E_{i+1,i}] \equiv [[E_{i+1,i}, E_{i,i-1}], E_{i+1,i}] = 0, \\ &[E_{i+1,i-1}, E_{i,i-1}] \equiv [[E_{i+1,i}, E_{i,i-1}], E_{i,i-1}] = 0. \end{aligned} \tag{67}$$

Checking the latter relations follows.

A.3. DEDUCTION OF THE RELATION (67)

By definition, $E_{i+1,i-1}$ is equal to $[E_{i+1,i}, E_{i,i-1}]$. Thus, taking into account that $a_{i-1,j}(\lambda - e_{ik}) = a_{i-1,j}(\lambda)$, we have:

$$E_{i+1,i-1}[\lambda] = \sum_{j,k} a_{i-1,j}(\lambda)\big(a_{ik}(\lambda - e_{i-1,j}) - a_{ik}(\lambda)\big)[\lambda - e_{i-1,j} - e_{ik}].$$

From this we get the following:

$$\begin{aligned} &[E_{i+1,i-1}, E_{i+1,i}][\lambda] \\ &\quad = \sum_{j,k,l} a_{i-1,j}(\lambda)\big(a_{il}(\lambda)a'_{ik}(\lambda - e_{il}) - a_{il}(\lambda - e_{i-1,j} - e_{ik})a'_{ik}(\lambda)\big) \times \\ &\qquad \times [\lambda - e_{i-1,j} - e_{i,l} - e_{ik}], \end{aligned} \tag{68}$$

where

$$a'_{ik}(\lambda) = \frac{a_{ik}(\lambda)}{\lambda_{ik} - \lambda_{i-1,j}};$$

$$\begin{aligned}
&[E_{i+1,i-1}, E_{i,i-1}][\lambda] \\
&\quad = \sum_{j,k,l}\big(a_{i-1,l}(\lambda)a_{i-1,j}(\lambda - e_{i-1,l}) \times \\
&\qquad\qquad \times \big(a_{ik}(\lambda - e_{i-1,j} - e_{i-1,l}) - a_{ik}(\lambda - e_{i-1,l})\big) - \\
&\qquad\qquad - a_{i-1,l}(\lambda - e_{i-1,j})a_{i-1,j}(\lambda)(a_{ik}(\lambda - e_{i-1,j}) - \\
&\qquad\qquad - a_{ik}(\lambda))\big)[\lambda - e_{i-1,j} - e_{ik}].
\end{aligned} \tag{69}$$

Let us prove the equality $[E_{i+1,i-1}, E_{i+1,i}] = 0$. Because of (68), it is sufficient to check that for any j the function

$$u_{kl}(\lambda) = a_{il}(\lambda)a'_{ik}(\lambda - e_{il}) - a_{il}(\lambda - e_{i-1,j} - e_{ik})a'_{ik}(\lambda)$$

is skew-symmetric with respect to k and l.

Since $a_{il}(\lambda) = (\lambda_{il} - \lambda_{i-1,j})a'_{il}(\lambda)$ and $a_{il}(\lambda - e_{i-1,l} - e_{il}) = (\lambda_{il} - \lambda_{i-1,j})a'_{il}(\lambda)$, it follows that

$$u_{kk}(\lambda) = (\lambda_{il} - \lambda_{i-1,j})(a'_{ik}(\lambda)a'_{ik}(\lambda - e_{ik}) - a'_{ik}(\lambda)a'_{ik}(\lambda - e_{ik}) = 0.$$

Further, for $k \neq l$ use the equality:

$$a'_{ik}(\lambda - e_{il}) = \frac{\lambda_{ik} - \lambda_{il}}{\lambda_{ik} - \lambda_{il} + 1}a'_{ik}(\lambda), \qquad a'_{il}(\lambda - e_{ik}) = \frac{\lambda_{il} - \lambda_{ik}}{\lambda_{il} - \lambda_{ik} + 1}a'_{il}(\lambda).$$

We obtain:

$$\begin{aligned}
u_{kl}(\lambda) &= (\lambda_{ik} - \lambda_{il})a'_{ik}(\lambda)a'_{il}(\lambda)\left(\frac{\lambda_{il} - \lambda_{i-1,j}}{\lambda_{ik} - \lambda_{[l]+1}} + \frac{\lambda_{il} - \lambda_{i-1,j} + 1}{\lambda_{il} - \lambda_{ik} + 1}\right) \\
&= \frac{\lambda_{il} + \lambda_{ik} - 2\lambda_{i-1,j} + 1}{1 - (\lambda_{il} - \lambda_{ik})^2}(\lambda_{ik} - \lambda_{il})a'_{ik}(\lambda)a'_{il}(\lambda).
\end{aligned}$$

From this it follows that $u_{kl}(\lambda) + u_{lk}(\lambda) = 0$.

Now let us prove the equality $[E_{i+1,i-1}, E_{i,i-1}] = 0$. Because of (69), it is sufficient to prove that for any k the function

$$\begin{aligned}
v_{jl}(\lambda) &= a_{i-1,l}(\lambda)a_{i-1,j}(\lambda - e_{i-1,l}) \times \\
&\quad \times(a_{ik}(\lambda - e_{i-1,j} - e_{i-1,l} - e_{i-1,l}) - a_{ik}(\lambda - e_{i-1,l})) - \\
&\quad - a_{i-1,l}(\lambda - e_{i-1,j})a_{i-1,j}(\lambda)(a_{ik}(\lambda - e_{i-1,j}) - a_{ik}(\lambda))
\end{aligned}$$

is skew-symmetric with respect to j and l.

Since

$$a_{ik}(\lambda - me_{i-1,j}) = (\lambda_{ik} - \lambda_{i-1,j} + m)a'_{ik}(\lambda),$$

we have for $j = l$:

$$v_{jj}(\lambda) = a_{i-1,j}(\lambda)a^2_{i-1,j}(\lambda - e_{i-1,j})a'_{ik}(\lambda)v'_j(\lambda),$$

where $v_l'(\lambda) = (\lambda_{ik} - \lambda_{i-1,j} + 2) - 2(\lambda_{ik} - \lambda_{i-1,j} + 1) + (\lambda_{ik} - \lambda_{i-1,j}) = 0$. Thus, $v_{jj}(\lambda) = 0$.

Further, for $j \neq l$ set:

$$a''_{ik}(\lambda) = (\lambda_{ik} - \lambda_{i-1,j})^{-1}(\lambda_{ik} - \lambda_{i-1,l})^{-1} a_{ik}(\lambda).$$

Then

$$\begin{aligned} a_{ik}(\lambda - e_{i-1,j}) &= (\lambda_{ik} - \lambda_{i-1,j} + 1)(\lambda_{ik} - \lambda_{i-1,l}) a''_{ik}(\lambda), \\ a_{ik}(\lambda - e_{i-1,j} - e_{i-1,l}) &= (\lambda_{ik} - \lambda_{i-1,j} + 1)(\lambda_{ik} - \lambda_{i-1,l} + 1) a''_{ik}(\lambda). \end{aligned}$$

Thus,

$$\begin{aligned} v_{jl}(\lambda) \ = \ & a''_{ik}(\lambda) a_{i-1,l}(\lambda) a_{i-1,j}(\lambda - e_{i-1,l})(\lambda_{ik} - \lambda_{i-1,l} + 1) \ - \\ & - a''_{ik}(\lambda) a_{i-1,l}(\lambda - e_{i-1,j}) a_{i-1,j}(\lambda)(\lambda_{ik} - \lambda_{i-1,l}). \end{aligned}$$

From this, since $a_{i-1,j}(\lambda - e_{i-1,l}) = \frac{\lambda_{i-1,j} - \lambda_{i-1,l}}{\lambda_{i-1,j} - \lambda_{i-1,l} + 1} a_{i-1,j}(\lambda)$, we get:

$$\begin{aligned} v_{jl}(\lambda) \ = \ & a''_{ik}(\lambda) a_{i-1,l}(\lambda) a_{i-1,j}(\lambda)(\lambda_{i-1,j} - \lambda_{i-1,l}) \ \times \\ & \times \left(\frac{\lambda_{ik} - \lambda_{i-1,l} + 1}{\lambda_{i-1,j} - \lambda_{i-1,l} + 1} + \frac{\lambda_{ik} - \lambda_{i-1,l}}{\lambda_{i-1,l} - \lambda_{i-1,j} + 1} \right) \\ = \ & -\frac{\lambda_{i-1,j} + \lambda_{i-1,l} - 2\lambda_{ik} + 1}{1 - (\lambda_{i-1,j} - \lambda_{i-1,l})^2} (\lambda_{i-1,j} - \lambda_{i-1,l}) a''_{ik}(\lambda) a_{i-1,l} \ \times \\ & \times (\lambda) a_{i-1,j}(\lambda). \end{aligned}$$

From this it follows that $v_{jl}(\lambda) + u_{lj}(\lambda) = 0$.

The relations (65) for $i < j$ and $i' < j'$ can be proved similarly.

A.4. RELATIONS (65) FOR $i > j$ AND $i' < j'$

Because the matrix elements of the operators $E_{i+1,i}$ and $E_{i,i+1}$ depend only on the rows λ_i, λ_{i-1} and the rows λ_i, λ_{i+1}, respectively, the following holds:

$$[E_{i+1,i}, E_{j,j+1}] = 0 \quad \text{for } |i - j| > 1 \quad \text{and } i = j - 1.$$

Let us prove that

$$[E_{i+1,i}, E_{i-1,i}] = 0.$$

Indeed,

$$\begin{aligned} [E_{i+1,i}, E_{i-1,i}][\lambda] = \sum_{k,l} & \big(b_{i-1,k}(\lambda) a_{il}(\lambda + e_{i-1,k}) - b_{i-1,k}(\lambda - e_{il}) a_{il}(\lambda)\big) \times \\ & \times [\lambda - e_{i-1,k} - e_{il}]. \end{aligned}$$

Since

$$a_{il}(\lambda + e_{i-1,k}) = \frac{\lambda_{il} - \lambda_{i-1,k} - 1}{\lambda_{il} - \lambda_{i-1,k}}, \qquad b_{ik}(\lambda - e_{il}) = \frac{\lambda_{i-1,k} - \lambda_{il} + 1}{\lambda_{i-1,k} - \lambda_{il}},$$

we have:

$$b_{i-1,k}(\lambda)a_{il}(\lambda + e_{i-1,k}) - b_{i-1,k}(\lambda - e_{il})a_{il}(\lambda) = 0.$$

The relation

$$[E_{i+1,i}, E_{i.i+1}] = E_{i+1,i+1} - E_{ii} \tag{70}$$

will be proved in the next subsection.

By induction on $|i - j|$ and $|i' - j'|$ the relations (65) for arbitrary $i > j$ and $i' < j'$ can be reduced to the presented relations.

A.5. PROOF OF THE RELATION (70)

Set

$$u_{kl}(\lambda) = b_{i,k}(\lambda)a_{il}(\lambda + e_{ik}) - b_{ik}(\lambda - e_{il})a_{il}(\lambda). \tag{71}$$

By definition of $E_{i+1,i}$ and $E_{i,i+1}$:

$$[E_{i+1,i}, E_{i,i+1}][\lambda] = \sum_{k,l} u_{kl}(\lambda)[\lambda + e_{ik} - e_{il}].$$

Thus the relation (70) is equivalent to the following one:

$$\begin{aligned} &u_{kl}(\lambda) = 0 \quad \text{for } k \neq l;\\ &\sum_{k=1}^{i} u_{kk} = \sum_{j=1}^{i+1} \lambda_{i+1,j} - 2\sum_{j=1}^{i} \lambda_{ij} + \sum_{j=1}^{i-1} \lambda_{i-1,j} + 1. \end{aligned} \tag{72}$$

The first relation is obvious. Indeed, for $k \neq l$ we have:

$$a_{il}(\lambda + e_{ik}) = \frac{\lambda_{il} - \lambda_{ik}}{\lambda_{il} - \lambda_{ik} - 1} a_{il}(\lambda), \qquad b_{ik}(\lambda - e_{il}) = \frac{\lambda_{ik} - \lambda_{il}}{\lambda_{ik} - \lambda_{il} + 1} a_{il}(\lambda).$$

From this it follows that $u_{kl} = 0$.

Now let us compute the sum $u(\lambda) = \sum_{k=1}^{i} u_{kk}(\lambda)$. By substituting the explicit expressions for $a_{il}(\lambda)$ and $b_{ik}(\lambda)$, we get:

$$\begin{aligned} u(\lambda) \ = \ \sum_{j=1}^{i} \Bigg(&- \frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij}) \prod_{k=1}^{i-1}(\lambda_{i-1,k} - \lambda_{ij} - 1)}{\prod_{k\neq j}(\lambda_{ik} - \lambda_{ij} - 1)(\lambda_{ik} - \lambda_{ij} - 1)} \ + \\ &+ \frac{\prod_{k=1}^{i+1}(\lambda_{i+1,k} - \lambda_{ij} + 1) \prod_{k=1}^{i-1}(\lambda_{i-1,k} - \lambda_{ij})}{\prod_{k\neq j}(\lambda_{ik} - \lambda_{ij})(\lambda_{ik} - \lambda_{ij} + 1)} \Bigg). \end{aligned}$$

Denote:

$$\begin{aligned} &\lambda_{i-1,k} = x_k, \quad k = 1, \ldots, i-1;\\ &\lambda_{i+i,k} = x_{i-1+k} - 1, \quad k = 1, \ldots, i+1; \qquad \lambda_{ik} = y_k, \quad k = 1, \ldots, i. \end{aligned}$$

In this notation the relation (72) takes the form:

$$\sum_{j=1}^{i}\left(\frac{\prod_{k=1}^{2i}(x_i - y_j)}{\prod_{k\neq j}(y_k - y_j)(y_k - y_j + 1)} - \frac{\prod_{k=1}^{2i}(x_k - y_j - 1)}{\prod_{k\neq j}(y_k - y_j)(y_k - y_j - 1)}\right)$$
$$= \sum_{k=1}^{2i} x_k - 2\sum_{j=1}^{i} y_j - i. \tag{73}$$

Finally the proof of the commutation relation (70) is reduced to a proof of the relation (73). Let us give a sketch of a proof of that relation.

It is easy to check that the function $u(x, y)$ from the left-hand side of the equality (73) is symmetric with respect to x_i and to y_j and has no singularities at the hyperplanes $y_i - y_j = 0$ and $y_i - y_j = \pm 1$, $i \neq j$. From this it follows that $u(x, y)$ is a symmetric polynomial in x_i and y_j, having degree not greater than 2.

Further, we have:

$$u(x, y) = x_1 - y_1 \quad \text{for } x_k = y_k,\ k = 2, \dots, i,\ x_{i+k} = y_k + 1,\ k = 1, \dots, i.$$

Hence $u(x, y)$ is a symmetric polynomial in x_i and in y_j of degree not greater than 1, i.e.,

$$u(x, y) = a\sum_{k=1}^{2i} x_k + b\sum_{k=1}^{i} y_k + c.$$

Substituting here $x_k = y_k$, $k = 2, \dots, i$ and $x_{i+k} = y_k + 1$, $k = 1, \dots, i$, we get:

$$x_1 - y_1 = a_1\left(x_1 - y_1 + 2\sum_{k=1}^{i} y_k + i\right) + b\sum_{k=1}^{i} y_k + c.$$

Thus, $a = 1$, $b = -2$, and $c = -i$, as had to be proved.

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