NONCOMMUTATIVE COMPLETE INTERSECTIONS

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Abstract. Several generalizations of a commutative ring that is a graded complete intersection are proposed for a noncommutative graded k-algebra; these notions are justified by examples from noncommutative invariant theory.

0. Introduction

Bass has noted that Gorenstein rings are ubiquitous [Ba]. Since the class of Gorenstein rings contains a wide variety of rings, it has proven useful to consider a tractable class of Gorenstein rings, and complete intersections fill that role for commutative rings. Similarly Artin-Schelter Gorenstein algebras, which are noncommutative generalizations of commutative Gorenstein rings, include a diverse collection of algebras, and finding a class of Artin-Schelter Gorenstein algebras that generalizes the class of commutative complete intersections is an open problem in noncommutative algebra. In this paper we use our work in noncommutative invariant theory to propose several notions of a noncommutative graded complete intersection. Moreover, the existence of noncommutative analogues of commutative complete intersection invariant subalgebras broadens our continuing project of establishing an invariant theory for finite groups acting on Artin-Schelter regular algebras that is parallel to classical invariant theory (see [KKZ1]–[KKZ5]).

When a finite group acts linearly on a commutative polynomial ring, the invariant subring is rarely a regular ring (the group must be a reflection group [ShT]), but Gorenstein rings of invariants are easily produced. For example, Watanabe's Theorem ([W1] or [Be, Theorem 4.6.2]) states that the invariant subring of $k[x_1, \cdots, x_n]$ under the natural action of a finite subgroup of $SL_n(k)$ is always Gorenstein, where k is a base field. In previous work we have shown there is a rich invariant theory for finite group (and even Hopf) actions on Artin-Schelter regular [Definition 3.1] (or AS regular, for short) algebras; for example there is a noncommutative version of Watanabe's Theorem [JZ, Theorem 3.3], providing conditions when the invariant subring is AS Gorenstein.

Cassidy and Vancliff defined a factor ring S/I of $S = k_{(q_{ij})}[x_1, \ldots, x_n]$, a skew polynomial ring, to be a complete intersection if I is generated by a regular sequence of length n in S (hence S/I is a finite dimensional algebra) [CV, Definition 3.7], and in [Va] Vancliff considered extending this definition to graded skew Clifford algebras. A few examples of noncommutative (or quantum) complete intersections have been constructed and studied along the line of factoring out a regular sequence of elements [BE, BO, Op]. Further, different kinds of generalizations of a commutative complete intersection have been proposed during the last fifteen years

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[Go, EG, BG, GHS]. Recent work on noncommutative (or twisted) matrix factorizations [CCKM], derived representation schemes [BFR], noncommutative versions of support varieties and finite generation of the cohomology ring of a Hopf algebra (ideas similar to [BWi, NWi]), as well as noncommutative crepant resolutions of commutative schemes [Da], advocate for a better understanding of noncommutative complete intersections. A satisfactory definition will have positive impact on several research areas.

In the commutative graded case a connected graded algebra A is called a *complete intersection* if one of the following four equivalent conditions holds [Lemma 1.7]

- (cci′) $A \cong k[x_1, \dots, x_d]/(\Omega_1, \cdots, \Omega_n)$, where $\{\Omega_1, \dots, \Omega_n\}$ is a regular sequence of homogeneous elements in $k[x_1, \dots, x_d]$ with $\deg x_i > 0$.
- (cci) $A \cong C/(\Omega_1, \cdots, \Omega_n)$, where C is a noetherian AS regular algebra and $\{\Omega_1, \dots, \Omega_n\}$ is a regular sequence of normalizing homogeneous elements in C.
- (gci) The Ext-algebra $E(A) := \bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_A(k, k)$ of A has finite Gelfand-Kirillov dimension.
- (nci) The Ext-algebra $E(A)$ is noetherian.

We call such an algebra A a *commutative complete intersection*. In the noncommutative case, unfortunately, these four conditions are not all equivalent, nor does (gci) or (nci) force A to be Gorenstein (Example 6.3), making it unclear which property to use as the proper generalization of a commutative complete intersection. A direct generalization to the noncommutative case is condition (cci) which involves considering regular sequences in *any* AS regular algebra (in the commutative case the only AS regular algebras are the polynomial algebras). Several researchers, including those whom we have mentioned earlier, have taken an approach that uses regular sequences. Though the condition (cci) seems to be a good definition of a noncommutative complete intersection, there are very few tools available to work with condition (cci) except for explicit construction and computation, and it is not easy to show condition (cci) fails since one needs to consider regular sequences in *any* AS regular algebra.

We consider both conditions (gci) and (nci) as possible definitions of a noncommutative complete intersection. One advantage of this approach is that it covers a large class of examples coming from noncommutative algebraic geometry and noncommutative ring theory. For example, let R be any noetherian Koszul algebra of finite Gelfand-Kirillov dimension (or GK-dimension, for short), then the Koszul dual $A := R^!$ satisfies both (gci) and (nci), since the Ext-algebra $E(A)$ of A is $A^! = R^{!!} = R$, which is noetherian of finite GK-dimension. We provide some information about the relationship between these different notions of noncommutative complete intersection in the following theorem.

Theorem 0.1. [Theorem 1.12(a)] *Let A be a connected graded noncommutative algebra. If A satisfies (cci), then it satisfies (gci).*

Example 6.3 shows that even both (gci) and (nci) do not imply (cci), and Example 6.2 shows that (gci) does not imply (nci).

We call an algebra A *cyclotomic* if its Hilbert series is a quotient of integral polynomials, all of whose roots are roots of unity. If A satisfies (cci) (including the case that A is a commutative complete intersection) then it is necessary (but not sufficient) for A to be cyclotomic. In the noncommutative case, under reasonable hypotheses, properties (gci) and (nci) also satisfy this necessary condition.

Theorem 0.2. [Theorem 1.12(b, c)] *If A satisfies (gci) or (nci), and if the Hilbert series of A is a rational function $p(t)/q(t)$ for some coprime integral polynomials $p(t), q(t) \in \mathbb{Z}[t]$ with $p(0) = q(0) = 1$, then A is cyclotomic.*

In Section 2 we use the cyclotomic condition to show that certain Veronese subrings are AS Gorenstein algebras that are not complete intersections in terms of any of our generalizations.

Many interesting examples arise from the noncommutative invariant theory of AS regular algebras under finite group actions, and the following question was one motivation for our work. Let $\mathrm{Aut}(A)$ denote the group of graded algebra automorphisms of A.

Question 0.3. Let A be a (noncommutative) noetherian connected graded AS regular algebra and G be a finite subgroup of $\mathrm{Aut}(A)$. Under what conditions is the invariant subring A^G a complete intersection?

When A is a commutative polynomial ring over $\mathbb{C}$, Question 0.3 was answered by Gordeev [G2] and Nakajima [N3, N4] independently. A very important tool in the classification of groups G, such that $k[x_1, \cdots, x_n]^G$ is a complete intersection, is a result of Kac and Watanabe [KW] and Gordeev [G1] independently; they prove that if G is a finite subgroup of $GL_n(k)$ and $k[x_1, \cdots, x_n]^G$ is a complete intersection, then G is generated by bireflections (i.e. elements $g \in G$ such that $\mathrm{rank}(g - I) = 2$). This leads us to the next natural question: what is a bireflection in the noncommutative setting? We seek notions of complete intersection and bireflection that lead to a generalization of the result of Kac-Watanabe-Gordeev.

In Section 3 we give the basic definitions in our noncommutative invariant theory. In the commutative case, we have [St3, p. 506]

$$\begin{array}{ccccc} \text{regular} & \Longrightarrow & \text{hypersurface} & \Longrightarrow & \text{complete intersection} \\ & \Longrightarrow & \text{Gorenstein} & \Longrightarrow & \text{Cohen-Macaulay.} \end{array}$$

In contrast to the commutative case, neither (gci) nor (nci) implies Gorenstein in the noncommutative case [Example 6.3]. On the other hand, when we work with invariant subrings A^G, we have a satisfactory situation. An algebra A is called *cyclotomic Gorenstein* if (i) A is AS Gorenstein and (ii) A is cyclotomic.

Theorem 0.4. [Theorem 3.4] *Assume that k is of characteristic zero. Let R be a connected graded noetherian Auslander regular algebra, and G be a finite subgroup of* $\mathrm{Aut}(R)$. *If $A = R^G$ satisfies any of (cci), (gci) or (nci) then R^G is cyclotomic Gorenstein.*

It follows from Theorem 0.4 that when A^G is not cyclotomic and AS Gorenstein we know that it does not satisfy any of our conditions for generalizations of complete intersections.

Our generalization of a reflection of a symmetric algebra to the notion of a quasi-reflection of a noncommutative AS regular algebra [KKZ1, Definition 2.2], suggests that a reasonable definition of a noncommutative bireflection is following: a graded algebra automorphism g of a noetherian AS regular algebra A of GK-dimension n

is called a *quasi-bireflection* if its trace has the form:

$$Tr_A(g,t) := \sum_{i=0}^{\infty} Tr(g|_{A_i})t^i = \frac{1}{(1-t)^{n-2}q(t)}$$

where $q(1) \neq 0$. As in the case of quasi-reflections, there are "mystic quasi-bireflections" (quasi-bireflections that are not bireflections of A_1) [Example 6.6]. We prove the following theorem in Section 4 by reducing to the commutative case.

Theorem 0.5. *Let A be the quantum affine space $k_q[x_1, \cdots, x_d]$ such that $q \neq \pm 1$. Let G be a finite subgroup of* $\mathrm{Aut}(A)$. *If A^G satisfies (gci), then G is generated by quasi-bireflections.*

Some partial results about $k_{-1}[x_1, \cdots, x_d]^G$ are given in [KKZ5], including the following theorem, that can be regarded as a converse of the Kac-Watanabe-Gordeev Theorem.

Theorem 0.6. [KKZ5, Theorem 5.4] *Assume that k is of characteristic zero. Let $A = k_{-1}[x_1, \cdots, x_n]$ and G be a subgroup of the permutation group S_n (acting naturally on A). If G is generated by quasi-bireflections, then A^G satisfies (cci).*

Further evidence that the definition of quasi-bireflection is a useful generalization comes from the study of invariants of noetherian graded down-up algebras, a class of AS regular algebras of global dimension 3. Let Q_3 be the finite subgroup $GL_2(k)$ generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where ϵ is a primitive nth root of unity for an odd integer $n \geq 3$. Let Q_4 be the finite subgroup $GL_2(k)$ generated by

$$\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$$

where ϵ is a primitive $4n$th root of unity.

Theorem 0.7. [KKZ4, Theorem 0.3 and Table 4] *Assume that k is of characteristic zero. Let A be a connected graded noetherian down-up algebra and G be a finite subgroup of* $\mathrm{Aut}(A)$.

(a) *If A^G satisfies (gci), then G is generated by quasi-bireflections.*
(b) *The following conditions are equivalent:*
   (i) *A^G satisfies (gci).*
   (ii) *A^G is cyclotomic Gorenstein.*
   (iii) *G is a finite subgroup of $GL_2(k)$ such that (iiia)* $\det g = 1$ *or* -1 *for each $g \in G$ and (iiib) G is not conjugate to Q_3 and Q_4 defined as above.*

Theorem 0.7 suggests that the "cyclotomic Gorenstein" property is closely related to our notions of complete intersection for noncommutative algebras of the form A^G where A is a noetherian AS regular algebra and $G \subset \mathrm{Aut}(A)$. For the generic 3-dimensional Sklyanin algebra $A := A(a, b, c)$, we also show that A^G is cyclotomic Gorenstein if and only if G is generated by quasi-bireflections [Theorem 5.5].

Section 6 contains examples and questions for further study.

1. Noncommutative versions of complete intersection

Throughout let k be a commutative base field of characteristic zero. Vector spaces, algebras and morphisms are over k.

In this section we propose several different, but closely related, definitions of a noncommutative complete intersection graded k-algebra. We begin by recalling some definitions that will be used in our work. The Hilbert series of an $\mathbb{N}$-graded A-module $M = \bigoplus_{i=0}^{\infty} M_i$ is defined to be the formal power series in t

$$H_M(t) = \sum_{i=0}^{\infty} (\dim M_i) t^i.$$

Definition 1.1. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a connected graded algebra.

(a) A has *exponential growth* if $\overline{\lim}_i (\dim A_i)^{1/i} > 1$.
(b) A has *sub-exponential growth* if $\overline{\lim}_i (\dim A_i)^{1/i} \leq 1$.
(c) The *Gelfand-Kirillov dimension* (or *GK-dimension*, for short) of A is

$$\operatorname{GKdim} A = \overline{\lim}_n \log_n (\dim \bigoplus_{i=0}^{n} A_i).$$

We refer to [KL] for the original definition of GK-dimension and its basic properties. Definition 1.1(c) agrees with the original definition given in [KL] when A is finitely generated, but it may differ otherwise.

Next are the proposed definitions of a noncommutative complete intersection, beginning with the definition of AS regularity; a commutative AS regular algebra is a commutative polynomial ring, and hence it is reasonable to replace a commutative polynomial ring by a noetherian AS regular algebra in our notions of complete intersection. By "an element in a graded algebra" we will mean a homogeneous element. A set of (homogeneous) elements $\{\Omega_1, \cdots, \Omega_n\}$ in a graded algebra A is called *a regular sequence of normalizing elements* if, for each i, the image of Ω_i in $A/(\Omega_1, \cdots, \Omega_{i-1})$ is regular (i.e. a non-zero-divisor in $A/(\Omega_1, \cdots, \Omega_{i-1})$) and normal in $A/(\Omega_1, \cdots, \Omega_{i-1})$ (i.e. if $\overline{\Omega_i}$ denotes the image of Ω_i in $\overline{A} = A/(\Omega_1, \cdots, \Omega_{i-1})$, then $\overline{\Omega_i}\, \overline{A} = \overline{A}\, \overline{\Omega_i}$).

Definition 1.2. Let A be a connected graded algebra. We call A *Artin-Schelter Gorenstein* (or *AS Gorenstein* for short) if the following conditions hold:

(a) A has injective dimension $d < \infty$ on the left and on the right,
(b) $\operatorname{Ext}^i_A({}_A k, {}_A A) = \operatorname{Ext}^i_A(k_A, A_A) = 0$ for all $i \neq d$, and
(c) $\operatorname{Ext}^d_A({}_A k, {}_A A) \cong \operatorname{Ext}^d_A(k_A, A_A) \cong k(l)$ for some l (where l is called the *AS index*).

If, in addition,

(d) A has finite global dimension, and
(e) A has finite Gelfand-Kirillov dimension,

then A is called *Artin-Schelter regular* (or *AS regular* for short) of dimension d.

Definition 1.3. Let A be a connected graded finitely generated algebra.

(a) We say A is a *classical complete intersection* (or *cci*, for short) if there is a connected graded noetherian AS regular algebra C and a regular sequence of normalizing elements $\{\Omega_1, \cdots, \Omega_n\}$ of positive degree such that $A \cong C/(\Omega_1, \cdots, \Omega_n)$.

(b) The *cci number* of A is defined to be

$$cci(A) = \min\{n \mid A \cong C/(\Omega_1, \cdots, \Omega_n) \text{ as in part (a)}\}.$$

(c) We say A is a *hypersurface* if $cci(A) \leq 1$.

Let $E(A)$ denote the Ext-algebra $\mathrm{Ext}^*_A(k,k)$ of A. It is $\mathbb{Z}^2$-graded and can be viewed as a connected graded algebra by using either the cohomological degree or the Adams grading (i.e. the grading inherited from A).

Definition 1.4. Let A be a connected graded finitely generated algebra.

(a) We say A is a *complete intersection of noetherian type* (or *nci*, for short) if the Ext-algebra $E(A)$ is a left and right noetherian ring.

(b) When A is a nci, the *nci number* of A is defined to be

$$nci(A) = \operatorname{Kdim} E(A)_{E(A)}$$

where Kdim is the Krull dimension (of a right module).

(c) We say A is an *n-hypersurface* if $nci(A) \leq 1$.

In the next definition, we consider $E(A) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_A(k,k)$ as a connected graded algebra by using the cohomological degree, so that the degree i piece is $E_i = \mathrm{Ext}^i_A(k,k)$.

Definition 1.5. Let A be a connected graded finitely generated algebra.

(a) We say A is a *complete intersection of growth type* (or *gci*, for short) if $E(A)$ has finite GK-dimension.

(b) The *gci number* of A is defined to be

$$gci(A) = \operatorname{GKdim}(E(A)).$$

(c) We say A is a *g-hypersurface* if $gci(A) \leq 1$.

(d) We say A is a *weak complete intersection of growth type* (or *wci*) if $E(A)$ has subexponential growth.

If $E(A)$ has finite GK-dimension then it has subexponential growth. Hence a gci must be a wci. We recall the χ-condition. Let k also denote the graded A-bimodule $A/A_{\geq 1}$.

Definition 1.6. [AZ, Definition 3.2] Let A be a noetherian, connected graded algebra. We say that the *χ-condition* holds for A if $\mathrm{Ext}^j_A(k,M)$ is finite dimensional over k for all finitely generated graded left (and right) A-modules M and all $j \geq 0$.

The following lemma is easy.

Lemma 1.7. *Let A be connected graded. Then the following are equivalent.*

(a) *A has finite global dimension and every term in the minimal free resolution of the trivial module k is finitely generated.*

(b) $\mathrm{Ext}^i_A(k,k)$ *is finite dimensional for all i and* $\mathrm{Ext}^i_A(k,k) = 0$ *for all $i \gg 0$.*

(c) $nci(A) = 0$.

(d) $gci(A) = 0$.

If, further, A is noetherian and satisfies the χ-condition, then the following are also equivalent to the above.

(e) *A is AS regular.*

(f) $cci(A) = 0$.

Proof. (a) $\iff$ (e): This follows from [Z1, Theorem 0.3]. The rest is straightforward. □

In the commutative case, the different versions of complete intersection are equivalent.

Lemma 1.8. *Let A be a connected graded finitely generated commutative algebra. Then the following are equivalent.*

(a) *$A \cong k[x_1, \dots, x_d]/(\Omega_1, \cdots, \Omega_n)$, where $\{\Omega_1, \dots, \Omega_n\}$ is a regular sequence of homogeneous elements in $k[x_1, \dots, x_d]$ with $\deg x_i > 0$.*
(b) *A is a cci.*
(c) *A is a nci.*
(d) *A is a gci.*

Proof. By [BH, FHT, FT, Gu, Ta], it is well-known that (a), (c) and (d) are equivalent for local commutative rings. The current graded version follows from [FT, Theorem IV.7] and [FHT, Theorem C].

(a) $\Longrightarrow$ (b): This is obvious.
(b) $\Longrightarrow$ (d): This follows from Theorem 1.11 which is to be proved later. □

In the noncommutative case, 1.8(a) and 1.8(b,c,d) are clearly not equivalent. Example 6.2 gives an algebra that is a gci, but is not a nci or a cci. Example 6.3 gives an algebra that is both a gci and a nci, but is not a cci.

Every commutative complete intersection ring $A := k[x_1, \cdots, x_d]/(\Omega_1, \cdots, \Omega_n)$ is Gorenstein and its Hilbert series is of the form
$$H_A(t) = \frac{\prod_{j=1}^{n}(1 - t^{\deg \Omega_j})}{\prod_{i=1}^{d}(1 - t^{\deg x_i})}.$$
We use a similar condition to define a related notion.

Definition 1.9. Let A be a connected graded finitely generated algebra.

(a) We say a rational function $p(t)/q(t)$, where $p(t)$ and $q(t)$ are coprime integral polynomials, is *cyclotomic* if all the roots of $p(t)$ and $q(t)$ are roots of unity.
(b) We say A is *cyclotomic* if its Hilbert series $H_A(t)$ is of the form $p(t)/q(t)$ and it is cyclotomic.
(c) The *cyc* number of A is defined to be
$$cyc(A) = \min\{m \mid H_A(t) = \frac{\prod_{i=1}^{m}(1 - t^{a_i})}{\prod_{j=1}^{n}(1 - t^{b_j})}\}.$$
(d) We say A is *cyclotomic Gorenstein* if the following conditions hold
   (i) A is AS Gorenstein;
   (ii) A is cyclotomic.

Let A be the commutative ring $k[x, y, z]/(xy)$ where $\deg x = \deg y = 1$ and $\deg z = 2$. Then the Hilbert series of A is $1/(1-t)^2$ and hence $cyc(A) = 0$. Then $E \cong k\langle x, y, z\rangle/(x^2, y^2, z^2, xz+zx, yz+zy)$ so that $cci(A) = nci(A) = gci(A) = 1$. It also is easy to construct algebras A such that $cyc(A) = 0$ and $cci(A) = nci(A) = gci(A)$ is any positive integer. The notion of cyclotomic Gorenstein is weaker than the notion of complete intersection, even in the commutative case, as seen in Example 6.4, an example given by Stanley. Thus we have the following implications in the commutative case.

(E1.9.1) $$\text{cci} \iff \text{gci} \iff \text{nci} \Longrightarrow \text{cyclotomic Gorenstein}.$$

Next we begin to relate these conditions in the noncommutative case.

Lemma 1.10. *Let A be connected graded and finitely generated. Suppose $E := E(A)$ is noetherian. Then $nci(A) = 1$ if and only if $gci(A) = 1$.*

Proof. If $gci(A) = \operatorname{GKdim} E = 1$ (and as E is noetherian), E is PI by a result of Small-Stafford-Warfield [SSW]. Hence $\operatorname{Kdim} E_E = \operatorname{GKdim} E = 1$ by [KL, Corollary 10.16]. Conversely, assume that $\operatorname{Kdim} E_E = nci(A) = 1$. By Lemma 1.7 it suffices to show that $\operatorname{GKdim} E \leq 1$. Since E is noetherian, we only need to show that $\operatorname{GKdim} E/\mathfrak{p} \leq 1$ for every prime ideal $\mathfrak{p}$ of E by [KL, Proposition 5.7]. When $R := E/(\mathfrak{p})$ is prime there is a homogeneous regular element f in R. Since $\operatorname{Kdim} R \leq \operatorname{Kdim} E = 1$, $\operatorname{Kdim} R/Rf = 0$. Thus R/Rf is finite dimensional. So R has GK-dimension 1. □

It is unknown if $nci(A) = 2$ is equivalent to $gci(A) = 2$ when E is noetherian.

We say a sequence of nonnegative numbers $\{b_n\}_{n=0}^{\infty}$ has subexponential growth (or the formal power series $\sum_{n=0}^{\infty} b_n t^n$ has subexponential growth) if $\overline{\lim}_i (b_i)^{1/i} \leq 1$. By [SZ, Lemma 1.1], $\{b_n\}_{n=0}^{\infty}$ has subexponential growth if and only if $\{c_n := \sum_{i=0}^{n} b_i\}_{n=0}^{\infty}$ has subexponential growth. Here is our first main theorem.

Theorem 1.11. *Suppose A is connected graded and finitely generated. Let Ω be a regular normal element in A of positive degree and let $B = A/(\Omega)$.*

(a) *$gci(A) \leq gci(B) \leq gci(A) + 1$, or equivalently,*
$$\operatorname{GKdim} \operatorname{Ext}_A^*(k,k) \leq \operatorname{GKdim} \operatorname{Ext}_B^*(k,k) \leq \operatorname{GKdim} \operatorname{Ext}_A^*(k,k) + 1.$$
(b) *A is a wci if and only if B is.*

(c) *If $cci(B) = 1$, then B is both a gci and a nci, and $gci(B) = nci(B) = 1$.*

Proof. (a) Let $a_n = \dim \operatorname{Tor}_n^A(k,k)$ and $b_n = \dim \operatorname{Tor}_n^B(k,k)$. Since $\operatorname{Tor}_n^A(k,k)^* \cong \operatorname{Ext}_A^n(k,k)$, we have have $a_n = \dim \operatorname{Ext}_A^n(k,k)$ and $b_n = \dim \operatorname{Ext}_B^n(k,k)$.

Since B is a factor ring of A, there is a graded version of the change-of-rings spectral sequence given in [Ro, Theorem 10.71]

$$\text{(E1.11.1)} \qquad {}^2E_{pq} := \operatorname{Tor}_p^B(k, \operatorname{Tor}_q^A(B,k)) \Longrightarrow_p \operatorname{Tor}_n^A(k,k).$$

Since $B = A/(\Omega)$ where Ω is a regular element in A, $\operatorname{Tor}_0^A(B,k) = k$, $\operatorname{Tor}_1^A(B,k) = k$ and $\operatorname{Tor}_i^A(B,k) = 0$ for $i > 1$. Hence the E^2-page of the spectral sequence (E1.11.1) has only two possibly non-zero rows; namely

$q = 0:$ $\operatorname{Tor}_p^B(k,k)$ for $p = 0,1,2,\cdots$, and

$q = 1:$ $\operatorname{Tor}_p^B(k,k)$ for $p = 0,1,2,\cdots$.

Since (E1.11.1) converges, we have $\operatorname{Tor}_0^B(k,k) = \operatorname{Tor}_0^A(k,k) = k$ and a long exact sequence

$$\begin{aligned} \cdots\cdots \longrightarrow \operatorname{Tor}_4^B(k,k) &\longrightarrow \operatorname{Tor}_2^B(k,k) \longrightarrow \\ \longrightarrow \operatorname{Tor}_3^A(k,k) \longrightarrow \operatorname{Tor}_3^B(k,k) &\longrightarrow \operatorname{Tor}_1^B(k,k) \longrightarrow \\ \longrightarrow \operatorname{Tor}_2^A(k,k) \longrightarrow \operatorname{Tor}_2^B(k,k) &\longrightarrow \operatorname{Tor}_0^B(k,k) \longrightarrow \\ \longrightarrow \operatorname{Tor}_1^A(k,k) \longrightarrow \operatorname{Tor}_1^B(k,k) &\longrightarrow 0. \end{aligned}$$

As a part of the above long exact sequence, we have, for each n, the following two exact sequences

$$\text{(E1.11.2)} \qquad \operatorname{Tor}_{n-1}^B(k,k) \longrightarrow \operatorname{Tor}_n^A(k,k) \longrightarrow \operatorname{Tor}_n^B(k,k),$$

and

$$\text{(E1.11.3)} \qquad \operatorname{Tor}^A_{n+2}(k,k) \longrightarrow \operatorname{Tor}^B_{n+2}(k,k) \longrightarrow \operatorname{Tor}^B_n(k,k) \longrightarrow \operatorname{Tor}^A_{n+1}(k,k).$$

Then (E1.11.2) implies that

$$\text{(E1.11.4)} \qquad a_n \leq b_n + b_{n-1}$$

for all n, and (E1.11.3) implies that

$$\text{(E1.11.5)} \qquad |b_{n+2} - b_n| \leq a_{n+2} + a_{n+1}$$

for all n. Using Definition 1.1(c), (E1.11.4) implies that

$$\operatorname{GKdim} \operatorname{Ext}^*_A(k,k) \leq \operatorname{GKdim} \operatorname{Ext}^*_B(k,k),$$

and (E1.11.5) implies that $b_n \leq \sum_{i=0}^n a_i$ for all n, whence

$$\operatorname{GKdim} \operatorname{Ext}^*_B(k,k) \leq \operatorname{GKdim} \operatorname{Ext}^*_A(k,k) + 1.$$

(b) If b_n has subexponential growth, then (E1.11.4) implies that a_n has subexponential growth. On the other hand, if a_n has subexponential growth, by [SZ, Lemma 1.1], $c_n := \sum_{i=0}^n a_i$ has subexponential growth. Now (E1.11.5) implies that b_n has subexponential growth. Thus the assertion in (b) follows.

(c) We prove this case assuming only that A has finite global dimension (we do not need to use the fact that A is AS regular).

Let $E^n(A) = \operatorname{Ext}^n_A(k,k)$ and $E^n(B) = \operatorname{Ext}^n_B(k,k)$ for every n. There is a spectral sequence for Ext-groups, similar to (E1.11.1), which gives rise to a long exact sequence, see [CS, Theorem 6.3 and the discussion afterwards],

$$\text{(E1.11.6)} \qquad E^{n-1}(A) \xrightarrow{\gamma} E^{n-2}(B) \xrightarrow{d_2} E^n(B) \xrightarrow{\phi^*} E^n(A).$$

When set $n = p$, this is the exact sequence [CS, (6.1)]. Then [CS, Lemma 6.1 and Theorem 6.3] show that (E1.11.6) can be interpreted as an exact triangle of right $E(B)$-modules. There is another sequence that can be interpreted as an exact triangle of left $E(B)$-modules, see [CS, (6.2)] and the discussion after [CS, Theorem 6.3]. For our purpose, it is enough to use that $d_2 : E(B) \to E(B)$ is a right $E(B)$-module homomorphism, which is special case of [CS, Lemma 6.1]. Therefore d_2 is a left multiplication l_x by some element $x \in E^2(B)$. Since A has finite global dimension $E^n(A) = 0$ for all $n \gg 0$. By the exact sequence (E1.11.6) and the fact that $E^n(A) = 0$ for all $n \gg 0$, we have that $E(B)/xE(B)$ is finite dimensional. Thus $E(B)$ is a finitely generated left $k[x]$-module. Therefore $E(B)$ is left noetherian of GK-dimension ≤ 1. By symmetry, $E(B)$ is right noetherian, too. Therefore $E(B)$ is left and right noetherian, and B is a nci.

By part (a) $gci(B) \leq 1$. By Lemma 1.8, $gci(B) \neq 0$. By Lemma 1.10, $gci(A) = nci(A) = 1$, and consequently A is both a n-hypersurface and a g-hypersurface. □

Under the hypotheses of Lemma 1.8 we have

$$\text{(E1.11.7)} \qquad cci(A) = 0 \iff gci(A) = 0 \iff nci(A) = 0.$$

By Lemma 1.10 and Theorem 1.11(c), if A is a nci, then

$$\text{(E1.11.8)} \qquad cci(A) = 1 \implies nci(A) = 1 \iff gci(A) = 1.$$

It is unknown if $gci(A) = 1$ implies that $nci(A) = 1$ (without assuming A is a nci a priori) and if $gci(A) = 1$ implies that $cci(A) = 1$. It would be nice to have examples to answer these basic questions.

Here is our second main result; part (a) is Theorem 0.1 and parts (b) and (c) are Theorem 0.2.

Theorem 1.12. *Let A be a connected graded finitely generated algebra.*

(a) *If A is a cci, then A is a gci, and $gci(A) \leq cci(A)$.*
(b) *If A is a gci or a nci, then A is a wci.*
(c) *Suppose that A is a noetherian wci and that the Hilbert series $H_A(t)$ is a rational function $p(t)/q(t)$ for some coprime integral polynomials $p(t), q(t) \in \mathbb{Z}[t]$ with $p(0) = q(0) = 1$. Then A is cyclotomic.*

Proof. (a) Write $A = C/(\Omega_1, \cdots, \Omega_n)$ for a noetherian AS regular algebra C and a regular sequence of normalizing elements $\{\Omega_1, \cdots, \Omega_n\}$ of positive degree. The assertion follows from Theorem 1.11(a) and the induction on n.

(b) Let $E = E(A)$. If A is a gci, then $\operatorname{GKdim} E < \infty$. This implies that E has subexponential growth. Hence A is a wci.

If A is a nci, then E is noetherian. By [SZ, Theorem 0.1], E has subexponential growth. Therefore A is a wci.

(c) Consider a minimal free resolution of the trivial module

$$\text{(E1.12.1)} \qquad \cdots \to P^i \to \cdots P^1 \to P^0 \to k \to 0$$

with $P^i = \bigoplus_{s=1}^{n_i} A(-d_s^i)$ for all $i \geq 0$. In particular, $P^0 = A$. The Hilbert series of P^i is

$$H_{P^i}(t) = \sum_{s=1}^{n_i} t^{d_s^i} H_A(t) = (\sum_{s=1}^{n_i} t^{d_s^i}) H_A(t).$$

Using the additive property of the k-dimension, the exact sequence (E1.12.1) implies that

$$1 = \sum_{i=0}^{\infty} (-1)^i H_{P^i}(t) = \sum_{i=0}^{\infty} (-1)^i (\sum_{s=1}^{n_i} t^{d_s^i}) H_A(t),$$

which implies that

$$H_A(t) = \frac{1}{\sum_{i=0}^{\infty} (-1)^i (\sum_{s=1}^{n_i} t^{d_s^i})} =: \frac{1}{Q(t)}.$$

Since $H_A(t) = p(t)/q(t)$, $Q(t) = q(t)/p(t)$. Since (E1.12.1) is a minimal resolution of k, we obtain that $\operatorname{Ext}_A^i(k,k) = \bigoplus_{s=1}^{n_i} k(d_s^i)$ for all i. Clearly the Ext-algebra $E := \operatorname{Ext}_A^*(k,k)$ is $\mathbb{Z} \times \mathbb{Z}$-graded and the every nonzero element in $k(d_s^i)$ has degree $(-d_s^i, i)$, where the first grading is the Adams grading coming from the grading of A and the second grading is the cohomological grading. Since (E1.12.1) is minimal, $d_s^i \geq i$ for all s. Reassigning degree by $\deg(k(d_s^i)) = (d_s^i, i)$, E becomes connected and $\mathbb{N} \times \mathbb{N}$-graded. This regrading is equivalent to letting $\operatorname{Ext}_A^i(k,k) = \bigoplus_{s=1}^{n_i} k(-d_s^i)$. In this case the $\mathbb{Z} \times \mathbb{Z}$-graded Hilbert series of E now is

$$H_E(t,u) = \sum_{i=0}^{\infty} (\sum_{s=1}^{n_i} t^{d_s^i}) u^i.$$

By definition of $Q(t)$, we have $Q(t) = H_E(t, -1)$. By hypotheses, A is a wci, so E has subexponential growth. Hence the Hilbert series $H_E(1,u)$ has subexponential growth. Write $H_E(1,u) = \sum_{i \geq 0} e_i u^i$, where $e_i = \dim \operatorname{Ext}_A^i(k,k)$, and let $F_n = \sum_{i=0}^n e_i$. By [SZ, Lemma 1.1(1)], $\{F_n\}_n$ has subexponential growth. Write $E(t,1) = \sum_{n \geq 0} f_n t^n$, where f_n is the number of d_s^i that are equal to n. Since each

$d_s^i \geq i$ for all $s = 1, \cdots, n_i$, $\sum_{i=0}^n f_n \leq F_n$ for all n. Since $\{F_n\}_n$ has subexponential growth, so does $\{f_n\}_n$. Since the absolute value of each coefficient of the power series $H_E(t, -1)$ is no more than (the absolute value of) each coefficient of the power series $H_E(t, 1)$, $H_E(t, -1)$ has subexponential growth. As noted before $Q(t) = H_E(t, -1)$, and we conclude that the coefficients of $Q(t)$ have subexponential growth. We have seen that $Q(t) = q(t)/p(t)$ and write $p(t) = \prod_{i=1}^d (1 - r_i t)$. By [SZ, Lemma 2.1], $|r_i| \leq 1$. Since $p(t)$ has integral coefficients, $|r_i| = 1$ for all i. The proof of [SZ, Corollary 2.2] shows that each r_i is a root of unity.

Since $H_A(t) = p(t)/q(t)$, by [SZ, Corollary 2.2], each root of $q(t)$ is a root of unity. Therefore A is cyclotomic. $\square$

By Theorem 1.12 we have the implications below.

(E1.12.2)

$$\begin{tikzcd} cci \arrow[d, "\text{Theorem 1.12(a)}"'] & nci \arrow[d, "\text{Theorem 1.12(b)}"] \\ gci \arrow[r, "\text{Theorem 1.12(b)}"] & wci \arrow[d, "\text{assume } H_A(t) \text{ is rational}"', "\text{Theorem 1.12(c)}"] \\ & cyclotomic \end{tikzcd}$$

2. Higher Veronese subrings are not cyclotomic

In many examples, one proves that an algebra A is not a complete intersection of any type by showing that A is not cyclotomic, see the diagram (E1.12.2). In this short section we show that most Veronese subrings of quantum polynomial rings are not cyclotomic. First we will use a very nice result of Brenti-Welker [BW, Theorem 1.1]. Recall from [BW] that for $a, b, c \in \mathbb{N}$ the partition number is defined by

$$C(a, b, c) = |\{(n_1, \cdots, n_b) \in \mathbb{N}^b \mid \sum_{i=1}^b n_i = c, 0 \leq n_i \leq a \ \forall\ i\}|.$$

For example, $C(1, 3, 1) = 3$.

Lemma 2.1. [BW, Theorem 1.1] *Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers such that for some $s, d \geq 0$ its generating series $f(t) := \sum_{n \geq 0} a_n t^n$ satisfies*

$$f(t) = \frac{h_0 + \cdots h_s t^s}{(1-t)^d}.$$

Set $f^{(r)}(t) = \sum_{n \geq 0} a_{rn} t^n$, for any integer $r \geq 2$. Then we have

$$f^{(r)}(t) = \frac{h_0^{(r)} + \cdots + h_m^{(r)} t^m}{(1-t)^d},$$

where $m := \max\{s, d\}$ and

$$h_i^{(r)} = \sum_{j=0}^s C(r-1, d, ir - j) h_j,$$

for $i = 0, \cdots, m$.

We will apply this lemma to the case when $f(t) = H_A(t)$ for a connected graded algebra A and $f^{(r)}(t) = H_{A^{(r)}}(t)$ where $A^{(r)}$ is the rth Veronese subring of A (i.e. the subring of elements of A with degree a multiple of r). The following lemma is easy to see.

Lemma 2.2. *Suppose $d \geq 2$.*

(a) *$C(a, d, 0) = 1$ for all $a \geq 0$.*
(b) *$C(r-1, 2, r) = r - 1$ for $r \geq 1$.*
(c) *If $d \geq 3$, then $C(r-1, d, r) \geq d$ for $r \geq 2$, and the inequality is strict unless $r = 2, d = 3$.*

Proposition 2.3. *Let A be a connected graded algebra with Hilbert series*

$$H_A(t) = \frac{1 + h_1 t \cdots + h_s t^s}{(1-t)^d}$$

where $h_i \geq 0$ for all i. Then the rth Veronese subring $A^{(r)}$ is not cyclotomic if one of the following conditions holds.

(a) *$r \geq 3$ and $H_A(t) = (1-t)^{-2}$.*
(b) *r satisfies the inequality $C(r-1, d, r) > \max\{s, d\}$.*
(c) *$r \geq 2$ and $H_A(t) = (1-t)^{-d}$ and $d \geq 3$.*

Proof. (a) In this case it is easy to see that $H_{A^{(r)}}(t) = \frac{1+(r-1)t}{(1-t)^2}$. Hence $A^{(r)}$ is not cyclotomic when $r \geq 3$.

(b) Let $m = \max\{s, d\}$. Then $m < C(r-1, d, r)$ by the hypothesis. By Lemma 2.1, $h_0^{(r)} = 1$ and

$$h_1^{(r)} = C(r-1, d, r) + \sum_{j=1}^{s} C(r-1, d, r-j) h_j \geq C(r-1, d, r) > m.$$

If $1 + h_1^{\langle r\rangle} t + \cdots + h_m^{(r)} t^m = \prod_{i=1}^m (1 + r_i t)$, then $\sum_{i=1}^m r_i = h_1^{(r)} > m$, so $h_0^{(r)} + \cdots + h_m^{(r)} t^m$ has a root with absolute value strictly greater than 1. Hence $A^{(r)}$ is not cyclotomic.

(c) In this case $s = 0$. The assertion follows from Lemma 2.2(c), noting the case $r = 2, d = 3$ gives the Hilbert series $(1+3t)/(1-t)^3$, which is not cyclotomic. $\square$

Quantum polynomial rings are noetherian AS regular domains whose Hilbert series are of the form $(1-t)^{-d}$. The following corollary can be used to state precisely when the rth Veronese subalgebra of a quantum polynomial ring is a complete intersection.

Corollary 2.4. *Let A be a connected graded algebra.*

(a) *Suppose $A = k[t]$ where $\deg t = 1$. For every $r \geq 2$, $A^{(r)} \cong k[x]$ where $\deg x = r$. So $A^{(r)}$ is AS regular, and consequently, $A^{(r)}$ is cyclotomic and a classical complete intersection.*
(b) *Suppose A is noetherian of global dimension 2 with Hilbert series $(1-t)^{-2}$.*
   (i) *$A^{(2)}$ is a classical complete intersection (and hence cyclotomic).*
   (ii) *For every $r \geq 3$, $A^{(r)}$ is not cyclotomic. Consequently, $A^{(r)}$ is not a complete intersection of any type.*
(c) *Suppose the Hilbert series of A is $(1-t)^{-d}$. If $d \geq 3$ and $r \geq 2$, then $A^{(r)}$ is not cyclotomic. Consequently, $A^{(r)}$ is not a complete intersection of any type.*

Proof. (a) This is obvious.

(bi) Quantum polynomial rings of dimension 2 are isomorphic to either $k_q[x,y] := k\langle x,y\rangle/(yx-qxy)$ or $k_J[x,y] := k\langle x,y\rangle/(xy-yx-x^2)$, and $A^{(2)}$ is the ring of invariants under the diagonal map sending $x \mapsto -x$ and $y \mapsto -y$. It follows from [KKZ4, Theorem 0.1] that each of these rings of invariants is a hypersurface in an AS regular ring of dimension 3.

(bii) This follows from Proposition 2.3(a).

(c) This follows from Proposition 2.3(c). □

Another special case is when A is AS regular of global dimension three and generated by two elements of degree 1.

Lemma 2.5. *Suppose $H_A(t) = \frac{1}{(1-t)^2(1-t^2)}$. If $r \geq 3$, then $A^{(r)}$ is not cyclotomic. Consequently, $A^{(r)}$ is not a complete intersection of any type.*

Proof. We compute the Hilbert series of $A^{(r)} = \sum_k a_{kr}t^{kr}$, when $\sum_j a_j t^j$ is the expansion of $\frac{1}{(1-t)^2(1-t^2)}$. When r is even the Hilbert series of $A^{(r)}$ is

$$\frac{(r^2-4r+4)t^{2r}+(r^2+4r-8)t^r+4}{4(1-t^r)^3}.$$

For the numerator to be symmetric it is necessary for r to be a positive integer with $r^2-4r+4 = \pm 4$ or 0, which happens only when $r=2$ or $r=4$. The Hilbert series for $r=4$ is

$$\frac{1+6t^4+t^8}{(1-t^4)^3},$$

which is not cyclotomic. When r is odd the Hilbert series of $A^{(r)}$ is

$$\frac{b_0+b_1t^r+b_2t^{2r}+b_3t^{3r}+b_4t^{4r}+b_5t^{5r}+b_6t^{6r}+b_7t^{7r}+b_8t^{8r}+b_9t^{9r}}{4(1-t^{2r})^5}$$

where

$$b_0=4, \quad b_1=3+4r+r^2, \quad b_2=-16+8r+4r^2,$$
$$b_3=-12-8r+4r^2, \quad b_4=24-24r-4r^2, \quad b_5=18-10r^2, \quad b_6=-16+24r-4r^2,$$
$$b_7=-12+8r+4r^2, \quad b_8=4-8r+4r^2, \quad b_9=3-4r+r^2.$$

For the numerator to be symmetric it is necessary that r is an odd positive integer ≥ 3 with $3-4r+r^2=\pm 4$ or 0, which could happen only when $r=3$. The Hilbert series for $r=3$ is

$$\frac{1+6t^3+11t^6-21t^{12}-18t^{15}+5t^{18}+12t^{21}+4t^{24}}{(1-t^6)^5},$$

which is not cyclotomic. □

Remark 2.6.

(a) When A is a noetherian AS regular ring of global dimension three that is generated by two elements of degree 1, then $H_A(t) = \frac{1}{(1-t)^2(1-t^2)}$ and $A^{(2)}$ is a cci by [VdB, Proposition 1.3].

(b) If A is noetherian and AS Gorenstein of dimension d then by [JZ, Theorem 3.6] $A^{(r)}$ is AS Gorenstein if and only if r divides ℓ (where ℓ is the AS index).

3. Invariant theory of AS regular algebras and quasi-bireflections

In this section we connect the study of complete intersections with noncommutative invariant theory. First we review some definitions.

Definition 3.1. Let A be a noetherian algebra.

(a) Given any A-module M, the j-number of M is defined by
$$j(M) = \min\{i \mid \operatorname{Ext}^i(M, A) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$
(b) A is called *Auslander Gorenstein* if
  (a) A has finite left and right injective dimension;
  (b) for every finitely generated left A-module M and for every right A-submodule $N \subset \operatorname{Ext}^i_A(M, A)$, $j(N) \geq i$,
  (c) the above condition holds when left and right are switched.

(c) A is called *Auslander regular* if A is Auslander Gorenstein and has finite global dimension.

(d) A is called *Cohen-Macaulay* if, for every finitely generated (left or right) A-module M, $\operatorname{GKdim}(M) + j(M) = \operatorname{GKdim} A < \infty$.

In a couple of places we also use dualizing complexes, as well as the Auslander and Cohen-Macaulay properties of dualizing complexes, all of which are quite technical. We refer to the paper [YZ] for these definitions. For a graded algebra, the existence of an Auslander Cohen-Macaulay dualizing complex was proved for many classes of noetherian algebras. It is conjectured that every noetherian AS regular algebra is Auslander regular.

The notion of the *homological determinant of* $g \in G$, for G a group of automorphisms of an AS-Gorenstein algebra A, was defined by Jørgensen and Zhang in [JZ, Definition 2.3], using the local cohomology of A. It is a particular group homomorphism from $\operatorname{Aut} A \to k^*$. We refer to [JZ] for the definition and properties of the homological determinant. For the examples in this paper the homological determinant can be computed using Lemma 3.6 below.

Lemma 3.2. *Let E be the* Ext*-algebra of A. Suppose A is a nci and E has an Auslander Cohen-Macaulay dualizing complex. Then $nci(A) \leq gci(A) < \infty$.*

Proof. By using [YZ, Corollary 2.18] (for $d_0 = 0$ and $\operatorname{Cdim} = \operatorname{GKdim}$), we obtain $\operatorname{Kdim} E \leq \operatorname{GKdim} E < \infty$, and the assertions follow. □

Combining Theorem 1.12 with Lemma 3.2, under some reasonable conditions, we have

(E3.3.1) $$cci(A) \geq gci(A) \geq nci(A).$$

Next we recall Molien's Theorem which will be used several times later. Let A be a noetherian connected graded AS regular algebra and g be a graded automorphism of A. The *trace* of g acting on A is defined to be the formal power series $Tr_A(g, t) := \sum_{i=0}^{\infty} Tr(g|_{A_i}) t^i$, where $Tr(g|_{A_i})$ is the trace of the linear map g restricted to the space of homogeneous elements of degree i. Traces can be used to compute the Hilbert series of fixed rings.

Theorem 3.3 (Molien's Theorem)**.** ([JiZ, Lemma 5.2]) *Let A be a connected graded k-algebra and let G be a finite group of graded automorphisms of A. Then the Hilbert* *series of the fixed subring is*

$$H_{A^G}(t)\frac{1}{|G|}\sum_{g\in G} Tr_A(g,t).$$

Now we are ready to prove Theorem 0.4.

Theorem 3.4. *Assume that k is of characteristic zero. Let R be a connected graded noetherian Auslander regular algebra and G be a finite subgroup of* $\operatorname{Aut}(R)$. *If R^G is a wci, then it is cyclotomic Gorenstein. In particular, if R^G is a (gci) or an (nci) then it is cyclotomic Gorenstein.*

Proof. By [Mo, Corollaries 1.12 and 5.9], R^G is noetherian (and R is finite over R^G). By Molien's Theorem (Theorem 3.3), the Hilbert series of R^G is of the form $\frac{1}{|G|}\sum_{g\in G} Tr_R(g,t)$ and each $Tr_R(g,t)$ is of the form $e_g(t)^{-1}$ by [JiZ, Theorem 2.3(4)]. By [KKZ1, Lemma 1.6(e)], the zeros of $e_g(t)$ are all roots of unity. So we can write $e_g(t)$ as $\frac{p_g(t)}{q_g(t)}$ where $p_g(0)=q_g(0)=1$ and $q_g(t)\in\mathbb{Z}[t]$. Let $q(t)$ be the common multiple of $q_g(t)$ for all g, then $H_{R^G}(t)=\frac{p(t)}{q(t)}$ where $q(0)=1$, $q(t)\in\mathbb{Z}[t]$ and $p(t)\in\mathbb{C}[t]$. Since $p(t)=q(t)H_{R^G}(t)$, $p(0)=1$ and $p(t)\in\mathbb{Z}[t]$.

Let σ be any graded algebra automorphism of R and let M be any finitely generated graded left R-module. Since R is noetherian and AS regular, by [JZ, Proposition 4.2], every σ-linear automorphism $f: M\to M$ is rational over k in the sense of [JZ, Definition 1.3]. By [JZ, Lemma 6.3], every module automorphism of a finitely generated graded left R^G-module is rational over k in the sense of [JZ, Definition 1.3]. As a special case (by taking the automorphism to be the identity map of R^G), the Hilbert series of R^G is a rational function. By Theorem 1.12(c), R^G is cyclotomic. It remains to show that R^G is AS Gorenstein.

Write $H_{R^G}(t)=p(t)/q(t)$ with $p(t),q(t)\in\mathbb{Z}[t]$ and $p(0)=q(0)=1$. Further assume that $p(t)$ and $q(t)$ are co-prime. By the proof of Theorem 1.12(c), every root of $p(t)$ is a root of unity. Since $p(t)$ is an integral polynomial, we have $p(t^{-1})=\pm t^{d_1}p(t)$ for some d_1. Since R^G has finite GK-dimension, every root of $q(t)$ is a root of unity (see the proof of [SZ, Corollary 2.2]). Hence $q(t^{-1})=\pm t^{d_2}q(t)$ for some d_2. It follows from [JZ, Theorem 6.4] that R^G is AS Gorenstein. □

When $A=R^G$ where R is a connected graded noetherian Auslander regular algebra and G is a finite subgroup of $\operatorname{Aut}(R)$, we can modify the diagram (E1.12.2) a little by changing "cyclotomic" to "cyclotomic Gorenstein".

This is a good place to mention a natural question.

Question 3.5. Let A be a noetherian connected graded algebra that is either a nci or a gci. Under what conditions must A be AS Gorenstein (or Gorenstein)?

Example 6.3 shows that A can be both a nci and a gci yet still not be Gorenstein. However Theorem 3.4 says that for $A=R^G$, where R is a noetherian Auslander regular algebra, then A must be AS Gorenstein.

Traces can also be used to compute the homological determinant of g.

Lemma 3.6. [JZ, Lemma 2.6] *Let A be AS Gorenstein of injective dimension d and let $g\in\operatorname{Aut}(A)$. Suppose g is k-rational in the sense of* [JZ, Definitions 1.3] (*e.g., if A is AS regular, or A is PI*). *Then the trace of g is of the form*

(E3.6.1) $$Tr_A(g,t)=(-1)^d(\operatorname{hdet} g)^{-1}t^l + \textit{lower degree terms},$$

when we express $Tr_A(g,t)$ as a Laurent series in t^{-1}.

If $A = k[x_1, \cdots, x_n]$ and $\deg(x_i) = 1$, then

$$Tr_A(g,t) = \frac{1}{\prod(1-\lambda_i t)}$$

where the product is taken over all n eigenvalues λ_i of g. Other basic properties of the trace of g can be found in [JiZ].

Definition 3.7. Let A be a noetherian connected graded AS regular algebra of GK-dimension n. Let $g \in \mathrm{Aut}(A)$.

(a) [KKZ1, Definition 2.2] We call g a *quasi-reflection* if its trace has the form:

$$Tr_A(g,t) = \frac{1}{(1-t)^{n-1}q(t)}$$

where $q(t)$ is an integral polynomial with $q(1) \neq 0$.

(b) We call g a *quasi-bireflection* if its trace has the form:

$$Tr_A(g,t) = \frac{1}{(1-t)^{n-2}q(t)}$$

where $q(t)$ is an integral polynomial with $q(1) \neq 0$.

As the classical case, a quasi-reflection may also viewed as a quasi-bireflection for convenience.

Definition 3.8. Let A be a noetherian AS regular algebra and let W be a subgroup of $\mathrm{Aut}(A)$.

(a) The *cci*-W-bound of A is defined to be

$$cci(W/A) = \sup\{cci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } cci(A^G) < \infty\}.$$

(b) The *gci*-W-bound of A is defined to be

$$gci(W/A) = \sup\{gci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } gci(A^G) < \infty\}.$$

(c) The *nci*-W-bound of A is defined to be

$$nci(W/A) = \sup\{nci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } nci(A^G) < \infty\}.$$

(d) The *cyc*-W-bound of A is defined to be

$$cyc(W/A) = \sup\{cyc(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } cyc(A^G) < \infty\}.$$

Example 3.9. Let A be a noetherian AS regular algebra of global dimension two that is generated in degree 1. Then by [KKZ4, Theorem 0.1],

$$cci(\mathrm{Aut}(A)/A) = gci(\mathrm{Aut}(A)/A) = nci(\mathrm{Aut}(A)/A) = cyc(\mathrm{Aut}(A)/A) = 1.$$

4. Graded twists

In this section we will prove Theorem 0.5 using the technique of graded twists developed in [Z2]. We refer to results in that paper, noting that [Z2] uses right modules, while we are using left modules.

Let A be an $\mathbb{Z}^d$-graded algebra and $\mathrm{Aut}_{\mathbb{Z}^d}(A)$ be the group of $\mathbb{Z}^d$-graded algebra automorphisms of A. Let $\{\tau_1, \cdots, \tau_d\}$ be a set of commuting elements in $\mathrm{Aut}_{\mathbb{Z}^d}(A)$. We define a twisting system of A by $\tau_g = \tau_1^{g_1} \cdots \tau_d^{g_d}$ for every $g = (g_1, \cdots, g_d) \in \mathbb{Z}^d$. Then $\tau = \{\tau_g \mid g \in \mathbb{Z}^d\}$ is a left (respectively, right) twisting system in the sense of [Z2, Definition 2.1]. Let A^τ be the graded twisted algebra by the twisting system τ [Z2, Proposition and Definition 2.3]. Let $A - \mathrm{Gr}$ be the category of graded left A-modules and for a graded left A-module M let M^τ be the associated twisted graded left A^τ-module [Z2, Proposition and Definition 2.6]. Then the assignment $F : M \mapsto M^\tau$ defines an equivalence of categories [Z2, Theorem 3.1]

$$A - \mathrm{Gr} \cong A^\tau - \mathrm{Gr}\,.$$

Let (g) denote the degree g shift of a graded module. Let M and N be finitely generated graded left modules over a left noetherian ring A. Then we have

$$\mathrm{Ext}^i_A(M,N) \cong \bigoplus_{g\in\mathbb{Z}^d} \mathrm{Ext}^i_{A-\mathrm{Gr}}(M, N(g)).$$

Lemma 4.1. *Let A be an $\mathbb{N}^d$-graded finitely generated algebra such that A becomes a connected $\mathbb{N}$-graded when taking the total degree. Suppose $\tau = \{\tau_1, \cdots, \tau_d\}$ is a set of commuting elements in $\mathrm{Aut}_{\mathbb{Z}^d}(A)$. Then A is a gci if and only if A^τ is.*

Proof. Let $B = A^\tau$. It suffices to show that $\mathrm{Ext}^i_A(k,k) \cong \mathrm{Ext}^i_B(k,k)$ as graded vector space. Let F be the functor sending M to M^τ. Since A is connected graded by using the total degree, one can check that the only simple graded module over A is $k(g)$ for $g \in \mathbb{Z}^d$. Using this fact we obtain that $F({}_Ak(g)) \cong_B k(g)$ for all $g \in \mathbb{Z}^d$. Hence

$$\begin{aligned}\mathrm{Ext}^i_A(k,k) = \bigoplus_{g\in\mathbb{Z}^d} \mathrm{Ext}^i_{A-\mathrm{Gr}}(k,k(g)) &\cong \bigoplus_{g\in\mathbb{Z}^d} \mathrm{Ext}^i_{B-\mathrm{Gr}}(F(k), F(k(g)))\\ &\cong \bigoplus_{g\in\mathbb{Z}^d} \mathrm{Ext}^i_{B-\mathrm{Gr}}(k,k(g)) = \mathrm{Ext}^i_B(k,k).\end{aligned}$$

□

Lemma 4.2. *Let A and τ be as in Lemma 4.1. Let G be a subgroup of $\mathrm{Aut}_{\mathbb{Z}^d}(A)$ such that every element $g \in G$ commutes with τ_i for all $i = 1, \cdots, d$. Then*

(a) *G is a subgroup of $\mathrm{Aut}_{\mathbb{Z}^d}(A^\tau)$ under the identification $A = A^\tau$ (as a graded vector space).*
(b) *The restriction of τ on A^G defines a twisting system of A^G, which we still denote by τ.*
(c) *$(A^\tau)^G$ is a graded twist of A^G by τ.*

Proof. Let $B = A^\tau$ with multiplication $*$. By definition, $A = B$ as graded vector spaces. The difference between A and B is in their multiplication, see [Z2, Proposition and Definition 2.3].

(a) For every $g \in G$, define $g_B : B \to B$ to be the graded vector space automorphism g. We need to show that this is an algebra homomorphism. For any $x, y \in B$,

$$\begin{aligned}g_B(x * y) &= g(\tau^{\deg y}(x)y) = g(\tau^{\deg y}(x))g(y)\\ &= \tau^{\deg y}(g(x))g(y) = g(x) * g(y) = g_B(x) * g_B(y).\end{aligned}$$

Therefore g_B is a graded algebra automorphism of B, and the assertion follows.

(b) Since τ_i commutes with G, each τ_i induces an automorphism of A^G by restriction. Since the restrictions of τ_i on A^G commute with each other, τ defines a twisting system of A^G.

(c) As graded subspaces of A, we have $(A^G)^\tau = A^G = (A^\tau)^G$. By using the twisting, we see that the multiplications of $(A^G)^\tau$ and $(A^\tau)^G$ are the same. In fact, $(A^G)^\tau = (A^\tau)^G$ as subalgebras of A^τ. □

Proposition 4.3. *Let (A, τ, G) be as in Lemma 4.2.*

(a) *$(A^\tau)^G$ is a gci if and only if A^G is.*
(b) *For every $g \in G$, $Tr_A(g_A, t) = Tr_{A^\tau}(g_{A^\tau}, t)$.*
(c) *$G \subset \mathrm{Aut}_{\mathbb{Z}^d}(A) \subset \mathrm{Aut}(A)$ is generated by quasi-bireflections of A if and only if $G \subset \mathrm{Aut}_{\mathbb{Z}^d}(A^\tau) \subset \mathrm{Aut}(A^\tau)$ is generated by quasi-bireflections of A^τ.*

Proof. (a) By Lemma 4.2(c), $(A^\tau)^G$ is a graded twist of A^G. The assertion follows from Lemma 4.1.

(b) This follows from the fact that $A = A^\tau$ as an $\mathbb{N}$-graded vector space and $g_A = g_{A^\tau}$ as a graded vector space automorphism.

(c) Follows from part (b) and the definition of quasi-bireflection. □

Now we consider the skew polynomial ring $B = k_{p_{ij}}[x_1, \cdots, x_d]$ which is generated by $\{x_1, \cdots, x_d\}$ and subject to the relations

$$x_j x_i = p_{ij} x_i x_j$$

for all $i < j$ where $\{p_{ij}\}_{1 \le i < j \le w}$ is a set of nonzero scalars.

This is a $\mathbb{Z}^d$-graded algebra when setting $\deg x_i = (0, \cdots, 0, 1, 0, \cdots, 0) =: e_i$, where 1 is in the ith position. We say an automorphism g of A is *diagonal* if $g(x_i) = a_i x_i$ for some $a_i \in k^\times$. Then every $\mathbb{Z}^d$-graded algebra automorphism of A is diagonal. As a consequence, $\mathrm{Aut}_{\mathbb{Z}^d}(A) = (k^\times)^d$.

Lemma 4.4. *Let B be the skew polynomial ring $k_{p_{ij}}[x_1, \cdots, x_d]$ and G a finite subgroup of $\mathrm{Aut}_{\mathbb{Z}^d}(A)$.*

(a) [KKZ2, Lemma 3.2] *B is a $\mathbb{Z}^d$-graded twist of the commutative polynomial ring by $\tau = (\tau_1, \cdots, \tau_d)$ where τ_i is defined by $\tau_i(x_s) = \begin{cases} p_{is} x_s & i < s \\ x_s & i \ge s \end{cases}$ for all s.*
(b) *G commutes with τ_i.*
(c) *If B^G is a gci, then G is generated by quasi-bireflections.*

Proof. (a,b) are checked directly.

(c) By part (a) $B = A^\tau$ where $A = k[x_1, \cdots, x_d]$. If B^G is a gci, by Proposition 4.3(a), A^G is a gci. Since $A = k[x_1, \cdots, x_d]$, by the Kac-Watanabe-Gordeev Theorem ([KW] or [G1]), $G \subset \mathrm{Aut}(A)$ is generated by bireflections (which are also quasi-bireflections in the sense of Definition 3.7(b)). By Proposition 4.3(c), $G \subset \mathrm{Aut}(B)$ is generated by quasi-bireflections of B. □

The following lemma is well-known, see for example [AC].

Lemma 4.5. *Let B be the skew polynomial ring $k_{p_{ij}}[x_1, \cdots, x_d]$. Suppose that $p_{ij} p_{st} \neq 1$ for any $i < j$ and $s < t$. Then $\mathrm{Aut}(B) = (k^\times)^d$.*

Proof. First of all $(k^\times)^d = \mathrm{Aut}_{\mathbb{Z}^d}(B) \subset \mathrm{Aut}(B)$. It remains to prove that every $\mathbb{Z}$-graded algebra automorphism g of B is diagonal. Since $p_{ij} \neq 1$ for all $i < j$, by [KKZ2, Lemma 3.5(e)] there is a permutation $\sigma \in S_d$ and $c_s \in k^\times$ for each s such that

$$g(x_i) = c_i x_{\sigma(i)} \qquad \text{for every} \qquad i = 1, \cdots, d.$$

Consequently, $p_{ij} = p_{\sigma(i)\sigma(j)}$ for all i, j. We need to show that σ is the identity. Suppose σ is not the identity. There are distinct integers $i_1, \cdots, i_k$ in the set $\{1, \cdots, d\}$ for some $k > 1$ such that

$$\sigma : i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k \to i_1.$$

Hence

$$p_{i_1 i_2} = p_{i_2 i_3} = \cdots = p_{i_{k-1} i_k} = p_{i_k i_1}.$$

Let i_t is the smallest among all $\{i_s\}_{s=1}^{k}$. Then $p_{i_t i_{t+1}} p_{i_t i_{t-1}} = 1$, a contradiction. Therefore σ is the identity, completing the proof. □

Now we are ready to prove Theorem 0.5.

Proof of Theorem 0.5. Let C denote the algebra $k_q[x_1, \cdots, x_d]$. Then C is a special case of $k_{p_{ij}}[x_1, \cdots, x_d]$ by taking $p_{ij} = q$ for all $i < j$. Since $q^2 \neq 1$ (or $q \neq \pm 1$), Lemma 4.5 says that $\mathrm{Aut}(C) = (k^\times)^d$. Let G be a finite subgroup of $\mathrm{Aut}(C)$. Then $G \subset (k^\times)^d$, and the assertion follows from Lemma 4.4(c). □

5. Some fixed subrings of the Sklyanin Algebra

Let $A = A(a, b, c)$ be the 3-dimensional Sklyanin algebra, i.e. the algebra generated by x, y, z with relations

$$ax^2 + byz + czy = 0$$

$$ay^2 + bzx + cxz = 0$$

$$az^2 + bxy + cyx = 0,$$

for $a, b, c \in k$. By [ATV] the algebra A is noetherian and AS regular except when $a^3 = b^3 = c^3$ or when two of the three parameters a, b, c are zero. We assume throughout that parameters a, b, c are generic enough so that none of a, b, c is zero and the cubes a^3, b^3, c^3 are not all equal so that $A(a, b, c)$ is AS regular, and we further assume that A is not PI. Then by [Sm, Example 10.1], Koszul dual $A^!$ of A is the three-dimensional algebra generated by X, Y, Z with quadratic relations:

$$\begin{aligned} cYZ - bZY &= 0 \qquad & bX^2 - aYZ &= 0 \\ cZX - bXZ &= 0 \qquad & bY^2 - aZX &= 0 \\ cXY - bYX &= 0 \qquad & bZ^2 - aXY &= 0, \end{aligned}$$

and cubic relations

$$XY^2 = XZ^2 = Y^2 Z = Z^2 X = 0.$$

Note that these cubic relations are consequence of the quadratic relations. A vector space basis for $A^!$ is $\{1, X, Y, Z, XY, ZX, YZ, XYZ\}$.

For a graded automorphism σ, the action of σ on the elements of degree 1, $\sigma|_{A_1}$ is given by a matrix, and the transpose of this matrix gives rise to an automorphism $\sigma^!$ of the dual algebra $A^!$. It follows that the groups $\mathrm{Aut}(A)$ and $\mathrm{Aut}(A^!)$ are anti-isomorphic (see discussion [JZ, p. 267]), and we can determine $\mathrm{Aut}(A)$ by computing $\mathrm{Aut}(A^!)$.

Let the matrix (α_{ij}) represent the linear map σ that takes

$$\begin{aligned} \sigma(X) &= \alpha_{11} X + \alpha_{21} Y + \alpha_{31} Z, \\ \sigma(Y) &= \alpha_{12} X + \alpha_{22} Y + \alpha_{32} Z, \\ \sigma(Z) &= \alpha_{13} X + \alpha_{23} Y + \alpha_{33} Z. \end{aligned}$$

Lemma 5.1. *The automorphisms of $A^!$ are of the following forms:*

$$g_1 = \begin{pmatrix} \alpha\omega & 0 & 0 \\ 0 & \alpha\omega^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \ g_2 = \begin{pmatrix} 0 & \alpha\omega & 0 \\ 0 & 0 & \alpha\omega^2 \\ \alpha & 0 & 0 \end{pmatrix}, \ g_3 = \begin{pmatrix} 0 & 0 & \alpha\omega \\ \alpha\omega^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix},$$

where α is arbitrary and ω satisfies $\omega^3 = 1$. The traces of these automorphisms are:

$$Tr_{A^!}(g_1, t) = 1 + \alpha(1+\omega+\omega^2)t + \alpha^2(1+\omega+\omega^2)t^2 + \alpha^3 t^3$$
$$Tr_{A^!}(g_i, t) = 1 + \alpha^3 t^3 \ \textit{for } i = 2, 3.$$

Hence the graded algebra automorphisms of A are transposes of these matrices, so of the same forms, and their homological determinants are all α^3.

Proof. Recall that we assume a, b, c are generic, and since A is not PI we may assume $b \neq c$. Since $b \neq 0$, without loss of generality we may assume that $b = 1$ and $c \neq 1$. Then the elements of $A^!$ of degree 2 form a 3-dimensional vector space with basis XY, YZ, and ZX. Let g be a graded automorphism of A and let $\sigma = g^!$ be the induced automorphism on $A^!$. The automorphism σ must preserve the equations: $cXY - YX = 0, cZX - XZ = 0$, and $cYZ - ZY = 0$ in $A^!$. Computing $\sigma(cXY - YX) = 0$ in terms of the basis elements XY, YZ, and ZX, and using the fact that $c \neq 1$, we see that the entries $\alpha_{i,j}$ of σ must satisfy the equations:

$$(1+c)\alpha_{1,2}\alpha_{2,1} + a\alpha_{3,1}\alpha_{3,2} = 0 \qquad \text{(coefficient of } XY)$$
$$(1+c)\alpha_{2,2}\alpha_{3,1} + a\alpha_{1,1}\alpha_{1,2} = 0 \qquad \text{(coefficient of } YZ)$$
$$(1+c)\alpha_{3,2}\alpha_{1,1} + a\alpha_{2,1}\alpha_{2,2} = 0 \qquad \text{(coefficient of } ZX).$$

Similarly the equation $\sigma(cZX - XZ) = 0$ gives the equations:

$$(1+c)\alpha_{2,3}\alpha_{1,1} + a\alpha_{3,1}\alpha_{3,3} = 0$$
$$(1+c)\alpha_{3,3}\alpha_{2,1} + a\alpha_{1,1}\alpha_{1,3} = 0$$
$$(1+c)\alpha_{1,3}\alpha_{3,1} + a\alpha_{2,1}\alpha_{2,3} = 0,$$

and the equation $\sigma(cYZ - ZY) = 0$ gives the equations:

$$(1+c)\alpha_{2,2}\alpha_{1,3} + a\alpha_{3,3}\alpha_{3,2} = 0$$
$$(1+c)\alpha_{3,2}\alpha_{2,3} + a\alpha_{1,3}\alpha_{1,2} = 0$$
$$(1+c)\alpha_{1,2}\alpha_{3,3} + a\alpha_{2,3}\alpha_{2,2} = 0.$$

Using the equations above and equations resulting from applying σ to the relations $X^2 - aYZ = 0, Y^2 - aZX = 0$ and $Z^2 - aXY = 0$, computations in Maple show that the only nonsingular matrices satisfying these equations are matrices of the forms:

$$g_1 = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \ g_2 = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & \beta \\ \alpha & 0 & 0 \end{pmatrix}, \ g_3 = \begin{pmatrix} 0 & 0 & \gamma \\ \beta & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}$$

with $\alpha^3 = \beta^3 = \gamma^3$. Taking $\alpha \neq 0$ arbitrary, these final three relations show that σ must be of one of the three forms indicated and it is easy to check that these linear maps are algebra automorphisms of A.

The traces of the g_i acting as automorphisms of the 3-dimensional algebra $A^!$ can be computed from the definition of trace, and the traces of the induced automorphisms on A can be computed using the formula [JZ, Corollary 4.4]

$$Tr_A(g, t) = \frac{1}{Tr_{A^!}(g^!, -t)}.$$

Using the $Tr_A(g_i, t)$ and Lemma 3.6 we see that $\operatorname{hdet}_A(g_i) = \alpha^3$. $\square$

If a, b, c are not generic, there are more graded algebra automorphisms of A. To specify ω and α, we also use the notation $g_i(\alpha, \omega)$ for the matrices (or the automorphisms) listed in Lemma 5.1.

By [KKZ1, Corollary 6.3], A has no quasi-reflection of finite order. By [KKZ3, Corollary 4.11], if G is a finite subgroup of $\operatorname{Aut}(A)$ and A^G is Gorenstein, then $\operatorname{hdet} g = 1$ for all $g \in G$. Hence for g_i of the forms above, if $g_i \in G$ with A^G AS Gorenstein then we must have $\alpha^3 = 1$ by Lemma 3.6. The following lemma follows this observation and direct computation.

Lemma 5.2. *Let $A = A(a, b, c)$ and $SL(A)$ be the subgroup of $\operatorname{Aut}(A)$ generated by g_1, g_2 and g_3 with $\alpha^3 = \omega^3 = 1$.*

(a) *The order of $SL(A)$ is 27.*
(b) $\operatorname{hdet} g = 1$ *for all $g \in SL(A)$.*
(c) *If G is a subgroup of $\operatorname{Aut}(A)$ and every $g \in G$ has trivial homological determinant, then G is a subgroup of $SL(A)$.*

Now we consider only $g \in G$ with trivial homological determinant. For g to be a quasi-bireflection we need $t = 1$ to be a root of $Tr_A(g, t)$ of multiplicity 1, which occurs when $t = -1$ is a root of $Tr_{A^!}(g^!, t)$ of multiplicity 1, and this happens in each case when $\alpha^3 = 1$ and in the first case when, in addition, we have $\omega \neq 1$ (i.e. we eliminate the case of a scalar matrix); then, in each case, the automorphism g is a quasi-bireflection with $Tr_A(g, t) = 1/(1 - t^3)$ and $\operatorname{hdet} g = 1$ by Lemma 3.6. We summarize these facts in the following lemma.

Lemma 5.3. *Retaining the notation above we have:*

(a) *g is a quasi-bireflection of A if and only if g is of the form g_1, g_2 or g_3 with $\alpha^3 = \omega^3 = 1$, and $\omega \neq 1$ when $g = g_1$. As a consequence, $g \in SL(A)$.*
(b) *If g is a quasi-bireflection of A, then $Tr_A(g, t) = (1 - t^3)^{-1}$.*

We can classify all subgroups of $SL(A)$.

Lemma 5.4. *The complete list of subgroups of $SL(A)$ is as follows:*

(a) $\{1\}$.
(b) *order 3 subgroups generated by single element, namely,*
$$\langle g_1(\alpha, \omega)\rangle, \quad \langle g_2(\alpha, \omega)\rangle, \quad \textit{and} \quad \langle g_3(\alpha, \omega)\rangle$$
for any pair (α, ω) with $\alpha^3 = \omega^3 = 1$. In the case of g_1, $(\alpha, \omega) \neq (1, 1)$.
(c) *order 9 subgroup $G_1 := \langle g_1(\alpha, \omega) \mid \alpha^3 = \omega^3 = 1\rangle$.*
(d) *order 9 subgroups*
$$G_2 := \langle g_1(\alpha \neq 1, \omega = 1), g_2\rangle, \quad \textit{and} \quad G_3 := \langle g_1(\alpha \neq 1, \omega = 1), g_3\rangle.$$
(e) *the whole group $SL(A)$.*

Proof. If the order of the subgroup is ≤ 3, then clearly we get cases (a) and (b). It is obvious that the subgroups G_1, G_2, G_3 in parts (c,d) are of order 9. Now assume that G is a subgroup of $SL(A)$ of order 9, which is not of the form in part (c). So G contains either g_2 or g_3. By symmetry, we assume that $g_2 \in G$. Since the order of every element in G is either 1 or 3, $G \cong C_3 \times C_3$. So G is abelian, so there are two elements of order g_2, say $a = g_2(\alpha_1, \omega_1)$ and $b = g_2(\alpha_2, \omega_2)$. Then $ab^{-1} = g_1(\alpha, \omega)$. Since the order of G is 9, $\omega = 1$. Thus we have the group G_2 or G_3. $\square$

Theorem 5.5. *Retaining the notation above, we have $cyc(\mathrm{Aut}(A)/A) = 1$, and the following are equivalent for subgroups $\{1\} \neq G \subset SL(A)$.*

(a) *G is not $\langle g_1(\alpha, 1)\rangle$.*
(b) *G is generated by quasi-bireflections.*
(c) *A^G is cyclotomic Gorenstein.*

Proof. Since $Tr_A(g_1(\alpha, 1), t) = (1 - \alpha t)^3$, $g_1(\alpha, 1)$ is not a quasi-bireflection when $\alpha \neq 1$. Hence $G = \langle g_1(\alpha, 1)\rangle$ is not generated by quasi-bireflections. The fixed subring A^G is the Veronese $A^{(3)}$. By Corollary 2.4(c), A^G is not cyclotomic.

It is straightforward to check that all other groups G are generated by quasi-bireflections. So it remains to show that A^G is cyclotomic Gorenstein in each of these cases. In each case we compute the Hilbert series of the fixed ring using Molien's Theorem (Theorem 3.3).

If G has order 3 with generator g where g is either g_2, g_3 or g_1 with $\omega \neq 1$, then the fixed ring A^g has Hilbert series

$$\frac{1}{3(1-t)^3} + \frac{2}{3(1-t^3)} = \frac{1-t+t^2}{(1-t)^2(1-t^3)} = \frac{1-t^6}{(1-t)(1-t^2)(1-t^3)^2}.$$

So A^G is cyclotomic Gorenstein. If G is the any group of order 9 in Lemma 5.4, then the Hilbert series of A^G is

$$\begin{aligned} H_{A^G}(t) &= \frac{1}{9}\left(\frac{1}{(1-t)^3} + \frac{1}{(1-\omega t)^3} + \frac{1}{(1-\omega^2 t)^3} + \frac{6}{(1-t^3)}\right) \\ &= \frac{1+t^3+t^6}{(1-t^3)^3} = \frac{1-t^9}{(1-t^3)^4}. \end{aligned}$$

where ω is a primitive 3rd root of unity. So A^G is cyclotomic Gorenstein.

Finally, if $G = SL(A)$, then the Hilbert series of A^G is

$$\begin{aligned} H_{A^G}(t) &= \frac{1}{27}\left(\frac{1}{(1-t)^3} + \frac{1}{(1-\omega t)^3} + \frac{1}{(1-\omega^2 t)^3} + \frac{24}{(1-t^3)}\right) \\ &= \frac{1-t^3+t^6}{(1-t^3)^3} = \frac{1+t^9}{(1-t^3)^2(1-t^6)} = \frac{1-t^{18}}{(1-t^3)^2(1-t^6)(1-t^9)}. \end{aligned}$$

So A^G is cyclotomic Gorenstein.

Finally, by the above Hilbert series, we see that $cyc(\mathrm{Aut}(A)/A) = 1$. □

We make the following conjecture.

Conjecture 5.6. *Let $A = A(a, b, c)$ where a, b, c are generic. Then the following are equivalent for any finite subgroup $G \subset \mathrm{Aut}(A)$.*

(a) *A^G is a cci.*
(b) *A^G is a gci.*
(c) *A^G is a nci.*
(d) *A^G is cyclotomic Gorenstein.*
(e) *G is generated by quasi-bireflections.*

6. Examples and questions

In this section we collect some examples and questions which indicate the differences between the commutative and the noncommutative situations. First we show that *cci*, *gci* and *nci* are different concepts.

Lemma 6.1. *Suppose A is a finite dimensional algebra. If $A \cong C/(\Omega_1, \cdots, \Omega_n)$ where C has finite global dimension and $\{\Omega_1, \ldots, \Omega_n\}$ is a regular sequence of normalizing elements in C, then C is noetherian, AS regular, Auslander regular and Cohen-Macaulay, and A is Frobenius.*

Proof. Since A is noetherian, C is noetherian by [Le, Proposition 3.5 (a)]. Furthermore C has enough normal elements in the sense of [Z1, p. 392]. By [Z1, Theorem 0.2], C is AS regular, Auslander regular and Cohen-Macaulay. By Rees's lemma [Le, Proposition 3.4(b)], A is Gorenstein of injective dimension 0. Hence A is Frobenius. □

Example 6.2. Let $A = k\langle x, y\rangle/(x^2, xy, y^2)$. This is a finite dimensional Koszul algebra with Hilbert series $1+2t+t^2$, and A is finitely generated and noetherian. The Ext-algebra $E = E(A)$ is isomorphic to $k\langle x, y\rangle/(yx)$ with Hilbert series $(1-t)^{-2}$. By definition, $gci(A) = \operatorname{GKdim} E = 2$ and A is a gci. It is well-known that E is not (left or right) noetherian. For example, if $I_i := Exy \oplus Exy^2 \oplus \cdots \oplus Exy^i$ and $J_i := xyE \oplus x^2yE \oplus \cdots \oplus x^iyE$, then I_i (respectively, J_i) gives an infinite ascending chain of left (respectively, right) ideals of E, so E is not (left or right) noetherian and $\operatorname{Kdim} E = \infty$. By definition, A is not a nci and $nci(A) = \infty$. Since A is finite dimensional and not Frobenius (A is local with 2 minimal right ideas xA and yxA), A is not a cci [Lemma 6.1] and $cci(A) = \infty$. Since $H_A(t) = \frac{(1-t^2)^2}{(1-t)^2}$, $cyc(A) = 2$.

Example 6.3. Let R be the connected graded Koszul noetherian algebra of global dimension four that is not AS regular given in [RS, Theorem 1.1]. Its Hilbert series is $H_R(t) = (1-t)^{-4}$. Let A be the Koszul dual of R. Since R is not AS regular, A is not Frobenius [Sm, Theorem 4.3 and Proposition 5.10]. Hence A is not a cci by Lemma 6.1. However, the Ext-algebra of A is R, which is noetherian and has GK-dimension 4. Consequently, A is both a gci and nci. It is easy to see that $gci(A) = cyc(A) = 4$.

An example of Stanley shows that, even in the commutative case, *cci* and "cyclotomic Gorenstein" are different concepts.

Example 6.4. [St2, Example 3.9] Let A be the connected commutative algebra

$$k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]/I$$

with $\deg(x_i) = 1$ where

$$I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5, x_1^2x_4 - x_5x_6x_7, x_1^3 - x_3x_5x_7).$$

This is a normal Gorenstein domain, but not a complete intersection (so $gci(A) = \infty$). Its Hilbert series is

$$H_A(t) = \frac{(1+t)^3}{(1-t)^4} = \frac{(1-t^2)^3}{(1-t)^7}.$$

Hence it is cyclotomic Gorenstein and $cyc(A) = 3$.

Many examples and Theorem 1.12 indicate that being a cci could be the strongest among all different versions of a noncommutative complete intersection. In this direction, we need to answer the following question.

Question 6.5. Suppose char $k = 0$. If A is a cci, then is A a nci?

By Theorem 1.11(c), this conjecture is true when $cci(A) \leq 1$, and is unsolved for $cci(A) \geq 2$. If Question 6.5 has a positive answer, then we have the following diagram for invariant subrings under a finite group acting on an AS-regular algebra.

(E6.5.1)

$$\begin{tikzcd}
cci \arrow[rr, "\not\leftarrow \text{Example 6.3}", "\text{Conjecture 6.5}"'] \arrow[d, "\text{Theorem 1.12(a)}"', "\not\uparrow \text{Example 6.3}"] & & nci \arrow[d, "\text{Theorem 1.12(b)}"] \\
gci \arrow[rr, "\text{Theorem 1.12(b)}"] & & wci \arrow[d, "\text{Theorem 3.4}"] \\
& & cyclotomic\ \ Gorenstein
\end{tikzcd}$$

where the implication from $wci \longrightarrow cyclotomic\ \ Gorenstein$ is valid only for fixed subrings of Auslander regular algebras [Theorem 3.4].

The next two examples concern the noncommutative version of a bireflection. We would like to show that quasi-bireflection is a good generalization in the noncommutative setting in order to answer Question 0.3.

Example 6.6. Let $B = k_{-1}[x, y, z]$ where all the variables (-1)-skew commute. Let $V = B_1 = kx + ky + kz$. Consider

$$g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which has eigenvalues $-1, i, -i$. Hence $g\mid_V$ is not a classical bireflection of the vector space V. By an easy computation,

$$Tr_A(g, t) = 1/((1+t)(1-t^2)) = Tr_A(g^2, t) = Tr_A(g^3, t)$$

and $\operatorname{hdet} g = 1$ by Lemma 3.6, and hence g is a quasi-bireflection. A computation using Molien's Theorem (Theorem 3.3) shows the Hilbert series of the fixed ring is

$$H_{A^{\langle g\rangle}}(t) = \frac{1-t^6}{(1-t^2)^3(1‘-t^3)},$$

and that A^g is isomorphic (under an isomorphism that associates $x^2 + y^2 \mapsto X$, $xy \mapsto Y$, $z^2 \mapsto Z$ and $x^2z - y^2z \mapsto W$) to

$$k[X, Y, Z, W]/(W^2 - (X^2 + 4Y^2)Z),$$

which is a commutative complete intersection. To match our terminology for quasi-reflections in [KKZ1], we might call g a mystic quasi-bireflection because g is a quasi-bireflection but not a classical bireflection of V.

Example 6.7. Let $B = k_{-1}[x, y, z, w]$ where all the variables (-1)-skew commute. Let $V = B_1 = kx + ky + kz + kw$. Let

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then g is a classical bireflection of the vector space V but g is not a quasi-bireflection, since its trace is $Tr(g, t) = 1/(1+t^2)^2$. The fixed subring is Gorenstein because $\operatorname{hdet} g = 1$ by Lemma 3.6. Using Molien's Thereom (Theorem 3.3) the Hilbert series of the fixed ring is

$$H_{B^g}(t) = \frac{1 - 2t + 4t^2 - 2t^3 + t^4}{(1-t)^4(1+t^2)^2}$$

so that B^g is Gorenstein, but not cyclotomic Gorenstein. Consequently, B^G is not a noncommutative complete intersection of any type.

In [WR] all groups G so that $\mathbb{C}[x,y,z]^G$ is a complete intersection are completely determined. In the noncommutative case, we can ask:

Question 6.8. Let A be a noetherian AS regular algebra of global dimension three that is generated in degree 1. Determine all finite subgroups $G \subset \operatorname{Aut}(A)$ such that A^G is a noncommutative complete intersection (of a certain type).

This question was answered for global dimension 2 in [KKZ4, Theorem 0.1]. The problem of determining which groups G of graded automorphisms of $A = k[x_1, \cdots, x_n]$ have the property that A^G is a hypersurface was solved in [N1, N2] (see also [NW, Theorem 7]). So we can ask

Question 6.9. Let A be a noetherian AS regular algebra of global dimension at most three that is generated in degree 1. Determine all finite subgroups $G \subset \operatorname{Aut}(A)$ such that A^G is a hypersurface (of a certain type).

Another closely related question is

Question 6.10. If $nci(A) = 1$, then is A cyclotomic Gorenstein?

The complete intersections A^G for $A = \mathbb{C}[x,y,z]$ and G an abelian subgroup of $GL(n, \mathbb{C})$ are computed in [W3]. We can ask, in the noncommutative case, what can be said when G is abelian?

In [WR] complete intersections $\mathbb{C}[x,y,z]^G$ are considered; here $\mathbb{C}[x,y,z]^G$ is a complete intersection if and only if the minimal number of algebra generators of $\mathbb{C}[x,y,z]^G$ is ≤ 5 [NW, Corollary p. 107]. A complete intersection A^G for $A = k[x_1, \cdots, x_n]$ always has $\leq 2n-1$ generators (but for n=4 there is an example of A^G not a complete intersection but the number of generators is 7). Are there noncommutative version(s) of these results? What nice properties distinguish noncommutative complete intersections from other AS Gorenstein rings?

Remark 6.11. To do noncommutative algebraic geometry and/or representation theory (e.g. suppport varieties and so on) related to noncommutative complete intersection rings sometimes it is convenient to assume that the Ext-algebra of A has nice properties. For example, one might want to assume that A is a nci and $E(A)$ has an Auslander Cohen-Macaulay rigid dualizing complex. Together with these extra hypotheses, this should be a good definition of noncommutative complete intersection. So we conclude with the following question.

Question 6.12. Let A be a connected graded noetherian algebra that is a nci. Suppose that the Ext-algebra $E(A)$ has an Auslander Cohen-Macaulay rigid dualizing complex. Then is A Gorenstein?

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