AN INTRODUCTION TO PERVERSE SHEAVES

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Introduction

The aim of these notes is to give an introduction to perverse sheaves with applications to representation theory or quantum groups in mind. The perverse sheaves that come up in these applications are in some sense extensions of actual sheaves on particular algebraic varieties arising in Lie theory (for example the nilpotent cone in a Lie algebra, or a Schubert variety, or the moduli space of representations of a Dynkin quiver). We will ultimately therefore be interested in perverse sheaves on algebraic varieties.

The origin of the theory of perverse sheaves is M. Goresky and R. MacPherson's theory of intersection homology [13, 14]. This is a purely topological theory, the original aim of which was to find a topological invariant similar to cohomology that would carry over some of the nice properties of homology or cohomology of smooth manifolds also to singular spaces (especially Poincaré duality). While the usual cohomology of a topological space can be defined sheaf theoretically as cohomology of the constant sheaf, the intersection homology turns out to be the cohomology of a certain complex of sheaves, constructed very elegantly by P. Deligne. This complex is the main example of a perverse sheaf.

The greatest part of these notes will be taken up by explaining Goresky and MacPherson's intersection homology and Deligne's complex. And throughout all of that our setting will be purely topological. Also we will always stay over $\mathbb{C}$. The deepest result in the theory of perverse sheaves on algebraic varieties, the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber [1], will only be stated. And the notes will end with an application, the intersection cohomology interpretation of the Kazhdan-Lusztig polynomials.

The overriding goal of this exposition (written by a non-expert) is to give a hopefully broadly accessible first introduction to perverse sheaves. It is intended more to give the flavor and some orientation without delving too much into technical details. The hope is that readers wishing to see something in more detail or greater generality should be able to orient themselves very quickly in the existing literature to find it. Moreover these lectures are very far from comprehensive, and the bibliography mostly just reflects the sources that I happened across and found useful. Many more pointers in all the different directions of the theory can be found in the

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Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 55N33; 14F43, 46M20.

Key words and phrases. perverse sheaves, intersection cohomology.

The author was funded by the Violette and Samuel Glasstone Foundation at Oxford throughout most of the preparation of these lectures, and is currently a Royal Society Dorothy Hodgkin Research Fellow at King's College, University of London.

references themselves. For a particularly nice overview of the many applications of perverse sheaves the two ICM addresses [24] and [25] are highly recommended.

These notes are based on seven lectures given first at Oxford University in the Spring of 2002, and then repeated in condensed form at the Fields Institute, as part of the ICRA X conference. I am grateful to the listeners in both places for their questions and comments, and to Professor Ringel for instigating these lectures. Special thanks go to Kevin McGerty and Catharina Stroppel for each sending me their helpful comments on the written notes.

1. Lecture - local systems, an introduction

Let X be a topological space. We will always assume X to be a nice, sensible topological space (locally compact, Hausdorff, paracompact, with a countable basis, locally simply connected). For example picture your favorite simplicial or CW complex, manifold or real/complex algebraic variety. We begin with a quick review of sheaves. Good references are for example [18, 19]. Any results from algebraic topology we use can be found in most textbooks, such as [10, 26, 27].

1.1. **Sheaves.** Let $\mathbb{K}$ be a field. A sheaf of $\mathbb{K}$-vector spaces, $\mathcal{F}$, on X is a contravariant functor

$$\begin{array}{ccc} \left(\begin{array}{c}\text{open sets in } X\\ \text{and inclusions}\end{array}\right) & \longrightarrow & \left(\begin{array}{c}\mathbb{K}\text{-vector spaces}\\ \text{and linear maps}\end{array}\right) \\ U & \longmapsto & \mathcal{F}(U) \\ V \hookrightarrow U & \longmapsto & r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V) : s \mapsto s|_V \end{array}$$

obeying the *sheaf axiom :* For any collection of open sets $\{V_i\}_{i\in I}$ in X and $s_i \in \mathcal{F}(V_i)$ that are compatible with one another in the sense that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all $i, j \in I$ there exists a unique $s \in \mathcal{F}(\bigcup_{i\in I} V_i)$ such that $s|_{V_i} = s_i$ for all $i \in I$. The elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U.

Loosely, the sheaf axiom says that sections are determined by their restrictions to any open cover. In a way a sheaf is a device for dealing with properties that are local in nature and measuring the transition from local to global. We list some common examples. These are widely spread, but mostly we will be interested in sheaves related to 4. below.

(1) The continuous $\mathbb{R}$-valued functions on a topological space X form a sheaf C^0_X. So $C^0_X(U)$ is the vector space of continuous maps $U \to \mathbb{R}$ and for $U \subset V$, r_{UV} is the usual restriction map.
(2) Similarly if X is a smooth manifold C^∞_X is the sheaf of smooth functions on X. A system of differential equations on X determines a subsheaf of C^∞_X with sections given by the local solutions. If X is an analytic manifold (with analytic gluing maps between the charts) then one also has the sheaf of analytic functions on X. For an algebraic variety there is the sheaf of regular (algebraic) functions.
(3) A vector bundle $E \to X$ gives rise to a sheaf of local sections. For example we have the sheaf of vector fields coming from the tangent bundle, or the sheaf of 1-forms associated to the cotangent bundle.
(4) A central role in topology is played by the *constant sheaf* $\underline{\mathbb{C}}_X$ on X, which is defined by

$$\underline{\mathbb{C}}_X(U) := \{\text{locally constant maps } U \to \mathbb{C}\}.$$

So if U has finitely many connected components we have

$$\underline{\mathbb{C}}_X(U) \cong \mathbb{C}^{\#\{\text{connected components of } U\}}.$$

Completely analogously there is a constant sheaf $\underline{V}_X$ for any vector space V. In particular we have the zero-sheaf, $\underline{0}_X$, with $\underline{0}_X(U) = 0$ for all U.

A morphism $\mathcal{F} \to \mathcal{G}$ between sheaves on X is defined to be a natural transformation of the functors, that is, a collection of compatible linear maps $\mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets U in X.

We denote the category of sheaves of $\mathbb{C}$-vector spaces on X by $\mathrm{Sh}(X)$. This is an abelian category. The zero object is given by the zero-sheaf $\underline{0}_X$. The direct sum $\mathcal{F} \oplus \mathcal{G}$ of two sheaves is just given in the obvious way by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ with the induced restriction maps. Also the kernel of a morphism $f : \mathcal{F} \to \mathcal{G}$ in the category may be defined by $\mathcal{K}er(f)(U) = \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$. Note that whether a section in $\mathcal{F}(U)$ is sent to zero under $f(U)$ is a property that can be checked locally, so $\mathcal{K}er(f)$ is indeed a sheaf.

Cokernels are a bit more tricky to define. The definition on individual spaces of sections is not local enough and only gives a presheaf (a contravariant functor of the same kind but that does not necessarily obey the sheaf axiom). However there is a standard way to 'sheafify', i.e. turn any presheaf into a sheaf by forcing the sheaf axiom, to get the correct definition. We'll skip the details, but see any textbook. In any case, an exact sequence of sheaves may be characterized by being exact for small enough neighborhoods, or by being exact on *stalks :*

Definition 1.1. Let $x \in X$ and $\mathcal{F}$ a sheaf (or presheaf) on X. The stalk of $\mathcal{F}$ at x is the $\mathbb{C}$-vector space

$$\mathcal{F}_x = \varinjlim_{\{U \subset X \text{ open } \mid \ x \in U\}} \mathcal{F}(U).$$

In other words an element in $\mathcal{F}_x$ can be represented by a pair (U, s) of an open neighborhood U of X and a section $s \in \mathcal{F}(U)$. And two such pairs represent the same element of the stalk if they agree restricted to some small enough neighborhood of x. This element is the *germ* of the section s and denoted by s_x.

Any map of sheaves $f : \mathcal{F} \to \mathcal{G}$ on X induces a map on stalks $f_x : \mathcal{F}_x \to \mathcal{G}_x$. For a section $s \in \mathcal{F}(U)$ the *support* of s is defined by $\mathrm{Supp}(s) = \{x \in U \mid s_x \neq 0 \text{ in } \mathcal{F}_x\}$. Note, using the definition of s_x, that $\mathrm{Supp}(s)$ is automatically closed(!) inside U.

A sequence, $\underline{0}_X \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to \underline{0}_X$, in $\mathrm{Sh}(X)$ is exact precisely if $0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$ is an exact sequence of vector spaces for all $x \in X$. In particular the functor from $\mathrm{Sh}(X)$ to vector spaces of taking stalks is exact. Also for any presheaf $\tilde{\mathcal{F}}$, the sheafification $\mathcal{F}$ is a sheaf with $\mathcal{F}_x = \tilde{\mathcal{F}}_x$ for all $x \in X$.

Remark 1.2 (Examples). For a sheaf of analytic functions the germs at x can be thought of as Taylor series around x. The stalk of the constant sheaf $\underline{V}_X$ at any point x is just the vector space V.

Let $x_0 \in X$. Then another example of a sheaf is the skyscraper sheaf $\mathcal{S}$ at x_0. Define $\mathcal{S}(U) = \mathbb{C}$ precisely if $x_0 \in U$ and otherwise zero, with the obvious restriction maps. Then $\mathcal{S}_{x_0} = \mathbb{C}$ while all other stalks vanish.

1.2. **A few functors and exactness.** The stalk functor is a special case of a *restriction functor*. Suppose $j = j_Z : Z \hookrightarrow X$ is the inclusion of a locally closed subset. Then there is an exact functor $j^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ defined by $j^*(\mathcal{F})(V) =$

$\varinjlim \mathcal{F}(U)$, where the limit is taken over $U \supset V$ which are open in X. When Z is open in X the limit is not required. One also writes $j^*(\mathcal{F}) = \mathcal{F}|_Z$. This functor has the property $j^*(\mathcal{F})_z = \mathcal{F}_z$ for all $z \in Z$.

Other important functors that we have straight away are

$$\begin{aligned} \Gamma\ &: \mathrm{Sh}(X) \to (\mathbb{C}\text{-vector spaces}) & \Gamma_c\ &: \mathrm{Sh}(X) \to (\mathbb{C}\text{-vector spaces}) \\ &\mathcal{F} \mapsto \Gamma(X,\mathcal{F}) := \mathcal{F}(X), & &\mathcal{F} \mapsto \Gamma_c(X,\mathcal{F}), \end{aligned}$$

where $\Gamma_c(X,\mathcal{F})$ is the space of compactly supported global sections. For open $U \subset X$ we also have the functors $\Gamma(U, \quad) = \Gamma \circ {j_U}^*$ and $\Gamma_c(U, \quad) = \Gamma_c \circ {j_U}^*$ of sections or compactly supported sections over U. These sections functors are in general only left exact.

Remark 1.3. The failure of right exactness of Γ comes from the problem that local properties need not be true globally. For example suppose X is a smooth manifold and Ω^i_X the sheaf of smooth i-forms on X, say with complex coefficients. Then the complex of sheaves

$$\cdots \to 0 \to \underline{\mathbb{C}}_X \to \Omega^0_X \to \Omega^1_X \to \cdots$$

with the usual differential is exact, since this is locally true by Poincaré's Lemma (every closed form on $\mathbb{R}^n$ is exact). However applying Γ to the degree ≥ 0 half of this complex we obtain the de Rham complex for computing $H^*(X,\mathbb{C})$.

In other words think of the cohomology of the de Rham complex as giving an obstruction for contractibility of a manifold X. Then this example comes down to the simple observation that while any manifold is locally contractible, it is not necessarily contractible globally.

1.3. **Local systems.**

Definition 1.4. A sheaf $\mathcal{L}$ on X is called *locally constant* if every $x \in X$ has a neighborhood U such that for all $y \in U$, the canonical map

$$\mathcal{L}(U) \to \mathcal{L}_y$$

is an isomorphism. A *local system* is a locally constant sheaf with finite dimensional stalks. If X is connected then all these stalks automatically have the same dimension. This dimension is called the rank of the local system.

Here are some examples.

(1) The standard example of a locally constant sheaf is of course the constant sheaf $\underline{V}_X$ for a vector space V.
(2) Suppose X is a connected n-manifold. The top degree cohomology with compact supports $H^n_c(X,\mathbb{C})$ detects whether X has a coherent orientation or not: if so it is one-dimensional, spanned by a so-called 'orientation class', otherwise zero. The orientation sheaf $\mathbb{O}_X$ may be defined by $\mathbb{O}_X(U) = H^n_c(U,\mathbb{C})^*$. Note that an open inclusion $U \hookrightarrow U'$ induces a map $H^n_c(U,\mathbb{C}) \to H^n_c(U',\mathbb{C})$ and therefore the dual is required to define a sheaf. $\mathbb{O}_X$ is a rank one local system which is constant precisely if X is orientable.
(3) The local solutions to a homogeneous system of differential equations on a manifold can form a locally constant sheaf. For example fix $\alpha \in \mathbb{C}$ and take the differential equation,

$$\frac{df}{dz} - \frac{\alpha}{z} f = 0, \tag{$*_\alpha$}$$

on the punctured plane $X = \mathbb{C}^*$. Then we may associate to this equation a sheaf $\mathcal{L}^\alpha$ on X by

$$\mathcal{L}^\alpha(U) = \{ \text{ complex analytic functions on } U \text{ satisfying } (*_\alpha)\}. \tag{1.1}$$

If U is simply connected then any branch of z^α gives a solution, and these are related to one another by a scalar multiple. So $\mathcal{L}^\alpha(U) = \langle z^\alpha \rangle_{\mathbb{C}}$. We have that $\mathcal{L}^\alpha$ is a rank one local system. $\mathcal{L}^\alpha$ is trivial (the constant sheaf) precisely if $\alpha \in \mathbb{Z}$. If the differential operator $\frac{d}{dz} - \frac{\alpha}{z}$ is replaced by $(\frac{d}{dz} - \frac{\alpha}{z})^m$ one obtains a rank m local system. This is the starting point of Riemann-Hilbert correspondence. See for example Chapter 7 of [23] and then [3].

1.4. **The monodromy representation.** Let $\pi_1(X, x_0)$ be the fundamental group of X with base point $x_0 \in X$. Any local system $\mathcal{L}$ gives rise to a representation of $\pi_1(X, x_0)$ on the stalk $\mathcal{L}_{x_0}$ called the *monodromy representation.* This representation is defined as follows.

Let $\gamma : [0,1] \to X$ be a continuous map, $\gamma(0) = \gamma(1) = x_0$, representing an element of $\pi_1(X, x_0)$. By compactness the image of γ may be covered with finitely many open sets $U_1, \dots, U_n$ on which $\mathcal{L}$ is trivial. And these may be chosen such that $U_j \cap U_{j+1} \neq \emptyset$ and $x_0 \in U_n \cap U_1$. Let $x_j \in U_j \cap U_{j+1}$. Then by the definition of locally constant sheaf we have a sequence of isomorphisms

$$\mathcal{L}_{x_0} \overset{\sim}{\leftarrow} \mathcal{L}(U_1) \overset{\sim}{\to} \mathcal{L}_{x_1} \overset{\sim}{\leftarrow} \mathcal{L}(U_2) \overset{\sim}{\to} \cdots \overset{\sim}{\leftarrow} \mathcal{L}(U_n) \overset{\sim}{\to} \mathcal{L}_{x_0},$$

which defines an automorphism (from left to right) of $\mathcal{L}_{x_0}$. It follows from the definition of locally constant sheaf that this automorphism $\mathcal{L}_{x_0} \to \mathcal{L}_{x_0}$ depends only on the homotopy type of γ. It is also clear that concatenation of paths corresponds to composition of maps, giving a representation of $\pi_1(X, x_0)$.

The monodromy representation defines a functor,

$$\begin{pmatrix} \text{local systems on } X, \text{ as} \\ \text{full subcategory of } \mathrm{Sh}(X) \end{pmatrix} \longrightarrow \begin{pmatrix} \text{finite dimensional rep's of } \pi_1(X, x_0), \\ \text{and equivariant homomorphisms} \end{pmatrix}.$$

The main observation is that most of the time these two categories are equivalent (similarly with local systems replaced by locally constant sheaves and the finite dimensionality constraint on the right hand side removed).

Proposition 1.5. *If X has a universal cover (i.e. X is path connected, locally path connected and locally simply connected), then the monodromy functor is an equivalence of categories.*

Proof. Suppose we are given a representation of $\pi_1(X) := \pi_1(X, x_0)$ on some (finite dimensional) vector space V. Let $p : \tilde{X} \to X$ be the universal cover of X. Then $\pi_1(X, x_0)$ acts on $\tilde{X}$ by deck transformations and p is the quotient map. Define a sheaf by

$$\mathcal{L}(U) := \left\{\text{locally constant, } \pi_1(X)\text{-equivariant maps } \phi : p^{-1}(U) \to V\right\}$$

and the obvious restriction maps (this clearly obeys the sheaf axiom).

If $x \in X$ and U is a connected, simply connected neighborhood of x, then $p^{-1}(U) \cong U \times \pi_1(X)$ with $\pi_1(X)$ acting from the right by right translation. In that case $\mathcal{L}(U) \cong V$ by $\phi \mapsto \phi(u, 1)$ (independent of u since ϕ is locally constant). The inverse isomorphism comes from reconstructing ϕ from $\phi(u, 1)$ using the group action. It follows that $\mathcal{L}$ is a locally constant sheaf with stalks isomorphic to

V. Moreover, it is easy to see from the construction of $\mathcal{L}$ that the monodromy representation is the representation we started with. $\square$

From now on let us always assume that any topological space X we will consider, if it is connected, has a universal cover.

Remark 1.6 (Example). Consider the rank one local system $\mathcal{L}^\alpha$ on $X = \mathbb{C}^*$ from (1.1). Here $\pi_1(X) \cong (\mathbb{Z}, +)$ and the monodromy representation ρ^α of $\mathcal{L}^\alpha$ is determined by the action of the generator $1 \in \mathbb{Z}$ on a (any) stalk. Then it is easy to check that $\rho^\alpha(1) = (\exp(2\pi i\alpha)) \in GL_1(\mathbb{C})$, assuming the generator 1 is represented by the path γ in X that winds around the origin once in positive, anti-clockwise, orientation.

1.5. **Extensions of local systems.** Suppose $U \subset X$ open and $\mathcal{L}$ is a local system on U. Then one can immediately ask, when is $\mathcal{L}$ the restriction of a (unique) local system $\tilde{\mathcal{L}}$ on X. For example for $\mathcal{L}^\alpha \in \mathrm{Sh}(\mathbb{C}^*)$ from (1.1) to come from a local system on $\mathbb{C}$ it must be constant. This is the case precisely if $\alpha \in \mathbb{Z}$ as we have already observed.

In general for $\mathcal{L}$ to extend to X it must have trivial monodromy around any loop γ in U which is contractible in X. Also U should be large enough in X so that $\pi_1(U) \to \pi_1(X)$ is surjective. Otherwise even if an extension exists it may not be unique. In a manifold this surjectivity is automatic if U has complement of codimension ≥ 2.

1.6. **Outlook : Perverse sheaves and intersection cohomology.** We have seen in Proposition 1.5 that the category of local systems on X is completely governed by the fundamental group $\pi_1(X)$. A common thing one might do given a local system $\mathcal{L}$ on X is to compute the cohomology of X with coefficients in $\mathcal{L}$. (For example this corresponds to group cohomology of $\pi_1(X)$ when the universal cover $\tilde{X}$ is contractible). When trying to study the topology of a manifold X, the standard cohomology comes from the constant sheaf. But it is also common to consider the orientation sheaf to get a more useful theory for non-orientable manifolds. The idea of choosing a suitable local system adapted to X when considering cohomology can be generalized to the case when X is not a manifold but has some singularities.

Suppose for example X is a complex algebraic variety, such as the nilpotent cone in a Lie algebra. Consider a stratification of X by smooth subvarieties with one open dense stratum U. Then associated to any local system $\mathcal{L}$ on U there is a natural 'perverse sheaf' on X extending $\mathcal{L}$, which gives rise to an appropriate analogue to cohomology with coefficients in $\mathcal{L}$ for the singular space X. This extension in a way assigns to $\mathcal{L}$ a collection of local systems in different degrees on the smaller strata by taking into close account the singularities of X and their severity. For example, the perverse sheaf extension of $\underline{\mathbb{C}}_U$ will not in general agree with $\underline{\mathbb{C}}_X$, unless X is a manifold. And the cohomology of the perverse extension can vary for two homotopic but not homeomorphic spaces whereas the cohomology of $\underline{\mathbb{C}}_X$ only depends on the homotopy type of X. When $\mathcal{L}$ is chosen to be the orientation sheaf on U this construction gives the intersection homology of X. For a description of the perverse sheaf extensions of the $\mathcal{L}^\alpha$ from (1.1) see [19, Section 2.9.14].

2. Lecture - Intersection Homology

2.1. **Stratified pseudomanifolds.** In this section we will give the definition of intersection homology of Goresky and MacPherson [13]. Some further references are [11, 23, 6]. One wants to deal with topological spaces that are somewhat like manifolds, but allowed to have singularities. So we will consider singular spaces that are stratified into a union of manifolds.

Definition 2.1. An n-dimensional *stratified pseudomanifold* is a topological space X with a filtration

$$X = X_0 \supset X_1 \supset X_2 \supset X_3 \supset \cdots \supset X_n \supset \emptyset$$

by closed subsets with the following properties.

- For any k the stratum $S_k := X_k \setminus X_{k+1}$ is a topological manifold of dimension $n-k$, or else empty (the index k may be thought of as the codimension). In particular $S_n = X_n$ is a discrete at most countable set.
- The stratum $S_1 = \emptyset$. Therefore $X_1 = X_2 =: \Sigma$, which we think of roughly as the singular locus.
- The open stratum $S_0 = X \setminus \Sigma$ is dense in X.
- *Local normal triviality along the strata :* Any $x \in S_k$ has an open neighborhood U and a compact stratified pseudomanifold of dimension $k-1$, $L = L_0 \supset L_2 \supset \cdots \supset L_{k-1}$, called the *link* of x, with compatible maps

$$\begin{array}{ccc} U & \xrightarrow{\sim} & \mathbb{R}^{n-k} \times \mathrm{cone}^\circ(L) \\ \uparrow & & \uparrow \\ U \cap X_i & \xrightarrow{\sim} & \mathbb{R}^{n-k} \times \mathrm{cone}^\circ(L_i) \\ \uparrow & & \uparrow \\ U \cap X_k & \xrightarrow{\sim} & \mathbb{R}^{n-k} \times \{\mathrm{vertex}\} \end{array}$$

  for any $0 \leq i < k$. Here $\mathrm{cone}^\circ(L) = L \times [0,\infty)/(L \times 0)$ is the open cone over L with vertex $L \times 0/(L \times 0)$. The vertical maps are the obvious inclusions.

Note that the indexing of the strata and the X_k by codimension differs from the one in most references. We use it because it is more clear when keeping track of dimensions of intersections. An intersection of a closed submanifold $Z \subset X$ with an S_k is dimensionally transverse if the dimension of Z is reduced by k (or if it is empty).

2.2. **Piecewise linear theory.** We begin with the 'simplicial' version of intersection homology theory, which is the historical route and familiar from usual homology theory. For this we need to consider the category of piecewise linear (pl) topological spaces.

Formally, a *pl-space* is a topological space X with a class $\mathcal{T}$ of locally finite triangulations such that for a triangulation T of X in the class any subdivision of T again lies in $\mathcal{T}$, and also any $T, T' \in \mathcal{T}$ have a common refinement in $\mathcal{T}$. Think for example of a simplicial complex along with all possible refinements. An advantage of having a whole class of triangulations is that any open set $U \subset X$ inherits a pl-structure. This is useful for constructing sheaves later on.

A map between pl-spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ should be a map $X \to Y$ for which there are triangulations $T_X \in \mathcal{T}_X$ and $T_Y \in \mathcal{T}_Y$ such that the image of any simplex $\Delta \in T_X$ lies inside a simplex of T_Y.

Definition 2.2. An *n-dimensional pl-pseudomanifold* is a pl-space which is the union of closed n-simplices in some admissible triangulation, and such that any $(n-1)$-simplex is the face of exactly two n-simplices.

Proposition 2.3. *Any pl-pseudomanifold has a filtration making it a stratified pseudomanifold in the category of pl-spaces and maps.*

Remark 2.4. The construction of the stratification is straightforward. Consider the triangulation from Definition 2.2. Then define X_k to be the union of the closed $(n-k)$-simplices for $k \geq 2$, perforce $X_1 = X_2$. Then $X_k \setminus X_{k-1}$ is obviously a manifold. That $X \setminus X_2$ is also a manifold follows from the pl-pseudomanifold condition that any $(n-1)$-simplex is the face of exactly two n-simplices. We omit the proof of local normal triviality, but see [16] Proposition 1.4.

2.3. **Borel-Moore Homology.** Let X be a pl-space from now on. Borel-Moore homology (also called homology with closed supports) is to usual homology what cohomology $H^*(X)$ is to cohomology $H_c^*(X)$ with compact supports, see for example [11] and [8, Section 2.6]. Following the conventions of [11] we will use the otherwise non-standard notation

$$\begin{aligned} H_*(X) &= \text{Borel Moore homology of } X \text{ over } \mathbb{C}, \\ H_*^c(X) &= \text{usual (e.g. singular or simplicial) homology of } X \text{ over } \mathbb{C}. \end{aligned}$$

The two notions coincide if X is compact.

Definition 2.5. For a triangulation T of X an i-chain is defined to be a formal $\mathbb{C}$-linear combination of (possibly infinitely many) oriented i-simplices. Denote by $C_i^T(X)$ the vector space of all such i-chains. For any refinement T' of T there is a map $C_i^T(X) \to C_i^{T'}(X)$ so that we can take the limit

$$C_i(X) = \varinjlim_{T \in \mathcal{T}} C_i^T(X).$$

The elements of $C_i(X)$ are called *geometric i-chains* or *Borel-Moore chains*. The usual boundary map defined on individual oriented simplices induces a boundary map $\partial : C_i(X) \to C_{i-1}(X)$. The *Borel-Moore homology* $H_*(X)$ of X is defined to be the homology of this complex. We call a geometric i-chain ξ with $\partial\xi = 0$ an i-cycle.

A geometric i-chain ξ is represented by an i-chain in some $C_i^T(X)$. Its support $|\xi| \subset X$ is the union of all closed i-simplices in T which occur with nonzero coefficient in ξ. It is closed in X, which is why Borel-Moore homology is also often called 'homology with closed supports'.

We may also consider the complex of compactly supported chains $C_i^c(X) \subset C_i(X)$, which coincides with the limit over compactly supported chains in $C_i^T(X)$. Since taking homology commutes with taking (inductive) limits, its homology computes the usual simplicial homology $H_*^c(X)$ of X. The dual complexes $C^i(X) := C_i^c(X)^*$ and $C_c^i(X) := C_i(X)^*$ then compute the usual cohomology $H^*(X)$ and its compactly supported version $H_c^*(X)$, respectively.

2.4. **Poincaré duality.** If X is an oriented n-manifold, then there is the well known Poincaré duality pairing for the homology of X. That is, there is a well-defined intersection product $H_j(X) \times H_k^c(X) \to H_{n-j-k}^c(X)$ such that the resulting map

$$H_i(X) \times H_{n-i}^c(X) \to H_0^c(X) \to \mathbb{C}$$

gives a perfect pairing. Here the final map is the augmentation homomorphism $H_0^c(X) \to H_0^c(\{\mathrm{pt}\}) \cong \mathbb{C}$ which is induced from mapping X to a point. The following two properties go into the definition of the intersection product.

- For any two pl-chains, it is possible to replace either one with a homologous one to make its support transversal to that of the other.
- For any two homology classes and transversal chains representing them, the class of the intersection is independent of the particular choice of transversal representatives.

We can see how Poincaré duality fails for pseudomanifolds in an example.

Remark 2.6 (Example). Let us consider a cylinder $X = \mathbb{R} \times S^1$ and a pinched cylinder, $Y = X/(0 \times S^1)$, where the circle at the origin is identified to a single point p_0. The homology is easy to compute :

i	$H_i(X)$	$H_i^c(X)$
0	0	$\mathbb{C}$
1	$\mathbb{C}$	$\mathbb{C}$
2	$\mathbb{C}$	0

i	$H_i(Y)$	$H_i^c(Y)$
0	0	$\mathbb{C}$
1	$\mathbb{C}$	0
2	$\mathbb{C} \oplus \mathbb{C}$	0

We see that Poincaré duality fails badly for Y. Let $\Sigma := \{p_0\} \subset Y$. The problem is apparent. Any geometric 1-chain representing a generator of $H_1(Y)$ passes through p_0. Hence it cannot be made transversal to Σ.

Therefore we need to restrict the allowed chains if there is a singular locus. For any geometric i-chain ξ with support $|\xi|$ we impose the *first intersection condition :*

$$\dim |\xi| \cap \Sigma \leq \dim |\xi| - 2. \tag{2.1}$$

For singular surfaces (2-dimensional pseudomanifolds) this condition suffices to define intersection homology and we get :

i	$IH_i(Y)$	$IH_i^c(Y)$
0	0	$\mathbb{C} \oplus \mathbb{C}$
1	0	0
2	$\mathbb{C} \oplus \mathbb{C}$	0

2.5. **Perversities and intersection homology.** So far we have introduced one intersection condition. Supposing the singular locus Σ is as large as it can be for a pseudomanifold (i.e. codimension 2), then it just says that any allowable chain must intersect Σ transversally. This however is just the tip of the iceberg, since intersection homology also takes into account the singularities within the singular set and so forth. There are different versions of intersection homology depending on the strictness of the intersection conditions with the various strata. These are encoded in what is called the perversity.

Definition 2.7. A *perversity* of a fixed dimension n is a map $\underline{p} : \{2, 3, \ldots, n\} \to \mathbb{Z}_{\geq 0}$ such that $\underline{p}(2) = 0$ and

$$\underline{p}(k) \in \left\{ \underline{p}(k-1)\, ,\, \underline{p}(k-1) + 1 \right\}.$$

The most important perversities are given by

$$\begin{array}{rcll} \underline{0} & = & (0,0,0,\ldots\ldots\ldots,0\,) & \text{0-perversity} \\ \underline{m} & = & (0,0,1,1,2,2,3,\ldots) & \text{lower middle perversity} \\ \underline{n} & = & (0,1,1,2,2,3,3,\ldots) & \text{upper middle perversity} \\ \underline{t} & = & (0,1,2,3,\ldots,n-2) & \text{top perversity.} \end{array}$$

Two perversities $\underline{p}$ and $\underline{q}$ are called *complementary* if $\underline{p}+\underline{q}=\underline{t}$ (e.g. $\underline{m}$ and $\underline{n}$).

Definition 2.8 (Intersection homology)**.** Let $\underline{p}$ be a fixed perversity. A geometric i-chain $\xi \in C_i(X)$ is called $\underline{p}$-allowable if

$$(*)_{\underline{p},k} \qquad \begin{array}{rcl} \dim(|\xi| \cap X_k) & \leq & i-k+\underline{p}(k), \\ \dim(|\partial\xi| \cap X_k) & \leq & i-1-k+\underline{p}(k) \end{array}$$

holds, for all $2 \leq k \leq n$. Notice that for all $\underline{p}$, $(*)_{\underline{p},2}$ coincides with the first intersection condition from (2.1) for ξ and $\partial\xi$.

Let us denote by

$$I_{\underline{p}}C_i(X) = \{\ \xi \in C_i(X) \mid \xi \text{ satisfies } (*)_{\underline{p},k}\ \}$$

the vector space of $\underline{p}$-allowable i-chains. The boundary map on geometric chains restricts to $\partial : I_{\underline{p}}C_i(X) \to I_{\underline{p}}C_{i-1}(X)$ to define a chain complex. The homology of this complex is denoted $I_{\underline{p}}H_*(X)$ and called the *intersection homology* of X with perversity $\underline{p}$.

Remark 2.9. If $\xi \in I_{\underline{0}}C_i$, then $|\xi|$ is transversal to all strata S_k of X. This is the strictest of the intersection conditions. In general if $\underline{p} < \underline{q}$, then $I_{\underline{p}}C_i(X) \subset I_{\underline{q}}C_i(X)$ and we have a map

$$I_{\underline{p}}H_i(X) \longrightarrow I_{\underline{q}}H_i(X).$$

In the weakest case, the top perversity $\underline{t}(k) = k-2$, the intersection conditions $(*)_{\underline{t},k}$ on a chain ξ are

$$\dim(|\xi| \cap X_k) \leq \dim|\xi| - 2,$$
$$\dim(|\partial\xi| \cap X_k) \leq \dim|\xi| - 3.$$

But for $k > 2$ these are automatic consequences of the first intersection condition, $(*)_{\underline{t},2}$. Therefore $I_{\underline{t}}C_*(X)$ is just the complex of chains ξ with $\dim(|\xi| \cap \Sigma) \leq \dim|\xi| - 2$, and the same condition for $\partial\xi$.

2.6. **Intersection homology and normality.**

Definition 2.10. An n-dimensional stratified pseudomanifold X is called *normal* if any $x \in \Sigma$ has a distinguished neighborhood U_x such that $U_x \setminus \Sigma$ is connected.

This definition of normality is a topological analogue of normality for algebraic varieties. Any pseudomanifold X has a *normalization* $\pi : \tilde{X} \to X$, where $\tilde{X}$ is normal, and π is 1-1 on the open stratum S_0 and finite to one on any other stratum. Explicitly $\#\{\pi^{-1}(x)\}$ is the number of connected components of $U_x \setminus \Sigma$. This map π also induces a map on chain complexes

$$C_*(\tilde{X}) \to C_*(X). \tag{2.2}$$

Lemma 2.11. *Any $\underline{p}$-allowable cycle σ of X is the image of a $\underline{p}$-allowable cycle $\tilde{\sigma}$ of $\tilde{X}$ under the map (2.2).*

Proof. For $\sigma \in C_i(X)$ there exists a geometric i-chain $\tilde{\sigma} \in C_i(\tilde{X})$ such that $|\tilde{\sigma}| = \pi^{-1}(|\sigma|)$. It follows from $\partial\sigma = 0$ that $|\partial\tilde{\sigma}| \subset \tilde{\Sigma}$. So we have

$$|\pi(\partial\tilde{\sigma})| \subset |\sigma| \cap \Sigma.$$

Here the left hand side is $(i-1)$-dimensional. But intersecting with Σ should cut down the dimension of σ by at least 2, by the first intersection condition. So $\partial\tilde{\sigma} = 0$.

Also $(*)_{\underline{p},k}$ for σ implies $(*)_{\underline{p},k}$ for $\tilde{\sigma}$. □

Corollary 2.12. *The map* (2.2) *induces an isomorphism*

$$I_{\underline{p}}H_*(\tilde{X}) \cong I_{\underline{p}}H_*(X).$$

The pinched cylinder from Remark 2.6, for example, is not normal. We have that the normalization $\tilde{Y}$ of the pinched cylinder consists of two separate hemispheres (the pinched cylinder is divided into two halves at the singular point which is split into two). Topologically $\tilde{Y}$ is a disjoint union $\mathbb{R}^2 \sqcup \mathbb{R}^2$. Thus the result in Remark 2.6 giving the intersection homology of Y is an illustration of Corollary 2.12. From that point of view, the example of the pinched cylinder just reduces to a calculation of classical homology. But see Goresky and MacPherson's Chapter III in [11] for a more sophisticated sample computation.

2.7. **Top- and $\underline{0}$-perversity intersection homology.** For the two extremal perversities intersection homology compares closely with Borel-Moore homology and usual cohomology, respectively. We compute the cohomology of X by the cochain complex with $C^i(X) = C_i^c(X)^*$. One can define two maps,

$$I_{\underline{t}}C_i(X) \to C_i(X), \tag{2.3}$$

$$C^{n-i}(X) \to I_{\underline{0}}C_i(X). \tag{2.4}$$

The first one is just the obvious inclusion. The second one is defined by Poincaré's dual cycle map. Any simplex in the support of a cochain is replaced by the complementary dimensional simplices in the barycentric subdivision which meet the center of the original simplex. See for example [26, §65]. The resulting chain is then automatically transversal to all the strata, hence lies in $I_{\underline{0}}C_i(X)$.

Proposition 2.13. *Suppose X is oriented and normal. The chain maps* (2.3) *and* (2.4) *induce isomorphisms*

(1) $I_{\underline{t}}H_*(X) \xrightarrow{\sim} H_*(X)$,
(2) $H^{n-i}(X) \xrightarrow{\sim} I_{\underline{0}}H^*(X)$.

Remark 2.14. Note that normality implies that $X \setminus X_3$ is still a topological manifold. The link L of any point in the stratum S_2 must be homeomorphic to S^1, and $\mathrm{cone}^\circ(S^1) \cong \mathbb{R}^2$. So any actual singularities occur in one codimension higher.

Recall that by Remark 2.9, $\underline{t}$-allowability just comes down to the first intersection condition for ξ and $\partial\xi$. So Proposition 2.13.(1) basically says that normality gives the amount of flexibility required to insure any i-cycle ξ is homologous to one transversal enough to $\Sigma = X_2$. See for example [13] for the proof. We will prove (2) in Remark 4.10.

2.8. **Poincaré duality pairing.** There is an analogue of the intersection product between intersection homology groups of different perversities.

Theorem 2.15. [13] *Let X be an n-dimensional pl-pseudomanifold.*

(1) *Let $\underline{p}, \underline{q}, \underline{r}$ be three perversities with $\underline{p} + \underline{q} \leq \underline{r}$. Then there is a well-defined pairing*

$$\cap \ : I_{\underline{p}}H_i(X) \times I_{\underline{q}}H_j^c(X) \longrightarrow I_{\underline{r}}H_{n-i-j}^c(X).$$

(2) *If $\underline{p} + \underline{q} = \underline{t}$, then*

$$I_{\underline{p}}H_i(X) \times I_{\underline{q}}H_{n-i}^c(X) \xrightarrow{\cap} I_{\underline{t}}H_0^c(X) \longrightarrow \mathbb{C}.$$

defines a perfect pairing.

Remark 2.16. The intersection pairing is defined by intersecting allowable cycles that are "sufficiently transverse". That is, the intersection must be in sufficiently general position with respect to each stratum (to give an element of the correct $I_{\underline{r}}C_*$). Then part (1) of the theorem says that any two homology classes can be represented by cycles with this property (for this $\underline{r}$ is not allowed to be too small), and then the homology class of the resulting intersection is independent of the choices. If $\underline{p} = \underline{0}$ and $\underline{q} = \underline{t}$, and X orientable and normal, then Theorem 2.15.(2) is just the classical cap product $H^{n-i}(X) \times H^c_{n-i}(X) \to \mathbb{C}$ (after applying Proposition 2.13).

2.9. **Middle intersection homology.** If $X_{k+1} = X_k$ in the filtration of X, then the intersection condition $(*)_{\underline{p},k+1}$ implies $(*)_{\underline{p},k}$. So $I_{\underline{p}}C_*(X)$ only depends on values $\underline{p}(k)$ for nonempty strata S_k. For example if all strata are of even dimension then the odd values of $\underline{p}$ do not matter. In particular in that case $I_{\underline{m}}C_*(X) = I_{\underline{n}}C_*(X)$, and one can speak of *middle intersection homology.*

Corollary 2.17. *If X has only even-dimensional strata, then the middle intersection homology $I_{\underline{m}}H_*(X)$ satisfies Poincaré duality. In other words*

$$I_{\underline{m}}H_i(X) \times I_{\underline{m}}H^c_{n-i}(X) \xrightarrow{\cap} I_{\underline{t}}H^c_0(X) \longrightarrow \mathbb{C}$$

is a non-degenerate pairing.

2.10. **Sheaves.** One feature of Borel-Moore chains is that for any open $U \hookrightarrow X$ we have restriction maps

$$C_i(X) \to C_i(U)$$
$$I_{\underline{p}}C_i(X) \to I_{\underline{p}}C_i(U).$$

For compactly supported chains C_i^c one has instead an inclusion $C_i^c(U) \hookrightarrow C_i^c(X)$.

Definition 2.18. Define sheaves $\mathcal{D}^i_X$ and $\mathcal{I}_{\underline{p}}\mathcal{C}^i_X$ on X by

$$\mathcal{D}^{-i}_X(U) := C_i(U),$$
$$\mathcal{I}_{\underline{p}}\mathcal{C}^{-i}_X(U) := I_{\underline{p}}C_i(U)$$

for all $U \subset X$ open. These sheaves form cochain complexes $\mathcal{D}^\bullet_X$ and $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X$ of sheaves on X, with differential induced from the boundary maps ∂ over each open set.

The original complexes of vector spaces are recovered (up to the change in indexing) by applying the functors of global sections or global sections with compact supports. So we have

$$H_i(X) = H^{-i}(\Gamma(\mathcal{D}^\bullet_X)), \qquad H^c_i(X) = H^{-i}(\Gamma_c(\mathcal{D}^\bullet_X)),$$
$$I_{\underline{p}}H_i(X) = H^{-i}(\Gamma(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X)), \qquad I_{\underline{p}}H^c_i(X) = H^{-i}(\Gamma_c(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X)).$$

We also have sheaves $\mathcal{C}^i_X$ on X defined by

$$\mathcal{C}^i_X(U) = C^i(U) := C^c_i(U)^*.$$

The corresponding cochain complex $\mathcal{C}^\bullet_X$ computes the cohomology of X by

$$H^i(X) = H^i(\Gamma(\mathcal{C}^\bullet_X)) \quad \text{and} \quad H^i_c(X) = H^i(\Gamma_c(\mathcal{C}^\bullet_X)).$$

Note that now $\mathcal{C}^i_X(U) \to C^{T,c}_i(U)^*$ for any admissible triangulation T. While the geometric i-chains were a direct limit over all triangulations, the cochains are necessarily an inverse limit.

3. Lecture - Derived functors and the derived category of sheaves

3.1. **Complexes and functors.** In the last lecture we introduced complexes of sheaves $\mathcal{D}_X^\bullet$ and $\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet$ on a pseudomanifold X. These are the objects we will be working with. Let us for the moment view them as objects in the category of complexes of sheaves with chain maps. Denote the i-th cohomology (sheaf) of a complex $\mathcal{F}^\bullet$ by $\mathcal{H}^i(\mathcal{F}^\bullet)$. We are more interested in the cohomology of our complexes than the complexes themselves.

Definition 3.1. A map of complexes

$$\begin{array}{cccccccccccc} \mathcal{F}^\bullet & : & \cdots \to & \mathcal{F}^{m-1} & \to & \mathcal{F}^m & \to & \mathcal{F}^{m+1} & \to & \mathcal{F}^{m+2} & \to \cdots \\ q\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \mathcal{G}^\bullet & : & \cdots \to & \mathcal{G}^{m-1} & \to & \mathcal{G}^m & \to & \mathcal{G}^{m+1} & \to & \mathcal{G}^{m+2} & \to \cdots \end{array}$$

is called a *quasi-isomorphism* if it induces isomorphisms $\mathcal{H}^i(q)\ :\mathcal{H}^i(\mathcal{F}^\bullet) \xrightarrow{\sim} \mathcal{H}^i(\mathcal{G}^\bullet)$ on the cohomology sheaves of the complex.

We would like to work in a category (of complexes of sheaves) where two quasi-isomorphic complexes are interchangeable. Moreover this category should 'extend' the category of sheaves in the sense that that the functors that one uses in the category of sheaves extend to functors between complexes.

The problem with these two demands is already apparent in the example of the global sections functor Γ. Consider the quasi-isomorphism from Remark 1.3,

$$\begin{array}{cccccccccccc} \underline{\mathbb{C}}_X[0] & : & \cdots \to & 0 & \to & \underline{\mathbb{C}}_X & \to & 0 & \to & \cdots & \to & 0 & \to \cdots \\ q\downarrow & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & \\ \Omega_X^\bullet & : & \cdots \to & 0 & \to & \Omega_X^0 & \to & \Omega_X^1 & \to & \cdots & \to & \Omega_X^n & \to \cdots, \end{array}$$

and apply Γ to it. The cohomology $H^i(\Gamma(\Omega_X^\bullet)) = H^i(X)$ of the lower complex will usually be nonzero in positive degree, for example if X is a compact manifold of positive dimension. So $\Gamma(q)$ is far from being a quasi-isomorphism. However we do recover the identity $\Gamma(\underline{\mathbb{C}}_X) = H^0(X)$ in degree zero.

These properties reflect the fact that Γ is left exact and not exact. The correct definition of the extension of a left exact functor to complexes may be illustrated by the above example. It says not to apply Γ to arbitrary complexes, but only to ones whose objects are 'Γ-acyclic'. The Ω_X^i are examples of Γ-acyclic sheaves. They are soft for example (see Section 3.8). Γ-acyclic sheaves have good local to global behaviour, making Γ act like an exact functor.[1] Any other complexes such as $\underline{\mathbb{C}}_X[0]$ should be replaced by quasi-isomorphic ones such as $\Omega_X^\bullet$ before applying Γ. Let us now explain these definitions more formally.

3.2. **Right derived functors and $\mathcal{D}^+(X)$.** Recall that an object I in an abelian category is called *injective* if $Hom(\ \ ,I)$ is an exact functor. Injective objects have the following useful property.

- If $q : I^\bullet \to J^\bullet$ is a quasi-isomorphism between complexes of injectives, then there is a quasi-isomorphism $p : J^\bullet \to I^\bullet$ such that $p \circ q : I^\bullet \to I^\bullet$ is homotopic to the identity.

[1] This may sound contradictory, since for example $\Omega_X^{n-1} \xrightarrow{d_n} \Omega_X^n \to 0$ is exact for an n-manifold X, while the global sections can give cohomology in degree n. But that can happen because the kernel of d_n need not be Γ-acyclic.

This implies that any additive functor F will act on injectives as if it were an exact functor, that is, preserving quasi-isomorphisms.

Let us introduce a slightly better category to work in, the homotopy category

$$\mathcal{K}^+(\mathrm{Sh}(X)) = \begin{pmatrix} \text{bounded below cochain complexes of sheaves in } \mathrm{Sh}(X) \\ \text{with cochain maps modulo homotopy equivalence} \end{pmatrix}.$$

Any two quasi-isomorphic complexes of injectives are isomorphic in $\mathcal{K}^+(\mathrm{Sh}(X))$. The (bounded below) derived category of $\mathrm{Sh}(X)$ is defined by

$$\mathcal{D}^+(X) := \mathcal{K}^+(\mathrm{Sh}(X))[\ (\text{quasi-isomorphisms})^{-1}\ ],$$

adjoining an inverse to any quasi-isomorphism in the homotopy category (localization). The category $\mathcal{D}^+(X)$ is no longer abelian but instead a 'triangulated category'. See for example [18] or [33].

Let $\mathcal{K}^+(Inj(X)) \subset \mathcal{K}^+(\mathrm{Sh}(X))$ be the full subcategory whose objects are complexes of injectives. Then we also have an inclusion $\iota : \mathcal{K}^+(Inj(X)) \hookrightarrow \mathcal{D}^+(X)$. To formally define a right derived functor of a functor F, we need a functor $I : \mathcal{D}^+(X) \to \mathcal{K}^+(Inj(X))$ assigning to any complex an injective resolution,

$$\begin{array}{ccccccccccc} \mathcal{F}^\bullet & : & \cdots \to & 0 & \to & \mathcal{F}^0 & \to & \mathcal{F}^1 & \to & \mathcal{F}^2 & \to \cdots \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ I(\mathcal{F}^\bullet) & : & \cdots \to & 0 & \to & \mathcal{I}^0 & \to & \mathcal{I}^1 & \to & \mathcal{I}^2 & \to \cdots , \end{array} \tag{3.1}$$

such that I and ι are inverse equivalences of categories. Such a functor exists for the category of sheaves, as we will see in the next section.

Definition 3.2. Suppose that $F : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is a left exact functor. Then define $RF(\mathcal{F}^\bullet) := F(I(\mathcal{F}^\bullet))$ for a complex of sheaves $\mathcal{F}^\bullet$. This defines a functor

$$RF : \mathcal{D}^+(X) \to \mathcal{D}^+(Y)$$

called a *right derived functor* of F. The i-th right derived functor

$$R^iF : \mathcal{D}^+(X) \to \mathrm{Sh}(Y)$$

is defined by $R^iF = \mathcal{H}^i \circ RF$.

Given two different injective resolutions of a complex one can always construct a quasi-isomorphism between them and uniquely up to homotopy, rendering them canonically isomorphic in the homotopy category. So the choice of the functor I hardly matters.

Suppose $\mathcal{F}[0]$ is the complex with $\mathcal{F}$ in degree 0 and zero elsewhere. If F is a left exact functor, then applying F to the quasi-isomorphism (3.1) induces an isomorphism $F(\mathcal{F}) \cong R^0F(\mathcal{F}[0])$. This is the sense in which RF is an extension of the functor F to complexes.

3.3. **Injectives in** $\mathrm{Sh}(X)$**.** In the category of vector spaces, that is the case of $\mathrm{Sh}(\{pt\})$, every object is injective and projective (the dual property) by an application of Zorn's Lemma. More generally there is the following characterization of injective sheaves due to N. Spaltenstein.

Lemma 3.3. *A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is injective precisely if for all open subsets $V \subset U$, the restriction map $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is a surjection.*

This says that for sheaves of vector spaces being injective is equivalent to being *flabby* (any section comes from a global section). See [11, V Lemma 1.13] for a more general formulation and the proof of this lemma. In the setting of sheaves of modules over a ring, say, one needs also the further requirements that the stalks and sections of $\mathcal{F}$ should be injective and the restriction maps be split.

The category of sheaves has "enough injectives", meaning that every sheaf embeds into an injective one. Explicitly, we may let $\mathcal{F} \hookrightarrow \mathcal{I}$ where

$$\mathcal{I}(U) := \prod_{x \in U} \mathcal{F}_x$$

with the obvious restriction (by projection) maps. Embedding the cokernel of this inclusion into another injective and so forth, one obtains an injective resolution of $\mathcal{F}$ called the *Godement resolution*. So any sheaf has an injective resolution. Viewed inside the derived category this resolution becomes an isomorphism (i.e. quasi-isomorphism of complexes)

$$\begin{array}{ccccccccc} \cdots \to & 0 & \to & \mathcal{F} & \to & 0 & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & \mathcal{I}^0 & \to & \mathcal{I}^1 & \to \cdots \end{array}$$

The existence of enough injectives also implies that any complex of sheaves $\mathcal{F}^\bullet$ is quasi-isomorphic to a complex of injectives $I(\mathcal{F}^\bullet)$. This quasi-isomorphic complex of injectives may be obtained as the total complex of a double complex constructed from the injective resolutions of all of the individual $\mathcal{F}^i$'s (the *Cartan-Eilenberg* resolution).

3.4. **The bounded derived category.** For our purposes it will be more convenient to consider complexes that are bounded in both directions. These form a full subcategory $\mathcal{D}^b(X)$ in $\mathcal{D}^+(X)$ called the *bounded derived category* of sheaves on X. One constructs bounded complexes from unbounded ones by *truncation*: Let $\tau_{\leq k} : \mathcal{D}^+(X) \to \mathcal{D}^b(X)$ be defined on objects by

$$\tau_{\leq k}(\mathcal{F}^\bullet) = (\cdots \to \mathcal{F}^{k-2} \to \mathcal{F}^{k-1} \to \ker(d) \xrightarrow{d} 0 \to 0 \to \cdots)$$

with the obvious extension to morphisms. $\tau_{\leq k}$ is called the (right) truncation functor at k since

$$\mathcal{H}^i(\tau_{\leq k}(\mathcal{F}^\bullet)) = \begin{cases} \mathcal{H}^i(\mathcal{F}^\bullet) & i \leq k, \\ 0 & i > k. \end{cases}$$

To define derived functors between bounded derived categories we need bounded injective resolutions. These may be obtained as follows. If $\mathcal{I}^\bullet \in \mathcal{D}^+(X)$ is an injective resolution of $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and X has dimension n, then $\tau_{\leq n+1}\mathcal{I}^\bullet$ is still an injective resolution of $\mathcal{F}^\bullet$, see [11, V Proposition 1.17].

Let us now introduce some explicit functors between derived categories of sheaves.

3.5. **Push-forwards.** Let $f : X \to Y$ be a continuous map. Most interesting functors on sheaves are left exact :

$$\begin{aligned} \Gamma, \Gamma_c &: \mathrm{Sh}(X) \to (\text{vector spaces}), \\ f_*, f_! &: \mathrm{Sh}(X) \to \mathrm{Sh}(Y). \end{aligned}$$

Here the first two functors are the global sections functors introduced in Section 1.2. The second are the push-forward (or direct image) functor f_* and the push-forward with proper supports $f_!$. They are defined by

$$f_*\mathcal{X}(U) = \mathcal{X}(f^{-1}(U)),$$
$$f_!\mathcal{X}(U) = \{s \in \mathcal{X}(f^{-1}(U)) \mid f : \mathrm{Supp}(s) \to U \text{ is proper}\},$$

where $\mathcal{X}$ is a sheaf on X. Recall that a map between topological spaces is called proper if the inverse image of a compact set is compact. For a composition of maps $f \circ g$ the push-forwards are $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)_! = f_! \circ g_!$ respectively. If $f = \pi : X \to \{pt\}$, then we recover $\pi_* = \Gamma$ and $\pi_! = \Gamma_c$. We have the derived functors

$$Rf_*, Rf_! : \mathcal{D}^b(X) \to \mathcal{D}^b(Y).$$

In the special case where $f = j : X \hookrightarrow Y$ is an inclusion of a locally closed subset,

$$(j_!\mathcal{X})_y = \begin{cases} 0 & y \notin X \\ \mathcal{X}_y & y \in X \end{cases}.$$

This functor is therefore called *extension by zero*, and it is exact. The extension by zero of the constant sheaf $\underline{\mathbb{C}}_{\{x_0\}}$ from $\{x_0\} \hookrightarrow X$ is the skyscraper sheaf $\mathcal{S}$ from Remark 1.2.

3.6. **Pull-backs.** Any continuous map $f : X \to Y$ gives rise to a pull-back (or inverse image) functor $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, which takes any $\mathcal{Y} \in \mathrm{Sh}(Y)$ to the sheaf on X associated to the presheaf

$$U \mapsto \varinjlim_{\{V \text{ open} \mid f(U) \subset V\}} \mathcal{Y}(V).$$

The key property of the pull-back is that

$$(f^*\mathcal{Y})_x = \mathcal{Y}_{f(x)}.$$

The restriction functors from Section 1.2 are pull-backs in the special case where f is an inclusion of a locally closed subset. The pull-back f^* defines a functor

$$f^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$$

directly without needing to be derived, since it is exact. For a composition of maps $f \circ g$ the pull-back is $(f \circ g)^* = g^* \circ f^*$.

Pull-back and push-forward are adjoint functors. I.e. there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*\mathcal{Y}, \mathcal{X}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{Y}, f_*\mathcal{X}).$$

For example the adjunction morphism $1 \to f_* \circ f^*$ is given explicitly (on $\mathcal{Y}$) by the natural maps $\mathcal{Y}(V) \to f^*(\mathcal{Y})(f^{-1}(V))$. In the derived category f^* becomes left adjoint to Rf_* using the above adjunction morphism together with the natural transformation coming from (3.1).

3.7. **Shifts.** Shifting the indices of a complex to the right defines a functor

$$[n] : \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

This functor takes $\mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet[n]$, where $(\mathcal{F}^\bullet[n])^i = \mathcal{F}^{i+n}$ and the differential of the complex is conventionally multiplied by a factor of $(-1)^n$.

3.8. **Computing derived functors.** Injective sheaves are a good theoretical tool, but not always practically useful. For computing right derived functors it tends to be more convenient to use resolutions that are not quite injective but still in a sense 'flexible enough' for the functor in question.

Definition 3.4. Let F be a left exact functor from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(Y)$. A sheaf $\mathcal{A} \in \mathrm{Sh}(X)$ is called *F-acyclic* if $R^iF(\mathcal{A}) = 0$ for all $i \neq 0$.

By the spectral sequence

$$E_2^{ij} = H^i(R^jF(\mathcal{A}^\bullet)) \Rightarrow R^{i+j}F(\mathcal{A}^\bullet)$$

for computing the cohomology of the Cartan-Eilenberg double complex we see that F-acyclic complexes are good enough for computing RF (up to isomorphism). In other words, if $\mathcal{F}^\bullet$ is quasi-isomorphic to an F-acyclic complex $\mathcal{A}^\bullet$, then $RF(\mathcal{F}^\bullet) \cong F(\mathcal{A}^\bullet)$ in $\mathcal{D}^b(X)$.

Definition 3.5. A sheaf $\mathcal{F}$ is called *soft* if the restriction maps

$$\Gamma(U, \mathcal{F}) \to \Gamma(K, \mathcal{F})$$

are surjective for all compact K and open $U \subset X$ containing K. Here $\Gamma(K, \mathcal{F}) := \Gamma(i^*\mathcal{F})$ for the inclusion $i : K \hookrightarrow U$.

In some references the definition of soft requires the surjectivity for all closed K, in which case the above version is called c-soft. However if X is a union of countably many compact sets (which is always the case for us) then the two definitions are equivalent.

A soft sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is automatically Γ_c-acyclic and Γ-acyclic, and the restriction of a soft sheaf to a locally closed subset is again soft. See for example [18, III Theorem 2.7 and IV Theorem 2.2] or [19, II 2.5].

Lemma 3.6. *$\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet$ (and $\mathcal{C}_X^\bullet, \mathcal{D}_X^\bullet$) are complexes of soft sheaves.*

Proof. Let K and $U \subset X$ be as in the definition of softness above. Suppose $s \in \Gamma(K, \mathcal{I}_{\underline{p}}\mathcal{C}_X^{-i})$ is represented by $\xi \in \Gamma(V, \mathcal{I}_{\underline{p}}\mathcal{C}_X^{-i})$ for some open neighborhood V of K. We may assume $V \subset U$. The problem with ξ is that it might not have closed support in U. Choose a triangulation of V fine enough such that there is a closed pl-neighborhood N of K entirely in V. Then the intersection of ξ with N lies in $\Gamma(U, \mathcal{I}_{\underline{p}}\mathcal{C}_X^{-i})$ and also represents s, and we are done. The proofs for $\mathcal{C}_X^\bullet$ and $\mathcal{D}_X^\bullet$ are similar. $\square$

As a consequence we see that the intersection homology is the *hypercohomology* (derived functor of Γ) of the complex $\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet$,

$$I_{\underline{p}}H_i(X) = R^{-i}\Gamma(\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet) =: \mathbb{H}^{-i}(X, \mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet),$$
$$I_{\underline{p}}H_i^c(X) = R^{-i}\Gamma_c(\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet) =: \mathbb{H}_c^{-i}(X, \mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet).$$

The complex of cochains $\mathcal{C}_X^\bullet$ on a pl-pseudomanifold X introduced in the end of Section 2.10 is a soft resolution of the constant sheaf $\underline{\mathbb{C}}_X[0]$. Therefore the cohomology of $\Gamma(\mathcal{C}_X^\bullet)$ is the sheaf cohomology (hypercohomology of a single sheaf) of $\underline{\mathbb{C}}_X$,

$$H^i(X) = R^i\Gamma(\mathcal{C}_X^\bullet) = \mathbb{H}^i(X, \underline{\mathbb{C}}_X[0]) =: H^i(X, \underline{\mathbb{C}}_X).$$

There are many useful formulas and identities for dealing with derived functors. We mention below some of the ones that will be needed later on, see for example V §10 in [11].

3.9. **Push-forward with proper supports.** For the projection $\pi : X \to \{pt\}$ and a complex of sheaves $\mathcal{X}^\bullet$ is on X we have $R^i\pi_!(\mathcal{X}^\bullet) = \mathbb{H}^i_c(X, \mathcal{X}^\bullet)$, by definition. Now let $f : X \to Y$ be any continuous map and $\mathcal{X}$ a sheaf on X. Then there is a natural isomorphism

$$(f_!\mathcal{X})_y \cong \Gamma_c(f^{-1}(y), \mathcal{X}|_{f^{-1}(y)})$$

for the stalk of the push-forward with proper supports. Applying this isomorphism for the sheaves in $\mathcal{I}^\bullet = I(\mathcal{X}^\bullet)$ in the definition of $Rf_!(\mathcal{X}^\bullet)$ gives the formula

$$(R^i f_!\mathcal{X}^\bullet)_y \cong H^i(\Gamma_c(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})) \cong \mathbb{H}^i_c(f^{-1}(y), \mathcal{X}^\bullet|_{f^{-1}(y)}). \tag{3.2}$$

Here the second isomorphism holds since $\mathcal{I}^\bullet|_{f^{-1}(y)}$ is a soft resolution of $\mathcal{X}^\bullet|_{f^{-1}(y)}$.

3.10. **Composition of derived functors.** The push-forward functor f_* takes injectives to injectives (for formal reasons, since it is right adjoint to the exact functor f^*). Therefore $R(g \circ f)_* \cong Rg_* \circ Rf_*$ for maps $f : X \to Y$ and $g : Y \to Z$. More generally, $R(G \circ F) \cong RG \circ RF$ if F turns injectives into G-acyclic objects.

3.11. **Base change.** Suppose we have a Cartesian square, that is, a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{h_1} & X_2 \\ f_1 \downarrow & & \downarrow f_2 \\ Y_1 & \xrightarrow{h_2} & Y_2 \end{array}$$

such that $(f_1, h_1) : X_1 \overset{\sim}{\to} \{(y, x) \in Y_1 \times X_2 \mid h_2(y) = f_2(x)\}$. Then there is a natural isomorphism of functors $\mathcal{D}^b(X_2) \to \mathcal{D}^b(Y_1)$,

$$h_2^* \circ R(f_2)_! = R(f_1)_! \circ h_1^*.$$

4. Lecture - Local intersection cohomology and Deligne's construction

4.1. **Intersection cohomology.** We will denote the hypercohomology of the complex $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X$ from Section 2.10 by

$$I_{\underline{p}}H^k(X) := \mathbb{H}^k(X, \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X),$$

and call it *intersection cohomology.* This differs from intersection homology only in the indexing: $I_{\underline{p}}H^k(X) = I_{\underline{p}}H_{-k}(X)$. Note that in these conventions intersection cohomology is concentrated in negative degrees. In the next few sections we want to investigate intersection cohomology locally in the neighborhood of a point.

Definition 4.1. Let $\mathcal{F}^\bullet \in \mathcal{D}^+(X)$ and $x \in X$. Then the *stalk cohomology* functor at x is the functor $\mathcal{H}^i_x : \mathcal{D}^+(X) \to$ (vector spaces) defined by

$$\mathcal{H}^i_x(\mathcal{F}^\bullet) := H^i(\mathcal{F}^\bullet_x).$$

An equivalent description is

$$\mathcal{H}^i_x(\mathcal{F}^\bullet) = \mathcal{H}^i(\mathcal{F}^\bullet)_x = \varinjlim_{\{U \text{ open} \mid x \in U\}} H^i(\mathcal{F}^\bullet(U)). \tag{4.1}$$

Remark 4.2. Intersection cohomology, unlike usual cohomology, *can* distinguish between spaces that are homotopic (but not homeomorphic) to one another. For example, as we shall see, even though every point in X has a contractible neighborhood, the local intersection cohomology and the stalks

$$\mathcal{H}_x^{-i}(\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet) = \varinjlim_{\{U \text{ open } \mid \, x \in U\}} I_{\underline{p}}H_i(U) \tag{4.2}$$

need not be trivial. Accordingly, $\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet$ is also not generally quasi-isomorphic to a single sheaf (unlike $\mathcal{C}_X^\bullet$).

Recall the notations from Definition 2.1 for the stratified pseudomanifold structure on X. Let x lie in a stratum S_k. Then we know that x has a distinguished neighborhood $U \cong \mathbb{R}^{n-k} \times \mathrm{cone}^\circ(L)$, where L is a $(k-1)$-dimensional compact pseudomanifold. To compute the stalk cohomology (4.2) it suffices to consider such distinguished neighborhoods U, since they form a neighborhood basis. Now $I_{\underline{p}}H_i(U)$ can be computed in terms of L in two steps.

4.2. **Suspension.** If X is a n-dimensional stratified pseudomanifold, then $\mathbb{R} \times X$ is an $(n+1)$-dimensional stratified pseudomanifold with respect to the filtration $(\mathbb{R} \times X)_i = \mathbb{R} \times X_i$. There is a natural suspension map

$$\mathbb{R} \times \_ \ : C_i(X) \to C_{i+1}(\mathbb{R} \times X)$$

on geometric chains satisfying $|\mathbb{R} \times \xi| = \mathbb{R} \times |\xi|$ on supports. Suppose $\underline{p}$ is an $(n+1)$-dimensional perversity giving rise to intersection conditions on both X and $\mathbb{R} \times X$. It is easy to see that the suspension $\mathbb{R} \times \xi$ of a $\underline{p}$-allowable chain is again $\underline{p}$-allowable. We have the following 'Künneth formula', see [11, Chapter II §2].

Proposition 4.3. *The suspension map*

$$\mathbb{R} \times \_ \ : C_i(X) \to C_{i+1}(\mathbb{R} \times X)$$

induces an isomorphism $I_{\underline{p}}H_i(X) \xrightarrow{\sim} I_{\underline{p}}H_{i+1}(\mathbb{R} \times X)$.

4.3. **Intersection homology of a cone.** Let $L \supset L_2 \supset \cdots \supset L_{k-1}$ be a compact $(k-1)$-dimensional stratified pseudomanifold, and $Y = \mathrm{cone}^\circ(L)$. So Y is a stratified pseudomanifold with

$$Y_i = \begin{cases} \mathrm{cone}^\circ(L_i) & 2 \le i < k, \\ \{\, v \,\} & i = k, \end{cases}$$

where v is the vertex of the cone. There is a natural map on chains,

$$\mathrm{cone}^\circ(\_) : C_i(L) \to C_{i+1}(\mathrm{cone}^\circ(L)),$$

satisfying $|\mathrm{cone}^\circ(\xi)| = \mathrm{cone}^\circ(|\xi|)$ on supports. Let $\underline{p}$ be a fixed perversity of dimension k.

Lemma 4.4. *If* $\xi \in I_{\underline{p}}C_{i-1}(L)$, *then*

$$\mathrm{cone}^\circ(\xi) \in I_{\underline{p}}C_i(\mathrm{cone}^\circ(L)) \iff \begin{cases} \textit{either} & i > k - \underline{p}(k), \\ \textit{or} & i = k - \underline{p}(k) \ \textit{and} \ \partial\xi = 0. \end{cases}$$

Proof. Note that $\operatorname{cone}^\circ(\xi)$ always meets the vertex, in other words the most singular stratum of $Y = \operatorname{cone}^\circ(L)$. So the dimension i of $\operatorname{cone}^\circ(\xi)$ must be big enough for this to be $\underline{p}$-allowable. That is,

$$0 \le \dim(|\operatorname{cone}^\circ(\xi)|) - k + \underline{p}(k), \tag{4.3}$$

$$0 \le \dim(|\partial(\operatorname{cone}^\circ(\xi))|) - k + \underline{p}(k). \tag{4.4}$$

This implies the necessary direction of the lemma. All other $\underline{p}$-allowability restrictions for $\operatorname{cone}^\circ(\xi)$ follow from the ones for ξ. □

Proposition 4.5. *The map* $\operatorname{cone}^\circ(\_) : C_i(L) \to C_{i+1}(\operatorname{cone}^\circ(L))$ *from above induces a quasi-isomorphism*

$$I_{\underline{p}}C_\bullet(L) \xrightarrow{\sim} \tau_{\le \underline{p}(k)}(I_{\underline{p}}C_{\bullet+1}(\operatorname{cone}^\circ(L))).$$

For a proof of this proposition see [11, Chapter II §3].

4.4. **Local intersection cohomology.** Let $x \in S_k \subset X$ and $U \cong \mathbb{R}^{n-k} \times \operatorname{cone}^\circ(L)$ be a distinguished neighborhood. Combining Propositions 4.3 and 4.5 one has

$$I_{\underline{p}}H^i(U) = \begin{cases} I_{\underline{p}}H^{n-k+1-i}(L) & -i \le -n + \underline{p}(k), \\ 0 & -i > -n + \underline{p}(k). \end{cases}$$

If $V \subset U$ is a compatible distinguished neighborhood of x, then the restriction map

$$\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X(U) \to \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X(V)$$

is a quasi-isomorphism, essentially since all cohomologically nontrivial cycles come from L. Therefore

$$\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X(U) \to (\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X)_x.$$

is a quasi-isomorphism and we have computed the stalk cohomology of $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X$.

Corollary 4.6. *Let $x \in X$ be a point lying in a stratum S_k with link L. The stalk cohomology $\mathcal{H}^i_x(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X)$ is concentrated in degrees $-n \le i \le -n + \underline{p}(k)$, where it is determined by the intersection homology of L as shown in the table below.*

i	$-n$	$-n+1$	$\cdots$	$-n+\underline{p}(k)$
$\mathcal{H}^i_x(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X)$	$I_{\underline{p}}H_{k-1}(L)$	$I_{\underline{p}}H_{k-2}(L)$	$\cdots$	$I_{\underline{p}}H_{k-\underline{p}(k)-1}(L)$

4.5. **A new beginning.** The rest of this lecture will be taken up with a recursive stratum by stratum construction of a new complex of sheaves $\mathcal{P}^\bullet$ with which to compute intersection cohomology. This construction is due to Deligne and appears in [14]. An immediate advantage the definition of $\mathcal{P}^\bullet$ has over $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X$ is that it does not depend on a pl-structure on X.

Let X be a fixed n-dimensional stratified pseudomanifold with all the usual notations. So $X \supset X_2 \supset X_3 \supset \cdots \supset X_n$, $\Sigma = X_2$ is the 'singular locus' and $S_k = X_k \setminus X_{k-1}$ is the codimension k stratum. Also define open sets

$$U_k := X \setminus X_k, \qquad 2 \le l \le n.$$

The smallest of these, U_2, agrees with the open stratum S_0. We have inclusions

$$S_k \xhookrightarrow{i_k} U_{k+1} \xhookleftarrow{j_k} U_k, \qquad \text{for } 2 \le k < n$$

where i_k is a closed embedding and j_k an open embedding.

4.6. **Extensions of complexes and the attachment map.** Let us consider the case of an open embedding, $j : V \hookrightarrow W$, later to be taken to be j_k from above. Suppose $\mathcal{F}_V^\bullet \in \mathcal{D}^b(V)$. Then one possible extension of $\mathcal{F}_V^\bullet$ to W is given by $Rj_*(\mathcal{F}_V^\bullet)$. Here $Rj_*(\mathcal{F}_V^\bullet)$ is an extension in the sense that $j^*(Rj_*(\mathcal{F}_V^\bullet)) \cong \mathcal{F}_V^\bullet$.

If $\mathcal{F}_W^\bullet$ is any other extension of $\mathcal{F}_V^\bullet$, then it maps to this one by what is called the *attachment map*,

$$\alpha : \mathcal{F}_W^\bullet \to Rj_*(\mathcal{F}_V^\bullet).$$

This is just the adjunction morphism $1 \to Rj_* \circ j^*$ from Section 3.6 applied to $\mathcal{F}_W^\bullet$. Explicitly, for any $U \subset W$ open, $Rj_*(\mathcal{F}_V^\bullet)(U) = I(\mathcal{F}_V^\bullet)(U \cap V)$, and $\alpha(U)$ is the composition of the natural map $\mathcal{F}_W^\bullet(U) \to I(\mathcal{F}_W^\bullet)(U)$ with the restriction map from U to $U \cap V$, followed by the isomorphism $\mathcal{F}_W^\bullet|_V \to \mathcal{F}_V^\bullet$. On V the attachment map α is clearly an isomorphism.

4.7. **The Deligne sheaf.** The restriction $\mathcal{I}_{\underline{p}}\mathcal{C}_X^\bullet|_{U_2}$ to the open stratum of X is always quasi-isomorphic to the (shifted) orientation sheaf $\mathbb{O}_{U_2}[n]$, where n is the dimension of X. This is simply because there are no intersection conditions on U_2, and we are in the case of usual Borel-Moore chains. And since the isomorphism $H^n(\mathbb{R}^n) \cong \mathbb{C}$ in Borel-Moore homology depends on a choice of orientation of $\mathbb{R}^n$, it follows that the local Borel-Moore homology $\mathcal{H}^{-n}(\mathcal{C}_{U_2}^\bullet)$ on the manifold U_2 is the orientation sheaf. The other local cohomology groups vanish of course, as they do for $\mathbb{R}^n$.

Definition 4.7. Let $\mathcal{L}$ be a local system on the open stratum $U_2 \subset X$ and $\underline{p}$ a fixed perversity. Define $\mathcal{P}_k^\bullet = \mathcal{P}_k^\bullet(\underline{p}, \mathcal{L}) \in \mathcal{D}^b(U_k)$ inductively as follows.

(1) Set $\mathcal{P}_2^\bullet := \mathcal{L}[n]$.
(2) For $k \geq 2$ set $\mathcal{P}_{k+1}^\bullet := \tau_{\leq -n+\underline{p}(k)} Rj_{k*}(\mathcal{P}_k^\bullet)$.

The complex $\mathcal{P}_{\underline{p}}^\bullet(\mathcal{L}) := \mathcal{P}_{n+1}^\bullet$ is called the *Deligne sheaf* corresponding to $\mathcal{L}$ and $\underline{p}$. If $\mathcal{L}$ is the orientation sheaf $\mathbb{O}_{U_2}$ we may also write $\mathcal{P}_{\underline{p}}^\bullet$ for $\mathcal{P}_{\underline{p}}^\bullet(\mathbb{O}_{U_2})$.

Proposition 4.8. *The attachment map corresponding to $j_k : U_k \to U_{k+1}$ gives rise to a quasi-isomorphism*

$$\alpha_k : \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet|_{U_{k+1}} \longrightarrow \tau_{\leq -n+\underline{p}(k)} Rj_{k*}(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet|_{U_k}).$$

Proof. Since $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet|_{U_k}$ is soft we have that

$$j_{k*}(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet|_{U_k}) \to Rj_{k*}(\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet|_{U_k})$$

is a quasi-isomorphism. Recall that $j_k : U_k \hookrightarrow U_{k+1} = U_k \sqcup S_k$. Let $x \in S_k$ and let $V \subset U_{k+1}$ be a distinguished neighborhood with

$$\begin{array}{ccc} V & \xrightarrow{\sim} & \mathbb{R}^{n-k} \times \operatorname{cone}^\circ(L) \\ \uparrow & & \uparrow \\ V \cap U_k & \xrightarrow{\sim} & \mathbb{R}^{n-k} \times (\operatorname{cone}^\circ(L) \setminus \{v\}). \end{array}$$

Note that topologically $\operatorname{cone}^\circ(L) \setminus \{v\} \cong \mathbb{R} \times L$. By a result analogous to Proposition 4.3 (see Chapter II, Proposition 3.4 in [11]) we have isomorphisms

$$I_{\underline{p}}H_i(L) \to I_{\underline{p}}H_{i+1}(\operatorname{cone}^\circ(L) \setminus \{v\})$$

induced on chains by $\xi \mapsto \operatorname{cone}^\circ(\xi) \setminus \{v\}$. As a consequence we have a quasi-isomorphism

$$I_{\underline{p}}C^\bullet(L)[n-k+1] \longrightarrow \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet(V \cap U_k) = (j_{k*}\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet)(V)$$

On the other hand Propositions 4.3 and 4.5 give a quasi-isomorphism

$$\tau_{\leq -n+\underline{p}(k)}(I_{\underline{p}}C^\bullet(L)[n-k+1]) \longrightarrow \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet(V).$$

The resulting diagram

$$\begin{array}{ccc} \tau_{\leq -n+\underline{p}(k)}(I_{\underline{p}}C^\bullet(L)[n-k+1]) & \longrightarrow & \tau_{\leq -n+\underline{p}(k)}(j_{k*}\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet)(V) \\ \downarrow & & \downarrow \\ \mathcal{I}_{\underline{p}}\mathcal{C}^\bullet(V) & \xrightarrow{\alpha_k(V)} & \tau_{\leq -n+\underline{p}(k)}(Rj_{k*}\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet)(V) \end{array}$$

commutes. Hence α_k is an isomorphism in $\mathcal{D}^b(U_{k+1})$. □

Corollary 4.9. *The complexes $\mathcal{I}_{\underline{p}}\mathcal{C}^\bullet_X$ and $\mathcal{P}^\bullet_{\underline{p}}$ are isomorphic in $\mathcal{D}^b(X)$. In particular we have*

$$I_{\underline{p}}H^*(X) = \mathbb{H}^*(\mathcal{P}^\bullet_{\underline{p}}(\mathbb{O}_{U_2})),$$

and intersection cohomology is independent of the choice of pl-structure on X. □

The definition of $I_{\underline{p}}H^*(X)$ can easily be generalized to have coefficients in a local system $\mathcal{L}$ on U_2. For any simplex Δ occurring in a $\underline{p}$-allowable chain the intersection conditions guarantee that $\Delta \cap U_2$ contains the interior Δ° of Δ, so that it makes sense to take the coefficient of Δ in $\Gamma(\Delta^\circ, \mathcal{L})$. With this definition the corollary generalizes to

$$I_{\underline{p}}H^*(X, \mathcal{L}) = \mathbb{H}^*(\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L} \otimes \mathbb{O}_{U_2})).$$

Remark 4.10. As another easy application let us give a proof of Proposition 2.13.(2), that $I_{\underline{0}}H_i(X) = H^{n-i}(X)$ for a normal, oriented pseudomanifold X. Note that since

$$\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L}) = \left(\tau_{\leq \underline{p}(n)} \circ Rj_{n*} \circ \cdots \circ \tau_{\leq \underline{p}(3)} \circ Rj_{3*} \circ \tau_{\leq \underline{p}(2)} \circ Rj_{2*}(\mathcal{L})\right)[n]$$

and $\tau_{\leq 0} \circ R(j_m)_* = (j_m)_*$ on $\mathrm{Sh}(U_m)$, we have

$$\mathcal{P}^\bullet_{\underline{0}}(\mathcal{L}) = j_*(\mathcal{L})[n],$$

where $j : U_2 \to X$ is the inclusion into X. So $\mathcal{P}^\bullet_{\underline{0}}(\mathbb{O}_{U_2}) = j_*(\mathbb{O}_{U_2})[n]$. Since U_2 is orientable, $\mathbb{O}_{U_2} = \underline{\mathbb{C}}_{U_2}$. Normality of X implies that $j_*(\underline{\mathbb{C}}_{U_2}) = \underline{\mathbb{C}}_X$ (since intersecting open sets with U_2 preserves the number of connected components). Therefore

$$I_{\underline{0}}H_i(X) = \mathbb{H}^{-i}(\mathcal{P}^\bullet_{\underline{0}}) = \mathbb{H}^{-i}(\underline{\mathbb{C}}_X[n]) = H^{n-i}(X).$$

5. Lecture - Verdier duality

5.1. **The dual of a local system.** Recall that a local system $\mathcal{L}$ on X (connected) corresponds to a representation V of the fundamental group, by monodromy. The dual local system $\mathcal{L}^\vee$ has a natural definition as the local system with monodromy representation V^* dual to V. Equivalently $\mathcal{L}^\vee$ may be given by

$$\mathcal{L}^\vee(U) = \mathrm{Hom}_{\mathrm{Sh}(U)}(\mathcal{L}|_U, \underline{\mathbb{C}}_U). \tag{5.1}$$

For example $\underline{\mathbb{C}}_X^\vee = \underline{\mathbb{C}}_X$.

This simple definition of duality is just the right one for orientable manifolds M in that we have the following version of Poincaré duality:

$$H^i_c(M, \mathcal{L})^* \cong H^{m-i}(M, \mathcal{L}^\vee).$$

This isomorphism comes from the usual Poincaré duality pairing just with coefficients in taken in the dual local systems $\mathcal{L}$ and $\mathcal{L}^\vee$, respectively.

5.2. **Duality for singular spaces.** Let X be a singular pseudomanifold. The origin of intersection cohomology's Poincaré duality on the level of $\mathcal{D}^b(X)$ is the duality functor of Borel and Moore [4] and J.-L. Verdier [32]. In rough outline, following Verdier's approach, this duality takes the following form. Given X one constructs a *dualizing complex* in $\mathcal{D}^b(X)$, which will turn out to be an injective resolution of the complex of Borel-Moore chains $\mathcal{D}_X^\bullet$. Let us call it $\hat{\mathcal{D}}_X^\bullet$. Then the dual of a complex $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ is given by

$$\mathbb{D}_X(\mathcal{F}^\bullet) = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \hat{\mathcal{D}}_X^\bullet),$$

with notation $\mathcal{H}om^\bullet$ defined in Definition 5.2 below.

5.3. **Versions of the Hom-functor.** Let us first define a sheaf from Hom between sheaves, following the example of the dual of a local system (5.1).

Definition 5.1. For $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$, define a $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathrm{Sh}(X)$ by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathrm{Sh}(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

with the obvious restriction maps. Homomorphisms between sheaves are locally determined, so this is indeed a sheaf. Note that

$$\Gamma \circ \mathcal{H}om = \mathrm{Hom}_{\mathrm{Sh}(X)}\,. \tag{5.2}$$

Definition 5.2. For two bounded complexes $\mathcal{A}^\bullet, \mathcal{B}^\bullet$ of sheaves let $\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ be the bounded complex associated to the double complex $\mathcal{H}om(\mathcal{A}^i, \mathcal{B}^j)$. Explicitly,

$$\mathcal{H}om^k(\mathcal{A}^\bullet, \mathcal{B}^\bullet) := \prod_{i \in \mathbb{Z}} \mathcal{H}om(\mathcal{A}^i, \mathcal{B}^{i+k}),$$

where $d^k : \mathcal{H}om^k(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \to \mathcal{H}om^{k+1}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ has $\mathcal{H}om(\mathcal{A}^i, \mathcal{B}^{i+k+1})$-component given by $d^k(\phi)_i = d_{\mathcal{B}}^{i+k} \circ \phi_k + (-1)^{i+1} \phi_{k+1} \circ d_{\mathcal{A}}^i$.

In order to get a well defined bifunctor on the derived category, say $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in \mathcal{D}^b(X)$, we need to replace $\mathcal{B}^\bullet$ by its (bounded) injective resolution $I(\mathcal{B}^\bullet)$. So

$$R\mathcal{H}om^\bullet : \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}^b(X),$$

is defined by $R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet) = \mathcal{H}om^\bullet(\mathcal{A}^\bullet, I(\mathcal{B}^\bullet))$.

Definition 5.3. The global version

$$\mathrm{RHom}^\bullet : \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}^b(\text{vector spaces}),$$

may be defined by $\mathrm{RHom}^\bullet = R\Gamma \circ R\mathcal{H}om^\bullet$.

The usual Hom in the derived category is recovered by

$$\mathbb{H}^i(R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]).$$

Definition 5.4 (Tensor products)**.** For two sheaves $\mathcal{F}, \mathcal{G}$ on X their tensor product $\mathcal{F} \otimes \mathcal{G}$ is defined as the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$. Since we are dealing with sheaves of vector spaces there are no difficulties with flatness and tensoring with a given sheaf is an exact functor. Setting $(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)^k := \bigoplus_{i+j=k} \mathcal{A}^i \otimes \mathcal{B}^j$ with the appropriate differential gives a well-defined bifunctor $\otimes : \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}^b(X)$.

Remark 5.5. $\mathcal{H}om(\mathcal{F}, \_)$ is right adjoint to the exact functor $\mathcal{F} \otimes \_$. This implies that $\mathcal{H}om(\mathcal{F}, \_)$ takes injectives to injectives.

5.4. **Pull-back with compact support.** We will follow the longer scenic route to constructing the dualizing complex via the introduction of a new pull-back functor. Let $f : X \to Y$ be a continuous map. Recall the two notions of push-forward, Rf_* and $Rf_!$. The usual pull-back functor f^* is related to Rf_* by the adjunction,

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(f^*\mathcal{Y}^\bullet, \mathcal{X}^\bullet) \cong \mathrm{Hom}_{\mathcal{D}^b(Y)}(\mathcal{Y}^\bullet, Rf_*\mathcal{X}^\bullet).$$

Duality should interchange cohomology with cohomology with proper supports and thus, more generally, Rf_* with $Rf_!$. Assuming that, there should also be a dual notion of pull-back and a dual adjunction formula. So we would like to construct a functor $f^! : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ such that

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{X}^\bullet, f^!\mathcal{Y}^\bullet) \cong \mathrm{Hom}_{\mathcal{D}^b(Y)}(Rf_!\mathcal{X}^\bullet, \mathcal{Y}^\bullet). \tag{5.3}$$

5.4.1. *The case of an open embedding.* If $j : U \hookrightarrow X$ is an open inclusion then j^* is already right adjoint to $j_!$,

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{X}, j^*\mathcal{Y}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(j_!\mathcal{X}, \mathcal{Y}),$$

with obvious adjunction morphisms $\mathcal{X} \to j^*j_!\mathcal{X}$ and $j_!j^*\mathcal{Y} \to \mathcal{Y}$. In this case we can simply set $j^!\mathcal{F}^\bullet = j^*\mathcal{F}^\bullet$. Note that since j^* and $j_!$ are both exact, the adjunction formula implies that j^* takes injectives to injectives and $j_!$ takes projectives to projectives.

5.4.2. *The case of a closed embedding.* Let us first introduce some simplifying notation. Suppose $\mathcal{F} \in \mathrm{Sh}(X)$ and $j_U : U \hookrightarrow X$ is an open inclusion. Define

$$\mathcal{F}_{U,X} := (j_U)_!(\mathcal{F}|_U) \in \mathrm{Sh}(X).$$

Explicitly, $\mathcal{F}_{U,X}(W) = \{s \in \mathcal{F}(U \cap W) \mid \mathrm{Supp}(s) \text{ is closed in } W\}$. For example $\underline{\mathbb{C}}_{U,X}(W) = \mathbb{C}^c$ where c is the number of connected components of W wholly contained in U.

Suppose $f = i : Z \to Y$ is the inclusion of a closed subset. Define a functor $i' : \mathrm{Sh}(Y) \to \mathrm{Sh}(Z)$ by

$$i'(\mathcal{Y})(U) := \mathrm{Hom}_{\mathrm{Sh}(Y)}(i_!\underline{\mathbb{C}}_{U,Z}, \mathcal{Y}) = \{\text{Sections } s \in \mathcal{Y}(V) \text{ supported in } Z\},$$

where V is an open set in Y such that $U = V \cap Z$. Then $i'(\mathcal{Y})$ really is a sheaf, since the property of being supported in Z can be checked locally. And i' is clearly left exact. Notice that this definition was engineered to give an adjoint to $i_!$,

$$\mathrm{Hom}_{\mathrm{Sh}(Z)}(\underline{\mathbb{C}}_{U,Z}, i'(\mathcal{Y})) = i'(\mathcal{Y})(U) = \mathrm{Hom}_{\mathrm{Sh}(Y)}(i_!\underline{\mathbb{C}}_{U,Z}, \mathcal{Y}).$$

The pull-back with compact support can be defined by $i^! = Ri'$.

5.4.3. *Construction of $f^!$ for general f.* Let $f : X \to Y$ be an arbitrary continuous map. Then the presheaf defined by

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}(Y)}(f_!\underline{\mathbb{C}}_{U,X}, \mathcal{Y}) \tag{5.4}$$

is not necessarily a sheaf. For example it may have nontrivial global sections while all of its stalks are zero. This is to do with the inflexibility of the sheaf $\underline{\mathbb{C}}_{U,X}$ along the fibers of f, which rarely allows for sections with proper support. This problem is solved by going over to the derived category and replacing $\underline{\mathbb{C}}_X$ with a fixed soft resolution $\mathcal{S}_X^\bullet$. For example we may take the soft resolution $\mathcal{C}_X^\bullet$ of $\underline{\mathbb{C}}_X$ from Section 2.10 if X is piecewise linear.

Theorem 5.6 ([18, VI 1.1], [11, VI]). *For $f : X \to Y$, the assignment*

$$f^!(\mathcal{Y}^\bullet)(U) := \mathrm{RHom}^\bullet(Rf_!\,\mathcal{S}^\bullet_{U,X}, \mathcal{Y}^\bullet)$$

defines a functor $f^! : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$. The functor $f^!$ is (up to isomorphism) independent of the choice of soft resolution $\mathcal{S}^\bullet_X$.

Note that the discussion around (5.4) implies that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ simply has no right adjoint in general. But $f^!$ does give a right adjoint functor to $Rf_!$ directly in the derived category. This, in a local version, is the main result of Verdier [32], see [18, VII.5 Theorem 5.2].

Theorem 5.7. *Let $\mathcal{X}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{Y}^\bullet \in \mathcal{D}^b(X)$. Then there is a natural isomorphism*

$$Rf_*\, R\mathcal{H}om^\bullet(\mathcal{X}^\bullet, f^!\mathcal{Y}^\bullet) \cong R\mathcal{H}om^\bullet(Rf_!\mathcal{X}^\bullet, \mathcal{Y}^\bullet). \tag{5.5}$$

The adjunction formula (5.3) is an immediate consequence.

Corollary 5.8. *Applying $\mathbb{H}^0$ to both sides of* (5.5) *we obtain a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{X}^\bullet, f^!\mathcal{Y}^\bullet) \cong \mathrm{Hom}_{\mathcal{D}^b(Y)}(Rf_!\mathcal{X}^\bullet, \mathcal{Y}^\bullet).$$

5.5. **The dualizing complex.** For the bounded derived category of sheaves on a point the dualizing object is just $\underline{\mathbb{C}}_{\{pt\}}$ viewed as a complex concentrated in degree zero.

Definition 5.9. The *dualizing complex* $\hat{\mathcal{D}}^\bullet_X \in \mathcal{D}^b(X)$ is defined by

$$\hat{\mathcal{D}}^\bullet_X := \pi^!(\underline{\mathbb{C}}_{\{pt\}}),$$

where $\pi = \pi_X : X \to \{pt\}$ is the projection to a point.

Let us compute the dualizing complex from the definition, taking as the soft resolution $\mathcal{S}^\bullet_X$ of $\underline{\mathbb{C}}_X$ the complex of geometric cochains $\mathcal{C}^\bullet_X$. We get

$$\pi^!\underline{\mathbb{C}}_{\{pt\}}(U) = \mathrm{Hom}^\bullet(\pi_!\mathcal{C}^\bullet_{U,X}, \mathbb{C}[0]) = \mathrm{Hom}^\bullet(\Gamma_c(\mathcal{C}^\bullet_{U,X}), \mathbb{C}[0]) = \Gamma_c(U, \mathcal{C}^{-\bullet}_X)^*, \tag{5.6}$$

where we used that $\underline{\mathbb{C}}_{\{pt\}} = \mathbb{C}$ is already injective. This recovers the dual of the constant sheaf in the sense of Borel and Moore [4].

5.6. **Comparison with $\mathcal{D}^\bullet_X$ and other properties.** The complex $\mathcal{D}^\bullet_X$ of Borel-Moore chains is related to the dualizing complex $\hat{\mathcal{D}}^\bullet_X$ by the following map. From the definition of $\mathcal{C}^i_X$, Section 2.10, one can see that there is a dual pairing $\Gamma_c(U, \mathcal{C}^{-i}_X) \times \Gamma(U, \mathcal{D}^i_X) \to \mathbb{C}$. Therefore we have an inclusion

$$\mathcal{D}^i_X(U) \hookrightarrow \Gamma_c(U, \mathcal{C}^{-i}_X)^*$$

of $\mathcal{D}^i_X(U)$ into its double dual. Since these complexes have finite dimensional cohomology, the inclusion induces isomorphisms there. So we get a quasi-isomorphism $\mathcal{D}^\bullet_X \to \hat{\mathcal{D}}^\bullet_X$.

One major advantage the dualizing complex has over the complex of Borel-Moore chains is that it is a complex of injectives, while $\mathcal{D}^\bullet_X$ was only soft. To see this intuitively, imagine for the sections of $\Gamma_c(U, \mathcal{C}^{-\bullet}_X)^*$ something like geometric chains but no longer necessarily with closed support. Then every restriction map is split by an extension by zero, and the corresponding sheaf is injective by Lemma 3.3. A proper, more general proof shows that there is a natural isomorphism $\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{A}, \hat{\mathcal{D}}^{-i}_X) = \Gamma_c(X, \mathcal{A} \otimes \mathcal{S}^i_X)^*$ generalizing (5.6) (e.g. [18, V Proposition

1.5]). Then since $\mathcal{S}_X^i$ is soft, and flat of course, $\mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{S}_X^i$ is an exact functor taking $\mathcal{A}$ to a soft sheaf. Since soft sheaves are Γ_c-acyclic, the combined functor $\mathrm{Hom}_{\mathrm{Sh}(X)}(\_, \hat{\mathcal{D}}_X^{-i})$ is exact.

Another advantage of $\hat{\mathcal{D}}_X^\bullet$ over $\mathcal{D}_X^\bullet$ is that it can be defined in more general settings. We may choose any soft resolution of $\underline{\mathbb{C}}_X$ to define $\pi^!$. In particular, X need not have a piecewise linear structure.

Note also that the dualizing complexes are inherently compatible with pull-back with compact supports. For a composition $f \circ g$ of two continuous maps there is a natural isomorphism $(f \circ g)^! = g^! \circ f^!$. And therefore

$$f^!(\hat{\mathcal{D}}_Y^\bullet) = f^! \circ \pi_Y^!(\mathbb{C}) = (\pi_Y \circ f)^!(\mathbb{C}) = \pi_X^!(\mathbb{C}) = \hat{\mathcal{D}}_X^\bullet.$$

This is analogous to the property $f^*(\underline{\mathbb{C}}_Y) = \underline{\mathbb{C}}_X$.

5.7. **Verdier duality.** The most important property of $\hat{\mathcal{D}}_X^\bullet$ is that it is a dualizing complex in the categorical sense for the duality for (constructible) complexes of sheaves introduced by Borel and Moore.

Definition 5.10. For $\mathcal{X}^\bullet \in \mathcal{D}^b(X)$ let

$$\mathbb{D}_X(\mathcal{X}^\bullet) := \mathcal{R}Hom^\bullet(\mathcal{X}^\bullet, \hat{\mathcal{D}}_X^\bullet).$$

This defines a contravariant functor $\mathbb{D}_X : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ which is called the *Verdier duality functor*.

Verdier's main theorem, the local adjunction formula (5.5), now implies

$$\mathbb{D}_X(\mathcal{X}^\bullet) = \mathcal{R}Hom^\bullet(R\pi_!(\mathcal{X}^\bullet), \underline{\mathbb{C}}_{\{pt\}}), \tag{5.7}$$

where $f = \pi : X \to \{pt\}$. The right hand side of (5.7) agrees with the definition of a dual of Borel and Moore [4]. Let us rewrite $R\pi_!$ as $R\Gamma_c$ and also apply $R\pi_* = R\Gamma$ to (5.7). So

$$R\Gamma(\mathbb{D}_X(\mathcal{X}^\bullet)) = R\,\mathrm{Hom}^\bullet(R\Gamma_c(\mathcal{X}^\bullet), \underline{\mathbb{C}}_{\{pt\}})$$

in $\mathcal{D}^b$(vector spaces). Then taking H^{-i} on both sides gives the very appealing global sense,

$$\mathbb{H}^{-i}(X, \mathbb{D}_X(\mathcal{X}^\bullet)) = \mathbb{H}_c^i(X, \mathcal{X}^\bullet)^*, \tag{5.8}$$

in which $\mathbb{D}_X(\mathcal{X}^\bullet)$ is dual to $\mathcal{X}^\bullet$.

If for example $\mathcal{X}^\bullet = \underline{\mathbb{C}}_X$, then $\mathbb{D}_X(\underline{\mathbb{C}}_X[0]) = \hat{\mathcal{D}}_X^\bullet \cong \mathcal{D}_X^\bullet$. Therefore (5.8) becomes

$$\begin{array}{ccc} \mathbb{H}^{-i}(X, \mathcal{D}_X^\bullet) & = & \mathbb{H}_c^i(X, \underline{\mathbb{C}}_X)^* \\ \| & & \| \\ H_i(X) & = & H_c^i(X)^*, \end{array}$$

which is a special case of the universal coefficient theorem for cohomology.

Remark 5.11. Here are some basic properties of Verdier duality.

(1) $\mathbb{D}_X : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is a contravariant functor. With respect to shifts, $\mathbb{D}_X(\mathcal{F}^\bullet[n]) = \mathbb{D}_X(\mathcal{F}^\bullet)[-n]$.

(2) Let $f : X \to Y$. Then

$$Rf_* \circ \mathbb{D}_X = \mathbb{D}_Y \circ Rf_!, \tag{5.9}$$

$$\mathbb{D}_X \circ f^* = f^! \circ \mathbb{D}_Y. \tag{5.10}$$

In particular, Verdier duality intertwines $R\Gamma = R\pi_*$ and $R\Gamma_c = R\pi_!$. Here the first formula follows from the adjunction (5.5). For the second one see [11, V §10].

(3) Suppose $\mathcal{L}$ is a local system on an m-dimensional manifold M. Then $\hat{\mathcal{D}}^\bullet_M \cong \mathbb{O}_M[m]$ and, since $\mathcal{L}$ is projective, $R\mathcal{H}om^\bullet(\mathcal{L}[m], \hat{\mathcal{D}}^\bullet_X) = \mathcal{H}om^\bullet(\mathcal{L}[m], \mathbb{O}_M[m])$. Therefore
$$\mathbb{D}_M(\mathcal{L}[m]) = \mathcal{L}^\vee \otimes \mathbb{O}_M.$$

5.8. **Constructibility.** For a local system on a manifold M the final property in Remark 5.11 implies that $\mathbb{D}_M(\mathbb{D}_M(\mathcal{L})) \cong \mathcal{L}$. This has an analogue for stratified spaces X.

Definition 5.12. Let σ be a pseudomanifold stratification of X. A complex $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ is called *σ-constructible* if for each stratum S we have that $\mathcal{H}^i(\mathcal{F}^\bullet|_S)$ is a local system. $\mathcal{F}^\bullet$ is called *constructible* if it is σ-constructible for some pseudomanifold stratification σ of X.

Note that all of the complexes we have been studying are constructible for the (given) stratification on X. For example the Deligne complex $\mathcal{P}^\bullet$ is constructible. This is essentially because we 'constructed' it from the constant sheaf, or a local system, using only truncation functors and push-forwards $R(j_k)_*$, where the inclusions j_k were compatible with the stratification (and the stratification obeys local normal triviality).

The dualizing complex $\hat{\mathcal{D}}^\bullet_X$ is in fact constructible for any pseudomanifold stratification σ of X. For example if $U \subset X$ open is a manifold, then we know that
$$\hat{\mathcal{D}}^\bullet_X|_U \cong \mathbb{O}_U[n]$$
by calculation of the local Borel-Moore homology. Similarly one can deduce that $\hat{\mathcal{D}}^\bullet_X$ has finite-dimensional locally constant cohomology along the strata of X, using local normal triviality. Moreover, if $\mathcal{F}^\bullet$ is a σ-constructible complex, then so is $\mathbb{D}_X(\mathcal{F}^\bullet)$. See [11, V §8].

Finally, if $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ is constructible, then the natural map
$$\mathcal{F}^\bullet \to \mathbb{D}_X \circ \mathbb{D}_X(\mathcal{F}^\bullet)$$
is an isomorphism in $\mathcal{D}^b(X)$. See [11, V Theorem 8.10].

6. Lecture - Topological invariance of intersection cohomology and Poincaré duality

The main aim of this section will be to apply the Verdier duality functor to the Deligne sheaf.

6.1. **Long exact sequences.** Suppose $\mathcal{F} \in \mathrm{Sh}(X)$ and $X = Z \sqcup U$ is a decomposition of X into a closed subset Z and an open one U. Denote the inclusions $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$. Let $\Gamma_Z(\mathcal{F}) := i_! i'(\mathcal{F}) \in \mathrm{Sh}(X)$. So
$$\Gamma_Z(\mathcal{F})(V) = \Gamma_Z(V, \mathcal{F}) = \{s \in \mathcal{F}(V) \mid \mathrm{Supp}(s) \subset Z\}.$$
Then the decomposition of X into Z and U gives rise to an exact sequence
$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \to j_* j^* \mathcal{F}$$
which extends to a short exact sequence $\to 0$ if $\mathcal{F}$ is injective.

Suppose now $\mathcal{F}^\bullet$ is a complex, and $\mathcal{I}^\bullet = I(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet$ its injective resolution. Then we get a sequence of chain maps

$$0 \to i_! i^! \mathcal{I}^\bullet \to \mathcal{I}^\bullet \to j_* j^* \mathcal{I}^\bullet \to 0$$

which is exact in every degree. In the language of triangulated categories, we obtain a 'distinguished triangle',

$$\begin{array}{ccc} i_! i^! \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^\bullet \\ {[1]}\nwarrow & & \swarrow \\ & Rj_* j^* \mathcal{F}^\bullet & \end{array} \quad , \tag{6.1}$$

where the map on the left hand side is a map to the shifted complex $Ri_! i^! (\mathcal{F}^\bullet)[1]$. Applying cohomology functors such as $\mathbb{H}^\bullet$, $\mathbb{H}^\bullet_c$, or $\mathcal{H}^\bullet_x$ gives a long exact sequence. For example,

$$\cdots \to \mathbb{H}^i(Z, i^! \mathcal{F}^\bullet) \to \mathbb{H}^i(X, \mathcal{F}^\bullet) \to \mathbb{H}^i(U, \mathcal{F}^\bullet) \to \mathbb{H}^{i+1}(Z, i^! \mathcal{F}^\bullet) \to \cdots .$$

The inclusions i and j also define another distinguished triangle,

$$\cdots \to j_! j^! \mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i_* i^* \mathcal{F}^\bullet \xrightarrow{[1]} \cdots .$$

6.2. **Costalks.** Let $i_x : \{x\} \to X$ be the inclusion of a point. The dual concept to the stalk functor i_x^* is the functor of costalks $i_x^! : \mathcal{D}^b(X) \to \mathcal{D}^b(\{pt\})$. Define

$$\mathcal{H}^j_{\{x\}}(\mathcal{F}^\bullet) := H^j(i_x^! \mathcal{F}^\bullet).$$

This is called the local cohomology supported in $\{x\}$.

Note that the stalk cohomology of $\mathcal{F}^\bullet$ at x can be described as

$$\mathcal{H}^j_x(\mathcal{F}^\bullet) = \varinjlim_{\{U \text{ open} \mid x \in U\}} \mathbb{H}^j(U, \mathcal{F}^\bullet),$$

by replacing $\mathcal{F}^\bullet$ by its injective resolution in (4.1). If $\mathcal{F}^\bullet$ is constructible, then an analogous description,

$$\mathcal{H}^j_{\{x\}}(\mathcal{F}^\bullet) = \varprojlim_{\{U \text{ open} \mid x \in U\}} \mathbb{H}^j_c(U, \mathcal{F}^\bullet),$$

for the costalk follows by Verdier duality. See also [11, V §3].

Lemma 6.1. *Suppose M is an m-manifold and $\mathcal{F}^\bullet \in \mathcal{D}^b(M)$ has finite rank, locally constant cohomology. Let $x \in M$ and $U \subset M$ an open ball containing x. Then*

(*a*) $\mathcal{H}^j_x(\mathcal{F}^\bullet) = \mathbb{H}^j(U, \mathcal{F}^\bullet)$.
(*b*) $\mathcal{H}^j_{\{x\}}(\mathcal{F}^\bullet) = \mathbb{H}^j_c(U, \mathcal{F}^\bullet) = \mathcal{H}^{j-m}_x(\mathcal{F}^\bullet)$.

In particular one has $j_x^!(\mathcal{F}^\bullet) \cong j_x^*(\mathcal{F}^\bullet)[-m]$.

Proof. To see (a), consider the spectral sequence

$$E_2^{ij} : H^i(U, \mathcal{H}^j \mathcal{F}^\bullet|_U) \Rightarrow \mathbb{H}^{i+j}(U, \mathcal{F}^\bullet).$$

Since $\mathcal{H}^j(\mathcal{F}^\bullet|_U)$ is a constant sheaf, by assumption, the cohomology $H^i(U, \mathcal{H}^j \mathcal{F}^\bullet|_U)$ is trivial for $i > 0$. Therefore the spectral sequence collapses to give $\mathbb{H}^j(U, \mathcal{F}^\bullet) = H^0(U, \mathcal{H}^j(\mathcal{F}^\bullet)) = \mathcal{H}^j_x(\mathcal{F}^\bullet)$. The second equality in (b) is proved in a similar way.

To get the identity $\mathcal{H}^j_{\{x\}}(\mathcal{F}^\bullet) = \mathbb{H}^j_c(U, \mathcal{F}^\bullet)$ apply Verdier duality to (a), in particular (5.8) and (5.10). $\square$

6.3. **Costalks and the Deligne sheaf.** As usual let X be an n-dimensional stratified pseudomanifold with a local system $\mathcal{L}$ on the open stratum $U_2 = X \setminus \Sigma$, and $\underline{p}$ a fixed perversity. We use the same notation as in Section 4.5. The Deligne sheaf $\mathcal{P}^\bullet = \mathcal{P}^\bullet_{\underline{p}}(\mathcal{L})$ in $\mathcal{D}^b(X)$ has the following properties (which also characterize it up to isomorphism).

($A1$) $\mathcal{P}^\bullet$ is constructible for the given stratification on X.
($A2$) $\mathcal{P}^\bullet|_{U_2} \cong \mathcal{L}[n]$.
($A3$) $\mathcal{H}^i_x(\mathcal{P}^\bullet) = 0$ for $i > -n + \underline{p}(k)$ and $x \in S_k$.
($A4$) Consider the inclusion $U_k \overset{j_k}{\hookrightarrow} U_{k+1}$. The attachment map

$$\alpha_k : \mathcal{P}^\bullet|_{U_{k+1}} \to R(j_k)_*(\mathcal{P}^\bullet|_{U_k})$$

is a quasi-isomorphism up to degree $\leq -n + \underline{p}(k)$.

Let us prove that $\mathcal{P}^\bullet$ also satisfies the following 'dual' property to $(A3)$:

($A3^!$) $\mathcal{H}^i_{\{x\}}(\mathcal{P}^\bullet) = 0$ for $x \in S_k$ and $i < -\underline{q}(k)$, where $\underline{q}$ is the complementary perversity to $\underline{p}$.

Lemma 6.2. $\mathcal{P}^\bullet$ *satisfies* $(A3^!)$

Proof. Let $\mathcal{P}^\bullet_k := \mathcal{P}^\bullet|_{U_k}$ and consider the inclusions

$$U_k \overset{j}{\hookrightarrow} U_{k+1} \overset{i}{\hookleftarrow} S_k.$$

Let $x \in S_k$. Applying $\mathcal{H}^l_x$ to the distinguished triangle (6.1) gives a long exact sequence

$$\cdots \to \mathcal{H}^l_x(i^!\mathcal{P}^\bullet_{k+1}) \to \mathcal{H}^l_x(\mathcal{P}^\bullet) \to \mathcal{H}^l_x(Rj_*j^*\mathcal{P}^\bullet_{k+1}) \to \mathcal{H}^{l+1}_x(i^!\mathcal{P}^\bullet_{k+1}) \to \cdots$$

To use this long exact sequence, we need to relate the terms $\mathcal{H}^l_x(i^!\mathcal{P}^\bullet_{k+1})$ to costalks. Let $i_x : \{x\} \to S_k$ denote the inclusion. Then one can use Lemma 6.1 to deduce

$$\begin{aligned}\mathcal{H}^l_x(i^!\mathcal{P}^\bullet_{k+1}) &= H^l(i_x^*i^!\mathcal{P}^\bullet_{k+1}) = H^{l+n-k}(i_x^!i^!\mathcal{P}^\bullet_{k+1}) \\ &= H^{l+n-k}((i \circ i_x)^!\mathcal{P}^\bullet_{k+1}) = \mathcal{H}^{l+n-k}_{\{x\}}(\mathcal{P}^\bullet_{k+1}).\end{aligned}$$

Recall that by property $(A3)$ we have $\mathcal{H}^l_x(\mathcal{P}^\bullet) = 0$ for all $l > m := -n + \underline{p}(k)$. So let us write down again the long exact sequence,

$$\begin{array}{ccccccc} & \cdots & \to & \mathcal{H}^{m-1}_x(\mathcal{P}^\bullet) & \to & \mathcal{H}^{m-1}_x(Rj_*j^*\mathcal{P}^\bullet_{k+1}) \\ \to & \mathcal{H}^{m+n-k}_{\{x\}}(\mathcal{P}^\bullet_{k+1}) & \to & \mathcal{H}^m_x(\mathcal{P}^\bullet) & \to & \mathcal{H}^m_x(Rj_*j^*\mathcal{P}^\bullet_{k+1}) \\ \to & \mathcal{H}^{m+n-k+1}_{\{x\}}(\mathcal{P}^\bullet_{k+1}) & \to & \mathcal{H}^{m+1}_x(\mathcal{P}^\bullet) & = 0. & \end{array} \tag{6.2}$$

From the property $(A4)$ of the attachment map we have that the maps $\mathcal{H}^l_x(\mathcal{P}^\bullet) \to \mathcal{H}^l_x(Rj_*j^*\mathcal{P}^\bullet_{k+1})$ are isomorphisms exactly for $l \leq m$. But therefore the long exact sequence (6.2) implies that

$$\mathcal{H}^{l+n-k}_{\{x\}}(\mathcal{P}^\bullet_{k+1}) = 0 \text{ for } l < -n + \underline{p}(k) + 2.$$

Since $\underline{q}(k) = k - 2 - \underline{p}(k)$, this is precisely the condition $(A3^!)$. □

Proposition 6.3. *The Deligne sheaf* $\mathcal{P}^\bullet = \mathcal{P}^\bullet_{\underline{p}}(\mathcal{L}) \in \mathcal{D}^b(X)$ *is uniquely determined up to isomorphism by the properties* $(A1)$, $(A2)$, $(A3)$ *and* $(A3^!)$.

This is proved basically by reading the proof of Lemma 6.2 in reverse. The vanishing of the costalks implies that the attachment map is a quasi-isomorphism in the required degrees by the same long exact sequence. So we have properties $(A1)-(A4)$. But $(A3)$ and $(A4)$ together with the constructibility property $(A1)$ imply that the attachment map gives a quasi-isomorphism

$$\mathcal{P}^\bullet_{k+1} \to \tau_{\leq -n+\underline{p}(k)} R(j_k)_* \mathcal{P}^\bullet_k.$$

So we recover the inductive construction of $\mathcal{P}^\bullet$.

Theorem 6.4. *Let $\mathcal{L}$ be a local system on the open stratum U_2 of an n-dimensional pseudomanifold X. Let $\underline{p}$ and $\underline{q}$ be complementary perversities. Then*

$$\mathbb{D}_X(\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L})) \cong \mathcal{P}^\bullet_{\underline{q}}(\mathcal{L}^\vee \otimes \mathbb{O}_{U_2})[-n]. \tag{6.3}$$

This theorem is proved by using the properties of Verdier duality to show that $\mathbb{D}_X(\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L}))[n]$ satisfies $(A1), (A2), (A3)$ and $(A3^!)$ for $\mathcal{L}^\vee \otimes \mathbb{O}_{U_2}$ and $\underline{q}$, and then applying Proposition 6.3.

If we apply $\mathbb{H}^i$ to (6.3), we obtain for the left hand side

$$H^i \circ R\Gamma \circ \mathbb{D}_X(\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L})) = \mathbb{H}^{-i}_c(\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L}))^*$$

using the property $R\Gamma \circ \mathbb{D}_X = \mathbb{D}_{\{pt\}} \circ R\Gamma_c$ from Remark 5.11. And the right hand side becomes $\mathbb{H}^{-n+i}(\mathcal{P}^\bullet_{\underline{q}}(\mathcal{L}^\vee \otimes \mathbb{O}_{U_2}))$.

Thus we obtain Theorem 2.15 as a corollary (in a slightly more general form).

Corollary 6.5 (Poincaré Duality)**.** *With notation as above,*

$$I_{\underline{p}} H^{-i}_c(X, \mathcal{L} \otimes \mathbb{O}_{U_2})^* \cong I_{\underline{q}} H^{-n+i}(X, \mathcal{L}^\vee).$$

6.4. **Variation of the stratification.** Let X be a pseudomanifold without a fixed stratification. And let $(U, \mathcal{L})$ be a pair consisting of an open dense submanifold $U \subset X$ with complement of codimension ≥ 2 and a local system defined on U. We may assume U is maximal (by Zorn's Lemma applied to the set of extensions of $\mathcal{L}$).

Theorem 6.6 ([11, V 4.15])**.** *Suppose we are given a pair $(U, \mathcal{L})$ as above, and assume that U contains the open stratum of some stratification of X.*[2] *Let $\underline{p}$ be a perversity. Then there exists a unique (up to isomorphism) complex $\tilde{\mathcal{P}}^\bullet \in \mathcal{D}^b(X)$ such that*

- $(B1)$ *$\tilde{\mathcal{P}}^\bullet$ is constructible.*
- $(B2)$ *$\tilde{\mathcal{P}}^\bullet|_V \cong \mathcal{L}|_V[n]$ for some open dense $V \subset U$ with codimension ≥ 2 complement.*
- $(B3)$ $\dim(\operatorname{Supp} \mathcal{H}^{-n+i}(\tilde{\mathcal{P}}^\bullet)) \leq n - \underline{p}^{-1}(i)$*, where $\underline{p}^{-1}(i) := min\{k \,|\, \underline{p}(k) \geq i\}$.*
- $(B3^!)$ $\dim(\operatorname{Cosupp} \mathcal{H}^{-i}(\tilde{\mathcal{P}}^\bullet)) \leq n - \underline{q}^{-1}(i)$*, where the cosupport of a sheaf is defined by* $\operatorname{Cosupp}(\mathcal{F}) := \{x \in X | \mathcal{F}_{\{x\}} \neq 0\}$.

[2] Borel indicates an example of a pair $(U, \mathcal{L})$ where U is maximal for $\mathcal{L}$ and does not contain the open stratum U_2 of any stratification, see [11, V Remark 4.14]. However the U in the example has no universal cover (not locally simply connected), and the definition of local system used seems to be as finite-dimensional representation of the fundamental group. For a local system in the locally constant sheaf sense I don't know whether it may not be possible to leave out this condition on U.

Suppose σ is a stratification of X adapted to $\mathcal{L}$, that is, with open stratum $U_2 \subset U$. Then one can check that the corresponding Deligne sheaf $\mathcal{P}^\bullet = \mathcal{P}^\bullet_{\underline{p}}(\mathcal{L})$ satisfies the above conditions. For example, property $(A3)$ in Section 6.3 immediately implies that $\mathrm{Supp}(\mathcal{H}^{-n+i}(\mathcal{P}^\bullet))$ cannot contain any stratum S_k having codimension less than $min\{k \,|\, \underline{p}(k) \geq i\}$, so property $(B3)$ follows. For $(B3^!)$ it is necessary to use also that Verdier duality preserves σ-constructibility.

The idea of the proof of Theorem 6.6 is to construct a coarsest possible stratification σ and σ-constructible Deligne sheaf $\mathcal{P}^\bullet_{\underline{p},\sigma}$, simultaneously. For any other adapted stratification σ', the strata of σ can be made up of unions of strata of σ'. Then one can show that $\mathcal{P}^\bullet_{\underline{p},\sigma}$ also satisfies the axioms $(A1) - (A3^!)$ with regard to σ' (using the σ-constructibility property). Therefore by Proposition 6.3 the two versions of the Deligne sheaf must be isomorphic.

Corollary 6.7. *The Deligne sheaf $\mathcal{P}^\bullet_{\underline{p}}(\mathcal{L}) \in \mathcal{D}^b(X)$ is independent of the stratification of X up to isomorphism. In particular, intersection cohomology is a topological invariant.*

6.5. **Intersection cohomology of varieties and perverse sheaves.** Let X be a quasi-projective algebraic variety over $\mathbb{C}$. Interesting examples in the context of intersection cohomology include affine varieties such as the nilpotent cone $\mathcal{N}$ in a Lie algebra, or projective varieties such as Grassmannians, flag varieties and Schubert varieties.

Any quasi-projective variety X has a stratification $X \supset X_2 \supset X_4 \supset \cdots \supset X_{2n}$ by Zariski closed subsets, such that the strata are smooth. A canonical such stratification is obtained for example by taking the singular locus of X and the singular locus of the singular locus, and so forth. A better stratification, a 'Whitney stratification', which can be shown also to satisfy local normal triviality can be obtained by refinement (by a combination of [34] and [11, Part IV], see also [23, §3.2]). As a consequence X has a pseudomanifold stratification by closed subvarieties. See also [19, Chapter 8] and the nice introduction to Whitney stratifications in [15, Part I Chapter 1]. In general references for the theory of perverse sheaves include [1], [19], [22], and the encyclopedic [12].

Let us restrict our attention to the middle perversity case, $\underline{m}(2k) = k - 1$. Since X and its strata are all even-dimensional there are better grading conventions to which we now switch. Let $n := \dim_{\mathbb{C}}(X) = \frac{1}{2}\dim_{\mathbb{R}}(X)$. Then shift the dualizing complex and Deligne sheaves by $[-n]$. These should become concentrated in degrees $-n \leq i \leq n$, rather than between $-2n$ and 0. The new version $\mathcal{P}^\bullet = \mathcal{P}^\bullet_{\underline{m}}(\mathcal{L}) \in \mathcal{D}^b(X)$ of the Deligne sheaf that can be constructed in this setting has the following properties.

- $(C1)$ $\mathcal{P}^\bullet$ is constructible for a Whitney stratification of X by Zariski closed subsets.
- $(C2)$ $\mathcal{P}^\bullet|_U \cong \mathcal{L}[\dim_{\mathbb{C}}(X)]$ for some Zariski open dense $U \subset X$.
- $(C3)$ $\dim_{\mathbb{C}}(\mathrm{Supp}^{-i}\,\mathcal{P}^\bullet) < i$ for all $i < \dim_{\mathbb{C}}(X)$.
- $(C3^!)$ $\dim_{\mathbb{C}}(\mathrm{Cosupp}^{i}\,\mathcal{P}^\bullet) < i$ for all $i < \dim_{\mathbb{C}}(X)$.

Definition 6.8. Let $\mathcal{D}^b_c(X)$ be the full subcategory of $\mathcal{D}^b(X)$ of complexes with cohomology constructible for a Whitney stratification of X by Zarsiki closed subsets.

A complex $\mathcal{G}^\bullet \in \mathcal{D}_c^b(X)$ is called a *perverse sheaf* if

$$\dim_{\mathbb{C}}(\mathrm{Supp}^{-i}\,\mathcal{G}^\bullet) \leq i, \tag{6.4}$$

$$\dim_{\mathbb{C}}(\mathrm{Cosupp}^{i}\,\mathcal{G}^\bullet) \leq i, \tag{6.5}$$

for all $i \in \mathbb{Z}$. Let $\mathcal{M}(X) \subset \mathcal{D}_c^b(X)$ be the full subcategory of perverse sheaves. If we fix a stratification σ of X then we may also consider the subcategory $\mathcal{M}_\sigma(X)$ of σ-constructible perverse sheaves.

This construction of a subcategory of $\mathcal{D}_c^b(X)$ is akin to the description of $\mathrm{Sh}(X)$ inside $\mathcal{D}^b(X)$ as $\mathcal{D}^{\leq 0}(X) \cap \mathcal{D}^{\geq 0}(X)$, complexes of sheaves concentrated in degree 0. Here (6.4) and (6.5) define the analogues of $\mathcal{D}^{\leq 0}(X)$ and $\mathcal{D}^{\geq 0}(X)$ – they can be shown to satisfy the axioms of a 't-structure' in the triangulated category. The resulting category $\mathcal{M}(X)$ is the 'core' of this t-structure and therefore an abelian category by general theory. We also have that $\mathcal{M}(X)$ and $\mathcal{M}_\sigma(X)$ are preserved under Verdier duality, thanks to the change of grading and the choice of middle perversity. And the simple objects are known. In summary :

Theorem 6.9 ([1], [22, Chapter III])**.** *Let X be a quasi-projective algebraic variety over $\mathbb{C}$ with a Whitney stratification by Zariski closed subsets σ.*

1. *The categories $\mathcal{M}(X)$ and $\mathcal{M}_\sigma(X)$ are abelian categories.*
2. *The Verdier duality functor defines an anti-equivalence of categories $\mathbb{D}_X : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$ which restricts to give anti-equivalences $\mathcal{M}(X) \to \mathcal{M}(X)$ and $\mathcal{M}_\sigma(X) \to \mathcal{M}_\sigma(X)$.*
3. *For a pair $(S, \mathcal{L})$ where $S \subset X$ is a closed irreducible subvariety and $\mathcal{L}$ a local system on a smooth Zariski open subset of S define*
$$\mathcal{IC}^\bullet_{S,X}(\mathcal{L}) := i_! \mathcal{P}^\bullet_{\underline{m}}(\mathcal{L}),$$
where $i : S \hookrightarrow X$ is the inclusion. If $\mathcal{L}$ is an irreducible local system, then $\mathcal{IC}^\bullet_{S,X}(\mathcal{L})$ is a simple object in $\mathcal{M}(X)$. All simple objects in $\mathcal{M}(X)$ are isomorphic to $\mathcal{IC}^\bullet_{S,X}(\mathcal{L})$ for some pair $(S, \mathcal{L})$ with $\mathcal{L}$ irreducible.

Definition 6.10. A complex $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ is called *semisimple* if it is of the form

$$\mathcal{F}^\bullet \cong \bigoplus_{(S,\mathcal{L}),k} \underline{V}^k_{(S,\mathcal{L})} \otimes \mathcal{IC}^\bullet_{S,X}(\mathcal{L})[k]$$

for some finite dimensional $\mathbb{Z}$-graded multiplicity vector spaces $V_{(S,\mathcal{L})} = \oplus_{k\in\mathbb{Z}} V^k_{(S,\mathcal{L})}$ which are nonzero for only finitely many pairs $(S, \mathcal{L})$.

We may now state the famous theorem of Beilinson, Bernstein, Deligne and Gabber [1].

Theorem 6.11 (Decomposition theorem)**.** *If $f : X \to Y$ is a proper morphism, then $Rf_*\underline{\mathbb{C}}_X$ is semisimple.*

Note that it is not known whether it is possible in the statement to replace the constant sheaf by an arbitrary local system. The only allowable local systems in this context are the ones that arise from the constant sheaf on an algebraic covering. In other words, suppose $p : \tilde{X} \to X$ is an unramified finite morphism. Then p_* is an exact functor, and $Rp_*\underline{\mathbb{C}}_{\tilde{X}} = p_*\underline{\mathbb{C}}_{\tilde{X}} = \mathcal{L}$ is a local system on X for which the decomposition theorem (obviously) holds.

The original proof [1] of this theorem involves base change to finite charactersitic and the study of eigenvalues of the Frobenius map on ℓ-adic cohomology. Another reference for the ℓ-adic theory of perverse sheaves (including the decomposition theorem) is [22]. A proof of the decomposition theorem without the base change was given by M. Saito [28]. We now turn to an application.

7. Lecture - Kazhdan-Lusztig polynomials

Let W be a Weyl group, for example the symmetric group S_n. Kazhdan and Lusztig [21] introduced polynomials $P_{u,w}(q)$ for pairs $u, w \in W$ and conjectured an interpretation of their value at 1 in terms of decomposition numbers of Verma modules in the representation theory of the corresponding complex Lie algebra. The proof of this conjecture was a major mathematical effort connecting these polynomials with intersection cohomology of Schubert varieties, then using Riemann-Hilbert correspondence to turn the intersection cohomology complexes into D-modules on the flag variety, and finally relating these to modules of the universal enveloping algebra of the Lie algebra by global sections and localization functors.

The last two steps in the proof of the conjecture were completed by Beilinson-Bernstein [2] and Brylinski-Kashiwara [7]. We will take this section to explain only the first step, which was done by Kazhdan and Lusztig in [20]. We follow the exposition of Springer [30], see also [29]. Background on algebraic groups can be found for example in the textbooks [5, 17, 31].

7.1. **Flag varieties and Schubert varieties.** Let G be a reductive linear algebraic group over $\mathbb{C}$ with fixed Borel subgroup B and an algebraic torus $T \subset B$. The Weyl group is $W = N_G(T)/T$. For GL_n we may take B to be the subgroup of upper-triangular matrices and T the diagonal matrices. Then $W = S_n$. Let $\dot{w} \in N_G(T)$ be a representative of $w \in W$, for example a permutation matrix in GL_n. Denote by $S \subset W$ the set of simple reflections generating W, so for example the adjacent transpositions in the symmetric group. The length $\ell(w)$ of $w \in W$ is the minimal number of factors required to write w as a product of simple reflections.

The flag variety X of G is defined to be the homogeneous space G/B. It is decomposed into B-orbits by the *Bruhat decomposition*,

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B,$$

where the individual orbits are affine cells $B\dot{w}B/B \cong \mathbb{C}^{\ell(w)}$.

The Schubert varieties are defined to be the closures of the Bruhat cells,

$$X_w := \overline{B\dot{w}B/B}.$$

The partial order on the Schubert varieties by inclusion gives rise to a partial order on W called the *Bruhat order*. It is a general fact that all Borel subgroups are conjugate to one another, and the normalizer of B is just B. Therefore the elements of the flag variety may be identified with Borel subgroups via the correspondence

$$gB \leftrightarrow g \cdot B := gBg^{-1}.$$

It will be useful to consider also the product $X \times X$. Let G act on $X \times X$ diagonally. By an application of Bruhat decomposition, the G-orbits in $X \times X$ are of the form

$$\mathcal{O}(w) := G\text{-orbit of } (B, \dot{w} \cdot B).$$

We say that two elements B_1, B_2 of the flag variety have *relative position* w if $(B_1, B_2) \in \mathcal{O}(w)$. Write in this case $B_1 \xrightarrow{w} B_2$. The basic properties of Bruhat decomposition imply

$$\begin{array}{lll} B_1 \xrightarrow{s} B_2 \xrightarrow{s} B_3 & \Rightarrow & B_1 \xrightarrow{s} B_3 \quad \text{or} \quad B_1 = B_3, \\ B_1 \xrightarrow{v} B_2 \xrightarrow{w} B_3 & \Rightarrow & B_1 \xrightarrow{vw} B_3 \quad \text{if} \quad \ell(vw) = \ell(v) + \ell(w), \end{array} \tag{7.1}$$

for $s \in S$ and $v, w \in W$. The set of B' in X such that $B \xrightarrow{w} B'$ is just the Bruhat cell $B\dot{w} \cdot B$.

7.2. **The Hecke algebra.** Let v be an indeterminate and let $\mathcal{H}$ be a $\mathbb{Z}[v, v^{-1}]$-module with basis indexed by W,

$$\mathcal{H} := \mathbb{Z}[v, v^{-1}] \otimes \langle T_w \mid w \in W \rangle_{\mathbb{Z}}\,.$$

There is a product on $\mathcal{H}$ with unit element T_1 such that

$$\begin{array}{rll} T_s T_s & = (v^2 - 1) T_s + v^2 T_1, & \text{for } s \in S \\ T_u T_w & = T_{uw}, & \text{for } u, w \in W \text{ with } \ell(uw) = \ell(u) + \ell(w). \end{array}$$

See for example the construction below, in Section 7.3. The algebra $\mathcal{H}$ determined by these relations is called the *Hecke algebra* associated to the Weyl group W. When v is specialized to 1 it is clearly just the group algebra $\mathbb{Z}[W]$ of W,

7.3. **Construction over $\mathbb{F}_q$.** There is a sense in which the Hecke algebra is the deformation of $\mathbb{Z}[W]$ one obtains when replacing W with the flag variety over $\mathbb{F}_q$ (where q corresponds to v^2). Let $\dot{G}, \dot{B}, \dot{X}$ be split forms of G, B and X over the finite field $\mathbb{F}_q$. For example $GL_n(\mathbb{F}_q)$. Then the definitions from Section 7.1 can be carried over to this case.

Let us consider the convolution algebra $\mathbb{Z}[\dot{X} \times \dot{X}]^{\dot{G}}$ of $\dot{G}$-invariant functions on $\dot{X} \times \dot{X}$. Let $T_w \in \mathbb{Z}[\dot{X} \times \dot{X}]^{\dot{G}}$ be the characteristic function of the $\dot{G}$-orbit $\dot{\mathcal{O}}(w)$. The convolution product on $\mathbb{Z}[\dot{X} \times \dot{X}]^{\dot{G}}$ is

$$S * T\,(\dot{B}_1, \dot{B}_3) = \sum_{\dot{B}_2 \in \dot{X}} S(\dot{B}_1, \dot{B}_2) T(\dot{B}_2, \dot{B}_3). \tag{7.2}$$

So for example if $s \in S$

$$T_s * T_s\,(\dot{B}_1, \dot{B}_3) = \begin{cases} q & \dot{B}_1 = \dot{B}_3 \\ q - 1 & \dot{B}_1 \xrightarrow{s} \dot{B}_3 \\ 0 & \text{otherwise.} \end{cases},$$

since there is a summand (of 1) for every $\dot{B}_2$ with $\dot{B}_1 \xrightarrow{s} \dot{B}_2 \xrightarrow{s} \dot{B}_3$. Therefore we find

$$T_s T_s = (q - 1) T_s + q T_1.$$

The relation $T_u T_w = T_{uw}$ if $\ell(u) + \ell(w) = \ell(uw)$ follows in a similar way using (7.1).

7.4. **The Hecke algebra and perverse sheaves.** Above we have recovered the relations of the Hecke algebra from the structure of the flag variety and its Bruhat decompositions over $\mathbb{F}_q$. We can get a more sophisticated model by replacing constructible functions with semisimple perverse sheaves. We return to working over $\mathbb{C}$. The stratification by Bruhat cells of a Schubert variety X_w and the G-orbit stratification of $\overline{\mathcal{O}(w)}$ are both Whitney stratifications, see [9, Proposition 1.4].

Let $\mathcal{A}^\bullet \in \mathcal{D}^b(X \times X)$ be constructible for the G-orbit stratification of $X \times X$. Define

$$h_{X\times X}(\mathcal{A}^\bullet) := \sum_{w\in W}\sum_{i\in\mathbb{Z}} \dim(\mathcal{H}^i_{(B,\dot{w}B)}(\mathcal{A}^\bullet))v^i T_w \in \mathcal{H}.$$

Similarly, for a bounded complex $\mathcal{B}^\bullet$ on X that is constructible for the Bruhat decomposition, define

$$h_X(\mathcal{B}^\bullet) := \sum_{w\in W}\sum_{i\in\mathbb{Z}} \dim(\mathcal{H}^i_{\dot{w}B}(\mathcal{B}^\bullet))\, v^i T_w \in \mathcal{H}.$$

We have of course,

$$h_{X\times X}(\underline{\mathbb{C}}_{\mathcal{O}(w),X\times X}) = h_X(\underline{\mathbb{C}}_{B\dot{w}\cdot B,X}) = T_w.$$

For another simple example let $w = s$ be a simple reflection. Then we get that $X_s = (B\dot{s}\cdot B) \sqcup (1\cdot B) \cong \mathbb{C}P^1$ and therefore $\mathcal{IC}^\bullet_{X_s,X} = \underline{\mathbb{C}}_{X_s,X}[1]$. So

$$h_X(\mathcal{IC}^\bullet_{X_s,X}) = v^{-1}(T_s + T_1).$$

Remark 7.1. Consider the projection $p_1 : \overline{\mathcal{O}(w)} \to X$ given by $p_1(B_1, B_2) = B_1$. This is a bundle with fiber $p_1^{-1}(B) = X_w$ (locally trivial by G-equivariance). Therefore $\overline{\mathcal{O}(w)}$ looks locally like a product $\mathbb{C}^N \times X_w$, where $N = \dim_{\mathbb{C}}(X)$. It follows that $\mathcal{IC}^\bullet_{\overline{\mathcal{O}(w)},X\times X}$ has stalks given by

$$\left(\mathcal{IC}^\bullet_{\overline{\mathcal{O}(w)},X\times X}\right)_{(B,\dot{v}B)} = \left(\mathcal{IC}^\bullet_{X_w,X}[N]\right)_{\dot{v}B},$$

by a local application of Proposition 4.3, for example. Therefore

$$h_{X\times X}(\mathcal{IC}^\bullet_{\overline{\mathcal{O}(w)},X\times X}) = v^{-N} h_X(\mathcal{IC}^\bullet_{X_w,X}).$$

7.5. **Convolution product.** Suppose $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in \mathcal{D}^b(X)$ are semisimple perverse sheaves constructible for the G-orbit stratification. Consider the projections

$$\begin{array}{ccccc} X\times X & & & & \\ & \overset{p_{12}}{\nwarrow} & & & \\ & & X\times X\times X & \xrightarrow{p_{13}} & X\times X. \\ & \underset{}{\overset{p_{23}}{\swarrow}} & & & \\ X\times X & & & & \end{array}$$

Then

$$\mathcal{A}^\bullet \circ \mathcal{B}^\bullet := R(p_{13})_*(p_{12}^*\mathcal{A}^\bullet \otimes p_{23}^*\mathcal{B}^\bullet)$$

is again a semisimple perverse sheaf by the decomposition theorem, and constructible for the stratification by G-orbits.

For the purpose of simplifying notation let us write w for the element $\dot{w}\cdot B$ in X or $(B, \dot{w}\cdot B) \in X \times X$, when considering stalks at these points.

Lemma 7.2. *Suppose $\mathcal{A}^\bullet \in \mathcal{D}^b(X \times X)$ is constructible with respect to G-orbits and has $\mathcal{H}^i(\mathcal{A}^\bullet) = 0$ for odd i. Then the same holds for $\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \circ \mathcal{A}^\bullet$ and*

$$h_{X\times X}(\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \circ \mathcal{A}^\bullet) = (T_s + 1) \cdot h_{X\times X}(\mathcal{A}^\bullet).$$

Proof. The coefficient of T_w in $(T_s+1)\cdot h_{X\times X}(\mathcal{A}^\bullet)$ is

$$\begin{cases} \sum_{i\in\mathbb{Z}} v^i(\dim \mathcal{H}^i_w(\mathcal{A}^\bullet) + \dim \mathcal{H}^{i-2}_{sw}(\mathcal{A}^\bullet)) & w < sw \\ \sum_{i\in\mathbb{Z}} v^i(\dim \mathcal{H}^i_{sw}(\mathcal{A}^\bullet) + \dim \mathcal{H}^{i-2}_{w}(\mathcal{A}^\bullet)) & sw < w. \end{cases}$$

We need to show, therefore, that

$$(7.3)\qquad \dim(\mathcal{H}^i_w(\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \circ \mathcal{A}^\bullet)) = \begin{cases} \dim \mathcal{H}^i_w(\mathcal{A}^\bullet) + \dim \mathcal{H}^{i-2}_{sw}(\mathcal{A}^\bullet) & w < sw \\ \dim \mathcal{H}^i_{sw}(\mathcal{A}^\bullet) + \dim \mathcal{H}^{i-2}_{w}(\mathcal{A}^\bullet) & sw < w. \end{cases}$$

Since p_{13} is proper, $(p_{13})_* = (p_{13})_!$ and we have by (3.2) that

$$(7.4)\qquad \mathcal{H}^i_w(R(p_{13})_!(p_{12}^*\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \otimes p_{23}^*\mathcal{A}^\bullet)) \\ = \mathbb{H}^i_c(p_{13}^{-1}(B, \dot w\cdot B)\,,\, p_{12}^*\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \otimes p_{23}^*\mathcal{A}^\bullet).$$

Define

$$D = \{(B_1,B_2,B_3) \in p_{13}^{-1}(B,\dot w\cdot B) \mid (B_1,B_2) \in \overline{\mathcal{O}(s)}\,\}.$$

Then $D \cong X_s \cong \mathbb{C}P^1$ and the right hand side in (7.4) is simply $\mathbb{H}^i_c(D, p_{23}^*\mathcal{A}^\bullet)$.

Let us now prove the identity (7.3) for $sw < w$. In that case D decomposes into

$$D_1 = \{(B,B',\dot w\cdot B) \in D \mid B' \xrightarrow{w} \dot w\cdot B\},$$
$$D_0 = \{(B, \dot s\cdot B, \dot w \cdot B)\},$$

where $D_1 \cong \mathbb{C}$. The inclusions

$$D_1 \overset{j}{\hookrightarrow} D \overset{i}{\hookleftarrow} D_0$$

give rise to a distinguished triangle, $\cdots \to j_!j^! \to 1 \to i_*i^* \xrightarrow{[1]} \cdots$, and a corresponding long exact sequence

$$\cdots \to \mathbb{H}^i_c(D_1, p_{23}^*\mathcal{A}^\bullet|_{D_1}) \to \mathbb{H}^i_c(D, p_{23}^*\mathcal{A}^\bullet) \to \mathbb{H}^i_c(D_0, p_{23}^*\mathcal{A}^\bullet|_{D_0}) \to \cdots.$$

Starting from the left, note that $p_{23}^*\mathcal{A}^\bullet$ has constant cohomology $\mathcal{H}^*_w(\mathcal{A}^\bullet)$ along D_1. Therefore by Lemma 6.1 we have $\mathbb{H}^i_c(D_1, p_{23}^*\mathcal{A}^\bullet|_{D_1}) = \mathcal{H}^{i-2}_w(\mathcal{A}^\bullet)$. Next $\mathbb{H}^i_c(D, p_{23}^*\mathcal{A}^\bullet) = \mathcal{H}^i_w(\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \circ \mathcal{A}^\bullet)$ as we saw above. Finally, $\mathbb{H}^i_c(D_0, p_{23}^*\mathcal{A}^\bullet|_{D_0})$ is trivially equal to $\mathcal{H}^i_{sw}(\mathcal{A}^\bullet)$. So we have a long exact sequence

$$\cdots \to 0 \to \mathcal{H}^{i-2}_w(\mathcal{A}^\bullet) \to \mathcal{H}^i_w(\underline{\mathbb{C}}_{\overline{\mathcal{O}(s)}, X\times X} \circ \mathcal{A}^\bullet) \to \mathcal{H}^i_{sw}(\mathcal{A}^\bullet) \to 0 \to \cdots,$$

using that $\mathcal{A}^\bullet$ has vanishing cohomology in odd degrees, and where i is assumed even. This implies the equality (7.3) in the $sw < w$ case. The other case is similar. □

Definition 7.3. Let $\iota : \mathcal{H} \to \mathcal{H}$ be the anti-involution defined on generators by

$$\begin{array}{lcl} v & \mapsto & v^{-1} \\ T_s & \mapsto & T_s^{-1} = v^{-2}T_s + (v^{-2}-1). \end{array}$$

Note that $\iota(T_s+1) = q^{-1}(T_s+1)$, where $q = v^2$.

Theorem 7.4 ([21]). *For any $w \in W$ there exists a unique element $C'_w \in \mathcal{H}$ such that*

(1) $\iota(C'_w) = C'_w$.
(2) *The element* C'_w *is expressed as*
$$C'_w = v^{-\ell(w)} \sum_{u \le w} P_{u,w}(v^2) T_u$$
for polynomials $P_{u,w} \in \mathbb{Z}[q]$ *satisfying* $\deg(P_{u,w}) \le \frac{1}{2}(\ell(w) - \ell(u) - 1)$ *if* $u < w$ *and such that* $P_{w,w} = 1$.

The polynomials $P_{u,w}$ are called the Kazhdan-Lusztig polynomials. They can be computed by recursive formulas.

Theorem 7.5. [20] $C'_w = h_X(\mathcal{IC}^\bullet_{X_w,X})$. *In particular the Kazhdan-Lusztig polynomials have nonnegative integer coefficients.*

The remainder of these notes will be concerned with proving this theorem. In other words, we want to show that $h_X(\mathcal{IC}^\bullet_{X_w,X})$ satisfies the conditions in Theorem 7.4. The main step is to show that the involution ι reflects the action of Verdier duality on the complexes of sheaves.

Let w be fixed, and with it a reduced expression $w = s_1 s_2 \cdots s_k$ of w in terms of the simple reflections. Consider the variety
$$Y = \{(B_1, \dots, B_{k+1}) \in X^{k+1} \mid (B_i, B_{i+1}) \in \overline{\mathcal{O}(s_i)}\},$$
and its subvariety $Y_0 = \{(B_1, \dots, B_{k+1}) \in Y \mid B_1 = B\}$. Then we have projection maps
$$\begin{array}{ccccc} Y_0 & \hookrightarrow & Y & & (B_1, \dots, B_{k+1}) \\ \pi_0 \downarrow & & \pi \downarrow & & \downarrow \\ X_w & \overset{i}{\hookrightarrow} & \overline{\mathcal{O}(w)} & & (B_1, B_{k+1}). \end{array} \tag{7.5}$$
Note that Y and Y_0 are smooth. The map π_0 is the *Bott-Samelson resolution* of the Schubert variety X_w.

Lemma 7.6. $R\pi_*(\underline{\mathbb{C}}_Y) = \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_1)},X\times X} \circ \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_2)},X\times X} \circ \cdots \circ \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_k)},X\times X}$.

Note that $\underline{\mathbb{C}}_Y = p_{12}^* \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_1)},X\times X} \otimes p_{23}^* \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_2)},X\times X} \otimes \cdots \otimes p_{k,k+1}^* \underline{\mathbb{C}}_{\overline{\mathcal{O}(s_k)},X\times X}|_Y$ and apply $R\pi_*$.

Proof of Theorem 7.5. Firstly we need to prove that $h_X(\mathcal{IC}^\bullet_{X_w,X}) \in \mathcal{H}$ is ι-invariant. If $w = 1$ this is trivial. Also for $s \in S$ the Schubert variety $X_s \cong \mathbb{C}P^1$ and we know ι-invariance by direct calculation. For general w the proof will be by induction. Let us assume $\iota(h_X(\mathcal{IC}^\bullet_{X_u,X})) = h_X(\mathcal{IC}^\bullet_{X_u,X})$ for all $u < w$.

Consider the diagram (7.5). By proper base change and the decomposition theorem,
$$i^* R\pi_*(\underline{\mathbb{C}}_Y[\ell(w)]) = R\pi_{0*}(\underline{\mathbb{C}}_{Y_0}[\ell(w)]) = \bigoplus_{u \le w, i \in \mathbb{Z}} \mathcal{IC}^\bullet_{X_u,X} \otimes \underline{V}_{u,i}[i]$$
for some multiplicity vector spaces $V_{u,i}$. The middle expression is Verdier self-dual, since π_0 is proper and Y_0 smooth. On the other hand, the $\mathcal{IC}^\bullet_{X_u,X}$ on the right hand side are each self-dual. Therefore we must have
$$V_{u,i} \cong V_{u,-i}.$$

Also since $\pi_0|_{\pi^{-1}(X_w)}$ is an isomorphism,

$$V_{w,i} = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Therefore

$$h_X(i^* R\pi_*(\underline{\mathbb{C}}_Y[\ell(w)])) = h_X(\mathcal{IC}^\bullet_{X_w,X}) + \sum_{u<w} h_X(\mathcal{IC}^\bullet_{X_u,X}) p_u(v)$$

for Laurent polynomials p_u with $p_u(v) = p_u(v^{-1})$. Now by Lemma 7.2 and Lemma 7.6 we have

$$h_X(i^* R\pi_*(\underline{\mathbb{C}}_Y[\ell(w)])) = v^{-\ell(w)}(T_{s_1}+1)\cdots(T_{s_k}+1). \tag{7.6}$$

Therefore it follows that $h_X(i^* R\pi_*(\underline{\mathbb{C}}_Y[\ell(w)]))$ is invariant under the involution ι. Since

$$h_X(\mathcal{IC}^\bullet_{X_w,X}) = h_X(i^* R\pi_*(\underline{\mathbb{C}}_Y[\ell(w)])) - \sum_{u<w} p_u(v) h_X(\mathcal{IC}^\bullet_{X_u,X}) \tag{7.7}$$

it follows that $h_X(\mathcal{IC}^\bullet_{X_w,X})$ is invariant under ι, by the induction hypothesis.

It remains to prove that $h_X(\mathcal{IC}^\bullet_{X_w,X})$ satisfies the second condition of Theorem 7.4. Define polynomials $\tilde{P}_{u,w}$ by

$$h_X(\mathcal{IC}^\bullet_{X_w,X}) = v^{-\ell(w)} \sum_{u\leq w} \tilde{P}_{u,w}(v) T_u.$$

Clearly, $\tilde{P}_{w,w} = 1$. From the axiom $(C3)$ for the Deligne sheaf on X_w it follows that $\mathcal{H}^{-i}_u(\mathcal{IC}^\bullet_{X_w,X}) = 0$ if $u < w$ and $-i > -\ell(w) + 2(\ell(w) - \ell(u))$. This implies that $\deg(\tilde{P}_{u,w}) \leq \ell(w) - \ell(u) - 1$. Moreover,

$$v^{\ell(w)} h_X(\mathcal{IC}^\bullet_{X_w,X}) \in \mathbb{Z}[v^2],$$

by equations (7.6) and (7.7) and induction. So $\tilde{P}_{u,w}(v) = P_{u,w}(v^2)$ where the $P_{u,w}$ are polynomials satisfying the axioms for the Kazhdan-Lusztig polynomials. □

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