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A NEW INTERPRETATION OF GELFAND–TZETLIN BASES

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To Yuri Ivanovich Manin on his 50th birthday

Introduction. In this paper we determine q-analogues of Gelfand-Tzetlin bases for a q-deformation $\mathcal{D}_N^q$ of the universal enveloping algebra $\mathcal{U}(\widetilde{\mathfrak{gl}}_N)$ for the Lie algebra $\widetilde{\mathfrak{gl}}_N = \mathfrak{gl}_N \otimes_{\mathbb{C}} \mathbb{C}[[x]]$. The definition of $\mathcal{D}_N^q$ (and its generalization) was given by V. G. Drinfeld and M. Jimbo (see [1,2]). We introduce $\mathcal{D}_N^q$ (§1) in a somewhat different way by means of R-matrix technique, developed by L. D. Faddeev and his collaborators (see e.g. [3]). This technique was applied to construct first some "elliptic" deformations of $\mathcal{U}(\mathfrak{gl}_2)$ [4], $\mathcal{U}(\mathfrak{gl}_N)$ [5] and then to introduce the "elliptic" R-algebras of level n [6]. The latter are deformations of $\widetilde{\mathfrak{gl}}_N \operatorname{mod}(x^n)$ and also natural generalizations of $\mathcal{D}_N^q$ as $n \to \infty$.

We use the results and methods of [6, 7] in this paper but can prove our "branching" theorems only in the special case of $\mathcal{D}_N^q$ since the branching properties for the elliptic R-algebras are of a more complicated nature than for $\mathcal{D}_N^q$. These properties are analoques of the well-known Young–Gelfand–Tzetlin rules for decomposing irreducible finite-dimensional representations of $\mathfrak{gl}_N$ or S_n under the action of $\mathfrak{gl}_K \subset \mathfrak{gl}_N$ or $S_k \subset S_n$ and are most important in our paper. It was shown recently by G. I. Olshansky that the Yangians, which were introduced by V. G. Drinfeld and are the limits of $\mathcal{D}_N^q$ as $q \to 1$, play an essential role in calculating the centralizers of $\mathcal{U}(\mathfrak{gl}_K)$ in $\mathcal{U}(\mathfrak{gl}_N)$. For S_n the degenerated Hecke algebras from [8] arise when we find the centralizers of $\mathbb{C}[S_k]$ in $\mathbb{C}[S_n]$ (cf. [6, 9]). We develop these results, generalizing the classic branching properties and extending them to $\mathcal{D}_N^q$ and affine Hecke algebras in §2, 3.

We apply the structural theorems for $\mathcal{D}_N^q$ obtained in this way to their irreducible representations, connected with the so-called skew Young diagrams. This class includes q-analogues of finite-dimensional irreducible representations of $\mathfrak{gl}_N$, constructed independently in [10, 11] for $\mathcal{D}_N^q$ of level = 1 and in [6, 7] for its elliptic generalization. As we prove (for sufficiently large N) at the end of the paper, only representations of the above type have the classic branching properties. We do not detail here the structure of the considered representations (e.g., do not describe the action of generators of $\mathcal{D}_N^q$ on Gelfand–Tzetlin bases). For further information we refer to [11] (where the latter problem was solved for level = 1) and the author's paper on the second Weyl character formula to be published soon (see also [6, 9] for Hecke algebras).

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Our technique of studying the representations of $\mathscr{D}_N^q$ (or more general R-algebras) has its origin in the "fusion process" by P. P. Kilish, N. Yu. Reshetikhin, E. K. Sklyanin and V. E. Korepin and in the notion of the quantum determinant (see [12]). We refer to accounts [12, 15], and recent papers by Drinfeld and Jimbo papers (see also [6] and, last but not least, new works of Leningrad's School of Faddeev) for the various results on R-matrices. The fusion process is similar to some constructive methods in the theory of representations of Hecke algebras by A. V. Zelevinsky and J. D. Rogawski (see [13]) since there exists a generalization of Weil duality for Yangians [8] and for $\mathscr{D}_N^q$ (see §3 below) with Hecke algebras instead of $\mathbf{C}[S_n]$. Another remarkable property of $\mathscr{D}_N^q$ is as follows.

We can obtain from $\mathscr{D}_N^q$ an algebra d_N of commutative polynomials with some Poisson bracket by some passage to limit. There is a maximal commutative subalgebra in $\mathscr{D}_N^q$, diagonalized by Gelfand-Tzetlin bases in any "good" representations. Its limit is a maximal involutive subalgebra $a_N \subset d_N$ relative to the Poisson bracket. Sufficiently general fibers of the morphism Specm $d_N \to$ Specm a_N of the corresponding affine (ind-) varieties are some open subsets of Jacobians of appropriate algebraic curves. These fibers are also orbits of the collection of flows with elements of a_N as Hamiltonians. These two statements (and their variants—see [14, 15]) generalize some classic results on integration of the equation of motion of the "rigid body" and many results on "finite-zoned" integration of soliton equations. Thus we have new and surprising connections between algebraic geometry and representation theory.

In studying the algebraic aspects of the Hamiltonian formalism and the "finite-zoned" integration, [16] and further works by Yu. I. Manin on these problems play an important role.

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§1. Trigonometric R-algebras. Let I^{lm} be the matrices with matrix elements $(I^{lm})_{ij} = \delta_{il}\delta_{jm}$ which form a basis in the algebra M_N of all $N \times N$-matrices over $\mathbf{C}$ (δ_{ij} is the Kronecker symbol). One defines inclusions of M_N, $M_N^{\otimes 2} = M_N \otimes M_N$ into $M_N^{\otimes n}$ by formulas ${}^iA = I^{\otimes(i-1)} \otimes A \otimes I^{\otimes(n-i)}$,

$$^{ij}(A \otimes B) = {}^iA\,{}^jB \quad \text{for } A, B \in M_N, 1 \leqslant i \neq j \leqslant n \quad \text{and} \quad I = \sum_{l=1}^{N} I^{ll}$$

being the matrix unit. Let us denote by $\mathscr{L}_N(u)$ the vector space of all functions

$$L(x) = \sum_{1 \leqslant i, j \leqslant N}^{0 \leqslant s \leqslant n} x^s \tilde{E}_{ij}^s I^{ji} \prod_{i=1}^{n} (x - q^{u_i})^{-1},$$

where $u = (u_i) \in \mathbf{C}^n$, $\tilde{E}_{ij}^s \in \mathbf{C}$ and $\tilde{E}_{ij}^n = 0 = \tilde{E}_{ji}^0$ for $i < j$. Here and below the parameter $q^{1/2} \in \mathbf{C} \setminus \{1\}$ is "sufficiently close" to $q^{1/2} = 1$ (in fact further results are valid for almost all q).

Let us introduce

$$\begin{aligned} R(x) &= R_N(x;q) \\ &= \sum_{i=1}^{N} I^{ii} \otimes \left(I + (q^{1/2}-1)I^{ii}\right) + \frac{q^{1/2}-q^{-1/2}}{x-1} \sum_{i \geqslant j} I^{ij} \otimes I^{ji} \\ &\quad + \frac{q^{1/2}-q^{-1/2}}{1-x^{-1}} \sum_{i<j} I^{ij} \otimes I^{ji} \\ &\in \mathscr{L}(0) \otimes_{\mathbf{C}} M_N \end{aligned}$$

(see [10, 15]). We define the quotient-algebra $\tilde{\mathscr{D}}_N^q(u)$ of the tensor algebra $T(\mathscr{L}_N^*(u))$ for $\mathscr{L}_N^*(u) = \mathrm{Hom}_{\mathbf{C}}(\mathscr{L}_N(u), \mathbf{C}) = \oplus \mathbf{C}\,\tilde{E}_{ij}^s$ by Yang-Baxter-Faddeev relations

$$^{12}R\left(x_1 x_2^{-1}\right)^1 L(x_1)^2 L(x_2) - {}^2L(x_2)^1 L(x_1)^{12} R\left(x_1 x_2^{-1}\right) = 0. \tag{1}$$

To be more exact $\tilde{\mathscr{D}}_N^q(u) = T(\mathscr{L}_N^*(u))/\tilde{K}_N^q(u)$, where $\tilde{K}_N^q(u)$ is the two-sided ideal generated by all matrix elements of the left-hand side of (1) for arbitrary values of $x_1, x_2 \in \mathbf{C}$. The subalgebra $\mathscr{D}_N^q(u)$ of a proper localization of $\tilde{\mathscr{D}}_N^q(u)$ generated by $\tilde{E}_{ij}^s(\tilde{E}_{ii}^n)^{-1} \overset{\text{def}}{=} E_{ij}^s$ is called the R-**algebra** (of level n), corresponding to $R(x;q)$. Later on $\mathscr{D}_N^q(\varnothing) = \mathbf{C} = \mathscr{D}_0^q(u)$.

We have to pass from $\tilde{\mathscr{D}}_N^q(u)$ to $\mathscr{D}_N^q(u)$ to eliminate the symmetry ${}^1\tilde{h}^2\tilde{h}R = R^1\tilde{h}^2\tilde{h}$ of $R(x;q)$, where $\tilde{h} = \sum_{i=1}^N q^{h_i} I^{ii}$, $h_i \in \mathbf{C}$. This symmetry involves an action τ_h: $L \to L\tilde{h} \Leftrightarrow \tau_h(\tilde{E}_{ij}^s) = q^{h_i}\tilde{E}_{ij}^s$ of the group $\mathbf{C}^N \ni (h_i) = h$ on the space $\mathscr{L}_N(u)$, the algebra $\tilde{\mathscr{D}}_N^q(u)$ and its localization $\tilde{\mathscr{D}}_N^q(u)'$ by $\{\tilde{E}_{ii}^n\}$.

PROPOSITION 1.1. *The algebra of invariants of $\tilde{\mathscr{D}}_N^q(u)'$ under the action τ of $\mathbf{C}^N$ is $\mathscr{D}_N^q(u)$.*

This statement follows from the identities

$$\tilde{E}_{ii}^n \tilde{E}_{jk}^s = q^{(\delta_{ij}-\delta_{ik})/2} \tilde{E}_{jk}^s \tilde{E}_{ii}^n$$

proved by a consideration of (1) at $x_1 = \infty$. □

PROPOSITION 1.2. *The algebra $\mathscr{D}_N^q(u)$ is generated by E_{ij}^s with the relations*

$$\begin{aligned} E_{ij}^r E_{kl}^s q^{(\delta_{ij}-\delta_{il})/2} - E_{kl}^s E_{ij}^r q^{(\delta_{ij}-\delta_{jk})} &= \left(q^{1/2}-1\right)\left(E_{kl}^s E_{ij}^r \delta_{ik} - E_{ij}^r E_{kl}^s \delta_{jl}\right) \\ &\quad + \left(q^{1/2}-q^{-1/2}\right)\left(\left(\theta_{sr}+\theta_{jl}-1\right)E_{il}^r E_{kj}^s + \left(1-\theta_{sr}-\theta_{ki}\right)E_{il}^s E_{kj}^r\right) \\ &\quad + \left(q^{1/2}-q^{-1/2}\right) \sum_{a<r<b}^{a+b=r+s} \left(E_{il}^a E_{kj}^b - E_{il}^b E_{kj}^a\right), \end{aligned} \tag{2}$$

where $1 \leqslant i, j \leqslant N$, $E_{ij}^n = 0 = E_{ji}^0$ *for* $i < j$, $E_{ij}^s = 0$ *for* $s < 0, s > n$, $E_{ii}^n = 1$, $\theta_{ij} = 1, 0 \overset{\text{def}}{\Leftrightarrow} i \geqslant j, i < j$. □

Let us enumerate some simple properties of $\mathscr{D}_N^q(u)$.

PROPOSITION 1.3. (i) *The mapping*

$$L(x) \to L(xq^d) \prod_{i=1}^{n} \frac{x - q^{u_i - d}}{x - q^{v_i}}$$

induces an isomorphism $\pi_u^v(d)$: $\mathscr{D}_N^q(u) \to \mathscr{D}_N^q(v)$ *for* $d \in \mathbf{C}$, $u, v \in \mathbf{C}^n$. (ii) *The natural inclusion* $\mathscr{L}_N(u') \to \mathscr{L}_N(u)$ *for* $u' = (u_i, i \leqslant n' < n) \in \mathbf{C}^{n'}$ *induces a surjective homomorphism of* $\mathscr{D}_N^q(u)$ *onto* $\mathscr{D}_N^q(u')$. (iii) *The mapping* $E_{ij}^1 \to E_{ij}^n$, $E_{ij}^0 \to E_{ij}^0$ *can be prolonged to an isomorphism of* $\mathscr{D}_N^q(0) \overset{\text{def}}{=} \mathscr{D}_N^q$ $(u = 0 \in \mathbf{C})$ *onto some subalgebra* $\mathscr{D}_N^q(u)_0 \subset \mathscr{D}_N^q(u)$. □

We will give below a posteriori reasons for such a definition for arbitrary u. As for $\mathscr{D}_N^q(0)$, if in (2) we put $q^{1/2} = 1 + \delta/2$, $E_{ij}^{0,1} = \delta e_{ij} - \delta_{ij}$ except $E_{ii}^1 = 1$, where $\delta^2 = 0$, then e_{ij} must obey the relations $[e_{ij}, e_{kl}] = e_{il}\delta_{jk} - e_{kj}e_{il}$. Hence, $\mathscr{D}_N^q(0)$ is a deformation of the universal enveloping algebra $U(\mathfrak{gl}_N)$ of $\mathfrak{gl}_N$. Moreover this deformation is flat (this statement requires a special proof—see Theorem 2.2).

Let us construct a $N \times N$-matrix

$$L_u^q(x) = {}^1L_u^q(x) = {}^{11'}R(xq^{-u_1}) \cdots {}^{1n'}R(xq^{-u_n})$$

with elements in ${}^{1'}M_N \cdots {}^{n'}M_N = M_N^{\otimes n}$ (we numerate by $1', \ldots, n'$ some extra components of tensor products, which are different from the components without primes). The main property of $\mathscr{D}_N^q(u)$ is as follows.

PROPOSITION 1.4. *The matrix* $L_u^q(x)$ *satisfies* (1) *and therefore determines a* $\mathscr{D}_N^q(u)$*-module* $V_u^q = (\mathbf{C}^N)^{\otimes n}$ *after the passage from* $\tilde{\mathscr{D}}_N^q(u)$ *to* $\mathscr{D}_N^q(u)$.

The proposition is easily deduced from its particular case $n = 1$ (see e.g., [3, 12]). For $n = 1$ one must verify the Yang–Baxter identity

$${}^{12}R(x_1x_2^{-1})\,{}^{13}R(x_1x_3^{-1})\,{}^{23}R(x_2x_3^{-1}) = {}^{23}R(x_2x_3^{-1})\,{}^{13}R(x_1x_3^{-1})\,{}^{12}R(x_1x_2^{-1}).$$

To compare $R(x; q)$ with the analogous R-matrix from [15], §3, let us note that the last equality is valid for a more general $\tilde{R}$ with $I^{ii} \otimes I^{jj}$ multiplied by $\rho_{i-j} = \rho_{j-i}^{-1} \in \mathbf{C}$ and $I^{ij} \otimes I^{ji}$ multiplied by $x^{a(i-j)}$, $a \in \mathbf{C}$ for $i \neq j$ in addition to the coefficients of R. □

Finally, we will introduce some "good" irreducible representations of $\mathscr{D}_N^q(u)$. See [6] for the general case.

The **skew Young diagram** $\mu = \{\alpha, \beta\}$ of degree n is given by two sequences $\alpha = (\alpha_i)$, $\beta = (\beta_i) \subset \mathbb{C}$ coinciding for almost all $i \in \mathbb{Z}$ with the properties (a) $\beta_i - \alpha_i \in \mathbb{Z}_+ \stackrel{\text{def}}{=} \{m \in \mathbb{Z} | m \geqslant 0\}$, (b) $i \leqslant k < j$, $\alpha_i - \alpha_j \in \mathbb{Z} \Rightarrow \alpha_k - \alpha_j$, $\beta_k - \beta_j \in \mathbb{Z}_+$. Let us construct by μ some ordered finite set $\tilde{\mu} = \{(i_k, y_k)\} \subset \mathbb{Z} \times \mathbb{C}$, where (a) $1 \leqslant k \leqslant n \stackrel{\text{def}}{=} = \Sigma_i(\beta_i - \alpha_i)$, (b) $\beta_{i_k} - \alpha_{i_k} > \beta_{i_k} - y_k \in \mathbb{Z}_+$, (c) if $l > k$ then $i_l > i_k$ or $y_l - y_k > 0 = i_l - i_k$. If $i \geqslant 1$, $\{\alpha_i, \beta_j\} \subset \mathbb{Z}$ then we obtain the standard definition. We put $u_k = i_k - y_k$, $\tilde{u}_k = u_k + (i_k - 1)v$, $v \in \mathbb{C}$, $R_{\tilde{u}} = \prod_{i<j} {}^{ij}R(q^{\tilde{u}_i - \tilde{u}_j})$, where $1 \leqslant i$, $j \leqslant n$ and (i, j) is before (i', j') in the product iff $i > i'$ or $j - j' > 0 = i - i'$. For this u we will sometimes write $u \sim \mu$.

PROPOSITION 1.5. (see [6, 7]). (i) *There exists a limit* $\lim_{v\to 0} R_{\tilde{u}} \stackrel{\text{def}}{=} R_\mu$. (ii) *The subspace* $V_\mu^q = R_\mu V_u^q$ *of* V_u^q *is an irreducible* $\mathcal{D}_N^q(u)$*-submodule, corresponding to* $L_\mu^q(x) = L_u(x)P_\mu$ *for a projector* P_μ *onto* V_μ^q. (iii) *The algebra* $\mathcal{D}_N^q(u)$ *acts on* V_μ^q *through its quotient-algebra* $\mathcal{D}_N^q(a)$, *where* $a = (\alpha_i + 1 - i | \alpha_{i-1} \neq \alpha_i \neq \beta_i)$. □

If we restrict our representations to $\mathcal{D}_N^q(u)_0 \simeq \mathcal{D}_N^q(0)$ and degenerate $\mathcal{D}_N^q(0)$ to $u(\mathfrak{gl}_N)$ by making $q^{1/2}$ tend to 1 (see above) then V_u^1 as $\mathfrak{sl}_N$-module will be the usual n-th tensor power of the fundamental N-dimensional representation and $V_\mu^1 = R_\mu(q = 1)V_u$ will be isomorphic to the $\mathfrak{sl}_N$-module arising from μ in the classic representation theory. If $\alpha_1 \neq \beta_1$, $\alpha_i = \beta_i$ for $i \leqslant 0$ (one can get this by a common shift of i, α_i, β_i) and $\alpha_i = 0$ for each i then V_μ^1 is the irreducible $\mathfrak{gl}_N$-module with the highest weight $(\beta_1, \beta_2, \ldots, \beta_i, \ldots)$. We further put:

$$L^q_{\{\alpha, \alpha\}}(x) = \mathbb{1} = L^q_\phi(x), \qquad V^q_{\{\alpha, \alpha\}} = \mathbb{C} = V^q_\phi.$$

§2. Structural theorems. We begin with the notion of "**quantum determinant**" of L in the spirit of [12] (see also [15, 10, 7]). For the universal solution L of (I) with $\tilde{E}_{ij}^s$ from $\tilde{\mathcal{D}}_N^q(u)$ (or for a concrete L according to the context) let us define

$$H^N(x) = Sp_{1,\ldots,N}({}^N L(xq^{-1}) \ldots {}^1 L(xq^{-N}) \mathcal{Q}_N^N) / \prod_{i=1}^N \tilde{E}_{ii}^n. \tag{3}$$

Here $Sp_{1,\ldots,K}$ denotes the matrix spur in $M_N^{\otimes K} \simeq M_{NK}$ over any algebra of coefficients, $\mathcal{Q}_N^K \stackrel{\text{def}}{=} 1/k! R_w$ for $w = (1, \ldots, K) \in \mathbb{C}^K$ (see §1), $Q_1^1 = Q_0^0 = 1$. A priori $H^N \in \mathcal{L}_1(u \uparrow N) \otimes_{\mathbb{C}} \mathcal{D}_N^q(u)$, where $u \uparrow K \stackrel{\text{def}}{=} (u_i + j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant K) \in \mathbb{C}^{nK}$. We will use later some properties of Q.

PROPOSITION 2.1. (i) *If* $q \to 1$ *then* Q_N^K *tends to the "antisymmetrizer"—the canonical projector onto the exterior power* $\Lambda^K \mathbb{C}^N \subset (\mathbb{C}^N)^{\otimes K}$; $2Q_N^2 = \Sigma_{i \neq j} I^{ii} \otimes I^{jj} - q^{1/2} \Sigma_{i>j} I^{ij} \otimes I^{ji} - q^{-1/2} \Sigma_{i<j} I^{ij} \otimes I^{ji}$, $rk(Q_N^K) = \dim_{\mathbb{C}} \Lambda^K \mathbb{C}^N$, $(Q_N^N)^2 =$

Q_N^N. (ii) *If* $C_i = I^{i'i'}$ *for some*

$$1 \leqslant 1', \ldots, K' \leqslant K \quad \textit{then} \quad Sp_{1,\ldots,K}\left({}^{1\ldots N}Q_N^{N1}C_1 \ldots {}^{K}C_K\right)$$

is equal to

$$\frac{(N-K)!}{N!}{}^{K+1\ldots N}Q_{N-K}^{N-K} \quad \text{if } i \neq j \Rightarrow i' \neq j'$$

and otherwise it is equal to zero; the analogous is true for $Sp_{K+1,\ldots,N}$ *and* $K < i' \leqslant N$. (iii) *The image of* Q_N^N *is the space* $\mathbf{C}\sum_\sigma(-q^{1/2})^{l(\sigma)}e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(N)}$, *where* e_i *is the first column of* I^{i1}, $\sigma \in S_N$, $l(\sigma)$*—the length of* σ *(see next* §3);

$$\prod_{i=1}^{N} \tilde{E}_{ii}^{n} \cdot H^N(x)$$
$$= \sum_{\sigma, r} \tilde{E}^{r_1}_{\sigma(N)N}\tilde{E}^{r_2}_{\sigma(N-1)N-1} \ldots \tilde{E}^{r_N}_{\sigma(1)1}(-q^{1/2})^{l(\sigma)} \prod_{j=1}^{N}(xq^{-j})^{r_j}\prod_{i=1}^{n}(xq^{-j} - q^{u_i})^{-1},$$

where $0 \leqslant r_i \leqslant n$ *(cf.* [11], *Prop.* 6.3). □

The symmetry of R (see §I) involves the following branching properties of $\mathscr{D}_N^q(u)$. The functions $[L]^K \stackrel{\text{def}}{=} \sum_{i,j=1}^K I^{ii}LI^{jj}$, $[L]_K = \sum_{i,j=N-K+1}^N I^{ii}LI^{jj}$ satisfy (1) and hence determine inclusions ρ^K: $\mathscr{D}_K^q(u) \to \mathscr{D}_N^q(u) \leftarrow \mathscr{D}_K^q(u)$: ρ_K. Let us denote by H^K, $H_K \in \mathscr{L}_1(u \uparrow K) \otimes_{\mathbf{C}} \rho(\mathscr{D}_K^q(u))$ the H-functions (3) constructed for $N := K$ and $[L]^K$, $[L]_K$ in place of L. Lataer on $H^0 = H_0 = 1$.

THEOREM 2.2 (cf. [6, 7]). (i) *The subspace* $[\mathscr{D}_N^q(u)]_p$ *of* $\mathscr{D}_N^q(u)$, *linearly generated by monomials of degree* $\leqslant p$ *in* E_{ij}^r, *is isomorphic to the space of all polynomials in* nN^2 *variables of the same degree.*

(ii) *The coefficients of the principal parts of* $H^1(\bar{x}), \ldots, H^N(x)$ *(in their expansions at points* $u_i + j$ *from* $u \uparrow N$*) are all algebraically independent and generate a maximal commutative subalgebra* $A_N^q(u)$ *in* $\mathscr{D}_N^q(u)$. *The analogue is true for* $H_1, \ldots, H_N$.

(iii) *The coefficients of* $H^N = H_N$ *generate the centre* $Z_N^q(u)$ *of* $\mathscr{D}_N^q(u)$.

Proof scheme. For every p one can find a skew diagram μ such that the homomorphism $\mathscr{D}_N^q(u) \to \mathrm{End}_{\mathbf{C}}V_\mu^q$ is injective on $[\mathscr{D}_N^q(u)]_p$. To do this we tend $q^{1/2}$ to 1 and construct a flat degeneration of $\mathscr{D}_N^q(u)$ (from a punctured neighbourhood of $q^{1/2} = 1$) to some quotient-algebra $\tilde{U}'$ of $\tilde{U} \stackrel{\text{def}}{=} U(\mathfrak{gl}_N[t]/(t^n))$ (see above and [6, 7]). We have: $\dim_{\mathbf{C}}[\tilde{U}']_p = \dim_{\mathbf{C}}[\mathscr{D}_N^q(u)]_p$. Next we use the representation theory of $U(\mathfrak{gl}_N)$ to find μ with the above property for $\tilde{U}$ and V_μ^1. This μ necessarily suit $\mathscr{D}_N^q(u)$ too. We have proved (i) and also the coincidence of $\tilde{U}$ and $\tilde{U}'$ since p is arbitrary. We remind the reader that $q^{1/2}$ is close to 1 here and everywhere.

It is not difficult to deduce the identities $[H^k(x), H^m(y)] = 0 = [H^N(x), E_{ij}^r]$, $x, y \in \mathbf{C}$ from Proposition 2.1, i) and (1) (see [12], [15] §3, [5, 6, 7]). To complete the proof of (ii), (iii) we pass to a quasi-classical limit. First we put $q^{1/2} = 1$ in (2), §1 and will come to the commutative algebra $d_N(u) \overset{\text{def}}{=} \mathbf{C}[E_{ij}^r]$. Next, by the formal substitution $q^{1/2} \to 1 + \delta/2$, $[E_{ij}^r, E_{kl}^s] \to \delta\{E_{ij}^r, E_{kl}^s\}$ in (2) and making $\delta = 0$ (after reducing by δ) one can define the brackets $\{E_{ij}^r, E_{kl}^s\}$ and then extend them to a Poisson bracket $\{\,,\}$ on $d_N(u)$. Let us denote $H^K(x; q = 1)$, $H_K(x; q = 1)$ by $h^K(x)$, $h_K(x)$. We have:

$$h^K(x) = \det\left([L(x)]^K\right) / \prod_{i=1}^{K} \tilde{E}_{ii}^n,\ h_K(x) = \det\left([L(x)]_K\right) / \prod_{i=N-K+1}^{N} \tilde{E}_{ii}^N.$$

Let $q_{K'+1}, q_{K'+2}, \ldots, q_{K'+nK}$ be the coefficients of the principal part of h^K, which are put in any order ($1 \leqslant K \leqslant N$, $K' = nK(K-1)/2$). The subalgebra $a_N(u) \overset{\text{def}}{=} \mathbf{C}[q_1, \ldots, q_{(N+1)'}]$ is commutative (involutive) relative to $\{\,,\}$.

LEMMA 2.3. (i) *The mapping $L \to (h^1, \ldots, h^N)$ determines a regular surjective morphism α: $\mathbf{C}^{N^2 n} \to \mathbf{C}^{N(N+1)n/2}$ where $\mathbf{C}^{N^2 n} =$* Specm $d_N(u)$, $\mathbf{C}^{N(N+1)n/2} =$ Specm $a_N(u)$. (ii) *Sufficiently general fibers of α are isomorphic to open submanifolds of some Jacobians of algebraic curves, which are the orbits of the entire collection of flows (differential equations)*

$$\partial E_{ij}^s / \partial t = \left\{h, E_{ij}^s\right\},$$

commuting with one another, h being an element of $a_N(u)$. All this holds good for $h_1, \ldots, h_N$.

Statement (i) is verified directly. As for (ii) we refer the reader to [14, 15]. □

The remaining part of the Theorem is deduced from the Lemma in the following way. If $h \in d_N(u)$ and $\{h, h^i(x)\} = 0$ for all i, x then h has to be constant along general fibers of α and hence is a polynomial in (q_i) (since α is surjective). By computation of the dimensions we obtain that sufficiently general fibers of the morphism ξ: $\mathbf{C}^{N^2 n} \to \mathbf{C}^{Nn} =$ Specm $z_N(u)$ induced by the mapping $L \to h^N$ have non-degenerate Poisson structures for $z_N(u) = \lim Z_N^q(u)$. Therefore the kernel of $\{\,,\}$ coincides with $z_N(u) = \mathbf{C}[q_{N'+1}, \ldots, q_{(N+1)'}]$. The subsequent lifting to $\mathscr{D}_N^q(u)$ uses statement (i) of the Theorem. □

Our next task is to calculate the centralizer $Z(\operatorname{Im} \rho^K)$ of $\mathscr{D}_K^q(u)$ in $\mathscr{D}_N^q(u)$. In solving this problem L-functions and R-algebras of arbitrary level n display their usefulness and add some new features to the classical results.

If L is our universal solution of (1) then $L(x)^{-1}$, $L^*(x) = H^N(xq)L(x)^{-1}$ and $\hat{L}(x) = L^*(x)/H^N(x)$ satisfy (1) for $\hat{R}(x) = R(x; q^{-1})$ (an easy computation). It follows (cf. [12]) from the definition and properties of H and Proposition 2.1, (ii) that

$${}^N L^*(x) \prod_{i=1}^{N} \tilde{E}_{ii}^n = S_{p_1, \ldots, N-1}\left({}^{N-1}L(xq^{-1}) \ldots {}^1 L(xq^{-N+1}) Q_N^N\right).$$

Hence $L^* \in \mathscr{L}_N(u \uparrow N-1) \otimes_{\mathbf{C}} \mathscr{D}_N^q(u)$ determines homomorphisms

$$\rho_K^*: \mathscr{D}_K^{q^{-1}}(u \uparrow N-1) \to \mathscr{D}_N^q(u)$$

corresponding to the mappings $L \to [L^*]_K$ (after the passage from $\tilde{\mathscr{D}}$ to $\mathscr{D}$). Let us define $\hat{H}_K$ by means of (3) for $\hat{L}(x)$ with $\hat{Q}_N^N$ constructed for $\hat{R}$ instead of R. One can prove with the aid of Proposition 2.1, (ii) the identities

$$H^K(x) = \hat{H}_{N-K}(xq^{-1})H^N(x), \tag{4}$$

which become the well-known relations for principal minors for $q = 1$.

THEOREM 2.4. *The centralizer* $Z(\operatorname{Im}\rho^K)$ *coincides with*

$$Z_N^q(u) \cdot \rho_{N-K}^*\left(\mathscr{D}_{N-K}^{q^{-1}}(u \uparrow N-1)\right).$$

Conversely, $Z(\operatorname{Im}\rho_{N-K}^*) = Z_N^q(u) \cdot \rho^K(\mathscr{D}_K^q(u))$.

Proof. We obtain from (1) the identity

$${}^{12}R^t(x_1x_2^{-1})^1L^t(x_1)^2L^*(x_2) = {}^2L^*(x_2)^1L^t(x_1)^{12}R^t(x_1x_2^{-1}),$$

where A^t is the transpose of $A \in M_N$, $(A \otimes B)^t = A^t \otimes B$. A moment's consideration shows that matrix elements of $[L]^K$ commute pairwise with matrix elements of $[L^*]_{N-K}$ (use the above identity and the special structure of R^t). To complete the proof we put $q = 1$ everywhere in the same way as above. We keep all the notation of Theorem 2.2.

For some sufficiently general stratum $F_c = \{q_{N'+i} = c_i\}$, $1 \leqslant i \leqslant Nn$ and a point $l \in F_c$ one can supplement $(q_1, \ldots, q_{N'})$ with such local functions $(p_1, \ldots, p_{N'})$ that $\tilde{q}_i = q_i - q_i(l)$, $\tilde{p}_j = p_j - p_j(l)$ form coordinates in a neighbourhood of l in F_c and $\{q_i, p_j\} = \delta_{ij}$, $\{p_i, p_j\} = 0$. We remind the reader that $\{q_i, q_j\} = 0$ and $(q_{N'+i})$ generate the kernel $z_N(u)$ of $\{\,,\}$—see Theorem 2.2. The functions $p_1, \ldots, p_{(K+1)'}$ belong necessarily (modulo $(q_{N'+i})$) to some formal completion of a localization of the algebra $\operatorname{Im}\rho^K \simeq d_K(u)$. The analogous case is true for $p_{(K+1)'+1}, \ldots, p_{N'}$ and $\operatorname{Im}\rho_{N-K}^*$ since $h^{K+i}(x) = \hat{h}_{N-K-i}(x)h^N(x)$ (see (4)). Here we use the fact that ρ^K and ρ_{N-K}^* preserve the appropriate Poisson brackets (i.e. are canonical maps).

Any local function f on F_c at l is a series of powers of $\tilde{q}_i, \tilde{p}_i$. Hence, $\{f, p_i\} = 0 = \{f, q_i\}$ for $1 \leqslant i \leqslant K'$ iff f does not depend on $p_1, \ldots, p_{K'}$ and belongs to a completion of a localization of the algebra $\operatorname{Im}\rho_{N-K}^* \cdot Z_N(u)$. Now let us consider the regular morphism $\mathbf{C}^{N^2n} \to \mathbf{C}^{(N-K)^2(N-1)n} \times \mathbf{C}^{Nn} = \operatorname{Specm} d_{N-K}(u \uparrow N-1) \times \operatorname{Specm} z_N(u)$ induced by $\rho_{N-K}^* \times \xi$. Its image is a normal affine variety. By means of this we eliminate the words "completion and localization" if the above f is from $d_N(u)$ and prove the quasi-classical version of the Theorem (the converse statement is quite analogous). The lifting of this limited statement to a general one with $q^{1/2} \neq 1$ uses Theorem 2.2, (i). □

We begin the description of the branching rules for representations of $\mathscr{D}_N^q(u)$ with the following proposition that is a formal consequence of the definition of

L_μ^q, $\hat{L}$. We remind the reader that L_μ^q are determined non-uniquely and depend on the choice of P_μ.

PROPOSITION 2.5. (i) *If we put* $\hat{\chi} = (\hat{\chi}_i)$, $\hat{\chi}_i = -\chi_{-i} - 1$ *for a set* $\chi = (\chi_i)$ *then*

$$\hat{L}^q_{\{\alpha,\beta\}}(x) = L^{q^{-1}}_{\{\hat{\beta},\hat{\alpha}\}}(x). \qquad \square$$

Let us define $\tilde{H}^K$ *by* (3) *with* Q_K^K *and* $\tilde{L}^K(x) \overset{\text{def}}{=} ([L^{-1}]_K)^{-1} H^{N-K}(x)/H^{N-K}(xq)$ ($\tilde{L}^K$ *satisfies* (I) *for* $R_K(x;q)$). *One can easily deduce from* (4) *that*

$$H^N(x) = H^{N-K}(x)\tilde{H}^K(x).$$

The mapping $L_\mu \to ([L_\mu]^K, \tilde{L}_\mu^{N-K})$ *gives us an action of* $\mathcal{D}_K^q(u) \times \mathcal{D}_{N-K}^q(u)$ *on* V_μ.

THEOREM 2.6. (i) *There exists such a basis* $\{\xi_\lambda\} \subset V_\mu$ *that*

$$H^K(x)\xi_\lambda = \xi_\lambda \prod_i \left(x - q^{i-\lambda_i^{(K)}}\right)/\left(x - q^{i-\alpha_i}\right),$$

where λ *ranges over the sequence of all collections* $\lambda = \{\lambda^{(0)}, \ldots, \lambda^{(N)}\}$, $\lambda^{(K)} = (\lambda_i^{(K)} \in \mathbb{C})$, $\lambda^{(0)} = \alpha$, $\lambda^{(N)} = \beta$ *with the following properties*: a) $\lambda_i^{(K)} - \lambda_i^{(K-1)} \in \mathbb{Z}_+$, b) $\mathbb{Z}_+ \ni \lambda_i^{(K)} - \lambda_{i-1}^{(K)} \Rightarrow \lambda_i^{(K-1)} - \lambda_{i-1}^{(K)} \in \mathbb{Z}_+$. *The vectors* ξ_λ *are unique up to scalers.*

(ii) *If we put* $W_\gamma = \bigoplus \mathbb{C}\xi_\lambda$, *where* $\gamma = (\gamma_i)$, $\lambda_i^{(K)} = \gamma_i$, *then* $V_\mu^q = \bigoplus_\gamma W_\gamma$ *and* $W_\gamma \simeq V^q_{\{\alpha,\gamma\}} \otimes V^q_{\{\gamma,\beta\}}$ *as* $\mathcal{D}_K^q(u) \times \mathcal{D}_{N-K}^q(u)$*-modules.*

To prove the Theorem we can restrict ourselves to the case $K = N - 1$ and then use the induction and Proposition 2.5. We note that statement (i) easily follows from (ii) and the formula for the action of H^N (see [6, 7, 11]), and the Theorem is verified by a direct computation for $n = 1$.

One can generalize to V_u^q considered as $\mathcal{D}_{N-1}^q(u)$-module the classic decomposition

$$(\mathbb{C}^N)^{\otimes n} = \bigoplus_{k=0}^{n} \mathbb{C}[S_n]V_k, \quad \text{where } V_k = e_N^{\otimes k} \otimes \left(\bigoplus_{i=1}^{N-1} \mathbb{C}e_i\right)^{\otimes(n-k)},$$

S_n acts naturally on $(\mathbb{C}^N)^{\otimes n}$, e_i are from Proposition 2.1. The Hecke algebra $H_n^q \subset M_N^{\otimes n}$ plays the role of $\mathbb{C}[S_n]$ in its q-analogue (see next §3). The branching properties of Hecke algebras (ibid.) allow us to decompose V_μ^q when N is "indefinite" (we note that the projector P_μ of §1 can be chosen from H_n^q). The restrictions for $\lambda_i^{(K)}$ (see (i)) arise from Proposition 2.5. and the following vanishing theorem:

$$V_\mu^q \neq \{0\} \Leftrightarrow i_k - i_l < N \quad \text{if } y_k = y_l.$$

We would like to outline another way of proving the Theorem. One can degenerate $\mathscr{D}_N^q(u)$ to some Yangian, while tending $q^{1/2}$ to 1, and put $n = 1$ to apply the classic results or the recent results by G. I. Olshansky about $U(\mathfrak{gl}_N)$ and Yangians. $\square$

§3. Hecke algebras. The Hecke algebra H_n^q is generated over $\mathbb{C}$ by $1, T_1, \ldots, T_{n-1}$ with the relations $[T_i, T_j] = 0$ whenever $i \neq j \pm 1$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and $(T_i - q)(T_i + 1) = 0$. As above $q \in \mathbb{C} \setminus \{1\}$ is sufficiently close to 1. Let S_n be the symmetric group acting on any set with n elements; $s_i = (i, i+1)$, s_0—the unit of S_n. Given any function f of arguments $x_1, \ldots, x_n$ permutations $w \in S_n$ act on f by the formula $({}^w f)(x_1, \ldots, x_n) = f(w^{-1}(x_1, \ldots, x_n))$. We denote by $l(w)$ the length of $w \in S_n$ (the length of a decomposition of w in a product of s_i of minimal length).

By adjoining to H_n^q pairwise commuting independent $X_1, \ldots, X_n$ with the relations $[X_i, T_j] = 0$ for $i \neq j+1, j$ and

$$X_{i+1}T_i - T_iX_i = (q-1)\,X_{i+1} = T_iX_{i+1} - X_iT_i \tag{5}$$

we shall get the **affine Hecke algebra** $\mathfrak{H}_n^q$ (see e.g. [13]). The localization of $\mathfrak{H}_n^q$ by $X_1, \ldots, X_n$ (generated by $\mathfrak{H}_n^q$ and $X_1^{-1}, \ldots, X_n^{-1}$) will be denoted by $\tilde{\mathfrak{H}}_n^q$.

PROPOSITION 3.1 [13]. (i) *The collection of H_n^q-valued functions $\varphi_i = \varphi_{s_i} = T_i + (q-1)/(x_ix_{i+1}^{-1} - 1)$ in $x_1, \ldots, x_n$ extends uniquely to a homogeneous 1-cocycle $\varphi_w(x_1, \ldots, x_n)$ with the following property: $\varphi_{ab} = \varphi_b{}^{b^{-1}}\varphi_a$ if $ab \geqslant b \overset{\text{def}}{\Leftrightarrow} l(ab) = l(a) + l(b)$ for $a, b \in S_n$.* (ii) *The set of elements $\Upsilon_i = \Upsilon_{s_i} = T_i + (q-1) \times (X_iX_{i+1}^{-1} - 1)^{-1} \in \tilde{\mathfrak{H}}_n^q$ extends uniquely to $\{\Upsilon_w, w \in S_n\}$ with the following properties: $X_i\Upsilon_w = \Upsilon_w X_{w(i)}$, $\Upsilon_{ab} = \Upsilon_b\Upsilon_a$ if $ab \geqslant b \in S_n$.* $\square$

In the notations of §1 given a skew μ let us define the permutation $\omega = \omega_\mu$ by means of the following rule: if $0 < y_{\omega(k)} - y_{\omega(l)} \in \mathbb{Z}_+$ or $i_{\omega(k)} - i_{\omega(l)} < 0$ when $y_{\omega(k)} - y_{\omega(l)} \in \{0, \mathbb{C} \setminus \mathbb{Z}\}$ then $l > k$. This ω reorders in a natural way the two-dimensional set $\tilde{\mu}$ when it has been reflected in a line $i + y =$ const. We put $\tilde{\varphi}_w = \varphi_w(q^{\tilde{u}_1}, \ldots, q^{\tilde{u}_n})$, $\tilde{u}_k = u_k + (i_k - 1)v$, $u_k = i_k - y_k$. There exist limits $\lim_{v \to 0}\tilde{\varphi}_w \overset{\text{def}}{=} f_w \in H_n^q$ for $w \geqslant \omega$ (see [6, 9]). Let $f_\omega = 1$ when $\mu = \{\alpha, \alpha\}$.

For an arbitrary $u = (u_i) \in \mathbb{C}^n$ we will denote below by $J_u^q \simeq H_n^q$ the only right $\mathfrak{H}_n^q$-module defined by relations $\langle z \rangle h = zh$, $\langle 1 \rangle X_i = q^{u_i}$, where $z, h, 1 \in H_n^q$, h takes z into $\langle z \rangle h$. Let $\mathfrak{H}_n^q(a) = \mathfrak{H}_n^q/A$, where $a = (a_i) \in \mathbb{C}^m$, A is the two-sided ideal $\mathfrak{H}_n^q(\prod_{i=1}^m (X_n - q^{a_i}))\mathfrak{H}_n^q$. One can prove that $\mathfrak{H}_n^q(a) = \oplus X_1^{i_1} \ldots X_n^{i_n} H_n^q$, where $0 \leqslant i_k < m$.

PROPOSITION 3.2 [6, 9]. (i) *The subspace $J_\mu^q \overset{\text{def}}{=} f_w H_n^q \subset J_u^q$ for $u \sim \mu$ does not depend on $w \geqslant \omega$ and is an irreducible $\mathfrak{H}_n^q$-module and is, in fact, a $\mathfrak{H}_n^q(a)$-module, where $a = (\alpha_i + 1 - i | \alpha_{i-1} \neq \alpha_i \neq \beta_i)$.* (ii) *The vectors $\{f_w, w \geqslant \omega\}$ form a basis of J_μ^q with the following property: $\langle f_w \rangle X_i = q^{w^{-1}(u_i)} f_w$. The last formulas determine each f_w uniquely up to a factor of proportionality.* $\square$

Let us denote below by $\hat{\mathfrak{H}}_k^q$ the subalgebra in $\mathfrak{H}_n^q$ generated by $1, T_{n-k+1}, \dots, T_{n-1}, X_{n-k+1}, \dots, X_n$, $0 \leqslant k \leqslant n$ and by $\tilde{\mathfrak{H}}_k^q$ the algebraic span of $1, T_1, \dots, T_{k-1}, X_1, \dots, X_k$ ($\hat{\mathfrak{H}}_k^q, \tilde{\mathfrak{H}}_k^q \simeq \mathfrak{H}_k^q$). We can connect with an arbitrary $w \in S_n$ two subsets $[w]^k = (w(n-k+1), \dots, w(n))$, $[w]_k = (w(1), \dots, w(k))$ of $(1, \dots, n)$ and two permutations $\hat{w}^k, \hat{w}_k \in S_{n-k}$ by throwing away respectively $[w]^k, [w]_k$ (e.g. $\hat{w}_k$: $(1, \dots, \widehat{w(i_1)}, \dots, \widehat{w(i_k)}, \dots, n) \to (\widehat{w(1)}, \dots, \widehat{w(k)}, w(k+1), \dots)$, where omitted terms are marked off by "$\hat{\ }$", $1 \leqslant i_l \leqslant k$). The following Theorem is deduced from [6, 9].

THEOREM 3.3. (i) *Let $\gamma = (\gamma_i)$, $\alpha_i \leqslant \gamma_i \leqslant \beta_i$ and both $\{\alpha, \gamma\}$ and $\{\gamma, \beta\}$ be skew diagrams of degree k and $n-k$ respectively. Define two disjoint sets $\gamma^* = \{1 \leqslant l \leqslant n \,|\, (i_l, y_l) \in \widetilde{\{\alpha, \gamma\}}\}$, $\gamma_* = \{1 \leqslant l \leqslant n \,|\, (i_l, y_l) \in \widetilde{\{\gamma, \beta\}}\}$ and a subset $[\gamma] = \{g \in S_n \,|\, [g]^k = \gamma^*\} = \{g \,|\, [g]_{n-k} = \gamma_*\}$. Then $J_\mu^q = \bigoplus_\gamma I_\gamma$, where $I_\gamma = \bigoplus_{g \in [\gamma]} \mathbb{C} f_g$.*

(ii) *We have: $g^k \geqslant \omega_{\{\alpha,\gamma\}}$, $g_{n-k} \geqslant \omega_{\{\gamma,\beta\}}$ for $g \in [\gamma]$. The linear map taking f_g, $g \in [\gamma]$ into $f_{g^k} \otimes f_{g_{n-k}}$ is an isomorphism of the $\mathfrak{H}_k^q \times \mathfrak{H}_{n-k}^q$-modules I_γ (which is invariant under the action of $\hat{\mathfrak{H}}_k^q \cdot \tilde{\mathfrak{H}}_{n-k}^q$) and $\mathscr{J}_{\{\alpha,\gamma\}}^q \otimes \mathscr{J}_{\{\gamma,\beta\}}^q$.* □

THEOREM 3.4. (i) *The subalgebra $\mathbb{C}[X_1, \dots, X_n]$ is a maximal commutative one in $\mathfrak{H}_n^q$. The centre Z_n^q of $\mathfrak{H}_n^q$ is generated by symmetric polynomials in $X_1, \dots, X_n$. The centralizer $Z(\hat{\mathfrak{H}}_k^q)$ of $\hat{\mathfrak{H}}_k^q \subset \mathfrak{H}_n^q$ coincides with $Z_n^q \tilde{\mathfrak{H}}_{n-k}^q$; conversely, $Z(\tilde{\mathfrak{H}}_k^q) = Z_n^q \hat{\mathfrak{H}}_{n-k}^q$.*

(ii) *Statements (i) are true for $\mathfrak{H}_n^q(0) = \mathfrak{H}_n^q / \mathfrak{H}_n^q (X_n - 1) \mathfrak{H}_n^q \simeq H_n^q$ and for the images $\tilde{\mathfrak{H}}_k^q(0)$ of $\tilde{\mathfrak{H}}_k^q$ in $\mathfrak{H}_n^q(0)$, where $X_1, \dots, X_n$ are replaced by their images*

$$X_n = 1, \qquad X_{n-1} = qT_{n-1}^{-2}, \dots, \qquad X_i = q^{n-i}\big(T_i^{-1} \dots T_{n-1}^{-1}\big)\big(T_{n-1}^{-1} \dots T_i^{-1}\big), \dots .$$

Proof. It follows from Proposition 3.1, (ii), that $\bar{\mathfrak{H}}_n^q$ is a simple algebra of dimension $(n!)^2$ over its centre $\mathbb{C}(Z_n^q)$. This allows us to prove (i) for $\mathfrak{H}_n^q$. The passage to $\mathfrak{H}_n^q$ is not difficult. Part (ii) follows from Theorem 3.3, since H_n^q is semi-simple (for q being sufficiently close to 1). The latter is a well-known fact. The formulas for X_i are the consequence of (5). □

Conjecture 3.5. Statements (i) of Theorem 3.4 are valid for arbitrary $\mathfrak{H}_n^q(a)$. □

Next we explain the similarity of the results of §2 and §3 and prove an analogue of Hermann Weyl duality and Drinfeld's theorem on Yangians from [8]. We discuss below the simplest version of this duality only. In a more functorial form (as in [8]) it can be proved by the same techniques.

According to M. Jimbo $q^{1/2}R(x)P = (T + q - 1)/(x - 1) I \otimes I$, where

$$T = qI \otimes I - \sum_{i>j} \big(qI^{ii} \otimes I^{jj} + I^{jj} \otimes I^{ii}\big) + q^{1/2} \sum_{i \neq j} I^{ij} \otimes I^{ji}$$

satisfies the equality

$$(T-q)(T+1)=0, \qquad P=\sum_{i,j=1}^{N} I^{ij}\otimes I^{ji}$$

is the permutation matrix. Let us further identify ${}^{ii+1}T, {}^{ii+1}P$ with $T_i \in H^q_{n+1}$, $s_i \in S_{n+1}$ for $1 \leqslant i \leqslant n$. Then we have ${}^{12}R(xq^{-u_1})\dots {}^{1n+1}R(xq^{-u_n}) = \varphi_{s_n\dots s_1}(x_0, x_1, \dots, x_n)(s_n \dots s_1)$ for $x_0 = x$, $x_i = q^{u_i}$ (by the way, $R_\mu = f_{w_0}w_0$ for $u \sim \mu$ and w_0: $(1,\dots,n) \to (n,\dots,1)$). Here and below we change the indices $1, 1', \dots, n'$ in the definition of L_u, V_u (see §1) for $1, 2, \dots, n+1$.

THEOREM 3.6. *Any $\mathscr{D}^q_N(u)$-invariant submodule of the $\mathscr{D}^q_N(u)$-module V^q_u for some arbitrary $u \in \mathbb{C}^n$ can be described as follows. If B is a right ideal in H^q_n which becomes $\mathfrak{H}^q_n$-invariant after H^q_n has been identified with the $\mathfrak{H}^q_n$-module $\mathscr{I}^q_u$ then BV^q_u is a $\mathscr{D}^q_N(u)$-submodule of V^q_u for the inclusion $T_i \to {}^{ii+1}T$ of H^q_n into $M^{\otimes n}_N$.*

Proof. We put

$$\begin{aligned}\varphi_{s_n\dots s_1}&(x, x_1, \dots, x_n)\prod_{i=0}^{n-1}(x_i - x_{i+1})\\ &= \left(xT_1 - qx_1T_1^{-1}\right)\dots\left(xT_{n-1} - qx_nT_n^{-1}\right) \overset{\text{def}}{=} F\\ &= \sum_{i=0}^{n} x^iF_{n-i}(-q)^{n-i}.\end{aligned}$$

One has $F_0 = T_1 \dots T_n$, $F_n = \prod_{i=1}^n x_iT_1^{-1}\dots T_n^{-1}$, $F_1 = x_1T_1^{-1}T_2\dots T_n + x_2T_1T_2^{-1}T_3\dots T_n + \dots + x_nT_1\dots T_{n-1}T_n^{-1}$ and so on. Let us define a projection $h \to \operatorname{res} h$ of $H^q_{n+1} \ni h$ onto $\hat H^q_n = H^q_{n+1} \cap \hat{\mathfrak{H}}^q_n$ ($\hat{\mathfrak{H}}^q_n$ is from Theorem 3.4) by the relations: $\operatorname{res}(\hat H^q_n T_n \hat H^q_n) = 0 = \operatorname{res} h - h$ for $h \in \hat H^q_n$. We need two lemmas on Hecke algebras.

LEMMA 3.7. *If $\operatorname{res}(hT_n^{-1}\dots T_{k+1}^{-1}T_{k-1}\dots T_1) = 0$ for $h \in H^q_{n+1}$ and $k = 1, \dots, n+1$ then $h = 0$.*

Proof. The equivalent condition for h is as follows: for such a sufficiently general $u' \in \mathbb{C}^{n+1}$ that $\mathscr{I}^q_{u'} = \oplus_{k=1}^{n+1}E_k$, where $E_k \overset{\text{def}}{=} \{z \in \mathscr{I}^q_{u'} | \langle z\rangle X_k = u'_i z\}$, the projections of h onto E_k vanish for $1 \leqslant k \leqslant n+1$. □

LEMMA 3.8. *For each $h \in \hat H^q_n$ we have the identities $(q-1)\langle h\rangle(X_{k+1} - x_{k+1}) = q^{1-k}\operatorname{res}((hF_1 - F_1h')T_n^{-1}\dots T_{k+1}^{-1}T_{k-1}\dots T_1)$, $h' = F_0^{-1}hF_0 = F_n^{-1}hF_n$, where the isomorphism $h \to h'$ from $\hat H^q_n$ onto $\tilde H^q_n = \tilde{\mathfrak{H}}^q_h \cap H^q_{n+1}$ transfers T_{k+1} to T_k for $1 \leqslant k \leqslant n$.* □

Any invariant submodule of V^q_u has the form SV^q_u for some $S = S^2 \in {}^2M_N{}^3M_N\dots{}^{n+1}M_N \simeq M_N^{\otimes^n}$ with the following property: $SFS' = FS'$, where

$S' = P_0^{-1}SP_0$ for $P_0 = s_1 \dots s_n$. In particular we have: $SF_0S' = F_0S'$, $SF_nS' = F_nS'$. These two relations are equivalent to the invariance of SV_u^q under the action of $\mathcal{D}_N^q(u)_0$ (see Proposition 1.3, (iii)). But $\mathcal{D}_N^q(u)_0 \simeq \mathcal{D}_N^q(0)$ is a deformation of $U(\mathfrak{gl}_N)$ (§I) and the $\mathcal{D}_N^q(u)_0$-decomposition of V_u is like that for $U(\mathfrak{gl}_N)$ and $(\mathbb{C}^N)^{\otimes n}$ (see the end of §1). Hence, by means of H. Weyl's duality and Lemma 3.8 (cf. [10], Theorem 1) we can assume S to belong to $\hat{H}_n^q$ and reduce the problem of describing S to Hecke algebras.

Next, by using the identities of Lemma 3.8 we obtain that if $SF_1S' = F_1S'$ for $S^2 = S \subset \hat{H}_n^q$ then $\langle S\rangle \hat{\mathfrak{H}}_n^q \subset S\hat{H}_n^q$. The converse statement is valid too, which is proved with the aid of the same Lemma, Lemma 3.7 and the formula

$$\operatorname{res}\big((SF_1 - F_1S')T_n \dots T_1\big) = (S-1)\operatorname{res}\big(F_1F_n^{-1}\big)S = 0.$$

The analogous implications are true for $F_{n-1}^{\vee}$ in place of F_1.

Thus we obtain that a subspace of V_u^q has the form BV_u^q for a $\mathfrak{H}_n^q$-invariant B iff it is invariant under the action of all matrix elements of F_0, F_1, F_{n-1}, F_n (we would remind the reader that F_i are $N \times N$-matrices with their elements in $M_N^{\otimes n} = {}^2M_N \dots {}^{n+1}M_N$). But $\{E_{ij}^s,\ s = 0, 1, n-1, n\}$ generate $\mathcal{D}_N^q(u)$ (see Proposition 1.2). □

If for some skew μ, μ' the sequence (u_k') can be obtained from (u_k) by a permutation of such segments $(u_{a_i}, u_{a_i+1}, \dots, u_{b_i})$ that $u_{a_i-1} - u_{a_i}, u_{b_i} - u_{b_i+1} \notin \mathbb{Z}$, $1 \leqslant a_i \leqslant b_i \leqslant n$, then $V_\mu^q \simeq V_{\mu'}^q$ and we will call μ, μ' to be equivalent. It follows from Theorem 3.6 that isomorphism classes of the $\mathcal{D}_N^q(u)$-modules $V_\mu^q \neq \{0\}$ up to $\pi_u^{u'}(0)$ from Proposition 1.3 are in one-to-one correspondence with the classes of μ modulo equivalence. This can be deduced from the results of §2 too. The automorphism $\pi_u^u(d)$ (ibid.) corresponds to the translation $u_k \to u_k + d$, $1 \leqslant k \leqslant n$.

Let us also use homomorphisms of the type (ii), (ibid.) to compare V_μ^q with μ of different degrees and identify representations modulo multiplications of their L-functions by rational scalar functions in x. Then if $V\mu$ is not one-dimensional it is equivalent to some $V_{\mu'}^q$ with the following μ': $y_k' = y_l' \Rightarrow i_k' - i_l' < N - 1$ (cf. Theorem 2.6). The mapping $\{\mu'\} \to \{V\mu'\}$ of equivalence classes is a one-to-one correspondence.

The following Theorem exposes to some extent the place of modules V_μ^q among all irreducible representations of $\mathcal{D}_N^q(u)$.

THEOREM 3.9. *In the notations of* §2 *we suppose the subalgebra* $A_N^q(u)$, *generated by the coefficients of* $H^k(x)$, *to act semi-simply and to have only simple eigenvalues on some irreducible* $\mathcal{D}_N^q(u)$-*composition factor* V *of* V_u^q, $u \in \mathbb{C}^n$ (*i.e.* V *is a quotient-module of a submodule of* V_u^q). *Then* $V \simeq V\mu$ *for some skew* μ *and the initial* u *is a permutation of* $u = (i_k - y_k) \sim \mu$ *if* N/n *is sufficiently large* (*e.g.* $N > n$).

Sketch of Proof. The next lemmas are valid for an arbitrary u, N.

LEMMA 3.10. (i) *In the notations of Theorem* 2.6 $V_u = \oplus_{k=0}^{n} H_n^q V_k$, *where* V_k *and* $H_n^q V_k = \mathbb{C}[S_n]V_k$ *are* $\mathscr{D}_{N-1}^q(u)$*-modules under the action of* $\mathscr{D}_{N-1}^q(u) = \rho^{N-1}(\mathscr{D}_{N-1}^q(u))$. (ii) *There exists a filtration of* $H_n^q V_k$ *by some* $\mathscr{D}_{N-1}^q(u)$*-modules with the modules* $V_{u'}$ *as its composition factors, where* u' *ranges over all the subsets* $u' \in \mathbb{C}^k$ *of* u. □

LEMMA 3.11 (cf. Theorem 3.6). *Every* $\mathscr{D}_{N-1}^q(u)$*-submodule of* $H_n^q V_k$ *has the form* $\hat{B}V_k$ *for* $\hat{B} \subset H_n^q$, *which is a submodule under the action of* $\hat{\mathfrak{H}}_k^q \subset \mathfrak{H}_n^q$ *on* $H_n^q \simeq \mathscr{I}_u^q$ *(any such* $\hat{B}V_k$ *is a* $\mathscr{D}_{N-1}^q(u)$*-submodule).* □

The proof of the Theorem is by induction on n/N with the case $1/(N-n+1)$ being clear. Firstly, we can suppose V to be not only a composition factor but a submodule of V_u^q (we use Theorem 3.6 and the theory of Hecke algebras). All the irreducible composition factors of $V' = V \cap H_n^q V_{N-1}$ are pairwise non-isomorphic $\mathscr{D}_{N-1}^q(u)$-submodules. It follows from the assumption of the Theorem. Therefore, if $V = BV_u^q$ for some $\mathfrak{H}_n^q$-submodule B of $H_n^q = \mathscr{I}_u^q$ then the irreducible composition factors B_i of B as an $\hat{\mathfrak{H}}_{n-1}^q$-module are pairwise non-isomorphic too. This is deduced from some corollary of Lemma 3.11 and implies a semi-simplicity of the action of X_1 on $B([X_1, \hat{\mathfrak{H}}_{n-1}^q] = 0)$. Next we apply the Lemmas and the statement of the Theorem for $(n-1)/(N-1)$ to show that each B_i corresponds to a skew diagram of $\deg = n-1$. Hence, other $X_2, \ldots, X_n$ act semi-simply on B and we can use Theorem 5 from [6]. □

We conjecture this Theorem to be true for any V, N modulo the above equivalence of representations (by homomorphisms of Proposition 1.3 and the proportionality of L-functions). To prove (or disprove) it in general one needs, to begin with, a description of irreducible representations of $\mathscr{D}_N^q(u)$ in terms of their "highest weights" like that of V. G. Drinfeld, V. O. Tarasov for Yangians [17] to show that every irreducible representation belongs to some V_u^q. Then one can use Theorem 3.6 and the classification of irreducible $\mathfrak{H}_n^q$-modules (A. V. Zelevinsky, J. D. Rogawski—see [13]) and can try to generalize the above branching properties of $\mathfrak{H}_n^q$.

Appendix: Yangians. We explain briefly how to specialize the above results to Yangians (see [8] for their general definition). We have to put $x = q^\lambda$, $\tilde{E}_{ii}^n = 1$ and tend $q^{1/2}$ to 1 in all the notations and formulas. After some substitutions we can degenerate

$$L(x) \text{ to } L(\lambda) = I + \sum_{s=1}^{n} \lambda^{n-s} e_{ij}^{s-1} I^{ji} \Big/ \prod_{i=1}^{n} (\lambda - u_i), \qquad 1 \leqslant i, j \leqslant n.$$

The relations of Proposition 1.2 will become as follows:

$$\left[e_{ij}^r, e_{kl}^s\right] = \delta_{jk} e_{il}^{r+s} - \delta_{li} e_{kj}^{r+s} + \sum_{a+b=r+s-1}^{a<r\leqslant b} \left(e_{ki}^a e_{il}^b - e_{kj}^b e_{il}^a\right).$$

The corresponding algebra $Y_N(u)$ is the Yangian of level n for $\mathfrak{gl}_N$ (note that $Y_N(0) \simeq U(\mathfrak{gl}_N)$). Drinfield's Yangian $Y(\mathfrak{sl}_N)$ is the quotient-algebra of $\lim_{\leftarrow u} Y_N(u)$ by the relation $H^N(\lambda) = 1$, where we replace xq^a by $\lambda + a$ in formula (3) for H^N and in any expressions with L, H-functions. For $u = 0$ Theorems 2.4, 2.6 are deduced from the results by G. I. Olshanski, Theorem 2.2 is a version of the classic theorems.

One obtains the degenerated Hecke algebra $\mathfrak{H}_n^1$ [8] by putting $X_i = q^{\chi_i}$, $T_i = s_i$ for $q \to 1$. This algebra is generated by $\mathbb{C}[S_n]$ and χ_i with relations (5) in the following form: $\chi_{i+1}s_i - s_i\chi_i = 1 = s_i\chi_{i+1} - \chi_i s_i$. Theorems 3.3, 3.4, (ii) (for χ_i instead of X_i) are deduced from [9]. The degenerated homomorphism $\mathfrak{H}_n^1 \to \mathfrak{H}_n^1(0) \simeq \mathbb{C}[S_n]$ was defined by Drinfeld. He conjectured in [8] the existence of q-versions of this homomorphism and some Weil-type duality (see Theorem 3.6) for $q \neq 1$.

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