P. M. Gadea
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Analysis and Algebra on Differentiable Manifolds:

A Workbook for Students and Teachers

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To Mary

Foreword

A famous Swiss professor gave a student's course in Basel on Riemann surfaces. After a couple of lectures, a student asked him, "Professor, you have as yet not given an exact definition of a Riemann surface." The professor answered, "With Riemann surfaces, the main thing is to UNDERSTAND them, not to define them."

The student's objection was reasonable. From a formal viewpoint, it is of course necessary to start as soon as possible with strict definitions, but the professor's answer also has a substantial background. The pure definition of a Riemann surface—as a complex 1-dimensional complex analytic manifold—contributes little to a true understanding. It takes a long time to really be familiar with what a Riemann surface is.

This example is typical for the objects of global analysis—manifolds with structures. There are complex concrete definitions but these do not automatically explain what they really are, what we can do with them, which operations they really admit, how rigid they are. Hence, there arises the natural question—how to attain a deeper understanding?

One well-known way to gain an understanding is through underpinning the definitions, theorems and constructions with hierarchies of examples, counterexamples and exercises. Their choice, construction and logical order is for any teacher in global analysis an interesting, important and fun creating task.

This workbook, using a very cleverly composed series of exercises and examples covering the whole area of manifolds, Lie groups, fibre bundles and Riemannian geometry, will enable the reader a deeper understanding and feeling for Riemann surfaces.

Jürgen Eichhorn
Greifswald University

Preface

This book is intended to cover the exercises of standard courses on analysis and algebra on differentiable manifolds for advanced undergraduate and graduate years, with specific focus on Lie groups, fibre bundles and Riemannian geometry. It will prove of interest for students in mathematics and theoretical physics, and in some branches of engineering.

It is not intended as a handbook on those topics, in the form of problems, but merely as a practical complement to the courses, often found in excellent books, as cited in the bibliography.

The prerequisites are linear and multilinear algebra, calculus on several variables and various concepts of point-set topology.

The first six chapters contain 375 solved problems sorted according to the aforementioned topics. These problems fall, "grosso modo," into four classes:

(1) Those consisting of mere calculations, mostly elementary, aiming at checking a number of notions on the subjects.

(2) Problems dedicated to checking some specific properties introduced in the development of the theory.

(3) A class of somewhat more difficult problems devoted to focusing the attention on some particular topics.

(4) A few problems introducing the reader to certain questions not usually explained. The level of these problems is quite varied, ranging from those handling simple properties to others that need sophisticated tools.

Throughout the book, differentiable manifolds, functions, and tensors fields are assumed to be of class C^∞, mainly to simplify the exposition. We call them, indiscriminately, either differentiable or C^∞.

Similarly, manifolds are supposed to be Hausdorff and second countable, though a section is included analysing what happens when these properties fail, aimed at a better understanding of the meaning of such properties.

The Einstein summation convention is used.

Chapter 7 provides a selection of the theorems and definitions used throughout the book, but restricted to those whose terminology could be misleading for the lack of universal acceptance. Moreover, to solve some types of problems, certain definitions and notations should be precisely fixed; recalling the exact statement of some theorems is often convenient in practice as well. However, this chapter has by no means the intention of being either a development or a digest of the theory.

Chapter 8 offers a collection of formulae and tables concerning spaces and groups frequent in differential geometry. Many of them are used throughout the book; others are not, but *they have been included since such a collection should be useful as an aide-mémoire, even for teachers and researchers*. As in Chapter 7, no effort to be exhaustive has been attempted.

We hope the book will render a good service to teachers and students of differential geometry and related topics. While no reasonable effort has been spared to ensure accuracy and precision, the attempt of writing such a book necessarily will contain misprints, and probably some errors. Any corrections, suggestions or comments helping to improve future editions will be highly appreciated. The reader is kindly requested to send his/her opinions to pmgadea@iec.csic.es or jaime@iec.csic.es

In this *corrected reprint* we have corrected a couple of dozens of typos, slightly modified the statement of Problem 1.1.13, and changed the proof of Problem 5.3.6, (2).

Madrid, August 2009 *The authors*

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Contents

Chapter 1
Differentiable manifolds

1.1 C^∞ manifolds

Problem 1.1.1. *Prove that the function* $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi(s) = s^3$, *defines a* C^∞ *differentiable structure on* $\mathbb{R}$ *different from the usual one* $($*that of the atlas* $\{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\})$.

Solution. Since $\varphi^{-1}(s) = \sqrt[3]{s}$, φ is a homeomorphism, so that $\{(\mathbb{R}, \varphi)\}$ is trivially an atlas for $\mathbb{R}$, with only one chart.

To see that the differentiable structure defined by $\{(\mathbb{R}, \varphi)\}$ is not the usual one, we must see that the atlases $\{(\mathbb{R}, \varphi)\}$ and $\{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ are not equivalent, i.e. that $\{(\mathbb{R}, \varphi), (\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ is not a C^∞ atlas on $\mathbb{R}$. In fact, although $\varphi \circ \mathrm{id}_{\mathbb{R}}^{-1} = \varphi$ is C^∞, the map $\mathrm{id}_{\mathbb{R}} \circ \varphi^{-1} = \varphi^{-1}$ is not differentiable at 0.

Let $\mathbb{R}_\varphi$ (resp. $\mathbb{R}_{\mathrm{id}}$) be the topological manifold $\mathbb{R}$ with the differentiable structure defined by the atlas $\{(\mathbb{R}, \varphi)\}$ (resp. $\{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$). Then, the map $\varphi\colon \mathbb{R}_\varphi \to \mathbb{R}_{\mathrm{id}}$ is a diffeomorphism. In fact, its representative map $\mathrm{id} \circ \varphi \circ \varphi^{-1}\colon \mathbb{R} \to \mathbb{R}$ is the identity map.

Problem 1.1.2. *Is the map* $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) = t^2$, *a chart?*

Solution. No, for any one of the following reasons:

(1) f is not injective.

(2) The image set is not an open subset of $\mathbb{R}$.

(3) The images by f of the open subsets containing the origin are not open subsets of $\mathbb{R}$. For example, $f(\mathbb{R}) = [0, \infty)$, $f((-a, a)) = [0, a^2)$.

REMARK. Notice that the situation is similar for any map of the type $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) = t^{2k}$, $k = 2, 3, \ldots$.

Problem 1.1.3. *Prove that if* $h\colon \mathbb{R}^n \to \mathbb{R}^n$ *is a homeomorphism, then the atlas* $\{(\mathbb{R}^n, h)\}$ *defines the usual differentiable structure on* $\mathbb{R}^n$ $($*that defined by the atlas* $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\})$ *if and only if* h *and* h^{-1} *are differentiable.*

Solution. If $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism such that the atlas $\{(\mathbb{R}^n, h)\}$ defines the usual differentiable structure on $\mathbb{R}^n$, then $h = h \circ \mathrm{id}_{\mathbb{R}^n}^{-1}$ and $h^{-1} = \mathrm{id}_{\mathbb{R}^n} \circ h^{-1}$ are differentiable. And conversely.

Problem 1.1.4. *For each real number $r > 0$, consider the map $\varphi_r\colon \mathbb{R} \to \mathbb{R}$, where $\varphi_r(t) = t$ if $t \leqslant 0$ and $\varphi_r(t) = rt$ if $t \geqslant 0$. Prove that the atlases $\{(\mathbb{R}, \varphi_r)\}_{r>0}$ define an uncountable family of differentiable structures on $\mathbb{R}$. Are diffeomorphic the corresponding differentiable manifolds?*

Solution. For each $r > 0$, φ_r is a homeomorphism, but φ_r and φ_r^{-1} are differentiable only when $r = 1$ ($\varphi_1 = \mathrm{id}_{\mathbb{R}}$). Thus $\{(\mathbb{R}, \varphi_r)\}$, for fixed $r \neq 1$, is an atlas defining a differentiable structure different from the usual one. Moreover we have

$$(\varphi_r \circ \varphi_s^{-1})(t) = \begin{cases} t & \text{if } t \leqslant 0 \\ (r/s)t & \text{if } t \geqslant 0. \end{cases}$$

So, if $r \neq s$ and both are different from 1, then $\varphi_r \circ \varphi_s^{-1}$ is not differentiable. Consequently, the atlases $\{(\mathbb{R}, \varphi_r)\}$ and $\{(\mathbb{R}, \varphi_s)\}$ define different differentiable structures and thus $\{(\mathbb{R}, \varphi_r)\}_{r>0}$ defines a family of different differentiable structures on $\mathbb{R}$.

All of them are diffeomorphic, though. In fact, given two differentiable manifolds $\mathbb{R}_{\varphi_{r_1}}$ and $\mathbb{R}_{\varphi_{r_2}}$ defined respectively from the differentiable structures obtained from the atlases $\{(\mathbb{R}, \varphi_{r_1})\}$ and $\{(\mathbb{R}, \varphi_{r_2})\}$, a diffeomorphism $\varphi\colon \mathbb{R}_{\varphi_{r_1}} \to \mathbb{R}_{\varphi_{r_2}}$ is given by the identity map for $t \leqslant 0$ and by $t \mapsto (r_1/r_2)t$ for $t \geqslant 0$. Indeed, the representative map $\varphi_{r_2} \circ \varphi \circ \varphi_{r_1}^{-1}$ is the identity map.

Problem 1.1.5. *Consider the open subsets U and V of the unit circle S^1 of $\mathbb{R}^2$ given by*

$$U = \{(\cos\alpha, \sin\alpha) : \alpha \in (0, 2\pi)\}, \qquad V = \{(\cos\alpha, \sin\alpha) : \alpha \in (-\pi, \pi)\}.$$

Prove that $\mathscr{A} = \{(U, \varphi), (V, \psi)\}$, where

$$\begin{aligned} &\varphi\colon U \to \mathbb{R}, \quad \varphi(\cos\alpha, \sin\alpha) = \alpha, \qquad \alpha \in (0, 2\pi), \\ &\psi\colon V \to \mathbb{R}, \quad \psi(\cos\alpha, \sin\alpha) = \alpha, \qquad \alpha \in (-\pi, \pi), \end{aligned}$$

is an atlas on S^1.

Solution. One has $U \cup V = S^1$ (see Figure 1.1). The maps φ and ψ are homeomorphisms onto the open subsets $(0, 2\pi)$ and $(-\pi, \pi)$ of $\mathbb{R}$, respectively, hence (U, φ) and (V, ψ) are local charts on S^1.

The change of coordinates $\psi \circ \varphi^{-1}$, given by

$$\begin{array}{ccccc} \varphi(U \cap V) & \xrightarrow{\varphi^{-1}} & U \cap V & \xrightarrow{\psi} & \psi(U \cap V) \\ \alpha & \mapsto & (\cos\alpha, \sin\alpha) & \mapsto & \begin{cases} \alpha & \text{if } \alpha \in (0, \pi) \\ \alpha - 2\pi & \text{if } \alpha \in (\pi, 2\pi), \end{cases} \end{array}$$

is obviously a diffeomorphism. Thus $\mathscr{A}$ is an atlas on S^1.

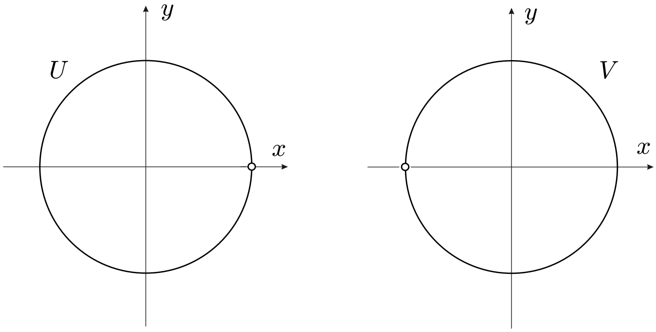

Fig. 1.1 An atlas on S^1 with two charts.

Problem 1.1.6. *Prove*:

(1) $\mathscr{A} = \{(U_1,\varphi_1),(U_2,\varphi_2),(U_3,\varphi_3),(U_4,\varphi_4)\}$, *where*

$$\begin{aligned}
U_1 &= \left\{(x,y)\in S^1 \,:\, x>0\right\}, &\quad \varphi_1 &: U_1 \to \mathbb{R}, &\quad \varphi_1(x,y) &= y,\\
U_2 &= \left\{(x,y)\in S^1 \,:\, y>0\right\}, &\quad \varphi_2 &: U_2 \to \mathbb{R}, &\quad \varphi_2(x,y) &= x,\\
U_3 &= \left\{(x,y)\in S^1 \,:\, x<0\right\}, &\quad \varphi_3 &: U_3 \to \mathbb{R}, &\quad \varphi_3(x,y) &= y,\\
U_4 &= \left\{(x,y)\in S^1 \,:\, y<0\right\}, &\quad \varphi_4 &: U_4 \to \mathbb{R}, &\quad \varphi_4(x,y) &= x,
\end{aligned}$$

is an atlas on the unit circle S^1 in $\mathbb{R}^2$.

(2) $\mathscr{A}$ *is equivalent to the atlas given in* Problem 1.1.5.

Solution. (1) We have $S^1 = \bigcup_i U_i$, $i = 1,2,3,4$ (see Figure 1.2), and each φ_i is a homeomorphism onto the open subset $(-1,1)$ of $\mathbb{R}$, thus each (U_i,φ_i) is a chart on S^1.

The change of coordinates $\varphi_1\circ\varphi_2^{-1}$, given by

$$\begin{array}{ccccc}
\varphi_2(U_1\cap U_2) = (0,1) & \to & U_1\cap U_2 & \to & \varphi_1(U_1\cap U_2) = (0,1)\\
t & \mapsto & (t,\sqrt{1-t^2}) & \mapsto & \sqrt{1-t^2},
\end{array}$$

is a C^∞ map, since $1-t^2>0$. Actually it is a diffeomorphism. The other changes of coordinates are also C^∞, as it is easily proved, thus $\mathscr{A}$ is a C^∞ atlas on S^1.

(2) To prove that the two atlases are equivalent, one must consider the changes of coordinates whose charts belong to different atlases. For example, for $\varphi\circ\varphi_1^{-1}$ we have $\varphi_1(U\cap U_1) = (-1,0)\cup(0,1)$, and the change of coordinates is given by

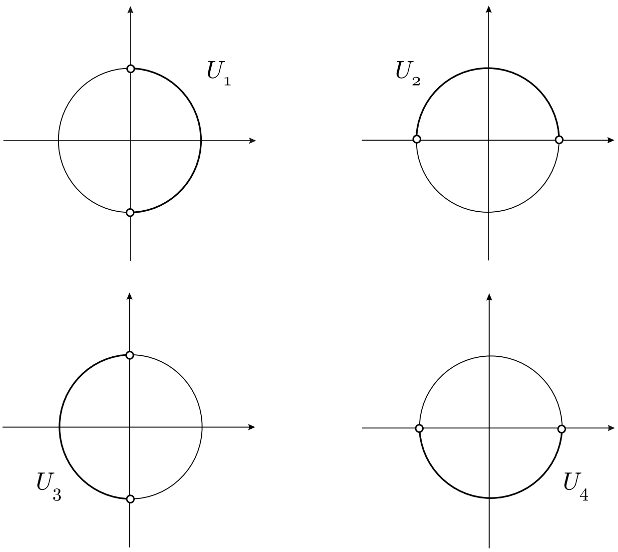

Fig. 1.2 An atlas on S^1 with four charts.

$$\begin{array}{ccccc} \varphi_1(U\cap U_1) & \to & U\cap U_1 & \to & \varphi(U\cap U_1) = \left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{3\pi}{2},2\pi\right) \\ t & \mapsto & (\sqrt{1-t^2},t) & \mapsto & \alpha = \arcsin t, \end{array}$$

which is a diffeomorphism of these intervals.

One can prove the similar results for the other cases.

Problem 1.1.7. *Consider the set* $\{(U_N,\varphi_N),(U_S,\varphi_S)\}$*, where*

$$U_N = \left\{(x,y)\in S^1 \,:\, y\neq 1\right\}, \qquad U_S = \left\{(x,y)\in S^1 \,:\, y\neq -1\right\},$$

φ_N *and* φ_S *being the stereographic projection* (*with the x-axis as image*) *from the north pole N and the south pole S of the sphere* S^1*, respectively* (*see* Figure 1.3).

(1) *Prove that* $\{(U_N,\varphi_N),(U_S,\varphi_S)\}$ *is a* C^∞ *atlas on* S^1.

(2) *Prove that the corresponding differentiable structure coincides with the differentiable structure on* S^1 *obtained in* Problem 1.1.6.

Solution. (1) The maps $\varphi_N\colon U_N\to\mathbb{R}$ and $\varphi_S\colon U_S\to\mathbb{R}$, respectively given by

$$\varphi_N(x,y) = \frac{x}{1-y}, \qquad \varphi_S(x,y) = \frac{x}{1+y},$$

are homeomorphisms. The inverse map φ_N^{-1} is given by

$$\varphi_N^{-1}(x') = (x,y) = \left(\frac{2x'}{1+x'^2},\frac{x'^2-1}{1+x'^2}\right).$$

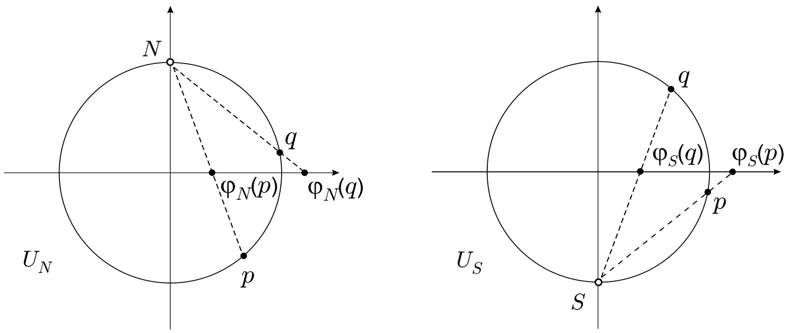

Fig. 1.3 Stereographic projections of S^1.

As for the change of coordinates

$$\varphi_S \circ \varphi_N^{-1} \colon \varphi_N(U_N \cap U_S) = \mathbb{R} - \{0\} \;\to\; \varphi_S(U_N \cap U_S) = \mathbb{R} - \{0\},$$

one has $(\varphi_S \circ \varphi_N^{-1})(t) = 1/t$, which is a C^∞ function on its domain. The inverse map is also C^∞. Thus, $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a C^∞ atlas on S^1.

(2) Consider, for instance,

$$U_2 = \left\{(x,y) \in S^1 : y > 0\right\}, \qquad \varphi_2 \colon U_2 \to (-1,1), \qquad \varphi_2(x,y) = x.$$

We have

$$\begin{array}{rcl} \varphi_N \circ \varphi_2^{-1} \colon (-1,0) \cup (0,1) & \to & (-\infty,-1) \cup (1,\infty) \\ t & \mapsto & t/(1-\sqrt{1-t^2}), \end{array}$$

which is C^∞ on its domain. Similarly, the inverse map $\varphi_2 \circ \varphi_N^{-1}$, defined by

$$\begin{array}{rcl} \varphi_N(U_N \cap U_2) = (-\infty,-1) \cup (1,\infty) & \to & \varphi_2(U_N \cap U_2) = (-1,0) \cup (0,1) \\ s & \mapsto & 2s/(1+s^2), \end{array}$$

is also C^∞. As one has a similar result for the other charts, we conclude.

Problem 1.1.8. *Can one construct an atlas on the sphere S^2 with only one chart?*

Solution. No, since the chart $\varphi \colon S^2 \to \mathbb{R}^2$ would be a homeomorphism onto an open subset of $\mathbb{R}^2$ and this is not possible. In fact, since S^2 is compact, $\varphi(S^2)$ would be a closed and open subset of $\mathbb{R}^2$; hence it would be $\varphi(S^2) = \mathbb{R}^2$. Absurd, since $\mathbb{R}^2$ is not compact. Every manifold admitting an atlas with only one chart should be homeomorphic to an open subset of a Euclidean space.

Problem 1.1.9. (1) *Define an atlas for the sphere*

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\},$$

using the stereographic projection with the equatorial plane as image plane.

(2) *Generalize this construction to S^n, $n \geqslant 3$.*

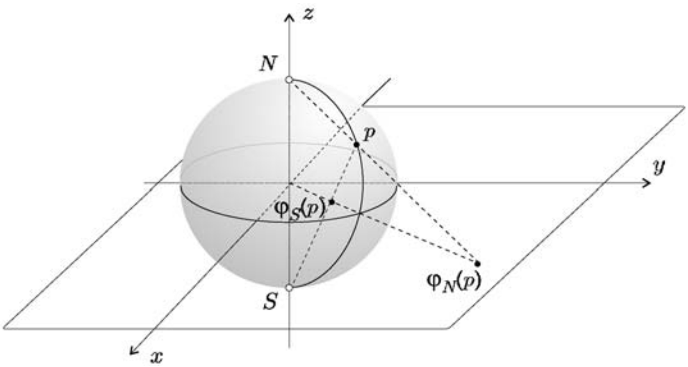

Fig. 1.4 Stereographic projections of S^2 onto the equatorial plane.

Solution. (1) Let us cover the sphere S^2 with the open subsets

$$U_N = \left\{(x,y,z) \in S^2 : z < a\right\}, \qquad U_S = \left\{(x,y,z) \in S^2 : z > -a\right\},$$

for $0 < a < 1$. One can consider the equatorial plane as the image plane of the charts of the sphere (see Figure 1.4).

We define $\varphi_N \colon U_N \to \mathbb{R}^2$ as the stereographic projection from the north pole $N = (0,0,1)$ and $\varphi_S \colon U_S \to \mathbb{R}^2$ as the stereographic projection from the south pole $S = (0,0,-1)$. If x', y' are the coordinates of $\varphi_N(p)$, with $p = (x,y,z)$, we have:

$$\begin{array}{rccl} \varphi_N \colon & U_N & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (x',y') = \left(\dfrac{x}{1-z}, \dfrac{y}{1-z}\right). \end{array}$$

Similarly,

$$\begin{array}{rccl} \varphi_S \colon & U_S & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (x'',y'') = \left(\dfrac{x}{1+z}, \dfrac{y}{1+z}\right). \end{array}$$

One has

$$\varphi_N(U_N) = \varphi_S(U_S) = B(0, 1/(1-a)) \subset \mathbb{R}^2.$$

Since the cases $z = 1$ or $z = -1$, respectively, have been dropped, φ_N and φ_S are one-to-one functions onto an open subset of $\mathbb{R}^2$. As a calculation shows, φ_N^{-1} is given by

$$\varphi_N^{-1}(x',y') = \left(\frac{2x'}{1+x'^2+y'^2}, \frac{2y'}{1+x'^2+y'^2}, \frac{x'^2+y'^2-1}{1+x'^2+y'^2}\right).$$

If $p \in U_N \cap U_S$, $p' = \varphi_N(p)$, and $p'' = \varphi_S(p)$, denoting by x', y' the coordinates of p' and by x'', y'' the coordinates of p'', we deduce that

$$\begin{aligned}(x'', y'') &= (\varphi_S \circ {\varphi_N}^{-1})(x', y') \\ &= \left(\frac{x'}{x'^2 + y'^2}, \frac{y'}{x'^2 + y'^2} \right).\end{aligned}$$

Hence $\varphi_S \circ {\varphi_N}^{-1}$ is a diffeomorphism.

(2) For arbitrary n, with the conditions similar to the ones for S^2, we have

$$\begin{aligned} S^n &= \left\{ (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}, \\ U_N &= \left\{ (x^1, \dots, x^{n+1}) \in S^{n+1} : x^{n+1} \neq 1 \right\}, \\ U_S &= \left\{ (x^1, \dots, x^{n+1}) \in S^{n+1} : x^{n+1} \neq -1 \right\}, \end{aligned}$$

$$\begin{array}{rccl} \varphi_N : & U_N & \to & \mathbb{R}^n \\ & (x^1, \dots, x^{n+1}) & \mapsto & \left(\dfrac{x^1}{1 - x^{n+1}}, \dots, \dfrac{x^n}{1 - x^{n+1}} \right), \end{array}$$

$$\begin{array}{rccl} \varphi_S : & U_S & \to & \mathbb{R}^n \\ & (x^1, \dots, x^{n+1}) & \mapsto & \left(\dfrac{x^1}{1 + x^{n+1}}, \dots, \dfrac{x^n}{1 + x^{n+1}} \right), \end{array}$$

$$\begin{aligned} \varphi_N^{-1}(y^1, \dots, y^n) &= (x^1, \dots, x^{n+1}) \\ &= \left(\frac{2y^1}{1 + \sum_i (y^i)^2}, \dots, \frac{2y^n}{1 + \sum_i (y^i)^2}, \frac{\sum_i (y^i)^2 - 1}{1 + \sum_i (y^i)^2} \right). \end{aligned}$$

So

$$(\varphi_S \circ \varphi_N^{-1})(y^1, \dots, y^n) = \left(\frac{y^1}{\sum_{i=1}^n (y^i)^2}, \dots, \frac{y^n}{\sum_{i=1}^n (y^i)^2} \right) = \frac{y}{|y|^2},$$

and similarly

$$(\varphi_N \circ \varphi_S^{-1})(y^1, \dots, y^n) = \frac{y}{|y|^2},$$

which are C^∞ in $\varphi_N(U_N \cap U_S) = \mathbb{R}^n - \{0\}$.

Notice that with the stereographic projections, the number of charts is equal to two, which is the lowest possible figure, since S^n is compact.

Problem 1.1.10. *Define an atlas on the* cylindrical surface

$$M = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2,\ 0 < z < h \right\},$$

where $h, r \in \mathbb{R}^+$.

Solution. We only need to endow the circle $S^1(r) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=r^2\}$ with an atlas. In fact, let (U,φ), (V,ψ) be an atlas as in Problem 1.1.5. This means that U,V are open subsets of $S^1(r) \subset \mathbb{R}^2$ such that $S^1(r) = U \cup V$, and $\varphi\colon U \to \mathbb{R}$, $\psi\colon V \to \mathbb{R}$ are diffeomorphisms. Then $U \times (0,h)$, $V \times (0,h)$ are open subsets in M and one defines an atlas on M by

$$\mathscr{A} = \{(U \times (0,h); \varphi \times \mathrm{id}),\ (V \times (0,h); \psi \times \mathrm{id})\}.$$

In fact, the map

$$(\psi \times \mathrm{id}) \circ (\varphi \times \mathrm{id})^{-1}\colon \varphi(U \cap V) \times (0,h) \to \psi(U \cap V) \times (0,h)$$

is a diffeomorphism, as it follows from the obvious formula

$$(\psi \times \mathrm{id}) \circ (\varphi \times \mathrm{id})^{-1} = (\psi \circ \varphi^{-1}) \times \mathrm{id}.$$

This construction is only an example of the general way of endowing a product of two manifolds with a differentiable structure. In fact, one can view M as the Cartesian product of $S^1(r)$ by an open interval.

Problem 1.1.11. (1) *Define an atlas on the* cylindrical surface *defined as the quotient set $A/\sim$, where A denotes the rectangle $[0,a] \times (0,h) \subset \mathbb{R}^2$, $a>0$, $h>0$, and $\sim$ stands for the equivalence relation $(0,y) \sim (a,y)$, where $(0,y),(a,y) \in A$.*

(2) *Relate this construction to the one in* Problem 1.1.10.

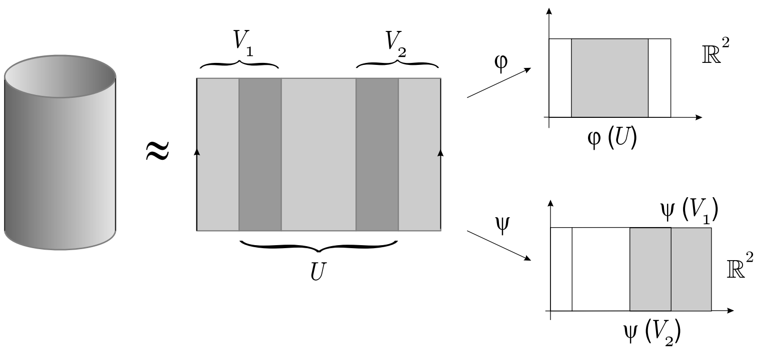

Fig. 1.5 Charts for the cylindrical surface.

Solution. (1) Denote by $[(x,y)]$ the equivalence class of (x,y) modulo $\sim$. Let $c,d,e,f \in \mathbb{R}$ be such that $0<c<e<f<d<a$. We define (see Figure 1.5) the charts (U,φ), (V,ψ) taking $U = \{[(x,y)] : c<x<d\}$, $V = V_1 \cup V_2$, where

$$V_1 = \{[(x,y)] : 0 \leqslant x < e\}, \qquad V_2 = \{[(x,y)] : f < x \leqslant a\},$$

$\varphi\colon U \to \mathbb{R}^2$, $\varphi([(x,y)]) = (x,y)$, and

$$\begin{aligned} \psi\colon \quad V \quad &\to \quad \mathbb{R}^2 \\ [(x,y)] \quad &\mapsto \quad \begin{cases} (x+a,y) & \text{if } (x,y)\in V_1 \\ (x,y) & \text{if } (x,y)\in V_2. \end{cases} \end{aligned}$$

It is obvious that $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms. The change of coordinates $\psi\circ\varphi^{-1}\colon \varphi(U\cap V) \to \psi(U\cap V)$ is given by

$$(\psi\circ\varphi^{-1})(x,y) = \begin{cases} (x+a,y) & \text{if } (x,y)\in\varphi(U\cap V_1) \\ (x,y) & \text{if } (x,y)\in\varphi(U\cap V_2), \end{cases}$$

which is trivially a diffeomorphism.

(2) Let

$$\begin{aligned} &\varphi\colon A = [0,a]\times(0,h) \to \mathbb{R}^3, \\ &\qquad \varphi(\alpha,z) = \left(r\cos\frac{2\pi\alpha}{a}, r\sin\frac{2\pi\alpha}{a}, z\right), \\ &\qquad\qquad 0\leqslant\alpha\leqslant a, \quad 0<z<h, \quad r=a/2\pi. \end{aligned}$$

From the very definition of φ it follows that $\varphi(A)=M$, where $M\subset\mathbb{R}^3$ is the submanifold defined in Problem 1.1.10. Then it is easily checked that $\varphi(\alpha,z)=\varphi(\alpha',z')$ if and only if $(\alpha,z)\sim(\alpha',z')$. Hence φ induces a unique homeomorphism $\widehat{\varphi}\colon A/{\sim} \to M$ such that $\widehat{\varphi}\circ p=\varphi$, where $p\colon A\to A/{\sim}$ is the quotient map.

Problem 1.1.12. *Define the* infinite Möbius strip *M as the topological quotient of* $[0,1]\times\mathbb{R}$ *by the equivalence relation $\sim$ which identifies the pairs* $(0,y)$ *and* $(1,-y)$ (*see* Figure 1.6). *Show that M admits a structure of C^∞ manifold consistent with its topology.*

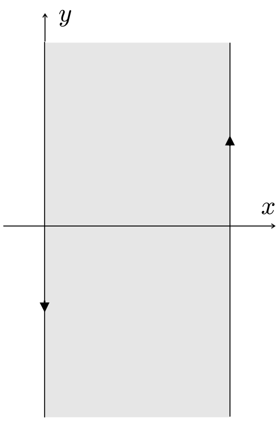

Fig. 1.6 The infinite Möbius strip.

Solution. Let $p\colon [0,1]\times\mathbb{R} \to M = ([0,1]\times\mathbb{R})/\!\sim$ be the quotient map. Consider the two open subsets of M given by

$$U = ((0,1)\times\mathbb{R})/\!\sim, \qquad V = \big(([0,1/2)\cup(1/2,1])\times\mathbb{R}\big)/\!\sim.$$

Every point $z\in U$ can be uniquely written as $z = p(x,y)$, with $(x,y)\in(0,1)\times\mathbb{R}$ and we can define a homeomorphism $\varphi\colon U \to \varphi(U)\subset\mathbb{R}^2$ by setting $\varphi(z) = (x,y)$. We also define $\psi\colon V \to \psi(V)\subset\mathbb{R}^2$ as follows: Set $z = p(x,y)$ with $(x,y)\in([0,1/2)\cup(1/2,1])\times\mathbb{R}$. Then,

$$\psi(z) = \begin{cases} (x+1,-y) & \text{if } x<1/2 \\ (x,y) & \text{if } x>1/2. \end{cases}$$

The definition makes sense as $\psi(p(0,y)) = \psi(p(1,-y)) = (1,-y)$, for all $y\in\mathbb{R}$. It is easily checked that ψ induces a homeomorphism between V and the open subset $(1/2,3/2)\times\mathbb{R}\subset\mathbb{R}^2$. The change of coordinates $\varphi\circ\psi^{-1}\colon \psi(U\cap V) \to \varphi(U\cap V)$, that is,

$$\varphi\circ\psi^{-1}\colon \big((1/2,1)\cup(1,3/2)\big)\times\mathbb{R} \to \big((0,1/2)\cup(1/2,1)\big)\times\mathbb{R},$$

is given by

$$(\varphi\circ\psi^{-1})(x,y) = \begin{cases} (x,y) & \text{if } 1/2<x<1 \\ (x-1,-y) & \text{if } 1<x<3/2, \end{cases}$$

which is a C^∞ map. Similarly, the change of coordinates

$$\psi\circ\varphi^{-1}\colon \varphi(U\cap V) \to \psi(U\cap V),$$

that is,

$$\psi\circ\varphi^{-1}\colon \big((0,1/2)\cup(1/2,1)\big)\times\mathbb{R} \to \big((1/2,1)\cup(1,3/2)\big)\times\mathbb{R},$$

is given by

$$(\psi\circ\varphi^{-1})(x,y) = \begin{cases} (x+1,-y) & \text{if } 0<x<1/2 \\ (x,y) & \text{if } 1/2<x<1, \end{cases}$$

which also is a C^∞ map.

Problem 1.1.13. (1) *Consider the circle in $\mathbb{R}^3$ given by $x^2+y^2=4$, $z=0$, and the open segment PQ in the yz-plane in $\mathbb{R}^3$ given by $y=2$, $|z|<1$. Move the center C of PQ along the circle and rotate PQ around C in the plane Cz, so that when C goes through an angle u, PQ has rotated an angle $u/2$. When C completes a course around the circle, PQ returns to its initial position, but with its ends changed.*

The surface so described is called the Möbius strip.

Consider the two parametrizations

$$\mathbf{x}(u,v) = (x(u,v),y(u,v),z(u,v))$$

$$= \left(\left(2 - v \sin \frac{u}{2} \right) \sin u,\ \left(2 - v \sin \frac{u}{2} \right) \cos u,\ v \cos \frac{u}{2} \right),$$
$$0 < u < 2\pi, \quad -1 < v < 1,$$

$$\begin{aligned} \mathbf{x}'(u,v) &= (x'(u,v), y'(u,v), z'(u,v)) \\ &= \left(\left(2 - v' \sin \left(\frac{\pi}{4} + \frac{u'}{2} \right) \right) \cos u', \right. \\ &\quad \left. - \left(2 - v' \sin \left(\frac{\pi}{4} + \frac{u'}{2} \right) \right) \sin u',\ v' \cos \left(\frac{\pi}{4} + \frac{u'}{2} \right) \right), \end{aligned}$$
$$\pi/2 < u' < 5\pi/2, \quad -1 < v' < 1.$$

Prove that the Möbius strip with these parametrizations is a 2*-dimensional manifold.*

(2) *Relate this manifold to the one given in* Problem 1.1.12.

Solution. (1) The coordinate neighborhoods corresponding to the parametrizations cover the Möbius strip. The intersection of these coordinate neighborhoods has the two connected components

$$U_1 = \{\mathbf{x}(u,v) : \pi < u < 2\pi\}, \qquad U_2 = \{\mathbf{x}(u,v) : 0 < u < \pi\},$$

and the changes of coordinates are given on U_1 and U_2, respectively, by

$$\begin{cases} u' = u - \dfrac{\pi}{2} \\ v' = v \end{cases}, \qquad \begin{cases} u' = u + \dfrac{3\pi}{2} \\ v' = -v \end{cases},$$

which are obviously C^∞.

(2) Let $\varphi\colon [0,2\pi] \times (-1,1) \to \mathbb{R}^3$ be the map given by

$$\varphi(u,v) = \left(\left(2 - v \sin \frac{u}{2} \right) \sin u,\ \left(2 - v \sin \frac{u}{2} \right) \cos u,\ v \cos \frac{u}{2} \right).$$

Note that the restriction of φ to $(0,2\pi) \times (-1,1)$ coincides with the first parametrization. Moreover, it is easy to see that $\operatorname{im} \varphi$ coincides with the Möbius strip.

Let $\alpha\colon [0,1] \times \mathbb{R} \to [0,2\pi] \times (-1,1)$ be the homeomorphism given by $\alpha(s,t) = (2\pi s, (2/\pi) \arctan t)$. Set $\psi = \varphi \circ \alpha$. Let $(s_1,t_1), (s_2,t_2) \in [0,1] \times \mathbb{R}$ be two distinct points such that $\psi(s_1,t_1) = \psi(s_2,t_2)$. As φ is a parametrization when restricted to $(0,2\pi) \times (-1,1)$, the assumption implies that

$$(s_1,t_1), (s_2,t_2) \in \partial([0,1] \times \mathbb{R}) = (\{0\} \times \mathbb{R}) \cup (\{1\} \times \mathbb{R}).$$

As $(s_1,t_1) \neq (s_2,t_2)$, either $(s_1,t_1) \in \{0\} \times \mathbb{R}$ and $(s_2,t_2) \in \{1\} \times \mathbb{R}$ or vice versa. In the first case, $\psi(0,t_1) = \psi(1,t_2)$ means

$$\varphi(0, (2/\pi) \arctan t_1) = \varphi(2\pi, (2/\pi) \arctan t_2).$$

So

$$(0,2,(2/\pi)\arctan t_1) = (0,2,-(2/\pi)\arctan t_2),$$

that is, $t_1 + t_2 = 0$. The other case is similar. This proves that the equivalence relation associated to ψ coincides with the equivalence relation $\sim$ defined in Problem 1.1.12.

Problem 1.1.14. *Let T^2 be a torus of revolution in $\mathbb{R}^3$ with center at $(0,0,0) \in \mathbb{R}^3$ and let $a\colon T^2 \to T^2$ be defined by $a(x,y,z) = (-x,-y,-z)$. Let K be the quotient space under the equivalence relation $p \sim a(p)$, $p \in T^2$, and let $\pi\colon T^2 \to K$ denote the map $\pi(p) = \{p, a(p)\}$. Assume T^2 is covered by parametrizations $\mathbf{x}_\alpha\colon U_\alpha \to T^2$ such that*

$$\mathbf{x}_\alpha(U_\alpha) \cap (a \circ \mathbf{x}_\alpha)(U_\alpha) = \emptyset,$$

where each U_α is an open subset of $\mathbb{R}^2$.

Prove that K is covered by the parametrizations $(U_\alpha, \pi \circ \mathbf{x}_\alpha)$ and that the corresponding changes of coordinates are C^∞.

K is called the Klein bottle.

Solution. The subsets $(\pi \circ \mathbf{x}_\alpha)(U_\alpha)$ cover K by assumption. Each of them is open in K as

$$\pi^{-1}\big((\pi \circ \mathbf{x}_\alpha)(U_\alpha)\big) = \mathbf{x}_\alpha(U_\alpha) \cup a(\mathbf{x}_\alpha(U_\alpha))$$

and $\mathbf{x}_\alpha(U_\alpha)$, $a(\mathbf{x}_\alpha(U_\alpha))$ are open subsets of T^2. Moreover, each map $\pi \circ \mathbf{x}_\alpha\colon U_\alpha \to K$ is a parametrization (that is, it is one-to-one) by virtue of the condition $\mathbf{x}_\alpha(U_\alpha) \cap (a \circ \mathbf{x}_\alpha)(U_\alpha) = \emptyset$. Finally, the changes of coordinates are C^∞. In fact, let

$$p \in \operatorname{domain}\big((\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha)\big).$$

Then $p \in U_\alpha$ and $(\pi \circ \mathbf{x}_\alpha)(p) \in (\pi \circ \mathbf{x}_\beta)(U_\beta)$; hence either $\mathbf{x}_\alpha(p) \in \mathbf{x}_\beta(U_\beta)$ or $\mathbf{x}_\alpha(p) \in (a \circ \mathbf{x}_\beta)(U_\beta)$. In the first case one has

$$(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha) = \mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$$

on a neighborhood of p; and in the second one we have

$$(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha) = (a \circ \mathbf{x}_\beta)^{-1} \circ \mathbf{x}_\alpha,$$

on a neighborhood of p. Since a is a diffeomorphism, $(\pi \circ \mathbf{x}_\beta)^{-1} \circ (\pi \circ \mathbf{x}_\alpha)$ is a diffeomorphism on a neighborhood of p. Thus, it is C^∞.

Problem 1.1.15. *Define an atlas on the topological space $M(r \times s, \mathbb{R})$ of all the real matrices of order $r \times s$.*

Solution. The map $\varphi\colon M(r \times s, \mathbb{R}) \to \mathbb{R}^{rs}$ defined by

$$\varphi(a_{ij}) = (a_{11}, \dots, a_{1s}, \dots, a_{r1}, \dots, a_{rs}),$$

is one-to-one and surjective. Now endow $M(r \times s, \mathbb{R})$ with the topology for which φ is a homeomorphism. So, $(M(r \times s, \mathbb{R}), \varphi)$ is a chart on $M(r \times s, \mathbb{R})$, whose domain is all of $M(r\times s,\mathbb{R})$. The change of coordinates is the identity, hence it is a diffeomorphism. So, $\mathscr{A}=\{(M(r\times s,\mathbb{R}),\varphi)\}$ is an atlas on $M(r\times s,\mathbb{R})$.

Problem 1.1.16. *Prove that there are C^∞ manifolds M for which there exist open subsets which are not domains of any coordinate system.*

Solution. It suffices that M be compact: M is then an open subset of itself and so it cannot be the domain of any chart.

Another example is a non-orientable connected manifold (see Section 3.1), as the connected open subsets in $\mathbb{R}^n$ are orientable. There are also proper open subsets having the property in the statement: For example, remove a point in the Möbius strip (Problems 1.1.12 and 1.1.13).

1.2 Differentiable Structures Defined on Sets

In the present section, and only here, we consider differentiable structures defined on sets.

Let S be a set. An n-dimensional chart on S is an injection of a subset of S onto an open subset of $\mathbb{R}^n$. A C^∞ atlas on S is a collection of charts whose domains cover S, and such that if the domains of two charts φ,ψ overlap, then the change of coordinates $\varphi\circ\psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$.

Hence, the manifold is not supposed to be *a priori* a topological space. It has the topology induced by the differentiable structure defined by the C^∞ atlas (see [6, 2.2]).

Problem 1.2.1. *Consider* $E=\{(\sin 2t,\sin t)\in\mathbb{R}^2 : t\in\mathbb{R}\}$ (*the* Figure Eight).

(1) *Prove that* $\{(E,\varphi)\}$*, where* $\varphi\colon E\to\mathbb{R}$ *is the injection of* E *onto an open interval of* $\mathbb{R}$*, defined by* $\varphi(\sin 2t,\sin t)=t$*,* $t\in(0,2\pi)$ (*see* Figure 1.7)*, is an atlas on the set* E*. Here* E *has the topology inherited from its injection in* $\mathbb{R}$*.*

(2) *Prove that, similarly,* $\{(E,\psi)\}$*, where* $\psi\colon E\to\mathbb{R}$*,* $\psi(\sin 2t,\sin t)=t$*,* $t\in(-\pi,\pi)$*, is an atlas on the set* E*.*

(3) *Do the two atlases define the same differentiable structure on* E*?*

REMARK. *Notice that the "Figure Eight" is not endowed with the topology inherited from* $\mathbb{R}^2$ *as, in this case, it would not be a differentiable manifold. The arguments here are similar to those given in studying the sets in* Problems 1.2.10, 1.2.11. *Instead, we endow it with the topology corresponding to its differentiable structure obtained from each of the atlases above.*

Solution. (1) φ is an injective map from E onto the open interval $(0,2\pi)$ of $\mathbb{R}$, whose domain is all of E. Consequently $\{(E,\varphi)\}$ is an atlas on E.

(2) Similar to (E,φ).

(3) The two atlases define the same differentiable structure if (E,φ) belongs to the structure defined by (E,ψ) and conversely. That is, the maps $\psi\circ\varphi^{-1}$ and $\varphi\circ\psi^{-1}$ must be C^∞. We have

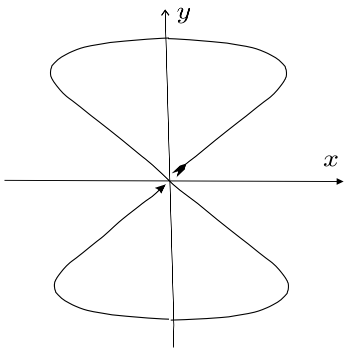

Fig. 1.7 The "Figure Eight" defined by (E,φ).

$$\begin{array}{rccccc} \psi\circ\varphi^{-1}\colon \varphi(E)=(0,2\pi) & \to & E & \to & \psi(E)=(-\pi,\pi) \\ t & \mapsto & (\sin 2t,\sin t) & \mapsto & \widetilde{\psi}(\sin 2t,\sin t), \end{array}$$

where

$$\widetilde{\psi}(\sin 2t,\sin t)=\begin{cases} t, & t\in(0,\pi)\\ 0, & t=\pi\\ \psi\big(\sin(2t-4\pi),\,\sin(t-2\pi)\big)=t-2\pi, & t\in(\pi,2\pi). \end{cases}$$

Thus, $\psi\circ\varphi^{-1}$ is not even continuous and the differentiable structures defined by these atlases are different.

Notice that the topologies induced on E by the two C^∞ structures are also different: Consider for instance the open subsets $\varphi^{-1}(U_\pi)$ and $\psi^{-1}(U_0)$, where U_π and U_0 denote small neighborhoods of π and 0 respectively.

Problem 1.2.2. *Consider the subset N of $\mathbb{R}^2$ (the* Noose*) defined (see* Figure 1.8*) by*

$$N=\big\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\big\}\cup\{(0,y) : 1<y<2\}.$$

(1) *Prove that the function*

$$\begin{array}{rcccl} \varphi\colon & N & \to & \mathbb{R} & \\ & (\sin 2\pi s,\cos 2\pi s) & \mapsto & s & \text{if}\quad 0\leqslant s<1,\\ & (0,s) & \mapsto & 1-s & \text{if}\quad 1<s<2, \end{array}$$

is a chart that defines a C^∞ structure on N.

(2) *Prove that the function*

$$\begin{array}{rccll} \psi\colon & N & \to & \mathbb{R} & \\ & (\sin 2\pi s, \cos 2\pi s) & \mapsto & 1-s & \textit{if} \quad 0 < s \leqslant 1, \\ & (0,s) & \mapsto & 1-s & \textit{if} \quad 1 < s < 2, \end{array}$$

also defines a C^∞ structure on N.

(3) *Prove that the two above structures are different.*

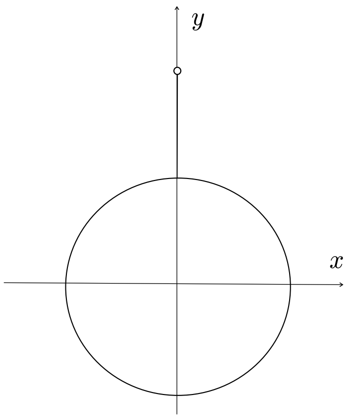

Fig. 1.8 The Noose.

Solution. (1) Obviously $\varphi\colon N \to (-1,1)$ is a one-to-one map. Endow N with the unique topology τ_a making φ a homeomorphism. Thus, the atlas $\{(N,\varphi)\}$, with the single chart φ, defines a C^∞ structure in N.

Notice that if N is endowed with the topology inherited from that of $\mathbb{R}^2$, then φ is not continuous at the point $(0,1)$.

(2) Proceed similarly to (1).

(3) If (N,ψ) would belong to the structure defined from (N,φ), then $\psi \circ \varphi^{-1}\colon (-1,1) \to (-1,1)$ should be C^∞, but it is not even continuous.

Problem 1.2.3. *Consider the sets*

$$U = \left\{(s,0) \in \mathbb{R}^2 : s \in \mathbb{R}\right\},$$
$$V = \left\{(s,0) \in \mathbb{R}^2 : s < 0\right\} \cup \left\{(s,1) \in \mathbb{R}^2 : s > 0\right\},$$

and the maps

$$\varphi\colon U \to \mathbb{R}, \qquad \varphi(s,0) = s,$$
$$\psi\colon V \to \mathbb{R}, \qquad \psi(s,0) = s, \qquad \psi(s,1) = s,$$

$$\gamma\colon V \to \mathbb{R}, \qquad \gamma(s,0) = s^3, \qquad \gamma(s,1) = s^3.$$

(1) *Prove that* $\{(U,\varphi),(V,\psi)\}$ *defines a* C^∞ *structure on the set* $M = U \cup V$ (*see* Figure 1.9).

(2) *Is* (V,γ) *a chart in the previous differentiable structure?*

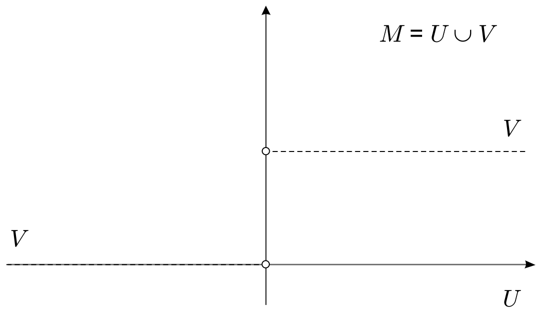

Fig. 1.9 An example of set with a C^∞ structure.

Solution. (1) The maps φ and ψ are injective, and we have $\varphi(U) = \mathbb{R}$, $\psi(V) = \mathbb{R} - \{0\}$, which are open subsets of $\mathbb{R}$. Moreover, both $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are the identity map on $\psi(U \cap V) = (-\infty,0) = \varphi(U \cap V)$, and $\varphi(U \cap V)$, $\psi(U \cap V)$ are open subsets of $\mathbb{R}$. Hence $\mathscr{A} = \{(U,\varphi),(V,\psi)\}$ is a C^∞ atlas on M.

(2) The map γ is injective, and $\gamma(V) = \mathbb{R} - \{0\}$, $\gamma(U \cap V) = (-\infty,0)$, are open subsets of $\mathbb{R}$. Moreover, the maps $\gamma \circ \varphi^{-1}$, $\varphi \circ \gamma^{-1}$, $\gamma \circ \psi^{-1}$, and $\psi \circ \gamma^{-1}$ are C^∞ maps. Thus, γ is, in fact, a chart of the above differentiable structure.

Problem 1.2.4. *Let*

$$S = \left\{(x,0) \in \mathbb{R}^2 : x \in (-1,+1)\right\} \cup \left\{(x,x) \in \mathbb{R}^2 : x \in (0,1)\right\}.$$

Let

$$U = \{(x,0) : x \in (-1,+1)\}, \qquad \varphi\colon U \to \mathbb{R}, \qquad \varphi(x,0) = x,$$
$$V = \{(x,0) : x \in (-1,0]\} \cup \{(x,x), x \in (0,1)\},$$
$$\psi\colon V \to \mathbb{R}, \qquad \psi(x,0) = x, \qquad \psi(x,x) = x$$

(*see* Figure 1.10). *Is* $\mathscr{A} = \{(U,\varphi),(V,\psi)\}$ *an atlas on the set* S*?*

Solution. We have $S = U \cup V$. Furthermore φ and ψ are injective maps onto the open subset $(-1,+1)$ of $\mathbb{R}$. Thus (U,φ) and (V,ψ) are charts on S. However, one has $\varphi(U \cap V) = \psi(U \cap V) = (-1,0]$, which is not an open subset of $\mathbb{R}$. Thus $\mathscr{A}$ is not an atlas on S.

Problem 1.2.5. *Consider on* $\mathbb{R}^2$ *the subsets*

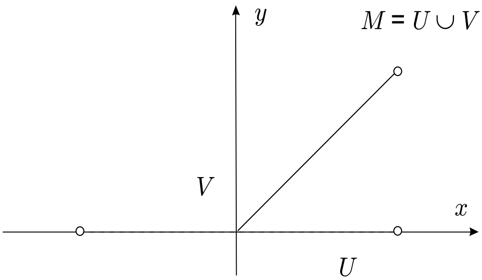

Fig. 1.10 Two charts which do not define an atlas.

$$E_1 = \left\{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}, \qquad E_2 = \left\{(x,1) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}.$$

Define on $E = E_1 \cup E_2$ an equivalence relation $\sim$ by

$$\begin{aligned} (x_1,0) \sim (x_2,0) &\iff x_1 = x_2, \\ (x_1,1) \sim (x_2,1) &\iff x_1 = x_2, \\ (x_1,0) \sim (x_2,1) &\iff x_1 = x_2 < 0. \end{aligned}$$

The classes of the quotient set $S = E/\sim$ are represented by the elements $(x,0)$ for $x < 0$, and the elements $(x,0)$ and $(x,1)$ for $x \geqslant 0$ (see Figure 1.11*). Prove that S admits a C^∞ atlas, but S is not Hausdorff with the induced topology.*

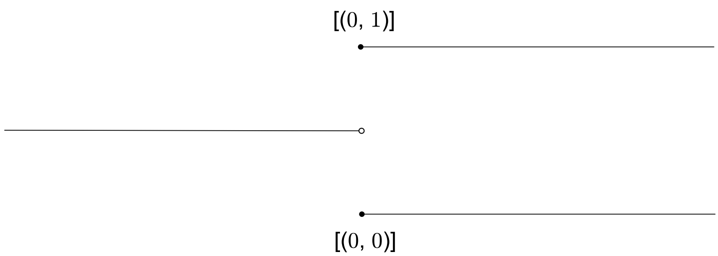

Fig. 1.11 A set with a C^∞ atlas, whose induced topology is not Hausdorff.

Solution. Denote by $[(x,y)]$ the class of (x,y). We can endow S with a manifold structure by means of the charts (U_1, φ_1) and (U_2, φ_2), where

$$U_1 = \{[(x,0)] : x \in \mathbb{R}\}, \qquad U_2 = \{[(x,0)] : x < 0\} \cup \{[(x,1)] : x \geqslant 0\},$$
$$\varphi_1([(x,0)]) = x, \qquad \varphi_2([(x,0)]) = \varphi_2([(x,1)]) = x.$$

One has $U_1 \cup U_2 = S$. Furthermore $\varphi_1(U_1) = \mathbb{R}$, $\varphi_2(U_2) = \mathbb{R}$ are open sets and

$$\begin{array}{rcccl} \varphi_1 \circ \varphi_2^{-1} : (-\infty,0) & \to & U_1 \cap U_2 & \to & (-\infty,0) \\ x & \mapsto & [(x,0)] & \mapsto & x \end{array}$$

is a diffeomorphism. Hence S admits a C^∞ atlas.

Nevertheless, the induced topology is not Hausdorff. The points $[(0,1)]$ and $[(0,0)]$ do not admit disjoint open neighborhoods. In fact, if U is an open subset of S containing $[(0,0)]$, then $\varphi_1(U \cap U_1)$ must be an open subset of $\mathbb{R}$. But $[(0,0)] \in U \cap U_1$, hence $\varphi_1(U \cap U_1)$ is an open subset of $\mathbb{R}$ that contains 0, thus it contains an interval of the form $(-\alpha,\alpha)$, with $\alpha > 0$. Therefore $\{[(x,0)] : -\alpha < x < 0\} \subset U$. Similarly, an open subset V of S containing $[(0,1)]$ can have a subset of the form $\{[(x,0)] : -\beta < x < 0, \beta > 0\}$. Thus U and V cannot be disjoint.

Problem 1.2.6. *Let S be the subset of $\mathbb{R}^2$ which consists of all the points of the set $U = \{(s,0)\}$, $s \in \mathbb{R}$, and the point $(0,1)$. Let U_1 be the set obtained from U replacing the point $(0,0)$ by the point $(0,1)$. We define the maps*

$$\varphi : U \to \mathbb{R}, \quad \varphi(s,0) = s, \qquad \varphi_1 : U_1 \to \mathbb{R}, \quad \begin{cases} \varphi_1(s,0) = s, & s \neq 0 \\ \varphi_1(0,1) = 0. \end{cases}$$

Prove that $\{(U,\varphi),(U_1,\varphi_1)\}$ is a C^∞ atlas on S, but S is not Hausdorff with the induced topology.

Solution. $U \cup V = S$, φ and φ_1 are injective maps in $\mathbb{R}$, and the changes of coordinates $\varphi \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi^{-1}$ are both the identity on the open subset $\mathbb{R} - \{0\}$. So, these two charts define a C^∞ atlas on S.

Let V be a neighborhood of $(0,0)$ and W a neighborhood of $(0,1)$ in S. Then $\varphi(U \cap V)$ and $\varphi_1(U_1 \cap W)$ are open subsets of $\mathbb{R}$ containing 0, and so they will also contain some point $a \neq 0$. The point $(a,0)$ belongs to $V \cap W$, hence the topology of S is not Hausdorff.

Problem 1.2.7. *Consider the set S obtained identifying two copies L_1 and L_2 of the real line except at a point $p \in \mathbb{R}$ (see* Figure 1.12*). Prove that S admits a C^∞ atlas but it is not Hausdorff with the induced topology.*

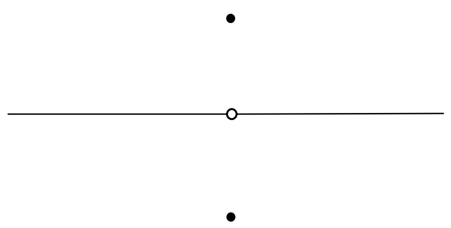

Fig. 1.12 The straight line with a double point.

Solution. Take the usual charts on L_1 and L_2, i.e. the identity map on $\mathbb{R}$. Then $S = L_1 \cup L_2$, and the change of coordinates on the intersection $L_1 \cap L_2$ is C^∞ as it is the identity map. Nevertheless, the points $p_1 \in L_1$ and $p_2 \in L_2$, where p_i, $i = 1, 2$, stands for the representative of p in L_i, are obviously not separable.

Problem 1.2.8. *Let $S = \mathbb{R} \times \mathbb{R}$, where in the first factor we consider the discrete topology, in the second factor the usual topology, and in S the product topology. Prove that S admits a C^∞ atlas and that S does not satisfy the second axiom of countability but is paracompact.*

Solution. For each $t \in \mathbb{R}$, let $L_t = \{(t, y) : y \in \mathbb{R}\} = \{t\} \times \mathbb{R}$, which is an open subset of S. The map $\varphi_t \colon L_t \to \mathbb{R}$, $\varphi_t(t, y) = y$, is a homeomorphism, hence $\{(L_t, \varphi_t)\}_{t \in \mathbb{R}}$ is a C^∞ atlas on S such that if $s \neq t$ then $L_t \cap L_s = \emptyset$. So that S is a locally Euclidean space of dimension 1 which admits a differentiable structure. The topological space S has uncountable connected components; thus it is not second countable with the induced topology. The space S is paracompact. In fact, S is Hausdorff as a product of Hausdorff spaces and if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of S, then, for some fixed t, $\{U_\alpha \cap L_t\}_{\alpha \in A}$ is an open covering of L_t (which is paracompact since it is homeomorphic to $\mathbb{R}$ with the usual topology) which admits a locally finite refinement $\{V^t_\lambda\}_{\lambda \in \Lambda}$. Thus $\{V^t_\lambda\}_{\lambda \in \Lambda, t \in \mathbb{R}}$ is a locally finite refinement of $\{U_\alpha\}_{\alpha \in A}$.

One could alternatively argue that S is paracompact since each connected component of S is second countable.

Problem 1.2.9. *Let S be a set with a C^∞ atlas and consider the topological space S with the induced topology.*

(1) *Is S locally compact, locally connected and locally connected by arcs as a topological space? Does it satisfy the first axiom of countability? Does it satisfy the separation axiom T_1?*

(2) *Does it satisfy the separation axiom T_2? And the second axiom of countability?*

(3) *Does it satisfy the separation axiom T_3? Is S a regular topological space? Is S pseudometrizable? Does it satisfy all separations axioms T_i? Is S paracompact? Can it have continuous partitions of unity?*

(4) *Does S satisfy the properties mentioned in* (3) *if we constrain it to be T_2 and to satisfy the second axiom of countability?*

HINT: *Consider*:

(i) *Urysohn's Theorem*: *If S verifies the second axiom of countability, then it is equivalent for S to be pseudometrizable and to be regular.*

(ii) *Stone's Theorem*: *If S is pseudometrizable, then it is paracompact.*

Solution. (1) S being locally Euclidean, it is locally compact, locally connected, locally connected by arcs, and satisfies the first axiom of countability.

S satisfies the separation axiom T_1. In fact, let p and q be different points of S. If they belong to the domain of some chart (U, φ) of S, we can choose disjoint open subsets V_1, V_2 in $\mathbb{R}^n$ (assuming $\dim S = n$), contained in $\varphi(U)$, and such that

$\varphi(p) \in V_1$, $\varphi(q) \in V_2$. Since φ is continuous, $\varphi^{-1}(V_1)$ and $\varphi^{-1}(V_2)$ are disjoint open subsets of S containing p and q respectively. If p and q do not belong to the domain of a given chart of S, there must be a chart whose domain U_1 contains p but not q, and one chart whose domain U_2 contains q but not p.

Notice that U_1 and U_2 are open subsets of S.

(2) It does not necessarily satisfy the separation axiom T_2, as it can be seen in the counterexamples given in the previous Problems 1.2.5, 1.2.6, 1.2.7. It does not necessarily satisfy the second axiom of countability, as the counterexample given in Problem 1.2.8 proves.

(3) Not necessarily, since S is not necessarily T_2.

(4) Yes, as we have:

(a) S is locally compact, as it follows from (1). As S is also T_2, it is T_3 and hence regular.

(b) By Urysohn's Theorem, S is pseudometrizable.

(c) S being pseudometrizable and T_2, it satisfies all the separation axioms.

(d) S being pseudometrizable, it is paracompact by Stone's Theorem.

(e) S being paracompact, it admits continuous partitions of unity.

Problem 1.2.10. *Consider the cone* $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ (*see* Figure 1.13) *with the topology induced by the usual one of* $\mathbb{R}^3$. *Prove that the algebraic manifold S is not even a locally Euclidean space.*

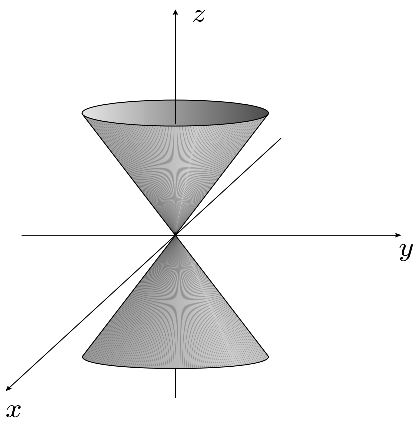

Fig. 1.13 The cone is not a locally Euclidean space because of the origin.

Solution. The point $(0,0,0) \in S$ does not have a neighborhood homeomorphic to an open subset of $\mathbb{R}^2$. In fact, if such a homeomorphism $h\colon U \to V$ exists between an open neighborhood U of $0 = (0,0,0)$ in S and an open subset V of $\mathbb{R}^2$, then, for small enough $\varepsilon > 0$, the open disk $B(h(0),\varepsilon)$ of center $h(0)$ and radius ε is contained in V. If we drop the point $h(0)$ in $B(h(0),\varepsilon)$ the remaining set is connected. So it suffices to see that if we drop 0 in any of its neighborhoods, the set $U-\{0\}$ is not connected. In fact, $U-\{0\}=U_+\cup U_-$, where

$$U_+=\{(x,y,z)\in U\,:\,z>0\},\qquad U_-=\{(x,y,z)\in U\,:\,z<0\},$$

so $U_+\cap U_-=\emptyset$, and U_+ and U_- are open subsets in the induced topology. Hence S is not even a locally Euclidean space.

Problem 1.2.11. *Let S be the topological space defined by the union of the two circles in $\mathbb{R}^2$ with radius 1 and centers $(-1,0)$ and $(1,0)$, respectively (see* Figure 1.14*), and the topology inherited from that of $\mathbb{R}^2$. Is S a locally Euclidean space?*

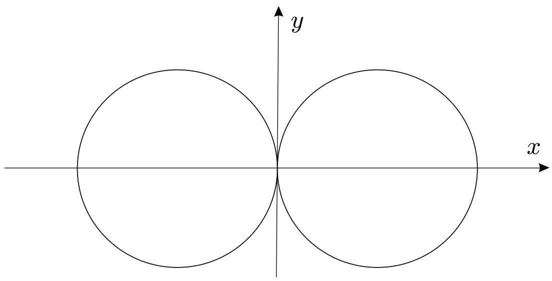

Fig. 1.14 Two tangent circles are not a locally Euclidean space.

Solution. No, as none of the connected neighborhoods in S of the point of tangency $(0,0)$ is homeomorphic to an open subset of $\mathbb{R}$. In fact, let V be a neighborhood of $(0,0)$ in S inside the unit open ball centered at the origin. If such a neighborhood V were homeomorphic to $\mathbb{R}$, then $V-\{(0,0)\}$ and $\mathbb{R}-\{0\}$ would be homeomorphic; but this is not possible, as $V-\{(0,0)\}$ has at least four connected components and $\mathbb{R}-\{0\}$ has only two.

1.3 Differentiable Functions and Mappings

Problem 1.3.1. *Consider the map*

$$f\colon\mathbb{R}^2\to\mathbb{R},\qquad (x,y)\mapsto x^3+xy+y^3+1.$$

(1) *Compute the map $f_*\colon T_p\mathbb{R}^2\to T_{f(p)}\mathbb{R}$.*

(2) *Which of the points $(0,0)$, $(\frac{1}{3},\frac{1}{3})$, $(-\frac{1}{3},-\frac{1}{3})$, is f_* injective or surjective at?*

Solution. (1)

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_p\right) = \frac{\partial f}{\partial x}(p)\ \left.\frac{\partial}{\partial t}\right|_{f(p)}$$
$$= (3x^2+y)(p)\,\left.\frac{\partial}{\partial t}\right|_{f(p)},$$

$$f_*\left(\left.\frac{\partial}{\partial y}\right|_p\right) = \frac{\partial f}{\partial y}(p)\ \left.\frac{\partial}{\partial t}\right|_{f(p)}$$
$$= (x+3y^2)(p)\,\left.\frac{\partial}{\partial t}\right|_{f(p)}.$$

(2)

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(0,0)}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_1, \qquad f_*\left(\left.\frac{\partial}{\partial y}\right|_{(0,0)}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_1,$$

hence $f_{*(0,0)}$ is neither surjective nor injective.

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(\frac{1}{3},\frac{1}{3})}\right) = \frac{2}{3}\ \left.\frac{\partial}{\partial t}\right|_{\frac{32}{27}} = f_*\left(\left.\frac{\partial}{\partial y}\right|_{(\frac{1}{3},\frac{1}{3})}\right),$$

hence $f_{*(\frac{1}{3},\frac{1}{3})}$ is surjective, but not injective.

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_{(-\frac{1}{3},-\frac{1}{3})}\right) = 0\cdot\left.\frac{\partial}{\partial t}\right|_{\frac{28}{27}} = f_*\left(\left.\frac{\partial}{\partial y}\right|_{(-\frac{1}{3},-\frac{1}{3})}\right),$$

hence $f_{*(-\frac{1}{3},-\frac{1}{3})}$ is neither surjective nor injective.

REMARK. f_* cannot be injective at any point since $\dim\mathbb{R}^2 > \dim\mathbb{R}$.

Problem 1.3.2. *Let*

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x,y)\mapsto (x^2-2y, 4x^3y^2),$$
$$g\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad (x,y)\mapsto (x^2y+y^2, x-2y^3, ye^x).$$

(1) *Compute* $f_{*(1,2)}$ *and* $g_{*(x,y)}$.

(2) *Find* $g_*\left(\left(4\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)_{(0,1)}\right)$.

(3) *Calculate the conditions that the constants* λ, μ, ν *must satisfy for the vector*

$$\left(\lambda\frac{\partial}{\partial x}+\mu\frac{\partial}{\partial y}+\nu\frac{\partial}{\partial z}\right)_{g(0,0)}$$

to be the image of some vector by g_*.

Solution. (1) $f_{*(1,2)} \equiv \begin{pmatrix} 2 & -2 \\ 48 & 16 \end{pmatrix}, \quad g_{*(x,y)} \equiv \begin{pmatrix} 2xy & x^2+2y \\ 1 & -6y^2 \\ ye^x & e^x \end{pmatrix}.$

(2)

$$g_*\left(\left(4\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)_{(0,1)}\right) \equiv \begin{pmatrix} 0 & 2 \\ 1 & -6 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -2 \\ 10 \\ 3 \end{pmatrix}_{g(0,1)} \equiv \left(-2\frac{\partial}{\partial x}+10\frac{\partial}{\partial y}+3\frac{\partial}{\partial z}\right)_{(1,-2,1)}.$$

(3) Since $g_{*(0,0)} \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, the image by g_* of $T_{(0,0)}\mathbb{R}^2$ is the vector subspace of $T_{(0,0,0)}\mathbb{R}^3$ of vectors of type $(0,\mu,\nu)$.

Problem 1.3.3. *The elements of $\mathbb{R}^4$ can be written as matrices of the form $A = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$. Let $A_0 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Let $T_\theta \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the differentiable transformation defined by $T_\theta(A) = A_0A$.*

(1) *Calculate $T_{\theta *}$.*

(2) *Compute $T_{\theta *}X$, where $X = \cos\theta\,\frac{\partial}{\partial x} - \sin\theta\,\frac{\partial}{\partial y} + \cos\theta\,\frac{\partial}{\partial z} - \sin\theta\,\frac{\partial}{\partial t}$.*

Solution. (1)

$$T_{\theta *} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

(2) It is immediate that $T_{\theta *}X = \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$. The result can also be obtained considering that if

$$X = \lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y} + \lambda_3 \frac{\partial}{\partial z} + \lambda_4 \frac{\partial}{\partial t}$$

is a vector field on $\mathbb{R}^4$, then $T_{\theta *} X = A_0 A$, where $A = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{pmatrix}$. We thus have

$$\begin{aligned} T_{\theta *} X &\equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \cos\theta \\ -\sin\theta & -\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial z}. \end{aligned}$$

Problem 1.3.4. *Consider the curve σ in $\mathbb{R}^2$ defined by $x = \cos t$, $y = \sin t$, $t \in (0, \pi)$, and the map $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2x + y^3$. Find the vector v tangent to σ at $\pi/4$ and calculate $v(f)$.*

Solution. We have $\sigma'(t) \equiv (-\sin t, \cos t)$, thus

$$\begin{aligned} \sigma'(\pi/4) &\equiv \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \\ &\equiv -\frac{\sqrt{2}}{2} \left. \frac{\partial}{\partial x} \right|_{(\sqrt{2}/2, \sqrt{2}/2)} + \frac{\sqrt{2}}{2} \left. \frac{\partial}{\partial y} \right|_{(\sqrt{2}/2, \sqrt{2}/2)}. \end{aligned}$$

Hence $\sigma'(\pi/4) f = -\sqrt{2}/4$.

Problem 1.3.5. *Consider the curve in $\mathbb{R}^2$ given by $\sigma(t) = (x(t), y(t)) = (t^2 - 1, t^3 - t)$. Find $\sigma(t)$ and $\sigma'(t)$ for $t = 1$ and $t = -1$. Compare $\sigma(1)$ with $\sigma(-1)$ and $\sigma'(1)$ with $\sigma'(-1)$.*

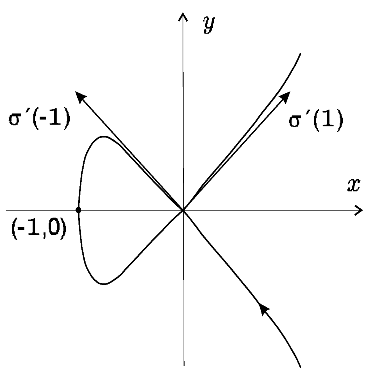

Fig. 1.15 The curve $\sigma(t) = (x(t), y(t)) = (t^2 - 1, t^3 - t)$.

Solution. We have $\sigma(1) = (0,0)$, $\sigma(-1) = (0,0)$, and

$$\sigma'(1) \equiv (2t, 3t^2 - 1)_{t=1} = (2,2), \qquad \sigma'(-1) \equiv (-2,2).$$

Hence, $\sigma(1) = \sigma(-1)$ but $\sigma'(1) \neq \sigma'(-1)$ (see Figure 1.15).

Problem 1.3.6. *Let E be the "Figure Eight" with its differentiable structure given by the global chart* $(\sin 2s, \sin s) \mapsto s,\ s \in (0, 2\pi)$ (*see* Problem 1.2.1). *Consider the vector $v = (\mathrm{d}/\mathrm{d}s)_0$ tangent at the origin $p = (0,0)$ to E and let $j\colon E \to \mathbb{R}^2$ be the canonical injection of E in $\mathbb{R}^2$.*

(1) *Compute $j_* v$.*

(2) *Compute $j_* v$ if E is given by the chart* $(\sin 2s, \sin s) \mapsto s,\ s \in (-\pi, \pi)$.

Solution. (1) The origin p corresponds to $s = \pi$, so

$$j_{*p} \equiv \begin{pmatrix} \dfrac{\partial \sin 2s}{\partial s} \\ \dfrac{\partial \sin s}{\partial s} \end{pmatrix}_{s=\pi} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

As $v = \left.\dfrac{\mathrm{d}}{\mathrm{d}s}\right|_p$ we have

$$\begin{aligned} j_{*p} v &\equiv \begin{pmatrix} 2 \\ -1 \end{pmatrix} (1) \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \equiv 2 \left.\frac{\partial}{\partial x}\right|_p - \left.\frac{\partial}{\partial y}\right|_p . \end{aligned}$$

(2) We now have

$$j_{*p} \equiv \begin{pmatrix} \dfrac{\partial \sin 2s}{\partial s} \\ \dfrac{\partial \sin s}{\partial s} \end{pmatrix}_{s=0} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

so $j_{*p} v = 2 \left.\dfrac{\partial}{\partial x}\right|_p + \left.\dfrac{\partial}{\partial y}\right|_p$.

Problem 1.3.7. *Consider the parametrization*

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta, \qquad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi,$$

of S^2. Let $f\colon S^2 \to S^2$ be the map induced by the automorphism of $\mathbb{R}^3$ with matrix

$$\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}.$$

Consider the coordinate neighborhood

$$U = \left\{ (x,y,z) \in S^2 : x+z \neq 0 \right\}.$$

Compute $f_* \left(\left.\frac{\partial}{\partial \theta}\right|_p \right)$ *and* $f_* \left(\left.\frac{\partial}{\partial \varphi}\right|_p \right)$ *for* $p \equiv (\theta_0, \varphi_0) \in U$ *such that* $f(p)$ *also belongs to* U*.*

Solution. This parametrization can be described by saying that we have a chart $\boldsymbol{\Phi}$ from U to an open subset of $A = (0,\pi) \times (0,2\pi)$ with

$$\boldsymbol{\Phi}^{-1}(u,v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u), \qquad u,v \in A,$$

and that we call

$$\theta = u \circ \boldsymbol{\Phi}, \qquad \varphi = v \circ \boldsymbol{\Phi},$$

where u and v are the coordinate functions on A. Then we need to compute $f_*((\partial/\partial\theta)_p)$ and $f_*((\partial/\partial\varphi)_p)$, where $p \in U$. As $f(p) \in U$, we have

$$\begin{aligned} f_* \left(\left.\frac{\partial}{\partial \theta}\right|_p \right) &= \left(\frac{\partial(\theta \circ f)}{\partial \theta} \right)(p) \left.\frac{\partial}{\partial \theta}\right|_{f(p)} + \left(\frac{\partial(\varphi \circ f)}{\partial \theta} \right)(p) \left.\frac{\partial}{\partial \varphi}\right|_{f(p)} \\ &= \left(\frac{\partial(\theta \circ f \circ \boldsymbol{\Phi}^{-1})}{\partial u} \right)(\boldsymbol{\Phi}(p)) \left.\frac{\partial}{\partial \theta}\right|_{f(p)} + \left(\frac{\partial(\varphi \circ f \circ \boldsymbol{\Phi}^{-1})}{\partial u} \right)(\boldsymbol{\Phi}(p)) \left.\frac{\partial}{\partial \varphi}\right|_{f(p)}, \end{aligned} \tag{$\star$}$$

$$\begin{aligned} f_* \left(\left.\frac{\partial}{\partial \varphi}\right|_p \right) &= \left(\frac{\partial(\theta \circ f)}{\partial \varphi} \right)(p) \left.\frac{\partial}{\partial \theta}\right|_{f(p)} + \left(\frac{\partial(\varphi \circ f)}{\partial \varphi} \right)(p) \left.\frac{\partial}{\partial \varphi}\right|_{f(p)} \\ &= \left(\frac{\partial(\theta \circ f \circ \boldsymbol{\Phi}^{-1})}{\partial v} \right)(\boldsymbol{\Phi}(p)) \left.\frac{\partial}{\partial \theta}\right|_{f(p)} + \left(\frac{\partial(\varphi \circ f \circ \boldsymbol{\Phi}^{-1})}{\partial v} \right)(\boldsymbol{\Phi}(p)) \left.\frac{\partial}{\partial \varphi}\right|_{f(p)}. \end{aligned}$$

Now,

$$\begin{aligned} (\theta \circ f \circ \boldsymbol{\Phi}^{-1})(u,v) &= (\theta \circ f)(\sin u \cos v,\ \sin u \sin v,\ \cos u) \\ &= \theta \left(\frac{\sqrt{2}}{2}(\sin u \cos v + \cos u),\ \sin u \sin v,\ \frac{\sqrt{2}}{2}(-\sin u \cos v + \cos u) \right) \end{aligned}$$

$$= \arccos\left(\frac{\sqrt{2}}{2}(-\sin u\cos v + \cos u)\right),$$

$$(\varphi \circ f \circ \Phi^{-1})(u,v) = \arctan\left(\sqrt{2}\,\frac{\sin u \sin v}{\sin u\cos v + \cos u}\right).$$

(Notice that, since $x+z \neq 0$ on U, the function arctan is well-defined on U.)

Then, we obtain by calculating and substituting the four partial derivatives in $(\star)$ above:

$$f_*\left(\left.\frac{\partial}{\partial\theta}\right|_p\right) = \frac{\sin\theta_0 + \cos\theta_0\cos\varphi_0}{\sqrt{1+\sin^2\theta_0\sin^2\varphi_0 + \sin 2\theta_0\cos\varphi_0}}\left.\frac{\partial}{\partial\theta}\right|_{f(p)} + \frac{\sqrt{2}\sin\varphi_0}{1+\sin^2\theta_0\sin^2\varphi_0+\sin 2\theta_0\cos\varphi_0}\left.\frac{\partial}{\partial\varphi}\right|_{f(p)},$$

$$f_*\left(\left.\frac{\partial}{\partial\varphi}\right|_p\right) = \frac{-\sin\theta_0\sin\varphi_0}{\sqrt{1+\sin^2\theta_0\sin^2\varphi_0 + \sin 2\theta_0\cos\varphi_0}}\left.\frac{\partial}{\partial\theta}\right|_{f(p)} + \frac{\sqrt{2}(\sin^2\theta_0 + \frac{1}{2}\sin 2\theta_0\cos\varphi_0)}{1+\sin^2\theta_0\sin^2\varphi_0+\sin 2\theta_0\cos\varphi_0}\left.\frac{\partial}{\partial\varphi}\right|_{f(p)}.$$

Problem 1.3.8. *Let M be a C^∞ manifold of dimension n and $\mathscr{F}(p)$ the set of C^∞ functions $f\colon M \to \mathbb{R}$ whose domains contain the point $p \in M$. Let $X \in T_pM$. Prove that if $f \in \mathscr{F}(p)$, then considering $\mathbb{R}$ as a C^∞ manifold with the chart* $(\mathbb{R}, \mathrm{id})$, *where* id *denotes the identity map of $\mathbb{R}$ with the coordinate t, we have $f_*X = a(\mathrm{d}/\mathrm{d}t)_{f(p)}$, where $a = Xf$.*

Solution. We have $f_*\colon T_pM \to T_{f(p)}\mathbb{R} \approx \mathbb{R}$. In general, given a C^∞ map h from the C^∞ manifold M to the C^∞ manifold N, and a tangent vector $X \in T_pM$, we have

$$h_*X = X(y^i \circ h)\left.\frac{\partial}{\partial y^i}\right|_{h(p)},$$

where $\{y^i\}$ denotes coordinate functions on a neighborhood of $h(p)$. In the present case,

$$\begin{aligned} f_*X &= X(t\circ f)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{f(p)} \\ &= (Xf)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{f(p)} \\ &= a\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{f(p)}. \end{aligned}$$

1.4 Critical Points and Values

Problem 1.4.1. *Consider the map*

$$\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad (x^1,x^2,x^3) \mapsto (y^1,y^2) = (x^1x^2, x^3).$$

(1) *Find the critical points of* φ .
(2) *Let* S^2 *be the unit sphere of* $\mathbb{R}^3$*. Find the critical points of* $\varphi|_{S^2}$.
(3) *Find the set* C *of critical values of* $\varphi|_{S^2}$.
(4) *Does* C *have zero measure?*

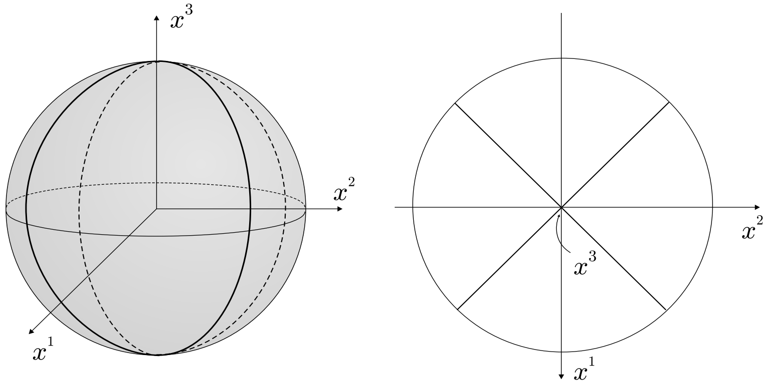

Fig. 1.16 The set of critical points of $\varphi|_{S^2}$.

Solution. (1) The Jacobian matrix $\varphi_* = \begin{pmatrix} x^2 & x^1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank $\varphi_* < 2$ if and only if $x^1 = x^2 = 0$; that is, the set of critical points of φ is the x^3-axis.

(2) Consider the charts defined by the parametrization

$$x^1 = \sin u \cos v, \qquad x^2 = \sin u \sin v, \qquad x^3 = \cos u,$$

for $u \in (0,\pi),\ v \in (0,2\pi)$ and $u \in (0,\pi),\ v \in (-\pi,\pi)$, respectively. We have

$$y^1 = \frac{1}{2}\sin^2 u \sin 2v, \qquad y^2 = \cos u.$$

So we can write

$$(\varphi|_{S^2})_* \equiv \begin{pmatrix} \dfrac{1}{2}\sin 2u \sin 2v & \sin^2 u \cos 2v \\ -\sin u & 0 \end{pmatrix};$$

thus $\operatorname{rank}(\varphi|_{S^2})_* < 2$ if and only if either $\sin u = 0$ or $\cos 2v = 0$.

We have $\sin u \neq 0$ in both charts. In the first chart, we have $\cos 2v = 0$ for $v = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$. In the second chart, one has $\cos 2v = 0$ for $v = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$. The sets of respective critical points coincide: They are the four half-circles in the Figure 1.16 excluding the poles, due to the parametrization. Now, we must add the poles as they are critical points for φ by virtue of (1) above.

Hence, the set of critical points of $\varphi|_{S^2}$ is given by the meridians corresponding to $v = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

(3) Since $\sin 2v = \pm 1$ for $v = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, the set of critical values of $\varphi|_{S^2}$ is

$$C = \left\{ (y^1, y^2) : y^1 = \pm\frac{1}{2}\sin^2 u,\ y^2 = \cos u \right\},$$

that is, the parabolas

$$2y^1 + (y^2)^2 = 1, \qquad 2y^1 - (y^2)^2 = -1.$$

Note that the images of the poles are included.

(4) A subset S of an n-manifold M has measure zero if it is contained in a countable union of coordinate neighborhoods U_i such that, φ_i being the corresponding coordinate map, $\varphi_i(U_i \cap S) \subset \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$. This is the case for $C \subset \mathbb{R}^2$, as it is a finite union of 1-submanifolds of $\mathbb{R}^2$.

Problem 1.4.2. (1) *Let $N = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $M = \mathbb{R}^2$. We define $f\colon M \to \mathbb{R}$ by $f(x,y) = y^2$. Prove that the set of critical points of $f|_N$ is the intersection with N of the set of critical points of f.*

(2) *Let $N = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $M = \mathbb{R}^2$. We define $f\colon M \to \mathbb{R}$ by $f(x,y) = x^2 + y^2$. Is the set of critical points of $f|_N$ the same as the one of f?*

Solution. (1) The set of critical points of f is N and $f|_N$ is the zero map. Thus all the points of N are critical for $f|_N$.

(2) No. In this case, the set of critical points of f is reduced to the origin, but $f|_N = 1$, so all the points of N are critical.

Problem 1.4.3. *Find the critical points and the critical values of the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2,\ (x,y,z) \mapsto (x+y^2, y+z^2)$.*

Solution. We have $f_* \equiv \begin{pmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \end{pmatrix}$. Since $\operatorname{rank} f_* = 2$, f has no critical points, hence it has no critical values.

Problem 1.4.4. *Consider the function*

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad (x,y,z) \mapsto x\sin y + y\sin z + z\sin x.$$

(1) *Prove that* $(0,0,0)$ *is a non-degenerate critical point of* f.

(2) *Calculate the index of* f *at* $(0,0,0)$.

Solution. (1)

$$f_{*(0,0,0)} \equiv (\sin y + z\cos x,\, x\cos y + \sin z,\, y\cos z + \sin x)_{(0,0,0)} = (0,0,0).$$

Thus rank $f_{*(0,0,0)} = 0$, so $(0,0,0)$ is a critical point. The Hessian matrix of f at $(0,0,0)$ is

$$H^f_{(0,0,0)} = \begin{pmatrix} -z\sin x & \cos y & \cos x \\ \cos y & -x\sin y & \cos z \\ \cos x & \cos z & -y\sin z \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since $\det H^f_{(0,0,0)} = 2 \neq 0$, the point $(0,0,0)$ is non-degenerate.

(2) The index of f at $(0,0,0)$ is the index of $H^f_{(0,0,0)}$, that is, the number of negative signs in a diagonal matrix representing the quadratic form $2(xy+xz+yz)$ associated to $H^f_{(0,0,0)}$. Applying Gauss's method of decomposition in squares, one has:

$$\begin{aligned} 2xy+2xz+2yz &= 2((x+z)(y+z)-z^2) \\ &= 2\left(\frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2\right) = \frac{1}{2}(x+y+2z)^2 - \frac{1}{2}(x-y)^2 - 2z^2. \end{aligned}$$

As two negative signs appear, the index of f at $(0,0,0)$ is 2.

Problem 1.4.5. *Consider the* C^∞ *manifold* $\mathbb{R}^n$ *and a submanifold* L *given by a vector subspace of* $\mathbb{R}^n$ *with* $\dim L \leqslant n-1$. *Prove that* L *has zero measure.*

Solution. Let $\dim L = k \leqslant n-1$. Consider the map

$$f\colon \mathbb{R}^k \to \mathbb{R}^n, \qquad f(x^1, x^2, \ldots, x^k) = x^i e_i,$$

where $\{e_i\}$ is a basis of L. By virtue of Sard's theorem, $f(\mathbb{R}^k) = L$ has zero measure.

Problem 1.4.6. *Let* M_1 *and* M_2 *be two* C^∞ *manifolds. Give an example of differentiable mapping* $f\colon M_1 \to M_2$ *such that all the points of* M_1 *are critical points and the set of critical values has zero measure.*

Solution. Let $f\colon M_1 \to M_2$ defined by $f(p) = q$, for every $p \in M_1$ and q a fixed point of M_2. Then the rank of f is zero, hence all the points of M_1 are critical. On the other hand, the set of critical values is reduced to the point q, and the set $\{q\}$ has obviously zero measure.

1.5 Immersions, Submanifolds, Embeddings and Diffeomorphisms

Problem 1.5.1. *Prove that the C^∞ map*

$$\Psi\colon \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (x,y) = (t^2, t^3),$$

(*see* Figure 1.17)*, is not an immersion.*

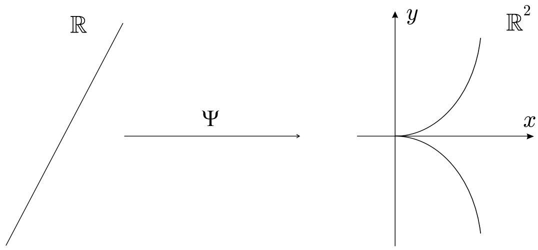

Fig. 1.17 The graph of the map $t \mapsto (t^2, t^3)$.

Solution.

$$\begin{aligned}\operatorname{rank} \Psi &= \operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t}\end{pmatrix} \\ &= \operatorname{rank}(2t \;\; 3t^2) \\ &= \begin{cases} 1 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}\end{aligned}$$

For $t = 0$, we have rank $\Psi = 0 < \dim \mathbb{R} = 1$, thus Ψ is not an immersion. Let us consider Ψ_{*t_0} in detail, as a map between tangent vector spaces. We have

$$\begin{array}{rccc} \Psi_*\colon & T_{t_0}\mathbb{R} & \to & T_{\Psi(t_0)}\mathbb{R}^2 \\ & \lambda \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} & \mapsto & \Psi_*\left(\lambda \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \end{array}$$

and

$$\begin{aligned}\Psi_*\left(\lambda \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) &= \lambda\left(\frac{\partial(x\circ\Psi)}{\partial t}(t_0)\left.\frac{\partial}{\partial x}\right|_{\Psi(t_0)} + \frac{\partial(y\circ\Psi)}{\partial t}(t_0)\left.\frac{\partial}{\partial y}\right|_{\Psi(t_0)}\right) \\ &= \lambda\left(2t_0 \left.\frac{\partial}{\partial x}\right|_{\Psi(t_0)} + 3t_0^2 \left.\frac{\partial}{\partial y}\right|_{\Psi(t_0)}\right)\end{aligned}$$

$$
\begin{aligned}
&\equiv (2\lambda t_0, 3\lambda t_0^2)\\
&= \begin{cases} (0,0) \in T_{(0,0)}\mathbb{R}^2 & \forall\lambda \text{ if } t_0 = 0\\ (0,0) \in T_{\Psi(t_0)}\mathbb{R}^2 & \text{if } \lambda = 0\\ \neq (0,0) \in T_{\Psi(t_0)}\mathbb{R}^2 & \text{if } t_0, \lambda \neq 0. \end{cases}
\end{aligned}
$$

That is, $\Psi_*(T_0\mathbb{R}) = (0,0) \in T_{(0,0)}\mathbb{R}^2$. The whole tangent space $T_0\mathbb{R}$ is applied by Ψ_{*0} onto only one point of the tangent space $T_{(0,0)}\mathbb{R}^2$.

Problem 1.5.2. *Let $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Define a C^∞ map by*

$$f\colon M \to \mathbb{R}^2, \qquad (x,y) \mapsto \left(\frac{y}{1-x^2-y^2}, \mathrm{e}^{x^2}\right).$$

(1) *Find the set S of points p of M at which f_{*p} is injective.*
(2) *Prove that $f(S)$ is an open subset of $\mathbb{R}^2$.*

Solution. (1) One has

$$\operatorname{rank} f_* < 2 \iff 2x\mathrm{e}^{x^2}\frac{1-x^2+y^2}{(1-x^2-y^2)^2} = 0 \iff x = 0 \text{ or } 1 = x^2 - y^2.$$

Since $1 > x^2 + y^2$, we have $1 > x^2 - y^2$, so $S = M - \{(0,y) : -1 < y < 1\}$.

(2) Consider the subset $\{(0,y) : -1 < y < 1\}$ of M. We have

$$
\begin{aligned}
f(\{(0,y) : -1 < y < 1\}) &= \left\{\left(\frac{y}{1-y^2}, 1\right)\right\}\\
&= (-\infty, \infty) \times \{1\} \subset \mathbb{R}^2.
\end{aligned}
$$

Thus $f(M) = \{(x,y) \in \mathbb{R}^2 : 1 \leqslant y < \mathrm{e}\}$, hence

$$f(S) = \{(x,y) \in \mathbb{R}^2 : 1 < y < \mathrm{e}\},$$

which is an open subset of $\mathbb{R}^2$.

Problem 1.5.3. *Let $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ be the C^∞ manifolds defined, respectively, by the differentiable structures obtained from the atlases $\{(\mathbb{R}, \mathrm{id})\}$ and $\{(\mathbb{R}, \varphi\}$ on $\mathbb{R}$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^3$. Prove that $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ are diffeomorphic* (*see* Problem 1.1.1).

Solution. To prove that $\mathbb{R}_{\mathrm{id}}$ and $\mathbb{R}_\varphi$ are diffeomorphic, we only have to give a map Φ such that its representative Ψ in the diagram

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\Phi} & \mathbb{R}\\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\varphi}\\
\mathbb{R} & \xrightarrow{\Psi} & \mathbb{R}
\end{array}
$$

be a diffeomorphism. Let $\Phi(t) = \sqrt[3]{t}$. One has $\Psi(t) = \varphi \circ \Phi \circ \mathrm{id}^{-1}(t) = t$.

Problem 1.5.4. *Consider the map* $f \colon \mathbb{R}^2 \to \mathbb{R}^2 - \{(0,0)\}$ *defined by* $x = \mathrm{e}^u \cos v$, $y = \mathrm{e}^u \sin v$.

(1) *Prove that the Jacobian determinant of f does not vanish at any point of the plane.*

(2) *Can f be taken as a local coordinate map on a neighborhood of any point?*

(3) *Is f a diffeomorphism?*

(4) *Given a point* $p_0 = (u_0, v_0)$*, give an example of a maximal open neighborhood of* p_0 *on which we can take f as a local coordinate map.*

Solution. (1) Notice that $x^2 + y^2 = \mathrm{e}^{2u} > 0$; so that $f(u,v) \in \mathbb{R}^2 - \{(0,0)\}$ for all $(u,v) \in \mathbb{R}^2$. We have

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^u \cos v & -\mathrm{e}^u \sin v \\[2ex] \mathrm{e}^u \sin v & \mathrm{e}^u \cos v \end{pmatrix},$$

hence $\partial(x,y)/\partial(u,v) = \mathrm{e}^{2u} > 0$ for all $(u,v) \in \mathbb{R}^2$.

(2) By (1), f is a local diffeomorphism at every point of $\mathbb{R}^2$. So f can be taken as a local coordinate map on a neighborhood of every point.

(3) The map f is not a diffeomorphism as it is not injective. We have $f(u,v) = f(u',v')$ if and only if $u = u'$ and $v - v' = 2k\pi,\ k \in \mathbb{Z}$. In fact, from the relations

$$\mathrm{e}^u \cos v = \mathrm{e}^{u'} \cos v', \qquad \mathrm{e}^u \sin v = \mathrm{e}^{u'} \sin v',$$

we obtain $\mathrm{e}^{2u} = \mathrm{e}^{2u'}$, and so $u = u'$. Then one has $\cos v = \cos v',\ \sin v = \sin v'$, hence the difference between v and v' is an integer multiple of 2π.

(4) The points having the same image as p_0 are the ones of the form $(u_0, v_0 + 2k\pi),\ k \in \mathbb{Z}$. The nearest ones to p_0 are $(u_0, v_0 \pm 2\pi)$. Hence such a neighborhood is $\mathbb{R} \times (v_0 - \pi, v_0 + \pi)$.

Problem 1.5.5. *Let V be a finite-dimensional real vector space. Consider the open subset* $\mathscr{E}$ *of* $\mathrm{End}_{\mathbb{R}} V$ *defined by*

$$\mathscr{E} = \{T \in \mathrm{End}_{\mathbb{R}} V : \det(I + T) \neq 0\},$$

where I denotes the identity endomorphism.

(1) *Prove that the map*

$$f \colon \mathscr{E} \to \mathrm{End}_{\mathbb{R}} V, \qquad T \mapsto (I - T)(I + T)^{-1},$$

is an involution of $\mathscr{E}$.

(2) *Consider on* $\mathscr{E}$ *the differentiable structure induced by* $\mathrm{End}_{\mathbb{R}} V$. *Prove that* $f \colon \mathscr{E} \to \mathscr{E}$ *is a diffeomorphism.*

Solution. (1) If $T \in \mathscr{E}$, then $I + f(T) = 2I(I+T)^{-1} \in \mathscr{E}$. Hence

$$\begin{aligned}\det(I + f(T)) &= \det(2I(I+T)^{-1}) \\ &= 2^{\dim V} / \det(I+T) \neq 0.\end{aligned}$$

It is easily checked that $f(f(T)) = T$.

(2) The map f is C^∞. In fact, the entries of $f(T)$ can be expressed as rational functions of the entries of T. As $f^{-1} = f$, we conclude.

Problem 1.5.6. *Prove that the function*

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) = (x\mathrm{e}^y + y,\, x\mathrm{e}^y - y),$$

is a C^∞ diffeomorphism.

Solution. Solving the system

$$x\mathrm{e}^y + y = x', \qquad x\mathrm{e}^y - y = y',$$

in x and y, we conclude that the unique solution is

$$x = \frac{x'+y'}{2\mathrm{e}^{(x'-y')/2}}, \qquad y = \frac{x'-y'}{2};$$

hence the map is one-to-one. Let us see that both f and f^{-1} are C^∞. We have

$$\begin{aligned} f\colon (x,y) &\mapsto (x\mathrm{e}^y + y,\, x\mathrm{e}^y - y), \\ f^{-1}\colon (x,y) &\mapsto \left(\frac{x+y}{2}\mathrm{e}^{(y-x)/2},\, \frac{x-y}{2} \right). \end{aligned}$$

Since the components of f and f^{-1} and their derivatives of any order are elementary functions, f and f^{-1} are C^∞. Thus f is a C^∞ diffeomorphism.

Problem 1.5.7. *Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by*

$$x' = \mathrm{e}^{2y} + \mathrm{e}^{2z}, \qquad y' = \mathrm{e}^{2x} - \mathrm{e}^{2z}, \qquad z' = x - y.$$

Find the image set $\varphi(\mathbb{R}^3)$ and prove that φ is a diffeomorphism from $\mathbb{R}^3$ to $\varphi(\mathbb{R}^3)$.

Solution. Solving, one has

$$x = z' + y, \qquad x' = \mathrm{e}^{2y} + \mathrm{e}^{2z}, \qquad y' = \mathrm{e}^{2z'}\mathrm{e}^{2y} - \mathrm{e}^{2z},$$

and so

$$\mathrm{e}^{2y} = \frac{x'+y'}{1+\mathrm{e}^{2z'}}, \qquad \mathrm{e}^{2z} = \frac{x'\mathrm{e}^{2z'} - y'}{1+\mathrm{e}^{2z'}}.$$

Hence it must be $x' > 0$, $x'+y' > 0$, $x'\mathrm{e}^{2z'} > y'$.

Thus,

$$\varphi(\mathbb{R}^3) = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x > 0,\ x+y > 0,\ x\mathrm{e}^{2z} > y\right\}.$$

The map φ is injective, since the above formulae give the unique point (x,y,z) having (x',y',z') as its image by φ.

In order to see that φ is a diffeomorphism from $\mathbb{R}^3$ to $\varphi(\mathbb{R}^3)$, it suffices to prove that the determinant of its Jacobian matrix J_φ never vanishes. We have

$$\begin{aligned}\det J_\varphi &= \det\begin{pmatrix} 0 & 2\mathrm{e}^{2y} & 2\mathrm{e}^{2z} \\ 2\mathrm{e}^{2x} & 0 & -2\mathrm{e}^{2z} \\ 1 & -1 & 0 \end{pmatrix} \\ &= -4(\mathrm{e}^{2y+2z} + \mathrm{e}^{2x+2z}) \neq 0.\end{aligned}$$

Problem 1.5.8. *Consider the C^∞ function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by*

$$f(x,y,z) = (x\cos z - y\sin z,\ x\sin z + y\cos z,\ z).$$

Prove that $f|_{S^2}$ is a diffeomorphism from the unit sphere S^2 onto itself.

Solution. For each $(x,y,z) \in S^2$, one has $f(x,y,z) \in S^2$, so that $(f|_{S^2})(S^2) \subset S^2$. Furthermore, given $(u,v,w) \in S^2$, we have to prove that there exists $(x,y,z) \in S^2$ such that $f(x,y,z) = (u,v,w)$, that is,

$$x\cos z - y\sin z = u, \qquad x\sin z + y\cos z = v, \qquad z = w.$$

Solving this system in x, y, z, we have

$$x = u\cos w + v\sin w, \qquad y = -u\sin w + v\cos w, \qquad z = w.$$

These equations are the ones of the components of the inverse function of $f|_{S^2}$, which is clearly C^∞, hence $f|_{S^2}$ is a diffeomorphism.

Problem 1.5.9. *Let $\{(E,\varphi)\}$ and $\{(E,\psi)\}$ be the atlases on the "Figure Eight" built in* Problem 1.2.1. *Exhibit a diffeomorphism between the differentiable manifolds E_φ and E_ψ defined, respectively, by the differentiable structures obtained from the atlases $\{(E,\varphi)\}$ and $\{(E,\psi)\}$.*

Solution. Let

$$f\colon E_\varphi \to E_\psi, \qquad f(\sin 2s, \sin s) = (\sin 2(s-\pi), \sin(s-\pi)).$$

Since $(\psi \circ f \circ \varphi^{-1})(s) = s - \pi$, it follows that f is a diffeomorphism.

Problem 1.5.10. *Let (N,φ), (N,ψ) be the atlases on the "Noose" built in* Problem 1.2.2. *Exhibit a diffeomorphism between the differentiable manifolds N_φ and N_ψ defined, respectively, by the differentiable structures obtained from the atlases $\{(N,\varphi)\}$ and $\{(N,\psi)\}$.*

Solution. The map $f\colon (N,\varphi) \to (N,\psi)$, $(x,y) \mapsto (-x,y)$, mapping a point to its symmetric with respect to the y-axis, is a diffeomorphism. One has

$$\begin{array}{ccccccc} (-1,1) & \stackrel{\varphi^{-1}}{\to} & (N,\varphi) & \stackrel{f}{\to} & (N,\psi) & \stackrel{\psi}{\to} & (-1,1) \\ s & \mapsto & \varphi^{-1}(s) & \mapsto & f(\varphi^{-1}(s)) & \mapsto & s. \end{array}$$

In fact,

$$\varphi^{-1}(s) = \begin{cases} (0, 1-s) & \text{if } -1 < s < 0 \\ (\sin 2\pi s, \cos 2\pi s) & \text{if } 0 \leqslant s < 1, \end{cases}$$

$$f(\varphi^{-1}(s)) = \begin{cases} (0, 1-s) & \text{if } -1 < s < 0 \\ (\sin 2\pi(1-s), \cos 2\pi(1-s)) & \text{if } 0 \leqslant s < 1, \end{cases}$$

and

$$(\psi \circ f \circ \varphi^{-1})(s) = s, \qquad \forall s \in (-1,1).$$

Problem 1.5.11. *Prove that the map*

$$p\colon \mathbb{R} \to S^1, \qquad t \mapsto (\cos 2\pi t, \sin 2\pi t),$$

is a covering map.

Solution. We must prove:

(1) p is C^∞ and surjective.

(2) For each $x \in S^1$, there exists a neighborhood U of x in S^1 such that $p^{-1}(U) = \bigcup U_i$, $i \in I$, where the U_i are disjoint open subsets of $\mathbb{R}$ such that, for each $i \in I$, $p\colon U_i \to U$ is a diffeomorphism.

Now, (1) is immediate. Moreover p is a local diffeomorphism.

As for (2), let $y \in \mathbb{R}$; then

$$p\colon (y-\pi, y+\pi) \to S^1 - p(y+\pi)$$

is a diffeomorphism and

$$p^{-1}(S^1 - p(y+\pi)) = \bigcup_{k\in\mathbb{Z}} (y+(2k-1)\pi, y+(2k+1)\pi).$$

Of course, one can take smaller intervals as domains of the diffeomorphisms.

Problem 1.5.12. *Consider the curves*:

$$\begin{array}{llcl} \text{(a)} & \sigma\colon \mathbb{R} & \to & \mathbb{R}^2 \\ & t & \mapsto & (t, |t|) \\ \text{(b)} & \sigma\colon \mathbb{R} & \to & \mathbb{R}^2 \\ & t & \mapsto & (t^3 - 4t, t^2 - 4) \end{array}$$

$$\text{(c)} \qquad \begin{aligned} \sigma\colon \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t, t) \end{aligned}$$

$$\text{(d)} \qquad \begin{aligned} \sigma\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t) \end{aligned}$$

$$\text{(e)} \qquad \begin{aligned} \sigma\colon (1,\infty) &\to \mathbb{R}^2 \\ t &\mapsto \left(\frac{1}{t}\cos 2\pi t, \frac{1}{t}\sin 2\pi t\right) \end{aligned}$$

$$\text{(f)} \qquad \begin{aligned} \sigma\colon (1,\infty) &\to \mathbb{R}^2 \\ t &\mapsto \left(\frac{1+t}{2t}\cos 2\pi t, \frac{1+t}{2t}\sin 2\pi t\right) \end{aligned}$$

$$\text{(g)} \qquad \begin{aligned} \sigma\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\to \left(2\cos\left(t-\frac{\pi}{2}\right), \sin 2\left(t-\frac{\pi}{2}\right)\right) \end{aligned}$$

$$\text{(h)} \qquad \begin{aligned} \sigma\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto \left(2\cos\left(f(t)-\frac{\pi}{2}\right), \sin 2\left(f(t)-\frac{\pi}{2}\right)\right) \end{aligned}$$

where $f(t)$ denotes a monotonically increasing C^∞ function on $-\infty < t < \infty$ such that $f(0) = \pi$, $\lim_{t\to-\infty} f(t) = 0$ and $\lim_{t\to\infty} f(t) = 2\pi$ (for instance, $f(t) = \pi + 2\arctan t$).

$$\text{(i)} \qquad \begin{aligned} \sigma\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto \begin{cases} (1/t, \sin \pi t) & \text{if } 1 \leqslant t < \infty \\ (0, t+2) & \text{if } -\infty < t \leqslant -1, \end{cases} \end{aligned}$$

where in addition one smoothly connects, for $-1 \leqslant t \leqslant 1$, the two curves $\sigma|_{(-\infty,-1]}$ and $\sigma|_{[1,\infty)}$ with a C^∞ curve (dotted in the Figure 1.22).

(1) *Is σ an immersion in* (a)*? (resp. in* (b), (d), (g))*?*

(2) *Is σ an injective immersion in* (b) *(resp. in* (d), (g), (h), (i))*?*

(3) *Is σ an embedding in* (c)*? (resp. in* (e), (f), (h), (i))*?*

Solution. (a) σ is not an immersion, as it is not a differentiable map at the origin (see Figure 1.18). We recall that

$$\sigma'(t_0) = \sigma_{*t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right),$$

that is, $\sigma'(t_0)$ is the image of the canonical vector at $t_0 \in \mathbb{R}$.

(b) σ is a differentiable map, and since $\sigma'(t) = (3t^2 - 4, 2t) \neq (0,0)$ for all t, the map σ is an immersion. But for $t = \pm 2$, it has a self-intersection, so it is not an injective immersion.

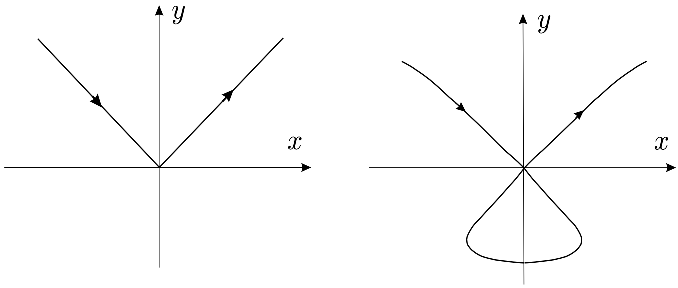

Fig. 1.18 (a) σ is not an immersion. (b) σ is a non-injective immersion.

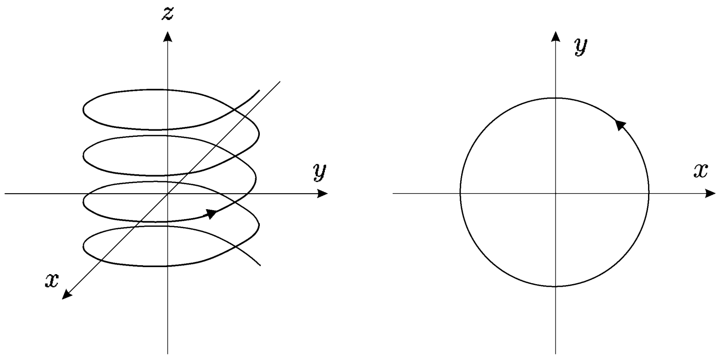

Fig. 1.19 (c) σ is an embedding. (d) σ is a non-injective immersion.

(c) σ is an immersion as

$$\sigma'(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 1) \neq (0,0,0), \qquad \forall t \in \mathbb{R}.$$

It is trivially injective and since the map $\sigma\colon \mathbb{R} \to \sigma(\mathbb{R})$ is open, σ is an embedding (see Figure 1.19).

(d) σ is an immersion since

$$\sigma'(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t) \neq (0,0)$$

for all t, but σ is obviously not injective. Nevertheless, $\sigma(\mathbb{R})$ is an embedded submanifold. (See Problems 1.1.5 and 1.6.1.)

(e) σ is an immersion as

$$\sigma'(t) = \left(-\frac{1}{t^2} \cos 2\pi t - \frac{2\pi}{t} \sin 2\pi t, -\frac{1}{t^2} \sin 2\pi t + \frac{2\pi}{t} \cos 2\pi t \right) = (0,0)$$

if and only if each component vanishes or, equivalently, the square of each component, or even the sum of those squares vanishes; that is, $(1/t^4) + 4\pi^2/t^2 = 0$, or

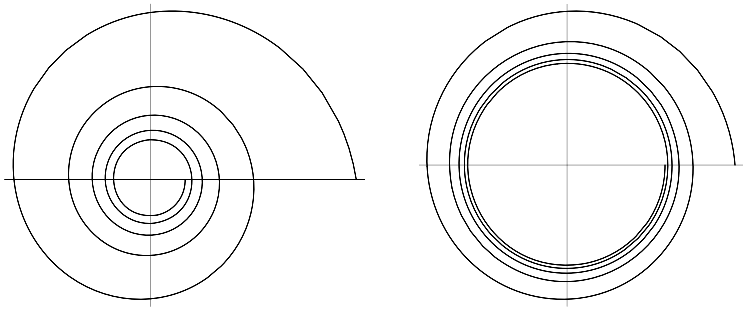

Fig. 1.20 (e) σ is an embedding. (f) σ is an immersion.

$1+4t^2\pi^2 = 0$, which is absurd. Since $\sigma\colon (1,\infty) \to \sigma(1,\infty)$ is an injective and open map, it follows that σ is an embedding (see Figure 1.20).

(f) σ is an immersion as

$$\sigma'(t) = \left(-\frac{\cos 2\pi t}{2t^2} - \frac{t+1}{t}\pi \sin 2\pi t, -\frac{\sin 2\pi t}{2t^2} + \frac{t+1}{t}\pi \cos 2\pi t\right) = (0,0)$$

if and only if the sum of the squares of the components vanishes, that is, if $(1/4t^4) + ((t+1)\pi/t)^2 = 0$, or $1+4t^2(t+1)^2\pi^2 = 0$, which is absurd. Finally, σ is an embedding, as $\sigma\colon (1,\infty) \to \sigma(1,\infty)$ is an open injective map.

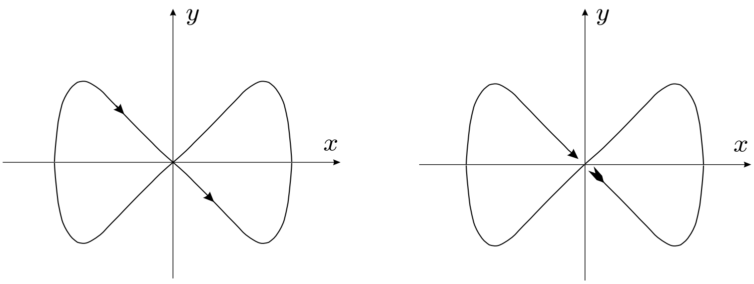

Fig. 1.21 (g) σ is an immersion. (h) σ is an injective immersion but not an embedding.

(g) The image is a "Figure Eight," whose image makes a complete circuit starting at the origin as t goes from 0 to 2π, in the sense shown in Figure 1.21 (g). The curve is an immersion as

$$\sigma'(t) = \left(-2\sin\left(t - \frac{\pi}{2}\right), 2\cos 2\left(t - \frac{\pi}{2}\right)\right) \neq (0,0)$$

for all t; but is not an injective immersion since $\sigma(\{0, \pm 2\pi, \pm 4\pi, \ldots\}) = \{(0,0)\}$.

(h) We have a "Figure Eight" as in (g), but the curve now passes through $(0,0)$ once only. Though it is an injective immersion, it is not an embedding, as the "Figure Eight" is compact and $\mathbb{R}$ is not (see Figure 1.21 (h)).

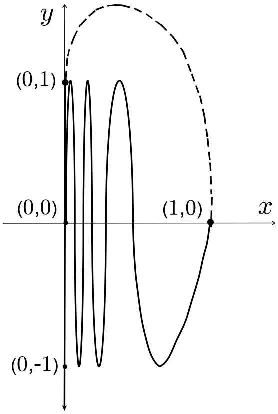

Fig. 1.22 (i) σ is not an embedding.

(i) σ is an injective immersion. It is not an embedding: In fact, take a point p on the vertical segment $\{0\} \times (-1,+1)$ of the graph of the curve. Then an open neighborhood of p in that vertical interval is never the intersection of an open neighborhood of p in $\mathbb{R}^2$ with the graph of the curve.

Problem 1.5.13. *Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball of the Euclidean space $\mathbb{R}^n$. Prove that the map*

$$f\colon U \to \mathbb{R}^n, \qquad f(x) = \frac{x}{1-|x|^2},$$

is a diffeomorphism.

Solution. As a computation shows, $x = f^{-1}(y) = 2y/(1+\sqrt{1+4|y|^2})$.

1.6 Constructing Manifolds by Inverse Image. Implicit Map Theorem

Problem 1.6.1. *Prove that the sphere S^n is a closed embedded submanifold of $\mathbb{R}^{n+1}$.*

Solution. The map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x^1,\dots,x^{n+1}) = \sum_{i=1}^{n+1}(x^i)^2$, is trivially C^∞ and has rank constant and equal to 1 on $\mathbb{R}^{n+1} - \{0\}$. Since $S^n = f^{-1}(1)$, S^n is a closed embedded submanifold of $\mathbb{R}^{n+1}$.

Problem 1.6.2. *Prove that each of the functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by*:

$$\begin{aligned} &\text{(a)} \quad f(x,y,z) = x^2 + y^2 - z^2 - 1, \\ &\text{(b)} \quad f(x,y,z) = x^2 - y^2 - z^2 - 1, \end{aligned}$$

defines the structure of a differentiable manifold on $f^{-1}(0)$. The corresponding manifolds are called one-sheet *and* two-sheet hyperboloids, *respectively. Find in each case a finite atlas defining its C^∞ structure.*

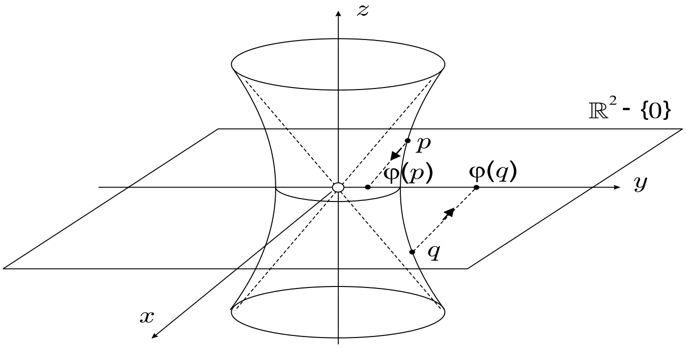

Fig. 1.23 An atlas with one chart for the one-sheet hyperboloid.

Solution. In the case (a), the rank of the Jacobian matrix of f is zero if and only if $x = y = z = 0$, but $(0,0,0) \notin f^{-1}(0)$. Thus the one-sheet hyperboloid is an embedded submanifold of $\mathbb{R}^3$.

In the case (b), one proceeds as in (a), now with the Jacobian matrix $J = \operatorname{diag}(2x, -2y, -2z)$. Thus the two-sheet hyperboloid is a C^∞ submanifold of $\mathbb{R}^3$.

As for the atlas, we prove below that the one-sheet hyperboloid is diffeomorphic to $\mathbb{R}^2 - \{0\}$ and hence it suffices to consider only one chart. This fact can be visualized by the map φ onto the plane $z = 0$ mapping each point p of the hyperboloid to the intersection $\varphi(p)$ with that plane of the straight line parallel to the asymptotic line by the meridian passing through the point (see Figure 1.23).

Notice that there is another choice, mapping the points of the hyperboloid with $z < 0$ to the interior of the disk $x^2 + y^2 < 1$ minus the origin, and the points with $z > 0$ to the points with $x^2 + y^2 > 1$.

The equations of φ are given by

$$x' = x\left(1 - \frac{z}{\sqrt{1+z^2}}\right),$$

$$y' = y\left(1 - \frac{z}{\sqrt{1+z^2}}\right),$$

and, as a computation shows, the inverse map φ^{-1} is given by

$$x = \frac{x'^2 + y'^2 + 1}{2(x'^2 + y'^2)} x',$$

$$y = \frac{x'^2 + y'^2 + 1}{2(x'^2 + y'^2)} y',$$

$$z = \frac{1 - x'^2 - y'^2}{2\sqrt{x'^2 + y'^2}}.$$

To have an atlas in the case (b), one needs at least two charts, as after finding x, y or z in the equation $x^2 - y^2 - z^2 - 1 = 0$, none of them is uniquely defined. Let $H = f^{-1}(0)$. Then the charts (U_1, φ_1), (U_2, φ_2), given by

$$\begin{aligned}
&U_1 = \{(x,y,z) \in H : x > 0\}, \qquad \varphi_1 : U_1 \to \mathbb{R}^2, \qquad \varphi_1(x,y,z) = (y,z),\\
&U_2 = \{(x,y,z) \in H : x < 0\}, \qquad \varphi_2 : U_2 \to \mathbb{R}^2, \qquad \varphi_2(x,y,z) = (y,z),
\end{aligned}$$

obviously define an atlas for the manifold.

Problem 1.6.3. *Let H be the two-sheet hyperboloid defined as in* Problem 1.6.2. *By using the charts defined there and proceeding directly, prove that the natural injection $j \colon H \to \mathbb{R}^3$ has rank 2 at every point.*

Solution. Take the atlas in Problem 1.6.2 (b). We have $U_1 = \varphi_1^{-1}(\mathbb{R}^2)$, $U_2 = \varphi_2^{-1}(\mathbb{R}^2)$, and the corresponding coordinate functions in $\mathbb{R}^3$ are given by the inclusion $j \colon H = U_1 \cup U_2 \to \mathbb{R}^3$, so that

$$\begin{aligned}
j \circ \varphi_1^{-1} \colon \quad \mathbb{R}^2 \quad &\to \quad \mathbb{R}^3\\
(y,z) \quad &\mapsto \quad (\sqrt{1+y^2+z^2}, y, z),
\end{aligned}$$

$$\begin{aligned}
j \circ \varphi_2^{-1} \colon \quad \mathbb{R}^2 \quad &\to \quad \mathbb{R}^3\\
(y,z) \quad &\mapsto \quad (-\sqrt{1+y^2+z^2}, y, z).
\end{aligned}$$

We have

$$\operatorname{rank} j_{*p} = \begin{cases} \operatorname{rank}(j \circ \varphi_1^{-1})_{*(y,z)} & \text{if } p \in U_1,\ (y,z) = \varphi_1(p) \\ \operatorname{rank}(j \circ \varphi_2^{-1})_{*(y,z)} & \text{if } p \in U_2,\ (y,z) = \varphi_2(p), \end{cases}$$

that is,

$$\operatorname{rank} j_{*p} = \operatorname{rank} \begin{pmatrix} \dfrac{y}{\sqrt{1+y^2+z^2}} & \dfrac{z}{\sqrt{1+y^2+z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2,$$

$$\operatorname{rank} j_{*p} = \operatorname{rank}\begin{pmatrix} \dfrac{-y}{\sqrt{1+y^2+z^2}} & \dfrac{-z}{\sqrt{1+y^2+z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2,$$

if $p \in U_1, (y,z) = \varphi_1(p)$, and $p \in U_2, (y,z) = \varphi_2(p)$, respectively.

Problem 1.6.4. *Prove that the function*

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad f(x,y,z) = x^3 + 2y^3 + z^3 + 6x^2y - 1,$$

defines the structure of a C^∞ *manifold on* $f^{-1}(0)$.

Solution. The Jacobian matrix

$$J = \begin{pmatrix} 3x^2 + 12xy & 6y^2 + 6x^2 & 3z^2 \end{pmatrix}$$

has rank 0 if and only if $(x,y,z) = (0,0,0)$, but this point does not belong to $H = f^{-1}(0)$, hence rank $J = 1$ and H admits a structure of C^∞ manifold.

Problem 1.6.5. *Prove that the subset H of the Euclidean space* $\mathbb{R}^3$ *of all the points* (x,y,z) *of* $\mathbb{R}^3$ *satisfying* $x^3 + y^3 + z^3 - 2xyz = 1$ *admits a* C^∞ 2*-manifold structure.*

Solution. The map

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad f(x,y,z) = x^3 + y^3 + z^3 - 2xyz - 1,$$

is C^∞ and its Jacobian matrix is

$$J = \begin{pmatrix} 3x^2 - 2yz & 3y^2 - 2xz & 3z^2 - 2xy \end{pmatrix},$$

which vanishes only if $(x,y,z) = (0,0,0)$. In fact, multiplying the identities

$$3x^2 = 2yz, \qquad 3y^2 = 2xz, \qquad 3z^2 = 2xy,$$

we get $27x^2y^2z^2 = 8x^2y^2z^2$, from which $xyz = 0$. If $x \neq 0$ then by the first of the three equations above we would have the absurd $y \neq 0$, $z \neq 0$. Thus $x = 0$. By the same reason one has $y = z = 0$; but $(0,0,0) \notin H$.

Problem 1.6.6. *Prove that the subset M of the Euclidean space* $\mathbb{R}^3$ *which consists of all the points* (x,y,z) *of* $\mathbb{R}^3$ *satisfying*

$$x^2 - y^2 + 2xz - 2yz = 1, \qquad 2x - y + z = 0,$$

admits a structure of C^∞ 1*-manifold.*

Solution. The functions

$$f_1(x,y,z) = x^2 - y^2 + 2xz - 2yz - 1, \qquad f_2(x,y,z) = 2x - y + z,$$

are C^∞ functions. The rank of the Jacobian matrix of f_1, f_2 with respect to x, y, z, is less than 2 if and only if $x - 2y - z = 0$, but the points satisfying this equation do not belong to $M = f^{-1}(0)$.

Problem 1.6.7. *Prove that, if $F\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is any differentiable function on $\mathbb{R}^{n-1}$, then the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined on $\mathbb{R}^n$ by*

$$f(x^1, \ldots, x^n) = F(x^1, \ldots, x^{n-1}) - x^n,$$

defines the structure of a C^∞ manifold on $f^{-1}(0)$. Prove that this manifold is diffeomorphic to $\mathbb{R}^{n-1}$. Illustrate the result considering the C^∞ manifolds on $\mathbb{R}^3$ thus determined by the functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\text{(a)} \quad f(x,y,z) = x^2 + y^2 - z, \qquad \text{(b)} \quad f(x,y,z) = x^2 - y^2 - z,$$

which are examples of paraboloids: Elliptic (of revolution) *in the case* (a), *and* hyperbolic *in the case* (b).

Solution. The rank of the Jacobian matrix of f is 1 everywhere, thus $f^{-1}(0)$ admits a structure of C^∞ manifold. Furthermore, it suffices to consider the chart $(f^{-1}(0), \varphi)$, where

$$\varphi\colon f^{-1}(0) \to \mathbb{R}^{n-1}, \qquad \varphi(x^1, \ldots, x^n) = (x^1, \ldots, x^{n-1}).$$

In the particular case of the paraboloids, taking into account the previous considerations, it is clear that:

Case (a): It is only necessary to consider the chart (U, φ) with

$$U = f^{-1}(0), \qquad \varphi\colon f^{-1}(0) \to \mathbb{R}^2, \qquad \varphi(x,y,z) = (x,y).$$

Case (b): Proceed as in (a).

Problem 1.6.8. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be any homogeneous polynomial function (with degree no less than one) with at least one positive value. Prove that the function $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = F(x) - 1$, defines on $f^{-1}(0)$ a structure of C^∞ manifold.*

Solution. The Jacobian matrix of f is

$$J_f = \begin{pmatrix} \dfrac{\partial F}{\partial x^1} & \cdots & \dfrac{\partial F}{\partial x^n} \end{pmatrix}.$$

If $\deg F = 1$, then at least one of the elements $(\partial F/\partial x^i)(p)$ does not vanish.

If $\deg F = r > 1$ and the matrix $((\partial F/\partial x^i)(p))$ is zero at a point $p = (x^1, \ldots, x^n)$, then $F(p)$ is also zero at that point. In fact, since F is homogeneous of degree r one has

$$rF(p) = x^1 \frac{\partial F}{\partial x^1}(p) + \cdots + x^n \frac{\partial F}{\partial x^n}(p).$$

Thus $f(p) = F(p) - 1 = -1$, hence on $f^{-1}(0)$ the Jacobian J_f does not vanish. That is, rank $J_f = 1$ on $f^{-1}(0)$, so that $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$. Notice that $f^{-1}(0)$

is not empty as if F has a positive value, then it also takes all the positive values, since $F(tp) = t^r F(p)$.

1.7 Submersions. Quotient Manifolds

Problem 1.7.1. *Let* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *be given by* $f(x,y,z) = x^2 + y^2 - 1$.

(1) *Prove that* $C = f^{-1}(0)$ *is an embedded* 2*-submanifold of* $\mathbb{R}^3$.

(2) *Prove that a vector*

$$v = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \right)_{(0,1,1)}$$

is tangent to C *if and only if* $b = 0$.

(3) *If* $j\colon S^1 \to \mathbb{R}^2$ *is the inclusion map, prove that* $j \times \mathrm{id}_{\mathbb{R}}\colon S^1 \times \mathbb{R} \to \mathbb{R}^3$ *induces a diffeomorphism from* $S^1 \times \mathbb{R}$ *to* C.

Solution. (1) f is a differentiable map and rank $f_* = \operatorname{rank}(2x\ 2y\ 0)$. Hence the rank of f is 1 at every point except on $\{(0,0,z) : z \in \mathbb{R}\}$, but these points do not belong to C. Thus, by virtue of the Implicit Map Theorem for submersions 7.1.11, C is a closed embedded submanifold of $\mathbb{R}^3$ and $\dim C = \dim \mathbb{R}^3 - \dim \mathbb{R} = 2$.

(2) Given $v \in T_p\mathbb{R}^3$, $p \in C$, one has $v \in T_pC$ if and only if $v(f) = 0$, but $v(f) = (2ax + 2by)_{(0,1,1)} = 2b$, thus $v \in T_pC$ if and only if $b = 0$.

(3) $\operatorname{im}(j \times \mathrm{id}_{\mathbb{R}}) = C$, as $(x,y) \in S^1$ if and only if $x^2 + y^2 = 1$, or similarly $(x,y,z) \in C$, for all $z \in \mathbb{R}$. Hence $F = j \times \mathrm{id}_{\mathbb{R}}\colon S^1 \times \mathbb{R} \to \mathbb{R}^3$ is a differentiable map (as it is a product of differentiable maps) that can be factorized by C, which is an embedded submanifold of $\mathbb{R}^3$. That is, there exists a differentiable map f_0 that makes commutative the diagram

$$\begin{tikzcd} S^1 \times \mathbb{R} \arrow[rr, "j\times \mathrm{id}_{\mathbb{R}}"] \arrow[dr, "f_0"'] & & \mathbb{R}^3 \\ & C \arrow[ur, "i"'] & \end{tikzcd}$$

where i denotes the embedding of C in $\mathbb{R}^3$. On the other hand, $j \times \mathrm{id}_{\mathbb{R}}$ is also an embedding, since j is. Thus the map f_0^{-1} that makes commutative the diagram

$$\begin{tikzcd} C \arrow[rr, "i"] \arrow[dr, "f_0^{-1}"'] & & \mathbb{R}^3 \\ & S^1 \times \mathbb{R} \arrow[ur, "j\times \mathrm{id}_{\mathbb{R}}"'] & \end{tikzcd}$$

is C^∞. Thus f_0 is a diffeomorphism.

Problem 1.7.2. *Let* $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ *be the map given by*

$$u = x^2 + y^2 + z^2 - 1, \quad v = ax + by + cz, \qquad a,b,c \in \mathbb{R}, \quad a^2 + b^2 + c^2 = 1.$$

(1) *Find the points at which* φ *is a submersion.*
(2) *Find* $\varphi^{-1}(0)$.
(3) *Find the points where* φ *is not a submersion, and its image.*

Solution. (1)

$$\operatorname{rank} \varphi_* = \operatorname{rank} \begin{pmatrix} 2x & 2y & 2z \\ a & b & c \end{pmatrix} = 2$$

at the points (x,y,z) in which the vector (x,y,z) is not a multiple of (a,b,c). Hence φ is a submersion on $\mathbb{R}^3 - \langle (a,b,c) \rangle$.

(2) Let $\langle (a,b,c) \rangle^\perp$ denote the plane through the origin orthogonal to the vector (a,b,c). Then:

$$\begin{aligned} \varphi^{-1}(0) &= \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ ax + by + cz = 0 \right\} \\ &= S^2 \cap \langle (a,b,c) \rangle^\perp. \end{aligned}$$

(3) The map φ is not a submersion at the points of $\langle (a,b,c) \rangle$, whose image is

$$\begin{aligned} \varphi(\langle a,b,c \rangle) &= \{ (\lambda^2 - 1, \lambda) \} \\ &\subset \mathbb{R}^2 = \left\{ (u,v) \in \mathbb{R}^2 : u = v^2 + 1 \right\}. \end{aligned}$$

Problem 1.7.3. *Consider the differentiable map* $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^2$ *given by*

$$\begin{aligned} u &= x^2 + y^2 + z^2 + t^2 - 1, \\ v &= x^2 + y^2 + z^2 + t^2 - 2y - 2z + 5. \end{aligned}$$

(1) *Find the set of points of* $\mathbb{R}^4$ *where* φ *is not a submersion, and its image.*
(2) *Calculate a basis of* $\operatorname{Ker} \varphi_{*(0,1,2,0)}$.
(3) *Calculate the image by* φ_* *of* $(1,0,2,1) \in T_{(1,2,0,1)}\mathbb{R}^2$ *and the image by* φ^* *of* $(\mathrm{d}u + 2\mathrm{d}v)_{(-1,5)} \in T^*_{(-1,5)}\mathbb{R}^2$, *choosing the point* $(0,0,0,0)$ *in* $\varphi^{-1}((-1,5))$.

Solution. (1) φ is not a submersion at the points of $\mathbb{R}^4$ where

$$\operatorname{rank} \varphi_* = \operatorname{rank} \begin{pmatrix} 2x & 2y & 2z & 2t \\ 2x & 2y-2 & 2z-2 & 2t \end{pmatrix} < 2.$$

Hence, the set is

$$A = \left\{ (x,y,z,t) \in \mathbb{R}^4 : x = 0,\ y = z,\ t = 0 \right\}.$$

Therefore,

$$\varphi(A) = \left\{(u,v) \in \mathbb{R}^2 : u = 2\lambda^2 - 1,\ v = 2\lambda^2 - 4\lambda + 5,\ \lambda \in \mathbb{R}\right\}.$$

(2) We have $\varphi_*\colon T_{(0,1,2,0)}\mathbb{R}^4 \to T_{(4,4)}\mathbb{R}^2$. Every vector $X \in T_{(0,1,2,0)}$ is of the type

$$X = \lambda_1 \left.\frac{\partial}{\partial x}\right|_p + \lambda_2 \left.\frac{\partial}{\partial y}\right|_p + \lambda_3 \left.\frac{\partial}{\partial z}\right|_p + \lambda_4 \left.\frac{\partial}{\partial t}\right|_p,$$

where $p = (0,1,2,0)$. Since $\varphi_{*(0,1,2,0)} \equiv \begin{pmatrix} 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, we have

$$\varphi_{*(0,1,2,0)}X \equiv (2\lambda_2 + 4\lambda_3) \left.\frac{\partial}{\partial u}\right|_{(4,4)} + 2\lambda_3 \left.\frac{\partial}{\partial v}\right|_{(4,4)}.$$

If $X \in \operatorname{Ker} \varphi_{*p}$, we deduce $\lambda_2 = \lambda_3 = 0$. Thus

$$\operatorname{Ker} \varphi_{*p} = \left\{\lambda \left.\frac{\partial}{\partial x}\right|_p + \mu \left.\frac{\partial}{\partial t}\right|_p : \lambda, \mu \in \mathbb{R}\right\}$$

and $\left\{ \left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial t}\right|_p \right\}$ is a basis of $\operatorname{Ker} \varphi_{*p}$.

(3) $\varphi_{*(1,2,0,1)}(1,0,2,1) = 4 \left.\frac{\partial}{\partial u}\right|_{(5,7)}$. Let $p = (0,0,0,0)$, so $\varphi(p) = (-1,5)$ and $\varphi^*_{(-1,5)}(\mathrm{d}u + 2\mathrm{d}v) = -4(\mathrm{d}y + \mathrm{d}z)_{(0,0,0,0)}$.

Problem 1.7.4. *We define an equivalence relation $\sim$ in the open subset $\mathbb{R}^{n+1} - \{0\}$ by the condition that two vectors of $\mathbb{R}^{n+1} - \{0\}$ are equivalent if they are proportional. The quotient space $\mathbb{R}P^n = (\mathbb{R}^{n+1} - \{0\})/\!\sim$ is the* real projective space *of dimension n.*

(1) *Prove that, giving $\mathbb{R}P^n$ the quotient topology induced by the previous equivalence relation, it is Hausdorff.*

(2) *Let $[x^1, \dots, x^{n+1}]$ be the equivalence class in $\mathbb{R}P^n$ of $(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} - \{0\}$. For each $i = 1, 2, \dots, n+1$, let U_i be the subset of points $[x^1, \dots, x^{n+1}]$ of $\mathbb{R}P^n$ such that $x^i \neq 0$. Prove that the functions $\varphi_i\colon U_i \to \mathbb{R}^n$ defined by*

$$\varphi_i([x^1, \dots, x^{n+1}]) = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right)$$

are homeomorphisms and that the changes of coordinates $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are differentiable. Hence the systems (U_i, φ_i), $i = 1, 2, \dots, n+1$, define a differentiable structure on the space $\mathbb{R}P^n$.

(3) *Prove that the projection map*

$$\begin{array}{rccc} \pi\colon & \mathbb{R}^{n+1}-\{0\} & \to & \mathbb{R}P^n \\ & (x^1,\dots,x^{n+1}) & \mapsto & [x^1,\dots,x^{n+1}] \end{array}$$

is a submersion. Hence $\mathbb{R}P^n$ is a quotient manifold of $\mathbb{R}^{n+1}-\{0\}$.

Solution. (1) The relation $\sim$ is open, i.e. given the open subset $U\subset\mathbb{R}^{n+1}-\{0\}$, then $[U]=\bigcup_{x\in U}[x]$ is an open subset of $(\mathbb{R}^{n+1}-\{0\})\times(\mathbb{R}^{n+1}-\{0\})$. In fact, since U is open, so is $U_\lambda=\{\lambda x, x\in U\}$, $\lambda\neq 0$ being fixed, and $[U]=\bigcup_{\lambda\neq 0}U_\lambda$. Moreover, the graph of $\sim$ is the subset

$$\Gamma=\big\{(x,\lambda x)\,:\,\lambda\in\mathbb{R}-\{0\}, x\in\mathbb{R}^{n+1}-\{0\}\big\}$$

of $(\mathbb{R}^{n+1}-\{0\})\times(\mathbb{R}^{n+1}-\{0\})$. Γ is closed, as if $(x_n,\lambda_n x_n)\mapsto(x,y)$ then (λ_n) is bounded. Thus it has a convergent subsequence (λ_{n_k}). Let $\lambda=\lim_{k\to\infty}\lambda_{n_k}$. Then $y=\lim_{n\to\infty}\lambda_n x_n=\lim_{k\to\infty}\lambda_{n_k}x_{n_k}=\lambda x$. So $(x,y)\in\Gamma$. We conclude that the quotient space is Hausdorff.

(2) It is obvious that the functions φ_i are homeomorphisms. As for the changes of coordinates

$$\varphi_{ij}=\varphi_i\circ\varphi_j^{-1}\colon\ \varphi_j(U_i\cap U_j)\to\varphi_i(U_i\cap U_j),$$

we clearly have for $i<j$:

$$\varphi_j(U_i\cap U_j)=\{(t^1,\dots,t^n)\in\mathbb{R}^n : t^i\neq 0\}.$$

Furthermore,

$$\varphi_j^{-1}(t^1,\dots,t^n)=[t^1,\dots,t^{j-1},1,t^j,\dots,t^n].$$

So, for $(t^1,\dots,t^n)\in\varphi_j(U_i\cap U_j)$ we have $\varphi_j^{-1}(t^1,\dots,t^n)$ as above and moreover

$$\begin{aligned}\varphi_{ij}(t^1,\dots,t^n)&=\left(\frac{t^1}{t^i},\dots,\ \frac{t^{i-1}}{t^i},\frac{t^{i+1}}{t^i},\dots,\ \frac{t^{j-1}}{t^i},\ \frac{1}{t^i},\ \frac{t^j}{t^i},\dots,\ \frac{t^n}{t^i}\right)\\&=(x^1,\dots,x^n).\end{aligned}$$

The equations

$$\begin{aligned}x^1=\frac{t^1}{t^i},\dots,\ x^{i-1}=\frac{t^{i-1}}{t^i},\ x^i=\frac{t^{i+1}}{t^i},\dots,\\ x^{j-2}=\frac{t^{j-1}}{t^i},\ x^{j-1}=\frac{1}{t^i},\ x^j=\frac{t^j}{t^i},\dots,\ x^n=\frac{t^n}{t^i},\end{aligned}$$

correspond to differentiable functions on U_{ij}.

(Note that we have supposed $i<j$, which is not restrictive.)

(3) $\mathbb{R}^{n+1}-\{0\}$ is an open submanifold of $\mathbb{R}^{n+1}$. Using the identity chart on $\mathbb{R}^{n+1}-\{0\}$ and an arbitrarily fixed chart φ_i as in (2) above on $U_i\subset\mathbb{R}P^n$, the projection map π has on $\pi^{-1}(U_i)$ (where $x^i\neq 0$) the representative map

$$
\begin{array}{rccc}
\varphi_i \circ \pi \circ \mathrm{id}^{-1}\colon & \pi^{-1}(U_i) & \to & \mathbb{R}^n \\
& (x^1,\dots,x^{n+1}) & \mapsto & \left(\dfrac{x^1}{x^i},\dots,\dfrac{x^{i-1}}{x^i},\dfrac{x^{i+1}}{x^i},\dots,\dfrac{x^{n+1}}{x^i}\right),
\end{array}
$$

which is easily seen to have rank n. Since i is arbitrary, π is a submersion, thus concluding.

Problem 1.7.5. *Construct an atlas on the real projective space $\mathbb{R}P^n$ considered as the quotient space of the sphere S^n by identification of antipodal points. Prove that the projection map $\pi\colon S^n \to \mathbb{R}P^n$ is a submersion. Hence $\mathbb{R}P^n$ is a quotient manifold of S^n.*

HINT: *Use the atlas given by the $2n+2$ open hemispheres defined by the coordinate axes, and the canonical projections.*

Solution. As we know, $\mathbb{R}P^n$ is the quotient space of the subspace $\mathbb{R}^{n+1}-\{0\}$ of $\mathbb{R}^{n+1}$, by the relation $\sim$ given by $x\sim y$ if there exists $\lambda\in\mathbb{R}-\{0\}$ such that $x=\lambda y$. The projection $\pi\colon \mathbb{R}^{n+1}-\{0\}\to\mathbb{R}P^n$ is an open mapping. On S^n the above relation is reduced to $x\sim\pm x$, that is, $[x]=\{x,-x\}$ for every $x\in S^n$. Hence on S^n the above relation corresponds to the antipodal identification.

Consider the restriction to S^n of the projection π, that we continue denoting by $\pi\colon S^n\to\mathbb{R}P^n$, $\pi(x)=[x]$, and which is still open and surjective. In fact, given $[x]\in\mathbb{R}P^n$, then $x/|x|\in S^n$ and $\pi(x/|x|)=\pi(x)=[x]$. Hence, $\mathbb{R}P^n$ can be considered (as a topological space) as the quotient space of S^n obtained by identifying antipodal points. From which it follows, since $\pi\colon S^n\to\mathbb{R}P^n$ is continuous, that $\mathbb{R}P^n$ is compact and connected.

Notice that if $U\subset S^n$ is contained in an open hemisphere and $x\in U$, then $-x\notin U$, hence $\pi|_U\colon U\to\pi(U)$ is injective; that is, $\pi|_U$ is a homeomorphism. This property allows us to construct an atlas in $\mathbb{R}P^n$ from an atlas in S^n whose coordinate domains are contained in open hemispheres of S^n. For instance, the atlas consisting in the $2n+2$ open hemispheres defined by the coordinate hyperplanes and the canonical projections. Let, for instance, $V_i^+=\{x\in S^n : x^i>0\}$, and

$$
\begin{array}{rccc}
h_i^+\colon & V_i^+ & \to & \mathbb{R}^n \\
& (x^1,\dots,x^{n+1}) & \mapsto & (x^1,\dots,x^{i-1},x^{i+1},\dots,x^{n+1}).
\end{array}
$$

We then define in $\mathbb{R}P^n$, $V_i=\pi(V_i^+)$ and $\varphi_i^+=h_i^+\circ(\pi|_{V_i^+})^{-1}\colon V_i\to\mathbb{R}^n$. The map φ_i^+ is a homeomorphism, since it is a composition of homeomorphisms. Considering $V_i^-=\{x\in S^n : x^i<0\}$, it follows that $\pi(V_i^+)=\pi(V_i^-)=V_i$ and the similar homeomorphism is $\varphi_i^-=h_i^-\circ(\pi|_{V_i^-})^{-1}\colon V_i\to\mathbb{R}^n$. Notice that $\varphi_i^-(V_i)=\varphi_i^+(V_i)$, but $\varphi_i^-\neq\varphi_i^+$; in fact, we have $\varphi_i^-([x])=-\varphi_i^+([x])$. Since $\varphi_i^-=-\varphi_i^+$ (they differ by the diffeomorphism $t\to -t$ of $\mathbb{R}^n$) we shall forget the charts (V_i,φ_i^-), and we shall consider only the charts (V_i,φ_i^+), $i=1,\dots,n+1$. If $i\neq j$, then $V_i\cap V_j\neq\emptyset$, and moreover

$$\varphi_i^+\circ(\varphi_j^+)^{-1}\colon \varphi_j^+(V_i\cap V_j)\to\varphi_i^+(V_i\cap V_j)$$

is given by

$$\begin{aligned}\varphi_i^+ \circ (\varphi_j^+)^{-1} &= h_i^+ \circ (\pi|_{V_i^+})^{-1} \circ (\pi|_{V_j^+}) \circ (h_j^+)^{-1} \\ &= h_i^+ \circ (\pi^{-1} \circ \pi)|_{V_i^+ \cap V_j^+} \circ h_j^{+\,-1} \\ &= h_i^+ \circ (h_j^+)^{-1},\end{aligned}$$

which is differentiable since it is a change of coordinates in S^n, known to be differentiable. By the above constructions, for a given i, the projection map π has locally the representative map

$$\varphi_i^+ \circ (\pi|_{V_i^+}) \circ (h_i^+)^{-1} \colon h_i^+(V_i^+) \to \varphi_i^+(V_i),$$

which is the identity map, so having rank n. Since i is arbitrary, π is a submersion, and we have finished.

Problem 1.7.6. (The real Grassmannian as a quotient manifold) *Let*

$$M \subset \mathbb{R}^n \times \overset{(k}{\cdots} \times \mathbb{R}^n$$

be the subset of k-tuples $(v_1, \dots, v_k)$ of linearly independent vectors of $\mathbb{R}^n$. Let $GL(k,\mathbb{R})$ act on M on the right by $(v_1, \dots, v_k) \cdot A = (v'_1, \dots, v'_k)$, where

$$v'_j = a^i_j v_i, \qquad A = (a^i_j) \in GL(k,\mathbb{R}), \quad i, j = 1, \dots, k.$$

Prove:

(1) *M is an open subset of $\mathbb{R}^n \times \overset{(k}{\cdots} \times \mathbb{R}^n$.*

(2) *If $\sim$ is the equivalence relation induced by this action, then the quotient manifold $M/\!\sim$ exists and can be identified to the Grassmannian $G_k(\mathbb{R}^n)$ of all k-planes in $\mathbb{R}^n$.*

Solution. (1) Let us denote by x^i_j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$, the natural coordinates on $\mathbb{R}^n \times \overset{(k}{\cdots} \times \mathbb{R}^n$. Given $(v_1, \dots, v_k) \in M$, we write

$$X = \begin{pmatrix} x^1_1(v_1, \dots, v_k) & \cdots & x^1_k(v_1, \dots, v_k) \\ \vdots & & \vdots \\ x^n_1(v_1, \dots, v_k) & \cdots & x^n_k(v_1, \dots, v_k) \end{pmatrix}, \tag{$\star$}$$

that is, $x^i_j(v_1, \dots, v_k)$ is the ith component of the column vector v_j.

Let $\Delta_{i_1 \dots i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, denote the determinant of the $k \times k$ submatrix of $(\star)$ defined by the rows $i_1, \dots, i_k$. The subset M is open as it is defined by the inequality

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \Delta^2_{i_1 \dots i_k} > 0.$$

(2) Let $((x_j^i),(y_s^r))$, $i,r=1,\dots,n$; $j,s=1,\dots,k$, be the natural coordinates on the product manifold $M\times M$, and let z_b^a, $a,b=1,\dots,k$, be the entries of a matrix in $GL(k,\mathbb{R})$.

The graph $\mathscr{G}$ of $\sim$ is the image of the differentiable map

$$\varphi\colon M\times GL(k,\mathbb{R})\to M\times M,\qquad \varphi(X,Z)=(X,XZ).$$

The graph $\mathscr{G}$ is closed in $M\times M$, as follows by taking into account that a pair $\big((v_1,\dots,v_k),(w_1,\dots,w_k)\big)\in M\times M$ belongs to $\mathscr{G}$ if and only if $w_i\in\langle v_1,\dots,v_k\rangle$, $1\leqslant i\leqslant k$, and that every vector subspace of $\mathbb{R}^n$ is a closed subset. Hence, by applying the Theorem of the Closed Graph 7.1.13, we only need to prove that $\mathscr{G}$ is an embedded submanifold.

Certainly, φ is injective as $\varphi(X,Z)=\varphi(X',Z')$ means $X=X'$, $XZ=X'Z'$, and since rank $X=k$, the latter equation implies $Z=Z'$.

Next we prove that $\varphi\colon M\times GL(k,\mathbb{R})\to\mathscr{G}$ is a homeomorphism. Assume

$$\begin{aligned}\lim_{h\to\infty}\varphi(X_h,Z_h)&=\lim_{h\to\infty}(X_h,X_hZ_h)\\&=(X,Y).\end{aligned}$$

Hence $\lim_{h\to\infty}X_h=X$. As $\mathscr{G}$ is closed in $M\times M$, there exists $Z\in GL(k,\mathbb{R})$ such that $Y=XZ$. We only need to prove that $\lim_{h\to\infty}Z_h=Z$. Set $X_h=(v_{1,h},\dots,v_{k,h})$, $X=(v_1,\dots,v_k)$. As the vectors $v_1,\dots,v_k$ are linearly independent, we can complete them up to a basis $(v_1,\dots,v_k,\,v_{k+1},\dots,v_n)$ in $\mathbb{R}^n$. Let

$$v_{j,h}=\sum_{i=1}^{k}a_{j,h}^iv_i+\sum_{i=k+1}^{n}b_{j,h}^iv_i,\qquad 1\leqslant j\leqslant k,\tag{$\star\star$}$$

be the expression of $v_{j,h}$ in this basis. As $\lim_{h\to\infty}v_{j,h}=v_j$, we obtain $\lim_{h\to\infty}a_{j,h}^i=\delta_{ij}$, for $i,j=1,\dots,k$, and $\lim_{h\to\infty}b_{j,h}^i=0$, for $k+1\leqslant i\leqslant n$, $1\leqslant j\leqslant k$. Set $\widehat{X}=(v_{k+1},\dots,v_n)$, and let A_h, B_h be the matrices of sizes $k\times k$, $(n-k)\times k$, respectively, given by

$$A_h=(a_{j,h}^i)_{j=1,\dots,k}^{i=1,\dots,k},\qquad B_h=(b_{j,h}^i)_{1\leqslant j\leqslant k}^{k+1\leqslant i\leqslant n}.$$

Then, ($\star\star$) can be rewritten as $X_h=XA_h+\widehat{X}B_h$; hence

$$X_hZ_h=XA_hZ_h+\widehat{X}B_hZ_h,$$

and passing to the limit we obtain

$$XZ=X\lim_{h\to\infty}(A_hZ_h)+\widehat{X}\lim_{h\to\infty}(B_hZ_h).$$

Taking components we have $Z=\lim_{h\to\infty}(A_hZ_h)$ and $\lim_{h\to\infty}(B_hZ_h)=0$. Since A_h goes to the $k\times k$ identity matrix $I_k=(\delta_{ij})$ as $h\to\infty$, we can conclude.

Let us compute φ_*. We have

$$\begin{aligned}
\xi^i_j &= \varphi_* \left(\left. \frac{\partial}{\partial x^i_j} \right|_{(X,Z)} \right) \\
&= \left(\frac{\partial}{\partial x^i_j} + \sum_{s=1}^{k} z^s_j \frac{\partial}{\partial y^i_s} \right)_{(X,XZ)}, \\
\zeta^a_b &= \varphi_* \left(\left. \frac{\partial}{\partial z^a_b} \right|_{(X,Z)} \right) \\
&= \sum_{r=1}^{n} x^r_a \left. \frac{\partial}{\partial y^r_b} \right|_{(X,XZ)},
\end{aligned}$$

where $1 \leqslant i \leqslant n$ and $j, a, b = 1, \ldots, k$.

We claim that the tangent vectors ξ^i_j, ζ^a_b, are linearly independent for every $(X,Z) \in M \times GL(k,\mathbb{R})$. In fact, if

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \lambda^i_j \xi^i_j + \sum_{a,b=1}^{k} \mu^a_b \zeta^a_b = 0, \tag{$\star\star\star$}$$

for some scalars λ^i_j, μ^a_b, then by applying the equation $(\star\star\star)$ to the function x^i_j, we obtain $\lambda^i_j = 0$. Hence, this equation reduces to

$$\sum_{a,b=1}^{k} \mu^a_b \sum_{r=1}^{n} x^r_a \left. \frac{\partial}{\partial y^r_b} \right|_{(X,XZ)} = 0,$$

or else,

$$\sum_{r=1}^{n} \sum_{b=1}^{k} \left(\sum_{a=1}^{k} x^r_a \mu^a_b \right) \left. \frac{\partial}{\partial y^r_b} \right|_{(X,XZ)} = 0.$$

Hence

$$\begin{pmatrix} x^1_1 & \cdots & x^1_k \\ \vdots & & \vdots \\ x^n_1 & \cdots & x^n_k \end{pmatrix} \begin{pmatrix} \mu^1_1 & \cdots & \mu^1_k \\ \vdots & & \vdots \\ \mu^k_1 & \cdots & \mu^k_k \end{pmatrix} = 0.$$

As $\operatorname{rank}(x^i_j) = k$, the previous equality implies $(\mu^a_b) = 0$.

Finally, let us show that $M/\sim$ can be identified to the Grassmannian. We have a natural surjective map

$$\Psi \colon M \to G_k(\mathbb{R}^n), \qquad \Psi(v_1, \ldots, v_k) = \langle v_1, \ldots, v_k \rangle.$$

We have $\dim \langle v_1, \ldots, v_k \rangle = k$ as $v_1, \ldots, v_k$ are linearly independent. Moreover,

$$\Psi(v_1, \ldots, v_k) = \Psi(v'_1, \ldots, v'_k) = V$$

if and only if $\{v_1,\dots,v_k\}$ and $\{v'_1,\dots,v'_k\}$ are two bases of V. Hence there exists $A \in GL(k,\mathbb{R})$ such that $(v'_1,\dots,v'_k) = (v_1,\dots,v_k)\cdot A$, thus proving that the fibres of Ψ are exactly the orbits of $GL(k,\mathbb{R})$.

Problem 1.7.7. *Let $\pi\colon M \to N$ be a differentiable map. Prove that π is a submersion if and only if it admits local sections through each point,* i.e. *for every $q_0 = \pi(p_0)$, $p_0 \in M$, there exist an open neighborhood V of q_0 in N, and a differentiable map $\sigma\colon V \to M$ such that*:

(1) $\sigma(q_0) = p_0$.

(2) $\pi \circ \sigma = \mathrm{id}_V$.

Solution. From (2) we have $\pi_{*p_0} \circ \sigma_{*q_0} = \mathrm{id}_{T_{q_0}N}$. Since the identity map is surjective, $\pi_*\colon T_{p_0}M \to T_{q_0}N$ is surjective. Conversely, if π is a submersion at p_0, by the Theorem of the Rank 7.1.8, there exist local coordinates $(x^1,\dots,x^m)$, $(y^1,\dots,y^n)$, centered at p_0, q_0 in M, N, respectively, such that $y^i \circ \pi = x^i$, $1 \leqslant i \leqslant n$. Notice that $m \geqslant n$, as π is a submersion. Hence we can define a map σ on the domain of $(y^1,\dots,y^n)$ by setting

$$x^i \circ \sigma = \begin{cases} y^i & \text{if } 1 \leqslant i \leqslant n \\ 0 & \text{if } n+1 \leqslant i \leqslant m. \end{cases}$$

Then, for every $i = 1,\dots,n$, we have

$$\begin{aligned} y^i \circ (\pi \circ \sigma) &= (y^i \circ \pi) \circ \sigma \\ &= x^i \circ \sigma \\ &= y^i, \end{aligned}$$

thus proving that σ is a local section of π.

1.8 The Tangent Bundle

Problem 1.8.1. *Prove that if σ is a C^∞ curve in the C^∞ manifold M, then the tangent vector field σ' is a C^∞ curve in the tangent bundle TM.*

Solution. Given a C^∞ curve $\sigma\colon \mathbb{R} \to M$, the tangent vector field σ' is, by definition, a map that we can write as $\sigma' = \sigma_* \circ \dfrac{\mathrm{d}}{\mathrm{d}t}\colon \mathbb{R} \to TM$, where $\mathrm{d}/\mathrm{d}t$ denotes the canonical vector field on $\mathbb{R}$ and hence it can be considered as a curve in TM (see Figure 1.24), so that for a coordinate neighborhood $U \subset M$ with coordinate functions $x^1,\dots,x^n$, one has

$$\sigma_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s\right) = \frac{\mathrm{d}(x^i \circ \sigma)}{\mathrm{d}t}(s)\, \left.\frac{\partial}{\partial x^i}\right|_{\sigma(s)}.$$

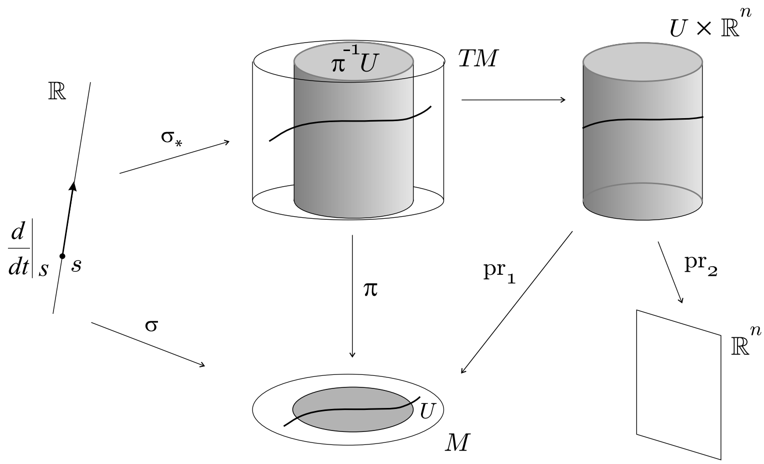

Fig. 1.24 The tangent vector field σ' to a curve σ as a curve in TM.

Thus,

$$\sigma'(s) = \left((x^1 \circ \sigma)(s), \ldots, (x^1 \circ \sigma)(s), \frac{\mathrm{d}(x^1 \circ \sigma)}{\mathrm{d}t}(s), \ldots, \frac{\mathrm{d}(x^n \circ \sigma)}{\mathrm{d}t}(s) \right).$$

The coordinate functions $\{x^i\}$ and σ are C^∞. Hence the composition $x^i \circ \sigma$ is C^∞ for each $i = 1, \ldots, n$; and the functions $\mathrm{d}(x^i \circ \sigma)/\mathrm{d}t$, which are coordinate functions on the open subset $\pi^{-1}(U)$ of TM, are also C^∞.

Problem 1.8.2. *Let M and N be C^∞ manifolds and let $p \in M$, $q \in N$. Prove that there exists a natural isomorphism*

$$T_{(p,q)}(M \times N) \approx T_pM \times T_qN.$$

Solution. Let $\text{pr}_1 \colon M \times N \to M$, $\text{pr}_2 \colon M \times N \to N$ denote the projection maps, and

$$i_q \colon M \to M \times N, \quad x \mapsto (x, q), \qquad i_p \colon N \to M \times N, \quad y \mapsto (p, y),$$

the inclusion maps. The map

$$\varphi \colon T_{(p,q)}(M \times N) \to T_pM \oplus T_qN, \qquad v \mapsto (\text{pr}_{1*}v, \text{pr}_{2*}v),$$

is an isomorphism. In fact, it is immediate that it is linear. Moreover, letting

$$\psi \colon T_pM \oplus T_qN \to T_{(p,q)}(M \times N), \quad (v_1, v_2) \mapsto i_{q*}v_1 + i_{p*}v_2,$$

we have

$$(\varphi \circ \psi)(v_1, v_2) = \big(\text{pr}_{1*}(i_{q*}v_1 + i_{p*}v_2), \text{pr}_{2*}(i_{q*}v_1 + i_{p*}v_2)\big).$$

From

$$\mathrm{pr}_1 \circ i_q = \mathrm{id}, \qquad \mathrm{pr}_1 \circ i_p = \mathrm{const}, \qquad \mathrm{pr}_2 \circ i_q = \mathrm{const}, \qquad \mathrm{pr}_2 \circ i_p = \mathrm{id},$$

it follows that $(\varphi \circ \psi)(v_1, v_2) = (v_1, v_2)$.

Since $\dim T_{(p,q)}(M \times N) = \dim (T_p(M) \oplus T_q(N))$, the conclusion follows.

Problem 1.8.3. *Assume that the manifold M admits a basis $\{X_1, \dots, X_n\}$ for the $(C^\infty M)$-module $\mathfrak{X}(M)$ of C^∞ vector fields on M.*

Prove that the map

$$\begin{array}{ccc} M \times \mathbb{R}^n & \xrightarrow{F} & TM = \bigcup_{p \in M} T_pM \\ (p, a^1, \dots, a^n) & \mapsto & F(p, a^1, \dots, a^n) = a^i \, X_i|_p \in T_pM \end{array}$$

is a diffeomorphism. That is, that TM is then trivial.

REMARK. *Compare* Problem 2.1.2.

Solution. To begin with, we prove that for every $p \in M$, the tangent vectors $X_1|_p, \dots, X_n|_p$ are linearly independent and hence they are a basis of T_pM. Let $(U, x^1, \dots, x^n)$ be a coordinate system defined on an open neighborhood U of p, and let $f \in C^\infty M$ be a function such that:

(a) $f = 1$ on an open neighborhood $V \subset U$.

(b) $\operatorname{supp} f \subset U$.

Then $f\partial/\partial x^i$ defines a global vector field. Hence there exists an $n \times n$ matrix with entries $f_i^h \in C^\infty M$ such that $f\partial/\partial x^i = f_i^h X_h$. Evaluating at p we obtain that

$$(\partial/\partial x^i)_p = f_i^h(p) X_h|_p. \qquad (\star)$$

Moreover, as $(\partial/\partial x^1)_p, \dots, (\partial/\partial x^n)_p$ is a basis of T_pM, there exist scalars λ_h^i such that $X_h|_p = \lambda_h^i (\partial/\partial x^i)_p$, and substituting this expression into $(\star)$ we obtain $(\partial/\partial x^j)_p = f_j^h(p)\lambda_h^i(\partial/\partial x^i)_p$. As $\{(\partial/\partial x^i)_p\}$ is a basis, we conclude $f_j^h(p)\lambda_h^i = \delta_j^i$, thus proving that the matrix $(f_j^h(p))$ is invertible.

Moreover,

(1) F is injective, as if

$$F(p, a^1, \dots, a^n) = F(p', \bar{a}^1, \dots, \bar{a}^n),$$

it follows that $p = p'$. Furthermore, $a^i X_i|_p = \bar{a}^i X_i|_p$, from which, since the $X_i|_p$ are a basis of T_pM, we have $a^i = \bar{a}^i$ for every $i = 1, \dots, n$.

(2) F is surjective, since each $v \in T_pM$ is of the form $v = \lambda^i X_i|_p$, that is, $v = F(p, \lambda^1, \dots, \lambda^n)$.

(3) F is differentiable. In fact, let (U, φ) be a chart around $p \in M$, with $\varphi = (x^1, \dots, x^n)$, and consider the associated chart $(\pi^{-1}(U), \Phi)$ in TM, where

$$\pi \colon TM \to M, \quad v \mapsto \pi(v) = p, \qquad v \in T_pM,$$

that is,

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & \varphi(U)\times\mathbb{R}^n \\ v=\lambda^i \left.\dfrac{\partial}{\partial x^i}\right|_p & \mapsto & \Phi(v)=(\varphi(p),\lambda^1,\dots,\lambda^n). \end{array}$$

Now, as $X_1,\dots,X_n$ are C^∞ vector fields, we have $X_i|_U = f_i^j \dfrac{\partial}{\partial x^j}$, where $f_i^j\colon U\to\mathbb{R}$ are C^∞ functions. Hence, given $t=(t^1,\dots,t^n)\in\varphi(U)$ such that $\varphi(p)=(t^1,\dots,t^n)$, one has

$$\begin{aligned}\big(\Phi\circ F\circ(\varphi\times\mathrm{id}_{\mathbb{R}^n})^{-1}\big)(t^1,\dots,t^n,a^1,\dots,a^n) &= (\Phi\circ F)(p,a^1,\dots,a^n)\\ &=\Phi\big(a^i X_i|_p\big)\\ &=\Phi\left(a^i f_i^j\left(\varphi^{-1}(t)\right)\left.\frac{\partial}{\partial x^j}\right|_{\varphi^{-1}(t)}\right)\\ &=\big(t^1,\dots,t^n,a^i f_i^1(\varphi^{-1}(t)),\dots,a^i f_i^n(\varphi^{-1}(t))\big).\end{aligned}$$

Thus $\Phi\circ F\circ(\varphi\times\mathrm{id}_{\mathbb{R}^n})^{-1}$ is C^∞, hence F is C^∞.

Moreover F^{-1} is C^∞. In fact,

$$\begin{aligned}&\big((\varphi\times\mathrm{id}_{\mathbb{R}^n})\circ F^{-1}\circ\Phi^{-1}\big)(t^1,\dots,t^n,\lambda^1,\dots,\lambda^n)\\ &\quad=\big((\varphi\times\mathrm{id}_{\mathbb{R}^n})\circ F^{-1}\big)\left(\lambda^i\left.\frac{\partial}{\partial x^i}\right|_{\varphi^{-1}(t)}\right)\\ &\quad=\big((\varphi\times\mathrm{id}_{\mathbb{R}^n})\circ F^{-1}\big)\big(\lambda^i\tilde f_i^j(\varphi^{-1}(t))\,X_j|_{\varphi^{-1}(t)}\big)\\ &\quad=(\varphi\times\mathrm{id}_{\mathbb{R}^n})\big(\varphi^{-1}(t),\lambda^i\tilde f_i^1(\varphi^{-1}(t)),\dots,\lambda^i\tilde f_i^n(\varphi^{-1}(t))\big)\\ &\quad=\big(t^1,\dots,t^n,\lambda^i\tilde f_i^1(\varphi^{-1}(t)),\dots,\lambda^i\tilde f_i^n(\varphi^{-1}(t))\big),\end{aligned}$$

where $(\tilde f_j^i)=(f_j^i)^{-1}$. Hence $(\varphi\times\mathrm{id}_{\mathbb{R}^n})\circ F^{-1}\circ\Phi^{-1}$ is C^∞ and thus F^{-1} is C^∞.

Problem 1.8.4. *Let $j\colon S^2\to\mathbb{R}^3$ be the natural inclusion map. Prove that the map $j_*\colon TS^2\to T\mathbb{R}^3$ is an embedding.*

Solution. Let $U=\mathbb{R}^3-\{(0,0,0)\}$. As $j(S^2)\subset U$, we have $j_*TS^2\subset TU$, and since U is open in $\mathbb{R}^3$, it suffices to prove that $j_*\colon TS^2\to TU$ is an embedding. Consider the map

$$\varphi\colon U\to S^2\times\mathbb{R}^+,\qquad \varphi(x)=\left(\frac{x}{|x|},|x|\right).$$

Then, φ is a diffeomorphism whose inverse map is $\varphi^{-1}(y,\lambda)=\lambda y$, $\lambda\in\mathbb{R}^+$, $y\in S^2$. One has $(\varphi\circ j)(y)=(y,1)$, for all $y\in S^2$. Hence $\varphi_*\circ j_*=(\varphi\circ j)_*$ establishes a diffeomorphism between TS^2 and $T(S^2\times\{1\})\subset TS^2\times T\mathbb{R}^+$. As φ_* is a diffeomorphism we conclude that j_* is a diffeomorphism between TS^2 and the closed submanifold $\varphi^{-1}\big(T(S^2\times\{1\})\big)\subset TU$.

1.9 Vector Fields

1.9.1 Working with Vector Fields

Problem 1.9.1. *Consider the vector fields*

$$X = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}, \qquad Y = y\frac{\partial}{\partial y},$$

on $\mathbb{R}^3$ *and the map* $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 y$. *Compute*:

$$(1)\ [X,Y]_{(1,1,0)}, \quad (2)\ (fX)_{(1,1,0)}, \quad (3)\ (Xf)(1,1,0), \quad (4)\ f_*(X_{(1,1,0)}).$$

Solution. (1)

$$[X,Y]_{(1,1,0)} = \left(-yx\frac{\partial}{\partial x}\right)_{(1,1,0)} = -\left.\frac{\partial}{\partial x}\right|_{(1,1,0)}.$$

(2)

$$(fX)_{(1,1,0)} = f(1,1,0)X_{(1,1,0)} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)_{(1,1,0)}.$$

(3)

$$(Xf)(1,1,0) = X_{(1,1,0)}f = \left(\frac{\partial f}{\partial x}\right)(1,1,0) = 2.$$

(4)

$$f_*(X_{(1,1,0)}) \equiv \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix}_{(1,1,0)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_1,$$

where t denotes the canonical coordinate on $\mathbb{R}$.

Problem 1.9.2. *Write in cylindrical coordinates the vector field on* $\mathbb{R}^3$ *defined by*

$$X = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$

Solution. The change from cylindrical coordinates (ρ, θ, z) to Cartesian coordinates is $x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = z$. The Jacobian of this transformation is

$$A = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The field X is written in cylindrical coordinates as

$$X = f_1(\rho,\theta,z)\frac{\partial}{\partial\rho} + f_2(\rho,\theta,z)\frac{\partial}{\partial\theta} + f_3(\rho,\theta,z)\frac{\partial}{\partial z}.$$

Therefore

$$\begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Hence

$$X = (2\cos\theta - \sin\theta)\frac{\partial}{\partial\rho} - \frac{2\sin\theta + \cos\theta}{\rho}\frac{\partial}{\partial\theta} + 3\frac{\partial}{\partial z}.$$

Problem 1.9.3. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the C^∞ function defined by $f(x,y,z) = x^2+y^2-1$, which defines a differentiable structure on $S = f^{-1}(0)$. Consider the vector fields on $\mathbb{R}^3$:*

$$\text{(a)}\quad X = (x^2-1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, \qquad \text{(b)}\quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}.$$

Are they tangent to S?

HINT: *If $p \in S$ and $X \in T_p\mathbb{R}^3$, X is tangent to the submanifold S if and only if $Xf = 0$.*

Solution. (a)

$$\begin{aligned} Xf &= (x^2-1)\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + xz\frac{\partial f}{\partial z} \\ &= 2x(x^2+y^2-1). \end{aligned}$$

Thus if $p = (x,y,z) \in S$, $X_p f = 0$. Hence X is tangent to S.

(b) $Yf = 2x^2 + 2y^2$. If $p = (x,y,z) \in S$, then $Y_p f = 2$, so Y is not tangent to S.

Problem 1.9.4. *Find the tangent plane to the one-sheet hyperboloid $H \equiv x^2+y^2-z^2 = 1$ at a generic point of itself.*

Solution. Consider the parametrization

$$x = \cosh u \sin v, \qquad y = \cosh u \cos v, \qquad z = \sinh u.$$

We have on the hyperboloid:

$$\begin{aligned} \frac{\partial}{\partial u} &= \sinh u \sin v\frac{\partial}{\partial x} + \sinh u \cos v\frac{\partial}{\partial y} + \cosh u\frac{\partial}{\partial z}, \\ \frac{\partial}{\partial v} &= \cosh u \cos v\frac{\partial}{\partial x} - \cosh u \sin v\frac{\partial}{\partial y}, \end{aligned}$$

that is, $\partial/\partial u$ and $\partial/\partial v$ are respectively the restrictions to the hyperboloid of the vector fields on $\mathbb{R}^3 - \{0\}$ given by

$$X = \frac{xz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} + \sqrt{x^2+y^2}\frac{\partial}{\partial z},$$

$$Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Hence, for $p = (x_0, y_0, z_0) \in H$,

$$\begin{aligned} T_pH &= \left\{\lambda \left.\frac{\partial}{\partial u}\right|_p + \mu \left.\frac{\partial}{\partial v}\right|_p, \ \lambda, \mu \in \mathbb{R}\right\} \\ &= \left\{aX_p + bY_p \, : \, a, b \in \mathbb{R}\right\} \\ &= \left\{\left(a\frac{x_0z_0}{\sqrt{x_0^2+y_0^2}} + by_0\right) \left.\frac{\partial}{\partial x}\right|_p + \left(\frac{y_0z_0}{\sqrt{x_0^2+y_0^2}} - bx_0\right) \left.\frac{\partial}{\partial y}\right|_p \right. \\ &\qquad \left. + a\sqrt{x_0^2+y_0^2}\left.\frac{\partial}{\partial z}\right|_p, \ a, b \in \mathbb{R}\right\}. \end{aligned}$$

Problem 1.9.5. *Find the tangent space at the point $p = (1,1,1)$ to the surface S in $\mathbb{R}^3$ defined by the equation $f \equiv x^3 - y^3 + xyz - xy = 0$.*

Solution. One has

$$\mathrm{d}f = (3x^2 + yz - y)\,\mathrm{d}x + (-3y^2 + xz - x)\,\mathrm{d}y + xy\,\mathrm{d}z.$$

So, $(\mathrm{d}f)p = (3\,\mathrm{d}x - 3\,\mathrm{d}y + \mathrm{d}z)_p$.

If $X = \left(\lambda_1\frac{\partial}{\partial x} + \lambda_2\frac{\partial}{\partial y} + \lambda_3\frac{\partial}{\partial z}\right)_p$ is a vector tangent to S, then $\mathrm{d}f(X) = 0$, and conversely. So, at $(1,1,1)$ we must have $\lambda_3 = -3\lambda_1 + 3\lambda_2$. Hence,

$$\begin{aligned} X_p &= \lambda_1 \left.\frac{\partial}{\partial x}\right|_p + \lambda_2 \left.\frac{\partial}{\partial y}\right|_p + (-3\lambda_1 + 3\lambda_2) \left.\frac{\partial}{\partial z}\right|_p \\ &= \lambda_1 \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial z}\right)_p + \lambda_2 \left(\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)_p, \end{aligned}$$

so the vectors $\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial z}\right)_p$ and $\left(\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)_p$ are a basis of the tangent space to S at $(1,1,1)$.

Problem 1.9.6. *Show that the vector fields X, Y, Z given by*

$$X_p = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \right)_p,$$

$$Y_p = \left(-z\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial t} \right)_p,$$

$$Z_p = \left(-t\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t} \right)_p,$$

where $p \in S^3 = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 = 1\}$, *define a global parallelization of* S^3.

Solution. The vector fields are tangent to S^3, as $\langle X_p, N_p \rangle = \langle Y_p, N_p \rangle = \langle Z_p, N_p \rangle = 0$, where N_p denotes the unit normal vector to S^3 at p,

$$N_p = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} \right)_p,$$

and $\langle\ ,\ \rangle$ denotes the Euclidean product of $T_p\mathbb{R}^4 \equiv \mathbb{R}^4$.

Furthermore, the fields are linearly independent, as

$$\operatorname{rank} \begin{pmatrix} -y & x & -t & z \\ -z & t & x & -y \\ -t & -z & y & x \end{pmatrix} < 3$$

if and only if $p = (0,0,0,0) \notin S^3$.

The fields X, Y, Z are the restriction to S^3 of the fields written similarly on $\mathbb{R}^4$, which are C^∞ on $\mathbb{R}^4$. Since S^3 is an embedded submanifold in $\mathbb{R}^4$, the vector fields given on S^3 are C^∞ on S^3.

Problem 1.9.7. *Give a C^∞ nonvanishing vector field on the sphere S^{2n+1}.*

Solution. $S^{2n+1} = \{p = (x^1,\dots,x^{2n+2}) \in \mathbb{R}^{2n+1} : \sum_{i=1}^{2n+1}(x^i)^2 = 1\}$.

The vector field X defined by

$$X_p = -x^2\,\frac{\partial}{\partial x^1}\bigg|_p + x^1\,\frac{\partial}{\partial x^2}\bigg|_p + \cdots - x^{2n+2}\,\frac{\partial}{\partial x^{2n+1}}\bigg|_p + x^{2n+1}\,\frac{\partial}{\partial x^{2n+2}}\bigg|_p,$$

where $p \in S^{2n+1}$, is tangent to S^2. In fact, it is clearly orthogonal to the normal vector

$$N_p = x^i\,\frac{\partial}{\partial x^i}\bigg|_p$$

at p with respect to the Euclidean product $\langle\ ,\ \rangle$ of $\mathbb{R}^{2n+1}$. Moreover, X is C^∞ since the functions x^i, $i = 1,\dots,2n+2$, are C^∞. Hence $X \in \mathfrak{X}(S^{2n+1})$.

Problem 1.9.8. *Find the general expression for $X \in \mathfrak{X}(\mathbb{R}^2)$ in the following cases*:

(1) $\left[\dfrac{\partial}{\partial x}, X\right] = X$ *and* $\left[\dfrac{\partial}{\partial y}, X\right] = X.$

(2) $\left[\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}, X\right] = X.$

HINT (to (2)): *Take new coordinates* $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$.

Solution. (1) Let $X = a(x,y)\dfrac{\partial}{\partial x} + b(x,y)\dfrac{\partial}{\partial y}$. Then,

$$\begin{aligned}\left[\frac{\partial}{\partial x}, X\right] &= \frac{\partial a(x,y)}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b(x,y)}{\partial x}\frac{\partial}{\partial y}\\ &= X,\\ \left[\frac{\partial}{\partial y}, X\right] &= \frac{\partial a(x,y)}{\partial y}\frac{\partial}{\partial x} + \frac{\partial b(x,y)}{\partial y}\frac{\partial}{\partial x}\\ &= X,\end{aligned}$$

from which

$$\frac{\partial a(x,y)}{\partial x} = a(x,y), \qquad \frac{\partial b(x,y)}{\partial x} = b(x,y), \tag{$\star$}$$

$$\frac{\partial a(x,y)}{\partial y} = a(x,y), \qquad \frac{\partial b(x,y)}{\partial y} = b(x,y). \tag{$\star\star$}$$

Solving, from ($\star$) we have

$$a(x,y) = Af(y)\mathrm{e}^x, \qquad b(x,y) = Bg(y)\mathrm{e}^x.$$

Substituting these expressions in ($\star\star$) one has

$$f'(y) = f(y), \qquad g'(y) = g(y),$$

from which $f(y) = C\mathrm{e}^y$, $g(y) = D\mathrm{e}^y$. Hence

$$a(x,y) = E\mathrm{e}^{x+y}, \qquad b(x,y) = F\mathrm{e}^{x+y},$$

and

$$X = \mathrm{e}^{x+y}\left(E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y}\right).$$

(2) Taking u and v as in the hint, we have $\dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$, and one can write $X = a(u,v)\dfrac{\partial}{\partial u} + b(u,v)\dfrac{\partial}{\partial v}$. We have

$$\left[\frac{\partial}{\partial u}, X\right] = \frac{\partial a(u,v)}{\partial u}\frac{\partial}{\partial u} + \frac{\partial b(u,v)}{\partial u}\frac{\partial}{\partial v} = X,$$

from which
$$\frac{\partial a(u,v)}{\partial u} = a(u,v), \qquad \frac{\partial b(u,v)}{\partial u} = b(u,v).$$
Hence, as in (1) above, we have $a(u,v) = f(v)\mathrm{e}^u$, $b(u,v) = g(v)\mathrm{e}^u$. So
$$X = f\left(\frac{1}{2}(x-y)\right)\mathrm{e}^{(x+y)/2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + g\left(\frac{1}{2}(x-y)\right)\mathrm{e}^{(x+y)/2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right),$$
that is,
$$X = \mathrm{e}^{(x+y)/2}\left\{h\left(\frac{1}{2}(x-y)\right)\frac{\partial}{\partial x} + k\left(\frac{1}{2}(x-y)\right)\frac{\partial}{\partial y}\right\},$$
for arbitrary C^∞ functions h,k.

Problem 1.9.9. *Consider the two vector fields on $\mathbb{R}^{n+1}$ defined by*
$$e_0 = \partial_0, \qquad e_1 = \sum_\alpha f_\alpha(x^i)\partial_\alpha,$$
where $\partial_0 = \partial/\partial x^0$ and
$$\partial_\alpha = \partial/\partial x^\alpha, \quad 1 \leqslant \alpha \leqslant n, \qquad f_\alpha(x^i) = f_\alpha(x^0,\dots,x^n), \quad 0 \leqslant i \leqslant n.$$
We define recursively $e_r = [e_0, e_{r-1}]$, $2 \leqslant r \leqslant n$. Then:

(1) *Compute e_r in terms of the vector fields ∂_α.*

(2) *Find functions $f_\alpha(x^i)$, such that $e_0,\dots,e_n$ are linearly independent.*

Solution. (1) $e_2 = \left[\partial_0, \sum_\alpha f_\alpha(x^i)\partial_\alpha\right] = \sum_\alpha \partial_0(f_\alpha(x^i))\partial_\alpha$.

We proceed by induction: Suppose $e_r = \sum_\alpha \partial_0^{r-1}(f_\alpha(x^i))\partial_\alpha$; then,
$$\begin{aligned} e_{r+1} &= [\partial_0, e_r] \\ &= \sum_\alpha \left[\partial_0, \partial_0^{r-1}(f_\alpha(x^i))\partial_\alpha\right] \\ &= \sum_\alpha \partial_0(\partial_0^{r-1}(f_\alpha(x^i))\partial_\alpha \\ &= \sum_\alpha \partial_0^r(f_\alpha(x^i))\partial_\alpha. \end{aligned}$$

(2) Take $f_\alpha(x^i) = (x^0)^{\alpha-1}$. Then
$$\begin{aligned} e_0 &= \partial_0, \\ e_1 &= \sum_\alpha f_\alpha(x^i)\partial_\alpha \\ &= \sum_\alpha (x^0)^{\alpha-1}\partial_\alpha \\ &= \partial_1 + x^0\partial_2 + \dots + (x^0)^{n-1}\partial_n, \\ e_2 &= \sum_\alpha \partial_0(f_\alpha(x^i))\partial_\alpha \end{aligned}$$

$$
\begin{aligned}
&= \sum_\alpha \partial_0((x^0)^{\alpha-1})\partial_\alpha \\
&= \sum(\alpha-1)((x^0)^{\alpha-2})\partial_\alpha \\
&= \partial_2 + 2x^0\partial_3 + 3(x^0)^2\partial_4 + \cdots + (n-1)(x^0)^{n-2}\partial_n, \\
e_3 &= \sum_\alpha \partial_0^2(f_\alpha(x^i))\partial_\alpha \\
&= \sum_\alpha (\alpha-1)(\alpha-2)(x^0)^{\alpha-3}\partial_\alpha, \\
&\ldots, \\
e_n &= \sum_\alpha (\alpha-1)(\alpha-2)\ldots(\alpha-n+1)(x^0)^{\alpha-n}\partial_\alpha \\
&= (n-1)!\,\partial_n.
\end{aligned}
$$

Problem 1.9.10. *Prove that if $(U,\varphi) = (U, x^1, \ldots, x^n)$ is a coordinate system on a C^∞ manifold M, then on U we have*

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \qquad i, j = 1, \ldots, n.$$

Solution. It suffices to prove $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_p (f) = 0$, for every function $f \in C^\infty U$, $p \in U$. If $t^1, \ldots, t^n$ denote the usual coordinates on $\mathbb{R}^n$, we recall that $\frac{\partial f}{\partial x^i}$ is defined as $\frac{\partial f}{\partial x^i} = \frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\circ\varphi$, hence

$$
\begin{aligned}
\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_p (f) &= \left.\frac{\partial}{\partial x^i}\right|_p \left(\frac{\partial(f\circ\varphi^{-1})}{\partial t^j}\circ\varphi\right) - \left.\frac{\partial}{\partial x^j}\right|_p \left(\frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\circ\varphi\right) \\
&= \left.\frac{\partial}{\partial t^i}\right|_{\varphi(p)} \left(\frac{\partial(f\circ\varphi^{-1})}{\partial t^j}\circ\varphi\circ\varphi^{-1}\right) \\
&\quad - \left.\frac{\partial}{\partial t^j}\right|_{\varphi(p)} \left(\frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\circ\varphi\circ\varphi^{-1}\right) \\
&= \left(\frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i\partial t^j} - \frac{\partial^2(f\circ\varphi^{-1})}{\partial t^j\partial t^i}\right)_{\varphi(p)} = 0,
\end{aligned}
$$

due to the independence of the order of partial derivatives.

Problem 1.9.11. *Let X, Y be vector fields on the C^∞ manifold M.*

(1) *Find the relation between $[fX, gY]$ and $fg[X,Y]$, where $f, g \in C^\infty M$.*

(2) *Find the expression in local coordinates of $[X,Y]$.*

Solution. (1)

$$[fX, gY]h = (fX)((gY)h) - (gY)((fX)h)$$

$$= fX(g)Yh + fgX(Yh) - gY(f)Xh - gfY(Xh)$$
$$= fX(g)Yh - gY(f)Xh + fg[X,Y]h.$$

Hence $[fX, gY] = fX(g)Y - gY(f)X + fg[X,Y]$.

(2) Let $(U, x^1, \dots, x^n)$ be coordinate system such that $X = X^i \dfrac{\partial}{\partial x^i}$ and $Y = Y^j \dfrac{\partial}{\partial y^j}$ on U. Then for any $f \in C^\infty M$, we have

$$\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j}\right) - Y^j \frac{\partial}{\partial x^j}\left(X^i \frac{\partial f}{\partial x^i}\right) \\
&= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\
&= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\frac{\partial f}{\partial x^i},
\end{aligned}$$

due to the independence of the order of partial derivatives. Hence,

$$[X,Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$$

with respect to $\{\partial/\partial x^i\}$.

1.9.2 Integral Curves

Problem 1.9.12. *Is every vector field on the real line $\mathbb{R}$ complete?*

Solution. Let $X = x^2 \mathrm{d}/\mathrm{d}x \in \mathfrak{X}(\mathbb{R})$. The integral curves are the solutions of the equation $x'(t) = x^2(t)$. That is, $x'(t)/x^2(t) = 1$, whose solution is $x(t) = -1/(t+A)$. The integral curve through x_0 verifies $x(0) = x_0$, hence $x_0 = -1/A$, thus it is the curve

$$x(t) = \frac{x_0}{1 - t x_0},$$

which is not defined for $t = 1/x_0$, so X is not complete.

Problem 1.9.13. *Compute the integral curves of the vector field on $\mathbb{R}^3$ given by*

$$X = y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}.$$

Solution. The tangent vector at a point p of an integral curve γ of the vector field X coincides with the value of X at p.

Let $\gamma(t) = (x(t), y(t), z(t))$. Hence, $\gamma'(t) = (x'(t), y'(t), z'(t))$, where

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = y, \qquad \frac{dz}{dt} = 2,$$

from which the integral curves are of the type $\gamma(t) = (Ae^t + B, Ae^t, 2t + C)$ and the curve passing through (x_0, y_0, z_0) for $t = 0$ is

$$\gamma(t) = \bigl(x_0 + y_0(e^t - 1),\ y_0 e^t,\ 2t + z_0\bigr).$$

Problem 1.9.14. *For each of the following vector fields find its integral curves and study whether it is complete or not*:

(1) $X = \dfrac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2 - \{0\})$, (2) $X = \dfrac{\partial}{\partial y} + e^x \dfrac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$,

(3) $X = e^{-x} \dfrac{\partial}{\partial x}$, (4) $X = y\dfrac{\partial}{\partial x}$, $Y = \dfrac{x^2}{2}\dfrac{\partial}{\partial y}$, $[X, Y]$,

(5) $X = x\dfrac{\partial}{\partial x}$, (6) $X = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$.

The last four vector fields belong to $\mathfrak{X}(\mathbb{R}^2)$.

Solution. (1) The integral curves are the solutions of the system

$$x'(t) = 1, \qquad y'(t) = 0;$$

thus,

$$x(t) = t + A, \qquad y(t) = B,$$

and the integral curve of X through a given point (x_0, y_0) is

$$x(t) = t + x_0, \qquad y(t) = y_0.$$

If $x_0 > 0$, the maximal integral curve through $(x_0, 0)$ is defined only for the interval $(-x_0, +\infty)$. Hence X is not complete.

(2) The integral curves are the solutions of the system

$$x'(t) = 0, \qquad y'(t) = 1, \qquad z'(t) = e^{x(t)};$$

thus,

$$x(t) = A, \qquad y(t) = t + B, \qquad z(t) = e^A t + C.$$

The integral curve of X through (x_0, y_0, z_0) is

$$x(t) = x_0, \qquad y(t) = t + y_0, \qquad z(t) = e^{x_0} t + z_0,$$

which is defined for $t \in \mathbb{R}$, so X is complete.

(3) The integral curves are the solutions of the system

$$e^{x(t)} x'(t) = 1, \qquad y'(t) = 0;$$

thus,

$$e^{x(t)} = t + A, \qquad y(t) = B;$$

that is,

$$x(t) = \log(t + A), \qquad y(t) = B.$$

The integral curve of X through (x_0, y_0) is

$$x(t) = \log(t + e^{x_0}), \qquad y(t) = y_0.$$

X is not complete as this curve is only defined for $t \in (-e^{x_0}, +\infty)$.

(4) The integral curves of X are the solutions of the system

$$x'(t) = y(t), \qquad y'(t) = 0.$$

The integral curve through (x_0, y_0) is $x(t) = y_0 t + x_0$, $y(t) = y_0$. Hence, X is complete.

Similarly, for Y we have:

$$x'(t) = 0, \qquad y'(t) = \frac{x^2(t)}{2}.$$

Hence $x(t) = x_0$. So $y'(t) = \frac{1}{2}x_0^2$ and thus $y(t) = \frac{1}{2}x_0^2 t + y_0$. Hence, Y is complete.

As for $[X,Y] = xy\frac{\partial}{\partial y} - \frac{x^2}{2}\frac{\partial}{\partial x}$, we have the system

$$x'(t) = -\frac{x^2(t)}{2}, \qquad y'(t) = y(t)x(t).$$

As in Problem 1.9.12 we obtain

$$x(t) = \frac{2x_0}{x_0 t + 2}.$$

So we have $y'(t)/y(t) = 2x_0/(x_0 t + 2)$, thus $\log y(t) = 2\log(x_0 t + 2) + \log B$. Since $y(0) = y_0$ it follows that $y_0 = 4B$. Therefore

$$y(t) = \frac{y_0}{4}(x_0 t + 2)^2.$$

Hence $[X,Y]$ is not complete as its integral curve is not defined for $t = -2/x_0$.

(5) The integral curves are the solutions of the system

$$x'(t) = x(t), \qquad y'(t) = 0.$$

Hence the integral curve through (x_0, y_0) is

$$x(t) = x_0 e^t, \qquad y(t) = y_0.$$

The graph is a horizontal half-line on $\mathbb{R}^2$ of exponential speed, with $x \in (-\infty, 0)$ or $(0, +\infty)$ depending on either $x_0 < 0$ or $x_0 > 0$, respectively. The graph is the point $(0, y_0)$ if $x_0 = 0$. X is complete.

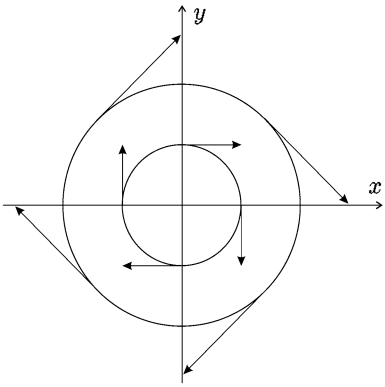

Fig. 1.25 Integral curves of the vector field $X = y\partial/\partial x - x\partial/\partial y$.

(6) The integral curves are the solutions of the system

$$x'(t) = y(t), \qquad y'(t) = -x(t).$$

That is,

$$x(t) = A\sin t + B\cos t, \qquad y(t) = -B\sin t + A\cos t.$$

As $x(0) = x_0 = B$, $y(0) = y_0 = A$, the integral curve through (x_0, y_0) is

$$x(t) = y_0 \sin t + x_0 \cos t, \qquad y(t) = -x_0 \sin t + y_0 \cos t.$$

Since $x^2(t) + y^2(t) = x_0^2 + y_0^2$, the integral curves are the circles with center at the origin (see Figure 1.25). The vector field is complete.

1.9.3 Flows

Problem 1.9.15. *For each $t \in \mathbb{R}$, consider the map $\varphi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$(x, y) \mapsto \varphi_t(x, y) = (x\cos t + y\sin t, -x\sin t + y\cos t).$$

(1) *Prove that φ_t is a 1-parameter group of transformations of $\mathbb{R}^2$.*

(2) *Calculate the associated vector field X.*

(3) *Describe the orbits.*

(4) *Prove that X is invariant by φ_t; that is, that $\varphi_{t*}X_p = X_{\varphi_t(p)}$.*

Solution. (1) Each φ_t is trivially C^∞. Furthermore:
(a) $\varphi_0(x,y) = (x,y)$, thus $\varphi_0 = \mathrm{id}_{\mathbb{R}^2}$.
(b)

$$\begin{aligned}(\varphi_t \circ \varphi_s)(x,y) &= \varphi_t(x\cos s + y\sin s,\ -x\sin s + y\cos s)\\ &= (x\cos(s+t) + y\sin(s+t),\ -x\sin(s+t) + y\cos(s+t))\\ &= \varphi_{t+s}(x,y).\end{aligned}$$

(2) We have $X = \lambda_1 \dfrac{\partial}{\partial x} + \lambda_2 \dfrac{\partial}{\partial y}$, with

$$\lambda_1(x,y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x\cos t + y\sin t) = y,$$

$$\lambda_2(x,y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (-x\sin t + y\cos t) = -x,$$

that is, $X = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$.

(3) The orbit through $p = (x_0, y_0)$ is the image of the map $\mathbb{R} \to \mathbb{R}^2$ given by

$$t \mapsto (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t),$$

that is, a circle centered at the origin and passing through $p = (x_0, y_0)$. If $p = (0,0)$, the orbit reduces to the point p.

(4) If $p = (x_0, y_0)$, then

$$X_{\varphi_t(p)} = (-x_0 \sin t + y_0 \cos t) \left.\frac{\partial}{\partial x}\right|_{\varphi_t(p)} - (x_0 \cos t + y_0 \sin t) \left.\frac{\partial}{\partial y}\right|_{\varphi_t(p)}.$$

Hence

$$\begin{aligned}\varphi_{t*}X_p &\equiv \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} y_0 \\ -x_0 \end{pmatrix}\\ &\equiv X_{\varphi_t(p)}.\end{aligned}$$

Problem 1.9.16. *Let TM be the tangent bundle over a differentiable manifold M. Let $\varphi\colon \mathbb{R} \times TM \to TM$ defined by $\varphi(t,X) = \mathrm{e}^t X$.*

(1) *Prove that φ is a 1-parameter group of transformations of TM.*

(2) *Calculate the vector field Y on TM associated to φ.*

(3) *Prove that Y is invariant under φ.*

Solution. Let $\varphi_t\colon TM \to TM$, $X \mapsto \mathrm{e}^t X$. Obviously $\varphi_0 = \mathrm{id}_{TM}$. Furthermore

$$\begin{aligned}(\varphi_t \circ \varphi_s)X &= \varphi_t(\mathrm{e}^s X)\\ &= \mathrm{e}^{s+t}X\\ &= \varphi_{t+s}X,\end{aligned}$$

so $\varphi_t \circ \varphi_s = \varphi_{t+s}$.

Let us see that φ is differentiable. Pick $(t_0, X_0) \in \mathbb{R} \times TM$. Let π denote the canonical projection from TM to M. Let $p = \pi(X_0) \in M$ and $(U, \psi = (x^1, \dots, x^1))$ be a coordinate system on a neighborhood of p. Let $(\pi^{-1}(U), \Psi)$ be the chart in TM built from (U, ψ); that is,

$$\Psi = (\psi \times \mathrm{id}_{\mathbb{R}^n}) \circ \tau \colon \pi^{-1}(U) \to \psi(U) \times \mathbb{R}^n,$$

with $\tau \colon \pi^{-1}(U) \to U \times \mathbb{R}^n$, where

$$\tau\left(\left(\lambda^1 \frac{\partial}{\partial x^1} + \cdots + \lambda^n \frac{\partial}{\partial x^n}\right)_x\right) = (x, \lambda^1, \dots, \lambda^n).$$

Let us denote $\Psi = (x^1, \dots, x^n, y^1, \dots, y^n)$. Then, given $Z_0 \in T_qM$, $q \in U$, such that

$$\begin{aligned}\Psi(Z_0) &= \big((x^1 \circ \pi)(Z_0), \dots, (x^n \circ \pi)(Z_0), y^1(Z_0), \dots, y^n(Z_0)\big)\\ &= (x^1(q), \dots, x^n(q), y^1(Z_0), \dots, y^n(Z_0))\\ &= (a^1, \dots, a^n, b^1, \dots, b^n),\end{aligned}$$

we have, taking on $\mathbb{R}$ the chart $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$:

$$\begin{aligned}&\big(\Psi \circ \varphi \circ (\mathrm{id}_{\mathbb{R}} \times \Psi)^{-1}\big)(t, a^1, \dots, a^n, b^1, \dots, b^n)\\ &\qquad = (\Psi \circ \varphi)(t, Z_0)\\ &\qquad = \Psi(\mathrm{e}^t Z_0)\\ &\qquad = (x^1(q), \dots, x^n(q), \mathrm{e}^t y^1(Z_0), \dots, \mathrm{e}^t y^n(Z_0))\\ &\qquad = (a^1, \dots, a^n, \mathrm{e}^t b^1, \dots, \mathrm{e}^t b^n).\end{aligned}$$

Hence φ is differentiable.

(2) Let Y be the vector field generated by φ. Let $X_0 \in TM$ and $p = \pi(X_0)$ and consider as before the charts (U, ψ) in p, with $\psi = (x^1, \dots, x^n)$ and $(\pi^{-1}(U), \Psi)$ in X_0, with $\Psi = (x^1, \dots, x^n, y^1, \dots, y^n)$.

Then $Y \colon TM \to TTM$ has at X_0 the expression

$$Y_{X_0} = Y_{X_0}(x^i)\left.\frac{\partial}{\partial x^i}\right|_{X_0} + Y_{X_0}(y^i)\left.\frac{\partial}{\partial y^i}\right|_{X_0}.$$

As Y is generated by φ, one has

$$Y_{X_0}(x^i) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\big(x^i(\varphi(t, X_0))\big), \qquad Y_{X_0}(y^i) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\big(y^i(\varphi(t, X_0))\big).$$

Thus

$$\begin{aligned} Y_{X_0}(x^i) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x^i(\mathrm{e}^t X_0)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (x^i(p)) = 0, \\ Y_{X_0}(y^i) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{e}^t y^i(X_0)) = y^i(X_0). \end{aligned}$$

So $Y_{X_0} = y^i(X_0) \left.\dfrac{\partial}{\partial y^i}\right|_{X_0}$, hence $Y = y^i \dfrac{\partial}{\partial y^i}$.

(3) It suffices to prove that $(\varphi_{t*})_{X_0} Y_{X_0} = Y_{\mathrm{e}^t X_0}$. We know that

$$Y_{\mathrm{e}^t X_0} = \mathrm{e}^t y^i(X_0) \left.\frac{\partial}{\partial y^i}\right|_{\mathrm{e}^t X_0}.$$

On the other hand,

$$\varphi_t(a^1,\dots,a^n,b^1,\dots,b^n) = (a^1,\dots,a^n,\mathrm{e}^t b^1,\dots,\mathrm{e}^t b^n),$$

so that the matrix associated to φ_{t*} is $\begin{pmatrix} I_n & 0 \\ 0 & \mathrm{e}^t I_n \end{pmatrix}$.

Since $Y_{X_0} = (0,\dots,0,y^1(X_0),\dots,y^n(X_0))$, we have

$$\begin{aligned} (\varphi_{t*})_{X_0}(Y_{X_0}) &= \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{e}^t I_n \end{pmatrix} {}^t(0,\dots,0,\ y^1(X_0),\dots,\ y^n(X_0)) \\ &= (0,\dots,0,\ \mathrm{e}^t y^1(X_0),\dots,\ \mathrm{e}^t y^n(X_0)) \\ &= Y_{\mathrm{e}^t X_0}, \end{aligned}$$

as expected.

1.9.4 Transforming Vector Fields

Problem 1.9.17. *Consider the projection $p\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$. Find the condition that must verify a vector field on $\mathbb{R}^2$ to be p-related to some vector field on $\mathbb{R}$.*

Solution. Let

$$X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

In order for X to be p-related to some vector field on $\mathbb{R}$, it must happen that for each couple of points (x_0,y_0), $(x_1,y_1) \in \mathbb{R}^2$ such that $p(x_0,y_0) = p(x_1,y_1)$ one has

$$p_*\left(\left(a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}\right)_{(x_0,y_0)}\right)=p_*\left(\left(a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}\right)_{(x_1,y_1)}\right). \tag{$\star$}$$

Since for the given pair of points we have $x_0 = x_1$, we can write such a couple of points in the form $(x,y_0),(x,y_1)$, and we have

$$\begin{aligned} p_*\left(\frac{\partial}{\partial x}\bigg|_{(x,y_0)}\right) &= p_*\left(\frac{\partial}{\partial x}\bigg|_{(x,y_1)}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_x, \\ p_*\left(\frac{\partial}{\partial y}\bigg|_{(x,y_0)}\right) &= p_*\left(\frac{\partial}{\partial y}\bigg|_{(x,y_1)}\right) = 0, \end{aligned}$$

where t is the canonical coordinate on $\mathbb{R}$. Substituting in ($\star$), we obtain the condition we are looking for: $a(x,y_0) = a(x,y_1)$, for all x, y_0, y_1.

Problem 1.9.18. *Let $M = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ be endowed with the natural differentiable structure as an open subset of $\mathbb{R}^2$, and let $f\colon M \to \mathbb{R}$, $(x,y) \mapsto x$.*

(1) *Prove that $X_{(x,y)} \equiv (x/r^3, y/r^3)$, where $r = \sqrt{x^2+y^2}$, is a C^∞ vector field on M.*

(2) *Is X f-related to a vector field on $\mathbb{R}$?*

Solution. (1) The functions $M \to \mathbb{R}$, given by $(x,y) \mapsto x/\sqrt{(x^2+y^2)^3}$, and $(x,y) \mapsto y/\sqrt{(x^2+y^2)^3}$, are C^∞ on M.

(2) No, as if X were f-related to a vector field on $\mathbb{R}$, then (as in Problem 1.9.17) $f_*X_p = f_*X_{p'}$ if $p = (x_0,y_0)$, $p' = (x_0,y_0')$, $y_0 \neq y_0'$, and this is not the case, as it is proved below.

The matrix associated to f_* with respect to the bases $\{\partial/\partial x, \partial/\partial y\}$ and $\{\mathrm{d}/\mathrm{d}t\}$, that is, the Jacobian matrix of $\mathrm{id}_{\mathbb{R}} \circ f \circ \mathrm{id}_M^{-1}$, is $(1\;0)$. Consequently, if $p = (x_0,y_0) \in M$, we have

$$f_*X_p = \frac{x_0}{\sqrt{(x_0^2+y_0^2)^3}}\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{x_0}.$$

Hence $f_*X_p \neq f_*X_{p'}$.

Problem 1.9.19. *Consider the 2-torus $T^2 = S^1 \times S^1$. Consider the submersion*

$$f\colon \mathbb{R}^2 \to T^2, \qquad f(\theta,\theta') = (\mathrm{e}^{\theta\mathrm{i}}, \mathrm{e}^{\theta'\mathrm{i}}),$$

and a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. Under which condition is X f-projectable onto a vector field Y on T^2?

Solution. It is immediate that the condition is $f_*(X_{(\theta+2k\pi,\theta'+2k'\pi)}) = f_*(X_{(\theta,\theta')})$. Equivalently, X must be invariant under the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ defined by $(k,k')\cdot(\theta,\theta') = (\theta+2k\pi, \theta'+2k'\pi)$ (see Figure 1.26).

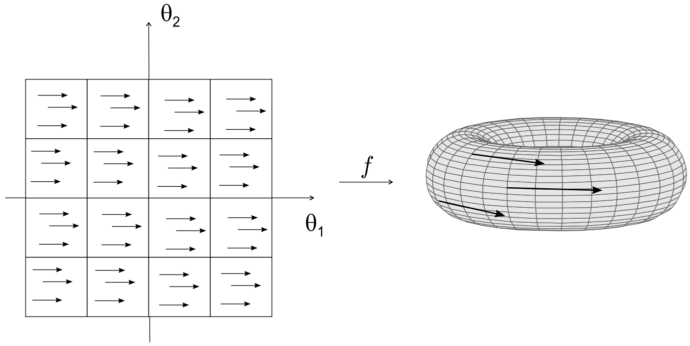

Fig. 1.26 A vector field on $\mathbb{R}^2$ inducing a vector field on T^2.

Problem 1.9.20. *Let $f\colon M \to N$ be a C^∞ map and X and Y be f-related C^∞ vector fields. Prove that f maps integral curves of X into integral curves of Y.*

Solution. For the integral curve of X through $p \in M$, $\sigma\colon (-\varepsilon, \varepsilon) \to M$, one has

(1) σ is C^∞.

(2) $\sigma(0) = p$.

(3) $\sigma_* \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0} \right) = X_{\sigma(t_0)}$, for all $t_0 \in (-\varepsilon, \varepsilon)$.

Then the map $f \circ \sigma\colon (-\varepsilon, \varepsilon) \to N$ satisfies:

(1) $f \circ \sigma$ is differentiable as a composition of differentiable maps.

(2) $(f \circ \sigma)(0) = f(p)$.

(3)

$$\begin{aligned}
(f\circ\sigma)_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0} \right) &= (f_{*\sigma(t_0)} \circ \sigma_{*t_0}) \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0} \right) \\
&= f_{*\sigma(t_0)} \left(\sigma_{*t_0} \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0} \right) \right) \\
&= f_*(X_{\sigma(t_0)}) \\
&= Y_{(f\circ\sigma)(t_0)}. \qquad (X \text{ and } Y \text{ are } f\text{-related})
\end{aligned}$$

That is, $f \circ \alpha$ is the integral curve of Y passing through $f(p)$.

Problem 1.9.21. *Let $\varphi\colon M \to N$ be a diffeomorphism between the C^∞ manifolds M and N. Given $X \in \mathfrak{X}(M)$, the* vector field image *$\varphi \cdot X$ of X is defined by*

$$(\varphi \cdot X)_x = \varphi_* \left(X_{\varphi^{-1}(x)} \right).$$

Prove:

(1) *In fact,* $\varphi \cdot X \in \mathfrak{X}(N)$.

(2) $\varphi \cdot [X,Y] = [\varphi \cdot X, \varphi \cdot Y],\ X,Y \in \mathfrak{X}(M)$.

Solution. (1) From the definition of φ_*, it is immediate that the image of a vector is a vector. Moreover, $\varphi \cdot X$ is C^∞, which follows from

$$\varphi \cdot X = \varphi_* \circ X \circ \varphi^{-1}.$$

Further, we have, denoting by π_{TM} (resp. π_{TN}) the projection map of the tangent bundle over M (resp. N), that $\pi_{TN} \circ (\varphi \cdot X) = \mathrm{id}$. In fact,

$$\pi_{TN} \circ \varphi_* \circ X \circ \varphi^{-1} = \varphi \circ \pi_{TM} \circ X \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \mathrm{id}.$$

(2) From the definition of $\varphi \cdot X$ it follows $(\varphi \cdot X)f = X(f \circ \varphi) \circ \varphi^{-1}$. Hence, for any $p \in N$, one has

$$\begin{aligned}
(\varphi \cdot [X,Y])_p f &= [X,Y]_{\varphi^{-1}(p)}(f \circ \varphi) \\
&= X_{\varphi^{-1}(p)}(Y(f \circ \varphi)) - Y_{\varphi^{-1}(p)}(X(f \circ \varphi)) \\
&= X_{\varphi^{-1}(p)}((\varphi \cdot Y)(f) \circ \varphi) - Y_{\varphi^{-1}(p)}((\varphi \cdot X)(f) \circ \varphi) \\
&= (\varphi \cdot X)_p((\varphi \cdot Y)f) - (\varphi \cdot Y)_p((\varphi \cdot X)f) \\
&= [\varphi \cdot X, \varphi \cdot Y]_p f.
\end{aligned}$$

Problem 1.9.22. *Let* $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \mathrm{e}^x$. *Find the vector field image* $f \cdot \partial/\partial x$.

Solution. The Jacobian of f is e^x. We have, for any fixed x_0, that

$$\begin{aligned}
\left(f \cdot \frac{\partial}{\partial x}\right)_{x_0} &= f_*\left(\left.\frac{\partial}{\partial x}\right|_{f^{-1}(x_0)}\right) \\
&= f_*\left(\left.\frac{\partial}{\partial x}\right|_{\log x_0}\right) \\
&= \left(x\frac{\partial}{\partial x}\right)_{x_0}.
\end{aligned}$$

Hence

$$f \cdot \frac{\partial}{\partial x} = x\,\frac{\partial}{\partial x}.$$

Chapter 2
Tensor Fields and Differential Forms

2.1 Vector Bundles

Problem 2.1.1. *Let* (E,π,M) *be a* C^∞ *vector bundle with fibre* $\mathbb{F}^n$*, where* $\mathbb{F} = \mathbb{R},\mathbb{C}$ *or* $\mathbb{H}$*. Prove that the homotheties*

$$h\colon \mathbb{F}\times E \to E, \qquad (\lambda,y)\mapsto h(\lambda,y)=\lambda y,$$

are C^∞*.*

Solution. Let U be an open subset of M. Let $\varphi\colon \pi^{-1}(U)\to U\times\mathbb{F}$ be a trivialization of (E,π,M), that is, a fibre-preserving diffeomorphism linear on the fibres, and φ a chart, that is, a diffeomorphism of the open subset $E_U=\pi^{-1}(U)$ of E onto $U\times\mathbb{F}^n$, linear on the fibres.

Then, $h|_U$ is the composition map

$$\begin{array}{ccccccc} \mathbb{F}\times E_U & \xrightarrow{\mathrm{id}_{\mathbb{F}}\times\varphi} & \mathbb{F}\times U\times\mathbb{F}^n & \xrightarrow{h'} & U\times\mathbb{F}^n & \xrightarrow{\varphi^{-1}} & E_U \\ (\lambda,y) & \longmapsto & (\lambda,p,x) & \longmapsto & (p,\lambda x) & \longmapsto & \lambda y. \end{array}$$

Since φ is a diffeomorphism and h' is C^∞, the map $h|_U$ is C^∞.

Problem 2.1.2. *Show that for a* C^∞ *vector bundle* $\xi=(E,\pi,M)$ *with fibre* $\mathbb{R}^n$*, triviality is equivalent to the existence of* n C^∞ *global sections, linearly independent at each point.*

Solution. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^n$. If we have a global trivialization

$$\begin{array}{ccc} E & \xrightarrow{\ u\ } & M\times\mathbb{R}^n \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ \mathrm{id}\ } & M \end{array}$$

then we have sections $\tilde{e}_i$ of $M \times \mathbb{R}^n$ given by $\tilde{e}_i = (\mathrm{id}, e_i)$. Thus, we have sections ξ_i of E defined by $\xi_i = u^{-1} \circ \tilde{e}_i$, which are linearly independent because u^{-1} is an isomorphism on each fibre.

Conversely, if σ_i are such linearly independent sections of E, we define the trivialization u by $u(\alpha) = (\pi(\alpha), \alpha^1, \dots, \alpha^n)$ with $\alpha = \alpha^i \xi_i(\pi(\alpha))$. Its inverse map is given by $u^{-1}(p, \alpha^1, \dots, \alpha^n) = \alpha^i \xi_i|_p$.

Problem 2.1.3. *Prove that the infinite Möbius strip M (see* Problem 1.1.12*) can be considered as the total space of a vector bundle over S^1. Specifically*:

(1) *Determine the base space, the fibre and the projection map π.*

(2) *Prove that the vector bundle (M, π, S^1) is locally trivial but not trivial.*

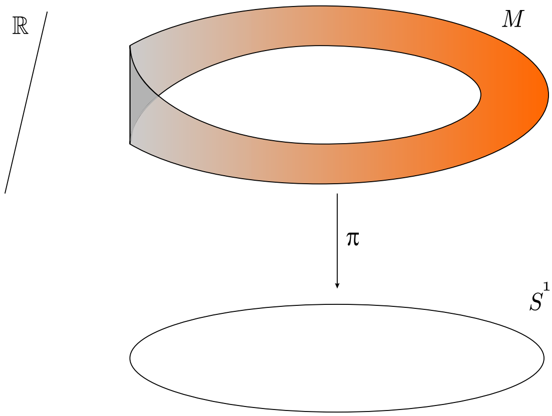

Fig. 2.1 The Möbius strip as the total space of a vector bundle.

Solution. (1) With the notations of Problem 1.1.12, we have that the base space is $S^1 \equiv ([0,1] \times \{0\})/\!\sim\, \subset M$, the fibre is $\mathbb{R}$ (see Figure 2.1), and the projection map is defined by

$$\pi([(x,y)]) = \begin{cases} [(x,0)] & \text{if } 0 < x < 1 \\ [(0,0)] = [(1,0)] & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(2) The charts in Problem 1.1.12 are in fact trivializations that cover S^1 entirely. Now suppose that there exists a nonvanishing global section $\sigma \colon S^1 \to M$, i.e. a continuous map such that $\pi \circ \sigma = \mathrm{id}_{S^1}$. This is equivalent to a continuous function $s \colon [0,1] \to \mathbb{R}$ such that $s(0) = -s(1)$. Since s must vanish somewhere, σ must also vanish somewhere. Contradiction.

Problem 2.1.4. (1) *Consider*

$$E = \left\{ (u,v) = (x,y,z,a,b,c) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1,\ \langle u, v\rangle = 0 \right\},$$

and the projection map on the unit sphere S^2 *given by* $\pi\colon E \to S^2$, $\pi(u,v) = u$. *Prove that* $\xi = (E,\pi,S^2)$ *is a locally trivial bundle over* S^2 *with fibre* $\mathbb{R}^2$.

(2) *Let* $\mathscr{A} = \{(U_i,\Phi_i)\}$, $i = 1,2,3$, *be as in the solution of* (1) *below. Prove that* $TS^2 = (E,\pi,S^2,\mathscr{A})$ *is a vector bundle* (*see* Definitions 7.2.1) *with fibre* $\mathbb{R}^2$.

Solution. (1) The open subsets U_1,U_2,U_3 of S^2 given by $|x| < 1$, $|y| < 1$, $|z| < 1$, respectively, are an open covering of S^2. Define local trivializations by

$$\begin{aligned}
\Phi_1 &: \pi^{-1}(U_1) \to U_1 \times \mathbb{R}^2, && (x,y,z,a,b,c) \mapsto (x,y,z,\, bz-cy, a),\\
\Phi_2 &: \pi^{-1}(U_2) \to U_2 \times \mathbb{R}^2, && (x,y,z,a,b,c) \mapsto (x,y,z,\, cx-az, b),\\
\Phi_3 &: \pi^{-1}(U_3) \to U_3 \times \mathbb{R}^2, && (x,y,z,a,b,c) \mapsto (x,y,z,\, ay-bx, c).
\end{aligned}$$

It is immediate that they are diffeomorphisms.

(2) As a computation shows, the changes of charts are given, for each $u = (x,y,z) \in S^2$, by

$$\begin{aligned}
g_{21}(u) &= \frac{-1}{y^2+z^2}\begin{pmatrix} xy & z \\ -z & xy \end{pmatrix},\\
g_{32}(u) &= \frac{-1}{z^2+x^2}\begin{pmatrix} yz & x \\ -x & yz \end{pmatrix},\\
g_{13}(u) &= \frac{-1}{x^2+y^2}\begin{pmatrix} zx & y \\ -y & zx \end{pmatrix}.
\end{aligned}$$

The cocycle condition is thus satisfied. Indeed, one has

$$\begin{aligned}
g_{21}(u)g_{13}(u) &= \frac{1}{x^2+y^2}\begin{pmatrix} -yz & x \\ -x & -yz \end{pmatrix}\\
&= (g_{32}(u))^{-1}\\
&= g_{23}(u),
\end{aligned}$$

and the similar identities for $g_{12}(u)g_{23}(u)$ and $g_{13}(u)g_{32}(u)$.

Moreover, for

$$\widehat{E} = \{\big((u,v),(u',v')\big) \in E \times E \,:\, u = u',\ \langle u,v\rangle = \langle u,v'\rangle = 0\},$$

the maps

$$\begin{aligned}
s\colon \widehat{E} \to E, &\qquad \big((u,v),(u',v')\big) \mapsto (u, v+v'),\\
h\colon \mathbb{R}\times E \to E, &\qquad (\lambda,(u,v)) \mapsto (u,\lambda v),
\end{aligned}$$

are C^∞ (as for h, see Problem 2.1.1) and they induce a structure of 2-dimensional vector space on each fibre of TS^2.

Problem 2.1.5. (1) *Let* $\{(U_\alpha,\varphi_\alpha)\}$ *be an atlas on a manifold* M, *where* $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$, $\varphi_\alpha = (x_\alpha^1,\dots,x_\alpha^n)$, $n = \dim M$. *Let* $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n,\mathbb{R})$ *be the map*

$$\left(g_{\alpha\beta}(p)\right)^h_i = \frac{\partial x^h_\alpha}{\partial x^i_\beta}(p), \qquad \forall p \in U_\alpha \cap U_\beta.$$

Prove that $\{g_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the tangent bundle TM.

(2) *Similarly, if the map $g^*_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n,\mathbb{R})$ is given by*

$$\left(g^*_{\alpha\beta}(p)\right)^h_i = \frac{\partial x^i_\beta}{\partial x^h_\alpha}(p), \qquad \forall p \in U_\alpha \cap U_\beta,$$

prove that $\{g^*_{\alpha\beta}\}$ is a cocycle on M whose associated vector bundle is the cotangent bundle T^*M.

Solution. (1) Let us define two linear frames at p:

$$u_\alpha = \left(\left.\frac{\partial}{\partial x^1_\alpha}\right|_p, \dots, \left.\frac{\partial}{\partial x^n_\alpha}\right|_p \right), \qquad u_\beta = \left(\left.\frac{\partial}{\partial x^1_\beta}\right|_p, \dots, \left.\frac{\partial}{\partial x^n_\beta}\right|_p \right).$$

According to the definition of $g_{\alpha\beta}(p)$ we have

$$\left.\frac{\partial}{\partial x^i_\beta}\right|_p = \left(g_{\alpha\beta}(p)\right)^h_i \left.\frac{\partial}{\partial x^h_\alpha}\right|_p.$$

Hence $u_\beta = u_\alpha \cdot g_{\alpha\beta}(p)$, where the dot on the right-hand side stands for the right action of $GL(n,\mathbb{R})$ on the bundle of linear frames FM (see Definitions 7.5.1). Accordingly,

$$\begin{aligned} u_\beta &= u_\gamma \cdot g_{\gamma\beta}(p) \\ &= (u_\alpha \cdot g_{\alpha\gamma}(p)) \cdot g_{\gamma\beta}(p) \\ &= u_\alpha \cdot (g_{\alpha\gamma}(p) g_{\gamma\beta}(p)). \end{aligned}$$

As $GL(n,\mathbb{R})$ acts freely on FM, we conclude that

$$g_{\alpha\beta}(p) = g_{\alpha\gamma}(p) g_{\gamma\beta}(p),$$

thus proving that $\{g_{\alpha\beta}\}$ is a cocycle.

Moreover, if $\pi\colon TM \to M$ is the tangent bundle, for every index α we have a trivialization

$$\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \qquad \Phi_\alpha(X) = (p, \lambda^1, \dots, \lambda^n),$$

$X = \lambda^i (\partial/\partial x^i_\alpha)_p \in T_p U_\alpha$, or in other words,

$$\Phi_\alpha(X) = (p, u_\alpha^{-1}(X)),$$

where u_α is understood as a linear isomorphism $u_\alpha\colon \mathbb{R}^n \to T_p M$.

In order to prove that the cocycle $\{g_{\alpha\beta}\}$ defines TM it suffices to see that the cocycle associated to these trivializations is $\{g_{\alpha\beta}\}$. In fact, if $\{e_i\}$ is the standard basis of $\mathbb{R}^n$, for $v = \lambda^i e_i$, $p \in U_\alpha \cap U_\beta$, we have

$$
\begin{aligned}
(\Phi_\alpha \circ \Phi_\beta^{-1})(p,v) &= \Phi_\alpha(u_\beta(v)) \\
&= (p, u_\alpha^{-1}(u_\beta(v))) \\
&= \left(p, u_\alpha^{-1}\left(\lambda^i \left.\frac{\partial}{\partial x_\beta^i}\right|_p\right)\right) \\
&= \left(p, u_\alpha^{-1}\left(\lambda^i \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p) \left.\frac{\partial}{\partial x_\alpha^h}\right|_p\right)\right) \\
&= \left(p, \lambda^i \frac{\partial x_\alpha^h}{\partial x_\beta^i}(p)\, u_\alpha^{-1}\left(\left.\frac{\partial}{\partial x_\alpha^h}\right|_p\right)\right) \\
&= \left(p, \lambda^i \big(g_{\alpha\beta}(p)\big)_i^h e_h\right) \\
&= (p, g_{\alpha\beta}(p)\cdot v).
\end{aligned}
$$

(2) We have

$$
\begin{aligned}
\big(g^*_{\alpha\beta}(p)\big)_j^h \big({}^t g^*_{\alpha\beta}(p)\big)_i^j &= \frac{\partial x_\beta^j}{\partial x_\alpha^h}(p) \frac{\partial x_\alpha^i}{\partial x_\beta^j}(p) \\
&= \frac{\partial x_\beta^j}{\partial x_\alpha^h}(p) \left.\frac{\partial}{\partial x_\beta^j}\right|_p (x_\alpha^i) \\
&= \left.\frac{\partial}{\partial x_\alpha^h}\right|_p (x_\alpha^i) \\
&= \delta_h^i.
\end{aligned}
$$

Hence $g^*_{\alpha\beta}(p) = ({}^t g_{\alpha\beta})^{-1}(p)$, and then

$$
\begin{aligned}
g^*_{\alpha\gamma}(p) g^*_{\gamma\beta}(p) &= ({}^t g_{\alpha\gamma})^{-1}(p)({}^t g_{\gamma\beta})^{-1}(p) \\
&= ({}^t g_{\alpha\beta})^{-1}(p) \\
&= g^*_{\alpha\beta}(p),
\end{aligned}
$$

thus proving that $\{g^*_{\alpha\beta}\}$ is a cocycle.

Finally, by proceeding as in (1) above, it is easily checked that $\{g^*_{\alpha\beta}\}$ is the cocycle attached to the trivializations of the cotangent bundle $\pi\colon T^*M \to M$ defined as follows:

$$\Psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n,$$

$$\Psi_\alpha(\omega) = (p, u_\alpha^*(\omega) = \left(p,\, \omega\left(\left.\frac{\partial}{\partial x_\alpha^1}\right|_p\right), \dots, \omega\left(\left.\frac{\partial}{\partial x_\alpha^n}\right|_p\right)\right),$$

$$\omega \in T_p^*M, \quad p \in U_\alpha \cap U_\beta,$$

where $u_\alpha^* \colon T_p^*M \to (\mathbb{R}^n)^*$ is the dual map to $u_\alpha \colon \mathbb{R}^n \to T_pM$.

Problem 2.1.6. (The tautological bundle over the real Grassmannian) *Denote by $\gamma^k(\mathbb{R}^n)$ the subset of pairs $(V,v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n$ such that $v \in V$ and let $\pi \colon \gamma^k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ be the projection $\pi(V,v) = V$. Prove that $\gamma^k(\mathbb{R}^n)$ is a C^∞ vector bundle of rank k.*

Solution. The fibres of π are endowed with a natural structure of vector space as $\pi^{-1}(V) = V$. Hence rank $\pi^{-1}(V) = k$, for all $V \in G_k(\mathbb{R}^n)$. The maps

$$\begin{aligned} \gamma^k(\mathbb{R}^n) \times_{G_k(\mathbb{R}^n)} \gamma^k(\mathbb{R}^n) &\to \gamma^k(\mathbb{R}^n), & ((V,v),(V,w)) &\mapsto (V, v+w), \\ \mathbb{R} \times \gamma^k(\mathbb{R}^n) &\to \gamma^k(\mathbb{R}^n), & (\lambda,(V,v)) &\mapsto (V, \lambda v), \end{aligned}$$

are differentiable as they are induced by the corresponding operations in $\mathbb{R}^n$. It remains to prove that $\gamma^k(\mathbb{R}^n)$ is locally trivial. Let us fix a point $V_0 \in G_k(\mathbb{R}^n)$ and let $\mathscr{U}$ be the set of k-planes V such that $\operatorname{Ker} p|_V = 0$, where p is the orthogonal projection onto V_0 relative to the decomposition $\mathbb{R}^n = V_0 \oplus V_0^\perp$. Certainly, $V_0 \in \mathscr{U}$ as $p|_{V_0} = \mathrm{id}$.

If $\{v_1^0, \dots, v_k^0\}$ is an orthonormal basis of V_0 and $\{v_1, \dots, v_k\}$ is a basis of V, then $V \in \mathscr{U}$ if and only if

$$\det\left(\langle v_i^0, v_j\rangle\right)_{i,j=1,\dots,k} \neq 0,$$

thus proving that $\mathscr{U}$ is an open neighborhood of V_0. For every $V \in \mathscr{U}$, the restriction $p|_V \colon V \to V_0$ is an isomorphism as $\operatorname{Ker} p|_V = 0$ and $\dim V = \dim V_0$. Hence we can define a C^∞ trivialization

$$\begin{array}{ccc} \mathscr{U} \times V_0 & \xrightarrow{\tau} & \pi^{-1}(\mathscr{U}) \subset \gamma^k(\mathbb{R}^n) \\ (V, v_0) & \mapsto & (V, (p|_V)^{-1}(v_0)). \end{array}$$

Problem 2.1.7. *Let $\Phi \colon E \to E'$ be a homomorphism of vector bundles over M with constant rank. Prove that* Ker Φ *and* im Φ *are vector subbundles of E and E', respectively.*

Solution. As the problem is local, we can assume that E, E' are trivial: $E = M \times \mathbb{R}^n$, $E' = M \times \mathbb{R}^m$. Then Φ is given by

$$\Phi(p,v) = (p, A(p)v),$$

where $A = (a_j^i)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $a_j^i \in C^\infty M$, is a C^∞ $m \times n$ matrix. Set $r = \operatorname{rank}_p \Phi$, for all $p \in M$. Given $p_0 \in M$, by permuting rows and columns in A, we can suppose

$$\det \begin{pmatrix} a_1^1(p_0) & \dots & a_r^1(p_0) \\ \vdots & & \vdots \\ a_1^r(p_0) & \dots & a_r^r(p_0) \end{pmatrix} \neq 0.$$

Hence there exists an open neighborhood U of p_0 such that

$$\det \begin{pmatrix} a_1^1(p) & \dots & a_r^1(p) \\ \vdots & & \vdots \\ a_1^r(p) & \dots & a_r^r(p) \end{pmatrix} \neq 0, \qquad p \in U.$$

As rank $A(p) = r$, for all $p \in U$, it is clear that $\mathrm{Ker}(\Phi|_U)$ is defined by the equations

$$a_j^i(p)v^j = 0, \qquad 1 \leqslant i \leqslant r,$$

where $v = v^j e_j$, $\{e_1, \dots, e_n\}$ being a basis of $\mathbb{R}^n$. By using Cramer's formulas we conclude that the previous system is equivalent to

$$v^h = \sum_{k=r+1}^{n} b_k^h(p)v^k, \qquad 1 \leqslant h \leqslant r.$$

Hence $(p, v) \in \mathrm{Ker}\ \Phi$ if and only if

$$v = \sum_{k=r+1}^{n} v^k \left(e_k + \sum_{h=1}^{r} b_k^h(p)e_h \right).$$

Define sections of E over U by

$$\sigma_k(p) = \begin{cases} e_k, & 1 \leqslant k \leqslant r \\ e_k + \sum_{k=1}^{r} b_k^h(p)e_h, & r+1 \leqslant k \leqslant n. \end{cases}$$

Then, $\{\sigma_1(p), \dots, \sigma_n(p)\}$ is a basis of E_p and $\{\sigma_{r+1}(p), \dots, \sigma_n(p)\}$ is a basis of $(\mathrm{Ker}\ \Phi)_p$, for all $p \in U$, thus proving that $\mathrm{Ker}\ \Phi$ is a subbundle of E.

Moreover, if $F \subset E$ is a subbundle, then $F^0 = \{w \in E^* : w|_F = 0\}$ is a subbundle of E^*, as if $\{\sigma_1, \dots, \sigma_n\}$ is a basis of sections of E over U and $\{\sigma_{r+1}, \dots, \sigma_n\}$ is a basis of sections of F, then the dual basis $\{\sigma_1^*, \dots, \sigma_n^*\}$ is a basis of sections of $E^*|_U$ and $\{\sigma_1^*, \dots, \sigma_r^*\}$ is a basis of sections of F^0. Furthermore, as Φ has constant rank, then the same holds for $\Phi^* \colon E'^* \to E^*$, as a matrix and its transpose have the same rank. We can conclude by remarking that $\mathrm{im}\, \Phi = (\mathrm{Ker}\ \Phi^*)^0$.

Finally, we give the following counterexample. Let $E = E' = \mathbb{R} \times \mathbb{R}$ be the trivial bundle over $\mathbb{R}$ with fibre $\mathbb{R}$, and let $\Phi \colon E \to E'$ be defined by $\Phi(p, \lambda) = (p, \lambda p)$. Then

$$(\mathrm{Ker}\ \Phi)_p = \begin{cases} 0 & \text{if } p \neq 0 \\ \mathbb{R} & \text{if } p = 0. \end{cases}$$

2.2 Tensor and Exterior Algebras. Tensor Fields

Problem 2.2.1. *Let V be a finite-dimensional vector space. An element $\theta \in \Lambda^{\bullet} V^*$ is said to be homogeneous of degree k if $\theta \in \Lambda^k V^*$, and a homogeneous element of degree $k \geqslant 1$ is said to be decomposable if there exist $\theta^1, \dots, \theta^k \in \Lambda^1 V^*$ such that $\theta = \theta^1 \wedge \cdots \wedge \theta^k$.*

(1) *Assume that $\theta \in \Lambda^k V^*$ is decomposable. Calculate $\theta \wedge \theta$.*

(2) *If $\dim V > 3$ and $\theta^1, \theta^2, \theta^3, \theta^4$ are linearly independent, is $\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$ decomposable?*

(3) *Prove that if $\dim V = n \leqslant 3$, then every homogeneous element of degree $k \geqslant 1$ is decomposable.*

(4) *If $\dim V = 4$, give an example of a non-decomposable homogeneous element of $\Lambda^{\bullet} V^*$.*

Solution. (1) It is immediate that $\theta \wedge \theta = 0$.

(2) No, since

$$(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) \wedge (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) = 2\,\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \neq 0,$$

so by virtue of (1) it is not decomposable.

(3) If $\dim V = 1$ or 2 the result is trivial. Suppose then $\dim V = 3$, and let $\{\alpha^1, \alpha^2, \alpha^3\}$ be a basis of V^*. If $\theta \in \Lambda^1 V^*$ the result follows trivially. If $\theta \in \Lambda^3 V^*$, then $\theta = a\,\alpha^1 \wedge \alpha^2 \wedge \alpha^3$, hence it is decomposable. Then suppose $\theta \in \Lambda^2 V^*$, so that $\theta = a\,\alpha^1 \wedge \alpha^2 + b\,\alpha^1 \wedge \alpha^3 + c\,\alpha^2 \wedge \alpha^3$. Assume $a \neq 0$. Then

$$\begin{aligned}\theta &= a\,\alpha^1 \wedge \left(\alpha^2 + \frac{b}{a}\,\alpha^3\right) + c\,\alpha^2 \wedge \alpha^3 \\ &= (a\,\alpha^1 - c\,\alpha^3) \wedge \left(\alpha^2 + \frac{b}{a}\,\alpha^3\right).\end{aligned}$$

If $a = 0$, then $\theta = (b\,\alpha^1 + c\,\alpha^2) \wedge \alpha^3$.

(4) The one given in (2) in the statement is such an example.

Problem 2.2.2. (1) *Let A, B be two $(1,1)$ vector fields on a C^∞ manifold M. Define S by*

$$\begin{aligned}S(X,Y) = [AX,BY] + [BX,AY] + AB[X,Y] + BA[X,Y] - A[X,BY] \\ - A[BX,Y] - B[X,AY] - B[AX,Y], \qquad X,Y \in \mathfrak{X}(M).\end{aligned}$$

Prove that S is a $(1,2)$ skew-symmetric tensor field on M, called the Nijenhuis torsion *of A and B.*

(2) *Let J be a tensor field of type $(1,1)$ on the C^∞ manifold M. The* Nijenhuis tensor *of J is defined by*

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y], \qquad X,Y \in \mathfrak{X}(M).$$

(a) *Prove that N_J is a tensor field of type $(1,2)$ on M.*

(b) *Find its local expression in terms of that of J.*

Solution. (1) From the formula

$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

it follows that $S(fX,gY) = fgS(X,Y)$, $f,g \in C^\infty M$. Since the Lie bracket is skew-symmetric, so is S.

(2) (a) The proof is similar to the one in the case (1).

(b) Let $x^1,\dots,x^n$ be local coordinates in which $J = J^i_j \dfrac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ and $N_J = N^i_{jk} \dfrac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$, so

$$J\frac{\partial}{\partial x^k} = J^i_k \frac{\partial}{\partial x^i}, \qquad N_J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = N^k_{ij} \frac{\partial}{\partial x^k}.$$

From the definition of the Nijenhuis tensor we obtain

$$N^i_{jk} = J^l_j \frac{\partial J^i_k}{\partial x^l} - J^l_k \frac{\partial J^i_j}{\partial x^l} + J^i_l \frac{\partial J^l_j}{\partial x^k} - J^i_l \frac{\partial J^l_k}{\partial x^j}.$$

Problem 2.2.3. *Compute the rank of the tensor field $J \in \mathscr{T}_1^{\,1}\mathbb{R}^2$ given by*

$$J = x\,\mathrm{d}x \otimes \frac{\partial}{\partial x} + \sqrt{2}y\left(\mathrm{d}x \otimes \frac{\partial}{\partial y} + \mathrm{d}y \otimes \frac{\partial}{\partial x}\right) + (x+y)\mathrm{d}y \otimes \frac{\partial}{\partial y},$$

at each point $p \in \mathbb{R}^2$.

Solution. The matrix of J is

$$A = \begin{pmatrix} x & \sqrt{2}y \\ \sqrt{2}y & x+y \end{pmatrix}.$$

We have $\det A = (x+2y)(x-y)$. Hence

$$\operatorname{rank} J_p = \begin{cases} 0 & \text{if } p = (0,0) \\ 1 & \text{if either } p = (x,x),\ x \neq 0, \text{ or } p = (-2y,y),\ y \neq 0 \\ 2 & \text{if } (x+2y)(x-y) \neq 0. \end{cases}$$

Problem 2.2.4. *Write the tensor field $J \in \mathscr{T}_1^{\,1}\mathbb{R}^3$ given by*

$$J = \mathrm{d}x \otimes \frac{\partial}{\partial x} + \mathrm{d}y \otimes \frac{\partial}{\partial y} + \mathrm{d}z \otimes \frac{\partial}{\partial z},$$

in the system of spherical coordinates given by

$$x = \rho\cos\varphi\cos\theta, \qquad y = \rho\cos\varphi\sin\theta, \qquad \rho = \sin\varphi,$$
$$\rho > 0, \quad -\pi/2 < \varphi < \pi/2, \quad 0 < \theta < 2\pi.$$

Solution. We have

$$J = \mathrm{d}\rho \otimes \frac{\partial}{\partial\rho} + \mathrm{d}\varphi \otimes \frac{\partial}{\partial\varphi} + \mathrm{d}\theta \otimes \frac{\partial}{\partial\theta},$$

as J represents the identity map in the natural isomorphism $T^*\mathbb{R}^3 \otimes T\mathbb{R}^3 \approx \operatorname{End} T\mathbb{R}^3$, and hence it has the same expression in any coordinate system.

2.3 Differential Forms. Exterior Product

Problem 2.3.1. *Consider on* $\mathbb{R}^2$:

$$X = (x^2 + y)\frac{\partial}{\partial x} + (y^2 + 1)\frac{\partial}{\partial y}, \qquad Y = (y-1)\frac{\partial}{\partial x},$$
$$\theta = (2xy + x^2 + 1)\mathrm{d}x + (x^2 - y)\mathrm{d}y,$$

and let f be the map

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2, \qquad (u,v,w) \mapsto (x,y) = (u - v, v^2 + w).$$

Compute: (1) $[X,Y]_{(0,0)}$. (2) $\theta(X)(0,0)$. (3) $f^*\theta$.

Solution. (1) $[X,Y] = (y^2 - 2xy + 2x + 1)\dfrac{\partial}{\partial x}$, so

$$[X,Y]_{(0,0)} = \left.\frac{\partial}{\partial x}\right|_{(0,0)}.$$

(2)

$$\theta(X)(0,0) = \big((2xy + x^2 + 1)(x^2 + y) + (x^2 - y)(y^2 + 1)\big)(0,0) = 0.$$

(3)

$$\begin{aligned} f^*\theta ={}&\{2(u-v)(v^2+w) + (u-v)^2 + 1\}\mathrm{d}u \\ &+ \{2v\big((u-v)^2 - v^2 - w\big) - 2(u-v)(v^2+w) - (u-v)^2 - 1\}\mathrm{d}v \\ &+ \{(u-v)^2 - v^2 - w\}\mathrm{d}w. \end{aligned}$$

Problem 2.3.2. *Consider the vector fields on* $\mathbb{R}^2$:

$$X = x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial y},$$

and let ω *be the differential form on* $\mathbb{R}^2$ *given by*

$$\omega = (x^2+2y)\,\mathrm{d}x + (x+y^2)\,\mathrm{d}y.$$

Show that ω *satisfies the relation*

$$\mathrm{d}\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$

between the bracket product and the exterior differential.

Solution. We have $[X,Y]=0$ and

$$\begin{aligned}\mathrm{d}\omega &= \left(\frac{\partial(x^2+2y)}{\partial x}\,\mathrm{d}x + \frac{\partial(x^2+2y)}{\partial y}\,\mathrm{d}y\right)\wedge \mathrm{d}x\\ &\quad + \left(\frac{\partial(x+y^2)}{\partial x}\,\mathrm{d}x + \frac{\partial(x+y^2)}{\partial y}\,\mathrm{d}y\right)\wedge \mathrm{d}y\\ &= -\mathrm{d}x\wedge \mathrm{d}y.\end{aligned}$$

From

$$\begin{aligned}X\omega(Y) &= xy + 2x^2y + 6xy^3,\\ Y\omega(X) &= 2xy + 2x^2y + 6xy^3,\\ \mathrm{d}\omega(X,Y) &= -(\mathrm{d}x\wedge \mathrm{d}y)\left(x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right) = -xy,\end{aligned}$$

one easily concludes.

Problem 2.3.3. *Find the subset of* $\mathbb{R}^2$ *where the differential forms*

$$\alpha = x\,\mathrm{d}x + y\,\mathrm{d}y, \qquad \beta = y\,\mathrm{d}x + x\,\mathrm{d}y,$$

are linearly independent and determine the field of dual frames $\{X,Y\}$ *on this set.*

Solution. We have $\det\begin{pmatrix} x & y\\ y & x\end{pmatrix} = x^2 - y^2 \neq 0$ on $\mathbb{R}^2 - \{(x,y) : x = \pm y\}$. Thus α and β are linearly independent on the subset of $\mathbb{R}^2$ complementary to the diagonals $x+y=0$ and $x-y=0$.

The dual field of frames

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, \quad Y = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}, \qquad a,b,c,d \in C^\infty\mathbb{R}^2,$$

must satisfy $X(\alpha) = Y(\beta) = 1,\ X(\beta) = Y(\alpha) = 0$. Hence,

$$\begin{cases} ax+by=1 \\ ay+bx=0 \end{cases} \qquad \text{and} \qquad \begin{cases} cx+dy=0 \\ cy+dx=1. \end{cases}$$

Solving these systems we obtain

$$X = \frac{x}{x^2-y^2}\frac{\partial}{\partial x} - \frac{y}{x^2-y^2}\frac{\partial}{\partial y}, \qquad Y = -\frac{y}{x^2-y^2}\frac{\partial}{\partial x} + \frac{x}{x^2-y^2}\frac{\partial}{\partial y}.$$

REMARK. The result also follows (here and in other problems below) from the general fact that, if $\left\{ e_i = \lambda_i^k \dfrac{\partial}{\partial x^k} \right\}$ is a basis of vector fields on a manifold and $\{\theta^j = \mu_l^j \,\mathrm{d}x^l\}$ denotes the dual basis, then, from $(\mu_l^j \,\mathrm{d}x^l)\left(\lambda_i^k \dfrac{\partial}{\partial x^k}\right) = \delta_{ij}$, one has

$$(\mu_j^i) = {}^t(\lambda_j^i)^{-1}.$$

Problem 2.3.4. *Let θ be the differential form on $\mathbb{R}^3$ defined by*

$$\theta = y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z.$$

If $\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$\psi(u,v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u),$$

compute $\psi^*\theta$.

Solution.

$$\begin{aligned} \psi^*\theta &= \sin u \sin v(\cos u \cos v\,\mathrm{d}u - \sin u \sin v\,\mathrm{d}v) \\ &\quad + \cos u(\cos u \sin v\,\mathrm{d}u + \sin u \cos v\,\mathrm{d}v) - \sin u \cos v \sin u\,\mathrm{d}u \\ &= \left(\frac{1}{4}\sin 2u \sin 2v + \cos^2 u \sin v - \sin^2 u \cos v\right)\mathrm{d}u \\ &\quad + \left(-\sin^2 u \sin^2 v + \frac{1}{2}\sin 2u \cos v\right)\mathrm{d}v. \end{aligned}$$

Problem 2.3.5. *Consider the three vector fields on $\mathbb{R}^3$:*

$$e_1 = (2+y^2)\mathrm{e}^z\frac{\partial}{\partial x}, \qquad e_2 = 2xy\frac{\partial}{\partial x} + (2+y^2)\frac{\partial}{\partial y},$$
$$e_3 = -2xy^2\frac{\partial}{\partial x} - y(2+y^2)\frac{\partial}{\partial y} + (2+y^2)\frac{\partial}{\partial z}.$$

(1) *Show that these vector fields are a basis of the module of C^∞ vector fields on $\mathbb{R}^3$.*

(2) *Write the elements θ^i of its dual basis in terms of* $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$.

(3) *Compute the Lie brackets $[e_i, e_j]$ and express them in the basis $\{e_i\}$.*

Solution. (1) The determinant of the matrix of coefficients is $(2+y^2)^3\mathrm{e}^z$, which is never null; hence the three fields are indeed a basis of $\mathfrak{X}(\mathbb{R}^3)$.

(2) We proceed by direct computation. One has $\theta^i(e_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. Hence, if

$$\theta^1 = A(x,y,z)\,\mathrm{d}x + B(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z,$$

we have

$$\begin{aligned}
1 &= \theta^1(e_1) = A(2+y^2)\mathrm{e}^z,\\
0 &= \theta^1(e_2) = A2xy + B(2+y^2),\\
0 &= \theta^1(e_3) = A(-2xy^2) + B(-y(2+y^2)) + C(2+y^2).
\end{aligned}$$

Solving the system we have

$$A = \frac{1}{(2+y^2)\mathrm{e}^z}, \qquad B = -\frac{2xy}{(2+y^2)\mathrm{e}^z}, \qquad C = 0.$$

Similarly, if $\theta^2 = D(x,y,z)\mathrm{d}x + E(x,y,z)\mathrm{d}y + F(x,y,z)\mathrm{d}z$, we deduce

$$D = 0, \qquad E = \frac{1}{2+y^2}, \qquad F = \frac{y}{2+y^2}.$$

Finally, if $\theta^3 = G(x,y,z)\,\mathrm{d}x + H(x,y,z)\,\mathrm{d}y + I(x,y,z)\,\mathrm{d}z$, we similarly obtain

$$G = 0, \qquad H = 0, \qquad I = \frac{1}{2+y^2}.$$

Hence,

$$\begin{aligned}
\theta^1 &= \frac{1}{(2+y^2)\mathrm{e}^z}\,\mathrm{d}x - \frac{2xy}{(2+y^2)^2\mathrm{e}^z}\,\mathrm{d}y,\\
\theta^2 &= \frac{1}{2+y^2}\,\mathrm{d}y + \frac{y}{2+y^2}\,\mathrm{d}z,\\
\theta^3 &= \frac{1}{2+y^2}\,\mathrm{d}z.
\end{aligned}$$

(3) Applying the formula

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y],$$

we deduce $[e_1, e_2] = 0$. Similarly, one gets

$$[e_1, e_3] = -(2+y^2)e_1, \qquad [e_2, e_3] = (y^2-2)e_2 + 2ye_3.$$

Problem 2.3.6. *Consider the three vector fields on* $\mathbb{R}^3$:

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (1+x^2)\frac{\partial}{\partial z}.$$

(1) *Show that these vector fields are a basis of the module of* C^∞ *vector fields of* $\mathbb{R}^3$.

(2) *Write the elements of the dual basis* $\{\theta^i\}$ *of* $\{e_i\}$ *in terms of* $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$.

(3) *Verify the Jacobi identity between* e_1, e_2 *and* e_3.

Solution. (1)

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1+x^2 \end{pmatrix} = 1 + x^2 \neq 0.$$

(2)

$$\begin{aligned} 1 &= \theta^1(e_1) = (A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z)(e_1) = A, \\ 0 &= \theta^1(e_2) = A + B, \\ 0 &= \theta^1(e_3) = A + B + (1+x^2)C. \end{aligned}$$

Solving the system we have $A = 1,\ B = -1,\ C = 0$. Hence $\theta^1 = \mathrm{d}x - \mathrm{d}y$. Similarly, we obtain $\theta^2 = \mathrm{d}y - \mathrm{d}z/(1+x^2)$ and $\theta^3 = \mathrm{d}z/(1+x^2)$.

(3) From $[e_1, e_2] = 0$ and $[e_1, e_3] = [e_2, e_3] = 2x\dfrac{\partial}{\partial z}$, we have

$$[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [e_3, e_1], e_2] = 0.$$

Problem 2.3.7. *Consider the vector fields*

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

on $\mathbb{R}^2$, *and let* $\psi\colon \mathbb{R}^2 \to \mathbb{R}^4$ *defined by*

$$u = x^2 - y^2, \qquad v = x^2 + y^2, \qquad w = x + y, \qquad t = x - y.$$

(1) *Compute* $[X, Y]$.

(2) *Show that* X, Y *are linearly independent on the open subset* $\mathbb{R}^2 - \{(0,0)\}$ *of* $\mathbb{R}^2$ *and write the basis* $\{\alpha, \beta\}$ *dual to* $\{X, Y\}$ *in terms of the standard basis* $\{\mathrm{d}x, \mathrm{d}y\}$.

(3) *Find vector fields on* $\mathbb{R}^4$, ψ*-related to* X *and* Y, *respectively.*

Solution. (1) $[X, Y] = 0$.

(2)

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0, \quad \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}.$$

Let

$$\alpha = a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y, \qquad \beta = c(x,y)\,\mathrm{d}x + d(x,y)\,\mathrm{d}y.$$

We thus have,

$$\begin{aligned}
1 = \alpha(X) &= a(x,y)\,\mathrm{d}x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + b(x,y)\,\mathrm{d}y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),\\
0 = \alpha(Y) &= a(x,y)\,\mathrm{d}x\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + b(x,y)\,\mathrm{d}y\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right).
\end{aligned}$$

That is, $1 = a(x,y)x + b(x,y)y$ and $0 = a(x,y)(-y) + b(x,y)x$, and one has $a(x,y) = x/(x^2+y^2)$, $b(x,y) = y/(x^2+y^2)$. Hence,

$$\alpha = \frac{x}{x^2+y^2}\,\mathrm{d}x + \frac{y}{x^2+y^2}\,\mathrm{d}y.$$

Similarly, we obtain $\beta = -\dfrac{y}{x^2+y^2}\,\mathrm{d}x + \dfrac{x}{x^2+y^2}\,\mathrm{d}y$.

(3)

$$\begin{aligned}
\psi_* X &\equiv \begin{pmatrix} 2x & -2y \\ 2x & 2y \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\
&\equiv (2x^2 - 2y^2)\left(\frac{\partial}{\partial u}\circ\psi\right) + (2x^2+2y^2)\left(\frac{\partial}{\partial v}\circ\psi\right)\\
&\quad + (x+y)\left(\frac{\partial}{\partial w}\circ\psi\right) + (x-y)\left(\frac{\partial}{\partial t}\circ\psi\right),\\
\psi_* Y &= -4xy\left(\frac{\partial}{\partial u}\circ\psi\right) + (x-y)\left(\frac{\partial}{\partial w}\circ\psi\right) + (-y-x)\left(\frac{\partial}{\partial t}\circ\psi\right).
\end{aligned}$$

Taking

$$\begin{aligned}
\widetilde{X} &= 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} + t\frac{\partial}{\partial t},\\
\widetilde{Y} &= (t^2 - w^2)\frac{\partial}{\partial u} + t\frac{\partial}{\partial w} - w\frac{\partial}{\partial t},
\end{aligned}$$

we have

$$\psi_* X = \widetilde{X}\circ\psi, \qquad \psi_* Y = \widetilde{Y}\circ\psi.$$

Problem 2.3.8. *Prove that the differential* 1*-forms* $\omega^1,\dots,\omega^k$ *on an n-manifold M are linearly independent if and only if* $\omega^1\wedge\cdots\wedge\omega^k \neq 0$.

Solution. If $\omega^1,\dots,\omega^k$ are linearly independent, then each T_pM, $p\in M$, has a basis $\{v_1,\dots,v_k,\dots,v_n\}$ such that its dual basis $\{\varphi^1,\dots,\varphi^k,\dots,\varphi^n\}$ satisfies $\varphi^i = \omega^i|_p$,

$1 \leqslant i \leqslant k$; hence $\omega^1 \wedge \cdots \wedge \omega^k$ is an element of a basis of $\Lambda^k M$ and so it does not vanish.

Conversely, suppose that such differential forms are linearly dependent. Then there exists a point $p \in M$ such that $\omega^i|_p = \sum_{j \neq i} a_j \omega^j|_p$, and thus

$$\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^i \wedge \cdots \wedge \omega^k = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \sum_{j \neq i} a_j \omega^j \wedge \cdots \wedge \omega^k = 0.$$

Problem 2.3.9. *Is $\alpha \wedge \mathrm{d}\alpha = 0$ for any differential 1-form α?*
HINT: *Take $\alpha = x\,\mathrm{d}y + \mathrm{d}z \in \Lambda^1 \mathbb{R}^3$.*

Solution. No, as the given counterexample shows. In fact,

$$\alpha \wedge \mathrm{d}\alpha = (x\,\mathrm{d}y + \mathrm{d}z) \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Problem 2.3.10. *Prove that the restriction to the sphere S^3 of the differential form*

$$\alpha = x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z$$

on $\mathbb{R}^4$, does not vanish.

Solution. Given $p \in S^3$, $(\alpha|_{S^3})_p = 0$ if and only if $\alpha_p(X) = 0$ for all

$$X \in T_p S^3 = \left\{ X \in T_p \mathbb{R}^4 \, : \, \langle X, N \rangle = 0 \right\},$$

where $\langle \, , \rangle$ stands for the Euclidean metric of $\mathbb{R}^4$ and

$$N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}$$

is the unit normal vector field to S^3. Define the differential form β by $\beta(X) = \langle X, N \rangle$. Thus $\beta = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z + t\,\mathrm{d}t$.

If $(\alpha|_{S^3})_p = 0$, then α_p and β_p vanish on $T_p S^3$. But two linear forms vanishing on the same hyperplane are proportional, thus $\alpha_p = \lambda \beta_p$, $\lambda \in \mathbb{R}$, or equivalently,

$$\frac{-y}{x} = \frac{x}{y} = \frac{-t}{z} = \frac{z}{t} = \lambda.$$

We find $x^2 + y^2 = 0$, $z^2 + t^2 = 0$, hence $x = y = z = t = 0$, which is not possible because $p \in S^3$.

Problem 2.3.11. *Let $\omega^1, \dots, \omega^r$ be differential 1-forms on a C^∞ n-manifold M that are independent at each point. Prove that a differential form θ belongs to the ideal $\mathscr{I}$ generated by $\omega^1, \dots, \omega^r$ if and only if*

$$\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

Solution. If $\theta \in \mathscr{I}$, then θ is a linear combination of exterior products where those forms appear as factors and hence $\theta \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0$.

Conversely, given a fixed point, complete $\omega^1,\dots,\omega^r$ to a basis

$$\omega^1,\dots,\omega^r,\omega^{r+1},\dots,\omega^n,$$

so

$$\theta = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f_{i_1\dots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}.$$

If $\theta\wedge\omega^1\wedge\cdots\wedge\omega^r=0$, then for each $\{i_1,\dots,i_k\}$ we have

$$f_{i_1\dots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}\wedge\omega^1\wedge\cdots\wedge\omega^r=0.$$

Then

$$\begin{aligned}
\{1,\dots,r\}\cap\{i_1,\dots,i_k\}\neq\emptyset \quad &\Longrightarrow \quad \omega^{i_1}\wedge\cdots\wedge\omega^{i_k}\wedge\omega^1\wedge\cdots\wedge\omega^r=0,\\
\{1,\dots,r\}\cap\{i_1,\dots,i_k\}=\emptyset \quad &\Longrightarrow \quad f_{i_1\dots i_k}=0.
\end{aligned}$$

Hence

$$\theta = \sum_{\{1,\dots,r\}\cap\{i_1,\dots,i_k\}\neq\emptyset} f_{i_1\dots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}.$$

Problem 2.3.12. *Let M be a C^∞ manifold. If $\{\omega^1,\dots,\omega^n\}$ is a basis of T_p^*M, $p\in M$, prove that there are coordinate functions $x^1,\dots,x^n$ around p such that $\mathrm{d}x^i|_p=\omega^i$, for all i.*

Solution. Let $(U,y^1,\dots,y^n)$ be a coordinate system around p. Since the differentials $\{\mathrm{d}y^1|_q,\dots,\mathrm{d}y^n|_q\}$ are a basis of T_q^*M for each $q\in U$, we can write $\omega^i=f^i_j\,\mathrm{d}y^j|_p$. Since $\{\omega^1,\dots,\omega^n\}$ is a basis of T_p^*M we have $\det(f^i_j)\neq 0$. Thus the system $(U,x^1,\dots,x^n)$ defined by $x^i(q)=f^i_jy^j(q)$ is a coordinate system, and one has $\mathrm{d}x^i|_p=f^i_j\mathrm{d}y^j|_p=\omega^i$.

Problem 2.3.13. *Prove*:

(1) *If α and β are closed differential forms, then $\alpha\wedge\beta$ is also closed.*

(2) *If moreover β is exact, then $\alpha\wedge\beta$ is also exact.*

Solution. (1) $\mathrm{d}(\alpha\wedge\beta)=\mathrm{d}\alpha\wedge\beta+(-1)^{\deg\alpha}\alpha\wedge\mathrm{d}\beta=0$.

(2) If $\beta=\mathrm{d}\omega$, then

$$\mathrm{d}(\alpha\wedge\omega)=\mathrm{d}\alpha\wedge\omega+(-1)^{\deg\alpha}\alpha\wedge\mathrm{d}\omega=(-1)^{\deg\alpha}\alpha\wedge\beta.$$

Hence, $\alpha\wedge\beta=\mathrm{d}((-1)^{\deg\alpha}\alpha\wedge\omega)$.

Problem 2.3.14. *Determine which of the following differential forms on $\mathbb{R}^3$ are closed and which are exact*:

(1) $\alpha=yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z$.

(2) $\beta=x\,\mathrm{d}x+x^2y^2\,\mathrm{d}y+yz\,\mathrm{d}z$.

(3) $\gamma=2xy^2\,\mathrm{d}x\wedge\mathrm{d}y+z\,\mathrm{d}y\wedge\mathrm{d}z$.

Solution. (1) $\alpha = \mathrm{d}(xyz)$; thus α is exact and hence closed.

(2) $\mathrm{d}\beta = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$; thus β is not closed, hence it is not exact.

(3) $\gamma = \mathrm{d}\omega$, where $\omega = \left(x^2y^2 - \frac{1}{2}z^2\right)\mathrm{d}y$; thus γ is exact, hence closed.

Remember that, by the Poincaré Lemma, every closed differential form on $\mathbb{R}^n$ is exact. Thus, another way to prove (1) and (3) is:

(1) $\mathrm{d}\alpha = 0$, thus α is closed and hence exact.

(3) $\mathrm{d}\gamma = 0$, thus γ is closed and hence exact.

Problem 2.3.15. *Let ω be a differential 1-form on a manifold M and consider a nowhere vanishing function $f\colon M \to \mathbb{R}$ such that $\mathrm{d}(f\omega) = 0$. Prove that $\omega \wedge \mathrm{d}\omega = 0$.*

Solution. We have $\mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}\omega$, and since $f(x) \neq 0$ for all $x \in M$, one has $\mathrm{d}\omega = -(1/f)\mathrm{d}f \wedge \omega$. As ω is a differential 1-form, we have $\omega \wedge \mathrm{d}\omega = -(1/f)\omega \wedge \mathrm{d}f \wedge \omega = 0$.

2.4 Lie Derivative. Interior Product

Problem 2.4.1. *Let $L_X T$ be the Lie derivative of the contravariant tensor field T with respect to the vector field X on a C^∞ manifold M. Prove that*

$$[L_X, L_Y]T = L_{[X,Y]}T, \qquad Y \in \mathfrak{X}(M).$$

Solution. Two derivations of the same degree of the contravariant tensor algebra are equal if and only if they are equal on the functions and on the vector fields, but we have:

(a) Let f be a function. Then

$$\begin{aligned}[L_X, L_Y](f) &= L_X(Yf) - L_Y(Xf)\\ &= X(Yf) - Y(Xf)\\ &= [X,Y]f\\ &= L_{[X,Y]}f.\end{aligned}$$

(b) Let Z be a vector field. Then

$$\begin{aligned}[L_X, L_Y]Z &= L_X L_Y Z - L_Y L_X Z\\ &= [X,[Y,Z]] - [Y,[X,Z]]\\ &= [[X,Y],Z]\\ &= L_{[X,Y]}Z.\end{aligned}$$

Problem 2.4.2. *Let α be a differential form on a C^∞ manifold M. Show that*

$$[L_X, L_Y](\alpha) = L_{[X,Y]}\alpha, \qquad X, Y \in \mathfrak{X}(M).$$

Solution. Two derivations of the same degree of the covariant tensor algebra coincide if and only if they coincide on the functions and on the differential 1-forms. We proved the result for functions in Problem 2.4.1, so we only have to prove it for differentials of functions.

Since the Lie derivative commutes with the exterior differential, one has:

$$\begin{aligned}
[L_X, L_Y]\mathrm{d}f &= L_X(L_Y\mathrm{d}f) - L_Y(L_X\mathrm{d}f) \\
&= \mathrm{d}(L_X L_Y f - L_Y L_X f) \\
&= \mathrm{d}([L_X, L_Y](f)) \qquad \text{(by Problem 2.4.1)} \\
&= \mathrm{d}(L_{[X,Y]} f) \\
&= L_{[X,Y]}\mathrm{d}f.
\end{aligned}$$

Problem 2.4.3. *Let M be a C^∞ manifold and let X, Y be vector fields on M; f, g, C^∞ functions on M; and ω a differential 1-form on M. Prove*:

(1) $L_{fX}Y = fL_X Y - \mathrm{d}f(Y)X.$

(2) $L_{fX}\omega = fL_X\omega + \omega(X)\mathrm{d}f.$

(3) $L_{fX}g = fL_X g.$

Solution. (1)

$$\begin{aligned}
L_{fX}Y &= [fX, Y] \\
&= f[X,Y] - (Yf)X \\
&= fL_X Y - \mathrm{d}f(Y)X.
\end{aligned}$$

(2)

$$\begin{aligned}
(L_{fX}\omega)(Y) &= L_{fX}(\omega(Y)) - \omega(L_{fX}Y) \\
&= (fX)(\omega(Y)) - \omega([fX,Y]) \\
&= (fX)(\omega(Y)) - \omega(f[X,Y] - (Yf)X) \\
&= f(X(\omega(Y))) - \omega([X,Y])) + (Yf)\omega(X) \\
&= f(L_X(\omega(Y)) - \omega L_X Y) + (\mathrm{d}f(Y))\omega(X) \\
&= f(L_X\omega)Y + \omega(X)(\mathrm{d}f)(Y) \\
&= (fL_X\omega + \omega(X)\mathrm{d}f)(Y).
\end{aligned}$$

(3)

$$\begin{aligned}
L_{fX}g &= (fX)g \\
&= f(Xg) \\
&= fL_X g.
\end{aligned}$$

Problem 2.4.4. *Let X and Y be vector fields on a C^∞ manifold M. Prove that if φ_t is the local 1-parameter group generated by X, we have for all $p \in M$:*

$$\varphi_{s*}\left((L_X Y)_{\varphi_s^{-1}(p)}\right) = \lim_{t\to 0}\frac{1}{t}\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*}Y_{\varphi_{s+t}^{-1}(p)}\right). \qquad (\star)$$

Solution. Since φ_t is the local 1-parameter group of X, one has $\varphi_s \cdot X = X$, where by definition $(\varphi_s \cdot X)_p = \varphi_{s*}(X_{\varphi_s^{-1}(p)})$. Then, applying Problem 1.9.21, we have

$$\begin{aligned}\varphi_s \cdot L_X Y &= \varphi_s \cdot [X,Y]\\ &= [\varphi_s \cdot X, \varphi_s \cdot Y]\\ &= [X, \varphi_s \cdot Y]\\ &= L_X(\varphi_s \cdot Y).\end{aligned}$$

Thus,

$$\begin{aligned}\varphi_{s*}\left((L_X Y)_{\varphi_s^{-1}(p)}\right) &= L_X\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)}\right)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\left(\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)}\right)_{\varphi_t^{-1}(p)}\right)\right)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{t*}\varphi_{s*}Y_{\varphi_s^{-1}(\varphi_t^{-1}(p))}\right)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\varphi_{s*}Y_{\varphi_s^{-1}(p)} - \varphi_{s+t*}Y_{\varphi_{s+t}^{-1}(p)}\right).\end{aligned}$$

Problem 2.4.5. *Let X and Y be vector fields on a C^∞ manifold M whose local 1-parameter groups are φ_t and ψ_s respectively. Prove that $[X,Y]=0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$.*

HINT: *Consider that a vector field Z is invariant under a diffeomorphism η of M (that is $\eta \cdot Z = Z$, where $(\eta \cdot Z)_x = \eta_* Z_{\eta^{-1}(x)}$), if and only if η commutes with the local 1-parameter group generated by Z.*

Solution. If $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$, then, since ψ_s is the local 1-parameter group of Y, Y is invariant by φ_t, that is $\varphi_t \cdot Y = Y$, thus $\varphi_{t*}Y_{\varphi_t^{-1}(p)} = Y_p$.

Hence for all $p \in M$,

$$\begin{aligned}[X,Y]_p &= (L_X Y)_p\\ &= \lim_{t\to 0}\frac{1}{t}\left(Y_p - \varphi_{t*}Y_{\varphi_t^{-1}(p)}\right) = 0.\end{aligned}$$

Conversely, assume that for all $p \in M$ one has $[X,Y]_p = 0$. Since the equation $(\star)$ in Problem 2.4.4 can be written as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s}(\varphi_t \cdot Y) = -\varphi_s \cdot [X,Y],$$

we deduce $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t \cdot Y) = 0$ for all t. So, since $\varphi_t \cdot Y$ is a constant vector for all t at each point, we have $\varphi_t \cdot Y = \varphi_0 \cdot Y = Y$, hence Y is invariant for any φ_t. Furthermore $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$.

Problem 2.4.6. *Let i_Y be the interior product with respect to the vector field Y on a C^∞ manifold M. Prove*

$$[L_X, i_Y] = i_{[X,Y]}, \qquad X \in \mathfrak{X}(M).$$

Solution. Since $i_{[X,Y]}$ and $[L_X, i_Y]$ are antiderivations on the algebra of differential forms, it suffices to prove the formula for functions and 1-forms:

(a) $[L_X, i_Y](f) = L_X i_Y f - i_Y L_X f = 0$, because i has degree -1 and f and $L_X f$ are functions; and one also has $i_{[X,Y]} f = 0$.

(b) $i_{[X,Y]}\omega = \omega([X,Y])$, and

$$\begin{aligned}
[L_X, i_Y](\omega) &= L_X i_Y \omega - i_Y L_X \omega \\
&= L_X(\omega(Y)) - (L_X\omega)(Y) \\
&= X(\omega(Y)) - X(\omega(Y)) + \omega([X,Y]) \\
&= \omega([X,Y]).
\end{aligned}$$

Problem 2.4.7. *Let f denote a diffeomorphism of the C^∞ manifold M. Prove*

$$i_X(f^*\alpha) = f^*(i_{f\cdot X}\alpha), \qquad X \in \mathfrak{X}(M), \quad \alpha \in \Lambda^* M$$

Solution. If $\alpha \in \Lambda^r M$, then for $X_1, \dots, X_{r-1} \in \mathfrak{X}(M)$, one has

$$\begin{aligned}
(i_X(f^*\alpha))_p(X_1|_p, \dots, X_{r-1}|_p) &= (f^*\alpha)_p(X_p, X_1|_p, \dots, X_{r-1}|_p) \\
&= \alpha_{f(p)}(f_* X_p, f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)) \\
&= \alpha_{f(p)}((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}),
\end{aligned}$$

and

$$\begin{aligned}
(f^*(i_{f\cdot X}\alpha))_p(X_1|_p, \dots, X_{r-1}|_p) &= (i_{f\cdot X}\alpha))_{f(p)}(f_*(X_1|_p), \dots, f_*(X_{r-1}|_p)) \\
&= \alpha_{f(p)}((f\cdot X)_{f(p)}, (f\cdot X_1)_{f(p)}, \dots, (f\cdot X_{r-1})_{f(p)}).
\end{aligned}$$

Problem 2.4.8. *Let M a C^∞ manifold, and consider $X \in \mathfrak{X}(M)$, $\omega \in \Lambda^r M$, $\omega' \in \Lambda^{r-1}M$, such that $i_X\omega = 0$, $i_X\mathrm{d}\omega = 0$, $i_X\mathrm{d}\omega' = 0$. Compute $i_X(\omega \wedge \mathrm{d}\omega')$ and $i_X\mathrm{d}(\omega \wedge \mathrm{d}\omega')$.*

Solution. Since i_X is an antiderivation, we have

$$i_X(\omega \wedge \mathrm{d}\omega') = (i_X\omega) \wedge \mathrm{d}\omega' + (-1)^r \omega \wedge i_X \mathrm{d}\omega' = 0,$$

$$\begin{aligned} i_X \mathrm{d}(\omega \wedge \mathrm{d}\omega') &= i_X(\mathrm{d}\omega \wedge \mathrm{d}\omega') \\ &= (i_X \mathrm{d}\omega) \wedge \mathrm{d}\omega' + (-1)^{r+1} \mathrm{d}\omega \wedge i_X \mathrm{d}\omega' = 0. \end{aligned}$$

Problem 2.4.9. *Consider on an open subset of* $\mathbb{R}^3$ *the differential* 1*-form*

$$\alpha = P_1(x)\,\mathrm{d}x^1 + P_2(x)\,\mathrm{d}x^2 + P_3(x)\,\mathrm{d}x^3,$$

where $x = (x^1, x^2, x^3)$.

(1) *Find the conditions under which* $i_X \mathrm{d}\alpha = 0$*, for*

$$X = X_1\,\partial/\partial x + X_2\,\partial/\partial y + X_3\,\partial/\partial z.$$

(2) *When do we have* $i_X \alpha = 0$ *and* $i_X \mathrm{d}\alpha = 0$*?*

Solution. (1) Let us compute $\mathrm{d}\alpha$. If we write $P_{ij} = \partial P_i/\partial x^j,\ Q_{ji} = P_{ji} - P_{ij}$, then

$$\begin{aligned} \mathrm{d}\alpha &= (P_{2,1} - P_{1,2})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + (P_{3,1} - P_{1,3})\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + (P_{3,2} - P_{2,3})\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j. \end{aligned}$$

Hence,

$$\begin{aligned} i_X \mathrm{d}\alpha = 0 &\Leftrightarrow i_X \mathrm{d}\alpha(Y) = 0, \quad \text{for all } Y \in \mathfrak{X}(\mathbb{R}^3) \\ &\Leftrightarrow \mathrm{d}\alpha\left(X, \frac{\partial}{\partial x^k}\right) = 0, \quad k = 1,2,3 \\ &\Leftrightarrow \sum_{i<j} Q_{ji}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \left(X_l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) \\ &\qquad = \sum_l \sum_{i<j} Q_{ji}(X_l \delta^i_l \delta^j_k - X_l \delta^i_k \delta^j_l) \\ &\qquad = \sum_l \left(\sum_{l<k} Q_{kl} X_l - \sum_{k<l} Q_{lk} X_l\right) \\ &\qquad = \sum_l Q_{kl} X_l = 0, \quad k = 1,2,3. \end{aligned}$$

(2) By (1),

$$i_X \mathrm{d}\alpha = 0 \quad \Leftrightarrow \quad \sum_{l=1}^{3} Q_{kl} X_l = 0, \quad k = 1,2,3,$$

and

$$i_X \alpha = \alpha(X) = 0 \quad \Leftrightarrow \quad (P_i\,\mathrm{d}x^i)\left(X^j \frac{\partial}{\partial x^j}\right) = 0 \quad \Leftrightarrow \quad P_i X^i = 0.$$

2.5 Distributions and Integral Manifolds. Frobenius' Theorem. Differential Ideals

Problem 2.5.1. *Consider on the octant of* $\mathbb{R}^3$ *of positive coordinates the vector fields*

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \qquad Y = xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}.$$

(1) *Prove that they span an involutive distribution on* $\mathbb{R}^3$.

(2) *Find the integral surfaces.*

HINT (to (2)): *Substitute* Y *by* $x^{-1}Y$.

Solution. (1) $[X,Y] = Y$.

(2) Since in the given domain x does not vanish, we can substitute $x^{-1}Y$ for Y, which, jointly with X, determines the same distribution. The integral curves of X are $(x_0 e^t, y_0 e^{-2t}, z_0)$ and that of $x^{-1}Y$ are $(x_0, y_0 e^s, z_0 e^{-s})$, so that the respective local flows are

$$\varphi_t(x,y,z) = (x e^t, y e^{-2t}, z), \qquad \psi_s(x,y,z) = (x, y e^s, z e^{-s}).$$

The map

$$\begin{aligned} (t,s) \in \mathbb{R}^2 \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0) &\\ &= \psi_s(x_0 e^t,\, y_0 e^{-2t},\, z_0) \\ &= (x_0 e^t,\, y_0 e^{-2t+s},\, z_0 e^{-s}), \end{aligned}$$

is the integral surface through (x_0, y_0, z_0). In fact, the point $(\psi_s \circ \varphi_t)(x_0, y_0, z_0)$ is obtained from (x_0, y_0, z_0) as follows: We first run an interval "t" from $p = (x_0, y_0, z_0)$ along the integral curve of X through p for $t = 0$, and then an interval "s" from $\varphi_t(p)$ along the integral curve of $x^{-1}Y$ through $\varphi_t(p)$ for $s = 0$. If we put

$$x(t,s) = x_0 e^t, \qquad y(t,s) = y_0 e^{-2t+s}, \qquad z(t,s) = z_0 e^{-s},$$

then we see that x^2yz is constant. Hence the integral surfaces are defined by $x^2yz =$ const. As a verification, observe that $X(x^2yz) = Y(x^2yz) = 0$.

Problem 2.5.2. *Consider on* $\mathbb{R}^3$ *the distribution* $\mathscr{D}$ *determined by*

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1+x^2+y^2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{2yz}{1+x^2+y^2}\frac{\partial}{\partial z}.$$

(1) *Calculate* $[X,Y]$ *and find whether* $\mathscr{D}$ *is involutive or not.*

(2) *Calculate the local flows of* X *and* Y.

(3) *If* $\mathscr{D}$ *is involutive, find its integral surfaces.*

Solution. (1) $[X,Y] = 0$, thus $\mathscr{D}$ is involutive.

(2) We have

$$\begin{cases} x' = 1 \\ y' = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = x_0 + t \\ y = y_0 \end{cases}$$

$$\begin{aligned} \frac{z'}{z} = \frac{2(x_0+t)}{1+(x_0+t)^2+y_0^2} \quad &\Leftrightarrow \quad \log z = \log A(1+(x_0+t)^2+y_0^2) \\ &\Leftrightarrow \quad z = A(1+(x_0+t)^2+y_0^2). \end{aligned}$$

For $t=0$, $z_0 = A(1+x_0^2+y_0^2)$, so $z = z_0\dfrac{1+(x_0+t)^2+y_0^2}{1+x_0^2+y_0^2}$.
Hence the local flow of X is

$$\varphi_t(x,y,z) = \left(x+t,\ y,\ z\frac{1+(x+t)^2+y^2}{1+x^2+y^2}\right).$$

Similarly, the local flow of Y is

$$\psi_s(x,y,z) = \left(x,\ y+s,\ z\frac{1+x^2+(y+s)^2}{1+x^2+y^2}\right).$$

(3) The integral manifolds can be written as $\psi(t,s) \mapsto (\psi_s \circ \varphi_t)(x_0, y_0, z_0)$. But let us see a better solution. We are looking for a differential 1-form annihilating X and Y. For example, we have as a solution:

$$\begin{aligned} \alpha &= 2xz\,\mathrm{d}x + 2yz\,\mathrm{d}y - (1+x^2+y^2)\mathrm{d}z \\ &= z\,\mathrm{d}(1+x^2+y^2) - (1+x^2+y^2)\mathrm{d}z \\ &= -(1+x^2+y^2)^2\mathrm{d}\left(\frac{z}{1+x^2+y^2}\right). \end{aligned}$$

Hence, the integral manifolds are $\dfrac{z}{1+x^2+y^2} = \text{const.}$

Problem 2.5.3. *The vector field* $X = x\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$*, defined on* $x>0$*,* $y>0$*,* $z>0$ *in* $\mathbb{R}^3$*, determines a* 2*-dimensional distribution given by the vector fields orthogonal to* X*. Is this distribution involutive?*

Solution. The vector fields $U = -y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$ and $V = -z\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial z}$ are orthogonal to X and linearly independent at each point. They span that distribution, but $[U,V] = -y\dfrac{\partial}{\partial z}$. Since

$$\begin{vmatrix} -y & 1 & 0 \\ -z & 0 & x \\ 0 & 0 & -y \end{vmatrix} = -yz$$

is not identically zero, we have $[U,V]_p \notin \langle U_p, V_p \rangle$. Hence the distribution is not involutive.

Problem 2.5.4. *Prove that*

$$X = \cos^2 x \frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y}$$

determines a foliation with non-Hausdorff quotient.

Solution. This vector field determines an integrable distribution of codimension 1 of $\mathbb{R}^2$. We have two kind of solutions: Integrating the equation that X determines,

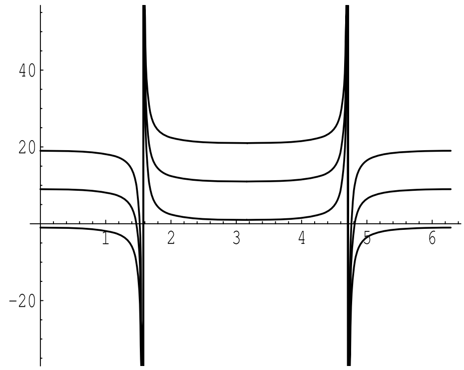

Fig. 2.2 An example of foliation with non-Hausdorff quotient manifold.

i.e.

$$\frac{\mathrm{d}x}{\cos^2 x} = \frac{\mathrm{d}y}{\sin x},$$

we obtain the curves

$$y = \sec x + A$$

(see Figure 2.2) for $x \neq (2k+1)\pi/2$, $k \in \mathbb{Z}$.

Moreover, we have the solutions with initial conditions of the type $((2k+1)\pi/2, y_0)$. That is, the straight lines $t \mapsto ((2k+1)\pi/2, (-1)^k t)$. Actually, if p and q are two non-separable points of the quotient, then each of them corresponds to a solution of this kind.

Take, for instance, the integral curve $x = -\pi/2$; a point on it, say $(-\pi/2, y_0)$; and an open disk around this point. This open disk intersects all the integral curves intersecting the y-axis at the points with ordinate greater than or equal to $A_0 > 0$. This phenomenon is also true for open disks around the point $(\pi/2, y_1)$. Such an open disk intersects all the integral curves which intersect the y-axis at points with ordinate greater than or equal to $A_1 > 0$. Now, the integral curves intersecting the y-axis at points with ordinate greater than $\max(A_0, A_1)$ intersect both open disks. Hence the projections of the two open disks on the quotient intersect, so that the projections of $x = -\pi/2$ and of $x = \pi/2$ cannot be separated. Consequently, the quotient manifold is not Hausdorff.

Problem 2.5.5. *Consider on* $\mathbb{R}^3$ *the vector fields*

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad Z = z\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

(1) *Prove that* X, Y, Z *define a* C^∞ *distribution* $\mathscr{D}$ *on* $\mathbb{R}^3$. *Which dimension is it? Is it involutive?*

(2) *Compute the set* $\mathscr{I}(\mathscr{D})$ *of forms which annihilate* $\mathscr{D}$. *Is it a differential ideal? Is the ideal* $\mathscr{I} = \langle \mathrm{e}^x\,\mathrm{d}y\rangle$ *a differential ideal?*

Solution. (1) X, Y, Z are not linearly independent because $Z = X - Y$. Hence $\mathscr{D}$ is a 2-dimensional C^∞ distribution spanned, for instance, by X and Y, which are linearly independent. $\mathscr{D}$ is not involutive, as $[X,Y] = -\dfrac{\partial}{\partial x}$, and $-\dfrac{\partial}{\partial x} \notin \mathscr{D}$, since if it were

$$-\frac{\partial}{\partial x} = az\frac{\partial}{\partial x} + a\frac{\partial}{\partial z} + b\frac{\partial}{\partial y} + b\frac{\partial}{\partial z},$$

we should have $az = -1$, $b = 0$, $b + a = 0$, which is absurd.

(2) $\{X, Y, \partial/\partial x\}$ is a basis of $\mathfrak{X}(\mathbb{R}^3)$. Therefore, if $\{\alpha, \beta, \omega\}$ is its dual basis of 1-forms, then $\mathscr{I}(\mathscr{D}) = \langle\omega\rangle$, where $\langle\omega\rangle$ stands for the ideal generated by ω.

Let us determine $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$, $f, g, h \in C^\infty\mathbb{R}^3$. From

$$0 = \omega(X) = fz + h, \qquad 0 = \omega(Y) = g + h, \qquad 1 = \omega\left(\frac{\partial}{\partial x}\right) = f,$$

it follows that $f = 1$. Thus $h = -z$, hence $g = z$; that is, $\omega = \mathrm{d}x + z\,\mathrm{d}y - z\,\mathrm{d}z$. Since $\mathscr{D}$ is not involutive, $\mathscr{I}(\mathscr{D})$ cannot be a differential ideal.

We can also prove this directly. One has $d\omega = \mathrm{d}z \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}z$. If it were, for $a, b, c \in C^\infty\mathbb{R}^3$,

$$\begin{aligned}\mathrm{d}\omega &= \omega \wedge (a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z) \\ &= (b - az)\,\mathrm{d}x \wedge \mathrm{d}y + (c + az)\,\mathrm{d}x \wedge \mathrm{d}z + (zc + zb)\,\mathrm{d}y \wedge \mathrm{d}z,\end{aligned}$$

we should have $b - az = 0$, $c + az = 0$, $zc + zb = -1$. From the first and second equations one has $b + c = 0$, in contradiction with the third equation. One can also conclude by applying Problem 2.3.11, as

$$\omega \wedge \mathrm{d}\omega = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0.$$

Finally, $\mathscr{I}$ is a differential ideal, for

$$\begin{aligned}\mathrm{d}(\mathrm{e}^x\,\mathrm{d}y) &= \mathrm{e}^x\,\mathrm{d}x \wedge \mathrm{d}y \\ &= \mathrm{e}^x\,\mathrm{d}y \wedge (-\mathrm{d}x).\end{aligned}$$

Problem 2.5.6. *Given on* $\mathbb{R}^4 = \{(x,y,z,t)\}$ *the* 1*-forms* $\alpha = \mathrm{d}x + z\mathrm{d}t$ *and* $\beta = \mathrm{d}z + \mathrm{d}t$ *let* $\mathscr{I}$ *be the ideal generated by* α *and* β*, and let* $\mathscr{D}$ *be the distribution associated to* $\mathscr{I}$*.*

(1) *Compute a basis for* $\mathscr{D}$*.*

(2) *Is* $\mathscr{D}$ *involutive?*

(3) *If* $p = (1,0,1,0) \in \mathbb{R}^4$*, do we have* $v_p = -3\left.\dfrac{\partial}{\partial y}\right|_p + z\left.\dfrac{\partial}{\partial x}\right|_p \in \mathscr{D}_p$*?*

(4) *If* $\omega = \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}z \wedge \mathrm{d}t$*, is* $\omega \in \mathscr{I}$*?*

(5) *Is* $y = \mathrm{const}$*,* $z = \mathrm{const}$*, an integral manifold of* $\mathscr{D}$*?*

Solution. (1) For $X, Y \in \mathscr{D}$ given by

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d\frac{\partial}{\partial t}, \qquad Y = e\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\frac{\partial}{\partial z} + h\frac{\partial}{\partial t},$$

for $a,b,c,d,e,f,g,h \in C^\infty\mathbb{R}^4$, it must be that

$$\alpha(X) = a + zd = 0, \qquad \alpha(Y) = e + zh = 0,$$
$$\beta(X) = c + d = 0, \qquad \beta(Y) = g + h = 0.$$

Thus, for instance, we can consider

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y}.$$

(2) $[X,Y] = 0$, hence $\mathscr{D}$ is involutive.

(3) No, as

$$\alpha_p(v_p) = (\mathrm{d}x + z\mathrm{d}t)_p\left(-3\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}\right)_p = 1 \neq 0.$$

(4) $\omega = \mathrm{d}x \wedge \beta + \mathrm{d}z \wedge \beta$, hence $\omega \in \mathscr{I}$.

(5) The tangent space is $\left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t} \right\rangle$, but $\alpha\left(\dfrac{\partial}{\partial x}\right) = 1$, so $y = \mathrm{const}$, $z = \mathrm{const}$, is not an integral manifold of $\mathscr{D}$.

Problem 2.5.7. *Prove that the* 1*-form* $\alpha = (1+y^2)(x\mathrm{d}y + y\mathrm{d}x)$*, defined on* $\mathbb{R}^2 - \{0\}$*, generates a rank-*1 *differential ideal and find the integral manifolds.*

Solution. Since $1+y^2$ does not vanish, α generates the same annihilator ideal as

$$\frac{\alpha}{1+y^2} = x\mathrm{d}y + y\mathrm{d}x$$
$$= \mathrm{d}(xy).$$

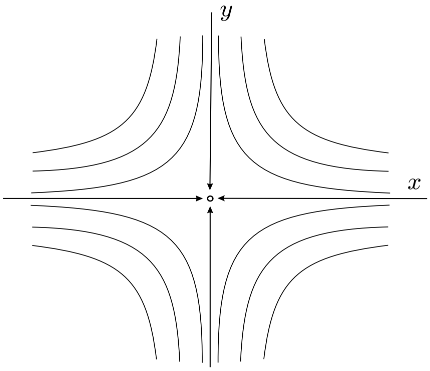

Fig. 2.3 Integral manifolds of $\alpha = (1+y^2)(x\,\mathrm{d}y + y\,\mathrm{d}x)$.

As $\mathrm{d}(x\,\mathrm{d}y + y\,\mathrm{d}x) = 0$, the ideal is differential. The integral manifolds are $xy = \text{const}$ (see Figure 2.3).

Problem 2.5.8. *Let $U = \mathbb{R}^3 -$ axes. Compute the integral surfaces of the distribution determined by the ideal of $\Lambda^* U$ generated by*

$$\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z.$$

Solution. We have $\alpha = \mathrm{d}(xyz)$. If X is annihilated by α, then we have $\alpha(X) = X(xyz) = 0$. Thus the integral surfaces are the surfaces $xyz = \text{const}$ (see Figure 2.4).

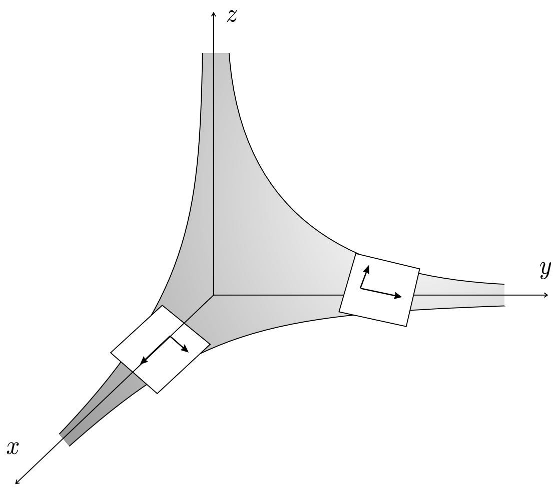

Fig. 2.4 The component in the first octant of an integral surface of the distribution $\alpha = yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z$.

Problem 2.5.9. *Consider the* $(1,1)$ *tensor field*

$$J = \frac{1}{\cosh x}\frac{\partial}{\partial y} \otimes \mathrm{d}x + \cosh x \frac{\partial}{\partial x} \otimes \mathrm{d}y$$

on $\mathbb{R}^2$ *and the distribution* $\mathscr{D}$ *defined by the condition*: $X \in \mathscr{D}$ *if and only if* $JX = X$.

(1) *Compute the integral curves of* $\mathscr{D}$.

(2) *Compute the fields* $X \in \mathscr{D}$ *for which* $L_X J = 0$.

Solution. (1) If $X = f\dfrac{\partial}{\partial x} + h\dfrac{\partial}{\partial y} \in \mathscr{D},\ f, h \in C^\infty\mathbb{R}^2$, then

$$\begin{aligned}\left(\frac{1}{\cosh x}\frac{\partial}{\partial y} \otimes \mathrm{d}x + \cosh x\frac{\partial}{\partial x} \otimes \mathrm{d}y\right)\left(f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}\right) &= \frac{f}{\cosh x}\frac{\partial}{\partial y} + h\cosh x\frac{\partial}{\partial x} \\ &= f\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}.\end{aligned}$$

Thus $f = h\cosh x$. Denoting by (x, y) the integral curves of $\mathscr{D}$, we have $\mathrm{d}x/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}t)\cosh x$. Hence $\mathrm{d}y = \mathrm{d}x/\cosh x$, and thus

$$y = \arctan\sinh x + A. \tag{$\star$}$$

That is, the integral curves of $\mathscr{D}$ are given by $(\star)$.

(2)

$$\begin{aligned}L_X J = {}&\left(h_x\cosh x - \frac{f_y}{\cosh x}\right)\left(\frac{\partial}{\partial x} \otimes \mathrm{d}x - \frac{\partial}{\partial y} \otimes \mathrm{d}y\right) \\ &+ (h_y\cosh x + f\sinh x - f_x\cosh x)\left(\frac{\partial}{\partial x} \otimes \mathrm{d}y - \frac{1}{\cosh^2 x}\frac{\partial}{\partial y} \otimes \mathrm{d}x\right) = 0.\end{aligned} \tag{$\star\star$}$$

Moreover, if $X \in \mathscr{D}$, then we have $f = g\cosh x$, and from this equation and $(\star\star)$ we conclude that we have to solve only the following equation:

$$\frac{\partial h}{\partial x}\cosh x = \frac{\partial h}{\partial y}.$$

Let $u = 2\arctan \mathrm{e}^x$. Then we have

$$\frac{\partial h}{\partial x} = \frac{1}{\cosh x}\frac{\partial h}{\partial u},$$

hence $\dfrac{\partial h}{\partial u} = \dfrac{\partial h}{\partial y}$. Taking $t = u + y$, $w = u - y$, we obtain

$$0 = \frac{\partial h}{\partial u} - \frac{\partial h}{\partial y} = 2\frac{\partial h}{\partial w}.$$

Thus $h = h(u+y) = h(2\arctan \mathrm{e}^x + y)$, and we finally have

$$f = h(2\arctan \mathrm{e}^x + y)\cosh x,$$

where $h(2\arctan \mathrm{e}^x + y)$ is an arbitrary differentiable function in that argument.

Problem 2.5.10. *Let M be a C^∞ n-manifold and let $\mathscr{D} \subset TM$ be an integrable distribution of rank p. By Frobenius' theorem, $\mathscr{D}$ is spanned by $\partial/\partial x^1, \dots, \partial/\partial x^p$ on an open subset U of M, for a certain coordinate system (U, x^i). We can consider local frames of M of the type*

$$(\partial/\partial x^1, \dots, \partial/\partial x^p, X_1, \dots, X_q), \qquad p + q = n = \dim M,$$

where $X_u = \partial/\partial x^{p+u} - f_u^a \partial/\partial x^a$, $1 \leqslant a \leqslant p$, $1 \leqslant u \leqslant q$, $f_u^a \in C^\infty M$.

Write the integrability condition of the complementary distribution $\mathscr{H}$ generated by $X_1, \dots, X_q$, on the open subset where these vector fields are defined.

Solution. In order for $\mathscr{H}$ to be integrable it must be $[X_u, X_v] \in \mathscr{H}$ for any $X_u, X_v \in \mathscr{H}$, $u, v = 1, \dots, q$. Then

$$\begin{aligned}[X_u, X_v] &= [\partial/\partial x^{p+u} - f_u^a \partial/\partial x^a, \partial/\partial x^{p+v} - f_v^b \partial/\partial x^b] \\ &= \left(\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b} \right) \frac{\partial}{\partial x^a} \in \mathscr{D}.\end{aligned}$$

As $[X_u, X_v] \in \mathscr{H}$, the last expression in parentheses must be zero, that is, the condition is

$$\frac{\partial f_u^a}{\partial x^{p+v}} - \frac{\partial f_v^a}{\partial x^{p+u}} + f_u^b \frac{\partial f_v^a}{\partial x^b} - f_v^b \frac{\partial f_u^a}{\partial x^b} = 0.$$

2.6 Almost Symplectic Manifolds

Problem 2.6.1. *Denote by $(q^1, \dots, q^n, p_1, \dots, p_n)$ the usual Cartesian coordinates of the space $\mathbb{R}^{2n}$, on which we consider*:

(a) *The 2-form $\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$.*

(b) *A hypersurface S defined by the implicit equation $H(q, p) = \mathrm{const.}$*

(c) *The vector field X such that $i_X \Omega = -\mathrm{d}H$.*

Prove that:

(1) *X is tangent to S.*

(2) *If ω_{2n-1} is a $(2n-1)$-form such that*

$$\Omega \wedge \overset{(n}{\cdots} \wedge \Omega = \omega_{2n-1} \wedge \mathrm{d}H,$$

then $L_X(\omega_{2n-1}|_S) = 0$.

Solution. (1) We have to prove that $XH = 0$, but

$$\begin{aligned} XH &= (\mathrm{d}H)(X) \\ &= -(i_X\Omega)(X) \\ &= -\Omega(X,X) = 0. \end{aligned}$$

(2) Considering the Lie derivative with respect to X of both sides of the equality $\Omega \wedge \overset{(n}{\cdots} \wedge \Omega = \omega_{2n-1} \wedge \mathrm{d}H$, we obtain

$$\sum_{i=1}^{n} \Omega \wedge \cdots \wedge \overset{i\text{th place}}{L_X\Omega} \wedge \cdots \wedge \Omega = (L_X\omega_{2n-1}) \wedge \mathrm{d}H + \omega_{2n-1} \wedge L_X \mathrm{d}H. \qquad (\star)$$

As

$$\begin{aligned} L_X\Omega &= i_X\,\mathrm{d}\Omega + \mathrm{d}i_X\Omega \\ &= \mathrm{d}i_X\Omega \\ &= -\mathrm{d}(\mathrm{d}H) = 0, \end{aligned}$$

and

$$\begin{aligned} L_X\mathrm{d}H &= \mathrm{d}L_X H \\ &= \mathrm{d}(XH) = 0, \end{aligned}$$

the equation $(\star)$ is reduced to

$$(L_X\omega_{2n-1}) \wedge \mathrm{d}H = 0. \qquad (\star\star)$$

Let $(x^1,\dots,x^{2n-1},H)$ be a local coordinate functions adapted to S; that is, such that $(\mathrm{d}H)|_S = 0$. Then, for some $(2n-2)$-form ω_{2n-2}, from $(\star\star)$ it follows that

$$L_X\omega_{2n-1} = \omega_{2n-2} \wedge \mathrm{d}H + \lambda\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n-1},$$

so

$$0 = (L_X\omega_{2n-1}) \wedge \mathrm{d}H = \lambda\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n-1} \wedge \mathrm{d}H,$$

and thus $\lambda = 0$. Hence $L_X\omega_{2n-1} = \omega_{2n-2} \wedge \mathrm{d}H$, so one has

$$L_X(\,\omega_{2n-1}|_S) = \omega_{2n-2}|_S \wedge (\mathrm{d}H)|_S = 0,$$

because $(\mathrm{d}H)|_S = 0$.

Problem 2.6.2. *Let $\pi\colon T^*M \to M$ be the cotangent bundle over a C^∞ n-manifold M. The* canonical 1-form ϑ *on T^*M is defined by*

$$\vartheta_\omega(X) = \omega(\pi_* X), \qquad \omega \in T^*M, \quad X \in T_\omega T^*M.$$

(1) *Compute the local expression of* ϑ *and prove that the* 2*-form* $\Omega = \mathrm{d}\vartheta$ *is nondegenerate*; *that is, that* $i_X\Omega = 0$ *implies* $X = 0$.

(2) *Show that* $\Omega \wedge \overset{(n}{\cdots} \wedge \Omega \neq 0$ *at each point. Hence* T^*M *is orientable.* Ω *is called the* canonical symplectic form *on* T^*M.

Let $H \in C^\infty(T^*M)$ *and let* $\sigma\colon (a,b) \to T^*M$ *be a* C^∞ *curve with tangent vector* σ'.

(3) *Write locally the differential equations*

$$i_{\sigma'}(\Omega \circ \sigma) + \mathrm{d}H \circ \sigma = 0 \qquad \text{(Hamilton equations)}.$$

(4) *Show that if* σ *is a solution,* $H \circ \sigma$ *is a constant function.*

(5) *Solve the Hamilton equations for the case* $M = \mathbb{R}^n$, *and*

$$H = \frac{1}{2}k(q^1)^2 + \frac{1}{2}m\sum_{i=1}^{n-1} p_i^2 + \frac{1}{2}p_n^2,$$

where k *and* m *stand for constants.*

Solution. (1) Given local coordinates $(q^1,\dots,q^n)$ on M, they induce local coordinates $(q^1,\dots,q^n,p_1,\dots,p_n)$ on T^*M putting $\omega_x = p_i(\omega_x)\,\mathrm{d}q^i\big|_x$ for $\omega_x \in T^*M$, $x \in M$. If

$$X = \lambda^i \left.\frac{\partial}{\partial q^i}\right|_{\omega_x} + \mu^i \left.\frac{\partial}{\partial p_i}\right|_{\omega_x} \in T_{\omega_x}T^*M,$$

then from the definition of ϑ it follows:

$$\begin{aligned}
\vartheta(X) &= \omega_x(\pi_* X_{\omega_x}) \\
&= \omega_x\left(\lambda^i \left.\frac{\partial}{\partial q^i}\right|_x\right) \\
&= \left(p_j(\omega_x)\,\mathrm{d}q^j\big|_x\right)\left(\lambda^i \left.\frac{\partial}{\partial q^i}\right|_x\right) \\
&= \lambda_i p_i(\omega_x) \\
&= p_i(\omega_x)\left(\mathrm{d}q^i\big|_{\omega_x}\right)(X) \\
&= (p_i\,\mathrm{d}q^i)(X),
\end{aligned}$$

and so $\vartheta = p_i\,\mathrm{d}q^i$; hence $\Omega = \mathrm{d}\vartheta = \mathrm{d}p_i \wedge \mathrm{d}q^i$, which is obviously nondegenerate.

(2) From (1) we have, $\mathfrak{S}_n$ being the group of permutations of order n, and $\operatorname{sgn}\sigma$ the sign of the permutation $\sigma \in \mathfrak{S}_n$:

$$\begin{aligned}
\Omega^n &= (\mathrm{d}p_1 \wedge \mathrm{d}q^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q^n) \wedge \cdots \wedge (\mathrm{d}p_1 \wedge \mathrm{d}q^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q^n) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{d}p_{\sigma(1)} \wedge \mathrm{d}q^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)} \wedge \mathrm{d}q^{\sigma(n)} \\
&= (-1)^{1+2+\cdots+n} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{d}q^{\sigma(1)} \wedge \cdots \wedge \mathrm{d}q^{\sigma(n)} \wedge \mathrm{d}p_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\frac{n(n+1)}{2}} \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn}\sigma)^2 \, \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \\
&= n!\,(-1)^{\frac{n(n+1)}{2}} \, \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \\
&\neq 0.
\end{aligned}
$$

(3) Put $\sigma_i = q^i \circ \sigma$, $\widetilde{\sigma}_i = p_i \circ \sigma$. We have $\sigma' = \dfrac{\mathrm{d}\sigma_i}{\mathrm{d}t}\dfrac{\partial}{\partial q^i} + \dfrac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t}\dfrac{\partial}{\partial p_i}$. Therefore

$$
\begin{aligned}
i_{\sigma'}(\Omega \circ \sigma) &= \sigma'(p_i)\, \mathrm{d}q^i\big|_\sigma - \sigma'(q^i)\, \mathrm{d}p_i|_\sigma \\
&= \frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t}\, \mathrm{d}q^i\big|_\sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t}\, \mathrm{d}p_i|_\sigma .
\end{aligned}
$$

Hence

$$
i_{\sigma'}(\Omega \circ \sigma) + \mathrm{d}H \circ \sigma = \left(\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} + \frac{\partial H}{\partial q^i} \circ \sigma \right) \mathrm{d}q^i\big|_\sigma + \left(\frac{\partial H}{\partial p_i} \circ \sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t} \right) \mathrm{d}p_i|_\sigma ,
$$

and the Hamilton equations are

$$
\frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} + \frac{\partial H}{\partial q^i} \circ \sigma = 0, \qquad \frac{\partial H}{\partial p_i} \circ \sigma - \frac{\mathrm{d}\sigma_i}{\mathrm{d}t} = 0.
$$

(4) If σ is a solution, then $\dfrac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q^i} \circ \sigma$ and $\dfrac{\mathrm{d}\sigma_i}{\mathrm{d}t} = \dfrac{\partial H}{\partial p_i} \circ \sigma$. So

$$
\frac{\mathrm{d}}{\mathrm{d}t}(H \circ \sigma) = \left(\frac{\partial H}{\partial q^i} \circ \sigma \right) \frac{\mathrm{d}\sigma_i}{\mathrm{d}t} + \left(\frac{\partial H}{\partial p_i} \circ \sigma \right) \frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} = 0.
$$

Thus $H \circ \sigma$ is a constant function.

(5)

$$
(a) \qquad \frac{\mathrm{d}\widetilde{\sigma}_1}{\mathrm{d}t} = -\frac{\partial H}{\partial q^1} \circ \sigma = -k\sigma_1, \qquad \frac{\mathrm{d}\widetilde{\sigma}_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} \circ \sigma = 0,
$$

for $i = 2, \dots, n$, hence:

(b) $\widetilde{\sigma}_i = A_i$, $i = 2, \dots, n$, with A_i constants.

(c) $\dfrac{\mathrm{d}\sigma_i}{\mathrm{d}t} = \dfrac{\partial H}{\partial p_i} \circ \sigma = m\widetilde{\sigma}_i$, for $i = 1, \dots, n-1$.

(d) $\dfrac{\mathrm{d}\sigma_n}{\mathrm{d}t} = \dfrac{\partial H}{\partial p_n} \circ \sigma = \widetilde{\sigma}_n$.

From (b) and (c) it follows that $\sigma_i = mA_i t + B_i$, for $i = 2, \dots, n-1$, $A_i, B_i \in \mathbb{R}$. From (a) and (c) we deduce that $\dfrac{\mathrm{d}\sigma_1}{\mathrm{d}t} = m\widetilde{\sigma}_1$ and $\dfrac{\mathrm{d}\widetilde{\sigma}_1}{\mathrm{d}t} = -k\sigma_1$, hence $\dfrac{\mathrm{d}^2\sigma_1}{\mathrm{d}t^2} + km\sigma_1 = 0$, and we have four cases:

(i) $k \neq 0$, $m = 0$, $\sigma_1 = A$, $\widetilde{\sigma}_1 = -kAt + B$,

(ii) $k=0,\ m\neq 0,\ \sigma_1 = mCt+D,\ \widetilde{\sigma}_1 = C,$

(iii) $km=\omega^2>0,\ \sigma_1 = E\cos\omega t + F\sin\omega t,\ \widetilde{\sigma}_1 = -\dfrac{\omega}{m}(E\sin\omega t - F\cos\omega t),$

(iv) $km=-\omega^2<0,\ \sigma_1 = G\cosh\omega t + H\sinh\omega t,$

$$\widetilde{\sigma}_1 = -\frac{\omega}{m}(G\sinh\omega t + H\cosh\omega t).$$

Finally, from (b) and (d) we have $\dfrac{\mathrm{d}\sigma_n}{\mathrm{d}t} = A_n$, thus $\sigma_n = A_n t + B_n$, for $A_n, B_n \in \mathbb{R}$.

Problem 2.6.3. *Consider the trivial principal bundle $\pi\colon M\times U(1)$ over the C^∞ n-manifold M. We use the same notations as in* Problem 5.2.3.

(1) *Let Φ_t be the flow of a vector field $X\in\mathfrak{X}(P)$. Prove that X is $U(1)$-invariant if and only if Φ_t is an automorphism of P, for all $t\in\mathbb{R}$.*

(2) *Let $p\colon T^*M\to M$ be the cotangent bundle over M. Each coordinate system $(U,q^1,\dots,q^n)$ on M induces a coordinate system $(p^{-1}(U),\ q^1,\dots,q^n,p_1,\dots,p_n)$ by setting $w=p_i(w)\mathrm{d}q^i|_x$ for all covector $w\in T_x^*M$.*

If Φ_t is the flow of a $U(1)$-invariant vector field $X\in\mathfrak{X}(P)$, then $\widetilde{\Phi}_t$ is a flow on T^*M, which generates a vector field $\widetilde{X}$. Prove that

$$\widetilde{X} = f^i\frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i}p_h\right)\frac{\partial}{\partial p_i},$$

where

$$X = f^i(q^1,\dots,q^n)\frac{\partial}{\partial q^i} + g(q^1,\dots,q^n)\frac{\partial}{\partial\alpha},$$

and α stands for the local coordinate on $U(1)$.

(3) *Let ϑ be the canonical form on T^*M and let $\Phi(x,\alpha)=(\phi(x),\alpha+\psi(x))$ be an automorphism of P. Compute $\Phi^*\vartheta$.*

(4) *Conclude that every automorphism of P leaves the canonical symplectic form $\mathrm{d}\vartheta$ invariant.*

(5) *Prove that $L_{\widetilde{X}}\mathrm{d}\vartheta=0$, for every $U(1)$-invariant vector field X.*

Solution. (1) The vector field X is $U(1)$-invariant if and only if for every $z\in U(1)$ we have $R_z\cdot X = X$. This means that R_z commutes with Φ_t; i.e. $R_z\circ\Phi_t = \Phi_t\circ R_z$, or equivalently, $\Phi_t(u)\cdot z = \Phi_t(u\cdot z)$, thus proving that Φ_t is an automorphism.

(2) If $\Phi(x,\alpha)=(\phi(x),\alpha+\psi(x))$, from (4) in Problem 5.2.3, we have

$$\widetilde{\Phi}(w) = (\phi^{-1})^*w - (\mathrm{d}(\psi\circ\phi^{-1}))_{\phi(x)}, \qquad w\in T_x^*M.$$

As $p\circ\widetilde{\Phi} = \phi\circ p$, we have

$$q^i\circ\widetilde{\Phi} = q^i\circ\phi. \tag{$\star$}$$

Moreover, from the very definition of the coordinates (p_i), we obtain

$$
\begin{aligned}
(p_i \circ \widetilde{\Phi})(w) &= p_i\big((\phi^{-1})^* w - (\mathrm{d}(\psi \circ \phi^{-1}))_{\phi(x)}\big) \\
&= p_h(w) p_i((\phi^{-1})^*(\mathrm{d}q^h|_x)) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^j}(\phi(x))\, p_i(\mathrm{d}q^j|_{\phi(x)}) \\
&= p_h(w) \frac{\partial(q^h \circ \phi^{-1})}{\partial q^i}(\phi(x)) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i}(\phi(x)).
\end{aligned}
$$

Hence

$$
p_i \circ \widetilde{\Phi} = p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi \right) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i} \circ \phi. \tag{$\star\star$}
$$

If $\Phi_t(x, \alpha) = (\phi_t(x), \alpha + \psi_t(x))$, then substituting $\widetilde{\Phi}_t$ for $\widetilde{\Phi}$ in ($\star$), ($\star\star$), taking derivatives with respect to t, and then $t = 0$, we obtain the formula for $\widetilde{X}$ in the statement.

(3) We have

$$
\begin{aligned}
\widetilde{\Phi}^* \vartheta &= \left(p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi \right) - \frac{\partial(\psi \circ \phi^{-1})}{\partial q^i} \circ \phi \right) \mathrm{d}(q^i \circ \phi) \\
&= p_h \left(\frac{\partial(q^h \circ \phi^{-1})}{\partial q^i} \circ \phi \right) \mathrm{d}(q^i \circ \phi) - \phi^* \mathrm{d}(\psi \circ \phi^{-1}) \\
&= p_h \,\mathrm{d}q^h - \phi^* \mathrm{d}(\psi \circ \phi^{-1}) \\
&= \vartheta - \mathrm{d}\psi.
\end{aligned}
$$

(4) From the previous formula, we have

$$
\Phi^* \mathrm{d}\vartheta = \mathrm{d}\vartheta.
$$

(5) It follows taking derivatives in $\Phi_t^* \,\mathrm{d}\vartheta = \mathrm{d}\vartheta$, for all $t \in \mathbb{R}$.

Problem 2.6.4. *Let ϑ be the canonical* 1*-form on the cotangent bundle T^*M over a C^∞ n-manifold M. Prove that* $\mathrm{d}\vartheta$ *is the only* 2*-form Ω on T^*M such that*:

(1) *The vertical bundle of the natural projection $p\colon T^*M \to M$ is a* Lagrangian foliation*; that is, the fibres of p are totally isotropic submanifolds.*

(2) *If η is a differential* 1*-form on M and we denote by τ_η the translation*

$$
\tau_\eta : T^*M \to T^*M, \quad \tau_\eta(w) = w + \eta(x), \qquad w \in T_x^*M, \quad x \in M,
$$

then

$$
\tau_\eta^* \Omega = \Omega + p^* \mathrm{d}\eta.
$$

(3) $L_{\widetilde{X}} \Omega = 0$*, for every $U(1)$-invariant vector field $X \in \mathfrak{X}(M \times U(1))$* (*see* Problem 5.2.3).

Solution. First we prove that $\mathrm{d}\vartheta$ satisfies (1), (2) and (3). Item (1) follows directly from the local expression $\Omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$, as the tangent space to the fibres of p is locally spanned by $\partial/\partial p_i$. As for (3), it follows from Problem 5.2.3.

Moreover, if $\eta = f_i\,\mathrm{d}q^i$, $f_i \in C^\infty M$, then the equations of τ_η are

$$q^i \circ \tau_\eta = q^i, \qquad p_j \circ \tau_\eta = p_j + f_j.$$

Hence

$$\begin{aligned}\tau_\eta^*\,\mathrm{d}\vartheta &= \mathrm{d}\tau_\eta^*\vartheta \\ &= \mathrm{d}((p_i \circ \tau_\eta)\,\mathrm{d}(q^i \circ \tau_\eta)) \\ &= \mathrm{d}(p_i + f_i) \wedge \mathrm{d}q^i \\ &= \mathrm{d}p_i \wedge \mathrm{d}q^i + \mathrm{d}f_i \wedge \mathrm{d}q^i \\ &= \mathrm{d}\vartheta + p^*\mathrm{d}\eta.\end{aligned}$$

Conversely, assume Ω satisfies (1)-(3). From (1) we have

$$\Omega = A_{hi}\,\mathrm{d}q^h \wedge \mathrm{d}q^i + B_i^h\,\mathrm{d}p_h \wedge \mathrm{d}q^i, \quad A_{hi} + A_{ih} = 0, \qquad A_{hi}, B_i^h \in C^\infty(T^*M).$$

Let us impose condition (2) on Ω. We have

$$\begin{aligned}\tau_\eta^*\Omega &= (A_{hi} \circ \tau_\eta)\mathrm{d}q^h \wedge \mathrm{d}q^i + (B_i^h \circ \tau_\eta)(\mathrm{d}p_h + \mathrm{d}f_h) \wedge \mathrm{d}q^i \\ &= A_{hi}\,\mathrm{d}q^h \wedge \mathrm{d}q^i + B_i^h\,\mathrm{d}p_h \wedge \mathrm{d}q^i + \mathrm{d}f_i \wedge \mathrm{d}q^i.\end{aligned}$$

Hence

$$A_{hi} \circ \tau_\eta + (B_i^j \circ \tau_\eta)\frac{\partial f_j}{\partial q^h} = A_{hi} + \frac{\partial f_j}{\partial q^h}\delta_{ij}, \tag{$\star$}$$

$$B_i^h \circ \tau_\eta = B_i^h.$$

Let $X = \partial/\partial q^l$ in (3). Then, we obtain

$$L_{\widetilde{X}}\Omega = \frac{\partial A_{hi}}{\partial q^l}\mathrm{d}q^h \wedge \mathrm{d}q^i + \frac{\partial B_i^h}{\partial q^l}\mathrm{d}p_h \wedge \mathrm{d}q^i = 0.$$

Accordingly,

$$\frac{\partial A_{hi}}{\partial q^l} = \frac{\partial B_i^h}{\partial q^l} = 0,$$

that is, A_{hi} and B_i^h depend only on $(p_1, \dots, p_n)$.

Next, let $X = q^l\partial/\partial\alpha$ in (3). Then we obtain $\widetilde{X} = -\partial/\partial p_l$, and

$$L_{\widetilde{X}}\Omega = -\frac{\partial A_{hi}}{\partial p_l}\mathrm{d}q^h \wedge \mathrm{d}q^i - \frac{\partial B_i^h}{\partial p_l}\mathrm{d}p_h \wedge \mathrm{d}q^i = 0.$$

Hence

$$\frac{\partial A_{hi}}{\partial p_l} = \frac{\partial B_i^h}{\partial p_l} = 0.$$

Therefore A_{hi} and B_i^h are constant functions.

Now, let us impose condition (3) for $X = q^k \partial/\partial q^l$, for two given indices k, l. We have

$$\widetilde{X} = q^k \frac{\partial}{\partial q^l} - p_l \frac{\partial}{\partial p_k}.$$

Hence

$$\begin{aligned} L_{\widetilde{X}} \Omega = &\sum_{k<i} (A_{li} - A_{il}) \mathrm{d}q^k \wedge \mathrm{d}q^i + \sum_{i<k} (A_{il} - A_{li}) \mathrm{d}q^i \wedge \mathrm{d}q^k \\ &- B_i^k \mathrm{d}p_l \wedge \mathrm{d}q^i + B_l^i \,\mathrm{d}p_i \wedge \mathrm{d}q^k = 0. \end{aligned}$$

Thus $A_{li} = A_{il}$. As $A_{il} + A_{li} = 0$, we have $A_{li} = 0$. Accordingly, the equation $(\star)$ now reads as

$$(B_i^j - \delta_{ij}) \frac{\partial f_j}{\partial q^h} = 0.$$

As the functions $\partial f_j / \partial q^h$ are arbitrary, we have $B_i^j = \delta_{ij}$, thus concluding.

Chapter 3
Integration on Manifolds

3.1 Orientable manifolds. Orientation-preserving maps

Problem 3.1.1. *Consider two charts (U,φ), (V,ψ) on an orientable manifold M, with U, V connected. Prove that the Jacobian determinant of the change of coordinates cannot change its sign on $\varphi(U \cap V)$.*

Solution. An orientation μ on M induces an orientation on U and an orientation on V. Since U is connected, $(x^1,\dots,x^n)$ is either positively oriented, i.e. $(\partial/\partial x^1|_p,\dots,\partial/\partial x^n|_p) \in \mu_p$, or negatively oriented, i.e. $(\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p) \notin \mu_p$, for every $p \in U$. Similarly for (V,ψ). If $\psi = (y^1,\dots,y^n)$, then $\partial/\partial x^j = (\partial y^i/\partial x^j)(\partial/\partial y^i)$. If both charts are positively oriented or both negatively oriented, then they are similarly oriented so $\det(\partial y^i/\partial x^j) > 0$. If one is positive and the other negative, then $\det(\partial y^i/\partial x^j) < 0$ on $\varphi(U \cap V)$. So, in any case, the determinant of the Jacobian of the change of coordinates cannot change its sign on $\varphi(U \cap V)$.

Problem 3.1.2. *Prove*:

(1) *The product of two orientable manifolds is orientable.*

(2) *The total space of the tangent bundle over any manifold is an orientable manifold.*

Solution. (1) A C^∞ manifold M is orientable if and only if (see Proposition 7.3.2,(2)) there is a collection Φ of coordinate systems on M such that

$$M = \bigcup_{(U,\varphi)\in\Phi} U \qquad \text{and} \qquad \det\left(\frac{\partial x^i}{\partial y^j}\right) > 0 \quad \text{on} \quad U \cap V$$

whenever $(U,x^1,\dots,x^n)$ and $(V,y^1,\dots,y^n)$ belong to Φ.

Suppose M_1 and M_2 are orientable. Denote by (U_1,x_1^i) and (U_2,x_2^j) two such coordinate systems on M_1 and M_2, respectively. With a little abuse of notation (that is, dropping the projection maps pr_1 and pr_2 from $M_1 \times M_2$ onto the factors M_1 and M_2), we can write the corresponding coordinate systems on $M_1 \times M_2$ as $(U_1 \times$

$U_2, x_1^i, x_2^j)$. As the local coordinates on each factor manifold do not depend on the local coordinates on the other one, the Jacobian matrix of the corresponding change of charts of the product manifold $M_1 \times M_2$ can be expressed in block form as

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_1^i}{\partial y_1^k} & 0 \\ 0 & \dfrac{\partial x_2^j}{\partial y_2^l} \end{pmatrix}.$$

Since $\det J_1$ and $\det J_2$ are positive, we have $\det J > 0$.

Alternatively, the question can be solved more intrinsically as follows: Given the nonvanishing differential forms of maximum degree ω_1 and ω_2 determining the respective orientations on M_1 and M_2, it suffices to consider the form $\omega = \mathrm{pr}_1^* \omega_1 \wedge \mathrm{pr}_2^* \omega_2$ on $M_1 \times M_2$.

(2) Let M be a differentiable n-manifold and let π be the projection map of the tangent bundle TM. For any coordinates $\{x^i\}$ on an open subset $U \subset M$, denote by $\{x^i, y^i\} = \{x^i \circ \pi, \mathrm{d}x^i\}$ the usual coordinates on $\pi^{-1}(U)$. Let $\{x'^i\}$ be another set of coordinates defined on a open subset $U' \subset M$ such that $U \cap U' \neq \emptyset$. The change of coordinates $x'^i = x'^i(x^j)$ on $U \cap U'$ induces the change of coordinates on $\pi^{-1}(U \cap V)$ given by

$$x'^i = x'^i(x^1, \dots, x^n), \quad y'^i = \frac{\partial x'^i}{\partial x^j} y^j, \qquad i, j = 1, \dots, n.$$

The Jacobian matrix of this change of coordinates is

$$J = \begin{pmatrix} \dfrac{\partial x'^i}{\partial x^j} & 0 \\ \dfrac{\partial^2 x'^i}{\partial x^k \partial x^j} y^k & \dfrac{\partial x'^i}{\partial x^j} \end{pmatrix}.$$

Since $\det J = \det \left(\dfrac{\partial x'^i}{\partial x^j} \right)^2 > 0$, it follows that TM is orientable.

Problem 3.1.3. *Prove that if a C^∞ manifold M admits an atlas formed by two charts (U, φ), (V, ψ), and $U \cap V$ is connected, then M is orientable. Apply this result to the sphere S^n, $n > 1$, with the atlas formed by the stereographic projections from the poles* (*see* Problem 1.1.9).

Solution. Let $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$ be the coordinate maps. If $\det(\partial x^i / \partial y^j) \neq 0$ on $U \cap V$ and $U \cap V$ is connected, we have either (a) $\det(\partial x^i / \partial y^j) > 0$ for all $U \cap V$; or (b) $\det(\partial x^i / \partial y^j) < 0$ for all $U \cap V$. In the case (a) it follows that M is orientable with the given atlas. In the case (b), we should only have to consider as coordinate maps $\varphi = (x^1, \dots, x^n)$ and $\psi = (-y^1, y^2, \dots, y^n)$.

For S^n, $n > 1$, considering the stereographic projections, we have the coordinate domains

$$U_N = \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq 1\},$$
$$U_S = \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq -1\}.$$

As

$$\begin{aligned} U_N \cap U_S &= \{(x^1, \dots, x^{n+1}) \in S^n : x^{n+1} \neq \pm 1\} \\ &= \varphi_N^{-1}(\mathbb{R}^n - \{0\}), \end{aligned}$$

is connected, we conclude that S^n is orientable.

Problem 3.1.4. *Study the orientability of the following C^∞ manifolds*:

(1) *A cylindrical surface of $\mathbb{R}^3$, with the atlas given in* Problem 1.1.11.

(2) *The Möbius strip, with the atlas given in* Problem 1.1.12.

(3) *The real projective space $\mathbb{R}P^2$, with the atlas given in* Problem 1.7.4.

Solution. (1) The Jacobian J of the change of the charts given in Problem 1.1.11 always has positive determinant; in fact, equal to 1. Thus the manifold is orientable.

(2) For the given atlas, the open subset $U \cap V$ decomposes into two connected open subsets W_1 and W_2, such that on W_1 (resp. W_2) the Jacobian of the change of coordinates has positive (resp. negative) determinant. Hence, by virtue of Problem 3.1.1, M is not orientable.

(3) With the notations in (2) in Problem 1.7.4, we have in the case of $\mathbb{R}P^2$ three charts (U_1, φ_1), (U_2, φ_2) and (U_3, φ_3), such that for instance

$$\begin{aligned} \varphi_1(U_1 \cap U_2) &= \varphi_1\left(\{[x^1, x^2, x^3] : x^1 \neq 0, x^2 \neq 0\}\right) \\ &= \{(t^1, t^2) \in \mathbb{R}^2 : t^1 \neq 0\} \\ &= V_1 \cup V_2, \end{aligned}$$

where $V_1 = \{(t^1, t^2) \in \mathbb{R}^2 : t^1 > 0\}$ and $V_2 = \{(t^1, t^2) \in \mathbb{R}^2 : t^1 < 0\}$ are connected. The change of coordinates on $\varphi_1(U_1 \cap U_2)$ is given by

$$\begin{aligned} (\varphi_2 \circ \varphi_1^{-1})(t^1, t^2) &= \varphi_2([1, t^1, t^2]) \\ &= \left(\frac{1}{t^1}, \frac{t^2}{t^1}\right), \end{aligned}$$

and the determinant of its Jacobian matrix is easily seen to be equal to $-1/(t^1)^3$, which is negative on V_1 and positive on V_2. Hence, by virtue of Problem 3.1.1, $\mathbb{R}P^2$ is not orientable.

Problem 3.1.5. *Consider the map*

$$\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (u, v) = (x\mathrm{e}^y + y, x\mathrm{e}^y + \lambda y), \qquad \lambda \in \mathbb{R}.$$

(1) *Find the values of λ for which φ is a diffeomorphism.*

(2) *Find the values of λ for which the diffeomorphism φ is orientation-preserving.*

Solution. (1) Suppose that

$$xe^y + y = x'e^{y'} + y', \qquad xe^y + \lambda y = x'e^{y'} + \lambda y'. \tag{$\star$}$$

Subtracting, we have $(1-\lambda)y = (1-\lambda)y'$. Hence, for $\lambda \neq 1$, we have $y = y'$. And from any of the two equations ($\star$), we deduce that $x = x'$.

The map φ is clearly C^∞ and its inverse map, given by

$$y = \frac{u-v}{1-\lambda}, \qquad x = \frac{\lambda u - v}{\lambda - 1} e^{\frac{u-v}{\lambda-1}},$$

is a C^∞ map if and only if $\lambda \neq 1$. Thus φ is a diffeomorphism if and only if $\lambda \neq 1$.

(2) Consider the canonical orientation of $\mathbb{R}^2$ given by $dx \wedge dy$, or by $du \wedge dv$. We have

$$\det \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \det \begin{pmatrix} e^y & xe^y + 1 \\ e^y & xe^y + \lambda \end{pmatrix}.$$

Therefore

$$\begin{aligned} du \wedge dv &= \frac{\partial(u,v)}{\partial(x,y)} dx \wedge dy \\ &= e^y(\lambda - 1)\, dx \wedge dy. \end{aligned}$$

That is, φ is orientation-preserving if $\lambda > 1$.

3.2 Integration on Chains. Stokes' Theorem I

Problem 3.2.1. *Compute the integral of the differential* 1*-form*

$$\alpha = (x^2 + 7y)dx + (-x + y\sin y^2)dy \in \Lambda^1\mathbb{R}^2$$

over the 1*-cycle given by the oriented segments going from* $(0,0)$ *to* $(1,0)$*, then from* $(1,0)$ *to* $(0,2)$*, and then from* $(0,2)$ *to* $(0,0)$.

Solution. Denoting by c the 2-chain (with the usual counterclockwise orientation) whose boundary is the triangle above, by Stokes' Theorem I (Theorem 7.3.3), we have

$$\int_{\partial c} \alpha = \int_c d\alpha$$

$$= -8\int_c \mathrm{d}x \wedge \mathrm{d}y$$
$$= -8\int_0^1 \left(\int_0^{2(1-x)} \mathrm{d}y\right)\mathrm{d}x = -8.$$

Problem 3.2.2. *Deduce from* Green's Theorem 7.3.4:

(1) *The formula for the area of the interior D of a simple, closed, plane curve* $[a,b] \to (x(t),y(t)) \in \mathbb{R}^2$:

$$A(D) = \int_D \mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_a^b \left(x(t)\frac{\mathrm{d}y}{\mathrm{d}t} - y(t)\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t.$$

(2) *The formula of change of variables for double integrals:*

$$\iint_D F(x,y)\mathrm{d}x\,\mathrm{d}y = \int_{\varphi^{-1}D} F(x(u,v),y(u,v))\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\,\mathrm{d}v,$$

corresponding to the coordinate transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $x = x(u,v)$, $y = y(u,v)$.

Solution. (1) follows directly from Green's Theorem by letting $g = x$, $f = -y$ in the formula 7.3.4.

(2) First, we let $f = 0$, $\partial g/\partial x = F$ in Green's formula. Then, from the formula for change of variables and again from Green's Theorem we obtain

$$\begin{aligned}
\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\partial D} g\,\mathrm{d}y \\
&= \int_{\varphi^{-1}(\partial D)} \varphi^*(g\,\mathrm{d}y) \\
&= \int_{\varphi^{-1}(\partial D)} (g\circ\varphi)\left(\frac{\partial y}{\partial u}u'(t) + \frac{\partial y}{\partial v}v'(t)\right)\mathrm{d}t \\
&= \int_{\varphi^{-1}(\partial D)} \left\{\left((g\circ\varphi)\frac{\partial y}{\partial u}\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \left((g\circ\varphi)\frac{\partial y}{\partial v}\right)\frac{\mathrm{d}v}{\mathrm{d}t}\right\}\mathrm{d}t \\
&= \iint_{\varphi^{-1}D} \left\{\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u}\right)\right\}\mathrm{d}u\,\mathrm{d}v. \qquad (\star)
\end{aligned}$$

Moreover, we have

$$\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right) = \left(\left(\frac{\partial g}{\partial x}\circ\varphi\right)\frac{\partial x}{\partial u} + \left(\frac{\partial g}{\partial y}\circ\varphi\right)\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v} + (g\circ\varphi)\frac{\partial^2 y}{\partial u\partial v},$$

$$\frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u}\right) = \left(\left(\frac{\partial g}{\partial x}\circ\varphi\right)\frac{\partial x}{\partial v} + \left(\frac{\partial g}{\partial y}\circ\varphi\right)\frac{\partial y}{\partial v}\right)\frac{\partial y}{\partial u} + (g\circ\varphi)\frac{\partial^2 y}{\partial v\partial u}.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial u}\left((g\circ\varphi)\frac{\partial y}{\partial v}\right)-\frac{\partial}{\partial v}\left((g\circ\varphi)\frac{\partial y}{\partial u})\right) &= (F\circ\varphi)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\\ &= (F\circ\varphi)\frac{\partial(x,y)}{\partial(u,v)}.\end{aligned}$$

Substituting this equality in $(\star)$ we have

$$\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\varphi^{-1}D}(F\circ\varphi)\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\,\mathrm{d}v.$$

Problem 3.2.3. *Let c_2 be a 2-chain in $\mathbb{R}^2$ and $f\in C^\infty\mathbb{R}^2$. Prove that*

$$\int_{\partial c_2}\frac{\partial f}{\partial y}\mathrm{d}x-\frac{\partial f}{\partial x}\mathrm{d}y=0$$

if f satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

Solution. From Stokes' Theorem I we have

$$\begin{aligned}\int_{\partial c_2}\frac{\partial f}{\partial y}\mathrm{d}x-\frac{\partial f}{\partial x}\mathrm{d}y &= \int_{c_2}\mathrm{d}\left(\frac{\partial f}{\partial y}\mathrm{d}x-\frac{\partial f}{\partial x}\mathrm{d}y\right)\\ &= -\int_{c_2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\wedge\mathrm{d}y=0.\end{aligned}$$

Problem 3.2.4. *Consider the 1-chain*

$$c_{r,n}\colon [0,1]\to\mathbb{R}^2-\{0\},\qquad c_{r,n}(t)=(x(t)=r\cos 2\pi nt,\ y(t)=r\sin 2\pi nt),$$

for $r\in\mathbb{R}^+$, $n\in\mathbb{Z}^+$.

Prove that $c_{r,n}$ is not the boundary of any 2-chain in $\mathbb{R}^2-\{0\}$.

Solution. Let θ be the angle function on $C=c_{r,n}([0,1])$. Then, $\mathrm{d}\theta$ is a globally defined differential 1-form on C, and we have

$$\int_{c_{r,n}}\mathrm{d}\theta=\int_{c_{r,n}}\mathrm{d}\arctan\left(\frac{y}{x}\right)=2\pi n.$$

Suppose $c_{r,n}=\partial c_2$ for a 2-chain $c_2\in\mathbb{R}^2-\{0\}$. Then, from Stokes' Theorem I, it follows that

$$\int_{c_{r,n}}\mathrm{d}\theta=\int_{c_2}\mathrm{d}(\mathrm{d}\theta)=0,$$

thus leading us to a contradiction.

3.3 Integration on Oriented Manifolds. Stokes' Theorem II

Problem 3.3.1. *Given on* $\mathbb{R}^3$ *the differential form*

$$\omega = (z - x^2 - xy)\,\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z - \mathrm{d}z \wedge \mathrm{d}x,$$

compute $\int_D i^*\omega$, *where* i *denotes the inclusion map of*

$$D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1,\ z = 0\right\}$$

in $\mathbb{R}^3$.

Solution. We have

$$\int_D i^*\omega = -\int_D (x^2 + xy)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Taking polar coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \qquad \rho \in (0,1), \quad \theta \in (0,2\pi),$$

one has

$$\frac{\partial(x,y)}{\partial(\rho,\theta)} = \det\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} = \rho.$$

Therefore, for $D_0 = D - \{[0,1) \times \{0\}\}$, one has

$$\begin{aligned}
\int_D i^*\omega &= -\int_{D_0} (x^2 + xy)\,\mathrm{d}x \wedge \mathrm{d}y \\
&= -\int_{D_0} \rho^2(\cos^2\theta + \sin\theta\cos\theta)\rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta \\
&= -\int_0^{2\pi}\int_0^1 \rho^3(\cos^2\theta + \sin\theta\cos\theta)\mathrm{d}\rho\,\mathrm{d}\theta \\
&= -\frac{1}{4}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2} + \frac{\sin 2\theta}{2}\right)\mathrm{d}\theta = -\frac{\pi}{4}.
\end{aligned}$$

Problem 3.3.2. *Let* (u,v,w) *denote the usual coordinates on* $\mathbb{R}^3$. *Consider the parametrization*

$$u = \frac{1}{2}\sin\alpha\cos\beta, \qquad v = \frac{1}{2}\sin\alpha\sin\beta, \qquad w = \frac{1}{2}\cos\alpha + \frac{1}{2}, \tag{$\star$}$$

$\alpha \in (0,\pi)$, $\beta \in (0,2\pi)$, *of the sphere* $S^2 \equiv u^2 + v^2 + \left(w - \frac{1}{2}\right)^2 = \frac{1}{4}$ *in* $\mathbb{R}^3$.

Let $N = (0,0,1)$ *be its north pole and* $\pi\colon S^2 - \{N\} \to \mathbb{R}^2$ *the stereographic projection onto the plane* $\mathbb{R}^2 \equiv w = 0$. *Let* $\rho = \mathrm{d}x \wedge \mathrm{d}y$ *be the canonical volume form on* $\mathbb{R}^2$ *and* $\sigma = \frac{1}{4}\sin\alpha\,\mathrm{d}\alpha \wedge \mathrm{d}\beta$ *the volume form on* S^2 *above. Write* $\pi^*\rho$ *in terms of* σ.

REMARK. *The* 2*-form* $\sigma = \frac{1}{4}\sin\alpha\,\mathrm{d}\alpha\wedge\mathrm{d}\beta$ *is called the* canonical volume form *on* S^2 *because one has* $\sigma(X,Y) = 1$, $X,Y \in \mathfrak{X}(S^2)$, *for* $\{X,Y,\mathbf{n}\}$ *an orthonormal basis of* $\mathbb{R}^3$, *where* $\mathbf{n}$ *denotes the exterior* (i.e. *pointing outwards*) *unit normal field on* S^2.

Solution. The given stereographic projection is the restriction to $S^2 - \{N\}$ of the map

$$\widetilde{\pi}\colon \mathbb{R}^3 - \{w=1\} \to \mathbb{R}^2, \qquad (u,v,w) \mapsto \left(\frac{u}{1-w}, \frac{v}{1-w}\right),$$

whose Jacobian matrix is

$$\begin{pmatrix} \dfrac{1}{1-w} & 0 & \dfrac{u}{(1-w)^2} \\ 0 & \dfrac{1}{1-w} & \dfrac{v}{(1-w)^2} \end{pmatrix}.$$

Hence

$$\begin{aligned} \widetilde{\pi}^*\rho &= \widetilde{\pi}^*(\,\mathrm{d}x\wedge\mathrm{d}y) \\ &= \widetilde{\pi}^*\,\mathrm{d}x\wedge\widetilde{\pi}^*\,\mathrm{d}y \\ &= \left(\frac{1}{1-w}\mathrm{d}u + \frac{u}{(1-w)^2}\mathrm{d}w\right)\wedge\left(\frac{1}{1-w}\mathrm{d}v + \frac{v}{(1-w)^2}\mathrm{d}w\right) \\ &= \frac{1}{(1-w)^2}\mathrm{d}u\wedge\mathrm{d}v + \frac{v}{(1-w)^3}\mathrm{d}u\wedge\mathrm{d}w - \frac{u}{(1-w)^3}\mathrm{d}v\wedge\mathrm{d}w. \end{aligned} \tag{$\star\star$}$$

Thus, substituting $(\star)$ in $(\star\star)$ we obtain after a computation

$$\begin{aligned} \pi^*\rho &= \widetilde{\pi}^*\rho \\ &= -\frac{\sin\alpha}{(1-\cos\alpha)^2}\,\mathrm{d}\alpha\wedge\mathrm{d}\beta \\ &= -\frac{4}{(1-\cos\alpha)^2}\,\sigma. \end{aligned}$$

Problem 3.3.3. *Compute the integral of* $\omega = (x-y^3)\mathrm{d}x + x^3\mathrm{d}y$ *along* S^1 *applying Stokes' Theorem* II.

Solution. Let D (resp. $\bar{D}$) be the open (resp. closed) unit disk of $\mathbb{R}^2$, and let $D_0 = D - \{\,[0,1)\times\{0\}\}$. Applying Stokes' Theorem II, we have

$$\begin{aligned} \int_{S^1}\omega &= \int_{\partial\bar{D}}\omega \\ &= \int_{\bar{D}}\mathrm{d}\omega \\ &= \int_{D_0}\mathrm{d}\omega \\ &= \int_{D_0} 3(x^2+y^2)\mathrm{d}x\wedge\mathrm{d}y. \end{aligned}$$

Taking polar coordinates we have as in Problem 3.3.1:

$$\begin{aligned}\int_{S^1} \omega &= \int_{D_0} 3\rho^3\,\mathrm{d}\rho \wedge \mathrm{d}\theta \\ &= 3\int_0^{2\pi}\left(\int_0^1 \rho^3 \mathrm{d}\rho\right)\mathrm{d}\theta = \frac{3\pi}{2}.\end{aligned}$$

Problem 3.3.4. *Let f be a C^∞ function on $\mathbb{R}^2$, and D a compact and connected subset of $\mathbb{R}^2$ with regular boundary ∂D such that $f|_{\partial D} = 0$.*

(1) *Prove the equality*

$$\int_D f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x \wedge \mathrm{d}y = -\int_D \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}\mathrm{d}x \wedge \mathrm{d}y.$$

(2) *Deduce from* (1) *that if* $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ *on D, then* $f|_D = 0$.

Solution. (1) By Stokes' Theorem II we have

$$\int_D \left\{ f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}\mathrm{d}x \wedge \mathrm{d}y = \int_{\partial D} \psi,$$

where ψ is a differential 1-form so that $\mathrm{d}\psi$ is equal to the 2-form in the left-hand side. One solution is given by

$$\psi = -f\frac{\partial f}{\partial y}\mathrm{d}x + f\frac{\partial f}{\partial x}\mathrm{d}y.$$

Since $f|_{\partial D} = 0$, we have

$$\int_{\partial D} \psi = 0,$$

from which the wanted equality follows.

(2) If $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$, by the equality we have just proved, one has

$$\int_D \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}\mathrm{d}x \wedge \mathrm{d}y = 0,$$

that is, $|\mathrm{d}f|$ being the modulus of $\mathrm{d}f$, we have $\int_D |\mathrm{d}f|^2 \mathrm{d}x \wedge \mathrm{d}y = 0$; thus f is constant on D, but since $f|_{\partial D} = 0$ we have $f|_D = 0$.

Problem 3.3.5. *Let* $\alpha = \dfrac{1}{2\pi}\dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2} \in \Lambda^1(\mathbb{R}^2 - \{0\})$.

(1) *Prove that α is closed.*

(2) *Compute the integral of α on the unit circle S^1.*

(3) *How does this result show that* α *is not exact?*

(4) *Let* $j\colon S^1 \hookrightarrow \mathbb{R}^2$ *be the canonical embedding. How can we deduce from* (3) *that* $j^*\alpha$ *is not exact?*

Solution.

(1) Immediate.

(2) Parametrizing S^1 as $x = \cos\theta$, $y = \sin\theta$, $\theta \in [0, 2\pi]$, one has

$$\int_{S^1} \alpha = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,\mathrm{d}\theta = 1.$$

(3) If it were $\alpha = \mathrm{d}f$ for a given function f, applying Stokes' Theorem II, it would be

$$\begin{aligned}\int_{S^1} \alpha &= \int_{S^1} \mathrm{d}f \\ &= \int_{\partial S^1} f = 0.\end{aligned}$$

(4) Let us suppose that $j^*\alpha$ is exact, i.e. $j^*\alpha = \mathrm{d}f$. Then we would have

$$\begin{aligned}\int_{S^1} j^*\alpha &= \int_{j(S^1)} \alpha \\ &= \int_{S^1} \alpha = 1.\end{aligned}$$

On the other hand, as $j^*d = \mathrm{d}\,j^*$, we have

$$\begin{aligned}\int_{S^1} j^*\alpha &= \int_{S^1} j^*\mathrm{d}f \\ &= \int_{S^1} \mathrm{d}\,j^*f \\ &= \int_{\partial S^1} j^*f \\ &= \int_{\emptyset} j^*f = 0,\end{aligned}$$

where $\emptyset$ denotes the empty set. Absurd.

Problem 3.3.6. *Consider*

$$\alpha = \frac{x\,\mathrm{d}y\wedge\mathrm{d}z - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}x\wedge\mathrm{d}y}{(x^2+y^2+z^2)^{3/2}} \in \Lambda^2(\mathbb{R}^3 - \{0\}).$$

(1) *Prove that* α *is closed.*

(2) *Compute* $\int_{S^2}\alpha$.

(3) *How does this prove that* α *is not exact?*

Solution. (1) Immediate.

(2) Consider the parametrization

$$x = \cos\varphi\cos\psi, \qquad y = \cos\varphi\sin\psi, \qquad z = \sin\varphi,$$

$\varphi \in [-\pi/2, \pi/2]$, $\psi \in [0, 2\pi]$, of S^2. Then $\alpha|_{S^2} = -\cos\varphi \,\mathrm{d}\varphi \wedge \mathrm{d}\psi$, and

$$\int_{S^2} \alpha = \int_0^{2\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos\varphi \,\mathrm{d}\varphi \right) \mathrm{d}\psi = -4\pi.$$

(3) If $\alpha = \mathrm{d}\beta$, by Stokes' Theorem II it would be

$$\int_{S^2} \alpha = \int_{S^2} \mathrm{d}\beta = \int_{\partial S^2} \beta = 0,$$

which contradicts the result in (2).

3.4 De Rham Cohomology

Problem 3.4.1. *Prove that the de Rham cohomology groups of the circle are*

$$H^i_{dR}(S^1, \mathbb{R}) = \begin{cases} \mathbb{R}, & i = 0, 1 \\ 0, & i > 1. \end{cases}$$

Solution. One has $H^0_{dR}(S^1, \mathbb{R}) = \mathbb{R}$, because S^1 is connected. Since $\dim S^1 = 1$, one has $H^i_{dR}(S^1, \mathbb{R}) = 0$ if $i > 1$.

As for $H^1_{dR}(S^1, \mathbb{R}) = \mathbb{R}$, every 1-form on S^1 is closed. Now, let ω_0 be the restriction to S^1 of the differential form $(-y\,\mathrm{d}x + x\,\mathrm{d}y)/(x^2 + y^2)$ on $\mathbb{R}^2 - \{(0,0)\}$. We locally have $\omega_0 = \mathrm{d}\theta$, θ being the angle function. Hence $\mathrm{d}\theta$ is nonzero at every point of S^1. (In spite of the notation, $\mathrm{d}\theta$ is not exact, *cf.* Problem 3.3.5.) Hence, if ω is any 1-form on S^1, then we have $\omega = f(\theta)\mathrm{d}\theta$, where f is differentiable and periodic with period 2π. To prove this, we only have to see that there is a constant $c$and a differentiable and periodic function $g(\theta)$ such that $f(\theta)\,\mathrm{d}\theta = c\,\mathrm{d}\theta + \mathrm{d}g(\theta)$. In fact, if this is so, integrating we have $c = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta$. We then define $g(\theta) = \int_0^\theta (f(t) - c)\,\mathrm{d}t$, where c is the constant determined by the previous equality. One clearly has that g is differentiable. Finally, we must see that it is periodic. Indeed:

$$\begin{aligned} g(\theta + 2\pi) &= g(\theta) + \int_\theta^{\theta+2\pi} (f(t) - c)\,\mathrm{d}t \\ &= g(\theta) + \int_\theta^{\theta+2\pi} f(t)\,\mathrm{d}t - \int_0^{2\pi} f(t)\,\mathrm{d}t \qquad (f \text{ is periodic}) \\ &= g(\theta). \end{aligned}$$

Problem 3.4.2. *Compute the de Rham cohomology groups of the annular region*

$$M = \left\{ (x,y) \in \mathbb{R}^2 : 1 < \sqrt{x^2+y^2} < 2 \right\}.$$

HINT: *Apply the following general result: if two maps $f,g\colon M \to N$ between two C^∞ manifolds are C^∞ homotopic, that is, if there exists a C^∞ map $F\colon M\times[0,1] \to N$ such that $F(p,0) = f(p)$, $F(p,1) = g(p)$ for every $p \in M$, then the maps*

$$f^*\colon H^k_{dR}(N,\mathbb{R}) \to H^k_{dR}(M,\mathbb{R}), \qquad g^*\colon H^k_{dR}(N,\mathbb{R}) \to H^k_{dR}(M,\mathbb{R}),$$

are equal for every $k = 0,1,\dots$.

Solution. Let $N = S^1(3/2)$ be the circle with center at the origin and radius $3/2$ in $\mathbb{R}^2$. Let $j\colon N \to M$ be the inclusion map and let r be the retraction $r\colon M \to N$, $p \mapsto \frac{3}{2}(p/|p|)$. Then, $r\circ j\colon N \to N$ is the identity on $S^1(3/2)$. The map $j\circ r\colon M \to M$, $p \mapsto \frac{3}{2}(p/|p|)$, although not the identity of M, is homotopic to the identity. In fact, we can define the homotopy by

$$H\colon M\times[0,1] \to M, \qquad (p,t) \mapsto tp + (1-t)\frac{3}{2}\frac{p}{|p|}.$$

Thus, for $k = 0,1,2$, we have

$$\begin{aligned} j^*\colon H^k_{dR}(M,\mathbb{R}) &\to H^k_{dR}(S^1(3/2),\mathbb{R}), \\ r^*\colon H^k_{dR}(S^1(3/2),\mathbb{R}) &\to H^k_{dR}(M,\mathbb{R}), \end{aligned}$$

so, applying the general result quoted in the hint, we have

$$\begin{aligned} r^*\circ j^* = (j\circ r)^* &= \text{ identity on } H^k_{dR}(M,\mathbb{R}), \\ j^*\circ r^* = (r\circ j)^* &= \text{ identity on } H^k_{dR}(S^1(3/2),\mathbb{R}). \end{aligned}$$

Hence, j^* and r^* are mutually inverse and it follows that

$$H^k_{dR}(M,\mathbb{R}) \approx H^k_{dR}(S^1(3/2),\mathbb{R}). \tag{$\star$}$$

Consequently, $H^0_{dR}(M,\mathbb{R}) = \mathbb{R}$ (as one can also deduce directly since M is connected). In fact, there are no exact 0-forms, and the closed 0-forms (that is, the differentiable functions f such that $\mathrm{d}f = 0$) are the constant functions, since M is connected.

As $\dim S^1(3/2) = 1$, from the isomorphism $(\star)$ we obtain $H^k_{dR}(M,\mathbb{R}) = 0$, $k \geqslant 2$.

Finally, $H^1_{dR}(M,\mathbb{R}) \approx H^1_{dR}(S^1(3/2),\mathbb{R}) = \mathbb{R}$, hence

$$H^k_{dR}(M,\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0,1 \\ 0, & k > 1. \end{cases}$$

Problem 3.4.3. (1) *Prove that every closed differential* 1*-form on the sphere* S^2 *is exact.*

(2) *Using de Rham's cohomology, conclude that the torus* T^2 *and the sphere are not homeomorphic.*

HINT: *Consider the parametrization*

$$x = (R + r\cos\theta)\cos\varphi, \qquad y = (R + r\cos\theta)\sin\varphi, \qquad z = r\sin\varphi,$$
$$R > r, \quad \theta, \varphi \in [0, 2\pi],$$

of the torus T^2*, and take the restriction to* T^2 *of the differential form* $\omega = \dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$ *on* $\mathbb{R}^3 - z$*-axis.*

Solution. Let ω be a closed 1-form on the sphere. We shall prove that it is exact. Let U_1 and U_2 be the open subsets of S^2 respectively obtained by removing two antipodal points. Then, writing $\omega_i = \omega|_{U_i}$, since U_i is homeomorphic to $\mathbb{R}^2$, there exist functions $f_i \colon U_i \to \mathbb{R}$, such that $\omega_i = \mathrm{d}f_i$. As $U_1 \cap U_2$ is connected, one has $f_1 = f_2 + \lambda$ on $U_1 \cap U_2$, for $\lambda \in \mathbb{R}$. The function $f \colon S^2 \to \mathbb{R}$ defined by $f|_{U_1} = f_1$, $f|_{U_2} = f_2 + \lambda$ is differentiable and $\mathrm{d}f = \omega$.

To prove that T^2 and S^2 are not homeomorphic, we only have to find a closed 1-form on the torus which is not exact. Let $j \colon T^2 \hookrightarrow \mathbb{R}^3 - \{0\}$ be the canonical injection map. The form $\omega = \dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$ on $\mathbb{R}^3 - \{0\}$ is closed. Since $\mathrm{d} \circ j^* = j^* \circ \mathrm{d}$, the form $j^*\omega$ on T^2 is also closed. To see that ω is not exact, by the Stokes theorem, we only have to see that there exists a closed curve γ on the torus such that $\int_\gamma j^*\omega \neq 0$. In fact, let γ be the parallel obtained taking $\theta = 0$ in the parametric equations above of the torus. Then

$$\int_\gamma j^*\omega = \int_0^{2\pi} \mathrm{d}\varphi = 2\pi \neq 0.$$

Problem 3.4.4. *Let* $z^0, \dots, z^n$ *be a homogeneous coordinate system on the complex projective space* $\mathbb{C}P^n$*, and let* U_α *be the open subset defined by* $z^\alpha \neq 0$*,* $\alpha = 0, \dots, n$*. Let us fix two indices* $0 \leqslant \alpha < \beta \leqslant n$*. Set* $u^j = z^j/z^\alpha$ *on* U_α*,* $v^j = z^j/z^\beta$ *on* U_β*.*

We define two differential 2*-forms* ω_α *on* U_α *and* ω_β *on* U_β*, by setting*

$$\omega_\alpha = \frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j}{\varphi} - \frac{\sum_{j,k} u^j \bar{u}^k \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j}{\varphi^2}\right),$$

$$\omega_\beta = \frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}v^j \wedge \mathrm{d}\bar{v}^j}{\psi} - \frac{\sum_{j,k} v^j \bar{v}^k \mathrm{d}v^k \wedge \mathrm{d}\bar{v}^j}{\psi^2}\right),$$

where $\varphi = \sum_{j=0}^n u^j \bar{u}^j$*,* $\psi = \sum_{j=0}^n v^j \bar{v}^j$*. Prove:*

(1) $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$.

(2) *There exists a unique differential* 2*-form* ω *on* $\mathbb{C}P^n$ *such that* $\omega|_{U_\alpha} = \omega_\alpha$*, for all* $\alpha = 0,\dots,n$.

(3) $\mathrm{d}\omega = 0$.

(4) $\omega \wedge \overset{(n}{\cdots} \wedge \omega$ *is a volume form.*

(5) ω *is not exact.*

REMARK. *Let* $a = [\omega]$ *be the* (*real*) *cohomology class of* ω. *It can be proved that* a *generates the real cohomology ring of* $\mathbb{C}P^n$; *specifically, that* $H^*_{dR}(\mathbb{C}P^n,\mathbb{R}) \approx \mathbb{R}[a]/a^{n+1}$.

Solution. (1) On $U_\alpha \cap U_\beta$ one has $v^j = u^j/u^\beta$ and hence $\varphi = \psi u^\beta \bar{u}^\beta$. We have

$$\begin{aligned}
\mathrm{d}v^k \wedge \mathrm{d}\bar{v}^j &= \frac{u^\beta \mathrm{d}u^k - u^k \mathrm{d}u^\beta}{(u^\beta)^2} \wedge \frac{\bar{u}^\beta \mathrm{d}\bar{u}^j - \bar{u}^j \mathrm{d}\bar{u}^\beta}{(\bar{u}^\beta)^2} \\
&= \frac{1}{(u^\beta)^2(\bar{u}^\beta)^2}\Big(u^\beta \bar{u}^\beta \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j - u^\beta \bar{u}^j \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta \\
&\quad - u^k \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^k \bar{u}^j \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\Big),
\end{aligned}$$

and substituting into the expression of ω_β, on $U_\alpha \cap U_\beta$ we obtain

$$\begin{aligned}
\mathrm{i}\omega_\beta = \frac{u^\beta \bar{u}^\beta}{\varphi} &\sum_j \frac{1}{(u^\beta)^2 \bar{u}^\beta)^2}\Big(u^\beta \bar{u}^\beta \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j - u^\beta \bar{u}^j \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta \\
&- u^j \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^j \bar{u}^j \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\Big) \\
&- \frac{(u^\beta)^2(\bar{u}^\beta)^2}{\varphi^2} \sum_{j,k} \frac{u^j \bar{u}^k}{u^\beta \bar{u}^\beta} \frac{1}{(u^\beta)^2(\bar{u}^\beta)^2}\Big(u^\beta \bar{u}^\beta \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j \\
&- u^\beta \bar{u}^j \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta - u^k \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j + u^k \bar{u}^j \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^\beta\Big).
\end{aligned}$$

Since the sum of the 1st and 5th summands above is $\mathrm{i}\omega_\alpha$, and moreover the 4th and 8th summands are easily seen to cancel, we have

$$\begin{aligned}
\mathrm{i}\omega_\beta = \mathrm{i}\omega_\alpha &\\
&- \frac{1}{\varphi u^\beta \bar{u}^\beta} \sum_j \Big(u^\beta \bar{u}^j \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta + u^j \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j\Big) \qquad (\star) \\
&+ \frac{1}{\varphi^2 u^\beta \bar{u}^\beta} \sum_{j,k} u^j \bar{u}^k \Big(u^\beta \bar{u}^j \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^\beta + u^k \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^j\Big).
\end{aligned}$$

Consider the last summand. Interchanging the indices j and k, since $\sum_j u^j \bar{u}^j = \sum_k u^k \bar{u}^k = \varphi$, this summand can be written as

$$\frac{1}{\varphi^2 u^\beta \bar{u}^\beta}\varphi\left(\sum_j \bar{u}^j u^\beta \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^\beta + \sum_k u^k \bar{u}^\beta \mathrm{d}u^\beta \wedge \mathrm{d}\bar{u}^k\right),$$

which is the opposite to the second summand in $(\star)$. Hence $\omega_\beta = \omega_\alpha$ on $U_\alpha \cap U_\beta$.

(2) Because of (1), we only need to prove that ω_α takes real values. In fact,

$$\bar{\omega}_\alpha = -\frac{1}{\mathrm{i}}\left(\frac{\sum_j \mathrm{d}\bar{u}^j \wedge \mathrm{d}u^j}{\varphi} - \frac{\sum_{j,k} \bar{u}^j u^k \mathrm{d}\bar{u}^k \wedge \mathrm{d}u^j}{\varphi^2}\right),$$

and permuting the indices j and k in the second summand, we obtain $\bar{\omega}_\alpha = \omega_\alpha$.

(3) On U_α we easily get

$$\begin{aligned}
\mathrm{i}\,\mathrm{d}\omega &= \mathrm{i}\,\mathrm{d}\omega_\alpha \\
&= -\sum_{j,k}\frac{1}{\varphi^2}\left(u^k \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j \wedge \mathrm{d}\bar{u}^k + \bar{u}^k \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j \wedge \mathrm{d}u^k\right) \\
&\quad -\sum_{j,k}\frac{1}{\varphi^2}\left(\bar{u}^k \mathrm{d}u^j \wedge \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j + u^j \mathrm{d}\bar{u}^k \wedge \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j\right) \qquad (\star\star) \\
&\quad + \frac{2}{\varphi^3}\mathrm{d}\varphi \wedge \sum_{j,k} u^j \bar{u}^k \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j.
\end{aligned}$$

The first two summands at the right-hand side of $(\star\star)$ cancel. The third summand vanishes, as $\mathrm{d}\varphi = \sum_h (u^h \mathrm{d}\bar{u}^h + \bar{u}^h \mathrm{d}u^h)$ yields

$$\begin{aligned}
\mathrm{d}\varphi \wedge \sum_{j,k} u^j \bar{u}^k \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j &= -\mathrm{d}\varphi \wedge \sum_{j,k}\left(u^j \mathrm{d}\bar{u}^j\right) \wedge \left(\bar{u}^k \mathrm{d}u^k\right) \\
&= -\mathrm{d}\varphi \wedge \left(\sum_j u^j \mathrm{d}\bar{u}^j\right) \wedge \left(\mathrm{d}\varphi - \sum_j u^j \mathrm{d}\bar{u}^j\right).
\end{aligned}$$

(4) As in (3) we have

$$\sum_{j,k} u^j \bar{u}^k \mathrm{d}u^k \wedge \mathrm{d}\bar{u}^j = \sum_k \bar{u}^k \mathrm{d}u^k \wedge \sum_j u^j \mathrm{d}\bar{u}^j.$$

Set

$$\nu = \frac{1}{\varphi}\sum_j \bar{u}^j \mathrm{d}u^j, \qquad \mu = \frac{1}{\varphi}\sum_j \mathrm{d}u^j \wedge \mathrm{d}\bar{u}^j.$$

Then $\mathrm{i}\omega = \mu - \nu \wedge \bar{\nu}$. Thus

$$\mathrm{i}^n \omega^n = \mu^n - \binom{n}{1}\mu^{n-1} \wedge \nu \wedge \bar{\nu}.$$

Now,

$$\mu^n = \frac{n!}{\varphi^n}\mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n$$

(we suppose that $\alpha = 0$, so that only the coordinates $u^1, \bar{u}^1, \ldots, u^n, \bar{u}^n$ are effective), and

$$\mu^{n-1} = \frac{(n-1)!}{\varphi^{n-1}} \sum_{k=1}^{n} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \widehat{\mathrm{d}u^k} \wedge \widehat{\mathrm{d}\bar{u}^k} \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n.$$

Hence

$$\begin{aligned} n\mu^{n-1} \wedge \nu \wedge \bar{\nu} &= \frac{n!}{\varphi^{n+1}} \sum_{k=1}^{n} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \\ &\quad \wedge \widehat{\mathrm{d}u^k} \wedge \widehat{\mathrm{d}\bar{u}^k} \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n \wedge \bar{u}^k \mathrm{d}u^k \wedge u^k \mathrm{d}\bar{u}^k \\ &= \frac{n!}{\varphi^{n+1}} \sum_{k=1}^{n} u^k \bar{u}^k \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n \\ &= \frac{n!(\varphi - 1)}{\varphi^{n+1}} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n, \end{aligned}$$

for

$$\begin{aligned} \sum_{k=0}^{n} u^k \bar{u}^k &= u^0 \bar{u}^0 + \sum_{k=1}^{n} u^k \bar{u}^k \\ &= 1 + \sum_{k=1}^{n} u^k \bar{u}^k = \varphi. \end{aligned}$$

Thus

$$\mathrm{i}^n \omega^n = \frac{n!}{\varphi^{n+1}} \mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 \wedge \cdots \wedge \mathrm{d}u^n \wedge \mathrm{d}\bar{u}^n,$$

which does not vanish on U_0, hence on $\mathbb{C}P^n$, as the same argument holds for any $\alpha = 0, \ldots, n$.

(5) Immediate from (4) and Stokes' Theorem.

Chapter 4
Lie Groups

4.1 Lie Groups and Lie Algebras

Problem 4.1.1. *Prove that the following are Lie groups*:

(1) *Each finite-dimensional real vector space with its structure of additive group. In particular* $\mathbb{R}^n$.

(2) *The set of nonzero complex numbers* $\mathbb{C}^*$ *with the multiplication of complex numbers.*

(3) $G \times H$, *where* G, H *are Lie groups, with the product* $(g,h)(g',h') = (gg',hh')$, $g, g' \in G$, $h, h' \in H$. *In general, if* G_i, $i = 1, \dots, n$, *is a Lie group, then* $G_1 \times \cdots \times G_n$ *is a Lie group.*

(4) T^n, *for* $n \geqslant 1$, (toral group).

(5) $\operatorname{Aut} V$, *where* V *is a vector space of finite dimension over* $\mathbb{R}$ *or* $\mathbb{C}$, *with the composition product, and in particular* $GL(n,\mathbb{R}) = \operatorname{Aut}_{\mathbb{R}} \mathbb{R}^n$ *and* $GL(n,\mathbb{C}) = \operatorname{Aut}_{\mathbb{C}} \mathbb{C}^n$.

(6) $K = \mathbb{R}^n \times GL(n,\mathbb{R})$, $n > 1$, *with the group structure defined by*

$$(x,A)(x',A') = (x + Ax', AA').$$

Solution. (1) Let V be a finite-dimensional real vector space. If $\dim V = n$, then V has a natural structure of C^∞ manifold, defined by the global chart (V, φ), $\varphi \colon V \to \mathbb{R}^n$, the coordinate functions being the dual basis to a given basis of V. The structure does not depend on the given basis, as it is easily checked. On the other hand, V has the structure of an additive group with the internal law, and the map $V \times V \to V$, $(v,w) \mapsto v - w$, is C^∞.

(2) $\mathbb{C}^*$ has a natural structure of a 2-dimensional manifold as an open subset of the 2-dimensional real vector space $\mathbb{C}$. $\mathbb{C}^*$ has the structure of a multiplicative group, and the map $\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(z,w) \mapsto zw^{-1}$, is C^∞, since if $z = a + b\mathrm{i}$, $w = c + d\mathrm{i}$, one has

P.M. Gadea, J. Muñoz Masqué, *Analysis and Algebra on Differentiable Manifolds*,

$$\begin{aligned} zw^{-1} &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i} \\ &\equiv \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2} \right) \in \mathbb{R}^2. \end{aligned}$$

(3) $G \times H$ is a Lie group with the structure of product manifold and the given product, since

$$((g,h),(g',h')) \mapsto (g,h)(g',h')^{-1} = (gg'^{-1}, hh'^{-1})$$

is C^∞.

(4) $T^n = S^1 \times \overset{(n}{\cdots} \times S^1$. Hence T^n is a Lie group as it is a finite product of Lie groups.

(5) $\operatorname{Aut} V$ is an open subset of $\operatorname{End} V$ because

$$\operatorname{Aut} V = \{A \in \operatorname{End} V \,:\, \det A \neq 0\}$$

and det is a continuous function. Therefore $\operatorname{Aut} V$ has a structure of C^∞ manifold (as an open submanifold of $\mathbb{R}^{n^2}$, $n = \dim V$). The multiplication in $\operatorname{Aut} V$ is the composition. Taking as its chart the map which associates to an automorphism its matrix in a basis, the product is calculated by multiplication of matrices. The map $\operatorname{Aut} V \times \operatorname{Aut} V \to \operatorname{Aut} V$, $(A,B) \mapsto AB^{-1}$, is C^∞, as the components of AB and B^{-1} are rational functions in the components of A and B. Hence $\operatorname{Aut} V$ is a Lie group. We have as particular cases the sets $GL(n,\mathbb{R}) = \operatorname{Aut}_{\mathbb{R}} \mathbb{R}^n$ and $GL(n,\mathbb{C}) = \operatorname{Aut}_{\mathbb{C}} \mathbb{C}^n$.

(6) $K = \mathbb{R}^n \times GL(n,\mathbb{R})$ $(n > 1)$ has the structure of a product manifold, and with the law $(x,A)\cdot(x',A') = (x + Ax', AA')$ it has the structure of a group. Let us see that K is a Lie group. In fact, the above product is C^∞, and the inverse of (x,A) is (y,B) such that $(y,B)\cdot(x,A) = (0,I)$. Hence the inverse of (x,A) is $(-A^{-1}x, A^{-1})$, so that the map $(x,A) \mapsto (x,A)^{-1}$, is C^∞. This is the Lie *group of affine transformations of* $\mathbb{R}^n$ (identify the element (v,A) of K with the affine transformations $x \mapsto v + Ax$ of $\mathbb{R}^n$). The multiplication in K corresponds with the composition of affine transformations of $\mathbb{R}^n$.

Problem 4.1.2. *Consider the product $T^1 \times \mathbb{R}^+$ of the one-dimensional torus by the multiplicative group of strictly positive numbers* (*that group is called the* group of similarities of the plane). *Let (θ, x) denote local coordinates. Show that the vector field*

$$\frac{\partial}{\partial \theta} + x\frac{\partial}{\partial x}$$

is left-invariant.

Solution. (1) Let $L_s \colon G \to G$ denote the left translation $L_s s_1 = s s_1$ on a Lie group G. A vector field Y on a Lie group is left-invariant if $L_{s*} Y_e = Y_s$ for all $s \in G$, where e stands for the identity element of G.

In the present case, let $(\alpha, a), (\theta, x)$ be in the coordinate domain with (α, a) arbitrarily fixed and any (θ, x). The left translation is given by $L_{(\alpha,a)}(\theta, x) = (\alpha + \theta, ax)$.

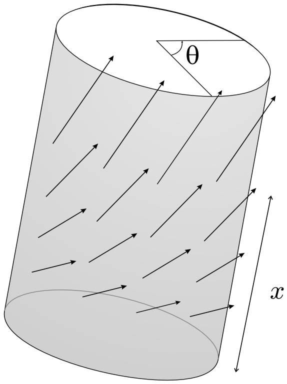

Fig. 4.1 The vector field $\partial/\partial\theta + x\partial/\partial x$ on the group of similarities of the plane.

Therefore one has

$$L_{(\alpha,a)*} = \begin{pmatrix} \dfrac{\partial(\alpha+\theta)}{\partial\theta} & \dfrac{\partial(\alpha+\theta)}{\partial x} \\[2ex] \dfrac{\partial(ax)}{\partial\theta} & \dfrac{\partial(ax)}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

A vector field Z on $T^1 \times \mathbb{R}^+$ is left-invariant if

$$L_{(\alpha,a)*}Z_{(0,1)} = Z_{(\alpha,a)}. \tag{$\star$}$$

For the vector field $X_{(\theta,x)} = \dfrac{\partial}{\partial\theta} + x\dfrac{\partial}{\partial x}$ (see Figure 4.1) we have

$$\begin{aligned} L_{(\alpha,a)*}X_{(0,1)} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\equiv \frac{\partial}{\partial\theta} + a\frac{\partial}{\partial x} \\ &= X_{(\alpha,a)}. \end{aligned}$$

Since (α, a) is arbitrary we have the condition $(\star)$ above at any point, and X is in fact left-invariant on the given coordinate domain. The prolongation to all of G is immediate.

Problem 4.1.3. *Using the coordinate vector fields $\partial/\partial x^i_j$, $1 \leqslant i, j \leqslant n$, on $GL(n,\mathbb{R})$, prove that the vector field Y on $GL(n,\mathbb{R})$ whose matrix of components at the identity is $A = (a^i_j)$, and whose matrix of components is equal to BA at the element $B = (b^i_j)$ of $GL(n,\mathbb{R})$, is a left-invariant vector field.*

Solution. We have $Y_I = \sum_{i,j=1}^n a^i_j(\partial/\partial x^i_j)_I$, where I denotes the identity element of $GL(n,\mathbb{R})$. Since $(L_{B*}Y_I)x^i_j = Y_I(x^i_j \circ L_B)$ and

$$\begin{aligned}(x^i_j \circ L_B)(C) &= x^i_j(BC)\\ &= b^i_k c^k_j,\end{aligned}$$

one has $x^i_j \circ L_B = b^i_k x^k_j$. Hence

$$\begin{aligned}Y_I(x^i_j \circ L_B) &= \sum_{h,l=1}^{n} a^h_l \left.\frac{\partial}{\partial x^h_l}\right|_I (b^i_k x^k_j)\\ &= a^h_j b^i_h,\end{aligned}$$

that is,

$$\begin{aligned}L_{B*}Y_I &= \sum_{i,j,h=1}^{n} b^i_h a^h_j \left.\frac{\partial}{\partial x^i_j}\right|_B\\ &= \sum_{i,j,h=1}^{n} (BA)^i_j \left.\frac{\partial}{\partial x^i_j}\right|_B\\ &= Y_B.\end{aligned}$$

Problem 4.1.4. *Show that the following are Lie algebras*:

(1) *The vector space $\mathfrak{X}(M)$ of C^∞ vector fields on a manifold M with the bracket of vector fields.*

(2) *Any vector space where all the brackets of vectors are zero* (*such a Lie algebra is called an Abelian Lie algebra*).

(3) *The vector space $\mathbb{R}^3$ with the vector product operation $\times$ of vectors.*

(4) *The set $M(n,\mathbb{R})$ of real $n \times n$ matrices, with the bracket $[A,B] = AB - BA$.*

(5) *The space* End V *of endomorphisms of a vector space V of dimension n, with the operation $[A,B] = AB - BA$.*

Solution. (1) Let $a, b \in \mathbb{R}$ and $X, Y \in \mathfrak{X}(M)$. Since

$$[aX_1 + bX_2, Y]f = a[X_1, Y]f + b[X_2, Y]f,$$

$[X,Y]$ is linear in the first variable. As $[X,Y] = -[Y,X]$, linearity on the first variable implies linearity on the second one. So $[X,Y]$ is $\mathbb{R}$-bilinear and anticommutative. The Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

is satisfied, as follows by adding

$$\begin{aligned}[[X,Y],Z]f &= [X,Y](Zf) - Z([X,Y]f)\\ &= X(Y(Zf)) - Y(X(Zf)) - Z(X(Yf)) + Z(Y(Xf)),\end{aligned}$$

and the two similar identities obtained by cyclic permutation of X, Y and Z.

(2) Immediate.

(3) One defines

$$v \times w = (bf - ce, -af + cd, ae - bd), \qquad v = (a,b,c), \quad w = (d,e,f).$$

Then we have:

(a) (bilinearity) $(\lambda v + \mu w) \times u = \lambda\, v \times u + \mu\, w \times u,\ \ \lambda, \mu \in \mathbb{R}$, and $u \times (\lambda v + \mu w) = \lambda\, u \times v + \mu\, u \times w$, as it is easily seen.

(b) (skew-symmetry) $u \times w + w \times u = 0$. Immediate from the definition of the vector product.

(c) (Jacobi identity) $(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$.

Using the formula of the relation between the vector product and the scalar product, we obtain:

$$\begin{aligned}
(u \times v) \times w &= (wu)v - (wv)u, \\
(v \times w) \times u &= (uv)w - (uw)v, \\
(w \times u) \times v &= (vw)v - (vu)w.
\end{aligned}$$

Adding these equalities and taking into account the skew-symmetry of the vector product, we obtain the Jacobi identity.

(4) $M(n,\mathbb{R})$ is a vector space over $\mathbb{R}$. Furthermore

$$\begin{aligned}
[a_1X_1 + a_2X_2, Y] &= (a_1X_1 + a_2X_2)Y - Y(a_1X_1 + a_2X_2) \\
&= a_1[X_1,Y] + a_2[X_2,Y], \qquad a_1, a_2 \in \mathbb{R}.
\end{aligned}$$

Similarly,

$$[X, b_1Y_1 + b_2Y_2] = b_1[X,Y_1] + b_2[X,Y_2], \qquad b_1, b_2 \in \mathbb{R},$$

and $[X,Y] = -[Y,X]$ is obvious. We obtain

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

as in (2).

(5) The map

$$\operatorname{End} V \times \operatorname{End} V \to \operatorname{End} V, \qquad (A,B) \mapsto [A,B] = AB - BA,$$

is bilinear, skew-symmetric and satisfies the Jacobi identity, as it is easily seen.

Problem 4.1.5. *Consider the set G of matrices of the form*

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R}, \quad x > 0,$$

with the structure of a C^∞ manifold defined by the chart mapping each element of G as above to $(x,y,z) \in \mathbb{R}^+ \times \mathbb{R}^2$.

(1) *Is G a Lie subgroup of* $GL(3,\mathbb{R})$*?*

(2) *Prove that*

$$\left\{ X = x\frac{\partial}{\partial x},\ Y = x\frac{\partial}{\partial y},\ Z = x\frac{\partial}{\partial z} \right\},$$

is a basis of left-invariant vector fields.

(3) *Find the structure constants of G with respect to the basis in* (2).

Solution. (1) The product of elements of G

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 & v \\ 0 & u & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} xu & 0 & xv+y \\ 0 & xu & xw+z \\ 0 & 0 & 1 \end{pmatrix} \in G,$$

and the inverse of an element

$$\begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & 0 & -y/x \\ 0 & 1/x & -z/x \\ 0 & 0 & 1 \end{pmatrix} \in G,$$

yield C^∞ maps $G \times G \to G$ and $G \to G$, respectively. Hence G is a Lie group, which is in addition an abstract subgroup of $GL(3,\mathbb{R})$. The inclusion $G \to GL(3,\mathbb{R})$ is an immersion, as its rank (that of the map $(x,y,z) \in \mathbb{R}^+ \times \mathbb{R}^2 \mapsto (x,0,y,0,x,z,0,0,1) \in \mathbb{R}^9$) is 3, so that G is a submanifold, hence a Lie subgroup, of $GL(3,\mathbb{R})$.

(2) Let $(a,b,c) \in G$ be arbitrarily fixed and any (x,y,z) in G. As the left translation by (a,b,c) is

$$L_{(a,b,c)}(x,y,z) = (ax,\, ay+b,\, az+c),$$

we have

$$L_{(a,b,c)*} \equiv \operatorname{diag}(a,a,a).$$

Let $e = (1,0,0)$ denote the identity element of G, we have

$$X_e = \left.\frac{\partial}{\partial x}\right|_e, \qquad Y_e = \left.\frac{\partial}{\partial y}\right|_e, \qquad Z_e = \left.\frac{\partial}{\partial z}\right|_e.$$

We deduce $L_{(a,b,c)*}X_e = X_{(a,b,c)}$ and similar expressions for Y and Z. Since X,Y,Z are C^∞ left-invariant vector fields which are linearly independent at e, they are a basis of left-invariant vector fields.

(3) Let $X_1 = X$, $X_2 = Y$, $X_3 = Z$. Then

$$[X_1,X_2] = X_2, \qquad [X_1,X_3] = X_3, \qquad [X_2,X_3] = 0,$$

so, with respect to that basis, the nonzero structure constants are

$$c_{12}^2 = -c_{21}^2 = c_{13}^3 = -c_{31}^3 = 1.$$

Problem 4.1.6. *Let*
$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

(1) *Show that H admits a structure of C^∞ manifold with which it is diffeomorphic to* $\mathbb{R}^3$.

(2) *Show that H with matrix multiplication is a Lie group* (*H is called the* Heisenberg group).

(3) *Show that* $B = \left\{ \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, x\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z} \right\}$ *is a basis of the Lie algebra* $\mathfrak{h}$ *of* H.

Solution. (1) The map
$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{R}^3 \\ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} & \mapsto & (x, y, z) \end{array}$$

is obviously bijective. Thus $\{(H, \varphi)\}$ is an atlas for H, which defines a C^∞ structure on H such that φ is a diffeomorphism with $\mathbb{R}^3$.

(2) H is a group with the product of matrices, because if $A, B \in H$, then $AB \in H$, and if
$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in H, \quad \text{then} \quad A^{-1} = \begin{pmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \in H.$$

Moreover, the maps
$$\begin{array}{ccc} H \times H & \xrightarrow{\Phi} & H \\ (A, B) & \mapsto & AB \end{array} \quad \text{and} \quad \begin{array}{ccc} H & \xrightarrow{\Psi} & H \\ A & \mapsto & A^{-1} \end{array}$$

are C^∞. Indeed, $\varphi \circ \Phi \circ (\varphi \times \varphi)^{-1} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, given by
$$(\varphi \circ \Phi \circ (\varphi \times \varphi)^{-1})((x, y, z), (a, b, c)) = (a + x, b + xc + y, c + z)$$

is obviously C^∞. Similarly,
$$\varphi \circ \Psi \circ \varphi^{-1} : \quad \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (-x, xz - y, -z) \end{array}$$

is also C^∞. Thus H is a Lie group.

One can also prove it considering H as the closed subgroup of the general linear group $GL(3, \mathbb{R})$, defined by the equations
$$x_1^1 = x_2^2 = x_3^3 = 1, \qquad x_1^2 = x_1^3 = x_2^3 = 0,$$

where x^i_j denote the usual coordinates of $GL(3,\mathbb{R}) \subset M(3,\mathbb{R}) \approx \mathbb{R}^9$. Hence, by Cartan's criterion on closed subgroups 7.4.2, G is a Lie subgroup of $GL(3,\mathbb{R})$.

(3) We have $\dim H = 3$. Thus $\dim \mathfrak{h} = 3$, and so we only have to prove that

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

are linearly independent, which is immediate; and that they are left-invariant, for which we shall write

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

We have to prove that for every $A \in H$ one has

$$(L_A)_{*B}(X_i|_B) = X_i|_{AB}, \qquad \text{for all } B \in H,\ i = 1,2,3. \tag{$\star$}$$

Let (a,b,c) be arbitrarily fixed and any (x,y,z) in H. As the left translation by (a,b,c) is $L_{(a,b,c)}(x,y,z) = (x+a, y+az+b, z+c)$, we have

$$L_{(a,b,c)*} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} (L_A)_{*B}\big(X_1|_B\big) &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &\equiv \left.\frac{\partial}{\partial x}\right|_{AB}, \end{aligned}$$

and similarly we obtain

$$(L_A)_{*B}\big(X_2|_B\big) = \left.\frac{\partial}{\partial y}\right|_{AB}, \qquad (L_A)_{*B}\,\big(X_3|_B\big) = (x+a)\left.\frac{\partial}{\partial y}\right|_{AB} + \left.\frac{\partial}{\partial z}\right|_{AB},$$

so the condition ($\star$) is satisfied.

Problem 4.1.7. *Find the structure constants of the Lie group $GL(n,\mathbb{R})$ with respect to the standard basis $\{E^i_j\}$ of matrices in $\mathfrak{gl}(n,\mathbb{R})$ with (i,j)th entry 1 and 0 elsewhere.*

Solution. The structure constants $c^{rs}_{ij,kl}$ are given by $[E^i_j, E^k_l] = c^{rs}_{ij,kl}E^r_s$. As $[E^i_j, E^k_l] = \delta^j_k E^i_l - \delta^l_i E^k_j$, we deduce

$$c^{rs}_{ij,kl} = \delta^r_i \delta^j_k \delta^s_l - \delta^r_k \delta^l_i \delta^s_j.$$

Problem 4.1.8. *Find the left- and right-invariant measures on*:

(1) *The* Euclidean group $E(2)$ *of matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ x & \cos\theta & \sin\theta \\ y & -\sin\theta & \cos\theta \end{pmatrix}.$$

(2) *The group of matrices of the form* $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$, $x, y > 0$.

(3) *The Heisenberg group* (*see* Problem 4.1.6).

(4) *The real general linear group* $GL(2,\mathbb{R})$.

REMARK. *Given a matrix of functions,* $A = (a^i_j)$, *we shall denote by* dA *the matrix* $\left(\mathrm{d}a^i_j\right)$.

Solution. Let A be a generic element of any of the above groups. We have [27, pp. 90–91] that one basis of left- (resp. right-) invariant 1-forms on G is given by a set of different elements of the matrix $A^{-1}\mathrm{d}A$ (resp. $(\mathrm{d}A)A^{-1}$). Then a left- (resp. right-) invariant measure is given by the wedge product of the given basis of left- (resp. right-) invariant 1-forms. In the present cases we obtain:

(1)

$$A^{-1}\mathrm{d}A = \begin{pmatrix} 0 & 0 & 0 \\ \cos\theta\,\mathrm{d}x - \sin\theta\,\mathrm{d}y & 0 & \mathrm{d}\theta \\ \sin\theta\,\mathrm{d}x + \cos\theta\,\mathrm{d}y & -\mathrm{d}\theta & 0 \end{pmatrix},$$

$$(\mathrm{d}A)A^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{d}x - y\,\mathrm{d}\theta & 0 & \mathrm{d}\theta \\ \mathrm{d}y + x\,\mathrm{d}\theta & -\mathrm{d}\theta & 0 \end{pmatrix},$$

hence the left- and right-invariant measures ω_L and ω_R are, up to a constant factor,

$$\omega_L = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\theta = \omega_R.$$

(2)

$$A^{-1}\mathrm{d}A = \begin{pmatrix} \dfrac{\mathrm{d}x}{x} & \dfrac{1}{x}\left(\mathrm{d}z - z\dfrac{\mathrm{d}y}{y}\right) \\ 0 & \dfrac{\mathrm{d}y}{y} \end{pmatrix}, \qquad (\mathrm{d}A)A^{-1} = \begin{pmatrix} \dfrac{\mathrm{d}x}{x} & \dfrac{1}{y}\left(\mathrm{d}z - z\dfrac{\mathrm{d}x}{x}\right) \\ 0 & \dfrac{\mathrm{d}y}{y} \end{pmatrix},$$

hence

$$\omega_L = \frac{1}{x^2 y}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

$$\omega_R = \frac{1}{x y^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

(3)

$$A^{-1}\mathrm{d}A = \begin{pmatrix} 0 & \mathrm{d}x & \mathrm{d}y - x\,\mathrm{d}z \\ 0 & 0 & \mathrm{d}z \\ 0 & 0 & 0 \end{pmatrix}, \qquad (\mathrm{d}A)A^{-1} = \begin{pmatrix} 0 & \mathrm{d}x & \mathrm{d}y - z\,\mathrm{d}x \\ 0 & 0 & \mathrm{d}z \\ 0 & 0 & 0 \end{pmatrix},$$

hence

$$\omega_L = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \omega_R.$$

(4) Given $A = (a_{ij}) \in GL(2,\mathbb{R})$, $i, j = 1,2$, let $A^{-1} = (\alpha^{ij})$ be its inverse. Then a basis of left-invariant 1-forms is given by the components of $A^{-1}\mathrm{d}A$. The left-invariant measure ω_L on $GL(2,\mathbb{R})$ is given by the wedge product of such components:

$$\begin{aligned} \omega_L &= (\alpha^{11}\mathrm{d}a_{11} + \alpha^{12}\mathrm{d}a_{21}) \wedge (\alpha^{11}\mathrm{d}a_{12} + \alpha^{12}\mathrm{d}a_{22}) \\ &\qquad \wedge (\alpha^{21}\mathrm{d}a_{11} + \alpha^{22}\mathrm{d}a_{21}) \wedge (\alpha^{21}\mathrm{d}a_{12} + \alpha^{22}\mathrm{d}a_{22}) \\ &= (\alpha^{11}\alpha^{22} - \alpha^{12}\alpha^{21})^2\, \mathrm{d}a_{11} \wedge \mathrm{d}a_{12} \wedge \mathrm{d}a_{21} \wedge \mathrm{d}a_{22} \\ &= \frac{1}{(\det A)^2}\, \mathrm{d}a_{11} \wedge \mathrm{d}a_{12} \wedge \mathrm{d}a_{21} \wedge \mathrm{d}a_{22}. \end{aligned}$$

One has $\omega_L = \omega_R$, as the computation of the components of $(\mathrm{d}A)A^{-1}$ shows.

Problem 4.1.9. *Let A be a finite-dimensional $\mathbb{R}$-algebra (not necessarily commutative). Set $n = \dim_{\mathbb{R}} A$. Let $\mathrm{Aut}_{\mathbb{R}} A \approx GL(n,\mathbb{R})$ be the group of all $\mathbb{R}$-linear automorphisms of A and let $G(A)$ be the group of $\mathbb{R}$-algebra automorphisms of A. Let* Der A *be the set of all $\mathbb{R}$-linear maps $X \colon A \to A$ such that*

$$X(a \cdot b) = X(a) \cdot b + a \cdot X(b), \qquad \forall a, b \in A.$$

Prove:

(1) Der A *is a Lie algebra with the bracket*

$$[X,Y](a) = X(Y(a)) - Y(X(a)).$$

(2) *$G(A)$ is a closed subgroup of $GL(n,\mathbb{R})$, hence it is a Lie group.*

(3) $\dim G(A) \leqslant (n-1)^2$.

(4) *The Lie algebra of $G(A)$ is isomorphic to* $(\mathrm{Der}\, A, [\ ,\ ])$.

Solution. (1) Certainly, Der A is a $\mathbb{R}$-vector space. Further, the bracket of two derivations is another derivation, as

$$\begin{aligned} [X,Y](a \cdot b) &= X(Y(a \cdot b)) - Y(X(a \cdot b)) \\ &= X\{Y(a) \cdot b + a \cdot Y(b)\} - Y\{X(a) \cdot b + a \cdot X(b)\} \\ &= \{X(Y(a)) \cdot b + Y(a) \cdot X(b) + X(a) \cdot Y(b) + a \cdot X(Y(b))\} \\ &\quad - \{Y(X(a)) \cdot b + X(a) \cdot Y(b) + Y(a) \cdot X(b) + a \cdot Y(X(b))\} \end{aligned}$$

$$
\begin{aligned}
&= \{X(Y(a)) - Y(X(a))\} \cdot b + a \cdot \{X(Y(b)) - Y(X(b))\} \\
&= [X,Y](a) \cdot b + a \cdot [X,Y](b).
\end{aligned}
$$

Accordingly, $\operatorname{Der} A$ is endowed with a skew-symmetric bilinear map

$$[\,,\,]\colon \operatorname{Der} A \times \operatorname{Der} A \to \operatorname{Der} A,$$

and the Jacobi identity follows from the following calculation:

$$
\begin{aligned}
&([[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y](a) \\
&\quad = ([X,Y](Z(a)) - Z([X,Y](a))) + ([Y,Z](X(a)) - X([Y,Z](a))) \\
&\qquad + ([Z,X](Y(a)) - Y([Z,X](a))) \\
&\quad = X(Y(Z(a))) - Y(X(Z(a))) - Z(X(Y(a))) + Z(Y(X(a))) \\
&\qquad + Y(Z(X(a))) - Z(Y(X(a))) - X(Y(Z(a))) + X(Z(Y(a))) \\
&\qquad + Z(X(Y(a))) - X(Z(Y(a))) - Y(Z(X(a))) + Y(X(Z(a))) = 0.
\end{aligned}
$$

(2) For every pair $a,b \in A$, let $\Phi_{a,b}\colon \operatorname{Aut}_{\mathbb{R}} A \to A$ be the map given by $\Phi_{a,b}(f) = f(a \cdot b) - f(a) \cdot f(b)$. Then we have

$$G(A) = \bigcap_{a,b \in A} \Phi_{a,b}^{-1}(0).$$

As each $\Phi_{a,b}$ is a continuous map we conclude that $G(A)$ is a closed subset in $\operatorname{Aut}_{\mathbb{R}} A$. Furthermore, $G(A)$ is an abstract subgroup as if $f,g \in G(A)$, then

$$
\begin{aligned}
(f \circ g)(a \cdot b) &= f(g(a \cdot b)) \\
&= f(g(a) \cdot g(b)) \\
&= f(g(a)) \cdot f(g(b)) \\
&= (f \circ g)(a) \cdot (f \circ g)(b).
\end{aligned}
$$

Hence $f \circ g \in G(A)$. Similarly $f^{-1} \in G(A)$, for

$$
\begin{aligned}
f(f^{-1}(a) \cdot f^{-1}(b)) &= f(f^{-1}(a)) \cdot f(f^{-1}(b))) \\
&= a \cdot b \\
&= f(f^{-1}(a \cdot b)),
\end{aligned}
$$

and since f is injective we conclude that $f^{-1}(a) \cdot f^{-1}(b) = f^{-1}(a \cdot b)$, for all $a,b \in A$.

(3) If $f \in G(A)$ then $f(1) = 1$. Hence each $f \in G(A)$ induces an automorphism $\bar{f} \in \operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R})$ by setting $\bar{f}(a \mod \mathbb{R}) = f(a) \mod \mathbb{R}$ and the map $h\colon G(A) \to \operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R})$, $f \mapsto \bar{f}$, is clearly a group homomorphism. We claim that h is injective. In fact, $f \in \operatorname{Ker} h$ if and only if $\bar{f}(a) \mod \mathbb{R} = a \mod \mathbb{R}$, for all $a \in A$, and this condition means $f(a) - a \in \mathbb{R}$, for all $a \in A$. Hence we can write $f(a) = a + \omega(a)$,

where $\omega\colon A \to \mathbb{R}$ is a linear form such that $\omega(1) = 0$. By imposing $f(a \cdot b) = f(a) \cdot f(b)$ we obtain

$$\omega(a \cdot b) = \omega(b)a + \omega(a)b + \omega(a)\omega(b).$$

Hence

$$\omega(b)a + \omega(a)b \in \mathbb{R}, \qquad \forall a, b \in A. \tag{$\star$}$$

If $\omega \neq 0$, then there exists $a \in A$ such that $\omega(a) = 1$, and from $(\star)$ it follows that $b \in \mathbb{R} + \mathbb{R}a$ for every $b \in A$. Hence $\dim A = 2$ and then either $A \approx \mathbb{R}[\varepsilon]$ or $A \approx \mathbb{R}[\mathrm{i}]$ or $A \approx \mathbb{R}[\mathrm{j}]$, with $\varepsilon^2 = 0$, $\mathrm{i}^2 = -1$, $\mathrm{j}^2 = 1$, thus leading us to a contradiction, as in these cases $\operatorname{Ker} h$ is the identity. Accordingly, $G(A)$ is a subgroup of $\operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R})$ so that $\dim G(A) \leqslant \dim \operatorname{Aut}_{\mathbb{R}}(A/\mathbb{R}) = (n-1)^2$.

(4) Let $\mathfrak{g}(A)$ be the Lie algebra of $G(A)$, which is a Lie subalgebra of $\operatorname{End}_{\mathbb{R}} A = \operatorname{Lie}(\operatorname{Aut}_{\mathbb{R}} A)$. We know that an element $X \in \operatorname{End}_{\mathbb{R}} A$ belongs to $\mathfrak{g}(A)$ if and only if for every $t \in \mathbb{R}$ we have $\exp(tX) \in G(A)$, or equivalently, $\exp(tX)(a \cdot b) = \exp(tX)(a) \cdot \exp(tX)(b)$. Differentiating this equation at $t = 0$ we conclude that X is a derivation of A.

Conversely, if X is a derivation, then by recurrence on k it is readily checked that

$$X^k(a \cdot b) = \sum_{h=0}^{k} \binom{k}{h} X^h(a) X^{k-h}(b).$$

Hence,

$$\begin{aligned}
\exp(tX)(a \cdot b) &= \sum_{k=0}^{\infty} t^k \frac{X^k(a \cdot b)}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{h=0}^{k} t^k \frac{1}{(k-h)!h!} X^h(a) X^{k-h}(b) \\
&= \exp(tX)(a) \cdot \exp(tX)(b),
\end{aligned}$$

thus proving that the Lie algebra of $G(A)$ is isomorphic to $(\operatorname{Der} A, [\,,\,])$.

Problem 4.1.10. *Prove that the Lie algebra* $\mathfrak{so}(3)$ *does not admit any* 2*-dimensional Lie subalgebra.*

Solution. Let $\{e_1, e_2, e_3\}$ be the standard basis; that is:

$$[e_1 e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2.$$

Assume $\mathfrak{g}$ is a 2-dimensional Lie subalgebra. Let $\{v = \lambda^i e_i,\, w = \mu^i e_i\}$ be a basis of $\mathfrak{g}$. As the rank of the 3×2-matrix

$$\begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \\ \lambda^3 & \mu^3 \end{pmatrix}$$

is 2, we can assume that

$$\det \begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \end{pmatrix} \neq 0.$$

By making a change of basis in $\mathfrak{g}$ we can thus suppose that

$$\begin{pmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$v = e_1 + \lambda^3 e_3, \qquad w = e_2 + \mu^3 e_3. \tag{$\star$}$$

As $\mathfrak{g}$ is a Lie subalgebra, we have $[v,w] = \alpha v + \beta w$. By using ($\star$) we obtain $\lambda^3 = -\alpha$, $\mu^3 = -\beta$, $\alpha\lambda^3 + \beta\mu^3 = 1$, and substituting the first two relations into the third one we obtain $\alpha^2 + \beta^2 + 1 = 0$, thus leading to a contradiction.

4.2 Homomorphisms of Lie Groups and Lie Algebras

Problem 4.2.1. *Let $\mathbb{R}$ be the additive group of real numbers and S^1 the multiplicative group of the complex numbers of modulus* 1. *Prove that*

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & S^1 \\ t & \mapsto & f(t) = \mathrm{e}^{2\pi \mathrm{i} t} \end{array}$$

is a homomorphism of Lie groups.

Solution. The map f is a homomorphism of groups, as

$$f(t_1 + t_2) = \mathrm{e}^{2\pi \mathrm{i}(t_1+t_2)} = \mathrm{e}^{2\pi \mathrm{i} t_1} \mathrm{e}^{2\pi \mathrm{i} t_2} = f(t_1) f(t_2),$$

and it is clearly C^∞.

Problem 4.2.2. *Consider the Lie group*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a > 0 \right\}$$

and the map $\varphi\colon G \to \mathbb{R}^3$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto (a, b, a-b)$. Is φ a homomorphism of Lie groups?

Solution. We have

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} \right) = (aa', ab' + ba', aa' - ab' - ba')$$

and

$$\varphi\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\varphi\begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = (a+a', b+b', a+a'-b-b'),$$

so φ is not even a homomorphism of groups.

Problem 4.2.3. *Consider the Heisenberg group* (cf. Problem 4.1.6) *and the map* $f\colon H \to \mathbb{R}$, $A \mapsto f(A) = x+y+z$.

(1) *Is f differentiable?*

(2) *Is it a homomorphism of Lie groups?*

Solution. Let

$$\psi\colon H \to \mathbb{R}^3, \qquad \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x,y,z),$$

be the usual global chart of H. The map

$$f \circ \psi^{-1}\colon \mathbb{R}^3 \to \mathbb{R}, \qquad (x,y,z) \mapsto x+y+z,$$

is C^∞ and thus f is C^∞. The additive group of real numbers $(\mathbb{R},+)$ is a Lie group.

Given

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

it is easy to see that $f(AB) \neq f(A) + f(B)$, so that f is not a homomorphism of Lie groups.

Problem 4.2.4. *Prove that one has*:

(1) *An isomorphism of Lie groups $SO(2) \approx U(1) \approx S^1$.*

(2) *A homeomorphism $O(n) \approx SO(n) \times \{-1,+1\}$.*

HINT (to (1)): *Consider the* real representation of the general linear group $GL(1,\mathbb{C})$*:*

$$\begin{aligned} \rho\colon GL(1,\mathbb{C}) &\to GL(2,\mathbb{R}) \\ a+b\mathrm{i} &\mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \end{aligned}$$

Solution. (1)

$$\begin{aligned} SO(2) &= \{A \in GL(2,\mathbb{R}) : {}^tAA = I,\ \det A = 1\} \\ &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2+c^2 = b^2+d^2 = 1,\ ab+cd = 0,\ ad-bc = 1\right\} \\ &\approx \left\{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \mathbb{R}\right\} \end{aligned}$$

and

$$\begin{aligned} U(1) &= \{A \in GL(1,\mathbb{C}) : {}^t\bar{A}A = 1\} \\ &= \{z \in \mathbb{C} - \{0\} : \bar{z}z = 1\} \\ &= \{z \in \mathbb{C} : z = \cos\alpha + \mathrm{i}\sin\alpha\} \\ &\approx \{(\cos\alpha, \sin\alpha) : \alpha \in \mathbb{R}\} = S^1. \end{aligned}$$

Let ρ be the real representation of $GL(1,\mathbb{C})$. If $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $a = 1$, $b = 0$; so ρ is injective. We have

$$\rho(U(1)) = \rho(\{\cos\alpha + \mathrm{i}\sin\alpha\}) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \right\} \approx SO(2).$$

Since ρ is injective, one obtains $U(1) \approx SO(2)$.

(2)

$$O(n) = \left\{A \in GL(n,\mathbb{R}) : {}^tAA = I\right\}.$$

Hence if $A \in O(n)$, then $\det {}^tAA = 1$. Consider the exact sequence

$$1 \to SO(n) \overset{j}{\hookrightarrow} O(n) \overset{\det}{\to} \{-1,+1\} \to 1,$$

where j denotes the inclusion map of $SO(n) = \{A \in O(n) : \det A = 1\}$ in $O(n)$. The map

$$\begin{aligned} \sigma\colon \{-1,+1\} &\to O(n) \\ 1 &\mapsto I \\ -1 &\mapsto \operatorname{diag}\{-1,1,\dots,1\} \end{aligned}$$

is a section of det, hence we have a homeomorphism

$$O(n) \approx SO(n) \times \{-1,+1\}, \qquad A \mapsto (\sigma(\det A)A, \det A).$$

Problem 4.2.5. *Let $\psi\colon G \to G$ be the diffeomorphism of the Lie group G defined by $\psi(a) = a^{-1}$, $a \in G$. Prove that ω is a left-invariant form if and only if $\psi^*\omega$ is right-invariant.*

Solution.

$$\begin{aligned} \psi(R_b x) &= \psi(xb) \\ &= (xb)^{-1} = b^{-1}x^{-1}, \end{aligned}$$

and

$$\begin{aligned} L_{b^{-1}}\psi(x) &= L_{b^{-1}}(x^{-1}) \\ &= b^{-1}x^{-1}. \end{aligned}$$

Thus $\psi \circ R_b = L_{b^{-1}} \circ \psi$, hence $(\psi \circ R_b)^* = (L_{b^{-1}} \circ \psi)^*$, that is $R_b^* \circ \psi^* = \psi^* \circ L_{b^{-1}}^*$. If ω is left-invariant, we have

$$\begin{aligned}(R_b^* \circ \psi^*)(\omega) &= (\psi^* \circ L_{b^{-1}}^*)(\omega) \\ &= \psi^* \omega,\end{aligned}$$

thus $\psi^* \omega$ is right-invariant. Conversely, if $\psi^* \omega$ is right-invariant, $\psi^* \omega = R_b^* \psi^* \omega = \psi^* L_{b^{-1}}^* \omega$, thus $\omega = L_{b^{-1}}^* \omega$, because ψ^* is an isomorphism, and ω is left-invariant.

Problem 4.2.6. *Let G be a compact, connected Lie group oriented by a left-invariant volume form ω. Prove that for every continuous function f on G and every $s \in G$ we have*

$$\int_G f\omega = \int_G (f \circ R_s)\omega,$$

where $R_s \colon G \to G$ denotes the right translation by s; that is, the left-invariant integral $f \mapsto \int_G f\omega$ is also right-invariant.

Solution. For every $s \in G$ there exists a unique scalar $\varphi(s) \in \mathbb{R}^*$ such that $R_s^* \omega = \varphi(s)\omega$. The map $\varphi \colon G \to \mathbb{R}^*$ is clearly differentiable and since $\varphi(e) = 1$ (where e denotes the identity element of G) and G is connected we have $\varphi(G) \subseteq \mathbb{R}^+$; hence R_s is orientation-preserving. By applying the formula of change of variables to the diffeomorphism $R_s \colon G \to G$ we obtain

$$\begin{aligned}\int_G f\omega &= \int_G R_s^*(f\omega) \\ &= \int_G (f \circ R_s) R_s^* \omega \\ &= \int_G (f \circ R_s)\varphi(s)\omega.\end{aligned}$$

Hence

$$\int_G f\omega = \varphi(s) \int_G (f \circ R_s)\omega.$$

Letting $f = 1$ and taking into account that $\int_G \omega \neq 0$, we conclude that $\varphi(s) = 1$, for all $s \in G$, and consequently

$$\int_G f\omega = \int_G (f \circ R_s)\omega.$$

We also remark that $\varphi(s) = 1$, for all $s \in G$, implies that ω is right-invariant.

Problem 4.2.7. *Let G be a compact, connected Lie group oriented by a left-invariant volume form ω and let $\psi \colon G \to G$ be the map $\psi(a) = a^{-1}$, for all $a \in G$. Prove that for every continuous function f on G we have*

$$\int_G f\omega = \int_G (f \circ \psi)\omega.$$

Solution. For every $s \in G$ we have from Problem 4.2.5:

$$\begin{aligned} R_s^*(\psi^*\omega) &= (\psi \circ R_s)^*\omega \\ &= (L_{s^{-1}} \circ \psi)^*\,\omega \\ &= \psi^*\left(L_{s^{-1}}^*\omega\right) \\ &= \psi^*\omega. \end{aligned}$$

Hence $\psi^*\omega$ is right-invariant and since ω is also right-invariant (see Problem 4.2.6), there exists $\varepsilon \in \mathbb{R}^*$ such that $\psi^*\omega = \varepsilon\omega$. Moreover, $\varepsilon^2 = 1$ as ψ is an involution. By applying the formula of change of variables to the diffeomorphism ψ (which may be orientation-reversing) we obtain

$$\begin{aligned} \int_G f\omega &= \varepsilon \int_G \psi^*(f\omega) \\ &= \varepsilon \int_G (f \circ \psi)\psi^*\omega \\ &= \varepsilon \int_G (f \circ \psi)\varepsilon\psi \\ &= \int_G (f \circ \psi)\omega. \end{aligned}$$

Problem 4.2.8. *Let λ be an irrational real number and let φ be the map*

$$\varphi\colon \mathbb{R} \to T^2 = S^1 \times S^1, \qquad \varphi(t) = (\mathrm{e}^{2\pi\mathrm{i}t}, \mathrm{e}^{2\pi\mathrm{i}\lambda t}).$$

(1) *Prove that it is an injective homomorphism of Lie groups.*

(2) *Prove that the image of φ is dense in the torus.*

Solution. (1) That φ is a homomorphism of Lie groups is immediate. We have $\varphi(t_1) = \varphi(t_2)$ if and only if $(\mathrm{e}^{2\pi\mathrm{i}t_1}, \mathrm{e}^{2\pi\mathrm{i}\lambda t_1}) = (\mathrm{e}^{2\pi\mathrm{i}t_2}, \mathrm{e}^{2\pi\mathrm{i}\lambda t_2})$, or equivalently if $t_1 - t_2$ and $\lambda(t_1 - t_2)$ are integers, which happens only if $t_1 = t_2$. Hence φ is injective.

(2) It suffices to show that the subgroup $\mathbb{Z} + \lambda\,\mathbb{Z}$ is dense in $\mathbb{R}$, since if this happens, given the real numbers t_1, t_2, there exists a sequence $m_j + \lambda\, n_j$ such that

$$t_2 - \lambda t_1 = \lim_{j\to\infty}(m_j + \lambda\, n_j),$$

that is,

$$t_2 = \lim_{j\to\infty}\bigl(m_j + \lambda(n_j + t_1)\bigr).$$

Hence

$$\begin{aligned} \varphi(n_j + t_1) &= (\mathrm{e}^{2\pi\mathrm{i}t_1}, \mathrm{e}^{2\pi\mathrm{i}\lambda(n_j+t_1)}) \\ &= (\mathrm{e}^{2\pi\mathrm{i}t_1}, \mathrm{e}^{2\pi\mathrm{i}(m_j+\lambda(n_j+t_1))}), \end{aligned}$$

and thus

$$\lim_{j\to\infty} \varphi(n_j + t_1) = (\mathrm{e}^{2\pi \mathrm{i} t_1}, \mathrm{e}^{2\pi \mathrm{i} t_2}).$$

Now, to prove that the subgroup $A = \mathbb{Z} + \lambda\,\mathbb{Z}$ is dense in $\mathbb{R}$, it suffices to see that the origin is an accumulation point in A, since in this case, given $x > 0$ and $0 < \varepsilon < x$, there exists $a \in A$ such that $0 < a < \varepsilon$, and if N denotes the greatest integer less than or equal to $(x-\varepsilon)/a$, then $Na \leqslant x-\varepsilon < (N+1)a$, which implies $(N+1)a < x+\varepsilon$, as in the contrary case we would have $x+\varepsilon \leqslant (N+1)a \leqslant x-\varepsilon+a$, that is $2\varepsilon \leqslant a < \varepsilon$. Contradiction. Thus we have $x-\varepsilon < (N+1)a < x+\varepsilon$, that is, $|(N+1)a - x| < \varepsilon$.

If the origin is not an accumulation point in A, it is an isolated point, and then every point in A is isolated as A is a subgroup. Hence A is a closed discrete subset of $\mathbb{R}$. In fact, if $\lim_{k\to\infty} x_k = x$, $x_k \in A$, then for k large enough, $x_k - x_{k+1}$ belongs to an arbitrarily small neighborhood of the origin. As $x_k - x_{k+1} \in A$ and the origin is isolated, we conclude $x_k = x_{k+1}$. Hence $x \in A$. Accordingly, $\mu = \inf\{x \in A : x > 0\}$ is a positive element in A. We prove that A is generated by μ, that is, $\mathbb{Z} + \lambda\,\mathbb{Z} = \mu\,\mathbb{Z}$. This will lead us to a contradiction, as λ is irrational.

Let $x \in A$ be a positive element. Let n denote the greatest integer less than or equal to x/μ, so that $n \leqslant x/\mu < n+1$. Hence $0 \leqslant x - n\mu < \mu$. As $x - n\mu \in A$, from the very definition of μ we conclude that $x - n\mu = 0$.

4.3 Lie Subgroups and Lie Subalgebras

Problem 4.3.1. *Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers.*

(1) *Prove that the map*

$$j\colon \mathbb{C}^* \to GL(2,\mathbb{R}), \qquad x + \mathrm{i}y \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

is a faithful representation of the Lie group $\mathbb{C}^*$ (faithful means that j is injective).

(2) *Find the Lie subalgebra* $\operatorname{Lie}(j(\mathbb{C}^*))$ *of* $\mathfrak{gl}(2,\mathbb{R})$.

Solution. (1) Since $\mathbb{C}^* \approx GL(1,\mathbb{C})$, this was proved in Problem 4.2.4.

(2)

$$\begin{aligned} \operatorname{Lie}(j(\mathbb{C}^*)) &= j_*(T_1\mathbb{C}^*) \\ &\approx \left\{ \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix},\ \lambda,\mu \in \mathbb{R} \right\}. \end{aligned}$$

Problem 4.3.2. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Consider the distribution $\mathscr{D}(s) = \{X_s : X \in \mathfrak{h}\}$, $s \in G$.*

(1) *Show that $\mathscr{D}$ is a C^∞ distribution of the same dimension as $\mathfrak{h}$. Is it involutive?*

(2) *Consider the 2-dimensional C^∞ distributions*

$$\mathscr{D}_1 = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle, \quad \mathscr{D}_2 = \left\langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle, \quad \mathscr{D}_3 = \left\langle \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle,$$

on the Heisenberg group (*see* Problem 4.1.6). *Are they involutive?*

(3) *Let* $\mathscr{I}(\mathscr{D}_i)$ *be the differential ideal corresponding to* $\mathscr{D}_i$, $i = 1,2,3$. *If* $\alpha = \mathrm{d}x \wedge \mathrm{d}z$, $\beta = \mathrm{d}x + \mathrm{d}z$. *Do we have* $\alpha, \beta \in \mathscr{I}(\mathscr{D}_1)$? *And* $\alpha, \beta \in \mathscr{I}(\mathscr{D}_3)$?

Solution. (1) The space $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$. Let $\{X_1, \dots, X_k\}$ be left-invariant vector fields which are a basis of $\mathfrak{h}$. Then,

$$\mathscr{D}(s) = \langle X_1|_s, \dots, X_k|_s \rangle, \qquad s \in G,$$

is a vector subspace of T_sG of dimension k. Hence $\mathscr{D}$ is a C^∞ k-dimensional distribution on G, because it is globally spanned by $X_1, \dots, X_k$, which are left-invariant vector fields, hence C^∞. Moreover, $\mathscr{D}$ is involutive. In fact, $X_1, \dots, X_k$ spans $\mathscr{D}$ and $\mathfrak{h}$ is a subalgebra, so $[X_i, X_j] \in \mathfrak{h}$.

(2) The Lie algebra $\mathfrak{h}$ of H is spanned (see Problem 4.1.6) by

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Since

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0, \qquad \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right] = \frac{\partial}{\partial y}, \qquad \left[\frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right] = 0,$$

it follows that $\mathscr{D}_1$ and $\mathscr{D}_3$ are involutive, but $\mathscr{D}_2$ is not.

(3) $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$ is a basis of the $(C^\infty H)$-module $\mathfrak{X}(H)$, with dual basis $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$, thus $\mathscr{I}(\mathscr{D}_1) = \langle \mathrm{d}z \rangle$. Hence $\alpha \in \mathscr{I}(\mathscr{D}_1)$ but $\beta \notin \mathscr{I}(\mathscr{D}_1)$. Also $\{\partial/\partial x, \partial/\partial y, x\partial/\partial y + \partial/\partial z\}$ is a basis of $\mathfrak{X}(H)$, with dual basis $\{\theta^1, \theta^2, \theta^3\}$, and we have $\mathscr{I}(\mathscr{D}_3) = \langle \theta^1 \rangle = \langle \mathrm{d}x \rangle$. Hence $\alpha \in \mathscr{I}(\mathscr{D}_3)$ and $\beta \notin \mathscr{I}(\mathscr{D}_3)$.

Problem 4.3.3. *Consider the set G of matrices of the form*

$$g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \qquad x, y \in \mathbb{R}, \quad x \neq 0.$$

(1) *Show that G is a Lie subgroup of* $GL(2,\mathbb{R})$.

(2) *Show that the elements of* $\omega = g^{-1}\mathrm{d}g$ *are left-invariant* 1*-forms.*

(3) *Since* $\mathfrak{g} \approx T_eG = \mathbb{R}^2$, *we have* $\dim \mathfrak{g} = 2$, *and we can choose* $\{\omega_1 = \mathrm{d}x/x,\ \omega_2 = \mathrm{d}y/x\}$ *as a basis of the space of left-invariant* 1*-forms. Compute the structure constants of G with respect to this basis.*

(4) *Prove that* ω *satisfies the relation* $\mathrm{d}\omega + \omega \wedge \omega = 0$.

REMARK. *Here* $\mathrm{d}\omega$ *denotes the matrix* $(\mathrm{d}\omega)^i_j = (\mathrm{d}\omega^i_j)$ *and* $\omega \wedge \omega$ *denotes the wedge product of matrices* $(\omega \wedge \omega)^i_j = (\omega^i_k \wedge \omega^k_j)$.

Solution. (1) Since

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xx' & xy'+y \\ 0 & 1 \end{pmatrix} \in G, \qquad \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y/x \\ 0 & 1 \end{pmatrix} \in G,$$

G is an abstract subgroup of $GL(2,\mathbb{R})$, and as both the product and the inverse are C^∞ maps, G is a Lie group.

Moreover, G is a closed subgroup of $GL(2,\mathbb{R})$, defined by the equations $x_2^2 = x_3^3 - 1 = 0$, x_j^i being the usual coordinates of $GL(2,\mathbb{R}) \subset M(2,\mathbb{R}) \approx \mathbb{R}^4$. Hence G is closed in $GL(2,\mathbb{R})$ and accordingly G is a Lie subgroup of $GL(2,\mathbb{R})$.

(2) If $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, one has

$$\omega = g^{-1}\mathrm{d}g = \frac{1}{x}\begin{pmatrix} \mathrm{d}x & \mathrm{d}y \\ 0 & 0 \end{pmatrix}.$$

We must prove $L_g^*\omega_g = \omega_e$. Let $s = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ be arbitrarily fixed. Proceeding similarly to Problem 4.1.2, we obtain $L_{s*} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Thus

$$\begin{aligned} L_g^*\omega_g &= \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1/x \\ 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 \\ 1/x \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \\ &\equiv \{\mathrm{d}x, \mathrm{d}y\} \\ &\equiv \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \\ 0 & 0 \end{pmatrix} \\ &= \omega_e, \end{aligned}$$

for $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(3) As

$$\mathrm{d}\left(\frac{\mathrm{d}x}{x}\right) = 0, \qquad \mathrm{d}\left(\frac{\mathrm{d}y}{x}\right) = -\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{x},$$

from Maurer-Cartan's equation $\mathrm{d}\omega_i = -\sum_{j<k} c_{jk}^i \omega^j \wedge \omega^k$, we deduce

$$c_{12}^2 = -c_{21}^2 = 1.$$

(4)

$$\mathrm{d}\omega = -\frac{1}{x^2}\begin{pmatrix} 0 & \mathrm{d}x \wedge \mathrm{d}y \\ 0 & 0 \end{pmatrix} = -\omega \wedge \omega.$$

Problem 4.3.4. *Let S be the set of matrices of the form*

$$M(u,v,w) = \begin{pmatrix} \cos w & \sin w & 0 & u \\ -\sin w & \cos w & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u,v,w \in \mathbb{R}.$$

(1) *Prove that S is a Lie subgroup of $GL(4,\mathbb{R})$.*

(2) *Let $\sigma\colon \mathbb{R}^3 \to GL(4,\mathbb{R})$, $(u,v,w) \mapsto M(u,v,w)$. Compute*

$$\sigma_* \frac{\partial}{\partial u}, \qquad \sigma_* \frac{\partial}{\partial v}, \qquad \sigma_* \frac{\partial}{\partial w},$$

and show that σ is an immersion.

(3) *Prove that the tangent space to S at the identity element $e \in S$ admits the basis*

$$\left\{ X_1 = \frac{\partial}{\partial x_4^1}\bigg|_e,\ X_2 = \frac{\partial}{\partial x_4^2}\bigg|_e,\ X_3 = \frac{\partial}{\partial x_2^1}\bigg|_e - \frac{\partial}{\partial x_1^2}\bigg|_e + \frac{\partial}{\partial x_4^3}\bigg|_e \right\}.$$

Solution. (1) For all $M(u,v,w) \in S$, one has $\det M(u,v,w) = 1$, so $S \subset GL(4,\mathbb{R})$. Moreover the product of two elements of S and also the inverse of any element, belong to S, as it follows by direct computation, so that S is a subgroup of $GL(4,\mathbb{R})$. Further, S can be considered as the closed subgroup of $GL(4,\mathbb{R})$ determined by the equations

$$x_1^1 = x_2^2 = \cos x_4^3, \qquad x_2^1 = -x_1^2 = \sin x_4^3, \qquad x_3^3 = x_4^4 = 1,$$
$$x_3^1 = x_3^2 = x_1^3 = x_2^3 = x_1^4 = x_2^4 = x_3^4 = 0,$$

x_j^i being the usual coordinates of $GL(4,\mathbb{R}) \subset M(4,\mathbb{R}) \approx \mathbb{R}^{16}$. Hence by Cartan's criterion on closed subgroups, S is a Lie subgroup of $GL(4,\mathbb{R})$.

(2) We have

$$\begin{aligned} \sigma_* \frac{\partial}{\partial u} &= \frac{\partial}{\partial x_4^1}, \\ \sigma_* \frac{\partial}{\partial v} &= \frac{\partial}{\partial x_4^2}, \\ \sigma_* \frac{\partial}{\partial w} &= -\sin w \frac{\partial}{\partial x_1^1} + \cos w \frac{\partial}{\partial x_2^1} - \cos w \frac{\partial}{\partial x_1^2} - \sin w \frac{\partial}{\partial x_2^2} + \frac{\partial}{\partial x_4^3}. \end{aligned}$$

Therefore σ is an immersion.

(3) The identity element of S, $e = I$, corresponds to $u = v = w = 2k\pi$. By (2), T_eS admits the basis in the statement.

Problem 4.3.5. *Let* $G=\left\{\begin{pmatrix} a & 0\\ b & 1\end{pmatrix} : a,b\in\mathbb{R},\, a>0\right\}$.

(1) *Prove that G admits a Lie group structure.*

(2) *Is G a Lie subgroup of* $GL(2,\mathbb{R})$*?*

(3) *Let* μ *be the map defined by*

$$G\to GL(2,\mathbb{R}),\qquad \begin{pmatrix} a & 0\\ b & 1\end{pmatrix}\mapsto\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}.$$

Is it differentiable? Is it a homomorphism of Lie groups? Is it an immersion? (cf. Problem 4.3.3.)

Solution. (1) The map

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & U=\left\{(a,b)\in\mathbb{R}^2 : a>0\right\}\\ \begin{pmatrix} a & 0\\ b & 1\end{pmatrix} & \mapsto & (a,b)\end{array}$$

is obviously bijective. Since U is open in $\mathbb{R}^2$, it is a 2-dimensional C^∞ manifold and thus there exists a unique differentiable structure on G such that $\dim G=2$ and φ is a diffeomorphism.

G is a group with the product of matrices, since given

$$A=\begin{pmatrix} a & 0\\ b & 1\end{pmatrix},\qquad B=\begin{pmatrix} a' & 0\\ b' & 1\end{pmatrix},$$

then

$$AB=\begin{pmatrix} aa' & 0\\ ba'+b' & 1\end{pmatrix}\in G,\quad\text{and}\quad A^{-1}=\begin{pmatrix} 1/a & 0\\ -b/a & 1\end{pmatrix}\in G.$$

Therefore G is a subgroup of $GL(2,\mathbb{R})$.

The operations

$$\begin{array}{ccc} G\times G & \xrightarrow{\Phi} & G\\ (A,B) & \mapsto & AB\end{array}\quad\text{and}\quad\begin{array}{ccc} G & \xrightarrow{\Psi} & G\\ A & \mapsto & A^{-1}\end{array}$$

are C^∞. In fact,

$$(\varphi\circ\Phi\circ(\varphi\times\varphi)^{-1})\big((a,b),(a',b')\big)=(aa',ba'+b'),\qquad a,a'>0,$$
$$(\varphi\circ\Psi\circ\varphi^{-1})(a,b)=(1/a,-b/a),\qquad a>0,$$

are C^∞.

(2) G is the closed submanifold of the open subset $x_1^1>0$ in $GL(2,\mathbb{R})$ given by the equations $x_2^1=0$, $x_2^2-1=0$, x_j^i being the usual coordinates of $GL(2,\mathbb{R})\subset M(2,\mathbb{R})\approx\mathbb{R}^4$. Thus G is a Lie subgroup of $GL(2,\mathbb{R})$.

Another way to prove that G is a Lie subgroup of $GL(2,\mathbb{R})$ is to observe that G is closed in $GL(2,\mathbb{R})$, as if the sequence

$$\begin{pmatrix} a_n & 0 \\ b_n & 1 \end{pmatrix}$$

goes to

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2,\mathbb{R}),$$

as $n \to \infty$, then it implies that $a_{11} \geqslant 0$, $a_{12} = 0$, $a_{22} = 1$; hence $a_{11} > 0$, and we can apply Cartan's criterion on closed subgroups.

(3) μ can be written in local coordinates as

$$(\psi \circ j \circ \mu \circ \varphi^{-1})(a,b) = (a,b,0,1), \qquad (a,b) \in U,$$

where ψ stands for the coordinate map of a local coordinate system on $GL(2,\mathbb{R})$ and j denotes the inclusion map $j\colon \mu(G) \to GL(2,\mathbb{R})$. As $\psi \circ j \circ \mu \circ \varphi^{-1}$ is a C^∞ map, μ is C^∞. On the other hand,

$$\mu\left(\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix}\right) = \mu\begin{pmatrix} aa' & 0 \\ ba'+b' & 1 \end{pmatrix} = \begin{pmatrix} aa' & ba'+b \\ 0 & 1 \end{pmatrix}$$

and

$$\mu\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\mu\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab'+b \\ 0 & 1 \end{pmatrix},$$

hence μ is not even a homomorphism of groups.

Finally, we have

$$\operatorname{rank} \mu_{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}} = \operatorname{rank}(\psi \circ \mu \circ \varphi^{-1})_{(a,b)} = 2.$$

Hence μ is an immersion.

Problem 4.3.6. *Prove that* $\mathfrak{sl}(n,\mathbb{C})$ *is an ideal of* $\mathfrak{gl}(n,\mathbb{C})$.

Solution. If $A,B \in \mathfrak{gl}(n,\mathbb{C})$, then $\operatorname{tr}[A,B] = 0$, hence $[A,B] \in \mathfrak{sl}(n,\mathbb{C})$. Therefore $\mathfrak{sl}(n,\mathbb{C})$ is an ideal.

Problem 4.3.7. (1) *Determine all the* 2*-dimensional Lie algebras. In fact, prove that there is a unique non-Abelian* 2*-dimensional Lie algebra.*

(2) *Prove that the map* ρ *of the non-Abelian* 2*-dimensional Lie Algebra* $\mathfrak{g}$ *to* $\operatorname{End}\mathfrak{g}$ *given by* $e \mapsto [e,\cdot]$ (*that is, the adjoint representation*) *is a faithful representation of* $\mathfrak{g}$.

(3) *Give a basis of left-invariant vector fields on the image of* ρ *and their bracket.*

(4) *Let $\mathscr{D}$ be the distribution on* Aut $\mathfrak{g}$ *spanned by the left-invariant vector fields on the image of* ρ. *Find a coordinate system* (u,v,w,z) *on* Aut $\mathfrak{g}$ *such that* $\partial/\partial z$, $\partial/\partial u$ *span* $\mathscr{D}$ *locally.*

(5) *Prove that the subgroup* $G_0 \subset GL(2,\mathbb{R})$ *determined by the subalgebra* $\mathfrak{g}$ *is the identity component* $(\beta > 0)$ *of the group*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} : \beta \neq 0 \right\}.$$

(6) *Prove that G can be viewed as the group* $\mathrm{Aff}(\mathbb{R})$ *of affine transformations of the real line* $\mathbb{R}$. *That is, the group of transformations*

$$t' = \beta t + \alpha, \qquad \beta \neq 0,$$

where $t = y/x, t' = y'/x'$ *are the affine coordinates.*

Solution. (1) Let $\mathfrak{g}$ be a 2-dimensional Lie algebra with basis $\{e_1, e_2\}$. The Lie algebra structure is completely determined, up to isomorphism, knowing the constants a and b in the only bracket

$$[e_1, e_2] = ae_1 + be_2.$$

If $a = b = 0$, the Lie algebra is Abelian, that is, $[e, e'] = 0$ for all $e, e' \in \mathfrak{g}$.

Otherwise, permuting e_1 and e_2 if necessary, we can suppose $b \neq 0$, so $\{e'_1 = (1/b)e_1, e'_2 = (a/b)e_1 + e_2\}$ is a basis of $\mathfrak{g}$ and one has

$$[e'_1, e'_2] = e'_2.$$

Hence there exist, up to isomorphisms, only two 2-dimensional Lie algebras.

(2) Let $\mathfrak{g} = \langle e'_1, e'_2 \rangle$ be the 2-dimensional non-Abelian Lie algebra. That the map ρ is a representation follows from Jacobi's identity. The representation is faithful (that is, the homomorphism is injective) as we have

$$\rho(ae'_1 + be'_2) = \begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix}$$

in the basis $\{e'_1, e'_2\}$.

(3) Fixing that basis, End $\mathfrak{g}$ can be identified to the space of 2×2 square matrices, which is the Lie algebra of $GL(2,\mathbb{R})$.

Let E^i_j be the $n \times n$-matrix with zero entries except the (i, j)th, which is 1. The left-invariant vector field $X_{i,j}$ associated to E^i_j generates the 1-parameter group $(\varphi^i_j)_t$ given by

$$(\varphi^i_j)_t X = X \cdot \exp(tE^i_j), \qquad X \equiv (x^k_l).$$

Now,

$$(E^i_j)^2 = \begin{cases} 0 & \text{if } i \neq j \\ E^i_j & \text{if } i = j. \end{cases}$$

Hence

$$\exp(tE^i_j) = \begin{cases} I + tE^i_j & \text{if } i \neq j \\ I + (\mathrm{e}^t - 1)E^i_j & \text{if } i = j, \end{cases}$$

where I denotes the identity matrix.

As a computation shows,

$$X^i_j = x^k_i \frac{\partial}{\partial x^k_j}.$$

So, in the present case we have

$$X^2_1 = x^1_2 \frac{\partial}{\partial x^1_1} + x^2_2 \frac{\partial}{\partial x^2_1}, \qquad X^2_2 = x^1_2 \frac{\partial}{\partial x^1_2} + x^2_2 \frac{\partial}{\partial x^2_2},$$

and

$$[X^2_1, X^2_2] = -X^2_1.$$

(4) Let us reduce X^2_1 to canonical form. The functions

$$u = \frac{x^2_1}{x^2_2}, \qquad v = x^1_1 - \frac{x^1_2 x^2_1}{x^2_2}, \qquad x^1_2, \qquad x^2_2,$$

are coordinate functions on the neighborhood defined by $x^2_2 \neq 0$ of the identity element I. In fact,

$$\frac{\partial(u, v, x^1_2, x^2_2)}{\partial(x^1_1, x^1_2, x^2_1, x^2_2)} = \frac{1}{x^2_2}.$$

In the new system we have

$$X^2_1 = \frac{\partial}{\partial u}, \qquad X^2_2 = -u\frac{\partial}{\partial u} + x^1_2 \frac{\partial}{\partial x^1_2} + x^2_2 \frac{\partial}{\partial x^2_2}.$$

Now, taking $w = x^1_2/x^2_2$, the functions (u, v, w, x^2_2) are coordinate functions on the neighborhood given by $x^2_2 \neq 0$, since

$$\frac{\partial(u, v, w, x^2_2)}{\partial(u, v, x^1_2, x^2_2)} = \frac{1}{x^2_2}.$$

In this system we have

$$X^2_1 = \frac{\partial}{\partial u}, \qquad X^2_2 = -u\frac{\partial}{\partial u} + x^2_2 \frac{\partial}{\partial x^2_2}.$$

Finally, defining $z = \log x^2_2$ in the neighborhood $x^2_2 > 0$ of the identity element, we obtain coordinate functions (u, v, w, z) in which

$$X^2_1 = \frac{\partial}{\partial u}, \qquad X^2_2 = -u\frac{\partial}{\partial u} + \frac{\partial}{\partial z}.$$

Thus, the involutive submodule $\mathscr{D}$ corresponding to the subalgebra $\mathfrak{g}$ is spanned by $\partial/\partial u$, $\partial/\partial z$, that is,

$$\begin{aligned}\mathscr{D} &= \langle X_1^2, X_2^2 \rangle \\ &= \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial z} \right\rangle.\end{aligned}$$

(5) From the above results, the integral submanifolds of $\mathscr{D}$ are defined by

$$v = x_1^1 - \frac{x_2^1 x_1^2}{x_1^2} = A, \qquad w = \frac{x_2^1}{x_2^2} = B,$$

where A, B denote arbitrary constants. In particular, the integral submanifold passing through the identity element I is obtained for $A = 1$, $B = 0$, that is, it is defined by

$$x_1^1 = 1, \qquad x_2^1 = 0.$$

Consequently, the subgroup G_0 of Aut $\mathfrak{g}$ defined by the subalgebra $\mathfrak{g}$ is the (identity component of the) one in the statement.

(6) The group G represents the transformations

$$x' = x, \qquad y' = \alpha x + \beta y.$$

The subgroup G admits a simple geometrical interpretation as the group of affine transformations of the real line $\mathbb{R}$ (see Problem 4.1.1). In fact, dividing we obtain

$$t' = \beta t + \alpha,$$

where $t = y/x$, $t' = y'/x'$ are the affine coordinates.

The group G has two components, defined by $\beta > 0$ and $\beta < 0$. The component passing through the identity element, which is the subgroup defined from $\mathfrak{g}$, is the first one.

4.4 The Exponential Map

Problem 4.4.1. *Prove that, up to isomorphisms, the only* 1*-dimensional connected Lie groups are* S^1 *and* $\mathbb{R}$.

Solution. The Lie algebra $\mathfrak{g}$ of such a Lie group G is a real vector space of dimension 1, hence isomorphic to $\mathbb{R}$. The exponential map is a homomorphism of Lie groups if the Lie algebra is Abelian, as in the present case. Consequently, here we have that exp is surjective since G is connected:

$$\exp\colon \mathbb{R} \to G, \qquad X \mapsto \exp X = \exp_X(1).$$

As Ker(exp) is a closed subgroup of $\mathbb{R}$, then either Ker(exp) $= 0$ or Ker(exp) $= a\mathbb{Z}$, $a \in \mathbb{R}$. Hence either

$$G = \mathbb{R}/\mathrm{Ker}(\exp) = \mathbb{R} \quad \text{or} \quad G = \mathbb{R}/a\mathbb{Z} \approx S^1.$$

Problem 4.4.2. *Prove that* $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ *is not of the form* e^A *for any* $A \in \mathfrak{gl}(2,\mathbb{R})$.

Solution. Suppose that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{e}^A$. Then, since $\mathrm{e}^A = \mathrm{e}^{A/2+A/2} = \mathrm{e}^{A/2}\mathrm{e}^{A/2}$, it would be $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = (\mathrm{e}^{A/2})^2$. That is, the matrix would have square root, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$; but a calculation shows that there is no real solution.

REMARK. Interestingly enough, $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *does* lie in the image of exp, as $\exp\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} = -I$. On the other hand, the square roots of $-I$ in $GL(2,\mathbb{R})$ are $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Problem 4.4.3. *Let X be an element of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the real special linear group $SL(2,\mathbb{R})$. Calculate* $\exp X$.

Solution. Since

$$\mathfrak{sl}(2,\mathbb{R}) = \{X \in M(2,\mathbb{R}) : \operatorname{tr} X = 0\},$$

if $X \in \mathfrak{sl}(2,\mathbb{R})$, it is of the form $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and

$$\exp X = \sum_{n \geqslant 0} \frac{1}{n!} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}^n.$$

It is immediate that $X^2 = (a^2 + bc)I = -(\det X)I$, hence

$$\begin{aligned} \exp X &= \left(\sum_{n \geqslant 0} \frac{(-\det X)^n}{(2n)!} \right) I + \left(\sum_{n \geqslant 0} \frac{(-\det X)^n)}{(2n+1)!} \right) X \\ &= \left(1 - \frac{\det X}{2!} + \frac{(\det X)^2}{4!} - \frac{(\det X)^3}{6!} + \cdots \right) I \\ &\quad + \left(1 - \frac{\det X}{3!} + \frac{(\det X)^2}{5!} - \frac{(\det X)^3}{7!} + \cdots \right) X. \end{aligned}$$

We have to consider three cases: (1) $\det X < 0$. Then

$$\begin{aligned}
\exp X &= \left(1 + \frac{|\det X|}{2!} + \frac{|\det X|^2}{4!} + \frac{|\det X|^3}{6!} + \cdots\right) I \\
&\quad + \left(1 + \frac{|\det X|}{3!} + \frac{|\det X|^2}{5!} + \frac{|\det X|^3}{7!} + \cdots\right) X \\
&= (\cosh\sqrt{-\det X})I + \left(\frac{\sinh\sqrt{-\det X}}{\sqrt{-\det X}}\right) X.
\end{aligned}$$

(2) $\det X = 0$. Hence $\exp X = I + X$.

(3) $\det X > 0$. Then $\exp X = (\cos\sqrt{\det X})I + \left(\frac{\sin\sqrt{\det X}}{\sqrt{\det X}}\right) X$.

Problem 4.4.4. *With the same definitions as in* Problem 4.3.1:

(1) *Prove that* exp *is a local diffeomorphism from* $\mathrm{Lie}(j(\mathbb{C}^*))$ *into* $j(\mathbb{C}^*)$.

(2) *Which are the* 1*-parameter subgroups of* $j(\mathbb{C}^*)$*?*

Solution. (1)

$$\begin{aligned}
\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} &= I + \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & t^3 \\ -t^3 & 0 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \cdots & -\left(t - \dfrac{t^3}{3!} + \cdots\right) \\ t - \dfrac{t^3}{3!} + \cdots & 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \cdots \end{pmatrix} \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
\end{aligned}$$

and since $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$ we have

$$\begin{aligned}
\exp\colon \mathrm{Lie}(j(\mathbb{C}^*)) &\to j(\mathbb{C}^*) \\
\begin{pmatrix} \lambda & -t \\ t & \lambda \end{pmatrix} &\mapsto \exp\lambda \exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = e^{\lambda}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
\end{aligned}$$

Hence $\exp\colon \mathrm{Lie}(j(\mathbb{C}^*)) \to j(\mathbb{C}^*)$ is a local diffeomorphism.

(2) A 1-parameter subgroup of $j(\mathbb{C}^*)$ is a homomorphism ρ from the additive group $\mathbb{R}$, considered as a Lie group, into $j(\mathbb{C}^*)$. As there exists a bijective correspondence between 1-parameter subgroups and left-invariant vector fields, that is, elements of the Lie algebra, the 1-parameter subgroups of $j(\mathbb{C}^*)$ are the maps

$$\rho : \mathbb{R} \to j(\mathbb{C}^*), \qquad t \mapsto \exp t\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix},$$

$a, b \in \mathbb{R}$. In fact, it is immediate that $\rho(t)\rho(t') = \rho(t + t')$.

Problem 4.4.5. *Let* $x = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2)$. *Verify the formula*

$$x = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n,$$

with $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$, *which justifies* (*as many other cases*) *the notation* $x = \exp tX$.

Solution. One has $\det tX = t^2 > 0$, so the results of Problem 4.4.3 apply. On the other hand, it is immediate that the powers of X with integer exponents from 1 on are cyclically equal to $X, -I, -X, I$. Hence

$$\begin{aligned}
\exp tX &= (\cos t)I + \frac{\sin t}{t} tX \\
&= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) X \\
&= I + tX + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-X) + \frac{t^4}{4!} I + \frac{t^5}{5!} X + \cdots \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n.
\end{aligned}$$

As $x = (\cos t)I + \dfrac{\sin t}{t} tX$, we are done.

Problem 4.4.6. *Let H be the Heisenberg group* (*see* Problem 4.1.6).

(1) *Determine its Lie algebra* $\mathfrak{h}$.

(2) *Prove that the exponential map is a diffeomorphism from* $\mathfrak{h}$ *onto* H.

Solution. (1) The Lie algebra $\mathfrak{h}$ of H can be identified to the tangent space at the identity element $e \in H$, that is,

$$\mathfrak{h} \equiv \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M(n, \mathbb{R}) \right\},$$

considered as a Lie subalgebra of $\operatorname{End} \mathbb{R}^3$.

(2) We have $\exp M = \sum_{n=0}^{\infty} \dfrac{M^n}{n!}$. Since

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

one has

$$\exp\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}^2$$
$$= \begin{pmatrix}1&a&b+\frac{1}{2}ac\\0&1&c\\0&0&1\end{pmatrix}.$$

Clearly exp is a diffeomorphism of $\mathfrak{h}$ onto H.

Problem 4.4.7. *Find the matrices* $X \in \mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ *such that* $\exp tX = \mathrm{e}^{tX}$ *is a* 1*-parameter subgroup of*

$$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\}.$$

Solution. Applying the formula $\det \mathrm{e}^X = \mathrm{e}^{\operatorname{tr} X}$, we have that if $\det \mathrm{e}^{tX} = 1$, then $\operatorname{tr}(tX) = 0$; that is, $\operatorname{tr} X = 0$. And conversely.

Problem 4.4.8. *Consider the next subgroups of the general linear group* $GL(n,\mathbb{C})$:

(a) $U(n) = \left\{A \in M(n,\mathbb{C}) : {}^t\bar{A}A = I\right\}$, unitary group (*the t means "transpose" and the bar indicates complex conjugation*).

(b) $SL(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : \det A = 1\}$, special linear group.

(c) $SU(n) = \{A \in U(n) : \det A = 1\}$, special unitary group.

(d) $O(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : {}^tAA = I\}$, complex orthogonal group.

(e) $SO(n,\mathbb{C}) = \{A \in O(n,\mathbb{C}) : \det A = 1\}$, complex special orthogonal group.

(f)

$$O(n) = U(n) \cap GL(n,\mathbb{R}) = O(n,\mathbb{C}) \cap GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : {}^tAA = I\},$$

orthogonal group.

(g) $SO(n) = \{A \in O(n) : \det A = 1\}$, special orthogonal group.

(h)

$$SL(n,\mathbb{R}) = SL(n,\mathbb{C}) \cap GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\},$$

real special linear group.

Then:

(1) *Prove that we have a diffeomorphism* $U(n) \approx SU(n) \times S^1$.

(2) *Compute the dimensions of each of the groups described above.*

Solution. (1) Consider the exact sequence

$$1 \to SU(n) \to U(n) \xrightarrow{\det} S^1 \to 1,$$

and let $\sigma\colon S^1 \to U(n)$ be the section of det given by $\sigma(u) = \begin{pmatrix} u & 0 \\ 0 & I_{n-1} \end{pmatrix}$. The map $f\colon SU(n) \times S^1 \to U(n)$ given by $f(A,u) = A\sigma(u)$ is clearly differentiable. We will show that f is one-to-one, calculating its inverse. If $B = A\sigma(u)$, then $\det B = \det \sigma(u) = u$. Hence $A = B(\sigma(\det B))^{-1}$. Hence $f^{-1}(B) = (B(\sigma(\det B))^{-1}, \det B)$.

(2) Let V and W be neighborhoods of 0 and I in $M(n,\mathbb{C})$ and $GL(n,\mathbb{C})$ respectively, such that the exponential map establishes a diffeomorphism between them. Moreover, we can suppose (taking smaller neighborhoods if necessary) that $A \in V$ implies $\bar{A}$, $-A$, ${}^t\!A \in V$ or $|\operatorname{tr} A| < 2\pi$.

(a) Suppose that $A \in V$ is such that $B = \mathrm{e}^A \in W \cap U(n)$. Then we have $B^{-1} = {}^t\bar{B}$, that is, $\mathrm{e}^{-A} = \mathrm{e}^{{}^t\bar{A}}$. Hence $A + {}^t\bar{A} = 0$, or equivalently ${}^t\!A + \bar{A} = 0$. Therefore, A is a skew-hermitian matrix. Conversely, if A is a skew-hermitian matrix belonging to V, then $\mathrm{e}^A \in W \cap U(n)$. Since the space of $n \times n$ skew-hermitian matrices has dimension n^2, it follows that $\dim_{\mathbb{R}} U(n) = n^2$.

(b) If $A \in V$ is such that $\mathrm{e}^A \in W \cap SL(n,\mathbb{C})$, then $\det \mathrm{e}^A = 1 = \mathrm{e}^{\operatorname{tr} A}$. Hence $\operatorname{tr} A = 2\pi \mathrm{i} k$, but $|\operatorname{tr} A| < 2\pi$, therefore $k = 0$, that is, $\operatorname{tr} A = 0$, so $\dim_{\mathbb{R}} SL(n,\mathbb{C}) = 2(n^2-1)$.

(c) For the unitary special group we can proceed as in (a) or (b). Alternatively, considering the above diffeomorphism $U(n) \approx SU(n) \times S^1$, we obtain $\dim_{\mathbb{R}} SU(n) = n^2 - 1$.

(d) Given $A \in V$, reasoning as in (a) above, except that one must drop the bars denoting complex conjugation in the corresponding matrices, we obtain that $\mathrm{e}^A \in W \cap O(n,\mathbb{C})$ if and only if A is skew-symmetric; that is, $A + {}^t\!A = 0$. Hence, $\dim_{\mathbb{R}} O(n,\mathbb{C}) = n(n-1)$.

(e) $\dim_{\mathbb{R}} SO(n,\mathbb{C}) = n(n-1)$ because $SO(n,\mathbb{C})$ is open in $O(n,\mathbb{C})$, since one has $SO(n,\mathbb{C}) = \operatorname{Ker} \det$, where $\det\colon O(n,\mathbb{C}) \to \{+1,-1\}$, and the last space is discrete.

(f) and (g): Proceeding as in (d) but with open subsets $V \subset M(n,\mathbb{R})$, $W \subset GL(n,\mathbb{R})$, we have $\dim_{\mathbb{R}} O(n) = n(n-1)/2$. Proceeding as in (e), we deduce $\dim_{\mathbb{R}} SO(n) = n(n-1)/2$.

(h) Obviously $\dim SL(n,\mathbb{R}) = n^2 - 1$.

Problem 4.4.9. *Compute* $\exp t \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

HINT: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ *commutes with* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Solution. $\mathrm{e}^{t\lambda} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

Problem 4.4.10. *Let G be an Abelian Lie group. Prove that $[X,Y] = 0$, for any left-invariant vector fields X and Y.*

Solution. The local flow generated by a left-invariant vector field X is given by $\varphi_t(x) = x \exp tX$. Moreover we know that $[X,Y]$ is the Lie derivative of Y with respect to X; hence

$$[X,Y]_x = \lim_{t\to 0}\frac{1}{t}\left(Y_x - \varphi_{t*}Y_{\varphi_{-t}(x)}\right).$$

Accordingly, $[X,Y]=0$ if $\varphi_{t*}Y_{\varphi_{-t}(x)} = Y_x$, that is, if Y is invariant by φ_t; and this is equivalent to saying (see Problem 2.4.5) that φ_t and ψ_s commute, where $\psi_s(x) = x\exp sY$ denotes the local flow of Y. As G is Abelian, we have

$$\begin{aligned}(\varphi_t\circ\psi_s)(x) &= x\exp sY\exp tX\\ &= x\exp tX\exp sY\\ &= (\psi_s\circ\varphi_t)(x).\end{aligned}$$

4.5 The Adjoint Representation

Problem 4.5.1. *Let G be the group defined by*

$$G=\left\{A\in GL(2,\mathbb{R}) : A\,{}^tA=\rho^2 I,\ \rho>0,\ \det A>0\right\}.$$

(1) *Find the explicit expression of the elements of G.*

(2) *Find its Lie algebra.*

(3) *Calculate the adjoint representation of G.*

Solution. (1) Let

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\in GL(2,\mathbb{R}).$$

By imposing $A\,{}^tA=\rho^2 I$, we obtain:

$$a_{11}^2+a_{21}^2 = a_{21}^2+a_{22}^2=\rho^2, \tag{$\star$}$$

$$a_{11}a_{12}+a_{21}a_{22}=0. \tag{$\star\star$}$$

From $(\star)$ we deduce

$$a_{11}=\rho\cos\alpha,\qquad a_{12}=\rho\cos\beta,\qquad a_{21}=\rho\sin\alpha,\qquad a_{22}=\rho\sin\beta,$$

and then equation $(\star\star)$ tells us that

$$0=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta).$$

Hence $\beta=\alpha+\dfrac{k\pi}{2}$, $k\in\mathbb{Z}$. Accordingly,

$$A=\begin{pmatrix}\rho\cos\alpha & (-1)^k\rho\sin\alpha\\ \rho\sin\alpha & (-1)^{k-1}\rho\cos\alpha\end{pmatrix},$$

from which $\det A = (-1)^{k-1}\rho^2$. Hence $A \in G$ if and only if k is odd, and we can write

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a^2 + b^2 = \rho^2,$$

where $a = \rho\cos\alpha$, $b = \rho\sin\alpha$. The elements of G are usually called the similarities of the plane, as they are the product of a rotation by a homothety, both around the origin, i.e.,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Hence, we have

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2,\mathbb{R}) : a_{11} - a_{22} = a_{12} + a_{21} = 0 \right\}.$$

(2) By (1), the tangent space at the identity element e is

$$\begin{aligned} T_eG &= \{X \in M(2,\mathbb{R}) : Xf = 0,\, f = a_{11} - a_{22} \text{ or } f = a_{12} + a_{21}\} \\ &\equiv \left\langle \frac{\partial}{\partial x_1^1}\bigg|_e + \frac{\partial}{\partial x_2^2}\bigg|_e, -\frac{\partial}{\partial x_2^1}\bigg|_e + \frac{\partial}{\partial x_1^2}\bigg|_e \right\rangle \\ &\equiv \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \end{aligned}$$

hence the Lie algebra of G is $\mathfrak{g} = \left\{ \begin{pmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{pmatrix} \in M(2,\mathbb{R}) \right\}$.

(3) For an arbitrary Lie group G with Lie algebra $\mathfrak{g}$, the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}\,\mathfrak{g}$ is given by

$$\mathrm{Ad}_s X = L_{s*} R_{s*}^{-1} X, \qquad s \in G, \quad X \in \mathfrak{g}.$$

For a matrix group we have

$$\mathrm{Ad}_s X = sXs^{-1}.$$

As the group G of similarities of the plane is Abelian, the adjoint representation is trivial; i.e.,

$$\mathrm{Ad}_s = \mathrm{id}_{\mathfrak{g}}, \qquad \forall s \in G.$$

Problem 4.5.2. *The algebra $\mathbb{H}$ of quaternions is an algebra of dimension 4 over the field $\mathbb{R}$ of real numbers. $\mathbb{H}$ has a basis formed by four elements e_0, e_1, e_2, e_3 satisfying*

$$e_0^2 = e_0, \qquad e_i^2 = -e_0, \qquad e_0 e_i = e_i e_0 = e_i, \qquad e_i e_j = -e_j e_i = e_k, \qquad (\star)$$

where (i,j,k) is an even permutation of $(1,2,3)$. If $q = \sum_{i=0}^3 a_i e_i \in \mathbb{H}$, the conjugate quaternion of q is defined by

$$\bar{q} = a_0e_0 - (a_1e_1 + a_2e_2 + a_3e_3),$$

and the real number $|q|^2 = \sum_{i=0}^{3} a_i^2$ *is called the norm of* q. *Let* $\mathbb{H}^*$ *denote the multiplicative group of nonzero quaternions.*

(1) *Prove that* $\mathbb{H}^*$ *is a Lie group.*

(2) *Consider the map* ρ *that defines a correspondence from each* $p \in \mathbb{H}^*$ *into the* $\mathbb{R}$*-linear automorphism of* $\mathbb{H}$ *defined by* $\rho(p)\colon q \mapsto \rho(p)q = pq$, $q \in \mathbb{H}$. *Which is the representative matrix of* $\rho(p)$ *with respect to the given basis of* $\mathbb{H}$? *Compute its determinant.*

(3) *Prove that* ρ *is a representation of* $\mathbb{H}^*$ *on* $\mathbb{H} \equiv \mathbb{R}^4$.

(4) *Find the adjoint linear group of* $\mathbb{H}^*$.

Solution. (1) To prove that $\mathbb{H}^*$ is an abstract group is left to the reader. Given $q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}^*$, applying the multiplication rules $(\star)$ we obtain

$$q^{-1} = \frac{1}{|q|^2}(a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3) = \frac{\bar{q}}{|q|^2},$$

and then, for $p \in \mathbb{H}^*$, we have

$$\begin{aligned} qp^{-1} = \frac{1}{|p|^2}\big\{&(a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)e_0 + (-a_0b_1 + a_1b_0 - a_2b_3 + a_3b_2)e_1 \\ &+ (-a_0b_2 + a_1b_3 + a_2b_0 - a_3b_1)e_2 + (-a_0b_3 - a_1b_2 + a_2b_1 + a_3b_0)e_3\big\}. \end{aligned}$$

Thus the map $\mathbb{H}^* \times \mathbb{H}^* \to \mathbb{H}^*$, $(q,p) \mapsto qp^{-1}$, is C^∞, hence $\mathbb{H}^*$ is a Lie group.

(2) Let $q \in \mathbb{H}, p \in \mathbb{H}^*$, written as in (1). Then it is easy to obtain

$$\rho(q)p = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

so the above matrix is the matrix of $\rho(q)$ with respect to the basis $\{e_0, e_1, e_2, e_3\}$. We have $\det \rho(q) = |q|^2$.

(3) A representation of $\mathbb{H}^*$ on $\mathbb{H} \equiv \mathbb{R}^4$ is a homomorphism from $\mathbb{H}^*$ to the group of automorphisms $GL(4,\mathbb{R})$ of $\mathbb{R}^4$. Since $\det \rho(q) = |q|^2 \neq 0$, $\rho(q)$ is invertible. Thus ρ sends $\mathbb{H}^*$ to $GL(4,\mathbb{R})$, and since $\rho(q^{-1})\rho(q)p = q^{-1}qp = p$ we have $\rho(q)^{-1} = \rho(q^{-1})$. Furthermore, we have $\rho(qq')p = qq'p = \rho(q)\rho(q')p$, that is $\rho(qq') = \rho(q)\rho(q')$.

(4) The adjoint linear group of $\mathbb{H}^*$ is the image of

$$\mathbb{H}^* \to \operatorname{Aut}\operatorname{Lie}(\mathbb{H}^*), \qquad q \mapsto \operatorname{Ad}_q,$$

where $\operatorname{Lie}(\mathbb{H}^*)$ stands for the Lie algebra of $\mathbb{H}^*$. We identify $\operatorname{Lie}(\mathbb{H}^*) \approx T_e\mathbb{H}^*$ to $\mathbb{H}$ and we consider the basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{H}$ above. Hence, the adjoint representation gives rise to a homomorphism $\mathbb{H}^* \to GL(4,\mathbb{R})$, $q \mapsto \operatorname{Ad}_q$.

We claim that the adjoint linear group of $\mathbb{H}^*$ is the subgroup $SO(3)$ embedded in $GL(4,\mathbb{R})$ as

$$\widetilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \qquad A \in SO(3).$$

As $\operatorname{Ad}_q Y_e = R_{q^{-1}*} L_{q*} Y_e = (R_{q^{-1}} \circ L_q)_* Y_e$, $Y \in \operatorname{Lie}(\mathbb{H}^*)$, the adjoint linear group is isomorphic to the group of matrices $(R_{q^{-1}} \circ L_q)_*$, $q \in \mathbb{H}$. Moreover, as R_q and L_q are linear maps on $\mathbb{H}$, we can identify $(R_{q^{-1}} \circ L_q)_*$ to $R_{q^{-1}} \circ L_q$; that is, $\operatorname{Ad}_q = R_{q^{-1}} \circ L_q$. With the same notations as above, we note that $L_q = \rho(q)$. Hence, $\det L_q = |q|^2$. Similarly, it can be proved that $\det R_q = |q|^2$. Hence $\det \operatorname{Ad}_q = \det R_{q^{-1}} \det L_q = (\det R_q)^{-1} \det L_q = 1$. Therefore, the adjoint linear group is contained in the special linear group $SL(4,\mathbb{R})$. Let $\langle\ ,\ \rangle$ denote the scalar product of vectors in $\mathbb{R}^4$. By using the formula for qp^{-1} in (1) we obtain $\langle q, p\rangle = |p|^2 \operatorname{Re}(qp^{-1})$, where $\operatorname{Re} q = \frac{1}{2}(q + \bar{q})$. Then, we have

$$\begin{aligned} \langle \operatorname{Ad}_q p_1, \operatorname{Ad}_q p_2 \rangle &= |\operatorname{Ad}_q p_2|^2 \operatorname{Re}(\operatorname{Ad}_q p_1 (\operatorname{Ad}_q p_2)^{-1}) \\ &= |q p_2 q^{-1}|^2 \operatorname{Re}(q p_1 p_2^{-1} q^{-1}) \\ &= |p_2|^2 \operatorname{Re}(p_1 p_2^{-1}) \\ &= \langle p_1, p_2 \rangle. \end{aligned}$$

It follows that Ad_q is an isometry and, consequently, it belongs to $O(4)$. Furthermore, $\operatorname{Ad}_q e_0 = e_0$. Hence Ad_q leaves invariant the orthogonal subspace $\langle e_0 \rangle^\perp = \langle e_1, e_2, e_3 \rangle$. Accordingly, every Ad_q is a matrix of the form $\widetilde{A}$ above. Therefore, $\operatorname{Ad}_q \in SO(3)$.

Moreover, the kernel of Ad is $\mathbb{R}^*$, the center of $\mathbb{H}^*$. We have $\mathbb{H}^*/\mathbb{R}^+ \approx S^3 = \{q \in \mathbb{H} : |q| = 1\}$. Hence $\mathbb{H}^*/\mathbb{R}^* \approx S^3/\{+1,-1\} = \mathbb{R}P^3$, which is compact and connected. Accordingly, the adjoint linear group of $\mathbb{H}^*$ is a compact, connected subgroup in $SO(3)$. Hence it necessarily coincides with $SO(3)$.

Problem 4.5.3. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If* ad *stands for the adjoint representation of $\mathfrak{g}$, that is, the differential of the adjoint representation $G \to \operatorname{Aut}\mathfrak{g}$, $s \mapsto \operatorname{Ad}_s$, prove:*

(1) $$(\exp \operatorname{ad} tX)(Y) = Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \cdots, \qquad X, Y \in \mathfrak{g}.$$

(2) $$\operatorname{Ad}_{\exp tX}(Y) = Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \cdots, \qquad X, Y \in \mathfrak{g}.$$

Solution. (1)

$$(\exp \operatorname{ad} tX)(Y) = (I + \operatorname{ad} tX + \frac{1}{2!}(\operatorname{ad} tX)^2 + \cdots)(Y)$$

$$= Y + [tX, Y] + \frac{1}{2!}[tX, [tX, Y]] + \cdots$$
$$= Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \cdots$$

(2) The expansion follows from the formula

$$\mathrm{Ad} \circ \exp = \exp \circ \,\mathrm{ad},$$

and (1) above.

Problem 4.5.4. *Consider the Lie algebra* $\mathfrak{g}$ *with a basis* $\{e_1, e_2, e_3\}$ *having nonvanishing brackets*

$$[e_1, e_3] = ae_1 + be_2, \quad [e_2, e_3] = ce_1 + de_2, \qquad ad - bc \neq 0, \quad a^2 + d^2 + 2bc \neq 0.$$

(1) *Compute the ideal* $[\mathfrak{g}, \mathfrak{g}]$. *Is* $\mathfrak{g}$ *Abelian? Is* $\mathfrak{g}$ *solvable?*
(2) *Compute* $\mathrm{ad}\, X$ *for any* $X = X^1 e_1 + X^2 e_2 + X^3 e_3$.
(3) *Compute* $\mathrm{tr}\,(\mathrm{ad}_X)^2$. *When is* $\mathrm{tr}\,(\mathrm{ad}_X)^2 = 0$*?*

Solution. (1) $[\mathfrak{g}, \mathfrak{g}] = \langle e_1, e_2 \rangle$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, then g is solvable but not Abelian.

(2)

$$\begin{pmatrix} -aX^3 & -cX^3 & aX^1 + cX^2 \\ -bX^3 & -dX^3 & bX^1 + dX^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(3) $\mathrm{tr}(\mathrm{ad}_X)^2 = (a^2 + d^2 + 2bc)(X^3)^2$, and $\mathrm{tr}(\mathrm{ad}_X)^2 = 0$ only if $X \in [\mathfrak{g}, \mathfrak{g}]$.

Problem 4.5.5. *Let* B *be the Killing form on* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ *with the standard basis*

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

(1) *Find the basis for* $\mathfrak{g}$ *dual to* $\{e, f, h\}$ *with respect to* B.
(2) *Determine the Casimir operator* C_{ad} *for the adjoint representation.*

Solution. The commutation relations are

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

so we have

$$\mathrm{ad}_e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the Killing form B is defined by $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, one obtains

$$\begin{aligned} &B(e,e)=0, && B(e,f)=4, && B(e,h)=0,\\ &B(f,e)=4, && B(f,f)=0, && B(f,h)=0,\\ &B(h,e)=0, && B(h,f)=0, && B(h,h)=8. \end{aligned}$$

Hence

$$\mathrm{ad}_e^* = \frac{1}{4}f, \qquad \mathrm{ad}_f^* = \frac{1}{4}e, \qquad \mathrm{ad}_h^* = \frac{1}{8}h,$$

and thus

$$\mathrm{C}_{\mathrm{ad}} = \mathrm{ad}_e\,\mathrm{ad}_{\frac{1}{4}f} + \mathrm{ad}_f\,\mathrm{ad}_{\frac{1}{4}e} + \mathrm{ad}_h\,\mathrm{ad}_{\frac{1}{8}h}.$$

Problem 4.5.6. *With the notations and terminology in* 7.4.7 (*see* [17]):

(1) *Prove that the roots of the general linear group* $GL(4,\mathbb{C})$ *are*

$$\begin{aligned} &\pm(\varepsilon_1-\varepsilon_2), && \pm(\varepsilon_1-\varepsilon_3), && \pm(\varepsilon_1-\varepsilon_4),\\ &\pm(\varepsilon_2-\varepsilon_3), && \pm(\varepsilon_2-\varepsilon_4), && \pm(\varepsilon_3-\varepsilon_4), \end{aligned}$$

each with multiplicity one.

(2) *Prove that the roots of the symplectic group* $Sp(\mathbb{C}^4,\Omega)$ *are*

$$\pm(\varepsilon_1-\varepsilon_2), \qquad \pm(\varepsilon_1+\varepsilon_2), \qquad \pm 2\varepsilon_1, \quad \pm 2\varepsilon_2,$$

each with multiplicity one.

(3) *Prove that the roots of the special orthogonal group* $SO(\mathbb{C}^5,B)$ *are*

$$\pm(\varepsilon_1-\varepsilon_2), \qquad \pm(\varepsilon_1+\varepsilon_2), \qquad \pm\varepsilon_1, \quad \pm\varepsilon_2,$$

each with multiplicity one.

(4) *Why* $2\varepsilon_i$, $i=1,2$, *is a root of* $Sp(\mathbb{C}^4,\Omega)$ *but not of* $SO(\mathbb{C}^4,B)$?

Solution. (1) Let E^i_j be the matrix with (i,j)th entry equal to 1 and zero elsewhere. For $A=\mathrm{diag}(a_1,\dots,a_4)\in\mathfrak{h}$ we have

$$\begin{aligned} [A,E^i_j] &= [\mathrm{diag}\,(a_1,a_2,a_3,a_4),E^i_j]\\ &= (a_i-a_j)E^i_j\\ &= \langle \varepsilon_i-\varepsilon_j, A\rangle E^i_j. \end{aligned}$$

Since the set $\{E^i_j\}$, $i,j\in\{1,\dots,4\}$, is a basis of $\mathfrak{g}=\mathfrak{gl}(4,\mathbb{R})=M(4,\mathbb{C})$, the roots are the ones in the statement, each with multiplicity 1.

(2) Label the basis for $\mathbb{C}^4$ as $\{e_1,e_2,e_{-2},e_{-1}\}$. Consider E^i_j for $i,j\in\{\pm1,\pm2\}$. Set

$$\begin{aligned} &X_{\varepsilon_1-\varepsilon_2} = E^1_2 - E^{-2}_{-1}, && X_{\varepsilon_2-\varepsilon_1} = E^2_1 - E^{-1}_{-2},\\ &X_{\varepsilon_1+\varepsilon_2} = E^1_{-2} + E^2_{-1}, && X_{-\varepsilon_1-\varepsilon_2} = E^{-2}_1 + E^{-1}_2,\\ &X_{2\varepsilon_1} = E^1_{-1}, && X_{-2\varepsilon_1} = E^{-1}_1,\\ &X_{2\varepsilon_2} = E^2_{-2}, && X_{-2\varepsilon_2} = E^{-2}_2. \end{aligned}$$

Then, for $A = \operatorname{diag}(a_1, a_2, a_{-2}, a_{-1}) \in \mathfrak{h}$, one has

$$[A, X_{\varepsilon_i - \varepsilon_j}] = \langle \varepsilon_i - \varepsilon_j, A \rangle X_{\varepsilon_i - \varepsilon_j}, \qquad [A, X_{\pm(\varepsilon_i + \varepsilon_j)}] = \pm \langle \varepsilon_i + \varepsilon_j, A \rangle X_{\varepsilon_i + \varepsilon_j}.$$

Hence the elements in $\mathfrak{h}^*$ in the statement are roots of $\mathfrak{sp}(\mathbb{C}^4, \Omega)$. Now,

$$\{X_{\pm(\varepsilon_1 - \varepsilon_2)}, X_{\pm(\varepsilon_1 + \varepsilon_2)}, X_{\pm 2\varepsilon_1}, X_{\pm 2\varepsilon_2}\}$$

is a basis for $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ mod $\mathfrak{h}$. So the given roots are all of the roots, each with multiplicity one.

(3) We embed $SO(\mathbb{C}^4, B)$ into $SO(\mathbb{C}^5, B)$ by using the map (7.4.1) for $r = 2$. Since $H \subset SO(\mathbb{C}^4, B) \subset SO(\mathbb{C}^5, B)$ via this embedding, the roots $\pm\varepsilon_1 \pm \varepsilon_2$ of $\operatorname{ad}(\mathfrak{h})$ on $\mathfrak{so}(\mathbb{C}^4, B)$ also occur for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^5, B)$. Label the basis of $\mathbb{C}^5$ as $\{e_{-2}, e_{-1}, e_0, e_1, e_2\}$. Consider E^i_j for $i, j \in \{0, \pm 1, \pm 2\}$. Then one can prove that the root vectors from $SO(\mathbb{C}^4, B)$ are

$$X_{\varepsilon_1 - \varepsilon_2} = E^1_2 - E^{-2}_{-1}, \qquad X_{\varepsilon_2 - \varepsilon_1} = E^2_1 - E^{-1}_{-2},$$
$$X_{\varepsilon_1 + \varepsilon_2} = E^1_{-2} - E^2_{-1}, \qquad X_{-\varepsilon_1 - \varepsilon_2} = E^{-2}_1 - E^{-1}_2.$$

Define

$$X_{\varepsilon_1} = E^1_0 - E^0_{-1}, \qquad X_{\varepsilon_2} = E^2_0 - E^0_{-2},$$
$$X_{-\varepsilon_1} = E^0_1 - E^{-1}_0, \qquad X_{-\varepsilon_2} = E^0_2 - E^{-2}_0.$$

Then we have $X_{\pm\varepsilon_i} \in \mathfrak{g}$, $i = 1, 2$, and $[A, X_{\pm\varepsilon_i}] = \pm\langle \varepsilon_i, A \rangle X_{\varepsilon_i}$ for $A \in \mathfrak{h}$. As $\{X_{\pm\varepsilon_i}\}$, $i = 1, 2$, is a basis for $\mathfrak{g}$ mod $\mathfrak{so}(\mathbb{C}^4, B)$, one concludes that the roots of $\mathfrak{so}(\mathbb{C}^5, B)$ are the ones in the statement, each with multiplicity one.

(4) Both $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ and $\mathfrak{so}(\mathbb{C}^4, B)$ have the same subalgebra of diagonal matrices $\operatorname{diag}(\lambda, \mu, -\mu, -\lambda)$ which give rise in both cases to the roots $-2\varepsilon_i$, $i = 1, 2$. For instance,

$$\left[\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 \end{pmatrix}.$$

However, the nonzero skew diagonal matrices

$$\begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \mu & 0 \\ 0 & \nu & 0 & 0 \\ \rho & 0 & 0 & 0 \end{pmatrix}$$

exist in $\mathfrak{sp}(\mathbb{C}^4, \Omega)$ and originate the roots $2\varepsilon_i$, $i = 1, 2$; but those matrices do not exist in $\mathfrak{so}(\mathbb{C}^4, B)$.

4.6 Lie Groups of Transformations

Problem 4.6.1. *Consider*:

(a) $M = (0,4\pi) \subset \mathbb{R}$, *with the differentiable structure induced by the usual one on* $\mathbb{R}$.

(b) S^1 *with the usual differentiable structure as a closed submanifold of* $\mathbb{R}^2$.

(c) *The map* $f\colon M \to S^1$, $s \mapsto f(s) = (\cos s, \sin s)$.

Prove:

(1) *The equivalence relation* $\sim$ *in* M *given by* $s \sim t$ *if and only if* $f(s) = f(t)$, *induces on the set* $M/\sim$ *a structure of quotient manifold diffeomorphic to* S^1.

(2) *The manifold* $M/\sim$ *cannot be obtained by the action of a group of transformations acting on* M.

Solution. (1) The differentiable map $\tilde{f}\colon M \to \mathbb{R}^2$ given by $\tilde{f}(s) = (\cos s, \sin s)$ is differentiable and defines the map $f\colon M \to S^1$. Since S^1 is an embedded submanifold of $\mathbb{R}^2$, f is differentiable.

Furthermore f is a submersion as the rank of f at any s is equal to the rank of the matrix $(-\sin s, \cos s)$, which is equal to 1. Moreover, the associated quotient manifold is diffeomorphic to S^1. In fact, as the equivalence relation is defined by $s \sim t$ if and only if $f(s) = f(t)$, we have to prove that on $M/\sim$ there is a differentiable structure such that the map $\pi\colon M \to M/\sim$ is a submersion. In fact, denote by $[s]$ the equivalence class of s under $\sim$. Then the map $h\colon M/\sim \to S^1$, $[s] \mapsto (\cos s, \sin s)$, is clearly bijective and thus $M/\sim$ admits only one differentiable structure with which $h\colon M/\sim \to S^1$ is a diffeomorphism. The following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & S^1 \\ \pi \searrow & & \nearrow h \\ & M/\sim & \end{array}$$

is obviously commutative, and since f is a submersion and h is a diffeomorphism we deduce that π is a submersion. Consequently $M/\sim$ is a quotient manifold of M.

(2) Let us suppose that there exists a group of transformations G acting on M by

$$\theta\colon G \times M \to M, \qquad (g,s) \mapsto \theta(g,s) = gs,$$

such that from this action we would have the previous quotient manifold. Then, given $g \in G$, as $gs \sim s$, it would be:

$$\begin{cases} gs = s \ \text{ or } \ s + 2\pi, & s \in (0,2\pi) \\ g(2\pi) = 2\pi & \\ gs = s \ \text{ or } \ s - 2\pi, & s \in (2\pi,4\pi). \end{cases} \tag{$\star$}$$

Consider the continuous map $h\colon M \to \mathbb{R}$, $s \mapsto h(s) = gs - s$. By $(\star)$ above, $h(M) \subset \{-2\pi, 0, 2\pi\}$. Moreover, we know that $0 \in h(M)$ because $h(2\pi) = 0$. But since M is connected and h continuous, $h(M)$ is connected. We conclude that $h(M) = 0$, that is, $gs = s$ for all $s \in M$. As this holds for every $g \in G$, the associated quotient manifold would be M, which cannot be homeomorphic to $M/\sim$, because $M/\sim$ is compact and M is not.

Problem 4.6.2. *Given* $\mathbb{R}^2$ *with its usual differentiable structure, show*:

(1) *The additive group* $\mathbb{Z}$ *of the integers acts on* $\mathbb{R}^2$ *as a transformation group by the action*:

$$\begin{array}{rrcl} \theta\colon & \mathbb{Z}\times\mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (n,(x,y)) & \mapsto & \theta(n,(x,y)) = (x+n,y). \end{array}$$

(2) *The quotient space* $\mathbb{R}^2/\mathbb{Z}$ *of* $\mathbb{R}^2$ *by that action admits a structure of quotient manifold.*

(3) $S^1\times\mathbb{R}$ *admits a structure of quotient manifold of* $\mathbb{R}^2$*, diffeomorphic to* $\mathbb{R}^2/\mathbb{Z}$ *as above.*

Solution. (1) $\mathbb{Z}$ acts on $\mathbb{R}^2$ as a transformation group by the given action. In fact, for each n, the map

$$\theta_n\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x,y) \mapsto \theta(x,y) = (x+n,y),$$

is C^∞. Moreover,

$$\begin{aligned} \theta\bigl(n_1,\theta(n_2,(x,y))\bigr) &= \theta\bigl(n_1,(x+n_2,y)\bigr) \\ &= (x+n_1+n_2,y) \\ &= \theta\bigl(n_1+n_2,(x,y)\bigr). \end{aligned}$$

(2) $\mathbb{Z}$ acts freely on $\mathbb{R}^2$, because if $\theta(n,(x,y)) = (x,y)$, i.e. $(x+n,y) = (x,y)$, we have $n = 0$, which is the identity element of $\mathbb{Z}$.

Furthermore, the action of $\mathbb{Z}$ is properly discontinuous. In fact, we have to verify the two conditions in Definition 7.4.9:

(i) Given $(x_0,y_0) \in \mathbb{R}^2$, let us consider $U = (x_0-\varepsilon,x_0+\varepsilon)\times\mathbb{R}$, with $0 < \varepsilon < \frac{1}{2}$. Then, if $(x_1,y_1) \in U\cap\theta_n(U)$, we have

$$x_0-\varepsilon < x_1 < x_0+\varepsilon, \qquad x_0+n-\varepsilon < x_1 < x_0+n+\varepsilon,$$

from which $|n| < 2\varepsilon$, that is, $n = 0$.

(ii) Let (x_0,y_0), $(x_1,y_1) \in \mathbb{R}^2$, such that $(x_0,y_0) \not\sim (x_1,y_1)$, that is, such that:

(a) $y_0 \neq y_1$, or (b) $y_0 = y_1$, $x_1 \neq x_0+n$, for all $n \in \mathbb{Z}$.

In the case (a), we have two different cases: $x_0 = x_1$, and $x_0 \neq x_1$, but the solution is the same: We only have to consider $U = \mathbb{R}\times(y_0-\varepsilon,y_0+\varepsilon)$ and $V = \mathbb{R}\times(y_1-\varepsilon,y_1+\varepsilon)$, with $0 < \varepsilon < |y_1-y_0|/2$. Since $\theta_n(V) = V$, we have $U\cap\theta_n(V) = U\cap V = \emptyset$.

In the case (b), we have $\alpha = |x_1-x_0| \notin \mathbb{Z}$. We can suppose $x_1 > x_0$. Thus $\alpha = x_1-x_0$. Let $m \in \mathbb{Z}$ be such that $m < \alpha < m+1$, and consider the value

$$0 < \varepsilon < \min\{(\alpha - m)/2, (m+1-\alpha)/2\}.$$

So, it suffices to consider $U = (x_0 - \varepsilon, x_0 + \varepsilon) \times \mathbb{R}$, $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \mathbb{R}$.

Let us see that $U \cap \theta_n(V) = \emptyset$, for all $n \in \mathbb{Z}$. It is clear that the only values of n that could give a non-empty intersection are $n = -m$ and $n = -(m+1)$.

If $n = -m$, then if $(x_2, y_2) \in U \cap \theta_{-m}(V)$, we have that

$$x_0 - \varepsilon < x_2 < x_0 + \varepsilon, \qquad x_1 - m - \varepsilon < x_2 < x_1 - m + \varepsilon,$$

so that $x_1 - m - \varepsilon < x_0 + \varepsilon$, hence $x_1 - x_0 - m < 2\varepsilon$, thus $\alpha - m < 2\varepsilon$. Absurd.

If $n = -(m+1)$ and $(x_2, y_2) \in U \cap \theta_{-(m+1)}(V)$, we have that

$$x_0 - \varepsilon < x_2 < x_0 + \varepsilon, \qquad x_1 - (m+1) - \varepsilon < x_2 < x_1 - (m+1) + \varepsilon.$$

Thus $x_0 - \varepsilon < x_1 - (m+1) + \varepsilon$, so $x_0 - x_1 + (m+1) < 2\varepsilon$, hence $(m+1) - \alpha < 2\varepsilon$. Absurd.

We conclude that $\mathbb{R}^2/\mathbb{Z}$ admits a structure of quotient manifold of dimension 2.

(3) We shall denote by $[(x,y)]$ the class of (x,y) under the previous action. It is immediate that the map

$$\begin{aligned} f\colon \mathbb{R} &\to S^1 \\ s &\mapsto f(s) = (\sin 2\pi s, \cos 2\pi s) \end{aligned}$$

is a local diffeomorphism and thus it is a submersion. Since the product of submersions is a submersion, it follows that $f \times \mathrm{id}_{\mathbb{R}} \colon \mathbb{R}^2 \to S^1 \times \mathbb{R}$ is a submersion. Consider the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr, "f\times \mathrm{id}_{\mathbb{R}}"] \arrow[dr, "\pi"'] & & S^1 \times \mathbb{R} \\ & \mathbb{R}^2/\mathbb{Z} \arrow[ur, "h"'] & \end{tikzcd}$$

$$\begin{tikzcd} (x,y) \arrow[rr, mapsto] \arrow[dr] & & (f(x),y) \\ & \pi(x,y) = [(x,y)] \arrow[ur, "h"'] & \end{tikzcd}$$

where h is defined by $h([(x,y)]) = (f(x), y)$. Note that the definition makes sense as if $(x_0, y_0) \sim (x_1, y_1)$, we have $y_0 = y_1$, $x_1 = x_0 + n$, and thus $f(x_0, y_0) = f(x_1, y_1)$, consequently h does not depend on the representative of a given equivalence class. Furthermore h is one-to-one. In fact:

(a) h is injective, because if $(f(x_0), y_0) = (f(x_1), y_1)$ then $\sin 2\pi x_0 = \sin 2\pi x_1$, $\cos 2\pi x_0 = \cos 2\pi x_1$, and $y_0 = y_1$, hence $x_0 = x_1 + n$, $y_0 = y_1$, so $[(x_0, y_0)] = [(x_1, y_1)]$.

(b) h is surjective, since $h \circ \pi$ is.

Problem 4.6.3. *Consider $M = \mathbb{R}^2$ with its usual differentiable structure and let $\mathbb{Z}$ be the additive group of integer numbers. Prove*:

(1) *$\mathbb{Z}$ acts on $\mathbb{R}^2$ as a transformation group by the C^∞ action*

$$\theta\colon \mathbb{Z}\times\mathbb{R}^2 \to \mathbb{R}^2, \qquad (n,(x,y)) \mapsto (x+n,(-1)^n y).$$

(2) $\mathbb{R}^2/\mathbb{Z}$ *is a quotient manifold.*

REMARK. $\mathbb{R}^2/\mathbb{Z}$ *is diffeomorphic to the infinite Möbius strip* (*see* Problem 1.1.12).

Solution. (1) θ is an action of $\mathbb{Z}$ on $\mathbb{R}^2$, because $\theta(0,(x,y)) = (x,y)$ and

$$\begin{aligned}\theta\big(n_1,\theta(n_2,(x,y)\big) &= (x+n_1+n_2,(-1)^{n_1+n_2}y)\\ &= \theta\big(n_1+n_2,(x,y)\big).\end{aligned}$$

Furthermore, the action is C^∞. In fact, since $\mathbb{Z}$ is a discrete group, we only have to prove that for each $n\in\mathbb{Z}$, the action $\theta_n\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y)\mapsto (x+n,(-1)^n y)$, is a diffeomorphism, but this is clear.

(2) Since $\mathbb{Z}$ is discrete, we only have to prove that the action θ is free and properly discontinuous.

(i) The action θ is free, because if $\theta(n,(x,y)) = (x,y)$, then $n=0$.

(ii) The action of $\mathbb{Z}$ is properly discontinuous. In fact:

(a) Given $(x_0,y_0)\in\mathbb{R}^2$, let $U=(x_0-\varepsilon,x_0+\varepsilon)\times\mathbb{R}$, with $0<\varepsilon<\frac{1}{2}$. Then, given $(x_1,y_1)\in U\cap\theta_n(U)$, one has that

$$x_0-\varepsilon<x_1<x_0+\varepsilon, \qquad x_0+n-\varepsilon<x_1<x_0+n+\varepsilon,$$

from which $n=0$.

(b) Now, let $(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2$ such that $(x_0,y_0)\not\sim(x_1,y_1)$, where $\sim$ denotes the equivalence relation given by the present action.

For the sake of simplicity we can assume that (x_0,y_0) and (x_1,y_1) are in the same quadrant of $\mathbb{R}^2$. We have two possibilities:

(α) $y_1\neq y_0$.

(β) $y_1=y_0$, $x_1\neq x_0+2n$, for all $n\in\mathbb{Z}$.

For the case (α), it suffices to consider $U=\mathbb{R}\times(y_0-\varepsilon,y_0+\varepsilon)$ and $V=\mathbb{R}\times(y_1-\varepsilon,y_1+\varepsilon)$, with $0<\varepsilon<|y_1-y_0|/2$. For, let $V^*=\big\{(x_0,y_0)\in\mathbb{R}^2 : (x_0,-y_0)\in V\big\}$. Then $U\cap\theta_n(V)\subset U\cap(V\cup V^*)=\emptyset$.

In the case (β), we can assume that $x_1>x_0$ and we have two possibilities:

(β_1) $x_1-x_0\neq n$, for all n.

(β_2) $x_1-x_0=n_0=$ an odd integer.

The case (β_1) admits a solution similar to that given for (b) in Problem 4.6.2 for the case $S^1\times\mathbb{R}$.

In the case (β_2), it suffices to consider the open balls $U=B((x_0,y_0),\varepsilon)$ and $V=B((x_1,y_0),\varepsilon)$, with $0<\varepsilon<\min(1/2,(x_1-x_0)/2,y_0/2)$. In fact, it is easily checked that if either $n=-n_0$ or $n\neq -n_0$, the wanted intersection is empty.

Consequently $\mathbb{R}^2/\mathbb{Z}$ is a quotient manifold.

Problem 4.6.4. *Find the* 1*-parameter subgroups of* $GL(2,\mathbb{R})$ *corresponding to*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute the corresponding actions on $\mathbb{R}^2$ *and their infinitesimal generators, from the natural action of* $GL(2,\mathbb{R})$ *on* $\mathbb{R}^2$.

Solution. The 1-parameter subgroup of $GL(n,\mathbb{R})$ corresponding to the element $X \in \mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ is $\mathbb{R} \to GL(n,\mathbb{R})$, $t \mapsto \mathrm{e}^{tX}$. Thus,

$$\mathrm{e}^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad \mathrm{e}^{tB} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The group $\{\mathrm{e}^{tA}\}$ acts on $\mathbb{R}^2$ and the orbit of the point (x_0,y_0) is the circle with center (0,0) and radius $r = \sqrt{x_0^2 + y_0^2}$.

The group $\{\mathrm{e}^{tB}\}$ acts on $\mathbb{R}^2$ giving as orbit of each point (x_0,y_0) the straight line $(x_0 + ty_0, y_0)$, which reduces to $(x_0,0)$ if $y_0 = 0$.

The infinitesimal generator of $(x,y) \mapsto (x\cos t + y\sin t, -x\sin t + y\cos t)$ is the vector field

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (x\cos t + y\sin t)\frac{\partial}{\partial x} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (-x\sin t + y\cos t)\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

and the infinitesimal generator of $(x,y) \mapsto (x + ty, y)$ is the vector field

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (x + ty)\frac{\partial}{\partial x} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 y\frac{\partial}{\partial y} = y\frac{\partial}{\partial x}.$$

Problem 4.6.5. *Find, in terms of the vector b, the matrix A, and its eigenvalues, when the Euclidean motion*

$$f\colon x \to Ax + b, \qquad A \in O(3), \quad b = (b^1, b^2, b^3),$$

of $\mathbb{R}^3$*, has a fixed point.*

Solution. The equation $f(x) = x$ for some $x \in \mathbb{R}^3$ is the same as $b = (I - A)(x)$, where I stands for the identity. Thus f has a fixed point if and only if $b \in \operatorname{im}(I - A)$. Then:

(1) If $+1$ is not an eigenvalue of A, then $\operatorname{Ker}(I - A) = \{0\}$ and so $I - A$ is an automorphism of $\mathbb{R}^3$. In this case $b \in \operatorname{im}(I - A)$.

(2) Suppose $Au = u$ for some nonzero $u \in \mathbb{R}^3$. We can assume that u is a unit vector. Then $\mathbb{R}^3 = \langle u \rangle \oplus \langle u \rangle^\perp$, where $\langle u \rangle = \{\lambda u : \lambda \in \mathbb{R}\}$; and A acts on the plane $\langle u \rangle^\perp$ as an isometry (in fact, from $Au = u$ it follows that $g(u,v) = g(u,Av)$, where g stands for the Euclidean metric of $\mathbb{R}^3$; thus, as $g(Au,Av) = g(u,v)$, $g(u,v) = 0$ implies $g(u,Av) = 0$). We have $b = \lambda u + b'$, $b' \in \langle u \rangle^\perp$, and $x = \alpha u + x'$ for all $x \in \mathbb{R}^3$. Thus $b = (I - A)(x)$ if and only if $\lambda = 0$ and $b' = (I - A)(x')$. Denote by

A' the restriction of A to $\langle u\rangle^{\perp}$. If $+1$ is not an eigenvalue of A', we are done. In the other case, making an orthonormal change of basis we will have $A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. That is, if $+1$ is an eigenvalue of A, then we have

(a) If $A = I$, then f has no fixed points, except for $b = 0$.

(b) If A is a mirror symmetry, f has no fixed points except when b is orthogonal to the plane of symmetry.

(c) If A is neither the identity nor a mirror symmetry, f has no fixed points except when $g(b,u) = 0$. That is, when b is orthogonal to the rotation axis of A (in this case).

Note that the multiplicity of the eigenvalue $+1$ is 3, 2 or 1, respectively, in the cases (a), (b) and (c).

Problem 4.6.6. *Let $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ be the upper half-plane and consider $(x,y) \in H^2$ as $z = x + \mathrm{i}y \in \mathbb{C}$ under the identification $\mathbb{R}^2 \approx \mathbb{C}$. Prove that the group of fractional linear transformations*

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1,$$

does not act freely on H^2.

HINT: *Compute, for instance, the isotropy group of* i.

Solution. The isotropy group is given by the condition $\dfrac{az+b}{cz+d} = z$, that is, $az + b = cz^2 + \mathrm{d}z$. For example, for $z = \mathrm{i}$, one has $a\mathrm{i} + b = d\mathrm{i} - c$, so we have $a = d,\ b = -c$, hence the isotropy group of i is the group of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with a, b integers such that $a^2 + b^2 = 1$. Hence, the solutions are $(a,b) = (1,0)$, $(-1,0)$, $(0,1)$ or $(0,-1)$, and the subgroup is not the identity.

Problem 4.6.7. (1) *Prove that the map*

$$\theta\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \qquad (a,x) \mapsto ax,$$

is a C^∞ action of $\mathbb{R}^+$ on $\mathbb{R}$. Is it free?

(2) *The action θ induces the equivalence relation $\sim$ in $\mathbb{R}$ defined by $x \sim y$, if there exists $a \in \mathbb{R}^+$ such that $\theta(a,x) = y$, or equivalently, if there exists $a \in \mathbb{R}^+$ such that $ax = y$. Prove that $\mathbb{R}/\mathbb{R}^+$ is not a quotient manifold of $\mathbb{R}$.*

Solution. (1) We have

$$\theta(1,x) = x, \qquad \theta(a, \theta(a',x)) = aa'x = \theta(aa',x).$$

Moreover, θ is C^∞, as $(\mathrm{id}_{\mathbb{R}} \circ \theta \circ (\mathrm{id}_{\mathbb{R}^+} \times \mathrm{id}_{\mathbb{R}})^{-1})(a,t) = at$ is C^∞.

The action θ is not free: For $x=0$ and any $a\in\mathbb{R}^+$ we have $ax=0$.

(2) If $\mathbb{R}/\mathbb{R}^+$ were a quotient manifold of $\mathbb{R}$, then the natural map $\pi\colon \mathbb{R}\to\mathbb{R}/\mathbb{R}^+$, $x\mapsto[x]$ should be a submersion. But $\mathbb{R}/\mathbb{R}^+$ has only three points: $[-1]$, $[0]$ and $[1]$. If it were a manifold, it would be discrete, so disconnected. Thus π cannot be even continuous.

Problem 4.6.8. *Show that*

$$(x,y)\mapsto \theta_t(x,y)=(x\mathrm{e}^{2t},y\mathrm{e}^{-3t}),$$

defines a C^∞ action of $\mathbb{R}$ on $\mathbb{R}^2$ and find its infinitesimal generator.

Solution. We have $\theta_0(x,y)=(x,y)$ and

$$\begin{aligned}\theta_{t'}\theta_t(x,y)&=(x\mathrm{e}^{2(t+t')},y\mathrm{e}^{-3(t+t')})\\&=\theta_{t+t'}(x,y),\end{aligned}$$

hence θ is a C^∞ action of $\mathbb{R}$ on $\mathbb{R}^2$. The infinitesimal generator X is

$$\begin{aligned}X&=\left.\frac{\mathrm{d}(x\mathrm{e}^{2t})}{\mathrm{d}t}\right|_{t=0}\frac{\partial}{\partial x}+\left.\frac{\mathrm{d}(y\mathrm{e}^{-3t})}{\mathrm{d}t}\right|_{t=0}\frac{\partial}{\partial y}\\&=2x\frac{\partial}{\partial x}-3y\frac{\partial}{\partial y}.\end{aligned}$$

Problem 4.6.9. *Let*

$$S^3=\{q=x+y\mathrm{i}+z\mathrm{j}+t\mathrm{k}\in\mathbb{H}:|q|=1\}$$

act on itself by right translations.

Prove that the fundamental vector fields $\mathrm{i}^*,\mathrm{j}^*,\mathrm{k}^*$ *associated to the elements* $\mathrm{i},\mathrm{j},\mathrm{k}\in\mathbb{H}$ *are, respectively,*

$$\begin{aligned}X&=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+t\frac{\partial}{\partial z}-z\frac{\partial}{\partial t},\\Y&=-z\frac{\partial}{\partial x}-t\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}+y\frac{\partial}{\partial t},\\Z&=-t\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}+x\frac{\partial}{\partial t}.\end{aligned}$$

Solution. Identify the vector space of purely imaginary quaternions to the tangent space T_1S^3. The flow generated by i^* is $R_{\exp(t\mathrm{i})}(q)$, $q\in S^3$. Hence

$$\begin{aligned}\mathrm{i}^*_q(x)&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(x\circ R_{\exp(t\mathrm{i})})(q)\\&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}x(q\exp(t\mathrm{i}))\end{aligned}$$

$$
\begin{aligned}
&= x(q\mathrm{i}) \\
&= x\{(x(q) + y(q)\mathrm{i} + z(q)\mathrm{j} + t(q)\mathrm{k})\mathrm{i}\} \\
&= -y(q).
\end{aligned}
$$

Similarly we obtain

$$
\mathrm{i}_q^*(y) = x(q), \qquad \mathrm{i}_q^*(z) = t(q), \qquad \mathrm{i}_q^*(t) = -z(q),
$$

so

$$
\mathrm{i}^* = X.
$$

The other cases are obtained analogously.

REMARK. The vector fields given in Problem 1.9.6 are ${}^*\mathrm{i}, {}^*\mathrm{j}, {}^*\mathrm{k}$, which are the fundamental vector fields with respect to left translations of S^3 on itself, instead of the right action above.

Problem 4.6.10. *Let $G \times M \to M$, $(g, p) \mapsto g \cdot p$, be a differentiable action of a Lie group G on a differentiable manifold M. Let $\sim$ be the equivalence relation induced by this action,* i.e.*,*

$$
p \sim q \quad \Leftrightarrow \quad \exists\, g \in G \text{ such that } q = g \cdot p.
$$

Let $N = \{(p,q) \in M \times M \,:\, p \sim q\}$. Assume that N is a closed embedded submanifold of $M \times M$. Prove that the map $\pi\colon N \to M$, $\pi(p,q) = p$, is a submersion.

REMARK. *According to the* Theorem of the closed graph 7.1.13*, this problem proves that the quotient manifold $M/G = M/\!\sim$ of a group action exists if and only if the graph of $\sim$ is a closed embedded submanifold of $M \times M$.*

Solution. Let $(p_0, q_0) \in N$ be an arbitrary point. Hence there exists $g \in G$ such that $q_0 = g \cdot p_0$. Let $s\colon M \to M \times M$ be the differentiable map $\sigma(p) = (p, g \cdot p)$. This map takes values in N and hence it induces, by virtue of the assumption, a differentiable map $\sigma\colon M \to N$, which is a section of π, i.e. $\pi \circ \sigma = \mathrm{id}_N$. As $\sigma(p_0) = (p_0, q_0)$, we conclude that π is a submersion at (p_0, q_0).

4.7 Homogeneous Spaces

Problem 4.7.1. *Prove that $O(n+1)/O(n)$ and $SO(n+1)/SO(n)$ are homogeneous spaces and that the sphere S^n is diffeomorphic to each of them.*

Solution. By means of the map

$$
O(n) \to O(n+1), \qquad A \mapsto \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right),
$$

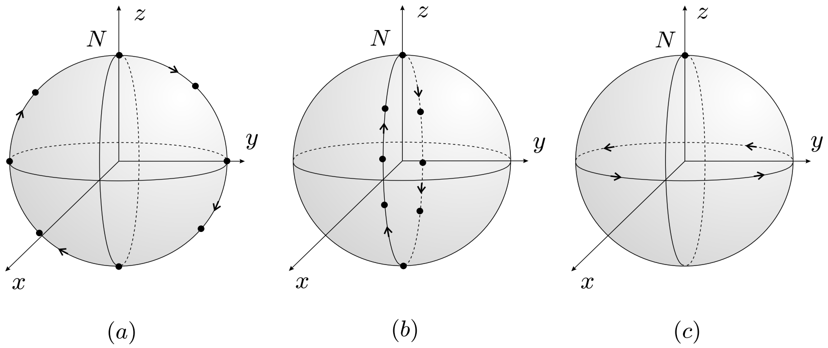

Fig. 4.2 The sphere S^2 viewed as the homogeneous space $SO(3)/SO(2)$. The north pole rotates under rotations around either the x- or the y-axis but not under rotations around the z-axis.

$O(n)$ is a closed Lie subgroup of $O(n+1)$, so that the quotient space $O(n+1)/O(n)$, with the usual C^∞ structure, is a homogeneous space.

We will prove:

(1) There exists a C^∞ action of $O(n+1)$ (resp. $SO(n+1)$) on S^n.

(2) This action is transitive.

(3) The isotropy group H_p is isomorphic to $O(n)$ (resp. $SO(n)$), for some $p \in S^n$.

Now, we have:

(1) The action $GL(n+1,\mathbb{R}) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(A,v) \mapsto Av$, is C^∞, and its restriction $O(n+1) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is also C^∞. As the action of the orthogonal group preserves the length of vectors, the restriction $O(n+1) \times S^n \to \mathbb{R}^{n+1}$ takes values in S^n and it is C^∞.

(2) Given any pair $p,q \in S^n$, there exists $A \in O(n+1)$ with $q = Ap$. For, let $\{e_i\}$, $\{\bar{e}_i\}$ be orthonormal bases with respect to the Euclidean metric of $\mathbb{R}^{n+1}$ satisfying $e_1 = p$, $\bar{e}_1 = q$. Then one takes as A the matrix of the change of basis, so that, in fact, $A \in O(n+1)$.

(3) We choose, for the sake of simplicity, $p = (1,0,\dots,0)$. By definition,

$$H_p = \{A \in O(n+1) : Ap = p\},$$

thus, if $A = (a_{ij})$, we have $a_{11} = 1,\ a_{i1} = 0,\ i = 2,\dots,n$. Moreover, as $A \in H_p \subset O(n+1)$, we have ${}^t\!AA = I$, hence $p = {}^t\!AAp = {}^t\!Ap$, so that $a_{1i} = 0,\ i = 2,\dots,n$. Thus

$$H_p = \left\{ \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array} \right) \in O(n+1) \right\},$$

but ${}^t\!AA = I$, so ${}^tBB = I$, i.e. $B \in O(n)$. Thus, $H_p \approx O(n)$.

Hence, one has a diffeomorphism $S^n \approx O(n+1)/O(n)$. One also has $S^n \approx SO(n+1)/SO(n)$, because the above arguments are valid taking orthonormal bases $\{e_i\}$ and $\{\bar{e}_i\}$ with the same orientation, satisfying $e_1 = p$, $\bar{e}_1 = q$, which is always possible.

Problem 4.7.2. *Prove that $U(n)/U(n-1)$ and $SU(n)/SU(n-1)$ are homogeneous spaces and that the sphere S^{2n-1} is diffeomorphic to each of them.*

Solution. By means of the map

$$U(n-1) \to U(n), \qquad A \mapsto \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right),$$

$U(n-1)$ is a closed subgroup of $U(n)$, and thus the quotient space $U(n)/U(n-1)$, with the usual C^∞ structure, is a homogeneous space.

Consider S^{2n-1} as the unit sphere of $\mathbb{C}^n$ with the usual Hermitian product $\langle\ ,\ \rangle$, that is, $\langle \lambda^i e_i, \mu^j e_j \rangle = \sum \lambda^i \bar{\mu}^i$, so

$$S^{2n-1} = \left\{ (z^1, \dots, z^n) \in \mathbb{C}^n : \sum z^i \bar{z}^i = 1 \right\}.$$

The isometry group of the metric $\langle\ ,\ \rangle$ is $U(n) = \left\{ A \in GL(n,\mathbb{C}) : {}^t\!\bar{A}A = I \right\}$. Hence, similarly to Problem 4.7.1 we have:

(1) The map $U(n) \times S^{2n-1} \to S^{2n-1}$, being the restriction of the C^∞ map $GL(n,\mathbb{C}) \times \mathbb{C}^n \to \mathbb{C}^n$, is differentiable.

(2) The action of $U(n)$ on S^{2n-1} is transitive.

(3) The isotropy subgroup of $p = (1,0,\dots,0) \in S^{2n-1}$ is isomorphic to $U(n-1)$.

Hence one has a diffeomorphism $S^{2n-1} \approx U(n)/U(n-1)$, and similarly to Problem 4.7.1, one proves that $S^{2n-1} \approx SU(n)/SU(n-1)$.

Problem 4.7.3. *Prove that S^1 and S^3 are Lie groups by two different methods: First, from* Problem 4.7.2. *Then, by using the fact that S^1 and S^3 can be respectively identified to the unit complex numbers and to the unit quaternions.*

Solution. From the diffeomorphisms $S^{2n-1} \approx U(n)/U(n-1) \approx SU(n)/SU(n-1)$ in Problem 4.7.2, for $n = 1$ one has $S^1 \approx U(1)$; and for $n = 2$ we have $S^3 \approx U(2)/U(1) = SU(2)$. So S^1 and S^3 are Lie groups.

That $S^1 \approx U(1)$ was already seen in Problem 4.2.4. As for S^3, we have

$$\begin{aligned} S^3 &= \left\{ (x,y,z,t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1 \right\} \\ &\equiv \{ q \in \mathbb{H} : |q| = 1 \}. \end{aligned}$$

Now, given $q, q' \in \mathbb{H}$, one can check that $|qq'| = |q|\,|q'|$, hence if $q, q' \in S^3$ as above, then $qq' \in S^3$. Moreover, from the rules of multiplication in $\mathbb{H}$ (Problem 4.5.2, (1)) we conclude that S^3 is a Lie group. One can also obtain this applying Cartan's criterion on closed subgroups of a Lie group to $S^3 \subset \mathbb{H}^*$.

Problem 4.7.4. *Let $V_k(\mathbb{R}^n)$ denote the set of k-frames $(e_1,\dots,e_k)$ in $\mathbb{R}^n$ which are orthonormal with respect to the Euclidean metric g of $\mathbb{R}^n$. Prove*:

(1) *$V_k(\mathbb{R}^n)$ is a closed embedded C^∞ submanifold of $\mathbb{R}^{nk}$ (called the* Stiefel manifold *of orthonormal k-frames in $\mathbb{R}^n$).*

(2) *$O(n)/O(n-k)$ is a homogeneous space diffeomorphic to $V_k(\mathbb{R}^n)$.*

Solution. (1) Let us denote by x^i_j, $i=1,\dots,n$, $j=1,\dots,k$, the coordinate functions on $\mathbb{R}^{nk}$; that is, $x^i_j(e_1,\dots,e_k)$ is the ith component of e_j in the standard basis $(v_1,\dots,v_n)$ of $\mathbb{R}^n$. The equations defining $V_k(\mathbb{R}^n)$ are $f_{ij} = \sum_{h=1}^n x^h_i x^h_j - \delta_{ij} = 0$, for $1 \leqslant i \leqslant j \leqslant k$.

We shall now prove that the differentials of the functions f_{ij} are linearly independent, so concluding. For this, we first consider that the action

$$O(n) \times V_k(\mathbb{R}^n) \to V_k(\mathbb{R}^n), \qquad (A,(e_1,\dots,e_k)) \mapsto (Ae_1,\dots,Ae_k), \tag{$\star$}$$

is transitive, since given two g-orthonormal k-bases of $\mathbb{R}^n$, they can be completed to two orthonormal bases of $\mathbb{R}^n$, and there is always a matrix $A \in O(n)$ which defines a correspondence between them.

Moreover, since $(f_{ij}+\delta_{ij})(e_1,\dots,e_k)$ is nothing but the scalar product of e_i and e_j, we clearly have

$$(f_{ij}+\delta_{ij})(A\cdot(e_1,\dots,e_k)) = (f_{ij}+\delta_{ij})(e_1,\dots,e_k),$$

for all $A \in O(n)$, $(e_1,\dots,e_k) \in V_k(\mathbb{R}^n)$. Thus, it suffices to see that the differentials of the functions f_{ij} are linearly independent at a point $(e_1,\dots,e_k) \in V_k(\mathbb{R}^n)$. Take the point represented by the $n \times k$ matrix whose first n rows are the identity matrix I_k and the other $n-k$ rows are zero; that is, $x^h_i(e_1,\dots,e_k) = \delta_{hi}$. Then, it is immediate that $(\mathrm{d}f_{ij})_{(e_1,\dots,e_k)} = (\mathrm{d}x^i_j + \mathrm{d}x^j_i)_{(e_1,\dots,e_k)}$. As $i \leqslant j$, we are done.

(2) By means of the map

$$O(n-k) \to O(n), \qquad A \mapsto \begin{pmatrix} I_k & 0 \\ 0 & A \end{pmatrix},$$

$O(n-k)$ is a closed Lie subgroup of $O(n)$, hence the quotient space $O(n)/O(n-k)$, with the usual C^∞ structure, is a homogeneous space.

We have

$$V_k(\mathbb{R}^n) = \Big\{(e_1,\dots,e_k) \in (\mathbb{R}^n)^k : g(e_i,e_j) = \delta_{ij}\Big\} \subset (S^{n-1})^k.$$

In particular, $V_1(\mathbb{R}^n) = S^{n-1} \approx O(n)/O(n-1)$, as we proved in Problem 4.7.1. The action $(\star)$ is obviously differentiable. We have seen that it is also transitive. To determine the isotropy group of a point we choose, for the sake of simplicity, the point $p = (e_1, \dots, e_k)$, where $e_i = (0, \dots, 0, 1, \ 0, \ \dots, 0)$, with 1 in the ith place. Then, since $H_p = \{A \in O(n) : Ap = p\}$, a calculation similar to that in Problem 4.7.5 shows that

$$H_p = \begin{pmatrix} I_k & 0 \\ 0 & A \end{pmatrix}, \qquad A \in O(n-k).$$

Consequently H_p is isomorphic to $O(n-k)$ and $V_k(\mathbb{R}^n)$ is diffeomorphic to $O(n)/O(n-k)$.

Problem 4.7.5. *Prove that $O(n)/O(k) \times O(n-k)$ is a homogeneous space diffeomorphic to the C^∞ manifold $G_k(\mathbb{R}^n)$ of k-planes through the origin of $\mathbb{R}^n$, called the* (*real*) Grassmann manifold *of k-planes in $\mathbb{R}^n$. Analyze the particular case $G_1(\mathbb{R}^n)$.*

Solution. By means of the map

$$O(k) \times O(n-k) \to O(n), \qquad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$O(k) \times O(n-k)$ is a closed Lie subgroup of $O(n)$ and thus the quotient space $O(n)/O(k) \times O(n-k)$, with the usual C^∞ structure, is a homogeneous space.

The map

$$V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \qquad p = \{e_1, \dots, e_k\} \mapsto \langle e^1, \dots, e^k \rangle,$$

which defines a correspondence between each k-basis of $\mathbb{R}^n$ and the k-plane it spans, is surjective, since given a k-plane, we always can choose a g-orthonormal k-basis, g being the Euclidean metric of $\mathbb{R}^n$. The map

$$O(n) \times G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \qquad (A, \langle e_1, \dots, e_k \rangle) \mapsto \langle Ae_1, \dots, Ae_k \rangle,$$

is C^∞. The action is transitive, as given two k-planes of $\mathbb{R}^n$, and a g-orthonormal k-basis in each one, we can complete both bases to g-orthonormal bases of $\mathbb{R}^n$; but there is always an element $A \in O(n)$ which transforms the one into the other, and thus it transforms the k-plane generated by the initial k-basis in the k-plane generated by the other k-basis.

In order to determine the isotropy group of a point, we choose $p = \langle e_1, \dots, e_k \rangle$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the ith place. It is easy to see that the elements of $O(n)$ leaving p invariant are those of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad A \in O(k), \quad B \in O(n-k).$$

Hence $H_p \approx O(k) \times O(n-k)$ and thus $G_k(\mathbb{R}^n) \approx O(n)/O(k) \times O(n-k)$. For $k = 1$, we have 1-planes, that is, straight lines through the origin of $\mathbb{R}^n$, and $G_1(\mathbb{R}^n)$ is then the real projective space $\mathbb{R}P^{n-1}$. We thus have

$$\begin{aligned} \mathbb{R}P^{n-1} &\approx G_1(\mathbb{R}^n) \\ &\approx O(n)/O(1) \times O(n-1) \end{aligned}$$

$$\approx O(n)/\mathbb{Z}_2 \times O(n-1)$$
$$\approx SO(n)/O(n-1),$$

where the last equivalence follows from an argument as in Problem 4.7.1. Hence, the real projective spaces are homogeneous spaces.

Problem 4.7.6. *Show that $GL(n,\mathbb{R})$ acts transitively on $\mathbb{R}P^{n-1}$ and determine the isotropy group of $[e_1]$, $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$.*

Solution. If the points $p,q \in \mathbb{R}P^{n-1}$ are given by $p = [\lambda]$, $q = [\mu]$, where $\lambda,\mu \in \mathbb{R}^n$ are two nonzero vectors, then there exists $A \in GL(n,\mathbb{R})$ such that $A\lambda = \mu$, as λ (resp. μ) can be completed to a basis $v_1 = \{\lambda, v_2, \dots, v_n\}$ (resp. $v'_1 = \{\mu, v'_2, \dots, v'_n\}$) of $\mathbb{R}^n$ and A is the isomorphism $Av_i = v'_i$, $i = 1,\dots,n$. The isotropy group of $[e_1]$ is the subgroup of $GL(n,\mathbb{R})$ of elements B such that $B(\lambda_1,0,\dots,0) = (\mu_1,0,\dots,0)$, that is

$$H = \left\{ B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{2n} & \cdots & b_{nn} \end{pmatrix} \in GL(n,\mathbb{R}) \right\}.$$

So $\mathbb{R}P^{n-1} = GL(n,\mathbb{R})/H$. (Note that $\dim \mathbb{R}P^{n-1} = \dim GL(n,\mathbb{R}) - \dim H = n-1$, as expected.)

Problem 4.7.7. *The punctured Euclidean space $\mathbb{R}^n - \{0\}$ is homogeneous since $GL(n,\mathbb{R})$ acts transitively on it.*

(1) *Determine the isotropy group H of $(1,0,\dots,0) \in \mathbb{R}^n - \{0\}$.*

(2) *Is the homogeneous space $GL(n,\mathbb{R})/H$ reductive?*

Solution (1) $H = \left\{ \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix} : v \in \mathbb{R}^{n-1}, B \in GL(n-1,\mathbb{R}) \right\}.$

(2) No, as we shall see giving two proofs.

1st *proof.* The Lie algebra $\mathfrak{h}$ of H is, as it is easily checked by using the exponential map,

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} : v \in \mathbb{R}^{n-1}, A \in \mathfrak{gl}(n-1,\mathbb{R}) \right\}.$$

Suppose $\mathfrak{gl}(n,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{m}$, with $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$. Since $\dim \mathfrak{h} = (n-1)n$, one has $\dim \mathfrak{m} = n$. Let $E^i_j \in \mathfrak{gl}(n,\mathbb{R})$ the matrix $(E^i_j)^h_k = \delta_{hi}\delta_{kj}$, so that $\{E^i_j\}^n_{i,j=1}$ is a basis of $\mathfrak{gl}(n,\mathbb{R})$.

First suppose $n = 2$. Then the matrix E^1_1 can be written as

$$E^1_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} + \begin{pmatrix} 1 & -v \\ 0 & -a \end{pmatrix}, \qquad a, v \in \mathbb{R},$$

with

$$E^{1\mathfrak{h}}_1 = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} \in \mathfrak{h}, \qquad E^{1\mathfrak{m}}_1 = \begin{pmatrix} 1 & -v \\ 0 & -a \end{pmatrix} \in \mathfrak{m}.$$

By virtue of the hypothesis, we have

$$[E_1^{1\mathfrak{h}}, E_1^{1\mathfrak{m}}] = \begin{pmatrix} 0 & -v \\ 0 & 0 \end{pmatrix} \in \mathfrak{m},$$

but this matrix also belongs to $\mathfrak{h}$, hence $v = 0$. Moreover, E_2^1 belongs to $\mathfrak{h}$. Consequently

$$\left[E_2^1, E_1^{1\mathfrak{m}}\right] = \begin{pmatrix} 0 & -(a+1) \\ 0 & 0 \end{pmatrix} \in \mathfrak{m},$$

and since this commutator also belongs to $\mathfrak{h}$, it follows that $a = -1$. Summarizing, one has $E_1^{1\mathfrak{m}} = I_2 \in \mathfrak{m}$. On the other hand, one has a decomposition

$$E_1^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -v \\ 1 & -a \end{pmatrix}, \qquad a, v \in \mathbb{R},$$

with

$$E_1^{2\mathfrak{h}} = \begin{pmatrix} 0 & v \\ 0 & a \end{pmatrix} \in \mathfrak{h}, \qquad E_1^{2\mathfrak{m}} = \begin{pmatrix} 0 & -v \\ 1 & -a \end{pmatrix} \in \mathfrak{m}.$$

Again from $E_2^1 \in \mathfrak{h}$ we deduce that

$$\left[E_2^1, E_1^{2\mathfrak{m}}\right] = \begin{pmatrix} 1 & -a \\ 0 & -1 \end{pmatrix} \in \mathfrak{m},$$

and since I_2 and $\left[E_2^1, E_1^{2\mathfrak{m}}\right]$ are linearly independent, one concludes that

$$\mathfrak{m} = \left\langle I_2, \left[E_2^1, E_1^{2\mathfrak{m}}\right] \right\rangle,$$

which is impossible as in this case the matrix $E_1^{2\mathfrak{m}}$ could not belong to $\mathfrak{m}$, since its $(2,1)$th entry is not null.

For $n \geqslant 3$, we have

$$E_1^2 = \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} + \begin{pmatrix} 0 & -v \\ u & -A \end{pmatrix},$$

$$A \in \mathfrak{gl}(n-1, \mathbb{R}), \qquad v \in \mathbb{R}^{n-1}, \qquad {}^t u = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1},$$

with

$$E_1^{2\mathfrak{h}} = \begin{pmatrix} 0 & v \\ 0 & A \end{pmatrix} \in \mathfrak{h}, \qquad E_1^{2\mathfrak{m}} = \begin{pmatrix} 0 & -v \\ u & -A \end{pmatrix} \in \mathfrak{m}.$$

As

$$E_3^1 = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \qquad w = (0, 1, \ldots, 0) \in \mathbb{R}^{n-1},$$

belongs to $\mathfrak{h}$, one has that

$$\left[E_3^1, E_1^{2\mathfrak{m}}\right] = \begin{pmatrix} 0 & -wA \\ 0 & -(u^i w^j) \end{pmatrix}$$

belongs to $\mathfrak{m}$, and since it also belongs to $\mathfrak{h}$, it is the null matrix. Contradiction, for the square matrix $(u^i w^j)$ of order $n-1$ never vanishes.

2^{nd} *proof.* Another proof, this time unified for $n \geqslant 2$, and which uses representation theory, is the following. First, we identify $\mathfrak{gl}(n-1,\mathbb{R})$ with the subalgebra

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(n-1,\mathbb{R}) \right\}.$$

Then $\mathfrak{gl}(n,\mathbb{R})$ decomposes as a $\mathfrak{gl}(n-1,\mathbb{R})$-module into

$$\mathfrak{gl}(n,\mathbb{R}) = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\} \oplus \mathfrak{gl}(n-1,\mathbb{R})$$
$$\oplus \mathbb{R}E_1^1 \oplus \left\{ \begin{pmatrix} 0 & 0 \\ {}^t v & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\},$$

which is a sum of four non-isomorphic irreducible $\mathfrak{gl}(n-1,\mathbb{R})$-modules.

Every $\mathfrak{h}$-submodule of $\mathfrak{gl}(n,\mathbb{R})$ is in particular a $\mathfrak{gl}(n-1,\mathbb{R})$-module, hence a direct sum of some of the four $\mathfrak{gl}(n-1,\mathbb{R})$-submodules above. Thus, the unique possibility for $\mathfrak{m}$ is

$$\mathfrak{m} = \mathbb{R}E_1^1 \oplus \left\{ \begin{pmatrix} 0 & 0 \\ {}^t v & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\},$$

but $[\mathfrak{h},\mathfrak{m}] \not\subseteq \mathfrak{m}$, from which we conclude that the space is not reductive.

Problem 4.7.8. *The complex projective space $\mathbb{C}P^n$, which is the set of complex lines through the origin in the complex $(n+1)$-space $\mathbb{C}^{n+1}$, is diffeomorphic to the homogeneous space $SU(n+1)/S(U(n)\times U(1))$.*

(1) *Does $SU(n+1)$ act effectively on $\mathbb{C}P^n$?*

(2) *Write $\mathbb{C}P^n$ as a homogeneous space G/H such that G acts effectively on $\mathbb{C}P^n$.*

REMARK. *We recall that the center $\mathbb{Z}_{n+1}$ of $SU(n+1)$ consists of the diagonal matrices* $\operatorname{diag}(\lambda,\ldots,\lambda)$*, λ being an $(n+1)$th root of* 1.

Solution. (1) The answer is no, since the isotropy group $S(U(n)\times U(1))$ contains the center $\mathbb{Z}_{n+1}$ of $SU(n+1)$.

(2) Let us compute the subgroup N. A matrix

$$s = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}, \qquad A \in U(n), \quad \lambda = \frac{1}{\det A},$$

belongs to N if and only if $g^{-1}sg \in S(U(n)\times U(1))$, for all $g \in SU(n+1)$. Let $\{v_1,\ldots,v_{n+1}\}$ be the standard basis of $\mathbb{C}^{n+1}$ and let $g \in SU(n+1)$ be the matrix given by

$$g(v_r) = (\cos\alpha)v_r + (\sin\alpha)v_{n+1},$$

$$g(v_{n+1}) = -(\sin\alpha)v_r + (\cos\alpha)v_{n+1},$$
$$g(v_i) = v_i, \qquad\qquad 1 \leqslant i \leqslant n, \quad i \neq r,$$

where $1 \leqslant r \leqslant n$ is a fixed index, and $\alpha \in \mathbb{R}$. Then, we must have $(g^{-1}sg)(v_{n+1}) = \mu v_{n+1}$ for some $\mu \in \mathbb{C}^*$, as $g^{-1}sg \in S(U(n) \times U(1))$, or equivalently, $s(g(v_{n+1})) = \mu g(v_{n+1})$, and expanding:

$$\begin{aligned} s(-(\sin\alpha)v_r + (\cos\alpha)v_{n+1}) &= -(\sin\alpha)A(v_r) + (\cos\alpha)\lambda v_{n+1} \\ &= \mu\big(-(\sin\alpha)v_r + (\cos\alpha)v_{n+1}\big). \end{aligned}$$

Hence $\lambda = \mu$, $Av_r = \lambda v_r$, for all $r = 1, \ldots, n$. Therefore, $A = \lambda I_n$, and since $1 = \lambda \det A = \lambda^{n+1}$ we conclude that $N = \mathbb{Z}_{n+1}$, which is the center of $SU(n+1)$. Accordingly, we can write

$$\mathbb{C}P^n = (SU(n+1)/\mathbb{Z}_{n+1})/(S(U(n) \times U(1))/\mathbb{Z}_{n+1}).$$

The group $G = SU(n+1)/\mathbb{Z}_{n+1}$ acts effectively on $\mathbb{C}P^n$.

Chapter 5
Fibre Bundles

5.1 Principal Bundles

Problem 5.1.1. *Denote by $E(F\,\mathbb{R},\mathbb{R})$ the bundle with fibre $\mathbb{R}$ associated to the frame bundle $F\,\mathbb{R} \approx \mathbb{R} \times (\mathbb{R} - \{0\})$. Show that the tangent bundle $T\,\mathbb{R} \approx \mathbb{R}^2$ is isomorphic to $E(F\,\mathbb{R},\mathbb{R})$.*

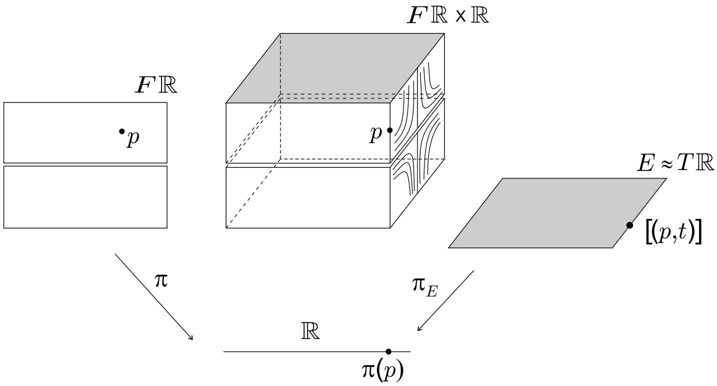

Fig. 5.1 The bundle $E(F\,\mathbb{R},\mathbb{R})$.

Solution. The structure group of $F\,\mathbb{R}$ is $GL(1,\mathbb{R})$. Thus we have to prove that $T\,\mathbb{R}$ is the quotient space $T\,\mathbb{R} = (F\,\mathbb{R} \times \mathbb{R})/GL(1,\mathbb{R})$, where $GL(1,\mathbb{R})$ acts on the manifold $F\,\mathbb{R} \times \mathbb{R}$ by

$$\begin{array}{rcl} (F\,\mathbb{R} \times \mathbb{R}) \times GL(1,\mathbb{R}) & \to & F\,\mathbb{R} \times \mathbb{R} \\ ((v,t),g) & \mapsto & (vg, g^{-1}t), \end{array}$$

that is, on the right on $F\,\mathbb{R}$ and on the left on $\mathbb{R}$ (by g^{-1}). Denoting $\pi\colon F\,\mathbb{R} \to \mathbb{R}$ and $\pi_E\colon E \to \mathbb{R}$, and once fixed $v \in F\,\mathbb{R}$, one has $\pi(v) = \pi(vg)$. Let $[(v,t)]$ be the class of $(v,t) \in F\,\mathbb{R} \times \mathbb{R}$ in E. Then $[(v,t)] = [(vg, g^{-1}t)]$ for all $g \in GL(1,\mathbb{R}) \approx \mathbb{R} - \{0\}$.

For every v, we have $vg = \lambda v \in T_{\pi(v)}\mathbb{R} \approx \mathbb{R}$, and $g^{-1}t = (1/\lambda)t \in \mathbb{R}$, $\lambda \neq 0$. Thus, each representative of the class $[(v,t)]$ has components with constant product for $vt = v\lambda \frac{1}{\lambda}t$. Consequently the class is a hyperbola in $(F\,\mathbb{R})_{\pi(v)} \times \mathbb{R}$, as one can see in the Figure 5.1. On the other hand, for $t = 0$, the class of (v,t) is the pair of half-lines

$$\pi^{-1}(\pi(v)) \times \{0\} = (F\,\mathbb{R})_{\pi(v)} \times \{0\} \subset F\,\mathbb{R} \times \mathbb{R}.$$

The set of classes is thus isomorphic to $\mathbb{R}$ in each fibre; and the total space of classes, $E = \mathbb{R}^2 \approx T\,\mathbb{R}$.

Problem 5.1.2. (Hopf bundles) *Set*

$$\begin{aligned}
S^1 &= \{x \in \mathbb{C} : |x| = 1\},\\
S^2 &= \left\{(x,t) \in \mathbb{C} \times \mathbb{R} : |x|^2 + t^2 = 1\right\},\\
S^3 &= \left\{(x,y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\right\},\\
S^7 &= \left\{(x,y) \in \mathbb{H}^2 : |x|^2 + |y|^2 = 1\right\}.
\end{aligned}$$

The spheres S^1 and S^3 are Lie groups with respect to the multiplication induced from $\mathbb{C}$ and $\mathbb{H}$, respectively (see Problem 4.7.3). *Let S^1 act on S^3 (resp. S^3 on S^7) by the formula*

$$(x,y)\cdot z = (xz, yz), \qquad (x,y) \in S^3, \quad z \in S^1 \quad (resp.\ (x,y) \in S^7,\ z \in S^3).$$

Let

$$\pi_{\mathbb{C}} \colon S^3 \to \mathbb{C} \times \mathbb{R}, \qquad \pi_{\mathbb{H}} \colon S^7 \to \mathbb{H} \times \mathbb{R},$$

be the maps given by

$$\begin{aligned}
\pi_{\mathbb{C}}(x,y) &= (2y\bar{x}, |x|^2 - |y|^2), \qquad (x,y) \in S^3,\\
\pi_{\mathbb{H}}(x,y) &= (2y\bar{x}, |x|^2 - |y|^2), \qquad (x,y) \in S^7.
\end{aligned}$$

Prove:

(1) $\pi_{\mathbb{C}}(S^3) = S^2$.

(2) $\pi_{\mathbb{H}}(S^7) = S^4$.

(3) *The induced map $\pi_{\mathbb{C}} \colon S^3 \to S^2$ is a principal S^1-bundle with respect to the action of S^1 on S^3 defined above.*

(4) *The induced map $\pi_{\mathbb{H}} \colon S^7 \to S^4$ is a principal S^3-bundle with respect to the action of S^3 on S^7 defined above.*

(5) $\mathbb{C}P^1 \approx S^2$.

(6) $\mathbb{H}P^1 \approx S^4$.

Solution. We solve the quaternionic case (2), (4), and (6). The same formulae solve the complex case (1), (3), and (5), too.

(2) First we check that $\pi_{\mathbb{H}}(S^7) \subseteq S^4$. In fact, if $(x,y) \in S^7$ we have $|x|^2 + |y|^2 = 1$ and then, since $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ for every $q_1, q_2 \in \mathbb{H}$, we have

$$\begin{aligned}|\pi_{\mathbb{H}}(x,y)|^2 &= 4|y|^2|x|^2 + (|x|^2 - |y|^2)^2 \\ &= (|x|^2 + |y|^2)^2 = 1.\end{aligned}$$

Let $(u,t) \in \mathbb{H} \times \mathbb{R}$ be a point in S^4, such that $|u|^2 + t^2 = 1$. If $u = 0$, then $t = \pm 1$, and we have $\pi_{\mathbb{H}}(1,0) = (0,1)$, $\pi_{\mathbb{H}}(0,1) = (0,-1)$. Hence we can assume $u \neq 0$. In this case $-1 < t < 1$, and one has

$$\pi_{\mathbb{H}}\left(\sqrt{\frac{1+t}{2}}\frac{\bar{u}}{|u|}, \sqrt{\frac{1-t}{2}}\right) = (u,t)$$

and

$$\left(\sqrt{\frac{1+t}{2}}\frac{\bar{u}}{|u|}, \sqrt{\frac{1-t}{2}}\right) \in S^7.$$

(4) First we have

$$\begin{aligned}\pi_{\mathbb{H}}((x,y)\cdot z) &= \pi_{\mathbb{H}}(xz, yz) \\ &= \left(2yz\overline{xz}, |xz|^2 - |yz|^2\right) \\ &= \left(2yz\bar{z}\bar{x}, |x|^2|z|^2 - |y|^2|z|^2\right) \\ &= \left(2y\bar{x}, |x|^2 - |y|^2\right) \\ &= \pi_{\mathbb{H}}(x,y),\end{aligned}$$

as $z\bar{z} = |z|^2 = 1$ for $z \in S^3$. Hence the orbit $(x,y)\cdot S^3$ is contained in the fibre $\pi_{\mathbb{H}}^{-1}(\pi_{\mathbb{H}}(x,y))$.

Conversely, if $\pi_{\mathbb{H}}(x_1,y_1) = \pi_{\mathbb{H}}(x_2,y_2)$, then

$$y_1\bar{x}_1 = y_2\bar{x}_2, \tag{$\star$}$$

$$|x_1|^2 - |y_1|^2 = |x_2|^2 - |y_2|^2. \tag{$\star\star$}$$

As $|x_1|^2 + |y_1|^2 = 1$, either $x_1 \neq 0$ or $y_1 \neq 0$. Hence we can assume $x_1 \neq 0$. Set $z = x_1^{-1}x_2 \in \mathbb{H}$. Hence $(\star)$ implies, since $|q_1||q_2| = |q_1q_2|$ for every $q_1, q_2 \in \mathbb{H}$,

$$\bar{y}_1 = z\bar{y}_2 \quad \text{i.e.} \quad y_1 = y_2\bar{z}. \tag{$\star\star\star$}$$

As $|x_1|^2 + |y_1|^2 = |x_2|^2 + |y_2|^2 = 1$, we have $|y_1|^2 = |y_2|^2$ by $(\star\star)$, from which $|y_2|^2|z|^2 = |y_1|^2$. If $y_2 = 0$ we should have from $(\star)$ that $y_1 = 0$ and so $|x_1|^2 = |x_2|^2 = 1 = |z|^2$. If $y_2 \neq 0$, then $|z| = 1$. That is, in both cases we have $|z| = 1$. Therefore, $z^{-1} = \bar{z}$ and from $(\star\star\star)$ we deduce $y_2 = y_1z$. In other words, $z \in S^3$ and $(x_1,y_1)\cdot z = (x_2,y_2)$, thus concluding.

(6) By definition $\mathbb{H}P^1$ is the quotient space $\mathbb{H}^2 - \{(0,0)\}/\sim$, where $(x,y) \sim (x',y')$ if and only if there exists $\lambda \in \mathbb{H}^*$ such that $x' = \lambda x$, $y' = \lambda y$. Moreover, the restriction to S^7 of the quotient map $q\colon \mathbb{H}^2 - \{(0,0)\} \to \mathbb{H}P^1$ is surjective as $q(x,y) = q(x/\rho, y/\rho)$, with $\rho = \sqrt{|x|^2 + |y|^2}$ and its fibres are the orbits of S^3. Hence

$\mathbb{H}P^1 \approx S^7/S^3$ and since we have a principal S^3-bundle $\pi_{\mathbb{H}}\colon S^7 \to S^4$, we conclude $\mathbb{H}P^1 \approx S^7/S^3 \approx S^4$.

Problem 5.1.3. *Parametrize S^3 by*

$$z_1 = \cos\frac{1}{2}\theta\, \mathrm{e}^{\psi_1 \mathrm{i}}, \quad z_2 = \sin\frac{1}{2}\theta\, \mathrm{e}^{\psi_2 \mathrm{i}}, \qquad 0 \leqslant \theta \leqslant \pi, \quad \psi_1,\, \psi_2 \in \mathbb{R}.$$

(1) *Find the expression of $\pi_{\mathbb{C}}(z_1,z_2)$ under the projection map of the Hopf bundle $\pi_{\mathbb{C}}\colon S^3 \to S^2$ given in* Problem 5.1.2*, in terms of that parametrization.*

(2) *Take as trivializing neighborhoods $U_1 = S^2 - S$ and $U_2 = S^2 - N$, where N,S stand for the north and south pole. Determine $\pi_{\mathbb{C}}^{-1}(U_k)$, $k = 1,2$.*

(3) *Define bundle trivializations*

$$f_k\colon \pi_{\mathbb{C}}^{-1}(U_k) \to U_k \times U(1), \quad f_k(z_1,z_2) = \left(\pi_{\mathbb{C}}(z_1,z_2), \frac{z_k}{|z_k|}\right), \qquad k = 1,2,$$

and put $f_{k,p} = f_k|_{\pi^{-1}(p)}$. Find the transition function $g_{21}\colon U_1 \cap U_2 \to U(1)$ of the bundle with respect to the given trivializations.

Solution. (1)

$$\begin{aligned}\pi_{\mathbb{C}}(z_1,z_2) &= (2\operatorname{Re}(z_2\bar{z}_1), 2\operatorname{Im}(z_2\bar{z}_1), |z_1|^2 - |z_2|^2)\\ &= (\sin\theta\cos(\psi_2-\psi_1), \sin\theta\sin(\psi_2-\psi_1), \cos\theta).\end{aligned}$$

(2) It is easily seen from the definitions of the trivializing neighborhoods that

$$\pi_{\mathbb{C}}^{-1}(U_k) = \left\{(z_1,z_2) \in S^3 \,:\, z_k \neq 0\right\}, \qquad k = 1,2.$$

(3) Given

$$p = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in U_1 \cap U_2, \qquad \mathrm{e}^{\alpha\mathrm{i}} \in U(1),$$

we obtain, on account of the parametrization of S^3 and the expression for the projection map π:

$$f_k(z_1,z_2) = (\sin\theta\cos(\psi_2-\psi_1), \sin\theta\sin(\psi_2-\psi_1), \cos\theta, \mathrm{e}^{\psi_k \mathrm{i}}),$$

so

$$f_{1,p}^{-1}(\mathrm{e}^{\alpha\mathrm{i}}) = \left(\cos\frac{1}{2}\theta\, \mathrm{e}^{\alpha\mathrm{i}}, \sin\frac{1}{2}\theta\, \mathrm{e}^{(\varphi+\alpha)\mathrm{i}}\right),$$

hence

$$(f_{2,p} \circ f_{1,p}^{-1})(\mathrm{e}^{\alpha\mathrm{i}}) = \mathrm{e}^{(\varphi+\alpha)\mathrm{i}}.$$

That is, the transition function for the given trivializations is

$$g_{21}\colon U_1 \cap U_2 \to U(1), \qquad (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \mapsto \mathrm{e}^{\varphi\mathrm{i}}.$$

Problem 5.1.4. *Let* (P, π, M, G) *be a principal fibre bundle and let* $\mathfrak{g}$ *be the Lie algebra of* G*. Every* $A \in \mathfrak{g}$ *induces a vector field* $A^* \in \mathfrak{X}(P)$ (*called the* fundamental vector field associated with A), *with flow*

$$\psi_t(u) = u\exp(tA), \qquad u \in P.$$

The map

$$\varphi\colon \mathfrak{g} \to \mathfrak{X}(P), \qquad \varphi(A) = A^*,$$

is $\mathbb{R}$*-linear, injective and satisfies* $[A, B]^* = [A^*, B^*]$, *for all* $A, B \in \mathfrak{g}$.

(1) *Prove that* $R_g \cdot A^* = \left(\mathrm{Ad}_{g^{-1}} A\right)^*$, *where* $g \in G$, $A \in \mathfrak{g}$.

(2) *Calculate the expression for* $\varphi(aX_1 + bX_2)$, *where* X_1 *and* X_2 *are the left-invariant vector fields on* $\mathbb{C}^*$ *given by*:

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

and φ *is the isomorphism associated to the principal fibre bundle*

$$(\mathbb{C}^{n+1} - \{0\}, \pi, \mathbb{C}P^n, \mathbb{C}^*),$$

where $\mathbb{C}P^n$ *stands for the complex projective space of real dimension* $2n$.

Solution. (1) For $u \in P$, denote by j_u the injection of G into P given by

$$j_u\colon G \to \pi^{-1}(\pi(u)), \qquad g \mapsto ug.$$

Let e be the identity element of G. It is clear that

$$\begin{aligned} A_u^* &= \psi_t(u)'(0) \\ &= j_{u*}(\exp(tA)'(0)) \\ &= j_{u*}A_e. \end{aligned}$$

Let $\iota_g\colon G \to G$ be the automorphism of G defined by $\iota_g(h) = ghg^{-1}$, and consider the composition map

$$\begin{array}{ccccc} G & \xrightarrow{j_{ug^{-1}}} & P & \xrightarrow{R_g} & P \\ h & \longmapsto & ug^{-1}h & \longmapsto & ug^{-1}hg = u\,\iota_{g^{-1}}(h) \end{array}$$

whose differential at e is

$$\begin{aligned} R_{g*}j_{ug^{-1}*}A_e &= (j_u \iota_{g^{-1}})_* A_e \\ &= \left(\varphi\left(\mathrm{Ad}_{g^{-1}}(A_e)\right)\right)_u \\ &= \left(\mathrm{Ad}_{g^{-1}} A\right)_u^*. \end{aligned}$$

Hence, the vector field image $R_g \cdot A^*$ is given at u by

$$
\begin{aligned}
(R_g \cdot A^*)_u &= R_{g*} A^*_{ug^{-1}} \\
&= R_{g*} j_{ug^{-1}*} A_e \\
&= \left(\mathrm{Ad}_{g^{-1}} A\right)^*_u,
\end{aligned}
$$

so $R_g \cdot A^* = (\mathrm{Ad}_{g^{-1}} A)^*$.

(2) Let $u^1, \ldots, u^{2n+2}$ be the real coordinates on $\mathbb{C}^{n+1} - \{0\}$ (that is, $\{z^j = u^{2j-1} + \mathrm{i}u^{2j}\}$ is the dual basis to the usual complex basis of $\mathbb{C}^{n+1}$). For $u = (u^1 + \mathrm{i}u^2, \ldots, u^{2n+1} + \mathrm{i}u^{2n+2}) \in \mathbb{C}^{n+1} - \{0\}$, the map j_u above is now given by $\mathbb{C}^* \to \pi^{-1}(\pi(u))$,

$$
\begin{aligned}
x + \mathrm{i}y \mapsto\ & (u^1 + \mathrm{i}u^2, \ldots, u^{2n+1} + \mathrm{i}u^{2n+2})(x + \mathrm{i}y) \\
&= (u^1 x - u^2 y + \mathrm{i}(u^2 x + u^1 y), \ldots) \\
&\equiv (u^1 x - u^2 y, u^2 x + u^1 y, \ldots).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\varphi &= j_{u*} \\
&= \begin{pmatrix} \dfrac{\partial(u^1 x - u^2 y)}{\partial x} & \dfrac{\partial(u^1 x - u^2 y)}{\partial y} \\ \dfrac{\partial(u^2 x + u^1 y)}{\partial x} & \dfrac{\partial(u^2 x + u^1 y)}{\partial y} \\ \vdots & \vdots \end{pmatrix} \\
&= \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \\ \vdots & \vdots \\ u^{2n+1} & -u^{2n+2} \\ u^{2n+2} & u^{2n+1} \end{pmatrix}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\varphi(sX_1 + tX_2))_u &= j_{u*}((sX_1 + tX_2)_e) \\
&= j_{u*}\left(s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\right), \qquad s, t \in \mathbb{R},
\end{aligned}
$$

as $e \equiv (x = 1, y = 0)$. So,

$$
\varphi(sX_1 + tX_2) \equiv \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}
$$

$$= \begin{pmatrix} u^1 s - u^2 t \\ u^2 s + u^1 t \\ \vdots \end{pmatrix}$$

$$\equiv (u^1 s - u^2 t)\frac{\partial}{\partial u^1} + (u^2 s + u^1 t)\frac{\partial}{\partial u^2} + \cdots$$

$$+ (u^{2n+1} s - u^{2n+2} t)\frac{\partial}{\partial u^{2n+1}} + (u^{2n+2} s + u^{2n+1} t)\frac{\partial}{\partial u^{2n+2}}.$$

Problem 5.1.5. *Let (FM, π, M) be the bundle of linear frames over the C^∞ n-manifold M. If $p \in M$ and $(x^1, \dots, x^n)$ is a coordinate system on a neighborhood U of p, we can define the map*

$$F_U \colon \pi^{-1}(U) \to GL(n, \mathbb{R}), \qquad z = (q, e_1, \dots, e_n) \mapsto (\mathrm{d}x^i(e_j)).$$

The functions $x^i = x^i \circ \pi$ and $x^i_j = x^i_j \circ F_U$, where x^i_j denote the standard coordinates on $GL(n, \mathbb{R})$, are a coordinate system on $\pi^{-1}(U)$ (see Figure 5.2*). If $z \in \pi^{-1}(U)$, prove that*

$$\pi_* \left(\frac{\partial}{\partial x^i}\bigg|_z \right) = Y^j_i(z) e_j,$$

where $(Y^i_j(z))$ stands for the inverse matrix of $(x^i_j(z))$.

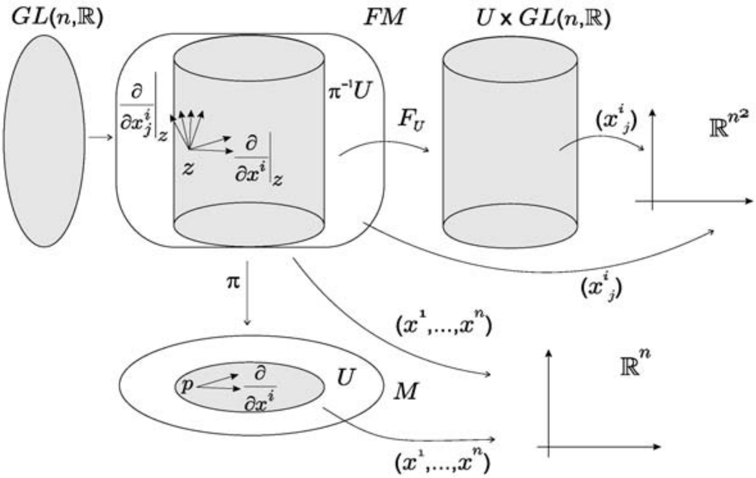

Fig. 5.2 The bundle of linear frames (FM, π, M) over M.

Solution. We have

$$\pi_* \left(\frac{\partial}{\partial x^i}\bigg|_z \right) = \frac{\partial (x^j \circ \pi)}{\partial x^i} \frac{\partial}{\partial x^j}\bigg|_q = \frac{\partial}{\partial x^i}\bigg|_q, \tag{$\star$}$$

but the coordinates of $\{e_j\}$ with respect to the canonical basis $\{(\partial/\partial x^i)_q\}$ are precisely $(x^i_j(z))$, that is,

$$(e_1,\dots,e_n) = \left(\left.\frac{\partial}{\partial x^i}\right|_q, \dots, \left.\frac{\partial}{\partial x^n}\right|_q \right) \begin{pmatrix} x^1_1(z) & \cdots & x^1_n(z) \\ \vdots & & \vdots \\ x^n_1(z) & \cdots & x^n_n(z) \end{pmatrix},$$

or equivalently $e_i = x^j_i(z) \left.\dfrac{\partial}{\partial x^j}\right|_q$. Thus

$$\left.\frac{\partial}{\partial x^i}\right|_q = Y^j_i(z) e_j. \qquad (\star\star)$$

From $(\star)$ and $(\star\star)$, it follows that $\pi_* \left(\left.\dfrac{\partial}{\partial x^i}\right|_z \right) = Y^j_i(z) e_j$.

Problem 5.1.6. *Let $\pi\colon P \to M$ be a principal G-bundle. Prove that a vector field $X \in \mathfrak{X}(P)$ is π-projectable if and only if for every $g \in G$, the vector field $R_g \cdot X - X$ is π-vertical.*

Solution. The vector field X is π-projectable if $\pi_* X_u = \pi_* X_v$ for all $u,v \in \pi^{-1}(p)$. As G acts transitively on $\pi^{-1}(p)$, there exists $g \in G$ such that $v = ug$. Hence $\pi_* X_u = \pi_* X_v$ means $\pi_* X_u = \pi_* X_{ug}$, and taking into account that $\pi \circ R_g = \pi$, we can rewrite the previous equation as $\pi_*(R_{g*} X_u - X_{ug}) = 0$. That is,

$$R_{g*} X_u - X_{ug} = (R_g \cdot X)_{ug} - X_{ug}$$

is a π-vertical tangent vector. The converse is immediate.

Problem 5.1.7. *Find the fundamental vector fields on the bundle of linear frames FM over a C^∞ n-manifold M.*

Solution. If A is an element of the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ then its value at the identity element e of $GL(n,\mathbb{R})$ is the tangent vector at e to the curve e^{tA}, and it has corresponding fundamental field A^* on FM, whose value at $z \in FM$ is A^*_z, the tangent vector to the curve $z\mathrm{e}^{tA}$ in FM at z. Let $\{x^i\}$ be local coordinates on M with domain U, and let $\{x^i_j\}$ be the canonical coordinates on $GL(n,\mathbb{R})$.

Then the coordinates of z are $x^i(z) = x^i(\pi(z))$, $x^i_j(z)$, as in Problem 5.1.5. Therefore

$$(A^* x^i)_z = \lim_{t\to 0} \frac{x^i(z\mathrm{e}^{tA}) - x^i(z)}{t} = 0,$$

because $\pi(z\mathrm{e}^{tA}) = \pi(z)$, and

$$(A^* x^i_j)_z = \lim_{t\to 0} \frac{x^i_j(z\mathrm{e}^{tA}) - x^i_j(z)}{t}$$

$$
\begin{aligned}
&= \lim_{t\to 0}\frac{1}{t}\left\{\left(x^i_j(z)+tx^i_j(zA)+\frac{t^2}{2!}x^i_j(zA^2)+\cdots\right)-x^i_j(z)\right\}\\
&= x^i_j(zA)\\
&= x^i_k(z)a^k_j,
\end{aligned}
$$

where $A=(a^i_j)$. Hence $A^*_z = x^i_k(z)a^k_j \left.\dfrac{\partial}{\partial x^i_j}\right|_z$ and $A^*|_{\pi^{-1}(U)} = a^k_j x^i_k \dfrac{\partial}{\partial x^i_j}$.

Problem 5.1.8. *Prove that a necessary and sufficient condition for a C^∞ $2n$-manifold M to admit an* almost tangent structure*, that is, a G-structure with group*

$$
G=\left\{\begin{pmatrix} A & 0\\ B & A\end{pmatrix}\in GL(2n,\mathbb{R}) : A\in GL(n,\mathbb{R})\right\},
$$

is that it admits a C^∞ tensor field J of type $(1,1)$ and rank n such that $J^2=0$.

Solution. First, suppose that M admits an almost tangent structure. Let (e_j), $j=1,\dots,2n$, be a frame adapted to the G-structure. One has $(e_j)=(e_\alpha,e_{\alpha^*})$, $\alpha=1,\dots,n$, $\alpha^*=n+1,\dots,2n$, such that if $(e_j)=(e_{\alpha'},e_{\alpha'^*})$ is another adapted frame, it is related to the previous one by the formulas

$$
e_{\alpha'} = A^\beta_{\alpha'}e_\beta + B^{\beta^*}_{\alpha'}e_{\beta^*}, \qquad e_{\alpha'^*} = A^\beta_{\alpha'}e_{\beta^*}.
$$

Hence, we can define a linear operator J of rank n, $J_p\colon T_pM\to T_pM$, $J_p(\lambda^\alpha e_\alpha + \lambda^{\alpha^*}e_{\alpha^*}) = \lambda^\alpha e_{\alpha^*}$, where (e_α,e_{α^*}) denotes an adapted frame, which is well defined because

$$
\begin{aligned}
J_p e_{\alpha'} &= J_p(A^\beta_{\alpha'}e_\beta + B^{\beta^*}_{\alpha'}e_{\beta^*})\\
&= A^\beta_{\alpha'}e_{\beta^*}\\
&= e_{\alpha'^*},\\
J_p e_{\alpha'^*} &= J_p(A^\beta_{\alpha'}e_{\beta^*}) = 0.
\end{aligned}
$$

Thus $J^2_p=0$. Furthermore, since $J_pe_{\alpha^*}=0$, (e_{α^*}) is a basis of $\operatorname{Ker}J_p$, and (e_α) is a basis of a vector subspace supplementary to $\operatorname{Ker}J_p$. That is, J_p is written in the adapted frames as $\begin{pmatrix} 0 & 0\\ I_n & 0\end{pmatrix}$.

Conversely, if there exists a C^∞ $(1,1)$ tensor field J of rank n on M such that $J^2=0$, then (e_α,e_{α^*}) is an adapted frame if (e_{α^*}) is a basis of $\operatorname{Ker}J_p$ and (e_α) is a basis of a vector subspace supplementary to $\operatorname{Ker}J_p$ (where J_p denotes the linear operator of rank n induced by J on each tangent space T_pM), in such a way that $J_pe_\alpha = e_{\alpha^*}$ and $J_pe_{\alpha^*}=0$. Consider another adapted frame $e_{\alpha'},e_{\alpha'^*}$. Then

$$
e_{\alpha'} = M^\beta_{\alpha'}e_\beta + N^{\beta^*}_{\alpha'}e_{\beta^*}, \qquad e_{\alpha'^*} = P^\beta_{\alpha'^*}e_\beta + Q^{\beta^*}_{\alpha'^*}e_{\beta^*}.
$$

Since $J_p e_{\alpha'} = e_{\alpha'^*}$, we have

$$\begin{aligned} J_p(M^{\beta}_{\alpha'} e_\beta + N^{\beta^*}_{\alpha'} e_{\beta^*}) &= M^{\beta}_{\alpha'} e_{\beta^*} \\ &= P^{\beta}_{\alpha'^*} e_\beta + Q^{\beta^*}_{\alpha'^*} e_{\beta^*}, \end{aligned}$$

so $M = Q$, $P = 0$, and the matrix of the change has the form $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$, $A \in GL(n,\mathbb{R})$.

5.2 Connections in Bundles

Problem 5.2.1. *Determine all the connections in the frame bundle $F\mathbb{R}$ over $\mathbb{R}$.*

Solution. Consider $F\mathbb{R} \approx \mathbb{R} \times GL(1,\mathbb{R}) = \mathbb{R} \times \mathbb{R}^*$ with coordinates (t,a). A connection in $F\mathbb{R}$ is given by a "horizontal subspace" $H_{(t,a)} \subset T_{(t,a)}(\mathbb{R} \times \mathbb{R}^*)$ at each point $(t,a) \in \mathbb{R} \times \mathbb{R}^*$, such that $H_{(t,a)}$ must be 1-dimensional and satisfy $\pi_*(H_{(t,a)}) = T_t\mathbb{R} \equiv \mathbb{R}$, where π stands for the projection map of $F\mathbb{R}$. Thus, we can put

$$H_{(t,a)} = \left\langle \left.\frac{\partial}{\partial t}\right|_{(t,a)} + h(t,a) \left.\frac{\partial}{\partial a}\right|_{(t,a)} \right\rangle, \qquad h \in C^\infty(\mathbb{R} \times \mathbb{R}^*).$$

Moreover, H must be invariant under right translations, i.e. if $b \in GL(1,\mathbb{R}) = \mathbb{R}^*$, then

$$\begin{aligned} R_{b*}(H_{(t,a)}) &= H_{R_b(t,a)} \\ &= H_{(t,ab)} \\ &= \left\langle \left.\frac{\partial}{\partial t}\right|_{(t,ab)} + h(t,ab) \left.\frac{\partial}{\partial a}\right|_{(t,ab)} \right\rangle. \end{aligned}$$

Since $R_b(t,a) = (t,ab)$, it is clear that

$$R_{b*}\left(\left.\frac{\partial}{\partial a}\right|_{(t,a)} \right) = b \left.\frac{\partial}{\partial a}\right|_{(t,ab)}.$$

Therefore $h(t,ab) = bh(t,a)$. Hence $h(t,a) = ah(t,1)$. Thus calling $f\colon \mathbb{R} \to \mathbb{R}$ the function given by $f(t) = h(t,1)$, the connection is given by the distribution $\mathscr{H}$ on $F\mathbb{R}$ generated by the vector field $\dfrac{\partial}{\partial t} + f(t)a\dfrac{\partial}{\partial a}$, that is,

$$\mathscr{H} = \left\langle \frac{\partial}{\partial t} + f(t)a\frac{\partial}{\partial a} \right\rangle, \qquad f \in C^\infty\mathbb{R}, \quad a \in \mathbb{R}^*.$$

Problem 5.2.2. *Let $\pi\colon P = M \times \mathbb{C}^* \to M$ be the trivial principal $\mathbb{C}^*$-bundle over the C^∞ manifold M. Prove that, in complex notation, every connection form ω_Γ on P can be written as*

$$\omega_\Gamma = z^{-1}\mathrm{d}z + \pi^*\omega, \qquad z \in \mathbb{C}^*,$$

where ω is a complex-valued differential 1-form on M; that is, $\omega \in \Lambda^1(M,\mathbb{C})$.

Solution. Let $\varphi_1, \varphi_2\colon \mathbb{R} \to \mathbb{C}^*$ be the homomorphisms $\varphi_1(t) = \mathrm{e}^t$, $\varphi_2(t) = \mathrm{e}^t\mathrm{i}$. These homomorphisms induce a basis $\{A_1, A_2\}$ of the Lie algebra of $\mathbb{C}^*$, which can be identified to $\mathbb{C}$ itself by $1 \mapsto A_1$, $\mathrm{i} \to A_2$. The fundamental vector fields attached to these vectors are:

$$A_1^* = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad A_2^* = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

For example, let us compute A_1^*. The flow generating A_1^* is

$$\begin{aligned}\psi_t(z) &= \varphi_1(t)z \\ &= \mathrm{e}^t(x + y\mathrm{i}) \\ &= \mathrm{e}^t x + \mathrm{e}^t y\mathrm{i} \\ &= \widetilde{x}_t + \widetilde{y}_t\mathrm{i}.\end{aligned}$$

Hence

$$A_1^*(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{x}_t = x, \qquad A_1^*(y) = \left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{y}_t = y.$$

Thus, by using the previous identification, ω_Γ can be written as

$$\omega_\Gamma = \eta^1 + \eta^2\mathrm{i},$$

where η^1, η^2 are differential 1-forms on P. By imposing that $\omega_\Gamma(A_k^*) = A_k$, $k = 1,2$, we obtain $\eta^j(A_k^*) = \delta_k^j$. Hence by using a coordinate system (x^h) on M, the forms η^1, η^2 can be written as

$$\eta^1 = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + f_h^1\,\mathrm{d}x^h, \qquad \eta^2 = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2} + f_h^2\,\mathrm{d}x^h,$$

where $f_h^1, f_h^2 \in C^\infty P$. We remark that

$$z^{-1}\mathrm{d}z = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}\mathrm{i}.$$

Hence

$$\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} \quad \text{and} \quad \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$$

are left-invariant differential forms on $\mathbb{C}^*$.

Moreover, as $\mathbb{C}^*$ is commutative, the condition $R_z^*\omega_\Gamma = \mathrm{Ad}_{z^{-1}} \circ \omega_\Gamma$ simply means that ω_Γ is right-invariant. Accordingly, this condition holds if and only if the functions f_h^1, f_h^2 are $\mathbb{C}^*$-invariant; that is, if and only if, $f_h^1, f_h^2 \in C^\infty M$. Hence $\omega^1 = f_h^1 \,\mathrm{d}x^h$, $\omega^2 = f_h^2 \,\mathrm{d}x^h$, are differential forms on M, and by setting $\omega = \omega^1 + \omega^2 \mathrm{i}$, we conclude.

Problem 5.2.3. *Consider the trivial bundle* $\pi\colon P = M \times U(1) \to M$. *Parametrize the fibre* $U(1)$ *as* $\exp(\mathrm{i}\alpha)$, $0 \leqslant \alpha \leqslant 2\pi$. *If* (q^j), $j = 1, \dots, n = \dim M$, *are local coordinates on* M, *then* (q^j, α) *are local coordinates on* P. *Let* $p\colon T^*M \to M$ *be the canonical projection of the cotangent bundle. Prove*:

(1) *For every connection form* ω_Γ *on* P *there exists a unique differential* 1*-form* ω *on* M *such that* $\omega_\Gamma = (\mathrm{d}\alpha + \pi^*\omega) \otimes A$, *where* $A \in \mathfrak{u}(1)$ *is the invariant vector field defined by the homomorphism* $\mathbb{R} \to U(1)$, $t \mapsto \exp(\mathrm{i}t)$.

(2) *Every automorphism* $\Phi\colon P \to P$ *can be described locally as* $\Phi(x,\alpha) = (\phi(x), \alpha + \psi(x))$, *where* $\phi\colon M \to M$ *is a diffeomorphism and* $\psi\colon M \to \mathbb{R}$ *is a differentiable map.*

(3) $(\Phi^{-1})^*\omega_\Gamma$ *is another connection form* $\omega_{\Gamma'}$ *on* P. *Set* $\omega_{\Gamma'} = (\mathrm{d}\alpha + \pi^*\omega') \otimes A$ *and compute* ω'.

(4) *There exists a unique diffeomorphism* $\widetilde{\Phi}\colon T^*M \to T^*M$ *such that*:

(i) $p \circ \widetilde{\Phi} = \phi \circ p$.

(ii) *If the differential forms* ω, ω' *on* M *are related as in* (3), *then* $\widetilde{\Phi} \circ \omega = \omega'$. *Here,* ω, ω' *are viewed as sections of the cotangent bundle.*

(5) *If* $\Psi\colon P \to P$ *is another automorphism, then* $(\Psi \circ \Phi)^{\sim} = \widetilde{\Psi} \circ \widetilde{\Phi}$. (*This property justifies the exponent* -1 *in defining* Γ' *in* (3).)

Solution. (1) As A is a basis of $\mathfrak{u}(1)$, it is clear that every connection form can be written as $\omega_\Gamma = \eta \otimes A$ for some differential 1-form η on P. Moreover, the fundamental vector field associated to A is readily seen to be $A^* = \partial/\partial\alpha$ and from the very definition of a connection form it must hold that $\omega_\Gamma(A^*) = \eta(A^*)A = A$. Hence $\eta(\partial/\partial\alpha) = 1$, and accordingly, $\eta = \mathrm{d}\alpha + f_j \mathrm{d}q^j$, for certain functions $f_j \in C^\infty P$. We now impose

$$R_z^*\omega_\Gamma = \mathrm{Ad}_{z^{-1}} \circ \omega_\Gamma, \qquad \forall z \in U(1), \tag{$\star$}$$

that is, the second property of a connection form. As $U(1)$ is Abelian, the adjoint representation is trivial and hence $(\star)$ simply means that η is invariant under right translations. As the forms $\mathrm{d}\alpha$ and $\mathrm{d}q^j$ are invariant, we conclude that η is invariant if and only if the functions f_j are invariant; that is, if each f_j does not depend on α, thus projecting to a function on M. Hence $\omega = f_j \mathrm{d}q^j$.

(2) A diffeomorphism $\Phi\colon P \to P$ is a principal bundle automorphism if Φ is equivariant; i.e. $\Phi(u \cdot z) = \Phi(u) \cdot z$, for all $u \in P$, for all $z = \exp(\mathrm{i}\alpha) \in U(1)$. We have $\Phi(x,w) = (\xi(x,w), \varphi(x,w))$, $(x,w) \in P$, where $\xi\colon P \to M$, $\varphi\colon P \to U(1)$ are the components of Φ. By imposing the condition of equivariance, we obtain $\Phi(x,wz) = \Phi((x,w)\cdot z) = \Phi(x,w)z$; that is,

$$(\xi(x,wz), \varphi(x,wz)) = (\xi(x,w), \varphi(x,w)z).$$

Letting $w = 1$, we have $\xi(x,z) = \xi(x,1)$ and $\varphi(x,z) = \varphi(x,1)z$. Hence ξ factors through π by means of a differentiable map $\phi \colon M \to M$ as follows: $\xi = \phi \circ \pi$, and, locally, we have $\varphi(x,1) = \exp(\mathrm{i}\psi(x))$. Then,

$$\begin{aligned}\varphi(x,z) &= \exp(\mathrm{i}\psi(x))\exp(\mathrm{i}\alpha)\\ &= \exp\big(\mathrm{i}(\alpha + \psi(x))\big).\end{aligned}$$

(3) As a simple computation shows, we have

$$\boldsymbol{\Phi}^{-1}(x,\alpha) = (\phi^{-1}(x), \alpha - (\psi \circ \phi^{-1})(x)).$$

Thus

$$\begin{aligned}(\boldsymbol{\Phi}^{-1})^*\omega_\Gamma &= \big((\boldsymbol{\Phi}^{-1})^*(\mathrm{d}\alpha + \pi^*\omega)\big) \otimes A\\ &= \big(\mathrm{d}\alpha - \mathrm{d}(\psi \circ \phi^{-1} \circ \pi) + \pi^*(\phi^{-1})^*\omega\big) \otimes A.\end{aligned}$$

Hence $\omega' = (\phi^{-1})^*\omega - \mathrm{d}\left(\psi \circ \phi^{-1}\right)$.

(4) Given a covector $w \in T_x^*M$, let ω be a differential 1-form on M such that $\omega(x) = w$. Then, from conditions (i), (ii) we obtain

$$\begin{aligned}\widetilde{\boldsymbol{\Phi}}(w) &= \widetilde{\boldsymbol{\Phi}}(\omega(x))\\ &= (\widetilde{\boldsymbol{\Phi}} \circ \omega)(x)\\ &= \omega'(x)\\ &= (\phi^{-1})^*\omega(x) - (\mathrm{d}(\psi \circ \phi^{-1}))_{\phi(x)}\\ &= (\phi^{-1})^*w - (\mathrm{d}(\psi \circ \phi^{-1}))_{\phi(x)},\end{aligned}$$

thus proving the existence and uniqueness of $\widetilde{\boldsymbol{\Phi}}$.

(5) Set $\omega_{\Gamma'} = (\boldsymbol{\Phi}^{-1})^*\omega_\Gamma$, $\omega_{\Gamma''} = (\boldsymbol{\Psi}^{-1})^*\omega_{\Gamma'}$. Then,

$$\begin{aligned}\omega_{\Gamma''} &= (\boldsymbol{\Psi}^{-1})^*(\boldsymbol{\Phi}^{-1})^*\omega_\Gamma\\ &= ((\boldsymbol{\Psi} \circ \boldsymbol{\Phi})^{-1})^*\omega_\Gamma.\end{aligned}$$

Hence $(\widetilde{\boldsymbol{\Psi}} \circ \widetilde{\boldsymbol{\Phi}}) \circ \omega = \omega''$ and $(\boldsymbol{\Psi} \circ \boldsymbol{\Phi})\tilde{} \circ \omega = \omega''$, so that $\widetilde{\boldsymbol{\Psi}} \circ \widetilde{\boldsymbol{\Phi}}$ and $(\boldsymbol{\Psi} \circ \boldsymbol{\Phi})\tilde{}$ satisfy the condition (ii) in (4). Moreover, we have

$$\begin{aligned}p \circ (\widetilde{\boldsymbol{\Psi}} \circ \widetilde{\boldsymbol{\Phi}}) &= (p \circ \widetilde{\boldsymbol{\Psi}}) \circ \widetilde{\boldsymbol{\Phi}}\\ &= (\psi \circ p) \circ \widetilde{\boldsymbol{\Phi}}\\ &= \psi \circ (p \circ \widetilde{\boldsymbol{\Phi}})\\ &= \psi \circ (\phi \circ p) = (\psi \circ \phi) \circ p.\end{aligned}$$

Hence the condition (i) in (4) holds.

Problem 5.2.4. *Let $z^k = x^k + \mathrm{i}y^k$, $0 \leqslant k \leqslant n$, be the standard coordinates on $\mathbb{C}^{n+1}$. Prove that the 1-form*

$$\omega = \sum_{k=0}^{n} (-y^k\,\mathrm{d}x^k + x^k\,\mathrm{d}y^k)|_{S^{2n+1}}$$

is a connection form on the principal $U(1)$-bundle $p\colon S^{2n+1} \to \mathbb{C}P^n$, where we identify the Lie algebra of $U(1)$ to $\mathbb{R}$ via the isomorphism $\lambda \mapsto \lambda(\partial/\partial\alpha)$, where α stands for the angle function on $U(1)$.

Solution. According to the definition of a connection form we must check the following properties:

(1) $\omega\left(\left(\lambda\dfrac{\partial}{\partial\alpha}\right)^*\right) = \lambda$, for all $\lambda \in \mathbb{R}$.

(2) $R_g^*\,\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$, for all $g \in U(1)$.

As the coordinates of the point $z \cdot \exp(\mathrm{i}\lambda\alpha)$ are

$$(x^k + \mathrm{i}y^k)(z \cdot \exp(\mathrm{i}\lambda\alpha)) = x^k\cos(\lambda\alpha) - y^k\sin(\lambda\alpha) + \mathrm{i}(x^k\sin(\lambda\alpha) + y^k\cos(\lambda\alpha)),$$

$0 \leqslant k \leqslant n$, we have

$$\left(\lambda\frac{\partial}{\partial\alpha}\right)^* = \lambda\sum_{k=0}^{n}\left(-y^k\frac{\partial}{\partial x^k} + x^k\frac{\partial}{\partial y^k}\right).$$

Hence

$$\begin{aligned}\omega\left(\left(\lambda\frac{\partial}{\partial\alpha}\right)^*\right) &= \sum_{k=0}^{n}(-y^k\mathrm{d}x^k + x^k\mathrm{d}y^k)\left(\lambda\sum_{l=0}^{n}\left(-y^l\frac{\partial}{\partial x^l} + x^l\frac{\partial}{\partial y^l}\right)\right)\\ &= \lambda\sum_{k=0}^{n}((y^k)^2 + (x^k)^2) = \lambda,\end{aligned}$$

at every point of the sphere, thus proving (1). As for (2), we first remark that the adjoint representation is trivial since $U(1)$ is Abelian, so (2) simply tells us that ω is invariant under right translations. In order to prove this, we note that

$$-y^k\mathrm{d}x^k + x^k\mathrm{d}y^k = ((x^k)^2 + (y^k)^2)\mathrm{d}\left(\arctan\frac{y^k}{x^k}\right),$$

and that $R_{\exp(\alpha\mathrm{i})}$ leaves the quadratic form $(x^k)^2 + (y^k)^2$ invariant. Working in polar coordinates we thus obtain

$$\begin{aligned}R^*_{\exp(\alpha\mathrm{i})}\left(-y^k\mathrm{d}x^k + x^k\mathrm{d}y^k\right) &= \left((x^k)^2 + (y^k)^2\right)R^*_{\exp(\alpha\mathrm{i})}\mathrm{d}\left(\arctan\frac{y^k}{x^k}\right)\\ &= \left((x^k)^2 + (y^k)^2\right)\mathrm{d}\left(\arctan\frac{y^k}{x^k} + \alpha\right)\end{aligned}$$

$$= \left((x^k)^2 + (y^k)^2\right) \mathrm{d}\left(\arctan \frac{y^k}{x^k}\right)$$
$$= -y^k \mathrm{d}x^k + x^k \mathrm{d}y^k.$$

5.3 Characteristic Classes

Problem 5.3.1. *Consider the trivial principal bundle* $(\mathbb{R}^3 - \{0\}) \times U(1)$ *over* $\mathbb{R}^3 - \{0\}$. *Then, for the connection with connection form described* (*on the open subset* $\mathbb{R}^3 - \{(0,0,z), z \geqslant 0\}$ *of the base manifold*) *by the* $\mathfrak{u}(1)$-*valued differential* 1-*form*

$$A_2 = \frac{\mathrm{i}}{2r(z-r)}(x\,\mathrm{d}y - y\,\mathrm{d}x), \qquad (\star)$$

where $r^2 = x^2 + y^2 + z^2$:

(1) *Calculate the curvature form* F *of the connection in terms of* A_2.

(2) *Write* A_2 *in spherical coordinates* (r, θ, φ), *given by*

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \qquad \theta \in [0,\pi], \quad \varphi \in [0, 2\pi),$$

and calculate

$$A_1 = A_2 + \gamma^{-1}\mathrm{d}\gamma,$$

γ *being the* $U(1)$-*valued function on* $\mathbb{R}^3 - z$-*axis defined by* $\gamma(p) = \mathrm{e}^{\varphi(p)\mathrm{i}}$.

(3) A_1 *and* A_2 *furnish well-defined differential forms on* $U_1 = S^2 - S$ *and* $U_2 = S^2 - N$, *respectively, where* N, S *denote the north and south pole.*

Consider the complex Hopf bundle H *studied in* Problems 5.1.2, 5.1.3 *and take real coordinates* $u^1, \dots, u^4$ *on* $\mathbb{C}^2 \equiv \mathbb{R}^4$, *such that*

$$S^3 = \left\{ \left(z_1 = u^1 + \mathrm{i}u^2, z_2 = u^3 + \mathrm{i}u^4\right) \in \mathbb{C}^2 \,:\, |z_1|^2 + |z_2|^2 = 1 \right\}.$$

Prove that

$$\omega = \mathrm{i}(u^1\mathrm{d}u^2 - u^2\mathrm{d}u^1 + u^3\mathrm{d}u^4 - u^4\mathrm{d}u^3)$$

is a connection form on the bundle. Show that $\sigma_1^*\omega = A_1$ *and* $\sigma_2^*\omega = A_2$, *where* σ_k *is the local section associated to the trivialization on* U_k, $k = 1, 2$, (*see* Problem 5.1.3) *by means of* $\sigma_k(p) = f_{k,p}^{-1}(1)$, *where* $1 \in U(1)$ *is the identity element. That is,* A_1 *and* A_2 *are local representatives of the connection in* H *with connection form* ω.

(4) *Compute the* (*only*) *Chern number of the bundle* H.

REMARK. *The above bundle is a particular case of a construction named in Physics a* Dirac magnetic monopole bundle. *Each of the given differential forms* A_1, A_2 *is called a* gauge potential *of a magnetic monopole at the origin of* $\mathbb{R}^3$, *the transformation in* (2) *is called a* gauge transformation, *and* F *is called the* field strength. *The general construction depends on an integer* n, *and the bundle of the problem corresponds to* $n = 1$.

Solution. (1)

$$\begin{aligned} F &= \mathrm{d}A_2 + A_2 \wedge A_2 \\ &= \mathrm{d}A_2 \\ &= \frac{\mathrm{i}}{2r^3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y). \end{aligned}$$

(2) Since $\gamma^{-1}\mathrm{d}\gamma = \mathrm{i}\,\mathrm{d}\varphi$, one has that

$$A_1 = \frac{\mathrm{i}}{2}(1-\cos\theta)\mathrm{d}\varphi, \qquad A_2 = \frac{\mathrm{i}}{2}(-1-\cos\theta)\mathrm{d}\varphi.$$

(3) Since

$$\mathfrak{u}(1) = \left\{X \in \mathfrak{gl}(1,\mathbb{C}) = \mathbb{C} : {}^tX + \bar{X} = 0\right\} = \mathbb{R}\mathrm{i},$$

one can identify $\mathfrak{u}(1)$ with the purely imaginary complex numbers. The fundamental vector field $X^* \in \mathfrak{X}(S^3)$ corresponding to $X \in \mathfrak{u}(1)$ is (see Problem 5.1.4) $X^*_{(z_1,z_2)} = j_{(z_1,z_2)*}X_1$, $(z_1,z_2) \in S^3$. According to the parametrization

$$(z_1,z_2) = \left(\cos\frac{1}{2}\theta\,\mathrm{e}^{\psi_1\mathrm{i}}, \sin\frac{1}{2}\theta\,\mathrm{e}^{\psi_2\mathrm{i}}\right)$$

of S^3 and the fibre action of S^1 by $\mathrm{e}^{\alpha\mathrm{i}}$, this action corresponds to (the same) changes in the parameters ψ_1 and ψ_2. In fact,

$$\begin{aligned} j_{(z_1,z_2)}(\mathrm{e}^{\alpha\mathrm{i}}) &= R_{\mathrm{e}^{\alpha\mathrm{i}}}(z_1,z_2) \\ &= (z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}}) \\ &= \left(\cos\frac{1}{2}\theta\,\mathrm{e}^{(\psi_1+\alpha)\mathrm{i}}, \sin\frac{1}{2}\theta\,\mathrm{e}^{(\psi_2+\alpha)\mathrm{i}}\right). \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial\psi_1} &= \frac{\partial u^1}{\partial\psi_1}\frac{\partial}{\partial u^1} + \frac{\partial u^2}{\partial\psi_1}\frac{\partial}{\partial u^2} \\ &= -\cos\frac{1}{2}\theta\,\sin\psi_1\frac{\partial}{\partial u^1} + \cos\frac{1}{2}\theta\,\cos\psi_1\frac{\partial}{\partial u^1} \\ &= -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \end{aligned}$$

and similarly

$$\frac{\partial}{\partial\psi_2} = -u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}.$$

So, we can take the vector

$$X^*_{(z_1,z_2)} = a\left(\frac{\partial}{\partial\psi_1} + \frac{\partial}{\partial\psi_2}\right)$$

$$= a\left(-u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}\right),$$

which is clearly tangent to S^3, as the tangent vector to the fibre at a generic point $(z_1, z_2) \in S^3$, image under $j_{(z_1,z_2)*}$ of $X \equiv \mathrm{i}a \in \mathfrak{u}(1)$. The vector field X^* is the fundamental vector field corresponding to X. In fact, since the Jacobian map of the map $\tau_{\mathrm{e}^{\alpha\mathrm{i}}} : (z_1, z_2) \mapsto (z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})$, is given, in terms of the real coordinates u^1, u^2, u^2, u^4, by

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

we deduce

$$\begin{aligned}
\tau_{\mathrm{e}^{\alpha\mathrm{i}}*}(X^*_{(z_1,z_2)}) = a\Bigg\{ & -(u^2(z_1)\,\cos\alpha + u^1(z_1)\,\sin\alpha)\;\frac{\partial}{\partial u^1}\bigg|_{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})} \\
& -(u^2(z_1)\,\sin\alpha - u^1(z_1)\,\cos\alpha)\;\frac{\partial}{\partial u^2}\bigg|_{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})} \\
& -(u^4(z_2)\,\cos\alpha + u^3(z_2)\,\sin\alpha)\;\frac{\partial}{\partial u^3}\bigg|_{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})} \\
& -(u^4(z_2)\,\sin\alpha - u^3(z_2)\,\cos\alpha)\;\frac{\partial}{\partial u^4}\bigg|_{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})}\Bigg\} \\
= a & \left(-u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4}\right)_{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})} \\
= X^* & _{(z_1\mathrm{e}^{\alpha\mathrm{i}}, z_2\mathrm{e}^{\alpha\mathrm{i}})}.
\end{aligned}$$

Next, we consider the properties of the form ω. It is clearly C^∞ and takes on S^3 imaginary values, which can be identified with elements of $\mathfrak{u}(1)$, as we have seen: It is immediate that

$$\omega(X^*_{(z_1,z_2)}) \equiv a\mathrm{i} \in \mathfrak{u}(1).$$

Moreover, we have

$$\begin{aligned}
R^*_{\mathrm{e}^{\alpha\mathrm{i}}}\,\omega = \mathrm{i}(& u^1\cos\alpha - u^2\sin\alpha)(-\sin\alpha\,\mathrm{d}u^1 - \cos\alpha\,\mathrm{d}u^2) \\
& -(u^1\sin\alpha + u^2\cos\alpha)(\cos\alpha\,\mathrm{d}u^1 - \sin\alpha\,\mathrm{d}u^2) \\
& +(u^3\cos\alpha - u^4\sin\alpha)(\sin\alpha\,\mathrm{d}u^3 + \cos\alpha\,\mathrm{d}u^4) \\
& -(u^3\sin\alpha + u^4\cos\alpha)(\cos\alpha\,\mathrm{d}u^3 - \sin\alpha\,\mathrm{d}u^4) \\
= \omega, &
\end{aligned}$$

and also, trivially, $\mathrm{Ad}_{\mathrm{e}^{-\alpha\mathrm{i}}}\,\omega = \omega$, hence

$$R^*_{e^{\alpha i}}\omega = \mathrm{Ad}_{e^{-\alpha i}}\,\omega.$$

Finally, we prove that the connection form ω has local representatives A_1 on U_1 and A_2 on U_2. In fact, the local sections corresponding to the trivializations over U_k are, respectively,

$$\sigma_1(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \left(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\, e^{\varphi i}\right), \qquad \theta < \pi,$$

$$\sigma_2(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \left(\cos\frac{1}{2}\theta\, e^{-\varphi i}, \sin\frac{1}{2}\theta\right), \qquad 0 < \theta.$$

Thus, it is immediate that the section σ_1 is given in terms of the real coordinates $u^1,\dots,u^4$ by

$$(u^1,u^2,u^3,u^4) = \left(\cos\frac{1}{2}\theta, 0, \sin\frac{1}{2}\theta\cos\varphi, \sin\frac{1}{2}\theta\sin\varphi\right).$$

Substituting in the expression for ω in the statement we easily obtain $\sigma_1^*\omega = A_1$. One proceeds similarly to obtain $\sigma_2^*\omega = A_2$.

(4)

$$\begin{aligned} c_{(1)}(H) &= \frac{i}{2\pi}\int_{S^2} F \\ &= -\frac{1}{4\pi}\int_{S^2} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy \\ &= -\frac{1}{4\pi}\int_{S^2} \sin\theta\, d\theta\wedge d\varphi = -1. \end{aligned}$$

Problem 5.3.2. (1) *Identify $SU(2)$ to the unit sphere S^3 in $\mathbb{H}$ and prove that there is an isomorphism $\mathfrak{su}(2) \approx \mathbb{H}'$ of the Lie algebra of $SU(2)$ onto the vector space of purely imaginary quaternions endowed with the Lie algebra structure given by $[a,b] = ab - ba$, for $a,b \in \mathbb{H}'$.*

(2) *Any connection in the principal $SU(2)$-bundle $P = \mathbb{R}^4 \times SU(2)$ over $\mathbb{R}^4$ can be expressed, by (1), in terms of an $\mathbb{H}'$-valued differential 1-form on $\mathbb{R}^4$. Let $q \in \mathbb{H}$ arbitrarily fixed, and let*

$$A_{\lambda,q}(x) = \mathrm{Im}\,\frac{(x-q)\,d\bar{x}}{\lambda^2 + |x-q|^2}, \qquad x \in \mathbb{H}, \quad 0 < \lambda \in \mathbb{R},$$

be an $\mathbb{H}'$-valued connection form. Prove that the curvature form of $A_{\lambda,q}$ is given by

$$F_{\lambda,q}(x) = \frac{\lambda^2\, dx\wedge d\bar{x}}{(\lambda^2 + |x-q|^2)^2}. \qquad (\star)$$

Solution. (1) We first remark that $(\mathbb{H}', [\,,\,])$ is a Lie algebra as $a \in \mathbb{H}'$ if and only if $a + \bar{a} = 0$, and for every $a, b \in \mathbb{H}'$ we have

$$\begin{aligned}
[a,b] + \overline{[a,b]} &= (ab - ba) + \overline{(ab - ba)} \\
&= (ab - ba) + (\bar{b}\bar{a} - \bar{a}\bar{b}) \\
&= (ab - ba) + ((-b)(-a) - (-a)(-b)) = 0.
\end{aligned}$$

Any quaternion can be written as

$$\begin{aligned}
q &= a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k} \\
&= (a_0 + a_1\mathrm{i}) + (a_2 + a_3\mathrm{i})\mathrm{j} \\
&= z_1 + z_2\,\mathrm{j},
\end{aligned}$$

with the rule $\mathrm{j}z = \bar{z}\mathrm{j}$, and, hence, q can be identified to the matrix

$$A_q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \qquad (\star\star)$$

In fact, for two quaternions q, q', we have

$$\begin{aligned}
A_q A_{q'} &= \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} z_1' & z_2' \\ -\bar{z}_2' & \bar{z}_1' \end{pmatrix} \\
&= \begin{pmatrix} z_1 z_1' - z_2 \bar{z}_2' & z_1 z_2' + z_2 \bar{z}_1' \\ -\bar{z}_1 \bar{z}_2' - \bar{z}_2 z_1' & \bar{z}_1 \bar{z}_1' - \bar{z}_2 z_2' \end{pmatrix} \\
&= A_{qq'},
\end{aligned}$$

where the last equality is immediate from the expression for the product of q and q'. Moreover, we have

$$SU(2) \equiv \left\{ A \in GL(2,\mathbb{C}) : A = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},\ \det A = |z_1|^2 + |z_2|^2 = 1 \right\},$$

so that $SU(2)$ can be identified to the quaternions of norm 1, which can be viewed as the 3-sphere in $\mathbb{H} \equiv \mathbb{R}^4$. The Lie algebra of the Lie group S^3 (see Problem 4.7.3) can be identified to the tangent space at the identity $(1,0,0,0) \in S^3$, that is, to the subspace of $\mathbb{R}^4$ orthogonal to the identity $1 \in S^3$, which is the vector space of purely imaginary quaternions $\mathbb{H}'$. The associated matrices $(\star\star)$ are thus written as

$$\begin{pmatrix} \mathrm{i}a & z_2 \\ -\bar{z}_2 & -\mathrm{i}a \end{pmatrix}, \qquad a \in \mathbb{R}.$$

Now, it is easily seen that these are exactly the matrices of $\mathfrak{su}(2)$. Finally, it is easily checked that the matrices

$$B_1 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad B_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B_3 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

are a basis of $\mathfrak{su}(2)$ (remark that $-2\mathrm{i}B_r$, $1 \leqslant r \leqslant 3$, are the *Pauli matrices)* such that

$$[B_1, B_2] = B_3, \qquad [B_2, B_3] = B_1, \qquad [B_3, B_1] = B_2.$$

Similarly, $b_1 = \frac{1}{2}\mathrm{i}$, $b_2 = \frac{1}{2}\mathrm{j}$, $b_3 = \frac{1}{2}\mathrm{k}$, is a basis of $\mathbb{H}'$ such that

$$[b_1, b_2] = b_3, \qquad [b_2, b_3] = b_1, \qquad [b_3, b_1] = b_2,$$

and we conclude. Notice that this isomorphism permits us to consider the $\mathfrak{su}(2)$-valued differential forms as $\mathbb{H}'$-valued differential forms.

(2) The quaternion differential is defined by

$$\begin{aligned} \mathrm{d}x = \mathrm{d}(x^0 + x^1\mathrm{i} + x^2\mathrm{j} + x^3\mathrm{k}) &= \mathrm{d}x^0 + \mathrm{d}x^1\,\mathrm{i} + \mathrm{d}x^2\,\mathrm{j} + \mathrm{d}x^3\,\mathrm{k}, \\ \mathrm{d}\bar{x} &= \mathrm{d}x^0 - \mathrm{d}x^1\,\mathrm{i} - \mathrm{d}x^2\,\mathrm{j} - \mathrm{d}x^3\,\mathrm{k}. \end{aligned}$$

We also use the following properties: If ω, η are two $\mathbb{H}$-valued differential forms and f is an $\mathbb{H}$-valued function, then

$$\omega f \wedge \eta = \omega \wedge f\eta. \tag{†}$$

Every $\mathbb{H}$-valued differential form ω can be decomposed as $\omega = \omega^0 + \omega'$, where ω^0 is an ordinary differential form and

$$\omega' = \mathrm{Im}\,\omega = \frac{1}{2}(\omega - \bar{\omega})$$

is an $\mathbb{H}'$-valued differential form. Hence, if the degree of ω is odd, then we have $\omega \wedge \omega = \omega' \wedge \omega'$, as $\omega^0 \wedge \omega^0 = 0$, $\omega^0 \wedge \omega' + \omega' \wedge \omega^0 = 0$. Therefore

$$\mathrm{Im}\,(\omega \wedge \omega) = (\mathrm{Im}\,\omega) \wedge (\mathrm{Im}\,\omega). \tag{††}$$

Setting

$$f_{\lambda,q}(x) = \frac{x - q}{\lambda^2 + |x - q|^2},$$

we have $A_{\lambda,q}(x) = \mathrm{Im}\{f_{\lambda,q}(x)\mathrm{d}\bar{x}\}$, and by using (††) we obtain

$$\begin{aligned} F_{\lambda,q}(x) &= \mathrm{d}A_{\lambda,q}(x) + A_{\lambda,q}(x) \wedge A_{\lambda,q}(x) \\ &= \mathrm{Im}\,\{\mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} + f_{\lambda,q}(x)\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\mathrm{d}\bar{x}\}. \end{aligned} \tag{‡}$$

Moreover, taking into account that $|x - q|^2 = (\bar{x} - \bar{q})(x - q)$, we have

$$\mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} = \left(\frac{\mathrm{d}x}{\lambda^2 + |x - q|^2} - (x - q)\frac{\mathrm{d}\bar{x}(x - q) + (\bar{x} - \bar{q})\mathrm{d}x}{(\lambda^2 + |x - q|^2)^2} \right) \wedge \mathrm{d}\bar{x}$$

$$= \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{\lambda^2 + |x-q|^2} - \frac{(x-q)\mathrm{d}\bar{x}(x-q) \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} - \frac{(x-q)(\bar{x}-\bar{q})\mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2},$$

and by using the formula (†), we obtain

$$\begin{aligned}
-\frac{(x-q)\mathrm{d}\bar{x}(x-q) \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} &= -\frac{(x-q)\mathrm{d}\bar{x} \wedge (x-q)\mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} \\
&= -f_{\lambda,q}(x)\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\mathrm{d}\bar{x}.
\end{aligned}$$

Hence, substituting into (‡), we obtain

$$\begin{aligned}
F_{\lambda,q}(x) &= \operatorname{Im}\{\mathrm{d}f_{\lambda,q}(x) \wedge \mathrm{d}\bar{x} + f_{\lambda,q}(x)\mathrm{d}\bar{x} \wedge f_{\lambda,q}(x)\mathrm{d}\bar{x}\} \\
&= \operatorname{Im}\left(\frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{\lambda^2 + |x-q|^2} - \frac{|x-q|^2 \mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2}\right) \\
&= \operatorname{Im}\frac{\lambda^2 \mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2} \\
&= \frac{\lambda^2 \mathrm{d}x \wedge \mathrm{d}\bar{x}}{(\lambda^2 + |x-q|^2)^2},
\end{aligned}$$

for it is immediate that $\mathrm{d}x \wedge \mathrm{d}\bar{x}$ is purely imaginary.

Problem 5.3.3. *Consider the quaternionic Hopf bundle* $\pi_{\mathbb{H}}\colon S^7 \to S^4$ (*see* Problem 5.1.2).

(1) *Prove that* ω *defined by*

$$\omega_{(a,b)}(X) = \operatorname{Im}(\bar{a}X_1 - \bar{X}_2 b),$$

where

$$(a,b) \in S^7 = \left\{(x,y) \in \mathbb{H} \times \mathbb{H} : |x|^2 + |y|^2 = 1\right\}$$

and

$$X = (X_1, X_2) \in T_{(a,b)}S^7 \hookrightarrow T_{(a,b)}(\mathbb{H} \times \mathbb{H}),$$

is a connection in the Hopf bundle.

(2) *Let* $N = (0,0,0,0,1) \in S^4$, $S = (0,0,0,0,-1) \in S^4$. *Consider the maps* (*inverse of the stereographic projections*)

$$\varphi_N^{-1}\colon \mathbb{H} \to S^4 - \{(0,0,0,0,1)\}, \qquad \varphi_S^{-1}\colon \mathbb{H} \to S^4 - \{(0,0,0,0,-1)\},$$

given by

$$\varphi_N^{-1}(x) = \left(\frac{2x}{x\bar{x}+1}, \frac{x\bar{x}-1}{x\bar{x}+1}\right), \qquad \varphi_S^{-1}(x) = \left(\frac{2x}{x\bar{x}+1}, \frac{1-x\bar{x}}{x\bar{x}+1}\right),$$

respectively. Denoting $U_N = S^4 - \{N\}$, $U_S = S^4 - \{S\}$, *we construct trivializations of* $\pi_{\mathbb{H}}\colon S^7 \to S^4$,

$$\psi_N \colon \pi_{\mathbb{H}}^{-1}(U_N) \to \mathbb{H} \times S^3, \qquad \psi_S \colon \pi_{\mathbb{H}}^{-1}(U_S) \to \mathbb{H} \times S^3,$$

by

$$\psi_N(x,y) = \left(\varphi_N(\pi_{\mathbb{H}}(x,y)), \frac{y}{|y|}\right), \qquad \psi_S(x,y) = \left(\varphi_S(\pi_{\mathbb{H}}(x,y)), \frac{x}{|x|}\right).$$

Consider the sections $\sigma_N, \sigma_S \colon \mathbb{H} \to S^7$ given by

$$\sigma_N(x) = \psi_N^{-1}(x,1), \qquad \sigma_S(x) = \psi_S^{-1}(x,1).$$

Let $r^2 = x\bar{x}$, $x \in \mathbb{H}$, and $\gamma \colon \mathbb{H} - \{0\} \to S^3$, $\gamma(x) = r^{-1}x$.

Prove that the local expressions of ω in terms of σ_N and σ_S are

$$\sigma_N^*\omega = -\frac{r^2}{1+r^2}\gamma^{-1}\mathrm{d}\gamma, \qquad \sigma_S^*\omega = \frac{r^2}{1+r^2}\gamma^{-1}\mathrm{d}\gamma.$$

Solution. (1) We have

$$\begin{aligned}\omega_{(a,b)}(X) &= \frac{1}{2}\left((\bar{a}X_1 - \bar{X}_2 b) - \overline{(\bar{a}X_1 - \bar{X}_2 b)}\right)\\ &= \frac{1}{2}(\bar{a}X_1 - \bar{X}_2 b - \bar{X}_1 a + \bar{b}X_2) \in \mathbb{H}.\end{aligned}$$

Since the Lie algebra of S^3 is identified to the purely imaginary quaternions, it follows that $\omega_{(a,b)}(X) \in T_1 S^3$. That is, ω takes its values in the Lie algebra of the Lie group $S^3 \hookrightarrow \mathbb{H}$.

The action of S^3 on S^7 is given by $R_z(a,b) = (az, bz)$ (see Problem 5.1.2). Then

$$\begin{aligned}(R_z^*\omega)_{(a,b)}(X) &= \omega_{(az,bz)}(R_{z*}X)\\ &= \omega_{(az,bz)}(X_1 z, X_2 z)\\ &= \frac{1}{2}\left(\bar{z}\bar{a}X_1 z - \bar{z}\bar{X}_2 bz - \bar{z}\bar{X}_1 az + \bar{z}\bar{b}X_2 z\right)\\ &= \bar{z}\,\omega_{(a,b)}(X)z\\ &= z^{-1}\omega_{(a,b)}(X)z\\ &= \left(\mathrm{Ad}_{z^{-1}} \circ \omega\right)(X).\end{aligned}$$

On the other hand, the fundamental vector field A^* corresponding to $A \in T_1 S^3$ is given by $A^*_{(a,b)} = j_{(a,b)*}A$, where

$$j_{(a,b)} \colon S^3 \to S^7, \qquad z \mapsto R_z(a,b) = (az, bz).$$

Hence

$$\begin{aligned}A^*_{(a,b)} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(a(1+At), b(1+At)\big)\\ &= (aA, bA).\end{aligned}$$

We thus have

$$\begin{aligned}\omega_{(a,b)}(A^*_{(a,b)}) &= \omega_{(a,b)}(aA, bA)\\ &= \frac{1}{2}(\bar{a}aA - \bar{A}\bar{b}b - \bar{A}\bar{a}a + \bar{b}bA)\\ &= (\bar{a}a + \bar{b}b)A\\ &= A,\end{aligned}$$

since $\bar{A} = -A$. We have thus proved that ω is in fact a connection in the bundle $\pi_{\mathbb{H}}\colon S^7 \to S^4$.

(2) In order to obtain the explicit expressions of σ_N and σ_S, we first suppose that $(u,v) \in S^7$ satisfies $\psi_S(u,v) = (x,1)$. We then have

$$\begin{aligned}\psi_S(u,v) &= \left(\varphi_S(2v\bar{u}, |u|^2 - |v|^2), \frac{u}{|u|}\right)\\ &= \left(\frac{2v\bar{u}}{1+|u|^2-|v|^2}, \frac{u}{|u|}\right) = (x,1).\end{aligned}$$

Then $u = k \in \mathbb{R}^+$, and after some computations one has

$$\sigma_S(x) = \left(\frac{1}{\sqrt{1+r^2}}, \frac{x}{\sqrt{1+r^2}}\right).$$

Denote by s the coordinate on $\mathbb{H}$ such that $s(x) = x$, $x \in \mathbb{H}$ and by r_1, r_2 the coordinates in $\mathbb{H}\times\mathbb{H}$ such that $r_1(x,y) = x$, $r_2(x,y) = y$. Then we can write

$$\omega = \frac{1}{2}(\bar{r}_1\,\mathrm{d}r_1 - (\mathrm{d}\bar{r}_2)r_2 - (\mathrm{d}\bar{r}_1)r_1 + \bar{r}_2\mathrm{d}r_2).$$

To compute $\sigma_S^*\omega$ we substitute

$$r_1 = \frac{1}{\sqrt{1+r^2}}, \qquad r_2 = \frac{s}{\sqrt{1+r^2}},$$

and after a calculation we obtain

$$\sigma_S^*\omega = \frac{1}{2(1+r^2)}(\bar{s}\mathrm{d}s - \mathrm{d}\bar{s}\cdot s),$$

which is well defined in all of $\mathbb{H}$.

Excluding the origin we can write $s = r\gamma$ so that $\bar{s} = r\gamma^{-1}$ and we obtain by computation

$$\sigma_S^* \omega = \frac{r^2}{1+r^2} \gamma^{-1} \mathrm{d}\gamma.$$

A similar calculation shows that we have the formula for $\sigma_N^* \omega$ in the statement.

Problem 5.3.4. *Let $[w]$ denote the standard generator of the group $H^2(\mathbb{C}P^1, \mathbb{Z}) \approx \mathbb{Z}$; that is, $\int_{\mathbb{C}P^1} w = 1$, where the canonical orientation as a complex manifold of $\mathbb{C}P^1$ is considered. Prove that the Chern class of the tautological line bundle E over $\mathbb{C}P^1$ is equal to $-[w]$.*

Solution. Let $(P = \mathbb{C}^2 - \{0\}, p, \mathbb{C}P^1, \mathbb{C}^*)$ be the principal bundle over $\mathbb{C}P^1$ with group $\mathbb{C}^*$ corresponding to the tautological line bundle E. The differential 1-form ω on P defined by

$$\omega_{z=(z^0,z^1)} = \frac{\bar{z}^0\,\mathrm{d}z^0 + \bar{z}^1\,\mathrm{d}z^1}{\bar{z}^0 z^0 + \bar{z}^1 z^1},$$

is a connection form on P. In fact, it takes values on the Lie algebra $\mathbb{C}$ of $\mathbb{C}^*$. Moreover, consider $\omega_U = \omega_{\sigma_U}$ and $\omega_V = \omega_{\sigma_V}$ for two sections σ_U, σ_V on two intersecting open subsets U, V of $\mathbb{C}P^1$. Then, if $\sigma_V = \lambda_{UV}\sigma_U$ on $U \cap V$, that is, $\lambda_{UV} \in \mathbb{C}^*$ is the transition function, we have

$$\begin{aligned}\omega_V &= \omega_{\sigma_U \lambda_{UV}} \\ &= \omega_U + \frac{\mathrm{d}\lambda_{UV}}{\lambda_{UV}},\end{aligned}$$

on $p^{-1}(U \cap V)$; that is, as λ_{UV} takes values in $\mathbb{C}^*$,

$$\begin{aligned}\omega_V &= \lambda_{UV}^{-1}\,\mathrm{d}\lambda_{UV} + \lambda_{UV}^{-1}\,\omega_U \lambda_{UV} \\ &= \lambda_{UV}^{-1}\,\mathrm{d}\lambda_{UV} + \mathrm{Ad}_{\lambda_{UV}^{-1}} \circ \omega_U.\end{aligned}$$

The curvature form of ω is

$$\begin{aligned}\Omega &= \mathrm{d}\omega + \omega \wedge \omega \\ &= \frac{1}{(\bar{z}^0 z^0 + \bar{z}^1 z^1)^2}\big\{(\bar{z}^0 z^0 + \bar{z}^1 z^1)(\mathrm{d}\bar{z}^0 \wedge \mathrm{d}z^0 + \mathrm{d}\bar{z}^1 \wedge \mathrm{d}z^1) \\ &\qquad - (\bar{z}^0 \mathrm{d}z^0 + \bar{z}^1 \mathrm{d}z^1) \wedge (z^0 \mathrm{d}\bar{z}^0 + z^1 \mathrm{d}\bar{z}^1)\big\}.\end{aligned}$$

Denote by U the open subset of $\mathbb{C}P^1$ defined by $z^0 \neq 0$, and set $w = z^1/z^0$. Then w can be taken as a local coordinate on U. Substituting $z^1 = z^0 w$ into the expression for the curvature form above, we have that

$$\Omega = \frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{1 + w\bar{w}^2}.$$

The first Chern form $c_1(E, \omega)$ can thus be written on U as

$$c_1(E, \omega) = \frac{\mathrm{i}}{2\pi} \frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{(1 + w\bar{w})^2}.$$

Taking polar coordinates, $w = r\mathrm{e}^{2\pi \mathrm{i} t}$, one obtains

$$\int_{\mathbb{C}P^1} c_1(E,\omega) = \int_0^1 \left(\int_0^\infty \frac{2r\,\mathrm{d}r}{(1+r^2)^2} \right) \mathrm{d}t = -1,$$

as wanted.

Problem 5.3.5. (Godbillon-Vey's exotic class for codimension 1 foliations) *Let M be a C^∞ n-manifold, and $\mathscr{F}$ a foliation of codimension* 1 (*that is, the leaves have dimension $n-1$*) *on M, defined by a nowhere vanishing global differential* 1*-form ω, which is integrable*; *that is, $\omega \wedge \mathrm{d}\omega = 0$. As $\omega(p) \neq 0$, for all $p \in M$, this condition can be written as*

$$\mathrm{d}\omega = \omega \wedge \omega_1, \tag{$\star$}$$

for certain ω_1. Consider the differential 3*-form*

$$\Xi = -\,\omega_1 \wedge \mathrm{d}\omega_1.$$

Prove:

(1) *The form Ξ defines a cohomology class $[\Xi] \in H^3_{dR}(M,\mathbb{R})$.*

(2) $[\Xi]$ *is an invariant of the foliation, that is, it does not change if either $\mathscr{F}$ is defined by $\omega' = f\omega$, with $f \in C^\infty M$ nowhere vanishing or if we take another form ω_1' satisfying* ($\star$).

Solution. (1) Taking the exterior derivative of both members of ($\star$) we obtain $0 = -\omega \wedge \mathrm{d}\omega_1$, from which

$$\mathrm{d}\omega_1 = \omega \wedge \omega_2,$$

and thus $\mathrm{d}\Xi = -\omega \wedge \omega_2 \wedge \omega \wedge \omega_2 = 0$.

(2) If $\mathscr{F}$ is defined by $\omega' = f\omega$, with $f \in C^\infty M$ nowhere vanishing, we have

$$\begin{aligned} \mathrm{d}\omega' &= \mathrm{d}f \wedge \omega + f\omega \wedge \omega_1 \\ &= \omega' \wedge \left(\omega_1 - \frac{\mathrm{d}f}{f} \right). \end{aligned}$$

Hence

$$\begin{aligned} \Xi' &= -\omega_1' \wedge \mathrm{d}\omega_1' \\ &= -\omega_1 \wedge \mathrm{d}\omega_1 - \frac{\mathrm{d}f}{f} \wedge \mathrm{d}\omega_1 \\ &= \Xi - \mathrm{d}(\log|f|\,\mathrm{d}\omega_1), \end{aligned}$$

from which $[\Xi'] = [\Xi]$.

If we choose another form, say ω_1', satisfying ($\star$), then from this equation and $\mathrm{d}\omega = \omega \wedge \omega_1'$ we have $\omega \wedge (\omega_1 - \omega_1') = 0$, that is, $\omega_1 - \omega_1'$ belongs to the ideal generated by ω. Hence, the general expression for such forms ω_1 is $\omega_1'' = \omega_1 + h\omega$, $h \in C^\infty M$. Now, we have

$$\begin{aligned}\omega_1'' \wedge \mathrm{d}\omega_1'' &= (\omega_1 + h\omega) \wedge (\mathrm{d}\omega_1 + \mathrm{d}h \wedge \omega + h\omega \wedge \omega_1) \\ &= \omega_1 \wedge \mathrm{d}\omega_1 + \mathrm{d}(h\,\mathrm{d}\omega),\end{aligned}$$

hence $[\Xi''] = [\Xi]$.

Problem 5.3.6. *Let (P,M,G) be a principal bundle over the C^∞ manifold M and $\Gamma,\widetilde{\Gamma}$ connections in P, whose connection forms (resp. curvature forms) can be described on trivializing open subsets of M by the $\mathfrak{g}$-valued differential 1-forms $A,\tilde{A}$ (resp. 2-forms $F,\widetilde{F}$).*

(1) *Let $I \in \mathscr{I}^r(G)$ be a G-invariant polynomial, and consider the global $2r$-forms $I(F^r)$ and $I(\widetilde{F}^r)$ (see* Definitions 7.6.11 and [13]). *Deduce from the* Chern-Simons Formula *in* Theorem 7.6.12 *for the difference $I(F^r) - I(\widetilde{F}^r)$, the formula for the particular case where G is a matrix group, $I(F^2) = \operatorname{tr}(F \wedge F)$, and $\tilde{A} = 0$:*

$$\operatorname{tr}(F \wedge F) = \mathrm{d}\operatorname{tr}\left((\mathrm{d}A) \wedge A + \frac{2}{3} A \wedge A \wedge A\right). \qquad (\star)$$

(2) *Let ξ be the map $\xi\colon \mathbb{R}^4 \to M(2,\mathbb{C})$,*

$$\xi(x) = \begin{pmatrix} x^4 - ix^3 & -x^2 - ix^1 \\ x^2 - ix^1 & x^4 + ix^3 \end{pmatrix}.$$

Consider on $\mathbb{R}^4$ with the Euclidean metric the differential forms

$$A_1 = \frac{r^2}{r^2 + c^2}\gamma^{-1}\mathrm{d}\gamma, \qquad A_2 = \gamma A_1 \gamma^{-1} + \gamma\,\mathrm{d}\gamma^{-1} = \frac{c^2}{r^2 + c^2}\gamma\,\mathrm{d}\gamma^{-1},$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$, $c \in \mathbb{R}$, and

$$\gamma\colon \mathbb{R}^4 - \{0\} \to SU(2), \qquad \gamma(x) = r^{-1}\xi(x),$$

identifies $S^3(r)$ with $SU(2)$.

The form A_1 is regular at $x = 0$ as it follows taking the formulas

$$A_1 = -\frac{r\,\mathrm{d}r}{r^2 + c^2} + \frac{r^2}{r^2 + c^2}\xi^{-1}\mathrm{d}\xi, \qquad \det\xi = r^2,$$

into account, but A_2 is singular at $x = 0$. Let N,S denote the north and south poles of S^4. Identify $\mathbb{R}^4$ with $U_1 = S^4 - \{S\}$ and on the other hand with $U_2 = S^4 - \{N\}$ under convenient stereographic projections (see [13, 10.7]). *Then one can define accordingly A_1 on U_1 and A_2 on U_2, since the singularity of A_2 at the origin manifests as a singularity at the north pole, which does not belong to U_2; in such a way that A_1 and A_2 are local representatives of a connection in a principal $SU(2)$-bundle P' over S^4, whose transition function g_{21} is γ.*

Express the Chern number $c_{(2)}(P')$ in terms of $\gamma^{-1}\mathrm{d}\gamma$, by means of the Chern-Simons formula $(\star)$.

REMARK. *In Physics, the differential form A with local representatives as in (2) is called an* instanton potential. *It solves the* Euclidean Yang-Mills equation; *that is, $D\star F = 0$, where $F = \mathrm{d}A + A\wedge A$ and $\star$ stands for the Hodge star operator (see* Problem 6.11.4).

Solution. (1) As G is a matrix group, we can write $F = \mathrm{d}A + \frac{1}{2}[A,A]$ as $F = \mathrm{d}A + A\wedge A$. In the particular case $I(F^2) = \mathrm{tr}(F\wedge F)$ we have

$$\mathrm{tr}\,(F\wedge F) - \mathrm{tr}\,(\widetilde{F}\wedge\widetilde{F}) = \mathrm{d}Q(A,\tilde{A}).$$

Putting $\alpha = A - \tilde{A}$, one has

$$\begin{aligned}
Q(A,\tilde{A}) &= 2\int_0^1 \mathrm{tr}\,\big(\alpha\wedge(\mathrm{d}\tilde{A} + t\,\mathrm{d}\alpha) + \alpha\wedge(\tilde{A}+t\alpha)\wedge(\tilde{A}+t\alpha)\big)\,\mathrm{d}t \\
&= 2\int_0^1 \mathrm{tr}\,\big(\alpha\wedge\mathrm{d}\tilde{A} + t\alpha\wedge\mathrm{d}\alpha + \alpha\wedge\tilde{A}\wedge\tilde{A} + t\alpha\wedge\tilde{A}\wedge\alpha \\
&\quad + t\alpha\wedge\alpha\wedge\tilde{A} + t^2\alpha\wedge\alpha\wedge\alpha\big)\,\mathrm{d}t \\
&= 2\,\mathrm{tr}\,\Big(\alpha\wedge\mathrm{d}\tilde{A} + \frac{1}{2}\alpha\wedge\mathrm{d}\alpha + \alpha\wedge\tilde{A}\wedge\tilde{A} + \frac{1}{2}\alpha\wedge\tilde{A}\wedge\alpha \\
&\quad + \frac{1}{2}\alpha\wedge\alpha\wedge\tilde{A} + \frac{1}{3}\alpha\wedge\alpha\wedge\alpha\Big) \\
&= \mathrm{tr}\,\left(2\alpha\wedge\widetilde{F} + \alpha\wedge\mathrm{d}\alpha + 2\alpha\wedge\tilde{A}\wedge\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right).
\end{aligned}$$

For $\tilde{A} = 0$, this expression reduces to the formula in the statement.

(2) Let S^4_+ (resp. S^4_-) denote the upper (resp. lower) hemisphere of S^4; that is, the subset with last coordinate $\geqslant 0$ (resp. $\leqslant 0$). Then, on account of

$$A_1 = \gamma^{-1}A_2\gamma + \gamma^{-1}\mathrm{d}\gamma,$$

$$F_1 = \mathrm{d}A_1 + A_1\wedge A_1, \qquad F_2 = \mathrm{d}A_2 + A_2\wedge A_2, \qquad F_1 = \gamma^{-1}F_2\,\gamma,$$

we can write

$$\begin{aligned}
c_{(2)}(P') &= \frac{1}{8\pi^2}\int_{S^4}\mathrm{tr}\,(F\wedge F) \\
&= \frac{1}{8\pi^2}\int_{S^4_+}\mathrm{tr}\,(F_1\wedge F_1) + \frac{1}{8\pi^2}\int_{S^4_-}\mathrm{tr}\,(F_2\wedge F_2) \\
&= \frac{1}{8\pi^2}\int_{S^3}\mathrm{tr}\,\left(\mathrm{d}A_1\wedge A_1 + \frac{2}{3}A_1\wedge A_1\wedge A_1\right) \\
&\quad - \frac{1}{8\pi^2}\int_{S^3}\mathrm{tr}\,\left(\mathrm{d}A_2\wedge A_2 + \frac{2}{3}A_2\wedge A_2\wedge A_2\right) \qquad \text{(by Stokes)} \\
&= \frac{1}{8\pi^2}\int_{S^3}\mathrm{tr}\,\left(F_1\wedge A_1 - \frac{1}{3}A_1\wedge A_1\wedge A_1 - F_2\wedge A_2 + \frac{1}{3}A_2\wedge A_2\wedge A_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8\pi^2}\int_{S^3} \operatorname{tr}\Big(\gamma^{-1}\mathrm{d}A_2\wedge A_2\,\gamma + \gamma^{-1}A_2\wedge A_2\wedge A_2\,\gamma \\
&\quad + \gamma^{-1}\mathrm{d}A_2\wedge \mathrm{d}\gamma + \gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge A_2\,\gamma - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}A_2\,\gamma - \frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}A_2\wedge A_2\,\gamma - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}A_2\wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}A_2\,\gamma - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad -\mathrm{d}A_2\wedge A_2 - A_2\wedge A_2\wedge A_2 + \frac{1}{3}A_2\wedge A_2\wedge A_2\Big) \\
&= \frac{1}{8\pi^2}\int_{S^3} \operatorname{tr}\Big(\gamma^{-1}\mathrm{d}A_2\wedge A_2\,\gamma + \gamma^{-1}A_2\wedge A_2\wedge A_2\,\gamma \\
&\quad + \gamma^{-1}\mathrm{d}A_2\wedge \mathrm{d}\gamma\gamma^{-1}\,\gamma + \gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge A_2\,\gamma - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma - \frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge A_2\wedge \mathrm{d}\gamma - \frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad -\mathrm{d}A_2\wedge A_2 - A_2\wedge A_2\wedge A_2 + \frac{1}{3}A_2\wedge A_2\wedge A_2\Big) \\
&= \frac{1}{8\pi^2}\int_{S^3} \operatorname{tr}\Big(\mathrm{d}A_2\wedge \mathrm{d}\gamma\gamma^{-1} - \gamma^{-1}A_2\wedge \mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma \\
&\quad - \frac{1}{3}\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\Big) \\
&= \frac{1}{8\pi^2}\int_{S^3}\Big(\mathrm{d}\left(\operatorname{tr}(A_2\wedge \mathrm{d}\gamma\gamma^{-1})\right) - \frac{1}{3}\operatorname{tr}(\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma)\Big) \\
&= -\frac{1}{24\pi^2}\int_{S^3}\operatorname{tr}(\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma\wedge\gamma^{-1}\mathrm{d}\gamma) \qquad \text{(by Stokes).}
\end{aligned}
$$

REMARK. The last expression for $c_{(2)}(P')$ is the opposite to a certain winding number (the *topological charge*), which is an element of the homotopy group $\pi_3(SU(2)) \approx \mathbb{Z}$, associated to a map from the equator S^3 in S^4 to $SU(2) \approx S^3$. It is important in Physics as it corresponds to a minimum of the Yang-Mills action functional.

5.4 Linear Connections

Problem 5.4.1. *Given a linear connection* ∇ *of the* C^∞ *manifold* M*, one defines the conjugate or opposite connection* $\widehat{\nabla}$ *of* M *by*

$$\widehat{\nabla}_X Y = \nabla_Y X + [X,Y], \qquad X,Y \in \mathfrak{X}(M).$$

(1) *Prove that* $\widehat{\nabla}$ *is a linear connection.*

(2) *Compute the local components* $\widehat{\Gamma}^i_{jh}$ *of* $\widehat{\nabla}$ *in terms of the components of* ∇.

Solution. (1) Since ∇ is a linear connection and from the expression

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X, \qquad f,g \in C^\infty M,$$

we deduce by some computations that $\widehat{\nabla}$ satisfies the properties:

$$\begin{aligned} &(\mathrm{a}) \quad \widehat{\nabla}_X(Y+Z) = \widehat{\nabla}_X Y + \widehat{\nabla}_X Z, \qquad &&(\mathrm{b}) \quad \widehat{\nabla}_{X+Y} Z = \widehat{\nabla}_X Z + \widehat{\nabla}_Y Z,\\ &(\mathrm{c}) \quad \widehat{\nabla}_{fX} Y = f\widehat{\nabla}_X Y, &&(\mathrm{d}) \quad \widehat{\nabla}_X fY = (Xf)Y + f\widehat{\nabla}_X Y, \end{aligned}$$

that is, $\widehat{\nabla}$ is a linear connection.

(2) One has $\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j} = \widehat{\Gamma}^k_{ij} \dfrac{\partial}{\partial x^k}$, in terms of the local coordinates $x^1, \ldots, x^n$ and also

$$\begin{aligned} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \\ &= \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \\ &= \Gamma^k_{ji} \frac{\partial}{\partial x^k}. \end{aligned}$$

That is $\widehat{\Gamma}^k_{ij} = \Gamma^k_{ji}$.

Problem 5.4.2. (1) *Let* ∇ *be a linear connection and* A *a tensor field of type* $(1,2)$ *on a* C^∞ *manifold* M*. Prove that* $\widetilde{\nabla}$*, defined by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + A(X,Y), \qquad X,Y \in \mathfrak{X}(M),$$

is a linear connection of M.

(2) *Let* ∇_0 *and* ∇_1 *be two linear connections of* M*. Prove that*

$$\nabla_t = (1-t)\nabla_0 + t\nabla_1$$

is a linear connection of M *for each* $t \in [0,1]$.

Solution. We prove only, in both cases, that $\nabla_X fY = (Xf)Y + f\nabla_X Y$ for a linear connection:

(1)

$$\begin{aligned}\widetilde{\nabla}_X fY &= \nabla_X fY + A(X, fY)\\ &= (Xf)Y + f\nabla_X Y + fA(X,Y)\\ &= (Xf)Y + f\widetilde{\nabla}_X Y.\end{aligned}$$

(2)

$$\begin{aligned}(\nabla_t)_X fY &= (1-t)(\nabla_0)_X fY + t(\nabla_1)_X fY\\ &= (1-t)(Xf)Y + (1-t)f(\nabla_0)_X Y + t(Xf)Y + tf(\nabla_1)_X Y\\ &= (Xf)Y + f(\nabla_t)_X Y.\end{aligned}$$

Problem 5.4.3. *Let $\varphi\colon M \to M'$ be a diffeomorphism. Given a linear connection ∇ of M, let $\nabla' = \varphi\cdot\nabla$ be defined by*

$$\nabla'_{X'}Y' = \varphi\cdot\left(\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot Y'\right), \qquad \forall X', Y' \in \mathfrak{X}(M').$$

Prove:

(1) ∇' *is a linear connection of* M'.

(2) *If φ_t is the flow of a vector field $X \in \mathfrak{X}(M)$ such that $\varphi_t\cdot\nabla = \nabla$, $\forall t \in \mathbb{R}$, then*

$$L_X \circ \nabla_Y - \nabla_Y \circ L_X = \nabla_{[X,Y]}, \qquad X, Y \in \mathfrak{X}(M). \tag{$\star$}$$

Solution. (1) We prove one property only: For any $f \in C^\infty M'$,

$$\begin{aligned}\nabla'_{X'} fY' &= \varphi\cdot\left(\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot(fY')\right)\\ &= \varphi\cdot\left(\nabla_{\varphi^{-1}\cdot X'}(f\circ\varphi)\varphi^{-1}\cdot Y'\right)\\ &= \varphi\cdot\left\{((\varphi^{-1}\cdot X')(f\circ\varphi))\varphi^{-1}\cdot Y' + (f\circ\varphi)\nabla_{\varphi^{-1}\cdot X'}\varphi^{-1}\cdot Y'\right\}\\ &= (X'f)Y' + f\nabla'_{X'}Y'.\end{aligned}$$

(2) Applying both sides of $(\star)$ to a function $f \in C^\infty M$, we obtain

$$X(Y(f)) - Y(X(f)) = [X,Y](f),$$

which trivially holds. Applying now both sides of $(\star)$ to a vector field Z, one has

$$\begin{aligned}L_X\nabla_Y Z &= \lim_{t\to 0}\frac{1}{t}\left(\nabla_Y Z - \varphi_t\cdot(\nabla_Y Z)\right)\\ &= \lim_{t\to 0}\frac{1}{t}(\nabla_Y Z - \nabla_{\varphi_t\cdot Y}Z) + \lim_{t\to 0}\frac{1}{t}\left(\nabla_{\varphi_t\cdot Y}Z - \nabla_{\varphi_t Y}(\varphi_t\cdot Z)\right)\end{aligned}$$

$$
\begin{aligned}
&= \nabla_{\lim_{t\to 0}\frac{1}{t}(Y-\varphi_t\cdot Y)} Z + \nabla_Y \left(\lim_{t\to 0} \frac{1}{t}\left(Z - \varphi_t \cdot Z\right)\right) \\
&= \nabla_{L_X Y} Z + \nabla_Y L_X Z \\
&= \nabla_{[X,Y]} Z + \nabla_Y L_X Z.
\end{aligned}
$$

As L_X and ∇_Y are type-preserving derivations that commute with contractions, for every tensor field T we have

$$(L_X \circ \nabla_Y - \nabla_Y \circ L_X)(T) = \nabla_{[X,Y]} T.$$

Problem 5.4.4. *Let M be a C^∞ n-manifold endowed with a torsionless linear connection. Prove that in a system of normal coordinates with origin p, all the Christoffel symbols at p vanish.*

Solution. In a system of normal coordinates $\{x^i\}$, $i = 1,\dots,n$, around p, the equations of the geodesics through p are given by $x^i = \lambda^i t$, with λ^i constants. These functions must satisfy the differential equations of the geodesics, i.e.,

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \qquad i = 1,\dots,n,$$

that now reduce to $\Gamma^k_{ij}(p)\lambda^i\lambda^j = 0$, for $k = 1,\dots,n$. As the connection is torsionless, it is immediate that $\Gamma^k_{ij}(p) = 0$, for $i,j,k = 1,\dots,n$.

5.5 Torsion and Curvature

Problem 5.5.1. *Consider a linear connection of a C^∞ manifold with components*

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\delta^i_k \theta_j,$$

where the Γ^i_{jk} are the components of another linear connection (it is said that they are projectively related connections*), θ is a differential 1-form, and δ^i_j denotes the Kronecker delta. Calculate the difference tensor $\widetilde{R}^i_{hjk} - R^i_{hjk}$ of their respective curvature tensors fields.*

Solution. Putting $\partial_j = \partial/\partial x^j$, we have

$$
\begin{aligned}
\widetilde{R}^i_{hjk} &= \partial_j \widetilde{\Gamma}^i_{kh} - \partial_k \widetilde{\Gamma}^i_{jh} + \widetilde{\Gamma}^r_{kh}\widetilde{\Gamma}^i_{jr} - \widetilde{\Gamma}^r_{jh}\widetilde{\Gamma}^i_{kr} \\
&= \partial_j(\Gamma^i_{kh} + 2\delta^i_h\theta_k) - \partial_k(\Gamma^i_{jh} + 2\delta^i_h\theta_j) + (\Gamma^r_{kh} + 2\delta^r_h\theta_k)(\Gamma^i_{jr} + 2\delta^i_r\theta_j) \\
&\quad - (\Gamma^r_{jh} + 2\delta^r_h\theta_j)(\Gamma^i_{kr} + 2\delta^i_r\theta_k) \\
&= R^i_{hjk} + 2\delta^i_h\partial_j\theta_k - 2\delta^i_h\partial_k\theta_j + 2\Gamma^i_{kh}\theta_j + 2\Gamma^i_{jh}\theta_k + 4\delta^i_h\theta_k\theta_j \\
&\quad - 2\Gamma^i_{jh}\theta_k - 2\Gamma^i_{kh}\theta_j - 4\delta^i_h\theta_j\theta_k
\end{aligned}
$$

$$= R^i_{hjk} + 2\delta^i_h(\partial_j\theta_k - \partial_k\theta_j),$$

from which we obtain

$$\widetilde{R}^i_{hjk} - R^i_{hjk} = 2\delta^i_h(\partial_j\theta_k - \partial_k\theta_j).$$

Problem 5.5.2. *Let M be a C^∞ manifold, with a linear connection having components Γ^i_{jk} with respect to a local coordinate system. Write the formulas for the covariant derivative of the following tensor fields on M:*

(1) *A vector field with components X^i.*

(2) *A differential 1-form with components θ_i.*

(3) *A $(1,1)$ tensor field with components J^i_j.*

(4) *A $(0,2)$ tensor field with components τ_{ij}.*

Moreover, prove:

(5) *If the given connection is torsionless, for a vector field with components X^i, one has*

$$X^i_{;jk} - X^i_{;kj} = -X^r R^i_{rjk},$$

where $X^i_{;jk} = (X^i_{;j})_{;k}$, and R^i_{jkl} are the components of the curvature tensor field of the given connection.

(6) *For a differential 1-form with components θ_i, one has*

$$\theta_{i;jk} - \theta_{i;kj} = \theta_r R^r_{ijk} - 2\theta_{i;r}T^r_{jk},$$

where T^i_{jk} and R^i_{jkl} are, respectively, the components of the torsion and curvature tensor fields of the given connection.

Solution. Let $\partial_j = \partial/\partial x^j$, where $\{x^j\}$ stand for local coordinates. Then:

(1)

$$\nabla_{\partial_j}(X^i\partial_i) = (\partial_j X^i)\partial_i + X^i\Gamma^r_{ji}\partial_r.$$

Hence

$$X^i_{;j} = \partial_j X^i + \Gamma^i_{jr}X^r.$$

(2)

$$\begin{aligned}
\theta_{j;i} &= (\nabla_{\partial_i}\theta)\partial_j \\
&= \nabla_{\partial_i}(\theta\partial_j) - \theta(\nabla_{\partial_i}\partial_j) \\
&= \nabla_{\partial_i}((\theta_l \mathrm{d}x^l)\partial_j) - \theta(\Gamma^r_{ij}\partial_r) \\
&= \partial_i\theta_j - \Gamma^r_{ij}\theta_r.
\end{aligned}$$

(3)

$$(\nabla_{\partial_i}J)\partial_j = \nabla_{\partial_i}J\partial_j - J\nabla_{\partial_i}\partial_j = \nabla_{\partial_i}J^r_j\partial_r - J\,\Gamma^r_{ij}\partial_r = (\partial_i J^r_j)\partial_r + J^r_j\Gamma^s_{ir}\partial_s - \Gamma^r_{ij}J^s_r\partial_s.$$

Hence

$$J^i_{j;k} = \partial_k J^i_j + J^r_j \Gamma^i_{kr} - \Gamma^r_{kj} J^i_r.$$

(4)

$$\begin{aligned}\tau_{ij;k} &= (\nabla_{\partial_i}\tau)(\partial_j, \partial_k)\\ &= \nabla_{\partial_i}\tau_{jk} - \tau(\Gamma^r_{ij}\partial_r, \partial_k) - \tau(\partial_j, \Gamma^r_{ik}\partial_r)\\ &= \partial_i \tau_{jk} - \Gamma^r_{ij}\tau_{rk} - \Gamma^r_{ik}\tau_{jr}.\end{aligned}$$

(5) We have

$$\begin{aligned}X^i_{;jk} &= \partial_k(\partial_j X^i + \Gamma^i_{jr}X^r) + X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\\ &= \partial_k\partial_j X^i + (\partial_k\Gamma^i_{jr})X^r + \Gamma^i_{jr}\partial_k X^r + X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\\ &= \partial_k\partial_j X^i + (\partial_k\Gamma^i_{jr})X^r + \Gamma^i_{jr}X^r_{;k} - X^s\Gamma^i_{jr}\Gamma^r_{ks} + X^r_{;j}\Gamma^i_{kr} - \Gamma^r_{kj}X^i_{;r}\\ &= \partial_k\partial_j X^i + X^r(\partial_k\Gamma^i_{jr} - \Gamma^i_{js}\Gamma^s_{kr}) + \Gamma^i_{jr}X^r_{;k} + \Gamma^i_{kr}X^r_{;j} - \Gamma^r_{kj}X^i_{;r},\end{aligned}$$

and

$$X^i_{;kj} = \partial_j\partial_k X^i + X^r(\partial_j\Gamma^i_{kr} - \Gamma^i_{ks}\Gamma^s_{jr}) + \Gamma^i_{kr}X^r_{;j} + \Gamma^i_{jr}X^r_{;k} - \Gamma^r_{jk}X^i_{;r},$$

hence

$$X^i_{;jk} - X^i_{;kj} = X^r(\partial_k\Gamma^i_{jr} - \partial_j\Gamma^i_{kr} + \Gamma^s_{jr}\Gamma^i_{ks} - \Gamma^s_{kr}\Gamma^i_{js}) = X^r R^i_{rkj}.$$

(6)

$$\theta_{i;jk} = (\partial_j\theta_i - \Gamma^r_{ji}\theta_r)_{;k} = \partial_k(\partial_j\theta_i - \Gamma^r_{ji}\theta_r) - \Gamma^r_{kj}(\partial_r\theta_i - \Gamma^s_{ri}\theta_s) - \Gamma^r_{ki}(\partial_j\theta_r - \Gamma^s_{jr}\theta_s).$$

Expanding this formula and the similar one for $\theta_{i;kj}$ we obtain

$$\begin{aligned}\theta_{i;jk} - \theta_{i;kj} &= (\partial_j\Gamma^r_{ki} - \partial_k\Gamma^r_{ji} + \Gamma^s_{ki}\Gamma^r_{js} - \Gamma^s_{ji}\Gamma^r_{ks})\theta_r + (\Gamma^r_{jk} - \Gamma^r_{kj})(\partial_r\theta_i - \Gamma^s_{ri}\theta_s)\\ &= R^r_{ijk}\theta_r + 2T^r_{jk}\theta_{i;r}.\end{aligned}$$

Problem 5.5.3. *Given a linear connection ∇ of the C^∞ manifold M, consider the linear connection $\overline{\nabla} = \frac{1}{2}(\nabla + \widehat{\nabla})$, where $\widehat{\nabla}$ denotes the conjugate connection of ∇* (*see* Problem 5.4.1). *Prove*:

(1) $\overline{\nabla}$ *is torsionless.*

(2) *If ∇ is torsionless, then $\nabla = \widehat{\nabla} = \overline{\nabla}$.*

Solution. (1)

$$\begin{aligned}T_{\overline{\nabla}}(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]\\ &= \frac{1}{2}\nabla_X Y + \frac{1}{2}\widehat{\nabla}_X Y - \frac{1}{2}\nabla_Y X - \frac{1}{2}\widehat{\nabla}_Y X - [X,Y]\\ &= \frac{1}{2}\nabla_X Y + \frac{1}{2}\nabla_Y X + \frac{1}{2}[X,Y] - \frac{1}{2}\nabla_Y X - \frac{1}{2}\nabla_X Y - \frac{1}{2}[Y,X] - [X,Y] = 0.\end{aligned}$$

(2) If $\nabla_X Y = \nabla_Y X + [X,Y]$, then

$$\widehat{\nabla}_X Y = \nabla_Y X + [X,Y] = \nabla_X Y,$$

hence $\overline{\nabla}_X Y = \frac{1}{2}\nabla_X Y + \frac{1}{2}\overline{\nabla}_X Y = \nabla_X Y$.

Problem 5.5.4. *Consider the linear connection of the half-plane $y > 0$ of $\mathbb{R}^2$ defined by the components $\Gamma^i_{jk} = 0$, except $\Gamma^1_{12} = 1$, with respect to the frame ($e_1 = \partial/\partial x$, $e_2 = \partial/\partial y$). Consider the frame*

$$(\bar{e}_1 = \partial/\partial x,\ \bar{e}_2 = x\partial/\partial x + y\partial/\partial y).$$

Compute the components of the connection and the components of the torsion tensor with respect to this frame.

Solution. We have $\nabla_{\bar{e}_j}\bar{e}_i = \bar{\Gamma}^k_{ij}\bar{e}_k$, and

$$\begin{aligned}
\nabla_{\bar{e}_1}\bar{e}_1 &= \nabla_{e_1} e_1 = 0,\\
\nabla_{\bar{e}_1}\bar{e}_2 &= \nabla_{e_1}(xe_1 + ye_2)\\
&= (1+y)e_1\\
&= (1+y)\bar{e}_1,\\
\nabla_{\bar{e}_2}\bar{e}_1 &= \nabla_{xe_1+ye_2} e_1 = 0,\\
\nabla_{\bar{e}_2}\bar{e}_2 &= \nabla_{xe_1+ye_2}(xe_1 + ye_2)\\
&= xe_1 + xye_1 + ye_2\\
&= xy\bar{e}_1 + \bar{e}_2.
\end{aligned}$$

Thus the nonvanishing components of ∇ with respect to the frame $(\bar{e}_1, \bar{e}_2)$ are

$$\bar{\Gamma}^1_{12} = 1 + y, \qquad \bar{\Gamma}^1_{22} = xy, \qquad \bar{\Gamma}^2_{22} = 1,$$

and the only nonvanishing component of the torsion tensor is $\bar{T}^1_{12} = y$.

Problem 5.5.5. *Let ∇ be a torsionless linear connection of the C^∞ manifold M. Prove that*

$$\mathrm{d}\omega(X,Y) = (\nabla_X \omega)Y - (\nabla_Y \omega)X, \qquad X,Y \in \mathfrak{X}(M), \quad \omega \in \Lambda^1 M.$$

Solution. By the relation between the bracket product and the exterior differential we have:

$$\begin{aligned}
\mathrm{d}\omega(X,Y) &= X\omega(Y) - Y\omega(X) - \omega([X,Y])\\
&= \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega(\nabla_X Y - \nabla_Y X)\\
&= (\nabla_X \omega)Y - (\nabla_Y \omega)X.
\end{aligned}$$

Problem 5.5.6. *Let M and N be C^∞ manifolds with linear connections ∇ and ∇', respectively. A C^∞ map $\varphi\colon M \to N$ is said to be connection-preserving if*

$$\varphi_*(\nabla_X Y)_p = \left(\nabla'_{X'} Y'\right)_{\varphi(p)}, \tag{$\star$}$$

for all $p \in M$, where X, Y are φ-related to X', Y', respectively. Prove that if φ is also a diffeomorphism, then:

(1) *$\varphi \cdot (R(X,Y)Z) = R'(\varphi \cdot X, \varphi \cdot Y)(\varphi \cdot Z)$, where R and R' are the curvature tensor fields of ∇ and ∇', respectively.*

(2) *$\varphi \cdot (T(X,Y)) = T'(\varphi \cdot X, \varphi \cdot Y)$, where T and T' stand for the torsion tensors of ∇ and ∇', respectively.*

Solution. (1) First, we remark (see also Problem 5.4.3) that if φ is a diffeomorphism, then the formula ($\star$) means

$$\varphi \cdot (\nabla_X Y) = \nabla'_{\varphi \cdot X} \varphi \cdot Y.$$

Thus,

$$\begin{aligned}
\varphi \cdot (R(X,Y)Z) &= \varphi \cdot (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
&= \varphi \cdot \nabla_X \nabla_Y Z - \varphi \cdot \nabla_Y \nabla_X Z - \varphi \cdot \nabla_{[X,Y]} Z \\
&= \nabla'_{\varphi \cdot X}(\varphi \cdot \nabla_Y Z) - \nabla'_{\varphi \cdot Y}(\varphi \cdot \nabla_X Z) - \nabla'_{\varphi \cdot [X,Y]} \varphi \cdot Z \\
&= \nabla'_{\varphi \cdot X} \nabla'_{\varphi \cdot Y} \varphi \cdot Z - \nabla'_{\varphi \cdot Y} \nabla'_{\varphi \cdot X} \varphi \cdot Z - \nabla'_{[\varphi \cdot X, \varphi \cdot Y]} \varphi \cdot Z \\
&= R'(\varphi \cdot X, \varphi \cdot Y)(\varphi \cdot Z).
\end{aligned}$$

(2)

$$\begin{aligned}
\varphi \cdot (T(X,Y)) &= \varphi \cdot (\nabla_X Y - \nabla_Y X - [X,Y]) \\
&= \varphi \cdot \nabla_X Y - \varphi \cdot \nabla_Y X - \varphi \cdot [X,Y] \\
&= \nabla'_{\varphi \cdot X} \varphi \cdot Y - \nabla'_{\varphi \cdot Y} \varphi \cdot X - [\varphi \cdot X, \varphi \cdot Y] \\
&= T'[\varphi \cdot X, \varphi \cdot Y].
\end{aligned}$$

Problem 5.5.7. *If ω is a differential r-form on a C^∞ manifold M equipped with a torsionless linear connection ∇, prove that*

$$\mathrm{d}\omega(X_0, \dots, X_r) = \sum_{i=0}^{r} (-1)^{\mathrm{i}} (\nabla_{X_i} \omega)(X_0, \dots, \widehat{X}_i, \dots, X_r),$$

$X_0, \dots, X_r \in \mathfrak{X}(M)$, where the hat symbol denotes that the corresponding vector field is dropped.

HINT: *If ω is a differential r-form, the formula relating the bracket product and the exterior differential is*

$$(\mathrm{d}\omega)(X_0,\dots,X_r) = \sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)) \\ + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_j,\dots,\hat{X}_i,\dots,X_r).$$

REMARK. *The more used case is that of differential* 2*-forms*:

$$\mathrm{d}\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y).$$

Solution.

$$\begin{aligned}
&\sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\omega)(X_0,\dots,\widehat{X}_i,\dots,X_r)\\
&\quad= \sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r))\\
&\qquad - \sum_{j<i}(-1)^i\omega(X_0,\dots,\nabla_{X_i}X_j,\dots,\widehat{X}_i,\dots,X_r)\\
&\qquad - \sum_{j>i}(-1)^i\omega(X_0,\dots,\widehat{X}_i,\dots,\nabla_{X_i}X_j,\dots,X_r)\\
&\quad= \sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r))\\
&\qquad - \sum_{j<i}(-1)^{i+j}\omega(\nabla_{X_i}X_j,X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_r)\\
&\qquad + \sum_{j>i}(-1)^{i+j}\omega(\nabla_{X_i}X_j,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r)\\
&\quad= \sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r))\\
&\qquad + \sum_{i<j}(-1)^{i+j}\omega(\nabla_{X_i}X_j-\nabla_{X_j}X_i,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r)\\
&\quad= \mathrm{d}\omega(X_0,\dots,X_r).
\end{aligned}$$

Problem 5.5.8. (1) *Prove that if ∇ is a flat connection of a connected manifold M whose parallel transport is globally independent of curves, then there exists a C^∞ global field of frames on M.*

(2) *Prove that if ∇ is a flat connection of a connected manifold M, then its curvature tensor field vanishes.*

Solution. (1) Let us fix a point $p_0 \in M$ and a basis $\{v_1,\dots,v_n\}$ of $T_{p_0}M$. Given an arbitrary point $p \in M$, there exists a differentiable arc $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p_0$, $\gamma(1) = p$. We define $X_i|_p = \tau_\gamma(v_i)$, $i = 1,\dots,n$, where $\tau_\gamma\colon T_{p_0}M \to T_pM$ is the parallel transport along γ. The definition makes sense by virtue of the hypothesis and $(X_1,\dots,X_n)$ is a frame as τ_γ is an isomorphism.

(2) According to the definition of a flat connection, given a point $p \in M$, there exist an open neighborhood U of p such that the parallel transport in U is independent of curves. Hence from (1) it follows that U admits a linear frame $(X_1,\dots,X_n)$ invariant under parallel transport; that is, $\nabla_X X_i = 0$, for all $X \in \mathfrak{X}(U)$. Then $R(X_i,X_j)X_k = 0$ and hence $R = 0$.

Problem 5.5.9. *Find the (equivalent) expression of Cartan's second equation of structure* $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, *that is, of* $\Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j$, *when one considers transpose matrices,* i.e. *when the upper index denotes the column and the lower index denotes the row of the corresponding matrix.*

Solution. When one considers the transpose matrices of ω and Ω, it is immediate to see that
$$\Omega^i_j = \mathrm{d}\omega^i_j + \omega^k_j \wedge \omega^i_k = \mathrm{d}\omega^i_j - \omega^i_k \wedge \omega^k_j,$$
that is, $\Omega = \mathrm{d}\omega - \omega \wedge \omega$.

REMARK. Some authors prefer to use this expression of Cartan's second equation of structure.

Problem 5.5.10. *Find the holonomy group of*:

(1) *The Euclidean space* $\mathbb{R}^n$.

(2) *The sphere* S^2 *with its usual connection.*

Moreover, prove:

(3) *The holonomy group, at any point, of a connection in the principal bundle*
$$(S^{2n+1}, \pi, \mathbb{C}P^n, S^1)$$
is S^1.

(4) *The holonomy group of a connection in the principal fibre bundle*
$$(S^{4n+3}, \pi, \mathbb{H}P^n, S^3)$$
is either S^1 *or* S^3.

Solution. (1) Let ∇ be the usual flat connection, then $\mathrm{Hol}(\nabla) = \{0\}$, as the parallel transport along any closed curve is the identity map.

(2) Let ∇ be the usual connection. As S^2 is orientable, the holonomy group $\mathrm{Hol}(\nabla)$ is a subgroup of $SO(2)$.

We shall see geometrically that $\mathrm{Hol}(\nabla) = SO(2)$. Consider, with no loss of generality, any orthonormal basis $\{e_1, e_2\}$ of $T_N S^2$, N being the north pole $(0,0,1)$, and do its parallel transport along the piecewise C^∞ curve in S^2, given (see Figure 5.3 for a certain vector tangent starting as (1) at the north pole) by the half-meridian determined by e_1 until the equator, then the curve along the equator by a rotation of angle β of the equatorial plane, and then the half-meridian of return to N. The net

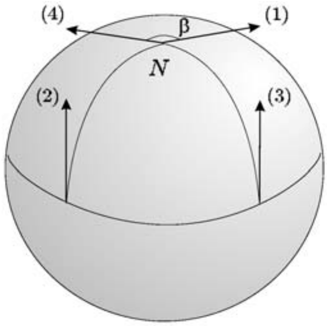

Fig. 5.3 An element of the holonomy group of S^2.

result of the transport is a rotation of angle β. As β can take any value $\beta \in [0, 2\pi]$, in fact $\mathrm{Hol}(\nabla) = SO(2)$.

(3) Let Γ be a connection in $(S^{2n+1}, \pi, \mathbb{C}P^n, S^1)$. Since $\mathbb{C}P^n$ is simply connected, the holonomy group $G = \mathrm{Hol}(\Gamma)$ at a point $u \in S^{2n+1}$ coincides with the corresponding restricted holonomy group $\mathrm{Hol}^0(\Gamma)$. Hence either $G = S^1$ or $G = \{1\}$. In the latter case, $\pi \colon S^{2n+1} \to \mathbb{C}P^n$ should admit a G-reduction $\pi \colon P \to \mathbb{C}P^n$, which should be trivial as $\mathbb{C}P^n$ is simply connected and the reduction P is a covering. Hence P admits a global section $\sigma \colon \mathbb{C}P^n \to P$ which induces a section of π, as $P \subset S^{2n+1}$. Consequently, the bundle $\pi \colon S^{2n+1} \to \mathbb{C}P^n$ should be trivial; that is, $S^{2n+1} \approx \mathbb{C}P^n \times S^1$. This leads to a contradiction as $H^2(S^{2n+1}, \mathbb{Z}) = 0$ while, by Künneth's Theorem, $H^2(\mathbb{C}P^n \times S^1, \mathbb{Z}) = \mathbb{Z}$.

(4) As in the previous case (3), $G = \mathrm{Hol}(\Gamma) = \mathrm{Hol}^0(\Gamma)$, since the quaternionic projective space $\mathbb{H}P^n$ is simply connected. Hence G cannot be discrete, because in this case, since $H^4(S^{4n+3}, \mathbb{Z}) = 0$, an argument similar to the one above applies. If $\dim G = 1$, then $G = S^1$. The case $\dim G = 2$ cannot occur by virtue of Problem 4.1.10, and $\dim G = 3$ implies $G = S^3$.

5.6 Geodesics

Problem 5.6.1. *Let $x^1 = x$, $x^2 = y$ be the usual coordinates on $\mathbb{R}^2$. Define a linear connection ∇ of $\mathbb{R}^2$ by $\Gamma^i_{jk} = 0$ except $\Gamma^1_{12} = \Gamma^1_{21} = 1$.*

(1) *Write and solve the differential equations of the geodesics.*

(2) *Is ∇ complete?*

(3) *Find the particular geodesic σ with*

$$\sigma(0) = (2, 1), \qquad \sigma'(0) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

(4) *Do the geodesics emanating from the origin go through all the points of the plane?*

(5) *If* σ *and* $\widetilde{\sigma}$ *are geodesics with* $\sigma(0) = \widetilde{\sigma}(0)$ *and* $\sigma'(0) = k\widetilde{\sigma}'(0),\ k \in \mathbb{R}$, *prove that* $\sigma(t) = \widetilde{\sigma}(kt)$ *for all possible* t.

Solution. (1) The differential equations are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0.$$

Now we obtain the equations of the geodesics through a given point (x_0, y_0); that is, such that $\sigma(0) = (x_0, y_0)$.

From the second equation we have $y = At + y_0$.

Let $A = 0$. Then the solutions are

$$x = Bt + x_0, \qquad y = y_0. \tag{$\star$}$$

Let $A \neq 0$. Then from $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2A\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$, that is, $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)\Big/\dfrac{\mathrm{d}x}{\mathrm{d}t} = -2A$, one has $\log\dfrac{\mathrm{d}x}{\mathrm{d}t} = -2At + C$, so that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = D\mathrm{e}^{-2At}$, $D \neq 0$. Therefore the equations are

$$x = \frac{D}{2A}(1 - \mathrm{e}^{-2At}) + x_0, \quad y = At + y_0, \qquad D \neq 0. \tag{$\star\star$}$$

(2) From equations ($\star$) and ($\star\star$) in (1), we see that ∇ is complete, because the geodesics are defined for $t \in (-\infty, +\infty)$.

(3) Since $\mathrm{d}y/dt = 1$, the geodesic is of the type $A \neq 0$, and one has

$$x_0 = 2, \qquad y_0 = 1, \qquad x'(0) = D = 1, \qquad y'(0) = A = 1,$$

hence

$$x = -\frac{1}{2}\mathrm{e}^{-2t} + \frac{5}{2}, \qquad y = t + 1.$$

(4) Suppose $A = 0$. Then such a geodesic is of the type $x = Bt$, $y = 0$. For $A \neq 0$ one has

$$x = \frac{D}{2A}\left(1 - \mathrm{e}^{-2At}\right), \qquad y = At.$$

That is,

$$x = \frac{D}{2A}(1 - \mathrm{e}^{-2y})$$

is the family of geodesics with $A \neq 0$ emanating from the origin. The points $(0, y)$, $y \neq 0$, are never reached from $(0,0)$. In fact, if $x = 0$, since $D/2A \neq 0$ we have $\mathrm{e}^{-2y} = 1$, thus $y = 0$. Obviously, those points are not reached either from $(0,0)$ with a geodesic such that $A = 0$.

(5) Suppose $A = 0$. Then from $\sigma(0) = \widetilde{\sigma}(0)$ it follows that

$$(x_0)_\sigma = (x_0)_{\widetilde{\sigma}}, \qquad (y_0)_\sigma = (y_0)_{\widetilde{\sigma}},$$

and from $\sigma'(0) = k\widetilde{\sigma}'(0)$ we deduce $B_\sigma = kB_{\widetilde{\sigma}}$. Hence

$$\begin{aligned}
\sigma(t) &= (B_\sigma t + (x_0)_\sigma,\, (y_0)_\sigma) \\
&= (kB_{\widetilde{\sigma}} t + (x_0)_{\widetilde{\sigma}},\, (y_0)_{\widetilde{\sigma}}) \\
&= \widetilde{\sigma}(kt).
\end{aligned}$$

Suppose now $A \neq 0$. Then from $\sigma(0) = \widetilde{\sigma}(0)$ it follows that

$$(x_0)_\sigma = (x_0)_{\widetilde{\sigma}}, \qquad (y_0)_\sigma = (y_0)_{\widetilde{\sigma}},$$

and from $\sigma'(0) = k\widetilde{\sigma}'(0)$ we deduce $A_\sigma = kA_{\widetilde{\sigma}}$, $D_\sigma = kD_{\widetilde{\sigma}}$. Thus

$$\begin{aligned}
\sigma(t) &= \left(\frac{D_\sigma}{2A_\sigma} \left(1 - \mathrm{e}^{-2A_\sigma t}\right) + (x_0)_\sigma,\, A_\sigma + (y_0)_\sigma \right) \\
&= \left(\frac{D_{\widetilde{\sigma}}}{2A_{\widetilde{\sigma}}} \left(1 - \mathrm{e}^{-2kA_{\widetilde{\sigma}} t}\right) + (x_0)_{\widetilde{\sigma}},\, kA_{\widetilde{\sigma}} t + (y_0)_{\widetilde{\sigma}} \right) \\
&= \widetilde{\sigma}(kt).
\end{aligned}$$

Problem 5.6.2. *Consider the linear connection* ∇ *of* $\mathbb{R}^2 = \{(x^1, x^2)\}$ *with components* $\Gamma_{ij}^k = 0$ *except* $\Gamma_{12}^1 = 1$*, and the curve*

$$\sigma(t) = \left(\sigma^1(t), \sigma^2(t)\right) = (-2\mathrm{e}^{-t} + 4, t + 1).$$

Compute the vector field obtained by parallel transport along σ *of its tangent vector at* $\sigma(0)$*. Is* γ *a geodesic curve?*

Solution. The tangent vector to σ at $\sigma(t)$ is

$$\sigma'(t) = 2\mathrm{e}^{-t} \left.\frac{\partial}{\partial x^1}\right|_{\sigma(t)} + \left.\frac{\partial}{\partial x^2}\right|_{\sigma(t)}.$$

Let

$$Y_{\sigma(t)} = Y^1(t) \left.\frac{\partial}{\partial x^1}\right|_{\sigma(t)} + Y^2(t) \left.\frac{\partial}{\partial x^2}\right|_{\sigma(t)}$$

be the requested vector field. The parallelism conditions are

$$\frac{\mathrm{d}\sigma^i(t)}{\mathrm{d}t} + \Gamma_{jk}^i \frac{\mathrm{d}\sigma^j(t)}{\mathrm{d}t} Y^k = 0,$$

that is

$$\frac{\mathrm{d}Y^1(t)}{\mathrm{d}t} + 2\mathrm{e}^{-t} Y^2(t) = 0, \qquad \frac{dY^2(t)}{\mathrm{d}t} = 0.$$

One easily obtains that $Y^1(t) = 2A\mathrm{e}^{-t} + B$. For $t = 0$, the vector $Y_{\sigma(0)}$ is, by hypothesis, $\sigma'(0)$. Thus $A = 1,\ B = 0$, and

$$Y = 2\mathrm{e}^{-t}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}.$$

Hence the curve is a geodesic.

Problem 5.6.3. *Let M be a C^∞ manifold with two linear connections ∇ and $\widetilde{\nabla}$ with Christoffel symbols Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$, respectively, such that $\Gamma^i_{jk} + \Gamma^i_{kj} = \widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^i_{kj}$.*

(1) *Have ∇ and $\widetilde{\nabla}$ the same geodesics?*

(2) *What intrinsic meaning has the previous condition?*

HINT (to (2)): *Use the difference tensor of ∇ and $\widetilde{\nabla}$.*

Solution. (1) The geodesics $\gamma(t) = (x^1(t),\dots,x^n(t))$ for ∇ and $\widetilde{\nabla}$ are given, respectively, by the systems of differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1,\dots,n.$$

We have $\Gamma^i_{jk}\dfrac{\mathrm{d}x^j}{\mathrm{d}t}\dfrac{\mathrm{d}x^k}{\mathrm{d}t} = \Gamma^i_{kj}\dfrac{\mathrm{d}x^j}{\mathrm{d}t}\dfrac{\mathrm{d}x^k}{\mathrm{d}t}$, from which

$$\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}.$$

If $\Gamma^i_{jk} + \Gamma^i_{kj} = \widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^i_{kj}$, then

$$\begin{aligned}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} &= \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}\\ &= \frac{1}{2}(\widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^i_{kj})\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}\\ &= \widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t},\end{aligned}$$

thus ∇ and $\widetilde{\nabla}$ have the same geodesics.

(2) It is immediate that the previous condition means that the difference tensor $A = \nabla - \widetilde{\nabla}$ is skew-symmetric.

Problem 5.6.4. *Let x^1, x^2 be the usual coordinates on $\mathbb{R}^2$. Consider the linear connection ∇ of $\mathbb{R}^2$ with components $\Gamma^k_{ij} = 0$ except $\Gamma^2_{22} = 2$, and the curve $\sigma(t) = (\mathrm{e}^{-4t} + 5, 3t + 7)$.*

(1) *Compute the vector field $Y_{\sigma(t)}$ obtained by parallel transport along σ of its tangent vector at $\sigma(2)$.*

(2) *Is σ a geodesic curve?*

Result. (1) $Y_{\sigma(t)} = -4\mathrm{e}^{-8}\dfrac{\partial}{\partial x^1} + 3\mathrm{e}^{12-6t}\dfrac{\partial}{\partial x^2}$. (2) No.

5.7 Almost Complex Manifolds

Problem 5.7.1. *An* almost complex structure *on a C^∞ manifold M is a differentiable map $J\colon TM \to TM$, such that:*

(a) *J maps linearly T_pM into T_pM for all $p \in M$.*

(b) *$J^2 = -I$ on each T_pM, where I stands for the identity map.*

Prove:

(1) *If M admits an almost complex structure (it is said that M is an* almost complex manifold*), then M has even real dimension $2n$.*

(2) *M admits an almost complex structure if and only if the structure group of the bundle of linear frames FM can be reduced to the* real representation of the general linear group *$GL(n,\mathbb{C})$, given by*

$$\begin{aligned} \rho\colon GL(n,\mathbb{C}) &\to GL(2n,\mathbb{R}) \\ A + \mathrm{i}B &\mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \end{aligned}$$

HINT: *Let f be the linear transformation of $\mathbb{R}^{2n}$ with matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Prove that the subset*

$$P = \left\{ z \in FM : f(\xi) = (z^{-1} \circ J \circ z)(\xi), \quad \forall \xi \in \mathbb{R}^{2n} \right\}$$

of the bundle of linear frames FM over M is a $GL(n,\mathbb{C})$-structure on M. The reference $z \in FM$ is viewed as an isomorphism $z\colon \mathbb{R}^{2n} \to T_{\pi(z)}M$.

Solution. (1) T_pM admits a structure of complex vector space defining a product by complex numbers by

$$(a + \mathrm{i}b)X = aX + bJ_pX, \qquad X \in T_pM, \quad a, b \in \mathbb{R}.$$

Thus the real dimension of T_pM is even, and so it is for M.

(2) We have

$$\rho(GL(n,\mathbb{C})) = \{\Lambda \in GL(2n,\mathbb{R}) : \Lambda f = f\Lambda\}.$$

In fact, decomposing Λ in $n \times n$ blocks,

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and by imposing $\Lambda f = f\Lambda$, we obtain $C = -B$, $D = A$. And conversely.

An almost complex structure J on M is a (1,1) tensor field on M such that $J^2 = -I$. The subset P of the bundle of linear frames over M, described in the hint above, determines a $GL(n,\mathbb{C})$-structure. In fact, a linear frame z at $p \in M$ is an isomorphism

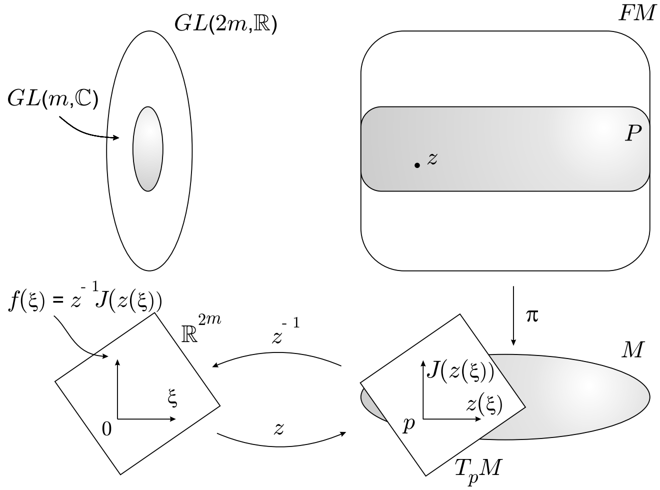

Fig. 5.4 Linear frames adapted to an almost complex structure.

$z\colon \mathbb{R}^{2n} \to T_pM$ (see Figure 5.4). Let us see that the acting group is $GL(n,\mathbb{C})$. In fact, given $z, z' \in P$, with $\pi(z) = \pi(z')$, we have

$$f = z^{-1} \circ J \circ z \quad \Leftrightarrow \quad z \circ f \circ z^{-1} = J.$$

Then $z^{-1} \circ z' \circ f \circ z'^{-1} \circ z = z^{-1} \circ J \circ z = f$, that is $z^{-1} \circ z' \in GL(n,\mathbb{C})$.

Conversely, given a $GL(n,\mathbb{C})$-structure P on M, we consider the operator J_p in T_pM such that

$$J_pX = (z^{-1} \circ J \circ z)(X), \qquad X \in T_pM, \quad z \in \pi^{-1}(p) \subset P.$$

By the definition of frame as an isomorphism of $\mathbb{R}^{2n}$ on T_pM, J_pX is an element of T_pM. J_pX does not depend, by the definition of $GL(n,\mathbb{C})$, on the element $z \in \pi^{-1}(p)$. In fact, if $z, z' \in \pi^{-1}(p)$, then there exists $g \in GL(n,\mathbb{C})$ such that $z' = zg$, and then

$$\begin{aligned} J'_pX &= \left(z' \circ f \circ z'^{-1}\right)(X) \\ &= \left(z \circ g \circ f \circ g^{-1} \circ z^{-1}\right)(X) \\ &= \left(z \circ f \circ z^{-1}\right)(X). \end{aligned}$$

Moreover, $J^2 = z \circ f \circ z^{-1} \circ z \circ f \circ z^{-1} = -I$.

Problem 5.7.2. (1) *Does the sphere S^2 admit a structure of complex manifold?*

(2) *And the sphere S^3?*

HINT: *Use the stereographic projections onto the equatorial plane, and identify this one with the complex plane $\mathbb{C}$.*

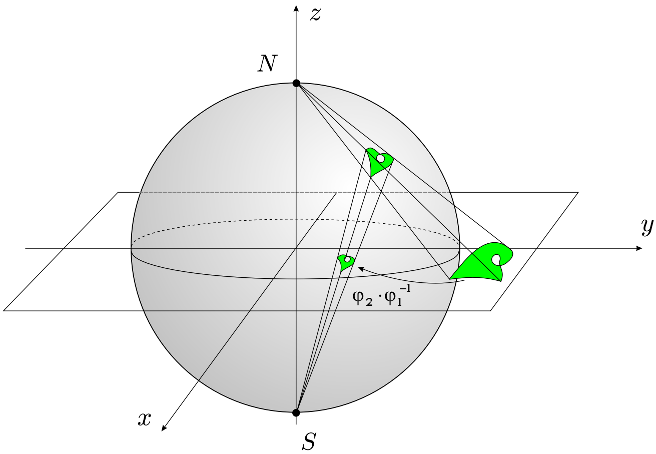

Fig. 5.5 The map $\varphi_2 \circ \varphi_1^{-1}$ changes the orientation.

Solution. (1) Let φ_1 be the stereographic projection onto the plane $z = 0$ from the north pole $N = (0,0,1) \in S^2$, and φ_2 the stereographic projection onto the plane $z = 0$ from the south pole $S = (0,0,-1) \in S^2$. We have (see Problem 1.1.9)

$$\varphi_1(a,b,c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right), \qquad \varphi_2(a,b,c) = \left(\frac{a}{1+c}, \frac{b}{1+c}\right),$$

so the changes of coordinates are

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1} : \mathbb{R}^2 - \{(0,0)\} &\to \mathbb{R}^2 - \{(0,0)\} \\ (a,b) &\mapsto \left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right). \end{aligned}$$

Identifying the plane $z = 0$ with $\mathbb{C}$, we can write

$$\varphi_1(a,b,c) = \frac{a}{1-c} + \mathrm{i}\frac{b}{1-c}, \qquad \varphi_2(a,b,c) = \frac{a}{1+c} + \mathrm{i}\frac{b}{1+c},$$

so

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1} : \mathbb{C} - \{0\} &\to \mathbb{C} - \{0\} \\ z = x + \mathrm{i}y &\mapsto \frac{x}{x^2+y^2} + \mathrm{i}\frac{y}{x^2+y^2}. \end{aligned}$$

To see that the changes of coordinates are holomorphic, we have to show that they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

where $u(x,y) = \dfrac{x}{x^2+y^2}$, $v(x,y) = \dfrac{y}{x^2+y^2}$. A computation shows that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

that is, the change of coordinates is anti-holomorphic, instead of holomorphic. This could be expected from the fact that $\varphi_2 \circ \varphi_1^{-1}$ changes the orientation (see Figure 5.5).

In order for the equations of Cauchy-Riemann to be satisfied we have to change the sign of one of the (real or imaginary) components of the change of coordinates. Consider, instead of φ_2, the new chart $\psi_2 = \bar{\varphi}_2$ given by

$$\psi_2(a,b,c) = \frac{a}{1+c} - \mathrm{i}\frac{b}{1+c}.$$

The map ψ_2 is a homeomorphism of $S^2 - \{N\}$ on $\mathbb{C} - \{0\}$, as it is the composition map

$$S^2 - \{N\} \quad \overset{\varphi_2}{\to} \quad \mathbb{C} - \{0\} \quad \overset{j}{\to} \quad \mathbb{C} - \{0\},$$

where j denotes the conjugation map. The new change of coordinates is

$$(\psi_2 \circ \psi_1^{-1})(x + \mathrm{i}y) = \frac{x}{x^2 + y^2} - \mathrm{i}\frac{y}{x^2 + y^2},$$

and it is immediate that they satisfy the Cauchy-Riemann equations.

(2) S^3 does not admit any complex structure, because a complex manifold necessarily has even real dimension (see Problem 5.7.1).

Problem 5.7.3. *Consider the torus $T^2 = S^1 \times S^1$ and let (x, y) be the canonical coordinates $(0 < x < 2\pi,\ 0 < y < 2\pi)$ on T^2. The corresponding coordinate fields define global fields denoted by $\partial/\partial x$, $\partial/\partial y$. Let J be the almost complex structure on T^2 given by*

$$J\frac{\partial}{\partial x} = -(1+\cos^2 x)\frac{\partial}{\partial y}, \qquad J\frac{\partial}{\partial y} = \frac{1}{1+\cos^2 x}\frac{\partial}{\partial x}.$$

(1) *Show that J is integrable.*

(2) *Find the corresponding chart of complex manifold.*

Solution. (1) A necessary and sufficient condition for a complex structure J to be integrable is that its Nijenhuis tensor N_J be identically zero. Since N_J is skew-symmetric in the covariant indices, we only have to show that $N_J\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ vanishes. Substituting, we have

$$\begin{aligned} N_J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= \left(\frac{\partial}{\partial x}(1+\cos^2 x)\right)\frac{1}{1+\cos^2 x}\frac{\partial}{\partial y} \\ &\quad + \left(\frac{\partial}{\partial x}\frac{1}{1+\cos^2 x}\right)(1+\cos^2 x)\frac{\partial}{\partial y} = 0. \end{aligned}$$

(2) We must find coordinates u,v such that $J\dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial v}$, $J\dfrac{\partial}{\partial v} = -\dfrac{\partial}{\partial u}$. We must have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}, \tag{$\star$}$$

and applying J in ($\star$),

$$\begin{cases} -(1+\cos^2 x)\dfrac{\partial}{\partial y} = \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial v} - \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial u} \\[2ex] \dfrac{1}{1+\cos^2 x}\dfrac{\partial}{\partial x} = \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial v} - \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial u}. \end{cases} \tag{$\star\star$}$$

From ($\star$) and ($\star\star$) it follows that

$$\begin{cases} \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial v} = (1+\cos^2 x)\dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial v} - (1+\cos^2 x)\dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial u} \\[2ex] \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial v} = -\dfrac{1}{1+\cos^2 x}\dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial v} + \dfrac{1}{1+\cos^2 x}\dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial u}. \end{cases} \tag{$\diamond$}$$

Both equations in ($\diamond$) imply:

$$\frac{\partial u}{\partial x} = -(1+\cos^2 x)\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{1}{1+\cos^2 x}\frac{\partial v}{\partial x} \tag{$\diamond\diamond$}$$

It suffices to give a particular solution of ($\diamond\diamond$). From the equations

$$v = y, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -(1+\cos^2 x),$$

one has a solution of ($\diamond\diamond$), given by

$$u = A - \frac{3}{2}x - \frac{1}{2}\sin x\cos x, \qquad v = y.$$

Problem 5.7.4. *Let $\pi\colon M \to N$ be a topological covering.*

(1) *Prove that if N is a complex manifold, then M also is a complex manifold. Equivalently, M has a unique structure of complex manifold such that π is a local diffeomorphism.*

(2) *If M is a complex manifold, is necessarily N another one?*

Solution. Let $p \in M$. We define the chart (U_p, Φ_p) around p in the following way: Let $x = \pi(p)$ and let U_x be a neighborhood of x such that U_x is the domain of a chart (U_x, φ_x) and $\pi\colon U \to \pi(U)$ is a homeomorphism. Then we define $\Phi_p = \varphi_x \circ (\pi|_{U_p})$, where U_p denotes the neighborhood of p homeomorphic to U_x by π. Thus we define an atlas on M, and we have to prove that the changes of coordinates in M

are holomorphic. Notice that $\dim M = \dim N$. Let $p, q \in M$, such that $U_p \cap U_q \neq \emptyset$, and let $x = \pi(p)$, $y = \pi(q)$. Then we have to prove that the map

$$\Phi_q \circ \Phi_p^{-1} \colon \Phi_p(U_p \cap U_q) \to \Phi_q(U_p \cap U_q)$$

is holomorphic. But

$$\begin{aligned} \Phi_q \circ \Phi_p^{-1} &= \varphi_y \circ (\pi|_{U_p \cap U_q}) \circ (\varphi_x \circ (\pi|_{U_p \cap U_q}))^{-1} \\ &= \varphi_y \circ (\pi|_{U_p \cap U_q}) \circ (\pi|_{U_p \cap U_q})^{-1} \circ \varphi_x^{-1} \\ &= \varphi_y \circ \varphi_x^{-1}, \end{aligned}$$

which is holomorphic because N is a complex manifold.

(2) It is not true in general. In fact, the map $\pi \colon S^2 \to \mathbb{R}P^2$ is a double covering. S^2 is a complex manifold, as we have seen in Problem 5.7.2, but $\mathbb{R}P^2$ is not, because it is not orientable (see Problem 3.1.4), and every complex manifold is orientable.

Problem 5.7.5. *Let X be a vector field on an almost complex manifold (M, J). Prove that the following conditions are equivalent*:

(1) $L_X J = 0$.

(2) $[X, JY] = J[X, Y], \qquad Y \in \mathfrak{X}(M)$.

Solution.

$$[X, JY] = L_X JY = (L_X J)Y + J L_X Y = (L_X J)Y + J[X, Y].$$

Problem 5.7.6. *Let (M, J) be an almost complex manifold. If ∇ is a linear connection of M whose torsion tensor T_∇ vanishes, define the linear connection $\widetilde{\nabla}$ by*

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{4}\big((\nabla_{JY} J)X + J(\nabla_Y J)X + 2J(\nabla_X J)Y\big).$$

(1) *Prove that $J\nabla_X J = -(\nabla_X J)J$.*

(2) *Compute the torsion tensor $T_{\widetilde{\nabla}}$ in terms of the Nijenhuis tensor of J.*

Solution. (1)

$$(\nabla_X J)J + J\nabla_X J = \nabla_X J^2 = \nabla_X(-I) = 0.$$

(2)

$$\begin{aligned} T_{\widetilde{\nabla}}(X, Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] \\ &= T_\nabla(X, Y) - \frac{1}{4}\big((\nabla_{JY} J)X - (\nabla_{JX} J)Y - J(\nabla_Y J)X + J(\nabla_X J)Y\big) \\ &= -\frac{1}{4}\big((\nabla_{JY} J)X - (\nabla_{JX} J)Y + (\nabla_Y J)JX - (\nabla_X J)JY\big) \\ &= -\frac{1}{4}(\nabla_{JY} JX - J\nabla_{JY} X - \nabla_{JX} JY + J\nabla_{JX} Y \end{aligned}$$

$$\begin{aligned}
&- \nabla_Y X - J\nabla_Y JX + \nabla_X Y + J\nabla_X JY) \\
&= -\frac{1}{4}([JY, JX] + [X,Y] + J[JX,Y] + J[X,JY]) \\
&= \frac{1}{4}N(X,Y),
\end{aligned}$$

where N denotes the Nijenhuis tensor of J and we have applied (1) above in the third equality.

Problem 5.7.7. *Let (M,J) be an almost complex manifold. Prove that the torsion tensor T and the curvature operator $R(X,Y)$ of an* almost complex linear connection ∇ (*that is, a linear connection such that $(\nabla_X J)Y = 0$, $X,Y \in \mathfrak{X}(M)$), satisfy the following identities*:

(1)

$$T(JX,JY) - JT(JX,Y) - JT(X,JY) - T(X,Y) = -N(X,Y),$$

where N denotes the Nijenhuis tensor of J.

(2) $R(X,Y)\circ J = J\circ R(X,Y)$.

Solution. (1)

$$\begin{aligned}
T(JX,JY) &- JT(JX,Y) - JT(X,JY) - T(X,Y) \\
&= J\nabla_{JX}Y - J\nabla_{JY}X - [JX,JY] - J\nabla_{JX}Y - \nabla_Y X + J[JX,Y] \\
&\quad + \nabla_X Y + J\nabla_{JY}X + J[X,JY] - \nabla_X Y + \nabla_Y X + [X,Y] \\
&= -N(X,Y).
\end{aligned}$$

(2)

$$\begin{aligned}
R(X,Y)JZ &= J\nabla_X\nabla_Y Z - J\nabla_Y\nabla_X Z - J\nabla_{[X,Y]}Z \\
&= JR(X,Y)Z.
\end{aligned}$$

Problem 5.7.8. *Let M be a complex manifold of complex dimension n. Let $\{z^k\}$, $k = 1,\dots,n$, be a system of complex coordinates around a given $p\in M$. If $z^k = x^k + \mathrm{i}y^k$, let $\{x^k, y^k\}$ be the corresponding system of real coordinates around p. Let T_pM, T_p^hM, and $T_p^{1,0}M$ be the* real tangent space at *p, the* holomorphic tangent space at *p, and the* space of vectors of type $(1,0)$ at *p, respectively* (*see* Definitions 7.5.9). *Prove that there exist unique $\mathbb{C}$-linear isomorphisms*

$$\Phi_p \colon T_pM \to T_p^hM, \qquad \Psi_p \colon T_p^hM \to T_x^{1,0}M,$$

with respect to the natural complex structure of each of these spaces given in 7.5.9*, such that for every system $\{z^k\}$, $k = 1,\dots,n$, we have*

$$\Phi_p\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = \left.\frac{\partial}{\partial z^k}\right|_p, \qquad k = 1,\dots,n,$$

$$\Psi_p\left(\left.\frac{\partial}{\partial z^k}\right|_p\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right)_p, \qquad k = 1,\dots,n. \tag{$\star$}$$

Solution. *Uniqueness.* An $\mathbb{R}$-basis of T_pM is $\{\partial/\partial x^k|_p, \partial/\partial y^k|_p\}$. As we have

$$\begin{aligned}\mathrm{i}\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) &= J\left(\left.\frac{\partial}{\partial x^k}\right|_p\right)\\ &= \left.\frac{\partial}{\partial y^k}\right|_p,\end{aligned}$$

it is clear that $\{\partial/\partial x^k|_p\}$ is a $\mathbb{C}$-basis of T_pM. Hence Φ_p is unique. Also Ψ_p is unique as $\{\partial/\partial z^k|_p\}$ is a $\mathbb{C}$-basis of T_p^hM and $\{\frac{1}{2}(\partial/\partial x^k - \mathrm{i}\partial/\partial y^k)_p\}$ is a $\mathbb{C}$-basis of $T_p^{1,0}M$ (see 7.5.9).

Existence. Each $X \in T_pM$ is an $\mathbb{R}$-derivation $X\colon C_p^\infty M \to \mathbb{R}$. Tensoring with $\mathbb{C}$ we obtain a $\mathbb{C}$-derivation

$$X \otimes 1\colon C_p^\infty M \otimes \mathbb{C} \to \mathbb{C}.$$

As $\mathscr{O}_pM \subset C_p^\infty M \otimes \mathbb{C}$, restricting $X \otimes 1$ to $\mathscr{O}_pM$, we obtain

$$\widetilde{X} := (X \otimes 1)|_{\mathscr{O}_pM} \in T_p^hM.$$

We define $\Phi_p\colon T_pM \to T_p^hM$ by $\Phi(X) = \widetilde{X}$. From the very definition of Φ_p we have

$$\Phi_p(X+Y) = \Phi_p(X) + \Phi_p(Y), \qquad \Phi_p(\lambda X) = \lambda\,\Phi_p(X),$$

for all $X, Y \in T_pM$, $\lambda \in \mathbb{R}$. Moreover, we have

$$\begin{aligned}(JX \otimes 1)z^k &= JX\,x^k + \mathrm{i}\,JX\,y^k\\ &= \mathrm{d}x^k|_p\,JX + \mathrm{i}\,\mathrm{d}y^k|_p\,JX\\ &= J^*(\mathrm{d}x^k|_p)X + \mathrm{i}\,J^*(\mathrm{d}y^k|_p)X\\ &= -\mathrm{d}y^k|_pX + \mathrm{i}\,\mathrm{d}x^k|_pX\\ &= \mathrm{i}\,\mathrm{d}z^k|_pX\\ &= \mathrm{i}((X \otimes 1)z^k).\end{aligned}$$

Hence, Φ_p is $\mathbb{C}$-linear.

Let us compute $\Phi_p(\partial/\partial x^k|_p)$. From the definition we obtain

$$\mathrm{d}z^\alpha|_p \left(\left.\frac{\partial}{\partial x^k}\right|_p \right)^{\sim} = \left(\left.\frac{\partial}{\partial x^k}\right|_p \right)^{\sim} (z^\alpha)$$
$$= \left(\left(\left.\frac{\partial}{\partial x^k}\right|_p \right) \otimes 1 \right) (z^\alpha) = \delta^k_\alpha.$$

Hence

$$\left(\left.\frac{\partial}{\partial x^k}\right|_p \right)^{\sim} = \delta^k_\alpha \left.\frac{\partial}{\partial z^\alpha}\right|_p = \left.\frac{\partial}{\partial z^k}\right|_p .$$

Let $\Theta_p \colon T^{1,0}_p M \to T^h_p M$ be the map given by

$$\Theta_p(Z) = \widetilde{Z} = Z|_{\mathscr{O}_p(M)}, \qquad \forall Z \in T^{1,0}_p M.$$

From its very definition it follows that Θ_p is $\mathbb{C}$-linear. Let us compute its expression in the standard basis (cf. 7.5.9). We have

$$\mathrm{d}z^\alpha|_p \left(\frac{1}{2} \left(\frac{\partial}{\partial x^k} - \mathrm{i} \frac{\partial}{\partial y^k} \right)_p \right)^{\sim} = \left(\frac{1}{2} \left(\frac{\partial}{\partial x^k} - \mathrm{i} \frac{\partial}{\partial y^k} \right)_p \right)^{\sim} (z^\alpha)$$
$$= \left(\frac{1}{2} \left(\frac{\partial}{\partial x^k} - \mathrm{i} \frac{\partial}{\partial y^k} \right)_p \right) (x^\alpha + \mathrm{i} y^\alpha) = \delta^k_\alpha.$$

Hence

$$\Theta_p \left(\frac{1}{2} \left(\frac{\partial}{\partial x^k} - \mathrm{i} \frac{\partial}{\partial y^k} \right)_p \right)^{\sim} = \delta^k_\alpha \left.\frac{\partial}{\partial z^\alpha}\right|_p = \left.\frac{\partial}{\partial z^k}\right|_p .$$

Therefore, $\Psi_p = \Theta_p^{-1}$. Moreover, the isomorphisms Φ_p and Ψ_p on each fibre extend naturally to complex vector bundle isomorphisms (see 7.5.9)

$$\Phi \colon TM \to T^h M, \qquad \Psi \colon T^h M \to T^{1,0} M.$$

We identify the bundles TM, $T^h M$ and $T^{1,0}M$ via Φ and Ψ. Under the isomorphisms Φ and Ψ, both TM and $T^{1,0}M$ are also holomorphic vector bundles.

Finally, we remark:

(a) The election in $(\star)$ is motivated by the fact that if f is a holomorphic function, i.e. $\partial f / \partial \bar{z}^k = 0$, then

$$\frac{\partial f}{\partial x^k} = \frac{\partial f}{\partial z^k} = \frac{1}{2} \left(\frac{\partial f}{\partial x^k} - \mathrm{i} \frac{\partial f}{\partial y^k} \right).$$

(b) The identification Φ is always tacitly assumed, i.e. one always writes

$$\frac{\partial}{\partial z^k} \quad \text{for} \quad \frac{1}{2} \left(\frac{\partial}{\partial x^k} - \mathrm{i} \frac{\partial}{\partial y^k} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^k} \quad \text{for} \quad \frac{1}{2} \left(\frac{\partial}{\partial x^k} + \mathrm{i} \frac{\partial}{\partial y^k} \right).$$

Chapter 6
Riemannian Geometry

6.1 Riemannian Manifolds

Problem 6.1.1. *Let* (M,g) *be a Riemannian n-manifold. Prove*:

(1) *Given* $\alpha, \beta \in T_p^*M$ *and an orthonormal basis* $\{e_i\}$, $i = 1, \ldots, n$, *of* T_pM, *and denoting by* g^{-1} *the contravariant metric associated to g, one has* $g^{-1}(\alpha,\beta) = \sum_i \alpha(e_i)\beta(e_i)$.

(2) *For* $X \in T_pM$, *one has* $g^{-1}(\alpha, X^\flat) = \alpha(X) = g(\alpha^\sharp, X)$, *where*

$$\begin{aligned} \flat &: T_pM \to T_p^*M, \qquad X^\flat = g(X,\cdot), \\ \sharp &: T_p^*M \to T_pM, \qquad \alpha^\sharp = g^{-1}(\alpha,\cdot), \end{aligned}$$

are the musical isomorphisms (*named "flat" and "sharp", respectively*) *associated to g*.

Solution. (1) In general, if $(g_{ij}(p))$ is the matrix of g with respect to $\{e_i\}$, then $(g^{ij}(p)) = (g_{ij}(p))^{-1}$ is the matrix of g^{-1} with respect to the dual basis $\{e^i\}$ to $\{e_i\}$ in T_p^*M. In this case, $(g_{ij}(p)) = (\delta_{ij})$ with respect to $\{e_i\}$, so

$$\begin{aligned} g^{-1}(\alpha,\beta) &= g^{ij}\alpha_i\beta_j \\ &= \delta^{ij}\alpha_i\beta_j \\ &= \sum_i \alpha_i\beta_i \\ &= \sum_i \alpha(e_i)\beta(e_i). \end{aligned}$$

(2)

$$\begin{aligned} g^{-1}(\alpha, X^\flat) &= g^{ij}(p)\alpha_i g_{kj}(p)X^k \\ &= \delta^i_k \alpha_i X^k \end{aligned}$$

$$
\begin{aligned}
&= \alpha_i X^i \\
&= \alpha(X), \\
g(\alpha^\sharp, X) &= g_{ij}(p) g^{ki}(p) \alpha_k X^j \\
&= \delta_j^k \alpha_k X^j \\
&= \alpha_j X^j \\
&= \alpha(X).
\end{aligned}
$$

Problem 6.1.2. *Let X_1 and X_2 be the coordinate vector fields for a set of orthogonal coordinates on a surface. Prove that there are* isothermal coordinates (*also called* conformal coordinates) *with the same domain of definition and the same coordinate curves* (*as images*) *if and only if* $X_2 X_1 \left(\log \dfrac{g_{11}}{g_{22}}\right) = 0$, *where* $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ *is the metric.*

Solution. We have orthogonal coordinates x^1, x^2 with coordinate fields $X_1 = \partial/\partial x^1$, $X_2 = \partial/\partial x^2$. Since $g(X_1, X_2) = 0$, the metric is

$$
g = g_{11}\,\mathrm{d}x^1 \otimes \mathrm{d}x^1 + g_{22}\,\mathrm{d}x^2 \otimes \mathrm{d}x^2.
$$

If there exist coordinates y^1, y^2 with the same coordinate curves (as images) it must be that

$$
\frac{\partial x^1}{\partial y^2} = \frac{\partial x^2}{\partial y^1} = \frac{\partial y^1}{\partial x^2} = \frac{\partial y^2}{\partial x^1} = 0
$$

and thus

$$
(\star) \qquad Y_1 = \frac{\partial}{\partial y^1} = \frac{\partial x^1}{\partial y^1}\frac{\partial}{\partial x^1}, \qquad Y_2 = \frac{\partial}{\partial y^2} = \frac{\partial x^2}{\partial y^2}\frac{\partial}{\partial x^2}.
$$

If the coordinates are isothermal, there exists ν such that

$$
\widetilde{g} = \nu \left(\mathrm{d}y^1 \otimes \mathrm{d}y^1 + \mathrm{d}y^2 \otimes \mathrm{d}y^2\right).
$$

That is, $\widetilde{g}_{11} = \widetilde{g}_{22} = \nu$, where $\widetilde{g}_{ij}$ are the components of g in the new coordinate system; but the change of metric is

$$
g_{ij} = \widetilde{g}_{kl} \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j},
$$

that is,

$$
\begin{aligned}
g_{11} &= \widetilde{g}_{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + \widetilde{g}_{22} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} \\
&= \left(\frac{\partial y^1}{\partial x^1}\right)^2 \widetilde{g}_{11} \\
&= \lambda(x^1) \widetilde{g}_{11},
\end{aligned}
$$

$$\begin{aligned}
g_{22} &= \widetilde{g}_{11} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} + \widetilde{g}_{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\
&= \left(\frac{\partial y^2}{\partial x^2} \right)^2 \widetilde{g}_{22} \\
&= \mu(x^2) \widetilde{g}_{22}.
\end{aligned}$$

Since $\widetilde{g}_{11} = \widetilde{g}_{22}$ it follows that $\dfrac{g_{11}}{g_{22}} = \dfrac{\lambda(x^1)}{\mu(x^2)}$. Since g is positive definite, g_{11} and $\widetilde{g}_{11}$ are positive, hence $\lambda > 0$, and similarly $\mu > 0$. Thus $\lambda(x^1)/\mu(x^2) > 0$. Taking logarithms, we have

$$\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^1} \log \frac{g_{11}}{g_{22}} = \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^1} \{\log \lambda(x^1) - \log \mu(x^2)\} = 0.$$

Conversely, if $X_2 X_1 \left(\log \dfrac{g_{11}}{g_{22}} \right) = 0$, then $\log \dfrac{g_{11}}{g_{22}} = \varphi(x^1) - \psi(x^2)$, for some functions φ, ψ, thus $\dfrac{g_{11}}{g_{22}} = \dfrac{\mathrm{e}^{\varphi(x^1)}}{\mathrm{e}^{\psi(x^2)}}$. We define

$$Y_1 = \frac{1}{\sqrt{\mathrm{e}^{\varphi(x^1)}}} X_1, \qquad Y_2 = \frac{1}{\sqrt{\mathrm{e}^{\psi(x^2)}}} X_2,$$

and coordinates y^1, y^2 such that $\dfrac{\partial}{\partial y^1} = Y_1$, $\dfrac{\partial}{\partial y^2} = Y_2$, or equivalently

$$\mathrm{d}y^1 = \sqrt{\mathrm{e}^{\varphi(x^1)}}\, \mathrm{d}x^1, \qquad \mathrm{d}y^2 = \sqrt{\mathrm{e}^{\psi(x^2)}}\, \mathrm{d}x^2.$$

The change of coordinates is possible, as the determinant of the Jacobian matrix is

$$\frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} = \frac{1}{\sqrt{\mathrm{e}^{\varphi(x^1) + \psi(x^2)}}} \neq 0.$$

In the new coordinates, the metric $\widetilde{g}$ is given by

$$\begin{aligned}
\widetilde{g}_{11} &= g_{11} \frac{\partial x^1}{\partial y^1} \frac{\partial x^1}{\partial y^1} + g_{22} \frac{\partial x^2}{\partial y^1} \frac{\partial x^2}{\partial y^1} \\
&= g_{11} \frac{1}{\mathrm{e}^{\varphi(x^1)}}, \\
\widetilde{g}_{22} &= g_{11} \frac{\partial x^1}{\partial y^2} \frac{\partial x^1}{\partial y^2} + g_{22} \frac{\partial x^2}{\partial y^2} \frac{\partial x^2}{\partial y^2} \\
&= g_{22} \frac{1}{\mathrm{e}^{\psi(x^2)}}, \\
\widetilde{g}_{12} &= \widetilde{g}_{21} = 0.
\end{aligned}$$

Hence $\dfrac{\widetilde{g}_{11}}{\widetilde{g}_{22}} = \dfrac{g_{11}}{g_{22}} \dfrac{\mathrm{e}^{\psi(x^2)}}{\mathrm{e}^{\varphi(x^1)}} = 1$. Thus y^1, y^2 are isothermal coordinates, with the same coordinate curves (as images) as x^1, x^2.

Problem 6.1.3. *Write the line element of* $\mathbb{R}^3 - \{0\}$ *in spherical coordinates, and identify* $\mathbb{R}^3 - \{0\}$ *as a* warped product.

Solution.

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= \big(\mathrm{d}(r\sin\theta\cos\varphi)\big)^2 + \big(\mathrm{d}(r\sin\theta\sin\varphi)\big)^2 + \big(\mathrm{d}(r\cos\theta)\big)^2 \\
&= \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \\
&\qquad\qquad r \in \mathbb{R}^+, \quad \theta \in [0,\pi], \quad \varphi \in [0,2\pi).
\end{aligned}$$

We have the diffeomorphism

$$\mathbb{R}^3 - \{0\} \to \mathbb{R}^+ \times S^2, \qquad v \mapsto \left(|v|, \frac{v}{|v|}\right),$$

and since for $r = 1$, $\mathrm{d}s^2$ furnishes the line element on S^2, we have, with the notation as in Definition 7.6.2,

$$\mathbb{R}^3 - \{0\} \approx \mathbb{R}^+ \times_r S^2,$$

where $\approx$ means "isometric to."

Problem 6.1.4. (The round metric on S^n) *Let* $\varphi_n \colon [-\frac{\pi}{2}, \frac{\pi}{2}]^{n-1} \times [-\pi,\pi] \to \mathbb{R}^{n+1}$ *be the map defined by the equations*:

$$\begin{cases}
x^1 = \sin\theta^1 \\
x^i = \left(\prod_{j=1}^{i-1} \cos\theta^j\right) \sin\theta^i, \quad i = 2,\dots,n \\
x^{n+1} = \prod_{j=1}^{n} \cos\theta^j,
\end{cases} \tag{$\star$}$$

with $-\frac{\pi}{2} \leqslant \theta^i \leqslant \frac{\pi}{2}$, $i = 2,\dots,n$; $-\pi \leqslant \theta^n \leqslant \pi$.

Prove:

(1) im $\varphi_n = S^n$.

(2) *The restriction of* φ_n *to* $(-\frac{\pi}{2}, \frac{\pi}{2})^n$ *is a diffeomorphism onto an open subset of the sphere.*

(3) *If* $g^{(n)} = \varphi_n^*\big((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^{n+1})^2\big)$, *then*

$$g^{(n)} = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \cos^2\theta^j\right) (\mathrm{d}\theta^i)^2, \quad \forall n \geqslant 1,$$

with $\prod_{j=1}^{k} \cos^2\theta^j = 1$ *for* $k < 1$.

Solution. (1) Let $\varphi_n = (\varphi_n^1, \dots, \varphi_n^{n+1})$ be the components of φ_n. From the very definition of this map it follows that

$$\begin{cases} \varphi_n^i = \varphi_{n-1}^i, \quad i = 1, \dots, n-1 \\ \varphi_n^n = \varphi_{n-1}^n \sin\theta^n \\ \varphi_n^{n+1} = \varphi_{n-1}^n \cos\theta^n. \end{cases} \tag{$\star\star$}$$

These formulas show, by induction on n, that $\operatorname{im}\varphi_n = S^n$, taking into account that for $n = 1$ we have $\varphi_1(\theta^1) = (\sin\theta^1, \cos\theta^1)$ and hence the statement holds obviously in this case.

(2) From the formulas $(\star)$ we obtain

$$\begin{aligned} \frac{\partial\left(x^1, \dots, x^n\right)}{\partial\left(\theta^1, \dots, \theta^n\right)} &= \begin{vmatrix} \cos\theta^1 & 0 & 0 & \cdots & 0 \\ * & \cos\theta^1\cos\theta^2 & 0 & \cdots & 0 \\ * & * & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \prod_{j=1}^n \cos\theta^j \end{vmatrix} \\ &= \cos\theta^1 \cdot (\cos\theta^1\cos\theta^2)\cdots\left(\prod_{j=1}^n \cos\theta^j\right) \\ &= \cos^n\theta^1 \cdot \cos^{n-1}\theta^2 \cdots \cos\theta^n. \end{aligned}$$

Hence on the open subset $(-\frac{\pi}{2}, \frac{\pi}{2})^{n-1} \times \left((-\pi, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (-\frac{\pi}{2}, \pi)\right)$ we have

$$\frac{\partial\left(x^1, \dots, x^n\right)}{\partial\left(\theta^1, \dots, \theta^n\right)} \neq 0.$$

Moreover, φ_n is injective on $(-\frac{\pi}{2}, \frac{\pi}{2})^{n-1} \times (-\pi, \pi)$, for $\varphi_n(\theta) = \varphi_n(\theta')$, with $\theta = (\theta^1, \dots, \theta^n)$, $\theta' = (\theta'^1, \dots, \theta'^n)$ means according to $(\star\star)$:

$$\varphi_n^i(\theta) = \varphi_n^i(\theta'), \quad i = 1, \dots, n-1, \tag{$\dagger$}$$

$$\varphi_{n-1}^n(\theta)\sin\theta^n = \varphi_{n-1}^n(\theta')\sin\theta'^n, \tag{$\dagger\dagger$}$$

$$\varphi_{n-1}^n(\theta)\cos\theta^n = \varphi_{n-1}^n(\theta')\cos\theta'^n. \tag{$\dagger\dagger\dagger$}$$

As $\varphi_{n-1}^n(\theta) > 0$, $\varphi_{n-1}^n(\theta') > 0$, from equations $(\dagger\dagger)$–$(\dagger\dagger\dagger)$ we obtain $\varphi_{n-1}^n(\theta) = \varphi_{n-1}^n(\theta')$; hence $\theta^n = \theta'^n$, and proceeding by recurrence on n, from equations $(\star)$ we conclude that $\theta = \theta'$.

(3) We have $g^{(1)} = (\mathrm{d}\theta^1)^2$ obviously. Hence the formula in (3) in the statement holds true in the case $n = 1$. Assume $n \geqslant 2$. We have

$$
\begin{aligned}
g^{(n)} &= (\mathrm{d}\varphi_n^1)^2 + \cdots + (\mathrm{d}\varphi_n^{n-1})^2 + (\mathrm{d}\varphi_n^n)^2 + (\mathrm{d}\varphi_n^{n+1})^2 \\
&= (\mathrm{d}\varphi_{n-1}^1)^2 + \cdots + (\mathrm{d}\varphi_{n-1}^{n-1})^2 + (\sin\theta^n \mathrm{d}\varphi_{n-1}^n + \varphi_{n-1}^n \cos\theta^n \mathrm{d}\theta^n)^2 \\
&\quad + (\cos\theta^n \mathrm{d}\varphi_{n-1}^n - \varphi_{n-1}^n \sin\theta^n \mathrm{d}\theta^n)^2 \\
&= (\mathrm{d}\varphi_{n-1}^1)^2 + \cdots + (\mathrm{d}\varphi_{n-1}^{n-1})^2 + (\mathrm{d}\varphi_{n-1}^n)^2 + (\varphi_{n-1}^n)^2(\mathrm{d}\theta^n)^2 \\
&= \varphi_{n-1}^*\big((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2\big) + (\varphi_{n-1}^n)^2(\mathrm{d}\theta^n)^2 \\
&= g^{(n-1)} + (\varphi_{n-1}^n)^2(\mathrm{d}\theta^n)^2 \\
&= \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1}\cos^2\theta^j\right)(\mathrm{d}\theta^i)^2 + \left(\prod_{j=1}^{n-1}\cos^2\theta^j\right)(\mathrm{d}\theta^n)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the induction hypothesis)} \\
&= \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\cos^2\theta^j\right)(\mathrm{d}\theta^i)^2.
\end{aligned}
$$

6.2 Riemannian Connections

Problem 6.2.1. *Let (M,g) be a Riemannian manifold and let ∇ denote the Levi-Civita connection. Prove the* Koszul formula:

$$
\begin{aligned}
2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
& + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y),
\end{aligned}
$$

$X, Y, Z \in \mathfrak{X}(M)$.

Solution. A linear connection ∇ of (M,g) is the Levi-Civita connection if ∇ parallelizes g and is torsionless. That is:

(a) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

(b) $\nabla_X Y - \nabla_Y X = [X,Y]$.

Now, one has

$$
\begin{aligned}
Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\
Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \\
-Zg(X,Y) &= -g(\nabla_Z X, Y) - g(X, \nabla_Z Y).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
&\qquad = g(\nabla_X Y, Z) + g([X,Z],Y) + g([Y,Z],X) + g(Z, \nabla_Y X),
\end{aligned}
$$

but $g(Z,\nabla_Y X) = g(Z,\nabla_X Y) - g(Z,[X,Y])$, so one has

$$Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z)\\ -g([Y,Z],X)+g([Z,X],Y)=2g(\nabla_X Y,Z).$$

Problem 6.2.2. *Let M be an n-dimensional Riemannian manifold, and Y a vector field defined along a curve $\gamma(t)$ in M. The covariant derivative $DY(t)/\mathrm{d}t$ of $Y(t) = Y_{\gamma(t)}$ is defined by*

$$\frac{DY(t)}{\mathrm{d}t} = \nabla_{\mathrm{d}\gamma/\mathrm{d}t} Y,$$

where ∇ denotes the Levi-Civita connection of the metric. If Y is given by $Y(t) = Y^i(t)(\partial/\partial x^i)_{\gamma(t)}$ in local coordinates x^i and $\gamma(t)$ is given by $\gamma(t) = \left(\gamma^1(t),\dots,\gamma^n(t)\right)$, then $\dfrac{\mathrm{d}\gamma}{\mathrm{d}t} = \dfrac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\dfrac{\partial}{\partial x^i}$, and

$$\frac{DY(t)}{\mathrm{d}t} = \left(\frac{dY^i(t)}{\mathrm{d}t} + \Gamma^i_{jk}\frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t}Y^k(t)\right)\frac{\partial}{\partial x^i}, \qquad (\star)$$

where Γ^i_{jk} are the Christoffel symbols of ∇ with respect to that local coordinate frame, given by

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}.$$

Let U be an open neighborhood of (u_0,v_0) in $\mathbb{R}^2$ with coordinates (u,v) and let $f\colon U \to M$ be a C^∞ map. Consider the two tangent vector fields $\partial f/\partial u$ and $\partial f/\partial v$ to the curves $v = \mathrm{const}$ and $u = \mathrm{const}$, respectively, and let $DX/\partial u$, $DX/\partial v$ be the covariant derivatives of any vector field X along these respective curves.

(1) *Using the previous expression $(\star)$ for DY/dt, prove by direct computation that $\dfrac{D}{\partial v}\dfrac{\partial f}{\partial u} = \dfrac{D}{\partial u}\dfrac{\partial f}{\partial v}$.*

(2) *Which property of the Levi-Civita connection does the equality in* (1) *correspond to?*

Solution. (1) We have $\dfrac{\partial f}{\partial u} = f_*\dfrac{\partial}{\partial u}$, $\dfrac{\partial f}{\partial v} = f_*\dfrac{\partial}{\partial v}$, and

$$\frac{D}{\partial v}\frac{\partial f}{\partial u} = \text{covariant derivative along } t \mapsto f(u,t) \text{ of } \frac{\partial f}{\partial u},$$

$$\frac{D}{\partial u}\frac{\partial f}{\partial v} = \text{covariant derivative along } t \mapsto f(t,v) \text{ of } \frac{\partial f}{\partial v}.$$

Let $\dfrac{\partial f}{\partial u} = \lambda^i\left(\dfrac{\partial}{\partial x^i}\circ f\right)$, $\dfrac{\partial f}{\partial v} = \mu^i\left(\dfrac{\partial}{\partial x^i}\circ f\right)$, where λ^i, μ^i are functions on U.

Then:

$$\begin{aligned}\frac{D}{\partial v}\frac{\partial f}{\partial u} &= \nabla_{\frac{\partial}{\partial v}}\lambda^i\left(\frac{\partial}{\partial x^i}\circ f\right)\\ &= \left(\frac{\partial \lambda^i}{\partial v}+(\Gamma^i_{jk}\circ f)\mu^j\lambda^k\right)\left(\frac{\partial}{\partial x^i}\circ f\right).\end{aligned}$$

Similarly:

$$\frac{D}{\partial u}\frac{\partial f}{\partial v} = \left(\frac{\partial \mu^i}{\partial u}+(\Gamma^i_{jk}\circ f)\lambda^j\mu^k\right)\left(\frac{\partial}{\partial x^i}\circ f\right).$$

But since

$$\begin{aligned}\frac{\partial f}{\partial u} &= \lambda^i\left(\frac{\partial}{\partial x^i}\circ f\right)\\ &= f_*\circ\frac{\partial}{\partial u}\\ &= \frac{\partial(x^i\circ f)}{\partial u}\left(\frac{\partial}{\partial x^i}\circ f\right),\end{aligned}$$

we have $\lambda^i = \dfrac{\partial(x^i\circ f)}{\partial u}$. Hence

$$\frac{\partial\lambda^i}{\partial v} = \frac{\partial^2(x^i\circ f)}{\partial v\partial u} = \frac{\partial\mu^i}{\partial u}.$$

Thus, as $\Gamma^i_{jk}=\Gamma^i_{kj}$, the claim proceeds.

(2) The property used is that ∇ is torsionless. The converse is immediate from the above local expressions of $\dfrac{D}{\partial v}\dfrac{\partial f}{\partial u}$ and $\dfrac{D}{\partial u}\dfrac{\partial f}{\partial v}$.

Problem 6.2.3. *Let (M,g) be a Riemannian manifold. Prove that for $X\in\mathfrak{X}(M)$ one has*

$$|L_Xg|^2 = 2|\nabla X|^2+2\,\mathrm{tr}\,(\nabla X\circ\nabla X)\in C^\infty M,$$

with respect to the extension of g to a metric on $T^*M\otimes T^*M$, where:

(a) $|L_Xg|$ *denotes the length of the Lie derivative L_Xg.*

(b) ∇ *denotes the Levi-Civita connection of g.*

(c) $|\nabla X|^2=\sum_i g(\nabla_{e_i}X,\nabla_{e_i}X)$, *where (e_i) is a g-orthonormal frame on a neighborhood of $p\in M$.*

(d) $\mathrm{tr}\,(\nabla X\circ\nabla X)=\sum_i g(\nabla_{\nabla_{e_i}X}X,e_i)$.

Solution. The extension of g to a metric on the fibre bundle $T^*M\otimes T^*M$ is the map

$$\langle\,,\rangle\colon(\otimes^2T^*M)\otimes(\otimes^2T^*M)\to\mathbb{R}$$

defined by

$$\langle \eta_1 \otimes \eta_2, \mu_1 \otimes \mu_2 \rangle = g(\eta_1^\sharp, \mu_1^\sharp) g(\eta_2^\sharp, \mu_2^\sharp), \qquad \eta_1, \eta_2, \mu_1, \mu_2 \in T^*M.$$

Given a g-orthonormal basis $\{e_i\}$ at $p \in M$, we have $g(X,Y) = \sum_i g(X,e_i)g(Y,e_i)$. Hence for any $\eta_1 \otimes \eta_2 \in T^*M \otimes T^*M$ we have

$$\begin{aligned}
\langle \eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2 \rangle &= \sum_{i,j} g(\eta_1^\sharp, e_i)^2 g(\eta_2^\sharp, e_j)^2 \\
&= \sum_{i,j} \eta_1(e_i)^2 \eta_2(e_j)^2 \\
&= \sum_{i,j} \big((\eta_1 \otimes \eta_2)(e_i, e_j) \big)^2,
\end{aligned}$$

so that for any $h \in T^*M \otimes T^*M$ one has $\langle h, h \rangle = \sum_{i,j} \big(h(e_i, e_j) \big)^2$.

In particular, the length of the Lie derivative of g with respect to a local orthonormal frame (e_i) in a neighborhood of $p \in M$, is given by

$$|L_X g|^2 = \sum_{i,j} \big((L_X g)(e_i, e_j) \big)^2.$$

Hence

$$\begin{aligned}
|L_X g|^2 &= \sum_{i,j} ((L_X g)(e_i, e_j))^2 \\
&= \sum_{i,j} (L_X g(e_i, e_j) - g(L_X e_i, e_j) - g(e_i, L_X e_j))^2 \\
&= \sum_{i,j} \big(X\delta_{ij} - g(\nabla_X e_i, e_j) + g(\nabla_{e_i} X, e_j) - g(e_i, \nabla_X e_j) + g(e_i, \nabla_{e_j} X) \big)^2 \\
&= \sum_{i,j} (g(\nabla_{e_i} X, e_j) + g(e_i, \nabla_{e_j} X))^2 \\
&= 2 \sum_{i,j} \big(g(\nabla_{e_j} X, e_i) g(\nabla_{e_j} X, e_i) + g(\nabla_{e_i} X, e_j) g(\nabla_{e_j} X, e_i) \big) \\
&= 2 \sum_i \left(\sum_j g\left(\nabla_{e_j} X, g(\nabla_{e_j} X, e_i) e_i\right) + g\left(\nabla_{\sum_j g(\nabla_{e_i} X, e_j) e_j} X, e_i \right) \right) \\
&= 2 \sum_i \Big(g(\nabla_{e_i} X, \nabla_{e_i} X) + g(\nabla_{\nabla_{e_i} X} X, e_i) \Big) \\
&= 2(|\nabla X|)^2 + 2 \operatorname{tr}(\nabla X \circ \nabla X).
\end{aligned}$$

Problem 6.2.4. *Let g be a* Hermitian metric *on an almost complex manifold (M, J),* i.e. *a Riemannian metric satisfying*

$$g(JX, JY) = g(X, Y), \qquad X, Y \in \mathfrak{X}(M).$$

We define a tensor field F of type $(0,2)$ on M by

$$F(X,Y)=g(X,JY), \qquad X,Y\in\mathfrak{X}(M).$$

Prove:

(1) *F is skew-symmetric (thus it is a 2-form on M, called the* fundamental 2-form *of the almost Hermitian manifold (M,g,J)).*

(2) *F is invariant by J, that is, $F(JX,JY)=F(X,Y)$.*

Suppose, moreover, that ∇ is any linear connection such that $\nabla g=0$. Then prove:

(3) $(\nabla_X F)(Y,Z)=g(Y,(\nabla_X J)Z)$.

(4) $g((\nabla_X J)Y,Z)+g(Y,(\nabla_X J)Z)=0$.

Solution. (1)

$$\begin{aligned}F(Y,X)&=g(Y,JX)\\&=g(JY,J^2X)\\&=-g(JY,X)\\&=-g(X,JY)\\&=-F(X,Y).\end{aligned}$$

(2)

$$\begin{aligned}F(JX,JY)&=g(JX,J^2Y)\\&=-g(JX,Y)\\&=-g(J^2X,JY)\\&=g(X,JY)\\&=F(X,Y).\end{aligned}$$

(3)

$$\begin{aligned}\nabla_X F(Y,Z)&=(\nabla_X F)(Y,Z)+F(\nabla_X Y,Z)+F(Y,\nabla_X Z),\\ \nabla_X g(Y,JZ)&=(\nabla_X g)(Y,JZ)+g(\nabla_X Y,JZ)+g(Y,\nabla_X JZ)\\&=g(\nabla_X Y,JZ)+g(Y,(\nabla_X J)Z)+g(Y,J\nabla_X Z)\\&=F(\nabla_X Y,Z)+g(Y,(\nabla_X J)Z)+F(Y,\nabla_X Z).\end{aligned}$$

Thus, $(\nabla_X F)(Y,Z)=g(Y,(\nabla_X J)Z)$.

(4) Since F is skew-symmetric, $\nabla_X F$ is also skew-symmetric. In fact,

$$\begin{aligned}(\nabla_X F)(Y,Z)&=\nabla_X F(Y,Z)-F(\nabla_X Y,Z)-F(Y,\nabla_X Z)\\&=-\nabla_X F(Z,Y)+F(Z,\nabla_X Y)+F(\nabla_X Z,Y)\\&=-(\nabla_X F)(Z,Y).\end{aligned}$$

Thus, by (3),

$$\begin{aligned} g((\nabla_X J)Y,Z) &= g(Z,(\nabla_X J)Y) \\ &= (\nabla_X F)(Z,Y) \\ &= -(\nabla_X F)(Y,Z) \\ &= -g(Y,(\nabla_X J)Z). \end{aligned}$$

6.3 Geodesics

Problem 6.3.1. *Compute the geodesics of* $\mathbb{R}^2$ *with the Euclidean metric.*

Solution. Since $\Gamma^i_{jk}=0$ for $g=\mathrm{d}x^1\otimes\mathrm{d}x^1+\mathrm{d}x^2\otimes\mathrm{d}x^2$, the differential equations of the geodesics are reduced to

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=0,$$

so the parametric equations of the geodesics are

$$x^1=a_1t+b_1,\quad x^2=a_2t+b_2,\qquad a_i,b_i\in\mathbb{R}.$$

That is, the geodesics are all the straight lines of $\mathbb{R}^2$ with that parametrization.

Problem 6.3.2. *Consider* $M=\mathbb{R}^2-\{(0,0)\}$ *with the usual metric* $g=\mathrm{d}x^2+\mathrm{d}y^2$ *and consider the* distance function d_g *given by*

$$\begin{array}{rcl} d_g\colon M\times M & \to & \mathbb{R}^+ \\ (p,q) & \mapsto & d_g(p,q)=\inf\displaystyle\int_0^1\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t, \end{array}$$

where γ *denotes a piecewise* C^∞ *curve with* $\gamma(0)=p$ *and* $\gamma(1)=q$.

(1) *Compute the distance between* $p=(-1,0)$ *and* $q=(1,0)$.

(2) *Is there a geodesic minimizing the distance between p and q?*

(3) *Is the topological metric space* (M,d_g) *complete?*

(4) *A Riemannian manifold is said to be* geodesically complete *if every geodesic* $\gamma(t)$ *is defined for every real value of the parameter* t. *Is in the present case M geodesically complete?*

(5) *Find an open neighborhood* U_p *for each point* $p\in M$, *such that for all* $q\in U_p$, *the distance* $d_g(p,q)$ *be achieved by a geodesic.*

Solution. (1) Let γ_a be the piecewise C^∞ curve obtained as the union of the line segment from $(-1,0)$ to $(0,a)$ and the line segment from $(0,a)$ to $(1,0)$. Since

$$d_g((-1,0),(0,a))=d_g((0,a),(1,0))=\sqrt{1+a^2},$$

we have

$$d_g((-1,0),(1,0)) \leqslant \inf_{a\to 0}\left\{2\sqrt{1+a^2}\right\} = 2.$$

On the other hand, as M is an open subset of $\mathbb{R}^2$, if $d_{\mathbb{R}^2}$ stands for the Euclidean distance, we have

$$d_g((-1,0),(1,0)) \geqslant d_{\mathbb{R}^2}((-1,0),(1,0)) = 2,$$

thus $d_g(p,q) = 2$.

(2) Since M is an open subset of $\mathbb{R}^2$, the geodesics of M are the ones of $\mathbb{R}^2$ intersecting with M. There is only one geodesic $\gamma_{\mathbb{R}^2}$ in $\mathbb{R}^2$ joining p and q, but $\gamma = \gamma_{\mathbb{R}^2} \cap M$ is not connected, and so the distance is not achieved by a geodesic.

(3) (M, d_g) is not complete. It is enough to give a counterexample: the sequence $\{(1/n, 1/n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (M, d_g) which is not convergent.

(4) (M, d_g) is not geodesically complete, because none of the lines passing (in $\mathbb{R}^2$) through the origin is a complete geodesic for the Levi-Civita connection. In fact, the geodesics $x = at,\ y = bt$ do define, for $t = 0$, no point of M.

(5) Given $p \in M$, take as U_p the open ball $B(p, |p|)$.

Problem 6.3.3. (1) *Find an example of a connected Riemannian manifold* (M, g) *to show that the property* "Any $p, q \in M$ can be joined by a geodesic whose arc length equals the distance $d_g(p,q)$" (*see* Problem 6.3.2) *does not imply that M is complete.*

(2) *Find an example of a connected Riemannian manifold to show that a minimal geodesic between two points need not be unique*; *in fact, there may be infinitely many.*

Solution. (1) The open ball

$$M = B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\} \subset (\mathbb{R}^n, g),$$

where g denotes the usual flat metric, and M is equipped with the inherited metric.

(2) The sphere (S^n, g), g being the usual metric. There exist infinitely many minimal geodesics joining two antipodal points.

Problem 6.3.4. *Consider on* $\mathbb{R}^3$ *the metric*

$$g = (1+x^2)\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{e}^z\,\mathrm{d}z^2.$$

(1) *Compute the Christoffel symbols of the Levi-Civita connection of the metric* g.

(2) *Write and solve the differential equations of the geodesics.*

(3) *Consider the curve* $\gamma(t)$ *with equations* $x = t$, $y = t$, $z = t$. *Obtain the parallel transport of the vector* $(a,b,c)_{(0,0,0)}$ *along* γ.

(4) *Is* γ *a geodesic?*

(5) *Calculate two parallel vector fields defined on γ, $X(t)$ and $Y(t)$, such that $g(X(t),Y(t))$ is constant.*

(6) *Are there two parallel vector fields defined on γ, $Z(t)$ and $W(t)$, such that $g(Z(t), W(t))$ is not constant?*

Solution. (1) We have

$$g \equiv \begin{pmatrix} 1+x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^z \end{pmatrix}, \qquad g^{-1} \equiv \begin{pmatrix} 1/(1+x^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-z} \end{pmatrix}.$$

Taking $x^1 = x$, $x^2 = y$, $x^3 = z$, the only nonvanishing Christoffel symbols are

$$\Gamma^1_{11} = \frac{x}{1+x^2}, \qquad \Gamma^3_{33} = \frac{1}{2}. \tag{$\star$}$$

(2) The differential equations of the geodesics are, by ($\star$),

$$\text{(a)} \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \text{(b)} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0, \qquad \text{(c)} \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = 0.$$

The solutions are:

(a) We can write $\dfrac{x''}{x'} + \dfrac{xx'}{1+x^2} = 0$, hence $\log x' + \dfrac{1}{2}\log(1+x^2) = \log A$, or equivalently $x' = \dfrac{A}{\sqrt{1+x^2}}$. We have $\sqrt{1+x^2}\,\mathrm{d}x = A\,\mathrm{d}t$ and

$$\begin{aligned} \int A\,\mathrm{d}t &= At + B \\ &= \int \sqrt{1+x^2}\,\mathrm{d}x \\ &= \frac{1}{2}\left(x\sqrt{1+x^2} + \log\left(x + \sqrt{1+x^2}\right)\right). \end{aligned}$$

(b) $y = Ct + D$.

(c) Let $p = \dfrac{\mathrm{d}z}{\mathrm{d}t}$. Then we have $\dfrac{\mathrm{d}p}{\mathrm{d}t} + \dfrac{p^2}{2} = 0$, from which $\dfrac{1}{p} = \dfrac{t}{2} + \dfrac{E}{2}$. Thus $\dfrac{2}{t+E} = \dfrac{\mathrm{d}z}{\mathrm{d}t}$, so that one has

$$\begin{aligned} z &= 2\log(t+E) + 2\log F \\ &= \log(Ft+G)^2. \end{aligned}$$

(3) The equations of parallel transport of the vector $X = (a^1, a^2, a^3)$ along a curve γ are $\nabla_{\gamma'} X = 0$; that is,

$$\frac{\mathrm{d}a^i}{\mathrm{d}t} + \Gamma^i_{jh}\frac{\mathrm{d}x^j}{\mathrm{d}t}a^h = 0, \qquad i = 1,2,3.$$

In this case, we have the equations:

$$\text{(a)} \quad \frac{\mathrm{d}a^1}{\mathrm{d}t} + \frac{x}{1+x^2}\frac{\mathrm{d}x}{\mathrm{d}t}a^1 = 0, \qquad \text{(b)} \quad \frac{\mathrm{d}a^2}{\mathrm{d}t} = 0, \qquad \text{(c)} \quad \frac{\mathrm{d}a^3}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}t}a^3 = 0,$$

along the curve $x = t$, $y = t$, $z = t$; that is, the previous equations are reduced to:

$$\text{(a)} \quad \frac{\mathrm{d}a^1}{\mathrm{d}t} + \frac{t}{1+t^2}a^1 = 0, \qquad \text{(b)} \quad \frac{\mathrm{d}a^2}{\mathrm{d}t} = 0, \qquad \text{(c)} \quad \frac{\mathrm{d}a^3}{\mathrm{d}t} + \frac{1}{2}a^3 = 0.$$

Integrating we have:

(a) $\log a^1 = -\frac{1}{2}\log(1+t^2) + \log A$, thus one has $a^1 = A/\sqrt{1+t^2}$, with $a^1(0) = a$, so $a^1 = a/\sqrt{1+t^2}$.

(b) $a^2 = A$, with $a^2(0) = A$; thus $a^2 = b$.

(c) $a^3 = A\mathrm{e}^{-t/2}$, with $a^3(0) = c = A$; thus $a^3 = c\,\mathrm{e}^{-t/2}$.

(4) The curve must verify the equations of the geodesics obtained in (2). Since $x(t) = t$, $y(t) = t$, $z(t) = t$, we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{x}{1+x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{t}{1+t^2} \neq 0$$

unless $t = 0$, so it is not a geodesic.

(5) We have obtained in (3) the vector field obtained by parallel transport from $(a,b,c)_{(0,0,0)}$; that is, $a^i(t) = (a/\sqrt{1+t^2}, b, c\mathrm{e}^{-t/2})$. Taking $X(0) = (1,0,0)$, $Y(0) = (0,1,0)$, one obtains under parallel transport the vector fields

$$X(t) = \left(\frac{1}{\sqrt{1+t^2}}, 0, 0\right), \qquad Y(t) = (0,1,0),$$

that satisfy $g(X(t),Y(t)) = 0$.

(6) No. In fact, consider the vectors $Z(0) = (a,b,c)$, $W(0) = (\lambda,\mu,\nu)$. Then the vector fields $Z(t)$, $W(t)$ obtained by parallel transport of the vectors along γ, satisfy

$$\begin{aligned} g(Z(t),W(t)) &= (1+t^2)\frac{a\lambda}{1+t^2} + b\mu + \mathrm{e}^t\frac{c\nu}{\mathrm{e}^t} \\ &= a\lambda + b\mu + c\nu, \end{aligned}$$

which is a constant function.

This can be obtained directly considering that ∇ is the Levi-Civita connection of g, and for all the Riemannian connections the parallel transport preserves the length and the angle.

Problem 6.3.5. *Prove with an example that there exist Riemannian manifolds on which the distance between points is bounded, that is, $d(p,q) < a$, for $a > 0$ fixed, but on which there is a geodesic with infinite length but that does not intersect with itself.*

Solution. The flat torus T^2 is endowed with the flat metric obtained from the metric of $\mathbb{R}^2$ by the usual identification $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. It is thus clear that the maximum distance is $\sqrt{2}/2$.

Nevertheless, the image curve of a straight line through the origin of $\mathbb{R}^2$ with irrational slope is a geodesic of infinite length which does not intersect itself in T^2 (see Problem 4.2.8).

Problem 6.3.6. *Give an example of a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ but such that none of its geodesics can be indefinitely extended.*

Solution. The open cube $(-1,1)^n \subset \mathbb{R}^n$, with center at $(0,\dots,0) \in \mathbb{R}^n$, is diffeomorphic to $\mathbb{R}^n$ by the map

$$\varphi\colon \mathbb{R}^n \to (-1,1)^n, \qquad (y^1,\dots,y^n) \mapsto (\tanh y^1,\dots,\tanh y^n).$$

In fact, φ is one-to-one and C^∞, and its inverse map on each component is also C^∞.

Take now on $(-1,1)^n$ the flat metric, restriction of $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$ on $\mathbb{R}^n$. It is obvious that none of the geodesics which are the connected open segments of straight lines of $\mathbb{R}^n$ in $(-1,1)^n$ can be indefinitely extended.

Problem 6.3.7. *Prove that the vertical lines $x = \mathrm{const}$ in the Poincaré upper half-plane H^2 are complete geodesics.*

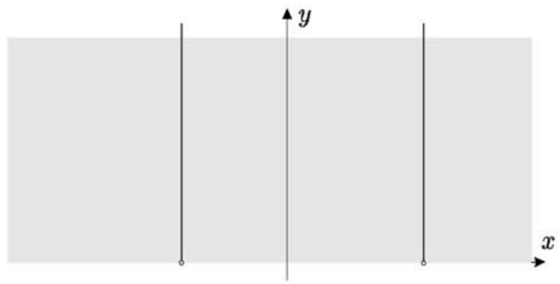

Fig. 6.1 The vertical lines of the Poincaré upper half-plane are geodesics.

Solution. We have the Riemannian manifold (M,g), where

$$M = \left\{(x,y) \in \mathbb{R}^2 \,:\, y > 0\right\}, \qquad g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

(see Figure 6.1). That is, $g_{ij} = (1/y^2)\delta_{ij}$ and $g^{ij} = y^2\delta^{ij}$, $i,j = 1,2$. Taking $x^1 = x$, $x^2 = y$, the nonvanishing Christoffel symbols are

$$\Gamma_{12}^1 = \Gamma_{21}^1 = -\Gamma_{11}^2 = \Gamma_{22}^2 = -1/y,$$

so the differential equations of the geodesics are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{2}{y}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{1}{y}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0.$$

Suppose $x(0) = x_0,\ y(0) = y_0,\ \mathrm{d}x/\mathrm{d}t = 0,\ \mathrm{d}y/\mathrm{d}t = 1$; that is, one considers the vertical line through (x_0, y_0). The previous equations are satisfied, and one has the equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2.$$

The conditions $x(0) = x_0$, $y(0) = y_0$, $(\mathrm{d}x/\mathrm{d}t)_0 = 0$, $(\mathrm{d}y/\mathrm{d}t)_0 = 1$, determine a unique geodesic. Integrating, we have $x = At + B$; and, from $y''/y' = y'/y$, one has $\log y' = \log y + C$ or equivalently $y = \mathrm{e}^{Ct+D}$. By the previous conditions, it follows that

$$x = x_0, \qquad y = y_0 \mathrm{e}^{t/y_0},$$

which proves $t \in (-\infty, +\infty)$; that is, the given geodesic is complete.

Problem 6.3.8. *Consider $\mathbb{R}^2$ with the usual flat metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$. Is the curve $\gamma(t)$ given by $x = t^3,\ \ y = t^3$, a geodesic?*

REMARK. *The fact that a curve is a geodesic depends both on its shape and its parametrization, as it is shown by the curve $\sigma(t) = (t, t)$ in $\mathbb{R}^2$ and the curve above.*

Solution. Write $\gamma(t) = (t^3, t^3)$. Then $\mathrm{d}\gamma/\mathrm{d}t = 3t^2\,\partial/\partial x + 3t^2\,\partial/\partial y$. As $\dfrac{D}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}$ in $(\mathbb{R}^2, g)$, we have

$$\begin{aligned}\frac{D}{\mathrm{d}t}\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(3t^2\frac{\partial}{\partial x} + 3t^2\frac{\partial}{\partial y}\right)\\ &= 6t\frac{\partial}{\partial x} + 6t\frac{\partial}{\partial y} \neq 0,\end{aligned}$$

hence $\gamma(t)$ is not a geodesic.

Another solution is as follows: Since γ is a geodesic curve, one should have $|\gamma'(t)| = \text{const}$, but actually $|\gamma'(t)| = 3\sqrt{2}t^2$.

6.4 The Exponential Map

Problem 6.4.1. *Consider on $\mathbb{R}^n$ with the Euclidean metric, the geodesic $\gamma(t)$ through p with unit initial velocity v_p, and let $(e_1, \dots, e_n)$ be an orthonormal frame along γ such that $e_1 = \gamma'(t)$. Compute the Fermi coordinates $(x^1, \dots, x^n)$ on $(\mathbb{R}^n, \gamma)$ relative to $(e_1, \dots, e_n)$ and p.*

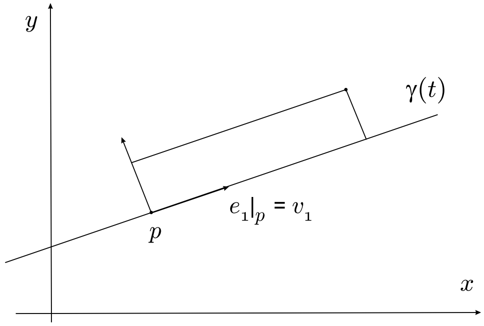

Fig. 6.2 A simple example of Fermi coordinates.

Solution. The geodesic $\gamma(t)$ through $p \in \mathbb{R}^n$ with initial velocity vector $v_p \in T_p\mathbb{R}^n$ is the straight line $\gamma(t) = p + tv_p$. Thus

$$\begin{array}{rcl} \operatorname{Exp}_p \colon T_p\mathbb{R}^n & \to & \mathbb{R}^n \\ v_p & \mapsto & \sigma(1) = p + v_p, \end{array}$$

hence

$$x^1\left(\operatorname{Exp}_{\gamma(t)}\left(\sum_{j=2}^n t^j e_j|_{\gamma(t)}\right)\right) = x^1\left(\gamma(t) + \sum_{j=2}^n t^j e_j|_{\gamma(t)}\right) = t,$$

$$x^i\left(\operatorname{Exp}_{\gamma(t)}\left(\sum_{j=2}^n t^j e_j|_{\gamma(t)}\right)\right) = t^i, \qquad 2 \leqslant i \leqslant n.$$

Since Exp_p is a global diffeomorphism, we have a new set of coordinates on $\mathbb{R}^n$. The first coordinate is the distance from the origin p along γ and the other coordinates are the orthogonal coordinates relative to $e_2, \dots, e_n$ (see Figure 6.2).

Problem 6.4.2. *Let M be an n-dimensional complete Riemannian manifold and let $q \in M$. Identify T_qM with $\mathbb{R}^n$ as a manifold by choosing an orthonormal basis at q. Then $\operatorname{Exp}_q \colon T_qM \to M$ is a C^∞ map of $\mathbb{R}^n$ onto M, mapping 0 to q.*

(1) *Suppose $M = S^n$, the unit sphere with its usual metric. Prove that*

$$\operatorname{rank}\left(\operatorname{Exp}_q\right)_{*X_q} < n, \qquad \textit{if} \quad |X_q| = k\pi,\ k = 1, 2, \dots,$$

without using Jacobi fields.

(2) *Find $(\operatorname{Exp}_N)_{*X}(e_1)$ and $(\operatorname{Exp}_N)_{*X}(e_2)$ for two orthonormal vectors $e_1, e_2 \in T_XT_NS^2$, and $X = \lambda e_1$, $e_1 \in T_NS^2$. In particular, find the values of the two above vectors if $\lambda = 0$, $\pi/2$, or π.*

Solution. (1) The geodesic through q with initial vector X_q is (see Problem 6.8.4) the great circle

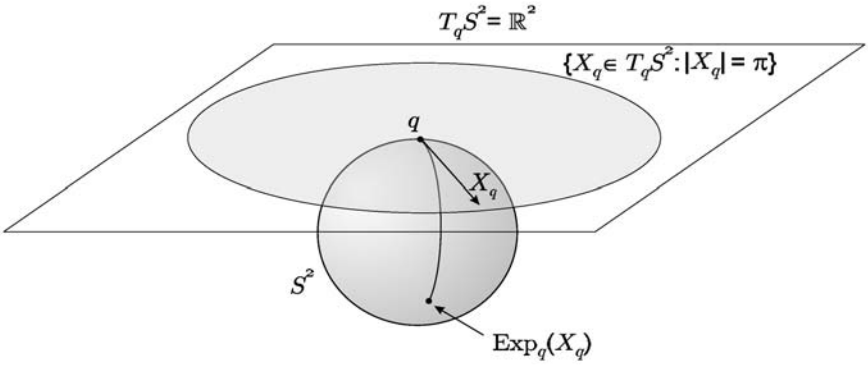

Fig. 6.3 The Exponential map on S^2 at q.

$$\gamma(t) = (\cos|X_q|t)q + (\sin|X_q|t)\frac{X_q}{|X_q|},$$

hence (see Figure 6.3)

$$\begin{aligned} \mathrm{Exp}_q(X_q) &= \gamma(1) \\ &= (\cos|X_q|)q + (\sin|X_q|)\frac{X_q}{|X_q|}. \end{aligned}$$

We can take, without loss of generality,

$$q = (0, \ldots, 0, 1) = N \in S^n \subset \mathbb{R}^{n+1}.$$

Thus, the map Exp_q is given by

$$\begin{array}{rcl} \mathrm{Exp}_N\colon T_NS^n & \to & S^n \\ X = (X_1, \ldots, X_n) & \mapsto & \left(\dfrac{\sin|X|}{|X|}X_1, \ldots, \dfrac{\sin|X|}{|X|}X_n, \cos|X|\right) \end{array}$$

(where we have simplified X_N to X) and has Jacobian matrix $(\mathrm{Exp}_N)_*$ given by

$$\begin{pmatrix} \frac{\sin|X|}{|X|}\left(1 - \frac{X_1^2}{|X|^2}\right) + \frac{\cos|X|}{|X|^2}X_1^2 & \left(\cos|X| - \frac{\sin|X|}{|X|}\right)\frac{X_1X_2}{|X|^2} & \cdots & \left(\cos|X| - \frac{\sin|X|}{|X|}\right)\frac{X_1X_n}{|X|^2} \\ \left(\cos|X| - \frac{\sin|X|}{|X|}\right)\frac{X_1X_2}{|X|^2} & \frac{\sin|X|}{|X|}\left(1 - \frac{X_2^2}{|X|^2}\right) + \frac{\cos|X|}{|X|^2}X_2^2 & \cdots & \left(\cos|X| - \frac{\sin|X|}{|X|}\right)\frac{X_2X_n}{|X|^2} \\ \vdots & & & \vdots \\ \left(\cos|X| - \frac{\sin|X|}{|X|}\right)\frac{X_1X_n}{|X|^2} & & \cdots & \frac{\sin|X|}{|X|}\left(1 - \frac{X_n^2}{|X|^2}\right) + \frac{\cos|X|}{|X|^2}X_n^2 \\ -\frac{\sin|X|}{|X|}X_1 & & \cdots & -\frac{\sin|X|}{|X|}X_n \end{pmatrix}$$

Suppose $|X| = k\pi$, $k = 1, 2, \ldots$, then

$$(\mathrm{Exp}_N)_{*X} = t\begin{pmatrix} \frac{(-1)^k}{k^2\pi^2}X_1^2 & \frac{(-1)^k}{k^2\pi^2}X_1X_2 & \cdots & \frac{(-1)^k}{k^2\pi^2}X_1X_n \\ \vdots & & & \vdots \\ \frac{(-1)^k}{k^2\pi^2}X_1X_n & \frac{(-1)^k}{k^2\pi^2}X_2X_n & \cdots & \frac{(-1)^k}{k^2\pi^2}X_n^2 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Since

$$\det\left(\frac{(-1)^k}{k^2\pi^2}X_iX_j\right) = X_1^2X_2^2\cdots X_n^2\left(\frac{(-1)^k}{k^2\pi^2}\right)^n \det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = 0,$$

we obtain $\mathrm{rank}(\mathrm{Exp}_N)_{*X} < n$. (2) We have $X = (X_1, X_2) = (\lambda, 0) = \lambda e_1 \in T_N S^2$,

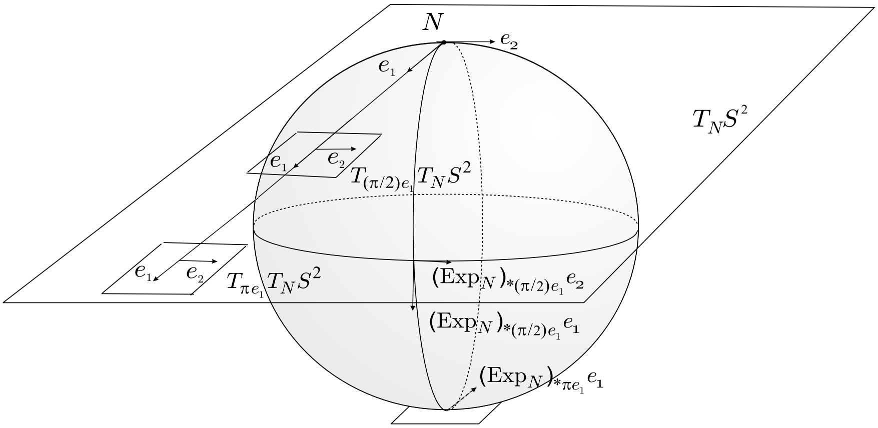

Fig. 6.4 The differential of the Exponential map on S^2 at the north pole.

hence

$$(\mathrm{Exp}_N)_{*\lambda e_1} = \begin{pmatrix} \cos\lambda & 0 \\ 0 & \frac{\sin\lambda}{\lambda} \\ -\sin\lambda & 0 \end{pmatrix}.$$

In particular:

$$(\mathrm{Exp}_N)_{*0e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\mathrm{Exp}_N)_{*\frac{\pi}{2}e_1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{\pi} \\ -1 & 0 \end{pmatrix}, \quad (\mathrm{Exp}_N)_{*\pi e_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

hence

$$
\begin{aligned}
(\mathrm{Exp}_N)_{*\lambda e_1}(e_1) &= \cos\lambda\, e_1 - \sin\lambda\, e_3 \\
&= \begin{cases} e_1 & \text{if } \lambda = 0 \\ -e_3 & \text{if } \lambda = \dfrac{\pi}{2} \\ -e_1 & \text{if } \lambda = \pi, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Exp}_N)_{*\lambda e_1}(e_2) &= \frac{\sin\lambda}{\lambda} e_2 \\
&= \begin{cases} e_2 & \text{if } \lambda = 0 \\ \dfrac{2}{\pi} e_2 & \text{if } \lambda = \dfrac{\pi}{2} \\ 0 & \text{if } \lambda = \pi, \end{cases}
\end{aligned}
$$

where the vectors in parentheses $e_1, e_2 \in T_{\lambda e_1}(T_N S^2)$ (see Figure 6.4).

Problem 6.4.3. *Show that if the Riemannian manifold (M,g) is complete and contains a point which has no conjugate points, then M is covered by $\mathbb{R}^n$.*

Solution. Let $p \in M$ be a point without conjugate points. As M is complete, the exponential map $\mathrm{Exp}_p \colon T_pM \to M$ is everywhere defined on the tangent space and it is surjective. Moreover, as is well known (see Definitions 7.6.5), $\mathrm{Exp}_p X$ is conjugate to p if and only if Exp_p is critical at X. Hence, by virtue of the hypothesis, Exp_p has no critical point. Accordingly, Exp_p is a surjective local diffeomorphism. Endow T_pM with the metric $\mathrm{Exp}_p^* g$ induced by Exp_p. Then it is clear that $\mathrm{Exp}_p \colon (T_pM, \mathrm{Exp}_p^* g) \to (M,g)$ is a local isometry. Since it applies each ray $t \mapsto tv$ to the geodesic curve γ_v, one deduces that these rays are geodesics, so that the manifold T_pM is complete at 0. The result thus follows from Theorem 7.6.14.

Problem 6.4.4. *Determine the cut locus of S^n.*

Solution. All the geodesics are minimizing before distance π. For a point $p \in S^n$, we have that Exp is a diffeomorphism on $U_p = B(p,\pi) \subset T_pS^n$ and that $\mathrm{Exp}_p(U_p) = S^n - \{-p\}$. Hence

$$\mathrm{Cut}(p) = \mathrm{Exp}(\partial U_p) = \{-p\},$$

that is, the cut locus is reduced to the antipodal point.

6.5 Curvature and Ricci Tensors

Problem 6.5.1. *Find the Riemann curvature tensor of the Riemannian manifold (U,g), where U denotes the unit open disk of the plane $\mathbb{R}^2$ and*

$$g = \frac{1}{1-x^2-y^2}(\mathrm{d}x^2+\mathrm{d}y^2).$$

Solution. We have

$$g^{-1} \equiv \begin{pmatrix} 1-x^2-y^2 & 0 \\ 0 & 1-x^2-y^2 \end{pmatrix}.$$

So, taking $x = x^1$, $y = x^2$, the Christoffel symbols are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \frac{x}{1-x^2-y^2},$$
$$-\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \frac{y}{1-x^2-y^2}.$$

Therefore,

$$\begin{aligned} R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) &= g\left(R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right) \\ &= -\frac{2}{(1-x^2-y^2)^2}. \end{aligned}$$

Problem 6.5.2. *Consider on $\mathbb{R}^3$ the metric*

$$g = \mathrm{e}^{2z}(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).$$

Compute $R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z},\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)$, where R denotes the Riemann curvature tensor.

Solution.

$$g^{-1} \equiv \begin{pmatrix} \mathrm{e}^{-2z} & 0 & 0 \\ 0 & \mathrm{e}^{-2z} & 0 \\ 0 & 0 & \mathrm{e}^{-2z} \end{pmatrix}.$$

So, taking $x = x^1,\ y = x^2,\ z = x^3$, the only nonvanishing Christoffel symbols are

$$\Gamma^1_{13} = \Gamma^2_{23} = -\Gamma^3_{11} = -\Gamma^3_{22} = \Gamma^3_{33} = 1.$$

Therefore

$$R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z},\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right) = g\left(R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z},\frac{\partial}{\partial x}\right) = 0.$$

Problem 6.5.3. *Let (M,g) be a Riemannian n-manifold. Consider an orthonormal basis $\{e_1,\ldots,e_{n-1},X\}$ of $T_pM,\ p\in M$. Let P_i be the plane section generated by e_i and X; $K(P_i)$ the sectional curvature of P_i; and r the Ricci tensor. Prove that*

$$r(X,X) = \sum_{i=1}^{n-1} K(P_i).$$

Solution. For $R(X,e_i)e_i$ and $R(X,e_i,X,e_i)$ as in Definition 7.6.4, we have

$$\begin{aligned} K(P_i) &= \frac{R(X,e_i,X,e_i)}{g(X,X)g(e_i,e_i)-g(X,e_i)^2} \\ &= \frac{g(R(X,e_i)e_i,X)}{g(X,X)g(e_i,e_i)-g(X,e_i)^2} \\ &= g(R(X,e_i)e_i,X). \end{aligned}$$

On the other hand, with respect to the given orthonormal basis we have

$$r(X,X) = \sum_{i=1}^{n-1} g(R(e_i,X)X,e_i).$$

Therefore, $r(X,X) = \sum_{i=1}^{n-1} K(P_i)$.

Problem 6.5.4. *Prove the following consequence of Bianchi's second identity on a Riemannian manifold* (M,g):

$$\mathrm{d}s = 2\,\mathrm{div}\, r,$$

where r *and* s *denote the Ricci tensor and the scalar curvature of the Levi-Civita connection.*

Solution. Let us fix a point $p \in M$ and consider the normal coordinates with origin p, associated to an orthonormal basis $\{\varepsilon_i\}$ of T_pM. We can get a local orthonormal moving frame (e_i) by parallel transport of $\{\varepsilon_i\}$ along radial geodesics, so $e_i|_p = \varepsilon_i$, and $\nabla e_i = 0$, along a radial geodesic; in particular, $(\nabla e_i)_p = 0$, where ∇ stands for the Levi-Civita connection.

Further, recall that $(R(X,Y)Z)(p)$, the curvature tensor field at any point p, depends only on the values of the vector fields X,Y,Z at p, so that if either X_p, or Y_p, or Z_p is zero, then $(R(X,Y)Z)(p) = 0$.

On the other hand, since ∇ is torsionless, the second Bianchi identity can be written as

$$g((\nabla_X R)(Y,Z)W,U) + g((\nabla_Y R)(Z,X)W,U) + g((\nabla_Z R)(X,Y)W,U) = 0,$$

$X,Y,Z,W,U \in \mathfrak{X}(M)$. Interchanging X and Y in the third summand and then contracting all the summands with respect to X and U, we have

$$\sum_i \big\{ g((\nabla_{e_i} R)(Y,Z)W,e_i) + g((\nabla_Y R)(Z,e_i)W,e_i) - g((\nabla_Z R)(Y,e_i)W,e_i) \big\} = 0. \quad (\star)$$

For the second summand in $(\star)$ we have

$$\begin{aligned} &\sum_i g((\nabla_Y R)(Z,e_i)W,\ e_i)(p) \\ &\qquad = \sum_i \big\{ g(\nabla_Y (R(Z,e_i)W),e_i) - g(R(\nabla_Y Z,e_i)W,e_i) \end{aligned}$$

$$
\begin{aligned}
&\quad - g(R(Z,\nabla_Y e_i)W,e_i) - g(R(Z,e_i)\nabla_Y W,e_i)\big\}(p) \\
&= \left\{\sum_i \big\{Y(g(R(Z,e_i)W,e_i)) - r(\nabla_Y Z,W) - r(Z,\nabla_Y W)\right\}(p) \\
&= \big\{Y(r(Z,W)) - r(\nabla_Y Z,W) - r(Z,\nabla_Y W)\big\}(p) \\
&= (\nabla_Y r)(Z,W)(p).
\end{aligned}
$$

Similarly, for the third summand in $(\star)$ we have

$$
-\sum_i g((\nabla_Z R)(Y,e_i)W,e_i)(p) = -(\nabla_Z r)(Y,W)(p).
$$

So, at the point p we can write $(\star)$ as

$$
\left\{\sum_i g((\nabla_{e_i} R)(Y,Z)W,e_i) + (\nabla_Y r)(Z,W) - (\nabla_Z r)(Y,W)\right\}(p) = 0. \qquad (\star\star)
$$

Contracting $(\star\star)$ with respect to Y and W, we obtain

$$
\sum_{i,j}\big\{g((\nabla_{e_i} R)(e_j,Z)e_j,e_i) + (\nabla_{e_j} r)(Z,e_j) - (\nabla_Z r)(e_j,e_j)\big\}(p) = 0,
$$

or equivalently

$$
\begin{aligned}
0 &= \left\{\sum_{i,j}\big\{g(\nabla_{e_i}(R(e_j,Z)e_j),e_i) - g(R(\nabla_{e_i}e_j,Z)e_j,e_i)\right. \\
&\quad - g(R(e_j,\nabla_{e_i}Z)e_j,e_i) - g(R(e_j,Z)\nabla_{e_i}e_j,e_i)\big\} + (\operatorname{div} r)Z \\
&\quad \left. - \sum_j \big\{Z(r(e_j,e_j)) - r(\nabla_Z e_j,e_j) - r(e_j,\nabla_Z e_j)\big\}\right\}(p) \\
&= \Big\{\sum_{j,j}\big\{e_i(r(Z,e_i)) - g(R(e_j,Z)e_j,\nabla_{e_i}e_i) - r(\nabla_{e_i}Z,e_i)\big\} \\
&\quad + (\operatorname{div} r)Z - Zs\Big\}(p) \\
&= \left\{\sum_i (\nabla_{e_i} r)(e_i,Z) + (\operatorname{div} r)Z - Zs\right\}(p) \\
&= \big\{2(\operatorname{div} r)Z - Zs\big\}(p),
\end{aligned}
$$

for every $p \in M$; that is, $((2\operatorname{div} r - \mathrm{d}s)Z)(p) = 0$ for all $Z \in \mathfrak{X}(M)$ and all $p \in M$.

6.6 Characteristic Classes

Problem 6.6.1. *Consider the complex projective space* $\mathbb{C}P^1$ *equipped with the Hermitian metric* $g = h(z)\,(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z)$, *where*

$$h(z) = \frac{1}{(1+|z|^2)^2}. \tag{$\star$}$$

If $w = 1/z$ *is the coordinate at infinity, then the metric is given by* $g = h(w)(\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w)$.

Prove that the Chern class of the tangent bundle $T\mathbb{C}P^1$ *is nonzero.*

Solution. Since the Chern classes of a complex vector bundle does not depend on the particular connection chosen to define them, we choose here the canonical Hermitian connection, which, for a given h defined by $h(z_0) = h\left(\left.\frac{\partial}{\partial z}\right|_{z_0}, \left.\frac{\partial}{\partial z}\right|_{z_0}\right)$, is the connection with connection form and curvature form relatives to the holomorphic moving frame $\partial/\partial z$, given by

$$\begin{aligned}
\widetilde{\omega} &= h^{-1}\partial h = h^{-1}\frac{\partial h}{\partial z}\mathrm{d}z,\\
\widetilde{\Omega} &= \bar{\partial}\widetilde{\omega} = \frac{\partial \widetilde{\omega}}{\partial \bar{z}}\mathrm{d}\bar{z},
\end{aligned}$$

respectively. Then we have, for the metric h in $(\star)$, the Chern form

$$\begin{aligned}
c_1(T\mathbb{C}P^1, \widetilde{\omega}) &= \frac{\mathrm{i}}{2\pi}\widetilde{\Omega}\\
&= \frac{\mathrm{i}}{2\pi}\bar{\partial}\frac{\partial h(z)}{h(z)}\\
&= \frac{\mathrm{i}}{2\pi}\bar{\partial}\left((1+|z|^2)^2\partial\frac{1}{(1+|z|^2)^2}\right)\\
&= \frac{\mathrm{i}}{\pi(1+|z|^2)^2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\\
&= \frac{2}{\pi(1+|z|^2)^2}\mathrm{d}x\wedge\mathrm{d}y.
\end{aligned}$$

By taking polar coordinates, it is easily seen that

$$\int_{\mathbb{C}P^1} c_1(T\mathbb{C}P^1, \widetilde{\omega}) = 2.$$

By Stokes' Theorem, the Chern form $c_1(T\mathbb{C}P^1, \widetilde{\omega})$ cannot be exact. Thus the Chern class is $c_1(T\mathbb{C}P^1) = 2\alpha \neq 0$, where α denotes the standard generator of the cohomology group $H^2(\mathbb{C}P^1, \mathbb{Z}) \approx \mathbb{Z}$; and $T\mathbb{C}P^1$ is thus a nontrivial complex line bundle.

Problem 6.6.2. *Prove that the Pontrjagin forms of a space M of constant curvature K vanish.*

Solution. The curvature forms Ω^i_j of the Levi-Civita connnection on the bundle of orthonormal frames are given in terms of the components θ^k of the canonical form on the bundle of orthonormal frames (which is the restriction of the canonical form on the bundle of linear frames) by

$$\Omega^i_j = K\theta^i \wedge \theta^j.$$

Hence, by the formula on page 406, the rth Pontrjagin form, denoted here by p_r, is given, for $r = 1,\dots,\dim M/4$, by

$$\begin{aligned} p^*(p_r) &= \frac{1}{(2\pi)^{2r}(2r)!}\,\delta^{j_1\dots j_{2r}}_{i_1\dots i_{2r}}\,\Omega^{i_1}_{j_1}\wedge\dots\wedge\Omega^{i_{2r}}_{j_{2r}} \\ &= \frac{K^{2r}}{(2\pi)^{2r}(2r)!}\sum_{\substack{i_1,\dots,i_{2r}\\ j_1,\dots,j_{2r}}}\delta^{j_1\dots j_{2r}}_{i_1\dots i_{2r}}\,\theta^{i_1}\wedge\theta^{j_1}\wedge\dots\wedge\theta^{i_{2r}}\wedge\theta^{j_{2r}}, \end{aligned}$$

where p denotes the projection map of the bundle of orthonormal frames.

The δ's vanish unless $j_1,\dots,j_{2r}$ is a permutation of $i_1,\dots,i_{2r}$, but then the wedge product of θ's has repeated factors, so $p^*(p_r)$ vanishes. As p^* is injective, p_r also vanishes.

Problem 6.6.3. *Let M be a 4-dimensional compact oriented C^∞ manifold. Let Ω^i_j, $i,j = 1,\dots,4$, be the curvature forms of a linear connection of M, and $\widetilde{\Omega}^i_j$ given by $\Omega^j_i = \sigma^*\widetilde{\Omega}^i_j$, the curvature forms relative to any fixed orthonormal moving frame σ on M. Prove that the signature $\tau(M)$ can be expressed by*

$$\tau(M) = -\frac{1}{24\pi^2}\int_M \widetilde{\Omega}^i_j \wedge \widetilde{\Omega}^j_i.$$

HINT: *Use Hirzebruch's formula in* Theorem 7.6.10.

Solution. To apply Hirzebruch's Theorem, we need to compute a representative form of the first Pontrjagin class of M; that is, of the first Pontrjagin class of the tangent bundle TM. The principal $GL(4,\mathbb{R})$-bundle corresponding to TM is the frame bundle (FM,p,M), where the given connection is defined. Now, by Weil's Theorem, the characteristic class does not depend on the chosen connection. Thus, as we can always reduce the structure group to the orthogonal group $O(4)$ (here, even to $SO(4)$, since M is oriented) or, equivalently, take a Riemannian metric on M, the matrix of curvature 2-forms of the connection is antisymmetric, as it takes values in the Lie algebra $\mathfrak{so}(4)$. We shall compute the Pontrjagin form $p_1(M)$ in terms of the curvature forms of the metric connection in two related ways, which is perhaps instructive. The form $p_1(M)$ is given by

$$\begin{aligned}
p^*(p_1(M)) &= \text{term of } \det\left(I - \frac{1}{2\pi}\Omega\right) \text{ quadratic in the } \Omega\text{'s} \\
&= \frac{1}{4\pi^2}(-\Omega_2^1 \wedge \Omega_1^2 - \Omega_3^1 \wedge \Omega_1^3 - \Omega_4^1 \wedge \Omega_1^4 \\
&\qquad - \Omega_3^2 \wedge \Omega_2^3 - \Omega_4^2 \wedge \Omega_2^4 - \Omega_4^3 \wedge \Omega_3^4) \\
&= \frac{1}{4\pi^2}\left(-\frac{1}{2}\operatorname{tr}(\Omega \wedge \Omega)\right).
\end{aligned}$$

We can also directly use the formula on page 406, as follows. Let (i_1, i_2) be an ordered subset of $\{1,2,3,4\}$, (j_1, j_2) a permutation of (i_1, i_2), and $\delta_{i_1 i_2}^{j_1 j_2}$ the sign of the permutation. Then

$$\begin{aligned}
p^*(p_1(M)) &= \frac{1}{(2\pi)^2 2!}\,\delta_{i_1 i_2}^{j_1 j_2}\,\Omega_{j_1}^{i_1} \wedge \Omega_{j_2}^{i_2} \\
&= \frac{1}{8\pi^2}(-\Omega_2^1 \wedge \Omega_1^2 - \Omega_3^1 \wedge \Omega_1^3 - \Omega_4^1 \wedge \Omega_1^4 - \Omega_1^2 \wedge \Omega_2^1 \\
&\qquad - \Omega_3^2 \wedge \Omega_2^3 - \Omega_4^2 \wedge \Omega_2^4 - \Omega_1^3 \wedge \Omega_3^1 - \Omega_2^3 \wedge \Omega_3^2 \\
&\qquad - \Omega_4^3 \wedge \Omega_3^4 - \Omega_1^4 \wedge \Omega_4^1 - \Omega_2^4 \wedge \Omega_4^2 - \Omega_3^4 \wedge \Omega_4^3) \\
&= -\frac{1}{8\pi^2}\operatorname{tr}(\Omega \wedge \Omega).
\end{aligned}$$

Furthermore, since for any given invariant polynomial in the curvature, the corresponding differential form on the base space (see, for instance, [22, p. 295], [26, vol. IV, L. 22]) does not depend on the chosen orthonormal moving frame, by applying Hirzebruch's formula, we can write:

$$\tau(M) = \frac{1}{3}\int_M p_1(M) = -\frac{1}{3}\int_M \frac{1}{8\pi^2}\widetilde{\Omega}_j^i \wedge \widetilde{\Omega}_i^j,$$

as stated.

Problem 6.6.4. *Let g be the bi-invariant metric on $SO(3)$. Calculate the Chern-Simons invariant $J(SO(3), g)$.*

REMARK. *For the related definitions and results, see* 7.6.11.

Solution. Let X_1, X_2, X_3 be the standard basis of TS^3, that is

$$\begin{aligned}
X_1 &= \frac{1}{2}\left(-x^1\frac{\partial}{\partial x^0} + x^0\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}\right), \\
X_2 &= \frac{1}{2}\left(-x^2\frac{\partial}{\partial x^0} - x^3\frac{\partial}{\partial x^1} + x^0\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3}\right), \\
X_3 &= \frac{1}{2}\left(-x^3\frac{\partial}{\partial x^0} + x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + x^0\frac{\partial}{\partial x^3}\right).
\end{aligned}$$

From Problem 4.6.9, it follows that X_1, X_2, X_3 are left-invariant vector fields, and as the Lie groups S^3 and $SO(3)$ have the same Lie algebra, X_1, X_2, X_3 can be considered as left-invariant vector fields on $SO(3)$.

We have

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2.$$

Let $\alpha^i_j = -\alpha^j_i$, $i, j = 1,2,3$, be the Maurer-Cartan forms on $SO(3)$, determined by

$$\alpha^3_2(X_i) = \delta_{1i}, \qquad \alpha^3_1(X_i) = \delta_{2i}, \qquad \alpha^2_1(X_i) = \delta_{3i},$$

so that the structure equations of the Lie group, are

$$\mathrm{d}\alpha^i_j = \alpha^i_k \wedge \alpha^k_j, \qquad i, j = 1,2,3. \tag{$\star$}$$

The bi-invariant metric on $SO(3)$ is given by

$$g = \alpha^1_2 \otimes \alpha^1_2 + \alpha^1_3 \otimes \alpha^1_3 + \alpha^2_3 \otimes \alpha^2_3.$$

In writing these equations one has chosen a basis of the Lie algebra of $SO(3)$ and hence, by right translations, a frame field on the manifold $SO(3)$. It is convenient to choose the notation so that the equations remain invariant under a cyclic permutation of $1,2,3$. Setting $\theta^i = \alpha^k_j$, $i, j, k =$ cyclic permutation of $1,2,3$, the invariant metric becomes

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3.$$

The connection and curvature forms $\omega^i_j = -\omega^j_i$, $\Omega^i_j = -\Omega^j_i$, are determined by Cartan's structure equations

$$\mathrm{d}\theta^i = -\omega^i_j \wedge \theta^j, \qquad \mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j.$$

Comparing these equations with the structure equations ($\star$) of the Lie group, one finds

$$\omega^i_j = \frac{1}{2}\theta^k, \qquad \Omega^i_j = -\frac{1}{4}\theta^i \wedge \theta^j.$$

Hence

$$\begin{aligned}\frac{1}{2} TP_1(\Omega) &= \frac{1}{8\pi^2} \sum_{1 \leqslant i < j \leqslant 3} \omega^i_j \wedge \Omega^i_j - \frac{1}{8\pi^2}\, \omega^1_2 \wedge \omega^2_3 \wedge \omega^3_1 \\ &= -\frac{1}{16\pi^2}\, \theta^1 \wedge \theta^2 \wedge \theta^3.\end{aligned} \tag{$\star\star$}$$

Let us compute the total volume of $SO(3)$ with its bi-invariant metric. We have

$$\mathrm{vol}_g\,(SO(3)) = \mathrm{vol}_g\,(\mathbb{R}P^3) = \frac{1}{2}\mathrm{vol}_g\,(S^3).$$

Hence we only need to calculate $\mathrm{vol}_g\,(S^3)$. Write

$$\theta^1 \wedge \theta^2 \wedge \theta^3 = \rho \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

By calculation we obtain

$$(\theta^1 \wedge \theta^2 \wedge \theta^3)(X_1, X_2, X_3) = \det(\theta^i(X_j)) = -1$$

and

$$(\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2)(X_1, X_2, X_3) = \frac{1}{8} \det \begin{pmatrix} -x^1 & -x^2 & -x^3 \\ x^0 & -x^3 & x^2 \\ x^3 & x^0 & -x^1 \end{pmatrix} = -\frac{x^3}{8}.$$

Hence $\rho = \dfrac{8}{x^3}$. By considering the standard parametrization of S^3, that is

$$x^0 = \sin u, \quad x^1 = \cos u \sin v, \quad x^2 = \cos u \cos v \sin w, \quad x^3 = \cos u \cos v \cos w,$$

with $u, v \in (-\pi/2, \pi/2)$, $w \in (-\pi, \pi)$, we compute

$$\begin{aligned} \int_{S^3} \theta^1 \wedge \theta^2 \wedge \theta^3 &= 8 \left(\int_{-\pi/2}^{\pi/2} \cos^2 u \, \mathrm{d}u \right) \left(\int_{-\pi/2}^{\pi/2} \cos v \, \mathrm{d}v \right) \left(\int_{-\pi}^{\pi} \mathrm{d}w \right) \\ &= 16\pi^2. \end{aligned}$$

Therefore

$$\operatorname{vol}_g(SO(3)) = 8\pi^2,$$

and from $(\star\star)$ we conclude

$$J(SO(3), g) = \frac{1}{2}.$$

REMARK. The reader can check that the metric g is really bi-invariant by proving that the forms θ^i are also right-invariant. This readily follows from formula $(\star\star)$ in Problem 4.5.2.

6.7 Isometries

Problem 6.7.1. *Let (M, g) be the Poincaré upper half-plane (see* Problem 6.9.1*). We define an action of $SL(2,\mathbb{R})$ on M as follows: Identifying $\mathbb{R}^2$ with $\mathbb{C}$ as usual, let $z = x + \mathrm{i}y$. Given $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, we define*

$$w = sz = \frac{az + b}{cz + d}.$$

(1) *Prove that $SL(2,\mathbb{R})$ is a group of isometries of (M, g) (called the* group of linear fractional transformations *of the Poincaré upper half-plane).*

(2) *Prove that under these isometries the half-circles with center at the x-axis are transformed either in the same type of half-circles or in vertical lines.*

HINT: *One can obtain an Iwasawa decomposition of $SL(2,\mathbb{R})$, writing each $s \in SL(2,\mathbb{R})$ as the product of a matrix of $SO(2)$ by a diagonal matrix with determinant equal to* 1 *by an upper triangular matrix with the elements of the diagonal equal to* 1.

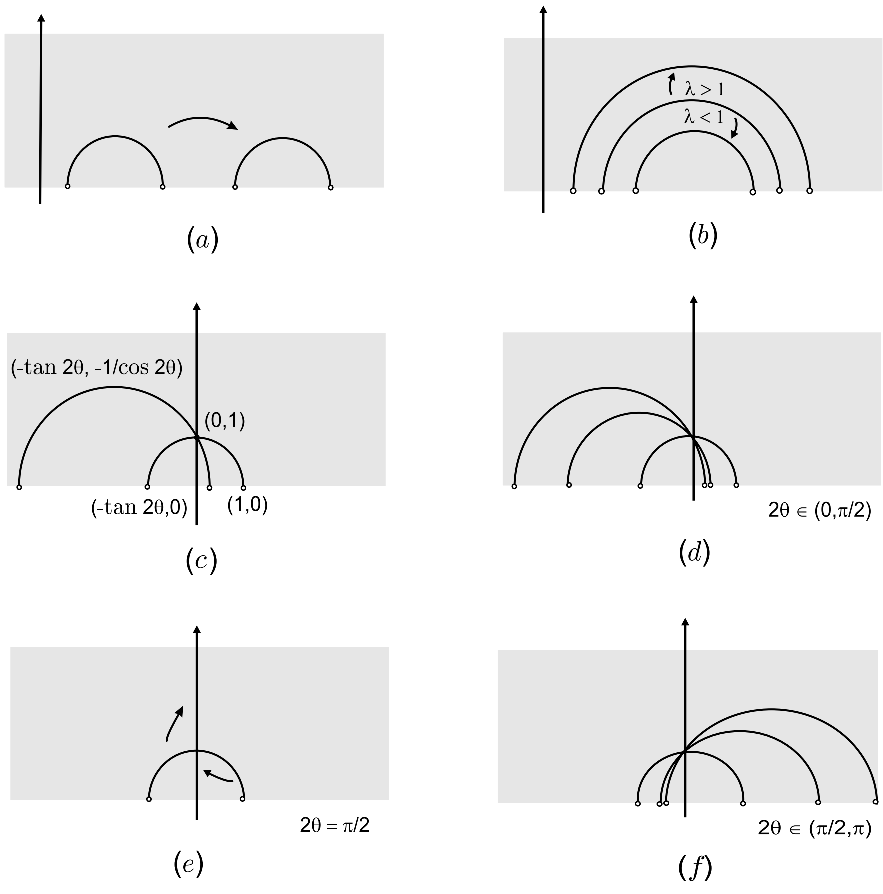

Fig. 6.5 Variations of the image of the half-circle C.

Solution. (1) Given $z = x + \mathrm{i}y \in M$ we have $y = \operatorname{Im} z > 0$. We also have $\operatorname{Im} w > 0$. In fact,

$$\operatorname{Im} w = \operatorname{Im} \frac{az+b}{cz+d}$$

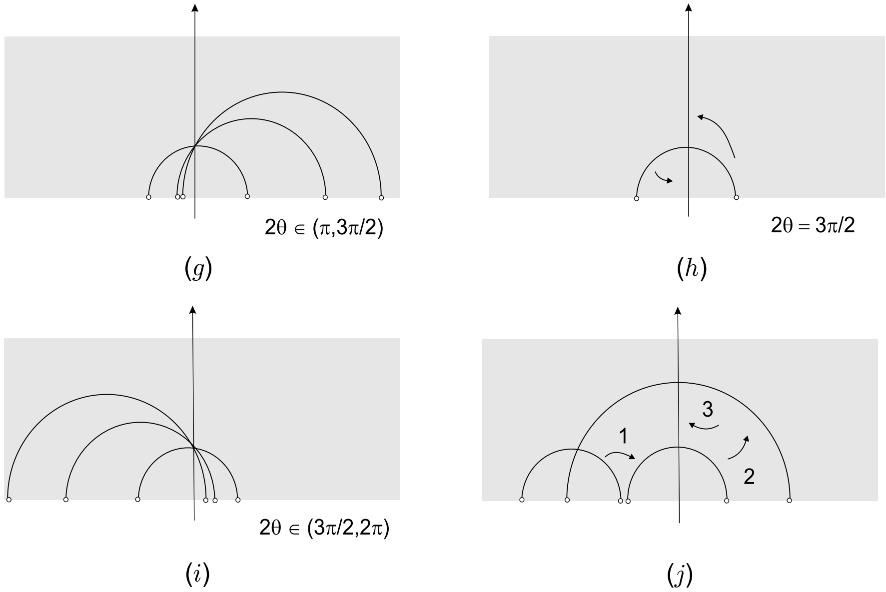

Fig. 6.6 Variations of the image of the half-circle C.

$$= \frac{\mathrm{Im}\, z}{|cz+d|^2} > 0,$$

and thus $w \in M$.

Moreover, $s_2(s_1 z) = (s_2 s_1)z$. In fact, putting

$$s_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad s_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

we have

$$\begin{aligned} s_2(s_1 z) &= \frac{(a'a+b'c)z+a'b+b'd}{(c'a+d'c)z+c'b+d'd} \\ &= (s_2 s_1)z. \end{aligned}$$

The metric g can be written on $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ as

$$g = \frac{1}{2}\,\frac{\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z}{(\mathrm{Im}\, z)^2}.$$

Moreover, it is easy to compute that

$$\frac{\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w}{(\mathrm{Im}\, w)^2} = \frac{\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z}{(\mathrm{Im}\, z)^2}.$$

As the expression of g is preserved in the new coordinates, the action of s is an isometry. Thus $SL(2,\mathbb{R})$ acts on M as a group of isometries.

(2) From the Iwasawa decomposition of $SL(2,\mathbb{R})$ in the Hint, we can write each element $s \in SL(2,\mathbb{R})$ uniquely as a product

$$s = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad \lambda \neq 0.$$

Considering the previous property $s_2(s_1 z) = (s_2 s_1)z$, in order to study the action of $SL(2,\mathbb{R})$ on a half-circle of M with center at the x-axis, it suffices to see the consecutive action of the elements of the previous decomposition. The action $z \mapsto \dfrac{az+b}{cz+d}$ by an element of the type $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is $z \mapsto z+a$. That is, the translation of the half-circle by the vector $a+0\,\mathrm{i}$, $a \in \mathbb{R}$ (see Figure 6.5 (a)).

The action by an element of the type $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$, $\lambda \in \mathbb{R} - \{0\}$, is $z \mapsto \lambda^2 z$. That is, a homothety of ratio $\lambda^2 \in \mathbb{R}^+$ (see Figure 6.5 (b)).

From these results, it follows that to study the whole action it suffices to consider the unit half-circle C with center at (0,0).

We can parametrize that half-circle as

$$(x,y) = (\cos\beta, \sin\beta), \qquad \beta \in (0,\pi).$$

The action of an element $s = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$ is

$$z \mapsto \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta} \equiv \left(\frac{x\cos 2\theta}{1+x\sin 2\theta}, \frac{y}{1+x\sin 2\theta} \right).$$

The limits when $y \to 0$, that is when $x \to \pm 1$, are

$$\left(-\frac{\cos 2\theta}{1-\sin 2\theta}, 0 \right), \qquad \left(\frac{\cos 2\theta}{1+\sin 2\theta}, 0 \right),$$

respectively. So the center of the half-circle image is at the point of the x-axis with abscissa

$$\frac{1}{2}\frac{\cos 2\theta}{1-\sin^2 2\theta}(1-\sin 2\theta - 1 - \sin 2\theta) = -\tan 2\theta.$$

Moreover, from

$$\left(\frac{x\cos 2\theta}{1+x\sin 2\theta} + \tan 2\theta \right)^2 + \frac{y^2}{(1+x\sin 2\theta)^2} = r^2,$$

one has $r = 1/\cos 2\theta$. The image of C is thus the half-circle of center $-\tan 2\theta$ and radius $1/\cos 2\theta$, if $\cos 2\theta \neq 0$. The image of $(0,1)$ is $(0,1)$ (see Figure 6.5 (c)).

Let us now see how the image of C varies as a function of $\theta \in [0,2\pi]$.

If $\theta = 0$, we have the identity $C \to C$.

For the interval $2\theta \in (0, \pi/2)$ we obtain as images a family of half-circles of the previous type, and $(0,1)$ is preserved (see Figure 6.5 (d)).

For $2\theta = \dfrac{\pi}{2}$ the image is $\left\{\left(0, \sqrt{\dfrac{1-x}{1+x}}\right)\right\}$, so the part at the first quadrant goes to the vertical segment from (0,0) to (0,1), and the part at the second quadrant into the vertical half-line $\{(0,y) : y \in (1,\infty)\}$. Thus C is transformed in the half-line $\{(0,y) : y \in (0,\infty)\}$ (see Figure 6.5 (e)).

For the interval $2\theta \in (\pi/2, \pi)$ we have the images obtained by reflection in the y-axis of the ones corresponding to the interval $(0, \pi/2)$, because $\cos 2\theta$ changes its sign, but $\sin 2\theta$ does not (see Figure 6.5 (f)).

For $2\theta = \pi$ the image is $(-x, y)$, that is $g(C) = C$ by reflection on the y axis.

For the interval $2\theta \in (\pi, 3\pi/2)$, the values of $\cos 2\theta$ and $\sin 2\theta$ change their sign with respect to their values when $2\theta \in (0, \pi/2)$. Therefore, changing the sign of x, the values of $x \sin 2\theta$ and $x \cos 2\theta$ are preserved and if the value of y does not change, we obtain that the image sets are the reflections with regard to the y-axis of the ones corresponding to the interval $(0, \pi/2)$ (see Figure 6.6 (g)).

For $2\theta = 3\pi/2$ one has $(x,y) \mapsto \left(0, \sqrt{\dfrac{1+x}{1-x}}\right)$; that is, again the half-line $\{(0,y) : y > 0\}$, but obtained from C in a different way, as we can see in Figure 6.6 (h).

For the interval $2\theta \in (3\pi/2, 2\pi)$, we have that $\sin 2\theta$ changes its sign with respect to $2\theta \in (0, \pi/2)$ and $\cos 2\theta$ preserves it. Changing x by $-x$, we have the symmetric situation with respect to the y-axis (see Figure 6.6 (i)).

For $2\theta = 2\pi$ we have $(x,y) \to (x,y)$; again the identity.

Summing up, from a half-circle of radius r with center at the x-axis we can obtain all the half-circles with center at the x-axis and any radius, and all the vertical lines (see Figure 6.6 (j)).

Problem 6.7.2. (1) *Prove that the isometry group $I(S^n)$ of S^n with the round metric, is $O(n+1)$.*

(2) *Prove that the isometry group of the hyperbolic space H^n, equipped with the canonical metric of negative constant curvature, is the proper Lorentz group $O_+(1,n)$, which is the group of all linear transformations of $\mathbb{R}^{n+1}$ which leave invariant the Lorentz product $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+1}$, defined by*

$$\big\langle (x^0, x^1, \dots, x^n), (y^0, y^1, \dots, y^n) \big\rangle = -x^0 y^0 + \sum_{i=1}^{n} x^i y^i,$$

and which also leave each component of the hyperboloid $\langle\ ,\ \rangle^{-1}(-1)$ invariant.

HINT: *Apply* Theorem 7.6.16.

Solution. (1) Let $\langle\ ,\ \rangle$ denote the Euclidean metric on $\mathbb{R}^{n+1}$. The round metric on S^n is defined by letting the embedding of S^n into $\mathbb{R}^{n+1}$ be isometric, i.e.,

$$\langle (p,u), (p,v) \rangle_{S^n} = \langle (p,u), (p,v) \rangle = \langle u, v \rangle \qquad (\star)$$

for $p \in S^n$ and (p,u), $(p,v) \in TS^n$. The group $O(n+1)$ acts on S^n by isometries. In fact:

$$\begin{aligned}
\langle a_*(p,u), a_*(p,v)\rangle_{S^n} &= \langle (ap,au),(ap,av)\rangle_{S^n} && (a \in O(n+1)) \\
&= \langle au, av\rangle && (\text{by } (\star)) \\
&= \langle u, v\rangle \\
&= \langle (p,u),(p,v)\rangle_{S^n} && (\text{by } (\star)).
\end{aligned}$$

Now let $a \in I(S^n)$. Fix $p = e_0 \in S^n$ and the orthonormal basis $\{(p,e_j)\}$ for T_pS^n. Let $(q,\tilde{e}_j) = a_*(p,e_j) \in T_qS^n$ for $q = a(p)$. Let $b \in O(n+1)$ take p to q and e_j to $\tilde{e}_j$, $j \geqslant 1$. Since $O(n+1)$ acts on S^n by isometries we have $b \in I(S^n)$. As moreover $b_{*p} = a_{*p}$, applying the Theorem 7.6.16, we obtain $a = b \in O(n+1)$.

(2) The proof is similar to the one in case (1), since the hyperbolic space H^n is the component

$$\left\{ (x^0, x^1, \dots, x^n) \in \mathbb{R}^{n+1} \,:\, -(x^0)^2 + \sum_{i=1}^{n} (x^i)^2 = -1,\ x^0 > 0 \right\}$$

of the hyperboloid $\langle\ ,\ \rangle^{-1}(-1)$.

Problem 6.7.3. *Let $f\colon (M,g) \to (M',g')$ be an isometry of Riemannian manifolds. Prove that f preserves the curvature tensor field and the sectional curvature.*

Solution. An isometry is a diffeomorphism and an affine map for the Levi-Civita connection, that is, $f \cdot \nabla_X Y = \nabla'_{f\cdot X} f \cdot Y$, where ∇ and ∇' denote the Levi-Civita connections of g and g', respectively. Hence, we conclude as in Problem 5.5.6, that for the respective tensors of curvature one has

$$R'(f\cdot X, f\cdot Y) f\cdot Z = f\cdot (R(X,Y)Z).$$

Moreover, an isometry preserves the metric by definition, that is:

$$g'(f\cdot X, f\cdot Y) = (f^*g')(X,Y)\circ f^{-1} = g(X,Y)\circ f^{-1}. \qquad (\star\star)$$

Put $p' = f(p)$. If $\{X_p, Y_p\}$ is a basis of the 2-plane P of T_pM, then $\{(f_*X)_{p'}, (f_*Y)_{p'}\}$ is a basis of the 2-plane $P = f_*P$ of $T_{p'}M'$. Then, taking

$$R(X_p, Y_p)Y_p, \qquad R'((f_*X)_{p'}, (f_*Y)_{p'})(f_*Y)_{p'}$$

according to Definition 7.6.4, from $(\star)$ and $(\star\star)$ above we obtain for the sectional curvature:

$$\begin{aligned}
K'(P) &= \frac{g'(R'((f_*X)_{p'}, (f_*Y)_{p'})(f_*Y)_{p'}, (f_*X)_{p'})}{g'((f_*X)_{p'}, (f_*X)_{p'})g'((f_*Y)_{p'}, (f_*Y)_{p'}) - g'((f_*X)_{p'}, (f_*Y)_{p'})^2} \\
&= \frac{g(R(X_p,Y_p)Y_p, X_p)}{g(X_p,X_p)g(Y_p,Y_p) - g(X_p,Y_p)^2} = K(P).
\end{aligned}$$

Problem 6.7.4. *Let (M,g) be a Riemannian manifold.*

(1) *Prove that if f is an isometry of (M,g) and ∇ denotes the Levi-Civita connection, then we have*

$$f^*\nabla_{e_j}\beta = \nabla_{f^{-1}\cdot e_j} f^*\beta, \qquad \beta \in \Lambda^1 M,$$

where (e_i) stands for a local orthonormal frame.

(2) *Prove that the* codifferential δ, *defined by*

$$\delta\beta = -\operatorname{div}\beta = -\sum_k i_{e_k}\nabla_{e_k}\beta, \qquad \beta \in \Lambda^* M, \tag{$\star$}$$

$\{e_k\}$ being an orthonormal basis, commutes with isometries.

Solution. (1) As f is an isometry, f preserves the Levi-Civita connection; that is, $f\cdot\nabla_X Y = \nabla_{f\cdot X} f\cdot Y$, so that $\nabla_{e_j}(f\cdot X) = f\cdot\nabla_{f^{-1}\cdot e_j}X$. Moreover, we recall that we have $(f^*\omega)(X)\circ f^{-1} = \omega(f\cdot X)$ for every $\omega\in\Lambda^1 M$, as it is readily checked. Letting $\omega = \nabla_{e_j}\beta$, we obtain that:

$$\begin{aligned}
(f^*\nabla_{e_j}\beta)(X)\ \circ f^{-1} &= (\nabla_{e_j}\beta)(f\cdot X)\\
&= \nabla_{e_j}(\beta(f\cdot X)) - \beta(\nabla_{e_j}(f\cdot X))\\
&= e_j((f^*\beta)(X)\circ f^{-1}) - \beta(f\cdot\nabla_{f^{-1}\cdot e_j}X)\\
&= \left\{(f^{-1}\cdot e_j)((f^*\beta)(X)) - (f^*\beta)\left(\nabla_{f^{-1}\cdot e_j}X\right)\right\}\circ f^{-1}\\
&= \left\{\nabla_{f^{-1}\cdot e_j}((f^*\beta)(X)) - (f^*\beta)\left(\nabla_{f^{-1}\cdot e_j}X\right)\right\}\circ f^{-1}\\
&= \left\{\left(\nabla_{f^{-1}\cdot e_j}f^*\beta\right)(X)\right\}\circ f^{-1}.
\end{aligned}$$

(2)

$$\begin{aligned}
f^*\delta\beta &= -f^*\operatorname{div}\beta && \text{(by definition of div, and locally)}\\
&= -\sum_j f^*\left(i_{e_j}\nabla_{e_j}\beta\right) && \text{(by } (\star)\text{)}\\
&= -\sum_j i_{f^{-1}\cdot e_j} f^*\nabla_{e_j}\beta\\
&= -\sum_j i_{f^{-1}\cdot e_j}\nabla_{f^{-1}\cdot e_j} f^*\beta && \text{(by part } (1) \text{ of this problem)}\\
&= -\operatorname{div} f^*\beta\\
&= \delta f^*\beta.
\end{aligned}$$

6.8 Homogeneous Riemannian Manifolds and Riemannian Symmetric Spaces

Problem 6.8.1. *Let G be a connected closed subgroup of the Lie group $E(n)$ of all the motions* (i.e. *isometries of the Euclidean metric*) *of* $\mathbb{R}^n$*, acting transitively on* $\mathbb{R}^n$*.*

Must G contain the full group of translations?

Solution. G need not contain the full group of translations, as the following counterexample shows. Let $n = 3$, and let Φ_t be the rotation around the z-axis through an angle t. Let X_t, Y_t, Z_t be the translations by $(t,0,0)$, $(0,t,0)$ and $(0,0,t)$, respectively. Let $\Psi_t = Z_t \circ \Phi_t$, so Ψ_t is a screw motion around the z-axis. Then, the group generated by the Ψ_t, X_t, and Y_t, as t varies over $\mathbb{R}$, acts simply transitively on $\mathbb{R}^3$ but does not contain the translation in the z-direction.

Problem 6.8.2. *Consider the action of the orthogonal group $O(n)$ on the Riemannian manifold $(\mathbb{R}^n, g)$, where g denotes the Euclidean metric.*

(1) *Is $(\mathbb{R}^n, g)$ a homogeneous Riemannian manifold with respect to that action?*

(2) *Describe the possible isotropy groups H_p.*

Solution. (1) No, because the action is not transitive. In fact, the origin is a fixed point (take the origin 0 as one of the points p, q of $\mathbb{R}^n$ such that there might exist $\sigma \in O(n)$ with $\sigma(p) = q$).

(2) $H_0 = O(n)$ and H_p are mutually conjugate subgroups isomorphic to $O(n-1)$ for every $p \neq 0$.

Problem 6.8.3. *Define a product on*

$$E(n) = \{(a,A) : a \in \mathbb{R}^n, A \in O(n)\}$$

by

$$(a,A)\cdot(b,B) = (a + Ab, AB).$$

Prove:

(1) $(E(n),\cdot)$ *is a Lie group* (*in fact, this is a semidirect product of the Abelian group* $(\mathbb{R}^n,+)$ *and* $O(n)$*, and it is called the* group of Euclidean motions *or simply the* Euclidean group *of* $\mathbb{R}^n$).

(2) *The subgroup of translations $T(n) = \{(a,I) : a \in \mathbb{R}^n\}$ is a normal subgroup of $E(n)$.*

Let $E(n)$ act on $\mathbb{R}^n$ by setting $(a,A)\cdot x = a + Ax$. Then:

(3) *Prove that the map $x \mapsto (a,A)\cdot x$ is an isometry of the Euclidean metric.*

(4) *Compute the isotropy of a point $x \in \mathbb{R}^n$. Are all these groups isomorphic? And conjugate in $E(n)$?*

(5) *Prove that $E(n)/O(n) \approx \mathbb{R}^n$.*

Let $p\colon E(n) \to \mathbb{R}^n$ *be the map* $p(a,A) = a$. *Prove*:

(6) *The map* p *is the projection map of a principal* $O(n)$*-bundle with respect to the action of* $O(n)$ *on* $E(n)$ *given by* $(a,A)\cdot B = (a,AB)$.

(7) *The bundle above can be identified to the bundle of orthonormal frames over* $\mathbb{R}^n$ *with respect to the Euclidean metric.*

Solution. (1a) Associativity:

$$\begin{aligned}((a,A)\cdot(b,B))\cdot(c,C) &= (a+Ab,AB)\cdot(c,C)\\ &= (a+Ab+(AB)c,(AB)C),\\ (a,A)\cdot((b,B)\cdot(c,C)) &= (a,A)\cdot(b+Bc,BC)\\ &= (a+A(b+Bc),A(BC))\\ &= (a+Ab+(AB)c,(AB)C).\end{aligned}$$

(1b) Identity element: $(a,A)\cdot(0,I) = (0,I)\cdot(a,A) = (a,A)$.

(1c) Inverse element: $(a,A)^{-1} = (-A^{-1}a,A^{-1})$.

We endow $E(n)$ with the differentiable structure $E(n) \approx \mathbb{R}^n \times O(n)$. As $O(n)$ is a Lie group, it follows from $(1c)$ and the very definition of the product law in $E(n)$ that $E(n)$ is also a Lie group.

(2) We have

$$\begin{aligned}(a,A)\cdot(b,I)\cdot(a,A)^{-1} &= (a+Ab,A)\cdot(-A^{-1}a,A^{-1})\\ &= (a+Ab+A(-A^{-1}a),I)\\ &= (Ab,I) \in T(n).\end{aligned}$$

(3) Trivial.

(4) The isotropy group $E(n)_x$ of a point $x \in \mathbb{R}^n$ is defined by

$$E(n)_x = \{(a,A) \in E(n) : (a,A)\cdot x = x\}.$$

So, $(a,A) \in E(n)_x$ if and only if $a + Ax = x$, or equivalently $(I-A)x = a$. In particular,

$$\begin{aligned}E(n)_0 &= O(n)\\ &= \{(0,A) : A \in O(n)\}.\end{aligned}$$

For every $A \in O(n)$ we have

$$(x,I)\cdot(0,A)\cdot(x,I)^{-1} = ((I-A)x,A) \in E(n)_x.$$

Hence the map $\psi\colon E(n)_0 \to E(n)_x$ is an isomorphism and all the isotropy groups are conjugate in $E(n)$, thus isomorphic.

(5) We have a diffeomorphism $x \mapsto (x, I) \mod O(n)$.

(6) For every $B \in O(n)$ we have $p((a,A)\cdot B) = p(a,AB) = a$. Hence $(a,A)\cdot O(n) = p^{-1}(a)$. Moreover, as $(a,A)\cdot B = (a,A)$ implies $B = I$, the $O(n)$-action is free. Thus, $p\colon E(n) \to \mathbb{R}^n$ is a principal $O(n)$-bundle.

(7) Let $\pi\colon \mathscr{O}(\mathbb{R}^n) \to \mathbb{R}^n$ be the bundle of orthonormal frames over $\mathbb{R}^n$ for the metric $g = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^n \otimes \mathrm{d}x^n$. Define a map $\varphi\colon E(n) \to \mathscr{O}(\mathbb{R}^n)$ by setting

$$\varphi(a,A) = \left(\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a \right)\cdot A.$$

It is immediate that $\pi \circ \varphi = p$. Moreover, we have

$$\begin{aligned}\varphi((a,A)\cdot B) &= \varphi(a,AB)\\ &= \left(\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a \right)\cdot (AB)\\ &= \left(\left(\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a \right)\cdot A\right)\cdot B\\ &= \varphi(a,A)\cdot B.\end{aligned}$$

Finally, $\varphi(a,A) = \varphi(b,B)$ means

$$\left(\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a \right)\cdot A = \left(\left.\frac{\partial}{\partial x^1}\right|_b, \dots, \left.\frac{\partial}{\partial x^n}\right|_b \right)\cdot B.$$

This implies $a = b$ and $A = B$, thus proving that φ is a principal bundle isomorphism.

Problem 6.8.4. *As the unit sphere in $\mathbb{R}^{n+1}$, $S^n \approx SO(n+1)/SO(n)$ is a symmetric space, with symmetry ζ at $o = (1,0,\dots,0)$ given (see* Figure 6.7*) by*

$$(t^0, t^1, \dots, t^n) \mapsto (t^0, -t^1, \dots, -t^n).$$

For the symmetric space S^n, find:

(1) *The involutive automorphism σ of $SO(n+1)$ such that*

$$SO(n+1)^\sigma_0 \subset SO(n) \subset SO(n+1)^\sigma,$$

where $SO(n+1)^\sigma$ denotes the closed subgroup of $SO(n+1)$ of fixed points of σ, and $SO(n+1)^\sigma_0$ its identity component.

(2) *The subspace*

$$\mathfrak{m} = \{X \in \mathfrak{so}(n+1) : \sigma_* X = -X\}.$$

(3) *The* $\mathrm{Ad}(SO(n))$*-invariant inner product on* $\mathfrak{m}$.

(4) *The geodesics.*

(5) *The isomorphism $p_*\colon \mathfrak{m} \approx T_o S^n$.*

(6) *The linear isotropy action.*

(7) *The curvature.*

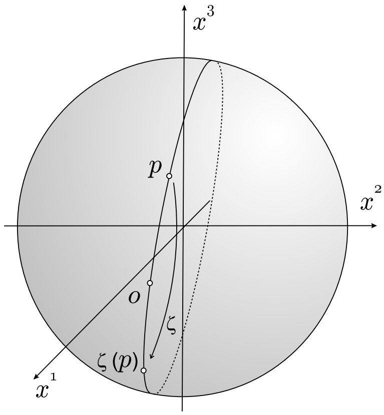

Fig. 6.7 The symmetry of S^2 at a point o.

Solution. (1) As $\zeta = \operatorname{diag}(1, -1, \ldots, -1)$, for $A \in SO(n+1)$ we have (see [24, Lemma 28, p. 315]):

$$\sigma(A) = \zeta A \zeta$$
$$= \left(\begin{array}{c|ccc} a_{00} & -a_{01} & \cdots & -a_{0n} \\ \hline -a_{10} & & & \\ \vdots & & (a_{ij}) & \\ -a_{n0} & & & \end{array}\right), \qquad 1 \leqslant i, j \leqslant n.$$

So $SO(n+1)^\sigma$ is $S(O(1) \times O(n))$, and $SO(n+1)^\sigma_0$ is the isotropy group $1 \times SO(n) \approx SO(n)$.

(2) As $\zeta = \zeta^{-1}$, we have $\sigma(A) = \zeta A \zeta^{-1}$, so that σ is conjugation by ζ. Thus, σ_* is also conjugation by ζ on the Lie algebra $\mathfrak{so}(n+1)$. Hence

$$\mathfrak{m} = \left\{ X = \begin{pmatrix} 0 & -{}^t x \\ x & 0 \end{pmatrix} \right\},$$

where x denotes any column vector, regarded as an element of $\mathbb{R}^n$. Write $X \leftrightarrow x$ for the resulting correspondence between $\mathfrak{m}$ and $\mathbb{R}^n$.

(3) Under $X \leftrightarrow x$, the dot product $x \cdot y$ on $\mathbb{R}^n$ corresponds to $B(X,Y) = -\frac{1}{2}\operatorname{tr} XY = \frac{1}{2} X \cdot Y$ on $\mathfrak{m}$, where $X \cdot Y$ denotes the scalar product in $\mathbb{R}^{(n+1)^2}$. B is thus a multiple of the Killing form on $\mathfrak{so}(n+1)$ (see table on page 387). One has $SO(n) \subset SO(n+1)$ and the Killing form is $\operatorname{Ad}(SO(n+1))$-invariant. It follows from (5) below that the corresponding metric tensor on S^n is the usual one. In fact,

$$B\left(\begin{pmatrix}0 & -{}^t x\\ x & 0\end{pmatrix}, \begin{pmatrix}0 & -{}^t y\\ y & 0\end{pmatrix}\right) = \sum_{i=1}^{n} x^i y^i = x\cdot y.$$

(4) Let γ be a geodesic of S^n starting at o. Since S^n is symmetric, we have (see [24, Prop. 31, p. 317])

$$\gamma(t) = \exp(tX)o$$

for some $X \in \mathfrak{m}$.
It is easily seen that

$$\begin{aligned}\exp tX &= \begin{pmatrix} 1-\dfrac{t^2}{2}x\cdot x+\cdots & * \\ tx-\dfrac{t^3}{6}(x\cdot x)x+\cdots & * \end{pmatrix}\\ &= \begin{pmatrix} \cos|x|t & * \\ (\sin|x|t)\dfrac{x}{|x|} & * \end{pmatrix},\end{aligned}$$

where $\begin{pmatrix}*\\ *\end{pmatrix}$ stands for an $((n+1)\times n)$-matrix which does not matter for our purpose. Thus,

$$\begin{aligned}(\exp tX)o &= (\exp tX)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\\ &= (\cos|x|t)o + (\sin|x|t)\frac{x}{|x|}.\end{aligned}$$

Hence γ is the great circle parametrization

$$(\cos|x|t)o + (\sin|x|t)\frac{x}{|x|},$$

where $X \leftrightarrow x$.

(5) In (3), $\mathbb{R}^n$ is assumed to be identified with the last n coordinate space of $\mathbb{R}^{n+1}$. Hence the canonical isomorphism identifies T_oS^n with $\mathbb{R}^n$. Then, according to (4), $x = \gamma'(0)$. But X is the initial velocity vector of the 1-parameter subgroup projecting to γ. Hence, the isomorphism $p_*\colon \mathfrak{m} \approx T_oS^n$ is $X \leftrightarrow x$.

(6) If $h = \begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix} \in H = SO(n)$ and $X \in \mathfrak{m}$, then

$$\mathrm{Ad}_h X = \begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\begin{pmatrix}0 & -{}^t x\\ x & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & A^{-1}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & -{}^t x A^{-1} \\ Ax & 0 \end{pmatrix}$$

(which is skew-symmetric since ${}^t A = A^{-1}$). Thus the linear isotropy action of H on $T_o S^n$ is, via the identifications, the usual action of $SO(n)$ on $\mathbb{R}^n$, i.e., $(A,x) \mapsto Ax$.

(7) In terms of the subspace $\mathfrak{m}$, we have

$$R(X,Y)Z = -[[X,Y],Z].$$

If x,y,z denote the corresponding vectors in $\mathbb{R}^n \approx T_o S^n$, we obtain

$$\begin{aligned} [X,Y] &= \begin{pmatrix} 0 & 0 \\ 0 & -(x^i y^j - x^j y^i) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -\widetilde{A} \end{pmatrix}, \end{aligned}$$

where $(\widetilde{A}^i_j) = (x^i y^j - x^j y^i)$, so

$$\begin{aligned} R(X,Y)Z = &- \begin{pmatrix} 0 & 0 \\ 0 & -(x^i y^j - x^j y^i) \end{pmatrix} \begin{pmatrix} 0 & -{}^t z \\ z & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -{}^t z \\ z & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -(x^i y^j - x^j y^i) \end{pmatrix} \\ = &\begin{pmatrix} 0 & -{}^t(\widetilde{A}z) \\ \widetilde{A}z & 0 \end{pmatrix}. \end{aligned}$$

Thus, under the identification, $R(X,Y)Z$ corresponds to $\widetilde{A}z$, that is to $(y \cdot z)x - (x \cdot z)y$. Hence S^n has constant curvature 1.

Problem 6.8.5. *The complex projective space*

$$\mathbb{C}P^n \approx U(n+1)/U(1) \times U(n) \approx SU(n+1)/S(U(1) \times U(n))$$

is a compact simply connected Hermitian symmetric space of dimension $2n$. *Find*:

(1) *The involutive automorphism* σ *of* $U(n+1)$ *such that*

$$U(n+1)^\sigma_0 \subset U(1) \times U(n) \subset U(n+1)^\sigma,$$

$U(n+1)^\sigma$ *being the closed subgroup of* $U(n+1)$ *of fixed points of* σ *and* $U(n+1)^\sigma_0$ *its identity component.*

(2) *The subspace*

$$\mathfrak{m} = \{X \in \mathfrak{u}(n+1) \,:\, \sigma_* X = -X\}.$$

(3) *The* $\operatorname{Ad}(U(1) \times U(n))$*-invariant inner product on* $\mathfrak{m}$.

(4) *The linear isotropy action.*

Moreover, prove that:

(5) *The scalar multiplication by* i *on* $\mathbb{C}^n \approx \mathfrak{m}$ *gives a corresponding complex structure* J_0 *on* $\mathfrak{m}$, *which is* $\mathrm{Ad}(U(1)\times U(n))$*-invariant, and so determines an almost complex structure* J *on* $\mathbb{C}P^n$ *making it a Kähler manifold.*

(6) $\mathbb{C}P^n$ *has thus constant holomorphic sectional curvature.*

Solution. (1) Let $\zeta = \operatorname{diag}(-1,1,\dots,1)$. The conjugation $\sigma\colon A \mapsto \zeta A \zeta^{-1}$ is an involutive automorphism of $U(n+1)$ whose fixed point set is $U(1)\times U(n)$, thus having

$$\begin{aligned} S(U(1)\times U(n)) &= U(n+1)^{\sigma}_0 \\ &\subset U(n+1)^{\sigma} \\ &= U(1)\times U(n). \end{aligned}$$

(2) The (-1)-eigenspace $\mathfrak{m}$ of σ_* consists of all the elements in $\mathfrak{u}(n+1)$ of the form $X = \begin{pmatrix} 0 & -{}^t\overline{x} \\ x & 0 \end{pmatrix}$, where x is an $n\times 1$ complex matrix.

(3) The inner product $B(X,Y) = -\frac{1}{2}\operatorname{tr} XY = \frac{1}{2} X\cdot \overline{Y}$ is a multiple of the Killing form on $\mathfrak{u}(n+1)$ (see table on page 387), and hence it is $\mathrm{Ad}(U(1)\times U(n))$-invariant. Because of the factor $\frac{1}{2}$, $B|_{\mathfrak{m}}$ corresponds under the identification $\mathfrak{m}\equiv\mathbb{C}^n$ to the real part of the natural Hermitian product $x\cdot\bar{y}$ in $\mathbb{C}^n$.

(4) We have

$$\mathrm{Ad}_{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}} \begin{pmatrix} 0 & -{}^t\overline{x} \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -{}^t\overline{(e^{-i\theta}Ax)} \\ e^{-i\theta}Ax & 0 \end{pmatrix}. \qquad (\star)$$

The linear isotropy action of $U(1)\times U(n)$ on $\mathfrak{m}\equiv\mathbb{C}^n$ thus corresponds to the action of $U(1)\times U(n)$ on $\mathbb{C}^n$ given by $(e^{i\theta},A)x = e^{-i\theta}Ax$.

(5) Scalar multiplication by i in $\mathbb{C}^n\equiv\mathfrak{m}$ gives a complex structure J_0 on $\mathfrak{m}$:

$$\begin{array}{ccc} \mathfrak{m}\equiv\mathbb{C}^n & \xrightarrow{J_0} & \mathfrak{m} \\ \begin{pmatrix} 0 & -{}^t\overline{x} \\ x & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} 0 & i\,{}^t\overline{x} \\ ix & 0 \end{pmatrix}, \end{array}$$

which is $\mathrm{Ad}(U(n)\times U(1))$-invariant. In fact, by $(\star)$ above we have with the usual notations, for any $X\in\mathfrak{m}$,

$$\begin{aligned} J_0\,\mathrm{Ad}_{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}} X &= \begin{pmatrix} 0 & -{}^t\overline{(i\,e^{-i\theta}Ax)} \\ ie^{-i\theta}Ax & 0 \end{pmatrix} \\ &= \mathrm{Ad}_{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}} J_0 X. \end{aligned}$$

Therefore (see [24, Prop. 43, p. 325]), J_0 determines an almost complex structure J on $\mathbb{C}P^n$ making it a Kähler manifold.

(6) (a) As $U(n)$ acts transitively on the complex lines in $\mathbb{C}^n$ (i.e. the holomorphic planes in $T_o(\mathbb{C}P^n)$), from $(\star)$ it follows that for $\theta = 0$, the action of the linear isotropy group is transitive on complex lines.

(b) Let $o \in \mathbb{C}P^n$ be the point corresponding to $(1,0,\ldots,0) \in \mathbb{C}^{n+1}$. The holomorphic sectional curvature is constant on $T_o(\mathbb{C}P^n)$, so by homogeneity it is constant everywhere. In fact, multiplying B by 4, we have $B(X,Y) = -2\,\mathrm{tr}\,XY$. Let $e_1, e_2 \in \mathfrak{m}$ correspond to elements of the natural basis of $\mathbb{C}^n \equiv \mathfrak{m}$. From (a) above, an arbitrary tangent plane on $\mathbb{C}P^n$ has sectional curvature $K(e_1,Y)$, where $Y = \cos\theta\, Je_1 + \sin\theta\, e_2$. We have

$$e_1 = \begin{pmatrix} 0 & -1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}, \qquad Je_1 = \begin{pmatrix} 0 & \mathrm{i} & & \\ \mathrm{i} & & & \\ & & & \\ & & & \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & -1 & \\ 0 & & & \\ 1 & & & \\ & & & \end{pmatrix}.$$

Thus, we deduce (see [24, Remark, p. 319]):

$$\begin{aligned} K(e_1,Y) &= \frac{B([e_1,Y],[e_1,Y])}{B(e_1,e_1)B(Y,Y) - B(e_1,Y)^2} \\ &= \frac{1}{16} B(\cos\theta\,[e_1,Je_1] + \sin\theta\,[e_1,e_2], \cos\theta\,[e_1,Je_1] + \sin\theta\,[e_1,e_2]) \\ &= \frac{1}{4}(1 + 3\cos^2\theta). \end{aligned}$$

Hence

$$\frac{1}{4} \leqslant K \leqslant 1.$$

Taking $\theta = 0$, so $Y = Je_1$, shows that $\mathbb{C}P^n$ has constant holomorphic sectional curvature 1.

Problem 6.8.6. *Let $G = O(p,q+1)$ and $H = O(p,q)$. Show that the homogeneous space $M = G/H$ is symmetric and can be represented as the hyperquadric*

$$Q = \big\{ x = (x^1,\ldots,x^{p+q+1}) \in \mathbb{R}^{p+q+1} : \\ (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - (x^{p+q+1})^2 = -1 \big\}.$$

HINT: *Show that the map*

$$\begin{aligned} \sigma\colon O(p,q+1) &\to O(p,q+1) \\ a &\mapsto \zeta\, a\, \zeta^{-1} \end{aligned}$$

where ζ is the matrix $\zeta = \begin{pmatrix} -I_{p+q} & 0 \\ 0 & 1 \end{pmatrix}$ in the canonical basis of $\mathbb{R}^{p+q+1}$, is an involutive automorphism (i.e. $\sigma^2 = \mathrm{id}$) *of $O(p,q+1)$.*

Solution. Since $\zeta^{-1} = \zeta$, the map σ is an involutive automorphism of $O(p,q+1)$. The closed subgroup of $O(p,q+1)$ of fixed points of σ is $G^\sigma = O(p,q+1)^\sigma = O(p,q)$, and thus

$$\begin{aligned} SO(p,q) &= O(p,q+1)_0^\sigma \\ &\subset O(p,q) \\ &\subset O(p,q+1)^\sigma \\ &= O(p,q), \end{aligned}$$

where $O(p,q+1)_0^\sigma$ denotes the identity component of $O(p,q+1)^\sigma$. Hence $M = G/H$ is a symmetric space.

The map

$$\begin{array}{rcl} \varphi\colon O(p,q+1)/O(p,q) & \to & Q \\ a\cdot O(p,q) & \mapsto & a\cdot x_0 \end{array}$$

where $a \in O(p,q+1)$ and $x_0 = (0,\dots,0,1)$, is a diffeomorphism of M with the orbit of x_0 under $O(p,q+1)$, which is the hyperquadric Q, since $O(p,q+1)$ is the group of linear transformations leaving invariant the quadratic form

$$\mathrm{q}(x) = \sum_{i=1}^{p}(x^i)^2 - \sum_{j=p+1}^{p+q+1}(x^j)^2.$$

Problem 6.8.7. *Find the involutive automorphism of $SL(n,\mathbb{R})$ making the homogeneous space $SL(n,\mathbb{R})/SO(n)$ into an affine symmetric space. Write the decomposition involving the corresponding Lie algebras.*

Solution. The usual definition $O(n) = \{a \in GL(n,\mathbb{R}) : {}^t aa = I\}$ suggests us to take the involutive automorphism σ given by

$$\begin{array}{rcl} \sigma\colon SL(n,\mathbb{R}) & \to & SL(n,\mathbb{R}) \\ b & \mapsto & {}^t b^{-1} \end{array}$$

for, then, the closed subgroup of $SL(n,\mathbb{R})$ of fixed points of σ,

$$SL(n,\mathbb{R})^\sigma = \{b \in SL(n,\mathbb{R}) : \sigma(b) = b\} = SO(n),$$

and its identity component $SL(n,\mathbb{R})_0^\sigma$ satisfy

$$SL(n,\mathbb{R})_0^\sigma = SO(n) = SL(n,\mathbb{R})^\sigma,$$

so $SL(n,\mathbb{R})/SO(n)$ is an affine symmetric space.

As for the Lie algebras, the differential

$$\sigma_*\colon \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R}), \qquad \sigma_* X = -{}^t X,$$

induces the decomposition in (± 1)-eigenspaces

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{R}) &= \sigma_{*+} \oplus \sigma_{*-} \\ &= \{X \in \mathfrak{sl}(n,\mathbb{R}) : \sigma_* X = X\} \oplus \{X \in \mathfrak{sl}(n,\mathbb{R}) : \sigma_* X = -X\} \\ &= \mathfrak{o}(n) \oplus \mathfrak{sym}(n,\mathbb{R}),\end{aligned}$$

where $\mathfrak{sym}(n,\mathbb{R})$ denotes the subset of traceless symmetric matrices in $\mathfrak{gl}(n,\mathbb{R})$.

6.9 Spaces of Constant Curvature

Problem 6.9.1. *Prove, using Cartan's structure equations, that the Poincaré upper half-plane, that is, the 2-dimensional manifold*

$$M = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\},$$

with the Riemannian metric

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2},$$

has constant curvature $K = -1$.

Solution. The orthonormal moving frame $(X_1 = y\,\partial/\partial x, X_2 = y\,\partial/\partial y)$ has dual moving coframe $(\tilde{\theta}^1 = \mathrm{d}x/y,\ \tilde{\theta}^2 = \mathrm{d}y/y)$. The first structure equation is $\mathrm{d}\tilde{\theta}^1 = -\widetilde{\omega}^1_2 \wedge \tilde{\theta}^2$, that is,

$$-\frac{1}{y^2}\,\mathrm{d}y \wedge \mathrm{d}x = -\widetilde{\omega}^1_2 \wedge \frac{1}{y}\,\mathrm{d}y,$$

hence

$$\widetilde{\omega}^1_2 = -\frac{\mathrm{d}x}{y}.$$

The second structure equation is $\mathrm{d}\widetilde{\omega}^1_2 = K\tilde{\theta}^1 \wedge \tilde{\theta}^2$, for certain differentiable real valued function K; that is,

$$\frac{1}{y^2}\,\mathrm{d}y \wedge \mathrm{d}x = K\frac{1}{y}\,\mathrm{d}x \wedge \frac{1}{y}\,\mathrm{d}y;$$

thus $K = -1$.

Problem 6.9.2. *Let a be any positive real number and let M be the subset of $\mathbb{R}^n$ such that $x^n > 0$. Prove, using Cartan's structure equations, that the Riemannian metric on M given by $g_{ij}(x) = (a^2/(x^n)^2)\delta_{ij}$ has constant curvature $K = -1/a^2$.*

HINT: *Take as connection forms*

$$\widetilde{\omega}^i_j = \frac{1}{a}(\delta_{ni}\tilde{\theta}^j - \delta_{nj}\tilde{\theta}^i),$$

where $\tilde{\theta}^i = a\,\mathrm{d}x^i/x^n$, for $i,j = 1,\dots,n$.

Solution. The frame

$$\sigma = \left(X_1 = \frac{x^n}{a}\frac{\partial}{\partial x^1}, \dots, X_n = \frac{x^n}{a}\frac{\partial}{\partial x^n} \right)$$

is an orthonormal moving frame, with dual moving coframe

$$\left(\tilde{\theta}^1 = a\frac{\mathrm{d}x^1}{x^n}, \dots, \tilde{\theta}^n = a\frac{\mathrm{d}x^n}{x^n} \right).$$

The forms $\widetilde{\omega}^i_j$ in the hint satisfy the conditions $\mathrm{d}\widetilde{\omega}^i = -\widetilde{\omega}^i_j \wedge \tilde{\theta}^j$ and $\widetilde{\omega}^i_j + \widetilde{\omega}^j_i = 0$. In fact,

$$\begin{aligned} \mathrm{d}\tilde{\theta}^i &= -\frac{a}{(x^n)^2}\,\mathrm{d}x^n \wedge \mathrm{d}x^i \\ &= -\sum_j \frac{1}{x^n}(\delta_{ni}\,\mathrm{d}x^j - \delta_{nj}\,\mathrm{d}x^i) \wedge \frac{a}{x^n}\,\mathrm{d}x^j, \end{aligned}$$

and the other condition is obvious. Thus, the forms $\widetilde{\omega}^i_j$ must be the connection forms relative to σ, since these are determined uniquely by the first structure equation. The second structure equation is

$$\begin{aligned} \widetilde{\Omega}^i_j &= \mathrm{d}\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j \\ &= \frac{1}{a}(\delta_{ni} d\tilde{\theta}^j - \delta_{nj}\mathrm{d}\tilde{\theta}^i) + \sum_k \frac{1}{a}(\delta_{ni}\tilde{\theta}^k - \delta_{nk}\tilde{\theta}^i) \wedge \frac{1}{a}(\delta_{nk}\tilde{\theta}^j - \delta_{nj}\tilde{\theta}^k) \\ &= -\frac{1}{a^2}\left(\sum_k \delta_{nk}\delta_{nk}\right)\tilde{\theta}^i \wedge \tilde{\theta}^j \\ &= -\frac{1}{a^2}\,\tilde{\theta}^i \wedge \tilde{\theta}^j. \end{aligned}$$

Problem 6.9.3. *Let (M,g) be a Riemannian manifold of constant curvature K. We define a metric $\widetilde{g}$ on $M \times M$ by*

$$\widetilde{g}\big((X_1,Y_1),(X_2,Y_2)\big) = g(X_1,X_2)\circ \mathrm{pr}_1 + g(Y_1,Y_2)\circ \mathrm{pr}_2,$$

where pr_1 *and* pr_2 *denote the projection map onto the first and the second factor, respectively. Is $(M \times M, \widetilde{g})$ a space of constant curvature?*

Solution. Let $(U,x^1,\dots,x^n)$ and $(V,x^{n+1},\dots,x^{2n})$ be coordinate systems on the first and second factor of $M \times M$, respectively. Hence, $(U \times V, x^1,\dots,x^{2n})$ is a coordinate system of $M \times M$. If $g = (g_{ij}(x))$ on U and $g = (g_{i+n,j+n}(y))$ on V, then $\widetilde{g}$ has matrix

$$\big(\widetilde{g}_{AB}(x,y)\big) = \begin{pmatrix} g_{ij}(x) & 0 \\ 0 & g_{i+n\,j+n}(y) \end{pmatrix}, \qquad A,B = 1,\dots,2n.$$

Computing the Christoffel symbols

$$\widetilde{\Gamma}^A_{BC} = \frac{1}{2}\widetilde{g}^{AD}\left(\frac{\partial \widetilde{g}_{DB}}{\partial x^C} + \frac{\partial \widetilde{g}_{DC}}{\partial x^B} - \frac{\partial \widetilde{g}_{BC}}{\partial x^D}\right), \qquad A,B,C,D = 1,\dots,2n,$$

it is easy to see that all of them vanish except perhaps $\widetilde{\Gamma}^i_{jk}(x,y) = \Gamma^i_{jk}(x)$ and $\widetilde{\Gamma}^{i+n}_{j+n,k+n}(x,y) \; = \Gamma^i_{jk}(y)$. Therefore, as one can easily compute, all the components of the curvature tensor field

$$\widetilde{R}^A_{BCD} = \frac{\partial \widetilde{\Gamma}^A_{BD}}{\partial x^C} - \frac{\partial \widetilde{\Gamma}^A_{BC}}{\partial x^D} + \widetilde{\Gamma}^A_{EC}\widetilde{\Gamma}^E_{BD} - \widetilde{\Gamma}^A_{ED}\widetilde{\Gamma}^E_{BC}$$

vanish except perhaps

$$\widetilde{R}^i_{jkl}(x,y) = R^i_{jkl}(x), \qquad \tilde{R}^{i+n}_{j+n,k+n,l+n}(x,y) = R^i_{jkl}(y).$$

Now, if $(M \times M, \widetilde{g})$ is a space of constant curvature, say $\widetilde{K}$, we have

$$\widetilde{R}^i_{jkl} = \widetilde{K}(\delta^i_k g_{jl} - g_{jk}\delta^i_l).$$

Hence, by the considerations above we deduce that, in particular,

$$0 = \widetilde{R}^{k+n}_{j,k+n,j} = \widetilde{K} g_{jj}.$$

Hence $(M \times M, \tilde{g})$ does not have constant curvature except when $\widetilde{K} = 0$.

Problem 6.9.4. *Prove that a Riemannian manifold of constant curvature K is an Einstein manifold.*

Solution. Let g denote the Riemannian metric, r the Ricci tensor and (e_i), $i = 1,\dots,n$, a local orthonormal frame. Given any $X,Y \in \mathfrak{X}(M)$, one has

$$\begin{aligned} r(X,Y) &= \sum_{i=1}^n g(R(e_i,Y)X,e_i) \\ &= \sum_{i,j,k=1}^n X^j Y^k g(R(e_i,e_k)e_j,e_i) \\ &= \sum_{i,j,k=1}^n X^j Y^k R_{ijik} \\ &= \sum_{i,j,k=1}^n X^j Y^k K(\delta_{ii}\delta_{jk} - \delta_{ik}\delta_{ji}) \\ &= K(n-1)g(X,Y). \end{aligned}$$

Problem 6.9.5. *Prove that a 3-dimensional Einstein manifold (M,g) is a space of constant curvature.*

Solution. Suppose $r = \lambda g$. Choose any plane $P \in T_pM$, and any orthonormal basis $\{e_1, e_2, e_3\}$ for T_pM such that $P = \langle e_1, e_2 \rangle$. Denote by P_{ij} the plane spanned by e_i and e_j for $i \neq j$, so that $P_{ij} = P_{ji}$. Then

$$r(e_i, e_i) = \sum_{j \neq i} K(P_{ij}),$$

where $K(P_{ij})$ stands for the sectional curvature determined by P_{ij}. Thus we have

$$r(e_1, e_1) + r(e_2, e_2) - r(e_3, e_3) = 2K(P).$$

As $r(e_i, e_i) = \lambda$ we obtain $K(P) = \frac{1}{2}\lambda$. As P is arbitrary, we conclude.

6.10 Left-invariant Metrics on Lie Groups

Problem 6.10.1. *Find, using Cartan's structure equations, the Levi-Civita connection, the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of the Heisenberg group H (see* Problem 4.1.6) *equipped with the left-invariant metric*

$$g = \mathrm{d}x^2 + \mathrm{d}z^2 + (\mathrm{d}y - x\,\mathrm{d}z)^2.$$

Solution. The moving coframe

$$\left(\tilde{\theta}^1 = \mathrm{d}x,\ \tilde{\theta}^2 = \mathrm{d}z,\ \tilde{\theta}^3 = \mathrm{d}y - x\,\mathrm{d}z\right), \tag{$\star$}$$

is dual to the orthonormal moving frame

$$\sigma = \left(X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial y}\right)$$

(see Problem 4.1.6).

The Levi-Civita connection forms $\widetilde{\omega}^i_j$ relative to σ satisfy Cartan's first structure equation

$$\mathrm{d}\tilde{\theta}^i = -\widetilde{\omega}^i_j \wedge \tilde{\theta}^j. \tag{$\star\star$}$$

From ($\star$) we have $\mathrm{d}\tilde{\theta}^1 = \mathrm{d}\tilde{\theta}^2 = 0$, $\mathrm{d}\tilde{\theta}^3 = -\tilde{\theta}^1 \wedge \tilde{\theta}^2$. Thus ($\star\star$) reduces to

$$\begin{aligned}
0 &= -\widetilde{\omega}^1_2 \wedge \mathrm{d}z - \widetilde{\omega}^1_3 \wedge (\mathrm{d}y - x\,\mathrm{d}z),\\
0 &= -\widetilde{\omega}^2_1 \wedge \mathrm{d}x - \widetilde{\omega}^2_3 \wedge (\mathrm{d}y - x\,\mathrm{d}z),\\
-\mathrm{d}x \wedge \mathrm{d}z &= -\widetilde{\omega}^3_1 \wedge \mathrm{d}x - \widetilde{\omega}^3_2 \wedge \mathrm{d}z.
\end{aligned}$$

The third equation is satisfied taking $\widetilde{\omega}^3_1 = -\frac{1}{2}\mathrm{d}z$, $\widetilde{\omega}^3_2 = \frac{1}{2}\mathrm{d}x$, and we have from the other equations that

$$0 = -\widetilde{\omega}^1_2 \wedge \mathrm{d}z - \frac{1}{2}\mathrm{d}z \wedge (\mathrm{d}y - x\,\mathrm{d}z), \qquad 0 = -\widetilde{\omega}^2_1 \wedge \mathrm{d}x + \frac{1}{2}\mathrm{d}x \wedge (\mathrm{d}y - x\,\mathrm{d}z),$$

which are satisfied if $\widetilde{\omega}^1_2 = \frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z)$. Since the forms $\tilde{\theta}^i$ determine uniquely a set of connection forms $\widetilde{\omega}^i_j$, we have that

$$\widetilde{\omega}^1_2 = \frac{1}{2}\tilde{\theta}^3, \qquad \widetilde{\omega}^1_3 = \frac{1}{2}\tilde{\theta}^2, \qquad \widetilde{\omega}^2_3 = -\frac{1}{2}\tilde{\theta}^1,$$

i.e.

$$(\widetilde{\omega}^i_j) = \begin{pmatrix} 0 & \frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z) & \frac{1}{2}\mathrm{d}z \\ -\frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z) & 0 & -\frac{1}{2}\mathrm{d}x \\ -\frac{1}{2}\mathrm{d}z & \frac{1}{2}\mathrm{d}x & 0 \end{pmatrix}.$$

From Cartan's second structure equation $\widetilde{\Omega}^i_j = \mathrm{d}\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j$ we obtain the curvature forms relative to σ:

$$\begin{aligned}
\widetilde{\Omega}^1_2 &= -\frac{1}{2}\mathrm{d}x \wedge \mathrm{d}z + \frac{1}{2}\mathrm{d}z \wedge \frac{1}{2}\mathrm{d}x \\
&= -\frac{3}{4}\tilde{\theta}^1 \wedge \tilde{\theta}^2, \\
\widetilde{\Omega}^1_3 &= \frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z) \wedge \left(-\frac{1}{2}\mathrm{d}x\right) \\
&= \frac{1}{4}\tilde{\theta}^1 \wedge \tilde{\theta}^3, \\
\widetilde{\Omega}^2_3 &= -\frac{1}{2}(\mathrm{d}y - x\,\mathrm{d}z) \wedge \frac{1}{2}\mathrm{d}z \\
&= \frac{1}{4}\tilde{\theta}^2 \wedge \tilde{\theta}^3.
\end{aligned}$$

Hence, from

$$\begin{aligned}
\widetilde{\Omega}^1_2 &= R^1_{212}\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R^1_{213}\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R^1_{223}\tilde{\theta}^2 \wedge \tilde{\theta}^3, \\
\widetilde{\Omega}^1_3 &= R^1_{312}\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R^1_{313}\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R^1_{323}\tilde{\theta}^2 \wedge \tilde{\theta}^3, \\
\widetilde{\Omega}^2_3 &= R^2_{312}\tilde{\theta}^1 \wedge \tilde{\theta}^2 + R^2_{313}\tilde{\theta}^1 \wedge \tilde{\theta}^3 + R^2_{323}\tilde{\theta}^2 \wedge \tilde{\theta}^3,
\end{aligned}$$

we deduce that the nonvanishing components of the Riemann curvature tensor are

$$R_{1212} = -\frac{3}{4}, \qquad R_{1313} = R_{2323} = \frac{1}{4}.$$

The Ricci tensor $r_{ij} = \sum_k R_{kikj}$ has thus components

$$r_{11} = r_{22} = -r_{33} = -1/2.$$

Finally, since $g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & x \\ 0 & x & 1 \end{pmatrix}$, the scalar curvature is given by

$$s = g^{ij} r_{ij} = -\frac{1}{2}(1+x^2).$$

Problem 6.10.2. *Let G be the Lie group defined by*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ v & tI \end{pmatrix} \in GL(n,\mathbb{R}) : v \in \mathbb{R}^{n-1},\ t > 0 \right\},$$

for an integer $n > 1$, where v denotes a column vector and I is the $(n-1)\times(n-1)$ identity matrix.

(1) *Prove that the Lie algebra $\mathfrak{g}$ of G consists of matrices of the form $\begin{pmatrix} 0 & 0 \\ v & tI \end{pmatrix}$, $v \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$.*

(2) *Let*

$$E_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \quad i < n, \qquad E_n = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where $\{e_i\}$ is the usual orthonormal basis of $\mathbb{R}^{n-1}$. Fix the left-invariant metric g on G so that $\{E_i\}$ is an orthonormal basis for $\mathfrak{g}$.

Prove that this metric is not bi-invariant on G considering that ad_{E_j} is not skew-symmetric for $j < n$.

(3) *Prove that (G, g) is a space of negative constant sectional curvature.*

Solution. (1)

$$\begin{aligned}
\exp\begin{pmatrix} 0 & 0 \\ v & tI \end{pmatrix} &= I + \begin{pmatrix} 0 & 0 \\ v & tI \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 \\ tv & t^2 I \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 0 \\ t^2 v & t^3 I \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right) v & \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) I \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ w & \mathrm{e}^t I \end{pmatrix} \in G.
\end{aligned}$$

Let e denote the identity element of G. It is easily checked that

$$\mathfrak{g} \equiv T_e G = \begin{pmatrix} 0 & 0 \\ v & tI \end{pmatrix}, \qquad v \in \mathbb{R}^{n-1}, \quad t \in \mathbb{R}.$$

(2)

$$[E_i, E_j] = \left[\begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e_j & 0 \end{pmatrix} \right] = 0, \qquad\qquad i, j < n,$$

$$[E_i, E_n] = \left[\begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right] = -\begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix} = -E_i.$$

Hence for $i, j < n$, one has

$$g(\mathrm{ad}_{E_i} E_n, E_j) + g(E_n, \mathrm{ad}_{E_i} E_j) = -\delta_{ij}.$$

(3) Let ∇ be the Levi-Civita connection of g. Then, as g is left-invariant, the Koszul formula for ∇ is reduced to

$$g(\nabla_X Y, Z) = \frac{1}{2}\big(g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\big),$$

$X, Y, Z \in \mathfrak{g}$. Thus for $i, j < n$ and for all k,

$$\nabla_{E_i} E_j = \delta_{ij} E_n, \qquad \nabla_{E_i} E_n = -E_i, \qquad \nabla_{E_n} E_k = 0.$$

Hence the sectional curvature is

$$\begin{aligned} K(E_i, E_j) &= g(R(E_i, E_j)E_j, E_i) \\ &= g(\nabla_{E_i}\nabla_{E_j}E_j - \nabla_{E_j}\nabla_{E_i}E_j - \nabla_{[E_i,E_j]}E_j, E_i) \\ &= \begin{cases} -g(E_i, E_i) = -1, & i, j < n \\ -g(E_i, E_i) = -1, & i < n,\ j = n, \end{cases} \end{aligned}$$

and (G, g) is in fact a space of constant sectional curvature -1.

Problem 6.10.3. *Let H be the Heisenberg group* (*see* Problem 4.1.6).

(1) *Compute the left-invariant Riemannian metric g on H built with the dual basis to the basis of left-invariant vector fields*

$$\mathscr{B} = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\}.$$

(2) *Find the Levi-Civita connection ∇ of g.*

(3) *Is (H, g) a space of constant curvature?*

Solution. (1) If $\{\beta_1, \beta_2, \beta_3\}$ is a basis of left-invariant 1-forms on H, then $g = \sum_{i=1}^3 \beta_i^2$ is a left-invariant metric. The dual basis to a basis of left-invariant vector fields, is a basis of left-invariant 1-forms. The dual basis $\{\beta_1, \beta_2, \beta_3\}$ of $\mathscr{B}$ is easily computed to be

$$\{\beta_1 = \mathrm{d}x,\ \beta_2 = \mathrm{d}y - x\,\mathrm{d}z,\ \beta_3 = \mathrm{d}z\}.$$

Therefore, the left-invariant metric on H is

$$g = \mathrm{d}x^2 + (\mathrm{d}y - x\,\mathrm{d}z)^2 + \mathrm{d}z^2.$$

(2) The Levi-Civita connection of g is given by

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \quad X,Y,Z \in \mathfrak{g}.$$

Now, since $\{X_1, X_2, X_3\}$ is a basis of $\mathfrak{h}$, to determine ∇ we only have to know $\nabla_{X_i} X_j$. The nonzero brackets $[X_i, X_j]$, for $i, j = 1,2,3$, are $[X_1, X_3] = -[X_3, X_1] = X_2$.

Hence, $g(\nabla_{X_1} X_1, X_i) = 0$, $i = 1,2,3$, so $\nabla_{X_1} X_1 = 0$;

$$g(\nabla_{X_1} X_2, X_1) = 0, \quad g(\nabla_{X_1} X_2, X_2) = 0, \quad 2g(\nabla_{X_1} X_2, X_3) = -1,$$

thus $\nabla_{X_1} X_2 = -\frac{1}{2} X_3$. So, as ∇ is torsionless, it follows that $\nabla_{X_2} X_1 = -\frac{1}{2} X_3$;

$$g(\nabla_{X_1} X_3, X_1) = 0, \quad 2g(\nabla_{X_1} X_3, X_2) = 1, \quad g(\nabla_{X_1} X_3, X_3) = 0,$$

therefore $\nabla_{X_1} X_3 = \frac{1}{2} X_2$, and $\nabla_{X_3} X_1 = -\frac{1}{2} X_2$;

$$g(\nabla_{X_2} X_2, X_i) = 0, \quad i = 1,2,3, \quad \text{so} \quad \nabla_{X_2} X_2 = 0;$$

$$2g(\nabla_{X_2} X_3, X_1) = 1, \quad g(\nabla_{X_2} X_3, X_2) = 0, \quad g(\nabla_{X_2} X_3, X_3) = 0,$$

and so $\nabla_{X_2} X_3 = \frac{1}{2} X_1$, and $\nabla_{X_3} X_2 = -\frac{1}{2} X_1$;

$$g(\nabla_{X_3} X_3, X_i) = 0, \quad i = 1,2,3, \quad \text{so} \quad \nabla_{X_3} X_3 = 0.$$

(3) Since $\{X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{h}$, $\{X_1|_p, X_2|_p\}$ is an orthonormal basis of a plane $P \subset T_p H$ and $\{X_1|_p, X_3|_p\}$ is an orthonormal basis of a plane $P' \subset T_p H$. The sectional curvatures $K(P)$ and $K(P')$ are thus

$$\begin{aligned} K(P) &= R(X_1, X_2, X_1, X_2)(p) \\ &= g(\nabla_{X_1} \nabla_{X_2} X_2 - \nabla_{X_2} \nabla_{X_1} X_2 - \nabla_{[X_1, X_2]} X_2, X_1)(p) = \frac{1}{4}, \end{aligned}$$

and $K(P') = -\frac{1}{4}$. Hence (H, g) is not a space of constant curvature.

Problem 6.10.4. (1) *Let G be a compact Lie group equipped with a bi-invariant metric g and $\mathfrak{g}$ its Lie algebra. If X and Y are left-invariant vector fields on G and ∇ is the Levi-Civita connection of g, prove that $\nabla_X Y = \frac{1}{2}[X,Y]$.*

(2) *Compute $R(X,Y)Z$, $X,Y,Z \in \mathfrak{g}$.*

(3) *Show that the sectional curvature of g is non-negative.*

HINT (*to* (1) *and* (3)): *If a metric is bi-invariant, each* ad_X, $X \in \mathfrak{g}$, *is skew-symmetric with respect to g* (*see* [25, p. 114]).

Solution. (1) By the result quoted in the hint, the Koszul formula is reduced to

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}(g([X,Y],Z) + g(\operatorname{ad}_Z Y, X) + g(\operatorname{ad}_Z X, Y)) \\ &= \frac{1}{2} g([X,Y],Z). \end{aligned}$$

That is, we have $\nabla_X Y = \frac{1}{2}[X,Y]$.

(2) From (1) one has for the curvature:

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}\left(\underset{XYZ}{\mathfrak{S}}[[X,Y],Z]\right) - \frac{1}{4}[[X,Y],Z] \\ &= -\frac{1}{4}[[X,Y],Z] \qquad \text{(by Jacobi identity).} \end{aligned}$$

(3) Let $X,Y \in \mathfrak{g}$ be orthonormal. Then, again by the result in the hint, we have for the sectional curvature at e, hence at all points:

$$\begin{aligned} K(X,Y) &= g(R(X,Y)Y,X) \\ &= -\frac{1}{4} g([[X,Y],Y],X) \\ &= \frac{1}{4} g(\mathrm{ad}_Y [X,Y],X) \\ &= -\frac{1}{4} g([X,Y],\mathrm{ad}_Y X) \\ &= \frac{1}{4} g([X,Y],[X,Y]) \\ &= \frac{1}{4} |[X,Y]|^2 \geqslant 0. \end{aligned}$$

Problem 6.10.5. *Consider the Lie group*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} : x,y \in \mathbb{R},\ y > 0 \right\}.$$

(1) *Prove that its Lie algebra is* $\mathfrak{g} = \langle y\partial/\partial x, y\partial/\partial y \rangle$.

(2) *Write the left-invariant metric on G built with the dual basis to that in* (1).

(3) *Determine the Levi-Civita connection* ∇ *of* g.

(4) *Is* (G,g) *a space of constant curvature?*

(5) *Prove* (*without using* (4)) *that* (G,g) *is an Einstein manifold.*

Solution. (1) G is a Lie group with the product of matrices and with only one chart:

$$(G,\varphi), \qquad G \xrightarrow{\varphi} U, \qquad \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \mapsto (x,y),$$

where U denotes the open subset $\{(x,y) \in \mathbb{R}^2 : y > 0\}$ of $\mathbb{R}^2$, hence $\dim G = 2$. Thus $\dim \mathfrak{g} = 2$ and so, to prove that $\mathfrak{g}$ is generated by $X_1 = y\partial/\partial x$, $X_2 = y\partial/\partial y$, we shall only have to prove that X_1, X_2 are linearly independent and left-invariant. They are linearly independent, as $y > 0$. To prove that they are left-invariant vector fields, we have to prove that for all $A \in G$, one has

$$(L_A)_{*B}(X_i|_B) = X_i|_{AB}\,, \qquad \text{for all} \quad B \in G, \quad i = 1,2. \tag{$\star$}$$

Let $A = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ x_0 & y_0 \end{pmatrix}$. As $AB = \begin{pmatrix} 1 & 0 \\ a + bx_0 & by_0 \end{pmatrix}$, the right-hand side of $(\star)$ is

$$X_1|_{AB} = by_0 \left.\frac{\partial}{\partial x}\right|_{AB}, \qquad X_2|_{AB} = by_0 \left.\frac{\partial}{\partial y}\right|_{AB}.$$

To determine the left-hand side of $(\star)$, we compute the Jacobian of L_A using the diagram

$$\begin{array}{ccc} G & \xrightarrow{L_A} & G \\ {\scriptstyle \varphi^{-1}}\uparrow & & \downarrow{\scriptstyle \varphi} \\ U & \xrightarrow{\varphi \circ L_A \circ \varphi^{-1}} & U \end{array}$$

with

$$\begin{array}{ccc} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} & \xrightarrow{L_A} & \begin{pmatrix} 1 & 0 \\ a + bx & by \end{pmatrix} \\ {\scriptstyle \varphi^{-1}}\uparrow & & \downarrow{\scriptstyle \varphi} \\ (x,y) & \xrightarrow{\varphi \circ L_A \circ \varphi^{-1}} & (a + bx, by). \end{array}$$

It follows that the Jacobian of $\varphi \circ L_A \circ \varphi^{-1}$ is $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, hence

$$\begin{aligned} (L_A)_{*B}\, X_1|_B &\equiv \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} y_0 \\ 0 \end{pmatrix} \\ &\equiv by_0 \left.\frac{\partial}{\partial x}\right|_{AB} \\ &= X_1|_{AB}\,, \\ (L_A)_*\, X_2|_B &\equiv \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ y_0 \end{pmatrix} \\ &\equiv by_0 \left.\frac{\partial}{\partial y}\right|_{AB} \\ &= X_2|_{AB}\,. \end{aligned}$$

(2) The dual basis $\{\beta_1, \beta_2\}$ to $\{X_1, X_2\}$ is $\{\beta_1 = \mathrm{d}x/y,\ \beta_2 = \mathrm{d}y/y\}$. Therefore, the left-invariant metric on G we are looking for is $g = (1/y^2)(\mathrm{d}x^2 + \mathrm{d}y^2)$.

(3) From the formula for the Levi-Civita connection of a left-invariant metric g on a Lie group and from

$$[X_1,X_1]=[X_2,X_2]=0, \qquad [X_1,X_2]=-[X_2,X_1]=-X_1,$$

we have:

$g(\nabla_{X_1}X_1,X_1)=0$, $2g(\nabla_{X_1}X_1,X_2)=2$; thus $\nabla_{X_1}X_1=X_2$;

$2g(\nabla_{X_1}X_2,X_1)=-2$, $g(\nabla_{X_1}X_2,X_2)=0$; so $\nabla_{X_1}X_2=-X_1$. As ∇ is torsionless, one has $\nabla_{X_2}X_1=0$;

$g(\nabla_{X_2}X_2,X_1)=g(\nabla_{X_2}X_2,X_2)=0$; that is, $\nabla_{X_2}X_2=0$.

(4)

$$\begin{aligned} R(X_1,X_2,X_1,X_2) &= g(\nabla_{X_1}\nabla_{X_2}X_2-\nabla_{X_2}\nabla_{X_1}X_2-\nabla_{[X_1,X_2]}X_2,X_1)\\ &= -g(X_1,X_1)=-1. \end{aligned}$$

Thus G is a space of constant curvature -1.

(5) Let $X,Y\in\mathfrak{X}(G)$, $X=f_1X_1+f_2X_2$, $Y=h_1X_1+h_2X_2$. Thus,

$$\begin{aligned} r(X,Y) &= R(X_1,Y,X_1,X)+R(X_2,Y,X_2,X)\\ &= R(X_1,h_2X_2,X_1,f_2X_2)+R(X_2,h_1X_1,X_2,f_1X_1)\\ &= (f_1h_1+f_2h_2)R(X_1,X_2,X_1,X_2)\\ &= -(f_1h_1+f_2h_2)\\ &= -g(X,Y). \end{aligned}$$

Therefore, G is an Einstein manifold.

REMARK. In Problem 6.9.4 it has been proved that every Riemannian manifold (M,g) of dimension n and constant curvature K is an Einstein manifold, with Ricci tensor $r(X,Y)=K(n-1)g(X,Y)$. Here we have a verification of this formula in this example.

Problem 6.10.6. *Let G be a Lie group and Γ a discrete subgroup of G which acts on the left on G. Denote by $\Gamma\backslash G$ the quotient space of right cosets*

$$\Gamma\backslash G=\{\Gamma g : g\in G\}.$$

Compute the de Rham cohomology of the compact quotient $\Gamma\backslash H$ of the Heisenberg group H (see Problem 4.1.6*) by the discrete subgroup*

$$\Gamma=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix},\ x,y,z\in\mathbb{Z}\right\}.$$

REMARK. *A* nilmanifold *is a manifold which is a quotient of a nilpotent Lie group.*

Solution. We know (see Problem 6.10.3) that $\{dx, dy, dz-x\,dy\}$ is a basis for the left-invariant differential 1-forms on H. In particular, they are preserved by Γ, and so they descend to 1-forms α, β, γ on the nilmanifold $\Gamma\backslash H$, that is, if π denotes the canonical projection map $\pi\colon H \to \Gamma\backslash H$,

$$\pi^*\alpha = \mathrm{d}x, \qquad \pi^*\beta = \mathrm{d}y, \qquad \pi^*\gamma = \mathrm{d}z - x\,\mathrm{d}y. \tag{$\star$}$$

From $(\star)$ we have

$$\mathrm{d}\alpha = \mathrm{d}\beta = 0, \qquad \mathrm{d}\gamma = -\alpha \wedge \beta. \tag{$\star\star$}$$

In fact, for instance we have $\pi^*(\mathrm{d}\alpha) = \mathrm{d}(\pi^*\alpha) = 0$. Moreover,

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}(\alpha \wedge \gamma) = \mathrm{d}(\beta \wedge \gamma) = 0.$$

By virtue of Nomizu's Theorem 7.6.19, from $(\star\star)$ we deduce that the de Rham cohomology groups are

$$\begin{aligned}
H^0_{dR}(\Gamma\backslash H, \mathbb{R}) &= \langle [1] \rangle, \\
H^1_{dR}(\Gamma\backslash H, \mathbb{R}) &= \langle [\alpha], [\beta] \rangle, \\
H^2_{dR}(\Gamma\backslash H, \mathbb{R}) &= \langle [\alpha \wedge \gamma], [\beta \wedge \gamma] \rangle, \\
H^3_{dR}(\Gamma\backslash H, \mathbb{R}) &= \langle [\alpha \wedge \beta \wedge \gamma] \rangle.
\end{aligned}$$

Notice that $\alpha \wedge \beta$ is closed but also exact. As for $\alpha \wedge \beta \wedge \gamma = -\mathrm{d}\gamma \wedge \gamma$, it is not exact.

6.11 Gradient, Divergence, Codifferential, Curl, Laplacian, and Hodge Star Operator on Riemannian Manifolds

Problem 6.11.1. *Let (M, g) be a Riemannian manifold, T_pM the tangent space at $p \in M$ and T_p^*M its dual space. The musical isomorphisms $\flat$ and $\sharp$ are defined (see* Problem 6.1.1*) by*

$$\flat\colon T_pM \to T_p^*M, \qquad X \mapsto X^\flat, \qquad X^\flat(Y) = g(X, Y),$$

and its inverse $\omega \mapsto \omega^\sharp$, respectively. The gradient *of a function $f \in C^\infty M$ is defined as*

$$\operatorname{grad} f = (\mathrm{d}f)^\sharp.$$

(1) *Prove that $g(\operatorname{grad} f, X) = Xf$, $X \in \mathfrak{X}(M)$.*

Given local coordinates $\{x^i\}$:

(2) *Compute $(\partial/\partial x^i)^\flat$.*

(3) *Calculate $(\mathrm{d}x^i)^\sharp$.*

(4) *Write $\operatorname{grad} f$ in local coordinates.*

(5) *Verify that in the particular case of $\mathbb{R}^3$ equipped with the Euclidean metric, we recover the classical expression of $\operatorname{grad} f$.*

Solution. (1) $g(\operatorname{grad} f, X) = g((\mathrm{d}f)^\sharp, X) = \mathrm{d}f(X) = Xf$.

(2) Since

$$\begin{aligned}\left(\frac{\partial}{\partial x^i}\right)^\flat \left(\frac{\partial}{\partial x^j}\right) &= g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\\ &= g_{ij}\\ &= g_{ik}\mathrm{d}x^k\left(\frac{\partial}{\partial x^j}\right),\end{aligned}$$

we have

$$\left(\frac{\partial}{\partial x^i}\right)^\flat = g_{ik}\mathrm{d}x^k.$$

(3) From (2) we have $\dfrac{\partial}{\partial x^i} = g_{ik}(\mathrm{d}x^k)^\sharp$, since $\flat$ and $\sharp$ are inverse maps. So we obtain

$$(\mathrm{d}x^j)^\sharp = g^{ji}\frac{\partial}{\partial x^i}.$$

(4)

$$\begin{aligned}\operatorname{grad} f &= (\mathrm{d}f)^\sharp\\ &= \left(\frac{\partial f}{\partial x^i}\mathrm{d}x^i\right)^\sharp\\ &= \frac{\partial f}{\partial x^i}(\mathrm{d}x^i)^\sharp\\ &= g^{ji}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}.\end{aligned}$$

(5) In this case,

$$\operatorname{grad} f = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}.$$

Problem 6.11.2. *Let M be a C^∞ manifold equipped with a linear connection ∇. Let $\{X_1, \dots, X_n\}$ be a basis of T_pM, and $\{\omega^1, \dots, \omega^n\}$ its dual basis. The* divergence *of $Z \in \mathfrak{X}(M)$ is defined by*

$$(\operatorname{div} Z)(p) = \omega^i(\nabla_{X_i} Z).$$

(1) *Prove that $(\operatorname{div} Z)(p)$ does not depend on the chosen basis.*

(2) *Show that the divergence of a C^∞ field on $\mathbb{R}^3$ is the same as the definition given in Advanced Calculus.*

Solution. (1) Given another basis $\left\{\widetilde{X}_j = a^i_j X_i\right\}$, its dual basis is given by $\{\widetilde{\omega}^j = b^j_i \omega^i\}$, so that $b^j_i a^i_k = a^j_i b^i_k = \delta^j_k$. Thus,

$$\begin{aligned}\widetilde{\omega}^j(\nabla_{\widetilde{X}_j}Z) &= b_i^j\,\omega^i(\nabla_{a_j^h X_h}Z)\\ &= a_j^h b_i^j\,\omega^i(\nabla_{X_h}Z)\\ &= \delta_i^h\,\omega^i(\nabla_{X_h}Z)\\ &= \omega^i(\nabla_{X_i}Z).\end{aligned}$$

(2) Given the basis $\{(\partial/\partial x^i)_p\}$ of $T_p\mathbb{R}^3$, its dual basis is $\{\mathrm{d}x^i|_p\}$, $i=1,2,3$, and we have for $Z = Z^i\partial/\partial x^i \in \mathfrak{X}(\mathbb{R}^3)$, since the Christoffel symbols of the flat connection on $\mathbb{R}^3$ vanish, that

$$\begin{aligned}(\operatorname{div} Z)(p) &= \mathrm{d}x^i|_p\left(\nabla_{\left.\frac{\partial}{\partial x^i}\right|_p}\left(Z^j\frac{\partial}{\partial x^j}\right)\right)\\ &= \mathrm{d}x^i|_p\frac{\partial Z^j}{\partial x^i}(p)\left.\frac{\partial}{\partial x^j}\right|_p\\ &= \sum_i\frac{\partial Z^i}{\partial x^i}(p).\end{aligned}$$

Problem 6.11.3. *Let (M,g) be a Riemannian manifold, and let*:

(a) ∇ *be the Levi-Civita connection of g.*

(b) $\operatorname{grad} f$ *be the gradient of the function $f \in C^\infty M$.*

(c) $\operatorname{div} X$ *be the divergence* (*see the previous problem*) *of the vector field $X \in \mathfrak{X}(M)$. For a local field of orthonormal frames (e_i), $i=1,\dots,n$, we have* $\operatorname{div} X = \sum_i g(\nabla_{e_i}X, e_i)$.

(d) H^f *be the* Hessian *of $f \in C^\infty M$, defined as the second covariant differential $\nabla(\nabla f)$, that is,*

$$H^f(X,Y) = XYf - (\nabla_X Y)f, \qquad X,Y \in \mathfrak{X}(M).$$

(e) Δf *be the* Laplacian *of $f \in C^\infty M$, defined by*

$$\Delta f = \operatorname{div}\operatorname{grad} f.$$

Moreover, suppose $\dim M = 3$. *Then*:

Prove the following formulas for $f,h \in C^\infty M$, $X,Y \in \mathfrak{X}(M)$:

(1) $\operatorname{grad}(fh) = f\operatorname{grad} h + h\operatorname{grad} f$.

(2) $\operatorname{div}(fX) = Xf + f\operatorname{div} X$.

(3) $H^{fh} = fH^h + hH^f + \mathrm{d}f\otimes\mathrm{d}h + \mathrm{d}h\otimes\mathrm{d}f$.

(4) $\Delta(fh) = f\Delta h + h\Delta f + 2g(\operatorname{grad} f, \operatorname{grad} h)$.

Solution. (1)

$$\begin{aligned} g(\operatorname{grad} fh, X) &= X(fh) \\ &= (Xf)h + fXh \\ &= g(\operatorname{grad} f, X)h + g(\operatorname{grad} h, X)f \\ &= g(h \operatorname{grad} f + f \operatorname{grad} h, X). \end{aligned}$$

(2)

$$\begin{aligned} \operatorname{div}(fX) &= \sum_i g(\nabla_{e_i} fX, e_i) \\ &= \sum_i g\left((e_i f)X + f\nabla_{e_i} X, e_i\right) \\ &= \sum_i g(X, e_i) e_i f + f \sum_i g(\nabla_{e_i} X, e_i) \\ &= Xf + f \operatorname{div} X. \end{aligned}$$

(3)

$$\begin{aligned} H^{fh}(X,Y) &= XYfh - (\nabla_X Y)fh \\ &= X((Yf)h + fYh) - ((\nabla_X Y)f)h - f(\nabla_X Y)h \\ &= (XYf)h + (Yf)(Xh) + (Xf)(Yh) \\ &\quad + fXYh - ((\nabla_X Y)f)h - f(\nabla_X Y)h \\ &= (fH^h + hH^f + \mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f)(X,Y). \end{aligned}$$

(4)

$$\begin{aligned} \Delta fh &= \operatorname{div} \operatorname{grad} fh \\ &= \operatorname{div}(f \operatorname{grad} h + h \operatorname{grad} f) \\ &= (\operatorname{grad} h)f + f\Delta h + (\operatorname{grad} f)h + h\Delta f \\ &= f\Delta h + h\Delta f + 2g(\operatorname{grad} f, \operatorname{grad} h). \end{aligned}$$

Problem 6.11.4. *Consider on* $\mathbb{R}^n$ *the metric* $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$ *and the volume element* $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

(1) *Prove that given a form* Ω_k *of degree* k *there is only one form* $\star\Omega_k$, *of degree* $n-k$, *such that*

$$(\star\Omega_k)(X_1, \ldots, X_{n-k})\, \omega = \Omega_k \wedge X_1^\flat \wedge \cdots \wedge X_{n-k}^\flat.$$

(2) *The* Hodge star operator

$$\star \colon \Lambda^k \mathbb{R}^n \to \Lambda^{n-k} \mathbb{R}^n$$

is defined by the previous formula. Prove that this operator satisfies the following equalities:

$$\star^2 = (-1)^{k(n-k)}, \qquad \star^{-1} = (-1)^{k(n-k)}\star, \qquad \Omega_k \wedge (\star\Theta_k) = \Theta_k \wedge (\star\Omega_k).$$

(3) *The* codifferential $\delta\colon \Lambda^k\mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^n$ *is defined by*

$$\delta = (-1)^{n(k+1)+1}\star \mathrm{d}\star.$$

Prove that δ *satisfies* $\delta^2 = 0$.

(4) *The* Laplacian $\Delta\colon \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ *is defined by*

$$\Delta = (\mathrm{d}+\delta)^2 = \mathrm{d}\,\delta + \delta\mathrm{d}.$$

Prove that if $f \in C^\infty\mathbb{R}^n$, *then* $\Delta f = -\sum_{i=1}^n \partial^2 f/\partial(x^i)^2$.

Solution. (1) We only have to prove the above properties for a basis of the exterior algebra. Let $\{X_1,\dots,X_n\}$ be an orthonormal basis of vector fields on $\mathbb{R}^n$ and $\{\theta^1,\dots,\theta^n\}$ its dual basis. Consider multi-indexes $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$. We have

$$\{\star(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k})\}(X_{j_1},\dots,X_{j_{n-k}})\,\omega = \theta^{i_1}\wedge\cdots\wedge\theta^{i_k}\wedge\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}}, \qquad (\diamond)$$

which vanishes if $(j_1,\dots,j_{n-k})$ is not the complement of $(i_1,\dots,i_k)$ in $(1,2,\dots,n)$. Denoting by $(j_1,\dots,j_{n-k})$ such ordered complement, we have

$$\star(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}) = \operatorname{sgn}(i_1,\dots,i_k,j_1,\dots,j_{n-k})\,\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}}, \qquad (\diamond\diamond)$$

(where sgn denotes the sign of a permutation). In fact, from $(\diamond)$ above we deduce

$$\{\star(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k})\}(X_{j_1},\dots,X_{j_{n-k}})\omega = \operatorname{sgn}(i_1,\dots,i_k,j_1,\dots,j_{n-k})\,\omega.$$

(2) From $(\diamond\diamond)$ above we have

$$\begin{aligned}
\star\{\star(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k})\} &= \star\{\operatorname{sgn}(i_1,\dots,i_k,j_1,\dots,j_{n-k})\,\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}}\}\\
&= \operatorname{sgn}(i_1,\dots,i_k,j_1,\dots,j_{n-k})\star(\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}})\\
&= \operatorname{sgn}(i_1,\dots,i_k,j_1,\dots,j_{n-k})\cdot\operatorname{sgn}(j_1,\dots,j_{n-k},i_1,\dots,i_k)\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}\\
&= (-1)^{i_1+\cdots+i_k-\frac{k(k+1)}{2}+j_1+\cdots+j_{n-k}-\frac{(n-k)(n-k+1)}{2}}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}\\
&= (-1)^{k(n-k)}\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}.
\end{aligned}$$

From $\star^2 = (-1)^{k(n-k)}$ it follows that $\star = (-1)^{k(n-k)}\star^{-1}$, and thus $\star^{-1} = (-1)^{k(n-k)}\star$.

Consider $\Omega_k = \theta^{i_1}\wedge\cdots\wedge\theta^{i_k}$ and $\Theta_k = \theta^{j_1}\wedge\cdots\wedge\theta^{j_k}$. Then

$$\Omega_k \wedge (\star\Theta_k) = \begin{cases} 0 & \text{if } \{i_1,\dots,i_k\} \neq \{j_1,\dots,j_k\}\\ \omega & \text{if } \{i_1,\dots,i_k\} = \{j_1,\dots,j_k\}. \end{cases}$$

Proceed similarly for $\Theta_k \wedge (\star\Omega_k)$.

(3) $\delta^2 = \star \mathrm{d} \star \star \mathrm{d} \star = (-1)^{(k-1)(n-k+1)} \star \mathrm{d}^2 \star = 0$, since $\mathrm{d}^2 = 0$.

(4) Since $\delta f = 0$ for $f \in C^\infty\mathbb{R}^n$, we have $\Delta f = \delta\,\mathrm{d}f$, hence

$$
\begin{aligned}
\Delta f &= \delta\,\mathrm{d}f \\
&= -\star \mathrm{d} \star \mathrm{d}f \\
&= -\star \mathrm{d} \star \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \\
&= -\star \mathrm{d}(-1)^{i-1} \frac{\partial f}{\partial x^i}\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= -\star (-1)^{i-1} \frac{\partial^2 f}{\partial x^i \partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \\
&= -\star \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2} \right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= -\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}.
\end{aligned}
$$

Problem 6.11.5. *Let* $\{1, \mathrm{d}x, \mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}y\}$ *be the standard basis of* $\Lambda^*\mathbb{R}^2$. *Consider* $\mathbb{R}^2$ *equipped with the usual flat metric, and let* $\star$ *denote the Hodge star operator.*

Compute $\star 1$, $\star \mathrm{d}x$, $\star \mathrm{d}y$, $\star(\mathrm{d}x \wedge \mathrm{d}y)$.

Solution. $\mathrm{d}x \wedge \mathrm{d}y$, $\mathrm{d}y$, $-\mathrm{d}x$ and 1, respectively.

Problem 6.11.6. *A differential* 2*-form* F *on* $\mathbb{R}^4$ *is said to be* autodual *if*

$$\star F = F,$$

where $\star$ *stands for the Hodge star operator. Prove that the curvature* 2*-form* $(\star)$ *in* Problem 5.3.2 *is autodual.*

REMARK. *A curvature form* F *satisfying* $\star F = F$ *is called an* instanton. *The instantons described in* Problem 5.3.2 *are called the* Belavin-Polyakov-Schwartz-Tyupkin instantons.

Solution. The Hodge star operator on the 2-forms on the Euclidean space $\mathbb{R}^4 \approx \mathbb{H}$ is easily seen to be defined from

$$
\begin{aligned}
\star(\mathrm{d}x^1 \wedge \mathrm{d}x^2) &= \mathrm{d}x^3 \wedge \mathrm{d}x^4, & \star(\mathrm{d}x^2 \wedge \mathrm{d}x^3) &= \mathrm{d}x^1 \wedge \mathrm{d}x^4, \\
\star(\mathrm{d}x^1 \wedge \mathrm{d}x^3) &= -\mathrm{d}x^2 \wedge \mathrm{d}x^4, & \star(\mathrm{d}x^2 \wedge \mathrm{d}x^4) &= -\mathrm{d}x^1 \wedge \mathrm{d}x^3, \\
\star(\mathrm{d}x^1 \wedge \mathrm{d}x^4) &= \mathrm{d}x^2 \wedge \mathrm{d}x^3, & \star(\mathrm{d}x^3 \wedge \mathrm{d}x^4) &= \mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned}
$$

Thus, the basis of autodual 2-forms is

$$\left\{ \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4,\ \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4,\ \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3 \right\}.$$

Now, due to the identification $\mathbb{R}^4 \equiv \mathbb{H}$, one has

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}\bar{x} &= (\mathrm{d}x^1 + \mathrm{d}x^2\,\mathrm{i} + \mathrm{d}x^3\,\mathrm{j} + \mathrm{d}x^4\,\mathrm{k}) \wedge (\mathrm{d}x^1 - \mathrm{d}x^2\,\mathrm{i} - \mathrm{d}x^3\,\mathrm{j} - \mathrm{d}x^4\,\mathrm{k}) \\
&= -2\{(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4)\mathrm{i} + (\mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4)\mathrm{j} \\
&\quad + (\mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3)\mathrm{k}\}.
\end{aligned}$$

Problem 6.11.7. *Define on* $\mathbb{R}^3$ *equipped with the usual flat metric* g:

(a) $\operatorname{div} X = \operatorname{div} X^\flat = -\delta X^\flat = \star\mathrm{d}\star X^\flat$, $X \in \mathfrak{X}(\mathbb{R}^3)$.

(b) $\operatorname{curl} X = (\star\mathrm{d}X^\flat)^\sharp$.

Prove the formulas:

(1) $\operatorname{curl}\operatorname{grad} f = 0$.

(2) $\operatorname{div}\operatorname{curl} X = 0$.

(3) $\Delta\omega = -(\operatorname{grad}\operatorname{div}\omega^\sharp + \operatorname{curl}\operatorname{curl}\omega^\sharp)^\flat$, $\omega \in \Lambda^1\mathbb{R}^3$.

(4) $\operatorname{curl}(fX) = (\operatorname{grad} f) \times X + f \operatorname{curl} X$, *where* $\times$ *denotes the usual vector product in* $\mathbb{R}^3$.

(5) $\operatorname{div}(fX) = (\operatorname{grad} f) \cdot X + f \operatorname{div} X$.

(6)

$$(\star(\operatorname{curl} X)^\flat)(Y,Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y).$$

(7) *Prove that* $\operatorname{curl} X$ *coincides with its classical expression and then*

$$\operatorname{div}(X \times Y) = X \cdot \operatorname{curl} Y + (\operatorname{curl} X) \cdot Y,$$

where the dot denotes the usual scalar product in $\mathbb{R}^3$.

Solution. (1)

$$\begin{aligned}
\operatorname{curl}\operatorname{grad} f &= \operatorname{curl}(\mathrm{d}f)^\sharp \\
&= (\star\mathrm{d}((\mathrm{d}f)^\sharp)^\flat)^\sharp \\
&= (\star\mathrm{d}\mathrm{d}f)^\sharp = 0.
\end{aligned}$$

(2)

$$\begin{aligned}
\operatorname{div}\operatorname{curl} X &= \operatorname{div}(\star\mathrm{d}X^\flat)^\sharp \\
&= \operatorname{div}((\star\mathrm{d}X^\flat)^\sharp)^\flat \\
&= -\delta\star\mathrm{d}X^\flat \\
&= \star\mathrm{d}\star\star\mathrm{d}X^\flat \\
&= (-1)^{2(3-2)}\star\mathrm{d}\mathrm{d}X^\flat = 0.
\end{aligned}$$

(3)

$$\begin{aligned}\Delta\omega &= (\mathrm{d}\delta+\delta\mathrm{d})\omega\\ &= \mathrm{d}(-\operatorname{div}\,\omega)+\delta\mathrm{d}\omega\\ &= -\mathrm{d}\operatorname{div}\,\omega^\sharp - \star\mathrm{d}\star\mathrm{d}\omega\\ &= -((\mathrm{d}\operatorname{div}\,\omega^\sharp)^\sharp)^\flat - \star\mathrm{d}((\star\mathrm{d}(\omega^\sharp)^\flat)^\sharp)^\flat\\ &= -(\operatorname{grad}\operatorname{div}\,\omega^\sharp)^\flat - \star\mathrm{d}(\operatorname{curl}\,\omega^\sharp)^\flat\\ &= -(\operatorname{grad}\operatorname{div}\,\omega^\sharp + \operatorname{curl}\operatorname{curl}\,\omega^\sharp)^\flat.\end{aligned}$$

(4)

$$\begin{aligned}\operatorname{curl}(fX) &= (\star\mathrm{d}(fX^\flat))^\sharp\\ &= (\star(\mathrm{d}f\wedge X^\flat + f\,\mathrm{d}X^\flat))^\sharp\\ &= (\star(\mathrm{d}f\wedge X^\flat))^\sharp + f\operatorname{curl}X\\ &= (\operatorname{grad}f)\times X + f\operatorname{curl}X,\end{aligned}$$

since

$$\begin{aligned}(\star(\mathrm{d}f\wedge X^\flat))^\sharp &= \left(\star\left(\frac{\partial f}{\partial x^i}\mathrm{d}x^i\wedge(X_j\mathrm{d}x^j)\right)\right)^\sharp\\ &= \left(\star\left\{\left(\frac{\partial f}{\partial x^1}X_2-\frac{\partial f}{\partial x^2}X_1\right)\mathrm{d}x^1\wedge\mathrm{d}x^2+\cdots\right\}\right)^\sharp\\ &= \left(\left(\frac{\partial f}{\partial x^1}X_2-\frac{\partial f}{\partial x^2}X_1\right)\mathrm{d}x^3+\cdots\right)^\sharp\\ &= \left(\frac{\partial f}{\partial x^2}X_3-\frac{\partial f}{\partial x^3}X_2\right)\frac{\partial}{\partial x^1}+\left(\frac{\partial f}{\partial x^3}X_1-\frac{\partial f}{\partial x^1}X_3\right)\frac{\partial}{\partial x^2}\\ &\quad+\left(\frac{\partial f}{\partial x^1}X_2-\frac{\partial f}{\partial x^2}X_1\right)\frac{\partial}{\partial x_3}\\ &= \operatorname{grad}f\times X.\end{aligned}$$

(5)

$$\begin{aligned}\operatorname{div}(fX) &= \operatorname{div}(fX)^\flat\\ &= -\delta(fX)^\flat\\ &= \star\mathrm{d}\star(fX)^\flat\\ &= \star\mathrm{d}(f(\star X^\flat))\end{aligned}$$

$$
\begin{aligned}
&= \star\big(\mathrm{d}f \wedge (\star X^\flat) + f\,\mathrm{d}(\star X^\flat)\big) \\
&= \star\big(X^\flat \wedge (\star \mathrm{d}f)\big) + f \star \mathrm{d} \star X^\flat \\
&= \star\big((\star \mathrm{d}f) \wedge X^\flat\big) + f \operatorname{div} X.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\star((\star \mathrm{d}f) \wedge X^\flat) &= \star\left(\left(\star \frac{\partial f}{\partial x^i}\mathrm{d}x^i\right) \wedge \big(X_j \mathrm{d}x^j\big)\right) \\
&= \star\left(\left(\frac{\partial f}{\partial x^1}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \frac{\partial f}{\partial x^2}\mathrm{d}x^1 \wedge \mathrm{d}x^3\right.\right. \\
&\qquad \left.\left. + \frac{\partial f}{\partial x^3}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\right) \wedge \big(X_j \mathrm{d}x^j\big)\right) \\
&= \star\left(\left(\frac{\partial f}{\partial x^i} X_i\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) = \operatorname{grad} f \cdot X.
\end{aligned}
$$

(6)

$$
\begin{aligned}
(\star(\operatorname{curl} X)^\flat)(Y,Z) &= (\star\star \mathrm{d}X^\flat)(Y,Z) \\
&= (\mathrm{d}X^\flat)(Y,Z) \\
&= Y X^\flat(Z) - Z X^\flat(Y) - X^\flat([Y,Z]) \\
&= Y g(X,Z) - Z g(X,Y) - g(X, \nabla_Y Z) + g(X, \nabla_Z Y) \\
&= g(\nabla_Y X, Z) - g(\nabla_Z X, Y).
\end{aligned}
$$

(7)

$$
\begin{aligned}
\operatorname{curl} X &= \operatorname{curl}\left(X_i \frac{\partial}{\partial x^i}\right) \\
&= (\star \mathrm{d}(X_i\, \mathrm{d}x^i))^\sharp \\
&= \left(\star\left\{\left(\frac{\partial X_1}{\partial x^2}\mathrm{d}x^2 + \frac{\partial X_1}{\partial x^3}\mathrm{d}x^3\right) \wedge \mathrm{d}x^1 + \cdots\right\}\right)^\sharp \\
&= \left(\star\left\{\left(-\frac{\partial X_1}{\partial x^2} + \frac{\partial X_2}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots\right\}\right)^\sharp \\
&= \left(\left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right)\mathrm{d}x^3 + \cdots\right)^\sharp \\
&= \left(\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3}\right)\frac{\partial}{\partial x^1} - \left(\frac{\partial X_3}{\partial x^1} - \frac{\partial X_1}{\partial x^3}\right)\frac{\partial}{\partial x^2} \\
&\quad + \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.
\end{aligned}
$$

From this, the formula

$$\operatorname{div}(X \times Y) = X \cdot \operatorname{curl} Y + (\operatorname{curl} X) \cdot Y$$

follows.

Problem 6.11.8. *Let g and $\widetilde{g}$ be conformally equivalent metrics on the C^∞ n-manifold M; that is, such that $\widetilde{g} = \mathrm{e}^{2f} g$, $f \in C^\infty M$. Find the relation between:*

(1) $\widetilde{\nabla}_X Y$ *and* $\nabla_X Y$*, where* $\widetilde{\nabla}$ *and* ∇ *denote, respectively, the Levi-Civita connections of* $\widetilde{g}$ *and* g*, and* $X, Y \in \mathfrak{X}(M)$.

(2) $\operatorname{div}_{\widetilde{g}} X$ *and* $\operatorname{div}_g X$, $X \in \mathfrak{X}(M)$.

Solution. (1) The Levi-Civita connection of $\widetilde{g}$ is given by the Koszul formula 7.6.3. Thus,

$$\begin{aligned} 2\mathrm{e}^{2f} g(\widetilde{\nabla}_X Y, Z) &= X\mathrm{e}^{2f} g(Y,Z) + Y\mathrm{e}^{2f} g(Z,X) - Z\mathrm{e}^{2f} g(X,Y) \\ &\quad + \mathrm{e}^{2f} g([X,Y],Z) - \mathrm{e}^{2f} g([Y,Z],X) + \mathrm{e}^{2f} g([Z,X],Y) \\ &= 2\mathrm{e}^{2f} \{ g(\nabla_X Y, Z) + (Xf) g(Y,Z) + (Yf) g(Z,X) - (Zf) g(X,Y) \}. \end{aligned}$$

Hence

$$\widetilde{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - g(X,Y) \operatorname{grad} f.$$

(2) Let (E_i) be a local g-orthonormal frame. Then the frame $(\mathrm{e}^{-f} E_i)$ is a $\widetilde{g}$-orthonormal local frame and we have locally, by definition of divergence and by (1):

$$\begin{aligned} \operatorname{div}_{\widetilde{g}} X &= \sum_i \widetilde{g}\left(\widetilde{\nabla}_{\mathrm{e}^{-f} E_i} X, \mathrm{e}^{-f} E_i \right) \\ &= \sum_i \mathrm{e}^{-2f} \widetilde{g}\left(\widetilde{\nabla}_{E_i} X, E_i \right) \\ &= \sum_i g(\widetilde{\nabla}_{E_i} X, E_i) \\ &= \sum_i \left(g(\nabla_{E_i} X, E_i) + (E_i f) g(X, E_i) + (Xf) g(E_i, E_i) - (E_i f) g(E_i, X) \right) \\ &= \operatorname{div}_g X + nXf. \end{aligned}$$

Problem 6.11.9. *Prove that the Laplacian $\Delta = \delta d + \mathrm{d}\delta$ and the Hodge star operator $\star$ on an oriented Riemannian manifold commute:*

$$\Delta \star = \star \Delta.$$

REMARK. *We recall that the codifferential δ, defined as the opposite of the divergence, satisfies $\delta\omega = (-1)^{n(k+1)+1} \star \mathrm{d} \star \omega$ for $\omega \in \Lambda^k M^n$.*

Solution. Suppose $\dim M = n$ and $\omega \in \Lambda^k M$, then

$$\begin{aligned}
\Delta\star\omega &= (\delta\mathrm{d}+\mathrm{d}\delta)\star\omega \\
&= (-1)^{n(n-k+2)+1}\star\mathrm{d}\star\mathrm{d}\star\omega + (-1)^{n(n-k+1)+1}\,\mathrm{d}\star\mathrm{d}\star\star\omega \\
&= (-1)^{n(n-k)+1}\star\mathrm{d}\star\mathrm{d}\star\omega + (-1)^{n(n-k+1)+1+k(n-k)}\mathrm{d}\star\mathrm{d}\omega, \\
\star\Delta\omega &= \star(\delta\mathrm{d}+\mathrm{d}\delta)\omega \\
&= \star(-1)^{n(k+2)+1}\star\mathrm{d}\star\mathrm{d}\omega + (-1)^{n(k+1)+1}\star\mathrm{d}\star\mathrm{d}\star\omega \\
&= (-1)^{nk+1+(n-k)k}\,\mathrm{d}\star\mathrm{d}\omega + (-1)^{n(k+1)+1}\star\mathrm{d}\star\mathrm{d}\star\omega.
\end{aligned}$$

Now, $(-1)^{n(n-k)+1} = (-1)^{n^2-nk+1} = (-1)^{n+nk+1} = (-1)^{n(k+1)+1}$, and on the other hand

$$(-1)^{n(n-k+1)+1+k(n-k)} = (-1)^{1-k} = (-1)^{nk+1+nk-k}.$$

Hence $\Delta\star = \star\Delta$.

Problem 6.11.10. *Prove that a parallel differential form on a Riemannian manifold (M,g) is harmonic.*

Solution. Let $\alpha \in \Lambda^* M$ be parallel; that is, if ∇ stands for the Levi-Civita connection of g, we have $\nabla\alpha = 0$.

Therefore α is closed. In fact, if $\alpha \in \Lambda^r M$, one has in general

$$\mathrm{d}\alpha(X_0,\dots,X_r) = \sum_{j=0}^{r} (-1)^j (\nabla_{X_j}\alpha)(X_0,\dots,\widehat{X}_j,\dots,X_r), \quad X_j \in \mathfrak{X}(M).$$

Moreover, α is coclosed ($\delta\alpha = 0$). In fact, we have in general

$$\begin{aligned}
(\delta\alpha)_p(v_1,\dots,v_{r-1}) &= (-\operatorname{div}\alpha)_p(v_1,\dots,v_{r-1}) \\
&= -\sum(\nabla_{e_i}\alpha)(e_i,v_1,\dots,v_{r-1}),
\end{aligned}$$

where $\{e_i\}$ is an orthonormal basis for T_pM, and $v_1,\dots,v_{r-1} \in T_pM$. Since $\Delta\alpha = (\mathrm{d}\delta+\delta\mathrm{d})\alpha$, we conclude.

Problem 6.11.11. *If the Riemannian n-manifold M is compact, prove*:

(1) *The codifferential δ is adjoint of the differential* d *with respect to the inner product of integration, that is*:

$$\int_M \langle\delta\alpha,\beta\rangle\omega = \int_M \langle\alpha,\mathrm{d}\beta\rangle\omega, \qquad \alpha,\beta \in \Lambda^r M, \quad r \in \{0,\dots,n\},$$

where ω denotes the volume form on the Riemannian manifold.

(2) *The Laplacian $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ on M is self-adjoint with respect to the inner product of integration, that is*:

$$\int_M \langle\Delta\alpha,\beta\rangle\omega = \int_M \langle\alpha,\Delta\beta\rangle\omega, \qquad \alpha,\beta \in \Lambda^r M, \quad r \in \{0,\dots,n\}.$$

Solution. (1) We have

$$\begin{aligned}
0 &= \int_M \mathrm{d}(\alpha \wedge \star\beta) && \text{(by Stokes' Theorem)}\\
&= \int_M (\mathrm{d}\alpha \wedge \star\beta + (-1)^r \alpha \wedge \mathrm{d}\star\beta)\\
&= \int_M \mathrm{d}\alpha \wedge \star\beta - \int_M \alpha \wedge \star\delta\beta && \text{(by definition of } \delta).
\end{aligned}$$

By the definition of inner product of integration, we conclude.

(2) By (1) above,

$$\begin{aligned}
\int_M \langle \Delta\alpha, \beta\rangle \omega &= \int_M \langle (\mathrm{d}\delta + \delta\mathrm{d})\alpha, \beta \rangle \omega\\
&= \int_M (\langle \delta\alpha, \delta\beta\rangle + \langle d\alpha, \mathrm{d}\beta\rangle)\omega\\
&= \int_M (\langle \alpha, \mathrm{d}\delta\beta\rangle + \langle \alpha, \delta\mathrm{d}\beta \rangle)\omega = \int_M \langle \alpha, \Delta\beta\rangle \omega.
\end{aligned}$$

Problem 6.11.12. *Prove that if a compact Riemannian n-manifold M admits a metric of constant positive curvature, then*

$$H^r_{dR}(M,\mathbb{R}) = 0, \qquad r = 1,\dots,n-1.$$

HINT: *Use*: (1) *Weitzenböck's formula for the Laplacian on a Riemannian manifold* $(M,\langle\ ,\ \rangle)$ *of constant sectional curvature c.*

(2) *The formula*

$$\int_M \Delta f\ \omega = 0, \qquad f \in C^\infty M,$$

which follows from (2) *in* Problem 6.11.11*, taking* $\alpha = f$ *and* β *to be a constant function.*

Solution. Integrating the two members of Weitzenböck's formula, we have in general:

$$\int_M \langle \Delta\alpha, \alpha\rangle \omega = \int_M \left(\frac{1}{2}\Delta|\alpha|^2 + |\nabla\alpha|^2 + r(n-r)c\,|\alpha|^2\right)\omega, \qquad \alpha \in \Lambda^r M,$$

where ω stands for the volume form on (M,g).

Let α be the harmonic representative of a class in $H^r_{dR}(M,\mathbb{R})$. Then $\Delta\alpha = 0$. Moreover, by (2) in the hint, $\int_M \Delta|\alpha|^2 \ = 0$. Hence

$$0 = \int_M \left(|\nabla\alpha|^2 + r(n-r)c\,|\alpha|^2\right)\omega.$$

If $r \neq 0, n$, from $c > 0$, it follows that $\alpha = 0$. Thus $H^r_{dR}(M,\mathbb{R}) = 0$, $r = 1,\dots,n-1$.

Problem 6.11.13. *Let α and β be n-forms on a compact oriented Riemannian n-manifold M such that*

$$\int_M \alpha = \int_M \beta.$$

Prove that α and β differ by an exact form.

HINT: *Use*:

(1) *Hodge's decomposition Theorem* 7.6.21.

(2) *Stokes' Theorem* 7.3.6.

Solution. Denote here the degree r of a differential form by the subindex r. By Hodge's decomposition Theorem, each r-form ω_r over such a manifold is decomposed in a unique way as

$$\omega_r = \mathrm{d}\omega_{r-1} + \delta\omega_{r+1} + \theta_r,$$

where θ_r is harmonic. In our case, the decomposition is reduced to

$$\alpha - \beta = \mathrm{d}\omega_{n-1} + \theta_n.$$

Applying Stokes' Theorem we have

$$\begin{aligned} 0 &= \int_M \alpha - \beta \\ &= \int_M \mathrm{d}\omega_{n-1} + \int_M \theta_n = \int_M \theta_n. \end{aligned}$$

As the n-form θ_n is harmonic and each cohomology class has a unique harmonic representative, from $\int_M \theta_n = 0$ it follows that $\theta_n = 0$. Thus $\alpha - \beta = \mathrm{d}\omega_{n-1}$.

6.12 Affine, Killing, Conformal, Projective, Jacobi, and Harmonic Vector Fields

Problem 6.12.1. *Find a non-affine projective vector field X on* $\mathbb{R}^3$.

HINT: *Let ∇ be the Levi-Civita connection of the Euclidean metric of $\mathbb{R}^3$. The vector field X is projective if*

$$(L_X\nabla)(Y,Z) = \theta(Y)Z + \theta(Z)Y, \qquad Y,Z \in \mathfrak{X}(\mathbb{R}^3), \tag{$\star$}$$

for some differential 1*-form $\theta \in \Lambda^1\mathbb{R}^3$, where*

$$(L_X\nabla)(Y,Z) = [X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z].$$

Moreover one has $\mathrm{d}(\operatorname{div} X) = (\dim\mathbb{R}^3 + 1)\theta = 4\theta$.

Solution. Let

$$X = 2x(x+y+z)\frac{\partial}{\partial x} + 2y(x+y+z)\frac{\partial}{\partial y} + 2z(x+y+z)\frac{\partial}{\partial z}.$$

Then

$$\begin{aligned}\theta &= \frac{1}{4}\mathrm{d}(\operatorname{div} X)\\ &= 2(\,\mathrm{d}x + \mathrm{d}y + \mathrm{d}z).\end{aligned}$$

Because of the symmetry of the vector field X and the differential form θ, it suffices to prove the formula ($\star$) for a couple of coordinate vector fields, for example $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$. We have

$$\theta\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} + \theta\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = 2\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x},$$

and

$$\begin{aligned}(L_X\nabla)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= -\nabla_{\frac{\partial}{\partial x}}\left[X, \frac{\partial}{\partial y}\right]\\ &= -\nabla_{\frac{\partial}{\partial x}}\left(-2x\frac{\partial}{\partial x} + (-2x-4y-2z)\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}\right)\\ &= 2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}.\end{aligned}$$

Problem 6.12.2. *Prove that the vector field $X = x^i\partial/\partial x^i$ on $\mathbb{R}^3$ with the Euclidean metric is affine but not Killing. Is X a vector field of homotheties?*

Solution.

$$\begin{aligned}L_X g &= \sum_{i,j} L_{x^i\frac{\partial}{\partial x^i}}(\mathrm{d}x^j \otimes \mathrm{d}x^j)\\ &= \sum_{i,j}\left(\mathrm{d}\left(x^i\frac{\partial x^j}{\partial x^i}\right)\otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}\left(x^i\frac{\partial x^j}{\partial x^i}\right)\right)\\ &= 2g.\end{aligned}$$

Hence $X = x^i\partial/\partial x^i$ is not Killing (see Figure 6.8).

Note that X is a conformal vector field, with the function $h \in C^\infty\mathbb{R}^n$, such that $L_X g = 2hg$, equal to 1, i.e. it is a constant function. It is said that a conformal vector field with $h = \text{const}$ is a vector field of homotheties.

Let us see if X is affine. As the Levi-Civita connection is torsionless and the curvature vanishes, the condition is $\nabla_Y\nabla X = 0$, $Y \in \mathfrak{X}(\mathbb{R}^n)$. Now, as

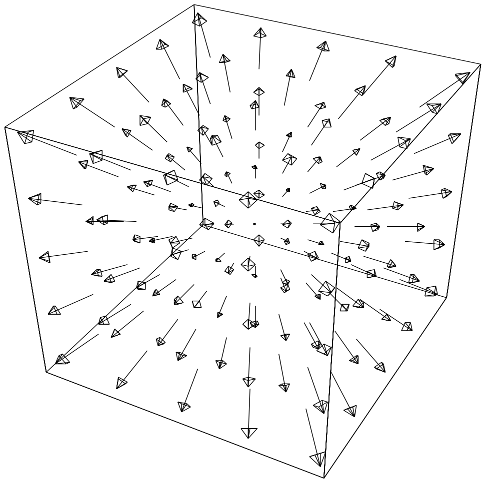

Fig. 6.8 A non-Killing affine vector field on $\mathbb{R}^3$.

$$\nabla_Y X = Y(x^i)\frac{\partial}{\partial x^i} = Y,$$

we have $\nabla X = I$, hence any covariant derivative under ∇ of ∇X vanishes. Thus X is affine.

Problem 6.12.3. *Let (M,g) be a Riemannian manifold. Prove that $X \in \mathfrak{X}(M)$ is a Killing vector field if and only if $L_X g = 0$.*

Solution. X is a Killing vector field if $\varphi_t^* g = g$ for every t, where φ_t is the local 1-parameter group generated by X. Hence

$$L_X g = \lim_{t\to 0} \frac{g - \varphi_t^* g}{t} = 0.$$

Conversely, assume $L_X g = 0$. For any tensor field K we know (see Proposition 7.2.10) that

$$\varphi_s \cdot (L_X K) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t \cdot K)\right)_{t=s}.$$

Hence, by virtue of the hypothesis, we have

$$0 = \varphi_s^*(L_X g) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^* g)\right)_{t=s},$$

and consequently $\varphi_t^* g$ does not depend on t. Therefore $\varphi_t^* g = \varphi_0^* g = g$.

Problem 6.12.4. *Show that the set of Killing vector fields of the Euclidean metric $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$ is the real Lie algebra generated by the vector fields*

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}, \qquad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \qquad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Solution. Let $X = \lambda^i \partial/\partial x^i$, where λ^i is a function of $x^1 = x$, $x^2 = y$ and $x^3 = z$. Then one has

$$L_X g = \sum_{i,j=1}^{3} \left(\frac{\partial \lambda^j}{\partial x^i} + \frac{\partial \lambda^i}{\partial x^j} \right) \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

If X is Killing, that is $L_X g = 0$, we deduce:

$$(1)\quad \frac{\partial \lambda^1}{\partial x^1} = 0, \qquad (2)\quad \frac{\partial \lambda^2}{\partial x^2} = 0, \qquad (3)\quad \frac{\partial \lambda^3}{\partial x^3} = 0,$$

$$(4)\quad \frac{\partial \lambda^1}{\partial x^2} + \frac{\partial \lambda^2}{\partial x^1} = 0, \qquad (5)\quad \frac{\partial \lambda^1}{\partial x^3} + \frac{\partial \lambda^3}{\partial x^1} = 0, \qquad (6)\quad \frac{\partial \lambda^2}{\partial x^3} + \frac{\partial \lambda^3}{\partial x^2} = 0.$$

From (1), (2) and (3) it follows that $\lambda^1 = \lambda^1(x^2, x^3)$, $\lambda^2 = \lambda^2(x^1, x^3)$, and $\lambda^3 = \lambda^3(x^1, x^2)$. Thus, from (4) and (5) one has

$$\frac{\partial^2 \lambda^1}{\partial x^2 \partial x^2} = 0, \qquad \frac{\partial^2 \lambda^1}{\partial x^3 \partial x^3} = 0,$$

from which

$$\lambda^1 = a_1 x^2 x^3 + b_1 x^2 + c_1 x^3 + d_1.$$

Similarly,

$$\lambda^2 = a_2 x^1 x^3 + b_2 x^3 + c_2 x^1 + d_2, \qquad \lambda^3 = a_3 x^1 x^2 + b_3 x^1 + c_3 x^2 + d_3.$$

On account of (4), (5) and (6) above these formulae reduce to

$$\lambda^1 = -c_2 x^2 + c_1 x^3 + d_1, \qquad \lambda^2 = -c_3 x^3 + c_2 x^1 + d_2, \qquad \lambda^3 = -c_1 x^1 + c_3 x^2 + d_3.$$

Hence the generators are indeed the ones in the statement. By using the property $L_{[X,Y]} = [L_X, L_Y]$ (see Problem 2.4.2), it is easily checked that $\{X : L_X g = 0\}$ is a Lie algebra.

Problem 6.12.5. *Calculate the divergence of a Killing vector field on a Riemannian manifold.*

Solution. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let $X \in \mathfrak{X}(M)$ be a Killing vector field. Since $L_X g = 0$, $\nabla g = 0$, and ∇ is torsionless, we have for any $Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} 0 &= (L_X g)(Y, Z) \\ &= Xg(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z) \\ &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \end{aligned}$$

Hence, for any $p \in M$, and any orthonormal basis $\{e_i\}$ of T_pM, one has

$$(\operatorname{div} X)(p) = \sum_i g(\nabla_{e_i} X, e_i) = 0,$$

that is, $\operatorname{div} X = 0$.

Problem 6.12.6. *Prove that a vector field X on a Riemannian manifold (M,g) is Killing if and only if the* Kostant operator *A defined by*

$$A_X = L_X - \nabla_X,$$

where ∇ stands for the Levi-Civita connection of g, satisfies

$$g(A_X Y, Z) + g(Y, A_X Z) = 0, \qquad Y, Z \in \mathfrak{X}(M).$$

REMARK. *Notice that as ∇ is torsionless, $A_X Y = -\nabla_Y X$.*

Solution. As $\nabla_X g = 0$ for all $X \in \mathfrak{X}(M)$, the condition $L_X g = 0$ is equivalent to $A_X g = 0$. Since A_X is the difference of two derivations of the algebra of tensor fields that commute with contractions, one has

$$A_X(g(Y,Z)) = (A_X g)(Y,Z) + g(A_X Y, Z) + g(Y, A_X Z), \qquad Y, Z \in \mathfrak{X}(M).$$

On the other hand,

$$A_X f = L_X f - \nabla_X f = Xf - Xf = 0, \qquad f \in C^\infty M,$$

thus $A_X(g(Y,Z)) = 0$. Hence $(A_X g)(Y,Z) = 0$ if and only if

$$g(A_X Y, Z) + g(Y, A_X Z) = 0,$$

as wanted.

Problem 6.12.7. *Consider $\mathbb{R}^2$ equipped with the metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$.*

(1) *Show that the vector field*

$$X = (ax - by)\partial/\partial x + (bx + ay)\partial/\partial y, \qquad a, b \in \mathbb{R},$$

is a conformal vector field.

(2) *Let $\mathbb{R}^3 - \{0\}$ with the usual metric and let ω denote the volume form. Write $L_Y \omega$, $Y \in \mathfrak{X}(\mathbb{R}^3 - \{0\})$, in cylindrical coordinates.*

Solution. (1)

$$\begin{aligned} & L_{(ax-by)\frac{\partial}{\partial x} + (bx+ay)\frac{\partial}{\partial y}}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) \\ &\quad = \mathrm{d}(ax - by) \otimes \mathrm{d}x + \mathrm{d}x \otimes \mathrm{d}(ax - by) + \mathrm{d}(bx + ay) \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}(bx + ay) \\ &\quad = 2a\, g. \end{aligned}$$

(2) One has

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \qquad z = z.$$

Hence, the volume form is $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}z$. Let

$$Y = F\frac{\partial}{\partial\rho} + G\frac{\partial}{\partial\theta} + H\frac{\partial}{\partial z},$$

where F, G, H are functions of ρ, θ, and z. Therefore,

$$\begin{aligned}
L_Y\omega &= \frac{1}{2}L_Y\left(\mathrm{d}(\rho^2) \wedge \mathrm{d}\theta \wedge \mathrm{d}z\right) \\
&= \frac{1}{2}\left\{\mathrm{d}\left(F\frac{\partial}{\partial\rho}\rho^2\right) \wedge \mathrm{d}\theta \wedge \mathrm{d}z + \mathrm{d}(\rho^2) \wedge \mathrm{d}\left(G\frac{\partial}{\partial\theta}\theta\right) \wedge \mathrm{d}z\right. \\
&\qquad \left. + \mathrm{d}(\rho^2) \wedge \mathrm{d}\theta \wedge \mathrm{d}\left(H\frac{\partial}{\partial z}z\right)\right\} \\
&= \left(\frac{F}{\rho} + \frac{\partial F}{\partial\rho} + \frac{\partial G}{\partial\theta} + \frac{\partial H}{\partial z}\right)\omega.
\end{aligned}$$

Problem 6.12.8. *Consider the* 1*-parameter group* φ_t, $t \in \mathbb{R}$, *of automorphisms of* $\mathbb{R}^2$ *defined by the equations*

$$x(t) = x\cos t + y\sin t, \qquad y(t) = -x\sin t + y\cos t.$$

(1) *Compute the infinitesimal generator* X *of* φ_t.

(2) *If* $g = \mathrm{d}x^2 + \mathrm{d}y^2$ *and* $\omega = \mathrm{d}x \wedge \mathrm{d}y$, *find the vector field* Y *on* $\mathbb{R}^2 - \{(0,0)\}$ *defined by*

$$g(Y,Y) = 1, \qquad g(X,Y) = 0, \qquad \omega(X,Y) > 0,$$

and prove that $[X,Y] = 0$.

(3) *Calculate* $L_X g$, $L_Y g$, $L_X\omega$, *and* $L_Y\omega$.

(4) *Compute the first integrals of* X *and* Y.

(5) *Prove that in a certain neighborhood of any point different from the origin there is a local coordinate system* (u,v) *such that* $X = \partial/\partial u$, $Y = \partial/\partial v$.

Solution. (1) Since $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = y(t)$ and $\dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = -x(t)$, we obtain

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(2) The vector field

$$Y = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

is a unit vector field with respect to g, which is normal to the circles with center the origin, so $g(X,Y)=0$. Moreover $\omega(X,Y)=\sqrt{x^2+y^2}>0$ if $(x,y)\neq(0,0)$. It is easily checked that $[X,Y]=0$.

(3) Let $\rho=\sqrt{x^2+y^2}$. Then:

$$L_X g = L_{y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y)=0.$$

$$\begin{aligned}L_Y g &= L_{\frac{x}{\rho}\frac{\partial}{\partial x}+\frac{y}{\rho}\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y)\\ &=\frac{2}{\rho^3}\big(y^2\mathrm{d}x\otimes\mathrm{d}x-xy(\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x)+x^2\mathrm{d}y\otimes\mathrm{d}y\big).\end{aligned}$$

$$L_X\omega = L_{y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}y-\mathrm{d}y\otimes\mathrm{d}x)=0.$$

$$L_Y\omega = L_{\frac{x}{\rho}\frac{\partial}{\partial x}+\frac{y}{\rho}\frac{\partial}{\partial y}}(\mathrm{d}x\otimes\mathrm{d}y-\mathrm{d}y\otimes\mathrm{d}x)=\frac{1}{\rho}\omega.$$

(4) The first integrals of X and Y are, respectively, $f(u^1)$, where $u^1=x^2+y^2$ and $f(v^1)$, where $v^1=y/x$.

(5) By (4), we have $X=\lambda\dfrac{\partial}{\partial v^1}$, $Y=\mu\dfrac{\partial}{\partial u^1}$. If moreover $X=\dfrac{\partial}{\partial u}$, we would have $Xu=1=\lambda\dfrac{\partial u}{\partial v^1}$, thus $\dfrac{\partial u}{\partial v^1}=\dfrac{1}{\lambda}$. Let us compute λ. One has

$$\begin{aligned}\lambda &= Xv^1\\ &=\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\frac{y}{x}\\ &=-\frac{y^2}{x^2}-1\\ &=-(1+(v^1)^2),\end{aligned}$$

that is $\dfrac{\partial u}{\partial v^1}=-\dfrac{1}{1+(v^1)^2}$, so $u=-\arctan v^1=-\theta$ (in polar coordinates). Hence $u=-\arctan(y/x)$. Similarly, if $Y=\dfrac{\partial}{\partial v}$ we have $Yv=1=\mu\dfrac{\partial v}{\partial u^1}$. Let us calculate μ. We have $\mu=Yu^1=2\rho=2\sqrt{u^1}$. Thus $\dfrac{\partial v}{\partial u^1}=\dfrac{1}{\mu}=\dfrac{1}{2\sqrt{u^1}}$, and $v=\sqrt{u^1}$. That is, $v=\sqrt{x^2+y^2}$.

Problem 6.12.9. *Find two linearly independent harmonic vector fields on the 2-torus T^2 with its usual embedding in $\mathbb{R}^3$ as a surface of revolution.*

Solution. Let us see if there exist $f(\varphi,\theta)\partial/\partial\varphi$ and $h(\varphi,\theta)\partial/\partial\theta$ harmonic, φ and θ being the parameters of the usual parametrization

$$x = (a + b\cos\varphi)\cos\theta, \qquad y = (a + b\cos\varphi)\sin\theta, \qquad z = b\sin\varphi,$$

$\varphi, \theta \in [0, 2\pi]$, and $f(\varphi, \theta)$, $h(\varphi, \theta)$ functions of these parameters. Such vector fields would obviously be linearly independent. If $j\colon T^2 \to \mathbb{R}^3$ denotes the usual embedding, the metric is

$$j^*(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) = b^2\mathrm{d}\varphi^2 + (a + b\cos\varphi)^2\mathrm{d}\theta^2.$$

If M is compact, as in our case, in order for a vector field Z to be harmonic (see Definition 7.6.20) it suffices to have $\mathrm{d}Z^\flat = 0$, $\delta Z^\flat = 0$.

Putting $X = f(\varphi, \theta)\dfrac{\partial}{\partial\varphi}$, $Y = h(\varphi, \theta)\dfrac{\partial}{\partial\theta}$, one has

$$X^\flat = b^2 f(\varphi, \theta)\mathrm{d}\varphi, \qquad Y^\flat = (a + b\cos\varphi)^2 h(\varphi, \theta)\mathrm{d}\theta.$$

Thus, from

$$\mathrm{d}X^\flat = b^2\frac{\partial f}{\partial\theta}\mathrm{d}\theta \wedge \mathrm{d}\varphi = 0,$$

we have $f = f(\varphi)$. Suppose similarly $h = h(\varphi)$. Then

$$\mathrm{d}Y^\flat = \frac{\partial\big((a + b\cos\varphi)^2 h(\varphi)\big)}{\partial\varphi}\mathrm{d}\varphi \wedge \mathrm{d}\theta = 0$$

implies

$$h(\varphi) = \frac{A}{(a + b\cos\varphi)^2}.$$

Hence, for

$$X = f(\varphi)\frac{\partial}{\partial\varphi}, \qquad Y = \frac{A}{(a + b\cos\varphi)^2}\frac{\partial}{\partial\theta}, \tag{$\star$}$$

we have $\mathrm{d}X^\flat = \mathrm{d}Y^\flat = 0$.

To compute $\delta X^\flat = -\operatorname{div} X^\flat$ and $\delta Y^\flat = -\operatorname{div} Y^\flat$ we use the formula, valid for any oriented manifold M,

$$L_Z\omega = (\operatorname{div} Z)\,\omega, \qquad Z \in \mathfrak{X}(M),$$

where ω denotes the volume element on M, which in our case is

$$\omega = \sqrt{g_{11}g_{22} - g_{12}^2}\,\mathrm{d}\varphi \wedge \mathrm{d}\theta = b(a + b\cos\varphi)\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

Applying moreover the general formula

$$L_Z\,(f\mathrm{d}\varphi \wedge \mathrm{d}\theta) = (Zf)\mathrm{d}\varphi \wedge \mathrm{d}\theta + f\,\mathrm{d}(Z\varphi) \wedge \mathrm{d}\theta + f\,\mathrm{d}\varphi \wedge \mathrm{d}(Z\theta),$$

$f \in C^\infty M$, to X and Y in $(\star)$, we obtain

$$(\operatorname{div} X)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta = L_{f(\varphi)\frac{\partial}{\partial\varphi}} b(a + b\cos\varphi)\mathrm{d}\varphi \wedge \mathrm{d}\theta$$

$$= \left(-f(\varphi)b^2 \sin\varphi + b(a+b\cos\varphi)\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi}\right)\mathrm{d}\varphi\wedge\mathrm{d}\theta,$$

$$(\operatorname{div} Y)\,\mathrm{d}\varphi\wedge\mathrm{d}\theta = L_{\frac{A}{(a+b\cos\varphi)^2}\frac{\partial}{\partial\theta}} b(a+b\cos\varphi)\mathrm{d}\varphi\wedge\mathrm{d}\theta = 0.$$

Hence, $\delta Y^\flat = 0$. And $\delta X^\flat = 0$ if

$$-f(\varphi)b\sin\varphi + (a+b\cos\varphi)\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\big(f(\varphi)(a+b\cos\varphi)\big) = 0,$$

that is, if

$$f(\varphi) = \frac{B}{a+b\cos\varphi}.$$

In this case, $\Delta X^\flat = (\mathrm{d}\delta+\delta\mathrm{d})X^\flat = 0$, $\Delta Y^\flat = (\mathrm{d}\delta+\delta\mathrm{d})Y^\flat = 0$, that is $X^\flat$ and $Y^\flat$ are harmonic forms, and

$$X = \frac{B}{a+b\cos\varphi}\frac{\partial}{\partial\varphi}, \qquad Y = \frac{A}{(a+b\cos\varphi)^2}\frac{\partial}{\partial\theta}$$

satisfy the conditions in the statement.

Notice that in order to compute $\delta X^\flat$ and $\delta Y^\flat$ we can instead use the definition $\operatorname{div} Z = \operatorname{tr}\nabla Z$ and thus the Christoffel symbols of g, as follows. Taking $x^1 = \varphi$, $x^2 = \theta$, since

$$g = \begin{pmatrix} b^2 & 0 \\ 0 & (a+b\cos\varphi)^2 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 1/b^2 & 0 \\ 0 & 1/(a+b\cos\varphi)^2 \end{pmatrix},$$

we deduce that the non-vanishing Christoffel symbols are

$$\Gamma^1_{22} = \frac{1}{b}(a+b\cos\varphi)\sin\varphi, \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{b\sin\varphi}{a+b\cos\varphi}.$$

Let us calculate $\delta X^\flat$ and $\delta Y^\flat$:

$$\begin{aligned}
\delta X^\flat &= -\operatorname{div} X^\flat \\
&= -\operatorname{div} X \\
&= -g\left(\nabla_{\frac{1}{b}\frac{\partial}{\partial\varphi}} f(\varphi)\frac{\partial}{\partial\varphi}, \frac{1}{b}\frac{\partial}{\partial\varphi}\right) \\
&\quad - g\left(\nabla_{\frac{1}{a+b\cos\varphi}\frac{\partial}{\partial\theta}} f(\varphi)\frac{\partial}{\partial\varphi}, \frac{1}{a+b\cos\varphi}\frac{\partial}{\partial\theta}\right) \\
&= -\frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} - f(\varphi)\left(-\frac{b\sin\varphi}{a+b\cos\varphi}\right),
\end{aligned}$$

since $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{21} = 0$.

$$\begin{aligned}
\delta Y^\flat &= -\operatorname{div} Y \\
&= -g\left(\nabla_{\frac{1}{b}\frac{\partial}{\partial\varphi}} \frac{A}{(a+b\cos\varphi)^2}\frac{\partial}{\partial\theta},\, \frac{1}{b}\frac{\partial}{\partial\varphi}\right) \\
&\quad -g\left(\nabla_{\frac{1}{a+b\cos\varphi}\frac{\partial}{\partial\theta}} \frac{A}{(a+b\cos\varphi)^2}\frac{\partial}{\partial\theta},\, \frac{1}{a+b\cos\varphi}\frac{\partial}{\partial\theta}\right) \\
&= 0,
\end{aligned}$$

since $\Gamma_{12}^1 = \Gamma_{22}^2 = 0$. That is, we obtain the same expressions as above.

Problem 6.12.10. *Let (M,g) be a Riemannian manifold. Prove that if $X \in \mathfrak{X}(M)$ is Killing and $Y \in \mathfrak{X}(M)$ is harmonic, then $g(X,Y)$ is a harmonic function.*

HINT: *Apply the following results*:

(1) *If $Z \in \mathfrak{X}(M)$ is Killing, then*:

$$g(\operatorname{tr}\nabla^2 Z, W) = -r(Z,W), \qquad W \in \mathfrak{X}(M),$$

where r denotes the Ricci tensor.

(2) *$Z \in \mathfrak{X}(M)$ is harmonic if and only if $g(\operatorname{tr}\nabla^2 Z, W) = r(Z,W)$.*

(3) *Let K be a symmetric* (i.e. *self-adjoint*) *transformation of an inner product space $(E,\langle\,,\,\rangle)$, and let L be skew-symmetric. Then we have $\langle K,L\rangle = 0$.*

Solution. Let (e_i) be an orthonormal frame on a neighborhood of the point $p \in M$. Then if ∇ denotes the Levi-Civita connection of g, we have

$$\begin{aligned}
(\Delta g(X,Y))(p) &= (\delta \mathrm{d} g(X,Y))(p) \\
&= -(\operatorname{div} \mathrm{d}(X,Y))(p) \\
&= -\sum_i \big((\nabla_{e_i} \mathrm{d} g(X,Y))(e_i)\big)(p) \\
&= -\sum_i \big\{\nabla_{e_i}(\mathrm{d} g(X,Y)(e_i)) - (\mathrm{d} g(X,Y))(\nabla_{e_i} e_i)\big\}(p) \\
&= -\sum_i \big\{\nabla_{e_i} e_i g(X,Y) - (\nabla_{e_i} e_i) g(X,Y)\big\}(p) \\
&= -\sum_i \big\{e_i g(\nabla_{e_i} X, Y) + e_i g(X, \nabla_{e_i} Y) - g(\nabla_{\nabla_{e_i} e_i} X, Y) \\
&\quad - g(X, \nabla_{\nabla_{e_i} e_i} Y)\big\}(p) \\
&= -\sum_i \big\{g(\nabla_{e_i}\nabla_{e_i} X, Y) + g(\nabla_{e_i} X, \nabla_{e_i} Y) + g(\nabla_{e_i} X, \nabla_{e_i} Y) \\
&\quad + g(X, \nabla_{e_i}\nabla_{e_i} Y) - g(\nabla_{\nabla_{e_i} e_i} X, Y) - g(X, \nabla_{\nabla_{e_i} e_i} Y)\big\}(p) \\
&= -\sum_i \big\{2g(\nabla_{e_i} X, \nabla_{e_i} Y) + g((\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})X, Y) \\
&\quad + g(X, (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}} e_i)Y)\big\}(p) \\
&= \{-2g(\nabla X, \nabla Y) - g(\operatorname{tr}\nabla^2 X, Y) - g(X, \operatorname{tr}\nabla^2 Y)\}(p)
\end{aligned}$$

$$= \{-2g(\nabla X, \nabla Y) - r(X,Y) + r(X,Y)\}(p)$$
$$= (-2g(\nabla X, \nabla Y))(p).$$

Now, since X is Killing, ∇X is skew-symmetric, i.e.,

$$g(\nabla_Z X, W) + g(Z, \nabla_W X) = 0$$

(see Problem 6.12.5) and as Y is harmonic, ∇Y is symmetric, i.e.,

$$g(\nabla_Z Y, W) = g(Z, \nabla_W Y),$$

each one with respect to g. Hence

$$g(\nabla_Z X, \nabla_W Y) = -g(Z, \nabla_{\nabla_W Y} X) = -g(\nabla_W Y, \nabla_Z X),$$

for $Z, W \in \mathfrak{X}(M)$, and we conclude that

$$\Delta g(X,Y) = 0.$$

Problem 6.12.11. *Determine the Jacobi fields on $\mathbb{R}^n$ with the Euclidean metric g.*

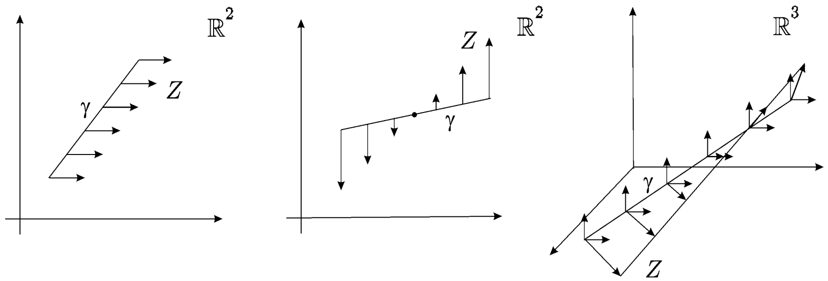

Fig. 6.9 Some simple Jacobi fields.

Solution. The geodesics of $(\mathbb{R}^n, g)$ are the straight lines parametrized as in Problem 6.3.1. Since the curvature vanishes, the Jacobi equation is reduced to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = 0.$$

The Jacobi fields along a straight line γ are the fields of the form $X = tY + Z$, where Y and Z are constant vector fields along γ (see Figure 6.9).

Problem 6.12.12. *Let*

$$\varphi(u,v) = (u\cos v,\, u\sin v,\, f(u))$$

be a parametric surface of revolution in $\mathbb{R}^3$, and let

$$Y_v|_{\varphi(u,v)} = \left.\frac{\partial}{\partial v}\right|_{\varphi(u,v)}.$$

Prove:

(1) *Y_v is a Jacobi field along meridians.*

(2) *If g denotes the metric and s the arc length, then*

$$\frac{\mathrm{d}^2|Y_v|}{\mathrm{d}s^2} = -K|Y_v|,$$

where K stands for the Gauss curvature.

HINT: *For such a surface of revolution one has* $K = \dfrac{f'f''}{u(1+(f')^2)^2}$.

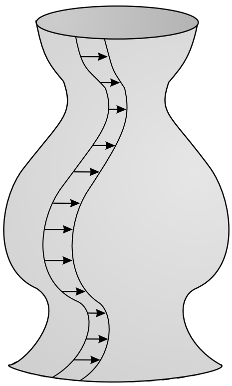

Fig. 6.10 A Jacobi field on a surface of revolution.

Solution. (1) The vector fields Y and γ' in the torsionless case of Definition 7.6.13 are here $Y = \dfrac{\partial}{\partial v}$ and

$$\gamma' = \frac{\partial}{\partial u}\Big/\left|\frac{\partial}{\partial u}\right| = \frac{1}{\sqrt{1+(f'(u))^2}}\frac{\partial}{\partial u}.$$

We must prove that

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_Y\nabla_{\gamma'}\gamma' - \nabla_{\gamma'}\nabla_Y\gamma' - \nabla_{[Y,\gamma']}\gamma' = 0.$$

Now, since ∇ is torsionless we have $\nabla_Y\gamma' - \nabla_{\gamma'}Y = [Y,\gamma']$; but it is immediate that in the present case $[Y,\gamma'] = 0$. On the other hand $\nabla_{\gamma'}\gamma' = 0$, as γ' is the tangent vector field to a geodesic curve. So we are done.

(2) We have $|Y_v| = u$. Moreover, the Gauss curvature of a surface of revolution is given by the expression in the hint, with $f' = \mathrm{d}f/\mathrm{d}u$.

The arc length $s(u)$ along the meridian is given, since v is constant, by

$$\begin{aligned} s(u) &= \int_0^u \sqrt{g\left(\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial u}\right)}\mathrm{d}u \\ &= \int_0^u \sqrt{1+(f'(u))^2}\mathrm{d}u, \end{aligned}$$

and thus

$$\begin{aligned} \frac{\mathrm{d}^2|Y_v|}{\mathrm{d}s^2} &= \frac{\mathrm{d}^2 u}{\mathrm{d}s^2} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\sqrt{1+(f')^2}} \\ &= \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{\sqrt{1+(f')^2}} \\ &= \frac{1}{\sqrt{1+(f')^2}}\frac{-f'f''}{\left(1+(f')^2\right)^{3/2}} \\ &= -Ku \\ &= -K|Y_v|. \end{aligned}$$

Notice that the lengths for the vector field Y_v are larger where the distance between the given geodesics (the meridians) grows, and are lower where that distance decreases (see Figure 6.10).

Problem 6.12.13. *Let* (M,g) *be an n-dimensional space of constant curvature c. Let* $\dot{\gamma}, X_1, \dots, X_{n-1}$ *be an orthonormal frame invariant by parallelism along a geodesic* γ *with unit tangent vector field* $\dot{\gamma}$.

Prove that the vector fields

$$\begin{aligned} &(1) \quad \dot{\gamma}, \qquad s\dot{\gamma}, \qquad Y_i = \sin(\sqrt{c}\,s)X_i, \qquad Z_i = \cos(\sqrt{c}\,s)X_i, \\ &(2) \quad \dot{\gamma}, \qquad s\dot{\gamma}, \qquad Y_i = \sinh(\sqrt{-c}\,s)X_i, \qquad Z_i = \cosh(\sqrt{-c}\,s)X_i, \\ &(3) \quad \dot{\gamma}, \qquad s\dot{\gamma}, \qquad X_i, \quad sX_i, \end{aligned}$$

$i = 1, \dots, n-1$, *where s denotes the arc length, are a basis of the space of Jacobi vector fields along the geodesic, for* $c > 0$ *in case* (1), $c < 0$ *in case* (2), *and* $c = 0$ *in case* (3).

Solution. That such $\dot{\gamma}$ and $s\dot{\gamma}$ are Jacobi fields is a general fact for Riemannian manifolds, and its proof is immediate from the Jacobi equation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y + R(Y,\dot{\gamma})\dot{\gamma} = 0.$$

In cases (1) and (2), since (M,g) is a space of constant curvature c, we have

$$\begin{aligned} R(Y_i,\dot{\gamma})\dot{\gamma} &= c\big(g(\dot{\gamma},\dot{\gamma})Y_i - g(\dot{\gamma},Y_i)\dot{\gamma}\big) \\ &= cY_i. \end{aligned}$$

In case (1) we have on the other hand,

$$\begin{aligned} \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y_i &= -c\sin(\sqrt{c}\,s)X_i + \sqrt{c}\cos(\sqrt{c}\,s)\nabla_{\dot{\gamma}}X_i \\ &\quad + \sqrt{c}\cos(\sqrt{c}\,s)\nabla_{\dot{\gamma}}X_i + \sin(\sqrt{c}\,s)\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X_i \qquad (\star) \\ &= -c\,Y_i. \qquad \text{(as } X_i \text{ is parallel)} \end{aligned}$$

Hence Jacobi's equation for a torsionless connexion,

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y_i + R(Y_i,\dot{\gamma})\dot{\gamma} = 0,$$

is satisfied, as wanted. The proof for Z_i (and Y_i, Z_i in the case (2)) is similar.

The case (3) is trivially true as $R = 0$ for $c = 0$.

Problem 6.12.14. *Determine the conjugate points and their orders for a point on an n-sphere of constant curvature c.*

Solution. From Problem 6.12.13, it follows that the only point conjugate to the point corresponding to $s = 0$ along a geodesic $\gamma(s)$ is the point corresponding to $s = \pi/\sqrt{c}$, with order $n-1$, as a basis of the Jacobi fields vanishing at $s = 0$ and $s = \pi/\sqrt{c}$ is given by the vector fields $Y_i = \sin(\sqrt{c}\,s)X_i$.

Problem 6.12.15. *Show that if M has nonpositive sectional curvature, then there are no conjugate points.*

Solution. Let Y be a Jacobi vector field along a geodesic $\gamma(t)$. From Jacobi's equation

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + R(Y,\gamma')\gamma' = 0,$$

we obtain by virtue of the hypothesis that

$$\begin{aligned} g(\nabla_{\gamma'}\nabla_{\gamma'}Y,Y) &= g(R(\gamma',Y)\gamma',Y) \\ &= -R(\gamma',Y,\gamma',Y) \geqslant 0, \end{aligned}$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\nabla_{\gamma'}Y,Y) = g(\nabla_{\gamma'}\nabla_{\gamma'}Y,Y) + \left|\nabla_{\gamma'}Y\right|^2 \geqslant 0. \qquad (\star)$$

The function $g(\nabla_{\gamma'}Y,Y)$ is thus monotonically increasing (strictly if $\nabla_{\gamma'}Y \neq 0$). If $Y(0) = Y(t_0) = 0$ for certain $t_0 > 0$, then $g(\nabla_{\gamma'}Y,Y)$ also vanishes at these points, hence it must vanish along the interval $[0,t_0]$. Thus, we have $Y(0) = (\nabla_{\gamma'}Y)(0) = 0$ by $(\star)$, so that Y vanishes identically, as Y is a solution of a second-order differential equation.

Problem 6.12.16. *Prove that the multiplicity of two points p and q conjugate along a geodesic γ in a manifold M is less than the dimension of the manifold.*

Solution. Let $\dim M = n$. Then the Jacobi vector fields vanishing on a given point $p \in M$ constitute a space of dimension n, but $(t-t_1)\dot{\gamma}$, where $p = \gamma(t_1)$, is a Jacobi vector field vanishing at p but not at q.

6.13 Submanifolds. Second Fundamental Form

Problem 6.13.1. *Prove*:

(1) *Every strictly conformal map is an immersion.*

(2) *If M is connected, then a strictly conformal map*

$$f\colon (M,g) \to (\overline{M},\overline{g})$$

of ratio λ transforms the Levi-Civita connection ∇ of g into the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$, if and only if $\lambda = \mathrm{const}$ and the second fundamental form of the immersed submanifold $f(M)$ vanishes.

(3) *If $\lambda = 1$, that is, f is an isometry, and the second fundamental form of $f(M)$ vanishes, then if R and $\overline{R}$ stand for the Riemann curvature tensors of M and $\overline{M}$, respectively, one has $f_*R = \overline{R}|_{f(M)}$.*

Solution. (1) Let $X \in T_pM$ such that $f_*X = 0$. Then

$$0 = \overline{g}(f_*X, f_*X) = \lambda(p)g(X,X).$$

As $\lambda(p) > 0$ for all $p \in M$, we have $X = 0$; that is, $\operatorname{Ker} f_{*p} = 0$ for all $p \in M$.

(2) As M is connected, we only need to prove that λ is locally constant. Thus we can assume that f is a diffeomorphism from M onto a submanifold $f(M)$ of $\overline{M}$. Denoting by $\overline{X}$ the vector field image $f \cdot X$ on $f(M)$ of $X \in \mathfrak{X}(M)$, we have that $X \mapsto \overline{X}$ is an isomorphism. Hence if f transforms ∇ into $\overline{\nabla}$, it follows that

$$\begin{aligned}
\overline{X}\overline{g}(\overline{Y},\overline{Z}) &= \overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z}) + g(\overline{Y},\overline{\nabla}_{\overline{X}}\overline{Z}) \\
&= \overline{g}(\overline{\nabla_X Y},\overline{Z}) + \overline{g}(\overline{Y},\overline{\nabla_X Z}) \\
&= \lambda g(\nabla_X Y, Z) + \lambda g(Y, \nabla_X Z) \\
&= \lambda X g(Y,Z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\overline{X}\overline{g}(\overline{Y},\overline{Z}) &= X\lambda g(Y,Z) \\
&= (X\lambda)g(Y,Z) + \lambda X g(Y,Z).
\end{aligned}$$

Hence $X\lambda = 0$ for all X. As M is connected, we deduce that λ is a constant function. Furthermore, as $\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y}$, it follows that $\overline{\nabla}_{\overline{X}}\overline{Y}$ is tangent to the submanifold $f(M)$, thus the second fundamental form of $f(M)$ vanishes.

Conversely, if we define on $f(M)$ the connection $\overline{\nabla}$ by $\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y}$, and prove that $\overline{\nabla}$ parallelizes the metric of $f(M)$ and has no torsion, then it will coincide with the Levi-Civita connection of the metric on $f(M)$. Let k be the constant function λ. One has:

(i)

$$\begin{aligned}\overline{X}\overline{g}(\overline{Y},\overline{Z}) &= X(kg(Y,Z))\\ &= k\{g(\nabla_X Y,Z)+g(Y,\nabla_X Z)\}\\ &= \overline{g}(\overline{\nabla_X Y},\overline{Z})+\overline{g}(\overline{Y},\overline{\nabla_X Z})\\ &= \overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z})+\overline{g}(\overline{Y},\overline{\nabla}_{\overline{X}}\overline{Z}).\end{aligned}$$

(ii)

$$\begin{aligned}\overline{\nabla}_{\overline{X}}\overline{Y}-\overline{\nabla}_{\overline{Y}}\overline{X}-[\overline{X},\overline{Y}] &= \overline{\nabla_X Y}-\overline{\nabla_Y X}-\overline{[X,Y]}\\ &= \overline{T_\nabla(X,Y)} = 0.\end{aligned}$$

(3) Since f is an isometry, it transforms the Riemann curvature tensor of M into the one of $f(M)$ (see Problem 6.7.3); but this one coincides with the restriction in $f(M)$ of the Riemann curvature tensor of $\overline{M}$, as the second fundamental form of $f(M)$ vanishes.

Problem 6.13.2. *Let M be an n-dimensional $(n \geqslant 3)$, totally umbilical submanifold of a $2m$-dimensional complex space form $(\widetilde{M},g,J)$ of holomorphic sectional curvature $c \neq 0$. Prove that M is one of the following submanifolds:*

(1) *A complex space form holomorphically immersed in $\widetilde{M}$ as a totally geodesic submanifold.*

(2) *A real space form* (i.e. *a—not necessarily simply connected—space of constant curvature*) *immersed in $\widetilde{M}$ as a totally real and totally geodesic submanifold.*

(3) *A real space form immersed in $\widetilde{M}$ as a totally real submanifold with nonvanishing parallel mean curvature vector.*

HINT. *See* Definitions 7.6.26 *and* Theorem 7.6.27.

Solution. As M is a totally umbilical submanifold, with the usual notations we have

$$\alpha(X,Y) = g(X,Y)\xi, \qquad X,Y \in \mathfrak{X}(M).$$

Thus the covariant derivative appearing in Codazzi's equation

$$\left(\widehat{\nabla}_X\alpha\right)(Y,Z) = \nabla^\perp_X(\alpha(Y,Z)) - \alpha(\nabla_X Y,Z) - \alpha(Y,\nabla_X Z),$$

reduces to

$$\left(\widehat{\nabla}_X\alpha\right)(Y,Z) = g(Y,Z)\nabla^\perp_X\xi,$$

so Codazzi's equation is written as

$$\nu\widetilde{R}(X,Y)Z = g(Y,Z)\nabla^{\perp}_X\xi - g(X,Z)\nabla^{\perp}_Y\xi. \tag{$\star$}$$

Since $\dim M \geqslant 3$, for each $X \in \mathfrak{X}(M)$ one can choose a unit vector field $Y \in \mathfrak{X}(M)$ orthogonal to X and JX. For such a Y, from $(\star)$ one has

$$\nu\widetilde{R}(X,Y)Y = \nabla^{\perp}_X\xi.$$

On the other hand, since $\widetilde{M}$ has constant holomorphic sectional curvature $c \neq 0$, we have

$$\widetilde{R}(X,Y)Z = \frac{c}{4}\left\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\right\},$$

from which we deduce $\widetilde{R}(X,Y)Y = \frac{c}{4}X$, so $\nu\widetilde{R}(X,Y)Y = 0$, hence

$$\nabla^{\perp}_X\xi = 0, \qquad \forall X \in \mathfrak{X}(M).$$

From $(\star)$ we then obtain

$$\nu\widetilde{R}(X,Y)Z = 0, \qquad X,Y,Z \in \mathfrak{X}(M).$$

Thus, by Proposition 7.6.27, M is either a complex or a totally real submanifold of $\widetilde{M}$. If M is a complex submanifold, then M is minimal, hence totally geodesic in $\widetilde{M}$. Therefore, from Gauss's equation

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + g\big(\alpha(X,Z),\alpha(Y,W)\big) - g\big((\alpha(Y,Z),\alpha(X,W)\big),$$

we obtain

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W).$$

That is, M is a complex space form of constant holomorphic sectional curvature c.

If M is a totally real submanifold, from Gauss's equation and from $\alpha(X,Y) = g(X,Y)\xi$, it follows that

$$R(X,Y,Z,W) = \left(\frac{c}{4} + g(\xi,\xi)\right)(g(X,Z)g(Y,W) - g(X,W),g(Y,Z)),$$

that is, M is a real space form of constant (ordinary) sectional curvature $\frac{c}{4} + g(\xi,\xi)$.

Problem 6.13.3. *Consider the flat torus $T^2 = \mathbb{R}^2/\Gamma$ defined by the lattice*

$$\Gamma = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2, \qquad v_1 = (-\pi,\pi), \quad v_2 = (0,2\pi).$$

(1) *Prove that*

$$f(u,v) = (\cos u \cos v,\, \cos u \sin v,\, \sin u \cos v,\, \sin u \sin v)$$

for $u,v \in [0,2\pi]$, is an isometric embedding of T^2 in the unit sphere S^3 of $\mathbb{R}^4$.

(2) *Prove that the total embedded curvature of* $f(T^2)$ *in* S^3 *is a constant.*
HINT: *Use the generalized Gauss* Theorema Egregium 7.6.23.

Solution. (1) We have $f(u,v) \in S^3$ since $|f(u,v)| = 1$. Moreover, $f(u,v) = f(u',v')$ if and only if $(u',v') - (u,v) \in \Gamma$. In fact, the previous equality is equivalent to

$$\sin(u+v) = \sin(u'+v'), \tag{$\star$}$$

$$\sin(u-v) = \sin(u'-v'), \tag{$\dagger$}$$

$$\cos(u+v) = \cos(u'+v'), \tag{$\star\star$}$$

$$\cos(u-v) = \cos(u'-v'). \tag{$\dagger\dagger$}$$

From ($\star$), ($\star\star$), and from ($\dagger$), ($\dagger\dagger$), we obtain

$$u'+v' = u+v+2k_1\pi, \quad u'-v' = u-v+2k_2\pi, \qquad k_1,k_2 \in \mathbb{Z},$$

respectively, from which

$$u' = u+h_1\pi, \quad v' = v+h_2\pi, \qquad h_1,h_2 \in \mathbb{Z}.$$

Now,

$$f(u+h_1\pi, v+h_2\pi) = (-1)^{h_1+h_2}(\cos u\cos v, \cos u\sin v, \sin u\cos v, \sin u\sin v).$$

So, $f(u,v) = f(u',v')$ if and only if $h_1 + h_2 = 2h$, thus

$$(u,v) \sim (u',v') \quad \Leftrightarrow \quad u'-u = k\pi, \quad v'-v = (2h-k)\pi.$$

Hence

$$\begin{aligned}(u',v') &= (u,v) + (k\pi,(2h-k)\pi)\\ &= (u,v) + k(\pi,-\pi) + h(0,2\pi)\\ &= (u,v) + kv_1 + hv_2.\end{aligned}$$

On the other hand, f is an immersion, as the rank of the Jacobian matrix

$$\begin{pmatrix} -\sin u\cos v & -\cos u\sin v\\ -\sin u\sin v & \cos u\cos v\\ \cos u\cos v & -\sin u\sin v\\ \cos u\sin v & \sin u\cos v \end{pmatrix}$$

is equal to 2, as it is easily seen.

Let j be the inclusion of S^3 in $\mathbb{R}^4$. Then the metric induced on T^2 by the embedding f, if $\widetilde{g}$ denotes the Euclidean metric on $\mathbb{R}^4$, is $f^*j^*\widetilde{g} = f^*\widetilde{g}$.

If (x,y,z,t) denote the coordinates on $\mathbb{R}^4$, then the Euclidean metric is

$$\widetilde{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2$$

and the metric induced on T^2 is $\mathrm{d}u^2 + \mathrm{d}v^2$. Hence $f\colon T^2 \to f(T^2) \subset S^3$ is an isometric embedding, since, as we have seen, it is an isometric immersion, and as T^2 is compact, it is homeomorphic to its image with the induced topology of S^3.

(2) The generalized Gauss *Theorema Egregium* applies in our case to $M = f(T^2)$ and $\widetilde{M} = S^3$. Now, since $\widetilde{M} = S^3$ has constant curvature equal to 1, one has $\widetilde{K}(P) = 1$. And as the metric on $f(T^2)$ is flat, we have $K(P) = 0$. So the equation

$$\widetilde{K}(P) = K(P) - \det L,$$

where L stands for the Weingarten map, is reduced to $\det L = -1$. Hence the total embedded curvature is equal to -1.

Problem 6.13.4. *Let M be a Riemannian n-manifold endowed with the metric*

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{nn}\,\mathrm{d}x^n \otimes \mathrm{d}x^n, \qquad i,j = 1,2,\dots,n-1,$$

with the condition $\partial g_{ij}/\partial x^n = 0$. Show that any geodesic in the hypersurface $x^n =$ const is a geodesic in M.

Solution. The metric given on M induces the metric

$$\widetilde{g} = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \qquad i,j = 1,\dots,n-1,$$

on the given hypersurface S. The geodesics in the hypersurface S are the curves having differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \widetilde{\Gamma}^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \qquad i,j,k = 1,\dots,n-1,$$

where $\widetilde{\Gamma}^i_{jk}$ are the Christoffel symbols of $\widetilde{g}$. We have to prove that the functions $x^1 = x^1(t),\dots,x^{n-1} = x^{n-1}(t)$, $x^n =$ const, satisfy the differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \qquad i,j,k = 1,\dots,n,$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of g. Consider first the case $i = n$ in the equation of the geodesics. As $x^n =$ const, we have $\dfrac{\mathrm{d}^2x^n}{\mathrm{d}t^2} = 0$. On the other hand, one has $\Gamma^n_{jk} = 0$ for $j,k = 1,\dots,n-1$, as

$$g \equiv \left(\begin{array}{ccc|c} & & & 0 \\ & g_{ij} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & g_{nn} \end{array}\right),$$

and moreover $\partial g_{ij}/\partial x^n = 0$ by hypothesis. So, by virtue of the condition $x^n =$ const, the functions defining the geodesics of S satisfy the case $i = n$ of the equation of the geodesics of M.

Consider now the cases $i = 1, \dots, n-1$. For $i, j, k = 1, \dots, n-1$, we have

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}), \qquad l = 1, \dots, n-1,$$

as $g^{in} = 0$; that is, $\Gamma^i_{jk} = \widetilde{\Gamma}^i_{jk}$. Finally, if $k = n$ (equivalently, $j = n$), one has

$$\Gamma^i_{jn} = \frac{1}{2} g^{ik} (g_{kj,n} + g_{kn,j} - g_{jn,k}) = 0,$$

by the hypotheses. We conclude that the geodesics in S are also geodesics in M.

Problem 6.13.5. *Let M_1 and M_2 be two hypersurfaces of $\mathbb{R}^n$ and γ a common geodesic curve which is not a geodesic of $\mathbb{R}^n$. Prove that M_1 and M_2 are tangent along γ.*

Solution. Consider the Gauss's equations

$$\widetilde{\nabla}_X Y = \nabla^i_X Y + \mathrm{II}^i(X,Y), \qquad i = 1, 2,$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection of the flat metric on $\mathbb{R}^n$, ∇^i the Levi-Civita connection of the metric on the hypersurface M_i, and II^i the second fundamental form of the hypersurface M_i.

Since $\nabla^i_{\gamma'} \gamma' = 0$, $i = 1, 2$, we have that $\widetilde{\nabla}'_\gamma \gamma'$ is normal to both M_1 and M_2. So, at any $p \in \gamma$ we have

$$T_p\mathbb{R}^n \;=\; T_pM_1 \oplus \widetilde{\nabla}_{\gamma'}\gamma' \;=\; T_pM_2 \oplus \widetilde{\nabla}_{\gamma'}\gamma' \,,$$

hence $T_pM_1 = T_pM_2$.

6.14 Surfaces in $\mathbb{R}^3$

Problem 6.14.1. *Let*

$$\mathbf{x} \colon U = (0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2 \to \mathbb{R}^3$$

be the parametrization of S^2 given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta \cos\varphi,\ \sin\theta \sin\varphi,\ \cos\theta).$$

(1) *Find the equation of the* loxodromic curves (*that is, the curves meeting the meridians at a constant angle*) *in the coordinate neighborhood $V = \mathbf{x}(U)$.*

(2) *Prove that a new parametrization of the coordinate neighborhood V is given by*

$$\mathbf{y}(u, v) = (\operatorname{sech} u \cos v,\ \operatorname{sech} u \sin v,\ \tanh u).$$

Find the expression of the metric on S^2 in terms of the coordinates u, v, and conclude that $\mathbf{y}^{-1} \colon V \subset S^2 \to \mathbb{R}^2$ is a conformal map transforming the meridians and parallels of S^2 into straight lines of the plane. This map is called the Mercator projection.

(3) *Consider a triangle on the unit sphere S^2 whose sides are segments of loxodromic curves without any of the poles. Prove that the sum of the internal angles of such a triangle is π.*

Solution. (1) The metric inherited on S^2 from the Euclidean metric on $\mathbb{R}^3$ is given by $g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. A loxodromic curve $\sigma(t)$ can be taken as the image under $\mathbf{x}$ of a curve $(\theta(t), \varphi(t))$ in the plane $\theta\varphi$. At the point $\mathbf{x}(\theta, \varphi)$ where the curve meets the meridian $\varphi = \text{const}$ at the angle, say, β we thus have

$$\begin{aligned}\cos\beta &= \frac{g(\mathbf{x}_\theta, \sigma'(t))}{|\mathbf{x}_\theta|\,|\sigma'(t)|}\\ &= \frac{g(\mathbf{x}_\theta, \theta'(t)\mathbf{x}_\theta + \varphi'(t)\mathbf{x}_\varphi)}{|\sigma'(t)|}\\ &= \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta\,\varphi'^2}}.\end{aligned}$$

From this one easily obtains $\tan^2\beta = \sin^2\theta\,\varphi'^2/\theta'^2$. Thus

$$\theta'/\sin\theta = \pm\cot\beta\,\varphi'.$$

Integrating, we obtain the equation of the loxodromic curves

$$\log\tan\frac{\theta}{2} = \pm\cot\beta(\varphi + A).$$

The integration constant A is determined when a point in the curve is given.

(2) It is immediate that the image points belong to S^2. The metric inherited from the Euclidean metric on $\mathbb{R}^3$ is now

$$\operatorname{sech}^2 u(\mathrm{d}u^2 + \mathrm{d}v^2).$$

The map $\mathbf{y}$ is a diffeomorphism which is clearly conformal. The meridians and parallels are the images of the coordinate lines $v = \text{const}$ and $u = \text{const}$, respectively.

The fact that the Mercator projection $\mathbf{y}^{-1}$ is conformal has been useful in cartography, since the angles are preserved.

(3) Under the Mercator projection the meridians are transformed into parallel straight lines of the plane. As the Mercator projection is conformal, the loxodromic curves are also transformed into straight lines. So, the asked sum is the same as that for a plane triangle.

Problem 6.14.2. *Prove that if two families of geodesics on a surface of $\mathbb{R}^3$ are cut at a constant angle, the surface is developable.*

Solution. Consider those families as local coordinate curves (u,v), and let X_u, X_v be the respective coordinate vector fields. Thus $[X_u,X_v]=0$ and $\nabla_{X_u}X_u=\nabla_{X_v}X_v=0$, where ∇ denotes the Levi-Civita connection of the metric g on the surface, inherited from the Euclidean metric on $\mathbb{R}^3$. Hence $\nabla_{X_u}X_v=\nabla_{X_v}X_u$. As $|X_u|,|X_v|$ are constant, it follows by the hypothesis of constant angle, say β, that one has

$$g(X_u,X_v)=|X_u|\,|X_v|\cos\beta=\text{const}.$$

Thus,

$$g(\nabla_{X_u}X_u,X_v)+g(X_u,\nabla_{X_u}X_v)=g(X_u,\nabla_{X_u}X_v)=0.$$

Similarly $g(X_v,\nabla_{X_u}X_v)=0$. So ∇ is flat, thus the Gauss curvature is zero, hence the surface is developable.

Problem 6.14.3. *Consider a surface of revolution around the z-axis in $\mathbb{R}^3$, the vector field $X=y\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial y}$ tangent to the parallels of the surface, and a unit vector field Y on that surface. Show that if $g(X,Y)=\text{const}$, where g denotes the metric on the surface, inherited from the Euclidean metric on $\mathbb{R}^3$, then Y is invariant by X, that is, $L_XY=0$.*

Solution. We have

$$(L_Xg)(X,Y)+g([X,X],Y)+g(X,[X,Y])=0,$$

but $L_Xg=0$ since X is the infinitesimal generator of the group of rotations; so that we have $g(X,[X,Y])=0$. On the other hand, as $g(Y,Y)=1$, one has

$$(L_Xg)(Y,Y)+g([X,Y],Y)+g(Y,[X,Y])=0,$$

that is, $g(Y,[X,Y])=0$. Therefore, $[X,Y]=L_XY=0$.

Problem 6.14.4. *Consider the following surfaces in $\mathbb{R}^3$:*

(a) *The catenoid C with parametric equations*

$$x=\cos\alpha\cosh\beta,\quad y=\sin\alpha\cosh\beta,\quad z=\beta,\qquad \alpha\in[0,2\pi),\quad \beta\in\mathbb{R};$$

that is, the surface of revolution obtained rotating the curve $x=\cosh z$ around the z-axis.

(b) *The helicoid H with parametric equations*

$$x=u\cos v,\quad y=u\sin v,\quad z=v,\qquad u,v\in\mathbb{R},$$

generated by one straight line parallel to the plane xy that intersects with the z-axis and the helix $x=\cos t,\ y=\sin t,\ z=t$ (see Figure 6.11*).*

Let $g=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$ be the Euclidean metric of $\mathbb{R}^3$ and denote by $i\colon C\hookrightarrow\mathbb{R}^3$ and $j\colon H\hookrightarrow\mathbb{R}^3$ the respective inclusion maps.

(1) *Compute i^*g and j^*g.*

(2) *Prove that (C, i^*g) and (H, j^*g) are locally isometric. Are they isometric?*

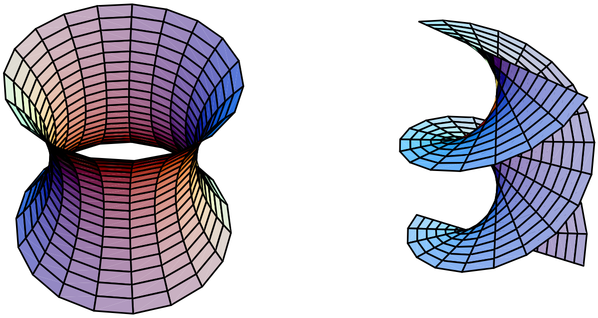

Fig. 6.11 The catenoid (left). The helicoid (right).

Solution. (1)

$$i^*g = \cosh^2\beta(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2), \qquad j^*g = \mathrm{d}u^2 + (1+u^2)\mathrm{d}v^2.$$

(2) The coefficient $1+u^2$ of $\mathrm{d}v^2$ in j^*g suggests that we try the change $u = \sinh\beta$, $v = \beta$, which is only a local isometry. There is a global isometry of the catenoid with the open submanifold of the helicoid corresponding to any interval $v \in (2k\pi, 2(k+1)\pi)$, $k \in \mathbb{Z}$.

Problem 6.14.5. *Let S be a surface of $\mathbb{R}^3$ with the metric induced from that of $\mathbb{R}^3$. Say if the following statements are true or not*:

(1) *The geodesics of S are the intersections of S with the planes of $\mathbb{R}^3$, and conversely.*

(2) *The geodesics of S are obtained intersecting S with some chosen planes.*

Solution. (1) No. For example, the geodesics of S^2 are only obtained when the plane goes through the origin.

(2) No. For example, the helices in the cylinder are not obtained in such a way.

Problem 6.14.6. *Prove that there is no Riemannian metric on the torus $T^2 = S^1 \times S^1$ with Gauss curvature either $K > 0$ in all points or $K < 0$ in all points.*

HINT: *Use the Gauss-Bonnet theorem.*

Solution. The Gauss-Bonnet theorem establishes that for a connected, compact and oriented 2-dimensional Riemannian manifold, one has

$$\int_M K = 2\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M. On the torus, since $\chi(M) = 0$, we have $\int_M K = 0$, and thus it follows that it is not possible either to be $K > 0$ for all $p \in T^2$, or $K < 0$ for all $p \in T^2$.

Problem 6.14.7. *Determine the volume form for the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^3$ on the unit sphere S^2 in $\mathbb{R}^3$, in terms of spherical coordinates (ρ, θ, φ) with $\rho = 1$. Compute the volume* $\operatorname{vol}(S^2)$.

Solution. The sphere S^2 with radius $\rho = 1$ can be parametrized as

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta, \qquad \theta \in [0,\pi], \quad \varphi \in [0,2\pi].$$

Hence, the metric induced on S^2 by the metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ of $\mathbb{R}^3$ is $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The volume element is

$$\omega = \sqrt{\det(g_{ij})}\,\mathrm{d}\theta \wedge \mathrm{d}\varphi = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi,$$

and

$$\begin{aligned}
\operatorname{vol}(S^2) &= \int_{S^2} \omega \\
&= \int_{S^2} \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\theta\,\mathrm{d}\theta \right) \mathrm{d}\varphi = 4\pi.
\end{aligned}$$

Problem 6.14.8. *Compute the volume form for the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$ on the torus T^2 on $\mathbb{R}^3$ obtained by rotating a circle with radius a and center $(b,0,0)$, $b > a > 0$, around the z-axis. Determine the volume* $\operatorname{vol}(T^2)$.

Solution. T^2 can be parametrized as

$$x = (b + a\cos\alpha)\cos\beta, \qquad y = (b + a\cos\alpha)\sin\beta, \qquad z = a\sin\alpha,$$

$\alpha \in [0,2\pi]$ and $\beta \in [0,2\pi]$. Hence, the metric induced by the metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$ is $g = a^2 d\alpha^2 + (b + a\cos\alpha)^2\mathrm{d}\beta^2$, the volume form is

$$\omega = \sqrt{g_{11}g_{22} - g_{12}^2}\,\mathrm{d}\alpha \wedge \mathrm{d}\beta = a(b + a\cos\alpha)\mathrm{d}\alpha \wedge \mathrm{d}\beta,$$

and the volume is

$$\begin{aligned}
\operatorname{vol}(T^2) &= \int_{T^2} \omega \\
&= a\int_{T^2} (b + a\cos\alpha)\mathrm{d}\alpha \wedge \mathrm{d}\beta \\
&= a\int_0^{2\pi}\int_0^{2\pi} (b + a\cos\alpha)\mathrm{d}\alpha\,\mathrm{d}\beta = 4\pi^2 ab.
\end{aligned}$$

Problem 6.14.9. (1) *Consider the flat torus* $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Prove that the map induced on T^2 *by the map* $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^4$ *defined by*

$$\Phi(x,y) = \frac{1}{2\pi}(\cos 2\pi x,\ \sin 2\pi x,\ \cos 2\pi y,\ \sin 2\pi y),$$

is an isometric embedding of T^2 *in* $\mathbb{R}^4$.

(2) *Let* $\Psi\colon \mathbb{R}^3 \to \mathbb{R}^4$ *be the map*

$$\Psi(x,y,z) = (x^2 - y^2, xy, xz, yz).$$

Since $\Psi(-x,-y,-z) = \Psi(x,y,z)$, *by restricting* Ψ *to the sphere* $S^2 \subset \mathbb{R}^3$ *and passing to the quotient,* Ψ *induces a map from the projective plane* $\mathbb{R}P^2 = S^2/\sim$ *into* $\mathbb{R}^4$. *Prove that this map is an embedding.*

(3) *Compute the length of the circles* $z = \text{const}$ *on* S^2 *with respect to the metric*

$$g = \Psi^*\left(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^4 \otimes \mathrm{d}x^4\right)|_{S^2}.$$

(4) *Prove that* (S^2, g), *where* g *is the metric in* (3), *is not isometric to* S^2 *with the standard round metric.* (*Actually, it is not even homothetic.*)

Solution. (1) Let $\tau\colon \mathbb{R}^2 \to T^2$ denote the quotient map. Since

$$\Phi(x+m, y+n) = \Phi(x,y), \qquad m,n \in \mathbb{Z},$$

the map

$$\varphi\colon T^2 \to \mathbb{R}^4, \quad \varphi(p) = \Phi(q), \qquad q \in \tau^{-1}(p),$$

is well defined. Since $\varphi \circ \tau = \Phi$ and $\tau\colon \mathbb{R}^2 \to T^2$ is a local diffeomorphism, φ is C^∞. Moreover,

$$\begin{aligned}\operatorname{rank} \varphi_* &= \operatorname{rank} \Phi_* \\ &= \operatorname{rank} \begin{pmatrix} -2\pi \sin 2\pi x & 0 \\ 2\pi \cos 2\pi x & 0 \\ 0 & -2\pi \sin 2\pi y \\ 0 & 2\pi \cos 2\pi y \end{pmatrix} = 2.\end{aligned}$$

Hence φ is an immersion. Let us see that it is isometric. We have, putting $\mathbb{R}^4 = \{(x^1, x^2, x^3, x^4)\}$:

$$\Phi^*\left(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 + \mathrm{d}x^4 \otimes \mathrm{d}x^4\right) = \mathrm{d}x^2 + \mathrm{d}y^2.$$

From the compactness of T^2 it follows that φ is an embedding. Hence, φ is an isometric embedding.

(2) As $\Psi(p) = \Psi(-p)$, the restriction of Ψ (again denoted by Ψ) to the unit sphere with center the origin of $\mathbb{R}^3$ induces a map $\psi\colon \mathbb{R}P^2 \to \mathbb{R}^4$ with $\psi(\tilde{p}) =$

$\Psi(p)$, where $\tilde{p}$ denotes the class of p in $\mathbb{R}P^2$. Let us see that Ψ (hence ψ) is an immersion. We have

$$\Psi_* = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}.$$

Hence rank $\Psi_* \neq 3$ implies $x = y = 0$. Given any $X \in T_{(0,0,\pm 1)}S^2$, then $X = {}^t(a,b,0)$ and if $\Psi_* X = \pm {}^t(0,0,a,b) = 0$ then $X = 0$. Thus, if $j\colon S^2 \to \mathbb{R}^3$ denotes the inclusion map, then $\Psi \circ j$ is an immersion.

The tangent bundle $T\mathbb{R}P^2$ can be defined as the set

$$T\mathbb{R}P^2 = \{\big((q,Y),(-q,-Y)\big),\ q \in S^2,\ Y \in T_q S^2\}$$

endowed with the differentiable structure inherited from the usual one of TS^2. Thus, from the diagram

$$\begin{array}{ccc} \mathbb{R}P^2 & \xrightarrow{\ \psi\ } & \mathbb{R}^4 \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle\Psi} \\ S^2 & \xrightarrow{\ j\ } & \mathbb{R}^3 \end{array}$$

we conclude that ψ is an immersion.

On the other hand, ψ is injective, as it follows from calculation, due to the condition $x^2 + y^2 + z^2 = 1$. From the compactness of $\mathbb{R}P^2$, it follows that ψ is an embedding.

(3) Consider the parametrization of the sphere

$$x = \cos\theta\cos\varphi, \qquad y = \cos\theta\sin\varphi, \qquad z = \sin\theta,$$

$-\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant \varphi \leqslant 2\pi.$

As a simple computation shows, we have

$$\begin{aligned} g &= \big((2x\,\mathrm{d}x - 2y\,\mathrm{d}y)^2 + (x\,\mathrm{d}y + y\,\mathrm{d}x)^2 + (x\,\mathrm{d}z + z\,\mathrm{d}x)^2 + (y\,\mathrm{d}z + z\,\mathrm{d}y)^2\big)|_{S^2} \\ &= \left(1 - \frac{3}{4}\sin^2 2\theta \sin^2 2\varphi\right)\mathrm{d}\theta^2 + \frac{3}{2}\cos^2\theta \sin 2\theta \sin 4\varphi \,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &\quad + \cos^2\theta(1 + 3\cos^2\theta\sin^2 2\varphi)\mathrm{d}\varphi^2. \end{aligned}$$

The length with respect to g of the circle C defined by $\theta = \theta_0$ is

$$l_g(C) = \cos\theta_0 \int_0^{2\pi} \sqrt{1 + 3\cos^2\theta_0 \sin^2 2\varphi}\,\mathrm{d}\varphi.$$

Making the change of variables

$$t = 2\varphi - \frac{\pi}{2},$$

we obtain $l_g(C)$ in terms of an elliptic integral of the second kind:

$$l_g(C) = \cos\theta_0\sqrt{1+3\cos^2\theta_0}\int_{-\pi/2}^{7\pi/2}\sqrt{1-k\sin^2 t}\,\mathrm{d}t,$$

where

$$k = \frac{\sqrt{3}\cos\theta_0}{\sqrt{1+3\cos^2\theta_0}}.$$

(4) The explicit expression of the Gauss curvature $K = K(\theta,\varphi)$ obtained by using the formula for the Gauss curvature of an abstract parametrized surface in page 416 is rather long, but, as a simple computation shows, K is not constant. In fact, we have

$$\begin{aligned}
K\left(\frac{\pi}{4},0\right) &= -2\cos\left(\frac{\pi}{4}\right)^2 + 9\cos\left(\frac{\pi}{4}\right)^4 - 3\cos\left(\frac{\pi}{4}\right)^8 + 3\cos\left(\frac{\pi}{4}\right)^{10} = \frac{37}{32},\\
K\left(\frac{\pi}{4},\frac{\pi}{4}\right) &= -\frac{1973}{2048},\\
K\left(\frac{\pi}{4},\frac{\pi}{3}\right) &= -\frac{16067107}{48234496}.
\end{aligned}$$

This proves that g is not isometric to the round metric.

6.15 Pseudo-Riemannian Manifolds

Problem 6.15.1. *Consider $M = \mathbb{R}^2 - \{0\}$ equipped with the metric*

$$g = \frac{\mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}x}{x^2+y^2}.$$

The multiplication by any nonzero real scalar is an isometry of M. Consider, in particular, the following isometry: $\lambda(x,y) = (2x,2y)$. The group $\Gamma = \{\lambda^n : n\in\mathbb{Z}\}$ generated by λ acts properly discontinuously. Hence $T = M/\Gamma$ is a Lorentz surface. Topologically, T is the closed ring $1 \leqslant r \leqslant 2$ with the points of the boundary identified by λ. Consequently T is a torus, named the Clifton-Pohl torus; *in particular it is compact.*

(1) *Show that T is not complete. According to* [24, p. 202], *it suffices to prove that M is not complete. For this, prove that the curve*

$$\sigma(t) = \left(\frac{1}{1-t},0\right)$$

is a geodesic.

(2) *Find a group of eight isometries and anti-isometries of M.*

(3) *Prove that* $s \to (\tan s, 1)$ *is a geodesic, and deduce that every null geodesic of* M *and* T *is incomplete.*

(4) *Prove that* $X = x\partial/\partial x + y\partial/\partial y$ *is a Killing vector field on* M.

(5) *If* $\sigma(s) = (x(s), y(s))$ *is a geodesic, then if* $r^2 = x^2 + y^2$, *prove that* $\dot{x}\dot{y}/r^2$ *and* $(x\dot{y} + y\dot{x})/r^2$ *are constant.*

(6) *Show that the curve* $\beta\colon s \to (s, 1/s)$ *is a pregeodesic of finite length on* $[1,\infty)$. (*A pregeodesic is a curve that becomes a geodesic by a reparametrization.*)

REMARK. *This example shows that for pseudo-Riemannian manifolds compactness does not imply completeness.*

Solution. (1) We have

$$g = \begin{pmatrix} 0 & \dfrac{1}{x^2+y^2} \\ \dfrac{1}{x^2+y^2} & 0 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 0 & x^2+y^2 \\ x^2+y^2 & 0 \end{pmatrix},$$

so the only nonvanishing Christoffel symbols are

$$\Gamma^1_{11} = -\frac{2x}{x^2+y^2}, \qquad \Gamma^2_{22} = -\frac{2y}{x^2+y^2},$$

and the differential equations of the geodesics are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{2x}{x^2+y^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{2y}{x^2+y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0,$$

which are easily seen to be satisfied by the given curve. The given geodesic is not defined for $t = 1$, hence M is not complete.

(2)

$$\begin{array}{lll} (x,y) \mapsto (x,y); & (x,y) \mapsto (-x,-y); & (x,y) \mapsto (-x,y); \\ (x,y) \mapsto (x,-y); & (x,y) \mapsto (y,x); & (x,y) \mapsto (-y,x); \\ (x,y) \mapsto (y,-x); & (x,y) \mapsto (-y,-x). & \end{array}$$

(3) We have

$$\frac{2\sin s}{\cos^3 s} - \frac{2\tan s}{\tan^2 s + 1}\,\frac{1}{\cos^4 s} = 0,$$

and the other equation of geodesics is trivially satisfied, for $y = 1$. The geodesic is incomplete because it is not defined for $\pm\pi/2$. The null curves are the ones satisfying $g_{ij}\dfrac{\mathrm{d}x^i}{\mathrm{d}s}\dfrac{\mathrm{d}x^j}{\mathrm{d}s} = 0,\ i,j = 1,2$, that is, $\dfrac{2}{x^2+y^2}\dfrac{\mathrm{d}x}{\mathrm{d}s}\dfrac{\mathrm{d}y}{\mathrm{d}s} = 0$, which are the curves $x = \text{const}$ or $y = \text{const}$. Due to the symmetry in x and y of the equations of geodesics, we can suppose $y = \text{const}$. Then the only equation is $\dfrac{\ddot{x}}{\dot{x}} = \dfrac{2x\dot{x}}{x^2+1}$, so $\log\dot{x} = \log A(x^2+1)$, thus $\arctan x = As + B$, that is $x = \tan(As+B)$. As the geodesic $s \mapsto (\tan(As+B), 1)$

is a model for the null geodesics, it follows that these are incomplete for M. So they are also incomplete for T.

(4)

$$L_X g = L_{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}} \frac{1}{x^2+y^2}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) = 0.$$

(5)

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{x}\dot{y}}{r^2} = \frac{1}{2}g(\dot{\sigma},\dot{\sigma}) = \text{const.}$$

As for $(x\dot{y}+y\dot{x})/r^2$, we have on account of the differential equations of the geodesics:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x\dot{y}+y\dot{x}}{r^2}\right) &= \frac{1}{r^4}\left(x^3\ddot{y}+x^2\ddot{x}y+xy^2\ddot{y}+y^3\ddot{x}-2xy\dot{y}^2-2xy\dot{x}^2\right)\\ &= \frac{1}{r^4}\left(x^3\frac{2y\dot{y}^2}{r^2}+x^2\frac{2x\dot{x}^2}{r^2}y+xy^2\frac{2y\dot{y}^2}{r^2}+y^3\frac{2x\dot{x}^2}{r^2}-2xy\dot{y}^2-2xy\dot{x}^2\right)=0.\end{aligned}$$

(6) We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{x'y''-x''y'}{x'^3}.$$

From the equations of the geodesics given in (1) it follows that

$$(x'y''-x''y')(x^2+y^2) = 2x'y'(yy'-xx').$$

Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x^2+y^2) = 2\frac{\mathrm{d}y}{\mathrm{d}x}\left(y\frac{\mathrm{d}y}{\mathrm{d}x}-x\right), \qquad (\star)$$

which are the equations of the geodesics for any parameter. The condition $(\star)$ is satisfied if $y = 1/x$, as it is easily seen.

The tangent vector along the curve is

$$\dot{\beta}(s) = \left.\frac{\partial}{\partial x}\right|_{\beta(s)} - \frac{1}{s^2}\left.\frac{\partial}{\partial y}\right|_{\beta(s)},$$

hence

$$\begin{aligned}|\dot{\beta}(s)| &= \sqrt{|g(\dot{\beta}(s),\dot{\beta}(s))|}\\ &= \frac{\sqrt{2}}{\sqrt{1+s^4}}.\end{aligned}$$

Since $1+s^{-4} \geqslant 1$ for $s>0$, we obtain

$$\begin{aligned}\int_1^\infty \frac{\sqrt{2}\,\mathrm{d}s}{\sqrt{1+s^4}} &= \int_1^\infty \frac{\sqrt{2}s^{-2}\,\mathrm{d}s}{\sqrt{1+s^{-4}}}\\ &\leqslant \int_1^\infty \sqrt{2}s^{-2}\mathrm{d}s = \sqrt{2}.\end{aligned}$$

Problem 6.15.2. *Consider on* $\mathbb{R}^6$ *the scalar product*

$$\langle\ ,\ \rangle = \mathrm{d}x^1 \otimes \mathrm{d}x^3 + \mathrm{d}x^3 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^4 + \mathrm{d}x^4 \otimes \mathrm{d}x^2 + \mathrm{d}x^5 \otimes \mathrm{d}x^5 + \mathrm{d}x^6 \otimes \mathrm{d}x^6,$$

and the tensor of type $(1,1)$ *given by*

$$J = \frac{\partial}{\partial x^1} \otimes \mathrm{d}x^1 + \frac{\partial}{\partial x^2} \otimes \mathrm{d}x^2 - \frac{\partial}{\partial x^3} \otimes \mathrm{d}x^3 - \frac{\partial}{\partial x^4} \otimes \mathrm{d}x^4 + \frac{\partial}{\partial x^6} \otimes \mathrm{d}x^5 - \frac{\partial}{\partial x^5} \otimes \mathrm{d}x^6.$$

(1) *Let* $W = \left\langle \dfrac{\partial}{\partial x^i} \right\rangle_{i=2,\dots,6}$. *Calculate* $W^\perp = \left\{ v \in \mathbb{R}^6 : v \perp W \right\}$.

(2) *Let* $W = \left\langle \dfrac{\partial}{\partial x^i} \right\rangle_{i=3,\dots,6}$. *Calculate* $W^\perp$.

(3) *Do we have* $\dim W + \dim W^\perp = 6$ *in* (1) *and* (2)*?*

(4) *Let* $U = \left\langle \dfrac{\partial}{\partial x^i} \right\rangle_{i=1,2}$ *and* $V = \left\langle \dfrac{\partial}{\partial x^i} \right\rangle_{i=3,4}$. *Prove that* $JX = X$, $X \in U$ *and* $JX = -X$, $X \in V$.

(5) *Calculate a vector* $X \notin U \cup V$ *such that* $\langle JX, JX \rangle = 0$.

Solution. (1) $W^\perp = \langle \partial/\partial x^3 \rangle$.

(2) $W^\perp = \langle \partial/\partial x^3, \partial/\partial x^4 \rangle$.

(3) Yes.

(4) Immediate.

(5) Take for instance $X = (1,0,1,0,0,\sqrt{2})$. Then

$$\langle JX, JX \rangle = \Big\langle (1,0,-1,0,-\sqrt{2},0),\ (1,0,-1,0,-\sqrt{2},0) \Big\rangle = 0.$$

Problem 6.15.3. *Consider the pseudo-Euclidean space* $\mathbb{R}^n_k$, *that is,* $\mathbb{R}^n$ *with the pseudo-Euclidean metric of signature* $(k, n-k)$:

$$g = -\sum_{i=1}^{k} \mathrm{d}x^i \otimes \mathrm{d}x^i + \sum_{i=k+1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^i. \tag{$\star$}$$

Compute the isometry group $I(\mathbb{R}^n_k)$ *of* $\mathbb{R}^n_k$. *For this prove*:

(1) *The linear isometries of* $\mathbb{R}^n_k$ (i.e. *the isometries of* $\mathbb{R}^n_k$ *which belong to* $GL(n,\mathbb{R})$) *form a subgroup* $O(k,n-k)$ *of* $I(\mathbb{R}^n_k)$.

(2) *The set* $T(n)$ *of all translations of* $\mathbb{R}^n_k$ *is an Abelian subgroup of* $I(\mathbb{R}^n_k)$ *and it is isomorphic to* $\mathbb{R}^n$ (*under vector addition*) *via* $\tau_x \leftrightarrow x$.

(3) *Each isometry* φ *of* $\mathbb{R}^n_k$ *has a unique expression as* $\tau_x \circ A$, *with* $x \in \mathbb{R}^n_k$ *and* $A \in O(k,n-k)$.

(4) *The composition law in* $I(\mathbb{R}^n_k)$ *is*

$$(\tau_x \circ A)(\tau_y \circ B) = \tau_{x+Ay} \circ AB.$$

HINT (to (3)): *Suppose first* $\varphi(0) = 0$.

Solution. (1) The group $O(k, n-k)$ of linear isometries of $\mathbb{R}^n_k$ can be viewed as the subgroup of matrices of $GL(n, \mathbb{R})$ which preserve the scalar product

$$\langle v, w \rangle = -\sum_{i=1}^{k} v^i w^i + \sum_{i=k+1}^{n} v^i w^i, \qquad v, w \in \mathbb{R}^n.$$

(2) Given $x_0 \in \mathbb{R}^n_k$, from $(\star)$ one has that the translation τ_{x_0} sending each $v \in \mathbb{R}^n_k$ to $v + x_0$ is an isometry. It is clear that $T(n)$ is an Abelian subgroup of $I(\mathbb{R}^n_k)$ isomorphic to $\mathbb{R}^n$.

(3) If $\varphi(0) = 0$, then the differential φ_{*0} at 0 is a linear isometry, hence it corresponds under the canonical linear isometry $T_0\mathbb{R}^n_k \approx \mathbb{R}^n_k$ to a linear isometry $A\colon \mathbb{R}^n_k \to \mathbb{R}^n_k$. But then $A_{*0} = \varphi_{*0}$ and thus $\varphi = A$ by Theorem 7.6.16.

Now, if $\varphi \in I(\mathbb{R}^n_k)$, let $x = \varphi(0) \in \mathbb{R}^n_k$. Thus $(\tau_{-x} \circ \varphi)(0) = 0$, so that by the above results, $\tau_{-x} \circ \varphi$ equals some $A \in O(k, n-k)$. Hence $\varphi = \tau_x \circ A$.

If $\tau_x \circ A = \tau_y \circ B$, then $x = (\tau_x \circ A)(0) = (\tau_y \circ B)(0) = y$, hence also $A = B$.

(4) Immediate.

Problem 6.15.4. (1) *Find the Exponential map for* $\mathbb{R}^n_k$.

(2) *Is* Exp_p*, for* $p \in \mathbb{R}^n_k$*, a diffeomorphism?*

(3) *Is* Exp_p *an isometry when* $T_p\mathbb{R}^n_k$ *has the metric induced by the canonical diffeomorphism* $T_p\mathbb{R}^n_k \approx \mathbb{R}^n_k$*?*

Solution. (1) The geodesic $\gamma(t)$ through p with initial velocity vector $v_p \in T_p\mathbb{R}^n_k$ is the straight line $\gamma(t) = p + tv$. Thus

$$\begin{aligned} \operatorname{Exp}_p\colon T_p\mathbb{R}^n_k &\to \mathbb{R}^n_k \\ v_p &\mapsto \gamma(1) = p + v. \end{aligned}$$

(2) Yes, as Exp_p is the composition of the canonical diffeomorphism $T_p\mathbb{R}^n_k \approx \mathbb{R}^n_k$ and the translation $\tau_p\colon x \mapsto x + p$.

(3) Yes, since both maps $T_p\mathbb{R}^n_k \approx \mathbb{R}^n_k$ and τ_p are isometries.

Problem 6.15.5. *Consider the open submanifold*

$$M = \left\{ (x, y) \in \mathbb{R}^2 : x + y > 0 \right\}$$

of the 2*-dimensional* Minkowski space

$$(\mathbb{R}^2, g = \mathrm{d}x^2 - \mathrm{d}y^2),$$

equipped with the inherited metric $g|_M$*, with which* M *is a flat simply connected Lorentz manifold.*

(1) *Prove that* $(M, g|_M)$ *is a non-complete G-homogeneous pseudo-Riemannian manifold, where G is the non-Abelian group* $G = \mathbb{R} \times \mathbb{R}$ *with product*

$$(u,v)(u',v') = (u + u'\mathrm{e}^{-v},\ v + v'),$$

under the action

$$(u,v)\cdot(x,y) = (x\cosh v + y\sinh v + u,\ x\sinh v + y\cosh v - u). \qquad (\star)$$

(2) *Does act G freely on M?*

(3) *Can we identify M and G?*

Solution. (1) (i) It is immediate that (M, g) is non-complete, since its geodesics are the restrictions of the geodesics of $(\mathbb{R}^2, g)$ to M, and these are the straight lines.

(ii) On the other hand, G acts on M: Writing $(x', y') = (u, v)\cdot(x, y)$, we have $x' + y' = \mathrm{e}^v(x + y) > 0$, hence $(x', y') \in M$.

(iii) The action is transitive: Given two points (x_1, y_1), $(x_2, y_2) \in M$, there exists $(u, v) \in G$ such that $(u, v)\cdot(x_1, y_1) = (x_2, y_2)$. In fact, take the parallels to the straight line $x + y = 0$ through (x_1, y_1) and (x_2, y_2), and let (x'_1, y'_1) and (x'_2, y'_2) be, respectively, the points of intersection with the branch of the hyperbola $x^2 - y^2 = 1$ passing through $(1, 0)$. Then it suffices to consider the composition of three transformations: The first one from (x_1, y_1) to (x'_1, y'_1), of type $(u_1, 0)$; the second one from (x'_1, y'_1) to (x'_2, y'_2) along the branch of hyperbola (with $u = 0$); and the third one from (x'_2, y'_2) to (x_2, y_2), again of type $(u_2, 0)$.

(2) If $(u, v)\cdot(x, y) = (x, y)$, it is clear that we must have $u = v = 0$.

(3) Yes, as the action $(\star)$ of G on M is simply transitive (that is, transitive and free, see Definition 7.4.10 and Theorem 7.4.12).

Problem 6.15.6. *Find, using Cartan's structure equations, the Gauss curvature of* $\mathbb{R}^2$ *endowed with the pseudo-Riemannian metric*

$$g = \frac{4}{c}\left(\cosh^2 2y\,\mathrm{d}x^2 - \mathrm{d}y^2\right), \qquad 0 \neq c \in \mathbb{R}.$$

Solution. We have the orthonormal moving frame on $\mathbb{R}^2$:

$$\sigma = \left(X_1 = \frac{\sqrt{|c|}}{2}\frac{1}{\cosh 2y}\frac{\partial}{\partial x},\ X_2 = \frac{\sqrt{|c|}}{2}\frac{\partial}{\partial y}\right).$$

That is, $g(X_i, X_i) = \varepsilon_i$, $i = 1, 2$, with $\varepsilon_1 = +1$, $\varepsilon_2 = -1$ if $c > 0$, and $\varepsilon_1 = -1$, $\varepsilon_2 = +1$ if $c < 0$. Its dual moving coframe is

$$\left(\tilde{\theta}^1 = \frac{2}{\sqrt{|c|}}\cosh 2y\,\mathrm{d}x,\ \tilde{\theta}^2 = \frac{2}{\sqrt{|c|}}\,\mathrm{d}y\right).$$

Let $\tilde{\theta}_i = \varepsilon_i \tilde{\theta}^i$ (no sum) and let $\widetilde{\omega}^i_j$ be the connection forms relative to σ. Then $\widetilde{\omega}_{ij} = \varepsilon_i \widetilde{\omega}^i_j$ (no sum) is the only set of differential 1-forms satisfying the first structure equation

$$\widetilde{\omega}_{ij} + \widetilde{\omega}_{ji} = 0, \qquad \mathrm{d}\tilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge (\varepsilon_j \tilde{\theta}_j).$$

We only have to calculate $\widetilde{\omega}_{12}$. From

$$\begin{aligned}
\mathrm{d}(\varepsilon_1 \tilde{\theta}_1) &= \varepsilon_1 \mathrm{d}\tilde{\theta}_1 \\
&= (\varepsilon_1)^2 \mathrm{d}\tilde{\theta}^1 \\
&= \frac{4}{\sqrt{|c|}} \sinh 2y \,\mathrm{d}y \wedge \mathrm{d}x \\
&= -\varepsilon_1 \widetilde{\omega}_{12} \wedge \varepsilon_2 \tilde{\theta}_2 \\
&= -\widetilde{\omega}_{12} \wedge \varepsilon_1 (\varepsilon_2)^2 \left(\frac{2}{\sqrt{|c|}} \mathrm{d}y \right) \\
&= -\widetilde{\omega}_{12} \wedge \varepsilon_1 \left(\frac{2}{\sqrt{|c|}} \mathrm{d}y \right),
\end{aligned}$$

one obtains that $\widetilde{\omega}_{12} = 2\varepsilon_1 \sinh 2y \,\mathrm{d}x$.

The Gauss curvature of the pseudo-Riemannian manifold $(\mathbb{R}^2, g)$ is the differentiable real valued function K defined by $\mathrm{d}\widetilde{\omega}_{12} = K\tilde{\theta}_1 \wedge \tilde{\theta}_2$; that is, by

$$4\varepsilon_1 \cosh 2y \,\mathrm{d}y \wedge \mathrm{d}x = \varepsilon_1 \varepsilon_2 K \frac{2}{\sqrt{|c|}} \cosh 2y \,\mathrm{d}x \wedge \frac{2}{\sqrt{|c|}} \mathrm{d}y.$$

Thus, $(\mathbb{R}^2, g)$ has constant Gauss curvature $K = c$, because

$$K = -\varepsilon_2 |c| = \begin{cases} c & \text{if } \ \varepsilon_2 = -1 \ \ (c > 0) \\ -|c| = c & \text{if } \ \varepsilon_2 = 1 \ \ (c < 0). \end{cases}$$

Problem 6.15.7. *Prove that the half-space*

$$H = \left\{ (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \, : \, x^1 > 0 \right\}$$

endowed with the pseudo-Riemannian metric

$$g = \frac{1}{K} \frac{\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4}{(x^1)^2}, \quad 0 \neq K \in \mathbb{R},$$

has constant curvature $-K$.

Solution. Applying Koszul's formula 7.6.3 to $e_i = \partial/\partial x^i$, $i = 1,2,3,4$, we obtain, on account of $[e_i, e_j] = 0$, for instance for $\nabla_{e_1} e_1$:

$$\begin{aligned}
2g(\nabla_{e_1}e_1,e_i) &= 2e_1 g(e_1,e_i) - e_i g(e_1,e_1) \\
&= 2e_1\left(\frac{\delta_{1i}}{K(x^1)^2}\right) - e_i\left(\frac{1}{K(x^1)^2}\right) \\
&= 2\frac{\delta_{1i}}{K}\left(-\frac{2}{(x^1)^3}\right) - \frac{1}{K}\left(-\frac{2\delta_{1i}}{(x^1)^3}\right) \\
&= -\frac{2\delta_{1i}}{K(x^1)^3},
\end{aligned}$$

from which $\nabla_{e_1}e_1 = -\dfrac{1}{x^1}e_1$. Similarly one obtains:

$$\begin{aligned}
&\nabla_{e_1}e_1 = -\nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = -\frac{1}{x^1}e_1, \\
&\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{x^1}e_2, \\
&\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\frac{1}{x^1}e_3, \\
&\nabla_{e_1}e_4 = \nabla_{e_4}e_1 = -\frac{1}{x^1}e_4, \\
&\nabla_{e_2}e_3 = \nabla_{e_3}e_2 = \nabla_{e_2}e_4 = \nabla_{e_4}e_2 = \nabla_{e_3}e_4 = \nabla_{e_4}e_3 = 0.
\end{aligned}$$

Thus

$$-R(e_1,e_2)e_2 = R(e_1,e_3)e_3 = R(e_1,e_4)e_4 = \frac{1}{(x^1)^2}e_1,$$

$$R(e_2,e_3)e_3 = R(e_2,e_4)e_4 = \frac{1}{(x^1)^2}e_2, \qquad R(e_3,e_4)e_4 = \frac{1}{(x^1)^2}e_3.$$

So

$$-R_{1212} = R_{1313} = R_{1414} = R_{2323} = R_{2424} = -R_{3434} = \frac{K}{(x^1)^4}.$$

Finally, the sectional curvature has values

$$K_{12} = K_{13} = K_{14} = K_{23} = K_{24} = K_{34} = -K.$$

Problem 6.15.8. *Let M be a pseudo-Riemannian manifold of dimension $n \geqslant 2$. Show, using Cartan's structure equations, that if there exist local coordinates x^i on a neighborhood of each $x \in M$ in which the metric is given by*

$$g = \frac{\varepsilon_i\,\mathrm{d}x^i \otimes \mathrm{d}x^i}{\left(1 + \dfrac{K}{4}\varepsilon_i (x^i)^2\right)^2}, \qquad \varepsilon_i = \pm 1, \quad i = 1,\dots,n, \quad K \in \mathbb{R},$$

then (M,g) has constant curvature K.

Solution. Let $r(x) = (\varepsilon_i (x^i)^2)^{1/2}$ and $A(x) = -\log(1 + (K/4)r^2)$. Then $\left(\mathrm{e}^{-A}\dfrac{\partial}{\partial x^i}\right)$ is an orthonormal moving frame, that is,

$$g\left(\mathrm{e}^{-A}\frac{\partial}{\partial x^i}, \mathrm{e}^{-A}\frac{\partial}{\partial x^j}\right) = \begin{cases} \varepsilon_i & \text{if } j = i \\ 0 & \text{if } j \neq i, \end{cases}$$

whose dual moving coframe is $(\tilde{\theta}^i = \mathrm{e}^A\,\mathrm{d}x^i)$. Therefore,

$$\begin{aligned}
\mathrm{d}\tilde{\theta}^i &= \mathrm{e}^A \mathrm{d}A \wedge \mathrm{d}x^i \\
&= \mathrm{e}^A \frac{\partial A}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
&= \tilde{\theta}^j \wedge \frac{\partial A}{\partial x^j}\,\mathrm{d}x^i \\
&= \sum_j \tilde{\theta}^j \wedge \left(\frac{\partial A}{\partial x^j}\,\mathrm{d}x^i - \varepsilon_i\varepsilon_j \frac{\partial A}{\partial x^i}\,\mathrm{d}x^j\right).
\end{aligned}$$

Let $\widetilde{\omega}^i_j$ denote the term in parentheses. One has

$$\begin{aligned}
\widetilde{\omega}_{ij} &= \varepsilon_i \widetilde{\omega}^i_j \\
&= \varepsilon_i \frac{\partial A}{\partial x^j}\,\mathrm{d}x^i - \varepsilon_j \frac{\partial A}{\partial x^i}\,\mathrm{d}x^j \\
&= -\widetilde{\omega}_{ji},
\end{aligned}$$

hence $\widetilde{\omega}^i_j$ are the connection forms relative to $\left(\mathrm{e}^{-A}\dfrac{\partial}{\partial x^i}\right)$. The second structure equation is thus

$$\begin{aligned}
\widetilde{\Omega}_{ij} &= \mathrm{d}\widetilde{\omega}_{ij} + \sum_k \varepsilon_k \widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik} \\
&= \varepsilon_i \frac{\partial^2 A}{\partial x^k \partial x^j}\,\mathrm{d}x^k \wedge \mathrm{d}x^i - \varepsilon_j \frac{\partial^2 A}{\partial x^k \partial x^i}\,\mathrm{d}x^k \wedge \mathrm{d}x^j \\
&\quad + \sum_k \varepsilon_k \left(\varepsilon_j \frac{\partial A}{\partial x^k}\,\mathrm{d}x^j - \varepsilon_k \frac{\partial A}{\partial x^j}\,\mathrm{d}x^k\right) \wedge \left(\varepsilon_i \frac{\partial A}{\partial x^k}\,\mathrm{d}x^i - \varepsilon_k \frac{\partial A}{\partial x^i}\,\mathrm{d}x^k\right).
\end{aligned}$$

Now, since

$$\frac{\partial A}{\partial x^i} = -\frac{K}{2}\varepsilon_i x^i \Big/ \left(1 + \frac{K}{4}r^2\right),$$

the three summands at the right hand side can be written, respectively, as:

$$\sum_k \varepsilon_i \frac{(1 + \frac{K}{4}r^2)(-\frac{K}{2}\varepsilon_j \delta_{jk}) + \frac{K}{2}\varepsilon_j x^j \frac{K}{2}\varepsilon_k x^k}{(1 + \frac{K}{4}r^2)^2}\,\mathrm{d}x^k \wedge \mathrm{d}x^i,$$

$$-\sum_k \varepsilon_j \frac{(1+\frac{K}{4}r^2)(-\frac{K}{2}\varepsilon_i\delta_{ik})+\frac{K}{2}\varepsilon_i x^i \frac{K}{2}\varepsilon_k x^k}{(1+\frac{K}{4}r^2)^2}\,\mathrm{d}x^k\wedge\mathrm{d}x^j,$$

and

$$\sum_k\left(\varepsilon_k\varepsilon_j\varepsilon_i \frac{\frac{K^2}{4}\varepsilon_k\varepsilon_k x^k x^k}{(1+\frac{K}{4}r^2)^2}\mathrm{d}x^j\wedge\mathrm{d}x^i-\varepsilon_k\varepsilon_j\varepsilon_k \frac{\frac{K^2}{4}\varepsilon_i\varepsilon_k x^i x^k}{(1+\frac{K}{4}r^2)^2}\mathrm{d}x^j\wedge\mathrm{d}x^k\right.$$
$$\left.-\varepsilon_k\varepsilon_k\varepsilon_i \frac{\frac{K^2}{4}\varepsilon_j\varepsilon_k x^j x^k}{(1+\frac{K}{4}r^2)^2}\mathrm{d}x^k\wedge\mathrm{d}x^i\right).$$

Substituting, we obtain

$$\begin{aligned}\widetilde{\Omega}_{ij}&=\frac{\varepsilon_i\varepsilon_j K}{\left(1+\frac{K}{4}r^2\right)^2}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\\&=K\varepsilon_i\tilde\theta^i\wedge\varepsilon_j\tilde\theta^j\\&=K\tilde\theta_i\wedge\tilde\theta_j.\end{aligned}$$

Problem 6.15.9. *Consider $\mathbb{R}^4$ as a spacetime with coordinates ρ,φ,ψ and t, where the first three are the usual spherical coordinates on $\mathbb{R}^3$, equipped with the metric*

$$g=-\left(1-\frac{2m}{\rho}\right)\mathrm{d}t^2+\left(1-\frac{2m}{\rho}\right)^{-1}\mathrm{d}\rho^2+\rho^2(\mathrm{d}\psi^2+\sin^2\psi\,\mathrm{d}\varphi^2),$$
$$0\leqslant\psi<\pi,\quad 0\leqslant\varphi<2\pi.$$

Prove, using Cartan's structure equations, that g is a solution (except at the singularity $\rho=0$) of the empty space Einstein field equations

$$r-\frac{1}{2}sg=0.$$

REMARK. *This solution, found by Schwarzschild, was the first one known to such field equations, and it is sometimes called Schwarzschild's "black hole" metric.*

Solution.

$$\sigma=\left(X_1=\left(1-\frac{2m}{\rho}\right)^{\frac12}\frac{\partial}{\partial t},\quad X_2=\left(1-\frac{2m}{\rho}\right)^{-\frac12}\frac{\partial}{\partial\rho},\right.$$
$$\left.X_3=\frac{1}{\rho}\frac{\partial}{\partial\psi},\quad X_4=\frac{1}{\rho\sin\psi}\frac{\partial}{\partial\varphi}\right),$$

is an orthonormal moving frame, that is,

$$g(X_1,X_1)=\varepsilon_1=-1,\qquad g(X_i,X_i)=\varepsilon_i=1,\quad i=2,3,4,$$

with dual moving coframe

$$\left(\tilde{\theta}_1 = \left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}} \mathrm{d}t,\ \tilde{\theta}_2 = \left(1-\frac{2m}{\rho}\right)^{-\frac{1}{2}} \mathrm{d}\rho,\ \tilde{\theta}_3 = \rho\,\mathrm{d}\psi,\ \tilde{\theta}_4 = \rho\sin\psi\,\mathrm{d}\varphi\right).$$

The first structure equation,

$$\mathrm{d}\tilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge (\varepsilon_j \tilde{\theta}_j)$$

gives us the nonvanishing connection forms relative to σ:

$$\widetilde{\omega}_{12} = -\widetilde{\omega}_{21} = \frac{m}{\rho^2}\mathrm{d}t, \qquad \widetilde{\omega}_{23} = -\widetilde{\omega}_{32} = -\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}} \mathrm{d}\psi,$$
$$\widetilde{\omega}_{24} = -\widetilde{\omega}_{42} = -\left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}} \sin\psi\,\mathrm{d}\varphi, \qquad \widetilde{\omega}_{34} = -\widetilde{\omega}_{43} = -\cos\psi\,\mathrm{d}\varphi.$$

The second structure equation

$$\widetilde{\Omega}_{ij} = \mathrm{d}\widetilde{\omega}_{ij} + \sum_k \varepsilon_k \widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik},$$

furnishes the nonvanishing curvature 2-forms relative to σ:

$$\widetilde{\Omega}_{12} = -\widetilde{\Omega}_{21} = \frac{2m}{\rho^3}\tilde{\theta}^1 \wedge \tilde{\theta}^2, \qquad \widetilde{\Omega}_{13} = -\widetilde{\Omega}_{31} = -\frac{m}{\rho^3}\tilde{\theta}^1 \wedge \tilde{\theta}^3,$$
$$\widetilde{\Omega}_{14} = -\widetilde{\Omega}_{41} = -\frac{m}{\rho^3}\tilde{\theta}^1 \wedge \tilde{\theta}^4, \qquad \widetilde{\Omega}_{23} = -\widetilde{\Omega}_{32} = -\frac{m}{\rho^3}\tilde{\theta}^2 \wedge \tilde{\theta}^3,$$
$$\widetilde{\Omega}_{24} = -\widetilde{\Omega}_{42} = -\frac{m}{\rho^3}\tilde{\theta}^2 \wedge \tilde{\theta}^4, \qquad \widetilde{\Omega}_{34} = -\widetilde{\Omega}_{43} = \frac{2m}{\rho^3}\tilde{\theta}^3 \wedge \tilde{\theta}^4.$$

From the equations

$$\widetilde{\Omega}_{ij} = \sum_{k<l} R_{ijkl}\tilde{\theta}^k \wedge \tilde{\theta}^l, \qquad r_{ij} = \sum_k R_{kikj},$$

one obtains that the Ricci tensor r vanishes. In fact:

$$r_{12} = R_{k1k2} = R_{3132} + R_{4142} = 0,$$
$$r_{22} = R_{k2k2} = R_{1212} + R_{3232} + R_{4242} = \frac{2m}{\rho^3} - \frac{m}{\rho^3} - \frac{m}{\rho^3} = 0.$$

The remaining calculations for the components r_{ij}, $i \neq j$, or r_{ii}, are similar.

Since the scalar curvature is given by $s = g^{ij}r_{ij}$, empty space Einstein's field equations are automatically satisfied.

Problem 6.15.10. (1) *Let V be an $(n+1)$-dimensional vector space, and V^* its dual space. We shall write $x+\alpha, y+\beta, \dots$, to denote the elements of $V \oplus V^*$. On the space $V \oplus V^*$ there exists a natural nondegenerate bilinear form $\langle\ ,\ \rangle$ given by*

$$\langle x+\alpha, y+\beta\rangle = \frac{1}{2}\big(\alpha(y)+\beta(x)\big),$$

and an involutive linear automorphism J_0 given by

$$J_0|_V = \mathrm{id}_V, \qquad J_0|_{V^*} = -\mathrm{id}_{V^*}.$$

The subgroup of the automorphisms group $GL(V \oplus V^*)$ of $V \oplus V^*$ preserving both $\langle\ ,\ \rangle$ and J_0 can be identified to the automorphisms group $GL(V)$ of V. In fact, if $A \in GL(V)$, we put $A(x+\alpha) = Ax + \alpha \cdot A^{-1}$.

Let us introduce on

$$(V \oplus V^*)_+ = \{x+\alpha \in V \oplus V^* : \langle x+\alpha, x+\alpha\rangle = \alpha(x) > 0\}$$

the equivalence relation $\sim$ defined by $x+\alpha \sim ax+b\alpha$ if $0 < a, b \in \mathbb{R}$, and define the paracomplex projective space *$P(V \oplus V^*)$ by*

$$P(V \oplus V^*) = (V \oplus V^*)_+/\!\sim .$$

Let π denote the natural projection $\pi\colon (V \oplus V^*)_+ \to P(V \oplus V^*)$. If $a, b \in \mathbb{R}^+$, we have $A(ax+b\alpha) = aAx + b(\alpha \cdot A^{-1})$, and so we can define an action of $GL(V)$ on $P(V \oplus V^*)$ in such a way that

$$A(\pi(x+\alpha)) = \pi(A(x+\alpha)), \quad A \in GL(V).$$

Then the identity component $GL_0(V)$ of $GL(V)$ acts transitively on the pseudosphere in $V \oplus V^*$,

$$S = \{x+\alpha \in (V \oplus V^*)_+ : \langle x+\alpha, x+\alpha\rangle = \alpha(x) = 1\}.$$

Prove that $P(V \oplus V^*)$ is a homogeneous space under the action of the group $GL_0(V)$, for $n \geqslant 1$.

(2) *We have a principal bundle $\pi\colon S \to P(V \oplus V^*)$ with group $\mathbb{R}^+$. The subgroup $\{aI \in GL_0(V) : a > 0\}$ of $GL_0(V)$ acts transitively on the fibres. The quotient of S by that action is $P(V \oplus V^*)$. Consider S equipped with the pseudo-Riemannian metric inherited from that of $V \oplus V^*$. Then, as $GL_0(V)$ acts on $V \oplus V^*$ by isometries, and preserves S, it also acts on S by isometries.*

Now, consider the formula

$$\langle Z, Z\rangle = \langle Z^h, Z^h\rangle, \qquad Z \in T_{\pi(x+\alpha)}P(V \oplus V^*), \tag{$\star$}$$

where $Z^h \in T_{x+\alpha}S$ is orthogonal to the fibre and satisfies $\pi_* Z^h = Z$. Show that this construction induces on $P(V \oplus V^*)$ a pseudo-Riemannian metric g such that π is a pseudo-Riemannian submersion.

(3) *The group $G = \mathbb{R}^+ \times \mathbb{R}^+$ acts on $(V \oplus V^*)_+$ by*

$$(a,b)(x+\alpha) = ax + b\alpha, \qquad (a,b) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

$P(V \oplus V^*)$ is the quotient space of this action. Let J_0 be the almost product structure (*that is, an automorphism such that $J_0^2 = I$*) *defined on $(V \oplus V^*)_+$ by*

$$J_0(v,\omega) = (v,-\omega), \qquad (v,\omega) \in T_{x+\alpha}(V \oplus V^*)_+.$$

Prove that J_0 passes to the quotient and gives an almost product structure J (*a $(1,1)$ tensor field with $J^2 = I$*) *on $P(V \oplus V^*)$ such that this manifold has a* para-Hermitian structure *with the metric in* (2) *and J* (*that is, we have $g(JX,Y) + g(X,JY) = 0$, where $X,Y, \in \mathfrak{X}(V \oplus V^*)$*).

(4) *Consider a basis $\{e_0,\dots,e_n\}$ of V, and the dual basis $\{e^0,\dots,e^n\}$ of V^*. We can consider the e_k, $k = 0,\dots,n$, as coordinates on V^* and the e^k as coordinates on V. Let U_0^+ be the open subset of $P(V \oplus V^*)$ given by*

$$U_0^+ = \left\{\pi(x+\alpha) : e^0(x) > 0,\, e_0(\alpha) > 0\right\}.$$

Let (x^i,y^i), $i = 1,\dots,n$, be coordinates on U_0^+ given by

$$x^i(\pi(x+\alpha)) = \frac{e^i(x)}{e^0(x)}, \qquad y^i(\pi(x+\alpha)) = \frac{e_i(\alpha)}{e_0(\alpha)}. \tag{$\star\star$}$$

Prove that the metric in terms of these coordinates on U_0^+ has the expression

$$g = \frac{1}{2(1+\langle x,y\rangle)}\left\{\mathrm{d}x^i \otimes \mathrm{d}y^i + \mathrm{d}y^i \otimes \mathrm{d}x^i - \sum_{i,j=1}^{n} \frac{x^i y^j}{1+\langle x,y\rangle}(\mathrm{d}y^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}y^i)\right\},$$

where $\langle x,y\rangle = \sum_i x^i y^i$.

(5) *Compute the almost product structure J on $P(V \oplus V^*)$ in terms of the coordinates (x^i,y^i).*

REMARK. *Since the metric admits locally that expression, it is said that the manifold $(P(V \oplus V^*), g, J)$ is a* para-Kähler manifold (*that is, the Levi-Civita connection of g parallelizes J*) *of constant paraholomorphic sectional curvature* (*equal to* 4), *which is an analog of the holomorphic sectional curvature.*

For such a space, the curvature tensor field R satisfies

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ.$$

Solution. (1) Let $x+\alpha$ be an arbitrarily fixed element of S. Then, for each $y+\beta \in S$, there exists an element $A \in GL_0(V)$ such that $A(y+\beta) = x+\alpha$. For, given a linearly independent set of elements $y_1,\dots,y_n$ of V such that $\beta(y_i)=0$, then $\{y,y_1,\dots,y_n\}$ is a basis of V. Similarly, if we have a linearly independent set of elements $x_1,\dots,x_n$ of V such that $\alpha(x_i)=0$, then $\{x,x_1,\dots,x_n\}$ is a basis of V. Take A such that $Ay=x$, $Ay_i=x_i$. Then

$$(\beta\cdot A^{-1})(x)=\beta(y)=1=\alpha(x), \qquad (\beta\cdot A^{-1})(x_i)=\beta(y_i)=0=\alpha(x_i),$$

and hence $\beta\cdot A^{-1}=\alpha$. Taking two bases with the same orientation we have $A\in GL_0(V)$ as desired.

(2) Denote by $\mathbf{n}$ and $\mathbf{v}$ the natural vector fields on $V\oplus V^*$ whose values at $x+\alpha$ are $\mathbf{n}_{x+\alpha}=x+\alpha$ and $\mathbf{v}_{x+\alpha}=x-\alpha$. Then one has $\pi_*\mathbf{n}=\pi_*\mathbf{v}=0$. In fact, $\mathbf{n}_{x+\alpha}$ is the vector tangent at $t=0$ to the curve $t\mapsto x+\alpha+t(x+\alpha)$ and $\mathbf{v}_{x+\alpha}$ is the vector tangent to the curve $t\mapsto x+\alpha+t(x-\alpha)$. As $\pi((1+t)(x+\alpha))=\pi(x+\alpha)$ and $\pi((1+t)x+(1-t)\alpha))=\pi(x+\alpha)$ for small t, the claim follows. Thus $\operatorname{Ker}\pi_*$ is spanned by $\mathbf{n}$ and $\mathbf{v}$. The vector $\mathbf{v}$ is tangent to the fibre and $\mathbf{n}$ is normal to S in $V\oplus V^*$. The process given in the statement of lifting a vector Z to such a vector Z^h has a unique solution if and only if the subspace orthogonal to the fibres has dimension equal to $2n$ or, equivalently, if and only if the restriction of $\langle\ ,\ \rangle$ to the subspace spanned by $\mathbf{v}_{x+\alpha}$ and $\mathbf{n}_{x+\alpha}$ is nondegenerate; but, indeed

$$\begin{pmatrix}\langle\mathbf{n},\mathbf{n}\rangle & \langle\mathbf{n},\mathbf{v}\rangle\\ \langle\mathbf{v},\mathbf{n}\rangle & \langle\mathbf{v},\mathbf{v}\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Consequently, we have the desired structure on $P(V\oplus V^*)$, which makes π a pseudo-Riemannian submersion.

(3) We have

$$(J_0\circ(a,b)_*-(a,b)_*\circ J_0)(v,\omega)=J_0(av,b\omega)-(a,b)_*(v,-\omega)=0.$$

Hence, J_0 passes to the quotient, giving an almost product structure J, which is easily seen to be para-Hermitian.

(4) After computation we have

$$\left.\frac{\partial}{\partial x^i}\right|^h_{\pi(x+\alpha)}=-e_i(\alpha)e^0(x)x+e^0(x)\left.\frac{\partial}{\partial e^i}\right|_{x+\alpha},$$
$$\left.\frac{\partial}{\partial y^i}\right|^h_{\pi(x+\alpha)}=-e^i(x)e_0(\alpha)\alpha+e_0(\alpha)\left.\frac{\partial}{\partial e_i}\right|_{x+\alpha}.$$

From this, on account of $(\star)$, one has

$$\left\langle\left.\frac{\partial}{\partial x^i}\right|^h_{\pi(x+\alpha)},\left.\frac{\partial}{\partial y^j}\right|^h_{\pi(x+\alpha)}\right\rangle=\frac{1}{2}e^0(x)e_0(\alpha)\big(\delta_i^j-e_i(\alpha)e^j(x)\big).$$

Now, from $(\star\star)$ we deduce

$$e^0(x)e_0(\alpha) = \frac{1}{1+\langle x,y\rangle}, \qquad e_i(\alpha)e^j(x) = \frac{y^i x^j}{1+\langle x,y\rangle},$$

hence

$$\left\langle \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial y^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = \frac{1}{2(1+\langle x,y\rangle)}\left(\delta_{ij} - \frac{y^k x^j}{1+\langle x,y\rangle}\right).$$

Similarly,

$$\left\langle \frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial x^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = 0, \qquad \left\langle \frac{\partial}{\partial y^i}\bigg|^h_{\pi(x+\alpha)}, \frac{\partial}{\partial y^j}\bigg|^h_{\pi(x+\alpha)} \right\rangle = 0.$$

Hence the metric on $P(V \oplus V^*)$ has on U_0^+ the expression given in the statement.

(5) We have

$$\begin{aligned} J\,\frac{\partial}{\partial x^i}\bigg|_{\pi(x+\alpha)} &= \pi_* J_0\left(\frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}\right) \\ &= \pi_*\left(\frac{\partial}{\partial x^i}\bigg|^h_{\pi(x+\alpha)}\right) \\ &= \frac{\partial}{\partial x^i}\bigg|_{\pi(x+\alpha)}, \end{aligned}$$

and similarly

$$J\,\frac{\partial}{\partial y^i}\bigg|_{\pi(x+\alpha)} = -\,\frac{\partial}{\partial y^i}\bigg|_{\pi(x+\alpha)}.$$

Hence

$$J = \frac{\partial}{\partial x^i}\otimes \mathrm{d}x^i - \frac{\partial}{\partial y^i}\otimes \mathrm{d}y^i.$$

Problem 6.15.11. *The* oscillator group *G is the simply connected* 4*-dimensional Lie group corresponding to the Lie algebra L with non-null Lie brackets*

$$[e_2,e_3] = e_1, \qquad [e_2,e_4] = e_3, \qquad [e_3,e_4] = -e_2.$$

G can be realized as $\mathbb{R}^4$ with the group operation $z = x\cdot y$ given by

$$\begin{aligned} z^1 &= x^1 + y^1 + \frac{1}{2}\big(x^2(-y^2\sin x^4 + y^3\cos x^4) - x^3(-y^2\cos x^4 + y^3\sin x^4)\big), \\ z^2 &= x^2 + y^2\cos x^4 + y^3\sin x^4, \\ z^3 &= x^3 - y^2\sin x^4 + y^3\cos x^4, \\ z^4 &= x^4 + y^4. \end{aligned}$$

Consider on L the family of Lorentz inner products given by

$$\langle a,b\rangle = \varepsilon a_1b_1 - a_1b_4 + a_2b_2 + a_3b_3 - a_4b_1 + \varepsilon a_4b_4, \qquad -1 < \varepsilon \leqslant 0.$$

Find the explicit expression of the family of corresponding left-invariant Lorentz metrics on G.

Solution. In general, if g denote a left-invariant metric on G, e the identity element of G and s an arbitrary element of G, one has

$$\begin{aligned} g_e\left(\left.\frac{\partial}{\partial x^i}\right|_e, \left.\frac{\partial}{\partial x^j}\right|_e\right) &= g_s\left(L_{s*}\left(\left.\frac{\partial}{\partial x^i}\right|_e\right), L_{s*}\left(\left.\frac{\partial}{\partial x^j}\right|_e\right)\right) \\ &= ({}^tL_{s*}g_sL_{s*})\left(\left.\frac{\partial}{\partial x^i}\right|_e, \left.\frac{\partial}{\partial x^j}\right|_e\right), \end{aligned}$$

that is,

$$g_s = {}^tL_{s*}^{-1}g_eL_{s*}^{-1}.$$

In the present case we have

$$\begin{aligned} L_{x*} &= \left(\frac{\partial z^i}{\partial y^j}\right)_{y=0} \\ &= \begin{pmatrix} 1 & -\frac{1}{2}(x^2\sin x^4 + x^3\cos x^4) & \frac{1}{2}(x^2\cos x^4 - x^3\sin x^4) & 0 \\ 0 & \cos x^4 & \sin x^4 & 0 \\ 0 & -\sin x^4 & \cos x^4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

so after computation we deduce

$$\begin{aligned} (g_\varepsilon)_x &= {}^tL_{x*}^{-1}(g_\varepsilon)_eL_{x*}^{-1} \\ &= \begin{pmatrix} \varepsilon & \dfrac{\varepsilon x^3}{2} & -\dfrac{\varepsilon x^2}{2} & -1 \\ \dfrac{\varepsilon x^3}{2} & \dfrac{\varepsilon(x^3)^2}{4}+1 & -\dfrac{\varepsilon x^2x^3}{4} & -\dfrac{x^3}{2} \\ -\dfrac{\varepsilon x^2}{2} & -\dfrac{\varepsilon x^2x^3}{4} & \dfrac{\varepsilon(x^2)^2}{4}+1 & \dfrac{x^2}{2} \\ -1 & -\dfrac{x^3}{2} & \dfrac{x^2}{2} & \varepsilon \end{pmatrix}. \end{aligned}$$

Problem 6.15.12. *Let*

$$G = \left\{ \begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(3,\mathbb{R}) \,:\, a>0 \right\}.$$

Prove:

(1) *G is a closed subgroup of $GL(3,\mathbb{R})$.*

(2) *G does not admit a pseudo-Riemannian bi-invariant metric.*

HINT (to (2)): *Show that $g = \lambda\, \omega_1 \otimes \omega_1$, where $\omega_1 = \mathrm{d}a/a$, is the general expression of a bi-invariant metric*; *but such a metric g is singular.*

Solution. (1) We have

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a' & 0 & 0 \\ 0 & a' & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/aa' & 0 & 0 \\ 0 & aa' & ab'+b \\ 0 & 0 & 1 \end{pmatrix} \in G \qquad (\star)$$

and

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1/a & -b/a \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Therefore, G is an abstract subgroup of $GL(3,\mathbb{R})$.

If a sequence in G of matrices $\begin{pmatrix} 1/a_n & 0 & 0 \\ 0 & a_n & b_n \\ 0 & 0 & 1 \end{pmatrix}$ goes as $n \to \infty$ to the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in GL(3,\mathbb{R}),$$

computing the limit we have

$$a_{12} = a_{13} = a_{21} = a_{31} = a_{32} = 0, \quad a_{33} = 1, \quad \lim_{n\to\infty} 1/a_n = a_{11}, \quad \lim_{n\to\infty} a_n = a_{22},$$

and thus $a_{11} \geqslant 0,\ a_{22} \geqslant 0,\ a_{11}a_{22} = 1$, so one has $a_{11} > 0,\ a_{22} > 0$, then $A \in G$. Hence, G is a closed subgroup of $GL(3,\mathbb{R})$.

(2) Suppose

$$X = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The equations of translations are

$$L_A \equiv \begin{cases} \bar{x} = ax \\ \bar{y} = ay + b \end{cases}, \qquad R_A \equiv \begin{cases} \bar{x} = ax \\ \bar{y} = bx + y. \end{cases}$$

A basis of left-invariant 1-forms is $\{\omega_1 = \mathrm{d}x/x,\ \omega_2 = \mathrm{d}y/x\}$. In fact,

$$L_A^*\omega_1 = \frac{\mathrm{d}\bar{x}}{\bar{x}} = \frac{\mathrm{d}x}{x} = \omega_1, \qquad L_A^*\omega_2 = \frac{\mathrm{d}\bar{y}}{\bar{x}} = \frac{\mathrm{d}y}{x} = \omega_2.$$

A basis of right-invariant differential 1-forms is

$$\{\bar{\omega}_1 = \omega_1,\ \bar{\omega}_2 = x\omega_2 - y\omega_1\}.$$

In fact,

$$\begin{aligned} R_A^*\bar{\omega}_1 &= \frac{\mathrm{d}\bar{x}}{\bar{x}} = \frac{\mathrm{d}x}{x} = \bar{\omega}_1, \\ R_A^*\bar{\omega}_2 &= \bar{x}\frac{\mathrm{d}\bar{y}}{\bar{x}} - \bar{y}\frac{\mathrm{d}\bar{x}}{\bar{x}} = \mathrm{d}y - \frac{y}{x}\,\mathrm{d}x = \bar{\omega}_2. \end{aligned}$$

Hence, the most general form of a left-invariant symmetric bilinear $(0,2)$ tensor is

$$g = \lambda\, \omega_1 \otimes \omega_1 + \mu(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + \nu\, \omega_2 \otimes \omega_2, \qquad \lambda, \mu, \nu \in \mathbb{R}.$$

Suppose g is also right-invariant. Since

$$\begin{aligned} R_A^*\omega_2 &= \frac{\mathrm{d}\bar{y}}{\bar{x}} \\ &= \frac{b\,\mathrm{d}x + \mathrm{d}y}{ax} \\ &= \frac{b}{a}\omega_1 + \frac{1}{a}\omega_2, \end{aligned}$$

we would have

$$\begin{aligned} R_A^*\,g = \lambda\, \omega_1 \otimes \omega_1 + \mu\left\{\omega_1 \otimes \left(\frac{b}{a}\omega_1 + \frac{1}{a}\omega_2\right) + \left(\frac{b}{a}\omega_1 + \frac{1}{a}\omega_2\right) \otimes \omega_1\right\} \\ + \nu\left(\frac{b}{a}\omega_1 + \frac{1}{a}\omega_2\right) \otimes \left(\frac{b}{a}\omega_1 + \frac{1}{a}\omega_2\right). \end{aligned}$$

If $R_A^*g = g$ for all $A \in G$, we necessarily have $\mu = \nu = 0$, thus $g = \lambda\ \omega_1 \otimes \omega_1$ is the most general expression of the bi-invariant metric. But it is singular.

Problem 6.15.13. *Let M be the pseudo-Euclidean space with metric $g = \sum_{i=1}^n \varepsilon_i \mathrm{d}x^i \otimes \mathrm{d}x^i$, $\varepsilon_i = \pm 1$, and let*

$$\Delta = -\sum_{i=1}^{n} \varepsilon_i \frac{\partial^2}{\partial (x^i)^2}$$

be the Laplacian on M.

Prove that the Laplace equation $\Delta f = 0$, $f \in C^\infty M$, has solution $f = \psi(\Omega)$, where

$$\psi(\Omega) = \begin{cases} A \log|\Omega| + B & \text{if } n = 2 \\ \dfrac{A}{|\Omega|^{\frac{1}{2}(n-2)}} + B & \text{if } n > 2, \end{cases}$$

and $\Omega = \frac{1}{2}\sum_{i=1}^n \varepsilon_i (x^i - x_0^i)^2$, in any neighborhood in which Ω does not vanish and has constant sign.

Solution. If $n = 2$, one has

$$\begin{aligned}
\Delta\psi(\Omega) &= -\sum_{i=1}^{2}\varepsilon_i \frac{\partial^2}{\partial(x^i)^2}(A\log|\Omega| + B) \\
&= -\sum_{i=1}^{2}\varepsilon_i A \frac{\partial}{\partial x^i}\frac{\varepsilon_i(x^i - x_0^i)}{|\Omega|} \\
&= -A\sum_{i=1}^{2}\frac{\Omega - (x^i - x_0^i)\varepsilon_i(x^i - x_0^i)}{\Omega^2} \\
&= -A\frac{2\Omega - 2\Omega}{\Omega^2} = 0.
\end{aligned}$$

For $n \geqslant 3$, we have

$$\begin{aligned}
\Delta\psi(\Omega) &= -\sum_{i=1}^{n}\varepsilon_i \frac{\partial^2}{\partial(x^i)^2}\left(\frac{A}{|\Omega|^{\frac{1}{2}(n-2)}} + B\right) \\
&= -A\sum_{i=1}^{n}\varepsilon_i \frac{\partial^2}{\partial(x^i)^2}\left|\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j(x^j - x_0^j)^2\right|^{-\frac{n}{2}+1} \\
&= -A\left(-\frac{n}{2}+1\right)\sum_{i=1}^{n}\varepsilon_i\frac{\partial}{\partial x^i}\left[\left(\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j(x^j - x_0^j)^2\right)^{-\frac{n}{2}}\cdot\left(\pm\varepsilon_i(x^i - x_0^i)\right)\right] \\
&= -A\left(-\frac{n}{2}+1\right)\sum_{i=1}^{n}\left[-\frac{n}{2}\varepsilon_i\left\{\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j(x^j - x_0^j)^2\right\}^{-\frac{n}{2}-1}\right. \\
&\qquad \left.\cdot\left\{\pm\varepsilon_i(x^i - x_0^i)\right\}\left\{\pm\varepsilon_i(x^i - x_0^i)\right\} + \left(\pm\frac{1}{2}\sum_{j=1}^{n}\varepsilon_j(x^j - x_0^j)^2\right)^{-\frac{n}{2}}(\pm 1)\right] \\
&= \begin{cases} -A(-\frac{n}{2}+1)(-n\Omega^{-\frac{n}{2}-1}\Omega + n\Omega^{-\frac{n}{2}}) = 0 & \text{if } \Omega > 0 \\ -A(-\frac{n}{2}+1)(n\Omega^{-\frac{n}{2}-1}\Omega - n\Omega^{-\frac{n}{2}}) = 0 & \text{if } \Omega < 0. \end{cases}
\end{aligned}$$

Problem 6.15.14. *Let* $(\mathbb{R}^2, g)$ *be the pseudo-Riemannian manifold with*

$$g = \frac{4}{c}(\cosh^2 y\,\mathrm{d}x^2 - \mathrm{d}y^2).$$

Calculate the Laplacian Δ *on functions* $f \in C^\infty\mathbb{R}^2$.

Solution. The nonvanishing Christoffel symbols are

$$\Gamma_{12}^1 = \tanh y, \qquad \Gamma_{11}^2 = \frac{1}{2}\sinh 2y.$$

Now applying the usual formula (valid in every local coordinate system)

$$\Delta f = -g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right),$$

we deduce

$$\Delta f = -\frac{c}{4}\left(\frac{1}{\cosh^2 y}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \tanh y \frac{\partial f}{\partial y}\right).$$

Problem 6.15.15. *Let $x\colon [a,b]\times(-\delta,\delta)\to M$ be a variation of a segment $\gamma(u) = x(u,0)$ of a geodesic on a Riemannian manifold (M,g). For each $v\in(-\delta,\delta)$, let $L_x(v)$ be the length of the longitudinal curve $u\to x(u,v)$. L_x is a real valued function, where $L_x(0)$ denotes the length of the given segment γ of the geodesic.*

Compute the second variation of arc length, $L_x''(0)$, by the usual formula and also directly from L_x, in the following cases (equipped with the respective usual metrics):

(1) *In S^2, $x(u,v) = (\cos v\cos u,\ \cos v\sin u,\ \sin v)$, $0\leqslant u\leqslant \pi$.*

(2) *In $\mathbb{R}^2$, $x(u,v) = (u\cosh v,\ v)$, $-1\leqslant u\leqslant 1$.*

(3) *In $\mathbb{R}^2$, $x(u,v) = \begin{cases}(u,\ vu) & \text{if } u\in[0,1]\\ (u,\ v(2-u)) & \text{if } u\in[1,2].\end{cases}$*

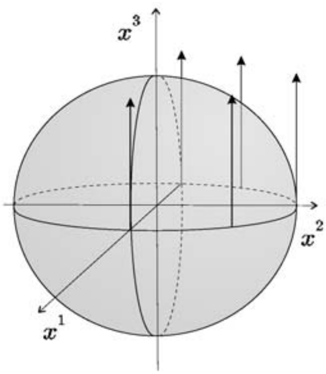

Fig. 6.12 The variation vector field on γ.

Solution. (1) The ends of $x(u,v)$ are $x(0,v) = (\cos v, 0, \sin v)$ and $x(\pi,v) = (-\cos v, 0, \sin v)$, for $v\in(-\delta,\delta)$. For $v=0$ we have the curve

$$\gamma : x(u,0) = (\cos u, \sin u, 0), \qquad 0\leqslant u\leqslant \pi,$$

with origin $(1,0,0)$ and end $(-1,0,0)$. For $v=-\delta$ we have the curve

$$x(u,-\delta) = (\cos\delta\cos u, \cos\delta\sin u, -\sin\delta),$$

with origin $(\cos\delta, 0, -\sin\delta)$ and end $(-\cos\delta, 0, -\sin\delta)$; for $v=\delta$,

$$x(u,\delta) = (\cos\delta\cos u, \cos\delta\sin u, \sin\delta).$$

The curve $x(u,0)$ is a segment of a geodesic. The length of $x(u,v)$, for a given v, is

$$\begin{aligned} L_x(v) &= \int_0^\pi \sqrt{(x_u(u,v))^2 \mathrm{d}u^2} \\ &= \int_0^\pi \cos v\, \mathrm{d}u \\ &= \pi\cos v. \end{aligned}$$

The length of γ is $L_x(0) = \pi$. The second variation of the arc on x is

$$\begin{aligned} L''(0) &= \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} \\ &= (-\pi\cos v)_{v=0} \\ &= -\pi, \end{aligned}$$

where $L = L_x$.

Since γ is a geodesic, it must be $L'(0) = 0$. In fact, we have

$$\begin{aligned} L'(0) &= (\mathrm{d}L/\mathrm{d}v)_{v=0} \\ &= (-\pi\sin v)_{v=0} = 0. \end{aligned}$$

As for Synge's formula (see page 416), since S^2 is a space of constant curvature 1, one has

$$g(R(V,\gamma')V,\gamma') = g(V,\gamma')g(V,\gamma') - g(V,V)g(\gamma',\gamma'),$$

where V denotes the variation vector field $V(u) = (\partial x/\partial v)_{v=0}$, given by

$$V(u) = (-\sin v\cos u,\ -\sin v\sin u,\ \cos v)_{v=0} = (0,0,1),$$

(see Figure 6.12) and $\gamma'(u) = (-\sin u, \cos u, 0)$, thus $c = |\gamma'| = 1$. Therefore

$$g(V,V) = 1, \qquad g(\gamma',\gamma') = 1, \qquad g(V,\gamma') = 0, \qquad g(R(V,\gamma')V,\gamma') = -1.$$

We have $V' = (0,0,0)$, thus $g(V'^{\perp}, V'^{\perp}) = 0$. On the other hand, the transverse acceleration vector field $A(u)$ on γ is given by

$$\begin{aligned} A(u) &= \left.\frac{\partial^2}{\partial v^2}\right|_{v=0} (\cos v\cos u,\ \cos v\sin u,\ \sin v) \\ &= (-\cos v\cos u,\ -\cos v\sin u,\ -\sin v)_{v=0} \\ &= (-\cos u,\ -\sin u,\ 0), \end{aligned}$$

from which $g(\gamma', A) = 0$ and $L''(0) = -\int_0^\pi \mathrm{d}u = -\pi$; that is, the same result as before.

(2) The ends of $x(u,v)$ are

$$x(-1,v) = (-\cosh v, v), \qquad x(1,v) = (\cosh v, v).$$

One has $v \in (-\delta, \delta)$. For $v = 0$ we have the curve in $\mathbb{R}^2$

$$\gamma\colon\, x(u,0) = (u,0), \qquad -1 \leqslant u \leqslant 1,$$

which is obviously a segment of a geodesic. For $v = -\delta$ we have the curve

$$x(u,-\delta) = (u\cosh\delta, -\delta),$$

with origin $(-\cosh\delta, -\delta)$ and end $(\cosh\delta, -\delta)$. For $v = \delta$, we have the curve

$$x(u,\delta) = (u\cosh\delta, \delta),$$

with origin $(-\cosh\delta, \delta)$ and end $(\cosh\delta, \delta)$. The length of $x(u,v)$ is

$$\begin{aligned} L_x(v) &= \int_{-1}^{1} (\cosh^2 v)^{\frac{1}{2}}\,\mathrm{d}u \\ &= 2\cosh v. \end{aligned}$$

The length of γ is $L_x(0) = 2$. The second variation of the arc on x is

$$\begin{aligned} L''(0) &= \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} \\ &= (2\cosh v)_{v=0} = 2. \end{aligned}$$

As γ is a geodesic, it must be $L'(0)$. In fact, we have

$$L'(0) = (2\sinh v)_{v=0} = 0.$$

As for Synge's formula, we have $c = |\gamma'| = 1$ and $R = 0$ as $M = \mathbb{R}^2$ with its usual metric. Moreover

$$\begin{aligned} V(u) &= \left.\frac{\partial}{\partial v}\right|_{v=0} (u\cosh v, v) \\ &= (u\sinh v, 1)_{v=0} = (0,1), \end{aligned}$$

and thus $V' = (0,0)$, so that $V'^{\perp} = (0,0)$. We have

$$\begin{aligned} A(u) &= \left.\frac{\partial^2}{\partial v^2}\right|_{v=0} x(u,v) \\ &= (u\cosh v, 0)_{v=0} = (u,0), \end{aligned}$$

so that one has $g(\gamma', A) = u$. Hence $L''(0) = [g(\gamma', A)]_{-1}^{1} = 2$, as before.

(3) For $v = 0$ we have the curve

$$\gamma\colon\, x(u,0) = (u,0), \qquad u \in [0,2],$$

which is a segment of a geodesic. For $v = -\delta$, we have the curve

$$x(u,-\delta) = \begin{cases} (u,-\delta u) & \text{if } u \in [0,1] \\ (u,-\delta(2-u)) & \text{if } u \in [1,2], \end{cases}$$

and for $v = \delta$ the symmetric one with respect to the u-axis. The length of $x(u,v)$ is

$$\begin{aligned} L_x(v) &= \int_0^1 (1+v^2)^{\frac{1}{2}}\,\mathrm{d}u + \int_1^2 (1+v^2)^{\frac{1}{2}}\,\mathrm{d}u \\ &= 2\sqrt{1+v^2}. \end{aligned}$$

The length of γ is $L_x(0) = 2$. The second variation of the arc on x is

$$\begin{aligned} L''(0) &= \left.\frac{\mathrm{d}^2 L}{\mathrm{d}v^2}\right|_{v=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}v}\right|_{v=0} \left(2\frac{v}{\sqrt{1+v^2}}\right) = 2. \end{aligned}$$

As γ is a geodesic, it must be $L'(0) = 0$. In fact,

$$L'(0) = \left(\frac{2v}{\sqrt{1+v^2}}\right)_{v=0} = 0.$$

As for Synge's formula, we have $c = |\gamma'| = 1$. Furthermore, one has $R = 0$, and

$$V = \left.\frac{\partial}{\partial v}\right|_{v=0} x(u,v) = \begin{cases} (0,u) & \text{if } u \in [0,1] \\ (0,2-u) & \text{if } u \in [1,2]. \end{cases}$$

$$V' = \begin{cases} (0,1) & \text{if } u \in [0,1] \\ (0,-1) & \text{if } u \in [1,2]. \end{cases}$$

$$g(V',\gamma') = \begin{cases} (0,1)\cdot(1,0) = 0 & \text{if } u \in [0,1] \\ (0,-1)\cdot(1,0) = 0 & \text{if } u \in [1,2]. \end{cases}$$

Thus $V'^{\perp} = V'$;

$$g(V'^{\perp},V'^{\perp}) = 1, \qquad A = \left.\frac{\partial^2}{\partial v^2}\right|_{v=0} x(u,v) = \begin{cases} (0,0) & \text{if } u \in [0,1] \\ (0,0) & \text{if } u \in [1,2]. \end{cases}$$

Hence, $L''(0) = \int_0^2 \mathrm{d}u = 2$, as before.

Problem 6.15.16. *Let M be an embedded submanifold of the paracomplex projective space $P(V \oplus V^*)$ (see* Problem 6.15.10*), such that the metric inherited on M from g is nondegenerate, and denote by $\mathscr{N}$ the normal bundle $\mathscr{N} = \bigcup_{p \in M} \mathscr{N}_p$, where $\mathscr{N}_p = (T_pM)^\perp$, which exists by the nondegeneracy of the induced metric. Such a submanifold is said to be totally umbilical if there exists $\xi \in \Gamma \mathscr{N}$ such that*

$$\alpha(X,Y) = g(X,Y)\xi, \qquad X,Y \in \mathfrak{X}(M),$$

where $\alpha(X,Y)$ is the second fundamental form and ξ is called the normal curvature vector field*. Then, for such a submanifold*:

(1) *Find the expression of Codazzi's equation.*

(2) *Find the expression of Ricci's equation.*

(3) *Prove, applying Gauss's equation, that if $J(TM) \subset \mathscr{N}$, then*

$$R(X,Y,Z,W) = \big(1 + g(\xi,\xi)\big)\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big).$$

Solution. (1) If ∇ denotes the Levi-Civita connection of any pseudo-Riemannian submanifold M, we have

$$\nabla_X Y = \tau \widetilde{\nabla}_X Y, \qquad \alpha(X,Y) = \nu \widetilde{\nabla}_X Y, \qquad A_\eta = -\tau \widetilde{\nabla}_X \eta, \qquad \nabla^\perp_X \eta = \nu \widetilde{\nabla}_X \eta,$$

where $X,Y \in \mathfrak{X}(M)$; $\eta \in \Gamma \mathscr{N}$; τ and ν denote the "tangential part" and the "normal part", respectively; $\widetilde{\nabla}$ is the Levi-Civita connection of $P(V \oplus V^*)$; $\nabla^\perp$ denotes the connection induced in $\mathscr{N}$; and

$$g(A_\eta X, Y) = g(\alpha(X,Y), \eta).$$

Codazzi's equation is written in general as

$$-\nu \widetilde{R}(X,Y)Z = \Big(\widehat{\nabla}_X \alpha\Big)(Y,Z) - \Big(\widehat{\nabla}_Y \alpha\Big)(X,Z),$$

where $\widehat{\nabla}_X \alpha$ is defined by

$$(\widehat{\nabla}_X \alpha)(Y,Z) = \nabla^\perp_X(\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

If the pseudo-Riemannian manifold M is moreover totally umbilical, then the previous equation reduces to

$$\begin{aligned}
(\widehat{\nabla}_X \alpha)(Y,Z) &= \nabla^\perp_X(g(Y,Z)\xi) - g(\nabla_X Y, Z)\xi - g(Y, \nabla_X Z)\xi \\
&= X(g(Y,Z))\xi + g(Y,Z)\nabla^\perp_X \xi - g(\nabla_X Y, Z)\xi - \big(g(Y, \nabla_X Z)\big)\xi \\
&= g(Y,Z)\nabla^\perp_X \xi + (\nabla_X g)(Y,Z) \\
&= g(Y,Z)\nabla^\perp_X \xi.
\end{aligned}$$

Hence, on account of the expression for the curvature of $P(V \oplus V^*)$ in the remark in Problem 6.15.10, we have for Codazzi's equation

$$\begin{aligned}
-\nu\widetilde{R}(X,Y)Z &= -\nu\big(g(X,Z)Y - g(Y,Z)X \\
&\qquad + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\big) \\
&= -g(X,JZ)\nu JY + g(Y,JZ)\nu JX + 2g(X,JY)\nu JZ \\
&= g(Y,Z)\nabla^{\perp}_{X}\xi - g(X,Z)\nabla^{\perp}_{Y}\xi.
\end{aligned}$$

That is,

$$g(X,\tau JZ)\nu JY - g(Y,\tau JZ)\nu JX + 2g(X,\tau JY)\nu JZ = g(Y,Z)\nabla^{\perp}_{X}\xi - g(X,Z)\nabla^{\perp}_{Y}\xi.$$

(2) Let $R_{\nabla^{\perp}}$ be the curvature tensor field of the connection $\nabla^{\perp}$ in $\mathscr{N}$. Then, Ricci's equation is is

$$\nu\widetilde{R}(X,Y)\eta = R_{\nabla^{\perp}}(X,Y)\eta - \alpha(A_\eta X,Y) + \alpha(A_\eta Y,X), \qquad X,Y \in \mathfrak{X}(M),\ \eta \in \mathscr{N}.$$

As

$$\begin{aligned}
g(A_\eta X,Y) &= g(\alpha(X,Y),\eta) \\
&= g(X,Y)g(\xi,\eta),
\end{aligned}$$

we have $A_\eta X = g(\xi,\eta)X$ and $\alpha(A_\eta X,Y) = g(\xi,\eta)g(X,Y)\xi$. Hence, Ricci's equation reduces to

$$\nu\widetilde{R}(X,Y)\eta = R_{\nabla^{\perp}}(X,Y)\eta.$$

(4) If $J(TM) \subset \mathscr{N}$, direct application of Gauss's equation gives us

$$\begin{aligned}
R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + g(\alpha(X,Z),\alpha(Y,W)) - g((\alpha(Y,Z),\alpha(X,W)) \\
&= g(X,Z)g(Y,W) - g(X,W)g(Y,Z) - g(X,JZ)g(Y,JW) \\
&\quad + g(X,JW)g(Y,JZ) - 2g(X,JY)g(Z,JW) \\
&\quad + g(X,Z)g(Y,W)g(\xi,\xi) - g(Y,Z)g(X,W)g(\xi,\xi) \\
&= \big(1+g(\xi,\xi)\big)\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big),
\end{aligned}$$

as wanted.

Chapter 7
Some Definitions and Theorems

7.1 Chapter 1. Differentiable Manifolds

Definitions 7.1.1. A *locally Euclidean space* is a topological space M such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$. If φ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, then U is called a *coordinate neighborhood*; φ is called a *coordinate map*; the functions $x^i = t^i \circ \varphi$, where t^i denotes the ith canonical coordinate function on $\mathbb{R}^n$, are called the *coordinate functions*; and the pair (U, φ) (or the set $(U, x^1, \dots, x^n)$) is called a *coordinate system* or a (local) *chart*. An *atlas* $\mathscr{A}$ of class C^∞ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying the following two properties:

(1) $\bigcup_{\alpha \in A} U_\alpha = M$.

(2) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^∞ for all $\alpha, \beta \in A$.

A *differentiable structure* (or *maximal atlas*) $\mathscr{F}$ on a locally Euclidean space M is an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of class C^∞, satisfying the above two properties (1) and (2) and moreover the condition:

(3) The collection $\mathscr{F}$ is maximal with respect to (2); that is, if (U, φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^∞, then $(U, \varphi) \in \mathscr{F}$.

A *topological manifold* of dimension n is a Hausdorff, second countable, locally Euclidean space of dimension n. A *differentiable manifold of class C^∞ of dimension n* (or simply *differentiable manifold of dimension n*, or *C^∞ manifold*, or *n-manifold*) is a pair $(M, \mathscr{F})$ consisting of a topological manifold M of dimension n, together with a differentiable structure $\mathscr{F}$ of class C^∞ on M.

The differentiable manifold $(M, \mathscr{F})$ is usually denoted by M, with the understanding that when one speaks of "the differentiable manifold" M one is considering the locally Euclidean space M with some given differentiable structure $\mathscr{F}$.

Let M and N be differentiable manifolds, of respective dimensions m and n. A map $\Phi \colon M \to N$ is said to be C^∞ provided that for every coordinate system (U, φ) on M and (V, ψ) on N, the composite map $\psi \circ \Phi \circ \varphi^{-1}$ is C^∞.

A *diffeomorphism* $\Phi\colon M \to N$ is a bijective C^∞ map such that the inverse map Φ^{-1} is also C^∞.

The tangent space T_pM to M at $p \in M$ is the space of real derivations of the local algebra $C^\infty_p M$ of germs of C^∞ functions at p, i.e. the $\mathbb{R}$-linear functions $X\colon C^\infty_p M \to \mathbb{R}$ such that

$$X(fg) = (Xf)g(p) + f(p)Xg, \qquad f, g \in C^\infty_p M.$$

Let $C^\infty M$ denote the algebra of differentiable functions of class C^∞ on M. The *differential map* Φ_{*p} of the C^∞ map $\Phi\colon M \to N$ is the map

$$\Phi_{*p}\colon T_pM \to T_{\varphi(p)}N, \quad (\Phi_{*p}X)(f) = X(f \circ \Phi), \qquad f \in C^\infty N.$$

Definitions 7.1.2. The *stereographic projection* from the north pole $N = (0,\dots,0,1)$ (resp. south pole $S = (0,\dots,0,-1)$) of the sphere

$$S^n = \left\{ (x^1,\dots,x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}$$

onto the equatorial plane $x^{n+1} = 0$ is the map sending $p \in S^n - \{N\}$ (resp. $p \in S^n - \{S\}$) to the point where the straight line through N (resp. S) and p intersects the plane $x^{n+1} = 0$.

The *inverse of the stereographic projection* is the map from $x^{n+1} = 0$ to $S^n - \{N\}$ (resp. $p \in S^n - \{S\}$) sending the point q in the plane $x^{n+1} = 0$ to the point where the straight line through q and N (resp. S) intersects S^n.

Other stereographic projections can be defined. For instance, that defined as above but for the sphere

$$S^n = \left\{ (x^1,\dots,x^{n+1} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x^i)^2 + (x^{n+1} - 1)^2 = 1 \right\},$$

from the north pole $N = (0,\dots,0,2)$ onto the plane $x^{n+1} = 0$. The inverse map is defined analogously to the previous case.

Definitions 7.1.3. Let $\Phi\colon M \to N$ be a C^∞ map. A point $p \in M$ is said to be a *critical point of* Φ if $\Phi_*\colon T_pM \to T_{\Phi(p)}N$ is not surjective. Let $f \in C^\infty M$. A point $p \in M$ is called a *critical point of* f if $f_{*p} = 0$. If we choose a coordinate system $(U, x^1,\dots,x^n)$ around $p \in M$, this means that

$$\frac{\partial f}{\partial x^1}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0.$$

The real number $f(p)$ is then called a *critical value of* f. A critical point is called *nondegenerate* if the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right)$$

is non-singular. The nondegeneracy does not depend on the choice of coordinate system.

If p is a critical point of f, then the *Hessian* H^f *of* f *at* p is a bilinear function on T_pM defined as follows. If $v, w \in T_pM$ and $X \in \mathfrak{X}(M)$ satisfies $X_p = v$, then

$$H^f(v,w) = w(Xf).$$

Definitions 7.1.4. Let S be a subset of $\mathbb{R}^n$. Then S has *measure zero* if for every $\varepsilon > 0$, there is a covering of S by a countable number of open cubes $C_1, C_2, \ldots$ such that the Euclidean volume $\sum_{i=1}^{\infty} \operatorname{vol}(C_i) < \varepsilon$.

A subset S of a differentiable n-manifold M has *measure zero* if there exists a countable open covering $U_1, U_2, \ldots$ of S and charts $\varphi_i \colon U_i \to \mathbb{R}^n$ such that $\varphi_i(U \cap S)$ has measure zero in $\mathbb{R}^n$.

Theorem 7.1.5. (Sard's Theorem) *Let* $\Phi \colon M \to N$ *be a* C^∞ *map. Then the set of critical values of* Φ *has measure zero.*

Definitions 7.1.6. Let $\Phi \colon M \to N$ be a C^∞ map. Then:

(1) Φ is an *immersion* if Φ_{*p} is injective for each $p \in M$.

(2) The pair (M, Φ) is a *submanifold* of N if Φ is a one-to-one immersion. If the inclusion map of M in N is a one-to-one immersion, then it is said that M is a submanifold of N.

(3) Φ is an *embedding* if Φ is a one-to-one immersion which is also a homeomorphism into; that is, the induced map $\Phi \colon M \to \Phi(M)$ is open when $\Phi(M)$ is endowed with the topology inherited from that of N.

(4) Φ is an *submersion* if Φ_{*p} is surjective for all $p \in M$.

Definition 7.1.7. Let $\Phi \colon M \to N$ be a C^∞ map, with $\dim M = m$, $\dim N = n$, and let $p \in M$. If (U, φ), (V, ψ) are coordinate systems around p and $\Phi(p)$, respectively, and $\Phi(U) \subset V$, then one has a corresponding expression for Φ in local coordinates, i.e.,

$$\widetilde{\Phi} = \psi \circ \Phi \circ \varphi^{-1} \colon \varphi(U) \to \psi(V).$$

The *rank of* Φ *at* p is defined to be the rank of Φ_{*p}, which is equal to the rank of the Jacobian matrix

$$\left(\frac{\partial f^i}{\partial x^j} (\varphi(p)) \right), \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m,$$

of the map

$$\widetilde{\Phi}(x^1, \ldots, x^m) = \left(f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m) \right),$$

expressing Φ in local coordinates.

Theorem 7.1.8. (Theorem of the Rank) *Let $\Phi\colon M \to N$ be a C^∞ map, with* $\dim M = m$, $\dim N = n$, *and* $\operatorname{rank} \Phi = r$ *at every point of M. If $p \in M$, then there exist coordinate systems (U,φ), (V,ψ) as above such that $\varphi(p) = (0,\dots,0) \in \mathbb{R}^m$, $\psi(\Phi(p)) = (0,\dots,0) \in \mathbb{R}^n$, and $\widetilde{\Phi} = \psi \circ \Phi \circ \varphi^{-1}$ is given by*

$$\widetilde{\Phi}(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

Moreover we can assume $\varphi(U) = C^m_\varepsilon(0)$, $\varphi(V) = C^n_\varepsilon(0)$, with the same ε, where $C^n_\varepsilon(0)$ denotes the cubic neighborhood centered at $0 \in \mathbb{R}^n$ *of edge* 2ε.

Theorem 7.1.9. (Inverse Map Theorem) *Let*

$$f = (f^1,\dots,f^n)\colon U \to \mathbb{R}^n$$

be a C^∞ map defined on an open subset $U \subseteq \mathbb{R}^n$. Given a point $x_0 \in U$, assume

$$\frac{\partial(f^1,\dots,f^n)}{\partial(x^1,\dots,x^n)}(x_0) \neq 0.$$

Then there exists an open neighborhood $V \subseteq U$ of x_0 such that:

(i) *$f(V)$ is an open subset of $\mathbb{R}^n$.*
(ii) *$f\colon V \to f(V)$ is one-to-one.*
(iii) *$f^{-1}\colon f(V) \to V$ is C^∞.*

Theorem 7.1.10. (Implicit Map Theorem) *Denote the coordinates on $\mathbb{R}^n \times \mathbb{R}^m$ by $(x^1,\dots,x^n,y^1,\dots,y^m)$. Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open subset, and let*

$$f = (f^1,\dots,f^m)\colon U \to \mathbb{R}^m$$

be a C^∞ map. Given a point $(x_0,y_0) \in U$, assume:

(i) $f(x_0,y_0) = 0$.
(ii)
$$\frac{\partial(f^1,\dots,f^m)}{\partial(y^1,\dots,y^m)}(x_0,y_0) \neq 0.$$

Then, there exist an open neighborhood V of x_0 in $\mathbb{R}^n$ and an open neighborhood W of y_0 in $\mathbb{R}^m$ such that $V \times W \subset U$, and there exists a unique C^∞ map $g\colon V \to \mathbb{R}^m$, such that for each $(x,y) \in V \times W$:

$$f(x,y) = 0 \quad \Leftrightarrow \quad y = g(x).$$

Theorem 7.1.11. (Implicit Map Theorem for Submersions) *Consider a surjective submersion $\pi\colon M \to N$. Then, for every $q \in \operatorname{im}\pi$, the fibre $\pi^{-1}(q)$ is a closed submanifold in M and* $\dim \pi^{-1}(q) = \dim M - \dim N$.

Definition 7.1.12. Let $\sim$ be an equivalence relation in M, and let $\pi\colon M \to M/{\sim}$ be the quotient map. Endow $M/{\sim}$ with the quotient topology τ, i.e.

$$U \in \tau \quad \Leftrightarrow \quad \pi^{-1}(U) \text{ is open in the topology of } M.$$

The *quotient manifold* of M modulo $\sim$ is said to exist if there is a (necessarily unique) differentiable manifold structure on $M/\!\sim$ such that π is a submersion.

The following criterion is often used to construct quotient manifolds:

Theorem 7.1.13. (Theorem of the Closed Graph) *Let $\sim$ be an equivalence relation in M and let $N \subset M \times M$ be the graph of $\sim$; that is,*

$$N = \{(p,q) \in M \times M \,:\, p \sim q\}.$$

The quotient manifold $M/\!\sim$ exists if and only if the following two conditions hold true:

(1) *N is a closed embedded submanifold of $M \times M$.*

(2) *The restriction $\pi\colon N \to M$ to N of the canonical projection* $\mathrm{pr}_1 : M \times M \to M$ *onto the first factor, is a submersion.*

Definitions 7.1.14. Let M be a differentiable n-manifold with differentiable structure $\mathscr{F}$. Let

$$TM = \bigcup_{p \in M} T_pM.$$

There is a natural projection $\pi\colon TM \to M$, given by $\pi(v) = p$ for any $v \in T_pM$. Let $(U,\varphi) = (U,x^1,\dots,x^n) \in \mathscr{F}$. Define $\widetilde{\varphi}\colon \pi^1(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(v) = \big((x^1 \circ \pi)(v),\dots,(x^n \circ \pi)(v),\mathrm{d}x^1(v),\dots,\mathrm{d}x^n(v)\big),$$

for all $v \in \pi^{-1}(U)$. Then the collection of such $(\pi^{-1}(U),\widetilde{\varphi})$ determines on TM a differentiable structure $\widetilde{\mathscr{F}}$ with which TM is called the *tangent bundle* over M.

A *vector field along a curve* $\gamma\colon [a,b] \to M$ in the differentiable manifold M is a C^∞ map $X\colon [a,b] \to TM$ satisfying $\pi \circ X = \gamma$. A *vector field X on M* is a C^∞ section $X\colon M \to TM$. If $f \in C^\infty U$, then Xf is the function on U whose value at $p \in M$ is X_pf. The vector fields on M are usually identified to the derivations of C^∞ functions, that is to the $\mathbb{R}$-linear maps $X\colon C^\infty M \to C^\infty M$ such that $X(fg) = (Xf)g + f(Xg)$. The $(C^\infty M)$-module of vector fields on M is denoted by $\mathfrak{X}(M)$.

If X and Y are vector fields on M, the *Lie bracket* $[X,Y]$ of X and Y is the vector field on M defined by

$$[X,Y]_p(f) = X_p(Yf) - Y_p(Xf), \qquad p \in M.$$

Let $X \in \mathfrak{X}(M)$. A C^∞ curve γ in M is said to be an *integral curve of X* if

$$\gamma'(t_0) = \gamma_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) = X_{\gamma(t_0)}.$$

Definitions 7.1.15. A vector field is said to be *complete* if each of its maximal integral curves is defined on the entire real line $\mathbb{R}$.

The *flow* or 1-*parameter group* of a complete vector field X on M is the map

$$\begin{array}{rcl}\varphi\colon M\times\mathbb{R} & \to & M\\ (p,t) & \mapsto & \varphi_t(p),\end{array}$$

where $t \mapsto \varphi_t(p)$ is the maximal integral curve of X with initial point p.

Definitions 7.1.16. Let $\Phi\colon M \to N$ be a C^∞ map. The vector fields $X, Y \in \mathfrak{X}(M)$ are said to be Φ*-related* if

$$\Phi_* X_p = Y_{\Phi(p)}, \qquad p \in M.$$

Let $\Phi\colon M \to N$ be a diffeomorphism. Given $X \in \mathfrak{X}(M)$, the *vector field image* $\Phi \cdot X$ of X is defined by

$$(\Phi \cdot X)_x = \Phi_* \left(X_{\Phi^{-1}(x)}\right).$$

That is, $\Phi \cdot X$ is a shortening for the section $\Phi_* \circ X \circ \Phi^{-1}$ of TN.

7.2 Chapter 2. Tensor Fields. Differential Forms

Definitions 7.2.1. Let $\xi = (E, \pi, M)$ be a locally trivial bundle with fibre F over M. A *chart* on ξ is a pair (U, Ψ) consisting of an open subset $U \subset M$ and a diffeomorphism $\Psi\colon \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \Psi = \pi$, where $\mathrm{pr}_1 : U \times F \to U$ is the first projection map. Ψ is called a *trivialization of* ξ *over* U.

Let V be real vector space of finite dimension n and let $\xi = (E, \pi, M)$ be a locally trivial bundle of fibre V. A structure of *vector bundle on* ξ is given by a family $\mathscr{A} = \{(U_\alpha, \Psi_\alpha)\}$ of charts on ξ satisfying:

(1) U_α is an open covering of the base space M.

(2) For each pair (α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$, one has

$$\left(\Psi_\beta \circ \Psi_\alpha^{-1}\right)(p, v) = (p, g_{\beta\alpha}(x)v), \qquad (p, v) \in (U_\alpha \cap U_\beta) \times V,$$

where $g_{\alpha\beta}$ is a C^∞ map from $U_\alpha \cap U_\beta$ to the group $GL(V)$ of automorphisms of V.

(3) If $\mathscr{A}' \supset \mathscr{A}$ is a family of charts on ξ satisfying the properties (1), (2) above, then $\mathscr{A}' = \mathscr{A}$.

Such a bundle $\xi = (E, \pi, M, \mathscr{A})$, or simply $\xi = (E, \pi, M)$, is called a (real) *vector bundle* of rank n. The C^∞ maps $g_{\alpha\beta} : M \to GL(V)$ are called the *changes of charts* of the atlas $\mathscr{A}$.

Proposition 7.2.2. *The changes of charts of a vector bundle have the property (called the* cocycle condition)

$$g_{\alpha\gamma}(p) g_{\gamma\beta}(p) = g_{\alpha\beta}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 7.2.3. Two vector bundles of rank n are said to be *equivalent* if they are isomorphic and have the same base space B.

One has the following converse to Proposition 7.2.2:

Theorem 7.2.4. *Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of a differentiable manifold M and let V be a finite-dimensional real vector space. Let $g_{\alpha\beta}\colon M \to GL(V)$, $U_\alpha \cap U_\beta \neq \emptyset$, be a family of C^∞ maps satisfying the cocycle condition (7.2.2). Then there exists a real vector bundle $\xi = (E,\pi,M,\mathscr{A})$, unique up to equivalence, such that the maps $g_{\alpha\beta}$ are the changes of charts of the atlas $\mathscr{A}$.*

Definition 7.2.5. The family $(U_\alpha, g_{\alpha\beta})$ is said to be a $GL(V)$-valued *cocycle* on M subordinated to the open covering $\mathscr{U}$.

Definitions 7.2.6. Let $\mathscr{T}_s{}^r(M)$ be the set of tensor fields of type (r,s) on a differentiable manifold M and write $\mathscr{T}(M) = \oplus_{r,s=0}^\infty \mathscr{T}_s{}^r(M)$. A *derivation D* of $\mathscr{T}(M)$ is a map of $\mathscr{T}(M)$ into itself satisfying:

(1) D is linear and satisfies

$$D_X(T_1 \otimes T_2) = D_X T_1 \otimes T_2 + T_1 \otimes D_X T_2, X \in \mathfrak{X}(M), \qquad T_1, T_2 \in \mathscr{T}(M).$$

(2) D_X is type-preserving: $D_X(\mathscr{T}_s{}^r(M)) \subset \mathscr{T}_s{}^r(M)$.

(3) D_X commutes with every contraction of a tensor field.

Let $\Lambda^r M$ be the space of differential forms of degree r on the n-manifold M, that is, skew-symmetric covariant tensor fields of degree r. With respect to the exterior product, $\Lambda^* M = \oplus_{r=0}^n \Lambda^r M$ is an algebra over $\mathbb{R}$. A *derivation* (resp. *anti-derivation*) of $\Lambda^* M$ is a linear map of $\Lambda^* M$ into itself satisfying

$$D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + \omega_1 \wedge D\omega_2, \qquad \omega_1, \omega_2 \in \Lambda^* M$$

(resp.

$$D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge D\omega_2, \qquad \omega_1 \in \Lambda^r M, \quad \omega_2 \in \Lambda^* M.)$$

A derivation or anti-derivation D of $\Lambda^* M$ is said to be *of degree k* if it maps $\Lambda^r M$ into $\Lambda^{r+k} M$ for every r.

Theorem 7.2.7. (Exterior differentiation) *There exists a unique antiderivation*

$$\mathrm{d}\colon \Lambda^* M \to \Lambda^* M$$

of degree $+1$ *such that*:

(1) $\mathrm{d}^2 = 0$.

(2) *Whenever $f \in C^\infty M = \Lambda^0 M$, $\mathrm{d}f$ is the differential of f.*

Definitions 7.2.8. Fix a vector field X on M and let φ_t be the local 1-parameter group of transformations associated with X. Let Y be another vector field on M. The *Lie derivative of Y with respect to X at $p \in M$* is the vector $(L_X Y)_p$ defined by

$$(L_X Y)_p = \lim_{t\to 0} \frac{Y_p - \varphi_{t*} Y_{\varphi_t^{-1}(p)}}{t} = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\varphi_{t*} Y_{\varphi_t^{-1}(p)}\right).$$

The *Lie derivative of a differential form ω with respect to X at p* is defined by

$$(L_X \omega)_p = \lim_{t\to 0} \frac{\omega_p - \varphi_t^* \omega_{\varphi_t(p)}}{t} = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\varphi_t^* \omega_{\varphi_t(p)}\right).$$

The *Lie derivative of a tensor field T of type (r,s) with respect to X at p* is defined by

$$(L_X T)_p = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_t \cdot T)_p,$$

where the dot denotes, for an arbitrary diffeomorphism Φ of M,

$$\begin{aligned}\Phi \cdot (X_1 \otimes \cdots \otimes X_r \otimes \theta_1 \otimes \cdots \otimes \theta_s)& \\ = \Phi \cdot X_1 \otimes \cdots \otimes \Phi \cdot X_r &\otimes (\Phi^{-1})^* \theta_1 \otimes \cdots \otimes (\Phi^{-1})^* \theta_s,\end{aligned}$$

$X_i \in \mathfrak{X}(M)$, $\theta_j \in \Lambda^1 M$.

In particular, the action of Φ on a differential form $\theta \in \Lambda^1 M$ is given by

$$(\Phi \cdot \theta)_p = \theta_{\Phi^{-1}(p)} \circ \left(\Phi^{-1}\right)_* = \left(\left(\Phi^{-1}\right)^* \theta\right)_p, \qquad p \in M.$$

For each $X \in \mathfrak{X}(M)$, the *interior product with respect to X* is the unique anti-derivation of degree -1 defined by $i_X f = 0$, $f \in C^\infty M$, and $i_X \theta = \theta(X)$, $\theta \in \Lambda^1 M$.

Theorem 7.2.9. *Let $X \in \mathfrak{X}(M)$. Then*:

(1) *$L_X f = Xf, \quad f \in C^\infty M$.*

(2) *$L_X Y = [X,Y], \quad Y \in \mathfrak{X}(M)$.*

(3) *L_X maps $\Lambda^* M$ to $\Lambda^* M$, and it is a derivation which commutes with the exterior differentiation* d.

(4) *On $\Lambda^* M$, we have*

$$L_X = i_X \circ \mathrm{d} + \mathrm{d} \circ i_X,$$

where i_X denotes the interior product with respect to X.

Proposition 7.2.10. *Let φ_t a local 1-parameter group of local transformations generated by a vector field X on M. For any tensor field T on M, we have*

$$\varphi_s \cdot (L_X T) = -\left(\frac{\mathrm{d}}{\mathrm{d}t} (\varphi_t \cdot T)\right)_{t=s}.$$

Definitions 7.2.11. Let m,n be integers, $1 \leqslant m \leqslant n$. An *$m$-dimensional distribution* $\mathscr{D}$ on an n-dimensional manifold M is a choice of an m-dimensional subspace $\mathscr{D}_p$ of $T_p M$ for each $p \in M$. $\mathscr{D}$ is C^∞ if for each $p \in M$ there are a neighborhood U of p and m vector fields $X_1, \ldots, X_m$ on U which span $\mathscr{D}$ at each point in U. A vector field is said to *belong to* (or *lie in*) the distribution $\mathscr{D}$, if $X_p \in \mathscr{D}_p$ for each $p \in M$. Then one writes $X \in \mathscr{D}$. A C^∞ distribution is called *involutive* (or *completely integrable*) if $[X,Y] \in \mathscr{D}$ whenever X and Y are vector fields lying in $\mathscr{D}$.

A submanifold (N,ψ) of M is an *integral manifold* of a distribution $\mathscr{D}$ on M if

$$\psi_*(T_qN) = \mathscr{D}_{\psi(q)}, \qquad q \in N.$$

Definitions 7.2.12. Let $\mathscr{D}$ be an r-dimensional C^∞ distribution on M. A differential s-form ω is said to *annihilate* $\mathscr{D}$ if, for each $p \in M$,

$$\omega_p(v_1,\dots,v_s) = 0 \quad \text{whenever} \quad v_1,\dots,v_s \in \mathscr{D}_p.$$

A differential form $\omega \in \Lambda^*M$ is said to annihilate $\mathscr{D}$ if each of the homogeneous parts of ω annihilates $\mathscr{D}$. Let

$$\mathscr{I}(\mathscr{D}) = \{\omega \in \Lambda^*M : \omega \text{ annihilates } \mathscr{D}\}.$$

A function $f \in C^\infty M$ is said to be a *first integral of* $\mathscr{D}$ if $\mathrm{d}f$ annihilates $\mathscr{D}$. An ideal $\mathscr{I} \subset \Lambda^*M$ is called a *differential ideal* if it is closed under exterior differentiation d; that is, $\mathrm{d}\mathscr{I} \subset \mathscr{I}$.

Proposition 7.2.13. *A C^∞ distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.*

Theorem 7.2.14. (Frobenius' Theorem) *Let $\mathscr{D}$ be an m-dimensional, involutive, C^∞ distribution on M. Let $p \in M$. Then through p there passes a unique maximal connected integral manifold of $\mathscr{D}$, and every connected integral manifold of $\mathscr{D}$ through p is contained in the maximal one.*

Definitions 7.2.15. In the conditions of Theorem 7.2.14 it is said that the involutive distribution $\mathscr{D}$ is a *foliation*, M is said to be *a foliated manifold*, the unique maximal connected integral manifold of $\mathscr{D}$ through each point is called a *leave* of the foliation, and the foliation is said to be of *codimension* $n-m$.

Definitions 7.2.16. Let T^*M be the cotangent bundle over a differentiable manifold M of dimension n and let $\pi\colon T^*M \to M$ be the natural projection. The *canonical* 1-*form* ϑ on T^*M is defined by

$$\vartheta_\omega(X) = \omega(\pi_*X), \qquad \omega \in T^*M, \quad X \in T_\omega T^*M.$$

An *almost symplectic manifold* is a differentiable manifold M endowed with a non-degenerate differential 2-form Ω. In this case, $\dim M = 2n$, and

$$\frac{(-1)^n}{n!}\,\Omega \wedge \overset{(n}{\cdots} \wedge \Omega$$

is a volume form on M, called the *standard volume form* associated with Ω. A *symplectic manifold* is an almost symplectic manifold whose corresponding 2-form is closed: $\mathrm{d}\Omega = 0$.

Theorem 7.2.17. (Darboux's Theorem) *If (M,Ω) is a symplectic manifold of dimension $2n$, then for every $p \in M$ there exists a chart $(U,x^1,\ldots,x^n,y^1,\ldots,y^n)$ centered at p such that*

$$\Omega|_U = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i.$$

7.3 Chapter 3. Integration on Manifolds

Definitions 7.3.1. Let V be a real vector space of dimension n. An *orientation* of V is a choice of component of $\Lambda^n V - \{0\}$.

A connected differentiable manifold M of dimension n is said to be *orientable* if it is possible to choose in a consistent way an orientation on T_p^*M for each $p \in M$. More precisely, let O be the "0-section" of the exterior n-bundle $\Lambda^n M^*$; that is,

$$O = \bigcup_{p\in M} \{0 \in \Lambda^n T_p^* M\}.$$

Then since $\Lambda^n T_p^*M - \{0\}$ has exactly two components, it follows that $\Lambda^n T^*M - O$ has at most two components. It is said that M is *orientable* if $\Lambda^n T^*M - O$ has two components; and if M is orientable, an *orientation* is a choice of one of the two components of $\Lambda^n T^*M - O$. It is said that M is *non-orientable* if $\Lambda^n T^*M - O$ is connected.

Let M and N be two orientable differentiable n-manifolds, and let $\Phi\colon M \to N$ be a differentiable map. It is said that Φ *preserves orientations* or that it is *orientation-preserving* if $\Phi_*\colon T_pM \to T_{\Phi(p)}N$ is an isomorphism for every $p \in M$, and the induced map $\Phi^*\colon \Lambda^n T^*N \to \Lambda^n T^*M$ maps the component $\Lambda^n T^*M - O$ determining the orientation of N into the component $\Lambda^n T^*M - O$ determining the orientation of M. Equivalently, Φ is orientation-preserving if Φ_* sends oriented bases of the tangent spaces to M to oriented bases of the tangent spaces to N.

Proposition 7.3.2. *Let M be a connected differentiable manifold of dimension n. Then the following are equivalent*:

(1) *M is orientable.*

(2) *There is a collection $\mathscr{C} = \{(U,\varphi)\}$ of coordinate systems on M such that*

$$M = \bigcup_{(U,\varphi)\in\mathscr{C}} U \quad \textit{and} \quad \det\left(\frac{\partial x^i}{\partial y^j}\right) > 0 \quad \textit{on} \quad U\cap V$$

whenever $(U,x^1,\ldots,x^n)$ and $(V,y^1,\ldots,y^n)$ belong to $\mathscr{C}$.

(3) *There is a nowhere-vanishing differential n-form on M.*

Theorem 7.3.3. (Stokes' Theorem I) *Let c be an r-chain in M, and let ω be a C^∞ $(r-1)$-form defined on a neighborhood of the image of c. Then*

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega.$$

Theorem 7.3.4. (Green's Theorem) *Let $\sigma(t) = (x(t)), y(t)),\ t \in [a,b]$ be a simple, closed, plane curve. Suppose that σ is positively oriented (that is, $\sigma|_{(a,b)}$ is orientation-preserving) and let D denote the bounded closed connected domain whose boundary is σ. Let $f = f(x,y)$ and $g = g(x,y)$ be real functions with continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial g/\partial x$, $\partial g/\partial y$ on D. Then*

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_\sigma \left(f\frac{\mathrm{d}x}{\mathrm{d}t} + g\frac{\mathrm{d}y}{\mathrm{d}t}\right) \mathrm{d}t.$$

Definition 7.3.5. Let M be a differentiable manifold. A subset $D \subseteq M$ is said to be a *regular domain* if for every $p \in \partial D$ there exists a chart $(U,\varphi) = (U, x^1, \ldots, x^n)$ centered at p such that

$$\varphi(U \cap D) = \{x \in \varphi(U) : x^n \geqslant 0\}.$$

Theorem 7.3.6. (Stokes' Theorem II) *Let D be a regular domain in an oriented n-dimensional manifold M, and let ω be a differential $(n-1)$-form on M such that* $\operatorname{supp}(\omega) \cap \bar{D}$ *is compact. Then*

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

Definitions 7.3.7. A differential r-form α on M is said to be *closed* if $\mathrm{d}\alpha = 0$. It is called *exact* if there is an $(r-1)$-form β such that $\alpha = \mathrm{d}\beta$. Since $\mathrm{d}^2 = 0$, every exact form is closed. The quotient space of closed r-forms modulo the space of exact r-forms is called the rth *de Rham cohomology group of M*:

$$H^r_{dR}(M,\mathbb{R}) = \{\text{closed } r\text{-forms}\}/\{\text{exact } r\text{-forms}\}.$$

If $\Phi\colon M \to N$ is differentiable, then $\Phi^*\colon \Lambda^*N \to \Lambda^*M$ transforms closed (resp. exact) forms into closed (resp. exact) forms. Hence Φ induces a linear map

$$\Phi^*\colon H^r_{dR}(N,\mathbb{R}) \to H^r_{dR}(M,\mathbb{R}).$$

7.4 Chapter 4. Lie Groups

Definitions 7.4.1. A *Lie group G* is a differentiable manifold endowed with a group structure such that the map $G \times G \to G$, $(s,t) \mapsto st^{-1}$, is C^∞.

Let G and H be Lie groups. A map $\Phi\colon G \to H$ is a *homomorphism of Lie groups* if it is a group homomorphism and a C^∞ map of differentiable manifolds. Φ is said to be an *isomorphism* if it is moreover a diffeomorphism. Let G and H be two Lie groups and consider a homomorphism of G into the abstract group of automorphisms of H, $\rho\colon G \to \operatorname{Aut} H$. The *semidirect product* $H \times_\rho G$ of H and G with respect to ρ is the product manifold $H \times G$, endowed with the Lie group structure given by

$$(h,g)(h',g') = (h\rho(g)h', gg'), \qquad (h,g)^{-1} = (\rho(g^{-1})h^{-1}, g^{-1}),$$

for $h,h' \in H$, $g,g' \in G$.

A *Lie algebra* over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the *bracket*) such that for all $x,y,z \in \mathfrak{g}$:

(1) $[X,Y] = -[Y,X]$ (anti-commutativity).

(2) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity).

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras* if it is linear and preserves brackets. φ is said to be an *isomorphism* if it is moreover one-to-one and surjective.

The *Lie algebra of the Lie group* G is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. There exists an isomorphism of vector spaces

$$\mathfrak{g} \to T_eG, \qquad X \mapsto X_e.$$

In other words, a left-invariant vector field is completely determined by its value at the identity. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and let $\rho\colon \mathfrak{g} \to \operatorname{End}\mathfrak{h}$ be a homomorphism such that every operator $\rho(Y)$, $Y \in \mathfrak{g}$, is a derivation of $\mathfrak{h}$. The *semidirect product* $\mathfrak{h} \oplus_\rho \mathfrak{g}$ of $\mathfrak{h}$ and $\mathfrak{g}$ with respect to ρ is the direct sum vector space $\mathfrak{h} \oplus \mathfrak{g}$, endowed with the Lie algebra structure given by the bracket

$$[(X,Y),(X',Y')] = ([X,X'] + \rho(Y)X' - \rho(Y')X, [Y,Y']),$$

for $X,X' \in \mathfrak{h}$, $Y,Y' \in \mathfrak{g}$.

Theorem 7.4.2. (Cartan's criterion on closed subgroups) *Let G be a Lie group, and let H be a closed abstract subgroup of G. Then H has a unique manifold structure which makes H into a Lie subgroup of G.*

Definitions 7.4.3. A Lie group G acts on itself on the left by inner automorphisms:

$$\iota\colon G \times G \to G, \qquad \iota(s,t) = sts^{-1} = \iota_s(t).$$

The map $s \mapsto \iota_{*s}|_{T_eG}$ is, under the identification as vector spaces of T_eG with the Lie algebra $\mathfrak{g}$ of G, a homomorphism of G into the group of automorphisms $\operatorname{Aut}\mathfrak{g}$ of the vector space $\mathfrak{g}$, called the *adjoint representation of G* and denoted by

$$\operatorname{Ad}\colon G \to \operatorname{Aut}\mathfrak{g}.$$

The differential map of Ad, denoted by ad, is a homomorphism of $\mathfrak{g}$ into the Lie algebra $\operatorname{End}\mathfrak{g}$ of endomorphisms of the vector space $\mathfrak{g}$, called the *adjoint representation of the Lie algebra* $\mathfrak{g}$. One has

$$\mathrm{ad}_X Y = [X,Y], \qquad X,Y \in \mathfrak{g}.$$

Definitions 7.4.4. Suppose $n = 2r$ is even. Let s_0 denote the $r \times r$ matrix

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

with 1 in the skew diagonal and 0 elsewhere. Set

$$J_+ = \begin{pmatrix} \mathbf{0} & s_0 \\ s_0 & \mathbf{0} \end{pmatrix}, \qquad J_- = \begin{pmatrix} \mathbf{0} & s_0 \\ -s_0 & \mathbf{0} \end{pmatrix},$$

and define the bilinear forms

$$B(z,w) = (z, J_+ w), \quad \Omega(z,w) = (z, J_- w), \qquad z,w \in \mathbb{C}^n.$$

The form B, with $B(z,w) = z^1 w^{2r} + \cdots + z^{2r} w^1$, is nondegenerate and symmetric. The form Ω, with

$$\Omega(z,w) = -z^1 w^{2r} - \cdots - z^r w^{r+1} + z^{r+1} w^r + \cdots + z^{2r} w^1,$$

is nondegenerate and skew-symmetric.

Proposition 7.4.5. *Let $SO(\mathbb{C}^{2r},B)$ be the Lie group of complex matrices preserving the bilinear form B and having determinant 1. The Lie algebra $\mathfrak{so}(\mathbb{C}^{2r},B)$ of $SO(\mathbb{C}^{2r},B)$ consists of all matrices*

$$A = \begin{pmatrix} a & b \\ c & -s_0\,{}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r,\mathbb{C})$, and b,c are $r \times r$ matrices such that

$${}^t b = -s_0 b s_0, \quad {}^t c = -s_0 c s_0$$

(*that is, b and c are skew-symmetric around the skew diagonal*).

Let $Sp(\mathbb{C}^{2r},\Omega)$ be the Lie group of complex matrices preserving the bilinear form Ω. The Lie algebra $\mathfrak{sp}(\mathbb{C}^{2r},\Omega)$ of $Sp(\mathbb{C}^{2r},\Omega)$ consists of all matrices

$$A = \begin{pmatrix} a & b \\ c & -s_0\,{}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r,\mathbb{C})$, b,c are $r \times r$ matrices such that ${}^t b = s_0 b s_0$, ${}^t c = s_0 c s_0$ (that is, b and c are symmetric around the skew diagonal).

Suppose now $n = 2r+1$. One then embeds the group $SO(\mathbb{C}^{2r},B)$ into the group $SO(\mathbb{C}^{2r+1},B)$, for $r \geqslant 2$, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}. \tag{7.4.1}$$

and one considers the symmetric bilinear form

$$B(z,w) = \sum_{i+j=n+1} z_i w_j, \qquad z,w \in \mathbb{C}^n.$$

One can write this form as $B(x,y) = (x,Sy)$, where the $n \times n$ symmetric matrix S has block form

$$\begin{pmatrix} 0 & 0 & s_0 \\ 0 & 1 & 0 \\ s_0 & 0 & 0 \end{pmatrix}.$$

Writing the elements of $M(n,\mathbb{C})$ in the same block form one has the following description (see [17]) of the Lie algebra of the complex orthogonal group in this case:

Proposition 7.4.6. *The Lie algebra $\mathfrak{so}(\mathbb{C}^{2r+1},B)$ of $SO(\mathbb{C}^{2r+1},B)$ consists of all matrices*

$$A = \begin{pmatrix} a & w & b \\ u & 0 & -{}^t w_0 \\ c & -s_0 {}^t u & -s_0 {}^t a s_0 \end{pmatrix},$$

where $a \in \mathfrak{gl}(r,\mathbb{C})$, b,c are $r \times r$ matrices such that

$${}^t b = -s_0 b s_0, \qquad {}^t c = -s_0 c s_0$$

(that is, b and c are skew-symmetric around the skew diagonal), w is an $r \times 1$ matrix (column vector), and u is a $1 \times r$ matrix (row vector).

Definitions 7.4.7. A *torus* is a Lie group T isomorphic to $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (r times). The integer r is called the *rank* of T. If G is a Lie group, then a torus $T \subset G$ is *maximal* if it is not contained in any larger torus of G. Let G be one of the following *classical Lie groups* of rank n:

$$GL(n,\mathbb{C}), \qquad SL(n+1,\mathbb{C}), \qquad Sp(\mathbb{C}^{2n},\Omega), \qquad SO(\mathbb{C}^{2n},B), \qquad SO(\mathbb{C}^{2n+1},B),$$

and let $\mathfrak{g}$ be its Lie algebra. The *rank* of any of such groups G is the rank of any maximal subgroup. The subgroup H of diagonal matrices in G is a maximal torus of rank n, and we denote its Lie algebra by $\mathfrak{h}$. Fix a basis for the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ as follows:

(1) Let $G = GL(n,\mathbb{C})$. Define the linear functional ε_i on $\mathfrak{h}$ by

$$\langle \varepsilon_i, A \rangle = a_i, \qquad A = \operatorname{diag}(a_1,\ldots,a_n).$$

Then $\{\varepsilon_1,\ldots,\varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

(2) Let $G = SL(n+1,\mathbb{C})$. Then $\mathfrak{h}$ consists of all diagonal traceless matrices. Define ε_i as in (1) as a linear functional on the space of diagonal matrices for $i = 1,\dots,n+1$. The restriction of ε_i to $\mathfrak{h}$ is then an element of $\mathfrak{h}^*$, again denoted as ε_i. The elements of $\mathfrak{h}^*$ can be written uniquely as

$$\sum_{i=1}^{n+1} \lambda_i \varepsilon_i, \qquad \lambda_i \in \mathbb{C}, \quad \sum_{i=1}^{n+1} \lambda_i = 0.$$

The functionals

$$\varepsilon_i - \frac{1}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}), \qquad i = 1,\dots,n,$$

are a basis for $\mathfrak{h}^*$.

(3) Let G be $Sp(\mathbb{C}^{2n},\Omega)$ or $SO(\mathbb{C}^{2n},B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A\rangle = a_i$ for $A = \operatorname{diag}(a_1,\dots,a_n,-a_n,\dots,-a_1) \in \mathfrak{h}$ and $i = 1,\dots,n$. Then $\{\varepsilon_1,\dots,\varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

(4) Let $G = SO(\mathbb{C}^{2n+1},B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A\rangle = a_i$ for $A = \operatorname{diag}(a_1,\dots,a_n,0,-a_n,\dots,-a_1) \in \mathfrak{h}$ and $i = 1,\dots,n$. Then $\{\varepsilon_1,\dots,\varepsilon_n\}$ is a basis for $\mathfrak{h}^*$.

For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [A,X] = \langle \alpha, A\rangle X,\, A \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$ then α is said to be a *root* and $\mathfrak{g}_\alpha$ is said to be a *root space*. if α is a root then a nonzero element of $\mathfrak{g}_\alpha$ is said to be a *root vector* for α. The set Φ of roots is said to be the *root system* of $\mathfrak{g}$. It depends on a choice of maximal torus, so one writes $\Phi = (\mathfrak{g},\mathfrak{h})$ to make the choice explicit.

Theorem 7.4.8. *Let G be one of the groups in* Definitions 7.4.7 *and let $H \subset G$ be a maximal torus. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the corresponding Lie algebras, and let $\Phi = (\mathfrak{g},\mathfrak{h})$ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$. If $\alpha \in \Phi$ then* $\dim \mathfrak{g}_\alpha = 1$ *and*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Definitions 7.4.9. The action of a Lie group G on a connected differentiable manifold M is said to be *effective* if $gp = p$ for all $p \in M$, implies $g = e$, the identity element of G.

The action of a Lie group G on a connected differentiable manifold M is said to be *free* if $gp = p$, for a point $p \in M$, implies $g = e$.

The action of a Lie group G on a connected differentiable manifold M is said to be *transitive* if for each two points $p,q \in M$, there exists $g \in G$ such that $gp = q$.

The action of a Lie group G on a connected differentiable manifold M is said to be *properly discontinuous* if the two following conditions hold:

(1) each point $p \in M$ has a neighborhood U such that $U \cap g(U)$ is empty unless $g = e$.

(2) any two points $p, p' \in M$ which are not equivalent have neighborhoods U, U' respectively such that $U \cap g(U')$ is empty for all $g \in G$.

Conditions (1), (2) together imply that M/G is a Hausdorff manifold of the same dimension as M.

Definition 7.4.10. A Lie group G is said to act *simply transitively* on a manifold M if the action is transitive and free.

Theorem 7.4.11. *Let H be a closed subgroup of a Lie group G. Then the quotient manifold G/H exists.*

Theorem 7.4.12. *Let $G \times M \to M$, $(s, p) \mapsto sp$, be a transitive action of the Lie group G on the differentiable manifold M on the left. Let $p \in M$, and let H be the isotropy group at p. Define a map*

$$\Phi\colon G/H \to M, \qquad \Phi(sH) = sp.$$

Then Φ is a diffeomorphism.

Proposition 7.4.13. *Let G/H be a homogeneous space, and let N be the maximal normal subgroup of G contained in H. Notice that N is a closed subgroup. Then $G' = G/N$ acts on G/H with isotropy subgroup $H' = H/N$ and G' acts effectively on $G/H = G'/H'$.*

Definition 7.4.14. A homogeneous space G/H is said to be *reductive* if there exists an $\mathrm{Ad}(H)$-invariant direct sum complement vector space $\mathfrak{m}$ to the Lie algebra $\mathfrak{h}$ of the isotropy group H at a point.

7.5 Chapter 5. Fibre Bundles

Definitions 7.5.1. A C^∞ *principal fibre bundle* (or simply a *principal bundle*) is a quadruple (P, π, M, G) where P, M are differentiable manifolds, G is a Lie group and π is a surjective submersion from P to M such that:

(1) G acts differentiably and freely on the right on P,

$$P \times G \to P.$$

For $g \in G$, one also writes $R_g\colon P \to P$ for the map $R_g u = ug$.

(2) M is the quotient space of P by equivalence under G, so that for $p \in M$, G acts simply transitively on $\pi^{-1}(p)$.

(3) P is locally trivial; that is, for any $p \in M$, there is an open neighborhood U of p and a C^∞ map $\Phi_U\colon \pi^{-1}(U) \to G$ such that Φ_U commutes with R_g for every $g \in G$ and the map $\pi^{-1}(U) \to U \times G$ given by $p \mapsto (\pi(p), \Phi_U(p))$ is a diffeomorphism.

P is called the *bundle space*, π the *projection map*, M the *base space*, and G the *structure group*. For $p \in M$, $\pi^{-1}(p)$ is called the *fibre* over p. Each fibre is diffeomorphic to G via the map $j_u\colon G \to \pi^{-1}(\pi(u)) \subset P$, defined by $j_u(g) = R_g u$.

Let G be a Lie group acting on a differentiable manifold M on the right. Each element $A \in \mathfrak{g}$ induces a vector field $A^* \in \mathfrak{X}(M)$, corresponding to the action of the 1-parameter group $a_t = \exp tA$ on M. A^* is called the *fundamental vector field corresponding to A*.

Given a differentiable n-manifold M, a *linear frame z* at a point p is an ordered basis $(X_1, \dots, X_n)$ of the tangent space T_pM. The set FM of all linear frames at all points of M is a principal bundle called the *bundle of linear frames over M*, with projection map π sending each ordered basis of T_pM to the point p, and with group $GL(n,\mathbb{R})$ acting on FM on the right.

There exists a natural $\mathbb{R}^n$-valued differential 1-form θ on FM called the *canonical form on the bundle of linear frames*, defined by

$$\theta(X) = z^{-1}(\pi_* X), \qquad z \in \pi^{-1}(p), \quad p \in M, \quad X \in T_z(FM),$$

where the linear frame z is viewed as an isomorphism $z\colon \mathbb{R}^n \to T_pM$.

A *G-structure* on a differentiable n-manifold M is a principal subbundle of the bundle of linear frames FM whose structure group is a Lie subgroup $G \subseteq GL(n,\mathbb{R})$.

Definition 7.5.2. Let (P,π,M,G) be a principal bundle, and let F be a manifold on which G acts on the left. The *fibre bundle associated to* (P,π,M,G) *with fibre F* is defined as follows. Let us consider the right action of G on the product $P \times F$ defined by $(u,f)g = (ug, g^{-1}f)$, where $p \in P$, $f \in F$, $g \in G$. The quotient space $E = (P \times F)/G$ under equivalence by G, is the bundle space of the associated fibre bundle.

The structure is as follows: The projection map $\pi_E\colon E \to M$ is defined by $\pi_E((u,f)G) = \pi(p)$. If $p \in M$, take a neighborhood U of p as in 7.5.1 (3), with $\Phi_U\colon \pi^{-1}(U) \to G$. Then we have $\Psi_U\colon \pi_E^{-1}(U) \to F$ given by $\Psi_U((u,f)G) = \Phi_U(u)f$, so that $\pi_E^{-1}(U)$ is diffeomorphic to the product $U \times F$.

Definitions 7.5.3. Let (P,π,M,G) a principal bundle. Denote by V_u the subspace of T_uP of vectors tangent to the fibre through $u \in P$. A *connection Γ in P* is an assignment of a subspace H_u of T_uP to each $u \in P$ such that:

(1) $T_uP = V_u \oplus H_u$.

(2) $H_{ug} = R_{*g} H_u$,

$u \in P$, $g \in G$. The subspaces V_u and H_u are respectively called the *vertical* and the *horizontal* subspace of T_uP. We denote by v and h respectively the projections of T_uP onto V_u and H_u.

Given the connection Γ, it defines a differential 1-form ω on P, called the *connection form of Γ*, which takes values in the Lie algebra $\mathfrak{g}$ of G and satisfies:

(1) $\omega(A^*) = A, \quad A \in \mathfrak{g}$.

(2) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega, \quad$ for all $g \in G$.

Given a $\mathfrak{g}$-valued 1-form on P satisfying the two conditions above, there is a unique connection Γ in P whose connection form is ω.

The *horizontal lift* of $X \in \mathfrak{X}(M)$ is the unique vector field $X^h \in \mathfrak{X}(P)$ which is horizontal and projects onto X, that is, $\pi_* X^h_u = X_{\pi(u)}$, for all $u \in P$.

Let ρ be a representation of G in a finite-dimensional real vector space V. Let α be a V-valued r-form on P such that $R^*_g \alpha = \rho(g^{-1})\,\alpha$, $g \in G$. The form $D\alpha$ defined by

$$
\begin{aligned}
(D\alpha)(X_1,\dots,X_{r+1}) &= ((\mathrm{d}\alpha)\circ h)(X_1,\dots,X_{r+1}) \\
&= \mathrm{d}\alpha(hX_1,\dots,hX_{r+1}),
\end{aligned}
$$

for $X_1,\dots,X_{r+1} \in T_uP$, is called the *exterior covariant derivative of* α and D is called the *exterior covariant differentiation*.

The *curvature form* of the connection form ω is defined by $\Omega = D\omega$.

Definition 7.5.4. A connection in the fibre bundle of linear frames FM over the manifold M is called a *linear connection* of M.

Definition 7.5.5. A differentiable manifold M is called *parallelizable* if there exists a linear connection ∇ of M for which parallel transport is locally independent of curves. Such a ∇ is called a *flat connection*.

Definition 7.5.6. Let M be a C^∞ manifold. The *normal coordinate system* associated with a linear frame $z = (X_1,\dots,X_n)\colon \mathbb{R}^n \to T_pM$ is defined by $(x^1,\dots,x^n) = z^{-1}\circ \mathrm{Exp}^{-1}$, on a neighborhood of $p \in M$ on which the Exponential map is invertible.

Definition 7.5.7. An *almost complex structure* on a differentiable manifold M is a differentiable map $J\colon TM \to TM$, such that:

(1) J maps linearly T_pM into T_pM for all $p \in M$.

(2) $J^2 = -I$ on each T_pM, where I stands for the identity map.

Definitions 7.5.8. A *complex manifold M* is defined similarly to a differentiable manifold, but taking homeomorphisms from open subsets of M to $\mathbb{C}^n$ instead of $\mathbb{R}^n$, and the changes of charts $\varphi_\alpha \circ \varphi_\beta^{-1}$ being holomorphic functions on $\mathbb{C}^n$. The number n is called the *complex dimension* of M and one writes $\dim_{\mathbb{C}} M = n$. The maximal set of charts is now called a *complex structure*. A complex manifold is a differentiable manifold, as it follows from the identification $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ obtained taking $z^k = x^k + \mathrm{i}y^k$, for $x^k, y^k \in \mathbb{R}$.

A complex manifold admits an almost complex structure J, taking the linear map J_p at any $p \in M$ defined by

$$
J_p\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = \left.\frac{\partial}{\partial y^k}\right|_p, \qquad J_p\left(\left.\frac{\partial}{\partial y^k}\right|_p\right) = -\left.\frac{\partial}{\partial x^k}\right|_p,
$$

where $z^k = x^k + \mathrm{i}y^k$ are the coordinate functions in a chart (U,φ) around p. The tensor field J does not depend on the chosen coordinates by virtue of the following result: A map f of an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ preserves the standard almost complex structures of $\mathbb{C}^n$ and $\mathbb{C}^m$ (i.e. $f_* \circ J = J \circ f_*$), if and only if f is holomorphic. The tensor field J is called *the almost complex structure* of the complex manifold M.

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$ and let g be a Riemannian metric on M as a differentiable manifold. If g and the almost complex structure J of M satisfy

$$g_p(J_p v, J_p w) = g_p(v, w), \qquad p \in M, \quad v, w \in T_p M,$$

then g is said to be a *Hermitian metric* and (M, J, g) is called a *Hermitian manifold*.

The tensor field F on such a manifold defined at any $p \in M$ by

$$F_p(v, w) = g_p(v, J_p w), \qquad v, w \in T_p M,$$

is called the *fundamental* (or *Kähler*) *form* of the Hermitian metric g. A *Kähler manifold* is a Hermitian manifold whose Kähler form is closed: $\mathrm{d}F = 0$. It can be proved that this is equivalent to $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g.

Definitions 7.5.9. Let M be a connected complex manifold of complex dimension n. Given $p \in M$, three definitions are usually considered of tangent space to M at p, of real dimension $2n$:

$T_p M$: *The real tangent space at* p. M has the underlying structure of a $2n$-dimensional differentiable manifold, and $T_p M$ refers to the tangent space of this underlying real structure, that is, to the space of real derivations of $C_p^\infty M$.

A basis of $T_p M$ can be exhibited as follows: let $z^1, \ldots, z^n$ be local complex coordinates near p and let $z^k = x^k + \mathrm{i} y^k$, $k = 1, \ldots, n$; then $x^1, \ldots, x^n, y^1, \ldots, y^n$ are real coordinates near p and

$$\left\{ \left.\frac{\partial}{\partial x^k}\right|_p, \left.\frac{\partial}{\partial y^k}\right|_p, \quad k = 1, \ldots, n \right\}$$

is a basis of $T_p M$ over $\mathbb{R}$.

The linear map J_p converts $T_p M$ into a complex space with $\dim_{\mathbb{C}} T_p M = n$ by the definition

$$(a + \mathrm{i}b) X = aX + bJ_p X, \qquad X \in T_p M, \quad a + \mathrm{i}b \in \mathbb{C}.$$

$T_p^h M$: *The holomorphic tangent space at* p, which is the complex vector space of all complex derivations of the local algebra $\mathscr{O}_p M$ of germs of holomorphic functions at p; that is, the $\mathbb{C}$-complex functions $Z\colon \mathscr{O}_p M \to \mathbb{C}$ such that

$$Z(fg) = (Zf) g(p) + f(p) Zg, \qquad f, g \in \mathscr{O}_p M.$$

With $z^1, \ldots, z^n$ as above,

$$\left\{ \left.\frac{\partial}{\partial z^k}\right|_p, \ k = 1, \ldots, n \right\}$$

is a basis of $T_p^h M$ over $\mathbb{C}$, where by definition

$$\left.\frac{\partial}{\partial z^k}\right|_p (f) = \frac{\partial f}{\partial z^k}(p),$$

for any holomorphic function f defined near p.

$T_p^{1,0}M$: *The space of vectors of type* $(1,0)$, which is the complex subspace of the complexification $T_p^c M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$ defined by the $(+\mathrm{i})$-eigenspace of the complexification of J. Then, $T_p^{1,0}M$ is spanned by the elements of the form $X - \mathrm{i}JX$, where $X \in T_p M$. That is, with z^k and x^k, y^k as above, since

$$J_p(\partial/\partial x^k)_p = (\partial/\partial y^k)_p, \qquad J_p(\partial/\partial y^k)_p = -(\partial/\partial x^k)_p,$$

a basis of $T_p^{1,0}M$ is given by

$$\left\{ \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right)_p, \; k = 1, \dots, n \right\}.$$

Note that every element $Z \in T_p^c M$ can be written as

$$Z = X + \mathrm{i}Y \approx X \otimes 1 + Y \otimes \mathrm{i}, \qquad X, Y \in T_p M.$$

Let

$$TM = \bigcup_{p \in M} T_p M, \qquad T^h M = \bigcup_{p \in M} T_p^h M, \qquad T^{1,0}M = \bigcup_{p \in M} T_p^{1,0}M,$$

be the bundles defined fibrewise. $T^h M$ has the obvious structure of a holomorphic vector bundle and it is called the *holomorphic vector bundle* of M.

7.6 Chapter 6. Riemannian Geometry

Definitions 7.6.1. Let V be a vector space of dimension n with a nondegenerate symmetric bilinear form. It is said that V has *signature* $(k, n-k)$ if, expressing the form as a sum of squares, there are k negative squares and $n-k$ positive squares.

A *metric tensor* g on a differentiable manifold M is a symmetric nondegenerate $(0,2)$ tensor field on M of constant signature. A (*pseudo*)-*Riemannian manifold* is a pair (M, g) of a differentiable manifold M and a metric tensor g on M. If there is no danger of confussion, one simply writes M.

Let $t^1, \dots, t^n$ be the canonical coordinates on $\mathbb{R}^n$, and let φ be a coordinate map with domain $U \subset M$, such that $x^i = t^i \circ \varphi$. If φ is a conformal map from U onto $\mathbb{R}^n$, with respect to the usual metric of $\mathbb{R}^n$, it is said that the coordinate system $(U, x^1, \dots, x^n)$ is *isothermal* or *conformal* (hence it is also orthogonal).

Definition 7.6.2. Let (M_1, g_1), (M_2, g_2) be pseudo-Riemannian manifolds, and let f be a C^∞ function on the manifold M_1. The *warped product* $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric

$$g = \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2,$$

where $\pi_i \colon M \to M_i$, $i = 1, 2$, denote the projection maps.

Theorem 7.6.3. (Koszul's formula for the Levi-Civita connection) *The only torsionless metric connection* ∇ *on a (pseudo)-Riemannian manifold* (M, g) *is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

Definitions 7.6.4. Let R denote the curvature tensor field of a linear connection ∇ of a differentiable manifold M. Given $X_p, Y_p, Z_p \in T_pM$, one defines $R(X_p, Y_p)Z_p$ by

$$R(X_p, Y_p)Z_p = \nabla_{X_p} \nabla_Y Z - \nabla_{Y_p} \nabla_X Z - \nabla_{[X,Y]_p} Z,$$

where X, Y, Z are vector fields on M whose values at p are respectively X_p, Y_p, Z_p. Similarly, if (M, g) is a (pseudo)-Riemannian manifold, one defines, given the vector fields $X, Y, Z, W \in \mathfrak{X}(M)$,

$$R(X_p, Y_p, Z_p, W_p) = g(R(Z_p, W_p)Y_p, X_p).$$

Definitions 7.6.5. Let N be a submanifold of M, and $\nu(N)$ its normal bundle. The exponential map of M gives, by restriction, a map $\mathrm{Exp}\colon \nu(N) \to M$, which is a diffeomorphism on a neighborhood of the zero section. For $p \in N$, let $\nu_p(N)$ be the fibre of $\nu(N)$ over p. Then $q \in \nu_p(N)$ is a *focal point of* N if Exp_* is singular at q. If ρ is the ray from 0 to q in $\nu_p(N)$, then $\mathrm{Exp}\,(q)$ is called a *focal point of* N *along* ρ, which is a geodesic perpendicular to N. When N is a single point, say p, so that $\nu(N) = T_pM$, then a focal point is called a *conjugate point to* p. The *order* of a focal point is the dimension of the linear space annihilated by Exp_*.

A *minimal segment* is a geodesic segment which minimizes arc length between its ends. A *minimal point* q of p along a geodesic γ is a point on γ such that the segment of γ from p to q is minimal but no larger segment from p is minimal. The set of all minimal points of p is called the *minimum* (or *cut*) *locus* of p.

Proposition 7.6.6. *Let N be the subset of the total space TM of the tangent bundle over M, such that if $(p,X) \in N$ then* $\mathrm{Exp}_p X$ *is defined, and define the map* $\mathrm{Exp}\colon N \to M$ *by* $\mathrm{Exp}(p,X) = \mathrm{Exp}_p X$. *Then N is an open set and* Exp *is C^∞ on N. Let TM_0 be the zero section of TM, that is, $TM_0 = \{(p,0) \in TM : p \in M\} \subset TM$; then there exists an open subset $\widehat{N}$ in TM such that $TM_0 \subset \widehat{N} \subset N$. Let $\Phi\colon \widehat{N} \to M \times M$ be defined by $\Phi(p,X) = (p, \mathrm{Exp}_p X)$. Then Φ is C^∞ and Φ_* is non-singular and surjective at all points of TM_0.*

Definition 7.6.7. Let γ be a C^∞ curve in the n-manifold M that is an injective map on the open interval $I \subset \mathbb{R}$. Let $e_1,\dots,e_n$ be vector fields on γ that are independent at each $\gamma(t)$ and with $e_n(t) = \gamma'(t)$ for all $t \in I$. Let $\{\theta^1,\dots,\theta^n\}$ be the basis dual to $\{e_1,\dots,e_n\}$ for each t. By the Proposition 7.6.6, there exists a neighborhood V of TM_0 such that the map Φ is a diffeomorphism of V onto a neighborhood U_M of the diagonal in $M \times M$. Let

$$U = \{(p,X) \in V \,:\, p = \gamma(t),\ \theta^n(X) = 0 \text{ for some } t \in I\}.$$

Then $\Psi = \Phi|_U$ is a one-to-one C^∞ map of the submanifold U into $M \times M$. Moreover, Ψ_* is non-singular at each point of U, so that Ψ is an embedding of U into $M \times M$. The map $\Upsilon = \mathrm{pr}_2 \circ \Psi$ then gives a one-to-one C^∞ map of U onto an open neighborhood W of the image set $\gamma(I)$. Define *Fermi coordinates* x^i on $q \in W$ by letting $\Upsilon^{-1}(q) = (\gamma(t),Y)$ in W and $x^i(q) = \theta^i(Y)$ for $i = 1,\dots,n-1$ and $x^n(q) = t$.

More special types of Fermi coordinates can be defined by taking $e_1,\dots,e_n$ to be parallel along a geodesic, and in the Riemannian case, one can take an orthonormal parallel basis along a geodesic.

Definition 7.6.8. Let $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ be the Laplacian on a Riemannian manifold (M,g). The elements of

$$H^r = \{\omega \in \Lambda^r M \,:\, \Delta\omega = 0\}$$

are called the *harmonic r-forms on M*.

Definition 7.6.9. Let M be a Riemannian $4n$-manifold. The Hodge star operator decomposes the space of harmonic forms H^{2n} on M into subspaces $H^{2n}_\pm$ with eigenvalues ± 1. The *Hirzebruch signature* is defined by

$$\tau(M) = \dim H^{2n}_+ - \dim H^{2n}_-.$$

This signature equals the usual topological signature.

Theorem 7.6.10. (Hirzebruch's Signature Formula (for dimension 4)) *The signature $\tau(M)$ of a 4-dimensional compact oriented differentiable manifold M is related to its first Pontrjagin form $p_1(M)$ by*

$$\tau(M) = \frac{1}{3}\int_M p_1(M).$$

Definition 7.6.11. Let (M,g) be a 3-dimensional compact orientable Riemannian manifold. Let $\Omega = (\Omega^i_j)$ denote the curvature form of the Levi-Civita connection ∇, and consider the closed form $TP_1(\Omega)$ on the bundle $\mathscr{O}_+(M)$ of positively-oriented orthonormal frames on M given by

$$\frac{1}{2}TP_1(\Omega) = \frac{1}{8\pi^2}\sum_{1\leqslant i<j\leqslant 3} \omega^i_j \wedge \Omega^i_j - \frac{1}{8\pi^2}\,\omega^1_2 \wedge \omega^2_3 \wedge \omega^3_1,$$

where ω^i_j and Ω^i_j denote, respectively, the connection forms and the curvature forms of the linear connection ∇.

The differential form $\frac{1}{2}TP_1(\Omega)$ gives rise to a *Chern-Simons invariant* $J(M,g) \in \mathbb{R}/\mathbb{Z}$ as follows: Since such an M is globally parallelizable, a section $s\colon M \to \mathscr{O}_+(M)$ exist. The integral

$$I(s) = \int_{s(M)} \frac{1}{2} TP_1(\Omega)$$

is a real number, and for another section s' the difference $I(s) - I(s')$ is an integer. The invariant $J(M,g)$ is defined to be $I(s)$ mod 1.

Let $\Gamma, \widetilde{\Gamma}$ be two connections in a principal bundle $P = (P,M,G)$. On a trivializing neighborhood, any such Γ can be described by a $\mathfrak{g}$-valued differential 1-form A and the corresponding curvature by

$$F = \mathrm{d}A + \frac{1}{2}[A,A].$$

Then, if $I \in \mathscr{I}^r(G)$ denotes a G-invariant polynomial on $\mathfrak{g}$, it can be proved that the differential $2r$-form $I(F^r)$ does not depend on the particular trivialization of P. Hence, the various locally defined differential forms $I(F^r)$ fit together to yield a differential 2-form on M, again denoted by $I(F^r)$, which is closed.

Let $\tilde{A}$, $\widetilde{F}$ be the connection form and the curvature form corresponding to $\widetilde{\Gamma}$. Then consider the connection 1-form

$$A_t = \tilde{A} + t(A - \tilde{A}), \qquad t \in [0,1],$$

with corresponding curvature form

$$F_t = \mathrm{d}A_t + \frac{1}{2}[A_t, A_t].$$

One has the following *transgression formula*, sometimes called Chern-Simons formula:

Theorem 7.6.12.

$$I(F^r) - I(\widetilde{F}^r) = \mathrm{d}Q(A, \widetilde{A}),$$

where $Q(A,\widetilde{A})$ *is defined by*

$$Q(A,\widetilde{A}) = r\int_0^1 I\left(A - \widetilde{A}, F_t, \ldots, F_t\right) \mathrm{d}t.$$

Definitions 7.6.13. Given a linear connection ∇ of a differentiable manifold M and a geodesic γ on M, a *Jacobi field* along γ is a vector field Y along γ satisfying

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_{\gamma'}(T(Y,\gamma')) + R(Y,\gamma')\gamma' = 0,$$

where T denotes the torsion tensor of ∇.

Theorem 7.6.14. *Let M,N be pseudo-Riemannian manifolds, with N connected, and let $\Phi\colon M \to N$ be a local isometry. Suppose that given any geodesic $\gamma\colon [0,1] \to N$ and a point $p \in M$ such that $\Phi(p) = \gamma(0)$, there exists a lift $\widetilde{\gamma}\colon [0,1] \to M$ of γ through Φ such that $\widetilde{\gamma}(0) = p$. Then Φ is a pseudo-Riemannian covering map.*

Definition 7.6.15. Let (P,π,M,G) a principal bundle. The *holonomy group* (resp. *restricted holonomy group*) of the connection Γ in P with reference point $p \in M$ is the group $\mathrm{Hol}_M(p)$ (resp. $\mathrm{Hol}_M^0(p)$) consisting of diffeomorphisms of the fiber $\pi^{-1}(p)$ onto itself obtained under parallel tranport along closed curves (resp. closed curves homotopic to zero) starting and ending at p. Since the holonomy groups at two points of a manifold are conjugate subgroups of G, we shall write simply $\mathrm{Hol}(\Gamma)$ or $\mathrm{Hol}^0(\Gamma)$ for a given manifold M and Γ as above.

Theorem 7.6.16. *Let $\Phi,\Psi\colon M \to N$ be isometries of pseudo-Riemannian manifolds. If M is connected and $\Phi_{*p} = \Psi_{*p}$ at some point $p \in M$, then $\Phi = \Psi$.*

Definition 7.6.17. An *affine symmetric space* is a triple (G,H,σ) consisting of a Lie group G, a closed subgroup H of G, and an involutive automorphism σ (that is, $\sigma^2 = \mathrm{id}$) of G such that

$$G_0^\sigma \subset H \subset G^\sigma,$$

where G^σ denotes the closed subgroup of G consisting of all the elements left fixed by σ, and G_0^σ stands for the identity component of G^σ.

Definition 7.6.18. A Riemannian manifold of constant sectional curvature is called a (real) *space form*.

Theorem 7.6.19. (Nomizu's Theorem on the cohomology of nilmanifolds) *Let G be a connected nilpotent Lie group with discrete subgroup Γ such that the space of left cosets $\Gamma\backslash G$ is compact. Then there is a natural isomorphism of cohomology groups*

$$H^*(\mathfrak{g}) \approx H^*_{dR}(\Gamma\backslash G,\mathbb{R}),$$

where $H^*(\mathfrak{g})$ denotes the cohomology of the Lie algebra $\mathfrak{g}$ of G.

Definition 7.6.20. A vector field X on a Riemannian manifold (M,g) is *harmonic* if the differential form dual with respect to the metric, $X^\flat$, defined at each $p \in M$ by $X_p^\flat(Y) = g_p(X_p,Y_p)$, $Y \in \mathfrak{X}(M)$, is harmonic.

Theorem 7.6.21. (Hodge Decomposition Theorem) *For each integer r with $0 \leqslant r \leqslant n$, the space H^r defined in 7.6.8 is finite-dimensional, and we have the following direct sum decompositions of $\Lambda^r M$:*

$$\begin{aligned}\Lambda^r M &= \Delta(\Lambda^r M) \oplus H^r \\ &= \mathrm{d}\delta(\Lambda^r M) \oplus \delta\mathrm{d}(\Lambda^r M) \oplus H^p = \mathrm{d}(\Lambda^{r-1} M) \oplus \delta(\Lambda^{r+1} M) \oplus H^p.\end{aligned}$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in \Lambda^r M$ if and only if the differential r-form α is orthogonal to the space of harmonic r-forms.

Corollary 7.6.22. (Corollary of Green's Theorem) *Let M be a compact Riemannian manifold. Then*

$$\int_M \Delta f \; \omega_g = 0, \qquad f \in C^\infty M.$$

Theorem 7.6.23. (Generalized Gauss's Theorema Egregium) *Let M be a hypersurface of a Riemannian manifold $\widetilde{M}$, let P be a subspace of dimension 2 of T_pM, $p \in M$, and let $K(P)$, $\widetilde{K}(P)$ be the sectional curvature of P in M and $\widetilde{M}$, respectively; then*

$$\widetilde{K}(P) = K(P) - \det L,$$

where L is the Weingarten map.

Remark 7.6.24. When $\widetilde{M}$ is 3-dimensional, the above theorem shows that the determinant of L is independent of the embedding (i.e. independent of L) and depends only on the Riemannian structure of $\widetilde{M}$ and M.

Definition 7.6.25. A C^∞ map $\Phi\colon (M,g) \to (\widetilde{M},\tilde{g})$ between Riemannian manifolds is said to be a *strictly conformal map of ratio* λ if there exists a strictly positive function $\lambda \in C^\infty M$ such that, for all $p \in M$ and $X, Y \in T_pM$, it satisfies $\tilde{g}(\Phi_* X, \Phi_* Y) = \lambda(p) g(X,Y)$.

Definitions 7.6.26. Let $(\widetilde{M}, g, J)$ be an almost Hermitian manifold with metric g and almost complex structure J. An isometrically immersed real submanifold M of $\widetilde{M}$ is said to be a *complex submanifold* (resp. a *totally real submanifold*) of $\widetilde{M}$ if each tangent space to M is mapped into itself (resp. into the subspace normal with respect to g) by the almost complex structure J.

Let Φ be an isometric immersion of the Riemannian manifold M into the Riemannian manifold $\widetilde{M}$. Then M is said to be an *invariant submanifold* of $\widetilde{M}$ if for all $X, Y \in TM$, the map $\widetilde{R}(\Phi_* X, \Phi_* Y)$, where $\widetilde{R}$ denotes the Riemann curvature tensor on $\widetilde{M}$, leaves the tangent space to $\Phi(M)$ invariant.

A Kähler manifold is called a *complex space form* if it has constant holomorphic sectional curvature.

Theorem 7.6.27. *An invariant submanifold M of a complex manifold $\widetilde{M}$ is either a complex or a totally real submanifold. If M is a complex submanifold, then it is a minimal submanifold.*

Theorem 7.6.28. *Let $\mathbf{x}_1 : U \subset \mathbb{R}^2 \to S_1$ and $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to S_2$ be two parametrizations of the surfaces S_1, S_2 in $\mathbb{R}^3$. If the metrics inherited on S_1 and S_2 by the usual metric of $\mathbb{R}^3$ are proportional with constant of proportionality $\rho > 0$, then the map $\mathbf{x}_2 \circ \mathbf{x}_1^{-1} : \mathbf{x}_1(U) \to S_2$ is locally conformal.*

Definition 7.6.29. A *pseudo-Riemannian submanifold N* of a pseudo-Riemannian manifold (M,g) is a submanifold such that the metric tensor inherited by g on N is non-degenerate.

Chapter 8
Some Formulas and Tables

Chapter 1

• Stereographic projection σ (from either the north pole or the south pole) of the sphere $S^n((0,\dots,0),1)$ with center $(0,\dots,0)\in\mathbb{R}^{n+1}$ and radius 1 onto the equatorial hyperplane:

$$\begin{array}{ccc} U_N & \xrightarrow{\sigma_N} & \mathbb{R}^n \\ (x^1,\dots,x^{n+1}) & \longmapsto & \left(\dfrac{x^1}{1-x^{n+1}},\dots,\dfrac{x^n}{1-x^{n+1}}\right) \end{array}$$

$$\begin{array}{ccc} U_S & \xrightarrow{\sigma_S} & \mathbb{R}^n \\ (x^1,\dots,x^{n+1}) & \longmapsto & \left(\dfrac{x^1}{1+x^{n+1}}\dots,\dfrac{x^n}{1+x^{n+1}}\right) \end{array}$$

where

$$S^n=\left\{(x^1,\dots,x^{n+1})\in\mathbb{R}^{n+1}\ :\ \sum_{i=1}^{n+1}(x^i)^2=1\right\}$$
$$U_N=\left\{(x^1,\dots,x^{n+1})\in S^{n+1}\ :\ x^{n+1}\neq 1\right\}$$
$$U_S=\left\{(x^1,\dots,x^{n+1})\in S^{n+1}\ :\ x^{n+1}\neq -1\right\}$$

• Inverse map σ_N^{-1} of the stereographic projection from the north pole of the sphere $S^n((0,\dots,0),1)$ onto the equatorial hyperplane:

$$\sigma_N^{-1}(y^1,\dots,y^n)=\left(\frac{2y^1}{|y|^2+1},\dots,\frac{2y^n}{|y|^2+1},\frac{|y|^2-1}{|y|^2+1}\right),\qquad |y|^2=\Sigma_{i=1}^n(y^i)^2$$

• Stereographic projection σ_N from the north pole of $S^n((0,\dots,0,r),r)\in\mathbb{R}^{n+1}$ with center $(0,\dots,0,r)\in\mathbb{R}^{n+1}$ and radius r onto the hyperplane $x^{n+1}=0$ tangent to the south pole:

$$\sigma_N(x^1,\dots,x^{n+1}) = \left(\frac{2rx^1}{2r-x^{n+1}},\dots,\frac{2rx^n}{2r-x^{n+1}}\right)$$

• Inverse map σ_N^{-1} of the stereographic projection from the north pole of the sphere $S^n((0,\dots,0,r),r)$ onto the hyperplane $x^{n+1}=0$:

$$\sigma_N^{-1}(y^1,\dots,y^n) = \left(\frac{4r^2y^1}{4r^2+|y|^2},\dots,\frac{4r^2y^n}{4r^2+|y|^2},\frac{2r|y|^2}{4r^2+|y|^2}\right)$$

• Differential of a map $\Phi\colon M\to N$ between differentiable manifolds at $p\in M$, in terms of coordinate systems $(U,x^1,\dots,x^m)$ and $(V,y^1,\dots,y^n)$ around p and $\Phi(p)$:

$$\Phi_{*p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial(y^j\circ\Phi)}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{\Phi(p)}, \qquad i=1,\dots,m$$

• A diffeomorphism between $\mathbb{R}^n$ and the open cube $(-1,1)^n\subset\mathbb{R}^n$:

$$\varphi\colon \mathbb{R}^n\to(-1,1)^n, \qquad (x^1,\dots,x^n)\mapsto(\tanh x^1,\dots,\tanh x^n)$$

• Usual local coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$ of the tangent bundle (TM,π,M) on a coordinate neighborhood $\pi^{-1}(U)$ of TM over a coordinate neighborhood U for a coordinate system $(U,x^1,\dots,x^n)$ around $p\in M$:

$$\begin{aligned}&(x^1,\dots,x^n,y^1,\dots,y^n)(v)\\ &\qquad = \big((x^1\circ\pi)(v),\dots,(x^n\circ\pi)(v),\mathrm{d}y^1(v),\dots,\mathrm{d}y^n(v)\big), \qquad v\in T_pM\end{aligned}$$

• A property of the bracket of vector fields ($f,g\in C^\infty M$; $X,Y\in\mathfrak{X}(M)$):

$$[fX,gY] = fg[X,Y]+f(Xg)Y-g(Yf)X$$

• Jacobi identity for vector fields:

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$$

• A parallelization of S^3 by unit vectors fields:

$$\begin{aligned}X_p &= \left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-t\frac{\partial}{\partial z}+z\frac{\partial}{\partial t}\right)_p\\ Y_p &= \left(-z\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}-y\frac{\partial}{\partial t}\right)_p\\ Z_p &= \left(-t\frac{\partial}{\partial x}-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}+x\frac{\partial}{\partial t}\right)_p\end{aligned}$$

• Image vector field $\Phi\cdot X\in\mathfrak{X}(N)$ of $X\in\mathfrak{X}(M)$ by the diffeomorphism $\Phi\colon M\to N$:

$$(\Phi\cdot X)_p = \Phi_*\left(X_{\Phi^{-1}(p)}\right), \qquad p\in N$$

- A nonvanishing vector field on the sphere S^{2n+1}:

$$X_p = -x^2 \left.\frac{\partial}{\partial x^1}\right|_p + x^1 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots - x^{2n+2} \left.\frac{\partial}{\partial x^{2n+1}}\right|_p + x^{2n+1} \left.\frac{\partial}{\partial x^{2n+2}}\right|_p$$

$\big(p \in S^3 = \big\{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 = 1\big\}\big)$

Chapter 2

- Nijenhuis torsion of two $(1,1)$ tensor fields A, B:

$$\begin{aligned} S(X,Y) = {} & [AX,BY]+[BX,AY]+AB[X,Y]+BA[X,Y] \\ & -A[X,BY]-A[BX,Y]-B[X,AY]-B[AX,Y] \end{aligned}$$

- Nijenhuis tensor of a $(1,1)$ tensor field J:

$$N(X,Y) = [JX,JY]-J[JX,Y]-J[X,JY]+J^2[X,Y]$$

$$N^i_{jk} = J^l_j \frac{\partial J^i_k}{\partial x^l} - J^l_k \frac{\partial J^i_j}{\partial x^l} + J^i_l \frac{\partial J^l_j}{\partial x^k} - J^i_l \frac{\partial J^l_k}{\partial x^j}$$

- Kulkarni-Nomizu product of two symmetric $(0,2)$ tensors h, k:

$$\begin{aligned} (h \cdot k)(X,Y,Z,W) = {} & h(X,Z)k(Y,W) \\ & + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W) \end{aligned}$$

- Exterior or “wedge” or “Grassmann” product of differential forms:

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p, \qquad p \in M, \quad \alpha \in \Lambda^r M, \quad \beta \in \Lambda^s M$$

$$\begin{aligned} (\alpha_p \wedge \beta_p)&(X_1,\dots,X_{r+s}) \\ &= \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} (\operatorname{sgn}\sigma)\, \alpha_p(X_{\sigma(1)},\dots,X_{\sigma(r)})\beta_p(X_{\sigma(r+1)},\dots,X_{\sigma(r+s)}) \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_{r+s} \\ \sigma(1)<\cdots<\sigma(r) \\ \sigma(r+1)<\cdots<\sigma(r+s)}} (\operatorname{sgn}\sigma)\, \alpha_p(X_{\sigma(1)},\dots,X_{\sigma(r)})\beta_p(X_{\sigma(r+1)},\dots,X_{\sigma(r+s)}), \end{aligned}$$

$$X_i \in T_p M, \quad i = 1,\dots,r+s$$

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha, \qquad \alpha \in \Lambda^r M, \quad \beta \in \Lambda^s M$$

- Exterior differential $\mathrm{d}\colon \Lambda^* M \to \Lambda^* M$:
  (1) If $f \in C^\infty M$, then $\mathrm{d}f \in \Lambda^1 M$ is the usual differential of f
  (2) d is a linear map such that $\mathrm{d}(\Lambda^r M) \subset \Lambda^{r+1} M$
  (3) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta \qquad$ (α homogeneous)
  (4) $\mathrm{d}^2 = 0$

• Relation between the bracket product of vector fields and the exterior differential of a differential 1-form:

$$(\mathrm{d}\omega)(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

• Relation between the bracket product of vector fields and the exterior differential of a differential r-form:

$$\begin{aligned}(\mathrm{d}\omega)(X_0,\dots,X_r) &= \sum_{i=0}^{r}(-1)^i X_i\big(\omega(X_0,\dots,\widehat{X}_i,\dots,X_r)\big)\\ &\quad + \sum_{0\leqslant i<j\leqslant r}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_r)\end{aligned}$$

• Induced (or pull-back of a) differential form $\Phi^*\theta$ of $\theta = f_i\,\mathrm{d}y^i$ for $\Phi\colon M\to N$ (in terms of local coordinates $(x^1,\dots,x^m)$, $(y^1,\dots,y^n)$ on M, N, respectively):

$$\Phi^*\theta \equiv \begin{pmatrix} \dfrac{\partial(y^1\circ\Phi)}{\partial x^1} & \cdots & \dfrac{\partial(y^n\circ\Phi)}{\partial x^1}\\ \vdots & & \vdots\\ \dfrac{\partial(y^1\circ\Phi)}{\partial x^m} & \cdots & \dfrac{\partial(y^n\circ\Phi)}{\partial x^m}\end{pmatrix}\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix} \equiv \frac{\partial(y^i\circ\Phi)}{\partial x^j}\,f_i\,\mathrm{d}x^j$$

• Basis of differential 1-forms $\{\mu^k = \mu^k_l\,\mathrm{d}x^l\}$ dual to the basis of vector fields $\{e_i = \lambda_i^j\partial/\partial x^j\}$:

$$(\mu^i_j) = {}^t(\lambda^i_j)^{-1}$$

• Some formulas for the Lie derivative:

$$\begin{aligned}L_X f &= Xf, \qquad f\in C^\infty M\\ (L_XY)_p &= \lim_{t\to 0}\frac{1}{t}\left(Y_p - \varphi_{t*}Y_{\varphi_t^{-1}(p)}\right), \quad \varphi_t = \text{ local flow of } X\\ L_XY &= [X,Y]\\ L_Y\big(\omega(X_1,\dots,X_r)\big) &= (L_Y\omega)(X_1,\dots,X_r)\\ &\quad + \sum_{i=1}^{r}\omega(X_1,\dots,X_{i-1},[Y,X_i],X_{i+1},\dots,X_r)\\ (L_XT)(\omega^1,\dots,\omega^r,Y_1,\dots,Y_s) &= X\left(T\left(\omega^1,\dots,\omega^r,Y_1,\dots,Y_s\right)\right)\\ &\quad - \sum_{i=1}^{r}T\left(\omega^1,\dots,L_X\omega^i,\dots,\omega^r,Y_1,\dots,Y_s\right)\\ &\quad - \sum_{i=1}^{s}T\left(\omega^1,\dots,\omega^r,Y_1,\dots,L_XY_i,\dots,Y_s\right)\\ L_X(T_1\otimes T_2) &= (L_XT_1)\otimes T_2 + T_1\otimes(L_XT_2)\end{aligned}$$

$$L_{[X,Y]} = [L_X, L_Y]$$
$$L_X \mathrm{d} = \mathrm{d} L_X$$

$(X, Y, X_i, Y_i \in \mathfrak{X}(M);\ \ \omega, \omega^i \in \Lambda^* M;\ \ T, T_i \in \mathscr{T}_s^r M)$

• Interior product:

$$(i_X \omega)(X_1, \dots, X_{r-1}) = \omega(X, X_1, \dots, X_{r-1}), \qquad \omega \in \Lambda^r M$$
$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^r \alpha \wedge i_X \beta, \qquad \alpha \in \Lambda^r M, \quad \beta \in \Lambda^s M$$
$$L_X \omega = i_X \mathrm{d}\omega + \mathrm{d} i_X \omega$$
$$[L_X, i_Y] = i_{[X,Y]}$$

• Canonical 1-form ϑ and canonical symplectic form Ω on the cotangent bundle (T^*M, π, M):

$$\vartheta_\omega(X) = \omega(\pi_* X), \qquad \omega \in T^*M, \quad X \in T_\omega T^*M$$
$$\vartheta = p_i \, \mathrm{d}q^i, \qquad \Omega = \mathrm{d}\vartheta = \mathrm{d}p_i \wedge \mathrm{d}q^i$$

($(q^1, \dots, q^n, p_1, \dots, p_n)$ = local coordinates on T^*M)

• Hamilton equations:
$$i_{\sigma'}(\Omega \circ \sigma) + \mathrm{d}H \circ \sigma = 0.$$
($H \in C^\infty(T^*M)$ and $\sigma \colon (a,b) \to T^*M$ a C^∞ curve with tangent vector σ').

Chapter 3

• Divergence of a vector field X on an oriented manifold M with fixed volume element ω:
$$(\operatorname{div} X)\,\omega = L_X \omega$$

• Stokes' Theorem I:
$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega$$
(see 7.3.3)

• Stokes' Theorem II:
$$\int_{\partial D} \omega = \int_D \mathrm{d}\omega$$
(see 7.3.6)

• Green's Theorem: X a vector field on an oriented compact manifold M with a fixed volume element ω:
$$\int_M (\operatorname{div} X)\,\omega = 0$$

Chapter 4

SOME USUAL LIE GROUPS

• General linear group:

$$GL(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) \,:\, \det A \neq 0\}$$

• Special linear group:

$$SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \,:\, \det A = 1\}$$

• Unitary group:

$$U(n) = \left\{A \in M(n,\mathbb{C}) \,:\, {}^t\bar{A}A = I\right\}$$

($t =$ transpose; bar = complex conjugation; $I =$ identity matrix)

• Special unitary group:

$$SU(n) = \{A \in U(n) \,:\, \det A = 1\}$$

• Complex orthogonal group:

$$O(n,\mathbb{C}) = \left\{A \in M(n,\mathbb{C}) \,:\, {}^tAA = I\right\}$$

• Complex special orthogonal group:

$$SO(n,\mathbb{C}) = \{A \in O(n,\mathbb{C}) \,:\, \det A = 1\}$$

• Symplectic group over $\mathbb{C}$:

$$Sp(n,\mathbb{C}) = \left\{A \in GL(n,\mathbb{C}) \,:\, {}^tA\Omega A = \Omega\right\}$$

$\left(\Omega = \sum_{k=1}^n \mathrm{d}x^k \wedge \mathrm{d}x^{n+k} \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \text{ symplectic 2-form on } \mathbb{R}^{2n}\right)$

• Real general linear group:

$$GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \,:\, \det A \neq 0\}$$

• Real special linear group:

$$SL(n,\mathbb{R}) = SL(n,\mathbb{C}) \cap GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \,:\, \det A = 1\}$$

• Orthogonal group:

$$\begin{aligned} O(n) &= U(n) \cap GL(n,\mathbb{R}) \\ &= O(n,\mathbb{C}) \cap GL(n,\mathbb{R}) \\ &= \left\{A \in GL(n,\mathbb{R}) \,:\, {}^t\!AA = I\right\} \end{aligned}$$

• Special orthogonal group:

$$SO(n) = \{A \in O(n) \,:\, \det A = 1\}$$

• Lorentz group:

$$O(k,n-k) = \left\{A \in GL(n,\mathbb{R}) \,:\, {}^t\!A \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} A = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}\right\}$$

• Symplectic group over $\mathbb{R}$:

$$Sp(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \,:\, {}^t\!A\Omega A = \Omega\}$$

• Quaternionic linear group:

$$GL(n,\mathbb{H}) = \{A\colon \mathbb{H}^n \to \mathbb{H}^n \,:\, A \text{ is right } \mathbb{H}\text{-linear and invertible}\}$$

• Quaternionic special linear group:

$$SL(n,\mathbb{H}) = \{A \in GL(n,\mathbb{H}) \,:\, \det A = 1\}$$

• Symplectic group:

$$\begin{aligned} Sp(n) &= Sp(n,\mathbb{C}) \cap SU(2n) \\ &= \left\{ \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \in GL(n,\mathbb{H}) \subset GL(2n,\mathbb{C}) \,:\, {}^t\!\bar{A}A = 1,\ {}^t\!A\Omega A = \Omega \right\} \\ &= \left\{A + \mathrm{j}B \in GL(n,\mathbb{H}) \,:\, {}^t(\overline{A+\mathrm{j}B})(A+\mathrm{j}B) = I\right\} \\ &= \{A + \mathrm{j}B \ \text{ preserving the symplectic inner product } \langle\ ,\ \rangle \text{ on } \mathbb{H}^n\} \end{aligned}$$

$\left(\mathbb{H} \equiv \mathbb{C} + \mathrm{j}\,\mathbb{C}\,;\ \langle u,v\rangle = \sum_r \bar{p}^r q^r,\ u = (p^1,\dots,p^n),\ v = (q^1,\dots,q^n) \in \mathbb{H}^n\right)$

• $Sp(n)Sp(1)$: Let V be a $4n$-dimensional real vector space. A quaternionic structure on V is a 3-dimensional space of $\operatorname{End} V$ given by

$$Q = \mathbb{R}J_1 + \mathbb{R}J_2 + \mathbb{R}J_3, \quad J_k^2 = -I, \quad J_3 = J_1J_2, \quad J_kJ_l = -J_lJ_k, \qquad k,l = 1,2,3.$$

Let
$$S(Q)=\left\{J=a_1J_1+a_2J_2+a_3J_3\in Q\,:\,a_1^2+a_2^2+a_3^2=1\right\}.$$
A Euclidean metric g on V is called Hermitian with respect to Q if $g(JX,JY)=g(X,Y)$, for $J\in S(Q)$, $X,Y\in V$. The pair (Q,g) is called a quaternionic Hermitian structure. Then, for V as a right module over $\mathbb{H}$:

$$Sp(n)Sp(1)=\mathrm{Aut}\,(Q,g)=\{\varphi\in GL(V)\,:\,\varphi\text{ preserves }(Q,g)\}.$$

REMARK. For other groups, as $U(p,q)$, $SU(p,q)$, $SO(p,q)$, $Sp(p,q)$, $SO^*(2n)$, $SU^*(2n)$ (group of complex matrices isomorphic to $SL(n,\mathbb{H})$), see tables beginning on page 389.

MISCELLANEOUS

- Euler angles (of rotations around the x,y,z-axes):

$$SO(3)=\{g(\varphi,\theta,\psi)=R_z(\varphi)R_y(\theta)R_z(\psi),\ 0\leqslant\varphi,\psi\leqslant 2\pi,\ 0\leqslant\theta\leqslant\pi,\},$$
$$R_z(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix},\qquad R_y(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$

SOME PROPERTIES OF SOME USUAL LIE GROUPS

Group	dim	Type	Group	dim	Type
$GL(n,\mathbb{C})$	$2n^2$	cn	$O(n,\mathbb{C})$	$n(n-1)$	2 cc
$SL(n,\mathbb{C})$	$2(n^2-1)$	cn, sc	$SO(n,\mathbb{C})$	$n(n-1)$	cn
$GL(n,\mathbb{R})$	n^2	2 cc	$O(n)$	$n(n-1)/2$	2 cc, cp
$SL(n,\mathbb{R})$	n^2-1	cn	$SO(n)$	$n(n-1)/2$	cn, cp
$U(n)$	n^2	cn, cp	$SO(p,q)$	$(p+q)(p+q-1)/2$	2 cc $(*)$
$SU(n)$	n^2-1	cn, sc, cp	$Sp(n,\mathbb{C})$	$2(2n^2+n)$	cn
$SU(p,q)$	$(p+q)^2-1$	cn	$Sp(n)$	$n(2n+1)$	cn, sc, cp
$SU^*(2n)$	$2(2n^2-1)$	cn	$Sp(p,q)$	$(p+q)(2(p+q)+1)$	cn
$SO^*(2n)$	$2n(n-1)$	cn	$Sp(n,\mathbb{R})$	$n(2n+1)$	cn

cn = connected; sc = simply connected; 2 cc = 2 connected components; cp = compact; $(*)\ 0<p<p+q$

SIMPLY CONNECTED COMPACT SIMPLE LIE GROUPS

G	$\dim G$	$\operatorname{rank} G$	$(*)$
$SU(n)$	n^2-1	$n-1$	$n \geqslant 2$
$\mathrm{Spin}(2n+1)$	$2n^2+n$	n	$n \geqslant 2$
$Sp(n)$	$2n^2+n$	n	$n \geqslant 3$
$\mathrm{Spin}(2n)$	$2n^2-n$	n	$n \geqslant 4$
G_2	14	2	
F_4	52	4	
E_6	78	6	
E_7	133	7	
E_8	248	8	

$(*)$ To avoid repetitions

POINCARÉ POLYNOMIALS OF THE COMPACT SIMPLE LIE GROUPS

$$p_{A_n}(t) = p_{SU(n+1)}(t) = (1+t^3)(1+t^5)\cdots(1+t^{2n+1})$$

$$p_{B_n}(t) = p_{SO(2n+1)}(t) = (1+t^3)(1+t^7)\cdots(1+t^{4n-1})$$

$$p_{C_n}(t) = p_{Sp(n)}(t) = (1+t^3)(1+t^7)\cdots(1+t^{4n-1})$$

$$p_{D_n}(t) = p_{SO(2n)}(t) = (1+t^3)(1+t^7)\cdots(1+t^{2n-1})(1+t^{4n-5})$$

$$p_{G_2}(t) = (1+t^3)(1+t^{11})$$

$$p_{F_4}(t) = (1+t^3)(1+t^{11})(1+t^{15})(1+t^{23})$$

$$p_{E_6}(t) = (1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$$

$$p_{E_7}(t) = (1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35})$$

$$p_{E_8}(t) = (1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59})$$

CENTER OF SOME USUAL LIE GROUPS

G	$Z(G)$	G	$Z(G)$
$U(n)$	$\{e^{2\pi i\theta}I : \theta \in \mathbb{R}/\mathbb{Z}\} \approx S^1$	$Sp(n)$	$\{\pm I\}$
$SU(n)$	$\{\omega I : \omega^n = 1\} \approx \mathbb{Z}_n$	$SO(2n,\mathbb{R}),\ n>1$	$\{\pm I\}$
$SO(2n+1,\mathbb{R})$	$\{I\}$	$SO(2,\mathbb{R})$	$SO(2,\mathbb{R})$

ISOMORPHISMS OF $\mathrm{Spin}(n)$ WITH SOME CLASSICAL GROUPS

$\mathrm{Spin}(2)$	$U(1)$	$\mathrm{Spin}(5)$	$Sp(2)$
$\mathrm{Spin}(3)$	$SU(2)$	$\mathrm{Spin}(6)$	$SU(4)$
$\mathrm{Spin}(4)$	$SU(2)\times SU(2)$		

UNIMODULAR 3-DIMENSIONAL LIE ALGEBRAS
AND THEIR CORRESPONDING LIE GROUPS

$$[e_2,e_3]=\lambda_1 e_1, \qquad [e_3,e_1]=\lambda_2 e_2, \qquad [e_1,e_2]=\lambda_3 e_3$$

Signs of $\lambda_1,\lambda_2,\lambda_3$	Associated Lie group	Description
$+,+,+$	$SU(2)$ or $SO(3)$	compact, simple
$+,+,-$	$SL(2,\mathbb{R})$ or $O(1,2)$	noncompact, simple
$+,+,0$	$E(2)$ $(*)$	solvable
$+,-,0$	$E(1,1)$ $(**)$	solvable
$+,0,0$	Heisenberg group	nilpotent
$0,0,0$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	Abelian

$(*)$ group of rigid motions of Euclidean 2-space
$(**)$ group of rigid motions of Minkowski 2-space, which is a semidirect product of subgroups isomorphic to $\mathbb{R}\oplus\mathbb{R}$ and $\mathbb{R}$, where each $t\in\mathbb{R}$ acts on $\mathbb{R}\oplus\mathbb{R}$ by the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$

MAURER-CARTAN EQUATION

$$d\omega(X,Y) = -\omega([X,Y]), \qquad \omega\in\mathfrak{g}^*,\ X,Y\in\mathfrak{g}$$
$$d\theta^i = -\textstyle\sum_{j<k} c^i_{jk}\,\theta^j\wedge\theta^k$$

($\{\theta^i\}$ = a basis of left-invariant differential 1-forms on the Lie group G with Lie algebra $\mathfrak{g}$; c^i_{jk} = structure constants with respect to that basis of differential forms)

KILLING FORM B FOR SOME LIE ALGEBRAS $\mathfrak{g}$

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y), \qquad X,Y \in \mathfrak{g}$$

$\mathfrak{g}$	$B(X,Y)$
$\mathfrak{gl}(n,\mathbb{R})$	$2n\operatorname{tr}(XY) - 2\operatorname{tr}(X)\operatorname{tr}(Y)$
$\mathfrak{sl}(n,\mathbb{R})$	$2n\operatorname{tr}(XY)$
$\mathfrak{su}(n)$	$2n\operatorname{tr}(XY)$
$\mathfrak{so}(n,\mathbb{C})$	$(n-2)\operatorname{tr} XY$
$\mathfrak{so}(n)$	$(n-2)\operatorname{tr} XY$
$\mathfrak{sp}(n,\mathbb{F})$	$(2n+2)\operatorname{tr}(XY) \qquad (\mathbb{F} = \mathbb{R},\mathbb{C})$

LIE ALGEBRAS OF SOME LIE GROUPS (NONVANISHING BRACKETS)

• Two-dimensional solvable non-Abelian Lie algebra with basis $\{X,Y\}$:

$$[X,Y] = X$$

• Special orthogonal $\mathfrak{so}(3)$ with basis $\{X,Y,Z\}$:

$$[X,Y] = Z, \qquad [Y,Z] = X, \qquad [Z,X] = Y$$

• Lie algebra $\mathfrak{h}$ with basis $\{X,Y,Z\}$ of the Heisenberg group:

$$[X,Y] = Z$$

• Lie algebra with basis $\{X,Y_1,\ldots,Y_{n-1}\}$ of the solvable Lie group which acts simply transitively on the real hyperbolic space $\mathbb{R}H^n$:

$$[X,Y_i] = Y_i$$

• Lie algebra with basis $\{X,Y_1,Z_1,\ldots,\, Y_{n-1},Z_{n-1},\, W\}$ of the solvable Lie group which acts simply transitively on the complex hyperbolic space $\mathbb{C}H^n$:

$$[X,Y_i] = \frac{1}{2}Y_i, \qquad [X,Z_i] = \frac{1}{2}Z_i, \qquad [X,W] = W, \qquad [Z_i,Y_j] = \delta_{ij}W$$

THE EXPONENTIAL MAP

• Product of exponentials:

$$\exp tX \cdot \exp tY = 1 + t(X+Y) + \frac{t^2}{2}[X,Y] - \frac{t^3}{12}\big([[X,Y],X] + [[Y,X],Y]\big) + \cdots$$

• Differential of the exponential map of a Lie group G with Lie algebra $\mathfrak{g}$ ($X \in \mathfrak{g}$; $e =$ identity element of G):

$$\exp_{*X} = (L_{\exp X})_{*e} \circ \frac{1 - \mathrm{e}^{-\operatorname{ad} X}}{\operatorname{ad} X}$$

Some isomorphisms of classical Lie algebras

$\mathfrak{su}(2) \approx \mathfrak{so}(3) \approx \mathfrak{sp}(1)$	$\mathfrak{sl}(2,\mathbb{R}) \approx \mathfrak{su}(1,1) \approx \mathfrak{so}(2,1) \approx \mathfrak{sp}(1,\mathbb{R})$
$\mathfrak{so}(5) \approx \mathfrak{sp}(2)$	$\mathfrak{so}(3,2) \approx \mathfrak{sp}(2,\mathbb{R})$
$\mathfrak{so}(4) \approx \mathfrak{sp}(1) \times \mathfrak{sp}(1)$	$\mathfrak{so}(4,1) \approx \mathfrak{sp}(1,1)$
$\mathfrak{su}(4) \approx \mathfrak{so}(6)$	$\mathfrak{so}(4) \approx \mathfrak{so}(3) \times \mathfrak{so}(3)$
$\mathfrak{sl}(4,\mathbb{R}) \approx \mathfrak{so}(3,3)$	$\mathfrak{su}^*(4) \approx \mathfrak{so}(5,1)$
$\mathfrak{su}(2,2) \approx \mathfrak{so}(4,2)$	$\mathfrak{su}(3,1) \approx \mathfrak{so}^*(6)$
$\mathfrak{so}^*(8) \approx \mathfrak{so}(6,2)$	$\mathfrak{so}(3,1) \approx \mathfrak{sl}(2,\mathbb{C})$
$\mathfrak{so}(2,2) \approx \mathfrak{sl}(2,\mathbb{R}) \approx \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}^*(4) \approx \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R})$

Systems of simple roots for the simple Lie algebras over $\mathbb{C}$

$\mathfrak{a}_n$ $(n \geqslant 1)$	$\{\varepsilon_i - \varepsilon_{i+1},\ i = 1,\dots,n\}$
$\mathfrak{b}_n$ $(n \geqslant 2)$	$\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n,\ i = 1,\dots,n-1\}$
$\mathfrak{c}_n$ $(n \geqslant 3)$	$\{\varepsilon_i - \varepsilon_{i+1}, 2\varepsilon_n,\ i = 1,\dots,n-1\}$
$\mathfrak{d}_n$ $(n \geqslant 4)$	$\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{n-1} + \varepsilon_n,\ i = 1,\dots,n-1\}$
$\mathfrak{g}_2$	$\{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$
$\mathfrak{f}_4$	$\{\varepsilon_1 - \frac{1}{2}\varepsilon^{(4)}, \varepsilon_4, \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3)\}$
$\mathfrak{e}_6$	$\{\varepsilon_1 + \varepsilon_8 - \frac{1}{2}\varepsilon^{(8)}, \varepsilon_2 + \varepsilon_1, \varepsilon_{i+1} - \varepsilon_i,\ i = 1,\dots,4\}$
$\mathfrak{e}_7$	$\{\varepsilon_1 + \varepsilon_8 - \frac{1}{2}\varepsilon^{(8)}, \varepsilon_2 + \varepsilon_1, \varepsilon_{i+1} - \varepsilon_i,\ i = 1,\dots,5\}$
$\mathfrak{e}_8$	$\{\varepsilon_1 + \varepsilon_8 - \frac{1}{2}\varepsilon^{(8)}, \varepsilon_2 + \varepsilon_1, \varepsilon_{i+1} - \varepsilon_i,\ i = 1,\dots,6\}$

$(\varepsilon^{(r)} = \varepsilon_1 + \cdots + \varepsilon_r)$

Real forms of the classical simple Lie algebras over $\mathbb{C}$ and their corresponding simple Lie groups

$\mathfrak{sl}(n,\mathbb{C}) \qquad (\sim \mathfrak{a}_{n-1}, n > 1)$

$$\mathfrak{su}(n) = \left\{A \in \mathfrak{gl}(n,\mathbb{C}) : A + {}^t\bar{A} = 0,\ \operatorname{tr} A = 0\right\}$$

$$SU(n) = \left\{A \in GL(n,\mathbb{C}) : {}^t\bar{A}A = I, \det A = 1\right\}$$

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) : \operatorname{tr} A = 0\}$$

$$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\}$$

$$\mathfrak{su}(p,q) = \left\{ \begin{pmatrix} A_1 & A_2 \\ {}^t\bar{A}_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{C}) : A_1 \in \mathfrak{gl}(p,\mathbb{C}),\ A_1 + {}^t\bar{A}_1 = 0, \right.$$
$$A_3 \in \mathfrak{gl}(q,\mathbb{C}),\ A_3 + {}^t\bar{A}_3 = 0,\ \operatorname{tr} A_1 + \operatorname{tr} A_3 = 0,$$
$$\left. A_2 \text{ arbitrary} \right\},\ p+q = n,\ p \geqslant q$$

$$SU(p,q) = \{A \in SL(p+q,\mathbb{C}) : Q(Az) = Q(z) = -z_1\bar{z}_1 - \cdots - z_p\bar{z}_p$$
$$+ z_{p+1}\bar{z}_{p+1} + \cdots + \cdots z_{p+q}\bar{z}_{p+q}\},\ (p+q = n,\ p \geqslant q)$$
$$= \left\{ A \in SL(p+q,\mathbb{C}) : {}^tAI_{p,q}\bar{A} = I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \right\}$$

(pseudo-unitary groups if $q \neq 0$; $SU(n)$ if $q = 0$)

$$\mathfrak{su}^*(2n) = \left\{ \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) : A_1, A_2 \in \mathfrak{gl}(n,\mathbb{C}),\ \operatorname{tr} A_1 + \operatorname{tr} \bar{A}_1 = 0 \right\}$$

$$SU^*(2n) = \left\{A \in SL(2n,\mathbb{C}) : A\tau = \tau A,\ \tau\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \right.$$
$$\left. \tau\colon (z_1,\ldots,z_{2n}) \to (\bar{z}_{n+1},\ldots,\bar{z}_{2n},-\bar{z}_1,\ldots,-\bar{z}_n)\right\}$$

$$SL(n,\mathbb{C})^{\mathbb{R}} = SL(n,\mathbb{C}) \text{ as a real Lie group}$$

$\mathfrak{so}(2n+1,\mathbb{C}) \qquad (\sim \mathfrak{b}_n, n \geqslant 1)$

$$\mathfrak{so}(2n+1) = \{A \in \mathfrak{gl}(2n+1,\mathbb{R}) : A + {}^tA = 0\}$$

$$SO(2n+1) = \{A \in GL(2n+1,\mathbb{R}) : {}^tAA = I\}$$

$$\begin{aligned}
\mathfrak{so}(p,q) &= \left\{ \begin{pmatrix} A_1 & A_2 \\ {}^tA_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{R}) : A_1 \in \mathfrak{gl}(p,\mathbb{R}), A_3 \in \mathfrak{gl}(q,\mathbb{R}), \right. \\
&\qquad \left. A_1 + {}^tA_1 = 0, A_3 + {}^tA_3 = 0,\ A_2 \text{ arbitrary} \right\}, \\
&\qquad p+q = 2n+1,\ p \geqslant q \\
SO(p,q) &= \left\{ A \in SL(p+q,\mathbb{R}) : Q(Ax) = Q(x) = -x_1^2 + \cdots - x_p^2 \right. \\
&\qquad \left. + x_{p+1}^2 + \cdots + x_{2n+1}^2 \right\},\ (p+q = 2n+1,\ p \geqslant q) \\
&= \left\{ A \in SL(p+q,\mathbb{R}) : {}^tAI_{p,q}A = I_{p,q} \right\} \\
SO(2n+1,\mathbb{C})^{\mathbb{R}} &= SO(2n+1,\mathbb{C}) \text{ as a real Lie group}
\end{aligned}$$

$\mathfrak{sp}(n,\mathbb{C})$ $\quad (\sim \mathfrak{c}_n,\ n \geqslant 1)$

$$\begin{aligned}
\mathfrak{sp}(n) &= \left\{ A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) : A + {}^t\bar{A} = 0,\ \operatorname{tr} A = 0, \right. \\
&\qquad \left. A_i \in \mathfrak{gl}(n,\mathbb{C}),\ A_2 = {}^tA_2,\ A_3 = {}^tA_3 \right\}, \\
&\qquad (\,\mathfrak{sp}(n) = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{su}(2n)\,) \\
Sp(n) &= Sp(n,\mathbb{C}) \cap SU(2n) \\
\mathfrak{sp}(n,\mathbb{R}) &= \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{R}) : A_i \in \mathfrak{gl}(n,\mathbb{R}),\ A_2 = {}^tA_2,\ A_3 = {}^tA_3 \right\} \\
Sp(n,\mathbb{R}) &= \{ A \in GL(2n,\mathbb{R}) : E(Ax) = E(x) \\
&\qquad = x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n} \} \\
&\qquad = \{ A \in GL(n,\mathbb{R}) : {}^tA\Omega A = \Omega \} \\
\mathfrak{sp}(p,q) &= \left\{ \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ {}^t\bar{A}_{12} & A_{22} & {}^tA_{14} & A_{24} \\ -\bar{A}_{13} & \bar{A}_{14} & \bar{A}_{11} & -\bar{A}_{12} \\ {}^t\bar{A}_{14} & -\bar{A}_{24} & -{}^tA_{12} & \bar{A}_{22} \end{pmatrix} \in \mathfrak{gl}(2(p+q),\mathbb{C}) : \right. \\
&\qquad A_{11}, A_{13} \in \mathfrak{gl}(p,\mathbb{C}),\ A_{12}, A_{14} \in M(p \times q,\mathbb{C}),\ A_{11} + {}^t\bar{A}_{11} = 0, \\
&\qquad \left. A_{22} + {}^t\bar{A}_{22} = 0, A_{13} = {}^tA_{13}, A_{24} = {}^tA_{24} \right\} \\
Sp(p,q) &= \left\{ A \in Sp(p+q,\mathbb{C}) : {}^tAI_{p,q,p,q}\bar{A} = I_{p,q,p,q} = \operatorname{diag}(-I_p, I_q, -I_p, I_q) \right\} \\
&\qquad (\,Sp(p) \text{ if } q = 0,\ Sp(n) = Sp(n,\mathbb{C}) \cap U(2n), \\
&\qquad Sp(p,q) = Sp(p+q,\mathbb{C}) \cap U(2p,2q)\,) \\
Sp(n,\mathbb{C})^{\mathbb{R}} &= Sp(n,\mathbb{C}) \text{ as a real Lie group}
\end{aligned}$$

$\mathfrak{so}(2n,\mathbb{C})$	$(\sim \mathfrak{d}_{n-1}, n \geqslant 1)$

$$\mathfrak{so}(2n) = \{A \in \mathfrak{gl}(2n,\mathbb{R}) : A + {}^tA = 0\}$$

$$\mathfrak{so}(p,q) = \left\{ \begin{pmatrix} A_1 & A_2 \\ {}^tA_2 & A_3 \end{pmatrix} \in \mathfrak{gl}(p+q,\mathbb{R}) \,:\, A_1 \in \mathfrak{gl}(p,\mathbb{R}),\, A_1 + {}^tA_1 = 0, \right.$$
$$\left. A_2 \text{ arbitrary},\, A_3 \in \mathfrak{gl}(q,\mathbb{R}),\, A_3 + {}^tA_3 = 0 \right\},\, p+q = 2n,\, p \geqslant q$$

$SO(p,q)$ $\quad p+q = 2n,\ p \geqslant q$ as $SO(p,q),\ p+q = 2n+1,\, p \geqslant q$

$$\mathfrak{so}^*(2n) = \left\{ \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) \,:\, A_1, A_2 \in \mathfrak{gl}(n,\mathbb{C}), \right.$$
$$\left. A_1 + {}^tA_1 = 0,\, A_2 = \bar{A}_2 \right\}$$

$$SO^*(2n) = \{A \in SO(2n,\mathbb{C}) \,:\, Q(Az) = Q(z) = -z_1\bar{z}_{n+1} + z_{n+1}\bar{z}_1$$
$$-z_2\bar{z}_{n+2} + z_{n+2}\bar{z}_2 + \cdots - z_n\bar{z}_{2n} + z_{2n}\bar{z}_n\}$$
$$= \left\{ A \in SO(2n,\mathbb{C}) \,:\, {}^tA \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \bar{A} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

$SO(2n,\mathbb{C})^{\mathbb{R}}$ $\quad$ as $SO(2n+1,\mathbb{C})^{\mathbb{R}}$

Lie groups G for the simple Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ and their compact real forms U

$\mathfrak{g}$	G	U	$Z(\widetilde{U})$ $(*)$	$\dim U$
$\mathfrak{a}_n$ $(n \geqslant 1)$	$SL(n+1,\mathbb{C})$	$SU(n+1)$	$\mathbb{Z}_{n+1}$	n^2+2n
$\mathfrak{b}_n$ $(n \geqslant 2)$	$SO(2n+1,\mathbb{C})$	$SO(2n+1)$	$\mathbb{Z}_2$	$2n^2+n$
$\mathfrak{c}_n$ $(n \geqslant 3)$	$Sp(n,\mathbb{C})$	$Sp(n)$	$\mathbb{Z}_2$	$2n^2+n$
$\mathfrak{d}_n$ $(n \geqslant 4)$	$SO(2n,\mathbb{C})$	$SO(2n)$	$\mathbb{Z}_4$ (n odd)	$2n^2-n$
			$\mathbb{Z}_2+\mathbb{Z}_2$ (n even)	
$\mathfrak{g}_2$	$G_2^{\mathbb{C}}$	G_2	$\mathbb{Z}_1$	14
$\mathfrak{f}_4$	$F_4^{\mathbb{C}}$	F_4	$\mathbb{Z}_1$	52
$\mathfrak{e}_6$	$E_6^{\mathbb{C}}$	E_6	$\mathbb{Z}_3$	78
$\mathfrak{e}_7$	$E_7^{\mathbb{C}}$	E_7	$\mathbb{Z}_2$	133
$\mathfrak{e}_8$	$E_8^{\mathbb{C}}$	E_8	$\mathbb{Z}_1$	248

$(*)$ $\widetilde{U}$ = universal covering group of U

Some usual homogeneous spaces

• Sphere:

$$S^n \approx O(n+1)/O(n) \approx SO(n+1)/SO(n), \qquad n \geqslant 1$$

$$S^{2n+1} \approx U(n+1)/U(n) \approx SU(n+1)/SU(n), \qquad n \geqslant 1$$

• Real Grassmann manifold of k-planes in $\mathbb{R}^n$:

$$G_k(\mathbb{R}^n) \approx O(n)/O(k) \times O(n-k)$$

• Real projective space:

$$\mathbb{R}P^n \approx G_1(\mathbb{R}^{n+1}) \approx O(n+1)/O(1) \times O(n) \approx SO(n+1)/O(n)$$

• Real Stiefel manifold of k-tuples of orthonormal vectors in $\mathbb{R}^n$:

$$V_k(\mathbb{R}^n) \approx O(n)/O(n-k)$$

• Complex projective n-space:

$$\begin{aligned}\mathbb{C}P^n &\approx U(n+1)/U(1) \times U(n)\\ &\approx SU(n+1)/S(U(1) \times U(n))\\ &\approx \big(SU(n+1)/\mathbb{Z}_{n+1}\big)/\big(S(U(1) \times U(n))/\mathbb{Z}_{n+1}\big)\end{aligned}$$

$\big(\mathbb{Z}_{n+1} = \text{center}(SU(n+1))\big)$

• Quaternionic projective space:

$$\mathbb{H}P^n \approx Sp(n+1)/Sp(1) \times Sp(n)$$

Compact connected Lie groups G acting effectively and transitively on some sphere

Sphere	G	Isotropy
S^{n-1}	$SO(n)$	$SO(n-1)$
S^{2n-1}	$U(n)$	$U(n-1)$
	$SU(n)$	$SU(n-1)$
S^{4n-1}	$Sp(n)Sp(1)$	$Sp(n-1)Sp(1)$
	$Sp(n)U(1)$	$Sp(n-1)U(1)$
	$Sp(n)$	$Sp(n-1)$
S^6	G_2	$SU(3)$
S^7	Spin(7)	G_2
S^{15}	Spin(9)	Spin(7)

Some inclusions of Lie groups, and their homogeneous spaces

$$F_4 \supset \mathrm{Spin}(9) \supset \mathrm{Spin}(8) \supset \mathrm{Spin}(7) \supset G_2 \supset SU(3) \supset S^3 \supset 1$$

(Respective dimensions: 52, 36, 28, 21, 14, 8, 3, 1)

$$\mathbb{C}\mathrm{ay}P^2 \approx F_4/\,\mathrm{Spin}(9) \text{ (Cayley projective plane)}$$

$$S^8 \approx \mathrm{Spin}(9)/\,\mathrm{Spin}(8), \quad S^7 \approx \mathrm{Spin}(8)/\,\mathrm{Spin}(7), \quad S^6 \approx G_2/SU(3)$$

$$S^5 \approx SU(3)/S^3, \quad S^{15} \approx \mathrm{Spin}(9)/\,\mathrm{Spin}(7), \quad V_2(\mathbb{R}^7) \approx G_2/S^3$$

Some more homogeneous spaces

$$\mathrm{Spin}(8)/G_2 \approx S^7 \times S^7, \qquad \mathrm{Spin}(7)/G_2 \approx S^7$$

Algebra $\mathbb{H}$ of quaternions

Basis: $\{e_0, e_1, e_2, e_3\}$ satisfying

$$e_0^2 = e_0, \quad e_i^2 = -e_0, \quad e_0 e_i = e_i e_0 = e_i, \quad e_i e_j = -e_j e_i = e_k$$

$\big((i,j,k) = \text{even permutation of } (1,2,3)\big)$

Conjugate quaternion of $q = \sum_{i=0}^{3} a_i e_i \in \mathbb{H}$, and relation with the product:

$$\bar{q} = a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3, \qquad \overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$$

Norm $|q|^2$ and inverse q^{-1} of q:

$$q\bar{q} = \textstyle\sum_{i=0}^{3} a_i^2 \in \mathbb{R}, \qquad q^{-1} = \dfrac{1}{|q|^2}(a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3) = \dfrac{\bar{q}}{|q|^2}$$

Algebra $\mathbb{O}$ of (the usual) octonions, multiplication table, and some associated spaces

$$\begin{aligned}\mathbb{O} = \Big\{ x = z + u \in \mathbb{C} \oplus \mathbb{C}^3 :\ & \\ & (z+u)(z'+v) = (zz' - \langle u,v\rangle) + zv + \bar{z}'u + u * v, \\ & \langle\ ,\ \rangle : \mathbb{C}^3 \to \mathbb{C}^3 \text{ the usual Hermitian product,} \\ & \langle u, v * w\rangle = \det(u,v,w),\ \alpha,\beta \in \mathbb{C},\ u,v,w \in \mathbb{C}^3 \Big\}\end{aligned}$$

Conjugate, trace and norm of $x = z + u \in \mathbb{O}$:

$$\bar{x} = \bar{z} - u, \qquad t(x) = x + \bar{x} \in \mathbb{R}, \qquad n(x) = x\bar{x} = |z|^2 + |u|^2 \in \mathbb{R}$$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_0	$-e_3$	$-e_2$	$-e_5$	$-e_4$	e_7	e_6
e_2	e_2	e_3	e_0	e_1	$-e_6$	$-e_7$	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	e_0	$-e_7$	$-e_6$	e_5	e_4
e_4	e_4	e_5	e_6	e_7	e_0	e_1	e_2	e_3
e_5	e_5	e_4	e_7	e_6	$-e_1$	e_0	$-e_3$	$-e_2$
e_6	e_6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	e_1
e_7	e_7	$-e_6$	e_5	$-e_4$	$-e_3$	e_2	$-e_1$	e_0

$$S^7 = \{x \in \mathbb{O} : n(x) = 1\}$$

$$S^6 = \{x \in \mathbb{O} : n(x) = 1,\ t(x) = 0\}$$

$$S^{15} = \{x \in \mathbb{O} \times \mathbb{O} : n(x) + n(y) = 1\}$$

$$\mathrm{Spin}(7) = \mathrm{Aut}(\mathbb{O}, \{\ ,\ ,\ \}), \qquad \{x,y,z\} = (x\bar{y})z$$

$$G_2 = \{\varphi \in \mathrm{Spin}(7) : \varphi(1) = 1\}, \qquad 1 \in S^7$$

Chapter 5

• Hopf bundles $\pi_{\mathbb{C}} : S^3 \subset \mathbb{C}^2 \to S^2$ and $\pi_{\mathbb{H}} : S^7 \subset \mathbb{H}^2 \to S^4$:

$$\pi(x,y) = (2y\bar{x},\ |x|^2 - |y|^2)$$

• Fundamental vector fields on the bundle of linear frames FM over M:

$$A^*_z = x^i_k(z) a^k_j \frac{\partial}{\partial x^i_j}$$

$\big(A = (a^i_j) \in M(n,\mathbb{R});\ z = (X_1,\dots,X_n) \in FM;\ x^i(z) = x^i(\pi(z));\ x^i_j(z) = \mathrm{d}x^i(X_j)\big)$

• Connection form ω on a principal bundle $P(M,G)$ in terms of forms $\omega_i = \sigma_i^*\omega$, with local sections σ_i, defined on open subsets U_i of M:

$$\omega_j = \mathrm{Ad}_{\psi_{ij}^{-1}}\, \omega_i + \theta_{ij} \quad \text{on } U_i \cap U_j$$

($\{U_i\} =$ open covering of M; $\psi_{ij}(U_i \cap U_j) \to G =$ transition functions; $\theta_{ij} = \mathfrak{g}$-valued 1-form $\psi_{ij}^*\theta$; $\theta =$ canonical 1-form on G: $\theta(X) = X$)

• Exterior covariant derivative $D\varphi$ of a tensorial 1-form of type $\mathrm{Ad}\,G$ with respect to a connection in the principal bundle P with connection form ω ($X,Y \in T_uP$, $u \in P$):

$$D\varphi(X,Y) = \mathrm{d}\varphi(X,Y) + [\varphi(X), \omega(Y)] + [\omega(X), \varphi(Y)]$$

• Cartan's structure equation (principal bundle).

ω the connection form of a connection in a principal G-bundle P, with curvature form Ω; $\omega = \omega^i \otimes e_i$, $\Omega = \Omega^i \otimes e_i$, $\{e_i\}$ basis of $\mathfrak{g}$; c^i_{jk} structure constants with respect to $\{e_i\}$:

$$\begin{aligned} \mathrm{d}\omega(X,Y) &= -[\omega(X),\omega(Y)] + \Omega(X,Y), \qquad X,Y \in T_uP, \quad u \in P \\ \mathrm{d}\omega &= -[\omega,\omega] + \Omega \qquad \text{(simplified expression)} \\ \mathrm{d}\omega^i &= -\sum_{j<k} c^i_{jk}\,\omega^j \wedge \omega^k + \Omega^i \end{aligned}$$

• Structure constants of $GL(n,\mathbb{R})$ with respect to the standard basis $\{E^i_j\}$ of $\mathfrak{gl}(n,\mathbb{R})$ (also for $\mathbb{C}$):

$$c^{rs}_{ij,kl} = \delta^r_i \delta^j_k \delta^s_l - \delta^r_k \delta^l_i \delta^s_j$$

• Cartan structure equation (vector bundle).

$E(M,\mathbb{F}^n,GL(n,\mathbb{F}),P)$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, a vector bundle associated to the principal fibre bundle P; $\{E^i_j\}$, $i,j = 1,\dots,n$, is a basis of $\mathfrak{gl}(n,\mathbb{F})$; $\omega = \omega^j_i \otimes E^i_j$ and $\Omega = \Omega^j_i \otimes E^i_j$ the connection form and the curvature form of a connection in P:

$$\mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j, \qquad i,j = 1,\dots,n$$

LINEAR CONNECTIONS

• Canonical 1-form θ on the frame bundle (FM,π,M):

$$\theta(X) = z^{-1}(\pi_* X), \qquad \theta^i = Y^i_j\,\mathrm{d}x^j$$

($X \in T_z(FM)$, $z \in FM$; $\{x^i, x^i_j\}$, $i = 1,\dots,n = \dim M$, local coordinates on FM; $Y = (x^i_j)^{-1}$)

• Components (or Christoffel symbols) Γ^i_{jk} of a linear connection ∇ of M with connection form $\omega = \omega^i_j \otimes E^j_i$; $\sigma = (X_1,\dots,X_n)$ a section of FM over an open subset U of M; $\omega_U = \sigma^*\omega$ (which is a $\mathfrak{gl}(n,\mathbb{R})$-valued 1-form on U). Then:

$$\omega_U = \Gamma^i_{jk}\,\mathrm{d}x^j \otimes E^k_i$$

Also, for local coordinate functions (x^i) on M,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

• Connection form ω of a linear connection with Christoffel symbols Γ^i_{jk} in terms of the local coordinates (x^i, x^j_k) on FM:

$$\omega^i_j = Y^i_k\left(\mathrm{d}x^k_j + \Gamma^k_{ml} x^l_j \mathrm{d}x^m\right), \quad i,j,k,l,m = 1,\dots,n = \dim M$$

• Structure equations (frame bundle).

∇ a linear connection of M, with connection form ω, torsion form $\Theta = (\Theta^i)$, and curvature form $\Omega = (\Omega^i_j)$; θ the canonical 1-form on FM; $X, Y \in T_z(TM)$; $i, j, k = 1, \dots, n = \dim M$:

$$\begin{aligned} \mathrm{d}\theta(X,Y) &= -(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)) + \Theta(X,Y) \\ \mathrm{d}\omega(X,Y) &= -[\omega(X),\omega(Y)] + \Omega(X,Y) \\ \mathrm{d}\theta^i &= -\omega^i_j \wedge \theta^j + \Theta^i \\ \mathrm{d}\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \Omega^i_j \end{aligned}$$

• Covariant differentiation ∇_X ($\mathscr{T}(M)$ = algebra of tensor fields on M):

(1) $\nabla_X \colon \mathscr{T}(M) \to \mathscr{T}(M)$ is a type-preserving derivation.

(2) ∇_X commutes with every contraction.

(3) $\nabla_X f = Xf, \quad f \in C^\infty M.$

(4) $\nabla_{X+Y} = \nabla_X + \nabla_Y, \quad X, Y \in \mathfrak{X}(M).$

(5) $\nabla_X(fK) = (Xf)K + f\nabla_X K, \quad K \in \mathscr{T}(M).$

• Covariant derivative of a $(0,r)$ tensor field Ψ:

$$(\nabla_Y \Psi)(X_1,\dots,X_r) = Y(\Psi(X_1,\dots,X_r)) - \sum_{i=1}^{r} \Psi(X_1,\dots,\nabla_Y X_i,\dots,X_r)$$

• Relation between exterior differential and covariant derivative for a differential r-form α:

$$\mathrm{d}\alpha(X_0,\dots,X_r) = \textstyle\sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\alpha)(X_0,\dots,\widehat{X_i},\dots,X_r)$$

• Second covariant derivative:

$$(\nabla^2 s)_{X,Y} = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s$$

• Torsion tensor and curvature tensor field of a linear connection in terms of covariant differentiation:

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \\ R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \end{aligned}$$

• Torsion tensor and curvature tensor field:

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$$

$$R(e_i,e_j)e_k = R^l_{kij} e_l$$

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^r_{lj}\Gamma^i_{kr} - \Gamma^r_{kj}\Gamma^i_{lr}$$

$$R^k_{lij} = -R^k_{lji}$$

$$R^k_{lij} = -R^l_{kij}$$

$$R^l_{kij} + R^l_{ijk} + R^l_{jki} = 0$$

$$R(X,Y)Z = -R(Y,X)Z$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

• Bianchi identities:

$$\text{(1st)} \qquad \underset{XYZ}{\mathfrak{S}} R(X,Y)Z = \underset{XYZ}{\mathfrak{S}} \{T(T(X,Y),Z) + (\nabla_X T)(Y,Z)\}$$

$$(\text{1st}, T=0) \qquad \underset{XYZ}{\mathfrak{S}} R(X,Y)Z = 0$$

$$\text{(2nd)} \qquad \underset{XYZ}{\mathfrak{S}} \{(\nabla_X R)(Y,Z) + R(T(X,Y),Z)\} = 0$$

$$(\text{2nd}, T=0) \qquad \underset{XYZ}{\mathfrak{S}} (\nabla_X R)(Y,Z) = 0$$

• Covariant and double covariant derivative of tensor fields for a linear connection Γ^i_{jk}, with torsion tensor T^i_{jk} and curvature tensor field R^i_{jkl}:

Vector field with components X^i:

$$X^i_{;j} = \partial_j X^i + \Gamma^i_{jr} X^r$$

Differential 1-form with components ω_i:

$$\omega_{j;i} = \partial_i \omega_j - \Gamma^r_{ij}\omega_r$$

$$\omega_{i;jk} - \omega_{i;kj} = R^r_{ijk}\omega_r + 2\,T^r_{jk}\omega_{i;r}$$

$(1,1)$ tensor field with components J^i_j:

$$J^i_{j;k} = \partial_k J^i_j + J^r_j \Gamma^i_{kr} - \Gamma^r_{kj} J^i_r.$$

$(0,2)$ tensor field with components τ_{ij}:

$$\tau_{ij;k} = \partial_i \tau_{jk} - \Gamma^r_{ij}\tau_{rk} - \Gamma^r_{ik}\tau_{jr}$$

(r,s) tensor field with components $K^{i_1\dots i_r}_{j_1\dots j_s}$:

$$K^{i_1\dots i_r}_{j_1\dots j_s;k} = \frac{\partial K^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^k} + \sum_{\alpha=1}^{r} \left(\Gamma^{i_\alpha}_{kl} K^{i_1\dots l\dots i_r}_{j_1\dots j_s} \right) - \sum_{\beta=1}^{s} \left(\Gamma^m_{kj_\beta} K^{i_1\dots i_r}_{j_1\dots m\dots j_s} \right)$$

A Ricci identity:

$$\nabla_l \nabla_k K^{i_1 \dots i_r}_{j_1 \dots j_s} - \nabla_k \nabla_l K^{i_1 \dots i_r}_{j_1 \dots j_s} = \sum_{\rho=1}^{r} K^{i_1 \dots i_{\rho-1} i i_{\rho+1} \dots i_r}_{j_1 \dots j_s} R^{i_\rho}_{ikl}$$

$$- \sum_{\sigma=1}^{r} K^{i_1 \dots i_r}_{j_1 \dots j_{\sigma-1} j j_{\sigma+1} \dots j_s} R^{j}_{j_\sigma kl} - \nabla_i \delta^{i_1 \dots i_r}_{j_1 \dots j_s} T^{i}_{kl}$$

• Components of the torsion and curvature forms. $\sigma = (X_1, \dots, X_n)$ a moving frame on an open subset U of M. $T(X_j, X_k) = T^i_{jk} X_i$, $R(X_k, X_l) X_j = R^i_{jkl} X_i$, the torsion and curvature tensors of a linear connection of M. Define $\widetilde{T}^i_{jk}$, $\widetilde{R}^i_{jkl} \in C^\infty(FM)$ by

$$\Theta^i = \sum_{j<k} \widetilde{T}^i_{jk} \theta^j \wedge \theta^k = \frac{1}{2} \widetilde{T}^i_{jk} \theta^j \wedge \theta^k, \qquad \widetilde{T}^i_{jk} = -\widetilde{T}^i_{kj}$$

$$\Omega^i_j = \sum_{k<l} \widetilde{R}^i_{jkl} \theta^k \wedge \theta^l = \frac{1}{2} \widetilde{R}^i_{jkl} \theta^k \wedge \theta^l, \qquad \widetilde{R}^i_{jkl} = -\widetilde{R}^i_{jlk}$$

Then

$$\sigma^* \widetilde{T}^i_{jk} = T^i_{jk}, \qquad \sigma^* \widetilde{R}^i_{jkl} = R^i_{jkl}$$

• Cartan's structure equations (moving frame).

$\sigma = (X_1, \dots, X_n)$ a moving frame defined on an open subset U of M; $\widetilde{\theta}^i = \sigma^* \theta^i$, $\widetilde{\omega}^i_j = \sigma^* \omega^i_j$. Then:

$$\mathrm{d}\widetilde{\theta}^i = -\widetilde{\omega}^i_j \wedge \widetilde{\theta}^j + \frac{1}{2} T^i_{jk} \widetilde{\theta}^j \wedge \widetilde{\theta}^k$$

$$\mathrm{d}\widetilde{\omega}^i_j = -\widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j + \frac{1}{2} R^i_{jkl} \widetilde{\theta}^k \wedge \widetilde{\theta}^l$$

• Structure equations (geodesic polar coordinates).

$\{e_1, \dots, e_n\}$ a basis of T_pM, $p \in M$; x^i normal coordinates defined by $\{e_i\}$ on a normal coordinate neighborhood U of p; $(X_1, \dots, X_n)$ the moving frame defined on U by parallel transport of $\{e_1, \dots, e_n\}$ along geodesic rays from p. Let F map an open set of $\mathbb{R}^{n+1}$ into U by $x^i(F(t; a^1, \dots, a^n)) = ta^i$. Define f^i, β^i, β^i_j by $F^* \widetilde{\theta}^i = f^i \, \mathrm{d}t + \beta^i$, $F^* \widetilde{\omega}^i_j = \beta^i_j$, where $\widetilde{\theta}^i = \sigma^* \theta^i$, $\widetilde{\omega}^i_j = \sigma^* \omega^i_j$, and β^i does not depend on $\mathrm{d}t$. Then $f^i(t; a^1, \dots, a^n) = a^i$; β^i_j does not depend on $\mathrm{d}t$; and we have (structure equations):

$$\frac{\partial \beta^i}{\partial t} = \mathrm{d}a^i + a^j \beta^i_j + T^i_{jk} a^j \beta^k, \qquad \frac{\partial \beta^i_j}{\partial t} = R^i_{jkl} a^k \beta^l,$$

with initial data $\beta^i(t; a^k, \mathrm{d}a^l)_{t=0} = 0 = \beta^j_i(t; a^k, \mathrm{d}a^l)_{t=0}$.

• Differential equations of geodesics ($i, j, k = 1, \ldots, n = \dim M$):

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0$$

• Covariant derivative on a vector bundle (E, π, M) over M. ($\Gamma E = (C^\infty M)$-module of C^∞ sections of E):

$$\begin{gathered}
\nabla \colon \mathfrak{X}(M) \times \Gamma E \to \Gamma E, \qquad (X, s) \mapsto \nabla_X s \\
\nabla_{fX+hY} s = f \nabla_X s + h \nabla_Y s, \qquad f, h \in C^\infty M, \quad X, Y \in \mathfrak{X}(M) \\
\nabla_X (s+t) = \nabla_X s + \nabla_X t, \qquad s, t \in \Gamma E \\
\nabla_X (fs) = (Xf)s + f \nabla_X s
\end{gathered}$$

Reductive homogeneous spaces

$M = G/H$ a homogeneous reductive space. G (with Lie algebra $\mathfrak{g}$) acts transitively and effectively on M. Reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}.$$

Isotropy representation $\lambda \colon H \to \mathrm{Aut}(T_o M)$, $\lambda(h) = (L_h)_{*o}$, $o \in G/H$ the origin.

Let P be a G-invariant K-structure over the reductive homogeneous space $M = G/H$ with reductive decomposition as above. There is a one-to-one correspondence between the set of G-invariant connections in P and the set of linear maps $\Lambda_{\mathfrak{m}} \colon \mathfrak{m} \to \mathfrak{k}$ such that $\Lambda_{\mathfrak{m}}(X) = \lambda(X)$ for $X \in \mathfrak{h}$ and

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}_h X) = \mathrm{Ad}_{\lambda(h)}(\Lambda_{\mathfrak{m}}(X)), \qquad X \in \mathfrak{m}, \quad h \in H.$$

• Torsion tensor and curvature operator at $o \in G/H$ for the invariant connection corresponding to $\Lambda_{\mathfrak{m}}$ ($X, Y \in \mathfrak{m}$):

$$\begin{aligned}
T(X,Y)_o &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}} \\
R(X,Y)_o &= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \lambda([X,Y]_{\mathfrak{h}})
\end{aligned}$$

• Curvature form Ω of the canonical invariant connection ω on G/H:

$$\Omega(X,Y) = -\frac{1}{2}[X,Y]_{\mathfrak{h}}, \qquad X, Y \in \mathfrak{m}$$

• Torsion tensor and curvature tensor field at $o \in G/H$ of the canonical connection ∇ ($\Lambda_{\mathfrak{m}} = 0$), ($X, Y, Z \in \mathfrak{m}$):

$$\begin{gathered}
T(X,Y)_o = -[X,Y]_{\mathfrak{m}} \\
(R(X,Y)Z)_o = -[[X,Y]_{\mathfrak{h}}, Z]
\end{gathered}$$

$$\nabla T = 0$$
$$\nabla R = 0$$

• Unique torsionless G-invariant connection on G/H with the same geodesics as the canonical connection:

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}}, \qquad X,Y \in \mathfrak{m}$$

ALMOST COMPLEX MANIFOLDS

• Canonical complex structure of $\mathbb{R}^{2n}$ induced from that of $\mathbb{C}^n$:

$$\begin{array}{ccc} \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ (x^1,\dots,x^n,y^1,\dots,y^n) & \mapsto & (y^1,\dots,y^n,-x^1,\dots,-x^n) \end{array}$$

Matrix with respect to the natural basis of $\mathbb{R}^{2n}$:

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

• Real representation of the general linear group:

$$\begin{array}{ccc} GL(n,\mathbb{C}) & \to & GL(2n,\mathbb{R}) \\ A+\mathrm{i}B & \mapsto & \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \end{array}$$

• V vector space with complex structure J; $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ complexified space of V; again J the extension of J to V^c. Eigenspaces of J in V^c:

$$V^{1,0} = \{Z \in V^c : JZ = \mathrm{i}Z\} = \{X - \mathrm{i}JX : X \in V\}$$
$$V^{0,1} = \{Z \in V^c : JZ = -\mathrm{i}Z\} = \{X + \mathrm{i}JX : X \in V\}$$

• Standard almost complex structure J on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. Defined by

$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}$$

• Cauchy-Riemann equations:

$f\colon U \subset \mathbb{C}^n = \{z^l = x^l + \mathrm{i}y^l\} \to \mathbb{C}^m = \{w^k = u^k + \mathrm{i}v^k\}$ holomorphic:

$$\frac{\partial u^k}{\partial x^l} = \frac{\partial v^k}{\partial y^l}, \qquad \frac{\partial u^k}{\partial y^l} = -\frac{\partial v^k}{\partial x^l}$$

• Torsion of an almost complex structure J:

$$N(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

• Basis of $T_p^{1,0}M$ and $T_p^{0,1}M$ for a complex manifold M:

$$\left\{ \frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + \mathrm{i}\frac{\partial}{\partial y^k}\right) \right\}$$

($z^1, \dots, z^n$, $z^k = x^k + \mathrm{i}y^k$ = complex local coordinate functions; and $\mathrm{d}z^k = \mathrm{d}x^k + \mathrm{i}\mathrm{d}y^k$, $\mathrm{d}\bar{z}^k = \mathrm{d}x^k - \mathrm{i}\mathrm{d}y^k$)

• Holomorphic vector field on a complex manifold of complex dimension n:

$$Z = f^k \frac{\partial}{\partial z^k}, \qquad f^k \text{ a holomorphic function } (\bar{\partial} f = 0), \quad k = 1, \dots, n$$

• Cartan structure equations (almost complex linear connection).

$C(M)$ the bundle of complex linear frames on an almost complex manifold M of real dimension $2n$; θ the canonical form on $C(M)$ = restriction of θ on FM to $C(M)$; ω = connection form of an almost complex linear connection with torsion form Θ and curvature form Ω; ω and Ω are valued on the subalgebra $\mathfrak{gl}(n,\mathbb{C})$ of $\mathfrak{gl}(2n,\mathbb{R})$. Set

$$\begin{aligned} &\varphi^\alpha = \theta^\alpha + \mathrm{i}\theta^{n+\alpha}, \quad \Phi^\alpha = \Theta^\alpha + \mathrm{i}\Theta^{n+\alpha}, \quad \alpha = 1, \dots, n \\ &\psi^\alpha_\beta = \omega^\alpha_\beta + \mathrm{i}\omega^\alpha_{n+\beta}, \quad \Psi^\alpha_\beta = \Omega^\alpha_\beta + \mathrm{i}\Omega^\alpha_{n+\beta}, \quad \alpha, \beta = 1, \dots, n \end{aligned}$$

($\varphi = (\varphi^\alpha)$ and $\Phi = (\Phi^\alpha)$ are $\mathbb{C}^n$-valued; $\psi = (\psi^\alpha_\beta)$ and $\Psi = (\Psi^\alpha_\beta)$ with values in $\mathfrak{gl}(n,\mathbb{C})$, as the Lie algebra of $n \times n$ complex matrices). Then, besides the real structure equations we can write:

$$\begin{aligned} \mathrm{d}\varphi^\alpha &= -\psi^\alpha_\beta \wedge \varphi^\beta + \Phi^\alpha, \qquad && \alpha = 1, \dots, n \\ \mathrm{d}\psi^\alpha_\beta &= -\psi^\alpha_\gamma \wedge \psi^\gamma_\beta + \Psi^\alpha_\beta, \qquad && \alpha, \beta = 1, \dots, n \end{aligned}$$

SOME PROPERTIES OF SPHERES

	S^1	S^2	S^3	S^4	S^5	S^6	S^7	$S^n(*)$
Lie group	**y**	n	**y**	n	n	n	n	n
Parallelizable	**y**	n	**y**	n	n	n	**y**	n
Almost complex	n	**y**	n	n	n	**y**	n	n

y = yes, n = no, $(*)$ $n > 7$

Chapter 6

• Musical isomorphisms associated to a metric g on M:

$$\flat\colon T_pM \to T_p^*M, \quad X^\flat(=\flat(X)) = g(X,\cdot)$$

$$\sharp\colon T_p^*M \to T_pM, \quad \alpha^\sharp(=\sharp(\alpha)) = g^{-1}(\alpha,\cdot)$$

• Arc length $L(\sigma)$ of a differentiable curve $\sigma = x_t$, $a \leqslant t \leqslant b$, in a Riemannian n-manifold (M,g) ($(x^1,\dots,x^n)$ local coordinates):

$$L(\sigma) = \int_a^b \sqrt{g(x_t',x_t')}\,\mathrm{d}t, \qquad L(\sigma) = \int_a^b \sqrt{g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t$$

• Energy of a curve $\sigma\colon [a,b] \to M$ in (M,g):

$$E(\sigma) = \frac{1}{2}\int_a^b |\sigma'(t)|^2\,\mathrm{d}t$$

• Poincaré upper half-plane:

$$M = \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}, \qquad \mathrm{d}s^2 = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$$

• Koszul formula for the Levi-Civita connection:

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned}$$

• Christoffel symbols:

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

• Geodesic through $p \in S^n$ (with the round metric) with initial velocity vector $v \in T_pS^n$:

$$\gamma(t) = (\cos|v|t)p + (\sin|v|t)\frac{v}{|v|}$$

RIEMANN CURVATURE TENSOR

• Riemann curvature tensor ($g_{ij} = g(e_i, e_j)$; (e_i) a local frame):

$$R(X,Y,Z,W) = g(R(Z,W)Y,X)$$

$$g(R(X,Y)Z,W) = -g(R(X,Y)W,Z) = -g(R(Y,X)Z,W)$$

$$g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$$

$$R_{ijkl} = R(e_i,e_j,e_k,e_l) = g(R(e_k,e_l)e_j,e_i) = g_{ih}R^h_{jkl}$$

$$R_{klij} = -R_{lkij} = -R_{klij}$$

$$R_{klij} = R_{ijkl}$$

$$R_{lkij} + R_{lijk} + R_{ljki} = 0$$

• Metric and Riemann curvature tensor near the origin p, $x^i(p) = 0$, of normal coordinates (x^i) (letting $R_{ikjl,r} = \partial R_{ikjl}/\partial x^r$):

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}x^kx^l - \frac{1}{3!}R_{ikjl,r}x^kx^lx^r$$
$$+ \frac{1}{5!}\left(-6R_{ikjl,rs} + \frac{4}{3}R_{iktl}R^t_{rjs}\right)x^kx^lx^rx^s + \cdots$$

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right)$$

• Sectional curvature for a plane $P \subset T_pM$:

$$K(P) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \qquad X,Y \text{ basis of } P$$

$$K(P) = R(X,Y,X,Y), \qquad X,Y \text{ orthonormal basis}$$

• Ricci tensor ((e_i) = a local orthonormal frame):

$$r(X,Y) = \text{trace of the map } Z \mapsto R(Z,X)Y \text{ of } T_pM$$

$$r(X,Y) = \textstyle\sum_i g(R(e_i,X)Y,e_i) = \sum_i R(e_i,X,e_i,Y)$$

$$r_{ij} = \sum_k R_{ikjk} = R^k_{jki}$$

• Scalar curvature:

$$s = g^{ij} r_{ij} = g^{ij} r(e_i, e_j) \qquad \big((e_i) \text{ a local frame}\big)$$

$$s = \sum r(e_i, e_i) \qquad \big((e_i) \text{ a local orthonormal frame}\big)$$

• Weyl conformal curvature tensor for a Riemannian n-manifold (M, g):

$$W(X,Y)Z = R(X,Y)Z + L(Y,Z)X - L(X,Z)Y + g(Y,Z)L^*X - g(X,Z)L^*Y$$

$$\left(L(X,Y) = -\frac{1}{n-2}\, r(X,Y) + \frac{s}{2(n-1)(n-2)}\, g(X,Y);\ g(L^*X,Y) = L(X,Y) \right)$$

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}(r_{jk}\delta^i_l - r_{jl}\delta^i_k + g_{jk}r^i_l - g_{jl}r^i_k) + \frac{s}{(n-1)(n-2)}(g_{jk}\delta^i_l - g_{jl}\delta^i_k)$$

• Weyl projective curvature tensor ($n > 1$):

$$P^i_{jkl} = R^i_{jkl} - \frac{1}{n-1}(r_{jk}\delta^i_l - r_{jl}\delta^i_k)$$

Kähler manifolds

• Hermitian metric on an almost complex manifold (M, J):

$$g(JX, JY) = g(X, Y),$$
$$g = 2\, g_{\alpha\bar\beta}\, \mathrm{d}z^\alpha\, \mathrm{d}\bar z^\beta$$

• (M, g, J) an almost Hermitian manifold. Holomorphic sectional curvature:

$$H(X) = g(R(X,JX)X, JX), \qquad X \in T_pM, \quad |X| = 1, \quad p \in M$$

• Fundamental 2-form of a Hermitian metric:

$$F(X,Y) = g(X, JY), \qquad F = -2\mathrm{i}\, g_{\alpha\bar\beta}\, \mathrm{d}z^\alpha \wedge \mathrm{d}\bar z^\beta$$

• Kähler metric:

$$\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar\beta}}{\partial z^\alpha} \quad \text{or} \quad \frac{\partial g_{\alpha\bar\beta}}{\partial z^{\bar\gamma}} = \frac{\partial g_{\alpha\bar\gamma}}{\partial z^{\bar\beta}}$$

• Curvature components:

$$R^\alpha_{\beta\gamma\bar\delta} = -\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial \bar z^\delta},$$

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z^\gamma \partial \bar{z}^\delta} - \sum_{\tau,\varepsilon} g^{\bar{\varepsilon}\tau} \frac{\partial g_{\alpha\bar{\varepsilon}}}{\partial z^\gamma} \frac{\partial g_{\bar{\beta}\tau}}{\partial \bar{z}^\delta}$$

• Ricci form:

$$\rho(X,Y) = r(X,JY),$$

$$\rho = -2\mathrm{i}\, r_{\alpha\bar{\beta}}\, \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta$$

• Structure equations (bundle of unitary frames).

$U(M)$ = bundle of unitary frames; θ = canonical form on $U(M)$; $\omega = (\omega^i_j)$, $i,j = 1,\dots,2n$ = connection form on $U(M)$ defining the Levi-Civita connection of the Kähler manifold M; Ω = curvature form (ω and Ω with values in the real representation of $\mathfrak{u}(n)$). Setting (for $\alpha,\beta = 1,\dots,n$)

$$\varphi^\alpha = \theta^\alpha + \mathrm{i}\theta^{n+\alpha}, \qquad \psi^\alpha_\beta = \omega^\alpha_\beta + \mathrm{i}\omega^\alpha_{n+\beta}, \qquad \Psi^\alpha_\beta = \Omega^\alpha_\beta + \mathrm{i}\Omega^\alpha_{n+\beta},$$

we have

$$\omega^\alpha_\beta = \omega^{n+\alpha}_{n+\beta}, \qquad \omega^\alpha_{n+\beta} = -\omega^{n+\alpha}_\beta, \qquad \omega^\alpha_\beta = -\omega^\beta_\alpha, \qquad \omega^\alpha_{n+\beta} = \omega^\beta_{n+\alpha}$$

$$\Omega^\alpha_\beta = \Omega^{n+\alpha}_{n+\beta}, \qquad \Omega^\alpha_{n+\beta} = -\Omega^{n+\alpha}_\beta, \qquad \Omega^\alpha_\beta = -\Omega^\beta_\alpha, \qquad \Omega^\alpha_{n+\beta} = \Omega^\beta_{n+\alpha}$$

Hence

$$\psi^\alpha_\beta = -\bar{\psi}^\beta_\alpha, \qquad \Psi^\alpha_\beta = -\bar{\Psi}^\beta_\alpha$$

• Riemann curvature tensor and curvature form on the bundle of unitary frames $U(M)$, of a Kähler manifold (M,g) of constant holomorphic sectional curvature c:

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{c}{2}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\bar{\beta}\gamma}\right)$$

$$\Psi^\alpha_\beta = \frac{c}{2}\left(\varphi^\alpha \wedge \bar{\varphi}^\beta + \delta_{\alpha\beta}\sum_\gamma \varphi^\gamma \wedge \bar{\varphi}^\gamma\right)$$

• Bochner curvature tensor for a Kähler manifold (M,g,J) of real dimension n:

$$\begin{aligned} B(X,Y,Z,W) = R(X,Y,Z,W) &- L(X,W)g(Y,Z) - L(X,Z)g(Y,W) \\ &+ L(Y,Z)g(X,W) - L(Y,W)g(X,Z) + L(JX,W)g(JY,Z) \\ &- L(JX,Z)g(JY,W) + L(JY,Z)g(JX,W) - L(JY,W)g(JX,Z) \\ &- 2L(JX,Y)g(JZ,W) - 2L(JZ,W)g(JX,Y) \end{aligned}$$

$$\left(L(X,Y) = -\frac{1}{n+4}r(X,Y) + \frac{s}{2(n+2)(n+4)}g(X,Y)\right)$$

CHARACTERISTIC FORMS

• rth Chern class $c_r(E)$ of a complex vector bundle E over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle $(P, p, M, GL(n,\mathbb{C}))$. Represented by the Chern form $\alpha_r \in \Lambda^{2r}M$:

$$p^*(\alpha_r) = \frac{(-1)^r}{(2\pi\mathrm{i})^r r!}\,\delta^{j_1\dots j_r}_{i_1\dots i_r}\,\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_r}_{j_r},$$

where one sums over all ordered subsets $(i_1,\dots,i_r)$ of $(1,\dots,n)$ and all permutations $(j_1,\dots,j_r)$ of $(i_1,\dots,i_r)$ and where $\delta^{j_1\dots j_r}_{i_1\dots i_r}$ denotes the sign of the permutation.

• rth Pontrjagin class $p_r(E)$ of a real vector bundle E over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle $(P, p, M, GL(n,\mathbb{R}))$. Represented by the Pontrjagin form $\beta_r \in \Lambda^{4r}M$:

$$p^*(\beta_r) = \frac{1}{(2\pi)^{2r}(2r)!}\,\delta^{j_1\dots j_{2r}}_{i_1\dots i_{2r}}\,\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_{2r}}_{j_{2r}},$$

where one sums over all the ordered subsets $(i_1,\dots,i_{2r})$ of $2r$ elements of $(1,\dots,n)$ and all permutations $(j_1,\dots,j_{2r})$ of $(i_1,\dots,i_{2r})$.

• Euler class $e(E)$ of an oriented real vector bundle E of rank $2r$ (with a fibre metric) over the differentiable manifold M in terms of the curvature form components Ω^i_j of a connection in the corresponding principal bundle $(P, p, M, SO(2r))$. Represented by $\gamma \in \Lambda^{2r}M$:

$$p^*(\gamma) = \frac{(-1)^r}{2^{2r}\pi^r\cdot r!}\sum_{i_1,\dots,i_{2r}}\varepsilon_{i_1\dots i_{2r}}\,\Omega^{i_1}_{i_2}\wedge\cdots\wedge\Omega^{i_{2r-1}}_{i_{2r}}$$

HOMOGENEOUS RIEMANNIAN MANIFOLDS

• $\pi\colon (\widetilde{M},\widetilde{g}) \to (M,g)$ a Riemannian submersion; X,Y orthonormal vector fields on M with horizontal lifts $\widetilde{X},\widetilde{Y}$; Z^v = vertical lift of $Z \in \mathfrak{X}(M)$. Sectional curvature:

$$K_M(X,Y) = K_{\widetilde{M}}(\widetilde{X},\widetilde{Y}) + \frac{3}{4}\,|\,[X,Y]^v|^2$$

• Levi-Civita connection of $(M = G/H, g)$ reductive homogeneous; $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ reductive decomposition; $\langle\ ,\ \rangle$ an $\mathrm{Ad}\,(H)$-invariant nondegenerate symmetric bilinear form on $\mathfrak{m}$ ($\langle X,Y\rangle = g_o(X,Y)$, $X,Y\in\mathfrak{m}$, $\mathfrak{m}\equiv T_oM$):

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y),$$

$U\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ defined by

$$2\langle U(X,Y),Z\rangle = \langle [Z,X]_{\mathfrak{m}},Y\rangle + \langle X,[Z,Y]_{\mathfrak{m}}\rangle, \qquad X,Y,Z\in\mathfrak{m}$$

• $M = (G/H, g)$ a naturally reductive homogeneous Riemannian manifold ($U \equiv 0$). Levi-Civita connection and Riemann curvature tensor at o ($X, Y \in \mathfrak{m}$):

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}$$

$$g_o(R(X,Y)Y,X) = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle - \langle [[X,Y]_{\mathfrak{h}}, Y], X\rangle$$

• $M = (G/H, g)$ a normal homogeneous Riemannian manifold (There exists an $\mathrm{Ad}(G)$-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\langle\ ,\ \rangle_{\mathfrak{h}}$ is nondegenerate); $\mathfrak{m}$ the orthogonal complement to $\mathfrak{h}$ for $\langle\ ,\ \rangle$; $X, Y \in \mathfrak{m}$. Sectional curvature:

$$g(R(X,Y)Y,X)_{\mathfrak{h}} = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle_{\mathfrak{m}} + \langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}\rangle_{\mathfrak{h}}$$

CURVATURE AND KILLING VECTOR FIELDS

• (M, g) Riemannian manifold; X, Y, Z, Killing vector fields; Levi-Civita connection:

$$2g(\nabla_X Y, Z) = g([X,Y],Z) + g([X,Z],Y) + g(X,[Y,Z])$$

• $(M = G/H, g)$ Riemannian homogeneous; $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$; X, Y, Killing vector fields in $\mathfrak{p}$; $\mathfrak{p} \equiv T_oM$; Levi-Civita connection and curvature at o:

$$\begin{aligned}
(\nabla_X Y)_o &= -\frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y) \\
g_o(R(X,Y)Y,X) &= -\frac{3}{4}|[X,Y]_{\mathfrak{p}}|^2 - \frac{1}{2}\langle [X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{p}}, Y\rangle \\
&\quad - \frac{1}{2}\langle [Y,[Y,X]_{\mathfrak{g}}]_{\mathfrak{p}}, X\rangle + |U(X,Y)|^2 - \langle U(X,X), U(Y,Y)\rangle
\end{aligned}$$

RIEMANNIAN SYMMETRIC SPACES

$$M = (G/H, g, \sigma), \qquad \mathfrak{h} = (\sigma_{*o})_+, \qquad \mathfrak{m} = (\sigma_{*o})_-, \qquad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}, \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$$

• Curvature tensor field ($X, Y, Z \in \mathfrak{m}$):

$$R(X,Y)Z = -[[X,Y],Z]$$

• Ricci tensor for G/H Hermitian symmetric with G semisimple and effective on G/H, and H compact, in terms of the Killing form B of $\mathfrak{g}$:

$$r(X,Y) = -\frac{1}{2}B(X,Y), \qquad X, Y \in \mathfrak{m}$$

IRREDUCIBLE RIEMANNIAN SYMMETRIC SPACES OF TYPE I AND III

	Compact	Noncompact	rank	dim
A I	$SU(n)/SO(n)$	$SL(n,\mathbb{R})/SO(n)$	$n-1$	$\frac{(n-1)(n+2)}{2}$
A II	$SU(2n)/Sp(n)$	$SU^*(2n)/Sp(n)$	$n-1$	$2n^2-n-1$
A III $(*)$	$\frac{SU(p+q)}{S(U(p)\times U(q))}$	$\frac{SU(p,q)}{S(U(p)\times U(q))}$	$\min(p,q)$	$2pq \quad (**)$
BD I $(*)$	$\frac{SO(p+q)}{SO(p)\times SO(q)}$	$\frac{SO_0(p+q)}{SO(p)\times SO(q)}$	$\min(p,q)$	pq
D III $(*)$	$SO(2n)/U(n)$	$SO^*(2n)/U(n)$	$[\frac{1}{2}n]$	$n(n-1)$
C I $(*)$	$Sp(n)/U(n)$	$Sp(n,\mathbb{R})/U(n)$	n	$n(n+1)$
C II	$\frac{Sp(p+q)}{Sp(p)\times Sp(q)}$	$\frac{Sp(p,q)}{Sp(p)\times Sp(q)}$	$\min(p,q)$	$4pq \quad (**)$
E I	$(\mathfrak{e}_6^{-78},\mathfrak{sp}(4))$	$(\mathfrak{e}_6^{6},\mathfrak{sp}(4))$	6	42
E II	$(\mathfrak{e}_6^{-78},\mathfrak{su}(6)+\mathfrak{su}(2))$	$(\mathfrak{e}_6^{2},\mathfrak{su}(6)+\mathfrak{su}(2))$	4	40
E III $(*)$	$(\mathfrak{e}_6^{-78},\mathfrak{so}(10)+\mathbb{R})$	$(\mathfrak{e}_6^{-14},\mathfrak{so}(10)+\mathbb{R})$	2	32
E IV	$(\mathfrak{e}_6^{-78},\mathfrak{f}_4)$	$(\mathfrak{e}_6^{-26},\mathfrak{f}_4)$	2	26
E V	$(\mathfrak{e}_7^{-133},\mathfrak{su}(8))$	$(\mathfrak{e}_7^{7},\mathfrak{su}(8))$	7	70
E VI	$(\mathfrak{e}_7^{-133},\mathfrak{so}(12)+\mathfrak{su}(2))$	$(\mathfrak{e}_7^{-5},\mathfrak{so}(12)+\mathfrak{su}(2))$	4	64
E VII $(*)$	$(\mathfrak{e}_7^{-133},\mathfrak{e}_6+\mathbb{R})$	$(\mathfrak{e}_7^{-25},\mathfrak{e}_6+\mathbb{R})$	3	54
E VIII	$(\mathfrak{e}_8^{-248},\mathfrak{so}(16))$	$(\mathfrak{e}_8^{8},\mathfrak{so}(16))$	8	128
E IX	$(\mathfrak{e}_8^{-248},\mathfrak{e}_7+\mathfrak{su}(2))$	$(\mathfrak{e}_8^{-24},\mathfrak{e}_7+\mathfrak{su}(2))$	4	112
F I	$(\mathfrak{f}_4^{-52},\mathfrak{sp}(3)+\mathfrak{su}(2))$	$(\mathfrak{f}_4^{4},\mathfrak{sp}(3)+\mathfrak{su}(2))$	4	28
F II	$(\mathfrak{f}_4^{-52},\mathfrak{so}(9))$	$(\mathfrak{f}_4^{-20},\mathfrak{so}(9))$	1	16
G	$(\mathfrak{g}_2^{-14},\mathfrak{su}(2)+\mathfrak{su}(2))$	$(\mathfrak{g}_2^{2},\mathfrak{su}(2)+\mathfrak{su}(2))$	2	8

$(*)$ Hermitian symmetric (for BD I, only if $q=2$)

$(**)$ $(p \leqslant q)$.

REMARK. The superindices for the exceptional simple Lie algebras denote the signature of the corresponding Killing form B, where the signature is defined here as the number of positive values minus the number of negative values when B is expressed in diagonal form (see [18]): $\mathfrak{e}_6$: $-78,-26,-14,2,6$; $\mathfrak{e}_7$: $-133,-25,-5,7$; $\mathfrak{e}_8$: $-248,-24,8$; $\mathfrak{f}_4$: $-52,-20,4$; $\mathfrak{g}_2$: $-14,2$.

Symmetric spaces G/H of classical type with noncompact isotropy group [1]

$G = SL(n,\mathbb{C})$	$G = SL(n,\mathbb{R})$	$G = SU(p,q)$
$SL(p,\mathbb{C})$	$SL(p,\mathbb{R})$	$SU(k,k+h)$
$\times SL(q,\mathbb{C}) \times \mathbb{C}$	$\times SL(q,R) \times \mathbb{R}$	$\times SU(p-k,n-k-h) \times U(1)$
$SL(n,\mathbb{R})$	$SO(p,q)$	$SO(p,q)$
$SO(n,\mathbb{C})$	$Sp(n/2,\mathbb{R})$	$Sp(p/2,q/2)$
$SU(p,q)$	$Sp(n/2,\mathbb{C}) \times \mathbb{R}$	$SO^*(n)$ $(*)$
$Sp(n/2,\mathbb{C})$		$Sp(n,\mathbb{R})$ $(*)$
$SU^*(n)$		$SL(n,\mathbb{C}) \times \mathbb{R}$ $(*)$

$G = SU^*(n)$	$G = SO(n,\mathbb{C})$	$G = SO(p,q)$
$SU^*(p)$	$SO(p,\mathbb{C})$	$SO(k,k+h)$
$\times SU^*(q) \times \mathbb{R}$	$\times SO(q,\mathbb{C})$ $(\dagger)$	$\times SO(p-k,n-k-h)$ $(**)$
$Sp(p/2,q/2)$	$SO(n-2) \times \mathbb{C}$	$SO(p-2,q) \times U(1)$
$SO^*(n)$	$SO(p,q)$	$SO(p-1,q-1) \times \mathbb{R}$
$SL(n,\mathbb{C}) \times U(1)$	$SL(n/2,\mathbb{C}) \times \mathbb{C}$	$SU(p/2,q/2) \times U(1)$
	$SO^*(n)$	$SL(n/2,\mathbb{R}) \times \mathbb{R}$ $(*)$
	$SO^*(n/2,\mathbb{C})$	$SO(n/2,\mathbb{C})$ $(*)$
	$SU(p,q) \times U(1)$	$SO(n-2) \times U(1)$ $(\ddagger)$

$G = Sp(n,\mathbb{C})$	$G = Sp(n,\mathbb{R})$	$G = Sp(p,q)$
$SL(n,\mathbb{C}) \times \mathbb{C}$	$Sp(p,\mathbb{R}) \times Sp(q,\mathbb{R})$	$Sp(k,k+h)$
		$\times Sp(p-k,n-k-h)$
$Sp(n,\mathbb{R})$	$SU(p,q) \times U(1)$	$SU(p,q) \times U(1)$
$Sp(p,\mathbb{C}) \times Sp(q,\mathbb{C})$	$SL(n,\mathbb{R}) \times \mathbb{R}$	
$Sp(p,q)$	$Sp(n/2,\mathbb{C})$	
	$SU^*(n) \times \mathbb{R}$ $(*)$	
	$Sp(n/2,\mathbb{C})$	

$(*)$ $p = q = n/2$, $(\dagger)$ $p = 1$ or $p,q > 2$, $(**)$ $k+h > 2$, $n-k-h > 2$, $(\ddagger)$ $p = 2$, $q = n-2$

[1] The isotropy groups for each group G are listed under it. In all cases but for $Sp(n,\mathbb{C})$ and $Sp(n,\mathbb{R})$, the expression of the first listed isotropy group has been broken in two lines.

Spaces of constant (ordinary or holomorphic) curvature

• Riemann curvature tensor for constant curvature c:

$$R(X,Y,Z,W) = c\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big)$$

• Metric of nonzero constant curvature $1/r$ on

$$M = \left\{(x^1,\dots,x^{n+1},t) \in \mathbb{R}^{n+1} \,:\, (x^1)^2 + \cdots + (x^n)^2 + rt^2 = r\right\}:$$

$$g = \frac{r\left\{(r+\sum(y^i)^2)(\sum(\mathrm{d}y^i)^2) - (\sum y^i \mathrm{d}y^i)^2\right\}}{(r+\sum(y^i)^2)^2}, \qquad y^i = x^i/t$$

• Riemann curvature tensor for constant holomorphic curvature c:

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{4}\big\{&g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &+ g(X,JZ)g(Y,JW) - g(X,JW)g(Y,JZ) + 2g(X,JY)g(Z,JW)\big\} \end{aligned}$$

• Fubini-Study metric of positive constant holomorphic sectional curvature c on the complex projective space $\mathbb{C}P^n$:

$$g = \frac{4}{c}\,\frac{(1+\sum z^k\bar{z}^k)(\sum \mathrm{d}z^k\,\mathrm{d}\bar{z}^k) - (\sum \bar{z}^k\,\mathrm{d}z^k)(\sum z^k\,\mathrm{d}\bar{z}^k)}{(1+\sum z^k\bar{z}^k)^2}$$

• Bergman metric of negative constant holomorphic sectional curvature c on the open unit ball $\left\{(z^1,\dots,z^n) \in \mathbb{C}^n \,:\, \sum z^k\bar{z}^k < 1\right\}$ (cf. [16, p. 73]):

$$g = -\frac{4}{c}\,\frac{(1-\sum z^k\bar{z}^k)(\sum \mathrm{d}z^k\,\mathrm{d}\bar{z}^k) + (\sum \bar{z}^k\,\mathrm{d}z^k)(\sum z^k\,\mathrm{d}\bar{z}^k)}{(1-\sum z^k\bar{z}^k)^2}$$

• Metric of negative constant holomorphic sectional curvature c on the Siegel domain

$$D = \left\{(z = x+\mathrm{i}y, u^1,\dots,u^n) \in \mathbb{C}^{n+1} \,:\, y - \textstyle\sum_{k=1}^n u^k\bar{u}^k > 0\right\}:$$

$$\begin{aligned} g = -\frac{1}{c(y-\sum u^k\bar{u}^k)^2}\Big\{&\mathrm{d}z\,\mathrm{d}\bar{z} + 4\left(y - \sum u^k\bar{u}^k\right)\sum \mathrm{d}u^k\,\mathrm{d}\bar{u}^k \\ &+ 2\mathrm{i}\left(\mathrm{d}z\sum u^k\mathrm{d}\bar{u}^k - \mathrm{d}\bar{z}\sum \bar{u}^k\mathrm{d}u^k\right) + 4\left(\sum \bar{u}^k\mathrm{d}u^k\right)\left(\sum u^k\mathrm{d}\bar{u}^k\right)\Big\}. \end{aligned}$$

Left-invariant metrics on Lie groups

• Koszul formula for the Levi-Civita connection of a left-invariant metric g on a Lie group G with Lie algebra $\mathfrak{g}$ ($X,Y,Z \in \mathfrak{g}$):

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

• Levi-Civita connection, curvature tensor field, and sectional curvature for a compact Lie group G with a bi-invariant metric g (X,Y,Z, left-invariant vector fields):

$$\nabla_X Y = \frac{1}{2}[X,Y]$$
$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$
$$K(\langle X,Y\rangle) = \frac{1}{4}g([X,Y],[X,Y]), \quad X,Y \text{ here orthonormal}$$

Basic Differential Operators

• The gradient:

$$\operatorname{grad} f = \sharp(\mathrm{d}f) = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$$

• Divergence of $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(\operatorname{div} X)(p) = \theta^i(\nabla_{e_i} Z)$$

($\{e_i\},\{\theta^i\}$ = dual bases for T_pM and T_p^*M, $p \in M$; $i = 1,\dots,n = \dim M$)

• Divergence of $X = X^i\partial/\partial x^i$ with respect to (the Levi-Civita connection of) a metric tensor g:

$$\sum_i \frac{1}{\sqrt{\det(g_{jk})}}\frac{\partial\sqrt{\det(g_{jk})X^i}}{\partial x^i}$$

• Divergence of a $(0,r)$ tensor α on (M,g):

$$(\operatorname{div}\alpha)_p(v_1,\dots,v_r) = \sum_i(\nabla_{e_i}\alpha)(e_i,v_1,\dots,v_{r-1})$$

(∇ = Levi-Civita connection; $v_i \in T_pM$; $\{e_i\}$ = orthonormal basis for T_pM, $p \in M$)

• The Hessian:

$$H^f(X,Y) = XYf - (\nabla_X Y)f$$

• Trace of the second covariant derivative:

$$\operatorname{tr}\nabla^2 X = \left(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}\right)X$$

• The Laplacian and the Laplacian on functions:

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta f = -g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial f}{\partial x^k}\right)$$

• Weitzenböck's formula for the Laplacian Δ on (M,g) ($n=\dim M$; $\{e_i\}$ = orthonormal basis of T_pM):

$$g(\Delta\alpha,\alpha)=\frac{1}{2}\Delta|\alpha|^2+|\nabla\alpha|^2+g(\rho_\alpha,\alpha),\qquad \alpha\in\Lambda^rM,$$

where $\rho_\alpha(v_1,\dots,v_r)=\sum_{i=1}^{n}\sum_{j=1}^{r}(R(e_i,v_j)\alpha)(v_1,\dots,v_{j-1},e_i,v_{j+1},\dots,v_r)$

Case of (M,g) of constant sectional curvature c:

$$g(\Delta\alpha,\alpha)=\frac{1}{2}\Delta|\alpha|^2+|\nabla\alpha|^2+r(n-r)c\,|\alpha|^2$$

• For $f,h\in C^\infty M,\ X,Y\in\mathfrak{X}(M)$:

(1) $\operatorname{grad}(fh)=f\operatorname{grad}h+h\operatorname{grad}f$

(2) $\operatorname{div}(fX)=Xf+f\operatorname{div}X$

(3) $H^{fh}=fH^h+hH^f+\mathrm{d}f\otimes\mathrm{d}h+\mathrm{d}h\otimes\mathrm{d}f$

(4) $\Delta(fh)=f\Delta h+h\Delta f+2g(\operatorname{grad}f,\operatorname{grad}h)$

(5) $\operatorname{curl}(\operatorname{grad}f)=0$

(6) $\operatorname{curl}X=\mathrm{d}\alpha$, where α is the 1-form dual metric to X

• Hodge's star operator $\star$: $\Lambda^rM\to\Lambda^{n-r}M$, $0\leqslant r\leqslant n$, on an oriented pseudo-Riemannian n-manifold (ω_g = volume form):

$$\alpha\wedge\star\beta=g^{-1}(\alpha,\beta)\,\omega_g,$$
$$\alpha_p\wedge(\star\beta_p)=\beta_p\wedge(\star\alpha_p)\qquad \alpha,\beta\in\Lambda^r_pM,\quad p\in M$$
$$\star^2=(-1)^{r(n-r)},\qquad \star^{-1}=(-1)^{r(n-r)}\star$$

Conformal changes of Riemannian metrics

• $\tilde g=\mathrm{e}^{2f}g$, $f\in C^\infty(M,g)$, $\dim M=n$, $|\mathrm{d}f|^2=g^{-1}(\mathrm{d}f,\mathrm{d}f)$:
Levi-Civita connection:

$$\widetilde\nabla_XY=\nabla_XY+\mathrm{d}f(X)Y+\mathrm{d}f(Y)X-g(X,Y)\operatorname{grad}f$$

Riemann curvature tensor ($\cdot$ = Kulkarni-Nomizu product):

$$\widetilde R=\mathrm{e}^{2f}\left(R-g\cdot(H^f-\mathrm{d}f\otimes\mathrm{d}f+\frac{1}{2}|\mathrm{d}f^2|^2g)\right)$$

Ricci tensor:

$$\tilde r=r-(n-2)(H^f-\mathrm{d}f\otimes\mathrm{d}f)+(\Delta f-(n-2)\,|\mathrm{d}f^2|^2)g$$

Scalar curvature:

$$\tilde{s} = \mathrm{e}^{-2f}(s + 2(n-1)\Delta f - (n-2)(n-1)|\mathrm{d}f^2|)$$

$(3,1)$ Weyl tensor:

$$\widetilde{W} = W$$

Volume element:

$$\omega_{\tilde{g}} = \mathrm{e}^{nf}\,\omega_g$$

Codifferential on r-forms:

$$\tilde{\delta}\alpha = \mathrm{e}^{-2f}\left(\delta\alpha - (n-2)i_{\operatorname{grad} f}\alpha\right)$$

Hodge's operator on r-forms (for oriented M):

$$\star_{\tilde{g}} = \mathrm{e}^{(n-2r)f}\star_g$$

Some geometric vector fields

• Affine $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(L_X\nabla)(Y,Z) = 0, \qquad Y,Z \in \mathfrak{X}(M)$$
$$(\nabla_Y\nabla X)Z = R(Y,X)Z \quad (\text{if } \nabla \text{ is torsionless})$$

$\big((L_X\nabla)(Y,Z) = [X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z];\ (\nabla_Y\nabla X)Z = \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z}X\big)$

• Projective $X \in \mathfrak{X}(M)$ with respect to a linear connection ∇ of M:

$$(L_X\nabla)(Y,Z) = \theta(Y)Z + \theta(Z)Y, \qquad Y,Z \in \mathfrak{X}(M), \quad \theta \in \Lambda^1 M$$

• Jacobi equation along a geodesic γ:

$$\nabla_{\gamma'}\nabla_{\gamma'}Y + \nabla_{\gamma'}(T(Y,\gamma')) + R(Y,\gamma')\gamma' = 0$$

Volumes of spheres and balls

• Volume of the sphere with the round metric, and volume of the closed unit ball $B^{n+1} = \bar{B}(0,1) \in \mathbb{R}^{n+1}$:

$$\operatorname{vol}(S^{2n+1}) = \frac{2\pi^{n+1}}{n!}, \quad \operatorname{vol}(S^{2n}) = \frac{(n-1)!\,(4\pi)^n}{(2n-1)!}, \quad \operatorname{vol}(B^{n+1}) = \frac{1}{n+1}\operatorname{vol}(S^n)$$

Riemannian Submanifolds

$M \hookrightarrow \widetilde{M}$ an immersion; $X,Y,Z,W \in \mathfrak{X}(M)$, $\xi \in \mathfrak{X}(M)^\perp$, ∇, $\widetilde{\nabla}$ = Levi-Civita connections.

• Gauss's formula:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y)$$
$$\text{where} \quad \nabla_X Y = \tau \widetilde{\nabla}_X Y, \qquad \alpha(X,Y) = \nu \widetilde{\nabla}_X Y$$

($\tau =$ tangential part, $\nu =$ normal part; $\alpha =$ second fundamental form of M for the given immersion)

• Weingarten's formula:

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

where

$$-A_\xi X = \tau \widetilde{\nabla}_X \xi, \quad \nabla^\perp_X \xi = \nu \widetilde{\nabla}_X \xi, \quad g(A_\xi X, Y) = g(\alpha(X,Y), \xi)$$
$$\nabla^\perp_X \colon \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \to \mathfrak{X}(M)^\perp \quad \text{(the normal connection)}$$
$$\nabla^\perp_X W = \nu \widetilde{\nabla}_X W, \qquad X \in \mathfrak{X}(M),\ W \in \mathfrak{X}(M)^\perp$$

• Gauss's equation:

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\alpha(Z,Y),\alpha(W,X)) - g(\alpha(W,Y),\alpha(Z,Y))$$

• Codazzi's equation:

$$\nu \widetilde{R}_{XY} Z = \left(\widehat{\nabla}_X \alpha\right)(Y,Z) - \left(\widehat{\nabla}_Y \alpha\right)(X,Z)$$

where

$$\left(\widehat{\nabla}_X \alpha\right)(Y,Z) = \nabla^\perp_X(\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

• Ricci's equation:

$$\nu \widetilde{R}_{XY} \xi = R^\perp_{XY} \xi - \alpha\left(A_\xi X, Y\right) - \alpha\left(X, A_\xi Y\right)$$

• Mean curvature normal:

$$\eta = \frac{1}{n} \sum_{i=1}^{r} (\operatorname{tr} A_i)\, \xi_i$$

($M = n$-dimensional Riemannian manifold isometrically immersed in an $(n+r)$-dimensional Riemannian manifold N; $\{\xi_1, \ldots, \xi_r\} =$ orthonormal basis in $(T_pM)^\perp$; $A_i = A_{\xi_i}$)

• Riemann curvature tensor on a complex submanifold M of a Kähler manifold $(\widetilde{M}, g, J)$ ($\alpha =$ second fundamental form; $\widetilde{R} =$ Riemann curvature tensor of $\widetilde{M}$, $X \in \mathfrak{X}(M)$):

$$R(X,JX,X,JX) = \widetilde{R}(X,JX,X,JX) - 2g(\alpha(X,X),\alpha(X,X))$$

HYPERSURFACES IN $\mathbb{R}^{n+1}$

M hypersurface in $\mathbb{R}^{n+1}$; $X,Y,Z \in \mathfrak{X}(M)$; ξ field of unit normal vectors defined locally, or globally if this is the case; ∇' = covariant differentiation in $\mathbb{R}^{n+1}$; $A = A_\xi$ = symmetric transformation of each T_pM corresponding to the symmetric bilinear function h on $T_pM \times T_pM$ defined by $\alpha(X,Y) = h(X,Y)\xi$.

• Gauss's formula for hypersurfaces:

$$\nabla'_X Y = \nabla_X Y + h(X,Y)\xi$$

• Weingarten's formula for hypersurfaces:

$$\nabla'_X \xi = -AX$$

• Gauss's equation for hypersurfaces:

$$R(X,Y)Z = g(AY,Z)AX - g(AX,Z)AY$$

• Codazzi's equation for hypersurfaces:

$$(\nabla_X A)Y = (\nabla_Y A)X$$

SURFACES

• Gauss-Bonnet's formula for a compact surface M:

$$\chi(M) = \frac{1}{4\pi}\int_M s\,\omega_g = \frac{1}{2\pi}\int_M K\,\omega_g$$

($\chi(M)$ = Euler characteristic; s = scalar curvature; K = Gauss curvature)

• A parametrization of $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$:

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta, \qquad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi$$

• A parametrization of the torus T^2 (with $R > r$, $\theta,\varphi \in [0,2\pi]$):

$$x = (r\cos\theta + R)\cos\varphi, \quad y = (r\cos\theta + R)\sin\varphi, \quad z = r\sin\theta.$$

• Gauss curvature K of an abstract parametrized surface with metric

$$g = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 = E\,\mathrm{d}u\otimes\mathrm{d}u + F\,\mathrm{d}u\otimes\mathrm{d}v + F\,\mathrm{d}v\otimes\mathrm{d}u + G\,\mathrm{d}v\otimes\mathrm{d}v,$$

$$K = -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & F & G \\ E_u & F_u & G_u \\ E_v & F_v & G_v \end{vmatrix} - \frac{1}{2\sqrt{EG-F^2}}\left\{\left(\frac{G_u - F_v}{\sqrt{EG-F^2}}\right)_u - \left(\frac{F_u - E_v}{\sqrt{EG-F^2}}\right)_v\right\}$$

(here a subindex u, v, denotes the derivative with respect to that variable)

PSEUDO-RIEMANNIAN MANIFOLDS

• Normal coordinates for a pseudo-Riemannian n-manifold (M,g). On a neighborhood of the origin $p \in M$:

(1) $g_{ij}(p) = \delta_{ij}\varepsilon_j$, $\varepsilon_j = \pm 1$

(2) Geodesics through p: $y^i = a^i t$, $i = 1,\dots,n$, $a^i = \text{const}$

(3) Christoffel symbols: $\Gamma^i_{jk}(p) = 0$

• Cartan's structure equations (for a pseudo-Riemannian metric).

$\sigma = (X_i)$ an orthonormal moving frame; $\varepsilon_i = g(X_i,X_i) = \pm 1$; $(\widetilde{\theta}^i)$ dual moving coframe; $\widetilde{\theta}^i = \sigma^*\theta^i$, with $\theta = (\theta^i)$ the canonical form on the bundle of orthonormal frames; $\widetilde{\omega}^i_j = \sigma^*\omega^i_j$, with ω^i_j the connection forms; $\widetilde{\theta}_i = \varepsilon_i\widetilde{\theta}^i$, $\widetilde{\omega}_{ij} = \varepsilon_i\widetilde{\omega}^i_j$; $\widetilde{\Omega}_{ij} = \varepsilon_i\widetilde{\Omega}^i_j$:

$$\mathrm{d}\widetilde{\theta}_i = -\sum_j \widetilde{\omega}_{ij} \wedge \left(\varepsilon_j\widetilde{\theta}_j\right), \qquad \widetilde{\omega}_{ij} + \widetilde{\omega}_{ij} = 0$$

$$\mathrm{d}\widetilde{\omega}_{ij} = -\sum_k \varepsilon_k\widetilde{\omega}_{jk} \wedge \widetilde{\omega}_{ik} + \widetilde{\Omega}_{ij}$$

(in the expression $\varepsilon_j\widetilde{\theta}_j$, no sum in j; in the expression $\varepsilon_k\widetilde{\omega}_{jk}$, no sum in k)

• Metric of constant curvature c. There exist coordinate functions x^i on a neighborhood of $p \in M$ such that:

$$g = \frac{\varepsilon_i \,\mathrm{d}x^i \otimes \mathrm{d}x^i}{\left(1 - \frac{c}{4}\varepsilon_i\,(x^i)^2\right)^2}, \qquad \varepsilon_i = \pm 1$$

• Pseudo-Riemannian metric of constant curvature c in normal coordinates x^i with origin q, at $p \neq q$:

$$g = \left(\frac{x^i x^j}{er^2} + \frac{\sin^2\left(r\sqrt{ec}\right)}{ecr^2}\left((g_{ij})_q - \frac{x^i x^j}{er^2}\right)\right)\mathrm{d}x^i\,\mathrm{d}x^j$$

(signature of $g = (\varepsilon_1,\dots,\varepsilon_n)$, $\varepsilon_i = \pm 1$; $(g_{ij})_q = \varepsilon_i\delta_{ij}$; $x_i = (g_{ij})_p x^j$; $e = x_i x^i/|x_i x^i|$ if $x_i x^i \neq 0$ and $e = 0$ if $x_i x^i \neq 0$; $r = \sqrt{e x_i x^i}$)

• First variation formula for a piecewise C^∞ curve segment $\sigma\colon [a,b] \to M$ with constant speed $c > 0$ and sign ε:

$$L'(0) = \frac{\varepsilon}{c}\left\{-\int_a^b g(\sigma'',V)\,\mathrm{d}u - \sum_{i=1}^k g(\Delta\sigma'(u_i),V(u_i)) + [g(\sigma',V)]_a^b\right\}$$

($V = V(u)$ = variation vector field; $u_1 < \cdots < u_k$ breaks of σ and its variation; $\Delta\sigma'(u_i) = \sigma'(u_i^+) - \sigma'(u_i^-)$)

• Synge's formula for the second variation of the arc of a geodesic segment $\sigma\colon [a,b] \to M$ of speed $c > 0$ and sign ε:

$$L''(0) = \frac{\varepsilon}{c}\left\{\int_a^b \left(g(V'^{\perp}, V'^{\perp}\right) + g(R(V,\sigma')V,\sigma')\}\mathrm{d}u + [g(\sigma', A)]_a^b\right\}$$

($V'^{\perp}$ = component of V' perpendicular to γ; A = tranverse acceleration vector field)

• Einstein's field equations:

$$r - \frac{1}{2}sg = T$$

(g = metric tensor; r = Ricci tensor; s = scalar curvature; T = stress-energy tensor)

• Schwarzschild's metric:

$$g = -\left(1 - \frac{2m}{R}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{R}\right)^{-1}\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2),$$

$$0 \leqslant \theta < \pi, \quad 0 \leqslant \varphi < 2\pi$$

• Kerr's metric for a fast rotating planet (cylindrically symmetric gravitational field; a = angular momentum):

$$g = -\mathrm{d}t^2 + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{2Mr\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right)^2}{r^2 + a^2\cos^2\theta}$$
$$+ \left(r^2 + a^2\cos^2\theta\right)\left(\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2 - 2Mr + a^2}\right)$$

• de Sitter's metric on S^4:

$$g = \frac{1}{\left(1 + \left(\frac{r}{2R}\right)^2\right)^2}\left(\mathrm{d}r^2 + r^2\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right)\right),$$

where R = radius of S^4;

$$\sigma_x = \frac{1}{r^2}(y\,\mathrm{d}z - z\,\mathrm{d}y + x\,\mathrm{d}t - t\,\mathrm{d}x),$$
$$\sigma_y = \frac{1}{r^2}(z\,\mathrm{d}x - x\,\mathrm{d}z + y\,\mathrm{d}t - t\,\mathrm{d}y),$$
$$\sigma_z = \frac{1}{r^2}(x\,\mathrm{d}y - y\,\mathrm{d}x + z\,\mathrm{d}t - t\,\mathrm{d}z).$$

• Robertson-Walker's metric:

$$g = -\mathrm{d}t^2 + f^2(t)\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\left(1 + \frac{k}{4}(x^2 + y^2 + z^2)\right)^2}$$

(3-dimensional space is fully isotropic; $f(t)$ = (increasing) distance between two neighboring galaxies in space; $k = -1, 0, +1$)

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